b-CHROMATIC NUMBER OF CORONA PRODUCT OF CROWN GRAPH AND COMPLETE BIPARTITE GRAPH WITH PATH GRAPH

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ABSTRACT

A b-coloring of a graph is a proper coloring where each color admits at least one node (called dominating node) adjacent to every other used color. The maximum number of colors needed to b-color a graph G is called the b-chromatic number and is denoted by φ(G). In this paper, we find the b-chromatic number and some of the structural properties of corona product of crown graph and complete bipartite graph with path graph.

Keywords: Corona product, crown graph, complete bipartite graph, path graph.

1. INTRODUCTION

A b-coloring by k-colors is a proper coloring of the vertices of graph G such that in each color classes there exists a vertex that has neighbors in all the other k-1 color classes. The b-chromatic number φ(G) is the largest number k for which G admits a b-coloring with k-colors (Irving and Manlove, 1999).

The corona G1 ◦ G2 of two graphs G1 and G2 is defined as a graph obtained by taking one copy of G1 (which has p1 vertices) and p1 copies of G2 and attach one copy of G2 at every vertex of G1 (Harary, 1972).

In this paper we find for which the largest number k for which corona product of crown graph and complete bipartite graph with path graph admits a b-coloring with k-colors. And also we find some of its structural properties (Venkatachalam and Vernold Vivin, 2010; Vernold Vivin and Venkatachalam, 2012; Vijayalakshmi and Thilagavathi, 2012).

2. Definition

2.1. Crown Graph

A crown graph on 2n vertices is an undirected graph with two sets of vertices u_i and v_j and with an edge from u_i to v_j whenever i ≠ j. The crown graph can be viewed as a complete bipartite graph from which the edges of a perfect matching have been removed (Wikipedia).

2.2. Complete Bipartite Graph

A complete bipartite graph is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. That is, it is a bipartite graph (V_1, V_2, E) such that for every two vertices v_1 ∈ V_1 and v_2 ∈ V_2, v_1v_2 is an edge in E. A complete bipartite graph with partitions of size |V_1|=m and |V_2|=n, is denoted K_{m,n} (Balakrishnan, 2004; Balakrishnan and Ranganathan, 2012).

2.3. Fan Graph

A Fan graph F_{m,n} is defined as the graph join K_{m} and P_{n} where K_{m} the empty graph on nodes is and P_{n} is the path on n nodes (Wikipedia).

2.4. Path Graph

The path graph P_{n} is a tree with two nodes of vertex degree 1, and the other n-2 nodes of vertex degree (Harary, 1972).

2.5. Corona Product

Corona product or simply corona of any graph G1 and graph G2, defined as the graph which is the disjoint union of one copy of G1 and |V1| copies of G2 (|V1| is the number of vertices of G1) in which each vertex of the copy of G1 is connected to all vertices of a separate copy of G2 (Harary, 1972).

2.6. b-coloring

A b-coloring of a graph is a proper coloring such that every color class contains a vertex that is adjacent to all other color classes. The b-chromatic number of a graph G, denoted by φ (G), is the maximum number t such that G admits a b-coloring with t colors (Irving and Manlove, 1999).
3. CORONA PRODUCT OF CROWN GRAPH WITH PATH GRAPH

3.1. b-chromatic number of corona product of Crown Graph with Path Graph

3.1.1. Theorem

For any \( n \geq 3 \), \( \varphi[S^0_n \circ P_n] = 2n \).

Proof: Let \( S^0_n \) be any Crown graph with vertices \( V = \{v_1, v_2, \ldots, v_n\} \) and \( V = \{v'_1, v'_2, \ldots, v'_n\} \) i.e. \( V(S^0_n) = V \cup V' \).

Let the edges of \( L_n \) be \( E(S^0_n) = \{e_j: 1 \leq j \leq n^2 - n\} \) where \( e_j \) is the edge connecting \( v_j \) and \( v_j' \) for every \( i \neq j \), \( i, j = 1, 2, \ldots, n \).

Let \( P_n \) be the path graph of length \( n-1 \) with \( n \)-vertices. \( V(P' \circ) = \{u_{ij}: 1 \leq i \leq 2n, 1 \leq l \leq n, 1 \leq j \leq n\} \) and every \( E(P_n) \) be \( \{e_i: 2n - 1 \leq i \leq n-1\} \).

By the definition of corona graph each vertex in \( S^0_n \) is adjacent to every vertex copy of \( P_n \) i.e. vertices of \( S^0_n \) is adjacent to every vertex copy of \( P_n \) i.e. vertices of \( V(L_n \circ P_n) = V(S^0_n) \cup V(P_n) \).

Let \( E[S^0_n \circ P_n] = E(S^0_n) \cup E(P_n) \cup \{e_i: 2n - 1 \leq i \leq n, i \neq 2n\} \).

Consider the color class \( C = \{c_1, c_2, \ldots, c_n, c_{n+1}, c_{n+2}, \ldots, c_{2n}\} \) to color the vertices of \( S^0_n \) and \( P_n \).

