Consider a periodic tiling of a plane by equal triangles obtained from the equilateral tiling by a linear transformation. We study a following tiling billiard: a ball follows straight segments and bounces of the boundaries of the tiles into neighboring tiles in such a way that the coefficient of refraction is equal to $-1$. We show that almost all the trajectories of such a billiard are either closed or escape linearly, and for closed trajectories we prove that their periods belong to the set $\mathbb{N}^+$. We also give a precise description of the exceptional family of trajectories (of zero measure): these trajectories escape nonlinearly to infinity and approach fractal-like sets. We show that this exceptional family is parametrized by the famous Rauzy gasket. This proves several conjectures stated previously on triangle tiling billiards. In this work, we also give a more precise understanding of fully flipped minimal exchange transformations on 3 and 4 intervals by proving that they belong to a special hypersurface. Our proofs are based on the study of Rauzy graphs for interval exchange transformations with flips.

1. Introduction

1.1. What are tiling billiards and how do they behave? Background and main result

Take a tiling (decomposition) of a plane by the shapes (possibly of infinite volume) with piece-wise smooth boundary. Consider a following billiard in such a tiling. A particle on the plane follows a straight line till it hits the boundary of one of the tiles. Then the trajectory continues in the neighboring tile, following the rule of negative refraction with coefficient $-1$. In other words, the oriented angle that the trajectory makes with the side of the tile, changes its sign but keeps the same absolute value. We call the dynamical system defined in this way a tiling billiard.

In this article, we will restrict ourselves to the case where the tiles are polygons or generalized polygons (infinite volume tiles with boundaries consisting of the union of straight-line segments or rays). Tiling billiards in the square tiling and the equilateral triangle tilings have been first studied in a preprint [Mascarenhas and Fluegel 15] by Mascarenhas and Fluegel from the point of view of physics of negative refraction of light. Unfortunately, this article has never been published and is not accessible on-line. Although, this study seems to be quite relevant since recently discovered materials can exhibit negative indices of refraction, see [Shelby et al. 01, Smith et al. 04, Valentine et al. 08].

Tiling billiards got their name and were first presented as an interesting mathematical object in [Davis et al. 16], where the first nontrivial case of periodic triangle tilings was considered. These tilings by congruent triangles are obtained by cutting the plane by three families of equidistant parallel lines. In this article, we mostly concentrate on the dynamics of such negative refraction billiards in these periodic triangle tilings: triangle tiling billiards. Our work is inspired by [Baird-Smith et al. 19], where the connection of these billiards with interval exchange transformations with flips is pointed out. In their work, Baird-Smith, Davis, Fromm, and Iyer show that the dynamics of triangle tiling billiards has a first integral: the (oriented) distance between a segment of a trajectory in each crossed triangle and its circumcenter. In particular, if a trajectory passes through a circumcenter of one of the triangles that it crosses, it passes through the circumcenters of all the crossed triangles.
The existence of the first integral is crucial in order to show the (very fruitful!) connection of these billiards with interval exchange transformations with flips. It also helps to prove following results about triangle tiling billiards: first, each trajectory passes through any tile at most once and second, all bounded trajectories are closed. Even more surprisingly, the authors manage to construct, as a corollary of results in [Lowenstein 07], a singular trajectory (with a branching point in some vertex of the tiling) in a triangle tiling billiard that exhibits fractal behavior and passes through all of the triangles in the tiling.

In this work, we give a full description of the qualitative behavior of trajectories of triangle tiling billiards. Our main result is the following

**Theorem 1.** Consider a trajectory $\delta$ of a triangle tiling billiard with the tiles congruent to the triangle $\Delta$. Suppose that $\Delta$ has the angles in its vertices equal to $\alpha$, $\beta$, and $\gamma$. Let $\mathcal{R}$ be the set of triangles $\Delta$ such that the point $p := (1 - \frac{2}{n} \alpha, 1 - \frac{2}{n} \beta, 1 - \frac{2}{n} \gamma) \in \mathbb{R}^3$ belongs to the Rauzy gasket, $p \in \mathcal{R}$. Let $C$ be the (well defined) set of trajectories that pass through the circumcenters of the crossed triangles. Then exactly one of the following four cases holds for $(\delta, \Delta)$:

1. a trajectory $\delta$ is closed and stable under perturbation and $(\delta, \Delta) \notin C \times \mathcal{R}$. Furthermore, the period of $\delta$ is equal to $4n + 2, n \in \mathbb{N}^*$;
2. a trajectory $\delta$ is drift-periodic (is linearly escaping with a translation symmetry), the angles of $\Delta$ are dependent over $\mathbb{Q}$ (and, automatically, $\Delta \notin \mathcal{R}$);
3. a trajectory $\delta$ is linearly escaping and its symbolic dynamics can be described as a Sturmian sequence and $(\delta, \Delta) \notin C \times \mathcal{R}$;
4. a trajectory $\delta$ is nonlinearly escaping, $\Delta$ is acute and $(\delta, \Delta) \in C \times \mathcal{R}$.

This result gives a positive answer to Conjectures 4.19 and 5.1 in [Baird-Smith et al. 19], about the behavior of exceptional trajectories as well as about the periods of all closed periodic trajectories.

As a corollary we get that almost any triangle tiling trajectory is either closed or linearly escaping. This property is related to the notion of integrability of an interval exchange transformation with flips that we define and study throughout this article. We borrowed the name from the terminology for the studies on Novikov's problem on the semiclassical motion of an electron [Dynnikov 97, Novikov 82, Zorich 84], since we think that these problems are related and hope to study them in future work.

The proof of our main theorem uses in a crucial way a powerful tool of the modified Rauzy induction for interval exchange transformations with flips. Such a modification of the Rauzy induction was first introduced by Nogueira in [Nogueira 89]. He proved that almost any interval exchange transformation with flips has a periodic sub-interval: Rauzy induction almost always stops. We implicitly use this result all along our work: triangle tiling billiards have abundant and stable closed trajectories exactly thanks to the phenomenon noticed by Nogueira. Nogueira's theorem shows how different interval exchange transformations with flips are from classic interval exchange transformations that do preserve orientation: the first are almost never minimal, the second are almost always minimal.

The proof of Theorem 1 proceeds in two major steps. First, we prove that there exists an invariant hyperspace for the Rauzy induction procedure that corresponds exactly to the space of trajectories hitting the circumcenters. This step uses as a key result Lemma 3 which we have proven by some (not very heavy but still...) computer assisted calculations of Rauzy graphs for interval exchange transformations with flips. For now, Lemma 3 seems quite miraculous and one of the major goals of our future work is to understand reasons behind its claim. The second step of the proof of Theorem 1 is based on a more precise understanding of the structure of permutations corresponding to the stopping points of Rauzy induction.

This article also contains links to the Rauzy graphs that were drawn by the program written by Paul Mercat to conclude the proof of Lemma 3 as well as to illustrate some of the arguments. We think that the study of the Rauzy graphs for interval exchange transformations with flips is a very interesting area for future research. As already mentioned above, these graphs are very different from Rauzy graphs for classical interval exchange transformations. The connections between these two worlds may be interesting to explore. We speak about this more in the last Section 7, as well as about other perspectives and open questions related to tiling billiards and interval exchange transformations with flips.

To conclude the introduction, we would like to say that the area of tiling billiards is a very young and

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1To be precise, the result in this form is not yet proven but we formulate it like this for simplicity of exposition. We indeed prove the necessary condition of point 4. For sufficient condition, we prove a little less than we would like to. The non-linearly escaping behavior holds for almost all points $(\delta, \Delta) \in C \times \mathcal{R}$ with respect to the natural measure on the Rauzy gasket but we strongly believe the nonlinearly escaping behavior holds for all points in $C \times \mathcal{R}$. See Proposition 17 for the exact statement.

2Except for the cases when they obviously are not! See classical Keane's theorem [Keane 75].
small (for now…) niche of dynamics of billiards. We find it very attractive. As far as we know, there are only a very few works on this subject. In addition to the works already mentioned, we are aware of the existence of two more works. First, in [Davis and Patrick Hooper 18] Davis and Hooper study tiling billiards on the trihexagonal tiling and their ergodic properties. Second, in a slightly larger setting, Glendinning in [Glendinning 16] studies the dynamics of tiling billiards on the standard infinite checkerboard. He supposes that the black and white tiles have different refraction coefficient indices, $k_1$ and $k_2$, and relates the dynamics to interval exchange transformations in the case when $\frac{k_2}{k_1} > \sqrt{2}$.

As far as we know, the first published (in 2016) works on tiling billiards are [Davis et al. 16] and [Glendinning 16]. Although, one could say that the story of tiling billiards starts almost thirty years earlier, in 1989 with the work of Nogueira [Nogueira 89] on interval exchange transformations with flips. Indeed, on the last page of his work, Nogueira defines the billiards with flips in polygons. One can see that the study of such a billiard in a square is equivalent to the study of the square periodic tiling billiard, and a study of such a billiard in a triangle is equivalent to the study of triangle tiling billiards. In this work this connection will be made explicit.\footnote{For the explanation of this connection, see the triangletangent system in paragraph 2.1.}  Although, the system of Nogueira can’t be generalized to the tilings with different tiles, for example.

### 1.2. Plan of the paper

Different dynamical systems equivalent to the triangle tiling billiard are defined in Section 2. One of them is a family of interval exchange transformations with flips (fully flipped 3-interval exchange transformations on $S^1$). Qualitative behavior of orbits in triangle tiling billiards are discussed in Section 2.3. A modified Rauzy induction for interval exchange transformations with flips is precisely defined in Section 3. This is a crucial tool in this work. Rauzy graphs are also introduced and the work of Nogueira is revisited. Section 4 gives a necessary condition for minimality. The proof uses the modified Rauzy induction. Section 5 explores integrability for interval exchange transformations with flips. In Section 6, properties of the orbits of triangle tiling billiards are derived from tools and ideas introduced in the previous sections: symbolic dynamics, periodic orbits, generic and exotic dynamics are considered. Further remarks and open questions are mentioned in Section 7.

We highlight that the statement of Theorem 1 is a union of statements of Propositions 13, 14, and 17. These propositions will be proved separately in the article.

### 2. Different approaches of triangle tiling billiards

#### 2.1. Four seemingly different dynamical systems

Take some triangle $\Delta$. Suppose that this triangle $\Delta$ has its angles equal to $\alpha$, $\beta$, and $\gamma$. From now on till the end of the article the corresponding vertices are denoted by $A$, $B$ and $C$ and the sides facing these vertices—by $a$, $b$, and $c$. This definition was given in the introduction but we repeat it here for completeness.

**Definition 1.** (Triangle tiling billiards). Consider a periodic tiling of the plane by triangular tiles which are all congruent to $\Delta$ which is obtained by cutting the plane by three families of equidistant parallel lines. A triangle tiling billiard is a dynamical system of a motion of a point particle in such a tiling defined in a following way. A particle follows a straight line till it hits the side of some tile. The trajectory continues in the neighboring tile, following the rule of negative refraction with coefficient $-1$, see Figure 1.

Throughout this article we will be interested in studying the dynamics of triangle tiling billiards for different triangles $\Delta$ and different initial conditions of the trajectory. Note that the tiles can be rescaled in such a way that $\Delta$ has area 1—the dynamics is invariant under homothety. The parameters of the dynamics are hence the angles $\alpha$, $\beta$, $\gamma$. We denote the sides of $\Delta$, corresponding to these angles, as $a$, $b$, and $c$.

Now, let us give three more definitions of other, seemingly unrelated, dynamical systems. Then we will clarify their connection to the triangle tiling billiards until the end of this Section.

**Definition 2** (Reflection in a circumcircle.). Consider a circle on the complex plane centered at the origin. Fix a triangle $\Delta$ that can be inscribed in this circle. For any $\tau \in (0, 1)$ define an oriented chord $l$ in the circle that connects the point with argument $2\pi \tau$ with the point of argument 0 and is headed to the later. Let us now define a following dynamical system on some subset of this circle. For any $X \in [0, 1)$, inscribe a triangle $\Delta_X$ congruent to $\Delta$ in such a way that the vertices $A$, $B$, and $C$ are placed on the circle in a...
counter-clockwise manner and that the argument of the vertex \( A \) as of a complex number is equal to \( 2\pi x \).

The dynamical system will be defined for a subset of such \( X \in [0, 1) \) such that the chord \( l \) intersects the corresponding triangle \( \Delta x \). Take the last (following the orientation of the chord \( l \)) side \( s_X \) of triangle \( \Delta x \) that the chord \( l \) intersects. Define \( \Delta_X' \) as a triangle congruent to \( \Delta_X \), inscribed in the same circle, sharing the side \( s_X \) with \( \Delta_X \) and having an opposite orientation. Now define \( \Delta_{x'} \) as a triangle obtained by reflecting \( \Delta_X' \) with respect to the diameter of the circle perpendicular to the chord \( l \). The orientation of \( \Delta_{x'} \) is the same as that of the initial triangle \( \Delta_X \). The map \( F_{\Delta X; l} : X \mapsto X', X \in S^1 \) is a reflection in a circumcircle, see Figure 2.

Let us make a couple of important remarks. First, for the map \( F_{\Delta X; l} : X \mapsto X \) defined analogically, we see that its square is equal to that of the map \( F_{\Delta X; l} \) of the reflection in a circumcircle: \( F^2 = F^2 \). Second, the reflection in a circumcircle is not necessarily defined on the full circle \( S^1 \). For example, for the obtuse triangle \( \Delta \), the map \( F_{\Delta X; l} \) is never defined on the full circle for any \( \tau \in [0, 1) \). On the contrary, for \( \Delta \) acute and \( \tau = \frac{1}{2} \), the map \( F_{\Delta X; l} \) is defined on the full circle. In this case \( l \) is a diameter.

**Definition 3 (Triangletangent system)**. Fix a triangle \( \Delta \) inscribed in its circumcircle, and fix a number \( \tau \in (0, 1) \) for a parameter. Consider a smaller circle \( \mathbb{T} \) homothetic to the initial circumcircle, with a coefficient of homothety equal to \( |\cos \pi \tau| \). The **triangletangent system** is a map defined on the (one-dimensional) space of oriented segments connecting two sides of the triangle \( \Delta \) and tangent to \( \mathbb{T} \). These segments are parametrized by a subset of points on the sides of the triangle \( \Delta \) which correspond to their end points. Then for any segment \( X \) one associates a segment with a point \( X' \) on the same side of the triangle but symmetrical with respect to the middle of the side. The map \( F_{\Delta X; \mathbb{T}} : X \mapsto X' \) defined in such a way on the space of oriented tangent segments is a reflection in a triangle with respect to a circle of tangency, see Figure 3.

The map \( F_{\Delta X; \mathbb{T}} \) here is well-defined. Indeed, if there exists a segment tangent to \( \mathbb{T} \) with an end-point in \( X \), then the segment tangent to \( \mathbb{T} \) with an end-point in a symmetrical point \( X' \) exists as well. This follows from the fact that the circle \( \mathbb{T} \) has its center in the circumcenter of \( \Delta \), and the circumcenter is placed on the intersection of line segment bisectors. Note that as in the case of the reflection in a circumcircle, the system
$F_{\Delta T}$ is not necessarily defined for any point $X$ on the sides of the triangle. For $\Delta$ acute, $F_{\Delta T}$ is defined everywhere if $\tau$ is small enough—smaller than any of the distances between the circumcenter to the sides of the triangle $\Delta$.

This system in this form was suggested to us by Shigeki Akiyama. This system is also a restriction on some subset of a billiard with flips that was defined by Nogueira for any polygon (and not only a triangle) in [Nogueira 89]. Although in his approach, Nogueira didn’t mention the first integral that appears in the case of triangle billiard (in this definition, this invariant is represented by the circle $T$). The triangletangent system is exactly the restriction of Nogueira’s billiard in a triangle to the subset of trajectories with the same value of the first integral.

