The effective Hamiltonian as a necessary basis of the open quantum optical system theory

A M Basharov

Moscow Institute of Physics and Technology, 141700 Dolgoprudny, Russia
National Research Centre «Kurchatov Institute», 123182 Moscow, Russia

E-mail: basharov@gmail.com; http://basharov.me

Abstract. Based on a model open system, the kinetic equation obtained from the initial exact Hamiltonian with anti-rotating terms is shown to differ from the kinetic equation from the approximate effective Hamiltonian without anti-rotating terms. The paper proves necessity of transition from the exact Hamiltonian to the effective Hamiltonian, with the main idea of its construction being the requirement for the absence of rapidly varying terms in the interaction representation. This requirement is necessary to obtain the correct kinetic equation.

1. Introduction

In order to study dynamics of systems that weakly interact with the environment – the so-called open systems – it is necessary to derive a kinetic equation for the density matrix of this open system. This problem can be solved by different methods – ranging from diagonalization of the Hamiltonian of an open system and its environment to perturbation theory variants over a small parameter, which characterize the interaction with the environment. Diagonalization variants of the Hamiltonian are also referred to as the global approach to the theory of open quantum systems, with the use of perturbation theory characterizing the local approach. Among the theories of open quantum systems based on the perturbation theory, we will consider the theory in accordance with quantum stochastic differential equations (SDEs) [1-3]. Hence, it is important to describe the conditions for applying SDEs to open quantum optical systems and prove necessity of a procedure for building an effective Hamiltonian for the system and its environment.

There exist different perceptions of the effective Hamiltonian and methods of its building. Regarding the effective Hamiltonian, we bear in mind the one whose construction is based on the requirement for the absence of rapidly time-varying terms in the interaction representation [3]. Below, we will prove necessity of imposing such a requirement for optical open systems and demonstrate that the conservation of rapidly varying terms in the initial Hamiltonian gives rise to a kinetic equation corresponding to a problem other than the initial one. If the perturbation theory is applied directly to a kinetic equation derived for any reason, then, as a rule, the result appears to be incorrect. These cases are analyzed in [4-9].

2. Specific character of optical systems

The specific feature of an optical system is that its characteristic frequencies belong to the range hz. This range is due only to processes within the elements of the optical system, such as atoms,
molecules, quantum dots, microcavities, and is not directly related to processes in the environment of the quantum system. For convenience, characteristic time can be interpreted as the revolution time of an electron around a nucleus in an atom. This interpretation can prove to be useful for comparison with the characteristic time of the environment. The environment is usually a multimode field. In our case we mean a broadband multimode field if the Nyquist approximation is possible [11], i.e. in the region of characteristic frequencies the distribution of energy over the frequencies of the field of the environment can be considered constant, inherent for each characteristic frequency. Then the broadband field can be considered delta-correlated according to the Wiener-Khinchin theorem. The zero correlation time of a broadband field is quite natural for such problems, and in this situation we mean the approximation of this field by a white noise [2, 11]. From the physical standpoint, the zero correlation time corresponds to the case when the real correlation time of processes in the external electromagnetic field surrounding the open system should be minimal in the given system. More precisely, subsequently deriving the kinetic equation, we could formalize our problem in such a way that even in the formal model the time would be as minimal as possible so that it could be set equal to zero. From a physical point of view, two characteristic times determine the necessary requirements for the formulation of a formal model: one of them is the correlation time, the other is the characteristic time of an open system. Comparison of the characteristic dimensions of the elements of the optical system with the dimensions in its environment naturally leads to the idea that, which also corresponds to experimental data and is consistent with other considerations. Thus, in open quantum optical systems, the original formulation of the problem should be added with that of the model based on the initial Hamiltonian, i.e. the formulation where the minimum characteristic times of the effective Hamiltonian of the open system would exceed the correlation time of the environment. Here, the effective Hamiltonian means the one from which the kinetic equation of an open system can be derived by standard procedures – the quantum stochastic limit, SDE, or some other methods. If the initial Hamiltonian is retained in the standard procedures for deriving the kinetic equation using the representation of the delta-correlated field for the environment, then in the case of optical systems the environment correlation time is not minimal compared with all characteristic times.

