Size Gap for Zero Temperature Black Holes in Semiclassical Gravity

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We show that a gap exists in the allowed sizes of all zero temperature static spherically symmetric black holes in semiclassical gravity when only conformally invariant fields are present. The result holds for both charged and uncharged black holes. By size we mean the proper area of the event horizon. The range of sizes that do not occur depends on the numbers and types of quantized fields that are present. We also derive some general properties that zero and nonzero temperature black holes have in all classical and semiclassical metric theories of gravity.

One of the unanswered questions in semiclassical gravity is how quantized fields alter the spacetime geometry near the event horizon of a black hole. This is important because the thermodynamic properties of a black hole are determined by the geometry at the event horizon. Some work has been done to answer this question using linearized semiclassical gravity and either analytical approximations or numerical computations of the stress-energy tensor in Schwarzschild spacetime. Further progress has been hampered by problems with the analytical approximations near the event horizons of other black holes and by the difficulty involved in numerically computing the stress-energy tensor for quantized fields in black hole spacetimes.

Given these difficulties and the importance which quantum effects may have in black hole spacetimes, it is useful to see if it is possible to deduce anything about the general properties of static spherically symmetric black holes without solving the full nonlinear set of semiclassical equations. We show that significant restrictions on the spacetime geometry near the event horizon of a black hole can be obtained by just requiring that no scalar curvature singularities exist at the horizon. These results apply to all classical and semiclassical metric theories of gravity. By requiring that the spacetime be a solution to the trace of the semiclassical backreaction equations we show that there exists a range of sizes for which no zero temperature black holes exist, so long as only conformally invariant fields are present. This result also applies to some possible nonzero temperature black holes. By size we mean the proper area of the event horizon. The range of excluded sizes depends on the number and types of quantized fields present.

Some previous work has been done to determine the general properties black holes must have in certain situations. Mayo and Bekenstein considered static spherically symmetric black hole solutions to Einstein’s equations for various types of matter fields. They found that black hole solutions exist if the stress-energy tensor is finite on the horizon and one component satisfies a certain inequality. The geometry near the horizon is of the same form as the Schwarzschild geometry except in the limit of an extreme black hole where the inequality becomes an equality. In this latter case they put constraints on the form of the geometry near the horizon. Recently Zaslavskii has used a power series expansion of the metric to determine the general form it takes near the horizons of near extreme and extreme charged black holes in a cavity when the grand canonical ensemble is utilized.

There has also been some previous evidence of excluded sizes for black holes when quantum effects are taken into account. Peleg, Bose, and Parker have found the existence of a mass gap in the formation of black holes in two-dimensional dilaton theories of gravity. Brady and Ottewill have recently found evidence that quantum effects can cause a mass gap in the formation of black holes in four dimensional spherically symmetric gravitational collapse.

We begin by examining the properties that result from requiring the spacetime curvature to be finite at the event horizon. These properties apply to black hole geometries in any classical or semiclassical metric theory of gravity. The metric for a static spherically symmetric spacetime can be written in the general form

\[ ds^2 = -f(r)dt^2 + \frac{1}{k(r)}dr^2 + r^2d\Omega^2. \]

(1)

If the spacetime has an event horizon then \( f \) vanishes on that horizon and the surface gravity is given by the formula

\[ \kappa = \frac{\nu}{2}(fk)^{1/2}. \]

(2)

The unique non vanishing components of the Riemann curvature tensor in an orthonormal frame are

\[ R_{\hat{t}\hat{r}\hat{t}\hat{r}} = \frac{v'}{2} + \frac{vk'}{4} + \frac{v^2k}{4}, \]

(3a)
where \( v \equiv f'/f \) and primes denote derivatives with respect to \( r \). The Kretschmann scalar for the metric (1) is

\[
R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = 4(R_{i\dot{i}j\dot{j}})^2 + 8(R_{i\dot{i}j\dot{k}})^2 + 8(R_{r\dot{r}i\dot{j}})^2 + 4(R_{\theta\dot{\theta}\phi\dot{\phi}})^2
\]  

Thus even though the coordinate system in (1) is singular on the event horizon, it is necessary that right hand sides of Eqs.(3a) - (3d) each be separately finite there to avoid a scalar curvature singularity.

