A phase-integral perspective on $\alpha$-decay

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Abstract

This paper applies the phase-integral method to the stationary theory of $\alpha$-decay. The rigorous form of the connection formulae, and their one-directional nature that was not widely known in the physical literature, are applied. The condition for obtaining $s$-wave metastable states affects the stationary state at large distance from the nucleus, which is dominated by the cosine of the phase integral minus $\frac{\pi}{4}$. Accurate predictions for the lowest $s$-wave metastable state and mean life of the radioactive nucleus are obtained in the case of Uranium. The final part of the paper describes the phase-integral algorithm for evaluating stationary states by means of a suitable choice of freely specifiable base function. Within this framework, an original approximate formula for the phase integrand with arbitrary values of the angular momentum quantum number is obtained.

1 Introduction

The detailed investigation of $\alpha$-decay is a topic that leads to a thorough understanding of the application of quantum mechanics to atomic and nuclear physics, since it is necessary to have a good knowledge of metastable states [1, 2] and of the Jeffreys-Wentzel-Kramers-Brillouin (hereafter, JWKB) method applied to the Schrödinger equation for stationary states [2, 3, 4]. In particular, in the description of the JWKB method, the introductory textbooks on quantum mechanics fail, even nowadays, to present the remarkable results obtained in Refs. [3, 4], which are written in a very clear and pedagogical style.
Our paper lies precisely within this framework. In section 2 we outline some relevant features of the phase-integral method and of the associated derivation of connection formulae. In section 3 we consider the basic equations for the elementary stationary theory of α-decay. Section 4 develops a more accurate model of the stationary theory in section 3, and improves the theoretical estimate of the lowest s-wave metastable state. The mean life of the radioactive nucleus is evaluated in section 5 for Uranium. Section 6 describes the phase-integral algorithm for evaluating stationary states by means of a suitable choice of freely specifiable base function [3, 4], while concluding remarks are presented in section 7.

2 Phase integral method and connection formulae

Both in one-dimensional problems and in the case of central potentials in three-dimensional Euclidean space, the Schrödinger equation for stationary states leads eventually to a second-order ordinary differential equation having the form [1, 2, 3, 4]

\[
\left[ \frac{d^2}{dz^2} + R(z) \right] \psi(z) = 0, \quad R(z) = \frac{2m}{\hbar^2} (E - V(z)),
\]

(2.1)

where \( V(z) \) is either the potential in one spatial dimension, or an effective potential that includes also the effects of angular momentum. The notation \( z \) for the independent variable means that one can study Eq. (2.1) in the complex field, restricting attention to real values of \( z \), denoted by \( x \), only at a later stage. In the phase-integral method, one looks for two linearly independent, exact solutions of Eq. (2.1) in the form

\[
\psi(z) = A(z)e^{\pm iw(z)}.
\]

(2.2)

Since the Wronskian of two linearly independent solutions of Eq. (2.1) is a non-vanishing constant, while the Wronskian of the functions (2.2) is \(-2iA^2 \frac{dw}{dz}\), for consistency one finds

\[
A = \text{constant} \frac{1}{\sqrt{\frac{dw}{dz}}},
\]

(2.3)

One can therefore write (up to a multiplicative constant)

\[
\psi(z) = \frac{1}{\sqrt{\frac{dw}{dz}}} e^{\pm iw(z)} = \frac{1}{\sqrt{q(z)}} e^{\pm i \int q(z) d\zeta},
\]

(2.4)
where \( w(z) \equiv \int^z q(\zeta) d\zeta \) is said to be the phase integral, while \( q(z) \) is the phase integrand \([3, 4]\).

In quantum mechanical problems, we shall agree to call classically forbidden the open interval of the independent variable where the energy \( E \) of the particle is strictly less than the potential \( V: E < V \). Conversely, if \( E > V \), we shall talk of classically allowed region. For the former, let \( x_1 \) be an internal point, where the stationary state takes the exact form (hereafter, the independent variable is always real)

\[
\psi(x_1) = c(x_1) \frac{1}{\sqrt{q(x_1)}} e^{i|w(x_1)|} + d(x_1) \frac{1}{\sqrt{q(x_1)}} e^{-|w(x_1)|},
\]

(2.5)

c and \( d \) being real-valued functions. For the latter, let \( x_2 \) be an internal point, where the stationary state takes the exact form

\[
\psi(x_2) = a(x_2) \frac{1}{\sqrt{q(x_2)}} e^{i|w(x_2)|} + b(x_2) \frac{1}{\sqrt{q(x_2)}} e^{-i|w(x_2)|}.
\]

(2.6)

From the detailed theory in section 2.4 of Ref. [4], one knows that (hereafter, the star denotes complex conjugation)

\[
a(x_2) = \left\{ \frac{1}{2\alpha} c(x_1) + i\alpha \left[ \gamma c(x_1) - d(x_1) \right] \right\} e^{\frac{i}{2}(\pi - \beta)},
\]

(2.7)

\[
b(x_2) = a^*(x_2).
\]

(2.8)

The exact expressions of the \( \alpha, \beta, \gamma \) parameters are known (see Ref. [4] and our appendix A) and they are not particularly enlightening. However, upon setting

\[
\chi(q(x)) \equiv q^{-\frac{2}{3}} \frac{d^2}{dx^2} q^{\frac{1}{3}} + \frac{R(x)}{q^2} - 1,
\]

(2.9)

\[
\mu(x, x_0) \equiv \left| \int_{x_0}^{x} \chi(q(x')) q(x') \ dx' \right|,
\]

(2.10)

if \( \mu << 1 \), one can use the approximate formulae [4]

\[
\alpha \sim 1 + O(\mu), \quad B \sim O(\mu), \quad \gamma \sim O(\mu)e^{2|w(x_1)|}.
\]

