Notes on Simple Modules over Leavitt Path Algebras

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Abstract
Given an arbitrary graph $E$ and any field $K$, a new class of simple modules over the Leavitt path algebra $L_K(E)$ is constructed by using vertices that emit infinitely many edges in $E$. The corresponding annihilating primitive ideals are also described. Using a Boolean subring of idempotents, bounds for the cardinality of the set of distinct isomorphism classes of simple $L_K(E)$-modules are given. We also append other information about the algebra $L_K(E)$ of a finite graph $E$ over which every simple module is finitely presented.

1 Introduction and Preliminaries

Leavitt path algebras were introduced in [1, 9] as algebraic analogues of graph C*-algebras and as natural generalizations of Leavitt algebras of type $(1,n)$ built in [17]. The various ring-theoretical properties of these algebras have been actively investigated in a series of papers (see, for e.g., [1, 4, 5, 9, 11, 13, 18, 19, 23]). In contrast, the module theory of Leavitt path algebras $L_K(E)$ of arbitrary directed graphs $E$ over a field $K$ is still at its infancy. The initial organized attempt to study $L_K(E)$-modules was done in [8] where, for a finite graph $E$, the simply presented $L_K(E)$-modules were described in terms of finite dimensional representations of the usual path algebras of the reverse graph $\bar{E}$ of $E$. As an important step in the study of modules over a Leavitt path algebra $L_K(E)$, the investigation of the simple $L_K(E)$-modules has recently received some attention (see [10, 11, 12, 14]). Following the ideas of Smith [21], Chen [14] constructed irreducible representations of $L_K(E)$ by using sinks and tail-equivalent classes of infinite paths in the graph $E$. Chen’s construction was expanded in [10] to introduce additional classes of non-isomorphic simple $L_K(E)$-modules.

In section 2 of this paper, we construct a new class of simple left $L_K(E)$-modules induced by vertices which are infinite emitters and at the same time streamline the process of construction of certain simple modules introduced in [10]. A description of the annihilating primitive ideals of these
simple modules shows that these new simple modules are distinct from (i.e., not isomorphic to) any of the previously constructed simple $L_K(E)$-modules in [10, 12] and [14]. In section 3, we adapt the ideas of Rosenberg [20] to show that the cardinality of any single isomorphism class of simple left $L_K(E)$-modules has at most the cardinality of $L_K(E)$. Using a Boolean subring of commuting idempotents induced by the paths in $L_K(E)$, we obtain a lower bound for the cardinality of the set of non-isomorphic simple $L_K(E)$-modules. In particular, if $L_K(E)$ is a countable dimensional simple algebra, then it will have exactly 1 or at least $2^\aleph_0$ distinct isomorphism classes of simple modules. In section 4, we include some improvements and simplification of the results of [6] dealing with the structure of Leavitt path algebras over which every simple module is finitely presented.

For the general notation, terminology and results in Leavitt path algebras, we refer to [1], [2], [9]. We give below a short outline of some of the needed basic concepts and results.

A (directed) graph $E = (E^0, E^1, r, s)$ consists of two sets $E^0$ and $E^1$ together with maps $r, s : E^1 \to E^0$. The elements of $E^0$ are called vertices and the elements of $E^1$ edges. All the graphs $E$ that we consider (excepting those studied in section 4) are arbitrary in the sense that no restriction is placed either on the number of vertices in $E$ or on the number of edges emitted by a single vertex. Also $K$ stands for an arbitrary field.

A vertex $v$ is called a sink if it emits no edges and a vertex $v$ is called a regular vertex if it emits a non-empty finite set of edges. An infinite emitter is a vertex which emits infinitely many edges. For each $e \in E^1$, we call $e^*$ a ghost edge. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. A path $\mu$ of length $n > 0$ is a finite sequence of edges $\mu = e_1e_2\cdots e_n$ with $r(e_i) = s(e_{i+1})$ for all $i = 1, \ldots, n - 1$. In this case $\mu^* = e_n^*\cdots e_2^*e_1^*$ is the corresponding ghost path. Any vertex is considered a path of length 0. The set of all vertices on the path $\mu$ is denoted by $\mu^0$.

A path $\mu = e_1\ldots e_n$ in $E$ is closed if $r(e_n) = s(e_1)$, in which case $\mu$ is said to be based at the vertex $s(e_1)$. A closed path $\mu$ as above is called simple provided it does not pass through its base more than once, i.e., $s(e_i) \neq s(e_1)$ for all $i = 2, \ldots, n$. The closed path $\mu$ is called a cycle if it does not pass through any of its vertices twice, that is, if $s(e_i) \neq s(e_j)$ for every $i \neq j$.

If there is a path from vertex $u$ to a vertex $v$, we write $u \geq v$. A subset $D$ of vertices is said to be downward directed if for any $u, v \in D$, there exists a $w \in D$ such that $u \geq w$ and $v \geq w$. A subset $H$ of $E^0$ is called hereditary if, whenever $v \in H$ and $w \in E^0$ satisfy $v \geq w$, then $w \in H$. A hereditary set is saturated if, for any regular vertex $v$, $r(s^{-1}(v)) \subseteq H$ implies $v \in H$.

Given an arbitrary graph $E$ and a field $K$, the Leavitt path algebra $L_K(E)$ is defined to be the $K$-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents together with a set of variables $\{e, e^* : e \in E^1\}$ which satisfy the following conditions:

1. $s(e)e = e = er(e)$ for all $e \in E^1$.
2. $r(e)e^* = e^* = e^*s(e)$ for all $e \in E^1$.
3. (The "CK-1 relations") For all $e, f \in E^1$, $e^*e = r(e)$ and $e^*f = 0$ if
e \neq f.

(4) (The "CK-2 relations") For every regular vertex \( v \in E^0 \),

\[ v = \sum_{e \in E^1, s(e) = v} ee^*. \]

For any vertex \( v \), the tree of \( v \) is \( T(v) = \{ w : v \succeq w \} \). We say there is a bifurcation at a vertex \( v \), if \( v \) emits more than one edge. In a graph \( E \), a vertex \( v \) is called a line point if there is no bifurcation or a cycle based at any vertex in \( T(v) \). Thus, if \( v \) is a line point, there will be a single finite or infinite line segment \( \mu \) starting at \( v \) (\( \mu \) could just be \( v \)) and any other path \( \alpha \) with \( s(\alpha) = v \) will just be an initial sub-segment of \( \mu \). It was shown in [12] that \( v \) is a line point in \( E \) if and only if \( vL_K(E) \) (and likewise \( L_K(E)v \)) is a simple left (right) ideal.

We shall be using the following concepts and results from [23]. A breaking vertex of a hereditary saturated subset \( H \) is an infinite emitter \( w \in E^0 \setminus H \) with the property that \( 1 \leq |s^{-1}(w) \cap r^{-1}(E^0 \setminus H)| < \infty \). The set of all breaking vertices of \( H \) is denoted by \( B_H \). For any \( v \in B_H \), \( v^H \) denotes the element \( v - \sum_{e=v, r(e) \notin H} ee^* \). Given a hereditary saturated subset \( H \) and a subset \( S \subseteq B_H \), \( (H,S) \) is called an admissible pair. Given an admissible pair \( (H,S) \), the ideal generated by \( H \cup \{ v^H : v \in S \} \) is denoted by \( I_{(H,S)} \). It was shown in [23] that the graded ideals of \( \text{Leavitt path algebra} \ L_K(E) \) are precisely the ideals of the form \( I_{(H,S)} \) for some admissible pair \( (H,S) \). Moreover, \( L_K(E)/I_{(H,S)} \cong L_K(E \setminus \{ H,S \}) \). Here \( E \setminus \{ H,S \} \) is the Quotient graph of \( E \) in which \( (E \setminus \{ H,S \})^0 = (E^0 \setminus H) \cup \{ v^H : v \in B_H \setminus S \} \) and \( (E \setminus \{ H,S \})^1 = \{ e \in E^1 : r(e) \notin H \} \cup \{ e^H : e \in E^1 \setminus B_H \setminus S \} \) and \( r, s \) are extended to \( (E \setminus \{ H,S \})^0 \) by setting \( s(e^H) = s(e) \) and \( r(e^H) = r(e) \).

