Brane-world generalizations of the Einstein static universe

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Abstract.
A static Friedmann brane in a 5-dimensional bulk (Randall-Sundrum type scenario) can have a very different relation between the density, pressure, curvature and cosmological constant than in the case of the general relativistic Einstein static universe. In particular, static Friedmann branes with zero cosmological constant and 3-curvature, but satisfying \( \rho > 0 \) and \( \rho + 3p > 0 \), are possible. Furthermore, we find static Friedmann branes in a bulk that satisfies the Einstein equations but is not Schwarzschild-anti de Sitter or its specializations. In the models with negative bulk cosmological constant, a positive brane tension leads to negative density and 3-curvature.

1. INTRODUCTION

At high enough energies, Einstein’s theory of general relativity breaks down and is likely to be a limit of a more general theory. In string theory/M theory, gravity is a truly higher-dimensional theory, becoming effectively 4-dimensional at lower energies. Recent developments may offer a promising road towards a quantum gravity theory [1]. In brane-world models inspired by string/M theory, the standard-model fields are confined to a 3-brane, while the gravitational field can propagate in \( 3 + d \) dimensions (the ‘bulk’). The \( d \) extra dimensions need not all be small, or even compact: recently Randall and Sundrum [2] have shown that for \( d = 1 \), gravity can be localized on a single 3-brane even when the fifth dimension is infinite. This noncompact localization arises via the exponential ‘warp’ factor in the non-factorizable metric:

\[
d\tilde{s}^2 = \exp(-2|y|/\ell) \left[ -dt^2 + d\vec{x}^2 \right] + dy^2.
\]

For \( y \neq 0 \), this metric satisfies the 5-dimensional Einstein equations with negative 5-dimensional cosmological constant, \( \Lambda \propto -\ell^{-2} \). The brane is located at \( y = 0 \), and the induced metric on the brane is a Minkowski metric. The bulk is a 5-dimensional anti-de Sitter metric, with \( y = 0 \) as boundary, so that \( y < 0 \) is identified with \( y > 0 \), reflecting the \( Z_2 \) symmetry, with the brane as fixed point, that arises in string theory.

Perturbation of the metric (1) shows that the Newtonian gravitational potential on the brane is recovered at lowest order:

\[
V(r) = \frac{GM}{r} \left( 1 + \frac{2\ell^2}{3r^2} \right) + \cdots
\]

Thus 4-dimensional gravity is recovered at low energies, with a first-order correction that is constrained by current sub-millimetre experiments. The lowest order term corresponds to the massless graviton mode, bound to the brane, while the corrections arise from massive Kaluza-Klein modes in the bulk. Generalizing the Randall-Sundrum model to allow for matter on the brane leads to a generalization of the metric (1), and in general to a breaking of conformal flatness, since matter on the brane in general induces Weyl curvature in the bulk. Indeed, the massive Kaluza-Klein modes that produce the corrective terms in Eq. (2) reflect the bulk Weyl curvature that arises from a matter source on the brane.
The 5-dimensional Einstein equations with a brane at \( y = 0 \) containing a general energy-momentum tensor are

\[
\bar{G}_{\mu\nu} = \kappa^2 \left[ -\Lambda \bar{g}_{\mu\nu} + \delta(y) \{ -\lambda g_{\mu\nu} + T_{\mu\nu} \} \right],
\]

where \( \kappa^2 = 8\pi/M_p^4 \), with \( M_p \) the fundamental 5-dimensional Planck mass, which is typically much less than the effective Planck mass on the brane, \( M_p = 1.2 \times 10^{19} \) GeV. The brane tension is \( \lambda \), and standard-model fields confined to the brane make up the brane energy-momentum tensor \( T_{\mu\nu} \), with \( T_{AB} n^B = 0 \), where \( n^A = g^{Aa} \) is the unit normal to the brane. Using the Gauss-Codazzi equations, the Darmois-Israel matching conditions and the \( Z_2 \) symmetry about the brane, one can derive the induced field equations on the brane:

\[
G_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa^2 T_{\mu\nu} + \bar{\kappa}^4 S_{\mu\nu} - \varepsilon_{\mu\nu},
\]

where \( \kappa^2 = 8\pi/M_p^4 \) and \( g_{AB} = \bar{g}_{AB} - n_A n_B \). The energy scales are related to each other via

\[
\lambda = \frac{6\kappa^2}{\bar{\kappa}^4}, \quad \Lambda = \frac{1}{2}\bar{\kappa}^2 \left( \bar{\Lambda} + \frac{1}{6}\bar{\kappa}^2 \lambda^2 \right).
\]

