On the nature of the phase transition in the three-dimensional random field Ising model

Vik S Dotsenko

LPTMC, Université Paris VI, 4 place Jussieu, 75252 Paris, France and
L D Landau Institute for Theoretical Physics, 117940 Moscow, Russia
E-mail: dotsenko@lptmc.jussieu.fr

Received 11 June 2007
Accepted 3 August 2007
Published 6 September 2007

Abstract. A brief survey of the theoretical, numerical and experimental studies of the random field Ising model (RFIM) during the last three decades is given. The nature of the phase transition in the three-dimensional RFIM with Gaussian random fields is discussed. Using simple scaling arguments it is shown that if the strength of the random fields is not too small (bigger than a certain threshold value), the finite temperature phase transition in this system is equivalent to the low temperature order–disorder transition which takes place with variations of the strength of the random fields. A detailed study of the zero-temperature phase transition in terms of simple probabilistic arguments and a modified mean field approach (which take into account nearest neighbor spin–spin correlations) is given. It is shown that if all thermally activated processes are suppressed, the ferromagnetic order parameter $m(h)$ as a function of the strength $h$ of the random fields becomes history dependent. In particular, the behavior of the magnetization curves $m(h)$ for increasing and decreasing $h$ reveals a hysteresis loop.

Keywords: phase diagrams (theory), disordered systems (theory), memory effects (theory)

ArXiv ePrint: 0706.0628
1. Introduction

The thermodynamic properties of the random field Ising model (RFIM) have remained controversial for more than 30 years now (for reviews see e.g. [1, 2]). Traditionally there have been three somewhat independent areas of research: theoretical, numerical and experimental. Within each of these, as well as between any two of them, one finds broad divergences in the results obtained and in their interpretations. The main issue in this controversy is the structure of the phase diagram of the system: what kinds of phase transitions and which thermodynamic phases are present there?

The random field model was proposed originally by Larkin [3] for modeling the defect pinning of vortices in superconductors. The simplest version of this model for systems with discrete Ising symmetry, the RFIM, can be described in terms of the Hamiltonian

\[ H = -\sum_{\langle i,j \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i \]  

where the Ising spins \( \{\sigma_i = \pm 1\} \) are placed at the vertices of a \( D \)-dimensional lattice with ferromagnetic interactions between the nearest neighbors, and the quenched random fields \( \{h_i\} \) are described either as a symmetric Gaussian distribution with \( \langle h_i h_j \rangle = \delta_{ij} h^2 \), or as a binary distribution in which the fields take two values \( \pm h \) with independent probabilities at each site.

1.1. Theory

The first round of controversies in the studies of the RFIM concerned its lower critical dimension. According to simple energy balance physical arguments advanced by Imry and Ma [4] it is expected that the dimension \( D_c \) above which the ferromagnetic ground state is stable at low temperatures (it is called the lower critical dimension) must be \( D_c = 2 \).
On the nature of the phase transition in the three-dimensional random field Ising model

(unkike for the pure Ising systems where $D_c = 1$). Later, the existence of ferromagnetic long range order in the 3D RFIM was confirmed by a rigorous proof [5]. On the other hand, a perturbative renormalization group study of the phase transition demonstrates the phenomenon of so-called dimensional reduction, such that the critical exponents of the random field system for the dimension $D$ appear to be the same as those of the ferromagnetic system without random fields for the dimension $d = D - 2$ [6], which implies that the lower critical dimension of the random field Ising model must be equal to 3, in contradiction to the rigorous results. Actually, the procedure of summation of the leading large scale divergences assumes that the Hamiltonian has only one minimum. However, one can easily see that as soon as the temperature is close enough to the putative critical point (as well as in the whole low temperature region), there are local values of the magnetic fields for which the free energy has more than one minimum [7,8], and therefore the dimensional reduction is not well grounded. Thus, the above arguments have settled the controversy about the lower critical dimension of the random field Ising model in favor of the value $D_c = 2$.

The phase diagram in the $(h, T)$ plane for the RF Ising system was first derived in terms of simple mean field theory for a model with infinite range ferromagnetic interactions (which corresponds to infinite space dimensionality) with Gaussian distribution of random fields [9]. In this system the ferromagnetic phase is separated from the paramagnetic phase by the line $T(h)$ of the second-order phase transition which decreases monotonically with $h$ from the point $T_c(0)$, for the pure system, down to the end point $T = 0$, at $h = h_c$. On the other hand, for the bimodal distribution [10] one finds that the line $T_c(h)$ contains a tricritical point $(T^*, h^*)$ such that at $h > h^*$ the phase transition becomes first order. Later on a simple scaling description of the RFIM phase transition in finite dimensions was developed assuming that it is of second order, and that it is controlled by the zero-temperature fixed point [11].

One could study, as another extreme case, instead of the Ising spins, the $m$-component vector spin system (the Ising model corresponds to $m = 2$) with Gaussian random fields. It can be shown [12] that in terms of the replica field theory in the limit $m \to \infty$ the statistics of interfaces in this system is described by the solution with broken replica symmetry (similar to that for spin glasses [13]) which is consistent with the low critical dimension $d_l = 2$. Later on, for the same limit $m \to \infty$, the instability of the replica symmetric state and the onset of the replica symmetry breaking scenario were demonstrated [14]. In terms of this approach it has been argued that in the phase diagram of the three-dimensional RFIM the paramagnetic and ferromagnetic phases are separated by a glassy phase in which the replica symmetry is broken [15].

An independent study of the $\phi^4$ replica field theory (which in the critical region is believed to describe the RFIM) in terms of the Legendre transforms technique and the virial-like relations also shows the presence of an intermediate RSB glassy phase below dimensions $D = 6$ [16]. Further renormalization group (RG) study of this field theory in dimensions $D < 6$ revealed no stable fixed points, signaling the onset of replica symmetry breaking [17].

Another line of theoretical study concerns the non-perturbative thermodynamic states which are apparently missing from the usual RG calculations and which may appear to be quite relevant for the nature of the phase transition in the RFIM. The importance of the non-perturbative phenomena was noted a long time ago [7,18,19]. In physical terms,
the non-perturbative configurations are rare spatial regions with ‘flipped’ (opposite to the background) magnetizations, and in terms of the replica field theory they are described as finite energy instanton configurations localized in space, in which the replica symmetry is broken. In recent investigations a systematic approach to the calculation of the non-perturbative contributions has been developed [21] and it has been shown that from the point of view of relevance the dimension \( D = 3 \) turns out to be marginal: formally these degrees of freedom produce finite (non-analytic in the strength of the field \( h \)) contributions only for dimension \( D \leq 3 \) [22].

Finally, the construction of an alternative approach, that of the functional renormalization group, which is supposed to take into account all non-perturbative degrees of freedom for the whole class of the random field \( O(N) \) spin model, is under way at this very time [23].

1.2. Numerics

Numerical investigations of the phase transition in the RFIM have given rise to a broad controversy as regards the results obtained and in particular their interpretation. The main point of the disagreements is as regards the nature of the phase transition: is it continuous, or is it of first order with a finite jump of the order parameter?

Originally, the analysis of the high temperature series expansion for the RFIM with a Gaussian distribution of random fields [24] indicated the existence of a fluctuation-driven first-order phase transition at sufficiently strong disorder below four dimensions (which suggests the existence of a tricritical point in the three-dimensional system). However, more recent study indicated a continuous transition, at least for weak disorder [25].

The first extensive Monte Carlo (MC) numerical simulations for the 3D RFIM were carried out by Young and Nauenberg [26], who reported that the transition may be first order. On the other hand, the numerical studies by Ogielski and Huse [27] and by Ogielski [28] show that the transition may be second order. More recent Monte Carlo simulations [29] can also be interpreted as describing a continuous transition but with a finite jump in the magnetization. However, later extensive MC calculations [30] for the energy and the magnetization distributions suggest a first-order transition.

