IMPROVED WEIGHTED RESTRICTION ESTIMATES IN $\mathbb{R}^3$

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Abstract. Suppose $0 < \alpha \leq n$, $H : \mathbb{R}^n \to [0,1]$ is a Lebesgue measurable function, and $A_\alpha(H)$ is the infimum of all numbers $C$ for which the inequality $\int_B H(x) dx \leq CR^n$ holds for all balls $B \subset \mathbb{R}^n$ of radius $R \geq 1$. After Guth introduced polynomial partitioning to Fourier restriction theory, weighted restriction estimates of the form $\|Ef\|_{L^p(B,Hdx)} \lesssim R^{\alpha A_\alpha(H)^{1/p}} \|f\|_{L^q(\sigma)}$ have been studied and proved in several papers, leading to new results about the decay properties of spherical means of Fourier transforms of measures and, in some cases, to progress on Falconer’s distance set conjecture in geometric measure theory. This paper improves on the known estimates when $E$ is the extension operator associated with the unit paraboloid $\mathcal{P} \subset \mathbb{R}^3$, reaching the full possible range of $p, q$ exponents (up to the sharp line) for $p \geq 3 + (\alpha - 2)/(\alpha + 1)$ and $2 < \alpha \leq 3$.

1. Introduction

This paper is concerned with the extension operator $E$ for the unit paraboloid $\mathcal{P} = \{ (\omega, |\omega|^2) : |\omega| \leq 1 \} \subset \mathbb{R}^3$, endowed with the measure $\sigma$ defined as the pushforward of Lebesgue measure under the map $\omega \mapsto (\omega, |\omega|^2)$ from the unit ball in $\mathbb{R}^2$ to $\mathcal{P}$. The operator $E$ assigns to each function $f \in L^1(\sigma)$ a function $Ef : \mathbb{R}^3 \to \mathbb{C}$ given by

$$Ef(x) = \int e^{-2\pi ix \cdot \xi} f(\xi) d\sigma(\xi) = \int_{|\omega| \leq 1} e^{-2\pi ix \cdot (\omega, |\omega|^2)} f(\omega, |\omega|^2) d\omega.$$

One of the central problems in harmonic analysis, asks for the best possible range of exponents $p$ and $q$ for which the estimate

$$\|Ef\|_{L^p(\mathbb{R}^3)} \lesssim \|f\|_{L^q(\sigma)} \tag{1}$$

holds. This problem is the three-dimensional version of the restriction conjecture, which (in $\mathbb{R}^3$) asserts that (1) is true whenever $p > 3$ and $4/p + 2/q \leq 2$.

We currently know that (1) is true whenever

$$p > 3 + \frac{1}{4} \quad \text{and} \quad \frac{4}{p} + \frac{2}{q} \leq 2. \tag{2}$$

The $p > 3 + 1/4$ and $q = \infty$ range was established by Guth in [10] using his breakthrough polynomial partitioning method. The $4/p + 2/q < 2$ range was then reached in [13], and the $4/p + 2/q = 2$ range in [9].

When $q = \infty$, the range of $p$ in (1) has been recently improved to $p > 3 + 3/13$ in [17].

In this paper, we use polynomial partitioning to prove weighted variants of (1): we replace the $L^p(\mathbb{R}^3) = L^p(dx)$ norm on the left-hand side of (1) by the $L^p(Hdx)$
norm for an appropriate non-negative Lebesgue measurable function $H$ on $\mathbb{R}^3$, and we obtain in return an improvement on the range of exponents in \((2)\). For example, if $0 < a \leq 1$ and $H_\alpha$ is the characteristic function of the set
\[
\bigcup_{m \in \mathbb{Z}} \left\{ [-10,10] \times \mathbb{R}^2 + ((\text{sgn } m)|m|^{1/a},0,0) \right\},
\]
where $\text{sgn } m = m/|m|$ if $m \neq 0$, and $\text{sgn } 0 = 1$, then we prove that
\[
\|Ef\|_{L^p(H_\alpha, dx)} \lesssim \|f\|_{L^q(\sigma)}
\]
for all $f \in L^q(\sigma)$ whenever
\[
p > 3 + \frac{a}{3 + a} \quad \text{and} \quad \frac{3 + a}{p} + \frac{2}{q} < 2.
\]
A simple Knapp example argument shows that the range $(3+a)/p + 2/q < 2$ in \((2)\) is sharp up to the critical line $(3+a)/p + 2/q = 2$, in the sense that $(3+a)/p + 2/q \leq 2$ is a necessary condition for \((3)\) to hold. We will show the details of this argument during the proof of Theorem \(2.4\) in Section 3. (We prove \((3)\) in the second paragraph following Remark \(2.1\) in Section 2.)

We note that when $a = 1$, \((4)\) becomes \((2)\) (minus the sharp line $4/p + 2/q = 2$). We also note that when $q = \infty$, the range $p > 3 + a/(3+a)$ in \((4)\) improves on the range $p > 3 + 3/13$ of \(17\) for $0 < a < 0.9$.

The best previously known range of exponents for which \(3\) holds is considerably smaller than \((4)\), and is the case $b = 1$ (and hence $\alpha = 2 + a$) of \(13\) Corollary 2.1. (See also Theorem \(2.3\) and the paragraph following the statement of Theorem \(2.2\) below.)

The weighted restriction estimates that we are interested in have been recently studied in \(13, 7,\) and \(14\). This paper uses the polynomial partitioning method from \(10\) as adapted to the weighted setting in \(14\) and \(7\) and obtains new and sharp (up to the critical line) restriction estimates (such as \(3\)).

In addition to Guth’s polynomial partitioning, \(7\) employs the refined Strichartz estimates that were proved in \(9\) using the Bourgain-Demeter $L^2$ decoupling theorem \(3\). In proving our main theorem, we also follow \(7\) in using the refined Strichartz estimates and combine this with some of the ideas of \(14\).

We would like to conclude this section by explaining one particular idea from \(13\) that we will use in this paper - continuing to use the weight function $H_\alpha$ from \(3\) as an example - hoping to provide the reader with a global view of the argument.

The function $H_\alpha$ satisfies the following dimensionality property:
\[
\int_{B(x_0, R)} H_\alpha(x) dx \leq A(H_\alpha) R^\alpha
\]
for all $x_0 \in \mathbb{R}^3$ and $R \geq 1$, where $\alpha = 2 + a$, $A(H_\alpha)$ is a constant that only depends on $\alpha$, and $B(x_0, R)$ is the ball of center $x_0$ and radius $R$.

To prove \(3\) following the strategy of \(13\), we first decompose the paraboloid $\mathcal{P}$ into caps $\{\tau\}$ each of radius $K^{-1}$ for some large constant $K$, write $Ef = \sum_\tau Ef_\tau$ with $f_\tau = \chi_\tau f$, and use Guth’s polynomial partitioning method to bound the $L^p(B(0,R), H_\alpha dx)$ norm of the broad part of $Ef$ (see Subsection 4.2) associated with the decomposition $\{\tau\}$ of $\mathcal{P}$. In this bound, the constant $A(H_\alpha)$ will appear raised to the power $3/(\alpha + 1) = 3/(a + 3) < 1$ (see \(15\) in Subsection 5.2).

Next, we bound the $L^p(B(0,R), H_\alpha dx)$ norm of each $Ef_\tau$ by performing parabolic scaling at scale $r = K^{-1}$ and using induction on the radius $R$. 


The parabolic scaling results in replacing the weight function $H_a$ by a weight $\tilde{H}_a$ that obeys the dimensionality property
\[
\int_{B(x_0,R)} \tilde{H}_a(x) dx \leq A(\tilde{H}_a) R^\alpha
\]
with the new constant $A(\tilde{H}_a)$ satisfying
\[
A(\tilde{H}_a) \leq (192)r^{3-\alpha} A(H_a)
\]
(for a proof of this, we refer the reader to [13, Equation (34)] or [7, Lemma 2.1]). Since the bound on the broad part has $A(\tilde{H}_a)$ raised to the power $3/(\alpha + 1) < 1$, the power of $r$ will consequently be reduced from $3 - \alpha$ to $3(3 - \alpha)/(\alpha + 1)$.

During the induction on the radius argument, however, one realizes that to reach the best possible range of $p, q$ exponents for which (3) holds, one needs to preserve the ‘original’ $3 - \alpha$ power of $r$.

It was realized in [13] that it is possible preserve the $3 - \alpha$ power of $r$, if we take advantage of the fact that the exponent of $R$ in (5) (which, starting from the next section, we will think of as the *dimension* of the weight) is invariant under localization. More precisely, if we tile $\mathbb{R}^3$ with cubes $\{Q_l\}$ each of side-length $K/3$, and define the function $H_{a,K}$ by
\[
H_{a,K}(x) = A(H_a)^{-1} K^{-\alpha} \int_{Q_l} H_a(y) dy \quad \text{for} \quad x \in Q_l,
\]
then
\[
\int_{B(x_0,R)} H_{a,K}(x) dx \leq K^{3-\alpha} R^\alpha
\]
for all $x_0 \in \mathbb{R}^3$ and $R \geq 1$.