(2.11)

If \( \mu \) is much bigger than \( e^{-2|w(x_1)|} \), one then finds from the third of Eqs. (2.11)

\[
\gamma \sim O(\mu)e^{2|w(x_1)|} >> 1.
\]

(2.12)
Since $\gamma$ is unknown and in general much bigger than 1, one can obtain approximate formulae for $a(x_2)$ and $b(x_2)$ only when

$$|\gamma c(x_1)| << |d(x_1)|.$$  

(2.13)

By virtue of the condition (2.12), the majorization (2.13) provides

$$\left| \frac{c(x_1)}{d(x_1)} \right| \leq e^{-2|w(x_1)|}.$$  

(2.14)

If the right-hand side of (2.14) is much smaller than 1, the exact formula (2.7) yields the remarkable approximate formula

$$a(x_2) \approx \left[ \frac{1}{2} c(x_1) - id(x_1) \right] e^{i(\frac{\pi}{4} - \mu)} 
\approx \left[ \frac{1}{2} c(x_1) - \frac{i}{2} d(x_1) \right] e^{i\frac{\pi}{4}} \approx -id(x_1)e^{i\frac{\pi}{4}} 
= d(x_1)e^{i\left(\frac{\pi}{4} - \frac{\pi}{2}\right)} = d(x_1)e^{-i\frac{\pi}{4}},$$  

(2.15)

while

$$b(x_2) = a^*(x_2) \approx d(x_1)e^{i\frac{\pi}{4}}.$$  

(2.16)

The exact formula (2.6) leads therefore to the approximate formula

$$\psi(x_2) \approx d(x_1)e^{-i\frac{\pi}{4}} \left| q^{-\frac{1}{2}}(x_2) \right| e^{i|w(x_2)|} + d(x_1)e^{i\frac{\pi}{4}} \left| q^{-\frac{1}{2}}(x_2) \right| e^{-i|w(x_2)|} 
= 2d(x_1) \left| q^{-\frac{1}{2}}(x_2) \right| \cos \left[ |w(x_2)| - \frac{\pi}{4} \right].$$  

(2.17)

From Eqs. (2.5) and (2.17) one gets therefore the connection formula

$$c \left| q^{-\frac{1}{2}}(x) \right| e^{i|w(x)|} + d \left| q^{-\frac{1}{2}}(x) \right| e^{-i|w(x)|} \longrightarrow 2d \left| q^{-\frac{1}{2}}(x) \right| \cos \left[ |w(x)| - \frac{\pi}{4} \right],$$  

(2.18)

which holds provided that the condition (2.13) is fulfilled. In the literature, the case $c = 0, d = 1$ is often considered for simplicity [4]. Remarkably, the connection formula (2.18) is one-directional. The work in Ref. [4] proves indeed that, if one is first given a stationary state in the classically allowed region having the form

$$\psi(x_2) = 2 \left| q^{-\frac{1}{2}}(x_2) \right| \cos \left[ |w(x_2)| - \frac{\pi}{4} \right],$$

the stationary state in the classically forbidden region does not reduce to the left-hand side of Eq. (2.18) with $c = 0$ and $d = 1$. This property is so important for our analysis that we prove it in appendix B, so that our paper becomes completely self-contained.
3 Elementary stationary theory of α-decay

As is well known, experiments in which a sufficiently large number of α-particles enter a chamber with a thin window and are collected show that α-rays correspond to positively charged particles whose charge-to-mass ratio is the one of doubly ionized helium atoms: He++. Their identification with He++ is made possible because, when a gas of α-particles produces light, it displays precisely the spectroscopic lines of He++. In the stationary theory of α-emission, one regards the α-particle as being pre-existent in the radioactive nucleus. Such a radioactive nucleus is therefore viewed as a metastable state consisting of the α-particle and the residual nucleus. The force acting on the α-particle is the joint effect of a short-range nuclear interaction and a long-range Coulomb repulsion. The long-range component is described by a potential \( 2(Z - 2)^2 \frac{e^2}{r} \), and the potential is assumed to obey the defining law

\[
V(r) = -U_0 \quad \text{if} \quad r \in [0, b[,
\]

\[
V(r) = 2(Z - 2)^2 \frac{e^2}{r} \quad r \in ]b, \infty[.
\]

The s-wave metastable states can be found by solving the equation (cf. Eq. (2.1))

\[
\left[ \frac{d^2}{dr^2} + \frac{2m}{\hbar^2}(E - V(r)) \right] y(r) = 0,
\]

in the three open intervals

\[
I_1 = [0, b[ , \quad I_2 = ]b, r_1[ , \quad I_3 = ]r_1, \infty[,
\]

where \( r_1 \) is the value of \( r \) for which the energy of the α-particle equals the Coulomb term, i.e.

\[
r_1 = 2(Z - 2)^2 \frac{e^2}{E}. \tag{3.5}
\]

