Explanation notes on the multipole expansions of the electromagnetic field

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Abstract

Starting from Jefimenko’s equations, we consider the multipole expansions of electric and magnetic fields for a confined system of charges and currents. We analyze and comment on the calculus of radiated power, on the consistent use of approximation criteria, on the invariance of physical results when changing the point of reference, as well as on the use of electric and magnetic moments described by symmetric and trace free tensors.
I. INTRODUCTION

Despite the successful and long history of the electromagnetic field theory, there are several topics open to new theoretical and pedagogical contributions. One of them concerns the formalism of the multipole expansion of fields, in general, and of the radiated one, in particular. Another issue is related to the importance of Jefimenko’s equations in the study of such problems. Motivated by recent publications on the topic, we discuss some features of this type of problems, the present paper being a revisited and completed version of a previous electronic preprint\textsuperscript{1}.

The multipole expansion of the electromagnetic field in Cartesian coordinates is exposed in electrodynamics textbooks, as the well-known Refs.\textsuperscript{2} and\textsuperscript{3}. Ordinarily, these expansions are calculated only in the first or second approximation, the higher-order terms being considered too complicated. As Jackson writes in his textbook, \textit{the labor involved in manipulating terms in the expansion of the vector potential becomes increasingly prohibitive as the expansion is extended beyond the electric quadrupole terms} (see Ref.\textsuperscript{3}, pp 415-416). For this reason and due to the applicability only in the long-wavelength range, another treatment, based on the spherical tensors and on the solutions of Helmholtz equation is preferred. This alternative has also a larger domain of applications. Actually, starting from the results obtained employing this calculation technique, the reader can verify what effort is involved when returning to the multipole Cartesian moments which offer a higher physical transparency (see Ref.\textsuperscript{4}). A relatively recent textbook\textsuperscript{5} and a paper\textsuperscript{6}, the last related to the importance of Jefimenko’s equations for expressing the electric and magnetic field when discussing the radiation theory, brought our attention on the necessity of some supplementary explanations. There are some prescriptions in the literature\textsuperscript{7,8,9} for calculating higher-order terms of the multipole series based on a simple algebraic formalism of tensorial analysis. One of the aims of the present paper is to show how one can hide, as much as possible, the higher-order tensors behind some vectors, reducing the calculation technique to the formalism of an ordinary vectorial algebra or analysis.

We start in section II by shortly presenting the notation convention we use by giving a general formalism for handling multipolar expansions in Cartesian coordinates. In section III we derive the electric and magnetic fields multipole expansions without using the retarded potentials, while in section IV we present some fundamental ideas of expressing the
multipole moments by symmetric trace free tensors. In section \( \text{V} \) we discuss the approximation criteria stressing the significance of \( d/\lambda \)-criterion. In section \( \text{VI} \) we further discuss some features of the calculation for the radiated power, with an emphasis on the \( 4-th \) order approximation in \( d/\lambda < 1 \). Section \( \text{VII} \) presents the problem of the translational invariance of the fields expansions. Finally, in appendix \( \text{A} \) we give the guidelines for the general tensorial calculus of the electric and magnetic moments and some general results in terms of symmetric trace free tensors. The last section is reserved for conclusions.

II. GENERAL FORMALISM

Let us write Maxwell’s equations with a “system free” notation (a notation independent of the unit system):

\[
\nabla \times B = \frac{\mu_0}{\alpha} \left( J + \varepsilon_0 \frac{\partial E}{\partial t} \right), \quad \nabla \times E = -\frac{1}{\alpha} \frac{\partial B}{\partial t}, \quad \nabla \cdot B = 0, \quad \nabla \cdot E = \frac{1}{\varepsilon_0} \rho \tag{1}
\]

where \( \varepsilon_0, \mu_0, \alpha \) are proportional factors depending on the system of units and satisfying the equation

\[
\frac{\alpha^2}{\varepsilon_0 \mu_0} = c^2. \tag{2}
\]

c is the vacuum light speed. Maxwell equations written in SI units are obtained from equations (1) for \( \alpha = 1 \) and the SI values of \( \varepsilon_0, \mu_0 \). For the Gauss system of units, \( \alpha = c, \varepsilon_0 = 1/4\pi, \mu_0 = 4\pi \).

The relations between fields and potentials are written as

\[
B = \nabla \times A, \quad E = -\nabla \Phi - \frac{1}{\alpha} \frac{\partial A}{\partial t}. \tag{3}
\]

Ordinarily, for the field calculation when the charge and the current distributions are known as functions of \( r \) and \( t \), one firstly determines the retarded potentials:

\[
A(r, t) = \frac{\mu_0}{4\pi \alpha} \int_{\mathcal{D}} \frac{J(r', t - \frac{R}{c})}{R} \, d^3x', \quad \Phi(r, t) = \frac{1}{4\pi \varepsilon_0} \int_{\mathcal{D}} \frac{\rho(r', t - \frac{R}{c})}{R} \, d^3x'. \tag{4}
\]

The fields \( E, B \) are then derived from equations (4). The domain \( \mathcal{D} \) includes the supports of \( \rho \) and \( J \), and \( R = r - r' \). The solutions (4) are indeed electromagnetic potentials verifying the Lorenz condition

\[
\nabla \cdot A + \varepsilon_0 \mu_0 \frac{\partial \Phi}{\alpha \partial t} = 0. \tag{5}
\]
This equation can be directly verified using the continuity equation at the retarded time \( \tau = t - R/c \):

\[
\frac{\partial \rho(r', \tau)}{\partial t} + [\nabla' \cdot J(r', t')]_{t' = \tau} = 0
\]  

and the relation between the retarded value of a spatial derivative of a function and the same spatial derivative of the retarded function:

\[
\partial_i' f(r', \tau) = [\partial_i f(r', t')]_{t' = \tau} + \frac{R_i}{cR} \frac{\partial f(r', \tau)}{\partial t}.
\]  

The last equation should be well-known for each student from a class of electrodynamics since when writing the retarded potentials as particular solutions of the wave equation, it is necessary to verify the Lorenz condition. Only in this way one can be convinced that the retarded potentials are indeed electromagnetic potentials. This verification can be realized in a direct calculation, a good exercise for the student.

An alternative approach for the field calculation is given by Jefimenko’s equations (see Ref. 3, section 6.5). The fields are obtained directly as retarded solutions of the following wave equations which are consequences of Maxwell’s equations:

\[
\Delta B - \frac{1}{c^2} \frac{\partial^2 B}{\partial t^2} = -\frac{\mu_0}{\alpha} \nabla \times J, \quad \Delta E - \frac{1}{\varepsilon_0} \frac{\partial^2 E}{\partial t^2} = \frac{1}{\varepsilon_0} \nabla \rho + \frac{\mu_0}{\alpha^2} \frac{\partial J}{\partial t}.
\]  

These equations allow the retarded solutions

\[
B(r, t) = \frac{\mu_0}{4\pi\alpha} \int_{\mathcal{D}} \frac{1}{R} [\nabla' \times J(r', t')]_{t' = \tau} \, d^3x',
\]

\[
E(r, t) = -\frac{1}{4\pi\varepsilon_0} \int_{\mathcal{D}} \frac{1}{R} [\nabla' \rho(r', t')]_{t' = \tau} \, d^3x' - \frac{\mu_0}{4\pi\alpha^2} \int_{\mathcal{D}} \frac{1}{R} \frac{\partial J(r', \tau)}{\partial t} \, d^3x'.
\]  

Using equation (10), the following expressions of the solutions (12) can be obtained:

\[
B(r, t) = \frac{\mu_0}{4\pi\alpha} \int_{\mathcal{D}} \nabla \times \frac{J(r', \tau)}{R} \, d^3x',
\]

\[
E(r, t) = -\frac{1}{4\pi\varepsilon_0} \int_{\mathcal{D}} \nabla \frac{\rho(r', \tau)}{R} \, d^3x' - \frac{\mu_0}{4\pi\alpha^2} \int_{\mathcal{D}} \frac{\dot{J}(r', \tau)}{R} \, d^3x',
\]  

where the dot signifies the time derivative. These last expressions can be obtained also introducing equations (11) in equations (3) and inverting the order of the derivatives and integrals. Considered as solutions of equations (8), the expressions (10) are known as Jefimenko’s equations. We think it is important to show that sometimes it is convenient to
perform a calculation directly on the fields represented by equations (10) instead of calculating firstly the potentials and after that, applying equations (3). In fact, in the present work we benefit from this circumstance. We owe our inspiration to Ref. 6.

III. MULTIPOLAR EXPANSIONS OF FIELDS

Let us consider the field $\mathbf{B}(r, t)$, given by equation (10), in a point from the exterior of the domain $D$ which includes the supports of $\rho$ and $\mathbf{J}$:

$$B(r, t) = \frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \int_D \frac{\partial_j J_k(r', \tau)}{R} d^3x' = \frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \int_D \left[ \frac{\partial_j J_k(r', t - \frac{|r - \xi|}{c})}{|r - \xi|} \right]_{\xi = r'} d^3x'. \quad (11)$$

e_i$ are the unit vectors of the Cartesian axes. Writing the Taylor series about $\xi = 0$ of the integrand, we can finally write

$$B(r, t) = \frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \int_D d^3x' \sum_{n \geq 0} \frac{(-1)^n}{n!} x'_{i_1} \ldots x'_{i_n} \frac{\partial_j \partial_{i_1} \ldots \partial_{i_n}}{r} J_k(r', \tau_0) \quad (12)$$

where $\tau_0 = t - r/c$. Here, we can apply a formula for the multiple derivative of a function of the type $F(\tau_0)/r$:

$$\partial_{i_1} \ldots \partial_{i_n} \frac{F(\tau_0)}{r} = \sum_{l=0}^{n} \frac{1}{c^{n+l+1}} C_{i_1 \ldots i_n}^{(n,l)} \frac{\partial^{n-l} F(\tau_0)}{\partial \tau^{n-l}}. \quad (13)$$

The coefficients are symmetric in $i_1, \ldots, i_n$ and can be expressed as

$$C_{i_1 \ldots i_n}^{(n,l)} = \sum_{k=0}^{[\frac{\beta}{2}]} D_k^{(n,l)} \delta_{i_1 i_2} \ldots \delta_{i_1 i_2 k} \nu_{2k+1} \ldots \nu_{i_n}. \quad (14)$$

In the last equation, $[\beta]$ is the integer part of $\beta$ and $\nu = r/r$. By $A_{i_1 \ldots i_n}$ we understand the sum over all the permutations of the symbols $i_q$ giving distinct terms. For the objective of the present paper ($n \leq 3$), the coefficients $D$ in equation (13) can be calculated directly from the successive derivative operations but, generally, one can establish recurrence relations.

