Investigating self-similar groups using their finite 
$L$-presentation

René Hartung

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Abstract

Self-similar groups provide a rich source of groups with interesting properties; e.g., infinite torsion groups (Burnside groups) and groups with an intermediate word growth. Various self-similar groups can be described by a recursive (possibly infinite) presentation, a so-called finite $L$-presentation. Finite $L$-presentations allow numerous algorithms for finitely presented groups to be generalized to this special class of recursive presentations. We give an overview of the algorithms for finitely $L$-presented groups. As applications, we demonstrate how their implementation in a computer algebra system allows us to study explicit examples of self-similar groups including the Fabrykowski-Gupta groups. Our experiments yield detailed insight into the structure of these groups.

Keywords. Recursive presentations; self-similar groups; Grigorchuk group; Fabrykowski-Gupta groups; coset enumeration; finite index subgroups; Reidemeister-Schreier theorem; nilpotent quotients; solvable quotients.

1 Introduction

The general Burnside problem is among the most influential problems in combinatorial group theory. It asks whether a finitely generated group is finite if every element has finite order. The general Burnside problem was answered negatively by Golod [22]. The first explicit counter-examples were constructed in [1, 23, 30]. Among these counter-examples is the Grigorchuk group $G$ which is a finitely generated self-similar group. The group $G$ is not finitely presented [27] but it admits a recursive presentation which could be described in finite terms using the action of a finitely generated monoid of substitutions acting on finitely many relations [39]. These recursive presentations are nowadays known as finite $L$-presentations [27] (or endomorphic presentations [2]) in honor of Lysënok’s work in [39] for the Grigorchuk group; see [2] or Section 2 for a definition.

Finite $L$-presentations allow computer algorithms to be employed in the investigation of the groups they define. A first algorithm for finitely $L$-presented groups is the nilpotent quotient algorithm [31, 5]. Recently, further algorithms for finitely $L$-presented groups were developed [33, 34, 35]. For instance, in [31], a coset enumeration process for finitely $L$-presented groups was described. This
is an algorithm which, given a finite generating set of a subgroup of a finitely \( L \)-presented group, computes the index of the subgroup in the finitely \( L \)-presented group provided that this index is finite. Usually index computations in self-similar groups have involved lots of tedious calculations (e.g., finding an appropriate quotient of the self-similar group; computing the index of the subgroup in this quotient; followed by a proof that the obtained index is correct; see, for instance, [6, Section 4] or [15, Chapter VIII]). The coset enumerator in [34] makes this process completely automatic and thus it shows the significance of finite \( L \)-presentations in the investigation of self-similar groups. Moreover, coset enumeration allows one to compute the number of low-index subgroups of finitely \( L \)-presented groups [34].

We demonstrate the application of the algorithms for finitely \( L \)-presented groups in the investigation of a class of self-similar groups \( \Gamma_p \) for \( 3 \leq p \leq 11 \). The group \( \Gamma_3 \) was introduced in [19]. It is a self-similar group with an intermediate word growth [19, 23, 8]. The groups \( \Gamma_p \), with \( p > 3 \), were introduced in [28]. They are known as Fabrykowski-Gupta groups. Their abelianization \( \Gamma_p/\Gamma'_p \cong \mathbb{Z}_p \times \mathbb{Z}_p \) was computed in [28]. Moreover, for \( p \geq 5 \), the groups \( \Gamma_p \) are just-infinite, regular branch groups [28]. The congruence subgroups of \( \Gamma_p \) for primes \( p > 3 \), were studied in [48]; see also [21]. The lower central series sections \( \gamma_c \Gamma_3/\gamma_{c+1}\Gamma_3 \) have been computed entirely in [3] while, for \( p > 3 \), parts of the lower central series sections \( \gamma_c \Gamma_p/\gamma_{c+1}\Gamma_p \) have been computed in [5]. So far, little more is known on the groups \( \Gamma_p \).

For \( p \geq 3 \), the Fabrykowski-Gupta group \( \Gamma_p \) admits a finite \( L \)-presentation [5]. We demonstrate how the implementations of the algorithms for finitely \( L \)-presented groups allow us to investigate the groups \( \Gamma_p \) for \( 3 \leq p \leq 11 \) in detail. For instance, we demonstrate the application of our algorithm

- to compute the isomorphism type of the lower central series sections \( \gamma_c \Gamma_p/\gamma_{c+1}\Gamma_p \) using improved (parallel) methods from [5, 31].
- to compute the isomorphism type of the Dwyer quotients \( M_c(\Gamma_p) \) of their Schur multiplier using the methods from [33].
- to determine the number of low-index subgroups of the groups \( \Gamma_p \) using the methods from [34].
- to compute the isomorphism type of the sections \( \Gamma_p^{(c)}/\Gamma_p^{(c+1)} \) of the derived series combining the methods from [35] and [5, 31].

We briefly sketch the algorithms available for finitely \( L \)-presented groups. Moreover, we compare our experimental results for the Fabrykowski-Gupta groups \( \Gamma_p \) with those results for the Grigorchuk group \( \mathfrak{G} \). The group \( \mathfrak{G} \) has been investigated for decades now. Even though a lot is known about its structure, various questions still remain open [29]. For further details on the Grigorchuk group \( \mathfrak{G} \), we refer to [15, Chapter VIII].

2 Self-Similar Groups

A self-similar group can be defined by its recursive action on a regular rooted tree: Consider the \( d \)-regular rooted infinite tree \( T_d \) as a free monoid over the
alphabet $X = \{0, \ldots, d-1\}$. Then a self-similar group can be defined as follows:

**Definition 2.1** A group $G$ acting faithfully on the free monoid $X^*$ is self-similar if for each $g \in G$ and $x \in X$ there exist $h \in G$ and $y \in X$ so that

$$(xw)^g = yw^h \text{ for each } w \in X^*.$$  

(1)

It suffices to specify the self-similar action in Eq. (1) on a generating set of a group. For instance, the Grigorchuk group $G = \langle a, b, c, d \rangle$ can be defined as a subgroup of the automorphism group of the rooted binary tree $T_2 = \{0, 1\}^*$ by its self-similar action:

$$(0 \, w)^a = 1 \, w \quad (1 \, w)^a = 0 \, w,$$

$$(0 \, w)^b = 0 \, w^a \quad (1 \, w)^b = 1 \, w^c,$$

$$(0 \, w)^c = 0 \, w^a \quad (1 \, w)^c = 1 \, w^d,$$

$$(0 \, w)^d = 0 \, w \quad (1 \, w)^d = 1 \, w^b.$$  

The Fabrykowski-Gupta group $\Gamma_3$ is another example of a self-similar group. It was introduced in [19] as a group with an intermediate word growth [20, 8]. The group $\Gamma_3$ was generalized in [28] to a class of self-similar groups $\Gamma_d$ acting on the $d$-regular rooted tree:

