APPLICATION OF SOFT COMPUTING

Numerical approach for differential-difference equations having layer behaviour with small or large delay using non-polynomial spline

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Accepted: 6 July 2021 / Published online: 18 August 2021 © The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2021

Abstract

A numerical approach is suggested for the layer behaviour differential-difference equations with small and large delays in the differentiated term. Using the non-polynomial spline, the numerical scheme is derived. The discretization equation is constructed using the first-order derivative continuity at non-polynomial spline internal mesh points. A fitting parameter is introduced into the scheme with the help of the singular perturbation theory to minimize the error in the solution. The maximum errors in the solution are tabulated to verify the competence of the numerical method relative to the other methods in literature. We also focussed on the impact of large delays on the layer behaviour or oscillatory behaviour of solutions using a special mesh with and without fitting parameter in the proposed scheme. Graphs show the effect of the fitting parameter on the solution layer.

Keywords Non-polynomial spline · Differential-difference equation · Layer behaviour · Delay · Fitting parameter · Difference approximation

1 Introduction

Differential-difference equations are problems in which the time evolution of the state variable can depend on specific past in some arbitrary way, i.e. the rate of physical system change depends not only on the state of the physical system but also on its history (Bellman and Cooke 1963). Such problems also occur when many practical phenomena are modelled such as thermo-elasticity (Bestehornand York and Grigorieva 2004), hybrid optical system (Derstine et al. 1982), in population dynamics (Kuang 1993), red blood cell system (Lasota and Wazewska 1976), in models for physiological processes (Mackey and Glass 1977), predator–prey models (Martin and Raun 2001), optimal control theory (Glizer 2000) and in the potential in nerve cells by synaptic inputs in dendrites (Stein 1967). Further analysis on mathematical aspects of the above group of modelling problems and of the singular perturbations is available from the collection of books to name but a few, Bellman and Cooke (Bellman and Cooke 1963), Doolan et al. (Doolan et al. 1980), Driver (Driver 1977), El’sgol’ts and Norkin (El’sgol’ts LE, Norkin SB 1973), Kokotovic et al. (Kokotovic et al. 1986), Miller et al. (Miller et al. 1996) and Smith (Smith 2010).

The authors in Lange and Miura 1994a; Lange and Miura 1994b provided asymptotic access for a class of layer behaviour differential equations. In (Lange and Miura 1994a), researchers provided a mathematical model of evaluating the expected time to produce potential action in nerve cells with random synaptic inputs in dendrites. The researchers in Lange and Miura 1994b illustrated the examples with quick oscillations. Kadalbajoo and his team initiated an extensive numerical work using finite differen-tiation techniques (Kadalbajoo and Sharma 2004, 2005, 2008; Kadalbajoo and Kumar 2010). Ravikanth and Murali (Ravikanth and Murali 2017) have proposed a fitted method to solve the problems through tension splines, which only contains a delay in the differentiated term.
2 Declaration of the problem

Consider the equation with a retarded term, i.e. negative shift or delay in a differentiated term
\[ \epsilon w''(s) + P(s)w'(s - \delta(x)) + Q(s)w(s) = R(s), 0 < s < 1, \]
subject to the boundary conditions
\[ w(s) = \phi(s), -\delta(s) \leq s \leq 0; w(1) = \gamma \]
where \( 0 < \epsilon \ll 1 \) is a perturbation parameter, \( P(s), Q(s), R(s) \) and \( \phi(s) \) are continuous functions in \([0, 1]\), \( \delta \) is a finite constant and \( \delta \) is the delay parameter. It is recognized that when \( \delta = 0 \), Eq. (1) is changes to a singularly perturbed equation exhibit layer behaviour and turning points depending upon the component of the respective convention and reaction term. The solution \( w(s) \) display layer on the left-end when \( P(s) \) is positive or on the right-end when \( P(s) \) is negative all through the interval \([0, 1]\). The solution’s layer behaviour is no longer preserved when \( \delta(s) \) is in order \( O(\epsilon) \), and the solution exhibits oscillatory behaviour.

3 Non-polynomial spline

With mesh length \( h = \frac{1}{N}, \) the domain \([0, 1]\) into \( N \) subdomains, so that \( s_i = ih, \forall i = 0, 1, \ldots, N \) with \( 0 = s_0, 1 = s_N \). Let \( w(s) \) be the exact solution and \( w_i \) be an approximation to \( w(s) \) by the non-polynomial spline \( \Psi_i(s) \) communicating through the \((s_i, w_i)\) and \((s_{i+1}, w_{i+1})\). We need not only \( \Psi_i(s) \) satisfy the interpolatory conditions at \( s_i \) and \( s_{i+1} \), but also to perform the continuity of the first derivative at the common nodes \((s_i, w_i)\). The cubic non-polynomial spline \( \Psi_i(s) \) has the form for each \( i \)th division (Ref. Jalil Rashidinia And Reza Jalilian 2010)
\[ \Psi_i(s) = A_i + B_i(s-s_i) + C_i \sin \tau(s-s_i) \\
+ L_i \cos \tau(s-s_i), \forall i = 0, 1, 2, \ldots, N-1, \]
where \( A_i, B_i, C_i \) and \( L_i \) are to be determine and \( \tau \) is a free parameter.

