Itô–Föllmer Calculus in Banach Spaces II: Transformations of Quadratic Variations

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Abstract

In this paper, we study properties of quadratic variations of càdlàg paths within the framework of the Itô–Föllmer calculus in Banach spaces. We prove a $C^1$-type transformation formula for quadratic variations. We also investigate relations between tensor and scalar quadratic variations.

Contents

1 Introduction 1
2 Settings 4
3 Linear transformations 9
4 Integral representation of quadratic variations by means of scalar quadratic variation 11
5 $C^1$-transformations 15
6 Quadratic variation of the Itô–Föllmer integrals 24
A Remarks on the Radon–Nikodym property 25
B Supplements on families of càdlàg paths 26
References 31

1 Introduction

The Itô–Föllmer calculus, which originated in Föllmer [22], is a deterministic counterpart to classical Itô’s stochastic calculus. Recently, the Itô–Föllmer calculus has seen increasing developments, receiving much attention from the viewpoint of its financial applications (see, e.g., Sondermann [53], Schied [50, 51], Mishura and Schied [41], Schied, Speiser, and Voloshchenko [52], Chiu and Cont [5], Hirai [28], and Cont and Perkowski [9]). In addition, in the spirit of functional Itô calculus (Dupire [20] and Cont and Fournié [6].

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If there is a càdlàg path map into another Banach space in stochastic calculus in Hilbert spaces. Next, we show quadratic variations are interpreted as a pathwise version of classical results (see, for example, Metivier [36]). Banach space with the Radon–Nikodym property (RNP). Both trace and integral representation formulae in a Hilbert space setting. We also show an integral representation formula of quadratic variation in a bilinear maps. We then prove the trace representation formula of scalar quadratic variations by tensor one projective tensor quadratic variation implies the existence of all quadratic variations with respect to bounded variations. We provide some linear transformation formulae, one of which implies that the existence of a càdlàg path

\[ \pi_n \]  defines quadratic variations. We also relaxed the assumption on the sequence of partitions along which we consider the quadratic variation.

In this paper, we study more detailed properties of quadratic variations, focusing on their transformations and representations. In particular, we first study the relations between tensor, scalar, and other quadratic variations. We provide some linear transformation formulae, one of which implies that the existence of projective tensor quadratic variation implies the existence of all quadratic variations with respect to bounded bilinear maps. We then prove the trace representation formula of scalar quadratic variations by tensor one in a Hilbert space setting. We also show an integral representation formula of quadratic variation in a Banach space with the Radon–Nikodym property (RNP). Both trace and integral representation formulae of quadratic variations are interpreted as a pathwise version of classical results (see, for example, Metivier [36]) in stochastic calculus in Hilbert spaces. Next, we show C^1-type transformation formulae for quadratic variations. Roughly speaking, these results claim that the path \( t \mapsto f(t, X_t) \) has quadratic variation whenever \( X \) has quadratic variation and \( f \) has C^1-smoothness in \( x \) and finite variation in \( t \). These can be interpreted as a method of generating new càdlàg paths having quadratic variation from the given path \( X \). Furthermore, we investigate quadratic variations of a path \( \int_0^1 Df(X_s)\,dX_s \) defined by the Itô–Föllmer integral. As an application, we obtain a rough-FV decomposition formula for a path of the form \( f(X) \).

Stochastic integration theories in infinite dimensions have been investigated in many studies (see, e.g., Kunita [33], Metivier [35], Pellaumail [44], Yor [54], Gravereaux and Pellaumail [23], Metivier and Pisette [38], Meyer [40], Metivier and Pellaumail [37], Gyöngy and Krylov [25, 26], Gyöngy [24], Metivier [36], Pratelli [46], Brooks and Dinculeanu [4], Dinculeanu [19], and van Neerven, Veraar, and Weis [42, 43]). Our method can be interpreted as a deterministic analogy of some of these classical theories. Note that Di Girolami, Fabbri, and Russo [13] and Di Girolami and Russo [16] treat quadratic covariations and stochastic integrations in Banach spaces with a different approach, called stochastic calculus via regularization. These are considered to be regularization counterparts of Föllmer’s discretization approach. See the survey by Russo and Vallois [47] for details of stochastic calculus via regularization. We also refer to Čoupek and Garrido-Atienza [10] for linear equations in a Hilbert space driven by irregular scalar noise, which works within the framework of Föllmer’s calculus.

Before describing the results of this paper, we give a brief outline of the Itô–Föllmer calculus in Banach spaces developed in our previous study [30]. Let \( (\pi_n)_{n \in \mathbb{N}} \) be a sequence of partitions of \( \mathbb{R}_{\geq 0} \) and let \( X: \mathbb{R}_{\geq 0} \to E \) be a càdlàg path in a Banach space \( E \). Moreover, let \( B: E \times E \to F \) be a bounded bilinear map into another Banach space \( F \). We say that \( X \) has strong (resp. weak) \( B \)-quadratic variation along \( (\pi_n) \) if there is a càdlàg path \( Q_B(X, X): \mathbb{R}_{\geq 0} \to F \) of finite variation satisfying the following conditions:

(i) the sequence \( \sum_{[r,s] \in \pi_n} B(X_{s\wedge t} - X_{r\wedge t}, X_{s\wedge t} - X_{r\wedge t}) \) converges to \( Q_B(X, X)_t \) in the norm (resp. weak) topology for all \( t \geq 0 \);

(ii) the equation \( \Delta[X, Y]_t = B(\Delta X_t, \Delta X_t) \) holds for all \( t \geq 0 \).
A typical example of the bilinear map $B$ in the definition above is the canonical bilinear map into a tensor product of Banach spaces. Let $\alpha$ be a reasonable crossnorm on the algebraic tensor product $E \otimes E$, and let $E \hat{\otimes}_\alpha E$ be the completion of $E \otimes E$ with respect to $\alpha$. Then the strong (resp. weak) quadratic variation $Q_\otimes(X, X)$, where $\otimes: E \times E \to E \hat{\otimes}_\alpha E$ is the canonical bilinear map, is called the strong (resp. weak) $\alpha$-tensor quadratic variation of $X$ and is denoted by $\langle X, X \rangle$. Next, we define the scalar quadratic variation $Q(X)$ of $X$ along $(\pi_n)$ as a nonnegative increasing path satisfying the following conditions:

(i) the sequence $\sum_{[s, r] \in \tau_n} \|X_{s \wedge t} - X_{r \wedge t}\|^2$ converges to $Q(X)_t$ for all $t \geq 0$;

(ii) the equation $\Delta Q(X)_t = \|\Delta X_t\|^2$ holds for all $t \geq 0$.

Note that the scalar quadratic variation coincides with the quadratic variation $Q(\cdot, \cdot)$ if $E$ is a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Moreover, we say that $X$ has finite $2$-variation along $(\pi_n)$ if

$$V((\pi_n); X)_t = \sup_{n \in \mathbb{N}} \sum_{[s, r] \in \tau_n} \|X_{s \wedge t} - X_{r \wedge t}\|^2 < \infty$$

for all $t \geq 0$. Now suppose that $(\pi_n)$ satisfies certain nice conditions associated with the path $X$, namely condition $(\text{C})$ and left-approximation (see Definition 2.1), and $X$ has strong or weak $\alpha$-tensor quadratic variation and finite $2$-variation along $(\pi_n)$. If a function $f: E \to G$ into a Banach space has $C^2$ smoothness in an appropriate sense (see Corollary 2.5 for the precise definition), then the composite function $f \circ X$ satisfies

$$f(X_t) - f(X_0) = \int_0^t Df(X_{s-})dX_s + \frac{1}{2} \int_0^t D^2f(X_{s-})d\alpha [X, X]_s + \sum_{0 \leq s \leq t} \{\Delta f(X_s) - Df(X_{s-})\Delta X_s\}.$$

The first integral on the right-hand side is defined by

$$\lim_{n \to \infty} \sum_{[u, v] \in \tau_n} Df(X_u)(X_{v \wedge t} - X_{u \wedge t}) = \int_0^t Df(X_{s-})dX_s,$$

where the limit exists in the strong or weak topology corresponding to the convergence of the quadratic variation. We call $\int_0^t Df(X_{s-})dX_s$ the Itô–Föllmer integral along $(\pi_n)$.

The remainder of this paper is organized as follows. In Section 2, we present the basic notation and terminology used throughout the paper. The remaining sections are divided into two parts. In the first part, which consists of Sections 3 and 4, we study relations between various quadratic variations introduced in our previous paper [30]. Sections 5 and 6 form the second part, where we focus on the formulation and the proof of our $C^1$-transformation formula and its consequences.

In Section 3, we provide some results related to linear transformations of paths and quadratic variations. It turns out that the projective tensor quadratic variation is the strongest notion in the sense of Proposition 3.2. We also show the trace representation formula of scalar quadratic variations in Hilbert spaces (Theorem 3.6), which claims that the scalar quadratic variation of a Hilbert space-valued path is the trace of the projective tensor quadratic variation. In Section 4, we show integral representation formulae of quadratic variations with respect to scalar quadratic variations. Namely, if $X$ has scalar quadratic variation and weak $B$-quadratic variation $Q_B$ in a Banach space $G$ with the RNP, there is a $G$-valued locally $Q(X)$-integrable function $q_B$ such that

$$Q_B(X, X)_t = \int_{[0, t]} q_B(s)dQ(X)_s, \quad \forall t \geq 0$$

3
For this purpose, we also show the absolute continuity of $Q_B(X, X)$ with respect to the scalar quadratic variation (Proposition 4.1).

The $C^1$-transformation formulae of quadratic variations are given in Section 5. Section 5.1 is the preliminary part of this section. To formulate our $C^1$-transformation formulae, we first introduce some concepts such as the variation of a family of càdlàg paths, conditions on a sequence of partitions (Condition (UC)), and spaces of functions with certain continuity or differentiability. We also provide auxiliary results regarding a càdlàg path in such a function space. The main results of the second part, $C^1$-transformation formulae for quadratic variations, are given in Section 5.2 (Theorem 5.9 and Corollaries 5.13 and 5.14). The statement of Corollary 5.13 is roughly as follows: Let $E$ and $F$ be Banach spaces and $\alpha$ be a uniform crossnorm. We consider two càdlàg paths $X: \mathbb{R}_{\geq 0} \to E$ and $f: \mathbb{R}_{\geq 0} \to C^1_{\alpha}(E, F)$, where $C^1_{\alpha}(E, F)$ is the space of Gâteaux differentiable functions from $E$ to $F$ satisfying additional conditions (see Definition 5.4). Assume that the family $(f(\cdot, x); x \in K)$ has uniformly finite variation for each compact set $K \subset E$ and that $(\pi_n)$ satisfies (UC) and is a left-approximation sequence for $X$ and $f$. If $X$ has strong (resp. weak) $\alpha$-tensor quadratic variation and finite 2-variation along a sequence of partitions $(\pi_n)$, then $f(\cdot, X)$ has the strong (resp. weak) $\alpha$-tensor quadratic variation given by

$$
\alpha [f(\cdot, X), f(\cdot, X)]_t = \int_{[0, t]} D_x f(s-, X_s-) \otimes^2 d\alpha [X, X]_s + \sum_{0 \leq s \leq t} (\Delta f(s, X_s)) \otimes^2.
$$

As a consequence of this and the Itô formula, we see that the path $Y_t = \int_0^t D f(A_s-, X_s-)dX_s$ has $\alpha$-tensor quadratic variation represented as

$$
\alpha [Y, Y]_t = \int_{[0, t]} D f(A_s-, X_s-) \otimes^2 d\alpha [X, X]_s
$$

(see Theorem 6.1), where $A$ is a càdlàg path of finite variation and $f$ is a function with a certain $C^{1,2}$-smoothness. We can directly deduce from this theorem that the càdlàg path $f(A, X)$ has the decomposition $f(X) = Y + C + D$, where $Y$ is the path defined above, $C$ is a continuous path of finite variation, and $D$ is a purely discontinuous path of finite variation (Corollary 6.2).

Some auxiliary results are provided in the appendices. Appendix A gives remarks on vector integration and the RNP. In Appendix B, we discuss the problem of controlling the oscillation of a family of càdlàg paths by a sequence of partitions. This discussion is related to Condition (UC) in Section 5.

2 Settings

The aim of this section is to present basic concepts in the Itô–Föllmer calculus in Banach spaces introduced in the preceding paper [30].

First, we introduce the basic notation and terminology used throughout this paper. Let $\mathbb{N} = \{0, 1, 2, \ldots \}$ and let $\mathbb{R}$ be the set of real numbers. In this article, the scalar field of a linear space is always assumed to be $\mathbb{R}$. Given two Banach spaces $E$ and $F$, let $\mathcal{L}(E, F)$ denote the space of bounded linear maps from $E$ to $F$. If, in addition, $G$ is another Banach space, we define $\mathcal{L}^{(2)}(E, F; G)$ as the space of bounded bilinear maps from $E \times F$ to $G$. As usual, we regard $\mathcal{L}(E, F)$ and $\mathcal{L}^{(2)}(E, F; G)$ as Banach spaces endowed with the following respective norms:

$$
\|A\| = \sup_{x \in E \setminus \{0\}} \frac{\|Ax\|}{\|x\|}, \quad \|B\| = \sup_{x \in E \setminus \{0\}, y \in F \setminus \{0\}} \frac{\|B(x, y)\|}{\|x\|\|y\|}.
$$
Let $[0, \infty[ = \mathbb{R}_{\geq 0} = \{ r \in \mathbb{R} \mid r \geq 0 \}$ and let $E$ be a Banach space. A càdlàg path in $E$ is a function $X: \mathbb{R}_{\geq 0} \to E$ that is right-continuous at every $t \geq 0$ and has a left limit at every $t > 0$. We also use the term right-regular to describe the same property. The symbol $D(\mathbb{R}_{\geq 0}, E)$ or $D([0, \infty[, E)$ denotes the set of all càdlàg paths in $E$. If $X$ is an $E$-valued càdlàg path, we define

$$X(t) = \lim_{s \to t^-} X(s), \quad \Delta X(t) = X(t) - X(t-).$$

We also use symbols $X_t$, $X_{t-}$, and $\Delta X_t$ to indicate the values $X(t)$, $X(t-)$, and $\Delta X(t)$, respectively. Next, let

$$D(X) = \{ t \in \mathbb{R}_{\geq 0} \mid \| \Delta X_t \| \neq 0 \}$$

$$D_\varepsilon(X) = \{ t \in \mathbb{R}_{\geq 0} \mid \| \Delta X_t \| > \varepsilon \}$$

$$D^\varepsilon(X) = D(X) \setminus D_\varepsilon(X) = \{ t \in \mathbb{R}_{\geq 0} \mid 0 < \| \Delta X_t \| \leq \varepsilon \}.$$

Given a set $D \subset [0, \infty[$ without accumulation points and a càdlàg path $X$, we define

$$J_D(X)_t = J(D; X)_t = \sum_{0 < s \leq t} \Delta X_t 1_D(s).$$

Then $J_D(X)$ is a càdlàg path of finite variation. For abbreviation, we often write $J_\varepsilon(X)$ instead of $J(D_\varepsilon(X); X)$.