So, to get the full range of exponents in (4), we go back and adjust the above argument by first localizing the weight $H_a$ at scale $K$, as described in the previous paragraph. Then we use the locally constant property of the Fourier transform and the fact that $f_\tau$ is supported on a cap of radius $r = K^{-1}$ to see that
\[
\int_{B(0,R)} |Ef_\tau(x)|^p H_a(x) dx \sim A(H_a) r^{3-\alpha} \int_{B(0,R)} |Ef_\tau(x)|^p H_{a,K}(x) dx.
\]
Performing parabolic scaling to the integral on the right-hand side will now result in replacing the weight $H_{a,K}$ by a weight $\tilde{H}_{a,K}$ that obeys the dimensionality property
\[
\int_{B(x_0,R)} \tilde{H}_{a,K}(x) dx \leq (192)r^{3-\alpha} K^{3-\alpha} R^\alpha = (192)R^\alpha
\]
for all $x_0 \in \mathbb{R}^3$ and $R \geq 1$, which will allow for no losses in the power of $r$. (For the full argument, we refer the reader to [13, Section 12].)

One consequence of the above discussion is that to prove a favorable restriction estimate for a particular weight (such as $H_a$) that satisfies a certain dimensionality property (such as (5)), one needs to establish the estimate for all (properly normalized) weight functions that obey the same dimensionality property. This brings us to the definition with which we start the next section.
2. Results

Suppose $0 < \alpha \leq n$ and $H$ is non-negative Lebesgue measurable function on $\mathbb{R}^n$ with $\|H\|_{L^{\infty}(\mathbb{R}^n)} \leq 1$. Following [13] and [14], we define

$$A_\alpha(H) = \inf \left\{ C : \int_{B(x_0,R)} H(x)dx \leq CR^\alpha \text{ for all } x_0 \in \mathbb{R}^n \text{ and } R \geq 1 \right\}.$$ 

We say $H$ is a weight of fractal dimension $\alpha$ if $A_\alpha(H) < \infty$. (For the motivation behind referring to $\alpha$ as a fractal dimension, see [14, Sections 4 and 8].)

Also, following [7], we denote by $F_{\alpha,n}$ the set of all non-negative Lebesgue measurable functions $H$ on $\mathbb{R}^n$ that satisfy

$$\int_{B(x_0,R)} H(x)dx \leq R^\alpha$$

for all $x_0 \in \mathbb{R}^n$ and $R \geq 1$.

We note that if $H$ is a weight of fractal dimension $\alpha$, then $A_\alpha(H) - 1$ $H \in F_{\alpha,n}$. Conversely, if $H \in F_{\alpha,n}$ and $N \in \mathbb{N}$, then

$$A_\alpha(N^{-1}\chi_{\{H \leq N\}}H) \leq \frac{1}{N}.$$ 

Thus, if $G : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function, then the inequality

$$(6) \quad \int |G(x)|H(x)dx \leq B \quad \text{ for all } H \in F_{\alpha,n}$$

is equivalent to the inequality

$$(7) \quad \int |G(x)|H(x)dx \leq A_\alpha(H)B \quad \text{ for all weights } H \text{ of fractal dimension } \alpha.$$ 

But, even with this equivalence, it is still more convenient to work with weights $H$ of fractal dimension $\alpha$ to prove the estimates that we are interested in. In fact, for such weights, the assumption $\|H\|_{L^{\infty}(\mathbb{R}^n)} \leq 1$ allows us to use Hölder’s inequality to get

$$\int_{B(0,R)} |G(x)|H(x)dx \leq \|G\|_{L^p(\mathbb{R}^n)} \left( \int_{B(0,R)} H(x)^{p'}dx \right)^{1/p'} \leq \|G\|_{L^p(\mathbb{R}^n)} \left( \int_{B(0,R)} H(x)dx \right)^{1/p'} \leq A_\alpha(H)^{1/p'} R^{\alpha/p'} \|G\|_{L^p(\mathbb{R}^n)}$$

for all $1 < p < \infty$. For $H \in F_{\alpha,n}$, however, the absence of a bound on $\|H\|_{L^{\infty}(\mathbb{R}^n)}$ prevents us from carrying out such calculations so easily (except if $G$ has compact Fourier support; see [7, Remark 1.3]).

In proving the main result of this paper (Theorem 2.1 below), the losses incurred as a result of applying Hölder’s inequality as above will be compensated for by the localization argument that was discussed in Section 1 (see also Subsection 5.3).

The equivalence between (6) and (7) will allow us to unify the notation of [13], [7], and [14] and present their weighted restriction estimates (in $\mathbb{R}^3$) in the following three theorems; the first of which describes the current state of affairs in low fractal dimensions.
Theorem 2-A ([13] Theorem 2.1]). (i) Suppose $0 < \alpha \leq 1$ and $p > 2$. Then
\[ \|Ef\|_{L^p(B(0,R),Hdx)} \lesssim A_\alpha(H)^{1/p}\|f\|_{L^2(\sigma)} \]
for all $f \in L^2(\sigma)$ and weights $H$ of fractal dimension $\alpha$.
(ii) Suppose $1 < \alpha \leq 3/2$ and $p > 2\alpha$. Then
\[ \|Ef\|_{L^p(B(0,R),Hdx)} \lesssim A_\alpha(H)^{1/p}\|f\|_{L^2(\sigma)} \]
for all $f \in L^2(\sigma)$ and weights $H$ of fractal dimension $\alpha$.

In intermediate fractal dimensions in $\mathbb{R}^3$, the estimates we know are local in the sense that the $L^p(Hdx)$ norm of $Ef$ is taken over a ball of radius $R$ rather than over the entire $\mathbb{R}^3$. For some weights (such as $H_\alpha$ in [3]) the local estimates can be turned into global ones by using Tao’s $\epsilon$-removal lemma from [15]. For the details of this argument, we refer the reader to [13] Proof of Corollary 2.1].

Theorem 2-B ([7] Theorem 1.4]). Suppose $3/2 < \alpha \leq 2$. Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that
\[ \|Ef\|_{L^3(B(0,R),Hdx)} \leq C_\epsilon R^\alpha A_\alpha(H)^{1/3}\|f\|_{L^2(\sigma)} \]
for all $f \in L^2(\sigma)$, weights $H$ of fractal dimension $\alpha$, and $R \geq 1$.

In high fractal dimensions, i.e. $2 < \alpha \leq 3$, the best previously known results are stated in the following theorem.

Theorem 2-C ([13] Theorem 5.1]). (i) Suppose $2 < \alpha < 5/2$ and
\[ p = 3 + \frac{2\alpha - 3}{2\alpha + 3} \quad \text{and} \quad q = \frac{2p}{3}. \]
Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that
\[ \|Ef\|_{L^p(B(0,R),Hdx)} \leq C_\epsilon R^\alpha A_\alpha(H)^{1/p}\|f\|_{L^q(\sigma)} \]
for all $f \in L^q(\sigma)$, weights $H$ of fractal dimension $\alpha$, and $R \geq 1$.
(ii) Suppose $5/2 \leq \alpha \leq 3$, $p = 13/4$, and $q > (11 - 2\alpha)/13$. Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that
\[ \|Ef\|_{L^p(B(0,R),Hdx)} \leq C_\epsilon R^\alpha A_\alpha(H)^{1/p}\|f\|_{L^q(\sigma)} \]
for all $f \in L^q(\sigma)$, weights $H$ of fractal dimension $\alpha$, and $R \geq 1$.

We alert the reader that in [13] Theorem 5.1], the restriction estimates are written in dual form: from $L^p(B(0,R),Hdx)$ to $L^q(\sigma)$ (see also Section 3 below). Also, part (i) of Theorem 2-C was in fact proved in [13] for $3/2 \leq \alpha < 5/2$, but for $\alpha = 3/2$ its result has been surpassed by [14] Theorem 2.1 and for $3/2 < \alpha \leq 2$ by [7] Theorem 1.4] as presented in Theorems 2-A and 2-B above.

The aim of this paper is to prove the following theorem.

Theorem 2.1. Suppose $2 < \alpha \leq 3$, $p = (4\alpha + 1)/(\alpha + 1)$, and $2 \leq \gamma < 2p - \alpha - 1$. Then to every $\epsilon > 0$ there is a constant $C_\epsilon$ such that
\[ \int_{B(0,R)} |Ef(x)|^p H(x)dx \leq C_\epsilon R^\alpha A_\alpha(H)\|f\|_{L^2(\sigma)}^p\|f\|_{L^\infty(\sigma)}^{p-\gamma} \]
for all $f \in L^\infty(\sigma)$, weights $H$ of dimension $\alpha$, and $R \geq 1$. 
Theorem 2.1 will allow us to extend the range of exponents for which our weighted estimates hold in high fractal dimensions until (but still not including) the sharp line \((\alpha + 1)/p + 2/q = 2\), and hence replace Theorem 2-C by the following result.

**Theorem 2.2.** Suppose \(2 < \alpha \leq 3\). Then:

(i) Suppose 
\[
p \geq 3 + \frac{\alpha - 2}{\alpha + 1} \quad \text{and} \quad \frac{\alpha + 1}{p} + \frac{2}{q} < 2.
\]
Then to every \(\epsilon > 0\) there is a constant \(C_\epsilon\) such that
\[
\|Ef\|_{L^p(B(0,R),Hdx)} \leq C_\epsilon R^\alpha A_\alpha(H)^{1/p}\|f\|_{L^q(\sigma)}
\]
for all \(f \in L^q(\sigma)\), weights \(H\) of fractal dimension \(\alpha\), and \(R \geq 1\).