The function \( \Psi_i(s) \) of the class \( C^2[0, 1] \) interpolates \( w(s_i) \) at mesh points \( s_i, \forall i = 0, 1, \ldots, N \), depending on a parameter \( \tau \) and transformed to the usual cubic spline as \( \tau \to 0 \).

To deduce the values of the coefficients in Eq. (3) in term of \( w_i, w_{i+1}, M_i \) and \( M_{i+1} \), define \( \Psi_i(s_i) = w_i, \Psi_i(s_{i+1}) = w_{i+1}, \Psi'(s_i) = M_i, \Psi'(s_{i+1}) = M_{i+1} \). Using the algebraic exercise, we get the following expression:
\[ A_i = s_i + \frac{M_i}{h}, B_i = \frac{s_{i+1} - s_i}{h} + \frac{M_{i+1} - M_i}{h}, C_i = \frac{M_i \cos \theta - M_{i+1}}{\sin \theta}. \]
Hence, \( A_i, B_i, C_i \) and \( L_i \) are depend on \( \tau \) and \( \theta \).

4 Numerical scheme with small delay

4.1 Left–end boundary layer

If the delay is of the small order of the perturbation parameter i.e. \( \delta = o(\epsilon) \), using Taylor’s series on the term in Eq. (1) containing the delay parameter, it reduces to
\[ (\epsilon - \delta P(s))w''(s) + P(s)w'(s) + Q(s)w(s) = R(s), \]
subject to
\[ w(0) = \phi(0) = \phi_0, w(1) = \gamma. \]
Assume that \( P(s) \geq M > 0 \) and \( (\epsilon - \delta P(s)) \geq 0 \) throughout the domain \([0, 1]\), where \( M \) is a positive constant. With this hypothesis, for small values of \( \epsilon \), Eq. (5) show layer behaviour at \( s = 0 \).

Let \( L^c \) be the differential operator for the problem Eq. (5), Eq. (6) which is defined for any smooth function \( \Omega(s) \in C^2 \) as
\[ L^c \Omega(s) = (\epsilon - \delta P(s))\Omega''(s) + P(s)\Omega'(s) + Q(s) \Omega(s). \]
Case-(i). When \( Q(s) \leq -\tau < 0, \) where \( \tau \) is positive constant.

Lemma 4.1 Continuous minimum principle Let \( \Omega(s) \) is smooth function satisfying \( \Omega_0 \geq 0 \) and \( \Omega_N \geq 0 \). Then, \( L^c \Omega(s) \leq 0, \forall i = 1, 2, \ldots, N - 1 \) implies that \( \Omega(s) \geq 0, \forall i = 0, 1, 2, \ldots, N \).
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\[ w_{i+1} = -w_{i+1} + 4w_i - 3w_{i-1} + O(h^2), \]
\[ w'_{i+1} = \frac{3w_{i+1} - 4w_i + w_{i-1}}{2h} + O(h^2), \]
\[ w_i = \frac{w_{i+1} - w_{i-1}}{2h} + O(h^2) \]

Substituting the values of \( M_{i-1}, M_i \) and \( M_{i+1} \) along with the above differences in Eq. (10), we get
\[
\frac{(e - \delta P(s_i))}{h^2}(w_{i+1} - 2w_i + w_{i-1}) = \left( -\frac{2\alpha P_{i+1}}{h} + \frac{\beta P_i}{h} - x Q_{i-1} + \frac{2\alpha P_{i-1}}{2h} \right) w_{i-1}
+ \left( \frac{2\alpha P_{i+1}}{h} - 2\beta Q_i - \frac{2\alpha P_{i-1}}{2h} \right) w_i
+ \left( \frac{-3\alpha P_{i+1}}{2h} - \frac{\beta P_i}{h} - x Q_{i+1} + \frac{2\alpha P_{i-1}}{2h} \right) w_{i+1}
+ (x R_{i+1} + 2\beta R_i + x R_{i-1}), \quad \forall i = 1, 2, \ldots, N - 1. \tag{11}
\]

