Smooth or shock: densities in active single file motion in closed systems

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We study the nonequilibrium steady states (NESS) in a unidirectional or active single file motion of a collection of active particles with hard-core repulsion in a closed system. For active propulsion that is smoothly varying in space with a few discontinuities, we show that, NESS are broadly classified into two types - states (i) where the steady state current depends explicitly on the total particle number $N_0$, and (ii) when it is independent of $N_0$, independent of any system detail, revealing a novel kind of universality. The transitions between these two states are controlled by the interplay between $N_0$ and the global minima of the position-dependent active propulsion. Our theory can be tested in laboratory experiments on self-propelled particles in a closed ring geometry.

Single file motion (SFM) implies particle motion along narrow channels where the particles cannot cross each other due to hardcore repulsion. This was originally introduced by Hodgkin and Keynes [1] to describe ion transport in biological channels; see also Ref. [2] for SFM in transport of biomolecules through cell membranes. SFM was subsequently used for statistical descriptions for pedestrian motions in quasi one-dimensional (1D) systems having bottlenecks with periodic boundary conditions [3, 4].

SFM with bidirectional diffusion (SFD) in 1D is an equilibrium process and distinctly different from normal diffusion due to the restricted geometry in 1D and hardcore repulsion leading to no mutual passage. This makes the dynamics slow. For instance, the mean square displacement $\Delta$ scales with time $t$ as $\sqrt{t}$, i.e., it is subdiffusive [5]. Experimental studies on SFD reveal its unusual properties [6, 7]. More recently, quenched disorder has been found to slow SFD further with $\Delta \sim \sqrt{\log t}$ [8]. In contrast to SFD, active or nonequilibrium systems consume energy for propulsion. These have received increasing attention in the recent past, both because of their wide range of applications and the host of new physical phenomena they describe. Prominent examples include orientable self-propelled particles, without or with quenched disorders [9, 10], and externally driven particles, such as probes in microrheology experiments [11].

Unlike the well-studied SFD, its active analog - unidirectional or active SFM (hereafter ASFM) is necessarily associated with a finite steady state particle current. We consider directed motion of particles with position-dependent propulsion speed and hardcore repulsion in closed geometries, such that the total number of particles $N_{tot}$ is a constant of motion. Our study is generic and applies to a host of physical systems, e.g., vehicular or pedestrian movement along closed network of roads with varying widths or speed restrictions or arbitrary blockages [12], ribosome translocations along mRNA loops [12, 13], spatially varying electric fields in closed arrays of quantum dots [14]. Our theory can be tested in carefully designed experiments on the collective motion of active particles along a nonuniform closed track.

We focus on the nonequilibrium steady state (NESS) densities and their dependences on $N_{tot}$ and position-dependence of the propulsion. In order to focus on the essential physics and extract generic results from a minimal description, we model ASFM by the well-known 1D totally asymmetric simple exclusion process (TASEP), where each site can accommodate at most one particle that can hop only in one direction if the neighboring site is empty. This was originally proposed to model the quasi-1D motion of molecular motors along microtubules in eukaryotic cells [15]. Later on, it was reinvented as a simple model for nonequilibrium phase transition in 1D open systems [16, 17].

In this Letter, we study closed 1D TASEP with $N$ sites as a model for ASFM. Space-dependent propulsion is described by quenched hopping rates assumed here to be spatially smoothly varying with finite number of discontinuities and single or multiple point global minima. We find that independent of the detailed form of the heterogeneous hopping rates, there are generically two types of NESS delineated by the steady state current $J$: (i) $J$ depends on mean density $n = N_{tot}/N$ ($0 < n < 1$) explicitly (hereafter smooth phase or SmP) for low or high $n$, and (ii) $J$ is independent of $N$, observed for intermediate values of $n$, and is characterized by phase separation with localized (LDW) or delocalized (DDW) domain walls (hereafter shock phase or ShP). The phases and the reentrant transitions between them are controlled by the interplay between $n$ and the global minima $q_{min}$ of the position-dependent hopping rate (defined below). The topology of the phase diagrams plotted as functions of $n$ and $q_{min}$ is universal, independent of the precise hopping rate functions. This universality is summarized in Fig. 1.

