The fast track to Löwner’s theorem

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Abstract

The operator monotone functions defined in the positive half-line
are of particular importance. We give a version of the theory in which
integral representations for these functions can be established directly
without invoking Löwner’s detailed analysis of matrix monotone func-
tions of a fixed order or the theory of analytic functions.

We found a canonical relationship between positive and arbitrary
operator monotone functions defined in the positive half-line, and this
result effectively reduces the theory to the case of positive functions.

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Key words and phrases: operator monotone function; integral rep-
resentation; Löwner’s theorem.

1 Introduction and preliminaries

The functional calculus is defined by the spectral theorem. Since we only
deal with matrices the function $f(x)$ of a hermitian matrix $x$ is defined for
any function $f$ defined on the spectrum of $x$.

**Definition 1.1.** Let $I$ be an interval of any type. A function $f: I \to \mathbb{R}$ is
said to be $n$-matrix monotone (or just $n$-monotone) if

$$x \leq y \Rightarrow f(x) \leq f(y)$$

for every pair of $n \times n$ hermitian matrices $x$ and $y$ with spectra in $I$. 

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**Definition 1.2.** Let $I$ be an interval of any type. A function $f : I \to \mathbb{R}$ is said to be $n$-matrix convex (or just $n$-convex) if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for every $\lambda \in [0, 1]$ and every pair of $n \times n$ Hermitian matrices $x$ and $y$ with spectra in $I$.

Note that the spectrum of the matrix $\lambda x + (1 - \lambda)y$ in the definition automatically is contained in $I$. The functional calculus on the left-hand side is therefore well-defined.

We realise that a point-wise limit of $n$-monotone ($n$-convex) functions is $n$-monotone ($n$-convex).

**Definition 1.3.** A function $f : I \to \mathbb{R}$ defined in an interval $I$ is said to be operator monotone (operator convex) if it is $n$-monotone ($n$-convex) for all natural numbers $n$.

We realise that a point-wise limit of operator monotone (operator convex) functions is operator monotone (operator convex).

### 1.1 Other proofs of Löwner’s theorem

Karl Löwner\[1\] analyses in great detail matrix monotone functions of a fixed order and then arrive at the characterisation of operator monotone functions by means of interpolation theory.

Wigner and von Neumann gives in \[16\] a new proof of Löwner’s theorem based on continued fractions which is almost never cited.

Bendat and Sherman \[2\] gives a new proof of Löwner’s theorem that relies on Löwner’s detailed analysis of matrix monotone functions of a fixed order but combines it with the Hamburger moment problem. They also rely on Kraus \[12\] to essentially prove that a function is operator convex if and only if the secant-slope function is operator monotone.

Korányi \[11\] gives a new proof of Löwner’s theorem by using a variant of Löwner’s characterisation of matrix monotone functions of a fixed order and spectral theory for unbounded self-adjoint operators.

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\[1\]Karel Löwner was a Czech known under his German name Karl Löwner. Fleeing the Nazis in 1939 he moved to the United States and changed his name to Charles Loewner.
The monograph of Donoghue [4] follows [13] closely but introduces some simplifications.

Sparr [15] gives a new proof of Löwner’s theorem that combines Löwner’s characterisation of matrix monotone functions of a fixed order with the theory of interpolation spaces.

The paper [7] by Pedersen and the author introduces the idea of first determining the extreme operator monotone functions and then obtain Löwner’s theorem by applying Krein-Milman’s theorem. The paper does not rely on Löwner’s detailed analysis of matrix monotone functions but uses algebraic methods based on Jensen’s operator inequality.

The proof in [7] is used in a number of other sources including the book of Bhatia [3].

Ameur [1] combines the techniques of applying Jensen’s operator inequality as in [7] with interpolation theory in the sense of Foiaş-Lions to obtain a new proof of Löwner’s theorem.

2 Matrix monotonicity and matrix concavity

There is a striking connection between matrix monotonicity and matrix concavity for functions defined in an interval extending to plus infinity.

**Theorem 2.1.** Let $f : (0, \infty) \to \mathbb{R}$ be a $2n$-monotone function where $n \geq 1$. Then $f$ is matrix concave of order $n$. In particular, $f$ is continuous.

**Proof.** Let $x_1, x_2$ be positive definite matrices of order $n$ and take $s \in [0, 1]$. We consider the unitary block matrix $V$ of order $2n \times 2n$ given by

$$V = \begin{pmatrix} s^{1/2} & -(1 - s)^{1/2} \\ (1 - s)^{1/2} & s^{1/2} \end{pmatrix}$$

and obtain by an elementary calculation that

$$V^* \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix} V = \begin{pmatrix} sx_1 + (1 - s)x_2 & s^{1/2}(1 - s)^{1/2}(x_2 - x_1) \\ s^{1/2}(1 - s)^{1/2}(x_2 - x_1) & (1 - s)x_1 + sx_2 \end{pmatrix}.$$

We set $d = -s^{1/2}(1 - s)^{1/2}(x_2 - x_1)$ and notice that to a given $\varepsilon > 0$ the
difference

\[
\begin{pmatrix}
 sx_1 + (1 - s)x_2 + \varepsilon & 0 \\
 0 & 2\lambda
\end{pmatrix} - V^* \begin{pmatrix}
 x_1 & 0 \\
 0 & x_2
\end{pmatrix} V
\geq \begin{pmatrix}
 \varepsilon & d \\
 d & \lambda
\end{pmatrix}
\]  
for \( \lambda \geq (1 - s)x_1 + sx_2 \).

