ON GLOBAL AXISYMMETRIC SOLUTIONS TO 2D COMPRESSIBLE FULL EULER EQUATIONS OF CHAPLYGIN GASES

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Abstract. For 2D compressible full Euler equations of Chaplygin gases, when the initial axisymmetric perturbation of a rest state is small, we prove that the smooth solution exists globally. Compared with the previous references, there are two different key points in this paper: both the vorticity and the variable entropy are simultaneously considered, moreover, the usual assumption on the compact support of initial perturbation is removed. Due to the appearances of the variable entropy and vorticity, the related perturbation of solution will have no decay in time, which leads to an essential difficulty in establishing the global energy estimate. Thanks to introducing a nonlinear ODE which arises from the vorticity and entropy, and considering the difference between the solutions of the resulting ODE and the full Euler equations, we can distinguish the fast decay part and non-decay part of solution to Euler equations. Based on this, by introducing some suitable weighted energies together with a class of weighted $L^\infty-L^\infty$ estimates for the solutions of 2D wave equations, we can eventually obtain the global energy estimates and further complete the proof on the global existence of smooth solution to 2D full Euler equations.

1. Introduction. In this paper we are concerned with the global existence of smooth axisymmetric solutions to 2D compressible Euler equations of Chaplygin gases with variable entropy and non-zero vorticity. The initial value problem of 2D full Euler equations is

\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0 \quad \text{(Conservation of mass)}, \\
\partial_t (\rho u) + \text{div}(\rho u \otimes u) + \nabla p &= 0 \quad \text{(Conservation of momentum)}, \\
\partial_t (\rho e + \frac{1}{2} \rho |u|^2) + \text{div}((\rho e + \frac{1}{2} \rho |u|^2 + p)u) &= 0 \quad \text{(Conservation of energy)}, \\
\rho(0, x) &= \bar{\rho} + \rho_0(x), \\
u(0, x) &= u_0(x), \\
S(0, x) &= \bar{S} + S_0(x),
\end{aligned}
\]

(1)

where $(t, x) = (t, x_1, x_2) \in \mathbb{R}^{1+2} := [0, \infty) \times \mathbb{R}^2$, $\nabla = (\partial_{x_1}, \partial_{x_2})$, and $u = (u_1, u_2)$, $\rho$, $p$, $e$, $S$ stand for the velocity, density, pressure, inner energy, entropy respectively. In addition, $\bar{\rho} > 0$ and $\bar{S}$ are constants, $\rho(0, x) > 0$, $u_0(x) = (u_0^1(x), u_0^2(x))$, and

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\[(\rho_0(x), u_0(x), S_0(x)) \in C^\infty(\mathbb{R}^2)\). Here, we point out that \(\text{supp}(\rho_0, u_0, S_0)\) does not require to be compact. Assume that both the pressure \(p = p(\rho, S)\) and internal energy \(e = e(\rho, S)\) are smooth on their arguments \((\rho, S)\), moreover, \(\partial_{\rho} p(\rho, S) > 0\) and \(\partial_{S} e(\rho, S) > 0\) for \(\rho > 0\).

For the polytropic gases (see [8]),
\[p(\rho, S) = A\rho^\gamma e^{S/c^\nu},\]  
(2)
where \(A, c^\nu\) and \(\gamma (1 < \gamma < 3)\) are some positive constants.

For the Chaplygin gases (see [8] or [11]),
\[p(\rho, S) = P_0 - B(S)\rho,\]  
(3)
where \(P_0 > 0\) is a positive constant, \(B(S) > 0\) is a smooth function of \(S\), and \(p(\rho, S) > 0\) for \(\rho > 0\). For notational convenience, we always assume \(\bar{\rho} = B(\bar{S}) = 1\) in the whole paper, where the local sound speed \(c(\rho, S) = \sqrt{\partial_{\rho} p(\rho, S)} = B(\bar{S})\rho\). Otherwise, let \(\tilde{\rho} = \rho/\bar{\rho}, \tilde{u} = u/c(\bar{\rho}, \bar{S}), \tilde{B}(S) = B(S)/B(\bar{S})\) and \(\tilde{p}(\tilde{\rho}, S) = P_0\rho/\bar{\rho} - B(S)\tilde{\rho}\) instead of \(\rho, u, B(S)\) and \(p\) in (1), respectively.

If \((\rho, u, S) \in C^1\) is a solution of (1) with \(\rho > 0\), then (1) is equivalent to the following form
\[
\begin{aligned}
\partial_t \rho + \text{div}(\rho u) &= 0, \\
\partial_t u + u \cdot \nabla u + \frac{\nabla \rho}{\rho} &= 0, \\
\partial_t S + u \cdot \nabla S &= 0,
\end{aligned}
\]  
(4)
where \(x \in \mathbb{R}\). It is easy to know that (5) is strictly hyperbolic, whose three eigenvalues and corresponding right eigenvectors are
\[\lambda_1 = u - c(\rho, S), \quad \lambda_2 = u, \quad \lambda_3 = u + c(\rho, S)\]
and
\[r_1 = (-\rho, c(\rho, S), 0)^T, \quad r_2 = (-\partial_S p(\rho, S), 0, c^2(\rho, S))^T, \quad r_3 = (\rho, c(\rho, S), 0)^T.\]

Direct computation yields
\[\nabla_{\rho, u, S} \lambda_i \cdot r_i^T = 0, \quad i = 1, 2, 3,\]
which means that (5) is totally linearly degenerate in the sense of P. D. Lax [20, page 557]. Moreover (5) will admit a global small perturbed smooth solution \((\rho, u, S)\) (see
For 2-D system in (4), if we let \((\rho, u, S)(t, x) = (\hat{\rho}, \hat{u}, \hat{S})(t, s)\) with \(s = x \cdot \omega\), \(\omega = (\omega_1, \omega_2)\) and \(\omega_1^2 + \omega_2^2 = 1\), then the system in (4) becomes

\[
\begin{cases}
\partial_t \hat{\rho} + (\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2) \partial_s \hat{\rho} + \omega_1 \partial_s \hat{u}_1 + \omega_2 \partial_s \hat{u}_2 = 0, \\
\partial_t \hat{u}_1 + (\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2) \partial_s \hat{u}_1 + \frac{\omega_1 c^2(\hat{\rho}, \hat{S})}{\hat{\rho}} \partial_s \rho + \frac{\omega_1 \partial_s p(\hat{\rho}, \hat{S})}{\hat{\rho}} \partial_s \hat{S} = 0, \\
\partial_t \hat{u}_2 + (\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2) \partial_s \hat{u}_2 + \frac{\omega_2 c^2(\hat{\rho}, \hat{S})}{\hat{\rho}} \partial_s \rho + \frac{\omega_2 \partial_s p(\hat{\rho}, \hat{S})}{\hat{\rho}} \partial_s \hat{S} = 0, \\
\partial_t \hat{S} + (\omega_1 \hat{u}_1 + \omega_2 \hat{u}_2) \partial_s \hat{S} = 0.
\end{cases}
\] (6)

It follows from direct computation that (6) has four eigenvalues
\[\hat{\lambda}_1 = \omega_1 \hat{u}_1 + \omega_2 \hat{u}_2 - c(\hat{\rho}, \hat{S}), \quad \hat{\lambda}_{2,3} = \omega_1 \hat{u}_1 + \omega_2 \hat{u}_2, \quad \hat{\lambda}_4 = \omega_1 \hat{u}_1 + \omega_2 \hat{u}_2 + c(\hat{\rho}, \hat{S})\]
and corresponding four right eigenvectors
\[\begin{align*}
\hat{r}_1 &= (-\hat{\rho}, c(\hat{\rho}, \hat{S})\omega_1, c(\hat{\rho}, \hat{S})\omega_2, 0)^T, \\
\hat{r}_2 &= (0, -\omega_2, \omega_1, 0)^T, \\
\hat{r}_3 &= (-\partial_s p(\hat{\rho}, \hat{S}), 0, 0, c^2(\hat{\rho}, \hat{S}))^T, \\
\hat{r}_4 &= (\hat{\rho}, c(\hat{\rho}, \hat{S})\omega_1, c(\hat{\rho}, \hat{S})\omega_2, 0)^T.
\end{align*}\]

It is not difficulty to verify that for any \((\omega_1, \omega_2)\) with \(\omega_1^2 + \omega_2^2 = 1\),
\[\nabla_{\hat{\rho}, \hat{u}, \hat{s}} \hat{\lambda}_i \cdot \hat{r}_i^T \equiv 0, \quad i = 1, 2, 3, 4.\]

Thus the 2D system in (4) is also totally linearly degenerate (see the definition in Page 89 of [25]). At this time, A. Majda posed the following conjecture on Page 89 of [25]:

**Conjecture 1** (A. Majda). *If the multidimensional nonlinear symmetric system is totally linearly degenerate, then it typically has smooth global solutions when the initial data are in \(H^s(\mathbb{R}^n)\) with \(s > \frac{n}{2} + 1\) unless the solution itself blows up in finite time. In particular, the shock wave formation never happens for any smooth initial data.*

By our knowledge, so far this conjecture has not been solved yet even for the small initial data. As illustrated in Page 89 of [25], the above conjecture is mainly of mathematical interest but its resolution would elucidate both the nonlinear nature of the conditions requiring linear degeneracy of each wave field and also might isolate the fashion in which the shock wave formation arises in quasilinear hyperbolic systems. In this paper, we focus on the global solution problem of A. Majda’s **Conjecture** when the initial axisymmetric perturbation in (4) is small.

Let \(\Omega := x_1 \partial_2 - x_2 \partial_1\) be the rotation operator. Set \(\Omega u_0(x) := (\Omega u_0^1(x), \Omega u_0^2(x)) + (u_0^3(x), -u_0^1(x))\). Throughout this paper, we always assume that \(\Omega p_0 = \Omega S_0 = 0\) and \(\Omega u_0 = 0\). In this case, one has \(\rho_0(x) = \rho_0(r), S_0(x) = S_0(r)\) and \(u_0(x) = f_0(r) \frac{x}{r} + g_0(r) \frac{x}{r} + x^1, \quad r = \sqrt{x_1^2 + x_2^2} \). Therefore, from (4) we know that the smooth solution \((\rho, u, S)\) will still admit such a form for any \(t \geq 0\):
\[\rho(t, x) = \rho(t, r), \quad S(t, x) = S(t, r)\]
and \(u(t, x) = f(t, r) \frac{x}{r} + g(t, r) \frac{x}{r} + x^1, \quad f, g\) stand for the radial speed and angular speed of gases, respectively. Denote by \(v := \frac{B(S)}{\rho} - 1\) (corresponding to \(-p(\rho, S)\)), \(h := B(S) - 1\) (the perturbation of entropy function \(B(S)\)) and \(v_0(r) := \frac{B(S_0(r))}{\rho_0(r)} - 1, \quad h_0(r) := B(S_0(r)) - 1\). Then
problem (4) together with the state equation of Chaplygin gases (3) admits the following equivalent form of (\(v, f, g, h\)):

\[
\begin{align*}
\partial_t v - (\partial_r f + \frac{1}{r} f) &= Q_1, \\
\partial_t f - \partial_r v &= Q_2, \\
\partial_t g + f \partial_r g + \frac{1}{r} fg &= 0, \\
\partial_t h + f \partial_r h &= 0,
\end{align*}
\]

where the nonlinearities \(Q\) are

\[
\begin{align*}
Q_1 := v(\partial_r f + \frac{1}{r} f) - f \partial_r v, \\
Q_2 := v \partial_r v - f \partial_r f - h(\partial_r f + f \partial_r f) + \frac{1 + h}{r} g^2.
\end{align*}
\]

For fixed integer \(N \geq N_0 = 11\), denote the size \(\varepsilon\) of the perturbed initial data in (4) by

\[
\varepsilon := \sum_{k=0}^{N} \| \langle r \rangle (r \partial_r)^k (\rho_0, u_0, S_0) \|_{L^2} + \sum_{k+l=0}^{N-2} \| \langle r \rangle^2 (r \partial_r)^k \nabla^l (\text{curl} u_0, S_0) \|_{L^2}
\]

\[
+ \sum_{k+l=0}^{N-1} \| \langle r \rangle (r \partial_r)^k \nabla^l (\nabla \rho_0, \text{div} u_0, \text{curl} u_0, \nabla S_0) \|_{L^2}
\]

\[
+ \sum_{k=0}^{N-4} \| \langle r \rangle^2 (r \partial_r)^k (\rho_0, f_0, S_0) \|_{L^1} + \sum_{k+l=0}^{N-4} \| \langle r \rangle^2 (r \partial_r)^k \nabla^l (\nabla \rho_0, \text{div} u_0, \nabla S_0) \|_{L^1},
\]

where \(\langle r \rangle = (1 + r^2)^{\frac{1}{2}} \geq 1\), and \(\| \phi \|_{L^p} := \left( \int_{\mathbb{R}^2} |\phi|^p dx \right)^{\frac{1}{p}}\) for \(p \geq 1\).

The main result in the paper is:

**Theorem 1.1.** There exists a constant \(\varepsilon_0 > 0\) such that if the initial data \((\rho_0(x), u_0(x), S_0(x)) \in C^\infty (\mathbb{R}^2)\) satisfy \(\Omega \rho_0(x) = \Omega S_0(x) = 0\) and \(\Omega u_0(x) = 0\), then for \(\varepsilon \leq \varepsilon_0\), problem (4) together with (3) admits a global smooth solution \((\rho(t), u(t), S(t))\) which fulfills \(\Omega \rho(t, x) = \Omega S(t, x) = 0\) and \(\Omega u(t, x) = 0\) for any fixed \(t \geq 0\).

Next, we give a brief survey on some remarkable works related to Theorem 1.1. In two or three space dimensions, it is well known that the smooth solution \((\rho, u, S)\) to problem (1) with the state equation of polytropic gases (2) will generally blow up in finite time. For examples,

- for a special class of initial data \((\rho(0, x), u(0, x), S(0, x))\), T. Sideris [27] proved that the smooth solution \((\rho, u, S)\) of (1) in two and three space dimensions can develop singularities in finite time.
- for 2D compressible isentropic Euler equations, when the rotationally invariant data are a perturbation of size \(\varepsilon > 0\) of a rest state, S. Alinhac in [1] proves that the small perturbed smooth solution blows up in finite time, moreover, the lifespan \(T_\varepsilon\) satisfies \(\lim_{\varepsilon \to 0} \varepsilon^2 T_\varepsilon = \tau_0^2 > 0\).
- for the 2D and 3D compressible Euler equations (1), there are other extensive literatures on the blowup or the blowup mechanism (including the formation of
shocks) of small perturbed smooth solution \((\rho, u, S)\), one can see [2], [6]–[7], [10], [12]–[13], [16], [24]–[26] and [28]–[30].

Up to now there are also many interesting results on the multidimensional compressible Euler equations of Chaplygin gases. For examples,

- when the Chaplygin gases are isentropic and irrotational in two or three space dimensions, (1) can be written as a second order quasilinear wave equation of the velocity potential function \(\Phi(t, x)\), where \(u = \nabla_x \Phi\). In 3D case, this quasilinear wave equation satisfies the null condition (its definition see [5] and [18]); in 2D case, the nonlinear equation verifies both the first and the second null condition (see their definitions in [3]). Consequently, according to the results in [5], [18] and [3], we know that the small perturbed smooth solution \((\rho, u)\) of (1) exists globally when 3D or 2D Chaplygin gases are isentropic and irrotational.

- when the Chaplygin gases are spherically symmetric and non-isentropic, P. Godin in [11] proved the global existence of smooth symmetric solution to the 3D non-isentropic compressible Euler equations under small perturbation of a rest state. Similar result in two space dimensions was established in [4].

- when the 3D Chaplygin gases are isentropic and irrotational, the authors in [22] proved the global existence of small perturbed smooth solutions in the exterior domain with slip boundary condition \(u \cdot \vec{n} = 0\), where \(\vec{n}\) stands for the unit outer normal of the boundary.

- when the 2D Chaplygin gases are axisymmetric and isentropic, if the rotationally invariant initial data are a compactly supported perturbation of a rest state, we show that the small perturbed smooth solution with non-zero vorticity exists globally in [14].

Next we give the comments on the proof of Theorem 1.1. When the first and the second null conditions of 2D quasilinear wave equations are both satisfied, S. Alinhac in [3] have established the global existence of small data smooth solution with compact support in the space variable by the “ghost weight” technique. Based on this, the global existence to 2D isentropic and irrotational Euler equations of Chaplygin gases (1) is a direct corollary of [3]. In addition, for the symmetric initial data, by introducing a new “ghost weight” and by utilizing the both null conditions and the variable entropy, the authors in [4] have established a global weighted energy estimate and obtained the global existence of smooth solutions for 2D full compressible Euler system under the small initial perturbation with compact support. In [14], we obtain the global smooth axisymmetric solutions to 2D compressible Euler equations of Chaplygin gases with non-zero vorticity under the small perturbation with compact support. In the present paper, for the non-isentropic and rotational case, we will study the global existence of (7) with non-zero vorticity and non-compact initial data. On the one hand, due to the strong effect of the term \((1+h)g^2\) (including the both entropy and vorticity) in \(Q_2\), the time-decay of \((v, f, g, h)\) becomes worse for problem (7). To overcome this essential difficulty, motivated by [14], we seek a suitable transformation \(v(t, r) = \tilde{v}(t, r) + G(t, r)\) so that the unknown function \(\tilde{v}\) will admit a better decay in time meanwhile the function \(G\) has some required “good” properties in the process of deriving energy estimates on \((\tilde{v}, f, g, h)\). For this purpose, we delicately choose \(G\) to satisfy the nonlinear ODE:

\[
(1 + v)\partial_r G + \frac{g^2}{r}(1 + h) = 0
\]
with $G(t, \infty) = 0$. On the other hand, all the proofs in [3], [4] and [14] heavily depend on the compactness of the initial data, namely, such a crucial Hardy type inequality $\int_{\mathbb{R}^n} \frac{|\psi(t,x)|^2}{1 + |r - x|^2} \, dx \leq C \int_{\mathbb{R}^n} |\nabla_x \psi|^2 \, dx$ with $\text{supp}_x \psi(t, x) \subset \{ x : r = |x| \leq M + t \}$ should be used. However, for the non-compact support case in Theorem 1.1, such a kind of Hardy type inequality does not hold, and we have to use other methods to overcome this essential difficulty (see Lemma 5.1). To this end, we carefully observe the following phenomena: by the transport equations of $g$ and $h$ in (7), $g$ and $h$ are expected to behave like $\langle r \rangle^{-\ell}$ with suitably large number $\ell > 0$; near the light cone $\{ r = t \}$, the flow will be “almost” irrotational and isentropic for large time since the vorticity curl $u = \partial_r g + \frac{1}{r} g$ and the perturbed entropy $h = B(S) - B(\hat{S})$ decay like $\langle t \rangle^{-\ell}$. Therefore, we can take the measure as follows: by introducing the radial velocity potential $\varphi$ (see (96) in Section 4.3), we get a nonlinear wave equation which approximately satisfies the null conditions. Through utilizing a class of weighted $L^\infty$ estimates of solution $\phi$ to the 2-D linear wave equation $\Box \phi = F(t, x)$ in [15] instead of the usual $L^2-L^\infty$ estimates used in [3] and so on (one can also see (49) which is derived by Sobolev embedding’s Theorem), we get a better time-decay rate of $\varphi$ (see (108)). Based by these, we will manage to derive the suitable global energy estimate for the solution to problem (7).

As in [14], the vector fields $\Gamma \in \{ \partial_t, X := t \partial_t + \nu \partial_r \}$ will be utilized. In addition, in order to distinguish the external and internal energy of solution to problem (7), we will introduce the auxiliary weighted energy near the light cone $\{ r = t \}$ for $k \in \mathbb{N}$

$$\mathcal{Y}_k(t) := \sum_{|a| \leq k-1} \left[ \langle r-t \rangle \chi_1 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a v(t, r) \right]_{L^2} + \langle r-t \rangle \chi_1 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a f(t, r) \right]_{L^2}$$

and away form the light cone $\{ r = t \}$

$$\mathcal{X}_k(t) := \langle t \rangle \sum_{|a| \leq k-1} \left[ \left( r - t \right) \chi_0 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a v(t, r) \right]_{L^2} + \langle r - t \rangle \chi_1 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a f(t, r) \right]_{L^2}$$

$$+ \langle t \rangle \sum_{|a| \leq k-1} \left[ \left( r - t \right) \chi_0 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a v(t, r) \right]_{L^2} + \langle r - t \rangle \chi_1 \left( \frac{r}{t} \right) \partial_r \partial_t \partial^a f(t, r) \right]_{L^2},$$

(13)

where $\langle t \rangle = (1 + t^2)^{\frac{1}{2}}$, $\chi_0(s)$ and $\chi_1(s)$ are smooth cutoff functions satisfying $0 \leq \chi_0 \leq 1$ with

$$\chi_0(s) + \chi_1(s) = 1, \quad \chi_0'(s) \leq 0, \quad \chi_0(s) = \begin{cases} 1, & s \leq \frac{1}{4}, \\ 0, & s \geq \frac{1}{2} \end{cases}$$

(14)

for $s \in \mathbb{R}$. Moreover, both $\frac{|\chi_0'|}{\chi_0}$ and $\frac{|\chi_1'|}{\chi_1}$ are bounded. Here, we should pay more attention on the possible singularity near $r = 0$ in $\mathcal{X}_k(t)$. Note that $\langle \partial_r + \frac{1}{r} \rangle \partial^a f(t, r) = \text{div} \Gamma^a u, \langle \partial_r + \frac{1}{r} \rangle \partial^a \partial_r \partial^b \hat{v}(t, r) = \Delta \Gamma^b \hat{v}$ and $\frac{1}{r} \partial_r \langle \partial_r + \frac{1}{r} \rangle \Gamma^b f(t, r) = \nabla \text{div} \Gamma^b u$, then each term in $\mathcal{X}_k(t)$ is finite for any fixed time $t \geq 0$ and for smooth solution $(v, f, g, h)$ of (7). On the other hand, in order to derive the $L^\infty$ estimate of $\partial_r g$ and $h$ by
Sobolev embedding theorem \( W^{1,3}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) instead of \( W^{2,2}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) (the latter one needs one more space derivatives), we also introduce the \( W^{1,3} \) higher order norm of \((w, h)\) as follows

\[
W_k(t) := \sum_{|\alpha| \leq k} \left( \| \nabla^\alpha w(t, r) \|_{L^2} + \| \langle r \rangle \partial_t \Gamma^\alpha w(t, r) \|_{L^2} \right)
\]

\[
+ \left( \| \nabla^\alpha h(t, r) \|_{L^2} + \| \langle r \rangle \partial_t \Gamma^\alpha h(t, r) \|_{L^2} \right),
\]

(15)

where \( w(t, r) := \frac{\text{cyl} w(t,x)}{\rho(t,r)} = (1 + v(t, r))(\partial_t + \frac{1}{r})g(t, r) \). As in [24], it is easy to know that \( w(t, r) \) satisfies

\[
\partial_t w + f\partial_r w = 0,
\]

(16)

which is the same as the equation of \( h \) in (7). Combining the preparations above with some rather involved analysis and delicate observations, the global energy estimates of \((v, f, g, h)\) can be eventually established and then the proof of Theorem 1.1 is completed by the continuation method.

This paper is organized as follows: In Section 2, we list or derive some basic results including the Sobolev-type embedding inequalities and Hardy-type inequalities. In Section 3, by the null condition structures of the nonlinearities, some good estimates for the \( L^\infty \) and \( L^2 \) norms of \( v + f \) and its derivatives near the light cone will be established for the smooth solution \((v, f, g, h)\) of (7). On the other hand, the pointwise estimates for the smooth solution \((v, f, g, h)\) of (7) and the improved decay rates of \( v + f \) near the light cone by the weighted \( L^\infty - L^\infty \) estimates are derived in Section 4. In Section 5, we obtain the estimates for the auxiliary weighted energy \( X_k(t) \) and \( Y_k(t) \) defined by (12) and (13). In Section 6, by using S. Alinhac’s “ghost weight” technique together with some other suitable multipliers, the \( L^2 \) estimates for \((v, f)\) in the energy \( E_k(t) \) are obtained, where

\[
E_k(t) := \sum_{|\alpha| \leq k} \left( \| \Gamma^\alpha v(t, r) \|_{L^2} + \| \Gamma^\alpha f(t, r) \|_{L^2} + \| \langle r \rangle \partial_t \Gamma^\alpha h(t, r) \|_{L^2} + \| \langle r \rangle \partial_r \Gamma^\alpha g(t, r) \|_{L^2} \right) + \sum_{|\alpha| \leq k} \left( \| \langle r \rangle \partial_t \Gamma^\alpha h(t, r) \|_{L^2} \right).
\]

The \( L^2 \) estimates for \((g, h)\) in \( E_k(t) \) and \( W^{1,3} \) estimates for \((w, h)\) are established in Section 7. The proof of Theorem 1.1 is finished in Section 8 by the local existence of smooth solution to problem (7) and the continuity argument method.

Through the whole paper, we shall use the following convention:

- \( \nabla = \{ \partial_{x_1}, \partial_{x_2} \} \) for \( x \in \mathbb{R}^2 \).
- \( \Gamma \in \{ \partial_t, X = t\partial_k + r\partial_r \} \) and \( \tilde{\Gamma} \in \{ \partial_t, \tilde{X} := X + 1 \} \).
- \( \chi_i \) represents \( \chi_i(\frac{x_i}{r}) \) \( (i = 1, 2) \), where \( \chi_i \) is defined in (14).
- For the multi-indices \( a, b, c \in \mathbb{N}_0^2 := \{0, 1, 2, \cdots \}^2 \), then \( \Gamma^a = \partial_t^{a_1} X^{a_2} \).
- \( a \leq b \) means \( a_1 \leq b_1 \) and \( a_2 \leq b_2 \), while \( a < b \) means \( a_1 < b_1 \) or \( a_2 < b_2 \) with \( a \leq b \).
- \( |a| = a_1 + a_2 \) and \( a! = a_1!a_2! \).
- Set \( \sigma_{ab} := \frac{(a+b)!}{a!b!} \) and \( \sigma_{abc} := \frac{(a+b+c)!}{a!b!c!} \).
- The \( L^p \) norm of scalar function \( \phi(t, x) \) \( (x \in \mathbb{R}^2, t \geq 0) \) is \( \| \phi \|_{L^p} = \| \phi(t, x) \|_{L^p} := \left( \int_{\mathbb{R}^2} |\phi(t, x)|^p dx \right)^{\frac{1}{p}} \).
- If \( \phi(t, x) \) is symmetric, i.e., \( \phi(t, x) = \phi(t, r) \), then \( \| \phi \|_{L^p} = \| \phi(t, r) \|_{L^p(\mathbb{R}^2)} = \left( \int_0^{\infty} |\phi(t, r)|^p r dr \right)^{\frac{1}{p}} \).
• For \( k \in \mathbb{N}_0 \) and the solution \((v, f, g, h)\) of (7), set
\[
E_k(t) := \sum_{|a| \leq k} \|\Gamma^a v(t, r)\|_{L^2} + \|\Gamma^a f(t, r)\|_{L^2} + \|\Gamma^a g(t, r)\|_{L^2} + \|\Gamma^a h(t, r)\|_{L^2} + \|\Gamma^a g(t, r)\|_{L^2} + \|\Gamma^a h(t, r)\|_{L^2} + \|\Gamma^a g(t, r)\|_{L^2} + \|\Gamma^a h(t, r)\|_{L^2}
\]
where \((\partial_r + \frac{1}{r})\Gamma^a g(t, r) = \text{curl} \Gamma^a u.\)

2. Some preliminaries. At first, it follows from direct computation that for any smooth functions \( \phi(t, r) \) and \( \psi(t, r) \),
\[
\begin{align*}
\tilde{X}(\phi \partial_r \psi) &= (X + 1)(\phi \partial_r \psi) = \phi \partial_r X \psi + X \phi \partial_r, \\
\tilde{X}(\frac{\phi \psi}{r}) &= \frac{1}{r} \phi X \psi + \frac{1}{r} \psi X \phi, \\
\tilde{X} \partial_r \psi &= \partial_r X \psi,
\end{align*}
\]
(18)
where \((\partial_r + \frac{1}{r})\Gamma^a g(t, r) = \text{curl} \Gamma^a u.\)

Through the paper, we always assume that for fixed integer \( N \geq N_0 = 11, \)
\[
\begin{align*}
E_N(t) + X_N(t) + Y_N(t) &\leq M \varepsilon (1 + t)^{M',} \quad M' \varepsilon_0 \leq 0.01, \\
E_{N-5}(t) + X_{N-5}(t) + Y_{N-5}(t) + W_{N-3}(t) &\leq M \varepsilon,
\end{align*}
\]
(19)
where the large positive constants \( M \) and \( M' \) will be chosen later.