**Definition 2.2** For $d \geq 3$, the Fabrykowski-Gupta group $\Gamma_d = \langle a, r \rangle$ is a self-similar group acting faithfully on the $d$-regular rooted tree $T_d = \{0, \ldots, d-1\}^*$ by

$$(x \, w)^a = x + 1 \text{ (mod } d) \, w, \quad \text{for } 0 \leq x \leq d - 1,$$

$$(0 \, w)^r = 0 \, w^a,$$

$$(x \, w)^r = x \, w, \quad \text{for } 1 \leq x < d - 1,$$

$$(d - 1 \, w)^r = d - 1 \, w^r.$$  

The groups $\mathfrak{G}$ and $\Gamma_d$ admit a finite $L$-presentation; that is, a finite $L$-presentation is a group presentation of the form

$$\langle \mathcal{X} \mid Q \cup \bigcup_{\sigma \in \Phi^*} R^\sigma \rangle,$$

(2)

where $\mathcal{X}$ is a finite alphabet, $Q$ and $R$ are finite subsets of the free group $F$ over $\mathcal{X}$, and $\Phi^*$ denotes the monoid of endomorphisms which is generated by the finite set $\Phi \subseteq \text{End}(F)$. The group defined by the finite $L$-presentation in Eq. (2) is denoted by $\langle \mathcal{X} \mid Q \mid F \mid R \rangle$. If $Q = \emptyset$ holds, the $L$-presentation in Eq. (2) is ascending. In this case, every endomorphism $\sigma \in \Phi^*$ induces an endomorphism of the group $G$.

The Grigorchuk group $\mathfrak{G}$ is an example of a self-similar group which is finitely $L$-presented [39]: the group $\mathfrak{G}$ satisfies

$$\mathfrak{G} \cong \langle \{a, b, c, d\} \mid \{a^2, b^2, c^2, d^2, bed\} \cup \bigcup_{i \geq 0} \{(ad)^4, (adaca)^4\}\sigma^i \rangle,$$

where $\sigma$ is the endomorphism of the free group $F$ over $\{a, b, c, d\}$ which is induced by the map $a \mapsto aca$, $b \mapsto d$, $c \mapsto b$, and $d \mapsto c$. A general method for computing a finite $L$-presentation for a class of self-similar groups was developed in [2] in order to prove
Theorem 2.3 (Bartholdi [2]) Each finitely generated, contracting, semi-fractal regular branch group is finitely $L$-presented; however, it is not finitely presented.

The constructive proof of Theorem 2.3 in [2] was used in [5] to compute the following finite $L$-presentation for the Fabrykowski-Gupta group $\Gamma_p$:

Theorem 2.4 (Bartholdi et al. [5]) For $d \geq 3$, the group $\Gamma_d$ is finitely $L$-presented by $\langle \{\alpha, \rho\} | \emptyset | \{\varphi\} | R \rangle$ where the iterated relations in $R$ are defined as follows: Writing $\sigma_i = \rho^{\alpha_i}$, for $1 \leq i \leq d - 1$, and reading indices modulo $d$, we have

$$R = \left\{ \alpha_d, [\sigma_i^{\alpha_{i-1}}, \sigma_j^{\alpha_{j-1}}], \sigma_i^{\alpha_{i-1}}\sigma_i^{\alpha_{i-1}}\sigma_i^{\alpha_{i-1}} \right\}_{1 \leq i, j \leq d, 2 \leq |i - j| \leq d - 2, 0 \leq k, \ell \leq d - 1}$$

The substitution $\varphi$ is induced by the map $\alpha \mapsto \rho^{\alpha - 1}$ and $\rho \mapsto \rho$.

It follows immediately from the $L$-presentation in Theorem 2.4 that the substitution $\varphi$ induces an endomorphism of the group $\Gamma_d$. Finite $L$-presentations $\langle X | Q | \Phi | R \rangle$ whose substitutions $\sigma \in \Phi$ induce endomorphisms of the group are invariant $L$-presentations. Each ascending $L$-presentation is invariant. It is also easy to see that the $L$-presentation for the Grigorchuk group $G$ above is invariant [26, Corollary 4].

A finite $L$-presentation allows us to define a group that is possibly infinitely presented in computer algebra systems such as GAP [22] or Magma [11]. Beside defining a self-similar group by its finite $L$-presentation, it can also be defined by its recursive action on a regular tree. A finite approximation of the recursive action of a self-similar group is often sufficient to study finite index subgroups since various self-similar groups have the congruence property: every finite index subgroup contains a level stabilizer (i.e., the stabilizer of some level of the regular tree). This often yields an alternative approach to investigate the structure of a self-similar group with the help of computer algebra systems [4]. However, there are self-similar groups that do not have the congruence property [9]. For these groups, their finite $L$-presentation may help to gain insight into the structure of the group. The groups $\Phi$ and $\Gamma_3$ have the congruence property [6].

In the following, we demonstrate how the finite $L$-presentation in Theorem 2.4 allows us to obtain detailed information on the structure of the groups $\Gamma_p$, for $3 \leq p \leq 11$. For further details on self-similar groups, we refer to the monograph by Nekrashevych [40].

3 A Nilpotent Quotient Algorithm

For a group $G$, the lower central series is defined recursively by $\gamma_1G = G$ and $\gamma_{c+1} = [\gamma_cG, G]$ for $c \in \mathbb{N}$. If $G$ is finitely generated, $G/\gamma_{c+1}G$ is polycyclic and therefore it can be described by a polycyclic presentation; i.e., a polycyclic presentation is a finite presentation whose generators refine a subnormal series with cyclic sections. A polycyclic presentation allows effective computations within the group it defines [17, Chapter 9].
A nilpotent quotient algorithm computes a polycyclic presentation for the factor group $G/\gamma_{c+1}G$ together with a homomorphism $G \to G/\gamma_{c+1}G$. Such an algorithm for finitely presented groups was developed in [42]. This nilpotent quotient algorithm was a first algorithm that could be generalized to finite $L$-presentations [31] [5]. The experimental results in this section were obtained with an improved, parallel version of the algorithm in [31] [5]. They extend the computational results in [5] significantly.

