On design-theoretic aspects of Boolean and vectorial bent functions

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Abstract—There are two construction methods of designs from \((n,m)\)-bent functions, known as translation and addition designs. In this paper we analyze, which equivalence relation for Boolean bent functions, i.e. \((n,1)\)-bent functions, and vectorial bent functions, i.e. \((n,m)\)-bent functions with \(2 \leq m \leq n/2\), is coarser: extended-affine equivalence or isomorphism of associated translation and addition designs. First, we observe that similar to the Boolean bent functions, extended-affine equivalence of vectorial \((n,m)\)-bent functions and isomorphism of addition designs are the same concepts for all even \(n\) and \(m \leq n/2\). Further, we show that extended-affine inequivalent Boolean bent functions in \(n\) variables, whose translation designs are isomorphic, exist for all \(n \geq 6\). This implies, that isomorphism of translation designs for Boolean bent functions is a coarser equivalence relation than extended-affine equivalence. However, we do not observe the same phenomenon for vectorial bent functions in a small number of variables. We classify and enumerate all vectorial bent functions in six variables and show, that in contrast to the Boolean case, one cannot exhibit isomorphic translation designs from extended-affine inequivalent vectorial \((6,m)\)-bent functions with \(m \in \{2,3\}\).

Index Terms—Bent Function, Extended-Affine Equivalence, Combinatorial Design, Linear Code, Difference Set, Relative Difference Set.

I. INTRODUCTION

Boolean and vectorial bent functions, also known as perfect nonlinear functions [19], [25], are special mappings of finite fields, which have the maximum Hamming distance from the set of all affine functions. Being optimal discrete structures, they have numerous applications in combinatorics, cryptography, coding and design theory. Particularly, the interaction between design theory and the theory of perfect nonlinear functions is of special interest. For instance, any new construction of bent functions may lead to a new construction of certain designs. On the other hand, combinatorial invariants of incidence structures constructed from functions over finite fields serve as good distinguishers between inequivalent functions and even classes of functions [10], [15], [28]. Before we briefly mention the main constructions of designs from bent functions and their most notable applications, we would like to point the reader’s attention, that the notation we use below for translation and addition designs of bent functions will be introduced in details in the following sections.

A translation design of a function \(F: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m\) (not necessarily perfect nonlinear) is defined as the development \(\text{dev}(A)\) of a certain set \(A\), which is constructed from the function \(F\) and has a nice combinatorial structure [9], [10]. The classical choice of a set \(A\) for a Boolean bent function \(f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2\) is either the support \(D_f\), a \((2^n, 2^{n-1} + 2^{n/2-1}, 2^{n-2} + 2^{n/2-1})\) difference set, or the graph \(G_f\), a \((2^n, 2^n, 2^{n-1})\) relative difference set, while for the vectorial function \(F: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m\) one considers only the graph \(G_F\), which is a \((2^n, 2^m, 2^n, 2^{m-n})\) relative difference set. The addition design \(D(F)\) of a perfect nonlinear function \(F: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m\) is defined as the design, supported by codewords of the minimum weight of the first-order Reed-Muller code, appended by the function \(F\), see [1], [6], [8], [11]. In this way, one can construct three designs from a Boolean bent function \(f\) on \(\mathbb{F}_2^n\): two translation designs \(\text{dev}(D_f), \text{dev}(G_f)\) and one addition design \(D(f)\). However, for a vectorial bent function \(F: \mathbb{F}_2^n \rightarrow \mathbb{F}_2^m\) there are only two options: one translation design \(\text{dev}(G_F)\) and one addition design \(D(F)\).

So far, translation and addition designs are used in the context of extended-affine equivalence of Boolean bent functions, which is known as the most general equivalence relation for Boolean functions. For instance, Weng et al. in [28, Theorem 5.11] used the 2-rank of the translation design \(\text{dev}(D_f)\) to prove, that almost every Desarguesian partial spread bent function is not extended-affine equivalent to a Maiorana-McFarland bent function. Recently, the authors of this paper in [21] used algebraic invariants of \(\text{dev}(D_f)\) and \(\text{dev}(G_f)\) to show inequivalence of certain homogeneous cubic bent functions. Bending in [1, Corollary 10.6] proved that extended-affine equivalence of bent functions coincides with isomorphism of addition designs. As a useful application of this result, one can use computer algebra systems, e.g. Magma [3] and GAP [26], to check effectively the equivalence of bent functions in a small number of variables via the isomorphism of addition designs. Bending in [1, Theorems 8.4, 8.13] used invariants of the addition design \(D(f)\) to derive a necessary condition for a bent function \(f\) on \(\mathbb{F}_2^n\) to be extended-affine equivalent to a Maiorana-McFarland bent function. Despite the fact, that the translation and the addition designs of vectorial bent functions are defined in the same way as for Boolean bent functions, there are no similar applications for vectorial bent functions so far.

The main goal of this paper is to compare Boolean and vectorial bent functions from the point of view of differences between extended-affine equivalence and isomorphism of the addition and the translation designs. For instance, in the Boolean bent case isomorphism of addition designs carries all the information about the extended-affine equivalence of Boolean bent functions, and vice versa. Our first objective is to prove, that the same phenomenon occurs for addition designs.
of vectorial bent functions. In general, isomorphic incidence structures do not necessarily come from equivalent difference sets; Edel and Pott in [9, Example 1] observed an example of extended-affine inequivalent Boolean bent functions \( f \) and \( f' \) on \( \mathbb{F}_2^n \), whose translation designs \( \text{dev}(D_f) \) and \( \text{dev}(D_{f'}) \) are isomorphic. However, these incidence structures do not have a proper generalization for the vectorial case. Our second objective is to extend the observation of Edel and Pott for translation designs \( \text{dev}(G_f) \) and \( \text{dev}(G_{f'}) \) of Boolean functions, i.e. find a pair of Boolean bent functions \( f, f' \) on \( \mathbb{F}_2^n \) for any \( n \geq 6 \), which are extended-affine inequivalent but their translation designs \( \text{dev}(G_F) \) and \( \text{dev}(G_{F'}) \) are isomorphic. Since the translation design \( \text{dev}(G_f) \) is invariant for the extended-affine equivalence, it will imply that isomorphism of translation designs \( \text{dev}(G_F) \) and \( \text{dev}(G_{F'}) \) of Boolean functions \( f \) and \( f' \) on \( \mathbb{F}_2^n \) is a coarser equivalence relation than extended-affine equivalence. The third objective of this paper is to show, that in contrast to the Boolean case, isomorphism of designs \( \text{dev}(G_F) \) and \( \text{dev}(G_{F'}) \) of vectorial bent functions in six variables coincides with the extended-affine equivalence.

After introducing the necessary background on bent functions and designs in Subsection I-A, we consider addition designs of vectorial bent functions. In Section II we prove that similarly to Boolean bent functions, extended-affine equivalence of vectorial bent functions coincides with the isomorphism of addition designs. In this way, we solve a recent open problem, addressed by Ding, Munemasa and Tonchev in [8, Note 24]. In Section III we first provide examples of extended-affine inequivalent Boolean bent functions \( f, f' \) on \( \mathbb{F}_2^n \), whose translation designs \( \text{dev}(G_f) \) and \( \text{dev}(G_{f'}) \) are isomorphic. Consequently, we prove that for any \( n \geq 6 \) the isomorphism of translation designs \( \text{dev}(G_f) \) and \( \text{dev}(G_{f'}) \) of Boolean bent functions \( f \) and \( f' \) on \( \mathbb{F}_2^n \) is a coarser than extended-affine equivalence. In Section IV we show that the similar phenomenon does not occur for vectorial bent functions in a small number of variables. We classify and enumerate all vectorial bent functions in six variables and observe, that in contrast to the Boolean case, vectorial bent functions \( F \) and \( F' \) on \( \mathbb{F}_2^n \) are extended-affine equivalent if and only if their translation designs \( \text{dev}(G_F) \) and \( \text{dev}(G_{F'}) \) are isomorphic. In Section V we give concluding remarks and raise some open problems on bent functions and their designs. In Appendix A we list algebraic normal forms of the obtained representatives of equivalence classes of vectorial bent functions in six variables together with their invariants.

