String triality, black hole entropy and Cayley’s hyperdeterminant

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ABSTRACT

The four-dimensional $N = 2$ STU model of string compactification is invariant under an $SL(2, Z)_S \times SL(2, Z)_T \times SL(2, Z)_U$ duality acting on the dilaton/axion $S$, complex Kahler form $T$ and the complex structure fields $U$, and also under a string/string/string triality $S \leftrightarrow T \leftrightarrow U$. The model admits an extremal black hole solution with four electric and four magnetic charges whose entropy must respect these symmetries. It is given by the square root of the hyperdeterminant introduced by Cayley in 1845. This also features in three-qubit quantum entanglement.
1 Introduction

An interesting subsector of string compactification to four dimensions is provided by the STU model whose low energy limit is described by $N = 2$ supergravity coupled to three vector multiplets. One may regard it as a truncation of an $N = 4$ theory obtained by compactifying the heterotic string on $T^6$ where $S, T, U$ correspond to the dilaton/axion, complex Kahler form and complex structure fields respectively. It exhibits an $SL(2, Z)_S$ strong/weak coupling duality and an $SL(2, Z)_T \times SL(2, Z)_U$ target space duality. By string/string duality, this is equivalent to a Type IIA string on $K3 \times T^2$ with $S$ and $T$ exchanging roles [1, 2, 3]. Moreover, by mirror symmetry this is in turn equivalent to a Type IIB string on the mirror manifold with $T$ and $U$ exchanging roles. Hence the truncated theory has a combined $[SL(2, Z)]^3$ duality and complete $S - T - U$ triality symmetry [4]. Alternatively, one may simply start with this $N = 2$ theory directly as an interesting four-dimensional supergravity in its own right, as described in section 2.

The model admits extremal black holes solutions carrying four electric and magnetic charges. In section 3 we organize these 8 charges into the $2 \times 2 \times 2$ hypermatrix and display the $S - T - U$ symmetric Bogomolnyi mass formula [4].

Associated with this hypermatrix is a hyperdeterminant, discussed in section 4, first introduced by Cayley in 1845 [5].

The black hole entropy, first calculated in [6], is quartic in the charges and must be invariant under $[SL(2, Z)]^3$ and under triality. The main result of the present paper, given in section 5, is to show that this entropy is given by the square root of Cayley’s hyperdeterminant.

The hyperdeterminant also makes it appearance in quantum information theory [7] as the measure of three qubit entanglement known as the 3-tangle [8], which we briefly review in section 6.

2 The STU model

Consider the three complex scalars axion/dilaton field $S$, the complex Kahler form field $T$ and the complex structure field $U$

$$
S = S_1 + iS_2, \\
T = T_1 + iT_2, \\
U = U_1 + iU_2.
$$

(2.1)

This complex parameterization allows for a natural transformation under the various $SL(2, Z)$ symmetries. The action of $SL(2, Z)_S$ is given by

$$
S \rightarrow \frac{aS + b}{cS + d},
$$

(2.2)

where $a, b, c, d$ are integers satisfying $ad - bc = 1$, with similar expressions for $SL(2, Z)_T$ and $SL(2, Z)_U$. Defining the matrices $\mathcal{M}_S$, $\mathcal{M}_T$ and $\mathcal{M}_U$ via

$$
\mathcal{M}_S = \frac{1}{S_2} \begin{pmatrix}
1 & S_1 \\
S_1 & |S|^2
\end{pmatrix},
$$

(2.3)
the action of $SL(2, \mathbb{Z})_{S}$ now takes the form
\[ \mathcal{M}_{S} \rightarrow \omega_{S}^{T} \mathcal{M}_{S} \omega_{S}, \quad (2.4) \]
where
\[ \omega_{S} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}, \quad (2.5) \]
with similar expressions for $\mathcal{M}_{T}$ and $\mathcal{M}_{U}$. We also define the $SL(2, \mathbb{Z})$ invariant tensors
\[ \epsilon_{S} = \epsilon_{T} = \epsilon_{U} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (2.6) \]

