Parameter Estimation Bounds Based on the Theory of Spectral Lines

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Abstract

Recent methods in the machine learning literature have proposed a Gaussian noise-based exogenous signal to learn the parameters of a dynamic system. In this paper, we propose the use of a spectral lines-based deterministic exogenous signal to solve the same problem. Our theoretical analysis consists of a new toolkit which employs the theory of spectral lines, retains the stochastic setting, and leads to non-asymptotic bounds on the parameter estimation error. The results are shown to lead to a tunable parameter identification error. In particular, it is shown that the identification error can be minimized through an optimal choice of the spectrum of the exogenous signal.

1 Introduction

Adaptive control has a history of over 40 years and is dedicated to the study of estimation and control of dynamical systems in the presence of parametric uncertainties [4, 18, 21, 28, 35]. Of particular importance in the adaptive control literature is learning the parameters of a time-varying dynamical system using the theory of spectral lines due to Boyd and Sastry [6, 7], in which a deterministic exogenous signal may be used to learn the unknown time-varying parameters of a dynamical system (c.f. [2, 3, 27, 43] also).

Recently, the machine learning community has focused on learning the constant parameters of a dynamical system using an independent Gaussian noise as the exogenous input [1, 9, 10, 13, 24, 33, 34, 37, 38, 39, 41]. Specifically, these works consider a system of the form

$$x_{k+1} = A_x x_k + B_x u_k + \eta_k,$$

where the observations $x_k \in \mathbb{R}^d$ are known, $A_x$ and $B_x$ are fixed but unknown, $\eta_k$ is unobserved, and the control input $u_k \in \mathbb{R}^m$ is either selected solely according to independent realizations of $\mathcal{N}(0, \sigma_u^2 I)$, or is perturbed by independent realizations of Gaussian noise.

While randomization of the control input allows for clean analysis of estimation using well-known tools, it is undesirable for several reasons. First, in mission-critical systems, the addition of intentional randomness of the exogenous input is undesirable as it may result in unexpected behavior of the system as a whole, endangering those who rely on the safety of the system [12, 16]. Second, Gaussian noise in principle is unbounded; realization of such an input $u_k$ may simply not be physically feasible [23]. Finally, the use of white noise in a dynamical system may perturb the system.
at undesirable frequencies, as unmodeled dynamics are always present in any system, and these dynamics can be excited by higher frequency content \([11]\). The true model of a dynamical system is rarely of the form of (1) which is somewhat idealized, but rather

\[
\begin{align*}
x_{k+1} &= A_s x_k + B_s u_k + w_k + \eta_k, & \quad \eta_k \text{ i.i.d. } \mathcal{N}(0, \sigma^2 I) \\
w_k &= f(x_0, w_{k-1}, w_{k-2}, \ldots, w_0, u_k, u_{k-1}, \ldots u_0).
\end{align*}
\] (2) (3)

When \(u_k\) is deterministic, \(w_k \in \mathbb{R}^n\) represents the effect of all deterministic unmodeled components, and is unknown. This may be due to higher-order linear dynamics, nonlinearities, and/or due to other effects that cannot be precisely accounted for \([11, 20]\). Typically \(w_k\) remains small for low amplitudes and/or low frequencies and large if either amplitudes or frequencies become large. It is therefore often desired to keep the frequency content of \(u_k\) small (see descriptions of \(f_m(\omega)\) in \([11]\) and amplitudes small \([20]\).

Our contributions in this work are as follows:

- We develop a new theory of spectral lines in a discrete-time and stochastic setting over a finite-time interval, bridging the gap between adaptive control and non-asymptotic estimation theory.
- We apply our results for the ideal system in (1), and achieve competitive estimation rates with deterministic inputs as compared to the stochastic inputs in previous literature \([1, 53]\). In particular, for the system (1), we show that the user of the system has the ability to tune the estimation rate by selecting the spectral lines of the deterministic control input appropriately. Such flexibility is not provided in the previous literature that advocates the use of a stochastic input \([1, 2, 10, 13, 24, 33, 34, 37, 38, 41]\).
- For the system in (2)-(3), we are able to provide explicit estimation rates which depend on the magnitude of the unmodeled \(w_k\), and in practice can be minimized through proper choice of the spectrum of the exogenous signal. Although the system in (2)-(3) is representative of true dynamics of physical systems, it has received little attention in recent literature. To our knowledge, we provide the first non-asymptotic estimation bound for a system of this form.

Throughout this work, we will make the following assumption on the systems (1) and (2)-(3).

**Assumption 1.** In (1) and (2)-(3), the matrix \(A_s\) is assumed to be Schur-stable.

To ensure that the signals of interest are well behaved, we make this standard assumption, commonly satisfied by physical systems. As articulated in \([22]\), closed-loop system identification based methods can be used to relax this requirement.

**Related Work** The recent machine learning literature has addressed parameter estimation in linear dynamical systems in the context of control \([9, 10, 24, 37]\) and system identification \([1, 17, 33, 34, 36, 38, 41]\). In the former, the goal is to show high-probability guarantees on the regret of the algorithm together with a Linear Quadratic Regulator in finite time \([9, 10, 24, 37]\). In the latter, papers such as \([17, 33, 36, 38, 41]\) have focused on system identification and provide high-probability guarantees of parameter estimates in finite time. The application of this work is in the latter setting, and will derive parameter estimation bounds achievable in finite-time.

A closer examination of the results of two specific papers in the above list, \([33, 41]\), is in order. Both of these works show estimation rates that decay as \(O(1/\sqrt{T})\). In \([33]\), the control input is taken to be a Gaussian input,

\[
\begin{align*}
u_k &= s_k, \\
s_k \text{ i.i.d. } \mathcal{N}(0, \sigma^2_s I), \\
\forall k &\in \{1, \ldots, T\}.
\end{align*}
\] (4)

In contrast, \([41]\) adds a deterministic, periodic \(\hat{u}_k\) in addition to \(s_k\). A numerical optimization procedure is used to actively determine \(\hat{u}_k\) in \([41]\) and it is shown that it is advantageous to the passive approach in \([33]\). In contrast to these methods, our approach entirely removes the stochastic component, and proposes a spectral-line based construction \([6, 7]\) of the control input. The advantage of choosing the spectral-line based input over the Gaussian input lies in robustness, avoidance of large inputs even at low probabilities, and active reduction in the parameter estimation error. The primary advantage of choosing the spectral line method of analysis over that suggested in \([41]\) is its...
generalization to an arbitrary dynamic system, nonlinear, with unmodeled higher-order dynamics, as in (2)-(3), that has a property of preserving spectral lines at its output. We leverage this property in providing estimation rates for this generalized class of dynamic systems.

**Organization** We begin with a discussion of preliminaries in Section 2 to introduce notions from adaptive control, including the theory of spectral lines and how they elicit excitation, as well as notions from non-asymptotic statistics, which have been used widely in recent literature [1] [10] [33] [34]. In Section 3 we then provide our new definitions regarding spectral lines in discrete-time, stochastic settings. Section 4 provides a key estimation theorem leveraging the new notions of deterministic spectral lines for the system in (1), and then Section 5 extends this result to the setting of (2)-(3) where there is also a deterministic, unobserved process noise.

**Notation** Let $\mathbb{R}$ denote the set of real numbers $\mathbb{R}^+$ denote the set of non-negative real numbers, and $\mathbb{R}^n$ denote the set of real-valued vectors of length $n$. We let $S^{d-1}$ denote the unit sphere in $\mathbb{R}^d$. The functions $\Re(\cdot)$ and $\Im(\cdot)$ represent the real and imaginary parts of their inputs, respectively. Let $\Omega_T = \{0, 1/T, \ldots, (T − 1)/T\}$ be the finite set of discrete frequencies for a sequence of length $T$. Given a real-valued, finite sequence $\{y_k\}_{k=0}^{T-1}$, we denote the Discrete Fourier Transform of the sequence using a bolded letter as $\tilde{y}(e^{j\omega}) = \sum_{k=0}^{T-1} y_k e^{-j2\pi k\omega}$, where $\omega \in \Omega_T$. Let $\|\cdot\|$ represent the Euclidean norm when its input is a vector, and the operator norm when its input is a matrix. Further, for a matrix $B \succeq 0$, we define $\|A\|_{B} = \|B^{1/2}A\|$.  

2 Preliminaries

In this section, we introduce the main technical tools employed in this paper. In Section 2.1 we discuss notions of persistent excitation and spectral lines for parameter convergence, as is common in the field of adaptive control theory. We then proceed to a discussion of non-asymptotic estimation tools from the recent statistic literature in Section 2.2.

