Critical travelling wave solution in one singularly perturbed parabolic equation

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Abstract. In the paper, we consider a new type of travelling waves, a profile of which is a pure slow heteroclinic canard. To construct the profile we use the so-called canard cascade. Such travelling waves are critical because they are a watershed between travelling waves of different types and simulate critical phenomena in various applied problems.

1. Introduction

Travelling wave solutions, as it is well known, are a kind of special solutions to partial differential equations characterized as solutions which are invariant to translation in space and hence are usually used to describe the process of transmission. Various travelling wave solutions including the so-called solitary wave, periodic wave, and kink (or named as front wave), even some singular wave solutions such as compacton, kinkon, and cuspon [1–4] have been found in many nonlinear wave equations by some different methods [1,2,5,6].

In this paper, we aim to introduce and study a new type of travelling waves in nonlinear singularly perturbed systems, a profile of which is a pure slow heteroclinic canard. Canard travelling waves are also named as critical travelling waves since they are a watershed between travelling waves of different types. They were first introduced and used under the analysis of combustion problems in [7].

A canard trajectory is a trajectory of a singularly perturbed system [8,9] of differential equations if it follows at first a stable invariant manifold, and then an unstable one [10]. In both cases, the distances travelled are not infinitesimally small. If a trajectory at first follows an unstable invariant manifold and then a stable one, it is called false canard. The term “canard” (or duck-trajectory) has been introduced by French mathematicians [11,12].

A canard cascade was introduced in [25] as a natural generalization of the term “canard”. For a case of planar system, if we take an additional function, whose arguments are a vector...
parameter and slow variable, we can glue the stable (attractive) and unstable (repulsive) slow invariant manifolds at several breakdown points at the same time. As a result, we obtain a canard cascade.

We point out that later the same term “critical travelling wave” was used in other sense, see [26] and references therein. It should be noted that W. Shen in [27, 28] used the more appropriate term a “generalized travelling wave”.

In the present work, we use the term “critical travelling wave” in the same sense as that in [7], i.e., this kind of waves may be considered as an object corresponding to a critical phenomenon in some applied problem. The geometric theory of singular perturbation [9] allows us to state the existence of this new type of travelling waves.

2. Singular perturbations and canards

Let us consider the following two-dimensional autonomous system:

\[ \frac{dx}{dt} = f(x, y, \mu), \]
\[ \varepsilon \frac{dy}{dt} = g(x, y, \mu), \]

where \( x, y \) are scalar functions of time, \( \varepsilon \) is a small positive parameter, \( \mu \) is an additional scalar parameter, and \( f \) and \( g \) are sufficiently smooth scalar functions. The set of points

\[ S_\mu = \{(x, y) : g(x, y, \mu) = 0\} \]

of the phase plane is called a slow curve of the system (1), (2).

We will need the following assumptions:

1) The curve \( S_\mu \) consists of ordinary points, i.e. at every such point \( (x, y) \in S_\mu \)

\[ [g_x(x, y, \mu)]^2 + [g_y(x, z, \mu)]^2 > 0. \]

2) Nonregular points, i.e. points at which

\[ g_y(x, y, \mu) = 0, \]

are isolated on \( S_\mu \) (all other points of \( S_\mu \) are called regular).

3) At nonregular points,

\[ g_{yy} \neq 0. \]

**Definition 2.1** [29] A nonregular point \( A \) of the slow curve \( S_\mu \) is called a jump point if

\[ g_{yy}(A)g_x(A)f(A) > 0. \]

**Definition 2.2** Part of \( S_\mu \) which contains only regular points are called regular. A regular part of \( S_\mu \), all points of which satisfy the inequality

\[ g_y(x, y, \mu) < 0 \quad (g_y(x, y, \mu) > 0), \]

is called stable (unstable).

In order to have a chance to consider more complicated situations when the nonregular point is not ordinary (self-intersections of the slow curve) or when it is ordinary but \( g_{yy}(x, y, \mu) = 0 \) at this point, we will use the following definition.

