ON THE SPECTRAL DISTRIBUTIONS OF DISTANCE-$k$ GRAPH OF FREE PRODUCT GRAPHS.

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Abstract. We calculate the distribution with respect to the vacuum state of the distance-$k$ graph of a $d$-regular tree. From this result we show that the distance-$k$ graph of a $d$-regular graphs converges to the distribution of the distance-$k$ graph of a regular tree. Finally, we prove that, properly normalized, the asymptotic distributions of distance-$k$ graphs of the $d$-fold free product graph, as $d$ tends to infinity, is given by the distribution of $P_k(s)$, where $s$ is a semicircle random variable and $P_k$ is the $k$-th Chebychev polynomial.

1. Introduction

In this paper we consider three problems on the distance-$k$ graphs, which generalize results of Kesten [11] (on random walks on free groups), McKay [13] (on the asymptotic distribution of $d$-regular graphs) and the free central limit of Voiculescu [15]. The first one is finding, for fixed $d$, the distribution w.r.t. the vacuum state of the distance-$k$ graphs of a $d$-regular tree. Then we consider two related problems which are in the asymptotic regime. On one hand, we show that the asymptotic distributions of distance-$k$ graphs of $d$-fold free product graphs, as $d$ tends to infinity, are given by the distribution of $P_k(s)$, where $s$ is a semicircle distribution and $P_k$ is the $k$-th Chebychev polynomial. On the other hand, we find the asymptotic spectral distribution of the distance-$k$ graph of a random $d$-regular graph of size $n$, as $n$ tends to infinity.

More precisely our first result is the following.

Theorem 1.1. For $d \geq 2$, $k \geq 1$, let $A_d^{[k]}$ be the adjacency matrix of distance-$k$ graph of the $d$-regular tree. Then the distribution with respect to the vacuum state of $A_d^{[k]}$ is given by the probability distribution of

$$T_k(b) = \sqrt{\frac{d-1}{d}} P_k \left( \frac{b}{2\sqrt{d-1}} \right) - \frac{1}{\sqrt{d(d-1)}} P_{k-2} \left( \frac{b}{2\sqrt{d-1}} \right),$$

where $P_k$ is the Chebyshev polynomial of order $k$ and $b$ is a random variable with Kesten-McKay distribution, $\mu_d$.

The spectrum of the distance-$k$ graph of the Cartesian product of graphs was first studied by Kurihara and Hibino [10] where they consider the distance-2 graph of $K_2 \times \cdots \times K_2$ (the $n$-dimensional hypercube). More recently, in a series of papers [7, 8, 9, 10, 12, 14] the asymptotic spectral distribution of the distance-$k$ graph of the $N$-fold power of the Cartesian product was studied. These investigations, finally lead to the following theorem which generalizes the central limit theorem for Cartesian products of graphs.

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Theorem 1.2 (Hibino, Lee and Obata [8]). Let $G = (V, E)$ be a finite connected graph with $|V| \geq 2$. For $N \geq 1$ and $k \geq 1$ let $G^{[N,k]}$ be the distance-$k$ graph of $G^N = G \times \cdots \times G$ (N-fold Cartesian power) and $A^{[N,k]}$ its adjacency matrix. Then, for a fixed $k \geq 1$, the eigenvalue distribution of $N^{-k/2}A^{[N,k]}$ converges in moments as $N \to \infty$ to the probability distribution of

$$
\left( \frac{2|E|}{|V|} \right)^{k/2} \frac{1}{k!} \tilde{H}_k(g),
$$

where $\tilde{H}_k$ is the monic Hermite polynomial of degree $k$ and $g$ is a random variable obeying the standard normal distribution $N(0,1)$.

In the same spirit, in [2], we consider the analog of Theorem 1.2 by changing the Cartesian product by the star product.

Theorem 1.3 (Arizmendi and Gaxiola [2]). Let $G = (V, E, e)$ be a locally finite connected graph and let $k \in \mathbb{N}$ be such that $G^{[k]}$ is not trivial. For $N \geq 1$ and $k \geq 1$ let $G^{[\star N,k]}$ be the distance-$k$ graph of $G^{\star N} = G \star \cdots \star G$ (N-fold star power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma = V^{[k]}_e$ be the number of neighbors of $e$ in the distance-$k$ graph of $G$, then the distribution with respect to the vacuum state of $(N\sigma)^{-k/2}A^{[\star N,k]}$ converges in distribution as $N \to \infty$ to a centered Bernoulli distribution. That is,

$$
\frac{A^{[\star N,k]}}{\sqrt{N\sigma}} \xrightarrow{\text{weakly}} \frac{1}{2} - \frac{1}{2} \delta_1,
$$

weakly.

Our second theorem is the free counterpart of the theorems above.

Theorem 1.4. Let $G = (V, E, e)$ be a finite connected graph and let $k \in \mathbb{N}$. For $N \geq 1$ and $k \geq 1$ let $G^{[\star N,k]}$ be the distance-$k$ graph of $G^{\star N} = G \star \cdots \star G$ (N-fold free power) and $A^{[\star N,k]}$ its adjacency matrix. Furthermore, let $\sigma = V^{[k]}_e$ be the number of neighbors of $e$ in the graph $G$. Then the distribution with respect to the vacuum state of $(N\sigma)^{-k/2}A^{[\star N,k]}$ converges in moments (and then weakly) as $N \to \infty$ to the probability distribution of

$$
P_k(s),
$$

where $P_k$ is the Chebychev polynomial of order $k$ and $s$ is a random variable obeying the semicircle law.

Finally, our third theorem considers the asymptotic spectral distribution of the distance-$k$ graph of $d$-regular random graphs.

Theorem 1.5. Let $d$, $k$ be fixed integers and, for each $n$, let $F_n(x)$ be the expected eigenvalue distribution of the distance-$k$ graph of a random regular graph with degree $d$ and order $2n$. Then, as $n$ tends to infinity, $F_n(x)$ converges to the distribution of $A_{d}^{[k]}$ with respect to the vacuum state, described in Theorem 1.1.

