OPTIMAL DESIGN OF AN OPTICAL LENGTH OF A ROD WITH THE GIVEN MASS

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Abstract. The optical length of an elastic rod appears to be important in problems of control of its longitudinal vibrations. We consider the problem of optimal design (optimal density distribution) of an elastic rod of a given variable modulus of elasticity with the optical length as a criterion and assuming that the total mass of the rod is given. The results provide some bounds on the optical length.

1. Introduction. Control theory for an elastic rod/string governed by the wave equation seems to be complete at the moment. The conditions of exact controllability for a non-homogeneous rod/string were described by H. O. Fattorini and D. L. Russell (see [6], [9] and references therein). In particular, the minimal time of control appears to be equal to twice the optical length of the rod, $T^*$. This is defined to be the time of wave propagation along the rod, and is given by $T^* = \int_0^l \sqrt{\rho(x)/k(x)} \, dx$, where $\rho(x)$ is the density and $k(x)$ is the modulus of elasticity. It is also known that the inclusion of the damping in the model does not change that time (see [4], [10]). Control theory for a string with a time dependent tension $p(t)$ was developed and the minimal time of control was found in [1]–[2]. We consider the problem of the optimal design (optimal density distribution) of a rod of a given length and total mass $M = \int_0^l \rho(x) \, dx$. We assume the modulus of elasticity is a given function of either coordinate or time. Specifically, we are looking for a density distribution $\rho(x)$ that minimizes/maximizes the time $T$ of control. We consider all models of the rod studied in the literature cited above and find maximum and/or minimum of $T$ (if possible). It appears that extrema do not necessarily exist, and so we construct the corresponding minimizing sequence.

2. Statement of the optimization problem. In the typical scenario of a controllability problem, we consider transverse oscillations of an elastic rod described by the partial differential equation (PDE)

$$\rho(x) y_{tt} = (p y_x)_x + g(x) f(t), \quad (x, t) \in Q_T = (0, l) \times (0, T) \quad (2.1)$$

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subject to the boundary conditions (fixed end points)
\[ y(0, t) = y(l, t) = 0 \] (2.2)
and the initial conditions
\[ y(x, 0) = y^0(x), \quad y_t(x, 0) = y^1(x). \] (2.3)

Here \( y(x, t) \) is the transversal displacement, \( \rho \) is the linear density of the rod, \( k \) is the modulus of elasticity, and \( g(x)f(t) \) is the exterior force. We proceed under the assumptions that the density, modulus of elasticity, initial data, and the exterior force are smooth enough, so that the initial-boundary value problem (1.1)–(1.3) has a unique solution in an appropriate functional class (see [2], [3], [4], [6], [9] for details). We further are looking for a time distribution of the force \( f(t) \) such that the solution of the problem (2.1)–(2.3) satisfies the additional conditions at the time \( t = T \),
\[ y(x, T) = y_t(x, T) = 0, \quad x \in [0, l]. \]
The uniqueness of the solution implies \( y(x, t) \equiv 0 \) for \( t \geq T \) if \( f(t) = 0 \) for \( t > T \), in other words, the force \( f(t) \) stops oscillations of the rod at the moment \( T \). The corresponding technique was suggested by D. L. Russell who proved that the minimal time of control, \( \min T \), is given in terms of the so-called optical length, \( \min T = 2T^* \), where
\[ T^* = \int_0^l \sqrt{\frac{\rho(x)}{k(x)}} \, dx. \] (2.4)

As we see, the time of control is a functional of the density and the modulus of elasticity of the rod. The total mass of the rod is given by
\[ M = \int_0^l \rho(x) dx. \] (2.5)

We discuss the following problem of the optimal design of the rod with the coordinate dependent density and modulus of elasticity.

**Problem 1.** Consider the control problem for a rod described by (2.1)–(2.3). Given the length of the rod \( l \) and the modulus of elasticity \( k(x) > 0 \), find a distribution of the density of material \( \rho(x) \) such that the time of control (2.4) is a maximal/minimal possible subject to the given mass of the rod (2.5).

Formally speaking, the problem of transversal oscillations of an elastic string is described by the same initial boundary value problem (2.1)–(2.3) with \( k(x) \) being the tension, and hence our consideration holds for that model as well.

We solve Problem 1 in Section 3. The main result is given by Theorems 1, 1a and 2, 2a. It appears that for Problem 1, there exists the unique max \( T^* \). The min \( T^* \) does not exist but \( \inf T^* = 0 \). We also discuss the optimal design problem that is dual to Problem 1 (finding a distribution of the density of material of the rod such that the mass of the rod is a maximal/minimal possible subject to the given time of control). It appears that there exists the unique min \( M \). The max \( M \) does not exist but \( \sup M = \infty \).