**Definition 4 (Fully flipped 3-interval exchange transformations on $S^1$).** Fix the numbers $\tau \in [0,1)$ and $l_j \in \mathbb{R}_{+}, j = 1, 2, 3$ such that $\sum_{j=1}^{3} l_j = 1$. Define a map $F : S^1 \to S^1$ by a following explicit formula:

$$F_{c_{\theta_1,\theta_2,\theta_3}}(x) := \begin{cases} 
-x + l_1 + \tau \mod 1 & \text{if } x \in I_2 := [0, l_1) \\
-x + l_2 + \tau \mod 1 & \text{if } x \in I_3 := [l_1, l_1 + l_2) \\
-x + l_3 + \tau \mod 1 & \text{if } x \in I_4 := [l_1 + l_2, 1) 
\end{cases}$$

We call such a transformation a fully flipped 3-interval exchange transformation on the circle with a trivial permutation of intervals. We denote the set of all such transformations by CET$_3^T$.

The letter C in the name CET$_3^T$ corresponds to the word circle, the letters E and T—to (interval) exchange transformation, and the number 3 to the number of continuity intervals on the circle, $\tau$ being a parameter. A map $F_{c_{\theta_1,\theta_2,\theta_3}} \in$ CET$_3^T$ is a continuous transformation of the circle outside a three-point set $\{l_1, l_1 + l_2, 1\} \in S^1$. Each one of the intervals of continuity is shifted by $\tau$ and then flipped. See Figure 4 for the illustration.

Let us make a couple of remarks: first, for $F \in$ CET$_3^T$, its square $T = F^2$ is a standard interval exchange transformation (without flips). For almost all the values of parameters, the map $T$ is a 6-interval exchange transformation on the circle $S^1$. Second, the space CET$_3^T$ is parametrized by a three-dimensional space which is a direct product of the simplex of lengths and the circle of parameter $\tau$.

### 2.2. The same system and its four faces: Different tools for understanding

Here are the four dynamical systems defined in a previous paragraph:

1. Triangle tiling billiards guided by the negative refraction in the tiling with tiles congruent to $\Delta$ (Definition 1);
2. reflection $F_{\Delta_1}$ of a triangle in its circumcircle by following an oriented fixed chord (Definition 2);
3. triangletangent system $F_{\Delta T}$ (Definition 3);
4. a family of maps in CET$_3^T$ (Definition 4).

What do these systems have in common? All of the systems use a triangle $\Delta$ as a parameter: for all of them except for the system 4 it is explicit, and for this one the lengths of the intervals of continuity $l_j, j = 1, 2, 3$ can be reparametrized to correspond to the angles of some triangle $\Delta$. The systems 2, 3 and 4) are one-dimensional systems (they are all defined on the circle or on its subsets), and all of them have a parameter $\tau$ in them (defining either the chord $l$, the circle $T$ or the shift $\tau$ in the action of a fully flipped IET). A triangle billiard 1 is a 2-dimensional system.

A simple and crucial remark (“folding observation”) from [Baird-Smith et al. 19] helps to reduce the triangle tiling billiards to a 1-dimensional system by finding the first integral $d(\delta)$ for its trajectories. Indeed, for any trajectory $\delta$ of a triangle tiling billiard and two consecutive triangles $\Delta, \Delta'$ that it crosses, these triangles can be folded one onto another (along...
the crossed edge). In this way, the segments of the trajectory fold onto one line and the triangles fold in such a way that their images have a common circumcircle. This permits to prove

**Proposition 1** ([Baird-Smith et al. 19]). In a triangle tiling billiard the following holds:

1. Every trajectory crosses each triangle in the tiling at most once;
2. the distance \( d(\delta, \Delta) \) between a segment of a trajectory in any triangle \( \Delta \) crossed by it and the circumcenter of \( \Delta \) is an invariant of the trajectory \( d(\delta) \) (doesn’t depend on \( \Delta \)). Moreover, the circumcenter of each crossed by the trajectory \( \delta \) triangle \( \Delta \) stays on the same side from the (oriented) segment of \( \delta \);
3. all bounded trajectories are closed.

This proposition gives an understanding of relationships between the first three dynamical systems in our list. For a fixed triangle \( \Delta \), the dynamics of the system 2 for the same \( \Delta \) and for all \( \tau \) depicts a general dynamics of the triangle tiling billiard 1. The parameter \( \tau(l) \) which is defining the chord \( l \) in the system 2 is the invariant of triangle tiling billiard trajectories, with the relationship \( d(\delta) = |\cos(\pi\tau(l))| \). For example, understanding the trajectories passing through the circumcenters of the crossed triangles \((d(\delta) = 0)\) is equivalent to the understanding of the system 2 for \( l \) being a diameter \((\tau = \frac{1}{2})\). The passage from the first system to the second is made by folding the triangles of the tiling along the trajectory \( \delta \). The third system is the same as the second: the connection can be seen by adding on the Figure 2 a circle of radius \(|\cos(\pi\tau(l))|\) concentric to an already drawn circle, and to notice that the chord \( l \) is tangent to it. In other words, in the system 2 the chord is fixed and a triangle moves, and in the system 3 a chord moves while the triangle is fixed. Then, Theorem 3.3 in [Baird-Smith et al. 19] shows that the system 2 written out explicitly as a map of the circle gives a map \( F_{l_1,l_2,l_3,l_4} \in \text{CET}_1^3 \) with parameters \( l_i \) corresponding to normalized angles and \( \tau = \tau(l) \) (at least, where the map from \( F_{A,l} \) is defined).

We are interested in the understanding of the behavior of trajectories in triangle tiling billiards. The other three dynamical systems of this Section give different approaches of this same problem. Most of our tools (Sections 3–6) are applied directly to the systems in 4.

### 2.3. Qualitative behavior of orbits in triangle tiling billiards

As a corollary of Proposition 1, one obtains a following classification (that we will use as well as definition) of different orbit behavior in a triangle tiling billiard.

**Definition 5** (Types of trajectories of triangle tiling billiards). Any trajectory of a triangle tiling billiard is one of these three types.

- **Closed (periodic) orbits.** A trajectory is closed if it is a closed piecewise linear curve in the plane, without self-intersections.
- **Drift-periodic orbits.** A trajectory is drift-periodic if it is invariant under a translation of the plane.
- **Escaping orbits.** A trajectory is escaping if it is not periodic (neither closed nor drift-periodic).

See Figure 5 for some examples of trajectories. Here are some remarks on the qualitative behavior that were clear from the previous work on triangle tiling billiards [Baird-Smith et al, 13, 19]. First, drift-periodic orbits occur only when \( \Delta \) has rationally dependent angles (i.e. the ratios of the angles \( \frac{a}{b}, \frac{b}{c} \in \mathbb{Q} \)). Even more, if the angles are rationally dependent, any orbit is periodic (closed or drift-periodic), see [Baird-Smith et al. 19]. Let us note that both closed and drift-periodic orbits correspond to the periodic orbits of the system 2. Second, an important corollary of the “folding observation” (see paragraph 2.2) is that closed orbits come in open families. If \( \delta \) is a closed trajectory then for a small enough perturbation of a point \((\delta, \Delta)\) the obtained trajectory is still closed. Indeed, an orbit close to a periodic one, by continuity, will continue the path in the same sequence of folded triangles (for a more quantitative explanation, see Theorem 4.2 in [Baird-Smith et al. 19]). Third, escaping orbits happen to have two types of qualitatively different behaviors (and these two behaviors do occur). We will distinguish between **linearly escaping orbits** and **nonlinearly escaping orbits**.

**Definition 6** (Escaping trajectories). A trajectory of a tiling billiard is **linearly escaping** if it stays at a bounded distance from some fixed line in the plane and is not drift-periodic. An escaping trajectory which is not linearly escaping, is **nonlinearly escaping**.

As we will show in this article, linearly escaping orbits come in sets of positive measure in the space \( \{(\delta, \Delta)\} \) of triangle tiling billiard trajectories, although nonlinearly escaping orbits are truly exceptional (and correspond to a zero measure set in this space). The first examples of nonlinearly escaping orbits was given in [Baird-Smith et al. 19] as a corollary of the noticed connection between triangle tiling billiards and Arnoux–Yoccoz 6-IET on the circle. These orbits were constructed as passing through the circumcenters of
the triangles in the tiling corresponding to \( \Delta \) with angles \(\frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2} \), where \( x + x^2 + x^3 = 1, x \in \mathbb{R} \) is the Tribonacci number. Our main goal in the following is to show that all of the nonlinearly escaping trajectories are passing through circumcenters of the crossed triangles in the tiling. And even more, the triangles \( \Delta \) that permit (as forms of tiles) the existence of nonescaping trajectories, are parametrized by the Rauzy gasket \( \mathcal{R} \) (see its Definition below, Definition 25). This proves the Conjecture 4.19 from [Baird-Smith et al. 19].

3. Modified Rauzy induction and its Rauzy graphs

The Rauzy induction is a powerful tool in the study of interval exchange transformations. It was introduced in [Rauzy 79] by Rauzy. For the study of IETs

Figure 5. Examples of trajectories in triangle tiling billiards. From left to right, from up to down: a periodic trajectory of period 10; a periodic trajectory of period 114; a drift-periodic trajectory (having a translation symmetry) of drift period 6 in the 30–60–90-triangle; linear escaping trajectory; another linear escaping trajectory (no translation symmetry, coding of crossed sides corresponds to a sturmian sequence, and a trajectory stays in a bounded distance from some fixed line in \( \mathbb{R}^2 \) with an irrational slope); a non-linearly escaping trajectory which spirals out to infinity, the first 15000 segments of the trajectory are drawn. These pictures are drawn by the program [Hooper] authored by P. Hooper and A. St Laurent, accessible on-line. Remark: finding the last trajectory by randomly adjusting the parameters on the computer is impossible—to find it, we had first to know how to search for it, see Theorem 17 first.
with flips the standard Rauzy induction procedure has
to be modified. This was first done by Nogueira in
[Nogueira 89]. The first two paragraphs of this
Section don’t contain any new results but only intro-
duces the core objects. Our presentation follows the
notations analog to those chosen by Delecroix in his
Lecture notes on IETs [Delecroix 16], where he in his
turn follows [Marmi et al. 10]. We adjust the not-
ations in order to present the modified Rauzy induc-
tion for interval exchange transformations with flips.

3.1. Interval exchange transformations with and
without flips and their dynamics

From now on, we denote
\[ I := (1, ..., 1) \in \mathbb{R}^n. \]

**Definition 7 (Interval exchange transformations).** An
interval exchange transformation (IET) of the interval
\([a, b)\) is a bijection \(T : [a, b) \to [a, b)\) such that there
exist the points \(a = a_0 < a_1 < \ldots < a_n = b\) on the interval
and the numbers \(t_1, ..., t_n \in \mathbb{R}\) (shifts) such that
\[ T_{\lfloor a_i, a_{i+1}\rfloor}(x) = x + t_i, 1 \leq i \leq n. \]

If \(F : [a, b) \to [a, b)\) is a bijection between the sets
\([a, b) \cup \{a_i\}\) and \([a, b) \cup F(a_i)\) and if \(k \neq I\),
then \(F\) is an interval exchange transformation with flips
(IETF). If \(k_i = -1\) (or 1), then we say that the interval
\([a_{i-1}, a_i)\) is flipped (or not flipped).

The set of all such transformations is denoted by
IETF\(^n\)\([a, b)\) or, in the case when the interval is not speci-
fied, simply IETF\(^n\). The vector \(k\) is called a vector of flips.

Note that the Definition 7 is a part of Definition 8
when \(k = I\).

**Definition 8 (Interval exchange transformations with
flips).** Fix the points \(a = a_0 < a_1 < \ldots < a_n = b\) on the interval
and the numbers \(t_1, ..., t_n \in \mathbb{R}\) as well as the vector \(k = (k_1, ..., k_n)\)
with \(k_i \in \{-1, 1\}\). Then define a map \(F : [a, b) \to [a, b)\) in a following way:
\[ F_{\lfloor a_i, a_{i+1}\rfloor}(x) = k_i x + t_i, 1 \leq i \leq n. \]

If \(F : [a, b) \to [a, b)\) is a bijection between the sets
\([a, b) \cup \{a_i\}\) and \([a, b) \cup F(a_i)\) and if \(k \neq I\), then \(F\) is an interval exchange transformation with flips
(IETF). If \(k_i = -1\) (or 1), then we say that the interval
\([a_{i-1}, a_i)\) is flipped (or not flipped).

The set of all such transformations is denoted by
IETF\(^n\)\([a, b)\) or, in the case when the interval is not speci-
fied, simply IETF\(^n\). The vector \(k\) is called a vector of flips.

Note that the Definition 7 is a part of Definition 8
when \(k = I\).

**Definition 9 (Fully flipped interval exchange trans-
formations).** We say that \(F \in \text{IETF}^n[a, b)\) is a fully flipped interval
exchange transformation (fully flipped IET) if \(k = -I\). We
denote the set of all such transformations by \(\text{FET}^n[a, b)\),
or simply FET\(^n\) for the set of all fully flipped IETs.

**Definition 10 (Associating permutation to an inter-
val exchange transformation (with or without flips)).** Any map \(F \in \text{IETF}^n[a, b) \cup \text{IETF}^n[a, b)\) is well defined
as a bijection between the sets \([a, b) \cup \{a_i\}\) and
\([a, b) \cup \{F(a_i)\}\). Define a labeling map \(\ell : \{[a_i, a_{i+1}), i = 0, ..., n-1\} \to \{1, ..., n\}\) such that
\(\ell([a_i, a_{i+1})) = i + 1\). Then, one associates to \(F\) a per-
mutation \(\sigma \in S_n\) in a natural way by reading off the
labels of the intervals in the set \([a, b) \cup \{F(a_i)\}\) in the
order of their positions on the interval \([a, b)\). A
standard graphic representation of a permutation is a
2 \times n matrix with a first line containing the numbers
1, ..., n in a natural order. For a map \(F \in \text{FET}^n[a, b)\) and
a permutation corresponding to it, we will add
additional information to this graphic representation
that is a graphic representation of the vector of flips \(k\)
by drawing bars over the labels which correspond to
the intervals which are flipped, in the first as well as
in the second row. In this way, a permutation corre-
sponding to a map in FET\(^n\)\([a, b)\) has a graphic repre-
sentation with bars over all the entries.

Any map \(F \in \text{IETF}^n\) defines the triple
\((\sigma, k, \ell)\) with \(\ell_i = a_i - a_{i-1}, i = 1, ..., n\) and vice-
versa: the combinatorial information and the lengths
of the intervals define an interval exchange transforma-
tion with flips.

The three definitions of classes of the maps on the
interval given above can be generalized in a natural
way to the \(n\)-interval exchanges on the circle (by
replacing \([a, b)\) with \(S^1\) and identifying \(a_0 = a_n\)). We
declare the obtained classes correspondingly as CET\(^n\)
(Definition 7), CET\(^n\)\(\sigma\) (Definition 8) and FCET\(^n\)
(Definition 9). The results of this work mostly concern
a subfamily of the set FCET\(^n\) of fully flipped interval exchange transformations on the circle that has already been defined for \(n=3\) in the previous
Section (see Definition 4) in relation to triangle tiling
billiards, and that we define in full generality now.