It is to be emphasized that while discussing the relationship between the characteristic times of the environment and the elements of an open system, the concept of the characteristic time of an open system as the reciprocal of the involved (for example, resonant) transition frequency can be compared with the characteristic time for establishing equilibrium in the environment, for example, in a multimode field. And the latter is determined not by the characteristic frequency of the oscillators of the multimode field, but by the energy of their interaction both with the elements of the open system and with the boundaries of the entire system. Therefore, both the time of establishing equilibrium in the environment \( \tau_{env} \) and the correlation time are large as compared to \( \tau_{opt} \).

The initial Hamiltonian of an open system and its environment consists of the Hamiltonian of the "isolated" open system \( H_{opt} \), the Hamiltonian of the environment \( H_{env} \) and their interaction operator \( V_{int} \). If we pass to the interaction representation, then the interaction operator \( V_{int}(t) \) will contain rapidly and slowly varying terms, with the time scale for rapidly varying terms being just \( \tau_{opt} \). As long as the Hamiltonian has such terms, it is impossible to introduce the white noise approximation; therefore, one should exclude the terms (the simplest way is to omit them) before the derivation of the kinetic equation of an open system based on the white noise approximation. This is how the rotating wave approximation appears, which is widely used in the approach based on quantum stochastic differential equations [1]. However, this approach lacks the conditions of its applicability, which led some authors [11], to an erroneous conclusion about the violation of the second law of thermodynamics for open quantum systems in the local approach as compared to the global approach. In this case, there is a requirement for a correct procedure to eliminate rapidly varying terms in the Hamiltonian and to construct a new Hamiltonian.

Since quantum theory has unitary symmetry, it is only natural to base a procedure of eliminating rapidly varying terms within the Hamiltonian by means of unitary transformation of the initial
Hamiltonian. In obtaining a new Hamiltonian, we thereby describe a new model of an open system and its interaction with the environment. If one does not consider the unitary transformation diagonalizing the Hamiltonian, which underlies the global approach to open systems, then it is convenient to expand the generators of the unitary transformation and the transformed Hamiltonian itself in a series in the coupling constant of the open system with its environment. There arises a peculiar perturbation theory where the above mentioned principles of selecting terms in the transformed Hamiltonian are in a sense analogous to averaging classical differential equations, which contain rapidly and slowly varying terms. The method of averaging classical differential equations is known as the Krylov-Bogolyubov-Mitropol’skii method [12, 13]. The ideas of selecting the terms of a perturbation series based on the Hamiltonian’s unitary transformation and the requirement that there were no rapidly time-varying terms in the interaction representation were first put forward in [14]. That is what is referred to as the concept of “construction of an effective Hamiltonian” in the paper. A survey of variations of perturbation theory based on the unitary transformation can be found in [4].

In order to understand better why the requirement for the absence of rapidly time-varying terms of the effective Hamiltonian is essential, let us first discuss the classical Langevin approach to the analysis of Brownian motion.

3. Langevin’s analysis of Brownian motion

Let a particle of mass execute one-dimensional Brownian motion. To take into account its interaction with environment particles, we introduce a noise term $X(t)$ to Newton’s equation for the coordinate $x(t)$ of a Brownian particle:

$$m \frac{d^2 x(t)}{dt^2} = F_{fr} + x(t), \quad F_{fr} = -6\pi \eta a \frac{dx}{dt}. \quad (1)$$

Here, the term (force) $F_{fr}$ takes into account the friction in the environment ($\eta$ is the viscosity coefficient, $a$ is the size of a Brownian particle).

The noise term is expected to change so rapidly in time that the mean $< x(t)X(t) >$ is zero. Angle brackets denote time averaging, e.g. $< x(t)X(t) > = \frac{1}{\tau_{av}} \int \limits_{t}^{t+\tau_{av}} x(t')X(t')dt'$, $\tau_{av}$ is the averaging time scale.

Let us see what conditions are necessary for the equality $< x(t)X(t) > = 0$.

Equation (1) is not difficult to solve, whence we obtain:

$$< x(t)X(t) >= \dot{x}(0)\Gamma^{-1} < X(t) > - \dot{x}(0)\Gamma^{-1} < e^{-\Gamma t} X(t) > -$$

$$-m^{-1}\Gamma^{-1} \int \limits_{0}^{t} e^{-\Gamma t'} X(t')X(t)e^{\Gamma t'} dt' + m^{-1}\Gamma^{-1} \int \limits_{0}^{t} < X(t')X(t) > dt'. \quad (2)$$

The parameter $\Gamma = \frac{6\pi \eta a}{m}$ determines the time scale $\tau_d$ of the dynamics of Brownian particles $\tau_d \sim \Gamma^{-1}$.

