Stable compactification and gravitational excitons from extra dimensions *

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Abstract

We study inhomogeneous multidimensional cosmological models with a higher dimensional space-time manifold \( M = M_0 \times \prod_{i=1}^{n} M_i \) \((n \geq 1)\) under dimensional reduction to \( D_0 \)-dimensional effective models. Stability due to different types of effective potentials is analyzed for specific configurations of internal spaces. Necessary restrictions on the parameters of the models are found and masses of gravitational excitons (small inhomogeneous excitations of the scale factors of the internal spaces near minima of effective potentials) are calculated.

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1 Introduction

Many modern theories beyond the Standard Model include the hypothesis that our space-time has a dimensionality of more than four. String theory [1] and its recent generalizations — p-brane, M- and F-theory [2, 3, 4] — widely use this concept and give it a new foundation. The most consistent formulations of these theories are possible in space-times with critical dimensions $D_c > 4$, for example, in string theory there are $D_c = 26$ or $10$ for the bosonic and supersymmetric version, respectively. Usually it is supposed that a $D$-dimensional manifold $M$ undergoes a "spontaneous compactification" [5]–[8]: $M \rightarrow M^4 \times B^{D-4}$, where $M^4$ is the 4-dimensional external space-time and $B^{D-4}$ is a compact internal space. It is clear that such compactifications necessarily lead to cosmological consequences. One way to study them is to investigate simplified multidimensional cosmological models (MCM) with topology

$$M = M_0 \times M_1 \times \ldots \times M_n,$$

(1.1)

where $M_0$ denotes the $D_0$ - dimensional (usually $D_0 = 4$) external space-time and $M_i \ (i = 1, \ldots, n)$ are $D_i$ - dimensional internal spaces. To make the internal dimensions unobservable at present time the internal spaces have to be compact and reduced to scales near Planck length $L_{Pl} \sim 10^{-33} cm$, i.e. scale factors $a_i$ of the internal spaces should be of order of $L_{Pl}$. In this case we cannot move in extra-dimensions and our space-time is apparently 4-dimensional.

Compact internal spaces can exist for any sign of scalar curvature [9]. There is no problem to construct compact spaces with positive curvature [10, 11]. (For example, every Einstein manifold with constant positive curvature is necessarily compact [12].) However, Ricci-flat spaces and negative curvature spaces can also be compactified. This can be achieved by appropriate periodicity conditions for the coordinates [13]–[17] or, equivalently, through the action of discrete groups $\Gamma$ of isometries related to face pairings and to the manifold’s topology. For example, 3-dimensional spaces of constant negative curvature are isometric to the open, simply connected, infinite hyperbolic (Lobachevsky) space $H^3$ [10, 11]. But there exist also an infinite number of compact, multiply connected, hyperbolic coset manifolds $H^3/\Gamma$, which can be used for the construction of FRW metrics with negative curvature [14, 15]. These manifolds are built from a fundamental polyhedron (FP) in $H^3$ with faces pairwise identified. The FP determines a tessellation of $H^3$ into cells which are replicas of the FP, through the action of the discrete group $\Gamma$ of isometries [16]. In [17] it is, e.g., shown that by this way one can construct tori with genus two or more.
The simplest example of Ricci-flat compact spaces is given by \( D \)-dimensional tori \( T^D = \mathbb{R}^D / \Gamma \). Thus, internal spaces may have nontrivial global topology, being compact (i.e. closed and bounded) for any sign of spatial curvature.

In the cosmological context, internal spaces can be called compactified, when they are obtained by a compactification \([\mathbb{R}]\) or factorization ("wrapping") in the usual mathematical understanding (e.g. by replacements of the type \( \mathbb{R}^D \to S^D \), \( \mathbb{R}^D \to \mathbb{R}^D / \Gamma \) or \( H^D \to H^D / \Gamma \)) with additional contraction of the sizes to Planck scale. The physical constants that appear in the effective 4-dimensional theory after dimensional reduction of an originally higher-dimensional model are the result of integration over the extra dimensions. If the volumes of the internal spaces would change, so would the observed constants. Because of limitation on the variability of these constants \([19, 20]\) the internal spaces are static or at least slowly variable since the time of primordial nucleosynthesis and as we mentioned above their sizes are of the order of the Planck length. Obviously, such compactifications have to be stable against small fluctuations of the sizes (the scale factors \( a_i \)) of the internal spaces. This means that the effective potential of the model obtained under dimensional reduction to a 4-dimensional effective theory should have minima at \( a_i \sim L_{Pl} \) \((i = 1, \ldots, n)\). Because of its crucial role the problem of stable compactification of extra dimensions was intensively studied in a large number of papers \([21]-[37]\). As result certain conditions were obtained which ensure the stability of these compactifications. However, position of a system at a minimum of an effective potential means not necessarily that extra-dimensions are unobservable. As we shall show below, small excitations of a system near a minimum can be observed as massive scalar fields in the external space-time. In solid state physics, excitations of electron subsystems in crystals are called excitons. In our case the internal spaces are an analog of the electronic subsystem and their excitations can be called gravitational excitons. If masses of these excitations are much less than Planck mass \( M_{Pl} \sim 10^{-5} g \), they should be observable confirming the existence of extra-dimensions. In the opposite case of very heavy excitons with masses \( m \sim M_{Pl} \) it is impossible to excite them at present time and extra-dimensions are unobservable by this way.
2 The model

We consider a cosmological model with metric

$$ g = g^{(0)} + \sum_{i=1}^{n} e^{2\beta_i(x)} g^{(i)}, $$

which is defined on manifold \( (1.1) \) where \( x \) are some coordinates of the \( D_0 \) - dimensional manifold \( M_0 \) and

$$ g^{(0)} = g^{(0)}_{\mu\nu}(x) dx^\mu \otimes dx^\nu. $$

Let manifolds \( M_i \) be \( D_i \) - dimensional Einstein spaces with metric \( g^{(i)} \), i.e.

$$ R_{mn} \left[ g^{(i)} \right] = \lambda^i g_{mn}, \quad m, n = 1, \ldots, D_i $$

and

$$ R \left[ g^{(i)} \right] = \lambda^i D_i \equiv R_i. $$

In the case of constant curvature spaces parameters \( \lambda^i \) are normalized as \( \lambda^i = k_i (D_i - 1) \) with \( k_i = \pm 1, 0 \). We note that each of the spaces \( M_i \) can be split into a product of Einstein spaces:

$$ M_i = \prod_{k=1}^{n_i} M^{k}_i $$

Here \( M^{k}_i \) are Einstein spaces of dimensions \( D^{k}_i \) with metric \( g^{(i)}_{(k)} \):

$$ R_{mn} \left[ g^{(i)}_{(k)} \right] = \lambda^i_{k} g^{(i)}_{mn}, \quad (m, n = 1, \ldots, D^{k}_i) \quad \text{and} \quad R \left[ g^{(i)}_{(k)} \right] = \lambda^i_{k} D^{k}_i. $$

Such a splitting procedure is well defined provided \( M^{k}_i \) are not Ricci-flat \([38, 39]\). If \( M_i \) is a split space, then for curvature and dimension we have respectively \([38]\): \( R \left[ g^{(i)} \right] = \sum_{k=1}^{n_i} R \left[ g^{(i)}_{(k)} \right] \) and \( D_i = \sum_{k=1}^{n_i} D^{k}_i \).

Later on we shall not specify the structure of the spaces \( M_i \). We require only \( M_i \) to be compact spaces with arbitrary sign of curvature.

With total dimension \( D = \sum_{i=0}^{n} D_i \), \( \kappa^2 \) a \( D \) - dimensional gravitational constant, \( \Lambda \) - a \( D \) - dimensional cosmological constant and \( S_{YGH} \) the standard York - Gibbons - Hawking boundary term \([40, 41]\), we consider an action of the form

$$ S = \frac{1}{2\kappa^2} \int_M d^D x \sqrt{|g|} \left\{ R[g] - 2\Lambda \right\} + S_{add} + S_{YGH}. $$

The additional potential term

$$ S_{add} = - \int_M d^D x \sqrt{|g|} \rho(x) $$

(2.6)
is not specified and left in its general form, taking into account the Casimir effect [21], the Freund-Rubin monopole ansatz [6], a perfect fluid [42, 43] or other hypothetical potentials [35, 37]. In all these cases \( \rho \) depends on the external coordinates through the scale factors \( a_i(x) = e^{\beta_i(x)} \) \( (i = 1, \ldots, n) \) of the internal spaces. We did not include into the action (2.5) a minimally coupled scalar field with potential \( U(\psi) \), because in this case there exist no solutions with static internal spaces for scalar fields \( \psi \) depending on the external coordinates [35].

After dimensional reduction the action reads

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^Dx \sqrt{|g(0)|} \prod_{i=1}^n e^{D_i\beta^i} \left\{ \frac{\kappa^2}{2\kappa_0^2} R[g(0)] - G_{ij}g^{(0)\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j + \sum_{i=1}^n R[g^{(i)}] e^{-2\beta^i} - 2\Lambda - 2\kappa^2 \rho \right\}, \tag{2.7}
\]

where \( \kappa_0^2 = \kappa^2/\mu \) is the \( D_0 \)- dimensional gravitational constant, \( \mu = \prod_{i=1}^n \mu_i = \prod_{i=1}^n \int_{M_i} d^Dy \sqrt{|g^{(i)}|} \) and \( G_{ij} = D_i \delta_{ij} - D_i D_j \) \( (i, j = 1, \ldots, n) \) is the midisuperspace metric [44, 45]. Here the scale factors \( \beta^i \) of the internal spaces play the role of scalar fields. Comparing this action with the tree-level effective action for a bosonic string it can be easily seen that the volume of the internal spaces \( e^{-2\Phi} \equiv \prod_{i=1}^n e^{D_i\beta^i} \) plays the role of the dilaton field [38, 45, 46]. We note that sometimes all scalar fields associated with \( \beta^i \) are called dilatons. Action (2.7) is written in the Brans-Dicke frame. Conformal transformation to the Einstein frame

\[
\hat{g}_{\mu\nu} = e^{-4\Phi} g_{\mu\nu}, \quad \frac{1}{2\kappa_0^2} \int_{M_0} d^Dx \sqrt{|\hat{g}(0)|} \left\{ \hat{R} \left[ \hat{g}(0) \right] - \bar{G}_{ij} \hat{g}(0)_{\mu\nu} \partial_\mu \beta^i \partial_\nu \beta^j - 2U_{eff} \right\}. \tag{2.8}
\]

The tensor components of the midisuperspace metric (target space metric on \( \mathbb{R}_T^n \) ) \( \bar{G}_{ij} \) \( (i, j = 1, \ldots, n) \), its inverse metric \( \bar{G}^{ij} \) and the effective potential are respectively

\[
\bar{G}_{ij} = D_i \delta_{ij} + \frac{1}{D_0 - 2} D_i D_j, \tag{2.10}
\]
\[ G^{ij} = \frac{\delta^{ij}}{D_i} + \frac{1}{2 - D} \]  
(2.11)

and

\[ U_{\text{eff}} = \left( \prod_{i=1}^{n} e^{D_i \beta^i} \right)^{-\frac{2}{n_D}} \left[ -\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2\beta^i} + \Lambda + \kappa^2 \rho \right]. \]  
(2.12)

We remind that \( \rho \) depends on the scale factors of the internal spaces: \( \rho = \rho (\beta^1, \ldots , \beta^n) \). Thus, we are led to the action of a self-gravitating \( \sigma- \) model with flat target space \( (\mathbb{R}^n, \bar{G}) \) (2.10) and self-interaction described by the potential (2.12).

Let us first consider the case of one internal space: \( n = 1 \). Redefining the dilaton field as

\[ \varphi \equiv \pm \sqrt{\frac{D_1 (D - 2)}{D_0 - 2}} \beta^1 \]  
(2.13)

we get for action and effective potential respectively

\[ S = \frac{1}{2\kappa_0^2} \int d^D_0 x \sqrt{|\bar{g}^{(0)}|} \left\{ \bar{R} \left[ \bar{g}^{(0)} \right] - \bar{g}^{(0)\mu \nu} \partial_\mu \varphi \partial_\nu \varphi - 2U_{\text{eff}} \right\} \]  
(2.14)

and

\[ U_{\text{eff}} = e^{2\varphi} \left[ \frac{D_1 (D - 2)}{D_0 - 2} \right]^{1/2} \left[ -\frac{1}{2} R_1 e^{2\varphi} \left[ \frac{D_1 (D - 2)}{D_0 - 2} \right]^{1/2} + \Lambda + \kappa^2 \rho(\varphi) \right], \]  
(2.15)

where in the latter expression we use for definiteness sign minus.

Coming back to the general case \( n > 1 \) we bring midisuperspace metric (target space metric) (2.10) by a regular coordinate transformation

\[ \varphi = Q \beta, \quad \beta = Q^{-1} \varphi \]  
(2.16)

to a pure Euclidean form

\[ \bar{G}_{ij} d\beta^i \otimes d\beta^j = \sigma_{ij} d\phi^i \otimes d\phi^j = \sum_{i=1}^{n} d\phi^i \otimes d\phi^i, \]  
(2.17)

\( \bar{G} = Q'Q, \quad \sigma = \text{diag}(1, 1, \ldots , 1) \).