Apart from this introduction the paper is organized as follows. In Section 2 we give the basic preliminaries on graphs, orthogonal polynomials and Non-Commutative Probability and Kesten-McKay distributions. Section 3 is devoted to prove Theorem 1.2. We prove Theorem 1.3 in Sections 4 and 5. Section 4 considers the case $k = 2$, while Section 5 considers the case $k \geq 3$. Finally, in Section 6 we use the results of Section 3 to prove Theorem 1.5.
2. Preliminaries

In this section we give very basic preliminaries on graphs, free product graphs, orthogonal polynomials, Jacobi parameters and non-commutative probability. The reader familiar with these objects may skip this section.

2.1. Graphs. By a rooted graph we understand a pair \((G, e)\), where \(G = (V, E)\), is an undirected graph with set of vertices \(V = V(G)\), and the set of edges \(E = E(G) \subseteq \{(x, x') : x, x' \in V, x \neq x'\}\) and \(e \in V\) is a distinguished vertex called the root. For rooted graphs we will use the notation \(V^0 = V \setminus \{e\}\). Two vertices \(x, x' \in V\) are called adjacent if \((x, x') \in E\), i.e. vertices \(x, x'\) are connected with an edge. Then we write \(x \sim x'\). Simple graphs have no loops, i.e. \((x, x) \notin E\) for all \(x \in V\). A graph is called finite if \(|V| < \infty\). The degree of \(x \in V\) is defined by \(\kappa(x) = |\{x' \in V : x' \sim x\}|\), where \(|I|\) stands for the cardinality of \(I\). A graph is called locally finite if \(\kappa(x) < \infty\) for every \(x \in V\). It is called uniformly locally finite if \(\sup\{\kappa(x) : x \in V\} < \infty\).

We define the free product of the rooted vertex sets \((V_i, e_i)\), \(i \in I\), where \(I\) is a countable set, by the rooted set \((\ast_{i \in I} V_i, e)\), where

\[\ast_{i \in I} V_i = \{e\} \cup \{v_1v_2 \cdots v_m : v_k \in V_{i_k}, \text{ and } i_1 \neq i_2 \neq \cdots \neq i_m, m \in \mathbb{N}\},\]

and \(e\) is the empty word.

**Definition 2.1.** The free product of rooted graph \((G_i, e_i)\), \(i \in I\), is defined by the rooted graph \((\ast_{i \in I} G_i, e)\) with vertex set \(\ast_{i \in I} V_i\) and edge set \(\ast_{i \in I} E_i\), defined by

\[\ast_{i \in I} E_i := \{(vu, v'u) : (v, v') \in \bigcup_{i \in I} E_i \text{ and } u, v, v'u \in \ast_{i \in I} V_i\} .\]

We denote this product by \(\ast_{i \in I} (G_i, e_i)\) or \(\ast_{i \in I} G\) if no confusion arises. If \(I = [n]\), we denote by \(G^n = (\ast_{i \in I} G, e)\).

Notice that for a fixed word \(u = v_1v_2 \cdots v_m\) with \(j \in I\) with \(v_1 \notin V_j\) the subgraph of \((\ast_{i \in I} G_i, e)\) induced by the vertex set \(\{wu : w \in V_j\}\) is isomorphic to \(G_j\). This motivates the following definition

**Definition 2.2.** If \(x, y \in \ast_{i \in I} V_i\), we say that \(x\) and \(y\) are in the same copy of \(G_i\) if \(x = vu\) and \(y = v'u\) for some \(u \in \ast_{i \in I} V_i\) and \(v, v' \in V_j^0\) for some \(j \in I\).

For a given graph \(G = (V, E)\), its distance-\(k\) graph \(G^{[k]} = (V, E^{[k]})\) is defined by

\[E^{[k]} = \{(x, y) : x, y \in V, \partial_G(x, y) = k\} .\]

For \(x \in V\), let \(\delta(x)\) be the indicator function of the one-element set \(\{x\}\). Then \(\{\delta(x), x \in V\}\) is an orthonormal basis of the Hilbert space \(L^2(V)\) of square integrable functions on the set \(V\), with the usual inner product.

The adjacency matrix \(A = A(G)\) of \(G\) is a 0-1 matrix defined by

\[(2.1) \quad A_{x,x'} = \begin{cases} 1 & \text{if } x \sim x' \\ 0 & \text{otherwise}. \end{cases}\]

We identify \(A\) with the densely defined symmetric operator on \(L^2(V)\) defined by

\[(2.2) \quad A\delta(x) = \sum_{x \sim x'} \delta(x')\]

for \(x \in V\). Notice that the sum on the right-hand-side is finite since our graph is assumed to be locally finite. It is known that \(A(G)\) is bounded if and only if \(G\) is
uniformly locally finite. If \( A(\mathcal{G}) \) is essentially self-adjoint, its closure is called the adjacency operator of \( \mathcal{G} \) and its spectrum is called the spectrum of \( \mathcal{G} \).

The unital algebra generated by \( A \), i.e. the algebra of polynomials in \( A \), is called the adjacency algebra of \( \mathcal{G} \) and is denoted by \( A(\mathcal{G}) \) or simply \( A \).

