List decoding of Convolutional Codes over integer residue rings

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Abstract

A convolutional code $C$ over $\mathbb{Z}_{p^r}[D]$ is a $\mathbb{Z}_{p^r}[D]$-submodule of $\mathbb{Z}_{p^r}^n[D]$ where $\mathbb{Z}_{p^r}[D]$ stands for the ring of polynomials with coefficients in $\mathbb{Z}_{p^r}$. In this paper we study the list decoding problem of these codes when the transmission is performed over an erasure channel, that is, we study how much information one can recover from a codeword $w \in C$ when some of its coefficients have been erased. We do that using the $p$-adic expansion of $w$ and particular representations of the parity-check matrix of the code. From these matrix representations we recursively select certain equations that $w$ must satisfy and have only coefficients in the field $p^{-1}\mathbb{Z}_{p^r}$. This yields a step by step procedure to obtain a list of possible codewords for a given corrupted codeword $w$. We show that such an algorithm actually computes all possible erased coordinates, that is, it provides a minimal list with the closest codewords to the vector $w$. Mathematically, this problem amounts to determine the set of all possible solutions of a set of linear equations over $\mathbb{Z}_{p^r}[D]$ that can be represented by a matrix with Toeplitz structure.

Keywords: Convolutional codes, finite rings, erasure channel

1. Introduction

Convolutional codes form a fundamental class of linear codes that are widely used in applications (see also the related notion of sequential cellular automata [2]). They are typically described by means of a generator matrix, which is a polynomial matrix with coefficients in a finite field or a finite ring, depending on the application. Yet, the mathematical theory of convolutional codes over finite fields is much developed and has produced many sophisticated classes of codes. On the other hand, very little is known about concrete optimal constructions of convolutional codes over finite rings. In the last few years there has been an increasing interest of the performance of convolutional codes over the erasure channel [13]. The decoding of convolutional codes is, in general, not easy. Probably the most prominent decoding algorithm is the Viterbi algorithm but its use is limited as its complexity grows exponentially with the size of the memory of the code. However, in [13] it was shown that the decoding of these codes requires only linear algebra when they are
used over an erasure channel, i.e., when the positions of the errors are known. Despite the fact that convolutional codes that possess optimal erasure correcting capabilities require large finite fields, the results in [13] allow to implement these codes in many practical situations and therefore attracted the interest of many researchers [8].

Following this thread of research and aiming to extend these results over finite fields to the context of finite rings, we consider in this paper convolutional codes \( C \) over \( \mathbb{Z}_{p^r}[D] \) and study the erasure correcting capabilities of these codes over the erasure channel. In particular, our goal is to retrieve as much information as possible from the received corrupted vector. The decoder proposed in this work is a maximum likelihood algebraic decoder and follows succinctly two main steps. Firstly, it searches for unique decoding, i.e., when there exists a unique most likelihood word, then, the decoder outputs such a word. It is well-known that this is possible when the number of erasures is smaller than the free distance of the code. When this is not possible the algorithm performs a list decoding algorithm, i.e., it computes a complete list of the most likelihood codewords for a given corrupted codeword.

For this problem, we shall use the parity-check matrix \( H(D) \) of \( C \) in a particular form. Then, the number of independent columns of specific submatrices of \( H(D) \) will determine the size of the list of possible codewords in the algorithm. The decoding problem treated here amounts to solving a system of linear equations over \( \mathbb{Z}_{p^r} \). The idea we used in this work is to multiply a certain subset of these equations by a power of \( p \) in such a way that we obtain a subset of equations with coefficients in \( p^{r-1}\mathbb{Z}_{p^r} \). Since \( p^{r-1}\mathbb{Z}_{p^r} \) is a field, isomorphic to \( \mathbb{Z}_p \), we can easily solve the system. Once we compute some of the coefficients that are involved in the equations, we can apply the same ideas to a different set of equations to recover another set of erased symbols. Using these ideas we develop a systematic procedure to recover all possible errors, obtaining a minimal set with all possible codewords.

The outline of this paper is as follows. In Section 2, we present basic results on convolutional codes over the finite ring \( \mathbb{Z}_{p^r} \), in particular about their generator matrices, which are important for decoding over the erasure channel. In Section 3, we present our erasure decoding algorithm for convolutional codes over \( \mathbb{Z}_{p^r} \) and illustrate it with an example. Finally, we conclude with some remarks in Section 4.

