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The minimax approach to the estimation of solutions to first order linear systems of ordinary differential periodic equations with inexact data.

Abstract

We consider first-order linear systems of ordinary differential equations with periodic coefficients. Supposing that right-hand sides of equations are not known and subjected to some quadratic restrictions, we obtain optimal, in certain sense, estimates of solutions to above-mentioned problems from indirect noisy observations of these solutions on a finite system of points and intervals.

Introduction

Estimation theory for systems with lumped and distributed parameters under uncertainty conditions was developed intensively during the last 30 years when essential results for ordinary and partial differential equations have been obtained. That was motivated by the fact that the realistic setting of boundary value problems describing physical processes often contains perturbations of unknown (or partially unknown) nature. In such cases the minimax estimation method proved to be useful, making it possible to obtain optimal estimates both for the unknown solutions (or right-hand sides of equations appearing in the boundary value problems) and for linear functionals from them, that is, estimates looked for in the class of linear estimates with respect to observations \[^1\], for which the maximal mean square error taken over all the realizations of perturbations from certain given sets takes its minimal value. Such estimates are called the guaranteed or minimax estimates.

Minimax estimation is studied in a big number of works; one may refer e.g. to [2]–[3] and the bibliography therein.

Let us formulate a general approach to the problem. If a state of a system is described by a linear ordinary differential equation

\[
\frac{dx(t)}{dt} = Ax(t) + Bv_1(t), \quad x(t_0) = x_0,
\]

and a function \(y(t)\) is observed in a time interval \([t_0, T]\), where \(y(t) = Hx(t) + v_2(t), x(t) \in \mathbb{C}^n, v_2 \in \mathbb{C}^m, y \in \mathbb{C}^m\), and \(A, B, H\) are known matrices, the minimax estimation problem consists in the most accurate determination of a function \(x(t)\) at the "worst"realization of unknown quantities \((x_0, v_1(\cdot), v_2(\cdot))\) taken from a certain set. N.N. Krasovskii was the first who stated this problem in [12]. Under different constraints imposed on function \(v_2(t)\) and for known function \(v_1(t)\) he proposed various methods of estimating inner products \((a, x(T))\) in the class of operations linear with respect to observations that minimize the maximal error. Later these estimates were called minimax a priori estimates (see [12], [3]).

Fundamental results concerning estimation under uncertainties were obtained by A. B. Kurzhanskii (see [3], [4]).

\[^1\] Here we understand observations of unknown solutions as the functions that are linear transformations of same solutions distorted by additive random noises.
The duality principle elaborated in [12], [3], and [2] proved its efficiency for the determination of minimax estimates [2]. According to this principle, finding minimax a priori estimates can be reduced to a certain problem of optimal control of the system adjoint to (1); this approach enabled one to obtain, under certain restrictions, recurrent equations, namely, the minimax Kalman–Bucy filter (see [2]).

The present paper is devoted to the problem of guaranteed estimation for systems described by first-order linear systems of ordinary differential periodic equations with inexact data. From indirect noisy observations of unknown solutions on a finite system of points and intervals, under quadratic restrictions on unknown right-hand sides of equations, we find the minimax estimates both for the unknown solutions and for linear functionals from them. It is proved that guaranteed estimates and estimation errors are expressed explicitly from the solutions of special systems of linear ordinary differential periodic equations, for which the unique solvability is established.

To do this, we reduce the guaranteed estimation problem to a certain optimal control problem. Solving this optimal control problem, we obtain uniquely solvable system of ODEs via whose solutions the minimax estimates are expressed.

**Preliminaries and auxiliary results**

Let vector-function \( x(t) \in \mathbb{C}^n \) be a solution of the following problem

\[
\frac{dx(t)}{dt} = A(t)x(t) + B(t)f(t), \quad t \in \mathbb{R},
\]

where \( A(t) = [a_{ij}(t)]_{i,j=1}^n \) is an \( n \times n \)-matrix and \( B(t) = [b_{ij}(t)], i = 1, \ldots, n, j = 1, \ldots, r \), is an \( n \times r \)-matrix with entries \( a_{ij}(t) \) and \( b_{ij}(t) \) which are continuous \( T \)-periodic functions, \( f(t) \in \mathbb{C}^r \) is a \( T \)-periodic vector-function such that \( f \in (L^2(0,T))^r \).

Denote by \( X(t) \) a matrix-valued function %\( X(t) = [x_1(t), \ldots, x_n(t)] \) whose columns are linearly independent solutions \( x_1(t), \ldots, x_n(t) \) of the homogeneous system

\[
\frac{dx(t)}{dt} = A(t)x(t)
\]

such that \( X(0) = E_n \), where \( E_n \) is the unit \( n \times n \)-matrix. In this case \( X(t) \) is said to be a normalized fundamental matrix of the equation (3).

Further we will assume that the following condition is valid

\[
\det(E_n - X(T)) \neq 0.
\]

Here a solution \( x(t) \) of equation (2) on the interval \( (0,T) \) is interpreted as a continuous solution of the integral equation

\[
x(t) = x(0) + \int_0^t (A(s)x(s) + B(s)f(s))ds
\]

or, equivalently, \( x(t) \) satisfies the condition \( x(0) = x(T) \) and is absolutely continuous on \([0,T]\) with its derivative \( x'(t) \) satisfying (2) on \((0,T)\) almost everywhere (except on a set of Lebesgue measure 0). Outside of the interval \([0,T]\), \( x(t) \) is supposed to be extended by periodicity to the whole real axis, i.e., \( x(t+T) = x(t) \forall t \in \mathbb{R} \).

Under the condition [4] the unique solvability of the problem

\[
\frac{dx(t)}{dt} = A(t)x(t) + B(t)f(t), \quad t \in (0,T),
\]
or, what is the same, the existence of a unique T-periodical solution of equation (2) is established, for example, in [7]-[10].

Simultaneously with problem (5), (6), the following problem: given vector-function \( g(t) \in \mathbb{C}^n \) such that \( g \in (L^2(0, T))^n, r_i \in \mathbb{C}^N, \) and \( t_i \in (0, T), \) find vector-function \( z(t) \in \mathbb{C}^n \) such that

\[
- \frac{dz(t)}{dt} = A^*(t)z(t) + g(t), \quad t \in (0, T), \quad t \neq t_i,
\]

is uniquely solvable. In fact, by Theorem 4.1 [9] problem (7)-(8) will have a unique solution then and only then

\[
\det(E_n - Z(T)) \neq 0,
\]

where \( Z(t) \) is a normalized fundamental matrix of the system

\[
- \frac{dz(t)}{dt} = A^*z(t)
\]

adjoint to (2). As is known \( Z(t) = [X^*(t)]^{-1}. \) Since \( \det(E_n - X(T)) \neq 0 \) then \( E_n - X^*(T) \) is a nonsingular matrix. Multiplying this matrix from the left by \(-[X^*(T)]^{-1}, \) we obtain that the matrix

\[
-[X^*(T)]^{-1}(E_n - X^*(T)) = E_n - Z(T)
\]

is also nonsingular, i.e., condition (2) is fulfilled.

