Nonlinear three-wave interaction in marine sediments

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Abstract

Nonlinear interaction of three acoustic waves in a sandy sediment is studied in the frequency range where there is a considerable wave velocity dispersion. The possibility of an experimental observation of the generation of a sound wave by two pump waves propagating at an angle to each other is estimated.
1 Introduction

Acoustic wave propagation is an effective tool for studying properties of porous media. Such media are known to exhibit a considerably stronger elastic nonlinearity as compared with homogeneous fluid or solid media [1, 2, 3, 4, 5]. This property stimulated a growing interest in studying nonlinear dynamics of poroelastic media (see, e.g. Refs. [6, 7, 8]).

In Refs. [6, 8] the generation of second-harmonic acoustic waves is studied theoretically for the one-dimensional case. In the frequency range where there is no sound velocity dispersion, the energy and momentum conservation laws,

$$\omega_1 + \omega_2 = \omega_3, \quad k_1 + k_2 = k_3 \quad (1)$$

are satisfied for nonlinear interactions of three waves propagating in one direction. Such interactions include harmonic generation of the fundamental wave as well as the generation of the waves with sum and difference frequencies of the fundamental and the excited waves. In this case, if the dissipation is not too high, the formation of a shock wave can start, and a particular three wave interaction could be hidden. For more explicit observation of nonlinear three wave interactions it is preferable to choose the frequency range where there is a significant amount of velocity dispersion. In this case the three waves that satisfy the conditions (1) will propagate in three different directions, and no shock formation would occur.

2 Theory

In the present work the nonlinear interaction of three acoustic waves propagating in a granular medium at an angle to each other is studied. It has been experimentally observed that marine sediments exhibit a noticeable acoustic wave velocity dispersion in some frequency ranges [9, 10, 11]. We shall use the dispersion velocity data [9] listed in Fig. 2 of Ref. [11]. The experimental data show a strong velocity dispersion within the frequency range between 1 kHz and 10 kHz. We assume that at the boundary of the porous medium two waves, $$(\omega_1, k_1)$$ and $$(\omega_2, k_2)$$, are excited at an angle to each other. They generate the wave $$(\omega_3, k_3)$$, and let it propagate along the x-axis. We may write

$$k_1 \cos \theta_1 + k_2 \cos \theta_2 = k_3 \quad (2)$$

where the angles $\theta_1$ and $\theta_2$ are the angles between the vectors $k_1$, $k_2$ and the x-axis correspondingly. The angles $\theta_1$ and $\theta_2$ depend on the choice of the frequencies $\omega_1$ and $\omega_2$. We shall choose them in the range of the maximum velocity dispersion in order the angles $\theta_1$ and $\theta_2$ are not too small and hence don’t fall in the dissipation spreading of the waves. We shall be based on Fig. 2 from Ref. [11] and choose the frequencies, $\omega_1 = 2\pi \cdot 2 \cdot 10^3 s^{-1}$, $\omega_2 = 2\pi \cdot 3 \cdot 10^3 s^{-1}$, for which the sum frequency equals $\omega_3 = \omega_1 + \omega_2 = 2\pi \cdot 5 \cdot 10^3 s^{-1}$. The corresponding acoustic wave velocities are the following: $c_1 \approx 1, 58 \cdot 10^5 cm/s$, $c_2 \approx 1, 62 \cdot 10^5 cm/s$, $c_3 \approx 1, 72 \cdot 10^5 cm/s$. These experimental data yield the values of the three wave vectors, which allows obtaining the values of the angles $\theta_1$ and $\theta_2, \theta_1 \approx 25^\circ, \theta_2 \approx 18^\circ$.

Our task is to find the amplitude of the wave $$(\omega_3, k_3)$$ generated by the waves $$(\omega_1, k_1)$$ and $$(\omega_2, k_2)$$ and to estimate if the intensity of the generated wave could reach a measurable value at a reasonable distance starting from the fluctuation level at the boundary. To solve the problem we start from the continuity equations for the densities and momenta of the liquid
and solid phases of a sediment composed of a rigid frame and pores filled with water, see Refs. [12, 13]. These equations are equivalent in the main features to the equations developed by Biot [14, 15, 16], but they are presented in a different form with a more explicit physical meaning. On the basis of these equations, in Ref. [7] the equations for the densities of the liquid and solid phases, \( \rho_f \) and \( \rho_s \), were derived (in this paper we don’t take diffraction into account):