### A. Preliminaries

Let \( \mathbb{F}_2 = \{ 0, 1 \} \) be the finite field with two elements and let \( \mathbb{F}_2^n \) be the vector space of dimension \( n \) over \( \mathbb{F}_2 \). Mappings \( F : \mathbb{F}_2^n \to \mathbb{F}_2^m \) are called \((n,m)\)-functions. The single-output case \( m = 1 \) corresponds to Boolean functions, while in the multi-output case \( m \geq 2 \) one deals with vectorial functions. The graph \( G_F \) of an \((n,m)\)-function \( F \) is the set \( G_F := \{ (x, F(x)) : x \in \mathbb{F}_2^n \} \). The support \( D_f \) of a Boolean function \( f \) on \( \mathbb{F}_2^n \) is the set \( D_f := \{ x \in \mathbb{F}_2^n : f(x) = 1 \} \). Any Boolean function \( f : \mathbb{F}_2^n \to \mathbb{F}_2 \) can be uniquely represented as a multivariate polynomial in the ring \( \mathbb{F}_2[x_1, \ldots, x_n]/(x_1 \oplus x_2^2, \ldots, x_n \oplus x_n^2) \). This representation is called the algebraic normal form (ANF for short) and given by

\[
f(x) = \bigoplus_{v \in \mathbb{F}_2^n} c_v \left( \prod_{i=1}^n x_i^{v_i} \right),
\]

where \( x = (x_1, \ldots, x_n) \in \mathbb{F}_2^n \), \( c_v \in \mathbb{F}_2 \) and \( v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n \).

There are several criteria, which an \((n,m)\)-function has to satisfy in order to be considered as a good cryptographic primitive, among them are high algebraic degree and high nonlinearity. The algebraic degree of a Boolean function \( f : \mathbb{F}_2^n \to \mathbb{F}_2 \), denoted by \( \text{deg}(f) \), is the algebraic degree of its ANF as a multivariate polynomial. This definition can essentially be extended to the vectorial case. Any vectorial function \( F : \mathbb{F}_2^n \to \mathbb{F}_2^m \) can be uniquely (up to the choice of basis of \( \mathbb{F}_2^m \)) associated with \( m \) coordinate Boolean functions \( f_i : \mathbb{F}_2^n \to \mathbb{F}_2 \) for \( 1 \leq i \leq m \) as a column-vector \( F(x) := (f_1(x), \ldots, f_m(x))^T \). In this way, the algebraic degree of an \((n,m)\)-function \( F \) is defined by \( \text{deg}(F) := \max_{1 \leq i \leq m} \text{deg}(f_i) \). Clearly, the algebraic degree of an \((n,m)\)-function \( F \) can not exceed \( n \).

The nonlinearity of a Boolean function \( f : \mathbb{F}_2^n \to \mathbb{F}_2 \) is a measure of distance between the function \( f \) and the set of all affine functions \( A_n := \{ : \mathbb{F}_2^n \to \mathbb{F}_2 \} \). Formally, it is defined as

\[
\text{nl}(f) := \min_{l \in A_n} d_H(f, l),
\]

where \( d_H(f, g) := |\{ x \in \mathbb{F}_2^n : f(x) \neq g(x) \}| \) is the Hamming distance between functions \( f \) and \( g \) on \( \mathbb{F}_2^n \). This definition can be extended for the vectorial case using the notion of component functions. Recall that for an \((n,m)\)-function \( F \), the component function \( F_b \) is the Boolean function \( F_b : \mathbb{F}_2^n \to \mathbb{F}_2 \), given by \( F_b(x) := \langle b, F(x) \rangle_m \), where \( \langle , \rangle_m \) is a non-degenerate bilinear form on \( \mathbb{F}_2^m \). In this way, the nonlinearity of a vectorial \((n,m)\)-function \( F \) is the minimum nonlinearity of all its component functions and is given by

\[
\text{nl}(F) := \min_{l \in A_n, b \in \mathbb{F}_2^m} d_H(F_b, l).
\]

The main tool to compute the nonlinearity of an \((n,m)\)-function \( F \) is the Walsh transform \( W_F : \mathbb{F}_2^n \times \mathbb{F}_2^m \to \mathbb{Z} \), defined by \( W_F(a, b) := W_{F_{b}}(a) \) and

\[
W_{F_b}(a) := \sum_{x \in \mathbb{F}_2^n} (-1)^{F_b(x) \langle b, a \rangle_m},
\]

for \( a \in \mathbb{F}_2^n \) and \( b \in \mathbb{F}_2^m \). Using the Walsh transform, the nonlinearity of an \((n,m)\)-function \( F \) can be computed as

\[
\text{nl}(F) := 2^{n-1} - \frac{1}{2} \max_{a \in \mathbb{F}_2^n, b \in \mathbb{F}_2^m} |W_F(a, b)|.
\]

The upper bound on nonlinearity of an \((n,m)\)-function \( F \) is given by \( \text{nl}(F) \leq 2^{n-1} - 2^{\frac{n-1}{2}} \) and functions, achieving this bound are called perfect nonlinear.

**Definition I.1.** An \((n,m)\)-function \( F \) is called bent or perfect nonlinear if \( \text{nl}(F) = 2^{n-1} - 2^{\frac{n-1}{2}} \).

**Remark I.2.** Throughout the paper we will call single-output bent functions, i.e. \( m = 1 \) Boolean bent functions, while multi-output bent functions, i.e. \( m \geq 2 \), vectorial bent functions. One can show that an \((n,m)\)-function \( F \) is bent if and only if for all \( a \in \mathbb{F}_2^n \) and all \( b \in \mathbb{F}_2^m \) with \( b \neq 0 \) the Walsh transform satisfies \( W_F(a, b) = \pm 2^{n/2} \). Boolean bent functions exist on \( \mathbb{F}_2^n \) if and only if \( n \) is even. Vectorial \((n,m)\)-bent functions
exist if and only if $m \leq n/2$, as it was shown by Nyberg in [19]. The algebraic degree of an $(n, m)$-bent function is at most $n/2$, see [25].

On the set of all $(n, m)$-functions we introduce an equivalence relation in the following way. We say that two $(n, m)$-functions $F, F'$ are extended-affine equivalent (EA-equivalent for short), if there exist a linear permutation $A_1$ of $\mathbb{F}_2^m$, an affine permutation $A_2$ of $\mathbb{F}_2^n$ and an affine function $A_3$: $\mathbb{F}_2^n \to \mathbb{F}_2^n$ such that $F = A_1 \circ F' \circ A_2 \circ A_3$. Further we will study equivalence of $(n, m)$-functions in connection with the equivalence of the associated linear codes and designs. We refer to [2] and [7] for extensive references on the subject.

A linear code $C$ over $\mathbb{F}_2$ is a vector subspace $C \subseteq \mathbb{F}_2^n$. Elements of a linear code $C$ are called codewords. The number of nonzero coordinates of a codeword $c \in C$ is called the weight of $c$ and is denoted by $\text{wt}(c)$. The minimum distance of a linear code is the minimum weight of its nonzero codewords. We say, that $C \subseteq \mathbb{F}_2^n$ is an $[n, k, d]$-linear code, if $C$ has dimension $k$ and the minimum distance $d$. The support of a codeword $c = (c_1, \ldots, c_n) \in C$ is defined by $\text{supp}(c) = \{1 \leq i \leq n : c_i \neq 0\} \subseteq \{1, 2, 3, \ldots, n\}$. Two linear codes $C$ and $C'$ are permutation equivalent if there is a permutation of coordinates which sends the code $C$ to $C'$.

An incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B})$ is called a 2-$(\nu, k, \lambda)$ design, if the cardinality of the point set $\mathcal{P}$ is $\nu$, the set of blocks $\mathcal{B}$ is a collection of $k$-subsets of $\mathcal{P}$ and every 2-subset of points $\{p, q\} \subset \mathcal{P}$ is contained in exactly $\lambda$ blocks of $\mathcal{B}$. There are several generalizations of 2-designs, one of them is a divisible design. For instance, the incidence structure $\mathcal{I} = (\mathcal{P}, \mathcal{B})$ is called a $(\mu, \nu, k, \lambda)$ divisible design, if the point set $\mathcal{P}$ with $|\mathcal{P}| = \nu = \mu \cdot \nu$ elements is divided into $\mu$ point classes of size $\nu$ each, the block set $\mathcal{B}$ is a collection of $k$-subsets of $\mathcal{P}$ and the number of blocks, containing any 2-subset $\{p, q\} \subset \mathcal{P}$ depends on the relation between points $p$ and $q$ in the following way: if $p$ and $q$ are in the same point class, the 2-subset $\{p, q\}$ is not contained in a block; otherwise it is contained in exactly $\lambda$ blocks. All the information about an incidence structure $\mathcal{I}$ is contained in its incidence matrix $M(\mathcal{I}) = (m_{i,j})$, which is a binary $b \times v$ matrix with $m_{i,j} = 1$ if $p_j \in B_i$ and $m_{i,j} = 0$ otherwise. In this way, two incidence structures $\mathcal{I}$ and $\mathcal{I}'$ are isomorphic, if there exist permutation matrices $P$ and $Q$ such that $M(\mathcal{I}) = P \cdot M(\mathcal{I}') \cdot Q$.

II. ADDITION DESIGNS OF BOOLEAN AND VECTORIAL BENT FUNCTIONS

Dillon and Schatz in [6] and Bending in [1, Corollary 10.6] proved independently that Boolean bent functions $f$ and $f'$ on $\mathbb{F}_2^n$ are extended-affine equivalent if and only if their addition designs $\mathbb{D}(f)$ and $\mathbb{D}(f')$ are isomorphic. In this section we show that similar to the Boolean case, vectorial $(n, m)$-bent functions $F$ and $F'$ are extended-affine equivalent if and only if their addition designs $\mathbb{D}(F)$ and $\mathbb{D}(F')$ are isomorphic. We also use the result of Bending [1, Theorem 9.6] to show, how one can construct an incidence matrix of the addition design of a vectorial bent function with the help of component functions and their duals. First, we give the definition of the addition design of a bent function.

**Definition II.1.** Let $F$ be an $(n, m)$-bent function and $C_F$ be an $(n + m + 1) \times 2^n$-matrix over $\mathbb{F}_2$, given by

$$C_F = \begin{pmatrix} \ldots & 1 & \ldots \\ \ldots & x & \ldots \\ \ldots & F(x) & \ldots \end{pmatrix}_{x \in \mathbb{F}_2^n}. \quad (II.1)$$

We define the linear code $C(F)$ over $\mathbb{F}_2$, as the row space of the matrix $C_F$. It is not difficult to check, that the linear code $C(F)$ is a $[2^n, n + m + 1, 2^{n-1} - 2^{n/2-1}]$-code. Further, we define two sets $\mathcal{P} = \{x : x \in \mathbb{F}_2^n\}$ and $\mathcal{B} = \{\text{supp}(f) : f \in C(F), \text{wt}(f) = 2^{n-1} - 2^{n/2-1}\}$. The addition design of an $(n, m)$-bent function $F$ is the incidence structure $\mathbb{D}(F) = (\mathcal{P}, \mathcal{B})$, which is supported by codewords of the minimum weight of the linear code $C(F)$. Ding, Munemasa and Tonchev in [8, Theorem 11] proved that the addition design $\mathbb{D}(F)$ of an $(n, m)$-bent function $F$ is a $2 \cdot (2^n, 2^{n-1} - 2^{n/2-1}, 2^{m-1} \cdot (2^{n-2} - 2^{n/2-1}))$ design.

**Remark II.2.** The designs $\mathbb{D}(F)$ for vectorial $(n, m)$-bent functions $F$ were introduced recently by Ding, Munemasa and Tonchev in [8]. Throughout the paper we will call these objects “addition design”, motivated by terminology introduced by Bending in his thesis [1] for the designs $\mathbb{D}(f)$ of Boolean bent functions $f$ on $\mathbb{F}_2^n$. The term “addition” means, that blocks of the design $\mathbb{D}(f)$ are formed by supports of bent functions, obtained via addition of the original bent function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ to those affine functions $l : \mathbb{F}_2^n \to \mathbb{F}_2$, which satisfy $\text{wt}(f \oplus l) = 2^{n-1} - 2^{n/2-1}$.

**Remark II.3.** An incidence matrix of the addition design $\mathbb{D}(F)$ of an $(n, m)$-vectorial bent function, similarly to the Boolean case, can be constructed without the use of the linear code $C(F)$. Recall that the dual of a Boolean bent function $f : \mathbb{F}_2^n \to \mathbb{F}_2$ is a bent function $\bar{f} : \mathbb{F}_2^n \to \mathbb{F}_2$, defined by $W_f(a) = 2^{n/2}(1/\sqrt{a})$. Bending in [1, Theorem 9.6] proved, that an incidence matrix of the design $\mathbb{D}(f)$ can be constructed with the help of the dual function $\bar{f}$ in the following way (without loss of generality we assume, that $f(0) = 0$):

$$M(\mathbb{D}(f)) = (m_{x,y})_{x,y \in \mathbb{F}_2^n},$$

$$m_{x,y} = f(x) \oplus \bar{f}(y) \oplus (x,y)_{l(0)} \oplus \bar{f}(0). \quad (II.2)$$

In this way, an incidence matrix of the addition design $\mathbb{D}(F)$ of an $(n, m)$-bent function $F$ (w.l.o.g. we assume $F(0) = 0$) can be constructed as the concatenation of incidence matrices of addition designs $\mathbb{D}(F_b)$ of the nonzero component functions $F_b$ of $F$, namely:

$$M(\mathbb{D}(F)) = \begin{bmatrix} M(\mathbb{D}(F_{b_1})) \\ \vdots \\ M(\mathbb{D}(F_{b_n})) \end{bmatrix}. \quad (II.3)$$

Recently, Ding, Munemasa and Tonchev conjectured [8, Note 24], that extended-affine equivalence of vectorial bent functions, similarly to the Boolean case [1], [6], coincides with the isomorphism of their addition designs. In the following theorem we show that this conjecture is true.
Theorem II.4. Let $F$ and $F'$ be two $(n,m)$-bent functions. Bent functions $F$ and $F'$ are extended-affine equivalent if and only if addition designs $\mathbb{D}(F)$ and $\mathbb{D}(F')$ are isomorphic.

Proof. First, let us recall the definition of the CCZ-equivalence (abbreviation from Carlet-Charpin-Zinoviev). Two $(n,m)$-functions $F$ and $F'$ are called CCZ-equivalent, if their graphs $G_F$ and $G_{F'}$ are affine equivalent, i.e., there exists an affine permutation $A$ of $\mathbb{F}_2^n \times \mathbb{F}_2^n$ s.t. $A(G_F) = G_{F'}$. The CCZ-equivalence is known as the most general equivalence relation for $(n,m)$-functions, however as it was shown in [5], [13] two $(n,m)$-bent functions $F, F'$ are extended-affine equivalent if and only they are CCZ-equivalent. By [4], Theorem 6.2 functions $F$ and $F'$ are CCZ-equivalent if and only if the linear codes $C(F)$ and $C(F')$ are equivalent. The proof of the statement now follows from [8, Corollary 14], since linear codes $C(F)$ and $C(F')$ of $(n,m)$-bent functions $F$ and $F'$ are equivalent if and only if the addition designs $\mathbb{D}(F)$ and $\mathbb{D}(F')$ are isomorphic, since an incidence matrix of the addition design $\mathbb{D}(F)$ is a generator matrix of the code $C(F)$. □

III. TRANSLATION DESIGNS OF BOOLEAN BENT FUNCTIONS

In this section we prove that isomorphism of translation designs $\text{dev}(G_f)$ and $\text{dev}(G_{f'})$ of Boolean bent functions $f, f': \mathbb{F}_2^n \to \mathbb{F}_2$ is a coarser equivalence relation for Boolean bent functions than extended-affine equivalence. First, we give a general definition of the translation design.