In addition, from the results containing in chapter 4 of [9] it follows that an a priori estimate

\[
\|z(t)\|_{\mathbb{C}^n} \leq K\left\{ \int_0^T \|g(s)\|_{\mathbb{C}^n} ds + \sum_{i=1}^N |r_i| \right\} \quad \forall t \in [0, T],
\]

holds, where \( K \) is a constant not depending on \( g \) and \( r_i. \)

Further, the following assertion will be frequently used. If vector-functions \( f(t) \in \mathbb{C}^n \) and \( g(t) \in \mathbb{C}^n \) are absolutely continuous on the closed interval \([t_1, t_2],\) then the following integration by parts formula is valid

\[
\left( f(t_2), g(t_2) \right)_n - \left( f(t_1), g(t_1) \right)_n = \int_{t_1}^{t_2} \left[ \left( f(t), \frac{dg(t)}{dt} \right)_n + \left( g(t), \frac{df(t)}{dt} \right)_n \right] dt,
\]

where by \((\cdot, \cdot)_n\) we denote here and later on the inner product in \( \mathbb{C}^n. \)

**Lemma 1.** Suppose \( Q \) is a bounded positive Hermitian (self-adjoint) operator in a complex (real) Hilbert space \( H \) with bounded inverse \( Q^{-1}. \) Then, the generalized Cauchy–Bunyakovsky inequality

\[
|\langle f, g \rangle_H| \leq (Q^{-1}f, f)_H^{1/2} (Qg, g)_H^{1/2} \quad (f, g \in H)
\]

is valid. The equality sign in (13) is attained at the element

\[
g = \frac{Q^{-1}f}{(Q^{-1}f, f)_H^{1/2}}.
\]

\(^2\)Here and in what follows, by \( \Lambda^* \) we will denote the matrix complex conjugate and transpose of a matrix \( \Lambda. \)

\(^3\)That is \( (Qf, f)_H > 0 \) when \( f \neq 0. \)
Proof. Introduce Hilbert space \( \tilde{H} \) consisting of elements of \( H \) endowed with inner product

\[
(u, v)_{\tilde{H}} := (Q^{-1}u, v)_{H}^{1/2}
\]

and norm

\[
\|v\|_{\tilde{H}} := (Q^{-1}v, v)_{H}^{1/2}
\]

generated by this inner product. Due to the properties of the operator \( Q \), the above inner product is well-defined. Setting in the Cauchy–Bunyakovsky inequality

\[
(u, v)_{\tilde{H}} \leq \|u\|_{\tilde{H}}^{1/2} \|v\|_{\tilde{H}}^{1/2}
\]

we obtain

\[
(f, Qg)_{\tilde{H}} \leq \|f\|_{\tilde{H}}^{1/2} \|Qg\|_{H}^{1/2} = (Q^{-1}f, f)_{H}^{1/2}(Q^{-1}Qg, Qg)_{H}^{1/2} = (Q^{-1}f, f)_{H}^{1/2}(Qg, g)_{H}^{1/2},
\]

where the equality is attained when

\[
g = \frac{Q^{-1}f}{(Q^{-1}f, f)_{H}^{1/2}}.
\]

\[\square\]

Problem statement

In this section we study minimax estimation problems in the case of point observations and deduce equations generated the minimax estimates of functionals from periodic solutions to the problem

\[
\frac{dx(t)}{dt} = Ax(t) + B(t)f(t), \quad t \in (0, T),
\]

\[
x(0) = x(T).
\]

Let \( t_i, i = 1, \ldots, N, 0 < t_1 < \cdots < t_N < T \) be a given system of points on the closed interval \([0, T]\) with \( t_0 = 0 \) and \( t_{N+1} = T \) and let \( \Omega_j, j = 1, \ldots, M, \) be a given system of subintervals of \([0, T]\). The problem is to estimate the expression

\[
l(x) = \int_{0}^{T} (x(t), l_0(t))_n dt,
\]

from observations of the form

\[
y_i = H_i x(t_i) + \xi_i, \quad i = 1, \ldots, N,
\]

\[
y_j(t) = H_j(t)x(t) + \xi_j(t), \quad t \in \Omega_j, \quad j = 1, \ldots, M,
\]

in the class of estimates

\[
\widehat{l}(x) = \sum_{i=1}^{N} (y_i, u_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (y_j(t), u_j(t))_l dt + c,
\]

linear with respect to observations (19), (20); here \( x(t) \) is the state of a system described by the Cauchy problem (5), (6), \( l_0 \in (L^2(0, T))^n \), \( H_i \) are \( m \times n \) matrices, \( H_j(t) \) are \( l \times n \) matrices with
the entries that are continuous functions on $\Omega_j$, $u_i \in \mathbb{C}^m$, $u_j(t)$ are vector-functions belonging to $(L^2(\Omega_j))^l$, $c \in \mathbb{C}$. We suppose that $f \in G_1$, where

$$G_1 = \left\{ \tilde{f} \in (L^2(0,T))^r : \int_0^T (Q(t)(\tilde{f}(t) - f_0(t)), \tilde{f}(t) - f_0(t))_r \, dt \leq 1 \right\},$$

(22)

$\xi := (\xi_1, \ldots, \xi_N, \xi_1(\cdot), \ldots, \xi_M(\cdot)) \in G_2$, where

$$\xi_i = \left( \xi^{(i)}_1, \xi^{(i)}_2, \ldots, \xi^{(i)}_m \right)$$

are realizations of random vectors $\xi_i = \xi_i(\omega) \in \mathbb{C}^m$ and random vector-functions $\xi_j(t) = \xi_j(\omega, t) \in \mathbb{C}^l$ and $G_2$ denotes the set of random elements $\xi = (\xi_1, \ldots, \xi_N, \xi_1(\cdot), \ldots, \xi_M(\cdot))$, whose components

$$\tilde{\xi}_i = \left( \tilde{\xi}^{(i)}_1, \tilde{\xi}^{(i)}_2, \ldots, \tilde{\xi}^{(i)}_m \right)$$

are uncorrelated, have zero means, $\mathbb{E} \tilde{\xi}_i = 0$, and $\mathbb{E} \tilde{\xi}_j(\cdot) = 0$, with finite second moments $\mathbb{E} |\tilde{\xi}_i|^2$ and $\mathbb{E} |\tilde{\xi}_j(\cdot)|^2$, and unknown correlation matrices $\hat{R}_i = \mathbb{E} \tilde{\xi}_i \tilde{\xi}_i^* = [r^{(i)}_{jk}]_{j,k=1}^m$ with entries $r^{(i)}_{jk} = \mathbb{E} \tilde{\xi}_i^{(i)} \tilde{\xi}_k^{(i)}$ and unknown correlation matrices $\hat{R}_j(t, s) = \mathbb{E} \tilde{\xi}_j(t) \tilde{\xi}_j^*(s)$ satisfying the conditions

$$\sum_{i=1}^N \text{Sp} [D_i \hat{R}_i] \leq 1,$$

(23)

and

$$\sum_{j=1}^M \int_{\Omega_j} \text{Sp} [D_j(t) \hat{R}_j(t, t)] \, dt \leq 1,$$

(24)

correspondingly, where $D_i = [d^{(i)}_{jk}]_{j,k=1}^m$ and $D_j(t)$ are Hermitian positive definite $m \times m$ and $l \times l$-matrices, respectively. Here in (22), $f_0 \in (L^2(0,T))^r$ is a prescribed vector, $Q(t)$ is a Hermitian positive definite matrix.

Set $u := (u_1, \ldots, u_N, u_1(\cdot), \ldots, u_M(\cdot)) \in \mathbb{C}^{N \times m} \times (L^2(\Omega_1))^l \times \cdots \times (L^2(\Omega_M))^l =: H$. Norm in space $H$ is defined by

$$\|u\|_H = \left\{ \sum_{i=1}^N \|u_i\|_2^2 + \sum_{j=1}^M \|u_j(\cdot)\|_{(L^2(\Omega_j))} \right\}^{1/2}.$$

Definition 1. The estimate

$$\hat{l}(x) = \sum_{i=1}^N (y_i, \hat{u}_i)_m + \sum_{j=1}^M \int_{\Omega_j} (y_j(t), \hat{u}_j(t))_l \, dt + \hat{c},$$

(25)

\[4\text{Sp } D := \sum_{i=1}^l d_{ii} \text{ denotes the trace of the matrix } D = \{d_{ij}\}_{i,j=1}^l\]
Lemma 2. Finding the minimax estimate of functional $\chi$ optimal control of the system (33) with the cost function $l$ in which elements $\hat{u}_i, \hat{u}_j(\cdot)$, and a number $\hat{c}$ are determined from the condition

$$\inf_{u \in H, c \in \mathbb{C}} \sigma(u, c) = \sigma(\hat{u}, \hat{c}),$$

(26)

where

$$\sigma(u, c) = \sup_{f \in G_1, \xi \in G_2} \mathbb{E}[l(\tilde{x}) - l(\hat{x})]^2,$$

$$\tilde{l}(\tilde{x}) = \sum_{i=1}^{N} (\tilde{y}_i, u_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{y}_j(t), u_j(t)) dt + c,$$

(27)

$$\tilde{y}_i = \tilde{H}_i \tilde{x}(t_i) + \tilde{\xi}_i, \quad i = 1, \ldots, N, \quad \tilde{y}_j(t) = \tilde{H}_j(t) \tilde{x}(t) + \tilde{\xi}_j(t), \quad j = 1, \ldots, M,$$

(28)

and $\tilde{x}(t)$ is the solution to the problem (3) at $f(t) = \tilde{f}(t)$, will be called the minimax estimate of expression (18).

