FAMILIES OF $p$-ADIC GALOIS REPRESENTATIONS AND
($\varphi, \Gamma$)-MODULES

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ABSTRACT. We investigate the relation between $p$-adic Galois representations and overconvergent ($\varphi, \Gamma$)-modules in families. Especially we construct a natural open subspace of a family of ($\varphi, \Gamma$)-modules, over which it is induced by a family of Galois-representations.

1. Introduction

In [BC], Berger and Colmez generalized the theory of overconvergent ($\varphi, \Gamma$)-modules to families parametrized by $p$-adic Banach algebras. More precisely their result gives a fully faithful functor from the category of vector bundles with continuous Galois action on a rigid analytic variety to the category of families of étale overconvergent ($\varphi, \Gamma$)-modules. This functor fails to be essentially surjective. However it was shown by Kedlaya and Liu in [KL] that this functor can be inverted locally around rigid analytic points.

It was already pointed out in our previous paper [He1] that the right category to handle these objects is the category of adic spaces (locally of finite type over $\mathbb{Q}_p$) as introduced by Huber, see [Hu]. Using the language of adic spaces, we show in this paper that given a family $\mathcal{N}$ of ($\varphi, \Gamma$)-modules over the relative Robba ring $\mathcal{B}^\dagger_{X, \text{rig}}$ on an adic space $X$ locally of finite type over $\mathbb{Q}_p$ (see below for the construction of the sheaf $\mathcal{B}^\dagger_{X, \text{rig}}$), one can construct natural open subspaces $X^{\text{int}}$ resp. $X^{\text{adm}}$, where the family $\mathcal{N}$ is étale resp. induced by a family of Galois representations.

Our main results are as follows. Let $K$ be a finite extension of $\mathbb{Q}_p$ and write $G_K$ for its absolute Galois group. Further we fix a cyclotomic extension $K_\infty = \bigcup K(\mu_{p^n})$ of $K$ and write $\Gamma = \text{Gal}(K_\infty/K)$.

Theorem 1.1. Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$, and let $\mathcal{N}$ be a family of ($\varphi, \Gamma$)-modules over the relative Robba ring $\mathcal{B}^\dagger_{X, \text{rig}}$.
(i) There is a natural open subspace $X^{\text{int}} \subset X$ such that the restriction of $\mathcal{N}$ to $X^{\text{int}}$ is étale, i.e. locally on $X^{\text{int}}$ there is a family of étale lattices $\mathfrak{N} \subset \mathcal{N}$.
(ii) The formation $(X, \mathcal{N}) \mapsto X^{\text{int}}$ is compatible with base change in $X$, and $X = X^{\text{int}}$ whenever the family $\mathcal{N}$ is étale.
In the classical theory of overconvergent \((\varphi, \Gamma)\)-modules, the slope filtration theorem of Kedlaya, \cite[Theorem 1.7.1]{Ke} asserts that a \(\varphi\)-module over the Robba ring admits an étale lattice if and only if it is purely of slope zero. The latter condition is a semi-stability condition which only involves the slopes of the Frobenius. The question whether there is a generalization of this result to \(p\)-adic families was first considered by R. Liu in \cite{Liu}, where he shows that an étale lattice exists locally around rigid analytic points. Here we show that Kedlaya’s result does not generalize to families. That is, we construct a family of \(\varphi\)-modules which is étale at all rigid analytic points but which is not étale as a family of \((\varphi, \Gamma)\)-modules (in the sense specified below).

On the other hand, we construct an admissible subset \(X_{\text{adm}} \subset X\) for a family of \((\varphi, \Gamma)\)-modules over \(X\). This is the subset over which there exists a family of Galois representations. It will be obvious that we always have an inclusion \(X_{\text{adm}} \subset X_{\text{int}}\).

**Theorem 1.2.** Let \(X\) be a reduced adic space locally of finite type over \(\mathbb{Q}_p\) and \(\mathcal{N}\) be a family of \((\varphi, \Gamma)\)-modules over the relative Robba ring \(B^\dagger_{X, \text{rig}}\).

(i) There is a natural open and partially proper subspace \(X_{\text{adm}} \subset X\) and a family \(\mathcal{V}\) of \(G_K\)-representations on \(X_{\text{adm}}\) such that \(\mathcal{N}|_{X_{\text{adm}}}\) is associated to \(\mathcal{V}\) by the construction of Berger-Colmez.

(ii) The formation \((X, \mathcal{N}) \mapsto (X_{\text{adm}}, \mathcal{V})\) is compatible with base change in \(X\), and \(X = X_{\text{adm}}\) whenever the family \(\mathcal{N}\) comes from a family of Galois representations.

(iii) Let \(X\) be of finite type and let \(\mathcal{X}\) be a formal model of \(X\). Let \(Y \subset X\) be the tube of a closed point in the special fiber of \(\mathcal{X}\). If \(Y \subset X_{\text{int}}\), then \(Y \subset X_{\text{adm}}\).

In a forthcoming paper we will apply the theory developed in this article to families of trianguline \((\varphi, \Gamma)\)-modules and give an alternative construction of Kisin’s finite slope space.

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2. Sheaves of period rings

In this section we define relative versions of the classical period rings used in the theory of \((\varphi, \Gamma)\)-modules and in \(p\)-adic Hodge-theory. Some of these sheaves were already defined in \cite[8]{He1}.

Let \(K\) be a finite extension of \(\mathbb{Q}_p\) with ring of integers \(\mathcal{O}_K\) and residue field \(k\). Fix an algebraic closure \(\overline{K}\) of \(K\) and write \(G_K = \text{Gal}(\overline{K}/K)\) for the absolute Galois group of \(K\). As usual we choose a compatible system
\( \epsilon_n \in \bar{K} \) of \( p^n \)-th root of unity and write \( K_\infty = \bigcup K(\epsilon_n) \). Let \( H_K \subset G_K \) denote the absolute Galois group of \( K_\infty \) and write \( \Gamma = \text{Gal}(K_\infty/K) \). Finally we denote by \( W = W(k) \) the ring of Witt vectors with coefficients in \( k \) and by \( K_0 = \text{Frac} W \) the maximal unramified extension of \( \mathbb{Q}_p \) inside \( K \).

2.1. The classical period rings. We briefly recall the definitions of the period rings, as defined in [Be1] for example. Write

\[
\hat{E}^+ = \lim_{x \to x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}.
\]

This is a perfect ring of characteristic \( p \) which is complete for the valuation \( \text{val}_E \) given by \( \text{val}_E(x_0, x_1, \ldots) = \text{val}_p(x_0) \). Let

\[
\hat{E} = \text{Frac} \hat{E}^+ = \hat{E}^+ \left[ \frac{1}{e} \right],
\]

where \( e = (e_1, e_2, \ldots) \in \hat{E}^+ \). Further we define

\[
\hat{A}^+ = W(\hat{E}^+) \quad \hat{A} = W(\hat{E}), \quad \hat{B}^+ = \hat{A}^+[1/p] \quad \hat{B} = \hat{A}[1/p].
\]

On all these ring we have an action of the Frobenius morphism \( \varphi \) which is induced by the \( p \)-th power map on \( \hat{E} \). Further we consider the ring \( A_K \) which is the \( p \)-adic completion of \( W((T)) \) and denote by \( B_K = A_K[1/p] \) its rational analogue. We embed these rings into \( \hat{B} \) by mapping \( T \) to \( \left[ \frac{1}{e} \right] - 1 \). The morphism \( \varphi \) induces the endomorphism \( T \mapsto (T + 1)^p - 1 \) on \( A_K \), resp. \( B_K \). Further \( G_K \) acts on \( A_K \) through the quotient \( G_K \to \Gamma \).

For \( r < s \in \mathbb{Z} \) we define

\[
A^{[r,s]} = \left\{ \sum_{n \in \mathbb{Z}} a_n T^n \bigg| a_n \in K_0, \begin{array}{l} 0 \leq \text{val}_p(a_n p^{n/r}) \to \infty, n \to -\infty \\ 0 \leq \text{val}_p(a_n p^{n/s}) \to \infty, n \to \infty \end{array} \right\},
\]

\[
\hat{A}^{[r,s]} = \left\{ \sum_{n \in \mathbb{Z}} [x_n] p^n \bigg| x_n \in \hat{E}, 0 \leq \text{val}_E(x_n) + \frac{prn}{p-1} \to \infty, n \to \infty \right\},
\]

\[
\hat{B}^{[r,r]} = \left\{ \sum_{n \in \mathbb{Z}} [x_n] p^n \bigg| x_n \in \hat{E}, \text{val}_E(x_n) + \frac{prn}{p-1} \to \infty, n \to \infty \right\}.
\]

The rings \( \hat{A}^{[r,s]} \) and \( \hat{B}^{[r,s]} \) are endowed with the valuation

\[
w_r : \sum p^k [x_k] \mapsto \inf_k \{ \text{val}_E(x_k) + \frac{prk}{p-1} \}.
\]

Using these definitions the usual period rings are defined as follows:

\[
\hat{B}^{[r,s]} \text{ rig} = \text{Frechet completion of } \hat{B}^{[r,s]} \text{ for the valuations } w_s, s' \geq s,
\]

(2.1) \[
\hat{B}^+ = \lim_{s \to -\infty} \hat{B}^{[r,s]},
\]

\[
\hat{B}^+ \text{ rig} = \lim_{s \to -\infty} \hat{B}^{[r,s]} \text{ rig}.
\]
\[ \begin{align*}
\mathcal{B}^{[r,s]} &= A^{[r,s]}[1/p], & A^{\dagger,r} &= \lim_{\leftarrow} A^{[r,s]}, \\
\mathcal{B}^{\dagger,s} &= \mathcal{B}_K \cap \hat{\mathcal{B}}^{\dagger,s}, & B^{\dagger,s} &= \lim_{\leftarrow} \mathcal{B}^{[r,s]}, \\
A^{\dagger} &= \mathcal{A}_K \cap \mathcal{B}^{\dagger}, & B^{\dagger} &= \lim_{\rightarrow} B^{\dagger,s}, \\
B^{\dagger} &= \lim_{\rightarrow} B^{\dagger,s},
\end{align*} \]

Note that these definitions equip all rings with a canonical topology. There is a canonical action of \( G_K \) on all of these rings. This action is continuous for the canonical topology. The \( H_K \)-invariants of \( \tilde{\mathcal{R}} \) for any of the rings in (2.1) are given by the corresponding ring without a tilde \( \mathcal{R} \) in (2.2), where \( \mathcal{R} \) is identified with a subring of \( \tilde{\mathcal{R}} \) by \( \mathcal{T} \mapsto \Gamma(\mathcal{B}^{[1/p-1]}_K, \mathcal{O}_K) \). Hence there is a natural continuous \( \Gamma \)-action on all the rings in (2.2). The Frobenius endomorphism \( \varphi \) of \( \hat{\mathcal{B}} \) induces a ring homomorphism

\[
A^{[r,s]} \rightarrow A^{[pr,ps]},
\]

for \( r, s \gg 0 \) and in the limit endomorphisms of the rings

\[
A^{\dagger}, B^{\dagger}, B^{\dagger}_{\text{rig}}, \hat{B}^{\dagger}, \hat{B}^{\dagger}_{\text{rig}}.
\]

These homomorphisms will be denoted by \( \varphi \) and commute with the action of \( \Gamma \), resp. \( G_K \).