**Definition 11.** Fix \(l_1, l_2, ..., l_n \in \mathbb{R}_{>0}\) such that
\(l_1 + \ldots + l_n = 1\). Take \(S^1 = \mathbb{R}/\mathbb{Z}\) and define
\(a_0, a_1, ..., a_n \in S^1\) as \(a_0 = 0, a_1 = l_1, a_2 = l_1 + l_2, ...,\)
\(a_i = l_1 + \ldots + l_i, ..., a_{n-1} = 1 - l_n, a_n = 0\). A map \(F \in \text{FCET}^n(0, 1)\) is said to belong to the set CET\(^n\)\(\tau\) if there
exists \(\tau \in S^1\) such that
\[ F(x) = -x + l_i + \tau \mod 1 \forall x \in [a_{i-1}, a_i) \forall i = 1, ..., n. \]

**Remark.** As we defined the circle interval exchange
transformations by demanding \(a_0 = a_n\), the circle has a
marked point \(a_0 = a_n\). Hence, the sets CET\(^n\), CET\(^n\)
and CET\(^n\)\(\tau\) (and CET\(^n\)) are in natural bijection with
the subsets of, correspondingly, IETF\(^{n+1}\), IETF\(^{n+1}\)
and CET\(^{n+1}\). Indeed, these bijections are obtained by cutting
the interval \([a_i, a_{i+1})\) such that \(a_0 = F(a_i, a_{i+1})\), into two
For a fixed $F \in \text{CET}^2$ on the circle represented here, after the marking $a_0 = a_n = 0 \in S^1$ and passing to an IETF on the interval, three cases can occur. First, as in a, when 0 belongs to the image of a non-flipped interval, one obtains a map $F_1 \in \text{IETF}^3|0, 1|$ with the combinatorics $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$. Second, when 0 belongs to the image of a flipped interval, the obtained map $F_2 \in \text{IETF}^3|0, 1|$ has the combinatorics $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$. Third, as in b, when 0 coincides with the image of one of the singularities, a corresponding map $F_3 \in \text{IETF}^2|0, 1)$, with combinatorics $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$ or $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

**Answer to the problem.** For all $x$.

The second author made a short movie about this problem, co-created with C. Gourdon [Gourdon and Paris-Romaskevich 18]. The cake problem is, of course, just a reformulation of the fact that a map from IETF\(^2\) presented on Figure 6 is completely periodic, see Proposition 2 in the following. The analog of this periodicity property holds in the family IETF\(^n\) even for $n > 2$.

**Theorem 2 ([Nogueira 89]).** Fix a permutation $\sigma \in S_n$ and a vector $k \in \{-1, 1\}^n, k \neq 1$. Then for almost any choice of points $a_0 = 0, a_1, ..., a_n = 1$ (with respect to Lebesgue measure), for the map $F \in \text{IETF}^n$ defined by $\sigma, k$ and $\{a_i\}_{i=0}^n$ (as in Definition 8) there exists $N \in \mathbb{N}$ and $I \subset [0, 1)$ such that the first return map of $F$ in restriction to $I$ coincides with $F^N$ and is a flip of the interval $I$ onto itself.

**Remark.** The conclusion of this theorem was recently sharpened by Skripchenko and Troubetzkoy who proved that the set of parameters with minimal dynamics does not have maximal Hausdorff dimension [Skripchenko and Troubetzkoy 18].

### 3.2. Modified Rauzy induction for interval exchange transformations with flips: Definition and notations

The proof of Theorem 2 is based on the tool of modified Rauzy induction that we introduce in this paragraph, and that we use extensively throughout the article.

When in the previous paragraph the permutations characterizing the IETs had a first row fixed and trivial (see Definition 10), in this paragraph we will label
the intervals by letters and not by numbers, and we will allow any order on the top and bottom line of the permutation. More formally, now let $\mathcal{A}$ be an alphabet of cardinality $n$. Let the maps $\sigma_{\text{top}}$ and $\sigma_{\text{bot}}$ be two orders on the alphabet $\mathcal{A}$.

**Definition 12.** A generalized permutation on the alphabet $\mathcal{A}$ corresponding to two bijections $\sigma_{\text{top}}, \sigma_{\text{bot}} : \mathcal{A} \rightarrow \{1, \ldots, n\}$ is defined as

$$
\sigma = \left( (\sigma_{\text{top}})^{-1}(1) \ldots (\sigma_{\text{top}})^{-1}(n) \right) \cdot \left( (\sigma_{\text{bot}})^{-1}(1) \ldots (\sigma_{\text{bot}})^{-1}(n) \right).
$$

We denote the set of all generalized permutations on the alphabet $\mathcal{A}$ with $n$ elements by $S_n^\mathcal{A}$.

Now let $\lambda := (\lambda_i)_{i \in \mathcal{A}}$ be a vector of positive real numbers, with $\lambda = |\lambda| := \sum_{i \in \mathcal{A}} \lambda_i$. Suppose $k = (k_i)_{i \in \mathcal{A}}, k_i \in \{0, 1\}$. Then, one associates a map $F \in \text{IET}^\mathcal{A}[0, \lambda] \cup \text{IETF}^\mathcal{A}[0, \lambda]$ to the data $(\sigma, k, \lambda)$ in a following way. First, set for every $j \in \{0, \ldots, n\}$ the following quantities:

$$
\lambda_j^\text{top} := \sum_{i : \sigma_{\text{top}}(i) \leq j} \lambda_i, \lambda_j^\text{bot} := \sum_{i : \sigma_{\text{bot}}(i) \leq j} \lambda_i.
$$

These quantities define the points that give two partitions of $[0, \lambda]$. Namely, we set for each $i \in \mathcal{A}$:

$$
I_i^\text{top} := \left( \lambda_{\sigma_{\text{top}}(i) - 1}^\text{top}, \lambda_{\sigma_{\text{top}}(i)}^\text{top} \right), I_i^\text{bot} := \left( \lambda_{\sigma_{\text{bot}}(i) - 1}^\text{bot}, \lambda_{\sigma_{\text{bot}}(i)}^\text{bot} \right).
$$

Define $F$ in restriction to each interval $I_i^\text{top}$ by first, $F(I_i^\text{top}) = I_i^\text{bot}$ and moreover, if $k_i=1$, it is a translation; and otherwise it is a translation in composition with a flip. This defines $F$ completely on $[0, \lambda]$ outside the extremities of the intervals $I_i^\text{top}, i \in \{1, \ldots, n\}$. We call $\lambda_i^\text{top}$ top singularities and $\lambda_i^\text{bot}$ bottom singularities of $F, i = 1, \ldots, n-1$. Any map $F \in \text{IET}^\mathcal{A}[0, \lambda] \cup \text{IETF}^\mathcal{A}[0, \lambda]$ is represented in such a way, and such a representation is unique (up to the re-labeling of the first row in $\sigma$). In the following, we will identify $F \in \text{IETF}^\mathcal{A}[0, \lambda]$ with the corresponding data $(\sigma, k, \lambda)$. The conventions of graphic representation are transmitted from the previous paragraph, see Figure 7 for illustration by example.

**Definition 13.** A generalized permutation $\sigma$ is called reducible if $\exists l \in \{1, \ldots, n-1\}$ such that

$$(\sigma_{\text{top}})^{-1}(\{1, 2, \ldots, l\}) = (\sigma_{\text{bot}})^{-1}(\{1, 2, \ldots, l\}).$$

A corresponding $F \in \text{IET}^\mathcal{A} \cup \text{IETF}^\mathcal{A}$ is in this case called reduced. If $\sigma$ is not reducible, it is called irreducible.

Automatically, reduced IETs and IETFs are not minimal.

Now we are ready to define the modified Rauzy induction. It is an algorithm that either associates to a map $F \in \text{IETF}^\mathcal{A}$ another map $\mathcal{R}F \in \text{IETF}^\mathcal{A}$ (defined on a smaller interval), or stops. One can look at the modified Rauzy induction as a map

$$
\mathcal{R} : \text{IET}^\mathcal{A} \rightarrow \text{IET}^\mathcal{A} \cup \{\square\},
$$

where $\square$ correspond to a stop of Rauzy induction.

**Definition 14 (Modified Rauzy induction).** For $F \in \text{IET}^\mathcal{A}[0, \lambda]$, if $\lambda_{n-1}^\text{top} = \lambda_{n-1}^\text{bot}$ then $\mathcal{R}F := \square$. Otherwise, the longest interval between $I_{(\sigma_{\text{top}})^{-1}(n)}^\text{top}$ and $I_{(\sigma_{\text{bot}})^{-1}(n)}^\text{bot}$ is called the winner and the shortest one the loser. Then $\mathcal{R}F$ is a first return map of $F$ on the interval $[0, \max(\lambda_{n-1}^\text{top}, \lambda_{n-1}^\text{bot})]$, i.e. $\mathcal{R}F$ is defined on the interval of the length different from the length of the initial interval by the length of the loser. In the case when the winner is on top (on bottom), the step of Rauzy induction is called top (bottom) induction.
When $F' = RF$ exists, it is an interval exchange transformation with flips, and its data $(\sigma', k', \lambda') \in S_n A \times \left(\{-1,1\}^n \setminus \{\text{I}\}\right) \times \mathbb{R}_+^n$ can be deduced from the analogical data $(\sigma, k, \lambda)$ of $F$.

In the case of IETs with flips, one step of Rauzy induction has a bigger number of combinatorial possibilities than one case of IETs without flips. Indeed, not only the lengths of the winner and loser are taken into account but also the values $k_{(\sigma k r)^{-1} r^{-1} k}$ and $k_{(\sigma k l)^{-1} l^{-1} k}$ of the vector $k$ (that take values in the set $\{-1,1\}$). This gives indeed 8 possibilities (instead of 2 for standard Rauzy induction for IETs), see Table 1.

### Remark
An important remark is that $\sigma'$ is defined on the same set of labels as is $\sigma$, and the labels change in a natural way (analogous to the standard Rauzy induction procedure). The winner is cut into two non-empty intervals by one of the extremities of the loser. The left of this interval is relabeled into the label of the winner as well as its image (if the winner was on top) or pre-image (if the winner was on bottom). The label of the loser is assigned to the part of the initial winner interval which doesn’t have a label in one of the lines (on top or on bottom, if the winner was on bottom or on top, correspondingly). In another line the label of the interval which was a loser doesn’t change. This remark can be considered as a part of Definition 14.

The Rauzy induction has been used extensively in the last 50 years as a powerful tool that unites combinatorics and linear algebra, see for example [Arnoux and Rauzy 91, Avila et al. 16, Boissy and Lanneau 08, 22, Skripchenko and Troubetzkoy 18]. Rauzy induction helps to understand the dynamics of IETs, as one can see for example, with this

**Lemma 1.** [Keane 75, Nogueira 89, Rauzy 79] The map $F \in \text{IETF}^n$ is minimal if and only if the modified Rauzy induction never stops, and the vector of the lengths of the intervals $\lambda^{(m)}$ obtained after $m$ iterations of the MRI, tends to zero:

$$||\lambda^{(m)}||_\infty = \max_{n, i} \lambda_i^{(m)} \to m \to \infty 0.$$

### 3.3. Rauzy graphs: Constructions, definitions, algorithms

Associated to the Rauzy induction on the space of n-IETs, one can define a combinatorial object which is called the Rauzy graph. Rauzy graphs were studied in a lot of detail for standard Rauzy induction [Kontsevich and Zorich 03, de Mourgues 17]. In this section, we define the Rauzy graphs for the modified Rauzy induction in an analogous way. The proof of our main result (Theorem 1) is based on one invariant of modified Rauzy graphs that we stumbled upon, and we believe that many more are to be discovered. The full understanding of modified Rauzy graphs is very far from being achieved.

Fix an alphabet $\mathcal{A}$ with $n$ letters. Denote $Y^X$ the set of maps from $X$ to $Y$, for any two finite sets $X$ and $Y$. Then any vector $k \in \{1, -1\}^n \setminus \{\text{I}\}$ with coordinates labeled by the elements of $\mathcal{A}$, can be seen as an element $k \in \{-1,1\}^n \setminus \{\text{I}\}$. Also note that $\sigma \in S_n^A$ is nothing else than a pair of bijective maps $\sigma^{\text{top}}, \sigma^{\text{bot}} : \mathcal{A} \to \{1, \ldots, n\}$.

**Definition 15.** Fix an alphabet $\mathcal{A}$ with $n$ letters.

Consider a set of all possibilities of combinatorial data of maps $F : \text{IETF}^n$:

$$V := \{(\sigma, k) | \sigma \in S_n^A, k \in \{-1,1\}^n \setminus \{\text{I}\}\}.$$

We call this set the set of Rauzy classes of interval exchange transformations with flips. Each element of this set is called a Rauzy class.

As we have seen above in Lemma 1, the minimality of a map $F : \text{IETF}^n$ can be formulated in terms of...
the modified Rauzy induction. The iterations \( \{ R^m F \} \)
define a path \((\sigma_n, k_n, \lambda^{(n)})\) in the modified Rauzy
graph \( G = (E, V) \). Suppose that this (possibly infinite)
path \( \gamma \) follows the sequence of edges \( e_1, \ldots, e_m, \ldots, e_j \in E \).
To any edge \( e \in E \) one associates a linear nonnegative
matrix \( A \), corresponding to the inverse transfor-
mation of the lengths in a following way. If the edge
\( e \) corresponds to the induction step where
\( \lambda_i^{(m)} > \lambda_j^{(m)} \), \( i \neq j \) with \( i, j \in A \),
then \( A_e := E + E_{ij} \). Here \( E_{ij} \) is a matrix
with all elements equal to zero except one in
the \( i \)-th row and \( j \)-th column which is equal to 1. Let
\( A_{1(m)} := A_{e_1} A_{e_2} \ldots A_{e_m} \). Then the lengths of
the intervals of continuity for an IET corresponding to
the \( m \)-step of the Rauzy induction are given by
\( \lambda_i^{(m)} = A_{1(m)} \lambda_i \) since
\( \lambda_i^{(m)} = A_{1(m)}^{-1} A_{1(m)} \lambda_i \),
by definition. We will study these
products more attentively in the following Section.

Now let us give some more combinatorial definitions.

**Definition 16.** The modified Rauzy graph or modified
Rauzy diagram is a finite oriented graph
\( G = (V, E) \) with the set of vertices being the set of Rauzy
classes of IETs. For the edges, \( e = (v, w) \in V \times V \in E \) if by
one step of Rauzy induction one can pass from the
combinatorial data \( v \) to the combinatorial data \( w \).

Since we will be working only with such graphs, we
will sometimes omit the world modified, and call
these graphs simply Rauzy graphs. Note that such
graphs (for different values of \( n \)) are not necessarily
connected. The number of outgoing edges from
each vertex belongs to the set \( \{0, 2\} \) as the number of
incoming arrows belongs to the set \( \{0, 1, 2\} \). Let us
define the following

**Definition 17** (Equivalence relationship on the set
of Rauzy classes). Two vertices \( v_1, v_2 \in V \) are equiva-
 lent \( (v_1 \sim v_2) \) if for \( v_1 = (\sigma_1, k_1) \) and \( v_2 = (\sigma_2, k_2) \),
there exists a bijective map \( h : A \to A \) such that
\[
\sigma_1^{\text{top}} = \sigma_2^{\text{top}} \circ h \\
\sigma_1^{\text{bot}} = \sigma_2^{\text{bot}} \circ h \\
k_1 = k_2 \circ h.
\]

The set of equivalent vertices to a vertex \( v_1 \) will be
denoted \( [v_1] \). If one wants to precise the map \( h \), one
also writes \( v_1 \sim_h v_2 \).

Obviously, this is a well-defined equivalence relation-
ship. Note that for two vertices \( v_1, v_2 \in G \) the bijection
\( h : A \to A \), if it exists, is uniquely defined. We consider
the quotients of the Rauzy graphs with respect to this
relationship, defined in a following way.

**Definition 18.** The quotient Rauzy graph (or, simply,
the quotient graph) is a finite oriented graph \( G =
(V, E) \) with the vertices being the equivalence classes
of Rauzy classes of IETF with respect to the equivalence
relationship \( \sim \). Two vertices \([v_1], [v_2] \) are connected
by an edge \( e \in G \) if there exist representatives of Rauzy
classes \( v_1 \in [v_1], v_2 \in [v_2] \) as well as a map \( h : A \to A \)
such that \( v_1 \sim_h v_2 \). Moreover, the quotient Rauzy graph
comes with a labeling map \( \ell : E \to H \), where \( H \) is a set
of bijections \( h : A \to A \). This labeling map is defined as
follows: \( e \mapsto h \), i.e. the edges of the quotient Rauzy graph
are labeled by the permutations of labels.

The quotient Rauzy graphs have a much smaller
number of vertices than Rauzy graphs (indeed, the class
\([v_1] \) contains \( n! \) Rauzy classes). Although, they carry all
the additional information contained in the Rauzy graph
in the labeling of the edges between the vertices.