The higher-dimensional modifications of the standard Einstein equations on the brane are of two forms: first, the matter fields contribute local quadratic energy-momentum corrections via the tensor \( S_{\mu\nu} \), which arise from the extrinsic curvature, and second, there are nonlocal effects from the free gravitational field in the bulk, transmitted via the projection onto the brane of the bulk Weyl tensor: \( \varepsilon_{\mu\nu} = \bar{C}_{ACBD}n^C n^D \), and \( \varepsilon_{\mu\nu} \) on the brane is given by the limit as \( y \to 0 \). The local corrections are given by

\[
S_{\mu\nu} = \frac{1}{12} T^a T_{\mu}^a - \frac{1}{4} T_{\mu}^{a\alpha} T^{\alpha \nu} + \frac{1}{4} g_{\mu\nu} \left[ 3T_{\alpha\beta} T^{\alpha\beta} - (T^a)^2 \right].
\]

2. STATIC FRIEDMANN BRANES

The generalized Friedmann equation on a spatially homogeneous and isotropic brane is

\[
H^2 = \frac{1}{3} \kappa^2 \rho \left( 1 + \frac{\rho}{2\lambda} \right) + \frac{1}{3} \Lambda - \frac{2 U_o}{\kappa^2 \lambda} \left( \frac{a_o}{a} \right)^4,
\]

where \( K = 0, \pm 1 \). The \( \rho^2/\lambda \) term is the \( S_{\mu\nu} \) contribution, which is significant only at high energies: \( \rho > \lambda > (100 \text{ GeV})^4 \). The \( U_o/a^4 \) term is the \( \varepsilon_{\mu\nu} \) contribution. General relativity is recovered in the limit \( \lambda^{-1} \to 0 \). The generalized Raychaudhuri equation becomes, for a Friedmann brane,

\[
\ddot{H} + H^2 = -\frac{1}{3} \kappa^2 \left[ \rho + 3p + (2\rho + 3p) \frac{\rho}{\lambda} \right] + \frac{1}{3} \Lambda - \frac{2 U_o}{\kappa^2 \lambda} \left( \frac{a_o}{a} \right)^4.
\]

Z\(_2\)-symmetric Friedmann branes can be embedded in 5-dimensional Schwarzschild-anti de Sitter (SAdS) space, with the mass parameter of the bulk black hole proportional to \( U_o \). These solutions include the special case of static Friedmann branes. As we show in the next section, there are other bulk solutions, which are not SAdS, but which do admit a \( Z_2 \)-symmetric static Friedmann brane.

For a static Friedmann brane, we have \( a = a_o \) and \( H = 0 \), so that Eqs. (7) and (8) become

\[
\kappa^2 \rho \left( 2 + \frac{\rho}{\lambda} \right) = 6 \frac{K}{a_o^2} - \frac{12 U_o}{\kappa^2 \lambda} - 2 \Lambda,
\]

\[
\kappa^2 \left[ 3p + \rho + \frac{\rho}{\lambda} (3p + 2\rho) \right] = -\frac{12 U_o}{\kappa^2 \lambda} + 2 \Lambda.
\]

The general relativity Einstein static universe is the case \( \lambda^{-1} = 0 \), which leads to

\[
\kappa^2 \rho = 3 \frac{K}{a_o^2} - \Lambda, \quad \kappa^2 p = -\frac{K}{a_o^2} + \Lambda,
\]

where \( \kappa^2 = 8\pi/M_p^4 \).
so that if \( \rho > 0 \) and \( \rho + 3p > 0 \), then \( K = +1 \) and \( \Lambda > 0 \).

