Canonical Dual Approach for Contact Mechanics Problems with Friction

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Abstract
This paper presents an application of Canonical duality theory to the solution of contact problems with Coulomb friction. The contact problem is formulated as a quasi-variational inequality which solution is found by solving its Karush-Kunt-Tucker system of equations. The complementarity conditions are reformulated by using the Fischer-Burmeister complementarity function, obtaining a non-convex global optimization problem. Then canonical duality theory is applied to reformulate the non-convex global optimization problem and define its optimality conditions, finding a solution of the original quasi-variational inequality.

We also propose a methodology for finding the solutions of the new formulation, and report the results on well known instances from literature.

Keywords
Global optimization, canonical duality theory, contact problems, KKT conditions

1 Introduction
Contact mechanics provides many challenging problems in both engineering and mathematics. The problem generally consists in analyzing the forces created when an elastic body comes in contact with a rigid obstacle and search for an equilibrium of such forces. In the moment the two bodies come in contact there are not only normal forces that prevent interpenetration between the two bodies, but also friction forces that prevent the elastic body to slide on the rigid obstacle.

One of the most popular application of this class of problems is the automate planning of tasks carried out by robots. Contact problems arise in such applications when a robotic arm has to come in contact with objects in its surrounding. In such cases it is necessary to find an equilibrium in the strength of the robotic arm so that the friction is sufficient to have a solid grip on the object without damaging it [1].

Early research focused on the frictionless contact between two or more bodies [2, 3], where a quadratic programming optimization problem or a variational inequality (like in [4]) is solved. However, in most cases, this formulation does not completely reflect the physical reality. For this reason, contact problems with Coulomb friction are studied and solved. Generally several mathematical programming methods are used for solving such problems (for more information refer to [5] and citations therein), and one of the most popular formulations for contact problems are Quasi-Variational Inequalities as reported in [6]. In the same book, the authors use a non-smooth newton method in order to compute a solution, finding satisfying results only for small values of the friction.
Quasi-variational inequalities (QVIs) are a powerful modeling tool capable of describing complex equilibrium situations that can appear in different fields such as generalized Nash games, mechanics, economics, statistics and so on (see e.g. [7, 8, 9, 10]). For what regards QVIs there are a few works devoted to the numerical solution of finite-dimensional QVIs (see e.g. [10, 11, 12, 13, 14, 15, 16]), in particular in the recent paper [17] a solution method for QVIs based on solving their Karush-Kuhn-Tucker (KKT) conditions is proposed.

In this work we propose a novel Canonical Duality approach for solving the QVI associated with the contact problem with Coulomb friction by presenting a deeper insight on said application of the theory already presented in [18]. In particular we show that the QVI associated with the problem belongs to a particular class of QVIs called Affine Quasi Variational Inequalities (AQVI). We search for a solution of the AQVI by determining a point that satisfies its KKT conditions. In order to find such point we reformulate the KKT conditions of the AQVI by using the Fisher-Burmeister complementarity function, obtaining a non-convex global optimization problem [19, 20]. By using Canonical Duality Theory it is possible to reformulate the obtained non-convex optimization problem and to find the conditions for a critical point to be a global solution.

The principal aim of this paper is to show the potentiality of canonical duality theory for this class of mechanics problems and propose a new methodology to solve them.

Canonical duality theory, developed from non-convex analysis and global optimization [21, 22], is a potentially powerful methodology, which has been successfully used for solving a large class of challenging problems in biology, engineering, sciences [23, 24, 25], and recently in network communications [26, 27], radial basis neural networks [28] and constrained optimization [29]. In this paper we use a canonical dual transformation methodology in order to formulate the Total Complementarity Function of the original problem which stationary points do not have any duality gap in respect to the corresponding solutions of the primal problem. With the proprieties of the total complementarity function it is also possible to find the optimality conditions of the original problem.

The paper is organized as follows: In Section 2 we present the problem from mechanics and then report its formulation as a quasi variational inequality. In Section 3 we use the dual canonical transformation to reformulate the global optimization problem as a total complementarity function and analyze its proprieties. Finally in Section 4 we report an optimization procedure based on the results obtained in the previous sections and numerical results on some instances of the contact friction problem.

