Abstract. The problem of space-efficient depth-first search (DFS) is reconsidered. A particularly simple and fast algorithm is presented that, on a directed or undirected input graph $G = (V, E)$ with $n$ vertices and $m$ edges, carries out a DFS in $O(n + m)$ time with $n + \sum_{v \in V_{\geq 3}} \left\lfloor \log_2 (d_v - 1) \right\rfloor + O(\log n) \leq n + m + O(\log n)$ bits of working memory, where $d_v$ is the (total) degree of $v$, for each $v \in V$, and $V_{\geq 3} = \{ v \in V \mid d_v \geq 3 \}$. A slightly more complicated variant of the algorithm works in the same time with at most $n + \left( \frac{4}{5} m + O(\log n) \right)$ bits. It is also shown that a DFS can be carried out in a graph with $n$ vertices and $m$ edges in $O(n + m \log^* n)$ time with $O(n)$ bits or in $O(n + m)$ time with either $O(n \log \log (4 + m/n))$ bits or, for arbitrary integer $k \geq 1$, $O(n \log^k n)$ bits. These results among them subsume or improve most earlier results on space-efficient DFS. Some of the new time and space bounds are shown to extend to applications of DFS such as the computation of cut vertices, bridges, biconnected components and 2-edge-connected components in undirected graphs.

Keywords: Graph algorithms, space efficiency, depth-first search, DFS.

1 Introduction and Related Work

Depth-first search or DFS is a very well-known method for visiting the vertices and edges of a directed or undirected graph [7,19]. DFS is set off from other ways of traversing the graph such as breadth-first search by the DFS rule: Whenever two or more vertices were discovered by the search and have unexplored incident (out)edges, an (out)edge incident on the most recently discovered such vertex is explored first. The DFS rule confers a number of structural properties on the resulting graph traversal that cause DFS to have a large number of applications. The rule can be implemented with the aid of a stack that contains those vertices discovered by the search that still have unexplored incident (out)edges, with more recently discovered vertices being located closer to the top of the stack. The stack is the main obstacle to a space-efficient implementation of DFS.

In the following discussion, let $n$ and $m$ denote the number of vertices and of edges, respectively, of an input graph. Let us also use the common picture according to which every vertex is initially white, becomes gray when it is discovered and pushed on the stack, and turns black when all its incident (out)edges have been explored and it leaves the stack. The study of space-efficient DFS was initiated by Asano et al. [2]. Besides a number of DFS algorithms whose running times were characterized only as polynomial in $n$ or worse, they described an algorithm that uses $O(m \log n)$ time and $O(n)$ bits and another algorithm that uses $O(nm)$ time and at most $(\log 3 + \epsilon)n$ bits, for arbitrary fixed $\epsilon > 0$, where “log”, here
and in the remainder of the paper, denotes the binary logarithm function \( \log_2 \). Their basic idea was, since the stack of gray vertices cannot be kept in full (it might occupy \( \Theta(n \log n) \) bits), to drop (forget) stack entries and to restore them in smaller or bigger chunks when they are later needed. Using the same idea, Elmasry, Hagerup and Kammer [9] observed that one can obtain the best of both algorithms, namely a running time of \( O((n + m) \log n) \) with \( (\log 3 + \epsilon)n \) bits. Assuming a slightly stronger representation of the input graph as a set of adjacency arrays rather than adjacency lists, they also devised an algorithm that runs in \( O(n + m) \) time with \( O(n \log \log n) \) bits or in \( O((n + m) \log \log n) \) time with \( O(n) \) bits, or anything in between with the same time-space product. The new idea necessary to obtain this result was, rather than to forget stack entries entirely, to keep for each gray vertex a little information about its entry on the stack and a little information about the position of that stack entry.

The space bounds cited so far may be characterized as density-independent in that they depend only on \( n \) and not on \( m \). If one is willing to settle for density-dependent space bounds that depend on \( m \) or perhaps on the multiset of vertex degrees, it becomes feasible to store with each gray vertex \( u \) an indication of the vertex immediately above it on the stack, which is necessarily a neighbor of \( u \) and therefore expressible in \( O(\log(d + 1)) \) bits, where \( d \) is the degree of \( u \). Since \( \log(d + 1) = O(d + 1) \), this yields a DFS algorithm that works in \( O(n + m) \) time with \( O(n \log n) \) bits, as observed in [3,15]. One can also use Jensen’s inequality to bound the space requirements of the pointers to neighboring vertices by \( O(n \log(2 + m/n)) \) bits. This was done in [9] for problems for which the authors were unable to obtain density-independent bounds. In the context of DFS, it was mentioned by Chakraborty, Raman and Satti [5].

Several applications of DFS relevant to the present paper can be characterized by means of equivalence relations on vertices or edges. Let \( G = (V, E) \) be a graph. If \( G \) is directed and \( u, v \in V \), let us write \( u \equiv_S^G v \) if \( G \) contains a path from \( u \) to \( v \) and one from \( v \) to \( u \). If \( G \) is undirected and \( e_1, e_2 \in E \), write \( e_1 \equiv_B^G e_2 \) (\( e_1 \equiv_E^G e_2 \), respectively) if \( e_1 = e_2 \) or \( e_1 \) and \( e_2 \) belong to a common simple cycle (a not necessarily simple cycle, respectively) in \( G \). Then \( \equiv_S^G \) is an equivalence relation on \( V \) and \( \equiv_B^G \) and \( \equiv_E^G \) are equivalence relations on \( E \). Each subgraph induced by an equivalence class of one of these relations is called a strongly connected component (SCC) in the case of \( \equiv_S^G \), a biconnected component (BCC) or block in the case of \( \equiv_B^G \), and a 2-edge-connected component (which we shall abbreviate to 2ECC) in the case of \( \equiv_E^G \). Sometimes a single edge with its endpoints is not considered a biconnected or 2-edge-connected component; adapting our algorithms to alternative definitions that differ in this respect is a trivial matter. Suppose that \( G \) is undirected. A cut vertex (also known as an articulation point) in \( G \) is a vertex that belongs to more than one BCC in \( G \); equivalently, it is a vertex whose removal from \( G \) increases the number of connected components. A bridge in \( G \) is an edge that belongs to no cycle in \( G \); equivalently, it is an edge whose removal from \( G \) increases the number of connected components.

For each of the three kinds of components introduced above, we may want the components of an input graph to be output one by one. Correspondingly, we will
speak of the SCC, the BCC and the 2ECC problems. Outputting a component may
mean outputting its vertices or edges or both. Correspondingly, we may describe
an algorithm as, e.g., computing the strongly connected components of a graph
with their vertices. We may either output special separator symbols between con-
secutive components or number the components consecutively and output vertices
and edges together with their component numbers; for our purposes, these two con-
ventions are equivalent. **Topologically sorting** a directed acyclic graph \( G = (V, E) \)
means outputting the vertices of \( G \) in an order such that for each \((u, v) \in E\), \( u \) is
output before \( v \).

Elmasry et al. [9] gave algorithms for the SCC problem and for topological sort-
ing that work in \( O(n + m) \) time using \( O(n \log \log n) \) bits. Their main tool was a
method for “coarse-grained reversal” of a DFS computation that makes it possible
to output the vertices of the input graph in *reverse postorder*, i.e., in the reverse
of the order in which the vertices turn black in the course of the DFS. Various
bounds for these problems were claimed without proof by Banerjee, Chakraborty
and Raman [3]: \( O(m \log n \log \log n) \) time with \( O(n) \) bits for the SCC problem and
\( O(n + m) \) time with \( m + 3n + o(n + m) \) bits as well as \( O(m \log \log n) \) time with
\( O(n) \) bits for topological sorting. For the BCC problem and the computation of cut
vertices, Kammer, Kratsch and Laudahn [15] described an algorithm that works
in \( O(n + m) \) time using \( O(n + m) \) bits and can be seen as an implementation
of an algorithm of Schmidt [18]. Essentially the same algorithm was sketched by
Banerjee et al. [3], who also applied it to the 2ECC problem and the computa-
tion of bridges. Space bounds of the form \( O(n \log(m/n)) \) for the same problems
were mentioned by Chakraborty et al. [5]. Essentially re-inventing an algorithm of
Gabow [10] and combining it with machinery from [9] and with new ideas, Kam-
mer et al. [15] also demonstrated how to compute the cut vertices in \( O(n + m) \)
time with \( O(n \log \log n) \) bits. Finally, decomposing the input graph into subtrees
and processing the subtrees one by one, Chakraborty et al. [5] were able to solve
the BCC problem and compute the cut vertices in \( O(m \log n \log \log n) \) time with
\( O(n) \) bits.

2 New Results and Techniques

The main thrust of this work is to establish new density-dependent and density-
independent space bounds for fast DFS algorithms. Let us begin by developing
simple notation that allows the results to be stated conveniently.

