STABILITY OF THE $L^p$-NORM OF THE CURVATURE TENSOR AT KÄHLER SPACE FORMS

SOMA MAITY

ABSTRACT. We consider the Riemannian functional defined on the space of Riemannian metrics with unit volume on a closed smooth manifold $M$ given by $R_p(g) := \int_M |R(g)|^p dv_g$ where $R(g), dv_g$ denote the corresponding Riemannian curvature, volume form and $p \in [2, \infty)$. We prove that $R_p$ restricted to the space of Kähler metrics attains its local minima at a metric with constant holomorphic sectional curvature.

1. Introduction

Let $M$ be a closed smooth manifold of dimension $n \geq 3$ and $\mathcal{M}_1$ be the space of Riemannian metrics with unit volume on $M$ endowed with the $C^{2,\alpha}$-topology for any $\alpha \in (0, 1)$. In this paper we study the following Riemannian functional:

$$R_p(g) = \int_M |R(g)|^p dv_g$$

where $R(g)$ and $dv_g$ denote the corresponding Riemannian curvature tensor and volume form, $p \in [2, \infty)$. Let $\mathcal{M}_k$ denote the space of Kähler metrics with unit volume on $M$. Consider $C^{2,\alpha}$-topology on it.

Theorem. Let $(M, g)$ be a closed Kähler manifold with constant holomorphic sectional curvature and $p \in [2, \infty)$. Then $(M, g)$ is a strict local minimizer for $R_p$ restricted to $\mathcal{M}_k$ i.e. there exists a neighborhood $U$ of $g$ in $\mathcal{M}_k$ such that for any $\tilde{g} \in U$

$$R_p(\tilde{g}) \geq R_p(g).$$

The equality holds if and only if there exists a biholomorphism $\phi$ of $M$ such that $\tilde{g} = \phi^* g$.

Remark : If $(M, g)$ is a Kähler manifold with constant negative holomorphic sectional curvature then there are finite number of biholomorphisms of $M$. Hence there exists a neighborhood $U_1$ of $g$ in $\mathcal{M}_k$ such that for any $\tilde{g} \in U_1$, $R_p(\tilde{g})$ is strictly greater than $R_p(g)$.

Definition : Let $(M, g)$ be a Kähler manifold. A symmetric 2-tensor $h$ on $M$ is called a Kähler variation of $g$ if there exists a one-parameter family of Kähler metrics $g(t)$ on $M$ with $g(0) = g$ and $\frac{d}{dt}g(t)|_{t=0} = h$.

Let $\mathcal{K}$ denote the space of Kähler variations of $g$ and $\mathcal{V}$ denote the space of symmetric 2-tensors orthogonal to the tangent space of the orbit of $g$ under the action of the group of diffeomorphisms of $M$. First we prove that the Hessian of $R_p$ has a positive lower bound when it is restricted to the space of unit vectors in $\mathcal{K} \cap \mathcal{V}$. The gradient of $R_p$ is given by

$$\nabla R_p = -p\delta^D d^*|R|^{p-2} R - p|R|^{p-2} \hat{R} + \frac{1}{2} |R|^p g + (\frac{p}{n} - \frac{1}{2}) ||R||^p g$$

Key words and phrases. Riemannian functional, critical point, stability, local minima.
For the notations we refer to Section 2. Every closed irreducible symmetric space is a critical point for $R_p$. The Hessian at a critical point of $R_p$ is given by

$$H(h_1, h_2) = \langle (\nabla R_p)'_g(h_1), h_2 \rangle \quad \forall \ h_1, h_2 \in S^2(T^*M)$$

where $S^2(T^*M)$ denotes the space of symmetric 2-tensor fields on $M$ and $(\nabla R_p)'_g(h_1)$ denotes the derivative of $\nabla R_p$ at $g$ along $h_1$. A Riemannian metric $g$ is called rigid if the kernel of $H$ restricted to $V \times V$ is zero and it is called stable if $H$ restricted to the products of unit vectors of $V$ has a positive lower bound. It is difficult to investigate rigidity and stability of $R_p$ for any arbitrary irreducible symmetric space. The strict stability of $R_p$ has been obtained for Riemannian manifolds with constant sectional curvature and their suitable products for certain values of $p$ [SM].

Let $\delta_g$ denote the divergence operator acting on $S^2(T^*M)$. The main result of this paper is the following.

**Proposition.** Let $(M, g)$ be a Riemannian manifold with constant holomorphic sectional curvature $c$. Then there exists a positive constant $k(c, p)$ such that for any $h \in K$ with $\text{tr}(h) = 0$ and $\delta_g(h) = 0$.

$$H(h, h) \geq k\|h\|^2$$

where $\|\cdot\|$ denotes the $L^2$-norm on $S^2(T^*M)$ defined by $g$. The condition $\delta_g(h) = 0$ and $\text{tr}(h) = 0$ implies that $h \in K \cap V$ ([AB] Lemma 4.57).