Introducing a fitting parameter \( \sigma(\rho) \) in Eq. (11), we have
\[
\frac{\sigma(\rho)(e - \delta P(s_i))}{h^2}(w_{i+1} - 2w_i + w_{i-1}) = \left( -\frac{2\alpha P_{i+1}}{h} + \frac{\beta P_i}{h} - x Q_{i-1} + \frac{2\alpha P_{i-1}}{2h} \right) w_{i-1}
+ \left( \frac{2\alpha P_{i+1}}{h} - 2\beta Q_i - \frac{2\alpha P_{i-1}}{2h} \right) w_i
+ \left( \frac{-3\alpha P_{i+1}}{2h} - \frac{\beta P_i}{h} - x Q_{i+1} + \frac{2\alpha P_{i-1}}{2h} \right) w_{i+1}
+ (x R_{i+1} + 2\beta R_i + x R_{i-1}), \quad \forall i = 1, 2, \ldots, N - 1. \tag{12}
\]

Using the procedure given in Doolan et al. (1980), we get
\[
\lim_{h \to 0} \frac{\sigma}{\rho}(w(ih + h) - 2w(ih) + w(ih - h)) = \frac{P(0)}{2} \lim_{h \to 0}(w(ih - h) - w(ih + h)) \tag{13}
\]

Using Eq. (9) into Eq. (13), we get the fitting parameter for the layer behaviour at the left-end point of the domain as:
\[
\lim_{h \to 0} \frac{\sigma}{\rho}(w(ih + h) - 2w(ih) + w(ih - h)) = \frac{P(0)}{2} \lim_{h \to 0}(w(ih - h) - w(ih + h)) \tag{14}
\]

where \( \rho = \frac{h}{e - \delta P(1)} \).

Exercising on Eq. (12), we get the following tridiagonal system
\[
E_{i-1}w_{i-1} + F_iw_i + G_{i+1}w_{i+1} = H_i, \quad \forall i = 1, 2, \ldots, N - 1, \tag{15}
\]

where
\[
E_i = -\sigma(e - \delta P_i) + \frac{3\alpha P_{i+1}}{2h} + \beta h P_i - \frac{3\alpha P_{i-1}}{2h} + \frac{3\alpha P_i}{h} - x P_i,
F_i = 2\sigma(e - \delta P_i) - 2xh P_{i-1} + 2xh P_{i+1} - 2\beta h^2 Q_i,
G_{i+1} = -\sigma(e - \delta P_i) + \frac{3\alpha P_{i+1}}{2h} + \beta h P_i - \frac{3\alpha P_{i-1}}{2h} - x h^2 Q_{i+1},
H_i = -h^2[2xR_{i+1} + 2\beta R_i + x R_{i-1}],
P(s_i) = P_i, Q(s_i) = Q_i, R(s_i) = R_i, \quad \forall i = 1, 2, \ldots, N - 1.
\]

The system Eq. (15) is resolved by applying the Thomas tridiagonal algorithm.

### 4.2 Right-end boundary layer

With the assumption \( P(s) \leq \mathcal{M} < 0 \) and \( (e - \delta P(s)) \geq 0 \) all over the domain \([0, 1] \), where \( \mathcal{M} \) is negative constant, for small values of \( e \), boundary layer for Eq. (5) exists at \( s = 1 \). From the singular perturbation theory, the solution of Eq. (5) and Eq. (6) is of the form
\[
w(s_i) = w_0(s_i) + (\phi(0) - w_0(1))e^{-\frac{P(1)}{2}(1-s)} + O(e) \tag{16}
\]

i.e.
\[
w(ih) = w_0(ih) + (\phi(0) - w_0(1))e^{-\frac{P(1)}{2}(1-i)} + O(e), \tag{17}
\]

\[
w(ih) = w_0(ih) + (\phi(0) - w_0(1))e^{-\frac{P(1)}{2}(1-i)} + O(e) \tag{18}
\]

Now, in the non-polynomial spline finite difference method Eq. (12), insert a fitting parameter \( \sigma(\rho) \) and apply the same procedure as in left boundary layer case, we get the fitting parameter
\[
\sigma = \rho(x + \beta)P(0)coth\left(\frac{P(1)\rho}{2}\right). \tag{19}
\]

In this case also, we have the tridiagonal system Eq. (15), where \( \sigma \) is given by Eq. (19).
5 Convergence analysis

To examine the convergence for the scheme for the left-end layer, consider the matrix form of Eq. (15) including the specified boundary conditions (Rashidinia et al. 2010)

\[
(\tilde{D} + \tilde{P}) \mathbf{w} + \tilde{Q} + T(h) = 0,
\]

where

\[
\tilde{D} = \begin{bmatrix}
2(\varepsilon - \delta P_1)\sigma & -(\varepsilon - \delta P_1)\sigma & 0 & 0 & 0 \\
-(\varepsilon - \delta P_2)\sigma & 2(\varepsilon - \delta P_2)\sigma & -(\varepsilon - \delta P_2)\sigma & \ldots & 0 \\
0 & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & \ldots \\
0 & \ldots & \ldots & \ldots & -(\varepsilon - \delta P_{N-1})\sigma & 2(\varepsilon - \delta P_{N-1})\sigma
\end{bmatrix},
\]

