Special Relativity and Theory of Gravity via Maximum Symmetry and Localization  
— In Honor of the 80th Birthday of Professor Qikeng Lu *

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Like Euclid, Riemann and Lobachevsky geometries are on an almost equal footing, based on the principle of relativity of maximum symmetry proposed by Professor Qikeng Lu and the postulate on invariant universal constants $c$ and $R$, the de Sitter/anti-de Sitter ($dS/AdS$) special relativity on $dS/AdS$-space with radius $R$ can be set up on an almost equal footing with Einstein’s special relativity on Minkowski-space as the case of $R \to \infty$.

Thus the $dS$-space is coin-like: A law of inertia in Beltrami atlas with Beltrami time simultaneity for the principle of relativity on one side. The proper-time simultaneity and a Robertson-Walker-like $dS$-space with entropy and an accelerated expanding $S^3$ fitting the cosmological principle on another.

If our universe is asymptotic to the Robertson-Walker-like $dS$-space of $R \simeq (3/\Lambda)^{1/2}$, it should be slightly closed in $O(\Lambda)$ with entropy bound $S \simeq 3\pi c^3 k_B/\Lambda G\hbar$. Contrarily, via its asymptotic behavior, it can fix on Beltrami inertial frames without ‘an argument in a circle’ and acts as the origin of inertia.

There is a triality of conformal extensions of three kinds of special relativity and their null physics on the projective boundary of a 5-d $AdS$-space, a null cone modulo projective equivalence $[N] \simeq \partial_P(AdS^5)$. Thus there should be a $dS$-space on the boundary of $S^5 \times AdS^5$ as a vacuum of supergravity.

In the light of Einstein’s ‘Galilean regions’, gravity should be based on the localized principle of relativity of full maximum symmetry with a gauge-like dynamics. Thus, this may lead to theory of gravity of corresponding local symmetry. A simple model of $dS$-gravity characterized by a dimensionless constant $g \simeq (\Lambda G\hbar/3c^3)^{1/2} \sim 10^{-61}$ shows the features on umbilical manifolds of local $dS$-invariance. Some gravitational effects out of general relativity may play a role as dark matter.

The dark universe and its asymptotic behavior may already indicate that the $dS$ special relativity and $dS$-gravity be the foundation of large scale physics.

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I. INTRODUCTION

As a famous mathematician, Professor Qikeng Lu’s contributions to physics concern various fields, such as dispersion relations, special and general relativity, theory of gravity, gauge theory, integrable systems, conformal field theory, so on and so forth. As one of his great contributions, he
has suggested that the principle of relativity should be generalized to constant curvature spacetimes with radius $R$, i.e. de Sitter ($dS$) and anti-de Sitter ($AdS$) spacetimes $^1$ and began to research the special relativity on these maximally symmetric spacetimes in 1970s $^1,^2$. Actually, based on the principle of relativity of maximum symmetry and the postulate on invariant universal constants of $c$ and $R$, the $dS/AdS$-invariant special relativity, the $dS/AdS$ special relativity for short, can be set up $^1,^2,^3,^4,^5,^6,^7,^8,^9,^10,^11,^12,^13,^14$ on an almost equal footing with Einstein’s special relativity on Minkowski($Mink$)-spacetime as the case of $R \to \infty$.

As is well known, Einstein’s theory of relativity including special relativity, general relativity and cosmology provides some of most important breakthroughs in the last century. Together with quantum theory, they constitute the foundation of modern physics although how to quantize gravity formulated in general relativity is still open. Recent observations $^15,^16$ show, however, that our universe is almost completely dark and accelerated expanding. It is not asymptotic to a $Mink$-spacetime, but possibly a $dS$-spacetime with a tiny positive cosmological constant $\Lambda > 0$. With plenty of $dS$-puzzles, these greatly challenge Einstein’s theory of relativity as a foundation of physics in large scale.

The cosmological constant is regarded as some quantum ‘vacuum’ energy in ordinary approach. This leads to a huge difference of $10^{-122}$ as real $\Lambda > 0$ is extremely tiny. According to Professor Lu’s proposal $^1$ and the $dS$ special relativity $^1,^2,^3,^4,^5,^6,^7,^8,^9,^10,^11,^12,^13,^14$, however, it should be one of the fundamental constants in the Nature like the speed of light $c$, Newton’s gravitational constant $G$ and Planck constant $\hbar$. Thus, the huge difference puzzle should transfer to another issue.

Why there should be three kinds of special relativity with maximum symmetry?

When Poincaré first introduced the principle of relativity as one of the most important principles in the Nature $^17$, he inherited the assumption from Newton that space and time be Euclidean. In his first paper on special relativity $^18$, Einstein also took this assumption and required that a rest rigid ruler be Euclidean. But, there is nowhere exact flat in either our universe or its asymptotical region except in the sense of Einstein’s ‘Galilean regions’ $^19$ where gravity and the dark energy can be completely ignored. Actually, just like weakening Euclid’s fifth axiom leads to non-Euclidean geometry, giving up the Euclidean assumption should first lead to two other kinds of $dS/AdS$ special relativity on an almost equal footing with Einstein’s special relativity.

In geometry, Euclid, Riemann and Lobachevsky geometries as three classes of constant curvature ones of maximum symmetries, there are Descartes coordinate systems for Euclid geometry or Beltrami coordinate systems $^20$ for non-Euclidean ones (see also $^21,^22,^23$) and all geodesics in these systems are simultaneously straight lines of linear form, respectively. These systems in four dimensions, for example, with points, straight lines and metric symmetrically transformed under the linear transformations of $ISO(4)$ for Euclid geometry or the fractional linear ones with a common denominator ($FLTs$) of $SO(5)$ for Riemann geometry and of $SO(1,4)$ for Lobachevsky geometry, respectively. Beltrami $^20$ introduced such coordinate in order to show the consistency of Lobachevsky plane. It is completed by Klein $^21$.

Changing signature by a Weyl unitary trick or an inverse Wick rotation, these spaces with corresponding coordinate systems become $ISO(1,3)/SO(1,4)/SO(2,3)$-invariant $Mink/dS/AdS$-
spacetime with Mink-systems and Beltrami systems, respectively\, [7]. At the same time, points, geodesics being straight lines and metrics turn out to be events, geodesics being straight world-liners and Mink or Beltrami metric of physical signature in relevant coordinate systems symmetrically transformed under corresponding transformations of group $ISO(1,3)/SO(1,4)/SO(2,3)$, respectively. In analogy with those on Mink-spacetime, the motions along straight world-lines and the Beltrami systems on $dS/AdS$-spacetime should be of inertia. Thus, there should be a law of inertia and a principle of relativity on $dS/AdS$-spacetime, respectively. All these properties are also true globally on the $dS/AdS$-spacetime with Beltrami coordinate atlas.

As was claimed by Klein: ‘Geometry of space is associated with mathematical group’\, [25], the idea of invariance of geometry under transformation group may imply that on some spacetimes of maximum symmetries there should be a principle of relativity, which requires the invariance of physical laws without gravity under transformations among inertial systems. This is just the key point of Lu’s proposal to generalize the invariance of maximally symmetry for physical laws without gravity to all maximally symmetric spacetimes\, [1]. Further, all other kinds of principle of relativity on corresponding space-time, such as Galilei principle of relativity on Newton’s space and time, Newton-Hooke/anti-Newton-Hooke principle of relativity\, [9] on Newton-Hooke/anti-Newton-Hooke space-time and even Poincaré principle of relativity on Mink-spacetime, can be regarded as certain contraction of the $dS/AdS$-invariant principle of relativity on $dS/AdS$-spacetime in different limiting case, respectively. Thus, the significance of Lu’s proposal for relativistical physics is more or less like Klein’s Erlangen program for geometry.

In the $dS$ special relativity, there are some very important issues.

For free particles and light signals, in addition to the law of inertia there is a set of conserved observables with a generalized Einstein’s formula on mass-energy-momentum-boost-angular momentum. The famous Einstein’s formula on mass-energy-momentum is the case of $R \to \infty$.

There are two kinds of simultaneity. For the principle of relativity with inertial observers and inertial law, there is Beltrami time simultaneity. The proper-time simultaneity is for these observers’ comoving-like observations. If the proper-time is taken as a temporal coordinate, inertial observers become comoving-like ones and the Beltrami metric transfers to its Robertson-Walker-like $dS$ counterpart with an accelerated expanding closed 3-cosmos $S^3$\, , which fits the cosmological principle with $dS$-symmetry. Thus, the $dS$-spacetime with both the principle of relativity and the cosmological principle is just like a coin with two sides. Actually, the maximum symmetry ensures that these principles do make sense in different sides of the coin. Thus, the Robertson-Walker-like $dS$-cosmos acts as the origin of the inertial law in Beltrami inertial frames. And the principle of relativity with the inertial law on Beltrami metric provides a benchmark for physics on $dS$-spacetime. Further qualitatively, due to the generalized Einstein formula and the $dS$-symmetry, all free moves of test objects such as celestial objects including the cosmic microwave background radiation (CMB) as a whole should have both conserved energy-momentum and angular momentum.

For the $dS$-horizon puzzle, i.e. why the $dS$-spacetime of constant curvature is like a black hole, there is another explanation. The $dS$-horizon in Beltrami systems is actually at $T = 0$ without entropy, while at the horizon in other systems, such as the static $dS$-universe and the Robertson-
Walker-like $dS$-spacetime, Hawking temperature and area entropy appear as non-inertial effects rather than that of gravity \cite{8}. Thus, $dS$-spacetime is completely different from a black hole.

Since our universe is asymptotic to a $dS$-spacetime, which should be such a Robertson-Walker-like $dS$-space \textit{with} $R \simeq (3/\Lambda)^{1/2}$, it should be an evolutional slightly closed \textit{3-dimensional} cosmos curved in the order of $\Lambda$, $O(\Lambda)$, with an upper entropy bound. The closeness with very tiny curvature of our universe is a simple but important prediction. It is more or less indicated by the data from WMAP recently \cite{16}. On the other hand, the evolution of our universe can fix on a kind of Beltrami inertial frames via their Robertson-Walker-like $dS$ counterpart as the fate of our universe \cite{13}. Thus, for the principle of inertia on $dS$-space there is no Einstein’s ‘argument in a circle’ \cite{19}. Further, if all other kinds of principle of inertia are regarded as contractions under different limits of the principle of inertia on $dS$-space, this is also true for all kinds of principle of inertia.

A scaling of $R$ leads to conformal extensions of the $dS/AdS$ special relativity on $dS/AdS$-spacetime. Together with conformal extension of Einstein’s special relativity on $Mink$-spacetime, in fact, all these conformal extensions are on a null cone modulo projective equivalence isomorphic to the projective boundary of a 5-dimensional $AdS$-spacetime, $[\mathcal{N}] \cong \partial_P(AdS^5)$. Thus, there is a triality of conformal extensions of three kinds of special relativity and null physics on $Mink/dS/AdS$-spacetimes. And certain Weyl mappings relate any two of them \cite{26}. Further, there should be a $dS$-spacetime on the boundary of $S^5 \times AdS^5$ as a vacuum of supergravity.

According to general relativity, there is no special relativity on $dS/AdS$-spacetime. Different from general relativity, in view of the $dS/AdS$ special relativity, there is no gravity on $dS/AdS$-spacetime. We should explain how to describe gravity in the universe.

In the light of Einstein’s ‘Galilean regions’ where his special relativity should hold locally since the regions are essentially ‘finite’ \cite{19}, it is the core of Einstein’s idea on spacetime with gravity that it should be curved with localized special relativity of local full Poincaré symmetry \cite{13}. In Einstein’s general relativity, however, there are local Lorentz frames of only local $SO(1,3)$ invariance rather than full local Poincaré invariance with local translations. Thus, in Einstein’s general relativity, the benchmark of physics for defining physical quantities and introducing laws of physics with gravity is not completely in consistency with that in Einstein’s special relativity \cite{13}. In addition, there is a ‘Gordian knot’ in dynamics (see, e.g., \cite{27}). These may cause some puzzles.

Taking into account the localization of special relativity, theory of gravity should be based on a generalized equivalence principle with full localized maximum symmetry of special relativity called the principle of localization. In consistency with this principle, it can be further expected that gravity be governed by a gauge-like dynamics with same local maximum symmetry. Thus, the localization of three kinds of special relativity leads to three kinds of theories of gravity with full local maximum symmetry. The Nature \textbf{should} prefer one of them.

How to realize mathematically the localization of three kinds of special relativity?

It is needed to localize $Mink/dS/AdS$-spacetime as maximally symmetric spacetime $\mathcal{S}$ with maximum symmetry group $\mathfrak{G}$ and to patch them together as a kind of differential manifolds $(\mathcal{M}, g, \Gamma)$ with metric $g$ and metric compatible connection $\Gamma$ valued in Lie algebra $\mathfrak{g}$ of $\mathfrak{G}$. That is,
in terminology of fibre bundle and connection theory \[28\], it is needed to set up a principal bundle \(P(\mathcal{M}, \mathfrak{G})\) over such an \(\mathcal{M}\) with \(\mathfrak{G}\) as a structure group and an associated bundle \(E(\mathcal{M}, \mathcal{S}, \mathfrak{G}, P)\) with \(\mathcal{S}\) as a typical fibre. In addition, there should be some associated bundles with certain irreducible representations of \(\mathfrak{G}\) as fibre fitted by the matter fields as sources, and so on. Since transformations on the Beltrami model of \(dS\)-spacetime are of \(\text{FLTs}\), these requirements may lead to some connection valued in \(g\) realized non-linearly. In fact, this is one of motivations for Lu to study the non-linear connection theory \[29\].

There are still some physical issues to be precisely set up for such geometric description of spacetimes with gravity based on the principle of localization, such as the relation between the metric with local maximum symmetry \(\mathfrak{G}\) and the connection valued in \(g\) and so on. However, there is a simple model of \(dS\)-gravity with a gauge-like action characterized by a dimensionless constant \(g \sim (\Lambda G h/3 c^3)^{1/2} \sim 10^{-61} [30, 31, 32, 33]\) on a kind of umbilical Riemann-Cartan manifolds with local \(dS\)-invariance \[31\]. It has partially shown these features and may present an explanation of the dark matter in terms of the gravitational effects out of general relativity, at least partially. Therefore, it may provide an alternative framework for data analysis in precise cosmology.

It should be notes that \(g^2\) is in the same order of the huge difference of \(\Lambda\) as so-called quantum ‘vacuum energy’. Then there are further questions: What is the origin of the dimensionless constant \(g\)? Is it with other dimensionless constants calculable?

This paper is arranged as follows. In section II, we explain why there should be two other kinds of special relativity with \(dS/AdS\)-invariance by means of the Beltrami model of Riemann sphere and its physical counterpart on \(dS\)-spacetime. Some historical remarks are also made. In section III, we briefly introduce the properties and cosmological significance of the \(dS\) special relativity. We explain how the evolution of our universe can fix on the inertial systems without Einstein’s ‘argument in a circle’. In section IV, we show that there is a triality of conformal extensions of three kinds of special relativity and null physics on 4-dimensional \(\text{Mink}/dS/AdS\)-spacetimes on the projective boundary of a 5-dimensional \(AdS\)-spacetime. In section V, in the light of Einstein’s ‘Galilean regions’ in spacetime with gravity, we explain why gravity should be based on the principle of localization, i.e. on the localized special relativity with full maximum symmetry. We also briefly introduce the simple model of \(dS\)-gravity with an action of gauge-like on umbilical manifolds. Finally, we end with some concluding remarks.

