VARIABLE HARDY SPACES

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ABSTRACT. We develop the theory of variable exponent Hardy spaces $H^{p(\cdot)}$. Analogous to the classical theory, we give equivalent definitions in terms of maximal operators. We also show that $H^{p(\cdot)}$ functions have an atomic decomposition including a “finite” decomposition; this decomposition is more like the decomposition for weighted Hardy spaces due to Strömberg and Torchinsky [28] than the classical atomic decomposition. As an application of the atomic decomposition we show that singular integral operators are bounded on $H^{p(\cdot)}$ with minimal regularity assumptions on the exponent $p(\cdot)$.

1. INTRODUCTION

Variable Lebesgue spaces are a generalization of the classical $L^p$ spaces, replacing the constant exponent $p$ with an exponent function $p(\cdot)$: intuitively, they consist of all functions $f$ such that

$$\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx < \infty.$$ 

These spaces were introduced by Orlicz [25] in 1931, but they have been the subject of more intensive study since the early 1990s, because of their intrinsic interest, for their use in the study of PDEs and variational integrals with nonstandard growth conditions, and for their applications to the study of non-Newtonian fluids and to image restoration. (See [4, 9] and the references they contain.)

In this paper we extend the theory of variable Lebesgue spaces by studying the variable exponent Hardy spaces, or more simply, the variable Hardy spaces $H^{p(\cdot)}$. The classical theory of $H^p$ spaces, $0 < p \leq 1$, is well-known (see [12, 14, 15, 22, 26]) and our goal is to replicate that theory as much as possible in this more general setting. This has been done in the context of analytic functions on the unit disk by Kokilashvili and Paatashvili [17, 18]. We are interested in the theory of real Hardy spaces in all dimensions. Here we give a broad overview of our techniques and results; we will defer the precise statement of definitions and theorems until the body of the paper.

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Given an exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, we define the space $L^{p(\cdot)}$; this is a quasi-Banach space. In the study of variable Lebesgue spaces it is common to assume that the exponent $p(\cdot)$ satisfies log-Hölder continuity conditions locally and at infinity. While these conditions will be sufficient for us, we prefer to work with a much weaker hypothesis: that there exists $p_0 > 0$ such that the Hardy-Littlewood maximal operator is bounded on $L^{p(\cdot)/p_0}$. This approach was first introduced in [6] and developed systematically in [4]. While in certain cases weaker hypotheses are possible, this appears to be the “right” universal condition for doing harmonic analysis in the variable exponent setting.

The variable Hardy space $H^{p(\cdot)}$ consists of all tempered distributions $f$ such that $\mathcal{M}_N f \in L^{p(\cdot)}$, where $\mathcal{M}_N$ is the grand maximal operator of Fefferman and Stein. We show that an equivalent definition is gotten by replacing the grand maximal operator with a maximal operator defined in terms of convolution with a single Schwartz function or with the non-tangential maximal operator defined using the Poisson kernel. This proof follows the broad outline of the argument in the classical case, but differs in many technical details. Here we make repeated use of the fact that the maximal operator is bounded on $L^{p(\cdot)/p_0}$.

We next prove an atomic decomposition for distributions in $H^{p(\cdot)}$. Given $p(\cdot)$ and $q$, $1 < q \leq \infty$, we say that a function $a(\cdot)$ is a $(p(\cdot), q)$ atom if there is a ball $B$ such that $\text{supp}(a) \subset B$,

$$\|a\|_q \leq |B|^{1/q} \|\chi_B\|_{p(\cdot)}^{1/p(\cdot)}; \quad \text{and} \quad \int a(x)x^\alpha \, dx = 0$$

for all multi-indices $\alpha$ such that $|\alpha|$ is not too large. We then show that $f \in H^{p(\cdot)}$ if and only if for $q$ sufficiently large there exist $(p(\cdot), q)$ atoms $a_j$ such that

$$f = \sum_j \lambda_j a_j, \quad (1.1)$$

and

$$\|f\|_{H^{p(\cdot)}} \approx \inf \left\{ \left. \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} : f = \sum_j \lambda_j a_j \right\}, \quad (1.2)$$

where the infimum is taken over all possible atomic decompositions of $f$. This is very different from the classical atomic decomposition; it is based on the atomic decomposition developed for weighted Hardy spaces by Strömberg and Torchinsky [28]. A comparable decomposition in the classical case is due to Uchiyama: see Janson and Jones [16]. Moreover, we are able to prove that for $q < \infty$, if the summation in (1.1) is finite, the infimum in (1.2) can be taken over finite decompositions. This “finite” atomic decomposition is a generalization of the result of Meda, et al. [23] in the classical case. As part of our work we also prove a finite atomic decomposition theorem for weighted Hardy spaces, extending the results in [28].
To construct our atomic decomposition we first adapt the Calderón-Zygmund decomposition of classical Hardy spaces to give a \((p(\cdot), \infty)\) atomic decomposition. Here a key tool is a vector-valued inequality for the maximal operator, which in turn depends on the boundedness of maximal operator. For the case \(q < \infty\) we also rely on the theory of weighted Hardy spaces and on the Rubio de Francia extrapolation theory for variable Lebesgue spaces developed in [6] (see also [4, 7]). Neither approach was sufficient in itself in this case. We were not able to extend the classical approach to prove half the equivalence in (1.2). On the other hand, while such an equivalence exists in the weighted case, extrapolation requires careful density arguments and we could not, \textit{a priori}, find the requisite dense subsets needed to prove both inequalities in (1.2). Again, in applying extrapolation the key hypothesis is the boundedness assumption on the maximal operator.

Finally, we prove that convolution type Calderón-Zygmund singular integral operators with sufficiently smooth kernels are bounded on \(H^{p(\cdot)}\). In our proof we make extensive use of the finite atomic decomposition in weighted Hardy spaces; this allows us to avoid the more delicate convergence arguments that are often necessary when using the “infinite” atomic decomposition (e.g., see [14]).

The remainder of this paper is organized as follows. In Section 2 we give precise definitions of variable Lebesgue spaces and state a number of results we will need in the subsequent sections. In Section 3 we characterize \(H^{p(\cdot)}\) in terms of maximal operators. In Sections 4 and 5 we prove two technical results: that \(L^{1}_{loc}\) is dense in \(H^{p(\cdot)}\) and the Calderón-Zygmund decomposition for distributions in \(H^{p(\cdot)}\). In Section 6 we construct the \((p(\cdot), \infty)\) atomic decomposition, and in Section 7 we construct the atomic decomposition for \(q < \infty\) and prove the finite atomic decompositions for both the variable and weighed Hardy spaces. This second decomposition is used in Section 8, where we prove that singular integrals are bounded on \(H^{p(\cdot)}\).

Remark 1.1. As we were completing this project we learned that the variable Hardy spaces had been developed independently by Nakai and Sawano [24]. They prove the equivalent definitions in terms of maximal operators using another approach. They also define an atomic decomposition but one which is weaker than ours. They show that \(\|f\|_{H^{p(\cdot)}}\) is equivalent to the infimum of

\[
\left( \sum_{j} \left( \frac{\lambda_{B_{j}}^{p_{*}}}{\|X_{B_{j}}\|_{p(\cdot)}} \right)^{1/p_{*}} \right)^{1/p_{(\cdot)}},
\]

where \(p_{*} = \min(1, \text{ess inf } p(x))\). In particular, if \(p(\cdot)\) takes on values less than 1, this quantity is larger than that in (1.2). They prove that this is equivalent to (1.2) only when \(q = \infty\) and with the further assumption that \(p(\cdot)\) is log-Hölder continuous. Using their atomic decomposition they prove that singular integrals are bounded, but again they must assume that \(p(\cdot)\) is log-Hölder continuous.
2. Preliminaries

In this section we give without proof some basic results about the variable Lebesgue spaces. Unless otherwise specified, we refer the reader to [4, 5, 9, 11, 20] for proofs and further information. Let \( \mathcal{P} = \mathcal{P}(\mathbb{R}^n) \) denote the collection of all measurable functions \( p(\cdot) : \mathbb{R}^n \to [1, \infty] \). Given a measurable set \( E \), let

\[
p_- (E) = \text{ess inf}_{x \in E} p(x), \quad p_+ (E) = \text{ess sup}_{x \in E} p(x).
\]

For brevity we will write \( p_- = p_- (\mathbb{R}^n) \) and \( p_+ = p_+ (\mathbb{R}^n) \). Define the set \( \Omega_\infty = \{ x \in \mathbb{R}^n : p(x) = \infty \} \). Then for \( p(\cdot) \in \mathcal{P} \), the space \( L^{p(\cdot)} = L^{p(\cdot)} (\mathbb{R}^n) \) is the collection of all measurable functions \( f \) such that for some \( \lambda > 0 \),

\[
\rho (f/\lambda) = \int_{\mathbb{R}^n \setminus \Omega_\infty} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} dx + \lambda^{-1} \| f \|_{L^\infty (\Omega_\infty)} < \infty.
\]

This becomes a Banach function space when equipped with the Luxemburg norm

\[
\| f \|_{p(\cdot)} = \inf \{ \lambda > 0 : \rho (f/\lambda) \leq 1 \}.
\]

Given \( p(\cdot) \in \mathcal{P} \), define the conjugate exponent \( p'(\cdot) \) by the equation

\[
\frac{1}{p(x)} + \frac{1}{p'(x)} = 1,
\]

with the convention that \( 1/\infty = 0 \).

**Lemma 2.1.** Given \( p(\cdot) \in \mathcal{P} \), if \( f \in L^{p(\cdot)} \) and \( g \in L^{p'(\cdot)} \),

\[
\int_{\mathbb{R}^n} |f(x)g(x)| dx \leq C(p(\cdot))\| f \|_{p(\cdot)}\| g \|_{p'(\cdot)}.
\]

Conversely for all \( f \in L^{p(\cdot)} \),

\[
\| f \|_{p(\cdot)} \leq C(p(\cdot)) \sup \int_{\mathbb{R}^n} f(x)g(x) dx,
\]

where the supremum is taken over all \( g \in L^{p'(\cdot)} \) such that \( \| g \|_{p'(\cdot)} \leq 1 \).

**Lemma 2.2.** Let \( E \subset \mathbb{R}^n \) be such that \(|E| < \infty\). If \( p(\cdot), q(\cdot) \in \mathcal{P} \) satisfy \( p(x) \leq q(x) \) a.e., then

\[
\| f \chi_E \|_{p(\cdot)} \leq (1 + |E|) \| f \chi_E \|_{q(\cdot)}.
\]

To define the variable Hardy spaces we need to extend the collection of allowable exponents. For simplicity we restrict ourselves to spaces where \( p(\cdot) \) is bounded. Let \( \mathcal{P}_0 \) denote the collection of all measurable functions \( p(\cdot) : \mathbb{R}^n \to (0, \infty) \) such that \( p_+ < \infty \). With the same definition of the modular \( \rho \) as above, we again define \( L^{p(\cdot)} \) as the collection of measurable functions \( f \) such that for some \( \lambda > 0 \), \( \rho (f/\lambda) < \infty \). We define \( \| \cdot \|_{p(\cdot)} \) as before; if
$p_- < 1$ (the case we are primarily interested in) this is not a norm: it is a quasi-norm and $L^{p(-)}$ becomes a quasi-Banach space. We will abuse terminology and refer to it as a norm.

The next four lemmas are proved exactly as in the case when $p_- \geq 1$.

**Lemma 2.3.** Given $p(\cdot) \in \mathcal{P}_0$, $p_+ < \infty$, then for all $s > 0$,

$$\|\|f|^{s}\|_{p(\cdot)} = \|f\|^{s}_{p(\cdot)}.$$  

**Lemma 2.4.** Suppose $p(\cdot) \in \mathcal{P}_0$. Given a sequence $\{f_k\} \subset L^{p(\cdot)}$,

$$\int_{\mathbb{R}^n} |f_k(x)|^{p(x)} \, dx \to 0$$

as $k \to \infty$ if and only if $\|f_k\|_{p(\cdot)} \to 0$.

**Lemma 2.5.** Suppose $p(\cdot) \in \mathcal{P}_0$. Given a sequence $\{f_k\}$ of $L^{p(\cdot)}$ functions that increase pointwise almost everywhere to a function $f$,

$$\lim_{k \to \infty} \|f_k\|_{p(\cdot)} = \|f\|_{p(\cdot)}.$$  

**Lemma 2.6.** Suppose $p(\cdot) \in \mathcal{P}_0$. Given $f \in L^{p(\cdot)}$, if $\|f\|_{p(\cdot)} \leq 1$,

$$\rho(f)^{1/p_-} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_+};$$

if $\|f\|_{p(\cdot)} \geq 1$,

$$\rho(f)^{1/p_+} \leq \|f\|_{p(\cdot)} \leq \rho(f)^{1/p_-}.$$

**Lemma 2.7.** Given $p(\cdot) \in \mathcal{P}_0$, $p_- \leq 1$, then for all $f, g \in L^{p(\cdot)}$,

$$\|f + g\|^{p_-}_{p(\cdot)} \leq \|f\|^{p_-}_{p(\cdot)} + \|g\|^{p_-}_{p(\cdot)}.$$  

**Proof.** Since $p(\cdot)/p_- \in \mathcal{P}$, by Lemma 2.3, convexity and Minkowski’s inequality for the variable Lebesgue spaces,

$$\|f + g\|^{p_-}_{p(\cdot)} = \|f + g|^{p_-}_{p(\cdot)/p_-}\| \leq \|f|^{p_-}_{p(\cdot)/p_-} + \|g|^{p_-}_{p(\cdot)/p_-} \leq \|f|^{p_-}_{p(\cdot)/p_-} + \|g|^{p_-}_{p(\cdot)/p_-} = \|f\|^{p_-}_{p(\cdot)} + \|g\|^{p_-}_{p(\cdot)}.$$  

**Remark 2.8.** This lemma is false if $p_- > 1$, but in this case $\|\cdot\|_{p(\cdot)}$ is a norm and so Minkowski’s inequality holds. This will cause minor technical problems in the proofs below; we will generally consider the case $p_- \leq 1$ in detail and sketch the changes required for the other case.
2.1. The Hardy-Littlewood maximal operator. Given a function \( f \in L^1_{\text{loc}} \) we define the maximal function of \( f \) by
\[
Mf(x) = \sup_{Q \ni x} \int_Q |f(y)| \, dy,
\]
where \( \int_Q g \, dy = |Q|^{-1} \int_Q g \, dy \), and the supremum is taken over all cubes whose sides are parallel to the coordinate axes. Throughout, we will make use of the following class of exponents.

Definition 2.9. Given \( p(\cdot) \in P_0 \), we say \( p(\cdot) \in M P_0 \) if \( p_- > 0 \) and there exists \( p_0, 0 < p_0 < p_- \), such that
\[
\|Mf\|_{p(\cdot)/p_0} \leq C(n, p(\cdot), p_0) \|f\|_{p(\cdot)/p_0}.
\]

A useful sufficient condition for the boundedness of the maximal operator is log-Hölder continuity: for a proof, see [4, 9].

Lemma 2.10. Given \( p(\cdot) \in P \), such that \( 1 < p_- \leq p_+ < \infty \), suppose that \( p(\cdot) \) satisfies the log-Hölder continuity condition locally,
\[
|p(x) - p(y)| \leq \frac{C_0}{\log(\|x - y\|)}, \quad |x - y| < 1/2,
\]
and at infinity: there exists \( p_\infty \) such that
\[
|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e + |x|)}.
\]
Then \( \|Mf\|_{p(\cdot)} \leq C(n, p(\cdot)) \|f\|_{p(\cdot)} \).

Remark 2.11. We want to stress that while in practice it is common to assume that the exponent \( p(\cdot) \) satisfies the log-Hölder continuity conditions, we will not assume this in our main results. For a further discussion of sufficient conditions for the maximal operator to be bounded, see [4, 9] and the references they contain.

Lemma 2.12. Given \( p(\cdot) \in P \), if the maximal operator is bounded on \( L^{p(\cdot)} \), then for every \( s > 1 \), it is bounded on \( L^{sp(\cdot)} \).

Proof. This follows at once from Hölder’s inequality and Lemma 2.3:
\[
\|Mf\|_{sp(\cdot)} = \|(Mf)^s\|^{1/s}_{p(\cdot)} \leq \|M(|f|^s)\|^{1/s}_{p(\cdot)} \leq C^{1/s}\|f|^s\|^{1/s}_{p(\cdot)} = C^{1/s}\|f\|_{sp(\cdot)}.
\]

The following necessary condition is due to Kopaliani [19]. It should be compared to the Muckenhoupt \( A_p \) condition from the study of weighted norm inequalities. (See [10, 14].)