We can write now the expansion (12) as

$$B(r, t) = \frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n+1} \frac{1}{c^{n+l+1} \tau^{l+1}} C_{j_{i_1 \ldots i_n}}^{(n+1,l)} \int_D d^3x' \frac{\partial^{n+1-l} J_k(r', \tau_0)}{\partial \tau^{n+1-l}}. \quad (15)$$

Next, we admit that in the previous equation one can invert the derivation with the integral operation on the domain $D$ and we have

$$B(r, t) = \frac{\mu_0}{4\pi\alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n+1} \frac{1}{c^{n+1-l} \tau^{l+1}} C_{j_{i_1 \ldots i_n}}^{(n+1,l)} \frac{\partial^{n+1-l} C_{j_{i_1 \ldots i_n}}^{(n+1,l)}}{\partial \tau^{n+1-l}} m_{i_1 \ldots i_n} k(\tau_0). \quad (16)$$
where it is introduced the \( n \)-th order magnetic moment \( \mathbf{M}^{(n)} \) with the Cartesian components:

\[
\mathbf{m}_{i_1...i_n}(t) = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_{n-1}} \, J_{in}(t) \, d^3x'. \tag{17}
\]

From equation (10), performing a similar calculation as in the magnetic field case, one obtains the multipole expansion of the electric field:

\[
\mathbf{E}(r, t) = -\frac{1}{4\pi \varepsilon_0} e_i \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n+1} \frac{1}{\varepsilon^{n+1-l}} C_{i_1...i_n}^{(n+1, l)} \frac{\partial^{n+1-l}}{\partial \mathbf{r}^{n+1-l}} \mathbf{P}_{i_1...i_n}(\tau_0) \\
- \frac{\mu_0}{4\pi \alpha^2} e_i \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n} \frac{1}{\varepsilon^{n-l+1}} C_{i_1...i_n}^{(n, l)} \frac{\partial^{n-l}}{\partial \mathbf{r}^{n-l}} \mathbf{m}_{i_1...i_n}(\tau_0). \tag{18}
\]

\( \mathbf{P}_{i_1...i_n} \) are the Cartesian components of the electric moment of order \( n \), \( \mathbf{P}^{(n)} \):

\[
\mathbf{P}_{i_1...i_n}(t) = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \, \rho(\mathbf{r}', t) \, d^3x'. \tag{19}
\]

We point out that the expressions from equations (16) and (18) are obtained without using commuting properties of the different operations except for the integration and the Taylor series expansion and, also, for the time derivation and the spatial integration. Of course, all the necessary convergence properties are supposed satisfied. In equation (16) and (18) one recognizes the multiple spatial derivative of \( \mathbf{m}_{i_1...i_n}(\tau_0)/r \) and \( \mathbf{P}_{i_1...i_n}(\tau_0)/r \) as, for example,

\[
\sum_{l=0}^{n+1} \frac{1}{\varepsilon^{n+1-l}} C_{j_1...j_n}^{(n+1, l)} \frac{\partial^{n+1-l}}{\partial \mathbf{r}^{n+1-l}} \mathbf{m}_{j_1...j_n}(\tau_0) = \partial_{j_1} \ldots \partial_{j_n} \left[ \frac{1}{r} \mathbf{m}_{i_1...i_n}(\tau_0) \right]. \tag{20}
\]

The fields \( \mathbf{B} \) and \( \mathbf{E} \) can be expressed as

\[
\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_j \partial_{i_1} \ldots \partial_{i_n} \frac{\mathbf{m}_{i_1...i_n}(\tau_0)}{r} \\
= \frac{\mu_0}{4\pi \alpha} \nabla \times \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \left[ \frac{\mathbf{m}^{(n+1)}(\tau_0)}{r} \right] \tag{21}
\]

and

\[
\mathbf{E}(\mathbf{r}, t) = -\frac{1}{4\pi \varepsilon_0} e_i \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_i \partial_{i_1} \ldots \partial_{i_n} \frac{\mathbf{P}_{i_1...i_n}(\tau_0)}{r} - \frac{\mu_0}{4\pi \alpha^2} e_i \sum_{n \geq 0} \frac{(-1)^n}{n!} \partial_i \ldots \partial_{i_n} \frac{\mathbf{m}_{i_1...i_n}(\tau_0)}{r} \\
= -\frac{1}{4\pi \varepsilon_0} \nabla \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \left[ \frac{\mathbf{P}^{(n)}(\tau_0)}{r} \right] - \frac{\mu_0}{4\pi \alpha^2} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \left[ \frac{\mathbf{m}^{(n+1)}(\tau_0)}{r} \right], \tag{22}
\]
where the following notation is introduced for the tensor contractions:

\[
(A^{(n)}||B^{(m)})_{i_1...i_{|n-m|}} = \begin{cases} 
A_{i_1...i_{n-m}j_1...j_m}B_{j_1...j_m}, & n > m \\
A_{j_1...j_n}B_{j_1...j_n}, & n = m \\
A_{j_1...j_n}B_{j_1...j_n i_1...i_{m-n}}, & n < m
\end{cases}
\] (23)

We recognize, comparing these equations with equations (3), the multipole expansions of the potentials:

\[
A(r,t) = \frac{\mu_0}{4\pi \alpha} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \frac{\mathcal{M}^{(n+1)}(\tau_0)}{r}, \quad \Phi(r,t) = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \frac{\mathcal{P}^{(n)}(\tau_0)}{r}.
\] (24)

Let us define

\[
a_k(r,t;\zeta,n) = \zeta_{i_1...i_n} \int_D x'_{i_1} \cdots x'_{i_n} J_k(r',t) \, d^3x' = \zeta_{i_1...i_n} M_{i_1...i_n k}(t),
\] (25)

with \(\zeta_{i_1...i_n}\) symmetric in the \(n\) indices. Considering the following consequence of the continuity equation

\[
J_k(r',\tau_0) = \nabla' \cdot \left[ x_k' \mathbf{J}(r',\tau_0) \right] + x_k' \frac{\partial \rho(r',\tau_0)}{\partial t},
\]

we can write

\[
a_k(r,\tau_0;\zeta,n) = -\zeta_{i_1...i_n} \int_D x_k' \mathbf{J}(r',\tau_0) \cdot \nabla'(x'_{i_1} \cdots x'_{i_n}) \, d^3x' + \zeta_{i_1...i_n} \dot{P}_{i_1...i_n k}(\tau_0).
\]

Partial integrations and vanishing surface integrals are employed for obtaining the last result. Further, because of the symmetry of \(\zeta\), we get

\[
a_k(r,\tau_0;\zeta,n) = -\zeta_{i_1...i_n} \dot{P}_{i_1...i_n k}(\tau_0) = -n \zeta_{i_1...i_n} \int_D x'_1 \cdots x'_{i_{n-1}} x'_k J_k(r',\tau_0) \, d^3x'
\]

\[
= -n \zeta_{i_1...i_n} \int_D x'_1 \cdots x'_{i_{n-1}} [x'_k J_k(r',\tau_0) - x'_i J_k(r',\tau_0)] \, d^3x' - n a_k(r,\tau_0;\zeta,n)
\]

\[
= -n \zeta_{i_1...i_n} \varepsilon_{ki_n q} \int_D x'_1 \cdots x'_{i_{n-1}} [r' \times \mathbf{J}(r',\tau_0)]_q \, d^3x' - n a_k(r,\tau_0;\zeta,n).
\]

Let us introduce the standard magnetic moment of order \(n\), \(\mathbf{M}^{(n)}\), by its Cartesian components

\[
M_{i_1...i_n}(t) = \frac{n}{(n+1)\alpha} \int_D x'_{i_1} \cdots x'_{i_{n-1}} [r' \times \mathbf{J}(r',t)]_{i_n} \, d^3x'.
\] (26)
We can write

\[ a_k(r, \tau_0; \zeta, n) \equiv \zeta_{i_1 \ldots i_n} M_{i_1 \ldots i_n} k(\tau_0) = \zeta_{i_1 \ldots i_n} \left( -\alpha \varepsilon_{kia} M_{i_1 \ldots i_{n-1}} q(\tau_0) + \frac{1}{n+1} \dot{P}_{i_1 \ldots i_n} k(\tau_0) \right). \]  

(27)

This result inserted in equation (16) gives

\[ B(r, t) = \frac{\mu_0}{4\pi} e_i \varepsilon_{ijk} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \varepsilon_{kia} \sum_{l=0}^{n+1} \frac{1}{c^{n+1-l} \tau_{l+1}} C^{(n+1, l)}_{i_1 \ldots i_n} \frac{\partial^{n+1-l}}{\partial \tau^{n+1-l}} M_{i_1 \ldots i_{n-1}} q(\tau_0) \]

\[ + \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \sum_{l=0}^{n} \frac{1}{c^{n-l} \tau_{l+1}} C^{(n, l)}_{i_1 \ldots i_n} \frac{\partial^n}{\partial \tau^n} \dot{P}_{i_1 \ldots i_n} k(\tau_0). \]

(28)

