Existence of periodic planar standing waves in phase-transitional viscoelasticity with surface energy

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Abstract

Extending investigations of Antman&Malek-Madani, Schecter&Shearer, Slemrod, Barker&Lewicka&Zumbrun, and others, we investigate viscoelasticity models with surface energy effects. We prove the existence of nonconstant planar periodic standing waves in viscoelasticity models with surface energy terms by variational methods and phase-plane analysis, for deformations of arbitrary dimension. For our variational analysis, we require that the mean vector of the unknowns over one period be in the elliptic region with respect to the corresponding inviscid (i.e., elastic) model. For our (1-D) phase-plane analysis, we have no such restriction, obtaining essentially complete information on the existence of nonconstant periodic waves and bounding homoclinic/heteroclinic waves. Our variational framework has implications also for time-evolutionary stability, through the link between the action potential for the traveling-wave ODE and the relative entropy for the time-evolutionary system.

Keywords: viscoelasticity, surface energy, periodic wave, Hamiltonian system.

1 Introduction

Viscoelasticity models are models taking both the viscosity and elasticity effects of the materials into account when we describe the physical phenomena in mathematical modeling. Since they have a wide range of applications, the study of viscoelasticity has been an important topic (see [AB, AM, BLeZ, D, FP, K, RZ, S1, S2, S3, SS, Z1, Z2], etc., and references therein). However, up to now, to the best of our knowledge, the study of phase-transitional viscoelasticity has been carried out only for phenomenological models [S1, S2, S3, SS, Z2] for one-dimensional shear flow, with classical double well potential and artificial viscosity–capillarity terms. The treatment of the general, physical, case was cited in [BLeZ], Appendix B, as an important direction for further study.

In this paper, continuing the work of Antman and Malek-Madani [AM], Slemrod [S1, S2, S3], Schecter and Shearer [SS], and Barker, Lewicka and Zumbrun [BLeZ], we study the

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existence of planar viscoelastic periodic traveling waves, compressible or incompressible, for
deformations of arbitrary dimension, starting from the most general form of the physical
equations. It turns out for general viscoelasticity model with surface energy terms that,
similarly as observed for the phenomenological models studied previously, the periodic trav-
eling waves can only be standing waves and the corresponding ordinary differential equation
(ODE) system exhibits Hamiltonian structure.

As we know, for a planar Hamiltonian system, we can use phase-plane analysis to study
its closed orbits. This corresponds to the case of one-dimensional deformations, for which
the unknowns are scalar in the ODE system (4.5) that we obtain. For higher dimensional
Hamiltonian systems, this method does not apply directly. In order to prove the existence
of non-constant periodic waves when the unknowns are vectors, we consider the problem
under the framework of calculus of variation. There are several problems to overcome.
First, we need to formulate the problem in proper Banach spaces. It turns out that the
proper space for this is the periodic Sobolev space with mean zero property. Working in
this framework amounts to prescribing the mean of the unknown over one periodic (no real
restriction, since each periodic wave has a mean as long as it exists).

Second, we need to make sure that the waves we find are not constant waves. We
overcome this issue by considering the equations satisfied by the difference between the
original unknown and its mean. This makes the 0 element in our working space always a
critical point, which helps to eliminate the possibility that the solutions we find are trivial.

Third, in the global model of viscoelasticity, we need to consider the physical assumption
(see Section 2 or 3), namely \(\tau_3 > 0\). This makes our variational problem a constrained one
(see Remark 6.3).

This kind of constraint usually leads to a variational inequality and is related to an
obstacle problem. Meanwhile, this constraint makes our admissible set (to make the wave
physically meaningful) not weakly closed. However, the asymptotic behavior of the elastic
potential function (see (2.4)) will help overcome the related problem. We note that in the
pure elastic case without surface energy and viscosity, this restriction on \(\tau_3\) imposes signifi-
cant challenge in the mathematical analysis (see the discussions in [Ba, AN]). However, the
additional regularity imposed by the surface energy term removes this difficulty.

Besides the Hamiltonian structure of the standing wave equations, we prove that for
the general physical model, there exist non-constant periodic waves no matter whether the
unknowns are scalar or not (see Theorem 6.12) under assumptions on the mean vector and
periodic of the wave. For some specific phase-transitional models, we given explicit condi-
tions under which the non-constant oscillatory waves exist (see Section 7). In particular,
for the one dimensional models, we use phase-plane analysis to get detailed information on
the wave phenomena (existence of periodic, homoclinic, heteroclinic waves, see Section 8).
The issue of their stability falls into the framework of [HZ, JZ] because the systems are
effectively strictly parabolic (in the sense that it may be transformed to a strictly parabolic
system; see Appendix B) and hence are completely understood.

Comparing our results with others (see the references of this paper), the problems here
is interesting enough even only from the modeling point of view, without even finding any
waves. In [AM], the authors treated the shear flow without the surface effect and with isotropic assumption preventing phase transition model (see discussion in [BLeZ]). Here we consider models with surface energy and the materials are anisotropic, which gives rich wave phenomena. Antmann, Slemrod and others (see [A, AM, S1, S2, S3] and references therein) have previously studied phenomenological 1D phase-transitional models with double-well potentials. Here we justify those types of qualitative models by direct derivation from the physical shear flow model; see Section 5.7.

It is well known that there are many outstanding open problems in elasticity and viscoelasticity ranging from steady state to dynamical ones (see [Ba]). Due to surface energy effect and associated higher regularity, we can give a neat treatment of our problem. It would be very interesting to explore by numerics cases that do not fit the hypotheses here (which are sufficient but by no means necessary in the non-scalar case) but nonetheless support periodic waves and also to explore either numerically or analytically the spectral stability of these waves. We hope to address these issues in a followup work [BYZ].

2 Viscoelasticity models with surface energy term

In this section, we will follow the presentations of [AN, Ba, BLeZ, NPT]. Let $\Omega$ be the reference configuration which models a viscous elastic body with constant temperature and density. A typical point in $\Omega$ will be denoted by $X$. We use $\xi : \Omega \times \mathbb{R}_+ \rightarrow \mathbb{R}^3$ to denote the deformation (i.e., the deformed position of the material point $X$). Consequently, the deformation gradient is given by $F = \nabla X \xi$, which we regard as an element in $\mathbb{R}^{3 \times 3}$.

Adopting the notations above, the equations of isothermal viscoelasticity with surface energy effect are given through the following balance of linear momentum modified with surface energy

\begin{equation}
\xi_{tt} - \nabla X \cdot \left( DW(\nabla \xi) + Z(\nabla \xi, \nabla \xi_t) - E(\nabla^2 \xi) \right) = 0.
\end{equation}

We make the following physical constraint on the deformation gradient (see [Ba, BLeZ] and [AN, NPT] for the physical background), prohibiting local self-impingement of the material:

\begin{equation}
\det F > 0.
\end{equation}

In (2.1), the operator $\nabla X \cdot$ stands for the divergence of an approximate field. As in [D, NPT], for a matrix-valued vector field, we use the convention that the divergence is taken row-wise. In what follows, we shall also use the matrix norm $|F| = (\text{tr}(F^T F))^{1/2}$, which is induced by the inner product: $F_1 : F_2 := \text{tr}(F_1^T F_2)$.

In view of the second law of thermodynamics (see [Ba, PB]), the Piola-Kirchhoff stress tensor $DW : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}^{3 \times 3}$ is expressed as the derivative of an elastic energy density $W : \mathbb{R}^{3 \times 3} \rightarrow \mathbb{R}_+$. Throughout the paper, we assume as in [AN, Ba, NPT] the elastic energy density function $W$ which is the stored-energy function for the viscoelastic material is frame-indifference. Let $SO(3)$ be the group of proper rotations in $\mathbb{R}^3$. Then the frame-indifference...
assumption can be formulated as

\[(2.3) \quad W(RF) = W(F), \quad \forall F \in \mathbb{R}^{3 \times 3}, \quad \forall R \in SO(3).\]

Also, the material consistency (to avoid interpenetration of matter, (2.2), [AN, Ba]) requires the following important assumption:

\[(2.4) \quad W(F) \to +\infty \quad \text{as} \quad \det F \to 0.\]

We emphasize that viscous stress tensor \(Z : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}\) depends on both the deformation gradient \(F\) and the velocity gradient \(Q = F_t = \nabla \xi_t = \nabla v\), where \(\tau = \xi_t\). From physical point of view, the stress tensor \(Z\) should also be compatible with principles of continuum mechanics: balance of angular momentum, frame invariance, and the Claussius-Duhem inequality. For the related mathematical description and corresponding stress forms see [AN, Ba, BLeZ].

The surface energy \(E\) is given by

\[E(\nabla^2 \xi) = \nabla X \cdot D\Psi(\nabla^2 \xi) = \left[ \sum_{i=1}^{3} \frac{\partial}{\partial X_i} \left( \frac{\partial}{\partial \xi_j \xi_k} \Psi(\nabla^2 \xi) \right) \right]_{j,k:1...3}\]

for some convex density \(\Psi : \mathbb{R}^{3 \times 3} \to \mathbb{R}\), compatible with frame indifference.

The corresponding inviscid part of system (2.1)

\[(2.5) \quad \xi_{tt} - \nabla X \cdot \left( D\Psi(\nabla^2 \xi) \right) = 0\]

can be written as

\[(2.6) \quad (F, \tau)_t + \sum_{i=1}^{3} \partial X_i \left( \tilde{G}_i(F, \tau) \right) = 0.\]

Above, \((F, \tau) : \Omega \to \mathbb{R}^{12}\) represents conserved quantities, while \(\tilde{G}_i : \mathbb{R}^{12} \to \mathbb{R}^{12}\) are given by

\[-\tilde{G}_i(F, \tau) = r^1 e_i \oplus r^2 e_i \oplus r^3 e_i \oplus \left[ \frac{\partial}{\partial F_{ki}} W(F) \right]_{k=1}^{3}, \quad i = 1...3\]

are the fluxes, and \(e_i\) denotes the \(i\)-th coordinate vector in \(\mathbb{R}^3\).

The convex density \(\Psi\) contributes to equation (2.1) the term

\[(2.7) \quad \nabla X \cdot \left( \mathcal{E}(\nabla^2 \xi) \right) = \nabla X \cdot \left\{ \nabla X \cdot D\Psi(\nabla^2 \xi) \right\}\]

In view of the orders of differentiation and convexity of \(\Psi\), we may assume that

\[\Psi \geq 0; \quad \Psi(0) = 0; \quad D\Psi(0) = 0; \quad \delta Id \leq D^2\Psi(\cdot) \leq MId\]
where $\delta, M$ are two positive real numbers. The mapping relations (ignoring physical constraints) are

$$s_i : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to \mathbb{R}_+;$$

$$D\Psi : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3};$$

$$D^2\Psi : \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \times \mathbb{R}^{3 \times 3} \to \mathcal{L}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$$

and $\text{Id}$ is an element in the space $\mathcal{L}(\mathbb{R}^{3 \times 3}, \mathbb{R}^{3 \times 3})$. When the operator $\nabla X \cdot$ reduces to the operator $\partial_x$ where $x$ is a one dimension variable, (2.7) takes the form

$$\partial_x \{\partial_x D\Psi(\partial_x^2 \xi)\}.$$ 

If we identify $\xi_x$ as $\tau$, then $\partial_x^2 \xi = \tau_x$ and (2.7) becomes:

$$\partial_x \{\partial_x D\Psi(\tau_x)\} = \partial_x \{D^2\Psi(\tau_x)\tau_{xx}\}.$$ 

Note that $D^2\Psi : \mathbb{R}^{3} \to \mathcal{L}(\mathbb{R}^{3}; \mathbb{R}^{3})$ when $\nabla X \cdot$ reduces to $\partial_x$. So we assume that $D^2\Psi(\cdot)$ as matrix function satisfy the following assumption:

$$\delta \text{Id} \leq D^2\Psi(\cdot) \leq M \text{Id} \quad \text{as operators.}$$

These mapping relations enable us to decide the proper form of surface energy term contributed to equation (2.1). Keeping this in mind, we study specific equations in the following sections.