Lemma 2.13. Given \( p(\cdot) \in P \), if the maximal operator is bounded on \( L^{p(\cdot)} \), then for every ball \( B \subset \mathbb{R}^n \),
\[
\|\chi_B\|_{p(\cdot)} \|\chi_B\|_{p'(\cdot)} \leq C|B|.
\]
The maximal operator also satisfies a vector-valued inequality. This result was proved using extrapolation in [6]. (See also [4, 7].)

**Lemma 2.14.** Given \( p(\cdot) \in \mathcal{P} \) such that \( p_+ < \infty \), if the maximal operator is bounded on \( L^{p(\cdot)} \), then for any \( r, 1 < r < \infty \),

\[
\left\| \left( \sum_k (Mf_k)^r \right)^{1/r} \right\|_{p(\cdot)} \leq C(n, p(\cdot), r) \left\| \left( \sum_k |f_k|^r \right)^{1/r} \right\|_{p(\cdot)}.
\]

Our final lemma is a deep result due to Diening [8, 9].

**Lemma 2.15.** Given \( p(\cdot) \in \mathcal{P} \) such that \( 1 < p_- \leq p_+ < \infty \), the maximal operator is bounded on \( L^{p(\cdot)} \) if and only if it is bounded on \( L^{p'(\cdot)} \).

### 3. The Maximal Characterization

In this section we define the variable Hardy spaces and give equivalent characterizations in terms of maximal operators. To state our results, we need a few definitions. Let \( \mathcal{S} \) be the space of Schwartz functions and let \( \mathcal{S}' \) denote the space of tempered distributions. We will say that a tempered distribution \( f \) is bounded if \( f^* \Phi \in L^\infty \) for every \( \Phi \in \mathcal{S} \). For complete information on distributions, see [13, 27]. Define the family of semi-norms on \( \| \cdot \|_{\alpha, \beta} \) by

\[
\| f \|_{\alpha, \beta} = \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)|,
\]

and for each integer \( N > 0 \) let

\[
\mathcal{S}_N = \{ f \in \mathcal{S} : \| f \|_{\alpha, \beta} \leq 1, |\alpha|, |\beta| \leq N \}.
\]

Given \( \Phi \) and \( t > 0 \), let \( \Phi_t(x) = t^{-n} \Phi(x/t) \). We define three maximal operators: given \( \Phi \in \mathcal{S} \) and \( f \in \mathcal{S}' \), define the radial maximal operator

\[
M_{\Phi,0} f = \sup_{t > 0} |f^* \Phi_t(x)|,
\]

and for each \( N > 0 \) the grand maximal operator,

\[
M_N f(x) = \sup_{\Phi \in \mathcal{S}_N} M_{\Phi,0} f(x).
\]

Finally, define the non-tangential maximal operator

\[
N f(x) = \sup_{|x-y| < t} |P_t - f(y)|,
\]

where \( P \) is the Poisson kernel

\[
P(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\pi^{\frac{n+1}{2}} (1 + |x|^2)^{\frac{n+1}{2}}}.
\]

Our main result in this section is the following.
Theorem 3.1. Given $p(\cdot) \in \mathcal{MP}_0$, for every $f \in S'$ the following are equivalent:

1. there exists $\Phi \in S$, $\int \Phi(x)dx \neq 0$, such that $M_{\Phi,0}f \in L^p(\cdot)$;
2. for all $N > n/p_0 + n + 1$, $M_Nf \in L^p(\cdot)$;
3. $f$ is a bounded distribution and $M_Nf \in L^p(\cdot)$.

Moreover, the quantities $\|M_{\Phi,0}f\|_{p(\cdot)}$, $\|M_Nf\|_{p(\cdot)}$ and $\|Nf\|_{p(\cdot)}$ are comparable with constants that depend only on $p(\cdot)$ and $n$ and not on $f$.

If we choose $N$ sufficiently large, then by Theorem 3.1 we can use any of these three maximal operators to given an equivalent definition of the variable Hardy spaces. To be definite we will use the grand maximal operator, but in the rest of the paper we will move between these three norms without comment.

Definition 3.2. Let $p(\cdot) \in \mathcal{MP}_0$. For $N > n/p_0 + n + 1$, define the space $H^p(\cdot)$ to be the collection of $f \in S'$ such that $\|f\|_{H^p(\cdot)} = \|M_Nf\|_{p(\cdot)} < \infty$.

Proof of Theorem 3.1. The proof is similar to that of the corresponding result for real Hardy spaces: cf. [22, 26]. The most difficult step is the implication (1) $\Rightarrow$ (2) which we will prove in Sections 3.1 and 3.2. We will then prove (2) $\Rightarrow$ (1) in Section 3.3 and (2) $\Rightarrow$ (3) $\Rightarrow$ (1) in Section 3.4.

3.1. The implication (1) $\Rightarrow$ (2). The proof requires two supplemental operators: the non-tangential maximal operator with aperture 1,

$$M_{\Phi,1}f(x) = \sup_{|x-y|<t \atop t>0} |f \ast \Phi_t(y)|,$$

and the tangential maximal operator,

$$M_{\Phi,T}f(x) = \sup_{y \in \mathbb{R}^n \atop t>0} |\Phi_t \ast f(x-y)| \left(1 + \frac{|y|}{t}\right)^{-T}.$$

Note that $T$ is a parameter in the definition of $M_{\Phi,T}$ and not just notation indicating the that this is a “tangential” operator.

We will prove this implication by proving three norm inequalities. First, if $N \geq T + n + 1$, we will show that

$$\|M_Nf\|_{p(\cdot)} \leq C(n, \Phi)\|M_{\Phi,T}f\|_{p(\cdot)}.$$  
(3.1)

Second, if $T > n/p_0$, we will show that

$$\|M_{\Phi,T}f\|_{p(\cdot)} \leq C(n, T, p(\cdot), p_0)\|M_{\Phi,1}f\|_{p(\cdot)}.$$  
(3.2)

Finally, we will show that

$$\|M_{\Phi,1}f\|_{p(\cdot)} \leq C(p(\cdot), T)\|M_{\Phi,0}f\|_{p(\cdot)}.$$  
(3.3)
To prove this we will first make the \textit{a priori} assumption that $M_{\Phi,1}f \in L^p(\cdot)$; we will then show that this is always the case by showing that if $M_{\Phi,0}f \in L^p(\cdot)$, then (3.3) holds with a constant that depends on $f$. This proof parallels the proof we just sketched; to emphasize this we will defer it to Section 3.2 and organize it similarly.

\textbf{Proof of inequality (3.1).} The proof requires a lemma from [22, Lemma 2.1].

\textbf{Lemma 3.3.} Let $\Phi \in \mathcal{S}$, $\int \Phi(x) \, dx \neq 0$. Then for any $\Psi \in \mathcal{S}$ and $T > 0$, there exist functions $\Theta^s \in \mathcal{S}$, $0 < s < 1$, such that

$$\Psi(x) = \int_0^1 \Phi_t \ast \Theta^s(x) \, dx$$

and for all $m \geq T + 1$,

$$\int_{\mathbb{R}^n} (1 + |x|)^T |\Theta^s(x)| \, dx \leq C(\Phi, n)s^T \|\Psi\|_{m+n,m}.$$

Fix $N \geq T + n + 1$ and fix $\Psi \in S_N$. Then by the definition of the tangential maximal operator, by making the change of variables $w = z/t$, and by Lemma 3.3, we get

$$|f \ast \Psi_t(x)| \leq \int_0^1 \int_{\mathbb{R}^n} |f \ast \Phi_{st}(x - z)||\Theta^s(z/t)|t^{-n} \, dz \, ds$$

$$= \int_0^1 \int_{\mathbb{R}^n} |f \ast \Phi_{st}(x - z)| \left(1 + \frac{|z|}{st}\right)^{-T} \times \left(1 + \frac{|z|}{st}\right)^T |\Theta^s(z/t)|t^{-n} \, dz \, ds$$

$$\leq M_{\Phi,T}f(x) \int_0^1 \int_{\mathbb{R}^n} \left(1 + \frac{|z|}{st}\right)^T |\Theta^s(z/t)|t^{-n} \, dz \, ds$$

$$\leq C(\Phi, n)M_{\Phi,T}f(x)\|\Psi\|_{T+n+1,T+1} \leq C(\Phi, n)M_{\Phi,T}f(x).$$

Given this pointwise inequality, we immediately get inequality (3.1).

\textbf{Proof of inequality (3.2).} Our proof is adapted from [22, Lemma 3.1]. Fix $x, y \in \mathbb{R}^n$ and $t > 0$. Then for all $z \in B(x - y, t)$,

$$|f \ast \Phi_t(x - y)| \leq M_{\Phi,1}f(z).$$

Let $q = n/T > 0$. Since $B(x - y, t) \subset B(x, |y| + t)$, we have that
\[ |f \ast \Phi_t(x - y)|^q \leq \int_{B(x-y,t)} M_{\Phi,1} f(z)^q \, dz \]
\[ \leq \frac{B(x,|y|+t)}{B(x-y,t)} \int_{B(x,|y|+t)} M_{\Phi,1} f(z)^q \, dz \leq \left( 1 + \frac{|y|}{t} \right)^n M(M_{\Phi,1}(f)^q)(x). \]

If we rearrange terms, then by our choice of \( q \) we have that
\[ \left| f \ast \Phi_t(x - y) \left( 1 + \frac{|y|}{t} \right)^{-T} \right|^q \leq M(M_{\Phi,1}(f)^q)(x). \]

Taking the supremum over all \( y \) and \( t \) we get that
\[ M_{\Phi,T} f(x)^q \leq M(M_{\Phi,1}(f)^q)(x). \]

Therefore, by Lemmas 2.3 and 2.12, since \( p(\cdot) \in MP_0 \) and \( q = n/T < p_0 \),
\[ \|M_{\Phi,T} f\|_{p(\cdot)} = \|(M_{\Phi,T} f)^q\|_{p(\cdot)/q}^{1/q} \leq \|M(M_{\Phi,1}(f)^q)\|_{p(\cdot)/q}^{1/q} \]
\[ \leq C(p(\cdot), p_0, n, q) \|M(M_{\Phi,1}(f)^q)\|_{p(\cdot)/q}^{1/q} = C(p(\cdot), p_0, n, q) \|M_{\Phi,1} f\|_{p(\cdot)}. \]

Since \( q \) depends on \( n \) and \( T \), this gives us inequality (3.2).

**Proof of inequality (3.3).** As we remarked above, we first assume that \( M_{\Phi,1} f \in L^{p(\cdot)} \). Our argument is very similar to that in Stein [26, pp. 95–98].

Let \( \lambda > 0 \) be some large number; the precise value will be fixed below. Define \( F = F_\lambda = \{ x : M_N f(x) \leq \lambda M_{\Phi,1} f(x) \} \). Then by inequalities (3.1) and (3.2),
\[ \|M_{\Phi,1}(f) \cdot \chi_{F^c}\|_{p(\cdot)} \leq \frac{1}{\lambda} \|M_N(f) \chi_{F^c}\|_{p(\cdot)} \leq \frac{1}{\lambda} \|M_N(f)\|_{p(\cdot)} \leq \frac{C_0}{\lambda} \|M_{\Phi,1}(f)\|_{p(\cdot)}, \]
where \( C_0 = C_0(n, \Phi, T, p(\cdot), p_0) \). Therefore, by Lemma 2.7 (if \( p_- < 1 \); the other case is treated similarly),
\[ \|M_{\Phi,1} f\|_{p(\cdot)}^{p_-} \leq \|M_{\Phi,1}(f) \cdot \chi_{F}\|_{p(\cdot)}^{p_-} + \|M_{\Phi,1}(f) \cdot \chi_{F^c}\|_{p(\cdot)}^{p_-} \]
\[ \leq \|M_{\Phi,1}(f) \cdot \chi_{F}\|_{p(\cdot)}^{p_-} + \left( \frac{C_0}{\lambda} \right)^{p_-} \|M_{\Phi,1} f\|_{p(\cdot)}^{p_-}. \]

Fix \( \lambda = 2^{1/p_-} C_0 \); since we assumed that \( M_{\Phi,1} f \in L^{p(\cdot)} \), we can rearrange terms to get
\[ \|M_{\Phi,1} f\|_{p(\cdot)} \leq 2 \|M_{\Phi,1}(f) \cdot \chi_{F}\|_{p(\cdot)}. \]

To estimate the right-hand side, we will use the fact that there exists \( c = c(p_0, \Phi, n, N, \lambda) \) such that for all \( x \in F \),
\[ M_{\Phi,1} f(x) \leq c M((M_{\Phi,0} f)^{p_0})(x)^{1/p_0}. \]
(See [26, p. 96].) Then again by Lemma 2.3 and since \( p(\cdot) \in MP_0 \),
\[ \|M_{\Phi,1} f \cdot \chi_{F}\|_{p(\cdot)} \leq c \|(M((M_{\Phi,0} f)^{p_0}))^{1/p_0}\|_{p(\cdot)}. \]
Finally, we will show that if \( (3.6) \) and the fact that \( M_{\Phi,0}^L \in L^{p(\cdot)} \), we show that there exists \( \lambda = \lambda(\Phi, n, T, p(\cdot), p_0) \) such that

\[
\| M_{\Phi,1}^L f \|_{p(\cdot)} \leq 2\| M_{\Phi,1} f \chi_F \|_{p(\cdot)}.
\]
Then we can use (3.7) to show that
\[ \|M_{\Phi,1}^\epsilon L f\|_{p(\cdot)} \leq C \|M_{\Phi,0} f\|_{p(\cdot)} < \infty. \]

The constant $C$ is independent of $\epsilon$, and so by Fatou's lemma, we get that
\[ \|M_{\Phi,1} f\|_{p(\cdot)} \leq C(f) \|M_{\Phi,0} f\|_{p(\cdot)} < \infty. \]

This completes the proof.

**Construction of the constant $L = L(f)$.** Since $f \in S'$, it is a continuous linear functional on $S$. In particular, arguing as in Folland [13, Proposition 9.10], we have that there exists $m > 0$ (depending only on $f$) such that
\[ |f * \Phi_t(y)| \leq C(\Phi, f) \left( 1 + \frac{|y|}{t} \right)^m. \]

Assume $L > 2m$. If $x$, $y$ and $t$ are such that $|x - y| < t < 1/\epsilon$, then we have that
\[ \left( 1 + \frac{|y|}{t} \right)^m \frac{t^L}{(t + \epsilon + \epsilon|y|)^L} \leq \epsilon^{-L} \left( \frac{1}{\epsilon} + \frac{|y|}{t} \right)^{m-L} \leq \epsilon^{-L} \left( \frac{1}{\epsilon} + \frac{|y|}{t} \right)^{-L/2}. \]

But by the triangle inequality,
\[ 1 + \epsilon|x| < 1 + \frac{|x|}{t} < 2 + \frac{|y|}{t} < \frac{1}{\epsilon} + \frac{|y|}{t}. \]

Combining these inequalities we get that
\[ |f * \Phi_t(y)| \frac{t^L}{(t + \epsilon + \epsilon|y|)^L} \leq \epsilon^{-L} C(\Phi, f)(1 + \epsilon|x|)^{-L/2}. \]

Fix $x$ and take the supremum over all such $y$ and $t$; this shows that
\[ M_{\Phi,1}^\epsilon L f(x) \leq \epsilon^{-L} C(\Phi, f)(1 + \epsilon|x|)^{-L/2}. \]

Finally, recall that
\[ (1 + \epsilon|x|)^{-n} \leq \epsilon^{-n} (1 + |x|)^{-n} \leq \epsilon^{-n} C(n) M(\chi_{B(0,1)})(x); \]

hence,
\[ M_{\Phi,1}^\epsilon L f(x) \leq \epsilon^{-3L/2} C(\Phi, f, n) M(\chi_{B(0,1)})(x)^{\frac{L}{2n}}. \]

Fix $L$ so that $L > 2m$. Since $p(\cdot) \in MP_0$, by Lemmas 2.3 and 2.12,
\[ \|M_{\Phi,1}^\epsilon L f\|_{p(\cdot)} \leq C \|M(\chi_{B(0,1)})\|_{\frac{L}{2n} p(\cdot)} \leq C \|\chi_{B(0,1)}\|_{\frac{L}{2n} p(\cdot)} < \infty, \]

where $C = C(\Phi, f, n, \epsilon, L, p(\cdot), p_0).$ Even though this constant depends on $\epsilon$ it does not effect the above argument, which only used the qualitative fact that $\|M_{\Phi,1}^\epsilon L f\|_{p(\cdot)} < \infty.$
Proof of inequality (3.5). We begin with an auxiliary estimate. Fix $s$, $0 < s < 1$ and as we did above, assume that $\epsilon < 1/2$. Then we have that

$$
\left(\frac{(st + \epsilon + \epsilon|x - z|)}{(st)^L}\right)^L \cdot \left(\frac{t^L}{(t + \epsilon + \epsilon|x|)^L}\right) = \left(\frac{(st + \epsilon + \epsilon|x - z|)}{s(t + \epsilon + \epsilon|x|)}\right)^L \leq s^{-L} \left(\frac{t + \epsilon + \epsilon|x|}{t + \epsilon + \epsilon|x|}\right)^L 
$$

Given this estimate we can argue exactly as in proof of inequality (3.5): if $N \geq T + L + n + 1$ and $\Psi \in S_N$, then we get that

$$
|f * \Psi_t(x)| \left(\frac{t^L}{(t + \epsilon + \epsilon|x|)^L}\right) \leq \int_0^1 \int_{\mathbb{R}^n} |f * \Psi_{st}(x - z)| \left(\frac{t^L}{(t + \epsilon + \epsilon|x|)^L}\right) \left(\frac{(st)^L}{(st + \epsilon + \epsilon|x - z|)^L}\right) \times \left(1 + \frac{|z|}{st}\right)^{-T(L + T)} \left(\frac{1 + |z|}{st}\right)^L \Theta^s(z/t)t^{-n} dz ds 
$$

$$
\leq M_{\Phi, T} f(x) \int_0^1 \int_{\mathbb{R}^n} \left(\frac{1 + |z|}{st}\right)^L \Theta^s(z/t)t^{-n} dz ds 
$$

$$
\leq C(\Phi, n) M_{\Phi, T} f(x) \|\Psi\|_{T + L + n + 1} \leq C(\Phi, n) M_{\Phi, T} f(x). 
$$

The desired inequality follows immediately.