The zero-temperature studies play an important role as regards understanding the nature of the phase transition in the RFIM. According to the zero-temperature fixed point hypothesis, the transitions which take place at \( T = 0 \) and at \( T \neq 0 \) are in the same universality class. The first zero-temperature Migdal–Kadanoff renormalization group calculations indicated a continuous transition which is characterized by a very small value of the magnetization critical exponent (\( \beta \approx 0.02 \)) [31]. Similar results were obtained in a finite temperature renormalization group study [32]. On the other hand, some of the more recent zero-temperature finite size scaling calculations [33] have demonstrated the existence of a first-order phase transition, although most of them [34,35] favor a continuous transition. Moreover, in one of the most recent calculations it has been demonstrated that for a given realization of disorder the ground states (\( T = 0 \)) and thermal states (\( T \neq 0 \)) near the critical line are strongly correlated, which may be interpreted as a concrete manifestation of the zero-temperature fixed point scenario [36]. The most precise results, obtained by Middleton and Fisher [35], are clearly consistent with the existence of a single phase transition (no intermediate spin glass phase has been observed) and it is characterized by a very small value of the magnetization critical exponent \( \beta = 0.017 \).
On the nature of the phase transition in the three-dimensional random field Ising model

Other numerical studies of the phase diagram of the RFIM in the plane \((T, h)\) \([37, 38]\) also find no glassy phase (with or without replica symmetry breaking) which would separate paramagnetic and ferromagnetic phases. On the other hand, for bimodal random field distributions a discontinuity in the magnetization at the phase transition line is observed \([38]\).

Finally, numerical simulations of the diluted antiferromagnet in a field (DAFF) designed to imitate the random field magnetic systems which are studied experimentally (see below) show that in equilibrium, the transition to the LRO state may be first order \([39, 40]\), and moreover it is claimed that at sufficiently strong dilution there is a first-order phase transition to a spin glass state \([41]\).

To conclude this brief survey of numerical studies, it should be noted that all numerical simulations in the RFIM are impeded by a dramatic slowing down on approaching a putative phase transition, which very often makes it very difficult to interpret the results obtained. Besides, the delicate point in all these studies is that the order parameter critical exponent \(\beta\) turns out to be very small, so it is very difficult to distinguish a second-order transition (for which the order parameter continuously goes to zero at \(T_c\)) from a first-order one (where the order parameter has a finite jump at \(T_c\)). On top of that, recent numerical study of the critical region of the 3D RFIM indicates that the situation could be even more complicated due to the absence of self-averaging of the correlation length, of the specific heat, and maybe of some other thermodynamical quantities \([42]\).

1.3. Experiments

From the point of view of experimental investigations an important theoretical breakthrough occurred when it was realized that the RFIM can be generated in dilute Ising antiferromagnets by application of a uniform external magnetic field \([43]\). This has opened the way for experimental studies of the RFIM, and in particular for the investigation of the phase transition.

In the very first experimental studies of the RFIM performed on the dilute antiferromagnets \(\text{Co}_x\text{Zn}_{1-x}\text{F}_2\), it was claimed that even the smallest magnetic fields destroy long range order at all temperatures in three dimensions \([44]\).

The first convincing experimental demonstrations of the existence of a phase transition in the 3D random field antiferromagnet \(\text{Fe}_x\text{Zn}_{1-x}\text{F}_2\) and its critical properties (assuming that it is of second order) were reported in \([45]\). An extensive study of the critical properties of the system \(\text{Fe}_{0.6}\text{Zn}_{0.4}\text{F}_2\) for temperatures \(T > T_c(h)\) was reported in \([46]\).

In fact, the behavior observed near the critical point depends on the field and temperature procedures used in the measurements. For the field cooled (FC) samples the transition appears to be rounded, and at low temperatures the system freezes in the metastable domain state. However, if the sample is cooled in zero field (ZFC), the transition appears to be much sharper, rounded only by slow dynamic effects \([2]\).

On the other hand, in another experimental study of the 3D RFIM (for weakly anisotropic \(\text{Mn}_{0.5}\text{Zn}_{0.5}\text{F}_2\)) \([47]\) it has been observed that on approaching the phase transition the correlation length reaches a finite size, at which point, it was assumed, further approach towards a putative second-order transition is interrupted by the occurrence of a first-order transition.

The most ‘slippery’ issue of the experimental studies of the phase transition in the RFIM is the measurement of the static critical behavior of the staggered magnetization.

doi:10.1088/1742-5468/2007/09/P09005
Like for the numerical studies, the fact that the critical exponent $\beta$ is experimentally very small (e.g. according to [48] $\beta \simeq 1/8$ or less; according to [49] $\beta \simeq 0.16$) makes it very difficult to reach any definite conclusion about the nature of the observed phase transition.

In the studies of the dilute antiferromagnet Mn$_{0.75}$Zn$_{0.25}$F$_2$ the field cooled data were interpreted as evidence for an equilibrium second-order transition at $T_{FC}$, while the zero-field cooled data were interpreted as showing a non-equilibrium trompe l’oeil transition in which the long range order diminishes continuously at $T_{ZFC} > T_{FC}$ [50]. Similar behavior was observed for Fe$_{0.5}$Zn$_{0.5}$F$_2$ [51]. On the other hand, all these results can be explained by a mean field first-order transition broadened by a cluster-flipping mechanism [52].

In the study of the random field antiferromagnet Fe$_{0.5}$Zn$_{0.5}$F$_2$ a notable broadening of the phase transition region has been observed, so the data may be described using a Gaussian distribution of effective transition temperatures with a width which scales with the strength of the applied field as $h^2$ [53]. Besides, the apparent critical behavior observed in these measurements represented a continuous evolution from metastable behavior towards equilibrium behavior. Moreover, according to the point of view of Birgeneau [54], so far no true equilibrium phase transition has been observed in the RFIM systems. Nevertheless, in the recent study of the high magnetic concentration Ising antiferromagnet Fe$_{0.93}$Zn$_{0.07}$F$_2$ (which does not exhibit severe critical scattering hysteresis [55]) it is claimed that in terms of general scaling assumptions the critical scattering analysis allows us to reveal equilibrium critical behavior and to obtain the corresponding critical exponents [56] (in particular the specific heat critical behavior is well fitted by the logarithmic singularity which corresponds to $\alpha = 0$ [57]).

In conclusion of this brief historic review, one can note that the situation with the phase transition in the three-dimensional RFIM remains far from clear. Theorists are mostly interested in their own problems (like $1/m$ expansion, or RSB in dimensions close to 6, or the two-loop approximation for the $O(N)$ sigma model with the number of spin components close to $N_c \simeq 2.835$), which are rather far from the original system, while people doing numerics and in particular experiments are facing realities full of numerous secondary and accessory effects which very often overshadow the main physical phenomena.

It is evident that one cannot hope to find the exact solution to this problem, and therefore some kinds of simplifications and approximations are unavoidable. In this paper, which is mostly devoted to the low temperature part of the phase diagram, it will be supposed that: (a) the finite temperature thermodynamical properties of the three-dimensional RFIM can be described in terms of the continuous Ginzburg–Landau $\phi^4$ field theory; (b) we can neglect the presence of randomness in the spin–spin interactions; and (c) in the study of the zero-temperature phase transition (with variation of the strength of the random fields) all thermally activated processes are suppressed.

The above reservations are rather essential.

(a) Ginzburg–Landau field theory. We all believe in universality, and therefore if we admit that the phase transition in a given system is of second order, then it is generally accepted that its critical properties can be described in terms of the corresponding Ginzburg–Landau Hamiltonian (with appropriate symmetry properties). However, if the system is considered at a finite distance from the putative critical point (which is the
case in the present paper) where the correlation length is finite, then the relevance of the continuous limit Ginzburg–Landau Hamiltonian for the original lattice Ising system becomes much less evident.

(b) Randomness in the spin–spin interactions. Apart from in the numerical investigations of DAFF [39,41], the effects of randomness in the spin–spin interactions are practically never taken into account in theoretical studies (there are enough headaches due to random fields ...). However, it is well known that the presence of this type of randomness (without random fields) may change the critical properties of a magnetic system, and in particular it definitely does in the three-dimensional Ising model. Moreover, in some statistical systems (e.g. in the Potts model) the presence of such disorder may turn the first-order transition (of the pure system) into a second-order one [58]. Therefore, it we admit for the moment that the phase transition in the ‘pure’ (without randomness in spin–spin interactions) 3D RFIM is of first order, it does not guarantee that in the corresponding experimental realizations of this system (where the randomness in the spin–spin interactions is inevitable) the phase transition would not turn into a second-order one.

