Dynamic Analysis and Vibrations Research on Multi-Degree of Freedom Shear Structures

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**ABSTRACT:** Vibration is everywhere in our daily life. This paper studies the effect of rectangular pulse and free vibration on multi-degree of freedom systems. An undamped three-floor shear frame structure subjected to rectangular pulse and a damping three-floor shear building considered free vibration is investigated with respect to natural frequencies, mode shapes, and displacements of each floor level. This paper assumes the three-story shear building structure discussed has orthogonal damping, and the fundamental frequency will not change over time, which means that the mode shape of the system does not also change with time. In addition, the modal superposition method and Duhamel integrals are useful to study the response of the various vibration systems. In this work, it can be concluded that the main response is due to the vibrations' first mode, even though all its mode shapes can lead to free vibration response. For amplitude of the motion, it will change with the frequency of excitation. When the non-periodic impulse is applied, the forced vibration will be different in the short duration of the impulse with the free vibration phase.

1. INTRODUCTION

Most human activities are related to vibration or its derivatives [1]. For example, the sound humans make is due to the vibrations of our vocal cords, and the heartbeat can also be considered a vibration with a particular frequency. From a micro point of view, every particle is vibrating all the time, but it is invisible to the naked eye. Through studies of vibration, people can explain phenomena in daily life, invent objects that work with vibrations, and give early warning of some natural disasters such as earthquakes and volcanic eruptions. The vibration in reality, in most cases, contains multiple degrees of freedom.

In the literature, extensive researches have been presented on investigating the response of various multi-degree of freedom (MDOF) systems. Elhelloty [2] adopted the modal and transient analysis on ANSYS16 to evaluate the effect of lateral load resisting systems on frame buildings. Displacement distribution and stress distribution are then visualized from time-displacement and time-stress response graphs of various systems with and without lateral load resisting systems. Baig [3] also investigated the displacements of a 15-story structure bare frame at different floor levels by using ANSYS.
Multiple advanced analytical approaches based on computational software were applied in the above research. The present research only uses simple numerical methods to establish two simplified models to simulate different multi-degree of freedom systems for obtaining the response of the shear structure with the effect of impulsive load and free vibration. This method, also known as the modal method, is the most popular and practical approach of structural analysis and identification from input and response measurements of multi-degree of freedom systems [4].

For two models created, one is an undamped building with three stories subjected to a rectangular pulse force on the third floor. According to the structure, formulate equations for the floor lateral displacements as functions of time, then the lateral story drift in the second story as a function of time is obtained. Moreover, to simplify the problem, lumped masses of each floor are placed in the middle.

Figure 1 Model 1: Undamped system subjected to the Rectangular Pulse Force

The other is also a three-story shear construction where the undamped system and damping system are compared. Start with the calculation of frequencies and mode shapes corresponded to mass and stiffness matrices without damping. Then add Rayleigh damping on each floor to discover the floor displacements when the initial condition is free vibration.

Figure 2 Model 2: the Free Vibration Model
2. PROCEDURE AND RESULTS

2.1. Orthogonality Properties
In this section, the third mode shape is calculated by the given first two natural vibration mode shapes according to orthogonality properties.

Making $\Phi_3 = (a \ b \ c)$

By Betti’s law, the normalized shapes:

$$\begin{align*}
\Phi_m^T m \Phi_n &= 0 \\
\Phi_m^T K \Phi_n &= 0
\end{align*}$$

Then get two equations:

$$(a \ b \ c) 600 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1.0 \\ 0.6667 \\ 0.3333 \end{bmatrix} = 0$$ (2)

$$(a \ b \ c) 600 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1.0 \\ -0.5 \\ -0.5 \end{bmatrix} = 0$$ (3)

By solving these matrixes, two three elements first order equations can be expressed:

$$\frac{1}{2} (a - b - c) = 0$$ (4)

$$\frac{1}{3} (a + b + c) = 0$$ (5)

Then the answer is: $a = 0$, and $b, c$ must be a pair of opposite numbers.