Proof of inequality (3.6). This proof is a straightforward modification of the proof of inequality (3.2). As before, for all $x, y \in \mathbb{R}^n$, $t > 0$ and $\epsilon > 0$, we have that

$$
|\Phi_t * f(x - y)| \left(\frac{t^L}{(t + \epsilon + \epsilon|x - y|)^L}\right) \leq M_{\Phi, 1} f(z) 
$$

for all $x \in B(x - y, t)$. The proof now proceeds as before.

Proof of inequality (3.7). Fix $x \in F = \{x : M_{\Phi, T} f(x) < \lambda M_{\Phi, 1} f(x)\}$. Then by the definition of the truncated maximal operator, there exists $(y, t)$ with $t < 1/\epsilon$ and $|x - y| < t$, such that

$$
M_{\Phi, 1} f(x) \leq 2 |f * \Phi_t(y)| \left(\frac{t^L}{(t + \epsilon + \epsilon|y|)^L}\right). 
$$

Let $r > 0$ be small; its precise value will be fixed below. If $x' \in B(y, rt)$, then by the Mean Value Theorem,

$$
|f(x', t) - f(y, t)| \leq rt \sup_{|z - y| < rt} |\nabla_z f(z, t)|, 
$$
where for brevity we write \( f(y, t) = \Phi_t * f(y) \).

Inequality (3.7) follows if we can prove that there exists \( c = c(N, L, n, \Phi) \) such that

\[
(3.9) \quad \sup_{|z - y| < rt} |\nabla_z f(z, t)| \leq c M_{N,M}^{c,L} f(x) \cdot \frac{(t + \epsilon + |y|)^L}{t^L}.
\]

For if (3.9) holds, then for \( x \in F \), and \( x' \in B(y, rt) \),

\[
|f(x', t) - f(y, t)| \leq c r M_{N,M}^{c,L} f(x) \cdot \frac{(t + \epsilon + |y|)^L}{t^L} \leq c r \lambda M_{\Phi,1}^{c,L} f(x) \cdot \frac{(t + \epsilon + |y|)^L}{t^L}.
\]

Now fix \( r = r(N, L, n, \Phi, \lambda) \) so small that \( c r \lambda \leq 1/4 \). Then we have

\[
|f(x', t)| \geq |f(y, t)| - c r \lambda M_{\Phi,1}^{c,L} f(x) \cdot \frac{(t + \epsilon + |y|)^L}{t^L} \geq \left( \frac{1}{2} - c r \lambda \right) M_{\Phi,1}^{c,L} f(x) \geq \frac{1}{4} M_{\Phi,1}^{c,L} f(x).
\]

We can now get (3.7) by taking the average over all such points \( x' \):

\[
M_{\Phi,1}^{c,L} f(x)^{p_0} \leq 4^{p_0} |f(x', t)|^{p_0}
\]

\[
= 4^{p_0} \frac{1}{|B(y, rt)|} \int_{B(y, rt)} |f(x', t)|^{p_0} dx'
\]

\[
\leq 4^{p_0} \left( \frac{r + 1}{r} \right)^n \frac{1}{|B(x, (1 + r)t)|} \int_{B(x, (1 + r)t)} |f(x', t)|^{p_0} dx'
\]

\[
\leq c(p_0, r, n) \frac{1}{|B(x, (1 + r)t)|} \int_{B(x, (1 + r)t)} M_{\Phi,0} f(x')^{p_0} dx'
\]

\[
\leq c(p_0, r, n) M(M_{\Phi,0} f)^{p_0}(x).
\]

To complete the proof it remains to show (3.9). We begin with some notation: if we set \( \Phi^{(i)} = \frac{\partial \Phi}{\partial z_i} \), and \( \Phi^{(i)}_t(z) = (\Phi^{(i)}_t)(z) \), then \( \frac{\partial}{\partial z_i} (\Phi_t)(z) = \frac{1}{t} \Phi^{(i)}_t(z) \). Since \( f * \Phi \in C^\infty \) whenever \( f \in S' \) and \( \Phi \in S \), differentiating the convolution gives

\[
\frac{\partial}{\partial z_i}[f(z, t)] = f * \frac{\partial}{\partial z_i} (\Phi_t)(z) = \frac{1}{t} f * \Phi^{(i)}_t(z).
\]

Hence, we can rewrite the gradient term as

\[
|t \nabla_z(f)(z, t)| = \left( \sum_{i=1}^{n} |f * \Phi^{(i)}_t(z)|^2 \right)^{1/2}.
\]
We multiply and divide the left-hand side by the terms needed to obtain the truncated operator:

\[ t|\nabla_z f(z, t)| = t|\nabla_z f(z, t)| \cdot \frac{t^L}{(t + \epsilon + \epsilon|z|)^L} \cdot \frac{(t + \epsilon + \epsilon|z|)^L}{(t + \epsilon + \epsilon|y|)^L} \cdot \frac{(t + \epsilon + \epsilon|y|)^L}{t^L}. \]

Recall that we have fixed \( x \in \mathbb{R}^n \) and \( (y, t) \) so that (3.8) holds, and fixed \( z \in B(y, rt) \). Without loss of generality, we may assume that \( r \leq 1 \). We first estimate \( S(z, t) \):

\[
S(z, t) = |t\nabla_z f(z, t)| \cdot \frac{t^L}{(t + \epsilon + \epsilon|z|)^L} = \left( \sum_{i=1}^{n} |f \ast \Phi_t^i(z)|^2 \right)^{1/2} \cdot \frac{t^L}{(t + \epsilon + \epsilon|z|)^L} \\
= \left( \sum_{i=1}^{n} |f \ast \Phi_t^i(z)|^2 \frac{t^L}{(t + \epsilon + \epsilon|z|)^L} \right)^{1/2} \\
= C(N, \Phi) \left( \sum_{i=1}^{n} |M_{\Phi_t^i}^{a, \alpha, \beta} f(x)|^2 \right)^{1/2} \leq c(N, \Phi, n) M^{c, \eta}_{N} f(x).
\]

To see the first inequality, define the set of functions \( \Psi = \Psi^{i,h} \) by \( \Psi^{i,h}(x) = \Phi(x + h) \), \( 1 \leq i \leq n, |h| \leq 2 \). Since \( z = x + th \) for some \( h \) such that \(|h| \leq 1 + r, 2 \), we have that \( f \ast \Phi_t^i(z) = f \ast \Psi^{i,h}_t(z) \). Moreover, since the collection of functions \( \Psi^{i,h} \) is sequentially compact in \( S \), there exists a constant \( c = c(\Phi, N) \) such that \( \|\Psi^{i,h}\|_{a, \alpha, \beta} \leq c \), \( |\alpha|, |\beta| \leq N \). Hence, \( c^{-1} \Psi^{i,h} \in S_N \) and the desired inequality follows.

To estimate \( R(z, y) \) we note that if \( z \in B(y, rt) \), then \(|z| < |y| + rt \). Then, since we may assume that \( \epsilon, r < 1 \),

\[
R(z, y) \leq \left( \frac{t + \epsilon + \epsilon(|y| + rt)}{t + \epsilon + \epsilon|y|} \right) = 1 + \frac{ert}{t + \epsilon + \epsilon|y|} \leq 1 + \frac{ert}{\epsilon r} = 1 + \epsilon r \leq 2.
\]

Taking the supremum over \( z \), we get

\[
\sup_{|z-y|<rt} t|\nabla_z f(z, t)| \leq C 2^L \mathcal{M}^{c, \eta}_{N} f(x) \cdot \frac{(t + \epsilon + \epsilon|y|)^L}{t^L},
\]

where \( C = C(N, \Phi, n) \). This gives us (3.9).

3.3. The implication (2) \( \Rightarrow \) (1). Given any \( \Phi \in S \), there exists \( c = c(\Phi) \) such that \( c \Phi \in S_N \). Therefore, the radial maximal operator is always dominated pointwise by a constant multiple of the grand maximal operator; hence,

\[
(3.10) \quad \|M_{\Phi,0}f\|_{p(\cdot)} \leq C(\Phi)\|\mathcal{M}_N f\|_{p(\cdot)}.
\]
3.4. The implication $2 \Rightarrow 3 \Rightarrow 1$. Suppose first that $(2)$ holds. Fix $f \in H^{p(\cdot)}$ and let $\Phi \in \mathcal{S}$; then by $(3.3)$ and $(3.10)$, $M_{\Phi,1}f \in L^{p(\cdot)}$. Moreover, for all $x \in \mathbb{R}^n$,

$$|f \ast \Phi(x)|^{p_0} \leq \inf_{|x-y| \leq 1} M_{\Phi,1}f(y)^{p_0} \leq C(n) \int_{B(x,1)} M_{\Phi,1}f(y)^{p_0} \, dy$$

$$\quad \leq C(n, p(\cdot), p_0) \|(M_{\Phi,1}f)^{p_0}\|_{p(\cdot)/p_0} \|\chi_{B(x,1)}\|_{(p(\cdot)/p_0)'}.$$

By Lemma 2.2, if $q = \text{ess sup}_x (p(x)/p_0)'$, then

$$\|\chi_{B(x,1)}\|_{(p(\cdot)/p_0)'} \leq (1 + |B(x,1)|) \|\chi_{B(x,1)}\|_q \leq C(n, p(\cdot), p_0).$$

Therefore, $f \ast \Phi \in L^{p(\cdot)}$, since this is the case for every $\Phi \in \mathcal{S}$, $f$ is a bounded distribution.

To show that $\mathcal{N}f \in L^{p(\cdot)}$, we use the fact [26, p. 98] that the Poisson kernel can be written as

$$P(x) = \sum_{k=0}^{\infty} 2^{-k} \Phi^k_2(x),$$

where $\{\Phi^k\}$ is a family functions in $\mathcal{S}$ with uniformly bounded seminorms. Fix $x$ and $y$ such that $|x-y| < t$. Then

$$|f \ast P_t(y)| \leq \sum_{k=0}^{\infty} 2^{-k} |f \ast \Phi^k_2(x)| \leq \sum_{k=0}^{\infty} 2^{-k} M_{\Phi^k,1}f(x).$$

Taking the supremum over all such $y$ and $t$ we get that $\mathcal{N}f$ is dominated by the right-hand side. Since the functions $\Phi^k$ are uniformly bounded, by the same argument as we used to prove $(3.10)$ we have that this inequality holds for $\Phi^k$ with a constant independent of $k$. Therefore, if $p_1 \leq 1$, by Lemma 2.7 and $(3.3)$,

$$\|\mathcal{N}f\|_{p(\cdot)}^{p_-} \leq \sum_{k=0}^{\infty} 2^{kp_0} \|M_{\Phi^k,1}f\|_{p(\cdot)}^{p_-} \leq C(n, p(\cdot), p_0) \|M_{\mathcal{N}}f\|_{p(\cdot)}^{p_-}.$$

If $p_- > 1$, the same argument holds if we omit $p_-$ and use Minkowski’s inequality.

Now suppose that $(3)$ holds. Then there exists $\Phi \in \mathcal{S}$, $\int \Phi(x) \, dx = 1$ such that $\mathcal{M}_{\Phi,0}f(x) \leq c\mathcal{N}f(x)$ (see [26, p. 99]). Condition $(1)$ follows immediately.

4. DENSITY OF $L^1$ IN $H^{p(\cdot)}$

To prove the atomic decomposition we need two facts about variable Hardy spaces that are of interest in their own right.

**Proposition 4.1.** Given $p(\cdot) \in M\mathcal{P}_0$, $H^{p(\cdot)}$ is complete with respect to $\| \cdot \|_{H^{p(\cdot)}}$.

**Proof.** First, if a sequence $\{f_k\}$ converges in $H^{p(\cdot)}$ with respect $\| \cdot \|_{H^{p(\cdot)}}$, then it converges in $\mathcal{S}'$. To see this, fix $\Phi \in \mathcal{S}$; then
\[ |\langle f, \Phi \rangle|^{p_0} = |f \ast \Phi(0)|^{p_0} \leq \inf_{|y| \leq 1} M_{\Phi,1} f(y)^{p_0} \leq C(n) \int_{B(0,1)} M_{\Phi,1} f(y)^{p_0} \, dy \]
\[ \leq C(n, p(\cdot), p_0) \|M_{\Phi,1} f\|_{p_0}^{p_0} \chi_{B(0,1)}(\rho(\cdot)/p_0) \leq C(n, p(\cdot), p_0) \|M_{\Phi,1} f\|_{p_0}^{p_0}. \]

Our assertion follows at once.

