Transmitting an analog Gaussian source over a Gaussian wiretap channel under SNR mismatch

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Abstract

In this work we study encoding/decoding schemes for the transmission of a discrete time analog Gaussian source over a Gaussian wiretap channel. The intended receiver is assumed to have a certain minimum signal to noise ratio (SNR) and the eavesdropper is assumed to have a strictly lower SNR compared to the intended receiver. For a fixed information leakage rate \( I_\epsilon \) to the eavesdropper, we are interested in minimizing the distortion in source reconstruction at the intended receiver, and we propose joint source channel coding (JSCC) schemes for this setup. For a fixed information leakage rate \( I_\epsilon \) to the eavesdropper, we also show that the schemes considered give a graceful degradation of distortion with SNR under SNR mismatch, i.e., when the actual channel SNR is observed to be different from the design SNR.

I. INTRODUCTION, SYSTEM MODEL AND PROBLEM STATEMENT

\[ W \sim \mathcal{N}(0, \sigma^2_a) \]

Let us consider the classical case of transmitting a Gaussian source over a Gaussian channel having an input power constraint, and we are interested in estimating the source at the intended receiver with the minimum possible distortion. In this case, both the uncoded scheme [1] of scaling the source to match the input power constraint and the separation based scheme of quantization followed by channel coding are both optimal for a given channel SNR. Moreover, there are an infinite family of schemes that can be shown to be optimal for the above setup [2]. Compared to the separation based scheme, the uncoded scheme provides a graceful degradation of distortion with SNR, though both schemes are optimal for a given design \( SNR_d \). In this work, we consider the same communication setup but with an eavesdropper present, and we are interested in minimizing the distortion for the intended receiver for a given information leakage to the eavesdropper. For a fixed information leakage \( I_\epsilon \) to the eavesdropper, we propose

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a scheme which achieves the minimal possible distortion for a given design $SNR_d$ and also provides a graceful degradation of distortion with SNR.

The source $V \sim N(0, \sigma^2_V)$ is an $n$-length discrete time real Gaussian source and the channel is a discrete time real Gaussian wiretap channel [3] as shown in Fig. 1. The source and channel bandwidths are assumed to be matched and hence the input to the channel is a $n$-length vector $x$, which is also assumed to have an average power constraint of $P$ expressed as $E[x^2] \leq P$. Henceforth in this sequel, boldface is used to denote $n$-dimensional vectors. The received signal at the eavesdropper $y_e$ can be expressed as

$$y_e = x + w_e.$$  

Here $w_e$ is the $n$-length additive white Gaussian noise (AWGN) vector having zero mean and noise variance of $\sigma^2_e$. The intended receiver receives the $n$-dimensional vector $y$ given by

$$y = x + w,$$

where the noise $w$ is the $n$-length additive white Gaussian noise vector having zero mean and noise variance of $\sigma^2_w$. The transmitter does not have an exact knowledge of $\sigma^2_w$ but knows that $\sigma^2_w \leq \sigma^2$, where $\sigma^2$ is the noise variance corresponding to some design SNR. The eavesdropper channel is a degraded version of the main channel and is assumed to have the lowest SNR i.e. $SNR_e < SNR < SNR_d$, where we define $SNR_e := P/\sigma^2_e$, $SNR_d := P/\sigma^2$ and $SNR_a := P/\sigma^2_a$. The receiver is assumed to have a perfect estimate of $SNR_a$, but the transmitter is assumed to be kept ignorant of this information.

We refer to the information leaked to the eavesdropper as the information leakage rate and is expressed as $I_e := \frac{1}{n} I(V; Y_e)$. The information leakage rate is the difference between the average source entropy $\frac{1}{n} h(V)$ and the equivocation rate $\frac{1}{n} h(V|Y_e)$ defined in [3] and [4]. It is fairly common to use equivocation rate in literature, but since we have a continuous source, the equivocation rate expressed as differential entropy may not be always positive. Therefore, we choose the metric as the mutual information between the source and the received signal which is always $\geq 0$. Notice that $I_e = 0$ corresponds to perfect secrecy, and this implies that the eavesdropper gains no information about the source.