The quantity

$$\sigma := \{\sigma(\hat{u}, \hat{c})\}^{1/2}$$

(29)

will be called the error of the minimax estimation of $l(x)$.

Main results

For any fixed $u = (u_1, \ldots, u_N, u_1(\cdot), \ldots, u_M(\cdot)) \in \mathbb{C}^{N \times m} \times (L^2(\Omega_1))^t \times \cdots \times (L^2(\Omega_M))^t = H$ introduce the vector-function $z(t; u)$ as a unique solution to the problem\footnote{Here and in what follows we assume that if a function is piecewise continuous then it is continuous from the left.}

$$- \frac{dz(t; u)}{dt} = A^*(t)z(t; u) + l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t)H_j^*(t)u_j(t), \quad t \in (0, T), \quad t \neq t_i,$$

(30)

$$\Delta z(\cdot; u) |_{t=t_i} = z(t_i + 0; u) - z(t_i; u) = H_i^* u_i, \quad i = 1, \ldots, N, \quad z(T; u) = z(0; u),$$

(31)

where $\chi_{\Omega}(t)$ is a characteristic function of the set $\Omega$.

Lemma 2. Finding the minimax estimate of functional $l(x)$ is equivalent to the problem of optimal control of the system (33) with the cost function

$$I(u) = \int_0^T (\tilde{Q}(t)z(t; u), z(t; u))_n dt + \sum_{i=1}^{N} (D_i^{-1} u_i, u_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (D_j^{-1}(t)u_j(t), u_j(t)) dt \rightarrow \inf_{u \in H},$$

(32)

where $\tilde{Q}(t) = B(t)Q^{-1}(t)B^*(t)$.

Proof. Denote by $z_i(t; u)$ restriction of function $z(t; u)$ to a subinterval $(t_{i-1}, t_i)$ of the interval $(0, T)$ and extend it from this subinterval to the ends $t_{i-1}$ and $t_i$ by continuity. Then

$$L^* z_1(t; u) = l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t)H_j^*(t)u_j(t), \quad 0 =: t_0 < t < t_1, \quad z_2(t_1; u) = z_1(t_1; u) + H_1^* u_1,$$

$$L^* z_2(t; u) = l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t)H_j^*(t)u_j(t), \quad t_1 < t < t_2, \quad z_3(t_2; u) = z_2(t_2; u) + H_2^* u_2,$$

$$L^* z_3(t; u) = l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t)H_j^*(t)u_j(t), \quad t_2 < t < T, \quad z(T; u) = z(T; u) + H_T^* u_T.$$
\[ L^* z_i(t; u) = l_0(t) - \sum_{j=1}^{M} \chi \Omega_j(t) H^*_j(t) u_j(t), \quad t_{i-1} < t < t_i, \quad z_{i+1}(t_i; u) = z_i(t_i; u) + H^*_i u_i, \quad (33) \]

\[ L^* z_N(t; u) = l_0(t) - \sum_{j=1}^{M} \chi \Omega_j(t) H^*_j(t) u_j(t), \quad t_{N-1} < t < t_N, \quad z_{N+1}(t_N; u) = z_N(t_N; u) + H^*_N u_N, \]

\[ L^* z_{N+1}(t; u) = L^* z_{N+1}(t) = l_0(t) - \sum_{j=1}^{M} \chi \Omega_j(t) H^*_j(t) u_j(t), \quad t_N < t < t_{N+1} := T, \quad z_{N+1}(T; u) = z_1(0; u), \]

where

\[ z_{i+1}(t_i; u) := z_{i+1}(t_i + 0; u), \quad z_i(t_i; u) := z_i(t_i - 0; u), \quad i = 1, \ldots, N, \]

\[ L^* z_i(t; u) := -\frac{dz_i(t; u)}{dt} - A^*(t) z_i(t; u), \quad t_{i-1} < t < t_i, \quad i = 1, \ldots, N + 1. \]

Obviously,

\[
 z(t; u) = \begin{cases} 
 z_1(t; u), & 0 = t_0 < t \leq t_1; \\
 \ldots & \\
 z_i(t; u), & t_{i-1} < t \leq t_i; \\
 \ldots & \\
 z_{N+1}(t), & t_N < t \leq t_{N+1} = T, 
\end{cases}
\]

Let \( \tilde{x} \) be a solution to problem (5), (6) at \( f(t) = \tilde{f}(t) \). From (18) with \( x = \tilde{x}, \ (28) \), and the integration by parts formula (12) with \( f(t) = \tilde{x}(t), \ g(t) = z(t; u) \), we obtain

\[
 l(\tilde{x}) - \tilde{l}(\tilde{x}) = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (\tilde{x}(t), l_0(t)) dt - \sum_{i=1}^{N} (\tilde{y}_i, u_i)_m - \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{y}_j(t), u_j(t)) dt - c \\
 = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (\tilde{x}(t), l_0(t)) dt - \sum_{j=1}^{M} \chi \Omega_j(t) H^*_j(t) u_j(t) dt - \sum_{i=1}^{N} (\tilde{x}(t_i), H^*_i u_i)_n - \sum_{i=1}^{N} (\tilde{\xi}_i, u_i)_m \\
 - \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{\xi}_j(t), u_j(t)) dt - c \\
 = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{dz_i(t; u)}{dt} - A^*(t) z_i(t; u) \right) dt - \sum_{i=1}^{N} (\tilde{x}(t_i), H^*_i u_i)_n - \sum_{i=1}^{N} (\tilde{\xi}_i, u_i)_m \\
 - \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{\xi}_j(t), u_j(t)) dt - c \\
 = \sum_{i=1}^{N+1} \left( (\tilde{x}(t_{i-1}), z_i(t_{i-1}; u))_n - (\tilde{x}(t_i), z_i(t_i; u))_n \right) + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{d\tilde{x}(t)}{dt} - A(t) \tilde{x}(t), z_i(t; u) \right)_n dt 
\]
\[-\sum_{i=1}^{N} (\bar{x}(t_i), z_{i+1}(t_i; u) - z_i(t_i; u))_n - \sum_{i=1}^{N} (\tilde{\xi}_i, u_i)_m - \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{\xi}_j(t), u_j(t))_l dt - c = (x(t_0), z_1(t_0; u))_n \]

\[-(z_1(x(t_1), t_1; u))_n + \sum_{i=2}^{N} (\bar{x}(t_{i-1}), z_i(t_{i-1}; u))_n - (\bar{x}(t_i), z_i(t_i; u))_n + (\bar{x}(t_N), z_{N+1}(t_N))_n \]

\[-(\bar{x}(t_{N+1}), z_{N+1}(t_{N+1}))_n + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (B(t)\tilde{f}(t), z_i(t; u))_n \]

\[-\sum_{i=1}^{N} (\bar{x}(t_i), z_{i+1}(t_i; u) - z_i(t_i; u))_n - \sum_{i=1}^{N} (\tilde{\xi}_i, u_i)_m - \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{\xi}_j(t), u_j(t))_l dt - c.\]