Since the last block matrix is positive semi-definite for \( \lambda \geq \varepsilon^{-1}\|d\|^2 \) we realize that

\[
V^* \begin{pmatrix}
 x_1 & 0 \\
 0 & x_2
\end{pmatrix} V \leq \begin{pmatrix}
 sx_1 + (1 - s)x_2 + \varepsilon & 0 \\
 0 & 2\lambda
\end{pmatrix}
\]

for a sufficiently large \( \lambda > 0 \). Since \( f \) is \( 2n \)-monotone we then obtain

\[
f \left( V^* \begin{pmatrix}
 x_1 & 0 \\
 0 & x_2
\end{pmatrix} V \right) \leq \begin{pmatrix}
 f(sx_1 + (1 - s)x_2 + \varepsilon) & 0 \\
 0 & f(2\lambda)
\end{pmatrix}
\]

for such \( \lambda \), and since

\[
f \left( V^* \begin{pmatrix}
 x_1 & 0 \\
 0 & x_2
\end{pmatrix} V \right) = V^* \begin{pmatrix}
 f(x_1) & 0 \\
 0 & f(x_2)
\end{pmatrix} V
\]

\[
= \begin{pmatrix}
 sf(x_1) + (1 - s)f(x_2) & s^{1/2}(1 - s)^{1/2}(f(x_2) - f(x_1)) \\
 s^{1/2}(1 - s)^{1/2}(f(x_2) - f(x_1)) & (1 - s)f(x_1) + sf(x_2)
\end{pmatrix}
\]

we realize that

\[
sf(x_1) + (1 - s)f(x_2) \leq f(sx_1 + (1 - s)x_2 + \varepsilon).
\]

Since \( f \) is monotone the right limit \( f^+ \) defined by setting

\[
f^+ (t) = \lim_{\varepsilon \downarrow 0} f(t + \varepsilon) \quad t > 0
\]

is well-defined. For positive numbers \( t_1, t_2 > 0 \) we obtain

\[
sf^+(t_1) + (1 - s)f^+(t_2) \leq sf(t_1 + \varepsilon) + (1 - s)f(t_2 + \varepsilon)
\]

\[
\leq f(st_1 + (1 - s)t_2 + 2\varepsilon),
\]

where the first inequality follows from the definition of the right limit and the second follows from inequality (1) by setting \( x_1 = t_1 + \varepsilon \) and \( x_2 = t_2 + \varepsilon \). By letting \( \varepsilon \) tend to zero we then obtain

\[
sf^+(t_1) + (1 - s)f^+(t_2) \leq f^+(st_1 + (1 - s)t_2),
\]
therefore $f^+$ is concave and thus continuous. Since $f$ is monotone increasing we have
\[ f^+(t - \varepsilon) \leq f(t) \leq f^+(t) \quad t > 0, \ 0 < \varepsilon < t, \]
and since $f^+$ is continuous we obtain $f = f^+$ by letting $\varepsilon$ tend to zero. Finally, since we established that $f$ is continuous, we may let $\varepsilon$ tend to zero in inequality (1) to obtain
\[ sf(x_1) + (1 - s)f(x_2) \leq f(sx_1 + (1 - s)x_2), \]
showing that $f$ is $n$-concave.  \textbf{QED}

The above theorem, with the added condition that $f$ is continuous, was proved by Mathias [14]. That a $4n$-monotone function defined in the positive half-line is $n$-concave already follows from [7] proofs of 2.5. Theorem and 2.1. Theorem. The idea of the above proof is taken from [8].

\textbf{Corollary 2.2.} An operator monotone function $f : (0, \infty) \to \mathbb{R}$ is automatically operator concave.

It is essential for the above result that the function is defined in an interval stretching out to infinity. Without this assumption there are easy counter examples.

\textbf{Theorem 2.3.} Let $f : (0, \infty) \to \mathbb{R}$ be a non-negative function which is $n$-concave for some $n \geq 1$. Then $f$ is also $n$-monotone.

\textit{Proof.} Let $x$ and $y$ be positive definite $n \times n$ matrices with $x < y$ and take $\lambda$ in the open interval $(0, 1)$. We may write
\[ \lambda y = \lambda x + (1 - \lambda)(\lambda(1 - \lambda)^{-1}(y - x)) \]
as a convex combination of two positive definite matrices. Since $f$ is $n$-concave we thus obtain
\[ f(\lambda y) \geq \lambda f(x) + (1 - \lambda)f(\lambda(1 - \lambda)^{-1}(y - x)) \geq \lambda f(x), \]
where we used that $f$ is non-negative. Since $f$ is continuous we obtain $f(x) \leq f(y)$ be letting $\lambda \to 1$. In the general case, where just $x \leq y$, we have
\[ \mu x < x \leq y \quad \text{for} \quad 0 < \mu < 1, \]
since $x$ is positive definite, and then obtain $f(\mu x) \leq f(y)$. The assertion now follows by letting $\mu \to 1$.  \textbf{QED}
The above proof is taken from [7, 2.5. Theorem].

**Corollary 2.4.** A function mapping the positive half-line into itself is operator monotone if and only if it is operator concave.

## 2.1 Regularization

The following regularization procedure is standard, cf. for example [4, Page 11]. Let \( \varphi \) be a positive and even \( C^\infty \)-function defined in the real line, vanishing outside the closed interval \([-1, 1]\) and normalized such that

\[
\int_{-1}^{1} \varphi(x) \, dx = 1.
\]

For any locally integrable function \( f \) defined in an open interval \((a, b)\), where possibly \( b = \infty \), we form, for small \( \varepsilon > 0 \), its regularization,

\[
f_\varepsilon(t) = \frac{1}{\varepsilon} \int_{a}^{b} \varphi \left( \frac{t - s}{\varepsilon} \right) f(s) \, ds \quad t \in (a + \varepsilon, b - \varepsilon),
\]

and realize that it is infinitely many times differentiable. We may also write

\[
f_\varepsilon(t) = \int_{-1}^{1} \varphi(s) f(t - \varepsilon s) \, ds \quad t \in (a + \varepsilon, b - \varepsilon).
\]

If \( f \) is continuous, then \( f_\varepsilon \) is eventually well-defined and converges uniformly towards \( f \) on any compact subinterval of \((a, b)\). In particular, for each \( t \in (a, b) \), the net \( f_\varepsilon(t) \) is well-defined for sufficiently small \( \varepsilon \) and converges to \( f(t) \) as \( \varepsilon \) tends to zero.