II. THREE KINDS OF SPECIAL RELATIVITY WITH MAXIMUM SYMMETRY

A. Beltrami model of Riemann sphere

Let us focus on the Beltrami model of Riemann sphere \(S^4_R\) \[7, 14\], since its physical counterpart is just the Beltrami model of \(dS\)-spacetime, denoted \(BdS\)-space. Similarly, we may consider the Beltrami model of Lobachevsky hypernoloid \(L^4\) and the one of \(AdS\)-spacetime as its physical counterpart. In the original Beltrami model \[20, 21\] for Lobachevsky geometry, it is of one coordinate chart (see, e.g. \[22, 23\]). For the case of Riemann sphere \(S^4\), however, one chart is
not enough. But, all fundamental properties can be generalized to a Beltrami coordinate atlas covering the sphere chart by chart.

A Riemann sphere $S^4_R$ with radius $R$ can be embedded in a 5-dimensional Euclidean space $E^5$

$$S^4_R : \quad \delta_{AB} \xi^A \xi^B = R^2 > 0, \quad A, B = 0, \cdots, 4, \quad (2.1)$$

$$ds^2_E = \delta_{AB} d\xi^A d\xi^B = d\xi d\xi', \quad (2.2)$$

where $I = (\delta_{AB}) = \text{diag}(1, \cdots, 1)$, $\xi = (\xi^0, \cdots, \xi^4)$. They are invariant under (linear) rotations:

$$\xi \to \xi' = \xi S, \quad SIS^t = I, \quad \forall S \in SO(5). \quad (2.3)$$

In the surface theory, an $S^2 \subset E^3$ is an umbilical one (see, e.g., [24]). This is also the case for the Riemann sphere $S^4_R \subset E^5$.

The Beltrami model provides an intrinsic geometry of $S^4_R$ on the Beltrami-space $B_R \cong S^4_R$ with Beltrami coordinates atlas. All properties of $S^4_R$ are well-defined in the atlas. For an orientable intrinsic geometry of $B_R$, it is needed an atlas with eight charts: $U_{\pm a} := \{ \xi \in S^4_R : \xi^a \geq 0 \}, a = 1, \cdots, 4$. In the chart $U_{+4}$, for instance, the Beltrami coordinates are

$$x^i|_{U_{+4}} = R \sum \frac{\xi^i}{\xi^4}, \quad i = 0, \cdots, 3; \quad \xi^4|_{U_{+4}} > 0. \quad (2.4)$$

In another chart $U_{+3}$, say,

$$y^j|_{U_{+3}} = R \sum \frac{\xi^j}{\xi^3}, \quad j = 0, \cdots, \hat{3}, \cdots, 4; \quad \xi^3|_{U_{+3}} > 0, \quad (2.5)$$

where $\hat{3}$ means omission of 3. It is important that the transition function $T_{+4,+3}$ on the intersection $U_{+4} \cap U_{+3}$ is of $\text{FLT}$: $T_{+4,+3} = \xi^3/\xi^4 = x^3/R = R/y^4$ so that $x^i = T_{+4,+3}y^j$ for $i = i' = 0, 1, 2$ and $x^3 = R^2/y^4$ are of $\text{FLT}$s.

In the chart $U_{+4}$, say, Riemann sphere (2.1) and metric (2.2) restricted on $B_R$ becomes domain condition and Beltrami metric, respectively:

$$B_R : \quad \sigma_E(x) := \sigma_E(x, x) = 1 + R^{-2} \delta_{ij} x^i x^j > 0, \quad (2.6)$$

$$ds^2_E = \{\delta_{ij} \sigma_E^{-1}(x) - R^{-2} \sigma_E^{-2}(x) \delta_{ij} x^i x^j \} dx^i dx^j, \quad (2.7)$$

which are invariant under the $\text{FLT}$s among Beltrami coordinates $x^i$ in a transitive form sending a point $A(a^i)$ to the origin $O(a^i = 0)$,

$$x^i \to \tilde{x}^i = \pm \sigma_E(a)^{1/2} \sigma_E(a, x)^{-1}(x^j - a^j) N^i_j, \quad N^i_j = O^i_j - R^{-2} \delta_{jk} a^k a^l (\sigma_E(a) + \sigma_E(a)^{1/2})^{-1} O^l_i, \quad (2.8)$$

$$O := (O^i_j)_{i,j=0,\cdots,3} \in SO(4).$$

There is an invariant for two points $A(a^i)$ and $X(x^i)$ on $B_R$, which corresponds to the cross ratio among the points together with the origin and the infinity in projective geometry approach:

$$\Delta^2_{E,R}(a, x) = R^2[\sigma_E(a) \sigma_E(x) - \sigma_E^2(a, x)]. \quad (2.9)$$
For two adjacent points $X(x^i)$ and $X'(x^i + dx^i)$, this invariant is just the Beltrami metric $\frac{2.7}{2.7}$.

The proper length between $A(a^i)$ and $B(b^i)$ is an integral of $ds_E$ over the geodesic segment $AB$:

$$L_E(a, b) = R \arcsin(|\Delta_E(a, b)|/R). \quad (2.10)$$

As was mentioned, there is an important property in the model: All geodesics of the Beltrami metric are straight lines linearly. This property of the Beltrami coordinates is different from other coordinates for the Riemann sphere and also different from other non-maximally symmetric spaces in Riemannian differential geometry in general.

In fact, the geodesics of the Beltrami metric are equivalent to

$$\frac{dq^i}{ds_E} = 0, \quad q^i := \sigma_E^{-1}(x^i) \frac{dx^i}{ds_E}. \quad (2.11)$$

Therefore,

$$q^i = \text{consts}. \quad (2.12)$$

Further, it is easy to see that the following rations are constants

$$\frac{q^\alpha}{q^0} = \frac{dx^\alpha}{dx^0} = \text{consts}, \quad \alpha = 1, 2, 3. \quad (2.13)$$

The eqn. $(2.11)$ can be integrated further to get the linear result:

$$x^i(s) = \alpha^i x^0(s) + \beta^i; \quad \alpha^i, \beta^i = \text{consts}. \quad (2.14)$$

Under the FLT's $(2.8)$ among Beltrami systems, all these properties are transformed among themselves. They are also well established globally chart by chart.

From the viewpoint of projective geometry, Beltrami coordinates are similar to inhomogeneous projective ones and antipodal identification should not be taken in order to preserve orientation.

**B. Beltrami model of $dS$-spacetime**

Via a Weyl unitary trick or an inverse Wick rotation of $E^5$, which turns $\xi^0$ to be time-like, the Riemann sphere $S^4_R \subset E^5$ and its Beltrami model $B_R$ becomes the $dS$-hyperboloid $H_{R^+} \subset M^{1,4}$ and its $BdS$-model, a $dS$-spacetime with the Beltrami atlas, respectively $[4, 5, 6, 7]$. In fact, a Weyl trick or an inverse Wick rotation changes all $(\delta_{AB}), (\delta_{ij})$ in all metrics to $(\eta_{AB}) := \text{diag}(1, -1, \cdots, -1), (\eta_{ij}) := \text{diag}(1, -1, -1, -1)$ and the sign of $R^2$ in all formulas. Then both $\xi^0$ and $x^0$ become time-like. In order to introduce the time coordinate, a universal constant $c$ of speed dimension is needed, say $x^0 = ct$. Thus, there are two universal constants $c$ and $R$.

Let us briefly review the $dS$-hyperboloid, its $BdS$-model and some physics on them.
1. $dS$-hyperboloid $H_{R^+} \subset M^{1,4}$ and uniform ‘great circular’ motion

The $dS$-hyperboloid can be embedded in a $4+1$-dimensional Mink-spacetime $H_{R^+} \subset M^{1,4}$ or simply $H_+ \subset M^{1,4}$:

$$H_{R^+}: \quad \eta_{AB}\xi^A\xi^B = -R^2 < 0, \quad A, B = 0, \cdots, 4,$$

$$(2.15)$$

$$ds^2_+ = \eta_{AB}d\xi^Ad\xi^B = d\xi Jd\xi^t,$$

$$(2.16)$$

$$\partial_P H_{R^+}: \quad \eta_{AB}\xi^A\xi^B = 0,$$

$$(2.17)$$

where $J = (\eta_{AB}) = \text{diag}(1, -1, -1, -1, -1)$, $\partial_P$ the projective boundary. They are invariant under (linear) transformations of $dS$-group $SO(1,4)$:

$$\xi \rightarrow \xi' = \xi S, \quad SJS^t = J, \quad \forall S \in SO(1,4).$$

$$(2.18)$$

It is clear that the $dS$-hyperboloid $H_+ \subset M^{1,4}$ (2.15) is also an umbilical hypersurface of constant curvature in the following sense: At any given point $\forall P \in H_+ \subset M^{1,4}$, the first and second fundamental forms are proportional to each other with a coefficient $R$. In addition, there are a tangent Mink-space $T_P(H_+)$ at $P$ and a radius vector $r_p$ opposite to the normal vector with respect to the tangent space, i.e. $r_P = -(N = Rn)_P$, where $n_P$ is a unit base of the normal space $N_P$, and $T_P(H_+) \times N_P \cong M^{1,4}$. This structure will be useful for the localization of the $H_+ \subset M^{1,4}$.

Corresponding to great circles as geodesics on the Riemann sphere $S^4_R$, there should be a kind of uniform ‘great circular’ motions for a particle with mass $m_R$ along geodesics on the $dS$-hyperboloid $H_+ \subset M^{1,4}$ defined by a conserved 5-dimensional angular momentum $L^{AB}$:

$$\frac{dL^{AB}}{ds_+} = 0, \quad L^{AB} := m_R(\xi^A d\xi^B - \xi^B d\xi^A).$$

$$(2.19)$$

And for the particle, there is an Einstein-like formula

$$-\frac{1}{2R^2}L^{AB} L_{AB} = m_R^2, \quad L_{AB} = \eta_{AC}\eta_{BD}L^{CD}.$$

$$(2.20)$$

For a massless particle or a light signal with $m_R = 0$, similar uniform ‘great circular’ motion can also be defined as long as the proper-time in $ds_+$ is replaced by an affine parameter $\lambda_+$ and there is no $m_R$ in the counterpart of $L^{AB}$ in (2.19), respectively. Namely,

$$\frac{dL^{AB}}{d\lambda_+} = 0, \quad L^{AB} := \xi^A K^B - \xi^B K^A, \quad K^A := \frac{d\xi^A}{d\lambda_+}.$$

$$(2.21)$$

There is also an Einstein-like formula for the massless case.

In order to make sense for these motions, simultaneity should be defined.

There are two time-like scales on the $dS$-hyperboloid, the coordinate-time $\xi^0$ and the proper-time $s_+$. For a pair of events $(P(\xi_P), Q(\xi_Q))$, they are simultaneous in the coordinate-time if and only if

$$\xi_P^0 = \xi_Q^0.$$

$$(2.22)$$
A simultaneous 3-hypersurface of $\xi^0 = \text{const}$ is an expanding $S^3$

$$\delta_{ab}\xi^a\xi^b = R^2 + (\xi^0)^2, \quad a, b = 1, \ldots, 4; \quad (2.23)$$

$$dl^2|_{\xi^0=\text{const}} = \delta_{ab}d\xi^ad\xi^b.$$ 

For a kind of observers $O_H$ at the point $O|_{\xi^0=0}$ with $\alpha$ takes three of $1, \ldots, 4$, which will become the spatial origin of a corresponding chart of the Beltrami atlas, this simultaneity is the same with respect to the proper-time simultaneity on $H_+ \subset M^{1,4}$.

The generators of the $dS$-algebra $\mathfrak{so}(1, 4)$ read:

$$i\hat{L}_{AB} = \xi_A \frac{\partial}{\partial \xi_B} - \xi_B \frac{\partial}{\partial \xi_A}, \quad \xi_A = \eta_{AB}\xi^B, \quad (2.24)$$

which are proportional to the Killing vectors on the $H_+ \subset M^{1,4}$. They form an $\mathfrak{so}(1, 4)$-algebraic relation and the 5-dimensional angular momentum (2.19) can also be viewed as a set of Noether’s charges of the particle with respect to these Killing vectors.

The first Cisimir operator of the algebra corresponding to the Einstein-like formula (2.19) is

$$\hat{C}_1 := -\frac{1}{2} R^{-2} \hat{L}_{AB}\hat{L}^{AB}, \quad \hat{L}^{AB} := \eta^{AC}\eta^{BD}\hat{L}_{CD}, \quad (2.25)$$

with eigenvalue $m_R^2$, which gives rise to the classification of the mass $m_R$.

2. Beltrami model of $dS$-spacetime and inertial motions

Let us now consider the $BdS$-spacetime and uniform motions along straight world-lines on it.

In order to preserve the orientation, for an intrinsic geometry of the $BdS$-space, it is also needed an atlas with eight charts $U_{\pm a} := \{ \xi \in H_+ : \xi^a \gtrless 0 \}; a = 1, \ldots, 4$. [4, 5].

In the charts $U_{\pm 4}$, for instance, the Beltrami coordinates are

$$x^i|_{U_{\pm 4}} = \frac{R^2}{\xi^4}, \quad i = 0, \ldots, 3; \quad \xi^4|_{U_{\pm 4}} = (\xi^0^2 - \sum_{\alpha=1}^{3} \xi^\alpha^2 + R^2)^{1/2} \gtrless 0. \quad (2.26)$$

In the charts $\{ U_{\pm a}, a = 1, 2, 3 \}$,

$$y^j|_{U_{\pm a}} = \frac{R^2}{\xi^a}, \quad j = 0, \ldots, \hat{a} \cdots, 4; \quad \xi^a|_{U_{\pm a}} \gtrless 0. \quad (2.27)$$

Then all transition functions are of FLT.