We use the following notation: \((a, b) \in \mathbb{R}^{n_a+n_b}\) indicates the column vector comprised by vectors \(a \in \mathbb{R}^{n_a}\) and \(b \in \mathbb{R}^{n_b}\); \(\mathbb{R}_+^n\) denotes the set of nonnegative numbers; \(\mathbb{R}_+^n \subset \mathbb{R}^n\) is the set of positive numbers; \(\text{sta}\{f(x) : x \in \mathcal{X}\}\) denotes the set of stationary points of function \(f\) in \(\mathcal{X}\); given a matrix \(Q \in \mathbb{R}^{n \times k}\) we indicate with \(Q_{i,:}\) its \(i\)-th row and with \(Q_{:,i}\) its \(i\)-th column; \(\text{diag}(a)\) denotes the (square) diagonal matrix whose diagonal entries are the elements of the vector \(a\); \(\circ\) denotes the Hadamard (component-wise) product operator; \(\mathbf{0}_n\) indicates the origin in \(\mathbb{R}^n\). The double dots product \(e : \sigma\) indicates \(\text{trace}(e^T \sigma)\) and is a standard notation in solid mechanics where \(e \in \mathbb{R}^n\) and \(\sigma \in \mathbb{R}^{n \times n}\).

2 Problem Formulation

Generally in contact problems, the Coulomb friction between the body and obstacle should be considered in order to have the most realistic representation of the mathematical modeling. A simple way to define this problem is to restrict the normal displacement of the boundary points of the elastic body by means of unilateral contact constraints.

Let us consider an elastic body which occupies a smooth, bounded simply-connected domain \(\Omega \subset \mathbb{R}^3\) with boundary \(\Gamma = \Gamma_g \cup \Gamma_u \cup \Gamma_c\), where

- \(\Gamma_g\): associated the Neumann boundary condition. The force that moves the object is applied on this surface.
- \(\Gamma_u\): associated to the homogeneous Dirichlet boundary conditions. This part of the body is considered fixed.
• $\Gamma_c$: associated with the unilateral boundary conditions. This is the part of the object in contact with the rigid obstacle.

The displacement of the elastic body $\Omega$ is a field function $u : \Omega \to \mathbb{R}^3$ that belongs to the following set of kinetically admissible space:

$$K = \left\{ v \in \mathcal{U}(\Omega) \left| \sum_{i=1}^{3} v_i n_i \leq 0 \text{ on } \Gamma_c \right. \right\},$$  \hspace{1cm} \text{(1)}$$

where $n_i$ for $i = 1, 2, 3$ denotes the outer normal to $\partial \Omega$ and $\mathcal{U}(\Omega)$ is a Sobolev space defined on $\Omega$ such as $v = 0$ on $\Gamma_u$. The function $u$ solves a contact problem with given friction if it is the solution of the following variational inequality:

Find $u$ such that: 
$$a(u, v - u) + \int_{\Gamma_c} \gamma(|v_i| - |u_i|) d\Gamma_c \geq \int_{\Gamma_g} F(v - u) d\Gamma_g \quad \forall v \in K,$$  \hspace{1cm} \text{(2)}$$

Where the bilinear form $a(\cdot, \cdot)$ is defined as 
$$a(u, v) = \int_{\Omega} e(u) : \mathcal{H} : e(v) d\xi,$$ where $e(u)$ is the infinitesimal strain tensor and $\mathcal{H}$ is a bounded, symmetric and elliptic mapping that expresses Hooke’s law. The function $F$ represent the external force on $\Gamma_g$ and $\gamma$ is a given friction function on $\Gamma_c$ such that $\gamma \geq 0$.

In order to solve problem (2) it is possible to use a finite element method by means of a discretization parameter $h$ and obtain:

Find $u$ such that: 
$$\langle Cu, v - u \rangle - \langle f, v - u \rangle \quad \forall v \in K,$$  \hspace{1cm} \text{(3)}$$

where $C$ is the stiffness matrix of the $r$ nodes considered on the contact surface $\Gamma_c$, $f$ is the right-hand side vector associated with the external force $F$ and $u$ and $v \in \mathbb{R}^N$, where $N$ is the number of variables of the problem that is composed by the normal and tangent components of the nodes on the contact surface $\Gamma_c$, that is $N = 2r$.

