On the classification of \((1+n)_{n\geq 2}\)-dimensional non-linear Klein-Gordon equation via Lie and Noether approach

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ABSTRACT
A complete group classification for the Klein-Gordon equation is presented. Symmetry generators, up to equivalence transformations, are calculated for each \( f(u) \) when the principal Lie algebra extends. Further, considered equation is investigated by using Noether approach for the general case \( n \geq 2 \). Conserved quantities are computed for each calculated Noether operator. At the end, a brief conclusion is presented.

Keywords
Klein-Gordon equation; Group classification; Noether approach; Conserved vectors.

SUBJECT CLASSIFICATION
35R01, 76M60

INTRODUCTION
The \((1+n)\)-dimensional Klein-Gordon equation

\[
u_{tt} = \Delta_x u + f(u), \quad f_{uu} \neq 0,
\]

where \( u = u(t, x_1, ..., x_n) \) with \( \Delta_x u = \sum_{i=1}^{n} u_{x_i x_i} \).

In the past, the authors of [3, 10] have studied Eq. (1) for different values of \( n \), for exact solutions, compatibility of the conditions for the reduction and reduced equations by consideration of an ansatz which reduces the dimension of the corresponding PDE (see [11]). In [9], the author discussed the symmetry properties and found particular solutions for some cases of Eq. (2). Tajiri [20] proposed some similarity and soliton solutions for the three-dimensional Klein-Gordon equation by means of similarity variables. Fushchych et al. [12] investigated the reductions and solutions by using the broken symmetry for Eq.(1) with \( n=3 \). In [8], Fedorchuk considered the reductions of Eq.(1) for \( n=4 \) by using decomposable subgroups of the generalized Poincare group P(1,4). Fushchych [10] invoked an ansatz of the form \( u = f(x)\phi(x) + g(x) \) to analyze exact solutions of Eq. (1). Description of such an ansatz for the Eq. (1) can be a difficult problem. That problem can be simplified by using symmetry methods.

Lie symmetry analysis is a systematic way to construct an ansatz which further reduces the dimension of the differential equation. The symmetry method also plays a central role in the algebraic analysis of the differential equation. There are nonlinear equations with arbitrary coefficients which possess nontrivial Lie point symmetries. Such nonlinear differential equations can be classified, with respect to unknown functions, according to the nontrivial Lie point symmetries they admit. This classification is known as group classification. The problem of group classification is one of the central aspects of modern symmetry analysis of differential equations. It was performed in the classical works of Lie.

For the nonlinear wave equation:

\[
u_{tt} = u_x x + f(t, x_1, u, u_x),
\]

the group properties are deduced by Ames [1]. Pucci [18] discussed the group classification of \( u_{tt} + u_{x_1 x_1} = f(u, u_x) \).
$u_n = u_{x,x_1} + u_{x,x_2} + u_{x,x_3} + f(u)$ was studied by Rudra [19]. The authors in [2] performed the group classification of the $(1+1)$-dimensional Klein-Gordon equation by using [7].

One of the classical aspects of the Lie theory is the computation of conservation laws. The existence of a large number of conserved quantities of a PDE or system of PDEs is a strong indication of its integrability. An efficient method to compute conservation laws is given by Noether [6,17]. The theorem states that there is a conservation law for the Noether symmetry of the differential equation. Conservation laws for the nonlinear $(1+1)$-dimensional wave equation $u_{tt} - (f(u)u_x)_x = 0$ are discussed in [15]. Bokhari et al. constructed the conservation laws [5] for the nonlinear $(1+n)$-dimensional wave equation $u_{ii} - (f(u)u_x)_x = 0$ via partial Noether approach. Conserved quantities for the $(1+1)$-dimensional nonlinear Klein-Gordon equation are reported in [14].

**Fundamental operators**

Consider the 2$^\text{nd}$ order PDE of the type

$$E(t,x_i,u_t,u_t,u_{x_i},u_{x_t},u_{x_{x_i}}) = 0$$

where $u$ is dependent variable, $t, x_i$ $(i = 1, 2, \ldots, n)$ are independent variables.

(I) The Euler operator is

$$\frac{\delta E}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - \sum_{i=1}^n D_{x_i} \frac{\partial}{\partial u_{x_i}} + \ldots$$

where

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_{x_i}} + \ldots$$

$$D_{x_i} = \frac{\partial}{\partial x_i} + u_i \frac{\partial}{\partial u} + u_{ij} \frac{\partial}{\partial u_{x_j}} + \ldots, \quad i = 1, \ldots, n$$

are known as the total derivative operators.