Suppose now that \( f \) is \( n \)-monotone in \((0, \infty)\) for \( n \geq 2 \). We notice that \( f \) is continuous by Theorem 2.1. It follows from the last integral representation that \( f_\varepsilon \) is \( n \)-monotone in the interval \((\varepsilon, \infty)\) for \( \varepsilon > 0 \). We realise that the restriction of \( f \) to any compact interval \( J \) in \((0, \infty)\) is the uniform limit of a sequence of \( n \)-monotone functions that are infinitely many times differentiable in a neighbourhood of \( J \).

A similar statement is obtained for \( n \)-convex functions defined in an open interval \((a, b)\). Notice that in this case the continuity is immediate.
3 Bendat and Sherman’s theorem

For a differentiable function $f : I \rightarrow \mathbb{R}$ the (first) divided difference $[t, s]_f$ for $t, s \in I$ is defined by

$$[t, s]_f = \begin{cases} \frac{f(t) - f(s)}{t - s} & t \neq s \\ f'(t) & t = s, \end{cases}$$

and the Löwner matrix $L(\lambda_1, \ldots, \lambda_n)$ is defined by setting

$$L(\lambda_1, \ldots, \lambda_n) = \left( [\lambda_i, \lambda_j]_f \right)_{i,j=1}^n$$

for $\lambda_1, \ldots, \lambda_n \in I$. Notice that a Löwner matrix is linear in the function $f$.

If $f$ is twice continuously differentiable the second divided difference $[t, s, r]_f$ for distinct numbers $s, t, r \in I$ is defined by setting

$$[t, s, r]_f = [t, s]_f - [s, r]_f$$

and the definition is then extended by continuity to arbitrary numbers $t, s, r \in I$. Notice that in this way $[t, t, t]_f = f''(t)/2$.

Divided differences are symmetric in the entries.

**Lemma 3.1.** Let $f$ be a real function in $C^1(I)$, where $I$ is an open interval, and let $x$ be an $n \times n$ diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n \in I$. The function $t \rightarrow f(x + th)$ is defined in a neighbourhood of zero for any hermitian $n \times n$ matrix $h = (h_{i,j})_{i,j=1}^n$ and

$$\frac{d}{dt} (f(x + th) \xi | \xi) \bigg|_{t=0} = (h \circ L(\lambda_1, \ldots, \lambda_n) \xi | \xi) \quad \xi \in \mathbb{C}^n,$$

where $h \circ L(\lambda_1, \ldots, \lambda_n)$ denotes the Hadamard (entry-wise) product of $h$ and $L(\lambda_1, \ldots, \lambda_n)$.

**Proof.** We first prove the lemma for a monomial $f(t) = t^m$, where $m \geq 1$ is an integer. The first divided difference

$$[\lambda_i, \lambda_j]_f = \frac{\lambda_i^m - \lambda_j^m}{\lambda_i - \lambda_j} = \lambda_i^{m-1} + \lambda_i^{m-2}\lambda_j + \cdots + \lambda_i\lambda_j^{m-2} + \lambda_j^{m-1}$$

$$= \sum_{a+b=m-1} \lambda_i^a \lambda_j^b$$

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and this holds also for $\lambda_i = \lambda_j$. Therefore,

$$
(h \circ L(\lambda_1, \ldots, \lambda_n)\xi | \xi) = \sum_{i=1}^{n}(h \circ L(\lambda_1, \ldots, \lambda_n)\xi)\bar{\xi}_i
$$

$$
= \sum_{i,j=1}^{n}h_{i,j}\sum_{a+b=m-1}^{\lambda_i^a\lambda_j^b\xi_j\bar{\xi}_i} = \sum_{a+b=m-1}^{(x^a h x^b | \xi)}
$$

which is the first order term of $((x + th)^m | \xi)$. By using linearity the statement of the lemma follows for arbitrary polynomials. The general case then follows by approximation. \textbf{QED}

**Theorem 3.2.** Let $f$ be a real function in $C^1(I)$, where $I$ is an open interval and take a natural number $n \geq 1$. Then $f$ is $n$-monotone if and only if the L"owner matrix $L(\lambda_1, \ldots, \lambda_n)$ is positive semi-definite for all sequences $\lambda_1, \ldots, \lambda_n \in I$.

**Proof.** It follows from classical analysis that $f$ is $n$-monotone if and only if

$$
\frac{d}{dt}(f(x + th)\xi | \xi)\bigg|_{t=0} \geq 0
$$

for every hermitian $x$ with spectrum in $I$, every positive semi-definite matrix $h$, and every $\xi \in \mathbb{C}^n$. We may now choose $x$ as a diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n \in I$. By choosing $h$ as the positive semi-definite matrix with $h_{i,j} = 1$ for $i, j = 1, \ldots, n$ we realise that $L(\lambda_1, \ldots, \lambda_n) \geq 0$ if $f$ is $n$-monotone. Since the Hadamard product of two semi-definite matrices is positive semi-definite (indeed, it is a principal submatrix of the tensor product), we realise that $f$ is $n$-monotone if all the L"owner matrices $L(\lambda_1, \ldots, \lambda_n) \geq 0$ for arbitrary $\lambda_1, \ldots, \lambda_n \in I$. \textbf{QED}

**Definition 3.3.** Take a function $f \in C^2(I)$, where $I$ is an open interval and (not necessarily distinct) numbers $\lambda_1, \ldots, \lambda_n \in I$. The associated Kraus \cite{12} matrices $H(1), \ldots, H(n)$ are defined by setting

$$
H(p) = 2\left([\lambda_p, \lambda_i, \lambda_j]_f\right)_{i,j=1}^{n}
$$

for $p = 1, \ldots, n$.

Notice that a Kraus matrix is linear in the function $f$. 