To show that \( H^{p(\cdot)} \) is complete we will consider the case \( p_- \leq 1 \); the case \( p_- > 1 \) is proved in essentially the same way. It will suffice to show that \( H^{p(\cdot)} \) has the Riesz-Fisher property: given any sequence \( \{f_k\} \) in \( H^{p(\cdot)} \) such that

\[ (4.1) \quad \sum_k \|f_k\|^{p_-}_{H^{p(\cdot)}} < \infty, \]

the series \( \sum f_k \) converges in \( H^{p(\cdot)} \). (Cf. Bennett and Sharpley [1]. The argument there is for normed spaces but holds for quasi-norms with the introduction of the exponent \( p_- \) in (4.1).) Let

\[ F_j = \sum_{k=1}^j f_k; \]

then by Lemma 2.7 and (4.1), the sequence \( \{F_j\} \) is Cauchy in \( H^{p(\cdot)} \) and so in \( S' \). Therefore, it converges in \( S' \) to a tempered distribution \( f \). Moreover, we have that

\[ \|f\|^{p_{\ast}}_{H^{p(\cdot)}} = \|\sum_k f_k\|^{p_{\ast}}_{H^{p(\cdot)}} \leq \sum_k \|f_k\|^{p_{\ast}}_{H^{p(\cdot)}} < \infty, \]

and so \( f \in H^{p(\cdot)} \). Finally,

\[ \|f - F_j\|^{p_{\ast}}_{H^{p(\cdot)}} \leq \sum_{k \geq j+1} \|f_k\|^{p_{\ast}}_{H^{p(\cdot)}}, \]

and since the right-hand side tends to 0 as \( j \to \infty \), the series converges to \( f \) in \( H^{p(\cdot)} \). \( \square \)

**Proposition 4.2.** Given \( p(\cdot) \in MP_0 \), \( H^{p(\cdot)} \cap L^1_{loc} \) is dense in \( H^{p(\cdot)} \).

**Proof.** Given \( f \in H^{p(\cdot)} \), by Theorem 3.1, \( f \) is a bounded distribution. Hence, for any \( t > 0 \), \( f \ast P_t \in C^\infty \subset L^1_{loc} \). Therefore, it will suffice to show \( f \ast P_t \to f \) in \( H^{p(\cdot)} \). Again by Theorem 3.1 it will be enough to show that as \( t \to 0 \),

\[ \|N(f \ast P_t - f)\|_{p(\cdot)} \to 0. \]

Since \( p_+ < \infty \), by Lemma 2.4 this will follow if

\[ (4.2) \quad \int_{\mathbb{R}^n} N(f \ast P_t - f)(x)^{p(x)} \, dx \to 0. \]

Since \( P_s \ast P_t = P_{s+t} \), we immediately have

\[ N(f \ast P_t - f)(x)^{p(x)} = \sup_{s>0} |P_s \ast (P_t \ast f)(x)|^{p(x)} \leq 2^{p_+} N f(x)^{p(x)} \in L^1. \]
Thus, (4.2) follows from the dominated convergence theorem if for almost every \( x \),

\[
\lim_{t \to 0} \left( \sup_{s > 0} |P_s * (P_t * f - f)(x)| \right) = 0.
\]

To prove this, let \( u(x, s) = P_s * f(x) \). Arguing as above we have that \( \mathcal{N} u \in L^p(\cdot) \), and so \( u(x, s) \) is non-tangentially bounded almost everywhere. Therefore, for almost every \( x \),

\[
\lim_{s \to 0} u(x, s)
\]

exists. (See [14, Theorem 4.21].) Moreover, by Lemmas 2.13 and 2.6 we have that for all \( s \) large,

\[
|u(x, s)|^p \leq \int_{B(x, s)} \mathcal{N} f(y)^p \, dy
\]

\[
\leq C(p(\cdot)) |B(x, s)|^{-1} \|N f\|_{p(\cdot)/p_0}^p \|\chi_{B(x, s)}\|_{(p(\cdot)/p_0)'}
\]

\[
\leq C(p(\cdot), p_0) \|\chi_{B(x, s)}\|_{p(\cdot)/p_0}^{1/p_0} \|N f\|_{p(\cdot)/p_0}^p
\]

\[
\leq C(p(\cdot), p_0) |B(x, s)|^{-p_0/p_+} \|N f\|_{p(\cdot)/p_0}^{p_0}
\]

\[
\leq C(n, p(\cdot), p_0) s^{-np_0/p_+} \|N f\|_{p(\cdot)/p_0}^p.
\]

Thus,

\[
\lim_{s \to \infty} u(x, s) = 0.
\]

These two limits, combined with the fact that \( u \) is continuous, show that for almost every \( x \), \( u(x, s) \) is uniformly continuous in \( s \). The limit (4.3) now follows immediately, and this completes our proof. \( \square \)

5. The Calderón-Zygmund decomposition

**Theorem 5.1.** Given \( p(\cdot) \in M\mathcal{P}_0 \), fix \( f \in L^1_{\text{loc}} \cap H^p(\cdot) \). For each \( \lambda > 0 \) define \( \Omega_\lambda = \{ x : \mathcal{M}_N f(x) > \lambda \} \). Then there exists a set of cubes \( \{Q_k^*\} \) such that

\[
\Omega_\lambda = \bigcup_k Q_k^*
\]

and

\[
\sum_k \chi_{Q_k^*}(x) \leq C.
\]

Moreover, we can write \( f = g + b \), where \( |g(x)| \leq c\lambda \), \( b = \sum b_k \), \( \text{supp}(b_k) \subset Q_k^* \), \( \int b_k \, dx = 0 \), and

\[
\|M_{\Phi_b} b_k\|_{L^p(\cdot)(\mathbb{R}^n)} \leq C \|\mathcal{M}_N f\|_{L^p(\cdot)(Q_k^*)}.
\]
Proof. Our proof is adapted from the proof for constant exponents in Stein [26]. Since \( \Omega_\lambda \) is open, we can form the Whitney decomposition of \( \Omega_\lambda \). This gives us a set of cubes \( \{Q_k\} \) with mutually disjoint interiors. Further, if we let \( x_k \) and \( \ell_k \) be the center and size length of \( Q_k \), then there exist constants \( 1 < a < a^* \) such that if \( Q_k = aQ_k \) and \( Q_k' = a^*Q_k \), then \( Q_k \subset \tilde{Q}_k \subset Q_k' \) and \((5.1)\) and \((5.2)\) hold. Let \( P_0 = [-1/2, 1/2]^n \) and let \( \zeta \) be a smooth function such that \( \zeta|_{Q_0} = 1 \) and \( \zeta = 0 \) outside \( \tilde{a}Q_0 \). Define \( \zeta_k(x) = \zeta(\frac{x - x_k}{\ell_k}) \) and \( \eta_k = \zeta_k/(\sum \zeta_j) \); then \( \{\eta_k\} \) is a partition of unity for \( \Omega_\lambda \) subordinate to the cover \( \{\tilde{Q}_k\} \). Lastly, define \( \tilde{\eta}_k = \eta_k/\int \eta_k \, dx \).

Let \( d = \lceil n(1/p_0 - 1) \rceil \). We first consider the case \( d \leq 0 \); then \( p_- > p_0 > \frac{n}{n + 1} \), and by Lemma 2.12 the maximal operator is bounded on \( L^{p(\cdot)\frac{n+1}{n}} \). Let \( c_k = \langle f, \tilde{\eta}_k \rangle \) and define \( b_k = (f - c_k)\eta_k \), \( b = \sum b_k \) and \( g = f - b \). Then \( \int b_k \, dx = 0 \). Moreover (see Stein [26, pp. 102-3]), \( |g(x)| \leq c(1 + d) \) and if \( x \in Q_k^* \),

\[
(M_{\Phi,0}b_k(x) \leq C M_N f(x); \tag{5.4}
\]

if \( x \in \mathbb{R}^n \setminus Q_k^* \),

\[
(M_{\Phi,0}b_k(x) \leq C \lambda \frac{\ell_k}{|x - x_k|^{n+1}}; \tag{5.5}
\]

It remains to prove \((5.3)\). By Lemma 2.7,

\[
\|M_{\Phi,0}b_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq \|M_{\Phi,0}b_k(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} + \|M_{\Phi,0}b_k(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} = I_1 + I_2.
\]

By \((5.4)\) we immediately get \( I_1 \leq C \|M_N f(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} = C \|M_N f(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-}. \)

To estimate \( I_2 \), let \( B_0 \) be the ball centered at \( x \) with radius \( c_n \|x - x_k\| \), with \( c_n \) a dimensional constant such that \( Q_k^* \subset B_0 \). Then by the definition of \( M \),

\[
M(\chi_{Q_k^*}(x)) \geq \frac{1}{|B_0|} \int_{B_0} \chi_{Q_k^*} \, dx = \frac{|Q_k^*|}{|B_0|} \geq c(n) \frac{\ell_k^n}{|x - x_k|^{n+1}}.
\]

Therefore, by inequality \((5.5)\), Lemma 2.3, the boundedness of the maximal operator, and the fact that \( Q_k^* \subset E_\lambda \),

\[
I_2 = \|M_{\Phi,0}b_k(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq C \lambda \|\chi_{Q_k^*} \|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq C \lambda \|M(\chi_{Q_k^*}(x)\chi_{\mathbb{R}^n \setminus Q_k^*})\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq C \lambda \|M(\chi_{Q_k^*}(x)\chi_{\mathbb{R}^n \setminus Q_k^*})\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-} \leq C \lambda \|M_N f(x)\|_{L^{p(\cdot)}(\mathbb{R}^n)}^{p_-}. \]

Now suppose \( d \geq 1 \); We modify the above construction as follows. We have that the maximal operator is bounded on \( L^{p(\cdot)\frac{1}{n-1}} \). Let \( f_{d} \) be the space of polynomials of degree at most \( d \), considered as a subspace of the Hilbert space \( L^2(Q_k, \tilde{\eta}_k \, dx) \). Let \( c_k \) be the
projection of $f$ onto $\mathcal{H}_d$: then for all $q \in \mathcal{H}_d$, $\langle f - c_k, q \eta_k \rangle = 0$. We again define $b_k = (f - c_k) \eta_k$, $b = \sum b_k$, and $g = f - b$. Then we have (see Stein [26, pp. 104-5]) that $|g(x)| \leq C\lambda$ and

$$M_{\Phi,0}b_k(x) \leq c\mathcal{M}_N f(x)$$

if $x \in Q_k^*$, and

$$M_{\Phi,0}b_k(x) \leq c\lambda \frac{\rho_n^{n+d+1}|x - x_k|^{n+d+1}}{n^{n+d+1}}$$

if $x \in \mathbb{R}^n \setminus Q_k^*$. We now repeat the argument above. The estimate for $I_1$ is the same; the estimate for $I_2$ is nearly so:

$$I_2 = \|M_{\Phi,0}b_k \cdot \chi_{Q_k^*} \|_{p-} \leq C\lambda \|\frac{\rho_n^{n+d+1}\chi_{Q_k^*}}{|x - x_k|^{n+d+1}}\|_{p-} \leq C\lambda \|M(\chi_{Q_k^*})\|_{p-}^{n+d+1} \leq C\lambda \|\chi_{Q_k^*}\|_{p-}^{n+d+1} \leq C\|M_{\Phi,0} \chi_{Q_k^*}\|_{p-}.$$ 

This completes the proof of (5.3). \qed

6. THE ATOMIC DECOMPOSITION: $(p(\cdot), \infty)$ ATOMS

We begin with the definition of atoms.

**Definition 6.1.** Given $p(\cdot) \in MP_0$, and $q$, $1 < q \leq \infty$, a function $a(\cdot)$ is a $(p(\cdot), q)$ atom if $\text{supp}(a) \subset B = B(x_0, r)$ and it satisfies

(i) $\|a\|_q \leq \rho \chi_B^{-1}$,  

(ii) $\int a(x)x^\alpha dx = 0$ for all $|\alpha| \leq \lfloor n(p_0^{-1} - 1) \rfloor$.

In (i) we interpret $1/\infty = 0$. These two conditions are called the size and vanishing moments conditions of atoms.

**Remark 6.2.** If $p_0 > 1$ (which can happen if $p_{-} > 1$), then $\lfloor n(p_0^{-1} - 1) \rfloor < 0$, and we interpret this to mean that no vanishing moments are required.

In the remainder of this section we consider the case $q = \infty$.

**Theorem 6.3.** Suppose $p(\cdot) \in MP_0$. Then a distribution $f$ is in $H^{p(\cdot)}(\mathbb{R}^n)$ if and only if there exists a collection $\{a_j\}$ of $(p(\cdot), \infty)$ atoms supported on balls $\{B_j\}$, and non-negative coefficients $\{\lambda_j\}$ such that

$$f = \sum \lambda_j a_j,$$
where the series converges in $H^{p(\cdot)}(\mathbb{R}^n)$. Moreover,

\begin{equation}
\|f\|_{H^{p(\cdot)}} \simeq \inf \left\{ \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} : f = \sum_j \lambda_j a_j \right\}.
\end{equation}

**Remark 6.4.** As an immediate corollary we get that $(p(\cdot), \infty)$ atoms are uniformly bounded in $H^{p(\cdot)}$. However, as we will see, unlike the classical case we will not use this fact to prove the boundedness of operators.

Theorem 6.3 follows from two lemmas whose proof we defer momentarily.

**Lemma 6.5.** Given $p(\cdot) \in MP_0$, suppose \(\{a_j\}\) is a sequence of $(p(\cdot), \infty)$ atoms, supported on $B_j = B(x_j, r_j)$, and \(\{\lambda_j\}\) is a non-negative sequence that satisfies

\begin{equation}
\left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} < \infty.
\end{equation}

Then the series $f = \sum_j \lambda_j a_j$ converges in $H^{p(\cdot)}$, and

\begin{equation}
\|f\|_{H^{p(\cdot)}} \leq C(n, p(\cdot), p_0) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{p(\cdot)}.
\end{equation}

**Lemma 6.6.** Let $p(\cdot) \in MP_0$. If $f \in H^{p(\cdot)}$, then there exist $(p(\cdot), \infty)$ atoms \(\{a_{k,j}\}\), supported on balls $B_{k,j}$, and non-negative coefficients \(\{\lambda_{k,j}\}\) such that

\begin{equation}
f = \sum_{k,j} \lambda_{k,j} a_{k,j}.
\end{equation}

Moreover,

\begin{equation}
\left\| \sum_{k,j} \lambda_{k,j} \frac{\chi_{B_{k,j}}}{\|\chi_{B_{k,j}}\|_{p(\cdot)}} \right\|_{p(\cdot)} \leq C(n, p(\cdot), p_0) \|f\|_{H^{p(\cdot)}}.
\end{equation}

**Proof of Theorem 6.3.** By Lemmas 6.5 and 6.6, $f \in H^{p(\cdot)}$ if and only if it has the desired atomic decomposition. Therefore, it remains to show that (6.1) holds. Given $f \in H^{p(\cdot)}$, there exists an atomic decomposition such that (6.5) holds. This shows that the $H^{p(\cdot)}$ norm of $f$ dominates the infimum of the atomic decomposition norms. To see the opposite inequality, given any decomposition $f = \sum_j \lambda_j a_j$, (6.3) holds. Since this is true for all atomic decomposition, we have that $\|f\|_{H^{p(\cdot)}}$ is majorized by the infimum of the atomic decomposition norms.

Throughout the rest of this section, let $d = \lfloor n(\frac{1}{p_0} - 1) \rfloor$ and $\gamma = (n + d + 1)/n$. Since $p(\cdot) \in MP_0$, $M$ is also bounded on $L^{p(\cdot)}$. For by definition, $d > n(\frac{1}{p_0} - 1) - 1$, and this is equivalent to $\frac{n+d+1}{n} > \frac{1}{p_0}$. Thus by Lemma 2.12 we get the boundedness of $M$. 

\[\square\]
Proof of Lemma 6.5. Fix $\Phi \in \mathcal{S}$ such that $\int \Phi \, dx \neq 0$ and supp($\Phi$) $\subset$ $B(0, 1)$. Fix atoms \{a$_j$\} with support \{B$_j$\} and coefficients \{λ$_j$\} such that (6.2) holds. Given $B = B(x_0, r)$, let $2B = B(x_0, 2r)$. We consider the case $p_- < 1$; if $p_- \geq 1$ the proof is essentially the same, omitting the exponent $p_-$. By Lemma 2.7,

$$M_{\Phi,0}f \|_{p(\cdot)}^{p_-} \lesssim \left\| \sum_j \lambda_j M_{\Phi,0}(a_j) \right\|_{p(\cdot)}^{p_-} \leq \left\| \sum_j \lambda_j M_{\Phi,0}(a_j) \cdot \chi_{2B_j} \right\|_{p(\cdot)}^{p_-} + \left\| \sum_j \lambda_j M_{\Phi,0}(a_j) \cdot \chi_{(2B_j)^{c}} \right\|_{p(\cdot)}^{p_-}.$$ 

We first estimate $I_1$. By the size condition on $(p(\cdot), \infty)$ atoms, we have that

$$M_{\Phi,0}a_j(x) \leq \|a_j\|_{\infty} \|\Phi\|_1 \leq c\|\chi_{B_j}\|_{p(\cdot)}^{-1}.$$ 

Define $g_j = (\|\chi_{B_j}\|_{p(\cdot)}^{-1} \lambda_j)^{p_0}\chi_{B_j}$. If $x \in \chi_{2B_j}$, then by the definition of the maximal operator,

$$Mg_j(x) \geq (\|\chi_{B_j}\|_{p(\cdot)}^{-1} \lambda_j)^{p_0} \frac{1}{2B_j} \int_{2B_j} \chi_{B_j} \, dx = 2^{-n}(\|\chi_{B_j}\|_{p(\cdot)}^{-1} \lambda_j)^{p_0}.$$ 

Then by Lemmas 2.3 and 2.14,

$$I_1 \leq C \left\| \sum_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \lambda_j \chi_{2B_j} \right\|_{p(\cdot)}^{p_-} \leq C \left\| \sum_j M(g_j)^{1/p_0} \right\|_{p(\cdot)}^{p_-} = C \left\| \left( \sum_j M(g_j)^{1/p_0} \right)^{p_0/p_0} \right\|_{p(\cdot)}^{p_-} = C \left\| \sum_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \lambda_j \chi_{B_j} \right\|_{p(\cdot)}^{p_-}.$$ 

To estimate $I_2$, let $a$ be an atom supported on $B = B(x_0, r)$. Then arguing as in [26, p. 106] we have for $x \in (2B)^{c}$ the pointwise estimate

$$M_{\Phi,0}a(x) \leq c \left( \frac{r}{|x - x_0|} \right)^{n+1+d} \int_{B} a(y) \, dy \leq c \left( \frac{r}{|x - x_0|} \right)^{n+1+d} \|a\|_{\infty} \leq c \left( \frac{r}{|x - x_0|} \right)^{n+1} \|\chi_{B}\|_{p(\cdot)}^{-1}.$$ 

Now arguing as we did in the proof of inequality (5.6), we have for each $j$ that

$$M_{\Phi,0}(a_j)(x) \leq c \left( \frac{r_j}{|x - x_j|} \right)^{n+1} \|\chi_{B_j}\|_{p(\cdot)}^{-1} \|\chi_{B_j}\|_{p(\cdot)}^{-1} \leq c \|\chi_{B_j}\|_{p(\cdot)}^{-1} \|\chi_{B_j}\|_{p(\cdot)}^{-1} M(\chi_{B_j})^\gamma.$$ 

We can now estimate $I_2$: by Lemmas 2.3 and 2.14,
Proof of Lemma 6.6. We will prove this result assuming \( f \in H^p(\cdot) \cap L^1_{loc} \), then by Proposition 4.2 and a density argument (cf. [26, p. 109]) we get it for arbitrary \( f \in H^p(\cdot) \).

