THE ARGUMENT SHIFT METHOD AND MAXIMAL COMMUTATIVE
SUBALGEBRAS OF POISSON ALGEBRAS

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INTRODUCTION

Let \( q \) be a Lie algebra over an algebraically closed field \( k \) of characteristic zero. The symmetric algebra \( \mathcal{S}(q) \) has a natural structure of Poisson algebra, and our goal is to present a sufficient condition for the maximality of Poisson-commutative subalgebras of \( \mathcal{S}(q) \) obtained by the argument shift method. Study of Poisson-commutative subalgebras of \( \mathcal{S}(q) \) has attracted much attention in the last years, see [2, 6, 14, 15, 16]. This is related to commutative subalgebras of the enveloping algebra \( \mathcal{U}(q) \), fine questions of symplectic geometry, and integrable Hamiltonian systems. Commutative subalgebras of \( \mathcal{U}(q) \) (e.g., the famous Gelfand-Zetlin subalgebra of \( \mathcal{U}(\mathfrak{sl}_n) \)) occur in the theory of quantum integrable systems and have interesting application in representation theory.

Let \( \mathcal{Z}(q) \) be the centre of the Poisson algebra \( \mathcal{S}(q) \). For \( \xi \in q^* \), let \( \mathcal{F}_\xi(\mathcal{Z}(q)) \) denote the algebra generated by the \( \xi \)-shifts of all \( f \in \mathcal{Z}(q) \) (see Subsection 2.2 for precise definitions). As is well-known, \( \mathcal{F}_\xi(\mathcal{Z}(q)) \) is a Poisson-commutative subalgebra of \( \mathcal{S}(q) \). Furthermore, \( \text{trdeg}(\mathcal{F}_\xi(\mathcal{Z}(q))) \leq \frac{\dim q + \text{ind} q}{2} =: b(q) \). We say that \( \mathcal{F}_\xi(\mathcal{Z}(q)) \) is of maximal dimension, if the equality holds. However, even in this case, it may happen that there is a strictly larger Poisson-commutative subalgebra (of the same transcendence degree). We say that \( \mathcal{F}_\xi(\mathcal{Z}(q)) \) is maximal, if it is maximal with respect to inclusion among the commutative subalgebras of \( \mathcal{S}(q) \). Let \( q^*_{\text{reg}} \) denote the set of regular elements of \( q^* \), i.e., those whose stabiliser in \( q \) has the minimal dimension. For the purposes of this introduction, we state our main result (Theorem 3.2) in a slightly abbreviated form:

**Theorem 0.1.** Suppose that

1. \( \mathcal{Z}(q) \) contains algebraically independent homogeneous polynomials \( f_1, \ldots, f_l \), where \( l = \text{ind} q \), such that \( \sum_{i=1}^l \deg f_i = b(q) \);
2. \( \text{codim}(q^* \setminus q^*_{\text{reg}}) \geq 3 \).

Then, for any \( \xi \in q^*_{\text{reg}} \), \( \mathcal{F}_\xi(\mathcal{Z}(q)) \) is a polynomial algebra of Krull dimension \( b(q) \) and it is a maximal Poisson-commutative subalgebra of \( \mathcal{S}(q) \).

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Obviously, Theorem 0.1 applies if \( q \) is semisimple, and we thus generalise results of A. Tarasov [16]. (He proved maximality if \( \xi \) is regular semisimple.) There are also other interesting classes of Lie algebras satisfying the conditions of this theorem, see Section 4.

A general criterion of Bolsinov [11] asserts that, for \( \xi \in q^*_{reg}, \) \( F_\xi(\mathcal{Z}(q)) \) is of maximal dimension if and only if \( \text{codim}(q^* \setminus q^*_{reg}) \geq 2 \). For the proof of Theorem 0.1 we need, however, a stronger result. Namely, we provide a precise description of pairs \( \xi, \eta \in q^* \) such that the differentials at \( \eta \) of all functions from \( F_\xi(\mathcal{Z}(q)) \) generate a subspace of dimension \( b(q) \), see Theorem 2.4.

Notation. If an algebraic group \( Q \) acts on an irreducible affine variety \( X \), then \( k[X]^Q \) is the algebra of \( Q \)-invariant regular functions on \( X \) and \( k(X)^Q \) is the field of \( Q \)-invariant rational functions. If \( k[X]^Q \) is finitely generated, then \( X/\!/Q := \text{Spec} k[X]^Q \), and the quotient morphism \( \pi_X : X \to X/\!/Q \) is the mapping associated with the embedding \( k[X]^Q \hookrightarrow k[X] \).

If \( V \) is a \( Q \)-module and \( v \in V \), then \( q_v \) is the stabiliser of \( v \) in \( q \). For the adjoint representation of \( q \), the stabiliser of \( x \in q \) is also denoted by \( \mathfrak{z}_q(x) \), and we say that \( \mathfrak{z}_q(x) \) is the centraliser of \( x \).

All topological terms refer to the Zariski topology. If \( M \) is a subset of a vector space, then \( \text{span}(M) \) denotes the linear span of \( M \); \( k^\times := k \setminus \{0\} \).

1. ON THE CODIM–n PROPERTY FOR THE COADJOINT REPRESENTATION

Let \( Q \) be a connected algebraic group with Lie algebra \( q \). We write \( S(q) \) for the symmetric algebra of \( q \). Recall that \( S(q) \simeq k[q^*] \) is a Poisson algebra, and the symplectic leaves in \( q^* \) are precisely the coadjoint orbits of \( Q \). Since each coadjoint orbit \( Q\cdot \xi \) is a symplectic variety, \( \dim Q\cdot \xi \) is even. Let \( \{\ , \ \} \) denote the Lie-Poisson bracket in \( S(q) \). Then the algebra of invariants \( k[q^*]^Q = S(q)^Q \) is the centre of \( (S(q), \{\ , \ \}) \). We also write \( \mathcal{Z}(q) \) for this centre.