(ii) Suppose \(1 \leq p, q \leq \infty\) are a pair of exponents for which the estimate in part (i) holds. Then
\[
p \geq \alpha \quad \text{and} \quad \frac{\alpha + 1}{p} + \frac{2}{q} \leq 2.
\]

**Remark 2.1.** The lower bound \(p \geq \alpha\) in part (ii) of Theorem 2.2 was proved by the author in [14, Theorem 2.3]. We are restating it here for obvious reasons.

We will prove Theorem 2.2 (using Theorem 2.1) in the next section. The proof of Theorem 2.1 will occupy all the sections of the paper following the next one.

Going back to (3), we know from (5) that \(H_a\) is a weight of fractal dimension \(\alpha = 2 + a\) with \(A_\alpha(H) \lesssim 1\), so applying part (i) of Theorem 2.2 and using Tao’s \(\epsilon\)-removal lemma (as discussed in the paragraph preceding the statement of Theorem 2-B above), we see that (3) holds for all pairs \(p, q\) of exponents that satisfy (4).

We would like to point out to the reader that [7, Theorem 1.4] also gives an estimate in the regime \(2 \leq \alpha \leq 3\):
\[
\|Ef\|_{L^p(B(0,R),Hdx)} \leq C_\epsilon R^{(\alpha - 2)/3}\|f\|_{L^q(\sigma)}
\]
for all \(f \in L^q(\sigma)\), \(H \in \mathcal{F}_{\alpha,3}\), and \(R \geq 1\), which the authors of [7] then use to obtain decay estimates on the \(L^2\) spherical means of the Fourier transforms of measures (see [7, Theorem 1.6]) that were new at the time of writing of their paper. Theorem 2.2 improves on the decay estimates of [7] in the regime \(2 < \alpha < 1 + (2/\sqrt{3})\). But all these decay estimates are now inferior to those of [8], so we omit the details.

### 3. Proof of Theorem 2.2

(i) We will establish the estimate in dual form. For \(g \in C_0(B(0,R))\), we let \(Rg = \tilde{g}H\), and our goal is to show that
\[
\|Rg\|_{L^q(\sigma)} \lesssim R^\alpha A_\alpha(H)^{1/p}\|g\|_{L^p(\sigma)}.
\]
To do this, we will use Theorem 2.1 in the following way:
\[
\left| \int \tilde{R}g(\xi)f(\xi)d\sigma(\xi) \right| = \left| \int Ef(x)g(x)H(x)dx \right|
\leq \|Ef\|_{L^p(B(0,R),Hdx)}\|g\|_{L^{p'}(Hdx)}
\lesssim R^\alpha A_\alpha(H)^{1/p}\|f\|_{L^q(\sigma)}\|f\|_{L^p(\sigma)}\|g\|_{L^p(\sigma)}.
for all $f \in L^1(\sigma)$. The key idea now (which the author has previously used in [13]
Theorem 5.1]) is that the above estimate allows us to estimate the $\sigma$-measure of the superlevel sets $\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}$ for $0 < \lambda \leq \|g\|_{L^1(\mathcal{H}d\lambda)}$ as follows.

Letting

$$S_{\lambda,l} = \{\xi \in \mathcal{P} : 2^{l-1}\lambda < |Rg(\xi)| \leq 2^l\lambda\}$$

and inserting $Rg_\chi S_{\lambda,l}$ for $f$ in the above estimate, we see that

$$\int_{S_{\lambda,l}} |Rg(\xi)|^2 d\sigma(\xi) \lesssim R^* A_\alpha(H)^{1/p}\|Rg\|_{L^2(\mathcal{H}d\lambda, d\xi)}^{\gamma/p}(2^l\lambda)^{(p-\gamma)/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)},$$

so that

$$(2^l\lambda)^2 \sigma(S_{\lambda,l}) \lesssim R^* A_\alpha(H)^{1/p}(2^l\lambda)^{\gamma/p} \sigma(S_{\lambda,l})^{\gamma/(2p)}(2^l\lambda)^{(p-\gamma)/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)},$$

so that

$$(2^l\lambda)^{2p} \sigma(S_{\lambda,l})^{2p-\gamma} \lesssim R^{2p\epsilon} A_\alpha(H)^2\|g\|_{L^{q'}(\mathcal{H}d\lambda)}^{2p}.$$ 

Letting $q' = 2p/(2p - \gamma)$, this becomes

$$\sigma(S_{\lambda,l}) \lesssim R^{q'-\epsilon} A_\alpha(H)^{q'/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}^{q'/(2l\lambda)^{-q'}}.$$

Thus

$$\sigma(\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}) \leq \sum_{l=1}^{\infty} \sigma(S_{\lambda,l}) \lesssim R^{q'-\epsilon} A_\alpha(H)^{q'/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}^{q'/(2l\lambda)^{-q'}}.$$

Of course, we also have the trivial bound $\sigma(\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}) \leq \sigma(\mathcal{P})$, so

$$\sigma(\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}) \leq \min\left[\sigma(\mathcal{P}), R^{q'-\epsilon} A_\alpha(H)^{q'/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}^{q'/(2l\lambda)^{-q'}}\right].$$

Having obtained a good bound on the size of the superlevel sets of $Rg$, we can now estimate its $L^{q'}(\sigma)$ norm:

$$\|Rg\|_{L^{q'}(\sigma)} = \int_0^{\lambda_1} \sigma(\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}) \lambda^{q'-1}d\lambda,$$

where $\lambda_1 = \|g\|_{L^1(\mathcal{H}d\lambda)}$. We also let

$$\lambda_0 = R^* A_\alpha(H)^{1/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}.$$

If $\lambda_0 \geq \lambda_1$, then the trivial part of (9) tells us that

$$\|Rg\|_{L^{q'}(\sigma)} \lesssim \int_0^{\lambda_0} \sigma(\mathcal{P}) \lambda^{q'-1}d\lambda \lesssim \lambda_0^{q'},$$

so that

$$\|Rg\|_{L^{q'}(\sigma)} \lesssim R^* A_\alpha(H)^{1/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}.$$ 

If $\lambda_0 \leq \lambda_1$, then, in view of the inequality before the last, we only have to estimate

$$\int_{\lambda_0}^{\lambda_1} \sigma(\{\xi \in \mathcal{P} : |Rg(\xi)| > \lambda\}) \lambda^{q'-1}d\lambda,$$

which, by the non-trivial part of (9), is

$$\lesssim \int_{\lambda_0}^{\lambda_1} R^{q'-\epsilon} A_\alpha(H)^{q'/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}\lambda^{-1}d\lambda = \left(\log \frac{\lambda_1}{\lambda_0}\right) R^{q'-\epsilon} A_\alpha(H)^{q'/p}\|g\|_{L^{q'}(\mathcal{H}d\lambda)}.$$
By Hölder’s inequality (applied with the measure $Hdx$ on the ball $B(0, R)$),

$$\lambda_1 \leq \left( \int_{B(0, R)} H(x) dx \right)^{1/p} \|g\|_{L^{p'}(Hdx)} \leq A_\alpha(H)^{1/p} R^{\alpha/p} \|g\|_{L^{p'}(Hdx)},$$

so

$$\log \frac{\lambda_1}{\lambda_0} \leq \log \frac{A_\alpha(H)^{1/p} R^{\alpha/p} \|g\|_{L^{p'}(Hdx)}}{R^\gamma \alpha^\gamma A_\alpha(H)^{1/p} \|g\|_{L^{p'}(Hdx)}} \lesssim \log R,$$

and so

$$\|Rg\|_{L^{p'}(\sigma)} \lesssim R^\gamma \alpha^\gamma A_\alpha(H)^{1/p} \|g\|_{L^{p'}(Hdx)}.$$

Recalling that $q' = 2p/(2p - \gamma)$ and $2 \leq \gamma < 2p - \alpha - 1$, we see that (8) is true whenever $1/p \leq 1/q < 1 - (\alpha + 1)/(2p)$, and hence (by Hölder) whenever $1/q < 1 - (\alpha + 1)/(2p)$, as promised.

(ii) As Remark 2.1 tells us, the lower bound $p \geq \alpha$ on the exponent $p$ comes from [13, Theorem 2.3], so we only need to show that $(\alpha + 1)/p + 2/q \leq 2$. For this we use a standard Knapp-example argument.

To every $R > 1$ there is a function $f_R$ on $\mathcal{P}$ such that $|f_R| = 1$ on the cap $\{(\omega, |\omega|^2) : |\omega| \leq R^{-1/2}\}$ and $|Ef_R| \gtrsim R^{-1}$ on the box $[-R^{1/2}, R^{1/2}] \times [-R^{1/2}, R^{1/2}] \times [-R, R]$.

