On superstatistics and black hole quasinormal modes.

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Abstract

It is known that using Boltzmann-Gibbs statistics, Bekenstein-Hawking entropy $S_{BH}$, and the quasinormal modes of black holes, one finds that the lowest value of spin is $j_{\text{min}} = 1$. In this paper, we determine $j_{\text{min}}$, using non-extensive entropies that depend only on the probability (known as Obregon’s entropies and have been derived from superstatistics). We also calculate $j_{\text{min}}$ for a set of non-extensive entropies that have free parameters and are written in terms of $S_{BH}$. We find that $j_{\text{min}}$ depends on the area and the non-extensive parameter.

For the non-extensive entropies that only depend on the probability, we find that the modification is only present for micro black holes. For classical black holes the results are the same as for the Boltzmann-Gibbs statistics.

1. Introduction

Black holes are one of the most enigmatic and mysterious objects in physics. Recently, direct verification of their existence was provided by the Event Horizon Telescope collaboration \cite{1}. Despite the amount of research on the subject, there are several unanswered questions on black hole physics.

Quantization of black holes was proposed in the pioneering work of Bekenstein \cite{2}. He suggested that the surface gravity is proportional to the temperature, and the area of the event horizon is proportional to its entropy. Moreover, he conjectured that the horizon area of non extremal black holes plays the role of a classic adiabatic invariant. Finally, he concluded that the horizon area should have a discrete spectrum with uniformly spaced eigenvalues,

$$A_n = \tilde{\gamma} l_p^2 n, \quad n = 1, 2, ...\quad (1)$$

where $\tilde{\gamma}$ is a dimensionless constant.

With the development of Loop Quantum Gravity (LQG), the correct spectrum of the area operator was obtained in \cite{3, 4}, being

$$A(j) = 8\pi\gamma l_p^2 \sqrt{j(j+1)},\quad (2)$$

where $\gamma$ is the Immirzi parameter \cite{5, 6}. It is a free parameter in Loop Quantum Gravity (LQG), and determines the value of the minimal area. As any fundamental constant in the theory we must find a way to determine its value, and this is where the entropy comes into play. The entropy is a quantity related to spectrum and therefore seems to be the main candidate to determine the value of $\gamma$. In \cite{7}, the author established a method to determine $\gamma$ using the quasinormal modes for the Schwarzschild black hole. This approach relates the area (derived in LQG) with the mass and area of the Schwarzschild black hole. Using Boltzmann-Gibbs statistics, obtains the expression $\gamma = \frac{\ln 3}{2\pi \sqrt{2}}$.

In this work following \cite{7}, we use quasinormal modes and Obregon’s entropies \cite{8} to determine the minimum value $j_{\text{min}}$. These entropies are generalizations to the Boltzmann-Gibbs entropy, and only depend on the probability. Calculation of $j_{\text{min}}$ has been done for Tsallis entropy \cite{9}. The parameter $q$ is used to fix $j_{\text{min}} = 1/2$. Finally, we also study no-extensive entropies that have free parameters and are explicitly a function of the Bekenstein-Hawking entropy. This approach was originally presented in \cite{10} for Barrow’s entropy.

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The paper is organized as follows. In section 2, we follow the approach in [7] for the non-extensive entropies that only depend on the probability. In section 3, we workout the case for other non-extensive entropies that have free parameters and are written in terms of $S_{BH}$. Section 4 is devoted discussion and final remarks.

2. Black holes and non-extensive entropies that only depend on the probability.

The loss of the extensive property is one of main complications that arise when generalizing Boltzmann’s entropy. Nonetheless, non-extensive entropies can be useful in the study of more general phenomena. There is a large set of non-extensive entropies that are related in Superstatistics. These entropies can be derived using different temperature distributions [11]. One of the most useful and better studied generalization is Tsallis entropy [12]. Tsallis entropy is non-extensive and has a free parameter $q$, known as the entropic index. The value of the entropic index is dependent on the physical system under study. For $q = 1$, Tsallis entropy reduces to the usual Boltzmann-Gibbs entropy.

