A note on Severi varieties of nodal curves on Enriques surfaces

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Abstract Let $|L|$ be a linear system on a smooth complex Enriques surface $S$ whose general member is a smooth and irreducible curve of genus $p$, with $L^2 > 0$, and let $V_{|L|,\delta}(S)$ be the Severi variety of irreducible $\delta$-nodal curves in $|L|$. We denote by $\pi : X \to S$ the universal covering of $S$. In this note we compute the dimensions of the irreducible components $V'$ of $V_{|L|,\delta}(S)$. In particular we prove that, if $C$ is the curve corresponding to a general element $[C]$ of $V$, then the codimension of $V$ in $|L|$ is $\delta$ if $\pi^{-1}(C)$ is irreducible in $X$ and it is $\delta - 1$ if $\pi^{-1}(C)$ consists of two irreducible components.

1 Introduction

Let $S$ be a smooth complex projective surface and $L$ a line bundle on $S$ such that the complete linear system $|L|$ contains smooth, irreducible curves (such a line bundle, or linear system, is often called a Bertini system). Let

$$p := p_\delta(L) = \frac{1}{2}L \cdot (L + K_S) + 1,$$

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be the arithmetic genus of any curve in $|L|$.

For any integer $0 \leq \delta \leq p$, consider the locally closed, functorially defined subscheme of $|L|$

$$V_{|L|,\delta}(S) \text{ or simply } V_{|L|,\delta}$$

parameterizing irreducible curves in $|L|$ having only $\delta$ nodes as singularities; this is called the **Severi variety** of $\delta$-nodal curves in $|L|$. We will let $g := p - \delta$, the geometric genus of the curves in $V_{|L|,\delta}$.

It is well-known that, if $V_{|L|,\delta}$ is non-empty, then all of its irreducible components $V$ have dimension $\dim(V) \geq \dim |L| - \delta$. More precisely, the Zariski tangent space to $V_{|L|,\delta}$ at the point corresponding to $C$ is

$$T_{[C]} V_{|L|,\delta} \cong H^0(L \otimes J_N) / <C >,$$

(1)

where $J_N = J_N|_S$ is the ideal sheaf of subscheme $N$ of $S$ consisting of the $\delta$ nodes of $C$ (see, e.g., [4, §1]). Thus, $V_{|L|,\delta}$ is smooth of dimension $\dim |L| - \delta$ at $[C]$ if and only if the set of nodes $N$ imposes independent conditions on $|L|$. In this case, $V_{|L|,\delta}$ is said to be regular at $[C]$. An irreducible component $V$ of $V_{|L|,\delta}$ will be said to be regular if the condition of regularity is satisfied at any of its points, equivalently, if it is smooth of dimension $\dim |L| - \delta$.

The existence and regularity problems of $V_{|L|,\delta}(S)$ have been studied in many cases and are the most basic problems one may ask on Severi varieties. We only mention some of known results. In the case $S \cong \mathbb{P}^2$, Severi proved the existence and regularity of $V_{|L|,\delta}(S)$ in [14]. The description of the tangent space is due to Severi and later to Zariski [15]. The existence and regularity of $V_{|L|,\delta}(S)$ when $S$ is of general type has been studied in [4] and [3]. Further regularity results are provided in [10]. More recently Severi varieties on K3 surfaces have received a lot of attention for many reasons. In this case Severi varieties are known to be regular (cf. [13]) and are nonempty on general K3 surfaces by Mumford and Chen (cf. [12], [2]).

As far as we know, Severi varieties on Enriques surfaces have not been studied yet, apart from [8, Thm. 4.12] which limits the singularities of a general member of the Severi variety $V^g_{|L|}$ of irreducible genus $g$ curves in $|L|$, and gives a sufficient condition for the density of the latter in the Severi variety $V_{|L|,p-g}$ of $(p - g)$-nodal curves. In particular, the existence problem is mainly open and we intend to treat it in a forthcoming article. The result of this paper is Proposition[11] which answers the regularity question for Severi varieties of nodal curves on Enriques surfaces.

## 2 Regularity of Severi varieties on Enriques surfaces

Let $S$ be a smooth Enriques surface, i.e., a smooth complex surface with nontrivial canonical bundle $\omega_S \cong O_S$, such that $\omega_S^{\otimes 2} \cong \omega_S$ and $H^1(O_S) = 0$. We denote linear (resp. numerical) equivalence by $\sim$ (resp. $\equiv$).