A useful observation is that every element \( a \) of \( L_K(E) \) can be written as

\[ a = \sum_{i=1}^n k_i \alpha_i \beta_i^* \]

where \( k_i \in K \), \( \alpha_i \), \( \beta_i \) are paths in \( E \) and \( n \) is a suitable integer. Moreover, \( L_K(E) = \oplus_{e \in E^0} L_K(E)v = \oplus_{e \in E^0} vL_K(E) \) (see [1]).

Even though the Leavitt path algebra \( L_K(E) \) may not have the multiplicative identity 1, we shall write \( L_K(E)(1-v) \) to denote the set \( \{ x- xv : x \in L_K(E) \} \). If \( v \) is an idempotent or a vertex, we get a direct decomposition \( L_K(E) = L_K(E)v \oplus L_K(E)(1-v) \).

2 A new class of simple modules

Let \( E \) be an arbitrary graph. Throughout this section, we shall use the following notation.

For \( v \in E^0 \) define

\[ M(v) = \{ w \in E^0 : w \succeq v \} \quad \text{and} \quad H(v) = E^0 \setminus M(v) = \{ u \in E^0 : u \neq v \}. \]

Clearly \( M(v) \) is downward directed. Also, for any vertex \( v \) which is a sink or infinite emitter, the set \( H(v) \) is a hereditary saturated subset of \( E \). If \( v \) is a
finite emitter, it might be that $H(v)$ is not saturated, and that $v$ belongs to the saturation of $H(v)$.

For convenience in writing, we shall denote the Leavitt path algebra $L_K(E)$ by $L$.

**Definition 2.1** The $L$-module $S_v^\infty$:

Suppose $v$ is an infinite emitter in $E$. Define $S_v^\infty$ to be the $K$-vector space having as a basis the set $B = \{ p : p$ a path in $E$ with $r(p) = v \}$. Following Chen [17], we define, for each vertex $u$ and each edge $e$ in $E$, linear transformations $P_u$, $S_e$ and $S_{e^*}$ on $S_v^\infty$ as follows:

For all paths $p \in B$,

$$
P_u(p) = \begin{cases} 
p, & \text{if } u = s(p) \\
0, & \text{otherwise} 
\end{cases}
$$

$$
S_e(p) = \begin{cases} 
e p, & \text{if } r(e) = s(p) \\
0, & \text{otherwise} 
\end{cases}
$$

$$
S_{e^*}(v) = 0
$$

$$
S_{e^*}(p) = \begin{cases} 
p', & \text{if } p = ep' \\
0, & \text{otherwise} 
\end{cases}
$$

Then it can be checked that the endomorphisms $\{ P_u, S_e, S_{e^*} : u \in E^0, e \in E^1 \}$ satisfy the defining relations (1) - (4) of the Leavitt path algebra $L$. This induces an algebra homomorphism $\phi$ from $L$ to $\text{End}_K(S_v^\infty)$ mapping $u$ to $P_u$, $e$ to $S_e$ and $e^*$ to $S_{e^*}$. Then $S_v^\infty$ can be made a left module over $L$ via the homomorphism $\phi$. We denote this $L$-module operation on $S_v^\infty$ by $\cdot$.

**Remark 2.2** The above construction does not work if $v$ is a regular vertex. Specifically, the needed CK-2 relation $\sum_{e \in s^{-1}(v)} S_e S_{e^*} = P_v$ does not hold. Because, on the one hand $\left( \sum_{e \in s^{-1}(v)} S_e S_{e^*} \right)(v) = 0$ but on the other hand $P_v(v) = v \neq 0$.

**Proposition 2.3** For each infinite emitter $v$ in $E$, $S_v^\infty$ is a simple left module over $L_K(E)$.

**Proof.** Suppose $U$ is non-zero submodule of $S_v^\infty$ and let

$$
0 \neq a = \sum_{i=1}^{n} k_i p_i \in U
$$

where $k_i \in K$ and the $p_i$ are paths in $E$ with $r(p_i) = v$ and we assume that the paths $p_i$ are all different.

By induction on $n$, we wish to show that $v \in U$. Suppose $n = 1$ so that $a = k_1 p_1$. Then $p_1^* \cdot a = k_1 v \in U$ and we are done. Suppose $n > 1$ and assume that $v \in U$ if $U$ contains a non-zero element which is a $K$-linear combination of less than $n$ paths. Among the paths $p_i$, assume that $p_1$ has the smallest length. If $p_1$ has length 0, that is, if $p_1 = v$ and, for some $s$, $p_s$ is a path of length $> 0$,
then since \( p_k^* \cdot v = 0, p_k^* \cdot a \in U \) will be a sum of less than \( n \) terms and so by induction \( v \in U \). Suppose \( p_1 \) has length \( > 0 \). Now \( p_1^* \cdot a = a' \in U \). If \( p_1^* \cdot a \) is a sum of less than \( n \) terms, we are done. Otherwise, \( a' = k_1 p_1^* \cdot p_1 + b = k_1 v + b \) where \( b \) is a sum of less than \( n \) terms with its first non-zero term, say, \( k_i p_i \). Then \( p_1^* \cdot a' = p_1^* \cdot b \in U \) and \( 0 \neq p_1^* \cdot b \) is a sum of less than \( n \) terms. Hence by induction, \( v \in U \) and we conclude that \( U = S_{v \infty} \). ■

The next proposition describes the annihilating primitive ideal of the simple module \( S_{v \infty} \).

**Proposition 2.4** Let \( v \) be an infinite emitter. Then

\[
\text{Ann}_{L_k(E)}(S_{v \infty}) = \left\{ \begin{array}{ll}
I(H(v), B_{H(v)}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| = 0; \\
I(H(v), B_{H(v)} \setminus \{v\}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| \neq 0 \text{ and finite} \\
I(H(v), B_{H(v)}), & \text{if } |s^{-1}(v) \cap r^{-1}(M(v))| \text{ is infinite.}
\end{array} \right.
\]

**Proof.** Let \( J = \text{Ann}_{L_k(E)}(S_{v \infty}) \). Clearly \( H(v) \subset J \) since for any \( u \in H(v) \), \( u \nRightarrow v \) and so \( u \cdot p = 0 \) for all \( p \) with \( r(p) = v \). Indeed \( J \cap E^0 = H(v) \).

Suppose \( |s^{-1}(v) \cap r^{-1}(M(v))| = 0 \) so that \( r(s^{-1}(v)) \subseteq H(v) \). Let \( u \in B_{H(v)} \).

