1 Overview

In GB (System of Gödel-Bernays), one starts with “class” variables (see Appendix A) and a unary relation M(X) meaning X is a set. The basic axiom for sethood is $Y \in X \rightarrow M(Y)$. We introduce a second unary relation SSC(X) meaning that X is a “set-sized” class, thus $Y \in X \rightarrow SSC(Y)$. These “set-sized” classes will then be subjected to a filtering process, which will determine which SSC’s are sets, i.e. explicated $(E(x))$, and which are not. As in GB, a class variable in lower case is automatically and implicitly an SSC. A dot above $\dot{x}$ denotes an SSC not yet explicated; if already a set, the dot disappears. Thus, $x \equiv \dot{x} \& E(x)$. It will be clear in the discussion of the axioms, which SSCs become sets and which are filtered out.

The basic idea of this axiom scheme (called EX) is that the SSCs which become sets are explicated by a finite sequence of “explicating axioms”, which closely follow the Axioms of ZF (System of Zermelo-Fraenkel), restricted to explicated SSCs. The goal of the filtering process is to filter out any SSC which doesn’t explicitly say, in a finite statement, what sets are its members. The mixing of GB and ZF is necessary. EX starts with GB, because ZF has only sets, while the SSC’s are needed to to make EX comprehensible. ZF is needed because the function making axioms of GB allow a non-explicated SSC to become explicated, while ZF’s can be structured so as not to allow that to happen.

See reference COHEN[1] for a complete treatment of GB and ZF.

The following suite of axioms is a complete treatment of EX. Items which may or may not be true in other axiom systems are not significant.
2 Axiom of Extension

\[ x = y \iff \forall u (u \in x \iff u \in y) \]. The Axiom of Extension reinforces the overall idea of EX in that an SSC which can’t explicitly say, in a finite statement, what elements are members, can’t be compared to another SSC, and is thus a poor candidate for being a set.

3 Set Explicating Axioms

1. Null Set
   \[ \exists \hat{x} (\forall y (y \notin \hat{x})) & E(x, NS) [x \text{ denoted by } \phi] \]

2. Unordered Pairs
   \[ \forall x, y \exists \hat{z} \forall w (w \in \hat{z} \iff (w = x) \lor (w = y)) \rightarrow E(z, UP, x, y) \]

3. Sum Set
   \[ \forall x \exists \hat{y} (\forall z (z \in \hat{y} \iff \exists t (z \in t \& t \in x))) \rightarrow E(y, SS, x) \]

4. Infinity
   \[ \exists \hat{x} (\phi \in \hat{x} \& \forall y (y \in \hat{x} \rightarrow y \cup \{y\} \in \hat{x})) \& E(x, IN, \phi) [x \text{ denoted by } \omega] \]
   The Sum Set makes sure that all integers are explicature.

5. Separation
   \[ \forall t_1, \ldots, t_k \forall x \exists \hat{y} (\forall z (z \in \hat{y} \iff (z \in x \& A_n(z, t_1, \ldots, t_k)))) \rightarrow E(y, SEP, x, n) \]
   For infinite subclasses, if they can’t Separate, they will be filtered out and will not become sets.

6. Replacement
   \[ \forall t_1, \ldots, t_k \forall x \exists ! y (A_n(x, y; t_1, \ldots, t_k)) \rightarrow \forall u \exists ! \hat{v} \& \langle u, \hat{v} \rangle \in \hat{f} \forall r (r \in \hat{v} \iff \exists s (s \in u \& A_n(s, r; t_1, \ldots, t_k))) \rightarrow \langle s, r \rangle \in \hat{f} \rightarrow (E(v, REP, x, y, u, n) \& E(f, REP, x, y, u, n)) \]
   Replacement implies Separation. The \( \langle u, v \rangle \)s are called set-explicating functions (hereafter SEF). They do not need to be 1-1.

7. Power Set
   \[ \forall x \exists \hat{y} (\forall z (z \in \hat{y} \iff z \subseteq x)) \rightarrow E(y, EPS, x) [y \text{ is denoted by } EPS(x)] \]
(a) “Not a Set” Axioms
Because in EX sets have an “only-if” condition, clarity can sometimes be added by saying what is not a set. Note that these “not a set axioms” don’t cause anything to be explicated.

\((x \text{ infinite}) \rightarrow \exists \hat{z}(\hat{z} \subset x \& \hat{z} \notin EPS(x))\).

In fact, the intuition is that there are an uncountable number of such SSCs, excluded from explication by the fact that their members can’t be given in a finite statement. Filtered out once in Separation and again here in the Power Set.

(b) Fate of \(P(\omega)\)
We remark at this point that the power class of \(\omega\) is not a set in EX. (As SSCs, \(EPS(\omega) \subset P(\omega)\)). \(EPS(\omega)\) must assume the role of the continuum (C) in EX.

It is easy, looking at the Axiom of Specification, to see that \(EPS(\omega)\) contains the normal model of the real numbers. That is an infinite subset of \(\omega\), controlled by a function, so that each member of the subset is specified.