**Lemma 2.1.** For any smooth function \( \phi(t, x) \), the following Sobolev inequalities hold
\[
\|\phi\|_{L^\infty} \leq C \|\nabla \phi\|_{L^2} \ln^{\frac{1}{2}}(2 + t) + (t)^{-1}(\|\phi\|_{L^2} + \|\nabla^2 \phi\|_{L^2}),
\]
(20)
\[
\|\phi\|_{L^\infty} \leq C \|\phi\|_{L^3} + \|\nabla \phi\|_{L^3},
\]
(21)
where and below \( A \leq C B_1 + B_2 \) means \( A \leq C(B_1 + B_2) \) with the generic positive constant \( C \) which is independent of \( t, \varepsilon, M \) and \( M' \).

**Proof.** For the first inequality (20), one can see (3.4) of [21]. Inequality (21) comes from the Sobolev embedding theorem \( W^{1,3}(\mathbb{R}^2) \subset L^\infty(\mathbb{R}^2) \) directly. This completes the proof of Lemma 2.1. \( \square \)

**Lemma 2.2.** For smooth function \( \phi(t, r) \) with \( \phi(t, 0) = \lim_{r \to 0^+} \phi(t, r) = 0 \), the following Hardy type inequalities hold
\[
\begin{align*}
\|\frac{\phi}{r}\|_{L^2} &\leq C \|\partial_r + \frac{1}{r}\phi\|_{L^2}, \\
\|\frac{\chi_0 \phi}{r}\|_{L^2} &\leq C \|\chi_0 \partial_r + \frac{1}{r}\phi\|_{L^2} + (t)^{-1}\|\phi\|_{L^2}, \\
\|\frac{\partial_r \phi}{r}\|_{L^2} &\leq C \|\phi\|_{L^2} + \|\partial_r + \frac{1}{r}\phi\|_{L^2},
\end{align*}
\]
(22)
(23)
(24)
where the definition of \( \chi_0 \) has been given in (14).

**Proof.** The first two inequalities (22)–(23) has been proved in [14, Lemma 2.4]. For the proof of (24), it is easy to get that
\[
\|\frac{(\partial_r)}{r}\phi\|_{L^2} \leq C \|\phi\|_{L^2} + \|\frac{1}{r}\phi\|_{L^2}.
\]
(25)
Applying (22) to the last term in (25) yields (24). This completes the proof of Lemma 2.2.

Lemma 2.3. For $a \in \mathbb{N}_0^3$, then the following weighted Sobolev type inequalities hold

\[
\langle r + t \rangle^{\frac{1}{2}}|\chi_1 |^{\Gamma^a f(t, r)}| \leq C E_{|a|}(t) + \mathcal{Y}_{|a|+1}(t), \tag{26}
\]
\[
\langle r + t \rangle^{\frac{1}{2}}|\chi_1 |^{\Gamma^a v(t, r)}| \leq C E_{|a|}(t) + \mathcal{Y}_{|a|+1}(t), \tag{27}
\]

where the definition of $\chi_1$ has been given in (14).

Proof. At first, we show

\[
\lim_{r \to \infty} \psi(t, r) = 0, \tag{28}
\]

where $\psi(t, r) = \langle r + t \rangle|\chi_1 |^{\Gamma^a f(t, r)}|^2$. Indeed, direct calculation implies that for large $r, s > 0$,

\[
|\psi(t, s) - \psi(t, r)| = \left| \int_r^s \partial_R \psi(t, R)dR \right|
\leq C \int_r^s \chi_1 |^{\Gamma^a f(t, R)}|^2 \langle (R + t) \rangle \chi_1 + \langle R - t \rangle \chi_1 + \langle R + t \rangle \langle R - t \rangle^{-1} |\chi_1 ||dR
+ \int_r^s \chi_1 |^{\Gamma^a f(t, R)}| |\partial_r \Gamma^a f(t, R) | \langle R + t \rangle \langle R - t \rangle dR
\leq C \int_r^s \| \Gamma^a f(t, R) \|^2 + \| \langle R - t \rangle \chi_1 \partial_r \Gamma^a f(t, R) \|^2 RdR \to 0 \quad \text{as} \ r, s \to \infty,
\]

where we have used the fact of $\langle (R + t) \rangle \chi_1$, $\langle R - t \rangle \chi_1$, and $\langle R + t \rangle \langle R - t \rangle^{-1} |\chi_1 |$ on supp $\chi_1$. Thus, we have proved the existence of $\lim_{r \to \infty} \psi(t, r)$.

If $\lim_{r \to \infty} \psi(t, r) \neq 0$, then there exists a positive function $C(t)$ for each $t$ such that

\[
|\Gamma^a f(t, r)|^2 \geq C(t) \int_r^\infty \| \Gamma^a f(t, R) \|^2 r dr \geq C(t) \int_1^\infty \frac{1}{r} dr = +\infty,
\]

which yields a contradiction. Thus (28) is proved.

Choosing $s = \infty$ in (29), then (26) is derived. Similarly, (27) can be proved.

Lemma 2.4. For $a, b, c \in \mathbb{N}_0^3$ and $|a| \leq N - 1, |b| \leq N - 2, |c| \leq N - 3$, if $(v, f, g, h)$ is a smooth solution of (7), then the following inequalities hold

1. $|\langle r \rangle^{\Gamma^a g(t, r)}| \leq C E_{|a|+1}(t)$.
2. $\frac{1}{r} \chi_0 \Gamma^a f(t, r) \leq C \| \chi_0 (\partial_r + \frac{1}{r}) \Gamma^a f(t, r) \|_{L^\infty}$.
3. $\langle t \rangle |\chi_0 \Gamma^b \tilde{v}(t, r)| \leq C E_{|b|}(t) + \mathcal{X}_{|b|+1}(t)$.
4. $\langle t \rangle^{0.99} |\chi_0 \Gamma^b \tilde{v}(t, r)| \leq C E_{|b|}(t) + \mathcal{X}_{|b|+1}(t) + \mathcal{Y}_{|b|+1}(t) + \| \Gamma^b G \|_{L^2} + \| \partial_r \Gamma^b \tilde{v} \|^2$, where $\tilde{v} = v - G$, and the definition of $G$ is given in (11).
5. $\langle r \rangle^{2} \Gamma^c \tilde{w}(t, r) + |\langle r \rangle \Gamma^c \tilde{h}(t, r)| \leq CW_{|c|}(t)$.

Furthermore, for any fixed $t \geq 0$, it holds that

6. $\Gamma^a h(t, 0) = \Gamma^a f(t, 0) = 0$ with $|a| \leq N - 1$ and $\partial_r \Gamma^b \tilde{v}(t, 0) = 0$ with $|b| \leq N - 2$.

Remark 1. In view of (6) in Lemma 2.4, Lemma 2.2 can be directly applied to the terms $\Gamma^a g(t, r), \Gamma^a f(t, r)$ and $\partial_r \Gamma^b \tilde{v}(t, r)$ for $|a| \leq N - 1$ and $|b| \leq N - 2$. 

Proof. At first, we prove (1). It is easy to check that
\[
|\langle r \rangle^{l} G(t, r)| = \frac{1}{r} \left| \int_{0}^{r} \partial_{r}(r' \langle r' \rangle^{l} G) \, dr' \right|
\leq \frac{1}{r} \left| \int_{0}^{r} \langle r' \rangle^{l} (\partial_{r} + \frac{1}{r'}) \Gamma^{a} \, dr' \right| + \frac{1}{r} \left| \int_{0}^{r} l \langle r' \rangle^{l-1} \Gamma^{a} \, dr' \right|
\leq \frac{1}{r} \left( \int_{0}^{r} r' \, dr' \right)^{\frac{1}{2}} \left\{ \left( \int_{0}^{r} \langle r' \rangle^{l} (\partial_{r} + \frac{1}{r'}) \Gamma^{a} \langle r' \rangle^{2} \, dr' \right)^{\frac{1}{2}} + \left( \int_{0}^{r} l \langle r' \rangle^{l-1} \Gamma^{a} \langle r' \rangle^{2} \, dr' \right)^{\frac{1}{2}} \right\},
\] (30)
then (1) is proved by choosing \( l = 1 \) in (30). In addition, the boundedness of \( \int_{0}^{\infty} |\langle r \rangle^{l} (\partial_{r} + \frac{1}{r'}) \Gamma^{a} g|^{2} \, dr \) implies that \( \int_{0}^{r} |\langle r' \rangle^{l} (\partial_{r} + \frac{1}{r'}) \Gamma^{a} g|^{2} \, dr' \to 0 \) as \( r \to 0 \).

Then we find \( \Gamma^{a} g(t, 0) = 0 \) by (1). With the same analysis as for \( l = 0 \), we also have \( \Gamma^{a} f(t, 0) = \partial_{r} \Gamma^{a} \tilde{v}(t, 0) = 0 \). Therefore, (6) is achieved.

Next, we deal with (2). Similarly to the proof of (1), we see that
\[
r \Gamma^{a} f(t, r) = \int_{0}^{r} \partial_{r}(r \Gamma^{a} f) \, dr' = \int_{0}^{r} (\partial_{r} + \frac{1}{r'}) \Gamma^{a} f(t, r') \, dr'.
\]
In addition, by \( \chi'_{0} \leq 0 \) in (14), we know that \( \chi_{0} \left( \frac{r}{t} \right) \leq \chi_{0} \left( \frac{r'}{t'} \right) \) holds for \( r' \leq r \).

Therefore,\[
\left| \frac{1}{r} \chi_{0} \Gamma^{a} f(t, r) \right| \leq \frac{1}{r^{2}} \int_{0}^{r} \chi_{0} \left( \partial_{r} + \frac{1}{r'} \right) \Gamma^{a} f \, r' \, dr' \leq \frac{1}{r^{2}} \int_{0}^{r} \partial_{r} \chi_{0} \left( \partial_{r} + \frac{1}{r'} \right) \Gamma^{a} f \, r' \, dr',
\]
which yields (2).

The proof of (3)–(4) is the same as in Lemma 2.5 of [14], here we omit it.

Finally, we turn our attention to (5). According to (21), we achieve
\[
\left| \langle r \rangle^{2} \Gamma^{a} h(t, r) \right| \leq C \left\| \langle r \rangle^{2} \Gamma^{a} h(t, r) \right\|_{L^{2}} + \left\| \partial_{r} \{ \langle r \rangle^{2} \Gamma^{a} h(t, r) \} \right\|_{L^{2}} \leq C W_{|a|}(t).
\] (31)
If \( h \) is replaced by \( w \), then (31) is also true. This completes the proof of Lemma 2.4.

Next we focus on the estimates of \( G \) in (11).

**Lemma 2.5.** If \((v, g, h)\) is a smooth solution of (7), let \( G(t, r) \) be defined in (11) and suppose that (19) holds. Then for \( a', a, b \in \mathbb{N}^{3}_{0} \) with \(|a'| \leq N\), \(|a| \leq N - 1 \) and \(|b| \leq N - 3\), the following inequalities hold

1. \( \lim_{r \to \infty} r^{2} \Gamma^{a'} G(t, r) = 0 \) for any fixed \( t \geq 0 \);
2. \( \langle r \rangle^{2} \Gamma^{a} G(t, r) \rangle \leq C \int_{0}^{\infty} \langle r \rangle^{2} \partial_{r} \Gamma^{a} G \, dr \leq C E_{|a|+1}(t) \);
3. \( \frac{1}{r} \Gamma^{a} g(t, r) \rangle \leq C \langle \langle \partial_{r} + \frac{1}{r} \rangle \Gamma^{b} g(t, r) \rangle \leq C E_{|b|+1}(t) + W_{|b|}(t) \);
4. \( \langle r \rangle^{3} \Gamma^{a} G(t, r) \rangle \leq C E_{|a|+1}(t) \);
5. \( \langle r \rangle^{2} \Gamma^{a} G(t, r) \rangle \leq C \langle \langle r \rangle^{2} \partial_{r} \Gamma^{a} G \rangle \leq C \langle \langle r \rangle^{2} \partial_{r} \Gamma^{a'} G \rangle \leq C E_{|a'|+1}(t) \).

**Proof.** Proof of (1): At first, we show that \( \lim_{r \to \infty} r^{2} \partial_{r}^{a} G(t, r) = 0 \) for any integer \( a_{1} \leq N \). Indeed, according to the definition of \( G \) and \( \lim_{r \to \infty} G(t, r) = 0 \), we arrive at
\[
G(t, r) = \int_{r}^{\infty} \frac{g^{2}(1 + h)}{1 + v} (t, r') \, dr' \, dr.
\] (32)
Applying $\partial_t^{a_1}$ on both sides of (32) yields
\[ |r^2 \partial_t^{a_1} G(t, r)| = \left| r^2 \int_r^\infty \partial_t^{a_1} \left( \frac{g^2(1+h)}{1+v} \right) (t, r') \frac{dr'}{r'} \right| \leq C \sum_{i+j+l+m=a_1} \mathcal{J}_1^{ijlm}(r), \] (33)
where
\[ \mathcal{J}_1^{ijlm}(r) := \int_r^\infty \left| \partial_t^i g \partial_t^j g \partial_t^l (1+h) \partial_t^m \left( \frac{1}{1+v} \right) \right| r' dr'. \]

For fixed $t \geq 0$ and $r \geq (t)$, by applying Cauchy-Schwartz inequality we have
\[ \sum_{i>N-3} \mathcal{J}_1^{ijlm}(r) \leq C \sum_{i>N-3} \left\| \partial_t^i g \partial_t^j g \left( \frac{1}{1+v} \right) \right\|_{L^\infty} \int_r^\infty (|\partial_t^m g|^2 + |\partial_t^m h|^2) r' dr' \] (34)
and
\[ \sum_{i \leq N-3} \mathcal{J}_1^{ijlm}(r) \leq C \sum_{i \leq N-3} (|\partial_t^i (1+h)||_{L^\infty} \left\{ \|g\|_{L^\infty} \int_r^\infty \left[ g^2 + \chi_1 \left| \partial_t^{a_1} \left( 1 + \frac{1}{v} \right) \right|^2 \right] r' dr' \right. \] \[ \left. + \int_r^\infty (|\partial_t^i g|^2 + |\partial_t^i h|^2) r' dr' \sum_{m < a_1} \left\| \partial_t^m \left( \frac{1}{1+v} \right) \chi_1 \right\|_{L^\infty} \right), \] (35)
where we have used the fact of $\chi \left( \frac{r}{(t)} \right) = 1$ for $r' \geq r \geq (t)$. From Lemma 2.3–2.4, we know that the $L^\infty$ norms on the right hand side of (34)–(35) are bounded for fixed $t \geq 0$. Hence, the boundedness of the integrations $\int_r^\infty \cdots dr'$ on the right hand side of (34)–(35) yields that $\lim_{r \to \infty} r^2 \partial_t^{a_1} G(t, r) = 0$.

Now we prove (1) for all $a' \in \mathbb{N}^2$ with $|a'| \leq N$ by induction method on $a_2$ and therefore assume that $\lim_{r \to \infty} r^2 \partial_t^{a_1} X^{a_2} G(t, r) = 0$ holds for any $a_1 + a_2 \leq N$ and fixed $t \geq 0$. Note that for $a'_1 + a_2 \leq N - 1$,
\[ r^2 \partial_t^{a'_1} X X^{a_2} G(t, r) \]
\[ = r^2 (t \partial_t + a'_1) \partial_t^{a'_1} X X^{a_2} G(t, r) + r^3 \partial_t^{a'_1} (1 + X)^{a_2} \partial_r G(t, r) \]
\[ = r^2 (t \partial_t + a'_1) \partial_t^{a'_1} X X^{a_2} G(t, r) - r^3 \partial_t^{a'_1} (1 + X)^{a_2} \left[ \frac{g^2(1+h)}{r(1+v)} \right] (t, r) \]
\[ =: J_1^1(r) + J_1^2(r). \]

According to the induction assumption, we have $\lim_{r \to \infty} J_1^1(r) = 0$. By the same treatments as in (34)–(35), we obtain the boundedness of the integration $\int_{(t)}^\infty |J_1^2(r)| dr$, and further derive $\lim_{r \to \infty} J_1^2(r) = 0$ as in (28). Therefore, we have proved $\lim_{r \to \infty} r^2 \partial_t^{a'_1} X X^{a_2} G(t, r) = 0$. Then (1) holds for all $|a'| \leq N$.

Proof of (2) and (3): According to (1), it holds that $\Gamma^a G(t, r) = \int_r^\infty \partial_t \Gamma^a G(t, r') dr'$, which yields the first inequality in (2). Next, we focus on the proof for the second inequality in (2). Applying the commutation identities
(18), we get the following equation for the higher order derivatives of $G$

$$(1 + v)\partial_t \Gamma^a G + \Gamma^a v \partial_v G + \sum_{c + d = a, c < a, d < a} \sigma_{cd}(\Gamma^c \tilde{v} + \Gamma^c G) \partial_c \Gamma^d G$$

$$= -\frac{1}{r} \sum_{c + d + e = a} \sigma_{cde} \Gamma^c g \Gamma^d g \Gamma^e (1 + h).$$

(36)

Thereafter, we arrive at

$$(1 - \|\chi_0\|_{L^\infty} - \|\chi_1 v\|_{L^\infty} - \|G\|_{L^\infty}) \int_0^\infty \langle r \rangle^2 |\partial_t \Gamma^a G|dr$$

$$\leq C \int_r^\infty \langle r \rangle^2 |\Gamma^a v \partial_v G|dr + \sum_{c + d + e = a, c < a, d < a} \int_0^\infty \langle r \rangle^2 |\Gamma^c g \Gamma^d g \Gamma^e (1 + h)| \frac{dr}{r}$$

$$+ \sum_{c + d + e = a, c < a, d < a} (\|\chi_0 \Gamma^c \tilde{v}\|_{L^\infty} + \|\chi_1 \Gamma^c v\|_{L^\infty} + \|\Gamma^c G\|_{L^\infty}) \int_0^\infty \langle r \rangle^2 |\partial_c \Gamma^d G|dr$$

for $|a| \neq 0$, and

$$(1 - \|\chi_0 \tilde{v}\|_{L^\infty} - \|\chi_1 v\|_{L^\infty} - \|G\|_{L^\infty}) \int_0^\infty \langle r \rangle^2 |\partial_t G|dr$$

$$\leq C (1 + \|h\|_{L^\infty}) \int_0^\infty \frac{\langle r \rangle^2 g^2}{r} dr.$$

(37)

Now, we deal with each term in the second line of (37). By the definition of $G$ and Cauchy-Schwartz inequality, we achieve

$$\int_r^\infty \langle r \rangle^2 |\Gamma^a v \partial_v G|dr \leq C \left( \|\chi_0 v\|_{L^\infty} \right) \int_0^\infty \langle r \rangle^2 \left( \|\Gamma^a v\|^2 + \frac{\langle r \rangle^2 g^2}{r^2} \right) dr$$

$$\leq C (1 + \|h\|_{L^\infty}) (\|g\|_{L^\infty} + \|\tilde{v} g \|_{L^\infty}) \int_0^\infty \langle r \rangle^2 \left( \|\Gamma^a v\|^2 + g^2 + |\partial_v g + \frac{1}{r} g|^2 \right) dr,$$

(39)

where we have used (24) in the last inequality.

In addition, set $J_{2de} := \int_0^\infty \langle r \rangle^2 |\Gamma^c g \Gamma^d g \Gamma^e (1 + h)| \frac{dr}{r}$. Then, by Cauchy-Schwartz inequality and Hardy inequality (24) again, we have

$$\sum_{c + d + e = a, |e| \leq N - 3} J_{2de}^c \leq C \sum_{c + d + e = a, |e| \leq N - 3} \|\Gamma^e (1 + h)\|_{L^\infty} \int_0^\infty \frac{\langle r \rangle^2 \Gamma^d g)^2}{r} dr$$

$$\leq C \sum_{c + d + e = a, |e| \leq N - 3} \left[ 1 + W_{|e|}(t) \right] \int_0^\infty \left\{ |\Gamma^c g|^2 + |\Gamma^d g|^2 \right\} \frac{r dr}{r}$$

$$+ \left( |\partial_v + \frac{1}{r} \right) \Gamma^c g \right|^2 + \left( |\partial_v + \frac{1}{r} \right) \Gamma^d g \right|^2 \} r dr$$

(40)

and

$$\sum_{c + d + e = a, |e| > N - 3} J_{2de}^c \leq C \sum_{c + d + e = a, |e| > N - 3} \frac{\langle r \rangle^2 g^2}{r^2} \int_0^\infty \left\{ \frac{\langle r \rangle^2 \Gamma^d g)^2}{r^2} + |\Gamma^e h|^2 \right\} r dr$$

$$+ \left( |\partial_v + \right) \Gamma^c g \right|^2 + \left( |\partial_v + \frac{1}{r} \right) \Gamma^d g \right|^2 \} r dr.$$
where ˜ the proof of (2) in Lemma 2.4, we achieve (19) we get which also yields the first inequality in (3).

In addition, it is easier to prove (2) for |a| ≤ N − 5. Therefore, for all |a| ≤ N − 1,
The proof of (4) is similar to that for (2). Indeed, by taking the
\[ \text{Collecting (37)-(46), we obtain (2) and (3) for sufficiently small } \varepsilon_0 > 0. \]

Proof of (4): The proof of (4) is similar to that for (2). Indeed, by taking the
\[ \text{Proof of (5): We conclude from (1) that } \]
\[ \| \langle r \rangle^3 \partial_r \Gamma^e G \|_{L^\infty} \leq C \sum_{c+d=b, \quad d<b} \| \langle r \rangle^3 \partial_r \Gamma^d G \|_{L^\infty} (E_{|b|+1}(t) + M \varepsilon) \]
\[ + \sum_{c+d+e=b} \| \frac{1}{r} \langle r \rangle^3 \Gamma^c g \Gamma^d g \Gamma^e (1 + h) \|_{L^\infty}, \]

where
\[ \sum_{c+d+e=b} \| \frac{1}{r} \langle r \rangle^3 \Gamma^c g \Gamma^d g \Gamma^e (1 + h) \|_{L^\infty} \leq C \sum_{c+d+e=b} \| \Gamma^c (1 + h) \|_{L^\infty} \| \frac{1}{r} \Gamma^c g \Gamma^d g \|_{L^\infty} + \| \langle r \rangle^2 \Gamma^c g \Gamma^d g \|_{L^\infty} \]
\[ \leq C E_{|b|+1}(t) [E_{M+1}(t) + W_{1/2}(t)] \]
\[ \leq C E_{|b|+1}(t). \]

Consequently, (4) is derived for small \( \varepsilon > 0 \).

Proof of (5): We conclude from (1) that
\[ \| \langle r \rangle \Gamma^a G \|^2_{L^2} = \int_0^\infty |\Gamma^a G|^2 (1 + r^2) dr = \int_0^\infty |\Gamma^a G|^2 d\left(\frac{r^2}{2} + \frac{r^4}{4}\right) \]
\[ = \left| \int \Gamma^a G \partial_r \Gamma^c G r^2 (1 + \frac{1}{2} r^2) dr \right| \leq C \| \langle r \rangle \Gamma^a G \|_{L^2} \| \langle r \rangle r \partial_r \Gamma^a G \|_{L^2}. \]

Note that for \( c + d = a' \) with \( |a'| \leq N \) and \( N \geq N_0 = 11 \), then \( |c| \leq N - 6 \) or \( |d| \leq N - 6 \) holds. This, together with (36), yields that
\[ \| \langle r \rangle \Gamma^a G \|_{L^2} \leq \| \langle r \rangle \partial_r \Gamma^a G \|_{L^2} \leq C \| \langle r \rangle^2 \partial_r \Gamma^a G \|_{L^2} \]
\[ \leq C \sum_{c+d=a'} \| \Gamma^c G \|_{L^2} \| \langle r \rangle \partial_r \Gamma^d G \|_{L^2} + \sum_{c+d+e=a'} \| \Gamma^c G \|_{L^2} \| \langle r \rangle^2 \partial_r \Gamma^d G \|_{L^2} \]
\[ \leq C \sum_{c+d=a', \quad |c| \leq N-6} (\| \Gamma^c G \|_{L^\infty} + M \varepsilon) \| \langle r \rangle^2 \partial_r \Gamma^d G \|_{L^2} + \sum_{c+d=a', \quad |d| \leq N-6} \| \Gamma^c G \|_{L^2} \| \langle r \rangle^2 \partial_r \Gamma^d G \|_{L^\infty}. \]
we have

\[ \text{Lemma 3.1.} \quad \text{For} \quad a, a' \in \mathbb{N}_0^2 \quad \text{with} \quad |a| \leq N - 3 \quad \text{and} \quad |a'| \leq N - 1, \quad \text{if} \quad (v, f, g, h) \quad \text{is a smooth solution of} \quad (7), \quad \text{then the following estimates for the} \quad L^\infty \quad \text{and} \quad L^2 \quad \text{norms of} \quad v + f \quad \text{hold}
\]

\[ \langle r + t \rangle^\frac{1}{2} \langle r - t \rangle^\frac{1}{2} |\chi_1 \partial_t \Gamma^a(v + f)| \leq C E_{|a|+1}(t) + 3 |a|+2(t), \quad (47) \]

\[ \| \langle r + t \rangle \chi_1 \partial_t \Gamma^a(v + f) \|_{L^2} \leq CE_{|a'|+1}(t), \quad (48) \]

\[ \langle r + t \rangle |\chi_1 \Gamma^a(v + f)| \leq CE_{|a'|+1}(t). \quad (49) \]

**Proof.** Acting \( \tilde{\Gamma}^a \) on two sides of the first two equations in (7) and applying (18), we have

\[ \begin{align*}
  \partial_t \Gamma^a v - (\partial_t \Gamma^a f + \frac{1}{r} \Gamma^a f) &= \tilde{\Gamma}^a Q_1, \\
  \partial_t \Gamma^a f - \partial_t \Gamma^a v &= \tilde{\Gamma}^a Q_2. 
\end{align*} \quad (50) \]

Direct computation shows that

\[ (t + r) \partial_t \Gamma^a(v + f) = t \partial_t \Gamma^a(v + f) + X \Gamma^a(v + f) - t \partial_t \Gamma^a(v + f) \]

\[ = -t \tilde{\Gamma}^a(Q_1 + Q_2) + X \Gamma^a v + X \Gamma^a f - \frac{t}{r} \Gamma^a f, \quad (51) \]

where

\[ \tilde{\Gamma}^a(Q_1 + Q_2) = \sum_{b+c=a} \sigma_{bc} [ \tilde{\Gamma}^b(v - f) \partial_t \Gamma^c(v + f) + \frac{1}{r} \tilde{\Gamma}^b v \Gamma^c f - \tilde{\Gamma}^b \partial_t \Gamma^c f ] 
\]

\[ + \sum_{b+c+d=a} \sigma_{bcd} [ \frac{1}{r} \tilde{\Gamma}^b g \Gamma^c \Gamma^d (1 + h) - \tilde{\Gamma}^b h \Gamma^c f \partial_t \Gamma^d f]. \quad (52) \]

Next we treat each term in the expression of (52). By \( \chi_1 = \chi_1(\chi_0 + \chi_1) \) and \( v = \tilde{v} + G \), we arrive at

\[ \langle r + t \rangle^\frac{1}{2} \langle r - t \rangle^\frac{1}{2} \sum_{b+c=a} |\chi_1 \frac{1}{r} \tilde{\Gamma}^b v \Gamma^c f| \]

\[ \leq C \langle r + t \rangle^\frac{1}{2} \langle r - t \rangle^\frac{1}{2} \sum_{b+c=a} \frac{1}{|b| \leq |c|} \left| (\chi_0 \frac{r}{(r)}^\frac{2}{2} G + \chi_0 \tilde{v} + \chi_1 \tilde{v}) \chi_1 \tilde{\Gamma}^c f \right|, \]
have easier. Indeed, by applying (5) in Lemma 2.4 to $\Gamma_{b}$, we will repeatedly make full use of the scaling operator $N$.

