Scaling and universality in the 2D Ising model with a magnetic field

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The scaling function of the 2D Ising model in a magnetic field on the square and triangular lattices is obtained numerically via Baxter’s variational corner transfer matrix approach. The use of the Aharony-Fisher non-linear scaling variables allowed us to perform calculations sufficiently away from the critical point to obtain very high precision data, which convincingly confirm all predictions of the scaling and universality hypotheses. The results are in excellent agreement with the field theory calculations of Fonseca and Zamolodchikov as well as with many previously known exact and numerical results for the 2D Ising model. This includes excellent agreement with the classic analytic results, Nickell, Guttmann and Perk.

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The principles of scaling and universality (see, e.g., [1]) play important roles in the theory of phase transition and critical phenomena. The scaling assumption asserts that observable quantities exhibit power law singularities in the variable $\Delta T = T - T_c$ in the vicinity of the critical temperature $T_c$, with coefficients being functions of certain dimensionless combinations of available parameters, e.g., the magnetic field $H$ and $\Delta T$. The universality hypothesis states that the leading singular part of the free energy is a universal scaling function which is same for all systems in a given “universality class”. In two dimensions classes of universal critical behaviour are well understood — they are classified by conformal field theory (CFT) [2]. The latter provides exact solutions for 2D systems at the critical point and, in particular, gives exact values of the scaling dimensions. However, the calculation of scaling functions describing off-critical behaviour is a hard problem which does not have any exact analytical solutions. From the field theory point of view this requires solving a CFT perturbed by at least two relevant operators. Direct numerical calculations and simulations in lattice models also face serious difficulties due to multiple long range correlations and the resulting poor convergence near the critical point.

It appears that despite numerous analytical and numerical results (cited below), the full picture of scaling and universality has never been convincingly demonstrated through numerical calculations in lattice models. Our aim is to do this. Here we consider the planar nearest-neighbour Ising model on the regular square and triangular lattices, which has already played a prominent role in the development of the theory of phase transition and critical phenomena [2-8]. Its partition function reads

$$Z = \sum_\sigma \exp \left\{ \beta \sum_{(ij)} \sigma_i \sigma_j + H \sum_i \sigma_i \right\}, \quad \sigma_i = \pm 1,$$

where the first sum in the exponent is taken over all edges, the second over all sites and the outer sum over all spin configurations $\{\sigma\}$ of the lattice. The constants $H$ and $\beta$ denote the (suitably normalized) magnetic field and inverse temperature. The specific free energy, magnetization and magnetic susceptibility are defined as

$$F = - \lim_{N \to \infty} \frac{1}{N} \log Z, \quad M = - \frac{\partial F}{\partial H}, \quad \chi = - \frac{\partial^2 F}{\partial H^2},$$

where $N$ is the number of lattice sites. The model exhibits a second order phase transition at $H = 0$ and $\beta = \beta_c$, where

$$\beta_c^{(s)} = \frac{1}{2} \log(1 + \sqrt{2}), \quad \beta_c^{(t)} = \frac{1}{4} \log 3,$$

for the square and triangular lattices, respectively.

The scaling and universality hypotheses predict that the leading singular part, $F_{\text{sing}}(\Delta \beta, H)$, of the free energy in the vicinity of the critical point, $\Delta \beta = \beta - \beta_c \sim 0$, can be expressed through a universal function $F(m, h)$,

$$F_{\text{sing}}(\Delta \beta, H) = F(m(\Delta \beta, H), h(\Delta \beta, H)),$$

where $\Delta \beta$ and $H$ enter the rhs only through non-linear scaling variables [10],

$$m = m(\Delta \beta, H) = O(\Delta \beta) + O((\Delta \beta)^3) + O(H^2) + \ldots, \quad h = h(\Delta \beta, H) = O(H) + O(\Delta \beta) + O(H^3) + \ldots,$$

which are analytic functions of $\Delta \beta$ and $H$. The coefficients in these expansions depend on the details of the microscopic interaction (for instance they are different for the square and triangular lattices), but the function $F(m, h)$ is the same for all models in the 2D Ising model universality class. It can be written as

$$F(m, h) = \frac{m^2}{8\pi} \log m^2 + h^{16/15} \Phi(\eta), \quad \eta = \frac{m}{h^{8/15}},$$