The intended receiver makes an estimate $\hat{v}$ of the analog source $v$ from the observed vector $y$. The distortion in estimating $V$ can be expressed as $D(SNR_a) = E[(V - \hat{V})^2]$, where $SNR_a = P/\sigma^2_a$. In this work for a fixed information leakage rate $I_e$, we are interested in schemes which are optimal for $SNR_d$ and also which provide a low source distortion for $SNR_a > SNR_d$. We are also interested in studying the graceful degradation in distortion with $SNR_a$. This can be captured mathematically by requiring the exponent of $D(SNR_a)$ to be $-1$, or precisely

$$\lim_{SNR_a \to \infty} \frac{\log D(SNR_a)}{SNR_a} = -1.$$  

The exponent of $-1$ is chosen, as it is the lowest possible exponent achievable for the matched bandwidth case [1], even in the absence of an eavesdropper.

Before introducing our proposed schemes, we would like to mention some prior work in this area. There has been a considerable amount of work in studying the graceful degradation of distortion with $SNR_a$ for both the bandwidth matched case [1] and the bandwidth mismatched case, and several joint source channel coding schemes have been proposed [5]. There also exists a considerable amount of literature on physical layer security. The wiretap channel was first introduced and studied by Wyner in [3], and the Gaussian wire-tap channel was studied by Leung and Hellman in [4]. In [6], Yamamoto studied the Shannon cipher system from a rate distortion perspective, where in one of the theorems it is shown that for a wiretap channel with a fixed SNR, a separation based approach of quantization followed by secrecy coding is optimal. However in this work, in contrast to [6] we are interested in JSCC scheme for the Gaussian wiretap channel under the SNR mismatch case. We are also interested in quantifying the graceful degradation of distortion with $SNR_a$ under the SNR mismatch case.

**II. A SEPARATION BASED SCHEME AND A SIMPLE SCALING SCHEME**

For the classical setup without the eavesdropper, both the uncoded scheme and the separation based scheme can be observed to be optimal for a given design $SNR_d$. However these schemes perform differently in the presence of a SNR mismatch. In this section we look at the performance of both the separation based scheme and the uncoded scheme in the presence of an eavesdropper.
A. Separation based scheme

Here we use a separation based scheme designed for a channel noise variance of $\sigma^2$ and information leakage $I_{e}$. We first design a vector quantizer of rate

$$R_v = C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right) + I_e,$$

where $C(x) := \frac{1}{2} \log(1 + x)$. We quantize $v$ to $v_q$ using the designed vector quantizer of rate $R_v$. $v_q$ is then mapped to $v_{sec}$ using a secrecy code as in [4]. The encoded $v_{sec}$ is transmitted over the channel. The eavesdropper obtains a maximum leakage rate of $I_e$. The proof of this claim is contained in the Appendix. The intended receiver achieves a distortion of $\sigma_v^2 - 2R_v$. Since we have quantized the source to a fixed rate, the achievable distortion is constant for all SNR’s above the designed SNR and is given by,

$$D(SNR_a) = \sigma_v^2 - 2R_v < \sigma_v^2.$$

For the given $I_e$, though we get some improvement in distortion, the exponent is 0 and we do not have a graceful degradation of distortion with $SNR_a$ as shown in Fig. 2.

B. A simple scaling scheme

Let us first consider a scheme that is optimal for all channel SNRs for the Gaussian wiretap channel in the absence of the eavesdropper. This reduces to the classical problem of point to point communication over a Gaussian channel. An optimal scheme is scaling the analog source $v$ by a constant $\kappa = \sqrt{P/\sigma_v^2}$ to match the transmit power constraint [1]. The transmitted vector is hence given by $x = \kappa v$. The receiver has perfect knowledge of $SNR_a$ and performs a minimum mean square estimate of $v$ from the observed $y$. The distortion obtained at the intended receiver is given by

$$D(SNR_a) = \frac{\sigma_v^2}{1 + SNR_a}.$$

The distortion exponent can be seen to be $-1$ for this scheme and also the obtained distortion is optimal for every given $SNR_a$. The information leakage rate is easily calculated to be

$$\frac{1}{n} I(V; Y_e) = I(V; Y_e) = \frac{1}{2} \log \left( 1 + \frac{\kappa^2 \sigma_v^2}{\sigma_e^2} \right) = \frac{1}{2} \log \left( 1 + \frac{P}{\sigma_e^2} \right).$$
From the above equation we can observe that the choice of \( \kappa = \sqrt{\frac{P}{\sigma_v^2}} \) results in a reasonably high information leakage rate. However, we can reduce the value of \( \kappa \) to satisfy our eavesdropper information requirement of \( I_e \) as follows by choosing
\[
I_e = \frac{1}{2} \log(1 + \frac{\kappa^2 \sigma_v^2}{\sigma_e^2})
\]
or
\[
\kappa^2 = \frac{\sigma_e^2}{\sigma_v^2} (2^{2I_e} - 1).
\]

