Tortoise coordinate and Hawking effect in a dynamical Kerr black hole

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Abstract

Hawking effect from a dynamical Kerr black hole is investigated using the improved Damour-Ruffini method with a new tortoise coordinate transformation. Hawking temperature of the black hole can be obtained point by point at the event horizon. It is found that Hawking temperatures of different points on the surface are different. Moreover, the temperature does not turn to zero while the dynamical black hole turns to an extreme one.

Keywords: Hawking effect, event horizon, Klein-Gordon equation, tortoise coordinate, dynamical Kerr black hole

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1 Introduction

It is well known that Hawking effect in a black hole is one of the most striking phenomena\cite{1, 2}. A black hole was recognized to have thermal property right after the four laws of black hole thermodynamics had been built successfully and Hawking radiation had been discovered. In 1976, Damour and Ruffini proposed a new method with which one can calculate Hawking radiation\cite{3}. Using this method, Liu, et al have proved that a Kerr-Newman black hole radiates Dirac particles\cite{4, 5}. In 1990’s, Z. Zhao, X. X. Dai and Z. Q. Luo, et al improved Damour-Ruffini method to study Hawking effect from some dynamical black holes. They only investigated several kinds of dynamical spherically symmetric black holes via the improved method\cite{6, 7}.

Many works are also focused on Hawking effect from dynamical black holes for these years\cite{8, 9}. Y. P. Zhang et al studied Hawking effect from a Vaidya black hole via Hamilton-Jacobi method\cite{10}. S. W. Zhou et al discussed the same problem through Parikh’s tunneling method\cite{11}. Recently we have improved Zhao’s tortoise coordinate. Using the new tortoise coordinate, Hawking effect in some dynamical spherically symmetric black holes has been investigated\cite{12}. We have got more accurate expressions of surface gravity and Hawking temperature.

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It is well known that a stationary black hole has identical temperature on the event horizon surface. A dynamical spherically symmetric black hole has also identical temperature at the same time while its temperature varies with time\cite{12}. However, the temperature at different points on the surface of an axisymmetric dynamical black hole is probably a point-dependent variable. Hawking effect in a dynamical Kerr black hole will be investigated in the following parts.

The organization is as follows. In Sec. 2, we will give a brief overview on a dynamical Kerr black hole. In Sec. 3, we will discuss Hawking effect in a dynamical Kerr black hole under the new tortoise coordinate transformation. In Sec. 4, some conclusion and discussion will be given.

## 2 The dynamical Kerr black hole

The line element of a dynamical Kerr black hole can be expressed in the advanced Eddington-Finkelstein time coordinate as \cite{13}

\[
ds^2 = -(1 - \frac{2mr}{\rho^2})dv^2 + 2dvdr - \frac{4mra\sin^2 \theta}{\rho^2}dvd\varphi - 2a\sin^2 \theta drd\varphi + \rho^2d\theta^2 + (r^2 + a^2 + \frac{2mra^2\sin^2 \theta}{\rho^2})\sin^2 \theta d\varphi^2, \]  

where \( \rho^2 = r^2 + a^2 \cos^2 \theta \), \( m = m(v), a = a(v) \).

The null hypersurface condition

\[
g^{\mu\nu} \frac{\partial f}{\partial x^\mu} \frac{\partial f}{\partial x^\nu} = 0 \]  

can be rewritten as

\[
r^2(1 - 2\dot{r}) - 2mr + a^2(1 - 2\dot{r} + r^2 \sin^2 \theta) + r^2 = 0, \]  

where \( \dot{r} = \frac{\partial r}{\partial v}, r' = \frac{\partial r}{\partial \theta} \).

The Eq.(3) determines the local event horizon of the dynamical Kerr black hole, whose solution is

\[
r_H = \frac{m \pm \sqrt{m^2 - (1 - 2\dot{r}_H)(1 - 2\dot{r}_H + \dot{r}_H^2 \sin^2 \theta)a^2 + r_H^2}}{1 - 2\dot{r}_H}. \]  

Obviously, it depends on the angular variable \( \theta \), and is different from the case of spherically symmetric black holes.