We briefly sketch the nilpotent quotient algorithm for finitely presented groups in the following. Let $G = \langle X \mid Q \mid \Phi \mid R \rangle$ be a finitely $L$-presented group. Denote by $F$ the free group over the alphabet $X$ and let $K$ be the normal closure $K = \langle \bigcup_{\sigma \in \Phi} \sigma^\sigma \rangle^F$. First, we assume that $Q = \emptyset$ holds. Then $K^\sigma \subset K$, for each $\sigma \in \Phi$, and $G = F/K$ hold. Therefore, each $\sigma \in \Phi$ induces an endomorphism of the group $G$. Furthermore, we have $G/\gamma_c G \cong F/K\gamma_c F$. The nilpotent quotient algorithm uses an induction on $c$ to compute a polycyclic presentation for $G/\gamma_c G$. For $c = 2$, we have

$$G/[G,G] \cong F/KF' \cong (F/F')/(KF'/F').$$

Since $G$ is finitely generated, $F/F'$ is free abelian with finite rank. The normal generators $\bigcup_{\sigma \in \Phi^c} R^\sigma$ of $K$ give a (possibly infinite) generating set of $KF'/F'$. From this generating set it is possible to compute a finite generating set $U$ with a spinning algorithm. The finite generating set $U$ allows us to apply the methods from [42] that eventually compute a polycyclic presentation for $F/KF'$ together with a homomorphism $F \to F/KF'$ which induces $G \to G/G'$. For $c > 2$, assume that the algorithm has already computed a polycyclic presentation for $G/\gamma_c G \cong F/K\gamma_c F$ together with a homomorphism $F \to F/K\gamma_c F$. Consider the factor group $H_{c+1} = F/[K\gamma_c F,F]$. Then $[K\gamma_c F,F] = [K,F]\gamma_{c+1}F$ and $H_{c+1}$ satisfies the short exact sequence

$$1 \to K\gamma_c F/[K\gamma_c F,F] \to H_{c+1} \to F/K\gamma_c F \to 1;$$

that is, $H_{c+1}$ is a central extension of a finitely generated abelian group by $G/\gamma_c G$. Thus $H_{c+1}$ is nilpotent and polycyclic. A polycyclic presentation for $H_{c+1}$ together with a homomorphism $F \to F/[K\gamma_c F,F]$ can be computed with the covering algorithm in [42]; for a proof that this algorithm generalizes to finite $L$-presentations we refer to [31]. Then $K\gamma_{c+1}F/[K\gamma_c F,F]$ is subgroup of $K\gamma_c F/[K\gamma_c F,F]$ and a (possibly infinite) generating set for $K\gamma_{c+1}F/[K\gamma_c F,F]$ can be obtained from the normal generators of $K$. Again, a finite generating set $U$ for $K\gamma_{c+1}F/[K\gamma_c F,F]$ can be computed with a spinning algorithm from the normal generators of $K$. The finite generating set $U$ allows us to apply the methods in [42] for computing a polycyclic presentation for $G/\gamma_{c+1}G \cong F/K\gamma_{c+1}F$ together with a homomorphism $F \to F/K\gamma_{c+1}F$. This finishes our description of the nilpotent quotient algorithm in the case where $Q = \emptyset$ holds.

If, on the other hand, $G$ is given by a finite $L$-presentation $\langle X \mid Q \mid \Phi \mid R \rangle$ with $Q \neq \emptyset$, the algorithm described above applies to the finitely $L$-presented group $H = \langle X \mid \emptyset \mid \Phi \mid R \rangle$. Write $H = F/K$ and $G = F/L$ for normal subgroups $K \leq L$. The nilpotent quotient algorithm applied to $H$ yields
a polycyclic presentation for $H/\gamma_{c+1}H$ together with a homomorphism $F \rightarrow F/K\gamma_{c+1}F$. This yields
\[
G/\gamma_{c+1}G \cong F/L\gamma_{c+1}F \cong (F/K\gamma_{c+1}F)/(L\gamma_{c+1}F/K\gamma_{c+1}F).
\]
The subgroup $L\gamma_{c+1}F/K\gamma_{c+1}F$ is finitely generated by the images of the relations in $Q$. Standard methods for polycyclic groups \cite{47} then give a polycyclic presentation for the factor group $G/\gamma_{c+1}G$ of the polycyclically presented group $H/\gamma_{c+1}H$ and a homomorphism $F \rightarrow G/\gamma_{c+1}G$.

3.1 Applications of the Nilpotent Quotient Algorithm

The nilpotent quotient algorithm allows us to compute within the lower central series quotients $G/\gamma_{c+1}G$ of a finitely $L$-presented group $G$. For instance, it allows us to determine the isomorphism type of the lower central series sections $\gamma_c G/\gamma_{c+1}G$. For various self-similar groups, the lower central series sections often exhibit periodicities. For instance, the Grigorchuk group $\Gamma$ satisfies

\textbf{Theorem 3.1 (Rozhkov \cite{46})} The lower central series sections $\gamma_c \mathcal{G}/\gamma_{c+1} \mathcal{G}$ are $2$-elementary abelian with the following $2$-ranks:

$$\text{rk}_2(\gamma_c \mathcal{G}/\gamma_{c+1} \mathcal{G}) = \begin{cases} 3 \text{ or } 2, & \text{ if } c = 1 \text{ or } c = 2, \text{ respectively } \\ 2, & \text{ if } c \in \{2 \cdot 2^m + 1, \ldots, 3 \cdot 2^m\} \\ 1, & \text{ if } c \in \{3 \cdot 2^m + 1, \ldots, 4 \cdot 2^m\} \end{cases} \text{ with } m \in \mathbb{N}_0.$$ 

The group $\mathcal{G}$ has finite width $2$.

Our implementation of the nilpotent quotient algorithm in \cite{32} allows a computer algebra system to be applied in the investigation of the quotients $G/\gamma_c G$ for a finitely $L$-presented group $G$. For instance, our implementation suggests that the group $\Gamma_d$ has a maximal nilpotent quotient whenever $d$ is not a prime-power. Based on this experimental observation, the following proposition was proved:

\textbf{Proposition 3.2 (Bartholdi et al. \cite{5})} If $d$ is not a prime-power, the group $\Gamma_d$ has a maximal nilpotent quotient. Its nilpotent quotients are isomorphic to the nilpotent quotients of the wreath product $\mathbb{Z}_d \wr \mathbb{Z}_d$.