(c) Thermal activations. In the studies of the zero-temperature properties of the system as a function of the effective strength of the random fields \( h \), one can consider procedures of two types. In the first one it is assumed that at any given \( h \) the system is in the ground state (which, for a given realization of the random fields, is assumed to be unique). In this case the variation of \( h \) to a new value \( h' \) means that either we admit that the temperature is actually slightly non-zero, and the system passes to a new ground state configuration (corresponding to \( h' \)) via thermally activated flips, or for each variation of \( h \) the ground state configuration is achieved by cooling the system (at fixed \( h' \)) from the high temperature disordered state.

The alternative procedure of the variation of \( h \) is essentially different. Here one has to fix the starting value of \( h \) (e.g. \( h = 0 \)), and then for any variation of \( h \) the spins are just following the directions of their local fields (which consist of the contributions due to the regular ferromagnetic interactions with the nearest neighbors plus the local value of the random field). In this procedure all thermal ‘jumps over energy barriers’ are prohibited, and therefore, most probably, due to the variation of \( h \) the system will get stuck in one of the metastable states.

Of course, from the experimental point of view, neither of these procedures can be implemented in the pure form. The first one (qualitatively, it corresponds to the field cooled (FC) measurements) requires a very long (formally infinite) waiting time to reach the thermal equilibrium for any variation of \( h \), while the second one (it corresponds to the zero-field cooled (ZFC) measurements) requires suppression of all thermally activated processes, which can be achieved only if the temperature is exactly equal to zero.

In the next section, using simple physical arguments, it will be shown that if the strength of the random fields is not too small then there exists a region of parameters where the critical fluctuations are irrelevant, and therefore the thermodynamic properties of the system can be described in terms of the saddle point equations of the corresponding (random) Ginzburg–Landau Hamiltonian. It is in this region that the ferromagnetic order breaks down. Moreover, using simple scaling arguments it will be shown that in this case the finite temperature phase transition in this system is equivalent to the low temperature phase transition with variation of the strength of the random fields.
In section 3 a detailed study of the zero-temperature order–disorder phase transition will be made, in terms of the discrete Ising model on the three-dimensional cubic lattice. Here we propose a modified (slightly improved) version of the mean field approach which takes into account nearest neighbor two-spin correlations. Then, considering the two regimes: increasing and decreasing strength of the random fields, and supposing that all thermally activated spin flips are suppressed, it will be demonstrated that the magnetization curve \( m(h) \) reveals a hysteresis loop. In particular, it is characterized by two different critical points \( h_c^- < h_c^+ \), where \( h_c^- \) is the strength of the random field at which \( m(h) \) becomes non-zero with decreasing \( h \), and \( h_c^+ \) is that for \( m(h) \) becoming zero with increasing \( h \). It is interesting to note that similar phenomena were discovered earlier in the zero-temperature RFIM in a uniform external magnetic field, where on sweeping the external field through zero, the model exhibits hysteresis and return point memory effects [59].

2. Heuristic arguments

For the sake of generality, before coming down to dimension 3, first, let us consider the continuous version of the \( D \)-dimensional Ising model in terms of the scalar field Ginzburg–Landau Hamiltonian:

\[
H[\phi(x), h(x)] = \int d^D x \left[ \frac{1}{2} (\nabla \phi(x))^2 - \frac{1}{2} \tau \phi(x)^2 + \frac{1}{4} g \phi^4(x) - h(x) \phi(x) \right].
\]  

(2)

Here \( \tau = (T_c - T)/T_c \) (\( \tau \ll 1 \)) is the reduced temperature parameter (for simplicity in what follows it will be supposed that \( T_c = 1 \)). According to this definition, positive values of \( \tau \) correspond to the low temperature ferromagnetic state. Random fields \( h(x) \) are described by the symmetric Gaussian distribution

\[
P[h(x)] = p_0 \exp \left( -\frac{1}{2h^2} \int d^D x h^2(x) \right),
\]

(3)

where \( h \) is the parameter which describes the strength of the random field, and \( p_0 \) is an irrelevant normalization constant. For a given realization of the random fields the partition function of this system is obtained by integration over all configurations of the scalar fields \( \phi(x) \):

\[
Z[h(x)] = \int \mathcal{D}\phi(x) \exp(-H[\phi(x), h(x)]).
\]

(4)

It is well known that if we consider such a system at temperatures not too close to the critical point (so that the parameter \( \tau \) is not too small) the thermal critical fluctuations are irrelevant, and the leading contributions to the partition function, equation (4), arise from the minima of the Hamiltonian, equation (2), described by the saddle point equation

\[
-\Delta \phi(x) - \tau \phi(x) + g\phi^3(x) = h(x).
\]

(5)

In the pure system (at \( h(x) = 0 \)) the restriction on the value of the parameter \( \tau \) is given by the Ginzburg–Landau condition

\[
\tau \gg g^{2/(4-D)} \equiv \tau_{GL}.
\]

(6)
It is clear that this requirement makes sense only if the coupling parameter $g$ is sufficiently small. Of course, formally for the three-dimensional Hamiltonian, equation (2), considered as the continuous limit representation of the original Ising model, this is not true (unlike for the corresponding systems in dimensions close to 4, where the effective (renormalized) value of the coupling parameter $g \sim \epsilon = (4-D) \ll 1$). Nevertheless, here we are going to consider the random field Ginzburg–Landau theory, equation (2), in which the coupling parameter $g$ is assumed to be sufficiently small, hoping that at the qualitative level the behavior of the system is not very sensitive to the actual value of this parameter.

The only relevant spatial scale in the system described by the Hamiltonian (2) is the correlation length $R_c(\tau)$ which under condition (6) has the scaling

$$R_c(\tau) \sim \tau^{-1/2}.$$  

In the absence of random fields the ferromagnetic ground state of the system is characterized by the homogeneous configuration $\phi_0 = \sqrt{\tau/g}$. To study the effects produced by the random fields on the ferromagnetic state of the system let us perform the following space and field rescaling:

$$\varphi(x) = \left(\frac{\tau}{g}\right)^{1/2} \varphi(x/R_c).$$

In terms of the new fields $\varphi(z)$, where $z \equiv x/R_c = \tau^{1/2} x$, the system is described by the rescaled Hamiltonian

$$H[\varphi(z), \tilde{h}(z)] = \frac{\tau(4-D)/2}{g} \int d^D z \left[ \frac{1}{2}(\nabla \varphi(z))^2 - \frac{1}{2} \varphi^2(z) + \frac{1}{4} \varphi^4(z) - \sqrt{\frac{g}{\tau^3}} \tilde{h}(z) \varphi(z) \right]$$

where the rescaled random fields

$$\tilde{h}(x/R_c) = R_c^{-D} \int_{|x′-x|<R_c} d^D x′ \ h(x′)$$

are described by the distribution function

$$P[\tilde{h}(z)] = \tilde{p}_0 \exp(\frac{1}{2h^2 \tau^{D/2}} \int d^D z \tilde{h}^2(z)), $$

(note that the integration here involves the ultraviolet cut-off length scale being equal to 1). Redefining the random fields again:

$$\tilde{h}(z) = \sqrt{\frac{\tau^3}{g}} \xi(z),$$

instead of equation (4), we obtain the partition function

$$Z[\xi(z)] = \int D\varphi(z) \exp(-\tilde{\beta} \tilde{H}[\varphi(z), \xi(z)])$$

which is controlled by the effective ‘inverse temperature’

$$\tilde{\beta} \equiv \left(\frac{\tau}{\tau_{GL}}\right)^{(4-D)/2} \gg 1$$

doi:10.1088/1742-5468/2007/09/P09005
(\(\tau_{\text{GL}}\) is the Ginzburg–Landau temperature, equation (6)) and which is defined by the new effective Hamiltonian

\[
H[\varphi(z), \xi(z)] = \int d^Dz \left[ \frac{1}{2} (\nabla \varphi(z))^2 - \frac{1}{2} \varphi^2(z) + \frac{1}{4} \varphi^4(z) - \xi(z) \varphi(z) \right]
\]