Definition III.1. For a subset $A$ of an additive group $(G, +)$ the development $\text{dev}(A)$ of $A$ is an incidence structure, whose points are the elements in $G$, and whose blocks are the translates $A + g := \{a + g : a \in A\}$. For a Boolean function $f$ on $\mathbb{F}_2^n$ there are two ways to construct a translation design, see [22, Section 3]:

- $\text{dev}(D_f)$, which is $2 \cdot (2^n, 2^{n-1} \pm 2^{n/2-1}, 2^{n-2} \pm 2^{n/2-1})$ design for a bent function $f$ on $\mathbb{F}_2^n$, with the “+” sign if $f(0) = 1$, and “−” otherwise;
- $\text{dev}(G_f)$, which is a $(2^n, 2^n, 2^{n-1})$ divisible design for a bent function $f$ on $\mathbb{F}_2^n$.

It seems, there is no proper generalization of the translation designs $\text{dev}(D_f)$ for vectorial bent functions, while the second design $\text{dev}(G_f)$ is defined in the same way. Thus, the translation design of an $(n,m)$-function $F$ is defined as:

- $\text{dev}(G_F)$, which is a $(2^n, 2^m, 2^n, 2^{n-m})$ divisible design for an $(n,m)$-bent function $F$.

Remark III.2. Despite translation and addition designs $\text{dev}(D_f)$ and $\mathbb{D}(f)$ of a Boolean bent function $f$ on $\mathbb{F}_2^n$ have the same parameters (up to a complement), in general they are non-isomorphic. However, for a quadratic bent function $f: \mathbb{F}_2^n \to \mathbb{F}_2$ the designs $\text{dev}(D_f)$ and $\mathbb{D}(f)$ are isomorphic, see [1, Theorem 11.9].

Further we denote by $J_{2^n}$ the all-one-matrix of order $2^n$ and by $A \otimes B$ the Kronecker product of matrices $A$ and $B$. In the following proposition we observe, that from isomorphism of designs $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ of Boolean (not necessarily bent) functions $f, f'$ on $\mathbb{F}_2^n$ follows the isomorphism of designs $\text{dev}(G_f)$ and $\text{dev}(G_{f'})$.

Proposition III.3. Let $f, f': \mathbb{F}_2^n \to \mathbb{F}_2$ be two Boolean functions. If $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ are isomorphic, then $\text{dev}(G_f)$ and $\text{dev}(G_{f'})$ are isomorphic too.

Proof. First, we denote the complement of a Boolean function $f$ by $\bar{f} := f + 1$ and by $M_f$ an incidence matrix of the translation design $\text{dev}(G_f)$, which can be computed as follows $M_f := (f(x) \oplus y)_{x,y \in \mathbb{F}_2^n}$, see [28]. With the use of incidence matrices $M_f$ and $M_{f'}$ of translation designs $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$, respectively, one can decompose the incidence matrix $M(\text{dev}(G_f))$ of a Boolean function $f: \mathbb{F}_2^n \to \mathbb{F}_2$ in the following way [21]:

$$M(\text{dev}(G_f)) = \begin{pmatrix} M_f & M_{\bar{f}} \\ \bar{M}_f & M_{\bar{f}} \end{pmatrix}.$$

Since $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ are isomorphic, there exist permutation matrices $P$ and $Q$, such that $M_f = P \cdot M_{f'} \cdot Q$. Clearly, $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ are isomorphic with the same permutation matrices $P$ and $Q$, as one can see from the following calculations

$$M_{\bar{f}} = M_f \oplus J_{2^n} = P \cdot M_{\bar{f}}' \cdot Q \oplus J_{2^n} = P \cdot (M_{\hat{f}} \oplus J_{2^n}) \cdot Q = P \cdot M_f \cdot Q.$$

Finally, since $\text{dev}(G_f) = (I_2 \otimes P) \cdot M(\text{dev}(G_{f'})) \cdot (I_2 \otimes Q)$, we conclude that $\text{dev}(G_f)$ and $\text{dev}(G_{f'})$ are isomorphic. □

Remark III.4. The converse of the previous statement is not true in general. A simple argument to see it, is that the design $\text{dev}(D_f)$ of a Boolean function $f$ on $\mathbb{F}_2^n$ is invariant for affine equivalence [28], that is $f(x) = f'(x \cdot A + b)$ for a non-degenerate $n \times n$ matrix $A$, but not extended-affine equivalence [12, Example 9.3.28]. In general, there are many examples of non-isomorphic translation designs $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$, obtained by addition of an affine (and even linear) function $f$ to a bent function $f$ on $\mathbb{F}_2^n$, as it was mentioned by Dempwolff to the second author of this paper in a private communication. At the same time, the design $\text{dev}(G_f)$ of an $(n,m)$-function $F$ is invariant for CCZ-equivalence and, hence, extended-affine equivalence, see [10]. In view of this remark we define isomorphic $(n,m)$-functions in the following way.

Definition III.5. Two $(n,m)$-functions $F, F'$ are isomorphic, if translation designs $\text{dev}(G_F)$ and $\text{dev}(G_{F'})$ are isomorphic.

Example III.6. Let $f$ be a quadratic and $f'$ be a cubic Maiorana-McFarland bent functions on $\mathbb{F}_2^n$, given by their ANFs

$$f(x) = x_1x_2 + x_3x_4 + x_5x_6,$$

$$f'(x) = x_1x_2 + x_3x_4 + x_5x_6 + x_1x_3x_5.$$
Theorem III.8. Boolean bent functions, which are extended-affine inequivalent but isomorphic exist on $\mathbb{F}_2^n$ for all $n \geq 6$.

Proof. Let $g$ be a quadratic bent function on $\mathbb{F}_2^n$ and let $f$ and $f'$ be bent functions from the Example III.6. By Proposition III.7 Boolean functions $f \oplus g$ and $f' \oplus g$ on $\mathbb{F}_2^n$ with $n = k + 6$ are isomorphic. Clearly, direct sums $f \oplus g$ and $f' \oplus g$ are bent, since all the functions $f, f'$ and $g$ are bent. Finally, since $\deg(f \oplus g) = 2$ and $\deg(f' \oplus g) = 3$, we get that functions $f \oplus g$ and $f' \oplus g$ are extended-affine inequivalent on $\mathbb{F}_2^n$. □

Remark III.9. Extended-affine inequivalent Boolean bent functions $f$ and $f'$ on $\mathbb{F}_2^n$ from Example III.6 define isomorphic designs $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ with 2-transitive automorphism group. According to Kantor [11, Theorem 1] any $2, (2^n, 2^{n-1} - 2^{n/2 - 1}, 2^{n-2} - 2^{n/2-1})$ design with a 2-transitive automorphism group is unique up to isomorphism. In general, if a design has a large automorphism group, it is more likely that it can be represented by several inequivalent difference sets (bent functions) due to the large symmetry. In this way, one may think that the reason why functions from Example III.6 have isomorphic translation designs is the 2-transitivity of the automorphism group. In the following example we show that isomorphic translation designs $\text{dev}(D_f)$ and $\text{dev}(D_{f'})$ of EA-inequivalent bent functions $f$ and $f'$ do not necessarily need to have a 2-transitive automorphism group.

Example III.10. Let $f, f'$ be two Maiorana-McFarland bent functions on $\mathbb{F}_2^{10}$ given by

$$f(x) = x_1 x_6 \oplus x_2 x_7 \oplus x_3 x_8 \oplus x_4 x_9 \oplus x_5 x_{10} \oplus x_1 x_2 x_3 x_4 x_5,$$

$$f'(x) = f(x) \oplus x_4 \oplus x_6 \oplus x_8 \oplus x_{10} \oplus x_1 x_2 \oplus x_2 x_3 \oplus x_1 x_2 x_3 \oplus x_2 x_4 x_5 \oplus x_2 x_3 x_4 x_5.$$
IV. TRANSLATION DESIGNS OF VECTORIAL BENT FUNCTIONS

In the previous section we showed that extended-affine inequivalent Boolean bent functions can give isomorphic translation designs. Further we show that the same phenomenon does not occur for the vectorial bent functions in 6 variables. We classify and enumerate all \((6, m)\)-vectorial bent functions and show, that two vectorial bent functions \(F, F'\) in six variables are EA-equivalent if and only if their translation designs \(\text{dev}(G_F)\) and \(\text{dev}(G_{F'})\) are isomorphic.