The averaging suggests the correlation function of noise and correlation time $\tau_{cor}$ as a characteristic scale of the change in a stochastic value $X(t)$. In case

$$\tau_{cor} << \tau_{av} << \tau_d, \quad (3)$$

then

$$< x(t)X(t) >= -m^{-1}\Gamma^{-1} \int \limits_{0}^{t} e^{-\Gamma t'} < X(t')X(t) > e^{\Gamma t'} dt' + m^{-1}\Gamma^{-1} \int \limits_{0}^{t} < X(t')X(t) > dt'. \quad (4)$$
A short correlation time $\tau_{cor}$, i.e. the smallest of all possible times in executing (3), further necessitates the fulfillment of the delta-correlation property of the value $X(t)$, $<X(t')X(t)> = X^2_0 \delta(t-t')$. Then the desired result follows from (4): $<x(t)X(t)> = 0$. It is this requirement, along with condition (3), that led Langevin to the results [15] consistent with some other studies of Brownian motion. 

Equations (2) and (4) clearly show the necessity of condition (3) to impose the requirement of delta-correlation on the value $X(t)$ and to obtain the equality $<x(t)X(t)> = 0$.

4. Interaction with a two-level quantum particle of a classic stochastic field

Let us now consider the simplest open system - a two-level atom with the Hamiltonian $H_a$ and energy levels $|E_i>$ and $|E_2>$: $H_a = \sum_{j=1,2} E_j |E_j><E_j|$. Here $\omega_{opt} = \hbar(E_2 - E_1)$. The energy levels of opposite parity are to be considered, and $d_{a}(t) = \sum_{jk} d_{jk} |E_j><E_k|$ in the operator of the dipole moment of the atom is regarded as $d_{jj} = 0$. In a conventional case, an atom interacts with its surrounding multimode quantized electromagnetic field. We simplify the case and assume the atom to interact with the classical electromagnetic field of intensity $E(t)$ in the electric dipole approximation. This field could be represented in the form of the Fourier integral

$$E(t) = E^*(t) = \int e^{-i\omega t} b(\omega) d\omega, \quad b(\omega) = \frac{1}{2\pi} \int e^{i\omega t} E(t) d\omega, \quad b(\omega) = b^*(-\omega).$$

To simulate the environment of an open system with such a field, the latter should be a stationary stochastic process. It is to be noted that stationarity entails uncorrelated (independence) spectral components $<b(\omega)b^*(\omega')> = \delta(\omega-\omega') S(\omega)$, Where $S(\omega)$ is the intensity spectrum. If we take the natural approximation $S(\omega) = \text{const}$ for a certain spectral range and extend it to the entire spectrum region (Nyquist approximation), then

$$<E(t)E(t')> = \text{D}\delta(t-t').$$

It would be possible to immediately impose the requirement of delta-correlation, but the approach based on the Nyquist theorem seems to be more clear, and the requirement for stationarity $E(t)$, i.e. dependence of the correlator $<E(t)E(t')>$ only on the time difference $t-t'$: $<E(t)E(t')> = g(t-t')$ is more natural.

In the interaction representation, the Hamiltonian of an atom interacting with the environment and the Schrödinger equation for the wave vector of an atom $|\Psi(t)>$ are given by the expressions (polarization effects, level degeneracy, and recoil are neglected):

$$V_{int}(t) = -E(t)d_{a}(t), \quad d(t) = \sum_{jk} d_{jk} e^{i\Omega_{jk} t} |E_j><E_k|, \quad \Omega_{jk} = (E_j - E_k)/\hbar, \quad \Omega_{jk} = (E_j - E_k)/\hbar, \quad \Omega_{jk} = (E_j - E_k)/\hbar, \quad \Omega_{jk} = (E_j - E_k)/\hbar,$$

$$i\hbar \frac{d}{dt} |\Psi(t)> = V_{int}(t) |\Psi(t)> + |\Psi(t)> U(t) |\Psi(0)> + i\hbar \frac{d}{dt} U(t) = V_{int}(t)U(t), \quad U(t) = 1. \quad (5)$$