This, together with (47), yields (48).

Analogously, we find that $|\chi_1 \partial_r \Gamma^c(v + f)|$ in the first line of (52) is much easier. Indeed, by applying (5) in Lemma 2.4 to $\Gamma^b$ with $|b| \leq |a| \leq N - 3$, we have

$$|\chi_1 \Gamma^b \partial_t \Gamma^c f| \leq C(r + t)^{-\frac{3}{2}}(r + t)^2 \Gamma^b h \|\chi_1 \Gamma^c f\|$$

(55)

The term $\frac{1}{r} \Gamma^b g \Gamma^c g \Gamma^d(1 + h)$ in the second line of (52) can be analogously treated as in (53):

$$|\chi_1 \frac{1}{r} \Gamma^b g \Gamma^c g \Gamma^d(1 + h)| \leq C(r + t)^{-\frac{3}{2}}(r + t)^2 \Gamma^b g \|\Gamma^d(1 + h)\|$$

(56)

To overcome the loss of the space regularity in the term $\Gamma^b h \Gamma^c f \partial_t \Gamma^d f$ since $\partial_t$ is not a “good derivative”, we will repeatedly make full use of the scaling operator as follows

$$r \partial_r \Gamma^d = X \Gamma^d - t \partial_t \Gamma^d.$$ 

(57)

Based on this, we get

$$\langle r + t \rangle^3 \langle r - t \rangle |\chi_1 \Gamma^b h \Gamma^c f \partial_t \Gamma^d f| \leq C(r + t)^3 \langle r - t \rangle |\chi_1 \Gamma^b h \Gamma^c f \Gamma^d f|$$

$$\leq C(r + t) \langle r - t \rangle (r + t)^2 \Gamma^b h \|\chi_1 \Gamma^c f\| [\|\chi_0 \Gamma^c f\| + |\chi_1 \Gamma^c f\|]$$

(58)

Consequently, substituting (53)–(58) into (51) yields (47).

The proof of (48) is similar. Indeed, from $|b| + |c| \leq |a| \leq N - 1$, one has $|b| \leq N - 2$ or $|c| \leq N - 3$. Therefore, we see that

$$\|\langle r + t \rangle \chi_1 \Gamma^b(v + f) \partial_r \Gamma^c(v + f)\|_{L^2}$$

$$\leq C \sum_{b + c = a', \atop |c| \leq N - 3} \|\langle r + t \rangle \chi_1 \partial_r \Gamma^c(v + f)\|_{L^\infty} \|\Gamma^b(v - f)\|_{L^2}$$

$$+ \langle t \rangle^{-\frac{1}{2}} \sum_{b + c = a', \atop |b| \leq N - 2} \|\chi_1 \partial_r \Gamma^c(v + f)\|_{L^2} \left[|E|_{b+1}(t) + |Y|_{|b|+2}(t) + |Y|_{|b|+1}(t)\right].$$

This, together with (47), yields (48).
Finally, by direct computation and Cauchy-Schwartz inequality, we achieve

\[ |⟨r' + t⟩χ1Γa'(v + f)(t, r')|^2 = \left| \int_{r'}^{\infty} \partial_r |⟨r + t⟩χ1Γa'(v + f)(t, r)|^2dr \right| \]

\[ \leq C \int_{r'}^{\infty} |χ1Γa'(v + f)(t, r)|^2rdr + \int_{r'}^{\infty} |χ1χ'_1||Γa'(v + f)(t, r)|^2rdr \]

\[ + \int_{r'}^{\infty} χ1^2|Γa'(v + f)(t, r)||⟨r + t⟩∂rΓa'(v + f)(t, r)|rdr \]

\[ \leq C \int_{r'}^{\infty} |Γa'(v + f)(t, r)|^2rdr + \int_{r'}^{\infty} |⟨r + t⟩χ1∂rΓa'(v + f)(t, r)|^2rdr, \]

which implies (49). This completes the proof of Lemma 3.1.

\[ \square \]

4. The pointwise estimates of \((v, f, g, h)\).

4.1. The pointwise estimates of \((v, f, g, h)\) away from the light cone. In this subsection, the pointwise estimates away from the light cone for the space or time derivatives of smooth solution \((v, f, g, h)\) will be achieved through the structure of the hyperbolic system (7).

**Lemma 4.1.** For \(a, a' \in \mathbb{N}_0^2\) with \(|a| \leq N - 3\) and \(|a'| \leq N - 4\), if \((v, f, g, h)\) is a smooth solution of (7), then for small \(\varepsilon_0 > 0\), the following inequalities hold

\[ |χ0(\partial_r + \frac{1}{r})Γa f(t, r)| + |χ0\partial_r Γa f(t, r)| + |χ0\partial_t Γa v(t, r)| + |χ0\partial_t Γa v(t, r)| \]

\[ + |⟨r⟩∂rΓa' G(t, r)| \leq C(r + t)^{-1.99}[E_{|a|+2}(t) + X_{|a|+3}(t) + Y_{|a|+2}(t)], \] (59)

\[ |⟨r⟩∂rΓa' h(t, r)| + |⟨r⟩∂rΓa' w(t, r)| + |∂tΓa g(t, r)| \]

\[ \leq C(r + t)^{-1.99}[E_{|a|+2}(t) + X_{|a|+3}(t) + Y_{|a|+2}(t)], \] (60)

\[ |⟨r⟩∂r∂tΓa' G| \leq C(r + t)^{-1.99}[E_{|a'|+2}(t) + X_{|a'|+3}(t) + Y_{|a'|+2}(t)]. \] (61)

**Proof.** Proof of (59): At first, it is easy to know that \((\tilde{v}, f)\) satisfies

\[ \begin{align*}
\partial_r \tilde{v} - (\partial_r f + \frac{1}{r} f) &= \tilde{Q}_1, \\
\partial_t f - \partial_r \tilde{v} &= \tilde{Q}_2,
\end{align*} \] (62)

where

\[ \tilde{Q}_1 := (\tilde{v} + G)(\partial_r f + \frac{1}{r} f) - f\partial_r \tilde{v} - \partial_t G - f\partial_r G, \]

\[ \tilde{Q}_2 := (\tilde{v} + G)\partial_r \tilde{v} - (1 + h)f\partial_r f - h\partial_tf. \] (63)

According to (18), we have

\[ \begin{align*}
\partial_t Γa \tilde{v} - (\partial_r Γa f + \frac{1}{r} Γa f) &= \tilde{Γ}a \tilde{Q}_1, \\
\partial_t Γa f - \partial_r Γa \tilde{v} &= \tilde{Γ}a \tilde{Q}_2; \\
\partial_t Γa g &= -\sum_{b+c=a} σ_{bc} Γb f\partial_r + \frac{1}{r} Γc g, \\
\partial_t Γa h &= -\sum_{b+c=a} σ_{bc} Γb f\partial_r Γc h. \end{align*} \] (65)
Direct computation yields
\[
(t^2 - r^2) \partial_t \Gamma^a \dot{v} = t^2 (\partial_t \Gamma^a f - \tilde{\Gamma}^a \tilde{Q}_2) - r X \Gamma^a \dot{v} + \text{tr} \partial_t \Gamma^a \dot{v} \\
= t X \Gamma^a f - \text{tr} \partial_t \Gamma^a f - t^2 \tilde{\Gamma}^a \tilde{Q}_2 - r X \Gamma^a \dot{v} + \text{tr} \partial_t \Gamma^a \dot{v} \\
= \text{tr} \tilde{\Gamma}^a \dot{Q}_1 - t^2 \tilde{\Gamma}^a \tilde{Q}_2 + t X \Gamma^a f - r X \Gamma^a \dot{v} + t \Gamma^a f.
\]  \hspace{1cm} (66)

Similarly, we also have
\[
(t^2 - r^2) \partial_t \Gamma^a \dot{v} = - r^2 \tilde{\Gamma}^a \tilde{Q}_1 + t r \tilde{\Gamma}^a \tilde{Q}_2 + t X \Gamma^a \dot{v} - r X \Gamma^a f - r \Gamma^a f,
\]  \hspace{1cm} (67)

Applying the Leibniz’s formula and (18), we achieve
\[
\tilde{\Gamma}^a \dot{Q}_1 = \sum_{b+c=a} \sigma_{bc} [\Gamma^c (\tilde{\dot{v}} + G) \partial_t + \frac{1}{r} \Gamma^c f - \Gamma^b f \partial_t \Gamma^c \dot{v} - \Gamma^b f \partial_t \Gamma^c G] - \partial_t \Gamma^a G
\]  \hspace{1cm} (68)

and
\[
\tilde{\Gamma}^a \dot{Q}_2 = \sum_{b+c=a} \sigma_{bc} [\Gamma^c (\tilde{\dot{v}} + G) \partial_t \Gamma^b - \Gamma^b \partial_t \Gamma^c f] \\
- \sum_{b+c+d=a} \sigma_{bcd} \Gamma^b f \partial_t \Gamma^c f \Gamma^d (1 + h).
\]  \hspace{1cm} (69)

We now deal with the term \( \partial_t \Gamma^a G \) in (68). Taking the time-derivative \( \partial_t \) on equation (11) yields that
\[
(1 + v) \partial_t \partial_t G + (\partial_t \tilde{\dot{v}} + \partial_t G) \partial_t G \\
= - \frac{1}{r} \partial_t [g^2 (1 + h)] = \frac{1}{r} \frac{d}{dr} [2(\partial_t g + \frac{1}{r} g)(1 + h) + g \partial_t h], \tag{70}
\]

where we have used the equations of \( g \) and \( h \) in (7). For \( a = (a_1, a_2) \in \mathbb{N}_0^2 \), by (18), acting \( \partial_t^a (X + 2)^{a_2} \) on (70) yields
\[
(1 + v) \partial_t \partial_t \Gamma^a G \\
= - \sum_{b+c=a, \ c < a} \sigma_{bc} (\Gamma^b \tilde{\dot{v}} + \Gamma^b G) \partial_t \partial_t \Gamma^c G - \sum_{b+c=a} \sigma_{bc} (\partial_t \Gamma^b \tilde{\dot{v}} + \partial_t \Gamma^b G) \partial_t \Gamma^c G \\
- \frac{1}{r} \sum_{b+c+d+e=a} \frac{a!}{b! c! d! e!} \Gamma^b f \Gamma^c g [2 \Gamma^d (1 + h)(\partial_t + \frac{1}{r}) \Gamma^e g + \Gamma^d g \partial_t \Gamma^e h]. \tag{71}
\]

Similarly to (37), we obtain
\[
(1 - \| \chi_0 \tilde{v} \|_{L^\infty} - \| \chi_1 v \|_{L^\infty} - \| G \|_{L^\infty}) \int_0^\infty \langle r + t \rangle^{1.99} \langle r \rangle \langle \partial_t \partial_t \Gamma^a G \rangle dr \\
\leq C \sum_{b+c=a, \ c < a} (\| \chi_0 \Gamma^b \tilde{\dot{v}} \|_{L^\infty} + \| \chi_1 \Gamma^b v \|_{L^\infty} + \| \Gamma^b G \|_{L^\infty}) \\
\times \int_0^\infty \langle r + t \rangle^{1.99} \langle r \rangle \langle \partial_t \partial_t \Gamma^a G \rangle dr.
\]
Γ

= 0. In addition, from inequality (72) and Lemma 2.5, we see that

\[ \| \langle r \rangle^a \chi_1 \partial_r \Gamma^b v \partial_r \Gamma^c G \|_{L^\infty} \leq C(t)^{-\frac{3}{2}} \| \langle r \rangle^{\frac{3}{2}} \chi_1 \Gamma^b v \|_{L^\infty} \| \langle r \rangle^2 \partial_r \partial_r \Gamma^c G \|_{L^\infty} \leq C(t)^{-\frac{3}{2}} E_{|c|+1}(t) [E_{|b|+1}(t) + \mathcal{Y}_{|b|+2}(t)]. \]

Applying (24) to the last integration in (72), we arrive at

\[ \int_0^\infty \langle r \rangle^2 |\Gamma^c g| \left\{ (\partial_r + \frac{1}{r} \Gamma^c g) + |r \partial_r \Gamma^c h| \right\} dr \leq C \int_0^\infty \frac{|\langle r \rangle |\Gamma^c g|^2}{r} dr + \int_0^\infty \left\{ |\langle r \rangle (\partial_r + \frac{1}{r} \Gamma^c g)|^2 + |\langle r \rangle r \partial_r \Gamma^c h|^2 \right\} dr \leq C \int_0^\infty \left\{ |\Gamma^c g|^2 + |(\partial_r + \frac{1}{r} \Gamma^c g)|^2 + |\langle r \rangle (\partial_r + \frac{1}{r} \Gamma^c g)|^2 + |\langle r \rangle r \partial_r \Gamma^c h|^2 \right\} r dr. \]

For the last term in the last line of (74), we conclude from $|c| \leq |a| \leq N - 3$, $r\partial_r = X - t\partial_t$ and the fourth equation in (65) that

\[ \int_0^\infty |\langle r \rangle r \partial_r \Gamma^c h|^2 dr \leq C \int_0^\infty \left\{ |\langle r \rangle X \Gamma^c h|^2 + \sum_{\epsilon' \leq \epsilon} |\langle r \rangle r \partial_r \Gamma^{\epsilon'} h|^2 |\chi_1 \Gamma^c f|^2 + |\langle r \rangle X \Gamma^c f|^2 \right\} dr \leq C \int_0^\infty \left\{ |\langle r \rangle X \Gamma^c h|^2 + M \varepsilon \sum_{\epsilon' \leq \epsilon} |\langle r \rangle r \partial_r \Gamma^{\epsilon'} h|^2 \right\} r dr. \]

We now treat the other nonlinearities in (68)-(69). It suffices only to deal with $\Gamma^b f \partial_r \Gamma^c G$, $\Gamma^c (v + G) \partial_r \Gamma^b v$ and $\Gamma^b h \partial_r \Gamma^b f$ since the treatments on the other left terms in (68)-(69) are similar or much easier. For the lower order case $|a| \leq N - 6$, by using assumption (19) directly, we obtain

\[ \langle r \rangle^1 \chi_0 \Gamma^b f \partial_r \Gamma^c G + |\chi_0 \Gamma^c (v + G) \partial_r \Gamma^b v| \leq C \langle r \rangle^1 \frac{1}{r} \chi_0 \Gamma^b f |r \partial_r \Gamma^c G| + |\chi_0 \partial_r \Gamma^b v| \times |\chi_1 \Gamma^c v|_{L^\infty} + |\chi_1 \Gamma^c G|_{L^\infty} \times |\chi_0 \Gamma^c v|_{L^\infty} + |\chi_0 \partial_r \Gamma^b v| \]

\[ \leq C \langle r \rangle^1 \chi_0 \Gamma^b f |r \partial_r \Gamma^c G| + |\chi_0 \partial_r \Gamma^b v| \times |\chi_1 \Gamma^c v|_{L^\infty} + |\chi_1 \Gamma^c G|_{L^\infty} \times |\chi_0 \Gamma^c v|_{L^\infty} + |\chi_0 \partial_r \Gamma^b v| \]

\[ \leq C M \varepsilon (t)^1 \chi_0 (\partial_r + \frac{1}{r} \Gamma^b f |L^\infty + |\chi_0 \partial_r \Gamma^b v| + M \varepsilon (t)^1.5 |\chi_0 \partial_r \Gamma^b v|. \]
Right now, we assume (59) holds for the lower order case $|a| \leq N - 6$. While for the higher order case $|a| \leq N - 3$, it always holds that $|b| \leq N - 8$ or $|c| \leq N - 6$. Then, we see that

\[
\langle r + t \rangle^{1.99} \sum_{b+c \leq a} \| \chi_0 \Gamma^b f \partial_r \Gamma^c G \| + \| \chi_0 \Gamma^c(\hat{v} + G) \partial_r \Gamma^b \hat{v} \| \\
\leq C \langle t \rangle^{1.99} \sum_{b+c \leq a} E_{|c|+1}(t) \| \chi_0 (\partial_r + \frac{1}{r}) \Gamma^b f \|_{L^\infty} + \| \chi_0 \partial_r \Gamma^b \hat{v} \| \\
+ \langle t \rangle^{1.49} \sum_{b+c \leq a} |\chi_0 \partial_r \Gamma^b \hat{v}| \sum_{|c| \leq N-6} E_{|c|}(t) + X_{|c|+2}(t) + Y_{|c|+1}(t) \\
\leq C \langle t \rangle^{1.99} \sum_{b+c \leq a, \ |b| \leq N-8} E_{|c|+1}(t) \| \chi_0 (\partial_r + \frac{1}{r}) \Gamma^b f \|_{L^\infty} + \| \chi_0 \partial_r \Gamma^b \hat{v} \| \\
+ M \varepsilon \langle t \rangle^{1.49+M^{'}) \sum_{b \leq a} \| \chi_0 \partial_r \Gamma^b \hat{v} \| \\
\leq C M \varepsilon \sum_{c \leq a} E_{|c|+1}(t) + M \varepsilon \langle t \rangle^{1.99} \sum_{b \leq a} \| \chi_0 (\partial_r + \frac{1}{r}) \Gamma^b f \|_{L^\infty} + \| \chi_0 \partial_r \Gamma^b \hat{v} \|. 
\]

Finally, we deal with $\Gamma^b \hat{v} \partial_r \Gamma^c f$. Since $|b| \leq |a| \leq N - 3$, applying (5) in Lemma 2.4 and (19) directly yields

\[
|\chi_0 \Gamma^b \hat{v} \partial_r \Gamma^c f| \leq C W_{|a|}(t) |\chi_0 \partial_r \Gamma^c f| \leq C W_{N-3}(t) |\chi_0 \partial_r \Gamma^c f| \leq C M \varepsilon |\chi_0 \partial_r \Gamma^c f|. 
\]

Collecting (66)–(69), (72)–(78) and Lemma 2.3–2.5 with the induction method, we get (59).

Proof of (60): At first, we start to prove (60). For this purpose, it follows from the fourth equation in (65) that

\[
|\partial_s \Gamma^a h| \leq C \sum_{b+c=a} |\Gamma^b f \partial_r \Gamma^c h|. 
\]

Then, by Lemma 2.3–2.4, (57) and (59) we conclude that

\[
|\langle r + t \rangle^{1.99} \langle \rangle \partial_s \Gamma^a h| \\
\leq C \langle r + t \rangle^{1.99} \langle \rangle \sum_{b+c=a'} |\frac{1}{r} \Gamma^b f \partial_r \Gamma^c h| \\
= C \langle r + t \rangle^{1.99} \langle \rangle \sum_{b+c=a'} |\frac{1}{r} \Gamma^b f (X - t \partial_r) \Gamma^c h| \\
\leq C t \sum_{b+c=a'} |\langle r + t \rangle^{1.99} \langle \rangle \partial_r \Gamma^a h| [\frac{1}{r} |\chi_0 \Gamma^b f| + |\frac{1}{r} \chi_1 \Gamma^b f|] \\
+ \sum_{b+c=a'} |\langle \rangle^2 X \Gamma^a h| [\langle \rangle^{1.99} \frac{1}{r} |\chi_0 \Gamma^b f| + |\chi_1 \Gamma^b f|] \\
\leq C M \varepsilon \sum_{c \leq a'} |\langle r + t \rangle^{1.99} \langle \rangle \partial_r \Gamma^c h| \\
+ \sum_{b+c=a'} W_{|c|+1}(t) [E_{|b|+2}(t) + X_{|b|+3}(t) + Y_{|b|+2}(t)].
\]
Combining this with the smallness of $\varepsilon_0 > 0$, one has

$$\langle r + t \rangle^{1.99} \| \partial_t G^a \| \leq C \sum_{b+c=a'} W_{|a'|+1} \left[ |E_{|a'|+2}(t) + X_{|a'|+3}(t) + Y_{|a'|+2}(t)| \right]$$

$$\leq C E_{|a'|+2}(t) + X_{|a'|+3}(t) + Y_{|a'|+2}(t),$$

where we have used the fact of $|a'| \leq N - 4$. Analogously, we can get the estimate of $w$ in (60) since equation (16) is the same as for $h$. By the analogous but much easier treatment, we can achieve the estimate of $|\partial_t \Gamma^a g(t, r)|$ in (60).

Proof of (61): Finally, we take the estimate of $|\langle r \rangle \partial_r \partial_t \Gamma^a G(t, r)|$. Acting $\partial_t$ on both sides of (36) yields

$$(1 + v) \partial_r \partial_t \Gamma^a G = - \sum_{b+c=a', \ c \leq a'} \sigma_{bc}(\Gamma^b \tilde{v} + \Gamma^b G) \partial_r \partial_t \Gamma^c G - \sum_{b+c=a'} \sigma_{bc}(\partial_r \Gamma^b \tilde{v} + \partial_r \Gamma^b G) \partial_r \Gamma^c G$$

$$- \frac{1}{r} \sum_{b+c+d=a'} \sigma_{bcd} \Gamma^b g \left[ 2 \partial_t \Gamma^c g \partial_t \Gamma^d (1+ h) + \Gamma^c g \partial_t \Gamma^d h \right].$$

From (79), one has

$$|\langle r \rangle \partial_r \partial_t \Gamma^a G| \leq C \sum_{b+c=a', \ c \leq a'} |\langle r \rangle \partial_r \partial_t \Gamma^c G| \left[ |\chi_0 \Gamma^b \tilde{v}| + |\chi_1 \Gamma^b v| + |\Gamma^b G| \right]$$

$$+ \sum_{b+c=a'} |\langle r \rangle^3 \partial_r \Gamma^c G| \left[ |\chi_0 \partial_t \Gamma^b \tilde{v}| + |\chi_1 \langle r \rangle^{-2} \partial_t \Gamma^b v| + |\partial_t \Gamma^b G| \right]$$

$$+ \sum_{b+c+d=a'} |\langle r \rangle \frac{1}{r} \Gamma^b g| \left[ |\partial_r \Gamma^c g \partial_t \Gamma^d (1+ h)| + |\Gamma^c g \partial_t \Gamma^d h| \right].$$

This, together with (59), (60) and Lemma 2.3–2.5, yields

$$|\langle r \rangle \partial_r \partial_t \Gamma^a G| \leq C \langle r + t \rangle^{-1.99} \left[ |E_{|a'|+2}(t) + X_{|a'|+3}(t) + Y_{|a'|+2}(t)| \right].$$

Therefore, we finish the proof of (61). Based on these, Lemma 4.1 is proved.