8. Rules for Explication
\(E(x) \iff (x = \phi) \lor (x = \omega) \lor (int(x)) \lor (E(x, UP, y, z)) \lor (E(x, SS, y)) \lor (E(x, SEP, y, x, \xi)) \lor (E(x, EPS, y)) \lor (E(x, REP, y, z, u, n)) \lor (E(f, REP, x, y, u, n))\)

And, of course, \(M(x) \iff E(x)\)

4 Axiom of Limited Well-Ordering
There is No Explicated Well-Ordering For EPS of Any Infinite Set.

Axiom - \((x \text{ infinite}) \rightarrow \forall y \neg(y \text{ Well - Orders}(EPS(x)))\)

1. Corollary - The Well-ordering Theorem is false in EX.
   If AC were true in EX, \(EPS(\omega)\) could be well-ordered, thus

2. Corollary - AC is false in EX. This allows us to filter out the unwanted, “class-sized” number of sets associated with AC being true, none of whose members, of course, can be given in a finite statement.
3. Corollary - No SEF maps $EPS(\omega)$ 1-1 into any ordinal, including $\aleph_1$. Thus,

4. Corollary - CH is false in EX, thus so is GCH, thus

5. Corollary - Since both AX and GCH are false, so is V=L. See Appendix B for further discussion.

6. Discussion - This axion is extremely well-grounded in intuition, as anyone who has spent time seeking a well-ordering for $P(\omega)$ will attest. In fact, AC was added to ZF for just this reason. As ZF developed, it was noticed that $P(\omega)$ was nothing like the power sets of finite sets. Instead of being naturally well-ordered, it strongly resisted well-ordering at all. Three possible choices for resolving this dilemma were: a) hold your nose to its lack of intuitive grounding and add AC; b) instead of AC, add the even less intuitive CH as an axiom; or, c) accept that $P(\omega)$ and the “smaller” $EPS(\omega)$ can not be well-ordered, and get along with the development of the theory. Choice a) was the one chosen by ZF; choice c) was taken by EX.

5 Developments, Conjectures and Sketches in EX

1. Regularity
   The axiom of Regularity is conjectured to be a theorem. The proof would start with the last explication sequence before regularity broke, then show that adding one more explicating axiom would not break it.

2. $EPS(\omega)$ is Uncountable in EX
   This follows immediately from Corollary 3 of Section 4. It also is shown by the usual approach of Cantor’s Theorem “set of all sets not members of the range of the assumed function from $\omega$ onto $EPS(\omega)$”.

3. The Usual Higher Well-ordered Cardinals Exist in EX
   The usual proof goes through in $EPS^4(\omega)$ with the unordered pairs, ordered pairs, partial or full-well-orderings, and equivalence classes of well-orderings by their ordinal length, combining to produce the set of all countable ordinals ($\aleph_1$).
4. Two Dimensions of Cardinality
EX thus has 2 “Dimensions” of cardinality. The well-ordered ones from $EPS^4(\omega)$ and the others from $EPS(\omega)$. This doesn’t cause any problems, if one remembers the cardinalities of the two dimensions don’t overlap.

5. Different Models for ZF-AC
Appendix B shows that the Constructable Sets (V=L) are a model for ZF-AC. In that model, both AC and GCH are theorems.

The explicaded sets (EX) are also a Model for ZF-AC, quite different from V=L. This is essentially clear, as each Axiom in ZF, except AC, corresponds to an Axion in EX, but for explicaded sets only. This model is interesting for two reasons. First, AC is false in EX; second, any other model of ZF (including ZF itself) with or without AC, must contain all the sets in EX.

6. EX Decides several Problems which Took a lot of Thinking in ZF.
Besides Regularity and a better solution for AC, we have $\neg AC \rightarrow (\neg CH \& \neg GCH \& \neg (V = L))$. Also it’s obvious in EX that there is no unreachable cardinal. All in all, EX is a cleaner, more-intuitive way to think about sets.

6 Appendix A - G.B. Axioms for Class Formations

1. $\exists X \forall a(a \in X \iff \exists b \exists c(a = \langle b, c \rangle \& b \in c))$
2. $\forall X \forall Y \exists Z \forall u(u \in Z \iff u \in X \& u \in Y)$
3. $\forall X \exists Y \forall u(u \in Y \iff \sim u \in X)$
4. $\forall X \exists Y \forall u(u \in Y \iff \exists v(\langle v, u \rangle \in X))$
5. $\forall X \exists Y \forall u(u \in Y \iff \exists r \exists s(u = \langle r, s \rangle \& s \in X))$
6. $\forall X \exists Y \forall a(a \in Y \iff \exists b, c(\langle b, c \rangle = a \& \langle c, b \rangle \in X))$
7. $\forall X \exists Y \forall u(u \in Y \iff \exists a, b, c(\langle a, b, c \rangle \in X \& \langle b, c, a \rangle \in Y \& \langle b, c, a \rangle = u))$
8. $\forall X \exists Y \forall u(u \in Y \iff \exists a, b, c(\langle a, b, c \rangle \in X \& \langle a, c, b \rangle \in y \& \langle a, c, b \rangle = u))$
7 Appendix B - Consis($ZF + AC + GCH$)

In 1958 Gödel proved that if ZF (without AC) is Consistent, it remains so if AC and GCH are added as Axioms. The blow by blow is given in COHEN[1], pp 85-99. Briefly, he showed that the Constructible Sets (L) are a class-sized model for ZF minus AC. In that Model(V=L), both GCH and AC are theorems. He uses a proof principle called “Trans-Finite Induction”, where each ordinal “stage” depends on the power class of the previous stage, or SUP of the previous stages for limit ordinals. Note that in ZF all axioms are “set-defining”. EX has the “no-such-explicated-set-type” Axiom of Limited Well-Ordering. That axiom would need to be true in any model of all EX’s axioms. Though not relavent, one could attempt an “Every Set is Constructible” model using only the set-explicating axioms of EX. Even that plan bogs down, because: 1) In EX, Trans-Finite Induction does not work to class-sized ordinals, at most only within a very large set; and 2) the $\omega$th stage of the construction fails because $P(\omega)$ is not a set.

8 References

[1] P. J. COHEN: "Set Theory and the Continuum Hypothesis,” Stanford University, 1966, W. A. Benjamin, Inc. Advanced Book Program, Reading, Massachusetts.

9 End of Article: Set Theory Axioms Using ”Explication”