**Remark 2.1.** Let us points out that some of the above rings have a geometric interpretation. We write \( \mathcal{B} \) for the closed unit disc over \( K_0 \) and \( \mathcal{U} \subset \mathcal{B} \) for the open unit disc. Then

\[
A^{[r,s]} = \Gamma(\mathcal{B}^{[1/p-1]}_K, \mathcal{O}_K^+), \quad B^{[r,s]} = \Gamma(\mathcal{B}^{[1/p-1]}_K, \mathcal{O}_K),
\]

\[
A^{\dagger,s} = \Gamma(\mathcal{U}^{[1/p-1]}_K, \mathcal{O}_K^+), \quad B^{\dagger,s} = \Gamma(\mathcal{U}^{[1/p-1]}_K, \mathcal{O}_K),
\]

where \( \mathcal{B}_{[a,b]} \subset \mathcal{B} \) is the subspace of inner radius \( a \) and outer radius \( b \) and \( \mathcal{U}_{\geq a} \subset \mathcal{U} \) is the subspace of inner radius \( a \).

### 2.2. Sheafification

Let \( X \) be an adic space locally of finite type over \( \mathbb{Q}_p \). Recall that \( X \) comes along with a sheaf \( \mathcal{O}_X^{+} \subset \mathcal{O}_X \) of open and integrally closed subrings.

Let \( A^+ \) be a reduced \( \mathbb{Z}_p \)-algebra topologically of finite type. Recall that for \( i \geq 0 \) the completed tensor products

\[
A^+ \otimes_{\mathbb{Z}_p} W_i(\mathcal{E}^+) \quad \text{and} \quad A^+ \otimes_{\mathbb{Z}_p} W_i(\mathcal{E})
\]

are the completions of the ordinary tensor product for the topology that is given by the discrete topology on \( A^+/p^i A^+ \) and by the natural topology on \( W_i(\mathcal{E}^+) \) resp. \( W_i(\mathcal{E}) \), see [He1] 8.1.
Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$. As in [He1, 8.1] we can define sheaves $\tilde{\mathcal{E}}_X^+, \tilde{\mathcal{E}}_X, \mathcal{A}_X^+$ and $\mathcal{A}_X$ by demanding

$$
\Gamma(\text{Spa}(A, A^+), \tilde{\mathcal{E}}_X^+) = A^+ \otimes_{\mathbb{Z}_p} \mathbb{E}_\tilde{\mathcal{E}},
$$

$$
\Gamma(\text{Spa}(A, A^+), \tilde{\mathcal{E}}_X) = A^+ \otimes_{\mathbb{Z}_p} \mathbb{E},
$$

$$
\Gamma(\text{Spa}(A, A^+), \mathcal{A}_X^+) = \lim_i A^+ \otimes_{\mathbb{Z}_p} W_i(\mathbb{E}^+),
$$

$$
\Gamma(\text{Spa}(A, A^+), \mathcal{A}_X) = \lim_i A^+ \otimes_{\mathbb{Z}_p} W_i(\mathbb{E}),
$$

for an affinoid open subset $\text{Spa}(A, A^+) \subset X$.

We define the sheaf $\mathcal{A}_{X,K}$ to be the $p$-adic completion of $(\mathcal{O}_X^+ \otimes_{\mathbb{Z}_p} W)((T))$. Further we set $\mathcal{B}_{X,K} = \mathcal{A}_{X,K}[1/p]$.

Let $A^+$ be as above and $A = A^+[1/p]$. We define

$$
A^+ \otimes_{\mathbb{Z}_p} A^{[r,s]} \quad \text{and} \quad A^+ \otimes_{\mathbb{Z}_p} \mathbb{A}^{t,s}
$$

to be the completion of the ordinary tensor product for the $p$-adic topology on $A^+$ and the natural topology on $A^{[r,s]}$ resp. $\mathbb{A}^{t,s}$. These completed tensor products can be viewed as subrings of $\Gamma(\text{Spa}(A, A^+), \mathcal{A}_{\text{Spa}(A,A^+)})$. For a reduced adic space $X$ locally of finite type over $\mathbb{Q}_p$, we define the sheaves $\mathcal{A}_X^{[r,s]}$ and $\mathcal{A}_X^{t,s}$ by demanding

$$
\Gamma(\text{Spa}(A, A^+), \mathcal{A}_X^{[r,s]}) = A^+ \otimes_{\mathbb{Z}_p} A^{[r,s]},
$$

$$
\Gamma(\text{Spa}(A, A^+), \mathcal{A}_X^{t,s}) = A^+ \otimes_{\mathbb{Z}_p} \mathbb{A}^{t,s},
$$

for an open affinoid $\text{Spa}(A, A^+) \subset X$. Similarly we define the sheaf $\mathcal{B}_X^{t,s}$. Finally, as in the case above, we can use these sheaves to define the sheafified versions of (2.2) and (2.1)

$$
\mathcal{B}_X^{t,s} = \text{Frechet completion of } \mathcal{B}_X^{t,s} \text{ for the valuations } w_{s'}, s' \geq s,
$$

$$
(2.3) \quad \mathcal{B}_X = \lim_{\to} \mathcal{B}_X^{t,s},
$$

$$
\mathcal{B}_{X,\text{rig}} = \lim_{\to} \mathcal{B}_{X,\text{rig}}^{t,s},
$$

$$
\mathcal{B}_X^{[r,s]} = \mathcal{A}_X^{[r,s]}[1/p], \quad \mathcal{B}_X^{t,s} = \mathcal{A}_X^{t,s},
$$

$$
(2.4) \quad \mathcal{B}_X^{t,s} = \mathcal{B}_{X,K} \cap \mathcal{B}_X^{t,s}, \quad \mathcal{B}_X^{t,s} = \lim_{\to} \mathcal{B}_X^{t,s},
$$

Note that all the rational period rings (i.e. those period rings in which $p$ is inverted) can also be defined on a non-reduced space $X$ by locally embedding the space into a reduced space $Y$ and restricting the corresponding period sheaf from $Y$ to $X$, compare [He1, 8.1].
Remark 2.2. As in the absolute case there is a geometric interpretation of some of these sheaves of period rings.

\[ A^{[r,s]}_X = \text{pr}_X^*(O_X^{\mathbb{B}_{[p-1/p-1]}}) \]  
\[ B^{[r,s]}_X = \text{pr}_X^*(O_X^{\mathbb{B}_{[p-1/p-1]}^\text{rig}}) \]  
\[ A^{[r,s]}_X = \text{pr}_X^*(O_X^{\mathbb{U}_{\geq 1/s}}) \]  
\[ B^{[r,s]}_X,_{\text{rig}} = \text{pr}_X^*(O_X^{\mathbb{U}_{\geq 1/s}}^\text{rig}) \]

Here \( \text{pr}_X \) denotes the projection from the product to \( X \). Especially we can define the sheaves \( A^{[r,s]}_X \), \( A^{[r,s]}_X \) and \( A^{[r,s]}_X \) on every adic spaces, not only on reduced spaces.

By construction all the sheaves \( \tilde{\mathcal{R}}_X \) (i.e. those of the period sheaves with a tilde) are endowed with a continuous \( O_X \)-linear \( \mathcal{G}_K \)-action and an endomorphism \( \varphi \) commuting with the Galois action. The sheaves \( \mathcal{R}_X \) (i.e. those period rings without a tilde) are endowed with a continuous \( \Gamma \)-action and an endomorphism \( \varphi \) commuting with the action of \( \Gamma \).

In the following we will use the notation \( X(\overline{\mathbb{Q}}_p) \) for the set of rigid analytic points of an adic space \( X \) locally of finite type over \( \mathbb{Q}_p \), i.e. \( X(\overline{\mathbb{Q}}_p) = \{ x \in X \mid k(x)/\mathbb{Q}_p \text{ finite} \} \).

Proposition 2.3. Let \( X \) be a reduced adic space locally of finite type over \( \mathbb{Q}_p \) and let \( R \) be any of the integral period rings (i.e. a period ring in which \( p \) is not inverted) defined above. Let \( \mathcal{R}_X \) be the corresponding sheaf of period rings on \( X \).

(i) The canonical map
\[ \Gamma(X, \mathcal{R}_X) \to \prod_{x \in X(\overline{\mathbb{Q}}_p)} k(x)^+ \otimes_{\mathbb{Z}_p} R \]
is an injection.

(ii) Let \( R' \subset R \) be another integral period ring with corresponding sheaf of period rings \( \mathcal{R}'_X \subset \mathcal{R}_X \) and let \( f \in \mathcal{R}_X \). Then \( f \in \mathcal{R}'_X \) if and only if
\[ f(x) \in k(x)^+ \otimes_{\mathbb{Z}_p} R' \subset k(x)^+ \otimes R \]
for all rigid analytic points \( x \in X \).

Proof. This is proven along the same lines as [He1, Lemma 8.2] and [He1, Lemma 8.6].

Corollary 2.4. Let \( X \) be an adic space locally of finite type over \( \mathbb{Q}_p \), then
\[ (\mathcal{B}^\dagger_{X,\text{rig}})^\varphi = \text{id} = O_X \]  
\[ (\mathcal{B}^\dagger_{X,\text{rig}})^{H_K} = \mathcal{B}^\dagger_{X,\text{rig}} \]  
\[ (\mathcal{B}^\dagger_X)^\varphi = \text{id} = O_X \]  
\[ (\mathcal{B}^\dagger_X)^{H_K} = \mathcal{B}^\dagger_X \]

Proof. If the space is reduced this follows from the above by chasing through the definitions. Otherwise we can locally on \( X \) choose a finite morphism to a reduced space \( Y \) (namely a polydisc) and study the \( \varphi \)- resp. \( H_K \)-invariants.
in the fibers over the rigid analytic points of $Y$, compare [He1] Corollary 8.4, Corollary 8.8]

□

Remark 2.5. Let $X$ be an adic space locally of finite type and $\mathcal{R}$ be any of the sheaves of topological rings defined above. If $x \in X$ is a point then we will sometimes write $\mathcal{R}_x$ for the completion of the fiber $\mathcal{R} \otimes k(x)$ of $\mathcal{R}$ at $x$ with respect to the canonical induced topology.

