Group classification of a family of generalized Klein-Gordon equations by the method of indeterminates.

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Abstract. A method for the group classification of differential equations we recently proposed is applied to the classification of a family of generalized Klein-Gordon equations. Our results are compared with other classification results of this family of equations labelled by an arbitrary function. Some conclusions are drawn with regards to the effectiveness of the proposed method.

1. Introduction
The Klein–Gordon equation is a relativistic wave equation that describes the behavior of spinless particles. It frequently occurs in several fields of applied physics, including particle physics, astrophysics, and classical mechanics [1, 2, 3]. A slight variant of this hyperbolic equation is the sine-Gordon equation which is one of the best known integrable models, and whose solutions have soliton properties [4, 5]. It differs from the former by a mere change of a non derivative term in the equation. Both equations can be seen as members of a general class of equations of the form

\[ u_{tt} - u_{xx} = Z(u), \]

where \( Z = Z(u) \) is an arbitrary function, \( u = u(t, x) \) and a variable in a subscript denotes partial derivative w.r.t. the variable, so that \( u_t = \partial u/\partial t, \ u_x = \partial u/\partial x, \ u_{tx} = \partial^2 u/(\partial t \partial x) \) and so on. A more general form of (1) which often occurs in applications [1, 2, 3] is the family of equations

\[ u_{tt} - u_{xx} = Z(t, x, u). \]

Under the change of variables \( t = (y - z)/2, \ x = (y + z)/2, \) and after renaming the new variables to the former as we shall often do, (2) is transformed to the form

\[ u_{tx} = F(t, x, u) \]

for a certain function \( F = F(t, x, u) \). The group classification of (2) with \( Z = Z(u) \) was performed in [6] by the direct method, coupled with a process of triangulation [7, 8]. Muatjetjeja [9] carried out the classification by the direct method of a more specific version of (2) modulo a slight change of variables, with the function \( Z = Z(x, u) \) depending only on some arbitrary constants. On the other hand the group classification of (3) was performed in [3] by a combination of the direct method and the well-known algebraic method.
In this paper, we illustrate the application of our newly proposed method of group classification termed the method of indeterminates to the family of nonlinear equations

\[ u_{tx} - F(x, u) = 0. \] (4)

where \( F = F(x, u) \) is an arbitrary function with \( F_{uu} \neq 0 \). Moreover, using this method we provide the complete group classification of some subfamilies of (4) including the subfamily of equations

\[ u_{tx} - F(uG(x)) = 0, \] (5)

where both \( F \) and \( G \) are arbitrary functions of their arguments, with \( F_{uu} \neq 0 \) as above.

2. The method of indeterminates

This method consists in treating whenever possible the classifying equations as a system of polynomials in whatever variables can be appropriately considered as associated indeterminates. Any extension of the principal algebra will then occur only for those particular values of the labeling parameters of the equation yielding a linear dependence of a set of indeterminates. The consideration of all possible linear dependence of indeterminates will yield the solution to the group classification problem.

Let us recall that the classifying equations in a group classification problem refer to the remainder of determining equations after the most general form of the symmetry algebra has been derived, without assigning any specific values to the labelling parameters of the equation. That is, while such parameters remain entirely arbitrary.

This vocabulary will become much clearer when the method is applied to specific cases of equations in the following lines.

3. Equivalence transformations

Given that equivalent equations under point transformations, in particular, have isomorphic symmetry groups, it is more meaningful and much efficient to perform the group classification of a family of equations under its equivalence group, that is, under the largest Lie pseudo-group of point transformations leaving invariant the family of equations.

All relevant facts about the equivalence group of (4) are summarized in the following results.