3 Equations and specific models

In this paper, we focus on the interesting subclass of planar solutions, which are solutions in the full 3D space that depend only on a single coordinate direction; that is, we investigate deformations $\xi$ given by

$$\xi(X) = X + U(z), \quad X = (x, y, z), \quad U = (U_1, U_2, U_3) \in \mathbb{R}^{3}.$$ 

Corresponding to the above deformation or displacement $\xi$, the deformation gradient is given by taking the gradient with respect to $X$

$$(3.1) \quad F = \begin{pmatrix} 1 & 0 & U_{1,z} \\ 0 & 1 & U_{2,z} \\ 0 & 0 & 1 + U_{3,z} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}.$$ 

We shall denote $V = (\tau, u) = (\tau_1, \tau_2, \tau_3, u_1, u_2, u_3)$, where $\tau_1 = U_{1,z}, \tau_2 = U_{2,z}, \tau_3 = 1 + U_{3,z}$ and $u_1 = U_{1,t}, u_2 = U_{2,t}, u_3 = U_{3,t}$ with the physical constraint

$$\tau_3 > 0,$$

corresponding to $\det F > 0$ in the region of physical feasibility of $V$. 
Writing $W(\tau) = W\left( \begin{bmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{bmatrix} \right)$, we see that for all $F$ as in (2.1) there holds

$$\nabla \cdot (DW(F)) = (D\tau W(\tau))_z.$$ 

That is, the planar equations inherit a vector-valued variational structure echoing the matrix valued variational structure (note that the left hand side is the divergence of $DW(F)$).

In this paper, we first study the problems (traveling wave ODEs, Hamiltonian ODEs, existence of standing waves) for general elastic potential energy and give a rather general abstract existence result. Then we study local models by specifying the related terms in system (2.1) as follows:

1. **Viscous elastic potential $W$.** A typical model, as described in [FP], has the form:

$$W(\tau) = |F^T F - C_-|^2 \cdot |F^T F - C_+|^2,$$

which is a potential for anisotropic material with material frame indifference property. We study models involving this phase-transitional elasticity potential (See Appendix B for related computations). It is important to notice that the elastic potential here does not satisfy the asymptotic behavior when $\det F \to 0^+$. Especially this is an irrelevant assumption for shear models. Hence the related models are local models for the real physics.

2. **Viscous stress tensor $Z$.** We use the following viscous stress tensor which is compatible with the principles of continuum mechanics (see [BLeZ]):

$$Z(\tau, Q) = 2(\det F) \text{sym}(Q F^{-1}) F^{-1} T.$$

We note that the related Cauchy stress tensor $T_2 = 2(\det F)^{-1} Z F^T = 2\text{sym}(Q F^{-1})$ is the Lagrangian version of the stress tensor $2\text{sym}\nabla v$ written in the Eulerian coordinates. For incompressible fluids $2\text{div}(\text{sym}\nabla v)$, giving the usual parabolic viscous regularization of the fluid dynamics evolutionary system.

3. **The surface energy $\mathcal{E}$.** A typical example of the energy density function is $\Psi(P) = \frac{1}{2} |P|^2$, so that

$$\mathcal{E}(\nabla^2 \xi) = \nabla \cdot \nabla^2 \xi = \triangle_X F,$$

which is an extension of the 1D case of [S1]. For simplicity, we take this surface energy to study the wave phenomena. We see that it is the surface energy term that makes the models have abundant wave phenomena.

**The system.** In order to be consistent with the usual presentation, we use $x \in \mathbb{R}^1$ as the space variable instead of $z$, i.e. $z \to x$. With the above choices, we have the system

$$\begin{cases}
\tau_t - u_x = 0; \\
u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(\tau_x)\tau_{xx})_x.
\end{cases} \tag{3.2}$$
Here \( \sigma = -D_x W(\tau) \), \( d(\cdot) = D^2 \Psi(\cdot) \),

\[
(3.3) \quad d(\cdot) = Id \quad \text{and} \quad b(\tau) = \tau_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\]

We are interested in the existence of periodic traveling waves of the above system. Note that our system (3.2) in the variable \((\tau, u) \in \mathbb{R}^n \oplus \mathbb{R}^n \) (\(n = 1, 2, 3\)) enjoys some interesting structures: it is a hyperbolic-parabolic system with partial parabolicity; it involves third order terms because of the surface energy effect. See Appendix B for discussions.

4 Traveling-wave ODEs

We seek a traveling wave solution of the system (3.2): \((\tau(x, t), u(x, t)) := (\tau(x-st), u(x-st))\) where \(s \in \mathbb{R}\) is the wave speed. Let us denote in the following \(\prime\) as differentiation with respect to \(x-st\). For convenience, we still use \(x\) to represent \(x-st\) (Indeed, we will show that \(s = 0\) is necessary for the existence of periodic or homoclinic waves). With further investigation in mind, we write the related equations for the general class of viscoelastic models. Now from system (3.2), we have the ODE system

\[
\begin{align*}
-s\tau' - u' &= 0; \\
-su' + \sigma(\tau)' &= (b(\tau)u')' - (d(\tau')\tau'')'.
\end{align*}
\]

Plugging the first equation into the second in the above system, we have the following second-order ODE in \(\tau\):

\[
(4.2) \quad s^2\tau' + \sigma(\tau)' = -(b(\tau)s\tau')' - (d(\tau')\tau'')'.
\]

In view of \(d(\cdot) = D^2 \Psi(\cdot)\), we readily see:

\[
(4.3) \quad s^2\tau' + \sigma(\tau)' = -(b(\tau)s\tau')' - (D^2 \Psi(\tau')\tau'')'.
\]

Choosing a specific space point, say \(x_0\), we integrate once to get:

\[
(4.4) \quad s^2\tau + \sigma(\tau) + q = -sb(\tau)\tau' - D \Psi(\tau')'
\]

Here \(q\) is an integral constant vector. Relating this with the elastic potential function \(W\), we have

\[
(4.5) \quad -DW(\tau) + s^2\tau + q = -sb(\tau)\tau' - D \Psi(\tau')'
\]

Carefully note that the integral constant vector is given by

\[
(4.6) \quad q = \{DW(\tau) - s^2\tau - sb(\tau)\tau' - D \Psi(\tau')'\} \bigg|_{x=x_0}.
\]
5 Hamiltonian systems

Defining $G(P) := \langle P, D\Psi(P) \rangle - \Psi(P)$, we see that $\frac{dG}{dP} = \langle P, D^2\Psi \rangle$. Here $P \in \mathbb{R}^n$ and $\Psi : \mathbb{R}^n \to \mathbb{R}$ is the surface energy functional (for our purpose in this paper, $n = 1, 2, 3$), $G : \mathbb{R}^n \to \mathbb{R}$ a different scalar potential type function. Now we are ready to state a structural property about the traveling wave ODE system (4.5).

**Proposition 5.1.** When $s = 0$, the system (4.5) is a Hamiltonian system with factor $(D^2\Psi(\tau'))^{-1}$, preserving the Hamiltonian integral

$$H(\tau, \tau') = -W(\tau) + q\tau + G(\tau') \equiv \text{constant}.$$ 

Here $W$ can be taken in particular as the phase-transitional elastic potential (see Appendix A and also p. 36 of [BLLeZ]), with $(a_1, a_2, a_3) = (\tau_1, \tau_2, \tau_3)$

$$W(\tau) = \left(2(\tau_1^2 + \tau_2^2 + \epsilon^2) + (|\tau|^2 - 1 - \epsilon^2)^2 - 16\epsilon^2\tau_2^2, $$

and $\Psi(p)$ as any convex function with $\Psi(0) = 0$ and $d\Psi(0) = 0$, simplest case $\Psi(P) = |P|^2/2$. (Note: in this simple case $G(P) = \Psi(P)$.)

**Proof.** When $s = 0$, the traveling wave ODE (4.5) becomes:

$$-dW(\tau) + q = -D\Psi(\tau')'$$

and the constant $q = \{DW(\tau) - D\Psi(\tau')'\} \bigg|_{x = x_0}$. Noticing the positive-definiteness of $D^2\Psi(\cdot)$, we may write the ODE as a first order system by regarding $\tau, \tau'$ as independent variables:

$$\tau' = \tau' = [D^2\Psi(\tau')]^{-1}D^2\Psi(\tau')\tau';$$

$$\tau'' = -[D^2\Psi(\tau')]^{-1}(-DW(\tau) + q)$$

Now, consider the energy surface given by:

$$H(\tau, \tau') := -W(\tau) + q\tau + G(\tau').$$

We see that

$$\frac{\partial}{\partial \tau'} H(\tau, \tau') = \frac{dG(\tau')}{d\tau'} = D^2\Psi(\tau')\tau';$$

$$\frac{\partial}{\partial \tau} H(\tau, \tau') = -DW(\tau) + q.$$
Comparing (5.2), (5.4), we see that the traveling wave ODE is a Hamiltonian system with factor \( \gamma := [D^2 \Psi(\tau')]^{-1} \). Thus, (4.5) preserves the Hamiltonian \( H \). We can see this also by the explicit computation, writing \( \zeta = x - st \):

\[
\frac{d}{d\zeta} H(\tau, \tau') = \frac{\partial}{\partial \tau} H(\tau, \tau') \tau' + \frac{\partial}{\partial \tau'} H(\tau, \tau') \tau'' = \gamma \frac{\partial}{\partial \tau} H(\tau, \tau') \frac{\partial}{\partial \tau'} H(\tau, \tau') + \gamma \frac{\partial}{\partial \tau'} H(\tau, \tau') \{ -\frac{\partial}{\partial \tau} H(\tau, \tau') \} = 0.
\]

From the above structural information, we easily get a necessary condition for the existence of periodic or homoclinic waves, extending results of [OZ] in a one-dimensional model case.

**Theorem 5.2.** For (4.5) with \( s \geq 0 \), there holds \( dH/d\zeta \leq 0 \), where

\[
H(\tau, \tau') := -W(\tau) + \frac{s^2}{2} |\tau|^2 + q\tau + G(\tau'),
\]

so that no homoclinic or periodic orbits can occur unless \( s = 0 \).

**Proof.** Considering the evolution of \( \frac{d}{d\zeta} H(\tau, \tau') \) along the flow of traveling wave ODE system (4.5), we have

\[
\frac{d}{d\zeta} H(\tau, \tau') = \frac{\partial}{\partial \tau} H(\tau, \tau') \tau' + \frac{\partial}{\partial \tau'} H(\tau, \tau') \tau''
= \langle -D_\tau W(\tau) + q + s^2, \tau' \rangle + \langle D(\tau'), \tau'' \rangle
= \langle -D_\tau W(\tau) + q + s^2 \tau, \tau' \rangle + \langle D^2 \Psi(\tau'), \tau', \tau'' \rangle
= \langle -D_\tau W(\tau) + q + s^2 \tau, \tau' \rangle + \langle D^2 \Psi(\tau'), \tau'', \tau' \rangle
= \langle -D_\tau W(\tau) + q + s^2 \tau + D\Psi(\tau'), \tau' \rangle
= \langle -s b(\tau) \tau', \tau' \rangle.
\]

The conclusion thus follows from the positive definiteness of \( b(\tau) \).

**The Hamiltonian system.** From the above analysis, we may set \( s = 0 \). The traveling wave ODE system reduces to the following form with an integral constant \( q \)

\[
\begin{align*}
-\tau'' &= -D_\tau W(\tau) + q; \\
q &= \{ D_\tau W(\tau) - \tau'' \} \big|_{x=x_0}.
\end{align*}
\]

If we take the Hamiltonian point of view, the corresponding Hamiltonian for the above system is

\[
H(\tau, \tau') = \frac{1}{2} |\tau'(x)|^2 + V(\tau, \tau'),
\]
where $V(\tau, \tau') := q \cdot \tau(x) - W(\tau(x))$. The periodic solutions of the system are confined to the surface $H(\tau, \tau') \equiv \text{constant}$. In the following, we list the elastic potential and related information for the phase-transitional models we deal with in this paper for completeness and future study. To get these models, we fix one or two directions of $\tau$ as zero, or, in the incompressible case, $\tau_3 \equiv 1$ (as described in [AM, BLeZ], the latter is an imposed constraint, that is compensated for in the $\tau_3$ equation by a Lagrange multiplier corresponding to pressure.