A. Extension invariants of bent functions

We denote by \(B_{n,m}\) the set of all \((n, m)\)-bent functions, by \(A_{n,m}\) the set of all \((n, m)\)-affine functions and by \(AB_{n,m}\) the set of affine-free \((n, m)\)-bent functions, i.e. any \(f \in AB_{n,m}\) contains no affine terms in its ANF. Since bentness is invariant with respect to the addition of affine terms, the cardinalities of these three sets are related as follows \(|B_{n,m}| = |AB_{n,m}| \cdot |A_{n,m}|\).

For the sake of convenience we denote by \(C_i^m\) an \(i\)-th EA-equivalence class of \((n, m)\)-bent functions. On the set \(\bigcup_{m=2}^{n} B_{n,m}\) we introduce the order relation “\(\preceq\)” in the following way. Let \(m < n\) and \(C_i^m\) and \(C_j^l\) be two equivalence classes of \((n, m)\)- and \((l, n)\)-bent functions, respectively. We say that a function \(F \in C_i^m\) is contained in \(G \in C_j^l\) and write \(F \preceq G\), if the first \(m\) coordinate functions of \(G(x) = (g_1(x), \ldots, g_m(x))^T\) form a function \(F\), that is \(F(x) = (g_1(x), \ldots, g_m(x))^T\). Similarly, we say that \(F \in C_i^m\) is contained in the equivalence class \(C_j^l\) and write \(F \preceq C_j^l\), if there exist a representative \(G \in C_j^l\) such that \(F \preceq G\).

Finally, we say that the equivalence class \(C_i^m\) is contained in \(C_j^l\) and denote it by \(C_i^m \preceq C_j^l\) if there exist \(F \in C_i^m\), such that \(F \preceq C_j^l\).

Definition IV.1. An \((n, m)\)-bent function \(F\) is called extendable, if the there exists a Boolean bent function \(f: \mathbb{F}_2^n \rightarrow \mathbb{F}_2\), such that the function \(G: x \in \mathbb{F}_2^n \mapsto (F(x), f(x))^T\) is \((n, m+1)\)-bent. If no such a bent function \(f\) exists, the function \(F\) is called non-extendable.

Remark IV.2. The problem of the existence of non-extendable bent functions \(F: \mathbb{F}_p^n \rightarrow \mathbb{F}_p^m\) has mostly been studied for the case \(p\) odd, see [20, Section 4]. The particular case of this problem, namely \(p = 2\) and \(m = 1\), is closely related to the Takahara’s conjecture [27, Hypothesis 1], that any Boolean function on \(\mathbb{F}_2^n\) of degree at most \(n/2\) can be represented as the sum of two Boolean bent functions on \(\mathbb{F}_2^n\). For instance, a single example of a non-extendable Boolean bent function would disprove the Takahara’s conjecture.

Definition IV.3. Let \(F\) be an \((n, m)\)-bent function. Further we define the following two sets

\[ \mathcal{F}(F) := \{ f \in AB_{n,1}: (F, f)^T \text{ is } (n, m+1)\text{-bent}\}, \]

\[ \text{Ext}(F) := \{ (F, f)^T : f \in \mathcal{F}(F) \}, \]

namely, \(\mathcal{F}(F)\) is the set of affine-free Boolean bent functions, which can extend an \((n, m)\)-bent function \(F\) to an \((n, m+1)\)-bent function and \(\text{Ext}(F)\) is the set of extensions of a function \(F\). Clearly, different extensions may lead to different equivalence classes. In this way, we define

\[ \mathcal{F}(F, C_j^{m+1}) := \{ f \in AB_{n,1}: (F, f)^T \in C_j^{m+1} \text{ is } (n, m+1)\text{-bent}\} \quad \text{(IV.2)} \]

as the set of affine-free Boolean bent functions, which can extend an \((n, m)\)-bent function \(F\) to the equivalence class \(C_j^{m+1}\). Similarly, we define the set of extensions of the function \(F\), which belong to the equivalence class \(C_j^{m+1}\), that is

\[ \text{Ext}(F, C_j^{m+1}) := \{ (F, f)^T : f \in \mathcal{F}(F, C_j^{m+1}) \}. \quad \text{(IV.3)} \]

Clearly, the collection of sets \(\text{Ext}(F, C_j^{m+1})\) forms a partition of \(\text{Ext}(F)\), namely

\[ \text{Ext}(F) = \bigcup_{j: F < C_j^{m+1}} \text{Ext}(F, C_j^{m+1}). \quad \text{(IV.4)} \]

Remark IV.4. Non-extendable \((n, m)\)-bent functions \(F\) are also called lonely, see [16]. In this way, it is essential to call the following sets:

- \(\mathcal{F}(F)\) — the set of bent friends of a bent function \(F\);
- \(\mathcal{F}(F, C_j^{m+1})\) — the set of bent friends of \(F\), leading to the equivalence class \(C_j^{m+1}\).

Indeed, according to Definition IV.3, a bent function \(F\) is lonely, if it has no bent friends, that is \(|\mathcal{F}(F)| = 0\). We also call \((n, n/2)\)-bent functions absolutely non-extendable (lonely), since \((n, m)\)-bent functions do not exist for \(m > n/2\) due to the Nyberg bound [19].

Definition IV.5. For an \((n, m+1)\)-function \(G\) we denote by \(S(G)\) a set of equivalence classes of \((n, m)\)-bent functions \(F, F'\), of the form \(F(x) : x \in \mathbb{F}_2^n \rightarrow A_F \circ G(x)\) and \(F'(x) : x \in \mathbb{F}_2^n \rightarrow A_{F'} \circ G(x)\), where \(A_F, A_{F'}: \mathbb{F}_2^{m+1} \rightarrow \mathbb{F}_2^m\) are surjective linear mappings and the equivalence relation “\(\sim\)” is defined as follows: \(F \sim F'\) if \{\(F_b \circ b \in \mathbb{F}_2^m\)\} = \{\(F'_b \circ b \in \mathbb{F}_2^m\)\}, i.e. functions \(F\) and \(F'\) have the same component functions. We will call the set \(S(G)\) the set of different \((n, m)\)-bent spaces of a function \(G\). Clearly, its cardinality is given by \(|S(G)| = \frac{2^{m+1}-1}{2} = 2^{m+1} - 1\). Finally, for an \((n, m)\)-bent function \(F < G\) we denote by \(S(F,G) := \{ H \in S(G) : H \text{ is EA-equivalent to } F \}\) the set of different \((n, m)\)-bent spaces of \(G\), which are EA-equivalent to \(F\).

Further, we show that cardinalities of the sets \(\mathcal{F}(F, C_j^{m+1})\) and \(S(F,G)\) do not depend on representatives of equivalence classes and thus are invariants for extended-affine equivalence.

Proposition IV.6. Let \(F, F' \in C_i^m \prec C_j^{m+1}\) be two \((n, m)\)-bent functions and \(G, G' \in C_i^{m+1} \prec C_j^{m+1}\) be two \((n, m+1)\)-bent functions. Then the following hold.

1) \(|\mathcal{F}(F, C_j^{m+1})| = |\mathcal{F}(F', C_j^{m+1})|;\)
2) \(|S(F, G)| = |S(F', G')|;\)

Proof. 1. Let \(F\) and \(F'\) be EA-equivalent, i.e. \(F = A_1 \circ F' \circ A_2 \oplus A_3\). Clearly, if \(f\) is a bent friend of \(F\), then \(f' := f \circ A_2\) is a bent friend of \(F'\). Moreover, the non-degenerate affine transformation \(A_2\) maps different bent friends to different ones.
Assume that $H \in S(F, G)$, i.e. there exist non-degenerate linear mapping $A_H : F_{m+1}^2 \rightarrow F_{n+1}^2$ such that $H = A_H \circ G = B_1 \circ F \circ B_2 \circ B_3$, since $H, F \in C_{m+1}$. Further, we may assume that $G' = A_1 \circ G \circ A_2 \circ A_3$, since $G, G' \in C_{m+1}$. Multiplying the latter equality by $A_H \circ A_1^{-1}$ from left and substituting it the second last, one gets $A_H \circ A_1^{-1} \circ G' = H \circ A_2 \circ A_H \circ A_1^{-1} \circ A_3$. Finally, denoting by $A_H' := A_H \circ A_1^{-1}$, we get that the function $H' := A_H \circ G'$ is EA-equivalent to $F'$, and hence $H' \in S(F', G')$.