The generalized or Lie Backlund operator is defined by:

$$Y = \tau \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x_i} + \phi \frac{\partial}{\partial u} + \phi^j \frac{\partial}{\partial u_{x_j}} + \phi^{x_i} \frac{\partial}{\partial u_{x_i}} + \ldots$$

(II) Suppose $L = L(t,x_i,u_t,u_{x_t},u_{x_i}) \in A$ (space of differential functions) is a differentiable function such that $L$ is said to be a standard Lagrangian if

$$\frac{\delta L}{\delta u} = 0.$$  

(III) The generalized operator (6) satisfying

$$Y(L) + L(D_{t}\tau + \sum_{i=1}^n D_{x_i}^{\xi^i}) = D_{t}B^{0} + \sum_{i=1}^n D_{x_i}B^{i}$$

is known as the Noether operator associated with a Lagrangian $L$.

In Eq. (8), $B^{i}$ for $i = 0, 1, 2, \ldots, n$ are known as the gauge terms.

(IV) The equation
\[ D_i T^0 + \sum_{i=1}^n D_i T^i = 0 \]
evaluated on the solution space given by (2) is known as the conservation law for Eq. (2) and vector 
\[ T = (T^0, T^1, \cdots, T^n) \]
is said to be a conserved vector.

(V) The conserved vectors of the system (2) associated with a Noether operator \( X \) can be determined from the formula
\[ T^i = B^i - N^i (L). \tag{9} \]
In Eq. (9),
\[ N^0 = \tau + W \frac{\delta}{\delta u_i}, \quad N^i = \xi^i + W \frac{\delta}{\delta u_i}, \]
where \( W \) is known as the Lie characteristic function and can be found from
\[ W = \phi - \tau u_t - \sum_{j=1}^n \xi^j u_j. \tag{10} \]

The outline of this paper is as follows. In Section 2, the group classification of the \((1+n)\) -dimensional Klein-Gordon equation is given. Section 3 is for the Noether symmetry operators and conserved vectors of Eq. (1). Finally, conclusions are summarized at the end.

### Lie point symmetries

In this section, we discuss the group classification for the \((1+n)\)-dimensional Klein-Gordon equation, i.e. Eq. (1) for arbitrary \( n \). We apply the 2nd prolongation vector i.e.
\[ Y^{(2)}(u_t - \sum_{i=1}^n u_{x_i} - f(u)) \frac{\delta}{\delta u_t} = 0. \tag{11} \]
Eq. (11) yields the following determining equations:
\[ (i) \xi^i = 0, \quad (ii) \tau^i = 0, \quad (iii) \phi_{t^i} = 0, \tag{12} \]
\[ (i) \xi^i_{x_j} + \xi^j_{x_i} = 0, \quad (ii) \tau^i - \xi^i_{x_j} = 0, \quad (iii) \phi_{t^i} - \tau^i_{x_j} = 0, \tag{13} \]
\[ -\tau^i + \sum_{j=1}^n \phi_{x_i x_j} - 2\phi_{x_i} = 0, \quad -\tau^i + \sum_{j=1}^n \tau_{x_i x_j} + 2\phi_{x_i} = 0, \tag{14} \]
\[ \phi_{t^i} - \sum_{i=1}^n \phi_{x_i x_j} - 2f \tau^i + f_{\phi_t} - f_{\phi_{x_i}} = 0. \tag{15} \]
Eq. (13) forms a set of equations for an infinitesimal conformal transformation on \( R^{n+1} \) with Lorentz metric and thus the unknowns appearing in these equations are quadratic polynomials of \( \tau, x^1, \cdots, x^n \) (see [21]) and Eq. (14) implies
\[ \phi_{x_i} = \text{constant}, \quad \phi_{t^i} = \text{constant}, \quad i = 1, 2, \cdots, n. \tag{16} \]
Differentiating Eq. (15) with respect to \( u \) and using the results given in Eq. (16), yields
\[ \left( \frac{f_{\phi_t}}{f_{\phi_{x_i}}} \right)_{t^i} = 0. \tag{17} \]
The solutions of Eq. (17) yield the following functions

(i) \( f(u) = ae^{bu} + c \),
(ii) \( f(u) = c \ln(au + b) \),
(iii) \( f(u) = (au + b)^m + c, \ m \neq 0,1 \).

### Lie algebra

In this section, we discuss the different forms of \( f(u) \), up to equivalence transformations, which lead to an extension of the principal Lie algebra of Eq. (1) for \( n = 2,3,\ldots \).