2. Preliminary results

In this section we present the elementary background required in the paper. Let \( \mathbb{Z}_{p^r}[D] \) denote the ring of polynomials with coefficients in \( \mathbb{Z}_{p^r} \) and let \( \mathcal{A} = \{0, 1, 2, \ldots, p - 1\} \) be the set of digits. We say that \( v(D) \) has order \( s \), and denoted by \( \text{ord}(v(D)) = s \), if \( p^{s-1}v(D) \neq 0 \) and \( p^{s-1}v(D) \in p^{r-1}\mathbb{Z}_{p^r}[D] \). Every element in \( v(D) \in \mathbb{Z}_{p^r}[D] \) admits a unique \( p \)-adic expansion as \( v(D) = a_0(D) + a_1(D)p + \cdots + a_{r-1}(D)p^{r-1} \), with \( a_i(D) \in \mathcal{A}[D] \), \( \text{ord}(a_i(D)) = r - 1 \) and \( i = 0, 1, \ldots, r - 1 \). We shall extensively use that \( p^{r-1}\mathbb{Z}_{p^r} \) is
generator matrix is called a polynomial matrix \( H \) where elements of the form \( pG \) such that \( C = \ker D \) is checked by simple multiplication by \( G \) and then we still can make use of the (componentwise) projection of \( A \) into \( Z_p \) for any matrix \( A \) with entries in \( Z_{p^r} \) or \( Z_{p^r}[D] \).

**Lemma 1.** Let \( C \) be a convolutional code of length \( n \). Then \( C \) admits a parity-check matrix if and only if it admits a generator matrix \( G(D) \) of the form

\[
G(D) = \begin{bmatrix}
G_0(D) \\
pG_1(D) \\
\vdots \\
p^{r-1}G_{r-1}(D)
\end{bmatrix}
\]

with \( G = \begin{bmatrix}
G_0(D) \\
G_1(D) \\
\vdots \\
G_{r-1}(D)
\end{bmatrix}_p \)

left prime over \( Z_p[D] \).

It is well-known that kernel representations are useful to detect errors introduced during transmission. If a word \( w(D) \) is received after channel transmission, the existence of errors is checked by simple multiplication by \( H(D) \): if \( H(D)w(D) = 0 \), it is assumed that no errors occurred. As Lemma 1 shows not all convolutional codes defined in \( Z_{p^r}[D] \) admit a parity-check matrix. Nevertheless we show next that there always exists a matrix \( H(D) \) such that \( C \subset \ker H(D) \), and then we still can make use of \( H(D) \) to decode when the transmission occurs over the erasure channel. In this channel the word can have only erasures (i.e., we know the position of the part of the codeword that is missing or erased) but no errors occur. In fact, if one considers the erasures as indeterminate, \( H(D)w(D) = 0 \) give rise to a system of linear equations. Solving this system amounts to decoding the received word \( w(D) \), as we explain in detail in the next section.

Given a convolutional code \( C \) defined in \( Z_{p^r}[D] \) with encoder \( G(D) \in \mathbb{Z}_{p^r}^{k \times n}[D] \), let us consider the set

\[
\tilde{C} = \{ G(D)^T u(D) : u(D) \in \mathbb{Z}_{p^r}^k((D)) \},
\]

where \( \mathbb{Z}_{p^r}((D)) \) denotes the ring of Laurent series over \( \mathbb{Z}_{p^r} \), i.e., \( \mathbb{Z}_{p^r}((D)) \) the set of elements of the form

\[
a(D) = \sum_{i=-\infty}^{+\infty} a_i D^i
\]

1A polynomial matrix \( A(D) \in \mathbb{Z}_p^{k \times n}[D] \) is left prime if it has a polynomial right inverse.
where the coefficients \( a_i \) are in \( \mathbb{Z}_{p^r} \) and only finitely coefficients with negative indices may be nonzero. It can be shown [4] that there always exists a polynomial matrix \( \tilde{H}(D) \) such that \( \tilde{C} = \{ w(D) \in \mathbb{Z}_{p^r}^n ((D)) : \tilde{H}(D)w(D) = 0 \} \) and therefore \( \tilde{C} \subset \ker \tilde{H}(D) \subset \mathbb{Z}_{p^r}^n [D] \) as in described in (1). Moreover, \( \tilde{H}(D) \) is the smallest observable convolutional code containing \( \tilde{C} \). For the sake of simplicity we consider only observable convolutional codes and develop are algorithms in terms of a parity-check polynomial matrix representing the code.