2.2. Model1: 3 DOF System Experenced Rectangular Pulse Force with Concentrated Mass
In the part, a function of time of the lateral story drift in the second story is formulated and the frame structure and relevant parameters are shown in Figure 1.

2.2.1. Equation of Motion
The equations of motion are presented as below:

$$m_1 \ddot{v}_1 + kv_1 + k(v_1 - v_2) = 0$$ (6)

$$m_2 \ddot{v}_2 + k(v_2 - v_1) + k(v_2 - v_3) = 0$$ (7)

$$m_3 \ddot{v}_3 + k(v_3 - v_2) = p(t)$$ (8)

In matrix form:

$$\begin{bmatrix} w & 0 & 0 \\ 0 & w & 0 \\ 0 & 0 & w/2 \end{bmatrix} \begin{bmatrix} \ddot{v}_1 \\ \ddot{v}_2 \\ \ddot{v}_3 \end{bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p_3(t) \end{bmatrix}$$ (9)

2.2.2. Analyze this equation
Use frequency equation $[k - \omega^2 \tilde{m}] \Phi = \tilde{0}$ to solve eigenvalues and eigenvectors.

The result is vibration frequencies and mode shapes:

$$\omega_1^2 = \left(2 - \sqrt{3}\right) \frac{k}{m}, \omega_2^2 = 2 \frac{k}{m}, \omega_3^2 = \left(2 + \sqrt{3}\right) \frac{k}{m}$$ (10)

$$\Phi_1 = \{\sqrt{3}, 0, 1\}, \Phi_2 = \{0, 1, 1\}, \Phi_3 = \{-1, -1, 1\}$$ (11)

Then use Duhamel integral as two parts:

When $0 \leq t \leq \frac{T_1}{2}$: $Y_n(t) = \frac{p_{n}}{M_n\omega_n} \int_{0}^{t} \sin \omega_n(t - \tau) d\tau$ (12)

To solve: $Y_1(t) = Y_2(t) = 0$ (13)

$$Y_3(t) = -0.613\cos(3.490t)$$ (14)

$$\tilde{v}(t) = [-\sqrt{3}(-0.613\cos(3.490t)]$$ (15)

$$v_2(t) = 1.062\cos(3.490t)$$ (16)

And when $t \geq \frac{T_1}{2}$: $Y_n(t) = \frac{p_{n}}{M_n\omega_n} \int_{0}^{T_{1}/2} \sin \omega_n(t - \tau) d\tau$ (17)
To solve: \( Y_1(t) = Y_2(t) = 0 \) (18)
\[
Y_3(t) = -0.613 \cos 3.490t + 0.613 \cos 3.490(t - 0.535) \quad (19)
\]
\[
\tilde{v}(t) = \left[ -\sqrt{3} \right](-0.613 \cos 3.490t + 0.613 \cos 3.490(t - 0.535)) \quad (20)
\]
\[
v_2(t) = 1.062 \cos 3.490t - 1.602 \cos 3.490(t - 0.535) \quad (21)
\]

2.3. Model 1: 3 DOF System Experienced Rectangular Pulse Force with Concentrated Mass

For the three-story shear building structure shown in Figure 2, firstly, the corresponding mass and stiffness matrices, the fundamental frequencies and mode shapes are formulated. Then, assume that a damping matrix to obtain the floor displacements as functions of time.

2.3.1. Equation of Motion

Stiffness Matrix: \( \bar{K} = \begin{bmatrix} 600 & -600 & 0 \\ -600 & 1800 & -1200 \\ 0 & -1200 & 3000 \end{bmatrix} \) Mass Matrix: \( \bar{m} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1.5 & 0 \\ 0 & 0 & 2.0 \end{bmatrix} \) (22)

