THE MUSKAT PROBLEM WITH $C^1$ DATA

KE CHEN, QUOC-HUNG NGUYEN, AND YIRAN XU

Abstract. In this paper we prove that the Cauchy problem of the Muskat equation is wellposed locally in time for any initial data in $C^1(R^d) \cap L^2(R^d)$.

1. Introduction

The Muskat equation is an important model in the analysis of free surface flows, which describes the dynamics of two incompressible and immiscible fluids with different densities and viscosities separated by a porous media whose velocities obey Darcy’s law (see [25], [32]). Its main feature is that it is a fractional degenerate parabolic equation. This feature is shared by several equations which have attracted a lot of attention in recent years, like the surface quasi-geostrophic equation, the Hele-Shaw equation and the fractional porous media equation.

Let us introduce the Muskat problem. We consider the dynamics of a time-dependent curve $\Sigma(t)$ separating two domains $\Omega_1(t)$ and $\Omega_2(t)$. Under the supposition that $\Sigma(t)$ is the graph of some function, we introduce the following notations

$$\Omega_1(t) = \{(x, y) \in R^d \times R; y > f(t, x)\},$$
$$\Omega_2(t) = \{(x, y) \in R^d \times R; y < f(t, x)\},$$
$$\Sigma(t) = \{(x, y) \in R^d \times R; y = f(t, x)\}.$$ Assume that each domain $\Omega_j$, $j = 1, 2$, is occupied by an incompressible fluid with constant density $\rho_j$ and denote $\rho = \rho_1 \mathbf{1}_{\Omega_1(t)} + \rho_2 \mathbf{1}_{\Omega_2(t)}$. Then the motion is determined by the incompressible porous media equations, where the velocity field $v$ is given by Darcy’s law:

$$\begin{cases}
\partial_t \rho + \operatorname{div}(\rho v) = 0, & \text{(transport equation)} \\
\operatorname{div}(v) = 0, & \text{(incompressible condition)} \\
v + \nabla(P + \rho gy) = 0, & \text{(Darcy’s law)}
\end{cases}$$

where $g > 0$ is the acceleration of gravity.

Changes of unknowns, reducing the problem to an evolution equation for the free surface parametrization, have been known for quite a long time (see [10], [26], [34]). These approaches were further developed by Córdoba and Gancedo [20] who obtained a beautiful compact formulation of the Muskat equation. Indeed, they
showed that the Muskat problem is equivalent to the following equation for the free surface elevation:

$$\partial_t f(t, x) = \frac{\rho_2 - \rho_1}{2\pi} \int_{\mathbb{R}^d} \frac{\alpha \cdot \nabla_x \Delta_\alpha f(t, x)}{\langle \Delta_\alpha f(t, x) \rangle^{d+1}} \, d\alpha,$$

where the integral is understood in the sense of principal values, $\Delta_\alpha f$ is the slope defined by $\Delta_\alpha f(x) = \frac{f(x) - f(x - a)}{|a|}$, and $\langle a \rangle = (1 + a^2)^{\frac{d}{2}}$.

At a linear level, the Muskat equation reads

$$\partial_t f(t, x) = \frac{\rho_2 - \rho_1}{2\pi} \int_{\mathbb{R}^d} \alpha \cdot \nabla_x \Delta_\alpha f(t, x) \, d\alpha - \frac{\rho_2 - \rho_1}{2} |D| f(t, x).$$

The problem is said to be in stable regimes (heavier fluid below) if $\rho_2 > \rho_1$ and unstable regimes (heavier fluid on the top) if $\rho_2 < \rho_1$. In this paper, we consider the problem in stable regimes, i.e. $\rho_2 > \rho_1$. In order to simplify the exposition we take $\frac{\rho_2 - \rho_1}{2\pi} = 1$. This leads to the equation

$$\partial_t f(t, x) = \int_{\mathbb{R}^d} \frac{\alpha \cdot \nabla_x \Delta_\alpha f(t, x)}{\langle \Delta_\alpha f(t, x) \rangle^{d+1}} \, d\alpha.$$

Recall that the Muskat equation is invariant under the change of unknowns:

$$f(t, x) \rightarrow f_\lambda(t, x) := \frac{1}{\lambda} f(\lambda t, \lambda x).$$

By a direct calculation, one verifies that the spaces $\dot{H}^{1+\frac{d}{2}}(\mathbb{R}^d)$, $\dot{W}^{1,\infty}(\mathbb{R}^d)$ are two critical spaces for the Cauchy problem of the Muskat equation (1.1).

The analysis of the Cauchy problem for the Muskat equation is now well-developed, including global existence results under mild smallness assumptions and blow-up results for some large enough initial data. Local well-posedness results can be traced back to the works of Yi [36], Ambrose [8-9], Córdoba and Córdoba, Gancedo [20-21], Córdoba, Córdoba and Gancedo [23], Cheng, Granero-Belinchón, Shkoller [16]. Local well-posedness results in the sub-critical spaces were obtained by Constantin, Gancedo, Shvydkoy and Vicol [19] for initial data in the Sobolev space $W^{2,p}(\mathbb{R})$ for some $p > 1$, Ables-Matioc [1] for $W^{s,p}$ with $s > 1 + \frac{1}{p}$, and Nguyen-Pausader [33], Matioc [30,31], Alazard-Lazar [2] for initial data in $H^s(\mathbb{R})$ with $s > 3/2$. Since the Muskat equation is parabolic, the proof of the local well-posedness results also gives global well-posedness results under a smallness assumption, see Yi [36]. The first global well-posedness results under mild smallness assumptions, namely assuming that the Lipschitz semi-norm is smaller than 1, was obtained by Constantin, Córdoba, Gancedo, Rodríguez-Piazza and Strain [17,19] (see also [11]). They also proved the existence of global classical solution when the initial data $f_0 \in H^1(\mathbb{R})$ with the Wiener norm $||f_0||_{L^{1,1}} = ||\xi f_0(\xi)||_{L^1(\mathbb{R})}$ less than some explicit constant (see also [17] which improved the constant to $\frac{3}{4}$). Note that there exists finite time blow up solution for the general non-graph interface (see [13,14]). However, it is possible to solve the Cauchy problem for initial data whose slope can be arbitrarily large. Deng, Lei and Lin in [24] obtained the first result in this direction, under the assumption that the initial data are monotone (see also Remark 2.2). Córdoba and Lazar in [22] proved global well-posedness for the 2D Muskat equation in critical space $H^{1+\frac{d}{2}} \cap W^{1,\infty}$ with $\dot{H}^{1+\frac{d}{2}}$ norm small, then Gancedo and Lazar [27] extended the result to 3D. Recently, Alazard and the second author [34,35] proved well-posedness for the Muskat equation with unbounded slopes. In particular, in [3
they obtained global well-posedness with $\dot{H}^{\frac{d}{2}}$ norm of initial data small and local well posedness for large data in $H^{\frac{d}{2}}$, where they used a null-type structure to compensate the degeneracy of the parabolic behavior. In 3D, the existence of global classical solutions was established when the initial data satisfies $\|f_0\|_{\mathcal{L}^1,1} \leq \frac{1}{C}$. We also note that this result has been extended in [28] to a more general scenario where the viscosity of the two fluids can be different. More recently, the global existence was proved in [12], under the assumption $\|\nabla f_0\|_{L^\infty} \leq \frac{1}{\sqrt{C}}$. The existence of self-similar solutions with small initial data can be found in [29]. We also refer interested readers to [7], [15] for other non-local parabolic equations.

At the moment, there is no result about local well-posedness for large data in $W^{1,\infty}(\mathbb{R}^d)$ or Wiener space $\mathcal{L}^{1,1}(\mathbb{R}^d)$. So it is quite interesting to study whether the problem is wellposed in the critical space $\dot{C}^1(\mathbb{R}^d)$ without any smallness assumptions on the data.

We state the main result of our paper as follows

**Theorem 1.1.** For any $c_0 > 0$, there exists $\sigma = \sigma(c_0) \in (0,1]$ such that for any initial data $f_0 \in L^2(\mathbb{R}^d) \cap W^{1,\infty}(\mathbb{R}^d)$ with $\|\nabla f_0\|_{L^\infty} \leq c_0$, and $f_0$ can be decomposed as

\begin{equation}
(1.2) \quad f_0 = f_{0,1} + f_{0,2}, \quad \text{with} \quad \|\nabla f_{0,1}\|_{L^\infty} \leq \sigma \quad \text{and} \quad f_{0,2} \in H^{10d},
\end{equation}

there exists a solution $f$ of the Cauchy problem (1.1) in $[0,T]$ for $T = T(\|f_{0,2}\|_{H^{10d}}, c_0) > 0$, satisfying

$$
\sup_{t \in [0,T]} \|f(t)\|_{L^2} \leq \|f_0\|_{L^2},
$$

$$
\sup_{t \in [0,T]} \|\nabla f(t)\|_{L^\infty} + \frac{1}{\sqrt{C}} \|f(t)\|_{\dot{C}_t^1} \leq C(\|f_{0,2}\|_{H^{10d}}, c_0).
$$

Moreover, the solution $f$ can be decomposed as

$$
\begin{align*}
f &= F_1 + F_2, \quad \text{with} \quad \|F_1\|_{L^\infty([0,T];W^{1,\infty})} \leq 10d\sigma \quad \text{and} \quad F_2 \in L^\infty([0,T], H^{10d}).
\end{align*}
$$

**Remark 1.2.** We remark that using the standard regularity theory, we can prove that the solution $f$ of (1.1) in the class $L^\infty_{loc}((0,T], C^{1+})$ will belong to $L^\infty_{loc}((0,T], C^{s})$. Therefore, the solution in Theorem 1.1 satisfies $f(t) \in C^{s}(\mathbb{R}^d)$ for any $0 < t \leq T$.

**Remark 1.3.** It is possible that $f_{0,2} \in H^s(\mathbb{R}^d)$ for $s > 1 + \frac{d}{2}$ is enough for our results. But in this paper we will not discuss this in detail.

Note that the decomposition (1.2) holds for any $f_0 \in L^2(\mathbb{R}^d) \cap \dot{C}^1(\mathbb{R}^d)$, hence we have

**Corollary 1.4.** The statement in Theorem 1.1 holds for initial data $f_0 \in L^2(\mathbb{R}^d) \cap \dot{C}^1(\mathbb{R}^d)$.