The associated truncated sliding parity-check matrix of \( H(D) = \sum_{i=0}^\nu H^i D^i \), is

\[
H^c_j = \begin{bmatrix}
H^0 & H^0 & \cdots \\
H^1 & H^0 & \cdots \\
\vdots & \vdots & \ddots \\
H^j & H^{j-1} & \cdots & H^0
\end{bmatrix}
\]

with \( H^j = 0 \) for \( j > \nu \). As any codeword \( w(D) \) of \( C \) satisfies \( H(D)w(D) = 0 \), if \( w(D) = \sum_{i \in \mathbb{N}_0} w^i D^i \), we have that, for all \( j \geq 0 \), \( \sum_{i=0}^j H^i w^{j-i} = 0 \), i.e.,

\[
\begin{bmatrix}
H^0 \\
H^1 \\
\vdots \\
H^j
\end{bmatrix}
\begin{bmatrix}
w^0 \\
w^1 \\
\vdots \\
w^j
\end{bmatrix} = 0.
\]

Two of the main notions of minimum distance of convolutional codes are the free distance and the column distance. Given \( w(D) = \sum_{i \in \mathbb{N}_0} w^i D^i \), we define its Hamming weight as

\[
\text{wt}(w(D)) = \sum_{j \in \mathbb{N}} \text{wt}(w^j).
\]

The free distance gives the correction capability of a convolutional code when considering whole codewords. In other words, there is no maximum degree \( j \) for a codeword considered by the free distance.

Given an \( (n, k) \) convolutional code \( C \subseteq \mathbb{Z}_{p^r}^n [D] \), we define its free distance as

\[
d_{\text{free}}(C) = \min \{ \text{wt}(w(D)) : w(D) \in C \text{ and } w(D) \neq 0 \}.
\]

In this work we shall focus on the sliding-window erasure correction capabilities of \( C \) within a time interval and this will be determined by the column distance of \( C \), which is defined as follows.

\[
d_j^c(C) = \min \{ \text{wt}((w^0, w^1, \ldots, w^j)) : \sum_{j \in \mathbb{N}_0} w^j D^j \in C, w^0 \neq 0 \}
\]

\[
= \{ \text{wt}((w^0, w^1, \ldots, w^j)) : (w^0, w^1, \ldots, w^j) \text{ satisfies [3] and } w^0 \neq 0 \}
\]

where the equality \( * \) holds for convolutional codes that satisfy Lemma [4]. Next, we present two preliminary results that we will need later on.
Lemma 2.  \cite{0} Let $Ax = b$ with $A \in \mathbb{Z}_p^{a \times s}$ and $b \in \mathbb{Z}_p^a$ be a linear system of equations in $x$. Then, it has unique solution if and only if $[A]_p$ is full column rank or equivalently, if the McCoy rank of $A$ is $a$.

Note that, as opposed to the fields case, a set of vectors in $\mathbb{Z}_p^r$ can be linearly dependent but none of them is in the $\mathbb{Z}_p^r$-span of the others. The following result states the erasure correcting capability of a convolutional code in terms of its column distance.

Lemma 3. Let $C = \text{Ker} H(D)$ and $j \in \mathbb{N}$. The following statements are equivalent:

1. the column distance $d_j^C(C) = d$;
2. none of the first $n$ columns of $[H_j^c]_p$ is contained in the $\mathbb{Z}_p$-span of any other $d - 2$ columns of $[H_j^c]_p$ and, moreover, one of the first $n$ columns of $[H_j^c]_p$ is in the $\mathbb{Z}_p$-span of other $d - 1$ columns of $[H_j^c]_p$;
3. if $(w^0, w^1, \ldots, w^j)$ contains up to $d - 1$ erasures then $w^0$ can be recovered and there exist $d$ erasures that make impossible to recover $w^0$.