**Theorem 1.** The equivalence group of (4) consists of invertible point transformations

\[ t = Jy + b, \quad x = S(z), \quad u = Kw + h(z), \] (6a)

where \( w = w(y, z) \) and \( J, K, \) and \( b \) are arbitrary constants with \( 0 \neq JK \), while \( S \) and \( h \) are arbitrary functions of their arguments. The transformed version of (4) under (6a) takes the form

\[ w_{yz} = F_1(z, w) \] (6b)

where

\[ F_1(z, w) = \frac{JS'(z)S(z) + F(S(z), h(z) + Kw)}{K}. \] (6c)

**Proof.** Let us assign to each monomial in \( u \) and its derivatives a weight equal to the total order of all derivatives occurring in the monomial, so that for instance \( u^3 \) has weight 0 while \( u_{tx}^2u_{xx} \) has weight 8. Based on this weight considerations and the fact that equivalence transformations
preserve maximal weight, it readily follows that any equivalence transformation of (4) should be sought in the form
\[ t = R(y, z), \quad x = S(y, z), \quad u = H(y, z, w), \quad \text{with } (R_z S_y - R_y S_z) H_w \neq 0. \] (7)

Next, requesting that the transformed version of (4) under (7) leaves the whole family of equations (7) invariant, and in particular that the function \( F_1 \) appearing in (6c) should depend only on \( z \) and \( w \) yields the required result.

The equivalence transformations (6a) clearly define an equivalence relation on the space of all functions \( F = F(z, w) \) acting on the product space of independent variables \( z \) and dependent variables \( w \). This is given precisely by the equality (6c) defining the equivalence between the function \( F \) and its transformed version \( F_1 \). Such an equivalence relation will be denoted \( F \sim F_1 \).

4. Preliminary group classification

Denote by
\[ v = \xi \partial_t + \eta \partial_x + \phi \partial_u \] (8)
a symmetry vector of an equation of the form (4), where \( \xi, \eta, \) and \( \phi \) are functions of \( t, x, u \) and by \( v^{[2]} \) the second prolongation of \( v \). The determining equations of (4) are given by the identical vanishing of the coefficients of the left-hand side of
\[ v^{[2]}(\Delta) \bigg|_{\Delta = 0} = 0. \] (9)

That is, the coefficients of the expression \( v^{[2]}(\Delta) \) taken on the solution surface \( \Delta = 0 \), and represented as a polynomial in the derivatives of the dependent variable \( u \) [10]. Here \( \Delta \) is the left-hand side of (4). Solving (9) shows that the generator \( v \) in (8) must be of the general form
\[ v = \xi(t) \partial_t + \eta(x) \partial_x + k_0 u + R(t, x), \] (10)
for some functions \( \xi = \xi(t), \eta = \eta(x), \) and \( R = R(t, x) \) and for an arbitrary constant \( k_0 \). The remaining determining equations, which we call the classifying equations are given in this case by the single equation
\[ R_{tx} + F(k_0 - \eta_x - \xi_t) - \eta F_x - (R + k_0 u) F_u = 0. \] (11)

When the function \( F = F(x, u) \) is assumed to be arbitrary, it follows from (11) that \( \xi = k_1 \) for some arbitrary constant \( k_1 \) and \( 0 = k_0 = R = \eta \). The principal algebra \( L \) of (4) is thus one-dimensional and generated by
\[ v = \partial_t. \] (12)

In view of the expression of the unknowns \( \xi, \eta, \) and \( \phi \) in (10), it turns out that the expressions \( u, F, F_x \) and \( F_u \) may be considered as indeterminates in (11). As already stated, any extension of the principal algebra may occur only if a subset of the resulting list \( M = \{1, u, F, F_x, F_u, u F_u\} \) of monomials is linearly dependent. Denote by \( \theta, 1 \leq \theta \leq 5 \) the number of distinct elements \( M_j \) of \( M \) in a given vanishing linear combination
\[ 0 = \sum_{j=0}^{\theta} \lambda_j M_j. \] (13)

Here the coefficients \( \lambda_j \) are assumed nonzero and considered as functions of \( x \) for \( F_x \neq 0 \). For each value of \( \theta \) we derive the corresponding value of \( F \) up to the equivalence transformations.
(6c). Solutions $F$ of (13) satisfying the preliminary conditions (such as $F_{uu} \neq 0$) will be termed \textit{admissible}, while an admissible function which does yield a particular symmetry class will be termed a \textit{particular function}. For $\theta = 1$, the only possible vanishing linear combination is $F_x = 0$, which yields $F = F(u)$. For other values of $\theta$ the admissible functions $F$ are given as follows, where $G, H, S, T$, and $V$ are arbitrary functions of their arguments.