Taking into account that

\[\sum_{i=2}^{N} (\bar{x}(t_{i-1}), z_i(t_{i-1}; u))_n + (\bar{x}(t_N), z_{N+1}(t_N))_n = \sum_{i'=1}^{N-1} (\bar{x}(t_{i'}), z_{i'+1}(t_{i'}; u))_n + (\bar{x}(t_N), z_{N+1}(t_N))_n \]

\[= \sum_{i=1}^{N} (\bar{x}(t_i), z_{i+1}(t_i; u))_n,\]

from latter equalities, we have

\[l(\bar{x}) - \bar{l}(\bar{x}) = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (B(t)\tilde{f}(t), z_i(t; u))_n \]

\[= \int_{0}^{T} (B(t)\tilde{f}(t), z(t; u))_n \]

\[= \int_{0}^{T} (\tilde{f}(t), B^*(t)z(t; u))_r \]

The latter equality yields

\[\mathbb{E}[l(\bar{x}) - \bar{l}(\bar{x})] = \int_{0}^{T} (\tilde{f}(t), B^*(t)z(t; u))_r dt - c. \quad (34)\]

Taking into consideration the known relationship

\[\mathbb{D}\eta = \mathbb{E}|\eta|^2 - |\mathbb{E}\eta|^2 \quad (36)\]

that couples the variance \(\mathbb{D}\eta = \mathbb{E}|\eta - \mathbb{E}\eta|^2\) of random variable \(\eta\) with its expectation \(\mathbb{E}\eta\), in which \(\eta\) is determined by right-hand side of (34) and noncorrelatedness of \(\tilde{\xi}_i = (\tilde{\xi}_i^{(1)}, \ldots, \tilde{\xi}_i^{(m)})^T\) and \(\tilde{\xi}_j(\cdot) = (\tilde{\xi}_j^{(1)}(\cdot), \ldots, \tilde{\xi}_j^{(m)}(\cdot))^T\), from the equalities (34) and (35) we find

\[\mathbb{E}|l(\bar{x}) - \bar{l}(\bar{x})|^2 \leq \left| \int_{0}^{T} (\tilde{f}(t), B^*(t)z(t; u))_r dt - c \right|^2 + \mathbb{E}\left| \sum_{i=1}^{N} (\tilde{\xi}_i, u_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (\tilde{\xi}_j(t), u_j(t))_l dt \right|^2 \]

\[= \left| \int_{0}^{T} (\tilde{f}(t) - f_0(t), B^*(t)z(t; u))_r dt + \int_{0}^{T} (f_0(t), B^*(t)z(t; u))_r dt - c \right|^2 \]
Calculate the last term on the right-hand side of (37). Applying Lemma 1, we have

\[ \inf_{c \in \mathcal{C}} \sigma(u, c) = \inf_{c \in \mathcal{C}} \sup_{\tilde{f} \in G_1, \tilde{z} \in G_2} \mathbb{E}[l(\tilde{x}) - l(\tilde{x})]^2 = \]

\[ \inf_{c \in \mathcal{C}} \sup_{\tilde{f} \in G_1} \left| \int_0^T \left( \tilde{f}(t) - f_0(t), B^*(t)z(t; u) \right)_r dt + \int_0^T \left( f_0(t), B^*(t)z(t; u) \right)_r dt - c \right|^2 \]

\[ + \sup_{\tilde{z} \in G_2} \left( \mathbb{E} \left| \sum_{i=1}^N (\tilde{\xi}_i, u_i)_m \right|^2 + \mathbb{E} \left| \sum_{j=1}^M (\tilde{\xi}_j(t), u_j(t))_m dt \right|^2 \right) \]

(37)

Set

\[ y := \int_0^T \left( \tilde{f}(t) - f_0(t), B^*(t)z(t; u) \right)_r dt, \]

\[ d = c - \int_0^T \left( f_0(t), B^*(t)z(t; u) \right)_r dt. \]

Then Lemma 1 and (22) imply

\[ |y| \leq \left| \int_0^T (Q^{-1}(t)B^*(t)z(t; u), B^*(t)z(t; u))_r dt \right|^{1/2} \left| \int_{t_0}^{t_1} (Q(t)(\tilde{f}(t) - f_0(t)), \tilde{f}(t) - f_0(t))_r dt \right|^{1/2} \]

\[ \leq \left| \int_0^T (\tilde{Q}(t)(t; u), z(t; u))_n dt \right|^{1/2} =: l. \]

(38)

The direct substitution shows that last inequality is transformed to an equality at \( f^{(0)} \in G_1 \), where

\[ f^{(0)} = f_0 \pm \frac{Q^{-1}B^*(\cdot; u)}{(Q^{-1}B^*(\cdot; u), B^*(\cdot; u))^{1/2}_{(t_0, t_1)}}. \]

Taking into account the equality

\[ \inf_{d \in \mathcal{C}} \sup_{|y| \leq l} |y - d|^2 = l^2, \]

we find

\[ \inf_{c \in \mathcal{C}} \sup_{\tilde{f} \in G_1} \left| \int_0^T \left( \tilde{f}(t) - f_0(t), B^*(t)z(t; u) \right)_r dt + \int_0^T \left( f_0(t), B^*(t)z(t; u) \right)_r dt - c \right|^2 \]

\[ = l^2 = \int_0^T (\tilde{Q}(t)(t; u), z(t; u))_n dt, \]

(39)

where the infimum over \( c \) is attained at

\[ c = \int_0^T \left( f_0(t), B^*(t)z(t; u) \right)_r dt. \]

(40)

Calculate the last term on the right-hand side of (37). Applying Lemma 1, we have

\[ \mathbb{E} \left| \sum_{i=1}^N (\tilde{\xi}_i, u_i)_m \right|^2 \leq \mathbb{E} \left[ \sum_{i=1}^N (D^{-1}u_i, u_i)_m \cdot \sum_{i=1}^N (D_i, \tilde{\xi}_i)_m \right] \]
Transform the last factor on the right-hand side of (41):

\[
\mathbb{E} \left[ \sum_{i=1}^{N} (D^{-1}_i \xi, \tilde{\xi}_i)_m \right] = \sum_{i=1}^{N} \mathbb{E} \left( \sum_{j=1}^{m} \sum_{k=1}^{m} d_{jk}^{(i)} \tilde{s}_{k}^{(i)} \tilde{\xi}_j \right) = \sum_{i=1}^{N} \sum_{j=1}^{m} \sum_{k=1}^{m} d_{jk}^{(i)} \mathbb{E} \tilde{s}_{k}^{(i)} \tilde{\xi}_j
\]

Analogously,

\[
\mathbb{E} \left[ \sum_{i=1}^{N} (D^{-1}_i \xi, \tilde{\xi}_i)_m \right] = \sum_{i=1}^{N} \mathbb{E} \left( \sum_{j=1}^{M} \int_{\Omega_j} (D^{-1}_j(t)u_j(t), u_j(t))_i dt \right)
\]

Taking into account (23) and (24) we deduce from (41)

\[
\mathbb{E} \left| \sum_{i=1}^{N} (\xi_i(t), u_i(t))_i dt \right|^2 \leq \sum_{i=1}^{N} \int_{\Omega_j} (D^{-1}_j(t)u_j(t), u_j(t))_i dt \cdot \mathbb{E} \left[ \sum_{j=1}^{M} \int_{\Omega_j} (D_j(t)\tilde{\xi}_j(t), \tilde{\xi}_j(t))_i dt \right] \]

and

\[
\mathbb{E} \left[ \sum_{j=1}^{M} \int_{\Omega_j} (D_j(t)\tilde{\xi}_j(t), \tilde{\xi}_j(t))_i dt \right] = \sum_{j=1}^{M} \int_{\Omega_j} \mathbb{E} \left[ D_j(t)\tilde{\xi}_j(t), \tilde{\xi}_j(t) \right] dt
\]