2.2. Orthogonal Polynomials and The Jacobi Parameters. Let \( \mu \) be a probability measure with all moments, that is \( m_n(\mu) := \int_R |x^n| \mu(dx) < \infty \). The Jacobi parameters \( \gamma_m = \gamma_m(\mu) \geq 0 \), \( \beta_m = \beta_m(\mu) \in \mathbb{R} \), are defined by the recursion

\[
x Q_m(x) = Q_{m+1}(x) + \beta_m Q_m(x) + \gamma_m Q_{m-1}(x),
\]

where the polynomials \( Q_{-1}(x) = 0, Q_0(x) = 1 \) and \( (Q_m)_{m \geq 0} \) is a sequence of orthogonal monic polynomials with respect to \( \mu \), that is,

\[
\int_R Q_m(x) Q_n(x) \mu(dx) = 0 \quad \text{if} \ m \neq n.
\]

**Example 2.3.** The Chebyshev polynomials of the second kind are defined by the recurrence relation

\[
P_0(x) = 1, \quad P_1(x) = x,
\]

and

\[
x P_n(x) = P_{n+1}(x) + P_{n-1}(x) \quad \forall n \geq 1.
\]

These polynomials are orthogonal with respect to the semicircular law, which is defined by the density

\[
d\mu = \frac{1}{2\pi} \sqrt{4-x^2} dx.
\]

The Jacobi parameters of \( \mu \) are \( \beta_m = 0 \) and \( \gamma_m = 1 \) for all \( m \geq 0 \).

2.3. Non-Commutative Probability Spaces. A \( C^* \)-probability space is a pair \((\mathcal{A}, \varphi)\), where \( \mathcal{A} \) is a unital \( C^* \)-algebra and \( \varphi : \mathcal{A} \to \mathbb{C} \) is a positive unital linear functional. The elements of \( \mathcal{A} \) are called (non-commutative) random variables. An element \( a \in \mathcal{A} \) such that \( a = a^* \) is called self-adjoint.

The functional \( \varphi \) should be understood as the expectation in classical probability.

For \( a_1, \ldots, a_k \in \mathcal{A} \), we will refer to the values of \( \varphi(a_{i_1} \cdots a_{i_k}) \), \( 1 \leq i_1, \ldots, i_k \leq k \), \( n \geq 1 \), as the joint moments of \( a_1, \ldots, a_k \). If there exists \( 1 \leq m, l, \leq n \) with \( i(m) \neq i(l) \) we call it a mixed moment.

For any self-adjoint element \( a \in \mathcal{A} \) there exists a unique probability measure \( \mu_a \) (its spectral distribution) with the same moments as \( a \), that is,

\[
\int_R x^k \mu_a(dx) = \varphi(a^k), \quad \forall k \in \mathbb{N}.
\]

We say that a sequence \( a_n \in \mathcal{A}_n \) converges in distribution to \( a \in \mathcal{A} \) if \( \mu_{a_n} \) converges in distribution to \( \mu_a \). In this setting convergence in distribution is replaced by convergence in moments. Let \( (\phi_n, \mathcal{A}_n) \) be a sequence of \( C^* \)-probability spaces and let \( a \in (\mathcal{A}, \varphi) \) be a selfadjoint random variable. We say that the sequence \( a_n \in (\phi_n, \mathcal{A}_n) \) of selfadjoint random variables converges to \( a \) in moments if

\[
\lim_{n \to \infty} \phi_n(a_n^k) = \varphi(a^k) \quad \text{for all} \ k \in \mathbb{N}.
\]

If \( a \) is bounded then convergence in moments implies convergence in distribution.

The following proposition is straightforward and will be used frequently in the paper. A sequence of polynomials \( \{P_n = \sum_{i=0}^l c_n(i)x^i\}_{n \geq 0} \) of degree at most \( l \geq k \)
is said to converge to a polynomial $P = \sum_{i=0}^{k} c_i x^i$ of degree $k$ if $c_{i,n} \to c_i$ for $0 \leq i \leq k$ and $c_{k,n} \to 0$ for $k < i \leq l$.

**Proposition 2.4.** Suppose that the sequence of random variables $\{a_n\}_{n>0}$ converges in moments to $a$ and the sequence of polynomials $\{P_n\}_{n>0}$ converges to $P$. Then the random variables $P_n(a_n)$ converges to $P(a)$.

In this work we will only consider the $C^*$-probability spaces $(\mathcal{M}_n, \varphi_1)$, where $\mathcal{M}_n$ is the set of matrices of size $n \times n$ and for a matrix $M \in \mathcal{M}_n$ the functional $\varphi_1$ evaluated in $M$ is given by

$$\varphi_1(M) = M_{11}.$$

Let $G = (V, E, 1)$ be a finite rooted graph with vertex set $\{1, \ldots, n\}$ and let $A_G$ be the adjacency matrix. We denote by $A(G) \subset \mathcal{M}_n$ be the adjacency algebra, i.e., the $\ast$-algebra generated by $A_G$.

It is easy to see that the $k$-th moment of $A$ with respect to the $\varphi_1$ is given by the number of walks in $G$ of size $k$ starting and ending at the vertex 1. That is,

$$\varphi_1(A^k) = |\{(v_1, \ldots, v_k) : v_1 = v_k = 1 \text{ and } (v_i, v_{i+1}) \in E\}|.$$

Thus one can get combinatorial information of $G$ from the values of $\varphi_1$ in elements of $A(G)$ and vice versa.

Let us recall the free central limit theorem for free product of graphs (see, e.g. [1]) which follows from the usual free central limit theorem for random variables [15].

**Theorem 2.5** (Free Central Limit Theorem for Graphs). Let $G = (V, E, e)$ be a finite connected graph. Let $A_N$ be the adjacency matrix of the $N$-fold free power $G^\ast N$, and let $\sigma$ be the number of neighbors of $e$ in the graph $G$. Then the distribution with respect to the vacuum state of $(N\sigma)^{-1/2} A_N$ converges in moments (and thus weakly) as $N \to \infty$ to the semicircular law.

For the rest of the paper we define an order which will become handy when estimating vanishing terms in Sections 4 and 5.

**Definition 2.6.** Let $A$ and $B$ be matrices (possibly infinite), we define the order $A \succeq B$ if $A_{ij} \geq B_{ij}$ for all entries $ij$.

