Fractional diffusions with time-varying coefficients

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Abstract

This paper is concerned with the fractionalized diffusion equations governing the law of the fractional Brownian motion $B_H(t)$. We obtain solutions of these equations which are probability laws extending that of $B_H(t)$. Our analysis is based on McBride fractional operators generalizing the hyper-Bessel operators $L$ and converting their fractional power $L^{\alpha}$ into Erdélyi–Kober fractional integrals. We study also probabilistic properties of the r.v.’s whose distributions satisfy space-time fractional equations involving Caputo and Riesz fractional derivatives. Some results emerging from the analysis of fractional equations with time-varying coefficients have the form of distributions of time-changed r.v.’s.

Keywords: Fractional Brownian motion; Grey Brownian motion; Fractional derivatives; Generalized Mittag–Leffler functions; Riesz fractional derivatives.

1 Introduction

In this paper we consider fractional diffusion equations with time-varying coefficients. This investigation is inspired by the fractional extension of the diffusion equation governing the law of the fractional Brownian motion $B_H(t)$, $t \geq 0$, $0 < H < 1$. Fractional Brownian motion is a zero-mean Gaussian process with covariance function

\[
\text{Cov}(B_H(s), B_H(t)) = \frac{1}{2} \left( |t|^{2H} + |s|^{2H} - |t - s|^{2H} \right).
\] (1.1)

We first deal with equations of the form

\[
\left( t^{1-2H} \frac{\partial}{\partial t} \right)^{\alpha} u_\alpha(x, t) = H^\alpha \frac{\partial^2}{\partial x^2} u_\alpha(x, t), \quad \alpha \in (0, 1), \ H \in (0, 1), \ x \in \mathbb{R},
\] (1.2)

where the operator appearing in (1.2) is a special case of the fractional power $L^{\alpha}$ of the hyper-Bessel-type operator

\[
L = t^{a_1} \frac{d}{dt} t^{a_2} \frac{d}{dt} \cdots \frac{d}{dt} t^{a_{n+1}}, \quad t > 0,
\] (1.3)

where $a_j, j = 1, \ldots, n + 1$, are real numbers and $n \in \mathbb{N}$.
For $\alpha = 1$, (1.2) coincides with the equation governing the one-dimensional probability law of $B_H(t)$, while for $H = 1/2$, (1.2) is the classical time-fractional diffusion equation. There is a considerably large literature about fractional diffusion equations with constant coefficients, dating back to the second half of the Eighties ([9, 35], for a detailed review see [20]). Amongst the more recent papers we refer for example to [12, 19, 31]. In some specific cases, the relationship between the iterated Brownian motion and fractional diffusion equations was established and fully investigated [1, 29].

The theory of powers of Bessel-type operators was developed by Dimovski [6], McBride [22, 23, 24], McBride and Lamb [18]. The fractional powers of Bessel operators were recently considered in [10] in the analysis of random motions at finite velocity.

The theory developed by McBride permits us to express the operator $L^\alpha$ in terms of products of Erdély–Kober fractional integrals $I^\alpha_m$, defined as

$$
(t^\alpha f)(t) = \frac{m}{\Gamma(\alpha)} \int_0^t (t^m - u^m)^{\alpha-1} u^{m-1} f(u) \, du, \quad \alpha > 0, t > 0, m > 0.
$$

As far as we are aware, the generalization of time-fractional equations with time-varying coefficients by means of the McBride theory have not been studied so far, neither from an analytical nor from a probabilistic point of view.

The fundamental solution of (1.2) can be written as

$$
u_a(x, t) = \frac{1}{2^{1-a/2} t^H a} W_{-a/2, 1-a/2} \left( -\frac{2^{a/2} |x|}{t^H a} \right), \quad x \in \mathbb{R},
$$

where

$$W_{\gamma, \beta}(x) = \sum_{k=0}^{\infty} \frac{x^k}{k! \Gamma(\gamma k + \beta)}, \quad \gamma > -1, \beta > 0, x \in \mathbb{R},
$$

(1.5)
is the Wright function.

Formula (1.5) shows that the solution $u_a(x, t)$ of (1.2) coincides with the fundamental solution of the time-fractional diffusion equation with the time-scale change $t \to t^{2H}$. We prove that $u_a(x, t)$ is the law of the r.v. $X_{a, H}(t)$ which is connected to the fractional Brownian motion by means of the relation

$$X_{a, H}(t) \overset{d}{=} X_{2a, H} \left( |B_H(t)|^{1/\alpha} \right), \quad 0 < \alpha < \frac{1}{2}.
$$

We note that the r.v. $X_{a, H}(t)$ has variance

$$\text{Var} X_{2a, H}(t) = \frac{t^{2H a}}{2^{\alpha-1} \Gamma(\alpha + 1)}
$$

and thus the presence of the fractional derivative has a decreasing effect on the variance.

In Section 3.2 we establish a relationship between solutions of fractional equations

$$
\left( t^{1-2H} \frac{\partial}{\partial t} \right)^{\alpha} u = H^\alpha \frac{\partial^2 u}{\partial x^2}
$$

and solutions of higher order diffusion equations

$$
t^{1-2H} \frac{\partial u}{\partial t} = (-1)^k H \frac{\partial^k u}{\partial x^k}, \quad k > 2.
$$

This type of relationship frequently appeared in the recent literature in the case of fractional and higher-order diffusion equations with constant coefficients (see e.g. [7]).