In the context of superstatistics, starting with a Gamma distribution for the temperature, in [8] the author derives non-extensive entropies that depend only on the probability. These entropies are known as Obregon's entropies and in contrast to Tsallis entropy, don’t have free parameters. Furthermore, Obregon's entropies in the limit of small probabilities (or equivalently, a large number of states), reduce to Boltzmann-Gibbs statistics. It is worth mentioning that these entropies have been used in connection with entropic gravity [13], and AdS/CFT [14].

Let us explicitly give the functional form of Obregon’s entropies. The first entropy is denoted by $S_+$ and is given by the expression

$$S_+ = \sum_{l=1}^{\Omega} \left(1 - p_l^q\right),$$

with the probabilities satisfying the usual constraint $\sum_l p_l = 1$.

There is also another entropy of the form

$$S_- = \sum_{l=1}^{\Omega} \left(p_l^q - 1\right),$$

and a third one defined from the sum of the previous entropies $S_\pm = \frac{1}{2} (S_+ + S_-)$. These three entropies are the only generalizations that depend only on the probability.

2.1. Relating entropy to black hole quasinormal modes

Let us briefly review the approach in [7], to relate the entropy and quasinormal modes of black holes. For a large imaginary part the frequency of the quasinormal modes [15] is

$$M \omega_n = \frac{\ln 3}{8\pi} + \frac{i}{4} \left(n + \frac{1}{2}\right),$$

the value of the real part of Eq.(5) was previously proposed in [16]. The energy spectrum is $\Delta M = l_0^2 w_n$ where we are using units $G = c = 1$ and we have defined $w_n = \frac{1}{\pi} Re(\omega_n) = \frac{\ln 3}{8\pi}$.

Considering that $A = 16\pi M^2$, when we introduce a change in the mass $\Delta M$ we get a change in the area $\Delta A = 4l_0^2 \ln 3$. Using Eq.(5), we find $\gamma$ as a function of $j_{\min}$

$$\gamma = \frac{\ln 3}{2\pi \sqrt{j_{\min}(j_{\min} + 1)}}$$

In terms of the probability, Tsallis entropy is given by $S_q = \frac{1}{q-1} \sum_{l=1}^{\Omega} p_l^q$.

Applications of Tsallis entropy are present in different areas of physics, ranging from high energy collisional experiments, velocity distributions in plasma, anomalous diffusion, quantum entangled systems, to name a few.

In the expressions for these entropies we are dealing with dimensionless quantities, thus we have already divided the entropy by the Boltzmann constant, $k_B$.  

