Structure of the Algebra of Effective Observables in Quantum Mechanics

R. Olkiewicz*
Institute of Theoretical Physics, University of Wrocław
PL-50-204 Wrocław, Poland

Abstract

A subclass of dynamical semigroups induced by the interaction of a quantum system with an environment is introduced. Such semigroups lead to the selection of a stable subalgebra of effective observables. The structure of this subalgebra is completely determined.

*Supported by the KBN research grant No 2 P03B 086 16
1 Introduction

One of the fundamental principles of quantum mechanics is the superposition principle which guarantees that any superposition of two distinct pure states is again a legitimate pure state. As an immediate consequence of this and the postulate that proportional vectors describe the same quantum state we obtain that pure states are in one-to-one correspondence with one-dimensional subspaces of a Hilbert space $\mathcal{H}$. This is usually taken as a basic ingredient of a mathematical description of a system, which ensures genuine quantum behavior of that system. Alternatively, we may say that physical quantities are in one-to-one correspondence with self-adjoint operators on $\mathcal{H}$. Since, without loss of generality, we may restrict to bounded operators so it implies that the von Neumann algebra $\mathcal{A}$ generated by observables equals to $\text{B}(\mathcal{H})$, the algebra of all bounded operators. However, it is evident that some superpositions of quantum pure states do not take place in the real world. Well known examples of such a phenomenon encompass the absence of superpositions of states with different electric charge or with integer and half-integer spin. This fact led to the introduction in 1952 of superselection rules [1], which axiomatically exclude certain superpositions from being observable. For a review of this subject see a recent paper by Wightman [2]. It follows that the connection between quantum states and rays in $\mathcal{H}$ should be changed to: every pure state is represented by a one-dimensional subspace of $\mathcal{H}$, but not every such a subspace represents a quantum state. Such a postulate has an immediate consequence for the algebra $\mathcal{A}$, since now the commutant $\mathcal{A}'$, which consists of superselection operators, is non-trivial.

Further, in 1960, Jauch [3] introduced a condition that there should exist at least one complete set of commuting observables in $\mathcal{A}$, which expressed more generally states that $\mathcal{A}$ should contain a maximal Abelian subalgebra. It implies that all superselection operators belong to $\mathcal{A}$ or, equivalently, that the center $\mathcal{Z}$ of $\mathcal{A}$ equals to $\mathcal{A}'$. Clearly, all superselection operators commute with each other since $\mathcal{A}'$ is Abelian in this case. Therefore, the existence of superselection rules makes the center $\mathcal{Z}$ non-trivial yielding a decomposition of Hilbert space $\mathcal{H}$ into coherent subspaces. In the discrete case it was concisely written by Wan [4] as follows:

Let $\mathcal{S}$ denote the set of all pure states. Then $\mathcal{H}$ may be decomposed into a direct sum of mutually orthogonal subspaces $\mathcal{H}_n$ such that $\mathcal{S} = \bigcup_n \text{CP}(\mathcal{H}_n)$, where $\text{CP}(\mathcal{H}_n)$ denotes the projective space over $\mathcal{H}_n$. There is no further
decomposition of $\mathcal{H}_n$.

It is worth noting that a superposition $\lambda|v> + \lambda'|w>$, $|\lambda|^2 + |\lambda'|^2 = 1$, of vectors from different coherent subspaces is empirically indistinguishable from the mixture $|\lambda|^2 P_v + |\lambda'|^2 P_w$, where $P_v = |v><v|$ and $P_w = |w><w|$. As a consequence, the algebra $\mathcal{A}$ consists of all operators $A \in \text{B}(\mathcal{H})$ such that $\sum_n P_n A P_n = A$, where $P_n$ denotes the orthogonal projector onto $\mathcal{H}_n$.

A more general situation can also occur. When we drop Jauch’s hypothesis we obtain that in principle $\mathcal{A}'$ has only a partial overlapping with $\mathcal{A}$. Hence, there are non-commuting superselection operators since $\mathcal{A}'$ cannot be Abelian now. It means that $\mathcal{A}$, when restricted to a coherent subspace $\mathcal{H}_n$, is still smaller than $\text{B}(\mathcal{H}_n)$. Therefore, some different and non-proportional vectors from $\mathcal{H}_n$ may still determine the same quantum state. Such a possibility was explicitly acknowledged by Messiah and Greenberg in 1964 [5], who introduced the term generalized ray for the set of such vectors. In such a case the lack of knowledge of the state vector is greater than in ordinary quantum mechanics. A generalized ray is represented by an $r$-dimensional sphere, $r$ being the dimension of an irreducible subspace of commuting physical observables. This inevitably puts additional constraints on the structure of algebra $\mathcal{A}$. We encounter such a situation when, for example, we want to study symmetry transformations called supersymmetry [6], which leave all the observables invariant. Let us recall that a unitary operator is a supersymmetry if it is not proportional to the identity operator and commutes with the set of all observables. Clearly, they and the identity form a unitary group of $\mathcal{A}'$, so-called gauge group.

Superselection rules was a useful postulate, but the question about an explanation of its appearance arose. It should be pointed out here that it is not a logical necessity of quantum theory. In 1982 Zurek [7] proposed a program of environment-induced superselection rules. He showed that when a quantum system is open, interacting with an environment, superselection rules do not need to be postulated. They arise naturally as a result of the decoherence process, which effectively destroys superpositions between macroscopically different states with respect to a local observer, so that the system appears to be in one or the other of those states. By the term “destroys superposition” we understand that the off-diagonal elements of the superposition are unavailable with respect to a specific set of observations. The idea was further developed in [8,9,10].

