Free Products of Generalized RFD C*-algebras

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Abstract

If \( k \) is an infinite cardinal, we say a C*-algebra \( A \) is residually less than \( k \) dimensional, \( R_{<k}D \), if the family of representations of \( A \) on Hilbert spaces of dimension less than \( k \) separates the points of \( A \). We give characterizations of this property, and we show that if \( \{A_i : i \in I\} \) is a family of \( R_{<k}D \) algebras, then the free product \( \ast_{i \in I} A_i \) is \( R_{<k}D \). If each \( A_i \) is unital, we give sufficient conditions, depending on the cardinal \( k \), for the free product \( \ast_{i \in I} A_i \) in the category of unital C*-algebras to be \( R_{<k}D \). We also give a new characterization of RFD, in terms of a lifting property, for separable C*-algebras.

1 Introduction

A C*-algebra \( A \) is residually finite dimensional (RFD) if the collection of all finite-dimensional representations of \( A \) separate the points of \( A \); equivalently, if there is a direct sum of finite-dimensional representations of \( A \) with zero kernel. It is clear that every commutative C*-algebra is RFD. Man-Duen Choi [4] showed that free group C*-algebras are RFD. Ruy Exel and Terry Loring [6] proved that the free product of two RFD algebras is RFD. The class of RFD C*-algebras plays an important role in the theory of C*-algebras, e.g., [1], [2], [3], [4], [5], [7], [11], [10].

In this paper we introduce a related notion. Suppose \( k \) is an infinite cardinal. We say that a C*-algebra \( A \) is residually less than \( k \)-dimensional, conveniently denoted by \( R_{<k}D \), if the class of representations of \( A \) on Hilbert spaces of dimension less than \( k \) separates the points of \( A \); equivalently, if there is a direct sum of such representations that has zero kernel. Note that when \( k = \aleph_0 \), we have \( R_{<k}D \) is the same as RFD. We give characterizations of \( R_{<k}D \) algebras that show that the free product of an arbitrary collection of \( R_{<k}D \) C*-algebras is \( R_{<k}D \). We also give conditions that ensure that the free product (amalgamated over \( \mathbb{C} \)) of unital C*-algebras in the category of unital C*-algebras is \( R_{<k}D \); this always happens when each of the algebras has a one-dimensional unital representation.
The proofs rely on a simple result (Lemma 1) and results of the author [8, 9] on approximate unitary equivalence and approximate summands of nonseparable representations of nonseparable C*-algebras.

Suppose \( k \) and \( m \) are infinite cardinals. We say that a C*-algebra \( A \) is \( m \)-generated if it is generated by a set with cardinality at most \( m \). For each cardinal \( s \), we let \( H_s \) be a Hilbert space whose dimension is \( s \). If \( \pi : A \to B(H) \) is a \( * \)-homomorphism, we say that the dimension of \( \pi \) is \( \dim \pi = \dim H \). We define \( \text{Rep}_k(A) \) to be the set of all representations \( \pi : A \to B(H_s) \) for some \( s < k \).

If \( A \) is a C*-algebra, then \( A^+ \) denotes the C*-algebra obtained by adding a unit to \( A \) (that is different from the unit in \( A \) if \( A \) is unital).

We end this section with our key lemma. Suppose \( H \) is a Hilbert space and \( P \) is a projection in \( B(H) \). We define \( M_P = PB(H)P \). Then \( M_P \) is a unital C*-algebra, but the unit is \( P \), not \( 1 \). However, \( M_P \) is a C*-subalgebra of \( B(H) \). A unitary element of \( M_P \) is an operator \( U \in B(H) \) such that \( UU^* = U^*U = P \), and is the direct sum of a unitary operator on \( P(H) \) with 0 on \( P(H)^\perp \). If \( P \neq 1 \), a unitary operator in \( M_P \) is never unitary in \( B(H) \).

We use the symbol \( * \)-SOT to denote the \( * \)-strong operator topology.