Clearly \( uH(v) \cdot p = 0 \) if \( p \) is a path with \( r(p) = v \) and \( s(p) \neq u \). On the other hand, if \( p \) is a path from \( u \) to \( v \) with \( p = ep' \) where \( e \) is an edge, then

\[
uH(v) \cdot p = (u - \sum_{f \in s^{-1}(u), r(f) \notin H(v)} ff^*) \cdot p = (e - e)p' = 0.
\]

This shows that \( I(H(v), B_{H(v)}) \subseteq J \). Now \( J \cap E^0 = H(v) \). If \( J \) were a non-graded ideal, then it follows from the proof of Theorem 3.12 (iii) of [18], that \( v \) will be the base of a cycle \( c \) with \( c^0 \subset M(v) \). In particular, there is an edge \( e \) with \( s(e) = v \) and \( r(e) \in M(v) \). But this is not possible since \( r(s^{-1}(v)) \subseteq H(v) \).

Thus \( J \) is a graded ideal with \( J \cap E^0 = H(v) \). Since \( I(H(v), B_{H(v)}) \) is the largest graded ideal for which \( I(H(v), B_{H(v)}) \cap E^0 = H(v) \), we conclude that \( J = I(H(v), B_{H(v)}) \).

Suppose \( |s^{-1}(v) \cap r^{-1}(M(v))| \neq 0 \) and is finite so that \( v \in B_{H(v)} \). If \( u \in B_{H(v)} \) with \( u \neq v \), then the arguments in the preceding paragraph shows that \( uH(v) \in J \). But \( vH(v) \notin J \) since

\[
vH(v) \cdot v = (v - \sum_{f \in s^{-1}(v), L(f) \notin H(v)} ff^*) \cdot v = v \cdot v = 0 = v \neq 0.
\]

This shows that \( I(H(v), B_{H(v)} \setminus \{v\}) \subseteq J \). Now \( J \) is a primitive ideal with \( J \cap E^0 = H(v) \). If \( J \) were a non-graded ideal, then from the description of the primitive ideals in Theorem 4.3 of [18], we will have \( I(H(v), B_{H(v)}) \subseteq J \) and this is not possible since \( vH(v) \notin J \). We then conclude that the graded ideal \( J \) must be equal to \( I(H(v), B_{H(v)} \setminus \{v\}) \).

Finally, suppose \( |s^{-1}(v) \cap r^{-1}(M(v))| \) is infinite. This means, in particular, there are infinitely many cycles in \( M(v) \) based at \( v \). As \( M(v) = E^0 \backslash H(v) \), it is then clear from Theorem 3.12 of [18] that \( J \) cannot be a non-graded ideal. Also, as \( v \notin B_{H(v)} \), the earlier arguments show that \( I(H(v), B_{H(v)}) \subseteq J \).

Observing that \( I(H(v), B_{H(v)}) \cap E^0 = H(v) = J \cap E^0 \), we then conclude that \( J = I(H(v), B_{H(v)}) \). ■
Before proceeding further, we shall review the construction of some of the simple modules introduced in [13] and [10] and refer them as simple modules of type 1, 2 or 3. In this connection, we wish to point out that the notation and terminology used by Chen in [13] is different from those used in papers on Leavitt path algebras such as [4] while we shall follow that of [13].

Type-1 Simple Module: Chen [14] defines an equivalence relation among infinite paths by using the following notation. If \( p = e_1 e_2 \cdots e_n \cdots \) is an infinite path where the \( e_i \) are edges, then for any positive integer \( n \), let \( \tau_{\leq n}(p) = e_1 e_2 \cdots e_n \) and \( \tau_{> n}(p) = e_{n+1} e_{n+2} \cdots \). Two infinite paths \( p \) and \( q \) are said to be tail equivalent, in symbols, \( p \sim q \), if there exist positive integers \( m \) and \( n \) such that \( \tau_{> n}(p) = \tau_{> m}(q) \). Then \( \sim \) is an equivalence relation.

Given an equivalence class of infinite paths \([p]\), let \( V_{[p]} \) denote the \( K \)-vector space having the set \( \{ q : q \in [p] \} \) as a basis. Then Chen [14] defines an \( L \)-module operation on \( V_{[p]} \) making \( V_{[p]} \) a left \( L \)-module similar to the way the module operation is defined on \( S_\infty \) above, except that the condition that \( S_\infty(v) = 0 \) for any edge \( e \) is dropped. Chen [14] shows that the module \( V_{[p]} \) becomes a simple \( L \)-module.

Type-2 Simple Module: Let \( w \) be a sink and \( N_w \) be a \( K \)-vector space having as a basis the set \( \{ p : p \text{ paths in } E \text{ with } L(p) = w \} \). Proceeding as was done above for \( S_\infty \), Chen [14] defines an \( L \)-module action on \( N_w \) and shows that \( N_w \) becomes a simple module.

Type-3 Simple Modules: These additional classes of simple \( L \)-modules, denoted respectively by \( N_v^{B_{H(v)}} \), \( N_v^{H(v)} \) and \( V_{[p]} \), were introduced in [10]:

(i) Suppose that \( v \) is an infinite emitter such that \( v \in B_{H(v)} \). Then we can build the primitive ideal \( P = I_{(H(v), B_{H(v)} \setminus \{ v \})} \) (see [18]) and the factor ring
\[
L_K(E)/P \cong L_K(F)
\]
where \( F = E \setminus (H(v), B_{H(v)} \setminus \{ v \}) \). Then \( F^0 = (E^0 \setminus H(v)) \cup \{ v' \} \),
\[
F^1 = \{ e \in E^1 : r(e) \notin H(v) \} \cup \{ e' : e \in E^1, r(e) = v \}
\]
and \( r \) and \( s \) are extended to \( F \) by \( s(e') = s(e) \) and \( r(e') = v' \) for all \( e \in E^1 \) with \( r(e) = v \). Note that \( v' \) is a sink in \( F \) and it is easy to see that \( M_F(v') = F^0 \).

Accordingly, we may consider the Type 2 simple module \( N_v^{B_{H(v)}} \) of \( L_K(F) \) introduced by Chen corresponding to the sink \( v' \) in \( F \). Using the quotient map \( L_K(E) \to L_K(F) \), we may view \( N_v^{B_{H(v)}} \) as a simple module over \( L_K(E) \). This simple \( L_K(E) \)-module is denoted by \( N_v^{B_{H(v)}} \).

(ii) Suppose \( v \) is an infinite emitter and such that \( r(s^{-1}(v)) \subseteq H(v) \). Then \( v \) is the unique sink in the graph \( G = E \setminus (H(v), B_{H(v)}) \). Let \( N_v \) be the corresponding Type 2 simple \( L_K(E \setminus (H(v), B_{H(v)}) \)-module introduced by Chen. It is clear that \( N_v \) is a faithful simple \( L_K(G) \)-module. Consider \( N_v \) as a simple \( L_K(E) \)-module through the quotient map \( L_K(E) \to L_K(G) \). This simple module is denoted by \( N_v^{H(v)} \).

(iii) For any infinite path \( p \), \( V_{[p]} \) is the twisted simple \( L_K(E) \)-module obtained from the simple \( L_K(E) \)-module \( V_{[p]} \). See [10] for details.
Proposition 2.5 If \( v \) is an infinite emitter such that \( s^{-1}(v) \cap r^{-1}(M(v)) \neq 0 \) and is finite, then \( S_{v_\infty} \cong N_v^{B_{H(v)}} \).