In order to study decoherence, the analysis of the evolution of the reduced
density matrix obtained by tracing out the environment variables is the most convenient strategy. For a large class of interesting physical phenomena the evolution of the reduced density matrix can be described by a dynamical semigroup, whose generator is given by a Markovian master equation. The loss of quantum coherence in the Markovian regime was established in a number of open systems [11,12] giving a clear evidence of environment-induced superselection rules. In a recent paper [13] a thorough mathematical analysis of the superselection structure induced by a dynamical semigroup which is also contractive in the operator norm was presented. It was achieved by the use of the isometric-sweeping decomposition, which singles out a subspace of density matrices, on which the semigroup acts in a reversible, unitary way, and sweeps out the rest of statistical states. The dual space of the isometric part of density matrices is a von Neumann algebra $\mathcal{M}$, which we call the algebra of effective observables. This algebra is stable with respect to the process of decoherence, i.e. its elements evolve in a unitary way according to Schrödinger dynamics in the Heisenberg picture. Other elements of $\mathcal{B}(\mathcal{H})$ decay in time to elements of $\mathcal{M}$. Therefore, when decoherence happens almost instantaneously, then $\mathcal{M}$ represents physical observables of the quantum system. The purpose of this paper is to describe the structure of $\mathcal{M}$.

The paper is organized as follows. In sec. 2 we introduced the notion of an environment-induced semigroup and discuss its properties. In sec. 3 we briefly recall some basic facts concerning superselection rules induced by the interaction with an environment. Finally, in sec. 4, we describe the structure of the algebra of effective observables.

2 Environment-induced semigroups

The irreversible behavior of the evolution of quantum statistical states (density matrices) is the main consequence of the assumption that they interact with their environments. As was mentioned in Introduction we restrict our considerations to the Markovian regime, and thus assume that the evolution of the reduced density matrix is given by a dynamical semigroup $T_t$. By a dynamical semigroup one usually means a strongly continuous semigroup of completely positive trace preserving and contractive operators acting on the Banach space of trace class operators $\text{Tr}(\mathcal{H})$ [14]. However, since the semigroup $T_t$ is to describe a measurement-like interaction with the envi-
ronment, the statistical entropy $S(\rho) = -\text{tr} \rho \log \rho$ of an evolving density matrix $\rho$ should not decrease. Here, by $\text{tr}$ we denote the usual trace on $\text{Tr}(\mathcal{H})$. For a measurement it follows from the following argument. Suppose that the properties of a quantum system are specified by probabilities $\{p_i\}$ for the outcomes of the measurement of a discrete observable $A$. Therefore, the state of such a system is a mixed state and reads $\rho = \sum_i p_i P_i$, where $P_i = |e_i><e_i|$ and $|e_i>$ are the corresponding eigenvectors of $A$. The statistical entropy of $\rho$ is a measure of our ignorance of the actual result of the measurement of $A$. Suppose further that we perform a measurement of another discrete observable $B$. According to the von Neumann projection postulate the state of the system changes to

$$\rho \rightarrow \rho' = \sum_j Q_j \rho Q_j$$

where $Q_j = |f_j><f_j|$, $|f_j>$ being eigenvectors of $B$. Therefore $\rho' = \sum_j p'_j Q_j$, where $p'_j = \sum_i p_i \text{tr}(Q_j P_i)$. Because coefficients $\text{tr}(Q_j P_i)$ form a doubly stochastic matrix so $S(\rho') \geq S(\rho)$. For a more general discussion of the entropy increase during the interaction with the environment see [15].

However, the concept of dynamical semigroup as defined above is too general to ensure the increase of the statistical entropy, as the following simple example shows. Consider an operator $L$ defined by

$$L \rho = A \rho A^* - \frac{1}{2} \{A^* A, \rho\}$$

where $\{\cdot, \cdot\}$ stands for the anticommutator and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Clearly, $L$ generates a dynamical semigroup $T_t$ on $2 \times 2$ complex matrices. By direct calculations we obtain that the evolution of the one-dimensional projector

$$P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is given by $T_t P = e^{-t} P + (1 - e^{-t}) P^\perp$, where $P^\perp = I - P$ and $I$ denotes the identity matrix. Hence the statistical entropy increases to its maximal value
when time approaches log 2, and then decreases to zero. Although such semigroups may also play some role in theoretical investigations of quantum open systems, they will be excluded from our current considerations. Therefore, we impose on $T_t$ an additional assumption, namely that $T_t$ is also contractive in the operator norm $\| \cdot \|_{\infty}$, and we call it environment-induced semigroup. For such a semigroup it follows that a maximal eigenvalue of a density matrix cannot increase during the evolution. Moreover, the following properties can be derived. First, notice that the linear entropy $S_{\text{lin}}(\rho) = \text{tr}(\rho - \rho^2)$, being a linear approximation of $S$ since $\log \rho = \log(I - (I - \rho)) = \rho - I + ...$, does not decrease. Indeed, for $t_1 \geq t_2$

$$S_{\text{lin}}(T_{t_1}\rho) - S_{\text{lin}}(T_{t_2}\rho) = \|T_{t_2}\rho\|_{2}^{2} - \|T_{t_1-t_2}(T_{t_2}\rho)\|_{2}^{2} \geq 0$$

since, by Lemma 4 in [13], $T_t$ is also contractive in the Hilbert-Schmidt norm $\| \cdot \|_{2}$. The statistical entropy does not decrease, either. For finite dimensional quantum systems it follows from the following argument. For a totally mixed state $\rho_0 = \frac{I}{\text{tr}I}$, all eigenvalues of $T_t(\rho_0)$ are not greater than $\frac{1}{\text{tr}I}$ and their sum is equal to 1. Hence $T_t(\rho_0) = \rho_0$ and so $T_t(I) = I$ for all $t \geq 0$. Thus the function $t \to S(T_t\rho)$ is non-decreasing for any density matrix $\rho$. We show that this property also holds in the infinite dimensional case.