Our model has a hopping rate $q_i \leq 1$ to a site $i$ that depends explicitly on $i$. The dynamics clearly conserves the total particle number $N_{tot} = \sum_{i=N} n_i$, where $n_i$ is the occupation of site $i$, and also respects particle-hole symmetry. The dynamics of TASEP is formally given by rate equations for every site, that involves nonlinear coupling with neighboring sites, and hence not closed [18]. We use analytical mean-field theory (MFT), complemented by
These intuitive physical pictures are corroborated by our change in 

branches, one more and the other less than 1/2, meet-

the NESS. The density is smooth everywhere except at the locations of 

x, the density at x; here ⟨...⟩ refers to temporal averages in the NESS.

Evidently, for very low (high) n, the model will be almost empty (full) and the steady state density everywhere is then less (more) than 1/2. For this, as we show below, steady state current J depends explicitly on n; density is smooth everywhere except at the locations of the discontinuities of q(x) - hence SmP. These are analogs of the low density (LD) and high density (HD) phases of TASEP [10, 21]. In contrast in ShP, ρ(x) consists of two branches, one more and the other less than 1/2, meeting discontinuously at x_w. Changing n further does not change J; rather x_w shifts to adjust for the change in n. These intuitive physical pictures are corroborated by our quantitative analyses below.

MFT begins by noting that in the NESS

\[ \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} [q(x)\rho(1 - \rho)] = 0, \]  

over a range of x in which q(x) is smooth and piece-wise continuous. This yields

\[ q(x)\rho(x)[1 - \rho(x)] = J, \]  

where J, a constant of integration, is the steady state current. Thus a local spatial variation in q(x) must be compensated by an equal and opposite spatial variation in ρ(x)[1 − ρ(x)] in NESS to keep J fixed. This evidently leads to nonuniform ρ(x). Equation (2) is a quadratic equation in ρ(x) that has two solutions ρ+(x) and ρ−(x):

\[ \rho_+(x) = \frac{1}{2} \left[ 1 + \sqrt{1 - 4J/q(x)} \right] \geq \frac{1}{2}, \]  

\[ \rho_-(x) = \frac{1}{2} \left[ 1 - \sqrt{1 - 4J/q(x)} \right] \leq \frac{1}{2}, \]  

for all x. Here J in (3) and (4) is still unknown. In order to ensure real and positive ρ(x) for all x, we must have 1 − 4J/q(x) ≥ 0. This implies for the constant steady state current

\[ J \leq q(x)/4, \]  

throughout the system. The maximum possible value of J is thus [22]

\[ J_{\text{max}} = q_{\text{min}}/4, \]  

whence ρ+(x_0) = ρ−(x_0); x = x_0 is the location of q_{\text{min}}. With these information, we now study the SmP and ShP below; see also Ref. [22].

Consider the low density limit of SmP. ρ−(x) can be arbitrarily close to zero as J/q(x) → 0. For n → 0, there are only a few particles in the system and thus J → 0 in that limit. Clearly, ρ−(x) gives the steady state density profile throughout the system. The physical picture for n → 1 can be easily obtained by using the particle-hole symmetry [23]. From (3) and (4), it is evident that in SmP both ρ+(x) and ρ−(x) are smooth functions for all x, except where q(x) itself is discontinuous.

Densities ρ−(x) and ρ+(x) depend upon J, still unknown. This may be fixed by using the particle number conservation (PNC):

\[ \int_0^1 \rho_a(x) = n, \ a = +, - \]  

that can be solved to obtain J. It is easy to see from [4] that the maximum (minimum) of ρ−(x) (ρ+(x)) coincides with the minimum (of q(x)), a fact borne out by our MCS studies. This is easy to understand: q_{\text{min}} effectively acts as a bottleneck, and as a result, particles tend to accumulate behind it. We illustrate SmP with q(x) = (x − 0.5)^2 + 0.5 and n = 0.2. Here, ρ−(x) describes the steady state densities. The value of J may be obtained numerically by using [4] with known q(x) and n; see Fig. [2] for a plot of ρ(x) versus x; see also Fig. [3], Fig. [4], Fig. [9], and Fig. [10] in Supplemental Material (SM) for plots of ρ(x) with different q(x).
As \( n \) increases from zero, \( J \) also rises and eventually reaches \( J_{\text{max}} \). Any further increase in \( n \) cannot make \( J \) increase any further, since it is already at its maximum value \( J_{\text{max}} \). Notice that for \( J = J_{\text{max}}, \rho_+(x_0) = \rho_-(x_0) \), where \( x_0 \) is the location of \( q_{\text{min}} \). If \( n \) is still increased, then the additional particles can no longer be accommodated by \( \rho_-(x) \); instead these are accommodated by \( \rho_+(x) \) as a combination of \( \rho_-(x) \) and \( \rho_+(x) \) which meet smoothly at \( x_0 \). Since particles are expected to accumulate behind the bottleneck at \( x_0 \), we expect that additional particles will go over to the high density solution represented by \( \rho_+(x) \). Since we have a closed system, the two solutions must meet at another point \( x_w \), such that \( \rho_+(x_w) > \rho_-(x_w) \) (since \( \rho_+(x) = \rho_-(x) \) only at \( x = x_0 \)), leading to a discontinuous jump in the form of a localized domain wall (LDW) in \( \rho(x) \) at \( x_w \), giving rise to the shock phase, with a jump