The following proposition implies that the solution in Theorem 1.1 is unique.

**Proposition 1.5.** For any $c > 0$, there exists $\sigma = \sigma(c) > 0$ such that, if $f, \tilde{f} \in L^\infty([0,T], W^{1,\infty})$ are solutions of the Muskat equation (1.1) with $\|\nabla f\|_{L^\infty_{t,x}} \leq c$ and the solution $\tilde{f}$ can be decomposed as

$$
\tilde{f} = \tilde{f}_1 + \tilde{f}_2, \quad \text{with} \quad \|\nabla \tilde{f}_1\|_{L^\infty([0,T]; W^{1,\infty})} \leq \sigma \quad \text{and} \quad \tilde{f}_2 \in L^\infty([0,T], H^{10d}).
$$

Then we have

$$
\sup_{t \in [0,T]} \|f(t) - \tilde{f}(t)\|_{L^\infty} \leq C(\|f - \tilde{f}\|_{L^\infty},
$$

where the constant $C$ depends on $c$ and $\|\nabla \tilde{f}_2\|_{L^\infty_{t,x}}.$
We note that unless specified, all the integrals in this paper are understood as principal value integrals over $\mathbb{R}^d$. We reformulate the equation as
\[
\partial_t f(t, x) = \int \frac{\alpha \cdot \nabla f(t, x) - \delta_\alpha f(t, x)}{(\Delta f(t, x))^{d+1}} \frac{d\alpha}{|\alpha|^{d+1}},
\]
where $\delta_\alpha f(t, x) = f(t, x) - f(t, x - \alpha)$.

We organize the paper as follows. In the rest of this section, we will introduce the regularized system and notations which will be used throughout the paper. In Section 2 we will establish the regularized system and notations which will be used throughout the paper. In Section 3 we will establish the regularized system and notations which will be used throughout the paper. In Section 4 we will establish the regularized system and notations which will be used throughout the paper. In Section 5 we will establish the regularized system and notations which will be used throughout the paper. Finally, we prove the uniqueness result Proposition 1.5 in Section 5. Finally, we prove the Proposition 1.6 in the appendix.

1.1. Regularization. In order to rigorously justify the computations, we want to deal with smooth solutions. To achieve this, we introduce an approximate Muskat equation depending on some parameters $\mu_1, \mu_2 \in (0, 1)$:

- Add a parabolic term of order 2 with a small viscosity of size $\mu_1$.
- Introduce a cut-off function in the singular integral to remove wave-length shorter than some parameter $\mu_2$.

More precisely, we introduce the following Cauchy problem
\[
\begin{cases}
\partial_t f(t, x) - \mu_1 \Delta f(t, x) = \int \frac{\alpha \cdot \nabla f(t, x) - \delta_\alpha f(t, x)}{(\Delta f(t, x))^{d+1}} (1 - \chi(\alpha/\mu_2)) \frac{d\alpha}{|\alpha|^{d+1}}, \\
f(0, x) = f_0(x),
\end{cases}
\]
where $\chi : \mathbb{R}^d \to [0, 1]$ is a smooth radial function such that $\chi(y) = 1$ if $0 \leq |y| \leq 1$, $\chi(y) = 0$ if $|y| \geq 2$. Denote $\chi'(y) = \partial_y \chi(y)$. It is easy to verify that $x \cdot \nabla \chi(x) = \chi'(x)|x|$. For simplicity we denote $\chi_{\mu_2}(y) = \chi\left(\frac{y}{\mu_2}\right)$. We also assume that $-2 \leq \chi'(x) \leq 0$. We have the following basic results [4, Proposition 2.1], [17, Proposition 2.3]:

**Proposition 1.6.** For any $\mu_1, \mu_2 \in (0, 1]$ and any initial data $f_0 \in L^2(\mathbb{R}^d) \cap C^1(\mathbb{R}^d)$, the Cauchy problem (1.3) has a unique global solution $f \in C^1([0, +\infty); H^\infty(\mathbb{R}^d))$.

**Proposition 1.7.** Let $f_0 \in H^{1,0}(\mathbb{R}^d)$ satisfy $\|f_0\|_{H^{1,0}(\mathbb{R}^d)} \leq c_0$. There exists $T = T(c_0)$ such that for any $\mu_1, \mu_2 \in (0, 1]$, the Cauchy problem (1.3) has a unique classical solution $f$ in $(0, T]$ satisfying $f(t) \in C^\infty(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ for any $t \in (0, T]$. In particular,
\[
\sup_{t \in [0, T]} \|f(t)\|_{H^{1,0}} \leq C(c_0).
\]

**Remark 1.8.** Let $x_t$ satisfy $f(t, x_t) = \inf_x f(t, x)$. Then we have $\nabla f(x_t) = 0$, $\Delta f(x_t) \leq 0$ and $\delta_\alpha f(x_t) \geq 0$. From (1.3) one has $\frac{df(x_t)}{dt} \leq 0$. A similar argument holds for $\inf_x f(t, x)$. Hence the $L^\infty$ norm of the solution is non-increasing. By standard interpolation, $\|f\|_{L^\infty}$ can be controlled by $\|f_0\|_{L^2}$ and $\|\nabla f_0\|_{L^\infty}$.
1.2. Notations. From Proposition 1.7 there exists a smooth solution of the equation (1.3) in $[0, T^\ast]$ with smooth initial data $f_{0,2}$, we denote it by $F_2$. We also denote $f$ as the solution with initial data $f_0$. Without loss of generality, we assume there exists a constant $R > 1$ depending on $\sigma$ such that

\begin{equation}
\sup_{t \in [0,T^\ast]} \|F_2(t)\|_{H^{10d}} \leq R.
\end{equation}

Denote

$$F_1 = f - F_2.$$  

The main idea in this paper is to estimate $F_1$. For any $t \in [0, T^\ast]$, denote

\begin{equation}
M_j(t) = \partial_j F_1(t, x_{t,j}) = \sup_x \partial_j F_1(t, x),
\end{equation}

\begin{equation}
m_j(t) = -\partial_j F_1(t, \tilde{x}_{t,j}) = \sup_x (-\partial_j F_1(t, x)), \quad j = 1, \ldots, d.
\end{equation}

We set

\begin{equation}
A(t) = \sum_{j=1}^d (|m_j(t)| + |M_j(t)|),
\end{equation}

\begin{equation}
B_j(t) = \int \frac{\Delta_\alpha \partial_j F_1(t, x_{t,j})}{(\Delta_\alpha f)^{d+1}} \left(1 - \chi_\alpha \left(\frac{\alpha}{\mu_2}\right)\right) \frac{d\alpha}{|\alpha|^d}.
\end{equation}

The key estimate of the Lipschitz norm is that for any $t \in (0, T^\ast]

\begin{equation}
\frac{dM_j}{dt} + \frac{1}{2} B_j \leq C_0 R^3(1 + A) + C_1 AB_j + CRA[\log(2 + \|\nabla F_1\|_{C^{d+1}})],
\end{equation}

where $C_0 = C(\|f_0\|_{L^2}, \|\nabla f_0\|_{L^\infty})$ and $C_1, C$ are constants depend on dimension $d$. We note that we can estimate $m_j(t)$ in a similar manner. The above estimate is established in Section 2.2. To control $\log(2 + \|\nabla F_1\|_{C^{d+1}})$ in $L^1$ in time, in Section 3 we use ideas in [3–6] to obtain an energy inequality (see (3.4)) in $H^s$ with $s$ large.

We note that without specified, we use $C$ to denote constants only depend on the dimension $d$, the value of $C$ may be different from line to line. We introduce the notation $a \lesssim b$, which means that there exists a constant $C$ such that $a \leq Cb$. We denote $a \lesssim_m b$ if the implicit constant also depends on $m$. We introduce the following elementary inequality which will be used frequently in our estimates.

**Lemma 1.9.** For any $a, b \in \mathbb{R}$ and any $m \geq 1$, we have the following inequalities

\begin{equation}
\left| \frac{a}{(a)^m} - \frac{b}{(b)^m} \right| \lesssim_m |a - b|.
\end{equation}

**Proof.** It is easy to verify that functions $\frac{1}{(z)^m}$ and $\frac{1}{(z)^{m+3}}$ are both uniformly Lipschitz on $\mathbb{R}$. Then we get the results. \qed

2. A priori estimates

In this section we establish a priori estimates for smooth solutions of the regularized Muskat equation (1.3).
2.1. Estimate of the $L^2$ norm.

**Lemma 2.1.** Assume $0 < \mu_2 \ll \mu_1 < 1$, let $f$ be a solution of the Cauchy problem \((1.3)\) with initial data $f_0$, then for any $t > 0$, there holds

\[
\|f(t)\|_{L^2} \leq e^t \|f_0\|_{L^2}.
\]

**Proof.** We refer the readers to \([17]\) and \([18]\) for the $L^2$ maximum principle of the original Muskat equation. For simplicity, we denote $\tilde{\chi}_{\mu_2} = 1 - \chi_{\left(\frac{\alpha}{\mu_2}\right)}$. We can rewrite \((1.3)\) as

\[
\partial_t f(x) - \mu_1 \Delta f(x) = \int (x - \alpha) \cdot \nabla_x (\Delta_{x - \alpha} f(x)) \tilde{\chi}_{\mu_2} (x - \alpha) \frac{d\alpha}{|x - \alpha|^d}
\]

with $G(a) = \int_0^a \frac{1}{(s + a)^d} ds$.