where $\Phi(\eta)$ is a universal scaling function of a single variable $\eta$ (the scaling parameter), normalized such that

$$F(m, 0) = \frac{m^2}{8\pi} \log m^2$$
The function $\mathcal{F}(m, h)$ has a concise interpretation in terms of 2D Euclidean quantum field theory. Namely, it coincides with the vacuum energy density of the “Ising Field Theory” (IFT) \(^{11}\). The latter is defined as a model of perturbed conformal field theory with the action

$$\mathcal{A}_{\text{IFT}} = A_{\text{c=1/2}} + \frac{m}{2\pi} \int \sigma(x) d^2x + \int c(x) d^2x, \quad (8)$$

where $A_{\text{c=1/2}}$ stands for the action of the $c = 1/2$ CFT of free massless Majorana fermions, $\sigma(x)$ and $c(x)$ are primary fields of conformal dimensions $1/16$ and $1/2$. The parameters $m$ and $h$ have the mass dimensions 1 and $15/8$, respectively, and the scaling parameter $\eta$ in \(^{(1)}\) is dimensionless.

The scaling function \(^{(1)}\) is of much interest as it controls all thermodynamic properties of the Ising model in the critical domain. Although there are many exact results (obtained through exact solutions of \(^{(8)}\) at the critical domain. Although there are many exact results (obtained through exact solutions of \(^{(8)}\) at the critical domain. Although there are many exact results (obtained through exact solutions of \(^{(8)}\) at the critical domain.

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Thus far, the scaling function \(^{(1)}\) has been calculated directly in the limit of an infinite lattice. Its accuracy depends on the magnitude of truncated eigenvalues of the corner transfer matrix (which is at our control), rather than on the size of the lattice. The original Baxter approach \(^{31}\) was enhanced by an improved iteration scheme \(^{32}\), known as the corner transfer matrix renormalization group (CTMRG). The use of the non-linear scaling variables \(^{33}\) allowed us to perform calculations insufficiently away from the critical point with a reliable convergence of the algorithm. The results for the scaling function $\Phi(\eta)$ are shown in Fig. 1. In total we have calculated about 10,000 data points for different values of the temperature and magnetic field on the square lattice and about 5,000 for the triangular one. As seen from the picture all points collapse on a smooth curve, shown by the solid line (as expected, the curve is the same for the square and triangular lattices). The spread of the points at any fixed value $\eta$ does not exceed $10^{-6}$ relative accuracy. This gives a convincing demonstration of the scaling and universality in the 2D Ising model. Furthermore, our numerical results for the function $\Phi(\eta)$ remarkably confirm the field theory calculations \(^{11}\), to within all six significant digits presented therein.

For further reference we write down expansions of the function $\Phi(\eta)$ for large values of $\eta$ on the real line

$$\Phi_{\text{low}}(\eta) = \eta^2 \sum_{k=1}^{\infty} G_k \eta^{-15k/8}, \quad \eta \to +\infty, \quad (9)$$

$$\Phi_{\text{high}}(\eta) = \eta^2 \sum_{k=1}^{\infty} G^2_k |\eta|^{-30k/8}, \quad \eta \to -\infty, \quad (10)$$

and for small values of $\eta$,

$$\Phi(\eta) = -\frac{\eta^2}{8\pi} \log \eta^2 + \sum_{k=0}^{\infty} \Phi_k \eta^k. \quad (11)$$

Some of the above expansion coefficients are known exactly. The coefficient $G_1$ has a simple explicit expression \(^{3}\); the coefficients $G_2$ and $G_2$ have integral expressions \(^{12, 13}\) involving solutions of the Painlevé III equation. They were numerically evaluated to very high precision (50 digits) in \(^{33}\). The coefficients $\Phi_0, \Phi_1$ were analytically calculated in \(^{15}\) and \(^{34}\), respectively. The numerical value of $\Phi_1$ (which requires certain quadratures) was found in \(^{28}\). The above values are quoted in the last column of Table 1.

In what follows we exclude the temperature variable $\beta$ in favour of a new variable

$$\tau = \begin{cases} 
(1 - \sinh^22\beta)/(2 \sinh2\beta), & \text{(sq. lat.)} \\
(e^{-\beta} - e^{\beta} \sinh2\beta)/(\sinh2\beta)^{1/2}, & \text{(tr. lat.)}
\end{cases} \quad (12)$$

which is vanishing for $\beta = \beta_c$ and positive for $\beta < \beta_c$.