Starting from the heterotic string, the bosonic action for the graviton $g_{\mu\nu}$, dilaton $\eta$, two-form $B_{\mu\nu}$ four $U(1)$ gauge fields $A_{5}^{a}$ and two complex scalars $T$ and $U$ is [4]
\[ I_{STU} = \frac{1}{16\pi G} \int d^{4}x \sqrt{-g} e^{-\eta} \left[ R_{g} + g_{\mu\nu} \partial_{\mu} \eta \partial_{\nu} \eta - \frac{1}{12} g^{\mu\lambda \nu\rho} g^{\rho\sigma} H_{\mu\lambda \rho} H_{\nu\tau \sigma} \right. \]
\[ + \frac{1}{4} \text{Tr}(\partial \mathcal{M}_{T}^{-1} \partial \mathcal{M}_{T}) + \frac{1}{4} \text{Tr}(\partial \mathcal{M}_{U}^{-1} \partial \mathcal{M}_{U}) \]
\[ - \frac{1}{4} F_{S\mu\nu}^{a} (\mathcal{M}_{T} \times \mathcal{M}_{U}) F_{S\mu\nu}^{a} \right], \quad (2.7) \]
where the metric $g_{\mu\nu}$ is related to the four-dimensional canonical Einstein metric $g_{\mu\nu}^{c}$ by
\[ g_{\mu\nu} = e^{\eta} g_{\mu\nu}^{c} \]
and where $H_{\mu\nu\rho} = 3(\partial_{[\mu} B_{\nu\rho]} - \frac{1}{2} A_{5}^{a} \epsilon^{T} (\epsilon_{T} \times \epsilon_{U}) F_{S\nu\rho})$. (2.8)

This action is manifestly invariant under $T$-duality and $U$-duality, with
\[ F_{S\mu\nu} \rightarrow (\omega_{T}^{-1} \times \omega_{U}^{-1}) F_{S\mu\nu}, \quad \mathcal{M}_{T/U} \rightarrow \omega_{T/U}^{T} \mathcal{M}_{T/U} \omega_{T/U}, \quad (2.9) \]
and with $\eta$, $g_{\mu\nu}$ and $B_{\mu\nu}$ inert. Its equations of motion and Bianchi identities (but not the action itself) are also invariant under $S$-duality (2.2), with $T$ and $g^{c}_{\mu\nu}$ inert and with
\[ \left( \begin{array}{c} F_{S\mu\nu}^{a} \\ F_{S\mu\nu}^{a} \end{array} \right) \rightarrow \omega_{S}^{-1} \left( \begin{array}{c} F_{S\mu\nu}^{a} \\ F_{S\mu\nu}^{a} \end{array} \right), \quad (2.10) \]
where
\[ \bar{F}_{S\mu\nu}^{a} = -S_{2}[(\mathcal{M}_{T}^{-1} \times \mathcal{M}_{U}^{-1})(\epsilon_{T} \times \epsilon_{U})]^{a}_{b} * F_{S\mu\nu}^{b} - S_{1} F_{S\mu\nu}^{a}, \quad (2.11) \]
where the axion field $a$ is defined by
\[ \epsilon^{\mu\nu\rho\sigma} \partial_{\sigma} a = \sqrt{-g} e^{-\eta} g^{\mu\lambda \nu\rho} g^{\rho\sigma} H_{\sigma\lambda \tau}, \quad (2.12) \]
and where $S = S_{1} + iS_{2} = a + ie^{-\eta}$.

Thus $T$-duality transforms Kaluza-Klein electric charges $(F_{S}^{3}, F_{S}^{4})$ into winding electric charges $(F_{S}^{1}, F_{S}^{2})$ (and Kaluza-Klein magnetic charges into winding magnetic charges), $U$-duality transforms the Kaluza-Klein and winding electric charge of one circle $(F_{S}^{3}, F_{S}^{2})$ into those of the other $(F_{S}^{4}, F_{S}^{1})$ (and similarly for the magnetic charges) but $S$-duality
transforms Kaluza-Klein electric charge \((F_S^3, F_S^4)\) into winding magnetic charge \((\tilde{F}_S^3, \tilde{F}_S^4)\) (and winding electric charge into Kaluza-Klein magnetic charge). In summary we have \(SL(2, Z)_T \times SL(2, Z)_U\) and \(T \leftrightarrow U\) off-shell but \(SL(2, Z)_S \times SL(2, Z)_T \times SL(2, Z)_U\) and an \(S-T-U\) interchange on-shell.