We multiply the above equation by $f$, integrate over $dx$, and use integration by parts to observe

\[
\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 + \mu_1 \|\nabla f\|_{L^2}^2 = - \int G(\Delta_{x - \alpha} f(x)) (x - \alpha) \cdot \nabla_x f(x) \tilde{\chi}_{\mu_2} (x - \alpha) \frac{d\alpha}{|x - \alpha|^d} 
+ \int \mu_2^{-1} \chi' \left( \frac{x - \alpha}{\mu_2} \right) G(\Delta_{x - \alpha} f(x)) \frac{d\alpha}{|x - \alpha|^d} dx 
=: K_1 + K_2.
\]

Define the function $H$ by $H(a) = \int_0^a G(s) ds$. It is easy to observe that

\[
K_1 = - \int (x - \alpha) \cdot \nabla_x H(\Delta_{x - \alpha} f(x)) \tilde{\chi}_{\mu_2} (x - \alpha) \frac{d\alpha}{|x - \alpha|^d} dx 
- \int G(\Delta_{x - \alpha} f(x)) \Delta_{x - \alpha} f(x) \tilde{\chi}_{\mu_2} (x - \alpha) \frac{d\alpha}{|x - \alpha|^d} dx.
\]

Integrate by parts we obtain

\[
K_1 = \int [H(\Delta_{\alpha} f(x)) - G(\Delta_{\alpha} f(x)) \Delta_{\alpha} f(x)] \tilde{\chi}_{\mu_2} (\alpha) \frac{d\alpha}{|\alpha|^{d-1}} dx 
- \int \mu_2^{-1} \chi' \left( \frac{\alpha}{\mu_2} \right) H(\Delta_{\alpha} f(x)) \frac{d\alpha}{|\alpha|^{d-2}} dx.
\]

Since $\sup_s (H(s) - sG(s)) \leq 0$, one gets

\[
K_1 \leq - \int \mu_2^{-1} \chi' \left( \frac{\alpha}{\mu_2} \right) H(\Delta_{\alpha} f(x)) \frac{d\alpha}{|\alpha|^{d-2}} dx.
\]

Then we have

\[
\frac{1}{2} \frac{d}{dt} \|f\|_{L^2}^2 + \mu_1 \|\nabla f\|_{L^2}^2 \leq \int \mu_2^{-1} \chi' \left( \frac{\alpha}{\mu_2} \right) \left| G(\Delta_{\alpha} f(x)) f(x) - H(\Delta_{\alpha} f(x)) \right| \frac{d\alpha}{|\alpha|^{d-2}} dx.
\]

Note that $G(s)$ is an odd function, by a change of variable we have

\[
\int \mu_2^{-1} \chi' \left( \frac{\alpha}{\mu_2} \right) G(\Delta_{\alpha} f(x)) f(x) \frac{d\alpha}{|\alpha|^{d-1}} = \frac{1}{2} \int \mu_2^{-1} \chi' \left( \frac{\alpha}{\mu_2} \right) G(\Delta_{\alpha} f(x)) \Delta_{\alpha} f(x) \frac{d\alpha}{|\alpha|^{d-2}}.
\]
Then we combine \( \sup_s (H(s) - sG(s)) \leq 0 \) and the fact that \( -2 \leq \chi' \left( \frac{\alpha}{\mu_2} \right) \leq 0 \) to conclude that
\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{L^2} + \mu_1 \| \nabla f \|^2_{L^2} \lesssim \int_{|\alpha| \leq 2\mu_2} \mu_2^{-1} G(\Delta_\alpha f(x)) \, d\alpha \frac{d\alpha}{|\alpha|^{d-2}} \\
\lesssim \mu_2 \int_{|\alpha| \leq 2\mu_2} |\delta_\alpha f(x)|^2 \frac{d\alpha}{|\alpha|^{d+2}} \\
\lesssim \mu_2 \| f \|^2_{H^{\frac{d}{2}}}
\]
Combining this with Sobolev interpolation inequality and Young’s inequality we obtain
\[
\frac{1}{2} \frac{d}{dt} \| f \|^2_{L^2} + \mu_1 \| f \|^2_{H^1} \leq C \mu_2 \| f \|^2_{H^1} + \frac{1}{2} \| f \|^2_{L^2}.
\]
Note that \( \mu_2 \ll \mu_1 \), we can absorb the contribution of \( C \mu_2 \| f \|^2_{H^1} \) by the left hand side, then
\[
\frac{d}{dt} \| f \|^2_{L^2} \leq \| f \|^2_{L^2}.
\]
By Gronwall’s inequality, we get (2.1). □

2.2. Estimate of the Lipschitz norm. In this subsection, we will prove the Lipschitz estimate (1.8).

For simplicity, we denote \( \tilde{\chi}_{\mu_2}(\alpha) = 1 - \chi \left( \frac{\alpha}{\mu_2} \right) \) and \( d\eta(\alpha) = \tilde{\chi}_{\mu_2}(\alpha) \frac{d\alpha}{|\alpha|} \). We define
\[
E_\alpha f(x) = \hat{\alpha} \cdot \nabla f(x) - \Delta_\alpha f(x).
\]
Assume \( t \in (0, T^*) \). By taking one derivative \( \partial_j = \partial_{x_j} \) in equation (1.3) we obtain
\[
\partial_i \partial_j f - \mu_1 \partial_j \Delta f = \int \frac{\hat{\alpha} \cdot \nabla \partial_j f}{\langle \Delta_\alpha f \rangle^{d+1}} \, d\eta(\alpha) - \int \frac{\Delta_\alpha \partial_j f}{\langle \Delta_\alpha f \rangle^{d+1}} \, d\eta(\alpha) \\
- (d+1) \int \frac{E_\alpha \Delta_\alpha f \partial_j \Delta_\alpha f}{\langle \Delta_\alpha f \rangle^{d+3}} \, d\eta(\alpha),
\]
where we denote \( \hat{\alpha} = \frac{\alpha}{|\alpha|} \). Now we look at the above equation with \( x = x_{t,j} \), where \( x_{t,j} \) is defined in (1.5). Recall the definition of \( B_j \) in (1.4), we obtain
\[
\frac{dM_j}{dt} + B_j \leq \int \frac{E_\alpha \partial_j F_2}{\langle \Delta_\alpha f \rangle^{d+1}} \, d\eta(\alpha) - (d+1) \int \frac{E_\alpha F_2 \Delta_\alpha f \partial_j \Delta_\alpha f}{\langle \Delta_\alpha f \rangle^{d+3}} \, d\eta(\alpha) \\
- (d+1) \int \frac{E_\alpha F_1 \Delta_\alpha f \partial_j \Delta_\alpha f}{\langle \Delta_\alpha f \rangle^{d+3}} \, d\eta(\alpha) - \frac{dF_2}{dt} + \mu_1 \partial_j \Delta F_2.
\]
where we used the fact that \( \nabla \partial_j F_1(t, x_{t,j}) = 0 \) and \( -\mu_1 \partial_j \Delta F_1(t, x_{t,j}) \geq 0 \). From (1.4) it is easy to check that
\[
\left| \frac{dF_2(t, x_{t,j})}{dt} \right| + |\mu_1 \partial_j \Delta F_2(t, x_{t,j})| \lesssim R.
\]
We denote the first three terms in (2.3) by \( J_1, J_2, J_3 \). For \( J_1 \), we have,
\[
\int_{|\alpha| \leq 1} \frac{|E_\alpha \partial_j F_2|}{\langle \Delta_\alpha f \rangle^{d+1}} \, d\alpha \lesssim \| F_2 \|_{L^3} \int \frac{d\alpha}{|\alpha|^{d-1}} \lesssim R.
\]
We denote the first three terms in (2.3) by \( J_1, J_2, J_3 \). For \( J_1 \), we have,
On the other hand, for $|\alpha| \geq 1$, we have
\[
\left| \int_{|\alpha| \geq 1} \frac{\hat{\alpha} \cdot \nabla \partial_j F_2}{(\Delta_{\alpha} f)^{d+1}} \, d\eta(\alpha) \right| \leq \frac{1}{2} \int_{|\alpha| \geq 1} \left| \frac{\hat{\alpha} \cdot \nabla \partial_j F_2}{(\Delta_{\alpha} f)^{d+1}} \right| \, d\eta(\alpha) \\
\lesssim R \int_{|\alpha| \geq 1} |\Delta_{\alpha} f - \Delta_{\alpha} f_1| \, d\alpha \\
\lesssim R \| f \|_{L^\infty},
\]
and
\[
\left| \int_{|\alpha| \geq 1} \frac{\Delta_{\alpha} \partial_j F_2}{(\Delta_{\alpha} f)^{d+1}} \, d\eta(\alpha) \right| \leq \| F_2 \|_{L^1} \int_{|\alpha| \geq 1} \frac{d\alpha}{|\alpha|^{d+1}} \lesssim R.
\]
By Remark 1.8 we have
\[
\| f \|_{L^\infty} \leq C_0 = C(\| f_0 \|_{L^2}, \| f_0 \|_{W^{1,\infty}}). \tag{2.4}
\]
Similarly, for any $\delta > 0$, it is easy to verify that
\[
|E_{\alpha} F_2 \Delta_{\alpha} \partial_j f| \lesssim |\alpha| R (R + \Delta_{\alpha} \partial_j F_1) 1_{|\alpha| \leq \delta} + |\alpha|^{-1} R (R + A) 1_{|\alpha| \geq \delta}.
\]
Hence we obtain
\[
J_2 \lesssim \delta R \int_{|\alpha| \leq \delta} \frac{|\Delta_{\alpha} \partial_j F_1|}{(\Delta_{\alpha} f)^{d+1}} \, d\eta(\alpha) + R^2 \int_{|\alpha| \leq \delta} \frac{d\alpha}{|\alpha|^{d+1}} + R (R + A) \int_{|\alpha| \geq \delta} \frac{d\alpha}{|\alpha|^{d+1}} \lesssim \delta R B_j + \delta R^2 + \delta^{-1} R (R + A). \tag{2.5}
\]
We can choose $\delta \leq CR^{-1}$ small enough such that the contribution of $\delta R B_j$ can be absorbed by $\frac{1}{2} B_j$ in the left hand side. For $J_3$ we have
\[
J_3 \lesssim \int_{|\alpha| \leq 1} \left| \frac{E_{\alpha} F_1 \Delta_{\alpha} \partial_j F_1}{(\Delta_{\alpha} f)^{d+1}} \right| \, d\eta(\alpha) + \left( \int_{|\alpha| \leq 1} + \int_{|\alpha| \geq 1} \right) \left| \frac{E_{\alpha} F_1 \Delta_{\alpha} \partial_j F_2}{(\Delta_{\alpha} f)^{d+1}} \right| \, d\eta(\alpha)
\]
\[
\lesssim AB_j + R \int_{|\alpha| \leq 1} |E_{\alpha} F_1| \frac{d\alpha}{|\alpha|^{d}} + RA. \tag{2.6}
\]
Combining (2.4), (2.5) and (2.6) we get
\[
\frac{dM_j}{dt} + \frac{1}{2} B_j \leq C_0 R^3 (1 + A) + C_1 AB_j + CR \int_{|\alpha| \leq 1} |E_{\alpha} F_1| \frac{d\alpha}{|\alpha|^{d}} \tag{2.7}
\]
for any $t \in (0, T^*)$, where $C_0 = C(\| f_0 \|_{L^2}, \| \nabla f_0 \|_{L^\infty})$ and $C_1, C$ are constants depend on dimension $d$. Combining (2.7) and Lemma 2.3 below we obtain (1.8).