For the static brane-world model we find

\[
\kappa^2 \rho \left( 1 + \frac{\rho}{2\Lambda} \right) = \frac{3K}{a_0^2} - \Lambda - \frac{6U_0}{\kappa^2 \lambda},
\]

(12)

\[
\kappa^2 \left[ p + \frac{\rho}{2\Lambda}(2p + \rho) \right] = -\frac{K}{a_0^2} + \Lambda - \frac{2U_0}{\kappa^2 \lambda}.
\]

(13)

The local (i.e., \( \rho/\lambda \)) and nonlocal (i.e., \( U_0/\lambda \)) terms introduce new possibilities compared with general relativity. If \( \rho > 0 \) and \( \rho + 3p > 0 \), then

\[
\Lambda > \frac{6U_0}{\kappa^2 \lambda} \quad 3\frac{K}{a_0^2} - \Lambda > \frac{6U_0}{\kappa^2 \lambda}.
\]

(14)

Thus we can satisfy \( \rho > 0 \) and \( \rho + 3p > 0 \) with \( K = 0 = \Lambda \), provided that \( U_0 < 0 \) and

\[
\rho - 3p = \frac{\rho}{\lambda}(\rho + 3p).
\]

(15)

This equation has no general relativity limit, since \( U_0 < 0 \), and it implies \( p < \frac{1}{3}\rho \).

The general static solution satisfies equations (12) and (13). This includes the solutions with \( \rho = (\gamma - 1)\rho \), \( \gamma \) constant, given in (3), which are saddle points in the dynamical phase space (like the Einstein static solution in general relativity).

### 3. NON-SAdS BULK WITH STATIC FRIEDMANN BRANE

In the 5-dimensional Einstein equation (3), we write

\[
\kappa^2 \tilde{\Lambda} = 3\epsilon \Gamma^2, \quad \Gamma > 0, \quad \epsilon = 0, \pm 1,
\]

(16)

where \( \Gamma \) gives the magnitude of the cosmological constant and \( \epsilon \) its sign; if \( \epsilon = 0 \), then \( \Gamma \) is a removable constant. Then we find the following solution of Eq. (3) for \( y > 0 \):

\[
\Gamma^2 d\tilde{s}^2 = -F(y; \epsilon) dt^2 + d\chi^2 + H^2(\chi; \epsilon) (d\theta^2 + \sin^2 \theta d\varphi^2) + dy^2,
\]

(17)

where

\[
F(y; \epsilon) = \begin{cases} 
A \cos \left( \sqrt{\gamma} y \right) + B \sin \left( \sqrt{\gamma} y \right), & \epsilon = 1, \\
A + \sqrt{\gamma} By, & \epsilon = 0, \\
A \cosh \left( \sqrt{\gamma} y \right) + B \sinh \left( \sqrt{\gamma} y \right), & \epsilon = -1,
\end{cases}
\]

and

\[
H(\chi; \epsilon) = \begin{cases} 
\sin \chi, & \epsilon = 1, \\
\chi, & \epsilon = 0, \\
\sinh \chi, & \epsilon = -1.
\end{cases}
\]

Either of the constants \( A \) or \( B \) can be absorbed into the coordinate \( t \), so that we have a one-parameter family of solutions. The derivatives of the metric functions obey (\( y > 0 \))

\[
(\partial_y F)^2 = \alpha \beta^2 - 2\epsilon F^2, \quad \partial_y F = -2\epsilon F, \\
(\partial_\chi H)^2 = 1 - \epsilon H^2, \quad \partial_\chi H = -\epsilon H,
\]

(18)

where \( \alpha = \text{sgn} (\epsilon A^2 + B^2) \) and \( \beta = \sqrt{2|\epsilon A^2 + B^2|} \). Note that \( \alpha = 0, -1 \) can occur for \( \epsilon = -1 \). We exclude the case \( B = 0 \), since, as is shown below, the vanishing of \( B \) is incompatible with matter on the brane (\( y = 0 \)).

