THE INFLUENCE OF THE PHYSICAL COEFFICIENTS OF A BRESSE SYSTEM WITH ONE SINGULAR LOCAL VISCOUS DAMPING IN THE LONGITUDINAL DISPLACEMENT ON ITS STABILIZATION

MOHAMMAD AKIL$^1$ AND HAIDAR BADAWI$^2$

Abstract. In this paper, we investigate the stabilization of a linear Bresse system with one singular local frictional damping acting in the longitudinal displacement, under fully Dirichlet boundary conditions. First, we prove the strong stability of our system. Next, using a frequency domain approach combined with the multiplier method, we establish the exponential stability of the solution if and only if the three waves have the same speed of propagation. On the contrary, we prove that the energy of our system decays polynomially with rates $t^{-1}$ or $t^{-\frac{1}{2}}$.

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$^1$ Université Savoie Mont Blanc, Laboratoire LAMA, Chambéry-France
$^2$ Université Polytechnique Hauts-de-France (UPHF-LAMAV), Valenciennes, France

E-mail addresses: mohammad.akil@univ-smb.fr, Haidar.Badawi@etu.uphf.fr.

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1. Introduction

In this paper, we investigate the stability of Bresse system with one discontinuous local frictional damping in the longitudinal displacement. More precisely, we consider the following system:

\[
\begin{align*}
\rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi + lw)_x - lk_3 (wx - l\varphi) = 0, & \quad (x, t) \in (0, L) \times (0, \infty), \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi + lw) = 0, & \quad (x, t) \in (0, L) \times (0, \infty), \\
\rho_1 w_{tt} - k_3 (wx - l\varphi)_x + lk_1 (\varphi_x + \psi + lw) + a(x)w_t = 0, & \quad (x, t) \in (0, L) \times (0, \infty),
\end{align*}
\]

with the following Dirichlet boundary conditions

\[
(1.2) \quad \varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = w(0, t) = w(L, t) = 0, \quad t > 0.
\]

and the following initial conditions

\[
(1.3) \quad \begin{cases}
\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad \psi(x, 0) = \psi_0(x), \quad x \in (0, L), \\
\psi_t(x, 0) = \psi_1(x), \quad w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in (0, L),
\end{cases}
\]

where \( \rho_1, \rho_2, k_1, k_2, k_3, l \) and \( L \) are positive real numbers. We suppose that there exists \( 0 < \beta < L \) and a positive constant \( a_0 \) such that

\[
(1.4) \quad a(x) = \begin{cases} 
a_0 & \text{if } x \in (0, \beta), \\
0 & \text{if } x \in (\beta, L).
\end{cases}
\]

\[\text{Figure 1. Geometric description of the function } a(x).\]

The notion of indirect damping mechanisms has been introduced by Russell in [35] and since this time, it retains the attention of many authors. In particular, the fact that only one equation of the coupled system is damped refers to the so-called class of "indirect" stabilization problems initiated and studied in [4, 5, 6] and further studied by many authors, see for instance [8, 28, 39] and the rich references therein.

The Bresse system is a model for arched beams, see [26, Chap. 6]. It can be expressed by the equations of motion:

\[
(1.5) \quad \begin{cases}
\rho_1 \varphi_{tt} = Q_x + lN, \\
\rho_2 \psi_{tt} = M_x - Q, \\
\rho_1 w_{tt} = N_x - lQ - a(x)\omega_t,
\end{cases}
\]

where \( N = k_3 (wx - l\varphi) \) is the axial force, \( Q = k_1 (\varphi_x + \psi + lw) \) is the shear force, and \( M = k_2 \psi_x \) is the bending moment. The functions \( \varphi, \psi, \) and \( w \) are respectively the vertical, shear angle, and longitudinal displacements. Here \( \rho_1 = \rho A, \rho_2 = \rho I, k_1 = kGA, k_3 = EA, k_2 = EI \) and \( l = \frac{R}{3} \), in which \( \rho \) is the density of the material, \( E \) the modulus of the elasticity, \( G \) the shear modulus, \( k \) the shear factor, \( A \) the cross-sectional area, \( I \) the second moment of area of the cross section, \( R \) the radius of the curvature, and \( l \) the curvature. Moreover, \( F_1, F_2, \) and \( F_3 \) are the external forces.
There are several publications concerning the stabilization of Bresse system with frictional or another kinds of damping (see [1], [2], [9], [10], [16], [17], [18], [19], [20], [22], [21], [23], [30], [29], [31], [32], [36] and [38]). We note that by neglecting $w (l \to 0)$ in (1.5), the Bresse system reduces to the following conservative Timoshenko system:

$$\rho_1 \ddot{\varphi} - k_1 (\varphi_x + \psi) = 0,$$

$$\rho_2 \ddot{\psi} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi) = 0.$$  

There are also several publications concerning the stabilization of Timoshenko system with different kinds of damping (see [3], [7], [12], [13] and [37]).

Among this vast literature let us recall some specific results on the Bresse systems.

In 2010, Wehbe and Youssef in [38] studied the stability of an elastic Bresse system with two locally distributed frictional dampings on shear angle and longitudinal displacements, under fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions; they showed that the system is exponential stable if and only if the equations of the vertical displacement and rotation angle have the same wave speeds of propagation. In case that the wave speeds of the equations are different, they obtained a polynomial decay rate. In 2011, Alabau et al. in [9] studied the stability of a Bresse system with one frictional damping on the shear angle displacement, under fully Dirichlet or Dirichlet-Neumann-Neumann conditions; they showed that the system is exponential stable if and only if the three equations have the same wave speeds of propagation. On the contrary, they proved that the solution of the system decays polynomially with rates $t^{-\frac{3}{4}+\epsilon}$ or $t^{-6+\epsilon}$, where $\epsilon > 0$. In 2012, Noun and Wehbe in [32] studied the stability of a Bresse system with one local frictional damping on the shear angle displacement, under fully Dirichlet or Dirichlet-Neumann-Neumann boundary conditions; they showed that the system is exponential stable if and only if the three equations have the same wave speeds of propagation. On the contrary, they proved that the energy of the system decays polynomially with different rates. In 2013, Soriano et al. in [36] studied the asymptotic stability of a Bresse system with a nonlinear frictional damping on the shear angle displacement, and nonlinear localized damping in the vertical and longitudinal displacement; they proved the asymptotic stability of the system. In 2015, Alves et al in [10] studied the stability of a Bresse system with two frictional dampings on vertical and longitudinal displacements, under Dirichlet-Neumann-Neumann boundary conditions; they showed that the system is exponential stable if and only if the equations of the vertical displacement and longitudinal displacement have the same wave speeds of propagation. In case that the wave speeds of the equations are different, they proved that the solution decays polynomially to zero with optimal decay rate. In 2018, Afilal et al. in [2] studied the stability of a Bresse system with global frictional damping in the longitudinal displacement, under mixed boundary conditions of the form

$$\begin{align*}
\varphi(0, t) &= \psi_x(0, t) = w_x(0, t) = 0, \quad \text{in } (0, \infty), \\
\varphi_x(1, t) &= \psi(1, t) = w(1, t) = 0, \quad \text{in } (0, \infty),
\end{align*}$$

they assumed that the curvature $l$ satisfies

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N} \quad \text{and} \quad l^2 \neq \frac{\rho_2 k_1 + \rho_1 k_2}{\rho_3 k_3} \left( \frac{\pi}{2} + m\pi \right)^2 + \frac{\rho_1 k_1}{\rho_2 (k_1 + k_2)}, \quad \forall m \in \mathbb{Z};$$

they showed that the system is exponential stable if and only if the equations have the same wave speeds of propagation. In case that the wave speeds of the equations are different, they established a polynomial energy decay rate of order $t^{-\frac{3}{4}}$.

In this paper, we extend the results in [2], by assuming that the frictional damping is locally distributed in the longitudinal displacement, under fully Dirichlet boundary conditions and without any condition on the curvature $l$, we also improve the polynomial energy decay rate.

But to the best of our knowledge, it seems that no result in the literature exists concerning the case of Bresse system with one discontinuous local frictional damping in the longitudinal displacement, especially under fully
Dirichlet boundary conditions and without any condition on the curvature $l$. The goal of the present paper is to fill this gap by studying the stability of system (1.1)-(1.3).

This paper is organized as follows: In Section 2, we prove the well-posedness of our system by using semigroup approach. In Section 3, we show the strong stability of our system. Finally, in Section 4, by using the frequency domain approach combining with a specific multiplier method, we establish the exponential stability of the solution if and only if the three waves have the same speed of propagation (i.e., $\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}$ and $k_1 = k_3$). On the contrary, we prove that the energy of our system decays polynomially with the rates:

$$
\begin{align*}
\begin{cases}
  t^{-1} & \text{if } \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \text{ and } k_1 \neq k_3, \\
  t^{-\frac{3}{2}} & \text{if } \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}.
\end{cases}
\end{align*}
$$

2. WELL-POSEDNESS OF THE SYSTEM

In this section, we will establish the well-posedness of system (1.1)-(1.3) by using semigroup approach. The energy of system (1.1)-(1.3) is given by

$$
E(t) = \frac{1}{2} \int_0^L \left( \rho_1 |\varphi|_x^2 + \rho_2 |\psi|_x^2 + \rho_1 |w_t|^2 + k_1 |\varphi|_x + \psi + lw|^2 + k_2 |\psi|_x^2 + k_3 |w_x - l\varphi|^2 \right) dx.
$$