Proof. From Proposition 2.4 above and Lemma 3.5 of [10], it is clear that both \( S_{v_\infty} \) and \( N_v^{B_{H(v)}} \) are annihilated by the same primitive ideal. Also the \( K \)-bases of \( S_{v_\infty} \) and \( N_v^{B_{H(v)}} \) are in bijective correspondence. Indeed if \( p = p'e \) is a path with \( r(p) = v \), then, in the graph \( F \) defined in type-3 (i) simple module above, \( r(e') = v' \) and \( s(e') = s(e) = r(p') \) and so \( p'e' \) is a path in \( F \) with \( r(p'e') = v' \). Then \( v \mapsto v' \) and \( p = p'e \mapsto p'e' \) is the desired bijection. It is then clear that the map \( \phi : S_{v_\infty} \rightarrow N_v^{B_{H(v)}} \) given by \( \phi(v) = v' \) and \( \phi(p'e) = p'e' \) extends to an isomorphism from \( S_{v_\infty} \) to \( N_v^{B_{H(v)}} \). \( \blacksquare \)

Proposition 2.6 If \( v \) is an infinite emitter for which \( |s^{-1}(v) \cap r^{-1}(M(v))| = 0 \), then \( S_{v_\infty} \cong N_v^{H(v)} \).

Proof. This is immediate after observing that these two simple modules have the same \( K \)-basis and the same annihilating primitive ideal. \( \blacksquare \)

Notation 2.7 In conformity with the notation used in [10], when \( v \) is an infinite emitter for which \( |s^{-1}(v) \cap r^{-1}(M(v))| \) is infinite, we shall denote the corresponding simple module \( S_{v_\infty} \) by \( N_{v_\infty} \).

Proposition 2.8 The new simple module \( N_{v_\infty} \) is not isomorphic to any of the previously defined simple \( L \)-modules of Type 1, 2 or 3.

Proof. For convenience, we list the simple modules of type 1, 2 and 3 as \( N_{v_1}^{B_{H(v_1)}} \), \( N_{v_2}^{H(v_2)} \), \( V'_1 \), \( V'_2 \). Now \( N_{v_\infty} \not\cong V'_1 \) since the annihilator of \( V'_1 \) is a non-graded primitive ideal ([10], Lemma 2.4) while, as we proved in Proposition 2.3, \( \text{Ann}_L(S_{v_\infty}) = I_{H(v_1),B_{H(v_1)}} \) is a graded ideal. The proof that \( N_{v_\infty} \not\cong V'_2 \) uses the same argument of Chen ([14], Theorem 3.7 (3)). We give the proof for completeness. Suppose \( \varphi : N_{v_\infty} \rightarrow V'_2 \) is an \( L \)-morphism. We claim \( \varphi = 0 \), that is, \( \varphi(v) = 0 \). Otherwise, write \( \varphi(v) = \sum_{i=1}^{n} k_i q_i \) where \( q_i \in [p] \) and assume that the \( q_i \) are all different. Choose \( n \) so that \( \tau_{\leq n}(q_i) \) are all pairwise different. Now in the definition of \( N_{v_\infty} \) as an \( L \)-module, \( e^* \cdot v = 0 \) for all \( e \in E^1 \) and so \( \tau_{\leq n}(q_1)^* \cdot v = 0 \), but \( \varphi(\tau_{\leq n}(q_1)^* \cdot v) = \tau_{\leq n}(q_1)^* \cdot \varphi(v) = k_1 \tau_{\geq n}(q_1) \neq 0 \), a contradiction. Hence \( N_{v_\infty} \not\cong V'_2 \).

Since the annihilators of \( N_{v_1}^{B_{H(v_1)}} \), \( N_{v_2}^{H(v_2)} \) and \( N_{v_\infty} \) are all graded ideals, it is enough if we can show that the set of vertices belonging to the annihilators of these modules are all different. We first show that \( N_{v_\infty} \not\cong N_{v_1}^{B_{H(v_1)}} \). Now the vertex set \( H(w) \neq H(v) \), since otherwise \( M(w) = M(v) \) and this is not possible since \( M(w) \) contains a sink (namely, \( w \)), while \( M(v) \) does not. Hence \( N_{v_\infty} \not\cong N_{v_1}^{B_{H(v_1)}} \). Likewise, \( H(v_2) \neq H(v) \), since otherwise \( M(v_2) = M(v) \) which will imply that \( v_2 \geq v \) in \( M(v_2) \) contradicting the fact that \( v_2 \) is a sink in \( M(v_2) \). So \( N_{v_\infty} \not\cong N_{v_2}^{H(v_2)} \). Finally, the annihilators of \( N_{v_1}^{B_{H(v_1)}} \) and \( N_{v_\infty} \) (being \( I_{H,B_{H}}(v_1) \)) and \( I_{H,B_{H}} \) respectively) are different and so \( N_{v_1}^{B_{H(v_1)}} \not\cong N_{v_\infty} \). \( \blacksquare \)
3 The cardinality of the set of simple $L_K(E)$-modules

As before, $E$ denotes an arbitrary graph with no restrictions on the cardinality of $E^0$ or $E^1$. We wish to estimate the size of the isomorphism classes of simple left $L_K(E)$-modules. In this connection, we follow the ideas of Rosenberg [20]. However, we need to modify his arguments for the case of Leavitt path algebras which, among other differences, do not always have multiplicative identities.

We first show that, given a fixed simple module $S$, the cardinality of the set of all maximal left ideals $M$ of $L_K(E)$ such that $L_K(E)/M \cong S$ is at most the cardinality of $L_K(E)$. Using a Boolean subring of idempotents induced by the paths in $L_K(E)$, we obtain a lower bound for the cardinality of the set of non-isomorphic simple $L_K(E)$-modules. In particular, if $L_K(E)$ is a countable dimensional simple algebra, then it will have either exactly 1 or at least $2^{\aleph_0}$ distinct isomorphism classes of simple modules.

As before, we shall denote the Leavitt path algebra $L_K(E)$ by $L$. We begin with a simple description of maximal left ideals of $L$.

**Lemma 3.1** Suppose $M$ is a maximal left ideal of $L$. Then for any idempotent $\epsilon \notin M$, $M\epsilon \subset M$ and $M$ can be written as $M = N \oplus L(1-\epsilon)$ where $N = M \cap L\epsilon = M\epsilon$. Every simple left $L$-module $S$ is isomorphic to $Lv/N$ for some $v \in E^0$ and some maximal $L$-submodule $N$ of $Lv$.

**Proof.** Let $M$ be a maximal left ideal of $L$ and $\epsilon = e^2 \in L \setminus M$. If $x \in M \cap L\epsilon$ then $x = xe\epsilon$ and so $M \cap L\epsilon \subset M\epsilon$. By maximality, $M \cap L\epsilon = M\epsilon$, so $M\epsilon \subset M$ for all idempotents $\epsilon$. Writing each $x \in M$ as $x = xe + (x-xe)$, we obtain $M = M\epsilon \oplus M(1-\epsilon) \subset M\epsilon \oplus L(1-\epsilon)$. By maximality, $M = N \oplus L(1-\epsilon)$ where $N = M\epsilon = M \cap L\epsilon$ is a maximal $L$-submodule of $L\epsilon$.

Suppose $S$ is a simple left $L$-module, say $S = L/M$ for some maximal left ideal of $L$. Since $L = \bigoplus_{v \in E^0} Lv$ and $M \neq L$, there is a vertex $v \notin M$. By the preceding paragraph, we can write $M = N \oplus L(1-v)$ where $N = M \cap Lv$. Then $S = [(Lv \oplus L(1-v))/N \oplus L(1-v)] \cong Lv/N$. $\blacksquare$

**Lemma 3.2** Suppose $Lv/N$ is a simple left $L$-module with $v \in E^0$. Then, for any vertex $u$, a simple module $Lu/N'$ is isomorphic to $Lv/N$ if and only if there is an element $a = uav \in Lv$ such that $a \notin N$ and $N'a \subset N$. In this case, $N' = \{y \in Lu : ya \in N\}$.