When \( G = (V, E) \) is a directed or undirected graph, \( d_v \) is the (total) degree of
\( v \) for each \( v \in V \) and \( k \) is an integer, let

\[
L_k(G) = \sum_{v \in V \atop d_v + k \geq 2} \lceil \log_2(d_v + k) \rceil.
\]

When \( G \) is directed, we use \( L^{\text{in}}_k(G) \) and \( L^{\text{out}}_k(G) \) to denote quantities defined in the
same way, but now with \( d_v \) taken to mean the indegree and the outdegree of \( v \),
respectively.
Lemma 2.1. Let $G$ be a directed or undirected graph with $n$ vertices and $m$ edges. Then

(a) $L_1(G) \leq n \log(1 + 4m/n)$;
(b) If $G$ is directed, then $L_1^{\text{in}}(G)$ and $L_1^{\text{out}}(G)$ are both bounded by $n \log(1+2m/n)$.

Proof. Let $G = (V,E)$ and, for each $v \in V$, denote by $d_v$ the (total) degree of $v$. To prove part (a), observe first that $\lceil \log(d+1) \rceil \leq \log(2d+1)$ for all integers $d \geq 0$. Since the function $d \mapsto \log(2d+1)$ is concave on $[0, \infty)$ and $\sum_{v \in V} d_v = 2m$, the result follows from Jensen’s inequality. Part (b) is proved in the same way, noting that the relevant vertex degrees now sum to $m$. \hfill $\square$

Our most accurate space bounds involve terms of the form $L_k(G)$. More convenient bounds can be derived from them with Lemma 2.1. Note that for all $a, b, c > 0$, the quantity $an + n \log(b + cm/n)$ can also be written as $n \log(2^a b + 2^a cm/n)$. The latter form will be preferred here.

Our first algorithm carries out a DFS of a graph $G$ with $n$ vertices and $m$ edges in $O(n + m)$ time using at most $n + L_{-1}(G) + O(\log n)$ bits. The number of bits needed, which can also be bounded by $n + m + O(\log n)$ and by $n \log(2 + 8m/n) + O(\log n)$, is noteworthy only for the constant factors involved. Comparable earlier space bounds were indicated only as $O(n + m)$ or $O(n \log(m/n))$ bits, and no argument offered in their support points to as small constant factors as ours. Moreover, all of the earlier algorithms make use of rank-select structures \cite{6}, namely to store variable-length information indexed by vertex numbers. Whereas asymptotically space-efficient and fast rank-select structures are known, it is generally accepted that in practice they come at a considerable price in terms of time and especially space (see, e.g., \cite{20}) and a certain coding complexity. In contrast, we view the algorithm presented here as the first truly practical space-efficient DFS algorithm.

The simple but novel idea that enables us to make do without rank-select structures is a different organization of the DFS stack. The vertices on the stack, in the order from the bottom to the top of the stack, always form a directed path, in $G$ itself if $G$ is directed and in the directed version of $G$ if not, that we call the gray path. Assume that $G$ is undirected. Instead of having a table that maps each vertex to how far it has progressed in the exploration of its incident edges, which in some sense distributes the stack over the single vertices and is what necessitates a rank-select structure, we return to using a stack implemented in contiguous memory locations and store there for each internal vertex $v$ on the gray path the distance in its adjacency array, considered as a cyclic structure, from the predecessor $u$ of $v$ to the successor $w$ of $v$ on the gray path. More intuitively, one can think of the stack entry as describing the “turn” that the gray path makes at $v$, namely from $u$ via $v$ to $w$. Knowing $w$, $v$ and the “turn value”, one can compute $u$. Provided that outside of the stack we always remember the current vertex $w$ of the DFS, the vertex on top of the DFS stack and at the end of the gray path, and the position in $w$’s adjacency array of the predecessor of $w$ on the gray path, if any, this allows us to pop from the stack in constant time, and pushing is equally easy. In the course of the processing of $v$, the “turn value” can be stepped from 1 (“after entering $v$ from $u$, take the next exit”) to $d - 1$, 

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where \(d\) is the (total) degree of \(v\) (directed edges that enter \(v\) are simply ignored). Aside from the somewhat unusual stack, the DFS can proceed as a usual DFS and complete in linear time. Handling vertices of small degree specially, we can lower the space bound to \(n + (4/5)m + \Theta(\log n)\) bits and solve the SCC problem in \(O(n + m)\) time with \(n \log 3 + (14/5)m + \Theta((\log n)^2)\) bits. Resorting to using rank-select structures, we describe linear-time algorithms for the SCC, BCC and 2ECC problems and for the computation of topological sortings, cut vertices and bridges with space bounds of the form \((an + bm)(1 + o(1))\) or \(an \log(b + cm/n)(1 + o(1))\) bits, where \(a, b\) and \(c\) are positive constants. Apart from minor tricks to reduce the values of \(a, b\) and \(c\), no new techniques are involved here.

Turning to space bounds that are independent on \(m\) or almost so, we first describe a DFS algorithm that works in \(O(n + m)\) time with \(O(n \log \log(4 + m/n))\) bits. The algorithm is similar to an algorithm of Elmasry et al. [9] that uses \(\Theta(n \log \log n)\) bits. Our superior space bound is made possible by two new elements: First, the algorithm is changed to use a stack of “turn values” rather than of “progress counters”, as discussed above. And second, when stack entries have to be dropped to save space, we keep approximations of the lost entries that turn out to work better than those employed in [9]. Our space bound is attractive because it unifies the earlier bounds of the forms \(O(n + m), O(n \log(2 + m/n))\) and \(O(n \log \log n)\) bits, being at least as good as all of them for every graph density and better than each of them for some densities.

Subsequently we show how to carry out a DFS in \(O(n + m \log^* n)\) time with \(O(n)\) bits or, with a slight variation, in \(O(n + m)\) time with \(O(n \log^{(k)} n)\) bits for arbitrary fixed \(k \in \mathbb{N} = \{1, 2, \ldots\}\). Here \(\log^{(k)}\) denotes \(k\)-fold repeated application of \(\log\), e.g., \(\log^{(2)} n = \log \log n\), and \(\log^* n = \min\{k \in \mathbb{N} \mid \log^{(k)} n \leq 1\}\). The main new idea instrumental in obtaining this result is to let each vertex \(v\) dropped from the stack record, instead of a fixed approximation of its stack position as in earlier algorithms, an approximation of that position that changes dynamically to become coarser when \(v\) is farther removed from the top of the stack. Adapting an algorithm of Kammer et al. [15] for computing cut vertices, we show that the time and space bounds indicated in this paragraph extend to the problems of computing biconnected and 2-edge-connected components, cut vertices and bridges of undirected graphs.

3 Preliminaries

We assume a representation of an undirected input graph \(G = (V, E)\) that is practically identical to the one used in [14]: For some known integer \(n \geq 1\), \(V = \{1, \ldots, n\}\), the degree of each \(u \in V\) can be obtained as \(\text{deg}(u)\), and for each \(u \in V\) and each \(i \in \{0, \ldots, \text{deg}(u) - 1\}\), \(\text{head}(u,i)\) and \(\text{mate}(u,i)\) yield the \((i + 1)\text{st}\) neighbor \(v\) of \(u\) and the integer \(j\) with \(\text{head}(v,j) = u\), respectively, for some arbitrary numbering, starting at 0, ordering of the neighbors of each vertex. The access functions \(\text{deg}, \text{head}\) and \(\text{mate}\) run in constant time. The representation of a directed graph \(G\) is similar in spirit: \(G\) is represented with in/out adjacency arrays, i.e., we can access the inneighbors as well as the outneighbors of a given
vertex one by one, and there are cross links, i.e., the function mate now, for each edge \((u, v)\), maps the position of \(v\) in the adjacency array of \(u\) to that of \(u\) in the adjacency array of \(v\) and vice versa.

A DFS of a graph \(G\) is associated with a spanning forest \(F\) of \(G\) in an obvious way: If a vertex \(v\) is discovered by the DFS when the current vertex is \(u\), \(v\) becomes a child of \(u\). \(F\) is called the DFS forest corresponding to the DFS, and its edges are called tree edges, whereas the other edges of \(G\) may be called nontree edges. At the outermost level, the DFS steps through the vertices of \(G\) in a particular order, called its root order, and every vertex found not to have been discovered at that time becomes the root of the next DFS tree in \(F\). The parent pointer of a given vertex \(v\) in \(G\) is an indication of the parent \(u\) of \(v\) in \(F\), if any. If the root order of a DFS is simply \(1, \ldots, n\) and the DFS always explores the edges incident on the current vertex \(u\) in the order in which their endpoints occur in the adjacency array of \(u\), the corresponding DFS forest is the lexicographic DFS forest of the adjacency-array representation.

The following lemmas describe two auxiliary data structures that we use repeatedly: the choice dictionary of Kammer and Hagerup \([12,13]\) and the ternary array of Dodis, Pătrașcu and Thorup \([8, \text{Theorem 1}]\).