The solution to equation (5) can be written as a series

$$U(t) = I + \left( -\frac{i}{\hbar} \right) \int_{0}^{t} V_{int}(t')dt' + \left( -\frac{i}{\hbar} \right)^2 \int_{0}^{t} \int_{0}^{t'} V_{int}(t')V_{int}(t'')dt'dt'' + \ldots = \exp \left( \frac{-i}{\hbar} \int_{0}^{t} V_{int}(t')dt' \right). \quad (6)$$

Integrals (6) contain the value $\int_{0}^{t} E(t')dt = W(t)$. Under delta-correlated conditions $E(t') s$, the function $W(t)$ is continuous, but is not differentiable. As can be shown in different ways, the density
of the probability distribution function $\rho(w, t)$ of the value $W(t)$ satisfies the kinetic equation (the Fokker-Planck equation [10]):

$$\frac{\partial \rho(w, t)}{\partial t} = D \frac{\partial^2 \rho(w, t)}{\partial w^2}$$

with the solution

$$\rho(w, t | w_0, t_0) = \frac{1}{\sqrt{2\pi D(t-t_0)}} e^{\frac{(w-w_0)^2}{2D(t-t_0)}}$$

for the initial condition $\rho(w, t_0 | w_0, t_0) = \delta(w-w_0)$. The value $W(t)$ is called the Wiener process. It is sometimes referred to as a non-standard Wiener process, with the case $D = 1$ being classified as a standard one.

Mathematically, equation (5) appears to be indefinite in the Nyquist approximation. But it is possible to strictly define its integral “solution” (6). This suggests that the Wiener process integral

$$\int_{t_0}^{t} G(t')dW(t')$$

(i.e. the stochastic integral, be reasonably defined.

The conventional definition of the integral $\int_{t_0}^{t} G(t')dW(t')$ as the limit of partial sums

$$S_n = \sum_{i=1}^{n} G(\tau_i)(W(\tau_i) - W(\tau_{i-1}))$$

leads to its dependence on the specific selection of intermediate points $\tau_i$ on the intervals of the partition: $t_{i-1} \leq \tau_i \leq t_i$. It can be evident from the integral $\int_{t_0}^{t} W(t')dW(t')$

when using the properties of the Wiener process $W(t)$ – the statistical independence of its increments $W(t_i) - W(t_{i-1})$ from each other. For convenience and self-consistency, the Ito is chosen $\tau_i = t_{i-1}$ for the so-called non-anticipating functions $G(t)$, which are statistically independent at a moment of time $t$ from the future behaviour of the Wiener process. In this case, the limit of integral sums is understood as the root-mean-square limit. The resulting stochastic integral is called the Ito stochastic integral.

Based on the stochastic Ito integral, we can introduce consistent Ito differentials:

$$dW(t)dW(t) = Ddt, \quad [dW(t)]^{N+1} = 0, \quad N > 0, \quad dW(t)dt = dtdt = 0, \quad <dW(t)>) = 0.$$ (7)

Here angle brackets denote averaging, and differential relations should be understood as the fulfillment for any non-anticipating functions of integral equalities of the following type:

$$\int_{t_0}^{t} G(t')dW(t')dW(t') = \int_{t_0}^{t} G(t')dt'.$$

A formal solution (6) for the evolution operator can be rewritten in terms of the Itô stochastic integral:

$$U(t) = \bar{T} \exp \left\{ \int_{0}^{t} \left( d_{12} e^{-i\Omega_{1}t} R_+ + d_{21} e^{i\Omega_{2}t} R_- \right) dW(t') \right\}, \quad R_+ = |E_2 > <E_1|, \quad R_- = |E_1 > <E_2| = R_+^\ast.$$ (8)

This step is not established mathematically – but otherwise the integral (6) for white noise cannot be defined. An alternative way is to define a stochastic integral in the sense of Stratonovich [10]. Comparison of the two approaches suggests that the resulting kinetic equation for the atom be the same in this case. From the physical point of view, the selection of the integral in the Ito sense could be reasonably explained by assuming that the results of calculations for a given moment of time should not depend on the future behavior of the white noise / Wiener process.