The interval \( I_2 \) corresponds to values of the energy \( E < V \), while the interval \( I_3 \) pertains to values of the energy \( E > V \). On defining

\[
p_0 \equiv \sqrt{2m_\alpha(E + U_0)}, \quad \bar{p}(r) \equiv \sqrt{2m_\alpha(V(r) - E)}, \quad p(r) \equiv \sqrt{2m_\alpha(E - V(r))},
\]

\[
1\text{Recall from the theory of resonance scattering} [1,2] \text{ that there exists a metastable state corresponding to a trapping of the particle in the region where the potential makes its effect manifest.}
\]
which are appropriate for $I_1, I_2, I_3$, respectively, one can write the solutions of Eq. (3.3) within such intervals (see comments after Eq. (3.13)) in the form [2]

\[
y_1(r) = \frac{C}{\sqrt{p_0}} \sin \left( \frac{p_0 r}{\hbar} \right) = \frac{C}{\sqrt{p_0}} \left\{ A_1 \sin \left[ \frac{p_0(b - r)}{\hbar} - \frac{\pi}{4} \right] + A_2 \cos \left[ \frac{p_0(b - r)}{\hbar} - \frac{\pi}{4} \right] \right\}, \quad (3.7)
\]

\[
y_2(r) = \frac{C}{\sqrt{\bar{p}(r)}} \left\{ A_3 \exp \left[ \frac{1}{\hbar} \int_b^r \bar{p}(r')dr' \right] + A_4 \exp \left[ -\frac{1}{\hbar} \int_b^r \bar{p}(r')dr' \right] \right\}, \quad (3.8)
\]

\[
y_3(r) = \frac{C}{\sqrt{p(r)}} \left\{ A_5 \cos \left[ \frac{1}{\hbar} \int_{r_1}^r p(r')dr' - \frac{\pi}{4} \right] + A_6 \sin \left[ \frac{1}{\hbar} \int_{r_1}^r p(r')dr' - \frac{\pi}{4} \right] \right\}, \quad (3.9)
\]

where [2]

\[
A_1 = -\cos \left( \frac{p_0 b}{\hbar} - \frac{\pi}{4} \right), \quad A_2 = \sin \left( \frac{p_0 b}{\hbar} - \frac{\pi}{4} \right), \quad (3.10)
\]

\[
A_3 = -A_1, \quad A_4 = \frac{1}{2} A_2, \quad (3.11)
\]

and, considering the parameter

\[
\theta \equiv \exp \left[ -\frac{1}{\hbar} \int_b^1 \bar{p}(r)dr \right] << 1, \quad (3.12)
\]

one finds the last two coefficients in the form

\[
A_5 = \frac{2A_3}{\theta}, \quad A_6 = -A_4 \theta. \quad (3.13)
\]

It should be stressed that only Eq. (3.7) provides an exact solution, in the open interval $I_1$, whereas Eqs. (3.8) and (3.9) provide approximate solutions in the open intervals $I_2$ and $I_3$, respectively. The coefficients $A_3$ and $A_4$ of Eq. (3.8) are obtained in Ref. [2] from a connection recipe, but not from the continuity condition of stationary states and their first derivative, unlike the work in Ref. [5].

The work in Ref. [2] points out that, for metastable states to occur, one has to maximize the derivative of the phase shift with respect to the energy, and this implies in turn that the ratio of values of the stationary state inside
and outside the potential well must be maximized. For this purpose, the authors of Ref. [2] set to zero $A_3$ (and hence $A_1$ and $A_5$), finding therefore, for the lowest $s$-wave metastable state,

$$\frac{p_0 b}{\hbar} - \frac{\pi}{4} = \frac{\pi}{2} \implies p_0 = \frac{3 \pi h}{4 b}. \quad (3.14)$$

The coefficient of the sin function in (3.9) is then found to be $-\frac{\theta}{2}$, from Eqs. (3.13) and (3.14). When the condition (3.13) is fulfilled, the derivative of the phase shift takes the approximate form [2]

$$\frac{d}{dp} \delta \approx \pi \int_0^b |y_1(r)|^2 dr. \quad (3.15)$$

In order to evaluate the integral on the right-hand side of (3.15), one has to evaluate the $C$ coefficient in the formulae for stationary states in the three intervals. For this purpose, one looks first at the interval $I_3$, where (see Eq. (3.9))

$$\frac{1}{h} \int_{r_1}^{r} p(r')dr' = \sqrt{\frac{2m_a E}{\hbar}} \int_{r_1}^{r} \left(1 - \frac{2(Z - 2)e_0^2}{E r'}\right)^{\frac{1}{2}} dr' \approx \frac{pr}{h} - m_a \frac{2(Z - 2)e_0^2}{hp} \log \left(\frac{2pr}{h}\right) + \ldots \quad (3.16)$$

Thus, at very large values of $r$, Ref. [2] finds

$$y_3(r) \approx -\frac{C\theta}{2\sqrt{p(r)}} \sin \left[\frac{1}{h} \int_{r_1}^{r} p(r')dr' - \frac{\pi}{4}\right]$$

$$\approx \frac{C\theta}{2\sqrt{p(r)}} \cos \left[\frac{pr}{h} - m_a \frac{2(Z - 2)e_0^2}{hp} \log \left(\frac{2pr}{h}\right) + \vartheta\right], \quad (3.17)$$

where the explicit form of $\vartheta$ is here inessential, but we can say that it is linearly related to the phase shift [2]. On the other hand, from the general analysis of Coulomb type potentials, one knows that [2]

$$y_3(r) \approx \sqrt{\frac{2}{\pi h}} \cos \left[\frac{pr}{h} - m_a \frac{2(Z - 2)e_0^2}{hp} \log \left(\frac{2pr}{h}\right) + \vartheta\right], \quad (3.18)$$

and hence by comparison of Eqs. (3.17) and (3.18) one finds

$$C = \sqrt{\frac{2}{\pi h} \frac{2\sqrt{p}}{\vartheta}}. \quad (3.19)$$
This implies in turn that

\[ y_1(r) = \sqrt{\frac{8}{\pi \hbar}} \sqrt{\frac{p}{p_0}} \sin \left( \frac{p_0 r}{\hbar} \right), \quad (3.20) \]

and therefore Eq. (3.15) yields

\[ \frac{d}{dp} \delta \approx \frac{8}{\hbar \theta^2} \frac{p}{p_0} \int_0^b \sin^2 \left( \frac{p_0 r}{\hbar} \right) dr \]
\[ = \frac{8}{\hbar \theta^2} \frac{p}{p_0} \left[ \frac{b}{2} - \frac{h}{4p_0} \sin \left( \frac{2p_0 b}{h} \right) \right] \]
\[ \approx \frac{4b}{\hbar \theta^2}. \quad (3.21) \]