For \( n = 1 \), the results are presented in [2].

#### When \( f(u) \) is arbitrary

The minimal algebra for the arbitrary case is:

\[
Y_0 = \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial x_i}, \quad Y_{n+i} = x_i \frac{\partial}{\partial t} + t \frac{\partial}{\partial x_i}, \quad Y_{2n+i} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}, \quad j > i
\]

and appeared in all the rest of the considered cases, thus we shall only present the additional algebra(s). The principal Lie algebra for this case is of dimension \( n(3+n)/2+1 \).

\[ f(u) = ae^{bu} + c \]

For \( c = 0 \) the principal algebra extends and additional generators will be:

\[
Y_{3n+p} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - 2 \frac{\partial}{\partial u}, \quad \text{where} \quad P = \begin{cases} 0, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd} \end{cases}
\]

The Lie algebra is of dimension \( n(n+3)/2+2 \).

\[ f(u) = c \ln(au + b) \]

There is no extension in the principal algebra.

\[ f(u) = (au + b)^m + c, \ m \neq 0,1 \]

In this case, for \( b = 0 = c \) leads to an extension of the principal algebra and additional generators will be:

\[
Y_{3n+p} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{2u}{m-1} \frac{\partial}{\partial u}, \quad \text{where} \quad p \text{ is defined in (20) and the Lie algebra is of dimension } n(n+3)/2+2.
\]

### Noether symmetries

In this section, we will use Noether approach for finding the conserved vectors of Eq. (1) for arbitrary \( n \), taking \( n \geq 2 \).

#### Case 1: \( n=1 \)
Noether operators and conserved vectors of Eq. (1) for \( n = 1 \) are reported in [14].

**Case 2: n=2**

The standard Lagrangian for Eq. (1) will be

\[
L = \frac{u_i^2}{2} - \sum_{i=1}^{n} \frac{u_{x_i}^2}{2} + F(u), \quad F'(u) = f(u).
\]  
\[
(22)
\]

The Noether determining equation (8) with the help of Eq. (22) after some lengthy manipulation gives the following set of determining equations:

\[
(i) \tau_i = 0, \quad (ii) \xi_{x_i} = 0, \quad (iii) \phi_{u_i} = \tau_i - \sum_{i=1}^{n} \xi_{x_i},
\]

\[
(23)
\]

\[
(i) \tau_{x_i} - \xi_{x_i} = 0, \quad (ii) \xi_{x_i} + \tau_{x_i} = 0, \quad (i) B_{u_i}^0 = \phi_i, \quad (ii) B_{u_i}^i = -\phi_i,
\]

\[
(24)
\]

Hence doing the routine calculation, Eq. (25) yields:

\[
\phi((2n + 1) f_{u u} f_{u u u u} - 2n f_{u u u u}^2) = 0.
\]

Eq. (26) further divides two cases and discussed in the following sections.

\[\phi = 0\]

For this case, the Noether operators will be:

\[Y_0, \quad Y_i, \quad Y_{n+i}, \quad Y_{2n+i}\].

This forms the minimal algebra and thus thus we shall only present the additional algebras in the next section.

\[\phi \neq 0\]

For this case, the additional Noether operators will be:

\[Y_{3n+i} = x_i t \frac{\partial}{\partial t} + \frac{1}{2} \left[ x_i^2 + t^2 - \sum_{j=1}^{n} x_j^2 \right] \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} x_i x_j \frac{\partial}{\partial x_j} - \frac{x_i u}{2} \frac{\partial}{\partial u}, \quad j \neq i,\]

\[Y_{4n+1} = t \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{1}{2} \left[ t^2 + \sum_{i=1}^{n} x_i^2 \right] \frac{\partial}{\partial t} - \frac{u t}{2} \frac{\partial}{\partial u}, \quad Y_{4n+2} = t \frac{\partial}{\partial t} + \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{u}{2} \frac{\partial}{\partial u}.\]

### Conserved quantities

**I** The vector \(T_0\) corresponding to \(Y_0\) has the following components:

\[T_0^0 = \frac{1}{2} \left[ u_i^2 + \sum_{j=1}^{n} u_{x_j}^2 \right] - F(u), \quad T_0^i = -u_{x_i} u_i.\]