We refer the reader to [BLeZ], Section 3, for details of the derivations of these models.

### 5.1 2D Incompressible Shear Model

This model corresponds to setting $\tau_3 = 1$.

\begin{align*}
W(\tau) &= \left(2\tau_1^2 + 2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - \varepsilon^2)^2\right)\left(2\tau_1^2 + 2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - \varepsilon^2)^2\right).
\end{align*}

Its gradient components are

\begin{align*}
D_{\tau_1}W(\tau) &= 8\tau_1(|\tau|^2 + 1 - \varepsilon^2)(2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2); \\
D_{\tau_2}W(\tau) &= 8\tau_2(|\tau|^2 + 1 - \varepsilon^2)(2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2) - 32\tau_2\varepsilon^2.
\end{align*}

The Hessian components are

\begin{align*}
w_{11} := D_{\tau_1\tau_1}W(\tau) &= 8(|\tau|^2 + 1 - \varepsilon^2 + 2\tau_1^2)(2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2) + 32\tau_1^2(|\tau|^2 + 1 - \varepsilon^2)^2; \\
w_{12} = w_{21} := D_{\tau_1\tau_2}W(\tau) &= 16\tau_1\tau_2(2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2) + 32\tau_1\tau_2(|\tau|^2 + 1 - \varepsilon^2)^2; \\
w_{22} := D_{\tau_2\tau_2}W(\tau) &= 8(|\tau|^2 + 1 - \varepsilon^2 + 2\tau_2^2)(2(|\tau|^2 + \varepsilon^2) + (|\tau|^2 - \varepsilon^2)^2) + 32|\tau_2|^2(|\tau|^2 + 1 - \varepsilon^2)^2 - \varepsilon^2] + 32|\tau_2|^2(|\tau|^2 + 1 - \varepsilon^2)^2 - \varepsilon^2].
\end{align*}

### 5.2 1D Shear Model I: $\tau_3 \equiv 1$; $\tau_2 \equiv 0$.

The elastic potential becomes

\begin{align*}
W(\tau) &= \left(2\tau_1^2 + 2\varepsilon^2 + (\tau_1^2 - \varepsilon^2)^2\right)^2.
\end{align*}

The first order derivative is

\begin{align*}
D_{\tau_1}W(\tau) &= 8\tau_1(\tau_1^2 + 1 - \varepsilon^2)(2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2); \\
D_{\tau_1\tau_1}W(\tau) &= 8(3\tau_1^2 + 1 - \varepsilon^2)(2(\tau_1^2 + \varepsilon^2) + (\tau_1^2 - \varepsilon^2)^2) + 32\tau_1^2(\tau_1^2 + 1 - \varepsilon^2)^2.
\end{align*}
5.3 1D Shear Model II: $\tau_3 \equiv 1; \tau_1 \equiv 0$.

Correspondingly, the elastic potential becomes

$$W(\tau) = \left(2(\tau_2 - \epsilon)^2 + (\tau_2^2 - \epsilon^2)^2\right) \times \left(2(\tau_2 + \epsilon)^2 + (\tau_2^2 - \epsilon^2)^2\right)$$

The first order derivative is

$$D_\tau W(\tau) = 8\tau_2(2(\tau_2^2 + \epsilon^2) + (\tau_2^2 - \epsilon^2)^2) - 32\tau_2\epsilon^2;$$

The second order derivative is

$$D_{\tau_2\tau_2} W(\tau) = 8(3(\tau_2^2 + 1 - \epsilon^2)\{2(\tau_2^2 + \epsilon^2) + (\tau_2^2 - \epsilon^2)^2\} + 32\tau_2^2(\tau_2^2 + 1 - \epsilon^2)^2).$$

**Remark 5.3** (1D vs. 2D shear solutions). Evidently, solutions of 1-D shear models I or II determine solutions of the full 2 shear model obtained by adjoining $\tau_2 \equiv 0$ or $\tau_1 \equiv 0$ respectively. It is worth noting that the structure $dW(\tau) = \tau f(|\tau|) + c(0, \tau_2)^T$ for $f$ a scalar-valued function and $c$ a scalar constant yields that the only solutions $\tau(x)$ of (5.6) with $\eta \cdot \tau \equiv c_2 = \text{constant}$ for some constant vector $\eta$ are those satisfying $c_2 f(|\tau|) + c\eta_2 \tau_2 \equiv \text{constant}$, which gives by direct computation $\tau \equiv \text{constant}$ or else $\eta \cdot \tau \equiv 0$ and $\eta_2 \tau_2 \equiv 0$, in which case $\tau_1 \equiv 0$ or $\tau_2 \equiv 0$. That is, the 1D systems derived here are the only solutions of the 2D shear model that are not genuinely two-dimensional in the sense that they are confined to a line in the $\tau$-plane. In particular, if the mean of $\tau_1$ or $\tau_2$ over one period is not zero, then we can be sure that the solution is genuinely two-dimensional.

5.4 1D Compressible Model III

In this case $\tau_1 = \tau_2 \equiv 0$ and we denote $\tau = \tau_3$. The potential and its derivatives are given below. The elastic potential becomes

$$W(\tau) = \left(2\epsilon^2 + (\tau_3^2 - 1 - \epsilon^2)^2\right)^2$$

The first order derivative is

$$D_{\tau_3} W(\tau) = 8\tau_3(2\epsilon^2 + (\tau_3^2 - 1 - \epsilon^2)^2);$$

And the second order derivative is

$$w_{33} := D_{\tau_3\tau_3} W(\tau) = 8(3\tau_3^2 - 1 - \epsilon^2)\{2\epsilon^2 + (\tau_3^2 - 1 - \epsilon^2)^2\} + 32\tau_3^2(\tau_3^2 - 1 - \epsilon^2)^2.$$

5.5 Another two 2D compressible models.

First, we consider the case \( \tau = (\tau_2, \tau_3)^T \in \mathbb{R}^2 \). The elastic potential \( W \) and derivatives are as follows.

\[
W(\tau) = \left( 2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right) \left( 2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right);
\]

The gradient components are

\[
D_{\tau_2} W(\tau) = 8\tau_2(2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2) - 32\tau_2\varepsilon^2.
\]

\[
D_{\tau_3} W(\tau) = 8\tau_3(|\tau|^2 - 1 - \varepsilon^2)(2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2);
\]

Similarly, we have the Hessian components

\[
w_{22} := D_{\tau_2 \tau_2} W(\tau) = 8(|\tau|^2 - \varepsilon^2 + 2\tau_2^2)(2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2) + 32\tau_2^2(|\tau|^2 - \varepsilon^2)^2 - \varepsilon^2.
\]

\[
w_{23} = w_{32} := D_{\tau_1 \tau_2} W(\tau) = 16\tau_2 \tau_3 \{2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_2 \tau_3 (|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2);
\]

\[
w_{33} := D_{\tau_3 \tau_3} W(\tau) = 8(|\tau|^2 - 1 - \varepsilon^2 + 2\tau_3^2)(2(\tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2) + 32\tau_3^2(|\tau|^2 - 1 - \varepsilon^2)^2.
\]

Second, if we fix the \( \tau_2 \) direction and let \( \tau := (\tau_1, \tau_3)^T \), we get another 2D compressible model. We omit the details here as the form is obvious.

5.6 The full 3D model.

In this case \( \tau = (\tau_1, \tau_2, \tau_3)^T \in \mathbb{R}^3 \). corresponding to the phase-transitional elastic potential function \( W \), we list the components of \( D^2 W(\tau) := (w_{ij})_{3 \times 3} \).

\[
w_{11} = 8(|\tau|^2 + 2\tau_1^2 - \varepsilon^2)\{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1^2(|\tau|^2 - \varepsilon^2)^2
\]

\[
w_{12} = w_{21} = 16\tau_1 \tau_2 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1 \tau_2 (|\tau|^2 - \varepsilon^2)^2
\]

\[
w_{13} = w_{31} = 16\tau_1 \tau_3 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_1 \tau_3 (|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2)
\]

\[
w_{23} = w_{32} = 16\tau_2 \tau_3 \{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_2 \tau_3 (|\tau|^2 - \varepsilon^2)(|\tau|^2 - 1 - \varepsilon^2)
\]

\[
w_{33} = 8(|\tau|^2 + 2\tau_3^2 - 1 - \varepsilon^2)\{2(|\tau|^2 - \tau_3^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2\} + 32\tau_3^2(|\tau|^2 - 1 - \varepsilon^2)^2
\]

\[
w_{22} = 8(|\tau|^2 - \varepsilon^2 + 2\tau_2^2)(2(\tau_1^2 + \tau_2^2 + \varepsilon^2) + (|\tau|^2 - 1 - \varepsilon^2)^2) + 32\tau_2^2(|\tau|^2 - \varepsilon^2)^2 - \varepsilon^2.
\]
Remark 5.4. Similarly as in Remark 5.3, we find that solutions of the 2D compressible models are genuinely two-dimensional, in the sense that they are not confined to a line in the $\tau$-plane, unless they are solutions of the 1D model derived above: in particular, the mean over one period of $(\tau_1, \tau_2)$ is zero. Likewise, solutions of the full 3D compressible model are genuinely three-dimensional in the sense that they are not confined to a plane, unless they are solutions of one of the 2D models derived above, in particular, the mean of $\tau_1$ or of $\tau_2$ over one period is zero.

5.7 Justification of phenomenological models

We note that, for the case of 1-D shear flow, the viscosity coefficient given by (3.3) become $b, d \equiv \text{constant}$, and the elastic potential $W$ is of a generalized double-well form. Thus, we recover from first principles the type of phenomenological model studied in [S1, S2, S3, SS, Z2], though with a slightly modified potential refining the quartic double-well potential assumed in the phenomenological models. The 2-D shear flow gives a natural extension to multidimensional deformations, which is also interesting from the pure Calculus of Variations point of view (see the following section), as a physically relevant example of a vectorial “real Ginzberg–Landau” problem of the type studied on abstract grounds by many authors. Finally, we note that the various compressible models give a different extension of the phenomenological models, to the case of “real” or nonconstant viscosity.

6 Calculus of Variations.

In this section, we formulate the problem in the framework of Calculus of Variations and give the proof of the existence result.

6.1 Space structure

As a first step, we recall the notions of Sobolev spaces involving periodicity and introduce the space structure we are going to use (see [MW]). For fixed real number $T > 0$, let $C_T^\infty$ be the space of infinitely differentiable $T$-periodic functions from $\mathbb{R}$ to $\mathbb{R}^n$ (for our purpose $n = 1, 2, 3$).

Lemma 6.1. Let $u, v \in L^1(0,T;\mathbb{R}^n)$. If the following holds: for every $f \in C_T^\infty$,

$$\int_0^T (u(t), f'(t))dt = -\int_0^T (v(t), f(t))dt,$$

then

$$\int_0^T v(s)ds = 0$$
and there exists a constant vector $c$ in $\mathbb{R}^N$ such that

$$u(t) = \int_0^t v(s)ds + c \quad a.e. \text{ on}[0,T].$$

The function $v := u'$ is called the weak derivative of $u$. Consequently, we have

$$u(t) = \int_0^t u'(l)dl + c,$$

which implies the following:

$$u(0) = u(T) = c;$$

$$u(t) = u(s) + \int_s^t u'(l)dl.$$

Proof. For the mean zero property, we could consider the specific test function $f = e_j$. For the integral formulation, we could consider the use of Fubini Theorem and Fourier expansion of $f$ to conclude ([MW]).