**Proof.** Suppose $\sigma : Lu/N' \to Lv/N$ is an isomorphism. Let $\sigma(u+N') = x + N$ for some $x \in Lv$. Now $\sigma(u+M) = \sigma(u(u+M)) = u(x+N) = ux + N$. Then $a = ux$ satisfies $a = uav$, $a \notin N$ and $\sigma(a+N') = a + N$. Moreover, $N'a \subset N$ because, for any $y \in N'$, we have $ya + N = y(a+N) = y\sigma(u+M) = \sigma(y+M) = \sigma(0+M) = 0 + N$. Note that the left ideal $I = \{y \in Lu : ya \in N\}$ contains $N'$ and $I \neq Lu$ since $u \notin I$ (as $ua = a \notin N$). Hence $I = N'$, by the maximality of $N'$.
Conversely, suppose \( N'a \subset N \) for some \( a \) satisfying \( a \notin N \) and \( a = uav \). Define \( f : Lu/N' \to Lv/N \) by \( f(y + N') = ya + N \). Now \( N'a \subset N \) implies that \( f \) is well-defined and is a homomorphism. Now \( f \neq 0 \) since \( f(u + N') = ua + N = a + N \neq N \). As both \( Lu/N' \) and \( Lv/N \) are simple modules, \( f \) is an isomorphism.

**Lemma 3.3** Let \( v \) be a vertex and \( A = Lv/N \) be a simple left \( L \)-module. Suppose, for \( u, w \in E^0 \), \( B = Lu/N_1 \) and \( C = Lw/N_2 \) are both isomorphic to \( A \) and \( b = ubv \) and \( c = wcv \) are the corresponding elements satisfying \( b, c \notin N \), \( N_1b \subset N \) and \( N_2c \subset N \) as established in Lemma 3.2. Then \( B \neq C \) implies \( b \neq c \).

**Proof.** Suppose, on the contrary, \( b = c \). First of all \( u = w \) since otherwise \( b = ub = uc = uwcv = 0 \), a contradiction. Thus \( u = w \) and \( N_1, N_2 \) are maximal submodules of \( Lu \). Then \( N_1b \subset N \) and \( N_2c \subset N \) implies \((N_1 + N_2)b = Lub \subset N \). Since \( b = ub \), we get \( b \in N \), a contradiction.

From the preceding Lemmas we get the following Proposition.

**Proposition 3.4** (a) Let \( Lv/N \) be a given simple left \( L \)-module, where \( v \in E^0 \). For any fixed vertex \( u \), the cardinality of the set of all maximal submodules \( N' \) of \( Lu \) (and thus the cardinality of all maximal left ideals of \( L \) of the form \( M = N' \oplus L(1-u) \) of \( L \)) for which \( Lv/N' \cong Lv/N \cong L/M \) is at most the cardinality of \( uLv \).

(b) Given a fixed simple left \( L \)-module \( Lv/N \), the cardinality of the set of maximal left ideals \( M \) of \( L \) for which \( L/M \cong Lv/N \) is at most the cardinality of \( L \).

For subsequent applications, we obtain a sharpened version of Lemma 3.2 as follows.

**Lemma 3.5** Let \( v \in E^0 \) and \( Lv/N \) be a simple left \( L \)-module. Then, the maximal left ideals \( M \) of \( L \) for which \( L/M \cong Lv/N \) are precisely the annihilators in \( L \) of non-zero elements \( a + N \) of \( Lv/N \) with \( a = uav \) for some vertex \( u \).

**Proof.** Suppose \( L/M \cong Lv/N \) for some maximal left ideal \( M \) of \( L \). By Lemma 3.1 we can write \( M = N' \oplus L(1-u) \) where \( u \) is a vertex, \( u \notin M \) and \( N' = M \cap Lu \). By Lemma 3.2 there is an element \( a = uav \notin N \) so that \( a + N \) is non-zero and \( N' = \{ y \in Lu : ya \notin N \} = \{ y \in Lu : y(a + N) = N \} \). It is then clear that \( M = \{ L \in L : L(a + N) = N \} \).

Conversely, suppose the left ideal \( I \) is the annihilator in \( L \) of some non-zero element \( a + N \) of \( Lv/N \), where \( a \in uLv \) for some vertex \( u \). Let \( N' = \{ y \in Lu : y(a + N) = N \} = \{ y \in Lu : ya \notin N \} \). Now \( N' \neq Lu \) since \( u \notin N' \) due the fact that \( ua = a \notin N \). Define \( \phi : Lu/N' \to Lv/N \) by \( \phi(ru + N') = rua + N \). Clearly \( \phi \) is a well-defined homomorphism and \( \phi \neq 0 \), as \( \phi(u) = ua + N = a + N \). If \( rua + N = N \), then \( ru(a + N) = N \), so \( ru \in N' \) and \( ru + N' = N' \). Thus \( \ker(\phi) = 0 \). Since \( Lv/N \) is simple, \( \phi : Lu/N' \to Lv/N \) is an isomorphism. In particular, \( N' \) is a maximal \( L \)-submodule of \( Lu \). Then \( M = N' \oplus L(1-u) \) is a
maximal left ideal of \( L \), \( L/M \cong Lv/N \) and \( M \subset I \). By maximality, \( M = I \), the annihilator of \( a + N \) in \( L \). ■

In the context of Proposition 3.4(b), our next goal is to investigate the size of the set of all non-isomorphic simple left \( L \)-modules. Towards this end, we consider maximal left ideals of \( L \) that arise from a specified Boolean subring of idempotents in \( L \).

A special Boolean subring \( B \) of \( L \): Let \( S = E^0 \cup \{ \alpha \alpha^* : \alpha \) a finite path in \( E \} \cup \{ 0 \} \). Observe that elements of \( S \) are commuting idempotents. Moreover, if \( a, b \in S \), then it is easy to see that \( ab \in S \). Let \( B \) be the additive subgroup of \( L \) generated by \( S \). Define, for any two elements \( a, b \in S \), \( a \triangle b = a + b - 2ab \) and \( a \cdot b = ab \). Then \( B \) becomes a Boolean ring under the operations \( \triangle \) and \( \cdot \).

Define a partial order \( \preceq \) on \( B \) by setting, for any two elements \( a, b \in B \), \( a \preceq b \) if \( ab = a \). Then \( B \) becomes a lattice under the operations, \( a \lor b = a + b - ab \) and \( a \land b = ab \).

**Proposition 3.6** (a) If \( M' \) is a maximal left ideal of \( L \), then \( M = M' \cap B \) is a maximal ideal of \( B \) and \( M'' = N \oplus L(1 - v) \) for some vertex \( v \not\in M' \) where \( N = M' \cap Lv = M'v \) and \( M = Mv \oplus B(1 - v) \).

(b) Every maximal ideal \( M \) of \( B \) embeds in a maximal left ideal \( P_M \) of \( L \) such that \( P_M \cap B = M \). Thus different maximal ideals \( M_1, M_2 \) of \( B \) give rise to different maximal left ideals \( P_{M_1}, P_{M_2} \).

**Proof.** (a) If \( M' \) is a maximal left ideal of \( L \), then clearly \( M = M' \cap B \) is an ideal of \( B \). To show that \( M \) is maximal, it is enough if we show that \( M \) is a prime ideal of \( B \). Suppose \( x, y \in B \) such that \( xy \in M \) and \( x \not\in M \). Since \( Lx + M' = L \), we can write \( y = rx + m' \) where \( r \in L \) and \( m' \in M' \). Then \( y = y^2 = rxy + m'y \). By Lemma 5.1, \( m'y \in M'y \subset M' \) and so \( y \in M' \cap B = M \).