**Definition 2.3** A nonregular point \( A \) of the slow curve \( S_\mu \) is called a turning point if it divides stable and unstable parts of the slow curve.
Stable and unstable parts of the slow curve are zeroth order approximations of the corresponding stable and unstable slow invariant manifolds. The invariant manifolds lie in an $\varepsilon$–neighbourhood of the slow curve, except near jump (turning) points (see [30], p. 155 and references therein).

**Definition 2.4** Trajectories which at first pass along the stable invariant manifold and then continue for a while along the unstable invariant manifold are called canards or duck-trajectories.

Note that the term “canards” admits of much broader interpretation, see *ducks with relaxation* in [10]. Recall that the duck with relaxation includes a jump segment between stable and unstable parts.

**Definition 2.5** Trajectories which at first pass along the unstable invariant manifold and then continue for a while along the stable invariant manifold are called false canard trajectories.

The availability of the additional scalar parameter $\mu$ provides a possibility of gluing stable and unstable slow invariant manifolds together. In the case that a turning point is unique, as a result we obtain a smooth stable-unstable slow invariant manifold which is a *longest canard*. If it is necessary to glue stable and unstable slow invariant manifolds at several turning points, we need several additional parameters and as a result we obtain a cascade of canards or *canard cascade*. Otherwise, the canard cascade may be bounded by neighbouring turning points.

**Remark 1** In the case when the turning point is a singular point or there exist singular points on stable or unstable branches of slow invariant manifold, there is no way of telling that we obtain a canard trajectory. However, the object obtained as a result of gluing stable and unstable slow invariant manifolds will be called a canard if the movement along it is made from a stable branch to an unstable one (false canard if the movement along it is made from an unstable branch to a stable one).

**Definition 2.6** The continuous slow invariant manifold of (1), (2) which contains at least two canards or false canards is called a *canard cascade*.

The van der Pol equation is the most popular model used to illustrate canard trajectories. Consider the following system:

$$
\dot{x} = y - \mu, \quad \varepsilon \dot{y} = \nu(y) - x, \tag{3}
$$

where $\nu(y) = -1/3y^3 + y$. The jump points $(-1, -2/3)$ and $(1, 2/3)$ divide the slow curve $x = \nu(y)$ into stable and unstable parts. System (3) has a singular point at $y = \mu$, $x = \nu(\mu)$. Elementary analysis shows that the singular point is unstable when $-1 < \mu < 1$ and stable when $\mu > 1$ or $\mu < -1$. When $\mu \in (-1, 1)$ there will be a limit cycle. When $\mu > 1$ or $\mu < -1$ there will be no limit cycle. The question is how does the limit cycle disappear when $\mu$ passes through the value $-1$ or 1. In what follows we shall concentrate on the case when $\mu$ passes $-1$, the other case is entirely similar. It has been shown [12] that there exists a value $\mu = \mu_c(\varepsilon)$ such that for $\mu$ in an asymptotical small neighborhood of $\mu_c$ the limit cycle deforms into a canard. As $\mu$ diminishes (still in the neighborhood of $\mu_c$) the head of the canard trajectory gets smaller and at the next stage one has a canard trajectory without a head. The canard trajectory continues to shrink as $\mu$ tends to $-1$ and then disappears. For $\mu < -1$ all solutions of the system tend to the stable steady state, see details in [12].

The canards and corresponding values of the parameter $\mu$ allow asymptotic expansions in powers of the small parameter $\varepsilon$. Near the slow curve the canards are exponentially close, and have the same asymptotic expansion in powers of $\varepsilon$. An analogous assertion is true for corresponding parameter values $\mu$. Namely, any two values of the parameter $\mu$ for which canards
exist have the same asymptotic expansions, and the difference between them is given by $e^{(-1/c \varepsilon)}$
where $c$ is some positive number.

It is clear that if we would like to glue slow invariant manifolds at both jump points we need
an additional parameter.

3. Examples

3.1. Simplest canard

As the simplest system with a canard we propose

$$
\dot{x} = 1, \ \varepsilon \dot{y} = xy + \mu.
$$

It is clear that for $\mu = 0$, the trajectory $y = 0$ is a canard. The left part ($x < 0$) of it is
attractive and the right part ($x > 0$) is repulsive. These two parts are divided by a turning
point $x = 0$.