Define the Hilbert space $H^1_T$ as usual (hence reflexive Banach space) with this inner product and corresponding norm: for $u, v \in H^1_T$,

$$\langle u, v \rangle := \int_0^T (u,v) + (u',v') ds;$$

$$\|u\|^2 := \int_0^T |u|^2 + |u'|^2 ds.$$

Next, we collect some facts for later use.

**Proposition 6.1.** (Compact Sobolev Imbedding property) $H^1_T \subset\subset C[0,T]$ compactly.

**Proposition 6.2.** If $u \in H^1_T$ and $(1/T) \int_0^T u(t) dt = 0$, then we have Wirtinger’s inequality

$$\int_0^T |u(t)|^2 dt \leq \frac{T^2}{4\pi^2} \int_0^T |u'(t)|^2 dt$$

and a Sobolev inequality

$$|u|_{\infty}^2 \leq \frac{T}{12} \int_0^T |u'(t)|^2 dt.$$

The Compact Sobolev imbedding property will give us the required weak lower semi-continuity property for the nonlinear functionals. The Wirtinger’s inequality supplies us equivalent norms in related Sobolev spaces with mean zero property (see [MW] for complete proofs).
6 CALCULUS OF VARIATIONS.

6.2 Variational formulation of the problems

Now given real positive constant $T$, we consider problem (5.6) in $H^1_T$
\[
\begin{cases}
-\tau'' = -D_\tau W(\tau) + q = -D_\tau(W(\tau) - q \cdot \tau); \\
\tau(0) - \tau(T) = 0; \tau'(0) - \tau'(T) = 0.
\end{cases}
\]

Let us first consider the cases and formulations without the physical restriction $\tau_3 > 0$. Assume that:
\[
\bar{\tau} := \frac{1}{T} \int_0^T \tau(x) dx = m.
\]
Here $m \in \mathbb{R}^n$, $n = 1, 2, 3$ and we will use bar to represent mean over one period similarly. Hence, we consider the following problem
\[
(6.1) \begin{cases}
\tau''(x) = DW(\tau) - q; \\
\tau(0) = \tau(T); \tau'(0) = \tau'(T); \\
\frac{1}{T} \int_0^T \tau(x) dx = m.
\end{cases}
\]

If we seek periodic solutions, $q$ can be determined by integrating the equations above over one period; that is,
\[
q = \frac{1}{T} \int_0^T DW(\tau(x)) dx.
\]
Define $v(x) = \tau(x) - m$. We see easily that $\frac{1}{T} \int_0^T v(x) dx = 0$, and $v(x)$ satisfies the system of equations:
\[
(6.2) \begin{cases}
v''(x) = DW(v + m) - q; \\
v(0) = v(T); v'(0) = v'(T); \\
\frac{1}{T} \int_0^T v(x) dx = 0.
\end{cases}
\]

For later convenience, we rewrite the above system as
\[
(6.3) \begin{cases}
v''(x) = DW(v + m) - DW(m) + DW(m) - q; \\
v(0) = v(T); v'(0) = v'(T); \\
\frac{1}{T} \int_0^T v(x) dx = 0.
\end{cases}
\]

Define $\tilde{W}(v) = W(v + m) - DW(m) \cdot v$ and $\tilde{q} = q - DW(m)$. We get the following problem
\[
(6.4) \begin{cases}
v''(x) = D\tilde{W}(v) - \tilde{q}; \\
v(0) = v(T); v'(0) = v'(T); \\
\frac{1}{T} \int_0^T v(x) dx = 0.
\end{cases}
\]

Here $\tilde{q}$ can be determined by integration: $\frac{1}{T} \int_0^T D\tilde{W}(v) dx = \tilde{q}$. 
Remark 6.3. Imposing the physical restriction $\tau_3 > 0$, we require that the $\tau_3$-component of the mean vector $m$ be positive: $m_3 > 0$. Hence we need the assumption $v_3 > -m_3$ for models involving $\tau_3$ direction on $[0, T]$.

Define $F(v) = W(v + m) - W(m) - DW(m) \cdot v$ and introduce the functional

$$\mathcal{I}(v) = \int_0^T \frac{1}{2} |v'|^2 \, dx + \int_0^T F(v) \, dx$$

on the space

$$H^1_{T,0} := \{ v \in H^1_T; \bar{v} = \frac{1}{T} \int_0^T v \, dx = 0 \}.$$

Proposition 6.4. 0 is always a critical point of the functional $\mathcal{I}$ defined above on $H^1_{T,0}$.

Proof. It is easy to verify that for $\phi \in H^1_{T,0}$, there holds

$$I'(v)\phi = \int_0^T v' \cdot \phi' + D\tilde{W}(v) \cdot \phi \, dx.$$

Taking $v = 0$ and noticing that $D\tilde{W}(0) = 0$, we get the desired result. □

Remark 6.5. By our formulation, we make 0 always a critical point and it corresponds to the constant solution. This geometric property supplies us nice way to exclude the possibility that the periodic solution we find is constant, i.e., to help prove the periodic waves we find are oscillatory.

Proposition 6.6. Without physical restriction on $\tau_3$, the critical point of $\mathcal{I}$ corresponds to the solution of (5.6).

Proof. This can be regarded as a simple consequence of Corollary 1.1 in [MW]. For completeness, we write the details here. First, assume $v$ solves

$$\begin{cases}
    v''(x) = D\tilde{W}(v) - \tilde{q} ; \\
    v(0) = v(T) ; v'(0) = v'(T) ; \\
    \frac{1}{T} \int_0^T v(x) \, dx = 0.
\end{cases} \tag{6.5}$$

Multiplying the equation by $\phi \in H^1_{T,0}$ and integrating to get

$$\int_0^T v' \phi' + D\tilde{W}(v) \cdot \phi \, dx = 0,$$

i.e $v$ is a critical point of $\mathcal{I}$.

Next, we assume $v$ is a critical point and $\phi \in H^1_T$. Then $\phi - \bar{\phi} \in H^1_{T,0}$. Hence we have

$$\int_0^T v' \cdot (\phi - \bar{\phi}) + \int_0^T D\tilde{W}(v) \cdot (\phi - \bar{\phi}) \, dx = 0$$
i.e.
\[ \int_0^T v' \cdot \phi' + \int_0^T D\tilde{W}(v) \cdot \phi - \int_0^T D\tilde{W}(v) \cdot \bar{\phi} \, dx = 0 \]

Noting that \( \bar{\phi} = \frac{1}{T} \int_0^T \phi \, dx \), we find that the left-hand side expression above is:
\[ \int_0^T v' \cdot \phi' + \int_0^T D\tilde{W}(v) \cdot \phi - \int_0^T D\tilde{W}(v) \cdot \left( \frac{1}{T} \int_0^T \phi \, dx \right) \, dx = 0 \]

Noticing that \( \frac{1}{T} \int_0^T D\tilde{W}(v) \, dx = \bar{q} \), we get
\[ \int_0^T v' \cdot \phi' + \int_0^T (D\tilde{W}(v) - \bar{q}) \cdot \phi \, dx = 0, \]
which implies \( v'' = D\tilde{W}(v) - \bar{q} \).

**Remark 6.7.** If we consider models involving the restriction \( v_3 > -m_3 \), we need to consider a variational problem with this constraint.

In view of the physical properties of the elastic potential function \( W \) (in particular, the polynomial structure of the phase-transitional potential function), we can apply the direct method of the calculus of variation to show existence. In order to deal with the integral constant \( \bar{q} \), we may restrict the admissible sets (or choose proper function space) on which we consider the functional or use Lagrange multiplier to recover it by adding restriction functional on the original space on which the functional is defined.

In the following, we state with proofs some propositions of general nonlinear functionals. These propositions and further materials can be found in [De, Ni, ZFC] and the references therein. We will use them to prove the existence of periodic waves.

**Proposition 6.8.** Let \( \mathcal{X} \) be a Banach space, \( I \) a real functional defined on \( \mathcal{X} \) and \( U \) be a sequentially weakly compact set in \( \mathcal{X} \). If \( I \) is weakly lower semi-continuous, then \( I \) attains its minimum on \( U \), i.e. there is \( x_0 \in U \), such that \( I(x_0) = \inf_{x \in U} I(x) \).

**Proof.** Let \( c := \inf_{x \in U} I(x) \). By definition of \( \inf \), there exists \( \{ x_n \} \subset U \) such that \( I(x_n) \to c \).
In view that \( U \) is sequentially weakly compact, \( \{ x_n \} \) admits a weakly convergent subsequence, still denoted by \( \{ x_n \} \). Denote \( x_0 \in \mathcal{X} \) the corresponding weak limit. Since \( U \) is weakly closed, we know \( x_0 \in U \). Noticing that weakly lower semi-continuity of \( I \), we have \( c = \lim_n I(x_n) \geq I(x_0) \). By the definition of \( c \), we in turn know \( I(x_0) = c > -\infty \), which completes the proof.

It is well-known that a bounded weakly closed set in a reflexive Banach space is weakly compact. In particular, a bounded closed convex set in reflexive Banach space is weakly compact since weakly close and close in norm are equivalent for convex sets. Hence we have the following corollaries:
Corollary 6.9. Let $U$ be a bounded weakly closed set in a reflexive Banach space $X$ and $I$ be a weakly lower semi-continuous real functional on $X$. Then there exists $x_0 \in U$ such that $I(x_0) = \inf_{x \in U} I(x)$.

Definition 6.10. A real functional $I$ on a Banach space $X$ is said to be coercive if

$$\lim_{|x|_X \to +\infty} I(x) = +\infty.$$ 

Corollary 6.11. Any coercive weakly lower semi-continuous real functional $I$ defined on a reflexive Banach space $X$ admits a global minimizer.

6.3 A general existence result

In this part, we first give a general result for models with the physical assumption $\tau_3 > 0$, i.e., $v_3 > -m_3$. We will assume the following conditions on the potential $W$

(A1) $W \in C^2$ and $W(\tau) \to +\infty$ as $\tau_3 \to 0^+$. For $\tau_3 \leq 0$, define $W(\tau) = +\infty$;

(A2) There exist a positive real number $C$ such that $W(\tau) \geq -C$ for $\tau \in \mathbb{R}^n$ ($n = 1, 2, 3$);

(A3) There exists a constant vector $m \in \mathbb{R}_+^3 := \{m \in \mathbb{R}^3; m_3 > 0\}$ such that $\sigma\{D^2W(m)\} \cap \mathbb{R}_-^1 \neq \emptyset$. Here $\sigma\{D^2W(m)\}$ is the spectrum set of $D^2W(m)$.

Because of our formulation, we only need that the potential is bounded from below. For a potential such an assumption is quite natural. Assumption (A3) amounts to saying that there is a point where the potential is concave. From the physical point of view, this is quite reasonable.

Theorem 6.12. If $(\frac{2\pi}{T})^2 < \lambda(m)$, then we have a physical nonconstant periodic wave solution for the problem (3.2) for which the mean over one period of $\tau$ is $m$. Here $-\lambda(m)$ is the smallest eigenvalue of $D^2W(m)$.

Define two subsets of $H^1_{T,0}$ by

$$A_1 := \{v \in H^1_{T,0}; v_3 > -m_3\};$$

$$A_2 := \{v \in H^1_{T,0}; v_3 \geq -m_3\}.$$ 

Remark 6.13. The admissible set $A_1$ is not weakly closed in $H^1_{T,0}$.

Lemma 6.14. Under assumptions (A1)-(A3), $I$ is a coercive functional on $H^1_{T,0}$.

Proof. By the definition of $I$, we just need to consider the part $\int_0^T F(v)\,dx$. By assumption (A2), we have
\[
\int_0^T F(v) \, dx = \int_0^T W(v + m) - W(m) - DW(m) \cdot v \, dx \\
= \int_0^T W(v + m) - W(m) \, dx \\
\geq \int_0^T -C - W(m) \, dx \\
= - (C + W(m)) T > -\infty.
\]

By the above lemma, we see for sufficient large \( R \) the minimizers of \( I \) on \( A_i \) are restricted to the sets \( \bar{A}_i := A_i \cap B_{H^1_{T,0}}[0, R] \) for \( i = 1, 2 \) where \( B_{H^1_{T,0}}[0, R] \) is the closed ball with center 0 and radius \( R \) in \( H^1_{T,0} \). Define \( S_i := \{ v \in A_i ; I(v) = \inf_{\tilde{v} \in A_i} I(\tilde{v}) \} \). Obviously, we have \( S_i := \{ v \in \bar{A}_i ; I(v) = \inf_{\tilde{v} \in A_i} I(\tilde{v}) \} \).

