TILING SPACES ARE INVERSE LIMITS

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Abstract. Let $M$ be an arbitrary Riemannian homogeneous space, and let $\Omega$ be a space of tilings of $M$, with finite local complexity (relative to some symmetry group $\Gamma$) and closed in the natural topology. Then $\Omega$ is the inverse limit of a sequence of compact finite-dimensional branched manifolds. The branched manifolds are (finite) unions of cells, constructed from the tiles themselves and the group $\Gamma$. This result extends previous results of Anderson and Putnam [AP], of Ormes, Radin and Sadun [ORS], of Bellissard, Benedetti and Gambaudo [BBG], and of Gähler [Gäh]. In particular, the construction in this paper is a natural generalization of Gähler’s.

1. Background

In the last few years, it has become clear that many spaces of tilings of $\mathbb{R}^d$ can be viewed as inverse limit spaces. Anderson and Putnam [AP] began this program for substitution tilings. Given a substitution, they showed that the corresponding space of tilings of $\mathbb{R}^d$ is the inverse limit of a branched $d$-manifold $K$ under an expansive map from $K$ to itself. If the substitution has a property called “forcing the border” [Kel], then the manifold $K$ is constructed by stitching all the tile types together along possible common boundaries. If the substitution does not force the border, then the construction is similar, only using collared tiles. (A collared tile is a tile that is labeled by the pattern of tiles that touch it). For this construction to work, the tilings must involve only a finite number of tile types (up to translation), meeting full-face to full-face. In particular, the construction does not apply to tilings like the pinwheel [Rad], where tiles appear in an infinite number of orientations.

Ormes, Radin and Sadun [ORS] extended the Anderson-Putnam construction to substitution tilings of $\mathbb{R}^d$ on which the entire Euclidean group acts continuously. Tiles may appear in arbitrary orientations, but there can only be a finite number of tile types up to Euclidean motion, and tiles must meet full-face to full-face. The branched manifold has dimension $d(d + 1)/2$, which is the dimension of the Euclidean group.

In this construction, a cell in the branched manifold $K$ is not a tile. Rather, a cell is the product of a (possibly collared) tile with $SO(d)$, modulo any (finite!) rotational symmetry that the tile might have. This gives a description of all the ways a tile containing the origin may be placed. The substitution (call it $\sigma$) replaces each oriented tile with a union of oriented tiles, giving a map from $K$ to itself. Such a union of tiles is called a supertile of order 1. The substitution applied to a supertile of order 1 gives a supertile of order 2, and so on.

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A point \((x_0, x_1, \ldots)\) in the inverse limit \(\leftarrow_{\sigma} K\) is a consistent description of a tiling, with \(x_0\) telling how the origin sits inside a tile, \(x_1\) telling how the origin sits inside a supertile of order 1, and \(x_n\) telling how the origin sits inside a supertile of order \(n\). If the substitution forces its border (or if we are using collared tiles), the sequence \((x_0, x_1, \ldots)\) gives a consistent description of a unique tiling of \(\mathbb{R}^d\).

More recently, Gähler [Gäh] and Bellissard, Benedetti and Gambaudo [BBG] have each applied inverse limit methods to tilings that need not be generated by a substitution. If \(x\) is a tiling of \(\mathbb{R}^d\) that has finitely many tile types up to translation, meeting full-face to full-face, then the continuous hull of \(x\) (i.e., the closure of the translational orbit of \(x\)) is the inverse limit of a sequence of compact branched manifolds \(K_0, K_1, K_2, \ldots\), under a sequence of maps \(\sigma_n : K_n \rightarrow K_{n-1}\), where each branched manifold \(K_n\) is the union of (marked) tiles from the original tiling. Of the two constructions, Gähler’s is conceptually simpler, but that of Bellissard, Benedetti and Gambaudo appears to be calculationally stronger, leading to results such as gap-labeling theorems [BBG].

This paper is an extension of Gähler’s construction to tilings of arbitrary Riemannian homogeneous spaces, with general symmetry group. The generalization of the Bellissard-Benedetti-Gambaudo approach to arbitrary spaces is being done independently by Benedetti and Gambaudo [BG].