Let $(\varphi, \varphi_t, \psi, \psi_t, w, w_t)$ be a regular solution of system (1.1)-(1.3). Multiplying the equations in (1.1) by $\varphi, \varphi_t$ and $w_t$ respectively, Then using the boundary conditions (2.13) and the definition of $a(x)$ (see (1.4) and Figure 1), we obtain

$$
E'(t) = -\int_0^L a(x)|w_t|^2 dx = -a_0 \int_0^3 |w_t|^2 dx \leq 0.
$$

From (2.1), system (1.1)-(1.3) is dissipative in the sense that its energy is non-increasing with respect to time. Now, we define the following Hilbert space $\mathcal{H}$ by:

$$
\mathcal{H} := (H^1_0(0, L) \times L^2(0, L))^3.
$$

The Hilbert space $\mathcal{H}$ is equipped with the following inner product

$$
(U, U^1)_{\mathcal{H}} = \int_0^L \left\{ k_1 (v_1^1 + v^3 + lv^5)(v_1^1 + v^3 + lv^5) + \rho_1 v^2 \overline{v}^2 + k_2 v_2^3 \overline{v}_x^3 + \rho_2 v^4 \overline{v}^4 + k_3 (v_5^5 - lv^1)(v_5^5 - lv^1) dx + \rho_1 v^6 \overline{v}^6 \right\} dx,
$$

where $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in \mathcal{H}$ and $\tilde{U} = (v_1, v_2, \overline{v}_1, v_3, \overline{v}_3, v_5, \overline{v}_5)^\top \in \mathcal{H}$. Now, we define the linear unbounded operator $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$ by:

$$
D(A) = \left[ (H^2(0, L) \cap H^1_0(0, L)) \times H^1_0(0, L) \right]^3
$$

and

$$
A \begin{pmatrix}
v^1 \\
v^2 \\
v^3 \\
v^4 \\
v^5 \\
v^6
\end{pmatrix} = \begin{pmatrix}
\frac{k_1}{\rho_1} (v_1^1 + v^3 + lv^5) + \frac{l k_1}{\rho_1} (v^5 - lv^1) \\
\frac{k_1}{\rho_1} (v^1 + v^3 + lv^5) + \frac{l k_1}{\rho_1} (v^5 - lv^1) \\
\frac{k_2}{\rho_2} v_3^3 - \frac{k_1}{\rho_2} (v_1^1 + v_3^3 + lv^5) \\
\frac{k_3}{\rho_1} (v_5^5 - lv^1) - \frac{l k_1}{\rho_1} (v_1^1 + v^3 + lv^5) - \frac{a(x)}{\rho_1} v^6
\end{pmatrix},
$$

for all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$.

In this sequel, $\| \cdot \|$ will denote the usual norm of $L^2(0, L)$.
Now, if $U = (\varphi, \varphi_1, \psi, \psi_1, w, w_1)^T$, then system (1.1)-(1.3) can be written as the following first order evolution equation
\begin{equation}
U_t = AU, \quad U(0) = U_0,
\end{equation}
where $U_0 = (\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1)^T \in \mathcal{H}$.

**Proposition 2.1.** The unbounded linear operator $A$ is m-dissipative in the Hilbert space $\mathcal{H}$.

**Proof.** For all $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A)$, we have
\begin{equation}
\mathcal{R}(AU, U)_{\mathcal{H}} = -\int_0^L a(x) |v^6|^2 \, dx = -a_0 \int_0^\beta |\varphi|^2 \, dx \leq 0.
\end{equation}
which implies that $A$ is dissipative. Let us prove that $A$ is maximal. For this aim, let $F = (f^1, f^2, f^3, f^4, f^5, f^6)^T \in \mathcal{H}$, we look for $U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A)$ unique solution of
\begin{equation}
- AU = F.
\end{equation}
Detailing (2.6), we obtain
\begin{align}
- v^2 &= f^1, \\
- k_1 (v^1 + v^3 + lv^5) - lk_3 (v^2 - lv^1) &= \rho_1 f^2, \\
- v^4 &= f^3, \\
- k_2 v^3 + k_1 (v^1 + v^3 + lv^5) &= \rho_2 f^4, \\
- v^6 &= f^5,
\end{align}
with the following boundary conditions
\begin{equation}
v^1(0) = v^1(L) = v^3(0) = v^3(L) = v^5(0) = v^5(L) = 0.
\end{equation}
Inserting (2.11) in (2.12), we obtain
\begin{equation}
- k_3 (v^5 - lv^1) +lk_1 (v^1 + v^3 + lv^5) = \rho_1 f^6 + a(x)f^5.
\end{equation}
Let $(\phi^1, \phi^2, \phi^3) \in (H^1_0(0, L))^3$. Multiplying (2.8), (2.10) and (2.14) by $\overline{\phi^1}$, $\overline{\phi^2}$ and $\overline{\phi^3}$ respectively, integrating over $(0, L)$, then using formal integrations by parts, we obtain
\begin{equation}
\mathcal{B}((v^1, v^3, v^5), (\phi^1, \phi^2, \phi^3)) = \mathcal{L}((\phi^1, \phi^2, \phi^3)), \quad \forall (\phi^1, \phi^2, \phi^3) \in (H^1_0(0, L))^3,
\end{equation}
where
\begin{align*}
\mathcal{B}((v^1, v^3, v^5), (\phi^1, \phi^2, \phi^3)) &= k_1 \int_0^L (v^1 + v^3 + lv^5)\overline{\phi^1} \, dx -lk_3 \int_0^L (v^2 - lv^1)\overline{\phi^3} \, dx + k_2 \int_0^L v^3 \overline{\phi^2} \, dx + k_1 \int_0^L (v^1 + v^3 + lv^5)\overline{\phi^3} \, dx \\
&\quad + k_3 \int_0^L (v^1 + v^3 + lv^5)\overline{\phi^1} \, dx + k_1 \int_0^L (v^2 - lv^1)\overline{\phi^3} \, dx \\
\mathcal{L}((\phi^1, \phi^2, \phi^3)) &= \rho_1 \int_0^L f^6 \overline{\phi^3} \, dx + \rho_2 \int_0^L f^4 \overline{\phi^2} \, dx + \rho_1 \int_0^L f^6 \overline{\phi^3} \, dx + \int_0^L a(x)f^5 \overline{\phi^3} \, dx.
\end{align*}
It is easy to see that, $\mathcal{B}$ is a sesquilinear, continuous and coercive form on $(H^1_0(0, L))^3 \times (H^1_0(0, L))^3$ and $\mathcal{L}$ is a antisilinear and continuous form on $(H^1_0(0, L))^3$. Then, it follows by Lax-Milgram theorem that (2.15) admits a unique solution $(v^1, v^3, v^5) \in (H^1_0(0, L))^3$. By taking test-functions $(\phi^1, \phi^2, \phi^3) \in (D(0, L))^3$, we see that (2.8), (2.10), (2.14), (2.13) hold in the distributional sense, from which we deduce that $(v^1, v^3, v^5) \in (H^2(0, L) \cap H^1_0(0, L))^3$. Consequently, $U = (v^1, -f^1, v^3, -f^3, v^5, -f^5)^T \in D(A)$ is a unique solution of (2.6). Then, $A$ is an isomorphism and since $\rho(A)$ is open set of $C$ (see Theorem 6.7 (Chapter III) in [25]), we easily get $\mathcal{R}(A) = \mathcal{H}$ for a sufficiently small $\lambda > 0$. This, together with the dissipativeness of $A$, imply that $D(A)$ is dense in $\mathcal{H}$ and that $A$ is m-dissipative in $\mathcal{H}$ (see Theorems 4.5, 4.6 in [33]). The proof is thus complete. □
According to Lumer-Phillips theorem (see [33]), Proposition 2.1 implies that the operator $A$ generates a $C_0$-semigroup of contractions $e^{tA}$ in $\mathcal{H}$ which gives the well-posedness of (2.4). Then, we have the following result:

**Theorem 2.1.** For all $U_0 \in \mathcal{H}$, system (2.4) admits a unique weak solution

$$U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, \mathcal{H}).$$

Moreover, if $U_0 \in D(A)$, then the system (2.4) admits a unique strong solution

$$U(t) = e^{tA}U_0 \in C^0(\mathbb{R}^+, D(A)) \cap C^1(\mathbb{R}^+, \mathcal{H}).$$

3. **Strong Stability**

In this section, we will prove the strong stability of system (1.1)-(1.3). The main result of this section is the following theorem.

**Theorem 3.1.** The $C_0$-semigroup of contraction $(e^{tA})_{t \geq 0}$ is strongly stable in $\mathcal{H}$; i.e., for all $U_0 \in \mathcal{H}$, the solution of (2.4) satisfies

$$\lim_{t \to +\infty} \|e^{tA}U_0\|_{\mathcal{H}} = 0.$$

**Proof.** Since the resolvent of $A$ is compact in $\mathcal{H}$, then according to Arendt-Batty theorem see (Page 837 in [11]), system (1.1)-(1.3) is strongly stable if and only if $A$ doesn’t have pure imaginary eigenvalues that is $\sigma(A) \cap i\mathbb{R} = \emptyset$. From Proposition 2.1, we have $0 \in \rho(A)$. We still need to show that $\sigma(A) \cap i\mathbb{R} = \emptyset$. For this aim, suppose by contradiction that there exists a real number $\lambda \neq 0$ and $U = (v^1, v^2, v^3, v^4, v^5, v^6)^{\top} \in D(A) \setminus \{0\}$ such that

$$AU = i\lambda U.$$  

Equivalently, we have the following system

$$\begin{align*}
(v^2) &= i\lambda v^1, \\
(k_1 v^1 + v^3 + tv^5)_x + \frac{lk_3}{\rho_1} (v^5 - tv^1) &= i\lambda v^2, \\
(v^4) &= i\lambda v^3, \\
\frac{k_2}{\rho_2} v^3_x - \frac{k_1}{\rho_1} (v^1_x + v^3 + tv^5) &= i\lambda v^4, \\
(v^6) &= i\lambda v^5, \\
\frac{k_3}{\rho_1} (v^6 - tv^1)_x - \frac{lk_1}{\rho_1} (v^1_x + v^3 + tv^5) + \frac{a(x)}{\rho_1} v^6 &= i\lambda v^6.
\end{align*}$$

From (2.5) and (3.1), we obtain

$$0 = \Re \langle i\lambda U, U \rangle_{\mathcal{H}} = \Re \langle AU, U \rangle_{\mathcal{H}} = - \int_0^L a(x) |v^6|^2 \, dx = -a_0 \int_0^\beta |v^6|^2 \, dx.$$

Thus, we have

$$v^6 = 0 \quad \text{in} \quad (0, \beta).$$

From (3.6), (3.9) and the fact that $\lambda \neq 0$, we get

$$v^5 = 0 \quad \text{in} \quad (0, \beta).$$

Now, from (3.9), (3.10), the regularity of $v^5$ and $v^6$, and the definition of $a(x)$, system (3.2)-(3.7) implies

$$\begin{align*}
\left(\lambda^2 - \frac{l^2 k_3}{\rho_1}\right) v^1 + \frac{k_1}{\rho_1} (v^1_x + v^3)_x &= 0 \quad \text{in} \quad (0, \beta), \\
\left(\lambda^2 - \frac{k_1}{\rho_2}\right) v^3 + \frac{k_2}{\rho_2} v^3_x - \frac{k_1}{\rho_2} v^1_x &= 0 \quad \text{in} \quad (0, \beta), \\
\frac{k_1}{\rho_1} (v^1_x + v^3) &= -\frac{k_3}{\rho_1} v^1, \quad \text{in} \quad (0, \beta).
\end{align*}$$
Inserting (3.13) in (3.11), we obtain
\[ s u^1 + v^1_{xx} = 0 \quad \text{in} \quad (0, \beta), \]
where \( s = \frac{l^2 k_3 - \rho_1 \lambda^2}{k_3} \). Let us introduce the following three cases.