**Proposition 2.1** Suppose $T_t$ is an environment-induced semigroup. Then $S(T_t\rho) \geq S(\rho)$ for any density matrix $\rho$.

**Remark.** Since $S$ takes values in $[0, \infty]$ so the above inequality means that if $S(T_t\rho) < \infty$, then also $S(\rho)$ is finite and not greater than $S(T_t\rho)$.

**Proof:** Let $\rho$ be a density matrix. Then $\rho = \sum_{i=1}^{n} p_i P_i$, $p_i > 0$, and $T_t\rho = \sum_{j=1}^{m} q_j Q_j$, $q_j > 0$, where $\{P_i\}$,$\{Q_j\}$ are spectral projectors of $\rho$ ($T_t\rho$) respectively. Clearly, all of them are finite dimensional. Projectors corresponding to zero eigenvalue we denote by $P_0$ and $Q_0$ respectively. Then

$$q_j = \frac{\text{tr}(Q_j T_t\rho)}{\text{tr}Q_j} = \sum_{i=1}^{n} p_i \alpha_j(i), \text{ where } \alpha_j(i) = \frac{\text{tr}(Q_j T_t P_i)}{\text{tr}Q_j}$$

By Lemma 4 in [13], $T_t$ has a normal extension to a contractive semigroup $\overline{T_t}$ on $B(H)$. Suppose $\{E_n\}$ is a sequence of mutually orthogonal one-dimensional projectors such that $\sum_{n} E_n = I$. Then

$$\overline{T_t}(I) = \lim_{n \to \infty} T_t(\sum_{k=1}^{n} E_k) \leq I$$

6
Therefore

\[ I \geq \mathcal{T}_t(I) = \mathcal{T}_t(P_0 + \sum_{i=1} P_i) = \mathcal{T}_t(P_0) + \sum_{i=1} T_t P_i \]

and so \( \sum_{i=1} \alpha_j(i) \leq 1 \). Let us define \( \alpha_j(0) = 1 - \sum_{i=1} \alpha_j(i) \). Then we have \( q_j = \sum_{i=0} p_i \alpha_j(i) \), where \( p_0 = 0 \) and \( \sum_{i=0} \alpha_j(i) = 1 \). Because function \( x \to x \log x \) is convex and continuous with \( f(0) = 0 \), so

\[ q_j \log q_j \leq \sum_{i=1} \alpha_j(i)p_i \log p_i \]

Assume now that \( S(T_t \rho) \) is finite. It means that \( -T_t \rho \log T_t \rho \) is a positive and trace class operator. Therefore

\[ S(T_t \rho) = -\text{tr}(\sum_{j=1} (q_j \log q_j)Q_j) = -\sum_{j=1} q_j \log q_j \text{tr}Q_j \geq \sum_{i,j=1} (-p_i \log p_i)\text{tr}(Q_j T_t P_i) \]

Because \( p_i P_i \leq \rho \) so \( T_t(p_i P_i) \leq T_t \rho \) and hence \( \text{tr}(Q_0 T_i P_i) = 0 \) for all \( i \geq 1 \). Therefore

\[ S(T_t \rho) \geq \sum_{i=1} (-p_i \log p_i)[\text{tr}(Q_0 T_i P_i) + \sum_{j=1} \text{tr}(Q_j T_i P_i)] \]

\[ = \sum_{i=1} (-p_i \log p_i)\text{tr}(T_i P_i) = S(\rho) \]

since \( T_t \) is trace preserving. □

Having discussed the properties of environment-induced semigroups, we now turn to a condition which guarantees that a dynamical semigroup \( T_t \) is also contractive in the operator norm. In order to avoid domain difficulties we restrict ourselves to a uniformly continuous semigroup. Then its generator

\[ L : \text{Tr}(\mathcal{H}) \to \text{Tr}(\mathcal{H}) \]

has the following standard form

\[ L\rho = -i[H, \rho] + \sum_{j=1} V_j \rho V_j^* - \frac{1}{2}(\sum_{j=1} V_j^*V_j, \rho) \quad (2) \]

where \( H = H^* \in \mathcal{B}(\mathcal{H}), V_j \in \mathcal{B}(\mathcal{H}) \) and \( \lim_n \sum_{j=1} V_j^*V_j = V \) in the strong topology.
Proposition 2.2 A uniformly continuous dynamical semigroup $T_t$ is contractive in the operator norm if and only if $\sum_j V_j V_j^* \leq \sum_j V_j^* V_j$.