In this way, for two equivalence classes $C_{m+1}$ and $C_{m+1}'$, we denote by $|F(C_{m+1}, C_{m+1}')|$, the number of Boolean bent functions, which can extend any representative of $C_{m+1}$ to the class $C_{m+1}$ and by $|S(C_{m+1}, C_{m+1}')|$, the number of different bent spaces contained in $C_{m+1}$, which represent the equivalence class $C_{m+1}$, that is

$$|F(C_{m+1}, C_{m+1}')| := |F(F, C_{m+1})|$$

$$|S(C_{m+1}, C_{m+1}')| := |S(F, C_{m+1})|$$

for $F \in C_{m+1}$. \hspace{1cm} (IV.5)

In the next subsection we will use the number of bent friends $|F(C_{m+1}, C_{m+1}')|$ in order to enumerate all vectorial bent functions in six variables and the number of bent spaces $|S(C_{m+1}, C_{m+1}')|$ in order to verify these computations.

**Proposition IV.7.** Let $C_{k} \prec C_{k+1} \prec C_{m+1}$ be all equivalence classes of $(n, m)$-bent functions, contained in $C_{m+1}$. Then the cardinality of the class $C_{m+1}$ is equal to

$$|C_{m+1}| = 2^{n+1} \sum_{i=1}^{k} |C_{m+1}| \cdot |F(C_{m+1}, C_{m+1})|. \hspace{1cm} (IV.6)$$

**Proof.** Any function $G \in C_{m+1}$ can be considered as an extension of a function $F \in C_{m+1}$, that is $G = (F, f)^T \in C_{m+1}$ for $f \in F(C_{m+1})$. There are $k$ ways to select an equivalence class $C_{i} \prec C_{m+1}$, such that $F \in C_{i}$, and there are $|C_{m+1}|$ ways to choose a representative $F$. Finally, for any representative $F \in C_{m+1}$ there exist exactly $2^{n+1} \cdot |F(C_{m+1}, C_{m+1})|$ ways to extend it to a function $G \in C_{m+1}$, since bentness is invariant with respect to addition of affine terms.

Further we summarize the above ideas in the form of a recursive algorithm. Applying the Algorithm IV.1 for Boolean bent functions in six variables, we obtain the main result of this section.

**Theorem IV.8.** For vectorial bent functions in 6 variables the following hold.

1. There are $23,392,233,361,244,160 \approx 2^{54.37}$ vectorial $(6, 2)$-bent functions, which are divided into 9 extended-affine equivalence classes.

2. There are $121,282,113,886,947,901,440 \approx 2^{66.71}$ vectorial $(6, 3)$-bent functions, which are divided into 13 extended-affine equivalence classes.

**Algorithm IV.1.** Classification and enumeration of all $(n, m)$-bent functions

**Input:** All pairs $(F_{i} \in C_{k}, |C_{i}|)$, where $B_{n,1} = \bigcup \{f : f \in C_{1}\}$.

**Output:** All pairs $(F_{m} \in C_{m}, |C_{m}|)$, where $B_{n,m} = \bigcup \{f : f \in C_{m}\}$ for all $2 \leq m \leq n/2$.

1. for $m = 1$ to $n/2 - 1$ do
2. for all equivalence classes $C_{m}$ do
3. Construct the set of extensions $Ext(F_{m})$.
4. Classify all $(n, m+1)$-bent functions from the set $Ext(F_{m})$ by constructing the partition $Ext(F_{m}) = \bigsqcup_{j \in C_{m}} Ext(F_{m}, C_{m+1})$.
5. Compute the number of bent friends $|F(C_{m}, C_{m+1})| := |F(F_{m}, C_{m+1})|$.
6. end for
7. Identify all equivalent classes $C_{m}$ and set $F_{m+1}$ to be a random representative of the equivalence class $C_{m+1}$.
8. Compute the number of bent spaces $|S(C_{m}, C_{m+1})| := |S(C_{m+1}, C_{m+1})|$ and cardinalities of equivalence classes $|C_{m+1}| = 2^{n+1} \sum_{i=1}^{k} |C_{m+1}| \cdot |F(C_{m}, C_{m+1})|$.
9. end for
10. Return pairs $(F_{m} \in C_{m}, |C_{m}|)$ for all $2 \leq m \leq n/2$.

Moreover, if a $(6, m)$-bent function $F$ is non-extendable, then $F$ is absolutely non-extendable, i.e. it has $m = 3$.

**Proof.** Further we discuss the main steps of the Algorithm IV.1 and explain how one can verify our computational results.

**Input.** For the input of the Algorithm IV.1 one has to provide the pairs $(F_{i} \in C_{1}, |C_{i}|)$ for all equivalence classes $C_{1}$, which form the partition of the set of Boolean bent functions $B_{6,1}$. The representatives of 4 equivalence classes are well-known and could be found in [25]. For the cardinalities of the equivalence classes we refer to [24, Table 8.7].

**Output.** For the computation of the collections $F(F_{m})$ one first has to construct all affine-free Boolean bent functions $AB_{6,1}$, which can be efficiently listed as described in [14], [17]. Further, for a given representative $F_{m} \in C_{m}$ we construct the set $F(F_{m})$, by checking directly the characteristic property in (IV.1). The classification of functions $G \in Ext(F_{m})$ is carried out with Magma [3], by checking equivalence of linear codes $C(G)$ introduced in Definition II.1.

In this way, Algorithm IV.1 constructs $n/2 - 1$ layers of the weighted Hasse diagram, given in Figure IV.1 as follows. For all $2 \leq m \leq n/2 - 1$ we draw an edge between equivalence classes $C_{m}$ and $C_{m+1}$ if $C_{m} \prec C_{m+1}$ and assign two weights with it. The first number closer to the equivalence class $C_{i}$ is the number of bent spaces $|S(C_{m}, C_{m+1})|$ and the second number, closer to $C_{m+1}$, is the number of bent friends.

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In the following table we list equivalence classes $C_i$ of vectorial bent functions in 6 variables.

| $C_i$ | 1,777,664 | 239,984,640 | 1,343,913,984 | 3,839,754,240 |
|-------|-------------|-------------|-------------|-------------|
| 1     | 5,700       | 51,840      | 768         | 10,752      |
|       | 3           | 1           | 2           | 2           |
| 1     | 2           | 3           | 1           | 3           |
| 1     | 7,761       | 7,761       | 7,761       | 7,761       |
|       | 1           | 1           | 1           | 1           |
|       | 1           | 1           | 1           | 1           |

We also checked that the only equivalence classes of Boolean bent functions, which lead to isomorphic translation designs are $C_1$ and $C_2$, as one can see from Example III.6 and Table A.1(a). Surprisingly, in contrast to the Boolean case, one cannot construct isomorphic translation designs from extended-affine inequivalent vectorial bent functions.

**Theorem IV.9.** Let $F$ and $F'$ be two $(6, m)$-bent functions with $m \geq 2$. The following statements are equivalent.

1. Bent functions $F$ and $F'$ are extended-affine equivalent.
2. Divisible designs $\text{dev}(G_F)$ and $\text{dev}(G_{F'})$ are isomorphic.

**Proof.** All computations about equivalence and isomorphism are carried out with Magma [3]. Invariants of equivalence classes and their translation designs are listed in Table A.1(b) and Table A.1(c).

**Remark IV.10.** It is well-known, that all Boolean bent functions in six variables up to EA-equivalence can be described by two classical constructions: Maiorana-McFarland $\mathcal{M}$, and Desarguesian partial spread $\mathcal{PS}_{ap}$, which have straightforward generalizations to the vectorial case, see [18, p. 309]. We endow $\mathbb{F}_2^{2n/2}$ with the structure of the finite field $\mathbb{F}_{2^{n/2}}$ and identify $\mathbb{F}_2$ with $\mathbb{F}_{2^{n/2}} \times \mathbb{F}_{2^{n/2}}$. The strict Maiorana-McFarland class $\mathcal{M}$ of vectorial bent functions is the set of $(n, m)$-functions $F$ of the form $F(x, y) = L(x \cdot \pi(y)) + G(y)$, where $L: \mathbb{F}_{2^{n/2}} \rightarrow \mathbb{F}_{2^{m}}$ is a linear or an affine function, $\pi: \mathbb{F}_{2^{n/2}} \rightarrow \mathbb{F}_{2^{n/2}}$ is a permutation, and $G: \mathbb{F}_{2^{n/2}} \rightarrow \mathbb{F}_{2^{m}}$ is an arbitrary $(n/2, m)$-function. The $\mathcal{PS}_{ap}$ class of vectorial bent functions is the set of $(n, m)$-bent functions $F$ of the form $F(x, y) := H(x \cdot y^{2^n/2} - 2) + H(x/y)$ with $x/y = 0$ if $y = 0$ for $x, y \in \mathbb{F}_{2^{n/2}}$ and $H$ is a balanced $(n/2, m)$-function (or, equivalently, permutation if $m = n/2$).