(15)

which contains no parameters. The new random fields \(\xi(z)\) are described by the Gaussian distribution

\[
P[\xi(z)] = p_0 \exp \left( -\frac{1}{2\lambda^2} \int d^Dx \xi^2(z) \right),
\]

(16)

characterized by the mean square value

\[
\lambda^2 = \left( \frac{gh^2}{\tau(6-D)/2} \right).
\]

(17)

The ground state configurations in terms of the new fields \(\varphi(z)\) are defined by the saddle point equation

\[
-\Delta \varphi(z) - \varphi(z) + \varphi^3(z) = \xi(z).
\]

(18)

In terms of the new fields, \(\varphi(z)\) and \(\xi(z)\), the transition from the ordered (ferromagnetic) to the disordered (paramagnetic) state looks as follows. In the absence of the random fields (\(\xi \equiv 0\)) the ground state is given by the trivial ferromagnetic solution \(\varphi_0(z) = 1\) (or \(\varphi_0(z) = -1\)). The presence of weak random fields (at \(\lambda \ll 1\)) introduces only small perturbations to this solution. However, if we increase the effective strength of \(\xi(z)\), which is controlled by the parameter \(\lambda\), equation (17), the ferromagnetic configuration becomes more and more perturbed, and finally, at a certain critical value \(\lambda_c\), the ferromagnetic ordering is destroyed.

One can note two important points in the above scenario:

First. According to equation (17), for fixed value of the parameter \(h\) (which is the strength of the original random fields \(h(x)\)), the effective strength of the random fields \(\xi(z)\) is controlled by the temperature parameter \(\tau\). According to its definition (\(\tau = (1 - T)\)), increasing the temperature \(T\) of the system means decreasing \(\tau\), which in turn produces increase of the parameter \(\lambda\). In other words, variations in temperature turn into variations of the effective strength of the random fields.

Second. According to the saddle point equation (18) the critical value of \(\lambda_c\) is of the order of 1. This means that the transition takes place at

\[
\tau \sim \tau_c = (gh^2)^{2/(6-D)}.
\]

(19)

At this point the value of the effective ‘inverse temperature’ parameter \(\tilde{\beta}\), equation (14), is

\[
\tilde{\beta}_c = \left( \frac{h^2}{g^2(4-D)^{1/(4-D)}} \right)^{(4-D)/(6-D)}.
\]

(20)

Thus, if the strength of random fields is not too small,

\[
h \gg h_{\text{c}}(g) = g^{1/(4-D)},
\]

(21)

doi:10.1088/1742-5468/2007/09/P09005
for dimensions $D < 4$ in the vicinity of the phase transition (when $\tau \sim \tau_c$) we have
\[
\tilde{\beta} \sim \tilde{\beta}_c = \left( \frac{h}{h^*(g)} \right)^{2(4-D)/(6-D)} \gg 1.
\]
(22)

In other words, in terms of the representation, equations (13)–(17), the original ferromagnetic–paramagnetic phase transition at any point of the critical curve $h_c(T)$ on the left of the point $(h^*, T^*)$ (figure 1) is equivalent to the low temperature order–disorder transition (far left extreme of the curve $h_c(T)$) for variation of the strength of the random fields. In particular, for the dimension $D = 3$,
\[
\tau_{GL} = g^2,
\tau_c = (gh^2)^{2/3},
\tilde{\beta}_c = \left( \frac{h}{g} \right)^{2/3},
\]
(23)
\[
h^*(g) = g.
\]

The conditions $\tau_c \gg \tau_{GL}, \tilde{\beta}_c \gg 1$ and $h \gg h^*$ discussed above are automatically satisfied through the only restriction on the strength of the random fields,
\[
h \gg g.
\]
(24)

In the next section, in an effort to understand the nature of the low temperature phase transition, we are going to consider its extreme version in the zero-temperature limit. In this case it is natural to consider the original discrete Ising model on a lattice instead of its continuous limit representation.

3. Zero-temperature phase transition

Let us consider the three-dimensional random field ferromagnetic Ising model on the cubic lattice with the Hamiltonian
\[
H = -\frac{1}{12} \sum_{\langle j \neq i \rangle} \sigma_i \sigma_j - \sum_i h_i \sigma_i.
\]
(25)
Here \( \langle j \neq j \rangle \) denotes the pairs of nearest neighbors, and the random fields \( h_i \) are described by the independent Gaussian distributions

\[
W_h[h_1, h_2, \ldots, h_N] = \prod_{i=1}^{N} \mathcal{P}_h(h_i)
\]

where

\[
\mathcal{P}_h(h_i) = \frac{1}{\sqrt{2\pi h^2}} \exp \left( -\frac{h_i^2}{2h^2} \right).
\]  

The ground state spin configuration of this system is defined by the set of \( N \) conditions

\[
\sigma_i = \text{sgn} \left( \frac{1}{6} \sum_{\alpha_i=1}^{6} \sigma_{\alpha_i} + h_i \right)
\]

where for every site \( i \) the summation goes over its six nearest neighbors.

First, let us discuss at the qualitative level what is going on if we increase the parameter \( h \) from zero to large values. At \( h = 0 \) the state of the system is ferromagnetic: the solution of the above equation (28) is trivial, \( \sigma_i = +1 \) (of course, there is another equivalent solution \( \sigma_i = -1 \), but in what follows we will suppose that in the ferromagnetic state the spins are directed ‘up’). At non-zero \( h \ll 1 \) almost all the spins are directed ‘up’, but in rare cases, when at a given site \( i \) the value of the local field is sufficiently negative, \( h_i < -1 \), the \( i \)th spin is flipped ‘down’. According to equation (27), the probability of this event (which is equal to the concentration of the negative spins) is

\[
\rho(h \ll 1) = \int_{-\infty}^{-1} dx \mathcal{P}_h(x) = \frac{1}{2} \text{erfc} \left( \frac{1}{\sqrt{2h}} \right)
\]  

where

\[
\text{erfc}(z) \equiv \frac{2}{\sqrt{\pi}} \int_{z}^{+\infty} dt \exp(-t^2)
\]

is the complementary error function (which exponentially goes to zero at \( z \gg 1 \)).

Note that in the above consideration we have neglected the configurations in which the flipped down spins appear to be the nearest neighbors. It is clear that at small \( h \) the concentration \( \rho(h) \) is exponentially small and therefore the probability of these events is negligible. However, on increasing the value of the parameter \( h \) the probability of finding two or more neighboring ‘down’ spins becomes non-small, and it is due to these configurations that the situation becomes rather complicated. First of all, one can note that if at neighboring sites the values of the random fields appear to be non-small (and negative), then the stable spin configurations can become ambiguous. As an example let us consider two neighboring sites \( i \) and \( i+1 \) (surrounded by spins which all are directed ‘up’) with the values of the random fields \(-1 < h_i < -2/3\) and \(-1 < h_{i+1} < -2/3\). One can easily see that in this case both the configuration \( \sigma_i = \sigma_{i+1} = +1 \) and the configuration \( \sigma_i = \sigma_{i+1} = -1 \) satisfy the stability conditions, equation (28). It is obvious that one could find similar phenomena in clusters consisting of larger numbers of spins.

The effects of the ambiguity of the stationary spin configurations due to the presence of the random fields and their (non-perturbative) contributions to the finite temperature

\[\text{doi:10.1088/1742-5468/2007/09/P09005}\]
thermodynamics were discussed earlier in terms of the continuous Ginzburg–Landau representation of the RFIM [22]. Here we will describe the consequences of the ambiguity of the stationary states for the phase transition at zero temperature. At the qualitative level it is tempting to suggest that the presence of such phenomena could be the origin of a phase transition of first order. Indeed, if upon increasing the strength of the random field the system gets stuck at ferromagnetic configurations, an alternative disordered state could become energetically preferable before the ferromagnetic state becomes unstable. Note that unlike the second-order phase transitions, which are characterized by the divergence of the correlation length (and which require us to find a way to study the system at large scales), the first-order phase transition is characterized by a finite value of the correlation length. For that reason one can hope that taking into account correlations only of the order of the lattice spacing could still give reasonable results.