**How the pictures of Rauzy graphs are drawn in
this article**

For this project, we collaborated with Paul Mercat,
who has written a code in Sage that draws Rauzy
graphs and quotient Rauzy graphs. For all the pictures
of these graphs, here are our assumptions on the
graphical representation that we have chosen.

A1. For the sake of the economy of place, we
draw quotient Rauzy graphs instead of the
Rauzy graphs.

A2. Sometimes only the connected components of
the quotient Rauzy graphs are drawn.

A3. For the vertices of quotient graphs, each class of
equivalence relationship \( \sim \) is represented by one
of the vertices in this class, i.e. by some general-
ized permutation \( \sigma \in S_n^A \). Of course, such a
graphic representation of the quotient Rauzy
graph is not unique because it depends on the
choice of the representatives for each class.

A4. The bars are put on the letters of \( \sigma \) that corres-
pond to flipped intervals. The letters that cor-
respond to the intervals which are not flipped,
are represented by a green color.

A5. Since we are mostly interested in the cycles in
the Rauzy graphs, we do not draw the vertices
\( [\sigma], \sigma \in S_n^A \) for which \( (\sigma^{\text{top}})^{-1}(n) = (\sigma^{\text{bot}})^{-1}(n) \).
Indeed, for any \( F \in \text{IETF}^n \) represented by such
a vertex, obviously, \( RF = \emptyset \).

A6. On each edge in the quotient Rauzy graph, we
write the labels of the winner and the loser for
the Rauzy induction on the corresponding
element of the equivalence class. The winner
and the loser are marked (e.g. \( C > D \) for \( I_C \) being
the winner and \( I_D \) being the loser).

A7. The arrows in the quotient Rauzy graph are
marked by a labeling map \( \ell \). In our pictures,
the arrows marked by the identity map, are
drawn in a standard way. Meanwhile, any arrow $e$ marked by $h = \ell(e) \neq \text{id}$ is dotted. Of course, the map $h$ can be reconstructed explicitly if one knows the connected vertices $[v_1], [v_2]$ (and their representatives), as well as the labels of the winner and the loser.

**Example.** On Figure 8 the reader can see the connected component of the permutation

$$\sigma = \begin{pmatrix} \bar{A} & \bar{B} & \bar{C} & \bar{D} \\ A & B & C & D \end{pmatrix}$$

in the quotient Rauzy graph for IETF$^4$. After applying the Rauzy induction to $\sigma$, one gets:

$$\sigma' = R_{C < D} \sigma = \begin{pmatrix} \bar{A} & \bar{B} & C & \bar{D} \\ B & C & D & A \end{pmatrix},$$

$$\sigma'' = R_{C > D} \sigma = \begin{pmatrix} \bar{A} & B & D & \bar{C} \\ B & D & \bar{A} & C \end{pmatrix}.$$

Only on one of these two permutations, $\sigma'$, the Rauzy induction can be continued. The class $[\sigma'']$ is hence not included into the graph on the picture, as in A5. The presented strongly connected component has two dotted arrows. For example, for the permutation

$$\delta = \begin{pmatrix} D & \bar{A} & \bar{B} & C \\ \bar{A} & \bar{C} & C & D \end{pmatrix},$$

the combinatorial Rauzy induction leads to

$$\delta' = R_{B < C} \delta = \begin{pmatrix} D & \bar{A} & B & \bar{C} \\ \bar{A} & \bar{B} & C & D \end{pmatrix}.$$

One can see that $\delta' \sim h\delta'$, where $\hat{\delta}' = \begin{pmatrix} D & \bar{A} & \bar{B} & \bar{C} \\ \bar{A} & \bar{B} & C & D \end{pmatrix}$ via the map $h: A \to A$ such that $h(A) = B, h(B) = C, h(C) = A, h(D) = D$. Hence the edge $e = (\delta, \hat{\delta}')$ is dotted, see A7.

### 4. Necessary condition for minimality in $\text{CET}^3_{\tau}$

In this Section we will prove the following

**Theorem 3.** If $F \in \text{CET}^3_{\tau}$ is minimal then $\tau = \frac{1}{2}$.

We will see in the following that this theorem is an important ingredient in order to qualify the non-linearly escaping behavior in triangle tilings billiards (Theorem 17).

#### 4.1. The existence of the invariant hyperplane $\tau = \frac{1}{2}$

In this paragraph we explain why the hyperspace $\tau = \frac{1}{2}$ is invariant under Rauzy induction in the families of $\text{CET}^3_{\tau}$.

Let $F \in \text{CET}^n_{\tau}$ acting on $S^1$ and $X_F$ be the cylinder $S^1 \times (-1, 1)$ with the following identifications on the horizontal boundaries: $(x, 1)$ is identified with $(F(x), -1)$. In plain terms, on each interval of continuity of $F$, an interval and its image are identified by flip. $X_F$ is a non orientable compact surface. The vertical flow is well-defined on $X_F$. It means that we consider an orientable foliation on a non orientable surface. By definition, the first return time on $S^1 \times \{1\}$ of this flow is the map $F$.

When $\tau = 1/2$, the surface $X_F$ possesses an additional symmetry. It is invariant by the antiholomorphic map $\phi: (x, y) \to (x + 1/2[1], -y)$. An elementary calculation shows that the quotient is a projective plane (see Figure 9).

The vertical flow descends to the quotient as a foliation. Since the action of Rauzy induction on $X_F$ is realized as a cut and paste of rectangles with horizontal and vertical sides, it preserves the symmetry $\phi$. Therefore, Rauzy induction preserves the hypersurface $\tau = 1/2$.

Unfortunately, this geometric remark doesn’t suffice to prove Theorem 3. We need one extra combinatorial remark that is based on the study of the invariants in the modified Rauzy graphs of IETFs. We hope that in the future we will be able to understand the geometric meaning of the invariant that we describe in the next paragraph.

#### 4.2. The combinatorial proof

**Lemma 2.** If $F \in \text{CET}^3_{\tau}$ is minimal hence its combinatorics as a map in $\text{FET}^3(0,1)$ where 0 corresponds to the end of one of the intervals of continuity, is encoded with a generalized permutation $\sigma$, defined by (3–2).
the interval is obviously not minimal since it has an open interval of 2-periodic points. Hence, if 0 is a left singularity of the interval \( I_a \) (as in Definition 4) and \( F \) is minimal then \( 0 \in F(I_b) \), see Figure 10.

**Remark.** One can explicitly write out all the possible combinatorial possibilities for the generalized permutations defining the maps in \( \text{CET}_3 \), only the first one of them being irreducible:

\[
\begin{pmatrix}
\bar{A} & \bar{B} & \bar{C} & \bar{D} \\
\bar{B} & \bar{D} & \bar{A} & \bar{C}
\end{pmatrix}, \quad
\begin{pmatrix}
\bar{A} & \bar{B} & \bar{C} & \bar{D} \\
\bar{A} & \bar{C} & \bar{D} & \bar{B}
\end{pmatrix},
\begin{pmatrix}
\bar{A} & \bar{B} & \bar{C} & \bar{D} \\
\bar{C} & \bar{A} & \bar{B} & \bar{D}
\end{pmatrix}.
\]

(4–1)

From Lemma 1 and Lemma 2 we see that the understanding of the minimal maps in \( \text{CET}_3 \) boils down to the understanding of the cycles of Rauzy induction in the connected component of the Rauzy graph of \( \sigma \). This connected component is the main hero of this Section, its full representation is very big although can be drawn explicitly which was done by Paul Mercat. Although, a representation of this component in the quotient graph has only 19 (irreducible) vertices and is sufficient for our needs. It is given on the Figure 8.

Note that the Rauzy induction can be defined as a map on \( \text{IETF}^0(0, 1) \) by renormalizing the interval of the definition of \( \mathcal{R} \) in order for it to have length 1. Then, the lengths of the intervals of continuity can be parameterized by homogeneous coordinates \( \{\lambda_i\}_{i \in A} \) with \( A = \{A, B, C, D\} \) and the permutation \( \sigma \) as in 2 define a map \( F \in \text{CET}_3 \). The goal is to prove that if the set \( \{\mathcal{R}(F)\} \) is infinite (i.e., the Rauzy induction doesn’t stop), then \( \lambda \in H \).

Consider the infinite path \( \gamma \) in the Rauzy graph \( G \) defined by \( \{\mathcal{R}^n(F)\}_{n=1}^\infty : \gamma = e_1 \ldots e_m \ldots \). This path is obviously contained in the connected component of \( \sigma \) in \( G \).

Since \( G \) is a finite graph, there exists a vertex \( \tilde{\sigma} \) in \( G \) such that the path \( \gamma \) passes by \( \tilde{\sigma} \) an infinite number of times: there exists a sequence of oriented edges \( e_k \in G, (k_i)_{i=1}^\infty \) such that each of them ends up at \( \tilde{\sigma} \).

\[{\lambda_i}_{i \in A} \text{ can be described in terms of the parameters } l_1, l_2, l_3, \tau:
\]

\[
\begin{pmatrix}
\lambda_A \\
\lambda_B \\
\lambda_C \\
\lambda_D
\end{pmatrix} =
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 1 & 1 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\begin{pmatrix}
l_1 \\
l_2 \\
l_3 \\
\tau
\end{pmatrix}.
\]

The equation \( \tau = \frac{1}{2} \) (or, \( 2\tau = l_1 + l_2 + l_3 \)), in the homogeneous coordinates \( \{\lambda_i\}_{i \in A} \) can be written as

\[
\lambda_A + \lambda_C = \lambda_B + \lambda_D.
\]

This equation defines a hyperplane \( H \subset \mathbb{P}^3 \) with an orthogonal vector \( \nu^\perp := (1, -1, 1, -1)^T \). It happens that the vector \( \nu^\perp \) is invariant under the Rauzy transformations.

**Lemma 3.** Fix any vertex \( \sigma' \) in the connected component of the Rauzy graph of the permutation \( \sigma \) defined by (3–2). Take a finite path \( \gamma = e_1 \ldots e_m \) in the graph connecting \( \sigma \) to \( \sigma' \). Then the vector \( A_{\gamma}^T \nu^\perp := A_{(m)}^T \nu^\perp \) doesn’t depend on the path taken.

**Proof.** The proof is an explicit verification by computer, see Figure 17 of the Appendix for illustration.

The vector \( A_{\gamma}^T \nu^\perp := A_{\sigma}^T \nu^\perp \) is then the invariant of this vertex.

We think that the invariance of the vector \( \nu^\perp \) in the sense of Lemma 3 has a geometric interpretation that we didn’t manage to find. The following is based on experimental data: for the vertices in the connected component of \( \sigma \) that belong to the cycles in the Rauzy graph and their assigned vectors \( A_{\gamma}^T \nu^\perp \), the nonflipped letters correspond to the coordinates equal to 0, and the flipped letters to coordinates equal to ±1. We hope to explore this in the future work.

On the Figure 17 of the Appendix we represent the graph from Figure 8 with the additional information on the values of vectors \( A_{[\sigma]}^T \nu^\perp \) for each of its vertices \([\sigma] \). Now we are ready to prove Theorem 3.

**Proof.** The vector of lengths \( \tilde{\lambda} = (\tilde{\lambda}_i)_{i \in A} \) with \( A = \{A, B, C, D\} \) and the permutation \( \sigma \) as in 2 define a map \( F \in \text{CET}_3 \). The goal is to prove that if the set \( \{\mathcal{R}(F)\} \) is infinite (i.e., the Rauzy induction doesn’t stop), then \( \tilde{\lambda} \in H \).

Here is the address: pa-ro.net/doc/2019-triangletilingbilliards/web_ABCD_BDAC_complete.pdf.
Then \( \lambda^{(k_m)} = A^{-1}_{(k_m)} \lambda \) for \( A_{k_m} = A_{e_1} \cdots A_{e_{m_n}} \). By Lemma 1, \( ||\lambda^{(k_m)}||_\infty \to 0 \) when \( k_m \to \infty \). Let

\[
\tilde{v}_j := A^T_{(k_j)} v^\perp = A_{e_1}^T \cdots A_{e_j}^T v^\perp
\]

be the image of \( v^\perp \) corresponding to the pre-cycle \( e_k, \ldots, e_j \). By Lemma 3, the vector \( \tilde{v}_j \) doesn’t depend on \( j \in \mathbb{N} \) and is an invariant of the vertex \( \tilde{\sigma} \) itself: \( \tilde{v}_j = A_{\tilde{\sigma}} v^\perp \).

Denote \( \tilde{H} := A^{-1}_{(k)} H \). The orthogonality of \( v^\perp \) and \( H \) is equivalent to the orthogonality of \( A_{\tilde{\sigma}} v^\perp \) and \( \tilde{H} \). Now define the products of matrices that correspond to the loops in \( G \) created by the path \( \gamma \) based at \( \tilde{\sigma} \). Let for any \( m \in \mathbb{N}^*, m \geq 2 \)

\[
\tilde{A}_{(m)} := A_{e_{m-1}}^{} \cdots A_{e_m}.
\]

Then, the calculation of the scalar products gives the result:

\[
\langle A_{\tilde{\sigma}} v^\perp, \lambda^{(k_j)} \rangle = \langle \left( \tilde{A}_{(m)} \right)^T A_{\tilde{\sigma}} v^\perp, \lambda^{(k_j)} \rangle = \langle A_{\tilde{\sigma}} v^\perp, \tilde{A}_{(m)}^{-1} \lambda^{(k_j)} \rangle = \langle A_{\tilde{\sigma}} v^\perp, \lambda^{(k_m)} \rangle \to 0, m \to \infty.
\]

Indeed, we see that \( \lambda^{(k_j)} \perp A_{\tilde{\sigma}} v^\perp \) which is equivalent to \( \lambda \in H \).

\section{5. Structure of non-minimal maps in \( \text{CET}_n \) and integrability}

In this Section we study a class of integrable interval exchange transformations with flips. For any map in this class, its suspension on a non-orientable surface has invariant cylinders and tori, foliated by linear foliations. This means that the dynamics of such IETs is very simple. We prove that a map in \( \text{IET}^n \) is integrable: always for \( n = 3 \) and almost always for \( n = 4 \). For \( n = 5 \), we find an open set of non-integrable dynamics.

\subsection{5.1. Simple and integrable interval exchange transformations with flips: definitions}

Any interval exchange transformation with flips \( F \) can be defined by a triple \((\sigma, k, \lambda)\) that contains combinatorial data (generalized permutation and the vector of flips) and the vector of lengths of the intervals of continuity, see paragraph 3.2. The generalized permutation \( \sigma \) encodes in itself two orders on alphabet \( A \) that correspond to two words \( \omega^{\text{top}}, \omega^{\text{bot}} \in A^n \) which correspond to the first and the second row of the matrix representation (3–1) of \( \sigma \).

\textbf{Definition 1.} The interval exchange transformation with (or without) flips \( F \in \text{IET}^n \cup \text{IET}^n \) defined by the data \((\sigma, k, \lambda)\) is called simple if \( \sigma \in \{1, \ldots, n\} \) such that \( \omega^{\text{top}} = \omega_1^{\text{top}} \cdots \omega_n^{\text{top}} \) and \( \omega^{\text{bot}} = \omega_1^{\text{bot}} \cdots \omega_n^{\text{bot}} \) with \( \omega_i^{\text{top}}, \omega_i^{\text{bot}} \) being nonempty words in the alphabet \( A \). And moreover, for any \( j \in \{1, \ldots, p\} \) exactly one of these possibilities holds:

1. (periodic cylinders) \( \omega_j^{\text{top}} = \omega_j^{\text{bot}} \),
2. (cylinders of rotation) there exist two words \( x \neq y \) such that \( \omega_1^{\text{top}} = xy \) and \( \omega_1^{\text{bot}} = yx \), and the coordinates of \( k \) corresponding to all the letters of \( A \) in the word \( \omega_j^{\text{top}} \) are equal to \( 1 \),
3. (cylinders of rotation with a marked singularity) there exist three different words \( x, y, \) and \( z \) such that \( \omega_j^{\text{top}} = xyz, \omega_j^{\text{bot}} = yxz \), \( \omega_j^{\text{top}} = xyz, \omega_j^{\text{bot}} = yzx \), and \( \omega_j^{\text{top}} = xyz, \omega_j^{\text{bot}} = yzx \), and the corresponding to \( \omega_j^{\text{top}} \) coordinates of \( k \) are equal to \( 1 \).