Let $L$ be a line bundle on $S$ such that $L^2 > 0$. It is well-known that $|L|$ contains smooth, irreducible curves if and only if it contains irreducible curves (see [5, Thm. 6.1].)
4.1 and Prop. 8.2); in other words, on Enriques surfaces the Bertini linear systems are the linear systems that contain irreducible curves. Moreover, by [6, Prop. 2.4], this is equivalent to $L$ being nef and not of the form $L \sim P + R$, with $|P|$ an elliptic pencil and $R$ a smooth rational curve such that $P \cdot R = 2$ (in which case $p = 2$). If $|L|$ is a Bertini linear system, the adjunction formula, the Riemann–Roch theorem, and Mumford vanishing yield that

$$L^2 = 2(p - 1) \quad \text{and} \quad \dim |L| = p - 1$$

(see, e.g., [5, 7]).

Let $K_S$ be the canonical divisor. It defines an étale double cover

$$\pi : X \longrightarrow S$$

where $X$ is a smooth, projective $K3$ surface (that is, $\omega_X = O_X$ and $H^1(O_X) = 0$), endowed with a fixed-point-free involution $\iota$, which is the universal covering of $S$. Conversely, the quotient of any $K3$ surface by a fixed-point-free involution is an Enriques surface.

Let $C \subset S$ be a reduced and irreducible curve of genus $g \geq 2$. We will henceforth denote by $\nu_C : \tilde{C} \rightarrow C$ the normalization of $C$ and define $\eta_C := O_C(K_S) = O_C(-K_S)$, a nontrivial 2-torsion element in $\text{Pic}^0 C$, and $\eta_{\tilde{C}} := \nu_\iota^* \eta_C$. The fact that $\eta_C$ is nontrivial follows from the cohomology of the restriction sequence

$$0 \longrightarrow O_S(K_S - C) \longrightarrow O_S(K_S) \longrightarrow \eta_C \longrightarrow 0,$$

which yields $h^0(\eta_C) = h^1(K_S - C) = h^1(C) = 0$, the latter vanishing as $C$ is big and nef. One has the fiber product

$$\begin{array}{ccc}
\pi^{-1}(C) \times_C \tilde{C} & \longrightarrow & \tilde{C} \\
\downarrow & & \downarrow \nu_C \\
\pi^{-1}(C) & \longrightarrow & C,
\end{array}$$

where $\pi_{\pi^{-1}(C)}$ and the upper horizontal map are the double coverings induced respectively by $\eta_C$ and $\eta_{\tilde{C}}$. By standard results on coverings of complex manifolds (cf. [1, Sect. I.17]), two cases may happen:

- $\eta_{\tilde{C}} \not\cong O_{\tilde{C}}$ and $\pi^{-1}C$ is irreducible, as in Fig. 1.
- $\eta_{\tilde{C}} = O_{\tilde{C}}$ and $\pi^{-1}C$ consists of two irreducible components conjugated by the involution $\iota$. These two components are not isomorphic to $C$, as $\eta_C$ is nontrivial, as in Fig. 2 (each component of $\tilde{C}$ is a partial normalization of $C$).

As mentioned in the Introduction, it is well-known that any irreducible component of a Severi variety on a $K3$ surface is regular when nonempty (see, e.g., [4, Ex. 1.3]; see also [8, §4.2]). The corresponding result on Enriques surfaces is the following.
First note that, in the above notation, the dimension of the Severi variety of genus $$g = p_g(C)$$ curves in $$|L| = |C|$$ at the point $$[C]$$ satisfies the inequality

$$\dim_{[C]}(V^g_{|L|}) \geq h^0(\omega_C \otimes \eta_C) = \begin{cases} 
  g - 1 & \text{if } \eta_C \neq O_C \\
  g & \text{if } \eta_C \simeq O_C 
\end{cases}$$

(see [8] Proofs of Thm. 4.12 and Cor. 2.7]). Our result implies that the latter is in fact an equality when $$C$$ is nodal, and gives a concrete geometric description of the situation in both cases.