**Lemma 1** Suppose \( \{P_\alpha\} \) is a net of projections in \( B(H) \) such that \( P_\alpha \to 1 \) (\( * \)-SOT) and let

\[
\mathcal{B} = \left\{ \{T_\alpha\} \in \prod_\alpha M_{P_\alpha} : \exists T \in B(H), T_\alpha \to T \ (\ * \text{-SOT} \ ) \right\},
\]

and

\[
\mathcal{J} = \{ \{T_\alpha\} \in \mathcal{B} : T_\alpha \to 0 \ (\ * \text{-SOT} \ ) \},
\]

and define \( \pi : \mathcal{B} \to B(H) \) by

\[
\pi(\{T_\alpha\}) = (\ * \text{-SOT} \)-lim_\alpha T_\alpha.
\]

Then

1. \( \mathcal{B} \) is a unital C*-algebra,
2. \( \mathcal{J} \) is a closed two-sided ideal in \( \mathcal{B} \),
3. If \( T \in \mathcal{B} \), then \( \pi(\{P_\alpha TP_\alpha\}) = T \),
4. \( \pi \) is a unital surjective \( * \)-homomorphism
5. If \( U \in B(H) \) is unitary, then there is a unitary \( \{U_\alpha\} \in \mathcal{B} \) such that

\[
\pi(\{U_\alpha\}) = U.
\]
Proof. Statements (1)-(4) are easily proved. To prove (5), note that if $U \in B(H)$ is unitary, then there is an $A = A^* \in B(H)$ such that $U = e^{iA}$. We can easily choose $A_\alpha = A^*_\alpha$ for each $\alpha$ so that $\pi(\{A_\alpha\}) = A$. Thus, if $U_\alpha = e^{iA_\alpha}$ (in $M_{\mathcal{P}_\alpha}$), then $\{U_\alpha\}$ is unitary in $B$ and $\pi(\{U_\alpha\}) = U$.

Here is a simple application that gives the flavor of our results.

Corollary 2 Every free group is RFD.

Proof. Suppose $F$ is a free group and $\mathcal{A} = C^*(F) = C^*(\{U_g : g \in F\})$. Choose a Hilbert space $H$ and a faithful representation $\rho : \mathcal{A} \to B(H)$. Choose a net $\{P_\alpha\}$ of finite-rank projections such that $P_\alpha \to 1$ (SOT). Applying Lemma 1, we have, for each $g \in F$, we can find a unitary element $\{U_{g,\alpha}\}$ in $B$ so that $\pi(\{U_{g,\alpha}\}) = U_g$. For each $\alpha$, we have a unitary group representation $\sigma_\alpha : F \to M_{\mathcal{P}_\alpha}$ defined by $\sigma_\alpha(g) = U_{g,\alpha}$. By the definition of $C^*(F)$, there is a $*$-homomorphism $\tau_\alpha : \mathcal{A} \to \mathcal{M}_\alpha$ such that $\tau_\alpha(U_g) = U_{g,\alpha}$. It follows that $\tau : \mathcal{A} \to B$ define by $\tau(U_g) = \{U_{g,\alpha}\}$ is a $*$-homomorphism such that $\pi \circ \tau = \rho$. Hence the direct sum of the $\tau_\alpha$’s is faithful, which shows that $\mathcal{A}$ is RFD.

The following corollary is from [3, Exercise 7.1.4].

Corollary 3 Every $C^*$-algebra is a $*$-homomorphic image of an RFD $C^*$-algebra.

Proof. Suppose $\mathcal{A}$ is a $C^*$-algebra. We can assume that $\mathcal{A} \subseteq B(H)$ for some Hilbert space $H$. Choose a net $\{P_\alpha\}$ of finite-rank projections converging $*$-strongly to 1, and let $\mathcal{B}$, $\mathcal{J}$ and $\pi$ be as in Lemma 1. Then $\mathcal{B}$, and thus $\pi^{-1}(\mathcal{A})$, is RFD and $\pi(\pi^{-1}(\mathcal{A})) = \mathcal{A}$. ■

2 $R_{<k}D$ Algebras

We now prove our main results on $R_{<k}D$ $C^*$-algebras. The following two lemmas contain the key tools.

Lemma 4 Suppose $\aleph_0 \leq k \leq m$, and $\mathcal{A}$ is $R_{<k}D$ and $m$-generated. Then

1. We can write $H_m = \bigoplus_{\lambda \in \Lambda} X_\lambda$ with $\text{Card} \Lambda = m$, and such that, for every $\lambda \in \Lambda$, $\dim X_\lambda < k$ and there is a unital representation $\pi_\lambda : \mathcal{A}^+ \to B(X_\lambda)$ such that the representation $\pi : \mathcal{A}^+ \to B(H_m)$ defined by $\pi = \bigoplus_{\lambda \in \Lambda} \pi_\lambda$ is faithful. Moreover, this can be done so that, for each $\lambda_0 \in \Lambda$, we have $\text{Card}(\{\lambda \in \Lambda : \pi_\lambda \approx \pi_{\lambda_0}\}) = m$.  