(The prime denotes the transposition.) An appropriate transformation \( Q: \beta^i \mapsto \phi^j = Q^i_j \beta^i \) is given e.g. by [44]

\[ \varphi^1 = -A \sum_{i=1}^{n} D_i \beta^i \]  
and

\[ \varphi^i = [D_{i-1}/\Sigma_{i-1} \Sigma_i]^{1/2} \sum_{j=i}^{n} D_j (\beta^j - \beta^{i-1}) \]  
(2.18)
where \( i = 2, \ldots, n \), \( \Sigma_i = \sum_{j=1}^{n} D_j \),

\[
A = \pm \left[ \frac{1}{D' D_0 - 2} \right]^{1/2},
\]

and \( D' = \sum_{i=1}^{n} D_i \). So we can write action \((2.9)\) as

\[
S = \frac{1}{2\kappa_0^2} \int_{M_0} d^D x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R} \left[ \hat{g}^{(0)} \right] - \sigma_{ik} \hat{g}^{(0)\mu\nu} \partial_\mu \varphi^i \partial_\nu \varphi^k - 2 U_{\text{eff}} \right\}
\]

with effective potential

\[
U_{\text{eff}} = e^{\frac{\kappa_0^2}{2} \varphi^2} \left( -\frac{1}{2} \sum_{i=1}^{n} R_i e^{-2(Q^{-1})_{ik} \varphi^k} + \Lambda + \kappa_0^2 \rho \right).
\]

### 3 Gravitational excitons as solutions of a linear \(\sigma\)-model

In general, the effective potential \((2.21)\) is a highly nonlinear function and it would be rather difficult to obtain explicit solutions \(\varphi^i\) of the Euler-Lagrange-equation for the corresponding \(\sigma\)-model action \((2.20)\) analytically. The situation crucially simplifies, when we are interested in small field fluctuations \(\xi^i\) around the minima of potential \((2.21)\) only.

Let us suppose that these minima are localized at points \(\bar{\varphi}_c, c = 1, \ldots, m\)

\[
\frac{\partial U_{\text{eff}}}{\partial \varphi^i} \bigg|_{\bar{\varphi}_c} = 0, \quad \xi^i \equiv \varphi^i - \varphi^i(c),
\]

\[
U_{\text{eff}} = U_{\text{eff}} (\bar{\varphi}_c) + \frac{1}{2} \sum_{i,k=1}^{m} \bar{a}_{(c)ik} \xi^i \xi^k + O(\xi^i \xi^k \xi^l)
\]

and that the Hessians

\[
\bar{a}_{(c)ik} := \left. \frac{\partial^2 U_{\text{eff}}}{\partial \xi^i \partial \xi^k} \right|_{\bar{\varphi}_c}
\]

are not vanishing identically. The action functional \((2.20)\) reduces then to a family of action functionals for the fluctuation fields \(\xi^i\)

\[
S_{(c)} = \frac{1}{2\kappa_0^2} \int_{M_0} d^D x \sqrt{|\hat{g}^{(0)}|} \left\{ \hat{R} \left[ \hat{g}^{(0)} \right] - 2 U_{\text{eff}} (\bar{\varphi}_c) - \sigma_{ik} \hat{g}^{(0)\mu\nu} \partial_\mu \xi^i \partial_\nu \xi^k - \bar{a}_{(c)ik} \xi^i \xi^k \right\}, \quad c = 1, \ldots, m.
\]
It remains to diagonalize the Hessians $\bar{a}_{(c)ik}$ by appropriate $SO(n)$-rotations $S_c: \xi \mapsto \psi = S_c \xi$, $S'_c = S^{-1}_c$

$$\bar{A}_c = S'_c M^2_c S_c, \quad M^2_c = \text{diag} (m^2_{(c)1}, m^2_{(c)2}, \ldots, m^2_{(c)n}), \quad (3.4)$$

leaving the kinetic term $\sigma_{ik} \hat{g}(0)^{\mu\nu} \partial_\mu \xi^i \partial_\nu \xi^k$ invariant

$$\sigma_{ik} \hat{g}(0)^{\mu\nu} \partial_\mu \xi^i \partial_\nu \xi^k = \sigma_{ik} \hat{g}(0)^{\mu\nu} \partial_\mu \psi^i \partial_\nu \psi^k, \quad (3.5)$$

and we arrive at action functionals for decoupled normal modes of linear $\sigma-$models in the background metric $\hat{g}(0)$ of the external space-time:

$$S = \frac{1}{2\kappa^2} \int_{M_0} d^D x \sqrt{|\hat{g}(0)|} \left\{ \tilde{R} \left[ \hat{g}(0) \right] - 2 \Lambda_{(c)\text{eff}} \right\} +$$

$$+ \sum_{c=1}^{n} \frac{1}{2} \int_{M_0} d^D x \sqrt{|\hat{g}(0)|} \left\{ -\hat{g}(0)^{\mu\nu} \epsilon^i_{\mu\nu} \psi^i \psi^j - m^2_{(c)i} \psi^i \psi^j \right\} , \quad (3.6)$$

where $c = 1, \ldots, m$, $\Lambda_{(c)\text{eff}} \equiv U_{\text{eff}} (\vec{\varphi}_c)$ and the factor $\sqrt{\mu/\kappa^2}$ has been included into $\psi$ for convenience: $\sqrt{\mu/\kappa^2} \psi \rightarrow \psi$.

Thus, conformal excitations of the metric of the internal spaces behave as massive scalar fields developing on the background of the external space-time. By analogy with excitons in solid state physics where they are excitations of the electronic subsystem of a crystal, the excitations of the internal spaces may be called gravitational excitons.

Before we turn to a discussion of concrete classes of effective potentials and physical conditions on the parameters of the model, which must be fulfilled for compatibility with observational data from our present time Universe, we consider some general features of the model.

First we note, that according to expansion (3.1) for $\bar{A}_c \neq 0$ and up to second order in $\xi^i$, the effective potential (2.21) has a minimum at a point $\vec{\varphi}_c$ iff

$$\xi' \bar{A}_c \xi = \sum_{i,k=1}^{n} \bar{a}_{(c)ik} \xi^i \xi^k \geq 0, \quad \forall \xi^k, \quad (3.7)$$

with exception of $\xi^1 = \xi^2 = \ldots = \xi^n = 0$. This condition is equivalent to the requirement that at least one of the exciton masses should be strictly positive, whereas the remaining could vanish

$$m^2_{(c)i} \geq 0, \quad m^2_{(c)k} > 0 \text{ for at least one } k. \quad (3.8)$$

In the following sections we focus on models with strictly positive exciton masses $m^2_{(c)i} > 0$, $\forall i$. In this case, according to the Sylvester
criterion, positivity of the quadratic form (3.7) is assured by the positivity of the principal minors of the matrix $A_c$:

$$
\begin{vmatrix}
\tilde{a}_{(c)11} & \tilde{a}_{(c)12} \\
\tilde{a}_{(c)21} & \tilde{a}_{(c)22}
\end{vmatrix}
> 0, \quad \ldots
$$

$$
\begin{vmatrix}
\tilde{a}_{(c)11} & \cdots & \tilde{a}_{(c)1n} \\
\tilde{a}_{(c)21} & \cdots & \tilde{a}_{(c)2n} \\
\vdots & \ddots & \vdots \\
\tilde{a}_{(c)n1} & \cdots & \tilde{a}_{(c)nn}
\end{vmatrix}
= \det \tilde{A}_c > 0.
$$

The consideration of Mexican-hat-type potentials, which correspond to degenerated minima ($m_2^2(c_1) = 0, m_2^2(c_2) > 0, \ldots$), yielding massless modes similar to Goldstone bosons we leave for a separate paper.

From a technical point of view, the explicit calculation of the exciton masses can be considerably simplified if one makes use of the equivalence of $\varphi$–representation and $\beta$–representation: Minima in $\varphi$–representation correspond to minima in $\beta$–representation. This property of the model is easily shown: Under the regular linear transformation (2.16) $\varphi = Q\beta$, which depends, according to (2.18) and (2.19), on the dimensional structure of the total midi-superspace $M$ only, extremum condition, Hessian and quadratic form transform as follows:

$$
\frac{\partial U_{\text{eff}}}{\partial \varphi^i} \bigg|_{\varphi_c} = \frac{\partial U_{\text{eff}}}{\partial \beta^k} \bigg|_{\beta_c} (Q^{-1})^i_k = 0, \quad \varphi_c = Q\beta_c, \quad (3.10)
$$

$$
a_{(c)ik} = \left. \frac{\partial^2 U_{\text{eff}}}{\partial \varphi^i \partial \varphi^k} \right|_{\varphi_c} = \left. \frac{\partial^2 U_{\text{eff}}}{\partial \beta^i \partial \beta^k} \right|_{\beta_c} \equiv Q^i_j \tilde{a}_{(c)jl} Q^l_k, \quad (3.11)
$$

$$
\xi = Q\eta, \quad \eta = Q^{-1}\xi, \quad \xi^i \equiv \varphi^i - \varphi^i_c, \quad \eta^i \equiv \beta^i - \beta^i_c \quad (3.12)
$$

$$
\eta^i A_c \eta = (Q^{-1}\xi)^i Q' A_c Q (Q^{-1}\xi) = \xi' \tilde{A}_c \xi. \quad (3.13)
$$

This means, first, that extrema in $\varphi$–representation correspond to extrema in $\beta$–representation. Second, (3.11) shows that $A_c$ and $\tilde{A}_c$ are congruent matrices [47]. Hence, their rank and signature coincide [47], and positive eigenvalues of $A_c$ correspond to positive eigenvalues of $\tilde{A}_c$. The equivalence of the representations is established.

Furthermore, it is easy to see from (2.17), (3.4) and (3.11) that eigenvalues of matrices $A_c$ coincide with eigenvalues of matrices $G^{-1}A_c$, so that exciton masses can be calculated without technical problems.
from the Hessian in $\beta-$representation directly. For two-scale-factor models ($n = 2$) we have, for example,

$$m_{(c)1,2}^2 = \frac{1}{2} \left[ Tr(B_c) \pm \sqrt{Tr^2(B_c) - 4 \det(B_c)} \right], \quad (3.14)$$

where

$$B_c = \bar{A}_c \quad \text{or} \quad B_c = \bar{G}^{-1} A_c. \quad (3.15)$$

It can be easily seen that $m_{(c)1,2}^2$ are positive iff $\bar{a}_{(c)11}, \bar{a}_{(c)22} > 0$ and $\bar{a}_{(c)11} \bar{a}_{(c)22} > \bar{a}_{(c)12}^2$.

As we will show explicitly in the next sections, models of the same type, e.g. for a one-component perfect fluid, may be unstable in the case of two independently varying scale factors ($\beta^1, \beta^2$), but become stable under scale factor reduction, i.e. when the scale factors are connected by a constraint $\beta^1 = \beta^2 = \beta$. The reason for this interesting behavior originates in the form of the effective potential $U_{\mathrm{eff}}$ at the extremum point.

Let us illustrate this situation with a reduction of an $n-$scale-factor model to a one-scale-factor model. In order to simplify our calculation we introduce the projection operator $P$ on the one-dimensional constraint subspace $\mathbb{R}_1^n = \{ \beta = (\beta^1, \ldots, \beta^n) \mid \beta^1 = \beta^2 = \ldots = \beta^n = \beta \}$ of the $n-$dimensional target space $\mathbb{R}_T^n$ of the $\sigma-$model:

$$P \mathbb{R}_T^n = \mathbb{R}_1^1 \subset \mathbb{R}_T^n. \quad (3.16)$$

Explicitly this projection operator can be constructed from the normalized base vector $\bar{e}$ of the subspace $\mathbb{R}_1^1$. With

$$\bar{e} = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \quad (3.17)$$

we have

$$P = \bar{e} \otimes \bar{e}' = \frac{1}{n} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & \cdots & 1 \\ \vdots \\ 1 & \cdots & 1 \end{pmatrix} = \frac{1}{n} \begin{pmatrix} \cdots \cdots \\ \vdots \\ \cdots \cdots \end{pmatrix} \quad (3.18)$$

and $P^2 = P$.