Let \( q^*_{reg} \) denote the set of all \( Q \)-regular elements of \( q^* \). That is,

\[
q^*_{reg} = \{ \xi \in q^* \mid \dim Q\cdot \xi \geq \dim Q\cdot \eta \text{ for all } \eta \in q^* \}.
\]

As is well-known, \( q^*_{reg} \) is a dense open subset of \( q^* \).

Definition 1. We say that the coadjoint representation of \( q \) has the codim–n property if \( \text{codim}(q^* \setminus q^*_{reg}) \geq n \).

If \( \xi \in q^*_{reg} \), then \( \dim q_\xi \) is called the index of \( q \), denoted \( \text{ind} q \). By Rosenlicht’s theorem, \( \text{trdeg} k[q^*]^Q = \text{ind} q \). It follows that if \( f_1, \ldots, f_r \in k[q^*]^Q \) are algebraically independent, then \( r \leq \text{ind} q \). Set \( b(q) = (\dim q + \text{ind} q)/2 \). If \( q \) is semisimple, then \( b(q) \) is the dimension of a Borel subalgebra.

Example. If \( q \) is reductive, then \( \text{ad} \simeq \text{ad}^* \) and \( \text{codim}(\mathfrak{g} \setminus \mathfrak{g}_{reg}) = 3 \). Hence the coadjoint representation of a reductive Lie algebra has the codim–3 property.
The following example pointed out by E.B. Vinberg shows that for any \( n \) there are noncommutative Lie algebras with codim–\( n \) property.

**Example 1.1.** Suppose \( s \in \mathfrak{gl}(V) \) is a semisimple linear transformation with nonzero rational eigenvalues. Let \( q \) be the semi-direct product of the 1-dimensional toral Lie algebra \( \mathbb{k}s \) and \( V \). The Lie bracket is given by
\[
[(\alpha s, v), (\beta s, v')] = (0, \alpha s(v') - \beta s(v)), \quad \alpha, \beta \in \mathbb{k}.
\]
It is easily seen that \( \text{ind} \ q = \dim q - 2 \). Moreover, let \( L \) be the annihilator of \( V \) in \( q^* \). Then the line \( L \) is precisely the set of \( Q \)-fixed points in \( q^* \), while \( \dim Q \cdot \xi = 2 \) for any \( \xi \in q^* \setminus L \). Thus, \( q \) has the codim–\( n \) property with \( n = \dim V \).

If \( f \in \mathcal{S}(q) \), then the differential of \( f \), \( df \), can be regarded as a polynomial mapping from \( q^* \) to \( q \), i.e., an element of \( \text{Mor}_Q(q^*, q) \cong \mathcal{S}(q) \otimes q \). More precisely, if \( f \in \mathcal{S}^{d}(q) \), then \( df \) is a polynomial mapping of degree \( d-1 \), i.e., an element of \( \mathcal{S}^{d-1}(q) \otimes q \). We write \( (df)_\xi \) for the value of \( df \) at \( \xi \in q^* \). Recall that \( (df)_\xi \) is an element of \( q \) that is defined as follows. If \( \nu \in q^* \) and \( \langle \cdot, \cdot \rangle \) denotes the natural pairing between \( q \) and \( q^* \), then
\[
\langle (df)_\xi, \nu \rangle := \text{the coefficient of } t \text{ in the Taylor expansion of } f(\xi + t\nu).
\]
The rôle of the codim–2 property is seen in the following result, see [11] Theorem 1.2.

**Theorem 1.2.** Suppose that \( (q, \text{ad}^*) \) has the codim–2 property and \( \text{trdeg} \mathbb{k}[q^*]^Q = \text{ind} \ q \). Let \( f_1, \ldots, f_{l} \in \mathbb{k}[q^*]^Q \) be arbitrary homogeneous algebraically independent polynomials. Then
\[
\begin{align*}
(\text{i}) & \quad \sum_{i=1}^{l} \deg f_i \geq b(q); \\
(\text{ii}) & \quad \text{If } \sum_{i=1}^{l} \deg f_i = b(q), \text{ then } \mathbb{k}[q^*]^Q \text{ is freely generated by } f_1, \ldots, f_{l} \text{ and } \xi \in q^*_{\text{reg}} \text{ if and only if } (df_1)_\xi, \ldots, (df_{l})_\xi \text{ are linearly independent.}
\end{align*}
\]

The second assertion in (ii) can be regarded as a generalisation of Kostant’s result for reductive Lie algebras [4] (4.8.2). Its geometric meaning is the following. Consider the quotient morphism \( \pi : q^* \to q^*/Q \cong \mathbb{A}^{\text{ind} \ q} \). Then \( \pi \) is smooth at \( \xi \in q^* \) if and only if \( \xi \in q^*_{\text{reg}} \).

2. **The argument shift method and Bolsinov’s criterion**

2.1. **Commutative subalgebras of \( \mathcal{S}(q) \).** Let \( \mathcal{A} \) be a subalgebra of the symmetric algebra \( \mathcal{S}(q) \). Then \( \mathcal{A} \) is said to be **Poisson–commutative** if the restriction of \( \langle \cdot, \cdot \rangle \) to \( \mathcal{A} \) is zero. Abusing the language, we will usually omit “Poisson” and merely say that \( \mathcal{A} \) is commutative. Notice that the words “subalgebra of \( \mathcal{S}(q) \)” always refer to the usual (associative and commutative) structure of the symmetric algebra, while “commutative” refers to the Poisson structure on \( \mathcal{S}(q) \).
For any subalgebra $A \subset S(q)$, we define the transcendence degree of $A$ as that of the quotient field of $A$. As is well-known, if $A$ is commutative, then $\text{trdeg} A \leq b(q)$. Indeed, if $f_1, \ldots, f_n \in A$ are algebraically independent, then $M := \text{span}\{(df_1)_\xi, \ldots, (df_n)_\xi\}$ is $n$-dimensional for generic $\xi$. Furthermore, $M$ is an isotropic subspace of $q$ with respect to the Kirillov form $K_\xi$. (Recall that $K_\xi(x, y) := \langle \xi, [x, y] \rangle$ and hence $\dim(\ker K_\xi) = \dim q_\xi$.)