Remark 2.2. We note that the Lipschitz estimate in 2D is simpler. In 2D, equation [24] reads
\[
\partial_t f_x + \int_{\mathbb{R}} \frac{\delta_{\alpha} f_x}{(\Delta_{\alpha} f)^{d+1}} \frac{d\alpha}{\alpha^2} = \int_{\mathbb{R}} \frac{f_{xx}}{(\Delta_{\alpha} f)^{d+1}} \frac{d\alpha}{\alpha^2} - 2 \int_{\mathbb{R}} \frac{E_{\alpha} f \Delta_{\alpha} f \delta_{\alpha} f_x}{(\Delta_{\alpha} f)^{d+1}} \frac{d\alpha}{\alpha^2},
\]
where $\Delta_{\alpha} f(x) = \frac{f(x) - f(x-\alpha)}{\alpha}$. Let $x_t \in \mathbb{R}$ such that $f_x(x_t) = \sup_{x \in \mathbb{R}} f_x(x)$. Then we have $f_{xx}(x_t) = 0$ and $\delta_{\alpha} f_x(x_t) \geq 0$. If $\| f \|_{L^p}$ is small, then $|E_{\alpha} f \Delta_{\alpha} f|$ is small. Hence the last term in the right hand side can be absorbed by the left hand side term, which leads to the desired estimate $\partial_t \| f_x \|_{L^\infty} \leq 0$. We also mention the result of [24]. For large monotone increasing initial data, we have $E_{\alpha} f(x_t) \geq 0$ and
Hence one obtains for Proposition 2.4.

\[ \nabla \]

Indeed, we can further split the integration into Proof. \[ F \]

We split the integral right hand side into small scales such that if \[ \min \]

Similar arguments hold for inf \[ \lambda \] where we take \[ \lambda = (1 + \| \nabla g \|_{C^1})^{-2} \].

We observe from the above lemma that, if we want to control the \( \hat{W}^{1,\infty} \) norm of \( F_1 \), we need to improve the regularity. Before that, we first estimate the \( L^\infty \) norm of \( \nabla f \). More precisely, we have the following result

**Proposition 2.4.** Assume \( \| \nabla f_{0,1} \|_{L^\infty} \leq \frac{1}{100(d+1)} \). There exists \( t_1 = t_1(R, \| \nabla f_0 \|_{L^\infty}) \) such that if \( \sup_{t \in [0,T_1]} \| \nabla F_1(t) \|_{L^\infty} \leq 10d \| \nabla f_{0,1} \|_{L^\infty} \) with \( 0 < T_1 \leq T^* \), then

\[ \sup_{t \in [0, \min(T_1,t_1)]} \| \nabla f(t) \|_{L^\infty} \leq 1 + 2 \| \nabla f_0 \|_{L^\infty} \].

**Proof.** Let \( y_{t,j} \) satisfy \( \partial_t f(t,y_{t,j}) = \sup_x \partial_t f(t,x) \). Then one has \( \nabla \partial_t f(y_{t,j}) = 0 \), \( \delta_x \partial_t f > 0 \), and \( \partial \Delta f(y_{t,j}) \leq 0 \). We substitute \( x = y_{t,j} \) in (2.2), then

\[ \partial_t \partial_t f + \int \frac{\delta_x \partial_t f}{\langle \Delta_x f \rangle^{d+1}} \frac{d\eta(\alpha)}{|\alpha|} \leq -(d+1) \int \frac{E_{\alpha} f \Delta_x f \Delta_x \partial_t f}{\langle \Delta_x f \rangle^{d+1}} d\eta(\alpha). \]

We split the integral right hand side into small scales \( |\alpha| \leq \kappa \) and large scales \( |\alpha| \geq \kappa \) for some \( \kappa > 0 \). As in previous discussions, we have

\[ |E_{\alpha} f \Delta_x \partial_t f| \leq C|\alpha|^{-1} \| \nabla f \|_{L^\infty}^2 \chi_{|\alpha| \geq \kappa} + (2 \| \nabla F_1 \|_{L^\infty} + \kappa R) |\Delta_x \partial_t f| \chi_{|\alpha| \leq \kappa}. \]

Hence one obtains for \( x = y_{t,j} \)

\[ \partial_t \partial_t f + \int \frac{\delta_x \partial_t f}{\langle \Delta_x f \rangle^{d+1}} \frac{d\eta(\alpha)}{|\alpha|} \leq C \kappa^{-1} \| \nabla f \|_{L^\infty}^2 + (20d \| \nabla f_{0,1} \|_{L^\infty} + \kappa R)(d+1) \int \frac{\delta_x \partial_t f}{\langle \Delta_x f \rangle^{d+1}} \frac{d\eta(\alpha)}{|\alpha|}. \]

Let \( \kappa = (4(d+1)R)^{-1} \), then \( (20d \| \nabla f_{0,1} \|_{L^\infty} + \kappa R)(d+1) \leq \frac{1}{2} \). Then, the last term of RHS can be absorbed by LHS, one obtains

\[ \partial_t \partial_t f \lesssim \| \nabla f \|_{L^\infty}^2 R. \]

Similar arguments hold for \( \inf_x \partial_t f(t,x) \). Take sum in \( j \) we get

\[ \partial_t \| \nabla f(t) \|_{L^\infty} \leq CR(\| \nabla f(t) \|_{L^\infty} + 1)^2. \]

Let \( t_1 > 0 \) satisfy \( CRt_1 = 1/(2(1 + \| \nabla f_0 \|_{L^\infty})) \). We observe that for any \( t \leq \min\{T_1,t_1\} \), one has

\[ \int_0^t \partial_t \left( \frac{-1}{1 + \| \nabla f(t) \|_{L^\infty}} \right) dt = \int_0^t \frac{\partial_t \| \nabla f(t) \|_{L^\infty}}{(\| \nabla f(t) \|_{L^\infty} + 1)^2} dt \leq CR t_1. \]
Hence we obtain
\[
\frac{1}{1 + ||\nabla f_0||_{L^\infty}} - \frac{1}{1 + ||\nabla f(t)||_{L^\infty}} \leq \frac{1}{2(1 + ||\nabla f_0||_{L^\infty})}.
\]
We get the estimate
\[
\sup_{\tau \in [0, \min\{T_1, t_1\}]} ||\nabla f(\tau)||_{L^\infty} \leq 1 + 2 ||\nabla f_0||_{L^\infty},
\]
which completes the proof.

3. Improve the Regularity

This section is devoted to improve the regularity of the solution, which helps us to control the remainder terms in the Lipschitz estimate (2.7). The main result is the following proposition

**Proposition 3.1.** For any \(r_0 > 0\), there exists \(\sigma = \sigma(r_0) \in (0, 1)\) such that, for any \(T \in [0, T^*]\), if
\[
\sup_{\tau \in [0, T]} ||\nabla f(\tau)||_{L^\infty} \leq r_0 \quad \text{and} \quad \sup_{\tau \in [0, T]} ||\nabla F_1(\tau)|| \leq \sigma,
\]
then
\[
\sup_{t \in [0, T]} t^{2(7d-2)}||\Delta^{2d} f(t)||_{L^2}^2 + \int_0^T s^{2(7d-2)}||\Delta^{2d} f(s)||_{L^2}^2 ds \leq C(r_0, R).
\]
In particular, we have
\[
\sup_{t \in [0, T]} t ||\nabla f(t)||_{C^{\frac{1}{2}}} \leq C(r_0, R),
\]
and
\[
\int_0^T \log(2 + ||\nabla F_1(t)||_{C^{\frac{1}{2}}})dt \leq C(r_0, R)T^{\frac{1}{2}}.
\]