Let us now calculate the exciton mass $m_{(c)0}$ for the reduced model. For this purpose we introduce the exciton Lagrangian, written according to (2.9), (2.17), (3.6) and (3.13) in terms of the fluctuation fields $\tilde{\eta} = (\eta^1, \ldots, \eta^n)$, $\eta^i \equiv \beta^i - \beta^i_c$:

$$\mathcal{L}_{\mathrm{exci}} = - \left[ \tilde{\eta} \bar{G} \tilde{K} \tilde{\eta} + \tilde{\eta} A_{(c)} \tilde{\eta} \right]. \quad (3.19)$$
\( \hat{K} := \overline{\partial}_\mu g^{(o)\mu\nu} \overline{\partial}_\nu \) denotes the pure kinetic operator. Under scale factor reduction \( \bar{\eta} = (\eta, \ldots, \eta) \) this Lagrangian simplifies to

\[
\mathcal{L}_{\text{exci}} = - \left[ \gamma_1 \bar{\eta} \hat{K} \bar{\eta} + \gamma_{(c)2} \bar{\eta}^2 \right],
\]

(3.20)

\[
\gamma_1 := n \bar{e} \tilde{G} \bar{e} = \sum_{i,j} \tilde{G}_{ij}, \quad \gamma_{(c)2} := n \bar{e} A_{(c)} \bar{e} = \sum_{i,j} A_{(c)ij}
\]

(3.21)

so that the substitution \( \eta = \gamma_1^{-1/2} \psi \) yields the effective one-scale-factor Lagrangian

\[
\mathcal{L}_{\text{exci}} = - \left[ \psi \hat{K} \psi + \psi m^2_{(c)0} \psi \right]
\]

with exciton mass

\[
m^2_{(c)0} = \gamma_{(c)2} / \gamma_1.
\]

Taking into account that \( \bar{e} A_{(c)} \bar{e} = \text{Tr}[PA_{(c)}] \)

\[
A_{(c)} = Q' S_c M^2_{(c)} S_c Q
\]

and

\[
M^2_{(c)} = \text{diag}(m^2_{(c)1}, \ldots, m^2_{(c)n})
\]

the needed relation between the exciton masses of the reduced and unreduced \( n \)-scale-factor models is now easily established as

\[
m^2_{(c)0} = n \gamma_1^{-1} \text{Tr}\left[QPQ' S_c M^2_{(c)} S_c\right].
\]

(3.22)

With use of explicit expressions for transformation matrix \( Q \) (2.18) and target space metric \( \tilde{G}_{ij} \) (2.10) we have

\[
[Q \bar{e}]_i = - \frac{AD'}{\sqrt{n}} \delta_{i1}, \quad Q P Q' = D' \frac{D - 2}{D - 2} \bar{P},
\]

(3.23)

\[
\bar{P}_{lk} := \delta_{il} \delta_{k1} \quad \text{and} \quad \gamma_1 = D' \frac{D - 2}{D - 2},
\]

so that relation (3.22) can be finally rewritten in terms of masses and components of the \( SO(n) \)-matrices \( S_c \) only

\[
m^2_{(c)0} = \text{Tr}\left[\bar{P} S_c M^2_{(c)} S_c\right] = \sum_{i=1}^{n} \left(S_{(c)i1}\right)^2 m^2_{(c)i}.
\]

(3.24)

In its compact form this mass formula implicitly reflects the behavior of the effective potential \( U_{\text{eff}} \) in the vicinity \( \Omega_{\beta_c} \subset \mathbb{R}_1^n \) of the extremum point \( \beta_c \). So, the exciton masses squared \( m^2_{(c)1}, \ldots, m^2_{(c)n} \) describe the potential as function over the \( n \)-dimensional \( \beta_c \)-vicinity \( \Omega_{\beta_c} \), whereas \( m^2_{(c)0} \) characterizes \( U_{\text{eff}} \) as function over the line interval \( \Omega_{\beta_c} \cap \mathbb{R}_1^n \) only. From (3.24) it is obvious that a positive exciton mass in the reduced model, corresponding to a minimum of the effective potential over the line interval \( \Omega_{\beta_c} \cap \mathbb{R}_1^n \), is not only possible for stable configurations of the unreduced model \( m^2_{(c)1} > 0, \ldots, m^2_{(c)n} > 0 \), but even in cases
when the potential $U_{\text{eff}}$ has a saddle point at $\vec{\beta}_c$ and the unreduced model is unstable. For the masses we have in these cases $m_{(c)i}^2 > 0$, $m_{(c)k}^2 < 0$, for some $i$ and $k$, and massive excitons in the reduced model correspond to exciton - tachyon configurations in the unreduced model.

As conclusion of this section we want to make a few remarks concerning the form of the effective potential. From the physical point of view it is clear that the effective potential should satisfy following conditions:

\begin{align*}
(i) \quad a_{(c)i} &= e^{\beta_c} \gtrsim L_{PI}, \\
(ii) \quad m_{(c)i} &\leq M_{PI}, \\
(iii) \quad \Lambda_{(c)\text{eff}} &\to 0.
\end{align*}

The first condition expresses the fact that the internal spaces should be unobservable at the present time and stable against quantum gravitational fluctuations. This condition ensures the applicability of the classical gravitational equations near positions of minima of the effective potential. The second condition means that the curvature of the effective potential should be less than Planckian one. Of course, gravitational excitons can be excited at the present time if $m_i \ll M_{PI}$. The third condition reflects the fact that the cosmological constant at the present time is very small:

$$|\Lambda| \leq 10^{-54}\text{cm}^{-2} \approx 10^{-120}\Lambda_{PI},$$

where $\Lambda_{PI} = L_{PI}^{-2}$. Thus, for simplicity, we can demand $\Lambda_{\text{eff}} = U_{\text{eff}}(\vec{\beta}_c) = 0$. (We used the abbreviation $\Lambda_{\text{eff}} \equiv \Lambda_{(c)\text{eff}}$.) Strictly speaking, in the multi-minimum case ($c > 1$) we can demand $a_{(c)i} \sim L_{PI}$ and $\Lambda_{(c)\text{eff}} = 0$ only for one of the minima to which corresponds the present universe state. For all other minima it may be $a_{(c)i} \gg L_{PI}$ and $|\Lambda_{(c)\text{eff}}| \gg 0$.

In the following sections we test several types of internal space configurations and effective potentials on their compatibility with physical conditions (3.25).

4 Pure geometrical potentials: $\rho \equiv 0$

In the case of an effective potential of pure geometric type ($\rho \equiv 0$) the condition for the existence of an extremum $\frac{\partial U_{\text{eff}}}{\partial \beta^c} = 0$ implies a
fine-tuning
\[ \frac{R_k}{D_k} e^{-2\beta^k_c} = \frac{2\Lambda}{D-2} \equiv \tilde{C}, \quad k = 1, \ldots, n \]
\[ \implies e^{\beta^k_c} = \left[ \frac{R_k D_k}{R_i D_i} \right]^{1/2} e^{\beta^k_c} \]
of the scale factors and sign \( \Lambda = \text{sign} R_i \). With the help of the explicit formula for the target space metric (2.10) we get for the Hessian
\[ a_{(c)ik} = \frac{\partial^2 U_{\text{eff},0}}{\partial \beta^i \partial \beta^k} \bigg|_{\beta^c} = -\frac{4\Lambda_{\text{eff}}}{D_0-2} \left[ \frac{D_i D_k}{D_0-2} + \delta_{ik} D_k \right] = -\frac{4\Lambda_{\text{eff}}}{D_0-2} \tilde{G}_{ik} \]
\[ = -\frac{4\Lambda_{\text{eff}}}{D_0-2} \tilde{G}_{ik} \exp \left[ -\frac{2}{D_0-2} \sum_{i=1}^n D_i \beta^i_c \right], \quad (4.2) \]
so that the auxiliary matrix \( \tilde{G}^{-1} A_c \) is proportional to the \( n \)-dimensional identity matrix \( I_n \)
\[ \tilde{G}^{-1} A_c = -\frac{4\Lambda_{\text{eff}}}{D_0-2} I_n \quad (4.3) \]
and exciton masses \( m^2 \), in the previous section defined as eigenvalues of \( A_c \) or \( G^{-1} A_c \), are simply given as
\[ m_1^2 = \ldots = m_n^2 = m^2 = -\frac{4\Lambda_{\text{eff}}}{D_0-2} \exp \left[ -\frac{2}{D_0-2} \sum_{i=1}^n D_i \beta^i_c \right] \]
\[ = 2 \left| \tilde{C} \right|^{D_0-2} \prod_{i=1}^n \left| \frac{D_i}{R_i} \right|^{D_0-2}. \quad (4.4) \]
From (4.1) and (4.4) we see that massive excitons can only occur when scalar curvature as well as bare and effective cosmological constant are negative: \( R_k, \Lambda, \Lambda_{\text{eff}} < 0 \). The additional requirement \( |\Lambda_{\text{eff}}| \leq 10^{-54} \text{ cm}^{-2} \approx 10^{-120} \text{ cm}^{-2} \) leads not only to an upper bound for the masses of excitons \( m \leq 10^{-60} M_{\text{Pl}} \sim 10^{-55} \text{ g} \approx M_{\text{Pl}} \sim 10^{-5} \text{ g} \), independently from the number of scale factors, but also strongly narrows the class of possible internal space configurations. Let us demonstrate the latter fact with the help of three models with a different number of scale factors.

- a) one-scale-factor model:

Assuming that for a space-time configuration \( M_0 \times M_1 \) with four-dimensional external space-time \( (D_0 = 4) \) and compact internal factor space \( M_1 = H^{D_1}/\Gamma \) with constant negative curvature \( R_1 = -D_1(D_1-1) \) there exists a minimum of the effective potential at \( a_c = 10^2 L_{\text{Pl}} \) we get \( m^2 = 2(D_1-1)10^{-2(D_1+2)} M_{\text{Pl}}^2 \) and \( \Lambda_{\text{eff}} = -(D_1-1)10^{-2(D_1+2)} \Lambda_{\text{Pl}} \). Thus, according to (3.26), \( |\Lambda_{\text{eff}}| \leq 10^{-120} \Lambda_{\text{Pl}} \), the dimension of the internal space should be at least \( D_1 = 59 \).
b) two-scale-factor model:

Extending the previous example, let us suppose that \( D_0 = 4; \ M_1 = H^{D_1}/\Gamma_1; \ R_1 = -D_1(D_1 - 1); \ D_1 = 2; \ a_{(c)1} = 10^2 L_{Pl} ; \ M_2 = H^{D_2}/\Gamma_2; \ R_2 = -D_2(D_2 - 1). \) Effective cosmological constant and fine-tuning condition (4.1) read in this case:

\[
\Lambda_{eff} = -(D_2 - 1)^{D_2/2} \cdot 10^{-2(D_2+4)} \Lambda_{Pl}, \\
a_{(c)2} = (D_2 - 1)^{1/2} a_{(c)1} = (D_2 - 1)^{1/2} 10^2 L_{Pl}.
\]

Thus, conditions (3.25) are fulfilled for internal spaces \( M_2 \) with dimensions \( D_2 \geq D_{2, \text{crit}} = 40. \) Indeed, in the case of \( D_2 = 40 \) we have \( \Lambda_{eff} \simeq -10^{-120} \Lambda_{Pl}, \ a_{(c)2} \simeq 6 \cdot 10^2 L_{Pl} \) and hence for \( D_2 > 40 \) there hold the relations \( m_i \ll M_{Pl}, \ |\Lambda_{eff}| < 10^{-120} \Lambda_{Pl}, \ a_{(c)i} \geq L_{Pl} \) as required in (3.25).

c) \( n \)-scale-factor model:

For simplicity we assume \( D_0 = 4 \) and an internal space consisting of \( n \) factor spaces of the same type \( M_i = H^{D_i}/\Gamma_i; \ R_i = -D_i(D_i - 1); \ D_i = 2; \ a_{(c)i} = 10^2 L_{Pl}. \) The effective cosmological constant is then given as

\[
\Lambda_{eff} = -10^{-4(n+1)} \Lambda_{Pl}, \tag{4.6}
\]

so that at least \( n = 29 \) spaces \( M_i = H^2/\Gamma_i \) are necessary to fulfill condition \( |\Lambda_{eff}| < 10^{-120} \Lambda_{Pl}. \)

Summarizing the three examples we can say that for an effective potential of pure geometrical type, according to observational data, gravitational excitons should be extremely light particles with masses \( m \leq 10^{-55} \text{g} \) caused by inhomogeneous scale factor fluctuations of a composite internal factor space with negative curvature and sufficiently high dimension greater than some critical dimension. The value of this critical dimension depends on the topological structure of the internal factor space.

As conclusion we note that a conformal transformation \( g^{(1)} \rightarrow D_1 g^{(1)} \) with fixed \( \kappa_0^2 = \kappa^2/\mu, \ \Lambda = 2R_1 \) and \( R_1 \) leads in the limit \( D_1 \rightarrow \infty \) to \( a_c \rightarrow L_{Pl} \) and \( \Lambda_{eff} \rightarrow 0. \) But at the same time the exciton mass vanishes \( (m \rightarrow 0) \) and the effective potential degenerates into a step function with infinite height: \( U_{eff} \rightarrow \infty \) for \( a < 1 \) and \( U_{eff} = 0 \) for \( a \geq 1. \) Thus, in the limit \( D_1 \rightarrow \infty \) there is no minimum at all. To satisfy the strong condition \( \Lambda_{eff} = 0 \) we should consider the case \( \rho \neq 0. \)

In the next sections we analyze three concrete types of nonvanishing potentials \( \rho, \) originating in the presence of additional fields — Casimir potential, perfect fluid potential and "monopole" potential.
5 Casimir and Casimir-like potentials $\rho$

Because of a nontrivial topology of the space-time, vacuum fluctuations of quantized fields result in a non-zero energy density.

- a) One-scale-factor model:

For internal spaces with only one scale factor this energy density has the form $[21, 24, 27, 48, 49, 50]

$$\rho = Ce^{-D\beta},$$

(5.1)

where $C$ is a constant that strongly depends on the topology of the model. For example, for fluctuations of scalar fields the constant $C$ was calculated to take the values:

$C = -8.047 \cdot 10^{-6}$ if $M_0 = \mathbb{R} \times S^3$, $M_1 = S^1$ (with $e^{\beta_0}$ as scale factor of $S^3$ and $e^{\beta_0} \gg e^{\beta_1}$) \[24\]; $C = -1.097$ if $M_0 = \mathbb{R} \times \mathbb{R}^2$, $M_1 = S^1$ \[18\] and $C = 3.834 \cdot 10^{-6}$ if $M_0 = \mathbb{R} \times S^3$, $M_1 = S^3$ (with $e^{\beta_0} \gg e^{\beta_1}$) \[24\].

For an effective potential with $\rho$-term (5.1) (and $n = 1$) the zero-extremum-conditions

$$\frac{\partial U_{\text{eff}}}{\partial \beta}\bigg|_{\beta_c} = 0 \quad \text{and} \quad \Lambda_{\text{eff}} = 0$$

lead to a fine tuning of the parameters of the model

$$R_1 e^{-2\beta_c} = \frac{2D}{D-2}\Lambda, \quad R_1 e^{(D-2)\beta_c} = \kappa^2 CD$$

(5.2)

which implies sign $R_1 = \text{sign} \Lambda = \text{sign} C$. We note that a similar fine tuning was obtained by different methods in papers \[27\] (for one internal space) and \[37\] (for $n$ identical internal spaces).