**Definition 2.** Let $A$ be a commutative subalgebra of $S(q)$. Then $A$ is said to be of maximal dimension, if $\text{trdeg} A = b(q)$; $A$ is said to be maximal, if it is maximal with respect to inclusion among the commutative subalgebras of $S(q)$.

We do not know of whether there exist maximal commutative subalgebras that are not of maximal dimension.

Suppose $A$ is commutative and of maximal dimension. If $A \subset A'$ and $A'$ is commutative, then each element of $A'$ is algebraic over $A$. Conversely, if $f \in S(q)$ is algebraic over $A$, then, for generic $\xi \in q^*$, $(df)_\xi$ belongs to $\text{span}\{(dF)_\xi \mid F \in A\}$, which is an isotropic subspace with respect to $K_\xi$. Hence $\{f, F\}(\xi) = 0$ for a generic $\xi$ and therefore $\{f, F\} \equiv 0$. Thus, $A$ is maximal if and only if it is algebraically closed in $S(q)$.

**2.2. The argument shift method.** Suppose $f \in S(q)$ is a polynomial of degree $d$. For any $\xi \in q^*$, we may consider a shift of $f$ in direction $\xi$: $f_{a, \xi}(\mu) = f(\mu + a\xi)$, where $a \in k$. Expanding the right hand side as polynomial in $a$, we obtain the expression $f_{a, \xi}(\mu) = \sum_{j=0}^{d} f^j_{\xi}(\mu)a^j$. Associated with this shift of argument, we obtain the family of polynomials $f^j_{\xi}$, where $j = 0, 1, \ldots, d - 1$. (Since $\deg f^j_{\xi} = d - j$, the value $j = d$ is not needed.) We will say that the polynomials $\{f^j_{\xi}\}$ are $\xi$-shifts of $f$. Notice that $f^0_{\xi} = f$ and $f^{d-1}_{\xi}$ is a linear form on $q^*$, i.e., an element of $q$. Actually, $f^{d-1}_{\xi} = (df)_{\xi}$. There is also an obvious symmetry with respect to $\xi$ and $\mu$: $f^j_{\xi}(\mu) = f^{d-j}_{\mu}(\xi)$.

The following observation is due to Mishchenko-Fomenko [7].

**Lemma 2.1.** Suppose that $h_1, \ldots, h_m \in Z(q)$. Then for any $\xi \in q^*$, the polynomials

$$\{h^i_{\xi, j} \mid i = 1, \ldots, m; \quad j = 0, 1, \ldots, \deg h_i - 1\}$$

pairwise commute with respect to the Poisson bracket.

Mishchenko and Fomenko used this procedure for constructing commutative subalgebras of maximal dimension in $S(q)$. Given $\xi \in q^*$ and an arbitrary subset $B \subset Z(q)$, let $F_\xi(B)$ denote the subalgebra of $S(q)$ generated by the $\xi$-shifts of all elements of $B$. Clearly, if $\hat{B}$ is the subalgebra generated by $B$, then $F_\xi(B) = F_\xi(\hat{B})$. By Lemma 2.1, all subalgebras $F_\xi(B)$ are commutative. In particular, subalgebras $F_\xi(Z(q))$ are natural candidates on the rôle of commutative subalgebras of maximal dimension.
For \( g \) semisimple, it is proved in [7] that there is an open subset \( \Omega \subset g^* \) such that \( \mathcal{F}_\xi(Z(g)) \) is of maximal dimension for any \( \xi \in \Omega \). The subalgebras of the form \( \mathcal{F}_\xi(Z(g)) \) are called Mishchenko-Fomenko subalgebras in [16, 17, 15].

**Remark 2.2.** The argument shift method is a particular case of a more general construction related to compatible Poisson brackets. Recall that two Poisson brackets on a commutative associative algebra \( S \) are said to be **compatible** if any linear combination of them is again a Poisson bracket. For \( S = S(q) \), we can consider the usual Lie-Poisson bracket \( (f, g) \rightarrow \{f, g\} \) and the bracket \( (f, g) \rightarrow \{f, g\}_\xi \) obtained by “freezing the argument”. Here \( f, g \in S(q) \) and \( \xi \in q^* \) is a fixed element. By definition, \( \{f, g\}(\eta) := \langle \eta, [(df)_\eta, (dg)_\eta] \rangle \) and \( \{f, g\}_\xi(\eta) := \langle \xi, [(df)_\eta, (dg)_\eta] \rangle \). A direct calculation shows that each linear combination \( a\{,\} + b\{,\}_\xi \) is again a Poisson bracket on \( S(q) \).

It is easily seen that if \( f \in Z(q) \) and \( f_{b,\xi}(\nu) := f(\nu + b\xi) \), then \( f_{b,\xi} \) is a central function with respect to \( \{,\}_\xi \) and \( b \). Furthermore, the assignment \( f \mapsto f_{b,\xi} \) is a bijection between two centres. It follows that \( \mathcal{F}_\xi(Z(q)) \) is the subalgebra of \( S(q) \) generated by the centres of all Poisson brackets \( \{,\}_\xi \).

### 2.3. BOLSINOV’S CRITERION AND ITS EXTENSION

A general criterion for \( \mathcal{F}_\xi(Z(q)) \) to be of maximal dimension is found by A.V. Bolsinov. Using our terminology, we can express it as follows.