2.1 Parameter Convergence in Discrete Time Linear Regression

To motivate a later discussion of spectral lines and excitation conditions, we first consider the problem of linear regression with time-varying regressors in discrete time, which has a large body of work in both the system identification and the adaptive control literature [3] [5] [15] [22] [28]. In Appendix A we provide a brief overview of well-studied tools in the continuous time setting, which we hope will inform future research on the subject in the context of machine learning.

The discrete time linear regression setting with time-varying parameters can be briefly stated as the estimation of the parameter $\theta_s$ in the regression relation

$$y_k = \theta^\top_s \phi_k,$$

where $y_k \in \mathbb{R}$ is an observed outcome, $\phi_k \in \mathbb{R}^N$ is the time-varying regressor, and $\theta_s \in \mathbb{R}^N$ is unknown. Such a time-varying regression setting has been common in the adaptive control literature, and encapsulates a variety of well-studied settings such as the ARMA model [15] [22] [42]. Given that $\theta_s$ is unknown, we formulate an estimator $\hat{\theta}_k = \theta^\top_k \phi_k$, where $\theta_k \in \mathbb{R}^N$ is an adjusted parameter and $\hat{y}_k \in \mathbb{R}$ is the predicted output. For example, in [13], the underlying estimator that is used is

$$\theta_{k+1} = \theta_k - \gamma \frac{1}{\mathcal{N}_k} (\theta^\top_k \phi_k - y_k),$$

where $\mathcal{N}_k$ is a suitably chosen normalization which guarantees that $\theta_k$ remains bounded for all initial conditions $\theta_0$, and $0 < \gamma < 2$ [15 Chapter 3]. The output prediction error is of the form

$$e_k = \hat{y}_k - y_k = \tilde{\theta}^\top_k \phi_k,$$

where $\tilde{\theta}_k = \theta_k - \theta_s$ is the parameter estimation error. A primary goal of time-varying regression is to ensure that, as $k \to \infty$, the parameter estimation error $\tilde{\theta} \to 0$ as well. A secondary goal would be to at least ensure that, as $k \to \infty$, the output prediction error $e_k \to 0$. While the primary goals ensures the latter, the converse implication is not necessarily true. We may, however, begin to bridge the gap between the two goals by introducing the following necessary and sufficient condition for these classes of problems.
**Definition 1 (Persistent Excitation [5]).** A regressor \( \{\phi_k\}_{k=1}^{\infty} \) is said to be persistently exciting if there exist strictly positive constants \( \rho_1 < \rho_2 \) and integers \( k_0, S \geq 1 \) such that

\[
\rho_2 I \geq \sum_{k=j}^{j+S} \phi_k \phi_k^\top \geq \rho_1 I ,
\]

for all \( j \geq k_0 \).

In particular, the definition above allows for the following result showing the importance of persistent excitation.

**Theorem 1 (Parameter Estimation and Persistent Excitation [15]).** Consider the estimator (6). If the regressor \( \{\phi_k\}_{k=1}^{\infty} \) satisfies the condition in Definition 1, then for any \( \epsilon > 0 \), there exists a \( T = O(\log \epsilon) \) such that

\[
\|\theta_k - \theta^*\| \leq \epsilon, \quad \forall k \geq T .
\]

That is, \( \theta_k \) corresponds to a parameter estimate at time \( k \), and as \( k \to \infty \), it can be shown that \( \theta_k \) converges exponentially fast to \( \theta^* \) [15, Chapter 3.4].

Further, there is an intimate relationship between whether or not an input satisfies the condition of persistent excitation and its spectral content. Specifically, we consider the notion of a spectral line.

**Definition 2 (Spectral Line [5]).** Consider \( \{\phi_k\}_{k=0}^{\infty} \) and a value \( \nu \). Then, \( \{\phi_k\}_{k=0}^{\infty} \) has a spectral line at \( \nu \in [-\pi, \pi] \) with amplitude \( \bar{\phi}(j\nu) \) if

\[
\lim_{M \to \infty} \frac{1}{M} \sum_{k=k_0+1}^{k_0+M} \phi_k e^{-j\nu k} = \bar{\phi}(j\nu) ,
\]

uniformly in \( k_0 \).

In particular, it is known that if the spectral content of a regressor has sufficiently many spectral lines with linearly independent amplitudes, then the persistent excitation condition in Definition 1 is immediate [5]. That is, if the spectral content of the regressor spans sufficiently many frequencies, then we are able to show that the parameter estimation error \( \hat{\theta} \) tends to 0 when \( \theta_k \) is updated according to (6) [3, 5].

**Definition 3 (Finite Excitation).** A regressor \( \{\phi_k\}_{k=1}^{i+S} \) is said to be finitely exciting from \( i \) to \( i + S \) if there exist strictly positive constants \( \rho_1 < \rho_2 \) such that

\[
\rho_2 I \geq \sum_{k=i}^{i+S} \phi_k \phi_k^\top \geq \rho_1 I .
\]

As we will see in Section 3, the notion of finite excitation helps to bridge the asymptotic results of adaptive control with the non-asymptotic setting which has been considered in recent literature in machine learning.

**2.2 Non-Asymptotic Estimation Bounds**

While the results in the previous section show convergence of parameter estimation errors to zero, they do not assume an unobserved external disturbance in the observation of each output. Even when an external disturbance is considered in the adaptive control literature, the results rely on the Martingale Convergence Theorem, which provide asymptotic results and hence may not apply to a setting with finite data [15, Chapter 8]. In order to avoid these issues, we leverage tools from recent advances in non-asymptotic statistics [40]. First, we note the definition of a sub-Gaussian random variable and a sub-Gaussian random vector.
Then, for any $\delta < 1$, with probability at least $1 - \delta$,

$$
\left\| (\bar{Y}_T)^{-1/2} S_T \right\| \leq \sigma \sqrt{8(n + m) \log \left( \frac{5 \det(\bar{Y}_T)^{1/(2(n+m))} \det(V)^{-1/(2(n+m))}}{\delta^{1/(n+m)}} \right)}.
$$

Since the system (1) is a special case of the system (2)-(3) where $w_k = 0$ for all $k$ in (3), we see the above proposition applies to both systems. With the toolkit for the theory of spectral lines from adaptive systems in Section 2.1 and recent results from non-asymptotic statistic in Section 2.2, we now proceed to the development of a new theory of discrete stochastic spectral lines in the following section.

### 3 A New Theory for Spectral Lines of Discrete Stochastic Signals

The focus of this paper is on the notion of spectral lines over a finite time-interval. Towards this end, we introduce Definition 6 that defines a spectral line based on finite sample of data in a stochastic setting.

**Definition 6** (Sub-Gaussian Spectral Line). A stochastic sequence $\{\phi_k\}_{k \geq 0}$ is said to have a sub-Gaussian spectral line from $i$ to $i + S$ at a frequency $\omega_0$ of amplitude $\bar{\phi}(\omega_0)$ and radius $R$ if

$$
\frac{1}{S + 1} \sum_{k=i}^{i+S} \phi_k e^{-j2\pi\omega_0 k} - \bar{\phi}(\omega_0) \sim \text{subG}(R^2/(S + 1)).
$$

Definition 6 determines the frequency content of a stochastic signal by decoupling the part of the signal which is deterministic from the part of the signal directly affected by the stochastic process noise. Hence, we are able to decouple our analysis and apply tools from adaptive control to the deterministic part of the signal while simultaneously using tools from non-asymptotic statistics to place bounds on the stochastic part of the signal with particular focus on bounding aberrant behavior with high probability.
Remark 1. Definition 2 pertains to the notion of a spectral line, and requires the underlying signal \( \{ \phi_k \}_{k \geq 0} \) to be specified for all \( k \in \mathbb{N} \). In contrast, Definition 6 introduces the notion of a sub-Gaussian spectral line where it suffices for \( \{ \phi_k \}_{k \geq 0} \) to be specified over a finite time-interval. This is a stronger condition on \( \{ \phi_k \}_{k \geq 0} \), as we require the signal to have the appropriate behavior over a finite duration.

We leverage Definition 6 in the following lemma, which relates the input and output of a system to one another in terms of their spectral content. The following new lemma is a discrete time, stochastic analogue to that provided by Boyd and Sastry [7, Lemma 3.3].

Lemma 1. Consider \( \{ u_k \}_{k=0}^{T-1}, \{ y_k \}_{k=0}^{T-1} \) as the input and output, respectively, of a discrete-time, stable linear time-invariant system with arbitrary initial conditions and an unobserved external disturbance. Specifically, with probability 1,

\[
\sum_{i=0}^{T-1} \phi_k^T \geq 1 - \delta \quad \text{and} \quad \phi_k \text{ is finitely exciting,}
\]

from \( i \) to \( i + S \) with probability at least \( 1 - e^{-\left( \frac{c}{2n} \right)^2} + 2n \log \delta \), where \( c \) is an absolute constant. Specifically, with probability 1 – \( \delta \), \( \phi_k \) will satisfy

\[
\sum_{k=i}^{i+S} \phi_k^T \phi_k^T \geq 1 \cdot \Phi^{-1} - 2 \delta I,
\]

so long as \( S \geq R^2 (\log(1/\delta) + \| \Phi^{-1} \|) \).