\[
\begin{align*}
(1 - \frac{m}{\rho_f c^2 G}) \frac{\partial \rho_f}{\partial \tau} - \frac{\nu}{\rho_s c^2 G} \frac{\partial \rho_s}{\partial \tau} &= -c \left(1 + \frac{m}{\rho_f c^2 G}\right) \frac{\partial \rho_f}{\partial x} - \frac{\nu}{\rho_s c G} \frac{\partial \rho_s}{\partial x} + \frac{1}{\rho_f} \frac{\partial \rho_f^2}{\partial \tau} + \frac{1}{\rho_f} \frac{\partial \rho_f}{\partial \tau} + \frac{1}{c^2} \frac{\partial P_n}{\partial \tau}; \\
(1 - \frac{m - k/\rho_s c^2}{\rho_s c^2}) \frac{\partial \rho_s}{\partial \tau} - \frac{m \nu}{\rho_f c^2 G} \frac{\partial \rho_f}{\partial \tau} &= -c \left(1 - \frac{m - k/\rho_s c^2}{\rho_s c^2}\right) \frac{\partial \rho_s}{\partial x} - \frac{m \nu}{\rho_f c G} \frac{\partial \rho_f}{\partial x} - \frac{1}{\rho_s} \frac{\partial \rho_s^2}{\partial \tau} + \frac{\nu}{c^2} \frac{\partial P_n}{\partial \tau} - \frac{1}{c^2} \frac{\partial \tilde{\sigma}_{xx}^n}{\partial \tau}.
\end{align*}
\]

(3) (4)

In these equations the following variables are used,

\[x' = \epsilon x\]

(5)

and the moving coordinate

\[\tau = t - x/c.\]

(6)

(In Eqs. (3), (4) and everywhere below the primes for \( x \) are omitted.) In the relation (5) the small parameter \( \epsilon \) is introduced as

\[
\epsilon \sim v_x/c \sim u_x/c \sim \delta \rho_f/\rho_f \sim \delta \rho_s/\rho_s,
\]

(7)

here \( c \) is the sound velocity in the sediment and \( v, u \) are the hydrodynamic velocities of the liquid and solid phases, \( \delta \rho_f, \delta \rho_s \) are the deviations from equilibrium values of the densities of the liquid and solid phases. In Eqs. (3), (4) the left-hand sides are of the order of \( \sim \epsilon \), the right-hand side terms are of the order of \( \sim \epsilon^2 \). The introduction of the new variables (5), (6) actually signifies the application of the method of slowly varying wave profile at the distance of the wave-length scale.

In the equations (3), (4) we wrote \( \rho_f, \rho_s \) instead of \( \delta \rho_f, \delta \rho_s \), \( m \) is the porosity;

\[
G = \frac{1}{k_s} - \frac{m}{k_f} - \frac{k}{k_s^2},
\]

where \( k_f, k_s \) and \( k \) are the bulk moduli of the fluid, mineral grains constituting the frame, and of the frame itself; \( \mu \) is the shear modulus of the frame; \( \nu = 1 - m - k/k_s \); \( P_n \) is the nonlinear part of the pressure in the fluid, \( \tilde{\sigma}_{xx}^n = \sigma_{xx}^n - k/k_s P_n \) where \( \sigma_{xx}^n \) is the nonlinear
part of the stress tensor of the frame. The nonlinear stress tensor is considered for the one-
dimensional case, which corresponds to the accepted approximations. The explicit forms for
$P^n$ and $\sigma^x_{xx}$ are given in Ref. [7] for some limiting cases.

Let us eliminate one of the variables, $\delta \rho_f$ or $\delta \rho_s$, from the linear parts of Eqs. (3), (4)
(let it be e.g. $\delta \rho_s$), by subtracting one equation from the other one. Note, that Eqs. (3), (4)
allow two independent longitudinal modes, the so called fast and slow waves. As it is shown
in Ref. [17], the slow wave (unlike the fast one) is a strongly attenuated diffusion mode, and
it does not contribute significantly to the sound field. In this single-mode approximation,
the elimination of $\delta \rho_s$ leads to the disappearance of the linear part of the equation provided
$c$ is the velocity of the fast wave. In the nonlinear parts the quantity $\delta \rho_s$ is expressed through
$\delta \rho_f$ with the formula which is valid to an accuracy $\sim \epsilon$,

$$\delta \rho_s = \left( \frac{\nu}{\rho_s c^2 G} \right)^{-1} \left( 1 - \frac{m}{\rho_f c^2 G} \right) \delta \rho_f. \quad (8)$$