In the chart $U_{+4}$, $\xi^4 > 0$, the observer $O_H|_{\xi^0=0}, (a = 1, 2, 3)$ on the $H_+ \subset M^{1,4}$ (2.15) now becomes an observer $O_I$ rest at the spatial origin $(x^a = 0)$. And there are domain condition, Beltrami metric and boundary condition as follows

$$BdS : \quad \sigma(x) := \sigma(x, x) = 1 - R^{-2}\eta_{ij}x^i x^j > 0, \quad (2.28)$$

$$ds^2_+ = [\eta_{ij}x^i x^j + R^{-2}\eta_{ik}\eta_{jk}x^i x^j x^k\sigma^{-2}(x)]dx^idx^j, \quad (2.29)$$

$$\partial P(BdS) : \quad \sigma(x) = 0, \quad (2.30)$$
where \((\eta_{ij})_{i,j=0,\ldots,3} = \text{diag}(1, -1, -1, -1)\). They are invariant under FLT's of \(SO(1, 4)\) sending a point \(A(a^i)\) to the origin \(O(a^i)\) with all coordinate \(a^i = 0\):

\[
\begin{align*}
    x^i \to \tilde{x}^i &= \pm \sigma^{1/2}(a)(a, x)(x^j - a^j)D^i_j, \\
    D^i_j &= L^i_j + R^{-2}\eta_{ij}a^j(\sigma(a) + \sigma^{1/2}(a))^{-1}L^i_k, \\
    L &= (L^i_j)_{i,j=0,\ldots,3} \in SO(1, 3).
\end{align*}
\]

(2.31)

For a free particle with mass \(m_R\), its uniform ‘great circular’ motion along a geodesic on the \(dS\)-hyperboloid now becomes a uniform motion along the time-like geodesic as a straight world-line on the \(BdS\)-spacetime. In fact, such a time-like geodesic is equivalent to

\[
\frac{dp^i}{ds_+} = 0, \quad p^i := m_R\sigma^{-1}(x)\frac{dx^i}{ds_+}.
\]

(2.32)

Thus, \(p^i = \text{consts}\). And the coordinate velocity components \(v^a = dx^a/dt\) are constants:

\[
\frac{p^a}{p^0} = \frac{dx^a}{dx^0} = : c^{-1}v^a = \text{consts}, \quad x^0 = ct, \quad a = 1, 2, 3.
\]

(2.33)

It can be integrated further to get linear result as a counterpart of (2.14).

For massless particles or light signals, similar issues hold as long as the proper-time \(s_+\) is replaced by an affine parameter \(\lambda_+\).

Under the FLT's (2.31) of \(SO(1, 4)\), all these properties together with Beltrami systems are transformed among themselves. And these properties are well defined chart by chart.

It should be noted that in principle we may also introduce two other sets of inhomogeneous projective coordinates without antipodal identification by \(\tilde{x}^j := R^j_0/j_0, j = 1, \cdots, 4\); or by \(\tilde{x}^j_\pm := R^j_0/(j_0 \pm j_4)\). However, if we require that under limit \(R \to \infty\) the coordinates and the transformations among them are back to the \(Mink\)-coordinates and their Poincaré transformations, only the Beltrami atlas with coordinates \(x^j\) in (2.26) survive.

C. Klein’s Erlangen program versus principle of relativity in all possible kinematics

As was emphasized, in analogy with that weakening Euclid’s fifth axiom leads to Riemann and Lobachevsky geometries on an almost equal footing with Euclid geometry, there should be two other kinds of \(dS/AdS\) special relativity on an almost equal footing with Einstein’s one.

In fact, there is a one-to-one correspondence between these geometries on maximally symmetric spaces \(\mathcal{G}_E\) with maximum symmetries \(\mathcal{G}_E\) and their physical counterparts on maximally symmetric spacetimes \(\mathcal{G}\) with maximum symmetries \(\mathcal{G}\). We list them in the following table:
Table 1. Correspondence between 4-d geometry and 3+1-d special relativity

| Geometry on $\mathcal{S}_E$ with $\mathcal{G}_E$ | Spacetime Physics on $\mathcal{G}$ with $\mathcal{G}$ |
|-----------------------------------------------|---------------------------------------------------|
| $E^4/S^1/L^4$ as $\mathcal{S}_E$              | $M^{1,3}/dS^{1,3}/AdS^{1,3}$ as $\mathcal{G}$    |
| $ISO(4)/SO(5)/SO(1,4)$ as $\mathcal{S}_E$   | $ISO(1,3)/SO(1,4)/SO(2,3)$ as $\mathcal{G}$     |
| Descartes/Beltrami systems                    | Minkowski/Beltrami systems                        |
| Points                                        | Events                                            |
| Geodesics as straight lines                   | Geodesics as straight world-lines                 |
| Principle of Invariance                       | Principle of Relativity                           |
| Erlangen Programm                             | Theory of Special Relativity                      |

From the viewpoint of $dS/AdS$ special relativity, all possible kinematics can be set up based on the corresponding principle of relativity and the corresponding postulate of universal constant(s), respectively, although for Newton’s theory these constants are all degenerate (see, e.g., [9]).

Actually, in view of geometrical and algebraic contractions, there are some important contraction relations among these kinematics. Namely, all other kinds of kinematics can be viewed as some contraction of the $dS/AdS$ special relativity under certain limit of the constant(s): Einstein’s special relativity on $Mink$-spacetime with Poincaré principle of relativity of Poincaré invariance under $R \to \infty$; Newton’s mechanics on Newton’s space and time with Galilei principle of relativity of Galilei invariance under $R, c, \to \infty$. Newton-Hooke/anti-Newton-Hooke mechanics on Newton-Hooke/anti-Newton-Hooke space-time with Newton-Hooke principle of relativity of Newton-Hooke/anti-Newton-Hooke symmetry under the Newton-Hooke limit: $R, c, \to \infty$, but the Newton-Hooke constant $\nu := c/R = const$, respectively [9].

Conversely, there are also some ‘deformation’ relations among them.

D. Historical remarks

It should be noted that the $dS$ geometry and physics are studied for long time in the framework of general relativity. However, the principle of inertia, the law of inertia and relevant physics on $dS$ spacetime had been missed, although Beltrami systems had been used or mentioned time after time in literatures.

As early as in 1917, de Sitter [34] introduced Beltrami-Klein coordinates for his solution in the debate with Einstein on ‘relative inertia’. Their debate also drew attentions from Klein and Weyl. A few years later, Pauli mentioned fractional linear transformations and the Beltrami model of 4-dimensional Riemann sphere in his famous book but ignored their possible physical applications [35]. Snyder [36] proposed a quantized space-time model in projective geometry approach, explained by Pauli, to $dS$-space of momenta. This is in fact the earliest and simplest model among the ‘doubly special relativity’ or the ‘deformed spatial relativity’ widely studied recently [37]. Although there is a simple one-to-one correspondence between Snyder’s model and the $dS$ special relativity [11], it had not been considered what should be the counterpart in coordinate picture of Snyder’s model in momentum space before. Schrödinger also proposed the
‘elliptic explanation’ of $dS$-spacetime concerning the antipodal identification [38], which has been also studied in [39]. However, there had been no study on such a key issue for long time that in either Beltrami coordinates or inhomogeneous projective ones there are uniform motions along time-like or null geodesics. Therefore, there should be the law of inertia on $dS$-spacetime and these coordinates should play a role of inertia.

On the other hand, Umov, Weyl and Fock (see, e.g., [40]) studied the FLT as most general transformations among inertial systems and inertial motions. But, they did not relate these FLTs to either Beltrami systems or the inertial motions on them. Otherwise, the inertial law on $dS/AdS$-spacetimes could be discovered long time ago.

Since 1950s, Hua and Lu develop the theory of classical domains and harmonic analysis on the domains [41]. As the Beltrami model of hyperboloid is a special case, Hua and Lu use the generalized Beltrami metric widely in their studies. In 1970, Lu [1] first noticed the key point in physics and began the research on the $dS/AdS$ special relativity later [1, 2]. Promoted by recent observations on the dark universe, further studies are made [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13].

III. PRINCIPLE OF RELATIVITY AND DE SITTER SPECIAL RELATIVITY

We now briefly introduce the properties of the $dS$ special relativity based on the principle of relativity and the postulate on invariant universal constants. We show its cosmological significance via the coin-like model of $dS$-space with both the principle of relativity and the cosmological principle. We also explain why our universe should be slightly closed if it is asymptotic to a $dS$-space and why its evolution can fix on a kind of Beltrami inertial frames together with all its contractions. Thus, our universe displays as the origin of inertia without Einstein’s ‘argument in a circle’ for the principle of inertia.

A. Transformations among inertial systems and principle of relativity

The existence of the $dS/AdS$ special relativity can also be prospected from another angle: What are the most general transformations among inertial motions and inertial systems? As was just mentioned, Umov, Weyl and Fock [40] studied this problem long time ago.

As in both Newton’s mechanics and Einstein’s special relativity, inertial motions can be defined as a kind of motions with uniform coordinate velocity along straight lines in a kind of coordinate systems. Namely, if in a system $S(x)$ for a free particle its motion satisfies

$$x^\alpha = x_0^\alpha + v^\alpha(t - t_0), \quad v^\alpha = \frac{dx^\alpha}{dt} = consts, \quad \alpha = 1, 2, 3,$$

the motion and the system are called inertial one, respectively.

Let us consider a transformed system $S'$, if the same particle is described by

$$x'^\alpha = x'_0^\alpha + v'^\alpha(t' - t'_0), \quad v'^\alpha = \frac{dx'^\alpha}{dt'} = consts,$$
the transformed system is also of inertia. What are the most general transformations between
these two inertial systems? Fock \[40\] showed that the most general form of transformations
\[x'^i = f^i(t, x^\alpha), \quad i = 0, \ldots, 3,\] (3.3)
which transform a uniform straight line motion in \(S\) with (3.1) to a motion of the same nature
in \(S'\) with (3.2) should be that four functions \(f^i\) are ratios of linear functions, all with the same
denominator. Thus, they are of the FLT-type.

As was mentioned, in general we may not assume that the proper-length of a ‘rigid’ ruler
and the proper-time of an ‘ideal’ clock be Euclidean. In other words, the spatial coordinates
themselves and the temporal coordinate itself are not assumed to be uniform in the Euclidean
sense, respectively. This is different from either Newton’s mechanics or Einstein’s special relativity.
Otherwise, the FLT-s should just be the linear ones in Newton’s mechanics or Einstein’s special
relativity. Fock just did so by assuming the wave front equation with Mink-metric so that the
FLT-s reduce to the transformations of Poincaré group.

As there is a Mink-metric on 4-dimensional Mink-spacetime invariant under transformations
of Poincaré group with ten parameters, we should require that there be a metric in the inertial
systems on 4-dimensional spacetime and the FLT-s form a group with ten parameters, like Galilei
group in Newton’s mechanics and Poincaré group in Einstein’s special relativity, including four for
spacetime ‘translations’, three for boosts, and rest three for space rotations. Thus, according to the
properties of maximally symmetric spaces (see, e.g., \[42\]), such kind of 4-dimensional spacetimes
should be the maximally symmetric spacetimes \(\mathcal{G}\) of positive/negative constant curvature with
radius \(R\) or zero curvature with \(R \to \infty\). Namely, they are just the \(dS/AdS/Mink\)-spacetime
being the maximally symmetric spacetime \(\mathcal{G}\) with \(SO(1, 4)/SO(2, 3)/ISO(1, 3)\)-invariance being
the maximum symmetry \(\mathcal{G}\), respectively.

From the viewpoint of projective transformations, these are obvious: those uniform motions
along straight lines are of projective and the transformations of projectively FLT-s. All the
maximally symmetric spacetimes with maximum symmetries, respectively, are of sub-geometries
of projective geometry. This is also in consistency with Klein’s program \[25\]. Of course, the
orientation should be preserved in physics.

As was mentioned, for the \(dS/AdS\)-spacetime Beltrami systems are indeed these systems and
the observer \(O_I\) at the spatial origin is of inertia. Therefore, on the \(BdS/anti-BdS\)-spacetime,
there are also the principle of relativity and the postulate on invariant universal constants. The
principle of relativity states: The physical laws without gravity are invariant under the group
transformations among inertial systems on the 4-dimensional \(dS/AdS\)-spacetime, respectively.
The postulate requires: In the inertial systems on 4-dimensional \(dS/AdS\)-spacetimes, there are
two invariant universal constants, the speed of light \(c\) and the curvature radius \(R\).

Based on the principle and the postulate, the \(dS/AdS\) special relativity can be set up \[4, 5, 6\].
B. Law of inertia, generalized Einstein formula, light cone and horizon

Thus, there is a Beltrami atlas of inertia on the $BdS$ and in each chart there are condition (2.28), metric (2.29) and FLTs (2.31) of $dS$-group.

In such a $BdS$, the generators of FLTs read

$$\hat{p}_i = (\delta^j_i - R^{-2}x_ix^j)\partial_j, \quad x_i := \eta_{ij}x^j,$$
$$\hat{L}_{ij} = x_i\hat{p}_j - x_j\hat{p}_i = x_i\partial_j - x_j\partial_i \in \mathfrak{so}(1,3),$$

and form an $\mathfrak{so}(1,4)$ algebra

$$[\hat{p}_i, \hat{p}_j] = R^{-2}\hat{L}_{ij}, \quad [\hat{L}_{ij}, \hat{p}_k] = \eta_{jk}\hat{p}_i - \eta_{ik}\hat{p}_j,$$
$$[\hat{L}_{ij}, \hat{L}_{kl}] = \eta_{jk}\hat{L}_{il} - \eta_{jl}\hat{L}_{ik} + \eta_{il}\hat{L}_{jk} - \eta_{ik}\hat{L}_{jl}. \quad (3.5)$$

For a free particle along a time-like geodesics being a straight world-line there is a set of conserved quantities $p^i$ in (2.32) and

$$L^{ij} = x^ip^j - x^j p^i, \quad \frac{dL^{ij}}{ds} = 0. \quad (3.6)$$

These are pseudo 4-momentum $p^i$, pseudo 4-angular-momentum $L^{ij}$ of the particle, which constitute the conserved 5-dimensional angular momentum as was shown in (2.19).

Thus, there is a law of inertia on $dS$/AdS: The free particles and light signals without undergoing any unbalanced forces should keep their uniform motions along straight world-lines in linear forms in Beltrami systems on $dS$/AdS-space, respectively.

The equation of motion for a forced particle can also be given [5, 6].

Further, all these conserved quantities satisfy a generalized Einstein formula on $BdS$-space from the Einstein-like formula (2.20):

$$E^2 = m^2c^4 + p^2c^2 + \frac{c^2}{R^2}j^2 - \frac{c^4}{R^2}k^2, \quad (3.7)$$

with energy $E = p^0$, momentum $p^\alpha$, $p_\alpha = \delta_{\alpha\beta}p^\beta$, ‘boost’ $k^\alpha$, $k_\alpha = \delta_{\alpha\beta}k^\beta$ and 3-angular momentum $j^\alpha$, $j_\alpha = \delta_{\alpha\beta}j^\beta$. And these observables may also be viewed as Noether’s charges of the particle with respect to the Killing vectors proportional to the generators in (3.4). Note that $m^2R$ now is the eigenvalue of first Casimir operator of $dS$-group, the same as the one in (2.25).

If we introduce the Newton-Hooke constant $\nu$ [9] and link the radius $R$ with the cosmological constant $R \simeq (3/\Lambda)^{1/2}$,

$$\nu := \frac{c}{R} \simeq c(3/\Lambda)^{-1/2}, \quad \nu^2 \sim 10^{-35} s^{-2}. \quad (3.8)$$

It is so tiny that all experiments that prove Einstein’s special relativity at ordinary scales cannot exclude the $dS$ special relativity. However, from the algebraic relation (3.5) and this important formula, it qualitatively follows that for all celestial objects including the CMB as test objects in
the cosmic scale, their free motions are always with both the conserved energy-momentum and the angular momentum.

