Numerical simulation of 2D and 3D compressible flows

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Abstract. The work deals with numerical solutions of 2D inviscid and laminar compressible flows in the GAMM channel and DCA 8\% cascade, and of 3D inviscid compressible flows in a 3D modification of the GAMM channel (Swept Wing). The FVM multistage Runge-Kutta method and the Lax-Wendroff scheme (Richtmyer’s form) with Jameson’s artificial dissipation were applied to obtain the numerical solutions. The results are discussed and compared to other similar results and experiments.

1. Introduction
The goal of the work was to achieve experience and knowledge in the field of numerical simulations of inviscid and viscous (laminar) compressible flows and their future applications in the field of turbulence modelling, etc.

2. Mathematical models
2.1. Navier-Stokes equations
A general two-dimensional flow of a compressible laminar fluid\textsuperscript{4} is described by the system of Navier-Stokes equations that was applied in the dimensionless form\textsuperscript{5}.

\[ W_t + F_x + G_y = \frac{Ma_\infty}{Re_\infty} (R_x + S_y) \]  (1)

where

\[ W = (\rho, \rho u, \rho v, e)^T, F = (\rho u, \rho u^2 + p, \rho uv, (e + p)u)^T, G = (\rho v, \rho uv, \rho v^2 + p, (e + p)v)^T \]  \quad \text{(2)}

\[ R = (0, \tau_{xx}, \tau_{xy}, u\tau_{xx} + v\tau_{xy} - q_x)^T, S = (0, \tau_{xy}, \tau_{yy}, u\tau_{xy} + v\tau_{yy} - q_y)^T \]

and

\[ p = (\gamma - 1) \left[ e - \frac{1}{2} (u^2 + v^2) \right], \quad \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \frac{1}{3} \delta_{ij} \frac{\partial u_k}{\partial x_k}, \quad q_j = -\frac{\gamma}{\gamma - 1} \frac{\mu}{Pr} \frac{\partial}{\partial x_j} \left( \frac{p}{\rho} \right) \]  (3)

\textsuperscript{3} If readers have questions or comments, they can write to jar.huml@gmail.com as well.
\textsuperscript{4} The authors assumed that the flowing medium was a calorically perfect gas – the Newtonian fluid.
\textsuperscript{5} The inflow variables (\cdot)_\infty were used as the reference variable for the transformation of the equations to the dimensionless form (e.g. \bar{\rho} = \rho/\rho_\infty).
2.2. Euler equations
Assuming an inviscid fluid ($\mu = 0$ in viscous fluxes $R, S$ in the 2D case), the system of Euler equations is used instead of the system of Navier-Stokes equations.

$$W_t + F_x + G_y + H_z = 0$$

(4)

where a vector of conservative variables $W$ and vectors of inviscid fluxes $F, G$ are extended by terms $\rho w, \rho uw$ and $\rho vw$ respectively. The third vector of the inviscid flux in the direction of $z$-axis is

$$H = \begin{pmatrix} \rho w, \rho uw, \rho vw, \rho w^2 + p, (e + p)w \end{pmatrix}^T.$$  

(5)

3. Numerical methods
For simulation of the flows mentioned previously, two finite volume method numerical schemes were applied to the cell centered form on non-orthogonal structured grids of quadrilateral or hexahedral cells.

- Lax-Wendroff scheme in Richtmyer’s predictor-corrector form ($LW$)
- multistage Runge-Kutta method ($RK$)

Both schemes were used for modelling the two-dimensional flow in the GAMM channel. For other mentioned simulations only the multistage Runge-Kutta method was used.

Each scheme was extended by including Jameson’s artificial dissipation to improve the stability of the method.

4. Formulation of the problems
The authors took into account the numerical simulations of 2D and 3D inviscid compressible flows in the GAMM channel and its 3D modification (denoted as Swept Wing) and 2D laminar compressible flows in a DCA 8% cascade. The outlines of all computational domains are shown in Figures 1 and 2.