It is possible to show that the solution of problem (2) satisfies the following contact conditions with given friction:

$$u_n \leq 0, \quad T_n \leq 0, \quad u_n T_n = 0, \quad |T_i| \leq \gamma, \quad (\gamma - |T_i|) u_t = 0, \quad u_t T_i \leq 0 \quad \text{on } \Gamma_c,$$  \hspace{1cm} \text{(4)}$$

where $u_n$ and $u_t$ indicate the tangent and the normal components of vector $u$, $T$ indicates the boundary stress vector on $\Gamma_c$ with normal component $T_n$ and tangent component $T_t$. The first relation in (4) indicates the standard unilateral contact conditions, and the second relation indicates that if the tangent component of the stress vector is lower than $\gamma$, then there is no displacement and once its value reaches $\gamma$ the object begins to slide on the obstacle.

For Coulomb friction, the function $\gamma$ depends linearly on the normal force, i.e. $\gamma = \Phi|T_n|$, where $\Phi$ is the coefficient of friction characterizing the physical properties of the surfaces in contact.

The solution of problem (3) can be found by solving a difficult fixed point problem which convergence has not been proven [6]. Moreover the solution of this problem yields the displacements, while a practitioner is generally interested to find the stress on the nodes of the contact surface. Computing the contact stress from the displacement is also considered a difficult task.

To avoid these issues, it is possible to use the reciprocal variational formulation of problem (3) with variables $\mu_j, j = 1, \ldots, N$. We assume that for $j = 1, 2, \ldots, N/2$ the components corresponding to the odd number $2j - 1$ are associated with the tangential components of the nodes on the contact surface and the even components are associated with the normal components of the nodes on the contact surface. Given a stress vector $\tau \in \mathbb{R}^N$, the following condition must be satisfied:

$$\tau \in \tilde{K}(\tau) = \{ \mu \in \mathbb{R}^N | \mu_{2j} \leq 0, \quad |\mu_{2j-1}| \leq \Phi|\tau_{2j}|, \quad j = 1, \ldots, N/2 \}. \hspace{1cm} \text{(5)}$$
It is easy to notice that conditions (5) represent the conditions reported in [4] and if we assume that the even components of \( \mu \) are lower bounded by a value \( l \), the bound constraints (5) can be rewritten in the following way:

\[
\tilde{K}(\tau) = \{ \mu \in \mathbb{R}^N \mid A\mu + B\tau - c \leq 0 \}
\]

with \( A, B \in \mathbb{R}^{m \times N}, c \in \mathbb{R}^m, m = 2N \) and

\[
(A)_{ij} = \begin{cases} 
1 & \text{if } i \text{ odd, } j = (i+1)/2 \\
-1 & \text{if } i \text{ even, } j = (i)/2 \\
0 & \text{otherwise}
\end{cases},
\]

\[
(B)_{ij} = \begin{cases} 
-\Phi & \text{if } [(i+1)/2] \text{ odd, } j = [(i+1)/2] + 1 \\
0 & \text{otherwise}
\end{cases},
\]

\[
c_i = \begin{cases} 
l & \text{if } \text{mod}(i,4) = 0 \\
0 & \text{otherwise}
\end{cases},
\]

The stress vector is the solution of the following quasi variational inequality:

\[
\text{Find } u \text{ such that } \langle D\tau, \mu - \tau \rangle + \langle e, \mu - \tau \rangle \geq 0 \quad \forall \mu \in \tilde{K}(\tau),
\]

(6)

Where \( D = C^{-1} \) and \( e = -C^{-1}f \).

Problem (6) is a Affine Quasi Variational Inequality AQVI(\( A, B, c, D, e \)) which equilibrium point can be found by satisfying its KKT conditions. We say that a point \( \tau \in \mathbb{R}^N \) satisfies the KKT conditions if multipliers \( \lambda \in \mathbb{R}^m \) exist such that

\[
D\tau + e + A^T\lambda = 0_N, \quad 0_m \leq \lambda \perp A\tau + B\tau - c \leq 0_m.
\]

(7)

It is quite easy to show the following result, whose proof we omit.