The interval between two events and light-cone can be well defined as the inverse Wick rotation counterparts of \((2.9)\) and \((2.10)\), respectively. In fact, for two separate events \(A(a^i)\) and \(X(x^i)\)

\[
\Delta^2_R(a, x) = R^2 \left[ \sigma^2(a, x) - \sigma(a)\sigma(x) \right]
\]  

is invariant under the \(FLT\)'s of \(SO(1, 4)\). Thus, the interval between \(A\) and \(B\) is time-like, null or space-like, respectively, according to

\[
\Delta^2_R(a, b) \geq 0.
\]  

The proper length of time-space-like interval between \(A\) and \(B\) is the integral of line element \(ds\) in \((2.29)\) over the geodesic segment \(AB\):

\[
S_{t\text{-like}}(a, b) = R \sinh^{-1}(\Delta(a, b)_R/R),
\] 

\[
S_{s\text{-like}}(a, b) = R \arcsin(|\Delta(a, b)_R|/R).
\]

The Beltrami light-cone at an event \(A\) with running events \(X\) is

\[
F_R := R\{\sigma(a, x) - [\sigma(a)\sigma(x)]^{1/2}\} = 0.
\]

It satisfies the null-hypersurface condition. At the origin \(a^i = 0\), the light cone becomes a \(Mink\)-one \(\eta_{ij}x^i x^j = 0\) and \(c\) is numerically the velocity of light in the vacuum.

There is also a horizon tangent to the boundary on \(BdS\) for the observers \(O_I\):

\[
\lim_{a \to a'} \sigma(a, x) = 0, \quad \lim_{a \to a'} \sigma(a) = 0.
\]

For the horizon in Beltrami systems, it is actually at \(T = 0\) without entropy. But, at the horizon in other \(dS\)-spacetimes, such as the static \(dS\)-universe and the Robertson-Walker-like \(dS\)-spacetime, Hawking temperature and area entropy appear as non-inertial effects rather than gravitational ones \([8]\). Thus, \(dS\)-spacetime is completely different from black hole.

C. Two kinds of simultaneity, principle of relativity and cosmological principle

In order to make measurements, simultaneity should be defined. As was mentioned, different from Einstein's special relativity, there are two kinds of simultaneity related to two kinds of measurements with respect to the principle of relativity and the cosmological principle, respectively. It is important that these two kinds of simultaneity together with corresponding principle are very closely related to each other just like a coin with two sides.

In the contraction \(R \to \infty\), however, they coincide with each other.
1. Beltrami simultaneity for principle of relativity

Let us first consider the Beltrami simultaneity with respect to the Beltrami time coordinate. For an inertial observer \( O_I \) at the spatial origin of the system, who is just the observer \( O_H \) for the uniform ‘great circular’ motion \((2.19)\) on the \( H_+ \subset M^{1,4} \), two events \((A, B)\) are simultaneous if and only if their Beltrami time coordinates are equal to each other

\[
a^0 := x^0(A) = x^0(B) =: b^0.
\] (3.15)

This simultaneity defines a 3 + 1 decomposition of the \( BdS \)-matric \((2.29)\)

\[
ds^2 = N^2(dx^0)^2 - h_{\alpha\beta}(dx^\alpha + N^\alpha dx^0)(dx^\beta + N^\beta dx^0), \quad \alpha, \beta = 1, 2, 3,
\] (3.16)

with lapse function, shift vector and induced 3-geometry on 3-hypersurface \( \Sigma_c \) in one coordinate chart, respectively

\[
N = \{\sigma_{\Sigma}(x)[1 - (x^0/R)^2]\}^{-1/2},
\]
\[
N^\alpha = x^0 x^\alpha [R^2 - (x^0)^2]^{-1},
\]
\[
h_{\alpha\beta} = \delta_{\alpha\beta}\sigma_{\Sigma}^{-1}(x) - [R\sigma_{\Sigma}(x)]^{-2}\delta_{\alpha\gamma}\delta_{\beta\delta}x^\gamma x^\delta,
\]
\[
\sigma_{\Sigma}(x) = 1 - (x^0/R)^2 + \delta_{\alpha\beta}x^\alpha x^\beta/R^2.
\] (3.17)

It is easy to see that at \( x^0 = 0 \), \( \sigma_{\Sigma}(x) = 1 + \delta_{\alpha\beta}x^\alpha x^\beta/R^2 \), \( N = \sigma_{\Sigma}^{-1/2}(x) \), \( N^\alpha = 0 \).

This simultaneity leads to a definition of non-Euclidean Beltrami ruler and its relation to spatial coordinate distance of two simultaneous events. A Beltrami ruler at \( x^0 \) is defined by

\[
dl_B^2|_{x^0} = -h_{\alpha\beta}|_{x^0}dx^\alpha dx^\beta.
\] (3.18)

In fact, all measurements in the Beltrami systems are in analogy with that on \( Mink \)-spacetime as long as it is are no longer Euclidean not only the Beltrami time and proper-time of a standard clock as well as their relation, but also the Beltrami spatial coordinates and the proper-length of a ruler as well as their relation.

2. Proper-time simultaneity and Robertson-Walker-like \( dS \)-space

Another simultaneity is the same as the one in \((2.22)\) for the observer \( O_H \).

The proper-time \( \tau \) of a clock rest at spatial origin \( x^a = 0 \) of Beltrami system relates the coordinate time \( x^0 \) as

\[
\tau = R \sinh^{-1}(R^{-1}\sigma^{-\frac{1}{2}}(x)x^0) + \tau_0,
\] (3.19)

where \( \tau_0 \) is a constant to be determined by physical consideration. For the sake of simplicity, we may simply take \( \tau_0 = 0 \). With respect to this proper-time \( \tau \), the proper-time simultaneity can be defined as: The events are simultaneous if and only if their proper time \( \tau \) is the same

\[
x^0\sigma^{-1/2}(x,x) = (\xi^0 :=)R \sinh(\tau/R) = \text{const.}
\] (3.20)
If $\tau$ is taken as a temporal coordinate together with the spatial Beltrami coordinates, the $BdS$-space becomes a Robertson-Walker-like $dS$-model with a metric having $\tau$ being a ‘cosmic’-time:

$$
\begin{align*}
    ds^2 &= d\tau^2 - dl^2 = d\tau^2 - \cosh^2(\tau/R)d\ell_0^2, \\
    dl_0^2 &= \{\delta_{\alpha\beta}\sigma_{\Sigma}^{-1}(x) - [R\sigma_{\Sigma}(x)]^{-2}\delta_{\alpha\gamma}\delta_{\beta\delta}x^\gamma x^\delta\}dx^\alpha dx^\beta, \\
    \sigma_{\Sigma}(x, x) &= 1 + R^{-2}\delta_{\alpha\beta}x^\alpha x^\beta > 0, \\
\end{align*}

(3.21)$$

where $dl_0^2$ a 3-dimensional Beltrami metric on an $S^3$ of radius $R$. This is an ‘empty’ cosmic model with an accelerated expanding and slightly closed cosmos of curvature in the order of $O(R^{-2})$.

Thermodynamically, from Eq. (3.19), it is easy to see that for the proper-time, there is a period in the imaginary proper-time that is inversely proportional to the Hawking-temperature $c\hbar/(2\pi Rk_B)$ at the horizon. If the temperature Green’s function can still be applied here, this should indicate that there are Hawking-temperature and ‘area’ entropy $S = \pi R^2 c^3 k_B/G\hbar$ at the horizon in the Robertson-Walker-like $dS$-space (3.21). But, they are not caused by gravity rather by non-inertial motions. This is also in analogy with relation between Einstein’s special relativity in Mink–space and the and the horizon in Rindler-coordinates. The temperature at the Rindler-horizon is caused by non-inertial motion rather than gravity [8].

Since there is a relation between two kinds of simultaneity for the principle of relativity and the cosmological principle, $dS$-spacetime provides a coin-like model for these two principles.

On one side, with Hawking temperature and ‘area’ entropy there is the Robertson-Walker-like $dS$-cosmos with cosmological constant fitting the cosmological principle. And on another side, at zero temperature without entropy there is the $BdS$-spacetime with the principle of inertia. Thus, the former should just display as the origin of law of inertia on the latter and the principle of inertia on the latter provides a benchmark for physics on $dS$-space including both the $BdS$-space and the Robertson-Walker-like $dS$-cosmos.

In other words, on $dS$-spacetime there is a kind of inertial-comoving-like observers, $O_{I-C}$, equipped a type of two-time-scale timers of Beltrami time and ‘cosmic’-time, as well as corresponding rulers. They may act as inertial observers $O_I$ or comoving-like ones $O_C$ in different experiments or observations, respectively, reflecting these principles and their important relation. Actually, once the observers would carry on experiments in their laboratories, they should switch on Beltrami time and off ‘cosmic’-time so that they act as inertial observers $O_I$ and all observations are of inertia. When they would take approximatively ‘cosmic’ observations on distant stars and cosmic objects other than the cosmological constant as test objects they should switch off Beltrami time and on ‘cosmic’-time again, so they should act as a kind of comoving observers $O_C$ as they hope. Namely, what should be done for those inertial-comoving-like observers $O_{I-C}$ is just to switch off ‘cosmic’-time and on Beltrami time once they want to be back to local experiments from their comoving-like observations and vice versa.

It is worth while to mention that in general there are other kinds of $dS$-comoving coordinates with flat or open 3-dimensional cosmos, respectively. However, from the viewpoint of $dS$ special relativity, the above Robertson-Walker-like $dS$-comoving coordinates in (3.21) with closed 3-dimensional cosmos is most natural and simplest among all of them.
D. Cosmological significance of de Sitter special relativity

If our universe is accelerated expanding and possibly asymptotic to a $dS$, its fate should be a Robertson-Walker-like $dS$-space. This is very natural from the viewpoint of $dS$ special relativity. Thus, there is remarkable cosmological significance for $dS$ special relativity different from the conventional approach in general relativity.

First, there is an important prediction.

If our universe is asymptotic to the Robertson-Walker-like $dS$-space of $R^2 \simeq 3\Lambda^{-1}$ with ‘area’ entropy, the 3-dimensional cosmic space of the dark universe should be closed and asymptotic to an accelerated expanding $S^3$ with an entropy bound $S \simeq 3\pi c^3 k_B/\Lambda G \hbar$. Its deviation from the flatness is in the order of the cosmological constant $O(\Lambda)$.

This is in consistency with recent data from WMAP [16] and can be further checked.

On the other hand, the evolution of our universe can determine the Beltrami inertial frames of the principle of inertia in $dS$ special relativity and all other kinds of inertial frames contracted from the Beltrami frames.

As is well known, according to Einstein, there is an ‘argument in a circle’ for the principle of inertia. In his most famous book, Einstein wrote: ‘The weakness of the principle of inertia lies in this, that it involves an argument in a circle: a mass moves without acceleration if it is sufficiently far from other bodies; we know that it is sufficiently far from other bodies only by the fact that it moves without acceleration. Are there at all any inertial systems for very extended portions of the space-time continuum, or, indeed, for the whole universe? We may look upon the principle of inertia as established, to a high degree of approximation, for the space of our planetary system, provided that we neglect the perturbations due to the sun and planets. Stated more exactly, there are finite regions, where, with respect to a suitably chosen space of reference, material particles move freely without acceleration, and in which the laws of the special theory of relativity, · · ·, hold with remarkable accuracy. Such regions we shall call “Galilean regions”.’ [19]

‘Are there at all any inertial systems · · · for the whole universe?’ Einstein raised such a severe question, but he did not answer.

With the help of asymptotic behavior of our universe and the $dS$ special relativity, this question can be definitely answered. In fact, for the principle of inertia on $dS$-spacetime, there is no Einstein’s ‘argument in a circle’ and the inertial frames of $BdS$-type do exist for the whole universe. Actually, without measuring any acceleration of a mass, all needed are the time arrow and approximative symmetry of our universe roughly described by the cosmological principle.

If our universe is asymptotic to the Robertson-Walker-like $dS$-space of $R^2 \simeq 3\Lambda^{-1}$, the time arrow and the homogeneous space of our universe should coincide with the ‘cosmic’-time arrow and tend to an accelerated expanding $S^3$ of the Robertson-Walker-like $dS$-space, respectively. These pick up the directions of the ‘cosmic’ temporal axis and the spatial axes for the Robertson-Walker-like $dS$-systems up to spatial rotations of $SO(4)$ among all them related by $dS$-transformations so that the $dS$-symmetry reduces to its subgroup $SO(4)$ of the Robertson-Walker-like $dS$-cosmos with a ‘cosmic’-time, the direction of which coincides with the time arrow of our universe. Then, via the important relation between Beltrami systems and the Robertson-Walker-like $dS$-model,
i.e. via the relation (3.19) between the ‘cosmic’-time and the Beltrami time, the directions of
the axes in a kind of Beltrami frames can be given. This is just like to flip a coin from one
side to another. In fact, the Beltrami temporal axis is related to the axis of ‘cosmic’-time on
the Robertson-Walker-like $dS$-space and the spatial axes of the Robertson-Walker-like $dS$-space
(3.21) are just the Beltrami spatial coordinates. Thus, the evolution of our universe can fix on
this kind of Beltrami frames in such a way that there is no Einstein’s ‘argument in a circle’, since
gravitational effects and acceleration of a mass do not explicitly play any roles here.

There are two invariant universal constants, $c$ and $R$, in the Beltrami frames. In order to set
up the real Beltrami frames, it is also needed to determine their value numerically. If so, how can
present experiments or observations nowadays determine their values in the fate of our universe?
How in these present experiments or observations we can neglect the gravitational effects?

In fact, although the Beltrami frames of inertia depend on the dimensions of these two invariant
universal constants, the property of inertia for the frames does not depend on their concrete values
unless for measurements of concrete physical processes. Of course, physically, their concrete values
are certainly needed and should be determined by two kinds of experiments or observations. Since
these constants are supposed to be invariant and universal approximately, the value of $c$ should still
be taken as the one in Einstein’s special relativity. Note that this also fixes on the origin of Beltrami
frames since the light cone (3.13) at the origin is just Minkowskian at present approximatively.
As for the value of $R$, it may also be taken as $R \simeq (3/\Lambda)^{1/2}$, where the cosmological constant
$\Lambda$ is given by the precise cosmology. Although the determination of $\Lambda$ may depend on some
gravitational effects nowadays and so does the value of $R$, this does not a matter in principle
for fixing on the inertial systems. In fact, changing the value of $R$ may lead to the conformal
extension of $dS$-spacetime, which will be explained later.

Since in all possible kinematics based on principle of inertia the inertial frames can be given
under certain contracting limit from the Beltrami frames, respectively, all different kinds of inertial
frames in the kinematics should also be fixed on by the evolution our universe without Einstein’s
‘argument in a circle’ so long as they are regarded as successors of the Beltrami systems. However,
if it is ignored this successive relation of the inertial systems, the coin-like relation between the
principle of relativity and the cosmological principle should no longer appear or becomes trivial
in Einstein’s special relativity and Newton’s mechanics, except the Newton-Hooke one.