\[
\tilde{P} = [x_i, y_i, t_i] = \begin{bmatrix}
y_1 & t_1 & 0 & 0 & \ldots & 0 \\
x_2 & y_2 & t_2 & 0 & \ldots & 0 \\
x_3 & y_3 & t_3 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & \ldots & \ldots & \ldots & 0 \end{bmatrix},
\]

where

\[
x_i = \frac{3hP_{i-1}}{2} + \beta h P_i - \frac{3hP_{i+1}}{2} - z h^2 Q_{i-1}, y_i = -2zhP_{i-1} + 2zhP_{i+1} - 2z h^2 Q_{i+1}, i = 1, 2, \ldots, N - 1 \text{ and } \tilde{Q} = \begin{bmatrix}
y_1 + (- (\varepsilon - \delta P_1)\sigma + x_1) \phi_0, y_2, y_3, \ldots, y_{N-2}, y_{N-1} + (- (\varepsilon - \delta P_{N-1})\sigma + t_{N-1}) \phi_0
\end{bmatrix}^T,
\]

where

\[
v_i = h^2 [x_{R_{i-1}} + 2 \beta R_i + z R_{i+1}]i = 1, 2, \ldots, N - 1, T(h) = O(h^4) \text{ and } W = [w_1, w_2, \ldots, w_{N-1}]^T, T(h) = [T_1, T_2, \ldots, T_{N-1}]^T, O = [0, 0, 0, \ldots, 0]^T \text{ are associated vectors of Eq. (20).}
\]

Let \( w = [w_1, w_2, \ldots, w_{N-1}]^T \cong \mathbf{w} \) which satisfies the equation

\[
(\tilde{D} + \tilde{P}) \mathbf{w} + \tilde{Q} = 0.
\]

Let \( e_i = w_i - W_i, \forall i = 1, 2, \ldots, N - 1 \) be the discretization error so that

\[
E = [e_1, e_2, \ldots, e_{N-1}]^T = w - W. \text{ Subtracting Eq. (20) from Eq. (21), we get the error equation}
\]

\[
(\tilde{D} + \tilde{P}) E = T(h).
\]

Let \(|P(s)| \leq C_1 \) and \(|Q(s)| \leq C_2\), where \( C_1, C_2 \) are positive constants. If \( \zeta_{ij} \) be the \((i, j)^{th}\) element of the matrix \((\tilde{D} + \tilde{P})\), then

\[
|\zeta_{i, i+1}| \leq (\varepsilon - \delta P_i) + h(x + \beta)C_1 + h^2 z C_2, \forall i = 1, 2, \ldots, N - 2,
\]

\[
|\zeta_{i, i-1}| \leq (\varepsilon - \delta P_i) + h(x + \beta)C_1 + h^2 z C_2, \forall i = 2, 3, \ldots, N - 1.
\]

Thus, for small values of \( h \), we have

\[
|\zeta_{i, i+1}| < (\varepsilon - \delta P_i), \forall i = 1, 2, \ldots, N - 2,
\]

\[
|\zeta_{i, i-1}| < (\varepsilon - \delta P_i), \forall i = 2, 3, \ldots, N - 1.
\]

Hence, \( (\tilde{D} + \tilde{P}) \) is irreducible (Varga 1962).

Let the sum of \((i)^{th}\) row elements of the matrix \((\tilde{D} + \tilde{P})\) be \( S_i \), then we have

\[
S_i = (\varepsilon - \delta P_i)\sigma - \frac{3zhP_{i-1}}{2} - \beta h P_i + \frac{3zhP_{i+1}}{2} - h^2 (2\beta Q_i + z Q_{i+1}), i = 1,
\]

\[
S_i = h^2 (3Q_{i-1} + 2\beta Q_i + z Q_{i+1}), \forall i = 2, 3, \ldots, N - 2,
\]

\[
S_i = (\varepsilon - \delta P_i)\sigma - \frac{3zhP_{i-1}}{2} - \beta h P_i + \frac{3zhP_{i+1}}{2} - h^2 (2\beta Q_i + z Q_{i+1}), i = N - 1.
\]

Let

\[
\tilde{C}_1 = \min_{1 \leq i \leq N} |P(s)| \text{ and } \tilde{C}_1^* = \max_{1 \leq i \leq N} |P(s)|, \tilde{C}_2 = \min_{1 \leq i \leq N} |Q(s)| \text{ and } \tilde{C}_2^* = \max_{1 \leq i \leq N} |Q(s)|.
\]