\[
\rho_+(x_w) - \rho_-(x_w) = \mathcal{P}, \tag{8}
\]

which is clearly controlled by \( n \) and the functional form of \( q(x) \). As more and more particles are added, \( x_w \) starts shifting in such a way to make the region of existence for \( \rho_+(x) \) larger and \( \rho_-(x) \) smaller. This indeed leaves \( J = J_{\text{max}} = q_{\text{min}}/4 \) unchanged. This continues till \( \rho_+(x) \) spans the full system. Thus, as \( n \) rises from the low to moderate values, a smooth-to-shock transition is encountered. Interestingly, independent of the form of \( q(x) \) this transition is reentrant - as \( n \) rises further, the system moves from ShP to SmP again. See Fig. 3 for a representative plot of the density in ShP, showing strong agreements between MCS and MFT results.

When there are additional local minima (but only one global minimum \( q_{\text{min}} \)) the physics of ShP with one

LDW is still controlled by \( q_{\text{min}} \), see Fig. 4. (However, the form of the LDW depends on any local minima through its dependence on the full form of \( q(x) \).) When there are multiple global minima, very different situations emerge in ShP. For instance, consider \( q(x) \) to have only two symmetrically placed global minima at \( x_1 \) and \( x_2 \): \( q(x_1) = q(x_2) = q_{\text{min}} \). Thus there are now two effective bottlenecks at \( x_1 \) and \( x_2 \) which split the ring into two identical segments, say \( T_A \) and \( T_B \) [21, 24], of length 1/2. Furthermore, from [3] and [4] we have

\[
\rho_-(x_1) = \rho_+(x_1) = \rho_-(x_2) = \rho_+(x_2). \tag{9}
\]
In each of $T_A$ and $T_B$, MFT for ShP applies. Therefore, two domain walls, say at $x_{w1}$ and $x_{w2}$, are expected. PNC then yields only a relation of $x_{w1}$ and $x_{w2}$ with $n$ and cannot determine both of them separately. This means that a shift in $x_{w1}$ can be balanced by an equivalent reverse shift in $x_{w2}$ that still satisfies PNC. Due to the inherent stochasticity of the system, all possible solutions of $x_{w1}$ and $x_{w2}$ that are consistent with PNC are visited by the system, if waited long enough. This leads to delocalized domain walls (DDW). Under long time averages, envelopes of the moving DDWs will be observed. The mean position of the DDW in each of $T_A$ and $T_B$ can however be calculated from PNC by noting that $\rho_A(x)$ and $\rho_B(x)$ are statistically same under long time averages, giving $\langle x_{w1}^A \rangle = 1/2 + \langle x_{w1}^B \rangle$. The full envelop of $\rho_A(x)$ or $\rho_B(x)$ can be calculated by considering the fluctuations of DW; see Fig. 5 and SM.

The movement of the two DDWs are perfectly synchronized - this is a consequence of PNC in the system. The synchronization is best represented in a kymograph; see Fig. 6. If there are more than two global minima of $q(x)$, then there will be as many DDWs. However, there should be no perfect synchro-}nization between any two of the DDWs; see Fig. 6 and SM.

If more particles are still added, eventually $\rho_+(x)$ will be the only valid solution with the system moving over to SmP again whose MFT has been discussed above.