The second derivative and mass squared read respectively

$$a_{11} = \frac{\partial^2 U_{\text{eff}}}{\partial \beta^2}\bigg|_{\beta_c} = (D-2)R_1\left(e^{-2\beta_c}\right)\frac{D-2}{D_0-2},$$

(5.3)

$$m^2 = \frac{D_0-2}{D_1}R_1\left(e^{-2\beta_c}\right)\frac{D-2}{D_0-2}.$$ (5.4)

Thus, the internal space should have positive curvature: $R_1 > 0$ (or for split space $M_1$ the sum of the curvatures of the constituent spaces $M^k_1$ should be positive).

Let us now perform an explicit test for a manifold $M$ with topology $M = \mathbb{R} \times S^3 \times S^3$, where $e^{\beta_0} \gg e^{\beta_1}$. Then \[24\] $C = 3.834 \cdot 10^{-6} > 0$ and, as $C, R_1 > 0$, the effective potential has a minimum provided $\Lambda > 0$. Normalizing $\kappa_0^2$ to unity, we get $\kappa^2 = \mu$ where $\mu = \frac{2\pi^{(d+1)/2}}{\Gamma(\frac{1}{2}(d+1))}$ is the
volume of the $d$-dimensional sphere. So we obtain $a_c \approx 1.5 \cdot 10^{-1}L_{Pl}$ and $m \approx 2.12 \cdot 10^2M_{Pl}$, and conditions (2.23) (i) and (ii) are not satisfied for this topology. For other topologies this problem needs a separate investigation.

b) Two-scale-factor models:

After these brief considerations on Casimir potentials for one-scale-factor models we turn now to some methods applicable for an analysis of two-scale-factor models with Casimir-like potentials. We proposed the use of such potentials of the general form

$$\rho = e^{-\sum_{i=1}^{n} D_i \beta_i} \sum_{k_1 < \ldots < k_n} |\epsilon_{k_1 k_2 \ldots k_n}| \sum_{\xi_1=0}^{D_{k_1}} \ldots \sum_{\xi_{n-1}=0}^{D_{k_n}} A^{(k_1)}_{\xi_1 \ldots \xi_{n-1}} \left( e^{\beta k_2} \right)^{\xi_1} \ldots \left( e^{\beta k_n} \right)^{\xi_{n-1}}$$

in our paper [37] in order to achieve a first crude insight into a possible stabilization mechanism of internal space configurations due to exact Casimir potentials depending on $n$ scale factors. (In (5.5) $\epsilon_{ik\ldots m}$ denotes the totally antisymmetric symbol ($\epsilon_{12\ldots n} = +1$) and $A^{(k_0)}_{\xi_0 \ldots \xi_{n-1}}$ are dimensionless constants which depend on the topology of the model.)

From investigations performed in the last decades (see e.g. [49, 50] and Refs. therein) we know that exact Casimir potentials can be expressed in terms of Epstein zeta function series with scale factors as parameters. Unfortunately, the existing integral representations of these zeta function series are not well suited for a stability analysis of the effective potential $U_{eff}$ as function over the total target space $\vec{\beta} \in \mathbb{R}^n_T$. The problems can be circumvented partially by the use of asymptotic expansions of the zeta function series in terms of elementary functions for special subdomains $\Omega_a$ of the target space $\Omega_a \subset \mathbb{R}^n_T$. According to [49, 50] potential (5.5) gives a crude approximation of exact Casimir potentials in subdomains $\Omega_a$. In contrast with other approximative potentials proposed in literature [26, 33] potential (5.5) shows a physically correct behavior under decompactification of factor space components [37]. The question, in as far (5.5) can be used in regions $\mathbb{R}^n_T \setminus \Omega_a$, needs an additional investigation. The philosophy of the proposed method consists in a consideration of potentials (5.5) on the whole target space $\mathbb{R}^n_T$, and testing of scale factors and parameters of possible minima of the corresponding effective potential on their compatibility with asymptotic approximations of exact Casimir potentials in $\Omega_a$. As a beginning, we describe in the following only some techniques, without explicit calculation and estimation of exciton masses.
Before we start our analysis of two-scale-factor models with Casimir-like potentials

\[
\rho = e^{-\sum_{i=1}^{2} D_i \beta_i} \left[ \sum_{i=0}^{D_2} A_{i}^{(1)} \frac{e^{i\beta_1}}{e^{(D_0+i)\beta_1}} + \sum_{j=0}^{D_1} A_{j}^{(2)} \frac{e^{j\beta_2}}{e^{(D_0+j)\beta_2}} \right]
\]  

(5.6)

let us introduce the following convenient (temporary) notations:

\( x := a_1 \equiv e^{\beta_1}, \ y := a_2 \equiv e^{\beta_2}, \ \kappa \xi := \kappa^2 A_\xi^{(1)}, \ P_\xi := \kappa^2 A_\xi^{(2)}. \) In terms of these notations the effective potential (2.12) reads

\[
U_{eff} = (xD_1 yD_2)^{-\frac{2}{D_0-2}} \left[ -\frac{R_1 y^2 x^2}{2} - \frac{R_2 y^2}{2} \right] + x^{-D_1} y^{-D_2} \left( \sum_{i=0}^{D_2} P_i y^i x^{-(D_0+i)} + \sum_{j=0}^{D_1} S_j y^j x^{-(D_0+j)} \right). 
\]  

(5.7)

For physically relevant configurations with scale-factors near Planck length

\[
0 < x, y < \infty 
\]  

(5.8)

we transform extremum conditions \( \partial_{\beta_1, \beta_2} U_{eff} = 0 \leftrightarrow \partial_{x, y} U_{eff} = 0 \) by factoring out of \( (xy)^{-D} \)-terms and taking combinations

\[ \partial_x U_{eff} \pm \partial_y U_{eff} = 0 \]

to an equivalent system of two algebraic equations in \( x \) and \( y \):

\[
I_{1+} = (xy)^{D-2-\frac{2}{D_0-2}} \left[ \frac{D_2 - D_1}{D_0-2} (R_1 y^2 + R_2 x^2) - \frac{2\Lambda}{D_0-2} D' x^2 y^2 \right] - \left( \frac{2D'}{D_0-2} + D \right) \left[ \sum_{i=0}^{D_2} P_i y^{D_0+D_1+i} x^{D_2-i} + \sum_{j=0}^{D_1} S_j y^{D_1-j} x^{D_0+D_2+j} \right] = 0
\]

(5.9)

\[
I_{1-} = (xy)^{D-2-\frac{2}{D_0-2}} \left[ \frac{D_1 - D_2}{D_0-2} (R_1 y^2 + R_2 x^2) + (R_1 y^2 - R_2 x^2) - \frac{2\Lambda}{D_0-2} (D_1 - D_2) x^2 y^2 \right] - \sum_{i=0}^{D_2} P_i \left[ D_0 \left( \frac{D_1 - D_2}{D_0-2} + 1 \right) + 2i \right] y^{D_0+D_1+i} x^{D_2-i} - \sum_{j=0}^{D_1} S_j \left[ D_0 \left( \frac{D_1 - D_2}{D_0-2} - 1 \right) - 2j \right] y^{D_1-j} x^{D_0+D_2+j} = 0.
\]

Thus, scale-factors \( a_1 \) and \( a_2 \) satisfying the extremum conditions are defined as common roots of polynomials (5.9). In the general case of
arbitrary dimensions \((D_0, D_1, D_2)\) and arbitrary parameters \(\{R_1, R_2, P_i, S_i\}\) these roots are complex, so that only a restricted subclass of them are real and fulfill condition (5.8). In the following we derive necessary conditions on the parameter set guaranteeing the existence of real roots satisfying (5.8). The analysis could be carried out using resultant techniques \([51]\) on variables \(x, y\) directly. The structure of \(I_{1\pm}\) suggests another, more convenient method \([52]\). Introducing the projective coordinate \(\lambda = y/x\) we rewrite (5.9) as \(I_{1\pm} = x^D I_{2\pm}(y, \lambda)\) with

\[
I_{2+} = -a_0(\lambda) + a_{D-2}(\lambda)y^{D-2} - y^D \Delta_+ = 0 \quad (a) \\
I_{2-} = -b_0(\lambda) + b_{D-2}(\lambda)y^{D-2} - y^D \Delta_- = 0 \quad (b)
\]

and coefficient-functions

\[
a_0(\lambda) = \left[\frac{2D'}{D_0-2} + D\right] \sum_{i=0}^{D_2} P_i \lambda^{D_0+D_1+i} + \sum_{j=0}^{D_1} S_j \lambda^{D_1-j} \\
a_{D-2}(\lambda) = \frac{D-2}{D_0-2} (R_1 \lambda^2 + R_2) \\
b_0(\lambda) = \sum_{i=0}^{D_2} P_i \left[D_0 \left(\frac{D_1-D_2}{D_0-2} + 1\right) + 2i\right] \lambda^{D_0+D_1+i} + \sum_{j=0}^{D_1} S_j \left[D_0 \left(\frac{D_1-D_2}{D_0-2} - 1\right) - 2j\right] \lambda^{D_1-j} \\
b_{D-2}(\lambda) = \frac{D_1-D_2}{D_0-2} (R_1 \lambda^2 + R_2) + (R_1 \lambda^2 - R_2) \\
\Delta_\pm = \frac{2\Lambda}{D_0-2} (D_1 \pm D_2).
\]

Equations (5.10) have common roots if the coefficient functions \(\{a_i(\lambda), b_i(\lambda)\}\) are connected by a constraint. This constraint is given by the vanishing resultant

\[
R_y[I_{2+}, I_{2-}] = w(\lambda) = 0. \quad (5.12)
\]

Now, the roots can be obtained in two steps. First, one finds the set of roots \(\{\lambda_i\}\) of the polynomial \(w(\lambda)\). Physical condition (5.8) on the affine coordinates \((x, y)\) implies here a corresponding condition on the projective coordinate \(\lambda = y/x\)

\[
Im(\lambda) = 0, \quad 0 < \lambda < \infty. \quad (5.13)
\]

Second, one searches for each \(\lambda_i\) solutions \(\{y_{ij}\}\) of (5.10). The complete set of physically relevant solutions of system (5.9) is then given in terms of pairs \(\{x_{ij} = y_{ij}/\lambda_i, \ y_{ij}\}\).
Because of the simple \( y \)-structure of equations (5.10) the polynomial \( w(\lambda) \) can be derived from (5.10) directly, without explicit calculation of the resultant. Taking
\[
\Delta_- l_2 - \Delta_+ l_2 = 0,
\]
and assuming \( y > 0 \) we get
\[
y^2 = \frac{L_3}{L_1}, \quad y^{D-2} = \frac{L_1}{L_2},
\]  
where
\[
L_1(\lambda) := \Delta_- a_0(\lambda) - \Delta_+ b_0(\lambda)
\]
\[
L_2(\lambda) := \Delta_- a_{D-2}(\lambda) - \Delta_+ b_{D-2}(\lambda)
\]
\[
L_3(\lambda) := a_0(\lambda)b_{D-2}(\lambda) - b_0(\lambda) a_{D-2}(\lambda)
\]
depend only on \( \lambda \). Excluding \( y \) from (5.14) yields the necessary constraint for the coefficient functions of equation system (5.10)
\[
w(\lambda) = L_2(\lambda)L_3^{D-2}(\lambda) - L_1^D(\lambda) = 0.
\]  
Together with condition (5.13), this polynomial of degree
\[
\deg_\lambda[w(\lambda)] = D^2
\]  
can be used for a first test of internal space configurations on stability of their compactification. If the corresponding parameters \( \{R_1, R_2, P_i, S_i\} \) allow the existence of positive real roots \( \lambda_i \), the space configuration is a possible candidate for a stable compactified configuration and can be further tested on the existence of minima of the effective potential \( U_{eff} \). Otherwise it belongs to the class of unstable internal space configurations.