**Proposition 1.** Let $$L$$ be a Bertini linear system, with $$L^2 > 0$$, on a smooth Enriques surface $$S$$. Then the Severi variety $$V_{|L|,0}(S)$$ is smooth and every irreducible component $$V \subseteq V_{|L|,0}(S)$$ has either dimension $$g - 1$$ or $$g$$; in the former case the component is regular. Furthermore, with the notation introduced above,

1. for any curve $$C$$ in a $$(g - 1)$$-dimensional irreducible component $$V$$, $$\pi^{-1}C$$ is irreducible (whence an element in $$V^g_{|\pi^*L|,\delta}(X)$$);
2. for any $$g$$-dimensional component $$V$$, there is a line bundle $$L'$$ on $$X$$ with $$(L')^2 = 2(p - d) - 2$$ and $$L' \cdot \iota^*L' = 2d$$ for some integer $$d$$ satisfying

$$\frac{p - 1}{2} \leq d \leq \delta,$$

such that $$\pi^*L \simeq L' \otimes \iota^*L'$$, and the curves parametrized by $$V \subseteq V_{|\pi^*L|,\delta}(S)$$ are the birational images by $$\pi$$ of the curves in $$V^g_{|\pi^*L|,\delta-d}(X)$$ intersecting their conjugates by $$\iota$$ transversely (in $$2d$$ points). In other words, for any $$[C] \in V$$, we have $$\pi^{-1}C = Y + \iota(Y)$$, with $$[Y] \in V^g_{|\pi^*L|,\delta-d}(X)$$ and $$[\iota(Y)] \in V^g_{|\pi^*L'|,\delta-d}(X)$$ intersecting transversely.

Furthermore, if $$L' \simeq \iota^*L'$$, which is the case if $$S$$ is general in moduli, then $$d = \frac{p - 1}{2}$$ and $$L \sim 2M$$, for some $$M \in \text{Pic } S$$ such that $$M^2 = d$$. 
We will henceforth refer to components of dimension $g-1$ as regular and the ones of dimension $g$ as nonregular. Note however that from a parametric perspective the Severi variety has the expected dimension and is smooth in both cases, as the fact that (3) is an equality indicates; we do not dwell on this here, and refer to [8] for a discussion of the differences between the parametric and Cartesian points of view (the latter is the one we adopted in this text).

Note that Proposition 1 does not assert that the Severi variety $V_{L,\delta}$ is necessarily non-empty: in such a situation, $V_{L,\delta}$ does not have any irreducible component and the statement is empty.

**Proof.** Pick any curve $C$ in an irreducible component $V$ of $V_{L,\delta}(S)$. Let $f : \tilde{S} \to S$ be the blow-up of $S$ at $N$, the scheme of the $\delta$ nodes of $C$, denote by $\epsilon$ the (total) exceptional divisor and by $\tilde{C}$ the strict transform of $C$. Thus $f|_{\tilde{C}} = \nu|_{\tilde{C}}$ and we have

$$K_{\tilde{S}} \sim f^*K_S + \epsilon \quad \text{and} \quad \tilde{C} \sim f^*C - 2\epsilon.$$

From the restriction sequence

$$0 \to \mathcal{O}_{\tilde{S}}(\epsilon) \to \mathcal{O}_{\tilde{S}}(\tilde{C} + \epsilon) \to \omega_{\tilde{C}}(\eta_{\tilde{C}}) \to 0$$

we find

$$\dim T_{[C]}V_{L,\delta}(S) = \dim |L \otimes \mathcal{I}_N| = h^0(L \otimes \mathcal{I}_N) - 1 = h^0(f^*L - \epsilon) - 1 = h^0(\mathcal{O}_{\tilde{S}}(\tilde{C} + \epsilon)) - 1 = h^0(\omega_{\tilde{C}}(\eta_{\tilde{C}}))$$

$$= \begin{cases} g - 1, & \text{if } \eta_{\tilde{C}} \not\cong \mathcal{O}_{\tilde{C}}, \\ g, & \text{if } \eta_{\tilde{C}} \cong \mathcal{O}_{\tilde{C}}. \end{cases} \quad (4)$$

In the upper case, by (1), we have that $V_{L,\delta}$ is smooth at $[C]$ of dimension $g - 1 = p - \delta - 1 = \dim |L \otimes \mathcal{I}_N|$.

