Propagation of Regular Singularities in a Complex Analytic 
Characteristic Initial Value Problem

By

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Abstract. We consider a characteristic initial value problem of a class of second order linear partial differential equation with regular singular initial data in the complex domain. We express the solution by means of series of hypergeometric functions, and clarify the behavior of the solution near the singularities. Our construction owes much to the contiguity relations of hypergeometric functions.

Key Words and Phrases. Initial value problem, Regular singularity, Hypergeometric function.

2010 Mathematics Subject Classification Numbers. 35A20, 33C05.

1. Introduction

In this paper, we consider a class of characteristic initial value problem of linear partial differential equations with regular singular initial data in the complex domain.

The well-known theorem of Cauchy-Kowalewski states that if the two conditions, (A) the initial hypersurface is non-characteristic; (B) the initial data are holomorphic, are satisfied, then we obtain locally a unique holomorphic solution. Then what happens if we break the condition (A) or (B)? Many mathematicians have asked such questions in the complex analytic category.

Hamada [2] considered the Cauchy problem breaking the condition (B). In fact, he considered the Cauchy problem with singular initial data (on non-characteristic initial hypersurface), and clarified where the singularity of the solution appear and how the solution behaves near the singularity.

On the other hand, Baouendi-Goulaouic [1] considered the initial value problem breaking the condition (A). The equations he considered are called Fuchsian partial differential equations.

From the viewpoint of complex analysis, we expect that, if we break both the conditions (A) and (B), interesting phenomena as analytic functions might occur. Ōuchi [7] studied such problem and clarified where the singularities of the solution appear. Recently, Watanabe-Urabe [9], [10] considered a
characteristic initial value problem with singular initial data for a class of second order linear partial differential equations and succeeded in expressing the solution by the use of hypergeometric functions.

In this paper, modeled after their study, we consider the following class of second order linear partial differential operators with holomorphic coefficients in two dimensional complex space.

Let \((t, x)\) be coordinates in a neighborhood of the origin in \(\mathbb{C}^2\). Introducing two vector fields \(X_p\) and \(X_q\) defined by

\[
X_p = \frac{\partial}{\partial t} + p(t, x) \frac{\partial}{\partial x}, \quad X_q = \frac{\partial}{\partial t} + q(t, x) \frac{\partial}{\partial x},
\]

where \(p(t, x)\) and \(q(t, x)\) are holomorphic functions in a neighborhood of \((0, 0)\) \(\in \mathbb{C}^2\), we consider the linear partial differential operator

\[
P = tX_pX_q + \{a + ta(t, x)\}X_p + \{b + tb(t, x)\}X_q + c(t, x),
\]

where \(a\) and \(b\) are complex parameters, and \(a(t, x)\), \(b(t, x)\) and \(c(t, x)\) are functions holomorphic in a neighborhood of \((0, 0)\) \(\in \mathbb{C}^2\).

For the operator \(P\), belonging to the so-called Fuchsian partial differential operator having regular singularity along \(\{t = 0\}\), we know, according to Baouendi-Goulaouic [1] or Kashiwara-Oshima [5], that, if \(a + \beta \notin \{0, -1, -2, \ldots\}\), given a local holomorphic initial data on the initial plane \(\{t = 0\}\), a local holomorphic solution is uniquely determined around that point.

The purpose of this paper is to consider the initial value problem with regular singular initial data

\[
\begin{align*}
P u(t, x) &= 0, \\
   u(0, x) &= f_\lambda(x) w(x),
\end{align*}
\]

where

\[
f_\lambda(x) = \frac{x^\lambda}{\Gamma(\lambda + 1)},
\]

\(\lambda\) being a complex parameter, and \(w(x)\) is a function holomorphic in a neighborhood of \(x = 0\).

Throughout this paper we assume that

\[
p(0, 0) \neq q(0, 0).
\]

We will show under the condition (1.4) that the solution of the initial value problem (1.3) is expressed by means of series of hypergeometric functions.

In order to state the main result, we consider the solutions \(\varphi(t, x)\) and \(\psi(t, x)\) of the following initial value problems for the characteristic equations for
$X_p$ and $X_q$ respectively, namely

$$
\begin{align*}
X_p \varphi(t, x) &= 0, \\
\varphi(0, x) &= x,
\end{align*}
$$

(1.5)

and

$$
\begin{align*}
X_q \psi(t, x) &= 0, \\
\psi(0, x) &= x.
\end{align*}
$$

(1.6)

Obviously $\varphi(t, x)$ and $\psi(t, x)$ are both uniquely determined holomorphic functions in a neighborhood of $(t, x) = (0, 0)$. Note that, as is easily seen from (1.4), (1.5) and (1.6), there is a non-vanishing holomorphic function $h(t, x)$ in a neighborhood of the origin such that

$$
\varphi(t, x) - \psi(t, x) = th(t, x).
$$

(1.7)

Using the functions $\varphi(t, x)$ and $\psi(t, x)$, we define the function

$$
\Phi^\alpha \beta \lambda(t, x) = \frac{\varphi(t, x)^\lambda}{F(\lambda + 1)} F\left(-\lambda, \beta + \beta; \frac{\varphi(t, x) - \psi(t, x)}{\varphi(t, x)}\right)
$$

(1.8)

having three complex parameters $\alpha$, $\beta$ and $\lambda$, where $F(\cdot, \cdot, \cdot, \cdot)$ denotes the Gauss hypergeometric function. We also define the function

$$
\delta^{-\lambda-\beta} \Phi^\alpha \beta \lambda(t, x) = \Phi^{\alpha-1, \beta+1} \lambda(t, x) - \Phi^\alpha \beta \lambda(t, x),
$$

(1.9)

which is obtained from $\Phi^\alpha \beta \lambda(t, x)$ by operating a difference operator to parameters.

We consider for a positive number $r$ the following form of neighborhood of the origin in $C^2$:

$$
\Omega_r = \{(t, x) \in C^2 \mid |\varphi(t, x)| < r, |\psi(t, x)| < r\}.
$$

(1.10)

We set

$$
\Sigma := \{t = 0\} \cup \{\varphi(t, x) = 0\} \cup \{\psi(t, x) = 0\}.
$$

(1.11)

The main theorem of this paper is stated as follows.

**Theorem 1.1** (Main Theorem). We assume that $\alpha + \beta \notin Z$, $\lambda + \alpha \notin Z$ and $\lambda + \beta \notin Z$. For sufficiently small $r > 0$, the characteristic initial value problem (1.3) admits a unique solution which is holomorphic in $\Omega_r \setminus \Sigma$, the universal covering space of $\Omega_r \setminus \Sigma$. Furthermore, the solution $u(t, x)$ is expressed as

$$
u(t, x) = \sum_{k=0}^{\infty} \{G_k(t, x) \Phi^{\alpha \beta, \lambda \lambda+k}(t, x) + H_k(t, x) \delta^{-\lambda-\beta} \Phi^{\alpha \beta, \lambda \lambda+k+1}(t, x)\},
$$

(1.12)
where the functions \( G_k(t, x) \) and \( H_k(t, x) \) (\( k = 0, 1, 2, \ldots \)) are holomorphic in \( \Omega_r \), and the series in the right-hand side of (1.12) is uniformly convergent on any compact subset in \( \Omega_r \setminus \Sigma \).

**Remark 1.1.** Of course, in the right-hand side of (1.12), the terms \( -H_k \Phi_{\lambda + k + 1}^{a, b} \) in the second part can be transferred to the first part, and we can write it as

\[
(1.12') \quad u(t, x) = \sum_{k=0}^{\infty} \{ G'_k(t, x) \Phi_{\lambda + k}^{a, b}(t, x) + H_k(t, x) \Phi_{\lambda + k + 1}^{a-1, b+1}(t, x) \},
\]

where \( G'_k = G_k - H_{k-1} \) (\( k \geq 1 \)), \( G'_0 = G_0 \). However, as the later discussion shows, the expression (1.12) clarifies the essence of things.

**Remark 1.2.** In the second terms of the series (1.12), we use the functions \( \delta^{-1} \Phi_{\lambda + k + 1}^{a, b} \), whereas Watanabe-Urabe [9], [10] used partial derivatives of hypergeometric functions. Here we employ the above functions obtained from \( \Phi_{\lambda}^{a, b} \) just by integral shift of parameters. The merit of using these functions is that, (1) each term in the series has an invariant meaning under coordinate transformation of \((t, x)\), hence there is no need to worry when coordinate transformations are considered; (2) we can get the behavior of the solution at a glance; (3) the recurrence relations for \( G_k \) and \( H_k \) (\( k = 0, 1, 2, \ldots \)) becomes simple, which will be clarified later.

### 2. Consequences of the Main Theorem

Before proceeding to the proof of the theorem, we will give several corollaries to the theorem.

We will first explain how we can get the behavior of the solution \( u(t, x) \) from the theorem. Set

\[
z = \frac{\varphi(t, x) - \psi(t, x)}{\varphi(t, x)}.
\]

As the relation (1.7) holds for nonvanishing holomorphic function \( h \), we have the correspondence

\[
\begin{cases}
  z = 0 & \iff t = 0, \\
  z = \infty & \iff \varphi = 0, \\
  z = 1 & \iff \psi = 0.
\end{cases}
\]

As is well-known (see Whittaker-Watson [11]), the Gauss hypergeometric function \( F(-\lambda, \beta, x + \beta; z) \) is written by the use of the Riemann \( P \)-function, which
shows exponents at singular points, as
\[
F(-\lambda, \beta, z + \beta; z) \in P \left\{ \begin{array}{ccc}
z = 0 & z = \infty & z = 1 \\
0 & -\lambda & 0 \\
1 - z - \beta & \beta & \lambda + \alpha \\
\end{array} \right\}.
\]
Thus the function \( \Phi_{z,\beta}(t, x) \) is also written in the same manner as
\[
\Phi_{z,\beta}(t, x) = \frac{\varphi^z}{T(\lambda + 1)} F(-\lambda, \beta, z + \beta; \frac{\varphi - \psi}{\varphi})
\]
\[
\in \frac{1}{T(\lambda + 1 + k + 1)} \hat{P} \left\{ \begin{array}{ccc}
t = 0 & \varphi = 0 & \psi = 0 \\
0 & 0 & 0 \\
1 - z - \beta & \lambda + \beta & \lambda + \alpha + k \\
\end{array} \right\},
\]
where instead of Riemann's \( P \) we used the letter \( \hat{P} \), for the scheme \( \hat{P} \{ \} \) only indicates exponents at singularities, having no longer a meaning of solution of ordinary differential equations. We have hence
\[
(2.1) \quad \Phi_{z+k}(t, x)
\]
\[
\in \frac{1}{T(\lambda + k + 1 + 1)} \hat{P} \left\{ \begin{array}{ccc}
t = 0 & \varphi = 0 & \psi = 0 \\
0 & 0 & 0 \\
1 - z - \beta & \lambda + \beta + k & \lambda + \alpha + k \\
\end{array} \right\},
\]
and also
\[
(2.2) \quad \Phi_{z+k+1}(t, x)
\]
\[
\in \frac{1}{T(\lambda + k + 2 + 1)} \hat{P} \left\{ \begin{array}{ccc}
t = 0 & \varphi = 0 & \psi = 0 \\
0 & 0 & 0 \\
1 - z - \beta & \lambda + \beta + k + 2 & \lambda + \alpha + k \\
\end{array} \right\}.
\]
Applying (2.1) and (2.2) to the expression (1.12), we can derive the behavior of the solution \( u(t, x) \) in a neighborhood of a point on \( \Sigma \cap \Omega \setminus \{(0, 0)\} \). For example, let \( (t_0, x_0) \) be a point (on any sheet of multi-valued function) on \( \{\varphi = 0\} \cap \Omega \setminus \{(0, 0)\} \). From theorem 1.1, using the behaviors (2.1) and (2.2), we observe easily that the solution \( u(t, x) \) has the form \( \varphi^{z+\beta} \cdot \hat{\varrho} + \hat{\varrho} \), where \( \hat{\varrho} \) denotes single-valued holomorphic function in a punctured neighborhood of \( (t_0, x_0) \) where \( \{\varphi = 0\} \) is removed. However, since the summation in the right-hand side of (1.12) is taken over nonnegative \( k \), we conclude easily that the function \( \hat{\varrho} \) has in fact only a removable singularity along \( \{\varphi = 0\} \), and we have accordingly \( u(t, x) = \varphi^{z+\beta} \cdot \hat{\varrho} + \hat{\varrho} \), where \( \hat{\varrho} \) denotes holomorphic function near \( (t_0, x_0) \). The same discussion holds for the other cases \( \{t = 0\} \) and \( \{\psi = 0\} \). Thus we have the following corollary.
Corollary 2.1. In addition to the conditions of Theorem 1.1, we assume that \( \lambda \notin \{-1, -2, \ldots\} \). Then the solution \( u(t, x) \) of the initial value problem (1.3) behaves locally near the three singular loci \( \{t = 0\} \), \( \{\varphi = 0\} \) and \( \{\psi = 0\} \) (in all sheets of the multi-valued function) as follows:

\[
\begin{align*}
t^{1-\lambda-\beta} \cdot \mathcal{O} + \mathcal{O} & \quad \text{locally along } \{t = 0\}\setminus\{(0,0)\}, \\
\varphi^{\lambda+\beta} \cdot \mathcal{O} + \mathcal{O} & \quad \text{locally along } \{\varphi = 0\}\setminus\{(0,0)\}, \\
\psi^{\lambda+\beta} \cdot \mathcal{O} + \mathcal{O} & \quad \text{locally along } \{\psi = 0\}\setminus\{(0,0)\},
\end{align*}
\]

where \( \mathcal{O} \) denotes functions holomorphic in a neighborhood of the corresponding point.

