ON DISTRIBUTION OF POLES OF EISENSTEIN SERIES AND THE LENGTH SPECTRUM OF HYPERBOLIC MANIFOLDS

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ABSTRACT. We extend results of [BR11] on a strong multiplicity one property for length spectrum to hyperbolic manifolds with cusps, showing that for two even dimensional hyperbolic manifolds of finite volume, if all but finitely many closed geodesics have the same length, then all closed geodesics have the same length. We also get some partial results showing that when the set exceptional lengths is infinite, but sufficiently sparse, the two manifolds must have the same volume, and in low dimension also the same number of cusps. A main ingredient in our proof is a generalization of a result of Selberg on the distribution of poles of Eisenstein series on hyperbolic manifolds.

INTRODUCTION

The length spectrum of a hyperbolic manifold is the set of lengths of primitive closed geodesics listed with their multiplicities. It is an interesting question how much of the geometry of the manifold, can be extracted from (partial) information on the length spectrum. In [BR11], Bhagwat and Rajan showed that if two compact even dimensional hyperbolic manifolds have the same multiplicities for all but possibly finitely many exceptional lengths, then they must have the same length spectrum. In [Kel11], we refined their result and showed that one can allow an infinite, but sparse, set of possible exceptional lengths.

The purpose of this note is to extend these results to hyperbolic manifolds with cusps. Using the results of Gon and Park [GP08, GP10] on the Selberg Zeta functions we can extend the result of [BR11] to this setting. However, the refinement in [Kel11] allowing an infinite exceptional set is more problematic. In this setting we only obtain a partial result showing that if the exceptional set is sufficiently sparse the two manifolds must have the same discrete Laplace spectrum and the same volume. In low dimensions we can also deduce that they have

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the same number of cusps. Even these partial results already require a detailed analysis of the distribution of poles of Eisenstein series. To describe our results in more detail we need to introduce some notation.

Let \( G \cong \text{SO}_0(d,1) \) denote the group of isometries of hyperbolic \( d \)-space, \( \mathbb{H}^d \). Any finite volume hyperbolic manifold is of the form \( X_\Gamma = \Gamma \backslash \mathbb{H}^d \) where \( \Gamma < G \) is a torsion free lattice. Given a hyperbolic manifold \( X_\Gamma \), for every \( \ell \in (0,\infty) \) we denote by \( m_{\Gamma}(\ell) \) the number of primitive (i.e., wrapping once around) closed geodesics of length \( \ell \) in \( X_\Gamma \). For any two lattices \( \Gamma_1, \Gamma_2 < G \) let

\[
D_L(\Gamma_1, \Gamma_2; T) = \sum_{\ell \leq T} |m_{\Gamma_1}(\ell) - m_{\Gamma_2}(\ell)|,
\]

and

\[
d_L(\Gamma_1, \Gamma_2) = \lim_{T \to \infty} \sup \frac{\log(D_L(\Gamma_1, \Gamma_2; T))}{T}.
\]

One can think of \( d_L(\Gamma_1, \Gamma_2) \) as measuring the scaled density of the exceptional set of lengths having different multiplicities in the two manifolds. If this exceptional set is finite then \( d_L(\Gamma_1, \Gamma_2) = 0 \), and the result of [Kel11] states that for two compact even dimensional hyperbolic manifolds the condition that \( d_L(\Gamma_1, \Gamma_2) < \frac{1}{2} \) implies that the two manifolds have the same length spectrum. Moreover, it was shown there, that (in any dimension) the weaker condition that \( d_L(\Gamma_1, \Gamma_2) < \frac{d-1}{2} \) already implies that the two manifolds have the same Laplace spectrum (and hence by Weyl’s law, also the same volume).

Our first result is to extend the result of [BR11] to finite volume non-compact hyperbolic manifolds. Since we rely on results of [GP10] we need to impose a certain technical condition on the cusp of \( X_\Gamma \). We say that \( X_\Gamma \) has neat cusps if for any parabolic subgroup \( P < G \) with unipotent radical \( N < P \) we have \( \Gamma \cap P = \Gamma \cap N \).

**Theorem 1.** Let \( X_{\Gamma_1}, X_{\Gamma_2} \) denote two even dimensional hyperbolic manifolds of finite volume with neat cusps. If \( m_{\Gamma_1}(\ell) = m_{\Gamma_2}(\ell) \) for all \( \ell \in \mathbb{R} \) except perhaps some finite exceptional set, then \( m_{\Gamma_1}(\ell) = m_{\Gamma_2}(\ell) \) for all \( \ell \).

Next, we want to generalize the results of [Kel11] allowing an infinite set of exceptions. The relation between the length spectrum and discrete Laplace spectrum follows by more or less the same arguments as in the compact case. We note that when the manifold is not compact, the corresponding Weyl law includes both discrete and continuous spectrum, hence, the fact that the two manifolds have the same discrete spectrum no longer implies that they have the same volumes. Nevertheless, we show:
Theorem 2. Let $X_{\Gamma_1}, X_{\Gamma_2}$ denote two $d$-dimensional hyperbolic manifolds of finite volume with neat cusps.

(1) If $d_L(\Gamma_1, \Gamma_2) < \frac{d-1}{2}$ then the two manifolds have the same discrete Laplace spectrum and the same volume.

(2) For $d = 2, 3$, if $d_L(\Gamma_1, \Gamma_2) < 1/4$ then the two manifolds have the same number of cusps.

The proof of Theorem 2 relies on results of Selberg \cite{Sel90} on the distribution of poles of Eisenstein series on hyperbolic surfaces and their generalization to hyperbolic manifolds. To describe these results we first recall some definitions and facts regarding Eisenstein series. Fix an Iwasawa decomposition, $G = NAK$, with $K$ maximal compact, $A$ Cartan, and $N$ unipotent, and let $P = NAM$ be a minimal parabolic with $M = Z_K(A)$. The cusps of $\Gamma \subseteq G$ are the $\Gamma$-conjugacy classes of minimal parabolic subgroups of $G$ intersecting $\Gamma$. Let $P_1, \ldots, P_\kappa$ denote a full set of representatives for these groups (that we fix once and for all). After maybe conjugating $\Gamma$ we assume that $P_1 = P = NAM$ is the standard parabolic subgroup and we fix $k_j \in K$ such that $P_j = k_j P k_j^{-1}$.

For each of the cusp $P_i = N_i A_i M_i$, let $\Gamma P_i = \Gamma \cap P_i$ and $\Gamma N_i = \Gamma \cap N_i$. The spherical Eisenstein series corresponding to the $i$’th cusp is the function on the upper half space defined for $\Re(s) > d - 1$ by the convergent series

$$E_i(s, z) = \sum_{\gamma \in \Gamma P_i \setminus \Gamma} y_i(\gamma . z)^s,$$

where we use the coordinates $z = (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}^+$ for the upper half space and set $y_i(z) = y(k_i^{-1} . z)$. The constant term of $E_i(s, z)$ with respect to the $j$’th cusp is defined by

$$E_{ij}(s, z) = \frac{1}{v_j} \int_{\Gamma N_j \setminus N_j} E_i(s, n . z) dn,$$

where $v_j = \text{vol}(\Gamma N_j \setminus N_j)$, and $dn$ is Haar measure on $N_j$, and satisfy

$$E_{ij}(s, z) = \delta_{ij} y_j(z)^s + \phi_{ij}(s) y_j(z)^{d-1-s},$$

where $\phi_{ij}(s)$ are the coefficients of the scattering matrix. The Eisenstein series $E_j(s, z)$, the scattering matrix $\phi(s) = (\phi_{ij}(s))$, and it’s determinant $\varphi(s) = \det(\phi(s))$ (a priori defined for $\Re(s) > d - 1$) have a meromorphic extension to the complex plane and satisfy the functional equation $\varphi(s) \phi(d - 1 - s) = I$. The poles of $\varphi(s)$ which are also the poles of the Eisenstein series, are all in the half plane $\Re(s) < \frac{d-1}{2}$ except for at most finitely many poles in the interval $(\frac{d-1}{2}, d - 1]$. In order to prove Theorem 2 we need a better understanding of the distribution...
of these poles. From the functional equation \( \varphi(s)\varphi(d - 1 - s) = 1 \), we can understand the distribution of these poles by looking at the zeroes of \( \varphi(s) \) in the half plane \( \Re(s) > \frac{d-1}{2} \). For these we show

**Theorem 3.** Let \( \rho = \beta + i\gamma \) denote the zeroes of the scattering determinant, \( \varphi(s) \), in the half plane \( \Re(s) > \frac{d-1}{2} \) listed with multiplicities.

1. There is a constant \( A_\Gamma \) such that

\[
\sum_{|\gamma|<T \atop \beta > \frac{d-1}{2}} (\beta - \frac{d-1}{2}) = \frac{\kappa(d-1)}{2\pi} T \log(T) + A_\Gamma T + O(\log(T))
\]

2. Let \( \alpha_0 = d - 1 - \frac{d-1}{2(d+1)} \) when \( d > 3 \) and \( \alpha_0 = \frac{3}{4} \) when \( d = 2 \), then for any \( \alpha \geq \alpha_0 \)

\[
\sum_{|\gamma|<T \atop \beta > \alpha} (\beta - \alpha) \ll T \log\left(\min\left\{\frac{1}{(\alpha - \alpha_0)}, \log T\right\}\right)
\]

**Remark 0.1.** Using the relation between the zeroes and poles, one can interpret this result as saying that a hundred percent of the poles \( \tilde{\rho} = \tilde{\beta} + i\gamma \) of \( \varphi(s) \) in the half plane \( \Re(s) < \frac{d-1}{2} \) are concentrated in the strip \( d-1 - \alpha_0 \leq \Re(s) < \frac{d-1}{2} \), in the sense that, for any \( \alpha < d - 1 - \alpha_0 \) as \( T \to \infty \)

\[
\sum_{|\gamma|<T \atop \beta > \frac{d-1}{2}} (\frac{d-1}{2} - \tilde{\beta}) \sim \sum_{|\gamma|<T \atop \beta < \frac{d-1}{2}} (\frac{d-1}{2} - \tilde{\beta}).
\]

**Remark 0.2.** Selberg [Sel90] proved this result for hyperbolic surfaces with the value of \( \alpha_0 = \frac{1}{4} \) which is best possible. Indeed, for \( \Gamma = \text{SL}_2(\mathbb{Z}) \) the scattering determinant can be computed explicitly in terms of the Riemann Zeta function and its poles are located at the zeroes of \( \zeta(1 - 2s) \), hence, a positive proportion\(^1\) are on the line \( \Re(s) = \frac{1}{4} \).

