Mechanical Calculation on the Flexible Bearing of Harmonic Driver

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Abstract. The flexible bearing is an important component and installed in the harmonic driver which transmits motion by elastic characteristics of flexible parts. The rings of the flexible bearing are forced to deform by the wave generator of harmonic driver in its unique working condition different from normal bearings. A large bending stress is generated in the both rings of the flexible bearing after assembly deformation in addition to the contact stress of normal bearings. And, the meshing state of the internal gear of the driver will also be affected greatly. In the present research, a statically indeterminate model is proposed by combining theory of normal bearings with mechanics of materials to solve the deformation characteristics and internal force state of the flexible bearing under the wave generator. The results show that the error of mathematical model is within 6.7%.

1. Introduction

In harmonic drivers, the destruction of flexible bearings is one of the main cause of failure[1]. Different from ordinary bearings, a large bending stress generated in the flexible bearing because of forced deformation, which would cause destruction of inner and outer rings. It will seriously affect the function and life of bearings[2]. Therefore, it is necessary to calculate the deformation and internal force state[3] of the flexible bearing.

Before working, a wave generator (a kind of cam) is pre-assembled in the flexible bearing. In this process, the inner ring will be deformed[4] and then leads to the deformation of the entire flexible bearing. The inner ring is completely fitted over the wave generator, but the outer ring’s deformation will not exactly equal to the inner ring due to supporting by discrete rolling balls after deformation. Although the current finite element simulation calculation can obtain an accurate calculation model, but there are large number of flexible bearing rolling elements with different contact states. It will cost much time and be difficult to converge to obtain the correct result because of large number of contact pairs in the model of simulation. Moreover, theoretical analysis of flexible bearing deformation is not enough at present. Therefore, it is a prerequisite for the mechanical analysis and load capacity calculation of the flexible bearings to establish a reasonable mathematical model to obtain the deformation characteristics and internal forces state of the inner and outer rings. It has a great significance to improve performance of the flexible bearing.

In this paper, a new theory for obtaining the deformation characteristics and internal forces state of the inner and outer rings of the flexible bearing is proposed basic on mechanics of materials. Combining the thin-walled ring theory and the deformation coordination equation, the displacement in
radial and circumferential directions of the inner and outer rings of the flexible bearing and its’ bending stress are obtained. Then the extra bending stress generated by the external load is solved through the three-moment equation. Finally, a finite element simulation of CSF-25-80 bearing is established and compared with the theoretical calculation results to verify the accuracy of the mathematical model.

2. Bending moment of outer ring under wave generator

In Fig. 1, flexible bearing’s deformation process\(^5,6,7\) under wave generator is:

\[ \Delta = \delta_{\text{max}} \cos 2\theta \]  

(1)

As shown in Fig. 2, the outer ring can be equivalent to a thin-walled ring and only quarter ring need to be analysed due to symmetry.

Normal angle change of section A and D on the X and Y axes are zero. Therefore, a model can be built: section A is simplified as a fixed end and section D as a free end\(^9,10\), and the relative normal angle change between section D and section A is zero. According this model, following equilibrium equation can be established\(^11\):

\[ \Delta_{M_1} + \Delta_{P_1} + \Delta_{Q_1} = \delta_1 M_1 + \delta_2 P_1 + \Delta Q_1 = 0 \]  

(2)

Where \( \Delta_{M_1}, \Delta_{P_1}, \) and \( \Delta_{Q_1} \) is the relative normal angle change between section D and section A while \( M_1, P_1, Q_1 \) is loaded independently. \( \delta_1 \) and \( \delta_2 \) is the relative normal angle change between sections D and section A while \( M_1 = -1, P_1 = 1 \) and loaded independently. (The moments caused by \( M_1 \) and \( P_1 \) for the model are: \( M_1 = -1 \) and \( M_2 = R(1 - \cos \theta) \).