**Case 1:** If \( \lambda^2 = \frac{l^2 k_3}{\rho_1} \). Then, from (3.14), we deduce that
\[ v^1(x) = c_1 x + c_2 \quad \text{in} \quad (0, \beta), \quad c_1, c_2 \in \mathbb{C}. \]
Using the fact that \( v^1(0) = 0 \), we get
\[ c_2 = 0 \quad \text{and consequently} \quad v^1(x) = c_1 x \quad \text{in} \quad (0, \beta). \]
Inserting (3.16) in (3.13), we get
\[ v^3(x) = -\left(1 + \frac{k_3}{k_1}\right)c_1 \quad \text{in} \quad (0, \beta). \]
Now, from (3.16), (3.17) and the fact that \( v^3(0) = 0 \), we get
\[ c_1 = 0, \quad v^1 = 0 \quad \text{in} \quad (0, \beta) \quad \text{and} \quad v^3 = 0 \quad \text{in} \quad (0, \beta). \]
Thus, from (3.2), (3.4), (3.9), (3.10), (3.18) and the fact that \( \lambda \neq 0 \), we obtain
\[ U = 0 \quad \text{in} \quad (0, \beta). \]
Let \( V = (v^1, v^1_x, v^3, v^3_x, v^5_x, v^5_{xx})^\top \). From (3.18) and the regularity of \( v^i, \ i \in \{1,3,5\} \), we get \( V(\beta) = 0 \). Now, by inserting (3.2), (3.4) and (3.6) in (3.3), (3.5) and (3.7) respectively, then system (3.2)-(3.7) can be written in \( (\beta, L) \) as the following
\[ V_x = AV \quad \text{in} \quad (\beta, L), \]
where
\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & l(1 - \frac{k_3}{k_1}) \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -\frac{k_1}{k_2} & \frac{\lambda^2 k_2}{\rho_1 k_1} & 0 & -\frac{k_1}{k_2} & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & -l(\frac{k_1}{k_2} + 1) & 0 & -\frac{k_1}{k_2} & \frac{\lambda^2}{\rho_1} - \frac{k_2}{k_1} & 0 \\
\end{pmatrix}.
\]
The solution of the differential equation (3.20) is given by
\[ V(x) = e^{A(x-\beta)}V(\beta). \]
Thus, from (3.21) and the fact that \( V(\beta) = 0 \), we get
\[ V = 0 \quad \text{in} \quad (\beta, L) \quad \text{and consequently} \quad U = 0 \quad \text{in} \quad (0, \beta). \]
Therefore, from (3.19) and (3.22), we obtain
\[ U = 0 \quad \text{in} \quad (0, L). \]

**Case 2:** If \( \lambda^2 > \frac{l^2 k_3}{\rho_1} \). Then, from (3.14), we deduce that
\[ v^1(x) = c_1 e^{\sqrt{-s}x} + c_2 e^{-\sqrt{-s}x} \quad \text{in} \quad (0, \beta), \quad c_1, c_2 \in \mathbb{C}. \]
Now, from (3.23) and the fact that \( v^1(0) = 0 \), we get
\[ c_2 = -c_1 \quad \text{and consequently} \quad v^1(x) = c_1 (e^{\sqrt{-s}x} - e^{-\sqrt{-s}x}) \quad \text{in} \quad (0, \beta). \]
Inserting (3.24) in (3.13), we get
\[ v^3(x) = -\left(1 + \frac{k_3}{k_1}\right) \sqrt{-s} \left(e^{\sqrt{-s}x} + e^{-\sqrt{-s}x}\right) c_1 \quad \text{in} \quad (0, \beta). \]
From (3.24), (3.25) and the fact that \( v^3(0) = 0 \), we obtain
\[ c_1 = 0, \quad v^1 = 0 \quad \text{in} \quad (0, \beta), \quad v^3 = 0 \quad \text{in} \quad (0, \beta) \quad \text{and consequently} \quad U = 0 \quad \text{in} \quad (0, \beta). \]
Similarly to Case 1, we get $U = 0$ in $(\beta, L)$ and consequently $U = 0$ in $(0, L)$.

**Case 3:** If $\lambda^2 < \frac{l^2 k_3}{\rho_1}$. Then, from (3.14), we deduce that

\[ v^1(x) = c_1 \cos(\sqrt{s}x) + c_2 \sin(\sqrt{s}x) \quad \text{in} \quad (0, \beta), \quad c_1, c_2 \in \mathbb{C}. \]

Now, from (3.27) and the fact that $v^1(0) = 0$, we get

\[ c_1 = 0 \quad \text{and consequently} \quad v^1(x) = c_2 \sin(\sqrt{s}x) \quad \text{in} \quad (0, \beta). \]

Inserting (3.28) in (3.13), we get

\[ v'^3(x) = -\left(1 + \frac{k_3}{k_1}\right) \sqrt{s} \cos(\sqrt{s}x)c_2 \quad \text{in} \quad (0, \beta). \]

From (3.29), (3.28) and the fact that $v'^3(0) = 0$, we obtain

\[ c_2 = 0, \quad v^1 = 0 \quad \text{in} \quad (0, \beta), \quad v^3 = 0 \quad \text{in} \quad (0, \beta) \quad \text{and consequently} \quad U = 0 \quad \text{in} \quad (0, \beta). \]

Similarly to Case 1, we get $U = 0$ in $(\beta, L)$ and consequently $U = 0$ in $(0, L)$. The proof is thus complete. □

4. **Exponential and Polynomial Stability**

In this section, we show the influence of the physical coefficients on the stability of system (1.1)-(1.3). The main results of this section are the following theorems.

**Theorem 4.1.** If

\[ \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{and} \quad k_1 = k_3, \]

then the $C_0$-semigroup $e^{tA}$ is exponentially stable; i.e. there exists constants $M \geq 1$ and $\epsilon > 0$ independent of $U_0$ such that

\[ \|e^{tA}U_0\|_H \leq Me^{-\epsilon t}\|U_0\|_H. \]

**Theorem 4.2.** If

\[ \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{and} \quad k_1 \neq k_3, \]

then there exists $C > 0$ such that for every $U_0 \in D(A)$, we have

\[ E(t) \leq \frac{C}{t}\|U_0\|_{D(A)}^2, \quad t > 0. \]

**Theorem 4.3.** If

\[ \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \]

then there exists $C > 0$ such that for every $U_0 \in D(A)$, we have

\[ E(t) \leq \frac{C}{\sqrt{t}}\|U_0\|_{D(A)}^2, \quad t > 0. \]

According to [24], [34] and Theorem 2.4 in [15] (see also [14] and [27]), a $C_0$-semigroup of contractions $(e^{tA})_{t \geq 0}$ on $H$ satisfy (4.1), (4.2) and (4.3) if

\[ i\mathbb{R} \subset \rho(A) \]

\[ \sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{\mathcal{L}(H)} = O(|\lambda|^\ell), \quad \text{with} \begin{cases} \ell = 0 \quad \text{for Theorem 4.1}, \\ \ell = 2 \quad \text{for Theorem 4.2}, \\ \ell = 4 \quad \text{for Theorem 4.3}. \end{cases} \]

Since $i\mathbb{R} \subset \rho(A)$ (see Section 3), then condition (M1) is satisfied. We will prove condition (M2) by a contradiction argument. For this purpose, suppose that (M2) is false, then there exists $\{(\lambda^n, U^n)\}_{n \geq 1} \subset \mathbb{R}^* \times D(A)$ with

\[ |\lambda^n| \to \infty \quad \text{and} \quad \|U^n\|_H = \|(v^{1,n}, v^{2,n}, v^{3,n}, v^{4,n}, v^{5,n}, v^{6,n})^T\|_H = 1, \]

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For simplicity, we drop the index \( n \). Equivalently, from (4.5), we have

\[
\begin{align*}
(\lambda^n f (i \lambda^n I - A) U^n &= F^n := \begin{pmatrix} f^{1,n} & f^{2,n} & f^{3,n} & f^{4,n} & f^{5,n} & f^{6,n} \end{pmatrix}^\top \rightarrow 0 \quad \text{in} \quad \mathcal{H}.
\end{align*}
\]

By inserting (4.6) in (4.7), (4.8) in (4.9) and (4.10) in (4.11), we deduce that

\[
\begin{align*}
\lambda^2 \rho_1 v^1 + k_1 (v_x^1 + v^3 + lv^5) + lk_3 (v_x^5 - lv^1) &= -\rho_1 \lambda^{-\ell} f^2 - i \rho_1 \lambda^{-\ell+1} f^1, \\
\lambda^2 \rho_2 v^3 + k_2 (v_x^3 + v^3 + lv^5) &= -\rho_2 \lambda^{-\ell} f^4 - i \rho_2 \lambda^{-\ell+1} f^3, \\
\lambda^2 \rho_1 v^5 + k_3 (v_x^5 - lv^1) &= -\rho_1 \lambda^{-\ell} f^6 - i \rho_1 \lambda^{-\ell+1} f^5.
\end{align*}
\]

Here we will check the condition \((M_2)\) by finding a contradiction with (4.4) by showing \( \| U \|_{\mathcal{H}} = o(1) \). For clarity, we divide the proof into several Lemmas. From the above system and the fact that \( \ell \in \{2, 4\} \), \( \| U \|_{\mathcal{H}} = 1 \) and \( \| F \|_{\mathcal{H}} = o(1) \), we remark that

\[
\| v^1 \| = O \left( |\lambda|^{-1} \right), \quad \| v^3 \| = O \left( |\lambda|^{-1} \right), \quad \| v^5 \| = O \left( |\lambda|^{-1} \right), \quad \| v^1_{xx} \| = O (|\lambda|), \quad \| v^3_{xx} \| = O (|\lambda|) \quad \text{and} \quad \| v^5_{xx} \| = O (|\lambda|).
\]