![FIG. 1: The scaling function $\Phi(\eta)$ in the three regions, separated by the dashed lines at $\eta \approx \pm 2.3$, can be approximated with high precision (up to $10^{-6}$) by the series (10), (11) and (9) with coefficients given in Table 1.](image-url)
(above the critical temperature). Another useful variable
\begin{equation}
k^2 = \begin{cases} 
16 \varepsilon^8 / (e^{\varepsilon^4} - 1)^4, & \text{(sq. lat.)} \\
16 \varepsilon^6 / ((e^{\varepsilon^4} - 1)^3 (e^{\varepsilon^3} + 3)), & \text{(tr. lat.)}
\end{cases}
\end{equation}

The lattice free energy for \( \tau, H \to 0 \),
\[ F(\tau, H) = F_{\text{sing}}(\tau, H) + F_{\text{reg}}(\tau, H) + F_{\text{sub}}(\tau, H), \]
contains leading universal part \[ \ref{14}, \] regular terms \( F_{\text{reg}}(\tau, H) \), which are analytic in \( \tau \) and \( H \), and subleading singular terms \( F_{\text{sub}}(\tau, H) \), which are non-analytic, but less singular than the first term in \[ \ref{14} \]. Therefore, to extract the universal scaling function from the lattice calculations one should be able to isolate and subtract these extra terms. Moreover, one needs to know the exact form of the non-linear scaling variables \[ \ref{5} \]. In principle, all this information can be determined entirely from numerical calculations (provided one assumes the values of exponents of the subleading terms, predicted by the analysis \[ \ref{33, 35} \] of the CFT irrelevant operators, contributing to the free energy \[ \ref{14} \]). Much more accurate results can be obtained if the numerical work is combined with known exact results. Namely, the zero-field free energy reads \[ \ref{3, 2} \]
\[ F^{(s)}(\tau, 0) = -\frac{1}{2} \log(4 \sinh \beta) - \frac{1}{8\pi^2} \int_0^{2\pi} d\phi_1 d\phi_2 \log(2\sqrt{1 + \tau^2} - \cos \phi_1 - \cos \phi_2), \]
\[ F^{(t)}(\tau, 0) = -\frac{1}{2} \log(4 \sinh \beta) - \frac{1}{8\pi^2} \int_0^{2\pi} d\phi_1 d\phi_2 \log(3 + \tau^2 - \cos \phi_1 - \cos \phi_2 - \cos(\phi_1 + \phi_2)), \]
where the superscripts \( (s) \) and \( (t) \) stand for the square and triangular lattices, respectively. Write the non-linear variables \[ \ref{5} \] in the form,
\[ m(\tau, H) = -C_\tau \tau a(\tau) + H^2 b(\tau) + O(H^4), \]
\[ h(\tau, H) = C_H H \left[c(\tau) + H^2 d(\tau) + O(H^4)\right], \]
where \( a(0) = c(0) = 1 \), \( h(\tau, H) = -h(\tau, -H) \) and write the regular part in \[ \ref{14} \] as,
\[ F_{\text{reg}}(\tau, H) = A(\tau) + H^2 B(\tau) + O(H^4). \]
As shown in \[ \ref{33} \], the most singular subleading term, contributing to \[ \ref{14} \] is of the order of \( \tau^{9/4} H^2 \sim m^6 \) for the square lattice and \( \tau^{13/4} H^2 \sim m^8 \) for the triangular lattice.