With the same calculation technique, one obtains the multipole expansion of the electric field:

\[ E(r, t) = -\frac{1}{4\pi \varepsilon_0} e_i \sum_{n \geq 0} \frac{(-1)^n}{n!} \sum_{l=0}^{n+1} \frac{1}{c^{n+l} \tau_{l+1}} C^{(n+1, l)}_{i_1 \ldots i_n} \frac{\partial^{n+1-l}}{\partial \tau^{n+1-l}} P_{i_1 \ldots i_n}(\tau_0) \]

\[ - \frac{\mu_0}{4\pi \alpha^2} e_i \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \sum_{l=0}^{n} \frac{1}{c^{n-l} \tau_{l+1}} C^{(n, l)}_{i_1 \ldots i_n} \frac{\partial^n}{\partial \tau^n} \dot{P}_{i_1 \ldots i_n} k(\tau_0) \]

\[ + \frac{\mu_0}{4\pi \alpha} e_i \sum_{n \geq 1} \frac{(-1)^n}{n!} \varepsilon_{kia} \sum_{l=0}^{n} \frac{1}{c^{n-l} \tau_{l+1}} C^{(n, l)}_{i_1 \ldots i_n} \frac{\partial^n}{\partial \tau^n} \dot{M}_{i_1 \ldots i_{n-1}} q(\tau_0). \]

(29)

Processing the expressions from equations (28) and (29) by the same procedure applied for obtaining equations (21) and (22), we obtain

\[ B(r, t) = \frac{\mu_0}{4\pi} \nabla \times \left[ \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \left( \nabla^{n-1} \frac{M^{(n)}(\tau_0)}{r} \right) + \frac{\alpha}{c^2} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \nabla^n \frac{P^{(n+1)}(\tau_0)}{r} \right]. \]

(30)

and

\[ E(r, t) = -\frac{1}{4\pi \varepsilon_0} \left[ \nabla \sum_{n \geq 0} \frac{(-1)^n}{n!} \nabla^n \frac{P^{(n)}(\tau_0)}{r} + \frac{1}{c^2} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \nabla^n \frac{\dot{P}^{(n+1)}(\tau_0)}{r} \right] + \frac{\alpha}{c^2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \left( \nabla^{n-1} \frac{\dot{M}^{(n)}(\tau_0)}{r} \right). \]

(31)

From equations (3), (30) and (31) one can immediately identify the multipole expansions of the potentials \( A \) and \( \Phi \).

The expansions from equations (30) and (31) can be easily processed by a vectorial calculus introducing the following vectors:

\[ \mathcal{P}(r, t; \xi, n) = e_i \xi_i \ldots \xi_{i_{n-1}} \frac{P_{i_1 \ldots i_{n-1}}(\tau_0)}{r} = \xi^{n-1} \frac{P^{(n)}(\tau_0)}{r}. \]

(32)
and

\[ \mathcal{M}(r, t; \xi, n) = e_i \xi_{i_1} \ldots \xi_{i_{n-1}} \frac{\mathcal{M}_{i_1 \ldots i_{n-1}}(\tau_0)}{r} = \xi^{n-1} \frac{\mathcal{M}^{(n)}(\tau_0)}{r}. \]  

(33)

Consequently, the expressions for the magnetic and electric field become

\[ B(r, t) = \frac{\mu_0}{4\pi} \nabla \times \left[ \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \mathcal{M}(r, t; \nabla, n) + \frac{1}{\alpha} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \dot{\mathcal{P}}(r, t; \nabla, n+1) \right] \]  

(34)

and

\[ E(r, t) = -\frac{1}{4\pi \varepsilon_0} \left[ \nabla \sum_{n \geq 0} \frac{(-1)^n}{n!} (\nabla \cdot \mathcal{P}(r, t; \nabla, n)) + \frac{1}{c^2} \sum_{n \geq 0} \frac{(-1)^n}{(n+1)!} \ddot{\mathcal{P}}(r, t; \nabla, n+1) \right. 

+ \left. \frac{\alpha}{c^2} \sum_{n \geq 1} \frac{(-1)^{n-1}}{n!} \nabla \times \mathcal{M}(r, t; \nabla, n) \right]. \]  

(35)

IV. EXPRESSING THE FIELDS BY SYMMETRIC TRACE-FREE Tensors

The general terms from the field expansions, given by equations (28) and (29) or (30) and (31), are apparently too complicated and they imply cumbersome calculations when the fields \( E \) and \( B \) are introduced in some physical interesting expressions as, for example, the radiation intensity. A considerable simplification of these calculations is obtained introducing the irreducible electric and magnetic moments defined as symmetric trace-free ("STF") Cartesian tensors. Let us consider a \( n - th \) order tensor \( T^{(n)} \) and the corresponding projections \( S(T^{(n)}) \) and \( T(T^{(n)}) \) on the subspaces of symmetric and STF tensors, respectively. For the electric moment \( \mathbf{P}^{(n)} \), we have a symmetric tensor and one has only to establish their trace free projection. Let us consider the simplest case of the quadrupolar moment \( \mathbf{P}^{(2)} \). Writing the components \( P_{ij} \) as

\[ P_{ij} = \Pi_{ij} + \Lambda \delta_{ij}, \]

there is a unique value of the parameter \( \Lambda \) such that \( \Pi^{(2)} = T(\mathbf{P}^{(2)}) \). For \( \Lambda = P_{qq}/3 \),

\[ \Pi_{ij} = P_{ij} - \frac{1}{3} P_{qq} \delta_{ij} = \int_{\mathcal{D}} \left( x_i x_j - \frac{1}{3} r^2 \delta_{ij} \right) \rho \, d^3 x. \]  

(36)

The STF projection of \( \mathbf{P}^{(3)} \) can be calculated searching the first order tensor \( \Lambda^{(1)} \) such that \( \Pi^{(3)} = T(\mathbf{P}^{(3)}) \) is given by the components

\[ \Pi_{ijk} = P_{ijk} - \delta_{i(j} \Lambda_{k)}. \]  

(37)
From the condition of vanishing traces of the tensor $\Pi^{(3)}$, one easily obtains:

$$\Lambda_i = \frac{1}{5} P_{qqi} = \frac{1}{5} \int_{\mathcal{D}} r^2 x_i \rho \, d^3 x. \quad (38)$$

Concerning the magnetic quadrupole moment $M^{(2)}$, we have a simple procedure for obtaining the STF projection. Let us write the identity

$$M_{ij} = \frac{1}{2} (M_{ij} + M_{ji}) + \frac{1}{2} (M_{ij} - M_{ji}),$$

where the first bracket represents the symmetric part of this tensor, and the second, the antisymmetric one. The symmetric part is, for this case ($n = 2$), a STF tensor $\Gamma^{(2)} = \mathcal{T}(M^{(2)})$. Therefore,

$$M_{ij} = \Gamma_{ij} + \frac{1}{2} \varepsilon_{ijk} N_k, \quad (39)$$

where

$$N_k = \varepsilon_{kij} M_{ij} = \frac{2}{3\alpha} \int_{\mathcal{D}} \left[ r \times (r \times J) \right]_k \, d^3 x = \frac{2}{3\alpha} \int_{\mathcal{D}} \left[ (r \cdot J) x_k - r^2 J_k \right] \, d^3 x. \quad (40)$$

We consider now the effect of the substitution

$$P^{(2)} \rightarrow \Pi^{(2)} : \ P_{ij} \rightarrow P_{ij} - \Lambda \delta_{ij} \quad (41)$$

in the multipole expansion of the field. From equation (32) we obtain the transformation of $\mathcal{P}(r, t; \nabla, 2)$ by the substitution (41):

$$\mathcal{P}(r, t; \nabla, 2) \mathcal{P}^{(2)} \rightarrow \Pi^{(2)} \mathcal{P}(r, t; \nabla, 2) - \nabla \frac{\Lambda(\tau_0)}{r}. \quad (42)$$

It is easy to see from equation (34) that $B$ is invariant to the substitution (42). In the expression (35) of the electric field $E$, the combination of the electric quadripolar terms from the first two sums gets

$$\frac{1}{8\pi \varepsilon_0} \nabla \left[ \Delta \frac{\Lambda(\tau_0)}{r} - \frac{1}{c^2} \frac{\ddot{\Lambda}(\tau_0)}{r} \right] = 0$$

since, for $r \neq 0$, $\Lambda(\tau_0)/r$ is a solution of the homogeneous wave equation. Therefore, the field $E$ is also invariant to the substitution (42).

The circumstances change dramatically for $n \geq 3$ in the cases of electric moments and for $n \geq 2$ in the magnetic ones. As we will see in the following, the electromagnetic field is not
invariant under the substitutions $P^{(3)} \rightarrow \Pi^{(3)}$ and $M^{(2)} \rightarrow \Gamma^{(2)}$. The general case, i.e. for $n > 3$ for electric moments and $n > 2$ for the magnetic ones, is developed in earlier works (see Ref. 9 and the literature cited in). We insert in the appendix some general results from these works, without demonstrations.