**Proof.** Multiply equation (1.3) by test function \(2\Delta^{2d} f\), and substitute \(g = f, g_1 = F_1, g_2 = F_2\) in Proposition 3.2 below, we obtain that for any \(t \in [0, T]\), there holds
\[
\frac{d}{dt}||\Delta^{2d} f||_{L^2}^2 + 2\mu_1 ||\Delta^{2d} \nabla f||_{L^2}^2 + \int \frac{||\delta_\alpha \Delta^{2d} f(x)||^2}{\langle r_0 \rangle^{d+1}} d\eta(\alpha) - dx
\]
\[
\leq r_0 \left(\sigma \frac{\langle r_0 \rangle}{\langle r_0 \rangle} + \epsilon\right) ||\Delta^{2d} f||_{L^2}^2 + F(R + \epsilon^{-1})
\]
for any \(\epsilon \in (0, 1)\), where we denote \(F : (1, \infty) \to (1, \infty)\) to be some functions increasing and \(F(r) \to \infty\) as \(r \to \infty\). The definition of \(F\) may be different from line to line. Note that
\[
\int \frac{||\delta_\alpha \Delta^{2d} f(x)||^2}{\langle r_0 \rangle^{d+1}} d\eta(\alpha) = \frac{1}{\langle r_0 \rangle^{d+1}} ||\Delta^{2d} f||_{L^2}^2 - \int \frac{||\delta_\alpha \Delta^{2d} f(x)||^2}{\langle r_0 \rangle^{d+1}} \frac{\chi}{\langle r_0 \rangle^{\frac{d}{2}}} d\alpha dx.
\]
It is easy to verify that
\[
\int \int \frac{|\delta_\alpha \Delta^2 f(x)|^2 \chi \left( \frac{\alpha}{|\alpha|^{d+1}} \right) d\alpha}{(\ell_0)^{d+1}} \leq \mu_2^\frac{1}{2} \int |\delta_\alpha \Delta^2 f(x)|^2 \frac{d\alpha}{|\alpha|^{d+1}} \\
\lesssim \mu_2^\frac{1}{2} \|\Delta^2 f(t)\|_{H^{1/2}} \|\Delta^2 f(t)\|_{H^1} \lesssim \mu_2^\frac{1}{2} \|\Delta^2 f(t)\|_{H^{1/2}}^2 + \mu_2^\frac{1}{2} \|\Delta^2 f(t)\|_{H^1}^2,
\]
where we also used Hölder’s inequality and Young’s inequality. Note that \(\mu_2 \ll \mu_1\), hence the contribution of \(\mu_2^\frac{1}{2} \|\Delta^2 f(t)\|_{H^1}^2\) can be absorbed by the viscosity terms. Then (3.2) leads to
\[
\frac{d}{dt} \|\Delta^2 f\|_{L^2}^2 + \frac{1}{2(\ell_0)^{d+1}} \|\Delta^2 f\|_{H^{1/2}}^2 \leq C(\ell_0)(\sigma \frac{1}{\sigma + r_0} + \varepsilon + \mu_2^\frac{1}{2}) \|\Delta^2 f\|_{H^{1/2}}^2 + \mathcal{F}(r_0 + r + \varepsilon^{-1}).
\]
We can take \(\sigma, \varepsilon, \mu_2\) small enough such that
\[
C(\ell_0)(\sigma \frac{1}{\sigma + r_0} + \varepsilon + \mu_2^\frac{1}{2}) \leq \frac{1}{2(\ell_0)^{d+1}}.
\]
Then
\[
\partial_t \|\Delta^2 f\|_{L^2}^2 + \frac{1}{2(\ell_0)^{d+1}} \|\Delta^2 f\|_{H^{1/2}}^2 \leq \mathcal{F}(r_0 + R)
\]
for any \(t \leq T\). For any \(0 \leq s < t \leq T\), integrate the above inequality in time we get
\[
||\Delta^2 f(t)||_{L^2}^2 + \frac{1}{2(\ell_0)^{d+1}} \int_s^t ||\Delta^2 f(\tau)||_{H^{1/2}}^2 d\tau \lesssim ||\Delta^2 f(s)||_{L^2}^2 + \mathcal{F}(r_0 + R) T.
\]
Then we multiply the above equation by \(s^{m-1}\) and integrate for \(s \in [0, t]\) to get
\[
\sup_{t \in [0, T]} t^m ||\Delta^2 f(t)||_{L^2}^2 \leq \frac{1}{2(\ell_0)^{d+1}} \int_0^t s^{m-1} ||\Delta^2 f||_{H^{1/2}}^2 ds \leq C \int_0^T s^{m-1} ||\Delta^2 f(s)||_{L^2}^2 ds + \mathcal{F}(r_0 + R) T^{m+1}.
\]
(3.5)
Applying the standard Sobolev interpolation inequality and the \(L^2\) estimates (2.1), one has
\[
||\Delta^2 f(s)||_{L^2}^2 \lesssim ||\Delta^2 f(s)||_{H^{1/2}}^2 ||f(s)||_{\dot{H}^{1/2}}^2 \lesssim ||\Delta^2 f(s)||_{H^{1/2}}^2 ||f_0||_{L^2}^2.
\]
By Hölder’s inequality and Young’s inequality,
\[
\int_0^T s^{m-1} ||\Delta^2 f(s)||_{H^{1/2}}^2 ds \leq \frac{1}{2(\ell_0)^{d+1}} \int_0^T s^{m-1} ||\Delta^2 f(s)||_{H^{1/2}}^2 ds + C(\ell_0) \int_0^T s^{m-8d-1} ds,
\]
Note that the first term can be absorbed by the left hand side of (3.5). To make the last term finite, we choose \(m = 2(7d - 2) > 8d\), which leads to (3.1).

The Gagliardo-Nirenberg inequality implies that
\[
||\nabla f(t)||_{L^2} \lesssim ||\Delta^2 f(t)||_{L^2} ||\nabla f(t)||_{L^2} \lesssim ||\Delta^2 f(t)||_{L^2} \lesssim ||\Delta^2 f||_{L^2}^2.
\]
Combining this with (3.1) and the definition of \(R\) in (1.4), we obtain (3.2) and (3.3). This completes the proof.
Denote
\[ N(f, g) = \int \frac{\hat{\alpha} \cdot \nabla f(x) - \Delta_{\alpha} f(x)}{(\Delta_{\alpha} g)^{q+1}} d\eta(\alpha). \]
(3.6)

We have the following proposition

**Proposition 3.2.** For any function \( g = g_1 + g_2 \) with \( \|g_1\|_{Lip} \leq 1 \) and \( \|\nabla g\|_{L^\infty} \leq r_0 \), there holds
\[
\left| \left| \Delta^2 d(N(g, g)) \Delta^2 g(x) dx + \frac{1}{2} \int \frac{|\delta_{\alpha} \Delta^2 g(x)|^2}{|\hat{\alpha} \cdot \nabla g(x)|^{q+1}} d\eta(\alpha) dx \right| \right|_{H^{\frac{q}{2}}} \leq r_0 \left( \|\nabla g_1\|_{L^\infty} + \varepsilon \right) \left( \|\Delta^2 g\|_{H^{\frac{q}{2}}}^2 + F(\|g_2\|_{H^{q+\frac{q}{2}}} + \varepsilon^{-1}) \right)
\]
for any \( \varepsilon \in (0, 1) \).

We postpone the proof in the Appendix.

4. **Complete the Proof of Theorem 1.1**

**Proof.** Assume \( \|\nabla f_0\|_{L^\infty} \leq \frac{1}{800C_1 + 1} \), where \( C_1 \) is the constant in the inequality (1.8). Let \( T_1 \leq T^* \) be such that
\[
(4.1) \quad \sup_{\tau \in [0, t_1]} \|\nabla F_1(\tau)\|_{L^\infty} \leq 10d\|\nabla f_0, 1\|_{L^\infty}.
\]

By Proposition 2.4 there exists \( t_1 \) independent of \( T_1 \) such that
\[
(4.2) \quad \sup_{\tau \in [0, \min\{T_1, t_1\}]} \|\nabla f(\tau)\|_{L^\infty} \leq 1 + 2\|\nabla f_0\|_{L^\infty}.
\]

By the Lipschitz estimate (1.8), we have for any \( 0 \leq t \leq T_1 \)
\[
\frac{dM_j}{dt} + \frac{1}{2} B_j \leq C_0 R^3 (1 + A) + C_1 AB_j + CRA \log(2 + \|\nabla F_1\|_{C^{\frac{1}{2}}}) \\
\leq 2C_0 R^3 + \frac{1}{10} B_j + CR \log(2 + \|\nabla F_1\|_{C^{\frac{1}{2}}}).
\]

Here we have used the definition of \( A \) in (1.6) to get
\[
A(t) \leq 2d\|\nabla F_1(t)\|_{L^\infty} \leq 20d^2\|\nabla f_0, 1\|_{L^\infty} \leq \frac{1}{40(C_1 + 1)}.
\]

So, for any \( 0 \leq t \leq T_1 \),
\[
\frac{dM_j}{dt} \leq 2C_0 R^3 + CR \log(2 + \|\nabla F_1\|_{C^{\frac{1}{2}}}).
\]

Note that the same arguments are valid for \( m_j(t) \). Recalling the definition (1.6) and summing in \( j \) we have
\[
\frac{dA}{dt} \leq 4dC_0 R^3 + CR \log(2 + \|\nabla F_1\|_{C^{\frac{1}{2}}}).
\]
with \( \sigma \), which also satisfy the above estimates. Thus we complete the proof of Theorem 1.1.

Pass the limit \( \mu \). Combining this with (3.2) and the standard compactness argument, we are able to obtain (4.2) implies

\[
\text{Set } \tau = \min \{ T^*, t_1, t_2 \}.
\]

Then we have

\[
\tau \leq A(0) + 4dC_0R^4t + CR \int_0^t \log(2 + \| \nabla F_1(\tau) \|_{C^1}) d\tau.
\]

(4.3)

\[
\leq 2d \| \nabla f_{0,1} \|_{L^\infty} + 4dC_0R^4t + CR \int_0^t \log(2 + \| \nabla F_1(\tau) \|_{C^1}) d\tau.
\]

Let \( r_0 = 1 + 2\| \nabla f_0 \|_{L^\infty} \) and \( \sigma_1 \) be in Proposition 3.1 associated to \( r_0 \). Assume \( \| \nabla f_{0,1} \|_{L^\infty} \leq \sigma_1 \). Then by (4.3)

\[
\sup_{\tau \in [0, T]} \| \nabla F_1(\tau) \|_{L^\infty} \leq 10d \| \nabla f_{0,1} \|_{L^\infty} \leq \sigma_1.
\]

Now we can apply Proposition 3.1 to \( T = \operatorname{min}\{T_1, t_1\} \) and obtain

\[
\int_0^t \log(2 + \| \nabla F_1(\tau) \|_{C^1}) d\tau \leq C(r_0, R)t^{\frac{1}{2}}.
\]

Combining this with (4.3) yields

\[
\sup_{\tau \in [0, t]} \| \nabla F_1(\tau) \|_{L^\infty} \leq 2d \| \nabla f_{0,1} \|_{L^\infty} + 4dC_0R^4t + C(r_0, R)t^{\frac{1}{2}}.
\]

Set

\[
t_2 = \frac{\| \nabla f_{0,1} \|_{L^\infty}^2}{4 \left( C(r_0, R) + R^2 + 10d(C_0 + 1) \right)^2},
\]

then we have

\[
4dC_0R^4t_2 + C(r_0, R)t_2^{\frac{1}{2}} \leq \| \nabla f_{0,1} \|_{L^\infty}.
\]

Thus,

\[
\sup_{\tau \in [0, \min\{T_1, t_1, t_2\}]} \| \nabla F_1(\tau) \|_{L^\infty} \leq (2d + 1) \| \nabla f_{0,1} \|_{L^\infty}. \tag{4.4}
\]

Now we will prove that

\[
\sup_{\tau \in [0, \min\{T^*, t_1, t_2\}]} \| \nabla F_1(\tau) \|_{L^\infty} \leq 10d \| \nabla f_{0,1} \|_{L^\infty}.
\]

In fact, set

\[
\tau_0 = \sup \left\{ t \in [0, \min\{T^*, t_1, t_2\}] : \sup_{\tau \in [0, t]} \| \nabla F_1(\tau) \|_{L^\infty} \leq 10d \| \nabla f_{0,1} \|_{L^\infty} \right\}.
\]

If \( \tau_0 < \min\{T^*, t_1, t_2\} \), using (4.3) and (4.4) with \( T_1 = \tau_0 \), we have

\[
\sup_{\tau \in [0, \tau_0]} \| \nabla F_1(\tau) \|_{L^\infty} \leq (2d + 1) \| \nabla f_{0,1} \|_{L^\infty} < 10d \| \nabla f_{0,1} \|_{L^\infty},
\]

which contradicts the definition that \( \tau_0 \) is a supremum. Hence \( \tau_0 = \min\{T^*, t_1, t_2\} \).