For 3-manifolds, when \( \Gamma = \text{SL}_2(\mathcal{O}_K) \) with \( \mathcal{O}_K \) the ring of integers of a quadratic complex number field, the poles of the scattering determinant are at the zeros of the Dedekind Zeta function \( \zeta_K(1-s) \) and (1.7) holds with \( \alpha_0 = \frac{1}{2} \). However, for general \( \Gamma < \text{PSL}_2(\mathbb{C}) \) our method only gives the weaker result with \( \alpha_0 = \frac{1}{4} \). For general \( d \geq 3 \) is is not clear what is the largest value, say \( \alpha_0(d) \), for which (1.7) holds.

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\(^1\)The Riemann hypothesis implies that all the zeroes are on that line.
1. Eisenstein series and Selberg Zeta functions

In this section we recall some of the theory of Eisenstein series in greater generality (we refer the reader to [War79] for more details), and present the results of Gon and Park [GP10] on the Zeta functions of Selberg and Ruelle.

1.1. Structure of $G$. Fix an Iwasawa decomposition $G = NAK$ and let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k}$ denote the corresponding decomposition of the Lie algebra $\mathfrak{g}$. Let $M, M^* \subseteq K$ denote the centralizer and normalizer of $A$ in $K$ respectively. Let $W = W(G, A) = M^*/M$ denote the baby Weyl group. In our case $W$ is of order two and we write $W = \{1, w\}$. We denote by $\Sigma = \Sigma(G, A)$ the set of restricted roots for the pair $(G, A)$ and by $\Sigma^+$ the set of positive restricted roots, then $\Sigma^+ = \{\alpha\}$. Let $\rho = \rho_G$ denote half the sum of the positive roots, that is, $\rho = \frac{d-1}{2}\alpha$. We fix (once and for all) an element $H_0 \in \mathfrak{a}$ with $\alpha(H_0) = 1$ and identify the dual spaces $\mathfrak{a}^* = \mathbb{R}$ and $\mathfrak{a}^*_C = \mathbb{C}$ via $\nu = \nu(H_0)$. With this identification $\rho = \frac{d-1}{2}$.

Let $\hat{K}$ and $\hat{M}$ denote the unitary duals of $K$ and $M$ respectively. The action of $M^*$ on $M$ (by conjugation) induces an action of the Weyl group $W = \{1, w\}$ on $\hat{M}$. We say that a representation $\sigma \in \hat{M}$ is ramified if $\sigma = w\sigma$ and unramified otherwise, and we recall that there are unramified $\sigma \in \hat{M}$ only when $d = 2n + 1$ is odd.

Let $\Gamma < G$ denote a torsion free lattice. The cusps of $\Gamma$ are defined as the $\Gamma$-conjugacy classes of minimal parabolic subgroups of $G$ intersecting $\Gamma$. Let $P_1, \ldots, P_\kappa$ denote a full set of representatives for these groups (that we fix once and for all). After maybe conjugating $\Gamma$ we assume that $P_1 = P = NAM$ is the standard parabolic subgroup and we fix $k_i \in K$ such that $P_i = k_i P k_i^{-1}$. For each of the cusps, let $P_i = N_i A_i M_i$ denote the corresponding Langlands decomposition and let $\Gamma_{P_i} = \Gamma \cap P_i$ and $\Gamma_{N_i} = \Gamma \cap N_i$. Note that $\Gamma \cap P_i \subseteq N_i M_i$ and that the quotient $\Gamma_{M_i} = \Gamma_{N_i} \backslash \Gamma_{P_i}$ is finite (since it is a discrete subgroup of the compact group $N_i \backslash N_i M_i \cong M$).

For each a pair of cusps, let $W(A_i, A_j)$ denote the set of inner automorphisms sending $A_i$ to $A_j$. There are exactly two inner automorphisms in $W(A_i, A_j)$: the standard one is conjugation by $k_i k_j^{-1}$ and the twisted one is conjugation by $k_i w k_j^{-1}$ with $w \in W(G, A)$ the nontrivial element. For each $t \in W(A_i, A_j)$ we let $t^* : \mathfrak{a}_{i,C}^* \to \mathfrak{a}_{j,C}^*$ denote the induced map.

1.2. Eisenstein series. For the definition of the Eisenstein series we follow Warner [War79], after making the appropriate modifications to
account for the fact that we are considering \( \Gamma \) acting on the left rather than on the right. We first define the Eisenstein series corresponding to the cusp at infinity \( P_1 = P \). The group \( M \) naturally acts on \( L^2(\Gamma_M \backslash M) \) and we can decompose

\[
L^2(\Gamma_M \backslash M) = \bigoplus_{\sigma \in \hat{M}} d_\Gamma(\sigma)\mathcal{H}_\sigma = \bigoplus_{\sigma \in W \backslash \hat{M}} \mathcal{H}(\sigma),
\]

where the action on \( \mathcal{H}_\sigma \) is irreducible (and isomorphic to \( \sigma \)), and

\[
\mathcal{H}(\sigma) = \begin{cases} 
d_\Gamma(\sigma)\mathcal{H}_\sigma & \text{\( \sigma \) is ramified} \\
d_\Gamma(\sigma)\mathcal{H}_\sigma \oplus d_\Gamma(w \sigma)\mathcal{H}_{w \sigma} & \text{\( \sigma \) is unramified}
\end{cases}
\]

For any \( \sigma \in W \backslash \hat{M} \) and \( \tau \in \hat{K} \) with \([\tau]_M : \sigma \neq 0\), let \( E(\sigma, \tau) \) denote the space of functions \( \Phi : \Gamma_M \backslash K \to \mathbb{C} \) such that for all \( x \in K \) the map \( m \mapsto \Phi(mx) \) is in \( \mathcal{H}(\sigma) \) and the map \( k \mapsto \Psi(xk) \) is in the \( \tau \)-isotopical subspace \( L^2(K, \tau) \) of \( L^2(K) \). The space \( E(\sigma, \tau) \) is finite dimensional with a Hilbert space structure given by the inner product

\[
(\Phi_1, \Phi_2) = \int_K \int_{\Gamma_M \backslash M} \Phi_1(mk)\overline{\Phi_2(mk)}dkdm.
\]

By identifying the quotients \( M \backslash K = NAM \backslash G \) we can think of functions in \( E(\sigma, \tau) \) as functions on \( G \) and define the Eisenstein series corresponding to \( \Phi \in E(\sigma, \tau) \) for \( \Re(s) > d - 1 \) by the series

\[
E(P, \Phi, s, g) = \sum_{\Gamma_P \backslash \Gamma} e^{-\langle \nu_s + \rho \rangle H(g)} \Phi(\gamma g).
\]

Here \( \nu_s \in a^*_s \cong \mathbb{C} \) is defined via our identification and \( H(g) \in a \) is defined by \( g \in N \exp(H(g))K \).

Making the obvious change of coordinates, one can define Eisenstein series \( E(P_i, \Phi_i, s, g) \) for each cusp \( P_i \) and \( \Phi_i \in E(\sigma^{(i)}, \tau) \) (here \( \sigma^{(i)} \in \hat{M}_i \) is obtained from \( \sigma \in \hat{M} \) via conjugation by \( k_i \)).

For any two cusps \( P_i, P_j \), the constant term of \( E(P_i, \Phi_i, s, g) \) along \( P_j \) is defined by

\[
E_{P_j}(P_i, \Phi_i, s, g) = \frac{1}{\text{vol}(\Gamma_{N_j} \backslash N_j)} \int_{\Gamma_{N_j} \backslash N_j} E(P_i, \Phi_i, s, ng)dn,
\]

and has an expansion of the form

\[
E_{P_j}(P_i, \Phi_i, s, g) = \sum_{t \in W(A_i, A_j)} e^{-\langle t^* \nu_s + \rho \rangle H_j(g)} (C_{ij}(t; s) \Phi_i)(g),
\]

where \( H_j(g) \) is defined as above with respect to the decomposition \( G = N_jA_jK \) and

\[
C_{ij}(t; s) : E(\sigma^{(i)}, \tau) \to E(\sigma^{(j)}, \tau),
\]
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is a certain linear transformation that is defined and holomorphic in the half plane $\Re(s) > \frac{d-1}{2}$.

The scattering operator $C_{\Gamma}(\sigma, \tau; s)$ is defined as the linear operator on the space $E_{\Gamma}(\sigma, \tau) = \sum_{i} E(\sigma_{i}, \tau)$, acting on $\Phi_{i} \in E(\sigma_{i}, \tau)$ by

$$C_{\Gamma}(\sigma, \tau; s)\Phi_{i} = \sum_{j} C_{ji}(k_{i}w^{-1}; k_{j}s)\Phi_{j}. \tag{1.4}$$

We recall the following fundamental facts about the Eisenstein series and the scattering operator: The scattering operator $C_{\Gamma}(\sigma, \tau, s)$ has a meromorphic continuation to the complex plane. It satisfies the functional equation $C_{\Gamma}(\sigma, \tau; s)C_{\Gamma}(\sigma, \tau; -s) = I$ and, in particular, since $C_{\Gamma}(\sigma, \tau, s)^{*} = C_{\Gamma}(\sigma, \tau, \bar{s})$, it is holomorphic and unitary on the imaginary axis. Moreover, for $\Phi \in \mathcal{E}_{\Gamma}(\sigma, \tau)$ the Eisenstein series

$$E(\Phi, s, g) = \sum_{i} E(P_{i}, \Phi_{i}, s, g),$$

has a meromorphic continuation to the whole complex plane with poles given by the poles of the scattering operator, and it satisfies the functional equation

$$E(\Phi, s, g) = E(C_{\Gamma}(\sigma, \tau; s)\Phi, -s, g).$$

The determinant of the scattering operator, denoted by

$$\Psi_{\Gamma}(\sigma, \tau; s) = \det C_{\Gamma}(\sigma, \tau; s) \tag{1.5}$$

is thus a meromorphic function on the complex plane satisfying the functional equation $\Psi_{\Gamma}(\sigma, \tau; s)\Psi_{\Gamma}(\sigma, \tau; -s) = 1$.