We now obtain the phase diagram in the $n - q_{min}$ plane when $q(x)$ has only one global minimum. At the phase boundary between SmP and ShP, current $J = J_{max} = q_{min}/4$ and $\rho(x) = \rho_-(x)$ (for $n < 1/2$) for all $x$, or for

$$\rho(x) = \rho_+(x)$$

for $n > 1/2$, $\rho_+(x)$ for all $x$. Thus

$$\int_0^1 dx \rho_+(x, J_{max}) = \int_0^1 dx \frac{1}{2} [1 + \sqrt{1 - \frac{q_{min}}{q(x)}}] = \text{(d0)}$$

give the boundaries between SmP and ShP in Fig. 6. The topology of the phase diagrams in Fig. 6 remains independent of the precise forms of $q(x)$ having same $q_{min}$ - even if there are multiple global minima with value $q_{min}$; ShP in this case corresponds to moving shocks. This is the universality in ASFM mentioned above.

For an open ASFM with position-dependent propulsion, modeled by an open TASEP with quenched disorder hopping rates and given entry ($\alpha$) and exit ($\beta$) rates, there is no strict PNC. Equation (24) and hence $\rho_+(x)$ and $\rho_-(x)$ as given by (3) and (4), respectively, should still hold in the bulk here. While obtaining a full phase diagram will require a quantitative analysis, we can already come to the following conclusions from our analyses above and based on the knowledge of an open uniform TASEP. We consider a single global minimum for $q(x)$ somehow in the bulk. For sufficiently low $\alpha < \beta$, there will be very few (many) particles and the density in NESS will be controlled by $\rho_-(x)$ ($\rho_+(x)$), which is the analogue of the LD (HD) phase for an open uniform TASEP. For low $\alpha = \beta$ the incoming and outgoing particle fluxes are same, a single DDW should be obtained - this is due to the lack of particle number conservation in an open TASEP and uncorrelated entry and exit of particles. The envelop of the DDW can be obtained following the logic outlined in SM. For sufficiently high $\alpha$ and $\beta$, an analogue of the maximal current (MC) phase of open TASEP should be obtained, where the bulk steady state density is independent of $\alpha$ and $\beta$. Unlike the (MC) phase of an open TASEP, density here should be space-dependent due to a space-varying $q(x)$. See Ref. (25) for detailed studies on similar models.
We have thus developed a theory for ASFM in a closed system with position-dependent propulsion by studying a closed TASEP with quenched hopping rates \( q(x) \). Assuming generic smooth \( q(x) \) with a finite number of discontinuities and global minimum, independent of the precise forms for \( q(x) \), \( \rho(x) \) belong to one of the two classes - SmP where \( \rho(x) \) is continuous except for where \( q(x) \) itself is discontinuous, and ShP where \( \rho(x) \) is discontinuous (i.e., a shock is formed) at a point where \( q(x) \) is continuous. This conclusions holds even when there are several local minima, so long as there is only one global minimum. For multiple global minima of \( q(x) \) with same value lead to moving shocks. MFT developed here is sufficiently general and applicable to any smoothly varying \( q(x) \) with finite number of discontinuities; it generalizes the analyses in Ref. \[20\]. It will be interesting to extend the boundary layer method developed in Refs. \[21 \, 22\] for this problem. From the perspectives of nonequilibrium systems, these results generalize the studies in Refs. \[23 \, 24\].

Our theory may be verified in model experiments on the collective motion of active particles with light-induced activity \([33]\) in a closed narrow circular channel \([7 \, 34]\). Unidirectionality of the motion can be ensured by suppressing rotational diffusion, e.g., by choosing ellipsoidal particles with the channel width shorter than the long axis of the particle everywhere, or by using dimer particles. Propulsion speed can be tuned by applying patterned or spatially varying illumination \([33]\). Steady state densities can be measured by microscopy with image processing. While technical challenges are anticipated in setting up appropriate experimental arrangements, we hope this can realized in near future.

We have confined ourselves only to characterizing the mean steady state nonuniform densities. It will be interesting to study fluctuations around these mean densities and examine the dynamics of a tagged particle, e.g., the behavior of mean and variance of the distance traveled by it, both of which should reflect the notion of universality illustrated here in some way not understood yet. We hope our work will trigger further research work along the directions.