Figure 1. GAMM channel and its three-dimensional modification Swept Wing

Figure 2. DCA 8% cascade

6 The numerical schemes and artificial dissipation used are presented in detail, e.g. in [4] or in [6].
4.1. Boundary conditions

- **inlet**: $\rho_1 = 1$, $u_1 = Ma_1 \cos \alpha$, $v_1 = Ma_1 \sin \alpha$, $(w_1 = 0)$, $p_1$ was extrapolated from the flow field and $e_1$ was calculated using the equation of state. The angle of attack $\alpha$ was non-zero only for the flows in the DCA 8% cascade.
- **outlet**: $p_2$ was prescribed and the other components were extrapolated from the flow field or calculated.
- **solid wall**: velocity components were prescribed so that the sum of velocity vectors equals zero (viscous flows – $u = v = 0$) or equals zero in their tangential component (inviscid flows – $(u,v,w)\vec{n} = 0$).
- **periodicity**: the value of the variable in the cell at the bottom/top part of the boundary corresponds to the value in the cell from the flow field near the top/bottom part of the boundary.

5. Numerical results and discussion

For numerical simulation of 2D inviscid compressible flows in the GAMM channel, the authors applied both numerical schemes (LW and RK) on a structured non-orthogonal mesh with $240 \times 50$ cells and compared with J. Fürst’s results (e.g. [5]) obtained by WLSQR scheme on two different computational grids with $150 \times 50$ and $300 \times 100$ cells – see Figure 3. If we compared the maximums of Mach number on the lower wall, we would found out that the values $Ma_{\text{max}} = 1.42$ (LW) and 1.39 (RK) are larger than the results by J. Fürst ($Ma_{\text{max}} = 1.29$ or 1.32 – the fine mesh) and those presented in literature (1.37).

**Figure 3.** Inviscid compressible flow in the GAMM channel at $Ma_{\infty} = 0.675$: Mach number isolines – RK scheme, mesh with $240 \times 50$ cells (left) and WLSQR scheme [5], mesh with $300 \times 100$ cells (right).

In the case of 3D inviscid compressible flows around the Swept Wing, RK scheme and a structured non-orthogonal grid with $110 \times 30 \times 10$ cells were used only and compared with J. Holman’s results ([3]) gained by WLSQR scheme on a mesh with $180 \times 35 \times 35$ cells (see Figure 5). The comparison shows a finer mesh should have been used for better detection of the Zierep singularity.

The authors applied RK scheme and a non-orthogonal structured grids with $170 \times 120$ cells for modelling of 2D inviscid and laminar transonic flows in the DCA 8% cascade at different values of inlet Mach $Ma_{\infty}$, Reynolds numbers $Re_{\infty}$ and angles of attack $\alpha$. The results were compared with P.Pořízková results ([7]) and the experiment carried out by R. Dvořák ([11]) at the Institute of Thermodynamics of Academy of Sciences of the Czech Republic.

Based on the comparisons with other similar numerical results or experiments, the software developed by the authors has proved itself (the good mutual agreement in all the simulated cases was achieved) and it can be used in the future.

6. Conclusions

This article presents a few results achieved by using the software with the implemented FVM Lax-Wendroff scheme and multistage Runge-Kutta method with Jameson’s artificial dissipation
**Figure 4.** Inviscid compressible flow around the Swept Wing at $Ma_\infty = 0.675$: Mach number isosurface – RK scheme, mesh with $110 \times 30 \times 10$ cells (left) and WLSQR scheme [3], mesh with $180 \times 35 \times 35$ cells (right).

**Figure 5.** Laminar compressible flow in the DCA 8% cascade at $Ma_\infty = 1.1$, $Re_\infty = 2.1 \cdot 10^6$ and $\alpha = 0^\circ$: Mach number isolines – RK scheme, mesh with $170 \times 120$ cells (left) and MacCormack scheme [7], mesh with $150 \times 30$ cells (right).

for the simulation of a 2D and 3D transonic flow of the inviscid and laminar compressible fluid in the GAMM channel and its 3D modification, and in the DCA 8% cascade. Numerical results show the good agreement with other numerical results (e.g. J. Holman [3], P. Pořízková [7]) and experimental results carried out at the Institute of Thermodynamics AS CZ.

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