In the second part of this paper, we consider fractional equations of the form

$$
\begin{align*}
(CD_v^{\alpha} u)(x, t) &= H t^{2H-1} \frac{\partial^2}{\partial x^2} u(x, t), \quad \nu \in (0, 1), x \in \mathbb{R}, t > 0, \\
U(x, 0) &= \delta(x),
\end{align*}
$$

(1.11)
where \( cD^\alpha_\nu \) is the \( \nu \)-th order Caputo time-fractional derivative. This problem was partially considered in [30]. The idea is to compare the two approaches (1.2) and (1.11).

The fundamental solution of (1.11) reads

\[
\int_{b}^{c} K^\alpha H\nu t \frac{\partial }{\partial t} u(x, t) \, dt = D_H t^\nu \frac{\partial }{\partial t} u(x, t), \quad \nu \in (0, 1), \quad x \in \mathbb{R}, \quad t > 0,
\]

where the function

\[
E_{\alpha, m, l}(z) = 1 + \sum_{k=0}^{\infty} \frac{1}{k!} \frac{1}{(\alpha + m + l + 1)_{k}} \Gamma(\alpha (jm + l + 1)), \quad z \in \mathbb{R},
\]

is the generalized three-index Mittag-Leffler function first introduced by Kilbas and Saigo in [13, 14, 15]; see also the recent monograph [11].

Our research is related to the recent papers [2, 3], where fractional generalizations of the heat equation for the fractional Brownian motion were inspired by some physical problems. In particular, in these papers the authors study

\[
cD^\nu_0, u(x, t) = D_H t^\nu \frac{\partial }{\partial t} u(x, t), \quad \nu \in (0, 1), \quad x \in \mathbb{R}, \quad t > 0,
\]

in order to take into account the joint action of trapping and correlated fluctuations for anomalous diffusions in heterogeneous media. In this framework we recover some of the analytical results discussed in [3]. For the special case \( H = 1/4 \) and \( \nu = 3/4 \), we are able to give an explicit and interesting form for the fundamental solution of (1.14) as the law of a time-changed Brownian motion.

## 2 Preliminaries

In this section we recall the main definitions and properties of fractional derivatives and fractional powers of hyper-Bessel-type differential operators for the convenience of the reader. There is a large literature about fractional powers of differential operators. We refer, for example, to the monographs [21] and [23]. We start our short review recalling the definitions of Riemann–Liouville fractional integrals and derivatives.

**Definition 2.1 (Riemann–Liouville integral).** Let \( f \in L^1_{\text{loc}}[a, b] \), where \(-\infty \leq a < t < b \leq \infty\), be a locally integrable real-valued function. The Riemann–Liouville integral is defined as

\[
I^\alpha_a f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t f(u) \left( t-u \right)^{1-\alpha} du = (f * K_\alpha)(t), \quad \alpha > 0,
\]

where \( K_\alpha(t) = t^{\alpha-1}/\Gamma(\alpha) \).

**Definition 2.2 (Riemann–Liouville derivative).** Let \( f \in L^1[a, b] \), \(-\infty \leq a < t < b \leq \infty\), and \( f * K_{m-\alpha} \in W^{m,1}[a, b] \), \( m = [\alpha] \), \( \alpha > 0 \), where \( W^{m,1}[a, b] \) is the Sobolev space defined as

\[
W^{m,1}[a, b] = \left\{ f \in L^1[a, b] : \frac{d^m}{dx^m} f \in L^1[a, b] \right\}.
\]

The Riemann–Liouville derivative of order \( \alpha > 0 \) is defined as

\[
D^\alpha_a f(t) = \frac{d^m}{dt^m} K_{m-\alpha} f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} f(s) ds.
\]
It is a simple matter to show that the Riemann–Liouville fractional derivative is rigorously the fractional power of the operator $D = d/dt$ in the classical theory of fractional powers of operators (see e.g. [3] and the references therein). We now recall that the so called Caputo derivative, widely used in applications, is in fact a regularization of the Riemann–Liouville derivative. For $n \in \mathbb{N}$, we denote by $AC^n[a, b]$ the space of real-valued functions $f(t)$, $t > 0$, which have continuous derivatives up to order $n - 1$ on $[a, b]$ such that $f^{(n-1)}(t)$ belongs to the space of absolutely continuous functions $AC[a, b]$:

$$AC^n[a, b] = \left\{ f : [a, b] \rightarrow \mathbb{R} : \frac{d^{n-1}}{dx^{n-1}}f(x) \in AC[a, b] \right\}. \quad (2.4)$$

**Definition 2.3** (Caputo derivative). Let $\alpha > 0$, $m = \lfloor \alpha \rfloor$, and $f \in AC^m[a, b]$. The Caputo derivative of order $\alpha > 0$ is defined as

$$cD^\alpha_{a^+} f(t) = \frac{t^{m-\alpha}}{\Gamma(m-\alpha)} \int_a^t (t-s)^{m-1-\alpha} \frac{d^m}{ds^m}f(s) \, ds. \quad (2.5)$$

Note that in the space of functions belonging to $AC^m[a, b]$ the relation between Riemann–Liouville and Caputo derivatives is given by the following

**Theorem 2.1.** For $f \in AC^m[a, b]$, $m = \lfloor \alpha \rfloor$, $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$, the Riemann–Liouville derivative of order $\alpha$ of $f$ exists almost everywhere and it can be written as

$$cD^\alpha_{a^+} f(t) = D^\alpha_{a^+} \left( f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^k}{k!} f^{(k)}(a^+) \right) = D^\alpha_{a^+} f(t) - \sum_{k=0}^{m-1} \frac{(t-a)^{k-\alpha}}{\Gamma(k+1-\alpha)} f^{(k)}(a^+). \quad (2.6)$$

We observe that the difference between the two definitions is given by the Riemann–Liouville derivative of the Taylor polynomial of order $m - 1$ centered in the lower extremum of integration $t = a$. Hereafter without loss of generality we will consider $a = 0$.