Assume the undamped forced vibrations. Therefore, the equation of motion is:
\[
\begin{bmatrix}
600 & -600 & 0 \\
-600 & 1800 & -1200 \\
0 & -1200 & 3000 
\end{bmatrix}
\begin{bmatrix}
\ddot{u}_1 \\
\ddot{u}_2 \\
\ddot{u}_3 
\end{bmatrix}
+ \begin{bmatrix}
1 & 0 & 0 \\
0 & 1.5 & 0 \\
0 & 0 & 2.0 
\end{bmatrix}
\begin{bmatrix}
\dddot{u}_1 \\
\dddot{u}_2 \\
\dddot{u}_3 
\end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \quad (23)
\]

2.3.2. Mode shapes

The fundamental frequencies and mode shapes can be determined by \( |\bar{K} - \omega^2 \bar{m}| = 0 \), which can be written as:
\[
\begin{bmatrix}
\frac{(600kips)}{in} & 1 - B & -1 \\
-1 & 3 - 1.5B & -2 \\
0 & -2 & 5 - 2B
\end{bmatrix}
= 0, \text{where } B = \frac{\omega^2}{600}
\]

The solutions of this equation give \( B_1 = 0.3515, B_2 = 1.6066, B_3 = 3.5420 \)

Therefore, the corresponding natural frequencies are as follows:
\[
\begin{bmatrix}
\frac{14.522}{15.048} \\
46.100
\end{bmatrix} \quad \text{rad/s}
\]

And the mode shapes are
\[
\Phi_1 = \begin{bmatrix} 1.0000 \\ 0.6485 \\ 0.3019 \end{bmatrix}, \quad \Phi_2 = \begin{bmatrix} 1.0000 \\ -0.6066 \\ -0.6790 \end{bmatrix}, \quad \Phi_3 = \begin{bmatrix} 1.0000 \\ 2.5418 \\ 2.4392 \end{bmatrix}
\]

2.3.3. Damping ratio and damping matrix

According to Rayleigh Damping: \( C_n = 2\xi_n \omega_n M_n = a_0 M_n + a_3 K_n \). Also, in damped uncoupled equation of motion, it defines \( K_n = \omega_n^2 M_n \). To find \( \xi_2 \), the equation of \( \xi_n \) is necessary:
\[
\xi_n = \frac{a_0}{2\omega_n} + \frac{a_3\omega_n}{2}. \quad (24)
\]

Knowns: \( \xi_1 = \xi_2 = 0.05, \omega_1 = 14.522, \omega_3 = 46.100 \). Unknowns: \( a_0, a_3 \) (25)

Results: \( a_0 = 1.10433 \), \( a_3 = 0.00165 \). Therefore, \( \xi_n = \frac{1.10433}{2\omega_n} + \frac{0.00165\omega_n}{2}, \quad \xi_2 = \frac{1.10433}{2\omega_2} + \frac{0.00165\omega_2}{2}. \quad (26)\)

Knowns: \( \omega_2 = 31.048 \) Results: \( \xi_2 = \frac{1.10433}{2\omega_2} + \frac{0.00165\omega_2}{2} = 0.043399 \). (27)

Damping matrix defined by Rayleigh Damping is \( \tilde{c} = a_0 \bar{m} + a_3 \bar{K} \). (28)

2.3.4. The formula of displacement of vibration

Use initial conditions to find \( A, B \) for each mode and solve the equation.
\[
Y_n = e^{-\xi_n \omega_n t} (A_n \sin \omega_n t + B_n \cos \omega_n t) \quad (29)
\]
\[
B_1 = Y_1(0) = 0.960 \quad A_1 = \frac{Y_1(0) + \xi_1 \omega_1 B_1}{\omega_1^2} = 0.589 \quad (30)
\]
\[
Y_1 = e^{-0.7261t}(0.589 \sin 14.504t + 0.960 \cos 14.504t) \quad (31)
\]
\[
B_2 = Y_2(0) = 0.110 \quad A_2 = \frac{Y_2(0) + \xi_2 \omega_2 B_2}{\omega_2^2} = -0.111 \quad (32)
\]
\[ Y_2 = e^{-1.3472t}(-0.111\sin31.017t + 0.110\sin31.017t) \quad (33) \]
\[ B_3 = Y_3(0) = 0.019 \quad A_3 = \frac{\gamma_3(0) + \xi_3\omega_3 B_3}{\omega_3} = -0.032 \quad (34) \]
\[ Y_3 = e^{-2.305t}(-0.032\sin46.042t + 0.019\sin46.042t) \quad (35) \]

