Existence and Non-Existence Criteria for Solutions of a Schrödinger Quasilinear System type

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Abstract
In this paper we analyze the existence of large positive radial solutions to some quasilinear elliptic systems. Also, a non-radially symmetric solution is obtained by using a lower and upper solution method. The equations are coupled by functions which are increasing with respect to all the variables.

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1 Introduction

In the present work we prove some existence and non-existence theorems for the solutions of a quasilinear elliptic pde’s system of Schrödinger type. Let \( m \in \{1, 2, 3, \ldots\} \) and \( r := |x| \) the Euclidean norm of \( x \in \mathbb{R}^N \). We begin now the precise statements of our existence theorems by assuming that the class of functions \( a_j, f_j \) (\( j = 1, \ldots, m \)) satisfy:

(A1) \( a_j : \mathbb{R}^N \to [0, \infty) \) are locally Hölder continuous functions of exponent \( \alpha \in (0, 1) \);

(C1) \( f_j : [0, \infty)^m \to [0, \infty) \) are continuously differentiable in each variable, \( f_j(0, \ldots, 0) = 0, f_j(s_1, \ldots, s_m) > 0 \) for \( s_i > 0 \) (\( i = 1, \ldots, m \));

(C2) \( f_j \) are increasing on \( [0, \infty)^m \) in each variable;

(C3) \( \int_1^\infty |F(s)|^{-1/p} ds = \infty \quad (F(s) = \int_0^s \sum_{i=1}^m f_i(t, \ldots, t) dt) \).

We are concerned with the following system of quasilinear elliptic partial differential equations in all of space

\[
\begin{cases}
\Delta_p u_1(x) = a_1(x) f_1(u_1(x), \ldots, u_m(x)), \\
\cdots \\
\Delta_p u_m(x) = a_m(x) f_m(u_1(x), \ldots, u_m(x)),
\end{cases}
\quad x \in \mathbb{R}^N
\]  

(1.1)

where \( N - 1 \geq p > 1 \), \( \Delta_p \) stands for the p-Laplacian operator defined by \( \Delta_p u := \text{div} \left( |\nabla u|^{p-2} \nabla u \right) \), \( 1 < p < \infty \).

Problems related to (1.1) have received an increased interest in the past several decades (see [1, 2, 3, 7, 10, 12, 15, 18, 19, 20] with their references). On the other hand, it is well known that the form (1.1) is
very simple and easy to be written, many important nonlinear partial differential equations arising from several areas of mathematics and various sciences including physics, the generalized reaction diffusion theory and chemistry sciences take this form. Particularly, one of the most important classes of such partial differential equations are the \textit{time-independent Schrödinger equation in quantum mechanics}

\[ \left( \frac{\hbar^2}{2p} \right) \Delta u = (V - E)u \]  

where \( \hbar \) is the Plank constant, \( p \) is the mass of a particle moving under the action of a force field described by the potential \( V \) whose wave function is \( u \) and the quantity \( E \) is the total energy of the particle, problems which falls into the class of equations discussed here. The equation (1.2) was invented at a time when electrons, protons and neutrons were considered to be the elementary particles (see [5] for details and some new device).

The modern structure of the nonlinear Schrödinger equation is much more complicated. Max Born says: \textit{"Who among us has not written the words ‘Schrödinger equation’ or ‘Schrödinger function’ countless times? The next generation will probably do the same, and keep his name alive"} which is true and in our case even we will refer to more general problem (1.1).

Our objective in the present work, in short, is to complete the principal results of [13] and [3] and other associated works (see for example, [1, 6, 9, 10, 11, 13, 15, 18, 20] and references therein) showing non-existence and existence of solutions for the similar problem (1.1).

Among the obtained results, two of them seem to be worth stressing. The first one is the problem of existence of a non-radially symmetric bounded solution to (1.1). The second one is to give a necessary and a sufficient condition for a positive radial solution of (1.1) to be large.