3. Coherent $\mathcal{O}_X$-modules and lattices

As the notion of being étale is defined by using lattices we make precise what we mean by (families of) lattices.

Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$. The space $X$ is endowed with a structure sheaf $\mathcal{O}_X$ and a sheaf of open and integrally closed subrings $\mathcal{O}_X^+ \subset \mathcal{O}_X$ consisting of the power bounded sections of $\mathcal{O}_X$. Recall that for any ringed space, there is the notion of a coherent module, see [EGA I] 5.3.

Definition 3.1. Let $X$ be an adic space (locally of finite type over $\mathbb{Q}_p$) and let $E$ be a sheaf of $\mathcal{O}_X^+$-modules on $X$.

(i) The $\mathcal{O}_X^+$ module $E$ is called of finite type or finitely generated, if there exist an open covering $X = \bigcup_{i \in I} U_i$ and for all $i \in I$ exact sequences

$$\left(\mathcal{O}_U^+\right)^{d_i} \rightarrow E|_{U_i} \rightarrow 0.$$ 

(ii) The module is called coherent, if it is of finite type and for any open subspace $U \subset X$ the kernel of any morphism $\left(\mathcal{O}_U^+\right)^d \rightarrow E|_U$ is of finite type.

Remark 3.2. Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$. Then locally on $X$ the sections $\Gamma(X, \mathcal{O}_X)$ as well as $\Gamma(X, \mathcal{O}_X^+)$ are noetherian rings. Hence the notion of a coherent $\mathcal{O}_X^+$-module and the notion of an $\mathcal{O}_X^+$-module of finite type coincide for these spaces. Especially, an $\mathcal{O}_X^+$-module which is locally associated with a module of finite type is coherent.

Remark 3.3. The same definition of course also applies to the sheaves of period rings that we defined above.

Let $X = \text{Spa}(A, A^+)$ be an affinoid adic space. Then any finitely generated $A^+$-module $M$ defines a coherent sheaf of $\mathcal{O}_X^+$-modules $E$ by the usual procedure

$$\Gamma(\text{Spa}(B, B^+), E) = M \otimes_{A^+} B^+$$

for an affinoid open subspace $\text{Spa}(B, B^+) \subset X$.

On the other hand it is not true that all coherent $\mathcal{O}_X^+$-modules on an affinoid space arise in that way, as shown by the following example. The reason is that the cohomology $H^1(X, E)$ of coherent $\mathcal{O}_X^+$-sheaves does not necessarily vanish on affinoid spaces.
Example 3.4. Let $X = \text{Spa}(\mathbb{Q}_p(T), \mathbb{Z}_p(T))$ be the closed unit disc. Let

$$U_1 = \{ x \in X \mid |x| \leq |p| \}$$

$$U_2 = \{ x \in X \mid |p| \leq |x| \leq 1 \}.$$ 

Define the $\mathcal{O}_X^+$-sheaf $E_1 \subset \mathcal{O}_X$ by glueing $\mathcal{O}_{U_1}^+$ and $p^{-1}T\mathcal{O}_{U_2}^+$ over $U_1 \cap U_2$ and $E_2 \subset \mathcal{O}_X$ by glueing $\mathcal{O}_{U_1}^+$ and $pT^{-1}\mathcal{O}_{U_2}$. Then $E_1$ and $E_2$ are coherent $\mathcal{O}_X^+$-modules. We have

$$\Gamma(X, E_1) = (1, p^{-1}T)\Gamma(X, \mathcal{O}_X^+),$$

$$\Gamma(X, E_2) = p\Gamma(X, \mathcal{O}_X^+).$$

Especially $E_2$ is not generated by global sections. If $\mathcal{X} = U_1 \cup U_2 \cong \hat{\mathbb{A}}^1_{\mathbb{Z}_p} \cup \hat{\mathbb{A}}^1_{\mathbb{Z}_p}$ is the canonical formal model of $X = U_1 \cup U_2$, then $E_2$ is defined by the coherent $\mathcal{O}_X^+$-sheaf which is trivial on the formal affine line and which is the twisting sheaf $\mathcal{O}(1)$ on the formal projective line, while $E_1$ is defined by the dual of the twisting sheaf $\mathcal{O}(-1)$ on the formal projective line.

Let $X$ be an adic space of finite type over $\mathbb{Q}_p$ (especially $X$ is quasi-compact) and $E$ be a coherent $\mathcal{O}_X^+$-module on $X$. As $E$ is not necessarily associated to an $\mathbb{A}^+$-module on an affinoid open $\text{Spa}(\mathbb{A}, \mathbb{A}^+) \subset X$, the sheaf $E$ does not necessarily have a model $\mathcal{E}$ over any formal model $\mathcal{X}$ of $X$: The sheaf $\mathcal{U} \mapsto \Gamma(\mathcal{U}^{\text{ad}}, E)$ does not define $E$ in the generic fiber in general. However there is a covering $X = \bigcup U_i$ of $X$ by finitely many open affinoids such that $E|_{U_i}$ is the sheaf defined by the finitely generated $\Gamma(U_i, \mathcal{O}_X^+)$-module $\Gamma(U_i, E)$. Hence there is a formal model $\mathcal{X}$ of $X$ such that $E$ is defined by a coherent $\mathcal{O}_X$-modules $\mathcal{E}$. Namely $\mathcal{X}$ is a formal model on which one can realize the covering $X = \bigcup U_i$ as a covering by open formal subschemes.

Let $E$ be a coherent $\mathcal{O}_X^+$-module on an adic space $X$ and let $x \in X$. Let $m_x \subset \mathcal{O}_{X,x}$ denote the maximal ideal of function vanishing at $x$ and write $m_x^+ = m_x \cap \mathcal{O}_{X,x}^+$, i.e. $\mathcal{O}_{X,x}/m_x^+ = k(x)^+$ is the integral subring of $k(x)$. We write $E \otimes k(x)^+$ for the fiber of $E$ at $x$, that is for the quotient of the $\mathcal{O}_{X,x}^+$-module

$$E_x = \lim_{\nu \ni x} \Gamma(U, E)$$

by the ideal $m_x^+$.

Let $\mathcal{X}$ be a formal model of $X$ and $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module defining $E$ in the generic fiber. Further let Spf $k(x)^+ \hookrightarrow \mathcal{X}$ denote the morphism defining $x$ in the generic fiber. Then $\mathcal{E} \otimes k(x)^+ = E \otimes k(x)^+$. If we write $\mathcal{X}$ for the special fiber of $\mathcal{X}$ and $\hat{\mathcal{E}}$ for the restriction of $\mathcal{E}$ to $\mathcal{X}$ and if $x_0 \in \mathcal{X}$ denotes the specialization of $x$, then it follows that

$$\hat{\mathcal{E}} \otimes k(x_0) = (\mathcal{E} \otimes k(x)^+) \otimes_{k(x)^+} k(x_0) = (E \otimes k(x)^+) \otimes_{k(x)^+} k(x_0).$$

Definition 3.5. Let $E$ be a vector bundle of rank $d$ on an adic space $X$, locally of finite type over $\mathbb{Q}_p$. A lattice in $E$ is a coherent $\mathcal{O}_X^+$-submodule
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$E^+ \subset E$ which is locally on $X$ free of rank $d$ and which generates $E$, i.e. the inclusion induces an isomorphism

$$E^+ \otimes_{O_X^+} O_X \cong E.$$  

Lemma 3.6. Let $X$ be an adic space of finite type over $\mathbb{Q}_p$ and let $E^+$ be a finitely generated $O_X^+$-submodule of $O_X^d$ which contains a lattice of $O_X^d$.

Then $E^+$ is a lattice if and only if $E^+ \otimes k(x)^+$ is $\varpi_x$-torsion free for all rigid points $x \in X$, where $\varpi_x \in k(x)^+$ is a uniformizer.

Proof. As $E^+$ is finitely generated, it is in fact coherent. Choose a formal model $\mathcal{X}$ and a coherent sheaf $\mathcal{E}^+$ on $\mathcal{X}$ which is a model for $E^+$. We may assume that $\mathcal{E}^+$ has no $p$-power torsion.

Let $\text{sp} : X \to \bar{X}$ denote the specialization map to the reduced special fiber $\bar{\mathcal{X}}$ of $\mathcal{X}$. Further we write $\bar{\mathcal{E}}^+$ for the restriction of $\mathcal{E}^+$ to $\bar{\mathcal{X}}$. Let $x_0 \in \bar{X}$ be a closed point and let $x \in X$ be a rigid analytic point (i.e. $k(x)$ is a finite extension of $\mathbb{Q}_p$) with $\text{sp}(x) = x_0$. Then we have

$$\bar{\mathcal{E}}^+ \otimes k(x_0) = (E^+ \otimes k(x)^+) \otimes_{k(x)^+} k(x_0).$$

On the other hand, as $E^+ \otimes k(x)^+$ has no $\varpi_x$-torsion, the $k(x)^+$-module $E^+ \otimes k(x)^+$ is a submodule of $O_{\mathcal{X}}^d \otimes k(x) = k(x)^d$. Further it is finitely generated and contains a basis of $k(x)^d$. Hence it is freely generated by $d$ elements. It follows that $\bar{\mathcal{E}}^+ \otimes k(x_0)$ has dimension $d$. As $\bar{\mathcal{X}}$ is reduced and $\mathcal{E}^+$ is a coherent sheaf it follows that it is a vector bundle. Locally on $\mathcal{X}$ we can lift $d$ generators of $\mathcal{E}^+$ to $\mathcal{E}^+$. By Nakayamas lemma these lifts generate $\mathcal{E}^+$ and hence they also generate the $O_{\mathcal{X}}^+$-module $E^+$. On the other hand $O_{\mathcal{X}}^d = E^+[1/p]$ is free on $d$ generators. Hence the generators of $E^+$ do not satisfy any relations. \hfill $\Box$

4. $(\varphi, \Gamma)$-modules over the relative Robba ring

In this section we define certain families of $\varphi$-modules that will appear in the context of families of Galois representations later on. The main results of this section are already contained in [He1, 6].

Definition 4.1. Let $X$ be an adic space and $\mathcal{R} \in \{\mathcal{A}_{X,K}, \mathcal{A}_{X}^\dagger\}$.

An étale $\varphi$-module over $\mathcal{R}$ is a coherent $\mathcal{R}$-module $N$ together with an isomorphism

$$\Phi : \varphi^* N \rightarrow N.$$

Definition 4.2. Let $X \in \text{Adl}_{\mathbb{Q}_p}^{\text{rig}}$ and

$$\mathcal{R} \in \{\mathcal{B}_{X,K}, \mathcal{B}_{X}^\dagger, \mathcal{B}_{X,\text{rig}}^\dagger\}.$$
Write $\mathcal{R}^+ \subset \mathcal{R}$ for the corresponding integral subring$^1$. 