If the estimate in part (i) of the theorem holds for all weights $H$ of fractal dimension $\alpha$, then, in particular, it will hold for the weight $H_\alpha$ from Section 1 with $a = \alpha - 2 \in (0, 1]$. Since $|m|^{1/a} \leq R^{1/2}$ for $\sim R^{a/2}$ integers $m$, we have

$$\int_{B(0, 10R)} |Ef_R(x)|^p H_\alpha(x) dx \gtrsim R^{-p} R^{\alpha/2} R^{3/2} = R^{\alpha + 1/2}.$$

Also, $\|f_R\|_{L^p(\sigma)} \sim R^{-1/4}$, so $R^{-p} R^{\alpha + 1/2} \lesssim R^p R^{-p/q}$ for all sufficiently large $R$, and so $(\alpha + 1)/p + 2/q \leq 2$.

4. Overview of Guth’s polynomial partitioning method

If $B$ is a ball in $\mathbb{R}^2$, then the cap corresponding to $B$ is defined to be the set $\Theta \subset \mathbb{R}^3$ given by $\Theta = \{(\omega, |\omega|^2) : \omega \in B^2\}$. If $B$ has center $\omega_0$ and radius $\rho$, then $\Theta$ has center $\xi_0 = (\omega_0, |\omega_0|^2)$ and radius $\rho$. Moreover, if $C$ is a constant, then $C\Theta$ is the cap of the same center as $\Theta$ and $C$ times its radius.

Let $m$ and $r$ be positive numbers and $\{\Theta\}$ be caps corresponding to a collection of balls that cover the unit ball $B^2 \subset \mathbb{R}^2$. If the centers of the balls are in $B^2$ and are $r$-separated, and the radius of each ball lies in the interval $[r, r\sqrt{m}]$, we say that $\{\Theta\}$ is a decomposition of the unit paraboloid $\mathcal{P}$ into caps of radius $\sim r$ and multiplicity at most $m$. (It is easy to see that if a point belongs to $M$ of the caps $\Theta$, then $M \leq cm$ for some absolute constant $c$.)

One reason polynomial partitioning works well for the restriction problem is the wave packet decomposition of $Ef$ and the way the wave packets interact with zero sets of polynomials.

4.1. The wave packet decomposition. The wave packet decomposition was first used to prove restriction estimates in Bourgain’s paper [2], and has since become a standard tool in Fourier restriction theory. The version that we are going to use in this paper comes from [10, Section 2] (see also [11, Section 3] and [13, Section 7]).
Proposition 4-A. Suppose $\delta > 0$, $R > 1$, $N$ is a positive integer, $\theta$ is a cap in $\mathcal{P}$ of center $\xi_0$ and radius $r = R^{-1/2}$, and $v(\theta)$ is the unit normal vector of $\mathcal{P}$ at $\xi_0$.

Then there is a countable collection $\mathcal{T}(\theta) = \{T\}$ of finitely overlapping tubes in $\mathbb{R}^3$ of radius $R^{(1/2)+\delta}$, which are parallel to $v(\theta)$, such that the following holds. To every function $f \in L^2(\sigma)$ with supp $f \subset \theta$ there is a sequence $\{f_T\}$ in $L^2(\sigma)$ with the following properties.

(i) Each $f_T$ is supported in $3\theta$, $f = \sum_{T \in \mathcal{T}(\theta)} f_T$ in $L^1(\sigma)$, and
\[
\sum_{T \in \mathcal{T}(\theta)} \int |f_T|^2 d\sigma \lesssim \|f\|_{L^2(\sigma)}^2.
\]

(ii) We have
\[
\sum_{T \in \mathcal{T}(\theta) : x \notin T} |Ef_T(x)| \lesssim R^{-N} \|f\|_{L^1(\mathcal{S})}
\]
for all $x \in \mathbb{R}^3$ with $|x| \leq R$.

(iii) If $T_1, T_2 \in \mathcal{T}(\theta)$ are disjoint, then
\[
\left| \int f_{T_1} f_{T_2} d\sigma \right| \lesssim R^{-N} \|f\|_{L^1(\sigma)}^2,
\]
where the implicit constant depends only on $N$.

(iv) Let $\mathcal{T}(\theta) = \{T \in \mathcal{T}(\theta) : T \cap B(0, R) \neq \emptyset\}$. Then
\[
|Ef(x) - \sum_{T \in \mathcal{T}(\theta)} Ef_T(x)| \lesssim R^{-N} \|f\|_{L^1(\sigma)}
\]
for all $x \in B(0, R)$.

We note that part (iv) of Proposition 4-A is an immediate consequence of parts (i) and (ii).

We fix a decomposition $\{\theta\}$ of the paraboloid $\mathcal{P}$ into caps of radius $\sim R^{-1/2}$ and multiplicity at most 1, as defined at the start of Section 4.

Any function $f$ on $\mathcal{P}$ can now be written as a sum $f = \sum_\theta f_\theta$ with $f_\theta$ supported in $\theta$ and such that the supports of $f_\theta$ and $f_{\theta'}$ are disjoint whenever $\theta \neq \theta'$. Also, if $f \in L^2(\sigma)$, then so are the functions $f_\theta$. Applying Proposition 4-A to each $f_\theta$, we get that
\[f = \sum_\theta \sum_{T \in \mathcal{T}(\theta)} (f_\theta)_T\]
with the sum converging in $L^1(\sigma)$. (Since the decomposition $\{\theta\}$ of $\mathcal{P}$ will be fixed for the rest of the paper, we will write $f_T$ for $(f_\theta)_T$ to simplify the notation.)

Letting $\mathcal{T} = \cup_\theta \mathcal{T}(\theta)$, we then see that
\[
Ef = \sum_{T \in \mathcal{T}} Ef_T
\]
with the sum converging uniformly.

Letting $\mathcal{T} = \{T \in \mathcal{T} : T \cap B(0, R) \neq \emptyset\} = \cup_\theta \mathcal{T}(\theta)$, part (iv) of Proposition 4-A tells us that, on $B(0, R)$, we can replace the infinite sum in (10) by a finite sum:
\[
Ef(x) = \sum_{T \in \mathcal{T}} Ef_T(x) + O(R^{-N} \|f\|_{L^1(\sigma)})
\]
(11)
for all $x \in B(0, R)$. This representation of $E f$ is often referred to as the wave packet decomposition of $E f$ on the ball $B(0, R)$. Part (ii) of Proposition 4-A tells us that each wave packet $E f_T$ (which is equal to $E(f_\theta)_T$ for some cap $\theta$ that comes from our fixed decomposition of the unit paraboloid) is essentially supported in $B(0, R) \cap T$, which is a tube in $\mathbb{R}^3$ of radius $R^{(1/2)+\delta}$ and length $\sim R$ and points in the direction $\nu(\theta)$ (since $T \in T(\theta)$).

Suppose $\{\theta_1, \ldots, \theta_L\}$ is a subset of our set $\{\theta\}$ of caps, and, for each $\ell$, $T_\ell$ is a subset of the set $T(\theta_\ell)$ of tubes corresponding to the cap $\theta_\ell$. For $1 \leq \ell \leq L$, part (i) of Proposition 4-A tells us that

$$
\sum_{T \in T_\ell} \int |f_T|^2 d\sigma \leq \sum_{T \in Y(\theta_\ell)} \int |f_T|^2 d\sigma \lesssim \int_{\theta_\ell} |f|^2 d\sigma
$$

(recall that for $T \in \mathcal{T}(\theta_\ell)$, $f_T = (f_{\theta_\ell})_T$). Also, writing

$$
\left| \sum_{T \in T_\ell} f_T \right|^2 = \sum_{T_1 \cap T_2 = \emptyset} f_{T_1} \overline{f_{T_2}} + \sum_{T_1 \cap T_2 \neq \emptyset} f_{T_1} \overline{f_{T_2}},
$$

where $(T_1, T_2)$ runs over $T \times T$, and using part (iii) of Proposition 4-A to get

$$
\sum_{T_1 \cap T_2 = \emptyset} \left| \int f_{T_1} \overline{f_{T_2}} d\sigma \right| \lesssim \int_{\theta_\ell} |f|^2 d\sigma,
$$

and the fact that

$$
\left| \int f_{T_1} \overline{f_{T_2}} d\sigma \right| \leq \frac{1}{2} \int |f_{T_1}|^2 d\sigma + \frac{1}{2} \int |f_{T_2}|^2 d\sigma
$$

to get

$$
\sum_{T_1 \cap T_2 \neq \emptyset} \left| \int f_{T_1} \overline{f_{T_2}} d\sigma \right| \lesssim \sum_{T \in T_\ell} \int |f_T|^2 d\sigma,
$$

we see that

$$
\int \left| \sum_{T \in T_\ell} f_T \right|^2 d\sigma \lesssim \int_{\theta_\ell} |f|^2 d\sigma.
$$

Since for each $\theta$, the set $J_\theta = \{1 \leq l \leq L : (3\theta) \cap (3\theta_\ell) \neq \emptyset\}$ has cardinality $\lesssim 1$, it follows that

$$
\int_{3\theta} \left| \sum_{l=1}^L \sum_{T \in T_l} f_T \right|^2 d\sigma \lesssim \sum_{l \in J_\theta} \int \left| \sum_{T \in T_l} f_T \right|^2 d\sigma \lesssim \sum_{l \in J_\theta} \int_{\theta_\ell} |f|^2 d\sigma
$$

(recall from part (i) of Proposition 4-A that if $T \in \mathcal{T}(\theta_\ell) \subset \mathcal{T}(\theta)$, then $f_T$ is supported in $3\theta_\ell$). But $l \in J_\theta$ implies $\theta_\ell \subset 10\theta$, so

$$
(12) \quad \int_{3\theta} \left| \sum_{l=1}^L \sum_{T \in T_l} f_T \right|^2 d\sigma \lesssim \int_{10\theta} |f|^2 d\sigma.
$$

4.2. The broad part of $Ef$. The main idea introduced by Bourgain and Guth in [2] concerning the restriction problem in $\mathbb{R}^3$ was to break down $Ef$ into a broad part and a narrow part, estimate the broad by combining the Bennett-Carbery-Tao multilinear restriction theorem from [1] with Wolff’s Kakeya result from [18] and the two-dimensional bilinear restriction theorem (see [16]), and estimate the narrow part by parabolic rescaling and induction. In [10], Guth replaced the multilinear theorem by polynomial partitioning and Wolff’s Kakeya result by a bound on the
number of the caps \( \{ \theta \} \) such that \( \mathfrak{T}(\theta) \) has at least one tube that intersects the zero set of the polynomial tangentially (see part (iii) of Proposition 4-B below) that he proved by adapting Wolff’s hairbrush argument from [18] to the polynomial partitioning setting. In this subsection, we introduce the broad part of \( Ef \) as defined in [10] and show how it connects to bilinear expressions.