Throughout this paper, the term partition of $\mathbb{R}_{\geq 0}$ always means a set of half-open intervals of the form $\pi = \{ [t_i, t_{i+1}] ; i \in \mathbb{N} \}$ that satisfies $0 = t_0 < t_1 < \cdots \to \infty$. The set of all partitions of $\mathbb{R}_{\geq 0}$ is denoted by $\text{Par}(\mathbb{R}_{\geq 0})$ or $\text{Par}([0, \infty[)$. Similarly, $\text{Par}([a, b])$ indicates the set of all partitions of the form $\pi = \{ [t_i, t_{i+1}] ; 0 \leq i \leq n - 1 \}$ with $a = t_0 < t_1 < \cdots < t_n = b$. If $\pi = \{ [t_i, t_{i+1}] ; i \in I \}$ is a partition of $\mathbb{R}_{\geq 0}$ or a compact interval, we set $\pi^b = \{ t_i ; i \in I \}$. In other words, $\pi^b$ is the set of all endpoints of intervals that belongs to $\pi$. As usual, define the mesh of a partition $\pi$ by $|\pi| = \sup_{r,s \in \pi} |r - s|$.

Let $\mathcal{F}$ be the semiring of subsets of $\mathbb{R}_{\geq 0}$ consisting of all intervals of the form $[a, b)$ $(a \leq b)$ and $\{0\}$. The difference of a path $X: \mathbb{R}_{\geq 0} \to E$, denoted by $\delta X$, is a function from $\mathcal{F}$ into $E$ defined by $\delta X([r, s]) = X(s) - X(r)$ for $0 \leq r \leq s$ and by $\delta X(\{0\}) = X_0$. In particular, we have $\delta X(\emptyset) = 0$. For each $t \in \mathbb{R}_{\geq 0}$, we also define $\delta X_t: \mathcal{F} \to E$ by $\delta X_t(I) = \delta X(I \cap [0, t])$. If $I = [r, s] \in \mathcal{F}$, then $\delta X_t(I) = X(s - t) - X(r - t)$. Next, consider a bilinear map $B: F \times E \to G$ between Banach spaces and another path $Y: \mathbb{R}_{\geq 0} \to F$. Then we define functions $B(Y, \delta X_t)$, $B(\delta Y_t, X)$, and $B(\delta Y_t, \delta X_t)$, from $\mathcal{F}$ to $G$ by the formulae

$$B(Y, \delta X_t)([r, s]) = B(Y_r, \delta X_t([r, s])),$$

$$B(\delta Y_t, X)([r, s]) = B(\delta Y_t([r, s]), X_r),$$

$$B(\delta Y_t, \delta X_t)([r, s]) = B(\delta Y_t([r, s]), \delta X_t([r, s]))$$

for $I = [r, s] \in \mathcal{F}$ and

$$B(Y, \delta X_t)(\{0\}) = B(\delta Y_t, X)(\{0\}) = B(\delta Y_t, \delta X_t)(\{0\}) = B(Y_0, X_0).$$

By this notation, the left-side Riemannian sum has a relatively shorter expression

$$\sum_{[r,s] \in \pi} B(Y_t, X_{t\wedge s} - X_{t\wedge r}) = \sum_{I \in \pi} B(Y, \delta X_t)(I).$$
Let $X$ be a càdlàg path in a Banach space $E$. We say that $X$ has finite variation if

$$V(X; [a, b]) = \sup_{\pi \in \text{Par}([a, b])} \sum_{I \in \pi} \|\delta X(I)\|_E < \infty$$

for all compact intervals $[a, b] \subset \mathbb{R}_2$. The set of all $E$-valued càdlàg paths is denoted by $FV(\mathbb{R}_{\geq 0}, E)$. For each $t \geq 0$, we define $V(X)_t = V(X; [0, t])$ and call $t \mapsto V(X)_t$ the total variation path of $X$. If $X$ has finite variation, then $V(X)$ is an increasing càdlàg path with $V(X)_0 = 0$.

**Definition 2.1.** Let $E, F, G$ be Banach spaces and let $B \in L^{(2)}(E, F; G)$. For a path $(X, Y) \in D(\mathbb{R}_{\geq 0}, E \times F)$ and a partition $\pi$ of $\mathbb{R}_{\geq 0}$, define the discrete quadratic covariation as

$$Q_B^n(X, Y)_t = \sum_{I \in \pi} B(\delta X_t, \delta Y_t)(I), \quad t \geq 0.$$ 

Now let $(\pi_n)$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$. We say that $(X, Y)$ has **strong $B$-quadratic covariation along** $(\pi_n)_{n \in \mathbb{N}}$ if there exists a $G$-valued càdlàg path $Q_B(X, Y)$ of finite variation that satisfies the following conditions:

(i) the sequence $(Q_B^n(X, Y))_{n \in \mathbb{N}}$ converges to $Q(X, Y)$ pointwise in the norm topology;

(ii) the jump of $Q_B(X, Y)$ satisfies $\Delta Q_B(X, Y)_t = B(\Delta X_t, \Delta Y_t)$ for all $t \geq 0$.

Then $Q_B(X, Y)$ is called the strong $B$-quadratic covariation of $(X, Y)$. If the convergence in (i) is replaced with weak convergence, we say that $X$ has the weak $B$-quadratic covariation $Q_B(X, Y)$.

If $E = F$ and $X = Y$, we simply call $Q_B(X, X)$ the strong or weak $B$-quadratic variation of $X$.

Let $\alpha$ be a reasonable crossnorm on $E \otimes F$ and $E \hat{\otimes}_\alpha F$ be the completion of normed space $(E \otimes F, \alpha)$. See Diestel and Uhl [18, pp. 221–222] or Ryan [48, p. 127] for the definition. For the strong (resp. weak) quadratic covariation with respect to the canonical bilinear map $\gamma: E \times F \to E \hat{\otimes}_\alpha F$, we use the symbol $\gamma[X, Y]$ and call it the strong (resp. weak) $\alpha$-tensor quadratic covariation. If $E = F$ and $X = Y$, the path $\gamma[X, X]$ is called the (strong/weak) $\alpha$-tensor quadratic variation. Recall that there is the greatest crossnorm $\gamma$, which is usually called the projective norm, on the tensor product of any two Banach spaces. For this special crossnorm, we often omit the symbol $\gamma$ and write $E \otimes E = E \hat{\otimes}_\gamma E$ and $[X, Y] = \gamma[X, Y]$. Then we call $[X, Y]: \mathbb{R}_{\geq 0} \to E \hat{\otimes}_\gamma E$ the projective tensor quadratic covariation.

Now we introduce a different type of quadratic variation, namely, scalar quadratic variation. First, define the discrete scalar quadratic variation of $X: \mathbb{R}_{\geq 0} \to E$ along a partition $\pi$ by

$$Q^\pi(X)_t = \sum_{I \in \pi} \|\delta X_t(I)\|^2.$$ 

**Definition 2.2.** Let $X$ be a càdlàg path in a Banach space $E$ and $(\pi_n)$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$.

(i) The path $X$ has **finite 2-variation along** $(\pi_n)$ if

$$V^{(2)}(X; (\pi_n))_t := \sup_{n \in \mathbb{N}} Q^\pi_n(X)_t < \infty$$

for all $t \in \mathbb{R}_{\geq 0}$.

(ii) The path $X: \mathbb{R}_{\geq 0} \to E$ has **scalar quadratic variation** if there exists a real-valued càdlàg increasing path $Q(X)$ satisfying the following conditions:
(a) the sequence $Q^\pi_n(X)$ converges to $Q(X)$ pointwise;
(b) the equality $\Delta Q(X)_t = \|\Delta X_t\|^2$ holds for all $t \geq 0$.

Then we call $Q(X)$ the scalar quadratic variation of $X$ along $(\pi_n)$.

Condition (i) is much weaker than asserting that $X$ has finite 2-variation in the usual sense, that is,

$$
\sup_{\pi \in \operatorname{Par}(a,b)} \sum_{I \in \pi} \|\delta X(I)\|^2 < \infty
$$

for all compact intervals $[a,b] \subset \mathbb{R}_{\geq 0}$. If $X$ has scalar quadratic variation along $(\pi_n)$, then it has finite 2-variation along the same sequence $(\pi_n)$. Note that if $E$ is a Hilbert space, the scalar quadratic variation coincides with the quadratic variation $Q(\cdot, \cdot)(X,X)$, where $(\cdot, \cdot): E \times E \to \mathbb{R}$ is the inner product of $E$.

A typical example of tensor and scalar quadratic variations is the tensor and scalar quadratic variations of a semimartingale in a Hilbert space. See, for example, Metivier and Pellaumail [37] and Metivier [36]. A Banach space-valued path with finite variation has tensor and scalar quadratic variations along any sequence of partitions satisfying $|\pi_n| \to 0$. It is also true for a partition satisfying a slightly general condition in Definition 2.3 below. See Hirai [30, Section 5] for a proof. Furthermore, for $\alpha > 1/2$, an $\alpha$-Hölder continuous path in a Banach space has tensor and scalar quadratic variations, which are identically zero, along a sequence with the condition $|\pi_n| \to 0$.

Next, we introduce conditions on a sequence of partitions that is required for some important theorems, including the Itô formula. Let $\pi \in \operatorname{Par}(\mathbb{R}_{\geq 0})$ and $t \in ]0, \infty[$. The symbol $\pi(i)$ denotes the element of $\pi$ that contains $t$. By definition, there exists only one such interval. Moreover, we define $\overline{\pi}(i) = \sup \pi(i)$ and $\underline{\pi}(i) = \inf \pi(i)$. Then we have $\pi(i) = [\underline{\pi}(i), \overline{\pi}(i)]$, and

$$
\delta X_t(\pi(s)) = X(\overline{\pi}(s) \land t) - X(\underline{\pi}(s) \land t), \quad \delta X_t(\pi(s)) = X(\overline{\pi}(s)) - X(\underline{\pi}(s))
$$

hold for all $s$ and $t$ in $]0, \infty[$.

Let $f: S \to E$ be a function into a Banach space and set

$$
\omega(f, A) = \sup_{x, y \in A} \|f(x) - f(y)\|_E
$$

for each subset $A$ of $S$. Using this notation, we define oscillations of $X \in D(\mathbb{R}_{\geq 0}, E)$ along $\pi \in \operatorname{Par}(\mathbb{R}_{\geq 0})$ by

$$
O_t^\pi(X, \pi) = \sup_{[r, s] \in \pi} \omega(X, [r, s] \cap [0, t]), \quad O_t^\pi(X, \pi) = \sup_{[r, s] \in \pi} \omega(X, [r, s] \cap [0, t]).
$$

By the right continuity, we have $O_t^\pi(X, \pi) \leq O_t^\pi(X, \pi)$. Two oscillations $O^-$ and $O^+$ coincide if $X$ is continuous.

In the Itô–Föllmer calculus, one often requires the sequence $(\pi_n)$ to satisfy either $\lim_{n \to \infty} |\pi_n| = 0$ or $O_t^\pi(X, \pi_n) \to 0$ for all $t \geq 0$. When considering càdlàg paths, neither includes the other. In this paper, we work with another assumption given in Definition 2.3 which unifies these two approaches.

**Definition 2.3** (Hirai [30]). Let $X$ be a càdlàg path in a Banach space $E$ and let $(\pi_n)_{n \in \mathbb{N}}$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$.

(i) The sequence $(\pi_n)$ satisfies Condition (C) for $X$ if it satisfies the following three conditions:
(C1) Let $t \in \mathbb{R}_{\geq 0}$ and $\epsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that $I \cap [0, t] \cap D_{x}(X)$ has at most one element for all $n \geq N$ and all $I \in \pi_{n}$.

(C2) The sequence $(\delta X_{t}(\pi_{n}(s)))_{n \in \mathbb{N}}$ converges to $\Delta X_{t}$ for all $s \in D(X)$ and $t \in [s, \infty[$.

(C3) For all $t \in \mathbb{R}_{\geq 0}$,
\[
\lim_{n} \lim_{n \to \infty} O_{t}^{t}(X - J_{x}(X); \pi_{n}) = 0.
\]

We say that $(\pi_{n})$ satisfies (C) for $X$ on $[0, T]$ if $(\pi_{n})$ satisfies (C) for the stopped path $X(\cdot \wedge T)$.

(ii) The sequence $(\pi_{n})$ approximates $X: \mathbb{R}_{\geq 0} \to E$ from the left if $\lim_{n \to \infty} X(\pi_{n}(t)) = X(t-)$ holds for all $t > 0$. Then we call $(\pi_{n})$ a left approximation sequence for $X$.

In the framework of the Itô–Föllmer calculus in Banach spaces described above, we have a $C^{1,2}$-type Itô formula. We first quote a generalized form of the theorem (Theorem 2.4), which treats general bilinear quadratic variations. In what follows, let $E$, $E_{1}$, $F$, and $G$ be Banach spaces and $B: E \times E \to E_{1}$ be a bounded bilinear map. The symbol $C_{\mathcal{K}}(F \times E, G')$ denotes the set of all functions $f: F \times E \to G'$ into a Banach space $G'$ such that $f|_{K}$ is continuous whenever $K \subset E \times E$ is compact.

As we can see in Corollary 2.5, a typical example of $B$ is the canonical bilinear map $\otimes: E \times E \to E\hat{\otimes}_{\alpha}E$ for some reasonable crossnorm $\alpha$. A more elementary case is given by $f = F \circ P$ with $P: E \to V$ being a finite-dimensional projection and $F$ is a $C^{2}$ function in the usual sense.

**Theorem 2.4** (Hirai [30]). Let $(A, X) \in D(\mathbb{R}_{\geq 0}, F \times E)$ and $(\pi_{n})$ be a sequence of partitions that satisfies Condition (C) for $(A, X)$ and approximate it from the left. Suppose that $X \in D(\mathbb{R}_{\geq 0}, E)$ has weak $B$-quadratic variation and finite $2$-variation along $(\pi_{n})$ and that $A \in D(\mathbb{R}_{\geq 0}, F)$ has finite variation. Consider a function $f: F \times E \to G$ that is twice Gâteaux differentiable in the second variable $x$ and once in the first variable $a$. In addition, suppose that $D_{a}f \in C_{\mathcal{K}}(F \times E, \mathcal{L}(F, G))$, $D_{x}f \in C_{\mathcal{K}}(F \times E, \mathcal{L}(E, G))$, and there exists $D^{2}_{B}f \in C_{\mathcal{K}}(F \times E, \mathcal{L}(E_{1}, G))$ such that $D^{2}_{B}f(a, x) = D^{2}_{B}f(a, x) \circ B$ for all $(a, x) \in F \times E$. Then the path $t \mapsto f(A_{t}, X_{t})$ satisfies

\[(2.1) \quad f(A_{t}, X_{t}) - f(A_{0}, X_{0}) = \int_{0}^{t} D_{a}f(A_{s-}, X_{s-})dA_{s}^{\alpha} + \int_{0}^{t} D_{x}f(A_{s-}, X_{s-})dX_{s}
\]
\[+ \frac{1}{2} \int_{0}^{t} D^{2}_{B}f(A_{s-}, X_{s-})dQ_{B}(X, X) + \sum_{0 \leq \Delta \leq t} \{\Delta f(A_{s}, X_{s}) - D_{x}f(A_{s}, X_{s})\Delta X_{s}\}
\]

for all $t \geq 0$, where the first and the third integrals on the right-hand side are usual vector Stieltjes integrals on $[0, t]$ and the second one is the Itô–Föllmer integral defined as the weak limit

\[(2.2) \quad \int_{0}^{t} D_{x}f(A_{s-}, X_{s-})dX_{s} = \lim_{n \to \infty} \sum_{I \in \pi_{n}} D_{x}f(A, X)\delta X_{I}(I).
\]

Moreover, if the quadratic variation $Q_{B}$ exits in the strong sense, the convergence of (2.2) holds in the norm topology of $G$.

