A VALUE REGION PROBLEM FOR STIELTJES TYPE CONTINUED FRACTIONS

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ABSTRACT. Motivated by applications in noncommutative geometry we prove several value range estimates for even convergents and tails, and odd reverse sequences of Stieltjes type continued fractions with bounded ratio of consecutive elements.

1. INTRODUCTION.

1.1. The problem considered in this paper asks about a region in the complex plane which contains values of the convergents of a continued fraction of a special type (see [9], Chapter VIII). More specifically in our “value region problem” we consider only even convergents for continued fractions of the Stieltjes type with bounded ratio of consecutive elements, whether the fractions are convergent or not. Even more generally, we also study the same question for even tail sequences and for what we call odd reverse sequences associated with a continued fraction. The main results we obtain in this paper show that two types of circles with sufficiently large radii form such value regions.

Our interest in this problem was motivated by an unrelated study in [5], where considerations in noncommutative geometry required estimates on certain sequences that could be interpreted as even tail and odd reverse sequences of a continued fraction. One of the most important features of the estimates was their independence on scaling all the elements by a constant factor, hence our interest in the ratio of two consecutive such elements. In this paper we look at a far more general situation than needed for [5], namely we consider continued fractions whose elements are complex numbers with positive real part and a full value region problem.

We obtain that circles centered at the origin and with sufficiently large radii are value regions for Van Vleck fractions with narrow sector angle $\theta < \pi/4$, assuming the ratio of two consecutive elements is bounded. We also provide an example illustrating that the results cannot be extended to sectors with the angle $\theta \geq \pi/4$. However, as described in our second main result, a shifted circle through the origin, centered on the real axis, and large enough radius is a value region for even convergents for a much larger class of continued fractions including all Van Vleck fractions with bounded ratio of two consecutive elements.

The material in this paper is divided into two parts. Section 2 contains our notation and the statements of the results while all the proofs are deferred to Section 3.

1.2. To put this problem into a perspective we will now describe a few results in the same direction, i.e. boundedness of continued fractions when all elements are scaled. One such result can be deduced from Stieltjes’ paper [8] using the following considerations. In what
follows we use standard abbreviated notation for continued fractions, reviewed in more detail in the following section. For a continued fraction of the form
\[
\frac{1}{zb_1 + b_2 + zb_3 + \cdots + zb_{2n-1} + b_{2n} + \cdots}
\]
with \(b_n > 0\) and \(\text{Re}(z) > 0\), Stieltjes proved that there are bounded, nondecreasing functions \(\phi_{2n}(t)\) on \([0, \infty)\) such that
\[
\int_0^\infty \frac{d\phi_{2n}(t)}{z + t} = \frac{1}{zb_1 + b_2 + zb_3 + \cdots + zb_{2n-1} + b_{2n}}.
\]

Writing \(z = w^2\), multiplying by \(w\) and using a simple equivalence transformation of the continued fraction we arrive at
\[
\int_0^\infty \frac{w}{w^2 + t} d\phi_{2n}(t) = \frac{1}{wb_1 + wb_2 + wb_3 + \cdots + wb_{2n-1} + wb_{2n}} =: f_{2n}.
\]

It follows, using an elementary inequality on the absolute value of the integrand, that for \(w\) such that \(|\text{Arg}(w)| \leq \theta < \pi/2\) we have
\[
\left| \frac{1}{wb_1 + wb_2 + wb_3 + \cdots + wb_{2n-1} + wb_{2n}} \right| \leq \frac{1}{2} \cos \theta \int_0^\infty \frac{d\phi_{2n}(t)}{\sqrt{t}},
\]
i.e. the even convergents stay bounded when scaling all elements by the parameter \(w\) in a sector. Stieltjes proved that the limit \(\lim_{n \to \infty} f_{2n} =: f\) always exist and so we have
\[
|f| \leq \frac{1}{2} \cos \theta \int_0^\infty \frac{d\phi(t)}{\sqrt{t}}
\]
for a bounded non-decreasing function \(\phi(t)\) on \([0, \infty)\), provided the last integral exists. Consequently, the estimates extend to the limit as \(n \to \infty\), i.e. we have that the ranges of the analytic functions \(f_{2n}\) and \(f\) are bounded for \(w\) in the sector \(|\text{Arg} w| \leq \theta < \pi/2\). Notice that the ratio of two consecutive elements of the continued fraction does not depend on \(w\).