Lemma 3.1. There is a data structure that, for every \(n \in \mathbb{N}\), can be initialized for universe size \(n\) in constant time and subsequently occupies \(n + O(n/\log n)\) bits and maintains an initially empty subset \(S\) of \(\{1, \ldots, n\}\) under insertion, deletion, membership queries and the operation choice (return an arbitrary element of \(S\)) in constant time as well as iteration over \(S\) in \(O(|S| + 1)\) time.

Lemma 3.2. There is a data structure that can be initialized with an arbitrary \(n \in \mathbb{N}\) in \(O(\log n)\) time and subsequently occupies \(n \log_2 3 + O((\log n)^2)\) bits and maintains a sequence drawn from \(\{0, 1, 2\}^n\) under constant-time reading and writing of individual elements of the sequence.

4 Density-Dependent Bounds

4.1 Depth-First Search

Theorem 4.1. A DFS of a directed or undirected graph \(G = (V, E)\) with \(n\) vertices and \(m\) edges can be carried out in \(O(n + m)\) time with at most any of the following numbers of bits of working memory:

(a) \(n + L_{-1}(G) + O(\log n)\);
(b) \(n + m + O(\log n)\);
(c) \(n \log(2 + 8m/n) + O(\log n)\).

Proof. We first show part (a) for the case in which \(G\) is undirected. The algorithm was described in Section 2, and it was argued there that it works in \(O(n + m)\) time. What remains is to bound the number of bits needed.

If an internal vertex \(v\) on the gray path has degree \(d\), its stack entry can be taken to be an integer in \(\{1, \ldots, d - 1\}\) that indicates the number of edges incident
on \( v \) that were explored with \( v \) as the current vertex. The stack entry can therefore be represented in \( \lceil \log(d - 1) \rceil \) bits, so that the entire stack never occupies more than \( L_{-1}(G) \) bits. In addition to the information on the stack, the DFS must know for each vertex \( v \) whether \( v \) is white; this takes \( n \) bits. (Unless an application calls for it, the DFS has no need to distinguish between gray and black vertices.) Finally the DFS must store a few simple variables in \( O(\log n) \) bits, for a grand total of \( n + L_{-1}(G) + O(\log n) \) bits. This concludes the proof of part (a) for undirected graphs.

If \( G \) is directed, we can pretend that the inneighbors and the outneighbors of each vertex are stored in the same adjacency array (whether or not this is the case in the actual representation of \( G \)). We can then use the same algorithm, except that an edge \((u, v)\) should not be explored in the wrong direction, i.e., when \( v \) is the current vertex of the DFS.

To show part (b) of the theorem, let \( d_v \) be the (total) degree of \( v \) for each \( v \in V \) and observe that \( \lceil \log(d - 1) \rceil \leq d/2 \) for all integers \( d \geq 3 \), so that \( L_{-1}(G) \leq (1/2) \sum _{v \in V} d_v = m \). Part (c) follows immediately from part (a) by an application of Lemma 2.1(a).

At the price of introducing a slight complication in the algorithm, we can obtain another space bound of \( c_1n + c_2m + O(\log n) \) bits for a smaller constant \( c_2 \). If \( c_1 \) is allowed to increase, it is also possible (but of little interest) to lower \( c_2 \) as far as desired towards 0 by treating vertices of small degree separately in the analysis.

**Theorem 4.2.** A DFS of a directed or undirected graph \( G \) with \( n \) vertices and \( m \) edges can be carried out in \( O(n + m) \) time with at most \( n + (4/5)m + O(\log n) \) bits of working memory.

**Proof.** The relation \( \lceil \log(d - 1) \rceil \leq (2/5)d \) is satisfied for all integers \( d \geq 3 \) except 4, 6 and 7. To handle the stack entries of vertices of degree 4, we divide these into groups of 5 and represent each group on the stack through a single combined entry of \( \lceil \log((4 - 1)^5) \rceil = 8 \) bits instead of 5 individual entries of \( \lceil \log(4 - 1) \rceil = 2 \) bits each. Since \( 8 \leq (2/5) \cdot 5 \cdot 4 \), the combined entry is small enough for the bound of the theorem. At all times, an incomplete group of up to 4 individual entries is kept outside of the stack in a constant number of bits. Similarly, groups of 3 entries for vertices of degree 6 are represented in \( \lceil \log((6 - 1)^3) \rceil = 7 \) bits, and groups of 3 entries for vertices of degree 7 are represented in \( \lceil \log((7 - 1)^3) \rceil = 8 \) bits. Since \( 7 \leq (2/5) \cdot 3 \cdot 6 \) and \( 8 \leq (2/5) \cdot 3 \cdot 7 \), this altogether yields a space bound of \( n + (2/5) \cdot 2m + O(\log n) \) bits.

The simplicity of the algorithm of Theorem 4.1 is demonstrated in Fig. 1, which shows an implementation of it for an undirected input graph \( G = (V, E) \). The description is given in complete detail except for items like the declaration of variables and for the specification of a bit stack \( S \) with the following two operations in addition to an appropriate initialization to being empty: \( S.push(\ell, d) \), where \( \ell \) and \( d \) are integers with \( d \geq 3 \) and \( 0 \leq \ell < 2^{\lceil \log(d - 1) \rceil} \), pushes on \( S \) the \( \lceil \log(d - 1) \rceil \)-bit binary representation of \( \ell \), and \( S.pop(d) \), where \( d \) again is an integer with \( d \geq 3 \),
correspondingly pops \([\log(d - 1)]\) bits from \(S\), interprets these as the binary representation of an integer \(\ell\) and returns \(\ell\). The task of the DFS is assumed to be the execution of certain user procedures at the appropriate times: \(\text{preprocess}(v)\) and \(\text{postprocess}(v)\), for each \(v \in V\), when \(v\) turns gray and when it turns black, respectively, \(\text{explore} \_\text{tree} \_\text{edge}(v, w)\), for \(\{v, w\} \in E\), when the edge \(\{v, w\}\) is explored with \(v\) as the current vertex and becomes a tree edge, \(\text{retreat} \_\text{tree} \_\text{edge}(v, w)\) when the DFS later withdraws from \(w\) to \(v\), and \(\text{handle} \_\text{back} \_\text{edge}(v, w)\), for \(\{v, w\} \in E\), when \(\{v, w\}\) is explored with \(v\) as the current vertex but does not lead to a new vertex. The code is made slightly more involved by a special handling of the first and last vertices of the gray path and by the fact that no stack entries are stored for vertices of degree 2. Timing experiments with an implementation of the algorithm of Fig. 1 showed it to be sometimes faster and sometimes slower than an alternative algorithm that also manages its own stack but makes no attempt at being space-efficient.

Fig. 1. The algorithm of Theorem 4.1 for an undirected input graph \(G = \langle V, E \rangle\).
4.2 Strongly Connected Components and Topological Sorting

**Theorem 4.3.** The strongly connected components of a directed graph with \(n\) vertices and \(m\) edges can be computed with their vertices and/or edges in \(O(n+m)\) time with \(n \log_2 3 + (14/5)m + O((\log n)^2)\) bits of working memory.

**Proof.** Let \(G\) be the input graph and let \(\tilde{G}\) be the directed graph obtained from \(G\) by replacing each edge \((u, v)\) by the antiparallel edge \((v, u)\). We use an algorithm attributed to Kosaraju and Sharir in [1] that identifies the vertex set of each SCC as that of a DFS tree constructed by a standard DFS of \(\tilde{G}\) that, however, employs as its root order the reverse postorder defined by an (arbitrary) DFS of \(G\).

Consider each vertex \(v\) in \(G\) to have a circular incidence array that contains all edges entering \(v\) as well as all edges leaving \(v\). A DFS of \(G\) can be viewed as entering each nonroot vertex \(v\) at a particular (tree) edge and each root \(v\) at a fixed position in its incidence array and eventually traversing \(v\)'s incidence array exactly once from that entry point, classifying certain edges out of \(v\) as tree edges and skipping over the remaining edges, either because they lead to vertices that were already discovered or because they enter \(v\), before finally, if \(v\) is a nonroot, retreating over the tree edge to \(v\)'s parent. During such a DFS of \(G\) that uses the root order \(1, \ldots, n\), we construct a bit sequence \(B\) by appending a 1 to an initially empty sequence whenever the DFS discovers a new vertex or withdraws over a tree edge and by appending a 0 whenever the DFS skips over an edge. The total number of bits in \(B\) is exactly \(2m\), and \(B\) can be seen to represent an Euler tour of each tree in the forest \(F\) defined by the DFS in a natural way. We also use an array \(A\) of \(n\) bits to mark those vertices that are roots in \(F\). Observe that the pair \((A, B)\) supports an Euler traversal that, in \(O(n+m)\) time and using only \(O(\log n)\) additional bits, enumerates the vertices in \(G\) in reverse postorder with respect to \(F\). In particular, whenever an Euler tour of a tree in \(F\) with root \(r\) has been followed backwards completely from end to start, \(A\) is used to find the end vertex of the next Euler tour, if any, as the largest root smaller than \(r\).