See the first paper in this series [30] for a proof. Note that in [30], we assumed the continuity of derivatives of $f$ on the whole space $F \times E$. We can indeed weaken the continuity of derivatives as in Theorem 2.4 without modifying the proof.

Applying Theorem 2.4 to $E_{1} = E\hat{\otimes}_{\alpha}E$ and $B = \otimes: E \times E \to E\hat{\otimes}_{\alpha}E$ for a reasonable crossnorm $\alpha$, we obtain the following corollary.
Corollary 2.5. Let \( X \in D(\mathbb{R}_{\geq 0}, E) \) be a path with weak \( \alpha \)-tensor quadratic variation and finite \( 2 \)-variation along \((\pi_n)\), and let \( A \in FV(\mathbb{R}_{\geq 0}, F) \). Assume that a sequence of partitions \((\pi_n)\) satisfies \((C)\) and is a left-approximation sequence for \((A, X)\). Let \( f : F \times E \to G \) be a function that is once Gâteaux differentiable in \( a \) and twice Gâteaux differentiable in \( x \). Suppose moreover that \( D_\alpha f \in C_X(F \times E, \mathcal{L}(F, G)) \), \( D_x f \in C_X(F \times E, \mathcal{L}(E, G)) \), and \( D_x^2 f : F \times E \to \mathcal{L}(E \otimes \alpha E, G) \) extends to a function in \( C_X(F \times E, \mathcal{L}(E \otimes \alpha E, G)) \). Then \( f(A, X) \) satisfies the following Itô formula:

\[
\begin{align*}
 f(A_t, X_t) - f(A_0, X_0) &= \int_0^t D_\alpha f(A_s, X_s) dA_s^\alpha + \int_0^t D_x f(A_s, X_s) dX_s \\
 & \quad + \frac{1}{2} \int_0^t D_x^2 f(A_s, X_s) d\tau^x [X, X]_s + \sum_{0 < \tau < t} \{ \Delta f(A_s, X_s) - D_x f(A_s, X_s) \Delta X_s \},
\end{align*}
\]

where the second integral on the right-hand side is the Itô–Föllmer defined as the weak limit. Moreover, if \( \alpha^x [X, X] \) is the strong \( \alpha \)-tensor quadratic variation, then the Itô–Föllmer integral converges in the norm topology.

If \( \alpha \) is the projective norm, then the assumption on the second derivative can be simply written as \( D_x^2 f \in C_X(F \times E, \mathcal{L}(E \otimes \alpha E, G)) \). This is because \( D_x^2 f(\alpha, x) \) is generally an element of \( \mathcal{L}(E \otimes \alpha E, G) \) but not of \( \mathcal{L}(E \otimes \alpha E, G) \).

The following lemma, which is essential for the proof of Theorem 2.4 will also be used in Section 5 to prove one of the main theorems of this paper.

Lemma 2.6. Let \((\pi_n)\) be a sequence in partitions and let \( X \in D(\mathbb{R}_{\geq 0}, E) \) be a path with strong or weak \( B \)-quadratic variation and finite \( 2 \)-variation along \((\pi_n)\). Assume that \((\pi_n)\) satisfies Condition \((C)\) for \( X \) and approximates \( a \in D(\mathbb{R}_{\geq 0}, \mathcal{L}(E_1, G)) \) from the left. Then for all \( t \in \mathbb{R}_{\geq 0} \),

\[
\lim_{n \to \infty} \sum_{I \in \pi_n} \xi_I \delta X_I(I) = \int_{[0,t]} \xi_s \, dQ_B(X, X)_s,
\]

holds in the respective topology.

3 Linear transformations

In this section, we give some simple results related to linear transformations of paths and quadratic variations.

Proposition 3.1. Let \( E, F, \) and \( G \) be Banach spaces and let \( B \in \mathcal{L}(E, F) \). Suppose that \((X, Y) \in D(\mathbb{R}_{\geq 0}, E \times F) \) has strong (resp. weak) \( B \)-quadratic covariation along a sequence of partitions \((\pi_n)\). Then for every Banach space \( G' \) and \( T \in \mathcal{L}(G, G') \), the pair \((X, Y)\) has strong (resp. weak) \( T \circ B \)-quadratic covariation along \((\pi_n)\), given by \( T \circ Q_B(X, Y) = T \circ Q_B(X, Y) \).

Proof. First, note that \( V(T \circ Q_B(X, Y); I) \leq \|T\|V(Q_B(X, Y); I) \) holds for all of compact interval \( I \), and therefore \( T \circ Q_B(X, Y) \) is a càdlàg path of finite variation. By the linearity and the continuity of \( T \), we see that

\[
\sum_{I \in \pi_n} T(B(\delta X_I, \delta Y_I))(I) = T \left( \sum_{I \in \pi_n} B(\delta X_I, \delta Y_I)(I) \right) \longrightarrow_{n \to \infty} T(Q_B(X, Y)_I)
\]

holds for all \( t \geq 0 \) in the suitable topology. Moreover, we have

\[
\Delta T(Q_B(X, Y)_t) = T(\Delta Q_B(X, Y)_t) = T(B(\Delta X_t, \Delta Y_t)).
\]
Hence, $T \circ Q_B(X,Y)$ is the $T \circ B$-quadratic covariation of $(X,Y)$. 

As a consequence of Proposition 3.1, we can reveal the relation between projective tensor quadratic variation and other quadratic variations with respect to bilinear maps. Now recall that there is an isometric isomorphism $\mathcal{L}^{(2)}(E,F;G) \cong \mathcal{L}(E \hat{\otimes} F, G)$.

**Proposition 3.2.** Let $(\pi_n)$ be a sequence of partitions and let $(X,Y) \in \mathcal{D}(\mathbb{R}_{\geq 0}, E \times F)$. Then the following conditions are equivalent:

(i) $(X,Y)$ has strong (resp. weak) projective tensor quadratic covariation.

(ii) $(X,Y)$ has strong (resp. weak) $B$-quadratic covariation for every $B \in \mathcal{L}^{(2)}(E,F;G)$.

If these conditions are satisfied, then $T_B \circ [X,Y] = Q_B(X,Y)$ holds for every $B \in \mathcal{L}(E,F;G)$, where $T_B$ is the unique bounded linear map that commutes the following diagram.

\[
\begin{array}{ccc}
E \times F & \xrightarrow{B} & G \\
\downarrow \circ & & \downarrow T_B \\
E \hat{\otimes} F & \xrightarrow{} & 
\end{array}
\]

**Proof.** First, assume that $X$ and $Y$ have tensor quadratic covariation along $(\pi_n)$. Then, by Proposition 3.1, $X$ and $Y$ have quadratic covariation with respect to $B = T_B \circ \circ$, given by $T_B \circ [X,Y] = Q_B(X,Y)$. Conversely, if condition (ii) holds, we get condition (i) by applying (ii) to the bounded bilinear map $\circ : E \times F \to E \hat{\otimes} F$. □

**Corollary 3.3.** Suppose that $(X,Y)$ has the weak projective tensor quadratic covariation along $(\pi_n)$. Then for any Banach space $G$, the map $B \mapsto Q_B(X,Y)$ from $\mathcal{L}^{(2)}(E,F;G)$ to $FV(\mathbb{R}_{\geq 0}, G)$ is continuous with respect to the topology of pointwise convergence in $FV(\mathbb{R}_{\geq 0}, G)$.

**Proof.** Recall that the canonical isomorphism $\mathcal{L}(E,F;G) \to \mathcal{L}(E \hat{\otimes} F, G)$ is isometric with respect to the operator norm. Combining this fact with Proposition 3.2, we see that

\[\|Q_B(X,Y)_t - Q_{B'}(X,Y)_t\| \leq \|B - B'\|_{\mathcal{L}^{(2)}(E,F,G)}\|\langle X,Y\rangle_t\|_{E \hat{\otimes} F}.\]

This shows the desired continuity. □

**Proposition 3.4.** Let $E_i, F_i$ ($i = 1, 2$), and $G$ be Banach spaces. We assume that $T_i \in \mathcal{L}(E_i, F_i)$ ($i \in \{1, 2\}$), $B \in \mathcal{L}^{(2)}(E_1, E_2; G)$, and $B' \in \mathcal{L}^{(2)}(F_1, F_2; G)$ satisfy $B' \circ (T_1 \times T_2) = B$.

\[
\begin{array}{ccc}
E_1 \times E_2 & \xrightarrow{T_1 \times T_2} & F_1 \times F_2 \\
\downarrow B & & \downarrow B' \\
G & & 
\end{array}
\]

If $(X_1, X_2) \in \mathcal{D}(\mathbb{R}_{\geq 0}, E_1 \times E_2)$ has strong (resp. weak) $B$-quadratic covariation, $(T_1 \circ X_1, T_2 \circ X_2)$ has the strong (resp. weak) $B'$-quadratic covariation, given by $Q_{B'}(T_1 \circ X_1, T_2 \circ X_2) = Q_B(X_1, X_2)$.

**Proof.** By a direct calculation, we get

\[B'(\delta(T_1 \circ X_1)_t, \delta(T_2 \circ X_2)_t)(I) = B(\delta(X_1)_t, \delta(X_2)_t)(I).\]
Therefore

\[
\lim_{n\to\infty} \sum_{[r,x] \in \pi_n} B'(\delta(T_1 \circ X_1)_r, \delta(T_2 \circ X_2)_r)(I) = Q_B(X_1, X_2)_t
\]

holds in the suitable topology. Moreover,

\[
\Delta Q_B(X_1, X_2)_r = B(\Delta(X_1)_r, \Delta(X_2)_r) = B'((\Delta T_1(X_1))_r, (\Delta T_2(X_2))_r)
\]

for all \( t \in \mathbb{R}_{\geq 0} \). Hence, \( Q_B \) is the \( B' \)-quadratic covariation of \( T_1 \circ X_1 \) and \( X_2 \circ X_2 \). \( \square \)

It follows from above results that every \((X, Y)\) with tensor quadratic variation has ‘cylindrical’ quadratic covariation.

**Corollary 3.5.** Let \( E \) and \( F \) be Banach spaces and let \( \alpha \) be a reasonable crossnorm on \( E \otimes F \). Suppose that \((X, Y) \in D(\mathbb{R}_{\geq 0}, E \times F)\) has weak \( \alpha \)-tensor quadratic variation along a sequence of partitions \((\pi_n)\). Then, for each \((x^*, y^*) \in E^* \times F^*\), real-valued paths \( x^*X \) and \( y^*Y \) have the quadratic covariation \([x^*X, y^*Y] = Q_{x^* \otimes y^*}(X, Y) = x^* \otimes y^*([X, Y])\) along \((\pi_n)\). Here, \( x^* \otimes y^*: X \otimes F \to \mathbb{R} \) is the bounded linear form defined to be \((x^* \otimes y^*)(y) = x^*(y^*)\) for all \((x, y) \in X \times Y\).

**Proof.** The equations \([x^*X, y^*Y] = Q_{x^* \otimes y^*}(X, Y)\) and \(Q_{x^* \otimes y^*}(X, Y) = x^* \otimes y^*([X, Y])\) follow from Propostions 3.4 and 3.1 respectively. \( \square \)

To derive the trace representation formula for the scalar quadratic variations, let us recall the definition of the trace operator. Let \((H, \langle \cdot, \cdot \rangle_H)\) be a Hilbert space. The trace operator \( \text{tr}_H: H \otimes H \to \mathbb{R} \) is a unique linear form satisfying \( \text{tr}_H(x \otimes y) = \langle x, y \rangle_H \) for all \( x, y \in H \). Since \( \text{tr}_H \) is contractive with respect to the projective norm on \( H \otimes H \), it can be uniquely extended to the completion \( H \hat{\otimes} H \).

**Proposition 3.6.** Let \( H \) be a Hilbert space and \( X: \mathbb{R}_{\geq 0} \to H \) a càdlâg path. If \( X \) has weak projective tensor quadratic variation along a sequence of partitions \((\pi_n)\), then it has the scalar quadratic variation given by \( Q(X)_t = \text{tr}_H([X, X]_t) \) for all \( t \).

**Proof.** Applying Proposition 3.2 to the bounded bilinear map \( \langle \cdot, \cdot \rangle_H = \text{tr}_H \circ \otimes \), we see that \( Q(X)_t = Q_{\langle \cdot, \cdot \rangle_H}(X, X)_t = \text{tr}_H([X, X]_t) \) holds for all \( t \geq 0 \). \( \square \)

### 4 Integral representation of quadratic variations by means of scalar quadratic variation

In this section, we show integral representation formulae of \( B \)-quadratic variations with respect to the scalar quadratic variation. For a similar result in the classical martingale theory in Hilbert spaces, see Metivier [36, Section 21]. Throughout this section, let \( E \) and \( G \) be Banach spaces and \( B: E \times E \to G \) be a bounded bilinear map. Moreover, let \((\pi_n)\) be a sequence of partitions of \( \mathbb{R}_{\geq 0} \).

**Proposition 4.1.** Suppose that \( X \in D(\mathbb{R}_{\geq 0}, E) \) has weak \( B \)-quadratic variation and finite 2-variation along \((\pi_n)\) and that \((\pi_n)\) satisfies Condition (C) for \( X \). Then

\[
(4.1) \quad \|Q_B(X, X)_t - Q_B(X, X)_s\|_G \leq \|B\|(Q(X)_t - Q(X)_s)
\]

for all \( s, t \in \mathbb{R}_{\geq 0} \) with \( s \leq t \). Consequently, \( Q_B(X, X) \) is absolutely continuous with respect to \( Q(X) \).

To prove Proposition 4.1, we present some technical lemmas.
Lemma 4.2. If \((\pi_n)\) satisfies (C) for \(X \in D(\mathbb{R}_{\geq 0}, E)\), then

\[
\lim_{n \to \infty} (X_{\pi_n(s)} - X_s) \otimes (X_s - X_{\pi_n(s)}) = 0
\]

holds for all \(s, t \in \mathbb{R}_{\geq 0}\) satisfying \(0 < s < t\).

**Proof.** Fix two real numbers \(s\) and \(t\) such that \(0 < s < t\).

**Case 1:** \(X\) is discontinuous at \(s\). If \(s\) is a discontinuous point of \(X\), Condition (C2) implies

\[
X_{\pi_n(s) \wedge t} - X_s = (X_{\pi_n(s)} - X_{\pi_n(s) \wedge t}) - (X_{\pi_n(s) \wedge s} - X_{\pi_n(s)}) \xrightarrow{n \to \infty} \Delta X_s - \Delta X_s = 0.
\]

**Case 2:** \(X\) is continuous at \(s\). First, note that \(\pi_n(s) \to s\) holds if \(D(X) \cap [s, s + \varepsilon]\) has infinitely many elements for every \(\varepsilon > 0\). This follows from Condition (C1). Similarly, \(\pi_n(s) \to s\) holds if \(D(X) \cap [s - \varepsilon, s]\) has infinitely many elements for every \(\varepsilon > 0\). In both cases, we have the desired convergence.