The intended receiver receives \( y = \kappa v + w \). Hence the distortion at the intended receiver is given by
\[
D(SNR_a) = \frac{\sigma_v^2}{1 + \frac{\kappa^2 \sigma_v^2}{\sigma_a^2}} = \frac{\sigma_v^2}{1 + \frac{\kappa^2 \sigma_v^2 \text{SNR}_a}{P}}.
\]

Hence for \( \text{SNR}_a = \text{SNR}_d \), \( D(SNR_d) > \sigma_v^2/(1 + \text{SNR}_d) \) as \( \kappa^2 \sigma_v^2 < P \). Though the above equation shows that the distortion exponent is \(-1\), we have a considerable loss in optimality at the intended receiver, as we have not used the full power \( P \) at the transmitter. This can be seen from Fig. 2 where for \( \text{SNR}_a = 20 \text{db} \), we see that the uncoded scheme has a higher distortion performance compared to the separation based scheme. However we can see from Fig. 2 that the uncoded scheme gives a graceful degradation in distortion, unlike the separation based scheme. Also if we need \( I_e = 0 \), then \( \kappa \) must be chosen to be 0 and hence \( D(SNR_d) = \sigma_v^2 \), which is the worst attainable distortion equivalent to simply estimating the source.

In the next section we show a scheme which is a combination of the schemes mentioned above, that for \( I_e \neq 0 \) gives both the optimal distortion for the designed SNR as well as a graceful degradation in distortion for all SNR’s above the designed SNR. In the case of \( I_e = 0 \), or perfect secrecy, it is however not clear if we can get a graceful degradation in distortion.

III. HYBRID SCHEME USING SECRECY CODING AND VECTOR QUANTIZATION

In this scheme we quantize the source \( v \) to get \( v_q \) given as follows,
\[
v = v_q + u.
\]
The quantized digital part is encoded using a secrecy code and the quantization error \( u \) is superimposed onto the secrecy code code and transmitted with some scaling. The transmitted vector \( x \) is given by
\[
x = v_{sec} + \kappa u.
\]
Here \( v_{sec} \) consists of a digital part that uses a secrecy code.

The digital part \( v_{sec} \) uses a power of \( \alpha P \) and the analog part a power of \( (1 - \alpha)P \).

The source \( v \) is quantized to \( v_q \) at a rate \( R(\alpha) \) chosen as
\[
R(\alpha) = C \left( \frac{\alpha P}{(1 - \alpha)P + \sigma^2} \right) - C \left( \frac{\alpha P}{(1 - \alpha)P + \sigma_v^2} \right)
\]

The encoder is the same encoder as in the Gaussian wire tap channel case [3] and [4]. We have \( \approx 2^{nR(\alpha)} \) bins and \( \approx 2^{nC \left( \frac{\alpha P}{(1 - \alpha)P + \sigma^2} \right)} \) codewords in each bin. Hence the transmitted vector \( v_{sec} \) has a rate of \( C \left( \frac{\alpha P}{(1 - \alpha)P + \sigma^2} \right) \). The intended receiver can decode the digital part and cancel \( v_{sec} \) from \( y \). It then forms a minimum mean square estimate (MMSE) of \( u \). Hence the obtained distortion can be expressed as
\[
D = \frac{\sigma_v^2 2^{-2R(\alpha)}}{1 + \frac{(1 - \alpha)P}{\sigma_v^2}}
\]
The eavesdropper obtains $y_e$ and we are interested in characterizing the information leakage rate $I_{\epsilon}$. $I_{\epsilon}$ can be bounded as follows.

$$I_{\epsilon} = \frac{1}{n}I(V; Y_e) \quad (a)$$

$$= \frac{1}{n}I(V_q; \kappa U; Y_e)$$

$$= \frac{1}{n}I(V_q; Y_e) + \frac{1}{n}I(\kappa U; Y_e | V_q) \quad (b)$$

$$= \frac{1}{n}I(\kappa U; Y_e | V_q)$$

$$\leq \frac{1}{n}h(\kappa U) - \frac{1}{n}h(\kappa U | Y_e, V_q) \quad (c)$$

$$= \frac{1}{n}h(\kappa U) - \frac{1}{n}h(\kappa U | Y_e, V_q, V_{sec}) \quad (d)$$

$$\leq \frac{1}{n}h(\kappa U) - \frac{1}{n}h((1 - \beta)\kappa U - \beta W_e) | Y_e, V_q, V_{sec}) \quad (e)$$

$$\leq \frac{1}{n}h(\kappa U) - \frac{1}{n}h((1 - \beta)\kappa U - \beta W_e) \quad (f)$$

$$= \frac{1}{n}h(\kappa U) - \frac{1}{n}h((1 - \beta)\kappa U - \beta W_e) \quad (g)$$