**Remark:** For any $p \in [2, \frac{n}{2})$, if the first eigenvalue of the Laplacian of a compact hyperbolic manifold satisfies a lower bound then it is a local minimizer for $R_p$ [SM]. In the spherical case, this holds for all $p \in [2, \infty)$. The extra condition in case of a compact hyperbolic manifold is required to prove the stability of $R_p$ restricted to the conformal metrics. Since the only Kähler metrics in the conformal class of $g$ are constant multiples of $g$, no extra condition is required to prove stability of $R_p$ restricted to the Kähler variations.

Next we prove the existence of a local minimizing neighborhood of $g$. The idea of proof is based on the work of Gursky and Viaclovsky in [GV]. Some crucial observations about the space of Kähler variations and Kähler metrics are required here which are given in Lemma 4 and 5. Finally we end up with the following question.

**Question:** Is $R_p$ stable for any compact Hermitian symmetric space?

2. **Proof**

2.1. **Preliminaries:** Let $\{e_i\}$ be an orthonormal basis at a point of $M$. $\hat{R}$ is a symmetric 2-tensor defined by $\hat{R}(x, y) = \sum R(x, e_i, e_j, e_k)R(y, e_i, e_j, e_k)$.

Define $\hat{\hat{R}} : S^2(T^*M) \rightarrow S^2(T^*M)$ by

$$\hat{\hat{R}}(h)(x, y) := \sum R(e_i, x, e_j, y)h(e_i, e_j)$$

Let $D$ and $D^*$ be the Riemannian connection and its formal adjoint. Let $\Gamma(V)$ denote the space of sections of a vector bundle $V$ and $\Lambda^p$ denote the space of $p$-forms on $M$. Next we define some differential operators.
The divergence operator $\delta_g$ on $S^2(T^*M)$ and its formal adjoint $\delta_g^*$ are defined by

$$\delta_g(h)(x) = -D_{e_i} h(e_i, x)$$

$$\delta_g^*\omega(x, y) := \frac{1}{2}(D_{xy} + D_{yx})$$

Let $g_t$ be a one-parameter family of metrics with $\frac{d}{dt}(g_t)_{t=0} = h$ and $T(t)$ be a tensor depending on $g_t$. Then $\frac{d}{dt}T(t)_{t=0}$ is denoted by $T'(h)$. Define $\Pi_h(x, y) = \frac{d}{dt}D_x y_{t=0}$ and $\bar{\tau}_h(x, y) := R'_g(h)(x, e_i, y, e_i)$ where $x, y$ are two fixed vector fields. The suffix $h$ will be omitted when there will not be any ambiguity.

$W$ is a 3-tensor defined by,

$$W_h(x, y, z) := \langle (D^*)'(h)(R)(x, y, z)$$

$$- \sum[R(y, z, \Pi_h(e_i, e_i, w) + R(y, z, e_i, \Pi_h(e_i, w)) + R(z, e_i, \Pi_h(y, e_i), w) + R(e_i, y, \Pi_h(z, e_i), w) + R(e_i, y, e_i, \Pi_h(z, w))]]$$

Next we prove some lemma which will be used in the proof of the proposition. Let $(M, g)$ be a symmetric two tensor field such that $\delta_g(h) = 0$ and $tr(h) = 0$. Then from the equation (4.1) in [SM] we have,

$$H(h, h) = p|R|^p - 2 \langle (\bar{\tau}_h, \delta^D d^D h) + \langle W_h, d^D h \rangle \rangle - \langle (\bar{R}_p)'(h), h \rangle + \frac{|R|^2}{n} ||h||^2$$

(2.1)

Next we compute each terms appeared in the above expression. We denote $T(e_{i_1}, e_{i_2}, ..., e_{i_k})$ by $T_{i_1i_2...i_k}$ where $T$ is a $k$-tensor. Let $h \in S^2(T^*M)$ with $\delta_g(h) = 0$ and $tr(h) = 0$. Then we have the following lemma.

**Lemma 1.** $\bar{\tau}_h = \frac{1}{2}(D^* Dh + 2\lambda h)$

**Proof.** From [AB] 1.174(c), we have,

$$2(R'_g(h))_{p_{q_{i_{j}}}} = [(D_{q_{i{j}}}^2 h)_{p_{i}} + (D_{p_{i}}^2 h)_{q_{i}} - (D_{q_{i}}^2 h)_{i_{i}} - (D_{i_{i}}^2 h)_{p_{q_{i}}} + h_{i_{j}} R_{p_{i_{j}}} - h_{q_{j}} R_{p_{i_{j}}}]$$

Therefore,

$$\bar{\tau}_h(x, y) = \frac{1}{2} \sum [D_{e_i, y}^2 h(x, e_i) + D_{x, e_i}^2 h(y, e_i) + D_{x, e_i}^2 h(x, e_i) - D_{e_i, e_i}^2 h(x, y)$$

$$+ h(R(x, e_i, y, e_i) - h(R(x, e_i, e_i), y)]$$

Applying Ricci identity we have,

$$D_{e_i, y}^2 h(x, e_i) - D_{y, e_i}^2 h(e_i, x) = h(R(e_i, y, x), e_i) + h(R(e_i, y, e_i), x)$$