In this approximation we arrive at the nonlinear equation for an acoustic wave in a
sediment,

$$\left\{ 2(1 - m) \left[ 1 - \left( \frac{c_f}{c} \right)^2 \right] - \frac{\nu^2 \rho_f}{m \rho_s} \left( \frac{c_f}{c} \right)^2 + \left[ 1 - m - \frac{k + 4/3 \mu}{\rho_s c^2} \right] \left[ 1 + \left( \frac{c_f}{c} \right)^2 \right] \right\} \frac{\partial \rho_f}{\partial x} +$$

$$\frac{1}{\rho_f c} \left\{ \nu \frac{\rho_f}{\rho_s} \left[ 1 - \left( \frac{c_f}{c} \right)^2 \right] - \left[ 1 - m - \frac{k + 4/3 \mu}{\rho_s c^2} \right] \left[ 1 + \left( \frac{c_f}{c} \right)^2 \right] \right\} \frac{\partial \rho_f^2}{\partial \tau} -$$

$$\frac{1}{c^4} \left[ \left( 1 - m - \frac{k + 4/3 \mu}{\rho_s c^2} \right) \frac{\partial P^n}{\partial \tau} - \nu \frac{\rho_f}{m \rho_s} \left( \frac{c_f}{c} \right)^2 \frac{\partial \tilde{\sigma}_{xx}}{\partial \tau} \right] + a_1 D_\tau \rho_f = 0. \quad (9)$$

To obtain these equations we took into account that in sand sediments the bulk modulus
$k_s$ of quartz grains is much greater than that of the pore water, and in this case $G$ can be
evaluated as $G \approx m/k_f$, provided $m$ is not close to zero.

In Eq. (9) the term $a_1 D_\tau \rho_f$ that accounts for dissipation is introduced. $D_\tau$ is the
dissipation linear operator in the variable $\tau$ which is characterized by the property

$$D_\tau e^{i\omega \tau} = \alpha(\omega) e^{i\omega \tau}, \quad (10)$$

where $\alpha$ is real and positive and it has the meaning of an amplitude attenuation coefficient if
the coefficient $a_1$ is taken to be equal to the coefficient at $\partial \rho_f / \partial x$. The relation (10) defines
the action of this operator on any function of the variable $\tau$ which can be represented by
a Fourier series or integral. An algebraic expression for $\alpha(\omega)$ is a combination of physical
parameters (complex bulk and shear frame moduli included) of a sediment, and it includes
the frequency correction function introduced by Biot [16].

Let us write Eq. (9) in a concise form,

$$a_1 \frac{\partial \rho_f}{\partial x} + a_3 \frac{\partial \rho_f^2}{\partial \tau} + a_1 D_\tau \rho_f = 0. \quad (11)$$

Note the following. The coefficient $a_3$ at the nonlinear term in this equation incorporates
contributions from the nonlinear stress-strain relations of the pore fluid, of the grains,
constituting the sediment frame, and of the sediment frame itself. To write Eq. (11) we
considered the nonlinear pressure $P^n$ and stress tensor $\tilde{\sigma}_{xx}$ from Eq. (9) as expansions in
powers of \( \delta \rho_{f,s} \), retaining quadratic terms. The coefficient \( a_3 \) includes all the coefficients at \( \partial \rho_f^2 / \partial \tau \), and this coefficient will be estimated basing on experimental data.

We shall use Eq. (11) to find the slowly varying amplitude of the generated wave \( \rho_3 \) (from now on we shall omit the index \( f \) at \( \rho \)). With the three interacting waves written in the form \( \rho_i = |\rho_i| \exp i(\mathbf{k}_i \mathbf{r} - \omega_i t + \varphi_i) \), \( i = 1, 2 \), and \( \rho_3 = |\rho_3| \exp i(k_3 x - \omega_3 t + \varphi_3) \), and taking into account the relations (11), we can write Eq. (11) in the form,

\[
\frac{d\rho_3}{dx} e^{i\varphi_3} + |\rho_3| i \frac{d\varphi_3}{dx} e^{i\varphi_3} - \frac{a_3}{a_1} i \omega_3 |\rho_1||\rho_2|e^{i(\varphi_1+\varphi_2)} + \alpha |\rho_3| e^{i\varphi_3} = 0, \tag{12}
\]

where \( \alpha \) is the amplitude attenuation coefficient of the wave \( \rho_3 \). The second term in Eq. (12) vanishes when the phase \( \varphi_3 \) attains its fixed value. The attenuation term \( \alpha |\rho_3| e^{i\varphi_3} \) can be omitted since the wave \( \rho_3 \) is generated and sustained by the waves \( \rho_1 \) and \( \rho_2 \) along the whole distance of their interaction. The attenuation of the exiting waves should be taken into account to estimate the real distance of the nonlinear interaction.