**Lemma 6.15.** \( \bar{A}_i \) is a weakly compact set in \( H^1_{T,0} \).

**Proof.** \( \bar{A}_2 \) is bounded by its definition. Since \( H^1_T \) is reflexive, we know \( \bar{A}_2 \) is weakly sequentially compact. Also, \( \bar{A}_2 \) is convex. Indeed, we can use the definition of convexity of a set to check this easily. An appeal to Sobolev embedding theorem yields that \( \bar{A}_2 \) is closed in norm topology of \( H^1_T \). For a convex set, closeness in norm topology and weak topology coincides, hence we have that \( \bar{A}_2 \) is weakly closed. Putting this information together, we have shown that \( \bar{A}_2 \) is weakly compact.

**Lemma 6.16.** \( I \) is a weakly lower semi-continuous functional on \( H^1_{T,0} \).

**Proof.** Let \( v^n \to v \) weakly in \( H^1_{T,0} \). By Sobolev imbedding, we have \( v^n \to v \) uniformly in \( [0, T] \). Hence we have \( \int_0^T F(v^n) \, dx \to \int_0^T F(v) \, dx \). Because of the mean zero property, \( \int_0^T |v'|^2 \, dx \) is of norm form, hence it is a weakly lower semi-continuous functional.

**Lemma 6.17.** \( S_2 \neq \emptyset \) and \( v_3 \geq -m_3 + \epsilon \) for \( v \in S_2 \) under the assumption of Theorem 6.12. Here \( \epsilon \) is a positive constant.

**Proof.** By Proposition 6.8, \( S_2 \neq \emptyset \). Note that \( 0 \in A_2, I(0) = 0 \) and hence \( I(v) \leq 0 \). Hence we will have \( v_3 \geq -m_3 + \epsilon \). Or else by Sobolev imbedding there would be a positive measurable set of \( [0, T] \) on which \( \tau_3 > 0 \) is vanishingly small. By assumption on the asymptotic behavior of \( W \) with respect to \( \tau_3 \to 0 \), we would have \( I(v) = \int_0^T (1/2)|v'|^2 \, dx + \int_0^T W(v + m) - W(m) \, dx > 0 = I(0) \), a contradiction.

**Lemma 6.18.** \( 0 \not\in S_1 = S_2 \) under the assumption of Theorem 6.12.
Proof. Consider the second variation. An easy computation shows that for \( v, \phi \) in \( H^1_{T,0} \)

\[
\mathcal{I}''(v) : (\phi \otimes \phi) = \int_0^T |\phi'|^2 \, dx + \int_0^T D^2W(v + m) : (\phi \otimes \phi) \, dx.
\]

To show \( 0 \not\in S_2 \), consider

\[
\mathcal{I}''(0) : (\phi \otimes \phi) = \int_0^T |\phi'|^2 \, dx + \int_0^T D^2W(m) : (\phi \otimes \phi) \, dx.
\]

Let \( \tilde{\phi}(x) = \eta \sin(\frac{2\pi x}{T}) \) for \( 0 < \eta < m_3 \) and \( v_0 \in \mathbb{R}^3 \) be a unit eigenvector corresponding to \(-\lambda(m)\). We see that \( \phi(x) := \tilde{\phi}(x)v_0 \in A_2 \). Since 0 is a critical point of \( \mathcal{I} \) on \( H^1_{T,0} \) and

\[
\mathcal{I}''(0) : (\phi v_0 \otimes \phi v_0) = \int_0^T \eta^2 (\frac{2\pi}{T})^2 (\cos(\frac{2\pi x}{T}))^2 \, dx - \lambda(m) \int_0^T \eta^2 (\sin(\frac{2\pi x}{T}))^2 \, dx
\]

\[
= \frac{\eta^2 T}{2} \left\{ \left( \frac{2\pi}{T} \right)^2 - \lambda(m) \right\} < 0.
\]

Hence we see that \( 0 \not\in S_2 \) and \( S_1 = S_2 \) is obvious. \( \square \)

Proof of Theorem 5.11. Combining Lemma 6.14-Lemma 6.18, we finish the proof of Theorem 6.12. \( \square \)

Remark 6.19. The condition \((\frac{2\pi}{T})^2 < \lambda(m)\) in Theorem 6.12 on the period \( T \), is readily seen by Fourier analysis to be the sharp criterion for stability of the constant solution \( \tau \equiv m, u \equiv 0 \). Equivalently, it is the Hopf bifurcation condition as period is increased, marking the minimum period of bifurcating periodic waves. Thus, it is natural, and no real restriction. On the other hand, there may well exist minimizers at whose mean \( m \) \( W \) is convex; this condition is sufficient but certainly not necessary. Likewise, there exist saddle-point solutions not detected by the direct approach.

6.4 Relation to standard results, and directions for further study

In the scalar case \( \tau \in \mathbb{R}^1 \), the condition that \( D^2W(m) \) have a negative eigenvalue is equivalent to convexity of the Hamiltonian \( H \) at the equilibrium \((m,0)\), under which assumption there are many results on existence of periodic solutions of all amplitudes; see, for example, [Ra] and later elaborations. Likewise in the vectorial case \( \tau \in \mathbb{R}^d, d > 1 \), if \( D^2W(m) < 0 \), then we may appeal to standard theory to obtain existence of periodic solutions by a variety of means; indeed, the convexity condition may be substantially relaxed for solutions in the large, as described in [Ra], and replaced by global conditions ensuring, roughly, star-shaped level sets of the Hamiltonian. On the other hand, review of the potentials considered here reveals that, typically, it is a single eigenvalue of \( D^2W \) that becomes negative and not all eigenvalues, and so these methods cannot be directly applied.

It is an interesting question to what extent such standard methods could be adapted to the situation of a Hamiltonian potential (in our case \(-W\)) with a single convex mode.
Existence of small amplitude periodic waves at least is treatable by Hamiltonian Hopf bifurcation analysis. The question is to what extent if any one can make global conclusions beyond what we have done here, in particular, to relax for large solutions the nonconvexity condition on $W$ at $m$. Finally, it would be interesting to find natural and readily verifiable conditions for existence of saddle-point solutions in this context.

7 Existence of periodic solutions for specific models

In this section, we focus on the existence of periodic waves for the incompressible models (i.e. models with $\tau_3 \equiv 1$), namely the 1D models I, II and the 2D incompressible model. First note that these models do not involve the $\tau_3$ direction. Hence we have no condition corresponding to (A1). However, the specific phase-transitional elastic potential energy function $W$ has good growth rate when $|\tau| \to +\infty$ for $\tau = \tau_1, \tau_2$ or $(\tau_1, \tau_2)$. This will make our functionals coercive. Hence we have the following:

**Theorem 7.1.** For the incompressible models, there exist non-constant periodic standing waves respectively if the mean $m$ (either vector or scalar) satisfies (A3) and $(\frac{2\pi}{T})^2 < \lambda(m)$. When $m$ is scalar, assumption (A3) means $m$ lies in the elliptic region of the viscoelasticity system (2.1).

**Proof.** It is easy to see the corresponding functionals are coercive, weakly lower semi-continuous functionals on the reflexive Banach spaces $H^1_{T,0}$. Hence Corollary 6.11 applies. The verification that the global minimizers respectively are not zero is entirely the same as in Lemma 6.18 by considering the second variation.

Next, we specify the corresponding conditions in Theorem 7.1 for these incompressible models.

7.1 1D Shear Model I

The condition is

$$ (\frac{2\pi}{T})^2 < -8(3m^2 + 1 - \varepsilon^2)(2(m^2 + \varepsilon^2) + (m^2 - \varepsilon^2)^2) - 32m^2(m^2 + 1 - \varepsilon^2)^2. $$

In particular, if $m = 0$, the condition reads

$$ (\frac{2\pi}{T})^2 < -8(1 - \varepsilon^2)(2\varepsilon^2 + \varepsilon^4). $$

Condition (7.2) illustrate that our assumption is not a void assumption. Also, in the mean zero case, (7.2) holds only if $\varepsilon > 1$. Comparing this with the existence result by phase-plane analysis (see in particular section 8.1), we see that these results match very well.
7.2 1D Shear Model II

The condition is
\[
(\frac{2\pi}{T})^2 < -8(3m^2 + 1 - \varepsilon^2)(2m^2 + \varepsilon^2) - (m^2 - \varepsilon^2)^2 - 32\{m^2(1 - \varepsilon^2) - \varepsilon^2\}.
\]

In particular, if \(m = 0\), the condition reads
\[
(\frac{2\pi}{T})^2 < -8(1 - \varepsilon^2)(2\varepsilon^2 + \varepsilon^4) + 32\varepsilon^2 = 8(\varepsilon^6 + \varepsilon^4 + 2\varepsilon^2).
\]

Condition (7.4) implies in particular that for any \(\varepsilon > 0\), we have long-periodic oscillatory waves. Similarly, for any given \(T > 0\), we have oscillatory waves as long as \(\varepsilon > 0\) large enough. Comparing with the phase-plane analysis (see section 8.2), we see that the related obtained wave phenomena match very well.

7.3 2D Incompressible Shear Model

In this case \(D^2W(m)\) is given by its components
\[
w_{11} := D_{\tau_1\tau_1}W(m) = 8(|m|^2 + 1 - \varepsilon^2 + 2m^2)\{2(|m|^2 + \varepsilon^2) + (|m|^2 - \varepsilon^2)^2\} + 32m_1^2(|m|^2 + 1 - \varepsilon^2);
\]
\[
w_{12} = w_{21} := D_{\tau_1\tau_2}W(m) = 16m_1m_2\{2(|m|^2 + \varepsilon^2) + (|m|^2 - \varepsilon^2)^2\} + 32m_1m_2(|m|^2 + 1 - \varepsilon^2);
\]
\[
w_{22} := D_{\tau_2\tau_2}W(m) = 8(|m|^2 + 1 - \varepsilon^2 + 2m_2^2)\{2(|m|^2 + \varepsilon^2) + (|m|^2 - \varepsilon^2)^2\} + 32m_2^2(|m|^2 + 1 - \varepsilon^2).
\]

The corresponding condition is \((\frac{2\pi}{T})^2 < \lambda(m)\). This is obviously a rather mild condition. To see this, we can consider in particular the mean \(m = (m_1, 0)^T\) or \((0, m_2)^T\). Then the results on the two 1D incompressible models readily give the conclusion because we have diagonal matrices.

Based on the analysis of these conditions, we have in particular (\(m = 0\) case):

**Theorem 7.2.** For the 2D shear model, 1D shear model I and II, we have the following existence result of periodic viscous traveling/standing waves:

1. Given any \(\varepsilon > \varepsilon_0 \geq 0\), for any \(T\) satisfying \(T > T(\varepsilon) > 0\), system (5.6) hence (2.1) has a nonconstant periodic solution with some appropriate integral constant \(q\); For the 1D model II we have \(\varepsilon_0 = 0\).

2. Given any \(T > 0\), for any \(\varepsilon\) satisfying \(\varepsilon > \varepsilon(T) > 0\), system (5.6) hence (2.1) has a nonconstant periodic solution with some appropriate integral constant \(q\).

**Remark 7.3.** From the above theorem, we see in particular indeed for the 2D shear model, we have infinitely many nontrivial periodic viscous traveling waves with appropriate corresponding \(q\) values. In particular, we have a sequence of waves with minimum positive period \(T \to +\infty\).
Remark 7.4. Remarks 5.3 and 5.4 show that solutions of the specific 3D model of Section 5 with \( m_1, m_2, \) and \( m_3 \) nonzero are genuinely 3-dimensional in the sense that they are not confined to a plane in \( \tau \)-space, and that solutions of the various 2D models of Section 5 are genuinely two-dimensional if the means of both components are nonzero. That is, we have constructed by the variational approach solutions that are not obtainable by the planar phase-portrait analysis of the 1D case (see just below). On the other hand, a dimensional count reveals that, generically, the periodic solutions nearby a 1D solution are all 1D, and likewise the periodic solutions nearby a 2D solution are all 2D.