## 3 Hawking effect from the event horizon

The Klein-Gordon equation in a dynamical Kerr space-time is

\[
a^2\sin^2 \theta \frac{\partial^2 \Phi}{\partial v^2} + 2(a\dot{v}\sin^2 \theta + r)\frac{\partial \Phi}{\partial v} + 2(r^2 + a^2)\frac{\partial^2 \Phi}{\partial v \partial r} \]
\begin{align*}
+2a\frac{\partial^2 \Phi}{\partial v \partial \varphi} + 2a\frac{\partial^2 \Phi}{\partial r \partial \varphi} + (r^2 + a^2 - 2mr)\frac{\partial^2 \Phi}{\partial r^2} \\
+2(r - m + a\dot{a})\frac{\partial \Phi}{\partial r} + \frac{\partial^2 \Phi}{\partial \theta^2} + \dot{a}\frac{\partial \Phi}{\partial \varphi} \\
+ \cot \theta \frac{\partial \Phi}{\partial \theta} + \frac{1}{\sin^2 \theta \partial \varphi^2} - \mu^2 \rho^2 \Phi = 0,
\end{align*}

(5)

where \(\mu\) is the mass of a Klein-Gordon particle.

It is well known that when Hawking effect from a Schwarzschild black hole is investigated using Damour-Ruffini method, the tortoise coordinate is defined as following\[3\]

\[r_* = r + 2M \ln \left[ \frac{r - 2M}{2M} \right].\]  

(6)

For a dynamical Kerr black hole, a new tortoise coordinate can be written as

\[r_* = r + \frac{1}{2\kappa(v_0, \theta_0)} \ln \left[ \frac{r - r_H(v, \theta)}{r_H(v, \theta)} \right],\]
\[v_* = v - v_0, \theta_* = \theta - \theta_0,\]

(7)

where both \(v_0\) and \(\theta_0\) are constants under tortoise coordinate transformation. At the same time, \(v_0\) is the moment when the particle escapes from the event horizon of the black hole and depicts the evolution of the black hole, \(\theta_0\) is the location where the particle escapes from the event horizon of the black hole and depicts the shape of the black hole. According to the tortoise coordinate transformation, we have

\[
\frac{\partial}{\partial r} = \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial}{\partial r_*},
\]
\[
\frac{\partial}{\partial v} = \frac{\partial}{\partial v_*} - \frac{\dot{r}_H}{2\kappa r_H(r - r_H)} \frac{\partial}{\partial r_*},
\]
\[
\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \theta_*} - \frac{\dot{r}_H}{2\kappa r_H(r - r_H)} \frac{\partial}{\partial r_*},
\]

\[
\frac{\partial^2}{\partial r^2} = \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right]^2 \frac{\partial^2}{\partial r_*^2} - \frac{1}{2\kappa(r - r_H)^2} \frac{\partial}{\partial r_*},
\]
\[
\frac{\partial^2}{\partial r \partial v} = \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial^2}{\partial r_* \partial v_*} + \frac{\dot{r}_H}{2\kappa(r - r_H)^2} \frac{\partial}{\partial r_*},
\]
\[
- \frac{\dot{r}_H}{2\kappa r_H(r - r_H)} \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial^2}{\partial r_*^2},
\]
\[
\frac{\partial^2}{\partial v \partial \varphi} = \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial^2}{\partial v_* \partial \varphi_*} - \frac{\dot{r}_H}{2\kappa r_H(r - r_H)} \frac{\partial^2}{\partial r_* \partial \varphi_*},
\]
\[
\frac{\partial^2}{\partial r \partial \varphi} = \frac{\partial^2}{\partial r_* \partial \varphi_*} - \left[ 1 + \frac{1}{2\kappa(r - r_H)} \right] \frac{\partial^2}{\partial r_* \partial \varphi_*},
\]

3
\[ \frac{\partial^2}{\partial v^2} = \frac{\partial^2}{\partial v_*^2} - \frac{r^2_H r_H (r - r_H) - r^2_H (r - r_H) + r^2_H r_H}{2r^2_H (r - r_H)^2} \frac{\partial}{\partial r_*} \]
\[ - \frac{2r^2_H}{2kr_H(r - r_H)} \frac{\partial^2}{\partial r_* \partial v_*} + \frac{4k^2 r^2_H (r - r_H)^2}{\partial r_*^2} \]
\[ \frac{\partial^2}{\partial \theta^2} = \frac{\partial^2}{\partial \theta_*^2} - \frac{r r'' H (r - r_H) - r r'' H (r - r_H) + r r'' H}{2k^2 r^2_H (r - r_H)^2} \frac{\partial}{\partial r_*} \]
\[ - \frac{2r r' H}{2kr_H(r - r_H)} \frac{\partial^2}{\partial r_* \partial \theta_*} + \frac{4k^2 r^2_H (r - r_H)^2}{\partial \theta_*^2} \]