Also, from Poincaré inequality and the fact that \( \| F \|_{\mathcal{H}} = o(1) \), we remark that

\[
\| f^1 \| = o(1), \quad \| f^3 \| = o(1) \quad \text{and} \quad \| f^5 \| = o(1).
\]

We define the following hypotheses:

\[(H_1) \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \quad k_1 = k_3 \quad \text{and} \quad \ell = 0; \]

\[(H_2) \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2}, \quad k_1 \neq k_3 \quad \text{and} \quad \ell = 2; \]

\[(H_3) \quad \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2} \quad \text{and} \quad \ell = 4; \]

\[
\text{Remark 4.1. According to Remark 3.8 in [31], the case of equal speed propagation (i.e., when \((H_1)\) holds) has only mathematical sound.} \]

\[
\text{Lemma 4.1. If \((H_1)\) or \((H_2)\) or \((H_3)\) holds, then the solution } U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(\mathcal{A}) \text{ of (4.6)-(4.11) satisfies the following estimations}
\]

\[
\int_0^\beta |v^6|^2 \, dx = o(\lambda^{-\ell}) \quad \text{and} \quad \int_0^\beta |v^5|^2 \, dx = o(\lambda^{-\ell-2}).
\]

\[
\text{Proof. First, taking the inner product of (4.5) with } U \text{ in } \mathcal{H} \text{ and using (2.5), we get}
\]

\[
\int_0^L a(x) \left| v^6 \right|^2 \, dx = a_0 \int_0^\beta |v^6|^2 \, dx = -\mathcal{R} (AU, U)_{\mathcal{H}} = \lambda^{-\ell} \mathcal{R} (F, U)_{\mathcal{H}} \leq \lambda^{-\ell} \| F \|_{\mathcal{H}} \| U \|_{\mathcal{H}}.
\]

\[
\text{Thus, from (4.16) and the fact that } \| F \|_{\mathcal{H}} = o(1) \text{ and } \| U \|_{\mathcal{H}} = 1, \text{ we obtain the first estimation in (4.15). From (4.10), we deduce that}
\]

\[
\int_0^\beta |v^5|^2 \, dx \leq \frac{1}{\lambda^2} \int_0^\beta |v^6|^2 \, dx + \frac{1}{\lambda^{2+2}} \int_0^\beta |f^5|^2 \, dx.
\]

\[
\text{Finally, from (4.17), the first estimation in (4.15), and the fact that } \ell \in \{0, 2, 4\}, \| f^5 \| = o(1), \text{ we get the second estimation in (4.15). The proof is thus complete.} \]
For all $0 < \varepsilon < \frac{\beta}{12}$, we fix the following cut-off functions

- $f_j \in C^2([0, L]), j \in \{1, \cdots, 6\}$ such that $0 \leq f_j(x) \leq 1$, for all $x \in [0, L]$ and
  \[
  f_j(x) = \begin{cases} 
  1 & \text{if } x \in [j\varepsilon, \beta - j\varepsilon], \\
  0 & \text{if } x \in [0, (j-1)\varepsilon] \cup [\beta + (1-j)\varepsilon, L]. 
  \end{cases}
  \]

- $g_1, g_2 \in C^1([0, L])$ such that $0 \leq g_1(x) \leq 1, 0 \leq g_2(x) \leq 1$ for all $x \in [0, L]$ and
  \[
  g_1(x) = \begin{cases} 
  1 & \text{if } x \in [0, \alpha_1], \\
  0 & \text{if } x \in [\alpha_2, L], 
  \end{cases}
  \text{ and } g_2(x) = \begin{cases} 
  1 & \text{if } x \in [\alpha_1], \\
  0 & \text{if } x \in [\alpha_2, L], 
  \end{cases}
  \text{ with } 0 < \alpha_1 < \alpha_2 < \beta < L.
  \]

Lemma 4.2. If $(H_1)$ or $(H_2)$ or $(H_3)$ holds, then the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$ of (4.6)-(4.11) satisfies the following estimations

- $\int_0^\beta -\varepsilon |v_x^5|^2 dx = o(1) = \frac{o(1)}{|\lambda|^{\min(\frac{4}{3}, \ell)}} = \begin{cases} 
  \frac{o(1)}{|\lambda|^{\ell}} & \text{if } \ell \in \{0, 2\}, \\
  \frac{o(1)}{|\lambda|^{\frac{4}{3}+1}} & \text{if } \ell \in \{2, 4\}. 
  \end{cases}$

Proof. First, multiplying (4.11) by $f_1 v^5$, integrating over $(0, \beta)$, and using the fact that $\|v^5\| = O(|\lambda|^{-1})$, $\|f^6\| = o(1)$, we obtain

\[
\begin{aligned}
& i\lambda\rho_1 \int_0^\beta f_1 v^6 \overline{v^5} dx - k_3 \int_0^\beta f_1 v_2^v \overline{v^5} dx + l k_3 \int_0^\beta f_1 v_2^v \overline{v^5} dx + l k_1 \int_0^\beta f_1 (v_1^v + v^3 + lv^5) \overline{v^5} dx \\
& + a_0 \int_0^\beta f_1 v^6 \overline{v^5} dx = o(\lambda^{-\ell}).
\end{aligned}
\]

Using integration by parts in the above equation and the fact that $\int_0^\beta f_1(0) = \int_0^\beta f_1(\beta) = 0$, we get

- $k_3 \int_0^\beta f_1 |v_x^5|^2 dx = -k_3 \int_0^\beta \overline{f_1 v_2^v v^5} dx - i\lambda\rho_1 \int_0^\beta f_1 \overline{v^6} v^5 dx - k_3 \int_0^\beta f_1 v_x^5 \overline{v^5} dx$

\[
\begin{aligned}
& -lk_1 \int_0^\beta f_1 (v_1^v + v^3 + lv^5) \overline{v^5} dx - a_0 \int_0^\beta f_1 \overline{v^6} \overline{v^5} dx + o(\lambda^{-\ell}).
\end{aligned}
\]

Using the above estimation, Lemma 4.1 and the fact that $v_1^v, v_5^v, (v_1^v + v^3 + lv^5)$ are uniformly bounded in $L^2(0, L), \ell \in \{0, 2, 4\}$, we obtain

- $k_3 \int_0^\beta f_1 |v_1^v|^2 dx = \frac{o(1)}{|\lambda|^{\min(\frac{4}{3}, \ell)}}$.

Finally, from the above estimation and the definition of $f_1$, we obtain (4.18). The proof is thus complete.

Lemma 4.3. If $(H_2)$ or $(H_3)$ holds, then the solution $U = (v^1, v^2, v^3, v^4, v^5, v^6)^\top \in D(A)$ of (4.6)-(4.11) satisfies the following estimations

- $\int_{2\varepsilon}^{\beta - 2\varepsilon} |v_x^1|^2 dx = o(1)$ and $\int_{3\varepsilon}^{\beta - 3\varepsilon} |\lambda v^1|^2 dx = o(1)$. 

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Proof. First, multiplying (4.11) by \( \overline{f_2 v_2^1} \), integrating over \((\varepsilon, \beta - \varepsilon)\), and using the fact that \( v_2^1 \) is uniformly bounded in \( L^2(0, L) \) and \( \|f^6\| = o(1) \), we get
\[
l(k_1 + k_3) \int_{\varepsilon}^{\beta - \varepsilon} f_2 |v_2^1|^2 dx = -i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^6 \overline{v_2^3} dx + k_3 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^5 \overline{v_2^2} dx - lk_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 (v^3 + lv^5) \overline{v_2^2} dx \\
- a_0 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^6 \overline{v_2^1} dx + o(|\lambda|^{-\ell}),
\]
using integration by parts and the fact that \( f_2(\varepsilon) = f_2(\beta - \varepsilon) = 0 \), we get
\[
l(k_1 + k_3) \int_{\varepsilon}^{\beta - \varepsilon} f_2 |v_2^1|^2 dx = -i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^6 \overline{v_2^3} dx + k_3 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^5 \overline{v_2^2} dx + k_3 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^5 \overline{v_2^2} dx \\
- lk_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 (v^3 + lv^5) \overline{v_2^2} dx - a_0 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^6 \overline{v_2^1} dx + o(|\lambda|^{-\ell}).
\]
Using the above equation, Lemmas 4.1-4.2 with \( \ell \in \{2, 4\} \), and the fact that \( v_2^1 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_2^1\| = O(|\lambda|) \), \( \|v^3\| = O(|\lambda|^{-1}) \), we obtain
\[
l(k_1 + k_3) \int_{\varepsilon}^{\beta - \varepsilon} f_2 |v_2^1|^2 dx = \frac{o(1)}{|\lambda|^{\frac{1}{2} - \frac{\ell}{2}}}.
\]
Thus, from the above estimation, the definition of \( f_2 \) and the fact that \( \frac{\ell}{2} - \frac{1}{2} \in \{0, \frac{1}{2}\} \), we obtain the first estimation in (4.20). Now, Multiplying (4.12) by \( f_3 \overline{v^1} \), integrating over \((2\varepsilon, \beta - 2\varepsilon)\), using integration by parts and the definition of \( f_3 \), then using the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), \( \|f^1\| = o(1) \) and \( \|f^2\| = o(1) \), we get
\[
\rho_1 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 |\lambda v^1|^2 dx = k_1 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 (v_2^1 v_2^1 + v_2^3 + lv_2^5 \overline{v_2^1}) dx + k_1 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 |v_2^1|^2 dx \\
+ k_1 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 (v_2^3 + lv_2^5) \overline{v_2^1} dx -lk_3 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 \overline{v_2^1} v_2^1 dx + l^2 k_3 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 |v_2^1|^2 dx + o(|\lambda|^{-\ell + 1}).
\]
From (4.21), Lemma 4.2, the first estimation in (4.20), and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), \( \|v^3\| = O(|\lambda|^{-1}) \), \( \|v^5\| = O(|\lambda|^{-1}) \) and \( \ell \in \{2, 4\} \), we obtain
\[
\rho_1 \int_{2\varepsilon}^{\beta - 2\varepsilon} f_3 |\lambda v^1|^2 dx = o(1).
\]
Finally, from the above estimation and the definition of \( f_3 \), we obtain the second estimation desired. The proof is thus complete. \( \square \)