Remark 2.1. Since \( \{t = 0\} \) is the initial plane, it is evident that on the initial sheet the solution \( u(t, x) \) is holomorphic on \( \{t = 0\}\setminus\{(0,0)\} \), however, after analytic continuation for example along a contour encircling the set \( \{\varphi = 0\} \) or \( \{\psi = 0\} \) (see Figure 1), generically the regular singular term \( t^{1-\lambda-\beta} \cdot \mathcal{O} \) appears, which follows immediately in view of the monodromy of the Gauss hypergeometric function.

The solution of the case of the problem where the initial data have pole at \( \{x = 0\} \) is obtained by differentiating the above solution with respect to the parameter \( \lambda \) at \( \lambda = \{-1, -2, \ldots\} \). Let us consider the initial value problem

\[
\begin{align*}
Pu(t, x) &= 0, \\
u(0, x) &= \frac{w(x)}{x^n},
\end{align*}
\]

\((n \in \mathbb{N}, w(x) \text{ being holomorphic in a neighborhood of } x = 0)\).

Fig. 1. Contour encircling \( \{\varphi = 0\} \)
The original initial value problem (1.3) reduces to a trivial problem if \( \lambda \in \{-1, -2, \ldots\} \), since in this case

\[
f_{\lambda}(x) = \frac{x^{\lambda}}{\Gamma(\lambda + 1)} = 0.
\]

However, as will be shown in this paper, since the solution \( u(t, x) \) of the original problem (1.3) is entire with respect to the parameter \( \lambda \), we can express and obtain the behavior of the solution of the problem (2.3) by considering differentiation in \( \lambda \) of the solution of the problem (1.3). In fact, note that

\[
\lim_{\lambda \to n} \frac{\partial}{\partial \lambda} f_{\lambda}(x) = \lim_{\lambda \to n} \frac{\partial}{\partial \lambda} \left( \frac{x^{\lambda}}{\Gamma(\lambda + 1)} \right) = (-1)^{n-1}(n-1)! \frac{1}{x^n}.
\]

Hence the solution of the initial value problem (2.3) is written as

\[
\frac{(-1)^{n-1}}{(n-1)!} \lim_{\lambda \to n} \frac{\partial}{\partial \lambda} u(t, x; \lambda),
\]

\( u(t, x; \lambda) \) denoting the solution of the original problem (1.3) with entire parameter \( \lambda \). Combining this expression with Theorem 1.1 and Corollary 2.1, we have the following result concerning the initial value problem (2.3).

**Corollary 2.2.** We assume that \( \alpha \notin \mathbb{Z}, \beta \notin \mathbb{Z} \) and \( \alpha + \beta \notin \mathbb{Z} \). For sufficiently small \( r > 0 \), the characteristic initial value problem (2.3) admits a unique solution which is holomorphic in \( \Omega \setminus \Sigma \). Furthermore, the solution \( u(t, x) \) behaves locally near the three singular loci \( \{ t = 0 \}, \{ \varphi = 0 \} \) and \( \{ \psi = 0 \} \) (in all sheets of the multi-valued function) as follows:

\[
t^{1-r-\beta} \cdot c + c \quad \text{locally along} \quad \{ t = 0 \} \setminus \{(0,0)\},
\]

\[
\varphi^{\beta-n} \cdot (c + c \log \varphi) + c \quad \text{locally along} \quad \{ \varphi = 0 \} \setminus \{(0,0)\},
\]

\[
\psi^{n-\beta} \cdot (c + c \log \psi) + c \quad \text{locally along} \quad \{ \psi = 0 \} \setminus \{(0,0)\},
\]

where \( c \) denotes functions holomorphic in a neighborhood of the corresponding point.

Moreover, applying higher order differentiation with respect to \( \lambda \), we can treat the following initial value problem having logarithmic singular initial data, namely

\[
\begin{cases}
Pu(t, x) = 0, \\
u(0, x) = \frac{w(x)}{x^n}(\log x)^k.
\end{cases}
\]
(n \in \mathbb{N}, k \in \mathbb{N}, w(x) \text{ being holomorphic in a neighborhood of } x = 0). \text{ In fact, note that}

\[
\lim_{\lambda \to -n} \left( \frac{\partial}{\partial \lambda} \right)^{k+1} f_\lambda(x)
= \frac{1}{x^n} \left\{ (-1)^{n-1} (n-1)! (k+1)(\log x)^k + \sum_{j=0}^{k-1} \mathcal{O}(\log x)^j \right\},
\]

where \( \mathcal{O} \) denotes functions holomorphic in a neighborhood of \( x = 0 \). Hence the function \( v(t, x) \) defined by

\[
v(t, x) := u(t, x) - \frac{(-1)^{n-1}}{(k+1)(n-1)!} \lim_{\lambda \to -n} \left( \frac{\partial}{\partial \lambda} \right)^{k+1} u(t, x; \lambda),
\]

where \( u(t, x) \) and \( u(t, x; \lambda) \) are the solutions of (2.4) and (1.3) respectively, satisfies the equation \( P v = 0 \) and the initial condition of the form

\[
v(0, x) = \frac{1}{x^n} \sum_{j=0}^{k-1} \mathcal{O}(\log x)^j,
\]

the right-hand side being reduced to a polynomial of degree \( k - 1 \) in \( \log x \). Thus we obtain the following corollary by induction on \( k \).

**Corollary 2.3.** We assume that \( \alpha \notin \mathbb{Z}, \beta \notin \mathbb{Z} \) and \( \alpha + \beta \notin \mathbb{Z} \). For sufficiently small \( r > 0 \), the characteristic initial value problem (2.4) admits a unique solution which is holomorphic in \( \Omega_r \setminus \Sigma \). The solution \( u(t, x) \) behaves locally near the three singular loci \( \{ t = 0 \}, \{ \varphi = 0 \} \) and \( \{ \psi = 0 \} \) (in all sheets of the multi-valued function) as follows:

\[
l^{1-\alpha-\beta} \cdot \mathcal{O} + \mathcal{O} \quad \text{locally along } \{ t = 0 \} \setminus \{(0,0)\},
\]

\[
\varphi^\beta \cdot \sum_{j=0}^{k+1} \mathcal{O}(\log \varphi)^j + \mathcal{O} \quad \text{locally along } \{ \varphi = 0 \} \setminus \{(0,0)\},
\]

\[
\psi^{\alpha} \cdot \sum_{j=0}^{k+1} \mathcal{O}(\log \psi)^j + \mathcal{O} \quad \text{locally along } \{ \psi = 0 \} \setminus \{(0,0)\},
\]

where \( \mathcal{O} \) denotes functions holomorphic in a neighborhood of the corresponding point.

3. **Reduction to normal form**

First of all, we will reduce the initial value problem (1.3) to a normal form by means of biholomorphic coordinate transformation in a neighborhood of \( (t, x) = (0,0) \).
For this purpose, we apply the coordinate transformation \((t, x) \rightarrow (t_1, x_1)\) defined by

\[
\begin{align*}
t_1 &= \frac{1}{2} \{\varphi(t, x) - \psi(t, x)\}, \\
x_1 &= \frac{1}{2} \{\varphi(t, x) + \psi(t, x)\},
\end{align*}
\]

or equivalently

\[
\begin{align*}
x_1 + t_1 &= \varphi(t, x), \\
x_1 - t_1 &= \psi(t, x).
\end{align*}
\]

This transformation is in fact biholomorphic in a neighborhood of \((t, x) = (0, 0)\) because its Jacobian at \((t, x) = (0, 0)\) is calculated by the use of (1.5) and (1.6) as

\[
\frac{\partial (t_1, x_1)}{\partial (t, x)} \big|_{(0, 0)} = \frac{1}{2} (q(0, 0) - p(0, 0)),
\]

which is nonzero from (1.4). Note that the first equality of (3.1) is also written as

\[
t_1 = t \cdot \frac{1}{2} h
\]

by (1.7).

Since

\[
\begin{align*}
\frac{\partial}{\partial t} &= \frac{1}{2} (\varphi_t - \psi_t) \frac{\partial}{\partial t_1} + \frac{1}{2} (\varphi_t + \psi_t) \frac{\partial}{\partial x_1}, \\
\frac{\partial}{\partial x} &= \frac{1}{2} (\varphi_x - \psi_x) \frac{\partial}{\partial t_1} + \frac{1}{2} (\varphi_x + \psi_x) \frac{\partial}{\partial x_1},
\end{align*}
\]

we have by the use of (1.5) and (1.6)

\[
X_p = -\frac{1}{2} (\psi_t + p\psi_x) \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial x_1}\right) = \frac{1}{2} (q - p) \varphi_x \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial x_1}\right),
\]

and similarly we have

\[
X_q = \frac{1}{2} (q - p) \varphi_x \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial x_1}\right).
\]

Thus, introducing the notation

\[
X^1_+ = \frac{\partial}{\partial t_1} - \frac{\partial}{\partial x_1}, \quad X^1_- = \frac{\partial}{\partial t_1} + \frac{\partial}{\partial x_1},
\]
and letting
\[ g_-(t, x) = \frac{1}{2}(q - p)\psi_x, \quad g_+(t, x) = \frac{1}{2}(q - p)\varphi_x, \]
we have
\[
\begin{align*}
\left\{ \begin{array}{l}
X_p = g_-(t, x)X_1^-, \\
X_q = g_+(t, x)X_1^+.
\end{array} \right.
\end{align*}
\]
(3.4)

Note that, by (1.4), (1.5) and (1.6), the multipliers \( g_- \) and \( g_+ \) are nonzero holomorphic functions with the property that
\[
g_-|_{t=0} = g_+|_{t=0} = \frac{1}{2}(q - p)|_{t=0}.
\]

Since \( t = 2h^{-1}t_1 \) by (3.3), we have
\[
P = 2h^{-1}t_1 \cdot g_- X_1^- \cdot g_+ X_1^+ + (x + t_1 2h^{-1} a)g_- X_1^-
+ (\beta + t_1 2h^{-1} b)g_+ X_1^+ + c
= 2h^{-1}g_- g_+ \cdot t_1 X_1^- X_1^+ + (x + t_1 2h^{-1} a)g_- X_1^-
+ \{\beta + t_1 2h^{-1}(b + g_-^{-1} X_1^- g_+)\} g_+ X_1^+ + c.
\]

On the other hand, we have from (1.7), (1.5) and (1.6)
\[
h|_{t=0} = (\varphi_t - \psi_t)|_{t=0} = (-p\varphi_x + q\psi_x)|_{t=0} = (q - p)|_{t=0}.
\]
Hence it holds that
\[
g_-|_{t=0} = g_+|_{t=0} = 2^{-1}h|_{t=0} = (2h^{-1}g_- g_+)|_{t=0}.
\]

Of course, this identity is also true for the restriction \( \{t = 0\} \) replaced by \( \{t_1 = 0\} \). Paying attention to this identity, we divide the above expression of \( P \) by \( 2h^{-1}g_- g_+ \), and we express the resulting operator also by \( P \) (note that, since the equation \( Pu = 0 \) is homogeneous, such division makes no change to the equation). Thus the operator \( P \) reduces to the normal form
\[
P = t_1 X_1^- X_1^+ + (x + t_1 a_1) X_1^- + (\beta + t_1 b_1) X_1^+ + c_1,
\]
where \( a_1, \ b_1, \) and \( c_1 \) are holomorphic functions in a neighborhood of \( (t_1, x_1) \).

We will mention several remarks. The initial plane \( \{t = 0\} \) is mapped to \( \{t_1 = 0\} \) by the transformation \( (t, x) \rightarrow (t_1, x_1) \). The expression (3.5) shows that the transformed operator \( P \) written in the new variables \( (t_1, x_1) \) still belongs to the same class of operators as (1.2), and moreover, the parameters \( \alpha \) and \( \beta \) remain unchanged. Furthermore, since the transformation \( (t_1, x_1) = (t(t, x), x) \),
4. Main theorem in the case of the normal form

We rewrite the normal form of the operator $P$ in (3.5) by omitting the script as

$$P = tX_-X_+ + (x + ta)X_- + (\beta + tb)X_+ + c,$$

where $a$, $b$ and $c$ are holomorphic functions in a neighborhood of $(0,0)$ and

$$X_- = \frac{\partial}{\partial t} - \frac{\partial}{\partial x}, \quad X_+ = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}.$$

By the words principal part of the operator $P$ we mean the operator

$$P_0 = tX_-X_+ + aX_- + \beta X_+.$$

First of all, we will give the explicit form of the solution to the initial value problem

$$\begin{cases}
P_0u(t,x) = 0, \\
u(0,x) = f_\lambda(x) \left( = \frac{x^2}{\Gamma(\lambda + 1)} \right).
\end{cases}$$

Since the principal symbol of the operator $P_0$ equals $\sigma(P_0) = t(\tau - \xi)(\tau + \xi)$, where $\tau$ and $\xi$ denotes the dual variables of $t$ and $x$ respectively, we have three characteristic lines passing through the origin, namely

$$t = 0, \quad x + t = 0, \quad x - t = 0$$

(4.4)

corresponding to the three factors $t$, $\tau - \xi$, $\tau + \xi$ respectively. We introduce the new variable

$$z = \frac{2t}{x + t}.$$

Then the three lines (4.4) correspond to the three points

$$z = 0, \quad z = \infty, \quad z = 1$$

respectively.