4.2. The pointwise estimates of $(v, f)$ near the light cone.

Lemma 4.2. For $a \in \mathbb{N}_0^3$ with $|a| \leq N - 3$, if $(v, f, g, h)$ is a smooth solution of (7), then for small $\varepsilon_0 > 0$, the following estimates of weighted $L^\infty$ norms hold

$$(r + t)^{\frac{1}{2}} \langle r - t \rangle^{\frac{3}{2}} \left[ |\chi_0 \partial_r \Gamma^a f(t, r)| + |\chi_1 \partial_r \Gamma^a f(t, r)| \right] \leq C \left[ E_{|a|+1}(t) + Y_{|a|+2}(t) \right].$$

(80)

$$(r + t)^{\frac{1}{2}} \langle r - t \rangle^{\frac{3}{2}} \left[ |\chi_0 \partial_t \Gamma^a v(t, r)| + |\chi_1 \partial_t \Gamma^a v(t, r)| \right] \leq C \left[ E_{|a|+1}(t) + Y_{|a|+2}(t) \right].$$

(81)

Proof. Similarly to (66)–(67), we achieve

$$(t^2 - r^2) \partial_r \Gamma^a v = tr \Gamma^a Q_1 - t \tilde{\Gamma}^a Q_2 + tX \Gamma^a f - rX \Gamma^a v + t \Gamma^a f,$$

$$(t^2 - r^2) \partial_r \Gamma^a v = -r \tilde{\Gamma}^a Q_1 + tr \tilde{\Gamma}^a Q_2 + tX \Gamma^a v - rX \Gamma^a f - r \Gamma^a f,$$

$$(t^2 - r^2) \partial_r \Gamma^a f = tr \tilde{\Gamma}^a Q_1 - t \tilde{\Gamma}^a Q_2 - rX \Gamma^a v + tX \Gamma^a f + t \Gamma^a f,$$

$$(t^2 - r^2) \partial_r \Gamma^a f = -t \tilde{\Gamma}^a Q_1 + tr \tilde{\Gamma}^a Q_2 + tX \Gamma^a v - rX \Gamma^a f - \frac{t^2}{r} \Gamma^a f.$$
By Leibniz’s formula and (18), we have
\[ \tilde{\Gamma}^a Q_1 = \sum_{b+c=a} \sigma_{bc} [\Gamma^b(v + f)\partial_r \Gamma^c f - \Gamma^b f \partial_r \Gamma^c(v + f) + \frac{1}{r} \Gamma^b v \Gamma^c f] \] (83)
and
\[ \tilde{\Gamma}^a Q_2 = \sum_{b+c=a} \sigma_{bc} [\Gamma^b v \partial_r \Gamma^c(v + f) - \Gamma^b(v + f) \partial_r \Gamma^c f - \Gamma^b h \partial_r \Gamma^c f] \]
\[ + \sum_{b+c+d=a} \sigma_{bcd} \frac{1}{r} \Gamma^b g \Gamma^c g \Gamma^d f \partial_r \Gamma^d f]. \] (84)

By Lemma 2.3–2.5, 3.1, 4.1 and an analogous analysis as in Lemma 3.1, we have such estimates for the following terms in (83)-(84):

\[ \Gamma^b(v + f)\partial_r \Gamma^c f: \]
\[ |(r - t)^{\frac{3}{2}} \chi_1 \Gamma^b(v + f)\partial_r \Gamma^c f| \leq C \chi_1 \Gamma^b(v + f)|[(r - t)^{\frac{1}{2}} \chi_1 \partial_r \Gamma^c f] + |(r - t)^{\frac{3}{2}} \chi_0 \partial_r \Gamma^c f]| \]
\[ \leq C \lambda^2 r^{-1} E_{[b]+1}(t)|[(r - t)^{\frac{1}{2}} \chi_1 \partial_r \Gamma^c f] + |(r - t)^{\frac{3}{2}} \chi_0 \partial_r \Gamma^c f]| + \lambda \chi_1 \Gamma^c f] \] (85)

\[ \frac{1}{r} \Gamma^b v \Gamma^c f: \]
\[ |(r - t)^{\frac{3}{2}} \chi_1 \Gamma^b v \Gamma^c f| \leq C \lambda^2 r^{-1} |(r - t)^{\frac{1}{2}} \chi_1 \Gamma^b v| |[(r - t)^{\frac{1}{2}} \chi_0 v | + |(r - t)^{\frac{3}{2}} \chi_0 \Gamma^c f]| \]
\[ \leq C \lambda^2 r^{-1} [E_{[b]+1}(t)|E_{[c]}(t) + \lambda |[E_{[c]}(t) + \lambda \chi_1 \Gamma^c f]| \] (86)

\[ \Gamma^b v \partial_r \Gamma^c f: \]
\[ |(r - t)^{\frac{3}{2}} \chi_1 \Gamma^b v \partial_r \Gamma^c(f + f)| \leq C |(r - t)^{\frac{1}{2}} \chi_1 \Gamma^b v| |(r - t)^{\frac{3}{2}} \chi_0 \Gamma^c v| + |(r - t)^{\frac{3}{2}} \chi_0 \Gamma^c f| \]
\[ \leq C \lambda^2 r^{-1} [E_{[b]+1}(t)|E_{[c]}(t) + \lambda |[E_{[c]}(t) + \lambda \chi_1 \Gamma^c f]| \] (87)

\[ \Gamma^b h \partial_r \Gamma^c f: \]
\[ |(r - t)^{\frac{3}{2}} \chi_1 \Gamma^b h \partial_r \Gamma^c f| \leq C |(r - t)^{\frac{1}{2}} \chi_1 \Gamma^b h| |(r - t)^{\frac{3}{2}} \chi_0 \partial_r \Gamma^c f| \]
\[ \leq C \lambda^2 r^{-1} 2E_{[b]}(t)|E_{[c]}(t) + \lambda |[E_{[c]}(t) + \lambda \chi_1 \Gamma^c f]| \] (88)

\[ \Gamma^b h \Gamma^c f \partial_r \Gamma^d f: \]
\[ |(r - t)^{\frac{3}{2}} \chi_1 \Gamma^b h \Gamma^c f \partial_r \Gamma^d f| \leq C |(r - t)^{\frac{1}{2}} \chi_1 \Gamma^b h| |(r - t)^{\frac{1}{2}} \chi_0 \Gamma^c f| + |\chi_1 \Gamma^c f| \]
\[ \leq C \lambda^2 r^{-1} 2E_{[b]}(t)|E_{[c]}(t) + \lambda |[E_{[c]}(t) + \lambda \chi_1 \Gamma^c f]| |(r - t)^{\frac{3}{2}} \chi_1 \partial_r \Gamma^d f|. \] (89)

Substituting (56) and (83)–(89) into (82) yields (80)–(81). This completes the proof of Lemma 4.2.
4.3. Better decay rates of \( v + f \) near the light cone. At first, we consider the following 2D linear Cauchy problem

\[
\begin{cases}
\Box \phi = F(t, x), \\
\phi(0, x) = \phi_0(x), \quad \partial_t \phi(0, x) = \phi_1(x),
\end{cases}
\]

where \( x \in \mathbb{R}^2 \), and \( \Box = \partial_t^2 - \partial_x^2 - \partial_y^2 \). By Poisson formula, we have

\[
\phi(t, x) = \phi_{hom}(t, x) + \phi_{inh}(t, x),
\]

where

\[
\phi_{hom}(t, x) = \phi_{0, hom}(t, x) + \phi_{1, hom}(t, x)
\]

\[
= \frac{1}{2\pi} \frac{\partial}{\partial t} \int_{|x-y| \leq t} \frac{\phi_0(y) dy}{\sqrt{t^2 - |x-y|^2}} + \frac{1}{2\pi} \int_{|x-y| \leq t} \frac{\phi_1(y) dy}{\sqrt{t^2 - |x-y|^2}}
\]

and

\[
\phi_{inh}(t, x) = \frac{1}{2\pi} \int_0^t \int_{|x-y| \leq t-t'} \frac{F(t', y) dy dt'}{\sqrt{(t-t')^2 - |x-y|^2}}.
\]

We then have the following results (see Section 3 of [15]):

**Lemma 4.3.** Let \( \phi_{hom}(t, x) \) be defined by (92), then

\[
\langle |x| + t \rangle^{\frac{1}{2}} \langle |x| - t \rangle^{\frac{1}{2}} \| \phi_{hom}(t, x) \| \leq C \| \langle |y| \rangle \phi_0(y) \|_{W^{2,1}} + \| \langle |y| \rangle \phi_1(y) \|_{W^{1,1}},
\]

where \( \| \cdot \|_{W^{k,p}} \) is the standard Sobolev norm.

**Lemma 4.4.** Let \( \phi_{inh}(t, x) \) be defined by (93). For any \( \ell_1 \in [0, \frac{1}{2}) \), \( \ell_2 \in (0, \frac{1}{2}) \) and \( \kappa > 0 \), then the following weighted \( L^\infty - L^\infty \) estimates hold

\[
\langle |x| + t \rangle^{\frac{1}{2} - \ell_1} \langle |x| - t \rangle^{\ell_2} \| \phi_{inh}(t, x) \| \leq C \sup_{(t', y) \in \mathbb{R}^{1+2}} \{ \langle |y| \rangle^{\frac{1}{2} (|y| + t')}^{1-\ell_1 + \ell_2} \langle |y| - t' \rangle^{1+\kappa} |F(t', y)| \}.
\]

**Remark 2.** For the case of \( \ell_1 = 0 \) in (95), it has been shown in Theorem 1.1 of [19]. For \( p \in \left( \frac{3+\sqrt{17}}{2}, 4 \right) \), R. Glassey [9] has proved (95) with \( \ell_1 = 0 \), \( \ell_2 = \frac{p-3}{2} \) and \( \kappa = \frac{p^2-3p-2}{4} > 0 \).

In order to apply Lemma 4.3 and Lemma 4.4, we need to find out the inherent wave equation from (7). To this purpose, we introduce the velocity potential function:

\[
\varphi(t, r) = - \int_r^\infty f(t', r') dr'.
\]

Here, we point out that \( \varphi(t, r) \) is well defined (see Remark 3 below). In this case, we have

**Lemma 4.5.** If \( (v, f, g, h) \) is a smooth solution of (7), then \( \varphi \) fulfills the following wave equation

\[
\begin{cases}
\Box \varphi = F(t, r), \\
\varphi(0, r) = \varphi_0(r), \quad \partial_t \varphi(0, r) = \varphi_1(r),
\end{cases}
\]

where the nonlinearity \( F \) is

\[
F := \partial_t H + 2(\tilde{v} + f)(\partial_t f + \frac{1}{r} f) - 2f[(1 + \tilde{v}) \partial_r (\tilde{v} + f) + \frac{1}{r} f] + (\tilde{v} + f)^2 \partial_r f
\]

\[
+ \frac{1}{r} f \tilde{v}^2 + (1 + \tilde{v})[G(\partial_r f + \frac{1}{r} f) - \partial_t G - f \partial_r G] - \frac{G - h(1 + \tilde{v})}{1 + h} f \partial_r \tilde{v},
\]
and $H$ is defined below by (104). In addition,
\[
\varphi_0(r) := -\int_r^\infty f_0(r')dr',
\]
(99)
\[
\varphi_1(r) := \frac{1}{2} \left[ \frac{B(S_0(r))}{\rho_0^2(r)} - 1 - f_0^2(r) \right] - \int_r^\infty \left\{ \frac{\partial_r B(S_0(r'))}{2\rho_0^2(r')} + \frac{g_0^2(r')}{r'} \right\} dr'.
\]
(100)

Proof. At first, we seek the initial data of $\varphi(t, r)$. Obviously, (99) holds if we set $t = 0$ in (96). In addition, from the second equation in (7), we arrive at
\[
\partial_t = 0 \text{ in (96).}
\]
In addition, from the second equation in (7), we arrive at
\[
\partial_t \varphi = -\frac{v}{\rho} \partial_r \varphi + \frac{1}{\rho^2} f^2 - \left( 1 + \frac{\rho}{2} \right) \frac{\rho}{h} \partial_r \varphi - G \partial_r v = 0.
\]
(101)

Integrating (101) with respect to the variable $r$ yields
\[
\partial_t \varphi(t, r) = \frac{1}{2} \left[ \frac{B(S(t, r))}{\rho^2(t, r)} - 1 - f^2(t, r) \right] - \int_r^\infty \left\{ \frac{\partial_r B(S(t, r'))}{2\rho^2(t, r')} + \frac{g^2(t, r')}{r'} \right\} dr',
\]
(102)
where we have used the definitions of $v = \frac{B(S)}{\rho} - 1$ and $h = B(S) - 1$. Let $t = 0$ in (102). Then (100) is derived.

Next, we look for the equation which is fulfilled by $\varphi(t, r)$. According to the second equation in (62), we achieve
\[
\partial_t (\partial_r \varphi + \frac{1}{2} f^2 - \frac{1}{2} v^2) + h \partial_r f + \rho \partial_r f - G \partial_r v = 0.
\]
(103)

Let $\mathcal{H}(t, r) := \int_r^\infty (h \partial_r f + \rho \partial_r f - G \partial_r v)(t, r')dr'$ with $\mathcal{H}(t, \infty) = 0$ and
\[
\partial_t \mathcal{H} + h \partial_r f + \rho \partial_r f - G \partial_r v = 0.
\]
(104)

Then integrating (103) with respect to the variable $r$ yields
\[
\partial_t \varphi + \frac{1}{2} f^2 - \mathcal{H} = \frac{1}{2} \frac{v}{r^2}.
\]
(105)

By acting $\partial_t$ on both sides of (105) and using the first equation in (62), we get
\[
\partial^2_t \varphi + \rho \partial_t f - \partial_t \mathcal{H} = (1 + \frac{\rho}{r}) \partial_r \varphi
\]
\[
= (1 + \frac{\rho}{r}) \left[ (1 + \frac{\rho}{r} + G) (\partial_r f + \frac{1}{r} f) - f \partial_r v - \partial_r G - f \partial_r G \right]
\]
\[
= (\partial_r + \frac{1}{r} \partial_r f + (2v + v^2)(\partial_r f + \frac{1}{r} f) - f (1 + v) \partial_r v
\]
\[
+ (1 + v) [G (\partial_r f + \frac{1}{r} f) - \partial_r G - f \partial_r G].
\]
(106)

On the other hand, by the second equation in (62) again, we find that
\[
\partial_t f = - f \partial_r f + \frac{1 + \rho + G}{1 + h} \partial_r v = - f \partial_r f + (1 + v) \partial_r v + \frac{G - h (1 + v)}{1 + h} \partial_r v.
\]
(107)

Substituting (107) into (106) yields
\[
\partial^2_t \varphi - (\partial_r + \frac{1}{r} \partial_r f = \mathcal{F},
\]
where $\mathcal{F}$ is defined by (98). Thus we complete the proof of Lemma 4.5. □
Lemma 4.6. Under condition (10) of Theorem 1.1, for \(a \in \mathbb{N}_0^d\) with \(|a| \leq N - 5\), if \((v, f, g, h)\) is a smooth solution of (7), there exists a positive constant \(C\) which is independent of \(t, \varepsilon, M\) and \(M'\) such that

\[
|\chi_1^\Gamma^a(v + f)(t, r)| \leq C\varepsilon (r + t)^{-\frac{d}{2}}(r - t)^{\frac{1}{2}}. \tag{108}
\]

Proof. According to (105), we deduce that

\[
(1 + \frac{1}{2} \tilde{v} - \frac{1}{2} f)(\tilde{v} + f) = (\partial_t + \partial_r)\varphi - \mathcal{H}. \tag{109}
\]

Applying \(\Gamma^a\) on both sides of (109) yields

\[
(1 + \frac{1}{2} \tilde{v} - \frac{1}{2} f)^\Gamma^a(\tilde{v} + f) = \frac{1}{2} \sum_{b+c=a\text{, }c \leq a} \Gamma^b(f - \tilde{v})^\Gamma^c(\tilde{v} + f) + \sum_{b \leq a} C^b_c(\partial_t + \partial_r)\Gamma^b\varphi - \Gamma^a\mathcal{H}, \tag{110}
\]

where \(C^b_c\) are some real constants that only depend on the multi-indices \(a\) and \(b\).

In addition, it is easy to check that

\[
(1 + t + r)(\partial_t + \partial_r)\Gamma^b\varphi = (1 + 2X)^\Gamma^b\varphi + (t - r)\partial_t\Gamma^b\varphi + (r - t)\partial_r\Gamma^b\varphi
\]

\[
= (1 + 2X)^\Gamma^b\varphi + \sum_{c \leq b} C^b_c(t - r)\Gamma^c\partial_t\varphi + \sum_{c \leq b} C^b_c(r - t)\Gamma^c\partial_r\varphi
\]

\[
= (1 + 2X)^\Gamma^b\varphi + \sum_{c \leq b} C^b_c(r - t)\Gamma^c f
\]

\[
+ \sum_{c \leq b} C^b_c(r - t)\Gamma^c(\tilde{v} + \frac{1}{2} \tilde{v}^2 - \frac{1}{2} f^2 + \mathcal{H}). \tag{111}
\]

Substituting (111) into (110) derives

\[
(1 - \frac{1}{2} \|\tilde{v} - f\|_{L^\infty})|\chi_1^\Gamma^a(\tilde{v} + f)|
\]

\[
\leq C \sum_{b+c=a\text{, }c \leq a} |\Gamma^b(\tilde{v} - f)| |\chi_1^\Gamma^c(\tilde{v} + f)| + \sum_{b \leq a} |\chi_1^\Gamma^b\mathcal{H}|
\]

\[
+ (r + t)^{-1} \sum_{b \leq a, \text{ }|c| \leq 1} \left\{|\chi_1^\Gamma^c\Gamma^b\varphi| + (r - t)|\chi_1^\Gamma^b f| + |\chi_1^\Gamma^b(\tilde{v} + \frac{1}{2} \tilde{v}^2 - \frac{1}{2} f^2)|\right\}. \tag{112}
\]

On the other hand, acting \(\tilde{\Gamma}^a\) on both sides of (104) yields

\[
\partial_r \Gamma^a\mathcal{H} = \sum_{b+c=a} \sigma_{bc}[\Gamma^b G \partial_r \Gamma^c \tilde{v} - \Gamma^b h \partial_t \Gamma^c f] - \sum_{b+c+d=a} \sigma_{bcd} \Gamma^b h \Gamma^c f \partial_r \Gamma^d f. \tag{113}
\]

By the similar analysis as in Lemma 2.5, we can prove \(\lim_{r \to \infty} \Gamma^a\mathcal{H}(t, r) = 0\). Consequently, we conclude from (113) that

\[
|\langle r \rangle^{\frac{1}{2}} \Gamma^a\mathcal{H}(t, r)| \leq C \int_r^\infty \langle r' \rangle^{\frac{1}{2}} \left\{ \sum_{b+c=a} |\Gamma^b G \partial_r \Gamma^c \tilde{v}| + |\Gamma^b h \partial_t \Gamma^c f| \right\} dr'. \tag{114}
\]
Next we only deal with $\Gamma^b h \partial_r \Gamma^e \tilde{v}$ add $\Gamma^b h \partial_r \Gamma^f f$ in (114) since the other term $\Gamma^b h \Gamma^c f \partial_r \Gamma^d f$ is more easily treated. Applying Lemma 2.5 and 4.1–4.2 yields that

$$\int_r^\infty \langle r' \rangle^{\frac{1}{2}} |\Gamma^b h \partial_r \Gamma^e \tilde{v}| dr' \leq C \int_r^\infty \langle r' \rangle^{\frac{1}{2}} \left[ \|\chi_0 \Gamma^b G \partial_r \Gamma^e \tilde{v}\| + |\chi_1 \Gamma^b G \partial_r \Gamma^e \tilde{v}| \right] dr'$$

$$\leq C\langle r + t \rangle^{-1.99} \|\langle r' \rangle^{1.99} \chi_0 \Gamma^e \tilde{v}\| L^\infty \int_r^\infty \langle r' \rangle^{-\frac{3}{2}} dr'$$

$$\leq C\langle r + t \rangle^{-1.99} E_{[b]+1}(t)[E_{[c]+2}(t) + \mathcal{X}_{[c]+3}(t) + \mathcal{Y}_{[c]+2}(t)] + \langle r + t \rangle^{-2} E_{[b]+1}(t)[E_{[c]+1}(t) + \mathcal{Y}_{[c]+1}(t)] \int_r^\infty \langle r' - t \rangle^{-\frac{3}{2}} dr'$$

$$\leq C\langle r + t \rangle^{-1.99} E_{[b]+1}(t)[E_{[c]+2}(t) + \mathcal{X}_{[c]+3}(t) + \mathcal{Y}_{[c]+2}(t)]$$

$$\leq CM^2 \varepsilon^2 \langle r + t \rangle^{M_\varepsilon-1.99} .$$

Analogously, through replacing $E_{[b]+1}(t)$ by $W_{[b]}(t)$ in the last line of (115), we arrive at

$$\int_r^\infty \langle r' \rangle^{\frac{1}{2}} |\Gamma^b h \partial_r \Gamma^e f| dr'$$

$$\leq C\langle r + t \rangle^{-1.99} W_{[b]}(t)[E_{[c]+2}(t) + \mathcal{X}_{[c]+3}(t) + \mathcal{Y}_{[c]+2}(t)]$$

$$\leq CM^2 \varepsilon^2 \langle r + t \rangle^{M_\varepsilon-1.99} .$$

Therefore, by substituting (115) and (116) into (114), we obtain

$$\| \langle r \rangle^{\frac{1}{2}} \Gamma^e \mathcal{H}(t, r) \| \leq CM^2 \varepsilon^2 \langle r + t \rangle^{M_\varepsilon-1.99} .$$

In addition, from the proofs of (115) and (116), we know that (117) also holds for $|a| \leq N - 3$. Collecting (112) with (117) and Lemma 2.3–2.5 implies

$$|\chi_1 \Gamma^a (\tilde{v} + f)| \leq C M_\varepsilon \sum_{c < a} |\chi_1 \Gamma^c (\tilde{v} + f)| + M^2 \varepsilon^2 \langle r + t \rangle^{M_\varepsilon-1.99}$$

$$+ \langle r + t \rangle^{-\frac{3}{2}} \langle r - t \rangle^{\frac{1}{2}} [E_{[a]}(t) + \mathcal{Y}_{[a]+1}(t)] + \langle r + t \rangle^{-1} \sum_{b \leq a, |c| \leq 1} |\chi_1 \Gamma^c \Gamma^b \varphi| .$$

Finally, we deal with the last term $|\chi_1 \Gamma^b \Gamma^e \varphi|$ in (118). Let $\Gamma^a' = \Gamma^a$ and we turn to the pointwise estimate for the top-order term $\Gamma^a \varphi$ in (118). According to (18) and Lemma 4.5, it is easy to find that $\Gamma^a \varphi$ satisfies:

$$\Box \Gamma^a \varphi = \partial_t \partial^i (X + 2) a^i \Box \varphi = \partial_t \partial^i (X + 2) a^i \mathcal{F} =: \mathcal{F}^{a'}$$

$$= \sum_{b + c \leq a'} C_{b}^{c} \left[ 2 \Gamma^b (\tilde{v} + f) (\partial_r + \frac{1}{r}) \Gamma^c f - \frac{2}{r} \Gamma^b f \Gamma^c f - \frac{1}{r} \Gamma^b f \Gamma^c \tilde{v} \right]$$

$$+ \sum_{b + c + d \leq a'} C_{b}^{c d} \left[ \Gamma^b (\tilde{v} + f) \Gamma^c (\tilde{v} + f) (\partial_r + \frac{1}{r}) \Gamma^d f + \frac{1}{r} \Gamma^b f \Gamma^c \tilde{v} \Gamma^d \tilde{v} \right]$$

$$- 2 \Gamma^d (\tilde{v} + f) \Gamma^b f \partial_r \Gamma^c (\tilde{v} + f) + \Gamma^b (1 + \tilde{v}) \Gamma^c G (\partial_r + \frac{1}{r}) \Gamma^d f$$

$$- \Gamma^b (1 + \tilde{v}) \Gamma^d f \partial_r \Gamma^c G - \Gamma^b \left[ \frac{G - h (1 + \tilde{v})}{1 + h} \right] \Gamma^c f \partial_r \Gamma^d \tilde{v} \right] + \sum_{b \leq a'} C_{b}^{a'} \partial_t \Gamma^b \mathcal{H},$$
where the constants $C_{ab}^{a'}$, $C_{bc}^{a'}$, $C_{cd}^{a'}$ may vary from line to line. By applying Lemma 4.3–4.4 with $\xi_1 = \frac{3}{2}$ and $\xi_2 = \kappa = \frac{1}{3}$ to (119), we obtain
\[
\langle r + t \rangle^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\Gamma^a \varphi| \leq C \| \langle r \rangle \Gamma^a \varphi(0, r) \|_{W^{2,1}} + \| \langle r \rangle \partial_t \Gamma^a \varphi(0, r) \|_{W^{1,1}}
+ \sup_{t', r \geq 0} \{ \langle r \rangle^{\frac{3}{4}} (r + t')^{\frac{1}{4}} (r - t')^{\frac{1}{4}} |F^a(t', r)| \}.
\]

Next we treat the nonlinearity $|F^a(t', r)|$ in the second line of (120). For convenience, we denote $t'$ by $t$ below.

By using (117) directly with $1 + |a'| \leq N - 3$ and choosing $M^2 \varepsilon_0 \leq 1$, we then arrive at
\[
r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\partial_t \Gamma^a \mathcal{H}| \leq CM^2 \varepsilon^2 (t)^{M'\varepsilon_0 - 0.1} \leq C \varepsilon.
\]

Applying Lemma 2.4–2.5 and 4.1 to the first term in the second line of (122) directly derives
\[
r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\chi^2 \Gamma^b(v + f)(\partial_r + \frac{1}{r}) \Gamma^c f|
\leq C(t)^{2+\frac{3}{4} - 2.98} |E_{|b|+1}(t) + X_{|b|+2}(t) + Y_{|b|+1}(t)|
\times |E_{|c|+2}(t) + X_{|c|+3}(t) + Y_{|c|+2}(t)|
\leq CM^2 \varepsilon^2 (t)^{2M'\varepsilon_0 - 0.6} \leq C \varepsilon.
\]

In addition, from (26), (49) and (80), we see that
\[
r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\chi^2 \Gamma^b(v + f)(\partial_r + \frac{1}{r}) \Gamma^c f|
\leq C r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\chi_1 \Gamma^b(v + f)| ||\chi_0 \partial_r \Gamma^c f| + (r + t)^{-\frac{3}{4}} |\chi_2 \Gamma^c f|
\leq C(r + t)^{-\frac{3}{4}} |E_{|b|+1}(t)| |E_{|c|+1}(t) + Y_{|c|+2}(t)|
\times |(r + t)^{-\frac{3}{4}} + (r + t)^{-\frac{3}{4}} (r - t)^{\frac{1}{4}}|
\leq CM^2 \varepsilon^2 (t)^{2M'\varepsilon_0 - 0.4} \leq C \varepsilon.
\]

By using (49) and (59) to the first term in the third line of (122), we achieve
\[
r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\chi_1 \Gamma^b(v + f)\chi_0(\partial_r + \frac{1}{r}) \Gamma^c f|
\leq C(r + t)^{-\frac{3}{4} - 0.99} |E_{|b|+1}(t)| |E_{|c|+2}(t) + X_{|c|+3}(t) + Y_{|c|+2}(t)|
\leq CM^2 \varepsilon^2 (t)^{2M'\varepsilon_0 - 0.6} \leq C \varepsilon.
\]

On the other hand, we conclude from Lemma 2.3–2.5 and 4.1–4.2 that
\[
r^{\frac{3}{4}} (r + t)^{\frac{3}{4}} (r - t)^{\frac{1}{4}} |\Gamma^b \mathcal{G}(\partial_r + \frac{1}{r}) \Gamma^c f|
\]
\[
\leq C \left( r^\frac{7}{2} (r + t)^\frac{7}{2} |\chi_0 \Gamma^b G(\partial_r + \frac{1}{r})^{1/2} \right| f \\
+ r^\frac{7}{2} (r + t)^\frac{7}{2} (r - t)^\frac{9}{2} |\chi_1 \Gamma^b G(\partial_r + \frac{1}{r})^{1/2} \right| f \\
\leq C \langle t \rangle^{\frac{7}{2}} |\langle r \rangle^{\frac{7}{2}} \Gamma^b G \chi_0 (\partial_r + \frac{1}{r})^{1/2} f \rangle \\
+ \langle r + t \rangle^{-\frac{7}{2}} |\langle r \rangle^{\frac{7}{2}} \Gamma^b G \chi_1 (\partial_r + \frac{1}{r})^{1/2} f \rangle \\
\leq C (r + t)^{\frac{7}{2}} \langle r \rangle^{-0.99} E[|\dot{c} + 1|] (E[|c| + 2] (t) + X[|c| + 3] (t) + Y[|c| + 2] (t)) \\
\leq C M^2 \varepsilon^2 (t)^{1/2} M' \leq 0.1 \leq C \varepsilon.
\]

\[\frac{1}{2} \Gamma^b f \Gamma^c f\] and \(\frac{1}{2} \Gamma^b f \Gamma^c \Gamma^d \ddot{\nu}^d;\) Here, we only need to deal with \(\frac{1}{2} \Gamma^b f \Gamma^c f\) since the term \(\frac{1}{2} \Gamma^b f \Gamma^c \Gamma^d \ddot{\nu}^d\) can be treated similarly.

\[
\langle r + t \rangle^\frac{7}{2} |\langle r \rangle^{\frac{7}{2}} \Gamma^b f \Gamma^c f | \\
\leq C (r + t)^{\frac{7}{2}} \langle r \rangle^{-0.99} E[|\dot{c} + 1|] (E[|c| + 2] (t) + X[|c| + 3] (t) + Y[|c| + 2] (t)) \\
\leq C M^2 \varepsilon^2 (t)^{1/2} M' \leq 0.1 \leq C \varepsilon.
\]

\[\Gamma^b (1 + \ddot{\nu}) \partial \gamma \Gamma^c G;\] By applying Lemma 2.4–2.5 and inequality (59) directly, we find that

\[
\langle r + t \rangle^\frac{7}{2} (r - t)^\frac{9}{2} |\partial \Gamma^c G | \\
\leq C (r + t)^{\frac{7}{2}} \langle r \rangle^{-0.99} E[|\dot{c} + 1|] (E[|c| + 2] (t) + X[|c| + 3] (t) + Y[|c| + 2] (t)) \\
\leq C M^2 \varepsilon^2 (t)^{1/2} M' \leq 0.1 \leq C \varepsilon.
\]

\[\Gamma^b (\ddot{\nu} + f) \Gamma^c (\ddot{\nu} + f) (\partial_r + \frac{1}{r}) \Gamma^d f;\) This term can be treated as for \(\Gamma^b (\ddot{\nu} + f) (\partial_r + \frac{1}{r}) \Gamma^c f\).

\[\Gamma^d (1 + \ddot{\nu}) \Gamma^b f \partial \gamma \Gamma^c (\ddot{\nu} + f):\] Note that

\[
\Gamma^b f \partial \gamma \Gamma^c (\ddot{\nu} + f) = \chi_0 \Gamma^b f \partial \gamma \Gamma^c (\ddot{\nu} + f) + \chi_1 \Gamma^b f \partial \gamma \Gamma^c (v + f - G).
\]

According to Lemma 2.3–2.4 and 4.1, we deduce that

\[
\langle r + t \rangle^\frac{7}{2} |\partial \Gamma^c (\ddot{\nu} + f) | \\
\leq C (r + t)^{\frac{7}{2}} \langle r \rangle^{-0.99} (r - t)^\frac{7}{2} [(r + t)^{\frac{1}{2}} + (r + t)^{-\frac{1}{2}} (r - t)^{-\frac{1}{2}}] \\
\times [E[|\dot{c} + 1|] (E[|c| + 2] (t) + X[|c| + 3] (t) + Y[|c| + 2] (t)) \\
\leq C M^2 \varepsilon^2 (t)^{1/2} M' \leq 0.6 \leq C \varepsilon.
\]

In addition, by Lemma 2.3–2.5 and inequality (47), we obtain

\[
\langle r + t \rangle^\frac{7}{2} |\partial \Gamma^c (\ddot{\nu} + f) | \\
\leq C (r + t)^{-\frac{1}{2}} |(r - t)^{\frac{7}{2}} (r + t)^{-\frac{1}{2}} (r - t)^{-\frac{1}{2}}| \\
\times [E[|\dot{c} + 1|] (E[|c| + 2] (t) + Y[|c| + 2] (t)) \\
\leq C M^2 \varepsilon^2 (t)^{1/2} M' \leq 0.4 \leq C \varepsilon.
\]
This term can be dealt with as in (126).