Let us consider the substitution $P^{(3)} \rightarrow \Pi^{(3)}$:

$$P_{ijk} \rightarrow P_{ijk} - \Lambda_i \delta_{jk},$$  \hspace{1cm} \text{(43)}

in the fields expressions. In this case,

$$\mathcal{P}(r, t; \nabla, 3) \rightarrow \mathcal{P}(r, t; \nabla, 3) - 2 \nabla \left[ \nabla \cdot \Lambda \left( \frac{\Lambda}{r} \right) \right] - \frac{1}{c^2} \frac{\ddot{\Lambda}(\tau_0)}{r},$$

$$= \mathcal{P}(r, t; \nabla, 3) - 2 \nabla \times \left( \nabla \times \Lambda \left( \frac{\Lambda}{r} \right) \right) - \frac{3}{c^2} \frac{\ddot{\Lambda}(\tau_0)}{r},$$ \hspace{1cm} \text{(44)}

where $\Lambda = e_i \Lambda_i$. Performing the substitution (44) in equation (34), we obtain the following transformation of $B$:

$$B(r, t) \rightarrow B(r, t) - \frac{\mu_0}{24 \pi \alpha c^2} \nabla \times \frac{\ddot{\Lambda}(\tau_0)}{r}.$$ \hspace{1cm} \text{(45)}

The same substitution in equation (35) gives

$$E(r, t) \rightarrow E(r, t) - \frac{1}{24 \pi \varepsilon_0 c^2} \nabla \left[ \nabla \cdot \frac{\ddot{\Lambda}(\tau_0)}{r} \right] + \frac{\mu_0}{24 \pi \alpha c^2} \frac{\dddot{\Lambda}(\tau_0)}{r}.$$ \hspace{1cm} \text{(46)}

Both, the modifications of the magnetic and electric field given in equations (45) and (46), respectively, can be compensated by the following transformation of the electric dipolar moment:

$$p \rightarrow p + \frac{1}{6c^2} \dddot{\Lambda}.$$ \hspace{1cm} \text{(47)}

Let be the substitution

$$M^{(2)} \rightarrow \Gamma^{(2)} : M_{ij} \rightarrow M_{ij} - \frac{1}{2} \varepsilon_{ijk} N_k,$$ \hspace{1cm} \text{(48)}

corresponding to equation (39). Then,

$$\mathcal{M}(r, t; \nabla, 2) \rightarrow \mathcal{M}(r, t; \nabla, 2) - \frac{1}{2} \nabla \times \frac{\dot{N}(\tau_0)}{r},$$

where $N = e_i N_i$, and

$$\nabla \times \mathcal{M}(r, t; \nabla, 2) \rightarrow \nabla \times \mathcal{M}(r, t; \nabla, 2) + \frac{1}{2} \nabla \left[ \nabla \cdot \frac{\dot{N}(\tau_0)}{r} \right] - \frac{1}{2c^2} \dddot{N}(\tau_0).$$ \hspace{1cm} \text{(49)}

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Performing the substitution (49) in equations (34) and (35), we obtain the transformations:

\[ B(r, t) \rightarrow B(r, t) + \frac{\mu_0}{16\pi c^2} \nabla \times \frac{\ddot{N}(\tau_0)}{r}. \]  

(50)

and

\[ E(r, t) \rightarrow E(r, t) + \frac{\mu_0}{16\pi \alpha} \nabla \left( \nabla \cdot \frac{\ddot{N}(\tau_0)}{r} \right) - \frac{\mu_0}{16\pi \alpha c^2} \frac{\dddot{N}(\tau_0)}{r}. \]  

(51)

The changes of \( B \) and \( E \) described by equations (50) and (51) are compensated by the transformation of electric dipolar moment, too:

\[ p \rightarrow p - \frac{\alpha}{4c^2} \dot{N}. \]  

(52)

Finally, substituting in the field expressions the momenta \( P^{(3)} \) and \( M^{(2)} \) by their STF-projections \( \Pi^{(3)} \) and \( \Gamma^{(2)} \), respectively, the corresponding changes of the fields can be compensated by the transformation

\[ p \rightarrow \tilde{p} = p - \frac{1}{c^2} \left( \frac{\alpha}{4} \ddot{N} - \frac{1}{6} \dddot{\Lambda} \right) = p - \frac{1}{c^2} \dot{t}, \]  

(53)

where by the vector \( t \) we understand

\[ t = \frac{\alpha}{4} \ddot{N} - \frac{1}{6} \dddot{\Lambda} = \frac{1}{10} \int_D \left( (r \cdot J) r - 2 r^2 J \right) d^3x. \]  

(54)

The last expression is obtained from equations (38) and (40) applying the continuity equation together with an operation of partial integration. This is the so-called electric toroidal dipolar moment, its presence in equation (53) leading to physical effects similar to the ones produced by an electric dipole moment. This fact was observed firstly in Ref.\(^1\) (see also Refs.\(^12, 13, 14\)).

V. APPROXIMATION CRITERIA

Generally, one can operate with the spectral decomposition of the charge and current distributions:

\[ \rho(r, t) = \int_{-\infty}^{\infty} \rho_\omega(r) e^{-i\omega t} d\omega, \quad J(r, t) = \int_{-\infty}^{\infty} J_\omega(r) e^{-i\omega t} d\omega. \]  

(55)

Correspondingly, one can introduce the spectral decomposition of the multipole moments:

\[ P^{(n)}(t) = \int_{-\infty}^{\infty} P_\omega^{(n)} e^{-i\omega t} d\omega, \quad M^{(n)}(t) = \int_{-\infty}^{\infty} M_\omega^{(n)} e^{-i\omega t} d\omega. \]  

(56)
The Fourier components are given by

\[ P(\omega)_{i_1 \ldots i_n} = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_n} \rho_\omega(r') \, d^3x', \quad M(\omega)_{i_1 \ldots i_n} = \int_{\mathcal{D}} x'_{i_1} \ldots x'_{i_{n-1}} (r' \times J_\omega(r'))_{i_n} \, d^3x'. \] (57)

Let us denote by \( d \) the linear dimension of the domain \( \mathcal{D} \). For estimating the order of magnitude of the different terms from the multipolar expansions of the fields, we notice the following relations:

\[ P^{(n)} \sim d^n, \quad M^{(n)} \sim d^{n+1}, \quad M^{(n)} \sim d^{n+1}. \] (58)

Considering the Fourier components in equations (16) and (18), we can estimate the contributions of the parameters \( d, \lambda = 2\pi c/\omega \) and \( r \) to the orders of magnitude of the different terms from the series expansions. For example, the estimation of the order of magnitude in equation (16) leads to the relations:

\[ S_{j i_1 \ldots i_n}^{(n)} k(\omega) = \sum_{l=0}^{n+1} \frac{c^{n+1-l} \rho_\omega}{\lambda^{n+1-l}} \frac{d^{n+1-l}}{c^{n+1-l} \rho_\omega} (M(\omega)_{i_1 \ldots i_n}) k e^{-i\omega t + i\omega r/c} \]

that is

\[ S_{j i_1 \ldots i_n}^{(n)} k(\omega) \sim \frac{1}{r} \left[ 1 + \frac{\lambda}{r} + \ldots \left( \frac{\lambda}{r} \right)^{n+1} \right] \left( \frac{d}{\lambda} \right)^{n+1}. \] (59)

From equation (59), we can conclude that for the convergence of the expansion series it is necessary to have the inequality

\[ \lambda > d \] (60)

satisfied. Consequently, for the spectral decomposition (55), the extension must be limited to the wave lengths verifying the last inequality. We can conclude that there are three distinct regions around an isolate source with corresponding approximations of fields: the near (static) region \( d < r << \lambda \), the intermediate (induction) region \( r \approx \lambda \) and the far (radiation) region \( r >> \lambda \), Ref.3. We consider here only the last region.

VI. THE RADIATION FIELD

For \( r >> \lambda \), the dominant term in the bracket from equation (59) is represented by the unity. The corresponding fields \( \mathbf{E} \) and \( \mathbf{B} \) are proportional to \( 1/r \) and represent the
parts contributing, as it is well known, to the energy and linear momentum radiated at large distances. The next term from the same bracket corresponds to the part of the field proportional to $1/r^2$ which contributes to the radiated angular momentum. In the present paper, as in most textbooks of electrodynamics, we consider only the radiated energy and therefore, we calculate the part of the field proportional to $1/r$. However, we point out that, as basic variables, the fields $E$ and $B$ must be defined such that all observables of the radiated field can be calculated. The angular momentum is one of these observables (for a complete calculation of the radiation field, see Refs. 4, 9 and literature cited in).

Let us calculate the first dominant terms, proportional to $1/r$, resulting from equations (16) and (18). For this purpose, we retain only the terms corresponding to the coefficients

$$C_{i_1\ldots i_n}^{(n,0)} = (-1)^n \nu_{i_1} \ldots \nu_{i_n}. \quad (61)$$

The expression from equation (16) becomes

$$B_{rad}(r, t) = -\frac{\mu_0}{4\pi \alpha} \frac{1}{r} \nu \times \sum_{n \geq 0} \frac{1}{n! c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \nu^n || \mathfrak{M}^{(n+1)}(\tau_0) \right) \quad (62)$$

or

$$B_{rad}(r, t) = -\frac{\mu_0}{4\pi \alpha} \frac{1}{r} \sum_{n \geq 0} \frac{1}{n! c^{n+1}} \left( \nu \times \frac{\partial^{n+1}}{\partial t^{n+1}} \mathfrak{m}(\tau_0; \nu, n + 1) \right), \quad (63)$$

where we introduced the vector

$$\mathfrak{m}(t; \xi, n) = \xi^{n-1} || \mathfrak{M}^{(n)}(t). \quad (64)$$

Similarly, equation (18) gets

$$E_{rad}(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r} \sum_{n \geq 1} \frac{1}{n! c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \nu^n || \mathfrak{P}^{(n)}(\tau_0) \right) \nu$$

$$- \frac{\mu_0}{4\pi \alpha^2} \frac{1}{r} \sum_{n \geq 0} \frac{1}{n! c^n \partial t^{n+1}} \left( \nu^n || \mathfrak{M}^{(n+1)}(\tau_0) \right), \quad (65)$$

since the source is an electric neutral system ($\dot{\mathfrak{P}}^{(0)} = 0$), or

$$E_{rad}(r, t) = \frac{1}{4\pi \varepsilon_0} \sum_{n \geq 1} \frac{1}{n! c^{n+1}} \frac{\partial^{n+1}}{\partial t^{n+1}} \left( \nu \cdot \pi(\tau_0; \nu, n) \right) \nu$$

$$- \frac{\mu_0}{4\pi \alpha^2} \frac{1}{r} \sum_{n \geq 0} \frac{1}{n! c^n \partial t^{n+1}} \mathfrak{m}(\tau_0; \nu, n + 1), \quad (66)$$
with
\[ \pi(t; \xi, n) = \xi^{n-1}||P^{(n)}(t)||. \]  
(67)

Based on equation (27), we can write the following relation for the vector \( \mathbf{m} \):
\[ \mathbf{m}(\tau_0; \nu, n + 1) = -\alpha \nu \times \mu(\tau_0; \nu, n) + \frac{1}{n+1} \pi(\tau_0; \nu, n + 1), \]  
(68)
where
\[ \mu(t; \xi, n) = \xi^{n-1}||M^{(n)}(t)||. \]  
(69)