Let \( K \) be a large constant and \( \{ \tau \} \) a decomposition of the unit paraboloid \( \mathcal{P} \) into caps of radius \( \sim K^{-1} \) and multiplicity at most \( m \). Any function \( f : \mathcal{P} \to \mathbb{C} \) can then be written as \( f = \sum_{\tau} f_{\tau} \) with \( \text{supp} f_{\tau} \subset \tau \) (but we do not insist this time that \( (\text{supp} f_{\tau}) \cap (\text{supp} f_{\tau'}) = \emptyset \) if \( \tau \neq \tau' \) as we did with the decomposition \( f = \sum_{\theta} f_{\theta} \) of the previous subsection).

Now let \( 0 < \beta \leq 1 \) and \( f \in L^4(\sigma) \). Given a point \( x \in \mathbb{R}^3 \), we say that \( x \) is \( \beta \)-broad for \( Ef \) if no single \( Ef_{\tau} \) dominates the value of \( Ef \) at \( x \). More precisely, \( x \) is \( \beta \)-broad for \( Ef \) if

\[
\max_{\tau} |Ef_{\tau}(x)| < \beta |Ef(x)|.
\]

The **broad part of \( Ef \)** is the function \( Br_{\beta} Ef \) defined as

\[
Br_{\beta} Ef(x) = \begin{cases} 
|Ef(x)| & \text{if } x \text{ is } \beta\text{-broad for } Ef, \\
0 & \text{otherwise.}
\end{cases}
\]

We note that

\[
|Ef(x)|^p \leq Br_{\beta} Ef(x)^p + \beta^{-p} \sum_{\tau} |Ef_{\tau}(x)|^p
\]

for all \( x \in \mathbb{R}^3 \), so that

\[
\int_{B(0,R)} |Ef(x)|^p H(x)dx 
\leq \int_{B(0,R)} Br_{\beta} Ef(x)^p H(x)dx + \beta^{-p} \sum_{\tau} \int_{B(0,R)} |Ef_{\tau}(x)|^p H(x)dx.
\]

In broad terms, Guth’s strategy consists of using

- polynomial partitioning to upgrade the two-dimensional bilinear restriction theorem into an estimate on

\[
\int_{B(0,R)} Br_{\beta} Ef(x)^p H(x)dx
\]

- parabolic rescaling and induction on the radius \( R \) to estimate

\[
\int_{B(0,R)} |Ef_{\tau}(x)|^p H(x)dx.
\]

It will be convenient at this time to introduce the following notation. For \( (R, K, m) \in [1, \infty) \times [1, \infty) \times [1, \infty) \), we let \( \Lambda(R, K, m) \) be the set of all functions \( f \in L^1(\sigma) \) such that \( f = \sum_{\tau} f_{\tau} \) for some decomposition \( \{ \tau \} \) of \( \mathcal{P} \) of multiplicity \( m \) with \( \text{supp} f_{\tau} \subset \tau \) and

\[
(13) \quad \int_{B(\xi_0, R^{-1/2}) \cap \mathcal{P}} |f_{\tau}(\xi)|^2 d\sigma(\xi) \leq \frac{1}{R}
\]

for all \( \xi_0 \in \mathcal{P} \). Since \( \mathcal{P} \) can be covered by \( \sim R \) of the balls \( B(\xi_0, R^{-1/2}) \) that appear in (13), it follows that

\[
\int |f_{\tau}(\xi)|^2 d\sigma(\xi) \lesssim 1
\]

for each cap \( \tau \).

(A first glance at Theorem 2.1 suggests that the natural way to normalize its estimate is to assume that \( \|f\|_{L^\infty(\sigma)} \leq 1 \). This condition is replaced by (13) because we are unable to control the \( L^\infty \) norms of the functions \( f_{\tau,T} \) that arise in the wave packet decomposition of \( Ef_{\tau,T} \).)
4.3. Polynomial partitioning. We denote by $\text{Poly}_D(\RR^n)$ the space of polynomials in $n$ real variables of degree at most $D$. If $P \in \text{Poly}_D(\RR^n)$, then we denote by $Z(P)$ the zero set of $P$. We say that $P$ is non-singular if $\nabla P(x) \neq 0$ for all $x \in Z(P)$.

Our starting point is the following result of Guth.

**Theorem 4-A ([10] Corollary 1.7).** To every non-negative function $F \in L^1(\RR^n)$, with $\|F\|_{L^1(\RR^n)} > 0$, and integer $D \in \mathbb{N}$ there is a polynomial $P \in \text{Poly}_D(\RR^n) \setminus \{0\}$, which is a product of non-singular polynomials, and a positive constant $C_n$, which only depends on the dimension $n$, such that:

(i) $\RR^n \setminus Z(P)$ is a disjoint union of open sets $O_i$.

(ii) For each $i$, we have

$$C_n^{-1} D^{-n} ||F||_{L^1(\RR^n)} \leq \int_{O_i} F(x)dx \leq C_n D^{-n} ||F||_{L^1(\RR^n)}.$$

The open sets $O_i$ in Theorem 4-A are called cells. Since they are disjoint and their union is $\RR^n \setminus Z(P)$, it follows that each cell is a union of connected components of $\RR^n \setminus Z(P)$. Moreover, since $Z(P)$ has Lebesgue measure zero, it follows from the inequalities in part (ii) of the theorem that

$$C_n^{-1} D^n \leq \# \{O_i\} \leq C_n D^n.$$

If $i \neq i'$, $x \in O_i$, and $y \in O_{i'}$, then $x$ and $y$ can not belong to the same connected component of $\RR^n \setminus Z(P)$, and hence the line segment $\{(1-t)x + ty : 0 \leq t \leq 1\}$ must intersect the zero set $Z(P)$. From this it follows that

- any line in $\RR^n$ can intersect at most $D+1$ of the cells $O_i$.

Let $2 < \alpha \leq 3$, $H$ be a weight in $\mathcal{F}_\alpha$, and $f$ a function in $\Lambda(R,K,m)$.

We apply Theorem 4-A in $\RR^3$ with the function $F$ defined by

$$F(x) = \begin{cases} 
\Br \beta E f(x)^p H(x) & \text{if } |x| \leq R, \\
0 & \text{if } |x| > R,
\end{cases}$$

and a degree $D$ that will be determined later in the argument.

In order for the resulting polynomial partitioning of $\RR^3$ to help us upgrade the two-dimensional bilinear restriction theorem into an estimate on $\int F(x)dx$, we need the above property about intersections of lines and cells to also hold for the tubes $T \in \mathcal{T}$ that are involved in the wave packet decomposition of $Ef$, as given by (11) (with $f$ replaced by $f_r$). But this is not true; in fact, it is possible for a tube $T$ to intersect all of the ~ $D^3$ cells $O_i$.

Guth mended the situation by replacing the zero set $Z(P)$ by the **cell-wall** $W$ that he defined as the $R^{(1/2)+\delta}$-neighborhood of $Z(P)$, and the cells $O_i$ by the **modified cells** $O'_i = O_i \setminus W$. If one of our tubes $T$ intersects more than $D+1$ of the modified cells $O'_i$, then its core line intersects more than $D+1$ of the cells $O_i$, which implies that the core line lies in $Z(P)$. Since the radius of $T$ is $R^{(1/2)+\delta}$, it follows that $T$ lies in $W$, which is a contradiction. Therefore,

- any tube $T \in \mathcal{T}$ can intersect at most $D+1$ of the modified cells $O'_i$.

We write

$$\int F(x)dx = \int_W F(x)dx + \int_{\cup_i O'_i} F(x)dx.$$

If $\int_W F(x)dx \leq \int_{\cup_i O'_i} F(x)dx$, we say we are in the **cellular case**. Otherwise, we are in the **algebraic case**.
In the cellular case, the above property about the intersection of tubes with modified cells will allow us to estimate our integral by induction on \( \sum_\tau \| f_\tau \|^2_{L^2(\sigma)} \).

The algebraic case is much more complicated.