Lemma 4.4. If (H₁) holds, then the solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A) \) of (4.6)-(4.11) satisfies the following estimations
\[
\int_{2\varepsilon}^{\beta - 2\varepsilon} |v_2^3|^2 dx = o(1) \quad \text{and} \quad \int_{2\varepsilon}^{\beta - 2\varepsilon} |\lambda v^1|^2 dx = o(1).
\]

Proof. First, take \( \ell = 0 \) in (4.12) and multiply it by \( f_2 (\overline{v_2^5 - lv^1}) \), integrating over \((\varepsilon, \beta - \varepsilon)\), and taking the real part, we get
\[
\Re \left\{ \lambda^2 \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^1 (\overline{v_2^5 - lv^1}) dx + k_3 \int_{\varepsilon}^{\beta - \varepsilon} f_2 v_2^1 (\overline{v_2^3 - lv^1}) dx + k_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 (v_2^3 + lv_2^5) (\overline{v_2^5 - lv^1}) dx \\
+ lk_3 \int_{\varepsilon}^{\beta - \varepsilon} f_2 |v_2^5 - lv^1|^2 dx \right\} = \Re \left\{ -\rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_2 (\overline{v_2^5 - lv^1}) dx - i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_1 (\overline{v_2^5 - lv^1}) dx \right\},
\]
using integration by parts and the fact that \( f_2(\varepsilon) = f_2(\beta - \varepsilon) = 0 \), then using Lemmas 4.1-4.2 with \( \ell = 0 \) and the fact that \( \|v^1\| = O(|\lambda|^{-1}) \), \( \|f^1\| = o(1) \), \( \|f^2\| = o(1) \), we get
\[
\Gamma_1 = \Re \left\{ \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_2 (\overline{v_2^5 - lv^1}) dx + i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_1 (\overline{v_2^5 - lv^1}) dx \\
- i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_1 (\overline{v_2^5 - lv^1}) dx + i \lambda \rho_1 \int_{\varepsilon}^{\beta - \varepsilon} f_2 f_1 (\overline{v_2^5 - lv^1}) dx = o(1),
\right.
\]
consequently, we obtain
\[
\Re \left\{ \lambda^2 \rho_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 v^1_i \overline{v^2_i} \, dx - l \rho_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 |\lambda v| \overline{v} \, dx + k_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 v^1_{ix} \overline{v^2_x} \, dx - l k_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 v^1_i \overline{v} \, dx \right\}
\]
\[
= \Re \left\{ - \rho_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 f^6 v^1_i \overline{v} \, dx - i \lambda \rho_1 \int_{\varepsilon}^{\beta+\varepsilon} f_2 f^5 v^i_x \overline{v^i_x} \, dx - \alpha_0 \right\} \int_{\varepsilon}^{\beta+\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx
\]
\[
(4.24)
\]
Now, multiplying \((4.8)\) by \(f_2 v^i_x\), integrating over \((\varepsilon, \beta - \varepsilon)\), taking the real part, we get
\[
\Re \left\{ \lambda^2 \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v^2_i} \, dx + k_3 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5_{ix} v^i_x \overline{v^i_x} \, dx - l (k_1 + k_3) \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^3 + l \overline{v^5} v^i_x \overline{v} \, dx
\]
\[
- \alpha_0 \right\} \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx = \Re \left\{ - \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 f^6 v^1_i \overline{v} \, dx - i \lambda \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 f^5 v^i_x \overline{v^i_x} \, dx - \alpha_0 \right\} \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx
\]
\[
:= I_2
\]
using integration by parts and the fact that \(f_2(x) = f_2(\beta - \varepsilon) = 0\), then using the fact that \(v^1_i\) is uniformly bounded in \(L^2(0, L)\), \(\|v^i\| = O(\chi^{-1})\), \(\|f^5\| = o(1)\), \(\|f^6\| = o(1)\), \(\|f^7\| = o(1)\), we get
\[
I_2 = \Re \left\{ - \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 f^6 v^1_i \overline{v} \, dx + i \lambda \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 f^5 v^i_x \overline{v^i_x} \, dx + \alpha_0 \right\} \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx = o(1),
\]
consequently, by using integration by parts in \((4.10)\) and the fact that \(f_2(x) = f_2(\beta - \varepsilon) = 0\), we obtain
\[
\Re \left\{ - \lambda^2 \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v} \, dx - \lambda^2 \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v} \, dx - k_3 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v} \, dx - k_3 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v} \, dx
\]
\[
- l (k_1 + k_3) \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^3 + l v^5 v^i_x \overline{v} \, dx - \alpha_0 \right\} \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx = o(1),
\]
(4.25)
Adding \((4.24)\) and \((4.25)\) and using the fact that \(k_1 = k_3\), we get
\[
l \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |\lambda v|^2 \overline{v} \, dx + 2 k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx + l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^1_{ix} \overline{v^2_x} \, dx = \Re \left\{ l k_3 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 - l v^1 \overline{v} \, dx
\]
\[
+ k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 (v^3 + l v^5) \overline{v^3 - l v^1} \overline{v} \, dx - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v^i_x} \, dx - k_3 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^5 v^i_x \overline{v^i_x} \, dx
\]
\[
- l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 (v^3 + l v^5) v^i_x \overline{v} \, dx - \alpha_0 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^6 v^1_i \overline{v} \, dx \right\} + o(1),
\]
using Lemmas 4.1-4.2 with \(\ell = 0\) and the fact that \(v^1_x, v^3_x\) are uniformly bounded in \(L^2(0, L)\) and \(\|v^i\| = O(|\chi|^{-1})\), \(\|v^3\| = O(|\chi|^{-1})\), we get
\[
l \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |\lambda v|^2 \overline{v} \, dx + 2 k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx + l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^1_{ix} \overline{v^2_x} \, dx + I_3 = o(1).
\]
(4.26)
Now, using integration by parts and the fact that \(f_2(x) = f_2(\beta - \varepsilon) = 0\), then using the fact that \(v^1_i\) is uniformly bounded in \(L^2(0, L)\), \(\|v^i\| = O(|\chi|^{-1})\), we get
\[
I_3 = - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 v^1_{ix} \overline{v^2_x} \, dx = - l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx + o(1),
\]
(4.27)
Inserting \((4.27)\) in \((4.26)\), we obtain
\[
l \rho_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |\lambda v|^2 \overline{v} \, dx + l k_1 \int_{\varepsilon}^{\beta-\varepsilon} f_2 |v^1|^2 \overline{v} \, dx = o(1).
\]
Finally, from the above estimation and the definition of \(f_2\), we obtain \((4.23)\). The proof is thus complete.  \(\square\)
Lemma 4.5. If \((H_3)\) holds, then the solution \(U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A)\) of (4.6)-(4.11) satisfies the following estimation
\[
\int_{4\varepsilon}^{3-4\varepsilon} |\lambda v^1|^2 dx = o(\lambda^{-2}) \quad \text{and} \quad \int_{4\varepsilon}^{3-4\varepsilon} |v^1_x|^2 dx = o(\lambda^{-2}).
\]

**Proof.** For clarity, we divide the proof into three steps:

**Step 1:** In this step, we will prove that:
\[
l_\rho_1 \int_{3\varepsilon}^{3-3\varepsilon} f_4|\lambda v^1|^2 dx + l k_3 \int_{3\varepsilon}^{3-3\varepsilon} f_4|v^1_x|^2 dx + R \left\{ -\rho_1 \lambda^2 \int_{3\varepsilon}^{3-3\varepsilon} f_4 v^5 v^1_x dx + k_3 \int_{3\varepsilon}^{3-3\varepsilon} f_4 v^5 v^1_x dx \right\} = o(1) \lambda^2.
\]

For this aim, take \(\ell = 4\) in (4.12) and multiply it by \(l f_4 v^1\), integrating over \((3\varepsilon, \beta - 3\varepsilon)\), using the fact that \(\|v^1\| = o(|\lambda|^{-1}),\|f^1\| = o(1)\) and \(\|f^2\| = o(1)\), then taking the real part, we get
\[
l_\rho_1 \int_{3\varepsilon}^{3-3\varepsilon} f_4|\lambda v^1|^2 dx + R \left\{ -\rho_1 \lambda^2 \int_{3\varepsilon}^{3-3\varepsilon} f_4 v^5 v^1_x dx + k_3 \int_{3\varepsilon}^{3-3\varepsilon} f_4 v^5 v^1_x dx \right\} = o(1) \lambda^4.
\]

From the above estimation, Lemma 4.2-4.3 with \(\ell = 4\), we obtain
\[
l_\rho_1 \int_{3\varepsilon}^{3-3\varepsilon} f_4|\lambda v^1|^2 dx + I_4 = o(1) \lambda^2.
\]

Using integration by parts and the definition of \(f_4\), we obtain
\[
I_4 = -R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3'(v^1_x + v^3 + lv^5) v^1 dx \right\} - R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3(v^1_x + v^3 + lv^5) v^1 dx \right\}
\]
\[
= -\frac{k_1}{2} \int_{2\varepsilon}^{3-2\varepsilon} f_3' \left(|v^1|^2\right)_x dx - R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3' v^5 v^1 dx \right\} - R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3 v^5 v^1 dx \right\}
\]
\[
- R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3(v^1_x + v^3 + lv^5) v^1 dx \right\}.
\]

Using integration by parts and the fact that \(f_3' (3\varepsilon) = f_3' (\beta - 3\varepsilon) = 0\), then using Lemma 4.3, we obtain
\[
-\frac{k_1}{2} \int_{2\varepsilon}^{3-2\varepsilon} f_3' \left(|v^1|^2\right)_x dx = \frac{k_1}{2} \int_{2\varepsilon}^{3-2\varepsilon} f_3' |v^1|^2 dx = o(1) \lambda^2.
\]