Since constraint of section D is released, the moment caused by force \( Q_1 \) between \( Q_1 \) and section D is zero. So, the moment caused by \( Q_i \) for the model is\(^12\):

\[ M_{Q_i} = \begin{cases} 0 & \text{if } 0 < \theta < \gamma_i \\ -Q_i R \sin (\theta - \gamma_i) & \text{if } \gamma_i < \theta < \frac{\pi}{2} \end{cases} \]  

(3)

According to Dummy-load method:

\[ \delta_1 = \int_0^{\frac{\pi}{2}} \frac{\pi M_1 M_1 R d\theta}{EI} = \frac{\pi R}{2EI} \]  

(4)

\[ \delta_2 = \int_0^{\frac{\pi}{2}} \frac{\pi M_1 M_2 R d\theta}{EI} = - \frac{R^2}{EI} \left( \frac{\pi}{2} - 1 \right) \]  

(5)

\[ \Delta Q_i = \int_{\gamma_i}^{\frac{\pi}{2}} \frac{\pi M_1 M_1 Q_i R d\theta}{EI} = \frac{Q_i R^2}{EI} (1 - \sin \gamma_i) \]  

(6)

For \( i > 1 \), according to static equilibrium conditions:

\[ P_i = Q_i \cos \left( \frac{\pi}{2} - \gamma_i \right) \]  

(7)

Solve equations above combined, following formula can be obtained:
The moment in the model under $Q_i$ is:

$$M_i(\theta) = \begin{cases} M_{P_i} - M_i & 0 < \theta < \gamma_i \\ M_{P_i} - M_i - M_{Q_i} & \gamma_i < \theta < \frac{\pi}{2} \end{cases}$$

(9)

While $Q_i = \bar{Q}_i = 1$, the moment under $\bar{Q}_i$ in the model is:

$$\bar{M}_Q = \bar{Q}_i R \left( \frac{1}{\pi} - \frac{\cos \theta}{2} \right) = R \left( \frac{1}{\pi} - \frac{\cos \theta}{2} \right)$$

(10)

According to Dummy-load method, the maximum deformation under $Q_i$ is:

$$\delta_i = \int_0^{\pi} \frac{\bar{M}_i(\theta)\bar{M}_Q R d\theta}{EI}$$

(11)

Then the overall deformation under $[Q_1, Q_2, \ldots, Q_i, \ldots, Q_L]$ is:

$$\delta_{\text{max}} = \sum_{i=1}^n \delta_i = \sum_{i=1}^n \int_0^{\pi} \frac{\bar{M}_i(\theta)\bar{M}_Q R d\theta}{EI}$$

(12)

According to reference[13], combined with the contour of typical wave generator:

$$Q_i = Q_1 \left[ 1 - \frac{1}{2} (1 - \cos 2\theta) \right]^n$$

(13)

For ball bearings: $n = 3/2$.

Combine the above calculations, the bending-moment of overall outer ring is: $M(\theta) = \sum M_i(\theta)$

3. Outer ring deformation calculation

3.1. Radial displacement calculation

In order to simplify the solution of the differential equation, the bending moment equation obtained by the above formula is modified to:

$$X_1 = P_i; X_2 = -\frac{2}{\pi} \left[ P_i R \left( \frac{\pi}{2} - 1 \right) - Q_i R (1 - \sin \gamma_i) \right] \text{ and } X_3 = -Q_i$$

(14)

Then, the moment of overall outer ring is:

$$M_{Q_i} = \begin{cases} X_1 R \cos (1 - \cos \theta) + X_2 & 0 < \theta < \gamma_i \\ (X_1 R \cos (1 - \cos \theta) + X_2 + X_3 R \sin (\theta - \gamma_i)) & \gamma_i < \theta < \frac{\pi}{2} \end{cases}$$

(15)

Then the above equation is substituted into the relationship between displacement and bending moment[14]:

$$\frac{d^2 u}{d\theta^2} + u = -\frac{M}{EI} R^2$$

(16)

And the solution is:

$$u_1 = \begin{cases} D_1 \sin \theta + D_2 \cos \theta + \frac{X_1 R^3}{2EI} \theta \sin \theta - \frac{(X_1 R + X_2) R^2}{EI} & 0 < \theta < \gamma_i \\ D_3 \sin \theta + D_4 \cos \theta + \frac{X_1 R^3}{2EI} \theta \sin \theta - \frac{(X_1 R + X_2) R^2}{EI} + \frac{X_3 R^3}{2EI} \theta \cos (\theta - \gamma_i) & \gamma_i < \theta < \frac{\pi}{2} \end{cases}$$

(17)

According to the symmetry of the thin-walled ring and continuity conditions, four boundary conditions can be obtained: 1) while $\theta = 0, du/d\theta = 0; 2$) while $\theta = \pi/2, du/d\theta = 0; 3$) while $\theta = \gamma_i, (du_1)/d\theta = (du_2)/d\theta; 4$) while $\theta = \gamma_i, u_1 = u_2$.