When $\tau = 1$ is the trivial representation, the space $\mathcal{E}_{\Gamma}(\sigma, 1)$ vanishes unless $\sigma$ is trivial in which case it is one dimensional and we retrieve spherical Eisenstein series defined in (0.3). Specifically, we have that

$$E(P_{j}, 1, s, g) = E_{j}(g.i, s + \frac{d-1}{2}),$$

and Theorem 3 gives the following result on the distribution of poles of $\Psi_{\Gamma}(s) = \Psi_{\Gamma}(1, 1, s)$: Let $S_{\Gamma}$ denote the set of poles of $\Psi_{\Gamma}$ in the half plane $\Re(s) < 0$, listed with multiplicities. For $T, c \geq 0$ denote by

$$S_{\Gamma}(T, c) = \{ \eta \in S_{\Gamma} : |\text{Im}\eta| < T, \Re\eta < -c \}.$$

and let $S_{\Gamma}(T) = S_{\Gamma}(T, 0)$. Theorem 3 is then equivalent to
Proposition 1.1. All poles $\eta \in S_\Gamma$ have $\Re(\eta) > -M$ for some $M = M_\Gamma > 0$ and satisfy asymptotic formula

\[
\sum_{\eta \in S_\Gamma(T)} |\Re\eta| = \frac{\kappa(d-1)}{2\pi} T \log(T) + A_\Gamma T + O(\log(T)).
\]

for some constant $A_\Gamma$. Moreover, if we let $c_0 = \frac{d(d-1)}{2(d+1)}$ for $d > 2$ and $c_0 = \frac{1}{4}$ when $d = 2$, then for all $c \geq c_0$

\[
\sum_{\eta \in S_\Gamma(T,c)} (|\Re\eta| - c) \ll T \log \left( \min \left\{ \frac{1}{c-c_0}, \log T \right\} \right)
\]

In the non-spherical case we know less about the location and distribution of the poles of the scattering matrix. Nevertheless, we have the following general result from [Müll89, Section 6].

Proposition 1.2. All the poles of $\Psi_\Gamma(\sigma, \tau, s)$ are in the half plane $\Re(s) < 0$ except for finitely many poles in the interval $(0, \frac{d-1}{2})$. If we denote by $S^\tau_\Gamma, \sigma$ the set of poles in $\Re(s) < 0$ listed with multiplicities then $\sum_{\eta \in S^\tau_\Gamma, \sigma} \frac{\Re(\eta)}{|\eta|^2}$ converges.

1.3. Selberg Zeta functions. For $k = 0, \ldots, \left[\frac{d-1}{2}\right]$, let $\sigma_k \in \hat{M}$ correspond to the irreducible representation of $M$ on the space $\bigwedge^k (\mathbb{C}^{d-1})$ (when $d = 2n + 1$ and $k = n$ we denote by $\sigma_n^\pm$ the two unramified irreducible representations acting on $\bigwedge^n (\mathbb{C}^{2n})$. Similarly, we denote by $\tau_k$ the irreducible representation of $K = \text{SO}(d)$ on $\bigwedge^k (\mathbb{C}^d)$ (and when $d = 2n$ we denote by $\tau_n^\pm$ the two unramified irreducible representations). To simplify notation, for each $k$ we denote by $\Psi_{\Gamma,k}(s)$ the determinant of the scattering matrix corresponding to the pair $(\sigma_k, \tau_k)$. Let $S^k_\Gamma$ denote the poles of $\Psi_{\Gamma,k}(s)$ in the half space $\Re(s) < 0$ and $\tilde{S}^k_\Gamma$ the finite set of poles in $(0, \frac{d-1}{2})$.

We recall the correspondence between closed geodesics on $X_\Gamma$ and hyperbolic conjugacy classes in $\Gamma$. Any hyperbolic $\gamma \in \Gamma$ is conjugated in $G$ to an element $m_\gamma a_{\ell_\gamma} \in MA^+$ where $A^+ = \{a_t | t > 0\}$. Here $\ell_\gamma$ is uniquely determined by the conjugacy class of $\gamma$, and $m_\gamma$ is determined up to conjugacy in $M$. The pair $(\ell_\gamma, m_\gamma)$ is then precisely the length and holonomy class of the closed geodesic corresponding to $\gamma$.

For $0 \leq k \leq \left[\frac{d-1}{2}\right]$ the Selberg Zeta function corresponding to $\sigma_k \in \hat{M}$ is defined on the half plane $\Re(s) > d-1$ by

\[
Z(\sigma_k, s) = \exp \left( - \sum_{\gamma \in \Gamma_k} \frac{\chi_{\sigma_k}(m_\gamma)}{j(\gamma) D(\gamma)} e^{-(s-d-1)\ell_\gamma} \right)
\]
where $\Gamma'_h$ denotes the set of primitive hyperbolic conjugacy classes and
\[
D(\gamma) = e^{\frac{d-1}{2} \ell_\gamma} \left| \det (\text{Ad}(m, a_{\gamma})^{-1} - \text{Id}) \right|.
\]
We recall the following result of Gon and Park

**Proposition 1.3.** \cite{GP10} Theorem 4.6] $Z(\sigma_k, s)$ has a meromorphic continuation to the complex plane. It has poles at the points $s = \frac{d-1}{2} - \ell$, $\ell \in \mathbb{N} \cup \{0\}$ (and additional zeroes/poles at negative integers when $d$ is even). It has spectral zeroes at the points $s = \frac{d-1}{2} \pm ir$ with $\lambda = r^2 + (\frac{d-1}{2} - k)^2$ an eigenfunction of $\triangle_k$, and residual zeroes at $s = \frac{d-1}{2} + \eta$ with $\eta \in S_k^h$. It also has residual poles at $s = \frac{d-1}{2} - \eta_j$ with $\eta_j \in S_k^h$. The order of the spectral zeroes equals the multiplicity of the corresponding eigenvalue, and the order of the residual poles and zeroes are equals to $\dim(\sigma_k)$ times the order of the corresponding pole of $\Psi_{\Gamma, k}$.

1.4. **Ruelle Zeta function.** The Ruelle Zeta function is defined in the half plane $\Re(s) > \frac{d-1}{2}$ by the Euler product
\[
R_\Gamma(s) = \prod_{\gamma \in \Gamma'_h} (1 - e^{-s \ell_\gamma}).
\]
As shown in \cite{GP10}, when $d = 2n$ is even it is related to the Selberg Zeta functions via

\[
R_\Gamma(s) = \prod_{k=0}^{n-1} \left[ \frac{Z_\Gamma(\sigma_k, s + k)}{Z_\Gamma(\sigma_k, s + 2n - 1 - k)} \right]^{(-1)^{k+1}},
\]
and when $d = 2n + 1$ is odd we have

\[
R_\Gamma(s) = \prod_{k=0}^{n} \left[ Z_\Gamma(\sigma_k, s + k) Z_\Gamma(\sigma_k, s + 2n - k) \right]^{(-1)^{k+1}},
\]
where in the odd case we denoted by $Z_\Gamma(\sigma_n, s) := Z_\Gamma(\sigma_n^+, s) Z_\Gamma(\sigma_n^-, s)$.

We also consider a variant of the Ruelle Zeta function that is similar to the Selberg Zeta function for surfaces, that is,

\[
Z_\Gamma(s) = \prod_{\gamma \in \Gamma'_h} \prod_{k=0}^{\infty} (1 - e^{-(s+k) \ell_\gamma}).
\]
When $d = 2n$ is even, a simple manipulation of (1.8) shows that this Zeta function can be expressed as a finite product of Selberg Zeta functions and their inverses as
\[
Z_\Gamma(s) = \prod_{k=0}^{n-1} \left[ \prod_{j=k}^{2n-2-k} Z_\Gamma(\sigma_k, s + j) \right]^{(-1)^{k+1}}.
\]
2. Proof of Theorem 1

Let $d = 2n$ be even and let $\Gamma_1, \Gamma_2 \subset \text{SO}(d, 1)$ be two torsion free lattices. Assume that $m_{\Gamma_1}(\ell) = m_{\Gamma_2}(\ell)$ for all $\ell \in \mathbb{R}$ except perhaps for a finite exceptional set $\{\ell_1, \ldots, \ell_N\}$. For these exceptional length let $\Delta m(\ell_j) = m_{\Gamma_1}(\ell_j) - m_{\Gamma_2}(\ell_j)$. We will show that $\Delta m(\ell_j) = 0$ as well.

To do this consider the quotient $F(s) = \frac{Z_{\Gamma_1}(s)}{Z_{\Gamma_2}(s)}$ of the corresponding Zeta functions given in (1.10), and note that for $\Re(s) > d - 1$ we have

$$F(s) = \prod_{j=1}^N \prod_{k=0}^\infty (1 - e^{-(s+k)\ell_j})^{\Delta m(\ell_j)}.$$ 

This product absolutely and uniformly converges on any compact set away from $\{s \in \mathbb{C} \mid e^{-(s+k)\ell_j} = 1, \ j = 1, \ldots, N, \ k \in \mathbb{Z}\}$. Consequently, $F(s)$ is a meromorphic functions with all it’s zeros and poles located at the points $\rho_{n,j,k} = -k + \frac{2\pi in}{\ell_j}$ with $n, k \in \mathbb{Z}, \ k \geq 0$. The order of the pole/zero $\rho_{n,j,k}$ does not depend on $k$ and is given by

$$\sum_{\{i \mid \ell_i \in \ell_j \mathbb{Z}\}} \Delta m(\ell_i).$$

Let $\ell_1$ be the largest for which $\Delta m(\ell_1) \neq 0$. If there are other $\ell_i$ with $\frac{\ell_i}{\ell_1} \in \mathbb{Q}$ and $\Delta m(\ell_i) \neq 0$ let $q \in \mathbb{N}$ denote their least common multiple. Now for all $n \in \mathbb{N}$ with $(n, q) = 1$ we have that $\ell_i n/\ell_1 \in \mathbb{Z}$ if and only if $\ell_i = \ell_1$, and hence $F(s)$ has poles/zeros at all points $\rho_{n,1,k} = -k + \frac{2\pi in}{\ell_1}$ with $(n, q) = 1$ of order $\Delta m(\ell_1) \neq 0$.