Since \( 0 < \varepsilon < 1 \) and \( \varepsilon \propto O(h) \), \( \tilde{C}_1, \tilde{C}_2 \) is monotone (Varga 1962; Young 1971). Hence \( (\tilde{D} + \tilde{P})^{-1} \) exists and \( (\tilde{D} + \tilde{P})^{-1} \geq 0 \). Thus, from Eq. (22), we have

\[
\|E\| \leq \| (\tilde{D} + \tilde{P})^{-1} \| T \|
\]

Let \( (\tilde{D} + \tilde{P})^{-1}_{ij} \) be the \((i, j)^{th}\) element of \((\tilde{D} + \tilde{P})^{-1} \) and define
\[ \| (\mathbf{D} + \mathbf{P}^{-1} \| = \max_{1 \leq i \leq N-1} \sum_{k=1}^{N-1} (\mathbf{D} + \mathbf{P})^{-1}_{i,k} \text{ and } \| T(h) \| = \max_{1 \leq i \leq N-1} |T(h)|. \]

(25a)

Since \((\mathbf{D} + \mathbf{P})^{-1}_{i,k} \geq 0 \text{ and } \sum_{k=1}^{N-1} (\mathbf{D} + \mathbf{P})^{-1}_{i,k} \leq 1, \forall i = 1, 2, \ldots, N - 1.

Hence,

\[ (\mathbf{D} + \mathbf{P})^{-1}_{i,k} \leq \frac{1}{\mathcal{S}_i} \leq \frac{1}{h^2 C_2}, \text{ } i = 1, \ldots, N - 1. \]

(25b)

Furthermore,

\[ \sum_{k=1}^{N-1} (\mathbf{D} + \mathbf{P})^{-1}_{i,k} \leq \frac{1}{\min_{2 \leq i \leq N-2} \mathcal{S}_i} \leq \frac{1}{h^2 C_2}, \forall i = 1, 2, \ldots, N - 1. \]

(25c)

By the help of Eqs. (25a) - (25d) and using Eq. (24) we get

\[ \| E \| \leq O(h^2). \]

(26)

Hence, the proposed scheme has second-order convergence.

### 6 Numerical examples

To illustrate the comparative efficiency of the proposed scheme, it is implemented on numerical experiments with a layer at left-end and right-end for the small \( \delta \) values. The maximum absolute errors in the examples considered were measured using the theory of double mesh \( E^N = \max_{0 \leq i \leq N} |w_i^N - w_i^{N^+}| \) and tabulated in Tables 1, 2, 3, 4, 5 for the examples considered. Computed errors are compared to the results given in Kadalbajoo and Kumar (2010); Ravi-kanth and Murali (2017; Reddy et al. 2012). It was observed that our method yields precise results than the method suggested in Kadalbajoo and Kumar (2010); Ravi-kanth and Murali (2017; Reddy et al. 2012).

**Example 6.1** \( e^{w}(s) + w'(s - \delta) - w(s) = 0 \) with \( w(s) = 1, -\delta \leq s \leq 0; w(1) = 1. \)

**Example 6.2** \( e^{w}(s) + (1 + s)\overline{w}(s - \delta) - e^{-s}w(t) = 1; w(s) = 0, -\delta \leq s \leq 0; w(1) = 1. \)

**Example 6.3** \( e^{w}(s) - w'(s - \delta) - w(s) = 0; \text{ } w(s) = 1, -\delta \leq s \leq 0; w(1) = -1. \)

**Example 6.4** \( e^{w}(s) - e^{w}(s - \delta) - sw(s) = 0, \text{ } w(s) = 1, -\delta \leq s \leq 0; w(1) = 1. \)

**Example 6.5** Consider the following singularly perturbed nonlinear delay differential equation.

\[ e^{w}_{\mu}'(s) + w(s)w'(s - \delta) - w(s) = 0, \]

under the interval and boundary conditions \( w(s) = 1, -\delta \leq s \leq 0, w(1) = 1. \)

### 7 Numerical scheme without fitting factor for big delay

If the delay \( \delta = O(\varepsilon) \), the layer behaviour of the solution is preserved in both the cases i.e. Taylor’s series for the term holding the shift parameter and the special mesh-type developed in Kadalbajoo and Sharma (2008). If the delay parameter \( \delta(\varepsilon) \) is of order \( O(\varepsilon) \), the layer behaviour of the solution is no longer preserved and oscillatory behaviour is shown in the solution.

For this reason, with the help of a special type mesh developed in Kadalbajoo and Sharma (2008), we are building a numerical scheme consisting of a non-polynomial spline method i.e. the term shift parameter is lying on the mesh point after discretization and selects the mesh as \( h = \frac{\delta}{m} \), where \( m = kl \), \( l \) is the mantissa of \( \delta \) and \( k \) is a positive integer.