We summarize in the next theorems the main objectives of the paper:

**Theorem 1.1.** Assume that \( a_j (j = 1, \ldots, m) \) satisfy (A1), \( f_j \) satisfy (C1)-(C3) and that there exists a positive number \( \varepsilon \) such that

\[ \int_0^t t^{1+\varepsilon} \left( \frac{1}{\sum_{j=1}^m \varphi_j(t)} \right)^{2/p} dt < \infty \]  

where \( \varphi_j(t) = \max_{|x|=t} |a_j(x)| \) and \( r^{p/(p-1)} \sum_{j=1}^m \varphi_j(r) \) is nondecreasing for large \( r \) then system (1.1) has a nonnegative nontrivial bounded solution on \( \mathbb{R}^N \). If, on the other hand, \( a_j \) satisfy

\[ \int_0^\infty t^{1/(p-1)} \left( \frac{1}{\sum_{j=1}^m \psi_j(t)} \right)^{1/(p-1)} dt = \infty \]  

where \( \psi_j(t) = \min_{|x|=t} |a_j(x)| \), then system (1.1) has no nonnegative nontrivial entire bounded radial solution on \( \mathbb{R}^N \).

**Theorem 1.2.** Suppose that \( a_j : [0, \infty) \to [0, \infty) \) (\( j = 1, \ldots, m \)) are continuous spherically symmetric functions (i.e. \( a_j(x) = a_j(|x|) \)). If \( f_j \) satisfy (C1)-(C3) then the problem (1.1) has a non-negative non-trivial entire radial solution. Suppose furthermore that \( r^{p/(p-1)} \sum_{j=1}^m a_j(r) \) are nondecreasing for large \( r \). If, in addition \( a_j \) satisfies

\[ \int_0^\infty \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} a_j(s) ds \right)^{1/(p-1)} dt = \infty \]  

for all \( j = 1, \ldots, m \) (1.5)
then any nonnegative nontrivial solution \((u_1,\ldots,u_m)\) of (1.1) is large. Conversely, if (1.1) has a nonnegative entire large solution, then \(a_j\) satisfy
\[
\int_0^\infty r^{1+\varepsilon} \left( \sum_{j=1}^m a_j (r) \right)^{2/p} \, dr = \infty,
\] for every \(\varepsilon > 0\).

We remark that in the case \(p \neq 2\) the above results are new even for the situation \(m = 1\) analyzed in the work \([20]\), where the authors have a problem in the proof when \(N - 1 = p\) which is solved here.

The contents of the paper are organized as follows: in Section 2 we establish a preliminary result. Section 3 deals with the proof of Theorem 1.1, Section 4 is devoted to proving Theorems 1.2.

2 Preliminary results

For the statement of the next result we need some additional definitions.

**Definition 2.1.** A function \((w_1,\ldots,w_m) \in \left[ C^{1+\alpha}_{loc} (\mathbb{R}^N) \right]^m (\alpha \in (0,1))\) is called a lower solution of the problem (1.1) if
\[
\Delta^p w_j (x) \geq a_1 (x) f_1 (w_1 (x),\ldots,w_m (x)) \text{ for } x \in \mathbb{R}^N, \\
\ldots \\
\Delta^p w_m (x) \geq a_m (x) f_m (w_1 (x),\ldots,w_m (x)) \text{ for } x \in \mathbb{R}^N.
\]

**Definition 2.2.** We will say that \((v_1,\ldots,v_m) \in \left[ C^{1+\alpha}_{loc} (\mathbb{R}^N) \right]^m (\alpha \in (0,1))\) is an upper solution of the problem (1.1) if
\[
\Delta^p v_1 (x) \leq a_1 (x) f_1 (v_1 (x),\ldots,v_m (x)) \text{ for } x \in \mathbb{R}^N, \\
\ldots \\
\Delta^p v_m (x) \leq a_m (x) f_m (v_1 (x),\ldots,v_m (x)) \text{ for } x \in \mathbb{R}^N.
\]

The proof of Theorem 1.1 is based on the results below, which can be proved as in \([6\), Theorem 5.1, pp. 146], \([16\), Lemma 1, pp. 15] or \([17\) in the same time with \([12\).

**Lemma 2.1.** Make the same assumptions on \(a_j\) and \(f_j (j = 1,\ldots,m)\) as in Theorem 1.1. If the problem (1.1) has a pair of upper and lower bounded solutions \((v_1,\ldots,v_m)\) and \((w_1,\ldots,w_m)\) fulfilling \(w_j (x) \leq v_j (x), (j = 1,\ldots,m), \forall x \in \mathbb{R}^N\) then there exists a bounded function \((u_1,\ldots,u_m)\) belonging to \([ C^{1+\alpha}_{loc} (\mathbb{R}^N) \right]^m (\alpha \in (0,1))\) with
\[
w_j (x) \leq u_j (x) \leq v_j (x), \forall x \in \mathbb{R}^N
\]
and satisfying (1.1).
3 Proof of the Theorem 1.1