**Remark 2.7.** 1) $\varphi_1(A^k) \geq \varphi_1(B^k)$ if $A \succeq B$.
2) For $G_1$ and $G_2$ graphs with $n$ vertices, $G_2$ is a subgraph of $G_1$ iff $A_{G_1} \succeq A_{G_2}$.
3) If $A \succeq B$ and $C \succeq D$ implies $AC \succeq BD$.

2.4. **Kesten-McKay Distribution.** As we know, by the free central limit theorem, if we have a sequence of $d$-regular trees, then the limiting spectral distribution of the sequence, as $d \to \infty$, converges to a semicircular law. However, if $d$ is fixed, and we consider a sequence of $d$-regular graphs, such that the number of vertices tends to infinity, then the limiting spectral distribution is not semicircular. These limiting spectral distributions, which are known as the Kesten-McKay distributions, were found by McKay [13] while studying properties of $d$-regular graphs and by Kesten [11] in his works on random walk on (free) groups.

Let $d \geq 2$ be an integer, we define Kesten-McKay distribution, $\mu_d$, by the density

$$d\mu_d = \frac{d\sqrt{4(d-1) - x^2}}{2\pi(d^2 - x^2)}dx.$$
The orthogonal polynomials and the Jacobi parameters of these distributions are well known. More precisely, for \(d \geq 2\), the polynomials defined by

\[T_0(x) = 1, \quad T_1(x) = x,\]

and the recurrence formula

\[(2.5) \quad xT_k(x) = T_{k+1}(x) + (d - 1)T_{k-1}(x),\]

are orthogonal with respect to the distribution \(\mu_d\). Thus, it follows that the Jacobi parameters of \(\mu_d\) are given by

\[\beta_m = 0, \quad \forall m \geq 0 \quad \text{and} \quad \gamma_0 = d, \quad \gamma_n = d - 1 \quad \forall n \geq 1.\]

**Remark 2.8.** If we define the following polynomials

\[\tilde{T}_k(x) = \begin{cases} 1, & k = 0 \\ \sqrt{\frac{d-1}{d}} P_k(x) - \frac{1}{\sqrt{d(d-1)}} P_{k-2}(x), & k = 1, 2, 3, \ldots, \end{cases}\]

then, \(T_k(x) = \tilde{T}_k(x/2\sqrt{d-1})\).

In Section 6 we will generalize the following theorem due to McKay [13] which gives a connection between large \(d\)-regular graphs and Kesten-McKay distributions.

**Theorem 2.9.** Let \(X_1, X_2, \ldots\) be a sequence of regular graphs with degree \(d \geq 2\) such that \(n(X_i) \to \infty\) and \(c_k(X_i)/n(X_i) \to 0\) as \(i \to \infty\) for each \(k \geq 3\), where \(n(X_i)\) is the order of \(X_i\) and \(c_k(X_i)\) is the number of \(k\)-cycles in \(X_i\). Then, the limiting distribution for the eigenvalues \(X_i\) as \(i \to \infty\) is given by \(\mu_d\).

### 3. Distance-\(k\) Graph of \(d\)-Regular Trees

The \(d\)-regular tree is the \(d\)-fold free product graph of \(K_2\), the complete graph with two vertices. Before we consider asymptotic behavior of the general case of the free product of graphs, we study the distance-\(k\) graph of a \(d\)-regular tree for fixed \(d\) and \(k\). This is an example where we can find the distribution with respect to the vacuum state in a closed form. Moreover, this example sheds light on the general case of the \(d\)-fold free product of graphs, in the same way as the \(d\)-dimensional cube was the leading example for investigations of the distance-\(k\) graph of the \(d\)-fold Cartesian product of graphs (Kurihara [9]).

As a warm up and base case, we calculate the distribution of the distance-2 graph with respect to the vacuum state.

For \(d \geq 2\), let \(A_d^{[2]}\) be the adjacency matrix of distance-2 graph of \(d\)-regular tree. We will sometimes omit the subindex \(d\) in the notation and write \(A^{[2]} = A\). Then we have the following equality, which expresses \(A^2\) in terms of the distance-2 graph and the identity matrix (see Figure 1). :

\[A^2 = A_d^{[2]} + dI.\]

Since \(A_d^{[2]} = A^2 - dI\) then the distribution is given by the law of \(x^2 - d\), where \(x\) is a random variable obeying the Kesten-McKay distribution of parameter \(d\), \(\mu_d\).

For \(k \geq 2\) we have the following recurrence formula.

**Lemma 3.1.** Let \(d \geq 1\) fixed, then \(A^{[1]} = A\), \(A^{[2]} = A^2 - dI\), and

\[(3.2) \quad AA^{[k]} = A^{[k+1]} + (d - 1)A^{[k-1]} \quad k = 2, \ldots, d - 1.\]
Proof. Let \( i \) and \( j \) be vertices of the \( d \)-regular tree, \( Y_d \). We have the following three cases.

Case 1. If \( \partial(i,j) = k + 1 \) then \((A^k)_{ij} = 1\), that is because, in this case, there is only one way to get from vertex \( j \) to vertex \( i \). Indeed, since this \( Y_d \) is a tree there is only one walk from \( i \) to \( j \) of size \( k + 1 \) in \( Y_d \). Thus, there is exactly one neighbor \( l \) of \( j \) at distance \( k \) from \( i \) and thus the only way to go across the distance-\( k \) graph and after across \( Y_d \) to reach \( j \) is through \( l \).

Case 2. When we have \( \partial(i,j) = k - 1 \), then \((A^k)_{ij} = d - 1\). In fact, for the vertex \( i \) there are \( d - 1 \) ways to arrive to \( j \) from a neighbor of \( j \) at distance \( k \) from \( i \). Thus, if we are in vertex \( i \), there are \( d - 1 \) ways to travel across the distance-\( k \) graph and finally go down one level in the \( d \)-regular tree to vertex \( j \).