3.1. Mean field approach

First, as a simple ‘warming up exercise’, let us consider how the zero-temperature phase transition looks if we study it in terms of the ordinary mean field approach (which neglects the spin–spin correlations completely).

Let us denote by $x$ the probability that a given spin $i$ points ‘up’. Then the probability of being ‘down’ is or course equal to $(1-x)$. In this case the ferromagnetic order parameter which is the global magnetization of the system is

$$m = \frac{1}{N} \sum_{i}^{N} \sigma_i = \bar{\sigma}_i = 2x - 1. \quad (31)$$

A given spin is surrounded by six nearest neighbors. If all the neighbors are pointing ‘up’ (the probability of this configuration is equal to $x^6$), then $\sigma_i = +1$ provided the local field $h_i > -1$. If one of the neighbors is ‘down’ (the probability is $6x^5(1-x)$), then $\sigma_i = +1$ provided the local field $h_i > -2/3$. If two of the neighbors are ‘down’ (the probability is $\frac{6 \times 5 \times 4}{3} x^4 (1-x)^2$), then $\sigma_i = +1$ provided the local field $h_i > -1/3$ etc. Collecting all these configurations one easily gets the following self-consistent equation for $x$:

$$x = x^6 \cdot P_6(h) + 6x^5(1-x) \cdot P_5(h) + 15x^4(1-x)^2 \cdot P_4(h) + 20x^3(1-x)^3 \cdot P_3(h) + 15x^2(1-x)^4 \cdot P_2(h) + 6x(1-x)^5 \cdot P_1(h) + (1-x)^6 \cdot P_0(h). \quad (32)$$

Here we have introduced the notation

$$P_6(h) = \int_{-1}^{\infty} dy \, \mathcal{P}_6(y) = 1 - \frac{1}{2} \text{erfc}\left(\frac{1}{\sqrt{2}h}\right)$$

$$P_5(h) = \int_{-2/3}^{\infty} dy \, \mathcal{P}_5(y) = 1 - \frac{1}{2} \text{erfc}\left(\frac{2}{3\sqrt{2}h}\right)$$

$$P_4(h) = \int_{-1/3}^{\infty} dy \, \mathcal{P}_4(y) = 1 - \frac{1}{2} \text{erfc}\left(\frac{1}{3\sqrt{2}h}\right)$$

$$P_3(h) = \int_{0}^{\infty} dy \, \mathcal{P}_3(y) = \frac{1}{2}$$

$$P_2(h) = \int_{1/3}^{\infty} dy \, \mathcal{P}_2(y) = \frac{1}{2} \text{erfc}\left(\frac{1}{3\sqrt{2}h}\right)$$

$$P_1(h) = \int_{1/3}^{\infty} dy \, \mathcal{P}_1(y) = \frac{1}{2} \text{erfc}\left(\frac{1}{3\sqrt{2}h}\right)$$

$$P_0(h) = \int_{1/3}^{\infty} dy \, \mathcal{P}_0(y) = \frac{1}{2} \text{erfc}\left(\frac{1}{3\sqrt{2}h}\right)$$
On the nature of the phase transition in the three-dimensional random field Ising model

Figure 2. Mean field solution for the order ferromagnetic order parameter \( m(h) \) \( (h_c \simeq 0.696) \).

![Figure 2](image)

\[ P_1(h) = \int_{2/3}^{\infty} dy \mathcal{P}_h(y) = \frac{1}{2} \text{erfc} \left( \frac{2}{3\sqrt{2h}} \right) \]
\[ P_0(h) = \int_{1}^{\infty} dy \mathcal{P}_h(y) = \frac{1}{2} \text{erfc} \left( \frac{1}{\sqrt{2h}} \right). \]

(33)

Figure 3. Log–log plot for the ferromagnetic order parameter in the vicinity of the critical point.

The solution of equation (32) for the ferromagnetic order parameter is shown in figure 2. We see that the phase transition from the ferromagnetic \( (m \neq 0) \) to the paramagnetic \( (m = 0) \) state takes place at \( h_c \simeq 0.696 \). One can easily check that in the vicinity of the critical point, as \( h \to h_c \), the ferromagnetic order parameter vanishes according to the scaling law

\[ m(h) \sim (h_c - h)^\beta \]

(34)

described by the usual mean field critical exponent \( \beta = 0.5 \) (see figure 3). This, of course,
is not surprising because if in equation (32) we substitute \( x = (m + 1)/2 \), the resulting equation for \( m \) (however complicated it may look) will still be the mean field equation for the order parameter, and its development in powers of small \( m(h) \) in the vicinity of the critical point cannot give anything but a mean field critical exponent.

It is clear that with the consideration presented above we cannot pretend to describe the true phase transition which takes place in the system under consideration. The reason for that is obvious: in such mean field calculations one misses all spin–spin correlations, which, as we have discussed at the beginning of this section, are crucial for the statistical properties of the present system.

Below we are going to modify the above approach so that it will take into account correlations between pairs of nearest neighbor spins. It turns out that even such rather limited improvement is sufficient to introduce rather dramatic changes in the scenario of the phase transition.

### 3.2. Modified mean field approach

Let us consider a pair of nearest neighbor spins \( \sigma_1 \) and \( \sigma_2 \) and let us denote by \( p(\uparrow\uparrow) \), \( p(\uparrow\downarrow) \), \( p(\downarrow\uparrow) \) and \( p(\downarrow\downarrow) \) the probabilities of the \((\sigma_1, \sigma_2)\) configurations \((+1,+1), (+1,-1), (-1,+1)\) and \((-1,-1)\) respectively. These four probabilities are, of course, bounded by the condition \( p(\uparrow\uparrow) + p(\uparrow\downarrow) + p(\downarrow\uparrow) + p(\downarrow\downarrow) = 1 \). As before, by \( x \) we denote the probability for a given spin to be ‘up’. One can easily see that

\[
x = p(\uparrow\uparrow) + p(\uparrow\downarrow) .
\] (35)

The two spins \( \sigma_1 \) and \( \sigma_2 \) are surrounded by \((5 + 5)\) neighbors (figure 4), and their orientations are defined by the equations

\[
\sigma_1 = \text{sgn} \left[ \frac{1}{6} \left( \sum_{\alpha_1=1}^{5} \sigma_{\alpha_1} + \sigma_2 \right) + h_1 \right]
\]

\[
\sigma_2 = \text{sgn} \left[ \frac{1}{6} \left( \sum_{\alpha_2=1}^{5} \sigma_{\alpha_2} + \sigma_1 \right) + h_2 \right]
\] (36)

where \( \sigma_{\alpha_1,\alpha_2} \) denote the neighbors of the spins \( \sigma_{1,2} \).
The idea is to compute $p(\uparrow\uparrow)$ and $p(\uparrow\downarrow)$ by summing over all possible configurations of these neighboring spins (with the corresponding probabilities defined by the parameter $x$) and integrating over local fields $h_1$ and $h_2$ (with the probability distribution $P_h(h_1)P_h(h_2)$). Then, substituting $p(\uparrow\uparrow)$ and $p(\uparrow\downarrow)$ (which will be functions of $x$ and $h$) into equation (35), we will get a self-consistent equation for $x$. It turns out, however, that this program cannot be implemented directly, because for every configuration of surrounding spins there are finite regions in the plane ($h_1$, $h_2$) where the orientations of spins $\sigma_1$ and $\sigma_2$ are not uniquely defined by the given values of $h_1$ and $h_2$.