The Ito differential $dU(t)$ is defined as $dU(t) = U(t + dt) - U(t)$. Then

$$dU(t) = \{ \exp(i(d_{12} e^{-i\Omega_{1}t} R_+ + d_{21} e^{i\Omega_{2}t} R_-) dW(t')) - 1 \} U(t).$$
In calculating the exponent by its expansion in a series and taking the Itô algebra (7) into account, the following classical SDEs \((U(t)\) is the matrix) are obtained:

\[
dU(t) = i(d_{12}e^{-\Omega_{21}t}R_+ + d_{21}e^{\Omega_{21}t}R_-)dW(t)U(t) - \frac{1}{2} D(d_{12}e^{-\Omega_{21}t}R_+ + d_{21}e^{\Omega_{21}t}R_-)^2 U(t)dt,
\]

At the same time (which can be easily verified by direct calculations), the Ito's differentiation rule is also satisfied

\[
d(U^+(t)U(t)) = (dU^+(t)U(t)) + U^+(t)dU(t)) + (dU^+(t))dU(t).
\]

This rule differs from the conventional Leibniz rule, but the choice of the Ito stochastic integral makes it possible to easily operate with SDE (7). If we had dealt with the stochastic Stratonovich integral, then the usual differentiation rule would have been obtained for differentials in the Stratonovich sense, but it would have been impossible to carry out the calculations presented below. Traditionally, the Stratonovich calculus is used to obtain SDE in the Stratonovich sense, but it is rewritten for further transformations in terms of Ito differentials [10]. In the author's opinion, this method is less universal for quantum theory than the approach proposed in the paper.

The study of the dynamics of the atom as an open system begins with the formulation of the kinetic equation for the density matrix \(\rho^a(t)\) of the atom. Here the calculations are standard and the kinetic equations are obtained as a result of the chain of transformations:

\[
dp^a(t) \equiv \rho^a(t + dt) - \rho^a(t) \quad \rho^a(t + dt) = \\
\langle \rho^a(t + dt) | U(t + dt) | \psi(0) \rangle | U^+(t + dt) | \psi(0) \rangle,
\]

\[
dp^a(t) = dU(t) | \psi(0) \rangle | U(t + U(t) + U(t)) | \psi(0) \rangle + dU^+(t) + \\
+ dU(t) | \psi(0) \rangle | U^+(t) | \psi(0) \rangle.
\]

For convenience, after averaging these equations can be written in the form

\[
\frac{d\rho^a(t)}{dt} = -\hat{\Gamma}_d \rho^a(t), \quad (9a)
\]

\[
\hat{\Gamma}_d \rho^a(t) = n[\rho^a(t)Y^+Y + Y^+\rho^a(t)Y - 2Y^+\rho^a(t)Y^+] + m[Y^+\rho^a(t)Y + \rho^a(t)Y^+Y^+ - 2Y^+\rho^a(t)Y^+] + m[Y^+\rho^a(t)Y + \rho^a(t)Y^+Y^+ - 2Y^+\rho^a(t)Y^+]. \quad (9b)
\]

The following notation has been introduced

\[
Y = d_{12}R_+ \quad Y^+ = d_{21}R_+ \quad n = D \quad m = D e^{i\Omega_{21}}.
\]

The operator \(\hat{\Gamma}_d\) is referred to as the relaxation operator.

The relation \(|m| = n\) emerges between the introduced parameters \(n\) and \(m\) for the considered case.

It should be emphasized that we considered a noise field as a noise wave without a carrier frequency and did not change the initial Hamiltonian of the atom, which had the form \(V_{int}(t)\) in the interaction representation. We expected to obtain a kinetic equation that would resemble an equation that is in good agreement with practice. Such an equation is obtained for a quantized electromagnetic field [11]. Based on the classical description of the electromagnetic field of the environment, equation (9) corresponds to the result [11] in the case of \(m = 0\).

Agreement with the experiment in [11] was reached owing to the rotating wave approximation. An important role of the rotating wave approximation in obtaining the correct kinetic equation for open systems was noticed long ago, see, for example, [16]. However, up to now the true reason for this
requirement has remained unclear. According to the analysis of Brownian motion given in the previous section, in order to use the white noise approximation and introduce the Wiener process, it is necessary that the correlation time of the electromagnetic field be the minimum characteristic time of the model. In keeping the initial Hamiltonian $V_{\text{int}}(t)$, we do not satisfy this requirement, since the minimum time is $\tau_{\text{opt}}$. The rotating wave approximation allows us to reject $\tau_{\text{opt}}$ in the interaction representation, which determined the correctness of the kinetic equations obtained in [1].