Other useful formulas for the above functions will be also employed later:

\[
\partial_y \log F + \alpha \beta^2 \int \frac{dy}{F^2} = -\sqrt{2} \frac{A}{B}, \quad \partial_\chi \log H + \int \frac{d\chi}{H^2} = 0.
\]

(19)

The projected part of the bulk Weyl tensor is

\[
\mathcal{E}_{AB} = \tilde{C}_{ACBD}u^C n^D - \frac{3}{2} \epsilon \Gamma^2 (u_A u_B + \frac{1}{3} h_{AB}),
\]

(20)
where \( u^A = \Gamma F^{-1} \delta^A_0 \) is the 4-velocity along the static Killing vector of the metric (17), and \( h_{AB} = g_{AB} + u_A u_B \). The general solution of the Killing equation is given in the appendix.

The \( \mathbb{Z}_2 \)-symmetric solution is given by Eq. (17) with \( y \) replaced by \(|y|\). On the brane \( y = 0 \), the induced metric is

\[
\Gamma^2 ds^2 = -A^2 dt^2 + d\chi^2 + H^2 (\chi; \epsilon) (d\vartheta^2 + \sin^2 \theta d\varphi^2),
\] (21)

which has Friedmann (with curvature index \( \epsilon \)) and static symmetry. The vectors \( K_{1-7} \) in Eq. (A1) are the Killing vectors for this metric. The scale factor is

\[
a_o = \frac{1}{\Gamma}.
\] (22)

By Eq. (17), the brane cosmological constant \( \Lambda \) is related to the brane tension and bulk cosmological constant via

\[
\Lambda = \frac{1}{2} (3\epsilon \Gamma^2 + \kappa^2 \lambda).
\] (23)

In the space-time given locally by the metric (17), the extrinsic curvature of any hypersurface \( y = \text{const.} \) has only one nonvanishing component

\[
[K_{\mu\nu}]^+ - [K_{\mu\nu}]^- = -\frac{2\sqrt{2}B}{\Gamma} \delta^0_\mu \delta^0_\nu.
\] (25)

Since \( [K_{\mu\nu}]^+ \propto T_{\mu\nu} + \frac{1}{3}(\lambda - T)g_{\mu\nu} \), it follows that the brane energy density is

\[
\rho = -\lambda.
\] (26)

Since the low energy limit of the modified Einstein equations implies \( \lambda > 0 \), our bulk solution has negative energy density on the brane, irrespective of the sign of the bulk cosmological constant.

4. COMPARISON WITH RELATED SOLUTIONS

Several recent works have tackled the issue of Friedmann-type branes embedded in 5-dimensional bulk spacetimes, and we can compare our solution with these to see if ours is simply a special case.

Equations (2), (23), (37) and (38) of [4] describe a bulk solution in Gaussian normal coordinates adapted to the Friedmann brane which is at \( y = 0 \). However in the limit of a static brane, their bulk metric function remains undefined.

The most general bulk solution in which a static, maximally symmetric 3-space is embedded, is claimed to be found in [10]. However the line element given by Eqs. (3) and (4) of [10] does not include our solution, Eq. (17). This can be seen as follows. After the transformation \( r = H(\chi; k) \), we see that the attempt to make the two line elements coincide requires that \( k = \epsilon = -1 \) and that the metric function \( f \) of [10] must be a constant, \( f = \Gamma^{-2} \). The latter can be achieved for the values \( \alpha = \beta = 0 \) of the parameters in [10]. But for these parameters the other metric function \( e \) is vanishing, and in consequence the bulk metric is singular.

In [7], the equivalence is proved between the bulk solution found in [4] and the SAdS bulk, given by

\[
d\tilde{s}^2 = -f(r; K) dt^2 + \frac{dv^2}{f(r; K)} + r^2 \left[ d\chi^2 + H^2 (\chi; K) (d\vartheta^2 + \sin^2 \theta d\varphi^2) \right],
\]

\[
f(r; K) = K + \frac{\Gamma^2}{2} r^2 - \frac{\mu}{r^2},
\] (27)

where the brane is described by a moving domain wall (see also [11]). Here \( K = 0, \pm 1 \) and \( \mu \) is a constant which is proportional to \( U_o \):