Now we seek the solution of the initial value problem (4.3) in the form

$$u = \frac{(x + t)^2}{\Gamma(\lambda + 1)} v \left( \frac{2t}{x + t} \right) = f_\lambda(x + t)v(z).$$
First of all, we have from the initial condition of (4.3)
\begin{equation}
(4.5)
\quad v(0) = 1.
\end{equation}
After an easy calculation by the use of the identities
\[ \frac{d}{d\xi} f_{\lambda}(\xi) = f_{\lambda-1}(\xi) \]
and
\[ \xi f_{\lambda-1}(\xi) = \lambda f_{\lambda}(\xi), \]
we have
\begin{equation}
(4.6)
\quad X_-u = \frac{2}{\lambda} f_{\lambda-1}(x + t)v'(z),
\end{equation}
\begin{equation}
(4.7)
\quad X_+u = 2f_{\lambda-1}(x + t)v(z) + \frac{2}{\lambda} f_{\lambda-1}(x + t)(1 - z)v'(z).
\end{equation}
Further calculation shows that
\begin{equation}
(4.8)
\quad tX_-X_+u = \frac{2(\lambda - 1)}{\lambda} f_{\lambda-1}(x + t)zv'(z) + \frac{2}{\lambda} f_{\lambda-1}(x + t)z(1 - z)v''(z).
\end{equation}
It follows from (4.6), (4.7) and (4.8) that the equation \( P_0u = 0 \) is equivalent to the ordinary differential equation
\begin{equation}
(4.9)
\quad z(1 - z)v''(z) + \{\alpha + \beta - (1 - \lambda + \beta)z\}v'(z) + \lambda \beta v(z) = 0.
\end{equation}
As the equation (4.9) is the Gauss hypergeometric differential equation, the solution satisfying the initial condition (4.5) is expressed by the use of the Gauss hypergeometric function as
\[ v(z) = F(-\lambda, \beta, \alpha + \beta; z). \]
We will denote by \( \Phi^{x,\beta}_{\lambda}(t, x) \) the unique solution of the initial value problem (4.3) for the principal part \( P_0 \). The above discussion gives the concrete expression
\begin{equation}
(4.10)
\quad \Phi^{x,\beta}_{\lambda}(t, x) = \frac{(x + t)^{\lambda}}{F(\lambda + 1)} \left[ F(-\lambda, \beta, \alpha + \beta; \frac{2t}{x + t}) \right].
\end{equation}
Observe that the function \( \Phi^{x,\beta}_{\lambda}(t, x) \) is holomorphic in the universal covering space of \( \mathbb{C}^2 \setminus \{ t = 0 \} \cup \{ x + t = 0 \} \cup \{ x - t = 0 \} \).
Now we proceed to consider the initial value problem for the operator \( P \) of the normal form (4.1):
\begin{equation}
(4.11)
\begin{cases}
Pu(t, x) = 0, \\
u(0, x) = f_{\lambda}(x)w(x),
\end{cases}
\end{equation}
where the function \( w(x) \) is holomorphic in a neighborhood of \( x = 0 \), and \( f_{\lambda}(x) = x^{\lambda}/\Gamma(\lambda + 1) \) as before. We set
\[
\Omega_r = \{(t, x) \in \mathbb{C}^2 \mid |x + t| < r, |x - t| < r\}.
\]
We also set
\[
\Sigma := \{t = 0\} \cup \{x + t = 0\} \cup \{x - t = 0\}.
\]
We use as before the notation
\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial t^2} + \frac{\partial}{\partial x} + \frac{\partial}{\partial \lambda}.
\]

Then, for the normal form operator \( P \), Theorem 1.1 takes the following form.

**Theorem 4.1.** We assume that \( \alpha + \beta \notin \mathbb{Z}, \lambda + x \notin \mathbb{Z} \) and \( \lambda + \beta \notin \mathbb{Z} \). For sufficiently small \( r > 0 \), the characteristic initial value problem (4.11) admits a unique solution which is holomorphic in \( \Omega_r \setminus \Sigma \), the universal covering space of \( \Omega_r \setminus \Sigma \). Moreover, the solution \( u(t, x) \) is expressed as
\[
(4.13) \quad u(t, x) = \sum_{k=0}^{\infty} \left\{ G_k(t, x) \Phi^{z, \beta}_{\lambda+k}(t, x) + H_k(t, x) \delta_{\lambda}^{z,-1} \Phi^{z, \beta}_{\lambda+k+1}(t, x) \right\},
\]
where the functions \( G_k(t, x) \) and \( H_k(t, x) \) (\( k = 0, 1, 2, \ldots \)) are holomorphic in \( \Omega_r \), and the series in the right-hand side of (4.13) is uniformly convergent on any compact subset in \( \Omega_r \setminus \Sigma \).

It is evident from the relation (3.2) that Theorem 4.1 implies Theorem 1.1. Observe that we may consider the coordinate transformation ‘termwise’ in the above series. This is the merit of our representation pointed out as merit (1) in Remark 1.2.

Thus all we have to do is to prove Theorem 4.1. Hence, hereafter, we mean the operator \( P \) the normal form (4.1), and \( P_0 \) its principal part.

5. **Formulas for the functions \( \Phi^{z, \beta}_{\lambda}(t, x) \)**

Recall that the function \( \Phi^{z, \beta}_{\lambda}(t, x) \) given by (4.10) is the solution of the initial value problem (4.3). We will show several formulas for these functions \( \Phi^{z, \beta}_{\lambda}(t, x) \). These formulas will be effectively used in the next section for the construction of the series expression (4.13).

**Proposition 5.1** (contiguity relation with respect to \( \lambda \)).
\[
(5.1) \quad \partial_{\lambda} \Phi^{z, \beta}_{\lambda} = \Phi^{z, \beta}_{\lambda-1}.
\]
Proof. Since it holds that \( P_0 \partial_x = \partial_x P_0 \) as operators, it follows from differentiating both sides of the equation \( P_0 \Phi_{\lambda, \beta}^{x, \beta} = 0 \) by \( x \) that
\[
P_0[\partial_x \Phi_{\lambda}^{x, \beta}] = 0.
\]

It also holds by differentiating both sides of the initial condition \( \Phi_{\lambda}^{x, \beta}(0,x) = f_\lambda(x) \) by \( x \) that
\[
\partial_x \Phi_{\lambda}^{x, \beta}(0,x) = f_{\lambda-1}(x).
\]

Hence the function \( \partial_x \Phi_{\lambda}^{x, \beta} \) satisfies the same initial value problem as \( \Phi_{\lambda}^{x, \beta} \). From the uniqueness of the solution, we conclude that \( \partial_x \Phi_{\lambda}^{x, \beta} = \Phi_{\lambda-1}^{x, \beta} \).

\( \square \)

**Proposition 5.2** (contiguity relation with respect to \( \alpha \) and \( \beta \) (i)).

\[
(tX_+ + \beta) \Phi_{\lambda}^{x, \beta} = \beta \Phi_{\lambda}^{x-1, \beta+1}.
\]

**Proof.** Since
\[
\Phi_{\lambda}^{x, \beta} = \frac{(x + t)^{\lambda}}{\Gamma(\lambda + 1)} F\left(-\lambda, \beta, \alpha + \beta, \frac{2t}{x + t}\right)
\]
\[
= \frac{1}{\Gamma(\lambda + 1)} \sum_{n=0}^{\infty} \frac{(-\lambda)_n (\beta)_n}{(\alpha + \beta)_n n!} (2t)^n (x + t)^{\lambda - n},
\]
we have
\[
(tX_+ + \beta) \Phi_{\lambda}^{x, \beta} = \frac{1}{\Gamma(\lambda + 1)} \sum_{n=0}^{\infty} (n + \beta) \frac{(-\lambda)_n (\beta)_n}{(\alpha + \beta)_n n!} (2t)^n (x + t)^{\lambda - n}
\]
\[
= \frac{1}{\Gamma(\lambda + 1)} \sum_{n=0}^{\infty} \beta \frac{(-\lambda)_n (\beta + 1)_n}{(\alpha + \beta + 1)_n n!} (2t)^n (x + t)^{\lambda - n}
\]
\[
= \beta \Phi_{\lambda}^{x-1, \beta+1}. \quad \square
\]

Proposition 5.2 shows that the partial differential operator \( tX_+ + \beta \) operates on \( \Phi_{\lambda}^{x, \beta} \) as a parameter shifting operator \( \alpha \to \alpha - 1, \beta \to \beta + 1 \). On the other hand, as the following proposition shows, the partial differential operator \( tX_+ + \alpha \) operates on \( \Phi_{\lambda}^{x, \beta} \) as a parameter shifting operator \( \alpha \to \alpha + 1, \beta \to \beta - 1 \).

**Proposition 5.3** (contiguity relation with respect to \( \alpha \) and \( \beta \) (ii)).

\[
(tX_+ + \alpha) \Phi_{\lambda}^{x, \beta} = \alpha \Phi_{\lambda}^{x+1, \beta-1}.
\]

**Proof.** The following discussion is known as the method of factor decomposition (Inui [4], however in Japanese). To begin with, we rewrite the operator \( P_0 \) as
\[ P_0 = tX_+X_+ + \alpha X_+ + \beta X_+ \]
\[ = (X_+t - 1)X_+ + \alpha X_+ + \beta X_+ \]
\[ = X_+(tX_+ + \beta) + (\alpha - 1)X_. \]

Hence we have
\[ tP_0 = tX_+(tX_+ + \beta) + (\alpha - 1)tX_+ \]
\[ = (tX_+ + \alpha - 1)(tX_+ + \beta) - (\alpha - 1)\beta. \]

Now we operate both sides of this identity on \( \Phi^{\alpha,\beta}_x \). Using the contiguity relation (5.2), we get the identity
\[ 0 = (tX_+ + \alpha - 1)\beta \Phi^{\alpha-1,\beta+1}_x - (\alpha - 1)\beta \Phi^{\alpha,\beta}_x. \]

Since all the appearing functions are analytic, we may divide the equality by \( \beta \), and accordingly we have
\[ (tX_+ + \alpha - 1)\Phi^{\alpha-1,\beta+1}_x = (\alpha - 1)\Phi^{\alpha,\beta}_x. \]

We obtain the desired identity by translating \( \alpha - 1 \to \alpha \) and \( \beta + 1 \to \beta \).

In addition to the notation (4.12), we will also use the notation
\[ \delta^{+, -}\Phi^{\alpha,\beta}_x(t, x) := \Phi^{\alpha+1,\beta-1}_x(t, x) - \Phi^{\alpha,\beta}_x(t, x). \]

Then the identities (5.2) and (5.3) are written as follows.

**Corollary 5.4.**

(5.5) \[ tX_- \Phi^{\alpha,\beta}_x = \beta \delta^{-, +}\Phi^{\alpha,\beta}_x, \]

(5.6) \[ tX_+ \Phi^{\alpha,\beta}_x = \alpha \delta^{+, -}\Phi^{\alpha,\beta}_x. \]

Thus, as operators operating on \( \Phi^{\alpha,\beta}_x \), the partial differential operator \( tX_- \) (resp. \( tX_+ \)) is identical to the parameter difference operator \( \delta^{-, +} \) (resp. \( \delta^{+, -} \)) multiplied by \( \beta \) (resp. \( \alpha \)).

There is also a relation not containing derivatives.

**Corollary 5.5.**

(5.7) \[ \alpha \delta^{+, -}\Phi^{\alpha,\beta}_x = \beta \delta^{-, +}\Phi^{\alpha,\beta}_x + 2t \Phi^{\alpha,\beta}_{t-1}. \]

**Proof.** Since it holds that \( X_+ = X_- + 2\hat{c}_x \), we obtain the desired identity from the relations (5.5), (5.6) and (5.1).

**Remark 5.1.** In Theorem 4.1 (and also in Theorem 1.1), the operator \( \delta^{-, +} \) was used to express the solution, however, in view of the identity (5.7), it is evident that we may use \( \delta^{+, -} \) instead of \( \delta^{-, +} \).
It is also necessary to know how the contiguous functions \( \Phi_{\lambda}^{x-1,\beta+1} \) and \( \Phi_{\lambda}^{x+1,\beta-1} \) change by operating \( P_0 \). For this purpose, we first show the following lemma.

**Lemma 5.6.**

(5.8) \[
P_0[tX_- \Phi_{\lambda}^{x,\beta}] = -2\beta\Phi_{\lambda-1}^{x-1,\beta+1},
\]

(5.9) \[
P_0[tX_+ \Phi_{\lambda}^{x,\beta}] = 2\alpha\Phi_{\lambda-1}^{x+1,\beta-1}.
\]

**Proof.** Since it holds as operators that

\[
X_+ t = tX_+, \quad X_- t = tX_-,
\]

we have

\[
P_0 \cdot tX_- = (tX_- + \alpha X_- + \beta X_+ + \beta X_-) \cdot tX_- = tX_- (tX_- + \alpha (tX_- + 1) X_- + \beta (tX_- + 1) X_-) = tX_- (tX_- + \alpha X_- + \beta X_+ + tX_- X_- + \alpha X_- + \beta X_-) = tX_- P_0 + P_0 - 2(tX_- + \beta) \xi,
\]

Hence we have by the use of the identities (5.1) and (5.2)

\[
P_0[tX_- \Phi_{\lambda}^{x,\beta}] = -2(tX_- + \beta)\Phi_{\lambda-1}^{x,\beta} = -2\beta\Phi_{\lambda-1}^{x-1,\beta+1},
\]

which proves (5.8).