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The number of microstates of the configurations in a punctured sphere is
\[ \Omega = \prod_{n=1}^{N} (2j_n + 1), \]  
where \( j_n \) is the spin label of such punctures, and \( N \) are the number of the punctures. Because the most important contributions come from configurations that satisfy \( j_n = j_{\min} \), Eq. (7) takes the simpler form
\[ \Omega = (2j_{\min} + 1)^N. \]  
The number of punctures \( N \) is given by the ratio \( A/\Delta A \),
\[ N = \frac{A}{4\pi l^2 \ln 3}. \]  
Now we can write \( \Omega \) in terms of \( j_{\min} \) and the area of the event horizon,
\[ \Omega = (2j_{\min} + 1)^{A/4\pi l^2 \ln 3}. \]  
Assuming a large number of states \( \Omega \), from Shannon’s entropy and considering equipartition, we get
\[ S = \ln \Omega. \]  
Moreover, using Eq. (7) and equating to the Bekenstein-Hawking entropy one finds that \( j_{\min} = 1 \). Finally, from Eq. (6) we can see that the Immirzi parameter is
\[ \gamma = \frac{\ln 3}{2\pi /\sqrt{2}}. \]  
As we are interested in determining \( j_{\min} \) for the non-extensive entropies that only depend on the probability, we simply follow the same approach, but instead of using Shannon’s entropy we use the entropy in Eq. (3). Working in the microcanonical ensemble and assuming equipartition, the probability of finding the system in a particular state is equal to the inverse of the number of states, \( p = \frac{l}{\Omega} = \Omega^{-1} \). Then Eq. (3) takes the form
\[ S = \Omega \sum_{l=1}^{\Omega} \left( 1 - \left( \frac{1}{\Omega} \right)^h \right) = \Omega \left( 1 - \left( \frac{1}{\Omega} \right)^h \right). \]  
With the same assumptions, we also rewrite \( S_- \) and \( S_+ \)
\[ S_+ = \frac{A}{4\pi l^2 p} \left( -p \ln p - \frac{1}{2} p^2 \ln^2 p - \frac{1}{6} p^3 \ln^3 p + \ldots \right), \]  
Solving for \( p \) gives the solution for \( \Omega \) and after substituting in Eq. (10) we obtain \( j_{\min} \). For large \( \Omega \) we can do a series expansion for small \( p \). To first order in \( p \), one simply reproduces the result derived from Shannon’s entropy. Deviations for \( j_{\min} \) are obtained by considering higher order terms in the expansion
\[ S \approx \frac{A}{4\pi l^2} \left( -p \ln p - \frac{1}{2} p^2 \ln^2 p - \frac{1}{6} p^3 \ln^3 p + \ldots \right). \]
this is the expansion for \( S_\pm \). As expected similar expressions are derived for \( S_- \) and \( S_+ \). Using the solution for \( p \) in Eq.\((7)\) we finally find \( j_{\text{min}} \) as a function of the area

\[
2j_{\text{min}} + 1 = \exp \ln 3 \left( 1 + \frac{1}{12} \frac{2A}{4l_p^2} e^{-\frac{A}{4l_p^2}} + \frac{1}{12} \left( \frac{A}{4l_p^2} \right)^2 \left( 4 - 3 \frac{A}{4l_p^2} \right) e^{-\frac{3A}{4l_p^2}} + \frac{1}{48} \left( \frac{A}{4l_p^2} \right)^3 \left( 10 - 24 \frac{A}{4l_p^2} + 9 \left( \frac{A}{4l_p^2} \right)^2 \right) e^{-\frac{15A}{4l_p^2}} + \ldots \right). \tag{14}
\]

We can see that higher order terms are exponentially suppressed and we have a good approximation using the first three terms. Following the same steps as before, we can calculate \( j_{\text{min}} \) using the entropy \( S_- \). From Eq.\((12b)\) we obtain

\[
2j_{\text{min}} + 1 = \exp \ln 3 \left( 1 + \frac{1}{6} \frac{A}{4l_p^2} e^{-\frac{A}{4l_p^2}} + \frac{1}{12} \left( \frac{A}{4l_p^2} \right)^2 \left( 4 - 3 \frac{A}{4l_p^2} \right) e^{-\frac{3A}{4l_p^2}} - \frac{1}{48} \left( \frac{A}{4l_p^2} \right)^3 \left( 10 - 24 \frac{A}{4l_p^2} + 9 \left( \frac{A}{4l_p^2} \right)^2 \right) e^{-\frac{15A}{4l_p^2}} + \ldots \right). \tag{16}
\]

Finally, for \( S_+ \) we have for \( j_{\text{min}} \)

\[
2j_{\text{min}} + 1 = \exp \ln 3 \left( 1 - \frac{1}{6} \frac{A}{4l_p^2} e^{-\frac{A}{4l_p^2}} + \frac{1}{360} \left( \frac{A}{4l_p^2} \right)^4 \left( 27 - 20 \frac{A}{4l_p^2} \right) e^{-\frac{15A}{4l_p^2}} - \frac{1}{64800} \left( \frac{A}{4l_p^2} \right)^6 \left( 2880 - 4860 \frac{A}{4l_p^2} + 1800 \left( \frac{A}{4l_p^2} \right)^2 \right) e^{-\frac{45A}{4l_p^2}} + \ldots \right). \tag{17}
\]