3.2. Simplest false canard

The simplest system with a false canard may be obtained by the modification of the previous
example.

$$
\dot{x} = 1, \ \varepsilon \dot{y} = -xy + \mu.
$$

For $\mu = 0$, the trajectory $y = 0$ plays the role of a false canard. The left part ($x < 0$) of it is
repulsive and the right part ($x > 0$) is attractive.

3.3. Simplest canard cascades

Consider now several examples of canard cascades. The differential system

$$
\dot{x} = 1, \ \varepsilon \dot{y} = x(x + 1)y
$$
gives us a simplest canard cascade $y = 0$ which consists of two unstable parts ($x < -1$ and
$x > 0$) and one stable part ($-1 < x < 0$ with two jump points $x = -1$ and $x = 0$.

For the next system

$$
\dot{x} = 1, \ \varepsilon \dot{y} = (x - 2)x(x + 2)y
$$

the canard cascade $y = 0$ consists of two unstable parts ($-2 < x < 0$ and $x > 2$) and two stable
parts ($x < -2$ and $0 < x < 2$) with three turning points $x = -2, x = 0,$ and $x = 2$.

The following example is an obvious generalization of two previous examples.

The system

$$
\dot{x} = 1, \ \varepsilon \dot{y} = (x - a_1)(x - a_2)\ldots(x - a_k)y
$$

possesses the simplest canard cascade $y = 0$ consisting of several repulsive and attractive parts
with $k$ turning points $x_j = a_j, \ j = 1, \ldots, k$.

It is possible to offer an example of system with an infinite number of turning points:

$$
\dot{x} = 1, \ \varepsilon \dot{y} = \cos(x)y.
$$

Consider now an example of a periodic canard cascade. For the planar differential system [30]

$$
\dot{x} = y, \ \varepsilon \dot{y} = x^2 + y^2 - \mu^2
$$

the circle $(x + \varepsilon/2)^2 + y^2 = \mu^2 - \varepsilon^2/4$ is a canard. The upper semicircle is repulsive
and the lower one is attractive. This canard (false canard) exists for any $\mu^2 > \varepsilon^2/4$. 
4. Canard cascade in a travelling wave problem

Consider the following singularly perturbed parabolic equation

\[ \varepsilon \frac{\partial u}{\partial t} = \varepsilon \delta \frac{\partial^2 u}{\partial x^2} - (u^2 + u + \varepsilon \gamma) \frac{\partial u}{\partial x} + (u - u^3)(u + \alpha), \]

where \( \alpha, \gamma, \) and \( \delta \) are positive constants. We are looking for solution to this equation of the type

\[ u = u(x - vt) = u(\xi), \]

i.e. we are interested in travelling wave solutions with speed \( v \), where \( \xi = x - vt \) is the phase of the wave. Such solution corresponds to a one-dimensional propagating wave. Substituting \( u = u(\xi) \) into the parabolic equation (4) we get

\[ -\varepsilon vu' = \varepsilon \delta u'' - (u^2 + u + \varepsilon \gamma)u' + (u - u^3)(u + \alpha), \]

where \( u' = du/d\xi \). This equation is equivalent to the system

\[ u' = w = f(u, w), \]
\[ \varepsilon \delta w' = (u^2 + u + \varepsilon \gamma - \varepsilon v)w + (u^3 - u)(u + \alpha) = g(u, w, \alpha, \gamma, \delta, v, \varepsilon), \]

which possesses the exact canard cascade of form \( w = 1 - u^2 \) under the critical values of \( v = v^* \) and \( \alpha = \alpha^* \), where

\[ v^* = \gamma, \quad \alpha^* = 1 + 2\varepsilon \delta. \]

Indeed, the slow curve of the system (5), (6) is described by the equation

\[ (u^2 + u + \varepsilon \gamma - \varepsilon v)w + (u^3 - u)(u + \alpha) = 0, \]

which for \( v = \gamma \) takes the form

\[ w = (1 - u)(u + \alpha). \]

Two real roots of the quadratic equation

\[ \frac{\partial g}{\partial w} = u^2 + u + \varepsilon(\gamma - v) = 0 \]

give the abscissas of turning points. These turning points, say, \( A_1 \) and \( A_2 \), divide the slow curve into three parts, stable and unstable, which are zeroth order approximations (\( \varepsilon = 0 \)) for the corresponding slow invariant manifolds of the system (5), (6). Figure 1 shows the slow curve for the case \( v = \gamma \). The part of the slow curve with \(-1 < u < 0\) is stable since

\[ \frac{\partial g}{\partial w} = u^2 + u < 0 \]

give here. Two other parts of the slow curve with \( u < -1 \) or \( u > 0 \) are unstable since \( \partial g/\partial w > 0 \).