Next we will use the fact that $F(s)$ is a finite product of quotients of Zeta functions to show that it can’t have that many poles. Specifically,

$$F(s) = \prod_{k=0}^{n-1} \left[ \prod_{j=k}^{2n-2-k} \frac{Z_{\Gamma}(\sigma_k, s+j)}{Z_{\Gamma}(\sigma_k, s)} \right]^{(-1)^{k+1}},$$

and hence all of it’s zeros and poles in the half plane $\Re(s) < 0$ come from poles of the functions $\Psi_{\Gamma,i,k}(s+j)$ with $0 \leq k \leq n-1$, and $k \leq j \leq 2n - 2 - k$, $i = 1, 2$. Since all $\rho_{n,k}$ with $(n, q) = 1$ are poles of $F(s)$ with the same multiplicity $\Delta m(\ell_1)$ we get a bound

$$|\Delta m(\ell_1)| \sum_{k=1}^{\infty} \sum_{(n,q)=1} \frac{k}{k^2 + 4\pi^2 n^2 / \ell_1^2} \leq \sum_{i=1}^{2} \sum_{k=0}^{n-1} \sum_{j=k}^{2n-2-k} \sum_{\eta \in S_{\Gamma,i,k}} \frac{|\Re(\eta - j)|}{|\eta - j|^2}.$$ 

But the right hand side converges by Proposition 1.2 and the sum on the left diverges implying that $\Delta m(\ell_1) = 0$ as claimed.
3. Proof of Theorem 2

To prove Theorem 2 consider the quotient of the Ruelle Zeta functions

\[ F(s) = \frac{R_{\Gamma_1}(s)}{R_{\Gamma_2}(s)} = \prod_{\ell \in \mathbb{L}} (1 - e^{-st_\ell})^{\Delta m(\ell)}. \]

A priori, this equality holds for \( \Re(s) > d - 1 \), however, if we know that

\[ \sum_{\ell \leq T} |\Delta m(\ell)| = O(e^{cT}), \]

then the right hand side absolutely converges for \( \Re(s) > c \) and hence defines an analytic function that has no zeroes or poles on that half plane.

The condition \( d_L(\Gamma_1, \Gamma_2) < \frac{d-1}{2} \) implies that (3.1) holds with some \( c < \frac{d-1}{2} \) and hence the zeroes and poles of \( R_{\Gamma_1}(s) \) and \( R_{\Gamma_2}(s) \) in the half pane \( \Re(s) > c \) must be the same. From the factorization of \( R_{\Gamma_i}(s) \) as a product of Selberg Zeta functions in (1.8) (1.9) together with the location of the poles and zeroes of the Zeta functions \( Z_{\Gamma}(\sigma_k, s) \) we see that \( R_{\Gamma_i}(s) \) has no zeroes in \( \Re(s) \geq \frac{d-1}{2} \) and it’s poles there all come from the spectral zeroes of \( Z(\sigma_0, s) \) with \( \sigma_0 \) the trivial representation.

That is, \( R_{\Gamma_i}(s) \) has poles at \( s = \frac{d-1}{2} + ir \) whenever \( \lambda = \frac{(d-1)^2}{4} + r^2 \) is an eigenvalue of \( \Delta \) with eigenfunction in \( L^2(X_{\Gamma_i}) \) with order given by the multiplicity of the eigenvalue. In particular, this shows that the discrete spectrum of \( \Gamma_1 \) and \( \Gamma_2 \) must be the same.

To see that the volumes are also equal we use the Weyl law, which for non-compact hyperbolic manifolds takes the form,

\[ N_{\Gamma}(T) - \frac{1}{2\pi} \int_{-T}^{T} \frac{\Psi'_{\Gamma}(it)}{\Psi_{\Gamma}(it)} dt = C_d \text{vol}(X_{\Gamma}) T^d + O(T^{d-1}) + O(T \log(T)), \]

where \( N_{\Gamma}(T) \) denotes the number of Laplace eigenvalues \( \lambda < \frac{(d-1)^2}{4} + T^2 \) and \( C_d \) is a constant depending only on \( d \). The term involving the scattering determinant can be further evaluated by counting poles. Specifically, by following the same argument as \cite[equation (0.15)]{Sel90}, we see that

\[ |S_{\Gamma}(T)| = -\frac{1}{2\pi} \int_{-T}^{T} \frac{\Psi'_{\Gamma}(it)}{\Psi_{\Gamma}(it)} dt + O(T), \]

and the Weyl law takes the form

\[ N_{\Gamma}(T) + |S_{\Gamma}(T)| = C_d \text{vol}(X_{\Gamma}) T^d + O(T^{d-1}) + O(T \log(T)). \]
Comparing this for the two lattices, noting that \(N_{\Gamma_1}(T) = N_{\Gamma_2}(T)\) and bounding all error terms by, say \(O(T^{d-1/2})\), we see that
\[
|S_{\Gamma_1}(T)| - |S_{\Gamma_2}(T)| = C_d(\text{vol}(X_{\Gamma_1}) - \text{vol}(X_{\Gamma_2}))T^d + O(T^{d-1/2})
\]
Let \(c' = \max\{c, \frac{d-3}{2}\}\), then \(\frac{R_{\Gamma_1}(s)}{R_{\Gamma_2}(s)}\) has no poles or zeros in \(c' < \Re(s) < \frac{d-1}{2}\). In this range all zeroes of \(R_{\Gamma_1}(s)\) are the residual zeros of \(Z_{\Gamma_1}(\sigma_0, s)\), and hence the residual zeroes of \(Z_{\Gamma_1}(\sigma_0, s)\) and \(Z_{\Gamma_2}(\sigma_0, s)\) in \(\Re(s) > c'\) must be the same. Recalling the relation between the residual zeroes and the poles of the scattering matrix we get that for all \(c_1 < \min\{1, \frac{d-1}{2} - c\}\)
\[
|S_{\Gamma_1}(T)| - |S_{\Gamma_2}(T)| = |S_{\Gamma_1}(c_1, T)| - |S_{\Gamma_2}(c_1, T)|.
\]
Finally, from (1.6) we get that for either lattice
\[
|S_{\Gamma_i}(c_1, T)| \leq \frac{1}{c_1} \sum_{\eta \in S_{\Gamma_i}(c_1, T)} |\Re(\eta)| \leq \sum_{\eta \in S_{\Gamma_i}(T)} |\Re(\eta)| \ll T \log(T)
\]
Thus, fixing \(0 < c_1 < \min\{1, \frac{d-1}{2} - c\}\) we get that
\[
C_d(\text{vol}(X_{\Gamma_1}) - \text{vol}(X_{\Gamma_2}))T^d = |(|S_{\Gamma_1}(\sigma, T)| - |S_{\Gamma_2}(\sigma, T)|)| + O(T^{d-1/2})
\]
\[
\ll T^{d-1/2},
\]
and hence \(\text{vol}(X_{\Gamma_1}) = \text{vol}(X_{\Gamma_2})\).

For the results on the number of cusps, let \(d \leq 3\) and assume that \(d_L(\Gamma_1, \Gamma_2) < \frac{1}{4}\), so that \(D_L(\Gamma_1, \Gamma_2, T) = O(e^{cT})\) with some \(c < \frac{1}{4}\).

Comparing the Zeta functions as before we see that the poles of \(\Psi_{\Gamma_1}\) and \(\Psi_{\Gamma_2}\) in the half plane \(\Re(s) > -c'\) with \(c' = \frac{d-1}{2} - c\) are the same. Hence
\[
\sum_{\eta \in S_{\Gamma_1}(T)} |\Re(\eta)| - \sum_{\eta \in S_{\Gamma_2}(T)} |\Re(\eta)| = \sum_{\eta \in S_{\Gamma_1}(T, c')} |\Re(\eta)| - \sum_{\eta \in S_{\Gamma_2}(T, c')} |\Re(\eta)|.
\]
By (1.6), the left hand side equals \(\frac{(\kappa_1 - \kappa_2)(d-1)}{2\pi}T \log(T) + O(T)\). On the other hand, the condition \(c < \frac{1}{4}\) implies that \(c' > c_0 = \{1/4 \ d = 2 \ 3/4 \ d = 3\}\), and using (1.7) we can bound,
\[
\sum_{\eta \in S_{\Gamma_1}(T, c')} |\Re(\eta)| \leq \frac{c'}{c' - c_0} \sum_{\eta \in S_{\Gamma_2}(T, c_0)} |\Re(\eta) - c_0| \ll T \log \log(T).
\]
Comparing the two we get that
\[
|\frac{(\kappa_1 - \kappa_2)(d-1)}{2\pi}T \log(T)| \ll T + T \log \log(T),
\]
and hence \(\kappa_1 = \kappa_2\) as claimed.
Remark 3.1. We note that, if we assume the bound $D(\Gamma_1, \Gamma_2, T) = O(e^{cT})$ with $c < \frac{1}{2}$, we get that all the poles and zeroes of $R_{\Gamma_1}(s)$ and $R_{\Gamma_2}(s)$ cancel out in the half plane $\Re(s) \geq \frac{1}{2}$. However, in contrast to the compact case, we cannot deduce from this that Zeta functions are the same because we cannot exclude the possibility that the spectral zeroes of $Z_{\Gamma_1}(\sigma_k, s)$ will cancel out with residual poles of $Z_{\Gamma_1}(\sigma_{k'}, s_j)$ for some $k' < k$.