Substituting equation (68) in equation (63), one gets
\[ B_{rad}(r, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \left\{ \sum_{n \geq 1} \frac{1}{n! c^{n+1} \partial^n \partial t^{n+1}} \left[ \nu \times (\nu \times \mu(\tau_0; \nu, n)) \right] \right. 
\]  
\[ - \frac{1}{\alpha} \sum_{n \geq 0} \frac{1}{(n+1)! c^{n+1} \partial^{n+2} \partial t^{n+2}} \left[ \nu \times \pi(\tau_0; \nu, n + 1) \right] \right\}. \]  
(70)
The change \( n \rightarrow n + 1 \) in the summation index of the second sum allows us to write:
\[ B_{rad}(r, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \nu \times \sum_{n \geq 1} \frac{1}{n! c^{n+1} \partial^n \partial t^{n+1}} \left[ \nu \times \mu(\tau_0; \nu, n) - \frac{c}{\alpha} \pi(\tau_0; \nu, n) \right]. \]  
(71)
Applying the same procedure to equation (66), we obtain
\[ E_{rad}(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r} \nu \times \sum_{n \geq 1} \frac{1}{n! c^{n+1} \partial^n \partial t^{n+1}} \left[ \nu \times \pi(\tau_0; \nu, n) + \frac{\alpha}{c} \nu \times \mu(\tau_0; \nu, n) \right] 
\]  
\[ - \frac{1}{c} \sum_{n \geq 0} \frac{1}{(n+1)! c^{n+1} \partial^{n+2} \partial t^{n+2}} \pi(\tau_0; \nu, n + 1) \right\}. \]  
(72)
The same change of the summation index \( n \rightarrow n + 1 \) in the last sum gives:
\[ E_{rad}(r, t) = \frac{1}{4\pi \varepsilon_0} \frac{1}{r} \nu \times \sum_{n \geq 1} \frac{1}{n! c^{n+1} \partial^n \partial t^{n+1}} \left[ \nu \times \pi(\tau_0; \nu, n) + \frac{\alpha}{c} \mu(\tau_0; \nu, n) \right]. \]  
(73)
Comparing equations (71) and (73), we notice the relation:
\[ E_{rad}(r, t) \rightarrow \pi(\tau_0; \nu, n) \leftarrow \mu(\tau_0; \nu, n) \rightarrow \mathbf{B}_{rad}(r, t). \]  
(74)
As seen from equations (71) and (73), the fields \( E_{rad} \) and \( B_{rad} \) are purely transverse fields satisfying the properties:
\[ \nu \cdot E_{rad} = 0, \quad \nu \cdot B_{rad} = 0, \quad E_{rad} = \frac{c}{\alpha} B_{rad} \times \nu, \quad \varepsilon_0 |E_{rad}|^2 = \frac{1}{\mu_0} |B_{rad}|^2. \]  
(74)
i.e. the characteristic wave properties.

Considering the wave region, the essential parameter characterizing the multipole expansion is $d/\lambda$. Coming back to the monochromatic case studied in Section V and introducing the notation $\zeta = \frac{d}{\lambda} \ll 1$, we can conclude that

$$
\frac{\partial^k}{\partial t^k} P^{(n)} \sim \begin{cases} 
\zeta^n, & k \geq n \\
\zeta^k, & k \leq n
\end{cases}, \\
\frac{\partial^k}{\partial t^k} M^{(n)} \sim \begin{cases} 
\zeta^{n+1}, & k \geq n \\
\zeta^{k+1}, & k \leq n
\end{cases}.
$$

(75)

In the field multipolar expansions in terms of primitive moments $P^{(n)}$ and $M^{(n)}$ as, for example, in equations (70) and (72), the series terms appear in an increasing order in the parameter $\zeta$. If the multipolar series are expressed in terms of the STF-tensors $\tilde{M}^{(n)}$ and $\tilde{P}^{(n)}$ as, for example, in equations (70) and (72), some caution is necessary. Replacing the pair $M^{(N-1)}$, $P^{(N)}$ by the corresponding STF-projections, one induces in the moments of the inferior order compensating terms of orders $\zeta^k$, with $k \leq N$. If we are interested in an approximation which includes the multipole moments up to a given rank $n < N$, we must discard the compensating terms from $\tilde{P}^{(m)}$, $\tilde{M}^{(m)}$, $m \leq n$ of orders $\zeta^k$ for $k > n$.

The task of writing finite multipolar sums in a well defined approximation becomes delicate when one considers products of field quantities as, for example, when defining the radiation intensities. The simplest, and most important, is the case of the radiated power. The Poynting vector can be written in terms of $E_{rad}$ and $B_{rad}$ with the help of equations (74) as:

$$
S = \frac{\alpha}{\mu_0} (E \times B) = \varepsilon_0 |E_{rad}|^2 c \nu + O\left(\frac{1}{r^3}\right) = \frac{1}{\mu_0} |B_{rad}|^2 c \nu + O\left(\frac{1}{r^3}\right).
$$

(76)

$B$ and $E$ are considered real vectors. The total radiated power may be written as the limit of a surface integral on a sphere centered in $O$, of radius $r$, for $r \to \infty$. Let us express the energy current in the radiation approximation corresponding to the sphere of radius $r$:

$$
N(S, \Sigma_r; t) = \int_{\Sigma_r} r^2 \nu \cdot S(r, t) \, d\Omega(\nu) = \frac{c}{\mu_0} \int_{\Sigma_r} r^2 |B(r, t)|^2 \, d\Omega(\nu) + O\left(\frac{1}{r}\right).
$$

(77)

The integrand from the last expression represents the angular distribution of the radiation. The quantity $N(S, \Sigma_r; t) \, dt$ represents the energy which crosses the sphere $\Sigma_r$ in the time interval $(t, t + dt)$ and is determined by the values of the multipole moments of the source in the interval $(t - r/c, t + dt - r/c)$. For large, but finite $r$, $r \gg \lambda$, this is an approximate expression obtained by neglecting the terms of order at least $1/r$. The expression from equation (77) can be employed for drawing conclusions on the electric charge distribution.
at the retarded time from observations, on the angular distribution or on the total radiated power. Since \( dt = d\tau_0 \), we can say that \( N(S; \Sigma_r; t) dt \) is the part of the energy emitted by the source in the given time interval which contributes to the energy intensity corresponding to \( \Sigma_r \) at the time \( t + \tau_0 \). If in equation (77) we put \( t \) instead \( \tau_0 \), then the quantity \( \lim_{r \to \infty} N(S; \Sigma_r; t) dt \) represents that part of the energy emitted by the source which contributes to the radiated energy or, shortly, radiated by the source. The situation changes when the support of the source depends on time, in particular, for the radiation of a moving point-like source (see Refs. 2, &73).

For calculating the angular distribution of the energy radiated or the total radiated power, one deals with the square of \( B_{rad} \) or \( E_{rad} \). Let us consider firstly that we operate with the series (70) or (72) expressed in terms of primitive electric and magnetic moments. For illustrating the kind of problems which appear in the approximation process, it is sufficient to consider the well known problem of the calculation of the electric and magnetic dipole and electric quadrupole field contributions to the radiation (see Refs. 2, 3). From the very beginning, we point out that here, this calculation is associated with the problem of the radiation of a complex system characterized by an arbitrary number of electric and magnetic multipoles. Once considered the contribution of the electric quadrupole field, from the expression of \( |B_{rad}|^2 \), with \( B \) considered real, the square of the electric quadrupolar moment \( P(2) \) is part of this contribution. This is a term of the order \( \zeta^4 \). There is a similar conclusion concerning the contribution of the magnetic dipolar moment \( m \). It is easy to see that in a consistent approximation procedure, together with the squares of \( P(2), m \) and \( p \), we must consider also the products of the electric dipolar moment \( p \) with the electric octopolar moment \( P(3) \) and the magnetic quadrupolar moment \( M^{(2)} \). They also give contributions of the order \( \zeta^4 \). Let us estimate these contributions in terms of primitive moments. In equation (70), to the contribution of \( (B_{rad}^{(0)} + B_{rad}^{(1)})^2 \), we have to add the fourth-order contribution of the product \( B_{rad}^{(0)} \cdot B_{rad}^{(2)} \). From equation (70) we can write, for \( n = 0, 1, 2 \):

\[
B_{rad}^{(0)}(r, t) = -\frac{\mu_0}{4\pi\alpha c} \frac{1}{r} \mathbf{\nu} \times \mathbf{\pi}(\tau_0; \mathbf{\nu}, 1), \tag{78}
\]

\[
B_{rad}^{(1)}(r, t) = \frac{\mu_0}{4\pi c^2} \frac{1}{r} \left\{ \mathbf{\nu} \times [\mathbf{\nu} \times \mathbf{\mu}(\tau_0; \mathbf{\nu}, 1)] - \frac{1}{2\alpha} \mathbf{\nu} \times \mathbf{\pi}(\tau_0; \mathbf{\nu}, 2) \right\}, \tag{79}
\]

\[
B_{rad}^{(2)}(r, t) = \frac{\mu_0}{4\pi c^3} \frac{1}{2r} \left\{ \mathbf{\nu} \times [\mathbf{\nu} \times \mathbf{\mu}(\tau_0; \mathbf{\nu}, 2)] - \frac{1}{3\alpha} \mathbf{\nu} \times \mathbf{\pi}(\tau_0; \mathbf{\nu}, 3) \right\}. \tag{80}
\]

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The angular distribution and the total power of the radiation are given by

\[ I(\nu) = \frac{c}{\mu_0} r^2 \langle B_{rad}^2 \rangle, \quad \mathcal{I} = \frac{4\pi c}{\mu_0} \langle r^2 (B_{rad})^2 \rangle, \] (81)

where

\[ \langle f(\nu) \rangle = \frac{1}{4\pi} \int f(\nu) \, d\Omega(\nu). \] (82)