Before we turn to the consideration of two-scale-factor models with factor spaces of the same topological type \( (M_1 = M_2) \) we note that for the coefficient functions (5.15), because of (5.8) and (5.14), there must hold
\[
\text{sign}(L_1)|_{\lambda_i} = \text{sign}(L_2)|_{\lambda_i} = \text{sign}(L_3)|_{\lambda_i}.
\]  
Furthermore we see from (5.16) that for even dimensions
\( D = \dim(M_1) + \dim(M_2) + \dim(M_0) \) of the product-manifold the polynomial \( w(\lambda) \) factors into two subpolynomials of degree \( D^2/2 \)
\[
w(\lambda) = \left[ L_2(\lambda)L_3^{\frac{D-2}{2}}(\lambda) + L_1^\frac{D}{2}(\lambda) \right] \left[ L_2(\lambda)L_3^{\frac{D-2}{2}}(\lambda) - L_1^\frac{D}{2}(\lambda) \right] = 0.
\]  
(5.19)
Two identical internal factor-spaces:

In the case of identical internal factor-spaces $M_1$ and $M_2$ we have $D_1 = D_2$, $P_i = S_i$, $R_1 = R_2$. If we assume additionally an external space-time $M_0$ with dim $M_0 = 4$ and, hence, $D = 2(D_1 + 2)$, then equations (5.10) and polynomial (5.19) can be rewritten as

\begin{align*}
I_{2+} &= -4(D_1 + 1)a_0(\lambda) + (D_1 + 1)R_1(\lambda^2 + 1)y^{D-2} - \\
&\quad -2D_1\Lambda y^D = 0 \quad (a) \\
I_{2-} &= (\lambda^2 - 1)\left[-2\bar{b}_0(\lambda) + R_1y^{D-2}\right] = 0 \quad (b)
\end{align*}

and

\begin{align*}
w(\lambda) &= 4\Lambda^2D_1^2 \left[2(\lambda^2 - 1)\right]^{2(D_1+2)} \times \\
&\quad \times \left[\bar{R}_1^{D_1+2}(2(D_1 + 1))^{D_1+1}\bar{L}_3^{D_1+1} + (-2\Lambda D_1)^{D_1+1}\bar{b}_0^{D_1+2}\right] \times \\
&\quad \times \left[\bar{R}_1^{D_1+2}(2(D_1 + 1))^{D_1+1}\bar{L}_3^{D_1+1} - (-2\Lambda D_1)^{D_1+1}\bar{b}_0^{D_1+2}\right] \\
&\quad \times \left[\bar{w}_+(\lambda)\right] \\
&\quad \times \left[\bar{w}_-(\lambda)\right] = 0 \quad (5.20)
\end{align*}

with the notations

\begin{align*}
\bar{L}_3 &= 2\bar{a}_0 - (\lambda^2 + 1)\bar{b}_0 \\
\bar{a}_0 &= \sum_{i=0}^{D_1} P_i \left[\lambda^{4+D_1+i} + \lambda^{D_1-i}\right] \quad (5.22) \\
\bar{b}_0 &= \sum_{i=0}^{D_1} P_i (2 + i)\lambda^{D_1-i}\sum_{j=0}^{i+1} \lambda^{2j}.
\end{align*}

From (5.20) and (5.21) we see that the constraint (5.12) is trivially satisfied for coinciding scale-factors $x = y$, i.e. $\lambda = 1$. Although (5.21) holds for all $\lambda$ corresponding to extrema of the effective potential, roots $y$ can be obtained from relations (5.14) only for $\lambda \neq 1$. In this nondegenerated case (5.14) reads

\begin{align*}
y^2 = -\frac{(D_1 + 1)R_1\bar{L}_3}{2\Lambda D_1\bar{b}_0}, \quad y^{D-2} = \frac{2\bar{b}_0}{R_1}. \quad (5.23)
\end{align*}
In the degenerated case $\lambda = 1$ relations (5.14) become undefined of type $0/0$ and the scale factor $y$ at the extremum point of the effective potential (5.7) must be found as a root of the polynomial $I_2+$ (5.20(a)) directly. For this polynomial we have now simply

$$I_2+(\lambda = 1) := \frac{\Lambda D_1}{D_1+1} y^{2(D_1+2)} - R_1 y^{2(D_1+1)} + 4 \sum_{i=0}^{D_1} P_i = 0. \quad (5.24)$$

Coming back to the general case of identical factor-spaces $M_1, M_2$ with coinciding or noncoinciding scale factors we note that there exists an interchange symmetry between $M_1$ and $M_2$, which becomes apparent in the root structure of the polynomial $w(\lambda)$. From

$$U_{\text{eff}} = (xy)^{-D_1} \left[ -\frac{R_i}{2} (x^2 - y^2) + \Lambda + (xy)^{-D_1} \sum_{i=0}^{D_1} P_i \left( y^i x^{-4-i} + x^i y^{-4-i} \right) \right] \quad (5.25)$$

we see that $x$ and $y$ enter (5.25) symmetrically. When one extremum of (5.25) is located at $\{x_i = a, y_i = b\}$ then because of the interchange symmetry $x \Leftrightarrow y$ there exists a second extremum located at $\{x_j = b, y_j = a\}$. So we have for the corresponding projective coordinates:

$$\lambda_i = y_i/x_i = b/a, \quad \lambda_j = y_j/x_j = a/b \implies \lambda_i = \lambda_j^{-1}. \quad (5.26)$$

By regrouping of terms in (5.21) it is easy to show that

$$w(\lambda^{-1}) = \lambda^{-D^2} w(\lambda) \quad (5.27)$$

and, hence, roots $\{\lambda_i \neq 0\}$ of $w(\lambda) = 0$ exist indeed in pairs $\{\lambda_i, \lambda_i^{-1}\}$. But there is no relation connecting this root-structure with a symmetry between $w_+(\lambda)$ and $w_-(\lambda)$ in (5.24) $w_+(\lambda^{-1}) \neq w_-(\lambda)$. For completeness, we note that relation (5.26) is formally similar to dualities recently investigated in superstring theory [53].

Before we turn to an analysis of minimum conditions for effective potentials $U_{\text{eff}}$ corresponding to special classes of solutions of $w(\lambda) = 0$
we rewrite the necessary second derivatives

\[
\partial_{xx}^2 U_{\text{eff}} = -\frac{R_1}{2} \left( \alpha_1 x^{-D_1-4} y^{-D_1} + \alpha_2 x^{-D_1-2} y^{-D_1-2} \right) + \\
+ \Lambda \alpha_2 x^{-D_1-2} y^{-D_1} + \\
+ \sum_{i=0}^{D_1} P_i \left( \alpha_3 x^{-2D_1} x^{-i-2D_1-6} + \alpha_4 x^{-2D_1-2} y^{-i-2D_1-4} \right)
\]

\[
\partial_{yy}^2 U_{\text{eff}} = \partial_{xx}^2 U_{\text{eff}} \bigg|_{x \leftrightarrow y}
\]

\[
\partial_{xy}^2 U_{\text{eff}} = -\frac{R_1}{2} \alpha_5 \left( x^{-D_1-3} y^{-D_1-1} + x^{-D_1-1} y^{-D_1-3} \right) + \\
+ \Lambda \alpha_6 x^{-D_1-1} y^{-D_1-1} + \sum_{i=0}^{D_1} P_i \alpha_7 \left( y^{-2D_1+1} x^{i-2D_1-5} + \right. \\
+ \left. \alpha_3 x^{-2D_1-1} y^{-i-2D_1-5} \right),
\]

where

\[
\alpha_1 = (D_1 + 2)(D_1 + 3) \quad \alpha_2 = D_1(D_1 + 1) \\
\alpha_3 = (2D_1 + i + 4)(2D_1 + i + 5) \quad \alpha_5 = D_1(D_1 + 2) \\
\alpha_4 = (2D_1 - i)(2D_1 - i + 1) \quad \alpha_6 = D_1^2 \\
\alpha_7 = (2D_1 + i + 4)(2D_1 - i),
\]

in the more appropriate form (notation \( \bar{\mu} = \lambda^{D_1} y^{2D_1} \))

\[
\partial_{xx}^2 U_{\text{eff}} = \lambda^2 \bar{\mu} \left[ -\frac{R_1}{2} \left( \alpha_1 \lambda^2 + \alpha_2 \right) y^{D-2} + \Lambda \alpha_2 y^D + \\
+ \sum_{i=0}^{D_1} P_i \left( \alpha_3 \lambda^{4+D_1+i} + \alpha_4 \lambda^{D_1-i} \right) \right]
\]

\[
\partial_{yy}^2 U_{\text{eff}} = \bar{\mu} \left[ -\frac{R_1}{2} \left( \alpha_1 + \alpha_2 \lambda^2 \right) y^{D-2} + \Lambda \alpha_2 y^D + \\
+ \sum_{i=0}^{D_1} P_i \left( \alpha_4 \lambda^{4+D_1+i} + \alpha_3 \lambda^{D_1-i} \right) \right]
\]

\[
\partial_{xy}^2 U_{\text{eff}} = \lambda \bar{\mu} \left[ -\frac{R_1}{2} \alpha_5 \left( \lambda^2 + 1 \right) y^{D-2} + \Lambda \alpha_6 y^D + \\
+ \sum_{i=0}^{D_1} P_i \alpha_7 \left( \lambda^{4+D_1+i} + \lambda^{D_1-i} \right) \right].
\]
Introducing the notations
\[
\tilde{A}_c := \begin{pmatrix}
\frac{\partial^2 U_{\text{eff}}}{\partial x^2} & \frac{\partial^2 U_{\text{eff}}}{\partial x \partial y} \\
\frac{\partial^2 U_{\text{eff}}}{\partial x \partial y} & \frac{\partial^2 U_{\text{eff}}}{\partial y^2}
\end{pmatrix}
\]
(5.31)
and
\[
w_{(c),1,2} := \frac{1}{2} \left[ Tr(\tilde{A}_c) \pm \sqrt{Tr^2(\tilde{A}_c) - 4 \det(\tilde{A}_c)} \right]
\]
(5.32)
the minimum conditions are given as
\[
w_{(c),1} > 0, \ w_{(c),2} \geq 0.
\]
(5.33)
In the degenerated case of coinciding scale-factors \(x = y, \lambda = 1\) there hold the following relations between the derivatives of effective potentials \(U_{\text{eff}}(x,y)\) and \(\tilde{U}_{\text{eff}}(y,y)\)
\[
\partial_y \tilde{U}_{\text{eff}} = \partial_x U_{\text{eff}} \big|_{x=y} + \partial_y U_{\text{eff}} \big|_{x=y}
\]
(5.34)
\[
\partial_{yy} \tilde{U}_{\text{eff}} = \partial_{xx} U_{\text{eff}} \big|_{x=y} + \partial_{yy} U_{\text{eff}} \big|_{x=y} + 2 \partial_{xy} U_{\text{eff}} \big|_{x=y}
\]
and minimum conditions reduce to
\[
\partial_y \tilde{U}_{\text{eff}} = 0, \ \partial_{yy} \tilde{U}_{\text{eff}} > 0
\]
(5.35)
with
\[
\partial_{yy} \tilde{U}_{\text{eff}} = 2y^{-2D+2} \left[-R_1(D_1 + 1)(2D_1 + 3)y^{D-2} + \Lambda D_1(2D_1 + 1)y^D + 4(D_1 + 1)(4D_1 + 5) \sum_{i=0}^{D_1} P_i \right].
\]
(5.36)
For convenience of the additional explicit calculations of constraint \(U_{\text{eff}}|_{\text{min}} = 0\) we rewrite also effective potential (5.25) in terms of variables \(y, \lambda\)
\[
U_{\text{eff}} = \lambda^{D_1}y^{-2D+4} \left[-\frac{R_1}{2} y^{D-2} (\lambda^2 + 1) + \Lambda y^D + \sum_{i=0}^{D_1} P_i \left( \lambda^{4+D_1+i} + \lambda^{D_1-i} \right) \right].
\]
(5.37)
The further analysis consists in a compatibility consideration of minimum conditions (5.33) and (5.35) with properties of the polynomial \(w(\lambda)\), expressions like (5.23) defining \(y^{D-2}\) and \(y^D = y^{D-2}y^2\) as functions of \(\lambda\) on the parameter-space \(\mathbb{R}_{\text{par}}^{D_1+3} = \{(R_1, \Lambda, P_i) | i = 0, \ldots, D_1\}\)
and the constraint $U_{eff}|_{\min} = 0$. As result we will get a first crude division of $\mathbb{R}^{D_1+3}_{par}$ in stability-domains allowing the existence of minima of the effective potential $U_{eff}$ and forbidden regions corresponding to instable internal space configurations.

After these general considerations we turn now to a more concrete analysis.

\textbf{- b.2) Noncoincident scale-factors ($\lambda \neq 1, R_1, \Lambda \neq 0$):}

First we consider the polynomial $w(\lambda)$. We know that stable internal space-configurations correspond to real projective coordinates $0 < \lambda < \infty$. So we have to test subpolynomials $w_{\pm}(\lambda)$ on the existence of such roots. The high degree $\deg_{\lambda}[w_{\pm}(\lambda)] = 2(D_1 + 2)(D_1 + 1) \geq 24$, $D_1 \geq 2$ (because of nonvanishing curvature of the factor-spaces $M_1, M_2$) allows only an analysis by techniques of number theory [54], the theory of ideals of commutative rings [51] or, for general parameter-configurations, numerical tests. In the latter case the number of effective test-parameters can be reduced by introduction of new coordinates in parameter-space

$$\mathbb{R}^{D_1+3}_{par} \to \mathbb{R}^{D_1+1}_{par} = \left\{ (\chi, p_i) \mid \chi = \left( \frac{2AD_1}{D_1+1} \right)^{D_1+1} \frac{2P_0}{R_1^{D_1+2}}, \quad p_i = \frac{P_i}{P_0}, \quad i = 1, \ldots, D_1 \right\}$$

(for $P_0 \neq 0, \quad P_0 = 1$ ; in the opposite case $P_0$ can be replaced by any nonzero $P_i$). Polynomials $w_{\pm}(\lambda) = 0$ transform then to $w_{\pm}(\lambda) = \frac{1}{2} R_1^{D_1+2} ((D_1 + 1)P_0)^{D_1+1} \bar{w}_{\pm}(\lambda) = 0$, where

$$\bar{w}_{\pm}(\lambda) := \tilde{L}_3^{D_1+1} \pm (-)^{D_1+1} \chi_0^{D_1+2} = 0$$

$$\tilde{L}_3 := \tilde{L}_3(P_0 = 1; P_1 = p_1, \ldots, P_{D_1} = p_{D_1})$$

$$\bar{b}_0 := \bar{b}_0(P_0 = 1; P_1 = p_1, \ldots, P_{D_1} = p_{D_1}).$$

(5.39)

Test are easy to perform with programs like \textsc{Mathematica} or \textsc{Maple}.

As a second step we have to consider minimum conditions (5.33).