Next, assume \([s - \alpha, s + \alpha]\) contains finitely many points of \(D(X)\), where \(0 < \alpha < s\). Then there are three cases to be considered:

A. \(\omega(X; [s, s + \varepsilon]) > 0\) for all \(\varepsilon \in ]0, \alpha[\);  
B. \(\omega(X; [s - \varepsilon, s]) > 0\) for all \(\varepsilon \in ]0, \alpha[\);  
C. \(\omega(X; [s - \varepsilon, s + \varepsilon]) = 0\) for some \(\varepsilon \in ]0, \alpha[\).

**Case A.** Take an arbitrary \(\varepsilon \in ]0, \alpha[\) satisfying \(\omega(X; [s, s + \varepsilon]) > \sup_{u \in [s, s + \varepsilon]} \|\Delta X_u\|\). The existence of such an \(\varepsilon\) follows from the assumption that \([s - \alpha, s + \alpha] \cap D(X)\) is finite. Since there are only finitely many elements in \([s, s + \varepsilon] \cap D(X)\), we can choose an \(N \in \mathbb{N}\) such that \(\pi_n(s) \cap D(X) \cap [s, s + \varepsilon]\) has at most one element for all \(n \geq N\). If \(n \geq N\) and \([s, s + \varepsilon] \subset \pi_n(s)\), we have

\[
\omega(X; [s, s + \varepsilon]) \leq \omega(X - J_\delta(X); [s, s + \varepsilon]) + \sup_{u \in [s, s + \varepsilon]} \|\Delta X_u\|\]

\[
\leq \omega(X - J_\delta(X); \pi_n(s) \cap [0, s + \alpha]) + \sup_{u \in [s, s + \varepsilon]} \|\Delta X_u\|
\]

for any positive \(\delta\), and hence

\[
O^+_{s+\alpha}(X - J_\delta(X); \pi_n) \geq \omega(X; [s, s + \varepsilon]) - \sup_{u \in [s, s + \varepsilon]} \|\Delta X_u\| > 0
\]

holds under the same condition. Combining this estimate with Condition (C3), we see that there are not infinitely many \(n\) satisfying \([s, s + \varepsilon] \subset \pi_n(s)\). In other words, we have \(\pi_n(s) \in [s, s + \varepsilon]\) for sufficiently large \(n\). Since \(\varepsilon\) is chosen arbitrarily, we obtain the convergence \(\pi_n(s) \to s\) in this case. Therefore, we obtain the desired convergence.

**Case B.** In this case, we can deduce that \(\pi_n(s) \to s\) by a discussion similar to that for Case A.

**Case C.** Set

\[
s' = \sup\{u \leq s \mid X(u) \neq X(s)\}, \quad s'' = \inf\{u \geq s \mid X(u) \neq X(s)\}.
\]

Then \(s' < s < s''\) holds by the assumption.

**Case C.1:** \(X\) is continuous at \(s'\). In this case, we can deduce by the same argument as that for Case B that for any \(\varepsilon > 0\), there is an \(N\) satisfying \(\pi_n(s) \in [s' - \varepsilon, s]\) for all \(n \geq N\). Therefore,

\[
\lim_{n \to \infty} (X_s - X_{\pi_n(s)}) = 0.
\]
Case C-2: $X$ is continuous at $s''$. Similarly, we have
\[ \lim_{n \to \infty} (X_{\pi_n(s)} - X_s) = 0. \]

Case C-3: $X$ is discontinuous at both $s'$ and $s''$. Let $\delta = ||\Delta X_{s'}|| \land ||\Delta X_{s''}||$. Then choose an $N \in \mathbb{N}$ such that $I \cap [0, t] \cap D_{\delta/2}(X)$ has at most one element for every $I \in \pi_n$ and every $n \geq N$. If $n \geq N$, we have either $\pi_n(s) \in [s, s'']$ or $\pi_n(s) \in [s', s]$. In both cases, we get
\[ (X_{\pi_n(s)} - X_s) \otimes (X_s - X_{\pi_n(s)}) = 0. \]

By the discussion above, we can conclude that
\[ \lim_{n \to \infty} (X_{\pi_n(s)} - X_s) \otimes (X_s - X_{\pi_n(s)}) = 0 \]
holds if $X$ is continuous at $s$. \hfill \Box

**Lemma 4.3.** Suppose that $(\pi_n)$ satisfies Condition (C) for $X \in D(\mathbb{R}_{\geq 0}, E)$.

(i) If $X$ has strong (resp. weak) B-quadratic variation along $(\pi_n)$, then
\[ Q_B(X, X)_t - Q_B(X, X)_s = \lim_{n \to \infty} \sum_{[r,u] \in \pi_n} B(X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}, X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}) \]
for all $t \geq s \geq 0$ in the norm (resp. weak) topology.

(ii) If $X$ has scalar quadratic variation along $(\pi_n)$, then $Q(X)$ satisfies
\[ Q(X)_t - Q(X)_s = \lim_{n \to \infty} \sum_{[r,u] \in \pi_n} \|X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}\|^2 \]
for all $s$ and $t$ with $s \leq t$.

**Proof.** We first show (i). Choose any two positive numbers $s$ and $t$ satisfying $s \leq t$. By direct calculation, we see that
\begin{align*}
Q_B^{\pi_n}(X, X)_t - Q_B^{\pi_n}(X, X)_s &= \sum_{[r,u] \in \pi_n} B(X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}, X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}) \\
&\quad + B(X_{\pi_n(s) \wedge t} - X_s, X_s - X_{\pi_n(s)}) + B(X_s - X_{\pi_n(s)}, X_{\pi_n(s)} \wedge t - X_s).
\end{align*}
This combined with Lemma 4.2 implies (4.2) in the corresponding topology.

Next, we consider (ii). Transforming the summation as
\begin{align*}
Q^{\pi_n}(X)_t - Q^{\pi_n}(X)_s &= \sum_{[r,u] \in \pi_n} \|X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}\|^2 - \|X_{\pi_n(s) \wedge t} - X_s\|^2 + \|\delta X_t(\pi_n(s))\|^2 - \|\delta X_s(\pi_n(s))\|^2,
\end{align*}
we see that
\begin{align*}
Q^{\pi_n}(X)_t - Q^{\pi_n}(X)_s &- \sum_{[r,u] \in \pi_n} \|X_{(u \wedge t) \wedge s} - X_{(r \wedge t) \wedge s}\|^2 \\
&\leq \|\delta X_t(\pi_n(s))\|^2 - \|X_{\pi_n(s) \wedge t} - X_s\|^2 + \|\delta X_s(\pi_n(s))\|^2 \leq 2\|X_{\pi_n(s) \wedge t} - X_s\| \|X_s - X_{\pi_n(s)}\|.
\end{align*}
Since the right-hand side converges to 0 as $n \to \infty$ by Lemma 4.2 we obtain (4.3). \hfill \Box
Proof of Proposition 4.1. Let \( t \geq s \geq 0 \) and take an arbitrary \( z^* \in G^* \) satisfying \( ||z^*|| \leq 1 \). Then

\[
\left| z^*, \sum_{\{r,a\} \in \mathcal{P}_n} B \left( X_{(u,\mathcal{I})} \cdot x - X_{(r,\mathcal{I})} \cdot x, X_{(u,\mathcal{I})} \cdot x - X_{(r,\mathcal{I})} \cdot x \right) \right| \leq ||B|| \sum_{\{r,a\} \in \mathcal{P}_n} ||X_{(u,\mathcal{I})} \cdot x - X_{(r,\mathcal{I})} \cdot x||^2,
\]

and therefore, by Lemma 4.3 we see that

\[
|\langle z^*, Q_B(X, X)_t - Q_B(X, X)_s \rangle| \leq ||B|| ||Q(X)_t - Q(X)_s||.
\]

By taking the supremum over all \( z^* \in G^* \) with \( ||z^*|| \leq 1 \), we obtain (4.1). \( \square \)

Remark 4.4. The assumption that \( Q_B(X, X) \) has finite variation is not used in the proof of Proposition 4.1. Therefore, if \( X \) has scalar quadratic variation and there is a path \( Q_B(X, X) \in D(\mathbb{R}_{\geq 0}, E \otimes E) \) satisfying Condition (i) of Definition 2.1, it necessarily satisfies (4.1). In this case, \( Q_B \) automatically has finite variation.

Corollary 4.5. Let \( \alpha \) be a reasonable crossnorm on \( E \otimes E \). Suppose that \( X \in D(\mathbb{R}_{\geq 0}, E) \) has both scalar and weak \( \alpha \)-tensor quadratic variations along \( (\mathcal{P}_n) \) and \( (\mathcal{P}_n) \) satisfies Condition (C) for \( X \). Then they satisfy

\[
||\alpha(X, X)_t - \alpha(X, X)_s|| \leq Q(X)_t - Q(X)_s
\]

for all \( s, t \in \mathbb{R}_{\geq 0} \) with \( s \leq t \). In particular, \( \alpha(X, X) \) is absolutely continuous with respect to \( Q(X) \).

Theorem 4.6. Suppose that \( X \in D(\mathbb{R}_{\geq 0}, E) \) has weak \( B \)-quadratic variation and scalar quadratic variation along \( (\mathcal{P}_n) \) and \( (\mathcal{P}_n) \) satisfies (C) for \( X \). If \( G \) has the RNP, then there is a \( q_B \in L^1_{\text{loc}}(Q(X); G) \) such that

\[
Q_B(X, X)_t = \int_{[0,t]} q_B(s)dQ(X)_s, \quad \forall t \geq 0,
\]

and \( ||q_B(s)||_E \leq ||B|| \) for \( dQ(X) \)-almost every \( s \). Moreover, we have

\[
V(Q_B(X, X))_t = \int_{[0,t]} ||q_B(s)||dQ(X)_s.
\]

Proof. By Proposition 4.1, we know that the path \( Q_B(X, X) \) is absolutely continuous with respect to \( Q(X) \). Then we can take a locally Bochner-integrable function \( q_B : \mathbb{R}_{\geq 0} \rightarrow G \) such that

\[
Q_B(X, X)_t - Q_B(X, X)_s = \int_{[s,t]} q_B(u)dQ(X)_u
\]

and

\[
\int_s^t ||q_B(u)||dQ(X)_u = V(Q_B(X, X), [s, t]) = V(Q_B(X, X))_t - V(Q_B(X, X))_s
\]

holds for all \( s, t \in \mathbb{R}_{\geq 0} \) with \( s \leq t \) (see Propositions A.1 and A.3). Equation (4.6) directly implies (4.5). Moreover, (4.6) combined with estimate (4.1) shows that \( ||q(u)|| \leq ||B|| \) holds \( dQ(X) \)-almost everywhere. \( \square \)
Corollary 4.7. Let \( a \) be a reasonable crossnorm on \( E \otimes E \). Suppose that \( X \in D(\mathbb{R}_{\geq 0}, E) \) has weak \( a \)-tensor quadratic variation and scalar quadratic variation along \((\pi_\alpha)\), and that \((\pi_\alpha)\) satisfies (C) for \( X \). If \( E \otimes_\alpha E \) has the RNP, then there is a \( q \in L^1_{\text{loc}}(Q(X); E \otimes E) \) such that

\[
q(X, X)_t = \int_{[0,t]} q(s) dQ(X)_s, \quad V(q(X, X))_t = \int_{[0,t]} ||q(s)|| dQ(X)_s
\]

for all \( t \geq 0 \) and \( ||q(s)||_E \leq 1 \) holds for \( Q(X) \)-almost every \( s \).

Remark 4.8. For the RNP of the projective tensor product of Banach spaces, see Diestel, Fourie, and Swart [17] and the references therein. If \( E = L^p(\mu) \) with \( 1 < p < \infty \) on some measure space \((\Omega, \mathcal{F}, \mu)\), then one can take a crossnorm \( a \) such that \( L^p(\mu) \otimes_\alpha L^p(\mu) \) and \( L^p(\mu \otimes \mu) \) are isomorphic Banach spaces. See Defant and Floret [14, Section 7] for details. In this case, \( L^p(\mu) \otimes_\alpha L^p(\mu) \) is reflexive and it has the RNP.

We can improve Corollary 4.7 when the state space \( E \) is a Hilbert space.

Corollary 4.9. Let \( H \) be a separable Hilbert space and \( X \in D(\mathbb{R}_{\geq 0}, H) \). Suppose that \( X \) has projective tensor quadratic variation along \((\pi_\alpha)\) and that \((\pi_\alpha)\) satisfies (C) for \( X \). Then the density \( q \in L^1_{\text{loc}}(Q(X); H) \) in Proposition 4.7 satisfies \( ||q(s)||_H = 1 \) for \( Q(X) \)-almost every \( s \). Consequently, \( V([X, X]) = Q(X) \).

Proof. First recall that there is an isomorphism \((H \otimes_\varepsilon H)^* \cong H \otimes H\), where \( \varepsilon \) denotes the injective tensor norm. See, for example, Schatten [39, Theorem 5.13] or Fabian [21, Proposition 16.40] for a proof. Moreover, the separability of \( H \) implies that \( H \otimes H \) is also separable. Therefore, \( H \otimes H \) has the RNP. By Proposition 4.7, there is a function \( q \in L^1_{\text{loc}}(Q(X); H) \) satisfying (4.7) for all \( t \geq 0 \). On the other hand, by Proposition 3.6, we have

\[
Q(X)_t = \text{tr}_H([X, X])_t = \int_{[0,t]} \text{tr}_H(q(s)) dQ(X)_s
\]

for all \( t \geq 0 \). Since the trace functional \( \text{tr}_H : H \otimes H \rightarrow \mathbb{R} \) is contractive, we see that \( |\text{tr}_H \circ q| \leq ||q|| \) holds almost everywhere. Thus, we obtain the inequality

\[
Q(X)_t \leq V([X, X])_t \leq Q(X)_t
\]

for all \( t \in \mathbb{R}_{\geq 0} \), which shows the assertion of the corollary.

\[\square\]

5 \( C^1 \)-transformations

In this section, we study quadratic variations of a path of the form \( f(t, X_t) \) defined by a càdlàg path \( X \) with quadratic variation and a sufficiently nice function \( f \). First, let us recall classical results in the theory of classical Itô’s stochastic calculus. If \( X \) is a semimartingale and \( f \) is of class \( C^{1,2} \), then \( f(\cdot, X) \) is still a semimartingale by the Itô formula and therefore it has quadratic variation. This result can partially be extended to a \( C^1 \) function \( f \) in the sense that \( f(\cdot, X) \) has quadratic variation (see, e.g., Meyer [39, Theorem 5 in Chapter VI]) while it is not necessarily a semimartingale. There are corresponding \( C^1 \)-transformation results in the Itô–Föllmer calculus in Euclidean spaces (see Sondermann [53] and Hirai [28]). We extend these previous results to infinite-dimensional paths. As we treat càdlàg paths, it is natural to assume that each path \( t \rightarrow f(t, x) \) is also càdlàg. In this case, the roles of variables \( t \) and \( x \) are no longer symmetric, and we regard \( f \) as a càdlàg path \( t \rightarrow f(t, \cdot) \) in a space of functions with appropriate \( C^1 \)-smoothness.