The Klein-Gordon Eq. (5) can be rewritten as

\[ \frac{a^2 r^2 H \sin^2 \theta - 2r H (r^2 + a^2) [1 + 2k(r - r_H)] r H}{2k r_H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \frac{\partial^2 H}{\partial r_*^2} \]
\[ + \frac{2r^2 H}{2k r_H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ + \frac{1}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ + \frac{2(a a \sin^2 \theta + r) \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ + \frac{2(a a \sin^2 \theta + r) \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + 2(a a \sin^2 \theta + r) \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ + 2(\frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + 2(a a \sin^2 \theta + r) \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} + \frac{\partial \partial H}{r H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}}) = 0. \] (8)

From the null hypersurface condition Eq. (3) of a dynamical Kerr space-time, the numerator of the coefficient of the term \( \frac{\partial^2 H}{\partial v_*^2} \) approaches to zero at the event horizon \( r_H \). Therefore we can calculate the limit of the coefficient using L’Hospital law. Let the limit to be equal to an undetermined constant \( K \)

\[ \lim_{r \to r_H, v \to \nu_0, \theta \to \theta_0} \frac{a^2 r^2 H \sin^2 \theta - 2r H (r^2 + a^2) [1 + 2k(r - r_H)] r H}{2k r_H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ + \lim_{r \to r_H, v \to \nu_0, \theta \to \theta_0} \frac{(r^2 + a^2 - 2mr) [1 + 2k(r - r_H)] r H^2 + r^2 H}{2k r_H (r - r_H) \{r H (r^2 + a^2) [1 + 2k(r - r_H)] - a^2 r H H \sin^2 \theta \}} \]
\[ = K, \]
while $\kappa$ is selected as

$$\kappa = \left(1 - 2\dot{r}_H\right)r_H - m + \frac{\dot{r}_H^2 a^2 \sin^2 \theta_0}{r_H} \left(\dot{r}_H^2 + a^2 - \dot{r}_H a^2 \sin^2 \theta_0\right),$$

we have $K = 1$.

From Eq. (4), external horizon $r_+$ is given by

$$r_+ = m + \sqrt{m^2 - (1 - 2\dot{r}_+)^2 \left(\dot{r}_+^2 + a^2 \sin^2 \theta_0\right) a^2 + r_+^2}.\quad (9)$$

When $r$ approaches to $r_+$, the Klein-Gordon Eq. (5) can be transformed into

$$\frac{\partial^2 \Phi}{\partial r^2_*} + 2 \frac{\partial^2 \Phi}{\partial v_* \partial r_*} + A \frac{\partial \Phi}{\partial r_*} + B \frac{\partial^2 \Phi}{\partial r_* \partial \varphi} + C \frac{\partial^2 \Phi}{\partial r_* \partial \theta_*} = 0,\quad (10)$$

where

$$A = \frac{2r_+ \dot{r}_+ - \dot{r}_+ a^2 \sin^2 \theta_0 + 2a\dot{a}(1 - \dot{r}_+ \sin^2 \theta_0) - r_+^2 - r_+ \cot \theta_0}{r_+^2 + a^2 (1 - \dot{r}_+ \sin^2 \theta_0)},$$

$$B = \frac{2a(1 - \dot{r}_+)}{r_+^2 + a^2 (1 - \dot{r}_+ \sin^2 \theta_0)},$$

$$C = \frac{-2r_+'}{r_+^2 + a^2 (1 - \dot{r}_+ \sin^2 \theta_0)}.$$ \quad (11)

Separate variables as following

$$\Phi = R(r_*) \Theta(\theta_*) e^{il\varphi - i\omega v_*},\quad (12)$$

where $\omega$ is the energy of Klein-Gordon particle, $l$ is the projection of angular momentum on $\varphi$-axis. We can get

$$\Theta' = \lambda \Theta,$$

$$R'' + (A + \lambda C + ilB - 2i\omega)R' = 0,\quad (13)$$

where the constant $\lambda$ is introduced by the separation of variables.