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2. It is possible to choose the decomposition in (1) so that, for each cardinal 
\( s < k \), there is a \( \lambda \in \Lambda \) such that \( \dim X_\lambda = s \).

**Proof.** Since \( \mathcal{A} \) is \( R_{<k} D \), there is a direct sum of representations in \( \text{Rep}_k (\mathcal{A}) \) whose direct sum is faithful. Suppose \( D \) is a generating set for \( \mathcal{A} \) and \( \text{Card} (D) \leq m \). We can replace \( D \) by the \(*\)-algebra over \( \mathbb{Q} + i\mathbb{Q} \) generated by \( D \) without making the cardinality exceed \( m \). For each \( a \in D \) we can choose a direct sum of countably many summands from our faithful direct sum that preserves the isometry on \( D \).

Suppose we can write \( X = \bigoplus_i X_i \) such that, for each cardinal \( s < k \), there is a unital one-dimensional representation, we know that, for every cardinal \( s < k \), there is a representation of \( \mathcal{A}^+ \) of dimension \( s \). If we take one such representation for each \( s < k \) and take a direct sum of \( m \) copies of all of them, we get a representation that has dimension at most \( m \), so we add this as a summand to the representation we constructed satisfying (1). □

**Lemma 5** Suppose \( \mathcal{A} \) is a \( \mathcal{C}^* \)-algebra and \( k \leq m \) are infinite cardinals and \( D \) is a generating set for \( \mathcal{A} \). Suppose we can write \( H_m = \bigoplus_{\lambda \in \Lambda} X_\lambda \) and \( \pi = \bigoplus_{\lambda \in \Lambda} \pi_\lambda \) as in part (1) of Lemma 4. If \( \rho : \mathcal{A}^+ \to B (H_m) \) is a unital representation, then, for every \( \varepsilon > 0 \), every finite subset \( W \subseteq D \) and every finite subset \( E \subseteq H_m \), there is a finite subset \( F \subseteq \Lambda \), such that, for every finite set \( G \) with \( F \subseteq G \subseteq \Lambda \), if \( Q_G \) is the orthogonal projection onto \( \bigoplus_{\lambda \in G} X_\lambda \), then there is a unitary \( U \in Q_G B (H_m) Q_G \) such that, for every \( a \in W \) and \( e \in E \), we have

\[
\| [\rho (a) - U_G^* \pi (a) U_G] e \| = \left\| \left[ \rho (a) - U_G^* \left( \sum_{\lambda \in G} \pi_\lambda \right) (a) U_G \right] e \right\| < \varepsilon .
\]

**Proof.** It follows that if \( a \in \mathcal{A} \) and \( a \neq 0 \), then \( \text{rank} \pi (a) = m = \text{rank} (\pi \oplus \rho) (a) \). Hence, by [8], \( \pi \) is approximately unitarily equivalent to \( \pi \oplus \rho \). However, by [9], \( \rho \) is a point-\( \ast \)-SOT limit of representations unitarily to \( \rho \). Hence there is a net \( \{ U_\alpha \} \) of unitary operators in \( B (H_m) \) such that, for every \( a \in \mathcal{A} \),

\[
( \ast-\text{SOT}) \lim_{\alpha} U_\alpha^* \pi (a) U_\alpha = \rho (a) .
\]

However, the net \( \{ Q_F : F \subseteq \Lambda, \text{ } F \text{ is finite} \} \) is a net of projections converging \( \ast \)-strongly to 1. Hence, by Lemma [10] each \( U_\alpha \) is a \( \ast \)-SOT limit of unitaries in the union of \( Q_F B (H_m) Q_F \) \( (F \subseteq \Lambda, \text{ } F \text{ is finite}) \). The result now easily follows. □

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**Theorem 6** Suppose $\aleph_0 \leq k \leq m$, and $\mathcal{A}$ is $m$-generated with a generating set $\mathcal{G}$ with $\text{Card}\mathcal{G} \leq m$. The following are equivalent.

1. $\mathcal{A}$ is $R_{<k} D$.