Assign the colors \( c_1, c_2, c_3, \ldots, c_n \) to \( v_i \) and \( v_i' \) respectively for every \( i \neq j \), \( i, j = 1, 2, \ldots, n \).

From the figure we see that, each \( v_i \) is adjacent to every \( v_i' \) for every \( i \neq j \) and vice versa. Hence both \( v_i \) and \( v_i' \) earns its adjacent color for every \( i \neq j \).

To make the above coloring to be b-chromatic proper coloring of \( V(P' \circ) \) by corresponding non-adjacent vertices of its \( v_i' \) and \( v_i \).

Thus each color has the neighbor in the every other color class. Thus \( \varphi(S^0_n \circ P_n) = 2n \).

Let us assume that \( \varphi[S^0_n \circ P_n] \geq 2n \), let it be \( \varphi[S^0_n \circ P_n] = 2n+1 \) then the graph \( S^0_n \circ P_n \) must requires \( 2n+2 \) vertices of degree \( 2n+1 \), all with distinct colors and each must have adjacent with all of the other color class, but at least one color class which does not have a color dominating vertex in \( S^0_n \circ P_n \), which invalidates the definition of b-coloring. Hence, \( \varphi[S^0_n \circ P_n] \geq 2n \), thus for any \( n \geq 3 \), the b-chromatic number of corona graph of crown graph with path graph is \( 2n \).

3.2. Illustration: b-coloring of corona product of Crown Graph with Path Graph

3.2.1. Theorem

For any \( n \geq 3 \), \( q[S^0_n \circ P_n] = 5n^2 - 3n \)

Proof: \( q[S^0_n \circ P_n] = \) Number of edges in \( S^0_n \circ P_n \)

\( = n^2 - n + 2n(2n - 1) \)

\( = n^2 - n + 4n^2 - 2n - 1 \)

\( = 5n^2 - 3n \).

3.2.3. Some Structural Properties of \( (S^0_n \circ P_n) n \geq 3 \)

| Property | Graphs | No. of Vertex | No. of Edges | Maximum Degree | Minimum Degree | Vertex Polynomial |
|----------|---------|--------------|-------------|----------------|----------------|-----------------|
| Path     | \( n \)  | \( n - 1 \)  | 2           | 1              | \( 2x(\frac{n}{n^2})^2 \) |
| Graph    | \( 2n \) | \( \frac{n}{n} \) | \( n - 1 \)  | \( n - 1 \)    | \( \frac{4n^2}{n} \) |

4. CORONA PRODUCT OF COMPLETE BIPARTITE GRAPH WITH PATH GRAPH

4.1. b-chromatic number on Corona Product of Complete Bipartite Graph with Path Graph
4.1.2. Theorem

For any \( n \geq 3 \), \( \varphi[\mathcal{K}_{m\times n} \circ P_n] = 2n \) \( m \neq n \)

Proof: Let \( K_{m,n} \) be any complete bipartite graph with vertices \( \mathcal{V} = \{ v_1, v_2, ..., v_n \} \) and \( P = \{ v_1, v_2, ..., v_m \} \) i.e.

\[
V(K_{m,n}) = \mathcal{V} \cup \mathcal{P}.
\]

Let the edges \( E(K_{m,n}) \) be

\[
E(K_{m,n}) = \{ e_j : 1 \leq j \leq n^2 \}
\]

where \( e_j \) is the edge connecting \( v_i \) and \( v_j \).

Let \( \mathcal{P}_n \) be an ant path graph of length \( n-1 \) with \( n \)-vertices. \( V(P_n) = \{ U \cup \{ v\} : 1 \leq i \leq m \} \) and \( E(P_n) \) be \( \{ e_i : 2n-1 \leq i \leq n-1 \} \).

By the definition of corona graph each vertex in \( K_{m,n} \) is adjacent to every vertex copy of \( P_n \), i.e., vertices of \( V(K_{m,n} \circ P_n) = V(K_{m,n}) \cup V(P_n) \).

Let the edges \( E(K_{m,n} \circ P_n) \) be

\[
E(K_{m,n} \circ P_n) = E(K_{m,n}) \cup E(P_n) \cup \{ e_i : 1 \leq i \leq 5n^2, i \leq n^2 \}
\]

where \( e_i \) is the edge connecting \( v_i \) and \( v_j \).

Consider the color class \( C = \{ c_1, c_2, c_3, ..., c_n, c_{n+1}, c_{n+2}, ..., c_{2n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \). The proof follows from the following cases.

Case (i) \( m=n \)

Consider the color class \( C = \{ c_1, c_2, c_3, ..., c_n, c_{n+1}, c_{n+2}, ..., c_{2n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \). Assign the colors \( c_1, c_2, c_3, ..., c_n \) to \( v_1, v_2, ..., v_n \), \( v_i \) respectively.