One may also consider the Type IIA action \(I_{TUS}\) and the Type IIB action \(I_{UST}\) obtained by cyclic permutation of the fields \(S, T, U\). Finally, one may consider an action [6] where the \(S\), \(T\) and \(U\) fields enter democratically with a prepotential

\[
F = STU
\]  

(2.13)

which off-shell has the full \(STU\) interchange but none of the \(SL(2, Z)\). All four versions are on-shell equivalent.

3 The Bogomolny Spectrum

Following [4], it is now straightforward to write down an \(S-T-U\) symmetric Bogomolny mass formula. Let us define electric and magnetic charge vectors \(\alpha_s^a\) and \(\beta_s^a\) associated with the field strengths \(F_S^a\) and \(\tilde{F}_S^a\) in the standard way. The electric and magnetic charges \(Q_s^a\) and \(P_s^a\) are given by

\[
F_{So}\sim \frac{Q_s^a}{r^2} \quad \star F_{So}\sim \frac{P_s^a}{r^2},
\]  

(3.1)

giving rise to the charge vectors

\[
\begin{pmatrix}
\alpha_s^a \\
\beta_s^a
\end{pmatrix} = \begin{pmatrix}
S_2^{(0)} M_T^{-1} \times M_U^{-1} \\
0
\end{pmatrix} \begin{pmatrix}
S_1^{(0)} \epsilon_T \times \epsilon_U \\
-\epsilon_T \times \epsilon_U
\end{pmatrix} \begin{pmatrix}
Q_s^b \\
P_s^b
\end{pmatrix}.
\]  

(3.2)

For our purpose it is useful to define a \(2 \times 2 \times 2\) array \(a_{ijk}\) via

\[
\begin{pmatrix}
a_{000} \\
a_{001} \\
a_{010} \\
a_{011} \\
a_{100} \\
a_{101} \\
a_{110} \\
a_{111}
\end{pmatrix} = \begin{pmatrix}
-\beta_1^1 \\
-\beta_2^1 \\
-\beta_3^1 \\
-\beta_4^1 \\
\alpha_1^1 \\
\alpha_2^1 \\
\alpha_3^1 \\
\alpha_4^1
\end{pmatrix},
\]  

(3.3)

transforming as

\[
a^{ijk} \rightarrow \omega_s^i \omega_t^j \omega_u^k a^{lmn}.
\]  

(3.4)

Then the mass formula is

\[
m^2 = \frac{1}{16} a^T (M_S^{-1} M_T^{-1} M_U^{-1} - M_S^{-1} \epsilon_T \epsilon_U - \epsilon_S M_T^{-1} \epsilon_U - \epsilon_S \epsilon_T M_U^{-1}) a.
\]  

(3.5)

This is consistent with the general \(N = 2\) Bogomolny formula [10]. Although all theories have the same mass spectrum, there is clearly a difference of interpretation with electrically charged elementary states in one picture being solitonic monopole or dyon states in the other.

This \(2 \times 2 \times 2\) array \(a_{ijk}\) is an example a “hypermatrix”, a term coined by Cayley in 1845 [5] where he also introduced a “hyperdeterminant”.
4 The Cayley hyperdeterminant

In analogy with the determinant of a $2 \times 2$ matrix $a^{ij}$

$$\det a_2 = \frac{1}{2} \epsilon^{ij} e^{lm} a_{il} a_{jm}$$

$$= a_{00} a_{11} - a_{01} a_{10} \quad (4.1)$$

the hyperdeterminant of $a_{ijk}$ is defined to be

$$\text{Det} \ a_3 = -\frac{1}{2} \epsilon^{ii'} \epsilon^{jj'} \epsilon^{kk'} e^{mm'} e^{nn'} e^{pp'} a_{i'j'k'm} a_{npk'n'p'm'}$$

$$= a_{000} a_{111} + a_{001} a_{110} + a_{010} a_{101} + a_{100} a_{011}$$

$$-2(a_{000} a_{001} a_{110} a_{111} + a_{000} a_{010} a_{101} a_{111} + a_{000} a_{100} a_{011} a_{110} + a_{000} a_{110} a_{101} a_{111})$$

$$+ 4 (a_{000} a_{111} a_{101} a_{110} + a_{000} a_{100} a_{011} a_{111}) \quad (4.2)$$