In view of this well-known relation between the two different fractional derivatives, we now recall the definition of fractional hyper-Bessel-type differential operators, studied in a series of works by McBride [22, 23, 24] and McBride and Lamb [18]. In [24] McBride considered the generalized hyper-Bessel operator

$$L = t^{a_n} \frac{d}{dt} t^{a_{n-1}} \ldots t^{a_1} \frac{d}{dt} t^{a_0}, \quad t > 0, \quad (2.7)$$

where $n$ is an integer number and $a_1, \ldots, a_{n+1}$ are real numbers.

The operator $L$ defined in (2.7) acts on the functional space

$$F_{p,\mu} = \left\{ f : t^{-\mu} f(t) \in F_p \right\}, \quad (2.8)$$

where

$$F_p = \left\{ f \in C^\infty : t^k \frac{d^k}{dt^k} f(t) \in L^p(0, \infty), \, k = 0, 1, \ldots \right\}, \quad (2.9)$$

for $1 \leq p < \infty$ and for any real number $\mu$ (see for details [22, 23]). The operator $L$ was first introduced and studied, as far as we know, by Dimovski [6].

Let us introduce the following constants related to the general operator $L$.

$$a_k = \sum_{k=1}^{n+1} a_k, \quad m = |a - n|, \quad b_k = \frac{1}{m} \left( \sum_{i=k+1}^{n+1} a_i + k - n \right), \quad k = 1, \ldots, n.$$

The definition of the fractional hyper-Bessel-type operator is given by
Definition 2.4. Let \( m = n - a > 0, f \in F_{p, \mu} \) and

\[
b_k \in A_{p, \mu, n} := \{ \eta \in \mathbb{R} : mn + \mu + m \neq \frac{1}{p} - ml, \ l = 0, 1, 2, \ldots \}, \quad k = 1, \ldots, n.
\]

Then

\[
L^\alpha f = m^a \ t^{-ma} \prod_{k=1}^{n} I_m^{b_k - a} f,
\]

where, for \( \alpha > 0 \) and \( mn + \mu + m > \frac{1}{p} \)

\[
I_m^{b_k-a} f = \frac{t^{-m\eta-ma}}{\Gamma(\alpha)} \int_0^t (t^m - u^m)^{\alpha-1} u^m f(u) \, d(u^m),
\]

and for \( \alpha \leq 0 \)

\[
I_m^{\eta+1} f = (\eta + \alpha + 1) I_m^{\eta+1} f + \frac{1}{m} I_m^{\eta+1} \left( \frac{d}{dt} f \right).
\]

For a full discussion about this approach to fractional operators we refer to [24], page 525.

A key-role in the next sections is played by the regularized Caputo-like counterpart of the operator \(2.10\). In analogy with Theorem 2.1 we introduce the following

Definition 2.5. Let \( \alpha \) be a positive real number, \( m = n - a > 0, f \in F_{p, \mu} \) is such that

\[
L^\alpha \left( f(t) - \sum_{k=0}^{b-1} \frac{t^k}{k!} f^{(k)}(0^+) \right)
\]

exists. Then we define \( ^{c}L^\alpha \) by

\[
^{c}L^\alpha f(t) = L^\alpha \left( f(t) - \sum_{k=0}^{b-1} \frac{t^k}{k!} f^{(k)}(0^+) \right),
\]

where \( b = \lfloor \alpha \rfloor \).

The relevance of this definition for the applications is due to the fact that for physical reasons we are interested in solving fractional Cauchy problems involving initial conditions on the functions (and their integer order derivatives), while in the case of Riemann–Liouville operators we must consider initial conditions involving fractional integrals and derivatives.

3 Fractional diffusions with time-dependent coefficients

Let us consider the fractional diffusion equation with time-dependent coefficients

\[
\left( t^{1-2H} \frac{\partial}{\partial t} \right)^\alpha u_a(x, t) = H^a \frac{\partial^2}{\partial x^2} u_a(x, t), \tag{3.1}
\]

where \( H \in (0, 1) \) is the Hurst parameter, \( \alpha \in (0, 1) \) and, according to Definition 2.4, the fractional operator is defined as

\[
\left( t^{1-2H} \frac{\partial}{\partial t} \right)^\alpha u_a(x, t) = (2H)^a t^{-2H} t_{2H}^{\alpha-a} u_a(x, t), \tag{3.2}
\]

where \( t_{2H}^{\alpha-a} \) is the Erdélyi–Kober fractional integral defined in (2.11). We can call (3.1) a bifractional equation, depending on the Hurst parameter \( H \) (which, in the case \( \alpha = 1 \), is related to the fractional
Brownian motion) and the real parameter \( \alpha \) (that is the fractional power of the operator). Notice that for \( \alpha = 1 \) we retrieve the heat equation with one time-dependent coefficient,

\[
t^{1-2H} \frac{\partial u}{\partial t} = H \frac{\partial^2 u}{\partial x^2},
\]

satisfied by the probability law of the fractional Brownian motion.

In view of definition 2.5 the regularized Caputo-like counterpart of the fractional operator (3.2) reads

\[
C \left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} u_a(x, t) = \left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} u_a(x, t) - (2H)^a \frac{t^{-2H\alpha}}{\Gamma(1-\alpha)} u_a(x, 0),
\]

in complete analogy with the classical theory of fractional derivatives.