Assuming $\lambda = \lambda_1 + i\lambda_2$, $\lambda_1, \lambda_2 \in R$, the Eq. (13) can be rewritten as

$$\Theta' = (\lambda_1 + i\lambda_2) \Theta,$$

$$R'' + [A + (\lambda_1 + i\lambda_2)C + ilB - 2i\omega]R' = 0,\quad (14)$$

and its solution is

$$\Theta = c_1 e^{(\lambda_1 + i\lambda_2) \theta_*},$$

$$R = c_2 e^{-[A + (\lambda_1 + i\lambda_2)C + ilB - 2i\omega]r_*} + c_3,\quad (15)$$

9
where \(c_1, c_2\) and \(c_3\) are integral constants, \(\theta_*\) is polar angle. Its radial ingoing and outgoing components are respectively

\[
\begin{align*}
\psi_{\text{in}} &= e^{-i\omega v_*}, \\
\psi_{\text{out}} &= e^{-i\omega v_*} e^{2i(\omega-\omega_0)r_*} e^{-(A+\lambda_1 C)r_*},
\end{align*}
\]

where

\[
\omega_0 = \frac{1}{2} l B - \frac{1}{2} \lambda_2 C = \frac{la(1 - \dot{r}_+) + \lambda_2 r_+'}{r_+^2 + a^2(1 - \dot{r}_+ \sin^2 \theta_0)}.
\]

The outgoing wave can be rewritten as

\[
\psi_{\text{out}} = e^{-i\omega v_*} e^{2i(\omega-\omega_0)r_*} e^{-A r_+ \left( \frac{r - r_+}{r_+} \right)} e^{i \frac{\pi}{2} \left( \frac{r - r_+}{r_+} \right)} e^{\frac{\pi (\omega - \omega_0)}{\kappa}},
\]

where \(\bar{A} = A + \lambda_1 C\). It is obvious that the outgoing wave is not analytical at the horizon \(r_+\). Extending the outgoing wave from outside to inside of the horizon analytically through the negative half complex plane, we get

\[
\tilde{\psi}_{\text{out}} = e^{-i\omega v_*} e^{2i(\omega-\omega_0)r_*} e^{-A r_+ \left( \frac{r - r_+}{r_+} \right)} e^{i \frac{\pi}{2} \left( \frac{r - r_+}{r_+} \right)} e^{\frac{\pi (\omega - \omega_0)}{\kappa}}.
\]

The scattering probability of outgoing wave at the horizon is

\[
\left| \frac{\psi_{\text{out}}}{\tilde{\psi}_{\text{out}}} \right|^2 = e^{-\frac{2\pi (\omega - \omega_0)}{\kappa}}.
\]

According to the explanation of Sannan[14], the outgoing wave has black body spectrum

\[
N_\omega = \frac{1}{e^{\frac{\omega - \omega_0}{k_B T}} - 1},
\]

\[
T = \frac{\kappa}{2\pi k_B}.
\]

4 Conclusion and Discussion

The Hawking effect from a dynamical Kerr black hole has been studied under a new tortoise coordinate transformation. It is found that Hawking temperature \(T\) not only depends on time but also varies with angle. Because there is temperature gradient between equator and pole, the heat fluid should exist on the black hole event horizon surface. It is an interesting issue which deserves to be studied further.

According to expression of local event horizon Eq.(14), the external horizon \(r_+\) and the internal horizon \(r_-\) are given respectively:

\[
\begin{align*}
\dot{r}_+ &= \frac{m + \sqrt{m^2 - (1 - 2\dot{r}_+)[(1 - 2\dot{r}_+ + r_+^2 \sin^2 \theta_0)a^2 + r_+^2]} \frac{1 - 2\dot{r}_+}{1 - 2\dot{r}_+}, \\
\dot{r}_- &= \frac{m - \sqrt{m^2 - (1 - 2\dot{r}_-)[(1 - 2\dot{r}_- + r_-^2 \sin^2 \theta_0)a^2 + r_-^2]} \frac{1 - 2\dot{r}_-}{1 - 2\dot{r}_-}.
\end{align*}
\]

6
When the external horizon coincides with the internal one, i.e. \( r_+ = r_- = r_H \), we will have
\[
(1 - 2 \dot{r}_H)r_H - m = 0,
\]
\[
m^2 - (1 - 2 \dot{r}_H)[(1 - 2 \dot{r}_H + \dot{r}_H^2 \sin^2 \theta)a^2 + \dot{r}_H^2] = 0.
\]
Now the surface gravity is equal to
\[
\kappa = \frac{\dot{r}_H a^2 \sin^2 \theta_0 - (r_H^2 + a^2) \dot{r}_H}{4mr_H - (1 - 2 \dot{r}_H)(r_H^2 + a^2) - \dot{r}_H a^2 \sin^2 \theta_0 - (1 - 2 \dot{r}_H)(r_H^2 + a^2) \dot{r}_H},
\]
and this is the temperature of the extreme dynamical black hole. When \( \dot{r}_H = 0 \), the temperature will be equal to zero, which is consistent with the case of a stationary Kerr black hole.

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