Then (4.4) implies

\[
\sup_{t \in [0, \tau_0]} \| \nabla f \|_{L^\infty} \leq 1 + 2 \| \nabla f_0 \|_{L^\infty}.
\]

Combining this with (3.2) and the standard compactness argument, we are able to pass the limit \( \mu_2 \to 0 \) and then \( \mu_1 \to 0 \) to get a solution of the Cauchy problem (1.1), which also satisfy the above estimates. Thus we complete the proof of Theorem 1.1 with \( \sigma = \frac{\sigma_1}{800d^2(C_1+1)} \).
5. Uniqueness

In this section, we give a proof of Proposition 1.5.

Proof. Set \( g = f - \tilde{f} \), then we have

\[
\partial_t g = \frac{E_0 g}{(D\tilde{f})} \frac{d\alpha}{|\alpha|^d} + \int \frac{E_0 \tilde{f}}{(D\tilde{f})} \left( \frac{1}{(D\tilde{f})} \frac{1}{|\alpha|^d} - \frac{1}{(D\tilde{f})} \frac{1}{|\alpha|^d} \right) \frac{d\alpha}{|\alpha|^d}.
\]

From Lemma 1.9 we have

\[
\partial_t g \leq \int \frac{E_0 g}{(D\tilde{f})} \frac{d\alpha}{|\alpha|^d} + C \int |E_0 \tilde{f}| |D_\alpha g| \frac{d\alpha}{|\alpha|^d}.
\]

Let \( x_t \) satisfy \( g(t) := g(t, x_t) = \sup_x g(t, x) \). Then we have \( \nabla g(x_t) = 0 \) and \( \delta_a g(x_t) \geq 0 \). Hence we have for \( x = x_t \)

\[
\frac{d}{dt} g \leq \frac{\dot{C}}{2} \int |\delta_a g| \frac{d\alpha}{|\alpha|^d} \leq \frac{\dot{C}}{2} \int |\bar{E} \tilde{f} \delta_a g| \frac{d\alpha}{|\alpha|^d}.
\]

where \( \dot{C} = 1/(||\nabla f||_{L_\infty})^{d+1}. \)

Recall that \( \bar{f} \) can be decomposed that \( \bar{f} = \bar{f}_1 + \bar{f}_2 \) with \( ||\bar{f}_1||_{W^{1,\infty}} \leq \sigma \) and \( \bar{f}_2 \in L_\infty([0, T], H^{10d}) \).

Then for \( \sigma \leq \frac{C}{8C} \), we have for \( x = x_t \)

\[
\frac{d}{dt} g \leq \frac{\dot{C}}{2} \int |\delta_a g| \frac{d\alpha}{|\alpha|^d} \leq \frac{\dot{C}}{2} \int |E_0 \tilde{f}_2| \frac{d\alpha}{|\alpha|^d}.
\]

Choosing \( \epsilon_0 > 0 \) small enough, the first term on the right hand side can be absorbed by the left hand side. Then one has

\[
\frac{d}{dt} g \leq C(||\nabla \tilde{f}_2||_{L_\infty}, ||\nabla f||_{L_\infty}) ||g||_{L_\infty}.
\]

Replace \( g \) by \(-g\), a similar discussion shows that the estimate holds for \( \inf_x g(t, x) \).

Thus we can conclude that

\[
\frac{d}{dt} ||g(t)||_{L_\infty} \leq C(||\nabla \tilde{f}_2||_{L_\infty}, ||\nabla f||_{L_\infty}) ||g(t)||_{L_\infty}.
\]

Finally, combining this with Gronwall’s inequality we get Proposition 1.5. \( \square \)

6. Appendix

In this section, we will prove Proposition 3.2. We first review some elementary results about Triebel-Lizorkin spaces following Triebel.

Definition 6.1. (Triebel-Lizorkin norms)

For any integer \( m \geq 0 \), and real number \( s \in (m, m + 1) \) and \( p, q \in [1, \infty) \), the homogeneous Triebel-Lizorkin space \( F^{s}_{p,q}(\mathbb{R}^d) \) consists of those tempered distributions \( f \) whose Fourier transform is integrable on a neighborhood of the origin and such that

\[
\|f\|_{F^{s}_{p,q}(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d} |\delta_\alpha D^m f(x)|^q \frac{d\alpha}{|\alpha|^{d+q(s-m)}} \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.
\]
We also define
\[
\|f\|_{\dot{F}^{s,\infty}_{p,q}(\mathbb{R}^d)} = \left( \int_{\mathbb{R}^d} \left( \sup_{\alpha} \frac{|\delta_\alpha D^m f(x)|}{|\alpha|^{s-m}} \right)^p \, dx \right)^{\frac{1}{p}}.
\]
We note that \( \| \cdot \|_{H^s} \) and \( \| \cdot \|_{\dot{F}^{s,\infty}_{p,q}} \) are equivalent. Moreover, for any \( 2 < p < \infty \) and \( 0 < q \leq \infty \), we have
\[
\|f\|_{\dot{F}^{s,\infty}_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{H^r(\mathbb{R}^d)} \quad \text{for} \ r = s - \frac{d}{p} + \frac{d}{2}.
\]

More generally, we introduce the following Gagliardo-Nirenberg interpolation inequality for the Triebel-Lizokin spaces:

Let \( 1 < q \leq \infty \) and \( s > 0 \). There holds
\[
(6.1) \quad \|f\|_{\dot{F}^{s+\theta}_{p,q}(\mathbb{R}^d)} \lesssim \|f\|_{H^s(\mathbb{R}^d)}^\theta \|f\|_{L^\infty}^{1-\theta}
\]
for any \( \theta \in (0,1) \).

To prove Proposition 3.2, we first observe that
\[
(6.2) \quad \left| \int \Delta^{2d}(N(g,g))\Delta^{2d}g(x) \, d\eta(\alpha) \right| dx + \frac{1}{2} \int \left| \frac{|\delta_\alpha \Delta^{2d}g(x)|^2}{\langle \hat{\alpha} \cdot \nabla g(x) \rangle^{d+1}} \frac{d\eta(\alpha)}{|\alpha|} \right| dx
\]
\[
\lesssim \int \left| \frac{E_\alpha(\Delta^{2d}g)(x) \, d\eta(\alpha)}{\langle \hat{\alpha} \cdot \nabla g(x) \rangle^{d+1}} \right| dx + \frac{1}{2} \int \left| \frac{|\delta_\alpha \Delta^{2d}g(x)|^2}{\langle \hat{\alpha} \cdot \nabla g(x) \rangle^{d+1}} \frac{d\eta(\alpha)}{|\alpha|} \right| dx
\]
\[
+ \left| \int \hat{\alpha} \cdot \nabla_x V(\alpha, x) \, d\eta(\alpha) \right| dx \Delta^{2d}g(x) \, dx + \left| \int M(\alpha, x) \, d\eta(\alpha) \Delta^{2d}g(x) \, dx \right|
\]
\[
=: I_1 + I_2 + I_3 + I_4,
\]
where
\[
M(\alpha, x) = \Delta^{2d} \left( \frac{E_\alpha g(x)}{\langle \Delta_\alpha g(x) \rangle^{d+1}} \right) - \frac{E_\alpha(\Delta^{2d}g)(x)}{\langle \Delta_\alpha g(x) \rangle^{d+1}}
\]
and
\[
V(\alpha, x) = \frac{1}{\langle \Delta_\alpha g(x) \rangle^{d+1}} - \frac{1}{\langle \hat{\alpha} \cdot \nabla g(x) \rangle^{d+1}}.
\]
In fact, recall the definition of \( N \) in (3.6), direct calculation leads to
\[
\Delta^{2d}(N(g,g)) = \int E_\alpha(\Delta^{2d}g)(x) \, d\eta(\alpha) + \int E_\alpha(\Delta^{2d}g)(x) V(\alpha, x) \, d\eta(\alpha)
\]
\[
+ \int M(\alpha, x) \, d\eta(\alpha).
\]
(6.5)

Note that
\[
\int E_\alpha(\Delta^{2d}g)(x) V(\alpha, x) \, d\eta(\alpha)
\]
\[
= \int \hat{\alpha} \cdot \nabla \Delta^{2d}g(x) V(\alpha, x) \, d\eta(\alpha) - \int \Delta_\alpha(\Delta^{2d}g)(x) V(\alpha, x) \, d\eta(\alpha).
\]
Take the $L^2$ inner product of (6.5) with $\Delta^{2d}g$ and integrate by parts, one has

$$
\iint \Delta^{2d}(N(g,g))\Delta^{2d}gdx
= \int \frac{E_\alpha(\Delta^{2d}g)(x)}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} d\eta(\alpha)\Delta^{2d}g(x)dx - \frac{1}{2} \iint \tilde{\alpha} \cdot \nabla \nabla \nabla V(\alpha, x) d\eta(\alpha) |\Delta^{2d}g(x)|^2 dx
- \int \Delta_\alpha(\Delta^{2d}g)(x)V(\alpha, x) d\eta(\alpha)\Delta^{2d}g(x)dx + \iint M(\alpha, x) d\eta(\alpha)\Delta^{2d}g(x)dx,
$$

which leads to (6.2).

To simplify the notations, we set

$$
p_1 = \frac{8d - 1}{4d - 1}, \quad p_2 = \frac{32d - 4}{16d - 3}, \quad p_3 = \frac{32d - 4}{16d - 3}, \quad p_4 = 16d - 2, \quad p_5 = \frac{4(8d - 1)}{5}.
$$

We recall that $\mathcal{F} : (1, \infty) \to (1, \infty)$ denote some increasing functions and $\mathcal{F}(r) \to \infty$ as $r \to \infty$. The definition of $\mathcal{F}$ may be different from line to line. To prove Proposition 6.2, it remains to estimate the four terms in the right hand side of (6.2). We finish this in Lemma 6.2–Lemma 6.5.