Note that Ananova and Cont [1] gives corresponding results for more general path-dependent functionals \( f \) in the case where \( X \) is finite-dimensional and continuous. Extending our results to such path-dependent functionals is also important, but beyond the scope of this article and thus not discussed here.
5.1 Preliminaries

In this subsection, we introduce some preliminary concepts that will be used for the main results of this section. First, we define a family of càdlàg paths of uniformly finite variation.

**Definition 5.1.** Let $E$ be a Banach space and $\mathcal{F}$ be a nonempty subset of $D(\mathbb{R}_{\geq 0}, E)$. For each compact interval $[a, b] \subset \mathbb{R}_{\geq 0}$, define

$$V(\mathcal{F}; [a, b]) := \sup_{f \in \mathcal{F}} V(f; [a, b]) = \sup_{f \in \mathcal{F}} \sup_{\pi \in \text{Part}([a, b])} \sum_{i \in \pi} \|\delta f(I)\|.$$ 

We say that $\mathcal{F}$ has uniformly finite variation if $V(\mathcal{F}; [a, b]) < \infty$ for all compact intervals $[a, b] \subset \mathbb{R}_{\geq 0}$. A parametrized family $(f_\lambda)_{\lambda \in \Lambda}$ of elements of $D(\mathbb{R}_{\geq 0}, E)$ has uniformly finite variation if the set $\{f_\lambda| \lambda \in \Lambda\}$ does.

By definition, $\mathcal{F}$ has uniformly finite variation if and only if $(V(f; I); f \in \mathcal{F})$ is bounded for all compact intervals $I \subset \mathbb{R}_{\geq 0}$. We simply write $V(\mathcal{F}_t) = V(\mathcal{F}; [0, t])$ for $t \geq 0$. Given a parametrized family $(f_\lambda)_{\lambda \in \Lambda}$ of càdlàg paths, set $V(f_\lambda; \lambda \in \Lambda)_t = V((f_\lambda; \lambda \in \Lambda))_t$.

Next, we introduce a variant of Definition 2.3 for a family of càdlàg paths. This condition will be used to formulate the main results of this section, such as Theorem 5.9 and Corollary 5.13. Given a subset $\mathcal{F}$ of $D(\mathbb{R}_{\geq 0}, E)$, define

$$D(\mathcal{F}) = \{s \in [0, \infty[ \mid \|\Delta f\|_s > 0 \text{ holds for some } f \in \mathcal{F}\},$$

$$D_\varepsilon(\mathcal{F}) = \{s \in [0, \infty[ \mid \|\Delta f\|_s > \varepsilon \text{ holds for some } f \in \mathcal{F}\},$$

$$D^\varepsilon(\mathcal{F}) = D(\mathcal{F}) \backslash D_\varepsilon(\mathcal{F}).$$

Note that $D_\varepsilon(\mathcal{F})$ can be an uncountable set in general, but, as a consequence of Proposition B.3, it is countable for every $\varepsilon > 0$ provided that $\mathcal{F}$ is equi-right-regular, as defined in Appendix B.

**Definition 5.2.** Let $\mathcal{F}$ be a subset of $D(\mathbb{R}_{\geq 0}, E)$ and $(\pi_n)$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$. We say that $(\pi_n)$ satisfies (UC) if it satisfies the following three conditions:

(UC1) For every $\varepsilon$ and every $t > 0$, there exists an $N$ such that $I \cap D_\varepsilon(\mathcal{F}) \cap [0, t]$ has at most one element for all $n \geq N$ and $I \in \pi_n$.

(UC2) For all $s \in D(\mathcal{F})$ and $t \geq s$, the sequence $(\delta f_i(\pi_n(s)))_{n \in \mathbb{N}}$ converges to $\Delta f(s)$ uniformly in $f \in \mathcal{F}$.

(UC3) For all $t \in [0, \infty[$

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \sup_{f \in \mathcal{F}} O^\varepsilon_t(f - J_\varepsilon(f), \pi_n) = 0.$$  

Although the assumption (UC) seems strong, we can always take a sequence of partitions $(\pi_n)$ satisfying (UC) if $\mathcal{F}$ is equi-right-regular (see Definition B.1), as in the following example.

**Example 5.3.** Let $E$ be a Banach space, $\mathcal{F}$ be a subset of $D(\mathbb{R}_{\geq 0}, E)$, and $(\pi_n)$ be a sequence of partitions of $\mathbb{R}_{\geq 0}$.

(i) If $\mathcal{F}$ is equi-right-regular and $|\pi_n| \to 0$ as $n \to \infty$, then $(\pi_n)$ satisfies (UC) for $\mathcal{F}$. This is a consequence of Proposition B.3.
(ii) For each \( t \geq 0 \) and \( n \in \mathbb{N} \), define
\[
O_t^-(\mathcal{T}, \pi_n) = \sup_{f \in \mathcal{T}} \sup_{[r,s] \in \pi_n} \omega(f, [r, s] \cap [0, t])
\]
If \( O_t^-(\mathcal{T}, \pi_n) \to 0 \) as \( n \to \infty \) for every \( t \geq 0 \), then \( (\pi_n) \) satisfies (UC) for \( \mathcal{T} \). One can always select such a sequence \((\pi_n)\) whenever \( \mathcal{T} \) is equi-right-regular.

Let us now introduce the function spaces used to formulate the \( C^1 \)-transformation formulae in the next subsection.

**Definition 5.4.**

(i) Let \( T \) and \( S \) be topological spaces. We say that a function \( f : T \to S \) belongs to \( C_{K}(T, S) \) if its restriction \( f|_{K} : K \to S \) is continuous for each compact topological subspace \( K \) of \( T \).

(ii) Let \( E \) and \( F \) be Banach spaces. We define \( C^1_{K}(E, F) \) to be the set of all functions \( f \in C_{K}(E, F) \) satisfying the following conditions:

(a) the function \( f \) is Gâteaux differentiable;

(b) the restriction of the Gâteaux derivative \( D_x f|_{K} : K \to \mathcal{L}(E, F) \) is continuous for each compact subset \( K \) of \( E \).

Here note that conditions (a) and (b) themselves imply that \( f \in C_{K}(E, F) \). Therefore we can simply restate that \( C^1_{K}(E, F) \) is the set of all Gâteaux differentiable functions with derivatives in \( C_{K}(E, \mathcal{L}(E, F)) \).

As usual, we regard \( C_{K}(E, F) \) as a locally convex Hausdorff topological vector space with the topology of uniform convergence on compact subsets. The topology of \( C_{K}(E, F) \) is generated by the family of seminorms \( \| \|_{\infty,K} \) defined by
\[
\| f \|_{\infty,K} := \sup_{x \in K} \| f(x) \|_F,
\]
where \( K \) runs over all compact subsets of \( E \). Similarly, we define a topology of \( C^1_{K}(E, F) \) using the seminorms
\[
\| f \|_{C^1_{K}} := \sup_{x \in K} \| f(x) \|_F + \sup_{x \in K} \| Df(x) \|_\mathcal{L}(E, F)
\]
indexed by all compact subsets \( K \) of \( E \).

Recall Ascoli’s theorem, which characterizes the total boundedness of a subset of \( C_{K}(E, F) \). Refer to Bourbaki [3] X.2.5 Theorem 2 for a proof. To state the theorem, we introduce the notion of uniform equicontinuity. Let \( A \) be a subset of \( E \). We say that \( \mathcal{T} \subset C(A, F) \) is uniformly equicontinuous if, for all \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that \( \| g(x) - g(y) \|_F < \varepsilon \) holds for all \( g \in \mathcal{T} \) and \( x, y \in A \) with \( \| x - y \| < \delta \).

**Theorem 5.5** (Ascoli). A subset \( \mathcal{T} \) of \( C_{K}(E, F) \) is totally bounded if and only if it satisfies the following conditions:

(i) the set \( \{ f(x) \mid f \in \mathcal{T} \} \) is totally bounded in \( F \) for each \( x \in E \);

(ii) the set \( \{ f|_{K} \mid f \in \mathcal{T} \} \) is uniformly equicontinuous for each compact set \( K \subset E \).

For two Banach spaces \( E \) and \( F \), we can consider the space \( D(\mathbb{R}_{\geq 0}, C_{K}(E, F)) \) of \( C_{K}(E, F) \)-valued càdlàg paths. Note that each \( f \in D(\mathbb{R}_{\geq 0}, C_{K}(E, F)) \) can be regarded as a function on \( \mathbb{R}_{\geq 0} \times E \), so we use two notations, \( f(t,x) \) and \( f(t, x) \), for the value of \( f \) in \( F \). In the remainder of this subsection, we study some properties of the path \( t \mapsto f(t, X_t) \) defined by an \( f \in D(\mathbb{R}_{\geq 0}, C_{K}(E, F)) \) and an \( X \in D(E, F) \). These properties will be used in the proof of the main theorem of Section 5.2.

We begin with the following preliminary theorem from the theory of general topology. See, for example, Kelley [3] 7.6 and 7.10 (e)] and Bourbaki [3] Remark X.2.5].
Lemma 5.6. Let $T$ be a Hausdorff topological space and $S$ be a metric space (or, more generally, a uniform space). Define the evaluation map $ev: T \times C_K(T, S) \to S$ by the formula $ev(x, f) = f(x)$. We regard $C_K(T, S)$ as a topological space endowed with the topology of uniform convergence on compact subsets.

(i) The restriction $ev|_{K \times C_K(T, S)}$ is jointly continuous for each compact subset $K$ of $T$.

(ii) Let $A$ be a compact subset of $C_K(T, S)$. Then the restriction $ev|_{K \times A}$ is jointly continuous relative to the topology of pointwise convergence on $A$ for each compact subset $K$ of $T$.

Lemma 5.7. If $f \in D(\mathbb{R}_{\geq 0}, C_K(E, F))$ and $X \in D(\mathbb{R}_{\geq 0}, E)$, then the path $t \mapsto f(t, X_t)$ is càdlàg and the left limits are given by

$$
\lim_{s \uparrow t} f(s, X_s) = f(t-, X_{t-}), \quad t \geq 0.
$$

Proof. To show the assertion, it suffices to verify that $t \mapsto f(t, X_t)$ is càdlàg on every compact interval of the form $[0, T]$. Fix $T > 0$ arbitrarily. Since the image $X([0, T])$ is totally bounded in $E$, one can regard the function $f(\cdot, X): [0, T] \to F$ as the composition of $(X, f)|_{[0, T]}$ and $ev|_{K \times C_K(E, F)}$ for some compact set $K \subset E$. The restricted evaluation map $ev|_{K \times C_K(E, F)}$ is continuous by Lemma 5.6, and therefore the composition $f(\cdot, X)|_{[0, T]} = ev|_{K \times C_K(E, F)} \circ (X, f)|_{[0, T]}$ is càdlàg. Moreover, again by the continuity of the evaluation map, we obtain

$$
\lim_{s \uparrow t} f(s, X_s) = ev|_{K \times C_K(E, F)}(X_{t-}, f_{t-}) = f(s-, X_{s-}).
$$

This completes the proof.

Lemma 5.8. Let $(X, f) \in D(\mathbb{R}_{\geq 0}, E \times C_K(E, F))$, $T > 0$, and $K$ be a compact subset satisfying $X([0, T]) \subset K$. Assume that a sequence of partitions $(\pi_n)$ approximate $X$ from the left on $[0, T]$ and so does $f(\cdot, x)$ for all $x \in K$. Then $(\pi_n)$ approximate $f(\cdot, X)$ from the left on $[0, T]$.

Proof. First, note that the restriction map $C_K(E, F) \to C_K(K, F)$ is continuous because $K$ is compact. We take a compact set $C \subset C_K(E, F)$ that includes the image $f([0, T])$. Then $C|_K := \{f|_K \mid f \in C\}$ is compact in $C_K(K, F)$ because it is the image of compact set $C$ by the restriction map $C_K(E, F) \to C_K(K, F)$. Therefore $ev|_{K \times C|_K}$ is continuous relative to the topology of pointwise convergence on $C_K(K, F)$ by Lemma 5.6. This continuity combined with the assumptions on $(\pi_n)$, $X$, and $f$ implies that

$$
\lim_{n \to \infty} f(\pi_n(s), X_{\pi_n(s)}) = ev|_{K \times C|_K}(X_{s-}, f_{s-}|_K) = f(s-, X_{s-}).
$$

This shows the assertion.

5.2 Main results of the section

In this subsection, let $E$, $E_1$, $F$, $F_1$, and $G$ be Banach spaces. Given $B \in \mathcal{L}^{(2)}(E, E_1)$ and $B' \in \mathcal{L}^{(2)}(F, F_1)$, we identify $B$ and $B'$ with elements of $\mathcal{L}(E \otimes E_1)$ and $\mathcal{L}(F \otimes F_1)$, respectively, by canonical isomorphisms. We also assume that there is a given sequence of partitions of $\mathbb{R}_{\geq 0}$, denoted by $(\pi_n)$.

Let us now give the first $C^1$-transformation formula. The following theorem, which seems somewhat complicated, provides a sufficient condition for the existence of $B'$-quadratic variation of a path $f(\cdot, X)$. It will later be used to derive the $C^1$-transformation formula for tensor quadratic variations.
Theorem 5.9. Let $X: \mathbb{R}_{\geq 0} \to E$ be a càdlàg path that has weak $B$-quadratic variation and finite 2-variation along $(\pi_n)$, and let $f: \mathbb{R}_{\geq 0} \to C^1_{\mathcal{F}}(E, F)$ a càdlàg path such that the family $(f(\cdot, x); x \in K)$ has uniformly finite variation for every compact set $K \subset E$. Assume that there is a càdlàg function $\Phi_f: \mathbb{R}_{\geq 0} \to C_{\mathcal{F}}(E, \mathcal{D}(E_1, F_1))$ that commutes the following diagram for all $(t, x) \in \mathbb{R}_{\geq 0} \times E$.

\[ \begin{array}{ccl} E \otimes E & \xrightarrow{D_{f(t,x)}} & F \otimes F \\ B \downarrow & & B' \downarrow \\ E_1 & \xrightarrow{\Phi_f(t,x)} & F_1 \end{array} \]

Let $T > 0$ and suppose that there is a compact convex set $K \subset E$ satisfying the following conditions:

(i) the image $X([0, T])$ is included in $K$;

(ii) the sequence $(\pi_n)$ approximates $(X, f(\cdot, x), \Phi_f(\cdot, x))$ from the left for all $x \in K$;

(iii) the sequence $(\pi_n)$ satisfies (UC) for the family of $E \times F$-valued càdlàg paths $(X, f(x))_{x \in K}$.

Under these assumptions, the path $[0, T] \ni t \mapsto f(t, X_t) \in F$ has the weak $B$-quadratic variation

\[ Q_B'(f(\cdot, X), f(\cdot, X)_t) = \int_0^t \Phi_f(s-, X_{s-}) dQ_B(X, X) + \sum_{0 < s \leq t} B'(\Delta f(s, X_s)) \]

for $t \in [0, T]$. If we assume, moreover, that $X$ has strong $B$-quadratic variation, then the path given by (5.1) is the strong $B'$-quadratic variation of $f(\cdot, X)$.

Lemma 5.10. Under the assumptions of Theorem 5.9, $(\pi_n)$ satisfies (C1) and (C2) of Definition 2.3 for $f(\cdot, X)$ on $[0, T]$.