Using Lemma 4.3 with \(\ell = 4\) and the fact that \(\|v^3\| = O(|\lambda|^{-1}),\|v^5\| = O(|\lambda|^{-1})\), we obtain
\[
-\frac{k_1}{2} \int_{2\varepsilon}^{3-2\varepsilon} f_3' v^5 v^1 dx = o(\lambda^{-2}) \quad \text{and} \quad -\frac{k_1}{2} \int_{2\varepsilon}^{3-2\varepsilon} f_3' v^5 v^1 dx = o(\lambda^{-2}).
\]

Inserting (4.32) and (4.33) in (4.31), we obtain
\[
I_4 = -R \left\{ k_1 \int_{2\varepsilon}^{3-2\varepsilon} f_3(v^1_x + v^3 + lv^5) v^1 dx \right\} + o(\lambda^{-2}).
\]

From (4.14), we deduce that
\[
-lk_1(v^1_x + v^3 + lv^5) = -\lambda^2 \rho_1 v^5 - k_3(v^5 - lv^1)_x + a(x)v^6 - \rho_1 \lambda^{-\ell} f^6 - i\rho_1 \lambda^{-\ell+1} f^5,
\]
Inserting the above equation in (4.34), then using the fact that $v^1_x$ is uniformly bounded in $L^2(0, L)$, $\|f^5\| = o(1)$, we obtain

$$I_4 = \Re \left\{ -\rho_1 \lambda^2 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx - k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} \left( f_4 v^5 - l v^1 \right) v^1_x dx + a_0 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx \right\} + o(|\lambda|^{-3}).$$

Using integration by parts and the definition of $f_4$, we obtain

$$I_5 = -k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + l k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 |v^1_x|^2 dx$$

using integration by parts and the fact that $f_4'(3\varepsilon) = f_4'(\beta - 3\varepsilon) = 0$, then using Lemma 4.1 with $\ell = 4$ and the fact that $v^1_x$ is uniformly bounded in $L^2(0, L)$, $\|v^1_x\| = O(|\lambda|)$, we obtain

$$I_6 = -k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4'' v^5 v^2 dx - k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f'_4 v^5 v^1_x dx = \frac{o(1)}{\lambda^2},$$

consequently, we obtain

$$I_6 = k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + l k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 |v^1_x|^2 dx + \frac{o(1)}{\lambda^2}.$$

Using Lemma 4.1 with $\ell = 4$ and the fact that $v^1_x$ is uniformly bounded in $L^2(0, L)$, we obtain

$$a_0 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx = \frac{o(1)}{\lambda^2}.$$ Inserting (4.38) and (4.39) in (4.35), we obtain

$$I_4 = \Re \left\{ -\rho_1 \lambda^2 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f'_4 v^5 v^1_x dx \right\} + l k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 |v^1_x|^2 dx + \frac{o(1)}{\lambda^2}.$$ Thus, by inserting the above equation in (4.30), we obtain (4.29).

**Step 2:** In this step, we will prove that:

$$\Re \left\{ k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx \right\} = \Re \left\{ \frac{\lambda^2 \rho_1 k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx \right\} + \frac{o(1)}{\lambda^2}.$$ For this aim, take $\ell = 4$ in (4.12) and multiply it by $\frac{k_1}{k_3} f_4 v^5$, integrating over $(3\varepsilon, \beta - 3\varepsilon)$, then using the fact that $v^5_x$ is uniformly bounded in $L^2(0, L)$, $\|f^4\| = o(1)$, $\|f^5\| = o(1)$, we obtain

$$k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^1_{xx} v^5_x dx = \frac{\lambda^2 \rho_1 k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx - k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f'_4 v^5 v^1_x dx$$

$$- k_3 \left( k_1 + k_3 \right) \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 |v^5|^2 dx + \frac{l k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + \frac{o(1)}{|\lambda|^{3/2}}.$$ From the above equation, Lemmas 4.2-4.3 with $\ell = 4$, we obtain

$$k_3 \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^1_{xx} v^5_x dx = I_7 + I_8 + \frac{o(1)}{|\lambda|^{3/2}}.$$ Using integration by parts and the definition of $f_4$, then using Lemmas 4.1, 4.3 with $\ell = 4$, we obtain

$$I_7 = \frac{\lambda^2 \rho_1 k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + \frac{\lambda^2 \rho_1 k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f'_4 v^5 v^1_x dx = \frac{\lambda^2 \rho_1 k_3}{k_1} \int_{3\varepsilon}^{\beta-3\varepsilon} f_4 v^5 v^1_x dx + \frac{o(1)}{\lambda^{2}}.$$
Using integration by parts and the definition of \( f_i \), then using Lemma 4.1 and the fact that \( v^3_x \) is uniformly bounded in \( L^2(0, L) \), \( \|v^3_{xx}\| = O(|\lambda|) \), we get

\[
I_8 = k_3 \int_{3\epsilon}^{\beta-3\epsilon} f_4 v^3_{xx} v^5 dx + k_3 \int_{3\epsilon}^{\beta-3\epsilon} f_4 v^5 dx = \frac{o(1)}{\lambda^2}.
\]

Inserting (4.44) and (4.43) in (4.42), then taking the real part, we obtain (4.40).

**Step 3:** In this step, we conclude the proof of (4.28). For this aim, inserting (4.40) in (4.29), then using Young’s inequality and Lemma 4.1 with \( \ell = 4 \), we deduce that

\[
l \rho_1 \int_{3\epsilon}^{\beta-3\epsilon} f_4 |\lambda v|^2 dx + l k_3 \int_{3\epsilon}^{\beta-3\epsilon} f_4 |v|^2 dx = \mathcal{R} \left\{ \rho_1 \lambda^2 \left( 1 - \frac{k_3}{k_1} \right) \int_{3\epsilon}^{\beta-3\epsilon} f_4 v^5 dx \right\} + \frac{o(1)}{\lambda^2}.
\]

\[
\leq \rho_1 \lambda^2 \left( \frac{k_3}{k_1} \int_{3\epsilon}^{\beta-3\epsilon} f_4 |v|^2 dx + \frac{o(1)}{\lambda^2} \right)
\]

\[
= \int_{3\epsilon}^{\beta-3\epsilon} \left( \frac{k_3}{k_1} \int_{3\epsilon}^{\beta-3\epsilon} f_4 |v|^2 dx + \frac{o(1)}{\lambda^2} \right)
\]

Thus, from the above estimation, we deduce that

\[
l \rho_1 \int_{3\epsilon}^{\beta-3\epsilon} f_4 |\lambda v|^2 dx + \frac{l k_3}{2} \int_{3\epsilon}^{\beta-3\epsilon} f_4 |v|^2 dx = \frac{o(1)}{\lambda^2}.
\]

Finally, from the above estimation and the definition of \( f_i \), we obtain (4.29). The proof is thus complete. \( \Box \)

**Lemma 4.6.** The solution \( U = (v^1, v^2, v^3, v^5, v^6) \) \( \in D(A) \) of (4.6)-(4.11) satisfies the following estimations

\[
\int_{3\epsilon}^{\beta-3\epsilon} |v^3|^2 dx = o(1) \quad \text{and} \quad \int_{4\epsilon}^{\beta-4\epsilon} |\lambda v^3|^2 dx = o(1) \quad \text{if} \quad (H_1) \quad \text{holds},
\]

\[
\int_{4\epsilon}^{\beta-4\epsilon} |v^3|^2 dx = o(1) \quad \text{and} \quad \int_{5\epsilon}^{\beta-5\epsilon} |\lambda v^3|^2 dx = o(1) \quad \text{if} \quad (H_2) \quad \text{holds},
\]

\[
\int_{5\epsilon}^{\beta-5\epsilon} |v^3|^2 dx = o(1) \quad \text{and} \quad \int_{6\epsilon}^{\beta-6\epsilon} |\lambda v^3|^2 dx = o(1) \quad \text{if} \quad (H_3) \quad \text{holds}.
\]

**Proof.** For clarity, we divide the proof into four steps:

**Step 1:** In this step, we assume that (H1) or (H2) or (H3) holds and we will prove that:

\[
\frac{k_1}{\rho_1} \int_{\omega_j} f_j |v^3_x|^2 dx = \left( \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} \right) \int_{\omega_j} f_j v^1_{xx} v^5 dx + \lambda^2 \int_{\omega_j} f'_j v^1 v^5 dx + \frac{k_2}{\rho_2} \int_{\omega_j} f_j v^1 v^5 dx
\]

\[
+ \frac{k_1}{\rho_2} \int_{\omega_j} f_j v^1_x (v^5_x + \lambda v^5) dx - \frac{l}{\rho_1} (k_1 + k_3) \int_{\omega_j} f_j v^1 v^5 dx + o(1)
\]

and

\[
\rho_2 \int_{\omega_j} f_j |\lambda v^3|^2 dx = k_2 \int_{\omega_j} f_j |v^3|^2 dx + o(1),
\]

\[15\]
where \( \omega_j := ((j-1)\varepsilon, \beta + (1-j)\varepsilon) \) and \( j \in \{1, \cdots, 6\} \). For this aim, multiplying (4.12) by \( \rho_1^{-1} f_j \overline{v_2} \), integrating over \( \omega_j \), using integration by parts and the definition of \( f_j \), we obtain

\[
\left(4.51\right)
\]

\[
\frac{k_1}{\rho_1} \int_{\omega_j} |f_j v_3^3|^2 \, dx = -\lambda^2 \int_{\omega_j} f_j v_1^2 \overline{v_2} \, dx - \frac{k_1}{\rho_1} \int_{\omega_j} f_j v_{1x}^2 \overline{v_2} \, dx - \frac{l}{\rho_1} (k_1 + k_3) \int_{\omega_j} f_j v_2^5 \overline{v_2} \, dx \\
+ \frac{l^2 k_3}{\rho_1} \int_{\omega_j} f_j v_1^2 \overline{v_2} \, dx - \rho_1 \lambda^{-\ell} \int_{\omega_j} f_j f_2 \overline{v_2} \, dx + \rho_1 \lambda^{-\ell+1} \int_{\omega_j} f_j f_2 v_3 \overline{v_2} \, dx.
\]