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Lemma 3.4. Let $f$ be a real function in $C^2(I)$, where $I$ is an open interval and let $x$ be an $n \times n$ diagonal matrix with diagonal elements $\lambda_1, \ldots, \lambda_n \in I$. The function $t \to f(x + th)$ is defined in a neighbourhood of zero for any hermitian $n \times n$ matrix $h = (h_{i,j})_{i,j=1}^n$ and

$$\frac{d^2}{dt^2}(f(x + th) \xi | \xi) \bigg|_{t=0} = \sum_{p=1}^n (H(p) \eta(p) | \eta(p)),$$

where

(i) $\xi = (\xi_1, \ldots, \xi_n)$ is a vector in $\mathbb{C}^n$.

(ii) $H(1), \ldots, H(n)$ are the Kraus matrices associated with $f$ and $\lambda_1, \ldots, \lambda_n$.

(iii) $\eta(p) = (\xi_1 h_{p,1}, \ldots, \xi_n h_{p,n})$ for $p = 1, \ldots, n$.

Proof. We first prove the lemma for a monomial $f(t) = t^m$, where $m \geq 2$ is an integer. Since

$$[\lambda_p, \lambda_i]_f - [\lambda_i, \lambda_j]_f = \lambda_{m-1}^i + \lambda_{m-2}^i + \lambda_{m-2}^j + \lambda_{m-1}^j - (\lambda_{m-1}^i + \lambda_{m-2}^i + \lambda_{m-2}^j + \lambda_{m-1}^j)$$

$$= \lambda_{i}^{m-2}(\lambda_p - \lambda_j) + \lambda_{m-3}^i(\lambda_p^2 - \lambda_j^2) + \lambda_{m-2}^i(\lambda_p - \lambda_j) + \lambda_{m-1}^i - \lambda_{m-1}^j$$

the second divided difference

$$[\lambda_p, \lambda_i, \lambda_j]_f = \frac{[\lambda_p, \lambda_i]_f - [\lambda_i, \lambda_j]_f}{\lambda_p - \lambda_j}$$

$$= \lambda_{i}^{m-2} + \lambda_{i}^{m-3}(\lambda_p + \lambda_j) + \lambda_{i}^{m-4}(\lambda_p^2 + \lambda_p \lambda_j + \lambda_j^2) + \lambda_{i}^{m-5}(\lambda_p^3 + \lambda_p^2 \lambda_j + \lambda_p \lambda_j^2 + \lambda_j^3) + \lambda_{i}^{m-6}(\lambda_p^4 + \lambda_p^3 \lambda_j + \lambda_p^2 \lambda_j^2 + \lambda_p \lambda_j^3 + \lambda_j^4)$$

$$= \sum_{a+b+c=m-2} \lambda_p^a \lambda_i^b \lambda_j^c,$$

where summation limits should be properly interpreted, and the case $\lambda_p = \lambda_j$
is handled separately. We then obtain
\[
\sum_{p=1}^{n} (H(p)\eta(p) | \eta(p)) = \sum_{p,i,j=1}^{n} 2[\lambda_p, \lambda_i, \lambda_j] f_{\xi_j} h_{p,j} \tilde{\xi}_i \bar{h}_{p,i} = 2 \sum_{a+b+c=m-2}^{n} \lambda_p^a \lambda_i^b \lambda_j^c \xi_j h_{p,j} \tilde{\xi}_i \bar{h}_{p,i} = 2 \sum_{a+b+c=m-2}^{n} (x^b h x c h \xi | \xi)
\]

which is the second order term in \( t \) of \( (x + th)^m \xi | \xi \). By using linearity the statement of the lemma follows for arbitrary polynomials. The general case then follows by approximation. QED

**Theorem 3.5.** Let \( f \) be a real function in \( C^2(I) \), where \( I \) is an open interval. Then \( f \) is \( n \)-convex if and only if the Kraus matrices associated with \( f \) and any choice of \( \lambda_1, \ldots, \lambda_n \in I \) are positive semi-definite.

*Proof.* The sufficiency of the conditions is obvious from the above lemma. Assume now that \( f \) is \( n \)-convex and choose \( \lambda_1, \ldots, \lambda_n \in I \).

Take a fixed \( p = 1, \ldots, n \) and a fixed vector \( \eta \in \mathbb{C}^n \). To a given \( \varepsilon > 0 \) we choose a vector \( \xi \) by setting \( \xi_i = \varepsilon^{-1} \) for \( i \neq p \) and \( \xi_p = 1 \). We then choose a vector \( a \) by setting
\[
a_i = \frac{\eta_i}{\xi_i} = \begin{cases} \varepsilon \eta_i & i \neq p \\ \eta_p & i = p \end{cases}
\]

We finally choose a self-adjoint (actually a positive semi-definite) matrix \( h \) by setting \( h_{i,j} = a_i a_j \) for \( i, j = 1, \ldots, n \) and calculate
\[
\eta(q)_i = \xi_i h_{q,i} = \xi_i a_q a_i \quad q = 1, \ldots, n.
\]

With these choices we obtain
\[
\eta(p) = \bar{\eta}_p \eta \quad \text{and} \quad \eta(q) = \varepsilon \bar{\eta}_q \eta \quad \text{for} \quad q \neq p.
\]

Therefore,
\[
\left. \frac{d^2}{dt^2} \left( (x + t h) \xi | \xi \right) \right|_{t=0} = |\eta_p|^2 (H(p) \eta | \eta) + \varepsilon^2 \sum_{q \neq p}^{n} |\eta_q|^2 (H(q) \eta | \eta)
\]

is non-negative, and since \( \eta \) is a fixed vector we obtain
\[
|\eta_p|^2 (H(p) \eta | \eta) \geq 0
\]

by letting \( \varepsilon \) tend to zero. In particular, \( (H(p) \eta | \eta) \geq 0 \) for all vectors \( \eta \in \mathbb{C}^n \) with \( \eta_p \neq 0 \). By continuity we finally realize that \( H(p) \) is positive semi-definite. QED
Theorem 3.6 (Bendat and Sherman). Let $f$ be a real function in $C^2(I)$, where $I$ is an open interval. Then $f$ is operator convex if and only if the function

$$g(t) = \begin{cases} \frac{f(t) - f(t_0)}{t - t_0} & t \neq t_0 \\ f'(t_0) & t = t_0 \end{cases}$$

is operator monotone for each $t_0 \in I$.