**Theorem 3.1.** The solution to the Cauchy problem

\[
\begin{align*}
C \left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} u_a(x, t) &= H^a \frac{\partial^2}{\partial x^2} u_a(x, t), \quad \alpha \in (0, 1), \ t > 0 \\
u_a(x, 0) &= \delta(x),
\end{align*}
\]

is given by

\[
u_a(x, t) = \frac{1}{2^{1-a/2}t^{Ha/2}} W_{-a/2,1-a/2} \left( -\frac{2^{a/2} |x|}{t^{Ha}} \right).
\]

**Proof.** By using (3.4) we obtain the equation

\[
\left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} u_a(x, t) = H^a \frac{\partial^2}{\partial x^2} u_a(x, t) + (2H)^a \frac{t^{-2H\alpha}}{\Gamma(1-\alpha)} \delta(x).
\]

The Fourier transform of (3.7) reads

\[
\left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} \widehat{u}_a(\beta, t) = -H^a \beta^2 \widehat{u}_a(\beta, t) + (2H)^a \frac{t^{-2H\alpha}}{\Gamma(1-\alpha)}.
\]

The solution to (3.8) can be easily determined as

\[
\widehat{u}_a(\beta, t) = E_{a,1} \left( -\frac{\beta^2 t^{2H\alpha}}{2^a} \right),
\]

where \( E_{a,1}(\cdot) \) is the Mittag–Leffler function. Hence, by using (3.2), we have that

\[
\left(t^{1-2H} \frac{\partial}{\partial t}\right)^{\alpha} \widehat{u}_a(\beta, t) = (2H)^a t^{-2H\alpha} \sum_{k=0}^{\infty} \left( -\frac{\beta^2 t^{2H\alpha}}{2^a} \right)^k \frac{1}{\Gamma(ak+1)} = (2H)^a \sum_{k=0}^{\infty} \left( -\frac{\beta^2 t^{2H\alpha}}{2^a} \right)^k \frac{t^\text{2Hmak-2Hak}}{\Gamma(ak+1-\alpha)}
\]

\[
= -H^a \beta^2 E_{a,1} \left( -\frac{\beta^2 t^{2H\alpha}}{2^a} \right) + (2H)^a \frac{t^{-2H\alpha}}{\Gamma(1-\alpha)} = -H^a \beta^2 \widehat{u}_a(\beta, t) + (2H)^a \frac{t^{-2H\alpha}}{\Gamma(1-\alpha)},
\]

where we used the fact that

\[
I^{0,-a}_{2H} t^{2aH} = \frac{\Gamma(ak+1)}{\Gamma(ak+1-\alpha)} t^{2aH}.
\]

By inverting the Fourier transform \( \widehat{u}(\beta, t) \) we obtain the claimed result.
We note that the time-change $t \to t^{2H}$ reduces equation (3.1) to the classical time-fractional diffusion equation
\begin{equation}
\frac{cD_{0}^{\alpha}}{\partial x^2} u(x,t) = \frac{1}{2^{\alpha}} \frac{\partial^2}{\partial x^2} u(x,t),
\end{equation}
with fundamental solution given by (see e.g. [29])
\begin{equation}
u_{\alpha}(x,t) = \frac{1}{2^{1-a/2}t^{\frac{\alpha}{2}}} W_{-\alpha/2,1-a/2} \left( -\frac{2^{\alpha/2}|x|}{t^{1/2}} \right).
\end{equation}
This is a particularly interesting fact. In the first part of the paper in fact we prove formally that suitably time-changed fractional diffusion processes can be analyzed by means of the McBride theory. The converse is not true. More general equations can be treated by applying the same theory but cannot be recovered by means of simple deterministic time-changes.

**Theorem 3.2.** The solution to
\begin{equation}
\begin{cases}
\frac{c}{\partial x^2} u_{\alpha}(x,t) = H^\alpha \frac{\partial^2}{\partial x^2} u_{\alpha}(x,t), & \alpha \in (0,1), \ x \in \mathbb{R}, \ t > 0, \\
u_{\alpha}(x,0) = \delta(x),
\end{cases}
\end{equation}
can be represented as
\begin{equation}
u_{\alpha}(x,t) = \frac{H\sqrt{2}}{\sqrt{\pi(\sqrt{H})^{1-2H}}} \int_{0}^{\infty} 2^{2H-1} e^{-\frac{t}{\sqrt{2}}} u_{2\alpha}(x,z) dz,
\end{equation}
where $u_{2\alpha}$ is the solution to
\begin{equation}
\begin{cases}
\frac{c}{\partial x^2} u_{2\alpha}(x,t) = H^{2\alpha} \frac{\partial^2}{\partial x^2} u_{2\alpha}(x,t), & \alpha \in (0,\frac{1}{2}], \\
u_{2\alpha}(x,0) = \delta(x),
\end{cases}
\end{equation}
and
\begin{equation}
\begin{cases}
\frac{c}{\partial x^2} u_{2\alpha}(x,t) = H^{2\alpha} \frac{\partial^2}{\partial x^2} u_{2\alpha}(x,t), & \alpha \in (\frac{1}{2},1), \\
u_{2\alpha}(x,0) = \delta(x),
\end{cases}
\end{equation}