$\Gamma^b(1 + \tilde{v}) \Gamma^c G(\partial_r + \frac{1}{r}) \Gamma^v f$: This term can be dealt with as in (126).

\[\Gamma^b(1 + \tilde{v}) \Gamma^v G \partial_r \Gamma^v G: \text{ We conclude from Lemma 2.3–2.5 that} \]

\[r^{\frac{1}{2}} (r + t)^{\frac{2}{3}} (r - t)^{\frac{4}{3}} |\Gamma^c f \partial_r \Gamma^v G| \leq C \langle t \rangle^{\frac{2}{3}} (r + t)^{-\frac{1}{3}} (r - t)^{-\frac{2}{3}} \langle r \rangle^2 |\chi_1 \Gamma^c f| |\langle r \rangle^2 r \partial_r \Gamma^v G| \]

\[\leq C \langle t \rangle^{-\frac{2}{3}} E_{|d|+1}(t) |E_{|v|-1}(t) + \mathcal{X}_{|v|+1}(t)| \mathcal{X}_{|v|+1}(t) \leq C M^2 \varepsilon^2 (t) 2^{M \varepsilon_0 - 0.1} \leq C \varepsilon. \]

\[\Gamma^b \left[ \frac{G - h(1 + \tilde{v})}{1 + h} \right] \Gamma^c f \partial_r \Gamma^v \tilde{v}: \text{ Similarly to (130), according to Lemma 2.3–2.4 and 4.1, we have} \]

\[r^{\frac{1}{2}} (r + t)^{\frac{2}{3}} (r - t)^{\frac{4}{3}} |\chi_1 \Gamma^c f | \partial_r \Gamma^v \tilde{v}| \leq C \langle t \rangle^{\frac{2}{3}} (r + t)^{-\frac{1}{3}} (r - t)^{-\frac{2}{3}} \]

\[\times |E_{|v|+1}(t) + \mathcal{X}_{|v|+2}(t)| |E_{|v|+1}(t) + \mathcal{X}_{|v|+2}(t) + \mathcal{Y}_{|v|+2}(t)| \leq C M^2 \varepsilon^2 (t) 2^{M \varepsilon_0 - 0.6} \leq C \varepsilon. \]

In addition, from Lemma 2.5 and 4.2, we see that

\[r^{\frac{1}{2}} (r + t)^{\frac{2}{3}} (r - t)^{\frac{4}{3}} |\chi_1 \partial_r \Gamma^v \tilde{v}| \leq C \langle t \rangle^{-\frac{1}{2}} |\chi_1 \partial_r \Gamma^v \tilde{v}| \]

\[\leq C \langle t \rangle^{-\frac{1}{2}} |\chi_1 \partial_r \Gamma^v \tilde{v}| \leq C M^2 \varepsilon^2 (t) 2^{M \varepsilon_0 - 1} \leq C \varepsilon. \]

At last, we turn to the initial data in the first line of (120). It is not hard to get that

\[\sum_{|\alpha| \leq N^{-2}} \left\{ \| \langle r \rangle \Gamma^v \partial_r \phi(0, r) \|_{W^{2,1}} + \| \langle r \rangle \partial_r \Gamma^v \phi(0, r) \|_{W^{1,1}} \right\} \leq C \sum_{|\alpha| \leq N^{-2}} \left\{ \| \langle r \rangle \Gamma^v \partial_r \phi(0, r) \|_{L^1} + \| \langle r \rangle \Gamma^v \partial_r \phi(0, r) \|_{L^1} \right\} \]

\[\leq C \sum_{|\alpha| \leq N^{-2}} \left\{ \| \langle r \rangle \Gamma^v \partial_r \phi(0, r) \|_{L^1} + \| \langle r \rangle \Gamma^v \partial_r \phi(0, r) \|_{L^1} \right\} \]

where the last inequality follows from

\[\int_0^\infty (r) |\Gamma^v \phi(0, r)| rdr = \frac{2}{3} \int_0^\infty |\Gamma^v \phi(0, r)| d[\langle (1 + r^2)^{\frac{2}{3}} - 1 \rangle] \leq C \int_0^\infty \langle r \rangle^2 |\partial_r \Gamma^v \phi(0, r)| rdr, \]
and \( \lim_{r \to \infty} r^3 \Gamma^{a'} \varphi(0, r) = 0 \) which is naturally satisfied under condition (10) (indeed, for the special case of \( a' = 0 \), one has \( r^3 |\varphi(0, r)| \leq C \int_r^\infty |R|^2 |f_0(R)| RdR \to 0 \) as \( r \to \infty \). For the general cases, one can apply an analogous argument by (1) of Lemma 2.5. In addition, by taking the divergence on the second equation of (4), we find that

\[
(\partial_t + u \cdot \nabla) \text{div} \mathbf{u} = - \sum_{i,j=1,2} \partial_t u_j \partial_j u_i - \frac{1}{\rho} \Delta P(\rho, S) + \frac{1}{\rho} \nabla \rho \cdot \nabla P(\rho, S). \tag{135}
\]

Then we conclude from the equation of \( \partial_t \text{div} \mathbf{u} \) in (135), (96) and (101) that

\[
\sum_{|a'| \leq N-4} \left\{ \| \langle r \rangle \Gamma^{a'} \varphi(0, r) \|_{W^{2,1}} + \| \langle r \rangle \partial_t \Gamma^{a'} \varphi(0, r) \|_{W^{1,1}} \right\} \\
\leq C \sum_{k=0}^{N-4} \left\{ \| \langle r \rangle^2 (\partial_t \rho) k(\rho_0 - 1, f_0, S_0 - S) \|_{L^1} + \| \langle r \rangle (\partial_t \rho) k g_0 \|_{L^2} \right\} \\
+ \sum_{k+l=0}^{N-4} \left\{ \| \langle r \rangle^2 (\partial_t \rho) k \nabla^l (\nabla \rho_0, \nabla u_0, \nabla S_0) \|_{L^2} + \| \langle r \rangle (\partial_t \rho) k \nabla^l (\text{div} u_0, \text{curl} u_0) \|_{L^2} \right\}. \tag{136}
\]

Substituting (10), (121)–(136) into (120) yields (108). Thus we complete the proof of Lemma 4.6.

\[\square\]

Remark 3. (96) is equivalent to the following ODE:

\[\partial_r \varphi(t, r) = f(t, r), \quad \varphi(t, \infty) = 0.\]

From (120), we know that

\[|X \varphi(t, r)| + |\partial_t \varphi(t, r)| \leq C(r)^{- \frac{t}{2}}.\]

This yields

\[|\partial_r \varphi(t, r)| = \frac{1}{r} |X \varphi(t, r) - t \partial_t \varphi(t, r)| \leq \frac{C t}{r} (r)^{- \frac{t}{2}}.\]

Thus \( \int_1^\infty \partial_r \varphi(t, r) dr \) is bounded for any fixed \( t \geq 0 \), and \( \varphi(t, r) \) in (96) is well defined.

5. Auxiliary energy estimates of \((v, f, g, h)\). In this section, we mainly establish the estimates for the auxiliary energies \( \mathcal{X}_k(t) \) and \( \mathcal{Y}_k(t) \) of the smooth solution \((v, f, g, h)\) to (7).

5.1. Auxiliary energy of \((v, f, g, h)\) near the light cone.

Lemma 5.1. For any smooth function \( \phi(t, x) \) with bounded norms \( \| \phi(t, x) \|_{L^2(\mathbb{R}^2)} \) and \( \| \chi \nabla \phi(t, x) \|_{L^2(\mathbb{R}^2)} \), where the definition of \( \chi \) is given in (14), then the following Hardy type inequality holds

\[
\| \chi \left( \frac{|x|}{|t|} \right) \phi(t, x) \|_{L^2(\mathbb{R}^2)}^2 \leq C \| \phi(t, x) \|_{L^2(\mathbb{R}^2)} \| \chi \nabla \phi(t, x) \|_{L^2(\mathbb{R}^2)} + C(t)^{-1} \| \phi(t, x) \|_{L^2(\mathbb{R}^2)}^2. \tag{137}
\]
Remark 4. For any large $r, s$, it holds that for each fixed $t$,
\[
\lim_{r \to \infty} \int_{S^1} r \phi^2(t, r\omega)d\omega = 0. \tag{138}
\]

Indeed, for fixed $t$, then $\chi_1\left(\frac{|x|}{t}\right) \equiv 1$ for large $r$. In this case, one has that for large $r, s \geq 1$, \[
\left|\int_{S^1} r \phi^2(t, r\omega)d\omega - \int_{S^1} s \phi^2(t, s\omega)d\omega\right| = \left|\int_s^r \int_{S^1} \frac{d}{dq}(r \phi^2(t, q\omega)d\omega)dy\right| \leq C \int_s^r \int_{S^1} \phi^2(t, q\omega)d\omega dq + \int_s^r \int_{S^1} \phi^2(t, q\omega)d\omega dq \to 0 \quad \text{as } r, s \to \infty.
\]

This means that \[
\lim_{r \to \infty} \int_{S^1} r \phi^2(t, r\omega)d\omega \text{ exists. If } \lim_{r \to \infty} \int_{S^1} r \phi^2(t, r\omega)d\omega \neq 0, \text{ then there exists a positive constant } C(t) \text{ such that for large } r, \lim_{r \to \infty} \int_{S^1} r \phi^2(t, r\omega)d\omega \geq C(t) > 0, \text{ which is contradictory with } \int_0^\infty \int_{S^1} r \phi^2(t, r\omega)d\omega < \infty. \text{ Thus, } (138) \text{ holds.}
\]

On the other hand, $\chi_1\left(\frac{|x|}{t}\right) \equiv 0$ holds for any $r \leq \frac{1}{4} \leq \frac{1}{4}(t)$ and $t \geq 0$. Thereafter, we conclude from the integration by parts and (138) that \[
\left|\int \chi_1^2 \frac{\phi^2(t, x)}{1 + (r - t)^2} dx\right| = \left|\int \chi_1^2 \frac{\phi^2(t, r\omega)}{1 + (r - t)^2} r\omega d\omega\right| \leq C \int_0^\infty \int \chi_1^2 \phi \phi^2(t, r\omega) r\omega d\omega + \int_0^\infty \phi \phi^2(t, r\omega) r\omega d\omega \leq C \int_0^\infty \chi_1^2 \phi \phi^2(t, r\omega) r\omega d\omega.
\]

This, together with the Cauchy-Schwartz inequality, yields the proof of Lemma 5.1. \hfill \Box

Remark 4. For any $\ell > 1$, (137) still holds with \[
\left\|\frac{\phi(t, x)}{|x|} - \frac{\phi(t, x)}{|x| - \ell}\right\|_{L^2(\mathbb{R}^2)} \text{ instead of } \left\|\chi_1\left(\frac{|x|}{\ell}\right) \phi(t, x)\right\|_{L^2(\mathbb{R}^2)} \text{ in the left hand side of } (137). \text{ The only difference in the proof lies in that the function “arctan}(r - t)^n)\text{ is replaced by } \int_{-\infty}^{r-t} \frac{ds}{(1 + s^2)^t}.
\]

Lemma 5.2. For any integer $k \in \mathbb{N}$ with $k \leq N$, if $(v, f, g, h)$ is a smooth solution of (7), then for small $\varepsilon_0 > 0$, the following auxiliary weighted energy inequality holds
\[
\mathcal{Y}_k(t) \leq CE_k(t). \tag{139}
\]

Proof. For $a \in \mathbb{N}^2_0$ with $|a| \leq N - 1$, set \[
\mathcal{E}_a^k(t) := \left\|\chi_1\partial_t \Gamma^a v(t, r)\right\|_{L^2} + \left\|\chi_1\partial_t \Gamma^a v(t, r)\right\|_{L^2} + \left\|\chi_1\partial_t \Gamma^a f(t, r)\right\|_{L^2} + \left\|\chi_1\partial_t \Gamma^a f(t, r)\right\|_{L^2}.
\]
where we have used the Cauchy-Schwartz inequality and (48). Recalling that

\[ \langle r \rangle \langle t \rangle \leq \langle r \rangle + \langle t \rangle \]

Then, we obtain

\[ \sum_{b+c=a} \| \langle r \rangle \chi_1 \Gamma^b(v + f) \partial_t \Gamma^c v \|_{L^2} \]

\[ \leq C \sum_{b+c=a, |b| \leq N-6} \| \langle r \rangle \chi_1 \partial_t \Gamma^c v \|_{L^2} \left[ \| \langle r \rangle \chi_1 \Gamma^b(v + f) \|_{L^\infty} \right] \]

\[ + \| \chi_0 \Gamma^b(\bar{v} + f) \|_{L^\infty} + \langle t \rangle^{-2} \| \langle r \rangle \Gamma^b G \|_{L^\infty} \]

\[ + \sum_{b+c=a, |b| > N-6} \left\{ \| \langle r \rangle \chi_1 \partial_t \Gamma^c v \|_{L^\infty} \left[ \| \langle r \rangle \chi_1 \Gamma^b(v + f) \|_{L^2} \right] \right\} \]

\[ + \| \chi_1 \Gamma^b(v + f) \|_{L^2} (\langle t \rangle \| \chi_0 \partial_t \Gamma^c \bar{v} \|_{L^\infty} + \langle t \rangle^{-2} \| \chi_0 \langle r \rangle^2 \partial_t \Gamma^c G \|_{L^\infty}) \right\}. \]

Note that \( \langle r + t \rangle \leq C \langle r - t \rangle + C \langle t \rangle \) holds. Applying Lemma 5.1 to

\[ \| \chi_1 \Gamma^b(v + f) \|_{L^2} \]

\[ \leq \| \langle r \rangle \chi_1 \Gamma^b(v + f) \partial_t \Gamma^c v \|_{L^2} \]

\[ \leq C \| \Gamma^b(v + f) \|_{L^2}^2 + \langle t \rangle \left[ \chi_1 \Gamma^b(v + f) \|_{L^2} \right]^2 \]

\[ \leq C \| \Gamma^b(v + f) \|_{L^2}^2 + \langle t \rangle \left[ \chi_1 \partial_t \Gamma^b(v + f) \|_{L^2} \right]^2 \]

\[ \leq C \| \chi_1 \Gamma^b(v + f) \|_{L^2} \]

where we have used the Cauchy-Schwartz inequality and (48). Recalling that \( |b| + |c| \leq |a| \leq N - 1 \) and \( N \geq N_0 = 11 \), if \( |b| > N - 6 \), then we have \( |c| + 2 \leq 6 \leq N - 5 \). Therefore, combining (140)–(141) with Lemma 2.3.2–2.5 derives

\[ \sum_{b+c=a} \| \langle r \rangle \chi_1 \Gamma^b(v + f) \partial_t \Gamma^c v \|_{L^2} \]

\[ \leq C \sum_{b+c=a, |b| \leq N-6} \mathcal{E}_b(t) \left[ E_{|b|+1}(t) + \langle t \rangle^{-1.99} [E_{|b|+2}(t) + \mathcal{Y}_{|b|+3}(t) + \mathcal{Y}_{|b|+2}(t)] \right] \]

\[ + \sum_{b+c=a, |b| > N-6} \left\{ E_{|b|+1}(t) [E_{|c|+1}(t) + \mathcal{Y}_{|c|+2}(t)] \right\} \]

\[ + \langle t \rangle^{-0.99} E_{|b|}(t) [E_{|c|+2}(t) + \mathcal{Y}_{|c|+3}(t) + \mathcal{Y}_{|c|+2}(t)] \right\} \]

\[ \leq CM \epsilon \left[ \sum_{c \leq a} \mathcal{E}_c(t) + E_{|a|+1}(t) \right]. \]

Now, we turn to the treatment of \( \frac{1}{r} \Gamma^b v \Gamma^c f \) in \( \tilde{\Gamma}^a Q_1 \). It is easy to get that

\[ \| \langle r \rangle \frac{1}{r} \chi_1 \Gamma^b v \Gamma^c f \|_{L^2} \leq C \| \chi_1 \Gamma^b v \Gamma^c f \|_{L^2} \leq C \| \chi_1 \Gamma^b v \|_{L^\infty} \| \Gamma^c f \|_{L^2} \]

\[ \leq C \langle t \rangle^{-\frac{3}{2}} E_{|c|}(t) [E_{|b|}(t) + \mathcal{Y}_{|b|+1}(t)] \leq CM \epsilon E_{|a|}(t), \]

where we have used \( |b| \leq |a| \leq N - 1 \).
By analogous but easier analysis for the term \( \Gamma^b v \partial_r \Gamma^c (v + f) \) in \( \tilde{\Gamma}^a Q_2 \) (the treatment of \( \Gamma^b v \partial_r \Gamma^c (v + f) \) in \( \tilde{\Gamma}^a Q_1 \) is similar), we have
\[
\sum_{b+c=a} \| (r + t) \chi_1 \Gamma^b v \partial_r \Gamma^c (v + f) \|_{L^2} \leq C \sum_{b+c=a, \ |b| > N-2} \| (r + t) \chi_1 \partial_r \Gamma^c (v + f) \|_{L^\infty} \| \Gamma^b v \|_{L^2} + \sum_{b+c=a, \ |b| \leq N-2} \| (r) \chi_1 \partial_r \Gamma^c (v + f) \|_{L^2} \| \chi_1 \Gamma^b v \|_{L^\infty} + \| \chi \partial_t \tilde{v} \|_{L^\infty} + \langle t \rangle^{-2} \| \langle t \rangle^2 \Gamma^b G \|_{L^\infty} \]
\[
\leq C \langle t \rangle^{-\frac{1}{2}} \sum_{b+c=a, \ |b| > N-2} E_{|b|} (t) [E_{|c|+1} (t) + \chi |c|+2 (t)] + \langle t \rangle^{-\frac{1}{2}} \sum_{b+c=a, \ |b| \leq N-2} E_{|c|+1} (t) [E_{|b|+1} (t) + \chi |b|+2 (t) + \chi |b|+1 (t)] 
\leq C M \varepsilon E_{|a|+1} (t). \tag{144}
\]

Next, we deal with the terms \( \Gamma^b h \partial_r \Gamma^c f \) and \( \Gamma^b h \Gamma^c \partial_r \Gamma^d f \) in \( \tilde{\Gamma}^a Q_2 \). Since the treatment on the cubic nonlinearity \( \Gamma^b h \Gamma^c \partial_r \Gamma^d f \) is much easier than the quadratic one \( \Gamma^b h \partial_r \Gamma^c f \), here we only estimate \( \Gamma^b h \partial_r \Gamma^c f \). It follows from direct computation that
\[
\sum_{b+c=a} \| (r + t) \chi_1 \Gamma^b h \partial_r \Gamma^c f \|_{L^2} \leq C \sum_{b+c=a, \ |b| \leq N-3} \| (r) \chi_1 \Gamma^b h \|_{L^\infty} \| \chi_1 \partial_r \Gamma^c f \|_{L^2} + \sum_{b+c=a, \ |b| > N-3} \| (r) \chi_1 \Gamma^b h \|_{L^2} \| \chi_1 \partial_r \Gamma^c f \|_{L^\infty} \tag{145}
\]
\[
\leq C \sum_{b+c=a, \ |b| \leq N-3} W_{|b|} (t) E_{|c|+1} (t) + \langle t \rangle^{-\frac{1}{2}} \sum_{b+c=a, \ |b| > N-3} E_{|b|} (t) [E_{|c|+1} (t) + \chi |c|+2 (t)] \leq C M \varepsilon E_{|a|+1} (t). \]

Finally, for the term \( \frac{1}{r} \Gamma^b g \Gamma^c g \Gamma^d (1 + h) \) in \( \tilde{\Gamma}^a Q_2 \), we have
\[
\sum_{b+c+d=a} \| (r + t) \frac{1}{r} \chi_1 \Gamma^b g \Gamma^c g \Gamma^d (1 + h) \|_{L^2} \leq C \sum_{b+c+d=a} \| \Gamma^b g \Gamma^c g \Gamma^d (1 + h) \|_{L^2} \leq C \sum_{b+c+d=a, \ |d| \leq N-3} [1 + W_{|d|} (t)] \| \Gamma^b g \Gamma^c g \|_{L^2} + \sum_{b+c+d=a, \ |d| > N-3} E_{|d|} (t) \| \Gamma^b g \Gamma^c g \|_{L^\infty} \tag{146}
\]
\[
\leq C E_{|a|} (t) E_{|a|+1} (t) + \sum_{b+c+d=a, \ |d| > N-3} E_{|b|+1} (t) E_{|c|+1} (t) E_{|d|} (t) \leq C M \varepsilon E_{|a|} (t). \]

Thus, collecting (142)–(146) leads to
\[
\| (r + t) \chi_1 \tilde{\Gamma}^a Q_1 \|_{L^2} + \| (r + t) \chi_1 \tilde{\Gamma}^a Q_2 \|_{L^2} \leq C M \varepsilon \sum_{c \leq a} E^c_1 (t) + E_{|a|+1} (t). \]

Therefore, for small \( \varepsilon > 0 \), we complete the proof of (139). \( \square \)
5.2. Auxiliary energy of \((v, f, g, h)\) away from the light cone.

Lemma 5.3. For any integer \(k \in \mathbb{N}\) with \(k \leq N\) and multi-index \(a \in \mathbb{N}_0^2\) with \(|a| \leq N - 1\), if \((v, f, g, h)\) is a smooth solution of (7), then for small \(\varepsilon_0 > 0\), the following weighted energy inequalities hold

\[
\mathcal{X}_k(t) \leq C E_k(t),
\]

\[
\|\partial_t \Gamma^n g\|_{L^2} + \|(r)\partial_t \Gamma^n h\|_{L^2} + \|(r)\partial_t \Gamma^n G\|_{L^2} + \|\partial_t \Gamma^n G\|_{L^2} \leq C(t)^{-1} E_{|a|+1}(t).
\]

Proof. We divide the proof of Lemma 5.3 into two parts.

Part I: The estimates including the first order space derivatives \(\partial_t \Gamma^n \tilde{v}(t, r)\) and \((\partial_t + \frac{1}{r}) \Gamma^n f(t, r)\).