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[1] A. L. Hodgkin and R. D. Keynes, J. Physiol. 128, 61 (1955).
[2] A.S. Stern, H.C. Berg, Biophys. J. 105, 182 (2013).
[3] V. Ziemer, A. Seyfried and A. Schadschneider, arXiv:1602.03053
[4] J. Krug and P. A. Ferrari, J. Phys. A: Math. Gen. 29 L465 (1996); D. Chowdhury, L. Santen and A. Schadschneider, Phys. Rep. 329, 199 (2000); D. Helbing, Rev. Mod. Phys. 73, 1067 (2001).
[5] T. E. Harris, J. Appl. Probab. 2, 323 (1965); D. G. Levitt, Phys. Rev. A 8, 3050 (1973); P. A. Fedders, Phys. Rev. B 17, 40 (1978); S. Alexander and P. Pincus, Phys. Rev. B 18, 2011 (1978); R. Arratia, Ann. Probab. 11, 362 (1983); L. Lizana, T. Ambojörnsson, A. Taloni, E. Barkai, and M. A. Lomholt, Phys. Rev. E 81, 051118 (2010); A. Taloni and M. A. Lomholt, Phys. Rev. E 78, 051116 (2008); G. Gradenberg, A. Puglisi, A. Sarracino, A. Vulpiani, and D. Villamaina, Phys. Scr. 86, 058516 (2012).
[6] V. Gupta, S. S. Ninavath, A. V. McCormick, and H. T. Davis, Chem. Phys. Lett. 247, 596 (1995); K. Hahn, J. Kärger, and V. Kulka, Phys. Rev. Lett. 76, 2762 (1996); T. Meersmann, J. W. Logan, R. Simonutti, S. Caldarelli, A. Comotti, P. Sozzani, L. G. Kaiser, and A. Pines, J. Phys. Chem. A 104, 11 665 (2000); B. Lin, M. Meron, B. Cui, S. A. Rice, and H. Diamant, Phys. Rev. Lett. 94, 216001 (2005).
[7] Q.-H. Wei, C. Bechinger, and P. Leiderer, Science 287, 625 (2000); C. Lutz, M. Kollmann, and C. Bechinger, Phys. Rev. Lett. 93, 026001 (2004).
[8] L. P. Sanders et al. New J. Phys. 16, 113050 (2014).
[9] J. Toner, Y. Tu, and S. Ramaswamy, Ann. Phys. (Amsterdam) 318, 170 (2005); Sriram RMP.
[10] J. Toner, N. Guttenberg and Y. Tu, arXiv:1805.03024 (2018): J. Toner, N. Guttenberg and Y. Tu, arXiv:1805.10326 (2018).
[11] L. G. Wilson and W. C. K. Poon, Phys. Chem. Chem. Phys. 13, 10 617 (2011).
[12] T. Chou, K Mallick, and R K P Zia Rep. Prog. Phys. 74 116601 (2011).
[13] S. E. Wells, E. Hillner, R. D. Vale and A. B. Sachs, Mol. Cell. 2, 135 (1998); S. Wang, K. S. Browning and W. A. Miller, EMBO J. 16, 4107 (1997); Y. Nakamura, T. Gojobori, and T. Ikemura, Nucleic Acids Res. 28 292 (2000).
[14] T. Karzig and F. von Oppen, Phys. Rev. B 81, 045317 (2010).
[15] J. MacDonald, J. Gibbs and A. Pipkin, Biopolymers 6, 1 (1968).
[16] B. Derrida, E. Domany, D. Mukamel, J. Stat. Phys. 69, 667 (1992); B. Derrida, S. A. Janowsky, J. L. Lebowitz, E. R. Speer, J. Stat. Phys. 78, 813 (1993); B. Derrida, M.R. Evans, J. Physique I 3, 311 (1993).
[17] J. Krug, Phys. Rev. Lett. 67, 1882 (1991).
[18] R.K.P. Zia, J.J. Dong, and B. Schmittmann, J. Stat. Phys. 144, 405 (2011).
[19] B. Schmittmann and R. Zia, in Phase Transitions and Critical Phenomena, edited by C. Domb and J. Lebowitz (Academic Press, London, 1995).
[20] T. Chou, K Mallick, and R K P Zia, Rep. Prog. Phys. 74, 116601 (2011).
[21] N. Sarkar and A. Basu, Phys. Rev. E 90, 022109 (2014).
[22] J. Krug, Brazilian J. of Phys. 30, 97 (2000).
[23] A. Parmeggiani, T. Franosch, and E. Frey, Phys. Rev. Lett. 90, 086001 (2003).
[24] P. Pierobon, M. Mobilia, R. Kouyos and E. Frey, Phys. Rev. E 74, 031906 (2006).
[25] P. Greulich and A. Schadschneider, J. Stat. Mech., online at stacks.iop.org/JSTAT/2008/P04009; J. Schmidt,
V. Popkov and A. Schadschneider, *Eur. Phys. Lett.* 110, 20008 (2015).