It is not difficult to check that here, the equality sign is attained at the element

\[
\xi \equiv (\xi_1, \ldots, \xi_N, \xi_1(\cdot), \ldots, \xi_M(\cdot)) \in G_2
\]

with

\[
\xi_i = \frac{\eta D^{-1}_i u_i}{\left( \sum_{i=1}^{N} (D^{-1}_i u_i, u_j)_m \right)^{1/2}}, \quad i = 1, \ldots, N,
\]

\[
\xi_j(t) = \frac{\eta D^{-1}_j(t)u_j(t)}{\left( \sum_{j=1}^{M} \int_{\Omega_j} (D^{-1}_j(t)u_j(t), u_j(t))_i dt \right)^{1/2}}, \quad j = 1, \ldots, M,
\]

where \(\eta\) is a random variable such that \(\mathbb{E}\eta = 0\) and \(\mathbb{E}|\eta|^2 = 1\). Hence,

\[
\sup_{\xi \in G_2} \left( \mathbb{E} \left| \sum_{i=1}^{N} (\xi_i, u_i)_m \right|^2 + \mathbb{E} \left| \sum_{j=1}^{M} \int_{\Omega_j} (\xi_j(t), u_j(t))_i dt \right|^2 \right)
\]

\[
= \sum_{i=1}^{N} (D^{-1}_i u_i, u_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (D^{-1}_j(t)u_j(t), u_j(t))_i dt. \quad (42)
\]

The statement of the lemma follows now from (37), (39), (40) and (42). The proof is complete. \(\square\)

Further in the proof of Theorem 1 stated below, it will be shown that solving the optimal control problem (30) − (32) is reduced to solving some system of differential equations.
Theorem 1. The minimax estimate \( \hat{l}(x) \) of expression \( l(x) \) has the form

\[
\hat{l}(x) = \sum_{i=1}^{N} (y_i, \hat{u}_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (y_j(t), \hat{u}_j(t))_{i} dt + \hat{c} = l(\hat{x}),
\]

where

\[
\hat{u}_i = D_i H_j p(t_i), \quad i = 1, \ldots, N, \quad \hat{u}_j(t) = D_j(t) H_j(t) p(t), \quad j = 1, \ldots, M, \quad \hat{c} = \int_{0}^{T} \left( f_0(t), B^* (t) \hat{z}(t) \right)_{i} dt,
\]

and vector-functions \( p(t) \), \( \hat{z}(t) \), and \( \hat{x}(t) \) are determined from the solution of the systems of equations

\[
\begin{align*}
- \frac{d\hat{z}(t)}{dt} &= A^* (t) \hat{z}(t) + l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_j^* (t) D_j(t) H_j(t) p(t), \quad t \in (0, T), \quad t \neq t_i, \\
\Delta \hat{z} |_{t=t_i} &= \hat{z}(t_i + 0) - \hat{z}(t_i) = H_i^* D_i H_j p(t_i), \quad i = 1, \ldots, N, \quad \hat{z}(T) = \hat{z}(0), \\
\frac{dp(t)}{dt} &= A(t) p(t) + \hat{Q}(t) \hat{z}(t), \quad t \in (0, T), \quad t \neq t_i, \\
\Delta p |_{t=t_i} &= p(t_i + 0) - p(t_i) = 0, \quad i = 1, \ldots, N, \quad p(0) = p(T)
\end{align*}
\]

and

\[
\begin{align*}
- \frac{d\hat{p}(t)}{dt} &= A^* (t) \hat{p}(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_j^* (t) D_j(t) [H_j(t) \hat{x}(t) - y_j(t)], \quad t \in (0, T), \quad t \neq t_i, \\
\Delta \hat{p} |_{t=t_i} &= \hat{p}(t_i + 0) - \hat{p}(t_i) = H_i^* D_i [H_i \hat{x}(t_i) - y_i], \quad i = 1, \ldots, N, \quad \hat{p}(T) = \hat{p}(0), \\
\frac{d\hat{x}(t)}{dt} &= A(t) \hat{x}(t) + \hat{Q}(t) \hat{p}(t) + B(t) f_0(t), \quad t \in (0, T), \quad t \neq t_i, \\
\Delta \hat{x} |_{t=t_i} &= \hat{x}(t_i + 0) - \hat{x}(t_i) = 0, \quad i = 1, \ldots, N, \quad \hat{x}(0) = \hat{x}(T),
\end{align*}
\]

respectively. Problems (44) - (47) and (48) - (51) are uniquely solvable. Equations (48) - (51) are fulfilled with probability 1.

The minimax estimation error \( \sigma \) is determined by the formula

\[
\sigma = [l(p)]^{1/2}.
\]

Proof. First notice that that functional \( I(u) \) can be represented in the form

\[
I(u) = \int_{0}^{T} (\hat{Q}(t) z(t; u), z(t; u))_{n} dt + \sum_{i=1}^{N} (D_i^{-1} \hat{u}_i, \hat{u}_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (D_j^{-1}(t) u_j(t), u_j(t))_{i} dt = \tilde{I}(u) + L(u) + C,
\]
where

\[
\overline{I}(u) = \int_0^T (Q(t)\tilde{z}(t; u), \tilde{z}(t; u))_n dt + \sum_{i=1}^N (D_i^{-1}u_i, u_i)_m + \sum_{j=1}^M \int_{\Omega_j} (D_j^{-1}(t)u_j(t), u_j(t))_1 dt,
\]

\[
L(u) = \int_0^T (Q(t)z^0(t), \tilde{z}(t; u))_n dt, \quad C = \int_0^T (Q(t)z^0(t), z^0(t))_n dt,
\]

\(\tilde{z}(t; u)\) is a solution of problem (33) at \(l_0(t) = 0\) and \(z^0(t)\) is a solution of the same problem at \(u = 0\).

From (11) we obtain

\[
\|\tilde{z}(\cdot; u)\|_{(L^2(0,T))^n} \leq c_1 \|u\|_H,
\]

whence,

\[
\overline{I}(u) \leq c_2 \left(\|\tilde{z}(\cdot; u)\|_{(L^2(0,T))^n}^2 + \|u\|_H^2\right) \leq c_3 \|u\|_H^2,
\]

where \(c_1, c_2,\) and \(c_3\) are constants. This inequality means that quadratic form \(\overline{I}(u)\) is continuous in the space \(H\). Analogously we can show that \(L(u)\) is a linear continuous functional in \(H\).

It follows from here that \(I(u)\) is a continuous strictly convex functional on \(H\). Then, by Corollary 1.8.3 from [6], \(I(u)\) is a weak lower semicontinuous strictly convex functional on \(H\). Therefore, since

\[
I(u) \geq \sum_{i=1}^N (D_i^{-1}u_i, u_i)_m + \sum_{j=1}^M \int_{\Omega_j} (D_j^{-1}(t)u_j(t), u_j(t))_1 dt \geq c_4 \|u\|_H^2 \quad \forall u \in H, \quad c=\text{const},
\]

then, by Theorems 13.2 and 13.4 (see [5]), there exists one and only one element \(\hat{u} \in H\) such that \(I(\hat{u}) = \inf_{u \in H} I(u)\). Hence, for any fixed \(v \in H\) and \(\tau \in \mathbb{R}\) the functions \(s_1(\tau) := I(\hat{u} + \tau v)\) and \(s_2(\tau) := I(\hat{u} + i\tau v)\) reach their minimums at a unique point \(\tau = 0\), so that

\[
\frac{d}{d\tau} I(\hat{u} + \tau v) \bigg|_{\tau=0} = 0 \quad \text{and} \quad \frac{d}{d\tau} I(\hat{u} + i\tau v) \bigg|_{\tau=0} = 0, \quad (54)
\]

where \(i = \sqrt{-1}\). Since \(z(t; \hat{u} + \tau v) = z(t; \hat{u}) + \tau\tilde{z}(t; v)\), where \(\tilde{z}(t; v)\) is a unique solution to problem (30), (31) at \(l_0 = 0\) and \(u = v\), from (32) and (54), we obtain