For \( S_+ , S_- \) and \( S_\pm \), when we consider the case of large area \( A \) we obtain \( j_{\text{min}} = 1 \). Since these entropies only depend on the probability, there are no free parameters we can use to fix the value of \( j_{\text{min}} \). We can see in Fig.\((1)\), that
for small \( A \) (micro black holes) we have a different value for \( j_{\text{min}} \). Even though \( j_{\text{min}} < 1 \) for \( S_- \) and \( S_+ \), we don’t have an area for which \( j_{\text{min}} = 1/2 \). This in contrast to the result from Tsallis entropy, where \( j_{\text{min}} \) is given by

\[
2j_{\text{min}} + 1 = \left[ (1 + (1 - q) \frac{A}{4lp^2})^{\frac{1}{1-q}} \right],
\]

the parameter \( q \) can be used to change the value of \( j_{\text{min}} \). The value for \( q \) is constrained to \(-\frac{4lp^2}{A} < (1 - q) < 1.37 \frac{4lp^2}{A} \). Although you can have \( j_{\text{min}} = 1/2 \), you have a different values of \( q \) for different black holes. For Obregon’s entropies, \( j_{\text{min}} < 1 \) for quantum black holes, and for classical black holes one recovers the prediction of Boltzmann-Gibbs statistics.

3. Non-extensive entropies with free parameters

The entropies studied in the previous section seem to be the only generalization to Shannon’s entropy that only depend on the probability. Nonetheless, there are several other generalizations that have free parameters besides Tsallis entropy. We will focus our attention to a set of entropies that have been used in connection to the black hole area entropy law. The expressions for these entropies are written in terms of the Bekenstein-Hawking entropy \( S_{\text{BH}} \), therefore we will follow [10] to simplify the derivation of \( j_{\text{min}} \).

3.1. Tsallis-Cirto entropy

The Tsallis-Cirto entropy was proposed in [17] to solve thermodynamic inconsistencies for the Schwarzschild black hole. This entropy is defined by the relation

\[
S_{TC} = (S_{BH})^\delta,
\]

in the limit \( \delta \to 1 \), the Bekenstein-Hawking entropy is recovered. Since the entropy in Eq. (19) is written in terms of \( S_{BH} \), we equate to the logarithm of the number of states (the number of states is given in Eq. (10)), to obtain \( j_{\text{min}} \)

\[
j_{\text{min}} = \frac{1}{2} \left[ \left( \frac{A}{4lp^2} \right)^{\frac{1}{1-q}} - 1 \right].
\]

Figure 2: The first plot is the predicted \( j_{\text{min}} \) for Tsallis-Cirto entropy. We can see that for a particular value of \( A \) there is a corresponding value of \( \delta \) that gives \( j_{\text{min}} = 1/2 \). The second plot corresponds to Barrow’s entropy. For \( \Delta > 0 \), we can find an equivalent description in terms of Tsallis-Cirto entropy.

The derivation and analysis is done in reference [9]. They find the range of values for \( q \) to satisfy \( j_{\text{min}} = 1/2 \).
For the Tsallis-Cirto entropy, we can satisfy $J_{\text{min}} = 1/2$, for particular values of $\delta$ and the black hole horizon area

$$\delta = \frac{\ln \frac{\ln 2}{\ln 3}}{\ln \frac{A}{4l_p^2}} + 1.$$  \hfill (21)

In Fig. 2 we show $J_{\text{min}}$ as a function of $\delta$ for values of the area. We can see that it is possible to have $J_{\text{min}} = 1/2$. If we take $\delta = 1 + \frac{1}{2}$ in Eq. (19) and Eq. (20), we obtain the results to Barrow’s entropy $S_{\text{Barrow}} = \left(\frac{A}{4l_p^2}\right)^{1+\Delta/2}$. This entropy is related to a fractal structure on the black hole surface, is of quantum origin and is encoded in the parameter $\Delta$. When using the Barrow’s interpretation, one gets $J_{\text{min}} > 1$ for $\Delta > 0$.