Using this mesh, Eq. (1) and (2) leads to

\[ e^{w}_{\mu} = r(s_i) - P(s_i)w_{i-m} - Q(s_i)w_i \]

(27)

with

\[ w_i = \phi_i, \forall i = -m, -m+1, \ldots, 0 \text{ and } w_N = \gamma. \]

(28)

Now use Eq. (4) from the non-polynomial spline together with the following finite difference approximations of the first-order derivatives,

\[ w'_{i+1} = -w_{i+1} + 4w_i - 3w_{i-1} + O(h^2), \]

\[ w'_{i+1} = \frac{3w_{i+1} - 4w_i + w_{i-1}}{2h} + O(h^2), \]

\[ w_i = \frac{w_{i+1} - w_{i-1}}{2h} + O(h^2), \]

we get

\[ \frac{e}{h^2} \left( w_{i+1} - 2w_i + w_{i-1} \right) = \left( \frac{-3P_{i+1}}{2h} + \frac{\beta P_i}{h} + \frac{3P_{i-1}}{2h} \right) \]

(29)

Exercising on Eq. (29), the difference scheme becomes
Table 1  Maximum absolute errors in Example 6.1 with $\varepsilon = 0.1$

| $N$ | $\delta = 0.3 \times \varepsilon$ | $\delta = 0.6 \times \varepsilon$ | $\delta = 0.9 \times \varepsilon$ |
|-----|---------------------------------|---------------------------------|---------------------------------|
|     | Our method                      | Results in (Kadalbajoo and Kumar 2010) | Our method                      | Results in (Kadalbajoo and Kumar 2010) | Our method                      | Results in (Kadalbajoo and Kumar 2010) |
| 64  | $9.30e-04$                      | $3.42e-03$                      | $1.40e-03$                      |
| 128 | $2.66e-04$                      | $1.66e-03$                      | $3.43e-03$                      |
| 256 | $6.93e-05$                      | $8.11e-04$                      | $1.43e-03$                      |
| 512 | $1.75e-05$                      | $4.03e-04$                      | $1.40e-03$                      |

Table 2  Maximum absolute errors in Example 6.2 with $\varepsilon = 0.1$

| $N$ | $\delta = 0.1 \times \varepsilon$ | $\delta = 0.4 \times \varepsilon$ |
|-----|---------------------------------|---------------------------------|
|     | Our method                      | Results in (Kadalbajoo and Kumar 2010) | Our method                      | Results in (Kadalbajoo and Kumar 2010) |
| 64  | $4.18e-04$                      | $5.19e-04$                      |
| 128 | $1.74e-04$                      | $2.58e-04$                      |
| 256 | $1.21e-04$                      | $2.21e-04$                      |
| 512 | $1.15e-04$                      | $2.33e-04$                      |

Table 3  Maximum absolute errors in Example 6.3 with $\varepsilon = 0.01$

| $\delta$ | $N = 10^2$ | $N = 10^3$ | $N = 10^4$ |
|----------|------------|------------|------------|
|          | Our method | Results in (Reddy et al. 2012) | Our method | Results in (Reddy et al. 2012) | Our method | Results in (Reddy et al. 2012) |
| 0.000    | $2.00e-03$ | $1.81e-01$ | $2.03e-04$ | $2.42e-02$ | $4.90e-06$ | $2.51e-03$ |
| 0.007    | $2.00e-03$ | $1.20e-01$ | $1.55e-04$ | $1.45e-02$ | $1.63e-06$ | $1.48e-03$ |
| 0.015    | $2.00e-03$ | $8.66e-02$ | $8.48e-05$ | $9.96e-03$ | $7.52e-07$ | $1.01e-03$ |
| 0.025    | $2.00e-03$ | $6.46e-02$ | $3.86e-05$ | $7.17e-03$ | $3.82e-07$ | $7.20e-04$ |

Table 4  Maximum absolute errors in Example 6.4 with $\varepsilon = 0.1$

| $\delta$ | $N = 10^2$ | $N = 10^3$ | $N = 10^4$ |
|----------|------------|------------|------------|
|          | Our method | Results in (Reddy et al. 2012) | Our method | Results in (Reddy et al. 2012) | Our method | Results in (Reddy et al. 2012) |
| 0.01     | $5.27e-04$ | $5.75e-03$ | $5.73e-06$ | $5.08e-04$ | $5.74e-08$ | $5.02e-05$ |
| 0.03     | $2.65e-04$ | $3.93e-03$ | $2.73e-06$ | $3.61e-04$ | $2.74e-08$ | $3.58e-05$ |
| 0.06     | $1.23e-04$ | $2.70e-03$ | $1.24e-06$ | $2.55e-04$ | $1.24e-08$ | $2.53e-05$ |
| 0.08     | $8.19e-05$ | $2.24e-03$ | $8.23e-07$ | $2.14e-04$ | $8.22e-09$ | $2.13e-05$ |
Table 5 Maximum errors in Example 6.5 for $\delta = 0.8$ and different values of $\varepsilon$ and $N$