Considering, for example, the angular distribution \( I(\nu) \), expanded up to the order \( \zeta^4 \), we have

\[ I(\nu) = \frac{c}{\mu_0} r^2 \left[ (B_{rad}^{(0)})^2 + 2B_{rad}^{(0)} \cdot B_{rad}^{(2)} \right] + \mathcal{O}(\zeta^5). \] (83)

In the following, for simplifying the notation, we understand by \( \pi(n) \) and \( \mu(n) \) the corresponding vectors with the arguments \( \tau_0 \) and \( \nu \). A simple algebraic calculation gets:

\[
\frac{c}{\mu_0} r^2 (B_{rad}^{(0)} + B_{rad}^{(1)})^2 = \frac{1}{(4\pi)^2 \varepsilon_0 c^3} \left[ (\nu \times \ddot{\pi}(1))^2 + \frac{\alpha^2}{c^2} (\nu \times \ddot{\mu}(1))^2 + \frac{1}{c^2} (\nu \times \ddot{\pi}(2))^2 \right. \\
+ \left. 2 \frac{\alpha}{c^2} \ddot{\mu}(1) \cdot (\nu \times \ddot{\pi}(2)) + \frac{1}{c} (\ddot{\pi}(1) \cdot \ddot{\pi}(2)) - \frac{1}{c} (\nu \cdot \ddot{\pi}(1)) (\nu \cdot \ddot{\pi}(2)) \right],
\] (84)

\[
\frac{2c}{\mu_0} r^2 B_{rad}^{(0)} \cdot B_{rad}^{(2)} = \frac{1}{(4\pi)^2 \varepsilon_0 c^5} \left[ \alpha \ddot{\mu}(2) \cdot (\nu \times \ddot{\pi}(1)) + \frac{1}{3} (\ddot{\pi}(1) \cdot \ddot{\pi}(3)) \\
- \frac{1}{3} (\nu \cdot \ddot{\pi}(1)) (\nu \cdot \ddot{\pi}(3)) \right].
\] (85)

One can determine the averaged quantities from the last two equations using formula 7:

\[ \langle \nu_1 \ldots \nu_n \rangle = \begin{cases} 
0, & n = 2k + 1, \\
\frac{1}{(2k+1)!} \delta_{i_1i_2} \ldots \delta_{i_{n-1}i_n}, & n = 2k, \quad k = 0, 1, \ldots.
\end{cases} \] (86)

The averaged expressions from equations (84) and (85) are given by

\[
\frac{c}{\mu_0} r^2 \langle (B_{rad}^{(0)} + B_{rad}^{(1)})^2 \rangle = \frac{1}{4\pi \varepsilon_0 c^3} \left( \frac{2}{3} \ddot{\pi}^2 + \frac{2\alpha^2}{3c^2} \ddot{\pi}^2 \right) + \frac{1}{20c^2} P_{ij} P_{ij} - \frac{1}{60c^2} P_{ii} \right),
\] (87)

and

\[
\frac{2c}{\mu_0} r^2 \langle B_{rad}^{(0)} \cdot B_{rad}^{(2)} \rangle = \frac{1}{4\pi \varepsilon_0 c^5} \left( \frac{\alpha}{3} \ddot{p}_k \varepsilon_{kij} \dddot{M}_{ij} + \frac{2}{45} \ddot{p}_k \dddot{P}_{kij} \right).
\] (88)

In equations (87) and (88) some standard notation for magnetic and electric moments is used. A simple calculation shows that

\[
\frac{2c}{\mu_0} r^2 \langle B_{rad}^{(0)} \cdot B_{rad}^{(2)} \rangle = -\frac{1}{4\pi \varepsilon_0 c^3} \frac{4}{3c^2} \ddot{p} \cdot \dddot{t},
\] (89)
where \( t \) is defined by equation (54). The total radiated power is then given by

\[
\mathcal{I} = \frac{1}{4\pi\varepsilon_0 c^3} \left( \frac{2}{3} \dddot{p}^2 + \frac{2\alpha^2}{3c^2} \dddot{m}^2 + \frac{1}{20c^2} \dddot{P}_{ij} \dddot{P}_{ij} - \frac{1}{60c^2} \dddot{P}_{ii}^2 - \frac{4}{3c^2} \dddot{p} \cdot \dddot{t} \right). \quad (90)
\]

Considering in equation (90) the multipolar moments at the time \( t \), we have the description of the source emission. This expression differs from the result given in Ref. 2 firstly by the fact that, whereas here the electric 4-polar moment is the primitive one, in this reference it is represented by the corresponding STF tensor and, consequently, the term containing the trace of the electric 4-polar moment is absent. The second difference consists in the presence in equation (90) of the toroidal moment \( t \).

For expressing \( \mathcal{I} \) by STF- tensors, we apply the invariance of the field to the substitutions of all multipole primitive tensors \( P \) and \( M \) by corresponding STF-tensors \( \tilde{P}, \tilde{M} \). These last tensors are, in general, different from the correspondent STF projections as it was shown in Section IV. Obviously, for the approximation corresponding to equation (90), it is sufficient to consider the STF-projections \( \tilde{P}^{(3)} = \mathcal{T}(P^{(3)}) \) and \( \tilde{M}^{(2)} = \mathcal{T}(M^{(2)}) \) and the induced transformations \( P^{(k)} \rightarrow \tilde{P}^{(k)} \) for \( k = 1, 2 \), and \( M^{(k)} \rightarrow \tilde{M}^{(k)} \) for \( k = 1 \) (see Section IV, equations (43)-(54)). Performing these substitutions in equations (87) and (88), we obtain for \( \mathcal{I} \):

\[
\mathcal{I} = \frac{1}{4\pi\varepsilon_0 c^3} \left( \frac{2}{3} (\dddot{p} - \frac{1}{c^2} \dddot{t})^2 + \frac{2\alpha^2}{3c^2} \dddot{m}^2 + \frac{1}{20c^2} \dddot{\Pi}_{ij} \dddot{\Pi}_{ij} \right) + \mathcal{O}(\zeta^5). \quad (91)
\]

The term \( t^2/c^4 \) must be eliminated since it is of order \( \zeta^6 \), such that the final expression of \( \mathcal{I} \) in the required approximation is given by

\[
\mathcal{I} = \frac{1}{4\pi\varepsilon_0 c^3} \left( \frac{2}{3} \dddot{p}^2 + \frac{2\alpha^2}{3c^2} \dddot{m}^2 + \frac{1}{20c^2} \dddot{\Pi}_{ij} \dddot{\Pi}_{ij} - \frac{4}{3c^2} \dddot{p} \cdot \dddot{t} \right). \quad (92)
\]

This is the correct expression of the radiated power in the \( \zeta^4 \)-approximation and it is obviously not represented by independent contributions of \( p, m, \Pi^{(2)} \). In Ref. 3, the contribution of the multipole moment \( P^{(2)} \) is calculated as an independent contribution. The corresponding result is a correct one if the radiation source is an elementary system characterized only by his electric 4-polar moment. The reader must be warned that in case one considers a complex system characterized by higher multipole moments, the correct evaluation of the contributions of the moments \( p, m, P^{(2)} \) cannot be identified with the sum of the independent contributions of these multipole moments. Particularly, this is the reason why in Ref. 3 (§ 71) one obtains an incomplete result when calculating the electric and magnetic...
dipolar and electric quadripolar radiation considered as a result of the multipole expansion of a complex system.

A simpler procedure for obtaining the expression from equation (92) is achieved if in equations (84) and (85) one substitutes \( \pi(t; \nu, n) \rightarrow \tilde{\pi}(t; \nu, n) = \nu^{n-1} ||\tilde{P}^{(n)}(t) \) and \( \mu(t; \nu, n) \rightarrow \tilde{\mu}(t; \nu, n) = \nu^{n-1} ||\tilde{M}^{(n)}(t) \). This procedure has the advantage of a simpler calculation of the \( \nu \)-averaged quantities for higher order terms of \( I \). In appendix A we give some general results for the radiated energy and linear and angular momenta (for detailed calculations see Ref.\(^9\) and the issues cited there).

VII. TRANSLATIONAL INVARIANCE

Let us consider a new point of reference \( O' \) specified in the Cartesian reference system \( \{O, (e_i)_{i=1+3}\} \) by the vector \( a = a_i e_i \). The new reference system in the affine space is defined as \( \{O', (e_i)_{i=1+3}\} \). Here, both systems are associated with the same basis in the vectorial space. For avoiding a possible confusion produced by the notation used, we denote by \( P \) the point where the field is calculated and by \( P' \) the current integration point. Correspondingly, instead of \( B(r, t) \) and \( J_k(r', \tau) \) in equation (11), the notation \( B(P, t) \) and \( J_k(P', \tau) \) will be used. Generally, the multipole moments depend on the point of reference chosen. As it is stressed in Ref.\(^15\), the “origin dependence” of multipole moments forms only a part of the effect of changing the point of reference. Considering consistently all the the changes in the calculation of an observable of the physical system, one can verify the invariance of this observable. We can write equation (11) in the new system of reference as

\[
B'(P, t) = \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \int_D \left( \frac{\partial_j J_k(P', t - \frac{|r - a - \xi|}{c})}{|r - a - \xi|} \right) d^3 x'', \tag{93}
\]

where \( r'' = x''_i e_i \) is the vector associated with the pair \( O'P' \). Writing the Taylor series about \( \xi = 0 \) and, after that, performing the substitution \( \xi = r'' \), one gets

\[
B'(P, t) = \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_D \left( x''_i \ldots x''_n \partial_j \partial_{i_1} \ldots \partial_{i_n} \right) \frac{J_k(P', t - \frac{|r - a|}{c})}{|r - a|} d^3 x''
\]

\[
= \frac{\mu_0}{4\pi \alpha} e_i \varepsilon_{ijk} \sum_{n \geq 0} \frac{(-1)^n}{n!} \int_D (x'_i - a_{i_1}) \ldots (x'_n - a_{i_n}) \partial_j \partial_{i_1} \ldots \partial_{i_n} \frac{J_k(P', t - \frac{|r - a|}{c})}{|r - a|} d^3 x'
\]
or

\[ B'(P, t) = \frac{\mu_0}{4\pi\alpha} e_{i\varepsilon_{ijk}} \sum_{n=0}^{n} \frac{(-1)^n}{n!} \sum_{m=0}^{m} \frac{n!}{m!(n-m)!} a_{i1} \ldots a_{im} \]

\times \int_{\mathcal{D}} \partial_j \partial_{i1} \ldots \partial_{in} x'_{i_{m+1}} \ldots x'_{in} \frac{J_k(P', t - \frac{|r-a|}{c})}{|r-a|} d^3x'.