Remark 3.2. The result on the number of cusps is restricted for dimensions $d = 2, 3$ for a similar reason. In general, for $d > 3$ our result shows that almost all poles of $Z_{\Gamma}(\sigma_0, s)$ are in the half plane $\Re(s) > \frac{1}{2} - \frac{1}{d+1}$, but even if we assume that $d_L(\Gamma_1, \Gamma_2)$ is smaller we cannot conclude the number of cusps is the same because of possible cancelation with zeroes of the other Zeta functions.

4. Distribution of Poles of Eisenstein series

In this section we give the proof of Theorem 3, generalizing the results of [Sel90] on the distribution of poles of Eisenstein series to hyperbolic manifolds. For the proof we follow the same lines of [Sel90], that is, we express $\varphi(s)$ as a certain Dirichlet series with positive coefficients and then use general results about such Dirichlet series to study the distribution of their zeroes.

4.1. Dirichlet series with positive coefficients. We first collect some results about general Dirichlet series with positive coefficients of the form,

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s},$$

with all $\lambda_n, a_n > 0$. What we need is the following generalization of Selberg’s [Sel90, Lemma 3]

Proposition 4.1. Let $f(s)$ be as in (4.1). Assume that the series absolutely converges for $\Re(s) > 1$, that it has an analytic continuation to $\Re(s) > 0$ with a simple pole at $s = 1$ and at most finitely many poles in the strip $0 < \Re(s) < 1$, all in the interval $(0,1)$. Further assume that $f(s)$ has a continuous extension to $\Re(s) = 0$ and it satisfies the growth condition

$$|f(\sigma + it)| = O(|t|^r), \quad 0 \leq \sigma < \frac{3}{2}, \quad t \gg 1.$$  

Let $\sigma_1 = 1 - \frac{1}{2(r+1)}$, then for all $\sigma \geq \sigma_1$ we have

$$\int_{1}^{T} |f(\sigma + it)|^2 dt \ll T \min\left(\frac{1}{(\sigma - \sigma_1)^2}, \log^2(T)\right).$$
Remark 4.1. For \( r = 1/2 \), this follows from Selberg’s [Sel90, Lemma 3], with a better value of \( \sigma_1 = 1/2 \). Our proof goes along the same line as his, except in one point where his argument seems to work only when \( r \leq 1/2 \) and we had to make some modifications, resulting in a slightly worse value \( \sigma_1 = 1/3 \).

Before we proceed with the proof we recall a couple of standard results on these Dirichlet series.

Lemma 4.2. Let \( f(s) \) be as above, and for a large parameter \( x > 0 \) and \( k \in \mathbb{N} \) let \( f^*(s) = f_{x,k}(s) \) be defined by the finite series

\[
f^*(s) = \sum_{\lambda_n \leq x} a_n \left( 1 - \frac{\lambda_n}{x} \right)^k \lambda_n^{-s}
\]

Let \( 0 < \sigma_0 < 1 \). Then for any \( \sigma_0 < \sigma < 2 \) and \( x^{\frac{1-\sigma_0}{r+1}} \leq t \leq x^{\frac{\sigma_0}{r}} \) we have

\[
f_{x,k}^*(s) = f(s) + O(1).
\]

Proof. For any \( c > 1 \) we can write

\[
f^*(s) = \frac{k!}{2\pi} \int_{c-i\infty}^{c+i\infty} \frac{x^{z-s}f(z)}{(z-s)(z-s+1)\cdots(z-s+k)} ds.
\]

Shifting the contour of integration to \( \Re(s) = \epsilon \) (and taking \( \epsilon \to 0 \)) the pole at \( z = s \) will contribute \( f(s) \) (if \( \sigma > 1 \) the pole at \( z = s-1 \) will contribute \( -\frac{kf(s-1)}{x} = O(t^r x^{-1}) \) and if \( \sigma = 1 \) we avoid the pole on the imaginary axis by integrating over half a circle centered at \( s-1 \) with some small fixed radius). The finitely many poles, \( \rho_j \in (0,1) \) each contribute \( c_j \frac{x^{\rho_j-s}(\rho_j-s)(\rho_j-s+1)\cdots(\rho_j-s+k)}{\rho_j} = O\left(\frac{x^{1-\sigma}}{t^{r+1}}\right) \) and using (4.2) we can bound the remaining integral by \( O(t^r x^{-\sigma}) \).

We thus get that

\[
f_{x,k}^*(s) = f(s) + O\left(\frac{x^{1-\sigma}}{tk+1}\right) + O(t^r x^{-\sigma}).
\]

Consequently, for any \( \sigma > \sigma_0 \) and \( x^{\frac{1-\sigma_0}{r+1}} \leq t \leq x^{\frac{\sigma_0}{r}} \) we have \( f_{x,k}^*(s) = f(s) + O(1) \) as claimed.

\[\square\]

Lemma 4.3. Let \( f(s) \) be as in (4.1) and let

\[
A_f(x) = \sum_{\lambda_n \leq x} a_n
\]

Then

\[
A_f(x) = \sum_{j=1}^{k} c_j \frac{x^{\rho_j}}{\rho_j} + O\left(\frac{x^{\frac{r+\sigma_0}{r}}}{t^{r+1}} \log(x)\right),
\]
where \( c_j = \text{Res}_{\rho_j} f \) with \( \rho_1, \ldots, \rho_k \) the poles of \( f(s) \) in \((0, 1]\).

**Proof.** This follows from standard techniques. Specifically, write

\[
\sum_{\lambda_n \leq x} a_n = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{f(s)x^s}{s} ds + O\left(\frac{x}{T}\right),
\]

with \( c = 1 + \frac{1}{\log x} \) and shift the contour of integration to \( \Re(s) = \frac{1}{2} \) picking up the contribution of the poles (which give the main term). Using (4.2) we bound the integrals

\[
\int_{1/2}^{\infty} f(s+iT) = O(\log(T)T^{2r(1-\sigma)})
\]

and the convexity bound

\[
f(\sigma + iT) = O\left(\log(T)T^{2r(1-\sigma)}\right)
\]

with a choice of \( T = x^{1/(r+1)} \) all error terms are bounded by \( O(x^{1/(r+1)} \log(x)) \).

**Proof of Proposition 4.1.** Let \( \sigma_0 = \frac{2r}{2r+1} < \sigma_1 \) and let \( f^*(s) = f_{x,k}^*(s) \) as in (4.3) with \( x = T^{(2r+1)/(2r-1)} \) and \( k = \lfloor r^{(2r+1)/(2r-1)} \sigma_0 \rfloor \). Then by Lemma 4.2 we have that \( f^*(s) = f(s) + O(1) \) for all \( \sigma > \sigma_0 \) and \( T^{1/(r+1)} \leq t \leq T \). We can thus bound for all \( \sigma > \sigma_0 \)

\[
\int_{1}^{T} |f(\sigma + it)|^2 = \int_{1}^{T} \frac{1}{T^{2r+1}} |f(\sigma + it)|^2 + \int_{T^{2r+1}}^{T} |f(\sigma + it)|^2 + O(T)
\]

\[
\ll \int_{T}^{T^{2r+1}} |f^*(\sigma + it)|^2 + O(T)
\]

where we used (4.2) to bound

\[
\int_{1}^{T^{2r+1}} |f(\sigma + it)|^2 \ll \int_{1}^{T^{2r+1}} t^{2r} dt \ll T.
\]

Next we bound

\[
\int_{-T}^{T} |f^*(\sigma + it)|^2 dt \leq 2 \int_{-T}^{T} |f^*(\sigma + it)|^2 \left(\frac{2T}{t} \sin\left(\frac{t}{2T}\right)\right)^2 dt
\]

\[
\leq 2 \int_{-\infty}^{\infty} |f^*(\sigma + it)|^2 \left(\frac{2T}{t} \sin\left(\frac{t}{2T}\right)\right)^2 dt
\]

\[
= 4T \sum_{\lambda_n, \lambda_m < x} \frac{a_n^* a_m^*}{(\lambda_n \lambda_m)^\sigma} \int_{-\infty}^{\infty} \left(\frac{\sin(t)}{t}\right)^2 e^{2itT |\log(\lambda_n/\lambda_m)|} dt
\]

\[
= 4\pi T \sum_{\lambda_n, \lambda_m < x} \frac{a_n^* a_m^*}{(\lambda_n \lambda_m)^\sigma} \left(1 - T |\log(\lambda_n/\lambda_m)|\right)
\]

\[
\sum_{\lambda_n, \lambda_m < x, |\log(\lambda_n/\lambda_m)| \leq T}.
\]
where \( a^*_n = a_n (1 - \frac{\lambda_n}{x})^k \leq a_n \). Next using the condition that \( |\log (\frac{\lambda_m}{\lambda_n})| \leq \frac{1}{T} \) we may restrict the double sum to an outer sum over \( \lambda_n \leq x \) and an inner sum over \( \lambda_n \leq \lambda_m \leq \lambda_n (1 + \frac{e}{T}) \), and note that in the inner sum we have \( \lambda_m \geq \lambda_n \). We thus get that

\[
\int_{-T}^{T} |f^*(\sigma + it)|^2 \ll T \sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n^\sigma} \sum_{\lambda_n \leq \lambda_m \leq \lambda_n (1 + \frac{e}{T})} \frac{a_m}{\lambda_m^\sigma},
\]

where \( A_f(x) \) is as in (4.5). Using Lemma 4.3 and taking differences we get that

\[
A_f(\lambda_n (1 + \frac{e}{T})) - A_f(\lambda_n)) \ll \frac{\lambda_n}{T} + \frac{\lambda_n}{T} \log(\lambda_n).
\]

Plugging this estimate back gives

\[
\int_{-T}^{T} |f^*(\sigma + it)|^2 \ll T \sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n} \lambda_n^{2-2\sigma} + \lambda_n^{-1} \log(\lambda_n),
\]

\[
\ll T \left( \frac{x'(2-2\sigma)}{T} + x' \sigma_{10}^{-1} + 1 - 2\sigma \log(x) \right) \sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n}.
\]

\[
\ll T \left( \frac{x}{\sigma_0}^{2r(1-\sigma)} - 1 + T \sigma_0^{r(1-2\sigma)} \right) \log^2(T),
\]

where we recall that \( x = T^{\frac{r}{\sigma_0}} \) and that

(4.7) \[
\sum_{\lambda_n \leq x} \frac{a_n}{\lambda_n} \ll \log(x),
\]

which follows from Lemma 4.3 and summation by parts.