The mean life of the radioactive nucleus is then

\[ \tau = \frac{\hbar}{2} \frac{d}{dE} \delta \approx \frac{2m_\alpha}{p} \frac{1}{\theta^2}. \quad (3.22) \]

Bearing in mind that the parameter \( \theta \) defined in Eq. (3.12) is the exponential of minus the integral

\[ \frac{1}{\hbar} \int_{r_1}^{r_2} \sqrt{2m_\alpha \left[ 2(Z - 2) \frac{e_0^2}{r} - E \right]} \] dr,

the stationary theory studied so far yields therefore the prediction

\[ \log(\tau) = \log \left( \frac{2bm_\alpha}{p} \right) - \log \theta^2 \]
\[ \approx \log(\tau_0) + 2(Z - 2)\pi \frac{e_0^2}{\hbar} \sqrt{\frac{2m_\alpha}{E}} \]
\[ + \frac{8}{\hbar} \sqrt{(Z - 2)e_0^2 m_\alpha b}. \quad (3.23) \]

4 A more accurate model

The careful reader might have noticed that the first line of Eq. (3.17) is at odds with the connection formula (2.18), whose left-hand side corresponds neatly to Eq. (3.8). In section 3 we stressed indeed that Eq. (3.8) is not an exact solution of the stationary Schrödinger equation in the interval \( I_2 \), because the coefficients \( A_3 = -A_1 \) and \( A_4 = \frac{1}{2}A_2 \) are obtained [4] from the connection recipe

\[ \cos \left[ |w(x)| - \frac{\pi}{4} \right] \to \frac{1}{2} e^{-|w(x)|}, \quad \sin \left[ |w(x)| - \frac{\pi}{4} \right] \to -e^{i|w(x)|}. \quad (4.1) \]
which contradicts the connection formulae in Refs. [3, 4]. On the other hand, the use of two consecutive connection formulae may be questionable as well. More precisely, on passing from the interval $I_1$ to the interval $I_2$, the work in section 3.12 of Ref. [4] would suggest using, instead of Eq. (4.1), the one-directional connection formula

$$\frac{a}{\sqrt{q(x)}} e^{i[w(x)]} + \frac{b}{\sqrt{q(x)}} e^{-i[w(x)]} \rightarrow \left[ a e^{-i\frac{\pi}{4}} + b e^{i\frac{\pi}{4}} \right] \frac{e^{i[w(x)]}}{\sqrt{q(x)}},$$

which is valid when the absolute value $|a e^{-i\frac{\pi}{4}} + b e^{i\frac{\pi}{4}}|$ is not too small compared to $|a| + |b|$ [4]. However, if in the interval $I_2$ only the increasing exponential $e^{i[w(x)]}$ survives, the connection formula (2.18) would tell us that we should expect a vanishing stationary state in the interval $I_3$. But this conclusion would be incorrect, as is shown from Eq. (3.18), which does not rely upon any form of connection formula. The deeper underlying reason might be, that the rigorous connection formulae in Ref. [4] hold for adjacent intervals, but their repeated use for a sequence of adjacent intervals requires further work. For this reason, and inspired in part by the work in Ref. [5], we consider hereafter the following method.

In the interval $I_1$, we write simply the exact solution of the $s$-wave stationary Schrödinger equation in the form displayed on the first line of Eq. (3.7). In the interval $I_2$, we look for a solution in the form (3.8), but with values of $A_3$ and $A_4$ not given by Eq. (3.11). We impose instead the continuity conditions for stationary state and its first derivative, which hold whenever the potential has a finite discontinuity [2, 6, 7]. Hence we require that

$$\lim_{r \to b} y_1(r) = \lim_{r \to b} y_2(r),$$

$$\lim_{r \to b} y_1'(r) = \lim_{r \to b} y_2'(r).$$

Equations (4.3)-(4.4) are solved by (cf. Eq. (9) in Ref. [5])

$$A_3 = \frac{1}{2} \left\{ \sqrt{\frac{p_0}{\bar{p}(b)}} \cos \left( \frac{p_0 b}{\hbar} \right) - \sqrt{\frac{\bar{p}(b)}{p_0}} \left[ 1 + \frac{\hbar}{2} \frac{\bar{p}'(b)}{\bar{p}(b)^2} \right] \sin \left( \frac{p_0 b}{\hbar} \right) \right\},$$

$$A_4 = \frac{1}{2} \left\{ -\sqrt{\frac{p_0}{\bar{p}(b)}} \cos \left( \frac{p_0 b}{\hbar} \right) + \sqrt{\frac{\bar{p}(b)}{p_0}} \left[ 1 - \frac{\hbar}{2} \frac{\bar{p}'(b)}{\bar{p}(b)^2} \right] \sin \left( \frac{p_0 b}{\hbar} \right) \right\}. \quad (4.5)$$