The proof is inspired by the corresponding ones for the $p = 2$ cases in [3] with some new ideas. Assume that (1.3) holds. We use the method of lower and upper solutions for the problem (1.1). We look for an upper solution $(w_1, ..., w_m)$ and a lower solution $(w_1, ..., w_m)$. To find a positive lower solution, we observe that an arbitrary positive solution $w_i$ ($i = 1, ..., m$) to the following auxiliary system

\[
\begin{align*}
\Delta_p w_1 (r) &= \varphi_1 (r) f_1 (w_1, ..., w_m) \quad \text{for } r := |x|, \quad x \in \mathbb{R}^N, \\
&\quad \quad \quad \vdots \\
\Delta_p w_m (r) &= \varphi_m (r) f_m (w_1, ..., w_m) \quad \text{for } r := |x|, \quad x \in \mathbb{R}^N,
\end{align*}
\]

(3.1)
is the best candidate. We shall only study the radial solutions of (3.1), hence always write (3.1) in the following radial version:

\[
(p - 1) w_1' (r)^{p-2} w_1'' + \frac{N - 1}{r} w_1' (r)^{p-1} = \varphi_1 (r) f_1 (w_1 (r), ..., w_m (r)), \\
\quad \quad \quad \vdots \\
(p - 1) w_m' (r)^{p-2} w_m'' + \frac{N - 1}{r} w_m' (r)^{p-1} = \varphi_m (r) f_m (w_1 (r), ..., w_m (r)).
\]

(3.2)

First we see that radial solutions of (3.2) are any positive solutions $(w_1, ..., w_m)$ of the integral equations

\[
w_1 (r) = \frac{1}{m} + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_1 (s) f_1 (w_1 (s), ..., w_m (s)) ds \right)^{1/(p-1)} dt,
\]

\[
\quad \quad \quad \vdots \\
w_m (r) = \frac{1}{m} + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_m (s) f_m (w_1 (s), ..., w_m (s)) ds \right)^{1/(p-1)} dt.
\]

To establish a solution to this system, we use successive approximation. Define, recursively, sequences $\{w_i^k\}_{i=1}^{k \geq 1}$ on $[0, \infty)$ by

\[
\begin{align*}
w_1^0 &= \ldots = w_m^0 = \frac{1}{m}, \quad r \geq 0, \\
w_1^k (r) &= \frac{1}{m} + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_1 (s) f_1 (w_1^{k-1} (s), ..., w_m^{k-1} (s)) ds \right)^{1/(p-1)} dt, \\
&\quad \quad \quad \vdots \\
w_m^k (r) &= \frac{1}{m} + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \varphi_m (s) f_m (w_1^{k-1} (s), ..., w_m^{k-1} (s)) ds \right)^{1/(p-1)} dt.
\end{align*}
\]

We remark that, for all $r \geq 0$, $i = 1, ..., m$ and $k \in \mathbb{N}$

\[
w_i^k (r) \geq \frac{1}{m},
\]

and that $\{w_i^k\}_{i=1}^{k \geq 1}$ are non-decreasing sequences on $[0, \infty)$. We note that $\{w_i^k\}_{i=1}^{k \geq 1}$ satisfy

\[
(p - 1) \left[ \left( w_i^k \right)^{p-2} w_i^k \right]'' + \frac{N - 1}{r} \left[ \left( w_i^k \right)^{p-1} \right] = \varphi_i (r) f_i (w_1^{k-1} (r), ..., w_m^{k-1} (r)), \\
\quad \quad \quad \vdots \\
(p - 1) \left[ \left( w_m^k \right)^{p-2} w_m^k \right]'' + \frac{N - 1}{r} \left[ \left( w_m^k \right)^{p-1} \right] = \varphi_m (r) f_m (w_1^{k-1} (r), ..., w_m^{k-1} (r)).
\]