Rewriting \[ \ref{15} \] in the form \[ \ref{14} \] plus regular terms, one obtains
\[ C^{(s)}_\tau = \sqrt{2}, \quad C^{(t)}_\tau = 3^{-1/4} \sqrt{2}, \]
and
\begin{align}
a^{(s)}(\tau) &= 1 - \frac{3}{10} \tau^2 + \frac{117}{1500} \tau^4 + O(\tau^6), \\
a^{(t)}(\tau) &= 1 - \frac{1}{24} \tau^2 + \frac{47}{4032} \tau^4 + O(\tau^6).
\end{align}
The contribution to the regular part reads
\[ A^{(s)}(\tau) = -\frac{2\tau}{\pi} - \frac{\log 2}{2} + \frac{1}{2} \tau - \frac{(1+5 \log 2)}{4\tau} \tau^2 - \frac{1}{\tau^3} + \frac{5(1+6 \log 2)}{6\tau^4} \tau + O(\tau^5), \]
\[ A^{(t)}(\tau) = -\frac{2\tau}{\pi} Cl_2 \left( \frac{\tau}{3} \right) - \frac{4}{3} \log 2 + \frac{1}{5} \tau - \frac{2+3 \log 12}{8\pi^3} \tau^3 + \frac{4+9 \log 12}{288 \pi^3} \tau^2 - \frac{1}{32} \tau + O(\tau^5). \]
Next, with the definition \[ \ref{13} \] the zero-field spontaneous magnetization has the same expression for both lattices
\[ M(\tau, 0) = (1 - k^2)^{1/8}, \quad \tau < 0. \]
Combining this with \[ \ref{2}, \ref{9}, \ref{14} \] and \[ \ref{13} \] one obtains
\[ C_h^{(s)} = -2^{3/16} \tilde{G}_1, \quad C_h^{(t)} = -2^{5/16} 3^{-3/32} \tilde{G}_1, \]
and
\begin{align}
c^{(s)}(\tau) &= 1 + \tau + \frac{15\tau^3}{128} - \frac{9\tau^3}{512} - \frac{4333\tau^4}{98304} + O(\tau^5), \\
c^{(t)}(\tau) &= 1 + \tau - \frac{5\tau^2}{6} + \frac{9\tau^3}{576} + \frac{727\tau^4}{165888} + O(\tau^5).
\end{align}
Finally, consider the zero-field susceptibility. The second field derivative of \[ \ref{14} \] at \( H = 0 \) gives
\[ \chi(\tau) = -\frac{2G C_h^2 c(\tau)^2}{(\sqrt{2} |a(\tau)|)^{7/4}} \left. \frac{\partial^2 F_{\text{sub}}}{\partial H^2} \right|_{H=0} - 2B(\tau) + \frac{\tau a(\tau) b(\tau)}{\sqrt{2} \pi} (1 + \log(2\tau^2 a(\tau))), \]
where \( G = G_2 \) for \( \tau > 0 \) and \( G = \tilde{G}_2 \) for \( \tau < 0 \). No simple closed form expression for the zero-field susceptibility \( \chi(\tau) \) is known. However, the authors of \[ \ref{33} \] obtained remarkable asymptotic expression of \( \chi(\tau) \) for the square lattice for small \( \tau \) to within \( O(\tau^{14}) \) terms with high-precision numerical coefficients. Using their results in \[ \ref{24} \], one obtains
\[ B^{(s)}(\tau) = 0.0520666225469 + 0.0769120341893 \tau + 0.0360200462309 \tau^2 + O(\tau^3), \]
and
\[ B^{(t)}(\tau) = 0.0247805582(2) + O(\tau^2), \quad \mu_h = -0.010475(1) \]
No similar expansion for \( \tau \approx 0 \) is available for the triangular lattice. We used our data for \( \tau = 0 \) to estimate
\[ \mu^{(s)}(\tau) = 0.071868670814 \]
and the coefficient \( d(\tau) = e_h + O(\tau) \) in \[ \ref{16} \]
\[ e^{(s)}_h = -0.00728(30), \quad e^{(t)}_h = +0.00129(1). \]
which is in agreement with $e_h^{(s)} = -0.00727(15)$ from \cite{25}.

The above expressions were used to analyze our extensive numerical data and extract the necessary information to obtain the universal scaling function. The results are summarized in Table 1. For convenience of comparison we quoted the field theory results from \cite{11}. Earlier exact numerical results for the same quantities are also quoted (whenever available).