One can see from Eq. (12), that the wave \( \rho_3 \) will "survive" and will be amplified in case \( \varphi_3 = \varphi_1 + \varphi_2 + \pi/2 \), and the equation for the slowly varying amplitude \( \rho_3 \) acquires the form,

\[
\frac{d\rho_3}{dx} - \frac{a_3}{a_1} i \omega_3 |\rho_1||\rho_2| = 0. \tag{13}
\]

Thus, the vertex (the second-order unharmonicity), that determines the interaction under consideration is equal to \( (a_3/a_1)\omega_3 \). From this equation we get

\[
\rho_3(l) - \rho_3(0) = \frac{a_3}{a_1} \omega_3 \rho_1 \rho_2 l, \tag{14}
\]

where \( \rho_3(0) \) is the amplitude (at the fluctuation level) at \( x = 0 \), and it may be neglected, since \( \rho_3(l) \) is supposed to be much higher.

Let us estimate the distance \( l \) at which the amplitude \( \rho_3 \) can reach a measurable value. This distance cannot exceed the dissipation lengths of the waves \( \rho_1 \) and \( \rho_2 \). The amplitude attenuation coefficients of these waves are equal correspondingly to \( \alpha_1 \approx 0.8 \cdot 10^{-3} \text{cm}^{-1} \) and to \( \alpha_2 \approx 3 \cdot 10^{-3} \text{cm}^{-1} \) (see Ref. [11]). This corresponds approximately to propagation distances \( \sim 1250 \text{ cm} \) and \( \sim 330 \text{ cm} \). This means, that \( l \) cannot exceed the distance that is a little more than \( 300 \text{ cm} \).

The coefficient \( (a_3/a_1) \) can be estimated if the second-order nonlinear parameter usually denoted as \( B/A \) is known: \( (a_3/a_1) \sim BA^{-1}/\rho c \), here \( \rho \) is the average density of the medium. Water-saturated porous media are known to have a stronger nonlinearity than a homogeneous fluid. For example, the parameter \( B/A \) can take values \( \sim 8 - 12 \) for sandy sediments [2-4], while in water it equals \( \sim 5 - 6 \). In Refs. [8, 19] the nonlinearity parameter of water-saturated sand was determined to be about 100. For such granular media as disordered packings of noncohesive elastic beads embedded in a fluid it is \( 10^2 - 10^3 \) times larger than in homogeneous fluids and solids [8]. One can also note that quadratic nonlinearity increases significantly in the presence of gas bubbles in water. Experiments [20] showed that the nonlinearity parameter of water containing gas bubbles can reach \( 10^4 - 10^5 \).

We shall make a numerical estimate for the distance \( l \) at which the ratio \( \rho_3(l)/\rho_{1,2}(0) \) can reach the value \( \sim (10^{-1} - 10^{-2}) \). We assume

\[
\rho_1(0) \sim \rho_2(0) \sim 10^{-4} \text{ g/cm}^3, \ \rho \sim 2 \text{ g/cm}^3, \ c = 1.7 \cdot 10^5 \text{ cm/s}, \ \omega_3 = 2\pi \cdot 5 \cdot 10^3 \text{ s}^{-1}, \ B/A \sim 10.
\]

These data yield the distance \( l \) equal approximately to \( \sim 300 \text{ cm} \), which falls within the attenuation length of the exciting waves.
3 Summary

Nonlinear interaction of three acoustic waves in a sandy sediment is studied. A significant amount of velocity dispersion in some frequency intervals allows momentum and energy conservation laws to be satisfied for the waves propagating not in one direction. This means that no shock formation would occur and hence this would not obscure a three-wave nonlinear process. Numerical estimates show that the wave generated by two pump waves propagating at an angle to each other in a sandy sediment can reach a measurable value at a distance realistic for an acoustic-wave experiment in a sediment.

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