In the same way, since

\[
P_0 \cdot tX_+ = (tX_+ + \alpha X_- + \beta X_+ + \beta X_-) \cdot tX_+ = tX_+ (tX_+ + \alpha (tX_+ + 1) X_+ + \beta (tX_+ + 1) X_+) = tX_+ (tX_+ + \alpha X_- + \beta X_+ + tX_+ X_+ + \alpha X_+ + \beta X_+) = tX_+ P_0 + P_0 + 2(tX_+ + \alpha) \xi,
\]

it follows from (5.1) and (5.3) that

\[
P_0[tX_+ \Phi_{\lambda}^{x,\beta}] = 2(tX_+ + \alpha)\Phi_{\lambda-1}^{x,\beta} = 2\alpha\Phi_{\lambda-1}^{x+1,\beta-1},
\]

concluding the lemma. \( \square \)

**Proposition 5.7.**

(5.10) \[
P_0[\Phi_{\lambda}^{x-1,\beta+1}] = -2\Phi_{\lambda-1}^{x-1,\beta+1},
\]

(5.11) \[
P_0[\Phi_{\lambda}^{x+1,\beta-1}] = 2\Phi_{\lambda-1}^{x+1,\beta-1}.
\]
Proof. Operating $P_0$ on both sides of (5.2), and using the identity (5.8), we obtain the equality

$$-2\beta \Phi^{x-1,\beta+1}_{\delta-1} = \beta P_0[\Phi^{x-1,\beta+1}_{\delta-1}].$$

Since both sides of the above are analytic, we may divide the equality by $\beta$, concluding (5.10). The identity (5.11) also follows from the same discussion using (5.3) and (5.9).

As an immediate consequence of Proposition 5.7, we have the following identities concerning $P_0[\delta^{-,-}\Phi^x_\delta]$ and $P_0[\delta^{+,-}\Phi^x_\delta]$.

**Corollary 5.8.**

(5.12) \hspace{1cm} P_0[\delta^{-,-}\Phi^x_\delta] = -2\Phi^{x-1,\beta+1}_{\delta-1},

(5.13) \hspace{1cm} P_0[\delta^{+,-}\Phi^x_\delta] = 2\Phi^{x+1,\beta-1}_{\delta-1}.

Besides the formulas (5.5) and (5.6), we need the following formulas, which implies that the functions $tX^{-,-}\Phi^x_{\delta+1}$ and $tX^{+,-}\Phi^x_{\delta+1}$ are expressed as linear combinations of three functions $\delta^{-,-}\Phi^x_{\delta+1}$, $\Phi^x_{\delta+1}$ and $\delta^{+,-}\Phi^x_{\delta+1}$.

**Proposition 5.9.**

(5.14) \hspace{1cm} tX^{-,-}\Phi^x_{\delta+1} = (1-\alpha-\beta)\delta^{-,-}\Phi^x_{\delta+1} - 2t\Phi^x_{\delta+1} - 2t\delta^{-,-}\Phi^x_{\delta+1},

(5.15) \hspace{1cm} tX^{+,-}\Phi^x_{\delta+1} = (1-\alpha-\beta)\delta^{+,-}\Phi^x_{\delta+1} - 2t\Phi^x_{\delta+1}.

**Proof.** We will first show the relation (5.15). The left-hand side of (5.15) is calculated by the use of (5.1), (5.5) and (5.6) as follows:

$$tX^{-,-}\Phi^x_{\delta+1} = tX_{+}\Phi^{x-1,\beta+1}_{\delta+1} - \Phi^x_{\delta+1}]$$

$$= tX_{+}\Phi^{x-1,\beta+1}_{\delta+1} - (t\delta^{-,-})\Phi^x_{\delta+1}$$

$$= (\alpha-1)\delta^{-,-}\Phi^{x-1,\beta+1}_{\delta+1} - \beta\delta^{-,-}\Phi^x_{\delta+1} - 2t\Phi^x_{\delta+1}$$

$$= (1-\alpha)\delta^{-,-}\Phi^x_{\delta+1} - \beta\delta^{-,-}\Phi^x_{\delta+1} - 2t\Phi^x_{\delta+1}$$

$$= (1-\alpha-\beta)\delta^{-,-}\Phi^x_{\delta+1} - 2t\Phi^x_{\delta+1}.

This proves (5.15). The equality (5.14) follows from (5.15), for

$$tX^{-,-}\Phi^x_{\delta+1} = t(X_{-} - 2\delta^{-,-})\Phi^x_{\delta+1}$$

$$= tX_{+}\Phi^{x-1,\beta+1}_{\delta+1} - 2t\delta^{-,-}\Phi^x_{\delta+1}.$$
We also use the following relations concerning the restriction to the initial plane \( \{ t = 0 \} \).

**Proposition 5.10.**

\[
\delta^{-,+} \Phi^x,\beta_{\lambda}(0, x) = 0,
\]

\[
\partial_t \Phi^x,\beta_{\lambda}(0, x) = \frac{\alpha - \beta}{\alpha + \beta} f_{\lambda - 1}(x),
\]

\[
\partial_t [\delta^{-,+} \Phi^x,\beta_{\lambda}](0, x) = - \frac{2}{\alpha + \beta} f_{\lambda - 1}(x).
\]

**Proof.** Since

\[
\Phi^x,\beta_{\lambda} = \frac{(x + t)^{\lambda}}{\Gamma(\lambda + 1)} \sum_{n=0}^{\infty} \frac{(-\lambda)_n (\beta)_n}{(\alpha + \beta)_n n!} \left( \frac{2t}{x + t} \right)^n
\]

and accordingly

\[
\Phi^{x-1,\beta+1} = \frac{(x + t)^{\lambda}}{\Gamma(\lambda + 1)} \left\{ 1 + \frac{(-\lambda)(\beta + 1)}{\alpha + \beta} \frac{2t}{x + t} + \text{higher order terms} \right\},
\]

it holds by subtraction that

\[
\delta^{-,+} \Phi^x,\beta_{\lambda} = \frac{(x + t)^{\lambda}}{\Gamma(\lambda + 1)} \left\{ \frac{-\lambda}{\alpha + \beta} \frac{2t}{x + t} + \text{higher order terms} \right\},
\]

from which the relation (5.16) follows immediately. The equality (5.17) follows from (5.19), for

\[
\partial_t \Phi^x,\beta_{\lambda}(0, x) = \frac{\lambda x^{\lambda - 1}}{\Gamma(\lambda + 1)} + \frac{x^{\lambda}}{\Gamma(\lambda + 1)} \cdot \left( \frac{-\lambda}{\alpha + \beta} \frac{2t}{x + t} \right)
\]

\[
= \frac{\lambda x^{\lambda - 1}}{\Gamma(\lambda + 1)} \left( 1 - \frac{2\beta}{\alpha + \beta} \right)
\]

\[
= \frac{\alpha - \beta}{\alpha + \beta} f_{\lambda - 1}(x).
\]

By the translation \( \alpha \to \alpha - 1, \beta \to \beta + 1 \) to this relation, we have

\[
\partial_t \Phi^{x-1,\beta+1}_{\lambda}(0, x) = \frac{\alpha - \beta - 2}{\alpha + \beta} f_{\lambda - 1}(x),
\]

and accordingly we obtain the desired relation (5.18).
6. Recurrence relations for $G_k$ and $H_k$

We assume formally that the solution $u(t, x)$ of the initial value problem (4.11) has the expression of the form (4.13), and seek for recurrence relations for the coefficient functions $G_k(t, x)$ and $H_k(t, x)$ ($k = 0, 1, 2, \ldots$).

To begin with, as a preparation, we try operating $P$, the normal form operator (4.1), to the product $uv$ of two arbitrary functions $u$ and $v$. From the trivial relations

$$\begin{align*}
(t X_+ X_)[uv] = t X_+ u \cdot v + u \cdot t X_+ v \\
(x + ta) X_- [uv] = (x + ta) X_- u \cdot v + u \cdot x X_- v + au \cdot t X_- v, \\
(\beta + tb) X_+ [uv] = (\beta + tb) X_+ u \cdot v + u \cdot \beta X_+ v + bu \cdot t X_+ v,
\end{align*}$$

it follows by adding together that

$$P[uv] = Pu \cdot v + u \cdot P_0 v + (X_- + b)u \cdot t X_+ v + (X_+ + a)u \cdot t X_- v.$$ 

Now we operate $P$ to the product $G\Phi_{\lambda}^{x, \beta}$, where $G$ is an arbitrary function.

**Lemma 6.1.**

$$P[G\Phi_{\lambda}^{x, \beta}] = P[G] \cdot \Phi_{\lambda}^{x, \beta} + \beta(2\partial_t + a + b)G \cdot \delta^{-, +} \Phi_{\lambda}^{x, \beta} + 2t(X_- + b)G \cdot \Phi_{\lambda-1}^{x, \beta}.$$ 

**Proof.** Since $P_0\Phi_{\lambda}^{x, \beta} = 0$, it follows from (6.1) that

$$P[G\Phi_{\lambda}^{x, \beta}] = P[G] \cdot \Phi_{\lambda}^{x, \beta} + (X_- + b)G \cdot t X_+ \Phi_{\lambda}^{x, \beta} + (X_+ + a)G \cdot t X_- \Phi_{\lambda}^{x, \beta},$$

and by the formulas (5.5) and (5.6)

$$= P[G] \cdot \Phi_{\lambda}^{x, \beta} + (X_- + b)G \cdot 2\delta^{+, -} \Phi_{\lambda}^{x, \beta} + (X_+ + a)G \cdot \beta \delta^{-, +} \Phi_{\lambda}^{x, \beta}.$$ 

We substitute in the right-hand side

$$2\delta^{+, -} \Phi_{\lambda}^{x, \beta} = 2\delta^{-, +} \Phi_{\lambda}^{x, \beta} + 2t\Phi_{\lambda-1}^{x, \beta}$$

by (5.7), and use the obvious identity

$$(X_- + b) + (X_+ + a) = 2\partial_t + a + b,$$

then we obtain the desired equality. 

We also operate $P$ to the product $H\delta^{+, -} \Phi_{\lambda+1}^{x, \beta}$, where $H$ is an arbitrary function.
Lemma 6.2.

\[ P[H\delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta}] = \{ P + (1 - \alpha - \beta)(2\hat{c}_r + a + b) \} H \cdot \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} \]

\[-2\{t(2\hat{c}_r + a + b) + 1\} H \cdot \Phi_{\lambda}^{\alpha,\beta} \]

\[-2(tX_+ + 1 + t\alpha) H \cdot \delta^{-\alpha} \Phi_{\lambda}^{\alpha,\beta}. \]

Proof. Since

\[ P_0[\delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta}] = -2\Phi_{\lambda}^{\alpha-1,\beta+1} \]

by (5.12), it follows from (6.1) that

\[ P[H\delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta}] = P[H] \cdot \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} - 2H \cdot \Phi_{\lambda+1}^{\alpha,\beta} - 2H \cdot \delta^{-\alpha} \Phi_{\lambda}^{\alpha,\beta} \]

\[ + (X_- + b) H \cdot tX_+ \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} + (X_+ + a) H \cdot tX_- \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta}, \]

and by the use of the formulas (5.14) and (5.15)

\[ = P[H] \cdot \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} - 2H \cdot \Phi_{\lambda+1}^{\alpha,\beta} - 2H \cdot \delta^{-\alpha} \Phi_{\lambda}^{\alpha,\beta} \]

\[ + (X_- + b) H \cdot \{(1 - \alpha - \beta)\delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} - 2t\Phi_{\lambda}^{\alpha,\beta}\} \]

\[ + (X_+ + a) H \cdot \{(1 - \alpha - \beta)\delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} - 2t\Phi_{\lambda}^{\alpha,\beta} - 2t\delta^{-\alpha} \Phi_{\lambda}^{\alpha,\beta}\} \]

\[ = \{ P + (1 - \alpha - \beta)(2\hat{c}_r + a + b) \} H \cdot \delta^{-\alpha} \Phi_{\lambda+1}^{\alpha,\beta} \]

\[-2\{t(2\hat{c}_r + a + b) + 1\} H \cdot \Phi_{\lambda}^{\alpha,\beta} \]

\[-2(tX_+ + 1 + t\alpha) H \cdot \delta^{-\alpha} \Phi_{\lambda}^{\alpha,\beta}. \]

Thus the lemma is proved. \( \square \)

Now, we assume that the solution \( u \) of the equation \( Pu = 0 \) is expressed by the series in the right-hand side of (4.13). Substituting the expression (4.13) to the equation \( Pu = 0 \), by a formal calculation using Lemma 6.1 and Lemma 6.2, we obtain the following equation:

\[ \sum_{k=0}^{\infty} P[G_k] \cdot \Phi_{\lambda+k}^{\alpha,\beta} \]

\[ + \sum_{k=0}^{\infty} \beta(2\hat{c}_r + a + b) G_k \cdot \delta^{-\alpha} \Phi_{\lambda+k}^{\alpha,\beta} \]
\[
+ \sum_{k=0}^{\infty} 2t(\partial_t - \partial_x + b)G_{k-1} \cdot \Phi_{\lambda + k-1}^{x,\beta} \\
+ \sum_{k=0}^{\infty} \left\{ P + (1 - \alpha - \beta)(2\partial_t + a + b) \right\} H_k \cdot \delta^{-,+} \Phi_{\lambda + k+1}^{x,\beta} \\
- \sum_{k=0}^{\infty} 2\left\{ t(2\partial_t + a + b) + 1 \right\} H_k \cdot \Phi_{\lambda + k}^{x,\beta} \\
- \sum_{k=0}^{\infty} 2\left\{ t(\partial_t + \partial_x) + 1 + ta \right\} H_k \cdot \delta^{-,+} \Phi_{\lambda + k}^{x,\beta} = 0.
\]

First, equating the coefficients of \( \Phi_{\lambda - 1}^{x,\beta} \), we obtain the equation
\[(\partial_t - \partial_x + b)G_0 = 0. \tag{6.2}\]