Proof: $\Leftarrow$ Suppose $\sum_j V_j V_j^* \leq V$. Then it converges strongly to some positive operator in $\mathrm{B}(\mathcal{H})$ and so $L$ extends to a bounded operator on $\mathrm{B}(\mathcal{H})$. It is clear that such an extension is a complete dissipation and so $T_t$ is contractive in the operator norm.

$\Rightarrow$ Suppose that $\|T_t \phi\|_\infty \leq \|\phi\|_\infty$ for all $\phi \in \mathrm{Tr}(\mathcal{H})$. Then $T_t$ extends to a contractive semigroup $\overline{T}_t$ on $\mathrm{K}(\mathcal{H})$, the space of compact operators. Clearly, $\overline{T}_t$ is strongly continuous with $\mathrm{Tr}(\mathcal{H}) \subset D(\overline{L})$, where $\overline{L}$ denotes the generator of $\overline{T}_t$. In order to show that $\sum_j V_j V_j^* \leq \sum_j V_j^* V_j$ it suffices to check that for any one-dimensional projector $P$ the following inequality holds

$$\mathrm{tr} P \sum_{j=1}^\infty V_j V_j^* \leq \mathrm{tr} P \sum_{j=1}^\infty V_j^* V_j$$

Let us fix projector $P$. Using the decomposition $\mathcal{H} = P\mathcal{H} \oplus P^\perp \mathcal{H}$ each $V_j$ can be written as

$$V_j = \begin{pmatrix} a_j & w_j^* \\ v_j & A_j \end{pmatrix}$$

where $a_j \in \mathbb{C}$, $v_j, w_j \in P^\perp \mathcal{H}$ and $A_j \in \mathrm{B}(P^\perp \mathcal{H})$. In consequence, the above inequality is equivalent to $\sum_j \|w_j\|^2 \leq \sum_j \|v_j\|^2$. Suppose now $\phi = P + E$, where $E$ is a finite dimensional subprojector of $P^\perp$. Clearly, $\phi \in D(\overline{L})$. Moreover, since $\mathrm{K}(\mathcal{H})^* = \mathrm{Tr}(\mathcal{H})$, $P$ is a normalized tangent functional to $\phi$ and so, by the assumption and Hille-Yosida theorem, $\mathrm{tr} \overline{L}(\phi) P \leq 0$ or, equivalently,

$$\mathrm{tr} P \sum_{j=1}^\infty V_j \phi V_j^* \leq \mathrm{tr} P \sum_{j=1}^\infty V_j^* V_j$$

Because

$$\mathrm{tr} P V_j \phi V_j^* = |a_j|^2 + < w_j, E w_j >$$

we obtain that $\sum_j \|E w_j\|^2 \leq \sum_j \|v_j\|^2$. Taking the supremum over $E$ ends the proof. $\square$

3 Environment-induced superselection rules
In this section we briefly present some basic facts concerning the superselection structure induced by the interaction with an environment. Clearly, there is a difference between them and the traditional superselection rules, which are said to operate between subspaces of a Hilbert space if the phase factors between vectors belonging to two distinct subspaces are unobservable. In the case of environment-induced superselection rules, phase coherence between vectors from some preferred set of pure states is being continuously destroyed by the interaction.

Suppose \( \hat{P} \) is a linear, bounded and positive operator on \( \text{Tr}(\mathcal{H}) \) such that \( \hat{P}^2 = \hat{P} \) and \( \text{tr} \hat{P} \phi \leq \text{tr} \phi \) for all \( \phi \in \text{Tr}(\mathcal{H})_+ \), the cone of positive elements in \( \text{Tr}(\mathcal{H}) \). We call such an operator the projection operator (when, in addition, \( \hat{P} \) preserves the trace, it is usually called the Zwanzig projection). Then space \( \text{Tr}(\mathcal{H}) \) splits into two linearly independent and closed subspaces \( \hat{P} \text{Tr}(\mathcal{H}) \) and \( (\text{id} - \hat{P}) \text{Tr}(\mathcal{H}) \). We start with the following general definition, see ref. 13.

**Definition 3.1** We say that the semigroup \( T_t \) induces a weak superselection structure on \( \text{Tr}(\mathcal{H}) \) if

a) there exists a projection operator \( \hat{P} \) such that

\[
T_t : \text{im} \hat{P} \rightarrow \text{im} \hat{P}, \quad T_t|_{\text{im} \hat{P}} = U_t \cdot U^*_t
\]

where \( U_t \) is a strongly continuous group of unitary operators,

b) \[
\lim_{t \to \infty} |\text{tr} A T_t \phi - \text{tr} \hat{P} (T_t \phi)| = 0
\]

holds for all \( \phi \in \text{Tr}(\mathcal{H}) \) and any \( A \) from some \( * \)-algebra \( \mathcal{B} \), which is strongly dense in \( \mathcal{B}(\mathcal{H}) \).

\( T_t \) induces a strong superselection structure if a) holds together with

b') \[
\lim_{t \to \infty} \| T_t \phi - \hat{P} (T_t \phi) \|_1 = 0 \quad \forall \phi \in \text{Tr}(\mathcal{H})
\]

where \( \| \cdot \|_1 \) is the trace norm. A weak(strong) superselection structure is said to be non-trivial if \( \hat{P} \neq \text{id} \), conservative, if \( \text{tr} \hat{P} \phi = \text{tr} \phi \) for all \( \phi \in \text{Tr}(\mathcal{H}) \).

It follows that environment-induced semigroups always induce (possibly trivial as they can be of purely unitary type) a superselection structure.