4.1. Admissible functions for $\theta = 2$

II.1 $F = H(u) + G(x)$

II.2 $F = G(x) - \ln(u)/V(x)$

II.3 $F = H[u + G(x)] \sim H(u)$

II.4 $F = G(x)H(u)$

II.5 $F = H[G(x)u]$\[H[x] = 0.5x^2, G(x) = \ln(x), V(x) = e^{x^2}]

II.6 $F = e^{uV(x)}G(x)$

II.7 $F = u^{V(x)}G(x)$.

In subsequent listings of admissible functions, attempts will be made to discard by inspection functions which up to equivalence transformations are redundant, for having appeared in a previous listing.

4.2. Admissible functions for $\theta = 3$

III.1 $F = G(x)H(u) + S(x)$

III.2 $F = H[G(x)u] + S(x)$

III.3 $F = S(x) + e^{uV(x)}G(x)$

III.4 $F = S(x) + u^{V(x)}G(x)$

III.5 $F = S(x)H[uG(x)]$

4.3. Admissible functions for $\theta = 4$

IV.1 $F = S(x)H[G(x)u] + V(x)$

For $\theta = 5$, the only admissible function has expression

$$F = S(x)H(G(x)u + T(x)) + V(x),$$

but this function is equivalent to the one given in 4.3, and is therefore not new.

It should be noted that the determination of linearly dependent monomials from $M$ has not given rise up to now to a complete group classification of (4). That only allowed the determination of all possible subfamilies of equations which may extend the principal Lie algebra, each of which is now labelled by functions of a single variable instead of the original function $F$ of two variables. In other words, the method of indeterminates involves a recursive process. Nevertheless, the complete group classification of subfamilies of (4) corresponding to many of the admissible functions listed in Subsections 4.1 to 4.3 is more or less straightforward. Some of these complete group classifications will be carried out in the next section, and this will include genuine cases of application of the method.

5. Complete group classification

We provide in this section the complete group classification of (4) for a number of admissible values of the function $F = F(x, u)$.
5.1. Case $F = H(u)$.
This case is much well-known and has been treated in a number of papers [11, 6, 1]. The
determining equations (9) are reduced to

$$R_{tx} + H(k_0 - \eta_x - \xi_t) - (R + k_0 u)H_u = 0. \tag{14}$$

This shows that for $F$ arbitrary, the principal algebra is generated in this case by

$$v = (k_1 t + k_2) \partial_t + (k_3 - k_1 x) \partial_x \tag{15}$$

and is therefore three-dimensional. Here and in the sequel, the $k_j$ represent arbitrary constants.
It also follows from (14) that under the equivalence transformations (6) the list of all admissible
functions of $H$ is given by

$$H = c - \ln(\alpha), e^\alpha, u^\alpha, e^u + \beta, \text{ or } u^\alpha + \beta,$$

where $\alpha$ and $\beta$ are arbitrary constants. It turns out however that the only particular functions
from this list are the functions $F = e^u$ and $F = u^\alpha$, ($\alpha \notin \{0, 1\}$). The corresponding symmetry generators are given by

$$v = f(t) \partial_t + g(x) \partial_x - [f(t) + g(x)] \partial_u, \text{ for } F = e^u \tag{16a}$$

$$v = (k_1 t + k_2) \partial_t + (k_3 x + k_4) \partial_x - \frac{(k_1 + k_3) u}{\alpha - 1} \partial_u \text{ for } F = u^\alpha (\alpha \neq 0, 1), \tag{16b}$$

where $f = f(t)$ and $g = g(x)$ are arbitrary functions. In particular, the symmetry algebra is
infinite dimensional for $F = e^u$, while it is 4-dimensional for $F = u^\alpha$, ($\alpha \neq 0, 1$).

We now consider the cases II.6 and II.7 outlined above, and corresponding to $F = e^{uV(x)} G(x)$
and $F = u^{V(x)} G(x)$, respectively. The application of our method of indeterminates to these two
cases can be done without carrying out any explicit calculations. Indeed, the unconditional linear
independence of monomials occurring in the corresponding classifying equation is self-evident.
The classification results for these two cases can be summarized as follows.