As an example let us consider again the simplest situation when all ten neighboring spins are directed ‘up’ (which has the probability $x^{10}$). In this case the condition for having both spins ‘up’ is $h_1 > -1$ and $h_2 > -1$ (so the corresponding contribution to $p(\uparrow\uparrow)$ must be given by the integration of $P_h(h_1)P_h(h_2)$ in the sector ($h_1 > -1, h_2 > -1$)). The contribution to the probability $p(\uparrow\downarrow)$ is given by the sector ($h_1 < -2/3, h_2 > -1$). The contribution to the probability $p(\downarrow\uparrow)$ is given by the sector ($h_1 > -1, h_2 > -2/3$). And finally the contribution to the probability $p(\downarrow\downarrow)$ is given by the sector ($h_1 < -2/3, h_2 < -2/3$), which overlaps with the sector corresponding to $p(\uparrow\uparrow)$. Thus in the square ($-1 < h_1 < -2/3; -1 < h_2 < -2/3$) we are facing a kind of frustration (figure 5).

One can propose two ways out of this situation. First of all, one can, of course, compare the energies of the two configurations, and then, if the spin orientations always correspond to the ground state, the ambiguity will be lifted (the configuration with both spins ‘up’ has lower energy in the triangle above the dotted line inside the ‘frustrated square’ in figure 5, and the one with both spins ‘down’ has lower energy below the dotted line). However, if we keep in mind the scenario in which in the course of variations of the external parameter $h$ each spin just follows the direction of the local field, while all thermally activated ‘jumps over barriers’ are suppressed, then we have to conclude that inside the ‘frustrated’ region the orientation of the two spins must depend on the history. Namely, for every point in the plane ($h_1$, $h_2$) we can assign the ‘trajectory’ which would demonstrate where these particular values of the random fields came from. For instance, in the situation when we study the evolution of the system for increasing value of $h$ (starting from zero), a particular point ($h_1$, $h_2$) has a ‘trajectory’ which is a straight line connecting this point with the origin. In the opposite case of decreasing $h$ (starting from infinity), the ‘trajectory’ of the point ($h_1$, $h_2$) is the straight line which comes from infinity and is directed towards the origin.

Figure 5. The structure diagram of the orientations of two neighboring spins for the case when all their (5 + 5) nearest neighbors are directed ‘up’.
On the nature of the phase transition in the three-dimensional random field Ising model

Figure 6. The orientation diagram of two neighboring spins for the case when all their \((5 + 5)\) nearest neighbors are directed ‘up’: (a) for increasing and (b) for decreasing strength \(h\) of the random fields.

Following these ‘rules of the game’, the ambiguity of the spin orientations is lifted provided one fixes which process is actually under study: increasing or decreasing of \(h\). In the study of the ferromagnetic \(\rightarrow\) paramagnetic transition (increasing \(h\)) one can readily note that the ‘trajectory’ to any point inside the ‘frustrated’ square \((-1 < h_1 < -2/3; -1 < h_2 < -2/3)\) comes from the region where both spins, \(\sigma_1\) and \(\sigma_2\), are directed ‘up’. Thus, in this case the square ‘belongs’ to the probability \(p(\uparrow\uparrow)\) (figure 6(a)). In the reverse process of decreasing \(h\) (the paramagnetic \(\rightarrow\) ferromagnetic transition) the only way to arrive inside this square is to come from the region where both spins are ‘down’, and in this case the square belongs to the probability \(p(\downarrow\downarrow)\) (figure 6(b)).

For a better understanding of how this works let us consider other examples. For the case when all 10 surrounding spins are directed ‘down’, the ‘orientation diagram’ for the spins \(\sigma_1\) and \(\sigma_2\) is as represented in figure 7. Here the ‘frustrated square’ is located in the region \((2/3 < h_1 < 1; 2/3 < h_2 < 1)\). Now, on increasing \(h\) one arrives in this square from the region where both spins are ‘down’ and therefore in this case it belongs to the probability \(p(\downarrow\downarrow)\) (figure 7(a)). Clearly, for the reverse process of decreasing \(h\) the square belongs to the probability \(p(\uparrow\uparrow)\) (figure 7(b)).

Let us consider the configuration in which all five neighbors of the spin \(\sigma_1\) are ‘up’, while all five neighbors of the spin \(\sigma_2\) are ‘down’. Using equations (36) one can easily build the corresponding orientation diagram (figure 8). We see that in this case both for increasing and for decreasing \(h\) one can arrive in the lower left triangle of the ‘frustrated square’ only by passing through the region where both spins are ‘down’. Similarly, for both increasing and decreasing \(h\) one can arrive in the upper right triangle only by passing through the region where both spins are ‘up’.

Finally, let us consider the configuration in which all five neighbors of the spin \(\sigma_1\) are ‘up’, and only four neighbors of the spin \(\sigma_2\) are ‘down’. Here the division of the ‘frustrated square’ between \(p(\uparrow\uparrow)\) and \(p(\downarrow\downarrow)\) is somewhat more complicated (figure 9). For increasing \(h\) one arrives in the section of the square below the line \(h_2(h_1) = -\frac{1}{2}h_1\) by passing through the region where both spins are ‘down’ and one arrives in the section above this line by passing through the region where both spins are ‘up’ (figure 9(a)). On the other hand, for decreasing \(h\) one arrives in the section of the square below the line

\[h_2(h_1) = -\frac{1}{2}h_1\]
Figure 7. The orientation diagram of two neighboring spins for the case when all their \((5 + 5)\) nearest neighbors are directed ‘down’: (a) for increasing and (b) for decreasing strength \(h\) of the random fields.

\[ h_2(h_1) = -\frac{2}{3}h_1 \]

by passing through the region where both spins are ‘down’ and one arrives in the section above this line by passing through the region where both spins are ‘up’ (figure 9(b)).

Now, with some patience and perseverance one can construct the structure of the frustrated squares for all 36 ‘analytically different’ (in terms of equations (36)) configurations of the surrounding spins. The resulting orientation diagrams for increasing and decreasing \(h\) are shown in figures 10 and 11 respectively.
On the nature of the phase transition in the three-dimensional random field Ising model

Figure 9. The orientation diagram of two neighboring spins for the case when all five neighbors of the spin $\sigma_1$ are directed ‘up’, and four neighbors of the spin $\sigma_2$ are directed ‘down’: (a) for increasing and (b) for decreasing strength $h$ of the random fields. The lines are: [1] $h_2(h_1) = -(1/2)h_1$; [2] $h_2(h_1) = -(2/3)h_1$.

Figure 10. The structure of the ‘frustrated squares’ for increasing strength $h$ of the random fields. The notation $(m, n)$ inside the square indicates that it corresponds to the configuration with $m$ neighbors of the spin $\sigma_1$ and $n$ neighbors of the spin $\sigma_2$ are directed ‘up’. Rose regions correspond to the state with both spins $\sigma_1$ and $\sigma_2$ directed ‘up’. Green regions are the ones where both spins $\sigma_1$ and $\sigma_2$ are directed ‘down’. The lines are: [1] $h_2(h_1) = -(1/2)h_1$; [2] $h_2(h_1) = -h_1$; [3] $h_2(h_1) = -2h_1$. 

doi:10.1088/1742-5468/2007/09/P09005
Figure 11. The structure of the ‘frustrated squares’ with decreasing strength $h$ of the random fields. The notation is the same as in figure 10. The lines are: [1] $h_2(h_1) = -(1/3)h_1$; [2] $h_2(h_1) = -(1/2)h_1$; [3] $h_2(h_1) = -(2/3)h_1$; [4] $h_2(h_1) = -h_1$; [5] $h_2(h_1) = -(3/2)h_1$; [6] $h_2(h_1) = -2h_1$; [7] $h_2(h_1) = -3h_1$.

Finally, after the regions corresponding to ‘up–up’ and ‘up–down’ orientations of the spins $\sigma_1$ and $\sigma_2$ are unambiguously defined, the derivation of the equations for the probabilities $p(\uparrow\uparrow)(x,h)$ and $p(\uparrow\downarrow)(x,h)$ is straightforward (although slightly onerous). The explicit forms of the equation (35) for increasing and for decreasing $h$ are given in the appendix.

The solution of these equations in terms of the ferromagnetic order parameter $m(h)$ is represented in figure 12. We see that the behavior of the magnetization exhibits a clear hysteresis phenomenon.