2. There is a faithful unital $*$-homomorphism $\rho : \mathcal{A}^+ \to B(H_m)$ such that, for every $\varepsilon > 0$, every finite subset $E \subseteq H_m$ and every finite subset $W \subseteq G$, there is a projection $P \in B(H_m)$ and a unital $*$-homomorphism $\tau : \mathcal{A} \to \mathcal{M}_P = PB(H_m)P$ such that, for every $e \in E$ and every $a \in W$ we have $||[\tau(a) - \rho(a)]e|| < \varepsilon$.

3. There is a faithful unital representation $\rho : \mathcal{A}^+ \to B(H_m)$ and a net $\{P_\alpha\}$ of projections in $B(H_m)$, each with rank less than $k$, such that $P_\alpha \to 1$ (SOT) and such that, for each $\alpha$, there is a representation $\pi_\alpha : \mathcal{A} \to \mathcal{M}_{P_\alpha}$ such that, for every $a \in \mathcal{A}$, we have $\pi_\alpha(a) \to \rho(a)$ (SOT).

4. For every unital representation $\rho : \mathcal{A}^+ \to B(H_m)$ there is a net $\{P_\alpha\}$ of projections in $B(H_m)$, each with rank less than $k$, such that $P_\alpha \to 1$ (SOT) and such that, for each $\alpha$, there is a representation $\pi_\alpha : \mathcal{A} \to \mathcal{M}_{P_\alpha}$ such that, for every $a \in \mathcal{A}$, we have $\pi_\alpha(a) \to \rho(a)$ (SOT).

**Proof.** (2) $\implies$ (1) Let $A$ be the set of triples $(\varepsilon, E, W)$ ordered by $(\geq, \subseteq, \subseteq)$. If $\alpha = (\varepsilon, E, W)$ let $\tau_\alpha : \mathcal{A} \to P_\alpha B(H_m)P_\alpha$ guaranteed by (2). Since $\mathcal{G} = \mathcal{G}^*$ we have $$(\text{SOT}) \lim_\alpha \tau_\alpha(a) = \rho(a)$$
for every $a \in \mathcal{G}$. Since $\rho$ and each $\tau_\alpha$ is a $*$-homomorphism, the set of $a \in \mathcal{A}$ for which $(\text{SOT}) \lim_\alpha \tau_\alpha(a) = \rho(a)$ is a unital $C^*$-algebra and is thus $\mathcal{A}^+$. Hence, for every $a \in \mathcal{A}^+$, we have
$$\|a\| = \|ho(a)\| \leq \sup \{\|\tau_\alpha(a)\| : \alpha \in \mathcal{A}\}.$$ Consequently, the direct sum of the $\tau_\alpha$’s is faithful and (1) is proved.

(3) $\implies$ (2). This is obvious.

(4) $\implies$ (3). It is clear that we need only show that there is a faithful unital representation $\rho : \mathcal{A}^+ \to B(H_m)$. Suppose $\tau : \mathcal{A}^+ \to B(M)$ is an irreducible representation, and suppose $D$ is a generating set with $\text{Card}(D) \leq m$. Let $\mathcal{A}_0$ be the unital subalgebra of $\mathcal{A}^+$ over the field $\mathbb{Q} + i\mathbb{Q}$ of complex rational numbers. Then $\mathcal{A}_0$ is norm dense in $\mathcal{A}$ and $\text{Card} \mathcal{A}_0 = \text{Card} D \leq m$. Suppose $f \in M$ is a unit vector. Since $\tau$ is irreducible, $\tau(\mathcal{A}_0)f$ must be dense in $M$. Suppose $B$ is an orthonormal basis for $M$, and, for each $e \in B$ let $U_e$ be the...
open ball centered at \( e \) with radius \( \sqrt{2}/2 \). Each \( U_e \) must intersect the dense set \( \tau(\mathcal{A}_0)f \), and since the collection \( \{U_e: e \in B\} \) is disjoint, we conclude that

\[
\dim M = \text{Card}B \leq \text{Card} \tau(\mathcal{A}_0)f \leq \text{Card} \mathcal{A}_0 \leq m.
\]

We know that for every \( x \in \mathcal{A}_0 \) there is an irreducible representation \( \tau_x: \mathcal{A}^+ \rightarrow B(M_x) \) such that \( \|\tau_x(x)\| = \|x\| \). Since \( \dim \sum_{x \in \mathcal{A}_0} M_x \leq m \cdot m = m \), there is a representation \( \rho: \mathcal{A}^+ \rightarrow B(H_m) \) that is unitarily to a direct sum of \( m \) copies of \( \sum_{x \in \mathcal{A}_0} \tau_x \). Hence \( \rho \) is isometric on the dense subset \( \mathcal{A}_0 \), which implies \( \rho \) is faithful.