We carry out a DFS of \(\tilde{G}\), interleaved with an execution of the Euler traversal that supplies new root vertices as needed, and output the vertex set of each resulting tree as an SCC. The total time spent is \(O(n+m)\). We could execute the algorithm using \(n\) bits for \(A\), \(2m\) bits for \(B\) and, according to Theorem 4.2, \(n + (4/5)m + O(\log n)\) bits for the depth-first searches. Recall, however, that the space bound of Theorem 4.2 is obtained as the sum of \(n\) bits for an array \(white\) and \((4/5)m + O(\log n)\) bits for the DFS stack and related variables. It turns out that we can realize \(A\) and \(white\) together through a single ternary array with \(n\) entries. To see this, it suffices in the case of the DFS of \(G\) to note that a vertex classified as a root certainly is not white. For the DFS of \(\tilde{G}\), assume first that we want to output only the vertices of the strongly connected components, as is standard. Then even a binary array would suffice—we could use the same binary value to denote both “root” and “not white”. The reason for this is, on the one hand, that when the Euler traversal has entered a tree \(T\) with root \(r\), it will never again need to inspect \(A[w]\) for any \(w \geq r\) and, on the other hand, that every vertex \(w\) reachable in \(\tilde{G}\) from a vertex \(v\) in \(T\) must satisfy \(w \geq r\)—otherwise \(v\) would belong to an earlier tree (with respect to the DFS of \(G\)) and not to \(T\).
Lemma 4.4. There is a data structure that can be initialized for a positive integer
and perhaps to highlight those edges whose endpoints belong to different strongly
connected components (the “inter-component” edges), we need to know for each
edge \((v, w)\) explored during the DFS of \(\tilde{G}\) whether \(w\) belongs to the DFS tree
under construction at that time (then \((v, w)\) is an edge of the current SCC) or
to an older DFS tree (then \((v, w)\) is an “inter-component” edge). We solve this
problem again resorting to a ternary array, splitting the value “not white” into
“not white, but in the current tree” and “in an older tree”. Whenever the DFS of
\(\tilde{G}\) completes a tree, we repeat the DFS of that tree, treating the color “not white,
but in the current tree” as “white” and replacing all its occurrences by “in an
older tree”. The space bound follows from Lemma 3.2. \(\Box\)

When \(m\) is larger relative to \(n\), it is advantageous, instead of storing the bit
vector \(B\), to store for each vertex \(v\) a parent pointer of \([\log(d + 1)]\) bits, where \(d\)
is the indegree of \(v\), that indicates \(v\)’s parent in the DFS forest of \(G\) or no parent
at all (i.e., \(v\) is a root). For this we need the standard static space allocation:

Lemma 4.4. There is a data structure that can be initialized for a positive integer
\(n\) and \(n\) nonnegative integers \(\ell_1, \ldots, \ell_n\) in \(O(n + N)\) time, where \(\ell_j = O(\log n)\)
for \(j = 1, \ldots, n\) and \(N = \sum_{j=1}^{n} \ell_j\), and subsequently occupies \((n + 2N)(1 +
O(\log \log n / \log n))\) bits and realizes an array \(A[1 \cdots n]\) of entries of \(\ell_1, \ldots, \ell_n\) bits
under constant-time reading and writing of individual entries in \(A\).

Proof. Maintain the entries of \(A\) in an array \(\tilde{A}[0 \cdots N - 1]\) of \(N\) bits and store
the sequence \(B = b_1 \cdots b_{n+N} = 0^{\ell_1} \cdots 0^{\ell_n} 1\) of \(n + N\) bits. For \(k = 1, \ldots, n\), \(A[k]\)
is located in \(\tilde{A}[\text{select}_B(k - 1) - (k - 1) \cdots \text{select}_B(k) - k - 1]\), where \(\text{select}_B(k) = \min\{j \in \{0, \ldots, n + N\} \mid \sum_{i=1}^{j} b_i = k\}\) for \(k = 0, \ldots, n\), and \(\text{select}_B\) can be
evaluated in constant time given \(O((n + N) \log \log n / \log n)\) bits of bookkeeping
information [11,17] that can be computed in \(O(n + N)\) time. \(\Box\)

Lemma 4.5. A representation of the parent pointers of the lexicographic DFS
forest of an adjacency-array representation of a graph \(G\) with \(n\) vertices and \(m\)
edges that allows constant-time access to the parent of a given vertex can be
stored in \((n + 2N)(1 + O(\log \log n / \log n))\) bits and computed in \(O(n + m)\) time
with \(n\) additional bits, where \(N = L_1(G)\) if \(G\) is undirected and \(N = L_1^m(G)\) if \(G\)
is directed.

Proof. The parent pointers themselves can be stored in \(N\) bits, and constant-
time access to them can be provided according to Lemma 4.4. To compute the
parent pointers, carry out a DFS of \(G\), using the \(n\) additional bits to store for each
vertex whether it is still white. When the DFS ends the processing at a vertex \(v\),
it follows the parent pointer of \(v\) to withdraw to \(v\)’s parent \(u\) in the DFS forest,
and from there proceeds to explore the edge that follows \((u, v)\) or \(\{u, v\}\) in \(u\)’s
incidence array, if any, and to store the appropriate new parent pointer if this
dge leads to a white vertex. The procedure to follow at the first exploration of
an edge from a newly discovered vertex is analogous. \(\Box\)
Theorem 4.6. The strongly connected components of a directed graph $G = (V,E)$ with $n$ vertices and $m$ edges can be computed in $O(n+m)$ time with at most $(2n + L_1(G) + 2L^u_1(G))(1 + O(\log \log n/\log n)) \leq 3n \log(2 + 4m/n)(1 + O(\log \log n/\log n))$ bits of working memory.

Proof. The parent pointers of a DFS of $G$ by themselves support the Euler traversal of the proof of Theorem 4.3 in $O(n+m)$ time, using $O(\log n)$ additional bits. To see this, observe that one can visit the children of a vertex $u$ by inspecting the outneighbors of $u$ one by one to see which of them indicate $u$ as their parent and that the array $A$ is superfluous since a vertex is a root in the DFS forest if and only if its parent pointer does not point to one of its neighbors—a value was reserved for this purpose. Thus first compute the parent pointers (Lemma 4.5) and then carry out a DFS of $G$ interleaved with the Euler traversal. The time needed is $O(n+m)$, and the number of bits is at most the sum of the bounds of Theorem 4.1 and Lemma 4.5. To prove the second bound, use Lemma 2.1. □

If the input graph $G$ happens to be acyclic, the algorithms of Theorems 4.3 and 4.6 output the vertices of $G$ in the order of a topological sorting. In the case of Theorem 4.3 this may yield the most practical algorithm. Better space bounds for topological sorting can, however, be obtained by implementing an alternative standard algorithm, due to Knuth [16], that repeatedly removes a vertex of indegree 0 while keeping track only of the indegrees of all vertices. This was also suggested by Banerjee et al. [3]. As mentioned in the discussion of related work, they indicated a space bound of $m + 3n + o(n+m)$ bits; it is not clear to this author, however, how such a bound is to be proved.

Theorem 4.7. A topological sorting of a directed acyclic input graph with $n$ vertices and $m$ edges can be computed in $O(n+m)$ time with at most any of the following numbers of bits:

(a) $(2n + 2L^u_0)(1 + O(\log \log n/\log n))$;
(b) $(2n + (4/3)m)(1 + O(\log \log n/\log n))$;
(c) $2n \log(2 + 4m/n)(1 + O(\log \log n/\log n))$.

Proof. Maintain the current set of vertices of indegree 0 in an instance of the choice dictionary of Lemma 3.1 which needs $n + O(n/\log n)$ bits. Also maintain the current indegrees according to Lemma 4.4. Since we can store an arbitrary value or nothing for vertices of current indegree 0, we need only distinguish between $d$ different values for a vertex of original indegree $d \geq 2$, so that $(n + 2L^u_0)(1 + O(\log \log n/\log n))$ bits suffice. With these data structures, the algorithm of Knuth [16] can be executed in $O(n+m)$ time. This proves part (a). Part (b) follows from part (a) since $\lceil \log d \rceil \leq (2/3)d$ for all integers $d \geq 2$, and part (c) follows from part (a) with Lemma 2.1(b). □

4.3 Biconnected and 2-Edge-Connected Components

In this subsection we will see that closely related algorithms can be used to compute the cut vertices, the bridges and the biconnected and 2-edge-connected components of an undirected graph. Our algorithms are similar to those of [3,5,15].
but whereas the earlier authors indicated the space bounds only as $O(n + m)$ or $O(n \log(m/n))$ bits, we will strive to obtain small constant factors and indicate these explicitly.

A simple but crucial fact is that for every DFS forest $F$ of an undirected graph $G$, every edge in $G$ joins an ancestor to a descendant within a tree in $F$. DFS is also known to interact harmoniously with the graph structures of interest in this subsection as exemplified, e.g., in the following lemma.