\[
0 = \text{Re} \left\{ \sum_{i=1}^{N+1} \int_{l_{i-1}}^{l_i} (\bar{Q}(t)z_i(t; \hat{u}), \tilde{z}_i(t; v))_n dt + \sum_{i=1}^N (D_i^{-1}\hat{u}_i, v_i)_m + \sum_{j=1}^M \int_{\Omega_j} (D_j^{-1}(t)\hat{u}_j(t), v_j(t))_1 dt \right\},
\]

and

\[
0 = \text{Im} \left\{ \sum_{i=1}^{N+1} \int_{l_{i-1}}^{l_i} (\bar{Q}(t)z_i(t; \hat{u}), \tilde{z}_i(t; v))_n dt + \sum_{i=1}^N (D_i^{-1}\hat{u}_i, v_i)_m + \sum_{j=1}^M \int_{\Omega_j} (D_j^{-1}(t)\hat{u}_j(t), v_j(t))_1 dt \right\},
\]

where \(\tilde{z}_i(t; v)\) have the same sense as \(z_i(t; u)\), \(i = 1, \ldots, N + 1\) (see page 7). Whence,

\[
0 = \sum_{i=1}^{N+1} \int_{l_{i-1}}^{l_i} (\bar{Q}(t)z_i(t; \hat{u}), \tilde{z}_i(t; v))_n dt + \sum_{i=1}^N (D_i^{-1}\hat{u}_i, v_i)_m + \sum_{j=1}^M \int_{\Omega_j} (D_j^{-1}(t)\hat{u}_j(t), v_j(t))_1 dt. \quad (55)
\]
Let $p(t)$ be a solution of the problem

$$\frac{dp(t)}{dt} = A(t)p(t) + \tilde{Q}(t)\tilde{z}(t; \tilde{u}), \quad t \in (0, T), \quad t \neq t_i, \quad (56)$$

$$\Delta p(t) \big|_{t=t_i} = p(t_i + 0) - p(t_i) = 0, \quad i = 1, \ldots, N, \quad p(0) = p(T) \quad (57)$$

and $p_i(t)$ be restriction of $p(t)$ to $(t_{i-1}, t_i)$ extended by continuity to $t_{i-1}$ and $t_i$, $i = 1, \ldots, N + 1$, satisfying the equations

$$Lp_1(t) = \tilde{Q}(t)z_1(t; \tilde{u}), \quad 0 = t_0 < t < t_1, \quad p_1(t_0) = p_{N+1}(t_{N+1}),$$

$$Lp_2(t) = \tilde{Q}(t)z_2(t; \tilde{u}), \quad t_1 < t < t_2, \quad p_2(t_1) = p_1(t_1),$$

$$\vdots$$

$$Lp_i(t) = \tilde{Q}(t)z_i(t; \tilde{u}), \quad t_{i-1} < t < t_i, \quad p_i(t_{i-1}) = p_{i-1}(t_{i-1}), \quad i = 1, \ldots, N + 1,$$

$$Lp_{N+1}(t) = \tilde{Q}(t)z_{N+1}(t; \tilde{u}), \quad t_{N+1} < t < t_N, \quad p_N(t_N) = p_{N-1}(t_{N-1}),$$

$$Lp_N(t) = \tilde{Q}(t)z_N(t; \tilde{u}), \quad t_N < t < t_{N+1} = T, \quad p_{N+1}(t_N) = p_N(t_N), \quad (58)$$

where

$$Lp_i(t) := \frac{dp_i(t)}{dt} - A(t)p_i(t), \quad t_{i-1} < t < t_i, \quad i = 1, \ldots, N + 1.$$

Then we have

$$\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (\tilde{Q}(t)z_i(t; \tilde{u}), \tilde{z}_i(t; v))_n dt = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{dp_i(t)}{dt} - A(t)p_i(t), \tilde{z}_i(t; v) \right)_n dt$$

$$= \sum_{i=1}^{N+1} \left( (p_i(t_i), \tilde{z}_i(t_i; v))_n - (p_i(t_{i-1}), \tilde{z}_i(t_{i-1}; v))_n \right)$$

$$+ \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( p_i(t), -\frac{d\tilde{z}_i(t; v)}{dt} - A^*(t)\tilde{z}_i(t; v) \right)_n dt$$

$$= \sum_{i=1}^{N} (p_i(t_i), \tilde{z}_i(t_i; v))_n + (p_{N+1}(t_{N+1}), \tilde{z}_{N+1}(t_{N+1}; v))_n - (p_0(t_0), \tilde{z}_1(t_0; v))_n$$

$$- \sum_{i=2}^{N+1} (p_i(t_{i-1}), \tilde{z}_i(t_{i-1}; v))_n = \sum_{i=1}^{N} (p_i(t_i), \tilde{z}_i(t_i; v))_n - \sum_{i=0}^{N} (p_{i+1}(t_i), \tilde{z}_{i+1}(t_i; v))_n$$

$$- \sum_{j=1}^{M} \int_{\Omega_j} (p(t), H_j^*(t)v_j(t))_n dt + \sum_{i=1}^{N} (p_i(t_i), \tilde{z}_i(t_i; v) - \tilde{z}_{i+1}(t_i; v))_n$$

$$= -\sum_{j=1}^{M} \int_{\Omega_j} (p(t), H_j^*(t)v_j(t))_n dt - \sum_{i=1}^{N} (p_i(t_i), H_i^*v_i)_n. \quad (59)$$

From (55) and (59), we find

$$\hat{u}_i = D_iH_i p_i(t_i), \quad i = 1, \ldots, N, \quad \hat{u}_j(t) = D_j(t)H_j(t)p(t), \quad j = 1, \ldots, M. \quad (60)$$
Setting
\[ u = \hat{u} = (D_1 H_{11} p_1(t_1), \ldots, D_i H_{1i} p_i(t_i), \ldots, D_N H_{1N} p_N(t_N), \]
\[ D_1(t) H_{11}(t) p(t), \ldots, D_j(t) H_{j1}(t) p(t), \ldots, D_M(t) H_{M1}(t) p(t) \]

in (40) and (33) and denoting \( \hat{z}(t) = z(t; \hat{u}) \), we see that \( \hat{z}(t) \) and \( p(t) \) satisfy system (44) – (47); the unique solvability of this system follows from the fact that functional \( \sigma(\hat{u}, \hat{c}) \) has one minimum point \( \hat{u} \).

Now let us establish that \( \sigma = [l(p)]^{1/2} \). Substituting expression (60) to (32), we obtain
\[
\sigma(\hat{u}, \hat{c}) = I(\hat{u}) = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (\hat{Q}(t) \hat{z}_i(t), \hat{z}_i(t))_n dt
\]
\[ + \sum_{i=1}^{N} (H_{i1} p_i(t_i), D_i H_{i1} p_i(t_i))_m + \sum_{j=1}^{M} \int_{\Omega_j} (H_j(t)p(t), D_j(t)H_j(t)p(t))_l dt. \quad (61) \]

But
\[
\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (\hat{Q}(t) \hat{z}_i(t), \hat{z}_i(t))_n dt = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{dp_i(t)}{dt} - A(t)p_i(t), \hat{z}_i(t) \right)_n dt
\]
\[ = \sum_{i=1}^{N+1} \left( (p_i(t_i), \hat{z}_i(t_i))_n - (p_i(t_{i-1}), \hat{z}_i(t_{i-1}))_n \right) + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( p_i(t), -\frac{d\hat{z}_i(t)}{dt} - A^*(t)\hat{z}_i(t) \right)_n dt
\]
\[ = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} (p_i(t), l_0(t)) - \sum_{i=1}^{N} (p_i(t_i), H_{i1}^* D_i H_{i1} p_i(t_i))_n - \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_{j1}^*(t) D_j(t) H_{j1}(t)p(t)_n dt
\]
\[ = l(p) - \sum_{i=1}^{N} (H_{i1} p_i(t_i), D_i H_{i1} p_i(t_i))_m - \sum_{j=1}^{M} \int_{\Omega_j} (H_j(t)p(t), D_j(t)H_j(t)p(t))_l dt \quad (62) \]

The representation (52) follows from (61) and (62).