We discuss the similar optimization problem for a rod with time dependent modulus of elasticity in Section 4 and for a rotating rod in Section 5.
3. Solution of the Problem 1. We use the standard notation $|| \cdot ||$ for the $L^2(0,l)$-norm. We introduce the (positive) functions
\[ z(x) \equiv \sqrt{\rho(x)}, \quad s(x) \equiv \frac{1}{\sqrt{k(x)}} \] (3.1)
so that the functionals under consideration are
\[ T^*[z] = \int_0^l s(x)z(x)dx \] (3.2)
and
\[ M[z] = \int_0^l z^2(x)dx. \] (3.3)
We thus have the following problem equivalent to Problem 1.

**Problem 1a.** Find a positive continuous function $z(x)$ such that the integral (3.2) is the maximal/minimal possible subject to the constraint (3.3) with the given $M > 0$.

The problem seems to be related to linear algebra. Introducing an orthonormal system of functions $\{f_n(x)\}$ in $L^2(0,l)$, representing both $z(x)$ and $s(x)$ as the series
\[ z(x) = \sum_n z_n f_n(x) \quad \text{and} \quad s(x) = \sum_n s_n f_n(x) \] (3.4)
with the Fourier coefficients $z_n$ and $s_n$, and using Parseval’s identity yields
\[ T^*[z] = \sum_n s_n z_n \] (3.5)
and
\[ M[z] = \sum_n z_n^2. \] (3.6)
Hence, Problem 1 is equivalent to the following.

**Problem 1 (equivalent version).** Find a sequence $\{z_n\}$ such that the series in (3.4) represent the positive functions on $[0,l]$ and the series (3.5) has a minimum/maximum value subject to the constraint (3.6).

We actually are looking a minimum of a linear function (3.5) of infinitely number of variables on the sphere (3.6). If by chance the series (3.4) for the given $s(x)$ contains a finite number of harmonics, we have exactly the problem from linear algebra and may try using Lagrange multipliers. Even in that case, it is not clear how to take the positivity of the series (3.4) that represent the functions $z(x)$ and $s(x)$ into consideration. Thus, we proceed differently.

**Theorem 1.** For Problem 1, there exists (the unique) $\max T^*$,
\[ \max T^*[z] = \sqrt{M} ||s|| \quad \text{as} \quad z(x) = \frac{\sqrt{M} s(x)}{||s||}. \] (3.7)

**Proof.** By the Cauchy-Schwarz inequality
\[ \int_0^l s(x)z(x)dx \leq ||s|| \cdot ||z|| = \sqrt{M} ||s||, \]
where equality holds if $z(x) = Cs(x)$ with $C$ a constant. Using the constraint (3.3) we find that

$$\max T^*[z] = \sqrt{M} ||s||, \text{ provided } z(x) = \frac{\sqrt{M} s(x)}{||s||}. \tag{3.8}$$

The physical meaning of this result becomes obvious if we go back to the original functions. That leads to the following version of Theorem 1.

**Theorem 1a.** For Problem 1, there exists $\max T^*$,

$$\max T^*[\rho] = \sqrt{M} \frac{1}{\sqrt{k}} ||\rho|| \text{ provided } \rho(x) = \frac{M}{k(x)||\frac{1}{\sqrt{k}}} \tag{3.9}$$

**Comment 1.** The density of the optimal rod is inversely proportional to the modulus of elasticity. In particular, if the modulus of elasticity is constant, the optimal density is constant as well.

We now consider the possibility of minimizing $T^*[z]$. We first give a heuristic explanation of why $\inf T^*[z] = 0$ and then prove the corresponding theorem. Introduce the sequence of functions

$$z_\epsilon(x) \equiv \begin{cases} \sqrt{\frac{M}{2\epsilon}}, & x \in (x_0 - \epsilon, x_0 + \epsilon), \\ 0, & \text{otherwise}, \end{cases} \tag{3.10}$$

with some fixed $x_0 \in (0, l)$. Note that $z_\epsilon(x)$ is not appropriate for $z(x)$. It does satisfy the constraint (3.3) because $\int_0^l z_\epsilon^2(x)dx = M$. Now using (3.2), we (formally) find the time of control to be

$$T^*[z_\epsilon] = \sqrt{\frac{M}{2\epsilon}} \int_{x_0 - \epsilon}^{x_0 + \epsilon} s(x)dx = \sqrt{\frac{M}{2\epsilon}} 2\epsilon s(x^*) \tag{3.11}$$

where the last equality is due to the continuity of $s(x)$ and $x^* \in (x_0 - \epsilon, x_0 + \epsilon)$. Obviously,

$$\lim_{\epsilon \to 0} T^*[z_\epsilon] = 0, \tag{3.12}$$

and, moreover,

$$T^*[z_\epsilon] = O(\sqrt{\epsilon}) \text{ as } \epsilon \to 0. \tag{3.13}$$

We now proceed in exact terms. Noting that $z_\epsilon^2(x)/M$ is a delta-sequence, we construct another delta-sequence that is smooth and hence appropriate as $z(x)$.