Fix such an \( f \) and for every \( j \in \mathbb{Z} \), let \( E_j = \{ x : \mathcal{M}_N f(x) > 2^j \} \). By Theorem 5.1 we have that \( f = g^j + b^j \), where \( |g^j(x)| \leq c 2^j \) and \( b^j = \sum_k b^j_k \), with each \( b^j_k \) supported on a cube \( Q^j_k \). These cubes have bounded overlap \( E_j = \bigcup_k Q^j_k \). Moreover, we have that

\[
\lim_{j \to \infty} \| b^j \|_{H^p(\cdot)} = 0. 
\]

To show this we proceed as in the proof of Lemma 6.5 (again only considering the case \( p_- < 1 \)):

\[
\| b^j \|_{H^p(\cdot)}^{p_-} \leq \left\| \sum_k M_{\phi,0}(b^j_k) \cdot \chi_{Q^j_k} \right\|_{p(\cdot)}^{p_-} + \left\| \sum_k M_{\phi,0}(b^j_k) \cdot \chi_{Q^j_k} \right\|_{p(\cdot)}^{p_-}.
\]

We first estimate \( I_1 \): by (5.4) we have that

\[
I_1 \leq c \left\| \sum_k M_{\phi,0}(b^j_k) \cdot \chi_{Q^j_k} \right\|_{p(\cdot)}^{p_-} \leq c \left\| M_{\phi,0} f \cdot \chi_{E_j} \right\|_{p(\cdot)}^{p_-}.
\]

The last term tends to 0 as \( j \to 0 \): this follows by Lemma 2.4 and the dominated convergence theorem.

To estimate \( I_2 \), let \( x_{k,j} \) and \( \ell_{k,j} \) be the center and side length of \( Q^j_k \). Then arguing as we did for inequality (5.6), if \( x \in (Q^j_k)^c \), then

\[
M(\chi_{Q^j_k})(x) \geq c \frac{\ell_{k,j}^n}{|x - x_k|^n}.
\]

Then by inequality (5.5) and Lemma 2.14,

\[
I_2 = \left\| \sum_k M_{\phi,0}(b^j_k) \cdot \chi_{Q^j_k} \right\|_{p(\cdot)}^{p_-} \leq c \left\| \sum_k 2^j \frac{\ell_{k,j}^{n+1+d}}{|x - x_k|^{n+d+1}} \cdot \chi_{Q^j_k} \right\|_{p(\cdot)}^{p_-}.
\]
As before, the last term goes to 0 as \( j \to \infty \). This proves the limit (6.9).

As a consequence of (6.9) we have that \( g_j \to f \) in norm (and so in \( S' \)) as \( j \to \infty \). Further, since \( g_j \to 0 \) uniformly as \( j \to -\infty \), we have that

\[
f = \sum_j (g_j^{j+1} - g_j^j).
\]

From the proof of Theorem 5.1, let \( \{\eta_k^j\} \) be the partition of unity for \( E_j \) with \( \text{supp}(\eta_k^j) \subset Q_{k,j}^{i^*} \). Since \( g_j^{j+1} - g_j^j = b_j^{j+1} - b_j^j \), \( \text{supp}(g_j^{j+1} - g_j^j) \subset E_j \). Therefore, we have that

\[
f = \sum_{j,k} (g_j^{j+1} - g_j^j) \eta_k^j.
\]

We now want to show that this expression can be rewritten as sum of atoms. Our argument follows Stein [26, pp.108–9], and since many details are the same, we omit them here. Again as in the proof of Theorem 5.1, define the projections \( \mathcal{P}_{k}^j : S' \to \mathcal{H}_d \), where \( \mathcal{H}_d \) is the space of polynomials of degree at most \( d \), thought of as a subspace of the Hilbert space \( L^2(Q_{k,j}^{i^*}, \eta_k^j \, dx) \). Define the polynomials \( c_k^j = \mathcal{P}_{k}^j(f) \) and \( c_{\ell}^{j+1} = \mathcal{P}_{\ell}^{j+1}(f) \), and let \( c_{k,\ell} = \mathcal{P}_{\ell}^{j+1}((f - c_{\ell}^{j+1}) \eta_k^j) \). For each \( j \), we can then write

\[
g^{j+1} - g^j = b^{j+1} - b^j = \sum_k (f - c_k^j) \eta_k^j - \sum_{\ell} (f - c_{\ell}^{j+1}) \eta_{\ell}^{j+1} = \sum_k A_k^j,
\]

where

\[
A_k^j = (f - c_k^j) \eta_k^j - \left( \sum_{\ell} (f - c_{\ell}^{j+1}) \eta_{\ell}^{j+1} \right) \eta_k^j + \sum_{\ell} c_{k,\ell} \eta_{\ell}^{j+1}.
\]

There exists a ball \( B_{k,j} = B(x_{k,j}, c_{\ell}^{k,j}) \) containing the cube \( Q_{k,j}^{i^*} \) such that \( |B_{k,j}| \leq c |Q_{k,j}^{i^*}| \). Moreover we have that \( |A_k^j| \leq c 2^j \) and \( A_k^j \) satisfies the moment conditions for \( (p(\cdot), \infty) \) atoms. Therefore, if we define

\[
a_{k,j} = A_k^j c_k^j 2^{-j} \|\chi_{B_{k,j}}\|_{p(\cdot)}^{-1}, \quad \lambda_{k,j} = c 2^j \|\chi_{B_{k,j}}\|_{p(\cdot)},
\]

the \( a_{k,j} \) are \( (p(\cdot), \infty) \) atoms and we have the decomposition (6.4). It converges in \( S' \), and so, arguing as in the proof of Proposition 4.2, it converges in \( H_{p(\cdot)}(\mathbb{R}^n) \).
Finally, we prove (6.5). We consider the case \( p_- < 1 \); if \( p_- \geq 1 \), modify the following argument by replacing \( 1/p_0 \) by \( q > 1 \). Since \( |B_{k,j}| \leq c|Q_k^*| \), \( M(\chi_{Q_k^*}) \geq c\chi_{B_{k,j}} = c\chi_{B_{k,j}^0} \). Therefore, by Lemmas 2.3 and 2.14,

\[
\left\| \sum_{k,j} \frac{\lambda_{k,j}}{\|X_{B_{k,j}}\|_{p(\cdot)}} \chi_{B_{k,j}} \right\|_{p(\cdot)} \leq C \left\| \sum_{k,j} \left( 2^{j/p_0} M(\chi_{Q_k^*}) \right)^{1/p_0} \right\|_{p(\cdot)}
\]

\[
= C \left\| \left( \sum_{k,j} M(2^{j/p_0} \chi_{Q_k^*})^{1/p_0} \right)^{p_0} \right\|^{1/p_0} \leq C \left\| \left( \sum_{k,j} 2^j \chi_{Q_k^*} \right)^{p_0} \right\|^{1/p_0}
\]

\[
= C \left\| \sum_{k,j} 2^j \chi_{Q_k^*} \right\|_{p(\cdot)} \leq C \left\| \sum_j 2^j \chi_{E_j} \right\|_{p(\cdot)}.
\]

If \( x \in \mathbb{R}^n \), there exists a unique \( j_0 \in \mathbb{Z} \) such that \( 2^{j_0} < M_N f(x) \leq 2^{j_0+1} \). Hence,

\[
\sum_j 2^j \chi_{E_j}(x) = \sum_{j \leq j_0} 2^j = 2^{j_0+1} \leq 2M_N f(x).
\]

If we combine this with the previous estimate, we get (6.5).

\[\square\]

### 7. The atomic decomposition: \((p(\cdot), q)\) atoms

In this section we consider the atomic decomposition when \( q < \infty \). Our first main result is that when \( q \) is sufficiently large, the analog of Theorem 6.3 holds. Furthermore, we show that in this case we can give a finite atomic decomposition of \( H^{p(\cdot)} \) (Theorem 7.8 below). Lastly, by minor modifications to the proof of Theorem 7.8, we give a finite atomic decomposition of the weighted Hardy space \( H^{p_0(w)} \) (Theorem 7.11 below). We use this to prove the boundedness of singular integral operators on \( H^{p(\cdot)} \) in Section 8.

#### 7.1. Infinite atomic decomposition using \((p(\cdot), q)\) atoms.

We extend Theorem 6.3 by giving an atomic decomposition using \((p(\cdot), q)\) atoms.

**Theorem 7.1.** Suppose \( p(\cdot) \in \mathcal{MP}_0 \). Then a distribution \( f \) is in \( H^{p(\cdot)} \) if and only if for \( q > 1 \) sufficiently large, there exists a collection \( \{a_j\} \) of \((p(\cdot), q)\) atoms supported on balls \( \{B_j\} \), and non-negative coefficients \( \{\lambda_j\} \) such that

\[
f = \sum_j \lambda_j a_j,
\]

where the series converges in \( H^{p(\cdot)}(\mathbb{R}^n) \). Moreover,

\[
\|f\|_{H^{p(\cdot)}} \simeq \inf \left\{ \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{p(\cdot)} : f = \sum_j \lambda_j a_j \right\}.
\]
Remark 7.2. Denote the norm of the maximal operator by \( \| M \|_{(p(\cdot)/p_0)'} \). Then it suffices to take \( q > \max(1, p_+, p_0(1 + 2^{n+3}\| M \|_{(p(\cdot)/p_0)'})) \).

One half of the proof of Theorem 7.1 is immediate: since for any \( q, 1 < q < \infty, |B|^{1/q}\| a \|_q \leq \| a \|_{\infty}, (p(\cdot), \infty) \) atoms are \( (p(\cdot), q) \) atoms. Therefore, by Lemma 6.6, every function \( f \in H^{p(\cdot)} \) can be written as the sum of \( (p(\cdot), q) \) atoms and \( \| f \|_{H^{p(\cdot)}} \) has the desired bound. Note that in this case there are no restrictions on \( q \). The heart of the proof, therefore, is to prove the converse.

Lemma 7.3. Given \( p(\cdot) \in MP_0 \), there exists \( q = q(p(\cdot), p_0, n) > \max(p_+, 1) \) such that if \( \{a_j\} \) is a sequence of \( (p(\cdot), q) \) atoms supported on \( B_j = B(x_j, r_j) \), and \( \{\lambda_j\} \) is a non-negative sequence that satisfies

\[
(7.2) \quad \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\| \chi_{B_j} \|_{p(\cdot)}} \right\|_{p(\cdot)} < \infty,
\]

then the series \( f = \sum_j \lambda_j a_j \) converges in \( H^{p(\cdot)}(\mathbb{R}^n) \), and

\[
(7.3) \quad \| f \|_{H^{p(\cdot)}} \leq C(n, p(\cdot), p_0, q) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\| \chi_{B_j} \|_{p(\cdot)}} \right\|_{p(\cdot)}.
\]

To prove Lemma 7.3 we will adapt the proof of Rubio de Francia extrapolation in the setting of variable Lebesgue spaces. This was first proved in [6] (see also [4, 7]). We need more careful control of the constants than was given in the original proof, and so we will reproduce the key steps.

To apply extrapolation we need a version of Lemma 7.3 for weighted \( H^p \) spaces. To state it we introduce some definitions and preliminary results. For complete information on the theory of weights, see [10, 14]. By a weight \( w \) we will always mean a non-negative, locally integrable function that is positive almost everywhere. We will say that \( w \in A_1 \) if

\[
[w]_{A_1} = \text{ess sup}_{x \in \mathbb{R}^n} \frac{M w(x)}{w(x)} < \infty.
\]

Equivalently, \( w \in A_1 \) if given any ball \( B \),

\[
\int_B w(y) \, dy \leq [w]_{A_1} \text{ess inf}_{x \in B} w(x).
\]

A weight satisfies the reverse Hölder inequality with exponent \( s > 1 \), denoted by \( w \in RH_s \), if for every cube \( B \),

\[
\left( \int_B w(x)^s \, dx \right)^{1/s} \leq C \int_Q w(x) \, dx;
\]

the best possible constant is denoted by \( [w]_{RH_s} \). Note that if \( w \in RH_s \), then by Hölder’s inequality, \( w \in RH_t \) for all \( t, 1 < t < s \), and \( [w]_{RH_t} \leq [w]_{RH_s} \). If \( w \in A_1 \), then \( w \in RH_s \), and we have sharp control over the exponent \( s \).
Lemma 7.4. Given \( w \in A_1 \), then \( w \in RH_s \), where \( s = 1 + (2^{n+2}[w]_{A_1}^{-1})^{-1} \).

Remark 7.5. This result is proved in [21] (see also [4]), where everything is done in terms of cubes instead of balls. However, because \( w \in A_1 \) is doubling, the reverse Hölder inequality holds for balls with same exponent; in this case the constant \([w]_{RH_s}\) depends on \( n \) and \([w]_{A_1}\).

Given a weight \( w \in A_1 \) and \( p_0 > 0 \), the weighted Hardy space \( H^{p_0}(w) \) consists of all tempered distributions \( f \) such that
\[
\|f\|_{H^{p_0}(w)} = \|M_{\Phi,0}f\|_{L^{p_0}(w)} = \left( \int_{\mathbb{R}^n} M_{\Phi,0}f(x)^{p_0}w(x) \, dx \right)^{1/p_0} < \infty.
\]

These spaces have an atomic decomposition: see Strömberg and Torchinsky [28]. We state their result in the form we need to apply it; see Remark 7.7 below.

Lemma 7.6. Given \( p(\cdot) \in \mathcal{MP}_0 \) and \( q > \max(p_0, 1) \), suppose \( \{a_j\} \) is a sequence of \((p(\cdot), q)\) atoms, \( \{\lambda_j\} \) is a non-negative sequence, and \( w \in A_1 \cap RH(q/p_0) \). If
\[
\left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{L^{p_0}(w)} < \infty.
\]
Then the series
\[
f = \sum_j \lambda_j a_j
\]
converges in \( H^{p_0}(w) \) and
\[
\|f\|_{H^{p_0}(w)} \leq C(p(\cdot), p_0, q, n, [w]_{A_1}, [w]_{RH(q/p_0)'} ) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(\cdot)}} \right\|_{L^{p_0}(w)}.
\]

Remark 7.7. In [28, Chapter VIII, Theorem 1] this result is stated for atoms \( \tilde{a}_j \) that (obviously) do not depend on a variable exponent \( p(\cdot) \). To pass between the two kinds of atoms, it suffices to take \( \tilde{a}_j = \|\chi_{B_j}\|_{p(\cdot)} a_j \) and \( \tilde{\lambda}_j = \lambda_j \|\chi_{B_j}\|_{p(\cdot)}^{-1} \). The atoms \( \tilde{a}_j \) are required to have vanishing moments for \(|\alpha| \leq \lfloor d/p - n \rfloor \), where \( d \) is a constant such that for all \( t \geq 1 \),
\[
w(B(x, tr)) \leq K t^d w(B(x, r)).
\]
If \( w \in A_1 \), then this is true with \( d = n \):
\[
w(B(x, tr)) \leq [w]_{A_1} |B(x, tr)| \, \text{ess inf}_{y \in B(x, tr)} w(y) \leq [w]_{A_1} t^n |B(x, r)| \, \text{ess inf}_{y \in B(x, r)} w(y) \leq [w]_{A_1} t^n w(B(x, r)).
\]