[26] R. B. Stinchcombe and S. L. A. de Queiroz, *Phys. Rev. E* 83, 061113 (2011).

[27] S. A. Janowsky and J. L. Lebowitz, *Phys. Rev. A* 45, 618 (1992).

[28] G. Tripathy and M. Barma, *Phys. Rev. E* 58, 1911 (1998).

[29] T. Banerjee, N. Sarkar and A. Basu, *JStat. Mech.: Theory and Experiment* P01024 (2015).

[30] R. J. Harris and R. B. Stinchcombe, *Phys. Rev. E* 70, 016108 (2004).

[31] S. Mukherji and S. M. Bhattacharjee, *J. Phys. A* 38, L285 (2005); S. Mukherji and V. Mishra, *Phys. Rev. E* 74, 011116 (2006); S. Mukherji, *Phys. Rev. E* 79, 041140 (2009).

[32] S. Mukherji, *Phys. Rev. E* 97, 032130 (2018).

[33] I. Buttino et al, *J. Phys. Condens. Matter* 24, 284129 (2012).

[34] C. Bechinger et al, *Rev. Mod. Phys.* 88, 045006 (2016).

**SUPPLEMENTAL MATERIAL**

**DENSITY PROFILES FOR DELocalized DOMAIN WALLS**

We calculate here on the steady state density profiles when \( q(x) \) has two symmetrically placed global minima of same value. The system then can be considered consist of two TASEP chains of equal size \( \frac{1}{2} \), say \( T_A \) (with \( 0 < x < \frac{1}{2} \)) and \( T_B \) (with \( \frac{1}{2} < x < 1 \)), each spanning from one global minimum of \( q(x) \) to the other. While the total particle number in the ring is conserved, the number of particles in each of \( T_A \) and \( T_B \) can fluctuate. We closely follow Ref. 2 in our analysis below.

Now consider one delocalized domain wall (DDW) in each of \( T_A \) and \( T_B \). Let \( x^A_w \) and \( x^B_w \) be the instantaneous positions of the DDWs in \( T_A \) and \( T_B \), respectively and the respective heights be \( \Delta_A(x^A_w) \) and \( \Delta_B(x^B_w) \). We note here that the DDW heights are explicit functions of their positions, since the steady state density is not uniform for an arbitrary \( q(x) \).

Now, increasing the number of particles in \( T_A \) by 1 would imply shifting \( x^A_w \) by an amount \( \delta x^A_w = \frac{1}{\Delta_A} \). Similarly, decrease of a particle would mean \( \delta x^A_w = \frac{-1}{\Delta_A} \).

In order to understand why this happens, we note that the height of the domain wall (DW) at \( x^A_w \) means that \( \Delta_A \) number of excess particles are needed to fill up one lattice spacing \( = \frac{1}{2} \), or to cause one lattice spacing leftward/rightward movement of the DW (and thus, the above values of \( \delta x^A_w \)). Let us now note that there are two basic processes which can alter the number of particles individually in \( T_A \) and \( T_B \), i.e., if a particle enters \( T_A \) through its left boundary (equivalent to a particle leaving \( T_B \) through its right boundary) and vice-versa.

For the following analysis, we will focus on \( T_A \). Let \( P(x^A_w, t) \) be the probability of finding a DW at \( x^A_w \) at time \( t \). For a given \( x^A_w \), one can evaluate \( x^B_w \) at time \( t \), uniquely, using total particle number conservation. The transition rate for a particle entering \( T_A \) through the left boundary can be written as, \( W = \alpha^A_e(1 - \alpha^A_e) \), \( \delta x^A_w = \frac{1}{\Delta_A} \).

Similarly, the transition rate for the particle leaving through the right boundary is given by \( W = \beta^A_e(1 - \beta^A_e) \), \( \delta x^A_w = \frac{1}{\Delta_A} \).

Here, \( \alpha_e \) and \( \beta_e \) are effective particle entry and exit rates in \( T_A \), e.g., \( \alpha_e = \rho(x=0)q(x=0) \) etc.

With these transition rates, we can calculate the average shift or the expectation value of the change, \( \langle \delta x^A_w \rangle \), which is given by the product of the increment (with sign) and the sum of the different transition rates:

\[
\langle \delta x^A_w \rangle = \frac{1}{\Delta_A(x^A_w)} \left[ \beta^A_e(1 - \beta^A_e) - \alpha^A_e(1 - \alpha^A_e) \right] \tag{11}
\]

It should be noted here, that the domain wall itself performs, a random walk about its mean position, \( x^A_w \).