**Lemma 4.8.** Let $F$ be a DFS forest of an undirected graph $G = (V, E)$ and let $e \in E$. Then the subgraph $F'$ of $F$ induced by the edges in $F$ equivalent to $e$ under $\equiv^B_G$ is a subtree of $F$ whose root has degree 1 in $F'$.

**Proof.** Every edge in $G$ is equivalent under $\equiv^B_G$ to an edge in $F$, so $F'$ is not the empty graph. Let us first prove that $F'$ is connected. Suppose for this that a simple path $\pi$ in $F$ contains the edges $e_1$, $e_2$ and $e_3$ in that order and that $e_1$ and $e_3$ belong to $F'$. To show that $e_2$ also belongs to $F'$, let $C$ be a simple cycle in $G$ that contains $e_1$ and $e_3$ and let $\pi'$ be the maximal subpath of $\pi$ that contains $e_2$ and whose internal vertices do not belong to $C$. The endpoints of $\pi'$ lie on $C$, so $\pi$ and a suitably chosen subpath of $C$ together form a simple cycle that contains $e_2$ and at least one of $e_1$ and $e_3$. Thus $F'$ is indeed a subtree of $F$ with a root $u$. A simple cycle in $G$ that contains two edges in $F'$ incident on $u$ must necessarily also contain a proper ancestor of $u$, contradicting the fact that the edge between $u$ and its parent in $F$, if any, does not belong to $F'$. Thus the degree of $u$ in $F'$ is 1. 

Let $F$ be a DFS forest of an undirected graph $G = (V, E)$ with $n$ vertices and $m$ edges. Let us call a subtree $F'$ of $F$ as in Lemma 4.8 a **BCC subtree** and its root a **BCC root**. A vertex common to two edge-disjoint subtrees of a rooted tree is a root in at least one of the subtrees. Therefore every cut vertex in $G$ is a BCC root. Conversely, a BCC root $u$ is also a cut vertex in $G$ unless $u$ is a root in $F$ with only one child. Every BCC of $G$ consists precisely of the vertices in a particular BCC subtree $F'$ and the edges in $G$ that join two such vertices, i.e., whose lower endpoint (with respect to $F'$) lies in $F'$ but is not the root of $F'$. An edge is a bridge exactly if, together with its endpoints, it constitutes a full BCC subtree. A 2ECC, finally, is either such a 1-edge BCC subtree or a maximal connected subgraph of $G$ with at least one edge and without bridges.

For each $w \in V$, denote by $P(w)$ the assertion that $w$ has a parent $v$ in $F$ and $G$ contains at least one edge between a descendant of $w$ and a proper ancestor of $v$. If $\{v, w\}$ is an edge in $F$ and $v$ is the parent of $w$, $P(w) = false$ exactly if $v$ is the root of the BCC subtree that contains $\{v, w\}$. We can compute $P(w)$ for all $w \in V$ by initializing all entries in a Boolean array $Q[1..n]$ to false and processing all nontree edges as follows: To process a nontree edge $\{x, y\}$, where $y$ is a descendant of $x$, start at $y$ and follow the path in $F$ from $y$ to $x$, setting $Q[z] := true$ for every vertex $z$ visited, but omitting this action for the last two vertices (namely $x$ and a child of $x$). Suppose that we process each nontree edge $\{x, y\}$, where $y$ is a descendant of $x$, when a preorder traversal of $F$ reaches $x$ and before it proceeds to children of $x$. Then we can stop the processing of $\{x, y\}$ once
we reach a vertex \( z \) for which \( Q[z] \) already has the value \( true \)—the same will be true for all outstanding vertices \( z \). Therefore the processing of all nontree edges can be carried out in \( O(n + m) \) time, after which \( Q[w] = P(w) \) for all \( w \in V \). To solve one of the problems considered in this subsection, compute the DFS forest \( F \) and traverse it to compute \( Q \), as just described, while executing the following additional problem-specific steps:

**Cut vertices:** Output each vertex \( v \) in \( F \) that has a child \( w \) with \( P(w) = false \) and is not a root in \( F \) or has two or more children.

**Bridges:** Output each tree edge \( \{u, v\} \), where \( u \) is the parent of \( v \), for which \( P(v) = false \) and \( P(w) = false \) for every child \( w \) of \( v \).

**Biconnected components:** Specialize the traversal of \( F \) to always visit a vertex \( w \) with \( P(w) = false \) before a sibling \( w' \) of \( w \) with \( P(w') = true \). To compute the biconnected components of \( G \) with their vertices and edges, when the traversal withdraws over a tree edge \( \{v, w\} \) from a vertex \( w \) to its parent \( v \), output the edges in \( G \) that have \( w \) as their lower endpoint (including \( \{v, w\} \) ), output \( w \) itself and, if \( P(w) = false \), also output \( v \) and wrap up the current BCC, i.e., except in the case of the very last component, output a component separator or increment the component counter. Visiting the children \( w \) of a vertex \( v \) with \( P(w) = true \) after those with \( P(w) = false \) ensures that the vertices and edges of the BCC that contains \( \{v, w\} \) are output together for each \( w \) without intervening vertices and edges of other biconnected components.

**2-edge-connected components:** Specialize the traversal of \( F \) so that for each vertex \( v \), a child \( w \) of \( v \) for which \( \{v, w\} \) is a bridge is always visited before a child \( w' \) of \( v \) for which \( \{v, w'\} \) is not a bridge. Suppose that the traversal withdraws from a vertex \( w \) to its parent \( v \). If \( w \) has at least one incident edge that is not a bridge, output \( w \) and, if \( \{v, w\} \) is a bridge, wrap up the current 2ECC. If \( \{v, w\} \) is a bridge, output \( v \), \( w \) and \( \{v, w\} \) and wrap up the current 2ECC. If \( \{v, w\} \) is not a bridge, output all nonbridge edges of which \( w \) is the lower endpoint (including \( \{v, w\} \) ). Finally, when the traversal withdraws from a root \( u \) with at least one incident edge that is not a bridge, output \( u \) and wrap up the current 2ECC. As above, visiting those children of a given vertex \( v \) that are adjacent to \( v \) via bridges before the other children of \( v \) ensures that the vertices and edges of the 2ECC that contains several edges incident on \( v \), if any, are output together without intervening vertices and edges of other 2-edge-connected components.

When the traversal of \( F \) reaches a vertex \( v \) with a child \( w \), \( Q[w] \) will have reached its final value, \( P(w) \), and will never again be written to. It is now obvious that we can test at that time whether \( v \) is a cut vertex and whether the edge between \( v \) and its parent in \( F \), if any, is a bridge in \( O(d + 1) \) time, where \( d \) is the degree of \( v \). It follows that each of the four problems considered above can be solved in \( O(n + m) \) time. In the most complicated case, that of 2-edge-connected components, in order to test during the processing of a vertex \( v \) whether an edge \( \{v, w\} \) is a bridge, where \( w \) is a child of \( v \) in \( F \), carry out a “preliminary visit” of the children of \( w \) in \( F \).

We can compute the DFS forest \( F \) with the algorithm of Lemma [4.5], which needs \( (n + 2L_1(G))(1 + O(\log \log n/\log n)) \) bits plus \( n \) bits that can be reused.
The subsequent traversal needs $n$ bits for the array $Q$. In addition, when the computation of $Q$ described above processes a nontree edge $\{x, y\}$, it needs to know whether $y$ is an ancestor or a descendant of $x$. We can use another Boolean array $A[1..n]$ to handle this issue, ensuring for each $y \in V$ that at all times $A[y] = true$ if and only if $y$ is an ancestor of the current vertex of the traversal of $F$ (i.e., if $y$ is gray). In some cases, however, we can make do with less space. Observe that in the computation of cut vertices and bridges, the value of $Q[v]$ is never again used after the arrival of the traversal of $F$ at $v$. When the traversal reaches $v$ and $Q[v]$ has been inspected, we can therefore set $Q[v] := true$ without detriment to the use of $Q$. Suppose that when processing a nontree edge $\{x, y\}$ in the computation of $Q$, we consult $Q[y]$ instead of $A[y]$ to know whether $y$ is an ancestor of the current vertex $x$. If $A[y] = true$, the artificial change to $Q$ introduced above ensures that we necessarily also have $Q[y] = true$, so that the algorithm proceeds correctly. If $A[y] = false$, we may have $Q[y] = true$, in which case the processing of $\{x, y\}$ stops immediately, but then $P(y) = true$ ($y$ has not yet been reached by the traversal, and so $Q[y]$ was not set artificially to true) and it is correct to do nothing.

If our goal is to compute the biconnected or 2-edge-connected components of $G$ with their vertices, but not with their edges, we are in an intermediate situation: We need to distinguish between three different combinations of $A[y]$ and $Q[y]$ (now with the original $Q$), but if $Q[y] = true$ the value of $A[y]$ is immaterial as above, and $Q[y]$ never changes from true to false. We can therefore represent $A$ and $Q$ together through a ternary array with $n$ entries. Altogether, we have proved the following result.