Here $(a)$ follows because of the Markov chain $V \rightarrow (V_q, \kappa U) \rightarrow X$. $(b)$ follows from the chain rule of mutual information. $(c)$ is obtained by the choice of our coding scheme for $V_q$, which is designed for perfect secrecy from the eavesdropper. Hence $H(V_q | Y_e) = H(V_q)$ or $I(V_q; Y_e) = 0$. We obtain the first term in $(d)$ since $V_q$ is independent of $U$, and the second term follows because conditioning reduces entropy. In $(e)$, $\beta$ is chosen as $\beta = \frac{(1-\alpha)P}{(1-\alpha)P + \sigma_e^2}$. $(f)$ follows because $(1 - \beta)\kappa U - \beta W_e$ is orthogonal to $Y_e, V_q$ and $V_{sec}$ and finally $(g)$ follows as all the terms are Gaussian.

The distortion as a function of SNR is plotted in Fig. 2 for $I_{\epsilon} = 0.01$, which shows that the hybrid scheme performs better than both the uncoded and the separation based scheme. Fig. 3 shows the performance of the hybrid scheme for different values of $I_{\epsilon}$ and this shows that the distortion exponent is $-1$ for $I_{\epsilon} > 0$. Also the distortion that can be achieved at the eavesdropper can be lower bounded by $\sigma_e^2 2^{-2I_{\epsilon}}$. This can be seen to be reasonable large when $I_{\epsilon}$ is small. Hence the eavesdropper gets only a poor estimate of the source.

A trivial outer bound for the problem can be obtained by assuming that the transmitter has knowledge of $SNR_d$. Fig. 3. Distortion vs SNR for different $I_{\epsilon}$. Here $P = 1, \sigma_e^2 = 1$ and $\sigma^2 = 0.01(SNR_d = 20db)$. 
[6] considers the rate distortion problem for the Shannon cipher system. It can be seen from [6, Theorem 1 with \( R_k = 0 \)], that the Shannon cipher system reduces to the wire-tap channel setup and the optimal distortion can be achieved by separate source coding followed by secrecy coding. We first quantize the source \( v \) to \( v_q \) at a rate \( R \).

For a maximal leakage rate of \( I_e \), the maximum value of \( R = C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right) + I_e \). This can be obtained by following the steps outlined in the Appendix. Hence we can achieve a distortion \( D = \sigma^2 2^{-2R} \) corresponding to the above \( R \). Fig. 3 shows the outer bound and we see that the achievable scheme has a constant gap from the outer bound.

IV. Conclusion

In this work we considered joint source channel coding schemes for transmitting an analog Gaussian source over a Gaussian wiretap channel. We analyzed the performance of a few schemes under SNR mismatch. We showed that for a fixed information leakage to the eavesdropper \( I_e \), we can be optimal for a design SNR and also can obtain a graceful degradation of distortion with SNR. A problem of interest is, wether it is still possible to get this graceful degradation when we enforce perfect secrecy at the eavesdropper (\( I_e = 0 \))? Another possible future work is the design of schemes for the source-channel bandwidth mismatch scenario.

APPENDIX

In this section we show that the information leakage rate is \( I_e \) for \( R_v = C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right) + I_e \). Applying [4, Theorem 1, (2), (3) and (17)] to our problem setup we obtain the following set of equations.

\[
R_v = \frac{H(V_q)}{n} \\
R_v \frac{H(V_q \mid Y_e)}{H(V_q)} \leq C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right)
\]

The left hand side term can be simplified as follows,

\[
R_v \frac{H(V_q \mid Y_e)}{H(V_q)} = R_v \frac{H(V_q \mid Y_e) - H(V_q) + H(V_q)}{H(V_q)} \\
\overset{(a)}{=} R_v \frac{H(V_q) - nI_e}{H(V_q)} \\
= R_v - I_e.
\]

In (a) we used the definition of \( I_e \). Thus we can bound \( R_v \) as,

\[
R_v \leq C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right) + I_e.
\]

Hence the maximal rate \( R_v \) for the vector quantizer can be obtained by choosing \( R_v = C \left( \frac{P}{\sigma^2} \right) - C \left( \frac{P}{\sigma_e^2} \right) + I_e \).

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