(1) \( \Rightarrow \) (3). Since \( \mathcal{A} \) is \( R_{<k}D \), we can choose a decomposition \( H_m = \oplus_{\lambda \in \Lambda} X_{s,\lambda} \) and representation \( \pi = \sum_{\lambda \in \Lambda} \pi_{s,\lambda} \) as in part (1) of Lemma 4. Now (3) follows from Lemma 5.}

We see that the class of \( R_{<k}D \) algebras is closed under arbitrary free products in the nonunital category of \( C^* \)-algebras.

**Theorem 7** Suppose \( k \) is an infinite cardinal and \( \{ \mathcal{A}_i : i \in I \} \) is a family of \( R_{<k}D \) \( C^* \)-algebras. Then the free product \( *_{i \in I} \mathcal{A}_i \) is \( R_{<k}D \).

**Proof.** Choose an infinite cardinal \( m \geq k + \sum_{i \in I} \text{Card}(\mathcal{A}_i) \). Since \( *_{i \in I} \mathcal{A}_i \) is generated by \( \mathcal{G} = \bigcup_{i \in I} \mathcal{A}_i \setminus \{0\} \subseteq *_{i \in I} \mathcal{A}_i \), clearly \( *_{i \in I} \mathcal{A}_i \) is \( m \)-generated. Choose a set \( \Lambda \) with \( \text{Card}(\Lambda) = m \) and let \( S \) be the set of cardinals less than \( k \). Write

\[
H_m = \sum_{s \in S} \sum_{\lambda \in \Lambda} X_{s,\lambda}
\]

where \( \dim X_{s,\lambda} = s \) for every \( s \in S \) and \( \lambda \in \Lambda \). It follows that, for each \( i \in I \), we can find a representation \( \pi^i: \mathcal{A}_i \rightarrow B(H_m) \) such that

\[
\pi^i = \sum_{s \in S} \sum_{\lambda \in \Lambda} \pi_{s,\lambda}^i
\]

satisfying (1) and (2) of Lemma 4. Suppose \( \varepsilon > 0 \), \( E \subseteq H_m \) is finite and \( W \subseteq \mathcal{G} \) is finite. We can write \( W \) as a disjoint union of \( W_1, \ldots, W_n \) with \( W_i = W \cap \mathcal{A}_i \). Let \( \rho_i \) be the restriction of \( \rho \) to \( \mathcal{A}_i \). Applying Lemma 5 to \( \mathcal{A}_{ij} \) and \( \rho_{ij} \) and \( \pi^i \) for \( 1 \leq j \leq n \), we can find one finite subset \( G \subseteq S \times \Lambda \) so that if \( P \) is the projection on \( \sum_{(s,\lambda) \in G} X_{s,\lambda} \), then there are unitary operators
\[U_{i_1}, \ldots, U_{i_n} \in \mathcal{M}_P = PB(H_m)P\] so that, for \(1 \leq j \leq n\), \(a \in W_j\), \(e \in E\), we have
\[\|\left[\rho_{ij}(a) - U_{ij}^* \pi^j(a)U_{ij}\right]e\| < \varepsilon.\]

Define \(\tau_{ij} : \mathcal{A}_{i_j}^+ \to \mathcal{M}_P\) by
\[\tau_{ij}(a) = U_{ij}^* \pi^j(a)U_{ij},\]
and for \(i \in \mathbb{I} \setminus \{i_1, \ldots, i_n\}\) define \(\tau_i : \mathcal{A}_i \to \mathcal{M}_P\) by
\[\tau_i(a) = P \pi^i(a)P.\]

Then, by the definition of free product, there is a representation \(\tau_\mathcal{A}^+ : \mathbb{I} \to \mathcal{M}_P\) such that \(\tau_\mathcal{A}^+ = \tau_i\) for every \(i \in \mathbb{I}\). It follows that, for every \(e \in E\) and every \(a \in W\),
\[\|\left[\rho(a) - \tau(a)\right]e\| < \varepsilon.\]

It follows from part (2) of Lemma 6 that \(* \mathbb{A}_i\) is \(R_{<k}D\). ■

**Corollary 8** Suppose \(k\) is an infinite cardinal and \(\{\mathcal{A}_i : i \in \mathbb{I}\}\) is a family of \(R_{<k}D\) \(C^*\)-algebras such that each \(\mathcal{A}_i\) has a one-dimensional unital representation. Then the unital free product \(* \mathbb{A}_i\) is \(R_{<k}D\).