**Proof.** The proof follows the main steps of Theorem 2.1 in [29]. In more detail, by applying the duplication formula of the Gamma function we have
\begin{equation}
\Gamma \left( -\frac{ak}{2} + 1 - \frac{a}{2} \right) = \sqrt{\pi} 2^{x(k+1)} \Gamma(1-a(k+1)) / \Gamma \left( \frac{1}{2} - \frac{a(k+1)}{2} \right)
\end{equation}
By substituting (3.18) in (3.6) we have
\begin{equation}
u_{\alpha}(x,t) = \frac{1}{2^{1-a/2}t^{Ha}} \sum_{k=0}^{\infty} \frac{(-|x|/t^{Ha})^k \Gamma((1-\alpha(k+1))/2)}{k! \sqrt{\pi} 2^{a(k+1)-1/2} \Gamma(1-a(k+1))}
\end{equation}
\begin{align*}
&\quad \quad = \frac{1}{2^{1+a/2} \sqrt{\pi} t^{Ha}} \sum_{k=0}^{\infty} \frac{(-|x|/t^{Ha})^k \int_{0}^{\infty} e^{-w} w^{-(a(k+1)/2)-1} \Gamma(1-a(k+1))}{k! 2^{a(k+1)/2} \Gamma(1-a(k+1))} dw \\
&\quad \quad = \frac{1}{2^{1+a/2} \sqrt{\pi} t^{Ha}} \int_{0}^{\infty} e^{-w} w^{a/2-1/2} \sum_{k=0}^{\infty} \frac{(-|x|/t^{Ha})^k \Gamma((1-a(k+1))/2)}{k! 2^{a(k+1)/2} \Gamma(1-a(k+1))} dw \\
&\quad \quad = \frac{2^{1-a}}{2^{1+a/2} \sqrt{\pi} t^{Ha}} \int_{0}^{\infty} e^{-w} w^{a/2-1/2} t^{Ha} (2^2 w)^{a/2} u_{2\alpha}(x, \sqrt{t^{2}w^{2/\alpha}}) dw.
\end{align*}
The solution of the Cauchy problem

\[ u(t, x) = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-w} w^{-1/2} u_{2a}(x, \sqrt{2}w^{1/2}) dw \]

\[ = [z = \sqrt{2}(2^{3/2})] \]

\[ = \frac{H \sqrt{2}}{\sqrt{\pi}t^{H}} \int_0^\infty z^{H-1} e^{-\frac{1}{4}z^2} u_{2a}(x, z) dz. \]

\[ \Box \]

Remark 3.1. In the special case \( \alpha = \frac{1}{2} \), the solution reads

\[ u_{1/2}(x, t) = \frac{H \sqrt{2}}{\sqrt{\pi}t^{1-2H}} \int_0^\infty z^{2H-1} e^{-\frac{1}{8}z^2} u_1(x, z) dz, \] (3.20)

where \( u_1(x, t) \) is the fundamental solution of the heat equation

\[ \left(t^{1-2H} \frac{\partial}{\partial t}\right) u_1(x, t) = H \frac{\partial^2}{\partial x^2} u_1(x, t). \] (3.21)

Therefore \( u_{1/2}(x, t) \) coincides with the probability density of the r.v. \( B_H^1 \left([B_H^1(t)]^{1/2}\right) \), where \( B_H^1 \) and \( B_H^2 \) are independent fractional Brownian motions. We note that \( \text{Var} B_H^1(t) = t^{2H} \) and \( \text{Var} B_H^2(t) = 4^t t^{2H} \).

If \( H = \frac{1}{2} \), we recover the result discussed in (3.3), i.e. \( u_{1/2}(x, t) \) coincides with the probability density of the iterated Brownian motion \( B^1 \left([B^1(t)]^{1/2}\right) \) \( (\text{Var} B^1(t) = t \) and \( \text{Var} B^2(t) = 4t) \).

Remark 3.2. We observe that, in the more general case, for \( \alpha \in (0, 1) \), the solution \( u_\alpha(x, t) \) coincides with the probability density of the r.v. \( X_{2a,H} \left([B_H(t)]^{1/2}\right) \), where \( X_{2a,H}(t), t \geq 0 \), is a r.v., independent from \( B_H(t) \) with density law equal to \( u_{2a}(x, t) \).

Remark 3.3. It is simple to find the moments of the process \( X_{\alpha,H}(t), t \geq 0 \), by means of its characteristic function (3.9). In particular, its variance is given by

\[ \text{Var} X_{\alpha,H}(t) = \frac{t^{2H\alpha}}{2^{\alpha-1} \Gamma(\alpha + 1)}. \] (3.22)

We observe that, for \( \alpha = 1 \) in (3.22), we recover the variance of the fractional Brownian motion; for \( H = \frac{1}{2} \) we have the variance of the time-fractional diffusion process.

More generally, from (3.9), we infer that

\[ \mathbb{E} X_{\alpha,H}^{2m}(t) = \frac{\Gamma(2m + 1)}{2^{m} \alpha^{m+1}} t^{2Hm}, \quad m \in \mathbb{N}. \] (3.23)

Remark 3.4. We now consider the more general time-fractional diffusion equation with time-varying coefficients

\[ C \left(t^{a_1} \frac{\partial}{\partial t}\right)^a u_a(x, t) = \left(1 - \frac{a_1}{2}\right)^a \frac{\partial^2}{\partial x^2} u_a(x, t), \quad a_1 < 1. \] (3.24)

In view of the McBride theory we have that in this case

\[ \left(t^{a_1} \frac{\partial}{\partial t}\right)^a f(t) = (1 - a_1)^a t^{-a(1-a_1)} \chi^{0-a}_{1-a} f(t). \] (3.25)