Proof. (1 $\iff$ 2) follows easily from \cite{6, Prop.2.1} and the fact that the a set of vectors in $\mathbb{Z}_p^r$ is linearly independent if and only if their projection over $\mathbb{Z}_p$ is linearly independent modulo $p$, which is basically Lemma \cite{2}.

(2 $\iff$ 3) readily follows from Theorem 3.1. and its proof in \cite{13}.

3. A decoding algorithm for erasures

In this section we state the problem using the notation presented in the previous section and then propose an efficient decoding algorithm to solve it. More concretely, we aim to recover erasures that may occur during the transmission of the information over an erasure channel using convolutional codes $C \subset \mathbb{Z}_p^r[D]$. If unique recovery is not possible, we derive a constructive step by step decoding algorithm to compute a minimal list with the closest codewords to the received vector. This is equivalent to solve a certain system of linear equation in $\mathbb{Z}_p^r$.

Suppose that $w(D) = \sum_{i \in \mathbb{N}_0} w^i D^i \in C$ is sent and assume that we have correctly received all coefficients up to an instant $i - 1$ and some of the components of $w^i$ are erased. The decoder tries to recover $w^i$ up to a given instant $i + T$ and if this is not possible it outputs a list with the closest vectors at time instant $T + i$. The parameter $T$ is called the delay constraint and represents the maximum delay the receiver can tolerate to retrieve $w^i$, see \cite{1, 9, 7}. For the sake of simplicity it will be assumed $T \leq \nu$, where $\nu$ is the degree of $H(D)$. The system of equations that involve $w^i$ up to time instant $i + T$ is

\footnote{The McCoy rank of a matrix is the largest size of a minor that is an invertible element in the ring, $A \setminus \{0\}$ in our case.}
We can take the columns of the matrix in (5) that correspond to the coefficients of the erased elements to be the coefficients of a new system. With the remaining columns we can compute the independent terms, denoted by \( b_i \).

We regard the erasures as to-be-determined variables and denote for \( i \in \mathbb{N}_0 \) by \( \tilde{w}^i \) the subvector of \( w^i \) that corresponds to the positions of the erasures. Similarly, denote by \( \tilde{H}^j_i \) the matrix consisting of the columns of \( H^j \) with indices corresponding to the erased positions in \( w^i \). Then, we obtain the following system of linear equations

\[
\begin{bmatrix}
\tilde{H}^0_i & \tilde{H}^1_i & \tilde{H}^i_{i+1} & \cdots & \tilde{H}^i_{i+T-1} & \tilde{H}^T_i
\end{bmatrix}
\begin{bmatrix}
\tilde{w}^i \\
\tilde{w}^{i+1} \\
\vdots \\
\tilde{w}^{i+T}
\end{bmatrix}
= \begin{bmatrix}
0 \\
0 \\
\vdots \\
0
\end{bmatrix}
\]

Hence, the problem of decoding is equivalent to solving the system of linear equations described in (6). As for the notation we note that generally we shall use the superscript to indicate the time instant, the subscript for the position within a matrix or vector and \( \tilde{A} \) for the submatrix of \( A \) corresponding to the position of the erasures .

It is easy to see ([14, 8, 4]) that \( H(D) \) can be written in the form

\[
H(D) = \begin{bmatrix}
H_0(D) \\
pH_1(D) \\
\vdots \\
p^{r-1}H_{r-1}(D)
\end{bmatrix}
\text{ with }
\begin{bmatrix}
H_0(D) \\
H_1(D) \\
\vdots \\
H_{r-1}(D)
\end{bmatrix}
\text{ full row rank.}
\]