Where $D_1, D_2, D_3$ and $D_4$ can be determined from the above four boundary conditions. Since the expression is too cumbersome, it will not be given here.
3.2. Calculation of circumferential displacement and normal angle change

According to the thin-walled ring theory, the relationship between the circumferential displacement and the radial displacement is determined by the formula: \( dv/d\theta = -u^{[15]} \). Substituting the radial displacement formula into it, the circumferential displacement of the ring can be obtained:

\[
v_1 = -\int_0^\theta u_1 d\theta = D_2 \sin\theta - D_1 (\cos\theta - 1) - \frac{1}{EI} \left( X_1 R^3 \theta + \frac{X_1 R^3 (\sin\theta - \theta \cos\theta)}{2} - X_2 R^2 \theta \right)
\]

\(0 < \theta < y_i\) (18)

\[
v_2 = -\int_{y_i}^\theta u_2 d\theta - \int_0^{y_i} u_1 d\theta = D_3 (\cos y_i - \cos\theta) - D_4 (\sin y_i - \sin\theta) + D_2 \sin y_i - D_1 (\cos y_i - 1) - \frac{X_1 R^3 \theta}{2EI} + \frac{X_1 R^3 (\sin y_i - \gamma_i \cos y_i)}{2EI} - \frac{X_2 R^2 \theta}{EI}
\]

\(\gamma_i < \theta < \frac{\pi}{2}\) (19)

Then the normal angle change of the section is determined by formula \( \varphi = \frac{1}{R} \left( v - \frac{d u_1}{d\theta} \right) \), and the solution is:

\[
\varphi_1 = \frac{1}{R} \left( v_1 - \frac{d u_1}{d\theta} \right) = \frac{1}{R} \left( v_1 - D_1 \cos\theta + D_2 \sin\theta - \frac{X_1 R^3 \sin\theta}{2EI} - \frac{X_1 R^3 \theta \cos\theta}{2EI} \right)
\]

\(0 < \theta < y_i\) (20)

\[
\varphi_2 = \frac{1}{R} \left( v_2 - \frac{d u_2}{d\theta} \right) = \frac{1}{R} \left( v_2 - D_3 \cos\theta + D_4 \sin\theta - \frac{X_1 R^3 \sin\theta}{2EI} - \frac{X_3 R^3 \cos\theta - \gamma_i}{2EI} - \frac{X_1 R^3 \theta \cos\theta}{2EI} \right)
\]

\(\gamma_i < \theta < \frac{\pi}{2}\) (21)

4. Deformation and stress calculation of the inner ring

According to the principle of non-elongation of the neutral surface in the design of the flexible bearing, the inner ring is completely fitted over the wave generator after deformation, the radial displacement is:

\[
u_i = \delta_{\text{max}} \cos \left( \frac{\pi}{2} - \theta \right)
\]

(22)

Substituting into formula (16), the following equation is obtained, which expresses the internal bending moment generated by the deformation of the inner ring:

\[
M_i = 3\delta_{\text{max}} EI \cos 2\theta / R^2
\]

(23)

Then, the circumferential displacement of the inner ring is:

\[
v_i = -\int_0^\theta u_i d\theta = \frac{\delta_{\text{max}} \sin 2\theta}{2}
\]

(24)

Finally, the circumferential and radial displacement formulas are used to get the normal angle change of the inner ring:

\[
\varphi_i = \frac{1}{R} \left( v_i - \frac{d u_i}{d\theta} \right) = -\frac{3\delta_{\text{max}} \sin 2\theta}{2R}
\]

(25)

5. Bending moment under load

The outer ring is supported on consecutive balls which is equivalent to a multi-span beam while flexible bearing is loaded.