Next, equating the coefficients of \( \delta^{-,+} \Phi_{\lambda}^{x,\beta} \), we get the equation
\[
\{ t(\partial_t + \partial_x) + 1 + ta \} H_0 = \beta \left( \partial_t + \frac{a + b}{2} \right) G_0. \tag{6.3}\]

Furthermore, by equating the coefficients of \( \Phi_{\lambda + k}^{x,\beta} \) for \( k = 0, 1, 2, \ldots \), we obtain the recurrence formula
\[(\partial_t - \partial_x + b)G_{k+1} = (2\partial_t + a + b)H_k + \frac{1}{t} \left( H_k - \frac{1}{2} P[G_k] \right). \tag{6.4}\]

Finally, by equating the coefficients of \( \delta^{-,+} \Phi_{\lambda + k+1}^{x,\beta} \) for \( k = 0, 1, 2, \ldots \), we get the recurrence formula
\[
\{ t(\partial_t + \partial_x) + 1 + ta \} H_{k+1} \\
= \beta \left( \partial_t + \frac{a + b}{2} \right) G_{k+1} + \left\{ \frac{P}{2} + (1 - \alpha - \beta) \left( \partial_t + \frac{a + b}{2} \right) \right\} H_k. \tag{6.5}\]

We remark that, apparently, the equation (6.4), which are the equations to determine \( G_{k+1} \ (k = 0, 1, 2, \ldots) \) inductively, seems to have singularity along \( \{ t = 0 \} \), however, as a matter of fact, the term \( t^{-1}(H_k - P[G_k]/2) \) in the right-hand side of (6.4) is holomorphic, namely we can show that
\[
\{ P[G_k] - 2H_k \}_{|_{t=0}} = 0. \tag{6.6}\]

We will prove the fact (6.6). Since
\[
P|_{t=0} = \alpha X_- + \beta X_+ + c(0, x),
\]
in order to show the assertion (6.6), it suffices to show that
\[
\{\alpha X_+ + \beta X_+ + c(0, x)\} G_k(0, x) - 2H_k(0, x) = 0
\]
for \(k = 0, 1, 2, \ldots\). In order to show (6.7), we restrict the equation \(Pu = 0\) to \(\{t = 0\}\), then we have
\[
\{\alpha X_+ + \beta X_+ + c(0, x)\} u(0, x) = 0,
\]
and restrict to \(\{t = 0\}\), then we have
\[
\{\alpha + \beta\partial_t - (\alpha - \beta)\partial_x + c(0, x)\} u(0, x) = 0.
\]
Now we differentiate the expression (4.13) by \(t\):
\[
\partial_t u(t, x) = \sum_{k=0}^{\infty} \left( \partial_t G_k \cdot \Phi^{\beta}_{\lambda+k} + G_k \cdot \partial_t \Phi^{\beta}_{\lambda+k} \right)
\]
\[
+ \sum_{k=0}^{\infty} \left( \partial_t H_k \cdot \delta^{- \beta + \beta \lambda} \Phi^{\beta}_{\lambda+k+1} + H_k \cdot \partial_t \delta^{- \beta + \beta \lambda} \Phi^{\beta}_{\lambda+k+1} \right),
\]
and restrict to \(\{t = 0\}\), then we have
\[
\partial_t u(0, x) = \sum_{k=0}^{\infty} \left\{ \partial_t G_k(0, x) \cdot f_{\lambda+k}(x) + G_k(0, x) \cdot \frac{x - \beta}{x + \beta} f_{\lambda+k-1}(x) \right\}
\]
\[
+ \sum_{k=0}^{\infty} H_k(0, x) \cdot \frac{-2}{x + \beta} f_{\lambda+k}(x)
\]
by Proposition 5.10. It also holds from the expression (4.13) that
\[
\partial_x u(0, x) = \sum_{k=0}^{\infty} \left\{ \partial_x G_k(0, x) \cdot f_{\lambda+k}(x) + G_k(0, x) f_{\lambda+k-1}(x) \right\}.
\]
Hence the left-hand side of (6.8) equals
\[
\sum_{k=0}^{\infty} \left\{ (\alpha + \beta) \partial_t G_k(0, x) - (\alpha - \beta) \partial_x G_k(0, x) \right\}
\]
\[
- 2H_k(0, x) + c(0, x) G_k(0, x) \} f_{\lambda+k}(x).
\]
Equating all coefficients to zero, we get the desired identity (6.7). Thus the relation (6.6) is established.

We will also derive the initial conditions for \(G_k\) \((k = 0, 1, 2, \ldots)\). Putting \(t = 0\) in the expression (4.13), we have
\[
u(0, x) = \sum_{k=0}^{\infty} G_k(0, x) f_{\lambda+k}(x).\]
Comparing with the initial condition of the Cauchy problem (4.11), we obtain
the initial conditions for \( G_k \) \((k = 0, 1, 2, \ldots)\) as

\[
G_0(0, x) = w(x), \tag{6.9}
\]

\[
G_{k+1}(0, x) = 0 \quad (k = 0, 1, 2, \ldots). \tag{6.10}
\]

We summarize how the functions \( G_k \) and \( H_k \) are determined.

First, the holomorphic function \( G_0 \) is uniquely determined in a neighborhood of the origin by the initial value problem (6.2)+(6.9). Next, the holomorphic function \( H_0 \) is uniquely determined in a neighborhood of the origin by the equation (6.3), because the left-hand side of (6.3) is a first order linear partial differential operator to \( H_0 \) having regular singularity along \( \{t = 0\} \) with characteristic exponent \(-1\) (see Kashiwara-Oshima [5] or Mandai [6] for corresponding theorem). Then we determine uniquely the local holomorphic functions \( G_{k+1} \) \((k = 0, 1, 2, \ldots)\) inductively by the initial value problem (6.4)+(6.10), and also \( H_{k+1} \) \((k = 0, 1, 2, \ldots)\) inductively by the partial differential equation (6.5) having regular singularity along \( \{t = 0\} \) with characteristic exponent \(-1\).

In short, we can uniquely determine the holomorphic functions \( G_k \) and \( H_k \) \((k = 0, 1, 2, \ldots)\) inductively in a neighborhood of the origin in the following order:

\[ G_0 \rightarrow H_0 \rightarrow G_1 \rightarrow H_1 \rightarrow G_2 \rightarrow H_2 \rightarrow \cdots. \]

We remark that all the functions \( G_k \) and \( H_k \) \((k = 0, 1, 2, \ldots)\) can be expressed inductively by integrations. In fact, first, the solution \( G_0 \) of the initial value problem (6.2)+(6.9) is expressed as

\[
G_0(t, x) = w(x + t) \exp \left\{ - \int_0^t b(\sigma, x + t - \sigma) d\sigma \right\}. \tag{6.11}
\]

Next, each of the equation (6.3) which defines \( H_0 \), or (6.5) which defines \( H_k \) \((k \geq 1)\), is expressed in all the same form as

\[
\{t(\partial_t + \partial_x) + 1 + ta\} H_k = \psi_k,
\]

where the function \( \psi_k \) is inductively known; an elementary discussion shows that the solution \( H_k \) \((k \geq 0)\) is expressed as

\[
H_k(t, x) = \frac{1}{t} \int_0^t \exp \left\{ - \int_0^\mu a(\mu, x - t + \mu) d\mu \right\} \psi_k(\sigma, x - t + \sigma) d\sigma. \tag{6.12}
\]

Furthermore, the initial value problem defining \( G_k \) \((k \geq 1)\) is written in all the same form as

\[
\begin{align*}
& \{\partial_t - \partial_x + b\} G_k = \varphi_k, \\
& G_k(0, x) = 0,
\end{align*}
\]
where the function $\varphi_k$ is inductively known; the solution $G_k$ is expressed as

\begin{equation}
G_k(t, x) = \int_0^t \exp \left\{ - \int_\sigma^t b(\mu, x + t - \mu) d\mu \right\} \varphi_k(\sigma, x + t - \sigma) d\sigma.
\end{equation}

We observe from the expression (6.11), (6.12) and (6.13) that, if the function $w(x)$ is holomorphic in $\{|x| < r\}$ and the functions $a(t, x), b(t, x)$ and $c(t, x)$ are holomorphic in $\Omega_r$, then the inductively determined functions $G_k$ and $H_k$ ($k = 0, 1, 2, \ldots$) are all holomorphic in $\Omega_r$. In fact, in order to see this, we have only to pay attention to the fact that the set $\Omega_r$ is convex, for we can take all paths of integration in (6.11), (6.12) and (6.13) to be line segments.

7. System of majorants

In order to estimate the functions $G_k$ and $H_k$ ($k = 0, 1, 2, \ldots$), we introduce the system of majorants following Hamada-Leray-Wagschal [3] or Watanabe-Urabe [9].

To begin with, we first consider the function

\begin{equation}
\phi(t, x) = \frac{1}{R - \rho t - x} = \sum_{v=0}^{\infty} \frac{1}{R^{v+1}} (\rho t + x)^v
\end{equation}

for $R, \rho > 0$. The last power series of (7.1), having all coefficients $\geq 0$, is absolutely convergent in any compact subset of $A_{R, \rho} := \{(t, x) \in C^2 | \rho|t| + |x| < R\}$. Let

$$
\phi_k(t, x) = \partial_x^k \phi(t, x)
$$

be the $k$-th derivative by $x$ of $\phi(t, x)$. The function $\phi_k(t, x)$ is also holomorphic in $A_{R, \rho}$, and explicitly written as

\begin{equation}
\phi_k(t, x) = \frac{k!}{(R - \rho t - x)^{k+1}} = \sum_{v=0}^{\infty} \frac{(v + 1)_k}{R^{v+k+1}} (\rho t + x)^v.
\end{equation}

Before stating properties of the functions $\phi_k(t, x)$ ($k = 0, 1, 2, \ldots$), we remember the definition of majorants. We say that the formal power series $F = \sum F_{\mu, v} t^\mu x^v$ ($F_{\mu, v} \geq 0$) majorants the formal power series $f = \sum f_{\mu, v} t^\mu x^v$ ($f_{\mu, v} \in C$), if $|f_{\mu, v}| \leq F_{\mu, v}$ for all $\mu$ and $v$, and then we use the notation $f \ll F$. 142 Toru Tsutsui
Lemma 7.1. (i) \( \partial_t \phi_k(t, x) = p \phi_{k+1}(t, x) \), \( \partial_x \phi_k(t, x) = \phi_{k+1}(t, x) \).
(ii) \( \phi_k(t, x) \ll R \phi_{k+1}(t, x) \).
(iii) \( (R' - pt - x)^{-1} \phi_k(t, x) \ll (R' - R)^{-1} \phi_k(t, x) \) \( (R' > R > |x|) \).
(iv) \( \phi_k(t, x) - \phi_k(0, x) \ll pt \phi_{k+1}(t, x) \).

Proof. (i) Evident from the definition.
(ii) The desired majorant inequality follows from the last expression of (7.2).
(iii) To begin with, we have the identity that
\[
\frac{1}{R' - R} \phi(t, x) = \frac{1}{R' - \rho t - x} \phi(t, x) + \frac{1}{(R' - R)(R' - \rho t - x)}.
\]
Since the second term is obviously expressed in a power series of nonnegative coefficients, it follows that
\[
\frac{1}{R' - \rho t - x} \phi(t, x) \ll \frac{1}{R' - R} \phi(t, x),
\]
which is the assertion for the case \( k = 0 \). Now we differentiate both sides of the above inequality \( k \)-times by \( x \), then we have
\[
\frac{1}{R' - \rho t - x} \phi_k(t, x) + \sum_{j=0}^{k-1} \binom{k}{j} \frac{j!}{(R' - \rho t - x)^{j+1}} \phi_j(t, x)
\ll \frac{1}{R' - R} \phi_k(t, x).
\]
Since the second term in the left-hand side is evidently expressed in power series of nonnegative coefficients, we obtain the desired majorant inequality.
(iv) It is evident that
\[
\frac{1}{R - x} \ll \frac{1}{R - \rho t - x}.
\]
Multiplying both side of the above by \( \rho t / (R - \rho t - x) \), we have
\[
\frac{\rho t}{(R - \rho t - x)(R - x)} \ll \frac{\rho t}{(R - \rho t - x)^2}.
\]
The left-hand side equals
\[
\frac{1}{R - \rho t - x} - \frac{1}{R - x} = \phi(t, x) - \phi(0, x),
\]
whereas the right-hand side equals $\rho t \phi_1(t,x)$. Hence we have
$$\phi(t,x) - \phi(0,x) \ll \rho t \phi_1(t,x).$$
Differentiating both sides $k$-times by $x$, we obtain the desired inequality. \square

8. Majorants for $G_k$ and $H_k$ ($k = 0, 1, 2, \ldots$)

We have obtained the recurrence formula for $G_k$ and $H_k$ ($k = 0, 1, 2, \ldots$) in section 6. The aim of this section is to construct majorants for these functions.

Before proceeding to the discussion, we give some preparatory majorations. For the functions $a(t,x)$, $b(t,x)$ and $c(t,x)$ in the operator $P$, there exist positive numbers $R'$, $\rho$ and $M$ such that

$$a(t,x), b(t,x), c(t,x) \ll \frac{M}{R' - \rho t - x}.$$ \hfill (8.1)

We may also assume that

$$w(x) \ll \frac{M}{R' - x}.$$ \hfill (8.2)

We remark that, by replacing $\rho$ satisfying (8.1) with larger $\rho$, the majoration (8.1) remains still valid. In the following discussion, we will take $\rho$ large enough; to what extent will be clarified later.