(3.3)
Using the monotonicity of \( \{w_i^k\}_{i=1,...,m} \) yields
\[
\varphi_1 (r) f_1 (w_1^{k-1} (r), ..., w_m^{k-1} (r)) \leq \varphi_1 (r) f_1 (w_1^k, ..., w_m^k) \leq \varphi_1 (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right),
\]
\[
\varphi_m (r) f_m (w_1^{k-1} (r), ..., w_m^{k-1} (r)) \leq \varphi_m (r) f_i (w_1^k, ..., w_m^k) \leq \varphi_m (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right),
\]
and, so
\[
(p - 1) \left[ (w_1^k) \right]^{p-1} (w_1^k)'' + \frac{N - 1}{r} \left[ (w_1^k) \right]^p \leq \varphi_1 (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
\[
(p - 1) \left[ (w_m^k) \right]^{p-1} (w_m^k)'' + \frac{N - 1}{r} \left[ (w_m^k) \right]^p \leq \varphi_m (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
which implies that
\[
(p - 1) \left[ (w_1^k) \right]^{p-1} (w_1^k)'' + \frac{N - 1}{r} \left[ (w_1^k) \right]^p \leq \varphi_1 (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
\[
(p - 1) \left[ (w_m^k) \right]^{p-1} (w_m^k)'' + \frac{N - 1}{r} \left[ (w_m^k) \right]^p \leq \varphi_m (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
We choose \( R > 0 \) so that \( r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^m \varphi_j (r) \) are non-decreasing for \( r \geq R \). First we prove that \( w_i^k (R) \) and \( (w_i^k (R))' \), both of which are nonnegative, are bounded above independent of \( k \). To do this, let
\[
\phi_i^R = \max \{ \varphi_i (r) : 0 \leq r \leq R \}, \quad i = 1, ..., m.
\]
Using this and the fact that \( (w_i^k)' \geq 0 \) \( (i = 1, ..., m) \), we note that (3.4) and (3.5) yields
\[
(p - 1) \left[ (w_1^k) \right]^{p-1} (w_1^k)'' + \frac{N - 1}{r} \left[ (w_1^k) \right]^p \leq \varphi_1 (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
\[
(p - 1) \left[ (w_m^k) \right]^{p-1} (w_m^k)'' + \frac{N - 1}{r} \left[ (w_m^k) \right]^p \leq \varphi_m (r) \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
which implies
\[
\left\{ \sum_{i=1}^m \left[ (w_i^k)' \right]^p \right\} \leq \frac{p}{p - 1} \sum_{i=1}^m \phi_i^R \sum_{i=1}^m f_i \left( \sum_{i=1}^m w_i^k \right) \left( \sum_{i=1}^m w_i^k \right),
\]
Integrate this equation from 0 to \( r \). We obtain
\[
\sum_{i=1}^m \left[ (w_i^k)' \right]^p \leq \frac{p}{p - 1} \sum_{i=1}^m \phi_i^R \int_0^r \sum_{i=1}^m f_i (s, ..., s) ds, 0 \leq r \leq R.
\]
Since \( p > 1 \) we know that
\[
(a_1 + ... + a_m)^p \leq m^{p-1} (a_1^p + ... + a_m^p)
\]
for any non-negative constants $a_i \ (i = 1, \ldots, m)$. Using this inequality in (3.7) we have
\[
m^{1-p} \left[ \sum_{i=1}^{m} \left( w_i^k (r) \right) \right]^{p} \leq \frac{p}{p-1} \sum_{i=1}^{m} \phi_i^R \int_{1}^{m} \sum_{i=1}^{m} w_i^k (r) \sum_{i=1}^{m} f_i (s, \ldots, s) \, ds, \quad 0 \leq r \leq R,
\]
which yields
\[
\left( \sum_{i=1}^{m} w_i^k (r) \right)^{1/p} \leq \sqrt{\frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \phi_i^R} \left( \int_{1}^{m} \sum_{i=1}^{m} w_i^k (r) \sum_{i=1}^{m} f_i (s, \ldots, s) \, ds \right)^{1/p}, \quad 0 \leq r \leq R. \tag{3.9}
\]
Integrating the above equation between 0 and $R$, we have
\[