(i) A $\varphi$-module over $\mathcal{R}$ is a locally free $\mathcal{R}$-module $N$ together with an isomorphism $\Phi: \varphi^* N \to N$. 

(ii) A $\varphi$-module over $\mathcal{R}$ is called étale if it is locally on $X$ induced from a locally free étale $\varphi$-module over $\mathcal{R}^+$. 

Recall that $K_\infty$ is a fixed cyclotomic extension of $K$ and $\Gamma = \text{Gal}(K_\infty/K)$ denotes the Galois group of $K_\infty$ over $K$.

**Definition 4.3.** Let $X \in \text{Ad}^\text{rig}_{Q_p}$ and $\mathcal{R}$ be any of the sheaves of rings defined above.

(i) A $(\varphi, \Gamma)$-module over $\mathcal{R}$ is a $\varphi$-module over $\mathcal{R}$ together with a continuous semi-linear action of $\Gamma$ commuting with the semi-linear endomorphism $\Phi$.

(ii) A $(\varphi, \Gamma)$-module over $\mathcal{R}$ is called étale if its underlying $\varphi$-module is étale.

4.1. **The étale locus.** If $X$ is an adic space (locally of finite type over $Q_p$) and $x \in X$ is any point, we will write $\iota_x : x \to X$ for the inclusion of $x$. If $\mathcal{R}$ is any of the sheaf of topological rings above and if $\mathcal{N}$ is a sheaf of $\mathcal{R}_X$-modules on $X$, we write

$$\iota_x^* \mathcal{N} = \iota_x^{-1} \mathcal{N} \otimes_{\mathcal{R}_X} \mathcal{R}_x$$

for the pullback of $\mathcal{N}$ to the point $x$. The following result is a generalization of [KL, Theorem 7.4] to the category of adic spaces.

**Theorem 4.4.** Let $X$ be a reduced adic space locally of finite type over $Q_p$ and $\mathcal{N}$ be a family of $(\varphi, \Gamma)$-modules over $\mathcal{B}^\dagger_{X,\text{rig}}$.

(i) The set 

$$X^\text{int} = \{ x \in X \mid \iota_x^* \mathcal{N} \text{ is étale} \} \subset X$$

is open.

(ii) There exists a covering $X^\text{int} = \bigcup U_i$ and locally free étale $\mathcal{A}^\dagger_{U_i}$-modules $N_i \subset \mathcal{N}|_{U_i}$ which are stable under $\Phi$ such that

$$N_i \otimes_{\mathcal{A}^\dagger_{U_i}} \mathcal{B}^\dagger_{U_i,\text{rig}} = \mathcal{N}|_{U_i},$$

i.e. $\mathcal{N}|_{X^\text{int}}$ is étale.

**Proof.** This is [He1, Corollary 6.11]. In loc. cit. we use a different Frobenius $\varphi$. However the proof works verbatim in the case considered here. \qed

**Theorem 4.5.** Let $f : X \to Y$ be a morphism of reduced adic spaces locally of finite type over $Q_p$. Let $\mathcal{N}_Y$ be a family of $(\varphi, \Gamma)$-modules over $\mathcal{B}^\dagger_{Y,\text{rig}}$ and write $\mathcal{N}_X$ for its pullback over $\mathcal{B}^\dagger_{X,\text{rig}}$. Then $f^{-1}(Y^\text{int}) = X^\text{int}$

**Proof.** This is [He1] Proposition 6.14]. Again the same proof applies with the Frobenius considered here. \qed

$^1$The integral subring of $\mathcal{B}^\dagger_{X,\text{rig}}$ is $\mathcal{A}^\dagger_{X}$. 
4.2. Existence of étale submodules. For later applications to Galois representations the existence of an étale lattice locally on \( X \) will not be sufficient. We cannot hope that the étale lattices glue together to a global étale lattice on the space \( X \). However we have a replacement which will be sufficient for applications.

**Convention:** Let \( X \) be a reduced adic space locally of finite type over \( \mathbb{Q}_p \) and let \( (N, \Phi) \) be an étale \( \varphi \)-module over \( \mathcal{B}_{X, K}^{\dagger, \text{rig}} \) and \( (\hat{N}, \hat{\Phi}) \) be an (étale) \( \varphi \)-module over \( \mathcal{A}_{X, K}^{\dagger} \). We say that \( (\hat{N}, \hat{\Phi}) \) is induced from \( (N, \Phi) \) if there exists a covering \( X = \bigcup U_i \) and étale \( \mathcal{A}_{U_i}^{\dagger} \) lattices \( N_i \subset N|_{U_i} \) such that

\[
(\hat{N}, \hat{\Phi})|_{U_i} = ((N_i, \Phi^\wedge)|_{U_i}^{|_{1/p}}).
\]

Note the every étale \( \varphi \)-module over \( \mathcal{B}_{X, K}^{\dagger, \text{rig}} \) gives rise to a unique \( \varphi \)-module over \( \mathcal{A}_{X, K}^{\dagger} \), as an étale \( \mathcal{A}_{X, K}^{\dagger} \)-lattice is unique up to \( p \)-isogeny.

**Proposition 4.6.** Let \( X \) be a reduced adic space of finite type and \( (\hat{N}, \hat{\Phi}) \) be a \( \varphi \)-module over \( \mathcal{B}_{X, K}^{\dagger, \text{rig}} \) which is induced from an étale \( \varphi \)-module \( (N, \Phi) \) over \( \mathcal{B}_{X, K}^{\dagger, \text{rig}} \). Then there exists an étale \( \varphi \)-submodule \( \hat{N} \subset \hat{N} \) over \( \mathcal{A}_{X, K}^{\dagger, \text{rig}} \) such that the inclusion induces an isomorphism after inverting \( p \).

**Proposition 4.7.** Let \( X \) be an reduced adic space of finite type over \( \mathbb{Q}_p \). Let \( N \) be a locally free \( \mathcal{B}_{X, K}^{\dagger, \text{rig}} \)-module, then there exists a coherent \( \mathcal{A}_{X, K}^{\dagger, \text{rig}} \)-submodule \( N \subset N \) which (locally on \( X \)) contains a basis of \( N \).

**Proof.** Let \( X = \bigcup_{i=1}^m U_i \) be a finite covering such that \( N|_{U_i} \) is free and write \( V_i = \bigcup_{j=1}^r U_j \). As obviously there exists an \( \mathcal{A}_{U_i}^{\dagger, \text{rig}} \)-lattice in \( N|_{U_i} \) it is enough to show that there is an extension of such a module from \( V_i \) to \( V_{i+1} \). This reduces the claim to the following lemma. \( \square \)

**Lemma 4.8.** Let \( X = \text{Spa}(A, A^+) \) be a reduced affinoid adic space and \( U \subset X \) an quasi-compact open subset. Let \( N_U \) be a finitely generated \( \mathcal{A}_{X, K}^{\dagger, \text{rig}} \)-submodule of \((\mathcal{B}_{X, K}^{\dagger, \text{rig}})^d\) which contains a basis. Then there exists a coherent \( \mathcal{A}_{X}^{\dagger, \text{rig}} \)-module \( N_X \subset (\mathcal{B}_{X, K}^{\dagger, \text{rig}})^d \) such that \( N_X \) contains a basis and such that \( N_X|_U = N_U \).

**Proof.** Let \( N', N'' \subset (\mathcal{B}_{X, K}^{\dagger, \text{rig}})^d \) be \( \mathcal{A}_{X}^{\dagger, \text{rig}} \)-lattices such that \( N''|_U \subset N_U \subset N'|_U \). After localizing we may assume that \( N' \) is free. Denote by \( j : U \hookrightarrow X \) the open embedding of \( U \). We define \( \tilde{N}_X \) by

\[
\tilde{N}_X = \ker(N' \longrightarrow (j \times \text{id})_*(N'_U/N_U)).
\]

This is easily seen to be a coherent sheaf on \( X \times U_{\geq p^{-1/r}} \). We define \( N_X \) by

\[
X \supset V \longmapsto \Gamma(V \times U_{\geq p^{-1/r}}, \tilde{N}_X).
\]
It is obvious that \( N'' \subset N_X \subset N' \) and hence \( N_X \) contains a basis of \( (\mathcal{A}^{1,r}_{X,\text{rig}})^d \). It remains to check that this sheaf is coherent. Let \( U = \bigcup U_i \) be a finite covering by open affinoids such that \( N_U \) is associated to a finitely generated \( \Gamma(U, \mathcal{A}^{1,r}_{X}) \)-module. Choose a covering \( X = \bigcup V_j \) by open affinoids such that \( V_j \cap U \subset U_{i_j} \) for some index \( i_j \). Then \( N_X \) is associated to the \( \Gamma(V_j, \mathcal{A}^{1,r}_{X}) \)-module

\[
\ker \left( \Gamma(V_j, N') \longrightarrow \Gamma(U_{i_j}, N_U/N_U) \otimes \Gamma(U_{i_j}, \mathcal{A}^{1,r}_{X}) \right) \Gamma(V_j \cap U, \mathcal{A}^{1,r}_{X}).
\]

Especially \( N_X \) is quasi-coherent. Finally \( N_X \) is coherent as the sections of \( \mathcal{A}^{1,r}_{X} \) are locally on \( X \) noetherian rings, and \( N_X \subset N' \).

**Proof of Proposition 4.6** As \( X \) is quasi-compact, we can choose a locally free model \( (N_r, \Phi_r) \) of \( (N, \Phi) \) over \( \mathcal{A}^{1,r}_{X,\text{rig}} \) for some \( r \gg 0 \). After enlarging \( r \) if necessary, we can assume that there exists a finite covering \( X = \bigcup U_i \) and étale lattices \( M_i \subset N_r|U_i \). By Proposition 4.7 there exist coherent \( \mathcal{A}^{1,r}_{X} \)-modules \( \tilde{M}_1 \subset N_0 \subset \tilde{M}_2 \subset N \) such that

\[
\tilde{M}_1|U_i \subset N_0|U_i \subset M_i \subset \tilde{M}_2|U_i.
\]

Let \( N_{r_i} \) denote the restriction of \( N_r \) to \( \mathcal{A}^{1,r}_{X,\text{rig}} \), where we write \( r_i = p^i r \).