Notice that

$$\sum_i D_{y, e_i}^2 h(e_i, x) = -D \delta_g h(y, x) = 0$$

$$\sum_i h(R(e_i, y, x), e_i) = -R(x, y)$$
and
\[ \sum_i h(R(e_i, y, e_i), x) = \sum_j r(y, e_j)h(x, e_j) = \lambda h(x, y). \]

We also have,
\[ \sum_i D^2_{e_i e_i} h(x, y) = -D^* Dh(x, y) \]
and
\[ \sum_i D^2_{x y} h(e_i, e_i) = Ddtr(h) = 0 \]

Combining all these results the lemma follows. \(\square\)

**Lemma 2.** \(\delta D^D h = 2D^* Dh + 2\lambda h - 2\circ R(h)\)

**Proof.** From the identity 2.8 in [MB] we have,
\[ \delta D^D h_{pq} = 2D^* Dh_{pq} - 2\delta_g^\circ h_{pq} + \sum_i (r_{pi}h_{iq} + r_{qi}h_{ip}) - 2\sum_{i,j} R_{piqj}h_{ij} \]
\[ = 2D^* Dh_{pq} + 2\lambda h_{pq} - 2\circ R(h)_{pq} \]
\(\square\)

**Lemma 3.** \(\delta W = 2[\lambda D^* Dh + \lambda^2 h + D^* D(\circ R(h)) - \circ R(\circ R(h))]\)

**Proof.** From the proof of Lemma 4.1 (ii) in [SM] we have,
\[ (W, d^D h) = 2\sum (R_{ij\Pi_{ikl}} - R_{li\Pi_{ikj}} - R_{lii \Pi_{kj}}) (d^D h)_{jkl} \]

Now,
\[ \sum R_{lii \Pi_{kj}} d^D h_{jkl} = \sum C_{kjm} R_{liim} d^D h_{jkl} \]
\[ = -\sum r_{lm} C_{kjm} d^D h_{jkl} \]
\[ = -\lambda \sum C_{kjm} d^D h_{jkm} \]
\[ = -\frac{\lambda}{2} \sum [C_{kjm} - C_{mjk} d^D h_{jkm}] \]
\[ = -\frac{\lambda}{2} d^D h_{jkm} d^D h_{jkm} \]

Therefore,
\[ -2\int_M \sum R_{lii \Pi_{kj}} d^D h_{jkl} d v_g = \|d^D h\|^2 \]

Using the previous lemma we have,
\[ -2\int_M \sum R_{lii \Pi_{kj}} d^D h_{jkl} d v_g = 2[\lambda \|Dh\|^2 + \lambda^2 \|h\|^2 - \lambda \langle h, \circ R(h) \rangle] \]

Now,
\[ \sum (R_{ij\Pi_{ikl}} - R_{li\Pi_{ikj}}) d^D h_{jkl} = \frac{1}{2} \sum D_k h_{mi} (R_{ijml} - R_{limj}) d^D h_{jkl} \]
Since $DR = 0$, $\sum_{m,i} D_k h_{mi} R_{ijml} = D_k \hat{R} (h)_{jl}$.

Therefore,

$$\sum (R_{ijkl} - R_{iklj}) d^D h_{jkl} = \sum D_k \hat{R} (h)_{jl} d^D h_{jkl} = (D \hat{R} (h), Dh) - D_k \hat{R} (h)_{jl} D_l h_{jk}$$

Applying integration by parts,

$$\int_M D_k \hat{R} (h)_{jl} D_l h_{jk} dv_g = - \int_M \hat{R} (h)_{jl} D^2 h_{ij} dv_g.$$ 

Since $\delta_g h = 0$, using Ricci identity,

$$\sum [D^2 h_{ij} - D^2 h_{ji}] = \sum [h_{ml} R_{ijml} + h_{mi} R_{ijml}] + D\delta_g h_{jl} = \lambda h_{jl} - \hat{R} (h)_{jl}$$

Therefore,

$$\langle \delta^D W, h \rangle = 2\lambda \|Dh\|^2 + \lambda^2 \|h\|^2 + \langle \hat{R} (h), D^* Dh \rangle - \| \hat{R} (h) \|^2$$

Hence the lemma follows. \hfill \Box

Combining Lemma 1-3 and using (2.1) we obtain that if $(M, g)$ is an irreducible symmetric space and $h$ is a symmetric two tensor with $\delta_g h = 0$ and $tr (h) = 0$ then

$$H(h, h) = p \|R\|^{p-2} \|D^* Dh\|^2 + \lambda \|Dh\|^2 - 3 \langle \hat{R} (h), D^* Dh \rangle + \frac{|R|^2}{n} \|h\|^2$$

$$- 2\lambda \langle h, \hat{R} (h) \rangle + 2 \| \hat{R} (h) \|^2 \] - p |R|^{p-2} \langle (\hat{R}) (h), h \rangle$$

Next we will study $H$ on the space of Kähler variations. For the definition of Kähler variation we refer to the introduction. A Kähler variation $h$ is characterized the following two equations.