Case 3. Suppose \( |\partial(i,j) - k| \neq 1 \), then \((A^k)_{ij} = 0\). To see this, we just note that, in the \( d \)-regular tree we can go up one-level or go down one-level, after going across the distances-\( k \) graph, this means that the distance between \( i \) and \( j \) would be \( k - 1 \) or \( k + 1 \), which is a contradiction. Therefore if \( |\partial(i,j) - k| \neq 1 \), there is no way to go from the vertex \( i \) to the vertex \( j \), going across the distance-\( k \) graph and after, across the \( d \)-regular tree in one step.

Thanks to the above, we obtain the following recurrence formula

\[
A^k A = A^{k+1} + (d - 1)A^{k-1}. \tag{3.3}
\]

From the equations (3.1) and (3.3) we can see that \( A^k \) is a polynomial in \( A \) for \( k \geq 1 \), and thus commutes with \( A \). Then we can rewrite equation (3.3) in the more convenient way as follows

\[
AA^k = A^{k+1} + (d - 1)A^{k-1}. \tag{3.3}
\]

Now we can calculate the distribution of the distance-\( k \) graph of the \( d \)-regular tree, for \( d \) fixed, which is exactly Theorem 1.1.

Proof of Theorem 1.1. From equation (3.2) we can see that \( A^k_d \) fulfills the same recurrence formula than \( T_k \) in (2.5). Since \( A \) is distributed as the Kesten-McKay distribution \( \mu_d \), we arrive to the conclusion.
To end this section we observe that from the considerations above, by letting $d$ approach infinity, we may find the asymptotic behavior of the distribution of the distance-$k$ graph of the $d$-regular tree. The same behavior is expected when changing the $d$-regular tree with the $d$-fold free product of any finite graph. We will prove this in Section 5 of the paper.

**Theorem 3.2.** For $d \geq 2$, let $A_d^{[k]}$ be the adjacency matrix of the distance-$k$ graph of the $d$-regular tree. Then the distribution with respect to the vacuum state of $d^{-k/2}A_d^{[k]}$ converges in moments as $d \to \infty$ to the probability distribution of

$$P_k(s),$$

where $P_k(s)$ is the Chebychev polynomial of degree $k$ and $s$ is a random variable obeying the semicircle law.

**Proof.** If we divide the equation (3.2) by $d^{(k+1)/2}$ we obtain

$$A_d A_d^{[k]} = \frac{A_d^{[k+1]} + A_d^{[k-1]} - 1}{d^{(k-1)/2}}.$$

We write $X = \frac{A_d}{\sqrt{d}}$, then we have

$$P^{(1)}(X) = X, \quad P^{(2)}(X) = X^2 - I,$$

and the recurrence

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X) - \frac{1}{d}P^{(k-1)}(X),$$

which when $d \to \infty$ becomes the recurrence formula

$$XP^{(k)}(X) = P^{(k+1)}(X) + P^{(k-1)}(X).$$

Thus $P^{(k)}(x)$ and $P_k(x)$ satisfy the same recurrence formula asymptotically and thanks to the free central limit theorem for graphs (Theorem 2.5) we have the convergence $X \xrightarrow{m} s$. Consequently, combining these two observations and using Lemma 2.4 we obtain the proof. $\square$

4. DISTANCE-2 GRAPH OF FREE PRODUCTS

In this section we derive the asymptotic spectral distribution of the distance-2 graph of the $n$-free power of a graph when $n$ goes to infinity.

In order to analyze the distance-2 graphs we give a simple, but useful, decomposition of the square of the adjacency matrix.

**Lemma 4.1.** Let $G$ be a simple graph with adjacency matrix $A$, we have the following decomposition of $A^2$:

$$A^2 = \tilde{A}^{[2]} + D + \Delta,$$

where $D$ is diagonal with $(D)_{ii} = \deg(i)$, $(\Delta)_{ij} = |\text{triangles in } G \text{ with one side } (i,j)|$ and $(\tilde{A}^{[2]})_{ij} = |\text{paths of size 2 from } i \text{ to } j|$, whenever $(A^{[2]})_{ij} = 1$ and $(\tilde{A}^{[2]})_{ij} = 0$ if $(A^{[2]})_{ij} = 0$.

**Proof.** Indeed $(A^2)_{ij}$ is zero if the distance between $i$ and $j$ is greater than 2. So $(A^2)_{ij} > 0$ implies that $\partial(i,j) = 0.1$ or 2. If $\partial(i,j) = 0$ then $i = j$ and since $(A^{[2]})_{ij} = \deg(i)$ we get $D$, a diagonal matrix with $(D)_{ii} = \deg(i)$. Next, if $\partial(i,j) = 1$ then each path of size 2 which forms a triangle with side $(i,j)$ will contribute to $(A^2)_{ij} = (\Delta)_{ij}$ where $(\Delta)_{ij} = |\text{triangles in } G \text{ with one side } (i,j)|$. 

Finally if $\partial(i, j) = 2$ then $(A^2)_{ij}$ equals the number of paths of size 2 from $i$ to $j$, which is non-zero exactly when $(\tilde{A}^2)_{ij} > 0$.

**Remark 4.2.** Notice in Lemma 4.1 that when $G$ is a tree then $\Delta = 0$, $\tilde{A}^2 = A^2$, therefore $A^2 = A^2 - D$.

Let $G = (V, E, e)$ be a rooted graph, $A_n = A_{G_n}$ and define $D_n$ and $\Delta_n$ by the decomposition (4.1) applied to $G^{\ast N} = G \ast \cdots \ast G$, i.e.

\[(4.2) \quad A^2_n = \tilde{A}^2_n + D_n + \Delta_n.\]

We will describe the asymptotic behavior of each of these matrices. First, we consider the diagonal matrix $D_n$.