At this stage, if we follow the physical requirement of Ref. [4] and our section 3 for obtaining $s$-wave metastable states, i.e., that the coefficient $A_3$ should vanish, we get the equation

$$\left[ 1 + \frac{\hbar}{2} \frac{\bar{p}'(b)}{\bar{p}(b)^2} \right] \tan \left( \frac{p_0 b}{\hbar} \right) = -\frac{p_0}{\bar{p}(b)}. \quad (4.7)$$
For example, in the case of Uranium \([5]\), the right-hand side of Eq. (4.7) equals \(-\frac{9}{50}\), and bearing in mind that

\[ 1 \gg \frac{\hbar}{2} \frac{\dot{\rho}(b)}{(\bar{\rho}(b))^2}, \]

the approximate root of Eq. (5.7) is equal to

\[ \frac{p_0 b}{\hbar} \equiv \rho \approx 2.963, \tag{4.8} \]

whereas the value (3.14) for \(\frac{p_0 b}{\hbar}\) is approximately equal to 2.356. We find therefore, for the energy of the lowest \(s\)-wave metastable state,

\[ E = \frac{(p_0)^2}{2m_\alpha} - U_0, \tag{4.9} \]

with \(p_0\) given by Eq. (4.8). The work in Ref. \([5]\) sets instead to zero the right-hand side of Eq. (4.7), which is not sufficiently accurate, at least in the case of Uranium.

At this stage, if we define

\[ \nu \equiv \frac{p_0}{\bar{\rho}(b)}, \tag{4.10} \]

we find from Eq. (4.6) a good approximation for \(A_4\) in the form

\[ A_4 \approx \frac{1}{2} \left[ -\sqrt{\nu} \cos(\rho) + \frac{1}{\sqrt{\nu}} \sin(\rho) \right], \tag{4.11} \]

where \(\rho\) solves the equation that ensures the vanishing of \(A_3\):

\[ \tan(\rho) = -\nu, \tag{4.12} \]

which implies (\(\cos(\rho)\) is negative since \(\rho\) is close to \(\pi\) by virtue of Eq. (4.8))

\[ \cos(\rho) = -\frac{1}{\sqrt{1 + \nu^2}}, \quad \sin(\rho) = \frac{1}{\sqrt{1 + \nu^2}}. \tag{4.13} \]

Hence we obtain

\[ 2A_4 = \frac{2}{\nu + \frac{1}{\nu}} = f(\nu), \tag{4.14} \]

where for Uranium, exploiting again the value of \(\nu = \frac{9}{50}\) from Ref. \([5]\), we find

\[ \sqrt{\nu + \frac{1}{\nu}} = \sqrt{\frac{2581}{450}} \approx 2.3949. \tag{4.15} \]
5 Mean life of the radioactive nucleus

By virtue of the connection formula (2.18), which can be used because the condition (2.13) is fulfilled having set $A_3 = 0$ in section 4, we can now write the stationary state in the interval $I_3$ in the approximate form

$$y_3(r) \approx 2A_4\theta \frac{C}{\sqrt{p(r)}} \cos \left[ \frac{1}{\hbar} \int_{r_1}^{r} p(r')dr' - \frac{\pi}{4} \right].$$ (5.1)

If we require that such a function should take the form (3.18) at large $r$, we find by comparison that, up to a sign,

$$C = \sqrt{\frac{2}{\pi\hbar} \frac{1}{2A_4}} = \sqrt{\frac{2}{\pi\hbar} \frac{2\sqrt{p}}{\theta} \frac{1}{2f(\nu)}}.$$ (5.2)

By comparison of Eqs. (5.2) and (3.19), and bearing in mind Eq. (3.22), our prediction for the mean life of the radioactive nucleus reads as

$$\log(\tau) = \log\left( \frac{2bm_\alpha}{p} \right) - 2\log(\theta) - 2\log(2f(\nu)).$$ (5.3)

whereas the work in Ref. [2] obtains (see Eq. (3.23))

$$\log(\tau) = \log\left( \frac{2bm_\alpha}{p} \right) - 2\log(\theta).$$ (5.4)

In light of Eq. (4.15), the difference between our result (5.3) and the theoretical prediction (5.4) is a constant factor which, for Uranium, equals $-1.025$.

6 Beyond s-wave

In the investigation of bigger values of angular momentum quantum number, it may be appropriate to exploit a further refined version of nuclear potential, and also the potentialities of the phase-integral method with unspecified base function [4], a concept that we are going to define shortly.

The models of current interest study the relative motion of the $\alpha$-particle and daughter nucleus in a central potential $U(r)$ built as follows. On considering the decay of nuclei surrounded by electrons, the $\alpha$-particle moves in the central potential [8]

$$U(r) = U_n(r) + U_C(r),$$ (6.1)

where $U_n(r)$ is the nuclear potential well, and $U_C(r)$ is the effective Coulomb potential. At small distances, when the $\alpha$-particle moves inside the nucleus
or under the barrier, the Coulomb contribution can be approximated (up to a correction \[9,10\] proportional to \(r^2\)) by

\[ U_C(r) \approx U_C^{(b)}(r) - E, \tag{6.2} \]

where \(U_C^{(b)}(r)\) is the Coulomb potential for bare uniformly charged nuclei (\(R\) being the nuclear radius):

\[ U_C^{(b)}(r) = (Z - 2)\frac{e^2}{R} \left(3 - \frac{r^2}{R^2}\right), \quad r \in [0, R], \tag{6.3} \]

\[ U_C^{(b)}(r) = 2(Z - 2)\frac{e^2}{r}, \quad r \in ]R, \infty[, \tag{6.4} \]

while \(E\) is the energy transferred to electrons. In non-metallic targets, \(E\) is the difference of electron binding energies of the parent and daughter atoms. Eventually, upon defining

\[ \kappa^2 \equiv \frac{2mE}{\hbar^2}, \quad \lambda \equiv l + \frac{1}{2}, \tag{6.5} \]

\[ v(r) \equiv \frac{2m}{\hbar^2} U(r), \tag{6.6} \]

stationary states are found by solving the stationary Schrödinger equation

\[ \left[ \frac{d^2}{dr^2} + \kappa^2 - \frac{\lambda^2}{r^2} - \frac{1}{4} - v(r) \right] w_\lambda(\kappa; r) = 0. \tag{6.7} \]