**Proof.** This follows from the fact that if \(\tau_i : \mathcal{A}_i \to \mathbb{C}\) is a unital \(*\)-homomorphism for each \(i \in \mathbb{I}\), then \(* \mathbb{A}_i\) is \(*\)-isomorphic to \(\left(* \ker \tau_i\right)^+\). ■

Without the condition on unital one-dimensional representations, the preceding corollary is false. For example, \(* \mathcal{M}_n(\mathbb{C})\) is not \(RFD\) (\(= R_{<\aleph_0}D\)), even though each \(\mathcal{M}_n(\mathbb{C})\) is \(RFD\). The reason is that each unital representation of the free product must be injective on each \(\mathcal{M}_n(\mathbb{C})\) and must have infinite-dimensional range. Call an infinite cardinal \(k\) a **limit cardinal**, if \(k\) is the supremum of all the cardinals less than \(k\).

However, there is something we can say about the general situation. If \(k\) is a limit cardinal, the cofinality of \(k\) is the smallest cardinal \(s\) for which there is a set \(E\) of cardinals less than \(k\) whose supremum is \(k\). Clearly, the cofinality of \(k\) is at most \(k\). If \(k\) is not a limit cardinal, then there is a cardinal \(s\) such that \(k\) is the smallest cardinal larger than \(s\), and if \(E\) is a set of cardinals less than \(k\), then \(\sup(E) \leq s < k\).

**Theorem 9** Suppose \(k\) is an infinite cardinal and \(\{\mathcal{A}_i : i \in \mathbb{I}\}\) is a family of unital \(R_{<k}D\) \(C^*\)-algebras. Then

1. If \(k\) is a limit cardinal and \(\text{Card} (\mathbb{I})\) is less than the cofinality of \(k\), then the free product \(* \mathbb{A}_i\) is \(R_{<k}D\).
2. If $k$ is not a limit cardinal, then the free product $\ast_{i \in I} A_i$ is $R_{<k} D$.

Proof. (1). Choose $m \geq k + \sum_{i \in I} \text{Card} (A_i)$, and choose a set $\Lambda$ with $\text{Card} (\Lambda) = m$. Using Lemma ?? we can, for each $i \in I$, find a faithful representation $\pi^i = \sum_{\lambda \in \Lambda} \pi^i_{\lambda, i}$ so that $\dim \pi^i = m$ and, for every $i \in I$ and $\lambda \in \Lambda$, we have $\dim \pi^i_{\lambda, i} < k$. Since $\text{Card} (I)$ is less than the cofinality of $k$, we have, for each $\lambda \in \Lambda$, a cardinal $s_\lambda < k$ such that $\sup_{i \in I} \dim \pi^i_{\lambda, i} \leq s_\lambda$. If we replace each $\pi^i_{\lambda, i}$ with a direct sum of $s_\lambda$ copies of itself, we get a new decomposition which we will denote by the same names such that, for each $i$ and each $\lambda$ we have $\dim \pi^i_{\lambda, i} = s_\lambda$. Hence we may write direct sum decompositions of the $\pi^i$’s with respect to a common decomposition $H_m = \sum_{\lambda \in \Lambda} X_\lambda$ where $\dim X_\lambda = s_\lambda$ for every $\lambda \in \Lambda$. The rest now follows as in the proof of Theorem 7.

(2) If $k$ is not a limit cardinal, there is a largest cardinal $s < k$. Repeat the proof of part (1) with $s_\lambda = s$ for every $\lambda \in \Lambda$. 

Remark 10 We cannot remove the condition on $\text{Card} (I)$ in part (1) of Theorem ???. Suppose $k$ is a limit cardinal and $I$ is a set of cardinals less than $k$ whose cardinality equals the cofinality of $k$ and such that $\text{sup} (I) = k$. For each infinite cardinal $m$, choose a set $\Lambda_m$ with cardinality $m$, and let $S_m$ denote the universal unital C*-algebra generated by $\{ v_\lambda : \lambda \in \Lambda_m \}$ with the conditions

1. $v_\lambda^* v_\lambda = 1$ for every $\lambda \in \Lambda_m$,
2. $v_{\lambda_1} v_{\lambda_1}^* v_{\lambda_2} v_{\lambda_2}^* = 0$ for $\lambda_1 \neq \lambda_2$ in $\Lambda_m$.