In addition, if it is further required that in the spacetimes with gravity there should exist locally
the principle of relativity everywhere and anytime and the values of $c$ and $R$ should be the same
as in the $dS$ special relativity, such kind of local inertial frames with the origin at present can
also be fixed on in the same manner by the evolution of our universe.

Thus, Beltrami systems of inertia and their localized version together with their contracting
forms do exist in the whole universe. In the sense that these systems can be fixed on by the evolu-
tion of our universe, the universe also plays a role as the origin of inertia in all these kinematics.

It should be noted that the Beltrami inertial frames determined by the evolution of our universe
are a kind of ‘preferred’ frames in the sense that their temporal axis is related to the time arrow
of our universe. These ‘preferred’ inertial frames still exist under different contractions. However,
this ‘preference’ does not break the principle of relativity that is for physical laws. In fact, the
‘preference’ only plays certain role when some comoving-like observations are taken, since its temporal axis is just transformed from the ‘cosmic’-time axis of the Robertson-Walker-like $dS$-space that coincides with the time arrow of our universe.

This is also true for the local Beltrami frames and all their contractions. Actually, even in general relativity once the cosmic observations or background should be taken into account such kind of local inertial frames should be taken that their time axis should coincide with the comoving time axis. In this sense, this kind of local inertial frames is ‘preferred’. But, in general relativity, the symmetry for local inertial frames is not the same as that in Einstein’s special relativity.

IV. CONFORMAL EXTENSIONS OF THREE KINDS OF SPECIAL RELATIVITY

We now consider conformal extensions of three kinds of special relativity and null physics on them as well as their relations via Weyl conformal mappings [26].

As is well known, in Einstein’s special relativity on $Mink$-space, massless particles and light signals move \textit{in inertia} along null geodesics satisfying $ds^2_M = 0$ invariant under conformal group transformations with fifteen parameters. Thus, symmetry of their motions should be enlarged from Poincaré group $ISO(1, 3)$ to conformal group. In the $dS/AdS$ special relativity, massless particles and light signals move also \textit{in inertia} along straight lines at constant coordinate velocities. Similarly, they also satisfy $ds^2_\pm = 0$, where $ds_\pm$ is given by (2.29), invariant under conformal group transformations as well. Thus, symmetry of their motions should also be enlarged from $dS/AdS$-group $SO(1, 4)/SO(2, 3)$ to conformal group.

In all these cases, the conformal extensions can be realized on a null cone $\mathcal{N}$ modulo projective equivalence in a $(4 + 2)$-dimensional $Mink$ space, $[\mathcal{N}] := \mathcal{N}/\sim \subset M^{2,4}$, invariant under the conformal group $SO(2, 4)/\mathbb{Z}_2$ with isometry subgroup $ISO(1, 3)/SO(1, 4)/SO(2, 3)$, respectively. Further, the null physics on $dS/AdS/Mink$-spaces can be mapped from one to another by Weyl conformal mappings. In this sense, there should be a triality of these conformal issues [26].

Since the projective boundary of a 5-dimensional $AdS$-space, $\partial(AdS^5)$, is just $[\mathcal{N}]$, 4-dimensional conformal $dS/AdS$-spaces can also be included in $\partial(AdS^5)$, in addition to conformal $Mink$-space. Thus, if the $AdS/CFT$ correspondence [44] is conjectured, there should be three versions of $AdS/CFT$ correspondence [26]. Further, there should be a $dS$-spacetime on the boundary of $S^5 \times AdS^5$ as a vacuum of supergravity.

A. Conformal extensions of $Mink/dS/AdS$-spaces on a null cone

Let us view the $dS/AdS$-space with radius $R$ as a 4-dimensional hyperboloid $H_{R\theta} (\theta = \pm 1)$ (or simply $H_{\theta}$) embedded in $M^{1,4}/M^{2,3}$, respectively:

\begin{align*}
H_{\theta} : \quad & \eta_{ij} \xi^i \xi^j - \theta (\xi^4)^2 = \eta_{AB} \xi^A \xi^B = -\theta R^2 \leq 0 \quad (4.1) \\
ds^2_{H_{\theta}} = \eta_{\theta AB} d\xi^A d\xi^B, \quad A, B = 0, \cdots, 4. \quad (4.2)
\end{align*}
The conformal extensions of $BdS/BAdS$-space can be realized via the conformal extensions of $dS/AdS$-hyperboloid $H_{\pm}$, respectively, first and back to the Beltrami coordinates afterwards.

Introducing a scaling variable $\kappa \neq 0$ and a set of coordinates $\zeta^A$, $A = 0, \cdots , 5$, respectively,

\begin{align}
    \zeta^i := \kappa \xi^i, \quad \zeta^4 := \kappa \xi^4, \quad \zeta^5 := \kappa R, \quad & \text{for } H_{R^+}; \\
    \zeta^i := \kappa \xi^i, \quad \zeta^4 := \kappa R, \quad \zeta^5 := \kappa \xi^4, \quad & \text{for } H_{R^-}.
\end{align}

(4.3) (4.4)

Then, under such a scaling, eq. (4.1) turns out to be

\[ \mathcal{N} : \eta_{\hat{A}\hat{B}} \zeta^\hat{A} \zeta^\hat{B} = 0, \quad \eta_{\hat{A}\hat{B}} = \text{diag}(J^{1,3}, -1, 1), \]

(4.5)

where $J^{1,3} = \text{diag}(1, -1, -1, -1)$, $\hat{\zeta} := (\zeta^\hat{A}) = (\zeta, \zeta^4, \zeta^5) \neq 0$. This is an $SO(2,4)$-invariant null cone $\mathcal{N} \subset M^{2,4}$ and there is the projective equivalence relation $\sim$ on $M^{2,4} - \{0\}$: $\hat{\zeta}^i \sim \hat{\zeta}$ if and only if there is a number $c \neq 0$ satisfying $\zeta'^i = c \zeta^\hat{A}$. The resulted quotient space $[\mathcal{N}] := \mathcal{N} / \sim$ is a 4-dimensional submanifold of $\mathbb{R}P^5$, homeomorphic to $S^1 \times S^3$. Intuitively, an equivalence class of $\hat{\zeta} \in \mathcal{N}$ can be viewed as the null straight line passing through both $\hat{\zeta}$ and the origin of $M^{2,4}$. The origin is not included in the equivalence class, however. In this sense, $[\mathcal{N}]$ consists of all the null straight lines through the origin. Thus, an $SO(2,4)/\mathbb{Z}_2$ transformation on $M^{2,4}$ induces a transformation on $[\mathcal{N}]$.

Since $H_{\pm}$ can be embedded into $[\mathcal{N}]$, when the metric on $M^{2,4}$ is pulled back to $[\mathcal{N}]$, it is conformal to $ds^2_{H_{\pm}}$ in (4.2):

\[ d\chi^2_{[\mathcal{N}]} := \eta_{\hat{A}\hat{B}} d\zeta^\hat{A} d\zeta^\hat{B}|_{[\mathcal{N}]} = \kappa^2 \cdot ds^2_{H_{\pm}}. \]

(4.6)

Consequently, an $SO(2,4)/\mathbb{Z}_2$ transformation on $[\mathcal{N}]$ induces the conformal transformation on $H_{\pm}$, respectively:

\[ ds^2_{H_{\pm}} = \rho^2 \cdot ds^2_{H_{\pm}}, \quad \rho = \frac{\kappa}{\kappa'} = \begin{cases} \zeta^5/\zeta'^5, & \text{for } H_{R^+}, \\ \zeta^4/\zeta'^4, & \text{for } H_{R^-}. \end{cases} \]

(4.7)

According to eqs. (4.3) and (4.4), $H_{\pm}$ can be viewed as an intersection of $\mathcal{N}$ and the hyperplanes $P_+ : \zeta^5 = R$ and $P_- : \zeta^4 = R$, respectively. Since $H_{\pm}$ is only part of $\mathcal{N}$ with $\zeta^5 \neq 0$ for $H_+$ or $\zeta^4 \neq 0$ for $H_-$, respectively, it is quite possible for an $SO(2,4)$ transformation to send a point in $H_{\pm}$, with nonzero $\zeta^5$ or $\zeta^4$, to another one with zero $\zeta^5$ or $\zeta^4$, and vice versa. Thus, $H_{\pm}$ are, in fact, not closed under the induced conformal transformations. To be closed, $H_{\pm}$ must be extended into the whole $[\mathcal{N}]$. Thus, $[\mathcal{N}]$ is the conformal extension of both $dS/AdS$-spaces.

It is clear that back to the Beltrami atlas, say (2.26) for the $BdS$, as inhomogeneous projective coordinates, the conformal $BdS/BAdS$-metric follows. It is straightforward to prove that all null geodesics of the conformal $BdS/BAdS$-metric are straight world lines, respectively. Thus, we get conformal extensions of $dS/AdS$ special relativity for those massless particles and light signals on $BdS/BAdS$-spacetime, respectively. Both them are defined on the same $[\mathcal{N}]$.

As is well-known, the conformal $Mink$-space can also be obtained from the the same null cone (see, e.g., [43]). To this end, it is needed to introduce a set of new coordinates

\[ \zeta^\pm := \frac{1}{\sqrt{2}}(\zeta^5 \pm \zeta^4) \]

(4.8)
and inhomogeneous projective coordinates

\[ x^i := R \zeta^i / \zeta^-, \quad x^+ := R \zeta^+ / \zeta^- \]  \hspace{1cm} (4.9)

to those points with \( \zeta^- \neq 0 \), where \( R \) is the same universal constant introduced before. In general, different \( R \) may be taken. Then eq. (4.5) becomes

\[ x^+ = -\eta_{ij} x^i x^j / (2R), \quad \text{metric } (4.6) \]

becomes

\[ ds^2_M := \eta_{ij} dx^i dx^j. \]  \hspace{1cm} (4.10)

Now an \( SO(2,4)/\mathbb{Z}_2 \) transformation on \( \mathcal{N} \) induces a conformal transformation on \( \text{Mink}-\text{space}: \)

\[ ds^2_M \to ds^2_M' = \rho^2 ds^2_M, \quad \rho = \zeta^- / \zeta'^-. \]  \hspace{1cm} (4.11)

Similarly, the \( \text{Mink}-\text{space} \) can be regarded as an intersection of \( \mathcal{N} \) and the hyperplane \( P_M : \zeta^- = R \) by identifying \( (x^i) \) with \( (x^i, (x^+ - R)/\sqrt{2}, (x^+ + R)/\sqrt{2}) \in \mathcal{N} \). The \( \text{Mink}-\text{space} \) is also not closed for these conformal transformations. Thus, the \( \text{Mink}-\text{space} \) needs to be extended, resulting in the space \( \mathcal{N} \cong S^1 \times S^3 \).

According to the above discussion, the \( \text{Mink}/dS/AdS\)-space and their conformal extensions can be related by Weyl conformal maps. An event on \( dS \), say, is first viewed as an event on \( P_+ \cap \mathcal{N} \). Then an event on \( P_- \cap \mathcal{N} \) equivalent to it could be found, in general. However, it is possible that an event on one space could not be mapped into another space, or could not find an inverse image on another space. But, this can be solved so that the map from the conformal extension of \( dS \) to that of \( AdS \) can be established. We will explain this issue in detail elsewhere.

For example, an event \( \xi_+ := (\xi^0_+, \ldots, \xi^4_+) \in H_+ \) can be mapped to an event on \( H_- \) with following Beltrami coordinates:

\[ x^i := R \zeta^i / \zeta^5 = \xi^i_+. \]  \hspace{1cm} (4.12)

As another example, the Weyl conformal map sending an event with coordinates \( (x^i) \) on the \( \text{Mink}-\text{space} \) to an event on the \( \text{BdS}-\text{space} \) with coordinates \( (x^i_+) \) reads

\[ x^i_+ = -\sqrt{2} x^i (1 + \frac{1}{2R^2} \eta_{jk} x^j x^k)^{-1}. \]  \hspace{1cm} (4.13)

This is just the conformally flat coordinate transformation for the \( \text{BdS}-\text{metric} \) (2.29) also known as a stereographic projection with an inverse transformation

\[ x^i = -\sqrt{2} x^i_+ (1 \mp \sqrt{\sigma(x^i_+)})^{-1}. \]  \hspace{1cm} (4.14)

The sign \( \mp \) is opposite to the sign of \( \xi^i_+ \geq 0 \) in the \( \text{BdS}-\text{space} \).

It is important that the normal vectors of \( P_+, P_- \) and \( P_M \) are time-like, space-like and null, respectively, and that \( P_+ \cap \mathcal{N}, P_- \cap \mathcal{N} \) and \( P_M \cap \mathcal{N} \) is \( dS/AdS/Mink\)-space, respectively. This can be generalized: given a hyperplane off the origin, its intersection with \( \mathcal{N} \) is \( dS/AdS/Mink\)-space if its normal vector is time-like, space-like or null, respectively.
B. Triality of null physics on conformal $Mink/dS/AdS$-spaces

We have shown that $Mink/dS/AdS$-spaces can all be conformally extended to the same $[N]$, so that they can be conformally mapped from one to another via Weyl mappings. A conformal transformation on one space is, in fact, also a conformal transformation on another space. And all these conformal transformations are induced from some transformations of $SO(2,4)/\mathbb{Z}_2$, due to the equivalence relation on $N$. Therefore, from the viewpoint of conformal transformations, these three kinds of spaces and CFTs on them are just same. We refer to this fact as a triality of conformal extensions of these spaces and the null physics on them including the $AdS/CFT$-correspondence. Thus, there should be also a triality for the conjecture.

1. Motion of free massless particles and light signals

As was mentioned, similar to a massive particle a free massless particle or a light signal in $dS$-spacetime is in the uniform ‘great circular’ motion with a conserved 5-d angular momentum (2.21). In terms of the Beltrami coordinates, the uniform ‘great circular’ motion turns out to be inertial motion along a null straight line $[4,5]$. These imply that a geodesic is the intersection of $\Sigma$ and $dS$-hyperboloid $H_+$ in (4.1), where $\Sigma$ is some 2-d plane passing through the origin of the 5-d $Mink$-space $M^{1,4}$. It can be proved that, when the geodesic is null, it is in fact a straight line in $M^{1,4}$, having the equation $\xi^A = \xi_0^A + \lambda^+ v^A$ for some constants $\xi_0^A$ and $v^A$, satisfying $\eta_{AB} \xi_0^A v^B = \eta_{AB} v^A v^B = 0$. Thus, the 5-d momentum $K^A = v^A$ of the null geodesic is also conserved.

Using the relations (4.3), we can obtain

$$L^{AB} = \frac{1}{\kappa^2} \frac{d\psi}{d\lambda} L^{AB}, \quad P^A = \frac{1}{\kappa^2 R^2} \frac{d\psi}{d\lambda} L^{5A},$$

(4.15)

where $\psi = \psi(\lambda)$ is a certain parameter and the 6-d angular momentum $L^{\hat{A}\hat{B}}$ is defined as

$$L^{\hat{A}\hat{B}} := \zeta^{\hat{A}} \frac{d\zeta^{\hat{B}}}{d\psi} - \zeta^{\hat{B}} \frac{d\zeta^{\hat{A}}}{d\psi}.$$  

(4.16)

It is conserved if

$$d\psi = \kappa^2 d\lambda.$$  

(4.17)

For a massless particle in the $AdS$-space, there are similar issues.