**Theorem 2.** For Problem 1, the $\min T^*$ does not exist but

$$\inf T^* = 0. \tag{3.14}$$

**Proof.** We construct the minimizing sequence $\{z_\epsilon(x)\}$ satisfying the constrain (3.3) exactly and such that

$$\lim_{\epsilon \to 0} T^*[z_\epsilon] = 0. \tag{3.15}$$

We introduce the sequence of functions

$$\omega_\epsilon(x) \equiv \frac{C_\epsilon}{\sqrt{\pi \epsilon}} e^{-\frac{(x-x_0)^2}{\epsilon}} \tag{3.16}$$
for some fixed $x_0 \in (0, l)$ and normalization factor $C_\epsilon$ such that
\begin{equation}
\int_0^l \omega_\epsilon(x)dx = 1.
\end{equation}
Clearly,
\begin{equation}
C_\epsilon = \frac{\sqrt{\pi \epsilon}}{\int_0^l e^{-\frac{(x-x_0)^2}{\epsilon}}dx}.
\end{equation}
We now estimate $C_\epsilon$ as $\epsilon \to 0$. First, we estimate the integral in (3.18). If we integrate over the whole real axis, the result is known,
\begin{equation}
\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{\epsilon}} dx = \sqrt{\pi \epsilon}.
\end{equation}
Hence, we need to estimate the integrals over $(-\infty, 0)$ and $(l, \infty)$. The asymptotic properties of these integrals are well-known (see [5], [8]),
\begin{align*}
\int_{-\infty}^{0} e^{-\frac{(x-x_0)^2}{\epsilon}} dx &= O(e^{-\frac{x_0^2}{\epsilon}} \sqrt{\epsilon}), \\
\int_{l}^{\infty} e^{-\frac{(x-x_0)^2}{\epsilon}} dx &= O(e^{-\frac{(l-x_0)^2}{\epsilon}} \sqrt{\epsilon}).
\end{align*}
Finally, we find the normalization factor
\begin{equation}
C_\epsilon = O(1) \quad \text{as } \epsilon \to 0.
\end{equation}
Note, without the factor $C_\epsilon$, the sequence $\omega_\epsilon(x)$ is the well-known infinitely smooth delta-sequence [7], and hence we proceed according to the same idea as in the previous heuristic derivation.

Let
\begin{align*}
z_\epsilon(x) &\equiv \sqrt{M} \omega_\epsilon(x)
\end{align*}
so that the constraint (3.3) is satisfied exactly. Then, the estimate (3.18) and the extended mean value theorem yield
\begin{align*}
T^*[z_\epsilon] &= \sqrt{M} \int_0^l s(x) \sqrt{\omega_\epsilon(x)} dx = \frac{\sqrt{MC_\epsilon}}{\sqrt{\pi \epsilon}} \int_0^l s(x) e^{-\frac{(x-x_0)^2}{\epsilon}} dx \\
&= \frac{\sqrt{MC_\epsilon}}{\sqrt{\pi \epsilon}} s(x^*) \frac{\sqrt{MC_\epsilon}}{\sqrt{\pi \epsilon}} O(\sqrt{\epsilon}) = O(\sqrt{\epsilon}).
\end{align*}
Hence, (3.15) follows.

Knowing that $\inf T^* = 0$ and assuming the existence of a function $z(x)$ such that $T^* = 0$, we immediately find from (3.2) that $z(x)s(x) = 0$, which is a contradiction.

We now briefly discuss the optimal design problem that is dual to Problem 1.

**Problem 1 (dual).** Given the length of the rod $l$ and the modulus of elasticity $k(x) > 0$, find a distribution of the density of material $\rho(x)$ such that the mass of the rod (2.5) is the maximal/minimal possible subject to the given time of control (2.4).

Using almost the same arguments as above yields the following results.