**Lemma 4.3.** $D_n/n \to \text{Ideg}(e)$ entrywise and in distribution w.r.t. the vacuum state.

**Proof.** For any $i \in G_n$,

\[
(D_n)_{ii} = \text{deg}(e) + (n-1)\text{deg}(e) - c_i \quad \text{for some } 0 < c_i < \max\text{deg}(G).
\]

Thus,

\[
\frac{(D_n)_{ii}}{n} = \frac{c_i}{n} + \frac{(n-1)\text{deg}(e)}{n} \to \text{deg}(e).
\]

\[\square\]

In order to consider the other matrices in the decomposition we will use the order $\succeq$ from Definition 2.6.

**Lemma 4.4.** The mixed moments of $A^2_n/n$ and $\Delta_n/n$ asymptotically vanish.

**Proof.** Note that the free product does not generate new triangles other than the ones in copies of the original graph. Thus, for $c = \max \text{deg}(G)$ the relation $cA_n \succeq \Delta_n$ holds. Hence, for $m_1, m_2, \ldots, m_s, l_1, l_2, \ldots, l_s \in \mathbb{N}$ and $l_1 > 0$, from Remark 2.7 we have that

\[
\phi_1 \left[ \left( \frac{A^2_n}{n} \right)^{m_1} \left( \frac{\Delta_n}{n} \right)^{l_1} \cdots \left( \frac{A^2_n}{n} \right)^{m_s} \left( \frac{\Delta_n}{n} \right)^{l_s} \right] \leq c \sum_{l_1, \ldots, l_s} \phi_1 \left[ \left( \frac{A^2_n}{n} \right)^{m_1} \left( \frac{A}{n} \right)^{l_1} \cdots \left( \frac{A^2_n}{n} \right)^{m_s} \left( \frac{A}{n} \right)^{l_s} \right].
\]

From Theorem 2.5 we have that $A^2/n$ and $A/n^{1/2}$ converge, then the right hand side of the preceding inequality converges to zero as $n$ goes to infinity.

\[\square\]

Since $\tilde{A}^2_n$ and $D_n$ are subgraphs of $A^2_n$ we have the following.

**Corollary 4.5.** The mixed moments of the pairs $(\tilde{A}^2_n/n, \Delta/n,)$ and $(D_n/n, \Delta/n)$ asymptotically vanish.

Finally, we consider the matrix $\tilde{A}^2_n$.

**Lemma 4.6.** $\tilde{A}^2_n$ converges to $A^2_n$ as $n$ goes to infinity.

**Proof.** Observe that we can write $A^2_n$ as

\[
\tilde{A}^2_n = A^2_n + \Box_n,
\]

where for $(i, j)$ at distance 2 in $G^{\ast n}$, the entry $(\Box_n)_{ij}$ exceeds in one the number of vertices $k$ such that $i \sim k$ and $k \sim j$. 

We will extend $G$ in the following way. For each $(i, j)$ such that $\square_{ij}$ is positive we put the edge $ij$ and call this new graph $G_{ext}$. Now notice that, by construction, $\Delta_{G_{ext}}^n \succeq \square$ and $A_{G_{ext}}^n \succeq A_{G^n}$. Finally, by the previous lemma the mixed moments of $\Delta_{G_{ext}}^n$ and $A_{G_{ext}}^2$ asymptotically vanish. But $A_{G_{ext}}^2 \succeq A_n^2$, so the mixed moments of $A_n^2$ and $\square_n$ also vanish in the limit. This of course means that $A_n^2$ and $A_n^2$ are asymptotically equal in distribution.

We have shown that asymptotically $D_n/n$ approximates $I$, $\tilde{A}_n^2$ approximates $A_n^2$ and that the joint moments between $A_n^2$ or $D_n$ and $\Delta_n$ vanish. Thus, we arrive to the following theorem.

**Theorem 4.7.** The asymptotic distributions of distance-2 graph of the $n$-fold free product graph, as $n$ tends to infinity, is given by the distribution of $s^2 - 1$, where $s$ is a semicircle.

**Proof.** From the equation (4.2), and thanks to Lemmas 2.4, 4.3, 4.6, Corollary 4.5 and Theorem 2.3 we have

$$A_n^2 \xrightarrow{D} \tilde{A}_n^2 \xrightarrow{D} A_n^2 - D_n - \Delta_n \xrightarrow{D} A_n^2 - I \xrightarrow{D} s^2 - 1.$$

\(\square\)

5. **Distance-$k$ graphs of free products**

This section contains the proof of Theorem 1.4 which describes the asymptotic behavior of the distance-$k$ graph of the $d$-fold free power of graphs. We will show that the adjacency matrix satisfies in the limit the recurrence formula (2.3). We start by showing a decomposition similar to the one seen above for $d$-regular trees which plays the role of Lemma 4.1 in the last section.

**Theorem 5.1.** Let $G$ be a simple finite graph, let $N, k \in \mathbb{N}$ with $N \geq 2$ and $k \geq 3$ and let $A = A_N$ denote the adjacency matrix of $G^*N$. Then, we have de following recurrence formula

$$(5.1) \quad A^k A = \tilde{A}^{k+1} + (N-1)\deg(e)A^{k-1} + D_N^{k-1} + \Delta_N^k,$$

where $$(\tilde{A}^{k+1})_{ij} = |\{l \sim j : \partial(i, l) = k\}| \quad \text{whenever} \quad \partial(i, j) = k + 1,$$

$$(D_N^{k-1})_{ij} = |\{l \sim j : \partial(i, l) = k, \text{and} \ j \text{and} \ l \text{are in the same copy of} \ G\}| \quad \text{if} \ \partial(i, j) = k - 1 \quad \text{and} \quad (\Delta_N^k)_{ij} = |\{l \sim j : \partial(i, l) = k\}| \quad \text{when} \ \partial(i, j) = k.$$

**Proof.** It’s easy to see that $(A^k A)_{ij}$ is zero if $|\partial(i, j)| - k \geq 2$. So $(A^k A)_{ij} > 0$ implies that $\partial(i, j) = k - 1$, $k$ or $k + 1$.