For \(a \in \mathbb{N}_0^2\) with \(|a| \leq N - 1\), set

\[
\mathcal{E}_0^a(t) := \|\chi_0 \partial_t \Gamma^n \tilde{v}(t, r)\|_{L^2} + \|\chi_0 \partial_t \Gamma^n \tilde{v}(t, r)\|_{L^2} + \|\chi_0 \partial_t \Gamma^n f(t, r)\|_{L^2} + \|\chi_0 \partial_t \Gamma^n f(t, r)\|_{L^2}.
\]

According to (66) and (67), we have

\[
\mathcal{E}_0^a(t) \leq C \langle t \rangle^{-1} E_{|a|+1}(t) + |\chi_0 \Gamma^n \tilde{T}_1\|_{L^2} + |\chi_0 \Gamma^n \tilde{T}_2\|_{L^2}.
\]

At first, we deal with the term \(\Gamma^n (\tilde{v} + G)(\partial_t + \frac{1}{r}) \Gamma^n f\) in \(\tilde{T}_1\) of (68). If \(|b| \leq N - 6\), we find that

\[
\|\chi_0 \Gamma^n (\tilde{v} + G)(\partial_t + \frac{1}{r}) \Gamma^n f\|_{L^2} \leq C \mathcal{E}_0^a(t)[\|\Gamma^n G\|_{L^\infty} + \|\chi_0 \Gamma^n \tilde{v}\|_{L^\infty} + \|\chi_1 \Gamma^n \tilde{v}\|_{L^\infty}]
\]

\[
\leq C \langle t \rangle^{-1} E_{|b|+1}(t) + \langle t \rangle^{-\frac{1}{2}} [\chi_0 \chi_1 + \chi_0 + \chi_1](t) \leq C M \mathcal{E}_0^a(t),
\]

where we have used (19), (27) and Lemma 2.4–2.5. For \(|b| > N - 6\), due to \(|b| + |c| = |a| \leq N - 1\) and \(N \geq N_0\), one has \(|c| \leq 4 \leq N - 3\). Therefore, we achieve

\[
\|\chi_0 \Gamma^n (\tilde{v} + G)(\partial_t + \frac{1}{r}) \Gamma^n f\|_{L^2} \leq C \|\Gamma^n \tilde{v}\|_{L^\infty} \|\chi_0 \partial_t + \frac{1}{r} \Gamma^n f\|_{L^\infty}
\]

\[
\leq C \langle t \rangle^{-1.99} E_{|b|}(t) E_{|c|+1}(t) \leq C \langle t \rangle^{-1} E_{|a|}(t).
\]

The treatments for \(\Gamma^n f \partial_t \Gamma^n \tilde{v}, \Gamma^n (\tilde{v} + G) \partial_t \Gamma^n \tilde{v}, \Gamma^n h \partial_t \Gamma^n f, \Gamma^n f \partial_t \Gamma^n f \Gamma^n(1 + h)\) in \(\tilde{T}_1\) and \(\tilde{T}_2\) are analogous. Then one has

\[
\|\chi_0 \Gamma^n \tilde{T}_1\|_{L^2} + \|\chi_0 \Gamma^n \tilde{T}_2\|_{L^2} \leq C M \varepsilon \sum_{c+a \leq a} \mathcal{E}_0^c(t) + \langle t \rangle^{-1} E_{|a|}(t)
\]

\[
+ \sum_{b+c=a} \|\chi_0 \Gamma^n f \partial_t \Gamma^n G\|_{L^2} + \|\partial_t \Gamma^n G\|_{L^2}.
\]

Now we deal with the terms in the last line of (151). By an analogous analysis for the multi-index \(b\) and \(c\) as in (150), we achieve

\[
\sum_{b+c=a} \|\chi_0 \Gamma^n f \partial_t \Gamma^n G\|_{L^2}
\]
\[ \leq C \sum_{b+c=a, \quad |c| > N-6} \| \chi_0 \frac{1}{r} \Gamma^b v \|_{L^\infty} \| r \partial_t \Gamma^c G \|_{L^2} + \sum_{b+c=a, \quad |c| \leq N-6} \| \chi_0 \frac{1}{r} \Gamma^b f \|_{L^2} \| r \partial_t \Gamma^c G \|_{L^\infty} \]

\[ \leq C \langle t \rangle^{-1.99} \sum_{b+c=a, \quad |c| > N-6} E_{|c|}(t) [E_{|b|+2}(t) + X_{|b|+3}(t) + Y_{|b|+2}(t)] \]

\[ + \sum_{b+c=a, \quad |c| \leq N-6} E_{|c|+1}(t) [c_0^b(t) + \langle t \rangle^{-1} E_{|b|}(t)] \tag{152} \]

\[ \leq C M \varepsilon \sum_{b \leq a} c_0^b(t) + \langle t \rangle^{-1} E_{|a|}(t). \]

Next we turn our attention to the last term \( \| \partial_t \Gamma^a G \|_{L^2} \) in (151). From Lemma 2.5 we know that \( \lim_{\langle r \rangle \to \infty} \rho^2 \partial_t \Gamma^a G(t, r) = 0 \). Thus, direct integration yields

\[ \| \partial_t \Gamma^a G \|_{L^2}^2 = \int_0^\infty \| \partial_t \Gamma^a G \|_{L^2}^2 \frac{dr}{r} = \left| \int \partial_t \Gamma^a G \partial_r \partial_t \Gamma^a G r^2 dr \right| \leq \| \partial_t \Gamma^a G \|_{L^2} \| r \partial_r \partial_t \Gamma^a G \|_{L^2}. \]

Then we achieve

\[ \| \partial_t \Gamma^a G \|_{L^2} \leq \| r \partial_r \partial_t \Gamma^a G \|_{L^2}. \tag{153} \]

Instead of \( \| r \partial_r \partial_t \Gamma^a G \|_{L^2} \) in (153), we will deal with the more general one \( \| \langle r \rangle \partial_r \partial_t \Gamma^a G \|_{L^2} \). Indeed, taking \( L^2 \) norms on both sides of (79) derives

\[ \| \langle r \rangle \partial_r \partial_t \Gamma^a G \|_{L^2} \leq C \sum_{b+c=a, \quad c < a} \| \Gamma^b v(r) \partial_r \partial_t \Gamma^c G \|_{L^2} + \sum_{b+c=a} \| \partial_t \Gamma^b v(r) \partial_r \Gamma^c G \|_{L^2} \]

\[ + \sum_{b+c+d=a} \| \| \langle r \rangle \partial_r \Gamma^c G \|_{L^2} | \chi_0 \Gamma^b \|_{L^\infty} + \| \Gamma^b G \|_{L^\infty} + \| \chi_1 \partial_t \Gamma^b v \|_{L^\infty} \|. \tag{154} \]

Then similarly to (152), we obtain

\[ \sum_{b+c=a, \quad c < a} \| \langle r \rangle \partial_r \partial_t \Gamma^c G \|_{L^2} \leq C \sum_{b+c=a, \quad |c| \leq N-4} \| \chi_0 \partial_t \Gamma^c G \|_{L^\infty} + \| \Gamma^b G \|_{L^\infty} + \| \chi_1 \partial_t \Gamma^b v \|_{L^\infty} \]

\[ \leq C M \varepsilon \sum_{c \leq a} \| \langle r \rangle \partial_r \partial_t \Gamma^c G \|_{L^2} + \langle t \rangle^{-1} E_{|a|}(t) \]

and

\[ \| \partial_t \Gamma^b v(r) \partial_r \Gamma^c G \|_{L^2} \]

\[ \leq C \sum_{b+c=a, \quad |c| \leq N-6} \| \langle r \rangle \partial_r \Gamma^c G \|_{L^\infty} \| \chi_0 \partial_t \Gamma^b \|_{L^2} + \| \partial_t \Gamma^b G \|_{L^2} + \langle t \rangle^{-2} \| \chi_1 \partial_t \Gamma^b v \|_{L^2} \]

\[ + \sum_{b+c=a, \quad |c| > N-6} \| \langle r \rangle \partial_r \Gamma^c G \|_{L^2} \| \chi_0 \partial_t \Gamma^b \|_{L^\infty} + \| \partial_t \Gamma^b G \|_{L^\infty} + \langle t \rangle^{-1} \| \chi_1 \partial_t \Gamma^b v \|_{L^\infty} \]

\[ \leq C M \varepsilon \sum_{b \leq a} c_0^b(t) + \| r \partial_r \partial_t \Gamma^b G \|_{L^2} + \langle t \rangle^{-1} E_{|a|}(t). \tag{156} \]
In addition, according to the fourth equation in (65) and (57), we deduce that
\[
\|\langle r \rangle \partial_r \Gamma^c h \|_{L^2} \leq C \sum_{d + e = c, |e| > N - 4} [\| \chi_0 \frac{1}{r} \Gamma^d f \|_{L^\infty} + \langle t \rangle^{-1} \| \chi_1 \Gamma^d f \|_{L^\infty} \| \langle r \rangle \chi \Gamma^c h \|_{L^2} + \langle r \rangle \| \partial_r \Gamma^c h \|_{L^2} ]
+ \sum_{d + e = c, |e| \leq N - 4} [\| \chi_0 \frac{1}{r} \Gamma^d f \|_{L^2} + \langle t \rangle^{-1} \| \chi_1 \Gamma^d f \|_{L^2} \| \langle r \rangle \chi \Gamma^c h \|_{L^\infty} + \langle r \rangle \| \partial_r \Gamma^c h \|_{L^\infty} ]
\leq C \langle t \rangle^{-\frac{3}{2}} \sum_{d + e = c, |e| > N - 4} [E_{|d|+2}(t) + \chi_{|d|+3}(t) + \mathcal{Y}_{|d|+2}(t)][E_{|e|+1}(t) + \langle r \rangle \| \partial_r \Gamma^c h \|_{L^2}] 
+ \sum_{d + e = c, |e| \leq N - 4} \left[ \mathcal{E}^c_0(t) + \langle t \rangle^{-\frac{1}{2}} E_{|d|+1}(t) \right]
\times \left\{ W_{|e|+1}(t) + \langle t \rangle^{-0.99} [E_{|e|+2}(t) + \chi_{|e|+3}(t) + \mathcal{Y}_{|e|+2}(t)] \right\},
\]
where we have used (23), Lemma 2.3–2.5 and Lemma 4.1. In addition, the treatment of \( \| \partial_t \Gamma^c g \|_{L^2} \) is similar. Consequently, we achieve
\[
\| \partial_t \Gamma^c g \|_{L^2} + \| \langle r \rangle \partial_r \partial_t \Gamma^c h \|_{L^2} \leq C \, M \varepsilon \mathcal{E}^c_0(t) + \langle t \rangle^{-1} E_{|e|+1}(t). \tag{157}
\]
This yields
\[
\sum_{b + c + d = a} \| \Gamma^b g \partial_t \Gamma^c g \Gamma^d (1 + h) \|_{L^2} + \| \Gamma^b g \partial_t \Gamma^c g \partial_r \Gamma^d h \|_{L^2} \leq C \, M \varepsilon \sum_{c \leq a} \mathcal{E}^c_0(t) + \langle t \rangle^{-1} E_{|a|+1}(t). \tag{158}
\]
Inserting (155), (156) and (158) into (154), we arrive at
\[
\| \partial_t \Gamma^a G \|_{L^2} + \| \langle r \rangle \partial_r \partial_t \Gamma^a G \|_{L^2} \leq C \, M \varepsilon \sum_{b \leq a} \mathcal{E}^b_0(t) + \langle t \rangle^{-1} E_{|a|+1}(t).
\]
This, together with (149), (151), (152) and (157), yields that for all \( |a| \leq N - 1 \),
\[
\mathcal{E}^a_0(t) + \| \partial_t \Gamma^a g \|_{L^2} + \| \langle r \rangle \partial_r \partial_t \Gamma^a h \|_{L^2} + \| \langle r \rangle \partial_r \partial_t \Gamma^a G \|_{L^2} 
+ \| \partial_t \Gamma^a G \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a|+1}(t). \tag{159}
\]

**Part II: The estimates including the second order space derivatives** \( \partial_r ( \partial_r + \frac{1}{r} ) \Gamma^{a'} f \) and \( ( \partial_r + \frac{1}{r} ) \partial_r \Gamma^{a'} v \).

Let \( |a'| \leq N - 2 \). From (65), we easily get
\[
\partial_r ( \partial_r + \frac{1}{r} ) \Gamma^{a'} f = \partial_r \partial_t \Gamma^{a'} v - \partial_r \Gamma^{a'} \tilde{Q}_1
\]
\[
( \partial_r + \frac{1}{r} ) \partial_r \Gamma^{a'} v = ( \partial_r + \frac{1}{r} ) \partial_t \Gamma^{a'} f - ( \partial_r + \frac{1}{r} ) \Gamma^{a'} \tilde{Q}_2,
\]
where \( \tilde{Q}_1 \) and \( \tilde{Q}_2 \) are given by
which yields
\[
\mathcal{V}^{a'}(t) := \|\chi_0 \partial_r (\partial_r + \frac{1}{r}) \Gamma^a f\|_{L^2} + \|\chi_0 (\partial_r + \frac{1}{r}) \partial_r \Gamma^a v\|_{L^2}
\]
\[
\leq C \|\chi_0 \partial_r (\partial_r + \frac{1}{r}) \partial_r \Gamma^a \bar{v}\|_{L^2} + \|\chi_0 (\partial_r + \frac{1}{r}) \partial_r \Gamma^a f\|_{L^2}
\]
\[
+ \|\chi_0 \partial_r \Gamma^a \bar{Q}_1\|_{L^2} + \|\chi_0 (\partial_r + \frac{1}{r}) \Gamma^a \bar{Q}_2\|_{L^2}.
\]

We now treat the last two nonlinearities \(\partial_r \Gamma^a \bar{Q}_1\) and \((\partial_r + \frac{1}{r}) \Gamma^a \bar{Q}_2\) in (160). It is not hard to check that
\[
\partial_r \Gamma^a \bar{Q}_1 = \sum_{b+c=d=a'} \sigma_{bc} [\Gamma^b (\bar{v} + G) \partial_r (\partial_r + \frac{1}{r}) \Gamma^c f - \Gamma^b f (\partial_r + \frac{1}{r}) \partial_r \Gamma^c \bar{v}]
\]
\[
+ 2 \Gamma^b f \left( \partial_r \Gamma^c \bar{v} + \frac{1}{r} \Gamma^b f \partial_r \Gamma^c G - \Gamma^b f \partial_r \Gamma^c G \right) - \partial_r \partial_r \Gamma^a G,
\]
and
\[
(\partial_r + \frac{1}{r}) \Gamma^a \bar{Q}_2 = \sum_{b+c=d=a'} \sigma_{bc} [\Gamma^b (\bar{v} + G) (\partial_r + \frac{1}{r}) \partial_r \Gamma^c \bar{v} + \partial_r \Gamma^b (\bar{v} + G) \partial_r \Gamma^c \bar{v}]
\]
\[
- \Gamma^b h (\partial_r + \frac{1}{r}) \partial_r \Gamma^c f - \partial_r \partial_r \Gamma^b h \partial_r \Gamma^c f - \sum_{b+c+d=a'} \sigma_{bcd} [\Gamma^b f \partial_r \Gamma^c f \partial_r \Gamma^d h]
\]
\[
+ \Gamma^d (1+h) [\Gamma^b f \partial_r (\partial_r + \frac{1}{r}) \Gamma^c f + (\partial_r + \frac{1}{r}) \Gamma^b f (\partial_r + \frac{1}{r}) \Gamma^c f - \frac{2}{r} \Gamma^b f \partial_r \Gamma^c f].
\]

Applying (23) to the first term \(\frac{1}{r} \partial_r \Gamma^c \bar{v}\) in the second line of (161), we achieve
\[
\|\chi_0 \partial_r \Gamma^a \bar{Q}_1\|_{L^2}
\]
\[
\leq C \sum_{b+c=a'} \|\mathcal{V}^{c}(t) + \langle t \rangle^{-1} E_{|c|+1}(t) \| E_{|b|}(t) + X_{|b|+1}(t) + Y_{|b|+1}(t) \|
\]
\[
+ \sum_{b+c=a'} \|\chi_0 \frac{1}{r} \Gamma^b f \partial_r \Gamma^c G\|_{L^2} + \|\chi_0 \frac{1}{r} \Gamma^b f \partial^2 \Gamma^c G\|_{L^2} + \langle t \rangle^{-1} E_{|a'|+1}(t).
\]

Next we start to treat each term in the last line of (163).
\[
\|\chi_0 \frac{1}{r} \Gamma^b f \partial_r \Gamma^c G\|_{L^2}:
\]
We conclude from Hardy inequality (23), Lemma 2.4–2.5 and 4.1 that
\[
\sum_{b+c=a'} \|\chi_0 \frac{1}{r} \Gamma^b f \partial_r \Gamma^c G\|_{L^2}
\]
\[
\leq C \sum_{b+c=a', \ |c| \leq N-6} \|\partial_r \Gamma^c G\|_{L^\infty} \|\chi_0 \frac{1}{r} \Gamma^b f\|_{L^2} + \sum_{b+c=a', \ |c| > N-6} \|\chi_0 \frac{1}{r} \Gamma^b f\|_{L^\infty} \|\partial_r \Gamma^c G\|_{L^2}
\]
\[
\leq C \langle t \rangle^{-1} \sum_{b+c=a', \ |c| \leq N-6} E_{|b|+1}(t) E_{|c|+1}(t)
\]
\[
+ \langle t \rangle^{-1.99} \sum_{b+c=a', \ |c| > N-6} E_{|c|}(t) [E_{|b|+2}(t) + X_{|b|+3}(t) + Y_{|b|+2}(t)]
\]
\[
\leq C \langle t \rangle^{-1} E_{|a'|+1}(t).
\]
Lemma 2.3–2.5, 3.1, 4.1–4.2 can be summarized as follows.

Consequently, we arrive at

\[ \| \chi_0 \Gamma^b f \partial_r \partial_t \Gamma^c G \|_{L^2} \leq C \| \chi_0 \Gamma^b f \partial_r (X - 1) \Gamma^c G \|_{L^2} + \langle t \rangle \| \chi_0 \Gamma^b f \partial_r \partial_t \Gamma^c G \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]

This, together with (63) and (64), yields

\[ \| \chi_0 (\partial_r + \frac{1}{r}) \tilde{Q}_1 \|_{L^2} \leq C M \varepsilon \sum_{c < a'} \mathcal{V}(t) + \langle t \rangle^{-1} E_{|a'|+1}(t). \quad (165) \]

With an analogous analysis to \( \| \chi_0 \partial_r \tilde{Q}_1 \|_{L^2} \), we achieve

\[ \| \chi_0 (\partial_r + \frac{1}{r}) \tilde{Q}_2 \|_{L^2} \leq C M \varepsilon \sum_{c < a'} \mathcal{V}(t) + \langle t \rangle^{-1} E_{|a'|+1}(t). \quad (166) \]

Substituting (165) and (166) into (157) yields

\[ \mathcal{V}(t) \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]

Thus, collecting Part I and Part II completes the proof of Lemma 5.3.

6. Energy estimates of \((v, f)\). According to estimates (139) and (147), then Lemma 2.3–2.5, 3.1, 4.1–4.2 can be summarized as follows.

Lemma 6.1. For \( a, b \in \mathbb{N}^2_0 \) with \(|a| \leq N - 1 \) and \(|b| \leq N - 3 \), if \((v, f, g, h)\) is a smooth solution of (7), then for small \( \varepsilon_0 > 0 \), the following inequalities hold

\[ \| v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]

Lemma 6.2. For \( a', b', b'' \in \mathbb{N}^2_0 \) with \(|a| \leq N - 1 \), \(|a'| \leq N - 2 \), \(|b| \leq N - 3 \) and \(|b'| \leq N - 4 \), if \((v, f, g, h)\) is a smooth solution of (7), then for small \( \varepsilon_0 > 0 \), the following inequalities hold

\[ \| v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]

\[ \| v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]

\[ \| v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-\frac{1}{2}} E_{|a|+1}(t), \]

\[ \| \partial_r \partial_r v(t, r) \|_{L^2} \leq C \langle t \rangle^{-1} E_{|a'|+1}(t). \]
6.1. Elementary energy estimates of \((v, f)\).

**Lemma 6.3.** For \(a \in \mathbb{N}_0^2\) with \(|a| \leq N\), if \((v, f, g, h)\) is a smooth solution of (7), then for small \(\varepsilon > 0\), there exists a positive constant \(C\) which is independent of \(t, t', \varepsilon, M\) and \(M'\) such that the following elementary energy inequality holds

\[
\|\Gamma^a v(t', r)\|_{L^2}^2 + \|\Gamma^a f(t', r)\|_{L^2}^2 + \int_0^{t'} \|\frac{\Gamma^a(v + f)}{(r - t)^{\frac{3}{2}}})\|^2 dt \\
\leq C \|\Gamma^a v(0, r)\|_{L^2}^2 + \|\Gamma^a f(0, r)\|_{L^2}^2 \\
+ M \epsilon \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_\varepsilon^2(t) dt + \left| \int_0^{t'} \int \delta r dt \right|
\]

(167)

where

\[
I := e^{g_f(1 + \bar{v})} \left\{ \frac{1}{r} \Gamma^a f \Gamma^a [g^2(1 + h)] - \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^b h \partial_r \Gamma^c f \\
- \sum_{b+c+d=a, \ d < a} \sigma_{bcd} \Gamma^b h \Gamma^c \frac{\Gamma^d f}{r} + \sum_{b+c=a, \ c < a} \sigma_{bc} \left[ \frac{1}{r} \Gamma^a v \Gamma^b v \Gamma^c f \right] \\
+ \partial_r \Gamma^c v(\Gamma^a f \Gamma^b v - \Gamma^a v \Gamma^b f) + \partial_r \Gamma^c f(\Gamma^a v \Gamma^b v - \Gamma^a f \Gamma^b f) \right\}
\]

with the smooth function \(q = q(r - t)\) satisfying \(q'(s) = \langle s \rangle^{-\frac{3}{2}}\) and \(\lim_{s \to -\infty} q(s) = 0\).

**Proof.** Multiplying (50) by \(e^{g_f} r^a v\) and \(e^{g_f} r^a f\) respectively, we obtain

\[
\frac{1}{2} \partial_t [e^{g_f} r^a v^2 + r^a f^2] - \partial_r (e^{g_f} r^a v^a f) + \frac{1}{2} (r - t)^{-\frac{3}{2}} e^{g_f} r^a (v + f)^2 \\
= e^{g_f} r^a v \tilde{\Gamma}^a Q_1 + r^a f \tilde{\Gamma}^a Q_2.
\]

We now treat the nonlinearities containing top-order derivative \(\partial_r \Gamma^a v\) and \(\partial_r \Gamma^a f\) in the right hand side of (169). To this end, (83)–(84) can be rewritten as follows:

\[
\tilde{\Gamma}^a Q_1 = v(\partial_r + \frac{1}{r}) \Gamma^a f - f \partial_r \Gamma^a v + \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^b v(\partial_r + \frac{1}{r}) \Gamma^c f - \Gamma^b f \partial_r \Gamma^c v
\]

and

\[
\tilde{\Gamma}^a Q_2 = v \partial_r \Gamma^a v - f \partial_r \Gamma^a f + \frac{1}{r} \Gamma^a [g^2(1 + h)] - h(\partial_r + f \partial_r) \Gamma^a f \\
- \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^b h \partial_r \Gamma^c f - \sum_{b+c+d=a, \ d < a} \sigma_{bcd} \Gamma^b h \Gamma^c \partial_r \Gamma^d f \\
+ \sum_{b+c=a, \ c < a} \sigma_{bc} [\Gamma^b v \partial_r \Gamma^c v - \Gamma^b f \partial_r \Gamma^c f].
\]

Direct computation yields that

\[
e^{g_f} r^a \tilde{\Gamma}^a Q_1 + r^a \tilde{\Gamma}^a Q_2 = v \partial_r [e^{g_f} r^a \epsilon^\Gamma a f] - (r - t)^{-\frac{3}{2}} e^{g_f} r^a v \Gamma^a f \\
- \frac{1}{2} r f \partial_r [e^{g_f} (\Gamma^a v^2 + r^a f^2)] + \frac{1}{2} (r - t)^{-\frac{3}{2}} e^{g_f} r^f ([\Gamma^a v]^2 + r^a f^2) \\
+ e^{g_f} r^a \Gamma^a [g^2(1 + h)] - \frac{1}{4} e^{g_f} h (\partial_r + f \partial_r) [\Gamma^a f]^2
\]
Substituting this into (169), we have
\[
\frac{1}{2} \frac{\partial_c}{\partial t} \left( e^{9r} \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right) + \frac{1}{2} \frac{\partial_r}{\partial t} \left( e^{9r} (1-f) \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right)
\]
\[
= \left( 1 + \frac{v}{1+v} \right) \partial_r \left( e^{9r} \frac{\|v\|}{(r-t)^\frac{2}{3}} \right) - \frac{1}{2} \frac{\partial_r}{\partial t} \left( e^{9r} \left( 1 + \frac{v}{1+v} \right) \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right)
\]
\[
- \left( r-t \right)^{-\frac{2}{3}} e^{9r} \left( v + f \right) \frac{\|f^a\|^2}{(r-t)^\frac{2}{3}} - \frac{1}{2} e^{9r} r \partial_n \left( \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right)
\]
(170)
where \( I \) is defined by (168).

Multiplying (170) by \( \frac{1+v}{1+v} \) and integrating it over \([0,t'] \times [0,\infty)\), we then see that
\[
\|\Gamma^a v(t',r)\|_{L^2}^2 + \|\Gamma^a f(t',r)\|_{L^2}^2 + \int_0^{t'} \|\Gamma^a (v + f)\|_{L^2}^2 dt
\]
\[
\leq C \|\Gamma^a v(0,r)\|_{L^2}^2 + \|\Gamma^a f(0,r)\|_{L^2}^2 + \int_0^{t'} \int_0^r I dr dt
\]
\[
+ \int_0^{t'} \int_0^r \left[ \partial_r \left( \frac{1+v}{1+v} \right) + \frac{\|f + v\|^2}{(r-t)^\frac{2}{3}} \right] \left( \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right) dr dt
\]
\[
+ \int_0^{t'} \int_0^r \left( \frac{1}{2} e^{9r} \left( \frac{r f(1+v)}{1+v} \right) \left( \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right) - e^{9r} \partial_r (1+v)^\frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right) dr dt
\]
\[
+ \int_0^{t'} \int_0^r e^{9r} \left( \partial_n + f \partial_r \right) \left( \frac{\|v\|^2 + \|f^a\|^2}{(r-t)^\frac{2}{3}} \right) dr dt
\]
(171)
First, we deal with the third line of (171). From Lemma 6.1–6.2, we know that
\[
\left| \partial_n \left( \frac{1+v}{1+v} \right) \right| = \left| \partial_n \left( 1 \frac{G}{1+v + G} \right) \right|
\]
\[
\leq C \frac{|\partial_n G|}{\|1+v + G\|_{L^\infty}} + \frac{|\partial_n (1+v + G)|}{\|1+v + G\|_{L^\infty}}
\]
\[
\leq C \frac{|\partial_n G|}{\|1+v + G\|_{L^\infty}} + \frac{|\partial_n G| + |\partial_n (1+v + G)|}{\|1+v + G\|_{L^\infty}} \leq C \left( t \right)^{-1.99} E_3(t),
\]
(172)
where we have used that
\[
\|1+v + G\|_{L^\infty} \leq \frac{1}{1 - \|\partial_n v\|_{L^\infty} - \|\Gamma^a v\|_{L^\infty} - \|G\|_{L^\infty}} \leq 2.
\]
By (108) and Lemma 6.2, we achieve
\[
\frac{|v + f|}{(r-t)^\frac{2}{3}} \leq C \left( t \right)^{-\frac{2}{3}} \left( v + f \right) + \frac{|\chi_1 (v + f)|}{(r-t)^\frac{2}{3}}
\]
\[
\leq C \left( t \right)^{-\frac{2}{3}} \left( E_2(t) + \varepsilon \right) \leq CM \varepsilon \left( t \right)^{-\frac{2}{3}},
\]
(173)
Next, we treat the fourth line of (171). It is easy to calculate that
\[
\frac{1}{2} \partial_r \left( \frac{rf(1+\tilde{v})}{1+v} \right) = \frac{1}{2} \frac{r(1-G)}{1+v} (\partial_r f + \frac{1}{r} f) + \frac{1}{2} R f G \partial_r \tilde{v} - \frac{1}{(1+v)^2}.
\]
Applying Lemma 6.2 again, we obtain
\[
\begin{align*}
&\int_0^t \int \chi_1 \left( \frac{1}{2} e^q \partial_r \left( \frac{rf(1+\tilde{v})}{1+v} \right) \right) \langle |\Gamma^a v|^2 + |\Gamma^a f|^2 \rangle - e^q r \partial_r \tilde{v} \Gamma^a v \Gamma^a f \right) \, dr dt \\
&\leq C \int_0^t \langle t \rangle^{-1.99} E_2(t) E^2_{|a|}(t) dt.
\end{align*}
\] (174)

On the other hand, by Lemma 6.1, we obtain
\[
\begin{align*}
&\int_0^t \int \chi_1 \left( \frac{1}{2} e^q \partial_r \left( \frac{rf(1+\tilde{v})}{1+v} \right) \right) \langle |\Gamma^a v|^2 + |\Gamma^a f|^2 \rangle - e^q r \partial_r \tilde{v} \Gamma^a v \Gamma^a f \right) \, dr dt \\
&\leq C \int_0^t \int \chi_1 \langle |\Gamma^a v|^2 + |\Gamma^a f|^2 \rangle \frac{1}{r} |f| + |G \partial_r f| + |G \partial_r \tilde{v}| + |f \partial_r G| \, r dr dt \\
&\quad + \int_0^t \int \chi_1 \langle |\partial_r f| |\Gamma^a (v + f)|^2 + |\partial_r (\tilde{v} + f) \Gamma^a v \Gamma^a f| \rangle \, r dr dt \\
&\leq C \int_0^t \int \langle \varepsilon (r - t)^{-\frac{3}{2}} |\Gamma^a (v + f)|^2 r dr dt + \int_0^t \langle t \rangle^{-\frac{3}{2}} E_2(t) E^2_{|a|}(t) dt.
\end{align*}
\] (175)

Finally, we turn our attention to the last line of (171). Applying the integration by parts to the fifth line of (171), we conclude from the equation of \( h \) in (7) that
\[
\begin{align*}
&\int_0^t \int e^q r (h(\partial_r + f \partial_r)) |\Gamma^a f|^2 \, dr dt \\
&\leq C \int_0^t \int \left\{ |h(\partial_r + \frac{1}{r}) f| + |\frac{h(1+|f|)}{r-t} | \right\} \langle |\Gamma^a f|^2 r dr dt \\
&\leq C M \int_0^t \langle t \rangle^{-\frac{3}{2}} E^2_{|a|}(t) dt.
\end{align*}
\] (176)

Therefore, substituting (172)–(176) into (171) derives (167). This completes the proof of Lemma 6.3.