To determine properties of \( k \) and \( f \) that arise from the requirement that the curvature be finite first note that, since \( f = 0 \) on the horizon, \( v = \infty \) there. Thus from Eq.(3b) one can infer that \( k = 0 \) at the horizon, while from Eq.(3c) it is clear that \( k' \) is finite there. The most significant constraints come from Eq.(3a). If the horizon is at \( r = r_0 \), then this equation can be written

\[
\frac{v' k}{2} + \frac{v k'}{4} + \frac{v^2 k}{4} = A(r)
\]  

with \( A(r) \) the function obtained by substituting specific functions \( k \) and \( f \) into Eq.(3a). Solving this equation for \( k' \) and noting that \( A \) is finite at the horizon, one finds that if \( k' = c_2 \) at the horizon, with \( c_2 \) some positive constant, then \( k = c_2(r - r_0) \) and the equation becomes to leading order

\[
1 = -(v + 2v'/v)(r - r_0)
\]  

The only solutions to this equation for which \( f \) vanishes are \( f = c_1(r - r_0) \), with \( c_1 \) a positive constant. Thus either

\[
f = c_1(r - r_0), \quad k = c_2(r - r_0)
\]  

near the horizon, or \( k' = 0 \) at the horizon. One can also formally integrate Eq.(5) to obtain

\[
k = \frac{B_0}{v^2 f} + \frac{4}{v^2 f} \int_{r_0}^{r} f'(r_1)A(r_1)dr_1.
\]  

Here \( B_0 \) is a nonnegative constant. Multiplying by \( v^2 f \), noting that the second term on the right is then zero at the horizon, and comparing with Eq.(2) gives \( B_0 = 4c_2 \). Multiplying Eq.(8) by \( v^2 \) shows that for zero temperature black holes \( kv^2 \) is finite and thus that \( kv = 0 \) at the horizon. For nonzero temperature black holes \( kv = 4c_2^2/f' \) at the horizon. Thus since \( kv \) must be finite, \( f' \) cannot vanish at the horizon. If \( f' \) diverges there then \( kv = 0 \). If \( f' \) approaches a constant then it is easy to see that Eq.(7) describes the behavior of \( f \) and \( k \) near the horizon. Thus for all black holes, either the geometry near the horizon is of the form (7) or at the horizon \( k' = kv = 0 \).

For nonzero temperature black holes the proper distance to the event horizon along a radial geodesic is finite. To see this note that this distance is

\[
\int_{r_0}^{r} dr_1/(k(r_1))^{1/2}.
\]  

Solving Eq.(2) for \( k \), substituting into (9) and integrating one finds that the integral is always finite. This result also allows one to make the change of variable \( dl = dr/k^{1/2} \). With this change of variable Eq.(5) becomes

\[
\frac{1}{2f} \frac{d^2 f}{dl^2} - \frac{1}{4f^2} \left( \frac{df}{dl} \right)^2 = A(l).
\]  

Multiplying by \( f/(df/dl) \) and integrating one finds that, since \( A \) is finite at the horizon, the only solutions for which \( f \) vanishes are of the form

\[
f = c_1(l - l_0)^2
\]  

near the horizon. Here \( l = l_0 \) at the horizon and \( c_1 \) is a positive constant.
Since the integral diverges,\( R = \alpha \) for the metric (1) the equation \( \Box R = \alpha(r) \) can be formally integrated with the result that to leading order near the horizon

\[
R = b_1 + b_2 \int_{r_0}^{r} dr_1 (k(r_1)f(r_1))^{-1/2} + a_0 \int_{r_0}^{r} dr_1 (k(r_1)f(r_1))^{-1/2} \int_{r_0}^{r_1} dr_2 (f(r_2)/k(r_2))^{1/2} .
\]  
(12)

Here \( b_1 \) and \( b_2 \) are arbitrary constants. For all zero temperature black holes (9) diverges and therefore \( R \) diverges unless \( b_2 = 0 \). Since \( kv^2 \) is finite on the horizon for zero temperature black holes, \( k^{-1/2} \geq c_2 v \) for some \( c_2 > 0 \). Substituting this inequality into Eq.(11) into Eq.(12), one finds to leading order the following equation for \( R \)

\[
|R - b_1| \geq 2|a_0|c_2 \int_{r_0}^{r} dr_1 (k(r_1))^{1/2} .
\]  
(13)

Since the integral diverges, \( R \) must diverge on the horizon which means there is a curvature singularity there.

For nonzero temperature black holes one can make the variable transformation \( dl = dr/k^{1/2} \) in Eq.(12). Assuming \( a = a_0 \) at the horizon and substituting Eq.(11) into Eq.(12), one finds to leading order the following equation for \( R \),

\[
R = -\frac{4}{r_0} \frac{d^2 r}{d^2 l^2} - \frac{4}{r_0 (l-l_0)} \frac{d r}{d l} + \frac{2}{r_0^2} = b_1 + \frac{a_0}{4} (l-l_0)^2 .
\]  
(14)

Note that \( b_2 \) must still be zero for \( R \) to be finite. This equation can be solved to give the only two forms the metric can have near the horizon that could actually result in \( \Box R = a_0 \) there. They are either Eq.(7) or

\[
f = c_1 (r-r_0)^{1/2}, \quad k = c_2 (r-r_0)^{3/2}
\]  
(15)

with \( c_1 \) and \( c_2 \) positive constants. All other possible black hole metrics result in \( \Box R \) being either divergent or zero at the horizon.