**Theorem 1** If a point \( \tau \), together with a suitable vector \( \lambda \in \mathbb{R}^m \) of multipliers, satisfies the KKT system (7), then \( \tau \) is a solution of the AQVI (\( A, B, c, D, e \)). Vice versa, if \( \tau \) is a solution of the AQVI (\( A, B, c, D, e \)) then multipliers \( \lambda \in \mathbb{R}^m \) exist such that the pair \( (\tau, \lambda) \) satisfies the KKT conditions (7).

Generally it is difficult to deal with the complementarity conditions in (7). Such conditions can be replaced by using complementarity functions. A complementarity function is a function \( \theta : \mathbb{R}^2 \to \mathbb{R} \) such that \( \theta(a,b) = 0 \) if and only if \( a \geq 0, b \geq 0, \) and \( ab = 0 \). One of the most prominent complementarity functions is the Fischer-Burmeister function:

\[
\theta_{FB}(a,b) = \sqrt{a^2 + b^2} - (a + b).
\]

We can then consider the following problem equivalent to the solution of system (7):

\[
(P) : \quad \min_{x,\lambda} P(x,\lambda) = \frac{1}{2} \left\| D\tau + e + A^T\lambda \right\|_2^2 = \frac{1}{2} \sum_{i=1}^m \theta_{FB}(\lambda_i, -g_i(\tau))^2 + \frac{1}{2}(\tau,\lambda)^T M(\tau,\lambda) - f^T(\tau,\lambda) =
\]

(8)

where for all \( i = 1, \ldots, m \):

\[
g_i(\tau) = A_i^*\tau + B_i^*\tau - c_i,
\]

and

\[
M = \begin{pmatrix} D^T \\ A \end{pmatrix}, \quad f = -\begin{pmatrix} D^T \\ A \end{pmatrix} e.
\]

It is easy to see that problem \( P \) is nonconvex, furthermore, since only the global minima of problem (8) correspond to a stress vector solutions, it is very hard to solve. In fact it is well known that not all critical points of \( P \) are solutions of the AQVI [19, 20].
3 Canonical Dual Transformation and Properties

The first step of a canonical dual transformation for problem \( \mathbf{8} \) is the introduction of operator \( \xi = \Lambda(\tau, \lambda) : \mathbb{R}^{N+m} \rightarrow \mathcal{E}_0 \equiv \mathbb{R}^m \), which is defined as
\[
\xi_i = \Lambda_i(\tau, \lambda) = \sqrt{\lambda_i^2 + g_i(\tau)^2} - \lambda_i + g_i(\tau), \quad i = 1, \ldots, m,
\]
(9)

note that each \( \xi_i \) is convex since it is defined as a composition of a convex function and a linear function. Furthermore we introduce a convex function \( V : \mathcal{E}_0 \rightarrow \mathbb{R} \) (associated with \( \xi \)), that is defined as
\[
V_0(\xi) = \frac{1}{2} \sum_{i=1}^m \xi_i^2.
\]
(10)

It is easy to see that
\[
W(\tau, \lambda) = V_0(\Lambda(x, \lambda)) = V_0(\xi).
\]
(11)

Furthermore, we introduce a dual variable
\[
\sigma = \nabla V_0(\xi) = \xi,
\]
(12)

which is defined on the range \( \mathcal{S}_0 \equiv \mathbb{R}^m \) of \( \nabla V_0(\cdot) \). Since the (duality) mapping \( \mathbf{12} \) is invertible, \( i.e. \) \( \xi \) can be expressed as a function of \( \sigma \), then the function \( V_0(\xi) \) is said to be a canonical function on \( \mathcal{E}_0 \), see \[21\].