A proof of Lemma 1 can be found in Appendix C. It is worth noting that, while Definition 6 can potentially be defined for any frequency, the use of the DFT in the transfer function restricts the application of the lemma to frequencies in the finite set \( \Omega_T \). Lemma 1 shows that the spectral content of the exogenous signal \( \{ u_k \}_{k=0}^{T-1} \) affects the spectral content of \( \{ y_k \}_{k=0}^{T-1} \) in a natural way, with the amplitude of the spectral line determined by the bandwidth of the system, and the radius being affected by external noise as well as any initial stochasticity of \( \{ u_k \}_{k=0}^{T-1} \) itself.

In order to relate the spectral content of a signal to the necessary and sufficient persistent excitation provided by Boyd and Sastry, we provide a new discrete time stochastic analogue of [7, Lemma 3.4]. This proposition relates the number of spectral lines with linearly dependent amplitudes of a sequence to whether or not such a sequence is finitely exciting. In order to show this claim, we must first define the expected information matrix of a sequence of stochastic vectors.

Definition 7 (Expected Information Matrix). Let \( \phi_k \in \mathbb{R}^n \) be a sequence of stochastic vectors. If \( \phi_k \) has \( n \) sub-Gaussian spectral lines at frequencies \( \{ \omega_1, \ldots, \omega_n \} := \Omega \) from \( i \) to \( i + S \) with amplitudes \( \{ \phi(\omega_1), \ldots, \phi(\omega_n) \} \), then the information matrix \( \Phi \) is defined

\[
\Phi = \left[ \begin{array}{c|c|c} \phi(\omega_1) & \cdots & \phi(\omega_n) \\ \hline \\ \vdots & \ddots & \vdots \\ \phi(\omega_n) & \cdots & \phi(\omega_1) \end{array} \right].
\]

The expected information matrix \( \Phi \) in Definition 7 represents the core idea in any system identification problem. It is well known in the deterministic setting that a full rank and numerically well conditioned \( \Phi \) results in fast estimation of unknown parameters in system identification.

With the information matrix defined, we can now show a clear relationship between the spectral content of a signal and whether it is finitely exciting.

Proposition 2. Let \( \phi_k \in \mathbb{R}^n \) be a sequence of stochastic vectors. If \( \phi_k \) has \( n \) spectral lines at frequencies \( \omega_1, \ldots, \omega_n \) from \( i \) to \( i + S \) with amplitudes \( \{ \phi(\omega_1), \ldots, \phi(\omega_n) \} \) which are linearly independent in \( \mathbb{C}^n \), and maximum radius \( R \) as defined in Definition 6 then \( \phi_k \) is finitely exciting from \( i \) to \( i + S \) with probability at least \( 1 - e^{-\left( \frac{c}{2n} \right)^2} + 2n \log \delta \), where \( c \) is an absolute constant.
The proof of Proposition may be found in Appendix D.

**Remark 2.** In the Sections 2 and 5 a deterministic choice of \( u_k \) will be shown to ensure that the expected information matrix \( \Phi \) is well-conditioned. In contrast, the recent literature (e.g., [2], [9], [10], [24], [33], [44], [53], [58], [44]) selects the control input as in \( (4) \), as opposed to a deterministic spectral-line based exogenous signal. This choice of such a zero-mean exogenous signal results in \( \Phi = 0 \). The analysis in this work quantifies the impact of \( u_k \) on \( \Phi \), which precisely allows for tunable estimation rates.

Proposition 2 allows us to bridge the gap between the discrete time and deterministic setting of [5] which has been considered widely in the adaptive control literature [15, 22, 28], with the more recently considered stochastic settings of the machine learning literature in which the exogenous signal is stochastic [1, 9, 10, 24, 33, 34, 37, 38, 59]. In particular, Proposition 2 can be viewed as the non-asymptotic, stochastic equivalent condition of persistent excitation, and leverages the notion of spectral lines as in [7]. The latter in turn helps us establish tunable estimation rates for parameters in both linear time-invariant systems and general dynamic systems. These are outlined in Sections 4 and 5, respectively.

### 4 Tunable Estimation Rates in Linear Time Invariant Systems

In this section, we consider the system in (1) and will provide a new analysis for estimation of \( A_* \in \mathbb{R}^{n \times n} \) and \( B_* \in \mathbb{R}^{n \times m} \) given the choice of a deterministic \( u_k \) and observations \( x_k \). The transfer function from \( u_k \) to \( \phi_k = [x_k^\top \ u_k^\top]^\top \) may be written as

\[
\phi(e^{j\omega}) = \begin{bmatrix} (e^{j\omega} I - A_*)^{-1} B_* \\ I_m \end{bmatrix} u(e^{j\omega}) + \begin{bmatrix} (e^{j\omega} I - A_*)^{-1} \\ 0 \end{bmatrix} \eta(e^{j\omega}).
\]

Hence, if \( u_k \) has a sub-Gaussian spectral line from 0 to \( T - 1 \) at frequency \( \omega_0 \), with radius 0 and amplitude \( \bar{u} (\omega_0) \), it follows from Lemma 1 that \( \phi_k \) has a sub-Gaussian spectral line from 0 to \( T - 1 \) at frequency \( \omega_0 \) with finite radius and amplitude

\[
\bar{\phi} (\omega_0) = \begin{bmatrix} (e^{j\omega} I - A_*)^{-1} B_* \\ I_m \end{bmatrix} \bar{u} (\omega_0),
\]

as long as \( \omega_0 \in \Omega_F \). With the expected information matrix of the regressor of system (1) defined, we now present the first main contribution of this work.

**Theorem 2.** Consider the dynamical system in (1), with expected information matrix \( \bar{\Phi} \) as in Definition 7 for \( \phi_k = [x_k^\top \ u_k^\top]^\top \). Let \((\hat{A}, \hat{B})\) be the least-squares estimates of \( A_* \) and \( B_* \),

\[
(\hat{A}, \hat{B}) = \arg\min_{A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}} \sum_{k=0}^{T-1} \|x_{k+1} - Ax_k - Bu_k\|_2^2.
\]

If the control input \( u_k \) is selected such that \( \phi_k \) has \( n + m \) spectral lines with linearly independent amplitudes and finite radii, then

\[
\max \left\{ \|\hat{A} - A_*\|, \|\hat{B} - B_*\| \right\} \leq \tilde{O} \left( \sqrt{\frac{1}{T} \|\Phi^{-1}\|^{-2}} \right),
\]

with probability at least \( 1 - \delta \), where the \( \tilde{O} \) notation discards constant and logarithmic terms.

Details of the proof of Theorem 2 may be found in Appendix F. Theorem 2 shows that the choice of deterministic exogenous signal matters in the context of system estimation. In particular, the estimation rate of the least-squares estimate of \( A_* \) and \( B_* \) will depend on the spectral content of the exogenous input \( \{u_k\}_{k=0}^{T-1} \). This is of particular importance for practitioners who may have arbitrary constraints on their choice of exogenous signal, for example those designing systems for uncertain but mission-critical systems. If a practitioner desires an efficient estimation rate for \( A_* \) and \( B_* \) but has a constrained set from which to select \( \{u_k\}_{k=0}^{T-1} \), then the practitioner may solve a constrained optimization problem to find the most efficient estimation rate given their constraints.
5 Tunable Estimation Rates in the Presence of Unmodeled Dynamics

In this section, we now consider the system (2)-(3), where the influence of the spectrum of \( u_k \) on \( x_k \) is more complex due to the presence of unmodeled dynamics \( w_k \). By exploiting the structure of (2)-(3), we may rewrite the system in the frequency domain as

\[
x(\epsilon j\omega) = (e^{j\omega}I - A_\ast)^{-1} B_\ast u(e^{j\omega}) + (e^{j\omega}I - A_\ast)^{-1} \left( w(e^{j\omega}) + \eta(e^{j\omega}) \right).
\]

In order for the above statement to hold, we make the following technical assumption.