\[
\frac{3\mu}{a^2} = -\mathcal{E}_{\mu\nu} u^\mu u^\nu.
\] (28)
This relation follows by comparing the generalized Friedmann equation on the moving brane in SAdS spacetime, Eq. (3) of [8], with the generic form of this equation, Eq. (3). Then Eqs. (20) and (22) imply that our bulk solution with negative cosmological constant is characterized by

$$\mu = -\frac{1}{2\Gamma^2}. \quad (29)$$

For this value of $\mu$ and vanishing Hubble constant, in the hyperbolic case $(K = -1)$ the metric functions of [8] become $\phi = 1$ and $\psi = \cosh(\sqrt{2}\Gamma w)$. By identifying $y = \Gamma w$, we recover the $\epsilon = -1$ case of our metric, Eq. (17), but with the constant $B = 0$. Therefore our generic bulk metric is not contained in the analysis of [8].

Alternatively, in the case $A > B$ we can identify our bulk solution with the metric in Eq. (23) of [8] by inserting $y = \Gamma w - \eta$, where $\sqrt{2}\eta = \tanh^{-1}(B/A)$. However, the brane is then confined to $w = \eta/\Gamma$ in our approach, and to $w = 0$ in [8].

SAdS spacetime admits an event horizon given by $f(r; K) = 0$:

$$r_h^2 = \frac{1}{\Gamma^2} \left( -K + \sqrt{K^2 + 2\Gamma^2\mu} \right). \quad (30)$$

We note that the coordinate transformation employed in [8] is singular at $r_h$, since the affine parameter $w$ diverges, so that it cannot be used as a new coordinate. (In our solution, $r_h = a_o$.) Thus we cannot repeat the procedure of [8] in order to transform our bulk metric (17) to the SAdS form (27).

An elegant approach in [8] leads to the generic constant curvature bulk solution containing a constant spatial curvature brane, generalizing Taub’s solution. By solving the 5-dimensional Einstein equations for the non-constant metric functions $B, \nu$, they find the generic solution in terms of derivatives of $B$. By introducing the radial coordinate $r = B^{1/3}$, they show that this is nothing other than the SAdS spacetime in 5 dimensions. The metric ansatz Eq. (3) of [8], found from symmetry considerations, does include our solution, Eq. (17), when their metric functions take the particular values $B^{2/3} = \Gamma^{-2} = a_o^2$ and $\epsilon = \Gamma^{-2}F(y)$, and their fifth coordinate $z$ is related to ours by $dz = dy/F(y)$. But their final result Eq. (20), containing derivatives of $B$, again does not contain our metric (17), as our metric is characterized by a constant $B$. Neither can $r = B^{1/3} = a_o$ be introduced as a new coordinate.

In some sense our solution (17) resembles the Bertotti-Robinson solution encountered in the conventional 4-dimensional Einstein theory. There the radius of the 2-spheres is also constant, so that it cannot be chosen as a new coordinate. However the Bertotti-Robinson solution can be interpreted as describing the infinite throat of the extreme Reissner-Nordstrom black hole. It would be interesting to find whether some relation between the metric (17) and the SAdS solution holds.

For this purpose first we note that the Killing vectors of the SAdS metric (27) are nothing but $K_{1-7}$ given by Eq. (A1), with $\alpha = 0$ and $H = H(\chi; K)$. Further, for $\epsilon = -1$ the following scalars agree for our bulk metric (17) and the SAdS metric (27): $\tilde{R} = \tilde{g}^{AB}\tilde{R}_{AB}$, $\tilde{R}_{AB}\tilde{R}^{AB}$. This, however happens for all solutions of the 5-dimensional Einstein equation (3), which implies $\tilde{R} = 10c\Gamma^2$ and $\tilde{R}_{AB} = 2c\Gamma^2\tilde{g}_{AB}$. However, there are other curvature scalars which are different, showing that our solution is not SAdS. Two examples are

$$\tilde{R}_{ABCD}\tilde{R}^{ABCD} = \begin{cases} 28\Gamma^4, & \text{our solution}, \\
10\Gamma^4 + 72\mu^2/\mu, & \text{SAdS}, \end{cases} \quad (31)$$

and

$$\tilde{C}_{ABCD}\tilde{C}^{ABCD} = \begin{cases} 18\Gamma^4, & \text{our solution}, \\
72\mu^2/\mu, & \text{SAdS}, \end{cases} \quad (32)$$