From the figure we assure that, each \( v \cdot s \) and \( v \cdot s \) are adjacent to every \( v_i \) for and vice versa. Hence both \( v \cdot s \) and \( v \cdot s \) earns its adjacent color. To make the above coloring to be \( b \)-chromatic proper coloring of \( V(p_n) \) by corresponding non-adjacent vertices of its \( v_i \) ‘s and \( v_i \) ‘s respectively and the remaining vertices are colored properly by the colors in the color class. Thus each color has the neighbor in the every other color class. Thus, \( \varphi[K_{m,n} \circ P_n] = 2n \).

Let us assume that \( \varphi[K_{m,n} \circ P_n] > 2n \), say \( \varphi[K_{m,n} \circ P_n] = 2n+1 \). The graph \( K_{m,n} \circ P_n \) must requires \( 2n+2 \) vertices of degree \( 2n+1 \), all with distinct color and each must have adjacent with all of the other color class which is not possible, since maximum degree of \( K_{m,n} \circ P_n \) is \( 2n \), hence at least one color class does not have the color dominating vertex, which contradicts the definition of \( b \)-coloring. Hence, \( \varphi[K_{m,n} \circ P_n] \) not equal to \( 2n+1 \), must be less than \( 2n+1 \) i.e. \( \varphi[K_{m,n} \circ P_n] = 2n \). Thus, for any \( n \geq 3 \), the \( b \)-chromatic number of corona graph of complete bipartite graph with path graph is \( 2n \) for each \( m = n \).

Case (ii) \( m \neq n \)

Consider the color class \( C = \{ c_1, c_2, c_3, ..., c_n, c_{n+1}, c_{n+2}, ..., c_{2n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \), \( m < n \). Assign the colors \( c_1, c_2, c_3, ..., c_m \) to \( v_1, v_2, ..., v_m \), \( v_i \) respectively.

The remaining proof of the theorem follows immediately from case (i). Hence \( \varphi[K_{m,n} \circ P_n] = 2n+1, m \neq n \).

Case (iii) \( m < n \)

Consider the color class \( C = \{ c_1, c_2, c_3, ..., c_m, c_{m+1}, c_{m+2}, ..., c_{2n+1} \} \) to color the vertices of \( K_{m,n} \circ P_n \), \( m < n \). Assign the colors \( c_1, c_2, c_3, ..., c_n \) to \( v_1, v_2, ..., v_n \), \( v_i \) respectively.

The remaining proof of the theorem follows immediately from case (i). Hence \( \varphi[K_{m,n} \circ P_n] = 2n+1, m < n \).

4.1.1. Illustration Corona Product of Complete Bipartite Graph with Path Graph

4.1.2. Theorem

For any \( m, n \geq 3 \), \( \varphi[K_{m,n} \circ P_n] = 2n^2+3mn-n \). Number of edges in \( K_{m,n} \circ P_n \) is \( 2n+1 \).

Proof: \( \varphi(K_{m,n} \circ P_n) = \text{Number of edges in Fan graph } F_{m,n} \)

\[
\text{Number of edges in Fan graph } F_{m,n} = mn+ (m+n) (2n-1)
\]

\[
= mn+2mn+2n^2-n.
\]

\[
= 2n^2+3mn-m-n.
\]
4.1.3. Theorem

For any m, n ≥ 3, the vertex polynomial of be( K_{m,n} ∗ P_n ) 4nx^2 + (2n^2 - 4n)x^3 + 2nx^{2n}, m=n.

Proof: V(K_{m,n} ∗ P_n; x) = ∆_G V x^k

= No of vertices having degree 2 x^2 +
No of vertices having degree 3 x^3 +
No of vertices having degree 2n x^{2n} +
= 4nx^2 + 2n^2 - 4n x^3 + 2n x^{n+2}.

4.1.4. Theorem

4.1.5. Some Structural Properties of ( K_{m,n} ∗ P_n ), n ≥ 3.

| Properties | Number of Vertex | Number of Edges | Maximum Degree | Minimum Degree | Vertex Polynomial |
|------------|-----------------|-----------------|----------------|----------------|------------------|
| PATH GRAPH | n               | n+1             | 2              | 1              | 2x+(n-2)x^2     |
| COMPLETE BIPARTITE GRAPH | m+n           | n^2             | max{m,n}       | min{m,n}       | (2n)x^n         |
| (K_{m,n} ∗ P_n) m = n | 2n^2 + 2n      | 5n^2 - 2n       | 2n             | 2              | 4nx^2 + (2n^2 - 4n)x^3 + 2nx^{2n} |
| (K_{m,n} ∗ P_n) m < n | 2n^2           | 5n^2            | 2n             | 2              | (4n - 2)x^2 + (2n^2 - 3n - 2)x^3 + 2x^{2n-1} |
| (K_{m,n} ∗ P_n) m > n | 2n^2           | 5n^2            | 2n+1           | 2              | (4n + 2)x^2 + (2n^2 - 3n - 2)x^3 + (2n + 1)x^{2n+1} |

5. CONCLUSION

In this paper we operated the graph operation corona product on crown graph and complete bipartite graph with path graph, we get corona product of crown graph with path graph and corona product of complete bipartite graph with path graph and also we find its b-chromatic number and some of its structural properties.

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