For a prime $p \geq 3$, the lower central series sections $\gamma_c \Gamma_p/\gamma_{c+1} \Gamma_p$ are $p$-elementary abelian. For $p = 3$, the lower central series sections $\gamma_c \Gamma_3/\gamma_{c+1} \Gamma_3$ were computed in \cite{3}:

\textbf{Proposition 3.3 (Bartholdi \cite{3})} The sections $\gamma_c \Gamma_3/\gamma_{c+1} \Gamma_3$ are $3$-elementary abelian with the following $3$-ranks:

$$\text{rk}_3(\gamma_c \Gamma_3/\gamma_{c+1} \Gamma_3) = \begin{cases} 2 \text{ or } 1, & \text{ if } c = 1 \text{ or } c = 2, \text{ respectively, } \\ 2, & \text{ if } c \in \{3^k + 2, \ldots, 2 \cdot 3^k + 1\}, \\ 1, & \text{ if } c \in \{2 \cdot 3^k + 2, \ldots, 3^{k+1} + 1\} \end{cases} \text{ with } k \in \mathbb{N}_0. \text{ The group } \Gamma_3 \text{ has finite width } 2.$$
For primes \( p > 3 \), little is known about the series sections \( \gamma_c \Gamma_p / \gamma_{c+1} \Gamma_p \) so far \([5]\). We use the following abbreviation to list the ranks of these sections: If the same entry \( a \in \mathbb{N} \) appears in \( m \) consecutive places in a list, it is listed once in the form \( a^m \). The sections \( \gamma_c \Gamma_p / \gamma_{c+1} \Gamma_p \) are \( p \)-elementary abelian. Their \( p \)-ranks are given by the following table:

| \( p \) | \( \text{rk}_p(\gamma_c \Gamma_p / \gamma_{c+1} \Gamma_p) \) | class |
|-------|---------------------------------|------|
| 3     | \( 2, 1^{[1]}, 2^{[1]}, 1^{[1]}, 2^{[3]}, 1^{[3]}, 2^{[9]}, 1^{[0]}, 2^{[27]}, 1^{[27]}, 2^{[65]} \) | 147  |
| 5     | \( 2, 1^{[3]}, 2^{[1]}, 1^{[13]}, 2^{[5]}, 1^{[65]}, 2^{[25]}, 1^{[26]} \) | 139  |
| 7     | \( 2, 1^{[5]}, 2^{[1]}, 1^{[33]}, 2^{[7]}, 1^{[68]} \) | 115  |
| 11    | \( 2, 1^{[9]}, 2^{[1]}, 1^{[97]}, 2^{[4]} \) | 112  |

These computational results were obtained with a parallel version of the nilpotent quotient algorithm in \([5, 31]\). They were intended to be published in \([18]\).

For 3 \( \leq \) \( c \) \( \leq \) 11, our implementation yields the following results:

- For \( d = 4 \), the Fabrykowski-Gupta group \( \Gamma_4 \) satisfies
  \[ \Gamma_4 / \Gamma'_4 \cong \mathbb{Z}_4 \times \mathbb{Z}_4 \quad \text{and} \quad \gamma_2 \Gamma_4 / \gamma_3 \Gamma_4 \cong \mathbb{Z}_4. \]

For 3 \( \leq \) \( c \) \( \leq \) 141, the sections \( \gamma_c \Gamma_4 / \gamma_{c+1} \Gamma_4 \) are 2-elementary abelian with 2-ranks: \( 2^{[4]}, 3^{[3]}, 2^{[13]}, 3^{[12]}, 2^{[2]}, 3^{[18]}, 2^{[7]} \).

- For \( d = 8 \), the Fabrykowski-Gupta group \( \Gamma_8 \) satisfies
  \[ \Gamma_8 / \Gamma'_8 \cong \mathbb{Z}_8 \times \mathbb{Z}_8, \quad \gamma_2 \Gamma_8 / \gamma_3 \Gamma_8 \cong \mathbb{Z}_8, \]
  and
  \[ \gamma_3 \Gamma_8 / \gamma_4 \Gamma_8 \cong \mathbb{Z}_4. \]

For 7 \( \leq \) \( c \) \( \leq \) 111, the sections \( \gamma_c \Gamma_8 / \gamma_{c+1} \Gamma_8 \) are 2-elementary abelian with 2-ranks: \( 2, 1, 2^{[2]}, 3, 2^{[2]}, 4, 3^{[8]}, 2^{[23]}, 3^{[5]}, 2^{[3]}, 1^{[8]}, 2^{[16]}, 3^{[8]}, 2^{[8]}, 3^{[16]}, 4 \).

- For \( d = 9 \), the Fabrykowski-Gupta group \( \Gamma_9 \) satisfies
  \[ \Gamma_9 / \Gamma'_9 \cong \mathbb{Z}_9 \times \mathbb{Z}_9, \quad \gamma_2 \Gamma_9 / \gamma_3 \Gamma_9 \cong \mathbb{Z}_9, \quad \text{and} \quad \gamma_3 \Gamma_9 / \gamma_4 \Gamma_9 \cong \mathbb{Z}_9. \]

For 4 \( \leq \) \( c \) \( \leq \) 117, the sections \( \gamma_c \Gamma_9 / \gamma_{c+1} \Gamma_9 \) are 3-elementary abelian with 3-ranks: \( 1^{[5]}, 2^{[6]}, 3, 2^{[17]}, 1^{[38]}, 1^{[47]} \).
4 Computing Dwyer Quotients of the Schur Multiplier

The Schur multiplier $M(G)$ of a group $G$ can be defined as the second homology group $H_2(G, \mathbb{Z})$ with integer coefficients. It is an invariant of the group which is of particular interest for infinitely presented groups because proving the Schur multiplier being infinitely generated proves that the group does not admit a finite presentation. This is due to the fact that the Schur multiplier of a finitely presented group is finitely generated abelian which can be seen as a consequence of Hopf's formula: If $F$ is a free group and $R \trianglelefteq F$ a normal subgroup so that $G \cong F/R$ holds, the Schur multiplier $M(G)$ satisfies

$$M(G) \cong (R \cap F')/[R, F].$$

However, a group with a finitely generated Schur multiplier is not necessarily finitely presented \[10\]. For further details on the Schur multiplier, we refer to \[44, Chapter 11\].

It is known that the Schur multiplier of a finitely $L$-presented group (and even the Schur multiplier of a finitely presented group) is not computable in general \[24\]. Nevertheless, the Schur multiplier of some self-similar groups has been computed in \[27, 9\]: For instance, the Grigorchuk group $\mathcal{S}$ satisfies

**Proposition 4.1 (Grigorchuk \[27\])** The Schur multiplier $M(\mathcal{S})$ is infinitely generated 2-elementary abelian. Therefore, the group $\mathcal{S}$ is not finitely presented.