To conclude, we have implemented Baxter’s variational corner transfer matrix approach to obtain the universal scaling function for the Ising model in a magnetic field on the square and triangular lattice, as shown in Fig. 1 and Table 1. The numerical data is seen to be in remarkable agreement with the field theory results obtained by Fonseca and Zamolodchikov \cite{11}. We also report a remarkable agreement (11 to 14 digits) between our numerical values for $G_1$, $G_2$ and $G_3$ and the classic exact results of Barouch, McCoy, Tracy and Wu \cite{5,12,13} and a similar agreement between the values $\Phi_0$ and $\Phi_1$ and the exact predictions \cite{15,24} of Zamolodchikov’s integrable $E_8$ field theory \cite{8}. Interestingly, this $E_8$ symmetry has now been observed in experiments on the transverse Ising chain \cite{35}.

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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|}
\hline
 & Tr. lat. CTM & Sq. lat. CTM & IFT \cite{11} & References \\
\hline $G_1$ & $-1.35738341706595(2)$ & $-1.357383417066(1)$ & $-1.35738335$ & \cite{5} \\
$G_2$ & $-0.048953289720(2)$ & $-0.048953289720(1)$ & $-0.0489589$ & \cite{12,13,33} \\
$\tilde{G}_4$ & $0.388639290(1)$ & $0.38863932(3)$ & $0.0388954$ & \cite{37}; \cite{28} \\
$G_4$ & $-0.068362121(1)$ & $-0.068362119(2)$ & $-0.06850601$ & \cite{37}; \cite{24} \\
$\tilde{G}_5$ & $0.18388371(1)$ & $0.18388370(1)$ & $0.184530$ & \cite{25} \\
$\tilde{G}_6$ & $-0.659170(1)$ & $-0.659171(1)$ & $-0.662150$ & \cite{25} \\
$G_7$ & $2.93763(2)$ & $2.937665(3)$ & $2.9520$ & \cite{25} \\
$\tilde{G}_8$ & $-15.57(2)$ & $-15.61(1)$ & $-15.690$ & \cite{25} \\
$G_9$ & $-1.84522807283(1)$ & $-1.845228072832(2)$ & $-1.8452282383$ & \cite{12,13,33} \\
$G_4$ & $8.3337117508(1)$ & $8.333711750(5)$ & $8.334100$ & \cite{25} \\
$G_6$ & $95.168997(3)$ & $95.16896(1)$ & $95.188400$ & \cite{25} \\
$G_8$ & $145.72(2)$ & $145.62(3)$ & $145.2100$ & \cite{25} \\
\hline $\Phi_0$ & $-1.197733383797993(1)$ & $-1.197733383797993(1)$ & $-1.19773200$ & \cite{15} \\
$\Phi_1$ & $-0.3188101248906(1)$ & $-0.318810124891(1)$ & $-0.31881920$ & \cite{29,34} \\
$\Phi_2$ & $0.1108861966832(3)$ & $0.110886196683(2)$ & $0.11089150$ & \cite{25} \\
$\Phi_3$ & $0.01642689465(1)$ & $0.01642689465(2)$ & $0.01642520$ & \cite{25} \\
$\Phi_4$ & $-2.6399783(1) \times 10^{-4}$ & $-2.6399781(1) \times 10^{-4}$ & $-2.640 \times 10^{-4}$ & \cite{25} \\
$\Phi_5$ & $-5.140526(1) \times 10^{-4}$ & $-5.140526(1) \times 10^{-4}$ & $-5.140 \times 10^{-4}$ & \cite{25} \\
$\Phi_6$ & $2.08866(1) \times 10^{-4}$ & $2.08865(1) \times 10^{-4}$ & $2.090 \times 10^{-4}$ & \cite{25} \\
$\Phi_7$ & $-4.481969(2) \times 10^{-5}$ & $-4.4819(1) \times 10^{-5}$ & $-4.480 \times 10^{-5}$ & \cite{25} \\
$\Phi_8$ & $3.194(1) \times 10^{-7}$ & $3.16 \times 10^{-7}$ & $3.160 \times 10^{-7}$ & \cite{25} \\
$\Phi_9$ & $4.313(1) \times 10^{-6}$ & $4.31 \times 10^{-6}$ & $4.310 \times 10^{-6}$ & \cite{25} \\
$\Phi_{10}$ & $-1.987(2) \times 10^{-6}$ & $-1.99 \times 10^{-6}$ & $-1.990 \times 10^{-6}$ & \cite{25} \\
$\Phi_{11}$ & $4.32(1) \times 10^{-7}$ & $4.32 \times 10^{-7}$ & $4.310 \times 10^{-7}$ & \cite{25} \\
\hline
\end{tabular}
\caption{Numerical values of the coefficients $G_n$, $\tilde{G}_n$, $\Phi_n$}
\end{table}