**Theorem 3.2** [13] Suppose \( T_t \) is an environment-induced semigroup. Then \( T_t \) induces a weak superselection structure. If moreover, \( T_t \) is relatively compact in the strong operator topology, then it induces a strong superselection
structure.

Sketch of proof: Let \( K \subset \text{HS}(\mathcal{H}) \) be a subspace of the Hilbert space of Hilbert-Schmidt operators given by

\[
K = \{ x \in \text{HS}(\mathcal{H}) : \| T_t x \|_2 = \| T^*_t x \|_2 = \| x \|_2 \forall t \geq 0 \}
\]

where \( T^*_t \) denotes the conjugate with respect to the scalar product in \( \text{HS}(\mathcal{H}) \).

Let \( \hat{P} : \text{HS}(\mathcal{H}) \to \text{HS}(\mathcal{H}) \) be the orthogonal projector onto \( K \). It turns out that \( \hat{P} \) maps trace class operators into trace class operators and \( \text{tr} \hat{P} \phi \leq \text{tr} \phi \) for any \( \phi \in \text{Tr}(\mathcal{H})_+ \). Therefore, \( \hat{P} \) induces a splitting \( \text{Tr}(\mathcal{H}) = \text{Tr}(\mathcal{H})_{\text{iso}} \oplus \text{Tr}(\mathcal{H})_a \) to the isometric and sweeping parts which fulfill the conditions from Definition 3.1. \( \square \)

Let \( \mathcal{M} \) be a von Neumann algebra having \( \text{Tr}(\mathcal{H})_{\text{iso}} \) as its predual space, that is \( \mathcal{M} = \text{im} \hat{P}^* \), where \( \hat{P}^* : B(\mathcal{H}) \to B(\mathcal{H}) \) is the conjugate projector. We call it algebra of effective observables. The action of the dual semigroup \( T^*_t : B(\mathcal{H}) \to B(\mathcal{H}) \), when restricted to \( \mathcal{M} \), is given by a unitary group of automorphisms. In the next section we describe the structure of \( \mathcal{M} \).

4 Algebra of effective observables

At first we show the following theorem.

Theorem 4.1 Suppose \( K \neq 0 \). Then

a) \( \hat{P} \) is completely positive i.e. \( \forall n \hat{P} \otimes \text{id}_{n \times n} : \text{Tr}(\mathcal{H}) \otimes M_{n \times n} \to \text{Tr}(\mathcal{H}) \otimes M_{n \times n} \) maps positive operators on \( \mathcal{H} \otimes \mathbb{C}^n \) into positive ones. Here \( M_{n \times n} \) denotes the algebra of \( n \times n \) complex matrices.

b) \( \| \hat{P} \|_{\infty, \infty} = 1 \), \( \| \hat{P} \|_{1, 1} = 1 \).

c) \( \hat{P} \) can be extended to a normal norm one projection \( \overline{P} : B(\mathcal{H}) \to B(\mathcal{H}) \) onto the von Neumann algebra \( \mathcal{M} \).

Proof: a) Let \( K_n := K \otimes M_{n \times n} \subset \text{HS}(\mathcal{H}) \otimes M_{n \times n} = \text{HS}(\mathcal{H} \otimes \mathbb{C}^n) \). If \( \tilde{x}, \tilde{y} \in K_n \), then also \( \tilde{x} \tilde{y} \in K_n \) and \( \tilde{x}^* \in K_n \), since \( K \) is a \( * \)-algebra. Suppose \( \tilde{x} = \tilde{x}^* \in K_n \). Because \( \tilde{x} \) is a Hilbert-Schmidt operator on \( \mathcal{H} \otimes \mathbb{C}^n \), so \( \tilde{x} = \sum_i a_i \tilde{e}_i, a_i \in \mathbb{R} \setminus \{0\} \). Since \( K_n \) is closed we obtain that \( \tilde{e}_i \in K_n \) for all \( i \).

Suppose now that \( \phi \in \text{Tr}(\mathcal{H} \otimes \mathbb{C}^n)_+ \). Then \( \tilde{\phi} = \tilde{\phi}_1 + \tilde{\phi}_2 \), where \( \tilde{\phi}_1 \in K_n \) and \( \tilde{\phi}_2 \in K^\perp_n = K^\perp \otimes M_{n \times n} \). Because \( \tilde{\phi}_1 \) is hermitian, so \( \tilde{\phi}_1 = \sum_i b_i \tilde{e}_i, b_i \neq 0 \) and \( \tilde{e}_i \in K_n \). Hence \( \text{tr} \tilde{\phi}_2 = 0 \) for any \( i \), what implies that \( b_i = \text{tr} \tilde{e}_i \tilde{\phi} / \text{tr} \tilde{e}_i \). Therefore \( \tilde{\phi}_1 \geq 0 \) and

\[
\text{tr} \tilde{\phi}_1 = \sum_i \text{tr} \tilde{e}_i \tilde{\phi} \leq \text{tr} \tilde{\phi}
\]
Hence \( \tilde{\phi}_1 \in \text{Tr}(\mathcal{H} \otimes \mathbb{C}^n)_+ \) and so \( \hat{P}^{(n)} : \text{Tr}(\mathcal{H} \otimes \mathbb{C}^n)_+ \to \text{Tr}(\mathcal{H} \otimes \mathbb{C}^n)_+ \), where \( \hat{P}^{(n)} \) denotes the orthogonal projection in \( \text{HS}(\mathcal{H} \otimes \mathbb{C}^n) \) onto \( K_n \). However, \( \hat{P}^{(n)} = \hat{P} \otimes \text{id}_{n \times n} \), what implies that \( \hat{P} \) is n-positive. Because \( n \) was arbitrary, the assertion follows.