There are various examples of self-similar groups for which nothing is known on their Schur multiplier. Even though the Schur multiplier $M(G)$ is not computable in general, it is possible to compute successive quotients of $M(G)$ provided that the group $G$ is given by an invariant finite $L$-presentation \[33\]. These quotients often exhibit periodicities as well: For instance, our experiments with the implementation of the algorithm in \[33\] suggest that the Schur multiplier of the Fabrykowski-Gupta groups $\Gamma_d$, for a prime-power $d = p^\ell$, is infinitely generated. The algorithm for computing successive quotients of $M(G)$ provides a first method to investigate the structure of the Schur multiplier of an invariantly finitely $L$-presented group (and even the Schur multiplier of a finitely presented group).

We briefly sketch the idea of this algorithm: Let $G$ be an invariantly finitely $L$-presented group. Write $G \cong F/K$ for a free group $F$ and a normal subgroup $K$. Then $G/\gamma_c G \cong F/K\gamma_c F$. We identify $M(G)$ with $(K \cap F')/[K, F]$ and $M(G/\gamma_c G)$ with $(K\gamma_c F \cap F')/[K\gamma_c F, F]$ and define

$$\varphi_c: M(G) \to M(G/\gamma_c G),\quad g[K, F] \mapsto g[K\gamma_c F, F].$$

Then $\varphi_c$ is a homomorphism of abelian groups. In the induction step of the nilpotent quotient algorithm, the algorithm computes a homomorphism $F \to F/[K\gamma_c F, F]$. This homomorphism allows us to compute the image of the Schur multiplier $M(G)$ in $M(G/\gamma_c G)$. In particular, it allows us to compute the
isomorphism type of the Dwyer quotient \( M_c(G) = M(G)/\ker \varphi_c \), for \( c \in \mathbb{N} \), where
\[
M(G) \geq \ker \varphi_1 \geq \ker \varphi_2 \geq \ldots.
\]
The algorithm for computing \( M_c(G) \) has been implemented in GAP. Its implementation allows us to compute the Dwyer quotients of various self-similar groups: Since the Schur multiplier of the Grigorchuk group \( \mathfrak{G} \) is 2-elementary abelian, the Dwyer quotients of \( \mathfrak{G} \) are 2-elementary abelian. We have computed the Dwyer quotients \( M_c(\mathfrak{G}) \) for \( 1 \leq c \leq 301 \). These quotients are 2-elementary abelian with the following 2-ranks:
\[
1, 2, 3^{[3]}, 5^{[6]}, 7^{[12]}, 9^{[24]}, 11^{[48]}, 13^{[96]}, 15^{[110]}.
\]
These experiments suggest that the Grigorchuk group satisfies
\[
M_c(\mathfrak{G}) \cong \begin{cases} 
\mathbb{Z}_2 \text{ or } (\mathbb{Z}_2)^2, & \text{if } c = 1 \text{ or } c = 2, \text{ respectively}, \\
(\mathbb{Z}_2)^{2m+3}, & \text{if } c \in \{3 \cdot 2^m, \ldots, 3 \cdot 2^{m+1} - 1\},
\end{cases}
\]
with \( m \in \mathbb{N}_0 \). For the Fabrykowski-Gupta groups \( \Gamma_d \), the algorithm in [33] yields first insight into the structure of \( M(\Gamma_d) \): We restrict ourselves to the groups \( \Gamma_d \) for prime powers \( d = p^x \) because, otherwise, the groups have a maximal nilpotent quotient by Proposition 372. For a prime \( p \in \{3, 5, 7, 11\} \), the Dwyer quotients \( M_c(\Gamma_p) \) are \( p \)-elementary abelian groups with the following \( p \)-ranks:

| \( p \) | \( \text{rk}_p(M_c(\Gamma_p)) \) |
|-------|----------------------------------|
| 3     | \( 0^{[2]}, 1^{[3]}, 2^{[0]}, 3^{[9]}, 4^{[1]}, 5^{[26]}, 6^{[4]}, 7^{[77]}, 8^{[13]}, 9^{[12]} \) |
| 5     | \( 0^{[1]}, 1^{[4]}, 2^{[2]}, 3^{[20]}, 4^{[10]}, 5^{[100]}, 6^{[1]} \) |
| 7     | \( 0^{[1]}, 1^{[2]}, 2^{[6]}, 3^{[2]}, 4^{[14]}, 5^{[42]}, 6^{[14]}, 7^{[34]} \) |
| 11    | \( 0^{[1]}, 1^{[2]}, 2^{[2]}, 3^{[2]}, 4^{[10]}, 5^{[2]}, 6^{[22]}, 7^{[22]}, 8^{[22]}, 9^{[27]} \) |

As noted by Bartholdi, these experimental results suggest that
\[
\text{rk}_3(M_{c+1}(\Gamma_3)) = \begin{cases} 
2 \lfloor \log_3 \left( \frac{2c-1}{10} \right) \rfloor + 3, & \text{if } \log_3(2c-1) \in \mathbb{Z}, \\
\lfloor \log_3(2c-1) \rfloor + \lfloor \log_3 \left( \frac{2c-1}{10} \right) \rfloor + 1, & \text{otherwise},
\end{cases}
\]
for \( c \geq 6 \). Our results for the Dwyer quotients \( M_c(\Gamma_d) \), for \( d \in \{4, 8, 9\} \), are shown in Table 4 where we list the abelian invariants of \( M_c(\Gamma) \). Here, a list \((\alpha_1, \ldots, \alpha_n)\) stands for the abelian group \( \mathbb{Z}_{\alpha_1} \times \cdots \times \mathbb{Z}_{\alpha_n} \). Again, we list the abelian invariants \((\alpha_1, \ldots, \alpha_n)^{[m]}\) just once if they appear in \( m \) consecutive places.

## 5 Coset Enumeration for Finite Index Subgroups

A standard algorithm for finitely presented groups is the coset enumerator introduced by Todd and Coxeter [49]. Coset enumeration is an algorithm that, given a finite generating set of a subgroup \( H \leq G \), computes the index \([G : H]\) provided that this index is finite. Its overall strategy is to compute a permutation representation for the group’s action on the right-cosets \( H \backslash G \). For finitely
For this purpose, we define $\Phi^\ell \ast G$ following. Let $G$ be a finitely presented group. Suppose that a subgroup $H \leq G$ is given by its finitely many generators $\{g_1, \ldots, g_n\}$. We consider the generators $g_1, \ldots, g_n$ as elements of the free group $F$ over $X$. Then $E = \langle g_1, \ldots, g_n \rangle \leq F$ satisfies $H \cong UK/K$ where $K = \langle Q \cup \bigcup_{p \in \Phi^\ell} R_p \rangle^F$ is the kernel of the free presentation. We are to compute the index $[G : H] = [F : UK]$. For this purpose, we define $\Phi^\ell = \{\sigma \in \Phi^* \mid \|\sigma\| \leq \ell\}$ where $\| \cdot \|$ denotes the usual word-length in the free monoid $\Phi^*$. Consider the finitely presented groups $G^\ell = F/K^\ell$ given by the finite presentation

$$G^\ell = \left\langle X \mid Q \cup \bigcup_{\sigma \in \Phi^\ell} R_\sigma \right\rangle.$$  