**Theorem 4.9.** Given an undirected graph $G$ with $n$ vertices and $m$ edges, we can compute the following in $O(n + m)$ time and with the number of bits indicated:

(a) The cut vertices and bridges of $G$ with $(2n + 2L_1(G))(1 + O(\log \log n/\log n))$
= $2n \log(2 + 8m/n)(1 + O(\log \log n/\log n))$ bits;

(b) The biconnected and 2-edge-connected components of $G$ with their vertices with $((1 + \log_2 3)n + 2L_1(G))(1 + O(\log \log n/\log n))$ bits;

(c) The biconnected and 2-edge-connected components of $G$ with their edges and possibly vertices with $(3n + 2L_1(G))(1 + O(\log \log n/\log n))$ bits.

Kammer et al. [15] consider the problem of preprocessing an undirected graph $G$ so as later to be able to output the vertices and/or edges of a single BCC, identified via one of its edges, in time at most proportional to the number of items output. Having available the parent pointers of a DFS forest $F$ and the array $Q$ corresponding to $F$, we can solve the problem in the following way, which is the translation of the procedure of Kammer et al. to our setting: Given a request to output the BCC $H$ that contains an edge $\{x, y\}$, first follow parent pointers in parallel from $x$ and $y$ until one of the searches hits the other endpoint or a root in $F$. This allows us to determine which of $x$ and $y$ is an ancestor of the other vertex in time at most proportional to the number of items to be output. Then traverse the subtree of $F$ reachable from the lower endpoint of $\{x, y\}$ without crossing any edge between a BCC root and its single child, producing the same output at each vertex.
as described above for the output of all biconnected components. In addition, at
the uniquely defined vertex \( w \) with \( P(w) = \text{false} \) visited by the search, also output
the parent \( v \) of \( w \) (but without continuing the traversal from \( v \)). In order to carry
out this procedure efficiently, we need a way to iterate over the edges in \( F \) incident
on a given vertex \( w \) and, if the edges of \( H \) are to be output, over the nontree edges
that have \( w \) as their lower endpoint. To this end we can equip each vertex \( w \) of
degree \( d \) with a choice dictionary (Lemma 3.1) for a universe size of \( d \) that allows
us to iterate over the relevant edges in time at most proportional to their number.
This needs another \( 2m + O(m/\log n) \) bits. Very similar constructions allow us to
output the vertices and/or the edges of a single 2-edge-connected component.

**Theorem 4.10.** There is a data structure that can be initialized for an undirected
graph \( G \) with \( n \) vertices and \( m \) edges in \( O(n + m) \) time, subsequently allows the
vertices and/or the edges of the biconnected or 2-edge-connected component that
contains a given edge to be output in time at most proportional to the number of
items output, and uses \( (3n + 2m + 2L_1(G))(1 + O(\log \log n/\log n)) \) bits.

5 The Density-Independent Case

5.1 Depth-First Search

Some aspects of the following proof are similar to those of [9, Lemma 3.2].

**Theorem 5.1.** A DFS of a directed or undirected graph with \( n \) vertices and \( m \) edges can be carried out in \( O(n + m) \) time with \( O(n \log \log (4 + m/n)) \) bits of
working memory.

**Proof.** Assume without loss of generality that \( m \geq n/2 \geq 1 \). We simulate the
algorithm of Theorem 4.1 but using asymptotically less space (unless \( m = O(n) \)).
Recall that the algorithm employs a stack \( S \) whose size is always bounded by \( nr \),
where \( r = O(\log (2 + m/n)) \). When a vertex is discovered by the DFS and enters
\( S \), it is permanently assigned an integer hue. The first vertices to be discovered are
given hue 1, the next ones receive hue 2, etc., and the vertices on \( S \) with a common
hue are said to form a segment. In general, a new segment is begun whenever the
current segment for the first time occupies more than \( n \) bits on \( S \). Thus no hue
larger than \( r \) is ever assigned.

As in [9], the algorithm does not actually store \( S \), which is too large, but only
a part \( S' \) of \( S \) consisting of the one or two segments at the top of \( S \). When a new
segment is begun and \( S' \) already contains two segments, the older of these is first
dropped to make room for the new segment. By construction, \( S' \) always occupies
\( O(n) \) bits.

The algorithm operates as that of Theorem 4.1 using \( S' \) in place of \( S \), except
when a pop causes \( S \) but not \( S' \) to become empty. Whenever this happens the
top segment of \( S \) is restored on \( S' \), as explained below, after which the DFS can
resume. Between two stack restorations a full segment disappears forever from \( S \),
so the total number of stack restorations is bounded by \( r \).
In order to enable efficient stack restoration, we maintain for each vertex \( v \) (a) its color—white, gray or black; (b) its hue; (c) whether it is currently on \( S' \); (d) the number of groups of \( \lceil m/n \rceil \) (out)edges incident on \( v \) that have been explored with \( v \) as the current vertex. The number of bits needed is \( O(1) \) for items (a) and (c), \( O(\log(2 + r)) \) for item (b) and \( O(\log(2 + dn/m)) \) for item (d), where \( d \) is the degree of \( v \). Summed over all vertices, this yields a bound of \( O(n\log(2 + r)) \) bits, as required. For each segment \( J \) on \( S' \), we also store on a second stack \( S_t \) the last vertex \( u \) of \( J \) (the vertex closest to the top of \( S \)) and the number of (out)edges incident on \( u \) explored by the DFS with \( u \) as the current vertex. The space occupied by \( S_t \) is negligible.

To restore a segment \( J \), we push the bottommost entry of \( J \) on \( S' \) and initialize accordingly the variables kept outside of \( S' \) to interpret entries of \( S' \) correctly. This can be done in constant time by consulting either the entry on \( S_t \) immediately below the top entry or separately remembered information concerning the root of the current DFS tree. We proceed to push on \( S' \) the remaining vertices in \( J \) one by one, stopping when the top entries on \( S' \) and \( S_t \) agree, at which point the restoration of \( J \) is complete and the normal DFS can resume. Each entry on \( S' \) above that of a vertex \( u \) is found by determining the first gray vertex in \( u \)'s adjacency array (counted cyclically from the position of \( u \)'s parent) that belongs to \( J \) (as we can tell from its hue) and is not already on \( S' \).

Because of item (d) of the information kept for each vertex, the search in the adjacency array of a vertex of degree \( d \) is easily made to spend \( O(\min\{d + 1, \lceil m/n \rceil\}) \) time on entries that were inspected before. Over all at most \( r \) restorations and over all vertices of degree at most \( \sqrt{m/n} \), this sums to \( O(rn\sqrt{m/n}) = O(m) \). Since a restoration involves \( O(n/r) \) vertices of degree larger than \( \sqrt{m/n} \), the sum over all restorations and over all such vertices is \( O(rn/r)(m/n)) = O(m) \). Altogether, therefore, the algorithm spends \( O(m) \) time on restorations and \( O(m) \) time outside of restorations.

□

Our remaining algorithms depend on the following lemma, which can be seen as a weak dynamic version of Lemma 4.4.

**Lemma 5.2.** For all \( n, N \in \mathbb{N} \), following an \( O(n) \)-time initialization, an array of \( n \) initially empty binary strings \( s_1, \ldots, s_n \) that at all times satisfy \( |s_i| = O(\log n) \) for \( i = 1, \ldots, n \) and \( \sum_{i=1}^{n} |s_i| \leq N \) can be maintained in \( O(n \log \log n + N) \) bits under constant-time reading and amortized constant-time writing of individual array entries.

**Proof.** Compute a positive integer \( h \) with \( h = \Theta((\log n)^2) \) and partition the strings into \( O(n/h) \) groups of \( h \) strings each, except that the last group may be smaller. For each group, store the strings in the group in \( O(\log n) \) piles, each of which holds all strings of one particular length in no particular order. For each string, we store its length and its position within the corresponding pile. Conversely, the entry for a string on a pile, besides the string itself, stores the number of the string within its group. The size of the bookkeeping information amounts to \( O(\log \log n) \) bits per string and \( O(n \log \log n) \) bits altogether.

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When a string changes, the string may need to move from one pile to another within its group. This usually leaves a "hole" in one pile, which is immediately filled by the entry that used to be on top of the pile. This can be done in constant time, which also covers the necessary update of bookkeeping information.

We are now left with the problem of representing \( O((n/h) \log n) = O(n/\log n) \) piles. Divide memory into words of \( \Theta(\log n) \) bits, each of which is large enough to hold one of the strings \( s_1, \ldots, s_n \). Rounding upwards, assume that each pile at all times occupies an integer number of words—this wastes \( O(n) \) bits. The update of a string may cause the sizes of up to two piles to increase or decrease by one word, but no operation changes the size of a pile by more than one word. Each pile is stored in a container, of which it occupies at least a quarter. For each pile we maintain its size, the size of its container, and the location in memory of its container, a total of \( O(\log n) \) bits per pile and \( O(n) \) bits altogether. We also maintain in a free pointer the address of the first memory word after the last container.