Obviously, in any of three cases, the lengths of \( \omega_j^{\text{top}} \) and \( \omega_j^{\text{bot}} \) coincide and these words consist of the same sets of letters. The simplicity of \( F \) doesn’t depend on \( \lambda \) but only on its combinatorial data \((\sigma, k)\).

\textbf{Remark.} Simplicity can be defined in purely geometric terms. For example, for a map \( F \in \text{IET}^n \), its square \( T = F^2 \) is an IET without flips. If \( F \) is simple then, a translation surface corresponding to \( T \) can be cut along the lines of the flow into the union of the invariant cylinders and tori. The flow preserves a linear foliation on the invariant tori and a trivial foliation by periodic leaves on the cylinders.
A map is a flip) and tori (on which a first-return map is integrable. Indeed, finding a Poincaré section with a rotation, possibly identical. Of course, integrability implies the absence of minimality.

5.2. Integrability of maps in IETF³

In this paragraph we show that all maps from IETF³ are integrable. Let us first remark that for $n=2$ it follows obviously from the following result, proven by Keane.

**Proposition 2** ([Keane 75]). All IETF² are integrable, and even more, completely periodic.

**Proof.** The study of the Rauzy graph gives a proof of this Theorem, see Figure 11. A Rauzy graph for IETF² doesn’t permit infinite loops for Rauzy induction and moreover, all the stop points correspond to simple maps.

The proof we give here is modern. Keane’s proof was done in 1975 by other methods, four years before the invention of the standard Rauzy induction. We prove now an analogous statement for the family IETF³.

**Proposition 3.** Any $F \in \text{IETF}^{0,1}$ is integrable, the corresponding Poincaré section can be chosen as a segment with one of its ends equal to 0, and the set $\{R^nF\}$ is finite (the modified Rauzy induction eventually stops).

**Proof.** The proof follows from the explicit study of the Rauzy graph for IETF³. One can easily see that for any cycle $\gamma$ in this graph there exists a letter from the alphabet $A = \{A, B, C\}$ such that the corresponding interval never wins along $\gamma$. This means that the Rauzy induction stops for any $F \in \text{IETF}^3$. See Figure 12 for one of such cycles in the Rauzy graph.

In general, the stop of the Rauzy induction reduces the study of the integrability of maps in IETF³ to the case of the integrability of IETs on a smaller number of intervals. Indeed, finding a Poincaré section with
an integrable map gives a (possibly, finer) Poincaré section with a simple map. The proof of integrability finishes by recurrence: in the set $\text{IETF}^2 \cup \text{IET}^2$ all maps are integrable, from Proposition 2 and the integrability of rotations. Here a more precise study of the Rauzy graph shows that the vertices on which the combinatorial Rauzy induction can’t be continued, are simple.

5.3. Generic integrability of maps in IETF$^4$

Proposition 4. Let $F \in \text{IETF}^4[0, 1)$ be such that the set \{R$^n$F\} is finite and the lengths $\lambda_i$ are rationally independent. Then $F$ is integrable.

Proof. Suppose that the Rauzy induction stopped after $n$ steps for $F \in \text{IETF}^4[0, 1)$. This means that the combinatorial Rauzy induction has stopped as well. Then $\exists x, y \in [0, 1], y < x$ and the segments $F_i = [x, y], I = [0, x)$ such that $I$ is a Poincaré section for $F$ and the restriction of the first-return map on $I$ is either an identity map or a flip. Denote by $G$ the restriction of the first-return map on $[0, y)$.

First, from the combinatorics of Rauzy induction follows that if $I$ is not flipped, then $F$ is reduced, $n = 1$ and $x = 1$ (since a non-flipped winner can’t come to the last place in a row). Moreover, $G \in \text{IETF}^3[0, y)$ by Proposition 3. $G$ is integrable with a corresponding Poincaré section $[0, \gamma)$ for some $\gamma \leq \beta$. Hence $F$ is integrable with a Poincaré section equal to the union $[0, \gamma) \cup [\beta, x)$.

Second, in the case when $I$ is flipped, then either $G \in \text{IETF}^3[0, y)$ or $G \in \text{IET}^3[0, y)$. The first case is treated as before. In the second case the proof is also finished since any map in IET$^3$ corresponds to a rotation with a marked singularity. \qed

In the proof of this Proposition, we construct explicitly a Poincaré section that provides a simple first-return map on it, with the help of the Rauzy induction (on the right) for the maps in IET$^3$. Let us note that this construction can be generalized for the maps in IET$^3[0, 1)$ for any $n$.

Definition 21. Take any $F \in \text{IETF}^n[0, 1)$ such that the lengths $\lambda_i$ are rationally independent. If \{R$^n$F\} is infinite, we define its standard Poincaré section to be $[0, 1)$ and its standard Poincaré map to be itself. Suppose now \{R$^n$F\} finite and $F$ not reduced. Then, analogically to the proof of Proposition 4, one defines $\lambda_1, \beta_1 \in [0, 1], J_1 \subseteq I_1 = [0, \lambda_1)$ and the first return map $G_1$. If $G_1$ is simple, then we stop the procedure. If $I$ is not simple, we reiterate the process with $F := G_1$. Then, the union of corresponding segments gives a so-called Poincaré-Rauzy section, and a Poincaré-Rauzy map (as a first-return map on it). For the reduced $F$, we proceed with the same construction for each of the reduced components, and then unite the resulting Poincaré sections.

As a direct corollary of Proposition 4 and Theorem 2, we obtain

Proposition 5. Almost any $F \in \text{IETF}^4$ is integrable.

5.4. Integrability of CET$^n$

Suppose that $F = F_{t,h,b} \in \text{CET}^3$ is such that the ratios $\frac{1}{2} \not\in \mathbb{Q}$ for $i \neq j$. Then the Rauzy induction stops as well. Then $F$ is integrable, from Proposition 4, $F$ is integrable, with a Poincaré–Rauzy map being simple.

Proposition 6. Suppose that $F = F_{t,h,b} \in \text{CET}^3$ is such as above. Then $G$ is an IET with flips and for any interval $I_i$ of continuity of $G$ such that $G(I_i) = I_i, k_i = -1$ (this interval is flipped).

Proof. Let $\sigma$ and $\delta$ be generalized permutations corresponding to $F$ and $G$. In a Rauzy graph, there exists a path connecting $\sigma$ to $\delta$, by construction. For some letter $X \in \mathcal{A} = \{A, B, C, D\}$ we have $(\delta_{\text{top}}^{-1}(4) = (\delta_{\text{bot}}^{-1}(4) = Y \in \mathcal{A}$ for $j = 1, 3$ (since the lengths are rationally independent). We will now obtain a combinatorial contradiction: there is no inverse Rauzy path connecting $\delta$ to $\sigma$. Indeed, first, if $j = 3$ then by following backward Rauzy induction, we obtain:

$\delta = \begin{cases} * & Y & \bar{X} \quad X > Y \Rightarrow \begin{cases} * & \bar{Y} & \bar{X} \\ * & \bar{Y} & \bar{X} \end{cases} =: \sigma_1 \leftarrow \emptyset. \end{cases}$

Inverse Rauzy induction can’t be continued, hence $\sigma_1 = \sigma$ but this is inconsistent with the possible combinatorics for the map $F$, see (4–1).

Second, if $j = 1$, $\delta \neq \sigma$ (since $\sigma$ is defined by one of three permutation (4–1) and $\delta$ has an invariant non-flipped cylinder on the left of the interval). Then, since any letter moves to the left in a row only if it is a lose in a step of the induction, we have a following chain in the connected component of $\sigma$:

$\delta = \begin{cases} Y & * & \bar{X} \quad Y > \bar{X} \Rightarrow \begin{cases} \bar{Y} & * & \bar{Z} \\ \bar{Y} & * & \bar{Z} \end{cases} =: \sigma_2 \leftarrow \emptyset. \end{cases}$
After $\sigma_2$, the inverse path can’t be continued and none of the permutations has one of the combinatorial types in (4–1).

As a direct corollary of Theorem 3 and Proposition 6, we have

**Proposition 7.** If $\tau \neq \frac{1}{2}$ then any $F_{l_1,l_2,l_3} \in \text{CET}_3^4$ is integrable (and not minimal). Moreover, if the ratios $\frac{1}{l_1}, \frac{1}{l_2}, \frac{1}{l_3} \not\in \mathbb{Q}$ for $i \neq j$ then for any periodic point of $F$ there exist an interval $I_i$ on the Poincaré–Rauzy section, containing it and flipped on itself by the Poincaré–Rauzy first-return map.

We have seen that the families IETF$^2$, IETF$^3$ consist only of integrable maps, and the families IETF$^4$ and CET$^5$ have almost all of their maps integrable. Although, for a bigger number of intervals, in the families IETF$^n$ for $n \geq 5$, the stop of the Rauzy induction doesn’t necessarily imply integrability.

**Proposition 8.** For any $n \in \mathbb{N}$, $n \geq 5$ there exists an open set of non-integrable maps in IETF$^n$.

**Proof.** For an alphabet $A = \{A, B, C, D, E\}$, take a generalized permutation

$$\delta := \begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ B & C & D & \bar{E} & \bar{A} \end{pmatrix}$$

and a subset $F$ of maps in FET$^5$ with such combinatorics $\delta$ and the following restrictions on the lengths: $\lambda_A \geq \max\{\lambda_B, \lambda_C, \lambda_D, \lambda_E\}$ and $\lambda_i$ being rationally independent, $i \in A$. This is obviously an open set.

For any map $F \in F$, the Rauzy induction will make the following four steps and then stop:

$$\begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ B & C & D & \bar{E} & \bar{A} \end{pmatrix} \xrightarrow{A \to \bar{E}} \begin{pmatrix} \bar{E} & \bar{A} & B & \bar{C} & D \\ B & C & D & \bar{E} & \bar{A} \end{pmatrix} \xrightarrow{A \to D} \begin{pmatrix} \bar{E} & D & C & B & \bar{A} \\ B & C & D & \bar{E} & \bar{A} \end{pmatrix} \xrightarrow{A \to B} \begin{pmatrix} E & D & C & B & \bar{A} \\ B & C & D & \bar{E} & \bar{A} \end{pmatrix}.$$

One can see that the map

$$\begin{pmatrix} E & D & C & B \\ B & C & D & \bar{E} \end{pmatrix}.$$ (5–1)

is not simple by looking at a corresponding foliation. This argument can be generalized for any value of $n \geq 5$.

We see that the class IETF$^5$ has open sets of non-integrable maps in it. But by restricting to a smaller subset of specific combinatorics, one observes integrable behavior.

**Proposition 9.** Consider the set $F$ of fully flipped interval exchange transformations obtained as the image of CET$^4$ under its natural inclusion in FET$^5$. Take any $F \in F$ such that $\lambda_i$ are independent over $\mathbb{Q}$ and the Rauzy induction stops. Then $F$ is integrable, and any periodic interval on the Poincaré–Rauzy section is flipped.

**Proof.** Here is the list of all possible combinatorics of the maps in $F$ depending on which of four intervals contains 0 in its image:

1. $\begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ \bar{A} & \bar{C} & D & \bar{E} & \bar{B} \end{pmatrix}$;
2. $\begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ B & D & \bar{E} & \bar{A} & \bar{C} \end{pmatrix}$;
3. $\begin{pmatrix} \bar{C} & \bar{B} & \bar{C} & D & \bar{E} \\ \bar{C} & \bar{E} & \bar{A} & B & D \end{pmatrix}$;
4. $\begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ D & \bar{A} & B & \bar{C} & \bar{E} \end{pmatrix}$.

Note that dynamically the cases 1. and 4. (as well as 2. and 3.) are equivalent—it suffices to change the orientation of the initial interval.

Note that in the case 1. (and 4.), the map $F$ has an invariant interval $I_A(I_B)$, and the restriction of $F$ on its complement belongs to FET$^4$. Hence, $F$ is integrable by Proposition 4.

Hence, the study of integrability of the maps in the family CET$^4$ is reduced to the study of integrability of the maps with the combinatorics

$$\begin{pmatrix} \bar{A} & B & \bar{C} & D & \bar{E} \\ B & D & \bar{E} & \bar{A} & \bar{C} \end{pmatrix}.$$ (5–2)

See Figure 13 for the illustration of the map with such combinatorics, when $0 \in F(I_B \cup I_C)$. Here, some additional work is needed to prove that the Rauzy induction can’t stop in such a way that the restriction $G$ is a non-integrable IET (as in Proposition 8). The only possibility for an IET on 4 (or less) intervals to be non-integrable is exactly to have the combinatorics (5–1). The Rauzy induction can always be continued on an IET with flips (except for the case of the invariant flipped cylinder on the right). Hence, one can indeed suppose that $G \in \text{IET}$. The end of the proof is computer assisted. We calculate explicitly the component of the Rauzy graph corresponding to the combinatorics 2. and we verify that the Rauzy induction never stops at the vertices of Rauzy graph of combinatorial type (5–1).

One can show that $F$ has all of its periodic intervals flipped, by repeating the argument of Proposition 6.

We have seen that almost all of the maps in CET$^3_4$, as well as in CET$^4_4$ are integrable. It turns out that
starting from \( n = 5 \), the dynamics of \( \text{CET}_n \) starts to become more complicated.

**Proposition 10.** There exists an open subset \( U \) in the set of parameters
\[
\mathcal{P} = \{(l_1, ..., l_n) \in [0, 1]^n | l_1 + ... + l_n = 1\} \times \{\tau | \tau \in S^1\}
\]
for the maps \( \text{CET}_n \), \( n \geq 5 \) such that the dynamics of corresponding maps is non-integrable.

**Proof.** Indeed, take \( U_0 := \{(l_1, ..., l_n, \tau) \in \mathcal{P} | l_1 > 1 - \tau > \max(l_2, ..., l_n)\} \). In this case the image of the first interval contains 0 and the combinatorics of a corresponding map is the following:
\[
\begin{pmatrix}
\tilde{I}_1 & \tilde{I}_1 & \tilde{I}_2 & ... & \tilde{I}_{n-1} & \tilde{I}_n \\
\tilde{I}_1 & \tilde{I}_2 & \tilde{I}_3 & ... & \tilde{I}_n & \tilde{I}_1
\end{pmatrix}.
\]

Let \( U \) be the intersection of \( U_0 \) with the set of the points in \( \mathcal{P} \) such that the lengths \( l_1 - 1 + \tau, 1 - \tau, l_2, ..., l_n \) don’t have any rational relationships except for trivial ones.

For \( F \in \text{CET}_n \) such that \( (l_1, ..., l_m, \tau) \in U \), the corresponding Rauzy induction follows the path
\[
\begin{pmatrix}
\tilde{I}_1 & \tilde{I}_1 & \tilde{I}_2 & ... & \tilde{I}_{n-1} & \tilde{I}_n \\
\tilde{I}_1 & \tilde{I}_2 & \tilde{I}_3 & ... & \tilde{I}_n & \tilde{I}_1
\end{pmatrix}.
\]

Thus, for \( n \geq 5 \), the obtained first return map has two invariant flipped intervals and one interval on which the dynamics is given by a non-simple \((n-1)\)-IET. Hence, the dynamics of \( F \) is not integrable.

### 6. The properties of the orbits in triangular tiling billiards

Now we get back to the properties of triangle tiling billiards and prove the results on their dynamics. For that, we will use the machinery prepared in previous sections as well as some additional arguments.