**Theorem 2.3** (cf. Bolsinov [1] Theorem 3.1]). Suppose that \( q \) satisfies the codim–2 property and \( \text{trdeg } Z(q) = \text{ind } q \). Then the algebra \( \mathcal{F}_\xi(Z(q)) \) is of maximal dimension for any \( \xi \in q^*_{\text{reg}} \).

The algebra \( \mathcal{F}_\xi(Z(q)) \) is of maximal dimension if and only if there is an \( \eta \in q^* \) such that the differentials at \( \eta \) of all polynomials in \( \mathcal{F}_\xi(Z(q)) \) span a subspace of dimension \( b(q) \). Clearly, such \( \eta \) form an open subset of \( q^* \). For our main result, we need, however, a more precise assertion. Here it is.

**Theorem 2.4.** Keep the assumptions of Theorem 2.3 Let \( P \subset q^* \) be a plane such that \( P \setminus \{0\} \subset q^*_{\text{reg}} \). Suppose that

\[
\dim \text{span}\{(df)_{\xi_0} \mid f \in Z(q)\} = \text{ind } q \text{ for some } \xi_0 \in P.
\]

Then \( \dim \text{span}\{(df)_\eta \mid f \in \mathcal{F}_\xi(Z(q))\} = b(q) \) for any linearly independent \( \xi \) and \( \eta \) in \( P \).

**Remark.** Condition (\( \ast \)) is open, hence it is satisfied on an open subset of \( P \). In many important cases, this condition follows from the other ones (see below). Therefore, there is not much harm in it.

**Proof.** We apply results of Bolsinov [1] (presented in Appendix A) to the compatible Poisson brackets \( \{,\} \) and \( \{,\}_\xi \) on \( q^* \), cf. Remark 2.2. For \( \eta \in q^* \), let \( A_\eta \) and \( B_\eta \) be the
corresponding skew-symmetric forms on $T^*_q(q^*) \cong q$. Explicitly, $A_q(x, y) = \langle \eta, [x, y] \rangle$ and $B_q(x, y) = \langle \xi, [x, y] \rangle$. It follows that $(aA_q + bB_q)(x, y) = \langle a\eta + b\xi, [x, y] \rangle$ and hence

$$\dim(\ker(aA_q + bB_q)) = \dim q_{a\eta + b\xi}.$$  

We will identify the 2-dimensional vector spaces $P = \text{span}\{A_q, B_q\}$ and $P = \text{span}\{\eta, \xi\} \subset q^*$ by taking $aA_q + bB_q$ to $a\eta + b\xi$.

Set $\mathcal{D} := \text{span}\{(df)_\eta \mid f \in \mathcal{F}_\xi(\mathbb{Z}(q))\}$. Our goal is to prove that $\dim \mathcal{D} = b(q)$. Recall that $\text{trdeg} S(q)^Q = \text{ind} q$. Therefore

$$\Omega := \{\nu \in q^* \mid \dim \text{span}\{(df)_\nu \mid f \in S(q)^Q\} = \text{ind} q\}$$

is a non-empty open subset of $q^*$. Note that $\Omega$ is conical, i.e., $\nu \in \Omega$ if and only if $t\nu \in \Omega$ for any $t \in \mathbb{k}_x$. By the assumption, $\Omega_P := \Omega \cap P \neq \emptyset$.

From Eq. (2.1), it follows that all nonzero forms in $P$ have the same rank. Applying Proposition $[A.4]$ to $V = q$ and $P = \text{span}\{A_q, B_q\}$ shows that $L = \sum_{(a, b) \neq (0, 0)} \ker(aA_q + bB_q)$ is a maximal isotropic subspace of $q$ with respect to any nonzero element of $P$. In particular, $\dim L = b(q)$. Furthermore, since $\Omega_P$ is a non-empty and conical subset of $P \setminus \{0\}$, we deduce from Lemma $[A.1]$ that

$$L = \sum_{(1, b) \in \Omega_P} \ker(A_q + bB_q),$$

where $(1, b)$ is regarded as the point $\eta + b\xi \in P$. Because $\dim \mathcal{D} \leq b(q)$, it suffices to prove that $L \subset \mathcal{D}$. Take any $(1, b) \in \Omega_P$ and let $C = \{, \} + b\{, \} \xi$ be the corresponding Poisson bracket on $q^*$. For any $f \in \mathbb{Z}(q)$, set $\tilde{f}(\nu) := f(\nu + b\xi)$. Then $(df)_\eta = (df)_{\eta + b\xi}$ and $f \mapsto \tilde{f}$ is a bijection between $\mathbb{Z}(q)$ and $\mathbb{Z}_C(q)$, the centre of the Poisson algebra $(S(q), C)$. Hence

$$\mathcal{H} := \text{span}\{(df)_{\eta + b\xi} \mid f \in \mathbb{Z}(q)\} = \text{span}\{(df)_{\eta} \mid f \in \mathbb{Z}_C(q)\} \subset \ker(A_q + bB_q).$$

Since $\eta + b\xi \in \Omega_P$, we have $\dim \mathcal{H} = \text{ind} q = \dim(\ker(A_q + bB_q))$. Hence $\text{span}\{(df)_{\eta} \mid f \in \mathbb{Z}_C(q)\} = \ker(A_q + bB_q)$. But each $(df)_{\eta}$ is a linear combination of differentials of elements of $\mathcal{F}_\xi$. Therefore $\ker(A_q + bB_q) \subset \mathcal{D}$ whenever $(1, b) \in \Omega_P$, and we conclude from Eq. (2.2) that $L \subset \mathcal{D}$. Hence $L = \mathcal{D}$, and we are done. \hfill \square

3. **Maximal Commutative Subalgebras of $S(q)$ and Flatness**

First, we prove an auxiliary geometric result. Let $V$ be a finite-dimensional vector space and $P \subset V$ a plane. Suppose $\Omega$ is a conical open subset of $V \setminus \{0\}$ such that $\text{codim}(V \setminus \Omega) \geq n \geq 2$. Let us say that $P$ is an $\Omega$-plane if $P \setminus \{0\} \subset \Omega$. Given $v \in \Omega$, let $\Omega_v$ be the set of all $u$ such that $\mathbb{k}v + \mathbb{k}u \subset V$ is an $\Omega$-plane.