When a pile outgrows its container, a new container, twice as large, is first allocated for it starting at the address in the free pointer, which is incremented correspondingly. The pile is moved to its new container, after which its old container is considered dead. Conversely, when a pile would occupy less than a quarter of its container after losing a string, the pile is first moved to a new container of size twice the size of the pile after the operation and also allocated from the address in the free pointer. If every operation that operates on a pile pays 5 coins to the pile, by the time when the pile needs to migrate to a new container, it will have accumulated enough coins to place a coin on every position in the old container and a coin on every element of the pile. In terms of an amortized time bound, the latter coins can pay for the migration of the pile to its new container.

When an operation would cause the size of the dead containers to exceed that of the live containers (the containers that are currently in use) plus \( n \) bits, we carry out a “garbage collection” that eliminates the dead containers and re-allocates the piles in tightly packed new live containers in the beginning of the available memory, where each new container is made twice as large as the pile that it contains, and the free pointer is reset accordingly. The garbage collection can be paid for by the coins left on dead containers.

A string can be read in constant time, and updating it with a new value takes constant amortized time, as argued above. Because every pile occupies at least a quarter of its (live) container and the dead containers are never allowed to occupy more space than the live containers, plus \( n \) bits, the number of bits occupied by the array of strings at all times is \( O(n + N) \).

**Corollary 5.3.** For all \( n, N \in \mathbb{N} \), following an \( O(n) \)-time initialization, an array of \( n \) initially empty binary strings \( s_1, \ldots, s_n \) that at all times satisfy \( |s_i| = O(\log n/\log \log n) \) for \( i = 1, \ldots, n \) and \( \sum_{i=1}^{n} |s_i| \leq N \) can be maintained in \( O(n + N) \) bits under constant-time reading and amortized constant-time writing of individual array entries.

**Proof.** Maintain groups of \( \Theta(\log \log n) \) strings with the data structure of the previous lemma. In more detail, compute a positive integer \( q \) with \( q = \Theta(\log \log n) \)
and partition the $n$ strings $s_1, \ldots, s_n$ into $\lceil n/q \rceil$ blobs of $q$ consecutive strings each, except that the last blob may be smaller. If a blob consists of the strings $s_i, \ldots, s_j$, let its label be the binary string $1^{\lceil s_i \rceil}0s_i \ldots 1^{\lceil s_j \rceil}0s_j$. Because the label of a blob is of $O(\log n)$ bits, given the label and the number of a string $s$ within the blob, we can extract $s$ in constant time by lookup in a table of $O(n)$ bits that can be computed in $O(n)$ time. Similarly, given a new value for $s$, we can update $s$ within the blob in constant time. Maintain the sequence of $\lceil n/q \rceil$ blobs of $O(\log n)$ bits each with the data structure of Lemma 5.2. The number of bits needed is $O((n/q) \log \log n + N) = O(n + N)$, and the operation times are as claimed. \hfill \square

**Theorem 5.4.** A DFS of a directed or undirected graph with $n$ vertices and $m$ edges can be carried out in $O(n + m \log^* n)$ time with $O(n)$ bits of working memory.

**Proof.** Assume without loss of generality that $m \geq n/2 \geq 1$. Compute a positive integer $t$ and a sequence $p_1, \ldots, p_t$ of $t$ powers of 2 with the following properties:

- $1 = p_t < p_{t-1} < \cdots < p_2 < p_1 = \Theta(\sqrt{\log n})$.
- For $i = 2, \ldots, t$, $p_i \geq \log p_{i-1}$.
- $t = O(\log^* n)$.

This is easy: Begin by computing $p_1$ as a power of 2 with $p_1 = \Theta(\sqrt{\log n})$, at least 1, take $t = 1$ and then, as long as $p_t > 1$, increment $t$ and let $p_t$ be the smallest power of 2 no smaller than $\log p_{t-1}$, i.e., $p_t = 2^{[\log \log p_{t-1}]}$.

As in the algorithm of Theorem 5.1, we operate with a conceptual stack $S$ and an actual stack $S'$ that contains the topmost one or two segments on $S$. By definition, a complete segment now contains exactly $\lceil n/p_i^2 \rceil$ vertices. The complete segments in turn are partitioned into stripes, each of which has a rank drawn from $\{1, \ldots, t\}$. For $i = 1, \ldots, t$, a stripe of rank $i$ comprises exactly $(p_1/p_i)^2$ segments and therefore at least $n/p_i^2$ vertices. A stripe of rank 1 can be identified with the single segment that it contains. From the bottom to the top of $S$, the stripes occur in an order of nonincreasing rank (informally, larger stripes are deeper in the stack). Taken in the same order, the stripes are also assigned the indices 0, 1, 2, \ldots, and each vertex is marked with the (current) index of its stripe. For $i = 2, \ldots, t$, since the number of stripes of rank at least $i - 1$ is bounded by $p_{i-1}^2$, the index of every such stripe is an integer of at most $2 \log p_{i-1} \leq 2p_i$ bits.

An additional stack records the ranks of all stripes in the order in which the stripes occur on $S$. Using this stack and $S_t$ to start at the bottom entry of the topmost stripe, we can step through the vertices of that stripe in the order in which they occur on $S$. If the stripe is of rank 1, in particular, this enables us to carry out a stack restoration. If the topmost stripe is of rank $i > 1$, we can carry out the stack restoration by splitting the stripe into $(p_{i-1}/p_i)^2$ stripes of rank $i - 1$ and proceeding recursively. Conversely, if at some point there are $2(p_{i-1}/p_i)^2$ stripes of rank $i - 1$ for $i = 2$, we join the bottommost $(p_{i-1}/p_i)^2$ of these into a stripe of rank $i$ and continue the conditional join recursively for $i = 3, \ldots, t$. In order to know when to stop the join, maintain for $i = 1, \ldots, t$ the number of stripes of rank $i$. 
For \( i = 2, \ldots, t \), there are never more than \( 2(p_{i-1}/p_i)^2 \) stripes of rank \( i - 1 \), and the number of vertices contained in stripes of rank \( i - 1 \) is \( O((n/p_i^2)(p_{i-1}/p_i)^2) = O(n/p_i^2) \). As noted above, every index of such a stripe is of \( O(p_i) \) bits, so the total number of bits consumed by indices of stripes of rank \( i - 1 \) is \( O(n/p_i) \). Summed over all values of \( i \), this yields \( O(n) \) bits occupied by stripe indices. Since the stripe indices are of \( O(\log \log n) \) bits by the upper bound on \( p_1 \), they can be maintained with the data structure of Corollary 5.3 at a cost of constant time per operation and a total of \( O(n) \) bits. The other data structures used by the algorithm are easily seen to fit in \( O(n) \) bits as well.

Fix \( i \in \{2, \ldots, t\} \), let \( N_0 \) be the number of vertices in a single stripe of rank \( i \), let \( N \) be the current number of vertices in stripes of rank at most \( i - 1 \) and define \( \Phi \) as \( |N - N_0| \). The split of a stripe of rank \( i \) into stripes of rank \( i - 1 \) reduces \( \Phi \) from \( N_0 \) to \( 0 \), the join of stripes of rank \( i - 1 \) into a stripe of rank \( i \) reduces \( \Phi \) by exactly \( N_0 \), and a push or pop on \( S \) increases \( \Phi \) by at most \( 1 \). A potential argument now shows the sum over all splits of stripes of rank \( i \) or joins of stripes to stripes of rank \( i \) of the number of vertices contained in the stripes concerned and the stripes above them to be \( O(n) \). Summing over all values of \( i \), this yields a bound of \( O(tn) \). If we again store with each vertex the number of completely explored groups of \( \lceil m/n \rceil \) incident (out)edges, the time needed for all splits and joins is \( O(tn(m/n)) = (m \log^* n) \), and all other parts of the algorithm work in \( O(m) \) time. \( \square \)

By using only the ranks \( 1, \ldots, k \), for some \( k \in \mathbb{N} \), i.e., by omitting all joins that would create stripes of rank \( k + 1 \) or more, we can lower the time bound of the previous algorithm to \( O(n + km) \), but at the price of having to store indices of stripes of rank \( k \) for almost all vertices.

**Theorem 5.5.** For every constant \( k \in \mathbb{N} \), a DFS of a directed or undirected graph with \( n \) vertices and \( m \) edges can be carried out in \( O(n + m) \) time with \( O(n \log^* n) \) bits of working memory.

### 5.2 Biconnected and 2-Edge-Connected Components

This subsection describes algorithms for the problems considered in Theorem 4.9 but with the resource bounds of Theorems 5.4 and 5.5. The approach uses elements of an algorithm of Kammer et al. [15] for computing cut vertices.