The solutions of Eq. (6.7) are discussed in Ref. [8], but here we would like to describe what new insight can be gained by using the phase-integral method, following Ref. [4]. For this purpose, we begin by remarking that, upon replacing \(r\) with \(z\), Eq. (6.7) is of the form (2.1). The latter is solved by \(\psi(z)\) having the form (2.2) provided that the exact phase integrand \(q(z)\) solves the differential equation

\[ \chi(q(z)) \equiv q^{-\frac{3}{2}} \frac{d^2}{dz^2} q^{-\frac{1}{2}} + \frac{R(z)}{q^2} - 1 = 0, \tag{6.8} \]

that is called the \(q\)-equation in Ref. [4]. Suppose now that it is possible to determine a function \(Q : z \to Q(z)\) that is an approximate solution of the \(q\)-equation (6.8). This means that \(\chi_0\), defined by

\[ \chi_0 \equiv \chi(Q(z)) = Q^{-\frac{3}{2}} \frac{d^2}{dz^2} Q^{-\frac{1}{2}} + \frac{R(z)}{Q^2} - 1, \tag{6.9} \]
must be much smaller than 1. The work in Ref. [4] proves that the phase integrand \( q(z) \) is related to the base function \( Q(z) \) by the asymptotic expansion

\[
q(z) \sim Q(z) \sum_{n=0}^{N} Y_{2n},
\]  

(6.10)

where, on defining the new independent variable (a sort of approximate phase integral)

\[
\zeta(z) \equiv \int_{z}^{z} Q(\tau)d\tau,
\]  

(6.11)

the first few \( Y_{2n} \) functions are given explicitly by [4]

\[
Y_{0} = 1,
\]

(6.12)

\[
Y_{2} = \frac{1}{2} \chi_{0},
\]

(6.13)

\[
Y_{4} = -\frac{1}{8} \left( \chi_{0}^{2} + \frac{d^{2}}{d\zeta^{2}} \chi_{0} \right),
\]

(6.14)

\[
Y_{6} = \frac{1}{32} \left[ 2 \chi_{0}^{3} + 5 \left( \frac{d\chi_{0}}{d\zeta} \right)^{2} + 6 \chi_{0} \frac{d^{2}}{d\zeta^{2}} \chi_{0} + \frac{d^{4}}{d\zeta^{4}} \chi_{0} \right].
\]

(6.15)

By virtue of Eqs. (6.1)-(6.4), the function \( R(z) \) can be written in the form

\[
R(z) = -\left( \frac{\lambda^{2} - 1}{z^{2}} \right) + \frac{a_{-1}}{z} + a_{0} + a_{1} z + O(z^{2}),
\]  

(6.16)

where the \( a \)'s are constants. Let us now assume that the square of the freely specifiable base function is given by

\[
Q^{2}(z) = \frac{b_{-2}}{z^{2}} + \frac{b_{-1}}{z} + b_{0} + b_{1} z + \ldots,
\]  

(6.17)

where the \( b \)'s are suitable constants. For the first-order phase-integral approximation to be valid close to the origin, one requires finiteness of the integral [4]

\[
\mu(z, z_{0}) \equiv \left| \int_{z_{0}}^{z} \chi_{0} Q(\tau) d\tau \right|
\]

(6.18)

as \( z \) approaches 0. After re-expressing \( \chi_{0} \) in (6.9) in terms of \( Q^{2} \) according to

\[
\chi_{0} = \frac{1}{16Q^{6}} \left[ 5 \left( \frac{dQ^{2}}{dz} \right)^{2} - 4Q^{2} \frac{d^{2}}{dz^{2}} Q^{2} \right] + \frac{R(z)}{Q^{2}} - 1,
\]

(6.19)
a patient calculation shows that \[4\]

\[
\chi_0 Q = -\frac{(\lambda^2 + b_{-2})}{\sqrt{b_{-2} z}} + \frac{\left[\lambda^2 + b_{-2} - \frac{1}{2}\right] b_{-1}}{2(b_{-2})^{\frac{3}{2}}} + \frac{(a_{-1} - b_{-1})}{\sqrt{b_{-2}}} + O(z). \tag{6.20}
\]

Thus, finiteness of \(\mu(z, z_0)\) as \(z\) approaches 0 requires the elimination of the non-integrable term proportional to \(\frac{1}{z}\) in Eq. (6.20). This is achieved if and only if

\[
b_{-2} = -\lambda^2, \tag{6.21}
\]

which implies in turn that

\[
\lim_{z \to 0} z^2 Q^2(z) = -\lambda^2 = -\left(t + \frac{1}{2}\right)^2, \tag{6.22}
\]

as well as \[4\]

\[
\lim_{z \to 0} z^2 \left[Q^2(z) - R(z)\right] = -\frac{1}{4}. \tag{6.23}
\]

The most convenient choice of \(Q^2(z)\) in order to obtain a stationary state that is regular at the origin at all orders of approximation is \[4\]

\[
Q^2(z) = R(z) - \frac{1}{4z^2}. \tag{6.24}
\]

The advantage of the freely specifiable base function \(Q(z)\) is that one has at disposal a new tool for finding approximate forms of the stationary states as the potential (6.1) is considered in greater detail, possibly including more involved terms. The JWKB method does not have such a flexibility, and higher orders of JWKB and phase-integral method may differ in a substantial way \[3, 4\].