3.2. Modified Rényi entropy

We now consider the modified Rényi entropy. In connection to the black hole area entropy law, this entropy is given by the relation

$$S_{\text{MR}} = \frac{1}{\lambda} \ln(1 + \lambda S_{\text{BH}}),$$  \hfill (22)

where $\lambda$ is a positive constant. In the limit $\lambda \to 0$ we recover the Bekenstein-Hawking entropy. As in the previous case, this entropy is written in terms of $S_{\text{BH}}$. Therefore we equate to the logarithm of the number of states of the black hole. Using the expression for the Bekenstein-Hawking entropy, the result for $J_{\text{min}}$ is

$$J_{\text{min}} = \frac{1}{2} \left[3f(A; \lambda) - 1\right],$$  \hfill (23)

where

$$f(A; \lambda) = \frac{4l_p^2}{\lambda A} \ln \left(1 + \lambda \frac{A}{4l_p^2}\right).$$  \hfill (24)

It is easy to verify that in the limit when $\lambda \to 0$, $J_{\text{min}} \to 1$, recovering the result for the Bekenstein-Hawking entropy. Looking for the value of $\lambda$ which gives us $J_{\text{min}} = 1/2$, we arrive at the next transcendental equation

$$\ln \left(1 + \frac{\lambda A}{4l_p^2}\right) = \ln \frac{2 \lambda A}{\ln 3 4l_p^2}. \hfill (25)$$

The case $\lambda = 0$, although a solution is excluded since for this value the entropy goes to Bekenstein-Hawking $J_{\text{min}} = 1$. For large $\lambda$ we find the approximate relation $\lambda = \frac{6l_p^2}{\ln 3}$. 

Figure 3: The first graph is a plot of $J_{\text{min}}$ derived from the modified Rényi entropy. We can see in this plot that to have $J_{\text{min}} = 1/2$ the value of $\lambda$ is negative. Therefore, for this entropy $J_{\text{min}} \geq 1$. The second plot is for the values of $\lambda$ and $A$ that give $J_{\text{min}} = 1/2$. The solid line corresponds to the numerical solution to Eq. (25), the dotted line for the approximation.
3.3. Sharma-Mittal entropy

The last entropy we consider is the Sharma-Mittal. This entropy was proposed to construct a new model for holographic dark energy \[19\]. Moreover, it is a generalization of both the Rényi and Tsallis entropy. In connection to the black hole area entropy law, is defined by the relation

\[
S_{SM} = \frac{1}{R} \left[ (1 + \delta S_{BH})^{R/\delta} - 1 \right].
\]  
(26)

The entropy interpolates between the modified Rényi entropy \((R \to 0)\) and Bekenstein-Hawking entropy \((R \to \delta)\). As in previous cases, we assume that \(S_{SM} = \ln \Omega\). Thus, the value for \(j_{\text{min}}\) is

\[
j_{\text{min}} = \frac{1}{2} \left[ 3^{g(A;R,\delta)} - 1 \right],
\]  
(27)

where

\[
g(A;R,\delta) = \frac{4l_p^2}{RA} \left( \frac{\delta A}{4l_p^2} + 1 \right)^{R/\delta} - 1.
\]  
(28)

The definition of this entropy in terms of these two parameters, allows for the possibility of finding a region where the value of \(j_{\text{min}}\) be equal to 1/2. In the next plot, we show that for certain values of the area which pairs of such values gives \(j_{\text{min}} = 1/2\).