Then $G^\ell$ naturally maps onto $G$ and we obtain a series of subgroups

$$UK_0 \leq UK_1 \leq \ldots \leq UK \leq F.$$ 

Table 1: Dwyer quotients of the Fabrykowski-Gupta groups $\Gamma_d$

| $d$ | $M_c(\Gamma_d)$ |
|-----|-----------------|
| 4   | $(1)^{[1]} (2)^{[1]} (2,2)^{[1]} (2,4)^{[4]} (2,2,2,4)^{[4]}$ |
|     | $(2,2,2,4)^{[4]} (2,2,2,4,4)^{[16]} (2,2,2,2,4,4)^{[1]} (2,2,2,2,4,4)^{[3]}$ |
|     | $(2,2,2,2,4,4)^{[16]} (2,2,2,2,2,4,4)^{[6]} (2,2,2,2,2,2,4,4)^{[5]}$ |
|     | $(2,2,2,2,2,2,2,4,4,4)^{[11]} (2,2,2,2,2,2,2,2,2,4,4,4)^{[26]}$ |
| 8   | $(1)^{[1]} (8)^{[2]} (4,8)^{[3]} (2,4,8)^{[4]} (2,8,8)^{[1]} (2,2,8,8)^{[2]}$ |
|     | $(2,2,2,8,8)^{[2]} (2,2,4,8,8)^{[2]} (2,4,4,8,8)^{[2]} (2,4,8,8,8)^{[2]}$ |
|     | $(2,8,8)^{[8]} (2,2,8,8,8)^{[4]} (2,4,8,8,8)^{[2]} (2,4,8,8,8)^{[2]} (2,4,8,8,8)^{[2]}$ |
|     | $(2,2,8,8,8)^{[7]} (2,2,2,8,8,8)^{[16]} (2,2,2,8,8,8)^{[16]} (2,2,2,4,8,8,8,8)^{[3]}$ |
| 9   | $(1)^{[1]} (9)^{[2]} (3,9)^{[2]} (3,3,9)^{[4]} (3,9,9)^{[2]}$ |
|     | $(9,9,9)^{[2]} (3,9,9,9)^{[2]} (3,3,9,9,9)^{[4]} (3,9,9,9)^{[2]}$ |
|     | $(9,9,9,9,9)^{[12]} (3,9,9,9,9)^{[18]} (3,9,9,9,9)^{[18]} (3,9,9,9,9)^{[36]}$ |
|     | $(3,9,9,9,9,9)^{[18]} (9,9,9,9,9,9)^{[17]} (3,9,9,9,9,9)^{[12]}$ |
Since $UK \leq F$ is a finite index subgroup of a finitely generated group, it is finitely generated by $u_1, \ldots, u_n$, say. Furthermore, we have $UK = \bigcup_{\ell \geq 0} UK_\ell$. For each $u_i \in UK$, there exists $n_i \in \mathbb{N}_0$ so that $u_i \in UK_{n_i}$. For $m = \max\{n_i | 1 \leq i \leq n\}$ we have $\{u_1, \ldots, u_n\} \subseteq UK_m$. Thus $UK = UK_m$. In fact, there exists a positive integer $m \in \mathbb{N}_0$ so that $H$ has finite index in the finitely presented group $G_m = \langle X | Q \cup \bigcup_{\sigma \in \Phi^m} R^\sigma \rangle$.

Coset enumeration for finitely presented groups allows us to compute a permutation representation $\pi: F \to \text{Sym}(UK_m \setminus F)$. The integer $m$ cannot be given a priori. However, various coset enumerators can be applied in parallel to the finitely presented groups $G_\ell$. In theory, termination is guaranteed for a sufficiently large integer $\ell$ if $[G : H]$ is finite. Suppose that one coset enumerator has terminated for $[G_\ell : H]$ and suppose that it has computed a permutation representation $\pi_\ell: F \to \text{Sym}(UK_\ell \setminus F)$. Then $[G : H] = [F : UK]$ divides the index $[G_\ell : H] = [F : UK_\ell]$. It suffices to check whether or not $\pi_\ell$ induces a group homomorphism $G \to \text{Sym}(UK_\ell \setminus F)$. In this case, we obtain $[G_\ell : H] = [G : H]$ and $\pi_\ell$ is a permutation representation for $G$'s action on the right-cosets $H \setminus G$. Otherwise, we have to enlarge the index $\ell$ and we would finally compute the index $[G : H]$ in this way. The following theorem was proved in [34]:

**Theorem 5.1** For a finitely $L$-presented group $G = \langle X | Q | \Phi | R \rangle$ and a homomorphism $\pi: F \to H$ into a finite group $H$, there exists an algorithm that decides whether or not $\pi$ induces a group homomorphism $G \to H$.

**Proof.** For an explicit algorithm, we refer to [34].

Coset enumeration for finitely $L$-presented groups allows various computations with finite index subgroups; e.g. computing the intersection of two finite index subgroups, computing the core of a finite index subgroup, solving the generalized word problem for finite index subgroups, etc. In the following, we demonstrate the application of our coset enumerator to the Fabrykowski-Gupta groups $\Gamma_p$. In particular, we show how to compute the number of finite index subgroups with a moderate index.

### 5.1 An Application of Coset Enumeration: Low-Index Subgroups

As an application of the coset enumeration process, we consider subgroups with small index in a finitely $L$-presented group. Since the finitely presented group $G_\ell$ from Eq. (4) naturally maps onto the finitely $L$-presented group $G$, it suffices to compute low-index subgroups of the finitely presented group $G_\ell$. These subgroups map to subgroups of $G$ with possibly smaller index. On the other hand, each finite index subgroup of $G$ has a full preimage with same index in $G_\ell$. Therefore it remains to remove duplicates from the list of subgroups obtained from the finitely presented group $G_\ell$. For finitely presented groups, an algorithm for computing all subgroups up to a given index was described in [16]. An implementation of this algorithm can be found in [17]. This implementation includes an algorithm for computing only the normal subgroups of a finitely presented group.
presented group \[14\]. The latter algorithm allows to deal with possibly larger indices than the usual low-index subgroup algorithms.