On the other hand the integral \(\int_0^\infty \frac{d\phi(t)}{\sqrt{t}}\) in the formula above is not easily interpretable in terms of the coefficients \(b_n\). Moreover similar considerations do not seem to work for tails and reverse sequences needed for \([5]\).

Another similar result called Limaçon Theorem is contained in \([7]\). It assumes that \(b_n \geq b > 0\). With the above notation it states that \(|f - c| \leq \sqrt{1 + c^2}\) for all positive \(c\) and \(w\) in the region \(|w + 2c/b| \geq 2\sqrt{1 + c^2}/b\). However for the purpose of applications in \([5]\), the condition \(b_n \geq b > 0\) is too restrictive.

Finally the inequality in \([3]\):
\[
\frac{1}{wb_1 + wb_2 + wb_3 + \cdots + wb_{2n-1} + wb_{2n}} \leq \sum_{i=1}^{n} \frac{wb_{2i-1}}{1 + w^2b_{2i-1}b_{2i}},
\]
for positive \(b_i\) and \(w\), requires extra convergence assumptions to be usable in the limit \(n \to \infty\).

The goal of this paper is, in a sense, to provide similar types of results for more general Stieltjes type fractions where the coefficients \(b_n\) are complex numbers with positive real part.
Given a sequence of nonzero complex numbers \( \{b_n\} \), we associate to it a continued fraction of the form

\[
\frac{1}{b_k} = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}} = \frac{1}{b_1 + \frac{1}{b_2 + \frac{1}{b_3 + \cdots}}}
\]

The sequence of convergents \( \{f_n\} = \{\frac{A_n}{B_n}\} \) (also referred to as the \( n^{th} \) approximant) is defined by

\[
f_n = A_nB_n - A_{n-1}B_{n-1} = (-1)^{n-1}, \quad n \geq 0
\]

\[
A_nB_{n-2} - A_{n-2}B_n = (-1)^n b_n, \quad n \geq 1.
\]

They also satisfy the following determinant relations.

\[
a_n = b_nA_{n-1} + A_{n-2} \quad \text{and} \quad B_n = b_nB_{n-1} + B_{n-2} \quad \text{for} \quad n \geq 2
\]

with \( A_0 = 0, \ A_1 = 1, \ B_0 = 1, \ B_1 = b_1. \) (2.1)

Consider the Möbius transformation \( s_n : \mathbb{C} \cup \{\infty\} \rightarrow \mathbb{C} \cup \{\infty\} \) defined by \( s_n(w) = \frac{1}{b_n + w}. \) It is easily seen by induction that

\[
s_1 \circ s_2 \circ \cdots \circ s_n(w) = \frac{A_n + wA_{n-1}}{B_n + wB_{n-1}} = f_n(w).
\]

Notice that \( s_1 \circ s_2 \circ \cdots \circ s_n(0) = f_n(0) = \frac{A_n}{B_n} = f_n. \) From now on we will write \( f_n \) for \( f_n(0). \)