For the fixed point of the random walk, i.e., the value of \( x^A_w \) for which \( \langle \delta x^A_w \rangle = 0 \), we obtain,

\[
\beta^A_e(1 - \beta^A_e) - \alpha^A_e(1 - \alpha^A_e) = 0, \tag{12}
\]

which implies, \( \alpha^A_e = \beta^A_e \), the well known condition for formation of domain walls. Here \( \alpha^A_e \) and \( \beta^A_e \) are the effective entry and exit rates in \( T_A \). In order to calculate the steady state profiles of the DDWs, we need to study the fluctuations in the DW positions that we do below.

Using the expressions for the transition rates defined above, we can write down the Master equation for \( P(x^A_w, t) \), the probability of finding the DW at \( x^A_w \) at time \( t \).

\[
\frac{dP(x^A_w, t)}{dt} = \sum_{\delta x^A_w} \left[ P(x^A_w + \delta x^A_w, t)W(x^A_w + \delta x^A_w \rightarrow x^A_w) - P(x^A_w, t)W(x^A_w \rightarrow x^A_w + \delta x^A_w) \right] \tag{13}
\]

To proceed further, we employ Kramers-Moyal expansion \( \delta x^A_w \) of the Master equation above around \( x^A_w \), up to second order in \( \delta x^A_w \). This gives,

\[
\frac{dP(x^A_w, t)}{dt} = -\frac{\partial}{\partial y} \left[ a(y)P(y, t) \right] + \frac{1}{2} \frac{\partial^2}{\partial y^2} \left[ b(y)P(y, t) \right], \tag{14}
\]

where, \( y = \delta x^A_w \), \( a(y) = \Sigma_y W(x^A_w + \delta x^A_w \rightarrow x^A_w) \) and \( b(y) = \Sigma_y y^2 W(x^A_w + \delta x^A_w \rightarrow x^A_w) \) Using the already known values for \( W \) and \( \delta x^A_w \), and Eq. 12 we arrive at the following results for \( a \) and \( b \):

\[
a(x^A_w) = \frac{1}{L\Delta_A(x^A_w)} \left[ -\alpha^A_e(1 - \alpha^A_e) + \beta^A_e(1 - \beta^A_e) \right] = 0 \tag{15}
\]

and

\[
b(x^A_w) = \frac{1}{L^2\Delta^2_A(x^A_w)} \left[ \alpha^A_e(1 - \alpha^A_e) + \beta^A_e(1 - \beta^A_e) \right] > 0 \tag{16}
\]
Thus up to this order
\[
\frac{dP(x,t)}{dt} = \frac{1}{2} \frac{\partial^2}{\partial x^2} [b(x)P(x)].
\] (17)
Since \( \alpha_e = \beta_e \), the DW position effectively follows detailed balance condition. This means the fluctuations in the DW position should follow an equilibrium distribution in the steady state. Hence, the probability current, given by \( J_{DW}(x) = \frac{\partial}{\partial x} [b(x)P(x)] = 0 \). This yields
\[
P(x) = \frac{C}{b(x)},
\] (18)
where \( C \) is a constant which can be evaluated by the normalization condition on \( P(x) \).

Construction of the density profiles

We can now construct the density profile \( \rho(x) \) with the knowledge about \( P(x) \). Since the long time averaged steady state density involves averaging over \( P(x) \), we argue that
\[
\frac{\partial \rho}{\partial x} = AP(x),
\] (19)
where \( A \) is a constant of proportionality. Clearly from Eq. (19), we can see if \( P(x) = \text{const.} \), as in the case for a DDW in an open TASEP, \( \rho(x) \) varies linearly with \( x \), a known result. The constant \( A \) in this example can be evaluated by the boundary conditions. In yet another example, for an LDW as \( P(x) \propto \delta(x - x_w) \), \( \rho(x) \) is a heaviside \( \Theta \)-function according to Eq. (12), whose height can be determined using the boundary conditions. We now obtain the DDW steady state density profiles.