Proof of Lemma 7.3. Fix \( p(\cdot) \in \mathcal{MP}_0 \); by Lemma 2.15 the maximal operator is bounded on \( L^{(p(\cdot))/p_0}(\mathbb{R}^n) \). Denote the norm of the maximal operator by \( \|M\|_{(p(\cdot))/p_0)'} \). Fix \( q > \max(1, p_+, p_0(1 + 2^{n+3}\|M\|_{(p(\cdot))/p_0)'}) \); the reason for this choice will be made clear below.
We will first show that if $a$ is a $(p(\cdot), q)$ atom with support $B$, then $a \in H^{p(.)}$. To do so we will show that $\|Mf,0a\|_{p(.)} < \infty$. By Lemma 2.7 (if $p_- < 1$; the case $p_- \geq 1$ is handled similarly),
\[ \|Mf,0a\|_{p(.)}^p \leq \|Mf,0(a)\chi_{2B}\|_{p(.)}^p + \|Mf,0(a)\chi_{(2B)^c}\|_{p(.)}^p = I_1 + I_2. \]
By Lemma 2.2, since $q > \max(p_+, 1)$ and $Mf,0$ is bounded on $L^q$,
\[ I_1 = \|Mf,0(a)\chi_{2B}\|_{p(.)} \leq (1 + |2B|)\|Mf,0(a)\chi_{2B}\|_q \leq C(1 + |2B|)\|a\|_q < \infty. \]
To show that $I_2$ is finite, by inequality (6.7) and the definition of $(p(\cdot), q)$ atoms, and arguing as we did for (6.8), for $x \in (2B)^c$ we have that
\[ Mf,0a(x) \leq c \left( \frac{r}{|x - x_0|} \right)^{n\gamma} |B|^{-1/q} \|a\|_q \]
\[ \leq \left( \frac{r}{|x - x_0|} \right)^{n\gamma} \|\chi_B\|_{p(.)}^{-1} \|\chi_B\|^{-1} M(\chi_B)(x)^\gamma, \]
where $x_0$ is the center of $B$ and $\gamma = (n+d+1)/n$. As we noted in the proof of Theorem 6.3, $M$ is bounded on $L^{p(\cdot)}$. Therefore, by Lemma 2.3,
\[ I_2 = \|Mf,0(a)\chi_{(2B)^c}\|_{p(.)} \leq c \|\chi_B\|_{p(.)}^{-1} \|M(\chi_B)\|_{p(.)} \leq c \|\chi_B\|_{p(.)}^{-1} \|\chi_B\|_{p(.)}^{\gamma} \leq \infty. \]
To construct our weight $w$, form the Rubio de Francia iteration algorithm with respect to $L^{p(\cdot)/p_0}'$. Given a function $h$, define
\[ Rh = \sum_{i=0}^{\infty} \frac{M^i f}{2^i \|M\|_{p(\cdot)/p_0}'}, \]
where $M^0 h = |h|$ and for $i \geq 1$, $M^i h = M \circ \cdots \circ M h$ is $i$ iterates of the maximal operator. Three facts follow at once from this definition (cf. [6, 7]):
1. $|h| \leq Rh$;
2. $R$ is bounded on $L^{p(\cdot)/p_0}'(\mathcal{R}^n)$ and $\|R h\|_{(p(\cdot)/p_0)'} \leq 2 \|h\|_{(p(\cdot)/p_0)'}$;
3. $R h \in A_1$ and $[R h]_{A_1} \leq 2 \|M\|_{(p(\cdot)/p_0)'} = C(p(\cdot), p_0, n)$.
By Lemma 7.4 we have that $Rh \in RH_s$, where $s = 1 + (2^{n+3}\|M\|_{(p(\cdot)/p_0)'}^{-1})$. Therefore, since $q \geq p_0(1 + 2^{n+3}\|M\|_{(p(\cdot)/p_0)'}^{-1})$, we have that $Rh \in RH_{(q/p_0)'}$ and $[R h]_{R H_{(q/p_0)'}} \leq C(p(\cdot), p_0, n)$. We stress that all of these constants are independent of $h$.
Fix a sequence of atoms $\{a_j\}$ and constants $\{\lambda_j\}$ as in the hypotheses. Let $f = \sum \lambda_j a_j$; a priori we do not know that this series converges in $H^{p(.)}$. To avoid this problem, define the functions
\[ f_k = \sum_{j=1}^{k} \lambda_j a_j. \]
Then $f_k \in H^{p(\cdot)}(\mathbb{R}^n)$: since $a_j \in H^{p(\cdot)}$, by Lemma 2.7 (if $p_- < 1$, the case $p_- \geq 1$ is handled similarly)
\[
\left\| M_{\Phi,0} f_k \right\|_{p(\cdot)}^{p_-} \leq \sum_{j=1}^k \lambda_j^{p_-} \left\| M_{\Phi,0} a_j \right\|_{p(\cdot)}^{p_-} < \infty.
\]
Furthermore, by Lemma 7.6, given any function $h$, $f_k \in H^{p_0}(\mathcal{R}h)$, and
\[
\tag{7.4}
\left\| f \right\|_{H^{p_0}(\mathcal{R}h)} \leq C(p(\cdot), p_0, n) \left\| \sum_{j=1}^k \lambda_j \frac{\chi_{B_j}}{\chi_{B_j}} \right\|_{L^{p_0}(\mathcal{R}h)}.
\]

We will now show that (7.3) holds for each $f_k$ with a constant independent of $k$. By Lemmas 2.1 and 2.3,
\[
\left\| M_{\Phi,0} f_k \right\|_{p(\cdot)}^{p_0} = \left\| (M_{\Phi,0} f_k)^{p_0} \right\|_{(p(\cdot))^{-1}} \leq C(p(\cdot), p_0) \sup_{h} \int_{\mathbb{R}^n} M_{\Phi,0} f_k(x)^{p_0} h(x) \, dx,
\]
where the supremum is taken over all $h \in L^{(p(\cdot))'}$ with $\left\| h \right\|_{(p(\cdot))'} \leq 1$. (We may assume that $h$ is non-negative.) Fix such a function $h$; we will estimate the integral on the right-hand side with a constant independent of $h$. By the properties of the Rubio de Francia iteration algorithm, (7.4) and Lemmas 2.1 and 2.3,
\[
\int_{\mathbb{R}^n} M_{\Phi,0} f_k(x)^{p_0} h(x) \, dx \leq \int_{\mathbb{R}^n} M_{\Phi,0} f_k(x)^{p_0} \mathcal{R}h(x) \, dx \leq C \int_{\mathbb{R}^n} \left( \sum_{j=1}^k \lambda_j \frac{\chi_{B_j}(x)}{\chi_{B_j}} \right)^{p_0} \mathcal{R}h(x) \, dx \leq C \left\| \left( \sum_{j=1}^k \lambda_j \frac{\chi_{B_j}}{\chi_{B_j}} \right)^{p_0} \right\|_{(p(\cdot))'} \mathcal{R}h \left\|_{(p(\cdot))'}^{p_0} \leq C \left\| \sum_{j=1}^k \lambda_j \frac{\chi_{B_j}}{\chi_{B_j}} \right\|_{p(\cdot)}^{p_0}.
\]
Inequality (7.3) for $f_k$ now follows and the constant depends only on $p(\cdot)$, $p_0$ and $n$.

To complete the proof we need to show that (7.3) holds for $f$. But the same argument that proved this inequality for $f_k$ shows that if $l > k$,
\[
\tag{7.5}
\left\| f_l - f_k \right\|_{H^{p(\cdot)}(\mathbb{R}^n)} \leq C \left\| \sum_{j=k+1}^l \lambda_j \frac{\chi_{B_j}}{\chi_{B_j}} \right\|_{p(\cdot)}.
\]
However, by hypothesis we have that
\[
\left\| \sum_{j=1}^\infty \lambda_j \frac{\chi_{B_j}}{\chi_{B_j}} \right\|_{p(\cdot)} < \infty
\]
and therefore the partial sums of this series are Cauchy in $L^p(\mathbb{R}^n)$. Hence, as $k, l \to \infty$, the right-hand side of (7.5) tends to 0. Therefore, the sequence $\{f_k\}$ is Cauchy in $H^{p(\cdot)}$ and so by Proposition 4.1 converges to $f$ in $H^{p(\cdot)}$. Therefore, by the monotone convergence theorem in variable Lebesgue spaces (Lemma 2.5) we have that

$$\|f\|_{H^{p(\cdot)}(\mathbb{R}^n)} = \lim_{k \to \infty} \|f_k\|_{H^{p(\cdot)}(\mathbb{R}^n)}$$

$$\leq C \lim_{k \to \infty} \left\| \sum_{j=1}^{k} \frac{\lambda_j}{\|\chi_{B_j}\|_{p(\cdot)}} \chi_{B_j} \right\|_{p(\cdot)} = C \left\| \sum_{j} \frac{\lambda_j}{\|\chi_{B_j}\|_{p(\cdot)}} \chi_{B_j} \right\|_{p(\cdot)}.$$

7.2. Finite atomic decompositions. Given $q < \infty$, let $H^{p(\cdot),q}_{fin}$ be the subspace of $H^{p(\cdot)}$ consisting of all $f$ that have decompositions as finite sums of $(p(\cdot), q)$ atoms. By Theorem 6.3, if $q$ is sufficiently large, $H^{p(\cdot),q}_{fin}$ is dense in $H^{p(\cdot)}$. Our next result shows that on this subspace the atomic decomposition norm, restricted to finite decompositions, is equivalent to the $H^{p(\cdot)}$ norm. This extends a result from [23] to the variable setting.

**Theorem 7.8.** Let $p(\cdot) \in M\mathcal{P}_0$ and fix $q$ as in Theorem 7.1. For $f \in H^{p(\cdot),q}_{fin}(\mathbb{R}^n)$, define

$$\|f\|_{H^{p(\cdot),q}_{fin}} = \inf \left\{ \left\| \sum_{j=1}^{k} \lambda_j \chi_{B_j} \right\|_{p(\cdot)} : f = \sum_{j=1}^{k} \lambda_j a_j \right\},$$

where infimum is taken over all finite decompositions of $f$ using $(p(\cdot), q)$ atoms $a_j$, supported on balls $B_j$. Then

$$\|f\|_{H^{p(\cdot)}} \simeq \|f\|_{H^{p(\cdot),q}_{fin}}.$$

Our argument is based on the proof of [3, Theorem 6.2]. It requires two lemmas. The first introduces a non-tangential variant of the grand maximal operator. A proof can be found in Bownik [2, Prop. 3.10].

**Lemma 7.9.** Define the non-tangential grand maximal function $\mathcal{M}_{N,1}$ by

$$\mathcal{M}_{N,1} f(x) \sup_{\Phi \in \mathcal{S}_N} \sup_{|y-x| < t} |\Psi_t * f(x)|.$$

Then for all $x \in \mathbb{R}^n$ and tempered distributions $f$,

$$\mathcal{M}_{N,1} f(x) \approx \mathcal{M}_N f(x),$$

where the constants depend only on $N$.

The second lemma is a decay estimate for the grand maximal operator.
Lemma 7.10. Given $p(\cdot) \in MP_0$, suppose $f \in H^{p(\cdot)}$ is such that $supp(f) \subset B(0, R)$ for some $R > 1$. Then for all $x \in B(0, 4R)^c$,

$$\mathcal{M}_N f(x) \leq C(N, p(\cdot), p_0) \| \chi_{B(0, R)} \|_{p(\cdot)}^{-1}.$$

Proof. To prove the desired estimate, it will suffice to show that for any $\Phi \in S_N$, $x \in B(0, 4R)^c$, and $t > 0$,

$$|f * \Phi_t(x)| \leq C \| \chi_{B(0, R)} \|_{p(\cdot)}^{-1},$$

where the constant $C$ is independent of $f$, $\Phi$, $x$ and $t$. We consider two cases, depending on the size of $t$.

Case 1: $t \geq R$. Given $x \in B(0, 4R)^c$ and $t \geq R$, we claim that there exists $\Psi \in S$ so that $f * \Phi_t(x) = f * \Psi_R(0)$. Let $\theta \in C_c^\infty$ be such that $supp(\theta) \subset B(0, 2)$ and $\theta = 1$ on $B(0, 1)$, and define $\Psi(z) = \Phi(\frac{x}{t} + \frac{Rz}{t}) \theta(z)(R/t)^p$. Then

$$f * \Phi_t(x) = \int f(y) t^{-n} \Phi \left( \frac{x-y}{t} \right) dy$$

$$= \int f(y) t^{-n} \Phi \left( \frac{x-y}{t} \right) \theta \left( \frac{y}{R} \right) dy = f * \Psi_R(0).$$

We actually have that $c \Psi \in S_N$, where $c = c(\theta, N)$. To see this, recall that since $\Phi \in S_N$, $\| \partial^\beta \Phi \|_\infty \leq c$ for all $|\beta| \leq N$. Fix $z \in supp(\Psi) = B(0, 2)$. Then for any multi-index $|\beta| \leq N$,

$$|\partial^\beta \Psi(z)| \leq \left( \frac{R}{t} \right)^n \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\partial^n \Phi \left( \frac{x + Rz}{t} \right) \left( \frac{R}{t} \right)^\gamma \partial^{\beta - \gamma} \theta \left( \frac{y}{R} \right) R^{-|\beta + |\gamma|}. $$

Since $t \geq R > 1$, we see that $|\partial^\alpha \psi(z)| \leq C(\theta, N)$. Hence,

$$\sup_{|\alpha|, |\beta| \leq N} \|\Psi\|_{\alpha, \beta} = \sup_{|\alpha|, |\beta| \leq N} \sup_{z \in B(0, 2)} |z^\alpha \partial^\beta \Psi(z)| \leq C(\theta, N).$$

Since $c(N, \theta) \Psi \in S_N$, by Lemma 7.9 we have the pointwise bound

$$|f * \Phi_t(x)| = |f * \Psi_R(0)| \leq \inf_{z \in B(0, R)} M_{\Psi, 1} f(z)$$

$$\leq C(N, \theta) \inf_{z \in B(0, R)} \mathcal{M}_{N, 1} f(z) \leq C(N, \theta) \inf_{z \in B(0, R)} \mathcal{M}_N f(z).$$

Therefore, by Lemmas 2.1, 2.3 and 2.13,

$$|f * \Phi_t(x)|^{p_0} \leq C \int_{B(0, R)} \mathcal{M}_N f(z)^{p_0} dz$$

$$\leq C \| B(0, R) \|^{-1} \| (\mathcal{M}_N f)^{p_0} \|_{p(\cdot)/p_0} \| \chi_{B(0, R)} \|_{p(\cdot)/p_0}$$

$$\leq C \| \mathcal{M}_N f \|_{p(\cdot)/p_0} \| \chi_{B(0, R)} \|_{p(\cdot)/p_0}^{-1} \| \leq C \| f \|_{H^{p(\cdot)}} \| \chi_{B(0, R)} \|_{p(\cdot)}^{-p_0}.$$
where \( C = C(N, p(\cdot), p_0, n) \).

**Case 2: \( t < R \).** This case is similar to the previous one but we need to construct \( \Psi \) differently as we need our estimate to hold at more points than the origin. Fix \( z \in B(0, R/2) \) and choose \( u \in B(0, R/2) \) such that \(|z - u| < t \). We claim there exists \( \Psi \) (depending on \( u \), \( t \), and \( R \)) such that \( f * \Phi_t(x) = f * \Psi_t(u) \). As before, let \( \theta \in C_c^\infty \) be supported on \( B(0, 2) \) and \( \theta = 1 \) on \( B(0, 1) \). Define \( \Psi \) by

\[
\Psi(z) = \Phi \left( \frac{x-u}{t} + z \right) \theta \left( \frac{u}{R} - \frac{t}{R} z \right).
\]

Then we have that

\[
f * \Phi_t(x) = \int f(y) \Phi_t(x-y) dy = \int f(y) \Phi_t(x-y) \theta(y/R) dy = f * \Psi_t(u).
\]

Assume for the moment that \( c(\theta, N) \Psi \in S_N \). Then by Lemma 7.9,

\[
|f * \Psi_t(u)| \leq M_{\Psi, 1} f(z) \leq C(\theta, N) M_{N, 1} f(z) \leq C(\theta, N) M_N f(z).
\]

Since this holds for every \( z \in B(0, R/2) \), we have that

\[
|f * \Phi_t(x)| \leq C(\theta, N) \inf_{z \in B(0, R/2)} M_N f(z),
\]

and we can repeat the above argument to get the desired estimate.