Given a continued fraction \( \frac{1}{b_k} \) we define a tail sequence \( \{t_n(w)\} \) starting with a constant \( w \in \mathbb{C} \cup \{\infty\}, \) see \[3\], recursively by

\[
t_{n-1}(w) = s_n(t_n(w)) = \frac{1}{b_n + t_n(w)}
\]

and \( t_0(w) = w. \) From equation (2.1) we see that \( -\frac{B_{n-1}}{B_n} = \frac{1}{b_n - \frac{b_n}{A_{n-1}}}. \) This gives an example of a tail sequence \( t_n(\infty) = -\frac{B_n}{B_{n-1}}. \) Similarly, \( t_n(0) = -\frac{A_n}{A_{n-1}} \) gives another example of a tail sequence. Next we define a reverse sequence \( \{r_n(w)\} \) starting with \( w \) recursively by

\[
r_{n+1}(w) = s_n(r_n(w)) = \frac{1}{b_n + r_n(w)}
\]

where \( r_1(w) = w \) is an arbitrary number in the extended complex plane. Two such examples of reverse sequences are \( r_{n+1}(0) = \frac{B_{n-1}}{B_n} \) and \( r_{n+1}(\infty) = \frac{A_n}{A_{n-1}}. \) Similar to (2.2), one can easily verify by induction that the tail and reverse sequences satisfy the following relations:
\[ t_n(w) = \frac{A_n - wB_n}{-A_{n-1} + wB_{n-1}} \quad \text{and} \quad r_{n+1}(w) = \frac{B_{n-1} + wA_{n-1}}{B_n + wA_n} = s_n \circ s_{n-1} \circ \cdots \circ s_1(w), \]

the last equality justifying the name: reverse sequences. It also follows that \( t_n(w) \) is the inverse function of \( f_n(w) \) and \( r_{n+1}(w) = -1/t_n(-1/w) \).

Below we describe three major types of continued fractions that we encounter in this paper.

- A continued fraction of the form \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) with \( \sum_{n=1}^{\infty} |b_n| < \infty \) is called a **Stern-Stolz fraction**.
- If there is \( \theta \) with \( 0 \leq \theta < \pi/2 \) such that \( b_n \in S_\theta = \{ z : |\text{Arg } z| \leq \theta \} \) for all \( n \) then \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) is called a **Van Vleck fraction**.
- A continued fraction of the form \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) where \( b_n \)'s are complex numbers with positive real parts is called a **Stieltjes type fraction**. In particular a Van Vleck fraction is a Stieltjes type fraction.

The following classical theorem regarding the convergence of a Stern-Stolz fraction states that the limits of numerators and denominators of the even and odd convergents exist even if the continued fraction itself is not convergent. As a consequence the even and odd convergents \( f_{2n}, f_{2n+1} \) will also converge to limits denoted by \( f_{\text{even}}, f_{\text{odd}} \) respectively.

**Theorem 2.1** (Stern-Stolz). If \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) is a Stern-Stolz fraction then the sequences \( \{A_{2n}\}, \{A_{2n+1}\}, \{B_{2n}\} \) and \( \{B_{2n+1}\} \) converge to the limits \( A_{\text{even}}, A_{\text{odd}}, B_{\text{even}} \) and \( B_{\text{odd}} \) respectively. Moreover, \( A_{\text{odd}}B_{\text{even}} - A_{\text{even}}B_{\text{odd}} = 1 \). Therefore \( f_{\text{even}} - f_{\text{odd}} \neq 0 \) and hence the continued fraction is not convergent.

For more details on the theorem see [2].

As a consequence of the above theorem and the formula (2.3) we have that for Stern-Stolz fractions \( t_{2n}(w), t_{2n+1}(w), r_{2n}(w) \) and \( r_{2n+1}(w) \) converge to finite limits \( t_{\text{even}}(w), t_{\text{odd}}(w), r_{\text{even}}(w) \) and \( r_{\text{odd}}(w) \) respectively. Moreover, \( t_{\text{even}}(w) = \frac{1}{t_{\text{odd}}(w)} \) and \( r_{\text{even}}(w) = \frac{1}{r_{\text{odd}}(w)} \).

The classical Van Vleck theorem below gives the necessary and sufficient condition for the convergence of a Van Vleck continued fraction.