In order to define the total complementarity function in both primal and dual variables \( (\tau, \lambda, \sigma) \) we use a Legendre transformation \[21\]. Specifically the Legendre conjugate \( V_0^*(\sigma) : \mathcal{S}_0 \rightarrow \mathbb{R} \) is defined in the following way
\[
V_0^*(\sigma) = \text{sta} \{ \xi^T \sigma - V_0(\xi) : \xi \in \mathcal{E}_0 \},
\]
which is equal to the function \( \xi^T \sigma - V_0(\xi) \) in which \( \xi \) is fixed to a stationary point. Since \( \xi^T \sigma - V_0(\xi) \) is a quadratic strictly concave function in \( \xi \), then it is easy to see that its (unique) stationary point is \( \xi = \sigma \), and then
\[
V_0^*(\sigma) = \xi^T \sigma - V_0(\xi) = \sigma^T \sigma - V_0(\xi(\sigma)) = \frac{1}{2} \sum_{i=1}^n \sigma_i^2,
\]
(13)

moreover we obtain that
\[
V_0(\xi) = \xi^T \sigma - V_0^*(\sigma).
\]
(14)

Since
\[
W(\tau, \lambda) \overset{\mathbf{11}}{=} V_0(\xi) \overset{\mathbf{13}}{=} \xi^T \sigma - V_0^*(\sigma)
\]
we obtain the total complementarity function:
\[
\Xi_0(\tau, \lambda, \sigma) = \sum_{i=1}^m \left[ \sigma_i \left( \sqrt{\lambda_i^2 + g_i(\tau)^2} - \lambda_i + g_i(\tau) \right) - \frac{1}{2} \sigma_i^2 \right] + \frac{1}{2} (\tau, \lambda)^T M(\tau, \lambda) - f^T(\tau, \lambda),
\]
(15)

where
\[
\hat{f}(\sigma) = f + \left( -\frac{(A^T + B^T)\sigma}{\sigma} \right).
\]
(16)

It is easy to see that the total complementarity function \( \Xi_0 \) is strictly concave in \( \sigma \) for all \( (\tau, \lambda) \). Moreover \( \Xi_0 \) is convex in \( (\tau, \lambda) \) (although non-smooth but only semi-smooth) for all \( \sigma \in \mathbb{R}^m \), since \( M \succeq 0 \) and each function \( \sqrt{\lambda_i^2 + g_i(\tau)^2} \) is convex in \( (x, \lambda) \).

Function \( \mathbf{15} \) has some interesting properties that can be exploited to find a global solution of problem \( \mathbf{P} \). In the following we report these properties omitting their proofs. The interested reader can refer to \[18\] for a detailed discussion on such properties. The first propriety shows the relations between the critical point of problem \( \mathbf{8} \) and \( \mathbf{15} \).
proximate solution of \( (6) \) for small values of \( \epsilon \).

It is easy to see that any point \((\bar{\tau}, \bar{\lambda}, \bar{\sigma})\) is a critical point for \(P(\bar{\tau}, \bar{\lambda})\) and

\[
P(\bar{\tau}, \bar{\lambda}) = \Xi_0(\bar{\tau}, \bar{\lambda}, \bar{\sigma})
\]

Theorem 2 proves that every critical point of \(\Xi_0\) has a corresponding critical point in \(P(\bar{\tau}, \bar{\lambda})\), furthermore they have the same value of the objective function. From now on we will indicate with:

\[
S_0^+ = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}_+^m.
\]

The next theorem characterizes the critical points of \(\Xi_0\) in a subset of the dual space:

**Theorem 3** Let a point \((\bar{\tau}, \bar{\lambda}, \bar{\sigma}) \in S_0^+\) be critical for \(\Xi_0\) then it is a saddle point for \(\Xi_0\).

**Theorem 4** Suppose that \((\tau^*, \lambda^*)\) exists such that \(P(\tau^*, \lambda^*) = -\frac{1}{2}e^T e\), that is \((\tau^*, \lambda^*)\) is a global minimum of the primal problem, then

1. \((\tau^*, \lambda^*, 0_m)\) is a critical point for \(\Xi_0\) and \(\Xi_0(\tau^*, \lambda^*, 0_m) = -\frac{1}{2}e^T e\);
2. all points \((\tau, \lambda, \sigma) \in \mathbb{R}^n \times \mathbb{R}^m \times \{\mathbb{R}_+^m \setminus \{0_m\}\}\) are not critical for \(\Xi_0\).