Assume next that we are in the lower case. Then, by the discussion prior to the proposition, we have $\pi^{-1}C = Y + \iota(Y)$ for an irreducible curve $Y$ on $X$, such that $\pi$ maps both $Y$ and $\iota(Y)$ birationally, but not isomorphically, to $C$. In particular, $Y$ and $\iota(Y)$ have geometric genus $p_g(Y) = p_g(\iota(Y)) = p_g(C) = p - \delta = g$. Set $L' := O_X(Y)$ and $2d := Y \cdot \iota(Y)$. Note that $d$ is an integer because, if $y = \iota(x) \in Y \cap \iota(Y)$, then $\iota(y) = x \in Y \cap \iota(Y)$. Since $Y \cong \iota(Y)$ and $\pi$ is étale, both $Y$ and $\iota(Y)$ are nodal with $\delta - d$ nodes and they intersect transversely at $2d$ points, which are pairwise conjugate by $\iota$, and therefore map to $d$ nodes of $C$. Hence $d \leq \delta$. We have

$$p_a(Y) = p_a(\iota(Y)) = g + \delta - d = p - \delta + \delta - d = p - d. \quad (5)$$

whence

$$(L')^2 = 2(p - 1 - d).$$

By the Hodge index theorem, we have
By the regularity of Severi varieties on $K3$ surfaces, any irreducible component of $V_{\ell|\delta-d}(X)$ has dimension $\dim |L|-(\delta-d)=p_g(Y)=g$. Hence, $V$ is $g$-dimensional; more precisely, the curves parameterized by $V$ are the (birational) images by $\pi$ of the curves in an irreducible component of $V_{\ell|\delta-d}(X)$ intersecting their conjugates by $\iota$ transversely (in $2d$ points). By (4), it also follows that $\dim V = \dim J_{(C)}V_{[L|\delta]}(S)$, so that $[C]$ is a smooth point of $V_{[L|\delta]}(S)$.

To prove the final assertion of the proposition, observe that, by the regularity of Severi varieties on $K3$ surfaces, we may deform $Y$ and $\iota(Y)$ on $X$ to irreducible curves $Y'$ and $\iota(Y')$ with any number of nodes $\leq \delta - d$ and intersecting transversally in $2d$ points; in particular, we may deform $Y$ and $\iota(Y)$ to smooth curves $Y'$ and $\iota(Y')$. Thus, $C' := \pi(Y')$ is a member of $V_{[L|\delta]}$, whence of geometric genus $p - d$. Since $\dim |Y'| = p_a(Y') = p_g(C') = p_a(C') - d = p - d$, the component of $V_{[L|\delta]}$ containing $[C']$ has dimension $\dim |L| - d + 1 = p - d$. We thus have $\dim |L| + J_{[C']}| = \dim |L| - d + 1$, where $N'$ is the set of $d$ nodes of $C'$, hence $N'$ does not impose independent conditions on $|L|$. Assume now that $L' = \iota L'$, which — as is well-known (see, e.g., [9 §11]) — is the case occurring for generic $S$, as then $\text{Pic} X$ is precisely the invariant part under $\iota$ of $H_2(X, \mathbb{Z})$. Then $2d = L' \cdot C' = (L')^2 = 2(p - 1 - d)$, so that $p - 1 = 2d$. Since $L' = 2(p - 1) = 4d$ and $N'$ does not impose independent conditions on $|L|$, by [11] Prop. 3.7] there is an effective divisor $D \subset S$ containing $N'$ satisfying $L - 2D \geq 0$ and

$$L \cdot D - d \leq D^2 \leq \frac{1}{2}L \cdot D \leq d,$$

with equality in (i) or (ii) only if $L = 2D$; moreover, since $L - 2D \geq 0$, the numerical equivalence $L \equiv 2D$ implies the linear equivalence $L \sim 2D$. Now since $N' \subset D$, we must have $L \cdot D = C' \cdot D \geq 2d$, hence the inequalities in (6) are all equalities, and thus $D^2 = d$ and $L \sim 2D$.

The following corollary is a straightforward consequence of Prop. [1] and the fact that the nodes on curves in a regular component in a Severi variety (on any surface and in particular on a $K3$ surface) can be independently smoothened.

**Corollary 1.** If a Severi variety $V_{[L|\delta]}$ on an Enriques surface has a regular (resp., nonregular) component, then for any $0 \leq \delta' \leq \delta$ (resp., $d \leq \delta' \leq \delta$, with $d$ as in Prop. [1]), also $V_{[L|\delta']}$ contains a regular (resp., nonregular) component.

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Severi varieties on Enriques surfaces

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