Proof. Using the symmetry of divided differences we realise that

$$[\lambda_i, \lambda_j]_f = [t_0, \lambda_i, \lambda_j]_f \quad i, j = 1, \ldots, n.$$ 

If $f$ is $(n + 1)$-convex then $g$ is $n$-monotone for each $t_0 \in I$ by Theorem 3.2 and Theorem 3.5. Conversely, if $g$ is $n$-monotone for each $t_0 \in I$ then $f$ is $n$-convex. QED

3.1 Further preparations

Theorem 3.7 (Bendat and Sherman). Let $f$ be an operator convex function defined in the positive half-line. Then $f$ is differentiable, and the function

$$g(t) = \begin{cases} \frac{f(t) - f(t_0)}{t - t_0} & t \neq t_0 \\ f'(t_0) & t = t_0 \end{cases}$$

is operator monotone for each $t_0 > 0$.

Proof. Suppose $f$ is operator convex, thus in particularly continuous. Using regularization (with $\varepsilon < t_0$) we obtain $f$ as the point-wise limit, for $\varepsilon \to 0$, of a sequence $(f_\varepsilon)_{\varepsilon > 0}$ of infinitely differentiable operator convex functions. The functions

$$g_\varepsilon(t) = \begin{cases} \frac{f_\varepsilon(t) - f_\varepsilon(t_0)}{t - t_0} & t \neq t_0 \\ f_\varepsilon'(t_0) & t = t_0 \end{cases}$$

are operator monotone in $(\varepsilon, \infty)$ by Theorem 3.6. In addition, $g_\varepsilon(t) \to g(t)$ for $t \neq t_0$. Since $f$ is convex the set of derivatives $\{f_\varepsilon'(t_0)\}$ is bounded for small $\varepsilon < t_0$. A subsequence of $(g_\varepsilon)_{\varepsilon > 0}$ therefore converges towards an operator monotone function which is continuous according to Theorem 2.1. But then $f$ is differentiable in $t_0$ and we conclude that $f'(t_0) = \lim_{\varepsilon \to 0} f_\varepsilon'(t_0)$. QED
In the above proof we also learn that \( f'(t) \to f'(t) \) for every \( t \in (0, \infty) \), where \( f_\epsilon \) is the regularization of \( f \). In connection with Corollary 2.4 we obtain

**Corollary 3.8.** An operator monotone or operator convex function \( f \) defined in the positive half-line is automatically differentiable and \( f_\epsilon'(t) \to f'(t) \) for every \( t \in (0, \infty) \), where \( f_\epsilon \) is the regularization of \( f \).

By applying regularization of \( f \) and then appealing to Theorem 3.2 and Corollary 3.8 we obtain:

**Corollary 3.9.** Let \( f \) be an operator monotone function defined in the positive half-line. The Löwner matrices \( L(\lambda_1, \ldots, \lambda_n) \) associated with \( f \) are well-defined and positive semi-definite for arbitrary \( \lambda_1, \ldots, \lambda_n \in I \).

**Corollary 3.10.** Let \( f \) be an operator monotone function defined in the open half-line. If the derivative \( f'(t) = 0 \) in any point \( t > 0 \), then \( f \) is a constant function.

**Proof.** The Löwner matrix

\[
L(t, s) = \begin{pmatrix}
    f'(t) & [t, s]_f \\
    [s, t]_f & f'(s)
\end{pmatrix}
\]

is well-defined and positive semi-definite by Corollary 3.9, thus

\[
f'(t)f'(s) \geq \left( \frac{f(t) - f(s)}{t - s} \right)^2.
\]

If \( f'(t) = 0 \), then necessarily \( f(s) = f(t) \) for every \( s > 0 \). QED

Notice that we for this result only need 2-monotonicity of \( f \), cf. [9].

### 4 The fast track to Löwner’s theorem

**Lemma 4.1.** Let \( f : (0, \infty) \to (0, \infty) \) be an operator monotone function. The function \( t \to t^{-1}f(t) \) is operator monotone decreasing.

**Proof.** For \( \epsilon > 0 \) the function \( f_\epsilon(t) = f(t + \epsilon) \) is defined in the open set \((-\epsilon, \infty)\) containing zero. Since \( f \) and hence \( f_\epsilon \) are operator monotone and
therefore operator concave by Corollary 2.2 we may use Theorem 3.7 (Bendat and Sherman) to obtain that the function

\[ t \rightarrow \frac{f_\varepsilon(t) - f_\varepsilon(0)}{t - 0} = \frac{f(t + \varepsilon) - f(\varepsilon)}{t} \]

is operator monotone decreasing. By using \( f(\varepsilon) > 0 \) and the identity

\[ \frac{f(t + \varepsilon)}{t} = \frac{f(t + \varepsilon) - f(\varepsilon)}{t} + \frac{f(\varepsilon)}{t} \]

we realize that the function \( t \rightarrow t^{-1}f(t + \varepsilon) \) is operator monotone decreasing when restricted to the positive half-line. The result now follows by letting \( \varepsilon \) tend to zero. \( \Box \) QED

**Corollary 4.2.** Let \( f: (0, \infty) \rightarrow (0, \infty) \) be an operator monotone function. The functions

\[ f^\sharp(t) = tf(t)^{-1} \quad \text{and} \quad f^*(t) = tf(t^{-1}) \]

are operator monotone in the positive half-line.

**Proof.** Since \( t \rightarrow f^\sharp(t)^{-1} = t^{-1}f(t) \) is operator monotone decreasing by the above lemma it follows that \( f^\sharp \) is operator monotone (increasing). The second assertion follows from the same argument by first replacing \( f \) with the operator monotone function \( t \rightarrow f(t^{-1})^{-1} \). QED

The corollary states that the mappings \( f \rightarrow f^\sharp \) and \( f \rightarrow f^* \) are involutions of the set of positive operator monotone functions defined in the positive half-line.