| $\varepsilon/N$ | 64   | 128  | 256  | 512  | 1024 | 2048 |
|----------------|------|------|------|------|------|------|
| $10^{-1}$     | 1.1651e-03 | 3.2884e-04 | 8.7186e-05 | 2.2435e-05 | 5.6893e-06 | 1.4325e-06 |
| $10^{-2}$     | 3.3515e-03 | 1.0682e-03 | 2.9959e-04 | 7.9176e-05 | 2.0341e-05 | 5.1542e-06 |
| $10^{-3}$     | 3.8819e-03 | 1.2730e-03 | 3.6165e-04 | 9.6155e-05 | 2.4775e-05 | 6.2866e-06 |
| $10^{-4}$     | 3.9415e-03 | 1.2967e-03 | 3.6893e-04 | 9.8159e-05 | 2.5299e-05 | 6.4208e-06 |
| $10^{-5}$     | 3.9476e-03 | 1.2991e-03 | 3.6967e-04 | 9.8363e-05 | 2.5352e-05 | 6.4344e-06 |
| $10^{-6}$     | 3.9482e-03 | 1.2994e-03 | 3.6974e-04 | 9.8383e-05 | 2.5358e-05 | 6.4358e-06 |
| $10^{-7}$     | 3.9482e-03 | 1.2994e-03 | 3.6975e-04 | 9.8385e-05 | 2.5358e-05 | 6.4359e-06 |
| $10^{-8}$     | 3.9482e-03 | 1.2994e-03 | 3.6975e-04 | 9.8385e-05 | 2.5358e-05 | 6.4359e-06 |

The system of Eq. (31) is solved using partial pivoting in the Gauss elimination method.

Using the expansion of Taylor’s series about $w_{i-m}$ on Eq. (30) and with Eq. (27), we get the local truncation error to the scheme as

$$T_i(h) = \varepsilon \left[ 1 - 2(\varepsilon + \beta)|h|^2 w_i^{(2)}(\xi_i) \right]$$

$$+ \left\{ \frac{\varepsilon}{12} [1 - 12\varepsilon] w_i^{(4)}(\xi_i) + \left[ \frac{1}{3} [-2\varepsilon + \beta p_i(\xi_i)] w_i^{(3)}(\xi_i) \right] \right\} |h|^4$$

$$+ O(h^6) \forall \xi_i \leq \xi_i \leq s_i + 1.$$

It is very clear that $T_i(h) = O(h^4)$ for any arbitrary choice of $\varepsilon$ and $\beta$ whose sum is equal to $\frac{1}{2}$. Thus, the above scheme is a second-order convergence.

8 Numerical scheme with a fitting parameter for big delay

The behaviour of the layer can modify its character and even get damaged if the delay is greater than that of a perturbation parameter, or oscillating behaviour can occur even in the case of a given type mesh (Kadalbajoo and Sharma 2008). In this respect, we have tried to introduce a fitting parameter into the non-polynomial spline scheme, which was defined in the previous section, with the special type mesh. The fitting parameter is determined by the
theory of singular perturbations. Now, inserting a fitting parameter $\sigma(\rho)$ in Eq. (29), we get

$$
\frac{\sigma(\rho)\varepsilon}{h^2} (w_{i+1} - 2w_i + w_{i-1}) = \left( -\frac{\alpha Q_{i+1}}{2h} + \frac{\beta P_i}{h} + \frac{3\alpha P_i - 1}{2h} \right) w_{i-m-1} + \left( \frac{2\alpha P_i}{h} - \frac{2\alpha P_i - 1}{h} \right) w_{i-m} + \left( -\frac{3\alpha P_i}{2h} - \frac{\beta P_i}{h} + \frac{\alpha P_i - 1}{2h} \right) w_{i-m+1} - \frac{\alpha Q_{i-1}}{2h} w_{i-1} + 2\beta Q_i w_i + \frac{x Q_{i+1} w_{i+1}}{2h} + (x R_{i+1} + 2\beta R_i + x R_{i-1}),
$$

(29)

\forall \ i = 1, 2, \ldots, N - 1.$
Fig. 1 Numerical solution in Example 9.1 for $\varepsilon = 0.01$ and $\delta = 1.5\varepsilon$ without fitting parameter.
Fig. 2 Numerical solution in Example 9.1 for \( \varepsilon = 0.01 \) and \( \delta = 1.5\varepsilon \) with fitting parameter

Fig. 3 Numerical solution in Example 9.1 for \( \varepsilon = 0.01 \) and \( \delta = 2.5\varepsilon \) without fitting parameter

Fig. 4 Numerical solution in Example 9.1 for \( \varepsilon = 0.01 \) and \( \delta = 2.5\varepsilon \) with fitting parameter
Fig. 5 Numerical solution in Example 9.2 for $\varepsilon = 0.01$ with different values of $\delta$ without fitting parameter