#### 6.1. Symbolic dynamics and combinatorial orbits

Fix a triangular tiling by the tiles congruent to the triangle \( \Delta \) with vertices \( A, B, C \) and sides \( a, b, c \) on the plane. To any trajectory \( \delta \) on this tiling one can associate a bi-infinite word \( \omega(\delta) = ...w_{-2}w_{-1}w_0w_1w_2... \) in the alphabet \( A = \{a, b, c\} \) by writing the labels of sides in the tiling intersected along \( \delta \). By fixing a starting triangle, one fixes a special slot in the symbolic word. Note that singular trajectories (passing by the vertices) may give several coding with common past or future since they branch out.

One can say that the symbolic sequence \( \omega(\delta) \) characterizes how the trajectory is seen with a naked eye (that can’t precisely measure the angles but sees the sequence of triangles that are crossed).

**Example.** On Figure 5 the first depicted trajectory is a closed trajectory corresponding to the sequence \( \tilde{\omega} \in \mathcal{A}^\mathbb{Z} \) where \( \omega = bcabababa \) and the names \( a, b, c \) are given to the sides in the decreasing order of lengths. One can see that by fixing the starting triangle (grey triangle on the Figure 5), one fixes the zero position symbol of the word \( \omega \) and we have \( w_0 = b \).

**Definition 22.** Define \( S \subset \mathcal{A}^\mathbb{Z} \) to be the set of all bi-infinite words in the alphabet \( A \) that correspond to symbolic dynamics of (possibly, singular) trajectories in triangle tiling billiards. Define \( S_\Delta \) the analogous set for the triangle tiling billiard in a tiling defined by the fixed triangle \( \Delta \). We call the elements in these sets symbolic orbits of triangle tiling billiards.

It is easy to see that \( S \) is strictly contained in \( \mathcal{A}^\mathbb{Z} \). Indeed, the words in \( S \) don’t contain \( aa, bb \) or \( cc \) as a subword. for since a trajectory can’t pass by the same side consequently. One can also prove that for \( c \) being the shortest side of \( \Delta \), the words in \( S_\Delta \) don’t contain the subwords \( cbc \) and \( cac \). In general, the set \( S_\Delta \) is a “small” set: the corresponding dynamics has zero topological entropy and linear complexity. The precise
understanding of the structure of $S_\Delta$ is not an easy question, see for example the discussion about the Conjecture 1.

For a fixed triangle $\Delta$, the study of the triangle tiling billiard with the tiles congruent to $\Delta$, is equivalent to that of the map $F_{l_1,l_2,l_3} \in \text{CET}_2^2$, restricted onto some subset of the circle (as explained in paragraph 2.2), with $l_j$ being the normalized angles of $\Delta$ and $\tau$ a parameter. Since $F$ is fully flipped, then its square $T_{l_1,l_2,l_3} = F^2 : S^1 \to S^1$ is an interval exchange transformation. An accelerated (two steps in one) triangle tiling trajectory (corresponding to $T$) goes through the triangles of the same orientation on the circumcircle. Define for any pair $w_0, w_1 \in A$ the following subset of $S^1$:

$$I_{w_0w_1} := \{ x \in S^1 | x \in I_{w_0}, F(x) \in I_{w_1} \}.$$  \hspace{1cm} (6–1)

Such a subset is a segment of continuity for $T$. Since $F$ has 3 singular points on $S^1$, then $T$ has at most 6 singular points on $S^1$. From this we see that at least 3 of these subsets are empty.

We associate to any symbolic orbit $w \in S_\Delta$, a word $\Omega$ in the alphabet of vertices $\{A, B, C\}$ simply by replacing small letters by capital ones. Then for those $x \in S^1$ that correspond to the trajectories of triangle tiling billiard, $\Omega_0\Omega_1$ considered as a vector on the plane, lies in a following set:

$$\mathcal{V} = \{ \pm AB, \pm BC, \pm AC \}.$$  \hspace{1cm} (6–2)

**Definition 23.** Fix a triangle $\Delta$ on the plane with positive orientation (such that the curve ABC has the triangle $\Delta$ on its left). Let the origin $(0, 0) \in \mathbb{R}^2$ be at the barycenter of $\Delta$. Consider a trajectory $\delta$ of a triangle tiling billiard starting in $\Delta$ and its symbolic orbit $w$ with an associated word $\Omega$ in the alphabet of vertices. Then the combinatorial orbit of $\delta$ is a piecewise linear curve that connects by straight segments the following points in the following order:

$$\cdots, \Omega_{j+1}, \Omega_{j+1} + \cdots, -\Omega_0, 0, \Omega_1, \cdots, \Omega_{j+1},$$

$$\cdots, \Omega_2, \sum_{j=0}^k \Omega_{j+1}, \cdots, \Omega_2, \cdots, \sum_{j=0}^k \Omega_{j+1}, \cdots$$

One can easily see that the points where the combinatorial orbit breaks (is not smooth) are exactly the barycenters of positively oriented triangles through which $\delta$ passes, and that the segments of the combinatorial orbit are parallel to the sides of the triangles in a tiling.

This curve was already studied by Hooper and Weiss in [Hooper and Weiss 18], where they were interested in rel leaves of some special class of translation surfaces, being one-parameter deformations of Arnoux–Yoccoz translation surface. By finding a triangle tiling billiard associated to this case (which was done in [Baird-Smith et al. 19]), one can see that its combinatorial orbits are abelianizations of other interesting curves, studied by McMullen [McMullen 15] and Arnoux [Arnoux 88, Arnoux et al. 11]. See more on this case in paragraph 6.4.

**Example.** A combinatorial orbit for the first trajectory of Figure 5 is a union of segments connecting the consecutive results of the sums of vectors $BC + AB + AB + CA + BA$.

Obviously, from Proposition 1 follows

**Lemma 4.** A combinatorial orbit of $\delta$ is a closed (drift-periodic, linearly escaping) curve if and only if the $\delta$ is closed (drift-periodic, linearly escaping).

### 6.2. Properties of periodic orbits

Closed and drift periodic trajectories both have their symbolic orbits periodic. Although, the balance properties of each periodic word $\omega(\delta) \in S$ show if $\delta$ is a periodic or a drift periodic trajectory. Indeed, take a finite word $s \in A^l$ of length $L$ corresponding to the geometric period of $\delta$. Then $\omega = \bar{s}$ and $L \in 2\mathbb{Z}$. The corresponding portion of a combinatorial orbit of length $\frac{L}{2}$ is a union of segments connecting the consecutive results in the sum $\Omega_0\Omega_1 + \Omega_2\Omega_3 + \cdots + \Omega_{l-1}\Omega_l$. Obviously,

**Proposition 11.** A trajectory $\delta$ in a triangle tiling billiard is closed (drift-periodic) if and only if a corresponding symbolic word $w(\delta)$ is periodic $w = \bar{s}$ and the sum $\Omega_0\Omega_1 + \Omega_2\Omega_3 + \cdots + \Omega_{l-1}\Omega_l$ is equal (not equal) to zero.

**Example.** For the third (drift-periodic) trajectory on Figure 5, the coding $\omega = \bar{s}$ with $s = bababc$, and $BA + BA + BC \neq 0$.

**Proposition 12.** Suppose that the symbolic coding of some trajectory $\delta$ in a triangle tiling billiard is periodic, i.e. there exists a word $s$ such that $\omega = \bar{s}$. Suppose additionally that $s$ has odd length. Then the trajectory $\delta$ is closed.

**Proof.** Indeed, $\omega = \bar{s} = s^2$ where $s = s_0 \ldots s_{l-1}$, $|s| = l, l \in 2\mathbb{Z} + 1$. Define $s_j$ for all $j = l, \ldots, 2l-1$ by $s_{j+l} := s_{j+1}$. In order to prove the statement, it suffices to show, by Lemma 4, that the sum of the vectors $S_l := \sum_{k=0}^{l-1} s_k s_{k+l}$ is equal to 0. Proof is by induction on $l$ and uses only simple linear algebra. For $l = 1$, we have $S_1 = \bar{s}_0\bar{s_0} = 0$. Suppose
the statement is proven for some odd \( l \) and \( \bar{S}_l = 0 \) for any word \( \omega \). Let us prove it for \( l + 2 \).

Indeed, now we have \( s^2 = s_0 \ldots s_{l+1} s_{l+2} \ldots s_{2l+1} \) with \( s_j = s_{j+l+2} \) for all \( j \in [0, l + 1] \). Then we have

\[
-S_{l+1} = \sum_{k=0}^{l+1} s_{2k}s_{2k+1} = \sum_{k=0}^{l-1} s_{2k}s_{2k+1} + s_{l-1}s_l + s_{l+1}s_0 + \sum_{k=l+2}^{l+1} s_{2k}s_{2k+1} + s_{l+2}s_{l+3}
\]

\[
= \left( \sum_{k=0}^{l-1} s_{2k}s_{2k+1} + s_{l-1}s_l + s_{l+1}s_0 + \sum_{k=l+2}^{l+1} s_{2k}s_{2k+1} \right) \frac{1}{s_l}
\]

\[
= \frac{-s_{l-1}s_l + s_{l-1}s_l + s_{l+1}s_0 + s_{l+2}s_{l+3}}{s_l} = 0.
\]

### 6.3. Generic behavior of trajectories in triangle tiling billiards

In this Section we use the results of previous Section 5 in order to characterize the qualitative behavior of triangle tiling billiards. The following Proposition describes the generic behavior of triangle tiling billiards and answers the following question. If one picks a random triangle and a random trajectory in a corresponding tiling, how does this trajectory look like? The answer is: it is either closed, or linearly escaping with an irrational slope.

**Proposition 13.** Let \( C = \{(l_1, l_2, l_3) \in \mathbb{R}_1^3 | l_1 + l_2 + l_3 = 1\} \) be a simplex of normalized angles of triangles \( \Delta \), \( l_j = \frac{2}{n} \). Consider a subset of \( C \subset C \) of lengths independent over \( \mathbb{Q} \). Consider the set of all pairs \((\delta, \Delta)\) such that the lengths of \( \Delta \) belong to \( C \) and the trajectory \( \delta \) does not pass through the circumcenters of the crossed triangles. Then the following holds:

1. The orbit \( \delta \) is not drift-periodic orbits.
2. If \( \delta \) is closed then it has a symboling coding \( \overline{s}^2 \) for some word \( s \) in the alphabet \( a, b, c \) of odd length. Consequently, its period is equal to \( 4n + 2 \) for some \( n \in \mathbb{N}^* \), and its combinatorial orbit has central symmetry.
3. If \( \delta \) is not closed, it is linearly escaping. Its symbolic orbit is described by an infinite word \( \omega \) that can be represented as an infinite concatenation of two finite words \( \omega_1 \) and \( \omega_2 \) in the alphabet \( A \) in some order corresponding to a Sturmian sequence.

**Proof.** Assertion 1. has been proven in [Baird-Smith et al. 19], see Proposition 2.15.

Then, for any point in the set \( C \), and any trajectory \( \delta \) not passing through the circumcenters, the Rauzy induction for a corresponding \( F = F_{l_1, l_2, l_3} \in \text{CET}_3 \) stops, by Theorem 3 since \( \tau \neq \frac{1}{2} \). By Theorem 7, \( F \) is integrable with a simple first-return map on the Poincaré–Rauzy section. A point \( x \in S^1 \) has a closed orbit if and only if it is contained in a flipped interval \( I \) on this section, or in its orbit by \( F \) (its Rokhlin tower). But \( F \) maps onto itself by the first return map which, in restriction to \( I \) is equal to \( F^d \), \( d = 2n + 1 \in 2\mathbb{N} + 1 \). Then the period of the billiard orbit is exactly \( 2d = 2(2n + 1) = 4n + 2 \), and its combinatorial orbit is centrally symmetric.

The escaping trajectories correspond to the orbits of the points on the Poincaré–Rauzy section which are not flipped. By integrability, they correspond to rotations. The symbolic coding of the first return map to the section Poincaré–Rauzy defines the words \( \omega_j, j = 1, 2 \). For the case of rotation with a marked point \((3-IET)\) as in point 3. of Definition 19, its symbolic dynamics can be reduced to the case of 2-IET by considering a smaller Poincaré section.

In paragraph 6.2 we have proven that the odd length codings correspond to the periodic orbits. The converse is true as well.

**Proposition 14.** Consider any closed trajectory \( \delta \) in a triangle tiling billiard. Then the minimal period of its symbolic coding \( \omega(\delta) \) has an odd length. Consequently, its period is equal to \( 4n + 2 \) for some \( n \in \mathbb{N}^* \).

**Proof.** The existence of a periodic trajectory implies that the Rauzy induction stops for the corresponding \( F = F_{l_1, l_2, l_3} \), by Lemma 1. If the Rauzy induction stops then either one is in the conditions of Proposition 13 (the lengths are independent over \( \mathbb{Q} \)), and there is nothing to prove, or the lengths \( l_j \) are rationally dependent. In this case, we perturb the point in \( C \) in such a way that this dependence is destroyed. If the perturbation is small enough, the symbolic coding (as well as the length) of the periodic trajectory stay fixed, by Proposition 1. One can then go back to applying Proposition 13.

**Remark.** The periods of drift-periodic orbits are not necessarily of length \( 4n + 2 \). For example, take a triangle with lengths of the sides \( a, b, c \), equal to 5, 7 and 8 and \( \tau \approx 1/2 \). One can verify that the orbit passing through the middle of the side \( b \) is drift-periodic with a drift period 24.
By looking at the simulations of trajectories, one can suggest a much stronger property of closed orbits which is the following conjecture which was first stated in [Baird-Smith et al. 19], see Conjecture 5.3.

**Conjecture 1** (Tree conjecture). Let $\Lambda$ be as a union of all vertices and edges of all drawn triangles in a periodic triangle tiling. Take any periodic closed trajectory $\gamma$ with minimal behavior. It incloses some bounded domain $U \subset \mathbb{R}^2$ in the plane, $\partial U = \delta$ and $U \cap \Lambda$ is an embedding of some graph in the plane. Then this graph is a tree.

In [Baird-Smith et al. 19] this conjecture is proven for obtuse triangles (the tree in question is in this case a chain). This conjecture has an even stronger form corresponding to the fact that any trajectory (not necessarily closed) fills in the subset of the plane that it occupies.

### 6.4. Description of the family CET$_{1/2}^3$ and Arnoux–Rauzy family

The generic behavior of a map $F_{\ell_1, \ell_2, \ell_3}$ from CET$_3$ has been understood thanks to Theorem 7: it is integrable, and corresponding billiard trajectories are either closed or linearly escaping. Although, there exists a subfamily of maps in CET$_3$ with minimal behavior. Indeed, the authors of [Baird-Smith et al. 19] remark that for $\tau = \frac{1}{2}$ and the vector of lengths being chosen as

$$ (l_1, l_2, l_3) = \left(\frac{1-x^3}{2}, \frac{1-x^2}{2}, \frac{1-x}{2}\right), \quad (6.3) $$

for $x$ the real solution of the algebraic equation

$$ x + x^2 + x^3 = 1, \quad (6.4) $$

the symbolic behavior of the corresponding triangle tiling billiard trajectories seems to be quite different from the expected generic behavior—they seem to escape to infinity non-linearly. The authors in [Baird-Smith et al. 19] notice that these trajectories “approach Rauzy fractal”. Unfortunately, this interesting remark has not been yet put in a form of a precise mathematical statement. The object obtained in [Baird-Smith et al. 19], i.e. the rescaled limit of combinatorial orbits for these parameters, is exactly the same than that defined in [Hooper and Weiss 18] outside the context of triangle tiling billiards.\footnote{For more on the relationship between the work [Hooper and Weiss 18] and triangle tiling billiards, see in [Baird-Smith et al. 19].}

Moreover, Hooper and Weiss already conjectured the convergence (up to rescaling and a uniform affine coordinate change) of these combinatorial orbits to the Rauzy fractal in Hausdorff topology, see the discussion at the end of Section 4 in [Hooper and Weiss 18]. This conjecture remains open, is one of our main motivations for future research, as well as the understanding of the non-linear escape in triangle tiling billiards. This motivates this paragraph’s discussion and the following

**Definition 24** ([Arnoux 88, Arnoux and Rauzy 91]). The Arnoux–Rauzy family is a 3-parametric family of the maps in CET$^3$ defined as follows. The circle is identified with the set $[0,1)$ and is cut into 6 intervals of lengths $\frac{x_j}{2}$, $j = 1, 2, 3$, $\sum_j x_j = 1$:

$$ [0, 1) = \left[0, \frac{x_1}{2}\right) \cup \left[\frac{x_1}{2}, \frac{x_1 + x_2}{2}\right) \cup \left[\frac{x_1 + x_2}{2}, 1\right) \cup \left[x_1 \frac{x_2}{2}, x_1 + x_2\right) \cup \left[1 - x_3, 1 - \frac{x_3}{2}\right) \cup \left[1 - \frac{x_3}{2}, 1\right). $$

Each transformation from Arnoux–Rauzy family is a composition of two maps. First, the intervals of equal lengths are exchanged. Second, the circle is rotated by $\frac{1}{2}$.