Then we inductively define coherent \( \mathcal{A}^{1,r_{i+1}}_{X,\text{rig}} \)-modules \( N_i \subset N_{r_i} \) by setting

\[
N_{i+1} = N_i \otimes_{\mathcal{A}^{1,r_i}_{X}} \mathcal{A}^{1,r_{i+1}}_{X} + \Phi(\varphi^* N_i).
\]

By assumption, we always have

\[
\tilde{M}_1|U_i \otimes_{\mathcal{A}^{1,r_i}_{X}} \mathcal{A}^{1,r_{i+1}}_{X} \subset N_j|U_i \subset M_i \otimes_{\mathcal{A}^{1,r_i}_{X}} \mathcal{A}^{1,r_{i+1}}_{X} \subset \tilde{M}_2|U_i \otimes_{\mathcal{A}^{1,r_i}_{X}} \mathcal{A}^{1,r_{i+1}}_{X}.
\]

Viewing \( N_i \) as coherent sheaves on \( X \times \bigcup \mathbb{Z}^{p^{-1/r_i}} \), we now define an \( \mathcal{A}^{1}_X \)-submodule \( N \subset N' \), by setting

\[
N = \left( \lim_{\longrightarrow} \varprojlim_{i \in \mathbb{N}} N_i \right) \otimes \mathcal{A}^{1}_X,
\]

where \( \varprojlim_{i \in \mathbb{N}} N_i : X \times \bigcup \mathbb{Z}^{p^{-1/r_i}} \to X \) is the projection to the first factor. By construction, this module satisfies

\[
\Phi(\varphi^* N) \subset N,
\]

\[
\tilde{M}_1 \otimes_{\mathcal{A}^{1,r}_{X}} \mathcal{A}^{1}_X \subset N \subset \tilde{M}_2 \otimes_{\mathcal{A}^{1,r}_{X}} \mathcal{A}^{1}_X.
\]

We then take \( \tilde{N} \) to be the completion of \( N \) with respect to the \( p \)-adic topology. If the module we started with is associated to a finitely generated module over an affinoid open \( U \subset X \), then the construction implies that \( \tilde{N} \) is also associated with a \( \Gamma(U, \mathcal{A}^{1,r}_{X,K}) \)-module which is contained in the finitely generated module \( \Gamma(U, \tilde{M}_2 \otimes_{\mathcal{A}^{1,r}_{X}} \mathcal{A}^{1}_X) \) and hence has to be finitely generated.
itself. It follows that \( \hat{N} \) is coherent, as claimed. Further the construction implies that \( \hat{N} \) is a submodule of \( \hat{N} \) and

\[
\hat{\Phi}(\varphi^*\hat{N}) \subset \hat{N},
\]

\[
\hat{N} \otimes_{\mathcal{O}_{X,K}} \mathcal{O}_{X,K} = \hat{N}.
\]

We need to show that \( \hat{\Phi}(\varphi^*\hat{N}) = \hat{N} \). To do so, we can work locally on \( X \) and hence assume that \( \hat{N} \) is contained in an étale \( \mathcal{O}_{X,K} \)-lattice \( \hat{M} \subset \hat{N} \). Further is is enough to assume that \( X = \text{Spa}(A, A^+) \) is affinoid and we need to show that for all maximal ideals \( m \subset A^+ \) we have

\[
(4.1) \quad \hat{\Phi}(\varphi^*\hat{N}) \otimes_k m = \hat{N} \otimes_k m,
\]

where \( k_m = A^+/m \) denotes the residue field at \( m \).

For a rigid analytic point \( x \in X \), the fiber \( \hat{N} \otimes k(x)^+ \) is a finitely generated module over the ring \( \mathcal{O}_{X,K} \otimes k(x)^+ \) which is (a product of) complete discrete valuation rings. Write

\[
(\hat{N} \otimes k(x)^+)_{\text{tors-free}} \subset \hat{N} \otimes k(x)^+
\]

for the submodule which is \( \varpi_x \)-torsion free. This submodule has to be free and

\[
(\hat{N} \otimes k(x)^+)_{\text{tors-free}} \otimes_{\mathcal{O}_X} k_x = (\hat{M} \otimes k(x)^+) \otimes_{\mathcal{O}_X} k_x = \hat{N}.
\]

It follows from Lemma 4.9 below that \( (\hat{N} \otimes k(x)^+)_{\text{tors-free}} \) is an étale \( \varphi \)-module, i.e. \( \hat{\Phi} \) is surjective. We reduce the inclusion \( \hat{\Phi} : \varphi^*\hat{N} \hookrightarrow \hat{N} \) modulo \( m \) and obtain a morphism

\[
\hat{\Phi} : \varphi^*\hat{N} \rightarrow \hat{N}.
\]

Assume there exists \( 0 \neq \bar{f} \in \ker \hat{\Phi} \). Then there exists a rigid point \( x \in X \) such that \( m_x^+ \subset m \) and a lift \( f \) of \( \bar{f} \) in the torsion-free part of \( \hat{N} \otimes k(x)^+ \) such that \( f \notin \varpi_x(\hat{N} \otimes k(x)^+) \), as \( \hat{N} \) is \( \mathbb{Z}_p \)-flat. It follows that \( \hat{\Phi}(f) \notin \varpi_x(\hat{N} \otimes k(x)^+) \) and hence \( f \in \varpi_x(\hat{N} \otimes k(x)^+) \), as \( \hat{\Phi} \) induces an isomorphism

\[
\varphi^*\left(\hat{N} \otimes k(x)^+\right)_{\text{tors-free}} \rightarrow (\hat{N} \otimes k(x)^+)_{\text{tors-free}}.
\]

We have shown that \( \ker \hat{\Phi} = 0 \) and hence \( \hat{\Phi} \) is injective modulo all maximal ideals. By comparing dimension, we find that (4.1) holds true for all maximal ideals \( m \subset A^+ \), and hence \( \hat{\Phi} \) induces an isomorphism on \( \hat{N} \).

**Lemma 4.9.** Let \( F \) be a finite extension of \( \mathbb{Q}_p \) and \( (\hat{N}, \hat{\Phi}) \) be a free étale \( \varphi \)-module over \( \mathcal{O}_{F,K} \). Let \( \hat{N}_1 \subset \hat{N} \) be a finitely generated submodule such that \( \hat{N}_1[1/p] = \hat{N}[1/p] \) and \( \hat{\Phi}(\varphi^*\hat{N}_1) \subset \hat{N}_1 \). Then \( (\hat{N}_1, \hat{\Phi}) \) is an étale \( \varphi \)-module, i.e.

\[
\hat{\Phi} : \varphi^*\hat{N}_1 \rightarrow \hat{N}_1.
\]

**Proof.** As \( \mathcal{O}_{F,K} \) is (a product of) discrete valuation rings, it is clear that \( \hat{N}_1 \) is free on \( d \) generators, where \( d \) is the \( \mathcal{O}_{F,K} \)-rank of \( \hat{N} \). Let \( b_1, \ldots, b_d \) be a basis of \( \hat{N} \) and \( e_1, \ldots, e_d \) be a basis of \( \hat{N}_1 \). Let \( A \) denote the change of basis...
matrix from $b$ to $e$ and denote by $\text{Mat}_b(\hat{\Phi})$ resp. $\text{Mat}_e(\hat{\Phi})$ the matrix of $\hat{\Phi}$ in the basis $\hat{b}$ resp. $\hat{e}$ of $\hat{N}[1/p] = \hat{N}_1[1/p]$. Then our assumptions imply that

$$\text{Mat}_e(\hat{\Phi}) \in \text{Mat}_{d \times d}(\mathcal{A}_{F,K}).$$

On the other hand

$$\text{Mat}_e(\hat{\Phi}) = A^{-1} \text{Mat}_b(\hat{\Phi}) \varphi(A)$$

and hence $\det \text{Mat}_e(\hat{\Phi}) \in \mathcal{A}_{F,K}^\times$, as $\hat{N}$ is étale and

$$\text{val}_p(\det A) = \text{val}_p(\det \varphi(A)).$$

□

5. Families of $p$-adic Galois representations

In this section we study the relation between Galois representations and $(\varphi, \Gamma)$-modules in families. This problem was first considered by Berger and Colmez in [BC], where they define a functor from families of $G_K$-representations to families of overconvergent $(\varphi, \Gamma)$-modules.

**Definition 5.1.** Let $G$ a topological group and $X$ an adic space locally of finite type over $\mathbb{Q}_p$. A family of $G$-representations over $X$ is a vector bundle $\mathcal{V}$ over $X$ endowed with a continuous $G$-action.

We write $\text{Rep}_X G$ for the category of families of $G$-representations over $X$. Recall that we write $G_K = \text{Gal}(\overline{K}/K)$ for the absolute Galois group of a fixed local field $K$. In this case Berger and Colmez define the functor

$$D^\dagger : \text{Rep}_X G_K \longrightarrow \{\text{étale } (\varphi, \Gamma)\text{-modules over } \mathcal{B}_X^\dagger\},$$

which maps a family $\mathcal{V}$ of $G_K$-representations on $X$ to the étale $(\varphi, \Gamma)$-module

$$D^\dagger(\mathcal{V}) = (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{B}_X^\dagger)^{H_K}.$$

More precisely they construct this functor if $X$ is a reduced affinoid adic space of finite type. As the functor $D^\dagger$ is fully faithful in this case and maps $\mathcal{V}$ to a free $\mathcal{B}_X^\dagger$-module it follows that we can consider $D^\dagger$ on the full category $\text{Rep}_X G_K$, whenever $X$ is reduced. In the following we will always assume that $X$ is reduced.

We will consider the variant

$$D^\dagger_{\text{rig}} : \mathcal{V} \mapsto (\mathcal{V} \otimes_{\mathcal{O}_X} \mathcal{B}_{X,\text{rig}}^\dagger)^{H_K} = D^\dagger(\mathcal{V}) \otimes_{\mathcal{B}_X^\dagger} \mathcal{B}_{X,\text{rig}}^\dagger.$$

Note that for an adic space $X$ of finite type over $\mathbb{Q}_p$, the $(\varphi, \Gamma)$-module $D^\dagger(\mathcal{V})$ is always defined over some $\mathcal{B}_X^{1,s} \subset \mathcal{B}_X^\dagger$, for $s \gg 0$. Especially an étale lattice can be defined over $\mathcal{B}_X^{1,s}$ for $s \gg 0$. 
5.1. The admissible locus. It is known that the functors $D^\dagger$ and $D^\dagger_{\text{rig}}$ are not essentially surjective. In [KL], Kedlaya and Liu construct a local inverse to this functor. More precisely, they show that if $\mathcal{N}$ is a family of $(\varphi, \Gamma)$-modules over $\mathcal{B}^\dagger_{X,\text{rig}}$, then every rigid analytic point at which $\mathcal{N}$ is étale has an affinoid neighborhood on which the family $\mathcal{N}$ is the image of a family of $G_K$-representations. However we need to extend this result to the setup of adic spaces in order to define a natural subspace over which such a family $\mathcal{N}$ is induced by a family of $G_K$-representations.

Theorem 5.2. Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and let $\mathcal{N}$ be a family of $(\varphi, \Gamma)$-modules of rank $d$ over $\mathcal{B}^\dagger_{X,\text{rig}}$.