The rotating wave approximation was introduced in [17] as discarding of the "harmful" terms. The interaction with atoms of coherent fields is correctly explained in [18] based on the Krylov-Bogolyubov-Mitropol'skii averaging method, in [2,14] on the basis of both unitary transformation of the initial Hamiltonian and construction of an effective Hamiltonian. It is worth recalling that the major requirement was the absence of rapidly time-varying terms.

Thus, the use of the initial Hamiltonian to obtain the kinetic equation led to an incorrect result (9). Let us see to what the requirement to reject rapidly time-varying terms in order to exclude time $\tau_{\text{opt}}$ from the model leads and to what real case the result corresponds (9).

5. Interaction of a stochastic field with a two-level quantum particle in the approximation of the effective Hamiltonian

Let us consider the interaction of a noise wave with a carrier frequency that coincides with the transition frequency of a two-level system in the rotating wave approximation. This paper does not describe the effective Hamiltonian method, as it can be found in the author’s works, see, for example, [3-9,14], but points out that its construction on the basis of unitary transformation is also called the algebraic perturbation theory following the work [19], in which the Krylov-Bogolyubov-Mitropol'skii averaging method is algebraized. We will start with the description of a noise wave with a carrier frequency equal to the transition frequency in a two-level atom. It makes no sense to consider the detuning of its frequency from the resonant transition, since the Nyquist approximation is used, where the power spectrum of the noise wave is the same in the entire frequency range near the transition frequency.

We introduce a (non-standard) complex Wiener process. With this purpose in view, we will first consider two independent non-standard Wiener processes $\tilde{W}_1(t)$ and $\tilde{W}_2(t)$ with algebra

$$
d\tilde{W}_1(t)d\tilde{W}_1(t) = D_1 dt, \quad d\tilde{W}_2(t)d\tilde{W}_2(t) = D_2 dt, \quad d\tilde{W}_1(t)dt = d\tilde{W}_2(t)dt = dtdt = 0.
$$

They make it possible to define a complex Wiener process as $\tilde{W}(t) = \tilde{W}_1(t) + i\tilde{W}_2(t)$, where we have the following Ito algebra

$$
d\tilde{W}(t)d\tilde{W}(t) = (D_1 - D_2)dt, \quad d\tilde{W}(t)d\tilde{W}(t) = (D_1 + D_2)dt, \quad d\tilde{W}(t)dt = d\tilde{W}(t)dt = dtdt = 0.
$$

A complex general Wiener process is defined as $\tilde{W}(t) = \tilde{W}_1(t) + e^{i\varphi}\tilde{W}_2(t)$, so that

$$
d\tilde{W}(t)d\tilde{W}(t) = (D_1 + D_2 + i2\varphi)dt = ndt, \quad d\tilde{W}(t)dt = d\tilde{W}(t)dt = dtdt = 0,
$$

$$
d\tilde{W}(t)d\tilde{W}(t) = (D_1 + D_2)e^{i\varphi}dt = mdt, \quad d\tilde{W}(t)d\tilde{W}(t) = (D_1 + D_2)e^{-i\varphi}dt = m^*dt.
$$

It is evident that

$$\quad |m| \leq n, \quad (11)$$

since

$$\quad |D_1 + D_2e^{i\varphi}| = \sqrt{(D_1 + D_2e^{i\varphi})(D_1 + D_2e^{-i\varphi})} = \sqrt{D_1^2 + D_2^2 + 2D_1D_2\cos(2\varphi)} \leq D_1 + D_2.$$

We can further introduce stochastic processes with independent increments satisfying conditions (10) with additional requirements (11) and

$$<d\tilde{W}(t)> = <d\tilde{W}(t)> = 0.$$

Condition (11) is necessary for the components of a complex process of this type $\tilde{W}(t)$ and $\tilde{W}^*(t)$ to have a Gaussian distribution.
The introduced complex Wiener processes with algebra (10), (11) are convenient to describe stochastic electromagnetic fields with a carrier frequency. A complex Wiener process of general form allows us to write the interaction operator of a classical electromagnetic field with a two-level system in the approximation of a rotating wave in the form

\[ \tilde{V}_{int}(t)dt = -d_{12}R_d dW^\prime(t') - d_{21}R_u dW(t') . \]