In the algebraic case, the wave packets \( Ef_{\tau,T} \) that contribute to our integral are those with ‘supporting’ tubes \( T \) that intersect the cell-wall \( W \) in the ball \( B(0, R) \). Guth separated such tubes into two groups depending on whether they lie in \( B(0, R) \cap W \) for a significant portion of their length, or they cut \( B(0, R) \cap W \) transversely. Here is the precise set-up.

We first recall that \( W \) is the \( R^{(1/2)+\delta} \)-neighborhood of the zero set \( Z(P) \) and we cover \( B(0, R) \cap W \) by a finitely overlapping family \( \{ B_j \} \) of balls each of radius \( R^{1-\delta} \). Next, we let \( Z_0(P) \) be the set of all non-singular points of \( Z(P) \), and, for \( z \in Z_0(P) \), we denote by \( T_z Z(P) \) the tangent space to \( Z(P) \) at the point \( z \). Also, for each tube \( T \in \mathbb{T} \), we denote by \( v(T) \) the unit vector in the direction of \( T \).

Given a tube \( T \in \mathbb{T} \), we say \( T \) cuts \( W \) tangentially in \( B_j \), and write \( T \in T_{j,\text{tang}} \), if

\[
\begin{cases}
    T \cap W \cap B_j \neq \emptyset \\
    \text{Angle}(v(T), T_z Z(P)) \leq R^{-(1/2)+2\delta} \quad \forall \ z \in Z_0(P) \cap 2B_j \cap 10T.
\end{cases}
\]

Also, we say \( T \) cuts \( W \) transversely in \( B_j \), and write \( T \in T_{j,\text{trans}} \), if

\[
\begin{cases}
    T \cap W \cap B_j \neq \emptyset \\
    \exists \ z \in Z_0(P) \cap 2B_j \cap 10T \text{ such that } \text{Angle}(v(T), T_z Z(P)) > R^{-(1/2)+2\delta}.
\end{cases}
\]

We also let

\[
f_{\tau,j,\text{tang}}(T) = \sum_{T \in T_{j,\text{tang}}} f_{\tau,T} \quad \text{and} \quad f_{j,\text{tang}} = \sum_{\tau} f_{\tau,j,\text{tang}},
\]

and

\[
f_{\tau,j,\text{trans}}(T) = \sum_{T \in T_{j,\text{trans}}} f_{\tau,T} \quad \text{and} \quad f_{j,\text{trans}} = \sum_{\tau} f_{\tau,j,\text{trans}}.
\]

Recall that \( \mathcal{P} \) is covered by \( \sim K^2 \) caps \( \tau \) of diameter \( \sim 1/K \). If \( I \) is any subset of these caps, we write

\[
f_{I,j,\text{trans}} = \sum_{\tau \in I} f_{\tau,j,\text{trans}}.
\]

We say that two caps \( \tau_1 \) and \( \tau_2 \) are non-adjacent if the distance between them is \( \geq 1/K \), and define

\[
\text{Bil}_{P,\delta} E f_{j,\text{tang}} = \sum_{\tau_1,\tau_2 \text{ non-adjacent}} \| E_{\tau_1,j,\text{tang}} \|^2 \| E_{\tau_2,j,\text{tang}} \|^2.
\]

We call the function \( \text{Bil}_{P,\delta} E f_{j,\text{tang}} \) the tangential part of \( Ef \) with respect to the polynomial \( P \) and the parameter \( \delta \).

The main properties of \( T_{j,\text{tang}} \), \( T_{j,\text{trans}} \), and \( \text{Bil}_{P,\delta} E f_{j,\text{tang}} \) are presented in the following proposition.

**Proposition 4-B (II).** (i) For each \( j \), \( T_{j,\text{tang}} \cap T_{j,\text{trans}} = \emptyset \).

(ii) A tube \( T \in \mathbb{T} \) can belong to at most \( D^{O(1)} \) different sets \( T_{j,\text{trans}} \).

(iii) For each \( j \), the number of different \( \theta \) such that \( T_{j,\text{tang}} \cap T(\theta) \neq \emptyset \) is at most \( D^2 R^{(1/2)+O(\delta)} \).
(iv) Suppose $0 < \epsilon \leq 2$, $K \geq \sqrt[4]{10}$, $K^{-\epsilon} \leq \beta \leq 1$, $\beta m \leq 10^{-5}$, and $R \geq CK^\epsilon$. If $x \in B_j \cap W$ and $C$ is sufficiently large, then

$$\text{Br}_\beta Ef(x) \leq \frac{5}{4} \sum_{\ell} \text{Br}_{2\beta} Ef_{j,\text{trans}}(x) + K^{100} \text{Bil}_{p,\delta} Ef_{j,\text{trans}}(x) + O\left(R^{-N+1} \sum_{\tau} \|f_\tau\|_{L^1(\sigma)}\right).$$

For a proof of part (i) of Proposition 4-B we refer the reader to the paragraph immediately following [10, Definitions 3.3 and 3.4].

Part (ii) of Proposition 4-B is [10, Lemma 3.5].

Part (iii) of Proposition 4-B is [10, Lemma 3.6]. (For the very interesting connection between this result and the Kakeya problem, we also refer the reader to [11, Conjecture B.1] and [12].)

Part (iv) is [10, Lemma 3.8]. (For a proof of this part in the precise way it is stated in Proposition 4-B, we refer to [13, Lemma 9-B].)

Part (iv) allows us to bound $\int_{B(0,R)} F(x)dx$ in the algebraic case by induction over the radius $R$ (recall that each $B_j$ has radius $R^{1-\delta}$) provided we have a bound on

$$\int_{B_j \cap W} \text{Bil}_{p,\delta} Ef_{j,\text{trans}}(x)^p H(x)dx.$$

The above discussion was formulated in [13] in the following theorem (which is in turn a reformulation of [10, Theorem 3.1] in the weighted setting). We remind the reader that the space $\Lambda(R, K, m)$ of functions was defined in the paragraph before the last of Subsection 4.2.

**Theorem 4-B** ([13, Theorem 9.1]). Let $3 < p \leq 4$, $\epsilon > 0$, $0 \leq q_1 \leq 1 \leq 2q_0$, $0 < q_2 < q_0$, and $H$ be a weight of fractal dimension $\alpha$. Also, let $\delta = \epsilon^2$, $\delta_{\text{deg}} = \epsilon^4$, and $\delta_{\text{trans}} = \epsilon^6$.

Suppose that

$$\int_{B_j \cap W} \text{Bil}_{p,\delta} Ef_{j,\text{trans}}(x)^p H(x)dx$$

(14)

$$\leq C_{\epsilon,K} R^{O(3)} R^{2\epsilon} A_{\alpha}(H)^{q_1} \left(\sum_{\tau} \|f_\tau\|_{L^2(S)}^2\right)^{(3/2)+\epsilon}$$

whenever $R \geq C$, $K \geq 100$, $m \geq 1$, $f \in \Lambda(R, K, m)$, and $P \in \text{Poly}_D(\mathbb{R}^3)$ with $D = R^{\delta_{\text{deg}}}$ and $P$ a product of non-singular polynomials.

Then there is a constant $c_0$, which is independent of $q_0$, $q_1$, and $p$, such that if $\epsilon \leq \min[c_0, (p-3)/2]$, then there is a $K = K(\epsilon)$ such that

$$\int_{B(0,R)} \text{Br}_\beta Ef(x)^p H(x)dx$$

(15)

$$\leq C_\epsilon R^{\delta_{\text{trans}}(K^\epsilon \beta m)} A_{\alpha}(H)^{q_1} \left(\sum_{\tau} \|f_\tau\|_{L^2(S)}^2\right)^{(3/2)+\epsilon} R^{\delta_{\text{trans}} \log(K^\epsilon \beta m)}$$

for all $\beta \geq K^{-\epsilon}$, $m \geq 1$, $R \geq 1$, and $f \in \Lambda(R, K, m)$. Moreover, $\lim_{\epsilon \to 0} K(\epsilon) = \infty$.

In [13, Theorem 9.1]), there is an additional parameter $b$. Theorem 4-B is the special case $b = 1$. 
5. Proof of Theorem 2.1

There are two known estimates that are key to proving Theorem 2.1. The first, which is part (i) of the next lemma, was proved in [10] by adapting Cordoba’s $L^4$ argument from [5] to neighborhoods of algebraic surfaces in $\mathbb{R}^3$. The second, which is part (ii) of the next lemma, was proved in [7] by adapting the refined bilinear Strichartz estimates from [6] to the weighted restriction setting.

**Lemma 5-A.** Suppose $B_j$ is one of the $R^{1-\delta}$-balls that cover $B(0, R) \cap W$, and $\tau_1$ and $\tau_2$ are non-adjacent $K^{-1}$-caps. Then:

(i) We have

$$\int_{B_j \cap W} |E f_{\tau_1, j, \text{tang}}|^2 |E f_{\tau_2, j, \text{tang}}|^2 \, dx \lesssim R^{O(\delta)} R^{-1/2} \|f_{\tau_1, j, \text{tang}}\|_{L^2(\sigma)}^2 \|f_{\tau_2, j, \text{tang}}\|_{L^2(\sigma)}^2 + \text{negligible}$$

for all $f \in \Lambda(R, K, m)$.