We first consider the Grigorchuk group \(G\): its lattice of normal subgroups is well-understood \[3,13\] while its lattice of finite index subgroups is widely unknown \[29\]. It is known that the Grigorchuk group has seven subgroups of index two \[29\]. In \[43\], it was shown that these index-two subgroups are the only maximal subgroups of \(G\). The implementation of our coset enumeration process allows us to compute the number of subgroups with index at most 64 in the group \(G\) \[34\]. Our computations correct the counts in \[7, Section 7.4\] and \[6, Section 4.1\]. The following list summarizes the number of subgroups (\(\leq\)) and the number of normal subgroups (\(\triangleleft\)) of \(G\):

| index | \(\leq\) | \(\triangleleft\) |
|-------|--------|---------|
| 1     | 1      | 1       |
| 2     | 7      | 7       |
| 4     | 31     | 7       |
| 8     | 183    | 7       |
| 16    | 1827   | 5       |
| 32    | 22931  | 3       |
| 64    | 378403 | 3       |

For the Fabrykowski-Gupta groups \(\Gamma_p\), where \(3 \leq p \leq 11\) is prime, we only found subgroups with prime-power index in \(\Gamma_p\). Their counts are as follows:

| index \(p\) | \(p = 3\) | \(p = 5\) | \(p = 7\) | \(p = 11\) |
|------------|----------|----------|----------|-----------|
| \(\leq\)   | \(\leq\) | \(\leq\) | \(\leq\) | \(\leq\)  |
| \(p^0\)    | 1        | 1        | 1        | 1         |
| \(p^1\)    | 4        | 4        | 6        | 6         |
| \(p^2\)    | 31       | 1        | 806      | 1         |
| \(p^3\)    | 1966     | 1        | ?        | ?         |
| \(p^4\)    | ?        | 4        | ?        | ?         |
| \(p^5\)    | ?        | 1        | ?        | ?         |
| \(p^6\)    | ?        | 1        | ?        | ?         |
| \(p^7\)    | ?        | 4        | ?        | ?         |

Here '?' denotes an index where our computations did not terminate within a reasonable amount of time. The only normal subgroups with index \(p^2\) are the derived subgroups since \(\Gamma_p/\Gamma_p' \cong Z_p \times Z_p\) holds \[28\]. For a prime power index \(d = p^\ell\) the groups \(\Gamma_d\) only admit subgroups with prime power index \(p^\ell\):

| index \(p^\ell\) \(= 2^2\) | \(p^\ell\) \(= 2^3\) | \(p^\ell\) \(= 3^2\) |
|---------------------------|---------------------|---------------------|
| \(\leq\)                  | \(\leq\)            | \(\leq\)            |
| \(p^0\)                   | 1                   | 1                   |
| \(p^1\)                   | 3                   | 3                   |
| \(p^2\)                   | 19                  | 7                   |
| \(p^3\)                   | 211                 | 7                   |
| \(p^4\)                   | 2419                | 11                  |

12
For the groups \( \Gamma_6 \) and \( \Gamma_{10} \), we obtain the following subgroup counts:

| index | \( \leq \leq \leq \leq \leq \) |
|-------|-------------------|
| 1     | 1 1 1 1 1         |
| 2     | 3 3 3 3 3         |
| 3     | 7 4 0 0 0         |
| 4     | 9 1 5 1           |
| 5     | 0 0 11 6          |
| 6     | 39 13 0 0         |
| 7     | 0 0 0 0           |
| 8     | 45 1 1 1          |
| 9     | 79 1 0 0          |
| 10    | 0 0 113 19        |

| index | \( \leq \leq \leq \leq \leq \) |
|-------|-------------------|
| 11    | 0 0 0 0           |
| 12    | 219 6 0 0         |
| 13    | 13 0 0 0          |
| 14    | 14 0 0 0          |
| 15    | 15 0 0 0          |
| 16    | 16 188 0 16       |
| 17    | 17 0 0 0          |
| 18    | 18 1299 7 0       |
| 19    | 19 0 0 0          |
| 20    | 20 0 0 ? ?        |

### 6 Computing Solvable Quotients

The coset enumeration process in [34] was used to prove the following version of the Reidemeister-Schreier theorem for finitely presented groups in [35]:

**Theorem 6.1** Each finite-index subgroup of a finitely \( L \)-presented group is finitely \( L \)-presented.

**Proof.** For a constructive proof, we refer to [35]. \( \square \)

The constructive proof of Theorem 6.1 allows us to apply the method for finitely \( L \)-presented groups to finite index subgroups of a finitely \( L \)-presented group. As an application of this method, we consider the successive quotients \( G/G^{(i)} \) of the derived series. This series is defined recursively by \( G^{(1)} = G' = [G,G] \) and \( G^{(i+1)} = [G^{(i)}, G^{(i)}] \) for \( i \in \mathbb{N} \). The isomorphism type of the abelian quotient \( G/G' \) can be computed with the methods from [5, 31] provided that \( G \) is given by a finite \( L \)-presentation. Moreover, it is decidable whether or not \( G' \) has finite index in \( G \); see [31, 5].

Suppose that \( G/G' \) is finite. Then the constructive proof of Theorem 6.1 allows us to compute a finite \( L \)-presentation for the finite index subgroup \( G' \leq G \). Then we can compute its abelianization and we can continue this process. In general, if \( G/G^{(i+1)} \) is finite, we can therefore compute the quotients \( G^{(i+1)}/G^{(i+2)} \) recursively. An alternative approach to compute the sections \( G^{(i)}/G^{(i+1)} \) could generalize the methods for finitely presented groups [38].

For the Grigorchuk group \( \mathfrak{G} \), the sections \( G^{(i)}/G^{(i+1)} \) of the derived series have been computed by Rozhkov [45]; see also [50]:

**Theorem 6.2 (Rozhkov [45])** The Grigorchuk group \( \mathfrak{G} \) satisfies \([\mathfrak{G} : \mathfrak{G}'] = 2^3\), \([\mathfrak{G} : \mathfrak{G}''] = 2^7\), and \([\mathfrak{G} : \mathfrak{G}^{(k)}] = 2^{2+2^k-2}\) for \( k \geq 3 \).