4. Discussion

In this paper the zero-temperature phase transition has been studied under the assumption that for any variations of the strength $h$ of the random fields the transformations of the spin configurations proceed under the constraint that all thermally activated spin flips are suppressed. It should be stressed that this situation is essentially different from that of the true equilibrium phase transition where at any given $h$ the system is supposed to be in the ground state (and which, in my view, is experimentally inaccessible, at least in the low temperature limit).

Comparing two possible scenarios of the order–disorder phase transition: continuous (second order) and discontinuous (first order), one should note that unlike for the continuous transition characterized by the divergence of the correlation length, for the first-order phase transition the correlation length remains finite. Thus, admitting that the transition is discontinuous one can hope that a theory which takes into account

doi:10.1088/1742-5468/2007/09/P09005
spin–spin correlations only at a limited scale (e.g. of the order of the lattice spacing) would still give a qualitatively correct description of the phase transition.

Usual mean field theory is useless here because it does not take spin–spin correlations into account at all. On the other hand, it has been demonstrated in this paper that even rather limited improvement of the mean field approach, which takes into account two nearest neighbor spin correlations, produces a rather dramatic effect on the scenario of the phase transition. After this modification the value of the ferromagnetic order parameter $m(h)$ as a function of the strength of the random fields becomes history dependent, exhibiting a clear hysteresis phenomenon (figure 12).

Unfortunately, in the framework of the present theory it is still difficult to reach a definite conclusion about the nature of the phase transition. The existence of the hysteresis loop (together with a clear understanding of the physical mechanism at its origin) is a strong argument in favor of a first-order phase transition. On the other hand, each of the curves $m_{+}(h)$ (for increasing $h$) and $m_{-}(h)$ (for decreasing $h$) in figure 12 shows a continuous transition which, of course, makes no sense, because for such a transition the correlation length diverges, while the present theory takes into account correlations only of the order of the lattice spacing. In this sense the present theory is not self-consistent, and it would be reasonable to suggest that the presence of the continuous transitions at $h_{c}^{(\pm)}$ is no more than an artifact of the proposed approach. These issues require further detailed study.

Appendix

Let us denote by $p_{+}(\uparrow\uparrow)$ and $p_{-}(\uparrow\uparrow)$ the probabilities for the two spins $\sigma_{1}$ and $\sigma_{2}$ to be both 'up' for increasing and decreasing variations of $h$ respectively. Note that the probabilities $p(\uparrow\downarrow)$ are the same for both (increasing and decreasing) cases. Then, using the orientation

Figure 12. Ferromagnetic order parameter $m(h)$ for increasing (red line) and decreasing (green line) strength $h$ of the random fields. $h_{c}^{(+)} \simeq 0.722; h_{c}^{(-)} \simeq 0.622.$
diagrams shown in figures 10 and 11 (after some work) we get

\begin{align}
p_+ (\uparrow \uparrow)(x, h) &= P_6(x)P_6(x)x^{10} + 10P_6(x)P_5(x)x^9(1 - x)^1 + [20P_6(x)P_4(x) \\
&+ 25P_5(x)P_5(x)]x^8(1 - x)^2 + [20P_6(x)P_3(x) + 10P_5(x)P_4(x)]x^7(1 - x)^3 \\
&+ [10P_6(x)P_2(x) - D_{51}^{(0)}(x)] + 100P_5(x)P_3(x) + 10P_4(x)P_4(x)]x^6(1 - x)^4 \\
&+ [2P_6(x)P_1(x) - 0.5(P_6(x) - P_5(x))(P_1(x) - P_6(x))] \\
&+ 50P_5(x)P_2(x) - 0.5(P_5(x) - P_4(x))(P_2(x) - P_1(x))] \\
&+ 200[P_4(x)P_3(x) - 0.5(P_4(x) - P_3(x))(P_3(x) - P_2(x))]x^5(1 - x)^5 \\
&+ [10P_5(x)P_1(x) - (P_5(x) - P_4(x))(P_1(x) - P_5(x)) + D_{30}^{(0)}(x)] \\
&+ 100[P_4(x)P_2(x) - (P_4(x) - P_3(x))(P_2(x) - P_4(x))]]x^4(1 - x)^6 \\
&+ [20P_4(x)P_1(x) - (P_4(x) - P_3(x))(P_1(x) - P_4(x)))]x^3(1 - x)^7 \\
&+ [20P_4(x)P_1(x) - (P_4(x) - P_3(x))(P_1(x) - P_4(x))]]x^2(1 - x)^8 \\
&+ [10P_2(x)P_1(x) - (P_2(x) - P_1(x))(P_1(x) - P_2(x)))]x(1 - x)^9 \\
&+ [P_1(x)P_1(x) - (P_1(x) - P_0(x))(P_1(x) - P_0(x))]x(1 - x)^10 \quad (A.1)
\end{align}

\begin{align}
p_-(\downarrow \downarrow)(x, h) &= [P_6(x)P_6(x) - (P_6(x) - P_5(x))(P_6(x) - P_5(x))]x^{10} \\
&+ 10[2P_6(x)P_6(x) - (P_6(x) - P_5(x))(P_6(x) - P_5(x))]x^9(1 - x)^1 \\
&+ [20P_6(x)P_6(x) - (P_6(x) - P_5(x))(P_6(x) - P_5(x))] \\
&+ 25[P_5(x)P_5(x) - (P_5(x) - P_4(x))(P_5(x) - P_4(x))]x^8(1 - x)^2 \\
&+ [20P_5(x)P_5(x) - (P_5(x) - P_4(x))(P_5(x) - P_4(x))]x^7(1 - x)^3 \\
&+ [10P_5(x)P_5(x) - (P_5(x) - P_4(x))(P_5(x) - P_4(x))]x^6(1 - x)^4 \\
&+ [2P_5(x)P_5(x) - (P_5(x) - P_4(x))(P_5(x) - P_4(x))]x^5(1 - x)^5 \\
&+ 50P_5(x)P_5(x) - 0.5(P_5(x) - P_4(x))(P_5(x) - P_4(x)) \\
&+ 200[P_4(x)P_4(x) - (P_4(x) - P_3(x))(P_4(x) - P_3(x))]x^4(1 - x)^6 \\
&+ [20P_4(x)P_4(x) - (P_4(x) - P_3(x))(P_4(x) - P_3(x))]x^3(1 - x)^7 \\
&+ [20P_4(x)P_4(x) - (P_4(x) - P_3(x))(P_4(x) - P_3(x))]x^2(1 - x)^8 \\
&+ 10P_2(x)P_2(x)x(1 - x)^9 + P_1(x)P_1(x)(1 - x)^10 \quad (A.2)
\end{align}

\begin{align}
p(\uparrow \downarrow)(x, h) &= P_5(x)P_0(x)x^{10} + [P_5(x)P_1(x) + 5P_4(x)P_0(x)]x^9(1 - x)^1 \\
&+ [10P_5(x)P_2(x) + 25P_5(x)P_1(x) + 10P_4(x)P_0(x)]x^8(1 - x)^2 \\
&+ [10P_5(x)P_2(x) + 10P_3(x)P_0(x) + 5P_3(x)P_1(x) + 10P_2(x)P_0(x)]x^7(1 - x)^3
\end{align}
where the functions \( P_k(h) \) \((k = 0, \ldots, 6)\) are defined in equation (33) and

\[
\begin{align*}
D_{51}^{(+)}(h) &= \int_{-1}^{-y_1/2} dy_1 \mathcal{P}_h(y_1) \int_{1/3}^{y_2} dy_2 \mathcal{P}_h(y_2) \\
D_{40}^{(+)}(h) &= \int_{2/3}^{1} dy_2 \mathcal{P}_h(y_2) \int_{-y_2/2}^{-1/3} dy_1 \mathcal{P}_h(y_1) \\
D_{52}^{(-)}(h) &= \int_{-1}^{-y_1/3} dy_1 \mathcal{P}_h(y_1) \int_{1/3}^{y_2} dy_2 \mathcal{P}_h(y_2) \\
D_{51}^{(-)}(h) &= \int_{-1}^{-2y_1/3} dy_1 \mathcal{P}_h(y_1) \int_{-2y_1/3}^{y_2} dy_2 \mathcal{P}_h(y_2) \\
D_{42}^{(-)}(h) &= \int_{-2y_1/3}^{1/3} dy_1 \mathcal{P}_h(y_1) \int_{y_1/2}^{1} dy_2 \mathcal{P}_h(y_2).
\end{align*}
\]