**Lemma 3.1.** $\Omega_v$ is an open subset of $V \setminus \{0\}$ and $\text{codim}(V \setminus \Omega_v) \geq n - 1.$
conclude that

Since

follows that

\[ \Omega = \eta \]

conclude that the subalgebra

\[ F \]

that \[ 9, \text{Theorem 1.1} \] applies to the polynomial subalgebra

\[ F \]

linearly independent.

is an open subset of \( q \)

Set

\[ \text{Proof.} \] If \( k \) \( \geq \) \( 2 \), then \( \text{trdeg} F \) is a polynomial algebra of Krull dimension \( b(q) \);

(ii) Furthermore, if \( (q, \text{ad}^*) \) has the codim–3 property, then \( F_{\xi}(\mathbb{Z}(q)) \) is a maximal commutative subalgebra of \( S(q) \).

\( \square \)

The following is our main result.

**Theorem 3.2.** Let \( q \) be an algebraic Lie algebra.

(i) Suppose \( (q, \text{ad}^*) \) has the codim–2 property and \( \mathbb{Z}(q) \) contains algebraically independent polynomials \( f_1, \ldots, f_l \), where \( l = \text{ind} q \), such that \( \sum_{i=1}^{l} \deg f_i = b(q) \). Then, for any \( \xi \in \mathfrak{q}_{\text{reg}}^{*} \), \( F_{\xi}(\mathbb{Z}(q)) = F_{\xi}(f_1, \ldots, f_l) \) is a polynomial algebra of Krull dimension \( b(q) \);

(ii) Furthermore, if \( (q, \text{ad}^*) \) has the codim–3 property, then \( F_{\xi}(\mathbb{Z}(q)) \) is a maximal commutative subalgebra of \( S(q) \).

\( \square \)

(i) It follows from the assumptions and Theorem 1.2 that \( \mathbb{Z}(q) = \mathbb{k}[f_1, \ldots, f_l] \). Hence \( F_{\xi} = F_{\xi}(f_1, \ldots, f_l) \). By Bolsinov’s criterion (Theorem 2.3), \( \text{trdeg} F_{\xi} = b(q) \) for any \( \xi \in \mathfrak{q}_{\text{reg}}^{*} \).

Set \( \Omega = \{ \xi \in \mathfrak{q}^{*} \mid (df_1)_{\xi}, \ldots, (df_l)_{\xi} \text{ are linearly independent} \} \). From Theorem 1.2(ii), it follows that \( \Omega = \mathfrak{q}_{\text{reg}}^{*} \). Hence \( \text{codim} (\mathfrak{q}^{*} \setminus \Omega) \geq 2 \).

Let \( P := \mathbb{k}\xi + \mathbb{k}\eta \subset \mathfrak{q}^{*} \) be a \( \mathfrak{q}_{\text{reg}}^{*} \)-plane, i.e., each nonzero element of it belongs to \( \mathfrak{q}_{\text{reg}}^{*} \).

Since \( \Omega = \mathfrak{q}_{\text{reg}}^{*} \), each nonzero point of \( P \) satisfies condition (s) of Theorem 2.4. Hence Theorem 2.4 guarantees us that, for any \( \eta \in P \setminus \mathbb{k}\xi \), the differentials of the \( \xi \)-shifts of \( f_1, \ldots, f_l \) at \( \eta \) span a subspace of dimension \( b(q) \). Next, in view of the equality \( \sum_{i=1}^{l} \deg f_i = b(q) \), the set of all \( \xi \)-shifts of the \( f_i \)'s consists of \( b(q) \) elements. It follows that the differentials

\[ \{ (df_{i,\xi})_{\eta} \mid i = 1, \ldots, l; \quad j = 0, 1, \ldots, \deg f_i - 1 \} \]

are linearly independent. This already proves that \( F_{\xi} \) is a polynomial algebra freely generated by the \( \{ f_{i,\xi} \} \)’s. We have also proved the following implication:

if \( \mathbb{k}\xi + \mathbb{k}\eta \) is a \( \mathfrak{q}_{\text{reg}}^{*} \)-plane, then the vectors \( \{ (df_{i,\xi})_{\eta} \mid i = 1, \ldots, l; \quad j = 0, 1, \ldots, \deg f_i - 1 \} \) are linearly independent.

(ii) Now \( \text{codim} (\mathfrak{q}^{*} \setminus \Omega) \geq 3 \). Applying Lemma 3.1 to \( V = \mathfrak{q}^{*}, \Omega = \mathfrak{q}_{\text{reg}}^{*} \), and \( v = \xi \), we conclude that

\[ \{ \nu \in \mathfrak{q}_{\text{reg}}^{*} \mid (df_{i,\xi})_{\nu} \text{ are linearly independent} \} \]

is an open subset of \( \mathfrak{q}^{*} \) whose complement is of codimension \( \geq 2 \). This means, in turn, that \( 9, \text{Theorem 1.1} \) applies to the polynomial subalgebra \( F_{\xi} \subset S(q) \). Therefore, we can conclude that the subalgebra \( F_{\xi} \) is algebraically closed in \( S(q) \).
Assume that $\mathcal{K}$ is a commutative subalgebra of $S(q)$ containing $\mathcal{F}_\xi$. Since $\mathcal{F}_\xi$ has the maximal possible Krull dimension, $\mathcal{F}_\xi \subset \mathcal{K}$ is an algebraic extension. Because $\mathcal{F}_\xi$ is algebraically closed in $S(q)$, we obtain $\mathcal{F}_\xi = \mathcal{K}$.