Since we assume \( \sigma \geq \sigma_1 > \sigma_0 \) we have that \( \frac{2r(1-\sigma)}{\sigma_0} \leq \frac{2r(1-\sigma_0)}{\sigma_0} \leq 1 \) and that \( \frac{r}{r+1} + 1 - 2\sigma \leq \frac{r}{r+1} + 1 - 2\sigma_1 \leq 0 \) hence

\[
\int_{-T}^{T} |f^*(\sigma + it)|^2 \ll T \log^2(T),
\]

as claimed.
When $\sigma > \sigma_1$, we can do better by setting $\delta = 2(\sigma - \sigma_1)$ and bounding
\[
\int_{-T}^{T} |f^*(\sigma + it)|^2 \ll T \sum_{\lambda_n \leq x} \frac{a_n \lambda_n^{2-2\sigma_1}}{\lambda_n^{1+\delta}} + \frac{\lambda_n^{\delta+1-2\sigma_1}}{T} \log(\lambda_n)
\]
\[
\ll T \sum_{\lambda_n \leq x} \frac{a_n \log(\lambda_n)}{\lambda_n^{1+\delta}}
\]
\[
\ll T f'(1 + \delta) \ll \frac{T}{\delta^2} \ll \frac{T}{(\sigma - \sigma_1)^2}.
\]

4.2. **Scattering determinant as a Dirichlet series.** In order to use the above result for the scattering determinant we need to express it as a Dirichlet series. To do this we explicitly compute the terms $y_i(\gamma.z)^s$ appearing in (4.3), and then integrate along the cusps.

Explicit formulas for the action of the group of isometries on hyperbolic space are well known in dimension $d = 2$ via the identification of the group of isometries $\text{SO}_0(2, 1) \cong \text{PSL}_2(\mathbb{R})$ acting on the upper half plane, $\mathbb{H}^2$, by linear fractional transformations. (There are similar nice formulas in dimensions 3 and 4 where one identifies the group of isometries with $\text{SL}_2(\mathbb{C})$ or with matrices with quaternion coefficients). In higher dimensions, the general formulas for the action of the isometry group on $\mathbb{H}^d$ are more complicated (and we could not find it written down explicitly in the literature). However, since we are only interested in the $y$ component, we have the following simple formula.

**Lemma 4.4.** Let $G \cong \text{SO}_0(d, 1)$ denote the group of isometries of hyperbolic space $\mathbb{H}^d$. For any $g \in G$ there is $\lambda = \lambda(g) \geq 0$ such that for any $z = (x, y) \in \mathbb{H}^d$

\[
y(z) = y(g.z) \begin{cases} 
\lambda(y^2 + \|x + \eta\|^2) & \lambda > 0 \\
\alpha & \lambda = 0
\end{cases} \quad \text{for some } \eta \in \mathbb{R}^{d-1} \text{ and } \alpha > 0 \text{ (depending only on } g\text{).}
\]

**Proof.** We start with the hyperboloid model for hyperbolic space

\[
\mathbb{L}^d = \{(\xi_0, \xi_1, \ldots, \xi_d) | \xi_0^2 + \ldots \xi_{d-1}^2 - \xi_d = -1, \xi_d > 0\}.
\]

In this model the group of isometries is just the group of linear maps sending $\mathbb{L}^d$ onto itself (see [CFKP97, Section 7] for more details on the various models of hyperbolic space and the action of the group of isometries on each). Explicitly, this is the identity component of the group

\[
\text{SO}(d, 1) = \{A \in \text{SL}_{d+1}(\mathbb{R}) | A^tJA = J = AJA^t\},
\]
with \( J = \text{diag}(1, \ldots, 1, -1) \), acting linearly on \( \mathbb{L}^d \subset \mathbb{R}^{d+1} \).

In order to see how this action looks like in the upper half space model we use the isometry \( \iota : \mathbb{L}^d \to \mathbb{H}^d \) given by
\[
(4.11) \quad \iota(\xi_0, \ldots, \xi_d) = \left( \frac{2\xi_1}{\xi_0 + \xi_d}, \ldots, \frac{2\xi_{d-1}}{\xi_0 + \xi_d}, \frac{2}{\xi_0 + \xi_d} \right) = (x, y)
\]
with inverse map given by
\[
(4.12) \quad \iota^{-1}(x, y) = \left( \frac{1 - \frac{1}{4}(y^2 + \sum_j x_j^2)}{y}, \frac{x_1}{y}, \ldots, \frac{x_{d-1}}{y}, \frac{1 + \frac{1}{4}(y^2 + \sum_j x_j^2)}{y} \right)
\]

Now fix some \( g \in \text{Isom}(\mathbb{H}^d) \) and let \( A \in \text{SO}_0(d, 1) \) denote the corresponding linear map acting on \( \mathbb{L}^d \). Given \( z = (x, y) \in \mathbb{H}^d \) let \( \xi = \iota^{-1}(z) \in \mathbb{L}^d \) and \( \xi = A\xi \) so that \( \iota(\tilde{\xi}) = (\tilde{x}, \tilde{y}) = g.z \). Using (4.11) we have that \( \tilde{y} = \frac{2}{\xi_0 + \xi_d} \), \( \tilde{x}_j = \xi_j \tilde{y} \),

and since \( A = (a_{i,j}) \) acts linearly we can write
\[
\tilde{\xi}_0 + \tilde{\xi}_d = \sum_{j=0}^d \alpha_j \xi_j, \text{ with } \alpha_j = a_{0,j} + a_{d,j}.
\]

Now use (4.12) to rewrite \( \xi_j \) back in terms of \( (x, y) \) to get
\[
\tilde{\xi}_0 + \tilde{\xi}_d = \frac{\alpha_0 + \alpha_d}{y} + \frac{\alpha_0 - \alpha_d}{4y} (y^2 + \sum_{j=1}^{d-1} x_j^2) + \sum_{j=1}^{d-1} \frac{\alpha_j x_j}{y}.
\]

We first, consider the case where \( \alpha_0 \neq \alpha_d \) and we define \( \lambda = \frac{\alpha_d - \alpha_0}{8} \) and \( \eta_j = \frac{2\alpha_j}{\alpha_d - \alpha_0} \). Completing the squares we get
\[
\tilde{\xi}_0 + \tilde{\xi}_d = \frac{1}{y} \left[ (\alpha_d + \alpha_0) - \frac{1}{\alpha_d - \alpha_0} \sum_{j=1}^{d-1} \alpha_j^2 + 2\lambda \left( y^2 + \sum_{j=1}^{d-1} (x_j + \eta_j)^2 \right) \right].
\]

A direct computation using the fact that \( A \in \text{SO}(d, 1) \) shows that
\[
(\alpha_d + \alpha_0) = \frac{1}{\alpha_d - \alpha_0} \sum_{j=1}^{d-1} \alpha_j^2,
\]
hence,
\[
\tilde{y} = \frac{2}{\xi_0 + \xi_d} = \frac{y}{\lambda (y^2 + \|x + \eta\|^2)}.
\]

Note that \( \lambda \) depends only on \( A \) (and hence only on \( g \)) and not on \( (x, y) \), in particular, the fact that \( \tilde{y} > 0 \) implies that \( \lambda > 0 \) as well.
Next, consider the case where \( \alpha_0 = \alpha_d = \alpha \) so that

\[
\tilde{\xi}_0 + \tilde{\xi}_d = \frac{2\alpha}{y} + \sum_{j=1}^{d-1} \frac{\alpha_j x_j}{y}.
\]

Using again that \( \tilde{y} > 0 \) we must have that \( \alpha > 0 \) and that \( \alpha_j = 0 \) for \( j = 1, \ldots, d-1 \), (otherwise one can always choose \( x_j \) to make \( \tilde{y} = \frac{2}{\tilde{\xi}_0 + \tilde{\xi}_d} < 0 \)). We thus see that \( \tilde{\xi}_0 + \tilde{\xi}_d = \frac{2\alpha}{y} \), hence, \( \tilde{y} = y/\alpha \), thus concluding the proof. \( \square \)

**Remark 4.2.** Note that \( \lambda(g) = 0 \) if and only if \( g \in P \). In this case, if we write \( g = n x_0 a_t m \in NAM \) then \( \alpha = e^{-t} \) and \( g.(x, y) = (e^t m x + x_0, e^t y) \) (where \( m \in M = SO(d-1) \) acts linearly on \( \mathbb{R}^{d-1} \)).

With this formula we can compute the scattering matrix and express it in terms of Dirichlet series as follows

**Proposition 4.5.** The coefficients of the scattering matrix can be written as

\[
\phi_{ij}(s) = c_j \frac{\Gamma(s - \frac{d+1}{2})}{\Gamma(s)} L_{ij}(s),
\]

where \( L_{ij}(s) \) is a Dirichlet series of the form

\[
L_{ij}(s) = \sum_{n=0}^{\infty} \frac{a_{ij}(n)}{\lambda_{ij}(n)^s},
\]

with \( a_{ij}(n) \in \mathbb{N} \) and \( \lambda_{ij}(n) > 0 \), converging in the half plane \( \Re(s) > d-1 \) with a simple pole at \( s = d - 1 \).