For the case \( \alpha = \alpha^* \), the system (5), (6) has the exact solution \( w = 1 - u^2 \) with multiple changes of its stability: the part with \( u < 0 \) is a false canard while the part with \( u > -1 \) is a canard. Thus, \( w = 1 - u^2 \) describes the canard cascade which lies in \( \varepsilon \)-neighborhood of the slow curve, see Figure 2. The canard cascade contains the solution of the system (5), (6) satisfying the boundary conditions

\[ \lim_{\xi \to -\infty} u(\xi) = -1, \quad \lim_{\xi \to -\infty} w(\xi) = 0; \]
\[ \lim_{\xi \to +\infty} u(\xi) = 1, \quad \lim_{\xi \to +\infty} w(\xi) = 0, \]
Figure 1. The slow curve of the system (5), (6) for the case \( v = \gamma \). The turning points \( A_1(-1; 2\alpha - 2) \) and \( A_2(0, \alpha) \) separate the stable part (II) and unstable parts (I and III) of the slow curve.

Figure 2. The slow curve (red) and the canard cascade (black) of the system (5), (6); \( \varepsilon = 0.1, \delta = 0.3 \).

that is, it corresponds to a heteroclinic trajectory of the singularly perturbed system connecting the equilibria \((-1, 0)\) and \((1, 0)\).

It should be noted that under the critical values of \( v = v^* \) and \( \alpha = \alpha^* \) the parabolic equation (4) has a critical travelling wave solution with the canard cascade profile, see Figure 3.

The term critical travelling wave is suggested for the following reason. Such travelling waves of a new type play the role of a watershed between travelling waves with essentially different dynamics [7]. In other words, critical travelling waves can be used for modelling of critical phenomena in various applied problems.

5. Conclusions

In [7], by the use of the geometric theory of singular perturbations the existence of the so-called canard travelling waves was stated in the study of combustion waves for an autocatalytic reaction in the non-adiabatic case. It was shown that it is possible to choose a control parameter in such a way that the projection of the associated heteroclinic trajectory is located in a small neighborhood of the canard trajectory characterizing the occurrence of a critical regime. The canard travelling wave separates two types of waves corresponding to the slow combustion regime and to the thermal explosion.

In the present paper, a canard cascade is used to find the critical travelling wave, which is a special type of kink wave or front wave. The main feature of the considered travelling wave is that its profile is a pure slow heteroclinic canard. It should be noted that slow/fast heteroclinic waves of canard type were also considered in [31].
In [9,25], it was suggested an approach for canard cascade constructing for the case when \(g\) in (1), (2) is an \(n\)th–degree polynomial in \(y\) and corresponding slow curve has \(k\) \((k \leq n - 1)\) jump points. According to this approach, to obtain a canard cascade which contains all these points, we need to have \(k\) independent parameters. For instance, we can consider \(\mu\) as a function \(\mu = \mu(x, \lambda)\) depending on the slow variable \(x\) and the \(k\)–vector \(\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)\), where vector \(\lambda\) is a function of \(\varepsilon\): \(\lambda = \lambda(\varepsilon)\). In [25], as a variant, it was suggested to consider the function \(\mu\) as a polynomial in \(x\), i.e., \(\mu = \lambda_k x^{k-1} + \lambda_{k-1} x^{k-2} + \ldots + \lambda_1\).

For system (5), (6), \(\alpha\) and \(v\) play the role of the gluing parameters because they are coefficients in different degrees of the fast variables \(w\).

Critical travelling waves can be used for modelling of critical phenomena in various applied problems.

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Figure 3. The \(u\)–profile of the critical travelling wave of the system (5), (6).
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