**Assumption 2.** In the system (2)-(3), \( f \) is Lipschitz [12].

In particular, Assumption 2 implies that, as long as \( \{u_k\}_{k=0}^{T-1} \) is Fourier integrable in the discrete setting, then so is \( \{w_k\}_{k=0}^{T-1} \). Under this condition, we see from (15) and Lemma 1 for \( \phi_k = [x_k^T w_k]^T \), the sequence \( \{\phi_k\}_{k=0}^{T-1} \) has a spectral line at frequency \( \omega_0 \) from 0 to \( T - 1 \) with amplitude

\[
\tilde{\phi}(\omega_0) = \left[(e^{j\omega}I - A_\ast)^{-1} B_\ast\right] u(\omega_0) + \left[(e^{j\omega}I - A_\ast)^{-1}\right] w(\omega_0),
\]

and finite radius. Hence, the information matrix for the system is clearly defined, leading to the following result.

**Theorem 3.** Consider the system (2)-(3) under Assumption 2 and suppose the input \( u_k \) is selected such that, with probability at least \( 1 - \delta \),

\[
\sum_{k=0}^{T-1} \|w_k\| \leq m_T,
\]

where \( m_T \) is finite. If the control input \( u_k \) is selected such that \( \phi_k = [x_k^T u_k]^T \) has \( n + m \) spectral lines with linearly independent amplitudes and finite radii, then the estimates \( \hat{A}, \hat{B} \) as defined in (14) will satisfy, with probability at least \( 1 - 2\delta \),

\[
\max \left\{ \| \hat{A} - A_\ast \|, \| \hat{B} - B_\ast \| \right\} \leq \tilde{O} \left( 1 + m_T^2 \sqrt{\frac{1}{T \| \Phi^{-1} \|}} \right).
\]

A proof of Theorem 3 can be found in Appendix [3]. In order to show the utility of Theorem 3, we must consider the form of the quantity \( m_T \), which the practitioner would wish to minimize. This quantity may not necessarily be small for all choices of \( f(\cdot) \) or arbitrary selections of \( u_k \). However, \( m_T \) can be shown to be of small order in the following two common settings. Clearly more research is needed to relax the condition in (16) to increase the scope of applicability of the proposed approach.

**Example 1** (High Pass Filter). Suppose the operator \( f(\cdot) \) in (3) is such that

\[
w_k = \sum_{i=1}^{n} a_i w_{k-i} + \sum_{i=1}^{m} b_i u_{k-i}
\]

and \( a_i, b_j \) are such that the transfer function from \( u_k \) to \( w_k \) is a high-pass filter. Then, a proper choice of \( \Omega \) as noted in Definition 2 will ensure that \( w_k \) will tend to zero exponentially with \( k \) [15], which will imply in turn that \( m_T = O(1) \).

**Example 2** (Small Non-Linearities). Suppose (3) is such that

\[
\begin{align*}
\bar{w}_k &= \sum_{i=1}^{n} a_i w_{k-i} + \sum_{i=1}^{m} b_i u_{k-i} \\
w_k &= f(\bar{w}_k).
\end{align*}
\]

If the function \( f(\cdot) \) is such that the choice of \( u_k \) will guarantee \( \|w_k\| = O(1/k^{1+\epsilon}) \) for any \( \epsilon > 0 \), then it implies that \( m_T = O(1) \). This can occur, for example, if \( f(\cdot) \) corresponds to higher-order terms, and \( u_k \) and initial conditions \( w_0 \) and \( x_0 \) are sufficiently small [20].
6 Conclusions

We develop tools for analysis of discrete-time and stochastic systems over a finite time which employs the theory of spectral lines. By providing a natural decoupling of deterministic and stochastic aspects of a signal in Definition 6, we provide an appropriate definition by which tools from adaptive control can be applied to recent settings considered in the machine learning literature.

To show the efficacy of our theoretical tools, we consider two applications. The first is in estimation of linear dynamical systems subject to external random noise, as widely considered in recent literature [1, 9, 10, 24, 33, 34, 41]. While the recent literature often selects the control input based on independent realizations of Gaussian noise, we propose a spectral lines based exogenous signal, and use a new theory of discrete stochastic spectral lines to analyze parameter estimation rates.

The second application is in estimation of a linear dynamical system subject to both external random noise and unobserved deterministic unmodeled dynamics, which has not yet been considered in recent literature but has far reaching applications. Using the tools of spectral lines, we are again able to show appropriate estimation rates in the finite-time setting with high probability.

Broader Impact

The central message of the paper builds on the notion of spectral lines and is applicable to all dynamic systems that preserve spectral lines. As such, the implications of this work are broad and of interest to all applications that can be modeled by dynamic systems and where parameter estimation is important. These include all physical applications where estimation and real-time control play a role including aerospace systems, automotive systems, energy systems, robotics, manufacturing, and process control systems. Beyond engineered systems, this may also benefit estimation in problems related to social networks and social systems engineering [29]. Whether in engineered or natural systems, the applicability of the central approach proposed here to both idealized systems as in [1] and non-idealized systems as in [2–3] will help facilitate parameter estimation with reduced bias. However, in both of these application areas, the tradeoffs between estimation and privacy remain to be investigated and are fundamental challenges.

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Appendix

Organization of the appendix. We discuss the continuous time analogue to Section 2.1 in Appendix A to help inform the reader of well-studied contexts for parameter convergence discussed in the adaptive control literature. Probabilistic inequalities are provided in Appendix B which are used in Appendices C, D, E, and F to provide detailed proofs of claims made in the paper.

A  Time-Varying Regression in Continuous Time

A.1  Parameter identification in a regression model

In this section, we provide a brief introduction to regression with time-varying regressors in continuous time. Consider a regression system of the form

\[ y(t) = \theta^T \phi(t), \quad (19) \]

where \( \theta^* \in \mathbb{R}^N \) represents an unknown constant parameter and \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^N \) represents a known time-varying regressor. The variable \( y : \mathbb{R}^+ \rightarrow \mathbb{R}^N \) represents a known time-varying output. Given that \( \theta^* \) is unknown, we formulate an estimator \( \hat{y}(t) = \theta^T(t) \phi(t) \), where \( \hat{y} : \mathbb{R}^+ \rightarrow \mathbb{R} \) is the predicted output and the unknown parameter is estimated as \( \hat{\theta} \), where \( \hat{\theta} : \mathbb{R}^+ \rightarrow \mathbb{R}^N \). This leads to an error model of the form

\[ e_y(t) = \hat{\theta}^T(t) \phi(t), \quad (20) \]

where \( e_y = \hat{y} - y \) is an output error, and \( \hat{\theta} = \theta - \theta^* \) is the parameter estimation error and the two errors are related through (20). The primary goal is to design a rule to adjust the parameter estimate \( \theta \) in a continuous manner using knowledge of \( \phi \) and \( e_y \) such that the unknown parameter is estimated, i.e., \( \hat{\theta}(t) \rightarrow 0 \) as \( t \rightarrow \infty \). The secondary goal is to at least ensure that the adjustment rule provides for the convergence of the output error \( e_y(t) \) towards zero. A gradient-flow based algorithm is often suggested for this purpose [28]. A squared output error loss function \( L = (1/2) e_y^2 \) is often used to lead to an update rule

\[ \hat{\theta}(t) = -\gamma \phi(t) e_y(t), \quad (21) \]

where \( \gamma > 0 \) is a user defined gain.

A careful application of stability theory easily guarantees that \( \theta(t) \), the solutions of (21), are bounded for any initial condition \( \theta(t_0) \). For any bounded regressor \( \phi(t) \) with a bounded time derivative, it can also be shown that the output error converges to zero, i.e., that the secondary goal is achieved. This raises the question as to when the primary goal of accurate parameter estimation is achieved. This problem is equivalent to determining the asymptotic stability of the equilibrium point \( \hat{\theta}(t) = 0 \) of the time-varying differential equation

\[ \dot{\hat{\theta}}(t) = -\gamma \phi(t) \phi^T(t) \hat{\theta}(t). \quad (22) \]

For this, we need the regressor to satisfy an additional condition known as persistent excitation (PE). This condition is expanded upon in the following section.

A.2  Persistent Excitation in Continuous Time

Definition 8 [28]. A function \( \phi : \mathbb{R}^+ \rightarrow \mathbb{R}^N \) is persistently exciting (PE) if there exists \( T > 0 \) and \( \alpha > 0, \beta > 0 \) such that

\[ \beta I \geq \int_{t}^{t+T} \phi(\tau) \phi^T(\tau) d\tau \geq \alpha I, \quad t \geq t_0. \]

Theorem 4. A persistently exciting regressor \( \phi \) is necessary and sufficient for the uniform asymptotic stability of \( \hat{\theta} = 0 \) in [22] [28].

We note that the preceding discussion revolves around deterministic signals. This together with the fact that the stability property in Theorem 4 is uniform and asymptotic imply that given an \( \epsilon > 0 \), there exists a finite \( T > 0 \) such that

\[ \| \hat{\theta}(t) \| \leq \epsilon, \quad \forall t \geq t_0 + T. \]

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We refer the reader to [28] and the original sources [26, 25] for the details of the proof of Theorem 4.