**Theorem 2.2** (Van Vleck). A Van Vleck fraction \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) is convergent if and only if \( \sum_{n=1}^{\infty} |b_n| = \infty \).

Several different proofs of this theorem are contained in [2], [4], [6].

**Remark:** It follows that for Van Vleck fractions \( f_{\text{even}}(0) \) and \( f_{\text{odd}}(0) \) always exist. If a Van Vleck fraction \( K_{n=1}^{\infty} \left( \frac{1}{b_n} \right) \) is convergent then \( f_{\text{even}}(0) = f_{\text{odd}}(0) =: f(0) \).

The first of our main theorems in this note is the following estimate for even Van Vleck continued fractions with bounded ratio of consecutive elements in the sector \( S_\theta \) for \( \theta < \pi/4 \). It shows that under those conditions the fractions stay inside an origin centered disk of sufficiently large radius.
Theorem 2.3. Suppose $C$ is a constant such that $C^2 \geq \sup_{1 \leq k \leq n} \left| \frac{p_k}{q_k} \right| \cdot \frac{1}{\cos 2\theta}$ where $p_k, q_k, z \in S$ with $0 \leq \theta < \pi/4$ and $p_k, q_k \neq 0$. If $|z| \leq C$ then
\[ \left| \frac{1}{q_1 + p_1 + q_2 + p_2 + \cdots + q_n + p_n + z} \right| \leq C. \]

Now we look at the consequences of this estimate. Since
\[ f_{2n}(w) = \frac{1}{b_1 + b_2 + b_3 + \cdots + b_{2n-1} + b_{2n} + w}, \]
\[ r_{2n+1}(w) = \frac{1}{b_{2n} + b_{2n-1} + b_{2n-2} + \cdots + b_1 + 1 + w}, \]
\[ t_{2n}(w) = \frac{1}{b_{2n+1} + b_{2n+2} + b_{2n+3} + \cdots + b_{2N-1} + b_{2N} + t_{2N}(w)}, \]
we immediately obtain the following information about the convergents, reverse and tail sequences.

Corollary 2.4. Let $K_{n=1}^{\infty}(\frac{1}{b_n})$ be a Van Vleck fraction with $b_n \in S$ where $0 \leq \theta < \pi/4$. Assume that $\sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| < \infty$ and let $C^2 \geq \sup_n \left| \frac{b_{2n-1}}{b_{2n}} \right| \cdot \frac{1}{\cos 2\theta}$. If $w \in S$ and $|w| \leq C$ then $f_{2n}(w) \in S$ and $|f_{2n}(w)| \leq C$.

A parallel result also holds for odd reverse sequences.

Corollary 2.5. Let $K_{n=1}^{\infty}(\frac{1}{b_n})$ be a Van Vleck fraction with $b_n \in S$ where $0 \leq \theta < \pi/4$. Assume that $\sup_n \left| \frac{b_{2n-1}}{b_{2n}} \right| < \infty$ and let $C^2 \geq \sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| \cdot \frac{1}{\cos 2\theta}$. Then the following holds: if $w \in S$ and $|w| \leq C$ then $r_{2n+1}(w) \in S$ and $|r_{2n+1}(w)| \leq C$.

Even tail sequences behave differently, as the later sequence terms determine the size of the previous terms as described in the following.

Corollary 2.6. Let $K_{n=1}^{\infty}(\frac{1}{b_n})$ be a Van Vleck fraction with $b_n \in S$ where $0 \leq \theta < \pi/4$. Assume that $\sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| < \infty$ and let $C^2 \geq \sup_n \left| \frac{b_{2n-1}}{b_{2n}} \right| \cdot \frac{1}{\cos 2\theta}$. If $t_{2N}(w) \in S$ and $|t_{2N}(w)| \leq C$ then $t_{2n}(w) \in S$ and $|t_{2n}(w)| \leq C$ for all $n \leq N$.

From the results above we can trivially, by taking the limit, deduce the following corollary.