Using the fact that \( v_3^3 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_1\| = O(|\lambda|^{-1}) \), \( \|v_3\| = O(|\lambda|^{-1}) \), \( \|f_1\| = o(1) \), \( \|f_2\| = o(1) \) and \( \|f_3\| = o(1) \), we get

\[
\left(4.52\right)
\]

\[
\frac{l^2 k_3}{\rho_1} \int_{\omega_j} f_j v_1^2 \overline{v_2} \, dx = o(1), \quad -\rho_1 \lambda^{-\ell} \int_{\omega_j} f_j f_2 \overline{v_2} \, dx = o(1), \quad \rho_1 \lambda^{-\ell+1} \int_{\omega_j} f_j f_2 v_3 \overline{v_2} \, dx = o(1) \quad \text{and} \quad \lambda^{-1} \int_{\omega_j} f_j v_3^3 \overline{v_2} \, dx = o(1).
\]

Inserting (4.52) in (4.51) and using the fact that \( \ell \geq 0 \), we get

\[
\left(4.53\right)
\]

\[
\frac{k_1}{\rho_1} \int_{\omega_j} |f_j v_3|^2 \, dx = -\lambda^2 \int_{\omega_j} f_j v_1 \overline{v_3} \, dx - \frac{k_1}{\rho_1} \int_{\omega_j} f_j v_{1x} \overline{v_3} \, dx - \frac{l}{\rho_1} (k_1 + k_3) \int_{\omega_j} f_j v_2^5 \overline{v_3} \, dx + o(1).
\]

Now, from (4.13), we deduce that

\[
\left(4.54\right)
\]

\[
\lambda^2 \pi_2 v_3 + k_2 v_3 + k_3 (v_3^3 + v_3^5 + l v_3^5) = -\rho_2 \lambda^{-\ell} f_2^2 + i \rho_2 \lambda^{-\ell} f_3^2.
\]

Multiplying (4.54) by \( \rho_2^{-1} f_j v_3^2 \), integrating over \( \omega_j \), using integration by parts and the definition of \( f_j \), then using the fact that \( v_3^2 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_1\| = O(|\lambda|^{-1}) \), \( \|f_3\| = o(1) \), \( \|f_2\| = o(1) \), \( \|f_4\| = o(1) \), we obtain

\[
\left(4.55\right)
\]

\[
\lambda^2 \int_{\omega_j} f_j v_3 \overline{v_3} \, dx + k_2 \int_{\omega_j} f_j v_{1x} \overline{v_3} \, dx - k_1 \int_{\omega_j} f_j v_3 \overline{(v_3^3 + v_3^5 + l v_3^5)} \, dx = -\rho_2 \lambda^{-\ell} \int_{\omega_j} f_j f_3 \overline{v_3} \, dx + o(\lambda^{-\ell})
\]

\[
-i \rho_2 \lambda^{-\ell} \int_{\omega_j} f_j f_3 \overline{v_3} \, dx = o(\lambda^{-\ell}).
\]

Using integration by parts to the first two terms in the above equation, we get

\[
\left(4.56\right)
\]

\[
-\lambda^2 \int_{\omega_j} f_j v_1^2 \overline{v_3} \, dx = \frac{k_2}{\rho_2} \int_{\omega_j} f_j v_{1x} \overline{v_2} \, dx + \frac{k_2}{\rho_2} \int_{\omega_j} f_j v_1 \overline{v_3} \, dx + \frac{k_2}{\rho_2} \int_{\omega_j} f_j v_3 \overline{v_3} \, dx
\]

\[
+ \frac{k_1}{\rho_2} \int_{\omega_j} f_j v_{1x} \overline{(v_3^3 + v_3^5 + l v_3^5)} \, dx + o(\lambda^{-\ell}).
\]

Inserting (4.56) in (4.53), we obtain (4.49). Next, multiplying (4.54) by \( f_j v_3^2 \), integrating over \( \omega_j \), using integration by parts and the definition of \( f_j \) and the fact that \( \|v_3\| = O(|\lambda|^{-1}) \), \( \|f_3\| = o(1) \) and \( \|f_4\| = o(1) \), we get

\[
\rho_2 \int_{\omega_j} f_j |v_3|^2 \, dx = k_2 \int_{\omega_j} f_j |v_3|^2 \, dx + k_2 \int_{\omega_j} f_j v_3 \overline{v_3} \, dx + k_1 \int_{\omega_j} f_j (v_3^3 + v_3^5 + l v_3^5) v_3 \overline{v_3} \, dx + o(\lambda^{-\ell}).
\]

From the above estimation, the first estimation in (4.47) and the fact that \( (v_3^2 + v_3^5 + l v_3^5) \), \( v_3^3 \) are uniformly bounded in \( L^2(0, L) \), \( \|v_3\| = O(|\lambda|^{-1}) \) and \( \ell \geq 0 \), we obtain (4.50).

**Step 2:** In this step, we assume that \((H_1)\) holds and we conclude the proof of (4.46). For this aim, take
\( j = 3 \) in (4.49) and using the fact that \( \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \), we get
\[
\frac{k_1}{\rho_1} \int_{2\varepsilon}^{3-\varepsilon} f_3 |v_3|^2 dx = \lambda^2 \int_{3\varepsilon}^{4\varepsilon} f_4 |v_3| dx + \frac{k_2}{\rho_2} \int_{3\varepsilon}^{4\varepsilon} f_4 |v_3| dx
\]
Using Lemma 4.2 with \( \ell = 0 \), Lemma 4.4, the fact that \( v_3^3 \), \( v_3^3 + v^3 + lv^5 \) are uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), and the definition of \( f_3 \), we get the first estimation in (4.46). Next, take \( j = 4 \) in (4.50), using the first estimation in (4.46) and the definition of \( f_4 \), we obtain the second estimation in (4.46).

**Step 3:** In this step, we assume that (H2) holds and we conclude the proof of (4.47). For this aim, take \( j = 4 \) in (4.49) and using the fact that \( \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \), we get
\[
\frac{k_1}{\rho_1} \int_{3\varepsilon}^{4\varepsilon} f_4 |v_3|^2 dx = \lambda^2 \int_{3\varepsilon}^{4\varepsilon} f_4 |v_3| dx + \frac{k_2}{\rho_2} \int_{3\varepsilon}^{4\varepsilon} f_4 |v_3| dx
\]
Using Lemma 4.2 with \( \ell = 2 \), Lemma 4.3, the fact that \( v_3^3 \), \( v_3^3 + v^3 + lv^5 \) are uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), and the definition of \( f_4 \), we get the first estimation in (4.47). Next, take \( j = 5 \) in (4.50), using the first estimation in (4.47) and the definition of \( f_5 \), we obtain the second estimation in (4.47).

**Step 4:** In this step, we assume that (H3) holds and we conclude the proof of (4.48). For this aim, take \( j = 5 \) in (4.49), we get
\[
\frac{k_1}{\rho_1} \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3|^2 dx = \left( \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} \right) \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3| dx + \frac{k_2}{\rho_2} \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3| dx
\]
Using Lemma 4.2 with \( \ell = 4 \), Lemma 4.5, the fact that \( v_3^3 \), \( v_3^3 + v^3 + lv^5 \) are uniformly bounded in \( L^2(0, L) \) and \( \|v^3\| = O(|\lambda|^{-1}) \), and the definition of \( f_4 \), we get
\[
\frac{k_1}{\rho_1} \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3|^2 dx = \left( \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} \right) \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3| dx + o(1).
\]
Using integration by parts in the above equation and the fact that \( f_5(4\varepsilon) = f_5(\beta - 4\varepsilon) = 0 \), we get
\[
\frac{k_1}{\rho_1} \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3|^2 dx = \left( \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} \right) \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3| dx + \left( \frac{k_2}{\rho_2} - \frac{k_1}{\rho_1} \right) \int_{4\varepsilon}^{\beta-4\varepsilon} f_5 |v_3| dx + o(1).
\]
From the above estimation, Lemma 4.5 and the fact that \( v_3^3 \) is uniformly bounded in \( L^2(0, L) \), \( \|v_3^3\| = O(|\lambda|) \), and the definition of \( f_5 \), we get the first estimation in (4.48). Finally, take \( j = 6 \) in (4.50), using the first estimation in (4.48) and the definition of \( f_6 \), we obtain the second estimation in (4.48). The proof is thus complete.

**Lemma 4.7.** The solution \( U = (v^1, v^2, v^3, v^4, v^5, v^6)^T \in D(A) \) of system (4.6)-(4.11) satisfies the following estimations
\[
J(4\varepsilon, \beta - 4\varepsilon) = o(1) \quad \text{if} \quad (H_1) \quad \text{holds},
\]
\[
J(5\varepsilon, \beta - 5\varepsilon) = o(1) \quad \text{if} \quad (H_2) \quad \text{holds},
\]
\[
J(6\varepsilon, \beta - 6\varepsilon) = o(1) \quad \text{if} \quad (H_3) \quad \text{holds},
\]
where
\[
J(\gamma_1, \gamma_2) := \int_0^\alpha \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x|^2 \right) dx \\
+ \int_{\alpha_2}^L \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x|^2 \right) dx + \rho_1 \int_\beta^L |\lambda v^5|^2 dx,
\]
for all \(0 < \alpha_1 < \alpha_2 < \beta < L\).