**Theorem 1 (dual).** For Problem 1 (dual), there exists $\min M$,
\begin{equation}
\min M[z] = \frac{T^*}{\|s\|^2} = \frac{T^*}{\|\frac{1}{\sqrt{\epsilon}}\|^2}.
\end{equation}
provided \( z(x) = \frac{T^*}{||s||^2} \) \( s(x) = \frac{T^*}{\sqrt{k(x)}||\frac{1}{\sqrt{k}}||^2} \).

\[ (3.22) \]

**Theorem 2 (dual).** For Problem 1 (dual), the \( \max M \) does not exist but \( \sup M = \infty \).

We omit the proofs.

4. **The optimization problem for a rod with time dependent modulus of elasticity.** The control problem of a rod with modulus of elasticity \( k(t) \) that is a function of time, not the space coordinate, is considered in [1]–[2]. The optical length is defined as the unique positive solution \( T^* \) of the equation

\[ \int_0^l \sqrt{\rho(x)} \, dx = \int_0^{T^*} \sqrt{k(t)} \, dt. \]  \[ (4.1) \]

Obviously the maximum/minimum of \( T^* \) corresponds to the maximum/minimum of \( \int_0^l \sqrt{\rho(x)} \, dx \). Using Theorem 1a and Comment 1 yields that, for the maximum of \( T^* \), the optimal density is constant.

**Theorem 3.** For a rod with time dependent density, the \( \max T^* \) is a (unique) positive solution of the equation

\[ \int_0^{T^*} \sqrt{k(t)} \, dt = \sqrt{ML}. \]  \[ (4.2) \]

The optimal density is constant, \( \rho(x) = \frac{M}{T^*} \).

**Theorem 4.** For a rod with time dependent density, the \( \min T^* \) does not exist but \( \inf T^* = 0 \).

Proof. Using the same construction as in the proof of Theorem 2 yields

\[ \inf \int_0^{T^*} \sqrt{k(t)} \, dt = 0, \]  \[ (4.4) \]

and hence \( \inf T^* = 0 \). Also, assuming that \( T^* = 0 \) leads to the contradiction in (4.2). \( \square \)

5. **Control problem for a rotating rod.** Though controllability of a rotating rod was considered in [3], only the case of the constant density \( \rho(x) \) was discussed there. For a rotating rod without a concentrated mass at the free end point, the modulus of elasticity \( k(x) \) must be equal to zero. We assume that

\[ k(x) = (l - x)q(x), \quad q(x) > 0, \quad x \in [0, l] \]  \[ (5.1) \]

for some smooth function \( q(x) \). Because of the structure of the coefficient \( k(x) \), the end point \( x = l \) is singular, and hence the boundary conditions (2.2) must be replaced by

\[ y(0, t) = 0 \quad \text{and} \quad \lim_{x \to l^-} |y(x, t)| < \infty, \quad \lim_{x \to l^-} |y_t(x, t)| < \infty. \]  \[ (5.2) \]

**Problem 3.** Consider Control Problem for a rotating rod described by (2.1), (5.2), (2.3). Given the length of the rod \( l \) and the modulus of elasticity \( k(x) = (l - x)q(x), \quad q(x) > 0 \) find a distribution of the density of material \( \rho(x) \) such that the time of control (2.4) is a maximal/minimal possible subject to the given mass of the rod (2.5).
If we attempt to use the results of Theorem 1, according to (3.1) we obtain,

\[ ||s|| = \left| \frac{1}{\sqrt{E(x)}} \right| = \infty, \] (5.3)

and hence the scheme has to be modified for \( s \notin L^2(0, l) \). Next, we describe the results and give brief description of the proof.

**Theorem 5.** For a rotating rod, (a) the \( \max T^* \) does not exist but \( \sup T^* = \infty \); (b) the \( \min T^* \) does not exist but \( \inf T^* = 0 \).

**Proof.** The proof of part (b) is exactly the same as in Theorem 2 because \( l - x_0 > 0 \).

For the proof of (a), introduce a sequence

\[ z_{\epsilon}(x) = \frac{C_{\epsilon}}{(l - x)^{1/2 - \epsilon}} \] (5.4)

with the normalization factor

\[ C_{\epsilon} = \frac{\sqrt{2M\epsilon}}{l^\epsilon} \]

such that

\[ \int_0^l z_{\epsilon}^2 \, dx = M. \]

It is easy to check that

\[ C_{\epsilon} = O\left( \sqrt{\epsilon} \right). \]

We find the time of control (see the proof of Theorem 2 for details) to be

\[ T^* = C_{\epsilon} \int_0^l \frac{1}{(l - x)^{1 + \epsilon}} \sqrt{q(x)} \, dx = C_{\epsilon} \frac{l^\epsilon}{\sqrt{q(x^*)} \epsilon} = O\left( \frac{1}{\sqrt{\epsilon}} \right), \]

and this completes the proof of (a).

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