Hence, it readily follows that one can rewrite equation (6), after appropriate row permutations, as
Exact decoding: Denote by $e^s$ the size of $\tilde{w}^s$, $s \in \{i, i + 1, \ldots, i + T\}$. Thus, it follows from Lemma 3 that $\tilde{w}^i$ is uniquely determined (i.e., unique decoding of $w^i$ is possible) if none of the first $e^i$ columns of

$$
\begin{bmatrix}
\tilde{H}^0_{i,0} & \tilde{H}^0_{i+1,0} \\
\tilde{H}^j_{i,0} & \tilde{H}^j_{i+1,0} \\
\vdots & \vdots \\
\tilde{H}^T_{i,0} & \tilde{H}^T_{i+1,0} \\
\end{bmatrix}
$$

is in the span of the remaining columns for a $j \in \{0, 1, \ldots, T\}$, which occurs if there exists a $j$ such that

$$
\sum_{s=i}^{i+j} e^s \leq d^j(C) - 1.
$$

List decoding: If exact decoding is not possible we aim to compute all possible solutions of (8). To this end we define the following matrix for all $0 \leq t \leq r - 1$,
\[ \tilde{\mathcal{H}}_t^c = \begin{bmatrix} \tilde{H}_{i,0}^0 & \tilde{H}_{i,1}^0 & \vdots & \tilde{H}_{i,r-t-1}^0 & \tilde{H}_{i+1,0}^0 & \tilde{H}_{i+1,1}^0 & \vdots & \tilde{H}_{i+1,r-t-1}^0 \\ \tilde{H}_{i,0}^1 & \tilde{H}_{i,1}^1 & \vdots & \tilde{H}_{i,r-t-1}^1 & \tilde{H}_{i+1,0}^1 & \tilde{H}_{i+1,1}^1 & \vdots & \tilde{H}_{i+1,r-t-1}^1 \\ \tilde{H}_{i,0}^T & \tilde{H}_{i,1}^T & \vdots & \tilde{H}_{i,r-t-1}^T & \tilde{H}_{i+1,0}^T & \tilde{H}_{i+1,1}^T & \vdots & \tilde{H}_{i+1,r-t-1}^T \end{bmatrix}, \tag{11} \]

and write

\[
\begin{bmatrix} \tilde{w}_i^0 \\ \tilde{w}_i^{i+1} \\ \vdots \\ \tilde{w}_i^{i+T} \end{bmatrix} = \begin{bmatrix} w_i^0 \\ w_i^{i+1} \\ \vdots \\ w_i^{i+T} \end{bmatrix} + p \begin{bmatrix} w_i^1 \\ w_i^{i+1} \\ \vdots \\ w_i^{i+T} \end{bmatrix} + \cdots + p^{r-1} \begin{bmatrix} w_i^{r-1} \\ \vdots \\ w_i^{r-1} \end{bmatrix}, \tag{12} \]

where \( w_j^t \) has entries in \( A_p = \{0, 1, \ldots, p-1\} \), for all \( j \in \{i, i+1, \ldots, i+T\} \) and \( t \in \{0, 1, \ldots, r-1\} \). We aim at computing the maximum number of coefficients \( w_j^t \) in (12).
**Step 1:** Find the solution \( (\hat{w}_0^i, \hat{w}_0^{i+1}, \ldots, \hat{w}_0^{i+T}) \) of the system

\[
\begin{bmatrix}
\tilde{H}_{i,0}^0 & \tilde{H}_{i,1}^0 & \cdots & \tilde{H}_{i,r-1}^0 \\
\tilde{H}_{i,0}^1 & \tilde{H}_{i,1}^1 & \cdots & \tilde{H}_{i,r-1}^1 \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{H}_{i,0}^{i} & \tilde{H}_{i,1}^{i} & \cdots & \tilde{H}_{i,r-1}^{i} \\
\tilde{H}_{i,0}^{i+1} & \tilde{H}_{i,1}^{i+1} & \cdots & \tilde{H}_{i,r-1}^{i+1} \\
\tilde{H}_{i,0}^{i+T} & \tilde{H}_{i,1}^{i+T} & \cdots & \tilde{H}_{i,r-1}^{i+T} \\
\end{bmatrix}
\begin{bmatrix}
\hat{w}_0^0 \\
\hat{w}_0^{i+1} \\
\vdots \\
\hat{w}_0^{i+T} \\
\end{bmatrix}
= \begin{bmatrix}
b_0^i \\
b_0^{i+1} \\
\vdots \\
b_0^{i+T} \\
\end{bmatrix}, \quad (13)
\]