In the following table we list equivalence classes $C_i$ of $(6, 3)$-bent functions, which can be described by $\mathcal{M}$ and $\mathcal{PS}_{ap}$ classes. Note that $(6, 2)$-bent functions from $\mathcal{M}$ and $\mathcal{PS}_{ap}$ can
be constructed as proper bent subspaces of \((6, 3)\)-bent from \(\mathcal{M}\) and \(\mathcal{PS}_{ap}\) classes.

**TABLE IV.2.** Equivalence classes of \((6, 3)\)-bent functions, described by classical constructions

| Class | Description |
|-------|-------------|
| \(C_{31}^1\) | \(x \cdot \pi_1(y) + G(y)\) |
| \(C_{31}^2\) | \(x \cdot \pi_1(y) + (y + y^2 + y^3 + y^6)\) |
| \(C_{31}^3\) | \(x \cdot \pi_1(y) + (y^3 + y^4 + y^5 + y^6)\) |
| \(C_{31}^4\) | \(x \cdot \pi_1(y)\) |
| \(C_{31}^5\) | \(x \cdot \pi_2(y)\) |

In this way, from Figure IV.1 and Table IV.2 one can see that the only “missing” equivalence classes of \((6, 3)\)-bent functions are \(C_{31}^1, C_{31}^3, C_{31}^7, C_{31}^9\) and of \((6, 2)\)-bent functions are \(C_{21}^1, C_{21}^2\). In view of this observation we conclude, that in contrast to the Boolean case, vectorial versions of the classical Maiorana-McFarland and Desarguesian partial spread constructions do not cover the whole set of vectorial bent functions in six variables.

**V. SUMMARY AND CONCLUDING REMARKS**

In this paper we compared different concepts of equivalence relations for Boolean and vectorial bent functions: extended-affine equivalence of functions, isomorphism of translation designs and isomorphism of addition designs. We summarize our results in the following table.

**TABLE V.1.** EA-equivalence vs. isomorphism of designs for bent functions

| Does isomorphism of designs coincide with EA-equivalence for \((n, m)\)-bent functions? | Translation Designs \(\text{dev}(\mathcal{F}_P)\) | Addition Designs \(\mathcal{D}(\mathcal{F})\) |
|---------------------------------|-----------------|-----------------|
| \(m = 1\) | No, isomorphism is coarser for all \(n \geq 6\). | Yes, for all \(n\). |
| \(m \geq 2\) | Yes, for \(n = 4, 6\). | |

Finally, we would like to mention some open problems on bent functions and their translation designs, which the reader is invited to attack.

**Open Problem V.1.** As one can see from Examples III.6 and III.10, it is possible to construct EA-inequivalent but isomorphic Boolean bent functions, by taking proper Maiorana-McFarland bent functions and extending them to infinite families using the Proposition III.7. So far, this approach does not seem to work for vectorial bent functions:

- there is only one up to EA-equivalence vectorial bent function in 4 variables, from what follows that all derived translation designs are isomorphic;
- by Theorem IV.9 all isomorphic vectorial bent functions in 6 variables are also EA-equivalent.

As one can see from Proposition III.7, a single example of EA-inequivalent but isomorphic vectorial bent functions will lead to an infinite family and, consequently, will prove that for vectorial bent functions the isomorphism of translation designs is a coarser equivalence relation than EA-equivalence. However, since one still does not have an example of such functions, it is essential to ask, whether EA-inequivalent but isomorphic vectorial bent functions may in general exist.

**Open Problem V.2.** There are very few symmetric designs with a 2-transitive automorphism group, as it was shown by Kantor in [11]. One of them is \(2^2(2^n, 2^n - 1, 2^n/2 - 1, 2^n/2 - 1)\) design, which can be constructed as the addition \(\mathcal{D}(f)\) or as the translation \(\mathcal{D}(\mathcal{D}_f)\) design of a bent function \(f\) on \(\mathbb{F}_2^n\). While in the case of addition designs \(\mathcal{D}(f)\) a bent function \(f\) on \(\mathbb{F}_2^n\) has to be quadratic, from Example III.6 one can see that for the translation design \(\mathcal{D}(\mathcal{D}_f)\) one still has some freedom to choose a function \(f\). We conjecture, that a translation design \(\mathcal{D}(\mathcal{D}_f)\) of a bent function \(f\) on \(\mathbb{F}_2^n\) has 2-transitive automorphism group if and only if function \(f\) is EA-equivalent to a Maiorana-McFarland bent function of the form \((x, y)^{n/2} + g(y)\) with \(\deg(g) \leq 3\).

**ACKNOWLEDGMENT**

The authors would like to thank anonymous referees for their comments that helped to improve the presentation of the results.

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For any equivalence class $C^m_i$ of $(6, m)$-bent functions we compute the following invariants:

- $|\text{Aut}(C^m_i)| := |\text{Aut}(F^m_i)|$, that is the order of the automorphism group of the linear code $C(F^m_i)$, for a representative $F^m_i \in C^m_i$ listed in Appendix A;

- $|\text{Aut}(\text{dev}(C^m_i))| := |\text{Aut}(\text{dev}(G^m_i))|$, that is the order of the automorphism group of the translation design $\text{dev}(G^m_i)$, for a representative $F^m_i \in C^m_i$ listed in Appendix A;

- $\text{SNF}(C^m_i) := \text{SNF}(F^m_i)$, that is the Smith normal form of the incidence matrix $M(\text{dev}(G^m_i))$, given by the multiset $\text{SNF}(F^m_i) = \{ s_1^{d_1}, \ldots, s_k^{d_k} \}$, where consecutive elementary divisors $d_k$ and $d_{k+1}$ satisfy $d_k | d_{k+1}$, and $e_k$ is the multiplicity of the elementary divisor $d_k$.

Note that the multiplicity of one in the Smith normal form $\text{SNF}(F^m_i)$ is the $\Gamma$-rank($F^m_i$) and similarly to [21, Proposition 2.4] one can show, that all elementary divisors $d_k$ in the $\text{SNF}(F)$ of an $(m, n)$-bent function $F$ are powers of two. We also observe that any two different equivalence classes $C^m_i$ and $C^m_j$ of bent functions in six variables have different pairs of invariants

$$|\text{Aut}(C^m_i)|, |\text{SNF}(C^m_i)| \neq |\text{Aut}(C^m_j)|, |\text{SNF}(C^m_j)|$$

In this way, the reader can be sure that all the representatives of equivalence classes listed in Appendix A are extended-affine inequivalent.