**Lemma 4.3.** We have the bound \( f(t) \leq t + 1 \) for any positive operator monotone function \( f \) defined in the positive half-line with \( f(1) = 1 \).

**Proof.** Since \( f \) is increasing we obviously have

\[ f(t) \leq f(1) = 1 \leq t + 1 \quad \text{for} \quad 0 < t \leq 1. \]

We also notice that \( f \) is concave by Theorem 2.1. It follows, for \( t > 1 \), that \( f(t) \) is bounded by the continuation of the chord between \((0, \lim_{\varepsilon \to 0} f(\varepsilon))\) and \((1, f(1)) = (1, 1)\). But the continuation of this chord is bounded by \( t + 1 \). QED

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Let \( \mathcal{P} \) denote the set of positive operator monotone functions defined in the positive half-line and consider the convex set
\[
\mathcal{P}_0 = \{ f \in \mathcal{P} \mid f(1) = 1 \}.
\]
We equip \( \mathcal{P}_0 \) with the topology of point-wise convergence and realize, by the preceding lemma, that \( \mathcal{P}_0 \) is compact in this topology.

**Theorem 4.4.** Let \( f: (0, \infty) \to \mathbb{R} \) be a non-constant operator monotone function. Then \( f \) can be written on the form
\[
f(t) = f(1) + f'(1) \frac{t-1}{t} (Tf)(t) \quad t > 0,
\]
where \( Tf \in \mathcal{P}_0 \) is given by
\[
(Tf)(t) = \frac{t}{f'(1)} \cdot \begin{cases} 
\frac{f(t) - 1}{t-1} & t \neq 1 \\
\frac{f'(1)}{t-1} & t = 1.
\end{cases}
\]

Notice that \( f'(1) > 0 \) by Corollary 3.10 since \( f \) is non-constant.

**Proof.** The function
\[
h_1(t) = \frac{1}{f'(1)} \cdot \frac{f(t) - f(1)}{t-1}
\]
is positive since \( f \) is strictly increasing, and \( h_1(1) = 1 \). Since \( f \) is operator monotone and thus operator concave the function \( h_1 \) is operator monotone decreasing by Theorem 3.7. By composing with the operator monotone decreasing function \( t \to t^{-1} \) we obtain that
\[
h_2(t) = h_1(t^{-1}) = \frac{1}{f'(1)} \cdot \frac{f(t^{-1}) - f(1)}{t^{-1}-1}
\]
is positive and operator monotone with \( h_2(1) = 1 \). By applying the involution \( h_2 \to h_2^* \) we finally obtain that the function
\[
(Tf)(t) = h_2^*(t) = th_2(t^{-1}) = \frac{t}{f'(1)} \cdot \frac{f(t) - f(1)}{t-1}
\]
is operator monotone by Corollary 4.2. It is also positive and \( (Tf)(1) = 1 \).
The assertion now follows by solving the equation for \( f \). **QED**
Lemma 4.5. The involution $f \mapsto f^*$ maps $\mathcal{P}_0$ into itself, and the operation $f \mapsto Tf$ maps the non-constant functions in $\mathcal{P}_0$ into $\mathcal{P}_0$.

Proof. Follows immediately from Corollary 4.2 and Theorem 4.4. QED

Lemma 4.6. The sum of the derivatives

$$
\left. \frac{d}{dt}f(t) \right|_{t=1} + \left. \frac{d}{dt}f^*(t) \right|_{t=1} = 1
$$

for any $f \in \mathcal{P}_0$.

Proof. The assertion follows from the calculation

$$
\frac{f(t) - 1}{t - 1} + \frac{f^*(t^{-1}) - 1}{t^{-1} - 1} = 1
$$

by letting $t$ tend to 1. QED

Both $f$ and $f^*$ are increasing functions. By Corollary 3.10 we therefore obtain:

Corollary 4.7. The derivative of $f$ satisfies

$$
0 < f'(1) < 1
$$

for any function $f \in \mathcal{P}_0$ different from the constant function $t \to 1$ or the identity function $t \to t$.

Lemma 4.8. An extreme point $f$ in $\mathcal{P}_0$ is necessarily of the form

$$
f(t) = \frac{t}{f'(1) + (1 - f'(1))t} \quad t > 0.
$$

Proof. Take first a function $f \in \mathcal{P}_0$ which is neither the constant function $t \to 1$ nor the identity function $t \to t$, thus $\lambda = f''(1) \in (0, 1)$ by the above corollary. An elementary calculation shows that

$$
(2) \quad \lambda Tf + (1 - \lambda)(Tf^*)^* = f.
$$

Indeed,

$$
\lambda(Tf)(t) = t \frac{f(t) - 1}{t - 1} \quad t \neq 1
$$
and

\[(1 - \lambda)(Tf)^*(t) = (1 - \lambda)t(Tf^*)(t^{-1}) = \frac{f^*(t^{-1}) - 1}{t^{-1} - 1} = \frac{f(t) - t}{1 - t}\]

from which the assertion follows. Consequently, if \(f\) is an extreme point in \(\mathcal{P}_0\) then \(Tf = f\) or

\[
\frac{t}{\lambda} \frac{f(t) - 1}{t - 1} = f(t) \quad t > 0
\]

from which it follows that

\[
f(t) = \frac{t}{\lambda + (1 - \lambda)t} \quad t > 0.
\]

Finally, the two functions we left out may also be written in this way. Indeed, the constant function \(t \to 1\) appears in the formula by setting \(\lambda = 0\) while the identity function \(t \to t\) appears by setting \(\lambda = 1\). \(\text{QED}\)

**Theorem 4.9.** Let \(f\) be a positive operator monotone function defined in the positive half-line. There is a bounded positive measure \(\mu\) on the closed interval \([0, 1]\) such that

\[
f(t) = \int_0^1 \frac{t}{\lambda + (1 - \lambda)t} d\mu(\lambda) \quad t > 0.
\]

Conversely, any function given on this form is operator monotone. The measure \(\mu\) is a probability measure if and only if \(f(1) = 1\).