By using Eq. (19) we write
\[
\rho(x) = \tilde{A} \int \frac{dx}{b(x)} + D = A_1 \int dx(\rho(x) - \rho(x)) + D, \] (20)
where, \( \tilde{A}, A_1 \) and \( D \) are constants, and we have substituted the value of the DW height \( \Delta A(x) = \rho(x) - \rho(x) \) in \( b(x) \). Using already derived expressions (in the main text) for \( \rho(x) \) and \( \rho(x) \), we finally arrive at the following expression for the DDW profile,
\[
\rho(x) = A_1 \int dx(1 - \frac{q_{min}}{q(x)}) + D. \] (21)

The value of the constants can be fixed using the boundary conditions on \( \rho(x) \). But there is one more undetermined quantity that we are yet to address. The DDW in general has certain extent of wandering in \( T_A \) that is less than the length of the channel, depending on the number density. We can have two situations: one, where \( \rho(x) \) shows a mix of LD and DDW profiles, or where it is a mix of DDW and HD profiles. Therefore, within \( x = [0,1/2] \), we can either have a situation with an LD phase from \( x = 0 \) to say, \( x = \overline{\tau} \), followed by the DDW from \( x = \overline{\tau} \) to \( x = 1/2 \); or the situation with the DDW from \( x = 0 \) to \( x = \overline{\tau} \), followed by an HD phase from \( x = \overline{\tau} \) to \( x = 1/2 \). Now, what determines the value of \( \overline{\tau} \) is the condition \( \int_0^{1/2} \rho_A(x)dx = n \), where \( \rho_A(x) \) is the complete density profile for \( T_A \) and \( n \) is the number density (notice that \( T_A \) and \( T_B \) are identical, and both must have the same average number density). If the DDW does not span the entire \( T_A \), an additional unknown parameter \( \overline{\tau} \) must be determined, which we fix numerically by using the particle number conservation. This in turn yields the complete density profile. We use this scheme to obtain \( \rho(x) \) for \( q(x) = 1 - x \) for \( x = [0,0.5] \) and \( q(x) = 1.5 - x \) for \( x = [0.5,1] \); see Fig. 3 in the main text. Good agreement with the MCS result is clearly visible, establishing our analytical framework.

**DENSITY PROFILES**

We show below some representative plots of \( \rho(x) \) versus \( x \) from MCS studies along with MFT predictions. MCS studies are generally done with \( N = 2000 \) with random sequential updates (except for Fig. 5 in the main text, where random updates have been used for reasons of limitations on computational resources).

**Density profiles in the smooth phase**

Here we show plots of \( \rho(x) \) versus \( x \) in SmP with various choices for \( q(x) \).
Density profiles in the shock phase
FIG. 9: Plot of \( \rho(x) \) versus \( x \) in SmP for \( q(x) = 1 - x, 0 \leq x < 0.5; q(x) = 1.5 - x, 0.5 \leq x < 1 \) (purple continuous line), \( n = 0.1 \). Continuous magenta line and overlapping blue points represent, respectively, MFT and MCS data.

FIG. 10: Plot of \( \rho(x) \) versus \( x \) in SmP for \( q(x) = 1 - x, 0 \leq x < 0.5; q(x) = 1.5 - x, 0.5 \leq x < 1 \) (purple continuous line), \( n = 0.9 \). Continuous magenta line and overlapping blue points represent, respectively, MFT and MCS data.

Density profile in the shock phase with four DDWs
FIG. 11: Plot of $\rho(x)$ versus $x$ in ShP for $q(x) = (x - 0.5)^2 + 0.5$ (purple continuous line), $n = 0.6$. Continuous magenta line and overlapping blue points represent, respectively, MFT and MCS data.

FIG. 12: Plot of $\rho(x)$ versus $x$ in ShP for $q(x) = 1 - 0.5x^2$ (purple continuous line), $n = 0.5$. Continuous magenta line and overlapping blue points represent, respectively, MFT and MCS data.

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[1] N. Sarkar and A. Basu, Phys. Rev. E 90, 022109 (2014).
[2] T. Reichenbach, T. Franosch, and E. Frey, Eur. Phys. J E 27, 47 (2008).
[3] U. Täuber, Critical Dynamics, Cambridge University Press (Cambridge, 2014).
FIG. 13: MCS plot (blue dashed lines) of $\rho(x)$ versus $x$ for $q(x) = 1 - 0.5 \cos^2(4\pi x)$, $n = 0.5$, $N = 1200$. The red solid lines and the blue dashed lines represent $q(x)$ and MCS data for $\rho(x)$ respectively.

FIG. 14: Kymograph for $q(x) = 1 - 0.5 \cos^2(4\pi x)$, $n = 0.5$, $N = 600$. Existence of four DDWs is clearly visible which move without any synchronization between any two of them (see main text).