b) By point a), \( \| \hat{P} \phi \|_1 \leq \| \phi \|_1 \) for all \( \phi \in \text{Tr}(\mathcal{H})_+ \). Hence \( \hat{P} \) is a bounded operator on \( \text{Tr}(\mathcal{H}) \) with \( \| \hat{P} \|_{1,1} \leq 2 \). Because \( \text{Tr}(\mathcal{H}) \subset K(\mathcal{H}) \) and \( K(\mathcal{H})^* = \text{Tr}(\mathcal{H}) \), so for any \( \phi \in \text{Tr}(\mathcal{H}) \),

\[
\| \hat{P} \phi \|_\infty = \text{tr}(\hat{P} \phi) = \text{tr}(\hat{P} \psi)
\]

for some \( \psi \in \text{Tr}(\mathcal{H}) \) with \( \| \psi \|_1 = 1 \). Hence

\[
\| \hat{P} \phi \|_\infty \leq \| \phi \|_\infty \| \hat{P} \psi \|_1 \leq 2 \| \phi \|_\infty
\]

Therefore, \( \hat{P} \) can be extended to a bounded operator on \( K(\mathcal{H}) \). Clearly such an extension is also completely positive. In particular, it is strongly positive and so, for any \( v \in \mathcal{H} \), \( \| v \| = 1 \), and any \( \phi \in \text{Tr}(\mathcal{H}) \) we have

\[
\| (\hat{P} \phi) v \|^2 = \langle v, (\hat{P} \phi)^* (\hat{P} \phi) v \rangle \leq \langle v, \hat{P} (\phi^* \phi) v \rangle
\]

\[
= \text{tr} P_v |\hat{P} (\phi^* \phi) \rangle = \text{tr} \hat{P} (P_v) \phi^* \phi \leq \| \hat{P} (P_v) \|_1 \| \phi^* \phi \|_\infty \leq \| \phi \|_\infty^2
\]

where \( P_v = | v > < v | \). However, \( \hat{P} \) is a non-zero projection, hence \( \| \hat{P} \|_{\infty, \infty} = 1 \). By duality, \( \| \hat{P} \|_{1,1} = 1 \), too.

c) The dual operator \( \hat{P}^* \) is a normal contraction on \( B(\mathcal{H}) \). It is also a projection. Suppose \( \phi, \psi \in \text{Tr}(\mathcal{H}) \). Then

\[
\text{tr}(\hat{P}^* \phi) = \text{tr}(\hat{P} \psi) = \text{tr}(\hat{P} \phi)\psi
\]

Hence \( \hat{P}^* |_{\text{Tr}(\mathcal{H})} = \hat{P} \). However, \( \text{Tr}(\mathcal{H}) \) is \( \sigma \)-weakly dense in \( B(\mathcal{H}) \) so \( \hat{P}^* \) is a normal extension of \( \hat{P} \) onto \( B(\mathcal{H}) \). We denote it by \( \mathcal{T} \). The image of \( \mathcal{T} \) equals to the \( \sigma \)-weak closure of \( \text{im} \hat{P} \) which coincides with the von Neumann algebra \( \mathcal{M} \). \( \square \)

Therefore, our task reduces to the description of the projection \( \mathcal{T} \). By Prop.13 and 14 in [13] we know that \( \mathcal{M} = \bigoplus_k \mathcal{M}_k = \bigoplus_k (\bigoplus_n \mathcal{M}_{kn}) \), where \( \mathcal{M}_{kn} \) are type I factors. Let \( E_{kn} \) denote the unit in \( \mathcal{M}_{kn} \) and let \( \mathcal{T}_{kn}(A) = E_{kn} \mathcal{T}(A) \). Then \( \mathcal{T}_{kn} \) is a projection onto \( \mathcal{M}_{kn} \) and \( \sum_k \sum_n \mathcal{T}_{kn} = \mathcal{T} \) since \( \sum_k \sum_n E_{kn} = E \), the unit in \( \mathcal{M} \). Hence

\[
\mathcal{T} = \sum_k \mathcal{T}_k = \sum_k (\sum_n \mathcal{T}_{kn}) \quad (6)
\]
where $\mathcal{P}_{kn}$ are normal, norm one, and pairwise orthogonal projections, i.e. $\mathcal{P}_{kn} \circ \mathcal{P}_{lm} = \mathcal{P}_{lm} \circ \mathcal{P}_{kn} = 0$ if $(kn) \neq (lm)$. Therefore, it suffices to determine the form of projections $\mathcal{P}_{kn}$. Let us recall that each $\mathcal{M}_{kn}$ has a minimal projector $e_n$ of a finite dimension $r(k)$ for all $n$, where $r(k)$ is a subsequence of natural numbers. Let $N = N(k, n)$ be the degree of homogeneity (possibly infinite) of $\mathcal{M}_{kn}$.