Fig. 6 Numerical solution in Example 9.2 for $\varepsilon = 0.01$ with different values of $\delta$ with fitting parameter

Fig. 7 Numerical solution in Example 9.3 for $\varepsilon = 0.01$ with different values of $\delta$ without fitting parameter
9 Numerical illustrations

To demonstrate the method competence, three problems are discussed. Using the principle of double mesh $E^N = \max_{0 \leq i \leq N} |w_i^N - w_{2N}^i|$, with and without fitting parameter, the maximum errors for the examples are calculated. The fitting parameter introduced in the scheme Eq. (32) is to handle the layer behaviour, when the shift is larger than the perturbation. The computed solutions of the problem are illustrated through the graphs, with and without fitting parameter for different values of and of δ. Tables 6, 7, 8, 9, 10, 11 display for the maximum absolute errors for the examples. The results in Kadalbajoo and Sharma (2008) are compared to the computed errors. It was found that the method proposed provides solutions that are more accurate than the approach suggested in Kadalbajoo and Sharma (2008).

Example 9.1 $\varepsilon w''(s) + w'(s - \delta) + w(s) = 0$ with $w(s) = 1, \forall - \delta \leq s \leq 0, w(1) = 1$.

Example 9.2 $w''(s) + 0.25w'(s - \delta) - w(s) = 0$ with $w(s) = 1, \forall - \delta \leq s \leq 0, w(1) = 0$.

Example 9.3 $\varepsilon w''(s) - w'(s - \delta) + w(s) = 0$ with $w(s) = 1, \forall - \delta \leq s \leq 0, w(1) = -1$.

10 Conclusions

Numerical treatment of second-order linear convention–diffusion equations with a small delay in the convention term having layer behaviour is considered. A finite difference scheme is constructed using the non-polynomial spline and its first-order derivative continuity condition at the common node. If the delay $\delta = o(\varepsilon)$, the term containing the delay is expanded in Taylor’s series and a fitted difference scheme is constructed for a layer at the left-end and right-end. The method is analysed for convergence. For the Examples 6.1–6.5, Tables 1, 2, 3, 4, 5 shows the maximum absolute errors in the solutions. Comparing the computed errors with the results suggested in Kadalbajoo and Kumar (2010); Ravikanth and Murali 2017; Reddy et al. (2012), it was found that the proposed method yielded accurate results relative to the approaches suggested in Kadalbajoo and Kumar (2010); Ravikanth and Murali 2017; Reddy et al. (2012).

When the delay parameter $\delta(\varepsilon)$ is of order $O(\varepsilon)$, the behaviour of the layer can alter its nature and even be demolished or the solution displays oscillatory behaviour. In this case, a special mesh is applied so that, the term having delay parameter lies on the mesh points after the discretization and then the non-polynomial method is applied. Tables 6, 7, 8, 9, 10, 11 of the numerical examples show the maximum absolute errors. Figures 1, 2, 3, 4, 5, 6, 7, 8 demonstrate the solutions of the examples for different values of the delay parameter. From Fig. 1, 3, 5 it is noticed that, when the value of delay is greater than the perturbation, the solution involves oscillations. Further, if we increase the $\delta$ further, the oscillations which are restricted in the layer region are extend over whole domain and even move from one side to another.

To deal with these oscillations in solutions, we tried a new scheme with the use of the special type mesh introduced in Kadalbajoo and Sharma (2008) by adding a fitting parameter in the non-polynomial spline method. Examples 9.1–9.3 are considered, and the maximum absolute error was determined using the principle of double mesh $E^N = \max_{0 \leq i \leq N} |w_i^N - w_{2N}^i|$. In Figs. 2, 4, 6 of the computed solution, we showed the graphs with the fitting parameter for different values of $\delta = O(\varepsilon)$ and compared to the graphs without fitting factor. It is noticed that oscillations are regulated in solutions and layer behaviour is preserved, while the layer behaviour of the solution is preserved in the case of the layer at right-end (Fig. 7, 8), although the delay
is of $O(\varepsilon)$. For the Examples 9.1–9.3, the maximum absolute errors with $\delta = O(\varepsilon)$, are tabulated in Tables 7, 9, 11. It is observed from the tabulated results that the maximum absolute errors are also decreasing as the mesh size $h$ decreases.

Based on the graphs of the solutions in Figs. 1, 2, 3, 4, 5, 6, 7, 8, it has also been concluded that the method proposed with a fitting parameter had a great advantage in regulating the oscillation in the solutions of the linear singularly perturbed differential delay equations.

**Declarations**

**Conflict of interest** The authors declares that they have no conflict of interest.

**Ethical approval** This article does not contain any studies with human participants or animals performed by any of the authors.

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