It remains to show that \( c(\theta, N) \Psi \in S_N \); it will suffice to show that for all \( \beta \) such that \( |eta| \leq N \),

\[
\sup_{z \in \mathbb{R}^n} |\partial^\beta \psi(z)|(1 + |z|)^N \leq C(\theta, N).
\]

Since \( \Phi \in S_N \), for all \( |eta| \leq N \), \( (1 + |y|)^N |\partial^\beta \Phi(y)| \leq c(N) \). Therefore, by the product rule, since \( t < R \),

\[
|\partial^\beta \Psi(z)| \leq \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} |\partial^\gamma \Phi \left( \frac{x-u}{t} + z \right)| \left( \frac{t}{R} \right)^{|\beta| - |\gamma|} |\partial^{\beta - \gamma} \theta \left( \frac{u}{R} - \frac{t}{R} z \right)| \leq \frac{C(\theta, N)}{(1 + |\frac{x-u}{t} + z|)^N}.
\]

To estimate the last term, note first that since \( x \notin B(0, 4R) \) and \( u \in B(0, R/2) \),

\[
\frac{|x-u|}{t} > \frac{7R}{2t}.
\]

Second, since the \( \theta \) term is non-zero only if \( |\frac{x-u}{t}| \leq 2 \), we must have that \( |\frac{u}{t} - z| < \frac{2R}{t} \), which implies \( |z| < \left| \frac{x}{t} \right| + \frac{2R}{t} = \frac{5R}{2t} \). Together these two estimates show that \( |\frac{x-u}{t} + z| > \frac{7R}{2t} \).
\[
\frac{7}{2} R^t - \frac{5}{2} R^t = \frac{R}{t}.
\]
Therefore, for \( z \in \text{supp}(\Psi) \),
\[
|\partial^j \psi(z)| (1 + |z|)^N \leq C \frac{(1 + |z|)^N}{(1 + \frac{z^a}{t^a} + z)^N} \leq C \left( \frac{1 + 3R/t}{1 + R/t} \right)^N \leq C(\theta, N).
\]

This completes the proof. \( \square \)

**Proof of Theorem 7.8.** Since the infimum over finite sums in (7.6) is larger than the infimum when taken over all possible atomic decompositions, by Theorem 7.1 we have that \( \|f\|_{H^p(\cdot), q} \leq C \|f\|_{H^p(\cdot), q}. \)

To prove the reverse inequality, fix \( f \in H^p_{\text{fin}}(\cdot, q) \). By homogeneity we may assume that \( \|f\|_{H^p(\cdot)} = 1; \) we will show that \( \|f\|_{H^p(\cdot), q} \leq C(N, p(\cdot), p_0, q, n). \) Since \( f \) has a finite atomic decomposition, there exists \( R > 1 \) such that \( \text{supp}(f) \subset B(0, R) \). By Lemma 7.10,
\[
(7.7) \quad \mathcal{M}_N f(x) \leq c \|\chi_{B(0,R)}\|_{p(\cdot)}^{-1}.
\]

Let \( \Omega_j = \{x : \mathcal{M}_N f(x) > 2^j\} \); define \( j' = j'(f, p(\cdot)) \) to be the smallest integer such that for all \( j > j' \), \( \Omega_j \subset B(0, 4R). \) By (7.7) it suffices to take \( j' \) to be the largest integer such that \( 2^j < c \|\chi_{B(0,R)}\|_{p(\cdot)}^{-1} \).

By Lemma 6.6 we can form the “canonical” decomposition of \( f \) in terms of \((p(\cdot), \infty)\) atoms:

\[
f = \sum_j \sum_k \lambda_{k,j} a_{k,j} = \sum_{j \leq j'} \sum_k \lambda_{k,j} a_{k,j} + \sum_{j > j'} \sum_k \lambda_{k,j} a_{k,j} = h + \ell.
\]

We will rewrite the sum \( h + \ell \) as a finite atomic decomposition in terms of \((p(\cdot), q)\) atoms. To do so, we will use the finer properties of the atoms \( a_{k,j} \) that are implicit in the proof of Lemma 6.6. First, \( \text{supp}(h), \text{supp}(\ell) \subset B(0, 4R) \). The atoms \( a_{k,j} \) are supported in \( \Omega_j \), so by our choice of \( j' \), \( \text{supp}(\ell) \subset B(0, 4R) \). Since \( \text{supp}(f) \subset B(0, R) \), we also have that \( \text{supp}(h) \subset B(0, 4R) \).

Second, \( h, \ell \in L^q \). Since \( f \) has a finite \((p(\cdot), q)\) atomic decomposition it is in \( L^q \); since \( q > 1 \) we also have that \( \mathcal{M}_N f \in L^q \). If we fix \( x \in \text{supp}(\ell) \), then there exists \( s > j' \) such that \( x \in \Omega_s \backslash \Omega_{s+1} \). By construction (see (6.10)) the sets \( \text{supp}(a_{k,j}) \) have bounded overlap and \( |\lambda_{k,j} a_{k,j}| \leq c 2^j \). Hence,
\[
(7.8) \quad \sum_{j > j'} \sum_k |\lambda_{k,j} a_{k,j}(x)| = \sum_{j < j' \leq s} \sum_k |\lambda_{k,j} a_{k,j}(x)| \leq c \sum_{j \leq s} 2^j = c 2^{s+1} \leq c \mathcal{M}_N f(x).
\]

Thus \( \ell \in L^q \), and so \( h = f - \ell \in L^q \) as well.

Third, \( h, \ell \) satisfy the vanishing moment condition for all \( |\alpha| \leq \lfloor n(1/p_0 - 1) \rfloor \). Since \( f \) is a finite sum of \((p(\cdot), q)\) atoms, it has vanishing moments for these \( \alpha \). Since \( \text{supp}(\ell) \subset \text{supp}(f) \subset B(0, R) \),
To show this we only need to check the size condition. Fix estimates for \( n \), and Lemma where the last follows by our choice of sum and integral to get that \( B \) has the same vanishing moments as each \( a_{k,j} \). Finally, since \( h = f - \ell \), \( h \) also has the same vanishing moments.

Fourth, there exists a constant \( c \) such that \( ch \) is a \( (p(\cdot), \infty) \) atom supported on \( B(0, 4R) \). To show this we only need to check the size condition. Fix \( x \in \mathbb{R}^n \); then by the same estimates for \( a_{k,j} \) we used above,

\[
|h(x)| \leq \sum_{j \leq j'} \sum_k |\lambda_{k,j}a_{k,j}(x)| \leq c \sum_{j \leq j'} 2^j \leq c2^j' \leq c\|\chi_{B(0,R)}\|_{L^p}^{-1},
\]

where the last follows by our choice of \( j' \).

Finally, we show that \( \ell \) can be rewritten as a finite sum of \( (p(\cdot), q) \) atoms. Let \( F_i = \{(j, k) : |j| + |k| \leq i\} \) and define the finite sum \( \ell_i \) by

\[
\ell_i = \sum_{F_i} \lambda_{k,j}a_{k,j}.
\]

Since the sum for \( \ell \) converges absolutely in \( L^q \), we can find \( i \) such that \( \|\ell - \ell_i\|_q \) is as small as desired. In particular, we can find \( i \) such that \( \ell - \ell_i \) is a \( (p(\cdot), q) \) atom.

Therefore,

\[
f = c(h/c) + (\ell - \ell_i) + \sum_{(j, k) \in F_i} \lambda_{k,j}a_{k,j}
\]

is a finite decomposition of \( f \) as \( (p(\cdot), q) \) atoms. To complete the proof we will use Lemma 6.6 to get the desired estimate on \( \|f\|_{H^p(\cdot)} \). Let \( \tilde{B} = B(0, 4R) \). By the definition of the finite atomic norm and Lemma 2.7 (if \( p_+ < 1 \),

\[
\|f\|_{H^p(\cdot)}^{p_-} \leq \left\| \frac{c\chi_{\tilde{B}}}{\|\chi_{\tilde{B}}\|_{p(\cdot)}} \right\|_{p(\cdot)}^{p_-} + \left\| \frac{\chi_{\tilde{B}}}{\|\chi_{\tilde{B}}\|_{p(\cdot)}} \right\|_{p(\cdot)} + \sum_{(j, k) \in F_i} \frac{\lambda_{k,j}\chi_{B_k,j}}{\|\chi_{B_k,j}\|_{p(\cdot)}} \right\|_{p(\cdot)}^{p_-} \leq C + \sum_{j,k} \frac{\lambda_{k,j}\chi_{B_k,j}}{\|\chi_{B_k,j}\|_{p(\cdot)}} \right\|_{p(\cdot)}^{p_-} \leq C + C\|f\|_{H^p(\cdot)(\mathbb{R}^n)} \leq C.
\]

This completes the proof. \( \square \)

7.3. Finite atomic decompositions for weighted Hardy spaces. We end this section by showing that a version of Theorem 7.8 holds for the weighted Hardy spaces. This result is of interest in its own right, but we give it primarily because we will need it in the next section to prove the boundedness of singular integrals on \( H^p(\cdot) \). For this reason we only
prove one particular case; we leave it to the interested reader to prove the more general result implicit in our work.

Let \( p(\cdot) \in MP_0 \), and let \( q > 1 \). Given \( w \in A_1 \), define \( H_{fin}^{p_0,q}(w) \) to be the set of all finite sums of \( (p(\cdot), q) \) atoms. By the proof of Lemma 7.3 we have that for \( q \) sufficiently large, \( H_{fin}^{p(\cdot),q}(\mathbb{R}^n) = H_{fin}^{p_0,q}(w) \) as sets. Given \( f \in H_{fin}^{p_0,q}(w) \), define a weighted atomic decomposition norm on \( H_{fin}^{p_0,q}(w) \) by

\[
\|f\|_{H_{fin}^{p_0,q}(w)} = \inf \left\{ \left\| \sum_{j=1}^k \lambda_j^{p_0} \chi_{B_j} \|\chi_{B_j}\|_{p(\cdot)}^{p_0} \right\|_{L^1(w)} : f = \sum_{j=1}^k \lambda_j a_j \right\},
\]

where the infimum is taken over all decompositions of \( f \) as a finite sum of \( (p(\cdot), q) \) atoms.

**Lemma 7.11.** Given \( p(\cdot) \in MP_0, \) fix \( q \) as in Theorem 7.1 and let \( w \in A_1 \cap L^{(p(\cdot)/p_0)'}(\mathbb{R}^n) \). Then there exists \( C = C(p(\cdot), p_0, [w]_{A_1}, \|w\|_{(p(\cdot)/p_0)'}^p) \) such that

\[
\|f\|_{H_{fin}^{p_0,q}(w)} \leq C\|f\|_{H_{fin}^{p_0}(w)}.
\]

**Remark 7.12.** We note in passing that Lemma 7.11 is not the same as [3, Theorem 6.2] because the atoms given there are defined using the weighted \( L^q \)-norm, and we cannot pass between these two types of atoms simply by multiplying by a constant.

**Proof.** The proof is very similar to the proof of Theorem 7.8; here we sketch the changes required. Fix \( f \in H_{fin}^{p_0,q}(w) \); then \( f \in H_{fin}^{p(\cdot),q}(\mathbb{R}^n) \), and is supported on a ball \( B = B(0, R) \) for some \( R > 1 \). Let \( \tilde{B} = B(0, 4R) \). By Lemma 7.10, for \( x \notin \tilde{B} \), we have \( \mathcal{M}_N f(x) \leq c\|\chi_B\|_{p(\cdot)}^{-1} \).

Assume that \( \|f\|_{H_{fin}^{p_0}(w)} = 1 \); we will show that \( \|f\|_{H_{fin}^{p_0,q}(w)} \leq C \). By the proof of [28, Chapter 8, Theorem 1] we have that

\[
f = \sum_{k,j} \lambda_{k,j} a_{k,j}
\]

where \( \{a_{k,j}\} \) are \( (p(\cdot), \infty) \) atoms supported on balls \( B_{k,j}, \) \( \{\lambda_{k,j}\} \) are non-negative, and

\[
\left\| \sum_{k,j} \lambda_{k,j}^{p_0} \chi_{B_{k,j}} \left\|\chi_{B_{k,j}}\right\|_{p(\cdot)}^{p_0} \right\|_{L^1(w)} \leq C\|f\|_{H_{fin}^{p_0}(w)}.
\]

(As we noted in Remark 7.7, this is a restatement of the results from [28] to our setting.) This decomposition is constructed in a fashion very similar to that of Lemma 6.6 and the atoms and coefficients have much the same properties. Therefore, if we let \( j' \) be the smallest integer such that \( 2^{j'} \leq \|\chi_B\|_{p(\cdot)}^{-1} \) and regroup the sum as

\[
f = \sum_{j \leq j'} \sum_k \lambda_{k,j} a_{k,j} + \sum_{j > j'} \sum_k \lambda_{k,j} a_{k,j} = h + \ell,
\]
the argument proceeds exactly as before. This allows us to write

$$f = c(h/c) + (\ell - \ell_i) + \sum_{F_i} \lambda_{k,j} a_{k,j},$$

where $h$ is a $(p(\cdot), \infty)$ atom, $i$ is chosen large enough that $(\ell - \ell_i)$ is $(p(\cdot), q)$ atom, and the sum is a finite sum of $(p(\cdot), \infty)$ atoms. Moreover, we have that

$$\|f\|_{H^{p_0,q}(w)} \leq C \bigg( \frac{\chi}{\|\chi\|_{p(\cdot)w}} + \sum_{k,j} \lambda_{k,j} \frac{\chi_{B_{k,j}}}{\|\chi_{B_{k,j}}\|_{p(\cdot)}w} \bigg)_{L^1(w)}.$$

By (7.9), since the $\lambda_{k,j}$ are non-negative, the last term is bounded by $C\|f\|_{H^{p_0}(w)} = C$. To bound the first term, note that by Lemmas 2.1 and 2.3,

$$w(\tilde{B}) = \int_{\tilde{B}} w(x)dx \leq C(p(\cdot), p_0)\|\chi_{\tilde{B}}\|_{p(\cdot)/p_0} \|w\|_{(p(\cdot)/p_0)'},$$

$$\leq C(p(\cdot), p_0, \|w\|_{(p(\cdot)/p_0)'}) \|\chi_{\tilde{B}}\|^{p_0}_{p(\cdot)}.$$

Hence, $\|f\|_{H^{p_0,q}(w)} \leq C(p(\cdot), p_0, [w]_{A_1}, \|w\|_{(p(\cdot)/p_0)'})$ and our proof is complete. \qed

8. BOUNDEDNESS OF OPERATORS ON $H^{p(\cdot)}$

In this section we show that convolution type Calderón-Zygmund singular integrals with sufficient regularity are bounded on $H^{p(\cdot)}$. Our two main techniques are the finite atomic decomposition from Section 7 and weighted norm inequalities. First we define the class of singular integrals we are interested in.

**Definition 8.1.** Let $K \in S'$. We say $Tf = K*f$ is a convolution-type singular integral operator with regularity of order $k$ if the distribution $K$ coincides with a function on $\mathbb{R}^n \setminus \{0\}$ and has the following properties:

1. $K \in L^\infty$;
2. for all multi-indices $0 \leq |\beta| \leq k + 1$ and $x \neq 0$, $|\partial^\beta K(x)| \leq \frac{C}{|x|^{n+|\beta|}}$.

Singular integrals that satisfy this definition are bounded on $L^p$, $1 < p < \infty$. More importantly, the pointwise smoothness conditions guarantee that they satisfy weighted norm inequalities. In particular, we have the following weighted Kolmogorov inequality; for a proof, see [10, 14].
Lemma 8.2. Let $T$ be a convolution-type singular integral operator as defined above. Given $w \in A_1$ and $0 < p < 1$, then for every ball $B$,\[
\int_B |Tf(x)|^p w(x) \, dx \leq C(T, n, p, [w]_{A_1}) w(B)^{1-p} \left( \int_{\mathbb{R}^n} |f(x)| w(x) \, dx \right)^p.
\]

Our main results in this section are the following two theorems.

Theorem 8.3. Given $p(\cdot) \in \mathcal{M}P_0$ and $q > 1$ sufficiently large (as in Theorem 7.1), let $T$ be a singular integral operator that has regularity of order $k \geq \lfloor n(\frac{1}{p_0} - 1) \rfloor$. Then\[
\|Tf\|_{p(\cdot)} \leq C(T, p(\cdot), p_0, q, n) \|f\|_{H^{p(\cdot)}}.
\]

Theorem 8.4. Given $p(\cdot) \in \mathcal{M}P_0$ and $q > 1$ sufficiently large (as in Theorem 7.1), let $T$ be a singular integral operator that has regularity of order $k \geq \lfloor n(\frac{1}{p_0} - 1) \rfloor$. Then\[
\|Tf\|_{H^{p(\cdot)}} \leq C(T, p(\cdot), p_0, q, n) \|f\|_{H^{p(\cdot)}}.
\]

We will prove both theorems as a consequence of a more general result for sublinear operators.

Theorem 8.5. Given $p(\cdot) \in \mathcal{M}P_0$ with $0 < p_0 < 1$, and $q > 1$ sufficiently large (as in Theorem 7.1), suppose that $T$ is a sublinear operator that is defined on $(p(\cdot), q)$ atoms. Then:

1. If for all $w \in A_1 \cap RH_{(q/p_0)}$ and every $(p(\cdot), q/p_0)$ atom $a(\cdot)$ with support $B$,\[
\|Ta\|_{L^{p_0}(w)} \leq C(T, p(\cdot), p_0, q, n, [w]_{A_1}, [w]_{RH_{(q/p_0)}}) \frac{w(B)^{1/p_0}}{\|\chi_B\|_{p(\cdot)}},
\]

then $T$ has a unique, bounded extension $\tilde{T} : H^{p(\cdot)} \rightarrow L^{p(\cdot)}$.