(k1) $h(Jx, Jy) = h(x, y)$

(k2) $J(\Pi h (x, y)) = \Pi h (x, Jy)$

Consider a closed Kähler manifold $(M, g)$ with constant holomorphic sectional curvature. We can choose an orthonormal basis of the form $\{e_1, Je_1, ..., e_m, Je_m\}$. With respect to this basis $R$ is given by

$$R(e_i, e_j, e_k) = R(e_i, Je_j, Je_k) = R(e_i, Je_j, e_k) = 0 \text{ if } k \notin \{i, j\}. $$

$$R(e_i, e_j, e_i, e_j) = R(e_i, Je_j, e_i, Je_j) = R(Je_i, Je_j, Je_i, Je_j) = c.$$

$$R(e_i, Je_i, e_j, Je_j) = 2c \text{ for } i \neq j.$$

$$R(e_i, Je_i, e_i, Je_i) = 4c \text{ where } c \text{ is a real number.}$$
Similarly we have, 

\[ \delta_h = \text{some expression} \]

Hence

\[ \text{For detail computation we refer to Lemma 4.1 (i) in [SM]. Define a (4,0) curvature tensor } Q \text{ as} \]

\[ Q(v_q, v_i, v_j, v_k) = D^2_{v_i,v_j} h(v_q, v_k) + D^2_{v_q,v_k} h(v_i, v_j) - D^2_{v_i,v_q} h(v_j, v_k) - D^2_{v_j,v_k} h(v_q, v_i) \]

To prove the proposition using the formula (2.2) one needs to compute the term \( \langle (\tilde{R}_g)'(h), h \rangle \). Let \( h \) be a Kähler variation with \( tr(h) = 0 \) and \( \delta_g(h) = 0 \).

\[ \tilde{R}_{pq} = g^{i_1 i_2} g^{j_1 j_2} g^{k_1 k_2} R_{pij_1 k_1} R_{qij_2 k_2} \]

Differentiating each terms and evaluating it in an orthonormal basis \( \{v_i\} \) and using

\[ (g^{ij})' = -g^{im} h_{mn} g^{nj} \]

we have,

\[ (\tilde{R}_g)'(h)_{pq} = -h_{mn} (R_{pmij} R_{qmnj} + R_{pinj} R_{qimj} + R_{pijm} R_{qijn}) + (R'_g h)_{pijk} R_{qijk} + R_{pijk} (R'_g h)_{qijk} \]

For detail computation we refer to Lemma 4.1 (i) in [SM]. Define a (4,0) curvature tensor \( Q \) as

\[ Q(v_q, v_i, v_j, v_k) = D^2_{v_i,v_j} h(v_q, v_k) + D^2_{v_q,v_k} h(v_i, v_j) - D^2_{v_i,v_q} h(v_j, v_k) - D^2_{v_j,v_k} h(v_q, v_i) \]

Hence

\[ R'_g(h)(v_q, v_i, v_j, v_k) = \frac{1}{2} Q(v_q, v_i, v_j, v_k) + \frac{1}{2} \sum_m [h(v_k, v_m) R(v_q, v_i, v_j, v_m) - h(v_m, v_j) R(v_q, v_i, v_k, v_m)] \]

Next define a (2,0) tensor,

\[ S(v_p, v_q) = \frac{1}{2} \sum_{i,j,k} R(v_p, v_i, v_j, v_k) Q(v_q, v_i, v_j, v_k) \]

Therefore,

\[ \langle (\tilde{R}_g)'(h), h \rangle = -\sum h_{pq} h_{mn} (R_{pmij} R_{qmnj} + R_{pinj} R_{qimj} + R_{pijm} R_{qijn}) + 2(h, S) + \sum h_{pq} h_{mk} R_{pijk} R_{qijk} \]

Next we switch to a basis of the form \( \{e_1, Je_1, e_2, Je_2, \ldots\} \) to use the nice form of the curvature tensor.

\[ S(e_p, e_q) = \sum_i R(e_p, e_i, e_p, e_i) Q(e_q, e_i, e_p, e_i) + \sum_{p \neq i} R(e_p, Je_i, e_p, Je_i) Q(e_q, Je_i, e_p, e_i) \]

\[ + \sum_{p \neq i} R(e_p, Je_i, e_p, e_i) Q(e_q, e_i, e_p, e_i) + \sum_{p \neq i} R(e_p, Je_i, e_p, e_i) Q(e_q, e_i, e_p, e_i) \]

\[ = c \sum_{i \neq p} [Q(e_q, e_i, e_p, e_i) + Q(e_q, e_i, e_p, e_i) + Q(e_q, e_i, e_p, e_i) + Q(e_q, e_i, e_p, e_i) - Q(e_q, e_i, e_p, e_i) - Q(e_q, e_i, e_p, e_i)] + 4cQ(e_q, e_p, e_p, e_p) \]