**Proof.** We divide the proof into two steps:

**Step 1:** Let \(h \in C^1([0, L])\) such that \(h(0) = h(L) = 0\). In this step, we assume that (H\(_1\)) or (H\(_2\)) or (H\(_3\)) holds and we will prove that:

\[
(4.63) \quad \int_0^L h' \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x|^2 + \rho_1 |\lambda v^5|^2 + k_3 |v_x|^2 \right) dx = o(1).
\]

For this aim, multiplying (4.12) by \(2\nu v_x^2\), integrating over \((0, L)\), taking the real part, using integration by parts and the definition of \(h\), then using the fact that \(v_x^2\) is uniformly bounded in \(L^2(0, L)\), \(\|v^3\| = O(|\lambda|^{-1})\), \(\|f^4\| = o(1)\), \(\|f^2\| = o(1)\) and \(\|f^3\| = o(1)\), we obtain

\[
(4.64) \quad \int_0^L h \left( \rho_1 |\lambda v|^2 + k_1 |v_x|^2 \right) dx + \mathcal{R} \left\{ 2k_1 \int_0^L h v_x^2 v_x dx \right\} + \mathcal{R} \left\{ 2k_1(k_2 + k_3) \int_0^L h v_x^2 v_x dx \right\} \\
= \mathcal{R} \left\{ -\rho_1 \lambda v^3 \int_0^L h f^3 v_x^3 dx \right\} + \mathcal{R} \left\{ \frac{i\rho_1}{\lambda x} \int_0^L h f^3 v_x^3 dx \right\} \\
= o(\lambda^{-1}).
\]

Now, multiplying (4.13) by \(2\nu v_x^2\), integrating over \((0, L)\), taking the real part, using integration by parts and the definition of \(h\), then using the fact that \(v_x^2\) is uniformly bounded in \(L^2(0, L)\), \(\|v^3\| = O(|\lambda|^{-1})\), \(\|v^5\| = O(|\lambda|^{-1})\), \(\|f^3\| = o(1)\), \(\|f^2\| = o(1)\) and \(\|f^4\| = o(1)\), we obtain

\[
(4.65) \quad \int_0^L h \left( |\rho_2 \lambda v^3|^2 + k_2 |v_x|^2 \right) dx + \mathcal{R} \left\{ 2k_1 \int_0^L h v_x^2 v_x dx \right\} \\
= \mathcal{R} \left\{ -\rho_2 \lambda x \int_0^L h f^4 v_x^4 dx \right\} + \mathcal{R} \left\{ \frac{i\rho_2}{\lambda x} \int_0^L h f^4 v_x^4 dx \right\} + \mathcal{R} \left\{ \frac{i\rho_2}{\lambda x} \int_0^L h f^4 v_x^4 dx \right\} \\
= o(\lambda^{-1}).
\]

Next multiplying (4.14) by \(2\nu v_x^2\), integrating over \((0, L)\), taking the real part, using integration by parts and the definition of \(h\), then using the definition of \(a(x)\), Lemma 4.1, the fact that \(v_x^2\) is uniformly bounded in \(L^2(0, L)\),
\[ \|v^3\| = O(|\lambda|^{-1}), \|v^5\| = O(|\lambda|^{-1}), \|f^5\| = o(1), \|f_x^5\| = o(1) \text{ and } \|f^6\| = o(1), \]

we obtain
\[
\int_0^L h \left( \rho_1 |\lambda v^1|^2 + k_3 |v_x^5|^2 \right) dx - \Re \left\{ 2l (k_1 + k_3) \int_0^L h v_x^5 v_x^2 dx \right\} - \Re \left\{ 2l k_1 \int_0^L h (v^3 + lv^5) v_x^2 dx \right\} = o(1)
\]
\[
\left(4.66\right)
\]
\[
\begin{align*}
-\Re \left\{ a_0 \int_0^\beta h v^6 v_x^2 dx \right\} &= \Re \left\{ -\frac{\rho_1}{\lambda} \int_0^L h f^6 v_x^2 dx \right\} + \Re \left\{ \frac{ip_1}{\lambda^{1/4}} \int_0^L h f^6 v_x^2 dx \right\} \\
=(\lambda^{-\frac{1}{4}}) &+ \Re \left\{ \frac{ip_1}{\lambda^{1/4}} \int_0^L h f^6 v_x^2 dx \right\} \\
= o(\lambda^{-\frac{3}{4}})
\end{align*}
\]
Adding \((4.64), (4.65), (4.66)\) and using the fact that \(\ell \in \{0, 2, 4\}\), then using integration by parts, we obtain \((4.63)\).

**Step 2:** In this step, we conclude the proof of Lemma 4.7. For this aim, take \(h = x g_1 + (x - L) g_2\) in \((4.63)\), we obtain
\[
\begin{align*}
\int_0^{\alpha_1} &\left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x^3|^2 + k_3 |v_x^5|^2 \right) dx \\
+ &\int_{\alpha_2}^L \left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x^3|^2 + k_3 |v_x^5|^2 \right) dx + \rho_1 \int_{\beta}^L |\lambda v^5|^2 dx \\
= &- \int_{\alpha_1}^{\alpha_2} (g_1 + x g_1') \left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x^3|^2 + k_3 |v_x^5|^2 \right) dx \\
- &\int_{\alpha_2}^{\alpha_1} (g_2 + (x - L) g_2') \left( \rho_1 |\lambda v^1|^2 + k_1 |v_x^1|^2 + \rho_2 |\lambda v^3|^2 + k_2 |v_x^3|^2 + k_3 |v_x^5|^2 \right) dx \\
+ &\rho_2 \int_{\alpha_1}^{\beta} g_1 |\lambda v^5|^2 dx + \rho_1 \int_{\alpha_1}^{\beta} g_2 |\lambda v^5|^2 dx.
\end{align*}
\]
Now, take \(\alpha_1 = 4\varepsilon\) and \(\alpha_2 = \beta - 4\varepsilon\) in the above equation, then using Lemmas 4.1, 4.2, 4.4 in case that \((H_1)\) holds and \((4.66)\), we obtain \((4.60)\). Next, take \(\alpha_1 = 5\varepsilon\) and \(\alpha_2 = \beta - 5\varepsilon\) in the above equation, then using Lemmas 4.1-4.3 in case that \((H_2)\) holds and \((4.47)\), we obtain \((4.61)\). Finally, take \(\alpha_1 = 6\varepsilon\) and \(\alpha_2 = \beta - 6\varepsilon\) in the above equation, then using Lemmas 4.1, 4.2, 4.5 in case that \((H_3)\) holds and \((4.48)\), we obtain \((4.62)\). The proof is thus complete.

**Proof of Theorem 4.1.** First, from Lemmas 4.1, 4.2, 4.4, \((4.46)\), and the fact that \(\ell = 0\), we obtain
\[
\begin{align*}
\left\{ \int_0^\beta |v^0|^2 dx = o(1), \int_0^\beta |v^3|^2 dx = o(1), \int_0^{\beta-\varepsilon} |v_x^5|^2 dx = o(1), \int_0^{\beta-2\varepsilon} |v_x^1|^2 dx = o(1) \right\} \\
\left\{ \int_0^{\beta-2\varepsilon} |\lambda v^1|^2 dx = o(1), \int_0^{\beta-3\varepsilon} |v_x^3|^2 dx = o(1) \right\} \\
\left\{ \int_0^{\beta-\varepsilon} |\lambda v^3|^2 dx = o(1) \right\} \\
\left\{ \int_0^{\beta-4\varepsilon} |\lambda v^5|^2 dx = o(1) \right\}
\end{align*}
\]
From \((4.67), (4.60)\) and the fact that \(0 < \varepsilon < \frac{\beta}{12}\), we deduce that \(\|U\|_{\mathcal{H}} = o(1)\), which contradicts \((M_2)\). This implies that
\[
\sup_{\lambda \in \mathbb{R}} \|(i\lambda I - A)^{-1}\|_{\mathcal{H}} = O(1).
\]
The proof is thus complete.\qed
**Proof of Theorem 4.2.** First, from Lemmas 4.1, 4.2, 4.3, (4.47), and the fact that \( \ell = 2 \), we obtain

\[
\int_0^\beta |v|^2 dx = o(\lambda^{-2}), \quad \int_0^\beta |v|^2 dx = o(\lambda^{-4}), \quad \int_\varepsilon^{\beta-\varepsilon} |v|^2 dx = o(\lambda^{-2}), \quad \int_{2\varepsilon}^{\beta-2\varepsilon} |v|^2 dx = o(1)
\]

(4.68)

\[
\int_{3\varepsilon}^{\beta-3\varepsilon} |\lambda v|^2 dx = o(1), \quad \int_{4\varepsilon}^{\beta-4\varepsilon} |v|^2 dx = o(1) \quad \text{and} \quad \int_{5\varepsilon}^{\beta-5\varepsilon} |\lambda v|^2 dx = o(1).
\]

From (4.68), (4.61) and the fact that \( 0 < \varepsilon < \frac{\beta}{12} \), we deduce that \( \|U\|_{\mathcal{H}} = o(1) \), which contradicts (M2). This implies that

\[
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{\mathcal{H}} = O \left( \lambda^2 \right).
\]

The proof is thus complete. \( \square \)

**Proof of Theorem 4.3.** First, from Lemmas 4.1, 4.2, 4.5, (4.48), and the fact that \( \ell = 4 \), we obtain

\[
\int_0^\beta |v|^2 dx = o(\lambda^{-4}), \quad \int_0^\beta |v|^2 dx = o(\lambda^{-6}), \quad \int_\varepsilon^{\beta-\varepsilon} |v|^2 dx = o(|\lambda|^{-2}), \quad \int_{4\varepsilon}^{\beta-4\varepsilon} |v|^2 dx = o(\lambda^{-2})
\]

(4.69)

\[
\int_{3\varepsilon}^{\beta-3\varepsilon} |\lambda v|^2 dx = o(\lambda^{-2}), \quad \int_{5\varepsilon}^{\beta-5\varepsilon} |v|^2 dx = o(1) \quad \text{and} \quad \int_{6\varepsilon}^{\beta-6\varepsilon} |\lambda v|^2 dx = o(1).
\]

From (4.69), (4.62) and the fact that \( 0 < \varepsilon < \frac{\beta}{12} \), we deduce that \( \|U\|_{\mathcal{H}} = o(1) \), which contradicts (M2). This implies that

\[
\sup_{\lambda \in \mathbb{R}} \| (i\lambda I - A)^{-1} \|_{\mathcal{H}} = O \left( \lambda^4 \right).
\]

The proof is thus complete. \( \square \)

5. Conclusion

We have studied the stabilization of a Bresse system with discontinuous local viscoelastic damping of Kelvin-Voigt type acting in the longitudinal displacement under fully Dirichlet boundary conditions. We proved the strong stability of the system. We established the exponential stability of the solution if and only if the three waves have the same speed of propagation (i.e., \( \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \) and \( k_1 = k_3 \)). On the contrary, we proved that the energy of our system decays polynomially with the rates

\[
\begin{cases}
  t^{-1} & \text{if} \quad \frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \quad \text{and} \quad k_1 \neq k_3, \\
  t^{-\frac{1}{2}} & \text{if} \quad \frac{k_1}{\rho_1} \neq \frac{k_2}{\rho_2}.
\end{cases}
\]

Moreover, it would be interesting to study system (1.1)-(1.3) with local internal frictional damping, in other words, by only assuming that \( a \) is positive on a non empty subinterval of \((0, L)\) that could be away from the boundary.

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