**Remark.** The two maps in this definition are non-commuting involutions.

The famous Arnoux–Yoccoz example (with the parameters $(x_1, x_2, x_3) = (x, x^2, x^3)$ with $x$ being the solution of (6.4)), belongs to this Arnoux–Rauzy family. The Arnoux–Rauzy family has been extensively studied [Arnoux 88, Arnoux and Rauzy 91, Avila et al. 16a, Avila et al. 16b] but, as far as we know, it was never noticed that this family has a natural square root in the set of fully flipped interval exchange transformations.

**Proposition 15. The set of the squares**

$$ \{ F^2 | F_{\ell_1, \ell_2, \ell_3} \in \text{CET}_3, \tau = \frac{1}{2}, l_1, l_2, l_3 < \frac{1}{2} \} $$

is exactly Arnoux–Rauzy family of interval exchange transformations on the circle.

**Proof.** The proof is a direct computation. Without loss of generality, we can suppose that $l_1 \leq l_2 \leq l_3$ by changing the orientation and possibly replacing $\tau$ by $1-\tau$. Consider a following subdivision of initial intervals of continuity:

\[O.P.-R.'s \ lecture \ based \ on \ this \ paper \ is \ available \ at \ the \ Youtube \ channel \ of \ Institut \ Fourier \ here: \ https://www.youtube.com/watch?v=I91c-g_BzbM.\]
In the Figure correspond to the iterations of the map $F$ represented on the Figure 14. One can see that $F^2 \in \text{CET}^6$ and $F^3 \in \text{IET}^7$. Indeed, one has to make a cut $J_2 = (\frac{1}{2} - l_3, l_3) \cup (l_3, 1)$ in order to pass from the CET on 6 intervals to an IET on 7 intervals. The intervals of the same length in the subdivision neighbor each other on the first level as well as on the last level, after applying $T$ but their order in a couple is reversed, exactly defining the Arnoux–Rauzy family.

$$I_1 = (0, \frac{1}{2} - l_2) \cup (\frac{1}{2} - l_2, l_1) =: J_2^+ \cup J_3^+$$
$$I_2 = (l_1, 1 - l_2) \cup (\frac{1}{2} + l_1, l_3 + l_2) =: J_1^+ \cup J_3^-$$
$$I_3 = (l_1, l_2, l_3, \frac{1}{2}) \cup (\frac{1}{2} + l_2, 1) =: J_1^+ \cup J_2^-.$$

Obviously, $F$ can be seen as IET with flips on the interval $[0, 1]$ with intervals of continuity $I_2^+, J_3^+, J_1^+, J_2^-, J_1^-$. These intervals are exchanged following the permutation

$$\left( \begin{array}{cccccc}
J_2^+ & J_3^+ & J_1^+ & J_2^- & J_1^- & J_3^- \\
J_3^+ & J_1^+ & J_2^- & J_1^- & J_3^- & J_2^+
\end{array} \right).$$

Applying once more the map $F$ to the image of the interval, one obtains the final result: $T = F^2$ is such that $F^2 \in \text{CET}^6$ and $F^3 \in \text{IET}^7$. The order of the corresponding permutation of 6 intervals on the circle is represented on the Figure 14.

Define $I_{j,k}$ as

$$I_{j,k} = \{x \in I | F(x) \in I_k\}.$$  \hfill (6-5)

for $j, k = 1, 2, 3$. Then one can see that exactly

$$I_1 = I_{2,3}, J_3^+ = I_{3,3};$$
$$I_2 = I_{3,1}, J_1^+ = I_{1,3};$$
$$I_3 = I_{1,2}, J_3^+ = I_{2,1}$$

and that these intervals come by couples of equal length:

$$|I_{1,2}| = |I_{2,1}| = \frac{1}{2} - l_3, |I_{1,3}| = |I_{3,1}| = \frac{1}{2} - l_2,$$
$$|I_{3,2}| = |I_{2,3}| = \frac{1}{2} - l_1.$$

Combinatorics and the lengths defined the map $T = F^2$ and show that this is exactly a map from the Arnoux–Rauzy family and that all the maps are represented. The parameters $x_j$ in Definition 24 are given by the affine change from the lengths of flipped intervals $I_j, j = 1, 2, 3$:

$$x_j = 1 - 2l_j, j = 1, 2, 3. \hfill (6-6)$$

Here $x_1 \geq x_2 \geq x_3$.

**Remark.** The fact that the intervals split into couples of equal length for the square of the map, is geometrically obvious from the symmetry, for any map in CET$^3$. Indeed, the lengths correspond to the lengths of the intervals that an inscribed polygon and its reflection with respect to its circumcenter cut out on the circumscribed circle (the intervals $I_{i,j}$ and $I_{j,i}$ being centrally symmetric, they have equal lengths). Although, the combinatorics has to be precised, see Proposition above.

Let us note that for any $F \in \text{CET}^3$ corresponding to an obtuse triangle its dynamics is completely periodic.

**Proposition 16.** For any $F \in \text{CET}^3$ such that there exists $l_j > \frac{1}{2}$, $F$ is completely periodic.

**Proof.** One can suppose $l_1 > \frac{1}{2}$. Then, the intervals $I_1 := (0, l_1 - \frac{1}{2})$ and $I_2 := (\frac{1}{2}, l_1)$ are invariant by $F$ and consist of periodic orbits of length 2. The dynamics of $F$ restricted on the interval $[0, 1) \setminus (I_1 \cup I_2)$ one can see that this dynamics, is equivalent to the dynamics of an 3-IED with flips with combinatorics

$$\left( \begin{array}{ccc}
\hat{A} & B & \hat{C} \\
\hat{B} & \hat{C} & A
\end{array} \right) $$  \hfill (6-7)
Such a system is integrable by Proposition 3 and, as can be explicitly verified in this case, via a more precise study of the corresponding component of the Rauzy graph, completely periodic. See Figure 15 for illustration.

### 6.5. Non-linearly escaping orbits and the Rauzy gasket

The hero of this paragraph is the following object.

**Definition 25** ([Arnoux and Starosta 13]). Consider a point \((x_1, x_2, x_3) \in \mathbb{R}^3_+\) such that \(x_1 + x_2 + x_3 = 1\). Apply the following algorithm:

1. If one of the entries, say \(x_1\), is greater than the sum of the two smaller ones, obtain a new point \((x_1 - x_2 - x_3, x_2, x_3)\) with positive entries. Analogically, for \(x_2 > x_1 + x_3\) (or \(x_3 > x_1 + x_2\)), one defines a new point \((x_1, x_2 - x_1 - x_3, x_3)\) (or \((x_1, x_2, x_3 - x_1 - x_2)\)).
2. Rescale a new point so that the sum of the entries is equal to 1.

The Rauzy gasket \(\mathcal{R}\) is the set of points in the simplex on which this algorithm can be applied infinitely many times, i.e. the step 1. is always possible.

The Rauzy gasket \(\mathcal{R}\) has zero Lebesgue measure [Arnoux and Starosta 13] and its properties were extensively studied in [Arnoux and Rauzy 91, Avila et al. 16a, Avila et al. 16b]. The relationship of the set \(\mathcal{R}\) to the triangle tiling billiards is the following. Proposition 15 shows that the study of the dynamics of trajectories of acute triangle tiling billiards passing through the circumcenter is equivalent to the study of the dynamics of Arnoux–Rauzy family. It is well known [Arnoux and Rauzy 91] that a corresponding \(F \in \text{CET}^3\) is minimal if and only if the corresponding triple belongs to the Rauzy gasket, \((x_1, x_2, x_3) \in \mathcal{R}\). Here \(x_j\) and \(l_j\) are related by \((6–6)\). One can ask ourselves, what is the dynamics of the trajectories corresponding to the parameters from the Rauzy gasket? We conjecture that all of these trajectories escape non-linearly to infinity, and we prove this for a full measure set of points in \(\mathcal{R}\) (with respect to the Avila-Hubert-Skripchenko measure defined in [Avila et al. 16]). Moreover, there are no other non-linearly escaping trajectories, other than that coming from the Rauzy gasket and passing through the circumcenters.

**Proposition 17.** Suppose that there exists a pair \((\delta, \Delta)\) such that a trajectory \(\delta\) in a triangle tiling billiard with the tiles congruent to \(\Delta\), is non-linearly escaping. Then

1. the triangle \(\Delta\) is acute,
2. the trajectory \(\delta\) passes through the circumcenters of all the triangles that it crosses,
3. the triple of \(x_j\) defined by \((6–6)\), where \(l_j\) are the normalized angles of \(\Delta\), belongs to the Rauzy gasket \(\mathcal{R}\).

Moreover, if the pair \((\delta, \Delta)\) are such as in \(1.–3.\) above and the corresponding map in CET\(^3\) is uniquely ergodic, then the trajectory \(\delta\) is non-linearly escaping.

**Proof.** By Proposition 16, \(\Delta\) is acute. The existence of a non-linearly escaping trajectory implies that \(\tau = \frac{1}{2}\) for a corresponding \(F \in \text{CET}^3\) by Proposition 13. The point 3. follows from [Arnoux and Rauzy 91] where it was proven that the only Arnoux–Rauzy minimal maps are those with parameters \(x_j\) in the set \(\mathcal{R}\).

Let us prove by contradiction that a trajectory is non-linearly escaping. Consider \(T = F^2\) and let us consider its coding as in equation \((6–2)\). Let denote by \(f\) the map with values in \(\mathbb{R}^2\) that is this approximated displacement only depending on two consecutive letters of the symbolic coding. If the trajectory is non-linearly escaping, there is a positive constant \(C\) such that

\[
\left| \sum_{j=0}^{N} \Omega_j \Omega_{j+1} \right| > CN. \quad (6–8)
\]

By unique ergodicity of the map \(T\), and by applying the ergodic theorem, we have

\[
\lim_{N \to \infty} \frac{1}{N} \left| \sum_{j=0}^{N} \Omega_j \Omega_{j+1} \right| = \int \! f(x) \, dx. \quad (6–9)
\]

Note that here, one can denote \(I_A := I_1, I_B := I_2, I_C := I_3\) and analogously to Definition \((11)\), one can denote the intervals \(I_{ij}\) with the labels \(i, j \in A\), for the alphabet \(A = \{A, B, C\}\). Now we calculate explicitly the right-hand side of \((6–9)\):

---

**Figure 15.** Connected component of the vertex \((13)\) in the Rauzy graph of \(\text{IETF}^3\).
\[ \int_{S^1} f(x) dx = \sum_{j \in A} \bar{\alpha}_j |I_j| = \sum_{j \in A} \left( \bar{\alpha}_j - \bar{\alpha}_j^t \right) |I_j| = \sum_j \bar{\alpha}_j \sum_j |I_j| - \sum \bar{\alpha}_j \sum_j |I_j| = 0, \]

since \( \sum_j |I_j| = I_j, \sum_j |I_j| = |I| \) by definition (6–5). Thus, this calculation and the relation (6–8) give a contradiction in the relation (6–9).

This Proposition proves, for example, that the trajectory with parameters 9 is indeed not linearly escaping. We didn’t manage to prove the non-linear escaping behavior for all the points in the set \( \mathcal{R} \) but only for those which correspond to uniquely ergodic maps.

7. Perspectives and open questions

7.1. Combinatorial orbits of non-linearly escaping trajectories

From the simulations, one can see that non-linearly escaping trajectories of triangle tiling billiards have a very interesting behavior. One can hope to understand it better even though Proposition 17 proves a spiraling behavior for most of these trajectories (with respect to an appropriate measure).

Problem. Describe the combinatorial orbits of non-linearly escaping trajectories of the triangle tiling billiards.

This direction of research was already discussed in paragraph 6.4 but we add it here for completeness. To some extent, a part of the work on this problem was done in \([\text{Avila et al. 16a}]\).

7.2. Squares of fully flipped interval exchange transformations and SAF invariant

For a fully flipped interval exchange transformation \( F \in \text{FET}^n[0,1) \) its square \( T = F^2 \) doesn’t have any of its intervals flipped, \( T \in \text{IET}^{2n-1}[0,1) \). Let

\[ \mathcal{F} := \{ T \in \text{IET}^{2n-1} | \exists F \in \text{FET}^n : F^2 = T \}. \]

Note that the set \( \mathcal{F} \) is a set of half of the dimension of \( \text{IET}^{2n-1}[0,1) \).

Problem. Describe the set \( \mathcal{F} \) in terms of the invariants of interval exchange transformations.

We still do not understand very well what exactly is this subset \( \mathcal{F} \) in the set of all interval exchange transformations \( \mathcal{F} \subset \text{IET}^{2n-1} \). In this paragraph, we prove that \( \mathcal{F} \) lies in the kernel of the SAF invariant.

Definition 26. Take \( T \) is an interval exchange transformation, \( T \in \text{IET}^n \) such that \( \lambda_i \) are the lengths of its intervals \( I_i \) of continuity, and for every interval \( I_i \) the map \( T \) is a translation by some number \( t_i \):

\[ T|_{I_i}(x) = x + t_i, 1 \leq i \leq n. \]

Then the Sah-Arnoux-Fathi invariant of \( T \) is a \( \mathbb{Q} \)-bilinear form which is given by

\[ \text{SAF}(T) = \sum_{i=1}^m l_i \otimes q_i, \quad (7–1) \]

The SAF invariant of an IET was defined by Arnoux in his thesis \([\text{Arnoux 81}]\), where he proved the following

Theorem 4. \([\text{Arnoux 81}]\) Consider \( T \in \text{IET}^n[0,1) \). Then its SAF-invariant has the following properties.

1. If \( T \) is periodic then \( \text{SAF}(T) = 0 \).
2. For \( T = T_x \) being a rotation of angle \( \alpha \),

\[ \text{SAF}(T_x) = 1 \otimes \alpha - \alpha \otimes 1 \]

And, consequently, \( T_x \) is periodic \((\alpha \in \mathbb{Q})\) if only if \( \text{SAF}(T_x) = 0 \).
3. Let \( T_1 \) and \( T_2 \) be two first return maps of a directional flow on some translation surface \( M \) with transversals \( \tau_1, \tau_2 \subset M \) correspondingly. Then \( \text{SAF}(T_1) = \text{SAF}(T_2) \). In other words, SAF is an invariant of the flow.
4. \( \text{SAF}(T) \) is invariant by the Rauzy induction.

A SAF invariant is a very powerful tool for working with interval exchange transformations and translation surfaces. For example, for a surface of genus 2, if \( \text{SAF} = 0 \) then the translation flow is not minimal, by the works of Arnoux, Boshernitzan and Caroll Calta, McMullen (see \([\text{Arnoux 88, Boshernitzan and Carroll 97, Calta 04, McMullen 15}]\)).

Although, in higher genus the nullity of SAF is not equivalent to the absence of minimality. Indeed, the Arnoux–Yoccoz example \([\text{Arnoux 88}]\) already mentioned in paragraph 6.4 is the first example of an IET that has zero SAF-invariant and is minimal. Its suspension dynamics corresponds to a translation flow on the surface of genus 3.

Proposition 18. For any map \( F \in \text{FET}^n \) its square \( T = F^2 \) has zero SAF invariant.