**Proof.** We first consider the top left coefficient \( \phi_{1,1} \), given by

\[
(4.13) \quad \frac{1}{\text{vol}(\Gamma_N \backslash N)} \int_{\Gamma_N \backslash N} E_1(s, n.z) dn = y^s + \phi_{1,1}(s)y^{d-1-s}.
\]

Note that \( \tau \in \Gamma_N \) acts on \( z = (x, y) \in \mathbb{H}^d \) via \( (x, y) \mapsto (x + u_\tau, y) \) and the set \( L_1 \subset \mathbb{R}^{d-1} \) of all \( u_\tau \)'s occurring in this way is a lattice in \( \mathbb{R}^{d-1} \). Equations (0.4) and (0.5) can then be written explicitly as

\[
(4.14) \quad \frac{1}{v_1} \int_{\mathcal{F}_1} E(s, (x, y)) dx = y^s + \phi_{1,1}(s)y^{d-1-s},
\]

where \( \mathcal{F}_1 \subset \mathbb{R}^{d-1} \) is a fundamental domain for \( L_1 \backslash \mathbb{R}^{d-1} \) and \( v_1 = \text{vol}(\mathcal{F}_1) \). Using the explicit formula (4.8) for \( y(\gamma.z) \) we can compute
this integral directly (for $\Re(s) > d - 1$) as
\[
\int_{\mathcal{F}_1} E(s, (x, y)) \, dx = y^s v_1 \left( \sum_{\gamma \in \Gamma_P \setminus \Gamma} \frac{1}{\alpha(\gamma)} \right) + \sum_{\gamma \in \Gamma_P \setminus \Gamma \setminus \Gamma_N \atop \lambda(\gamma) > 0} \frac{y^s}{\lambda(\gamma)^s} \int_{\mathcal{F}_1} \frac{dx}{(y^2 + \|x + \eta\|^2)^s}
\] (4.15)

For any $\gamma \in \Gamma_P \setminus \Gamma$ with $\lambda(\gamma) > 0$ and $\tau \in \Gamma_N$ we have that $\gamma$ and $\gamma\tau$ are distinct classes in $\Gamma_P \setminus \Gamma$ with $\lambda(\gamma) = \lambda(\gamma\tau)$. Indeed, a direct computation using (4.8) shows that $y(\gamma.z) = y(\gamma\tau.z)$ for all $z$ if and only if $\tau = 1$, hence, $\gamma\tau\gamma^{-1} \notin \Gamma_P$ when $\tau \neq 1$. To see that $\lambda(\gamma) = \lambda(\gamma\tau)$, use (4.8) to get the identity
\[
\frac{1}{\lambda(\gamma\tau)} = y(z) \cdot y(\gamma\tau.z) = \frac{y^2}{\lambda(\gamma)(y^2 + \|x + u_\tau + \eta(\gamma)\|^2)} = \frac{1}{\lambda(\gamma)(1 + y^{-2} \|x + u_\tau + \eta(\gamma)\|^2)}
\]
and take $y \to \infty$.

We can thus write the second sum in (4.15) as
\[
\sum_{\gamma \in \Gamma_P \setminus \Gamma / \Gamma_N \atop \lambda(\gamma) > 0} \sum_{u \in L_1} \frac{y^s}{\lambda(\gamma)^s} \int_{\mathcal{F}_1} \frac{dx}{(y^2 + \|x + u + \eta\|^2)^s}
= \sum_{\gamma \in \Gamma_P \setminus \Gamma / \Gamma_N \atop \lambda(\gamma) > 0} \frac{y^s}{\lambda(\gamma)^s} \int_{\mathbb{R}^{d-1}} \frac{dx}{(y^2 + \|x\|^2)^s}
= \sum_{\gamma \in \Gamma_P \setminus \Gamma / \Gamma_N \atop \lambda(\gamma) > 0} \frac{y^{d-1-s}}{\lambda(\gamma)^s} \int_{\mathbb{R}^{d-1}} \frac{dx}{(1 + \|x\|^2)^s}
\]
The integral no longer depends on $y$ or $\gamma$ and can be evaluated in terms of $\Gamma$-functions as
\[
\int_{\mathbb{R}^{d-1}} \frac{dx}{(1 + \|x\|^2)^s} = c(d) \frac{\Gamma(s - \frac{d-1}{2})}{\Gamma(s)}
\] (4.16)

Consider the set
\[
\Lambda_{11} = \{ \lambda(\gamma) \mid \gamma \in \Gamma_P \setminus \Gamma / \Gamma_N \} \cap (0, \infty).
\]
From the discreteness of $\Gamma$ we get that $\Lambda_{11} \subset (0, \infty)$ is discrete and we can order it
\[ \Lambda_{11} = \{ \lambda_{11}(0) < \lambda_{11}(1) < \lambda_{11}(2) < \cdots \}, \]
and let
\[ a_{1,1}(n) = \# \{ \gamma \in \Gamma \backslash \Gamma / \Gamma_N : \lambda(\gamma) = \lambda_{1,1}(n) \}. \]
With these notation we get that
\[ (4.17) \]
\[ \frac{1}{v_1} \int_{\mathcal{F}_1} E_1(s, z)\,dx = y^s \left( \sum_{\gamma \in \Gamma \backslash \Gamma \atop \lambda(\gamma) = 0} \frac{1}{\alpha(\gamma)} \right) + \frac{c(d)}{v_1} \frac{\Gamma(s - \frac{d-1}{2})}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{a_{11}(n)}{\lambda_{11}(n)^s} \]
Comparing (4.14) with (4.17) we see that
\[ \sum_{\gamma \in \Gamma \backslash \Gamma \atop \lambda(\gamma) = 0} \frac{1}{\alpha(\gamma)} = 1, \]
and that
\[ \phi_{1,1}(s) = c_1 \frac{\Gamma(s - \frac{d-1}{2})}{\Gamma(s)} L_{11}(s), \]
with $c_1 = \frac{c(d)}{v_1}$. Finally, since the Eisenstein series $E_1(s, z)$ absolutely converges for $\Re(s) > d - 1$ and has a simple pole at $s = d - 1$, the series $L_{11}(s)$ also absolutely converges in this region and has a simple pole at $s = d - 1$.

The formula for the other coefficients $\phi_{ij}(s)$ follows from the same arguments where we denote by $\lambda_{ij}(\gamma) = \lambda(k^{-1}_i \gamma k_j)$ and let
\[ \Lambda_{ij} = \{ \lambda_{ij}(\gamma) > 0 | \gamma \in \Gamma \backslash \Gamma / \Gamma_{N_j} \} \]
\[ = \{ \lambda_{ij}(0) < \lambda_{ij}(1) < \lambda_{ij}(2) < \cdots \}, \]
and $a_{ij}(n) = \# \{ \gamma \in \Gamma \backslash \Gamma / \Gamma_{N_j} : \lambda_{ij}(\gamma) = \lambda_{ij}(n) \}$. Note that the fact that the cusps are distinct implies that when $i \neq j$ we have that $\lambda_{ij}(\gamma) > 0$ for all $\gamma \in \Gamma$ and the first sum in (4.17) vanishes. \[ \square \]

Since sums and products of Dirichlet series with positive coefficients give a Dirichlet series with positive coefficients, the scattering determinant has an expression of the form
\[ (4.18) \]
\[ \varphi(s) = \left( \frac{\Gamma(s - \frac{d-1}{2})}{\Gamma(s)} \right)^\kappa L(s), \]
where $L(s)$ is another a Dirichlet series with positive coefficients. We can also rewrite it as
\[ (4.19) \]
\[ L(s) = ab^{d-1-2s} L^*(s) \]

for some \(a, b > 0\) with
\[
(4.20) \quad L^*(s) = 1 + \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}
\]
with all \(a_n > 0\) and \(1 < \lambda_1 < \lambda_2 < \ldots\)

We can now apply our results for Dirichlet series with positive coefficients to give some estimates on \(L^*(s)\).

**Proposition 4.6.** The function \(L^*(s)\) is holomorphic in \(\Re(s) > \frac{d-1}{2}\) except for finitely many poles in \((\frac{d-1}{2}, d - 1]\) and satisfies there
\[
(4.21) \quad |L^*(\sigma + it)| = 1 + O(e^{-c\sigma}),
\]
for some \(c > 0\) and all \(\sigma >> 1\) sufficiently large,
\[
(4.22) \quad L^*(\sigma + it) = O(|t|^{(d-1)\frac{s}{2}}),
\]
for \(\sigma > \frac{d-1}{2}\) and \(t >> 1\) sufficiently large, and
\[
(4.23) \quad |L^*(\frac{d-1}{2} + it)| = a^\Gamma \left| \frac{\Gamma(\frac{d-1}{2} + it)}{\Gamma(it)} \right|^\kappa.
\]

**Proof.** The first part follows from the holomorphic extension of \(\varphi(s)\). The bound \((4.21)\) follows from the expansion \((4.1)\) which absolutely converges for \(\sigma > d - 1\) (one can take the constant \(c = \log(\lambda_1)\)).

Next, \((4.19)\) and \((4.18)\) give
\[
(4.24) \quad L^*(s) = (ac)^{-1}b^{2s+1-d} \left( \frac{\Gamma(s)}{\Gamma(s - \frac{d-1}{2})} \right)^\kappa \varphi(s),
\]
and since \(|\varphi(\frac{d-1}{2} + it)| = 1\) we get \((4.23)\).

Finally, from the the Maass-Selberg relations (see [Sel89, Equation (7.44)]) we get that for \(Y > 0\) sufficiently large (but fixed), if we let
\[
E^Y_i(z, s) = \begin{cases} E_i(z, s) & y_j(z) < Y, j = 1, \ldots, \kappa \\ E_i(z, s) - \delta_{ij}y_j^s - \phi_{ij}(s)y_j^{d-1-s} & y_j(z) \geq Y \end{cases},
\]
and let \(E^Y(z, s)\) denote the column vector with components \(E^Y_i(z, s)\). Then for \(s = \sigma + it\) we get the matrix equation
\[
\int_{F\Gamma} E^Y(z, s)E^Y(z, s)^*dz = \frac{1}{2\sigma - d - 1} \left( Y^{2\sigma+1-d}I - Y^{d-1-2\sigma}\phi(s)\phi(s)^* \right) + \frac{\phi(s)^*Y^{2it} - \phi(s)Y^{-2it}}{2it},
\]
where $I$ is the identity matrix. Since the matrix on the left hand side is positive, so is the matrix on the right, which in turn implies that

$$\phi(s)\phi(s)^* \leq Y^{4\sigma - 2n} \left( \sqrt{1 + \left( \frac{2\sigma + 1 - d}{2t} \right)^2} + \frac{2\sigma + 1 - d}{2t} \right)^2 I.$$ 

Hence for $\frac{d-1}{2} < \sigma < d$ we have the uniform bound

$$|\varphi(\sigma + it)| \lesssim \left( \sqrt{1 + \left( \frac{2\sigma + 1 - d}{2t} \right)^2} + \frac{2\sigma + 1 - d}{2t} \right)^n.$$ 

In particular, $\varphi(\sigma + it) = O(1)$ for $\sigma \in (\frac{d-1}{2}, d)$ and $|t| > 1$. Combining this with (4.24) for $\frac{d-1}{2} < \sigma \leq d$ gives the bound

$$|L^*(\sigma + it)| \lesssim \left| \frac{\Gamma(\sigma + it)}{\Gamma(\sigma - \frac{d-1}{2} + it)} \right|^\kappa \lesssim |t|^{\kappa (\frac{d-1}{2})}.$$ 

Since $L^*(\sigma + it) = O(1)$ for $\sigma \geq d$ (from the series expression) this concludes the proof of (4.22).