In the $Mink$-case, the 4-d momentum $k^i_M$ and the angular momentum $l^i_M$ are conserved for a light signal

$$k^i_M := \frac{dx^i}{d\lambda}, \quad l^i_M := x^i k^j_M - x^j k^i_M.$$  

(4.18)

Similarly, there is a 6-d angular momentum

$$L^{ij} = \frac{d\lambda}{d\psi} \kappa^2 l_M^{ij}, \quad L^{4j} = \frac{1}{\sqrt{2}} (L^{+j} - L^{-j}),$$

$$L^{5j} = \frac{1}{\sqrt{2}} (L^{+j} + L^{-j}), \quad L^{45} = L^{+-}.$$  

(4.19)

(4.20)
where
\[ L^{-j} = \frac{d\lambda}{d\psi} \kappa^2 R P^j, \quad L^{+j} = \frac{d\lambda}{d\psi} \kappa^2 (x^+ P^j - x^j dx^+), \quad L^{+-} = -\frac{d\lambda}{d\psi} \kappa^2 R \frac{dx^+}{d\lambda}. \] (4.21)

If eq. (4.17) is satisfied, then the above 6-d angular momentum is also conserved.

For a massless free particle, its equation of motion in \([\mathcal{N}]\) is not unique in terms of \(\zeta^\hat{A}\), because \(\zeta^\hat{A} = \zeta^{\hat{A}}(\psi)\) and \(\zeta^{\hat{A}} = \zeta^{\prime\hat{A}}(\psi') := \rho(\psi') \zeta^{\hat{A}}(\psi(\psi'))\) are equivalent, with \(\psi = \psi(\psi')\) a re-parameterization. Formally, there are the angular momenta \(L^{\hat{A}\hat{B}}\) and \(L'^{\hat{A}\hat{B}}(\psi')\) for the same particle. But, a re-parameterization can always be chosen so that \(L'^{\hat{A}\hat{B}}(\psi')\) is still conserved.

Consequently, the world-line is lying in a 2-d plane \(\Sigma\) passing through the origin of \(M^{2,4}\), which is also contained in \(\mathcal{N} \subset M^{2,4}\) except for the origin. Thus, the world-line \(\Sigma - \{0\}/\sim\) is a projective straight line in \([\mathcal{N}]\): in the Beltrami coordinate on \(dS/AdS\), or in \(Mink\)-coordinate, its equations look like
\[ x^i(s) = x_0^i + \tau c^i, \] (4.22)
where \(x_0^i\) and \(c^i\) are some constants while \(\tau\) is the curve parameter. Hence, the world-line is a null geodesic \([4, 5]\). The relation of its 5-d angular momentum and \(L^{\hat{A}\hat{B}}\) is as shown in eqs. (4.15), etc. This coincides with the well known fact that null geodesics are conformally invariant up to a re-parameterization.

2. On CFT and AdS/CFT correspondence

Let us consider other conformal issues and their relations on conformal \(Mink/dS/AdS\)-spaces. The generators of the conformal group on \(Mink\)-space are
\[ \hat{\rho}_i := \partial_i, \quad \hat{l}_{ij} := x_i \partial_j - x_j \partial_i, \]
\[ \hat{D} := x^l \partial_l, \quad \hat{s}_i := -x \cdot x \partial_i + 2x_i x^l \partial_l. \] (4.23)
(4.24)

A CFT on \(Mink\)-space must be invariant under action of these generators. Coordinates \(x^i\) can be extended to be a set of coordinates \((x^i, \kappa, \phi)\) on \(M^{2,4} - \{\zeta^- = 0\}\), where \(\kappa\) is the scaling factor introduced before
\[ \kappa = \frac{\zeta^-}{R}, \quad \phi := \eta^{\hat{A}\hat{B}} \zeta^\hat{A} \zeta^\hat{B}. \] (4.25)

Thus, the \(Mink\)-space is described by \(\kappa = 1\) and \(\phi = 0\). Then it can be verified that
\[ \hat{\rho}_i = \frac{1}{R} \hat{\mathcal{L}}_{+i}, \quad \hat{l}_{ij} = \hat{\mathcal{L}}_{ij}, \]
\[ \hat{D} = \hat{\mathcal{D}} + \hat{\mathcal{L}}_{-+}, \quad \hat{s}_i = 2x_i \hat{\mathcal{D}} + 2R \hat{\mathcal{L}}_{-i}, \] (4.26)
(4.27)
where
\[ \hat{\mathcal{D}} := \zeta^\hat{A} \frac{\partial}{\partial \zeta^\hat{A}} \] (4.28)
is the generator of scaling in $M^{2,4}$, while

$$\hat{L}_{A\hat{B}} := \zeta_A \frac{\partial}{\partial \zeta^B} - \zeta_B \frac{\partial}{\partial \zeta^A}$$

(4.29)

are generators of $\mathfrak{so}(2,4)$ (up to a fact $i$). Since $\hat{D}$ is commutative with each $\hat{L}_{A\hat{B}}$, it does not matter that the conformal generators of the Mink-space differ from those of $M^{2,4}$ by a vector field along $\hat{D}$ (see, eqs. (4.24)). This coincides with (i) the idea that the equivalence relation $\sim$ will be considered on $N$, and (ii) the fact that conformal transformations on the Mink-space are induced from, but not the same as, $SO(2,4)$-transformations on $N$. In fact, a quantity on the Mink-space can be realized by homogeneous function of degree zero on $M^{2,4} - \{0\}$. In this way $\hat{D}$ somehow could be dropped directly.

Generators of conformal transformations on $dS/AdS$-spaces, or specially on $BdS/BAdS$-spaces, can also be given as the ones of $\mathfrak{so}(2,4)$. Thus, they can be related by the Weyl conformal mappings such as (4.12) and (4.13). Correspondingly, the CFTs in these spaces are also related by these mappings. Since the Maxwell equations are the simplest CFT, as an illustration, we show how the sourceless Maxwell equations

$$d F = 0, \quad \star d \star F = 0,$$

(4.30)

where $\star$ is the Hodge dual operator, are related among them.

Consider the Weyl conformal mapping $\psi : M^{1,3} \rightarrow dS^4$ as shown in eq. (4.13):

$$\psi^* g = \Omega^2 \eta, \quad \Omega = \sqrt{2} \left(1 - \frac{1}{2R^2} \eta_{ij} x^i x^j\right)^{-1},$$

(4.31)

with $g$ the metric (2.29) of $BdS^4$, $\eta$ the one in (4.10). If $F_{dS}$ is the Maxwell field on $dS$, its equations follow

$$d F_{dS} = 0, \quad \star d \star F_{dS} = 0,$$

(4.32)

where $\star$ is the dual operator with respect to $g$. We pull $F_{dS}$ back to the Mink-space, resulting in

$$F = \psi^* F_{dS}.$$

(4.33)

Thus, $d F = d (\psi^* F_{dS}) = \psi^* d F_{dS} = 0$ is satisfied. It can be verified that

$$\psi^* (\star d \star F_{dS}) = \Omega^{-2} [\star d \star F].$$

Therefore, on the Mink-space, $F$ as in eq. (4.33) is a sourceless electromagnetic field: eqs. (4.30) are satisfied. In this way the Weyl conformal mapping $\psi : M^{1,3} \rightarrow BdS^4$ relates a sourceless electromagnetic field $F_{dS}$ on the $BdS$-space to a sourceless $F$ on the Mink-space.

Similarly, this approach can be applied to other CFTs between $dS$ and $AdS$-spaces, $AdS$ and Mink-spaces and so on. Basically, the CFTs on Mink/$dS$/AdS-spaces, in which all the relevant fields are assumed to behave well as the infinity points are approached, can be unified together. The former is merely a realization of the latter.

For the $AdS/CFT$ correspondence, there should also be a triality.
A 5-dimensional $AdS$-space with radius $R_5$ can be embedded into $M^{2,4}$ as a hypersurface $\mathcal{S}$:

$$\mathcal{S} : \eta_{\hat{A}\hat{B}} \zeta^\hat{A} \zeta^\hat{B} = R_5^2.$$  \hspace{1cm} (4.34)

If antipodal points on $\mathcal{S}$ are identified, the resulted space, denoted $\mathcal{S}/\mathbb{Z}_2$, is still homoemorphic to $\mathcal{S} \cong AdS_5$. In the projective space $\mathbb{R}P^5 = M^{2,4} - \{0\}/\sim$, the quotient space of those $\zeta^\hat{A}$ satisfying $\eta_{\hat{A}\hat{B}} \zeta^\hat{A} \zeta^\hat{B} > 0$ are homeomorphic to $\mathcal{S}/\mathbb{Z}_2 \cong AdS_5$. Identifying $AdS_5$ with this quotient space, its boundary is just the null cone modulo a projective equivalence

$$\partial_P(AdS_5) \cong [\mathcal{N}].$$  \hspace{1cm} (4.35)

Thus, due to the triality of the $CFT$s in conformal $Mink/dS/AdS$-spaces, there should be three $AdS/CFT$ correspondences starting from the well-known $AdS/CFT$ correspondence [44]. Namely, there should be the $AdS/CFT$ correspondence between $AdS_5$ and $dS_4/AdS_4$, respectively, in addition to that between $AdS_5$ and $Mink$-space. Clearly, this triality of the $AdS/CFT$ correspondence can be generalized to any dimensions whenever the $AdS/CFT$ correspondence is conjectured.

V. theory of gravity with localization of maximum symmetry

In this section, we explain why gravity should be based on the localization of special relativity with full maximum symmetry and be governed by some gauge-like dynamics of the same local maximum symmetry. We also construct a kind of umbilical manifolds with local $dS$-invariance and briefly introduce a simple model of $dS$-gravity with a gauge-like dynamics characterized by a dimensionless coupling constant $g \sim (G\hbar\Lambda/3c^3)^{1/2} \sim 10^{-61}$. Although this model is quite simple, it may still shed light on why our universe is so dark.

A. from the equivalence principle to the principle of localization

As was quoted before, right after explain why there is ‘an argument in a circle’ for the principle of inertia and raised a severe question on the existence of inertial systems, Einstein claimed that ‘there are finite regions, ⋯ in which the laws of the special theory of relativity ⋯ hold with remarkable accuracy. Such regions we shall call “Galilean regions”’ [19]. Then Einstein explained why the spacetimes with gravity should be curved. This is the most remarkable and most successful point of view in Einstein’s general relativity, although his argument on rotating disc is fallacious.

Let us analyze Einstein’s above statement from both physical and geometrical viewpoints.

Firstly, since all these regions are ‘finite’, ‘in which the laws of the special theory of relativity, ⋯, hold with remarkable accuracy,’ it is important to note that the Poincaré symmetry of the laws of special relativity on these ‘finite regions’ should be eventually local. Although in practice, Poincaré symmetry in these regions may still be regarded as global symmetry approximately.

Secondly, let us consider how to pass from one ‘Galilean region’ to another at different but nearby positions in the spacetime with gravity and what kind of local symmetry should be for the
curved spacetime with gravity. According to Einstein, there should be gravity in-between these ‘regions’. Therefore, in order to transit from one to another, some paths on curved spacetime with gravity in-between should be passed. Since there is local Poincaré symmetry in these ‘regions’, in order to transit along these paths in-between, the curved spacetime with gravity should also be of some local symmetry. It would be better still the local Poincaré symmetry. Otherwise, it is hard to transit consistently from one ‘region’ to another if Poincaré symmetry cannot be maintained locally in the course of transition along certain path in-between. For any number of such ‘finite regions’, it is the same.

This may also be seen from another angle more mathematically. Each of the finite ‘Galilean regions’ is essentially a portion of a Mink-space with Poincaré symmetry isomorphic to an $R^4$, so that there are intersections among these Mink-spaces with different ‘finite regions’ at different positions and the transition functions on these intersections should also be valued in Poincaré symmetry. Further, in terminology of differential geometry, these Mink-spaces with ‘finite regions’ may be viewed as tangent spaces at different positions of a curved manifold as the spacetime with gravity and the transition functions in the intersections of different coordinate charts on the manifold should be valued in local Poincaré symmetry.

Thus, it is the core of Einstein’s idea on gravity that the theory of gravity should be based on the localization of his special relativity with full Poincaré symmetry anywhere and anytime on some curved spacetimes. For the sake of definiteness, we name this principle as the local principle of relativity or the principle of localization.

Since there are three kinds of special relativity of Poincaré/dS/AdS-invariance, there should be also three kinds of gravitational effects with full local Poincaré/dS/AdS-symmetry, respectively. The principle of localization states: On spacetimes with gravity, there always exist local relativity-frames of local Mink/dS/AdS-spacetime, physical laws must take the gauge covariant versions of their special-relativistic forms with respect to the local Poincaré/dS/AdS-symmetry, respectively.

In general relativity, however, the principle of equivalence requires: ‘In any and every local Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of physics must take on their familiar special-relativistic forms.’ It is clear that on $3 + 1$-dimensional pseudo-Riemannian geometry $(M, g)$ with metric $g$ of signature $-2$ as spacetime with gravity, there is no local translation symmetry in local Lorentz space as tangent space (see, e.g., and for some earliest references, see, e.g.,). Actually, the definitions for mass, spin and other physical quantities of particles and fields as test objects or gravitational sources as well as the physical laws they obeyed in general relativity are merely made formally ‘on their familiar special-relativistic forms’ in local Lorentz frames. As far as the local symmetry is concerned in general relativity, it is $GL(4, R)$ or its subgroup $SO(1, 3)$.

For example, a rank-$(r, s)$ tensor $T(x)$ is defined as at a point

$$T(x) := T^{i_1, \cdots, i_r}_{j_1, \cdots, j_s}(x) \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_r}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_s}.$$  

(5.1)

In the same coordinate chart, it is invariant under the transformations of bases of the tangent space and its dual, i.e. $(\frac{\partial x'^{i}}{\partial x^{j}})_{i, j = 0, \cdots, 3} \in GL(4, R)$ at the point. It is also invariant from one chart to another on an intersection of two charts since transition functions are also valued in $GL(4, R)$. 
In Einstein’s special relativity, however, the full Poincaré symmetry plays a central role for the principle of inertia as the benchmark for physics. In fact, the mass and the spin, which characterize systems invariant under Poincaré group \([45]\), are related to the eigenvalues of two Casimir operators, in which translation generators always appear, of Poincaré algebra \(\mathfrak{iso}(1,3)\):

\[
C_1 := \eta^{jk} \hat{p}_j \hat{p}_k, \quad C_2 := \eta^{jk} \hat{w}_j \hat{w}_k,
\]

where \(\hat{w}_j := \epsilon_{jklm} \hat{p}^k \hat{l}^m\) is the Pauli-Lubanski vector, \(\hat{p}^j := \eta^{jk} \hat{p}_k, \hat{l}^m := \eta^{lrs} \hat{l}_{rs}, \hat{p}_j, \hat{l}_{jk}\) generators of translations and homogeneous Lorentz algebra \(\mathfrak{so}(1,3)\), respectively. It was Wigner \([45]\) who found that although spin also corresponds to the rotation group symmetry \(SU(2)\) as a subgroup of homogeneous Lorentz group \(SO(1,3)\), but only if \(m^2 > 0\). In the case \(m^2 = 0\), the spin is no longer described by \(SU(2)\) and this, in fact, is why the polarization states of a massless particle with spin \(s\) are \(s_z = \pm s\) only. For example, physical photons do not exist in a \(s_z = 0\) state, whereas massive spin 1 particles do (see, e.g., \([46]\)). This is also the case that there is no longitudinal component for the electromagnetic wave in the vacuum.