Similarly we have,

\[ S(e_p, Je_q) = c \sum_{i \neq p} [Q(Je_q, e_i, e_p, e_i) + Q(Je_q, e_i, e_p, e_i) + Q(Je_q, e_i, e_p, e_i) \]

\[ - Q(Je_q, e_i, e_p, e_i) + 2Q(Je_q, e_p, e_i, e_i)] + 4cQ(e_q, e_p, e_p, e_p) \]
\[ S(Je_p, e_q) = c \sum_{i \neq p} [Q(e_q, Je_i, Je_p, Je_i) + Q(Je_q, e_i, Je_p, e - i) + Q(e_q, Je_i, e_p, e_i) \\
- Q(Je_q, e_i, e_p, Je_i) + 2Q(e_q, e_p, Je_i, e_i)] + 4cQ(e_q, e_p, Je_p, e_p) \]

\[ S(Je_p, Je_q) = c \sum_{i \neq p} [Q(Je_q, Je_i, Je_p, Je_i) + Q(Je_q, e_i, Je_p, e_i) + Q(Je_q, Je_i, e_p, e_i) \\
- Q(Je_q, e_i, e_p, Je_i) + 2Q(Je_q, e_p, Je_i, e_i)] + 4cQ(Je_q, e_p, Je_p, e_p) \]

Define \( r_Q \) by trace of \( Q \) in 2nd and 3rd entries. Therefore,

\[ (S, h) = c \sum h(e_p, e_q)[Q(e_q, e_i, Je_p, Je_i) - Q(e_q, Je_i, Je_p, e_i) + 2Q(e_q, Je_p, e_i, Je_i)] \\
+ c \sum h(Je_p, Je_q)[Q(Je_q, e_i, Je_p, e_i) - Q(Je_q, e_i, Je_p, e_i) + 2Q( Je_q, Je_p, e_i, Je_i)] \\
+ c \sum h(Je_p, Je_q)[Q(Je_q, Je_i, e_p, e_i) - Q( Je_q, Je_i, e_p, e_i) + 2Q( Je_q, e_p, Je_i, e_i)] \\
+ c \sum h(Je_p, e_q)[Q(e_q, Je_i, e_p, e_i) - Q( e_q, e_i, e_p, Je_i) + 2Q( e_q, e_p, Je_i, e_i)] \\
+ c(r_Q, h) \]

\[ \sum_i Q(e_q, Je_p, e_i, Je_i) = \sum_i \left[ D_{Je_p, e_i}^2 h(e_q, Je_i) + D_{Je_p, Je_i}^2 h(e_q, e_i) - D_{Je_q, e_i}^2 h(Je_p, Je_i) - D_{Je_q, Je_i}^2 h(e_q, e_i) \right] \]

By a simple calculation we have,

\[ D_{x,y}^2 h(Jw, Jz) = D_{x,y} h(w, z) \] (2.3)

Therefore,

\[ \sum_i \left[ D_{Je_p, e_i}^2 h(e_q, Je_i) - D_{Je_p, Je_i}^2 h(e_q, e_i) \right] = \sum_i \left[ -D_{Je_q, e_i}^2 h(Je_q, e_i) - D_{Je_p, Je_i}^2 h(Je_q, Je_i) \right] \\
= 2\delta_y^* \delta_g h(Je_p, Je_q) \\
= 0 \]

and

\[ \sum_i \left[ D_{Je_q, e_i}^2 h(Je_p, e_i) - D_{Je_q, Je_i}^2 h(Je_p, Je_i) \right] = -\sum_i \left[ D_{Je_q, e_i}^2 h(e_p, Je_i) + D_{e_q, e_i}^2 h(e_p, e_i) \right] \\
= 2\delta_y^* \delta_g h(e_q, Je_p) \\
= 0 \]

Hence,

\[ \sum_i Q(e_q, Je_p, e_i, Je_i) = 0 \]

Similarly,

\[ \sum_i Q(Je_q, Je_p, e_i, Je_i) = \sum_i Q(e_q, e_p, Je_i, e_i) = \sum_i Q(e_q, e_p, Je_i, e_i) = 0 \]
Next,
\[ \sum_i [Q(Je_q, e_i, Je_p, Je_i) - Q(Je_q, Je_i, Je_p, e_i)] \]
\[ = \sum_i [D_{e_i, Je_q} h(e_i, Je_i) + D_{Je_q, Je_i} h(e_i, Je_p) - D_{e_i, Je_i} h(Je_q, Je_p)] \]
\[ + D_{Je_i, e_i} h(Je_q, Je_p) - D_{Je_q, e_i} h(Je_i, Je_p) - D_{Je_i, Je_p} h(Je_q, e_i)] \]