Notice that for each neighbor $l$ of $j$ at distance $k$ from $i$, there is one edge from $i$ to $l$ in $A^k$ and one from $l$ to $j$ in $A$. Thus each of these neighbors adds 1 to $(A^k A)_{ij}$ and there is no further contribution.

First, if $\partial(i, j) = k - 1$ there are two types of neighbors $l$ at distance $k$ in $G^*N$. The first ones come from the $(N-1)$ copies of $G$ in $G^*N$ which have $j$ as a root and contribute to the matrix $A^{k-1}$ by $(N-1)\deg(e)$ and the second ones in which $j$ is in the same copy that $l$, which contribute to $D_N^{k-1}$.

Secondly, if $\partial(i, j) = k$ and $(A^k A)_{ij} > 0$ is the number of neighbors of $j$ which are at distance $k$ from $i$, then we get $\Delta_N^k$. 


Finally, if we have \( \partial(i, j) = k + 1 \), so there exists at least one minimal path from \( i \) to \( j \), which contains itself a neighbor of \( j \) which is at distance \( k \) from \( i \), therefore this path contributes to \( \tilde{A}^{[k+1]} \).

\[ \Box \]

**Proof of Theorem 1.4.** We now proceed to prove Theorem 1.4 in various steps. While the steps are very similar as the one for the case \( k = 2 \) there are some non trivial modifications to be done for the general case.

We will use induction over \( k \). First, observe that for \( k = 2 \) we obtained the conclusion in the last section. Now, suppose that the fact holds for all \( l \leq k \). In order to complete the proof we need the following lemmas and corollaries.

**Lemma 5.2.** The mixed moments of \( A^{[k]}A/N^{k+1} \) and \( \Delta^{[k]}N/N^{k+1} \) asymptotically vanish.

**Proof.** By definition, since the free product does not generate new cycles,

\[ \Delta^{[k]}N \leq \max \deg \left( A^{[k]} \right) . \]

Hence, for \( m_1, m_2, \ldots, m_s, n_1, n_2, \ldots, n_s \in \mathbb{N} \) and \( n_1 > 0 \)

\[ \varphi_1 \left( \left( \frac{A^{[k]}A}{N^{k+1}} \right)^{m_1} \left( \frac{\Delta^{[k]}N}{N^{k+1}} \right)^{n_1} \cdots \left( \frac{A^{[k]}A}{N^{k+1}} \right)^{m_l} \left( \frac{\Delta^{[k]}N}{N^{k+1}} \right)^{n_l} \right) \leq (\max \deg) \sum n_i \varphi_1 \left( \left( \frac{A^{[k]}A}{N^{k+1}} \right)^{m_1} \left( \frac{\Delta^{[k]}N}{N^{k+1}} \right)^{n_1} \cdots \left( \frac{A^{[k]}A}{N^{k+1}} \right)^{m_l} \left( \frac{\Delta^{[k]}N}{N^{k+1}} \right)^{n_l} \right) . \]

By induction hypothesis \( \left( \frac{A^{[k]}A}{N^{k+1}} \right) \) and \( \left( \frac{\Delta^{[k]}N}{N^{k+1}} \right) \) converge and, therefore, the right hand side of the last inequality goes to zero.

Since \( \tilde{A}^{[k+1]} \) and \( D^{[k-1]}N \) are subgraphs of \( A^{[k]}A \), the following is a direct consequence of the previous lemma.

**Corollary 5.3.** The mixed moments of \( \left( \frac{\tilde{A}^{[k+1]}N}{N^{k+1}}, \frac{\Delta^{[k]}N}{N^{k+1}} \right) \) and \( \left( \frac{D^{[k-1]}N}{N^{k+1}}, \frac{\Delta^{[k]}N}{N^{k+1}} \right) \) asymptotically vanish.

**Corollary 5.4.** The matrices \( \Delta^{[k]}N/N^{k+1} \) and \( D^{[k-1]}N/N^{k+1} \) converge to zero as \( N \) tends to infinity.

**Proof.** In the proof of Lemma 5.2 by taking \( m_i = 0 \) for all \( 0 \leq i \leq s \) we obtain the conclusion for \( \Delta^{[k]}N/N^{k+1} \) and \( D^{[k-1]}N/N^{k+1} \) for \( k \) is a cycle of even length smaller than \( 2k \), we add all the possible edges between the vertices of this cycle. Note that \( G_{\text{ext}(2)} = G_{\text{ext}} \). It is important to notice the fact that \( (G_{\text{ext}(2)})^*N = (G^*N)_{\text{ext}(2)} \).

**Lemma 5.5.** Let \( k \geq 2 \), then \( \lim_{N \to \infty} \frac{\tilde{A}^{[k+1]}N}{N^{k+1}} = \frac{A^{[k+1]}N}{N^{k+1}} = 0 \).
Proof. Let \( i, j \in \ast \ s \in [N] \) be such that \( \left( \tilde{A}^{[k+1]} \right)_{ij} > 0 \). Let

\[
C^{[k+1]}_{ij} = \{\text{cycles of even length in a path of length } k + 1 \text{ from } i \text{ to } j\},
\]

notice that

\[
|C^{[k+1]}_{ij}| \leq (\max \deg(G))^{k+1}.
\]

Here, it is important to observe that the right side bound does not depend on \( i, j \) neither \( N \), because the free product of graph does not produce new cycles. Then we can write

(5.2)

\[
\tilde{A}^{[k+1]} - A^{[k+1]} \preceq (\max \deg(G))^{k+1} \left( A^{[k]}_{\text{Gext}(k+1)} + A^{[k-1]}_{\text{Gext}(k+1)} + \cdots + A_{\text{Gext}(k+1)} \right).
\]

Then, we obtain from (5.2)

\[
\left( \frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}} \right) \preceq (\max \deg(G))^{k+1} \left( \frac{A^{[k]}_{\text{Gext}(k+1)}}{N^{k/2}} + \frac{A^{[k-1]}_{\text{Gext}(k+1)}}{N^{k/2}} + \cdots + \frac{A_{\text{Gext}(k+1)}}{N^{k/2}} \right).
\]

By induction hypothesis we have that \( (A^{[i]}_{\text{Gext}(k)}/N^{1/2}) \) converges for all \( i \leq k \). Therefore we have

\[
\left( \frac{\tilde{A}^{[k+1]} - A^{[k+1]}}{N^{(k+1)/2}} \right) \rightarrow_{N \rightarrow \infty} 0,
\]

which completes the proof.