A spectral lines-based necessary and sufficient condition for a regressor to satisfy the persistent excitation condition in Definition 8 may alternatively be provided, as discussed in [6, 7]. Before proceeding to this theorem, we recall the definition of a spectral line.

**Definition 9 (Spectral Line [7]).** A function $u : \mathbb{R}^+ \rightarrow \mathbb{R}^n$ is said to have a spectral line at a frequency $\omega_0$ of amplitude $\bar{u}(\omega_0) \in \mathbb{C}^n$ iff

$$
\frac{1}{T} \int_t^{t+T} u(\tau) e^{-j\omega_0 \tau} d\tau
$$

converges to $\bar{u}(\omega_0)$ as $T \to \infty$ uniformly in $t$.

The following theorem uses the definition of a spectral line to relate the persistent excitation condition in Definition 8 to spectral lines of an input of a dynamical system.

**Theorem 5 ([6]).** Suppose that $\phi$ is given by the state of a linear dynamic system,

$$\dot{\phi} = A\phi + b r(t),$$

where $(A, b)$ is controllable, $A \in \mathbb{R}^{N \times N}$ and $b \in \mathbb{R}^N$. Then, $\phi$ satisfies the PE condition unless the spectrum of $r(t)$ is concentrated on $k < N$ lines.

### B Probabilistic Inequalities

**Proposition 3 (Sum of Dependent sub-Gaussian Random Variables).** Let $X \sim \text{subG}(\sigma^2)$ and $Y \sim \text{subG}(\tau^2)$ be two arbitrarily dependent sub-Gaussian random variables. Then,

$$X + Y \sim \text{subG}((\sigma + \tau)^2).$$

**Proof.** By Hölder’s inequality,

$$\mathbb{E}[e^{\lambda(X+Y)}] \leq \left(\mathbb{E}[e^{\lambda X}]\right)^{1/p} \left(\mathbb{E}[e^{\lambda Y}]\right)^{1/q}
$$

$$\leq e^{\frac{\lambda^2}{p} \left(\frac{\sigma^2}{q} + \frac{\tau^2}{q}\right)}.$$

The claim then follows by setting $q = \frac{\sigma^2}{\tau^2} + 1$. 

In particular, by induction, the above proposition states that for random variables $X_1, \ldots, X_n$, where $X_i \sim \text{subG}(\sigma_i^2), X_1 + \cdots + X_n \sim \text{subG} \left(\sum_{i=1}^n \sigma_i^2\right)$.

**Proposition 4 (Operator Norm of Matrix with Dependent sub-Gaussian Entries).** Let $M \in \mathbb{R}^{n \times n}$ be a random matrix with dependent sub-Gaussian entries with variance proxy $\hat{R}^2$. Then,

$$\mathbb{P}(\|M\| > t) \leq 9^2 n e^{-\frac{ct^2}{2n}},$$

for a universal constant $c$.

**Proof.** First, we note

$$\|M\| = \sup_{x \in \mathcal{S}^{n-1}, y \in \mathcal{S}^{n-1}} x^T M y.$$

Let $\mathcal{E}^n$ be a $1/4$-net of $\mathcal{S}^{n-1}$, which has size at most $9^n$ [40]. Then, we see:

$$\|M\| \leq 2 \sup_{w \in \mathcal{E}^n, z \in \mathcal{E}^n} w^T M z$$

Hence, for arbitrary $w, z \in \mathcal{E}^n$, we see

$$\mathbb{P}(\|M\| > t) \leq 9^{2n} \mathbb{P} \left( w^T W z > \frac{t}{2} \right).$$
Finally, we may note that $w^\top Mz \sim \text{subG}(nR^2)$. This follows from Proposition 5 and the fact that $\sum_{i,j} |w_i||z_j| \leq n$ when $w, z \in \mathcal{S}^{n-1}$. Therefore, for an absolute constant $c$,

$$\mathbb{P}(\|M\| > t) \leq 9^{2n} e^{-\frac{ct}{nR^2}},$$

as desired. \hfill \Box

**Proposition 5 ([40]).** Let $M \in \mathbb{R}^{n \times d}$ be a random matrix. Then, for any $\epsilon < 1$ and any $w \in \mathcal{S}^{d-1}$,

$$\mathbb{P}(\|M\| > z) \leq \left(1 + \frac{2\sqrt{\epsilon}}{\epsilon}\right)^d \mathbb{P}(\|Mw\| > (1 - \epsilon)z).$$

A proof of the above claim may be found in [40].

Similar to [33], we will be able to use the above claim to help us analyze self-normalized martingale terms, as initially discussed in [1] with the following theorem. We will also be able to apply similar arguments to the above proposition to the setting of random matrices with arbitrarily dependent sub-Gaussian entries.

**Theorem 6 (Theorem 1 in [1]).** Let $\{F_k\}_{k=1}^\infty$ be a filtration. Let $\{\eta_k\}_{k=1}^\infty$ be a sequence of real-valued random variables such that $\eta_k$ is $F_{k+1}$-measurable and conditionally $R$ sub-Gaussian given $F_k$, i.e.

$$\mathbb{E}[e^{\lambda \eta_k} | F_k] \leq e^{\frac{\lambda^2 R^2}{2}} \quad \forall \lambda > 0.$$

Let $\{x_k\}_{k=1}^\infty$ be a sequence of random vectors such that $x_k \in \mathbb{R}^d$ is $F_k$ measurable, and let $V \in \mathbb{R}^{d \times d}$ be an arbitrary positive definite deterministic matrix, and define

$$\bar{V}_T = V + \sum_{k=1}^T x_k x_k^\top, \quad S_t = \sum_{k=1}^T x_k \eta_k.$$

Then, for any $\delta > 0$, with probability at least $1 - \delta$, for all $T > 0$,

$$\|S_T\|_{\bar{V}_T}^2 \leq 2R^2 \log \left(\frac{\det(\bar{V}_T)^{1/2} \det(V)^{-1/2}}{\delta}\right).$$

**Proposition 1 (Self-Normalized Martingale Bound).** Consider the system (2), (3) where $w_k, \eta_k, x_k \in \mathbb{R}^n$, $\eta_k \sim \text{subG}(\sigma^2)$, and $u_k \in \mathbb{R}^m$ are deterministic. For an arbitrary deterministic matrix $V \succ 0$, define

$$\phi_k = \begin{bmatrix} x_k^\top \\ u_k \end{bmatrix}, \quad \bar{Y}_T = V + \sum_{k=1}^T \phi_k \phi_k^\top, \quad S_T = \sum_{k=1}^T \phi_k \eta_k \phi_k^\top.$$

Then, for any $0 < \delta < 1$, with probability at least $1 - \delta$,

$$\left\| (\bar{Y}_T)^{-1/2} S_T \right\| \leq \sigma \sqrt{8(n + m) \log \left(\frac{5 \det(\bar{Y}_T)^{1/2} \det(V)^{-1/2}}{\delta^{1/(n+m)}}\right)}.$$

**Proof.** From Proposition 5 setting $\epsilon = 1/2$, we note that for any $y$,

$$\mathbb{P} \left( \left\| (\bar{Y}_T)^{-1/2} S_T \right\| > y \right) \leq 5^{n+m} \mathbb{P} \left( \left\| (\bar{Y}_T)^{-1/2} S_T w \right\| > \frac{y^2}{2} \right)$$

$$= 5^{n+m} \mathbb{P} \left( \left\| (\bar{Y}_T)^{-1/2} S_T w \right\|^2 > \frac{y^2}{4} \right).$$

Note that $S_T w = \sum_{k=1}^T \phi_k \eta_k^\top w$, and that $\eta_k^\top w \sim \text{subG}(R^2)$ by the definition of a sub-Gaussian random vector. Letting $F_k$ be the $\sigma$-algebra generated by $(x_1, \ldots, x_k, \eta_{k-1}, \ldots, \eta_0)$, we are then able to apply Theorem 1 where $\eta_k^\top w$ now corresponds to $\eta_k$, and the claim follows by setting

$$y^2 = 8\sigma^2 \log \left(\frac{\det(\bar{Y}_T)^{1/2} \det(V)^{-1/2}}{5^{-(n+m)}}\right),$$

as this choice will ensure $\mathbb{P} \left( \left\| (\bar{Y}_T)^{-1/2} S_T w \right\|^2 > \frac{y^2}{4} \right) \leq 5^{-(n+m)} \delta$. \hfill \Box
Remark 3. The above proposition may then be applied to an arbitrary system, assuming there is a deterministic upper bound on \( \hat{Y}_T \) with high probability. We will show this indeed the case for both (1) and (2)–(3) with the following results.