2. Theorem and Proof

Before stating and proving the result, we must establish some notation. Let \(M\) be a Riemannian homogeneous space (such as \(\mathbb{Z}^d, \mathbb{R}^d, \mathbb{H}^2, \mathbb{H}^2 \times \mathbb{R}^3\), etc.), and pick a point to be the origin. Let \(G\) be the group of isometries of \(M\), let \(\Gamma\) be a closed subgroup of \(G\), and let \(\Gamma_0\) be the subgroup of \(\Gamma\) that fixes the origin. Let \(\Omega\) be a collection of tilings of \(M\). We give \(\Omega\) the topology that two tilings are \(\epsilon\)-close if they agree on a ball of size \(1/\epsilon\) around the origin, up to the action of an \(\epsilon\)-small element of \(\Gamma\). We assume that \(\Omega\) is closed under the action of \(\Gamma\) (i.e., \(\Omega\) is a union of \(\Gamma\)-orbits), and that \(\Omega\) is compact. This implies that \(\Omega\) has finite local complexity, up to the action of \(\Gamma\).

**Theorem.** \(\Omega\) is the inverse limit of a sequence of compact branched manifolds \(K_1, K_2, \ldots\) and continuous maps \(\sigma_n : K_n \rightarrow K_{n-1}\). The dimension of the branched manifold is the dimension of \(\Gamma\).

The idea of the proof is quite simple. A point in the \(n\)-th approximant \(K_n\) is a description of a tile containing the origin, its nearest neighbors (sometimes called the “first corona”), its second nearest neighbors (the “second corona”) and so on out to the \(n\)-th nearest neighbors. (For these purposes, tiles that meet at a point are considered nearest neighbors.) The map \(\sigma_n : K_n \rightarrow K_{n-1}\) simply forgets the \(n\)-th corona. A point in the inverse limit is then a consistent prescription for constructing a tiling out to infinity. In other words, it is a tiling.

What remains is to actually construct \(K_n\) out of geometric pieces and show that \(K_n\) is a branched manifold.

First suppose that the tiles are polytopes that meet full-face to full-face. We consider two tiles \(t_1, t_2\) in (possibly different) tilings of \(M\) to be equivalent if a patch of the first tiling, containing \(t_1\) and its first \(n\) coronas, is identical, up the the action of \(\Gamma\), to a
similar patch around \( t_2 \). Since \( \Omega \) has finite local complexity, there are only finitely many equivalence classes, each of which is called an \( n \)-collared tile.

For each \( n \)-collared tile \( t_i \), we consider how such a tile can be placed around the origin. Let \( s_i \subseteq t_i \) be the set of points where the origin may sit. By finite local complexity, there can only be a finite number of connected components to \( s_i \), and each component is a submanifold of \( t_i \) with the same dimension as \( \Gamma/\Gamma_0 \). If \( t_i \) does not admit any symmetry, then for each point \( p \in s_i \), \( \Gamma_0 \) acts simply transitively on the ways to place \( t_i \) down with the spot \( p \) landing at the origin. The set of ways to place \( t_i \) is therefore a principal \( \Gamma_0 \) bundle over \( s_i \), which we denote \( E_i \). The cell \( C_i \subseteq K_n \) associated with \( t_i \) is then exactly \( E_i \).

If there are no topological obstructions to trivializing this bundle, we make the identification

\[
C_i = E_i = s_i \times \Gamma_0.
\]

If \( M \) is flat, then there is a \emph{canonical} trivialization of the frame bundle, and this descends to a canonical product (1). If \( \Gamma \) acts transitively on \( M \), then \( s_i = t_i \) is contractible, and the decomposition (1), while not canonical, is guaranteed to exist. Although there do exist tilings where neither of these conditions are met, the author knows of no examples where \( C_i \) fails to be trivializable.