\int_{1}^{m} \sum_{i=1}^{m} \left( \sum_{i=1}^{m} w_i^k (r) \right)^{1/p} \left[ \int_{1}^{m} \sum_{i=1}^{m} f_i (s, \ldots, s) \, ds \right]^{-1/p} \, dt \leq \sqrt{\frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \phi_i^R} R. \tag{3.8}
\]
By the assumption C3), we now conclude that $\sum_{i=1}^{m} w_i^k (R)$ is bounded above independent of $k$ and using this fact in (3.9) shows that the same is true of $\left( \sum_{i=1}^{m} w_i^k (R) \right)^{1/p}$. Thus, the sequences $w_i^k (R)$ and $(w_i^k (R))^{1/p}$ are bounded above independent of $k$. Now let us verify that the non-decreasing sequences $w_i^k$ is bounded for all $r \geq 0$ and all $k$. Multiplying (3.9) by $\frac{p}{p-1} \frac{\rho(N-1)}{\rho(N-1)}$ and summing we have
\[
\left\{ r^{\frac{(N-1)}{p} \sum_{i=1}^{m} \left( w_i^k (r) \right)^{p}} \right\} \leq \frac{pr^{\frac{(N-1)}{p} \sum_{i=1}^{m} \varphi_i (r)}}{p-1} \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (r), \ldots, \sum_{i=1}^{m} w_i^k (r) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p}.
\]
and integrating gives
\[
\int_{R}^{r} \left\{ s^{\frac{(N-1)}{p} \sum_{i=1}^{m} \left( w_i^k (s) \right)^{p}} \right\} \, ds \leq \int_{R}^{r} \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (s) \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (s), \ldots, \sum_{i=1}^{m} w_i^k (s) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p} \, ds. \tag{3.10}
\]
Hence (3.8) in (3.10) gives
\[
\int_{R}^{r} \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \left( w_i^k (r) \right)^{p} \leq R \int_{R}^{r} \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \left( w_i^k (R) \right)^{p}
\]
\[
\leq \int_{R}^{r} \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (s) \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (s), \ldots, \sum_{i=1}^{m} w_i^k (s) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p} \, ds.
\]
and thus
\[
r^{\frac{(N-1)}{p} \sum_{i=1}^{m} \left( w_i^k (r) \right)^{p}} \leq \sum_{i=1}^{m} \varphi_i (r) \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (r), \ldots, \sum_{i=1}^{m} w_i^k (r) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p}
\]
\[
+ \int_{R}^{r} \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (s) \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (s), \ldots, \sum_{i=1}^{m} w_i^k (s) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p} \, ds.
\]
for $r \geq R$. Noting that, by the monotonicity of $s^{\frac{(N-1)}{p} \sum_{i=1}^{m} \varphi_i (s)}$ for $r \geq s \geq R$, we get
\[
r^{\frac{(N-1)}{p} \sum_{i=1}^{m} \left( w_i^k (r) \right)^{p}} \leq C + \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (r) \sum_{i=1}^{m} f_i \left( \sum_{i=1}^{m} w_i^k (r), \ldots, \sum_{i=1}^{m} w_i^k (r) \right) \left( \sum_{i=1}^{m} w_i^k \right)^{1/p},
\]
where
\[ C = R \frac{m(1-N)}{m^{p-1}} \sum_{i=1}^{m} \left[ \left( w_i^k (R) \right)^p \right]^{-1/p} , \]
which yields
\[ \left( \sum_{i=1}^{m} w_i^k \right)'^p \leq \left[ C R^{(1-N)/p} + \frac{p m^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (r) F \left( \sum_{i=1}^{m} w_i^k (r) \right) \right]^{1/p} . \]
(3.11)