5.2. Case $F = e^{uV(x)} G(x)$
In this case, when $V_x \neq 0$, the symmetry algebra is three-dimensional and generated by

$$v = (k_1 + k_2 t) \partial_t - k_3 \frac{V}{V_x} \partial_x + \left[ - \frac{k_2 - 2 k_3}{V} + k_3 \left( u + \frac{G_x}{G V_x} - \frac{V_x}{V_x^2} \right) \right] \partial_u.$$ 

Otherwise, when $V_x = 0$, one has $F \sim e^u G(x)$ in this case and the symmetry algebra is infinite
dimensional and generated by

$$v = f(t) \partial_t + g(x) \partial_x - \left( f(t) + g(x) \frac{G_x}{G(x)} \right) \partial_u.$$ 

5.3. Case $F = u^{V(x)} G(x)$
This case is similar to the above case 5.2. Indeed, for $V_x \neq 0$, there is no extension of the
principal algebra and thus the symmetry algebra is generated by $v = \partial_t$ just as in (12). On the
other hand, for $V_x = 0$ the symmetry algebra is 4-dimensional and generated by

$$v = (k_1 + k_2 t) \partial_t + \frac{1}{G(x)} \left( k_4 - \int [k_2 + k_3 (V - 1)] G(x) dx \right) \partial_x + k_3 u \partial_u.$$
Table 1. Symmetry classes of Equation (4) with \( F = H[u \, G(x)] \). In the table, \( f = f(t), g = g(x)\), and \( 0 \neq G = G(x) \) are arbitrary functions. The functions \( f \) and \( g \) occur in generators of infinite dimensional symmetry algebras. The \( k_j \) for \( j = 1, 2, 3, \ldots \) are arbitrary constants whose number in a symmetry generator specifies the corresponding finite dimension of the symmetry algebra. Also, \( c, d \in \{0, 1\} \) and \( \alpha \) is an arbitrary constant. Any constraints on \( \alpha \) is specified in the table.

| Particular functions \( F \) | Conditions | Symmetry generators |
|-----------------------------|-----------|---------------------|
| \( e^u \)                  | \( f(t) \partial_t + g(x) \partial_x - (f_t + g_x) \partial_u \) |
| \( u^\alpha \)             | \( \alpha \notin \{0, 1\} \) | \( (k_1 t + k_2) \partial_t + (k_3 x + k_3) \partial_x - \frac{(k_1 + k_2) u}{u-1} \partial_u \) |
| \( e^u G(x) \)             | \( G \neq 0 \) | \( (k_1 + k_2 t) \partial_t - k_3 \frac{G}{x} \partial_x + \left[ -\frac{k_2 - k_3}{u} + k_3 \left( u - \frac{G(x)}{x} \right) \right] \partial_u \) |
| \( u^\alpha G(x) \)        | \( \alpha \notin \{0, 1\} \) | \( (k_1 + k_2 t) \partial_t + \frac{1}{k_4} \left( k_4 - f(k_2 + k_3(\alpha - 1)) G dx \right) \partial_x + k_3 u \partial_u \) |
| \( c + \ln(u) \)           | \( (k_1 + k_2 t) \partial_t + (k_3 - k_2 x) \partial_x \) |
| \( c + \ln(u \, e^{\alpha x}) \) | \( \alpha \neq 0 \) | \( (k_1 + k_2 t) \partial_t - \frac{k_3}{x} \partial_x + k_2 u \partial_u \) |
| \( c + \ln(u \, (d + x \, \alpha)^\alpha) \) | \( \alpha \neq 0 \) | \( k_1 + k_2 t \left( 1 + \frac{1}{x} \right) \partial_t - \frac{k_2 (d + \alpha x)}{\alpha^2} x \partial_x + k_3 u \partial_u \) |
| All Others                  | \( \partial_t \) |