Since $S_m$ is $m$-generated, it follows that every irreducible representation of $S_m$ is at most $m$-dimensional (see the proof of (4) $\Rightarrow$ (3) in Theorem 6). Hence $S_m$ is separated by $m$-dimensional representations. On the other hand, if $\pi$ is a unital representation of $S_m$, then $\{ \pi (v_\lambda v_\lambda^*) : \lambda \in \Lambda_m \}$ is an orthogonal family of nonzero projections, which implies that the dimension of $\pi$ is at least $m$. It follows that each $S_s$ is $R_{<k} D$ for $s \in I$. However, any unital representation $\pi$ of the free product $\ast_{s \in I} S_s$ must induce a unital representation of each $S_s$, so its dimension is at least $\text{sup} I = k$. Hence $\ast_{s \in I} S_s$ is not $R_{<k} D$.

3 Separable RFD Algebras

In this section we show that for a separable C*-algebra being RFD is equivalent to a lifting property.
Suppose \( \{e_1, e_2, \ldots \} \) is an orthonormal basis for a Hilbert space \( \ell^2 \), and, for each integer \( n \geq 1 \), let \( P_n \) be the projection onto \( sp(\{e_1, \ldots, e_n\}) \). Let \( M_n = P_n B(\ell^2) P_n \) for \( n \geq 1 \), and, following Lemma 1, let

\[
B = \left\{ \{T_n\} \in \prod_{n=1}^{\infty} M_n : \exists T \in B(\ell^2) \text{ with } T_n \to T \ (\ast \text{-SOT}) \right\},
\]

and let

\[
J = \{ \{T_n\} \in B : T_n \to 0 \ (\ast \text{-SOT}) \}.
\]

Then, by Lemma 1, we have that \( B \) is a unital C*-algebra, \( J \) is a closed ideal in \( B \) and

\[
\pi (\{T_n\}) = (\ast \text{-SOT})- \lim_{n \to \infty} T_n
\]

defines a unital surjective \( \ast \)-homomorphism from \( B \) to \( B(H) \) whose kernel is \( J \).

We can now give our characterization of RFD for separable C*-algebras.

**Theorem 11** Suppose \( A \) is a separable C*-algebra. The following are equivalent.

1. \( A \) is RFD
2. For every unital \( \ast \)-homomorphism \( \rho : A^+ \to B(\ell^2) \) there is a unital \( \ast \)-homomorphism \( \tau : A^+ \to B \) such that \( \pi \circ \tau = \rho \).

**Proof.** The implication (2) \( \implies \) (1) is clear.

(1) \( \implies \) (2). Suppose \( A = C^*(\{a_1, a_2, \ldots \}) \) is RFD and \( \rho : A^+ \to B(\ell^2) \) is a unital \( \ast \)-homomorphism. It follows from Theorem 3 that there is an increasing sequence \( \{n_k\} \) of positive integers and unital \( \ast \)-homomorphisms \( \tau_k : A \to M_{n_k} \) such that

\[
\| (\tau_k (a_j) - \rho(a_j)) e_i \| < 1/k
\]

for \( 1 \leq i, j \leq k \). It follows that \( \tau_{n_k} (a) \to \rho(a) \ (\ast \text{-SOT}) \) for every \( a \in A^+ \). If \( n_k < n < n_{k+1} \), we define \( \tau_n : A^+ \to M_n \) by

\[
\tau_n (a) = \begin{pmatrix} \tau_{n_k} (a) \\ \beta (a) \\ \vdots \\ \beta (a) \end{pmatrix},
\]

where \( \beta : A^+ \to \mathbb{C} \) is the unique \( \ast \)-homomorphism with \( \ker \beta = A \), relative to the decomposition

\[
P_n (\ell^2) = P_{n_k} (\ell^2) \oplus C e_{1+n_k} \oplus \cdots \oplus C e_{-1+n_{k+1}}.
\]

It is easily seen that \( \tau_n (a) \to \rho(a) \ (\ast \text{-SOT}) \) for every \( a \in A^+ \). If we define \( \tau : A \to B \) by

\[
\tau (a) = \{ \tau_n (a) \},
\]

we see that \( \pi \circ \tau = \rho \). \( \blacksquare \)
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