The solution of the Cauchy problem

\[ \begin{cases} C \left(t^{a_1} \frac{\partial}{\partial t}\right)^a u_a(x, t) = \left(1 - \frac{a_1}{2}\right)^a \frac{\partial^2}{\partial x^2} u_a(x, t), \quad a \in (0, 1), \\ u_a(x, 0) = \delta(x), \end{cases} \] (3.26)
is given by
\[ u_a(x,t) = \frac{1}{2^{1-a/2} t^{\frac{1}{2-a}}} W_{-a/2,1-a/2} \left( -\frac{2^{a/2}|x|}{t^{\frac{1}{2-a}}} \right). \]  
(3.27)

**Remark 3.5.** If we consider the complete McBride operator of order one, that is \((t^{a_1} \frac{1}{t} t^{a_2})\), with \(a_1 + a_2 < 1\), it is easy to prove that the function
\[ h_a(x,t) = \frac{1}{2^{1-a/2} t^{\frac{1}{2-a}}} W_{-a/2,1-a/2} \left( -\frac{2^{a/2}|x|}{t^{\frac{1}{2-a}}} \right) \]  
(3.28)
solves the non-homogeneous pde
\[ \left( t^{a_1} \frac{\partial}{\partial t} t^{a_2} \right)^{\alpha} h_a(x,t) = \frac{\partial^2}{\partial x^2} h_a(x,t) + \frac{t^{-\alpha(1-a_1-a_2)} - a_2}{(1 - a_1 - a_2)^\alpha} \delta(x). \]  
(3.29)

Indeed, by taking the Fourier transform of (3.29) we have that
\[ \left( t^{a_1} \frac{\partial}{\partial t} t^{a_2} \right)^{\alpha} \hat{h}_a(\beta,t) = -\beta^2 \hat{h}_a(\beta,t) + \frac{t^{-\alpha(1-a_1-a_2)} - a_2}{(1 - a_1 - a_2)^\alpha} \frac{1}{\Gamma(1 - \alpha)}, \]  
(3.30)
whose solution is given by
\[ \hat{h}_a(\beta,t) = \frac{1}{t^{a_2} E_{a,1}} \left( -\frac{\beta^2 t^{2(1-a_1-a_2)}}{(1 - a_1 - a_2)^2} \right). \]  
(3.31)

In view of the fact that
\[ \left( t^{a_1} \frac{\partial}{\partial t} t^{a_2} \right)^{\alpha} t^{\gamma} = (1 - a_1 - a_2)^{\alpha} t^{\gamma} \]  
(3.32)
we have that
\[ \left( t^{a_1} \frac{\partial}{\partial t} t^{a_2} \right)^{\alpha} \frac{1}{t^{a_2} E_{a,1}} \left( -\frac{\beta^2 t^{2(1-a_1-a_2)}}{(1 - a_1 - a_2)^2} \right) \]  
(3.33)

\[ = (1 - a_1 - a_2)^{\alpha} \sum_{k=0}^{\infty} \frac{(-\beta^2)^k}{(1 - a_1 - a_2)^{2k}} \frac{t^{\alpha(1-a_1-a_2)k-a(1-a_1-a_2)} - a_2}{\Gamma(ak + 1 - \alpha)} \]  
\[ = -\sum_{k=1}^{\infty} \frac{(-1)^k (\beta)^{2k} \Gamma(ak + 1)}{(1 - a_1 - a_2)^{2k}} \frac{t^{\alpha(1-a_1-a_2)k-a_2}}{\Gamma(ak + 1)} \]  
\[ = -\beta^2 \hat{u}_a + \frac{t^{-\alpha(1-a_1-a_2)-a_2}}{(1 - a_1 - a_2)^\alpha} \frac{1}{\Gamma(1 - \alpha + 1)}. \]

By inverting the Fourier transform we obtain the claimed result. The case \(a_1 + a_2 = 1\) can be treated by using the results discussed by Lamb and McBride in [18]. In this case, a logarithmic deterministic time-change leads to the solution. Finally, in the case \(a_1 + a_2 > 1\) complex coefficients appear and therefore we neglect this case.

If we consider the space-time fractional equation
\[ C \left( t^{1-2\alpha} \frac{\partial}{\partial t} \right)^{\alpha} u_a(x,t) = H^{\alpha} \frac{\partial^\nu}{\partial |x|^\nu} u_a(x,t), \quad \nu \in (0,2), \]  
(3.34)
where $\partial^{\nu}/\partial|x|^{\nu}$ denotes the Riesz fractional derivative
\[
\frac{\partial^{\nu}}{\partial|x|^{\nu}} g(x) = -\frac{1}{\Gamma(m - \nu)} \frac{1}{2\cos \frac{\nu \pi}{2}} \frac{\partial^{m}}{\partial x^{m}} \int_{-\infty}^{+\infty} \frac{1}{|x-s|^{1+\nu-m}} g(s) ds, \quad m - 1 < \nu < m, \ x > 0
\] (3.35)

where $m$ is the ceiling of $\nu$.

The expression $u_a(x,t)$ coincides with the distribution of the time-changed r.v. $Y_{\nu/\alpha}$ where $Y_{\nu/\alpha}$ is a symmetric stable r.v. of order $0 < \nu/\alpha < 2$ and $W_\gamma(t)$ is the Lamperti r.v. with parameter $\alpha$ (that is the ratio of two independent positively skewed stable r.v.’s of order $0 < \alpha < 1$), possessing density
\[
P\{\mathcal{W}_\gamma \in dr\}/dr = \frac{\sin \alpha \pi}{\pi} \frac{r^{\alpha-1}}{r^{2\alpha} + 2r^\alpha \cos \alpha \pi + 1} \int \frac{e^{-|\beta|^{\gamma/\delta}} \beta}{2} d\beta.
\] (3.37)

### 3.1 Relation with the generalized grey Brownian motion (ggBm)

The generalized grey Brownian motion (ggBm) was recently introduced and studied by A. Mura and coauthors in [24, 27, 28] as a family of non-Markovian stochastic processes for anomalous fast or slow diffusions.