Performing the Taylor expansion of the fraction \( J_k(P', t - |r-a|/c)/|r-a| \) as a function of \( a \) about \( a = 0 \), it results:

\[ B'(r, t) = \frac{\mu_0}{4\pi\alpha} e_{i\varepsilon_{ijk}} \sum_{n=0}^{n} \frac{(-1)^n}{n!} \sum_{m=0}^{m} \frac{n!}{m!(n-m)!} \frac{(-1)^l}{l!} a_{j1} \ldots a_{j_l} \]

\times \partial_j \partial_{i1} \ldots \partial_{in} \partial_{j1} \ldots \partial_{jl} \left( \frac{1}{r} \int_{\mathcal{D}} x'_{m+1} \ldots x'_{in} J_k(r', t - \frac{r}{c}) \right) d^3x'. \quad (94)

This equation can be written, in a more compact form, as:

\[ B'(r, t) = \frac{\mu_0}{4\pi\alpha} e_{i\varepsilon_{ijk}} \partial_j \sum_{n=0}^{n} \sum_{m=0}^{m} \frac{(-1)^{n-m}}{m!(n-m)!} (a \cdot \nabla)^m \sum_{l=0}^{l} \frac{(-1)^l}{l!} (a \cdot \nabla)^l \left( \nabla^{n-m} \frac{m^{(n-m+1)}(\tau_0)}{r} \right)_k \]

\[ = \frac{\mu_0}{4\pi\alpha} e_{i\varepsilon_{ijk}} \partial_j \sum_{n=0}^{n} \sum_{m=0}^{m} \frac{(-1)^{n-m}}{m!(n-m)!} (a \cdot \nabla)^m e^{-a \cdot \nabla} \left( \nabla^{n-m} \frac{m^{(n-m+1)}(\tau_0)}{r} \right)_k. \quad (95) \]

Obviously, considering the infinite series corresponding to the multipole expansion of the magnetic field, we have no dependence of the choice of the reference point since the Taylor series gives the same values of a function no matter what is the point about the expansion is taken. Indeed, considering the last expression of \( B' \), for the sums after \( n \) and \( m \), we have:

\[ B'(r, t) = \sum_{n=0}^{n} \sum_{m=0}^{m} f_{nm} = \sum_{n=0}^{n} \sum_{m=0}^{m} \theta(n-m) f_{nm} = \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} f_{nm}, \]

i.e. the possibility of inverting the order of the summation after \( n \) and \( m \) in the case of infinite series. Using this property in equation (95) and introducing a new summation index \( q = n - m \), we obtain

\[ B'(r, t) = \frac{\mu_0}{4\pi\alpha} e_{i\varepsilon_{ijk}} \partial_j \sum_{m=0}^{\infty} \sum_{q=0}^{q} \frac{(a \cdot \nabla)^m}{m!} \frac{(-1)^q}{q!} e^{-a \cdot \nabla} \left( \nabla^q \frac{m^{(q+1)}(\tau_0)}{r} \right)_k. \]

Recognizing in the last expression the series of \( \exp(a \cdot \nabla) \) and the expansion (21) of \( B(r, t) \), one can conclude that \( B' = B \), i.e. the translation invariance of the magnetic field multipole.
expansion. For practical calculations, we have to take into account the expression in equation (95) as a double series expansion upon the parameters $a$ and $d$. Since $a \leq d$, the parameters $a$ and $d$ are associated with the subunit and adimensional parameter $\zeta = d/\lambda \sim a/\lambda$.

Obviously, the translation invariance cannot be verified by each term of the field expansion but, for a given approximation defined by a maximum power of the parameter $\zeta$, it is verified for a well defined partial sum from the series. For illustrating how this invariance property is working, we write equation (95) as

$$B'(r, t) = \sum_{n \geq 0} \sum_{m=0}^n \frac{(a \cdot \nabla)^m}{m!} e^{-a \cdot \nabla} B^{(n-m)}(r, t).$$

Let us consider the first three terms from the field series:

$$B^{(0)} = e^{-a \cdot \nabla} B^{(0)} = [1 - (a \cdot \nabla) + \frac{1}{2}(a \cdot \nabla)^2 - \frac{1}{6}(a \cdot \nabla)^3 + \ldots] B^{(0)},$$

$$B^{(1)} = [(a \cdot \nabla) - (a \cdot \nabla)^2 + \frac{1}{2}(a \cdot \nabla)^3 + \ldots] B^{(0)} + [1 - (a \cdot \nabla) + \frac{1}{2}(a \cdot \nabla)^2 + \ldots] B^{(1)},$$

$$B^{(2)} = [\frac{1}{2}(a \cdot \nabla)^2 - \frac{1}{2}(a \cdot \nabla)^3 + \ldots] B^{(0)} + [(a \cdot \nabla) - (a \cdot \nabla)^2 + \ldots] B^{(1)} + [1 - (a \cdot \nabla) + \ldots] B^{(2)}.$$

Since $(a \cdot \nabla)$ is associated to the parameter $\zeta$ and $B^{(0)}$, $B^{(1)}$ and $B^{(2)}$ are of the orders $\zeta$, $\zeta^2$ and $\zeta^3$, respectively, we can express the partial sums up to the order $\zeta^4$ inclusively:

$$B^{(0)} + B^{(1)} = B^{(0)} + B^{(1)} + [- \frac{1}{2}(a \cdot \nabla)^2 + \frac{1}{3}(a \cdot \nabla)^3] B^{(0)} + [- (a \cdot \nabla) + \frac{1}{2}(a \cdot \nabla)^2] B^{(1)} + O(\zeta^5),$$

$$B^{(0)} + B^{(1)} + B^{(2)} = B^{(0)} + B^{(1)} + B^{(2)} - \frac{1}{6}(a \cdot \nabla)^3 B^{(0)} - \frac{1}{2}(a \cdot \nabla)^2 B^{(1)} - (a \cdot \nabla) B^{(2)} + O(\zeta^5).$$

From equations (96) and (97) it is seen that the sum $B^{(0)} + B^{(1)}$ is translational invariant up to the order $\zeta^2$ and the sum $B^{(0)} + B^{(1)} + B^{(2)}$, up to the order $\zeta^3$. An interesting circumstance appears in case one analyzes the second powers of these sums as, for example, when one calculates the radiation intensities. Let us consider the squared sum from equation (96):

$$\left( B^{(0)} + B^{(1)} \right)^2 = \left( B^{(0)} + B^{(1)} \right)^2 + O(\zeta^4).$$
Terms of the orders $\zeta^3$ and $\zeta^4$ are present in $(B^{(0)} + B^{(1)})^2$ such that we have to ensure the translational invariance up the order four in the parameter $\zeta$. Therefore, the contribution of $B^{(2)}$ must be added up since this term introduces the product $B^{(0)} \cdot B^{(2)} \sim \zeta^4$:

$$
(B^{(0)} + B^{(1)} + B^{(2)})^2 = (B^{(0)} + B^{(1)})^2 + 2B^{(0)} \cdot B^{(2)} + O(\zeta^5).
$$

Equation (99)

In terms of the multipole moments $\mathbf{M}$, we conclude that, if one includes the square of the magnetic moment $\mathbf{M}^{(1)}$, then, for ensuring the translation invariance, the inclusion of a partial contribution of $\mathbf{M}^{(2)}$ is necessary, too. This partial contribution is expressed in terms of the contraction of the tensors $\mathbf{M}^{(2)}$ and $\mathbf{M}^{(0)}$. Correspondingly, including the squares of the electric 4-polar moment $\mathbf{P}^{(2)}$ and of the magnetic dipolar moment $\mathbf{m}$ we have to include also partial contributions of the moments $\mathbf{M}^{(2)}$ and $\mathbf{P}^{(3)}$ represented by contractions of the corresponding tensors with the electric dipolar moment $\mathbf{p}$. This property explains why in Ref. 16, introducing in the expression of the radiated power only the sum of the moments $\mathbf{p}$, $\mathbf{m}$ and $\mathbf{P}^{(2)}$ squared, finally, for establishing a translation invariant result one performs a correction by a term represented by contractions of $\mathbf{M}^{(2)}$ and $\mathbf{P}^{(3)}$ with $\mathbf{p}$. In such a way one introduces in fact a contribution of the dipole toroidal moment $\mathbf{t}$.

The procedure can be continued for higher orders. In the following step, for example, the contribution of the square of $B^{(0)} + B^{(1)} + B^{(2)}$ up to the order $\zeta^6$ is obtained as

$$
(B^{(0)} + B^{(1)} + B^{(2)})^2 = (B^{(0)} + B^{(1)} + B^{(2)})^2 + 2(B^{(0)} \cdot B^{(3)} + B^{(1)} \cdot B^{(3)}).
$$

Equation (100)

Besides the contributions of the multipole moments $\mathbf{p}$, $\mathbf{m}$, $\mathbf{P}^{(2)}$, $\mathbf{M}^{(2)}$, $\mathbf{P}^{(3)}$, we have to include also some partial contributions from $\mathbf{P}^{(4)}$ and $\mathbf{M}^{(3)}$ represented by the last two terms from equation (100).

Combining the above observations with the results of the previous section, we can conclude here that all the invariance properties are ensured if a consistent approximation procedure is used. For this reason, we suspect some inconsistencies in considering the approximation criteria when, performing multipole expansion, one concludes that a correction is necessary for ensuring the translational invariance (see Refs. 16, 15, 5).