We find it appropriate to end this section with an original calculation suggested by Eqs. (6.10)-(6.24). For this purpose, we assume to have chosen \(Q^2(z)\) in the form (6.24), where \(R(z)\) takes the form (6.18) with vanishing \(O(z^2)\) term (for simplicity). We then find the approximate phase integrand \(q(z)\) with arbitrary values of angular momentum quantum number in the form

\[
q(z) \sim -\frac{\lambda^2}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z \left[1 + \frac{\chi_0}{2}\right], \tag{6.25}
\]
where, by virtue of (6.19) and (6.24),

\[
\chi_0 = \frac{1}{16} \left( -\frac{\lambda^2}{z^2} + \frac{a_{-1}}{z} + a_0 + a_1 z \right)^{-3}
\]

\[
\times \left[ -\frac{4\lambda^4}{z^6} + \frac{12\lambda^2 a_{-1}}{z^5} + \frac{(24\lambda^2 a_0 - 3(a_{-1})^2)}{z^4} \right.
\]

\[
+ \left( \frac{44\lambda^2 a_1 - 8a_0 a_{-1}}{z^3} - \frac{18a_1 a_{-1}}{z^2} + 5(a_1)^2 \right)\right]
\]

\[
+ \frac{1}{4} \left( -\lambda^2 + a_{-1} z + a_0 z^2 + a_1 z^3 \right)^{-1}.
\]

This formula yields in turn the asymptotic expansion of the stationary state \(\psi(z)\) by means of Eq. (2.4).

### 7 Concluding remarks

Following the important findings in Refs. [11, 12], there has been valuable work on \(\alpha\)-decay for almost a century by now [13, 14, 15, 16, 5, 9, 10, 17, 18, 19, 20, 8]. In particular, the work in Ref. [5] performs a very enjoyable presentation of four methods: complex eigenvalue, scattering state method, semiclassical path integral, instanton method. However, even the author of Ref. [5], who was more familiar with the work in Ref. [21], was not aware of the one-directional nature of connection formulae. Thus, our investigation is truly original, since it has applied the work of Refs. [3, 4] to a nuclear physics problem in which several generations of research workers were not aware of the proof of one-directional nature of connection formulae.

Our original result (4.8) for the lowest \(s\)-wave metastable state improves the values obtained in Refs. [2, 5]. The authors of Ref. [2] find \(\rho = \frac{\pi}{2}\) because they use in Eq. (3.8) the coefficients \(A_3\) and \(A_4\) enforced by the wrong connection formulae (4.1). The work in Ref. [5] finds instead \(\rho = \pi\) because it approximates the solutions of the equation

\[\tan(\rho) = -\nu\]

by integer multiples of \(\pi\). Moreover, our formula (5.3) for the logarithm of the mean life of the radioactive nucleus yields a correction factor equal to \(-1.025\) for the value obtained in Ref. [2], and this prediction can be checked against observation.

As far as we can see, our sources in the physics-oriented literature did an excellent work but were misled by their lack of knowledge of one-directional
nature of connection formulae (cf. [3, 4]). The main open problem is now the application of the phase-integral perspective to the involved models of modern nuclear physics. Our section 6 has prepared the ground for this purpose, describing in detail the logical steps that are in order. Our original result for the approximate form (6.25)-(6.26) of the phase integrand with arbitrary values of the angular momentum quantum number provides, as far as we can see, encouraging evidence in favour of new tools being available for investigating $\alpha$-decay from a phase-integral perspective.

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A The $F$-matrix method

Let us assume that Eq. (2.1) is given, with the associated phase-integral functions (2.4). Following Ref. [4], we consider the $a$-coefficients $a_1(z)$ and $a_2(z)$, which are uniquely determined by the requirement that any exact solution $\psi$ of Eq. (2.1) can be written in the form

$$\psi(z) = a_1(z)f_1(z) + a_2(z)f_2(z),$$

(A.1)

with first derivative given by

$$\frac{d\psi}{dz} = a_1(z)\frac{df_1}{dz} + a_2(z)\frac{df_2}{dz}.$$  

(A.2)

For Eq. (A2) to be satisfied, we have to impose that [4]

$$f_1(z)\frac{da_1}{dz} + f_2(z)\frac{da_2}{dz} = 0.$$  

(A.3)

Interestingly, Eq. (2.1) can be now replaced by a system of two coupled differential equations of first order, which can be written in matrix form as [4]

$$\frac{d}{dz}\begin{pmatrix}a_1(z) \\ a_2(z)\end{pmatrix} = M(z)\begin{pmatrix}a_1(z) \\ a_2(z)\end{pmatrix},$$

(A.4)

having defined (see Eq. (2.9))

$$M(z) = \frac{i}{2}\chi(z)q(z)\begin{pmatrix}1 \\ -\frac{e^{i\omega(z)}}{e^{-2i\omega(z)}} \\ -1\end{pmatrix}.$$  

(A.5)
Equation (A4) can be replaced by the integral equation
\[
\begin{pmatrix} a_1(z) \\ a_2(z) \end{pmatrix} = \begin{pmatrix} a_1(z_0) \\ a_2(z_0) \end{pmatrix} + \int_{z_0}^{z} d\tau M(\tau) \begin{pmatrix} a_1(\tau) \\ a_2(\tau) \end{pmatrix},
\]
whose solution can be obtained in closed form by an iteration procedure that yields
\[
\begin{pmatrix} a_1(z) \\ a_2(z) \end{pmatrix} = F(z, z_0) \begin{pmatrix} a_1(z_0) \\ a_2(z_0) \end{pmatrix},
\]
where \( F(z, z_0) \) is a 2 \( \times \) 2 matrix given by a convergent series [4]. Such a matrix is the particular solution of the differential equation
\[
\frac{\partial}{\partial z} F(z, z_0) = M(z) F(z, z_0),
\]
that is equal to the 2 \( \times \) 2 unit matrix for \( z = z_0 \). The \( F \)-matrix satisfies the general relations [4]
\[
\det F(z, z_0) = 1, \quad F(z, z_0) = F(z, z_1) F(z_1, z_0), \quad F(z_0, z) = [F(z, z_0)]^{-1} = \begin{pmatrix} F_{22}(z, z_0) & -F_{12}(z, z_0) \\ -F_{21}(z, z_0) & F_{11}(z, z_0) \end{pmatrix}.
\]