The marginal density function of the ggBm is given by
\[
P(x, t) = \frac{1}{2\gamma^{1/2}} \frac{\gamma}{\delta} P(x, 0) + \frac{1}{\Gamma(\delta)} \tau^{\delta-1} \int_{0}^{\tau} \tau^{1-\delta} \delta^{-1} \frac{\partial^{2}}{\partial x^{2}} P(x, \tau) d\tau.
\] (3.39)

Equation (3.39) is an integro-differential equation involving the Erdélyi–Kober integral.

We now show the equivalence between the fractional equation
\[
\frac{c}{\tau^{1-2H}} \frac{\partial}{\partial \tau} \frac{a}{P(x, \tau)} = H^a \frac{\partial^{2}}{\partial x^{2}} P(x, \tau)
\] (3.40)

and the master equation (3.39) introduced by A. Mura, in the case of $\delta = \alpha$ and $\gamma/2 = H\alpha$.

From (2.13) and by using (3.2), we have that
\[
\frac{c}{\tau^{1-2H}} \frac{\partial}{\partial \tau} \frac{a}{P(x, \tau)} = \left(t^{1-2H} \frac{\partial}{\partial t} \right)^a (P(x, \tau) - P(x, 0))
\] (3.41)

\[= (2H)^a t^{-2H} \frac{\partial^{H}}{\partial \tau^{H}} (P(x, \tau) - P(x, 0))
\]
\[
H^n \frac{\partial^2}{\partial x^2} P(x, t),
\]
and therefore,
\[
I_{2H}^{0,-\alpha} (P(x, t) - P(x, 0)) = \frac{t^{2H\alpha}}{2^n} \frac{\partial^2}{\partial x^2} P(x, t).
\] (3.42)

We now recall the following property of the Erdélyi–Kober integral (see [24], Theorem 2.7, page 523)
\[
\left( I_m^{\eta, a} \right)^{-1} = I_m^{\eta+a,-a}.
\] (3.43)

Thus, in our case, we have that the inverse of the operator appearing in the left hand side of (3.42) is given by
\[
\left( I_{2H}^{0,-\alpha} \right)^{-1} = I_{2H}^{-\alpha, a}.
\] (3.44)

Finally, by applying the inverse operator (3.44) to both sides of (3.42), we arrive at
\[
P(x, t) - P(x, 0) = \frac{2^{1-a} H}{\Gamma(a)} \int_0^t \tau^{2H-1} \left( t^{2H} - \tau^{2H} \right)^{a-1} \frac{\partial^2}{\partial x^2} P(x, t) \, d\tau,
\]
which coincides with (3.39) for \( \delta = \alpha \) and \( \gamma = 2H\alpha \), up to a multiplicative constant.

A further different approach highlighting the relation between (3.39) and equations involving Erdélyi–Kober derivatives was recently discussed by Pagnini in [32]. To conclude, we remark that the general McBride theory for fractional powers of hyper-Bessel-type operators represents a convenient framework to derive the governing equation of the ggBm considered as a fractional generalization of the diffusion equation governing the fractional Brownian motion.

### 3.2 Relation with higher order heat equations with time-varying coefficients

In this section we consider the relation between the solutions of Theorem 3.1 and the fundamental solutions of higher order heat-type equations.

We start from a special case of (3.5), corresponding to \( \alpha = 2/3 \). We know, from [29] (Remark 4.1, pages 231–232), that the fundamental solution to the fractional equation involving Caputo derivatives
\[
{\cD}^{2/3}_0 u = \lambda^2 \frac{\partial^2 u}{\partial x^2}, \quad \lambda > 0, \ t \geq 0,
\] (3.46)
is related to the fundamental solution of the linearized Korteweg–DeVries equation, that is the third order heat equation
\[
\frac{\partial v}{\partial t} = -\lambda^3 \frac{\partial^3 v}{\partial x^3}, \quad \lambda > 0, \ t \geq 0.
\] (3.47)

By following the same idea, we establish here a relationship between fractional equations and higher-order heat-type equations with time-varying coefficients.

**Theorem 3.3.** The solution \( u_{2/k}(x, t) \) of (3.5), for any \( k \in \mathbb{N} : k > 2 \), coincides with the fundamental solution \( v(x, t) \) of the higher order diffusion equation with time-dependent coefficients
\[
t^{1-2H} \frac{\partial v}{\partial t} = (-1)^k H \frac{\partial^k v}{\partial x^k}, \quad x > 0, \ t \geq 0.
\] (3.48)
Proof. We first recall that in the special case $a = 2/k$ the solution of (3.5) is given for all $x \in \mathbb{R}$ by
\[
    u_{2/k}(x, t) = \frac{1}{2^{1/2}t^{2H/k}} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{2^n/k}{t^{Hn}} x^n.
\] (3.49)

By direct calculation we have, for $x > 0$, that
\[
    t^{1-2H} \frac{\partial u_{2/k}}{\partial t} = 2^{3/2}H \sum_{n=0}^{\infty} \frac{(-1)^n2^n/k}{n!} \frac{2^{Hn-k/2}}{t^{Hn-k/2-1}} x^n
\] (3.50)
and this coincides with $(-1)^{k}H$ times the $k$-th order derivative with respect to $x$, as claimed.