Corollary 2.7. Let $K_{n=1}^{\infty}(\frac{1}{b_n})$ be a Van Vleck fraction with $b_n \in S$ where $0 \leq \theta < \pi/4$. Then the following results hold:

1. Assume that $\sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| < \infty$ and let $C^2 \geq \sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| \cdot \frac{1}{\cos 2\theta}$. If $w \in S$, $|w| \leq C$ then $f_{even}(w) \in S$ and $|f_{even}(w)| \leq C$.
2. If $\sum_{n=1}^{\infty} |b_n| = \infty$, i.e. the fraction is convergent, then $|f(0)|^2 \leq \sup_n \left| \frac{b_{2n}}{b_{2n-1}} \right| \cdot \frac{1}{\cos 2\theta}$.
3. Assume that $\sum_{n=1}^{\infty} |b_n| < \infty$ and $\sup_n \left| \frac{b_{2n-1}}{b_{2n}} \right| < \infty$, and let $C^2 \geq \sup_n \left| \frac{b_{2n-1}}{b_{2n}} \right| \cdot \frac{1}{\cos 2\theta}$. If $w \in S$, $|w| \leq C$ then $r_{odd}(w) \in S$ and $|r_{odd}(w)| \leq C$. 


(4) Assume that \( \sum_{n=1}^{\infty} |b_n| < \infty \) and \( \sup_n \left| b_{2n-1} \right| < \infty \), and let \( C^2 \geq \sup_n \left| b_{2n-1} \right| \cdot \frac{1}{\cos 2\theta} \). If \( t_{\text{even}}(w) \in S_\theta \) and \( |t_{\text{even}}(w)| \leq C \) then \( t_{2n}(w) \in S_\theta \) and \( |t_{2n}(w)| \leq C \) for all \( n \).

Next we will describe similar results but for different circles as value regions, namely circles through the origin. The results are stronger in this case and can be applied to Stieltjes type fractions as well as arbitrary Van Vleck fractions with coefficients in arbitrary sectors.

**Theorem 2.8.** Suppose \( C \) is a constant such that \( C^2 \geq \frac{1}{4} \sup_{1 \leq k \leq n} \frac{1}{\Re(q_{k}) \Re \left( \frac{1}{b_{2n-1}} \right)} \) where \( p_k, q_k, z \in S_\theta \) with \( 0 \leq \theta < \pi/2 \) and \( p_k, q_k \neq 0 \). If \( |z - C| \leq C \) then

\[
\left| \frac{1}{q_1} + \frac{1}{q_2} + \frac{1}{q_3} + \cdots + \frac{1}{q_n} + p_n + z \right| \leq C.
\]

As before we have the following list of consequences.

**Corollary 2.9.** Let \( K_{n=1}^{\infty} \left( \frac{1}{b_{n}} \right) \) be a Stieltjes type continued fraction. Let \( C \) be a constant such that \( C^2 \geq \frac{1}{4} \sup_n \frac{1}{\Re(b_{2n-1}) \Re \left( \frac{1}{b_{2n}} \right)} \), assuming that \( \sup_n \frac{1}{\Re(b_{2n-1}) \Re \left( \frac{1}{b_{2n}} \right)} < \infty \). Then the following results hold:

1. If \( |w - C| \leq C \) then \( |f_{2n}(w) - C| \leq C \).
2. If \( \sum_{n=1}^{\infty} |b_n| = \infty \) then \( |f_{\text{even}}(w) - C| \leq C \) whenever \( |w - C| \leq C \).

Similarly for odd reverse sequences we have the following results.

**Corollary 2.10.** Let \( K_{n=1}^{\infty} \left( \frac{1}{b_{n}} \right) \) be a Stieltjes type fraction with \( \sup_n \frac{1}{\Re(b_{2n}) \Re \left( \frac{1}{b_{2n-1}} \right)} < \infty \). Let \( C^2 \geq \frac{1}{4} \sup_n \frac{1}{\Re(b_{2n}) \Re \left( \frac{1}{b_{2n-1}} \right)} \). Then the following is true.