Proof. Assume that $T > 0$ and a compact set $K \subset E$ satisfies (i)–(iii) of Theorem 5.9. First, we check the following estimate

\[ \|\Delta f(s, X_s)\| \leq \sup_{x \in K} \|f(s, x) - f(s-, x)\| + \int_0^1 \|D f(s-, X_s + \theta \Delta X_s) \Delta X_s\| d\theta \]

\[ \leq \|\Delta f_s\|_{\infty, K} + \sup_{(r, x) \in [0, T] \times K} \|D f(r, x)\| \|\Delta X_s\|. \]

Then, by setting $C = 1 \vee \sup_{(r, x) \in [0, T] \times K} \|D_x f(r, x)\|$, we have

\[ \|\Delta f(s, X_s)\| \leq C(\|\Delta X_s\| + \|\Delta f(s)\|_{\infty, K}). \]

This shows $D_f(\cdot, X) \subset D_{C^{-1}}(X, f(\cdot, x); x \in K)$, which implies Condition (C1).

To show (C2), let $s \in D(f(\cdot, X)) \cap [0, T]$ and $t \in [s, T]$. The discussion in the previous paragraph implies $D(f(\cdot, X)) \subset D((X, f), \|\cdot\| + \|\cdot\|_{\infty, K})$. Therefore, by assumption,

\[ \lim_{n \to \infty} X(\pi_n(s) \wedge t) = X_s, \quad \lim_{n \to \infty} X(\pi_n(s) \wedge t) = X_{s-}, \]

\[ \lim_{n \to \infty} f(\pi_n(s) \wedge t, x) = f(s, x), \quad \lim_{n \to \infty} f(\pi_n(s), x) = f(s-, x) \]

holds for all $x \in K$. This combined with Lemma 5.7 implies

\[ \lim_{n \to \infty} \{f(\pi_n(s) \wedge t, X_{\pi_n(s) \wedge t}) - f(\pi_n(s) \wedge t, X_{\pi_n(s) \wedge t})\} = f(s, X_s) - f(s-, X_{s-}), \]

which completes the proof. \qed
Proof of Theorem 5.9. Let $K$ be a compact convex set satisfying (i)--(iii) of the theorem. For convenience, let

$$\mathcal{F}_K = \{ f(\cdot, x) \mid x \in K \} \subset D(\mathbb{R}_{\geq 0}, F), \quad \mathcal{F}_{X,K} = \{ (X, f(\cdot, x)) \mid x \in K \} \subset D(\mathbb{R}_{\geq 0}, E \times F).$$

Step 1: Convergence of jumps. First, we check the absolute convergence of the jump part of (5.1).

Observing the estimate

$$\sum_{0 < s \leq t} \| \Delta f(s, X_s) \|_2^2 \leq 2 \sup_{(s, x) \in [0, T] \times K} \| D_x f(s, x) \|_2^2 \sum_{0 < s \leq t} \| \Delta X_s \|_2^2 + 2 \sup_{(s, x) \in [0, T] \times K} \| f(s, x) \| V(\mathcal{F}_K)_t,$$

we see that $B(\Delta f(\cdot, X) \otimes 2)$ is absolutely summable on $[0, T]$. Note that

$$\sup_{(s, x) \in [0, T] \times K} \| f(s, x) \| = \sup_{x \in [0, K]} \| f_s \|_{\infty, K} < \infty,$$

$$\sup_{(s, x) \in [0, T] \times K} \| D_x f(s, x) \| = \sup_{x \in [0, K]} \| D_x f_s \|_{\infty, K} < \infty$$

because $f$ is a $C^1_{\text{loc}}(E, F)$-valued càdlàg path. Moreover, we have

$$\sum_{0 < s \leq t} \| \Omega_f(\cdot, X_s)B(\Delta X_s, \Delta X_s) \|_2 \leq \| B' \| \sup_{(s, x) \in [0, T] \times K} \| D_x f(s, x) \|_2 \sum_{0 < s \leq t} \| \Delta X_s \|_2^2 < \infty.$$

Therefore, Equation (5.1) is equivalent to

$$\tag{5.2} Q_B(f(X), f(X))_t = \int_0^t \Phi_f(s, X_s)\partial Q_B(X, X)_s + \sum_{0 < s \leq t} B'(\Delta f(s, X_s) \otimes 2) - \sum_{0 < s \leq t} \Phi_f(s, X_s)B(\Delta X_s, \Delta X_s).$$

We shall show (5.2) instead of (5.1) in the rest of this proof.

Step 2: The Taylor expansion. Let $I = [r, s] \in \pi_{\mathcal{H}}$. Then by the first order Taylor expansion, we obtain

$$\delta f(\cdot, X)_{r}(I) = f(s \land t, X_{s \land t}) - f(r \land t, X_{r \land t}) + D_x f(\cdot, X)\delta X_t(I) + R_t(I),$$

where $R_t(I)$ is defined by

$$R_t(I) = \int_{[0,1]} \{ D_x f(r \land t, X_{r \land t} + \theta\delta X_t(I)) - D_x f(r \land t, X_{r \land t}) \} d\theta.$$

For notational convenience, let

$$\delta'_{f, X}(I) = \text{ev}(\delta f_t(I), X_{s \land t}) = f(s \land t, X_{s \land t}) - f(r \land t, X_{s \land t}).$$

By (5.3) and bilinearity, we see that

$$\tag{5.4} B'(\delta f(\cdot, X) \otimes 2) = B'(\delta'_{f, X}(\cdot) \otimes 2) + \Phi_f(\cdot, X)B(\delta X_t \otimes 2)(I) + B'(R^1_t)(I) + T^1_t(I) + T^2_t(I) + T^3_t(I),$$

where

$$T^1_t(I) = B'[\delta'_{f, X}(I) \otimes D_x f(\cdot, X)\delta X_t(I) + D_x f(\cdot, X) \delta X_t(I) \otimes \delta'_{f, X}(I)],$$

$$T^2_t(I) = B'[\delta'_{f, X}(I) \otimes R_t(I) + R_t(I) \otimes \delta'_{f, X}(I)],$$

$$T^3_t(I) = B'[D_x f(\cdot, X)\delta X_t(I) \otimes R_t(I) + R_t(I) \otimes D_x f(\cdot, X)\delta X_t(I)].$$
Here we introduce the notation

$$e_k^1(A) = \begin{cases} 1 & \text{if } E \cap A \neq \emptyset, \\ 0 & \text{if } E \cap A = \emptyset, \end{cases} \quad e_k^2 = 1 - e_k^1.$$  

Moreover, let $D = D(\mathcal{F}X, X)$, $D^e = D^e(\mathcal{F}X, X)$, and $D^e \setminus D^e$ for each $e > 0$. Notice that $D = \bigcup_{e > 0} D^e$. Then we can deduce from (5.4) that

\begin{equation}
B'(\delta f(\cdot, X)^{\otimes 2}) - e_k^1 B'(\delta f(\cdot, X)^{\otimes 2})
= \Phi_f(\cdot, X)B(\delta X^{\otimes 2}) - e_k^1 \Phi_f(\cdot, X)B(\delta X^{\otimes 2})
+ e_k^2 B'(\delta f(\cdot, X)^{\otimes 2}) + e_k^2 B'(R^{\otimes 2}) + e_k^2 T_1 T_i^2 + e_k^2 T_i^3.
\end{equation}

Summing each term of (5.5) over the partition $\pi_n$, we obtain

\begin{equation}
\sum_{I \in \pi_n} B'(\delta f(\cdot, X)^{\otimes 2})(I)
= \sum_{I \in \pi_n} e_k^1 B'(\delta f(\cdot, X)^{\otimes 2})(I) + \sum_{I \in \pi_n} e_k^2 B'(\delta f(\cdot, X)^{\otimes 2})(I)
+ \sum_{I \in \pi_n} \Phi_f(\cdot, X)B(\delta X^{\otimes 2})(I) - \sum_{I \in \pi_n} e_k^1 \Phi_f(\cdot, X)B(\delta X^{\otimes 2})(I)
+ \sum_{I \in \pi_n} e_k^2 B'(R_s(I)^{\otimes 2}) + \sum_{I \in \pi_n} e_k^2 T_1 T_i^2(I) + \sum_{I \in \pi_n} e_k^2 T_i^3(I)
= I_1^{(n)} + I_2^{(n)} + I_3^{(n)} - I_4^{(n)} + I_5^{(n)} + I_6^{(n)} + I_7^{(n)} + I_8^{(n)}.
\end{equation}

Therefore, it suffices to observe the limit of each $I_i^{(n)}$.

**Step 3: Behavior of $I_i^{(n)}$'s.** We first treat $i \in \{1, 3, 4\}$. By Lemma 5.8 and assumptions (C1) and (C2) for $X$, we see that

$$\lim_{n \to \infty} I_4^{(n)} = \lim_{n \to \infty} \sum_{s \in D^e \cap [0, t]} \Phi_f(\pi_n(s), X_{\pi_n(s)})B(\delta X^{\otimes 2})(\pi_n(s)) = \sum_{s \in D^e \cap [0, t]} \Phi_f(s - X_{s -})B(\Delta X^{\otimes 2}).$$

By Lemma 5.10 and the assumption, $(\pi_n)$ and $f(\cdot, X)$ satisfies (C1) and (C2) on $[0, t]$. Therefore, we can deduce that

$$\lim_{n \to \infty} I_1^{(n)} = \sum_{s \in D^e \cap [0, t]} \Delta f(s, X_s).$$

Lemma 2.6 implies that

$$\lim_{n \to \infty} I_3^{(n)} = \int_{[0, t]} \Phi_f(s - X_{s -}) \, dQ_B(X, X)s$$

holds in the weak topology.

It remains to observe the behaviour of residual terms $I_i^{(n)}$'s for $i \in \{2, 5, 6, 7, 8\}$. For convenience, set

$$\alpha(e, n) = O^e_t(X - J_e(X; \pi_n)), \quad \beta(e, n) = \sup_{x \in K} O^e_t(f(x) - J_e(f(x); \pi_n)), \quad C = \sup_{(s, x) \in [0, t] \times K} ||D_x f(s, x)||.$$

21
Then we notice that the estimates
\[
e_1^2 I \leq \alpha (\varepsilon, n), \quad e_2^2 I \leq \beta (\varepsilon, n)
\]
hold for \(I \in \pi_n\). This leads to the inequalities
\[
\| I_1^n \| \leq \| B' \| V(\mathcal{F}_K) \beta (\varepsilon, n),
\]
\[
\| I_2^n \| \leq V^2 (X; \Pi), \quad \sup_{x \in [0, t], y \in \mathcal{K}, \| x - y \| \leq \alpha (\varepsilon, n)} \| D_x f (s, x) - D_x f (s, y) \|,
\]
\[
\| I_3^n \| \leq 2 C \beta (\varepsilon, n),
\]
\[
\| I_4^n \| \leq 4 C \beta (\varepsilon, n),
\]
\[
\| I_5^n \| \leq 2 C \beta (\varepsilon, n),
\]
\[
\| f (n) \| \leq 2 C \beta (\varepsilon, n). \]

Therefore,
\[
\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} I_i^{(n)} = 0, \quad i \in \{2, 5, 6, 7, 8\}.
\]

Note that here we have used
\[
\lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \| D_x f (s, x) - D_x f (s, y) \| = 0.
\]

To deduce (5.7), recall that \(D_x f : \mathbb{R}^n \rightarrow C \phi (E, \mathcal{L} (E, F)) \) is càdlàg by assumption and therefore it has totally bounded range on \([0, T]\). Hence, by Ascoli’s theorem (Theorem 5.5), the family \((D_x f_i)_{i \in [0, T]}\) is uniformly equicontinuous, which guarantees (5.7).

**Step 4: Conclusion.** Combining all the estimates obtained in step 3, we obtain for all \(z^* \in G^*\) with \(\| z^* \| = 1\)
\[
\lim_{n \rightarrow \infty} | (z^*, \text{RHS of (5.2)}) - (\text{RHS of (5.6)}) | 
\]
\[
\leq \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \| B' \| \sum_{s \in D^+} \| \Delta f (s, X_s) \|^2 + \lim_{\varepsilon \downarrow 0} \lim_{n \rightarrow \infty} \sum_{s \in D^+} \| \Phi_f (s^-, X_{s^-}) B (\Delta X^{s^2}) \| = 0,
\]
which completes the proof for the weak case. If \(Q_B (X, X)\) is the strong \(B\)-quadratic variation of \(X\), then the convergence of \(I_4^{(n)}\) holds in the norm topology. In this case, we obtain the norm convergence of discrete \(B^t\)-quadratic variation by replacing (5.8) with a similar norm estimate.

**Corollary 5.11.** In addition to the assumptions in Theorem 5.9, assume that there is an increasing divergent sequence of positive numbers \((T_n)\) and an increasing sequence of compact convex sets \((K_n)\) such that \((T_n, K_n)\) satisfies conditions (i)-(iii) in Theorem 5.9 for each \(n \in \mathbb{N}\). Then (5.1) holds for all \(t \in \mathbb{R}^+.\)

**Remark 5.12.** The assumption on \((\pi_n)\) in Corollary 5.11 is always satisfied whenever \(|\pi_n| \rightarrow 0\). If \((T_n, K_n)\) satisfies \(X ([0, T_n]) \subset K_n\) and \(\lim_{n \rightarrow \infty} \sup_{x \in K_n} O^* \phi ((X_f, \cdot, x), \pi_k) = 0\) for all \(n\), then the assumption on \((\pi_n)\) in Corollary 5.11 is also satisfied. See Example 5.3 and Appendix B.
To derive $C^1$ transformation formula for tensor quadratic variations, recall the definition of a uniform crossnorm. Let $\alpha$ be a system that defines a reasonable crossnorm on $E \otimes F$ for each pair of Banach spaces $(E, F)$. We say that $\alpha$ is a uniform crossnorm if for arbitrary four Banach spaces $E_1, E_2, F_1, F_2$ and for all $(T, S) \in \mathcal{L}(E_1, E_2) \times \mathcal{L}(F_1, F_2)$, the tensor product $T \otimes S$ defines a bounded operator $E_1 \otimes \alpha E_2 \to F_2 \otimes \alpha F_2$ satisfying $||T \otimes S|| \leq ||T|| ||S||$. In the remainder of this section, let $\alpha$ be a fixed uniform crossnorm.

**Corollary 5.13.** Let $X : \mathbb{R}_{\geq 0} \to E$ be a càdlàg path and $f : \mathbb{R}_{\geq 0} \to C^1_{\mathcal{K}}(E, F)$ be a càdlàg path such that the family $(f(\cdot, x); x \in K)$ has uniformly finite variation for all compact sets $K \subset E$. Assume that there is a sequence $0 < T_0 < T_1 < \cdots < T_n < \cdots \to \infty$ and an increasing sequence of compact convex subsets of $E$, denoted by $(K_n)$, such that $(T_n, K_n)$ satisfies conditions (i)–(iii):

(i) the image $X([0, T_n]) \subset K_n$ is included in $K$;
(ii) the sequence $(\pi_n)$ approximates $(X, f(\cdot, x), D_s f(\cdot, x))$ from the left for all $x \in K_n$;
(iii) the sequence $(\pi_n)$ satisfies (UC) for the family $(X, f(\cdot, x))_{x \in K_n}$.