**Proof.** We noticed that \(\mathcal{P}_0\) is convex and compact in the topology of pointwise convergence of functions. Therefore, by Krein-Milman’s theorem, it is generated by its extreme points \(\text{Ext} (\mathcal{P}_0)\) in the sense that \(\mathcal{P}_0\) is the closure \(\mathcal{P}_0 = \overline{\text{conv}} (\text{Ext} (\mathcal{P}_0))\) of the convex hull of \(\text{Ext} (\mathcal{P}_0)\). By Lemma 4.8 the convex hull of \(\text{Ext} (\mathcal{P}_0)\) consists of functions of the form

\[
f(t) = \int_0^1 \frac{t}{\lambda + (1 - \lambda)t} d\mu(\lambda) \quad t > 0,
\]

where \(\mu\) is a discrete probability measure on \([0, 1]\). A function \(f\) in \(\mathcal{P}_0\) is therefore the limit of a net of functions \((f_j)_{j \in J}\) written on the form \((3)\) in terms of discrete probability measures \((\mu_j)_{j \in J}\). Since the set of probability measures on \([0, 1]\) is compact in the weak topology there exists an accumulation measure \(\mu\) such that \(f\) is expressed as in the statement of the theorem. \(\text{QED}\)
Notice that a possible atom in zero of the measure \( \mu \) in the above theorem contributes with the constant term \( \mu \{0\} \) in the integral. A possible atom in 1 contributes with the term \( \mu \{1\} t \).

A brief outline of the theory presented in Theorem 4.4, Lemma 4.8 and Theorem 4.9 was given in the authors’ PhD thesis [10, Page 12-13]. It is illuminating to consider the linear mapping

\[
\Lambda(f)(t) = \begin{cases} 
  t \frac{f(t) - f(1)}{t - 1} & t \neq 1 \\
  f'(1) & t = 1
\end{cases}
\]

defined for differentiable functions \( f : (0, \infty) \to \mathbb{R} \). It is closely related to the non-linear transformation \( T \) introduced in Theorem 4.4. Indeed,

\[
\Lambda(f) = f'(1) T f \quad \text{for} \quad f \in \mathcal{P}_0
\]

and \( \Lambda \) is thus a transformation of \( \mathcal{P} \). However, we cannot replace \( T \) by \( \Lambda \) in the proof of Lemma 4.8 since \( \Lambda \) does not map \( \mathcal{P}_0 \) into itself, and we cannot alternatively work directly with \( \mathcal{P} \) since \( \mathcal{P} \) is not compact.

**Theorem 4.10.** The measure \( \mu \) appearing in Theorem 4.9 is uniquely defined by the operator monotone function \( f \in \mathcal{P} \).

**Proof.** The action of \( \Lambda \) on functions in \( \mathcal{P} \) is calculated by noticing that

\[
\Lambda \left( \frac{t}{\lambda + (1 - \lambda)t} \right) = \frac{\lambda t}{\lambda + (1 - \lambda)t}
\]

and thus

\[
p(\Lambda) \left( \frac{t}{\lambda + (1 - \lambda)t} \right) = \frac{p(\lambda)t}{\lambda + (1 - \lambda)t}
\]

for any polynomial \( p \). For a function \( f \in \mathcal{P} \) we thus have

\[
p(\Lambda)(f) = \int_0^1 \frac{t}{\lambda + (1 - \lambda)t} p(\lambda) \, d\mu(\lambda),
\]

where \( \mu \) is the representing measure for \( f \). This identity recovers the measure \( \mu \) from \( f \) by using Weierstrass’s polynomial approximation theorem. **QED**
5 Other integral representations

Corollary 5.1. Let $f$ be a positive operator monotone function defined in the positive half-line. There is a bounded positive measure $\mu$ on the closed extended half-line $[0, \infty]$ such that

$$f(t) = \int_0^\infty \frac{t(1 + \lambda)}{t + \lambda} \, d\mu(\lambda) \quad t > 0.$$ 

Conversely, any function given on this form is operator monotone. The measure $\mu$ is a probability measure if and only if $f(1) = 1$.

Proof. The assertion follows from the previous theorem by applying the transformation

$$\lambda \to \alpha = \lambda(1 - \lambda)^{-1}$$

which maps the closed interval $[0, 1]$ onto the closed extended half-line $[0, \infty]$, and by noticing the identity

$$\frac{t}{\lambda + (1 - \lambda)t} = \frac{t(1 - \lambda)^{-1}}{\lambda(1 - \lambda)^{-1} + t} = \frac{t(1 + \alpha)}{t + \alpha}$$

which is valid also in the end points of the two intervals. QED

We are finally able to give an integral formula for the operator monotone functions defined in the positive half-line. There are various ways of doing so, but the following formula establishes the connection between operator monotone functions and the theory of Pick functions [4] in complex analysis.

Theorem 5.2. Let $f : (0, \infty) \to \mathbb{R}$ be an operator monotone function. There exists a positive measure $\nu$ on the closed positive half-line $[0, \infty)$ with $\int (1 + \lambda^2)^{-1} \, d\nu(\lambda) < \infty$ such that

$$f(t) = \alpha t + \beta + \int_0^\infty \left( \frac{\lambda}{1 + \lambda^2} - \frac{1}{t + \lambda} \right) \, d\nu(\lambda) \quad t > 0,$$

where $\alpha \geq 0$ and $\beta \in \mathbb{R}$. Conversely, any function given on this form is operator monotone.