Using (5.23) we substitute

$$y^{D-2} = \frac{2\bar{b}_0}{R_1}; \quad y^D = -\frac{(D_1 + 1)\tilde{L}_3}{AD_1}$$

(5.40)
into (5.30) and transform (5.33) to the following equivalent inequalities

\[ 2(D_1 + 1)(D_1 + 2)Q_1(\lambda) + Q_2(\lambda) > (D_1 + 2)(\lambda^2 + 1)\bar{b}_0(\lambda) \]

\[ [2(D_1 + 2)Q_1(\lambda) - (\lambda^2 + 1)\bar{b}_0(\lambda)][Q_2(\lambda) - (\lambda^2 + 1)\bar{b}_0(\lambda)] \geq (D_1 + 1)(\lambda^2 - 1)^2\bar{b}_0^2(\lambda) \]  

(5.41)

with notations

\[ Q_1(\lambda) := \sum_{i=0}^{D_1} P_i \left( \lambda^{i+D_1} + \lambda^{D_1-i} \right) \equiv \bar{a}_0(\lambda) \]

\[ Q_2(\lambda) := \sum_{i=0}^{D_1} P_i(i+2)^2 \left( \lambda^{i+D_1} + \lambda^{D_1-i} \right). \]  

(5.42)

Stability-domains in parameter-space \(\mathbb{R}^{D_1+3}_{\text{par}}\), corresponding to minima of the effective potential are given as intersections of domains defined by (5.41) with domains which allow the existence of physical relevant roots of \(\bar{w}_\pm(\lambda) = 0\). So numerical tests on minima are easy to perform. If we additionally assume that \(U_{\text{eff}}|_{\text{min}} = 0\) then the class of possible stability domains narrows considerably. Substitution of (5.40) into (5.37) transforms this constraint to

\[ (\lambda^2 + 1)\bar{b}_0(\lambda) = (D_1 + 2)Q_1(\lambda) \]

(5.43)

and inequalities (5.41) to

\[ D_1(\lambda^2 + 1)\bar{b}_0(\lambda) + Q_2(\lambda) > 0 \]

\[ (\lambda^2 + 1)[Q_2(\lambda) - (\lambda^2 + 1)\bar{b}_0(\lambda)] \geq (D_1 + 1)(\lambda^2 - 1)^2\bar{b}_0(\lambda) \geq 0. \]  

(5.44)

From (5.41) we get additional analytical insight into the minimum structure of the effective potential. Taking the limit \(\lambda \rightarrow 1\) we have

\[ Q_1(1) = 2\sum_{i=0}^{D_1} P_i \]

\[ Q_2(1) = [(\lambda^2 + 1)\bar{b}_0(\lambda)]_{\lambda=1} = 2\sum_{i=0}^{D_1} P_i(i+2)^2 \]  

(5.45)

so that inequalities (5.41) reduce to

\[ 2(D_1 + 2)\sum_{i=0}^{D_1} P_i > \sum_{i=0}^{D_1} P_i(i+2)^2 \]

(5.46)

and for \(U_{\text{eff}}|_{\text{min}} = 0\) even to

\[ (D_1 + 2)\sum_{i=0}^{D_1} P_i = \sum_{i=0}^{D_1} P_i(i+2)^2 > 0. \]  

(5.47)
From (5.30), (5.32) and (5.45) it is easy to see that in the case \( \lambda = 1 \) the eigenvalues \( w_{(c)1,2} \) of the Hessian \( \tilde{A}_c \) (5.31) are given as

\[
w_{(c)1} = (D_1 + 1) \left[ 2(D_1 + 2)Q_1(1) - Q_2(1) \right] > 0, \quad w_{(c)2} = 0 \quad (5.48)
\]

and the minimum of the effective potential in quadratic approximation (3.1) becomes degenerated.

- b.3) Coinciding scale-factors \( \lambda = 1, R_1, \Lambda \neq 0 \):

In this case extrema of the effective potential (5.25) are given by the roots of polynomial \( I_{2+}(\lambda = 1) \) (5.24). From the structure of \( I_{2+}(\lambda = 1) \) immediately follows:

1. Because \( I_{2+}(\lambda = 1) \) contains only terms with even degree in \( y \), there exist no real roots — and hence no extrema of the effective potential \( U_{\text{eff}} \) — for parameter combinations with:

\[
\text{sign} \left( \sum_{i=0}^{D_1} P_i \right) = \text{sign}(\Lambda) \neq \text{sign}(R_1). \quad (5.49)
\]

2. For arbitrary parameters \( \Lambda, R_1, \bar{\Delta} := \sum_{i=0}^{D_1} P_i \) roots of \( I_{2+}(\lambda = 1) \) can be found by analytical methods up to dimensions \( D_1 \leq 2 \) performing a substitution \( z := y^2 \) and using standard techniques for polynomials of degree \( \deg z I_{2+}(\lambda = 1) \leq 4 \). Because of \( R_1 \neq 0 \Leftrightarrow D_1 \geq 2 \) such considerations are restricted to the case \( D_1 = 2 \).

3. There exist no general mathematical methods to obtain roots of polynomials with degree \( \deg z I_{2+}(\lambda = 1) > 4 \) and arbitrary coefficients analytically. For special restricted classes of coefficients techniques of number theory [54], are applicable. We do not use such techniques in the present paper. For polynomials \( I_{2+}(\lambda = 1) \) and dimensions \( \dim M_1 = \dim M_2 = D_1 > 2 \) this implies that arbitrary parameter sets should be analyzed numerically or parameters \( \Lambda, R_1, \bar{\Delta} \) should be fine tuned — chosen ad hoc in such a way that \( I_{2+}(\lambda = 1) = 0 \) is fulfilled.

In the following we derive a necessary condition for the existence of a minimum of the effective potential with fine-tuned parameters. Using the ansatz

\[
\frac{\Lambda D_1}{D_1 + 1} = \sigma_1 y_0^{-2} ; \quad \bar{\Delta} := \sum_{i=0}^{D_1} P_i = \sigma_2 y_0^{D-2} \quad (5.50)
\]

equation (5.24) reduces to

\[
(\sigma_1 - R_1 + 4\sigma_2)y_0^{D-2} = 0. \quad (5.51)
\]
Without loss of generality we choose $\sigma_2$ as free parameter, and hence $\sigma_1 = R_1 - 4\sigma_2$, so that from relations (5.50)

$$y_0^{D-2} = \frac{\Delta}{\sigma_2}, \quad y_0^D = \frac{D_1 + 1}{\Lambda D_1} \Delta \left( \frac{R_1}{\sigma_2} - 4 \right) \quad (5.52)$$

and (5.36) minimum condition (5.35) reads

$$\partial_{yy}^2 \tilde{U}_{eff} \bigg|_{\text{min}} = 4y_0^{-2D+2}(D_1 + 1) \left[ 4(D_1 + 2) - \frac{R_1}{\sigma_2} \right] \Delta > 0 \quad (5.53)$$

or

$$(2D - \frac{R_1}{\sigma_2}) \sum_{i=0}^{D_1} P_i > 0. \quad (5.54)$$

We see that there exists a critical value $\sigma_c = \frac{R_1}{2D}$ which separates stability-domains with different signs of $\Delta$

$$\Delta = \sum_{i=0}^{D_1} P_i > 0 \iff |\sigma_2| > |\sigma_c|$$

$$\Delta = \sum_{i=0}^{D_1} P_i < 0 \iff |\sigma_2| < |\sigma_c|. \quad (5.55)$$

To complete our considerations of the degenerated case ($\lambda = 1$), $R_1$, $\Lambda \neq 0$ we derive the constraint $U_{eff} \big|_{\text{min}} = 0$. By use of (5.37) and (5.52) this is easily done to yield $\sigma_2 = \frac{R_1}{D} = 2\sigma_c$. So the constraint fixes the free parameter $\sigma_2$. Remembering that according to our temporary notation $y := a_2 \equiv e^{\beta_2}$ the value $y_0$ defines the scale factor of the internal spaces at the minimum position of the effective potential, we get now for the fine-tuning conditions (5.50)

$$\Lambda = \frac{(D-2)R_1}{D a_{(c)2}^2}, \quad \bar{\Delta} = \frac{R_1 a_{(c)2}^{D-2}}{D}, \quad \bar{\Delta}^2 = \frac{R_1^2 (D - 2)^{D-2}}{D^D \Lambda^{D-2}} \quad (5.56)$$

— the well-known conditions widely used in literature [33]. From (5.55), (5.56) and $a_{(c)2} > 0$ we see that for $\sigma_2 = \frac{R_1}{D} = 2\sigma_c$ the stability-domain in parameter-space $\mathbb{R}_{\text{par}}^{m+3}$ is narrowed to the sector

$$\bar{\Delta} = \sum_{i=0}^{D_1} P_i > 0, \quad R_1 > 0, \quad \Lambda > 0. \quad (5.57)$$
Vanishing curvature-scalars \((R_1 = 0), \Lambda \neq 0\):

For vanishing curvature scalars equations (5.20) reduce to

\[
I_{2+} = -2(D - 2)\bar{a}_0(\lambda) - (D - 4)\Lambda y^D = 0 \quad (a)
\]

\[
I_{2-} = -2(\lambda^2 - 1)\bar{b}_0(\lambda) = 0. \quad (b)
\]

Extrema of the effective potential are given by roots of \(I_{2-} = 0\) with scale-factors defined as

\[
y^D = -\frac{2(D_1 + 1)\bar{a}_0(\lambda)}{\Lambda D_1} = -\frac{2(D_1 + 1)Q_1(\lambda)}{\Lambda D_1}. \quad (5.59)
\]

Substitution of (5.59) into minimum-conditions (5.33) yields the following inequalities

\[
Q_1(\lambda) \geq 0, \quad Q_2(\lambda) \geq 0, \quad Q_1(\lambda) + Q_2(\lambda) > 0
\]

\[
8(D_1 + 1)(D_1 + 2)Q_1(\lambda)Q_2(\lambda) \geq (4D_1 + 5)(\lambda^2 - 1)^2\bar{b}_0^2(\lambda). \quad (5.60)
\]

From (5.59) and (5.60) we see that for even \(D\) positive \(y\) are only allowed when the bare cosmological constant \(\Lambda\) is negative: \(\Lambda < 0\).

As in the case of nonvanishing curvature scalars so also roots of \(I_{2-} = 0\) split into two classes. For nondegenerated physical relevant configurations \((\lambda \neq 1)\) the corresponding \(\lambda_i\) must satisfy equation

\[
\bar{b}_0(\lambda) = \sum_{i=0}^{D_1} P_i (2 + i) \lambda^{D_1 - i} \sum_{j=0}^{i+1} \lambda^{2j} = 0. \quad (5.61)
\]

For \(\lambda > 0\) this is only possible when there exist \(P_i\) with different signs.

In the case of degenerate configurations \((\lambda = 1)\) equation \(I_{2-} = 0\) is trivially satisfied and the scale-factor at the minimum of the effective potential given by

\[
y_0^D = -\frac{4(D_1 + 1)\sum_{i=0}^{D_1} P_i}{\Lambda D_1} \quad (5.62)
\]

with additional condition

\[
\sum_{i=0}^{D_1} P_i \geq 0, \quad \sum_{i=0}^{D_1} P_i (2 + i)^2 \geq 0, \quad \sum_{i=0}^{D_1} P_i [(2 + i)^2 + 1] > 0. \quad (5.63)
\]

From inequalities (5.60) and (5.63) immediately follows that effective potentials with parameters \((P_0 < 0, \ldots, P_{D_1} < 0)\) are not stable.
- b.5) Vanishing curvature scalars and vanishing cosmological constants \((R_1 = 0, \Lambda = 0)\):

In this case equations (5.20) contain only the projective coordinate \(\lambda = y/x\)

\[
I_{2+} = -2(D - 2)\bar{a}_0(\lambda) = 0 \quad (a)
\]

\[
I_{2-} = -2(\lambda^2 - 1)\bar{b}_0(\lambda) = 0. \quad (b)
\]

Corresponding physical configurations are possible for domains in parameter space given by

\[
\lambda = 1, \quad \sum_{i=0}^{D_1} P_i = 0 \quad (5.65)
\]

or

\[
\lambda \neq 1, \quad R_\lambda[\bar{a}_0(\lambda), \bar{b}_0(\lambda)] = 0. \quad (5.66)
\]

Minima of the effective potential are localized at lines \(\{\lambda_i = y/x\}\) and must be stabilized by additional terms. Otherwise we get an unstable "run-away" minimum of the potential.

- c) Generalization to \(n\)-scale-factor models:

The analytical methods used in the above considerations on stability conditions of internal space configurations with two scale-factors can be extended to configurations with 3 and more scale-factors by techniques of the theory of commutative rings [51]. In this case constraints, similar to polynomial (5.12) \(w(\lambda)\), follow from resultant systems on homogeneous polynomials. We note that in master equations (5.9) we can pass from affine coordinates \(\{x, y\}\) to projective coordinates \(\{X, Y, Z \mid x = X/Z, y = Y/Z\}\) and transform polynomials \(I_{1\pm}\) to homogeneous polynomials in \(\{X, Y, Z\}\) so that these generalizations are immediately to perform. A deeper insight in extremum conditions can be gained by means of algebraic geometry [52]. Polynomials \(I_{1\pm}\) define two algebraic curves on the \(\{x, y\}\)-plane and solutions of system (5.9) \(I_{1\pm}(x, y) = 0\) correspond to intersection-points of these curves. For \(n\) scale-factors extremum conditions \(\{\partial_{a_i} U_{\text{eff}} = 0\}_{i=1}^{n}\) would result in \(n\) polynomials \(I_n(x_1, \ldots, x_n) = 0\) defining \(n\) algebraic varieties on \(\mathbb{R}^n\). The sets of solutions of system \(I_n(x_1, \ldots, x_n) = 0\), or equivalently, the intersection points of the corresponding algebraic varieties, define the extremum points of \(U_{\text{eff}}\).