**Remark 3.3.** The codim–3 property is essential for the maximality of $\mathcal{F}_\xi(Z(q))$, see Example 4.1.

It would be interesting to find general conditions that guarantee us that the family of $\xi$-shifts of the free generators of $Z(q)$ form a regular sequence in $S(q)$. In the geometric language, this means that we are interested in the property that the natural morphism $q^* \to \text{Spec}(\mathcal{F}_\xi(Z(q))) \simeq A^{n(q)}$ is flat. It is likely that the assumptions of Theorem 3.2 are sufficient for this. However, we unable to prove this as yet.

**Remark 3.4.** One can use deformation arguments for proving flatness. We mention an affirmative result for $sl_n$, which is obtained by combining work of several authors. For an arbitrary reductive $\mathfrak{g}$, there is a general procedure of obtaining new commutative subalgebras of $S(\mathfrak{g})$ as limits of Mishchenko-Fomenko subalgebras $\mathcal{F}_\xi(Z(\mathfrak{g}))$, where $\xi$ runs inside a fixed Cartan subalgebra of $\mathfrak{g}$, see [13]. In particular, for $\mathfrak{g} = sl_n$, there is a special limit subalgebra that is the associated graded algebra of the Gelfand-Zetlin subalgebra of $U(sl_n)$, see [17, §6]. In [8], it is proved that the free generators of the latter form a regular sequence in $S(sl_n)$. This implies that if $\xi \in (sl_n)^* \simeq sl_n$ is regular semisimple, then the free generators of $\mathcal{F}_\xi(Z(sl_n))$ form a regular sequence.

4. **Applications**

4.1. **Some Lie algebras with codim–3 property.** Here we describe several classes of Lie algebras, where Theorem 3.2 applies.

1) If $\mathfrak{g}$ is reductive, then the assumptions of Theorem 3.2 are satisfied. This follows from the classical results of Kostant [4]. Therefore, for any $\xi \in \mathfrak{g}_{\text{reg}}$, $\mathcal{F}_\xi(Z(\mathfrak{g}))$ is a polynomial algebra, and it is a maximal commutative subalgebra of $S(\mathfrak{g})$. For the regular semisimple $\xi$, this has already been proved by Tarasov [16].

2) Following [13], recall the definition of a (generalised) Takiff Lie algebra (modelled on $q$). The infinite-dimensional $k$-vector space $q_\infty := q \otimes k[T]$ has a natural structure of a Lie algebra such that $[x \otimes T^i, y \otimes T^k] = [x, y] \otimes T^{i+k}$. Then $q_{\geq (n+1)} = \bigoplus_{j \geq n+1} q \otimes T^j$ is an ideal of $q_\infty$, and $q_\infty/q_{\geq (n+1)}$ is a generalised Takiff Lie algebra, denoted $q(n)$. If $\mathfrak{g} = \mathfrak{g}$ is semisimple, then $\mathfrak{g}(n)$ satisfies all the assumptions of Theorem 3.2, see [13]. For $n = 1$, one obtains the semi-direct product $\mathfrak{g} \ltimes \mathfrak{g}$. This case was studied by Takiff in 1971.

3) Let $e \in sl_n$ be a nilpotent element. Set $q = \mathfrak{z}_{sl_n}(e)$. Then $\text{ind } q = \text{rk } (sl_n) = n-1$ [19] and $S(q)^Q$ is a polynomial algebra of Krull dimension $n-1$ such that the sum of the
degrees of free generators equals \( b(q) \) [9, Theorem 4.2]. The second author can prove that here \((q, \text{ad}^*)\) have codim–3 property. (This will appear elsewhere.) Thus, \( \mathfrak{z}_{sl_n}(e) \) satisfies all the assumptions of Theorem 3.2.

4) Let \( q \) be a \( \mathbb{Z}_2 \)-contraction of a simple Lie algebra \( g \). It is known that \( \text{trdeg} \mathbb{Z}(q) = \text{ind} q \) [11, Lemma 2.6] and \((q, \text{ad}^*)\) has the codim–2 property [11, Theorem 3.3]. However, the stronger codim–3 property is not always satisfied. Recall the relevant setup.

Let \( g = g_0 \oplus g_1 \) be a \( \mathbb{Z}_2 \)-grading of \( g \). Then the semi-direct product \( q = g_0 \ltimes g_1 \) is called a \( \mathbb{Z}_2 \)-contraction of \( g \). Here \( \text{ind} q = \text{ind} g = \text{rk} g \), hence \( b(q) = b(g) \). For most \( \mathbb{Z}_2 \)-gradings, it is proved that \( \mathbb{Z}(q) \) is polynomial and the sum of degrees of free generators equals \( b(g) \), see [11, Sect. 4 & 5]. It follows that, for such \( \mathbb{Z}_2 \)-contractions, the commutative subalgebras \( \mathcal{F}_\xi(\mathbb{Z}(q)), \xi \in q^*_\text{reg} \) are polynomial and of maximal dimension. However, these are not always maximal.

**Example 4.1.** Let \( g = g_0 \oplus g_1 \) be a \( \mathbb{Z}_2 \)-grading such that \( g_1 \) contains a Cartan subalgebra of \( g_1 \). It is equivalent to that \( \dim g_1 = b(g) \). Then \( S(q)^Q = S(g_1)^{G_0} \simeq S(g)^G \). (This clearly shows that the sum of degrees of free generators of \( S(q)^Q \) equals \( b(g) \).) By the assumption, \( g_1 \) contains regular elements of \( g \) and, hence, of \( q \). Let \( \xi \in g_1 \) be such an element. Then \( \mathcal{F}_\xi(\mathbb{Z}(q)) = \mathcal{F}_\xi(S(g_1)^{G_0}) \) is a proper subalgebra of \( S(g_1) \). Indeed, the family of \( \xi \)-shifts of the generators contains \( b(g) \) elements, but not all of them are of degree 1. On the other hand, the subspace \( g_1 \) is a commutative Lie subalgebra of \( q \), hence \( S(g_1) \) is a commutative subalgebra of \( S(q) \). (Actually, it is a maximal commutative subalgebra!) Thus, \( \mathcal{F}_\xi(\mathbb{Z}(q)) \) is a commutative subalgebra of \( S(q) \) of maximal dimension, but not maximal.