![Figure 4: Plot for the values of \(R\) and \(\delta\) in order to have \(j_{\text{min}}\) for the the Sharma-Mittal entropy.](image)

4. Final remarks

In this paper we have used quasinormal modes to determine the minimum value \(j_{\text{min}}\) for non-extensive entropies. These entropies are generalizations of the Boltzmann-Gibbs entropy. We worked out two classes of non-extensive entropies, the first class only depend on the probability. The second class have free parameters and are written in terms of Bekenstein-Hawking entropy.

With respect to the non-extensive entropies that only depend on the probability, the only entropies that satisfy this requirement are the so called Obregon’s entropies. Assuming equipartition we find \(j_{\text{min}}\) for these entropies. For \(S_\ldots\) and \(S_+\) we see from Fig.7 that the minimum value is less than one, but for large area we recover the usual value. Therefore, we can conclude that the modification from using these entropies is only present for micro black holes. This is consistent with results obtained for fluids \[20\], where it is showed that the effects of these entropies are not
present for classical systems. Moreover, for $A > 8l^2$, the result is the same as BG, therefore we conclude that that the effects of using non extensive entropies are only present on micro black holes.

Of the non-extensive entropies that have free parameters and are function of $S_{BH}$. The free parameters on these entropies, allows us to fix $j_{\text{min}} = 1/2$, for particular values of the $A$. Moreover, for specific values of their respective parameters, they reproduce the Boltzmann-Gibbs results. Using this class of non-extensive entropies $SO(3)$ and $SU(2)$ spin networks are valid. It is worth mentioning that we can have $j_{\text{min}} > 1$ and therefore these entropies generalize the value of $j_{\text{min}}$.

In summary, the use of non-extensive entropies modify $j_{\text{min}}$. For Obregon’s entropies, $j_{\text{min}} \neq 1$ for micro black holes. Therefore, there is a possibility that non-extensive statistics modifies the dynamics of quantum gravity. Of particular interest are the effects on cosmology, consequently the effects of non-extensive entropies could change the dynamics of the very early Universe. This is work in progress and will be reported elsewhere.

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Appendix

To derive $j_{\text{min}}$ as a function of the black hole area, we must solve for $p$ in terms $A$. We begin by rewriting Eq. (12a)

$$
\frac{A}{4l^2p} = \frac{1}{p} (1 - p^p) = - \ln p - \sum_{n=2}^{\infty} p^{n-1} \ln^n p,
$$

or equivalently

$$
\ln p = - \frac{A}{4l^2p} - \sum_{n=2}^{\infty} p^{n-1} \ln^n p = - \frac{A}{4l^2p} - \sum_{n=2}^{\infty} \ln^n p e^{(n-1) \ln p}.
$$

Figure 5: Plot of $j_{\text{min}}$ as a function of the black hole area, derived from non-extensive entropies.
Using inversion theorem [21], we can find a series solution for \( y \). This equation is of the form

\[
y = x + \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{d}{dx} \right)^{n-1} \{ f(x)^n \}.
\]

Using \( n = 3 \) in the formula, the result is

\[
\ln p = -\frac{A}{4l_p^2} - \frac{1}{2} \left( \frac{A}{4l_p^2} \right)^2 e^{-\frac{2A}{l_p}} - \frac{1}{12} \left( \frac{A}{4l_p^2} \right)^3 \left( 4 - 3 \frac{A}{4l_p^2} \right) e^{-\frac{2A}{l_p}} - \frac{1}{48} \left( \frac{A}{4l_p^2} \right)^4 \left( 10 - 24 \frac{A}{4l_p^2} + 9 \left( \frac{A}{4l_p^2} \right)^2 \right) e^{-\frac{2A}{l_p}} - \ldots,
\]

the terms become exponentially suppressed for higher order terms, therefore we good accuracy for \( n = 3 \).

We apply the same method for the entropies (11b) and (11c).

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