Since \((1/p) < 1\) we know that
\[ (b_1 + b_2)^{1/p} \leq b_1^{1/p} + b_2^{1/p} \]
for any non-negative constants \(b_i \ (i = 1, \ldots, m)\). Therefore, by applying this inequality in (3.11) we get
\[ \left( \sum_{i=1}^{m} w_i^k \right)'^p \leq \sqrt{C} R^{(1-N)/(p-1)} + \sqrt{C} R^{(1-N)/(p-1)} \left[ F \left( \sum_{i=1}^{m} w_i^k (r) \right) \right]^{1/p} \]
Integrating the above inequality, we get
\[ \frac{m}{m^r} \int_{\sum_{i=1}^{m} w_i^k (R)}^r \left[ F (t) \right]^{-1/p} dt \leq \sqrt{C} R^{(1-N)/(p-1)} \left[ F \left( \sum_{i=1}^{m} w_i^k (r) \right) \right]^{-1/p} + \left( \frac{p m^{p-1}}{p-1} \sum_{i=1}^{m} \varphi_i (r) \right)^{1/p} . \]
(3.12)

Integrating (3.12) and using the fact that
\[ \left( \sum_{i=1}^{m} \varphi_i (s) \right)^{1/p} = \left( s^{p(1+\epsilon)/2} \sum_{i=1}^{m} \varphi_i (s) s^{-p(1+\epsilon)/2} \right)^{1/p} \leq \left( \frac{1}{2} \right)^{1/p} \left[ s^{1+\epsilon} \left( \sum_{i=1}^{m} \varphi_i (r) \right)^{2/p} + s^{-1-\epsilon} \right] \]
for each \(\epsilon > 0\), we have
\[ \int_{\sum_{i=1}^{m} w_i^k (R)}^r \left[ F (t) \right]^{-1/p} dt \leq \sqrt{C} \int_{R}^r t^{1-N/p} \left[ F \left( \sum_{i=1}^{m} w_i^k (t) \right) \right]^{-1/p} dt \]
\[ + \left( \frac{1}{2} \right)^{1/p} \left[ \frac{p m^{p-1}}{p-1} \int_{R}^r t^{1+\epsilon} \left( \sum_{i=1}^{m} \varphi_i (t) \right)^{2/p} dt + \int_{R}^r t^{-1-\epsilon} dt \right] \]
\[ \leq \sqrt{C} \left[ F \left( \sum_{i=1}^{m} w_i^k (R) \right) \right]^{-1/p} + \left( \frac{1}{2} \right)^{1/p} \left[ \frac{p m^{p-1}}{p-1} \int_{R}^r t^{1+\epsilon} \left( \sum_{i=1}^{m} \varphi_i (t) \right)^{2/p} dt + \frac{1}{\epsilon R^\epsilon} \right] . \]
(3.13)

Since the right side of this inequality is bounded independent of \(k\) (note that \(w_i^k (t) \geq 1/m\), so is the left side and hence, in light of C3), the sequence \(\left\{ \sum_{i=1}^{m} w_i^k \right\}_{k\geq1} \) is a bounded sequence and so \(\left\{ w_j^k \right\}_{j=1,...,m} \) are bounded sequence. Thus, for every \(x \in \mathbb{R}^N\), it makes sense to define \(w_j (|x|) := \lim_{k \to \infty} w_j^k (|x|) \) for all \(j = 1, \ldots, m\) and so \((w_1, \ldots, w_m)\) is a positive solution of (3.11).

Since, we have found upper bounds for \(\left\{ w_j \right\}_{j=1,...,m} \) we can let \(M\) be the least upper bound of \(\sum_{i=1}^{m} w_i\) and note that
\[ M = \lim_{r \to \infty} \sum_{i=1}^{m} w_i (r) . \]
Now let $\psi_i(t) = \min_{|x|=t} a_i(x)$ and $v_i (i = 1, ..., m)$ be the positive increasing bounded solutions of

$$
v_1 (r) = M + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \psi_1 (s) f_1 (v_1 (s), ..., v_m (s)) ds \right)^{1/p-1} dt,
$$

...\hspace{1cm}

$$
v_m (r) = M + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} \psi_m (s) f_m (v_1 (s), ..., v_m (s)) ds \right)^{1/p-1} dt,
$$

which, of course, satisfies (3.1) with $w_i$ replaced with $v_i$ and $\varphi_i$ replaced with $\psi_i$. It is also clear that $v_i \geq M$ (i = 1, ..., m). (The proof of the existence of $v_i$ and that it has the properties mentioned is similar to that given in the proof for $w_i$ and is therefore omitted). Thus we have an upper solution ($v_1, ..., v_m$) and a lower solution ($w_1, ..., w_m$). Then the standard upper-lower solution principle (see Lemma 2.1) implies that the problem (1.1) has a solution ($u_1, ..., u_m$).

We end this section analyzing the non-existence of solutions. For this, assume that (1.4) holds. Arguing by contradiction, let us assume that the system (1.1) has nonnegative non-trivial entire bounded solution ($u_1, ..., u_m$) on $\mathbb{R}^N$. Assuming $M_i = \sup_{x \in \mathbb{R}^N} u_i(x)$ (i = 1, ..., m) and knowing that $u'_i \geq 0$, we get $\lim_{r \to \infty} u_i(r) = M_i$ (i = 1, ..., m). Thus there exists R > 0 such that $u_i \geq M_i/2$ for $r \geq R$. From conditions of $f_i$, it follows that

$$
f_1^{1/(p-1)} (u_1, ..., u_m) \geq f_1^{1/(p-1)} (M_1/2, ..., M_m/2) := c_0^1
$$

(3.14)

$$
f_m^{1/(p-1)} (u_1, ..., u_m) \geq f_m^{1/(p-1)} (M_1/2, ..., M_m/2) := c_0^m
$$

for $r \geq R$. On the other hand

$$
u_1 \geq u_1 (0) + \int_0^r \left( t^{1-N} \int_0^t s^{N-1} \psi_1 (s) f_1 (u_1 (s), ..., u_m (s)) ds \right)^{1/(p-1)} dt,
$$