8 1D existence by phase-plane analysis.

In this section, we discuss how to generate periodic waves for 1D models. In (5.6), the integral constant \( q = D_\tau W(\tau_-) - \tau'' \). Here \( (\tau_-, \tau'') \) is the vector evaluated at some specific space value \( x_0 \). If there indeed exist periodic-T waves, \( q = \frac{1}{T} \int_0^T D(\tau(x)) dx \).

By the variational formulation and the usual bootstrap argument, we conclude that the periodic waves are classical solutions of the system (5.6). For the possible homoclinic/heteroclinic solutions of (5.6), standard elliptic regularity theory also guarantees that these waves are classical solutions. There must be points (say \( x_0, x_1 \)) in a period \([0, T]\) such that \( \tau'(x_0) = 0 \) and \( \tau''(x_1) = 0 \), etc., if such periodic solution did exist for \( \tau \) scalar. The reason is that \( \tau(x) \) cannot be always monotone and convex in view of periodicity (this applies to all derivatives). Hence we can make the integration constant have the form \( q = D_\tau W(\tau(x_1)) \) for convenience a priori. Then we can show existence, which in turn guarantees the a priori assumption. Hence, we could assume \( q = D_\tau W(\tau_-) \) to show existence.

We adopt this convention in the following analysis.

The guiding principle is that the ODE systems are planar Hamiltonian systems. To get complete and clear pictures of the phase-portraits, we just need to specify the “potential energy” term \( V(\tau, \tau_-) \) in the Hamiltonian \( H(\tau, \tau') \).

8.1 1D Shear Model I: \( \tau_2 \equiv 0; \tau_3 \equiv 1 \).

In this section, we denote \( \tau = \tau_1 \). We use similar notation in other sections. Recall the elastic potential \( W(\tau) = \left( 2\tau_1^2 + 2\xi^2 + (\tau_1^2 - \xi^2)^2 \right)^2 \) and its first and second order derivatives

\[
D_{\tau_1} W(\tau) = 8\tau_1(\tau_1^2 + 1 - \xi^2)(2(\tau_1^2 + \xi^2) + (\tau_1^2 - \xi^2)^2);
\]

\[
D_{\tau_1^2} W(\tau) = 8(3\tau_1^2 + 1 - \xi^2)\left[ 2(\tau_1^2 + \xi^2) + (\tau_1^2 - \xi^2)^2 \right] + 32\tau_1^2(\tau_1^2 + 1 - \xi^2)^2.
\]

The traveling wave ODE and corresponding Hamiltonian system are

\[
\tau'' = W'(\tau) - W'(\tau_-).
\]

\[
\begin{aligned}
\tau' &= \tau' ; \\
\tau'' &= W'(\tau) - W'(\tau_-).
\end{aligned}
\]
We write the Hamiltonian system as follows:

\[
\frac{|\tau'|^2}{2} = H(\tau, \tau') - V(\tau; \tau_-) \equiv E - V(\tau, \tau_-).
\]

Here \( E \) are constants corresponding to energy level curves of \( H(\tau, \tau') \) and \( V(\tau, \tau_-) := q\tau - W(\tau) \).

First, we determine the number of equilibria of the Hamiltonian system, hence focus on the solution of \( W'(\tau) = q \).

Note that \( W'(\tau) \) is an odd function on the real line, hence we just need to study its graph on the interval \((0, \infty)\). In view of the expression of \( W'(\tau) \), we need consider the cases:

1. \( 0 < \epsilon \leq 1 \);
2. \( \epsilon > 1 \).

For the case \( 0 < \epsilon < 1 \), we have \( W''(\tau) > 0 \) for \( \tau \) real, hence \( W'(\tau) \) is strictly monotone increasing and

\[
W'(0) = 0; \quad W'(\tau) > 0, \quad \text{for } \tau > 0; \quad W'(\tau) < 0, \quad \text{for } \tau < 0;
\]

Hence in this case, for any given \( \tau_- \), the solution of \( W'(\tau) = W'(\tau_-) \) is \( \tau_- \) and unique.

Similar analysis holds true for \( \epsilon = 1 \). Considering the definition of \( V(\tau, \tau_-) \), we have

**Proposition 8.1.** When \( 0 < \epsilon \leq 1 \), \( V(\tau; \tau_-) \) has exactly one critical point \( \tau_- \), which must be a global maximum.

**Remark 8.2.** In this case, our Hamiltonian system admits no periodic orbit for any \( \tau_- \) (or equivalently, for any \( q \)).

Next, consider the case \( \epsilon > 1 \). In this case, we can see from the expression of \( W'(\tau) \) that \( W'(\tau) \) has three distinct zeros: \(-\sqrt{\epsilon^2 - 1}, 0, \sqrt{\epsilon^2 - 1}\). A qualitative graph of \( W'(\tau_1) \) is as follows:
The corresponding graph for the potential $V(\tau_1, \tau_{1,-})$ is as follows

![Graph of $V(\tau_1, \tau_{1,-})$](image-url)

**Proposition 8.3.** The function $W'(\tau)$ has exactly two critical points.

**Proof.** By symmetry, we do the following computations: Denote $\tau_1^2 := X$ and $\varepsilon^2 := a > 1$. We want to show that the function

$$f(X) := (3X + 1 - a)[2(X + a) + (X - a)^2] + 4X(X + 1 - a)^2$$

has exactly one zero when $X > 0$.

First, noting that $f(0) < 0$ and $f(\frac{a}{3}) > 0$, we know that $f(X)$ has a root on $(0, \frac{a}{3})$.

Also note that $f(X) > 0$ on $[\frac{a}{3}, \infty)$, hence we just need to show that $f(X)$ admits a unique zero on $(0, \frac{a}{3})$. Computing the derivative, we have

$$f'(X) = 3[7X^2 + 10(1-a)X + a^2 + 2a + 2(1-a)^2].$$

Denote

$$\Delta = 100(1-a)^2 - 28[a^2 + 2a + 2(1-a)^2].$$

If $\Delta \leq 0$, we know that $f'(X) \geq 0$, hence $f(X)$ is monotone increasing, which implies that $f(X)$ admits a unique zero;

If $\Delta > 0$, we will have two positive roots for $f'(X) = 0$ and the smaller one is $\frac{10(a-1)-\sqrt{\Delta}}{14}$. However, we can show that $\frac{10(a-1)-\sqrt{\Delta}}{14} \geq \frac{a-1}{3}$, hence the function $f(X)$ is monotone increasing on the interval $(0, \frac{a-1}{3})$, which also implies the uniqueness of the zero. \hfill \Box

Now we have a clear picture on the potential $W'(\tau)$ (see the graph for $W'(\tau_1)$).

**Proposition 8.4.** When $\varepsilon > 1$, the function $W'(\tau)$ is an odd function with 3 zeros and 2 critical points and goes to infinity when $\tau \to +\infty$. 
Denote the two critical values of $W'(\tau)$ as $q^* > 0$ and $-q^*$, for convenience denoting $Q = q^*$. Then we have the following property:

**Proposition 8.5.** Assume $\varepsilon > 1$. When $|q| > Q$, the equation $W'(\tau) = q$ has exactly one solution; When $|q| = Q$, the equation $W'(\tau) = q$ has exactly 2 solutions; When $|q| < Q$, the equation $W'(\tau) = q$ has exactly 3 solutions.

As the solutions of $W'(\tau) = q$ correspond to the critical points of $V(\tau; \tau_-)$, we have:

**Theorem 8.6.** For $|q| \geq Q$, the Hamiltonian system admits no periodic orbit; For $|q| < Q$, the Hamiltonian system admits a family of nontrivial periodic orbits. Further if $q = 0$, the Hamiltonian system also admits a heteroclinic orbit.

**Proof.** For $|q| \geq Q$, we know that $V(\tau; \tau_-)$ has a global maximum and hence the Hamiltonian admits no periodic orbit. For the case $|q| < Q$, we know that the potential $V(\tau; \tau_-)$ must have exactly 3 critical points with 2 local maxima and 1 local minimum. Also, we know that $V(\tau; \tau_-)$ has strictly lower energy at the local minimum than at the two local maxima. Hence the existence of a family of periodic orbits follows. Further, when $q = 0$, the energies at the two local maxima of $V(\tau; \tau_-)$ are the same, hence we get an heteroclinic orbit.

**Remark 8.7.** We may compare the two energy values of $V(\tau; \tau_-)$ at the two local maxima. If they are equal (when $q = 0$ in particular), we have a heteroclinic orbit. In general, they are not equal to each other, which yields a homoclinic orbit.

### 8.2 1D Shear Model II: $\tau_1 \equiv 0; \tau_3 \equiv 1$.

Recall the elastic potential

$$W(\tau) = \left(2(\tau^2 - \varepsilon)^2 + (\tau^2 - \varepsilon^2)^2\right) \times \left(2(\tau^2 + \varepsilon)^2 + (\tau^2 - \varepsilon^2)^2\right)$$

and the relevant derivatives

$$D_{\tau_2} W(\tau) = 8\tau_2 (\tau^2 + 1 - \varepsilon^2) \{2(\tau^2 + \varepsilon^2) + (\tau^2 - \varepsilon^2)^2\} - 32\tau_2 \varepsilon^2;$$

$$D_{\tau_2^2} W(\tau) = 8(3\tau^2 + 1 - \varepsilon^2) \{2(\tau^2 + \varepsilon^2) + (\tau^2 - \varepsilon^2)^2\} + 32\{\tau^2 (\tau^2 + 1 - \varepsilon^2)^2 - \varepsilon^2\}.$$ 

As above, we list the Hamiltonian system and the potential $V(\tau; \tau_-)$. The system is:

\[
\begin{align*}
\tau' &= \tau' \\
\tau'' &= W'(\tau) - W'(\tau_-)
\end{align*}
\] 

(8.4)

The potential is $V(\tau; \tau_-) = q\tau - W(\tau)$. 
Remark 8.8. There is a slight difference with the 1D shear model I in the function $W'(\tau)$. Because of this difference, we do not need to restrict the positive number $\varepsilon$ to get periodic orbits for the parameter $q$ in proper range. The conclusions are completely the same when $1 \geq \varepsilon > 0$ as in 1D shear model I when $\varepsilon > 1$.

We have the following:

**Proposition 8.9.** When $1 \geq \varepsilon > 0$, the behavior of the function $W'(\tau)$ is the same as that of the function $W'(\tau)$ in the 1D shear model I when $\varepsilon > 1$. In fact, $W''(\tau)$ is monotone increasing in this case for $\tau > 0$.

**Remark 8.10.** For the range $\varepsilon > 1$, numerics suggest that the behaviors are also the same as we may show that the function $W'(\tau)$ has exactly three solutions and two critical points. We have a small problem to verify this by direct computation though we just need to show that $f(X) > 0$ evaluated at the larger root of $f'(X)$ ($f(X)$ is defined similar as in 1D shear model I as the second derivatives of the two potentials differ with a constant $32\varepsilon^2$. Even without this, we still can conclude the existence of periodic orbits since the potential $V(\tau; \tau_-)$ admits a minimum. Together with the existence obtained by variational argument, we know that there are still infinitely many nontrivial periodic waves for any $\varepsilon > 0$.

### 8.3 1D Compressible Model III

For this model, we need to pay special attention to the physical restriction $\tau_3 > 0$ when we do the phase-plane analysis. To find physical waves, we use a continuity argument and a simple comparison criterion.