2. If for all $w \in A_1 \cap RH_{(q/p_0)}$ and every $(p(\cdot), q/p_0)$ atom $a(\cdot)$ with support $B$,\[
\|Ta\|_{H^{p_0}(w)} \leq C(T, p(\cdot), p_0, q, n, [w]_{A_1}, [w]_{RH_{(q/p_0)}}) \frac{w(B)^{1/p_0}}{\|\chi_B\|_{p(\cdot)}},
\]

then $T$ has a unique, bounded extension $\tilde{T} : H^{p(\cdot)} \rightarrow H^{p(\cdot)}$.

Remark 8.6. The additional hypothesis that $0 < p_0 < 1$ is not a real restriction, since by Lemma 2.12 we may take $p_0$ as small as desired.

Remark 8.7. Note that when $p(\cdot)$ is constant and $w \equiv 1$, then conditions (8.1) and (8.2) reduce to showing that $T$ is uniformly bounded on atoms, which is the condition used to prove singular integrals are bounded on classical Hardy spaces.
Proof. First suppose that (8.1) holds. Fix \( f \in H^p_{\text{fin}}(\cdot, q/p_0) \); by Theorem 7.1 this set is dense in \( H^p(\cdot) \). Since \( T \) is well-defined on the elements of \( H^p_{\text{fin}}(\cdot, q/p_0) \), it will suffice to prove that

\[
\| Tf \|_{L^p(\cdot)} \leq C(T, p(\cdot), p_0, q, n) \| f \|_{H^p(\cdot)}.
\]

(8.3)

For in this case by a standard density argument there exists a unique bounded extension \( \tilde{T} \) such that \( \tilde{T} : H^p(\cdot) \to L^p(\cdot) \).

To prove (8.3) we will use the extrapolation argument in Lemma 7.6 to reduce the variable norm estimate to a weighted norm estimate. Arguing as we did in that proof, we have that

\[
\| Tf \|^p_{L^p(\cdot)} \leq \sup \int |Tf(x)|^p \mathcal{R}g(x) dx,
\]

with the supremum taken over all \( g \in L^p(\cdot)/p_0' \) with \( \|g\|_{(p(\cdot)/p_0)'} \leq 1 \). Suppose for the moment that we can prove that for all \( f \in H^p_{\text{fin}}(\cdot, q/p_0) \),

\[
\| Tf \|_{L^p(\mathcal{R}_g)} \leq C(T, p(\cdot), p_0, q, n) \| f \|_{H^p(\mathcal{R}_g)}.
\]

(8.4)

(In particular, the constant is independent of \( g \).) Then we can continue the argument as in the proof of Lemma 7.6 to get

\[
\| Tf \|^p_{L^p(\mathcal{R}_g)} \leq C\| f \|^p_{H^p(\mathcal{R}_g)} \leq C \int \mathcal{M}_N f(x)^p \mathcal{R}g(x) dx
\]

\[
\leq C \| (\mathcal{M}_N f)^p \|_{p(\cdot)/p_0} \| \mathcal{R}g \|_{(p(\cdot)/p_0)'} \leq C \| \mathcal{M}_N f \|_{p(\cdot)} \leq C \| f \|_{H^p(\cdot)}.
\]

This gives us (8.3).

To complete the proof we will show (8.4). Recall that as sets, \( H^p_{\text{fin}}(\cdot, q/p_0) \cap \mathcal{R}_g = H^p_{\text{fin}}(\cdot, q/p_0) \). Therefore, let

\[
f = \sum_{j=1}^{k} \lambda_j a_j
\]

be an arbitrary finite decomposition of \( f \) in terms of \( (p(\cdot), q/p_0) \) atoms. Since, \( 0 < p_0 < 1 \), by the sublinearity of \( T \), convexity and (8.1),

\[
\| Tf \|^p_{L^p(\mathcal{R}_g)} = \int |Tf(x)|^p \mathcal{R}g(x) dx \leq \sum_{j=1}^{k} \lambda_j^{p_0} \int_{B_j} |Ta_j(x)|^p \mathcal{R}g(x) dx
\]

\[
\leq C \sum_{j=1}^{k} \lambda_j^{p_0} \| \mathcal{R}g(B_j) \|_{p(\cdot)} = C \left\| \sum_{j=1}^{k} \lambda_j \| \mathcal{R}g \|_{p(\cdot)} \right\|_{L^1(\mathcal{R}_g)}.
\]

This is true for any such decomposition of \( f \). Therefore, since \( \mathcal{R}g \in A_1 \cap L^{p(\cdot)/p_0'} \) by construction, by Lemma 7.11 we can take the infimum over all such decompositions to get
\[ \| Tf \|_{L^p(\mathbb{R}^d)} \leq C \| f \|_{H^{p_0}(\mathbb{R}^d)}, \text{ where } C = C(T, p(\cdot), p_0, q, n). \] This proves (8.4) for all \( f \in H^p_{\text{fin}}. \)

We now consider the case when condition (8.2) holds. The proof is essentially the same as before, except instead of proving (8.4), we need to prove that for all \( f \in H^p_{\text{fin}}, \)

\[ (8.5) \quad \| Tf \|_{H^{p_0}(\mathbb{R}^d)} \leq C(T, p(\cdot), p_0, q, n) \| f \|_{H^{p_0}(\mathbb{R}^d)}. \]

Given this, we can then repeat the extrapolation argument as before. To prove (8.5) we use the same argument used to prove (8.4), replacing \( Tf \) with \( M_{\Phi,0}(Tf) \) where \( \Phi \in \mathcal{S} \) with \( \int \Phi \, dx = 1 \), and using (8.2) instead of (8.1).

**Proof of Theorem 8.3.** By Theorem 8.5 it will suffice to show that condition (8.1) holds for all \((p(\cdot), q/p_0)\) atoms and all \( w \in A_1 \cap RH_{(q/p_0)}'.\)

Fix such an atom \( a(\cdot) \) with support \( B = B(x_0, r) \). Let \( 2B = B(x_0, 2r) \) and write

\[ \| Ta \|_{L^p(w)} = \int |Ta(x)|^{p_0} w(x) \, dx \]

\[ = \int_{2B} |Ta(x)|^{p_0} w(x) \, dx + \int_{(2B)^c} |Ta(x)|^{p_0} w(x) \, dx = I_1 + I_2. \]

We first estimate \( I_1 \). By Lemma 8.2 there exists a constant \( C = C(T, n, p_0, [w]_{A_1}) \) such that

\[ \int_{2B} |Ta(x)|^{p_0} w(x) \, dx \leq C w(B)^{1-p_0} \left( \int_{\mathbb{R}^n} |a(x)| w(x) \, dx \right)^{p_0} \]

\[ \leq C w(B)^{1-p_0} |B|^{p_0} \left( \int_{B} |a|^{q/p_0} \, dx \right)^{1/q} \left( \int_{B} w(x)^{q/p_0} \, dx \right)^{p_0/(q/p_0)'}. \]

Since \( a(\cdot) \) is a \((p(\cdot), q/p_0)\) atom and \( w \in RH_{(q/p_0)}' \), we get that

\[ I_1 \leq C \left[ w \right]_{RH_{(q/p_0)}'} w(B)^{1-p_0} |B|^{p_0} \| \chi_B \|_{L^{p(\cdot)}} |B|^{-p_0} w(B)^{p_0} = C \left[ w \right]_{RH_{(q/p_0)}'} w(B) \| \chi_B \|_{L^{p(\cdot)}}^{-p_0}. \]

To estimate \( I_2 \), we start with a pointwise estimate. Let \( d = \lfloor n \left( \frac{1}{p_0} - 1 \right) \rfloor \). We claim that there exists a constant \( C = C(T, n) \) such that for all \( x \in (2B)^c \),

\[ (8.6) \quad |Ta(x)| \leq C \frac{|B|^{1+(d+1)/n}}{\| \chi_B \|_{p(\cdot)}} \cdot \frac{1}{|x - x_0|^{n+d+1}}. \]

To prove this, let \( P_d \) be the Taylor polynomial of \( K \) of degree \( d \) centered at \( x - x_0 \). By our definition of \( d \) and our assumption on \( k, d + 1 \leq k + 1 \). Therefore, the remainder \( |K(x - y)| - P_d(y)\) can be estimated by Condition (2) in Definition 8.1. Hence, by the vanishing moment and size conditions on \( a(\cdot) \) and Hölder’s inequality,

\[ |Ta(x)| \leq \int |K(x - y) - P_d(y)| |a(y)| \, dy \]
\[
\begin{align*}
\leq & \frac{C}{|x - x_0|^{n+d+1}} \int_{B(x_0,r)} |y - x_0|^{d+1} |a(y)| \, dy \\
\leq & \frac{C}{|x - x_0|^{n+d+1}} \int_{B} a(y) \, dy \\
\leq & \frac{C |B| |B|^{-p_0/q} \|a\|_{q/p_0}}{|x - x_0|^{n+d+1}} \\
\leq & \frac{C |B|^{1 + \frac{d+1}{n}}}{\|\chi_B\|_{p(\cdot)}}, \quad 1 \frac{1}{|x - x_0|^{n+d+1}}.
\end{align*}
\]

Given (8.6) we have that
\[
\int (2B)^c |T a(x)|^{p_0} w(x) \, dx \leq C \frac{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}}{\|\chi_B\|_{p(\cdot)}} \int (2B)^c \frac{w(x)}{|x - x_0|^{p_0(n+d+1)}} \, dx.
\]

To complete the proof we will show that there exists a constant \( C = C(n, p_0) \) such that
\[
(8.7) \quad J \leq C \left[ \frac{[w]_{A_1} w(B)}{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}} \right].
\]
The proof of this is standard; for the convenience of the reader we sketch the details. Write
\[
(2B)^c = \bigcup_{i=1}^{\infty} (2^{i+1}B \setminus 2^iB);
\]
then for \( x \in 2^{i+1}B \setminus 2^iB \), we have \(|x - x_0| \simeq 2^i r \simeq 2^i |B|^{1/n} \). Since \( w \in A_1 \) and \( p_0(n + d + 1) > n \), we can estimate as follows:
\[
J = \sum_{i=1}^{\infty} \int_{2^{i+1}B \setminus 2^iB} \frac{w(x)}{|x - x_0|^{p_0(n+d+1)}} \, dx \\
\leq & \frac{C}{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}} \sum_{i=1}^{\infty} 2^{n(i+1)} \int_{2^{i+1}B \setminus 2^iB} w(x) \, dx \\
= & \frac{C}{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}} \sum_{i=1}^{\infty} 2^{n(i+1)} |B| \int_{2^{i+1}B} w(x) \, dx \\
\leq & \frac{C 2^n [w]_{A_1}}{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}} \sum_{i=1}^{\infty} \frac{1}{2^{n(i+1)}} \left( |B| \inf_{x \in B} \right) \\
= & \frac{C [w]_{A_1} w(B)}{|B|^{p_0 \left( \frac{n+d+1}{n} \right)}}.
\]
This gives us (8.7) and so completes the proof.
Proof of Theorem 8.4. Our argument is similar to the proof of Theorem 8.3. By Theorem 8.5 it will suffice to show that condition (8.2) holds for an arbitrary \((p(\cdot), q/p_0)\) atom \(a(\cdot)\) with support \(B = B(x_0, r)\), and all \(w \in A_1 \cap RH(q/p_0)'\). Fix \(\Phi \in \mathcal{S}\) with \(\int \Phi = 1\); then we can estimate \(\|Ta\|_{H^{p_0}(w)}^p\) as follows:

\[
\|Ta\|_{H^{p_0}(w)}^p \lesssim \int_{2B} M_{\Phi,0}(Ta)(x)^{p_0} w(x) dx + \int_{(2B)^c} M_{\Phi,0}(Ta)(x)^{p_0} w(x) dx = J_1 + J_2.
\]

To estimate the \(J_1\) we first use the fact that \(M_{\Phi,0}(Ta) \leq cM(Ta)\). Moreover, we have that since \(w \in A_1\), the Hardy-Littlewood maximal operator also satisfies Kolmogorov’s inequality (see \([10, 14]\)):

\[
J_1 \leq C w(2B)^{1-p_0} \left( \int_{\mathbb{R}^n} |Ta(x)| w(x) dx \right)^{p_0}.
\]

To get the desired estimate for \(J_1\) it will suffice to show that

\[
L = \int_{\mathbb{R}^n} |Ta(x)| w(x) dx \leq \frac{w(B)}{\|X_B\|_{p(\cdot)}}.
\]

To prove this, we again split the integral:

\[
L = \int_{2B} |Ta(x)| w(x) dx + \int_{(2B)^c} |Ta(x)| w(x) dx = L_1 + L_2.
\]

To estimate \(L_1\) we apply Hölder’s inequality, the boundedness of \(T\) on \(L^{q/p_0}\), and the fact that \(w \in RH(q/p_0)'\) to get

\[
L_1 = \int_{2B} |Ta(x)| w(x) dx \leq \left( \int_{2B} |Ta(x)|^{q/p_0} dx \right)^{p_0/q} \left( \int_{2B} w(x)^{(q/p_0)'} dx \right)^{1/(q/p_0)'}
\]

\[
\leq \|a\|_{L^{q/p_0}} |2B|^{1/(q/p_0)'} \left( \int_{2B} w(x)^{(q/p_0)'} dx \right)^{1/(q/p_0)'} \leq C(n, [w], [w]_{RH(q/p_0)'}) \frac{w(B)}{\|X_B\|_{p(\cdot)}}.
\]

To estimate \(L_2\) we repeat the argument we used to estimate \(L_2\) in the proof of Theorem 8.3, replacing the exponent \(p_0\) by 1. Then using the pointwise estimate for \(Ta\) and the decomposition argument, we have that

\[
L_2 \leq C \frac{|B|^{n+d+1}}{\|X_B\|_{p(\cdot)}} \left( \int_{(2B)^c} \frac{w(x)}{|x - x_0|^{n+d+1}} dx \right)
\]

\[
\leq C \frac{|B|^{n+d+1}}{\|X_B\|_{p(\cdot)}} \cdot \frac{w(B)}{|B|^{n+d+1}} \cdot \left( \sum_{i=0}^{\infty} \frac{2^{ni}}{2^{(n+d+1)}} \right) \leq C \frac{w(B)}{\|X_B\|_{p(\cdot)}}.
\]

To estimate \(J_2\), we will prove a pointwise bound for \(M_{\Phi,0}(Ta_j)(x)\) for \(x \in (2B_j)^c\) similar to (8.6). Define \(K^{(t)} = K \ast \Phi\); then \(K^{(t)}\) satisfies condition (3) of Definition 8.1.
uniformly for all \( t > 0 \). Moreover, for \( x \in (2B)^c \), the integral for \( K * a(x) \) converges absolutely, so \( |\Phi_t * (K * a)(x)| = |K^{(t)} * a(x)| \).

Again let \( d = \lfloor n\left(\frac{1}{p_0} - 1\right) \rfloor \) and fix \( t > 0 \). If \( P_d \) is the Taylor polynomial of \( K^{(t)} \) centered at \( x - x_0 \), we can argue exactly as we did to prove (8.6) to get

\[
|K^{(t)} * a(x)| = \left| \int [K^{(t)}(x - y) - P_d(y)]a(y) \, dy \right|
\]

\[
\leq C \frac{1}{|x - x_0|^{n+d+1}} \int_{B(x_0,r)} |y - x_0|^{d+1} |a(y)| \, dy
\]

\[
\leq C \left[ B^{\frac{n+d+1}{n}} B^{-\frac{p_0}{q}} \right] |x - x_0|^{\frac{n}{n+d+1}} \|a\|_{L^{q/p_0}}
\]

\[
\leq C \left[ B^{\frac{n}{n+d+1}} \right] \frac{1}{\|\chi_B\|_{L^{p_0}(\cdot)}} |x - x_0|^{n+d+1}.
\]

The final constant is independent of \( t \), and so we can take the supremum over all \( t \) to get

\[
M_{\Phi,0}(T a)(x) \leq C \left[ B^{\frac{n}{n+d+1}} \right] \frac{1}{\|\chi_B\|_{L^{p_0}(\cdot)}} |x - x_0|^{n+d+1}.
\]

Then arguing as we did before, by (8.7) we have that \( J_2 \leq w(B)/\|\chi_B\|_{L^{p_0}(\cdot)} \). This completes the proof. \( \square \)

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