(i) The subset $X^{\text{adm}} = \{ x \in X \mid \dim_{k(x)} ((\mathcal{N} \otimes \mathcal{B}^\dagger_{X,\text{rig}}) \otimes k(x))^{\Phi=\text{id}} = d \}$ is open.

(ii) There exists a family of $G_K$-representations $\mathcal{V}$ on $X^{\text{adm}}$ such that there is a canonical and functorial isomorphism

$$D^\dagger_{\text{rig}}(\mathcal{V}) \cong \mathcal{N}|_{X^{\text{adm}}}.$$ 

(iii) Let $\mathcal{V}$ be a family $G_K$-representations on $X$ such that $D^\dagger_{\text{rig}}(\mathcal{V}) = \mathcal{N}$. Then $X^{\text{adm}} = X$.

Let $A$ be a complete topological $\mathbb{Q}_p$-algebra and let $A^+ \subset A$ be a ring of integral elements. Assume that the completed tensor products $A^+ \hat{\otimes} \hat{\mathbb{A}}^\dagger$ and $A \hat{\otimes} \mathbb{B}^\dagger_{\text{rig}}$ are defined. In this case the following approximation Lemma of Kedlaya and Liu applies.

Lemma 5.3. Let $\hat{\mathcal{N}}$ be a free $(\varphi, \Gamma)$-module over $A \hat{\otimes} \mathbb{B}^\dagger_{\text{rig}}$ such that there exists a basis on which $\Phi$ acts via $\text{id} + B$ with

$$B \in p \text{Mat}_{d \times d}(A^+ \hat{\otimes} \hat{\mathbb{A}}^\dagger).$$

Then $\hat{\mathcal{N}}^{\Phi=\text{id}}$ is free of rank $d$ as an $A$-module.

**Proof.** This is [KL, Theorem 5.2].

Corollary 5.4. Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$ and $\hat{\mathcal{N}}$ be a family of $(\varphi, \Gamma)$-modules over $\mathcal{B}^\dagger_{X,\text{rig}}$. Let $x$ in $X$, then

$$\dim_{k(x)}(\iota_x^* \hat{\mathcal{N}})^{\Phi=\text{id}} = d \iff \dim_{k(x)} ((\mathcal{N} \otimes \mathcal{B}^\dagger_{X,\text{rig}}) \otimes k(x))^{\Phi=\text{id}} = d.$$

**Proof.** The proof is the same as the proof of [He1, Proposition 8.20 (i)].

---

2The examples we consider here, are $\Gamma(X, \mathcal{O}_X)$ for an affinoid adic space of finite type and the completions of $k(x)$ for a point $x \in X$. In the latter case the completed tensor product is the completion of the fiber of $\mathcal{O}^\dagger$ resp. $\mathcal{B}^\dagger_{\text{rig}}$ at the point $x$. 

Proof of Theorem 5.2. Let \( x \in X^{\mathrm{adm}} \) and denote by \( Z \) the Zariski-closure of \( x \), that is, the subspace defined by the ideal of all functions vanishing at \( x \). This is an adic space locally of finite type. Let \( U \subseteq Z \) be an affinoid neighborhood of \( x \) in \( Z \) such that a basis of the \( \Phi \)-invariants extends to \( U \). It follows from Lemma 5.3 that

\[
\mathcal{V}_U = (\mathcal{N}|_Z \otimes_{\hat{\mathcal{B}}_{U,\mathrm{rig}}^\dagger} \hat{\mathcal{B}}_{U,\mathrm{rig}}^\dagger)_{\Phi = \mathrm{id}}
\]

is free of rank \( d \) over \( \mathcal{O}_U \). On this sheaf we have the diagonal \( G_K \)-action given by the natural action on \( \hat{\mathcal{B}}_{U,\mathrm{rig}}^\dagger \) and the \( \Gamma \)-action on \( \mathcal{N} \). It is a direct consequence of the construction that

\[
D^\dagger_{\mathrm{rig}}(\mathcal{V}_U) = \mathcal{N}|_U.
\]

Especially we have shown that \( X^{\mathrm{adm}} \subseteq X^{\mathrm{int}} \).

Now let \( x \in X^{\mathrm{adm}} \) and let \( U \) denote a neighborhood of \( x \) to which we can lift a basis of \( \Phi \)-invariants. As \( \mathcal{N} \) is known to be étale, we can shrink \( U \) such that we are in the situation of Lemma 5.3.

It follows that \( X^{\mathrm{adm}} \) is open and that

\[
(N \otimes_{\hat{\mathcal{B}}_{X,\mathrm{rig}}^\dagger} \hat{\mathcal{B}}_{X,\mathrm{rig}}^\dagger)_{\Phi = \mathrm{id}}
\]

gives a vector bundle \( \mathcal{V} \) on \( X^{\mathrm{adm}} \). Again, we have the diagonal action of \( G_K \). As above we find that

\[
D^\dagger_{\mathrm{rig}}(\mathcal{V}) = \mathcal{N}|_{X^{\mathrm{adm}} \cap X^{\mathrm{int}}}.
\]

Finally (iii) is obvious by the construction of [BC].

Theorem 5.5. Let \( f : X \to Y \) be a morphism of adic spaces locally of finite type over \( \mathbb{Q}_p \) with \( Y \) reduced. Further let \( \mathcal{N}_Y \) be a family of \( (\varphi, \Gamma) \)-modules over \( \hat{\mathcal{B}}_{Y,\mathrm{rig}}^\dagger \) and write \( \mathcal{N}_X \) for the pullback of \( \mathcal{N}_Y \) to \( X \). Then \( f^{-1}(Y^{\mathrm{adm}}) = X^{\mathrm{adm}} \) and \( f^*\mathcal{N}_Y = \mathcal{V}_X \) on \( X^{\mathrm{adm}} \).

Proof. Using the discussion above, the proof is the same as the proof of [He1, Proposition 8.22].

Proposition 5.6. Let \( X \) be a reduced adic space locally of finite type over \( \mathbb{Q}_p \) and let \( \mathcal{N} \) be a family of \( (\varphi, \Gamma) \)-modules over \( \hat{\mathcal{B}}_{X,\mathrm{rig}}^\dagger \). Then the inclusion

\[
f : X^{\mathrm{adm}} \to X
\]

is open and partially proper.

Proof. We have already shown that \( f \) is open. Especially it is quasi-separated and hence we may apply the valuative criterion for partial properness, see [Hu, 1.3]. Let \( (x, A) \) be a valuation ring of \( X \) with \( x \in X^{\mathrm{adm}} \) and let \( y \in X \) be a center of \( A, x \). We need to show that \( y \in X^{\mathrm{adm}} \). As \( y \) is a specialization
of $x$, the inclusion $i : k(y) \hookrightarrow k(x)$ identifies $k(y)$ with a dense subfield of $k(x)$. Especially

$$
\tilde{N}_y := N \otimes_{\mathfrak{B}_{k(y),\text{rig}}} \tilde{\mathfrak{B}}_{k(y),\text{rig}} \to N \otimes_{\mathfrak{B}_{k(x),\text{rig}}} \tilde{\mathfrak{B}}_{k(x),\text{rig}} =: \tilde{N}_x
$$

is dense. Let $e_1, \ldots, e_d$ be a basis of $\tilde{N}_x$ on which $\Phi$ acts as the identity. We may approximate this basis by a basis of $\tilde{N}_y$. Thus we can choose a basis of $\tilde{N}_y$ on which $\Phi$ acts by $\text{id} + A$ with

$$
A \in \text{Mat}_{d \times d}(\tilde{\mathfrak{B}}_{k(y),\text{rig}})
$$
sufficiently small. For example we can choose

$$
A \in \mathfrak{p} \text{Mat}_{d \times d}(\tilde{\mathfrak{B}}_{k(y)})
$$

By Lemma 5.3 and Corollary 5.4 it follows that $y \in X^{\text{adm}}$. □

5.2. Existence of Galois representations. In this section we link deformations of Galois representations and deformations of étale $\varphi$-modules.

In the following $(R, \mathfrak{m})$ will denote a complete local noetherian ring, topologically of finite type over $\mathbb{Z}_p$. As above we have the notion of an étale $\varphi$-module over $\mathfrak{m}$.

A Galois representation with coefficients in $R$ (or a family of Galois representations on $\text{Spf} R$) is a continuous representation

$$
G \to \text{GL}_d(R),
$$

where $G$ is the absolute Galois group of some field $L$. The relation between Galois representations and étale $\varphi$-modules with coefficients in local rings was first considered by Dee, see [Dec, 2].

**Theorem 5.7.** Let $X$ be a reduced adic space of finite type over $\mathbb{Q}_p$ and and let $(N, \Phi)$ be a family of étale $\varphi$-modules over $\mathfrak{B}_{X,\text{rig}}$. Let $x_0 \in \bar{X}$ be a closed point in the special fiber of some formal model $\bar{X}$ of $X$ and let $Y \subset X$ denote the tube of $x_0$. Then $(N, \Phi)|_Y$ is associated to a family of $H_K$-representations on the open subspace $Y$.

**Proof.** It follows from Proposition 4.6 that there exist an étale $\varphi$-module $\tilde{N}$ over $\mathfrak{A}_{X,K}$ such that $\tilde{N} \subset \bar{N}$ as $\varphi$-modules and such that $\tilde{N}$ contains a basis of $\bar{N}$. Here $\bar{N}$ is the $\mathfrak{B}_{X,K}$-module induced from $N$. Choose an affine neighborhood $U = \text{Spf}(A^+)$ of $x_0$ and write $U$ for its generic fiber. We write $\mathfrak{m} \subset A^+$ for the maximal ideal defining $x_0$ and write $\mathfrak{m}$ for the $\mathfrak{m}$-adic completion of $A^+$. Then $Y$ is the generic fiber of $\text{Spf} R$. Write $\mathfrak{R} = \Gamma(U, \tilde{N})$, then this is a finitely generated $\Gamma(U, \mathfrak{A}_{X,K})$-module on which $\tilde{\Phi}$ induces a semi-linear isomorphism. Especially $\mathfrak{R} = \Gamma(U, \tilde{N})$ is a finitely generated étale $\varphi$-module over $\Gamma(Y, \mathfrak{A}_{X,K}) = R \otimes_{\mathbb{Z}_p} A_K$. Hence, by [Dec], there is a
finitely generated $R$-module $E$ with continuous $H_K$ action associated with $\mathfrak{R}$. Then

$$Y \supset V \mapsto E \otimes_R \Gamma(V, \mathcal{O}_X)$$

defines the desired family of Galois representations on $Y$. \hfill \Box

**Corollary 5.8.** Let $X$ be a reduced adic space locally of finite type over $\mathbb{Q}_p$ and $\mathcal{N}$ be a family of étale $(\varphi, \Gamma)$-modules on $X$. Let $x_0 \in X$ be a closed point in the special fiber of some formal model $\mathcal{X}$ of $X$ and let $Y \subset X$ denote the tube of $x_0$. Then $\mathcal{N}|_Y$ is associated to a family of $G_K$-representations on the open subspace $Y$.