...\hspace{1cm}

$$
u_m \geq u_m (0) + \int_0^r \left( t^{1-N} \int_0^t s^{N-1} \psi_m (s) f_m (u_1 (s), ..., u_m (s)) ds \right)^{1/(p-1)} dt.
$$

(3.15)

Rearranging the terms, and by using the conditions (3.14) in (3.15) follows

$$
\sum_{i=1}^m u_i (r) \geq mc_1 + c_2 \left( \frac{1}{N} \right)^{1/(p-1)} \int_0^r t^{1-N} \sum_{i=1}^m \psi_i^{1/(p-1)} (t) dt \to \infty \text{ as } r \to \infty,
$$

where $c_1 = \min\{u_1 (0), ..., u_m (0)\}$ and $c_2 := \min\{c_0^1, ..., c_0^m\}$ (see also (1)). A contradiction to the boundedness of $\sum_{i=1}^m u_i (r)$. This concludes the proof. ■

4 Proof of the Theorem 1.2

We first notice that from [20] Theorem 2] the problem

$$
\Delta_p z (r) = \sum_{i=1}^m a_i (r) \sum_{i=1}^m f_i (z (r), ..., z (r)) \text{ for } r := |x|, x \in \mathbb{R}^N
$$

(4.1)

has a non-negative non-trivial entire solution. Moreover, for each $R > 0$, there exists $c_R > 0$ such that $z (R) \leq c_R$. Due to the fact that $z$ is radial, we have

$$
z (r) = z (0) + \int_0^r \frac{1}{t^{N-1}} \left( \int_0^t s^{N-1} \sum_{i=1}^m a_i (s) \sum_{i=1}^m f_i (z (s), ..., z (s)) ds \right)^{1/(p-1)} dt \text{ for all } r \geq 0.
$$
We choose $\beta_1 \in (0, z(0))$. Define the sequences $\{u^k_i\}_{i=1,...,m}^{k \geq 1}$ on $[0, \infty)$ by

$$
u_i^0 = ... = u^0_m = \beta_1 \text{ for all } r \geq 0$$

$$
u_i^0 (r) = \beta_1 + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} a_i (s) f_i (u_1^0 (s), ..., u_m^0 (s)) \right)^{(p-1)/2} ds dt,$$

$$\vdots$$

$$
u_i^k (r) = \beta_1 + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} a_i (s) f_i (u_1^k (s), ..., u_m^k (s)) \right)^{(p-1)/2} ds dt.$$

With the same arguments as in the proof of Theorem 1.1, we obtain that $\{u^k_i\}_{i=1,...,m}^{k \geq 1}$ are non-decreasing sequence on $[0, \infty)$. Because $z^\prime (r) \geq 0$ follows $0 < \beta_1 \leq z(0) \leq z(r)$ for all $r \geq 0$ and so

$$u_i^k (r) = \beta_1 + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} a_i (s) f_i (u_1^k (s), ..., u_m^k (s)) \right)^{(p-1)/2} ds dt$$

$$\leq z(0) + \int_0^r \left( \frac{1}{t^{N-1}} \int_0^t s^{N-1} a_i (s) f_i (z(s), z(s)) \right)^{(p-1)/2} ds dt = z(r).$$

Thus $u_i^k (r) \leq z(r)$ ($i = 1, ..., m$). Similar arguments show that $u_i^k (r) \leq z(r)$ for all $r \in [0, \infty)$ and $k \geq 1$.

Then, we may assume

$$(u_1 (|x|), ..., u_m (|x|)) := \left( \lim_{k \to \infty} u_i^k (|x|), ..., \lim_{k \to \infty} u_m^k (|x|) \right), \text{ for every } x \in \mathbb{R}^N$$

is an entire radial solution of system (1.1).

Now, let $(u_1, ..., u_m)$ be any non-negative non-trivial entire radial solution of (1.1) and suppose that $a_i$ ($i = 1, ..., m$) satisfies (1.5). Since $u_i$ ($i = 1, ..., m$) is nontrivial and non-negative, there exists $R > 0$ so that $u_i (R) > 0$. Since $u_i' \geq 0$, we get $u_i (r) \geq u_i (R)$ for $r \geq R$ and thus from

$$u_i (r) = u_i (0) + \int_0^r \left( \int_0^t s^{N-1} a_i (s) f_i (u_1 (s), ..., u_m (s)) \right)^{(p-1)/2} ds dt,$$

we obtain

$$u_i (r) = u_i (0) + \int_0^r \left( \int_0^t s^{N-1} a_i (s) f_i (u_1 (s), ..., u_m (s)) \right)^{(p-1)/2} ds dt$$

$$\geq u_i (R) + f_i^{1/(p-1)} (u_1 (R), ..., u_m (R)) \int_R^r \left( \int_t^R s^{N-1} a_i (s) \right)^{(p-1)/2} ds dt \to \infty \text{ as } r \to \infty,$$

for all $i = 1, ..., m$.