### TABLE A.1: Invariants of inequivalent $(6, m)$-bent functions

#### A.1(a) Boolean $(6, 1)$-bent functions

| $C^1_i$ | $|\text{Aut}(C^1_i)|$ | $|\text{Aut}(\text{dev}(C^1_i))|$ | $|\text{SNF}(C^1_i)|$ |
|---------|-----------------|-------------------------------|-----------------|
| $C^1_1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $[1^1, 2^1, 3^3, 5^2, 7^1, 11^1]$ |
| $C^1_2$ | $2^5$·$3^1$·$7^1$ | $2^5$·$3^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^1_3$ | $2^3$·$3^1$·$5^1$·$7^1$ | $2^3$·$3^1$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^1_4$ | $2^3$·$5^1$·$7^1$ | $2^3$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |

#### A.1(b) Vectorial $(6, 2)$-bent functions

| $C^2_i$ | $|\text{Aut}(C^2_i)|$ | $|\text{Aut}(\text{dev}(C^2_i))|$ | $|\text{SNF}(C^2_i)|$ |
|---------|-----------------|-------------------------------|-----------------|
| $C^2_1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $[1^1, 2^1, 3^3, 5^2, 7^1, 11^1]$ |
| $C^2_2$ | $2^5$·$3^1$·$7^1$ | $2^5$·$3^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^2_3$ | $2^3$·$3^1$·$5^1$·$7^1$ | $2^3$·$3^1$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^2_4$ | $2^3$·$5^1$·$7^1$ | $2^3$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |

#### A.1(c) Vectorial $(6, 3)$-bent functions

| $C^3_i$ | $|\text{Aut}(C^3_i)|$ | $|\text{Aut}(\text{dev}(C^3_i))|$ | $|\text{SNF}(C^3_i)|$ |
|---------|-----------------|-------------------------------|-----------------|
| $C^3_1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $2^6$·$3^3$·$5^2$·$7^1$ | $[1^1, 2^1, 3^3, 5^2, 7^1, 11^1]$ |
| $C^3_2$ | $2^5$·$3^1$·$7^1$ | $2^5$·$3^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^3_3$ | $2^3$·$3^1$·$5^1$·$7^1$ | $2^3$·$3^1$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |
| $C^3_4$ | $2^3$·$5^1$·$7^1$ | $2^3$·$5^1$·$7^1$ | $[1^1, 2^1, 3^1, 5^1, 7^1]$ |

For quadratic vectorial bent functions from equivalence classes $C^m_i$ with $m = 2, 3$ we have $|\text{Aut}(\text{dev}(C^m_i))| = 2^{n+m} |\text{Aut}(C^m_i)|$ · $7^{|1|}$, where $n = 6$. For the rest of vectorial $(n, m)$-bent functions in six variables we have $|\text{Aut}(\text{dev}(C^m_i))| = 2^{m+n} |\text{Aut}(C^m_i)|$. In this way, translation designs of vectorial bent functions from equivalence classes $C^m_i$ can be distinguished by the orders of their automorphism groups, despite the Smith normal forms coincide.

Finally, we list algebraic normal forms of representatives of EA-equivalence classes of bent functions in 6 variables. The representatives $F^m_i \in C^m_i$ and $F^{m+1}_j \in C^{m+1}_j$ are selected in such a way, that $F^m_i \prec F^{m+1}_j$ as on Figure IV.1. We abbreviate $1 \leq i \leq 6$ for the variable $x_i$.

### APPENDIX

For any equivalence class $C^m_i$ of $(6, m)$-bent functions we compute the following invariants:

- $|\text{Aut}(C^m_i)| := |\text{Aut}(F^m_i)|$, that is the order of the automorphism group of the linear code $C(F^m_i)$, for a representative $F^m_i \in C^m_i$ listed in Appendix A;

- $|\text{Aut}(\text{dev}(C^m_i))| := |\text{Aut}(\text{dev}(G^m_i))|$, that is the order of the automorphism group of the translation design $\text{dev}(G^m_i)$, for a representative $F^m_i \in C^m_i$ listed in Appendix A;
TABLE A.2. Algebraic normal forms of inequivalent \((6, m)\)-bent functions

A.2(a) Boolean \((6, 1)\)-bent functions

| \(F_i^1\) | Algebraic normal form of \(F_i^1 \in C_i^1\) |
|----------|----------------------------------|
| \(F_1^1\) | \(14 \oplus 25 \oplus 36\) |
| \(F_2^1\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_3^1\) | \(14 \oplus 26 \oplus 35 \oplus 45 \oplus 123 \oplus 245\) |
| \(F_4^1\) | \(14 \oplus 26 \oplus 34 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |

A.2(b) Vectorial \((6, 2)\)-bent functions

| \(F_i^2\) | Algebraic normal form of \(F_i^2 \in C_i^2\) |
|----------|----------------------------------|
| \(F_1^2\) | \(14 \oplus 25 \oplus 36\) |
| \(F_2^2\) | \(15 \oplus 16 \oplus 24 \oplus 25 \oplus 34\) |
| \(F_3^2\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_4^2\) | \(15 \oplus 16 \oplus 24 \oplus 25 \oplus 34\) |
| \(F_5^2\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_6^2\) | \(12 \oplus 13 \oplus 16 \oplus 26 \oplus 45 \oplus 56 \oplus 156 \oplus 235\) |
| \(F_7^2\) | \(12 \oplus 14 \oplus 26 \oplus 35 \oplus 45 \oplus 123 \oplus 245\) |
| \(F_8^2\) | \(13 \oplus 23 \oplus 24 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |
| \(F_9^2\) | \(14 \oplus 26 \oplus 34 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |

A.2(c) Vectorial \((6, 3)\)-bent functions

| \(F_i^3\) | Algebraic normal form of \(F_i^3 \in C_i^3\) |
|----------|----------------------------------|
| \(F_1^3\) | \(14 \oplus 25 \oplus 36\) |
| \(F_2^3\) | \(15 \oplus 16 \oplus 24 \oplus 25 \oplus 34\) |
| \(F_3^3\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_4^3\) | \(15 \oplus 16 \oplus 24 \oplus 25 \oplus 34\) |
| \(F_5^3\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_6^3\) | \(15 \oplus 16 \oplus 24 \oplus 25 \oplus 34\) |
| \(F_7^3\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_8^3\) | \(13 \oplus 15 \oplus 23 \oplus 46 \oplus 124\) |
| \(F_9^3\) | \(13 \oplus 14 \oplus 26 \oplus 45\) |
| \(F_{10}^3\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_{11}^3\) | \(13 \oplus 15 \oplus 23 \oplus 46 \oplus 124\) |
| \(F_{12}^3\) | \(12 \oplus 14 \oplus 16 \oplus 34 \oplus 46 \oplus 56 \oplus 126 \oplus 136 \oplus 246\) |
| \(F_{13}^3\) | \(14 \oplus 25 \oplus 36 \oplus 123\) |
| \(F_{14}^3\) | \(13 \oplus 15 \oplus 23 \oplus 46 \oplus 124\) |
| \(F_{15}^3\) | \(12 \oplus 13 \oplus 24 \oplus 25 \oplus 35 \oplus 45 \oplus 56 \oplus 125 \oplus 345\) |
| \(F_{16}^3\) | \(12 \oplus 14 \oplus 26 \oplus 35 \oplus 45 \oplus 123 \oplus 245\) |
| \(F_{17}^3\) | \(13 \oplus 23 \oplus 24 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |
| \(F_{18}^3\) | \(16 \oplus 23 \oplus 26 \oplus 35 \oplus 36 \oplus 45 \oplus 56 \oplus 123 \oplus 124 \oplus 256\) |
| \(F_{19}^3\) | \(12 \oplus 14 \oplus 26 \oplus 35 \oplus 45 \oplus 123 \oplus 245\) |
| \(F_{20}^3\) | \(13 \oplus 23 \oplus 24 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |
| \(F_{21}^3\) | \(16 \oplus 25 \oplus 26 \oplus 35 \oplus 36 \oplus 45 \oplus 56 \oplus 123 \oplus 124 \oplus 234 \oplus 256 \oplus 346\) |
| \(F_{22}^3\) | \(14 \oplus 26 \oplus 34 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |
| \(F_{23}^3\) | \(12 \oplus 35 \oplus 36 \oplus 124 \oplus 134 \oplus 235 \oplus 236 \oplus 245\) |
| \(F_{24}^3\) | \(12 \oplus 13 \oplus 24 \oplus 25 \oplus 35 \oplus 45 \oplus 56 \oplus 123 \oplus 124 \oplus 234 \oplus 256 \oplus 346\) |
| \(F_{25}^3\) | \(12 \oplus 13 \oplus 24 \oplus 25 \oplus 35 \oplus 36 \oplus 45 \oplus 46 \oplus 123 \oplus 245 \oplus 346\) |
| \(F_{26}^3\) | \(12 \oplus 35 \oplus 36 \oplus 124 \oplus 134 \oplus 235 \oplus 236 \oplus 245\) |
| \(F_{27}^3\) | \(12 \oplus 15 \oplus 16 \oplus 23 \oplus 26 \oplus 35 \oplus 45 \oplus 46 \oplus 125 \oplus 126 \oplus 135 \oplus 136 \oplus 145 \oplus 256\) |
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