**Proof.** By the above theorem it follows that $Y = Y^{\text{adm}}$. The claim follows from Theorem 5.2. \hfill \Box

**Conjecture 5.9.** The claim of the theorem (and the corollary) also holds true if we replace $x_0$ by a (locally) closed subscheme of the special fiber over which there exists a Galois representation that is locally associated with the reduction of an étale lattice.

### 5.3. Local constancy of the reduction modulo $p$.

Let $L$ be a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$, uniformizer $\varpi_L$ and residue field $k_L$. Let $V$ be a $d$-dimensional $L$-vector space with a continuous action of a compact group $G$. We choose a $G$-stable $\mathcal{O}_L$-lattice $\Lambda \subset V$ and write $\overline{\Lambda} = \Lambda / \varpi_L \Lambda$ for the reduction modulo the maximal ideal of $\mathcal{O}_L$. Then $\overline{\Lambda}$ is a (continuous) representation of $G$ on a $d$-dimensional $k_L = \mathcal{O}_L / \varpi_L \mathcal{O}_L$-vector space. The representation $\overline{\Lambda}$ depends on the choice of a $G$-stable lattice $\Lambda \subset V$, however it is well known that its semisimplification $\overline{\Lambda}^{\text{ss}}$ (i.e. the direct sum of its Jordan-Hölder constituents) is independent of $\Lambda$ and hence only depends on the representation $V$. In the following we will write $\overline{V}$ for this representation and refer to it as the reduction modulo $\varpi_L$ of the representation $V$.

The aim of this section is to show that the reduction modulo $\varpi_L$ is locally constant in a family of $p$-adic representations of $G$. In the context of families of Galois representations this was shown by Berger for families of 2-dimensional crystalline representations of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ in a weaker sense: Berger showed that every rigid analytic point has a neighborhood on which the reduction is constant, see [Be2].

Let $X$ be an adic space locally of finite type over $\mathbb{Q}_p$ and $E$ a vector bundle on $X$ endowed with a continuous $G$-action. If $x \in X$, then we write

$$\left( E \otimes k(x) \right)^{\text{ss}}$$

Note that we do not claim that locally on $Y$ the integral representation $E$ is associated with an étale lattice in $(\mathcal{N}, \Phi)$. This is only true up to $p$-isogeny.

4This seems to be a well known fact, at least in the context of pseudo-characters. As we do not want to assume $p > d$ here, we give a different proof.
for the semisimplification of the $G$-representation in the special fiber $\overline{k(x)} = k(x)^+/(\varpi x)$ of $k(x)$.

We first claim that (up to semisimplification) there are no nontrivial families of representations of a finite group on varieties over $\mathbb{F}_p$.

**Proposition 5.10.** Let $H$ be a finite group. Let $X$ be a connected $\mathbb{F}_p$-scheme of finite type and $\mathcal{E}$ a vector bundle on $X$ endowed with an $H$-action. Then there is a semi-simple $H$-representation $\mathcal{E}$ on a finite dimensional $\overline{\mathbb{F}}_p$-vector space such that for all $x \in X$ there is an isomorphism of $H$-representations

$$(\mathcal{E} \otimes k(x))^\text{ss} \cong \mathcal{E}.$$  

**Proof.** For $h \in H$ consider the morphism

$$f_h : x \mapsto \text{charpoly}(h|\mathcal{E} \otimes k(x)) \in k(x)^d,$$

mapping $x \in X$ to the coefficients of the characteristic polynomial of $h$ acting on $\mathcal{E} \otimes k(x)$, where $d$ is the rank of $\mathcal{E}$. This gives a morphism of schemes $X \to \mathbb{A}^d$. As there are only finitely many isomorphism classes of semi-simple $H$-representations of fixed rank (there are only finitely many irreducible representations), this map has finite image and hence it has to be constant, as $X$ is connected. It follows that for all $h \in H$ we have the equality

$$\text{charpoly}(h|\mathcal{E} \otimes k(x)) = \text{charpoly}(h|\mathcal{E} \otimes k(y))$$

for all $x, y \in X$. Then [CR, Theorem 30.16] implies the claim. \hfill $\square$

**Lemma 5.11.** Let $X$ be an adic space locally of finite type and $\mathcal{E}$ be a vector bundle on $X$ endowed with a continuous action of a compact group $G$. Then locally on $X$ there exists a $G$-stable $\mathcal{O}_X^+$-lattice $E^+ \subset \mathcal{E}$.

**Proof.** We may assume that $\mathcal{E} \cong \mathcal{O}_X^d$ is trivial and hence there is a lattice $E_1^+ = (\mathcal{O}_X^d)^d \subset \mathcal{E}$. As $G$ is compact the entries of the matrices of $g \in G$ acting on the standard basis have a common bound. Hence the $\mathcal{O}_X^+$-submodule $E^+ \subset \mathcal{E}$ which is generated by the $G$-translates of $E_1^+$ is contained in $p^{-N}E_1^+$ for some large integer $N$. Especially it is coherent. We need to show that it is a lattice, and hence by Lemma 3.6 we only need to show that the stalks are torsion free. But if $e_1, \ldots, e_d$ are generators of $(k(x)^+)^d = E_1^+ \otimes k(x)^+$, then the translates of $e_1, \ldots, e_d$ under the action of $G$ generate the stalk $E^+ \otimes k(x)^+$. It follows that the stalks are torsion free. \hfill $\square$

**Corollary 5.12.** Let $X$ be an adic space locally of finite type and let $E$ be a vector bundle on $X$ endowed with a continuous action of a compact group $G$. Then the semi-simplification of the reduction $E \otimes \overline{k(x)}$ is locally constant.

**Proof.** As the statement is local on $X$, we may assume that $X$ is quasi-compact and admits a $G$-stable $\mathcal{O}_X^+$-lattice $E^+ \subset \mathcal{E}$. Let $\mathcal{X}$ be a $\mathbb{Z}_p$-flat formal model of $X$ such that there exists a model $\mathcal{E}^+$ of $E^+$ on $\mathcal{X}$. Then
defines a continuous $G$-representations on the special fiber $\tilde{E}^+$ which is a vector bundle on the special fiber $\tilde{X}$ of $X$. As $G$ is compact and the representations is continuous, the representation on $\tilde{E}^+$ has to factor over some finite quotient $H$ of $G$. Now the claim follows from Proposition 5.10, as $X$ is connected if and only if $\tilde{X}$ is connected.

6. A REMARK ON SLOPE FILTRATIONS

In this section we give an explicit example of a family $(\mathcal{N}, \Phi)$ of $\varphi$-modules over the relative Robba ring which is not étale, but étale at all rigid analytic points (and hence $(\mathcal{N}, \Phi)$ is purely of slope zero). For this section we use different notations. Let $K$ be a totally ramified quadratic extension of $\mathbb{Q}_p$. Fix a uniformizer $\pi \in \mathcal{O}_K$ and a compatible system $\pi_n \in \bar{K}$ of $p^n$-th roots of $\pi$. Let us write $K^\infty = \bigcup K(\pi_n)$ and $G_{K^\infty} = \text{Gal}(\bar{K}/K^\infty)$ for this section. Further let $E(u) \in \mathbb{Z}_p[u]$ denote the minimal polynomial of $\pi$. Finally we adapt the notation from [He1] and write

$$B_R^X = B^\dagger_X, \text{rig} \quad \text{and} \quad B^{(0,1)}_X = \text{pr}_{X,\ast} \mathcal{O}_X \times U.$$  

We consider the following family $(D, \Phi, \mathcal{F}^\bullet)$ of filtered $\varphi$-modules on $X = \mathbb{P}^1_K \times \mathbb{P}^1_K$. Let $D = \mathcal{O}_X^2 = \mathcal{O}_X e_1 \oplus \mathcal{O}_X e_2$ and $\Phi = \text{diag}(\varpi_1, \varpi_2)$, where $\varpi_1$ and $\varpi_2$ are the zeros of $E(u)$. We consider a filtration $\mathcal{F}^\bullet$ of $D_K = D \otimes_{\mathbb{Q}_p} K$ such that $\mathcal{F}^0 = D_K$ and $\mathcal{F}^2 = 0$. Fix an isomorphism $D \otimes_{\mathbb{Q}_p} K \cong \mathcal{O}_X^2 \oplus \mathcal{O}_X^2$ and let the filtration step $\mathcal{F}^1$ be the universal subspace on $X$. This is a family of filtered $\varphi$-modules in the sense of [He1]. One easily computes that $X^\text{wa} = X \setminus \{(0,0), (\infty, \infty)\}$, where $X^\text{wa} \subset X$ is the weakly admissible locus defined in [He1 4.2]. Generalizing a construction of Kisin [Ki] the family $(D, \Phi, \mathcal{F}^\bullet)$ defines a family $(\mathcal{M}, \Phi)$ consisting of a vector bundle on $X^\text{wa} \times \mathbb{U}$ and an injection $\Phi : \varphi^\ast \mathcal{M} \to \mathcal{M}$ such that $E(u) \text{coker} \Phi = 0$ (see [He1 Theorem 5.4]).

We define the family $(\mathcal{N}, \Phi)$ over $B^R_{X^\text{wa}}$ as

$$(\mathcal{N}, \Phi) = (\mathcal{M}, \Phi) \otimes_{B^{(0,1)}_{X^\text{wa}}} B^R_{X^\text{wa}}.$$  

This is obviously a family of $\varphi$-modules over the Robba ring which is étale at all rigid analytic points.

**Proposition 6.1.** The family $(\mathcal{N}, \Phi)$ over $B^R_{X^\text{wa}}$ defined in (6.1) is not étale.

**Lemma 6.2.** There exists a covering $X^\text{wa} = \bigcup U_i$, where each $U_i$ is a closed disc or a closed annulus. Further there exists $x_i$ in the special fiber of the canonical formal model of $U_i$ such that $X^\text{wa} = \bigcup V_i$, where $V_i \subset U_i$ is the tube of $x_i$. 