VIII. CONCLUSION

Along this paper, we mainly exposed the fundamental grounds for the calculus procedure of the multipole expansion of the electromagnetic field.
Firstly, the direct approach in the expansion of the quantities $E$ and $B$ from Jefimenko’s equations revealed that this method is neither more difficult, nor easier than the one based on the expansion of the potentials $A$ and $\Phi$. An advantage of the current approach can be the elimination, at least partly, of the difficulties related to the inversion of different operations, mostly of derivation and integration. Another advantage is offered by the direct examination of the invariance of the fields $E$ and $B$ to the transformations required by the manipulation of the electric and magnetic moment tensors (sections III and IV). In this case, one does not need to employ gauge invariance.

Secondly, in section V we specify the approximation criteria according to which one has to consider three approximation intervals for the field associated to a localized source, as mentioned in Refs.2 and 2. Then, in the next two sections, for radiation calculus, we consistently apply the criterion $d/\lambda$. The consequences are the ones mainly known from Ref.16.

We hope that the discussion from section VII regarding the invariance of the physical results when changing the point of reference, although without an extension to the general multipole terms, is convincing enough. We concluded that the translational invariance is automatically ensured if one respects the approximation criteria when employing multipole expansions.

Since in the main part of the current work we wanted to avoid the generalization to superior orders of the multipole series, in the Appendix we summarized for the interested reader, the main known results in the literature.

We believe that in an electrodynamics lecture, there should exist some remarks on the contributions of the toroidal moments, at least for the quadrupole electric moment. In this approximation, as one can see from sections IV and VI the highlight of the contributions is not involving a considerable computational effort.

**APPENDIX A: SOME BASIC FORMULAE FOR GENERALIZING THE PROCEDURE OF REDUCING THE ELECTRIC AND MAGNETIC MOMENTUM TENSORS**

In this section we list the required formulae for the generalized tensor reducing procedure. The equations are given without the related proofs since they can be found in the literature.
Let $S^{(n)}$ be a symmetric tensor of rank $n$. Its STF-projection results from the following formula:

$$T(S^{(n)})_{i_1...i_n} = S_{i_1...i_n} - \delta_{\{i_1i_2}\}(S^{(n)})_{i_3...i_n}. \quad (A1)$$

The operator $\Lambda$ is defined on the account of a formula for the STF-projection of a symmetric tensor given in Ref. 7 (the book 17 is cited as the origin of this formula):

$$\Lambda(S^{(n)})_{i_1...i_n-2} = \frac{[n/2-1]}{(m+1)(2n-1)!!} \sum_{m=0}^{n-1} (-1)^m [2n - 1 - 2(m + 1)]!! \times \delta_{\{i_1i_2...i_{2m-1}i_{2m}\}} S^{(n,m+1)}_{i_{2m+1}...i_{n-2}}. \quad (A2)$$

The proof can be found in Ref. 18. The notation $S^{(n,p)}$ indicates $p$ pairs of contracted indices.

In the following, for simplifying the notation, all arguments of the operator $\Lambda$ should be considered symmetric tensors, i.e. $\Lambda(T^{(n)}) = \Lambda(S(T^{(n)}))$ for any tensor $T^{(n)}$. The same applies to the operator $T$: $T(T^{(n)}) = T(S(T^{(n)}))$, $S$ being the symmetrization operator.

In the symmetrization process we have to calculate the symmetric projections of some tensors $L^{(n)}$ of the magnetic moment type: they are symmetric in the first $n - 1$ indices and the contraction of $i_n$ with any index $i_q$, $q = 1...n - 1$ gives a null result. For the symmetric projection of such a tensor, we introduce the formula:

$$(S(L^{(n)}))_{i_1...i_n} = L_{i_1...i_n} - \frac{1}{n} \sum_{\lambda=1}^{n-1} \varepsilon_{i_1i_2...i_n} \mathcal{N}^{(\lambda)}_{i_1...i_{n-1}q} (L^{(n)}), \quad (A3)$$

where $\mathcal{N}^{(\lambda)}_{i_1...i_{n-1}}$ is the component with the $i_{\lambda}$ index suppressed. The operator $\mathcal{N}$ defines a correspondence between $L^{(n)}$ and a tensor of rank $(n - 1)$ of the same type from the symmetry point of view. Particularly,

$$\mathcal{N}^{2k}(M^{(n)}) = \frac{(-1)^k n}{(n+1)\alpha} \int_{\mathcal{D}} (r^2)^k r^{n-2k-1} \times J \, d^3x, \quad (A4)$$

where $a \times b$ is the tensor defined by the components $(a \times b)_{i_1...i_n} = a_{i_1}...a_{i_n-1} (a \times b)_{i_n}$. Let us consider the process of STF reducing for the multipole tensors starting with the rank $N + 1$ in the electric case, and $N$ in the magnetic one. A general formula for the resulting
tensors $\tilde{P}^{(n)}$, for $n = 1, \ldots, N$ and $\tilde{M}^{(n)}$, for $n = 1, \ldots, N - 1$ is given below. It includes all compensating terms obtained in this process

$$
\tilde{P}^{(n)} = \mathcal{T}(\tilde{P}^{(n)}) + \sum_{k=1}^{\lceil(N-n)/2 \rceil} \frac{(-1)^k}{e^{2k}} \frac{\partial^{2k-1}}{\partial t^{2k-1}} T_k^{(n)},
$$

$$
T_k^{(n)} = (-1)^k c^{2k} \mathcal{T}\left(A_k^{(n)} \Lambda^k (\tilde{P}^{(n+2k)}) + \sum_{l=0}^{k-1} B_{k-l}^{(n)} \Lambda^l \Lambda^{2k-2l-1}(\tilde{M}^{(n+2k-1)})\right), \quad (A5)
$$

and

$$
\tilde{M}^{(n)} = \mathcal{T}(\tilde{M}^{(n)}) + \mathcal{T}\left(\sum_{k=1}^{\lceil(N-n)/2 \rceil} \frac{\partial^{2k}}{\partial t^{2k}} \sum_{l=0}^{k} C_{k-l}^{(n)} \Lambda^l \Lambda^{2k-2l}(\tilde{M}^{(n+2k)})\right). \quad (A6)
$$

The coefficients are given in a compact algebraic form in Ref.2:

$$
A_k^{(n)} = \frac{1}{2^k c^{2k}} \frac{n}{n + 2k},
$$

$$
B_{k,l}^{(n)} = \frac{(-1)^{k-l+1} \alpha}{2^l c^{2k+2}} \frac{n(n+2l)!}{(n+2k+1)(n+2k+1)!},
$$

$$
C_{k,l}^{(n)} = \frac{(-1)^{k-l}}{2^l c^{2k}} \frac{n(n+2l)!}{(n+2k)(n+2k)!}. \quad (A7)
$$

The reader is encouraged to apply these formulae for the cases $N \geq 4$ and to find convenient intermediary calculation. We point out that in equation (A5), the normalization of the quantities $T^{(n)}$ is chosen such that the quantities $T_1^{(n)}$ coincide with the electric toroidal moments given in literature at least for $n$ up to 3. We present the results given in Ref.2 concerning the total radiated power $\mathcal{I}$, the recoil force $F_r$ and angular momentum loss $dL/dt$:

$$
\mathcal{I} = \frac{\alpha^2}{4 \pi \varepsilon_0 c^3} \sum_{n \geq 1} \frac{n+1}{n!(2n+1)!!} \left[ (\tilde{M}^{(n)}_{n+1} \cdot \tilde{M}^{(n)}_{n+1}) + \frac{c^2}{\alpha^2} (\tilde{P}^{(n)}_{n+1} \cdot \tilde{P}^{(n)}_{n+1}) \right], \quad (A8)
$$

$$
F_R = -\frac{\mu_0}{2 \pi c^3} \sum_{n \geq 1} \frac{1}{c^{2n}} \left\{ \frac{n+2}{(n+1)!(2n+3)!!} (\tilde{M}^{(n)}_{n+1} \cdot \tilde{M}^{(n+1)}_{n+2}) + \frac{c^2}{\alpha^2} (\tilde{P}^{(n)}_{n+1} \cdot \tilde{P}^{(n+1)}_{n+2}) \right\}
$$
$$
\quad - \frac{c^2}{\alpha n! n(2n+1)!!} \varepsilon_{ij} \varepsilon_{jk} \left( \tilde{M}^{(n)}_{i+n+1} \cdot n - 1, \tilde{P}^{(n)}_{i+n+1} \right)_{jk}, \quad (A9)
$$

with the obvious notation,

$$
(A^{(n)} \cdot n - 1 \cdot B^{(n)})_{jk} = A_{i_1,\ldots,i_{n-1}j} B_{i_1,\ldots,i_{n-1}k}
$$
and

\[ f_{,k} = \frac{\partial^k}{\partial t^k} f, \]

\[
\frac{dL}{dt} = \frac{\mu_0}{4\pi c} \sum_{n \geq 1} \frac{1}{c^{2n}} \left\{ -\frac{n+1}{n!(2n+1)!!} \varepsilon_i \varepsilon_{ijk} \times \left[ \left( \frac{c^2}{\alpha^2} \left( P^{(n)}_n \cdot n - 1 \cdot \tilde{P}^{(n)}_{n+1} \right) \right)_{jk} + \left( M^{(n)}_n \cdot n - 1 \cdot \tilde{M}^{(n)}_{n+1} \right) \right] \right. \\
\left. \quad + \frac{n+2}{an!(2n+3)!!} \left[ \tilde{P}^{(n+1)}_{n+1} \cdot \tilde{M}^{(n)}_{n+1} - \tilde{M}^{(n+1)}_{n+1} \cdot \tilde{P}^{(n)}_{n+1} + \tilde{M}^{(n)}_n \cdot \tilde{P}^{(n+1)}_{n+2} - \tilde{P}^{(n)}_n \cdot \tilde{M}^{(n+1)}_{n+2} \right] \right\}.
\] (A10)

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