**Lemma 4.7.** There are constants $B_\Gamma, C_\Gamma$ depending on $\Gamma$ such that

$$\frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |L^*(\frac{d-1}{2} + it)| dt = \frac{\kappa (d-1)}{4\pi} T^2 \log T + B_\Gamma T^2 + C_\Gamma T + O(\log(T)).$$

**Proof.** Let $d - 1 = 2m + \delta$ with $\delta \in \{0, 1\}$, and expand

$$\Gamma(\frac{d-1}{2} + it) = \prod_{j=1}^{m} (\frac{d-1}{2} - j + it) \Gamma(\frac{\delta}{2} + it).$$

Using (4.23) we evaluate the integral

$$\frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |L^*(\frac{d-1}{2} + it)| dt = \frac{\log(a_\Gamma)}{2\pi} T^2$$

$$+ \frac{\kappa}{\pi} \sum_{j=1}^{m} \int_{0}^{T} (T - t) \log |it + \frac{d-1}{2} - j| dt$$

$$+ \frac{\kappa}{\pi} \int_{0}^{T} (T - t) \log \left| \frac{\Gamma(\frac{\delta}{2} + it)}{\Gamma(it)} \right| dt.$$
The third term on the right hand side of (4.26) can be evaluated as

\[
\frac{\kappa}{\pi} \sum_{j=1}^{m} \int_{0}^{T} (T - t) \log |it + \frac{d-1}{2} - j| dt =
\]

\[
= \frac{\kappa m}{2\pi} T^2 \log T - \frac{3\kappa m}{4\pi} T^2 + \frac{\kappa m (d - m - 2)}{8} T + O(\log(T)),
\]

and the last term is given by

\[
\frac{\delta \kappa}{\pi} \int_{0}^{T} (T - t) \log \left| \Gamma \left( \frac{1}{2} + it \right) \Gamma(it) \right| dt = \frac{\delta \kappa}{4\pi} T^2 \log(T) - (\frac{3\delta \kappa}{8\pi} + \frac{\delta \kappa \log(\pi)}{4\pi}) T^2 - \frac{\delta \kappa}{16} T + O(1).
\]

Plugging these back in (4.26) proves the claim with

\[
B_{\Gamma} = \frac{4 \log(\alpha_{T} - \kappa (d - 1 + 2\delta \log \pi)}{8\pi},
\]

and \( C_{\Gamma} = \frac{\kappa (2m(m + \delta - 1) - \delta)}{16} \).

**Lemma 4.8.** For \( \alpha \geq \alpha_{0} = \frac{(d-1)(2d+1)}{2d+2} \) we have

\[
\int_{-T}^{T} \log |L^{*}(\alpha + it)| dt \lesssim T \log(\min\{\frac{1}{\alpha - \alpha_{0}}, \log(T)\})
\]

**Proof.** First, from (4.18) and (4.25) we get that for \( |t| < 1 \) and \( \frac{d-1}{2} < \alpha < d \)

\[
\log |L^{*}(\alpha + it)| \ll 1 + \log(1 + \frac{1}{t}),
\]

so it is enough to bound \( \int_{-T}^{T} \log |L^{*}(\alpha + it)| dt. \)

Next, since \( L(s) = \det(L_{ij}(s)) \) we get the bound

\[
|L^{*}(\alpha + it)| \leq \frac{b^{2\alpha + 1 - d}}{a} \left( \frac{1}{\kappa} \sum_{i,j} |L_{ij}(\alpha + it)|^2 \right)^{\kappa/2},
\]

and from the inequality between geometric and arithmetic mean we get

(4.27) \( \frac{1}{T} \int_{1}^{T} \log(|L^{*}(\alpha + it)|) dt \leq (2\alpha + 1 - d) \log b - \log a \)

\[
+ \frac{\kappa}{2} \log \left( \frac{1}{T} \int_{1}^{T} \frac{1}{\kappa} \sum_{i,j} |L_{ij}(\alpha + it)|^2 dt \right)
\]

Now, for each of the functions \( L_{ij}(s) \), consider \( f(s) = L_{ij}(\frac{d-1}{2}(s+1)) \).

This function is still given by a Dirichlet series with positive coefficients, it converges absolutely for \( \Re(s) > 1 \) with a simple pole at \( s = 1 \) and has
an analytic continuation to $\Re(s) \geq 0$ except perhaps finitely many poles in $(0, 1)$. The bound (4.22) implies that $|f(\sigma + it)| = O(t^{\frac{d-1}{2}})$ for $\sigma \geq 0$ and $t > 1$, hence, we can apply Lemma 4.1 to $f(s) = L_{ij}(\frac{d-1}{2}(s + 1))$ and get that for $\alpha \geq \alpha_0$,

$$\frac{1}{T} \int_{1}^{T} |L_{ij}(\alpha + it)|^2 dt \ll \min \left( \frac{1}{(\alpha - \alpha_0)^2}, \log^2(T) \right).$$

Using this bound with (4.27) gives

$$\frac{1}{T} \int_{1}^{T} \log(|L^*(\alpha + it)|)dt \leq (2\alpha + 1 - d) \log b - \log a$$

$$+ \frac{\kappa}{2} \log \min \left( \frac{1}{(\alpha - \alpha_0)^2}, \log^2(T) \right)$$

$$\ll \log \min \left( \frac{1}{\alpha - \alpha_0}, \log(T) \right)$$

as claimed. □

**Lemma 4.9.** For any $\alpha > \frac{d-1}{2}$ we have

$$\int_{\alpha}^{\infty} \arg(L^*(\sigma + iT))d\sigma = O(\log(T)).$$

**Proof.** Since $L^*(\sigma + iT) = 1 + O(e^{-c\sigma})$ as $\sigma \to \infty$ we have $\arg(L^*(\sigma + iT)) \ll e^{-c\sigma}$, so for $\sigma_1 > d$ large enough (but fixed), we have

$$\int_{\alpha}^{\infty} \arg(L^*(\sigma + iT))d\sigma = \int_{\alpha}^{\sigma_1} \arg(L^*(\sigma + iT))d\sigma + O(1).$$

For the remaining integral, using the bound $|L^*(\sigma + iT)| = O(T^{\frac{d-1}{2}})$ and Titchmarsh’s [Tit86, Lemma 9.2] we get that $\arg(L^*(\sigma + iT)) = O(\log(T))$ and hence the whole integral is bounded by $O(\log(T))$. □

We now have all the ingredients to give the

**Proof of Theorem.** The zeroes and poles of $\varphi(s)$ in $\Re(s) > \frac{d-1}{2}$ are the same as the zeroes and poles of $L^*(s)$ in $\Re(s) > \frac{d-1}{2}$. Using Proposition 4.6 we see that $L^*(s)$ satisfies all the assumptions needed for [Sel90, Lemma 2,3], stating that for any $\alpha \geq \frac{d-1}{2}$,

$$(4.28)\sum_{\beta > \alpha} (T - |\gamma|)(\beta - \alpha) = \frac{1}{2\pi} \int_{-T}^{T} (T - |t|) \log |L^*(\alpha + it)|dt$$

$$+ T \sum_{\sigma_j > \alpha} (\sigma_j - \alpha) + O(\log(T))$$

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Let $F(\alpha, T)$ denote the left hand side of (4.28), and let
\[(4.29) \quad F_1(\alpha, T) = \sum_{|\gamma| \leq T, \beta > \alpha} (\beta - \alpha).\]

One easily sees that
\[(4.30) \quad F(\alpha, T) - F(\alpha, T - 1) \leq F_1(\alpha, T) \leq F(\alpha, T + 1) - F(\alpha, T).\]

From (4.28) with $\alpha = \frac{d-1}{2}$, together with Lemma 4.7 we get
\[F(\frac{d-1}{2}; T) = \kappa(d-1) T^2 \log T + B \Gamma T + O(\log(T)),\]
which together with (4.30) implies that
\[F_1(\frac{d-1}{2}; T) = \kappa(d-1) T \log T + A \Gamma T + O(\log(T)),\]
with $A \Gamma = 2(C \Gamma + \sum_{\sigma_j > \alpha} (\sigma_j - \frac{d-1}{2})) + B \Gamma$. This confirms (0.6)

Next, to show (0.7), we use Littlewood’s formula
\[\sum_{|\gamma| \leq T, \beta > \alpha} (\beta - \alpha) = \frac{1}{2\pi} \int_{-T}^{T} \log |L^*(\alpha + it)| dt + \frac{1}{\pi} \int_{\alpha}^{\infty} \arg(L^*(\sigma + iT)) d\sigma + \sum_{\sigma_j > \alpha} (\sigma_j - \alpha).\]

For $\alpha \geq \alpha_0 = \frac{(2d+1)(d-1)}{2(2d+1)}$, we can use Lemma 4.8 to bound the first integral, Lemma 4.9 to bound the second integral, and bound the sum over the poles by $O(1)$ to get that
\[\sum_{|\gamma| \leq T, \beta > \alpha} (\beta - \alpha) \ll T \log(\min\{\frac{1}{\alpha - \alpha_0}, \log(T)\})\]
confirming (0.7). \qed

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