If $X$ has strong (resp. weak) $\alpha$-tensor quadratic variation and finite 2-variation along $(\pi_n)$, then $f(X)$ has the strong (resp. weak) $\alpha$-tensor quadratic variation, given by

$$\alpha[f(\cdot, X), f(\cdot, X)]_t = \int_0^t D_s f(s, X_s)^{\otimes 2} d\gamma_s [X, X]^c_s + \sum_{0 < s \leq t} \Delta f(s, X_s)^{\otimes 2}, \quad t \geq 0.$$

Next, we consider a case in which $f$ of Corollary 5.13 is represented as $f(t, x) = \tilde{f}(A_t, x)$ for some function $\tilde{f}$ and a càdlàg path $A$ of finite variation.

**Corollary 5.14.** Let $X : D(\mathbb{R}_{\geq 0}, E)$ and $A \in \text{FV}(\mathbb{R}_{\geq 0}, E)$. Moreover, let $f : F \times E \to G$ be a function satisfying the following conditions:

(i) the restriction of $f$ to each compact subset of $F \times E$ is Lipschitz continuous;
(ii) the map $x \mapsto f(a, x)$ is Gâteaux differentiable and $D_s f \in C^1_{\mathcal{K}}(F \times E, \mathcal{L}(E, G))$.

Suppose that $(\pi_n)$ satisfies either $\lim_{n \to \infty} ||\pi_n|| = 0$ or $\lim_{n \to \infty} O^T((A, X), \pi_n) = 0$ for all $t \geq 0$. If $X$ has strong (resp. weak) $\alpha$-tensor quadratic variation along $(\pi_n)$, then $f(A, X)$ has the strong (resp. weak) $\alpha$-tensor quadratic variation, given by

$$\alpha[f(A, X), f(A, X)]_t = \int_0^t D_s f(A_s, X_s)^{\otimes 2} d\gamma_s [X, X]^c_s + \sum_{0 < s \leq t} \Delta f(A_s, X_s)^{\otimes 2}. \quad t \geq 0.$$

Notice that Condition (C) for $(A, X)$ is not sufficient to prove the corollary because a large jump of $A$ is not necessarily a large jump of $t \mapsto f(A_t, x)$.

**Proof.** Let $g(t, x) = f(A_t, x)$. Then, it suffices to check that $g$ and $X$ satisfy the assumptions in Theorem 5.9 and Corollary 5.11. We see that $\alpha \mapsto f(a, \cdot)$ is a function of the class $C^1_{\mathcal{L}}(F, C^1_{\mathcal{K}}(E, G))$ by conditions (i) and (ii), and therefore $t \mapsto g(t, \cdot)$ belongs to $D(\mathbb{R}_{\geq 0}, C^1_{\mathcal{K}}(E, G))$. Condition (i) and the fact that $A$ has finite variation shows that $(g(t, \cdot))_{t \in K}$ has uniformly finite variation for each compact $K \subset E$. Moreover, since the canonical bilinear map $\mathcal{L}(E, F) \times \mathcal{L}(E, F) \to \mathcal{L}(E \otimes \alpha E, F \otimes \alpha F)$ is continuous, the path $t \mapsto D_s g(t, \cdot)^{\otimes 2}$ is càdlàg as a $C^1_{\mathcal{K}}(E, \mathcal{L}(E \otimes \alpha E, F \otimes \alpha F))$-valued path.

It remains to show that the conditions on $(\pi_n)$ hold. If $||\pi_n|| \to 0$, then $(\pi_n)$ satisfies the expected assumptions by Remark 5.12. Otherwise, we suppose $O^T((X, g(\cdot, x)), \pi_n) = 0$ as $n \to \infty$. Then there is a $C > 0$ such that $\sup_{x \in K} O^T((X, g(\cdot, x)), \pi_n) \leq C O^T((X, A), \pi_n)$ for each $t > 0$ and compact set $K \subset E$. This implies the desired conditions again by Remark 5.12. \qed
6 Quadratic variation of the Itô–Föllmer integrals

Let $E$ and $G$ be Banach spaces and $X$ be a càdlàg path that has tensor quadratic variation and finite 2-variation along a sequence of partitions $(\pi_n)$. Then, by the Itô–Föllmer formula (Theorem 2.4),
\[
\int_0^t D_x f(X_s) dX_s = f(X_t) - f(X_0) - \frac{1}{2} \int_0^t D_x^2 f(X_s) d[X,X]_s + \sum_{0 < s \leq t} (\Delta f(X_s) - D_x f(X_s) \Delta X_s)
\]
holds for a function $f : E \to G$ with $C^2$-smoothness in an appropriate sense. In this section, we study the properties of the path $t \mapsto \int_0^t D_x f(X_s) dX_s$, focusing on its quadratic variation. In semimartingale theory, the stochastic integral of a suitable predictable process with respect to a semimartingale is again a semimartingale, and its quadratic variation can be explicitly computed by using the quadratic variation of the integrator. We aim to prove a corresponding formula in the Itô–Föllmer calculus in Banach spaces. Such formulae have been already given by Sondermann \[53\] and Schied \[50\] for the finite-dimensional and continuous case and by Hirai \[28\] for the finite-dimensional and càdlàg case. Ananova and Cont \[1\] gave a corresponding result, which they call the pathwise isometry formula, in a more general situation where the function $f$ is a path-dependent one.

**Theorem 6.1.** Let $\alpha$ be a uniform crossnorm and assume that $X, A, f,$ and $(\pi_n)$ satisfy the same conditions as those in Corollary \[2.5\]. Define a $G$-valued càdlàg path $Y$ by
\[
Y_t = \int_0^t D_x f(A_s, X_s) dX_s, \quad t \geq 0.
\]
Then $Y$ correspondingly has the strong or weak $\alpha$-tensor quadratic variation given by
\[
[\alpha [Y,Y]]_t = \int_0^t D_x f(A_s, X_s) \otimes^2 d[\alpha [X,X]]_s, \quad t \geq 0.
\]

**Proof.** We show the assertion for weak tensor quadratic variations. By Corollary \[5.14\], the path $f(A, X)$ has the weak $\alpha$-tensor quadratic variation
\[
[\alpha [f(A, X), f(A, X)]]_t = \int_0^t D_x f(A_s, X_s) \otimes^2 d[\alpha [X,X]]_s + \sum_{0 < s \leq t} (\Delta f(A_s, X_s)) \otimes^2.
\]
Now let
\[
B_t = \int_0^t D_x f(A_s, X_s) dA_s^c + \frac{1}{2} \int_0^t D_x^2 f(A_s, X_s) d[X,X]_s^c + \sum_{0 < s \leq t} (\Delta f(A_s, X_s) - D_x f(A_s, X_s) \Delta X_s).
\]
Then by the Itô formula, we have $f(A_t, X_t) = Y_t + B_t$. According to Corollary 5.10 of Hirai \[30\], we see that $Y$ has the weak $\alpha$-tensor quadratic variation
\[
[\alpha[Y,Y]]_t = [\alpha[f(A, X), f(A, X)]]_t - [\alpha[f(A, X), Y]]_t - [\alpha[Y, f(A, X)]]_t + [\alpha[B,B]]_t.
\]
By direct calculations of $[f(A, X), Y], [Y, f(A, X)],$ and $[B, B]$ using Corollary 5.9 in \[30\], we consequently obtain formula \[6.1\].

24
Corollary 6.2. Under the assumptions of Theorem 6.1, the càdlàg path \( f(A, X) \) admits a decomposition
\[
f(A_t, X_t) = Y_t + C_t + D_t,
\]
where \( Y \) is a càdlàg path having the \( \alpha \)-tensor quadratic variation given by (6.1), \( C \) is a continuous path of finite variation, and \( D \) is a purely discontinuous path of finite variation.

A  Remarks on the Radon–Nikodym property

Let \( \mathcal{R} \) be a ring of subsets of \( \Omega \) and \( \mu: \mathcal{R} \to E \) be a finitely additive vector measure. Here, recall that the variation of \( \mu \) on \( A \subseteq \Omega \) is defined as
\[
|\mu|(A) = \sup \left\{ \sum_i \|\mu(A_i)\| \mid (A_i) \text{ is a finite disjoint family of elements of } \mathcal{R} \text{ and } \bigcup A_i \subseteq A \right\}
\]
Then the function \( |\mu|: \mathcal{R} \to [0, \infty] \) defines a finitely additive measure. We say that \( \mu \) has bounded variation if \( |\mu|(|\Omega|) < \infty \) and finite variation if \( |\mu|(A) < \infty \) for all \( A \in \mathcal{R} \). If, moreover, \( \mu \) is countably additive on \( \mathcal{R} \), then the variation \( |\mu| \) is also countably additive on \( \mathcal{R} \).

We quote the following proposition in the theory of vector integration. See Dinculeanu [19, Theorem 2.29] for a proof.

Proposition A.1. Given a positive measure space \( (\Omega, \mathcal{A}, \mu) \) and a \( f \in L^1(\mu; E) \), define
\[
(f \cdot \mu)(A) = \int_A f \, d\mu
\]
for each \( A \in \mathcal{A} \). Then \( f \cdot \mu \) is an \( E \)-valued countably additive measure of bounded variation. Moreover, the variation is given by \( |f \cdot \mu| = \|f\| \cdot |\mu| \)

Definition A.2. Let \( (\Omega, \mathcal{A}, \mu) \) be a finite positive measure space. A Banach space \( E \) has the Radon–Nikodym property (RNP) with respect to \( (\Omega, \mathcal{A}, \mu) \) if for every \( \mu \)-absolutely continuous vector measure \( \nu: \mathcal{A} \to E \) of bounded variation, there exists a \( g \in L^1(\mu; E) \) such that
\[
\nu(A) = \int_A g(\omega) \, d\mu(\omega)
\]
for all \( A \in \mathcal{A} \). The Banach space \( E \) has the RNP if it has the RNP with respect to every finite positive measure space.

It is known that every reflexive Banach space and every separable dual space have the RNP (see Diestel and Uhl [18]).

Proposition A.3. Let \( \mathcal{R} \) be a \( \delta \)-ring of subsets of \( \Omega \) and let \( \mu: \mathcal{R} \to [0, \infty] \) be a \( \sigma \)-finite measure. Suppose that the Banach space \( E \) has the RNP and \( \nu: \mathcal{R} \to E \) is a vector measure of finite variation. If \( \nu \) is \( \mu \)-absolutely continuous, then there is a unique strongly measurable function \( f: \Omega \to E \) such that \( f 1_A \in L^1(\mu) \), and
\[
\nu(A) = \int_A f \, d\mu
\]
for all \( A \in \mathcal{R} \).
Proof. First, take a $\mathcal{R}$-measurable partition $(\Omega_n)_{n \in \mathbb{N}}$ of $\Omega$ such that $\mu(\Omega_n) < \infty$ for all $n$. Moreover, define a sequence of measures of bounded variation $(\nu_n)$ by $\nu_n(A) = \nu(A \cap \Omega_n)$. Since each $\Omega_n \cap \mathcal{R}$ is a $\delta$-algebra (and hence a $\sigma$-algebra) and $\nu_n$ is a measure of bounded variation, there is a function $f_n \in L^1(\Omega_n, \mu|_{\mathcal{R}/\Omega_n}; E)$ such that

$$\nu_n(\Omega) = \int_{A \cap \Omega_n} f_n \, d\mu, \quad A \in \mathcal{R}.$$ 

Next, set $f(\omega) = f_n(\omega)$ for $\omega \in \Omega_n$. Then the function $f$ is strongly $\sigma(\mathcal{R})$-measurable. Here, note that for $A \in \mathcal{R}$, $f$ is $\mu$-integrable on $A$. Indeed, we have

$$\int_A f(\omega) \, d\mu = \sum_{n \in \mathbb{N}} \int_{A \cap \Omega_n} f_n(\omega) \, d\mu = \sum_{n \in \mathbb{N}} |\nu_n|(A \cap \Omega_n) = \sum_{n \in \mathbb{N}} |\nu|(A \cap \Omega_n) = |\nu|(A) < \infty.$$ 

By the $\sigma$-additivity of $\nu$ and $\mu$ on $\mathcal{R}$, we can justify the following calculation:

$$\int_A f(\omega) \, d\mu = \sum_{n \in \mathbb{N}} \int_{A \cap \Omega_n} f(\omega) \, d\mu = \sum_{n \in \mathbb{N}} \nu_n(A \cap \Omega_n) = \sum_{n \in \mathbb{N}} \nu(A \cap \Omega_n) = \nu(A).$$

Thus, we obtain the assertion. \hfill \qed

B \quad \textbf{Supplements on families of càdlàg paths}

In this section, we consider the problem of uniformly controlling the oscillation of a family of càdlàg paths by a sequence of partitions. To observe that, we first introduce the notion of equi-right-regularity.

\textbf{Definition B.1.} Let $E$ be a Banach space and $\mathcal{F}$ a subset of $D(\mathbb{R}_{\geq 0}, E)$.

(i) The set $\mathcal{F}$ is \textit{equi-right-continuous} at $t \in \mathbb{R}_{\geq 0}$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\|f(t) - f(s)\|_F < \varepsilon$$

holds for all $f \in \mathcal{F}$ and $s \in [t, t + \delta]$.

(ii) The set $\mathcal{F}$ is \textit{equi-right-regular} at $t > 0$ if it is equi-right-continuous at $t$ and, for every $\varepsilon > 0$, there is a $\delta \in ]0, t[$ such that

$$\|f(t-) - f(s)\|_F < \varepsilon$$

holds for all $f \in \mathcal{F}$ and $s \in ]t - \delta, t[$.

(iii) The set $\mathcal{F} \subset D(\mathbb{R}_{\geq 0}, E)$ is \textit{equi-right-regular} if it is equi-right-regular at every $t > 0$ and equi-right continuous at 0.

A parametrized family $(f_i)_{i \in I}$ of càdlàg paths in $E$ is said to be equi-right-regular if the set $\{f_i \mid i \in I\} \subset D(\mathbb{R}_{\geq 0}, E)$ is equi-right-regular. The notion of equi-right-regularity is an analogue of the equicontinuity of the set of continuous functions. A subset $\mathcal{F}$ of the space $C(\mathbb{R}_{\geq 0}, E)$ is equi-right-regular if and only if it is equicontinuous. We can prove an Arzelà–Ascoli-like theorem for the space $D(\mathbb{R}_{\geq 0}, E)$ with the topology of uniform convergence on compact sets. Note that like equicontinuity, equi-right-regularity can be characterized in the language of uniform convergence.

Let us give some examples of equi-right-regular families of paths.
Example B.2. Let $E$ and $F$ be Banach spaces.

(i) Every finite subset of $D(\mathbb{R}_{\geq 0}, E)$ is equi-right-regular.

(ii) Let $\mathcal{G}$ be a family of subsets of $E$ and $C(\mathcal{G}, E)$ be the set of all functions whose restriction to each $S \in \mathcal{G}$ are continuous. We regard $C(\mathcal{G}, E)$ as a topological space by the topology of uniform convergence on each member of $\mathcal{G}$, i.e., the topology of $\mathcal{G}$-convergence. See, for example, Bourbaki [3, Section X.1] for details about this topology. Then the family of paths $(f(\cdot, x); x \in S)$ is equi-right-regular for every $f \in D(\mathbb{R}_{\geq 0}, C(\mathcal{G}, E))$ and $S \in \mathcal{G}$.

If a family of càdlàg path $\mathcal{F}$ is equi-right-regular, its oscillation can be controlled uniformly by a partition. The following lemma is a generalization of Lemma 1 in Section 12 of Billingsley [2, p. 122] to equi-right-regular families.