The hyperdeterminant vanishes iff the following system of equations in six unknowns $u^i, v^j, w^k$ has a nontrivial solution, not allowing any of the pairs to be both zero:

$$a_{ijk} u^i v^j = 0$$

$$a_{ijk} u^i w^k = 0$$

$$a_{ijk} v^j w^k = 0 \quad (4.3)$$

Other useful identities are provided by the polynomial symmetric under permutation of the four indices [11]:

$$P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_4^2 y_4^2$$

$$-4 x_1 x_2 x_3 x_4 - 4 y_1 y_2 y_3 y_4$$

$$-2 x_1 y_1 x_2 y_2 - 2 x_1 y_1 x_3 y_3 - 2 x_1 y_1 x_4 y_4 - 2 x_2 y_2 x_3 y_3 - 2 x_2 y_2 x_4 y_4 - 2 x_3 y_3 x_4 y_4 \quad (4.4)$$

which obeys

$$P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_1 y_1 + x_2 y_2 + x_3 y_3 + x_4 y_4)^2$$

$$-4 (x_1 x_2 + y_3 y_4) (x_3 x_4 + y_1 y_2) \quad (4.5)$$

and

$$y_1^2 P(x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4) = (x_1 y_1^2 - x_2 y_2 y_1 - x_3 y_3 y_1 - x_4 y_4 y_1 - 2 x_2 x_3 x_4)^2$$

$$-4 (x_2 x_3 + y_1 y_4) (x_2 x_4 + y_1 y_3) (x_3 x_4 + y_1 y_2) \quad (4.6)$$

Comparison with (4.2) yields

$$\text{Det} \ a_3 = P(-a_{000}, a_{110}, a_{101}, a_{011}, -a_{111}, a_{001}, a_{010}, a_{100}) \quad (4.7)$$

For our purposes, the important properties of the hyperdeterminant are that it is a quartic invariant under $[SL(2, Z)]^3$ and under triality.
5 Black hole entropy

The STU model admits extremal black hole solutions satisfying the Bogomolnyi mass formula. As usual, their entropy is given by one quarter the area of the event horizon. However, to calculate this area requires evaluating the mass not with the asymptotic values of the moduli, but with their frozen values on the horizon which are fixed in terms of the charges [9]. This ensures that the entropy is moduli-independent, as it should be. The relevant calculation was carried out in [6] for the model with the STU prepotential. The electric and magnetic charges of that paper are denoted \((p^0, q_0), (p^1, q_1), (p^2, q_2), (p^3, q_3)\) with \(O(2,2)\) scalar products

\[
p^2 = (p^0)^2 + (p^1)^2 - (p^2)^2 - (p^3)^2,
q^2 = (q_0)^2 + (q_1)^2 - (q_2)^2 - (q_3)^2,
p \cdot q = (p^0 q_0) + (p^1 q_1) + (p^2 q_2) + (p^3 q_3).
\]

In these variables, the entropy is given by

\[
S = \pi \left( W(p^A, q_\Lambda) \right)^{1/2}
\]

where

\[
W(p^A, q_\Lambda) = -(p \cdot q)^2 + 4((p^1 q_1)(p^2 q_2) + (p^1 q_1)(p^3 q_3) + (p^3 q_3)(p^2 q_2)) - 4p^0 q_1 q_2 q_3 + 4q_0 p^1 p^2 p^3.
\]

The function \(W(p^A, q_\Lambda)\) is symmetric under transformations: \(p^1 \leftrightarrow p^2 \leftrightarrow p^3\) and \(q_1 \leftrightarrow q_2 \leftrightarrow q_3\). For the solution to be consistent we have to require \(W > 0\), otherwise the model is not defined.