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[1] J. Cardy, \textit{Scaling and Renormalization in Statistical Physics} (Cambridge University Press, 1996).
[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. B \textbf{241}, 333 (1984).
[3] L. Onsager, Phys. Rev. \textbf{65}, 117 (1944).
[4] C. N. Yang, Phys. Rev. \textbf{85}, 808 (1952).
[5] B. McCoy and T. T. Wu, \textit{The Two-Dimensional Ising
Model (Harvard University Press, 1973).

[6] R. J. Baxter, Exactly Solved Models in Statistical Mechanics (Academic, London, 1982).

[7] A. Aharony and M. E. Fisher, Phys. Rev. B 27, 4394 (1983).

[8] A. B. Zamolodchikov, Int. J. Mod. Phys. A 4, 4235 (1989).

[9] G. F. Newell, Physical Review 79, 876 (1950).

[10] A. Aharony and M. E. Fisher, Phys. Rev. Lett. 45, 679 (1980).

[11] P. Fonseca and A. Zamolodchikov, J. Stat. Phys. 110, 527 (2003).

[12] E. Barouch, B. M. McCoy, and T. T. Wu, Phys. Rev. Lett. 31, 1409 (1973).

[13] C. A. Tracy and B. M. McCoy, Phys. Rev. Lett. 31, 1500 (1973).

[14] T. T. Wu, B. M. McCoy, C. A. Tracy, and E. Barouch, Phys. Rev. B 13, 316 (1976).

[15] V. A. Fateev, Phys. Lett. B 324, 45 (1994).

[16] G. Delfino and G. Mussardo, Nucl. Phys. B 455, 724 (1995).

[17] S. O. Warnaar, B. Nienhuis, and K. A. Seaton, Phys. Rev. Lett. 69, 710 (1992).

[18] S. O. Warnaar, P. A. Pearce, K. A. Seaton, and B. Nienhuis, J. Stat. Phys. 74, 469 (1994).

[19] V. V. Bazhanov, B. Nienhuis, and S. O. Warnaar, Phys. Lett. B 322, 198 (1994).

[20] A. V. Smilga, Phys. Rev. D 55, R443 (1997).

[21] M. T. Batchelor and K. A. Seaton, J. Phys. A 30, L479 (1997).

[22] G. Delfino, J. Phys. A 37, R45 (2004).

[23] J. W. Essam and D. L. Hunter, J. Phys. C 1, 392 (1968).

[24] S.-Y. Zinn, S.-N. Lai, and M. E. Fisher, Phys. Rev. E 54, 1176 (1996).

[25] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Phys. A 34, 2923 (2001).

[26] M. Caselle and M. Hasenbusch, Nucl. Phys. B 579, 667 (2000).

[27] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Phys. A 33, 8171 (2000).

[28] S. B. Rutkevich, Phys. Rev. B 60, 14525 (1999).

[29] V. V. Mangazeev, M. T. Batchelor, V. V. Bazhanov, and M. Yu. Dudalev, J. Phys. A 42, 042005 (2009).

[30] R. J. Baxter, J. Math. Phys. 9, 650 (1968).

[31] R. J. Baxter, J. Stat. Phys. 19, 461 (1978).

[32] T. Nishino and K. Okunishi, J. Phys. Soc. Japan 66, 3040 (1997).

[33] W. P. Orrick, B. Nickel, A. J. Guttmann, and J. H. H. Perk, J. Stat. Phys. 102, 795 (2001).

[34] V. Fateev, S. Lukyanov, A. Zamolodchikov, and Al. Zamolodchikov, Nucl. Phys. B 516, 652 (1998).

[35] M. Caselle, M. Hasenbusch, A. Pelissetto, and E. Vicari, J. Phys. A 35, 4861 (2002).

[36] R. Coldea, D. A. Tennant, E. M. Wheeler, E. Wawrzynska, D. Prabhakaran, M. Telling, K. Habicht, P. Smeibidl, and K. Kiefer, Science 327, 177 (2010).

[37] B. M. McCoy and T. T. Wu, Phys. Rev. B 18, 4886 (1978).