over the field \( \mathbb{Z}_p \). Let \( e = \sum_{s=0}^{i+T} e^s \). Then, the “integer” part of the set of solutions, i.e. the vector 
\[
\begin{bmatrix}
w_0^i \\
w_0^{i+1} \\
\vdots \\
w_0^{i+T} \\
\end{bmatrix}
\] in (12), is given by:

\[
S_0 = \left\{ \begin{bmatrix}
w_0^i \\
w_0^{i+1} \\
\vdots \\
w_0^{i+T} \\
\end{bmatrix} \in \mathcal{A}^e : \begin{bmatrix}
w_0^i \\
w_0^{i+1} \\
\vdots \\
w_0^{i+T} \\
\end{bmatrix}_p = \begin{bmatrix}
\hat{w}_0^i \\
\hat{w}_0^{i+1} \\
\vdots \\
\hat{w}_0^{i+T} \\
\end{bmatrix} \text{ with } \begin{bmatrix}
\hat{w}_0^i \\
\hat{w}_0^{i+1} \\
\vdots \\
\hat{w}_0^{i+T} \\
\end{bmatrix} \text{ satisfying (13)} \right\}.
\]

It is straightforward to see that the size of \( S_0 \) is given by

\[ |S_0| = p^e \text{–rank} \tilde{h}_0^e. \]

To compute the remaining vectors, if necessary, in the \( p \)-adic decomposition of (12), we recursively apply the following algorithm in the next step.

**Step 2:** Let \( b_{s,0}^j = b_{s}^j, j = i, i+1, \ldots, i+T, s = 0, 1, \ldots, r-1. \)

For \( t = 1, \ldots, r-1 \) do
1. For \( j = i, i + 1, \ldots, i + T \), consider the solutions 
\[
\begin{bmatrix}
\hat{b}_{0,t}^j \\
\hat{b}_{1,t}^j \\
\vdots \\
\hat{b}_{r-t-1,t}^j
\end{bmatrix} = 
\begin{bmatrix}
p^{i-1}b_{0,t-1}^j \\
p^{i}b_{1,t-1}^j \\
\vdots \\
p^{r-2}b_{r-t-1,t-1}^j
\end{bmatrix} - 
\begin{bmatrix}
p^{i-1}\hat{H}_{i,0}^{j-i} \\
p^{i}\hat{H}_{i,1}^{j-i} \\
\vdots \\
p^{r-2}\hat{H}_{i,r-t-1}^{j-i}
\end{bmatrix} 
\begin{bmatrix}
w_{i-1}^j \\
w_{i+1}^j \\
\vdots \\
w_{i+t-1}^j
\end{bmatrix}
\] 
over \( \mathbb{Z}_p \) and let 
\[
S_t = \left\{ \begin{bmatrix}
w_{i}^j \\
w_{i+1}^j \\
\vdots \\
w_{i+t-1}^j
\end{bmatrix} : \begin{bmatrix}
\hat{w}_{i}^j \\
\hat{w}_{i+1}^j \\
\vdots \\
\hat{w}_{i+t-1}^j
\end{bmatrix} = \begin{bmatrix}
\tilde{w}_{i}^j \\
\tilde{w}_{i+1}^j \\
\vdots \\
\tilde{w}_{i+t-1}^j
\end{bmatrix} \right\}
\] with 
\[
\begin{bmatrix}
w_{i}^j \\
w_{i+1}^j \\
\vdots \\
w_{i+t-1}^j
\end{bmatrix} \in S_t, \ t = 0, 1, \ldots, r - 1 \right\}
\]

Output data:
The size of the list decoding is
\[
\prod_{t=0}^{\ell} |S_t|.
\]
where each $|S_t|$ is given by
\[
|S_t| = p^{(e_i + e_{i+1} + \cdots + e_{i+t}) - \text{rank} \bar{H}_t^c}.
\] (15)

Note that Steps 1 and 2 deal with systems of linear equations over fields. The fact that these steps yield the set of all solution follows from [11, Theorem 3].