(ii) We have

$$\int_{B_j} |E f_{\tau_1, j, \text{tang}}|^2 |E f_{\tau_2, j, \text{tang}}|^2 \mathcal{H} \, dx \lesssim R^{O(\delta)} A_\alpha(H) R^{(\alpha-2)/4} \|f_{\tau_1}\|_{L^2(\sigma)}^{3/2} \|f_{\tau_2}\|_{L^2(\sigma)}^{3/2}$$

for all $f \in \Lambda(R, K, m)$.

Part (i) of Lemma 5-A is the estimate immediately preceding [10, Equation (43)]. Part (ii) of Lemma 5-A is [7, Equation (4.19)]. We alert the reader, however, that [7, Equation (4.19)] is stated as

$$\int_{B_j} |E f_{\tau_1, j, \text{tang}}|^2 |E f_{\tau_2, j, \text{tang}}|^2 \mathcal{H} \, dx \lesssim R^{O(\delta)} R^{3\gamma_3^0/2} \|f_{\tau_1}\|_{L^2(\sigma)}^{3/2} \|f_{\tau_2}\|_{L^2(\sigma)}^{3/2}$$

with $3\gamma_3^0 = \alpha - 2$ and $\mathcal{H} \in F_{\alpha, 3}$, but its proof gives the better exponent $(\alpha - 2)/4$ of $R$ that appears in the above lemma (see the last line of [7, Section 4]). Also, the equivalence between [10] and [7] allows us to replace the $\mathcal{H} \in F_{\alpha, 3}$ by a weight $H$ of fractal dimension $\alpha$.

To prove our theorem, we use Lemma 5-A to establish the estimate (4) on the tangential part of $Ef$, which we then turn into an estimate on the broad part of $Ef$ via Theorem 4-B. We then localize the weight function as described in Section 1 and use parabolic scaling and induction on the radius to turn the estimate on the broad part of $Ef$ into the desired estimate of Theorem 2.1.

5.1. Estimate on the tangential part. The function we are interested in estimating is

$$\text{Bil}_{P, B} Ef_{j, \text{tang}} = \sum_{\tau_1 \neq \tau_2 \text{ non-adjacent}} |E f_{\tau_1, j, \text{tang}}|^{1/2} |E f_{\tau_2, j, \text{tang}}|^{1/2}.$$

To simplify the notation a little, we write

$$F = \text{Bil}_{P, B} Ef_{j, \text{tang}} \quad \text{and} \quad G = |E f_{\tau_1, j, \text{tang}}| \, |E f_{\tau_2, j, \text{tang}}|.$$

Since the constant $C_{\epsilon, K}$ in (14) in Theorem 4-B is allowed to depend on $K$, and there are $\sim K^4$ pairs $\tau_1, \tau_2$, we have

$$\int_{B_j \cap W} F^p H \, dx \lesssim \sum_{\tau_1, \tau_2 \text{ non-adjacent}} \int_{B_j \cap W} G^{p/2} H \, dx.$$
Our plan is to apply Hölder’s inequality to $\int_{B_j \cap W} G^{p/2} H dx$ and then use both parts of Lemma 5-A. In order to do this, we need to find positive numbers $a$ and $b$ such that $a + b = p/2$, $aq = 2$, and $bq' = 3/2$ for some $1 < q < \infty$. This means $a$ and $b$ must satisfy

$$\begin{cases} \frac{a}{2} + \frac{2b}{3} = 1 \\ \frac{a}{2} + \frac{b}{3} = \frac{p}{2} \end{cases}$$

which gives $a = 2(p - 3)$ and $b = (3/2)(4 - p)$, and hence

$$\int_{B_j \cap W} G^{p/2} H dx = \int_{B_j \cap W} G^{2(p-3)} G^{(3/2)(4-p)} H dx.$$

Applying Hölder’s inequality with $q = 2/a = 1/(p - 3)$, we get

$$\int_{B_j \cap W} F^p H dx \lesssim \sum_{\tau_1, \tau_2 \text{ non-adjacent}} \left( \int_{B_j \cap W} G^{2} H dx \right)^{p-3} \left( \int_{B_j \cap W} G^{3/2} H dx \right)^{4-p}. $$

Writing $G_0 = \| f_{\tau_1, j, \text{tang}} \|_{L^2(\sigma)} \| f_{\tau_2, j, \text{tang}} \|_{L^2(\sigma)}$, Lemma 5-A tells us that

$$\int_{B_j \cap W} G^2 H dx \lesssim R^{O(\delta)} R^{-1/2} G_0^2$$

(because $\| H \|_{L^\infty(\mathbb{R}^3)} \leq 1$) and

$$\int_{B_j \cap W} G^{3/2} H dx \lesssim R^{O(\delta)} A_\alpha(H) R^{(\alpha-2)/4} G_0^{3/2}.$$  

Therefore,

$$\int_{B_j \cap W} F^p H dx \lesssim R^{O(\delta)} A_\alpha(H)^{4-p} R^{-(p-3)/2} R^{(\alpha-2)(4-p)/4} \sum_{\tau_1, \tau_2} G_0^{p/2}. $$

Following [10], we now use the fact that only few of the caps $\theta$ contribute to $G_0$. Suppose $\tau$ is one of our $K^{-1}$-caps, and let

$$\{ \theta_1, \ldots, \theta_L \} = \{ \theta : \theta \cap \tau \neq \emptyset \text{ and } T_{\tau, j, \text{tang}} \cap \mathbb{T}(\theta) \neq \emptyset \}$$

and $\mathcal{T}_i = T_{\tau, j, \text{tang}} \cap \mathbb{T}(\theta_i)$. Then

$$f_{\tau, j, \text{tang}} = \sum_{l=1}^{L} \sum_{T \in \mathcal{T}_i} f_{\tau, T}$$

and it follows by (12) that

$$\int_{3\delta_l} |f_{\tau, j, \text{tang}}|^2 d\sigma \lesssim \int_{10\delta_l} |f_{\tau}|^2 d\sigma$$

for all $1 \leq l \leq L$. By part (i) of Proposition 4-A we know that $f_{\tau, j, \text{tang}}$ is supported in $\cup_{l=1}^L 3\delta_l$, so

$$\int |f_{\tau, j, \text{tang}}|^2 d\sigma \lesssim \sum_{l=1}^{L} \int_{10\delta_l} |f_{\tau}|^2 d\sigma.$$