From Theorem 4 the critical points of \(\Xi_0\) corresponding to the solutions of the Coulomb friction problem are located in \(S_0^+\). The property that the stationary points must all have \(\|\sigma^+\| = 0\) means that the point \((\tau^*, \lambda^*)\) satisfies the KKT complementarity conditions as \(\sigma^+_i = \theta_F B(\lambda^*_i, g_i(\tau^*))\), for \(i = 1, \ldots, m\).

4 **Results**

In this section two instances of the contact problem with friction are solved. The two instances are taken from problem 11.1 in [3] and are called CPCF31 and CPCF41. In the case the Coulomb friction \(\Phi = 10\) the two instances correspond to problems OutKZ31 and OutKZ41 of QVILIB [31], a collection of test problems from diverse sources that gives a uniform basis on which algorithms for the solution of QVIs can be tested and compared. The two problems have the same rigid obstacle and elastic body, but the segmentation of the obstacle is different. In CPCF31 the obstacle is divided into 30 segments, while in CPCF41 the obstacle is divided in 40 segments.

The approach we use in order to find a solution of \(\Xi_0\) is based on the results of Theorem 4. In particular we search a solution of the following problem:

\[
(\mathcal{SP}) : \quad \min_{(\tau, \lambda, \sigma)} \max_{\sigma \in S_0^+} \Xi_0(\tau, \lambda, \sigma).
\]

Since the total complementarity function is non-smooth because of the term due to the Fisher-Burmeister, we apply a simple smoothing procedure and obtain the following smoothed total complementarity function:

\[
\Xi_e(\tau, \lambda, \sigma) = \sum_{i=1}^m \left[ \sigma_i \left( \sqrt{\lambda_i^2 + g_i(\tau)^2} + e^2 - \lambda_i + g_i(\tau) \right) - \frac{1}{2} \sigma_i^2 \right] + \frac{1}{2} (\tau, \lambda)^T M(\tau, \lambda) - f^T(\tau, \lambda),
\]

\(\Xi_e(\tau, \lambda, \sigma)\) still retains its properties of convexity in respect to \((\tau, \lambda)\) for all \(\sigma \in \mathbb{R}_+^m\) and concavity in respect to \(\sigma\) for all \((\tau, \lambda)\), but differently from \(\Xi_0\) it is continuously differentiable in \((\tau, \lambda)\).

If we define the following operator:

\[
H_e(\tau, \lambda, \sigma) = \begin{pmatrix} \nabla_{\tau, \lambda} \Xi_e(\tau, \lambda, \sigma) \\ -\nabla_{\sigma} \Xi_e(\tau, \lambda, \sigma) \end{pmatrix},
\]

It is easy to see that any point \((\tau^*, \lambda^*, \sigma^*)\) such that \(H_e(\tau^*, \lambda^*, \sigma^*) = 0_{n+2m}\) in \(S_0^+\) is an approximate solution of \(\Xi_0\) for small values of \(e\). Furthermore this operator has some favorable
the value of the Coulomb friction $\Phi$. The convergence of the algorithm proposed in [17] has been proved, when applied to this kind of problems varying the coefficient of friction $\Phi$ from 1 to 3. In the case the algorithm stops to a value that does not satisfy the termination criterion it is labelled as failure. Starting points are taken with $\sigma^0$, $\lambda^0$ and $\sigma^0$ with all zero entries. The sequence $\{\epsilon^k\}$ is defined by $\epsilon^1 = 1e - 4$ and $\epsilon^{k+1} = 10^{-k+1}\epsilon^k$, and we set $\delta^0 = 0.1, \gamma = 0.1$.

In order to have an exhaustive analysis on the problem, we run several tests on the considered problems varying the coefficient of friction $\Phi$ from $1e - 3$ to $1e5$. The coefficient of friction is important because it is the parameter that determines the difficulty of the problem. As a matter of facts in several works [32, 33] the existence of a solution for contact problems with Coulomb friction has been proved only for small values of the friction coefficient. Furthermore the convergence of the algorithm proposed in [17] has been proved, when applied to this kind of problems, only for small value of the friction. In other words the analyzed examples with the value of the Coulomb friction $\Phi \geq 10$ can be considered difficult friction contact problem instances. In Table I we list