1. If \( |w - C| \leq C \) then \( |r_{2n+1}(w) - C| \leq C \).
2. If \( \sum_{n=1}^{\infty} |b_n| < \infty \) then \( |r_{\text{odd}}(w) - C| \leq C \) whenever \( |w - C| \leq C \).

As before, the even tail sequences behave a bit differently.

**Corollary 2.11.** Let \( K_{n=1}^{\infty} \left( \frac{1}{b_{n}} \right) \) be a Stieltjes type continued fraction. Let \( C \) be a constant such that \( C^2 \geq \frac{1}{4} \sup_n \frac{1}{\Re(b_{2n-1}) \Re \left( \frac{1}{b_{2n}} \right)} \), assuming that \( \sup_n \frac{1}{\Re(b_{2n-1}) \Re \left( \frac{1}{b_{2n}} \right)} < \infty \). Then the following results hold:

1. If \( |t_{2N} - C| \leq C \) then \( |t_{2n}(w) - C| \leq C \) for all \( n \leq N \).
2. If \( \sum_{n=1}^{\infty} |b_n| < \infty \) then \( |t_{2n}(w) - C| \leq C \) whenever \( |t_{\text{even}}(w) - C| \leq C \).

Corollaries 2.9, 2.10 and 2.11 yield in turn the following results regarding the size of even convergents, odd reverse sequences, and even tail sequences of a Van Vleck fraction \( K_{n=1}^{\infty} \left( \frac{1}{b_{n}} \right) \) with the coefficients \( b_n \) now in a bigger sector \( S_\theta \) with \( 0 \leq \theta < \pi/2 \).

**Corollary 2.12.** Let \( K_{n=1}^{\infty} \left( \frac{1}{b_{n}} \right) \) be a Van Vleck fraction with \( b_n \in S_\theta \) where \( 0 \leq \theta < \pi/2 \). Then the following results hold:
(1) Assume that \( \sup_n |b_{2n}| < \infty \) and let \( C^2 \geq \frac{1}{4} \sup_n |b_{2n}| \cdot \frac{1}{\cos^2 \theta} \). If \( w \in S_\theta \), \( |w - C| \leq C \) then \( f_{\text{even}}(w) \in S_\theta \) and \( |f_{\text{even}}(w) - C| \leq C \).

(2) If \( \sum_{n=1}^{\infty} |b_n| = \infty \), then \( |f(0)|^2 \leq \sup_n |b_{2n}| \cdot \frac{1}{\cos^2 \theta} \).

(3) Assume that \( \sum_{n=1}^{\infty} |b_n| < \infty \) and \( \sup_n |b_{2n-1}| < \infty \) and let \( C^2 \geq \frac{1}{4} \sup_n |b_{2n-1}| \cdot \frac{1}{\cos^2 \theta} \). If \( w \in S_\theta \), \( |w - C| \leq C \) then \( r_{\text{odd}}(w) \in S_\theta \) and \( |r_{\text{odd}}(w) - C| \leq C \).

(4) Assume that \( \sum_{n=1}^{\infty} |b_n| < \infty \) and \( \sup_n |b_{2n-1}| < \infty \) and let \( C^2 \geq \frac{1}{4} \sup_n |b_{2n-1}| \cdot \frac{1}{\cos^2 \theta} \). If \( t_{\text{even}}(w) \in S_\theta \) and \( |t_{\text{even}}(w) - C| \leq C \) then \( t_{2n}(w) \in S_\theta \) and \( |t_{2n}(w) - C| \leq C \) for all \( n \).

3. Proofs of main theorems

The proof of Theorem 2.3 relies completely on the lemmas below. They describe the behavior of Möbius transformations from two-step continued fractions. First we remark that, the sector \( S_\theta \) is closed under addition and reciprocation, i.e., for any \( z, w \in S_\theta \) we also have \( \frac{1}{z}, z + w \in S_\theta \).