Conversely, if $f_i$ ($i = 1, ..., m$) satisfy (C1)-(C3) and $(w_1, ..., w_m)$ is a nonnegative entire large solution of (1.1), then $w_i$ satisfy

$$(p - 1) w_i^\prime (r)^{p-2} w_i'' + \frac{N-1}{r} w_i^\prime (r)^{p-1} = a_1 (r) f_1 (w_1, ..., w_m),$$

$$\vdots$$

$$(p - 1) w_m^\prime (r)^{p-2} w_m'' + \frac{N-1}{r} w_m^\prime (r)^{p-1} = a_m (r) f_i (w_1, ..., w_m).$$
Then, using the monotonicity of $r^{\frac{p(N-1)}{p-1}} \sum_{j=1}^{m} a_j(r)$ we can apply similar arguments used in obtaining Theorem 1.1 to get

$$\left( \sum_{i=1}^{m} w_i(r) \right)' \leq \left[ Cr^{\frac{p(1-N)}{p-1}} + \frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} a_i(r) F \left( \sum_{i=1}^{m} w_i(r) \right) \right]^{1/p}$$

$$\leq \sqrt[p]{C} r^{\frac{1-N}{p-1}} + \sqrt[p]{\frac{pm^{p-1}}{p-1} \sum_{i=1}^{m} a_i(r) \left[ F \left( \sum_{i=1}^{m} w_i(r) \right) \right]^{1/p}},$$

and hence as in (3.13), we get

$$\int_{\sum_{i=1}^{m} w_i(r)} \left[ F(t) \right]^{-1/p} dt$$

$$\leq \frac{\sqrt[p]{C}}{\left[ F \left( \sum_{i=1}^{m} w_i(R) \right) \right]^{1/p}} \int_{R}^{r} t^{1-\epsilon} dt + \sqrt[p]{\frac{1}{2} \left( \int_{R}^{r} t^{1+\epsilon} \left( \sum_{i=1}^{m} a_i(t) \right)^{2/p} dt + \int_{R}^{r} t^{-1-\epsilon} dt \right)}$$

$$\leq \sqrt[p]{C} \left[ F \left( \sum_{i=1}^{m} w_i(R) \right) \right]^{-1/p} \left( \frac{p-1}{p-N} R^{\frac{p-N}{p-1}} \right) + \sqrt[p]{\frac{1}{2} \left( \int_{R}^{r} t^{1+\epsilon} \left( \sum_{i=1}^{m} a_i(t) \right)^{2/p} dt + \frac{1}{\epsilon R^{\epsilon}} \right)}$$

$$\leq C_R + \int_{R}^{r} t^{1+\epsilon} \left( \sum_{i=1}^{m} a_i(t) \right)^{2/p} dt,$$

where

$$C_R = \sqrt[p]{C} \left[ F \left( \sum_{i=1}^{m} w_i(R) \right) \right]^{-1/p} \left( \frac{p-1}{p-N} R^{\frac{p-N}{p-1}} \right) + \frac{1}{\epsilon R^{\epsilon}}.$$

Passing to the limit as $r \to \infty$, we find that $a_j (j = 1, \ldots, m)$ satisfies (1.6). □

**Remark 4.1.** If (C1)-(C3) are satisfied then

$$\int_{1}^{\infty} \left( \int_{0}^{s} f_i(t, \ldots) dt \right)^{-1/p} ds = \infty, \ i = 1, \ldots, m.$$

**Remark 4.2.** (see [2]) If (C1)-(C2) and

$$\int_{1}^{\infty} \left( \sum_{i=1}^{m} f_i(s, \ldots, s) \right)^{-1/(p-1)} ds = \infty,$$

are satisfied, then

$$\int_{1}^{\infty} \left( \int_{0}^{s} \sum_{i=1}^{m} f_i(s, \ldots) ds \right)^{-1/p} dt = \infty.$$

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