It is to be emphasized that this expression makes sense only for substitution into operator (6) of the evolution of a quantum system in the form of T-exponent

\[ U(t) = \tilde{T} \exp \left( i \int_0^t \left( d_{12}R_d dW^\prime(t') + d_{21}R_u dW(t') \right) \right) . \]

In expanding the exponential in the expression \( dU(t) \) and using the algebra of differentials (10), we obtain

\[ dU(t) = i(d_{12}R_d dW^\prime(t') + d_{21}R_u dW(t'))U(t) - \frac{1}{2} (d_{12}^2 R_d R_u m^2 + |d_{12}|^2 R_d R_u n + |d_{12}|^2 R_u R_u n + d_{21}^2 R_d R_u m) U(t). \]

As a result of the described calculations we obtain the kinetic equation for the density matrix of a two-level atom in the form (9), while the parameters \( n \) and \( m \) are no longer connected by the relation \(| m | = n \) as in the absence of any resonance approximation, but obey the general relation (11) and are determined by the equations (10).

The case \( m \neq 0 \) corresponds to the so-called compression of the electromagnetic field, which requires special modulation conditions. The limiting case \(| m | = n \) corresponds to the maximum compression and can be interpreted differently in the case of a two-level system, as presented in the previous section.

The quantum case of limiting compression satisfies the condition \(| m | = \sqrt{n(n+1)} \). In [20], in the interaction problems, a broadband quantized electromagnetic field (both squeezed and unsqueezed) with a nonzero photon density with localized open optical systems was proposed to be modeled by a superposition of a quantized field with a zero photon density and a classical noise field. As a result, in the first order of the algebraic perturbation theory, allowance for the broadband quantized field is determined by the relaxation operator \( \Gamma_q \), as in [1], the classical noise field is determined by the relaxation operator \( \Gamma_{cl} \). Thus a kinetic equation is obtained which takes the form

\[ \frac{dp^\alpha(t)}{dt} = -\Gamma_q p^\alpha(t) - \Gamma_{cl} p^\alpha(t), \quad \Gamma_q p^\alpha(t) = p^\alpha(t)Y^\dagger Y + Y^\dagger Y p^\alpha(t) - 2Yp^\alpha(t)Y^+. \]

Equality of the compression parameter to zero \( m = 0 \) corresponds to the usual noise classical electromagnetic wave, which has a carrier frequency.

Thus, the execution of the resonance approximation or the refusal to execute it for the purpose of considering the supposedly "general" case is actually determined by the nature of the external influence. If the conditions for representing the external action are satisfied using the Wiener process, then the equation for the evolution operator in the form of a T-exponent is immediately solved, and the derivative of the kinetic equation for the density matrix of a quantum system serves as an example of analytical calculations with such expressions. If the resonance conditions are satisfied, it is necessary to develop a consistent algebraic perturbation theory [2], in the first order of which the above-considered rotating wave approximation appears. If one confines himself to solving the problem at the level of the kinetic equation for the density matrix of a quantum system, then within the framework of the resonant approximation all the features of the noise field, including its compression, are considered, while outside the framework of the resonant algebraic perturbation theory only the limiting case of the state of the noise field is described, but at the same time the constraint of the model of a multilevel quantum system with its two levels of a noise field without a carrier frequency...
often makes no physical meaning. It is also problematic to prove an SDE in the case when the characteristic time scale of a random change in the field is of the order of the inverse frequency of the quantum transition. To prove an SDE, it is necessary that the scale of the random field change be much less than the inverse frequency of the quantum transition, which seems unrealistic in the optical frequency range in terms of physics of electromagnetic fields. Nevertheless, in the case of a two-level system formed by sublevels of a quantum level, i.e. when the transition frequency in a two-level system is low, it is possible to consider the kinetic equation beyond the resonance approximation, provided that other levels of the quantum system are either absent or located far enough to apply the Nyquist approximation to the spectrum of the noise wave. Then formula (9) with $|m|=n$ produces the kinetic equation for the quantum system beyond the resonance approximation.

Acknowledgments
The author expresses his gratitude to Prof. V.V. Samartsev for the invitation to give a lecture at the School for Young Scientists "Coherent Optics and Optical Spectroscopy". This work is supported in part by the Russian Foundation for Basic Research (grant No. 19-02-00234a).

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