Our implementation of the Reidemeister-Schreier Theorem 6.1 yields that
\[
\mathfrak{G}/\mathfrak{G}' \cong (\mathbb{Z}_2)^3, \quad \mathfrak{G}'/\mathfrak{G}'' \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4, \quad \text{and} \quad \mathfrak{G}''/\mathfrak{G}^{(3)} \cong (\mathbb{Z}_2)^2 \times (\mathbb{Z}_4)^3 \times \mathbb{Z}_8.
\]
Since the abelianization $\Gamma_p/\Gamma'_p \cong \mathbb{Z}_p \times \mathbb{Z}_p$ of the Fabrykowski-Gupta group $\Gamma_4$ is finite [28], the derived subgroup $\Gamma'_p$ satisfies $[\Gamma_p : \Gamma'_p] = p^2$. A finite $L$-presentation for $\Gamma'_p$ can be computed with the methods in [35]. We obtain that

$$\Gamma'_3/\Gamma''_3 \cong (\mathbb{Z}_3)^2, \quad \Gamma''_3/\Gamma'''_3 \cong (\mathbb{Z}_3)^4, \quad \text{and} \quad \Gamma'''_3/\Gamma^{(4)}_3 \cong (\mathbb{Z}_3)^{10}$$

as well as $\Gamma'_d/\Gamma''_d \cong (\mathbb{Z}_d)^2$,

$$\Gamma''_d/\Gamma^{(3)}_d \cong \mathbb{Z}_2 \times (\mathbb{Z}_4)^2 \times \mathbb{Z}_8, \quad \text{and} \quad \Gamma^{(3)}_d/\Gamma^{(4)}_d \cong (\mathbb{Z}_2)^3 \times (\mathbb{Z}_4)^9 \times (\mathbb{Z}_8)^3.$$

For $5 \leq d \leq 41$, our computations suggest the following

**Proposition 6.3** For $d \geq 5$, $\Gamma_d$ satisfies $\Gamma_d/\Gamma'_d \cong (\mathbb{Z}_d)^2$ and $\Gamma_d/\Gamma''_d \cong (\mathbb{Z}_d)^{d-1}$.

**Proof.** It was already shown in [28] that $\Gamma_d/\Gamma'_d \cong \mathbb{Z}_d \times \mathbb{Z}_d$ holds. For the second statement, we combine the methods from [21] and [28]. For primes $p$, the structure of the congruence subgroups $\Gamma_p/\text{Stab}_p(n)$, $n \in \mathbb{N}$, were studied in [21]. Moreover, it was shown in [28] that, for $d \geq 5$, the index $[\Gamma'_d : \Gamma''_d]$ is finite.

Let $d \geq 5$ be given. Denote by $\text{Stab}_d(1)$ the first level stabilizer in $\Gamma_d$. Then $\Gamma_d = \text{Stab}_d(1) \times \langle a \rangle$ and $\text{Stab}_d(1) = \langle r^{\ell}, r^{\ell - 1}, \ldots, r^{\ell - k-1} \rangle$ hold. Since $\Gamma'_d = \langle [a, r] \rangle \Gamma_d = \langle r^{-a}r \rangle \Gamma_d$, we have that $\Gamma'_d \leq \text{Stab}_d(1)$ and, as $\Gamma_d/\Gamma'_d \cong \mathbb{Z}_d \times \mathbb{Z}_d$ holds, we have that $[\text{Stab}_d(1) : \Gamma'_d] = d$. More precisely, we have $\text{Stab}_d(1) = \Gamma'_d \leq \langle r \rangle$.

For each $0 \leq i < d$, we write $g_i = r^{a_i}$. In the following, indices are read modulo $d$. If $0 \leq \ell \leq d$, $g_i^{(d)}$ decomposes as $(1, \ldots, 1, a_\ell, 1, \ldots, 1)$ where $a_\ell$ is at position $i$. If $|\ell - k| > 1$, the commutator $[g_i^{(d)}, g_k^{(d)}]$ is trivial; otherwise, the commutator $[g_i^{(d)}, g_k^{(d)}]$ decomposes as $(1, \ldots, [a_\ell, r^k], 1, \ldots, 1)$ with $[a_\ell, r^k]$ at position $i$. Since $[a_\ell, r^k] \in \text{Stab}_d(1)$, we have that $[g_i^{(d)}, g_k^{(d)}] \in \text{Stab}_d(2)$. Thus, $\text{Stab}_d(1)/\text{Stab}_d(2)$ is abelian and it is generated by the images of the elements $g_0, \ldots, g_{d-1}$. Because $[a_\ell, r^k] = a^{-k} r^{-k} a^k r^k = g_{k-\ell}^{(d)} g_0^{(d)}$, we have that $[g_i^{(d)}, g_k^{(d)}] \in \text{Stab}_d(3)$ if and only if $\ell k \equiv 0 \pmod{d}$. Therefore $\text{Stab}_d(1)/\text{Stab}_d(2) = \mathbb{Z}_d \times \cdots \times \mathbb{Z}_d$ and $\Gamma_d/\text{Stab}_d(2) \cong \mathbb{Z}_d \times \mathbb{Z}_d$. Since $\text{Stab}_d(1)/\text{Stab}_d(2)$ is abelian, we have that $\text{Stab}_d(1)'/\text{Stab}_d(2)$. Because each generator of $\text{Stab}_d(1) \leq \text{Stab}_d(2)$ has order $d$, the largest abelian quotient $\text{Stab}_d(1)/\text{Stab}_d(1)'$ has order at most $d^d$. It follows that $\text{Stab}_d(2) = \text{Stab}_d(1)'$. Moreover, we have $\text{Stab}_d(2) = \text{Stab}_d(1)' \leq \Gamma'_d$ and, since $\Gamma'_d \leq \text{Stab}_d(1)$, it follows that $\Gamma''_d \leq \text{Stab}_d(2)$. The proofs in [28, 5] yield that $\text{Stab}_d(2) \leq \Gamma''_d$ if $d \geq 5$. Therefore $d^{d-1} = |\Gamma'_d/\text{Stab}_d(2)| = |\Gamma'_d/\Gamma''_d|$ and $\Gamma'_d/\Gamma''_d \cong \mathbb{Z}_d \times \cdots \times \mathbb{Z}_d$. □

The constructive proof of Theorem 6.1 in [35] yields a finite $L$-presentation over the Schreier generators of the subgroup. By the Nielsen-Schreier theorem (as, for instance, in [44, 6.1.1]), a subgroup $H$ with index $m = [G : H]$ in an $n$-generated finitely $L$-presented group $G$ has $nm + 1 - m$ Schreier generators. The Fabrykowski-Gupta groups are $2$-generated and therefore, the subgroup $\Gamma^{(3)}_3$ satisfies $[\Gamma_3 : \Gamma^{(3)}_3] = 3^{16}$. Thus $\Gamma^{(3)}_3$ has $3^{16} - 1$ Schreier generators as a subgroup of the $2$-generated group $\Gamma_3$. Therefore, computing the sections $\Gamma^{(i)}_3/\Gamma^{(i+1)}_3$, $i \geq 4$, with the above method is hard in practice.
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René Hartung, Mathematisches Institut, Georg-August Universität zu Göttingen, Bunsenstrasse 3–5, 37073 Göttingen, Germany

Email: rhartung@uni-math.gwdg.de