The ferromagnetic order parameters \( m_\pm(h) \) as a function of \( h \) are obtained from the relation

\[
m_\pm(h) = 2x_\pm(h) - 1
\]

where \( x_+(h) \) and \( x_-(h) \) are the corresponding solutions of the equations

\[
x = p_\pm(\uparrow\uparrow)(x, h) + p(\downarrow\downarrow)(x, h).
\]

\[\text{(A.5)}\]

\[\text{(A.6)}\]

\textbf{References}

[1] Nattermann T and Villain J, 1988 \textit{Phase Transi.} 115

[2] Jaccarino V and King A R, 1990 \textit{Physica A} 163 291

[3] Larkin A I, 1970 \textit{Sov. Phys. JETP} 31 784

[4] Imry Y and Ma S-K, 1975 \textit{Phys. Rev. Lett.} 35 1399

[5] Imbrie J, 1984 \textit{Phys. Rev. Lett.} 53 1747

[6] Aharony A, Imry Y and Ma S-K, 1976 \textit{Phys. Rev. Lett.} 37 1364

[7] Parisi G, 1984 \textit{Proceedings of Les Houches 1982, Session XXXIX} ed J B Zuber and R Stora (Amsterdam: North-Holland)

\textbf{doi:10.1088/1742-5468/2007/09/P09005}
On the nature of the phase transition in the three-dimensional random field Ising model

[8] Guagnelli M, Marinari E and Parisi G, 1993 J. Phys. A: Math. Gen. 26 5675
[9] Schneider T and Pytte E, 1977 Phys. Rev. B 15 1519
[10] Aharony A, 1978 Phys. Rev. B 18 3318
[11] Grinstein G, 1976 Phys. Rev. Lett. 37 944
[12] Mezard M and Parisi G, 1990 J. Phys. A: Math. Gen. 23 L1229
[13] Parisi G, 1980 J. Phys. A: Math. Gen. 13 1887
[14] Aharony A, 1978 Phys. Rev. B 18 653
[15] Mezard M and Montasson R, 1994 Phys. Rev. B 50 7199
[16] De Dominicis C, Orland H and Temisvari T, 1995 J. Physique I 5 987
[17] Brezin E and De Dominicis C, 1998 Europhys. Lett. 44 13
[18] Parisi G and Dotsenko V S, 1992 J. Phys. A: Math. Gen. 25 3143
[19] Schneider T and Pytte E, 1977 Phys. Rev. B 15 1519
[20] Aharony A, 1978 Phys. Rev. B 18 3318
[21] Grinstein G, 1976 Phys. Rev. Lett. 37 944
[22] Mezard M and Parisi G, 1990 J. Phys. A: Math. Gen. 23 L1229
[23] Parisi G, 1980 J. Phys. A: Math. Gen. 13 1887
[24] Aharony A, 1978 Phys. Rev. B 18 653
[25] Mezard M and Montasson R, 1994 Phys. Rev. B 50 7199
[26] De Dominicis C, Orland H and Temisvari T, 1995 J. Physique I 5 987
[27] Brezin E and De Dominicis C, 1998 Europhys. Lett. 44 13
[28] Parisi G and Dotsenko V S, 1992 J. Phys. A: Math. Gen. 25 3143
[29] Schneider T and Pytte E, 1977 Phys. Rev. B 15 1519
[30] Aharony A, 1978 Phys. Rev. B 18 653
[31] Mezard M and Montasson R, 1994 Phys. Rev. B 50 7199
[32] De Dominicis C, Orland H and Temisvari T, 1995 J. Physique I 5 987
[33] Brezin E and De Dominicis C, 1998 Europhys. Lett. 44 13
[34] Parisi G and Dotsenko V S, 1992 J. Phys. A: Math. Gen. 25 3143
[35] Dotsenko V S, 1994 J. Phys. A: Math. Gen. 27 3397
[36] Dotsenko V S, 2006 Physica A 361 463
[37] Dotsenko V S, 2006 J. Stat. Mech. P06003
[38] Feldman D E, 2002 Phys. Rev. Lett. 88 177202
[39] Brezin E, 2001 Eur. Phys. J. B 19 467
[40] Malakis A and Fytas N G, 2006 Phys. Rev. E 73 016109
[41] Fishman S and Aharony A, 1979 J. Phys. C: Solid State Phys. 12 L729
[42] Cardy J, 1984 Phys. Rev. B 30 505
[43] Fishman S and Aharony A, 1979 J. Phys. C: Solid State Phys. 12 L729
[44] Yoshizawa H, Cowley R A, Shirane G, Birgeneau R G, Guggenheim H J and Ikeda H, 1982 Phys. Rev. Lett. 48 438
[45] Hagen M, Cowley R A, and Saito K, Yoshizawa H, Shirane G, Birgeneau R G and Guggenheim H J, 1983 Phys. Rev. B 28 2602
[46] Belanger D P, King A R, Jaccarino V and Cardy J L, 1983 Phys. Rev. B 28 2552
[47] Belanger D P, King A R and Jaccarino V, 1985 Phys. Rev. B 31 4538
[48] Belanger D P, King A R and Jaccarino V, 1985 Phys. Rev. B 31 4538
[49] Birgeneau R J, Cowley R A, Shirane G and Yoshizawa H, 1985 Phys. Rev. Lett. 54 2147
[50] Belanger D P, King A R, Jaccarino V and Resende S M, 1988 J. Physique 49 C8 1241

doi:10.1088/1742-5468/2007/09/P09005
On the nature of the phase transition in the three-dimensional random field Ising model

[49] Ye F, Zhou L, Larochelle S, Lu L, Belanger D P, Greven M and Lederman D, 2002 Phys. Rev. Lett. 89 157202
[50] Hill J P, Thurston T R, Ervin R W, Ramstad M J and Birgeneau R J, 1991 Phys. Rev. Lett. 66 3291
[51] Birgeneau R J, Feng Q, Harris Q J, Hill J P and Ramirez A P, 1996 Phys. Rev. Lett. 77 2342
[52] Wong P-Z and Cable J W, 1983 Phys. Rev. B 28 5361
Wong P-Z, 1996 Phys. Rev. Lett. 77 2338
[53] Hill J P, Feng Q, Harris Q J, Birgeneau R J, Ramirez A P and Cassanho A, 1997 Phys. Rev. B 55 356
[54] Birgeneau R J, 1998 J. Magn. Magn. Mater. 177 1
[55] Slanic Z, Belanger D P and Fernandez-Baca J A, 1998 J. Magn. Magn. Mater. 177 171
[56] Slanic Z, Belanger D P and Fernandez-Baca J A, 1999 Phys. Rev. Lett. 82 426
Slanic Z, Belanger D P and Fernandez-Baca J A, 2001 J. Phys.: Condens. Matter 13 1711
[57] Slanic Z and Belanger D P, 1998 J. Magn. Magn. Mater. 186 65
[58] Aizenman M and Wehr J, 1989 Phys. Rev. Lett. 62 2503
Hui K and Berker A N, 1989 Phys. Rev. Lett. 62 2507
Uzelac K, Hasmy A and Jullien R, 1995 Phys. Rev. Lett. 74 422
Cardy J and Jacobsen J, 1997 Phys. Rev. Lett. 79 4063
Timonin P N, 2004 Phys. Rev. B 69 092102
[59] Sethna J P, Dahmen K, Kartha S, Krumhansl J A, Roberts B W and Shore J D, 1993 Phys. Rev. Lett. 70 3347
Perez-Reche F J and Vives E, 2004 Phys. Rev. B 70 214422

doi:10.1088/1742-5468/2007/09/P09005