**Lemma 3.1.** Suppose \( p, q \) and \( z \in S_\theta \) with \( p, q \neq 0 \) and \( 0 \leq \theta < \pi/4 \). If \( |z| \leq C \) for some constant \( C \) then

\[
\left| \frac{1}{q + \frac{1}{p + z}} \right| \leq \frac{1}{|q| e^{i\theta} + \frac{e^{-i\theta}}{|p| + C}}.
\]

**Proof.** First we note that for \( p, q \in \mathbb{C} \) we have \( \text{Re}(pq) + \text{Re}(p\overline{q}) = 2\text{Re}(p)\text{Re}(q) \). Since any \( z \in S_\theta \) satisfies \( |z| \cos \theta \leq \text{Re} z \leq |z| \) it follows that

\[
\text{Re}(pq) \geq 2|p||q| \cos^2 \theta - |p||q| = |p||q| \cos 2\theta.
\]

Thus,

\[
\text{Re}\left( \frac{q}{p + z} \right) \geq \frac{|q| \cos 2\theta}{|p + z|} \geq \frac{|q| \cos 2\theta}{|p| + C}.
\]

Now we see that

\[
\left| q e^{i\theta} + \frac{e^{-i\theta}}{|p| + C} \right|^2 = |q|^2 + \frac{|q| \cos 2\theta}{|p| + C} + \frac{1}{(|p| + C)^2}
\]

\[
\leq |q|^2 + \text{Re}\left( \frac{q}{p + z} \right) + \frac{1}{|p + z|^2}
\]

\[
= \left| q + \frac{1}{p + z} \right|^2
\]

from which the result follows. \( \square \)
Using this lemma we will solve the following value region problem, namely we will show that for large enough \( C > 0 \) the map \( \frac{1}{q + \frac{1}{z}} \) preserves the circle centered at the origin with radius \( C \), given \( p, q, z \in S_\theta \) with \( 0 \leq \theta < \pi/4 \).

**Lemma 3.2.** Let \( p, q, z, \theta \) be as in Lemma 3.1 and \( C \) be a constant such that \( C^2 \geq \frac{|p|}{|q| \cos 2\theta} \).

If \( |z| \leq C \) then \( \frac{1}{q + \frac{1}{p + z}} \leq C \).

**Proof.** Since \( |q| (1 - \cos^2 2\theta) \geq 0 \) for \( 0 \leq \theta < \pi/4 \) we have that

\[
|q| + \frac{2 \cos 2\theta}{|p| + C} \geq |q| \cos^2 2\theta + \frac{2 \cos 2\theta}{|p| + C}.
\]

Therefore,

\[
\left| q e^{i\theta} + \frac{e^{-i\theta}}{|p| + C} \right|^2 = |q| \left( |q| + \frac{2 \cos 2\theta}{|p| + C} \right) + \frac{1}{(|p| + C)^2} 
\geq |q| \left( |q| \cos^2 2\theta + \frac{2 \cos 2\theta}{|p| + C} \right) + \frac{1}{(|p| + C)^2} 
= \left( \cos 2\theta |q| + \frac{1}{(|p| + C)} \right)^2,
\]

showing \( |q e^{i\theta} + \frac{e^{-i\theta}}{|p| + C}| \geq \cos 2\theta |q| + \frac{1}{|p| + C} \).

Hence,

\[
\left| q + \frac{1}{p + z} \right| \geq \left| q e^{i\theta} + \frac{e^{-i\theta}}{|p| + C} \right| \geq \cos 2\theta |q| + \frac{1}{|p| + C} \geq \frac{C^2 |q|^2 \cos 2\theta + C|q| \cos 2\theta + 1}{C(1 + C|q| \cos 2\theta)} = \frac{1}{C}.
\]

Now we can proceed with the proof of Theorem 2.3.