**Lemma 6.2.** *(Estimate for $I_1$)*

Let $g, r_0$ as defined in Proposition 6.2 there holds

$$
\left| \iint \frac{E_\alpha(\Delta^{2d}g)(x)}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} d\eta(\alpha)\Delta^{2d}g(x)dx + \frac{1}{2} \iint \frac{\delta_\alpha(\Delta^{2d}g)(x)^2}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} \left| \frac{d\eta(\alpha)}{\alpha} \right| dx \right|
\lesssim r_0 (||\nabla g||_{L^\infty} + \varepsilon) ||\Delta^{2d}g||_{H^{d+\frac{1}{2}}}^2 + \mathcal{F}(||g||_{H^{d+\frac{1}{2}}} + \varepsilon^{-1})
$$

for any $\varepsilon \in (0, 1)$.

**Proof.** Set $h = \Delta^{2d}g$. Note that $\int \frac{\tilde{\alpha} \cdot \nabla h(x)}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} d\eta(\alpha) = 0$, hence

$$
\iint \frac{E_\alpha h(x)}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} d\eta(\alpha)h(x)dx = -\frac{1}{2} \iint \Delta_\alpha h(x)\delta_\alpha \left( \frac{h(\cdot)}{\langle \tilde{\alpha} \cdot \nabla g(\cdot) \rangle^{d+1}} \right)(x) d\eta(\alpha)dx.
$$

Then one has

$$
|I_1| = \left| \iint \frac{E_\alpha h(x)}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} d\eta(\alpha)h(x)dx + \frac{1}{2} \iint \frac{\delta_\alpha h(x)^2}{\langle \tilde{\alpha} \cdot \nabla g(x) \rangle^{d+1}} \left| \frac{d\eta(\alpha)}{\alpha} \right| dx \right|
\lesssim \left| \iint \delta_\alpha h(x)h(x - \alpha)\delta_\alpha \left( \frac{1}{\langle \tilde{\alpha} \cdot \nabla g(\cdot) \rangle^{d+1}} \right)(x) \left| \frac{d\eta(\alpha)}{\alpha} \right| dx \right|.
$$

By Lemma 1.9 we obtain

$$
|I_1| \lesssim \iint |\delta_\alpha h(x)||h(x - \alpha)||\delta_\alpha \nabla g(x)| \frac{d\alpha dx}{|\alpha|^{d+1}}
\lesssim \iint |\delta_\alpha b(x)||\delta_\alpha \nabla g(x)||h(x)| \frac{d\alpha dx}{|\alpha|^{d+1}} + \iint |\delta_\alpha h(x)||\delta_\alpha \nabla g(x)| \frac{d\alpha dx}{|\alpha|^{d+1}}
:= I_{1,1} + I_{1,2}.
$$

For the first term $I_{1,1}$, we apply the Hölder’s inequality to get

$$
I_{1,1} \leq ||\Delta^{2d}g||_{L^p} ||\Delta^{2d}g||_{H^{d+\frac{1}{2}}} \left( ||\nabla g_1||_{L^p \to L^{\infty}} + ||\nabla g_2||_{L^p \to L^{\infty}} \right).
$$
We apply H"older’s inequality again to $I_{1,2}$, then
\[
I_{1,2} \leq \int_{\mathbb{R}^d} |\delta_\alpha h(x)|^2 |\delta_\alpha \nabla g_1(x)| \frac{d\alpha dx}{|\alpha|^{d+1}} + \int_{\mathbb{R}^d} |\delta_\alpha h(x)|^2 |\delta_\alpha \nabla g_2(x)| \frac{d\alpha dx}{|\alpha|^{d+1}} \\
\lesssim \|\nabla g_1\|_{L^\infty} \|\Delta^{2d} g\|_{H^{\frac{d}{2}}}^{2} + \|\Delta^{2d} g\|_{F^{\frac{d}{2},4}} \|\nabla g_2\|_{F^{\frac{d}{2},4}}.
\]
By the interpolation inequality \eqref{interpolation}, one obtains for any $1 < q \leq +\infty$
\[
|\Delta^{2d} g|_{L^q} \lesssim \|\nabla g\|_{L^{\frac{d}{2}}} \|g\|_{H^{d+\frac{1}{2}}}^{\frac{d-2}{4}}, \quad \|\Delta^{2d} g\|_{F^{\frac{d}{2},q}} \lesssim \|\nabla g\|_{L^{\frac{d}{2}}} \|g\|_{H^{d+\frac{1}{2}}}^{\frac{1}{2}}, \\
|\nabla g|_{F^{\frac{d}{2},q}} \lesssim \|\nabla g\|_{L^{\infty}} \|g\|_{H^{d+\frac{1}{2}}}^{\frac{d-2}{4}}, \quad \|\nabla g\|_{F^{\frac{d}{2},q}} \lesssim \|\nabla g\|_{L^{\infty}} \|\Delta^{2d} g\|_{H^{\frac{d}{2}}}^{\frac{d-2}{4}}.
\]
Note that $\|\nabla g\|_{L^\infty} \leq r_0$, thus we have
\[
I_{1,1} \lesssim \|\nabla g\|_{L^{\frac{d}{2}}} \|g\|_{H^{d+\frac{1}{2}}}^{\frac{d-2}{4}} \left( \|\nabla g_1\|_{L^{\frac{d}{2}}} \|g_1\|_{H^{d+\frac{1}{2}}}^{\frac{1}{2}} + \|\nabla g_2\|_{L^{\frac{d}{2}}} \|g_2\|_{H^{d+\frac{1}{2}}}^{\frac{1}{2}} \right) \\
\lesssim r_0 \|g\|_{H^{d+\frac{1}{2}}}^{\frac{3d-7}{4}} \left( \|\nabla g_1\|_{L^{\infty}} \|g_1\|_{H^{d+\frac{1}{2}}^{\frac{3d-5}{2}}} + \|\nabla g_2\|_{L^{\infty}} \|g_2\|_{H^{d+\frac{1}{2}}^{\frac{3d-5}{2}}} \right) \\
\lesssim r_0 \|g\|_{H^{d+\frac{1}{2}}}^{2} \|\nabla g_1\|_{L^{\infty}} + \|g_1\|_{H^{d+\frac{1}{2}}^{\frac{3d-5}{2}}} \|\Delta^{2d} g_2\|_{H^{\frac{d}{2}}}^{\frac{3d-5}{2}}.
\]
and
\[
I_{1,2} \lesssim \|\nabla g_1\|_{L^{\infty}} \|\Delta^{2d} g\|_{H^{\frac{d}{2}}}^{2} + \|g_1\|_{H^{d+\frac{1}{2}}^{\frac{3d-5}{2}}} \|\Delta^{2d} g_2\|_{H^{\frac{d}{2}}}^{\frac{1}{2}}.
\]
Combining the above results with Young’s inequality we obtain
\[
I_1 \leq \left( \|\nabla g_1\|_{L^{\infty}}^{\frac{3d-5}{2}} + \varepsilon \right) \|g_1\|_{H^{d+\frac{1}{2}}}^{\frac{3d-5}{2}} + \mathcal{F}(\|g_2\|_{H^{d+\frac{1}{2}}^{\frac{3d-5}{2}}} + \varepsilon^{-1})
\]
for any $\varepsilon \in (0,1)$.

\[\square\]

**Lemma 6.3.** (Estimate for $I_2$)

Let $g, r_0$ as defined in Proposition \ref{proposition} and $V$ as defined in \eqref{definition}, there holds
\[
\left| \int_{\mathbb{R}^d} \hat{\alpha} \cdot \nabla V(\alpha,x) |\Delta^{2d} g(x)|^2 d\eta(\alpha) dx \right| \\
\lesssim r_0 (\|\nabla g_1\|_{L^{\infty}}^{\frac{3d-5}{4}} + \varepsilon) \|\Delta^{2d} g\|_{H^{\frac{d}{2}}}^{2} + \mathcal{F}(\|g_2\|_{H^{d+\frac{1}{2}}}^{\frac{3d-5}{2}} + \varepsilon^{-1})
\]
for any $\varepsilon \in (0,1)$.