If \( t_i \) admits a discrete symmetry (e.g., is a regular \( n \)-gon in a tiling of \( \mathbb{R}^2 \) or \( \mathbb{H}^2 \)), then more than one point in \( s_i \times \Gamma_0 \) may describe the same placement of a tile containing the origin. In that case, the cell associated to \( t_i \) is the quotient of the \( \Gamma_0 \) bundle \( E_i \) by the symmetry. That is,

\[
C_i = E_i/\Gamma_{t_i} \quad (= s_i \times_{\Gamma_{t_i}} \Gamma_0, \text{ if } E_i \text{ is trivializable}),
\]

where \( \Gamma_{t_i} \subseteq \Gamma_0 \) is the group of symmetries of \( t_i \). Since \( t_i \) is a collared tile, \( \Gamma_{t_i} \) must be a discrete subgroup of \( \Gamma_0 \). (Even if a tile had a continuous symmetry, its first corona could not.) By construction, \( \Gamma_{t_i} \) acts without fixed points on \( E_i \), so the interior of \( C_i \) is indeed a manifold. (For instance, if \( M = \mathbb{R}^2 \) and \( \Gamma \) is the Euclidean group, then \( C_i \) is a Seifert fibered space. There may be multiple fibers over points of symmetry, but the total space is smooth.)

A patch of a tiling in which the origin is on the boundary of two or more tiles is described by points on the boundary of two or more cells, and these points must be identified. The branched manifold \( K_n \) is the disjoint union of the cells \( C_i \), modulo this identification. Since we are using \( n \)-collared tiles with \( n \geq 1 \), each of the points being identified carries complete information about the placement of all the tiles that meet the origin, together with their first \( n - 1 \) coronas.

We must show that a neighborhood of such a branch point is the union of topological disks whose tangent spaces may be identified. Each such disk is obtained by taking a patch of a tiling in which the above data is actually realized, and considering its orbit under the action of a neighborhood of the identity in \( \Gamma \). This shows that the dimension of \( K_n \) is the dimension of \( \Gamma \).

Finally, we remove the assumption that the tiles are polytopes that meet full-face to full-face. To a tiling by other shapes we may associate a pattern of marked points, where
a special point is chosen from each tile and labeled by the type of that tile. The Voronoi
cells of those points are then polytopes whose faces, properly subdivided, meet full-face to
full-face. The original tiling and the tiling by Voronoi cells are mutually locally derivable
[BSJ], and so are described by the same topological space, and hence by the same inverse
limit structure. ■

In this construction, the group \( \Gamma_0 \) acts naturally on each space \( K_n \), and the maps \( \sigma_n \)
are equivariant, from which it follows that

**Corollary.** The space \( \Omega/\Gamma_0 \) of tilings modulo rotation is the inverse limit of a sequence
of compact branched orbifolds \( K_n/\Gamma_0 \).

### 3. Examples

1. If \( M = \Gamma = \mathbb{Z}^d \), then we have a \( \mathbb{Z}^d \) subshift. The total space \( \Omega \) is a Cantor
set. The \( n \)-th approximant \( K_n \) is a finite collection of points, corresponding to
a decomposition of the Cantor set into a finite number of clopen sets. This
decomposition becomes finer as \( n \to \infty \), and the Cantor set is recovered as the
inverse limit.

2. If \( M = \mathbb{R}^d \) and \( \Gamma = \mathbb{Z}^d \), then (up to a fixed translation) \( \Omega \) is a space of tilings of
\( \mathbb{R}^d \) by square tiles centered at the lattice points. This is a different description of
the previous example. In these examples, note that \( \Omega \) does not have to be the hull
of a single tiling, and that the \( \mathbb{Z}^d \) action need not be minimal. The \( \mathbb{Z} \) subshift on
two letters, in which one of the letters appears at most twice, is neither minimal
nor the closure of a single orbit, but is an inverse limit space.

3. The \((d\text{-fold})\) suspension of a \( \mathbb{Z}^d \) subshift has \( M = \Gamma = \mathbb{R}^d \). This is a space of tilings of
\( \mathbb{R}^d \) by unit cubes oriented parallel to the coordinate axes.