Applying Ricci identity we have,
\[ \sum_i [D_{e_i, Je_q} h(Je_q, Je_i) - D_{e_i, Je_i} h(Je_q, Je_p)] \]
\[ = \sum_{i,j} [h(Je_j, e_p)R(Je_i, e_i, eq, Je_j) + h(Je_j, eq)R(e_i, e_i, ep, Je_i)] \]
\[ = \sum_i \sum_j [h(Je_q, e_p)R(Je_i, e_i, eq, Je_q) + h(Je_p, eq)R(Je_i, e_i, ep, Je_p)] \]
\[ = 0 \]
\[ \sum_i [D_{e_i, ep} h(Je_q, Je_i) - D_{Je_i, ep} h(Je_q, e_i)] \]
\[ = \sum_i [D_{e_i, ep} h(Je_q, Je_i) - D_{Je_i, ep} h(Je_q, e_i)] \]
\[ + \sum_i [D_{Je_i, e_i} h(Je_q, e_i) - D_{Je_i, ep} h(Je_q, e_i)] \]
\[ = \sum_{i,j} [h(e_i, e_j)R(e_i, ep, Je_q, e_j) + h(Je_q, e_j)R(e_i, ep, Je_i, e_j)] \]
\[ + \sum_{i,j} [h(e_i, Je_q)R(Je_p, Je_i, Je_q, Je_j) + h(e_j, Je_q)R(Je_p, Je_i, e_i, e_j)] \]
\[ = -2c(m + 1)h(e_p, Je_q) \]
\[ \sum_i [D_{Je_i, e_i} h(e_i, Je_p) - D_{Je_i, e_i} h(Je_i, Je_p)] = -\sum_i [D_{Je_i, e_i} h(Je_i, Je_p) + D_{Je_q, e_i} h(e_i, Je_p)] \]
\[ = \delta_g \delta_g h(Je_p, Je_q) \]
\[ = 0 \]

Hence,
\[ \sum_i [Q(Je_q, e_i, Je_p, Je_i) - Q(Je_q, Je_i, Je_p, e_i)] = -2c(m + 1)h(e_p, Je_q) \]

Following similar computation we have,
\[ \sum_i [Q(e_q, e_i, Je_p, Je_i) - Q(e_q, Je_i, Je_p, e_i)] = -2c(m + 1)h(e_p, e_q) \]
\[ \sum_i [Q(Je_q, Je_i, e_p, e_i) - Q(Je_q, e_i, ep, Je_i)] = -2c(m + 1)h(Je_p, Je_q) \]

and
\[ \sum_i [Q(e_q, Je_i, e_p, e_i) - Q(e_q, e_i, ep, Je_i)] = -2c(m + 1)h(Je_p, e_q) \]
Hence

\( (S, h) = c(r_Q, h) - 2c^2(m + 1)|h|^2 \)

Next we will compute the term \((r_Q, h)\).

\[
\begin{align*}
 r_Q(e_p, e_q) &= \sum_i [Q(e_p, e_i, e_q, e_i) + Q(e_p, Je_i, e_q, Je_i)] \\
 &= \sum_i [D^2_{e_i, e_q} h(e_p, e_i) + D^2_{e_p, e_i} h(e_i, e_q) + D^2_{Je_i, e_q} h(e_p, Je_i) + D^2_{e_p, Je_i} h(e_q, e_i)] \\
 - \sum_i [D^2_{e_p, e_q} (e_i, e_i) + D^2_{e_i, e_i} h(e_p, e_q) + D^2_{e_p, e_q} h( Je_i, e_i)] \\
- \sum_i [D^2_{e_p, e_i} h(e_i, e_q) + D^2_{e_p, Je_i} h( e_q, Je_i)] = 2\delta_y^*\delta_y h(e_p, e_q) = 0 \\
\end{align*}
\]

Now applying Ricci identity to the remaining terms after adding the term \(2\delta_y^*\delta_y h = 0\), we have

\[
\begin{align*}
\sum_i [D^2_{e_i, e_q} h(e_p, e_i) + D^2_{Je_i, e_q} h(e_p, Je_i)] \\
= \sum_i [R(e_i, e_q, e_p, e_j)h(e_j, e_i) + R(e_i, e_q, e_i, e_j)h(e_j, e_p)] \\
+ R(Je_i, e_q, e_p, Je_j)h( Je_j, Je_i) + R(Je_i, e_q, Je_i, e_j)h(e_j, e_p)] \\
= (\lambda + 6c)h(e_p, e_q) \\
= 2c(m + 4)h(e_p, e_q)
\end{align*}
\]

Hence

\[
 r_Q(e_p, e_q) = D^* Dh(e_p, e_q) + 2c(m + 4)h(e_p, e_q)
\]

Similarly computing the other coefficients of \(r_Q\) we get

\[
 r_Q = D^* Dh + 2(m + 4)h
\]

Therefore,

\[
 (S, h) = c|Dh|^2 + 6c^2|h|^2
\]