**Proposition 6** (Markov’s Inequality [40]). Consider an integrable random variable \( X \) defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) such that for all \( \omega \in \Omega \), \( X(\omega) \geq 0 \). Then, for any \( \delta > 0 \),

\[
\mathbb{P} \left( X \geq \frac{\mathbb{E}[X]}{\delta} \right) \leq \delta.
\]

**Proof.** For all \( \omega \in \Omega \), and any \( a \), note \( X(\omega) \geq a \mathbb{1}_{X(\omega) \geq a} \). Thus,

\[
\mathbb{E}[X] \geq \mathbb{E}[a \mathbb{1}_{X(\omega) \geq a}] = a \mathbb{P}(X \geq a).
\]

The claim follows by rearranging the inequality above and setting \( a = \mathbb{E}[X]/\delta \).

In order to show the upper bounds on \( \hat{Y}_T \), we must also define the following two system-dependent matrices.

**Definition 10** (Gramian Matrix). For a matrix \( A \in \mathbb{R}^{n \times n} \), the Gramian of \( A \) is defined

\[
\Gamma_k(A) = \sum_{i=0}^{k} A^i (A^i)^\top.
\]

**Definition 11** (Controllability Gramian). For matrices \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \), the Controllability Gramian of the pair \((A, B)\) is defined

\[
\Gamma_k(A, B) = \sum_{i=0}^{k} A^i B B^\top (A^i)^\top.
\]

**Proposition 7** (Deterministic Upper Bound for LTI Systems). Consider the system

\[
x_{k+1} = A_x x_k + B_x u_k + \eta_k,
\]

where \( \eta_k, x_k \in \mathbb{R}^n \), \( \eta_k \sim \text{subG}(\sigma^2) \), and \( u_k \in \mathbb{R}^m \) are deterministic for each \( k \). Define

\[
\phi_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad Y_T = \sum_{k=0}^{T-1} \phi_k \phi_k^\top.
\]

Then, for any \( \delta > 0 \), with probability at least \( 1 - \delta \),

\[
Y_T \preceq \left( \frac{\sigma^2 T \text{tr}(\Gamma_{T-1}(A_x)) + T u_0^2 \text{tr}(\Gamma_{T-1}(A_x, B_x)) + T u_0^2}{\delta} \right) I.
\]

**Proof.** We first define the quantities

\[
\hat{A} = \begin{bmatrix} I & 0 & \ldots & 0 \\ A_x & I & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_x^{T-1} & A_x^{T-2} & \ldots & I \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_x & 0 & \ldots & 0 \\ A_x B_x & B_x & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_x^{T-1} B_x & A_x^{T-2} B_x & \ldots & B_x \end{bmatrix},
\]

\[
\hat{F} = \begin{bmatrix} \hat{A} & \hat{B} \\ 0 & I \end{bmatrix}, \quad \hat{\eta} = [\eta_0 \ldots \eta_{T-1} u_0 \ldots u_{T-1}]^\top,
\]

and note that

\[
\hat{F} \hat{\eta} = [x_1 \ldots x_T u_1 \ldots u_T]^\top.
\]

Then, for any realization \( \{\phi_k\}_{k=0}^{T-1} \),

\[
\left\| \sum_{k=0}^{T-1} \phi_k \phi_k^\top \right\| \preceq \sum_{k=1}^{T} \phi_k^\top \phi_k = (\hat{F} \hat{\eta})^\top (\hat{F} \hat{\eta}) = \text{tr} (\hat{F} \hat{\eta} (\hat{F} \hat{\eta})^\top).
\]
We then note, letting $U = [u_1 \ldots u_{T}]^{T} [u_1 \ldots u_{T_1}]$, 

$$
\mathbb{E}[\tilde{\eta}] = \begin{bmatrix} R^2 I & 0 \\ 0 & U \end{bmatrix}.
$$

which implies

$$
\mathbb{E}\left[\left\| \sum_{k=0}^{T-1} \phi_k \phi_k^{\top} \right\| \right] \leq \text{tr}\left( \tilde{\mathcal{F}} \mathbb{E} [\tilde{\eta}] \tilde{\mathcal{F}}^{\top} \right) \leq \text{tr}\left( \left[ \begin{array}{cc} \sigma^2 \tilde{A} \tilde{A}^{\top} + \tilde{B} U \tilde{B}^{\top} & U \\ BU & U \end{array} \right] \right) = \sigma^2 \text{tr}\left( \tilde{A} \tilde{A}^{\top} \right) + \text{tr}\left( \tilde{B} U \tilde{B}^{\top} \right) + \text{tr}(U)
$$

Letting $u_M = \max_{1 \leq t \leq T} u_M$, we then have

$$
\text{tr}\left( \tilde{A} \tilde{A}^{\top} \right) = \sum_{k=0}^{T-1} \text{tr}(\Gamma_k(A_*)) \leq T \text{tr}(\Gamma_{T-1}(A_*))
$$

$$
\text{tr}\left( \tilde{B} U \tilde{B}^{\top} \right) \leq u_M^2 \sum_{k=0}^{T-1} \text{tr}(\Gamma_k(A_* B_*)) \leq T u_M^2 \text{tr}(\Gamma_{T-1}(A_* B_*))
$$

Hence, by Markov’s inequality,

$$
\mathbb{P}\left( \left\| \sum_{k=0}^{T-1} \phi_k \phi_k^{\top} \right\| > \frac{\sigma^2 T \text{tr}(\Gamma_{T-1}(A_*)) + T u_M^2 \text{tr}(\Gamma_{T-1}(A_* B_*)) + T u_M^2}{\delta} \right) \leq \mathbb{P}\left( \text{tr}\left( \tilde{F} \tilde{\eta} \tilde{F}^{\top} \right) > \frac{\sigma^2 T \text{tr}(\Gamma_{T-1}(A_*)) + T u_M^2 \text{tr}(\Gamma_{T-1}(A_* B_*)) + T u_M^2}{\delta} \right) \leq \delta
$$

That is, with probability at least $1 - \delta$,

$$
Y_T = \sum_{k=0}^{T-1} \phi_k \phi_k^{\top} \leq \frac{\sigma^2 T \text{tr}(\Gamma_{T-1}(A_*)) + T u_M^2 \text{tr}(\Gamma_{T-1}(A_* B_*)) + T u_M^2}{\delta} I.
$$

\[\Box\]

**Proposition 8** (Refined Deterministic Upper Bound for Systems with Unmodeled Dynamics). Consider the system in \ref{2} - \ref{3} where $w_k, \eta_k, x_k \in \mathbb{R}^n$, $\eta_k \sim \text{subG}(\sigma^2)$, and $u_k \in \mathbb{R}^m$ are deterministic for each $k$. Define

$$
\phi_k = \begin{bmatrix} x_k \\ u_k \end{bmatrix}, \quad Y_T = \sum_{k=0}^{T-1} \phi_k \phi_k^{\top}.
$$

Further, suppose it is known

$$
\sum_{k=0}^{T-1} \|w_k\| \leq m_T.
$$

Then, for any $\delta > 0$, with probability at least $1 - \delta$,

$$
Y_T \leq C T u_M^2 m_T^2 \sigma^2 \left( \Gamma(A_*) + \Gamma(A_* B_*) \right) \left( 1 + \log(1/\delta) \right) I,
$$

for a universal constant $C$.

**Proof.** We first define the quantities

$$
\tilde{\Gamma} = \begin{bmatrix} I & 0 & \cdots & 0 \\ A_* & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_*^{T-1} & A_*^{T-2} & \cdots & I \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} B_* & 0 & \cdots & 0 \\ A_* B_* & B_* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_*^{T-1} B_* & A_*^{T-2} B_* & \cdots & B_* \end{bmatrix},
$$

$$
\tilde{A} = \begin{bmatrix} I & 0 & \cdots & 0 \\ A_* & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_*^{T-1} & A_*^{T-2} & \cdots & I \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_* & 0 & \cdots & 0 \\ A_* B_* & B_* & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_*^{T-1} B_* & A_*^{T-2} B_* & \cdots & B_* \end{bmatrix}.
$$

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\[
\hat{E} = \begin{bmatrix} \hat{A} & \hat{A} & \hat{B} \\ 0 & 0 & I \end{bmatrix}, \quad \tilde{\eta} = [\eta_0 \ldots \eta_{T-1} \ u_0 \ldots \ u_{T-1}]^\top,
\]
and note that
\[
\hat{E}\tilde{\eta} = [x_1 \ldots x_T \ u_1 \ldots \ u_T]^\top.
\]

Then, again for any realization \(\{\phi_k\}_{k=0}^T\),
\[
\left\|\sum_{k=0}^{T-1} \phi_k \tilde{\phi}_k^\top \right\| \leq \sum_{k=1}^T \phi_k^2 \phi_k = (\hat{E}\tilde{\eta})^\top (\hat{E}\tilde{\eta}) = \text{tr}\left((\hat{E}\tilde{\eta})^\top \hat{E}^\top\right).
\]