Of course, the reason for such a “bad” behaviour is that \( \text{codim} (q^* \setminus q^*_\text{reg}) = 2 \). This can also be proved directly using invariant-theoretic properties of the \( G_0 \)-module \( g_1 \) [5].

**Example 4.2.** We have verified that the codim–3 property holds for \( \mathbb{Z}_2 \)-contractions associated with the following symmetric pairs \((g, g_0)\): \((\mathfrak{sl}_{2n}, \mathfrak{sp}_{2n}); (\mathfrak{sl}_{n+1}, \mathfrak{gl}_n), n \geq 2; (\mathfrak{so}_n, \mathfrak{so}_{n-1}); (\mathfrak{E}_6, \mathfrak{F}_4); (\mathfrak{F}_4, \mathfrak{B}_4)\). However, the complete list is not known yet. For items 2, 3, and 5, it is shown in [11] that \( \mathbb{Z}(q) \) is polynomial and the sum of degrees of the free generators equals \( b(q) \). Hence Theorem 3.2 applies there.

**Remark 4.3.** In [2], Joseph and Lamrou consider the so-called truncated parabolic subalgebras of maximal index in \( \mathfrak{sl}_n \). In this case, \( \mathbb{Z}(q) \) is a polynomial algebra and the equality \( \sum \text{deg } f_i = b(q) \) holds. Furthermore, they prove the maximality of \( \mathcal{F}_\xi(\mathbb{Z}(q)) \). It would be interesting to check whether the codim–3 property also holds there.

### 4.2. Semi-direct products and the codim–3 property

Example 4.1 can be put in a more general context. Suppose \( G \) is semisimple and \( V \) is a finite-dimensional \( G \)-module. Set \( m = \max_{\xi \in V^*} \dim G \cdot \xi \). Form the semi-direct product \( q = g \ltimes V \).
Proposition 4.4. Suppose that (a) \( S(V)^G = \mathbb{k}[V^*]^G \) is a polynomial algebra and (b) \( m = \dim \mathfrak{g} \). Then \((q, \text{ad}^*)\) does not satisfy the codim–3 property and the commutative subalgebras \( \mathcal{F}_\xi(Z(q)) \) are not maximal.

Proof. It follows from assumption (b) and Raïs’ formula \[12\] that \( \text{ind} \mathfrak{q} = \dim V - \dim \mathfrak{g} \) and therefore \( b(\mathfrak{q}) = \dim V \). Also, assumption (b) implies that \( \mathbb{k}[\mathfrak{q}]^G = \mathbb{k}[V^*]^G \) \[10\]. Theorem 6.4]. Thus, \( Z(q) = S(q)^G = \mathbb{k}[\mathfrak{q}]^G \) is a polynomial algebra. Since \( G \) has no rational characters, \( \mathbb{k}[V^*]^G \) is the quotient field of \( \mathbb{k}[V^*]^G \). Hence \( \text{trdeg} \mathbb{k}[V^*]^G = \text{ind} \mathfrak{q} \). Let \( d \) be the sum of degrees of free generators of \( \mathbb{k}[V^*]^G \). By \[3\] Korollar 6], \( d \leq \dim V \). Assume that \((q, \text{ad}^*)\) has the codim–3 property. Then \( d \geq b(\mathfrak{q}) = \dim V \) (Theorem 1.2]. Hence \( d = b(\mathfrak{q}) \) and by Theorem 3.2, \( \mathcal{F}_\xi(Z(q)) \) is a maximal commutative subalgebra of \( S(q) \) for any \( \xi \in \mathfrak{q}_{\text{reg}}^* \). Since \( Z(q) \) is a subalgebra of \( S(V) \), \( \mathcal{F}_\xi(Z(q)) \) is a subalgebra of \( S(V) \), too. Furthermore, \( \mathcal{F}_\xi(Z(q)) \) is generated by \( \dim V \) elements, and not all of them are of degree 1. Thus, \( \mathcal{F}_\xi(Z(q)) \) is a proper subalgebra of \( S(V) \), and the latter is a (maximal) commutative subalgebra of \( S(q) \). This contradiction shows that the codim–3 property cannot be satisfied for \((q, \text{ad}^*)\). The above argument also proves the second assertion. \( \square \)

Remark 4.5. Set \( V_{\text{sing}}^* = \{ \nu \in V^* \mid \dim G \cdot \nu < m \} \). (This closed subset plays an important rôle in theory developed in \[3\].) It is easily seen that if \( m = \dim G \) and \( \text{codim} V_{\text{sing}}^* \geq n \), then \( \text{codim} q^* \setminus q_{\text{reg}}^* \geq n \). Hence, under the assumptions of Proposition 4.4, we have \( \text{codim} V_{\text{sing}}^* \leq 2 \), and according to \[3\] Korollar 2], \( \text{codim} V_{\text{sing}}^* = 2 \) if and only if \( d = b(\mathfrak{q}) \).

Appendix A. Some results on skew-symmetric bilinear forms

Here we present some general facts concerning skew-symmetric bilinear forms that are needed for the proof of Theorem 2.4. All these results are extracted from \[11\], but we present them in a more systematic and algebraic form.