Next a simple calculation gives,

\[
 - \sum h_{pq} h_{mn} (R_{pimj} R_{qni} + R_{pimj} R_{qni} + R_{pijm} R_{qijn}) + \sum h_{pq} h_{mn} R_{pimj} R_{qijn} = - \sum [h_{mn} h_{pq} R_{pimj} R_{qijn} + h_{mn} h_{pq} R_{pimj} R_{qjni}]
\]
Since $h, R$ are invariant under the action of $J$,

$$\sum h_{mn}h_{pq}R_{prij}R_{qijn}$$

\[= 4\sum h(e_p, e_q)h(e_m, e_n)[R(e_p, e_m, e_i, e_j)e_i, e_n, e_i, e_j] + R(e_p, J e_m, e_i, J e_j)R(e_q, J e_n, e_i, J e_j)]
\[+ 4\sum h(J e_p, e_q)h(J e_m, e_n)[R(e_p, e_m, e_i, e_j)e_i, e_n, e_i, e_j] + R(J e_p, e_m, J e_i, e_j)R(e_q, e_n, J e_i, J e_j)]
\[= 16(m + 1)c^2 \sum [h^2(e_p, e_q) + h^2(J e_p, e_q)]
\[= 8c^2(m + 1)|h|^2$$

Therefore,

$$\langle (\tilde{R}^i(h), h) = 2c||Dh||^2 - 12c^2m||h||^2$$

From the formula (2.2) we obtain,

$$H(h, h) = p|R|^{p-2}||D^* Dh||^2 + 2c(m - 3)||Dh||^2 + 4c^2m(4m + 5)||h||^2$$

\[= p|R|^{p-2}||D^* Dh + (m - 3)ch||^2 + (15m^2 + 14m + 6)c^2||h||^2$$

where $k$ is a positive constant. Hence, the proof of the Proposition follows. \hfill \Box

2.3. Proof of the main Theorem: Next we prove the existence of a local minimizing neighborhood around $g$. Let $\mathcal{H}$ denote the space of holomorphic vector fields on $M$. If $(M, g)$ is a closed Kähler Einstein manifold then by a theorem by Lichnerowicz the dimension of $\mathcal{H}$ is finite. If $(M, g)$ is a Kähler manifold with constant negative holomorphic sectional curvature then $\mathcal{H} = 0$ [AB] Proposition 2.138.

**Lemma 4.** Let $(M, g)$ be a Kähler Einstein manifold with positive scalar curvature. Then

$$\mathcal{K}_g = \mathbb{R} \oplus \delta^*(\mathcal{H}) \oplus (\mathcal{K}_g \cap \delta^{-1}(0) \cap tr^{-1}(0)).$$

**Proof.** It is easy to see that $\delta^*(\mathcal{H}) \subset \mathcal{K}_g$. Let $h \in \mathcal{K}_g$ and $x, y, z$ are three vector fields. Then

$$\Pi_h(J x, J y) = -\Pi_h(x, y).$$

Hence

$$\langle \Pi_h(e_i, e_i), z \rangle + \langle \Pi_h(J e_i, J e_i), z \rangle = 0.$$
This implies,
\[ D_zh(e_i, e_i) + D_z(Je_i, Je_i) = 2[D_{Je_i}h(z, Je_i) + D_{e_i}h(z, e_i)] \]

Therefore,
\[ dtrh(z) + 2\delta g h(z) = 0 \] (2.4)

When \( h = \delta_g^* \omega \) for some one form \( \omega \) then
\[ 2\delta g \delta_g^* \omega = d\delta \omega \] (2.5)

Any one form \( \omega \) also satisfies the following identity.
\[ 2\delta g \delta_g^* \omega + \delta d \omega = 2D^*D \omega \] (2.6)

Using Böchner technique on the space of one forms we have
\[ \Delta \omega = D^*D \omega + \lambda \omega. \] (2.7)

where \( \lambda \) denotes the Einstein constant. Lichnerowicz’s criterion for holomorphic vector fields says that \( \omega^\sharp \) is a holomorphic vector field if and only if
\[ \frac{1}{2} \Delta \omega = \lambda \omega \] (2.8)

Combining the equations (2.5)-(2.8) we have that \( \delta_g^* \omega \) is a Kähler variation if and only if \( \omega^\sharp \) is a holomorphic vector field. We also have that if \( \omega^\sharp \) is a holomorphic vector field then so is \( \delta_g^* \omega \). (2.4) implies that if \( fg \) is a Kähler variation for some smooth function \( f \) on \( M \) then \( f \) is a constant function. Now the proof follows from the decomposition given in [AB] Lemma 4.57.

The following lemma is analogous to the Lemma 2.10 in [GV].