Remark 5. The multiplier function \( e^q(r-t) \) in (169) is called the “ghost weight” by S. Alinhac in [3], which was first used to treat the global small data solution problem for the 2D quasilinear wave equation when both the null conditions are fulfilled.

6.2. The treatment on \( I \) near the light cone. In this subsection, we deal with \( \int_0^t \int \chi_1 \, dr dt \).

Lemma 6.4. For \( a \in \mathbb{N}_0^2 \) with \( |a| \leq N \), if \( (v, f, g, h) \) is a smooth solution of (7), then for small \( \varepsilon_0 > 0 \), there exists a positive constant \( C \) which is independent of
$t, t', \varepsilon, M$ and $M'$ such that the following inequality holds
\begin{equation}
\left| \int_0^{t'} \int \chi_1 I dr dt \right| \leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_N(t)E_{[a]}(t) dt + \sum_{b \leq a} \int_0^{t'} E_{N-5}(t) \left\| \frac{\Gamma^b(v + f)}{(r-t)^{\frac{3}{2}}} \right\|_{L^2}^2 dt + Q_a(t'),
\end{equation}
where
\begin{equation}
Q_a(t') := \begin{cases}
\int_0^{t'} \langle t \rangle^{-1} E_{[a]}(t)[E_{N-5}(t) + W_{N-3}(t)] dt, & |a| \leq N, \\
\int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_{[a]}^2(t)(E_N(t) + \varepsilon) dt, & |a| \leq N - 5.
\end{cases}
\end{equation}

**Proof.** Rewrite the nonlinearity $I$ defined by (168) as follows
\[ I = I_{11} + I_{12} + I_{13}, \]
where
\begin{align*}
I_{11} &:= \frac{e^q r(1 + \tilde{v})}{1 + \tilde{v}} \left\{ \frac{1}{r} \Gamma^a f \Gamma^c [g^2(1 + h)] + \sum_{b+c=a, \ c < a} \sigma_{bc} \frac{1}{r} \Gamma^a v \Gamma^b v \Gamma^c f \\
& \quad - \left\{ \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^a f \Gamma^b h \partial_t \Gamma^c f + \sum_{b+c+d=a, \ d < a} \sigma_{bcd} \Gamma^a f \Gamma^b h \Gamma^c f \partial_t \Gamma^d f \right\}, \\
I_{12} &:= \frac{e^q r(1 + \tilde{v})}{1 + \tilde{v}} \sum_{b+c=a, \ c < a} \sigma_{bc} \partial_t \Gamma^c (v + f) (\Gamma^a f \Gamma^b v - \Gamma^c v \Gamma^b f), \\
I_{13} &:= \frac{e^q r(1 + \tilde{v})}{1 + \tilde{v}} \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^a (v - f) \Gamma^b (v + f) \partial_t \Gamma^c f.
\end{align*}

By using Lemma 6.1–6.2 to $I_{11}$ directly yields
\begin{equation}
\left| \int_0^{t'} \int \chi_1 I_{11} dr dt \right| \leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_N(t)E_{[a]}^2(t) dr dt.
\end{equation}

Now we deal with $I_{12}$. For $|a| \leq N$, we conclude from Lemma 6.1–6.2 that
\begin{align*}
&\sum_{b+c=a, \ c < a} \| \chi_1 \Gamma^b v \partial_t \Gamma^c (v + f) \|_{L^2} \\
\leq C \sum_{b+c=a, \ |b| > N-6} \| \chi_1 \partial_t \Gamma^c (v + f) \|_{L^\infty} \| \Gamma^b v \|_{L^2} + \sum_{b+c=a, \ |b| \leq N-6, c < a} \| \chi_1 \partial_t \Gamma^c (v + f) \|_{L^2} \| \Gamma^b v \|_{L^\infty} \\
\leq CE_{[a]}(t)[(t)^{-\frac{3}{2}} E_N(t) + (t)^{-1} E_{N-5}(t)],
\end{align*}
(180)
While for $|a| \leq N - 5$, we obtain
\[
\sum_{b+c=a, \ c \leq a} \| \chi_1 \Gamma^b \varphi \partial_r \Gamma^c (v + f) \|_{L^2} \leq C \sum_{b+c=a, \ |c| < |a| \leq N-5} \| \chi_1 \partial_r \Gamma^c (v + f) \|_{L^\infty} \| \Gamma^b \varphi \|_{L^2} \leq C (t)^{-\frac{3}{2}} E_N(t) E_{|a|}(t). \tag{181}
\]

The treatment on the other term $\partial_r \Gamma^c (v + f) \Gamma^a \varphi \Gamma^b f$ in $I_{12}$ is the same.

Finally, we treat $I_{13}$. For $N \geq N_0 = 11$ and $b + c = a$ with $|a| \leq N$, it always holds that $|c| \leq N - 7$ or $|b| \leq N - 5$. In the former case, by applying Cauchy-Schwartz inequality, we see that
\[
\sum_{b+c=a, \ |c| \leq N-7} \int_0^t \chi_1 |\Gamma^a (v - f) \Gamma^b (v + f) \partial_r \Gamma^c f| r dr dt \leq C \sum_{b+c=a, \ |c| \leq N-7} \int_0^t \| \Gamma^a (v - f) \|_{L^2} \| \frac{\Gamma^b (v + f)}{(r-t)} \|_{L^2} \| (r-t) \chi_1 \partial_r \Gamma^c f \|_{L^\infty} dt \tag{182}
\]
\[
\leq C \sum_{b \leq a, \ |a| \leq N-5} \int_0^t E_{N-5}(t) \left| \frac{\Gamma^b (v + f)}{(r-t)} \right|_{L^2}^{2} dt + \int_0^t (t)^{-\frac{1}{2}} E_{N-5}(t) E_{|a|}^2(t) dt.
\]

While in the latter case $|b| \leq N - 5$ or the lower-order energy case $|a| \leq N - 5$, we conclude from (108) that
\[
\sum_{b+c=a, \ c \leq a, |b| \leq N-5} \int_0^t \chi_1 |\Gamma^a (v - f) \Gamma^b (v + f) \partial_r \Gamma^c f| r dr dt \leq \sum_{b+c=a, \ c \leq a, |b| \leq N-5} \int_0^t E_{|a|}(t) \left| \frac{\chi_1 \Gamma^b (v + f)}{(r-t)} \right|_{L^\infty} \| (r-t) (\chi_0 + \chi_1) \partial_r \Gamma^c f \|_{L^2} dt \tag{183}
\]
\[
\leq C \int_0^t (t)^{-\frac{3}{2}} E_{|a|}^2(t) (E_N(t) + \varepsilon) dt.
\]

Collecting (179), (180), (182) and (183) yields (177) with $|a| \leq N$. While, combining (179), (181), (182) and (183) implies (177) with $|a| \leq N - 5$. This completes the proof of Lemma 6.4. \hfill \Box

6.3. The treatment on $I$ away from the light cone. In this subsection, we establish the estimates for $\int_0^t \int \chi_0 I dr dt$.

**Lemma 6.5.** For $a \in \mathbb{N}_0^3$ with $|a| \leq N$, if $(v, f, g, h)$ is a smooth solution of (7), then for small $\varepsilon_0 > 0$, the following inequality holds
\[
\left| \int_0^t \int \chi_0 I dr dt \right| \leq C M \varepsilon E_{|a|}^2(t') + M \varepsilon \int_0^t (t)^{-1.49} E_{|a|}^2(t) dt + Q_a(t'), \tag{184}
\]
where $Q_a(t')$ is defined by (178).
Proof. At first, by \( v = \tilde{v} + G \), the nonlinearity \( I \) defined by (188) can be rewritten as

\[
I = I_{01} + I_{02} + I_{03},
\]

where

\[
I_{01} := -\frac{e^q r (1 + \tilde{v})}{1 + \tilde{v} + G} \left[ \sum_{b+c=a, \ c < a} \sigma_{bc} \Gamma^a f \Gamma^b \phi \Gamma^c \partial_t f \right] + \sum_{b+c+d=a, \ d < a} \sigma_{bcd} \Gamma^a f \Gamma^b \phi \Gamma^c \partial_t \Gamma^d \eta f,
\]

\[
I_{02} := \frac{e^q r (1 + \tilde{v})}{1 + \tilde{v} + G} \sum_{b+c=a, \ c < a} \sigma_{bc} \left[ \partial_t \Gamma^c \tilde{v} (\Gamma^a f \Gamma^b v - \Gamma^a v \Gamma^b f) + \Gamma^a v \Gamma^b v (\partial_t + \frac{1}{r}) \Gamma^c f \right]
\]

\[
I_{03} := \frac{e^q r (1 + \tilde{v})}{1 + \tilde{v} + G} \left\{ \frac{1}{r} \Gamma^a f \Gamma^b \phi [g^2 (1 + h)] \right\} \sum_{b+c=a, \ b < a, c < a} \sigma_{bc} \Gamma^c G (\Gamma^a f \Gamma^b v - \Gamma^a v \Gamma^b f).
\]

Next, we deal with \( I_{01} \). Note that \( |c| \leq N - 3 \) or \( |b| \leq N - 3 \) always holds for \( b + c = a \) with \( |a| \leq N \). If \( |a| \leq N \), by using Lemma 5.3 and 6.2 to the first term \( \Gamma^a f \Gamma^b \phi \Gamma^c \partial_t f \) in \( I_{01} \) directly, we have

\[
\sum_{b+c=a, c < a} \| \chi_0 \Gamma^b \phi \Gamma^c \partial_t f \|_{L^2} \leq C \sum_{b+c=a, \ |c| \leq N-3} \| \Gamma^b \phi \|_{L^2} \| \chi_0 \partial_t \Gamma^c f \|_{L^\infty} + \sum_{b+c=a, \ |b| \leq N-3, c < a} \| \Gamma^b \phi \|_{L^2} \| \chi_0 \partial_t \Gamma^c f \|_{L^2}
\]

\[
\leq C \langle t \rangle^{-1.99} E_N (t) E_{|a|} (t) + \langle t \rangle^{-1} W_{N-3} (t) E_{|a|} (t).
\]

While \( |a| \leq N - 5 \), we arrive at

\[
\sum_{b+c=a, c < a} \| \chi_0 \Gamma^b \phi \Gamma^c \partial_t f \|_{L^2} \leq C \sum_{b+c=a, \ |c| < |a| \leq N-5} \| \Gamma^b \phi \|_{L^2} \| \chi_0 \partial_t \Gamma^c f \|_{L^\infty}
\]

\[
\leq C \langle t \rangle^{-1.99} E_N (t) E_{|a|} (t).
\]

In addition, for the term \( \Gamma^a f \Gamma^b \phi \Gamma^c \partial_t \Gamma^d f \) in \( I_{01} \), we have

\[
\sum_{b+c+d=a, \ d < a} \| \chi_0 \Gamma^b \phi \Gamma^c \partial_t \Gamma^d f \|_{L^2} \leq C \langle t \rangle^{-\frac{3}{2}} E_N (t) E_{|a|} (t).
\]

Next, we only treat the term \( \Gamma^a f \Gamma^b \phi \partial_t \Gamma^c \tilde{v} \) in \( I_{02} \) since the other left terms can be analogously estimated. In this case, it always holds that \( |b| \leq N - 6 \) or \( |c| \leq N - 3 \).

Therefore, for \( |a| \leq N \), we obtain

\[
\sum_{b+c=a, c < a} \| \chi_0 \Gamma^b \phi \partial_t \Gamma^c \tilde{v} \|_{L^2} \leq C \sum_{b+c=a, \ |c| \leq N-3} \| \Gamma^b \phi \|_{L^2} \| \chi_0 \partial_t \Gamma^c \tilde{v} \|_{L^\infty}
\]

\[
+ \sum_{b+c=a, \ |b| \leq N-6, c < a} \| \chi_0 \partial_t \Gamma^c \tilde{v} \|_{L^2} \| \chi_0 \Gamma^b \phi \|_{L^\infty} + \| \chi_1 \Gamma^b \phi \|_{L^\infty} + \| \Gamma^b G \|_{L^\infty}
\]

\[
\leq C \langle t \rangle^{-1.99} E_N (t) E_{|a|} (t) + \langle t \rangle^{-1} E_{|a|} (t) E_{N-5} (t) + \langle t \rangle^{-\frac{3}{2}} E_N (t)
\]

\[
\leq C \langle t \rangle^{-\frac{3}{2}} E_N (t) E_{|a|} (t) + \langle t \rangle^{-1} E_{N-5} (t) E_{|a|} (t).
\]
While, for $|a| \leq N - 5$, we achieve
\[
\sum_{b+c=a, \quad c < a} \| \chi_0 \Gamma^b v \partial_r \Gamma^c v \|_{L^2} \leq C \sum_{b+c=a, \quad |c| < |a| \leq N - 5} \| \Gamma^b v \|_{L^2} \| \chi_0 \partial_r \Gamma^c v \|_{L^\infty}
\]
(190)
\[
\leq C(t)^{-1.99} E_N(t) E_{|a|}(t).
\]

Collecting (186)–(190) yields
\[
\left| \int_0^t \int \chi_0 (I_{01} + I_{02}) dr dt \right| \leq C \int_0^t (t)^{-\frac{3}{2}} E_N(t) E_{|a|}(t) dt
\]
\[
+ \left\{ \int_0^t (t)^{-1} E_{|a|}(t) [E_{N-5}(t) + W_{N-3}(t)] dt, \quad |a| \leq N,
\right. \\
\left. \int_0^t (t)^{-\frac{3}{2}} E_N(t) E_{|a|}(t) dt, \quad |a| \leq N - 5. \right. 
\]
(191)

Finally, we turn to the treatment of $I_{03}$. At first, we estimate the last term $\Gamma^b v^d \partial_r \Gamma^c G$ in $I_{03}$. In fact, it is not hard to check that for $|b| \leq N - 3$ or the lower order case of $|a| \leq N - 5$,
\[
\| \partial_r \Gamma^c G \Gamma^b f \|_{L^2} \leq C \| r \partial_r \Gamma^c G \|_{L^2} \| \frac{1}{r} \Gamma^b f \|_{L^\infty} \leq C(t)^{-\frac{3}{2}} E_N(t) E_{|a|}(t).
\]
(192)

While in the remaining case of $|b| > N - 3$, it holds $|c| \leq 2 \leq N - 6$. Consequently, we have
\[
\| \partial_r \Gamma^c G \Gamma^b f \|_{L^2} \leq C \| r \partial_r \Gamma^c G \|_{L^2} \frac{1}{r} \| \Gamma^b f \|_{L^2}
\]
\[
\leq C E_{|c|+1}(t) \| \frac{1}{r} \Gamma^b f \|_{L^2} \leq C(t)^{-1} E_{N-5}(t) E_{|a|}(t).
\]
(193)

Next we estimate the other two terms $\Gamma^a f \Gamma^c g^2 (1 + h)$ and $\Gamma^a f \Gamma^b v \partial_r \Gamma^c G$ in $I_{03}$. For $\Gamma^a = \partial^a X^{a2}$ with $a_1 \geq 1$, let $\partial_r \Gamma^a = \Gamma^a$ with $|a'| = |a| - 1 \leq N - 1$. It is easy to find that
\[
\Gamma^a f \Gamma^a [g^2 (1 + h)] = \partial_r \Gamma^a f \partial_r \Gamma^a [g^2 (1 + h)] = \sum_{b+c+d=a'} \sigma_{bcd} \partial_r \Gamma^a f \partial_r \Gamma^b g \Gamma^c g \Gamma^d (1 + h)]
\]
\[
= \sum_{b+c+d=a'} \sigma_{bcd} \partial_r \Gamma^a f [2 \partial_r \Gamma^b g \Gamma^c g \Gamma^d (1 + h) + \Gamma^b g \Gamma^c g \partial_r \Gamma^d h].
\]

Subsequently, we achieve
\[
\sum_{b+c+d=a'} \left[ \frac{1}{r} \Gamma^b g \Gamma^c g \partial_r \Gamma^d h \|_{L^2} \right] + \frac{1}{r} \partial_r \Gamma^b g \Gamma^c g \Gamma^d (1 + h) \|_{L^2} \right]
\]
\[
\leq C \sum_{b+c+d=a', \quad |d| \leq N - 4} \frac{1}{r} \Gamma^b g \Gamma^c g \|_{L^2} \partial_r \Gamma^d h \|_{L^\infty} + \sum_{b+c+d=a', \quad |d| > N - 4} \frac{1}{r} \Gamma^b g \Gamma^c g \|_{L^\infty} \partial_r \Gamma^d h \|_{L^2}
\]
\[
+ \sum_{b+c+d=a', \quad |b| \leq N - 3} \| \partial_r \Gamma^b g \|_{L^\infty} \left\| \frac{\Gamma^c g \Gamma^d (1 + h)}{r} \right\|_{L^2} + \sum_{b+c+d=a', \quad |b| > N - 3} \| \partial_r \Gamma^b g \|_{L^2} \left\| \frac{\Gamma^c g \Gamma^d (1 + h)}{r} \right\|_{L^\infty}
\]
\[
\leq C(t)^{-1} E_{|a'|+1}(t) [E_N(t) + W_{N-3}(t)].
\]
which implies

where we have used Lemma 2.5, 6.2 and (148).

Consequently, we achieve that for \( \Gamma \)

Next, we treat the case of \( \Gamma \)

Analogously, for the term \( \Gamma f \Gamma v \partial_r \Gamma G \) in \( I_{03} \), one has

which implies

where we have used Lemma 2.5, 6.2 and (148).

Consequently, we achieve that for \( \Gamma = \partial_t^a, S^a \) with \( a_1 \geq 1 \),

Next, we treat the case of \( \Gamma = X^{a_2} = X^l \) with \( l = |a| = a_2 \geq 0 \). When \( l = |a| = 0 \), the second term \( \sum_{b+c=a, \ b<a, c<a} \sigma_{bc} e^{\eta} \Gamma f \Gamma v \partial_r \Gamma G \) in \( I_{03} \) does not appear.

While \( l \geq 1 \), rewrite the scaling operator as \( X^l f = (t \partial_t + r \partial_r) X^{l-1} f \). Then applying the integration by parts with respect to \( t \), we arrive at that for all \( m = 1, \cdots, l - 1 \),

\[
\begin{align*}
\int_0^t \int_0^r & \left| \chi_0 e^{\eta} \left( 1 + \frac{v}{1 + v} \right) X^m v \partial_r X^{l-m} Grdrdt \right| \\
\leq & C \int_0^t \int_0^r \left| \chi_0 e^{\eta} \left( X^{l-1} f \right) X^m vr \partial_r X^{l-m} G \right| rdr \\
+ & \int_0^t \int_0^r \left| \chi_0 \partial_r X^{l-1} f X^m vr \partial_r X^{l-m} G \right| rdrdt \\
+ & \int_0^t \left| \partial_t \left[ \chi_0 e^{\eta} \left( 1 + \frac{v}{1 + v} \right) X^m vr \partial_r X^{l-m} G \right] \right| X^{l-1} f \left( \frac{r}{1 + v} \right) rdrdt.
\end{align*}
\]
Note that \( l - m \leq N - 6 \) or \( m \leq N - 6 \) always holds for \( N \geq N_0 = 11 \). Then, for the first integration in the second line of (195), we have

\[
\sum_{m=1}^{l-1} \int_0^{l'} \chi_0 t' \frac{X^{l-1}f}{r} X^m v \partial_r X^{l-m} G \left| r \right| dr \\
\leq C E_1^2 (t') \left[ \sum_{l-m \leq N-6} \left| r \partial_r X^{l-m} G \right|_{L^\infty} \right] \\
+ \sum_{m \leq N-6} \left( \| X^m G \|_{L^\infty} + \| \chi_0 X^m \tilde{v} \|_{L^\infty} + \| \chi_1 X^m v \|_{L^\infty} \right) \\
\leq CE_1^2 (t') [E_{N-5}(t') + \langle t' \rangle^{-\frac{1}{2}} E_N(t')] .
\]

Similarly to the treatment for the second term

\[
\int_0^{l'} \int_0^{l'} \chi_0 e^q \frac{1 + \tilde{v}}{1 + v} X^l f X^m v \partial_r X^{l-m} G r dr dt \\
\leq C E_2^2 (t') [E_{N-5}(t') + \langle t' \rangle^{-\frac{1}{2}} E_N(t')] + \int_0^{l'} \langle t \rangle^{-1.99} E_N(t) E_2^2 (t') dt \\
+ \left\{ \begin{array}{ll}
\int_0^{l'} \langle t \rangle^{-1.99} E_{N-5}(t) E_2^2 (t') dt, & |a| \leq N, \\
\int_0^{l'} \langle t \rangle^{-1.99} E_{N-5}(t) E_2^2 (t') dt, & |a| \leq N - 5,
\end{array} \right.
\]

Next, we deal with the first term \( X^l f X^l [g^2 (1 + h)] \) in \( I_{03} \). When \( l = |a| = 0 \), we achieve

\[
\left| \int_0^{l'} \int_0^{l'} \chi_0 \frac{1 + \tilde{v}}{1 + v} e^q X^l f X^l [g^2 (1 + h)] dr dt \right| \leq C \int_0^{l'} \frac{1}{r} \| \chi_0 f \|_{L^\infty} E_0^2 (t) dt \\
\leq C \int_0^{l'} \langle t \rangle^{-1.99} E_3 (t) E_0^2 (t) dt.
\]

Similarly to the treatment for the second term \( X^l f X^m v \partial_r X^{l-m} G \) in \( I_{03} \), we have that

\[
\left| \int_0^{l'} \int_0^{l'} \chi_0 \frac{1 + \tilde{v}}{1 + v} e^q X^l f X^l [g^2 (1 + h)] dr dt \right| \leq C I_{03}^1 + I_{03}^2 + I_{03}^3 + I_{03}^4 + I_{03}^5 .
\]

where

\[
I_{03}^1 := \left| \int_0^{l'} \int_0^{l'} \chi_0 t [1 + \tilde{v}] e^q X^{l-1} f [2g (1 + h)] \partial_r X^l g + g^2 \partial_r X^l h] r dr dt \right| , \\
I_{03}^2 := \int_0^{l'} \chi_0 t |X^l[f |g^2 (1 + h)] \frac{X^{l-1} f}{r} r dr , \\
I_{03}^3 := \int_0^{l'} \int_0^{l'} \chi_0 \partial_r X^{l-1} f X^l [g^2 (1 + h)] r dr dt , \\
I_{03}^4 := \int_0^{l'} \int_0^{l'} \partial_t \left( \chi_0 t e^q \frac{1 + \tilde{v}}{1 + v + G} \right) X^l [g^2 (1 + h)] \frac{X^{l-1} f}{r} r dr dt \right| , \\
I_{03}^5 := \int_0^{l'} \int_0^{l'} \chi_0 t [1 + \tilde{v}] e^q X^l f X^l [g^2 (1 + h)] \frac{X^{l-1} f}{r} r dr dt \right| ,
\]
We now deal with (199). It is easy to see that
\[
\frac{\partial_r}{\partial_t} X^l g = -X^l f (\partial_r g + \frac{1}{r} g) - f(\partial_r + \frac{1}{r}) X^l g - \sum_{m=1}^{l-1} X^{l-m} f(\partial_r + \frac{1}{r}) X^m g.
\]
Substituting this into \( I_{03}^1 \) in (198) yields
\[
\left| \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^q g(1 + h) X^{l-1} f \partial_r X^l g \, dr \, dt \right| \leq C I_{03}^{11} + I_{03}^{12} + I_{03}^{13},
\]
where
\[
I_{03}^{11} := \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^q g(1 + h)(\partial_r g + \frac{1}{r} g) X^l f X^{l-1} f \, dr \, dt,
\]
\[
I_{03}^{12} := \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^q f X^l (\partial_r + \frac{1}{r}) X^l g \, dr \, dt,
\]
\[
I_{03}^{13} := \sum_{m=1}^{l-1} \int_0^{t'} \int \chi_0 t^1 g(\partial_r + 1) X^m g \frac{X^{l-m} f X^{l-1} f}{r} \, dr \, dt.
\]

Analogously, we have that for the other term in the second line of (198)
\[
\left| \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^g g^2 X^{l-1} f \partial_r X^l h \, dr \, dt \right|
\]
\[
\leq C \left| \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^g g^2 \partial_r h X^l f X^{l-1} f \, dr \, dt \right|
\]
\[
+ \left| \int_0^{t'} \int \chi_0 t^1 \frac{1 + \tilde{v}}{1 + v} e^g g^2 \partial_r h X^l f X^{l-1} f \, dr \, dt \right|
\]
\[
+ \sum_{m=1}^{l-1} \int_0^{t'} \int \chi_0 t^1 g(\partial_r + 1) X^m h \frac{X^{l-m} f X^{l-1} f}{r} \, dr \, dt.
\]

We now deal with (199). It is easy to see that
\[
X^l f X^{l-1} f = \frac{1}{2} \partial_t \left( |X^{l-1} f|^2 \right) + r \partial_r X^{l-1} f X^{l-1} f. \]
Then we derive
\[
I_{03}^{11} \leq C \left| \int_0^{t'} \int \partial_t \left[ \chi_0 t^2 \frac{1 + \tilde{v}}{1 + v} e^q g(\partial_r g + g) \right] \frac{X^{l-1} f}{r} \, dr \, dt \right|^2 r \, dr
\]
\[
+ \left| \int \chi_0 t^2 g(\partial_r g + g) \left| \frac{X^{l-1} f}{r} \right|^2 r \, dr \right|
\]
\[
+ \int_0^{t'} \int \chi_0 t^1 g(\partial_r g + g) \partial_r X^{l-1} f \frac{X^{l-1} f}{r} \, dr \, dt.
\]
Thanks to (57), we deduce that
\[
\partial_r \partial_t g = -r \partial_r [f \partial_r g + \frac{1}{r} f g] = -r \partial_r X g + \frac{1}{r} r \partial_r \partial_t g - \partial_r f (\partial_r g + g) + \frac{1}{r} f g.
\]
This, together with
\[ ||r \partial_r X^m g||_{L^\infty} \leq C \|X^{m+1} g\|_{L^\infty} + t \| \partial_r X^m g \|_{L^\infty} \leq C E_{m+2}(t) + (t)^{-0.99} E_{m+3}(t), \]
yields
\[ ||r \partial_r \partial_t g||_{L^\infty} \leq C (t)^{-1.99} E_3(t) E_4(t). \] (200)

Therefore, we achieve
\[
I_{03}^1 \leq C \left( E_{[a]}^2 (t') [E_2(t') + W_1(t')] + \int_0^{t'} (t)^{-1.99} E_N(t) E_{[a]}^2 (t) dt \right)
+ \left\{ \begin{array}{ll}
\int_0^{t'} (t)^{-1.99} E_{N-5}(t) E_{[a]}^2 (t) dt, & |a| \leq N, \\
\int_0^{t'} (t)^{-1.99} E_N(t) E_{[a]}^2 (t) dt, & |a| \leq N - 5.
\end{array} \right. \] (201)

For \( I_{03}^2 \), applying the integration by parts with respect to \( r \), we arrive at
\[
I_{03}^2 \leq C \left[ \int_0^{t'} \chi \left| X^l \frac{1 + \hat{\nu}}{1 + \nu} (1 + h) X^l g \frac{\partial_r (f X^{l-1} f)}{r} \right| rdr dt \\
+ \int_0^{t'} \chi \left| \frac{g (1 + h) X^l g f X^{l-1} f}{r} \right| rdr dt \\
+ \int_0^{t'} \left| r \partial_r \left[ \chi \frac{1 + \hat{\nu}}{1 + \nu} (1 + h) \right] X^l g f X^{l-1} f \right| rdr dt \right] \] (202)
\[
\leq C \int_0^{t'} (t)^{-1.99} E_3(t) E_{[a]}^2 (t) dt.
\]

For the other terms \( I_{03}^{13}, I_{03}^2, I_{03}^3, I_{03}^4, I_{03}^5 \), applying the same analysis on the multi-index as in the above, we can obtain
\[
I_{03}^{13} + I_{03}^2 + I_{03}^3 + I_{03}^4 + I_{03}^5 \leq C E_{[a]}^2 (t') [E_{N-5}(t') + (t')^{-\frac{1}{2}} E_N(t') + W_{N-3}(t')] \\
+ \int_0^{t'} (t)^{-\frac{1}{2}} E_{[a]}^2 (t) [E_N(t) + W_{N-3}(t)] dt + Q_a (t'). \] (203)

Collecting (185)–(203), then (184) is proved. This completes the proof of Lemma 6.5. \( \square \)

7. Estimates of \( (g, h, w) \).

7.1. \( L^2 \) estimates of \( (g, h, w) \). In this subsection, we derive the \( L^2 \) estimate of \( (g, h, w) \) in (17).