**Example 1.** Let $C \subset \mathbb{Z}_8[D]$ be the convolutional code with parity-check matrix $H(D) = H^0 + H^1 D + H^2 D^2 \in \mathbb{Z}_8[D]$ where $H^0 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 0 & 2 \\ 4 & 4 & 0 & 4 & 4 \end{bmatrix}$, $H^1 = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 \\ 4 & 0 & 4 & 4 \end{bmatrix}$

and $H^2 = \begin{bmatrix} 3 & 5 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \end{bmatrix}$. It is easy to check that $w(D) = w^0 + w^1 D + w^2 D^2 + w^3 D^3$ with $w^0 = [5, 5, 0, 6, 0]$, $w^1 = [6, 6, 4, 3, 6]$, $w^2 = [2, 1, 1, 2, 0]$ and $w^3 = [2, 6, 4, 0, 0]$ is a codeword of $C$. Assume that one receives $w^0 = [5, w^0_1, w^0_2, 6, w^0_3]$, $w^1 = [6, 6, 4, w^1_1, 6]$, $w^2 = [2, 1, w^2_1, w^2_2, w^2_3]$ and $w^3 = [2, w^3_1, 4, 0, 0]$ where $w^0_1, w^0_2, w^0_3, w^1_1, w^2_1, w^2_2, w^2_3, w^3_1$ are erasures.

Let the delay constraint for the decoding be $T = 2$. Exact decoding of $w^0$ is not possible and we start our list decoding algorithm. One has

$$
\bar{H}_0^c = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 5 & 7 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}
$$

and $$
\bar{H}_0^c |_2 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.
$$

We write $w^i_j = w^i_{j,0} + 2w^i_{j,1} + 4w^i_{j,2}$ for $j = 1, 2, 3$ and $i = 0, \ldots, 3$. Solving the linear system

$$
[\bar{H}_0^c]_2 \cdot [w^0_{1,0}, w^0_{2,0}, w^0_{3,0}, w^1_{1,0}, w^1_{2,0}, w^2_{1,0}, w^2_{2,0}, w^3_{0}]^T = [5, 0, 1, 5, 0, 1, 4, 0, 1]^T = [1, 0, 1, 1, 0, 1, 0, 0, 1]^T
$$

over $\mathbb{Z}_2$ gives the (unique) solution $[w^0_{1,0}, w^0_{2,0}, w^0_{3,0}, w^1_{1,0}, w^1_{2,0}, w^2_{1,0}, w^2_{2,0}, w^3_{0}] = [1, 0, 0, 1, 1, 0, 0, 1, 0]$.
i.e. $S_0 = \{ [1, 0, 0, 1, 1, 0, 0] \}$. Then, in step 2.1 and step 2.2 of the algorithm, one computes

$$\begin{align*}
\left( \frac{\hat{b}_{0,1}}{\hat{b}_{1,1}} \right) &= \left( \frac{5}{0} \right) - \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 0 & 2 & 2 \end{array} \right] \left( \begin{array}{c} 1 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 4 \\ 0 \\ 0 \end{array} \right) \quad \Rightarrow \quad \left( \frac{\hat{b}_{0,1}}{\hat{b}_{1,1}} \right) = \left( \frac{2}{0} \right) \\
\hat{b}_{0,1} &= 5 - \left[ \begin{array}{ccc} 1 & 0 & 0 \\ 2 & 0 & 1 \end{array} \right] \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 4 \\ 0 \\ 0 \end{array} \right) \quad \Rightarrow \quad \hat{b}_{0,1} = 1 \\
\left( \frac{\hat{b}_{0,1}}{\hat{b}_{1,1}} \right) &= \left( \frac{4}{0} \right) - \left[ \begin{array}{ccc} 5 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{array} \right] \left( \begin{array}{c} 0 \\ 1 \\ 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 6 \\ 4 \end{array} \right) \quad \Rightarrow \quad \left( \frac{\hat{b}_{0,1}}{\hat{b}_{1,1}} \right) = \left( \frac{3}{1} \right)
\end{align*}$$

Afterwards, according to step 2.3, one has to solve the system of linear equations

$$\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1
\end{bmatrix} \begin{bmatrix}
w_{1,1}^0 \\
w_{2,1}^0 \\
w_{3,1}^0 \\
w_{1,1}^1 \\
w_{2,1}^1 \\
w_{3,1}^1 \\
w_{2,1}^2 \\
w_{3,1}^2
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
1 \\
1 \\
1
\end{bmatrix}$$

over $\mathbb{Z}_2$, which yields $[w_{1,1}^0, w_{2,1}^0, w_{3,1}^0, w_{1,1}^1, w_{2,1}^1, w_{3,1}^1] = [0, c_1, c_1, 1, c_1 + c_2, 1, c_2]$ with free parameters $c_1, c_2 \in \mathbb{Z}_2$, i.e. $S_1 = \{ [0, c_1, c_1, 1, c_1 + c_2, 1, c_2], \ c_1, c_2 \in \mathbb{A}_2 \}$.