Now we can finish the proof of Theorem 1.4. From (5.1), we have that

(5.3)

\[
\frac{A^{[k+1]}_{N}}{(\deg(e)N)^{\frac{k+1}{2}}} = \frac{A^{[k]}_{N}A_{N}}{(\deg(e)N)^{\frac{k+1}{2}}} - \frac{A^{[k-1]}_{N}}{(\deg(e)N)^{\frac{k+1}{2}}} - C(N, k + 1),
\]

where

\[
C(N, k + 1) = \frac{\deg(e)A^{[k-1]}_{N} + \Delta^{[k]}_{N} + D^{[k-1]}_{N} - (\tilde{A}^{[k+1]} - A^{[k+1]})}{(\deg(e)N)^{\frac{k+1}{2}}}.
\]

Due to the induction hypothesis we have, \( \deg(e)A^{[k-1]}_{N}/(\deg(e)N)^{\frac{k+1}{2}} \) converging to zero, furthermore by Corollary 5.4 and Lemma 5.5

\[
\frac{\Delta^{[k]}_{N} + D^{[k-1]}_{N} - (\tilde{A}^{[k+1]} - A^{[k+1]})}{(\deg(e)N)^{\frac{k+1}{2}}},
\]

converges to zero. Hence

(5.4)

\[
C(N, k + 1) \rightarrow 0.
\]
Finally, using the induction hypothesis we can see that
\[
A_N^{[k]} A_N = \frac{A_N^{[k-1]}}{(\deg(e)N)^{\frac{k-1}{2}}} \longrightarrow P_k(s) - P_{k-1}(s) = P_{k+1}(s),
\]
where the last equality is given by (2.3). Thus, mixing (5.3) with (5.4) and (5.5) we obtain that
\[
\frac{A_N^{[k+1]}}{(\deg(e)N)^{\frac{k+1}{2}}} \longrightarrow P_{k+1}(s).
\]

6. $d$-Regular Random Graphs

Apart from the Erdos-Renyi models [5, 6], possibly, the random $d$-regular graphs are possibly the most studied and well understood random graphs.

In the original paper by McKay [13], he proved that the asymptotical spectral distributions of $d$-regular random graph are exactly the ones appearing in (2.4). Heuristically, the reason is that, locally, large random $d$-regular graphs look like the $d$-regular tree and thus asymptotically their spectrum should coincide. This turns out to remain true for the distance-$k$ graph and thus we shall expect to get a similar result. In this section we formalize this intuition.

Let $X$ be a $d$-regular graph with vertex set \{1, 2, \ldots, n(X)\}. For each $i \geq 3$ let $c_i(X)$ be the number of cycles of length $i$. Let $A^{[k]}(X)$ be the adjacency matrix of the distance-$k$ graph of $X$. The following is a generalization of the main theorem in McKay [13].

**Theorem 6.1.** For $d \geq 2$ fixed, let $X_1, X_2, \ldots$ be a sequence of $d$-regular graphs such that $n(X_i) \to \infty$ and $c_j(X_i)/n(X_i) \to 0$ as $i \to \infty$ for each $j \geq 3$. Then the distribution with respect to the normalized trace of $A^{[k]}(X_i)$ converges in moments, as $i \to \infty$, to the distribution of $A_d^{[k]}$ with respect to the vacuum state.

**Proof.** We follow the original proof of McKay [13] with simple modifications. Let $n_r(X_i)$ denote the number of vertices $v$ of $X_i$ such that the subgraph of $X_i$ induced by the vertices at distance at most $r = mk$, where $m \in \mathbb{N}$, from $v$ is acyclic. By hypothesis we have that $n_r(X_i)/n(X_i) \to 1$ as $i \to \infty$. The number of closed walks of length $m$ in the distance-$k$ graph of the $d$-regular graph starting at each such vertex is $\varphi(A_d^{[k]})$. For each of the remaining vertices the number of closed walks of length $m$ is certainly less than $d^r$. Then, for each $m$, there are numbers $\varphi_m(X_i)$ such that $0 \leq \varphi_m(X_i) \leq d^r$, and

\[
\varphi_{tr} \left( A_d^{[k]}(X_i) \right) = \frac{\varphi(A_d^{[k]}) n_r(X_i)}{n(X_i)} + \frac{(n(X_i) - n_r(X_i)) \varphi_m(X_i)}{n(X_i)}
\]

\[
\longrightarrow \varphi(A_d^{[k]}) \quad \text{as } i \to \infty.
\]

$\square$

Fix $d > 0$. Let $s_1 < s_2, < \ldots$ be the sequence of possible cardinalities of regular graphs with degree $d$. For each $n$ define $R_n$ to be the set of all labeled regular graphs with degree $n$ and order $s_i$.

In order to consider the $d$-regular uniform random graphs we use the following lemma of Wormald [17].
Lemma 6.2. For each $k > 3$ define $c_{k,n}$ to be the average number of $k$-cycles in the members of $R_n$. Then for each $k$, $c_{k,n} \to (d-1)^k/2k$ as $n \to \infty$.

Proof of Theorem 1.5. Consider a graph $G_n$ which consist a disjoint union of the all the labelled graphs of size $s_n$. The eigenvalue distribution of $G_n$ coincides with the expected eigenvalue distribution of $R_n$. Now by Lemma 6.2 $G$ satisfies the assumptions of Theorem 6.1 and thus we arrive to the theorem. □

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