According to Ivanov’s experiment, bearing’s external load is shown in Fig. 3(left)\(^{[16]}\):

\[
\left\{ \begin{array}{l}
q_\varphi = q_{\text{max}} \cos \left[ \frac{\pi (\varphi - \varphi_1)}{2\varphi_2} \right] \\
q_r = q_\varphi \tan \alpha \\
\varphi_1 = 0, \quad \varphi_2 = \frac{\pi}{3}, \quad q_{\text{max}} = \frac{\pi T}{2\varphi_2 d_e b_d}
\end{array} \right.
\]

(26)

(27)
Three-moment equation of continuous beam can be used to solve this multi-span beam. Decomposes the constraint of each support section into a pair of equal moments and on opposite direction, as shown in Fig. 3(right). Moments of any three adjacent fulcrum can be taken to establish the equation\(^\text{17}\):

\[
M_{n-1}l_n + 2M_n(l_n + l_{n+1}) + M_{n+1}l_{n+1} = -\frac{6\omega_n a_n}{l_n} - \frac{6\omega_{n+1} b_{n+1}}{l_{n+1}}
\]  

\((28)\)

![Figure 3](image3.png)

**Figure 3.** External force distribution

Where \(l_n, l_{n+1}\) are the spans between adjacent support sections, \(\omega_n, \omega_{n+1}\) are moment diagram area under \(q_n(\theta)\), \(q_{n+1}(\theta)\) in span; \(a_n\) and \(b_{n+1}\) are distance from the centroid of moment diagram to the left and right ends.

All three-moment equations can be established at each fulcrum. Solving the equations, maximum bending moment under load at fulcrum can be obtained.

6. Calculation results

Based on the established mathematical model. A MATLAB program was compiled and flexible bearing of CSF-25-80 is used as an example for calculation.

First, the bending moment calculated is converted into stress while the maximum deformation is \(\delta_{\text{max}} = 0.4\text{mm}\). Due to the bilaterally symmetrical state of bearing, here we give the calculation results of the quarter ring where the maximum stress located.

![Figure 4](image4.png)

**Figure 4.** Deformation bending stress

![Figure 5](image5.png)

**Figure 5.** Outer ring displacement
In order to verify the mathematical model, a FEA simulation model has been established based on ANSYS\cite{18,19}. Then several important points from the figure above are extracted and compared with corresponding values in FEA model in the following Table 1.

|                  | Simulation | Theory     | error |
|------------------|------------|------------|-------|
| Max Bending Stress (Inner) | 221.6 Mpa | 219.21 Mpa | 1.09% |
| Max Bending Stress (Outer)  | 341.8 Mpa | 366.33 Mpa | 6.7%  |
| Deformation of out ring at 0° | -0.372 mm | -0.384 mm | 3.13% |
| Max Angle change (Inner)   | 1.46°     | 1.52°      | 3.95% |
| Max Angle change (Outer)   | 1.09°     | 1.12°      | 2.68% |
| Max Bending stress under load | 323.7 MPa | 344.516 MPa | 6.04% |

From the figure and table above, maximum stress of outer ring is 219.216 Mpa, while inner ring’s is greater which is 366.359 Mpa in deformation phase, the stress determines the magnitude of deformation of the flexible bearing. Then the outer ring’s displacement at 0° is 0.384 mm, it is smaller than the inner ring’s which is 0.4 mm because of the effect of multi-support. As a result, a 0.016 mm gap is formed between the inner, outer rings and balls at 0° which will influence the engagement of gears of harmonic drive. Normal angle change the degree of deformation of both rings. In the FEA model, the load step\cite{20} is continued and the maximum bending stress under load is extracted to compared with corresponding values obtained through three-moment equation which will determines bearing’s the loading capacity.

7. Conclusion

(1) A method for solving the deformation characteristics and bending stress of the CSF-25-80 bearing after matched with a cam is proposed. The maximum bending stress under the load is solved by the three-moment equation. It affords a basis for subsequent improvement working of flexible bearings and harmonic driver.

(2) Large deformation and bending stress of the inner and outer rings of flexible bearings are produced after assembly and loading. The maximum stress is 344.516 MPa and 366.33 MPa, it is necessary to reduce those stress in order to increase the life of bearings.