5.4. Case \( F = H[u \, G(x)] \)

Letting \( F = H[u \, G(x)] \) in the classifying equations (11) and then setting \( w = u \, G(x) \) yields

\[
-G^2 R H w + w H w (-G k_0 - \eta G x) + G H (k_0 - \eta x - \xi t) + G R t x = 0. \tag{17}
\]

The form of monomials occurring in (17) shows that admissible functions \( H = H(w) \) are the same as for the case \( F = H(u) \) given in Subsection 5.1. Reverting back to the original variables \( u \) and \( x \) through \( w = u \, G(x) \) then yields the following list

\[
\mathcal{A} = \left\{ c + \ln[u \, G(x)], e^u G(x), e^{u \, G(x)} + \beta, u^\alpha G(x), u^\alpha G(x) + \beta \right\} \tag{18}
\]

of admissible functions up to equivalence transformations, where we may assume \( \beta = 1 \) and \( c \in \{0, 1\} \), while \( \alpha \) is an arbitrary constant. The group classification of (4) corresponding to \( F = u^\alpha G(x) \) and \( F = e^{u \, G(x)} \) have already been carried out in 5.2 and 5.3. The procedure is similar for the functions \( F = e^{u \, G(x)} + \beta \) and \( F = u^\alpha G(x) + \beta \). We therefore discuss only the case \( F = c + \ln[u \, G(x)] \) that occurs in the list \( \mathcal{A} \).

The substitution \( F = c + \ln[u \, G(x)] \) in the classifying equations (11) shows that the symmetry generator should be of the form

\[
v = (k_1 t + k_2) \partial_t + [(k_0 - k_1)x + k_3] \partial_x + k_0 u \partial_u \tag{19}
\]
while (11) is reduced to
\[-k_0 G - k_3 G_x + (k_1 - k_0)x G_x = 0. \tag{20}\]

Therefore, for \( G = G(x) \) arbitrary the principal algebra is given by (12). Applying the method of indeterminates to the polynomial (20) in the indeterminates \( x, G, G_x \) shows that the principal algebra may be extended in the actual case only if \( G \) equals either \( \beta e^{\alpha x}, \beta x^\alpha \), or \( \beta(1 + x\alpha)^\alpha \), for some arbitrary constants \( \alpha \) and \( \beta \). With these particular values for \( G \) at hand the complete classification for the case \( F = c + \ln[u G(x)] \) easily follows. The complete classification of (4) corresponding to \( F = H[u G(x)] \) is given in Table 1.

6. Concluding remarks

In this paper we have endeavoured to explain our method of indeterminates for group classification largely through a specific example, by applying it to the preliminary group classification of (4). More exactly, we have given the principal algebra for (4) and all possible subfamilies of this class of equations that may yield an extension of the principal algebra. In each such subfamily, the labelling parameter function \( F = F(x, u) \) now depends only of one variable instead of two as in the original equation. By considering some cases of these subfamilies, we have shown in Subsections 5.1 to 5.4 how an application of the method yields a complete group classification of a given class of equations.

It should be recalled that the case \( F = H(u) \) treated in Subsection 5.1 has been considered in a number of papers. In particular, Azad et al [6] performed a group classification of this subfamily of equations by a combination of the direct method and the powerful process of triangulation [7, 8]. This entails solving in particular certain third order differential equations to find the admissible functions as opposed to solving only first order ones in our case. But even so, no explanation is given in [6] about the process through which the list of admissible functions is ultimately obtained. For instance, that paper does not make use of equivalence transformations and often arbitrarily reduces functions to simpler ones, also without justification. The earlier treatment of the case \( F = H(u) \) considered in Subsection 5.1 was done by S. Lie [11] through a direct method of analysis.

The method of indeterminate turns out to be simple and systematic. It is also algorithmic and can be, at least partly, translated into computer codes. It can in principle easily handle the case of several labelling functions \( F \) of several variables through a tree process. However, the amount of operations to be carried out for the complete classification grows quickly with the number of arguments in each parameter function \( F \). Typically, admissible functions are found by solving first order differential equations (des), or even algebraic equations. Nevertheless, the order of these des may increase if the arbitrary labelling functions appear in the equation together with their derivatives. But such an increase is of course not specific to the method of indeterminates, and will occur regardless of the method of group classification used.

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