Semiclassical gravity can be used to place further constraints on the geometry near the event horizon of a static spherically symmetric black hole in the case that only conformally invariant free quantized fields are present. The semiclassical backreaction equations can be written in the general form

\[
G_{\mu\nu} + uU_{\mu\nu} + wW_{\mu\nu} = 8\pi (T_{\mu\nu})_{cl} + 8\pi < T_{\mu\nu} >
\]  
(16)

Here \( U_{\mu\nu} \) and \( W_{\mu\nu} \) are tensors which result from the variation of an \( R^2 \) term and a \( C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} \) term in the gravitational Lagrangian, with \( C_{\alpha\beta\gamma\delta} \) the Weyl tensor. Their coefficients, \( u \) and \( w \), are arbitrary and must in principle be determined by experiment or observation. \( (T_{\mu\nu})_{cl} \) is the stress-energy tensor for any classical fields. We shall only be concerned here with the classical electromagnetic field \[14\]. The trace of \( W_{\mu\nu} \) is identically zero as is the trace of the stress-energy tensor for the classical electromagnetic field. The trace of \( U_{\mu\nu} \) is equal to \(-6\Box R \) \[15\]. For conformally invariant fields the trace of \(< T_{\mu\nu} >\) is equal to the trace anomaly \[15\]. Thus the trace equation is

\[
- \Box R = 8\pi [\alpha \Box R + \beta (R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3} R^2) + \gamma C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta}]
\]  
(17)

with

\[
\alpha = [N(0) + 6N(1/2) - 18N(1)]/2880\pi^2 \tag{18a}
\]
\[
\beta = [N(0) + 11N(1/2) + 62N(1)]/2880\pi^2 \tag{18b}
\]
\[
\gamma = [N(0) + \frac{7}{2} N(1/2) - 13N(1)]/2880\pi^2 . \tag{18c}
\]

Here \( N(0), N(1/2), \) and \( N(1) \) are the number of scalar, four component spin 1/2, and vector fields respectively.

Since all of the other terms in Eq.(17) are finite at the horizon, \( \Box R \) must be finite as well. Then for all zero temperature black holes and for all nonzero temperature ones with metrics near the horizon other than those given
by Eqs. (7) and (15), $k' = kv = \Box R = 0$ at the horizon. The components of the Riemann tensor there are then $R_{t\hat{t}t\hat{t}} = A(r_0) = 0$, $R_{t\hat{t}\theta\theta} = 0$, $R_{\hat{t}\theta\theta\hat{t}} = 0$, and $R_{\hat{t}\theta\hat{t}\theta} = 1/r_0^2$. The terms in Eq. (17) can be computed at the horizon from these components. After some algebra we find the following equation for $A_0$.

$$A_0^2 \left( \frac{16\pi}{3} (\beta + 2\gamma) \right) + A_0 \left( -2 + \frac{64\pi}{3r_0^2} (\beta - \gamma) \right) + \left( \frac{2}{r_0^3} + \frac{16\pi}{3r_0^2} (\beta + 2\gamma) \right) = 0 \quad (19)$$

Since $\beta + 2\gamma > 0$ for all fields there is no solution to this equation if $A_0 = 0$. For zero temperature black holes this result along with Eq. (8) implies that near the horizon

$$k = 4A_0/v^2 \quad \text{(20)}$$

For $A_0 \neq 0$, Eq. (19) can be solved with the result that

$$A_0 = \frac{1}{16\pi(\beta + 2\gamma)r_0^2} \left[ 3r_0^2 - 32\pi(\beta - \gamma) \pm (768\pi^2\beta^2 - 3072\pi^2\beta\gamma - 288\pi\beta r_0^2 + 9r_0^4)^{1/2} \right]. \quad (21)$$

$A_0$ is not real (and therefore there can be no solutions to the semiclassical backreaction equations with $k' = kv = \Box R = 0$ at the horizon) if

$$r_- < r_0 < r_+$$

$$r_\pm = 4(\pi\beta)^{1/2} \left[ 1 \pm \left( \frac{2}{3\beta} \right)^{1/2} (\beta + 2\gamma)^{1/2} \right]^{1/2} \quad (22)$$

Thus only nonzero temperature black holes with the behaviors (7) or (15) near the horizon can exist with sizes in this range. All other black holes including all zero temperature black holes cannot have sizes in this range. For all allowed values of $\beta$ and $\gamma$ $r_+$ is real. If $\beta \leq 4\gamma$ then $r_-$ is imaginary or zero and solutions only occur for $r_0 \geq r_+$. If $\beta > 4\gamma$ then solutions also occur for $0 < r_0 < r_-$.

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[14] In fact the results will be valid for any classical stress-energy tensor whose trace vanishes.
[15] See for example N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Freeman, San Francisco, 1973).