Finally, the displacement of vibration can be obtained as follows:
\[ \ddot{v} = \left( \begin{array}{c} 1 \\ 0.6485 \\ 0.3019 \end{array} \right) e^{-0.7261t}(0.589\sin14.504t + 0.960\sin14.504t) + \\
\left( \begin{array}{c} 1 \\ -0.6066 \\ -0.6790 \end{array} \right) e^{-1.3472t}(-0.111\sin31.017t + 0.110\sin31.017t) + \\
\left( \begin{array}{c} 1 \\ -2.5418 \\ 2.4392 \end{array} \right) e^{-2.305t}(-0.032\sin46.042t + 0.019\sin46.042t) \]

3. DISCUSSIONS

3.1. 3 DOF system experienced rectangular pulse force

![Figure 3. Amplitude of Free Vibration Displacement in Two Phases](image)

Use MATLAB to plot the amplitude of displacements in two phases. As shown in figure 3, after the structure experienced the rectangular impulse, the displacement amplitude of the free vibration phase will be larger than when the impulse was still applied. It is different from the single degree of freedom system. To be more specific, for the single degree of freedom system, when \( t_1 \leq T_1/2 \), the max amplitude will occur in phase 1, where the impulse is applied, which is opposite of one of multi-degree of freedom system.
3.2 Three-Story Shear Building Structure

Figure 4. Mode shapes of different frequencies

Figure 5. Amplitude of Free Vibration Displacement in Each Floor
Figure 4 presents the mode shapes corresponded to three fundamental frequencies. By comparing the results, it is evident that the actual vibration will behave more like Mode Shape 1 than Mode Shape 2 and Mode Shape 3. Figure 5 illustrates that the top floor has the largest displacement amplitude, and the lowest floor has the smallest displacement amplitude.

4. CONCLUSIONS
The research is aimed to discuss the differences between undamped and damping shear systems. Two simple three-floor shear structures are created to simulate the actual buildings, and various impulse imitates different vibrations in real life. Comparing conditions of the two models, the damped condition in model 2 is more following the actual circumstance than the undamped one in model 1, and it is more stable during vibration.

Concerning the three-story shear building structure, the equation of motion of one floor is the same form with E.O.M of a single degree of freedom system, and mode shape does not change with time under the assumption that models have orthogonal damping, and the fundamental frequency will not change over time.

When the nonperiodic impulse is applied, the forced vibration will be different in the short duration of the impulse with the free vibration phase. Also, the multi-degree of freedom system has different responses from the single degree of freedom system when the rectangular impulse is applied.

For multi-degree of freedom systems, the vibration response of the whole system will be generated if the excitation is given to only one part of the system, and the response intensity is not necessarily the largest in part affected by the direct excitation. And the corresponding amplitude of the motion changes with the excitation frequency. Though all three mode shapes lead to free vibration response, the main response is caused by the first mode because it is similar to the initial condition.

Furthermore, in general, there is a phase difference between the response of acceleration, velocity, and displacement, and the phase of acceleration is the most advanced.

REFERENCES
[1] Singiresu, S. R. (1995). Mechanical vibrations. Boston, MA: Addison Wesley.
[2] Elhelloty, A. (2017). Effect of Lateral Loads Resisting Systems on Response of Buildings Subjected to Dynamic Loads. International Journal of Engineering Inventions, 6(10), 62-76.
[3] Baig, S. S., Mogali, M., & Hampali, M. P. (2014). Harmonic Response Analysis of Multi-Storey Building. International Journal of Current Engineering and Technology, 4(4), 2387-2391.
[4] Rice, H. J., & Fitzpatrick, J. A. (1991). A procedure for the identification of linear and non-linear multi-degree-of-freedom systems. Journal of Sound and Vibration, 149(3), 397-411.