For simplicity, write \(u = [u_0 \ldots u_{T-1}]^\top\), \(w = [w_0 \ldots w_{T-1}]^\top\), and \(\eta = [\eta_0 \ldots \eta_{T-1}]^\top\). Expanding the definition of \(\text{tr}\left((\hat{E}\tilde{\eta})^\top \hat{E}^\top\right)\), we then see
\[
\text{tr}\left((\hat{E}\tilde{\eta})^\top \hat{E}^\top\right) = \text{tr}(\hat{A}\eta\eta^\top \hat{A}^\top)
+ 2\text{tr}(\hat{A}\eta w^\top \hat{A}^\top)
+ 2\text{tr}(\hat{A}\eta u^\top \hat{B}^\top)
+ \text{tr}(\hat{A}w w^\top \hat{A}^\top) + \text{tr}(\hat{B}w u^\top \hat{B}^\top)
\]

We then see that \(\text{tr}(\hat{A}\eta\eta^\top \hat{A}^\top)\) can be bounded with high probability using [33] Proposition 9.4, and that \(\text{tr}(\hat{A}\eta w^\top \hat{A}^\top)\) and \(\text{tr}(\hat{A}\eta u^\top \hat{B}^\top)\) are sub-Gaussian random variables with variance proxy \(TT(A_\sigma)\sigma^2 m_\sigma^2\) and \(TT(A_\sigma, B_\sigma)\sigma^2 u_M^2\), respectively. Hence, with probability at least \(1 - 3\delta\), we will have
\[
\text{tr}(\hat{A}\eta\eta^\top \hat{A}^\top) \leq \sigma^2 (TT(A_\sigma)) (1 + c_1 \log(1/\delta))
+ 2\text{tr}(\hat{A}\eta w^\top \hat{A}^\top) \leq c_2 TT(A_\sigma) \sigma^2 m_\sigma^2 \log(1/\delta)
+ 2\text{tr}(\hat{A}\eta u^\top \hat{B}^\top) \leq c_3 T(\Gamma(A_\sigma) + \Gamma(A_\sigma, B_\sigma)) \sigma^2 u_M^2 \log(1/\delta),
\]
where \(u_M = \max_{0 \leq i \leq T} |u_i|\). We further note, by expanding definitions, that \(\text{tr}(\hat{A}w w^\top \hat{A}^\top) \leq T m_\sigma^2 \Gamma_T(A_\sigma)\) and \(\text{tr}(\hat{B}w u^\top \hat{B}^\top) \leq T u_M^2 \Gamma_T(A_\sigma, B_\sigma)\). Hence, with probability at least \(1 - 3\delta\), there is a universal constant \(C\) such that
\[
Y_T \leq \left\|\sum_{k=0}^{T-1} \phi_k \tilde{\phi}_k^\top \right\| I \leq CT u_M^2 m_\sigma^2 \sigma^2 (\Gamma(A_\sigma) + \Gamma(A_\sigma, B_\sigma)) \left(1 + \log(1/\delta)\right) I,
\]
implying the claim above. \(\square\)

### C Proof of Lemma 1

We first note
\[
\frac{1}{T} \sum_{k=0}^{T-1} y_k e^{-j2\pi\omega_0 k} = \frac{1}{T} \sum_{k=0}^{T-1} H(e^{j\omega_0}) u_k e^{-j2\pi\omega_0 k} + \frac{1}{T} \sum_{k=0}^{T-1} \eta_k e^{-j2\pi\omega_0 k},
\]
by the definition of the Discrete Fourier Transform, since \(y(e^{j\omega}) = H(e^{j\omega}) u(e^{j\omega}) + \eta(e^{j\omega})\) and \(\omega_0 \in \Omega_T\). Rearranging, we have:
\[
\frac{1}{T} \sum_{k=0}^{T-1} y_k e^{-j2\pi\omega_0 k} - H(e^{j\omega_0}) \bar{u}(\omega_0) =
H(e^{j\omega_0}) \left(\frac{1}{T} \sum_{k=0}^{T-1} u_k e^{-j2\pi\omega_0 k} - \bar{u}(\omega_0)\right) + \frac{1}{T} \sum_{k=0}^{T-1} \eta_k e^{-j2\pi\omega_0 k}.
\]
From Proposition 33 the claim then follows immediately.
D Proof of Proposition 2

Note that for any unit vector \( z \), and any realization \( \{ \phi_k \}_{k=i}^{i+S} \),
\[
z^\top \left( \frac{1}{S+1} \sum_{k=i}^{i+S} \phi_k \phi_k^\top \right) z = \frac{1}{S+1} \left( \frac{i+S}{1} \sum_{k=i}^{i+S} (\phi_k z)^2 \right) \geq \left| \frac{1}{S+1} \sum_{k=i}^{i+S} \phi_k z e^{-j2\pi\omega_k} \right|^2,
\]
by Jensen’s inequality. Then, we see
\[
z^\top \left( \frac{1}{S+1} \sum_{k=i}^{i+S} \phi_k \phi_k^\top \right) z \geq \frac{1}{n} \sum_{t=1}^{n} \left| \frac{1}{S+1} \sum_{k=i}^{i+S} \phi_k z e^{-j2\pi\omega_k} \right|^2 \geq \frac{1}{n} \left| (\Phi + W)z \right|^2.
\]
Here, \( W \) is a random matrix for which each column is \( R/(S+1) \) sub-Gaussian. Continuing, we have:
\[
z^\top \left( \frac{1}{S+1} \sum_{k=i}^{i+S} \phi_k \phi_k^\top \right) z \geq \frac{1}{n} \left( \|\Phi^{-1}\|^{-2} - \|W\|^2 \right).
\]
Finally, we see that because \( W \) has (possibly dependent) entries which are \( R/(S+1) \) sub-Gaussian, the claim follows from Proposition 4 setting \( t = \|\Phi^{-1}\|^{-1}/2 \).

E Proof of Theorem 2

First, we rewrite \( (1) \) as
\[
\begin{bmatrix} x_{k+1} \\ u_{k+1} \end{bmatrix} = \begin{bmatrix} A_* & B_* \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} \eta_k \\ u_{k+1} \end{bmatrix},
\]
noting that, as stated in Theorem 2, \( u_k \) is selected such that \( \phi_k = [x_k^\top u_k^\top] \) has \( n + m \) spectral lines of linearly independent amplitudes.

Considering this as a multi-dimensional regression problem, we note that \( (\hat{A}, \hat{B}) \) in \( (14) \) have a closed form solution
\[
[\hat{A} \quad \hat{B}] = ((\Phi^\top \Phi)^\dagger \Phi^\top E)^\top + [A_* \quad B_*],
\]
where
\[
\Phi = \begin{bmatrix} x_0^\top & u_0^\top & \vdots & \vdots \\ x_{T-1}^\top & u_{T-1}^\top \end{bmatrix}, \quad E = \begin{bmatrix} \eta_0^\top \\ \vdots \\ \eta_{T-1}^\top \end{bmatrix}.
\]
Defining \( \hat{F} = [\hat{A} \quad \hat{B}] \) and \( F_* = [A_* \quad B_*] \), we then see
\[
\hat{F} - F_* = ((\Phi^\top \Phi)^\dagger \Phi^\top E)^\top.
\]
Defining the quantities
\[
Y_T = \Phi^\top \Phi = \sum_{k=0}^{T-1} \phi_k \phi_k^\top, \quad S_T = \Phi^\top E,
\]
we then see
\[
\max \left\{ ||\hat{A} - A_*||, ||\hat{B} - B_*|| \right\} \leq ||\hat{F} - F_*||_2 \leq \|((Y_T^\dagger)^{1/2})_2\| \|((Y_T^\dagger)^{1/2})_2\|_2. \tag{29}
\]
Similar to [33], we then proceed in two steps. First, we wish to define, for two matrices \( V_{dn} \) and \( V_{up} \), and some value \( T_0 \), the event
\[
\mathcal{E}_0 = \{ 0 \prec V_{dn} \preceq Y_T \preceq V_{up}, T \geq T_0 \}.
\]
With \( V_{dn} \) as a fixed matrix, we may also define the event \( \mathcal{E}_1 \), as
\[
\mathcal{E}_1 = \left\{ \| S_T \|_{(Y_T + V_{dn})^{-1}} \leq \sigma \sqrt{8(n + m) \log \left( \frac{5 \det(Y_T + V_{dn})^{1/2(n+m)} \det(V_{dn})^{-1/2(n+m)}}{\delta^{1/(n+m)}} \right)} \right\},
\]
with \( \delta = n + m \), and which from Proposition[1] we know occurs with probability at least \( 1 - \delta \). Under \( \mathcal{E}_0 \cap \mathcal{E}_1 \), we then see, since \( \mathcal{E}_0 \) implies \( (Y_T + V_{dn})^{-1} \geq \frac{1}{2} Y_T^{-1/2} \), that
\[
\| S_T \|_{Y_T^{-1}} \leq \sqrt{2} \| S_T \|_{(Y_T + V_{dn})^{-1}} \leq \sigma \sqrt{16(n + m) \log \left( \frac{5 \det(V_{up}^{-1} + I)^{1/2(n+m)}}{\delta^{1/(n+m)}} \right)} \tag{30}
\]
Hence, all that remains is to find the \( V_{dn} \) and \( V_{up} \) which define \( \mathcal{E}_0 \), at which point we may appropriately bound both \( \| (Y_T^1)^{1/2} \|_2 \) and \( \| (Y_T^1)^{1/2} S_T \|_2 \). In particular, as opposed to using techniques from [1][33] to find \( V_{dn} \), we use tools from adaptive control and Section[3]