Since the scalar $E_{AB}u^Au^B$ coincides for the two metrics when the condition $\mu = -1/(2\Gamma^2)$ holds, we find that the scalars in Eqs. (31) and (32) agree at the particular radial coordinate value $r = a_o$. Of course in 5 dimensions there are 40 independent curvature scalars to compare, but still we have a serious indication that our solution with $\epsilon = -1$ is related to the event horizon of that particular SAdS space-time, hyperbolic case $(K = -1)$, which has the smallest possible horizon area.
5. CONCLUDING REMARKS

We have shown in Sec. 2 how branes of static Friedmann type in general allow many more possibilities than the special limiting case of the general relativistic Einstein static universe. We have also found, in Sec. 3, a family of bulk solutions of the 5-dimensional Einstein equations with high symmetry, admitting a $Z_2$-symmetric brane of static Friedmann type. These solutions are not Schwarzschild-anti de Sitter (SAdS), as confirmed by calculating the bulk curvature scalars $\tilde{R}_{ABCD}$ and $\tilde{C}_{ABCD}$.

Up to now, the known $Z_2$-symmetric Friedman branes in Randall-Sundrum type models have all been embedded in an SAdS bulk. One could easily assume that all such Friedman branes must be embedded in SAdS spacetime. However, our new solutions show that this is not true; there are static $Z_2$-symmetric Friedmann branes that are embedded in the non-SAdS bulk given by Eq. (17). This is the main significance of our new solutions. These solutions have negative matter energy density, $\rho$. However, the total energy density on the brane is $\rho_{\text{brane}} = \rho + \lambda$, and by Eq. (26) this is zero: $\rho_{\text{brane}} = 0$. By Eq. (23), the effective cosmological constant on the brane is

$$\Lambda = \frac{1}{2} \left[ \kappa^2 \lambda + \epsilon \kappa^2 |\tilde{\Lambda}| \right],$$

(33)

which is positive for $\epsilon \geq 0$. In the case $\epsilon = -1$, one can always choose the bulk cosmological constant $\tilde{\Lambda}$ so that $\Lambda$ is positive or zero. Unlike the general relativity Einstein static universe, the brane-world for $\epsilon = -1$ has negative curvature; it is kept static by negative nonlocal energy density:

$$\frac{6 \mathcal{U}_6}{\kappa^2 \lambda} = -\frac{1}{2} \kappa^2 |\tilde{\Lambda}|.$$  

(34)

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Appendix A. Killing vectors

From the general solution of the Killing equation for the metric (17), we find the independent Killing vectors. In the coordinates $x^A = (x^\mu, y)$, they are ($y > 0$):

$$K_1 = (0, 0, 0, 1, 0),$$
$$K_2 = (0, 0, -\cos \varphi, \cot \theta \sin \varphi, 0),$$
$$K_3 = (0, 0, \sin \varphi, \cot \theta \cos \varphi, 0),$$
$$K_4 = (0, -\cos \theta, \partial_\chi (\log H) \sin \theta, 0, 0),$$
$$K_5 = \left(0, \sin \theta \sin \varphi, \partial_\chi (\log H) \cos \theta \sin \varphi, \partial_\chi (\log H) \frac{\cos \varphi}{\sin \theta}, 0\right),$$
$$K_6 = \left(0, \sin \theta \cos \varphi, \partial_\chi (\log H) \cos \theta \cos \varphi, -\partial_\chi (\log H) \frac{\sin \varphi}{\sin \theta}, 0\right),$$
$$K_7 = [\alpha_0^2 + \delta_0^2]^{-1} (1, 0, 0, 0, 0),$$
$$K_8 = \frac{1}{\sqrt{2}} \left[ \alpha_0^2 + \frac{A}{B} \delta_0^2 \right] \left( -\delta_0^0 \frac{\partial_y (\log F)}{2\sqrt{2}F^2} - \partial_y (\log F) \int L dt, 0, 0, 0, L \right),$$
$$K_9 = \frac{1}{\sqrt{2}} \left[ \alpha_0^2 + \frac{A}{B} \delta_0^0 \right] (-\partial_y (\log F) L, 0, 0, 0, \partial_t L),$$(A1)
Table A1. Killing algebras for different values of $\epsilon$ and $\alpha$