**Theorem 4.2** For $A \in B(\mathcal{H})$

$$\mathcal{P}_{kn}(A) = \int_{U(N_{kn})} d\mu(U) U(E_{kn}AE_{kn})U^* (7)$$

where $\mathcal{N}_{kn} = (\mathcal{M}_{kn})E_{kn}$ is the commutant of $\mathcal{M}_{kn}$ in $B(E_{kn}\mathcal{H})$, $U(\mathcal{N}_{kn})$ is the group of unitary operators in $\mathcal{N}_{kn}$, and $d\mu$ is a unique normalized Haar measure on $U(\mathcal{N}_{kn})$.

**Proof:** First, notice that for any projector $e$ in $\mathcal{M}$ the von Neumann algebras $e\mathcal{M}e$ and $\mathcal{M}e$ are isomorphic. We use this identification in the proof. Let $V : E_{kn}\mathcal{H} \to \tilde{\mathcal{H}}_{kn} = C^N \otimes C^{r(k)}$ be a unitary isomorphism. Here $\tilde{\mathcal{H}}_{kn}$ denotes a Hilbert space being the direct sum of $N = N(k, n)$ copies of range $e_n$. $\mathcal{M}_{kn}$ is isomorphic to the matrix algebra $M_{N \times N}(e_n \mathcal{M}_{kn} e_n)$. Because $e_n$ is minimal, so $e_n \mathcal{M}_{kn} e_n = C e_n$. Therefore, $\alpha(\mathcal{M}_{kn}) = B(\mathcal{H}_{kn}) \otimes I_{r(k)}$, where $\alpha(A) = VAV^*$ for $A \in B(E_{kn}\mathcal{H})$ and $I_{r(k)}$ is the identity $r(k) \times r(k)$ matrix. The projection $\alpha \circ \mathcal{P}_{kn}|_{B(E_{kn}\mathcal{H})} \circ \alpha^{-1}$ is the conditional expectation from $B(\mathcal{H}_{kn}) \otimes M_{r(k) \times r(k)}$ onto the first factor. Hence, for any $B \in B(\mathcal{H}_{kn}) \otimes M_{r(k) \times r(k)}$,

$$\alpha \circ \mathcal{P}_{kn}|_{B(E_{kn}\mathcal{H})} \circ \alpha^{-1}(B) = \int_{U(r(k))} d\mu(U) (1 \otimes U) B(1 \otimes U^*) \int_{U(r(k))} d\mu(U) V^*(1 \otimes U)(VAV^*)(1 \otimes U^*)V^*$$

and so

$$\mathcal{P}_{kn}(A) = \int_{U(r(k))} d\mu(U) V^*(1 \otimes U)(VAV^*)(1 \otimes U^*)V^*$$

for any $A \in B(E_{kn}\mathcal{H})$. However $V^*(1 \otimes U)V$ is a unitary operator in $\mathcal{N}_{kn}$ and $\mathcal{N}_{kn}$ is isomorphic to $M_{r(k) \times r(k)}$. Thus $U(\mathcal{N}_{kn})$ is isomorphic to $U(r(k))$. 

12
For a general $A \in \mathcal{B}(\mathcal{H})$ there is
\[
\mathcal{P}_{kn}(A) = \int_{\mathcal{U}(\mathcal{N}_{kn})} d\mu(U) U (E_{kn} AE_{kn}) U^* \quad \square
\]

Let us consider some particular cases of the projection $\mathcal{P}$.

**Corollary 4.3** If $r(1) = 1$, then $\mathcal{P}_1$ is given by
\[
\mathcal{P}_1(A) = \sum_n \mathcal{P}_{1n}(A) = \sum_n E_{1n} AE_{1n}
\]
where $\{E_{1n}\}$ is a sequence (possibly finite) of pairwise orthogonal projectors of arbitrary dimensions. Let $J = \{(kn) : N(k, n) = 1\}$. Then for any $(kn) \in J$, $\dim E_{kn} < \infty$ and
\[
\sum_J \mathcal{P}_{kn}(A) = \sum_J \text{tr}(E_{kn} A) \frac{E_{kn}}{\dim E_{kn}}
\]
where $\{E_{kn}\}_J$ is a sequence of pairwise orthogonal finite dimensional projectors.

**Proof:** For $r(1) = 1$ any $U \in \mathcal{U}(\mathcal{N}_{1n})$ is of the form $U = e^{ia} E_{1n}$. Hence (6) and (7) implies (8). If $N(k, n) = 1$, then $E_{kn}$ is a minimal projector in $\mathcal{M}_{kn}$ with $\dim E_{kn} = r(k)$. Therefore, $\mathcal{P}_{kn}|_{\mathcal{B}(E_{kn}\mathcal{H})}$ is the conditional expectation onto $CE_{kn}$ and formula (9) follows. $\square$

Thus, if $r(1) = 1$ we recover the Wan scheme [4], while for any $r(k) > 1$ the minimal projectors in a corresponding coherent subspace are of dimension $r(k)$ and so we meet the case of generalized rays. The restriction of the gauge group to subspace $E_{kn}\mathcal{H}$ is isomorphic to the unitary group $\mathcal{U}(r(k))$. Therefore, the whole gauge group equals to $\oplus_{r(k) > 1} \mathcal{U}(r(k))$. In particular, if $N(k, n) = 1$ for some $r$ and $k$ such that $r(k) > 1$, then the restriction of algebra $\mathcal{M}$ to $E_{kn}\mathcal{H}$ consists only of numbers, i.e. $\mathcal{M}E_{kn} = CE_{kn}$. Hence we recover in formula (9) the coarse graining projection [16].