In this case $\tau_1 = \tau_2 \equiv 0$; let $\tau = \tau_3$. The potential and its derivatives are given below. The elastic potential becomes $W(\tau) = \left(2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2\right)^2$. Its first and second order derivative are $D_{\tau_3}W(\tau) = 8\tau_3(\tau_3^2 - 1 - \varepsilon^2)(2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2)$ and $w_{33} := D_{\tau_3\tau_3}W(\tau) = 8(3\tau_3^2 - 1 - \varepsilon^2)(2\varepsilon^2 + (\tau_3^2 - 1 - \varepsilon^2)^2) + 32\tau_3^2(\tau_3^2 - 1 - \varepsilon^2)^2$.

We write $V = V(\tau, q, \varepsilon) := q\tau - W(\tau)$ in this section to emphasize the analytical dependence of $V$ on the parameters $q$ and $\varepsilon$ (because $V$ is a polynomial). As in previous sections, we see:

**Proposition 8.11.** (1) $W'(\tau) = q$ always has one, two or three roots when $|q| > Q$, $|q| = Q$ or $|q| < Q$ for some positive $Q$. In the case that $W'(\tau) = q$ has 3 distinct roots, we denote them from small to large by $\tau_1, \tau_m$, and $\tau_r$.

(2) $W'(\tau)$ has exactly two critical points.

As before, in order to analyze the existence of periodic or homoclinic/heteroclinic waves, we just need to consider the potential energy $V(\tau, q, \varepsilon)$. Further, in order to have physical waves, we need necessarily that $-Q < q < 0$. In this situation, the two roots $\tau_m$ and $\tau_r$ of $W'(\tau) = q$ are positive. Noticing that $\tau = \tau_m$ is a local minimizer of $V(\tau, q, \varepsilon)$, there is a periodic annulus around $\tau_m$. Hence we have the following proposition:
Proposition 8.12. When $-Q < q < 0$, there always exists a periodic annulus.

To show existence of physical homoclinic orbit, we just need to compare the values of $V(0, q, \varepsilon) =: V(0)$ and $V(\tau, q, \varepsilon) =: V(r)$. We have

Proposition 8.13. Let $-Q < q < 0$. If $V(0) > V(r)$, there is a physical homoclinic orbit; If $V(0) \leq V(r)$, there is no physical homoclinic orbit.

In particular, for the case $q = 0$, the 3 distinct roots of $W'(\tau) = q$ are easily seen to be $\tau_l = -\sqrt{1 + \varepsilon^2}$, $\tau_m = 0$ and $\tau_r = -\tau_l$. So $V(0, 0, \varepsilon) = -(2\varepsilon^2 + (1 + \varepsilon^2)^2)^2 < V(\tau_r, 0, \varepsilon) = -(2\varepsilon^2)^2$. By continuity, we have the following conclusion:

Proposition 8.14. There exists a constant $\eta > 0$ such that if $-\eta < q \leq 0$, then there exists no physical homoclinic orbit.

Proof. When $q = 0$, $V(0, 0, \varepsilon) < V(\tau_r, 0, \varepsilon)$. Thus, by continuous dependence and Propositions 8.11 and 8.13, we have the relation $V(0) < V(m)$ holds when $q < 0$ is small and the conclusion holds.

Next, we study the existence of physical homoclinic waves when $-Q < q < 0$ is large. For this purpose, we first set $\varepsilon = 0$ then proceed by a perturbation argument. When $\varepsilon = 0$, the corresponding elastic energy function and its derivatives are:

$$\tilde{W}(\tau) = (\tau^2 - 1)^4;$$

$$\tilde{W}'(\tau) = 8\tau(\tau^2 - 1)^3;$$

$$\tilde{W}''(\tau) = 8(\tau^2 - 1)^2(7\tau^2 - 1).$$

Note that $\tilde{W}''(\tau) = 0$ has roots $\tau = \pm 1, \pm \sqrt{1/7}$ (this can be easily computed).

For this potential $\tilde{V}(\tau, q, \varepsilon = 0) := q\tau - \tilde{W}(\tau)$, we need $|q| < \tilde{W}'(-\sqrt{1/7}) = 8(\sqrt{\frac{6}{7}})^3\sqrt{\frac{1}{7}}$ to have a homoclinic wave. For physical ones, we need $-8(\sqrt{\frac{6}{7}})^3\sqrt{\frac{1}{7}} < q < 0$. Consider the case $q \to -8(\sqrt{\frac{6}{7}})^3\sqrt{\frac{1}{7}}$ from the right, we see the largest root $\tau_m$ of $\tilde{W}''(\tau) = q$ tends to $\sqrt{1/7}$. Consequently, $\tilde{V}(\tau = 0, q, \varepsilon = 0) \to -\tilde{W}(0) = -1$ and the right local maximum value of $\tilde{V}(\tau_m, q, \varepsilon = 0) \to -8(\sqrt{\frac{6}{7}})^3\sqrt{\frac{1}{7}} - \tilde{W}(\sqrt{1/7}) < -1$. By Proposition 8.11 and Proposition 8.13, we have:

Proposition 8.15. For the compressible 1D model, assume $-Q < q < 0$ ($Q$ as in Proposition 8.11). Then, when $\varepsilon > 0$ and $q + Q$ are small, we have a physical homoclinic orbit.

9 Time-evolutionary stability

We conclude by discussing briefly the question of time-evolutionary stability of viscoelastic traveling waves, up to now not considered.
9 TIME-EVOLUTIONARY STABILITY

9.1 Spectral vs. nonlinear stability

A very useful observation regarding the earlier phenomenological models $\tau_t - u_x = 0$, $u_t + dW(\tau)_x = bu_{xx} - d\tau_{xx}$, $b, d > 0$ constant, for 1-D shear flow, made by Schecter and Shearer in [SS], was that for a wide range of $b, d$, specifically, $d < b^2/4$, the system can be transformed by the change of independent variable $u \rightarrow \tilde{u} := u - c\tau_x$, $c(b - c) = d$ to the fully parabolic system

$$\tau_t - \tilde{u}_x = c\tau_{xx},$$
$$\tilde{u}_t + dW(\tau)_x = (b - c)\tilde{u}_{xx},$$

thus allowing the treatment of nonlinear stability by standard parabolic techniques, taking into account, for example, sectorial structure, parabolic smoothing, etc.

Quite recently, this observation has been profoundly generalized by M. Kotschote [K], who showed that a somewhat different transformation in similar spirit may be used to convert viscoelastic or fluid-dynamical equations with surface energy (resp. capillarity) effects to quasilinear fully parabolic form, in complete generality, not only to the cases $4d > b^2$ previously uncovered for the phenomenological model, but to the entire class of physical models considered here. For further discussion/description of this transformation, see Appendix B.

This reduces the question of nonlinear stability to a standard format already well studied. In particular, it follows that (except possibly in nongeneric boundary cases of neutrally stable spectrum) **nonlinear stability is equivalent to spectral stability**, appropriately defined. This follows for heteroclinic and homoclinic waves by the analysis of [HZ], and for periodic waves by the analysis of [JZ]. For precise definitions of the notions of spectral stability, we refer the reader to those references; in the shock wave (heteroclinic or homoclinic) case, see also the discussion of [BLeZ]. Spectral stability may be efficiently determined numerically by Evans function techniques, as in for example [BHRZ, BLeZ, BHZ, BJNRZ1, BJNRZ2]. We intend to carry out such a numerical study in a followup work [BYZ].

9.2 Variational vs. time-evolutionary stability

More fundamentally, perhaps, there is relation between variational stability of periodic waves and their time-evolutionary stability as solutions of (2.1). In particular, the energy functional that we minimized in constructing periodic solutions is essentially the self-same functional that defines the mechanical energy of the system, a Lyapunov functional that decreases with the flow of (2.1); for further discussion, see Appendix C. This gives a strong link between the two notions of stability. Indeed, it can be used to directly show that the periodic waves constructed as minimizers of the associated variational problem are time-evolutionarily stable with respect to co-periodic perturbations (Remark C.2, Appendix C).

Moreover, as discussed further in [BYZ], the indirect spectral arguments of [Z2] on 1-D heteroclinic and homoclinic waves extend to to the general-dimensional and or periodic

---

1The analysis of [JZ] concerns modulational stability, or stability with respect to localized perturbations on the whole line; co-periodic stability may be treated by standard semigroup techniques [He]. Spectral analyses of [OZ, BYZ, PSZ] suggest that modulational stability occurs rarely if ever for viscoelastic waves.
case, yielding the much stronger result that variational stability in each of these contexts is necessary and sufficient for time-evolutionary stability (co-periodic stability, in the case of periodic waves, and variational stability constrained by a prescribed mean). Moreover, these arguments yield at the same time the curious fact that unstable spectra of the linearized operator about the wave must, if it exists, be real. These properties give additional insight, and additional avenues by which time-evolutionary stability may be studied.

In particular, this shows that the waves we have constructed are the (co-periodically) stable ones. However, these are not necessarily the only stable waves, as we did not construct all minimizers of the variational problem, but only those with mean satisfying a nonconvexity condition. Just recently (in particular, after the completion of the analysis of this paper), there has been introduced in [PSZ] a different, more direct argument showing equivalence of variational and time-evolutionary stability, which yields at the same time concise conditions for variational stability. These yield in particular that the sharp condition for stability is not the condition of nonconvexity of $W$ at $m$, defined as the mean over one period of $\tau$, but rather the “averaged” condition of nonconvexity of the Jacobian with respect to $m$ of the mean over one period of $DW(\tau)$. See [PSZ] for further details.

A Appendix: Phase-transitional elasticity

In this appendix, we collect some computations for the phase-transitional elasticity ([BLeZ],[FP] and references therein).\footnote{For the convenience of the reader, we note that the vector $(a_1,a_2,a_3)$ in [BLeZ] corresponds to $(\tau_1,\tau_2,\tau_3)$ here.}

$$W(F) := \left| F^T F - C_- \right|^2 \left| F^T F - C_+ \right|^2,$$

where

$$C_\pm = (F^T F)_\pm := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm \varepsilon \\ 0 & \pm \varepsilon & 1 + \varepsilon^2 \end{pmatrix}, \quad F_\pm = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \pm \varepsilon \\ 0 & 0 & 1 \end{pmatrix}.$$

Evidently, $W$ is minimized among planar deformations at the two values $A_\pm = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \pm \varepsilon \\ 0 & 0 & 0 \end{pmatrix}$.

Indeed, we have then

$$W(F) = \left( 2\tau_1^2 + 2(\tau_2 - \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right) \left( 2\tau_1^2 + 2(\tau_2 + \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \right),$$

where, as in [BLeZ], $A = \begin{pmatrix} 0 & 0 & \tau_1 \\ 0 & 0 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}$ and $F = \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ 0 & 0 & \tau_3 \end{pmatrix}$, so that

$$F^T F = \begin{pmatrix} 1 & 0 & \tau_1 \\ 0 & 1 & \tau_2 \\ \tau_1 & \tau_2 & |\tau|^2 \end{pmatrix}.$$
and

\begin{equation}
W(F) = \left| \begin{pmatrix} 0 & 0 & \tau_1 \\ 0 & 0 & \tau_2 + \varepsilon \\ \tau_1 & \tau_2 + \varepsilon & |\tau|^2 - 1 - \varepsilon^2 \end{pmatrix} \right| \left| \begin{pmatrix} 0 & 0 & \tau_1 \\ 0 & 0 & \tau_2 - \varepsilon \\ \tau_1 & \tau_2 - \varepsilon & |\tau|^2 - 1 - \varepsilon^2 \end{pmatrix} \right|^2
\end{equation}

\begin{equation}
(F^T F - C_\pm)^2 = \begin{pmatrix} \tau_1^2 & * & * \\ * & (\tau_2 \mp \varepsilon)^2 & * \\ * & * & \tau_1^2 + (\tau_2 \mp \varepsilon)^2 + (|\tau|^2 - 1 - \varepsilon^2)^2 \end{pmatrix}.
\end{equation}

\section{Appendix: The structure of the viscoelastic model.}

In this section, we verify the structural properties of the viscoelastic model and related modified systems obtained by the approach of Kotschote [K]. To be clear, we collect these related systems. The original system is

\begin{equation}
\begin{cases}
\tau_t - u_x = 0 \\
u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(\tau_x)\tau_{xx})_x.
\end{cases}
\end{equation}

Here \( \sigma = -D_\tau W(\tau), d(\cdot) = D^2\Psi(\cdot), d(\cdot) = Id \) and

\begin{equation}
b(\tau) = \tau_3^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}
\end{equation}

Introducing the phase variable \( z := \tau_x \), we may write as a quasilinear second-order system

\begin{equation}
\begin{cases}
\tau_t + z_x - u_x = \tau_{xx} \\
z_t = u_{xx} \\
u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(z)z_x)_x.
\end{cases}
\end{equation}

Remark B.1. This transformation, introduced in [K], is similar in spirit to but more general than the one\(^3\) introduced by Slemrod [S1, S2, S3] and used in [OZ] for an artificial viscosity/capillarity model.