**II** The components of the vector \(T_i\) are:

\[T_i^0 = u_{x_i} u_i, \quad T_i^i = \frac{1}{2} \left[ u_i^2 + u_{x_i}^2 - \sum_{j=1}^{n} u_{x_j}^2 \right] - F(u), \quad T_i^j = -u_{x_i} u_j, \quad j \neq i.\]
For $Y_{n+i}$ the components of $T_{n+i}$ are:

\[ T_{n+i}^0 = \frac{x_i}{2} \left( \sum_{j=1}^{n} u_j^2 - 2F(u) \right) + tu_i u_e, \quad T_{n+i}^i = \frac{-f}{2} \left[ u_i^2 + u_e^2 - \sum_{j=1, j \neq i}^{n} u_j^2 + 2F(u) \right] - x_i u_e u_j, \]

\[ T_{n+i}^j = -(x_i u_j + tu_e u_j) u_j, \quad j \neq i. \]

For $Y_{2n+j}$ with $j > i$ the conserved vector $T_{2n+j}$ has the following components:

\[ T_{2n+i}^0 = (x_j u_j - x_i u_i) u_e, \quad T_{2n+i}^i = \frac{-x_j}{2} \left[ u_j^2 + u_e^2 - \sum_{k=1, k \neq i}^{n} u_k^2 + 2F(u) \right] + x_j u_i u_j, \]

\[ T_{2n+i}^j = \frac{x_j}{2} \left[ u_j^2 + u_e^2 - \sum_{k=1, k \neq j}^{n} u_k^2 + 2F(u) \right] - x_j u_i u_j, \quad k \neq i, j. \]

For $Y_{3n+i}$ the components of $T_{3n+i}$ are:

\[ T_{3n+i}^0 = \frac{x_i}{2} \left[ u_i^2 + \sum_{k=1}^{n} u_k^2 \right] + \frac{x_i}{2} \left[ \sum_{j=1}^{n} x_j^2 \right] u_i + \sum_{j=1}^{n} x_j x_i u_j, \]

\[ T_{3n+i}^i = \frac{1}{2} \left[ u_i^2 + \sum_{j=1}^{n} x_j^2 \right] u_i - \frac{x_i}{2} \sum_{j=1}^{n} x_j u_j, \]

\[ T_{3n+i}^j = \frac{1}{2} \left[ u_i^2 + \sum_{m=1}^{n} u_m^2 \right] u_j - \frac{x_i}{2} \sum_{m=1}^{n} x_m u_m, \quad m \neq i, j. \]

For $Y_{4n+1}$ the components of $T_{4n+1}$ are:

\[ T_{4n+1}^0 = -\frac{u_i^2}{4} + \frac{1}{4} \left( r^2 + \sum_{i=1}^{n} x_i^2 \right) \left[ u_i^2 + \sum_{i=1}^{n} u_i^2 \right] + \left[ \frac{u_e}{2} + x_i u_e \right] u_i, \]

\[ T_{4n+1}^i = -\frac{x_i}{2} \left[ u_i^2 + u_e^2 - \sum_{j=1}^{n} u_j^2 \right] - \frac{u_e}{2} \left( r^2 + \sum_{i=1}^{n} x_i^2 \right) u_i + \sum_{i=1}^{n} x_i u_i u_j, \]

\[ T_{4n+1}^j = -\frac{x_j}{2} \left[ u_j^2 + u_e^2 - \sum_{j=1}^{n} u_j^2 \right] - \frac{u_j}{2} + tu_j + \sum_{j=1}^{n} x_j u_j u_i, \quad j \neq i. \]

For $Y_{4n+2}$, the vector $T_{4n+2}$ has the following components:

\[ T_{4n+2}^0 = \frac{t}{2} \left[ u_i^2 + \sum_{i=1}^{n} u_i^2 \right] + \left[ \frac{u_i}{2} + \sum_{i=1}^{n} x_i u_i \right] u_i, \]

\[ T_{4n+2}^i = -\frac{x_i}{2} \left[ u_i^2 + u_e^2 - \sum_{j=1}^{n} u_j^2 \right] - \frac{u_i}{2} + tu_i + \sum_{j=1}^{n} x_j u_j u_i, \]

\[ T_{4n+2}^j = -\frac{x_j}{2} \left[ u_j^2 + u_e^2 - \sum_{j=1}^{n} u_j^2 \right] - \frac{u_j}{2} + tu_j + \sum_{j=1}^{n} x_j u_j u_i, \]

Where $p$ is defined in (20).