If we now make the identifications

\[
\begin{pmatrix}
a_{000} \\
a_{001} \\
a_{010} \\
a_{011} \\
a_{100} \\
a_{101} \\
a_{110} \\
a_{111}
\end{pmatrix} = \begin{pmatrix}
-p^0 \\
-p^1 \\
-p^2 \\
-q_3 \\
p^3 \\
q_2 \\
q_1 \\
-q_0
\end{pmatrix},
\]

we recognize from (4.2) that

\[
W = -\text{Det } a_3
\]

and hence the black hole entropy is given by

\[
S = \pi \sqrt{-\text{Det } a_3}
\]

Some examples of supersymmetric black hole solutions [12] are provided by the electric Kaluza-Klein black hole with \(\alpha = (1, 0, 0, 0)\) and \(\beta = (0, 0, 0, 0)\); the electric winding black hole with \(\alpha = (0, 0, 0, -1)\) and \(\beta = (0, 0, 0, 0)\); the magnetic Kaluza-Klein black hole with
\( \alpha = (0, 0, 0, 0) \) and \( \beta = (0, -1, 0, 0) \); the magnetic winding black hole with \( \alpha = (0, 0, 0, 0) \) and \( \beta = (0, 0, -1, 0) \). These are characterized by a scalar-Maxwell coupling parameter \( a = \sqrt{3} \). By combining these 1-particle states, we may build up 2-, 3- and 4-particle bound states at threshold [12, 4]. For example \( \alpha = (1, 0, 0, -1) \) and \( \beta = (0, 0, 0, 0) \) with \( a = 1 \); \( \alpha = (1, 0, 0, -1) \) and \( \beta = (0, -1, 0, 0) \) with \( a = 1/\sqrt{3} \); \( \alpha = (1, 0, 0, -1) \) and \( \beta = (0, -1, -1, 0) \) with \( a = 0 \). The 1-, 2- and 3-particle states all yield vanishing contributions to \( \text{Det} \ a_3 \). A non-zero value is obtained for the 4-particle example, however, which is just the Reissner-Nordstrom black hole.

### 6 3-qubit quantum entanglement

Interestingly enough, Cayley’s hyperdeterminant also makes its appearance in quantum information theory.

Let the system ABC be in a pure state \( |\Psi\rangle \), and let the components of \( |\Psi\rangle \) in the standard basis be \( a_{ijk} \):

\[
|\Psi\rangle = \sum_{ijk} a_{ijk} |ijk\rangle \tag{6.1}
\]

or

\[
|\Psi\rangle = a_{000}|000\rangle + a_{001}|001\rangle + a_{010}|010\rangle + a_{011}|011\rangle + a_{100}|100\rangle + a_{101}|101\rangle + a_{110}|110\rangle + a_{111}|111\rangle \tag{6.2}
\]

In this context the \( a_{ijk} \) are complex numbers rather than integers and the symmetry is \( [SL(2, C)]^3 \) rather than \( [SL(2, Z)]^3 \). The three way entanglement of the three qubits A, B and C is given by the 3-tangle of Coffman, Kondu and Wooters [8]

\[
\tau_{ABC} = 2|\epsilon^{ii'} \epsilon^{jj'} \epsilon^{kk'} \epsilon^{mm'} \epsilon^{nn'} \epsilon^{pp'} a_{ijk}a_{i'j'm}a_{nkp}a_{n'p'm'}| \tag{6.3}
\]

The 3-tangle is maximal for the GHZ state \( |000\rangle + |111\rangle \) [13] and vanishes for the states \( p|100\rangle + q|010\rangle + r|001\rangle \). The relation between three qubit quantum entanglement and the Cayley hyperdeterminant was pointed out by Miyake and Wadati [7].

Thus Cayley’s hyperdeterminant provides an interesting connection, at least at the level of mathematics, between string theory and quantum entanglement. Other mathematical similarities are provided by the division algebras [14] and by twistors [15]. What about physics? The near horizon geometry of the black holes is \( AdS^2 \times S^2 \) and one might expect a relation between the black hole entropy and the entanglement entropy of the conformal quantum mechanics that lives on the boundary [16], although the nature of this particular AdS/CFT duality is not well-understood [17]. In any event, the 3-tangle is not the same as the entropy of entanglement [18]. So the appearance of the Cayley hyperdeterminant in these two different contexts of stringy black hole entropy and 3-qubit quantum entanglement remains, for the moment, a purely mathematical coincidence.

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