Proof. We first use Theorem 4.4 to write $f$ on the form

$$f(t) = f(1) + f'(1) \frac{t - 1}{t} (Tf)(t) \quad t > 0,$$
where $Tf$ is a positive and normalized operator monotone function. We can then apply Corollary 5.1 to obtain a probability measure $\mu$ on the closed extended half-line $[0, \infty]$ such that

$$f(t) = f(1) + f'(1) \int_{0}^{\infty} \frac{t(1 + \lambda)}{t + \lambda} d\mu(\lambda), \quad t > 0.$$ 

We explicitly remove a possible atom in $\infty$ to obtain

$$f(t) = f(1) + f'(1)(\{\infty\})(t - 1) + f'(1) \int_{0}^{\infty} \frac{(t - 1)(1 + \lambda)}{t + \lambda} d\tilde{\mu}(\lambda),$$

where $\tilde{\mu}$ is a positive finite measure on the closed half-line $[0, \infty)$. We then make use of the identity

$$\frac{(t - 1)(1 + \lambda)}{t + \lambda} = (1 + \lambda)^2 \left( \frac{\lambda}{1 + \lambda^2} - \frac{1}{t + \lambda} \right) + \frac{1 - \lambda^2}{1 + \lambda^2}$$

to obtain

$$f(t) = \alpha t + \beta + f'(1) \int_{0}^{\infty} (1 + \lambda)^2 \left( \frac{\lambda}{1 + \lambda^2} - \frac{1}{t + \lambda} \right) d\tilde{\mu}(\lambda),$$

where $\alpha = f'(1)\mu(\{\infty\}) \geq 0$ and

$$\beta = f(1) - \mu(\{\infty\})f'(1) + f'(1) \int_{0}^{\infty} \frac{1 - \lambda^2}{1 + \lambda^2} d\tilde{\mu}(\lambda)$$

is finite since the integrand is bounded between $-1$ and $1$. The assertion now follows by setting $d\nu(\lambda) = f'(1)(1 + \lambda)^2 d\tilde{\mu}(\lambda)$ and noticing that

$$1 \leq (1 + \lambda^2)/(1 + \lambda^2) \leq 2$$

for $0 \leq \lambda < \infty$. QED

**Remark 5.3.** The unicity of the representing measure $\mu$ in Theorem 4.9 readily implies unicity of the representing measures in Corollary 5.1 and Theorem 5.2.
5.1 Löwner’s theorem

We learn from the integral expression in the previous theorem that an operator monotone function \( f \) defined in the positive half-line can be continued to an analytic function defined in \( \mathbb{C}\{(-\infty, 0] \). Since the imaginary part

\[
\Im\left(-\frac{1}{z + \lambda}\right) = \frac{\Im z}{|z + \lambda|^2}
\]

we also learn that the analytic continuation of \( f \) to the complex upper half-plane has non-negative imaginary part. In fact, the imaginary part of the continuation is positive if \( f \) is not constant.

**Theorem 5.4** (Löwner). Let \( f : I \to \mathbb{R} \) be a function defined in an open interval which is either finite \( I = (a, b) \) or infinite of the form \( (a, \infty) \). Then \( f \) is operator monotone if and only if it allows an analytic continuation to the upper half-plane with non-negative imaginary part.

**Proof.** The case where \( I \) is the positive half-line follows from Theorem 5.2 and from the Theory of Pick functions [4], and the case \( I = (a, \infty) \) then follows by a simple translation. The remaining cases may be similarly reduced to the case \( I = (0, 1) \). The function,

\[
h(t) = \frac{t}{t + 1} \quad t > 0,
\]

is a bijection between \((0, \infty)\) and the interval \((0, 1)\). It is operator monotone, and the inverse function,

\[
h^{-1}(t) = \frac{1}{1 - t} = \frac{1}{t^{-1} - 1} \quad 0 < t < 1,
\]

is also operator monotone. Both functions have analytic continuations which map the complex upper half-plane into itself. Composition with \( h \) therefore establishes a bijection between the operator monotone functions defined in the two intervals \((0, \infty)\) and \((0, 1)\). It also establishes a bijection between the functions defined in each of the two intervals, that allow an analytic continuation into the complex upper half-plane with non-negative imaginary part. \( \text{QED} \)
5.2 The representing measure

**Theorem 5.5.** Let $f: (0, \infty) \to \mathbb{R}$ be an operator monotone function, and let $\nu$ be the representing measure as given in Theorem 5.2. Let $\tilde{\nu}$ be the measure obtained from $\nu$ by removing a possible atom in zero. Then

$$
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_0^\infty \Im f(-t+i\varepsilon)g(t) \, dt = \frac{g(0)}{2} \nu(\{0\}) + \int_0^\infty g(\lambda) \, d\tilde{\nu}(\lambda)
$$

for every continuous, bounded and integrable function $g$ defined in $[0, \infty)$.

**Proof.** By applying Theorem 5.2 we obtain

$$
I_\varepsilon = \frac{1}{\pi} \int_0^\infty \Im f(-t+i\varepsilon)g(t) \, dt
$$

$$
= \frac{1}{\pi} \int_0^\infty \left( \varepsilon \alpha + \int_0^\infty \frac{\varepsilon}{(\lambda-t)^2 + \varepsilon^2} \, d\nu(\lambda) \right) g(t) \, dt.
$$

By Fubini’s theorem we may then write

$$
I_\varepsilon = \varepsilon \alpha \int_0^\infty g(t) \, dt + \frac{1}{\pi} \int_0^\infty \int_0^\infty \frac{\varepsilon}{(\lambda-t)^2 + \varepsilon^2} g(t) \, dt \, d\nu(\lambda).
$$

Since

$$
\frac{1}{\pi} \int_{-\infty}^\infty \frac{\varepsilon}{(\lambda-t)^2 + \varepsilon^2} \, dt = 1,
$$

we obtain by Lebesgue’s convergence theorem

$$
\lim_{\varepsilon \to 0} \frac{1}{\pi} \int_0^\infty \frac{\varepsilon}{(\lambda-t)^2 + \varepsilon^2} g(t) \, dt = g(\lambda) \quad \text{for} \quad \lambda > 0.
$$

For $\lambda = 0$ we only obtain $g(0)/2$. **QED**

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