6  Perfect fluid potentials

In the case of a multicomponent perfect fluid the energy density reads

$$\rho = \sum_{a=1}^{m} \sum_{i=1}^{n} A_a \exp \left( - \sum_{i=1}^{n} \alpha_i^{(a)} D_i \beta_i \right),$$  \hspace{1cm} (6.1)

where $A_a$ are arbitrary positive constants. This formula describes an $m$-component perfect fluid with equations of state $P_i^{(a)} = (\alpha_i^{(a)} - 1) \rho^a$ in the internal space $M_i$ ($i = 1, \ldots, n$). In the external space each component corresponds to vacuum: $\alpha_i^{(a)} = 0$ ($a = 1, \ldots, m$). Physical values of $\alpha_i^{(a)}$ are restricted to

$$0 \leq \alpha_i^{(a)} \leq 2.$$  \hspace{1cm} (6.2)

It is easy to see that the case $\alpha_i^{(a)} = 0 \ \forall a, i$ corresponds to the vacuum in spaces $M_i$ and contributes to the bare cosmological constant $\Lambda$. Therefore we shall not consider this case here, because it leads to the pure geometrical potential of section 4. The other limiting case $\alpha_i^{(a)} = 2 \alpha_i^{(a)}$, $m = n$ formally coincides with the ”monopole” potential, which will be considered in the next section.

-a) One-scale-factor model:

Let us first analyze a one-component perfect fluid living in a one-scale-factor model. In this case energy density (6.1) reads $\rho = Ae^{-aD_1 \beta}$ and for a vanishing effective cosmological constant $\Lambda_{eff} = 0$ the extremum condition leads to

$$R_1 e^{(\alpha D_1 - 2) \beta c} = \kappa^2 \alpha D_1 A$$  \hspace{1cm} (6.3)

and

$$R_1 e^{-2 \beta c} = \frac{2 \alpha D_1}{\alpha D_1 - 2} \Lambda.$$  \hspace{1cm} (6.4)

For the second derivative of the effective potential in the minimum we obtain:

$$a_{11} = \left. \frac{\partial^2 U_{eff}}{\partial \beta^2} \right|_{\beta c} = (\alpha D_1 - 2)R_1 \left( e^{-2 \beta c} \right)^{D_0 - 2}.$$  \hspace{1cm} (6.5)

Because of $\alpha, A > 0$, equation (6.3) shows that the internal space $M_1$ should have positive curvature: $R_1 > 0$. From eq. (6.5) we see that
there exists a minimum if $\alpha > 2/D_1$. The corresponding mass squared of the exciton is given as

$$m^2 = \frac{(D_0 - 2)(\alpha D_1 - 2)}{D_1(D - 2)}R_1 \left( e^{-2\beta_c} \right) \frac{\alpha - 2}{D_0 - 2}. \quad (6.6)$$

For the critical value of $\alpha$ at $\alpha = 2/D_1$ the model becomes degenerated: $U_{\text{eff}} \equiv 0$.

As illustration, let $M_1$ be a 3-dimensional sphere and $a_c = 10L_{Pl}$.

This minimum can be achieved for

$$A = \left( \alpha \pi^2 \right)^{-1} \cdot 10^{\alpha D_1 - 2}.$$ 

Thus, $\frac{3}{\pi^2} < A \leq 5 \cdot 10^2$ and $0 < m^2 \leq 16 \cdot 10^{-5}$ for $2/D_1 < \alpha \leq 2$ and $D_0 = 4$. So, conditions (3.25) are satisfied.

-b) **Two-scale-factor model:**

For a multicomponent perfect fluid with energy density (6.1) living in a two-scale-factor model the effective potential reads

$$U_{\text{eff}} = \left( \prod_{i=1}^{m} e^{D_i \beta_i} \right)^{-\frac{2}{D_0 - 2}} \left[ -\frac{1}{2} \sum_{i=1}^{m} R_i e^{-2\beta_i} + \Lambda + \kappa^2 \sum_{a=1}^{m} A_a \exp \left( -\sum_{k=1}^{m} \alpha_k(a) D_k \beta_k \right) \right]. \quad (6.7)$$

Introducing the abbreviations

$$u_k^{(a)} := \alpha_k^{(a)} + 2 \sum_{\alpha_1=1}^{m} \frac{\alpha_1^{(a)} D_1}{D_1 - 2}, \quad v_k^{(a)} := \bar{h}_a \alpha_k^{(a)}, \quad c_k := \frac{2A D_k}{D_1 - 2},$$

$$h_a := \kappa^2 A_a e^{-\alpha_1^{(a)} D_1 \beta_1^1 - \alpha_2^{(a)} D_2 \beta_2^2} > 0,$$

$$\bar{h}_a := h_a \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{m} D_i \beta_i^1 \right]. \quad (6.8)$$

extremum condition and Hessian can be calculated to yield

$$\frac{\partial U_{\text{eff}}}{\partial \beta_k} = 0, \quad k = 1, 2 \Rightarrow$$

$$I_k := c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} e^{-\alpha_1^{(a)} D_1 \beta_1^1 - \alpha_2^{(a)} D_2 \beta_2^2} - R_k e^{-2\beta_c} = 0 \quad (6.9)$$

and

$$a_{(c)ik} \equiv \frac{\partial^2 U_{\text{eff}}}{\partial \beta_i \partial \beta_k} \bigg|_{\beta_c}$$

$$= -\frac{4A_{\text{eff}}}{D_0 - 2} \left( \frac{D_0 D_k}{D_0 - 2} + \delta_{ik} D_k \right) + \sum_{a=1}^{m} \bar{h}_a \alpha_k^{(a)} D_k \left( \alpha_i^{(a)} D_i - 2\delta_{ik} \right) \quad (6.10)$$
From the auxiliary matrix
\[
\left[ \tilde{G}^{-1} A_c \right]_{ik} = -\frac{4\Lambda_{\text{eff}}}{D_0 - 2} \delta_{ik} + J_{ik}, \quad J_{ik} = \sum_{a=1}^{m} v_k^{(a)} (D_k u_i^{(a)} - 2\delta_{ik}) \quad (6.11)
\]
we get then the exciton masses squared as
\[
m_{1,2}^2 = -\frac{4\Lambda_{\text{eff}}}{D_0 - 2} + \frac{1}{2} \left[ Tr(J) \pm \sqrt{\text{Tr}^2(J) - 4 \det(J)} \right]. \quad (6.12)
\]
Similar to the case of the Casimir-like potential, considered in section 3, extremum condition (6.1) has the form of a system of equations in variables \(z_1 = e^{-\beta_1}, \ z_2 = e^{-\beta_2}\)
\[
I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} z_1^{\alpha_1^{(a)}} z_2^{\alpha_2^{(a)}} - R_k z_k^2 = 0, \quad k = 1, 2 \quad (6.13)
\]
and for a given point \(p = \{\Lambda, R_1, R_2, A_1, \ldots, A_m, \alpha_1^{(1)}, \ldots, \alpha_2^{(m)}\}\) in parameter space \(\mathbb{R}_{\text{par}}^{2(m+1)}\) positions of extrema should be found as solutions of this system. In contrast with (5.9), the powers of \(z_i\) are real \((\alpha_i^{(a)} \in [0, 2] \subset \mathbb{R})\) so that in the general case the solutions should be found numerically. Partially analytical methods can be applied, e.g. for \(\alpha_i^{(a)}\) rational \((\alpha_i^{(a)} \in \mathbb{Q})\). In this case the representation \(\alpha_i^{(a)} D_i = \frac{n_i^{(a)}}{d_i^{(a)}}\) holds with natural numerator \(n_i^{(a)} \in \mathbb{N}\) and denominator \(d_i^{(a)} \in \mathbb{N}^+\), and \(n_i^{(a)}, d_i^{(a)}\) relative prime, \(\text{GCD}(n_i^{(a)}, d_i^{(a)}) = 1\). Introducing the least common multiple of the denominators \(l = \text{LCM}(d_1^{(1)}, \ldots, d_2^{(m)})\) and the natural numbers \(\varrho_i^{(a)} := \frac{l}{d_i^{(a)} n_i^{(a)}}\) one has \(\alpha_i^{(a)} D_i = \frac{\varrho_i^{(a)}}{\varrho_i^{(a)}}\). Eqs. (6.13) transform then to a system of polynomials
\[
I_k = c_k + D_k \kappa^2 \sum_{a=1}^{m} A_a u_k^{(a)} y_1^{\varrho_i^{(a)}} y_2^{\varrho_i^{(a)}} - R_k y_k^{2l} = 0, \quad k = 1, 2 \quad (6.14)
\]
in the new variables \(y_k = z_k^{1/l}\), which can be analyzed by algebraic methods \[51, 52\] and for rational parameters by methods of number theory \[54\]. So, for common roots of equations \(I_1 = 0, \ I_2 = 0\) the resultants \[54\] \(R_{y_1} [I_1, I_2], \ R_{y_2} [I_1, I_2]\) must necessarily vanish
\[
R_{y_1} [I_1, I_2] = w(y_2) = 0, \ R_{y_2} [I_1, I_2] = w(y_1) = 0 \quad (6.15)
\]
and the analysis of \((6.13)\) can be reduced to an analysis of the polynomials \(w(y_1), w(y_2)\) of degree

\[
\deg[w(y_1)], \quad \deg[w(y_2)] \leq \left[ l \max_a (\alpha_1^{(a)} D_1 + \alpha_2^{(a)} D_2, 2) \right]^2 \tag{6.16}
\]
in only one of the variables \(y_1\) and \(y_2\) respectively. In contrast with equation system \((5.9)\) for the Casimir-like potential, for arbitrary \(\vartheta^{(a)}\) the sum-term in \((6.14)\) cannot be factorized and the explicit calculation of resultants \((6.15)\) cannot be circumvented. So, the further analysis should be performed by computer-algebraic programs.

We turn now to some concrete subclasses of perfect fluids, which allow analytical considerations.

- b.1) \textbf{m–component perfect fluid with} \(\alpha_1^{(a)} = \alpha^{(a)}\):

In this case there exist no massive excitons for vanishing effective cosmological constants \(\Lambda_{\text{eff}} = 0\). Indeed, \(m_{1,2}^2 > 0\) and eq. \((6.12)\) imply \(\text{Tr}(J) > 0, \det(J) > 0\) which with

\[
J_{ik} = D_k W_1 - 2 \delta_{ik} W_2, \quad W_1 := \sum_{a=1}^m u^{(a)} v^{(a)}, \quad W_2 := \sum_{a=1}^m v^{(a)} \tag{6.17}
\]

read \(\text{Tr}(J) = D' W_1 - 4 W_2 > 0, \det(J) = 2 W_2 (2 W_2 - D' W_1) > 0\). But because of \(v^{(a)} = \tilde{h}_a \alpha^{(a)} > 0\) and hence \(W_2 > 0\) this leads to a contradiction. Thus, for the existence of massive excitons \(m_{1,2}^2 > 0\) the effective cosmological constant must be negative \(\Lambda_{\text{eff}} < 0\).

- b.2) \textbf{One-component perfect fluid with} \(\alpha_1 \neq \alpha_2\):

Again massive excitons are possible for negative effective cosmological constants \(\Lambda_{\text{eff}} < 0\) only. For \(\Lambda_{\text{eff}} = 0\) we have here at one hand \(\det(J) = -2 v_1 v_2 \delta D_0^2 \delta - 2 > 0, \delta := D_1 \alpha_1 + D_2 \alpha_2 - 2\) and hence \(\delta < 0\). On the other hand from \(\text{Tr}(J) > 0\) follows \(\delta (\alpha_1 + \alpha_2 - \frac{\delta + 2}{D_0 - 2}) > 0\) and hence \(0 > (D_0 - 2)(\alpha_1 + \alpha_2) + D_1 \alpha_1 + D_2 \alpha_2\). Because of \(\alpha_k > 0\) this is impossible and so should be \(\Lambda_{\text{eff}} < 0\).

- b.3) \textbf{One-component perfect fluid with} \(\alpha_1 = \alpha_2 = \alpha\):

For this subclass of b.1) extremum conditions \((6.9)\) can be considerably simplified to yield

\[
h = \kappa^2 A e^{-\alpha (D_1 \beta_1^2 + D_2 \beta_2^2)} = \frac{1}{(D_0 - 2)\alpha + 2} \left( \frac{D - 2}{D_k} R_k e^{-2 \beta_k^2} - 2 \Lambda \right) \tag{6.18}
\]

and the same fine-tuning condition as in the case of a pure geometrical potential

\[
\tilde{C} = \frac{R_1}{D_1} e^{-2 \beta_1^2} = \frac{R_2}{D_2} e^{-2 \beta_2^2}. \tag{6.19}
\]
An explicit estimation of exciton masses and effective cosmological constant can be easily done. Using (6.8), (6.12), (6.17) we rewrite the exciton masses squared as
\[
\left( \frac{m_1^2}{m_2^2} \right) = \frac{1}{D^2 - 2} \left\{ -4\Lambda + h [(D_0 - 2)\alpha + 2] \left( \frac{D'\alpha}{0} - 2 \right) \right\} \times 
\exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i \right]
\]
and transform with (6.18) inequalities \(m_1^2, m_2^2 > 0, h > 0\) to the following equivalent condition
\[
\frac{2}{D - 2} \Lambda < \tilde{C} < 0.
\]
Hence stable space configurations with massive excitons are only possible for internal spaces with negative curvature \(R_k < 0\). Reparametrizing \(\Lambda\) according to (6.21) as
\[
\Lambda = \frac{D - 2}{2} \left( \tilde{C} - \tau \right),
\]
with \(\tau > 0\) — a new parameter, we get for exciton masses squared and effective cosmological constant
\[
\left( \frac{m_1^2}{m_2^2} \right) = \left[ \left( \frac{D'\alpha\tau}{0} \right) - 2\tilde{C} \right] \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i \right],
\]
\[
\Lambda_{eff} = -\frac{D_0 - 2}{2} \left[ \frac{(D - 2)\alpha}{(D_0 - 2)\alpha + 2} - \tilde{C} \right] \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_i \right].
\]
According to definition (5.22) and equations (5.18), (1.14) the parameter \(\tau\) can be expressed in terms of \(\tilde{C}\) and \(R_k\) as
\[
\tau = \kappa^2 A \left( \frac{D_0 - 2}\alpha + 2 \right) \left[ \frac{\tilde{C}}{D} \right] \prod_{k=1}^{2} \left| \frac{D_k}{R_k} \right|^\frac{D_k\alpha}{2}.
\]
Comparison of equations (6.23), (6.24) with formula (4.4) shows that for \(\tau \ll \tau_0 \equiv \left| \tilde{C} \right| \min \left( \frac{1}{D_{\alpha}}, \frac{1}{(D_0 - 2)\alpha + 2} \right)\) we return to the pure geometrical potential considered in section 4. So physical conditions (3.25) are fulfilled for internal space configurations with sufficiently high dimensions greater than some critical dimension \(D_{crit}\). From (6.23) and
4.24 we see that depending on the value of $\tau$ this critical dimension $D_{\text{crit}}$ can only be larger than that for the pure geometrical model. According to (4.25) there exist excitons for any positive and finite values of the fluid parameter $A$, but than larger $A$ for fixed $\alpha$ than larger would be the critical dimension $D_{\text{crit}}$. (Here we take into account that $\kappa^2 = \mu$ and that the volume $\mu$ of the compact internal factor spaces with constant negative curvature is finite.)