We meet another interesting case when $N(k, n) = r(k)$. Then, $E_{kn}\mathcal{H}$ is isomorphic to $\mathbb{C}^{r(k)} \otimes \mathbb{C}^{r(k)}$ and the restriction of $\mathcal{M}$ to $E_{kn}\mathcal{H}$ is isomorphic to $M_{r(k) \times r(k)} \otimes I_{r(k)}$, that is effective observables act on the first factor, and so are isomorphic with their commutant in $\mathcal{B}(E_{kn}\mathcal{H})$. Such a situation was discussed by Giulini [17] in the context of the quantization of a system whose classical configuration space is not simply connected.
It is worth pointing out that all cases discussed above are of the discrete type, that is all self-adjoint superselection operators have discrete spectral decompositions.

Finally, we discuss the conservativeness of the induced superselection structure.

**Proposition 4.4** The induced superselection structure is conservative if and only if $I \in \mathcal{M}$.

**Proof:** $\Leftarrow$ If the identity operator belongs to $\mathcal{M}$, then $\overline{P}(I) = I$. Hence $\text{tr}\hat{P}\rho = \text{tr}\overline{P}(I)\rho = \text{tr}\rho$ for all $\rho \in \text{Tr}(\mathcal{H})$.

$\Rightarrow$ Suppose now that $\text{tr}\hat{P}\rho = \text{tr}\rho$ for all $\rho \in \text{Tr}(\mathcal{H})$. Let us assume on the contrary that $\mathcal{M}$ does not contain $I$. Then $\sum_{k,n} E_{kn}$ is a non-trivial projector. Let $E^\perp = I - \sum_{k,n} E_{kn}$. For a state $\rho_0 = E^\perp \rho E^\perp$ we have that $\hat{P}(\rho_0) = 0$ and so $\text{tr}\hat{P}(\rho_0) = 0$, the contradiction. \(\Box\)

**Corollary 4.5** Suppose $T_t$ is relatively compact in the strong operator topology. Then the induced superselection structure is conservative.

**Proof:** By Prop. 4.4 it suffices to show that $I \in \mathcal{M}$. Suppose on the contrary that $I$ does not belong to $\mathcal{M}$. Then again for a state $\rho_0 = E^\perp \rho E^\perp$ we have that $\hat{P}(\rho_0) = 0$. However, by (5)

$$\lim_{t \to \infty} \|T_t \rho_0 - \hat{P}(T_t \rho_0)\|_1 = \lim_{t \to \infty} \|T_t \rho_0\|_1 = 0$$

the contradiction, since $\|T_t \rho_0\|_1 = \text{tr}T_t \rho_0 = 1$ for all $t \geq 0$. \(\Box\)

Consequently, a strong superselection structure is always conservative and so the projection $\overline{P}$ is a tr-compatible conditional expectation from $B(\mathcal{H})$ onto $\mathcal{M}$.

**References**

[1] Wick, G.C., Wightman, A.S., Wigner, E.P.: The intrinsic parity of elementary particles. Phys. Rev. 88, 101-105 (1952)

[2] Wightman, A.S.: Superselection rules; old and new. Il Nuovo Cimento B 110, 751-769 (1995)

[3] Jauch, J.: System of observables in Quantum Mechanics. Helv. Phys. Acta 33, 711-726 (1960)

[4] Wan, K.: Superselection rules, quantum measurement, and Schrödinger’s cat. Canadian J. Phys. 58, 976-982 (1980)

[5] Messiah, A.M.L., Greenberg, O.W.: Symmetrization postulate and its experimental foundation. Phys. Rev. 136B, 248-267 (1964)
[6] Jauch, J.M., Misra, B.: Sypersymmetries and essential observables. Helv. Phys. Acta 34, 699-709 (1961)
[7] Zurek, W.H.: Environment-induced superselection rules. Phys. Rev. D 26, 1862-1880 (1982)
[8] Joos, E., Zeh, H.D.: The emergence of classical properties through interaction with the environment. Z. Phys. B 59, 223-243 (1985)
[9] Paz, J.P., Zurek, W.H.: Environment-induced decoherence, classicality, and consistency of quantum histories. Phys. Rev. D 48, 2728-2738 (1993)
[10] Joos, E.: Decoherence through interaction with the environment. In: Giulini, D. et al. (eds.) Decoherence and the appearance of a classical world in quantum theory. Berlin: Springer 1996
[11] Unruh, W.G., Zurek, W.H.: Reduction of a wave packet in Quantum Brownian motion. Phys. Rev. D 40, 1071-1094 (1989)
[12] Twamley, J.: Phase-space decoherence: a comparison between consistent histories and environment-induced superselection. Phys. Rev. D 48, 5730-5745 (1993)
[13] Olkiewicz, R.: Environment-induced superselection rules in Markovian regime. Commun. Math. Phys. (in press)
[14] Davies, E.B.: Quantum Theory of Open Systems. Academic Press: London (1976)
[15] Partovi, M.H.: Irreversibility, reduction and entropy increase in quantum measurement. Phys. Lett. A 137, 445-450 (1989)
[16] Kupsch, J.: Open quantum systems. In: Giulini, D. et al. (eds.) Decoherence and the appearance of a classical world in quantum theory. Berlin: Springer 1996
[17] Giulini, D.: Quantum Mechanics on spaces with finite fundamental group. Helv. Phys. Acta 68, 438-469 (1995)