Prove that
\[
\overline{\overline{l(x)}} = l(\hat{x}). \quad (63) \]

We should note, first of all that unique solvability of problem (48) – (51) at realizations \( y_i, i = 1, \ldots, N \), that belong with probability 1 to the space \( \mathbb{R}^m \) can be proved similarly as to the problem (44) – (47).

Denote by \( \hat{p}(t) \) and \( \hat{x}(t) \) restrictions of \( \hat{p}(t) \) and \( \hat{x}(t) \), respectively, to \( (t_{i-1}, t_i), i = 1, \ldots, N + 1 \), extended by continuity to their ends. Using (25), (43), and (49), we have
\[
\overline{\overline{l(x)}} = \sum_{i=1}^{N} (y_i, \hat{u}_i)_m + \sum_{j=1}^{M} \int_{\Omega_j} (y_j(t), \hat{u}_j(t))_l dt + \hat{c}
\]
\[ = \sum_{i=1}^{N} (y_i, D_i H_{i1} p_i(t_i))_m + \sum_{j=1}^{M} \int_{\Omega_j} (y_j(t), D_j(t)H_j(t)p(t))_l dt + \hat{c} \]
\[
= \sum_{i=1}^{N} (H_i^* D_i y_i, p_i(t_i))_n + \int_{0}^{T} \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_j^*(t) D_j(t)y_j(t), p(t)) dt + \hat{c}
\]
\[
= \sum_{i=1}^{N} (\hat{p}_i(t_i) - \hat{p}_i(t_{i+1}) + H_i^* D_i H_i \hat{x}(t_i), p_i(t_i)) + \int_{0}^{T} \left( B^*(t) \hat{z}(t), f_0(t) \right)_r dt \\
+ \int_{0}^{T} \left( -\frac{d\hat{p}_i(t)}{dt} - A^*(t)\hat{p}_i(t) + \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_j^*(t) D_j(t)H_j(t)\hat{x}(t), p_i(t) \right)_n dt 
\]
(64)

From (44) – (47) and (48) – (51), we obtain
\[
\int_{0}^{T} \left( -\frac{d\hat{p}_i(t)}{dt} - A^*(t)\hat{p}_i(t), p(t) \right) dt = \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( -\frac{d\hat{p}_i(t)}{dt} - A^*(t)\hat{p}_i(t), p_i(t) \right)_n dt
\]
\[
= \sum_{i=1}^{N+1} \left( (\hat{p}_i(t_{i-1}), p_i(t_{i-1}))_n - (\hat{p}_i(t_i), p_i(t_i))_n \right) + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \hat{p}_i(t), \frac{dp_i(t)}{dt} - A(t)p_i(t) \right)_n dt
\]
\[
= (\hat{p}_1(t_0), p_1(t_0))_n + \sum_{i=2}^{N} (\hat{p}_i(t_{i-1}), p_i(t_{i-1}))_n - \sum_{i=1}^{N} (\hat{p}_i(t_i), p_i(t_i))_n - (\hat{p}_{N+1}(t_{N+1}), p_{N+1}(t_{N+1}))_n \\
+ \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \hat{p}_i(t), Q(t)\hat{z}_i(t) \right)_n dt 
\]
\[
= \sum_{i=1}^{N} (\hat{p}_{i+1}(t_i), p_i(t_i))_n - \sum_{i=1}^{N} (\hat{p}_i(t_i), p_i(t_i))_n + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \hat{Q}(t)\hat{p}_i(t), \hat{z}_i(t) \right)_n dt \\
= \sum_{i=1}^{N} (\hat{p}_{i+1}(t_i) - \hat{p}_i(t_i), p_i(t_i))_n \\
+ \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{d\hat{x}_i(t)}{dt} - A(t)\hat{x}_i(t), \hat{z}_i(t) \right)_n dt - \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( B(t)f_0(t), \hat{z}_i(t) \right)_n dt. 
\] (65)

But
\[
\sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \frac{d\hat{x}_i(t)}{dt} - A(t)\hat{x}_i(t), \hat{z}_i(t) \right)_n dt
\]
\[
= \sum_{i=1}^{N+1} \left( (\hat{x}_i(t_i), \hat{z}_i(t_i))_n - (\hat{x}_i(t_{i-1}), \hat{z}_i(t_{i-1}))_n \right) + \sum_{i=1}^{N+1} \int_{t_{i-1}}^{t_i} \left( \hat{x}_i(t), -\frac{d\hat{z}_i(t)}{dt} - A^*(t)\hat{z}_i(t) \right)_n dt \\
= \sum_{i=1}^{N} (\hat{x}_i(t_i), \hat{z}_i(t_i))_n - \sum_{i=2}^{N} (\hat{x}_i(t_{i-1}), \hat{z}_i(t_{i-1}))_n + \int_{t_0}^{T} (\hat{x}(t), l_0(t) - \sum_{j=1}^{M} \chi_{\Omega_j}(t) H_j^*(t) D_j(t)H_j(t)p(t))_n dt \\
\]
\[
= l(\hat{x}) + \sum_{i=1}^{N} (\hat{x}_i(t_i), \hat{z}_i(t_i) - \hat{z}_{i+1}(t_i))_n - \sum_{j=1}^{M} \int_{\Omega_j} (\hat{x}(t), H_j^*(t)D_j(t)H_j(t)p(t))_n dt \\
= l(\hat{x}) - \sum_{i=1}^{N} (\hat{x}_i(t_i), H_i^* D_i H_i p(t_i))_n - \sum_{j=1}^{M} \int_{\Omega_j} (\hat{x}(t), H_j^*(t)D_j(t)H_j(t)p(t))_n dt. 
\] (66)

The representation (63) follows from (64) – (66).
If observations are only pointwise, i.e., $H_j(t) = 0$ and $\xi_j(t) = 0$, $j = 1, \ldots, M$, in \cite{20}, the systems that generate minimax estimates take the form

$$- \frac{d\hat{z}(t)}{dt} = A^*(t)\hat{z}(t) + l_0(t), \quad t \in (0, T), \quad t \neq t_i,$$

$$\Delta\hat{z} \big|_{t=t_i} = H_i^*D_iH_p(t_i), \quad i = 1, \ldots, N, \quad \hat{z}(T) = \hat{z}(0),$$

$$\frac{dp(t)}{dt} = A(t)p(t) + \tilde{Q}(t)\hat{z}(t), \quad t \in (0, T), \quad t \neq t_i,$n

$$\Delta p \big|_{t=t_i} = 0, \quad i = 1, \ldots, N, \quad p(0) = p(T)$$

and

$$- \frac{d\hat{p}(t)}{dt} = A^*(t)\hat{p}(t), \quad t \in (0, T), \quad t \neq t_i,$$

$$\Delta \hat{p} \big|_{t=t_i} = H_i^*D_i[H_i\hat{x}(t_i) - y_i], \quad i = 1, \ldots, N, \quad \hat{p}(T) = \hat{p}(0),$$

$$\frac{d\hat{x}(t)}{dt} = A(t)\hat{x}(t) + \tilde{Q}(t)\hat{p}(t) + B(t)f_0(t), \quad t \in (0, T), \quad t \neq t_i,$$

$$\Delta\hat{x} \big|_{t=t_i} = 0, \quad i = 1, \ldots, N, \quad \hat{x}(0) = \hat{x}(T),$$

respectively. Show that in the case the determination of functions $p(t)$ and $\hat{z}(t)$ from \eqref{71} – \eqref{77} can be reduced to solving some system of linear algebraic equations. To do this, let us represent function $\hat{z}(t)$ in the form