4. A \( \mathbb{Z}^d \) subshift may be suspended in some directions but not in others. For instance,
the suspension of a \( \mathbb{Z}^2 \) subshift in the \( x \) direction is a space of tilings of \( \mathbb{R}^2 \) by
square tiles, meeting full-face to full-face, whose centers have integral \( y \) coordinate.
In this case \( M = \mathbb{R}^2 \) and \( \Gamma = \mathbb{R} \times \mathbb{Z} \).

5. The Penrose tiling space, or any other tiling of \( \mathbb{R}^d \) with a finite set of prototiles
up to translation, has \( M = \Gamma = \mathbb{R}^d \). Since \( \Gamma_0 \) is trivial and \( \Gamma \) is the full trans-
lation group, the cells \( C_i \) can be identified with the collared tiles \( t_i \) themselves.
This is precisely Gähler’s construction. As was shown in [SW], such a space is
homeomorphic to the suspension of a \( \mathbb{Z}^d \) subshift.

6. The pinwheel tiling space [Rad] has \( M = \mathbb{R}^2 \) and \( \Gamma \) the 2-dimensional Euclidean
group [ORS].

7. In tiling hyperbolic space, there are a number of interesting choices for \( \Gamma \). If \( \Gamma \) is
a discrete group, then we have the analog of a subshift, associating letters to a
discrete set of points in the space being tiled. At the other extreme, one can take
\( \Gamma \) to be the entire group of isometries of \( \mathbb{H}^n \).

8. One dimensional orientable hyperbolic attractors are either solenoids or one di-
menstional tiling spaces [WAP]. However, the dyadic solenoid can be viewed as a
tiling space, of \( \mathbb{H}^2 \) rather than \( \mathbb{R}^1 \), following a construction of Penrose [Pen]. See
figure 1. In the upper-half-plane model, the basic tile looks like a rectangle, with
the sides of the rectangle geodesics, with the top and bottom edges horocyclic, and with the size chosen such that the bottom edge has twice the length of the top edge. Here the group is $\Gamma = \mathbb{Z} \ltimes \mathbb{R}$, acting on $\mathbb{H}^2$ by $(n, t)(x, y) = (t + 2^n x, 2^n y)$.

More generally, any geometric substitution in $\mathbb{R}^d$ gives rise to a space of tilings of $\mathbb{H}^{d+1}$, with group $\Gamma = \mathbb{Z} \ltimes \mathbb{R}^d$. As with the dyadic solenoid, it doesn’t matter whether the substitution is invertible, since the $\mathbb{Z}$ action enforces the hierarchy. Chaim Goodman-Strauss has adapted this construction to produce a strongly aperiodic set of prototiles for $\mathbb{H}^2$ [G-S1], and to develop a general formalism for describing tilings of hyperbolic space [G-S2].

4. Conclusions and open problems

The inverse limit structure of $\Omega$ implies that the Čech cohomology $H^*(\Omega)$ is the direct limit of $H^*(K_n)$ under the pullback maps $\sigma_n^*$. Every element of $H^*(\Omega)$ is the pullback, under the natural projection $\pi_n : \Omega \to K_n$, of a cohomology class in $K_n$, for $n$ sufficiently large. If (and only if) $H^*(\Omega)$ is finitely generated, then for $n$ large enough the entire cohomology of $\Omega$ is the quotient of $H^*(K_n)$ by the kernel of $\pi_n^*$.

To make effective use of this principle, however, requires specific knowledge of the tiling space in question. For substitution tilings, it is easiest to work with the Anderson-Putnam inverse limit construction, rather than that constructed here, although in fact the two are shift equivalent. For cut-and-project tilings with sufficiently nice “windows”, Gähler
has shown that $\pi^*_n$ is actually an isomorphism in cohomology for $n$ sufficiently large, with the required size of $n$ computable from the geometry of the window.

The inverse limit structure of tiling spaces is related to a possible fiber bundle structure. Locally, $\Omega$ looks like a piece of $\Gamma$ times a Cantor set. Can these neighborhoods be stitched together to yield a fiber bundle (with Cantor set fiber) over a compact manifold? Is that manifold the quotient of the identity component of $\Gamma$ by a co-compact subgroup? When $M = \Gamma = \mathbb{R}^d$, the answer to both questions is yes [SW], but the general case is not known.

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