Comparing the results of this subsection with the results for the one-scale-factor model at the beginning of the section we see that there exists a different behavior of the perfect fluid models in the case of vanishing effective cosmological constant $\Lambda_{\text{eff}} = 0$. For the one-scale-factor model massive excitons are allowed for $\Lambda_{\text{eff}} = 0$, whereas in the two-scale-factor model they cannot occur. This means that, according to the explanations of section 3, the $\Lambda_{\text{eff}} = 0$ extremum of the effective potential must be a saddle point and we are explicitly confronted with the specifics of the scale factor reduction described by eq. (3.24).

7 "Monopole" potentials

The "monopole" ansatz [6] consists in the proposal that the antisymmetric tensor field $F^{(i)}$ of rank $D_i$ is not equal to zero only for components corresponding to the internal space $M_i$. The energy density of these fields reads [25, 26]

$$\rho = \sum_{k=1}^{n} (f_k)^2 e^{-2D_k\beta_k},$$

(7.1)

where $f_k$ are arbitrary constants (free parameters of the model). Energy density (7.1) formally coincides with the energy density (6.1) of a multicomponent perfect fluid with parameters $\alpha_i^{(a)} = 2\delta_i^{(a)}$, $m = n$ and $A_k = (f_k)^2$, so that the calculations parallel that of the previous section.

Extremum condition (3.1) leads in the case of vanishing effective cosmological constant $\Lambda_{\text{eff}} = 0$ to a fine-tuning of the scale factors

$$\frac{R_k}{2\kappa^2 D_k(f_k)^2} = e^{-2\beta_k(D_k-1)}$$

(7.2)

and

$$\Lambda = \frac{1}{2} \sum_{k=1}^{n} R_k e^{-2\beta_k} D_k - 1 \frac{D_k}{D_k},$$

(7.3)
so that extrema are only possible iff $R_k > 0$, $\Lambda > 0$.

For a one-scale-factor model the exciton mass squared reads

$$m^2 = \frac{2(D_0 - 2)(D_1 - 1)}{D_1(D - 2)} R_1 \left( e^{-2\beta_c} \right)^{D_0 - 2}. \tag{7.4}$$

Condition (i) is satisfied if

$$f^2 \gtrsim R_1 / 2\kappa^2 D_1. \tag{7.5}$$

Let $M_1$ be a 3-dimensional sphere, then $R_1 = 6$ and $\kappa^2 = 2\pi^2$. To get a minimum of the effective potential for a scale factor $a_c = 10L_{Pl}$ we should take $f^2 \approx 5 \cdot 10^2$. For this value of $a_c$ and for $D_0 = 4$ the mass squared is $m^2 = \frac{16}{5} 10^{-5} \ll M_{Pl}^2$. Thus, all three conditions (3.25) are satisfied.

For a two-scale-factor model the exciton masses are given by (6.12)

$$m_{1,2}^2 = \frac{1}{2} \left[ Tr(J) \pm \sqrt{Tr^2(J) - 4 \det(J)} \right], \tag{7.6}$$

where in terms of abbreviations (6.8) matrix $J$ reads

$$J_{ik} = 4\tilde{h}_k(D_k - 1) \left[ \delta_{ik} - \frac{D_k}{D - 2} \right]. \tag{7.7}$$

One immediately verifies that $Tr(J) > 0$, $\det(J) > 0$, $Tr^2(J) - 4 \det(J) \geq 0$ for dimensions $D_1 > 1$, $D_2 > 1$ and hence $0 < m_2^2 \leq \frac{1}{2} Tr(J) \leq m_1^2 < Tr(J)$. This means that physical conditions (3.25) are satisfied if $Tr(J) \leq M_{Pl}^2$ and $e^{\beta_c} \gtrsim L_{Pl}$. Substituting

$$\tilde{h}_k = \frac{R_k}{2D_k} e^{-2\beta_c} \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_c^i \right] \tag{7.8}$$

into (7.7) we get the matrix trace as

$$Tr(J) = \frac{2}{D - 2} \left[ \sum_{k=1}^{2} \frac{(D_k - 1)}{D_k} R_k (D - 2 - D_k) e^{-2\beta_c^k} \times \exp \left[ -\frac{2}{D_0 - 2} \sum_{i=1}^{2} D_i \beta_c^i \right] \right]. \tag{7.9}$$

With this formula at hand we have e.g. for an internal space configuration $M_1 \times M_2 : M_1 = S^3$, $a_{(c)1} = 10L_{Pl}$; $M_2 = S^5$, $a_{(c)2} = 10^2L_{Pl}$ the estimate $Tr(J) \approx 56 \cdot 10^{-14} M_{Pl}^2 \ll M_{Pl}^2$ and all conditions (i) - (iii) of (3.25) are satisfied.
8 Cosmological stability of compactified internal space configurations

In section 3 we showed that inhomogeneous scale factor fluctuations of internal factor spaces have the form of massive scalar fields in the external space-time. These scalar fields are coupled with the gravitational field $\hat{g}^{(0)}_{\mu\nu}$ of the external space-time. Thus the energy of the external gravitational field can enlarge scalar field perturbations during the universe evolution. The type of reasoning that was used by Maeda [27] shows that for an expanding external space scalar field perturbations decrease with time.

To show it we consider an external space-time metric in the form

$$\hat{g}^{(0)} = -e^{2\hat{\gamma}_0} d\tau \otimes d\tau + \hat{a}_0^2(\tau) \bar{g}(0),$$

(8.1)

where $\hat{a}_0 = e^{\hat{\beta}_0}$ is the scale factor of the external space in the Einstein frame and $\bar{g}(0)$ is $D_0$-dimensional constant curvature space :

$$R[\bar{g}(0)] = k D_0 (D_0 - 1) \equiv R_0, k = \pm 1, 0.$$  

(8.2)

Since we are interested in cosmological solutions, we restrict our model to homogeneous scalar fields. The behavior of such model is described by the Lagrangian

$$L = e^{\hat{\gamma}_0} e^{D_0 \hat{\beta}_0} \left\{ e^{-2\hat{\beta}_0} R_0 + e^{-2\hat{\gamma}} D_0 (1 - D_0) \left( \dot{\hat{\beta}}_0 \right)^2 + e^{-2\hat{\gamma}} \sum_{i=1}^{n} (\dot{\phi}_i)^2 - 2 U_{eff} \right\} + 2 D_0 \frac{d}{d\tau} \left( e^{-\hat{\gamma}_0} e^{D_0 \hat{\beta}_0} \dot{\hat{\beta}}_0 \right)$$

(8.3)

with constraint

$$\frac{\partial L}{\partial \hat{\gamma}} = 0 \Rightarrow \rho = \frac{1}{2} e^{-2\hat{\gamma}} \sum_{i=1}^{n} (\dot{\phi}_i)^2 + U_{eff} = \frac{1}{2} \left[ e^{-2\hat{\beta}_0} R_0 + D_0 (D_0 - 1) e^{-2\hat{\gamma}} \left( \dot{\hat{\beta}}_0 \right)^2 \right],$$

(8.4)

where the overdot denotes differentiation with respect to time $\tau$ and $\rho = -T^{0}_{0}$ is the energy density of the system. The equations of motion for the scalar fields are

$$\ddot{\phi}_i + (D_0 \hat{\beta}_0 - \dot{\hat{\gamma}}) \dot{\phi}_i + e^{2\hat{\gamma}} \frac{\partial U_{eff}}{\partial \phi_i} = 0.$$  

(8.5)
It can be easily seen that the energy density $\rho$ satisfies following equation

$$\dot{\rho} \equiv \frac{d\rho}{d\tau} = -D_0 \dot{\beta}^0 e^{-2\dot{\beta}^0} \sum_{i=1}^{n} (\dot{\varphi}_i)^2. \quad (8.6)$$

We see that in a synchronous system $\dot{\gamma} = 0$ of an expanding external space ($H \equiv \dot{\beta}^0 = \frac{da}{dt} > 0$), the energy density $\rho$ decreases with time. Thus in the comoving system (Einstein frame) our model is always dissipative and the universe can reach the effective potential minimum.

Additionally to $\rho$ we can consider the quantity $E = v_0 \rho$, where $v_0 = e^{D_0 \dot{\beta}^0}$ is proportional to the volume of the external space. For closed external spaces $E$ plays the role of a total energy, which varies in time as

$$\dot{E} = D_0 \dot{\beta}^0 \left( E - v_0 e^{-2\dot{\beta}^0} \sum_{i=1}^{n} (\dot{\varphi}_i)^2 \right) \quad \text{(8.7)}$$

and near minima according to (8.6) as

$$\dot{E} = D_0 \dot{\beta}^0 v_0 \left( -\frac{1}{2} e^{-2\dot{\beta}^0} \sum_{i=1}^{n} (\dot{\varphi}_i)^2 + U_{eff} \right),$$

and near minima according to (8.7) as

$$\dot{E} = D_0 \dot{\beta}^0 v_0 \left( \Lambda_{eff} - \frac{1}{2} e^{-2\dot{\beta}^0} \sum_{i=1}^{n} (\dot{\varphi}_i)^2 + \frac{1}{2} \sum_{i=1}^{n} m_{(c)i}^2 (\dot{\psi}_i)^2 \right). \quad (8.8)$$

In contrast with (8.6) the total energy can for $\dot{\beta}^0 > 0$ decrease as well as increase.

Relevant for directly observable physical characteristics like cross sections, transition probabilities etc. is the energy density $\rho = -T^0_0$. It also enters the Einstein equations and defines by this way the dynamics of the scale factor $\dot{a}_0 = e^{\dot{\beta}^0}$ of the external space. The total energy $E$ is not less interesting, because it is connected with the Wheeler-deWitt equation and plays a role in a quantized midisuperspace model.

9 Conclusion

In the present lecture we studied stability conditions for compactified internal spaces. Starting from a multidimensional cosmological model we performed a dimensional reduction and obtained an effective four-dimensional theory in Brans-Dicke and Einstein frames. The Einstein frame was considered here as the physical one \cite{55}. In this frame we derived an effective potential. It was shown that small excitations of
the scale factors of internal spaces near minima of the effective potential have a form of massive scalar particles (gravitational excitons) developing in the external space-time. The exciton masses strongly depend on the dimensions and curvatures of the internal spaces, and possibly present additional fields living on the internal spaces. These fields will contribute to the effective potential, e.g. due to the Casimir effect, and by this way affect the dynamics of the scale factor excitations. So, the detection of the scale factor excitations can not only prove the existence of extra dimensions, but also give additional information about the dimension of the internal spaces and about fields possibly living on them.

For some particular classes of effective potentials with one, two and \( n \) scale factors we calculated exciton masses as functions of parameters of the internal spaces and derived stability criterions necessary for the compactification of the spaces.

Our analysis shows that conditions for the existence of stable configurations may depend not only on dimension and topology of the internal spaces, and additional fields contributing to the effective potential, but also on the number of independently oscillating scale factors. For example, \( n \)-scale-factor models with a saddle point as extremum of the effective potential \( U_{\text{eff}}(\beta_1, \ldots, \beta_n) \) would lead to an unstable configuration. Masses of the corresponding excitations would be positive (excitons) as well as negative (tachyons). Under scale factor reduction to an \( m \)-scale factor model with \( m < n \), i.e. when we connect some of the scale factors by constraints \( \beta_i = \beta_k \), the saddle point may, for certain potentials, reduce to a stable minimum point of the new effective potential \( U_{\text{eff}}(\beta_1, \ldots, \beta_m) \). As result all masses of excitations would be positive (excitons). We demonstrated this "stabilization via scale factor reduction" explicitly on a model with one-component perfect fluid.

In the present lecture we did not consider the case of degenerated minima of the effective potential, for example, self-interaction-type potentials or Mexican-hat-type potentials. In the former case one obtains massless fields with self-interaction. In the latter case one gets massive fields together with massless ones. Here, massless particles can be understood as analog of Goldstone bosons. This type of the potential was described in [34].

Another possible generalization of our model consist in the proposal that the additional potential \( \rho \) may depend also on the scale factor of the external space. It would allow, for example, to consider a perfect fluid with arbitrary equation of state in the external space.
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