(3) This paper provided a comparison between theoretical methods and finite element simulation analysis methods. The results between two methods are relatively close, and the maximum error is within 6.7%, which verifies the accuracy of theoretical calculations.

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References

[1] Ueura K, Slatter R. ACTUATORS: Development of the harmonic drive gear for space applications[J]. European Space Agency, 1999.

[2] Zhang C, Wang S, Wang Z, et al. An accelerated life test model for harmonic drives under a segmental stress history and its parameter optimization[J]. Chinese Journal of Aeronautics, 2015, 28(6):1758-1765.

[3] HD Taghirad, PR Belanger. Modeling and Parameter Identification of Harmonic Drive Systems[J]. Journal of Dynamic Systems Measurement & Control, 1997, 120(4): 439–444.

[4] Timothy D, Tuttle and Warren P. Seering. A Nonlinear Model of a Harmonic Drive Gear Transmission[J]. Ieee Transactions on robotics and automation, 1996, 12(3) :368-374.

[5] Gravagno F, Mucino V H, Pennestri E. Influence of wave generator profile on the pure kinematic error and centroids of harmonic drive[J]. Mechanism & Machine Theory, 2016, 104:100-117.

[6] Hsia L M. The Analysis And Design Of Harmonic Gear Drives[C]// IEEE International Conference on Systems, Man, and Cybernetics. IEEE, 1988:616-619.

[7] Maiti R, Roy A K. Minimum tooth difference in internal-external involute gear pair[J]. Mechanism & Machine Theory, 1996, 31(4):475-485.

[8] Guan Chongfu. Calculation for the Contact Pressure of Flexible Bearings under Dynamic Load[J]. Journal of Yanshan University, 1994(3):220-226. (in Chinese with English abstract)

[9] Alessandro B, Beatrice C, Bertotti G, et al. Thin-wall dynamics and Barkhausen effect in metallic ferromagnetic materials. I. Theory: Journal of Applied Physics, Vol. 68, No. 6, pp. 2901–2907 (15 Sep. 1990)[J]. Journal of Applied Physics, 1990, 68(6):2901-2907.

[10] Fichter WB. A theory for inflated thin-wall cylindrical beams. NASA TN D-3466. 1966.

[11] Liu Hongwen. Mechanics of Materials [M]. High Education Press. 1985.5. (in Chinese with English abstract)

[12] CHEN Xiaoxia, LIU Yusheng, et al. Neutral Line Stretch of Flexspline in Harmonic Driver [J]. Journal of Mechanical Engineering, 2014, 50(21):189-196. (in Chinese with English abstract)

[13] Harris T A. Rolling Bearing Analysis (2nd Edition)[J]. A Wiley-Interscience Publica-tion, John Wiley&Sons, 1984.

[14] WANG Jingjing. The Research on Fatigue Life and Structure Optimization of the Flexible Bearing in a Harmonic Reducer Wave Generator[D]. Hunan University, 2016. (in Chinese with English abstract)

[15] M.H.Иванов. Harmonic gear drive[M]. Nation Defense Industry Press, 1987.

[16] Kayabasi O, Erzincanli F. Shape optimization of tooth profile of a flexspline for a harmonic drive by finite element modelling[J]. Materials & Design, 2007, 28(2):441-447.

[17] Liu Zhenglin. Analysing Stress of Flexible Bearing under Harmonic Gear Drive[J]. Journal of Dalian University, 1995(4):512-517. (in Chinese with English abstract)

[18] Ostapski W. Analysis of the stress state in the harmonic drive generator-flexspline system in relation to selected structural parameters and manufacturing deviations[J]. Bulletin of the Polish Academy of Sciences Technical Sciences, 2010, 58(4):683-698.

[19] Ostapski W, Mukha I. Stress state analysis of harmonic drive elements by FEM[J]. Bulletin of the Polish Academy of Sciences Technical Sciences, 2007, 55(1):115-123.

[20] Tang Z, Sun J. The Contact Analysis for Deep Groove Ball Bearing Based on ANSYS[J]. Procedia Engineering, 2011, 23:423–428.