On the one hand, this gives

$$\int |f_{\tau, j, \text{tang}}|^2 d\sigma \lesssim \int |f_{\tau}|^2 d\sigma.$$
On the other hand, from the definition of $\Lambda(R, K, m)$ (see [13]) we know that
\[
\int_{10\delta} |f_\tau|^2 d\sigma \lesssim \frac{1}{R},
\]
so
\[
\int |f_{\tau,j,\text{tang}}|^2 d\sigma \lesssim \sum_{l=1}^{L} \int_{10\delta} |f_\tau|^2 d\sigma \lesssim \frac{L}{R}.
\]
Part (iii) of Proposition [14] also tells us that $L \lesssim R^{O(\epsilon)} R^{1/2}$, so
\[
\int |f_{\tau,j,\text{tang}}|^2 d\sigma \lesssim R^{O(\epsilon)} R^{-1/2}.
\]
Therefore,
\[
G_0 = \|f_{\tau_1,j,\text{tang}}\|_{L^2(\sigma)} \|f_{\tau_2,j,\text{tang}}\|_{L^2(\sigma)} \lesssim R^{O(\epsilon)} R^{-1/2}.
\]
Going back to (16), we now have
\[
G_0^{p/2} = G_0^{(p-3-2\epsilon)/2} G_0^{(3/2)+\epsilon} \lesssim R^{O(\epsilon)} R^{-(p-3-2\epsilon)/4} G_0^{(3/2)+\epsilon},
\]
and (16) becomes
\[
\int_{B_j \cap W} F^p H dx \lesssim R^{O(\epsilon)} R^{\epsilon/2} A_\alpha(H)^{4-p} R^{-3(p-3)/4} R^{(\alpha-2)(4-p)/4} \sum_{\tau_1, \tau_2} G_0^{(3/2)+\epsilon}.
\]
By easy considerations,
\[
\sum_{\tau_1, \tau_2} G_0^{(3/2)+\epsilon} = \left( \sum \|f_{\tau,j,\text{tang}}\|_{L^2(\sigma)}^{(3/2)+\epsilon} \right)^2 \lesssim \left( \sum \|f_{\tau,j,\text{tang}}\|_{L^2(\sigma)}^2 \right)^{(3/2)+\epsilon},
\]
and so, by (17),
\[
\sum_{\tau_1, \tau_2} G_0^{(3/2)+\epsilon} \lesssim \left( \sum \|f_\tau\|_{L^2(\sigma)}^2 \right)^{(3/2)+\epsilon}.
\]
Also,
\[
R^{-3(p-3)/4} R^{(\alpha-2)(4-p)/4} = 1 \quad \text{if} \quad p = \frac{4\alpha + 1}{\alpha + 1}.
\]
Therefore, with this value of $p$, we have
\[
\int_{B_j \cap W} F^p H dx \lesssim R^{O(\epsilon)} R^{\epsilon/2} A_\alpha(H)^{4-p} \left( \sum \|f_\tau\|_{L^2(\sigma)}^2 \right)^{(3/2)+\epsilon}.
\]
5.2 Estimate on the broad part. In the previous subsection, we established (14) with $p = (4\alpha + 1)/(\alpha + 1)$, $q_1 = 4 - p$, and $q_2 = 1/2$. Inserting these values into Theorem [14] and choosing $q_0 = 1$, we conclude that there is a constant $K_0$ such that to every $\epsilon \leq \min(c_0, (\alpha - 2)/((2\alpha + 2)/2))$ there are constants $K = K(\epsilon)$ and $C_\epsilon$ such that $\lim_{\epsilon \to 0} K(\epsilon) = \infty$ and
\[
\int_{B(0, R)} |B_{\epsilon} f(x)|^p H(x) dx \leq C_\epsilon R^\epsilon A_\alpha(H)^{3/(\alpha+1)} \left( \sum \|f_\tau\|_{L^2(\sigma)}^{(3/2)+\epsilon} \right) R^{\epsilon/2} \log(K^\epsilon K^\epsilon)
\]
for all $\beta \geq K^{-\epsilon}$, $m \geq 1$, $R \geq 1$, and $f \in \Lambda(R, K, m)$. 
Suppose \( f \in L^\infty(\sigma) \) is a non-zero function, and define the function \( g : \mathcal{P} \to \mathbb{C} \) by 
\[
g = \pi^{-1/2} \|f\|^{-1}_{L^\infty(\sigma)} f,
\]
so that 
\[
\int_{B(\xi_0, R^{-1/2}) \cap \mathcal{P}} |g|^2 \, d\sigma \leq \frac{1}{R}
\]
for all \( \xi_0 \in \mathcal{P} \). Writing \( g = \sum \tau g_\tau \) with \( \text{supp} g_\tau \subset \tau \) and \( \text{supp} g_\tau \cap \text{supp} g_{\tau'} = \emptyset \) if \( \tau \neq \tau' \), we see that \( g \in \Lambda(R, K, m) \) and 
\[
\sum \|g_\tau\|_{L^2(\sigma)}^2 = \|g\|_{L^2(\sigma)}^2.
\]
Applying the above estimate on the broad part with \( \beta = K^{-\epsilon} \) and \( m = 1 \), we obtain
\[
\int_{B(0, R)} \text{Br}_{K^{-\epsilon}} E g(x)^p H(x) \, dx \leq C \epsilon R' A_\alpha(H)^{3/(\alpha+1)} \|g\|_{L^2(\sigma)}^{3+2\epsilon},
\]
Since \( \|g\|_{L^2(\sigma)} \leq 1 \), we have \( \|g\|_{L^2(\sigma)}^{3+2\epsilon} \leq \|g\|_{L^2(\sigma)}^3 \). So, replacing \( g \) by \( f \), and letting \( c = \min[c_\alpha, (\alpha - 2)/(2\alpha + 2)] \), we see that to every \( \epsilon \leq c \) there are constants \( K = K(\epsilon) \) and \( C \), such that \( \lim_{\epsilon \to 0} K(\epsilon) = \infty \) and
\[
\int_{B(0, R)} \text{Br}_{K^{-\epsilon}} E f(x)^p H(x) \, dx \leq \tilde{C} \epsilon R' A_\alpha(H)^{3/(\alpha+1)} \|f\|_{L^2(\sigma)}^3 \|f\|_{L^\infty(\sigma)}^{p-3}
\]
for all functions \( f \in L^\infty(\sigma) \), weights \( H \) of fractal dimension \( \alpha \), and radii \( R \geq 1 \).

### 5.3. Estimate on \( E f \)

Moving from the estimate on the \( L^p \)-norm of \( \text{Br}_{K^{-\epsilon}} E f(x) \) against the weight \( H \) into an estimate on the \( L^p \)-norm of \( E f \) itself involves localizing the weight \( H \) at scale \( r = K^{-1} \), followed by parabolic scaling, followed by an induction on the radius argument. These three arguments were carried out in detail by the author in [13] and resulted in the following theorem that we now borrow from that paper.

**Theorem 5-A ([13, Theorem 12.1]).** Let \( 0 < \alpha \leq 3 \), \( 3 \leq p \leq 4 \), \( 2 \leq \gamma \leq 3 \), \( 0 \leq q_1 \leq 1 \), and \( c > 0 \). Suppose that we have the following estimate on the broad part of \( E f \): to every \( \epsilon \in (0, c) \) there are constants \( K(\epsilon) \) and \( C_\epsilon \) such that \( \lim_{\epsilon \to 0} K(\epsilon) = \infty \) and
\[
\int_{B(0, R)} \text{Br}_{K^{-\epsilon}} E f(x)^p H(x) \, dx \leq \tilde{C} \epsilon R' A_\alpha(H)^{q_1} \|f\|_{L^2(\sigma)}^\gamma \|f\|_{L^\infty(\sigma)}^{p-\gamma}
\]
for all radii \( R \geq 1 \), weights \( H \) of dimension \( \alpha \), and functions \( f \in L^\infty(\sigma) \).

If \( 2p - \alpha - 1 - \gamma > 0 \), then there is a constant \( c' \), which only depends on \( \alpha, p \), and \( \gamma \), such that for \( 0 < \epsilon < c' \) we have
\[
\int_{B(0, R)} |Ef(x)|^p H(x) \, dx \leq C_\epsilon R' \left( \max \left[A_\alpha(H), A_\alpha(H)^{q_1} \right] \right) \|f\|_{L^2(\sigma)}^\gamma \|f\|_{L^\infty(\sigma)}^{p-\gamma},
\]
with
\[
C_\epsilon = 2(\tilde{C}_\epsilon + 4^4 \sigma(S)^4),
\]
for all radii \( R \geq 1 \), weights \( H \) of dimension \( \alpha \), and functions \( f \in L^\infty(\sigma) \).

Since
\[
\|f\|_{L^2(\sigma)} = \|f\|_{L^2(\sigma)}^\gamma \|f\|_{L^\infty(\sigma)}^{3-\gamma} \leq \|f\|_{L^2(\sigma)}^\gamma \|f\|_{L^\infty(\sigma)}^{3-\gamma},
\]
the estimate (18) implies (19) with \( p = (4\alpha + 1)/(\alpha + 1) \) and \( q_1 = 4 - p \), and hence Theorem 5-A tells us that the estimate
\[
\int_{B(0, R)} |Ef(x)|^p H(x) \, dx \leq C_\epsilon R' \left( \max \left[A_\alpha(H), A_\alpha(H)^{q_1} \right] \right) \|f\|_{L^2(\sigma)}^\gamma \|f\|_{L^\infty(\sigma)}^{p-\gamma}
\]
holds whenever \(2p - \alpha - 1 - \gamma > 0\). So, to finish the proof of Theorem \ref{thm:main}, we just have to show that this estimate remains true if one replaces \(\max[A_\alpha(H), A_\alpha(H)^q]\) by \(A_\alpha(H)\):

\[
\int_{B(0, R)} |Ef(x)|^p H(x)dx \leq C_\epsilon R^\epsilon A_\alpha(H)
\|
\begin{array}{c}
\hat{f}L^2(\sigma)\\
\hat{f}L^\infty(\sigma)
\end{array}
\|
\|
\begin{array}{c}
p^{-\gamma}\\p
\end{array}
\|
\]

In fact, since the unit paraboloid \(P\) is compact, we can find a \(C_0^\infty\) function \(\phi\) on \(\mathbb{R}^3\) that satisfies \(|\phi| \geq 1\) on \(P\) and \(\hat{\phi}\) is supported in the unit ball. So, if we define the function \(g\) on \(P\) by \(g = f/\phi\) and observe that \(|g| \leq |f|, Ef = (Eg) \ast \hat{\phi}\), and \(|Ef|^p \lesssim |Eg|^p \ast \hat{\phi}|\), we see that

\[
\int |Ef(x)|^p H(x)dx \lesssim \int |Eg(y)|^p \int |\hat{\phi}(x - y)|H(x)dxdy
\]

\[
\lesssim A_\alpha(H) \int |Eg(y)|^p \hat{H}(y)dy
\]

\[
\lesssim A_\alpha(H) R^\epsilon \max[A_\alpha(\hat{H}), A_\alpha(\hat{H})^q]\|g\|L^2(\sigma)\|g\|L^\infty(\sigma)
\]

\[
\lesssim A_\alpha(H) R^\epsilon \max[A_\alpha(\hat{H}), A_\alpha(\hat{H})^q]\|\hat{f}\|L^2(\sigma)\|f\|L^\infty(\sigma)
\]

where

\[
\hat{H}(y) = A_\alpha(H)^{-1}\|\hat{\phi}\|L_\infty(\mathbb{R}^3)^{-1} \int |\hat{\phi}(x - y)|H(x)dx
\]

is clearly a weight of fractal dimension \(\alpha\) with \(A_\alpha(\hat{H}) \lesssim 1\). Therefore,

\[
\int_{B(0, R)} |Ef(x)|^p H(x)dx \leq C_\epsilon R^\epsilon A_\alpha(H)\|\hat{f}\|L^2(\sigma)\|f\|L^\infty(\sigma)
\]

whenever \(p = (4\alpha + 1)/(\alpha + 1)\) and \(2p - \alpha - 1 - \gamma > 0\), as desired.

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