The main differences to the density-dependent setting of Theorem 4.9 can be explained in terms on a nontree edge \( \{x, y\} \), where \( y \) is a descendant of \( x \). In the density-dependent case, because a DFS forest is computed in a first pass and remembered, in a second pass \( \{x, y\} \) can be processed when a preorder traversal of the forest reaches \( x \). This makes the processing of \( \{x, y\} \), which is essentially the marking of a number of vertices, efficient, because the marking can stop as soon as it reaches a vertex that is already marked.

In the density-independent setting there is only a single pass, and \( \{x, y\} \) must be processed when it is explored by the DFS in that pass, at which time the current vertex of the DFS is \( y \). The vertices to be marked all lie on the gray path.

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between \( x \) and \( y \), but they may be few and distributed irregularly, and we cannot afford to follow the gray path backwards all the way from \( y \) to \( x \) in order to find them. To solve this problem we could keep on a separate stack \( S_u \) only those gray vertices that have not yet been marked and let the processing of \{ \( x, y \) \} pop and mark vertices from \( S_u \) until, loosely speaking, \( x \) is reached.

The approach outlined in the previous paragraph would be correct, but the stack \( S_u \) of unmarked gray vertices would take up too much space. Indeed, just as we can keep only one or two segments, the surface segments, of the virtual stack \( S \) on an actual stack \( S' \), we can keep only the part of \( S_u \) that contains vertices in the surface segments on an actual stack that, for convenience, we continue to call \( S_u \). Because of this, we cannot mark vertices in the other segments, the buried segments. The marking must still be carried out, but it can be postponed until the segments in question once again become surface segments. In order to realize this, we use a special “propagating mark” that marks a vertex but also calls for the marking to be extended towards the top of the stack once the propagating mark is no longer buried (other, normal, marks may have become buried and should not be propagated in the same way).

The “scope” of a propagating mark could extend to the vertex \( y \) that was the current vertex of the DFS when the propagating mark was placed. Since this is not easy to handle, we instead stipulate that the “scope” of a propagating mark ends at the end of its stripe. As a consequence, the processing of a non-tree edge \{ \( x, y \) \} may require several buried stripes to receive propagating marks. For efficiency reasons we must prevent stripes for which this already happened from being processed again, which we can do by placing buried stripes, represented by their bottommost vertices, on \( S_u \), processing stripes only when they are on \( S_u \), and removing them from there when this happens. Since the number of stripes is small, storing distinct stripes on \( S_u \) does not violate the space bound.

The remaining details are given in the following proof. In particular, it is shown how one can know when to stop when popping from \( S_u \).

**Theorem 5.6.** Given an undirected graph \( G = (V, E) \) with \( n \) vertices and \( m \) edges, in \( O(n + m \log^* n) \) time and with \( O(n) \) bits of working memory, we can compute the biconnected components and the 2-edge-connected components with their vertices and/or their edges, the cut vertices and the bridges of \( G \). For every constant \( k \in \mathbb{N} \), the same problems can also be solved in \( O(n + m) \) time with \( O(n \log^{(k)} n) \) bits of working memory.

**Proof.** Let us change the DFS algorithm of Theorem 5.4 or Theorem 5.5 to make it carry out a stack restoration not when the actual stack \( S' \) is empty, but already when it contains only two vertices (without loss of generality, full segments comprise at least three vertices). Let us call this modified stack restoration *eager restoration*. We show how to augment the algorithm with steps that maintain a Boolean array \( R[1..n] \) such that whenever the gray path of the DFS contains three vertices \( u, v \) and \( w \) in that order, \( R[u] = true \) exactly if the part of the graph explored so far contains a cycle through the two edges \{ \( u, v \) \} and \{ \( v, w \) \}. With \( P(w) \) defined as in Subsection 4.3, it is clear that \( P(w) \) can be read off \( R[u] \)
when the search is about to withdraw over \{v, w\}. This enables us to compute a table of \(P\), and the arguments given in Subsection 4.3 show how to solve the problems indicated in the theorem using only \(O(n + m)\) additional time and \(O(n)\) additional bits.

\(R\) is in fact a virtual array implemented via an actual array \(\overline{R}[1 \ldots n]\), each of whose entries takes values in \{false, true, propagating-true\}. The connection is as follows: For \(u \in V\), \(R[u] = true\) if and only if \(\overline{R}[u] \in \{true, propagating-true\}\) or \(\overline{R}[u'] = propagating-true\) for some \(u' \in V\) that belongs to the same stripe as \(u\) and precedes \(u\) within that stripe (thus \(u'\) was pushed before \(u\)). Moreover, we stipulate that \(R[u] = \overline{R}[u]\) for all \(u\) currently stored on \(S'\) (i.e., belonging to a surface segment), which implies that we can determine \(R[u]\) in constant time whenever we need it.

For each \(u \in V\) currently on \(S'\), we store with \(u\) its position on \(S'\). We also use an additional stack \(S_u\) that contains some of the vertices on \(S\) in the same order as on \(S\). If \(u\) is stored on \(S'\), it is also present on \(S_u\) exactly if it is followed on the gray path by at least two vertices and \(R[u] = false\)—informally, if \(u\)'s gray child has not yet been “covered” by a nontree edge. A vertex \(u\) on \(S\) but not on \(S'\) can be stored on \(S_u\) only if it is the bottommost vertex in its stripe; if so, it is certain to be stored on \(S_u\) if \(R[u] = false\) for at least one vertex \(u'\) in the same stripe as \(u\). Informally, \(u\) now represents its stripe and records the fact that the stripe may contain one or more “uncovered” vertices.

What remains is to describe the manipulation of \(\overline{R}\) and \(S_u\), which must respect the invariants introduced above. Initially \(\overline{R}[u] = false\) for all \(u \in V\) and \(S_u\) is empty. When the DFS discovers a vertex \(w\) over a tree edge \{\(v, w\)\} and \(v\) has a parent \(u\), we set \(\overline{R}[u] := false\) and push \(u\) on \(S_u\). When the DFS withdraws from a vertex \(w\) to its parent \(v\) and \(v\) has a parent \(u\), we pop \(u\) from the stack \(S_u\) if it is present there—if so, it is the top entry. When processing a nontree edge \{\(x, y\)\}, where \(y\) is the current vertex and \(x\) is an ancestor of \(y\), we pop all vertices from \(S_u\) that are equal to \(x\) or closer than \(x\) to the top of \(S\). Informally, some of these vertices—those on \(S'\)—represent only themselves, while each of the remaining vertices represents a whole stripe. Because we know the position on \(S'\) of every vertex stored on \(S'\) and the stripe index of every vertex, the process can happen in constant time plus constant time per vertex popped. For each vertex \(u\) popped, we set \(\overline{R}[u] := true\) if \(u\) is stored on \(S'\) (if \(u\) belongs to a surface segment) and \(\overline{R}[u] := propagating-true\) if not. When a stripe is split or several stripes are joined, the entries on \(S_u\) are changed correspondingly: The bottommost vertex of each stripe that disappears is popped from \(S_u\) (if present), and the bottommost vertex of each new stripe is pushed on \(S_u\).

When a segment is restored, an invariant demands that the value propagating-true be eliminated from its vertices. This is done in a simple scan of the segment from bottom to top: If the value propagating-true is ever encountered, the value of \(\overline{R}\) is set to true for the relevant vertex and all vertices that follow it, in accordance with the relation between \(\overline{R}\) and \(R\) described above. Moreover, all vertices \(u\) with \(R[u] = false\) after the bottommost vertex are pushed on \(S_u\) (because of the eager restoration, every such vertex is followed on the gray path by at least two vertices).
When a segment is dropped from $S'$, all entries of its vertices on $S_u$ are replaced by a single entry for its bottommost vertex.

The mapping of the $O(n/\log n)$ vertices on $S'$ to their positions on $S'$ can be maintained in a way described by Kammer et al. [15]: When the first vertex of a segment is pushed on $S'$, the DFS is first executed without this positional information until it has computed the set $U$ of vertices in the segment. Then $\Theta(\log n)$ bits are allocated to each vertex in $U$ using static space allocation, and finally the relevant part of the DFS is repeated, at which point the position of each new vertex on $S'$ can be recorded. A similar procedure is followed when a segment is restored. It is shown in [15] how table lookup allows the static space allocation to happen sufficiently fast, namely in $O(n/\log n)$ time—in essence, it suffices to mark each of $O(n/\log n)$ regularly spaced vertices in $V$ with the number of smaller vertices belonging to $U$. An alternative is to appeal to the fast construction of rank-select structures of Baumann and Hagerup [4]. In either case, it is easy to see that the total number of bits needed is $O(n)$. The fact that the usual stack restoration has been replaced by eager restoration does not invalidate the bound established in its proof for splitting and joining stripes. The other steps described above are no more expensive, to within a constant factor. In particular, the number of pops from $S_u$ is bounded by the number of pushes on $S_u$. Therefore the asymptotic time and space bounds demonstrated for the DFS are valid also for the entire computation.

□

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