Thus \( M \) is a maximal ideal of \( B \). Let \( v \) be a vertex with \( v \not\in M' \). By Lemma 5.1, \( M'' = N \oplus L(1 - v) \) where \( N = M'v \). Note that \( v \in B \) and \( Mv \subset M \), as \( M \) is an ideal. Thus \( M = Mv \oplus M(1 - v) \subset Mv \oplus B(1 - v) \). By maximality, \( M = Mv \oplus B(1 - v) \).

(b) Let \( M \) be a maximal ideal of \( B \). Then there is at least one vertex \( v \not\in M \). Because if \( E^0 \subset M \), then for every path \( \alpha \) with, say \( s(\alpha) = u, \alpha \alpha^* = u\alpha \alpha^* \in M \), as \( M \) is an ideal of \( B \). This implies \( M = B \), a contradiction. We now claim that the left ideal \( LMv \neq Lv \). Suppose, by way of contradiction, assume that \( v \in LMv \), so that \( v = \sum_{i=1}^{k} r_i m_i v \) where \( m_i \in M \) and \( r_i \in L \). Observing that \( m = m_1 v \cdots m_k \) belongs to the ideal \( M \) and satisfies \( m_i m = m_i \) for all \( i \), we get \( mv = vm = \sum_{i=1}^{k} r_i m_i m v = \sum_{i=1}^{k} r_i m_i v = v \). This is not possible, since \( mv \in M \) while \( v \not\in M \). Thus \( LMv \) is a proper \( L \)-submodule of \( Lv \) and hence can be embedded in a maximal \( L \)-submodule \( N \) of \( Lv \). Writing each element \( x \in M \) as \( x = xv + (x - xv) \) we see that \( M \) embeds in the maximal left ideal \( P_M = N \oplus L(1 - v) \). By the maximality of \( M \), it is clear that \( P_M \cap B = M \). This implies that if \( M_1 \neq M_2 \) are maximal ideals of \( B \) embedding, as above, in maximal left ideals \( P_{M_1} \) and \( P_{M_2} \) of \( L \), then \( P_{M_1} \neq P_{M_2} \). ■
Corollary 3.7 The cardinality of the set of all maximal left ideals of $L$ is at least the cardinality of the set of all maximal ideals of $B$.

For each maximal ideal $M$ of $B$, choose one maximal left ideal $P_{M} = N \oplus L(1-v)$ where $N = P_{M}v$ as constructed in Proposition 3.6(b). Let $T$ denote the set of all such maximal left ideals $P_{M}$ of $L$. We shall call such $P_{M}$ a Boolean maximal left ideal corresponding to the maximal ideal $M$ of $B$ and call the simple module $L/P_{M}$ a Boolean simple module.

From Proposition 3.6 it is clear that for each vertex $v$ there is a Boolean maximal left ideal $P_{M} = P_{M}v \oplus L(1-v)$ not containing $v$. Because, given $v$ we can find a maximal left ideal $Q$ of $L$ not containing $v$. Clearly $Q \cap B = M$ is a maximal ideal in $B$ not containing $v$. Then proceed as on Proposition 3.6(b), to construct the Boolean maximal left ideal $P_{M}$ corresponding to $M$ and, as noted there, $P_{M} = P_{M}v \oplus L(1-v)$.

Proposition 3.8 Let $Lv/N$ be a fixed simple left $L$-module where $v \in E^{0}$ and $N$ is a maximal $L$-submodule of $L$. Let $S_{v,N} = \{P_{M} \in T: L/P_{M} \cong Lv/N\}$. Let $\sigma = |S_{v,N}|$ and write $S_{v,N} = \{P_{M_{\alpha}} = P_{M_{\alpha}}v_{\alpha} \oplus L(1-v_{\alpha}) : v_{\alpha} \in E^{0}, \alpha < \sigma\}$. Then

(a) $|S_{v,N}|$ satisfies $\dim_{K}(Lv/N)$.

(b) the cardinality of the set of all maximal left ideals $P$ of $L$ such that $L/P \cong Lv/N$ is $\leq \sum_{\alpha < \sigma} |v_{\alpha}Lv_{\alpha}|$.

Proof. (a) By Lemma 3.3, each $P_{M_{\alpha}} \in S_{v,N}$ annihilates an element $x_{\alpha} = a_{\alpha} + N \in Lv/N$. Regarding $Lv/N$ as a $K$-vector space, we claim that these elements $x_{\alpha}$ (corresponding to the various $P_{M_{\alpha}} \in S_{v,N}$) must be $K$-independent. To justify this, suppose a finite subset of the elements $x_{\alpha}$, with $j = 1, ..., n + 1$, satisfy

$$\sum_{j=1}^{n+1} k_{j}x_{j} = 0 \quad (\ast)$$

where, for each $j$, $k_{j} \in K$ and the maximal ideal $P_{M_{j}}$ is the corresponding annihilator of the element $x_{j}$. Observe that the maximal ideals of $B$ satisfy the Chinese remainder theorem and so, corresponding to the finite set $M_{1}, ..., M_{n}, M_{n+1}$ of maximal ideals of $B$, there is an element $b \in \cap_{i=1}^{n} M_{i}$ such that $b \notin M_{n+1}$ so that the ideal generated by $\cap_{i=1}^{n} M_{i}$ and $M_{n+1}$ is $B$. Since the vertex set $E^{0} \subset B$, we then see that $(\cap_{i=1}^{n} P_{M_{j}}) + P_{M_{n+1}} = L$ and so there is an element $a \in \cap_{j=1}^{n} P_{M_{j}}$, but $a \notin P_{M_{n+1}}$. Since $a$ annihilates $x_{1}, ..., x_{n}$, multiplying the equation $(\ast)$ on the left by the element $a$, we get $k_{n+1}a_{n+1} = 0$ which implies $k_{n+1} = 0$. Proceeding like this, we establish the independence of the elements $x_{j}$. Thus the elements $x_{j}$ can be regarded as part of a basis of $Lv/N$. Since distinct maximal left ideals $P_{M_{j}}$ correspond to different such elements $x_{j}$ in a basis of $Lv/N$ (Lemma 3.3), we conclude that $|S_{v,N}| \leq \dim_{K}(Lv/N)$.

(b) Now, for a fixed $\alpha$, Proposition 3.4(a) implies that the cardinality of the set of all the maximal left ideals $P$ with $P \cap B = P_{M_{\alpha}} \cap B$ (so $P = P_{v_{\alpha}} \oplus L(1-v_{\alpha})$) and satisfying $L/P \cong L/P_{M_{\alpha}} \cong Lv/N$ is $\leq |v_{\alpha}Lv_{\alpha}|$. So the cardinality of the set of all maximal left ideals $P$ such that $L/P$ is isomorphic to $Lv/N$ is $\leq \sum_{\alpha < \sigma} |v_{\alpha}Lv_{\alpha}|$.
Lemma 3.9 If Soc(L) = 0 and the graph E satisfies Condition (L), then the Boolean ring B is atomless, that is, it has no minimal elements.