Useful estimates of the matrix elements of \( F(z, z_0) \) have been obtained in Ref. [3] under the assumption that the points \( z \) and \( z_0 \) can be connected by a path in the complex \( z \)-plane along which the absolute value of \( e^{iw(z)} \) increases monotonically, in the non-strict sense, in the direction from \( z_0 \) to \( z \). Upon defining (cf. Eq. (2.10))
\[
\mu = \mu(z, z_0) \equiv \left| \int_{z_0}^{z} |\chi(q(z')) q(z')| dz' \right|,
\]
these basic estimates read as [4]
\[
|F_{11}(z, z_0) - 1| \leq \frac{\mu}{2} + \text{higher powers of } \mu,
\]
\[
|F_{12}(z, z_0)| \leq \left| e^{-2i\nu(z_0)} \right| \left( \frac{\mu}{2} + \text{higher powers of } \mu \right), \quad (A.14)
\]
\[
|F_{21}(z, z_0)| \leq \left| e^{2i\nu(z_0)} \right| \left( \frac{\mu}{2} + \text{higher powers of } \mu \right), \quad (A.15)
\]
\[
|F_{22}(z, z_0) - 1| \leq \frac{\mu}{2} + \left| e^{2i[w(z)-w(z_0)]} \right| \left( \frac{\mu^2}{4} + \text{higher powers of } \mu \right). \quad (A.16)
\]
The parameters occurring in Eqs. (2.7) and (2.8) are real-valued and can be defined as follows in terms of the \( F \)-matrix \[4\]:

\[
\alpha = \alpha(x_1, x_2) = |F_{11}(x_1, x_2)|, \tag{A.17}
\]

\[
\beta = \beta(x_1, x_2) = \pm \arg F_{11}(x_1, x_2), \tag{A.18}
\]

\[
\gamma = \gamma(x_1, x_2) = \text{Re} \left[ \frac{F_{21}(x_1, x_2)}{F_{11}(x_1, x_2)} \right]. \tag{A.19}
\]

Strictly speaking, the possibility of writing \( \gamma \) in the form (A19) results from the simple but non-obvious property, according to which \[4\]

\[
F_{21}(x_1, x_2)F_{11}^*(x_1, x_2) \mp \frac{i}{2}
\]

is real-valued.

B One-directional nature of the connection formula (2.18)

Suppose that, upon setting \( d = 1 \) on the right-hand side of Eq. (2.18), we are given a stationary state that, at a point \( x_2 \) of the classically allowed region, reads as

\[
\psi(x_2) = 2 \left| q^{-\frac{1}{2}}(x_2) \right| \cos \left[ |w(x_2)| - \frac{\pi}{4} \right]
= a(x_2) \left| q^{-\frac{1}{2}}(x_2) \right| e^{i|w(x_2)|} + b(x_2) \left| q^{-\frac{1}{2}}(x_2) \right| e^{-i|w(x_2)|}, \tag{B.1}
\]

where \( a(x_2) = e^{-i\frac{\pi}{4}}, b(x_2) = e^{i\frac{\pi}{4}} \). The stationary state at a point \( x_1 \) in the classically forbidden region reads therefore

\[
\psi(x_1) = c(x_1) \left| q^{-\frac{1}{2}}(x_1) \right| e^{|w(x_1)|} + d(x_1) \left| q^{-\frac{1}{2}}(x_1) \right| e^{-|w(x_1)|}, \tag{B.2}
\]

where the technique of Ref. [4] yields the formulae (the approximate forms of the parameters \( \alpha, \beta, \gamma \) being the ones given in our Eq. (2.11))

\[
c(x_1) = \alpha e^{-i(\frac{\pi}{4} - \beta)}a(x_2) + \alpha e^{i(\frac{\pi}{4} - \beta)}b(x_2)
= 2\alpha \sin \beta, \tag{B.3}
\]

\[
d(x_1) = \left( \frac{\alpha \gamma + \frac{i}{2\alpha}}{\alpha} \right) e^{-i(\frac{\pi}{4} - \beta)}a(x_2)
+ \left( \frac{\alpha \gamma - \frac{i}{2\alpha}}{\alpha} \right) e^{i(\frac{\pi}{4} - \beta)}b(x_2)
= \frac{\cos \beta}{\alpha} + 2\alpha \gamma \sin \beta. \tag{B.4}
\]
By virtue of Eqs. (B2)-(B4), one finds

$$\psi(x_1) = \left|q^{-\frac{1}{2}}(x_1)\right| e^{-|w(x_1)|} \times \left\{2\alpha \sin \beta e^{2|w(x_1)|} + \frac{\cos \beta}{\alpha} + 2\alpha \gamma \sin \beta \right\}.$$  \hspace{1cm} (B.5)

By virtue of the approximate formulae (2.11) for $\alpha, \beta, \gamma$, the sum of terms within curly brackets on the second line of (B5) can never approach 1, and hence the stationary state in Eq. (B5) can never approach the left-hand side of Eq. (2.18) with $c = 0$ and $d = 1$. Thus, the connection formula (2.18) is one-directional [4].

As is stressed in Ref. [4], the connection formula has the same form for every order of the phase-integral approximation.

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