With \( \Phi \) defined as in Definition[7] we see from Proposition[2] that, we have
\[
\sum_{k=0}^{T-1} \phi_k \phi_k^\top \succcurlyeq \frac{1}{2(n + m)} \| \Phi^{-1} \|^{-2} TI := V_{dn}, \tag{31}
\]
with probability at least \( 1 - e^{-c \| \Phi^{-1} \|^{-1}_T 2(n + m) \log 9} \), where \( c \) is an absolute constant and \( A_\sigma = \max_{\Omega} \| e^{-j\omega \Omega} I - A_* \| \) represents the maximum variance of the external disturbance in the frequency domain, where \( \Omega \) is defined as in Definition[7]. If we ensure \( T \geq (\log \frac{1}{\delta} + 2(n + m) \log 9) \frac{2(n + m) A_\sigma^2}{\| \Phi^{-1} \|^{-1}_T} := T_\Phi(\delta) \), we may ensure that this event occurs with probability at least \( 1 - \delta \).

We may then bound \( V_{up} \) using Proposition[7] and find with probability at least \( 1 - \delta \),
\[
Y_T \preceq \left( \sigma^2 \text{tr}(\Gamma_T^{-1}(A_*)) + u_M^2 \text{tr}(\Gamma_T^{-1}(A_*, B_*)) + u_M^2 \right) TI := V_{up}.
\]
Combining the results above, we see with probability at least \( 1 - 3\delta \), both \( \mathcal{E}_0 \) and \( \mathcal{E}_1 \) will occur, such that
\[
\max \left\{ \| \bar{A} - A_* \|, \| \bar{B} - B_* \| \right\} \leq \| F - F_* \|_2
\]
\[
\leq \frac{2(n + m)}{T \| \Phi^{-1} \|^{-2} \gamma \left( A_*, B_*, \Phi \right)}
\]
where
\[
\gamma(A_*, B_*, \Phi) = \left( \log \left( \frac{5}{\delta^{1/(n+m)}} \right) + \frac{1}{2} \log \left( \left( \sigma^2 \text{tr}(\Gamma_T^{-1}(A_*)) + u_M^2 \text{tr}(\Gamma_T^{-1}(A_*, B_*)) + u_M^2 \right) \frac{2(n + m)}{\delta \| \Phi^{-1} \|^{-1}_T + 1} \right) \right)^{1/2},
\]
which implies the claim in Theorem[2]
We can then see, considering this as a multi-dimensional regression problem, defining

$$\vec{x}_{k+1} = \begin{bmatrix} A_* & B_* \end{bmatrix} \begin{bmatrix} x_k \\ u_k \end{bmatrix} + \begin{bmatrix} w_k + \eta_k \\ u_{k+1} \end{bmatrix},$$

where

$$\max_{\text{variance of the external disturbance in the frequency domain, with}}$$

Similar to the proof of Theorem 2, we first rewrite (2) as

$$F - F_* = \left((\Phi^{\top} \Phi)^{1/2} \Phi^\top (W + E)\right)^\top,$$

where

$$\Phi = \begin{bmatrix} \vec{x}_0^\top \\ \vec{u}_0^\top \\ \vdots \\ \vec{x}_{T-1}^\top \\ \vec{u}_{T-1}^\top \end{bmatrix}, \quad E = \begin{bmatrix} \eta_0 \\ \vdots \\ \eta_{T-1} \end{bmatrix}, \quad W = \begin{bmatrix} \vec{w}_0^\top \\ \vdots \\ \vec{w}_{T-1}^\top \end{bmatrix}.$$}

Defining the quantities

$$Y_T = \Phi^\top \Phi, \quad S_T = \Phi^\top E,$$

we then see, similar to Appendix [2],

$$\max \left\{ \|\hat{A} - A_*\|, \|\tilde{B} - B_*\| \right\} \leq \left( (Y_T^1)^{1/2} \right)_2 \left( (Y_T^1)^{1/2} S_T \right)_2 + \left( (Y_T^1)^{1/2} \Phi^\top W \right)_2.$$

We then define two events, similar to the proof of Theorem [2]

$$\mathcal{E}_0 = \{V_{dn} \leq Y_T \leq V_{up}, T \geq T_0\}$$

$$\mathcal{E}_1 = \left\{ \|S_T\|_{(Y_T + V_{dn})^{-1}} \leq \sigma \sqrt{8(n + m) \log \left( \frac{5 \det(Y_T + V_{dn})^{1/2} \det(V_{dn})^{-1/2}}{\delta^{1/(n+m)}} \right)} \right\}$$

(33)

(34)

From Proposition [2] and Proposition [8] we see that with probability at least $1 - 2\delta$, $\mathcal{E}_0$ is satisfied with

$$V_{dn} = \frac{1}{2(n + m)} \|\Phi^{-1}\|_T I$$

$$V_{up} = C T u_M^2 n_T \sigma^2 (\Gamma_T(A_*) + \Gamma_T(A_*, B_*)) \left( 1 + \log(1/\delta) \right)$$

$$T_0 = \left( \log \frac{1}{\delta} + 2(n + m) \log 9 \right) \frac{2(n + m) A_* \sigma^2}{c \|\Phi^{-1}\|_T^{-1}},$$

(35)

(36)

(37)

where $C$ is a universal constant and $A_* = \max_{i \in \Omega} \|e^{i \omega_i} I - A_*\|^{-1}$ represents the maximum variance of the external disturbance in the frequency domain, with $\Omega$ defined as in Definition [7].

Further, we see that by Proposition [11] with probability at least $1 - \delta$, $\mathcal{E}_1$ will hold for $V_{dn}$ as in (35). Combining these statements as in (30), we conclude that with probability at least $1 - 3\delta$,

$$\max \left\{ \|\hat{A} - A_*\|, \|\tilde{B} - B_*\| \right\} \leq \left( (Y_T^1)^{1/2} \right)_2 \left( (Y_T^1)^{1/2} S_T \right)_2$$

$$+ \left( (Y_T^1)^{1/2} \right)_2 \left( (Y_T^1)^{1/2} \Phi^\top W \right)_2$$

$$\leq \left( V_{dn}^{-1/2} \right)_2 \sigma \sqrt{16(n + m) \log \left( \frac{5 \det(V_{up} V_{dn}^{-1} + I)^{1/2}}{\delta^{1/(n+m)}} \right)}$$

$$+ \left( V_{dn}^{-1/2} \right)_2 \left( V_{dn}^{-1/2} \right)_2 \left( V_{up} \right)_2 m_T$$

(38)

(39)
Applying (35) and (36), we see that with probability at least $1 - 3\delta$, for $T$ satisfying $T \geq T_0$ as given in (37),
\[
\max \left\{ \|A - A_0\|, \|B - B_0\| \right\} \leq \sqrt{\frac{n + m}{T} \| \Phi^{-1} \|^{-2}} \left( \gamma_1(A_0, B_0, \Phi) + m^2 \gamma_2(A_0, B_0, \Phi) \right),
\]
where, for universal constants $c_1$ and $c_2$,
\[
\gamma_1(A_0, B_0, \Phi) = c_1 \sigma \sqrt{(n + m) \log \left( \frac{5}{\delta^3/(n+m)} \right) + \frac{1}{2} \log \det (V_{dn}^{-1} V_{up})}
\]
\[
\gamma_2(A_0, B_0, \Phi) = c_2 u M \sigma \sqrt{(\Gamma_T(A_0) + \Gamma_T(A_0, B_0)) \left( 1 + \log \left( \frac{1}{\delta} \right) \right)}. \]

Dividing the failure probability above by 3, and noting the event (16) occurs with probability at least $1 - \delta$, the claim of Theorem 3 follows.