Proof. Since Soc(L) = 0, the graph E cannot have any line points and, in particular, has no sinks. Suppose, by way of contradiction, B has a minimal element m so that, for all b ∈ B, either mb = 0 or mb = m. In order to reach a contradiction, we first claim that m can be taken to be a monomial of the form \( \gamma \gamma^* \) for some path \( \gamma \). To see this, if \( m = v \) is a vertex, then as \( v \) is not a sink, it will be the source of some path \( \alpha \) and in that case \( m = m \alpha \alpha^* = v \alpha \alpha^* = \alpha \alpha^* \). Likewise, suppose \( m = \sum_{i=1}^k t_i \alpha_i \alpha_i^* \) where \( t_i \in K \) and \( \alpha_i \alpha_i^* \not= \alpha_j \alpha_j^* \) for all \( i,j \). Assume, without loss of generality that \( \alpha_1 \alpha_1^* \not= 0 \) implies that \( \alpha_1 \alpha_1^* \not= \alpha_1 \alpha_1^* \); we conclude that \( m = \alpha_1 \alpha_1^* \). We thus conclude that \( m = \gamma \gamma^* \) for some path \( \gamma \) with \( s(\gamma) = u \). Suppose \( r(\gamma) = w \) (\( w \) may be equal to \( u \)). Since \( w \) cannot be a line point, there is a vertex in \( T(w) \) which is either a bifurcation vertex or is the base of a cycle in \( T(w) \). Since every cycle has an exit (due to Condition (L)), \( T(w) \) will always contain a bifurcation vertex \( w' \) with \( w' = s(e) = s(f) \) for some edges \( e \not= f \). Denoting a path from \( w \) to \( w' \) by \( \delta \), we obtain, by the minimality of \( m = \gamma \gamma^* \),

\[
\gamma \gamma^* = \gamma \gamma^* \gamma \delta e^* \delta^* \gamma^*.
\]

Multiplying on the right by \( \gamma \delta f \), we get \( \gamma \gamma^* \gamma \delta f = \gamma \gamma^* \gamma \delta e^* \delta^* \gamma^* \gamma \delta f \) and from this we get \( \gamma \delta f = \gamma \gamma^* \gamma \delta e^* f = 0 \), a contradiction. This proves that the Boolean ring B has no minimal element. □

Theorem 3.10 Let E be an arbitrary graph satisfying Condition (L). If \( L = L_K(E) \) is a countable dimensional \( K \)-algebra with Soc(L) = 0, then \( L \) has at least \( 2^{2^{80}} \) distinct isomorphism classes of simple left \( L \)-modules.

Proof. Consider the Boolean ring \( B \) of \( L \) defined earlier. By Lemma 3.9, the Boolean ring \( B \) has no minimal elements. Thus \( B \) is a countable atomless Boolean ring without identity. In this case, it is well-known (see [16] or Theorems 1, 8 and 13 in [22]) that the space \( X \) of all maximal ideals of \( B \) is a locally compact totally disconnected Hausdorff space with no isolated points. Let \( X^* \) be the one-point compactification of \( X \) obtained by the adjunction of a single non-isolated point to \( X \). Now \( X^* \) is homeomorphic to the Cantor set (see [22]) and so \( X^* \), and hence \( X \), has cardinality \( 2^{80} \). Thus \( B \) has \( 2^{80} \) distinct maximal ideals. From Corollary 3.7 we conclude that there is a set \( T \) of \( 2^{80} \) distinct Boolean maximal left ideals of \( L \) obtained from the ideals of \( B \). Now for any given maximal ideal \( P = N \oplus L(1 - v) \in T \), the set \( SP = \{ Q \in T : L/Q \cong L/P \} \) is countable since \( L \) and hence \( L_0/N \) has countable \( K \)-dimension and, by Proposition 3.8 \( |SP| \leq \dim_K(L_0/N) \). Since \( |T| = 2^{80} \), \( L \) has \( 2^{80} \) non-isomorphic Boolean simple \( L \)-modules of the form \( L/P \) where \( P \in T \). Consequently, \( L \) has at least \( 2^{80} \) non-isomorphic simple left \( L \)-modules. □
Corollary 3.11  Let $E$ be an arbitrary graph. If $L = L_K(E)$ is a simple countable dimensional $K$-algebra, then $L$ has either exactly 1 or at least $2^\aleph_0$ distinct isomorphism classes of simple left $L$-modules.

Proof. If $\text{Soc}(L) \neq 0$, then, by simplicity, $L = \text{Soc}(L)$ and all the simple left $L$-modules are isomorphic. Suppose $\text{Soc}(L) = 0$, then the simplicity of $L$ implies that the graph $E$ satisfies Condition (L) (see [1]). We then obtain the desired conclusion from Theorem 3.10. 

Remark 3.12 The method of proof of Theorem 3.10 breaks down if $E$ is an uncountable graph. Because, unlike the case of a countable atomless Boolean ring, the set of maximal ideals of an uncountable atomless Boolean ring $B$ may not have the desired larger cardinality than $|B|$, unless some conditions such as completeness of $B$ holds, or if $B$ has an independent subset of cardinality $|B|$ (I am grateful to Professor Stefan Geschke for this remark. See [17] for details). When $E$ is an uncountable graph, the Boolean ring $B$ that we constructed in the proof of Theorem 3.10 need not be complete and also need not have a large enough independent subset.

4  Finitely presented simple modules

Let $E$ be a finite graph. It was shown in [10] that every simple left $L_K(E)$-module is finitely presented if and only if distinct cycles in $E$ are disjoint, that is, they have no common vertex. Interestingly, in [5], this same condition on the graph $E$ is shown to be equivalent to the algebra $L_K(E)$ having finite Gelfand-Kirillov dimension. Further, Theorem 1 of [6] shows that if the graph $E$ has the stated property, then $L = L_K(E)$ is the union of a finite ascending chain of ideals

$$0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L$$

where $I_0$ is a direct sum of finitely many matrix rings $M_n(K)$ over $K$ with $n \in \mathbb{N} \cup \{\infty\}$ and, for $j \geq 1$, each successive quotient $I_j/I_{j-1}$ is a direct sum of finitely many matrix rings $M_n(K[x,x^{-1}])$ over $K[x,x^{-1}]$ with $n \in \mathbb{N} \cup \{\infty\}$.

In this section, we show that the converse of the above statement holds and obtain an improved version of the statement and proof of Theorem 1 of [6] (see Theorem 4.2 below). An easy proof of Theorem 2 of [6] is also pointed out.

We begin with the following easily derivable Lemma which was implicit in [1] and was proved in [13].

Lemma 4.1 Let $E$ be any graph and let $H$ be a hereditary subset of vertices in $E$. If $w$ is the base of a closed path and if $w \in \bar{H}$ the saturated closure of $H$, then $w \in H$.

In addition to proving the converse of Theorem 1 of [6], the next theorem consolidates the various properties of the algebra $L_K(E)$ where the graph $E$ has the mentioned property.
Theorem 4.2 Let $E$ be a finite graph and let $K$ be any field. Then the following are equivalent for the Leavitt path algebra $L = L_K(E)$:

(i) No two distinct cycles in $E$ have a common vertex;
(ii) Every simple left $L$-module is finitely presented;
(iii) $L$ has finite Gelfand-Kirillov dimension;
(iv) $L$ is the union of a finite ascending chain of graded ideals

\[ 0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L \quad (\ast) \]

with $H_j = I_j \cap E^0$, where $I_0 = \text{Soc}(L)$ and, for each $j \geq 0$, identifying, $L/I_j$ with $L_K(E\setminus H_j)$, $I_{j+1}/I_j$ is the ideal generated by the vertices in all the cycles without exits in $E\setminus H_j$ and $\text{Soc}(L/I_j) = 0$.

(v) $L$ is the union of a finite ascending chain of graded ideals

\[ 0 \subset I_0 \subset I_1 \subset \cdots \subset I_m = L \]

where $I_0$ is a direct sum of finitely many matrix rings of the form $M_n(K)$ where $n \in \mathbb{N} \cup \{\infty\}$ and for each $j > 1$, $I_j/I_{j-1}$ is a direct sum of finitely many matrix rings of the form $M_n(K[x,x^{-1}])$ where $n \in \mathbb{N} \cup \{\infty\}$.