8.1. Majorants for $G_0$ and $H_0$

Recall that the function $G_0(t,x)$ is determined as the solution of the initial value problem (6.2)+(6.9). We consider, instead of the equation (6.2), the majorant inequality

$$\partial_t \tilde{G}_0(t,x) \gg \partial_x \tilde{G}_0(t,x) + \frac{M}{R' - \rho t - x} \tilde{G}_0(t,x),$$ \hfill (8.3)

and, instead of the initial condition (6.9), the majorant inequality

$$\tilde{G}_0(0,x) \gg \frac{M}{R' - x}.$$ \hfill (8.4)

Since the estimates (8.1) and (8.2) are assumed, as is easily seen from the well-known discussion (for example, well-known in the classical proof of the Cauchy-Kowalewski theorem), for any formal power series $\tilde{G}_0(t,x)$ that satisfies (8.3)+(8.4), it holds the majorant inequality

$$\tilde{G}_0(t,x) \gg G_0(t,x).$$
Let $R$ be an arbitrary positive number smaller than $R'$. We will first prove that we can find a positive number $\rho$ such that the function

$$\tilde{G}_0(t, x) = M(\phi + \rho t \phi_1)$$

satisfies the majorant inequalities \(8.3\)+\(8.4\). For the function $\tilde{G}_0(t, x)$ given by \(8.5\), the left-hand side of \(8.3\) is calculated as

$$\partial_t \tilde{G}_0(t, x) = M \rho (2\phi_1 + \rho t \phi_2)$$

from Lemma 7.1 (i). On the other hand, the right-hand side of \(8.3\) is evaluated as

$$\partial_x \tilde{G}_0(t, x) + \frac{M}{R' - \rho t - x} \tilde{G}_0(t, x)$$

$$= M \partial_x (\phi + \rho t \phi_1) + \frac{M^2}{R' - \rho t - x} (\phi + \rho t \phi_1)$$

$$\ll M (\phi_1 + \rho t \phi_2) + \frac{M^2}{R' - R} (\phi + \rho t \phi_1)$$

by Lemma 7.1 (iii), and furthermore

$$\ll M (\phi_1 + \rho t \phi_2) + \frac{M^2 R}{R' - R} (\phi_1 + \rho t \phi_2)$$

by Lemma 7.1 (ii), and accordingly

$$= M (1 + R'') (\phi_1 + \rho t \phi_2),$$

where we put

$$R'' = \frac{MR}{R' - R}.$$ 

Comparing the coefficients of $\phi_1$ and $\rho t \phi_2$ respectively, we conclude that, if

$$\rho > 1 + R'',$$

then the inequality \(8.3\) holds. Concerning the initial condition, the inequality \(8.4\) is automatically satisfied by the definition \(8.5\), for it is evident that

$$\tilde{G}_0(0, x) = \frac{M}{R - x} \gg \frac{M}{R' - x}.$$

Therefore, it is verified that, as long as $\rho$ satisfies the condition \(8.7\), the function $\tilde{G}_0$ given by \(8.5\) is a majorant of $G_0$. 

**Propagation of Regular Singularities**
Next, instead of the recurrence equation (6.3) which defines $H_0$, we consider the majorant inequality

$$(8.8) \quad (t\hat{\sigma}_t + 1) \tilde{H}_0 \gg t \left( \partial_x + \frac{M}{R' - \rho t - x} \right) \tilde{H}_0 + |\beta| \left( \partial_t + \frac{M}{R' - \rho t - x} \right) \tilde{G}_0.$$ 

From the assumption (8.1), we can easily verify as before that, for any formal power series $\tilde{H}_0(t, x)$ satisfying (8.8), it holds the majorant inequality

$$\tilde{H}_0(t, x) \gg H_0(t, x).$$

We will now prove that we can find a positive number $\kappa$ such that the function

$$(8.9) \quad \tilde{H}_0(t, x) = M\kappa \phi_1(t, x)$$

satisfies the majorant inequality (8.8), where $\rho$ in the expression $\phi_1$ is assumed to satisfy (8.7).

For the function $\tilde{H}_0(t, x)$ given by (8.9), the left-hand side of (8.8) is calculated by Lemma 7.1 (i) as

$$(t\hat{\sigma}_t + 1) \tilde{H}_0 = M\kappa (1 + t\hat{\sigma}_t) \phi_1 = M\kappa (\phi_1 + \rho t\phi_2).$$

On the other hand, the right-hand side of (8.8) is expressed by substituting (8.5) as

$$M\kappa \left( \partial_x + \frac{M}{R' - \rho t - x} \right) \phi_1 + M|\beta| \left( \partial_t + \frac{M}{R' - \rho t - x} \right) (\phi + \rho t\phi_1),$$

and evaluated by the use of Lemma 7.1 (iii) as

$$\ll \frac{M\kappa}{\rho} \rho t \left( \phi_2 + \frac{M}{R' - \rho t - x} \phi_1 \right) + M|\beta| \left\{ 2\rho \phi_1 + \rho^2 t\phi_2 + \frac{M}{R' - \rho t - x} (\phi + \rho t\phi_1) \right\},$$

and furthermore evaluated by Lemma 7.1 (ii) and using $R''$ defined by (8.6) as

$$\ll \frac{M\kappa}{\rho} (1 + R'') \rho t\phi_2 + M|\beta| \left\{ 2\rho \phi_1 + \rho (\rho t\phi_2) + R''(\phi_1 + \rho t\phi_2) \right\}$$

$$= M|\beta| (2\rho + R'') \phi_1 + M \left\{ \frac{\kappa}{\rho} (1 + R'') + |\beta|(\rho + R'') \right\} \rho t\phi_2.$$

Hence, comparing the coefficients of $\phi_1$ and $\rho t\phi_2$ respectively, we obtain the following two inequalities as sufficient condition for the inequality (8.8) to be satisfied, that is,

$$(8.10) \quad \kappa > |\beta|(2\rho + R''),$$
The above two inequalities (8.10) and (8.11) can be combined to a single condition for \( \kappa \) as

\[
\kappa > \frac{K}{p} (1 + R'') + |\beta| (\rho + R'') \quad \text{and} \quad \kappa > \max \left\{ |\beta|(2\rho + R''), |\beta| (\rho + R'') \left( 1 - \frac{1 + R''}{\rho} \right)^{-1} \right\}.
\]

We conclude accordingly that, as long as the positive number \( \kappa \) satisfies the condition (8.12), the function \( \tilde{H}_0 \) given by (8.9) is a majorant of \( H_0 \).

### 8.2. Majorants for \( G_k \) and \( H_k \) (\( k \geq 1 \))

We will construct majorants for \( G_k \) and \( H_k \) also for \( k \geq 1 \).

**Proposition 8.1.** Let \( \rho \) and \( \kappa \) satisfy the conditions (8.7) and (8.12) respectively. Then there exists a positive constant \( C \) such that the functions \( G_k \) and \( H_k \) (\( k = 0, 1, 2, \ldots \)) satisfies the following majorant inequalities:

\[
G_k(t, x) \ll MC^k (\phi_k + \rho t \phi_{k+1}),
\]

\[
H_k(t, x) \ll M\kappa C^k \phi_{k+1}.
\]

The proof of the proposition is by induction on \( k \).

We see from (8.5) and (8.9) that we have the majorant inequalities (8.13) and (8.14) for \( k = 0 \).

Now we assume the inequalities (8.13) and (8.14) for \( k \), and we will prove these two inequalities for \( k + 1 \).

For the sake of simplicity, we will denote the right-hand sides of (6.4) and (6.5) by \( \varphi_{k+1}(t, x) \) and \( \psi_{k+1}(t, x) \) respectively, namely

\[
\varphi_{k+1}(t, x) = (2\partial_t + a + b)H_k + \frac{1}{t} \left( H_k - \frac{1}{2} P[G_k] \right),
\]

\[
\psi_{k+1}(t, x) = \beta \left( \partial_t + \frac{a + b}{2} \right) G_{k+1} + \left\{ \frac{P}{2} + (1 - \alpha - \beta) \left( \partial_t + \frac{a + b}{2} \right) \right\} H_k.
\]

First of all, we estimate the function \( \varphi_{k+1}(t, x) \) under the induction hypothesis that \( G_k \) and \( H_k \) satisfy the inequalities (8.13) and (8.14) respectively.
Lemma 8.2. Let $G_k$ and $H_k$ satisfy (8.13) and (8.14) respectively. Then the function $\varphi_{k+1}$ is estimated by majorants as

$$\varphi_{k+1}(t, x) \ll MC^k \{Q_1(\rho) + \kappa Q_2(\rho)\} \phi_{k+2} + MC^k Q_3(\rho) \cdot \rho t \phi_{k+3},$$

where $Q_j(\rho)$ ($j = 1, 2, 3$) are polynomials in $\rho$ given by

$$Q_1(\rho) = \left(\frac{1 + \rho}{2} + R''\right)(1 + 3\rho) + \frac{|x| + |\beta|}{2} \rho(2 + 3\rho),$$

$$Q_2(\rho) = 3\rho + 2R'',$$

$$Q_3(\rho) = (1 + \rho)\left(\frac{1 + \rho}{2} + R''\right).$$

Proof. We begin by evaluating the first term of the right-hand side of (8.15). Using (8.1) and (8.14), we have

$$\left(2\partial_t + a + b\right)H_k \ll 2M\kappa C^k \left(\partial_t + \frac{M}{R' - \rho t - x}\right) \phi_{k+1},$$

and accordingly, by the use of Lemma 7.1 (i), (iii),

$$\ll 2M\kappa C^k \left(\rho \phi_{k+2} + \frac{M}{R' - R} \phi_{k+1}\right).$$

Hence, moreover applying Lemma 7.1 (ii) and using (8.6), we get the majorant inequality

$$\left(2\partial_t + a + b\right)H_k \ll 2M\kappa C^k (\rho + R'') \phi_{k+2}.\quad(8.18)$$

We will next evaluate the second term of the right-hand side of (8.15). Since the operator $P$ is rewritten as

$$P = t(X+ X_+ + aX_+ + bX_+) + (\xi X_+ + \beta X_+ + c),$$

the second term of (8.15) is written as

$$\frac{1}{t} \left(H_k - \frac{1}{2} P[G_k]\right) = -\frac{1}{2} (X_+ X_+ + aX_+ + bX_+) G_k + \frac{1}{t} \left\{H_k - \frac{1}{2} (\xi X_+ + \beta X_+ + c) G_k\right\}.\quad(8.19)$$

We remark that, in the right-hand side of (8.19), it follows from (6.6) that the second term is also holomorphic. As for the first term of the right-hand side of (8.19), we have, by the use of (8.1) and (8.13), and by replacing all the signature


\[ (X_-X_+ + aX_- + bX_+)G_k \]

\[ \ll MC^k \left( X_+ + \frac{M}{R' - \rho t - x} X_+ + \frac{M}{R' - \rho t - x} X_+ \right) (\phi_k + \rho t\phi_{k+1}) \]

\[ = MC^k \left( X_+ + \frac{2M}{R' - \rho t - x} \right) X_+(\phi_k + \rho t\phi_{k+1}). \]

Because

\[ X_+ \phi_k = (1 + \rho)\phi_{k+1} \]

from Lemma 7.1 (i), we have

\[ X_+(\phi_k + \rho t\phi_{k+1}) = (1 + 2\rho)\phi_{k+1} + (1 + \rho)\rho t\phi_{k+2}. \]

Therefore we have

\[ (X_-X_+ + aX_- + bX_+)G_k \]

\[ \ll MC^k \left( X_+ + \frac{2M}{R' - \rho t - x} \right) \{(1 + 2\rho)\phi_{k+1} + (1 + \rho)\rho t\phi_{k+2}\} \]

\[ \ll MC^k \left( X_+ + \frac{2M}{R' - \rho} \right) \{(1 + 2\rho)\phi_{k+1} + (1 + \rho)\rho t\phi_{k+2}\} \]

by Lemma 7.1 (iii), and furthermore

\[ \ll MC^k \{(1 + \rho)(1 + 3\rho)\phi_{k+2} + (1 + \rho)^2 \rho t\phi_{k+3}\} \]

\[ + MC^k \cdot 2R'' \{(1 + 2\rho)\phi_{k+2} + (1 + \rho)\rho t\phi_{k+3}\} \]

by (8.20) again and by Lemma 7.1 (ii) with (8.6). Hence we get the majorant (ii) with (8.6). Therefore we have

\[ \frac{1}{2} (X_-X_+ + aX_- + bX_+)G_k \]

\[ \ll MC^k \left\{ \frac{(1 + \rho)(1 + 3\rho)}{2} + R''(1 + 2\rho) \right\} \phi_{k+2} \]

\[ + MC^k (1 + \rho) \left( \frac{1 + \rho}{2} + R'' \right) \rho t\phi_{k+3}. \]