$$\hat{z}(t) = z(t; \hat{u}) = \bar{z}^{(0)}(t) + \sum_{i=1}^{N} \bar{z}^{(i)}(t; \hat{u}),$$

where functions $\bar{z}^{(0)}(t)$ and $\bar{z}^{(i)}(t; \hat{u})$, $i = 1, \ldots, N$, solve the problems

$$- \frac{d\bar{z}^{(0)}(t)}{dt} = A^*(t)\bar{z}^{(0)}(t) + l_0(t), \quad t \in (0, T), \quad \bar{z}^{(0)}(T) = \bar{z}^{(0)}(0),$$

and

$$- \frac{d\bar{z}^{(i)}(t; \hat{u})}{dt} = A^*(t)\bar{z}^{(i)}(t; \hat{u}), \quad t \in (0, T), \quad t \neq t_i,$$

$$\bar{z}^{(i)}(t_i + 0; \hat{u}) - \bar{z}^{(i)}(t_i; \hat{u}) = H_i^*\hat{u}_i, \quad \bar{z}^{(i)}(T; \hat{u}) = \bar{z}^{(i)}(0; \hat{u}),$$

respectively. It is easy to see that

$$\bar{z}^{(0)}(t) = Z(t)(E_n - Z(T))^{-1} \int_0^T Z(T)Z^{-1}(s)l(s) \, ds + \int_0^t Z(t)Z^{-1}(s)l(s),$$

$$\bar{z}^{(i)}(t; \hat{u}) = Z(t)M_i(t)Z^{-1}(t_i)H_i^*\hat{u}_i,$$

where $Z(t) = [X^*(t)]^{-1}$, $M_i(t) = (E_n - Z(T))^{-1}Z(T) + \chi(t, T)E_n$, $\chi(t, T)(t)$ is the characteristic function of the interval $(t_i, T)$. In fact, \eqref{76} is the direct corollary of the Cauchy formula. For proof of \eqref{77}, we note that if $0 < t < t_i$ then

$$\bar{z}^{(i)}(t; \hat{u}) = Z(t)\bar{z}^{(i)}(0; \hat{u})$$
and \( \bar{z}^{(i)}(t_i - 0; \bar{u}) = Z(t_i)\bar{z}^{(i)}(0; \bar{u}) \). If \( t_i < t < T \) then

\[
\bar{z}^{(i)}(t; \bar{u}) = Z(t)Z^{-1}(t_i)\bar{z}^{(i)}(t_i + 0; \bar{u}) = Z(t)Z^{-1}(t_i)(\bar{z}^{(i)}(t_i - 0; \bar{u}) + H_i^*\bar{u}_i)
\]

\[
= Z(t)Z^{-1}(t_i)Z(t_i)\bar{z}^{(i)}(0; \bar{u}) + Z(t)Z^{-1}(t_i)H_i^*\bar{u}_i
\]

\[
= Z(t)\bar{z}^{(i)}(0; \bar{u}) + Z(t)Z^{-1}(t_i)H_i^*\bar{u}_i. 
\]  
(79)

Setting in (79) \( t = T \) we find

\[
\bar{z}^{(i)}(T; \bar{u}) = Z(T)\bar{z}^{(i)}(0; \bar{u}) + Z(T)Z^{-1}(t_i)H_i^*\bar{u}_i
\]

Due to the equality \( \bar{z}^{(i)}(T; \bar{u}) = \bar{z}^{(i)}(0; \bar{u}) \), it follows from here that

\[
\bar{z}^{(i)}(0; \bar{u}) = Z(T)\bar{z}^{(i)}(0; \bar{u}) + Z(T)Z^{-1}(t_i)H_i^*\bar{u}_i
\]

and

\[
\bar{z}^{(i)}(0; \bar{u}) = (E_n - Z(T))^{-1}Z(T)Z^{-1}(t_i)H_i^*\bar{u}_i.
\]

Substituting this expression into (78) and (79), we obtain

\[
\bar{z}^{(i)}(t; \bar{u}) = Z(t)(E_n - Z(T))^{-1}Z(T)Z^{-1}(t_i)H_i^*\bar{u}_i, \quad \text{if} \quad 0 < t < t_i 
\]  
(80)

and

\[
\bar{z}^{(i)}(t; \bar{u}) = Z(t)[(E_n - Z(T))^{-1}Z(T) + E_n]Z^{-1}(t_i)H_i^*\bar{u}_i, \quad \text{if} \quad t_i < t < T. 
\]  
(81)

Combining (80), and (81) we get (77).

Further, using the Cauchy formula, (75), (76), and (77) we obtain from equations (69) and (70) that

\[
p(t) = X(t)(E - X(T))^{-1} \int_0^T X(T)X^{-1}(s)\tilde{Q}(s)\bar{z}(s) ds + \int_0^t X(t)X^{-1}(s)l(s)\tilde{Q}(s)\bar{z}(s) ds 
\]

\[
= X(t)(E - X(T))^{-1} \int_0^T X(T)X^{-1}(s)\tilde{Q}(s)\bar{z}^{(0)}(0) ds + \int_0^t X(t)X^{-1}(s)l(s)\tilde{Q}(s)\bar{z}^{(0)}(0) ds 
\]

\[
+X(t)(E - X(T))^{-1}X(T) \sum_{k=1}^N \int_0^T X^{-1}(s)\tilde{Q}(s)Z(s)M_k(s) dsZ^{-1}(t_k)H_k^*\bar{u}_k 
\]

\[
+X(t) \sum_{k=1}^N \int_0^t X^{-1}(s)\tilde{Q}(s)Z(s)M_k(s) dsZ^{-1}(t_k)H_k^*\bar{u}_k 
\]

\[
= X(t)C_0(t) + X(t) \sum_{k=1}^N [(E - X(T))^{-1}X(T)C_k(T) + C_k(t)]Z^{-1}(t_k)H_k^*\bar{u}_k, 
\]  
(83)

where

\[
C_0(t) = (E - X(T))^{-1} \int_0^T X(T)X^{-1}(s)\tilde{Q}(s)\bar{z}^{(0)}(0) ds + \int_0^t X^{-1}(s)l(s)\tilde{Q}(s)\bar{z}^{(0)}(0) ds, 
\]

\[
C_k(t) = \int_0^t X^{-1}(s)\tilde{Q}(s)Z(s)M_k(s) ds.
\]
Setting in (83) $t = t_i$ and $\hat{u}_k = D_k H_k p(t_k)$, $i = 1, \ldots, N$, $k = 1, \ldots, N$, we arrive at the following system of linear algebraic equations for determination of unknown quantities $p(t_i)$:

$$p(t_i) = X(t_i)C_0(t_i) + X(t_i) \sum_{k=1}^{N} [(E - X(T))^{-1}X(T)C_k(T) + C_k(t_i)]Z^{-1}(t_k)H_k^*D_k H_k p(t_k)$$

or

$$p(t_i) + \sum_{k=1}^{N} \alpha_{ik} p(t_k) = b_i, \quad i = 1, \ldots, N,$$

where

$$\alpha_{ik} = -X(t_i) \sum_{k=1}^{N} [(E - X(T))^{-1}X(T)C_k(T) + C_k(t_i)]Z^{-1}(t_k)H_k^*D_k H_k, \quad b_i = X(t_i)C_0(t_i).$$

Finding $p(t_i)$ from (84) we determine $\hat{u}_i$, $\bar{z}^{(i)}(t; \hat{u})$, $i = 1, \ldots, N$, $\bar{z}^{(0)}(t)$, $\hat{z}(t)$, $p(t)$, and $c$ according to (43), (77), (76), (75), and (82), respectively.

In a similar way we can deduce a system of linear algebraic equations via whose solution the functions $\hat{x}(t)$ and $\hat{p}(t)$ satisfying (48) – (51) are expressed.

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