(vi) $E^0$ is the union of a finite ascending chain of hereditary saturated subsets

\[ H_0 \subset \cdots \subset H_m = E^0 \]

where $H_0$ is the hereditary saturated closure of all the line points in $E$ and, for each $j \geq 0$, $E\setminus H_j$ has no line points and $H_{j+1}\setminus H_j$ is the hereditary saturated closure of the set of vertices in all the cycles without exits in the graph $E\setminus H_j$.

Proof. The equivalence of (i) and (ii) was proved in [10] and that of (i) and (iii) was proved in [5].

Now (i) $\Rightarrow$ (iv) follows from the proof of Theorem 1 of [9]. We give a slightly different streamlined proof. Let $I_0 = \text{Soc}(L)$, so $I_0$ is the ideal generated by all the line points in $E$ (12). Now, for any graded ideal $J$ containing $\text{Soc}(L)$, $\text{Soc}(L/J) = 0$. This is because, as the hereditary saturated set $J \cap E^0 = H$ contains all the line points in $E$, the finiteness of $E$ implies that the quotient graph $E/H$ contains no sinks and hence no line points. Suppose for $n \geq 0$ we have defined the graded ideal $I_n \supseteq I_0$. Let $H_n = I_n \cap E^0$. Then $E\setminus H_n$ satisfies the same hypothesis as $E$ and has no sinks, so that every vertex in it connects to a cycle. Moreover, we claim that $E\setminus H_n$ contains cycles without exits. To see this, for any given two cycles $c, c'$ in $E\setminus H_n$, define $c \geq c'$ if there is a path from a vertex in $c$ to a vertex in $c'$. Since no two cycles in $E\setminus H_n$ have a common vertex, $\geq$ is antisymmetric and hence a partial order. Clearly every cycle which is minimal in this partial order has no exits in $E\setminus H_n$. Now $L/I_n \cong L_K(E\setminus H_n)$ and $\text{Soc}(L/I_n) = 0$. Define $I_{n+1}/I_n$ to be the ideal generated by the vertices in all the cycles without exits in $E\setminus H_n$. It is clear that $I_{n+1}$ is a graded ideal of $L$. By induction on $n$, after a finite number of steps, we then obtain the chain (\ast) with the desired properties.

(iv) $\Rightarrow$ (v) By Theorem 5.6 of [12], $I_0$ is a direct sum of finitely many matrix rings of the form $M_n(K)$ where $n \in \mathbb{N} \cup \{\infty\}$ and, for each $j \geq 0$, $I_{j+1}/I_j$ is,
by Proposition 3.7 of [4], a direct sum of finitely many matrix rings of the form $M_n(K[x, x^{-1}])$ where $n \in \mathbb{N} \cup \{\infty\}$.

(v) $\Rightarrow$ (vi). Obvious from the proof of (iii) $\Rightarrow$ (iv).

Assume (vi). Suppose, by way of contradiction, there is a vertex $w$ which is the base of two distinct cycles $g, h$. Now $w \notin H_0$ since otherwise, by Lemma 4.1, $w$ will be a line point in $E$, a contradiction. Let $t \geq 0$ be the smallest integer such that $w \notin H_t$, so $w \in H_{t+1} \setminus H_t$. Now $H_{t+1} \setminus H_t$ is the saturated closure of the set $S_t$ of all the vertices on cycles without exits in the quotient graph $E \setminus H_t$. Then, by Lemma 4.1, $w \in S_t$, a contradiction. This proves (i).

Following the ideas in [10], we illustrate Theorem 4.2 by the simplest example of the Toeplitz algebra.

Example 4.3 Let $E$ be the graph with two vertices $v, w$, an edge $f$ with $s(f) = v, r(f) = w$ and a loop $c$ with $s(c) = v = r(c)$. Since $w$ is the only line point, the socle of $L = L_K(E)$ is $S = \langle w \rangle$ (see [12]) and there is an epimorphism $L \rightarrow K[x, x^{-1}]$ with kernel $S$ mapping $v$ to $1$, $c$ to $x$ and $c^*$ to $x^{-1}$. Thus $L/S \cong K[x, x^{-1}]$ and, moreover, $S = Lw \oplus \bigoplus_{n=0}^{\infty} Lwf^*(c^*)^n$ is the direct sum of simple left ideals in $L$. We wish to show that every simple left $L$-module $A = L/M$ is cyclically (hence finitely) presented, where $M$ is a maximal left ideal of $L$.

If $S \nsubseteq M$, then we have a direct decomposition $S = (S \cap M) \oplus T$ and $L = M \oplus T$. If $1 = e + e'$ with $e \in M$ and $e' \in T$, then $M = Le$ is cyclic.

Suppose $S \subseteq M$. Then there is an irreducible polynomial $p(x) = 1 + k_1 x + \cdots + k_m x^m \in K[x, x^{-1}]$ such that $M/S = \langle p(x) \rangle$. So $M = Lp(c) + S = Lp(c) + Lw + \bigoplus_{n=0}^{\infty} Lwf^*(c^*)^n = Lp(c) + \bigoplus_{n=0}^{\infty} Lf^*(c^*)^n$ as $wp(c) = w$. Let $N = \bigoplus_{i=0}^{m-1} Lf^*(c^*)^i$. Suppose $r \geq m - 1$ and that $f^*(c^*)^t \in Lp(c) + N$ for all $t \leq r$. Then, $f^*(c^*)^{r+1} = f^*(c^*)^r p(c) - k_1 f^*(c^*)^{r-1} - \cdots - k_m f^*(c^*)^{r+m} \in Lp(c) + N$. Thus we conclude that $Lp(c) + S = Lp(c) + N$. Observing that $\{f^*(c^*)^i : i = 0, \ldots, m-1\}$ is a set of mutually orthogonal elements, we get $N = \bigoplus_{i=0}^{m-1} Lf^*(c^*)^i = Lb$ where $b = f^* + f^* c^* + \cdots + f^*(c^*)^{n-1}$. Further, $p(c) f^*(c^*)^i = 0$ for all $i$ and that $p(c) w \in Lw$. Consequently, $M = Lp(c) + S = L(v + b)$ is cyclic. This proves that the simple module $L/M$ is cyclically presented.

REMARK: Observe that, in our proof above, we never used the fact that the polynomial $p(x)$ is irreducible. Since $K[x, x^{-1}]$ is a principal ideal domain, the same argument shows that every left ideal $A \nsubseteq S$ in $L$ is a principal left ideal. Also, if $A$ is a left ideal such that $S \not\subseteq A$ and $A \not\subseteq S$, then decomposing $S = (S \cap A) \oplus T$, we see that the left ideal $A + S = A \oplus T \not\subseteq S$. Thus $A \not\subseteq T$ and hence $A$ is a principal left ideal in $L$. On the other hand, if $A \subseteq S$, $A$ need not be a principal left ideal. This is clear if $A = S$, as $S$ is a direct sum of infinitely many simple left ideals. In particular, $S$ is not a direct summand of $L$. Thus we obtain an easy proof of the following proposition which occurs as Theorem 2 in [6].
Proposition 4.4 Let $E$ be a graph with two vertices $v, w$ and two edges $c, f$ with $s(c) = v = r(c), s(f) = v, r(f) = w$. If $S = \langle w \rangle$ is the two-sided ideal generated by $w$, then $S$ cannot be a direct summand of $L = L_K(E)$ as a left $L$-module.

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