We can slightly modify the above system in the second equation. Then we have the following system

\begin{equation}
\begin{cases}
\tau_t + z_x - u_x = \tau_{xx} \\
z_t + z_x = u_{xx} + \tau_{xx} \\
u_t + \sigma(\tau)_x = (b(\tau)u_x)_x - (d(z)z_x)_x.
\end{cases}
\end{equation}

\(^3\)A transformation \((\tau, u) \to (\tau, u - c\tau_x)\) reducing the model to a parabolic system of the same size.
If we write the above system (C.3) and (C.4) in matrix form

\[ U_t + f(U)_x = (B(U)U_x)_x \]

using the variable \( U := \begin{pmatrix} \tau \\ z \\ u \end{pmatrix} \), the corresponding matrix \( B \) becomes:

\[
\begin{pmatrix}
I_3 & 0_3 & 0_3 \\
0_3 & I_3 & 0_3 \\
0_3 & -d(z) & b(\tau)
\end{pmatrix} ; \\
\begin{pmatrix}
I_3 & 0_3 & 0_3 \\
I_3 & 0_3 & I_3 \\
0_3 & -d(z) & b(\tau)
\end{pmatrix} .
\]

Also, the corresponding matrix \( Df(U) \) for the two systems are:

\[
\begin{pmatrix}
0_3 & I_3 & -I_3 \\
0_3 & 0_3 & 0_3 \\
D\sigma(\tau) & 0_3 & 0_3
\end{pmatrix} ; \\
\begin{pmatrix}
0_3 & I_3 & -I_3 \\
0_3 & I_3 & 0_3 \\
D\sigma(\tau) & 0_3 & 0_3
\end{pmatrix} .
\]

**Proposition B.2.** (Strict parabolicity) Systems (B.3) and (B.4) are both strictly parabolic systems in the sense that the spectrum of \( B \) have positive real parts.

**Proof.** Comparing the two matrices \( B \) above, we know they have the same spectrum. We prove this proposition for \( \tau, z, u \in \mathbb{R}^3 \). The lower dimension cases becomes easier and the computations are totally the same. Pick one of the \( B \)'s, say

\[
\begin{pmatrix}
I_3 & 0_3 & 0_3 \\
0_3 & 0_3 & I_3 \\
0_3 & -d(z) & b(\tau)
\end{pmatrix} .
\]

To compute the spectrum of \( B \), consider the characteristic polynomial \( \det(\lambda I_3 - B) = 0 \), which is

\[
\det \begin{pmatrix}
(\lambda - 1)I_3 & 0_3 & 0_3 \\
0_3 & \lambda I_3 & -I_3 \\
0_3 & +d(z) & \lambda - b(\tau)
\end{pmatrix} = 0 .
\]

Doing Laplace expansion and elementary column transformation, we get

\[
\det\{\lambda - 1\} I_3 \det\{\lambda^2 I_3 - \lambda b(\tau) + d(z)\} = (\lambda - 1)^3 (\lambda^2 - \frac{\lambda}{\tau} + 1)^2 (\lambda^2 - \frac{2\lambda}{\tau} + 1) = 0 .
\]
From the first factor of the above degree 9 polynomial, we get three equal root 1 which has positive real parts. The other 6 roots also have positive real parts noticing that $\tau_3 > 0$.

**Proposition B.3.** (Nonzero characteristic speeds) The corresponding first order systems of (C.4) has nonzero characteristic speed at $\tau$ where the matrix $D^2W(\tau)$ are strictly positive definite.

**Proof.** Again, we prove this for $\tau, z, u \in \mathbb{R}^3$. To prove the corresponding first order system is non-characteristic, we consider the spectrum of the matrix

$$
\begin{pmatrix}
0_3 & I_3 & -I_3 \\
0_3 & I_3 & 0_3 \\
D\sigma(\tau) & 0_3 & 0_3
\end{pmatrix}
$$

We get the following system for the characteristic speed:

$$
\det(\lambda I_9 - Df(U)) = 0.
$$

By direct computation, we know:

$$
(B.5) \quad \det(\lambda I_9 - Df(U)) = \det((\lambda - 1)I_3) \det(-D\sigma(\tau) - \lambda^2 I_3) = 0.
$$

It is easy to see we have three roots 1 which is not 0. The other roots satisfy the algebraic equation: $\det(-D\sigma(\tau) - \lambda^2 I_3) = 0$. Hence the proposition follows.

**Proposition B.4.** (Same spectrum) For system (B.4) and its first order system, the matrix $Df(U)$ and $B^{-1}Df(U)$ have the same spectrum.

**Proof.** We prove this for the variables $\tau, z, u \in \mathbb{R}^3$. It is easy to verify that

$$
B^{-1} = \begin{pmatrix}
I_3 & 0_3 & 0_3 \\
-b(\tau) & b(\tau) & -I_3 \\
-I_3 & I_3 & 0_3
\end{pmatrix}
$$

Since

$$
Df(U) = \begin{pmatrix}
0_3 & I_3 & -I_3 \\
0_3 & I_3 & 0_3 \\
D\sigma(\tau) & 0_3 & 0_3
\end{pmatrix},
$$

we immediately get

$$
B^{-1}Df(U) = \begin{pmatrix}
0_3 & I_3 & -I_3 \\
-D\sigma(\tau) & 0_3 & b(\tau) \\
0_3 & 0_3 & I_3
\end{pmatrix}.
$$
Considering the corresponding eigenvalue problem, we have:

\[ \det \left( \lambda I_3 - B^{-1} D f(U) \right) = \det \begin{pmatrix} \lambda I_3 & -I_3 & I_3 \\ D\sigma(\tau) & \lambda I_3 & -b(\tau) \\ 0_3 & 0_3 & (\lambda - 1)I_3 \end{pmatrix} = 0. \]

Doing Laplace expansion and performing basic transformation, we get:

\[ \det \left( \lambda I_3 - I_3 \right) \det \left( \lambda^2 I_3 + D\sigma(\tau) \right) = \det \left( \lambda I_3 - I_3 \right) \det \left( \lambda^2 I_3 - D^2 W(\tau) \right) = 0, \]

which implies the conclusion by noticing (C.5).

\[ \square \]

C  The principle of least action and relative entropy

In this section, we discuss further the relation between variational stability and time-evolutionary stability with respect to co-periodic perturbations of periodic waves. It is well-known that one of the fundamental laws in nature is the principle of least action (also known as Hamilton’s principle, Maupertuis’ principle). The most general formulation of the law governing the motion of mechanical system can be derived from this principle.

Consider a system which occupies at instants \( t_1 \) and \( t_2 \) positions given by the generalized co-ordinates vectors \( q^1, q^2 \in \mathbb{R}^N \). Then the principle says that the system moves between such positions that makes the action functional \( S := \int_{t_1}^{t_2} L(q(t), q'(t), t) \, dt \) takes the least possible value. Here the function \( L \) in terms of the time \( t \), the generalized co-ordinates \( q \) and velocities \( q'(t) \) is called the Lagrangian of the system.

Based on the principle of least action, one can give a unified formulation of laws of motion in mechanics in terms of differential equations (for example, wave equation, membrane equation, small oscillations, equations in fluid dynamics, etc). In particular, one can derive the Hamilton-Jacob theory of mechanics from which one could obtain systems of conservation laws (See [Lan]). Also, this principle is the foundation of calculus of variation and critical point theory (see [MW]). For more related details on this topic see [Lan, MW, PB] and the references therein.

To be self-contained, we recall the system (3.2) (which is (C.1) below now)

\[ \begin{aligned} \tau_t - u_x &= 0; \\ u_t + \sigma(\tau)_x &= (b(\tau)u_x)_x - (d(\tau_x)\tau_{xx})_x \end{aligned} \]

Here the functions \( b, d \) are the same as before. From [BLeZ] we know the following is the associated mechanical entropy

\[ \eta(\tau, u) = \frac{u^2}{2} + W(\tau) + \Psi(\tau_x) \]

Consider the following mechanical energy for a given positive period \( T \)
(C.3) \[ E = \int_0^T \eta(\tau, u) dx = \int_0^T \frac{u^2}{2} + W(\tau) + \Psi(\tau_x) dx \]

In any case, the mechanical energy \( E \) decreases along the flow for periodic boundary condition, with dissipation as follows

\[
\frac{d}{dt} E(\tau, u) = \int_0^T (uu_t + DW(\tau)\tau_t + D\Psi(\tau_x)\tau_{xt}) dx
\]

\[
= \int_0^T -u_x b(\tau) u_x + u_x d(\tau_x)\tau_{xx} + u_x \sigma(\tau) + DW(\tau) u_x + D\Psi(\tau_x)u_{xx} dx
\]

\[
= -\int_0^T u_x b(\tau) u_x dx \leq 0.
\]

Hence we see immediately that periodic traveling wave solutions must have \( u \equiv \text{constant vector} \), from which we then find easily the speed \( s = 0 \) in view of the relation \( u_t = -su_x = 0 \) and \( \tau_t - u_x = 0 \).

Next, we adopt the periodic Sobolev space framework to discuss the relation between the least action functional we defined and the relative entropy

(C.4) \[ E(\tau, u; < \tau >, < u >) := E(\tau, u) - E(< \tau >, < u >) - D\mathcal{E}(< \tau >, < u >) \cdot (\tau, u), \]

where \( < \tau > := \frac{1}{T} \int_0^T \tau(x) dx \) as in physics literature and similarly for \( u \).

After a brief computation, we get that the relative entropy is given by

\[
E(\tau, u; < \tau >, < u >) = \int_0^T \Psi(\tau_x) + W(\tau) - DW(< \tau >) \cdot \tau - W(< \tau >) dx
\]

\[ + \int_0^T \frac{1}{2} |u|^2 - \frac{1}{2} |< u >|^2 - < u > \cdot u dx. \]

From the system (3.2) we know that the structure is preserved under the transformation \( u \to u + c \) where \( c \) is an arbitrary constant vector. Hence without loss of generality, we can let \( u \equiv 0 \). Hence we get the following expression by further choosing \( < u > = m \) as before

\[
E(\tau, 0; 0, m) = \int_0^T \Psi(\tau_x) + W(\tau) - DW(m) \cdot \tau - W(m) dx
\]

\[
= \int_0^T \frac{1}{2} |\tau_x|^2 + W(\tau) - DW(m) \cdot \tau - W(m) dx
\]

Defining the translated variable \( v(x) = \tau(x) - m \), we get the relation between the least action functional and the relative entropy

\[
E(v, 0; 0, m) = \mathcal{I}(v) + \text{constant},
\]

where the constant is given by \( T(DW(m) \cdot m) \).
Remark C.1. The relative entropy is a rather common construction, meant to be stationary about the reference configuration (in our case $<\tau> = m$ and $<u> = 0$).

Remark C.2. The discussion above also sheds some light on the relation between time-evolutionary properties and variational structure of the Hamiltonian structure of our problem (See [Z2] for further discussions). By the method of [GSSI, GSSI], we see that the mean-constraint minimizers we constructed are necessarily stable in the time-evolutionary sense with respect to co-periodic perturbation if they are stable in the variational sense.

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