Proof. Divide the initial interval of definition of \( F \) into the intervals \( I_i \) of continuity of \( F \). On each of these intervals, one writes \( F|_{I_i} = -x + \tau_j, \tau_j \in \mathbb{R} \). We subdivide each of these intervals into the subsets \( I_{j,k} \) (which are either segments or empty sets) defined as in (6–5). We have then \( \bigcup_{k=1}^m I_{j,k} = I_j \) and \( \bigcup_{j=1}^m I_{j,k} = F^{-1}(I_k) \). Also, we have \( \sum_k |I_{j,k}| = \lambda_j \) and also...
The combinatorial study of the modified Rauzy graphs in connection to minimality seems a promising area of research, see e.g. [de Mourges 17] for the combinatorial approach of standard Rauzy graphs. The modified Rauzy graphs are very different from standard Rauzy graphs: for example, if there exists a path from one vertex to another in a modified graph, there is not necessarily a path backward (which was yet true for the standard Rauzy graphs).

We would like to formulate a general

**Conjecture 2.** For any \( n \), if \( F \in \text{CET}_n \) is minimal on \( S^1 \) then \( \tau = \frac{1}{2} \).

**Theorem 5.** Conjecture 2 is true for \( n = 4 \).

**Proof.** Proof is analogical to the case when \( n = 3 \), and also uses an analog of Lemma 3 whose proof is computer assisted.
7.4. Quadrilateral tilings

The Definition 2 of a system of reflections in a circumcircle can be generalized in an obvious way to any inscribed $n$-polygon. Although, not any polygon tiles the plane. A tiling by congruent polygons in the plane can be achieved only for $n = 3, 4, 5,$ and $6$. All of these tilings are classified, with a final achievement by M. Rao in proving the classification of all pentagonal tiling families into 15 (already known) classes [Rao 17]. In this paragraph, we concentrate on the case of quadrilateral tilings but we mention this interesting

**Problem.** Study the behavior of tiling billiards on all the tilings of a plane by congruent pentagons (hexagons).

**Problem.** Any quadrilateral tiles a plane in a periodic way. Study the behavior of tiling billiards in such a quadrilateral tiling. For example, the parallelogramm tiling.

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Figure 16. Periodic cyclic quadrilateral tiling billiards. The two trajectories on the top are closed and the two on the bottom are linearly escaping. A program simulating quadrilateral tiling billiards dynamics has been written by the second author in collaboration with Ilya Schurov.
The initial permutation (2) has its corresponding vector of the components is either equal to 0, 1 or 1 with the set of lengths $l_j$ such that $l_j \neq 1$. The non-flipped intervals have their corresponding components equal to ±1. The trajectory in a folded system, as before.

Here, we restrict ourselves to the case of the quadrilaterals inscribed in circles (cyclic quadrilaterals), in order to the analog of Theorem 1 to hold. The class of tilings we study is periodic cyclic quadrilateral tilings on the plane, defined as follows: for any cyclic quadrilateral in a tiling a neighboring tile is obtained by a central symmetry in the middle of the edge, all the tiles being congruent. As before, one can associate to any such tiling the map $F_{l_1, l_2, l_3, l_4} \in \text{CET}_4$. The lengths $l_j$ correspond to the lengths of the sides of the inscribed quadrilateral, and $\tau$ to the direction of the trajectory in a folded system, as before.

Following the strategy of the proof of Proposition 13, by applying Theorem 5 and Proposition 9, we show that qualitatively periodic cyclic quadrilateral tilings have the same behavior as triangle tiling billiards, see Figure 16.

**Theorem 7.** Consider the set of all cyclic quadrilaterals with the set of lengths $l_j$ such that $l_j, j = 1, ..., 4$ are independent over $\mathbb{Q}$. Consider any trajectory $\delta$ in a corresponding tiling that doesn’t pass through circumcenters of the tiles. Then the following holds:

1. The trajectory $\delta$ is not drift-periodic;
2. if $\delta$ is closed, its symbolic orbit is equal to $\tilde{s}^2$ for some word $s$ in the alphabet of sides $A := \{a, b, c, d\}$ of odd length. Consequently, the period of $\delta$ is equal to $4n + 2$ for some $n \in \mathbb{N}$;
3. if $\delta$ is not closed, it is linearly escaping. Its symbolic coding can be described by an infinite word $\omega$ that can be represented as an infinite concatenation of the words $w_1$ and $w_2$ forming a Sturmian sequence.

As we have seen in Section 5, starting from $n = 5$, the family $\text{CET}_n^\circ$ exhibits non-integrable behavior in open sets. Maybe this is related to the fact, that most of the inscribed pentagons never tile the plane? ...

For the quadrilaterals, one can ask the same questions about exceptional behavior of the trajectories (related to the problems discussed in the previous paragraph), as well as generalize Conjecture 1.

**Problem.** Study the set of exceptional (possibly non-linearly escaping) trajectories in the periodic cyclic quadrilateral tilings. This set is a subset of the trajectories passing through the circumcenters of the tiles. In other words, are the parameters $l_1, l_2, l_3, l_4$ for which the maps in $F_{l_1, l_2, l_3, l_4} \in \text{CET}_4$ are minimal?

The answer to this question can probably be given in the terms of Theorem 7 and introduce a higher dimensional analog of the Rauzy gasket.

**Conjecture 3** (Tree conjecture for quadrilateral tilings). Let $\Lambda$ be as a union of all vertices and edges of all drawn triangles in a periodic cyclic quadrilateral tiling. Take any periodic closed trajectory $\delta$ of a corresponding billiard. It incloses some bounded domain $U \subset \mathbb{R}^2$ in the plane, $\partial U = \delta$ and $U \cap \Lambda$ is an embedding of some graph in the plane. Then this graph is a tree.

**Appendix: the Rauzy graphs**

This Appendix includes a few examples of modified Rauzy graphs (or parts of these graphs) that we constructed for this work, in collaboration with Paul Mercat. We put the links for downloading some of them which
are too big to be inserted in the paper in the following list.

**Rauzy graphs of interval exchange transformations with flips: Some examples**

For the study of triangle tiling billiards and the minimality properties of the associated interval exchange transformations with flips, we studied the full connected component of the permutation $\sigma$, defined by $(3–2)$, in the Rauzy graph of IET$^3$. In Figure 8 we represent a quotient of this connected component. Here are some additional figures:

- In Figure 17, we represent this quotient with an additional information on the values of the invariant of the vertex of the graph given by a vector with entries in the set $\{0, 1, -1\}$ (see the discussion in Section 4),
- for downloading the full connected component of $\sigma$, follow the link pa-ro.net/doc/2019-triangletilingbillards/web_ABCD_BDAC_complete.pdf,
- for the same full connected component of $\sigma$ with the additional information on the combinatorial invariant, follow the link pa-ro.net/doc/2019-triangletilingbillards/vec_ABCD_BDAC.pdf.

In order to understand the behavior of the maps in the family CET$^4$ and prove the integrability of almost all the maps in the family (Proposition 9), one needs to study the connected component of the permutation $\sigma$ defined by $(5–2)$. The number of the (irreducible) vertices in this component is equal to 8222. For obvious reasons, we do not draw the entire graph here. The corresponding component in the quotient graph is much smaller (it has only 130 vertices) and can be downloaded here, pa-ro.net/doc/2019-triangletilingbillards/vec_ABCD_BDAC.pdf, with the additional combinatorial data associated to it.

**Acknowledgments**

We would like to thank the organizers of the conferences Teichmüller Space, Polygonal Billiard, Interval Exchanges at CIRM in February 2017 (where the work on the project started) as well as Teichmüller Dynamics, Mapping Class Groups and Applications at Institut Fourier in June 2018 (where the work on the project continued) that provided wonderful environment for research and exchange. We are both very grateful to Diana Davis for her enthusiastic talk in February 2017 in CIRM that introduced us to tiling billiards as well as for the graphic representation of IET’s with flips (as in Figure 4) that we use throughout this article.

The second author benefits from the support of the French government “Investissements d’Avenir” program ANR-11-LABX-0020-01 - Centre Henri Lebesgue & Région Bretagne - dispositif SAD. During the work on this article she was also supported by the grant L’Oréal-UNESCO for Women in Science 2016. This article wouldn’t exist without the help of Paul Mercat who wrote the program that drew modified Rauzy graphs for the maps in CET$^2$ and CET$^3$. The proof of Lemma 3 which is crucial for our work, is for now computer assisted. The needed calculations were done by the program that Paul wrote. The second author thanks Ilya Schurov for his help on the program for the trajectories in quadrilateral tilings. We would like to thank Pat Hooper and Alexander St Laurent for their program that draws the tiling billiard trajectories, accessible on-line [Hooper], and Shigeki Akiyama for suggesting us a new representation of tiling billiards as the systems of tangent reflections.

**References**

[Arnoux 81] P. Arnoux. “Un invariant pour les échanges d’intervalles et les flots sur les surfaces.” Doctoral thesis, (1981)
[Arnoux 88] P. Arnoux. “Un exemple de semi-conjugaison entre un échange d’intervalles et une translation sur le tore.” Bull. Soc. Math. France 116 (1988), 489–500
[Arnoux and Rauzy 91] P. Arnoux, G. Rauzy. “Représentation géométrique des suites de complexité $2n + 1$.” Bull. de la SMF. 119:2 (1991), 199–215
[Arnoux and Starosta 13] P. Arnoux and S. Starosta. *The Rauzy Gasket. Further Developments in Fractals and Related Fields Trends in Mathematics* pp. 1–23. New York: Birkhäuser Boston, Springer Science, 2013.
[Arnoux et al. 11] P. Arnoux, J. Bernat, and X. Bressaud. “Geometrical Models for Substitutions.” Exp. Math. 20 (2011), 97–127.
[Avila et al. 16a] A. Avila, P. Hubert, and A. Skripchenko. “Diffusion for Chaotic Plane Sections of 3-Periodic Surfaces.” Invent. Math. 206:1 (2016), 109–146
[Avila et al. 16b] A. Avila, P. Hubert, and A. Skripchenko. “On the Hausdorff Dimension of the Rauzy Gasket.” Bull. Soc. Math. France. 144:3 (2016), 539–568.
[Baird-Smith et al. 19] P. Baird-Smith, D. Davis, E. Fromm, and S. Iyer. “Tiling Billiards on Triangle Tilings, and Interval Exchange Transformations.” preprint, (http://www.swarthmore.edu/NaSci/ddavis3/triangle_tiling_billiards.pdf), 2018.
[Boissy and Lanneau 08] C. Boissy and E. Lanneau. “Dynamics and Geometry of the Rauzy-Veech Induction for Quadratic Differentials.” Ergodic Theory Dyn. Syst. 29 (2008), 767–816.
[Boshernitzan and Carroll 97] M. Boshernitzan and C. Carroll. “An Extension of Lagrange’s Theorem to Interval Exchange Transformations over Quadratic Fields.” J. Anal. Math. 72 (1997), 21–44.
[Calta 04] K. Calta. “Veech Surfaces and Complete Periodicity in Genus Two.” J. Amer. Math. Soc. 17:4 (2004), 871–908.
[Davis and Patrick Hooper 18] D. Davis and W. Patrick Hooper. Periodicity and ergodicity in the trihexagonal
tiling, accepted pending revision in Commentarii Mathematici Helvetici (2018).

Davies et al. 16 D. Davies, K. DiPietro, J. Rustad, and A. St Laurent. “Negative Refraction and Tiling Billiards, to appear in Advances in Geometry.” (2016)

Delecroix 16 V. Delecroix. “Interval Exchange Transformations.” Lecture Notes, Salta (Argentina) (2016).

de Mourgues 17 Q. de Mourgues. “A Combinatorial Approach to Rauzy-Type Dynamics.” Université Paris 13, thesis, 2017.

Dynnikov 97 I. Dynnikov. “Semiclassical Motion of the electron. A proof of the Novikov Conjecture in General Position and Counterexamples.” In: Solitons, Geometry and Topology: On the Cross Road, Translations of the AMS, Ser. 2, 179, pp. 45–73, Providence: AMS (1997).

Glendinning 16 P. Glendinning. “Geometry of Refractions and Reflections Through a Biperiodic Medium.” SIAM J. Appl. Math. Soc. Ind. Appl. Math. 76:4, 1219–1238 (2016)

Gourdon and Paris-Romaskevich 18 Je voudrais vous parler de mathématiques..., short film co-created by C. Gourdon and O. Paris-Romaskevich, for a competition Symbiose 48 hour film project, scientific documentary festival Pariscience2018. (https://vimeo.com/297265239), 2018.

Hooper P. Hooper. “Alexander St Laurent, Negative Snell Law Tiling Billiards Trajectory Simulations.” (http://aws-tlaur.github.io/negsnel/).

Hooper and Weiss 18 W. P. Hooper and B. Weiss. “Rel Leaves of the Arnoux-Yoccoz Surfaces.” Sel. Math. 24:2 (2018), 875–934.

Keane 75 M. Keane. “Interval Exchange Transformations.” Math. Z. 141 (1975), 25–31.

Kontsevich and Zorich 03 M. Kontsevich and A. Zorich. “Connected Components of the Moduli Spaces of Abelian Differentials with Prescribed Singularities.” Invent. Math. 153:3 (2003), 631–678.

Lowenstein 07 J. H. Lowenstein, G. Poggiaspalla, and F. Vivaldi. “Interval Exchange Transformations Over Algebraic Number Fields: The Cubic Arnoux-Yoccoz Model.” Dyn. Syst. 22:1 (2007), 73–106

Marmi et al. 10 S. Marmi, P. Moussa, and J.-C. Yoccoz. “Affine Interval Exchange Maps with a Wandering Interval.” Proc. London Math. Soc. 100:3 (2010), 639–669.

Mascarenhas and Fluegel 15 A. Mascarenhas and B. Fluegel. “Antisymmetry and the Breakdown of Bloch’s Theorem for Light”, unpublished draft. (2015).

McMullen 15 C. McMullen. “Cascades in the Dynamics of Measured Foliations.” Ann. Sci. l’École Normale Supérieure 48:1 (2015), 1–39.

Nogueira 89 A. Nogueira. Almost all Interval Exchange Transformations with Flips are Nonergodic.” Ergod. Theory Dyn. Syst. 9:3 (1989), 515–525.

Novikov 82 S. P. Novikov. “The Hamiltonian Formalism and Multivalued Analogue of Morse Theory.” (Russian) Uspekhi Mat. Nauk. 37:5 (1982), 3–49; translated in Russian Math. Surveys 37:5 (1982) 1–56.

Rao 17 M. Rao. "Exhaustive Search of Convex Pentagons which tile the Plane." (2017)

Rauzy 79 G. Rauzy. “Échanges d’intervalles et transformations induites.” Acta Arith. 34:4 (1979), 315–328.

Shelby et al. 01 R. A. Shelby, D. R. Smith, and S. Schultz. “Experimental Verification of a Negative Index of Refraction." Science. 292:5514 (2001), 77–79.

Strenner 18 B. Strenner. “Lifts of Pseudo-Anosov Homeomorphisms of Nonorientable Surfaces have Vanishing SAF Invariant.” Math. Res. Lett. 25 (2018), 2

F. Smith et al. 04 D. Smith, J. Pendry, and M. Wiltshire Metamaterials and negative refractive index, Science. 305 (2004), 788–792

Skripchenko and Troubetzkoy 18 A. Skripchenko and S. Troubetzkoy, On the Hausdorff Dimension of Minimal Interval Exchange Transformations with Flips." J. Lond. Math. Soc. 97:2 (2018), 149–169.

Valentine et al. 08 J. Valentine, S. Zhang, T. Zentgraf, E. Ulin-Avila, D. A. Genov, G. Bartal and X. Zhang. “Three-Dimensional Optical Metamaterial with a Negative Refractive Index.” Nature. 455 (2008), 376.

Zorich 84 A. Zorich. “A Problem of Novikov on the Semiclassical Motion of an Electron in a Uniform Almost Rational Magnetic Field.” Russ. Math. Surv. 39:5 (1984), 287–288.