**Lemma 7.1.** For \( a, a' \in \mathbb{N}_0 \) with \( |a| \leq N \) and \( |a'| \leq N - 1 \), if \((v, f, g, h)\) is a smooth solution of (7), then for small \( \varepsilon > 0 \), the following energy inequalities hold
\[
||\Gamma^a g(t', r) ||_{L^2}^2 \leq C ||\Gamma^a g(0, r) ||_{L^2}^2 + M \varepsilon E_{[a]}^2 (t') + Q_a (t'), \] (204)
\[
||\langle r \rangle \Gamma^a h(t', r) ||_{L^2}^2 \leq C \langle \langle r \rangle \Gamma^a h(0, r) ||_{L^2}^2 + M \varepsilon E_{[a]}^2 (t') + Q_a (t'), \] (205)
\[
||\langle r \rangle \Gamma^a w(t', r) ||_{L^2}^2 \leq C \langle \langle r \rangle \Gamma^a w(0, r) ||_{L^2}^2 + M \varepsilon E_{[a]}^2 (t') + Q_{a'} (t'), \] (206)
where \( Q_a (t') \) is defined by (178).
Proof. Proof of (204) and (205): The proof of (204) is much easier than that for (205). Thereby, we only deal with (205). It is easy to get
\[
\frac{d}{dt}(e^{q(t-t')}|\langle r \rangle \Gamma^a h(t,r)|^2) + e^q \frac{\langle r \rangle \Gamma^a h(t,r)}{\langle r-t \rangle^{\frac{3}{2}}} = 2\langle r \rangle e^{q} \Gamma^a h(t,r) \partial_t \Gamma^a h(t,r),
\]
where \( q'(s) = 2 \) and \( \lim_{s \to -\infty} q(s) = 0 \). Then integrating (207) over \([0,t'] \times [0,+\infty)\) yields
\[
\| \langle r \rangle \Gamma^a h(t',r) \|_{L^2}^2 + \int_0^{t'} \| \frac{\langle r \rangle \Gamma^a h(t,r)}{\langle r-t \rangle^{\frac{3}{2}}} \|_{L^2}^2 \, dt \\
\le C \| \langle r \rangle \Gamma^a h(0,r) \|_{L^2}^2 + \int_0^{t'} \int \langle r \rangle^2 e^{q} \Gamma^a h \partial_t \Gamma^a h r dr dt.
\]
(208)

At first, we deal with the case of \(|a| \le N\) in (208). According to the equation of \( h \) in (7), we see that
\[
\left| \int_0^{t'} \int \langle r \rangle^2 e^{q} \Gamma^a h \partial_t \Gamma^a h r dr dt \right| \le C J_1 + J_2 + J_3,
\]
where
\[
J_1 := \left| \int_0^{t'} \int \langle r \rangle^2 e^{q} f \Gamma^a h \partial_r \Gamma^a h r dr dt \right|,
\]
\[
J_2 := \left| \int_0^{t'} \int \langle r \rangle^2 e^{q} \Gamma^a h \Gamma^b f \partial_r \Gamma^c h r dr dt \right|,
\]
\[
J_3 := \sum_{b+c=a, b \prec a, c \prec a} \int_0^{t'} \int \langle r \rangle^2 \Gamma^a h \Gamma^b f \partial_r \Gamma^c h r dr dt.
\]
For \( J_1 \), applying the integration by parts with respect to \( r \), we have
\[
J_1 \le C \int_0^{t'} \int \langle r \rangle^2 |\Gamma^a h|^2 |\chi_1 \partial_r f| + |\chi_0 \partial_r f| + |\frac{f}{r}| + \frac{1}{r^2} |f| + \frac{1}{\langle r-t \rangle^{\frac{3}{2}}} |f| r dr dt
\]
\[
\le C \int_0^{t'} \int \langle t \rangle^{-\frac{3}{2}} E_3(t) E^2_{|a|}(t) dt + \int_0^{t'} \int \langle r \rangle^2 |\Gamma^a h|^2 |\chi_1 \partial_r f| + \frac{|f|}{\langle r-t \rangle^{\frac{3}{2}}} r dr dt
\]
(209)
\[
\le C \int_0^{t'} \int \langle t \rangle^{-\frac{3}{2}} E_3(t) E^2_{|a|}(t) dt + \int_0^{t'} \left| \langle r \rangle \Gamma^a h(t,r) \right|_{L^2}^2 E_3(t) dt.
\]

Comparing estimate (204) with the first term \( \Gamma^a f \Gamma^a [g^2(1+h)] \) in \( I_{03} \) of (185), the main difficulty for the estimates of \( J_2 \) and \( J_3 \) lies in the lack of the \( L^\infty \) and \( L^2 \) norm of \( \partial_r \Gamma^c h \). However, our key observation is: the scaling operator \( X = t \partial_t + r \partial_r \) can be used to overcome the loss of regularity on the variable \( r \), and we shall firstly take the estimate of \( t \partial_t \Gamma^c h \). Indeed, by the equation of \( h \) in (7), we arrive at for \(|c| \le N-1\),
\[
t \partial_t \Gamma^c h = - \sum_{c' + c'' = c} \frac{t \Gamma^c f}{r} [X \Gamma^{c''} h - t \partial_t \Gamma^{c''} h].
\]
(210)
In addition, we conclude from Lemma 6.1 and 6.2 that
\[
\frac{\|\Gamma'f\|}{r} \leq C \frac{|\chi_0 \Gamma'f|}{r} + (t)^{-1} |\chi_1 \Gamma'f| \leq C (t)^{-\frac{3}{2}} E|\alpha|+3(t). \tag{211}
\]
Substituting (211) into (210) yields
\[
|t\partial_t \Gamma^c h| \leq C \sum_{c' + c'' \leq c} \frac{t|\Gamma'f|}{r} |\chi \Gamma^c h| + \sum_{c' + c'' \leq c, |c'| \geq N-2} \frac{t|\Gamma'f|}{r} |t\partial_t \Gamma^c h|. \tag{212}
\]
Especially, if $|c| \leq N - 3$, instead of (212) we have
\[
|t\partial_t \Gamma^c h| \leq C \sum_{c' + c'' \leq c} \frac{t|\Gamma'f|}{r} |\chi \Gamma^c h|. \tag{213}
\]
On the other hand, we also get from (213) that for $|c| \leq N - 3$,
\[
|\langle r \rangle^{2}r\partial_t \Gamma^c h| \leq C \langle \langle r \rangle^{2} \chi \Gamma^c h \rangle + \langle \langle r \rangle^{2} t\partial_t \Gamma^c h \rangle
\]
\[
\leq C \langle \langle r \rangle^{2} \chi \Gamma^c h \rangle + \sum_{c' + c'' \leq c} \frac{t|\Gamma'f|}{r} \langle \langle r \rangle^{2} \chi \Gamma^c h \rangle \leq C \langle \langle r \rangle^{2} \chi \Gamma^c h \rangle \tag{214}
\]
and for $|c| \leq N - 4$,
\[
|\langle r \rangle^{2}r\partial_t \Gamma^c h| \leq C \langle \langle r \rangle^{2} \chi \Gamma^c h \rangle \leq CW_{|c|+1}(t). \tag{215}
\]
Now we turn to the treatment of $J_2$. For $|\alpha| \leq N - 5$, we conclude from (215) that
\[
\int \langle r \rangle^{2}|\Gamma^a h \Gamma^a f \partial_t h| rdr \leq C \int \langle r \rangle^{2} \frac{\Gamma^a f}{r} \Gamma^a h X h |r| dr
\]
\[
\leq C \left\| \frac{\Gamma^a f}{r} \right\|_{L^\infty} \int \langle r \rangle^{2} |\Gamma^a h| |X h| rdr \leq C (t)^{-\frac{3}{2}} E_{|\alpha|+3}(t) E_{|\alpha|}(t) E_1(t). \tag{216}
\]
Next, we consider the higher order case of $|\alpha| > N - 5$. Note that $\Gamma^a = \partial_t \Gamma^a'$ or $\Gamma^a = \chi \Gamma^a = (t\partial_t + r\partial_r) \Gamma^a$ holds with $|\alpha'| = |\alpha| - 1$. The case of $\Gamma^a = \partial_t \Gamma^a'$ is much easier than the other one and we omit it here. In the latter case, the treatment is similar to that for the first term $\Gamma^a f \Gamma^a [g^2(1 + h)]$ in $I_{03}$ of (185). Then we achieve
\[
J_2 = \left| \int_0^{t'} \int (r)^{2} e^t X \Gamma^a' f \partial_t h rdr dt \right|
\]
\[
\leq C \int \langle r \rangle^{2} t |X \Gamma^a' f \partial_t h| rdr + \int_0^{t'} \int (r)^{2} e^t t \partial_t h X \Gamma^a' f \partial_t h rdr dt
\]
\[
+ \int_0^{t'} \int (r)^{2} t |X \Gamma^a' f| d((r - t)^{-\frac{3}{2}} |\partial_t h| + |\partial_t \partial_t h|) rdr dt
\]
\[
+ \int_0^{t'} \int (r)^{2} |X \Gamma^a' \partial_t h \Gamma^a f r \partial_t h| rdr dt
\]
\[
=: J_{21} + J_{22} + J_{23} + J_{24}. \tag{217}
\]
By using (23), (215) and Lemma 6.2 to \( J_{21} \) directly, we obtain
\[
J_{21} \leq C \| (r) r \partial_r h \|_{L^\infty} \| \langle r \rangle X \Gamma^a f \|_{L^2} \left\| \frac{t' \Gamma^a f}{r} \right\|_{L^2} \\
\leq C \| (r) r \partial_r h \|_{L^\infty} \| \langle r \rangle X \Gamma^a f \|_{L^2} \left[ t' \left\| \frac{\chi_0 \Gamma^a f}{r} \right\|_{L^2} + \left\| \chi_1 \Gamma^a f \right\|_{L^2} \right] \tag{218}
\]
Next we estimate \( J_{22} \). It is not hard to see that
\[
\partial_t X \Gamma^a h = -X \Gamma^a f \partial_r h - f \partial_r X \Gamma^a h - \sum_{b+c=a', \ b < a'} (X \Gamma^b f \partial_r \Gamma^c h + \Gamma^b f \partial_r X \Gamma^c h). \tag{219}
\]
Substituting (219) into \( J_{22} \) yields
\[
J_{22} \leq C \left[ \int_0^t \int_0^t (r)^2 e^{q(t)} (t \partial_t + r \partial_r) \Gamma^a f \Gamma^a f \partial_r h \| r dr dt \right. \\
+ \left. \int_0^t \int_0^t (r)^2 e^{q(t)} \Gamma^a f \partial_r X \Gamma^c h \partial_r h \| r dr dt \right] \\
+ \sum_{b+c=a', \ b < a'} \int_0^t \int_0^t (r)^2 t \Gamma^a f \partial_r X \Gamma^b h \partial_r h \| r dr dt \\
+ \sum_{b+c=a', \ c < a'} \int_0^t \int_0^t (r)^2 t \Gamma^a f \partial_r X \Gamma^b h \partial_r h \| r dr dt \\
=: J_{122} + J_{22}^2 + J_{22}^3 + J_{322}^4.
\]
For \( J_{122} \), applying integration by parts with respect to \( t \), one has
\[
J_{122} \leq C \int_0^t \int_0^t (r)^2 t'^2 \| \Gamma^a f \partial_r h \|^2 r dr dt \\
+ \int_0^t \int_0^t (r)^2 t \| \partial_r h \|^2 (|r \partial_r \Gamma^a f| + \| \Gamma^a f \|) r dr dt \\
+ \int_0^t \int_0^t (r)^2 \| \Gamma^a f \|^2 t^2 (r - t)^{-\frac{1}{2}} \| \partial_r h \|^2 + t^2 \| \partial_r h \| \| \partial_r \partial_r h \| r dr dt \\
=: J_{122} + J_{22}^2 + J_{322}^3.
\]
The treatment of the term \( J_{122} \) is the same as for \( J_{21} \) (218), then
\[
J_{122} \leq C \| (r) r \partial_r h \|_{L^\infty} \left\| \frac{t' \Gamma^a f}{r} \right\|_{L^2} \leq CW_1^2(t') E^2_{a|}(t'). \tag{222}
\]
Analogously for \( J_{22}^2 \), we obtain
\[
J_{22}^2 \leq C \int_0^t \int_0^t (r)^2 \| \partial_r h \|_{L^\infty} \left[ \| \partial_r \Gamma^a f \|_{L^2}^2 + \left\| \frac{\Gamma^a f}{r} \right\|_{L^2}^2 \right] dt \\
\leq C \int_0^t (t)^{-1} W_1^2(t) E^2_{a|}(t) dt \tag{223}
\]
Before dealing with $J_{22}^{13}$, we start to treat $\partial_r \partial_h$ as follows

$$\langle \partial_x \rangle t \partial_t \partial_x \partial_h = \langle \partial_x \rangle t \partial_t \partial_x \partial_h \leq C \langle \partial_x \rangle t \partial_t \partial_x \partial_h + \langle \partial_x \rangle t \partial_t \partial_x \partial_h + \langle \partial_x \rangle t \partial_t \partial_x \partial_h$$

$$\leq C \langle \partial_x \rangle t^{-0.99} [E_4(t) + W_1(t)] + \frac{t f}{r} \langle \partial_x \rangle t \partial_t \partial_x \partial_h + \langle \partial_x \rangle t \partial_t \partial_x \partial_h$$

$$\leq C \langle \partial_x \rangle t^{-0.99} [E_4(t) + W_1(t)] + M \epsilon \langle \partial_x \rangle t \partial_t \partial_x \partial_h + \langle \partial_x \rangle t \partial_t \partial_x \partial_h$$

$$\leq C \langle \partial_x \rangle t^{-0.99} [E_4(t) + W_1(t)] + M \epsilon \langle \partial_x \rangle t \partial_t \partial_x \partial_h + \langle \partial_x \rangle t \partial_t \partial_x \partial_h$$

$$\leq C \langle \partial_x \rangle t^{-\frac{1}{2}} [E_4(t) + W_1(t)] + M \epsilon \langle \partial_x \rangle t \partial_t \partial_x \partial_h.$$  

(224)

This, together with

$$\langle \partial_x \rangle t^2 |r - t|^{-\frac{3}{2}} |r \partial_x h| \leq C \langle t \rangle \frac{1}{2} |\partial_x^2 r \partial_x h|^2$$

yields

$$J_{22}^{13} \leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_{a_1}^2(t) [E_4(t) + W_1(t)] dt.$$  

(225)

For $J_{22}^{3}$, by using integration by parts with respect to $r$ again, we achieve

$$J_{22}^{3} \leq C \int_0^{t'} \int (\langle t \rangle^2 t |X \Gamma \partial_x h||\partial_x^2 h||r^2 \partial_x h|) \, r \, dr \, dt$$

$$\leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_{a_1}^2(t) [E_4(t) + W_1(t)] dt,$$

(226)

where we have used $|\langle t \rangle |^2 \partial_x^2 h| \leq C |\langle t \rangle |^2 X \partial_x h| |\partial_x^2 h| + |\langle t \rangle | r \partial_x h| |\partial_x h| \leq C E_3(t) + W_1(t).$

Analogously for the remaining terms $J_{22}^{3}, J_{23}^{2}, J_{24}^{2}, J_{24}^{1}$ and $J_3$, we conclude from (214) that for the higher order case of $N - 5 < |a| \leq N$, we have

$$J_{22}^{3} + J_{22}^{3} + J_{23}^{2} + J_{24}^{2} + J_3 \leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_{a_1}^2(t) [E_{N-5}(t) + W_{N-3}(t)] dt$$  

(227)

and for the lower order case $|a| \leq N - 5$,

$$J_{22}^{3} + J_{22}^{3} + J_{23}^{2} + J_{24}^{2} + J_3 \leq C \int_0^{t'} \langle t \rangle^{-\frac{3}{2}} E_{a_1}^2(t) E_{N-5}(t) [E_N(t) + W_{N-3}(t)] dt.$$  

(228)

Collecting (208)–(228) yields (205).

Proof of (206): Now, we deal with estimate (206). Note that $\|\langle t \rangle \Gamma^x w\|_{L^2} = \|\langle t \rangle \Gamma^x ([1 + \nu](\partial_x g + \frac{1}{r^2} g))\|_{L^2} \leq C E_{|a|+1}(t)$. Since the equation of $w$ (16) is the same as for $h$ (see the fourth equation in (7)), the proof (206) is similar to (205).

Thus we complete the proof of Lemma 7.1. □
7.2. $W^{1,3}$ estimates of $(w, h)$. In this subsection, the $W^{1,3}$ estimate $(15)$ on $(w, h)$ will be shown.

**Lemma 7.2.** If $(v, f, g, h)$ is a smooth solution of $(7)$, then for small $\varepsilon_0 > 0$, the following energy inequalities for $(w, h)$ hold

$$W^3_{k-3}(t') \leq C \int_0^{t'} (t')^{-\frac{3}{2}} E_N(t) W^3_{k-3}(t) dt, \quad (229)$$

where $W_k(t)$ is defined by $(15)$.

**Proof.** We only focus on the proof for $h$ since the equation $(16)$ of $w$ is the same as the one of $h$ in $(7)$, thereafter the proof of $w$ is also the same.

Similar to Lemma 7.1, it is not hard to obtain

$$\frac{d}{dt} \left( e^{q(t-r)} \langle r \rangle^2 \Gamma^a h(t, r) \right) + e^q \left( \frac{\langle r \rangle^2 \Gamma^a h(t, r)}{\langle r - t \rangle^\frac{3}{2}} \right)^3 = 3e^q \langle r \rangle^6 |\Gamma^a h(t, r)| \Gamma^a h(t, r) \partial_t \Gamma^a h(t, r),$$

where $q'(s) = s^{-\frac{3}{2}}$ and $\lim_{s \to -\infty} q(s) = 0$. Integrating $(230)$ over $[0, t'] \times [0, +\infty)$ derives

$$\|\langle r \rangle^2 \Gamma^a h(t', r)\|_{L^3}^3 + \int_0^{t'} \left\| \langle r \rangle^2 \Gamma^a h(t, r) \right\|_{L^3}^3 dt$$

$$\leq C \left\| \langle r \rangle^2 \Gamma^a h(0, r) \right\|_{L^3}^3 + \int_0^{t'} \int (r)^6 e^q |\Gamma^a h| \Gamma^a h \partial_t \Gamma^a h r dr dt.$$

According to the equation of $\partial_t \Gamma^a h$ in $(65)$, we achieve

$$\left| \int_0^{t'} \int (r)^6 e^q |\Gamma^a h| \Gamma^a h \partial_t \Gamma^a h r dr dt \right|$$

$$\leq C \int_0^{t'} \int (r)^6 |\Gamma^a h|^3 |\partial_r f| + \frac{1}{r} |f| + \langle r - t \rangle^{-\frac{3}{2}} |f| r dr dt \quad (231)$$

and

$$+ \sum_{b+c=a} \int_0^{t'} \int (r)^6 |\Gamma^a h|^3 |r \partial_r \Gamma^a h| \left( \frac{\Gamma^b f}{r} \right)_{L^\infty} r dr dt.$$

Comparing with Lemma 7.1, the key point in the proof of this lemma lies in that we have to use the $L^\infty$ norm of $f$ and its derivative in the nonlinearities due to the lack of the $L^3$ norm of $f$. To this end, applying $(214)$ to the last integration in $(231)$, we obtain

$$\left| \int_0^{t'} \int (r)^6 |\Gamma^a h| \Gamma^a h \partial_t \Gamma^a h r dr dt \right|$$

$$\leq C \int_0^{t'} (t')^{-\frac{3}{2}} E_N(t) W^3_{k-3}(t) dt + \int_0^{t'} (t')^{-\frac{3}{2}} (r - t)^{-\frac{3}{2}} E_3(t) \|\langle r \rangle^2 \Gamma^a h(t, r)\|^3 r dr dt,$$

which yields

$$\|\langle r \rangle^2 \Gamma^a h(t', r)\|_{L^3}^3 \leq C \|\langle r \rangle^2 \Gamma^a h(0, r)\|_{L^3}^3 + \int_0^{t'} (t')^{-\frac{3}{2}} E_N(t) W^3_{k-3}(t) dt. \quad (232)$$
Next we turn to the treatment of $\partial_r \Gamma^a h(t, r)$. By repeating the above steps for $\Gamma^a h(t, r)$, we achieve

$$\| (r) \partial_r \Gamma^a h(t', r) \|_{L^3}^3 + \int_0^{t'} \left\| \frac{(r) \partial_r \Gamma^a h(t, r)}{(r - t)^\frac{3}{2}} \right\|_{L^3}^3 dt \leq C \| (r) \partial_r \Gamma^a h(0, r) \|_{L^3}^3 + \left| \int_0^{t'} \int (r) \partial_r \Gamma^a h | \partial_r \Gamma^a h | \partial_t \Gamma^a hrdrdt \right|.$$  \hspace{1cm} (233)

It is easy to find that $\partial_r \partial_t \Gamma^a h$ satisfies

$$\partial_r \partial_t \Gamma^a h = - \sum_{b+c=a} \sigma_{bc} [\partial_r \Gamma^b f \partial_r \Gamma^c h + \Gamma^b f \partial_r^2 \Gamma^c h].$$

Then we have

$$\left| \int_0^{t'} \int (r) \partial_r \Gamma^a h | \partial_r \Gamma^a h | \partial_t \Gamma^a h \partial_r \Gamma^a hrdrdt \right| \leq C \tilde{J}_1 + \tilde{J}_2,$$

where

$$\tilde{J}_1 := \int_0^{t'} \int (r) \partial_t \Gamma^a h^2 | \partial_r f | + \frac{1}{r} | f | + (r - t)^{-\frac{3}{2}} | f | drdt$$

$$+ \sum_{b+c=a, c \neq a} \int_0^{t'} \int (r) \partial_r \Gamma^a h^2 | \partial_r \Gamma^b f | | \partial_t \Gamma^c h | drdt,$$

$$\tilde{J}_2 := \sum_{b+c=a, c \neq a} \int_0^{t'} \int (r) \partial_r \Gamma^a h^2 | \Gamma^b f | | \partial_r^2 \Gamma^c f | drdt.$$  

By using (214) to $\tilde{J}_1$ again yields

$$\tilde{J}_1 \leq C \sum_{c \leq a} \int_0^{t'} (t)^{-\frac{3}{2}} (r - t)^{-\frac{3}{2}} E_N(t) | (r) \partial_r \Gamma^c h(t, r) |^3 drdt$$

$$+ \sum_{b+c=a, c \neq a} \int_0^{t'} \int (r) \partial_t \Gamma^a h^2 | \partial_r f | | \partial_r \Gamma^b f | | \partial_t \Gamma^c h | drdt$$

$$+ \sum_{b+c=a, c \neq a} \int_0^{t'} \int (r) \partial_r \Gamma^a h^2 | (t)^{-\frac{3}{2}} \partial_r \Gamma^b f | | \partial_r \Gamma^c h | drdt$$

$$\leq C \sum_{c \leq a} \int_0^{t'} (t)^{-\frac{3}{2}} (r - t)^{-\frac{3}{2}} E_N(t) | (r) \partial_r \Gamma^c h(t, r) |^3 drdt$$

$$+ \int_0^{t'} (t)^{-\frac{3}{2}} E_N(t) W^3_{N-3}(t) dt.$$
For $J_2$, it is easy to check that

$$|\partial_r r^2 \Gamma^c| \leq C |\partial_r X \Gamma^c| + |\partial_r \Gamma^c| + \sum_{c'+c''=c} |t \dot{\partial}_r \Gamma^c|, \text{ which derives}$$

$$J_2 \leq C \sum_{c \leq 0} \int_0^t \langle t \rangle^{-\frac{1}{2}} \langle r-t \rangle^{-\frac{3}{2}} E_N(t) \langle r \rangle^3 |\partial_r \Gamma^c| dr dt$$

which implies

$$J_2 \leq C \sum_{c \leq 0} \int_0^t \langle t \rangle^{-\frac{1}{2}} \langle r-t \rangle^{-\frac{3}{2}} E_N(t) \langle r \rangle^3 |\partial_r \Gamma^c| dr dt$$

Then (229) is achieved. Thus we complete the proof of Lemma 7.2. \(\square\)

8. Proof of Theorem 1.1.

Proof of Theorem 1.1. Note that for $|a| \leq N-1$,

$$\| (\partial_r + \frac{1}{r}) \Gamma a g(t,r) \|_{L^2} = \| \Gamma^a (\partial_r + \frac{1}{r}) g(t,r) \|_{L^2} = \| \Gamma^a \left( \frac{w}{1+w+G} \right) \|_{L^2}. $$

This, together with (167), (177), (184), Lemma 7.1–7.2 and assumption (19) implies

$$E_N^2(t) \leq C E_N^2(0) + M \varepsilon \int_0^t \langle t \rangle^{-1} E_N^2(t) dt,$$

$$E_{N-5}^2(t) \leq C E_{N-5}^2(0) + M \varepsilon \int_0^t \langle t \rangle M \varepsilon^{-\frac{1}{2}} E_{N-5}^2(t) dt,$$

$$W_{N-3}^3(t) \leq C W_{N-3}^3(0) + M \varepsilon \int_0^t \langle t \rangle M \varepsilon^{-\frac{1}{2}} W_{N-3}^3(t) dt.$$

From the assumption of the initial data (10), we know that $E_N(0) + W_{N-3}(0) \leq C \varepsilon$ holds. Then applying the Gronwall’s inequality to the above inequalities and using (139) and (147) yields

$$E_N(t) + X_N(t) + Y_N(t) \leq C \varepsilon (1 + t)^M \varepsilon, \text{ and}$$

$$E_{N-5}(t) + X_{N-5}(t) + Y_{N-5}(t) + W_{N-3}(t) \leq C \varepsilon M \varepsilon \leq \frac{1}{2} M \varepsilon,$$

where $M, M'$ are sufficiently large but fixed positive constants, and $\varepsilon_0 > 0$ is small enough. Consequently, by the continuity argument and the local existence of smooth solution to (4) with (3) we derive Theorem 1.1. \(\square\)

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