We will next consider the second term of the right-hand side of (8.19). Using (8.1), (8.13) and (8.14), replacing \( X_- \) by \( X_+ \), and replacing \( \alpha, \beta \) by \( |\alpha|, |\beta| \) respectively, we have
\begin{align*}
H_k & - \frac{1}{2}(xX_- + \beta X_+ + c)G_k \\
& \ll M \kappa C_k^k \phi_{k+1} + MC_k^k \left\{ \frac{1}{2} \left( |x| + |eta| \right) X_+ + \frac{M}{R' - p t} \right\} (\phi_k + pt \phi_{k+1}),
\end{align*}

and by the use of (8.21) and Lemma 7.1 (iii)

\begin{align*}
& \ll M \kappa C_k^k \phi_{k+1} + MC_k^k \frac{|x| + |eta|}{2} \left\{ (1 + 2 \rho) \phi_{k+1} + (1 + \rho) pt \phi_{k+2} \right\} \\
& + MC_k^k \frac{1}{2} \frac{M}{R' - R} (\phi_k + pt \phi_{k+1}),
\end{align*}

and furthermore by using Lemma 7.1 (ii) with (8.6)

\begin{align*}
& \ll M \kappa C_k^k \phi_{k+1} + MC_k^k \frac{|x| + |eta|}{2} \left\{ (1 + 2 \rho) \phi_{k+1} + (1 + \rho) pt \phi_{k+2} \right\} \\
& + MC_k^k \frac{R''}{2} (\phi_{k+1} + pt \phi_{k+2}) \\
& = MC_k^k \left\{ \kappa + \frac{R''}{2} + \frac{|x| + |eta|}{2} (1 + 2 \rho) \right\} \phi_{k+1} \\
& + MC_k^k \left\{ \frac{R''}{2} + \frac{|x| + |eta|}{2} (1 + \rho) \right\} pt \phi_{k+2}.
\end{align*}

However, as mentioned before, since the second term of the right-hand side of (8.19) is holomorphic, we have

\begin{align*}
\left\{ H_k - \frac{1}{2}(xX_- + \beta X_+ + c)G_k \right\} |_{t=0} = 0,
\end{align*}

and accordingly, in the last expression of the above majoration, we can omit the non-vanishing term with respect to \( t \), hence get the majorant inequality as

\begin{align*}
H_k & - \frac{1}{2}(xX_- + \beta X_+ + c)G_k \\
& \ll MC_k^k \left\{ \kappa + \frac{R''}{2} + \frac{|x| + |eta|}{2} (1 + 2 \rho) \right\} \left\{ \phi_{k+1}(t, x) - \phi_{k+1}(0, x) \right\} \\
& + MC_k^k \left\{ \frac{R''}{2} + \frac{|x| + |eta|}{2} (1 + \rho) \right\} pt \phi_{k+2}.
\end{align*}

Now Lemma 7.1 (iv) is applicable to the first term of the right-hand. Thus we obtain the inequality that
Hence, adding together the inequalities (8.18), (8.22) and (8.23), we obtain the majorant inequality as

\[
\phi_{k+1}(t, x) \ll MC^k \{ Q_1(\rho) + \kappa Q_2(\rho) \} \phi_{k+2} + MC^k Q_3(\rho) \cdot \rho t \phi_{k+3},
\]

where \( Q_j(\rho) \) \( (j = 1, 2, 3) \) are the polynomials as given in the statement of the lemma. The proof of the lemma is thus completed.

Now we proceed to the demonstration of the inequality (8.13) for \( k + 1 \). We denote by \( \tilde{\phi}_{k+1}(t, x) \) the right-hand side of (8.17). We consider, instead of the equation (6.4), the majorant inequality

\[
\partial_t \tilde{G}_{k+1}(t, x) \gg \partial_x \tilde{G}_{k+1}(t, x) + \frac{M}{R' - \rho t - x} \tilde{G}_{k+1}(t, x) + \tilde{\phi}_{k+1}(t, x),
\]

and, instead of the initial condition (6.10), the majorant inequality

\[
\tilde{G}_{k+1}(0, x) \gg 0.
\]

As is easily seen as before, for any formal power series \( \tilde{G}_{k+1}(t, x) \) satisfying (8.24) and (8.25), it holds the majorant inequality

\[
\tilde{G}_{k+1}(t, x) \gg G_{k+1}(t, x).
\]

We will prove that we can find a positive constant \( C \) (not depending on \( k \)) such that the function

\[
\tilde{G}_{k+1}(t, x) = MC^{k+1}(\phi_{k+1} + \rho t \phi_{k+2})
\]

satisfies the majorant inequality (8.24), for the condition (8.25) is automatically satisfied.

For the function \( \tilde{G}_{k+1}(t, x) \) given by (8.26), the left-hand side of (8.24) is calculated by Lemma 7.1 (i) as

\[
\partial_t \tilde{G}_{k+1}(t, x) = MC^{k+1} \partial_t (\phi_{k+1} + \rho t \phi_{k+2}) = MC^{k+1}(2\rho \phi_{k+2} + \rho^2 t \phi_{k+3}).
\]
On the other hand, the right-hand side of the inequality (8.24) is evaluated by Lemma 7.1 (iii) as
\[ q x ~ GGk + 1(t, x) + \phi_{k+1}(t, x) \]
and furthermore evaluated by Lemma 7.1 (ii) with (8.6) as
\[ MCk + 1(\phi_{k+2} + \rho t \phi_{k+3}) + MCk + 1 \frac{M}{R' - R} (\phi_{k+1} + \rho t \phi_{k+2}) + \phi_{k+1}(t, x), \]
and by substitution of \( \phi_{k+1}(t, x) \) (namely the right-hand side of (8.17)),
\[ MCk + 1 \frac{R''}{C} \phi_{k+2} \]
Thus, comparing the coefficients of \( \phi_{k+2} \) and \( \rho t \phi_{k+3} \) respectively, we obtain the following two inequalities as sufficient condition for the inequality (8.24) to be satisfied, that is,
\[ 2 \rho \geq 1 + R'' + \frac{Q_1(\rho)}{C} + \kappa \frac{Q_2(\rho)}{C}, \]
and
\[ \rho \geq 1 + R'' + \frac{Q_3(\rho)}{C}. \]
Recall that \( \rho > 1 + R'' \) by (8.7). It is therefore evident that, the conditions (8.27) and (8.28) are satisfied if the constant \( C \) is sufficiently large.
We will next prove the inequality (8.14) for \( k+1 \) under the induction hypothesis that the inequality (8.13) holds for \( k+1 \) and that the inequality (8.14) holds for \( k \). We will estimate the function \( \psi_{k+1} \) under these conditions.

**Lemma 8.3.** Let \( G_{k+1} \) satisfy (8.13) for \( k+1 \) and let \( H_k \) satisfy (8.14). Then the function \( \psi_{k+1} \) is estimated by majorants as
\[ \psi_{k+1}(t, x) \ll MC^{k+1} \left\{ \frac{\beta}{2\rho + R''} + \frac{Q_4(\rho)}{C} \right\} \phi_{k+2} \]
\[ + MC^{k+1} \left\{ \frac{\beta}{\rho + R''} + \frac{Q_5(\rho)}{C} \right\} \rho t \phi_{k+3}, \]
where
\[
Q_4(\rho) = \frac{|x| + |\beta|}{2} (1 + \rho) + \frac{R''}{2} + (1 + |x| + |\beta|)(\rho + R''),
\]
\[
Q_5(\rho) = \frac{1 + \rho}{\rho} \left( \frac{1 + \rho}{2} + R'' \right).
\]

**Proof.** We begin by evaluating the first term of the right-hand side of (8.16). From (8.1) and (8.13) for $k + 1$, we have
\[
\beta \left( \frac{\partial}{\partial t} + \frac{a + b}{2} \right) G_{k+1} \ll MC^{k+1} |\beta| \left( \frac{\partial}{\partial t} + \frac{M}{R' - \rho t - x} \right) (\phi_{k+1} + \rho \phi_{k+2}),
\]
and by the use of Lemma 7.1 (i), (iii)
\[
\ll MC^{k+1} |\beta| \left\{ (2 \rho \phi_{k+2} + \rho^2 \phi_{k+3}) + \frac{M}{R' - R}(\phi_{k+1} + \rho \phi_{k+2}) \right\},
\]
and accordingly by Lemma 7.1 (ii) with (8.6)
\[
\ll MC^{k+1} |\beta| \left\{ (2 \rho + R'')(\phi_{k+2} + (\rho + R'') \rho \phi_{k+3}) \right\},
\]
We will next estimate the second term of the right-hand side of (8.16). To begin with, since
\[
P = tX_-.X_+ + (a + t)X_+ + (b + t)X_+ + c,
\]
the function $PH_k$ is estimated, by the use of (8.1) and (8.14), and by replacing $X_-$ by $X_+$, as
\[
PH_k \ll MC^k
\]
\[
\times \left\{ tX_+X_+ + (|x| + |\beta|)X_+ + t \frac{2M}{R' - \rho t - x} X_+ + \frac{M}{R' - \rho t - x} \right\} \phi_{k+1}.
\]
Concerning the first two terms of the right-hand side, we have
\[
\{ tX_+X_+ + (|x| + |\beta|)X_+ \} \phi_{k+1}
\]
\[
\ll \frac{(1 + \rho)^2}{\rho} \rho \phi_{k+3} + (|x| + |\beta|)(1 + \rho) \phi_{k+2}
\]
by Lemma 7.1 (i). On the other hand, concerning the last two terms of the right-hand side, we have

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by Lemma 7.1 (i), (iii), and furthermore
\[ \ll 2R'' \frac{1 + \rho}{\rho} \rho t \phi_{k+3} + R'' \phi_{k+2} \]
by Lemma 7.1 with (8.6). Hence we have
\[ PH_k \ll M^k C^k \{(|\alpha| + |\beta|)(1 + \rho) + R''\} \phi_{k+2} \]
\[ + M^k C^k \frac{1 + \rho}{\rho} \rho t \phi_{k+3} \]
by (8.1) and (8.14), and therefore
\[ \ll M^k C^k (1 + |\alpha| + |\beta|) \left( \partial_t + \frac{M}{\rho' - \rho} \right) \phi_{k+1} \]
by (8.1) and (8.6), and accordingly
\[ \ll M^k C^k (1 + |\alpha| + |\beta|) \left( \rho \phi_{k+2} + \frac{M}{\rho' - \rho} \phi_{k+1} \right) \]
by Lemma 7.1 (i), (iii), and
\[ \ll M^k C^k (1 + |\alpha| + |\beta|)(\rho + R'') \phi_{k+2} \]
by Lemma 7.1 (ii) with (8.6).
Adding all the obtained inequalities together, we obtain the estimate as
\[ \psi_{k+1}(t, x) \ll MC^{k+1} \left\{ \left| \beta \right| (2 \rho + R'') + \frac{Q_4(\rho)}{C} \right\} \phi_{k+2} \]
\[ + MC^{k+1} \left\{ \left| \beta \right| (\rho + R'') + \frac{Q_5(\rho)}{C} \right\} \rho t \phi_{k+3}, \]
where \( Q_4(\rho) \) and \( Q_5(\rho) \) are as given in the statement of the lemma. Thus the lemma is established. \( \square \)

Now we proceed to the demonstration of the inequality (8.14) for \( k + 1 \).
We denote by \( \tilde{\psi}_{k+1}(t, x) \) the right-hand side of (8.29). We consider, instead of
the equation (6.5), the majorant inequality

\[(8.30) \quad (t\tilde{\partial}_t + 1)\tilde{H}_{k+1}(t,x) \gg t\tilde{x}_t\tilde{H}_{k+1}(t,x) + \frac{Mt}{R' - pt - x}\tilde{H}_{k+1}(t,x)\]

\[+ \tilde{\psi}_{k+1}(t,x).\]

As is easily seen as before again, for any formal power series \(\tilde{H}_{k+1}(t,x)\) satisfying (8.30), it holds the majorant inequality

\[\tilde{H}_{k+1}(t,x) \gg H_{k+1}(t,x).\]

We will prove that we can find a positive constant \(C\) (that is, besides the already obtained conditions (8.27) and (8.28), we will derive additional conditions on \(C\)) such that the function

\[(8.31) \quad \tilde{H}_{k+1}(t,x) = M\kappa C^{k+1} \phi_{k+2}(t,x)\]

satisfies the majorant inequality (8.30).

For the function \(\tilde{H}_{k+1}(t,x)\) given by (8.31), the left-hand side of (8.30) is calculated by Lemma 7.1 (i) as

\[(t\tilde{\partial}_t + 1)\tilde{H}_{k+1}(t,x) = M\kappa C^{k+1}(1 + t\tilde{\partial}_t)\phi_{k+2}\]

\[= M\kappa C^{k+1}(\phi_{k+2} + \rho t\phi_{k+3}).\]

On the other hand, the right-hand side of (8.30) is evaluated as

\[t\tilde{x}_t\tilde{H}_{k+1}(t,x) + \frac{Mt}{R' - pt - x}\tilde{H}_{k+1}(t,x) + \tilde{\psi}_{k+1}(t,x)\]

\[\ll M\kappa C^{k+1}t\phi_{k+3} + M\kappa C^{k+1}\frac{Mt}{R' - R}\phi_{k+2} + \tilde{\psi}_{k+1}(t,x)\]

by Lemma 7.1 (i), (iii), and furthermore

\[\ll M\kappa C^{k+1}t\phi_{k+3} + M\kappa C^{k+1}R''t\phi_{k+3} + \tilde{\psi}_{k+1}(t,x)\]

\[= M\kappa C^{k+1}\frac{1 + R''}{\rho}\rho t\phi_{k+3} + \tilde{\psi}_{k+1}(t,x)\]

by Lemma 7.1 (ii) with (8.6), and substituting \(\tilde{\psi}_{k+1}(t,x)\), namely the right-hand side of (8.29), after all

\[\ll MC^{k+1}\left\{|\beta|(2\rho + R'') + \kappa \frac{Q_s(\rho)}{C}\right\}\phi_{k+2}\]

\[+ MC^{k+1}\left\{|\beta|(\rho + R'') + \kappa \frac{Q_s(\rho)}{C}\right\}(1 + R'')\rho t\phi_{k+3}.\]
Hence, comparing the coefficients of $f_{k+2}$ and $\rho f_{k+3}$ respectively, we obtain the following two inequalities as sufficient condition for the inequality (8.30) to be satisfied, that is,

\begin{equation}
\kappa \geq |\beta|(2\rho + R'') + \kappa \frac{Q_3(\rho)}{C},
\end{equation}

and

\begin{equation}
\kappa \geq |\beta|(\rho + R'') + \frac{\kappa}{\rho}(1 + R'') + \kappa \frac{Q_5(\rho)}{C}.
\end{equation}

Remember that $\kappa$ satisfies the condition (8.12), which was equivalent to the two inequalities (8.10) and (8.11). Compare (8.32) with (8.10), and also compare (8.33) with (8.11). It is therefore obvious that if $C$ is sufficiently large, the conditions (8.32) and (8.33) are satisfied.