Proof. We can cover the weakly admissible set $X^{wa} = X_1 \cup X_2 \cup X_3 \cup X_4$, where

$$X_1 = ((\mathbb{P}^1 \setminus \{\infty\}) \times (\mathbb{P}^1 \setminus \{\infty\})) \setminus \{(0,0)\} \cong \mathbb{A}^2 \setminus \{0\},$$

$$X_2 = (\mathbb{P}^1 \setminus \{\infty\}) \times (\mathbb{P}^1 \setminus \{0\}) \cong \mathbb{A}^2,$$

$$X_3 = (\mathbb{P}^1 \setminus \{0\}) \times (\mathbb{P}^1 \setminus \{\infty\}) \cong \mathbb{A}^2,$$

$$X_4 = ((\mathbb{P}^1 \setminus \{0\}) \times (\mathbb{P}^1 \setminus \{0\})) \setminus \{(\infty, \infty)\} \cong \mathbb{A}^2 \setminus \{0\}.$$ 

Then obviously each of the $X_i$ can be covered by open subsets of the form

$$U \cong \{x \in \mathbb{A}^2 | ||x|| \leq s\} \text{ for some } s > 0,$$

$$U \cong \{x = (x_1, x_2) \in \mathbb{A}^2 | |x_1| \leq s_1, s_1 \leq |x_2|\} \text{ for some } 0 < s_1 \leq s_2,$$

$$U \cong \{x = (x_1, x_2) \in \mathbb{A}^2 | |x_1| \leq s_2, s_1 \leq |x_2|\} \text{ for some } 0 < s_1 \leq s_2,$$

with $s, s_1, s_2 \in p^\mathbb{Q}$.

Choose a covering $U_i$ of $X^{wa}$, where each $U_i$ is of the form described above, and let $\mathcal{U}_i$ be the canonical formal model of $U_i$ with special fiber $\mathbb{A}^2$ resp. $\mathbb{A}^1 \times (\mathbb{A}^1 \cup \mathbb{A}^1)$. Here the two affine lines $\mathbb{A}^1 \cup \mathbb{A}^1$ are glued together along the zero section. Let $V_i \subset U_i$ be the tube of the zero section. Then the $V_i$ also cover $X^{wa}$. □

Claim: If the family $(\mathcal{N}, \Phi)$ over $X^{wa}$ defined by (6.1) was étale, then there would exist a family of crystalline $G_K$-representations $\mathcal{E}$ on $X^{wa}$ such that

$$D_{\text{cris}}(\mathcal{E}) = (D, \Phi, \mathcal{F}^\bullet).$$

Proof of claim. Assume that the family $(\mathcal{N}, \Phi)$ is étale. By Theorem 6.4 there exists a family of $G_{K,\infty}$-representations on each of the $V_i$ which gives rise to the restriction of $\mathcal{M} \otimes \mathcal{B}_X^R$ to $V_i$. Hence, by [He1] Theorem 8.25, there exists a family of crystalline $G_K$-representations on each of the $V_i$ giving rise to the restriction of our family of filtered isocrystals to $V_i$. By the construction in [He1] this family is naturally contained in $D \otimes \mathcal{O}_{V_i}(\mathcal{O}_{V_i} \hat{\otimes} B_{\text{cris}})$ and in fact identified with

$$\text{Fil}^0(D \otimes \mathcal{O}_{V_i}(\mathcal{O}_{V_i} \hat{\otimes} B_{\text{cris}}))_{\Phi=\text{id}}.$$ 

Further the space $X^{wa}$ can be covered by the $V_i$. It follows that we can glue these families of crystalline representations to a family $\mathcal{E}$ of crystalline $G_K$-representations on $X^{wa}$ such that $D_{\text{cris}}(\mathcal{E}) = (D, \Phi, \mathcal{F}^\bullet)$. □

Proof of Proposition 6.1 Assume by way of contradiction that the family $(\mathcal{N}, \Phi)$ is étale. By the above claim there exists a family of $G_K$-representations $\mathcal{E}$ on $X^{wa}$ such that $V_{\text{cris}}(D \otimes k(x)) = \mathcal{E} \otimes k(x)$ for all $x \in X$.

\footnote{Strictly speaking Theorem 6.7 does not apply here, as we use a slightly different kind of Frobenius in [He1]. However the proof of the theorem works verbatim with the Frobenius used in loc. cit.}
Now the space $X^{\text{wa}}$ contains $K$-valued points $x_1$, $x_2$ and $x_3$ such that

$$(M, \Phi) \otimes k(x_1) \cong \left( \mathcal{O}_{U_K}^2, \begin{pmatrix} 0 & -E(u) \\ 1 & \omega_1 + \omega_2 \end{pmatrix} \right)$$

$$(M, \Phi) \otimes k(x_2) \cong \left( \mathcal{O}_{U_K}^2, \begin{pmatrix} 0 & -(u - \omega_1) \\ (u - \omega_2) & \omega_1 + \omega_2 \end{pmatrix} \right)$$

$$(M, \Phi) \otimes k(x_3) \cong \left( \mathcal{O}_{U_K}^2, \begin{pmatrix} -(u - \omega_1) & 0 \\ 0 & -(u - \omega_2) \end{pmatrix} \right).$$

The semi-simplifications of the reduction modulo $\pi$ of these $\varphi$-modules are

$$(M, \Phi) \otimes \overline{k(x_1)} \cong \left( \mathbb{F}_p[[u]]^2, \begin{pmatrix} 0 & -u^2 \\ 1 & 0 \end{pmatrix} \right),$$

$$(M, \Phi) \otimes \overline{k(x_2)} \cong \left( \mathbb{F}_p[[u]]^2, \begin{pmatrix} 0 & -u \\ u & 0 \end{pmatrix} \right),$$

$$(M, \Phi) \otimes \overline{k(x_3)} \cong \left( \mathbb{F}_p[[u]]^2, \begin{pmatrix} -u & 0 \\ 0 & -u \end{pmatrix} \right).$$

Using Caruso’s classification [Ca, Corollary 8] of those $\varphi$-modules we find that they are all non-isomorphic. After inverting $u$ these $\varphi$-modules correspond (up to twist) under Fontaine’s equivalence of categories to the restriction to $G_{K_\infty}$ of the reduction modulo $\pi$ of the constructed Galois representations $\mathcal{E} \otimes k(x_i)$. By [Br, Theorem 3.4.3] this restriction is fully faithful and hence we find that

$$\overline{\mathcal{E} \otimes k(x_1)} \not\cong \overline{\mathcal{E} \otimes k(x_j)}$$

as $G_K$-representations for $i \neq j$.

However, by Proposition 5.12 we know that the reduction modulo $p$ of the $G_K$-representation on the fibers of the family $\mathcal{E}$ has to be constant. This is a contradiction, and hence the family $(N, \Phi)$ is not étale. \(\square\)

Remark 6.3. This situation seems to be typical for the weakly admissible locus in the fibers over some fixed Frobenius $\Phi$. If we fix the Frobenius (or even its semi-simplification) then the main result of [He2] shows that the weakly admissible locus $X^{\text{wa}}$ is a Zariski open subset of some flag variety $X$. It should be possible to cover $X^{\text{wa}}$ by open subspaces $U_i$ such that there exists formal models $U_i$ and (locally) closed subschemes $Z_i \subset U_i$ such that the tubes $V_i \subset U_i$ of $Z_i$ also cover $X^{\text{wa}}$. If Conjecture 5.9 holds true, then the same argument as in this section shows that the weakly admissible family is not étale unless all residual representations are isomorphic.

This result also clarifies the relation between certain subspaces of a stack of filtered $\varphi$-modules considered in [He1]. In loc. cit. we construct three open substacks of the stack $\mathcal{D}$ of filtered $\varphi$-modules

$$\mathcal{D}^{\text{adm}} \subset \mathcal{D}^{\text{int}} \subset \mathcal{D}^{\text{wa}} \subset \mathcal{D}.$$
The stack $\mathcal{D}^{\text{wa}}$ parametrizes those families of filtered $\varphi$-modules which are weakly admissible, while the stack $\mathcal{D}^{\text{int}}$ parametrizes those filtered $\varphi$-modules such that the associated family of $\varphi$-modules on the open unit disc admits an étale lattice after restriction to the (relative) Robba ring. Finally $\mathcal{D}^{\text{adm}}$ is the maximal (open) subspace over which there exists a family of crystalline Galois representations giving rise to the restriction of the universal family of filtered $\varphi$-modules. In [He1], the stacks $\mathcal{D}^{\text{int}}$ and $\mathcal{D}^{\text{adm}}$ are only constructed in the case of Hodge-Tate weights in $\{0, 1\}$.

As already stressed in [He1], the inclusion $\mathcal{D}^{\text{adm}} \subset \mathcal{D}^{\text{int}}$ is strict, as is already shown by the family of unramified characters for example. The above example shows that the inclusion $\mathcal{D}^{\text{int}} \subset \mathcal{D}^{\text{wa}}$ is strict as well.

References

[Be1] L. Berger, *Représentations $p$-adiques et équations différentielles*, Inv. Math. 148 (2002), 219-284.
[Be2] L. Berger, *Local constancy for the reduction mod $p$ of 2-dimensional crystalline representations*, preprint 2010.
[BC] L. Berger, P. Colmez, *Familles de représentations de de Rham et monodromie $p$-adique*, Astérisque 319 (2008), 303-337.
[Br] C. Breuil, *Integral $p$-adic Hodge theory*, Algebraic Geometry 2000, Azumino, Adv. Studies in Pure Math. 36, 2002, 51-80.
[Ca] X. Caruso, *Sur la classification de quelques $\varphi$-modules simples*, Moscow Math. J. 9 (2009), no. 3, 562-568.
[CR] C. W. Curtis, I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Wiley Classics Libary.
[Dec] J. Dec, $\varphi$-$\Gamma$-modules for families of Galois representations, Journal of Algebra 235 (2001), 636-664.
[EGA I] A. Grothendieck, J. Dieudonne, *Eléments de géométrie algébrique I*, Publ. Math. IHES 4, 1960.
[He1] E. Hellmann, *On arithmetic families of filtered $\varphi$-modules and crystalline representations*, preprint 2011, [arXiv:1010.4577v2](http://arxiv.org/).
[He2] E. Hellmann, *On families of weakly admissible filtered $\varphi$-modules and the adjoint quotient of $GL_d$*, Documenta Math. 16 (2011), 969-991.
[Hu] R. Huber, *Étale Cohomology of rigid analytic varieties and adic spaces*, Aspects of Math. 30, Vieweg & Sohn, 1996.
[Ke] K. Kedlaya, *Slope filtrations for relative Frobenius*, in: Représentations $p$-adiques I: représentations galoisiennes et $(\varphi, \Gamma)$-modules, Astérisque 319 (2008), 259-301.
[Ki] M. Kisin, *Crystalline representations and $F$-crystals*, in: Algebraic geometry and number theory, Prog. in Math. 253, 459-496, Birkhäuser, 2006.
[KL] K. Kedlaya, R. Liu, *On families of $(\varphi, \Gamma)$-modules*, Algebra & Number Theory 4 No. 7 (2010), 943-967.
[Liu] R. Liu, *Slope filtrations in families*, to appear in J. Inst. Math. Jussieu.