Let \( \mathcal{P} \) be a two-dimensional linear space of (possibly degenerate) skew-symmetric bilinear forms on a finite-dimensional vector space \( V \). Set \( m = \max_{A \in \mathcal{P}} \text{rk} A \), and let \( \mathcal{P}_{\text{reg}} \subset \mathcal{P} \) be the set of all forms of rank \( m \). For each \( A \in \mathcal{P} \), let \( \ker A \subset V \) be the kernel of \( A \). Our main object of interest is the subspace \( L := \sum_{A \in \mathcal{P}_{\text{reg}}} \ker A \).

Lemma A.1. For any nonempty open subset \( \Omega \subset \mathcal{P}_{\text{reg}} \), we have \( \sum_{A \in \Omega} \ker A = L \).

Proof. Set \( r = \dim V - m \) and \( M = \sum_{A \in \Omega} \ker A \subset L \). Take any \( C \in \mathcal{P}_{\text{reg}} \setminus \Omega \). Then \( \ker C \) is a point of the Grassmannian \( \text{Gr}_r(V) \). Because \( \mathcal{P} \) is irreducible, \( \Omega = \mathcal{P} \) and there is a curve \( \varphi : \mathbb{k}^* \to \Omega \) such that \( \lim_{t \to 0} \varphi(t) = C \). Hence \( \lim_{t \to 0} (\ker \varphi(t)) = \ker C \), where the last limit is taken in \( \text{Gr}_r(V) \). Since \( \ker \varphi(t) \in \text{Gr}_r(M) \) for \( t \neq 0 \) and \( \text{Gr}_r(M) \) is closed in \( \text{Gr}_r(V) \), we obtain \( \ker C \subset M \). Thus, \( M = L \). \( \square \)
For $A \in \mathcal{P}$, let $\hat{A}$ denote the corresponding linear map from $V$ to $V^*$. Then $\ker A = \ker \hat{A}$.

**Lemma A.2.** For all $A, B \in \mathcal{P} \setminus \{0\}$, we have $\hat{A}(L) = \hat{B}(L)$.

**Proof.** Clearly, we may assume that $A$ and $B$ are linearly independent. By virtue of Lemma A.1, $L$ is spanned by some $L_{a,b} := \ker (aA+bB)$ with $ab \neq 0$. Since $(a\hat{A}+b\hat{B})(L_{a,b}) = 0$, we obtain $(a\hat{A})(L_{a,b}) = (b\hat{B})(L_{a,b})$ and hence $\hat{A}(L_{a,b}) = \hat{B}(L_{a,b})$. The result follows. $\square$

For $A \in \mathcal{P} \setminus \{0\}$, let $\tilde{L} \subset V$ denote the annihilator of $\hat{A}(L) \subset V^*$. By Lemma A.2, $\tilde{L}$ does not depend on the choice of $A$. Note also that $\tilde{L} = \{v \in V \mid A(v, L) = 0\}$. Since $\ker A \subset \tilde{L}$ for each nonzero $A$, $L$ is a subspace of $\tilde{L}$.

**Lemma A.3.** Suppose that $B \in \mathcal{P}$ and $A \in \mathcal{P}_{reg}$. Then

(i) $\tilde{B}(\tilde{L}) \subset \hat{A}(\tilde{L})$;

(ii) Associated with $A$ and $B$, there is a natural linear operator $\Phi_{A,B} = \Phi : \tilde{L}/L \to \tilde{L}/L$.

**Proof.** (i) Let $M_A$ and $M_B$ be the the annihilators of $\hat{A}(\tilde{L})$ and $\hat{B}(\tilde{L})$, respectively. Since $M_A = \ker A + L = L$ and $M_B = \ker B + L$, we obtain $M_A \subset M_B$.

(ii) Take any $v \in \tilde{L}$. Since $\hat{B}(\tilde{L}) \subset \hat{A}(\tilde{L})$, where is $w \in \tilde{L}$ such that $\hat{A}(w) = \hat{B}(v)$. Letting $\Phi(v + L) := w + L$, we have to check that there is no ambiguity in this. To this end, assume that $\hat{A}(w') \in \hat{B}(v + L) = \hat{A}(w) + \hat{B}(L)$. Since $\hat{B}(L) = \hat{A}(L)$, we obtain $\hat{A}(w' - w) \in \hat{A}(L)$. Hence $w - w' \in L + \ker A = L$. Thus, given $v = v + L \in \tilde{L}/L$, there is a unique $w = w + L \in \tilde{L}/L$ such that $\tilde{B}(v) = \hat{A}(w)$. The claim follows. $\square$

**Proposition A.4.** If $\mathcal{P}_{reg} = \mathcal{P} \setminus \{0\}$, then $L = \tilde{L}$; in other words, $L$ is a maximal isotropic subspace of $V$ with respect to any nonzero $A \in \mathcal{P}$.

**Proof.** Take linearly independent $A$ and $B$, as in Lemma A.3. We use the operator $\Phi : \tilde{L}/L \to \tilde{L}/L$ introduced in Lemma A.3(ii). Since $k$ is algebraically closed, $\tilde{L}/L = \{0\}$ if and only if all eigenvectors of $\Phi$ are zero. Assume that $v + L \in \tilde{L}/L$ is a $\lambda$-eigenvector of $\Phi$. Then expanding the definition of $\Phi$ yields $(\tilde{B} - \lambda \hat{A})v \in \hat{A}(L)$. Since $\hat{A}(L) = (\tilde{B} - \lambda \hat{A})(L)$ by Lemma A.2 we get $(\tilde{B} - \lambda \hat{A})(v) \in (\tilde{B} - \lambda \hat{A})(L)$ and, hence, $v \in L + \ker (B - \lambda A)$. If $v \notin L$, then $\ker (B - \lambda A) \nsubseteq L$ and therefore $(B - \lambda A) \notin \mathcal{P}_{reg}$. A contradiction! $\square$

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