| $\epsilon$ | 1 | 0 | $-1$ |
|------------|---|---|-----|
| $\alpha$  | 1 | 1 | 0, $\pm 1$ |
| $K_{1-7}$  | $so(4) \oplus \mathbb{R}$ | $\epsilon(3) \oplus \mathbb{R}$ | $so(1,3) \oplus \mathbb{R}$ |
| $K_{1-9}$  | $so(4) \oplus so(1,2)$ | $\epsilon(3) \oplus ( \epsilon(1,1)$ | $so(1,3) \oplus so(1,2)$ |

where we have introduced the function

$$L(t; \alpha) = \begin{cases} 
\beta^{-1} \cosh \beta t, & \alpha = 1, \\
t, & \alpha = 0, \\
\beta^{-1} \cos \beta t, & \alpha = -1,
\end{cases}$$

(A2)

obeying

$$(\partial_t L)^2 = 1 + \alpha \beta^2 L^2, \quad \partial_t^2 L = \alpha \beta^2 L,$$

(A3)

and we have used

$$\int dy \frac{F^2}{F^2} = -\frac{1}{2\sqrt{2}F^2}$$

(A4)

for $\alpha = 0$.

Among the Killing vectors $K_{1-6}$ are obviously spacelike and $K_7$ is timelike, while for $K_{8,9}$ we find that

$$\bar{g}(K_8, K_8) = \frac{1}{2\Gamma^2} \begin{cases} 
2\epsilon(F/\beta)^2[1 + \alpha \beta^2 L^2] - \alpha, & \alpha \neq 0, \\
-E^2 < 0, & \alpha = 0,
\end{cases}$$

$$\bar{g}(K_9, K_9) = \frac{1}{2\Gamma^2} [2\epsilon F^2 L^2 + 1],$$

(A5)

where we have defined

$$E^2 = \frac{1}{2F^2} \left[ 4F^4 + 2\left( 1 + 2F^2 L^2 \right)^2 + \left( 1 - \frac{|A|}{|B|} \right) F^2 L^2 \right].$$

The vector $K_9$ is spacelike for $\epsilon = 1, 0$ and time-like for large $|t|$ in the cases $(\epsilon, \alpha) = (-1, 1), (0, 0).$ Otherwise its causal character depends strongly on the actual values of $t$ and $y$. The vector $K_8$ is timelike in the cases $(\epsilon, \alpha) = (0, 1), (-1, 1), (-1, 0),$ while for $(\epsilon, \alpha) = (1, 1)$ it becomes spacelike for large $|t|$. For all other cases the causal character of $K_8$ changes with $t$ and $y$.

The Killing algebra is given by

$$[K_1, K_j] = \epsilon_{ijk} K_k,$$

$$[K_{3+i}, K_{3+j}] = \epsilon \epsilon_{ijk} K_k,$$

$$[K_1, K_{3+j}] = \epsilon_{ijk} K_{3+k},$$

$$[K_{6+i}, K_j] = 0 = [K_{6+i}, K_{3+j}],$$

$$[K_7, K_8] = K_9,$$

$$[K_8, K_9] = -\epsilon \alpha K_7 + \delta^0 \alpha K_8,$$

$$[K_9, K_7] = -\alpha K_8 + \delta^0 \alpha K_7.$$  

(A6)

The Killing algebra for the different admissible values of the parameters $\epsilon$ and $\alpha$ is classified in Table A1. The Killing vectors $K_{8,9}$ have components in the bulk $(y)$ direction, while $K_{1-7}$ are confined to the $y =$ const sections, which is the reason we list their algebras separately in Table A1.
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