In the last iteration, one computes

$$\begin{align*}
\hat{b}_{0,2}^0 &= p \cdot 2 - p \cdot \left[ \begin{array}{ccc} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \left( \begin{array}{c} 0 \\ c_1 \end{array} \right) = 4 - 4c_1 \quad \Rightarrow \quad \hat{b}_{0,2}^0 = 1 - c_1 \\
\hat{b}_{0,2}^1 &= p \cdot 1 - p \cdot \left[ \begin{array}{ccc} 2 & 0 & 0 \\ 2 & 0 & 0 \\ 2 & 0 & 0 \end{array} \right] \left( \begin{array}{c} 0 \\ r_1 \\ r_1 \end{array} \right) = 0 \quad \Rightarrow \quad \hat{b}_{0,2}^1 = 0 \\
\hat{b}_{0,2}^2 &= p \cdot 3 - p \cdot \left[ \begin{array}{ccc} 5 & 7 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array} \right] \left( \begin{array}{c} 0 \\ c_1 \\ c_1 + c_2 \end{array} \right) = 4 - 4c_2 \quad \Rightarrow \quad \hat{b}_{0,2}^2 = 1 - c_2
\end{align*}$$
and afterwards solve the system of linear equations

\[
\begin{bmatrix}
1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
w_1^0 \\
w_2^0 \\
w_3^0 \\
w_1^2 \\
w_2^2 \\
w_3^2
\end{bmatrix}
= \begin{bmatrix}
1 - c_1 \\
0 \\
1 - c_2
\end{bmatrix}
\]

over \( \mathbb{Z}_2 \), which yields

\[
[w_1^{0}, w_2^{0}, w_3^{0}, w_1^{1}, w_2^{1}, w_2^{2}, w_3^{2}] = [1 + c_2 + c_3 + c_4 + c_5 + c_6, c_3, c_1 + c_2 + c_4 + c_5 + c_6, 0, c_4, c_5, c_6]
\]

with free parameters \( c_3, c_4, c_5, c_6 \in \mathbb{Z}_2 \), i.e.

\( S_2 = \{ [1 + c_2 + c_3 + c_4 + c_5 + c_6, c_3, c_1 + c_2 + c_4 + c_5 + c_6, 0, c_4, c_5, c_6], c_3, c_4, c_5, c_6 \in A_2 \} \).

In summary, all solutions for the erased positions are given by

\[
\begin{bmatrix}
w_1^0 \\
w_2^0 \\
w_3^0 \\
w_1^2 \\
w_2^2 \\
w_3^2
\end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ c_1 \\ c_1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 1 + c_2 + c_3 + c_4 + c_5 + c_6 \\ c_3 \\ c_1 + c_2 + c_4 + c_5 + c_6 \\ 0 \\ c_4 \\ c_5 \\ c_6 \end{bmatrix}
\]

with \( c_1, c_2, c_3, c_4, c_5, c_6 \in A_2 \).

Note that for \( c_1 = c_2 = c_3 = c_4 = c_5 = c_6 = 0 \), one gets the solution that leads to the original codeword we started with. Because of the constraint \( T = 2 \), the vector \( w_3^3 \) is not recovered yet. However, since all other erasures are recovered, the remaining erasure \( w_1^3 \) can now easily be recovered with exact decoding.

Of course the smaller the size of the output the better. This holds if \( \text{rank } \mathcal{H}_t^c \) is maximal.

4. Conclusion

In this paper, we presented an erasure decoding algorithm for convolutional codes over the finite ring \( \mathbb{Z}_{p^r} \). This algorithm can be applied to any convolutional code but leads to better results if the parity-check matrix of the code has certain properties. It is the aim of future research to construct convolutional codes over \( \mathbb{Z}_{p^r} \) with these properties.
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