**Proof of Theorem 2.3.** Since \( S_\theta \) is closed under addition and reciprocation we can now apply the previous lemma to obtain

\[
\left| q_n + \frac{1}{p_n + z} \right| \leq C.
\]

By \( n \) successive applications of the above step we get the desired result.

The following example illustrates that the angle \( \theta = \pi/4 \) in Theorem 2.3 is optimal.
Example: Let $p = q = te^{in/4}$ and $z = \frac{1}{t^1} e^{i\pi/4}$ where $0 < t \leq 1/2$. We will show that $\frac{1}{[q + p]} \geq |z|$. Since $|z|$ can be arbitrarily large for $\theta = \pi/4$ this will then show that statement of Lemma 3.2 does not hold for this case.

First we compute that

$$\left|q + \frac{1}{p + z}\right|^2 = \left|ti + \frac{1}{t + \frac{1}{t^{1/3}}}ight|^2 = t^2 + \frac{t^{2/3}}{(t^{4/3} + 1)^2}.$$ 

Let $s = t^{4/3}$. Since $0 < s \leq 1/2$ we see that $s(s^2 + s - 1) < s(1/4 + 1/2 - 1) < 0$. Thus,

$$\left|q + \frac{1}{p + z}\right|^2 = t^2 + \frac{t^{2/3}}{(t^{4/3} + 1)^2} = s^{3/2} + \frac{s^{1/2}}{(s + 1)^2} = s^{1/2} \left[\frac{s(s^2 + s - 1)}{(s + 1)^2} + 1\right] < t^{2/3} = \frac{1}{|z|^2}.$$ 

In order to prove Theorem 2.8 we need the following key lemma on Möbius transformations.

**Lemma 3.3.** Suppose $p, q, z \in S_\theta$ where $0 \leq \theta < \pi/2$ and $\text{Re} q, \text{Re} p, \text{Re} z > 0$. Let $C$ be such that $C^2 \geq \frac{1}{4\text{Re}(q)\text{Re}\left(\frac{1}{p}\right)}$. If $|z - C| \leq C$ then $\left|\frac{1}{q + p + z} - C\right| \leq C$.

**Proof.** First, we notice that for any $w \in \mathbb{C}$ and constant $K > 0$

$$|w - K| \leq K \quad \text{if and only if} \quad \text{Re} \left(\frac{1}{w}\right) \geq \frac{1}{2K}.$$ 

Thus, it suffices to show that $\text{Re} (q) + \text{Re} \left(\frac{1}{p + z}\right) \geq \frac{1}{2C}$ whenever $|z - C| \leq C$.

Since the condition on $C$ implies that $4C^2\text{Re}(q)\text{Re} \left(\frac{1}{p}\right) \geq 1 > 1 - 2C\text{Re}(q)$, we have $\text{Re} \left(\frac{1}{p}\right) \geq \frac{1 - 2C\text{Re}(q)}{4C^2\text{Re}(q)}$. Thus,

$$\text{Re} \left(\frac{1}{p}\right) \geq \frac{1 - 2C\text{Re}(q)}{4C^2\text{Re}(q)} = \frac{1}{2 \left(\frac{C}{1 - 2C\text{Re} q} - C\right)} = \frac{1}{2(B - C)}$$

where $B = \frac{C}{1 - 2C\text{Re} q}$, which is equivalent to, $|p - (B - C)| \leq B - C$. Thus,

$$|p + z - B| \leq |z - C| + |p - (B - C)| \leq C + B - C = B,$$

which then is equivalent to

$$\text{Re} \left(\frac{1}{p + z}\right) \geq \frac{1}{2B} = \frac{1}{2C} - \text{Re} q$$

from which the result follows. □
Proof of Theorem 2.8: Applying the above lemma once we obtain,

\[ \left| \frac{1}{q_n + \frac{1}{p_{n+1}}} - C \right| \leq C. \]

By recursively applying the lemma \( n \) times, we get the result. \( \square \)

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