Conclusion
A complete classification of the \((1 + n)\)-dimensional Klein-Gordon equation is reported. The procedure is carried out for arbitrary \(n\). A class of functions is obtained which possess the non-trivial Lie point symmetries. It is observed that obtained class of function does not depend on the number of independent variables. Extensions of the principal algebras up to equivalence transformations are constructed for each \(f(u)\). These symmetry algebras can be further used to construct ansatz or similarity variables.

It should be noted that in the entire obtained Lie point symmetries, the coefficients of \(\partial_t\) and \(\partial_{x_i}\) are independent of the dependent variable \(u\) and hence the obtained symmetries are fiber preserving or projective transformations. Such transformations allow one to calculate the expression for the group transformations on the actual function \(u(t, x_1, \ldots, x_n)\) with less difficulty. Noether operators and conservation laws of the considered equation are computed by using Noether approach for \(n \geq 2\).

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REFERENCES
1. W. F. Ames, R. J. Lohner and E. Adams, Group properties of \(u_{tt} = (f(u)u_x)_x\), Int. J. Nonlinear Mech., 16 (1981) 439-447.
2. H. Azad, M. T. Mustafa and M. Ziad, Group classification, optimal system and optimal reductions of a class of Klein-Gordon equations, Commun. Nonlinear Sci. Numer. Simul., 15(5) (2009) 1132-1147.
3. L. F. Barannyk, A. F. Barannyk and W. I. Fushchych, Reduction of multi-dimensional Poincare invariants nonlinear equation to two-dimensional equations, Ukrain. Math. J., 43(10) (1991) 1311-1323.
4. G. Bluman and S. Kumei, Symmetries and Differential Equation, Springer-Verlag, New York, 1989.
5. A. H. Bokhari, A. Y. Al-Dweik, F. M. Mahomed and F. D. Zaman, Conservation laws of a nonlinear \((n + 1)\)-wave equation, Nonlinear Analysis: Real World Applications, 11(4) (2010) 2862-2870.
6. N. Byers, E. Noether's discovery of the deep connection between symmetries and conservation laws, Isr. Math. Conf. Proc., 12, (1999) 67 – 82.
7. P. A. Clarkson and E. L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equation, Phys. D, 70(3) (1993) 250-288.
8. V. M. Fedorchuk, Symmetry reduction and exact solutions of a nonlinear five-dimensional wave equation, Ukrain. Math. J., 48(4) (1996) 573-576.
9. W. I. Fushchych, On symmetry and exact solution of multi-dimensional nonlinear wave equation, Ukrain. Math. J., 39(1) (1987) 116-123.
10. W. I. Fushchych, The symmetry of mathematical physics problems, in Algebraic-Theoretical Studies in Mathematical Physics, Kiev, Mathematical Institute, (1981) 6-28.
11. W. I. Fushchych, Ansatz-95, J. Nonlinear Math. Phys., 2 (1995) 216-235.
12. W. I. Fushchych and I. M. Tsyfra, On a reduction and exact solutions of nonlinear wave equations with broken symmetry, Scientific Works, 3 (2001) 251-255.
13. P. E. Hydon, Symmetry Methods for Differential Equations, Cambridge University Press, Cambridge, 2000.
14. A. Jhangeer and S. Sharif, Conservation laws for the nonlinear Klein-Gordon equation, Afr. Mat., 25(3) (2014) 833-840.
15. A. G. Johnpillai, A. H. Kara and F. M. Mahomed, Conservation laws of a nonlinear \((1 + 1)\)-dimensional wave equation, Nonlinear Anal. B, 11(4) (2010) 2237-2242.
16. V. Lahno, R. Zhdanov and O. Magda, Group classification and exact solutions of nonlinear wave equation, Acta Appl. Math., 91 (2006) 253-313.
17. E. Noether, Invariante Variationsprobleme, Nacr. Konig. Gesell. Wissen., Gottingen, Math.-Phys. Kl. Heft, 2 (1918) 235 - 257. (English translation in Transport Theory and Statistical Physics, 1(3), 186-207).

18. E. Pucci, Group analysis of the equation $u_{tt} + \lambda u_{xx} = g(u, u_x)$, Riv. Mat. Univ. Parma, 4 (1987) 71-87.

19. P. Rudra, Symmetry group of the nonlinear Klein-Gordon equation, J. Phys. A: Math. Gen., 19(13) (1986) 2499-2504.

20. M. Tajiri, Some remarks on similarity and soliton solutions of nonlinear Klein-Gordon equations, J.Phys. Soc. Japan, 53 (1984) 3759-3764.

21. P. J. Olver, Applications of Lie Groups to Differential Equations, Springer-Verlag, New York, 1986.