Thus, the benchmarks for physics in Einstein’s special relativity and general relativity seem to be not completely in consistency with each other in symmetry and its localization. This may lead to some potential problems. In order to get rid of this kind of problems, it is reasonable to require an enhanced equivalence principle with localization of special relativity of full symmetry, the principle of localization, as was proposed above.

How to describe the general spacetimes with gravity based upon the principle of localization?

As was mentioned earlier, firstly, \(\mathcal{M}\) should be a kind of 3+1-dimensional manifolds with metric \(g\) of local relativity-frames in corresponding special relativity. Secondly, in order to describe that there is localized full symmetry in the corresponding special relativity at each event on \(\mathcal{M}\), a kind of bundles \(E(\mathcal{M}, \mathcal{S}, \mathfrak{G}, P)\) is needed with \(\mathcal{M}\) as base manifold, the maximally symmetric spacetime \(\mathcal{S}\), one of the Mink/dS/AdS-spacetimes, as typical fibre and the maximum symmetry \(\mathfrak{G}\), one of \(ISO(1,3)/SO(1,4)/SO(2,3)\), as structure group. And there should be also a principal bundle \(P(\mathcal{M}, \mathfrak{G})\). Thirdly, gravity with localized full symmetry should be described by the matric \(g\) or its local frames and a kind of connections \(\Gamma\) valued in the Lie algebra \(g\) of \(\mathfrak{G}\). It is important that in principle these bundles with required connections can be constructed.

As was mentioned, however, the pseudo-Riemann manifolds with local Lorentz frames in general relativity is just a special case: the bundle \(E(M, M^{1,3}, G, P)\) with pseudo-Riemann manifold \(M\) as base manifold and the Mink-spacetime \(M^{1,3}\) as fibre. It is clear that such a geometrical description is not complete from the viewpoint of the principle of localization, since the structure group \(G\) is just \(GL(4, R)\) or its subgroup \(SO(1,3)\).

**B. Principle of localization and gravitational dynamics**

In general relativity, Einstein-Hilbert equation reads symbolically \([27]\)

\[
G = 8\pi G T,
\]

(5.3)
where $G$ is Einstein tensor, $T$ energy-momentum tensor of source and $G$ Newton’s gravitational constant. The Einstein-Cartan ‘moment of rotation’ $G$ \[^{27}\] is made of Riemann-Christoffel curvature. From the viewpoint of holonomy theorem, however, the curvature is basically related to local homogeneous Lorentz rotation (see, e.g., \[^{48},^{49},^{50}\]). But, $T$ is in a same form with the stress-energy tensor related to the translation invariance of matter on the Mink-spacetime in view of Noether’s theorem (see, e.g., \[^{48},^{49},^{50}\]).

Although by means of variational principle, Einstein-Cartan ‘moment of rotation’ $G$ is derived from variation of Einstein-Hilbert action with respect to metric or coefficients of Lorentz frame, which may be regarded as a kind of ‘translation’ connection from the viewpoint of Cartan’s structure equation or as canonical affine connection (see, e.g., \[^{28}\]). Thus, it seems more or less still reasonable to connect it with the stress-energy tensor $T$, which is also given by the variation of the matter’s action with respect to the same variable(s), metric or coefficients of Lorentz frame. However, in connection theory (see, e.g., \[^{28}\]), the coefficients of Lorentz frame can be regarded as a kind of ‘translation’ connection for what is called the canonical affine connection. Namely, there should be an affine structure locally on the spacetimes with gravity. This is just in consistency with the principle of localization with respect to Poincaré invariance. Therefore, the spacetimes with gravity should be in general pseudo-Riemann-Cartan manifolds with torsion rather than pseudo-Riemann manifolds without torsion.

On the other hand, a spinning particle with mass $m$ moves with a curvature-spinning current force in general relativity \[^{27},^{42}\]:

$$m \frac{D^2 x^k}{ds^2} = f R^{kl}_{ab} S^{ab}_l,$$

where $R^{kl}_{ab} := e^i_a e^j_b R^{ij}_{kl}$, $e^i_a$ coefficients of Lorentz frame, $R^{ij}_{kl}$ Riemann curvature, $S^{ab}_l$ spinning current of the particle and $f$ a free parameter. It is important to note that although $f$ may be very tiny, the coupling is like the Lorentz-force of a charged particle moving in electromagnetic field, which is of gauge coupling. Therefore, in general relativity there are two kinds of couplings between gravity and matter: The one in Einstein-Hilbert equation (5.3) and that in (5.4).

Thus, some questions can be raised: Why does the dynamics connect geometry with matter in different (local) symmetry in field equation? Why gravitational fields should not be described by the both curvature and torsion? Why the spinning current as a property of the matter with respect to spacetime symmetry does undergo an action from curvature as gravity, but cannot effect gravity as a kind of source?

Cartan suggested that Einstein-Hilbert equation should be generalized by what is called Einstein-Cartan equations now \[^{47},^{48},^{49},^{50},^{51},^{52}\], which read symbolically:

$$G_\Gamma = 8\pi G T, \quad Y = 8\pi G S,$$

where $G_\Gamma$ is Einstein-like tensor of Cartan’s connection $\Gamma$ or $B^{ab}_j \in \mathfrak{so}(1,3)$, $Y$ con-torsion of the connection and $S$ spin-current of gravitational source. However, there is still another kind of gauge-like coupling in the equation of motion for test (spinning) particles. Thus, from the viewpoint of
holonomy theorem and Noether’s theorem, the questions on connect between geometric quantities and physical quantities are still there \[49, 50\].

According to the principle of localization, it seems reasonable to require further that geometry and matter should be connected in same local symmetry and the gravitational dynamics be of local invariance of the principle of localization. Namely, the gravitational dynamics should be in consistency with the principle of localization. This also indicates that gravitational field equations be of gauge-like with localized symmetry of the principle of localization (see, e.g. \[49, 50\]). Of course, correct equations should pass observation tests for general relativity at least.

C. Localization of \(dS\)-hyperboloid and umbilical manifold

Simply speaking, the spacetimes with gravity of local \(dS\)-invariance may be described as a kind of \(3 + 1\)-dimensional umbilical manifolds \(\mathcal{M}^{1,3} := \mathcal{H}^{1,3}\) as sub-manifolds of \(4 + 1\)-dimensional manifolds \(\mathcal{M}^{1,4}\). This reflects a localization of the \(dS\)-hyperboloid \(H_+ \subset M^{1,4}\) \[31\].

Let us illustrate how to construct such an \(\mathcal{M}^{1,3} := \mathcal{H}^{1,3} \subset \mathcal{M}^{1,4}\).

Suppose there is an local \(H_+ \subset M^{1,4}\) anywhere and anytime tangent to the \(\mathcal{M}^{1,4}\) such that at a point \(p \in \mathcal{H}^{1,3}\), the radius vector \(r_p\) with norm \(R\) of the \(H_+ \subset M^{1,4}\) is oppositely normal to the tangent \(Mink\)-space of \(\mathcal{H}^{1,3}\), i.e. \(r_p = -N_p\), at the point. Since this local \(Mink\)-space is also tangent to the \(H_+ \subset M^{1,4}\) at the point, which is umbilical for the \(H_+ \subset M^{1,4}\) in \(\mathcal{M}^{1,4}\). Thus, \(\mathcal{H}^{1,3}\) consists of all these points, which are umbilical in the above sense, and is a sub-manifold of the \(\mathcal{M}^{1,4}\), i.e., \(\mathcal{H}^{1,3} \subset \mathcal{M}^{1,4}\). Such a kind of Riemann-Cantan manifolds \(\mathcal{H}^{1,3}\) are called umbilical manifolds with an umbilical structure of \(H_+ \subset M^{1,4}\) anywhere and anytime.

This construction can also be given in an opposite manner: Given a point \(p\) on \(\mathcal{H}^{1,3}\), there is a local \(Mink\)-space as the tangent space at the point, \(T_p(\mathcal{H}^{1,3})\), and given a vector \((N = Rn)_p\) of norm \(R\) at the point with an \(n_p\) as the unit base of space \(N^1_p\) normal to \(T_p(\mathcal{H}^{1,3})\) with a metric of \(dS\)-signature in \(\mathcal{M}^{1,4}\). Then the space \(T_p \times N^1_p \cong M^{1,4}_p\) is tangent to \(\mathcal{M}^{1,4}\) at the point. Thus, under local \(dS\)-transformations on \(T_p \times N^1_p \cong M^{1,4}_p\) there is a local hyperboloid structure \(H_R \subset M^{1,4}_p\) isomorphic to the \(dS\)-hyperboloid \(H_+ \subset M^{1,4}\) in \(\mathcal{M}^{1,4}\) at the point \(p\) as long as \(Rn_p = -r_p\) is taken. In fact, all these points consist of the umbilical manifold \(\mathcal{M}^{1,3} := \mathcal{H}^{1,3} \subset \mathcal{M}^{1,4}\).

Therefore, on the co-tangent space \(T_p^*\) at the point \(p \in \mathcal{H}^{1,3}\) there is a Lorentz frame 1-form:

\[
\theta^b = e^b_j dx^j, \quad \theta^b(\partial_j) = e^b_j; \quad e^a_j e^j_b = \delta^a_b, \quad e^a_j e^k_a = \delta^k_j; \tag{5.6}
\]

with respect to a Lorentz inner product:

\[
< \partial_j, \partial_k > = g_{jk}, \quad < e_a, e_b > = \eta_{ab}, \quad \eta_{ab} = diag(1, -1, -1, -1). \tag{5.7}
\]

Here, \(\partial_j\) the base of the tangent space \(T_p\). The line-element on \(\mathcal{H}^{1,3}\) can be expressed as

\[
ds^2 = g_{jk} dx^j dx^k = \eta_{ab} \theta^a \theta^b; \quad g_{jk} = \eta_{ab} e^a_j e^b_k. \tag{5.8}
\]

There is a Lorentz covariant derivative a la Cartan:

\[
\nabla e_a e_b = \theta^c_b (e_a) e_c, \quad \theta^b_j = B^a_{bj} dx^j, \quad \theta^b_j(\partial_j) = B^a_{b,j}. \tag{5.9}
\]
$B^a_{c_j} \in so(1,3)$ are connection coefficients of the Lorentz connection 1-form $\theta^{ab} = \eta^{bc} \theta^a_c$. The torsion and curvature can be defined as

$$\Omega^a = d\theta^a + \theta^c_b \wedge \theta^b = \frac{1}{2} T^a_{jk} dx^j \wedge dx^k$$
$$T^a_{jk} = \partial_j e^a_k - \partial_k e^a_j + B^a_{c_j} e^c_k - B^a_{ck} e^c_j; \quad (5.10)$$

$$\Omega^b = d\theta^b + \theta^c_a \wedge \theta^c = \frac{1}{2} F^a_{bjk} dx^j \wedge dx^k$$
$$F^a_{bjk} = \partial_j B^a_{bk} - \partial_k B^a_{bj} + B^a_{c_j} B^c_{bk} - B^a_{ck} B^c_{bj}. \quad (5.11)$$

They satisfy corresponding Bianchi identities.

It is easy to get a metric compatible affine connection $\Gamma^i_{jk}$ from the requirement

$$g_{jk/l} = 0, \quad \Leftrightarrow \quad e^a_{j/j/k} = 0 = \partial_k e^a_j - \Gamma^i_{jk} e^a_i + B^a_{ck} e^c_j. \quad (5.12)$$

As was just mentioned, at the point $p \in H^{1,3}$, there are a space $N^1_p$ and its dual $N^{1*}_p$ normal to $H^{1,3}$ with a normal vector $n$ and its dual $\nu$ on $T_p(M^{1,4})$ and $T^*_p(M^{1,4})$, respectively. Namely, $\{\partial_j, n; dx^j, \nu\}$ and $\{e_a, n; \theta^b, \nu\}$ span $M^{1,4}_p = T^1_{p13} \times N^1_p$ and $M^{1,4*}_p = T^1_{p13} \times N^{1*}_p$, respectively. Let these bases satisfy the following conditions in addition to (5.7)

$$dx^j(n) = \theta^b(n) = 0, \quad \nu(\partial_j) = \nu(e_a) = 0, \quad n(\nu) = 1; \quad (5.13)$$
$$< e_a, n >= 0, \quad < n, n >= -1. \quad (5.14)$$

Then, the $ds$-Lorentz base $\{\hat{E}_A\}$ and their dual $\{\hat{\Theta}^B\}$ can be defined as:

$$\{\hat{E}_A\} = \{e_a, n\}, \quad \{\hat{\Theta}^B\} = \{\theta^b, \nu\}. \quad (5.15)$$

And (5.7) and (5.13) can be expressed as

$$\hat{\Theta}^B(\hat{E}_A) = \delta^B_A, \quad < \hat{E}_A, \hat{E}_B >= (\eta_{AB})_{A,B=0,...,4} = diag(1, -1, -1, -1, -1). \quad (5.16)$$

Introduce a normal vector $N = Rn$ with norm $R$:

$$N = Rn = \xi^A \hat{E}_A, \quad (\xi^A) = (0, 0, 0, 0, R), \quad < N, N >= -R^2. \quad (5.17)$$

For the $ds$-Lorentz base, there are

$$g_{jk} = \eta_{AB} \hat{E}_j^A \hat{E}_k^B, \quad \eta_{AB} \xi^A \hat{E}_j^B = 0, \quad \eta_{AB} \xi^A \xi^B = -R^2, \quad (5.18)$$

where

$$\hat{E}_j^A = \hat{\Theta}^A(\partial_j), \quad \{\hat{E}_j^A\} = \{e^a_j, 0\}. \quad (5.19)$$

The transformations, which maps $M^{1,4}_p$ to itself and preserves the inner product, are

$$\hat{E}_A \rightarrow E_A = S^B_A \hat{E}_B, \quad \hat{\Theta}^A \rightarrow \Theta = S^{-1}_B^A \hat{\Theta}^B, \quad SJS^t = J, \quad (5.20)$$
where \( J = (\eta_{AB}) = \text{diag}(1, -1, -1, -1, -1) \), \( S = (S^A_4) \in SO(1, 4) \), \( \ast^t \) denotes the transpose. The transformed base is defined as the \( dS \)-base and its dual \( E_A, \Theta^B \), respectively:

\[
\Theta^A(E_B) = \delta^A_B, \quad \Theta^A(\partial_j) = E^A_j, \quad <E_A, E_B> = \eta_{AB}. \tag{5.21}
\]

\[
g_{jk} = \eta_{AB}E^A_jE^B_k; \quad \eta_{AB}\xi^A_E^B = 0, \quad \eta_{AB}\xi^A_E^B = -R^2, \tag{5.22}
\]

where \( E^B_j \) are the \( dS \)-frame coefficients. Obviously, these formulas reflect the local \( dS \)-invariance on \( \mathcal{H}^{1,3} \) and \( (5.22) \) show that there is a local 4-dimensional hyperboloid \( H^{1,3}_p \subset M^{1,4}_p \) tangent to \( \mathcal{H}^{1,3} \) at the point \( p \). Thus, \( (5.22) \) may be called the local \( dS \)-hyperboloid condition.