- the value of the friction $\Phi$;
- the number of iterations, which is equal to the number of VIIs solved;
- the number of crash, major and minor iterations of the PATH solver;
- the number of evaluations of $H$;
- the number of evaluations of $JH$;
- elapsed CPU time in seconds;
- the value of the KKT violation measure $\|Y(x, \lambda)\|_\infty$ at termination.
Table 1: Numerical results of Algorithm 1 for the contact problems.

| Problem  | $\Phi$  | Iter | $(\text{crash}, \text{maj}, \text{min})$ | $H$ | $J$ | $H$ | $\|Y\|_\infty$ |
|----------|---------|------|--------------------------------|-----|-----|-----|----------------|
| CPCF31   | $10^{-3}$ | 2    | (2, 4, 4)                        | 8   | 8   | 0.1376 | 5.11300e-07  |
| CPCF31   | $10^{-2}$ | 2    | (2, 4, 4)                        | 8   | 8   | 0.1338 | 5.03590e-07  |
| CPCF31   | $10^{-1}$ | 1    | (1, 5, 5)                        | 7   | 7   | 0.1116 | 4.19531e-07  |
| CPCF31   | $10^{0}$  | 1    | (2, 7, 7)                        | 11  | 11  | 0.1689 | 6.81777e-07  |
| CPCF31   | $10^{1}$  | 1    | (2, 10, 10)                      | 35  | 10  | 0.2547 | 3.54464e-07  |
| CPCF31   | $10^{2}$  | 2    | (2, 7, 7)                        | 14  | 10  | 0.2235 | 1.50271e-05  |
| CPCF31   | $10^{3}$  | 1    | (0, 13, 310)                     | 44  | 14  | 0.3597 | 3.19657e-08  |
| CPCF31   | $10^{4}$  | 2    | (1, 20, 26)                      | 96  | 23  | 0.4406 | 4.48908e-06  |
| CPCF41   | $10^{-3}$ | 1    | (1, 6, 6)                        | 8   | 8   | 0.2001 | 1.14243e-05  |
| CPCF41   | $10^{-2}$ | 1    | (1, 6, 6)                        | 8   | 8   | 0.1705 | 1.30477e-06  |
| CPCF41   | $10^{-1}$ | 1    | (1, 6, 6)                        | 8   | 8   | 0.1788 | 1.09641e-06  |
| CPCF41   | $10^{0}$  | 1    | (1, 8, 8)                        | 10  | 10  | 0.2215 | 4.32535e-05  |
| CPCF41   | $10^{1}$  | 1    | (1, 6, 6)                        | 8   | 8   | 0.2136 | 4.78402e-05  |
| CPCF41   | $10^{2}$  | 2    | (2, 11, 11)                      | 20  | 15  | 0.4600 | 2.01671e-07  |
| CPCF41   | $10^{3}$  | 2    | (2, 14, 15)                      | 28  | 18  | 0.5676 | 9.80390e-05  |
| CPCF41   | $10^{4}$  | failure |                                          |     |     |     |                 |
| CPCF41   | $10^{5}$  | failure |                                          |     |     |     |                 |

From Table 1 it is possible to see that the method based on canonical duality reaches a good approximation of the stress vector solution in 16 instances on 18. The only instances in which the algorithm fails are those of CPCF41 with really big values of the coefficient of friction. The solution is reached in less than a second in all the proposed instances, and it is possible to notice that the running time substantially increases when the value of the coefficient exceeds 10, showing that the instances with big values of the friction coefficient are indeed difficult to solve.

5 Conclusions

In this paper we presented a canonical duality approach to the solution of contact problem in mechanics with Coulomb friction. We formulated the contact friction problem as a quasi-variational inequality and then exploited the Fisher-Burmeister complementarity function in order to obtain a global optimization problem. Such global optimization problem is non-convex, but with canonical duality theory it is possible to define the optimality conditions of the problem and create a simple strategy that converges to a stress vector solution of the contact problem. We also presented results on some instances of such problems varying the values of the friction coefficient on a vast range, obtaining encouraging results.

In our future research we will improve both the theory and the algorithms in order to extend such approach to other problems in mechanics and improve the methods used to find a solution.

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