Hence, after all, our conclusion is summarized as follows.

First, take $\rho$ according to the condition (8.7). Next, take $\kappa$ according to the condition (8.12). Finally, take $C$ sufficiently large according to the conditions (8.27), (8.28), (8.32) and (8.33). Then all the induction steps proceed.

The proof of Proposition 8.1 is thus completed.

Estimates by absolute value of the functions $G_k$ and $H_k$ are obtained from Proposition 8.1. In fact, substituting (7.2) of the expression of $f_k(t, x)$ to the majorant inequalities (8.13) and (8.14), we easily obtain the following corollary.

**Corollary 8.4.** Let $\rho$ satisfy the condition (8.7). Then, for any compact subset $K$ in $\Delta_{R, \rho}$, there exist positive constants $M_1$, $M_2$ and $B$ such that the following two estimates hold on $K$:

\begin{equation}
|G_k(t, x)| \leq M_1 B^k(k + 1)!,
\end{equation}

\begin{equation}
|H_k(t, x)| \leq M_2 B^k(k + 1)!.\end{equation}

9. **Estimates for $\Phi_{k+1}^{x, \beta}$ and $\delta^{\gamma-\lambda} \Phi_{k+1}^{x, \beta}$**

We have next to estimate the functions $\Phi_{k+1}^{x, \beta}$ and $\delta^{\gamma-\lambda} \Phi_{k+1}^{x, \beta}$ ($k = 0, 1, 2, \ldots$).

To begin with, it holds from the Euler integral representation of the Gauss hypergeometric function (by putting $z = 2t/(x + t)$ as before) that

\begin{align*}
\Phi_{k+1}^{x, \beta}(t, x) &= \frac{(x + t)\lambda}{\Gamma(\lambda + 1)} F(-\lambda, \beta, \alpha + \beta; z) \\
&= \frac{(x + t)^\lambda}{\Gamma(\lambda + 1)} \frac{\Gamma(\alpha + \beta)}{\Gamma(-\lambda)\Gamma(\lambda + \alpha + \beta)} \int_0^1 \zeta^{-\lambda-1}(1 - \zeta)^{\lambda+x+\beta-1}(1 - z\zeta)^{-\beta} d\zeta.
\end{align*}
It is also expressed by the use of the well-known Pochhammer's double loop (see Yoshida [8] for example) \( C = C(z) \) as

\[
\Phi^x_\lambda(t, x) = \frac{(x + t)^\lambda}{\Gamma(\lambda + 1) \Gamma(-\lambda) \Gamma(\lambda + k) \Gamma(\lambda + k + 1)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + k)} \int_C \frac{1}{(1 - e^{-2\pi i \lambda})(1 - e^{2\pi i \lambda})} \zeta^{-\lambda - 1} (1 - \zeta)^{-\alpha - 1} (1 - z\zeta)^{-\beta} d\zeta,
\]

where we have written

\[
A := \lambda + \alpha + \beta
\]

for the sake of simplicity. We assume that, as the point \( z(= 2t/(x + t)) \) moves on the universal covering space of \( \mathbb{P} \setminus \{0, 1, \infty\} \) (where \( \mathbb{P} \) denotes the complex projective line), the double loop \( C(z) \) deforms, avoiding the four points \( 0, 1, \infty \) and \( 1/z \), which are the singular points of the integrand (see Figure 2).

Replacing \( \lambda \rightarrow \lambda + k \) in the above expression, we have

\[
\Phi^x_{\lambda+k}(t, x) = \frac{(x + t)^{\lambda+k}}{\Gamma(\lambda + k + 1) \Gamma(-\lambda - k) \Gamma(\lambda + k + 1)} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha + \beta + k)} \int_C \frac{1}{(1 - e^{-2\pi i \lambda})(1 - e^{2\pi i \lambda})} \zeta^{-\lambda-k-1} (1 - \zeta)^{-\alpha+k-1} (1 - z\zeta)^{-\beta} d\zeta,
\]

Since it holds from the well-known identity of \( \Gamma \)-function that

\[
\Gamma(\lambda + k + 1) \Gamma(-\lambda - k) = \frac{(-1)^{k-1} \pi}{\sin \pi \lambda},
\]

and it also holds that

\[
1 - e^{-2\pi i \lambda} = \frac{2i \sin \pi \lambda}{e^{\pi i \lambda}},
\]
it follows that
\[
\frac{1}{\Gamma(\lambda + k + 1)} \frac{1}{\Gamma(-\lambda - k)} \frac{1}{1 - e^{2\pi i \lambda}} = \frac{(-1)^{k-1} e^{\pi i \lambda}}{2\pi i}.
\]
Hence we obtain the expression of \( \Phi_{\lambda+k}^{x,y}(t,x) \) as

\[
(9.1) \quad \Phi_{\lambda+k}^{x,y}(t,x) = -\frac{1}{2\pi i} \frac{e^{\pi i \lambda} \Gamma(x + \beta) (x + t)^{\lambda+k}}{1 - e^{2\pi i A} \Gamma(A + k)} \times \int_C \frac{\zeta^{-\lambda-1} (1 - \zeta)^{A-1} (1 - z\zeta)^{-\beta} \left( \frac{\zeta - 1}{\zeta} \right)^k}{\zeta} d\zeta.
\]

Now, let \( \tilde{K} \) be a compact subset in \( \Omega \setminus \Sigma \), and let \( (t,x) \) be a movable point on \( \tilde{K} \). We will estimate \( |\Phi_{\lambda+k}^{x,y}(t,x)| \) under this situation. We note that, in the right-hand side of (9.1), only the factors \((x + t)^{\lambda+k} / \Gamma(A + k)\) and \((\zeta - 1)^k / \zeta\) in the integrand depend on \( k \).

First of all, if \( (t,x) \in \tilde{K} \), then \( |x + t| < r \), and therefore

\[
|\Phi_{\lambda+k}^{x,y}(t,x)| \leq r^k \quad ((t,x) \in \tilde{K}).
\]

For the estimate of \( \Gamma \)-function, we will make use of

**Stirling’s formula.** If \( |\arg z| \leq \pi - \delta \ (\delta > 0) \), then we have

\[
(9.3) \quad \Gamma(z) \sim \sqrt{2\pi z} e^{-z} \left( \frac{z}{e} \right)^z \quad (z \to \infty).
\]

Using this, we have

\[
(9.4) \quad \Gamma(A + k) \sim \sqrt{\frac{2\pi}{A + k}} \left( \frac{A + k}{e} \right)^{A+k} \quad (k \to \infty).
\]

Dividing both sides by \( \Gamma(k) \), and using (9.3) again, we have

\[
\frac{\Gamma(A + k)}{\Gamma(k)} \sim \sqrt{\frac{k}{A + k}} e^{-A} \left( \frac{A + k}{k} \right)^k (A + k)^A \sim (A + k)^A \sim k^A \quad (k \to \infty),
\]

because \( \lim_{k \to \infty} ((A + k)/k)^k = e^A \). Hence we have

\[
\Gamma(A + k) \sim k^A \Gamma(k) \sim k^{A-2}(k+1)!. \]

It therefore follows that

\[
(9.5) \quad \left| \frac{1}{\Gamma(A + k)} \right| \leq \text{const.} \frac{k^{2 - \Re(A)}}{(k + 1)!}.
\]
Next, we will estimate the integration in the right-hand side of (9.1). Let
\[ \pi : \mathbb{C}^2 \setminus \Sigma \to \mathbb{P}\{0, 1, \infty\} \]
be the mapping defined by \( \pi(t, x) = 2t/(x + t) \). As \((t, x)\) moves on \( \tilde{K} \), \( z = \pi(t, x) \) moves on \( \pi(\tilde{K}) \). Since \( \tilde{K} \) is a compact subset in \( \Omega \setminus \Sigma \), \( \pi(\tilde{K}) \) is also compact in the universal covering space of \( \mathbb{P}\{0, 1, \infty\} = \mathbb{C}\setminus\{0, 1\} \). As \( z \) moves on \( \pi(\tilde{K}) \), we must deform the contour \( C = C(z) \) avoiding the three points 0, 1 and \( 1/z \) in the complex plane \( \mathbb{C} \). The collection of all such \((z, \zeta)\), namely the set
\[ (9.6) \quad \bigcup_{z \in \pi(\tilde{K})} \{z\} \times C(z) \]
is also compact in \((z, \zeta)\)-space. As \( z \) moves on \( \pi(\tilde{K}) \), it is possible to deform the contour \( C(z) \) keeping the condition that the distance of \( C(z) \) from the three points 0, 1 and \( 1/z \) is always not less than a certain positive number throughout the deformation. In fact, since the set
\[ \pi(\tilde{K})^\gamma := \{1/z \mid z \in \pi(\tilde{K})\} \]
is a compact subset in the universal covering space of \( \mathbb{P}\{0, 1, \infty\} = \mathbb{C}\setminus\{0, 1\} \), the distance
\[ \delta := \text{dist}(\pi(\tilde{K})^\gamma, \{0, 1\}) \]
is certainly a positive number. Hence, it is possible to deform the contour \( C(z) \) keeping the condition
\[ \text{dist}(C(z), \pi(\tilde{K})^\gamma \cup \{0, 1\}) \geq \frac{\delta}{2}. \]
It follows from this observation that there exists a positive constant \( C_1 \) such that
\[ (9.7) \quad |z^{-\lambda - 1}(1 - \zeta)^{\lambda + \mu - 1}(1 - z\zeta)^{-\beta}| \leq C_1 \]
as long as \((z, \zeta)\) belongs to the compact set (9.6). It also holds by the same reason that there exists a positive constant \( C_2 \) such that
\[ (9.8) \quad \left| \frac{\zeta - 1}{\zeta} \right| \leq C_2 \]
as long as \( \zeta \in \bigcup_{z \in \pi(\tilde{K})} C(z) \). Since \( \ell(C(z)) \), the length of the contour \( C(z) \), is a continuous function of \( z \in \pi(\tilde{K}) \), it also holds that there exists a positive
constant $C_3$ such that
\begin{equation}
(9.9) \quad \sup_{z \in \pi(K)} |\ell(C(z))| = C_3 < \infty.
\end{equation}

By (9.2), (9.5), (9.7), (9.8) and (9.9), we can estimate the right-hand side of (9.1) by absolute value. Thus we obtain the estimate of $|\Phi^z_{A+k}(t,x)|$, which is stated as (9.10) in the following proposition.

We also derive the estimates for $|\delta^{-i+} \Phi^z_{A+k+1}(t,x)|$. We obtain the expression of $\Phi^z_{A+k+1}(t,x)$ easily from (9.1) as
\begin{equation}
\delta^{-i+} \Phi^z_{A+k+1}(t,x) = -\frac{1}{2\pi i} \frac{e^{i\pi \alpha}}{(1 - e^{2\pi i })} \Gamma(A + k + 1) \times \int_C \zeta^{-j} (1 - \zeta)^{A-1} (1 - z\zeta)^{-\beta -1} z^{k+1} d\zeta.
\end{equation}

Hence, by the same discussion as above, we obtain the estimate by absolute value of this function, which is stated as (9.11) in the following proposition.

**Proposition 9.1.** Let $r > 0$ be sufficiently small, and let $\bar{K}$ be a compact subset of $\Omega_r \setminus \Sigma$. Then there exist positive constants $M_1$, $M_2$ and $A$ such that the following two inequalities hold for all $(t,x) \in \bar{K}$:
\begin{align}
(9.10) \quad |\Phi^z_{A+k}(t,x)| &\leq \frac{M_1 A^k}{(k+1)!} r^k, \\
(9.11) \quad |\delta^{-i+} \Phi^z_{A+k+1}(t,x)| &\leq \frac{M_2 A^{k+1}}{(k+2)!} r^{k+1}.
\end{align}

Now we proceed to the final stage of the proof of convergence of the series (4.13).

Let $r > 0$ be sufficiently small, and let $\bar{K}$ be a compact subset of $\Omega_r \setminus \Sigma$. Since $r > 0$ is sufficiently small, the inequalities (8.34) and (8.35) hold for all $(t,x) \in \bar{K}$. Multiplying (8.34) by (9.10), we obtain the inequality that
\begin{equation}
|G_k(t,x) \Phi^z_{A+k}(t,x)| \leq M_1 M_1' (ABr)^k
\end{equation}
for all $(t,x) \in \bar{K}$. Similarly, multiplying (8.35) by (9.11), we also obtain the inequality that
\begin{equation}
|H_k(t,x) \delta^{-i+} \Phi^z_{A+k+1}(t,x)| \leq \frac{M_2 M_2'}{(k+2)!} (ABr)^{k+1}.
\end{equation}

Hence we conclude that the series (4.13) converges uniformly on any compact subset $\bar{K}$ contained in $\Omega_r \setminus \Sigma$ if $r > 0$ is sufficiently small.

We have therefore completed the proof of Theorem 4.1.
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(Ricevita la 25-an de septembro, 2012)