Proposition B.3. Let $\mathcal{F}$ be an equi-right-regular subset of $D(\mathbb{R}_{\geq 0}, E)$.

(i) For each $t > 0$ and $\varepsilon > 0$, there is a partition $\pi \in \text{Par}([0, t])$ that satisfies

$$O^-(f, \pi) := \sup_{[r, s] \in \pi} \omega(f, [r, s]) < \varepsilon \quad \text{for all } f \in \mathcal{F}.$$  

(ii) The set $\{s \in [0, t] \mid \|\Delta f(s)\| \geq C \text{ for some } f \in \mathcal{F} \}$ is finite for every $t > 0$ and $C > 0$.

(iii) Let $t \geq 0$ and suppose that $\|f(s)\| < C$ holds for all $s \in [0, t]$ and $f \in \mathcal{F}$. Then for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|f(s) - f(r)\|_E < C + \varepsilon$ for all $f \in \mathcal{F}$ whenever $s, r \in [0, t]$ satisfies $|s - r| < \delta$.

Proof. (i) Fix $t > 0$. Define

$$T = \{t' \in [0, t] \mid \text{there exists a partition } \pi \in \text{Par}[0, t'] \text{ satisfying } O^-(f, \pi) < \varepsilon \text{ for all } f \in \mathcal{F} \}$$

and $t^* = \sup T$. Then it suffices to show that $t^* = t$ and $t^* \in T$.

Note that $T \neq \emptyset$ and hence $t^* > 0$ by the equi-right continuity of $\mathcal{F}$ at 0. Since $\mathcal{F}$ is equi-right-regular at $t^*$, we can take a $\delta \in (0, t^*[t]$ such that $\|f(s) - f(t^*)\|_E < \varepsilon$ for all $f \in \mathcal{F}$ and $s \in [t^* - \delta, t^*[$. Next, choose a $t' \in [t^* - \delta, t^*] \cap T$ and a $\pi' \in \text{Par}([0, t'])$ so that $O^-(f, \pi') < \varepsilon$. Then the partition $\pi'' = \pi' \cup \{[t', t^*]\} \in \text{Par}([0, t^*])$ satisfies $O^-(f, \pi'') < \varepsilon$. Hence $t^* \in T$.

Now assume that $t^* < t$. Since $\mathcal{F}$ is equi-right-regular at $t^*$, we can take a $\delta > t^*$ such that $|f(t^*) - f(s)| < \varepsilon$ for all $f \in \mathcal{F}$ and $s \in [t^*, t[$. Moreover, choose a $\pi \in \text{Par}([0, t^*])$ satisfying $O^-(f, \pi) < \varepsilon$ for all $f \in \mathcal{F}$. Then the new partition $\pi = \pi' \cup \{[t^*, t]\}$ satisfies $O^-(f, \pi) < \varepsilon$ for all $f \in \mathcal{F}$ and thus we obtain $t \in T$. This contradicts the definition of $t^*$.

(ii) Let $C > 0$ and $t > 0$. Then, by (i), we can choose a $\pi \in \text{Par}([0, t])$ satisfying $O^-(f, \pi) < \varepsilon$ for all $f \in \mathcal{F}$. Then we see that

$$\{s \in [0, t] \mid \|\Delta f(s)\| \geq C \text{ for some } f \in \mathcal{F} \} \subset \pi^0.$$  

Since $\pi^0$ is finite, by definition, the set of jumps of $\mathcal{F}$ greater than $C$ is also finite.

(iii) Fix $\varepsilon > 0$ arbitrarily and choose a $\pi \in \text{Par}([0, t])$ such that $O^-(f, \pi) < \varepsilon/2$ for all $f \in \mathcal{F}$. Moreover, take a $\delta \in [0, \pi[. Then, each nonempty interval $[r, s] \subset [0, t]$ satisfying $|s - r| < \delta$ contains at most one element of $\pi^0$. If there is no element of $\pi^0$ in $[r, s]$, we have

$$\|f(s) - f(r)\|_E \leq O^-(f, \pi) < \varepsilon$$

27
Then the uniformity of uniform convergence on compact sets on \( D(\mathbb{R}_{\geq 0}, E) \) defines the entourage \( \mathcal{O} \) for all \( f \in \mathcal{F} \). This completes the proof.  

The notion of equi-right-regularity is useful to characterize the compactness in the space \( D(\mathbb{R}_{\geq 0}, E) \) with the topology of uniform convergence on compact subsets.

**Proposition B.4.** For \( \mathcal{F} \subset D(\mathbb{R}_{\geq 0}, E) \), the following conditions are equivalent:

(i) \( \mathcal{F} \) is relatively compact with respect to the topology of uniform convergence on compact subsets.

(ii) \( \mathcal{F} \) satisfies the following conditions:

(a) For any \( t \geq 0 \), the set \( \{ f(t) \mid f \in \mathcal{F} \} \) is relatively compact in \( E \);

(b) the family \( \mathcal{F} \) is equi-right-regular.

**Proof.** For convenience, let \( \mathcal{O}_p \) and \( \mathcal{O}_c \) denote the topology of pointwise convergence and that of uniform convergence on compact sets on \( D(\mathbb{R}_{\geq 0}, E) \), respectively. Given an \( \varepsilon > 0 \) and a compact interval \([0, t]\), define the entourage \( U_{\varepsilon, t} \) in \( D(\mathbb{R}_{\geq 0}, E) \) as

\[
U_{\varepsilon, t} = \{ (f, g) \in D(\mathbb{R}_{\geq 0}, E) \times D(\mathbb{R}_{\geq 0}, E) \mid \| f(s) - g(s) \| < \varepsilon \text{ for all } s \in [0, t] \}.
\]

Then the uniformity of uniform convergence on compact sets on \( D(\mathbb{R}_{\geq 0}, E) \) is generated by the family \( \{ U_{\varepsilon, t}; \varepsilon, t > 0 \} \).

\( i) \implies (ii). \) Suppose that \( \mathcal{F} \) is relatively compact in \( D(\mathbb{R}_{\geq 0}, E) \) with respect to \( \mathcal{O}_c \). Because every evaluation mapping \( ev : D(\mathbb{R}_{\geq 0}, E) \to E \) is continuous with respect to \( \mathcal{O}_p \), and hence to \( \mathcal{O}_c \), the image \( ev(F) \) is relatively compact in \( E \).

Next, we show that \( \mathcal{F} \) is equi-right-regular. Let \( s \geq 0 \) and choose a positive number \( t \) with \( t > s \). Given an \( \varepsilon > 0 \) and \( f \in \mathcal{F} \), set

\[
U_{\varepsilon, t}(f) = \{ g \in D(\mathbb{R}_{\geq 0}, E) \mid (f, g) \in U_{\varepsilon, t} \}.
\]

Since \( \mathcal{F} \) is totally bounded, there are finite elements \( f_1, \ldots, f_N \in \mathcal{F} \) such that

\[
\mathcal{F} \subset \bigcup_{1 \leq i \leq N} U_{\varepsilon/3, i}(f_i).
\]

For each \( i \in \{1, \ldots, N\} \), choose a \( \delta_i \in ]0, t - s[ \) such that

\[
u \in [s, s + \delta_i] \implies \| f(u) - f(s) \| < \frac{\varepsilon}{3}.
\]

Next, we define \( \delta = \bigwedge_{1 \leq i \leq N} \delta_i \). For \( f \in \mathcal{F} \), choose an \( i \in \{1, \ldots, N\} \) such that \( f \in U_{\varepsilon/3, i}(f_i) \). Then for all \( u \in [s, s + \delta] \),

\[
\| f(u) - f(s) \| \leq \| f(u) - f_i(u) \| + \| f_i(u) - f_i(s) \| + \| f_i(s) - f(s) \|
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Similarly, if $s > 0$ we can choose a $\gamma > 0$ that fulfills
\[
\|f(u) - f(s-))\| < \varepsilon
\]
for all $f \in \mathcal{F}$ and $u \in [s - \gamma, s]$. Thus, we can conclude that $\mathcal{F}$ is equi-right-regular.

(ii) $\implies$ (i). Assume that $\mathcal{F}$ satisfies conditions (a) and (b).

Take an arbitrary ultralimit and $\mathcal{F}$. Condition (a) implies that $\prod_{t \in \mathbb{R}^+} \mathcal{F}_{t}$ is totally bounded with respect to the uniformity of pointwise convergence on $\mathcal{F}$. Therefore, the filter generated by $\mathcal{F}$ on $\mathcal{F}$, which is indeed an ultralimit, converges to an element $g$ of $\mathcal{F}$.

Now let us show that $g \in D(\mathbb{R}^+, \mathcal{F})$. Fix an arbitrary $\varepsilon > 0$ and choose a corresponding $\delta > 0$ such that
\[
\|f(t) - f(s)\| < \frac{\varepsilon}{3}
\]
holds for all $f \in \mathcal{F}$ and all $s \in [t, t + \delta]$. For an arbitrary $s \in [t, t + \delta]$, choose a $B \in \mathcal{B}$ satisfying both $\mathcal{F}_{t}(B) \subset U(g(t), \varepsilon / 3)$ and $\mathcal{F}_{s}(B) \subset U(g(s), \varepsilon / 3)$. Such a $B$ indeed exists, because $g(t)$ and $g(s)$ are cluster points of $\mathcal{F}_{t}(\mathcal{B})$ and $\mathcal{F}_{s}(\mathcal{B})$, respectively, and $\mathcal{B}$ is a filter. Then,
\[
\|g(t) - g(s)\| \leq \|g(t) - f(t)\| + \|f(t) - f(s)\| + \|f(s) - g(s)\|
\]
\[
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]

Since $s$ is an arbitrary element of $[t, t + \delta]$, we can conclude that $g$ is right continuous at $t$.

Next, we show that $g$ has a left limit at every $t > 0$. Let $t > 0$ and $\varepsilon > 0$, and choose a $\gamma \in ]0, t]$ such that
\[
\|f(s) - f(t-))\| < \frac{\varepsilon}{4}
\]
holds for all $f \in \mathcal{F}$ and all $s \in ]t - \gamma, t]$. For an arbitrarily chosen $u, s \in ]t - \gamma, t]$, choose a $B \in \mathcal{B}$ that meets both $\mathcal{F}_{u}(B) \subset U(g(u), \varepsilon / 4)$ and $\mathcal{F}_{s}(B) \subset U(g(s), \varepsilon / 4)$. Then, using a $f \in B$, we obtain the estimate
\[
\|g(u) - g(s)\| \leq \|g(u) - f(u)\| + \|f(u) - f(t-)\| + \|f(t-) - f(s)\| + \|f(s) - g(s)\|
\]
\[
< \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \varepsilon.
\]
This estimate, along with the completeness of $\mathcal{F}$, implies the existence of the left limit $\lim_{t \uparrow \gamma} g(s)$.

It remains to show that the filter base $\mathcal{B}$, considered as a filter base on $D(\mathbb{R}^+, \mathcal{F})$, converges to $g$ with respect to $\mathcal{F}_{t}$. For $\varepsilon > 0$ and $t > 0$, take a finite sequence
\[
0 = t_0 < t_1 < \cdots < t_N = t
\]
such that
\[
\omega(f; [t_i, t_{i+1}]) < \frac{\varepsilon}{3}
\]
for all $i \in \{0, \ldots, N - 1\}$ and $f \in \mathcal{F} \cup \{g\}$. Moreover, choose a $B \in \mathcal{B}$ such that $\mathcal{F}_{t_i}(B) \subset B(g(t_i), \varepsilon / 3)$ for all $i \in \{0, \ldots, N\}$. Let $f \in B$ and arbitrarily choose an $s \in [0, t]$. If $s \in \{t_0, \ldots, t_N\}$, we have
\[
\|f(s) - g(s)\| < \frac{\varepsilon}{3}
\]
\footnote{This coincides with the restriction of product uniformity.}
by the choice of \( B \). On the other hand, if \( s \in ]t_i, t_{i+1}[ \) for some \( i \in \{0, \ldots, N - 1\} \), we see that
\[
\|f(s) - g(s)\| \leq \|f(s) - f(t_i)\| + \|f(t_i) - g(t_i)\| + \|g(t_i) - g(s)\| < \varepsilon.
\]
Therefore, \( f \in U_{\varepsilon,1}(g) \). The arbitrariness of \( f \) leads to \( B \subset U_{\varepsilon,1}(g) \). Thus, the filter base \( \mathcal{B} \) on \( D(\mathbb{R}_{\geq 0}, E) \) converges to \( g \).

**Proposition B.5.** Suppose that \( \mathcal{F} \subset D(\mathbb{R}_{\geq 0}, E) \) is relatively compact with respect to the topology of uniform convergence on compact sets. If a sequence \( (\pi_n) \) of partitions of \( \mathbb{R}_{\geq 0} \) satisfies (UC1), (UC2), and (C3) for \( \mathcal{F} \), then it satisfies (UC) for \( \mathcal{F} \).

**Proof.** Given an arbitrary \( t \geq 0 \) and a \( \delta > 0 \), choose finite elements \( f_1, \ldots, f_N \) of \( \mathcal{F} \) such that
\[
\mathcal{F} \subset \bigcup_{1 \leq i \leq N} U_{t_i, \delta/5}(f_i).
\]
By assumption, we can choose an \( \varepsilon_0 \) such that
\[
\sup_{\varepsilon < \varepsilon_0} \lim_{n \to \infty} O^\dagger_t(f_i - J_\varepsilon(f_i), \pi_n) < \frac{\delta}{5}
\]
for all \( 1 \leq i \leq N \). Moreover, for \( \varepsilon < \varepsilon_0 \), choose an \( N_\varepsilon \in \mathbb{N} \) such that
- \( [0, t] \cap D_\varepsilon(\mathcal{F}) \cap I \) contains at most one element for all \( I \in \pi_n \), and
- \( O^\dagger_t(f_i - J_\varepsilon(f_i), \pi_n) < \delta/5 \) for \( i \in \{1, \ldots, N\} \)
whenever \( n \geq N_\varepsilon \). Let \( n \geq N_\varepsilon \), and arbitrarily choose \( I \in \pi_n \) and \( f \in \mathcal{F} \). Then for any \( u, v \in I \), we have
\[
\|(f - J_\varepsilon(f))(u) - (f - J_\varepsilon(f))(v)\|
\]
\[
\leq O^\dagger_t(f_i - J_\varepsilon(f_i)) + 2 \sup_{s \in [0, t]} \|f_i(s) - f(s)\| + \sum_{s \in I \cap D_\varepsilon} \|\Delta f(s) - \Delta f_i(s)\|
\]
\[
\leq O^\dagger_t(f_i - J_\varepsilon(f_i)) + 4 \sup_{s \in [0, t]} \|f_i(s) - f(s)\|
\]
\[
\leq \delta.
\]
Therefore, we can see that
\[
\lim_{n \to \infty} \sup_{f \in \mathcal{F}} O^\dagger_t(f - J_\varepsilon(f); \pi_n) \leq \delta
\]
holds for all \( \varepsilon \leq \varepsilon_0 \). This shows that
\[
\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} \sup_{f \in \mathcal{F}} O^\dagger_t(f - J_\varepsilon(f); \pi_n) = 0,
\]
which is the assertion of the proposition.

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