Now the \( dS \)-covariant derivative a la Cartan can be introduced

\[
\nabla_{E_A} E_B = \Theta^C_B(E_A)E_C. \tag{5.23}
\]

\( \Theta^A_C \in \mathfrak{so}(1, 4) \) is the \( dS \)-connection 1-form. In the local coordinate chart \( \{ x^j \} \),

\[
\nabla_{\partial_j} E_B = \Theta^C_B(\partial_j)E_C = B^C_{Bj}E_C, \tag{5.24}
\]

\( B^A_{Cj} \) denote the \( dS \)-connection coefficients. There are also the \( dS \)-torsion \( \Omega^A \), curvature 2-forms \( \Omega^A_B \) and their Bianchi identities.

In the light of Gauss formula and Weingarten formula in the surface theory \cite{24}, from the \( dS \)-covariant derivative of the \( dS \)-Lorentz base \( (5.15) \) with properties of \( \theta^a, \theta^4_b \), it follows a generalization of Gauss formula and Weingarten formula

\[
\nabla_{\partial_j} e_a = \theta^b_a(\partial_j)e_b - b_{ab}\theta^b(\partial_j)n, \quad \nabla_{\partial_j} n = b^a_b\theta^b(\partial_j)e_a. \tag{5.25}
\]

Here, \( b_{ab} \) denotes a second fundamental form of the hypersurface. Since \( \mathcal{H}^{1,3} \) is supposed to be an umbilical hypersurface, where every point satisfies the umbilical condition on \( H_R \subset M^{1,4} \)

\[
g_{jk} = Rb_{jk}, \tag{5.26}
\]

these formulas read on \( \mathcal{H}^{1,3} \)

\[
\nabla_{\partial_j} e_a = \theta^b_a(\partial_j)e_b - R^{-1}\theta^a(\partial_j)n, \quad \nabla_{\partial_j} n = R^{-1}\theta^a(\partial_j)e_a. \tag{5.27}
\]

On the other hand, for the \( dS \)-Lorentz base from \((5.23)\) there are

\[
\nabla_{\partial_j} e_a = \check{\theta}^b_a(\partial_j)e_b + \check{\Theta}^a_4(\partial_j)n, \quad \nabla_{\partial_j} n = \check{\Theta}^a_4(\partial_j)e_a, \tag{5.28}
\]

where \( \check{\Theta} \) denotes the \( dS \)-connection \( \Theta \) in the \( dS \)-Lorentz gauge.

Comparing with \((5.27)\), it follows

\[
\check{\Theta}^{ab}(\partial_j) = \theta^{ab}(\partial_j) = B^{ab}_j; \quad \check{\Theta}^{a4}(\partial_j) = R^{-1}\theta^{a4}(\partial_j) = R^{-1}e^a_j; \tag{5.29}
\]

\[
\check{B}^{ab}_j = B^{ab}_j; \quad \check{B}^{a4}_j = R^{-1}e^a_j.
\]

Namely, the \( dS \)-connection in the \( dS \)-Lorentz gauge may be written as

\[
(\check{B}^{AB}_j) = \begin{pmatrix} B^{ab}_j & R^{-1}e^a_j \\ -R^{-1}e^b_j & 0 \end{pmatrix} \in \mathfrak{so}(1, 4). \tag{5.30}
\]
This is just the connection introduced in \[30, 31, 32, 33\]. Here, it is recovered from the umbilical manifolds with local $dS$-invariance.

The corresponding curvature reads:

$$\tilde{\mathcal{F}} := d\tilde{B} + \tilde{B} \wedge \tilde{B} = \frac{1}{2} \tilde{F}_{jk} dx^j \wedge dx^k,$$

$$\tilde{F}_{jk} \equiv (\tilde{F}^{AB}_{jk}) = \left( \begin{array}{cc} F^{ab}_{jk} + 2R^{-1}c^{ab}_{jk} & R^{-1}T^{a}_{jk} \\ -R^{-1}T^{b}_{jk} & 0 \end{array} \right) \in \mathfrak{so}(1,4),$$

where $c^{ab}_{jk} = \frac{1}{2}(e^{c}_{j}e^{b}_{k} - e^{a}_{k}e^{b}_{j})$, $e_{b_{j}} = \eta_{ab}e_{a_{j}}$, $F^{ab}_{jk}$ and $T^{a}_{jk}$ are curvature (5.11) and torsion (5.10).

D. A simple model of $dS$-gravity

For the $dS$-connection (5.30), a simple model of $dS$-gravity can be introduced \[30, 31, 32, 33\].

The total action of the model with source may be taken as

$$S_T = S_{GYM} + S_m,$$  \hspace{1cm} (5.32)

where $S_m$ is action of source with minimum coupling, and $S_{GYM}$ the Yang-Mills-like action of the model as follows (in the $dS$-Lorentz gauge):

$$S_{GYM} = \frac{\hbar}{4g^2} \int_{M^{1,3}} d^4x e \text{Tr}_{dS}(\tilde{F}_{jk} \tilde{F}^{jk})$$

$$= \int_{M^{1,3}} d^4x \left[ \frac{c^3}{16\pi G} (F - 2\Lambda) - \frac{\hbar}{4g^2} F^{\mu\nu} F_{\mu\nu} + \frac{c^3}{32\pi G} T^{a}_{\mu\nu} T^{a}_{\mu\nu} \right].$$  \hspace{1cm} (5.33)

Here $e = \det(e_{ij}^a)$, a dimensionless constant $g$ should be introduced as usual in the gauge theory to describe the self-interaction of the gauge field, $F = \frac{1}{2} F^{ab}_{jk} e^{jk}$ the scalar curvature of Cartan connection, the same as the action in Einstein-Cartan theory. In order to make sense in comparing with Einstein-Cartan theory, we should take $g^2 \approx G\hbar \Lambda / 3 c^3 \sim 10^{-122}$.

It is natural to see that the gravitational field equations now should be of gauge-like. But, different from ordinary gauge theory, there is some energy-momentum-like tensor $T^a_{Gj}$ for gravity itself as source from variation with respect to the coefficients of Lorentz frame of the third and last term in the action (5.33), respectively:

$$T^a_{Gj} := g^{-2} T^a_{Fj} + 2\chi T^a_{Tj}, \quad T^a_{*j} = T_{*kj}e^{ak}, \quad \chi = c^3 / G\hbar$$

$$T_{Fjk} := \text{Tr}(F_{jl} F_{k}^l) - \frac{1}{4} g_{jk} \text{Tr}(F_{lm} F_{lm}),$$

$$T_{Tjk} := T^a_{jl} T^l_{a} + \frac{1}{4} g_{jk} T^a_{lm} T^l_{a} T^m_{a},$$

For the case of spinless for matter and torsion-free for gravity, the field equations become Einstein-Yang equations \[53\] with $\Lambda$-term (in what follows, we take unit of $c = \hbar = 1$).

$$R^a_j - \frac{1}{2} e^a_j R + \Lambda e^a_j = -8\pi G (T^a_{mj} + g^{-2} T^a_{Rj}),$$  \hspace{1cm} (5.37)

$$R^{ab}_{jk} = 0,$$  \hspace{1cm} (5.38)
where $\nabla$ is the double covariant derivative with respect to Christoffel and Ricci rotation coefficients $\gamma^a_{\ jb}$, $T^a_{\ mj} = e^a_k T^{\ k}_{mj}$ the energy-momentum tensor of matter, and $T^a_{\ Rj} = e^a_k T^{\ Rjk}_{\ k}$ the energy-momentum-like tensor of Riemann curvature $R^a_{\ jk} \in \mathfrak{so}(1,3)$

$$T^k_{\ Rj} = R_{abjl} R^a_{\ kl} - \frac{1}{4} \delta^k_j (R_{ablm} R^a_{\ blm}),$$

$$= 2 C^m_{\ ljk} R^l_m + \frac{R}{3} (R^k_j - \frac{1}{4} R^k_{\ ij}),$$

(5.39)

where $C_{l j m k}$ is Weyl tensor. For the last equation in (5.39), the Géheniau-Debever decomposition for Riemann curvature is used. It is clear that if Ricci tensor vanishes, i.e., $R_{jk} = 0$, this energy-momentum-like tensor of Riemann curvature (5.39) vanishes so that the vacuum solutions in general relativity do satisfy the Einstein-Yang equations (5.37) and (5.38) without $\Lambda$-term [54].

It is easy to prove that for $dS$-spacetime the ‘energy-momentum’-like tensor in (5.39) vanishes as well, so $dS$-spacetime also satisfies eqs (5.37) and (5.38). It can also be proved that all solutions of vacuum Einstein equation with $\Lambda$-term do satisfy these equations, so this simple model does pass the observation tests in solar-scale. Further, it is shown [55] that some simplest cosmic models may have ‘Big Bang’ but differ from general relativity, as $T^a_{\ Rj}$ could play a role as a kind of the ‘dark stuffs’. Since the general equations are of gauge-like, there are gravitational potential waves of the both metric and Cartan’s connection including the gravitational metric waves in general relativity.

It is important that the $dS$-gravity in this model is characterized by a dimensionless coupling constant $g$ like in ordinary gauge theory. This is one of reasons why the model is renormalizable [33]. It is also interesting that it is of an $SO(5)$ gauge-like Euclidean action with the Riemann sphere being an instanton. Thus, the quantum tunneling scenario may support $\Lambda > 0$. For the gauge-like gravity, asymptotic freedom may indicate the coupling constant $g$ should be very tiny and link the cosmological constant $\Lambda$ with the Planck length $\ell_P$ properly, since $\Lambda$ and $\ell_P$ as a fixed point should provide an infrared and an ultraviolet cut-off, respectively [11, 14].

This model presents some important indications to why the universe is so dark. First, the cosmological constant $\Lambda$ as a fundamental constant is introduced from the ‘gauge’ symmetry so that it is not just a ‘dummy’ constant at classical level put in by hand in general relativity. And it should play a role of the simplest dark energy. In addition, there are some candidates for the dark matter from $dS$-gravity itself, such as the ‘energy-momentum-like tensors’ for gravity and so on. In fact, by means of the relation between Cartan’s connection $B^a_{\ j}$ and Ricci rotational coefficients $\gamma^a_{\ j}$, Einstein-Hilbert action can be picked up from the first term in (5.33), all other terms except the cosmological constant $\Lambda$, which is the simplest form of the dark energy, are all the dark matter from the viewpoint of general relativity. Thus, this model should provide an alternative framework for the dark-data analysis in precise cosmology.

VI. CONCLUDING REMARKS

In the last century physics, symmetry, its localization and symmetry breaking play very important roles. For physics in the large scale, it should be also the case. Namely, the maximally
symmetric spacetime with maximum symmetry and their localization should play a central role.

Initiated by Professor Lu’s proposal \[1\], there are three kinds of special relativity \[1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\] based on the principle of relativity on \(dS/AdS\)-spacetimes, or Poincaré principle of relativity as its \(Mink\)-contraction \(R \to \infty\), and the postulate on invariant universal constants, or its \(Mink\)-contraction. All other kinematics with the principle of relativity should be their contractions \[9\].

From the viewpoint of the \(dS\) special relativity, the dark energy is at least mainly the cosmological constant \(\Lambda\) and \(dS\)-spacetime provides an important model: There is the principle of relativity and a law of inertia in Beltrami coordinate atlas with Beltrami simultaneity. The proper-time simultaneity flips it to another side of a Robertson-Walker-like \(dS\)-space with an accelerated expanding \(S^3\) fitting the cosmological principle. If our universe is asymptotic to such a Robertson-Walker-like \(dS\)-space, it should be slightly closed in \(O(\Lambda)\) with \(R \simeq (3/\Lambda)^{1/2}\) and all celestial objects including the CMB in the cosmic scale should be rotated qualitatively. On the other hand, the universe can fix on Beltrami systems via its evolution. Therefore, for the principle of inertia on \(dS\)-spacetime and its all contractions there should be no Einstein’s ‘argument in a circle’ \[13\] and the universe just acts as the origin of inertia \[6, 12, 13\].

For null physics of three kinds of special relativity, symmetry should be enlarged to conformal group realized on the same projective null cone isomorphic to the projective boundary of a 5-dimensional \(AdS\)-space, i.e., \([\mathcal{N}] \cong \partial_p(AdS^5) \subset M^{2,4}\). Thus, there is a triality for conformal extensions of null physics on \(Mink/dS/AdS\)-spacetimes including the \(AdS/CFT\) correspondence. And there should be a \(dS\)-spacetime on the boundary of \(S^5 \times AdS^5\) as a vacuum of supergravity.

Gravity should be based on the principle of localization with localized principle of relativity of full maximum symmetry. Thus, the localization of special relativity leads to corresponding theory of gravity with local maximum symmetry. For \(dS\)-gravity, its dynamics should be gauge-like in consistency with the principle of localization characterized by a dimensionless constant \(g \simeq (\Lambda G \hbar /3c^3)^{1/2} \sim 10^{-61}\). A simple model \[30, 31, 32, 33\] shows the features on a kind of umbilical Riemann-Cartan manifolds of local \(dS\)-invariance \[31\]. Some gravitational effects in this model that cannot be included in general relativity should play the role as the dark matter.

What are the benchmarks for physics? Whether these benchmarks are consistent each other? These are most important and fundamental issues.

If the principle of relativity should be generalized to all maximally symmetric spacetimes and if gravity should be described based on the localized principle of inertia with full maximum symmetries, the benchmark for physics with gravity is in consistency with the one without gravity of special relativity \[13\].

Some seventy years ago, Einstein claimed: ‘Physics constitutes a logical system of thought which is in a state of evolution’. ‘Evolution is proceeding in the direction of increasing simplicity of the logical basis (principles).’ ‘We must always be ready to change these notions - that is to say, the axiomatic basis of physics - in order to do justice to perceived facts in the most perfect way logically.’ \[56\]. This has greatly enlightened us how to understand evolution of physics in the past and how to look forward the direction of its evolution. Especially, how to deal with the theory of relativity as a kind of ‘principle theory’ in the face of the challenges from the dark universe.
It seems that study on special relativity and theory of gravity via maximum symmetry and its localization is in a right direction: increasing simplicity of the principles, ‘in order to do justice to perceived facts in the most perfect way logically’.

The dark universe and its asymptotic behavior may already indicate that the \( dS \) special relativity and the \( dS \)-gravity should be the foundation of physics in the large scale.

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