**Proof.** Using Hölder’s inequality we obtain
\[
I_2 \leq \|\Delta^{2d} g\|_{L^2}^2 \left\| \int_{\mathbb{R}^d} \hat{\alpha} \cdot \nabla V(\alpha,\cdot) d\eta(\alpha) \right\|_{L^{\frac{d-1}{2}}}.
\]
Note that
\[
\int_{\mathbb{R}^d} \hat{\alpha} \cdot \nabla \left( \frac{1}{(\hat{\alpha} \cdot \nabla g)(x)^{d+1}} \right) d\eta(\alpha) = 0.
\]
Recall the definition \(6.3\) of \(V\) and Lemma \(1.3\) we directly have
\[
\left| \int \hat{\alpha} \cdot \nabla V(\alpha, x) \eta(\alpha) \right| \lesssim \int \frac{\Delta_\alpha g(x)}{(\Delta_\alpha g(x))^{d+3}} \hat{\alpha} \cdot \nabla \Delta_\alpha g(x) \eta(\alpha)
\lesssim \int |E_\alpha g(x)| \left| \Delta_\alpha \nabla g(x) \right| \frac{d\alpha}{|\alpha|^{d+1}} + \int \frac{\hat{\alpha} \cdot \nabla g(x)}{(\hat{\alpha} \cdot \nabla g(x))^{d+3}} \hat{\alpha} \cdot \nabla \Delta_\alpha g(x) \eta(\alpha)
=: L_1 + L_2.
\]
Using Hölder’s inequality one has
\[
L_1 \lesssim \left( \int |E_\alpha g(x)|^2 \frac{d\alpha}{|\alpha|^{d+1}} \right)^\frac{1}{2} \left( \int \left| \delta_\alpha \nabla g(x) \right|^2 \frac{d\alpha}{|\alpha|^{d+1}} \right)^\frac{1}{2}.
\]
By standard interpolation one has
\[
(6.9) \quad \left( \int |E_\alpha g(x)|^2 \frac{d\alpha}{|\alpha|^{d+1}} \right)^\frac{1}{2} + \left( \int \left| \delta_\alpha \nabla g(x) \right|^2 \frac{d\alpha}{|\alpha|^{d+1}} \right)^\frac{1}{2} \lesssim \|\nabla g\|_{L^\infty} \left( \sup_{\alpha} \frac{|E_\alpha g(x)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2}.
\]
Use the condition \(\|\nabla g\|_{L^\infty} \leq r_0\) we have
\[
(6.10) \quad \|L_1\|_{L^{d-1}} \lesssim r_0 \left( \sup_{\alpha} \frac{|\delta_\alpha \nabla g(\cdot)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2} \lesssim r_0 \|\nabla g\|_{F_{p_1,\infty}^{\frac{d}{2}}}^\frac{1}{2}.
\]
Similarly, we have
\[
L_2 \lesssim \left( \sup_{\alpha} \frac{|\delta_\alpha \nabla g(x)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2} \left( \sup_{\alpha} \frac{|E_\alpha \nabla g(x)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2}.
\]
Applying Hölder’s inequality again, one has
\[
(6.11) \quad \|L_2\|_{L^{d-1}} \lesssim \left( \sup_{\alpha} \frac{|\delta_\alpha \nabla g(x)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2} \left( \sup_{\alpha} \frac{|E_\alpha \nabla g(x)|}{|\alpha|^\frac{d}{2}} \right)^\frac{1}{2} \lesssim \|\nabla g\|_{L^{\infty}} \|\nabla^2 g\|_{F_{p_1,\infty}^{\frac{d}{2}}} \|\Delta^{2d} g\|_{\dot{H}^\frac{1}{2}} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}.
\]
By the Gagliardo-Nirenberg interpolation inequality \(6.1\), we have
\[
\|\nabla^2 g\|_{F_{p_1,\infty}^{\frac{d}{2}}} \lesssim \|\nabla g\|_{L^\infty}^{\frac{16d-2}{11}} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}^\frac{5}{8} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}.
\]
Combining this with \(6.7\), \(6.8\), \(6.10\) and \(6.11\), we obtain
\[
I_2 \lesssim r_0 \left( \|\nabla g_1\|_{L^{\infty}}^{\frac{2}{11}} \|\Delta^{2d} g_1\|_{H^{\frac{3}{2}}}^{\frac{16d-4}{11}} + \|\nabla g_2\|_{L^{\infty}} \|\Delta^{2d} g_2\|_{H^{\frac{3}{2}}} \right) \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}
\lesssim r_0 \|\nabla g_1\|_{L^{\infty}} \left( \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}^\frac{2}{7} + \|\Delta^{2d} g_2\|_{H^{\frac{3}{2}}} \right) + \|\Delta^{2d} g_2\|_{H^{\frac{3}{2}}} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}}.
\]
Applying Young’s inequality, we get
\[
I_2 \lesssim r_0 \left( \|\nabla g_1\|_{L^{\infty}} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}} + \|\nabla g_2\|_{L^{\infty}} \|\Delta^{2d} g\|_{H^{\frac{3}{2}}} + \mathcal{F}(\|g_2\|_{H^{4d+\frac{1}{2}}} + \varepsilon)^{-1} \right).
\]
Lemma 6.4. (Estimate for $I_3$)
Let $g$, $r_0$ as defined in Proposition 6.3 and $M$ as defined in 6.3, there holds
\[
\left| \int M(\alpha, x) d\eta(\alpha) \Delta^{2d} g(x) dx \right| 
\lesssim r_0 \left( \| \nabla g_1 \|_{L^\infty} + \varepsilon \right) \| \Delta^{2d} g \|_{H^{\frac{1}{2} + \varepsilon}}^2 + \mathcal{F}(\| g_2 \|_{H^{4d + \varepsilon} + \varepsilon^{-1}})
\]
for any $\varepsilon \in (0, 1)$.

Proof. Applying H"older’s inequality one has
\[
\left| \int M(\alpha, x) d\eta(\alpha) \Delta^{2d} g(x) dx \right| \lesssim \| \Delta^{2d} g \|_{L^{p_1}} \int M(\alpha, x) d\eta(\alpha) \left| \frac{1}{(\Delta g)^{d+1}} \right|^{\frac{m_2}{m_2}}.
\]
Recall the definition of $M(\alpha, x)$ in 6.3, we have
\[
|M(\alpha, x)| \lesssim \sum_{m_1 + m_2 = d, m_2 > 0} \sum_{k=1}^{m_2} |\alpha| \| \Delta_{\alpha} D^{1+m_1} g(x) \| \left| D^{m_2} \left( \frac{1}{(\Delta g)^{d+1}} \right) \right|^{\frac{m_2}{m_2}}.
\]
Applying H"older’s inequality one obtains
\[
\left| \int M(\alpha, x) d\eta(\alpha) \right| \left| \frac{1}{L^{\frac{d-1}{4d}}} \right| \lesssim \sum_{m_1 + m_2 = d, m_2 > 0} \sum_{k=1}^{m_2} \| D^{1+m_1} g \|_{F^{p_\beta}_{p_{\beta}, \frac{d}{2}}} \| D^\beta g \|_{F^{p_{\beta}, p_{\beta}}_{p_{\beta}, \frac{d}{2}}} \left| \frac{1}{(\Delta g)^{d+1}} \right|^{\frac{m_2}{m_2}}.
\]
where
\[
p_\beta = \frac{2(8d-1)}{2m_1 + 1}, \quad p_{\beta, p_{\beta}} = \frac{2(8d-1)m_2}{k(2m_2 - 1)}, \quad \beta = 1 - \frac{k}{2m_2}.
\]
By the interpolation inequality (6.1), we know that
\[
\| D^{1+m_1} g \|_{F^{p_{\beta}}_{p_{\beta}, \frac{d}{2}}} \lesssim \| \nabla g \|_{L^{\infty}} \| \Delta^{2d} g \|_{H^{\frac{d}{2}}}^{\frac{2m_1 + 1}{2m_1}},
\]
\[
\| D^k g \|_{F^{p_{\beta}}_{p_{\beta}, \frac{d}{2}}} \lesssim \| \nabla g \|_{L^{\infty}} \| \Delta^{2d} g \|_{H^{\frac{d}{2}}}^{\frac{(2m_2 - 1)k}{2m_2}} \lesssim \| \nabla g \|_{L^{\infty}} \| \Delta^{2d} g \|_{H^{\frac{d}{2}}}^{\frac{(2m_2 - 1)k}{2m_2}}.
\]
Then one has
\[
\left| \int M(\alpha, x) d\eta(\alpha) \right| L^{\frac{d-1}{4d}} \lesssim \| \Delta^{2d} g \|_{H^{\frac{d}{2}}}^{\frac{8d}{8d-2}} \| \nabla g \|_{L^{\infty}}^{\frac{8d-2}{8d-4}}.
\]
Combining this with (6.7), we have
\[
I_3 \lesssim \left( \| \nabla g_1 \|_{L^{\infty}} \| \Delta^{2d} g_1 \|_{H^{\frac{d}{2}}}^{\frac{8d-2}{8d}} + \| \nabla g_2 \|_{L^{\infty}} \| \Delta^{2d} g_2 \|_{H^{\frac{d}{2}}}^{\frac{8d-2}{8d}} \right) \| \Delta^{2d} g \|_{H^{\frac{d}{2}}}^{\frac{8d}{8d-2}} \| \nabla g \|_{L^{\infty}}^{\frac{8d-2}{8d-4}}.
\]

Applying Young’s inequality we have
\[ I_3 \lesssim r_0 \left( \| \nabla g_1 \|_{L^\infty} + \varepsilon \right)^2 \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^2 + \mathcal{F}(\| g_2 \|_{H^{4d+\frac{1}{2}}} + \varepsilon^{-1}), \]
which completes the proof. \(\square\)

**Lemma 6.5. (Estimate for \(I_4\))**

Let \(g\) as defined in Proposition 3.2 and \(V\) as defined in (6.4), there holds
\[
\left| \int \Delta_{\alpha}(\Delta^{2d} g)(x)V(\alpha,x)d\eta(\alpha)\Delta^{2d} g(x)dx \right| \\
\lesssim r_0 \left( \| \nabla g_1 \|_{L^\infty} + \varepsilon \right)^2 \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^2 + \mathcal{F}(\| g_2 \|_{H^{4d+\frac{1}{2}}} + \varepsilon^{-1})
\]
for any \(\varepsilon \in (0,1)\).

**Proof.** Applying Lemma 1.3 Holder’s inequality and (6.9) one has
\[ I_4 \lesssim \| \Delta^{2d} g \|_{L^1} \| \Delta^{2d} g \|_{H^{\frac{1}{2}}} \| \nabla g \|_{F_{2,1}} \| \nabla g \|_{L^\infty}. \]
By the interpolation inequalities (6.7) we obtain
\[
I_4 \lesssim r_0 \left( \| \nabla g_1 \|_{L^\infty} \| \Delta^{2d} g_1 \|_{H^{\frac{1}{2}}}^{\frac{8d-2}{H^{\frac{1}{2}}}} + \| \nabla g_2 \|_{L^\infty} \| \Delta^{2d} g_2 \|_{H^{\frac{1}{2}}}^{\frac{8d-2}{H^{\frac{1}{2}}}} \right) \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^{\frac{8d}{H^{\frac{1}{2}}}}
\lesssim r_0 \| \nabla g_1 \|_{L^\infty} \left( \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^{\frac{8d}{H^{\frac{1}{2}}}} + \| \Delta^{2d} g_2 \|_{H^{\frac{1}{2}}}^{\frac{8d}{H^{\frac{1}{2}}}} \| \Delta^{2d} g \|_{H^{\frac{1}{2}}} \right)
+ \| \Delta^{2d} g_2 \|_{H^{\frac{1}{2}}}^{\frac{8d-2}{H^{\frac{1}{2}}}} \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^{\frac{8d}{H^{\frac{1}{2}}}}.
\]
Applying Young’s inequality we have
\[ I_4 \lesssim r_0 \left( \| \nabla g_1 \|_{L^\infty} + \varepsilon \right)^2 \| \Delta^{2d} g \|_{H^{\frac{1}{2}}}^2 + \mathcal{F}(\| \Delta^{2d} g_2 \|_{H^{\frac{1}{2}}} + \varepsilon^{-1}), \]
which completes the proof. \(\square\)

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**Fudan University, 220 Handan Road, Shanghai, 200433, China.**

*Email address*: kchen18@fudan.edu.cn

**ShanghaiTech University, 393 Middle Huaxia Road, Shanghai, 201210, China.**

*Current address*: Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China.

*Email address*: qhnguyen@amss.ac.cn

**Fudan University, 220 Handan Road, Shanghai, 200433, China.**

*Email address*: yrxu20@fudan.edu.cn