**Lemma 5.** For Kähler metric \( \tilde{g} = g + \theta_1 \) in a sufficiently small \( C^{l+1,\alpha} \)-neighborhood of the Kähler Einstein metric \( g \) \( (l \geq 1) \), there is an automorphism \( \phi \) of \( M \) and a constant \( c \) such that

\[ \tilde{\theta} = e^c \phi^* \tilde{g} - g \]

satisfies, \( \delta_g \tilde{\theta} = 0 \) and \( tr(\tilde{\theta}) = 0 \)

**Proof.** Consider the map \( \mathcal{N} : \mathcal{H} \times \mathbb{R} \times \mathcal{K} \to \mathcal{H} \times \mathcal{R} \) given by,

\[ \mathcal{N}(x, t, \theta) = \mathcal{N}_0(x, t) = (\delta_g [\phi_{x,1}^*(g + \theta)], e^t \int_M tr_g [\phi_{x,1}^*(g + \theta)]dv_g - nV(g)) \]

where \( \phi_{x,1} \) denotes the diffeomorphism obtained by following the flow generated by the vector field \( x \) for unit time. Linearizing this map in \( (x, t) \) at \( (0, 0, 0) \) we obtain,

\[ \mathcal{N}'_0(y, s) = (\delta_g L_g(y), snV(g)) \]

\[ = \frac{1}{2} \delta_g \delta_g^*(y), snV(g) \]

where \( L \) denotes Lie derivative. It is easy to see that \( \mathcal{N}' \) is surjective. By implicit function theorem, given a \( \theta_1 \) small enough there exists \( x \in \mathcal{H} \) and a real number \( t \) such that

\[ \delta_g [\phi_{x,1}^*(g + \theta_1)] = 0 \quad \text{and} \quad e^t \int_M tr_g [\phi_{x,1}^*(g + \theta_1)] = nV(g) \]

Let \( \phi = \phi_{x,1} \) and \( \tilde{\theta} = e^t \phi^*(g + \theta_1) - g \). Then \( \phi \) and \( \tilde{\theta} \) satisfies the conditions given in the Lemma. □
Let $\mathcal{M}$ denote the space of Riemannian metrics on $M$, $\mathcal{R}_p$ denote the Riemannian functional defined by volume normalization of $\mathcal{R}_p$ on $\mathcal{M}$ and $\tilde{H}_g$ its 2nd derivative at $g$. Let $\mathcal{W} = (\mathcal{K}_g \cap \delta_{\tilde{g}}^{-1}(0) \cap \text{tr}^{-1}(0))$. To complete the proof of the main theorem we recall some results from [SM].

Let $g$ be a Riemannian metric on $M$ with unit volume. There exists a neighborhood $U$ of $g$ in $\mathcal{M}_1$ such that for any $\tilde{g} \in U$ and $h \in \mathcal{W}$,

$$|\tilde{H}_g(h, h) - \tilde{H}_g(h, h)| \leq C \|\tilde{g} - g\|_{C^{2,\alpha}}^4 \|h\|_{L^2}^2$$

where

$$\|h\|_{L^2}^2 = \|D^2h\|^2 + \|Dh\|^2 + \|h\|^2.$$

Using the Proposition one can prove that if $(M, g)$ is a metric with constant holomorphic sectional curvature then there exists a positive constant $k_1$ such that for every $h \in (\mathcal{K}_g \cap \delta_{\tilde{g}}^{-1}(0) \cap \text{tr}^{-1}(0))$

$$\tilde{H}_g(h, h) \geq k_1 \|h\|_{L^2}^2.$$

Now consider a neighborhood $U$ of $g$ in the space of Kähler metrics where Lemma 5 and 6 holds. Let $\tilde{g} \in U$. Using Lemma 5 we have a Kähler metric $g_0$ in $U$ such that $g - g_0 \in \mathcal{W}$. Now consider $\gamma(t) = tg_0 + (1 - t)g$. Since $\tilde{H}_\gamma(t)$ restricted to the space $\mathcal{W}$ is positive definite integrating $\tilde{H}$ along $\gamma(t)$ we obtain the main theorem. For detail description we refer to the proof of Proposition 3 in [SM].

**Remark :** If $(M, g)$ is a Kähler manifold with constant negative holomorphic sectional curvature then there are finitely many biholomorphisms of $M$. So the space of Kähler metrics intersect the orbit of group of diffeomorphisms of $g$ in finitely many points. In this case Lemma 4 and 5 are not required to prove the main theorem.

**Acknowledgement:** I wish to thank Harish Seshadri for several discussions. This work is financially supported by UGC Center for Advanced Studies.

**References**

[AB] Arthur L. Besse, *Einstein manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3)[Results in Mathematics and Related Areas (3)], Volume 10, Springer-Verlag, Berlin, (1987).

[GV] Matthew J. Gursky and Jeff A. Viaclovsky, *Rigidity and stability of Einstein metrics for quadratic functionals*, arXiv:1105.4648v1 [math.DG] 23 May 2011.

[MB] M. Berger, *Quelques formules de variation pour une structure riemannienne*, Ann. Sci. Ecole Norm. Sup. 4ème série, Volume 3 (1970), 285-294.

[SM] Soma Maity, *On the stability of the $L^p$-norm of curvature tensor*, arXive:1201.1691[math.DG] 15 March 2012.

**Department of Mathematics, Indian Institute of Science, Bangalore-12, India**

**E-mail address:** somamaity@math.iisc.ernet.in