**Remark 3.6.** As a special case of interest of Theorem 3.3 we observe that the solution $u_{2/k}(x, t)$ of (3.5) coincides for $x > 0$ with the fundamental solution $v(x, t)$ of the linearized Korteweg–DeVries equation with time-dependent coefficients
\[
    t^{1-2H} \frac{\partial v}{\partial t} = -H \frac{\partial^3 v}{\partial x^3}.
\] (3.51)

In particular, it is possible to prove that (3.49) can be represented in terms of Airy functions, following the same arguments of Theorem 4.1 in [29], since the fundamental solution to (3.51) can be written as
\[
    v(x, t) = \frac{1}{\sqrt{\frac{2}{3}t^{2H}}} \text{Ai} \left( \frac{x}{\sqrt{\frac{2}{3}t^{2H}}} \right).
\] (3.52)

4 Diffusion equation with time-dependent coefficients involving Caputo derivatives

In this section we adopt a different fractional generalization of the time-varying diffusion equation, involving Caputo time-fractional derivatives. Our aim is then to compare analytical and probabilistic results relative to these two different approaches.

In more detail, here we consider the time-fractional diffusion equation
\[
    \begin{cases}
        D_{t}^{\nu}u(x, t) = Ht^{2H-1} \frac{\partial^2}{\partial x^2} u(x, t), & v \in (0, 1], \ x \in \mathbb{R}, \ t > 0, \\
        u(x, 0) = \delta(x),
    \end{cases}
\] (4.1)
with $0 < H < 1$. The derivative operator appearing in (4.1) is the Caputo fractional derivative.

For the utility of the reader, we first recall the following result (see [16], pag.232-233).

**Lemma 4.1.** The function
\[
    f(t) = E_{v,1+\gamma,\gamma} (\lambda t^v) = 1 + \sum_{k=0}^{\infty} (\lambda t^v)^k \prod_{j=0}^{k-1} \frac{\Gamma(vj+\gamma+j+1)}{\Gamma(vj+\gamma+j+v+1)},
\] (4.2)
is a solution to the Cauchy problem
\[
    \begin{cases}
        D_{t}^{\nu} f(t) = \lambda t^v f(t), & t \geq 0, \ v \in (0, 1], \ \gamma > -v, \\
        f(0) = 1.
    \end{cases}
\] (4.3)

**Proof.** We have that
\[
    D_{t}^{\nu} f(t) = \sum_{k=1}^{\infty} \lambda^k D_{t}^{\nu} \left[ t^{k(v+\gamma)} \right] \prod_{j=0}^{k-1} \frac{\Gamma(vj+\gamma+j+1)}{\Gamma(vj+\gamma+j+v+1)}
\] (4.4)
Theorem 4.1. Since Equation (4.6) is solved by (see Lemma 4.1 and \[ \text{and using the well-known multiplication formula for the Gamma function} \]

By taking the Fourier transform of (4.1) we obtain the differential equation

Proof. Moreover, the solution is proved to be unique for 1/2 ≤ H < 1 (see [16], page 233). By taking the inverse Fourier transform we obtain the claimed result.

Remark 4.1. An interesting case of (4.7) is given by the choice H = 1/4, ν = 3/4. In this case, from (4.23) we have that

Since

and using the well-known multiplication formula for the Gamma function

we have that

\[ \sum_{k=1}^{\infty} \frac{\Gamma \left( \frac{1}{4} + \frac{k}{2} \right)}{\Gamma \left( \frac{3}{4} + \frac{k}{2} \right)} = \frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(k+2)\Gamma(k+3)\Gamma(k+4)} \]

\[ \Gamma(z)\Gamma \left( z + \frac{1}{m} \right) \Gamma \left( z + \frac{2}{m} \right) \ldots \Gamma \left( z + \frac{m-1}{m} \right) = (2\pi)^{(m-1)/2}m^{\frac{1}{2}-m}\Gamma(mz), \]

13
Therefore,

\[ U(\beta, t) = E_{\frac{1}{2}, -\frac{3}{2}} \left( -\frac{1}{4} \beta^2 t^{1/4} \right) = 1 + \frac{\Gamma(3/4)}{\pi \sqrt{2}} \sum_{k=1}^{\infty} \left( -\beta^2 u^4 \right)^k \frac{\Gamma(k+1/4)}{k!} \]  \hspace{1cm} (4.11)

\[ = 1 + \frac{\Gamma(3/4)}{\pi \sqrt{2}} \int_{0}^{\infty} e^{-w^{-3/4}} \sum_{k=1}^{\infty} \left( -\beta^2 w^{1/4} t^{1/4} \right)^k \frac{\Gamma(k+1/4)}{k!} dw \]

\[ = \frac{\Gamma(3/4)}{\pi \sqrt{2}} \int_{0}^{\infty} e^{-w^{-3/4}} e^{-\beta^2 t^{1/4}} dw. \]

Finally, we have

\[ u(x, t) = \frac{\Gamma(3/4)}{\pi \sqrt{2}} \int_{0}^{\infty} e^{-w^{-3/4}} \frac{e^{-x^2/\pi \sqrt{w}}}{\sqrt{4 \pi \sqrt{w}}} dw \]

\[ = \frac{\sqrt{2} \Gamma(3/4)}{\pi \sqrt{\pi}} \int_{0}^{\infty} \frac{e^{-x^2/\pi \sqrt{w}}}{\sqrt{2 \pi w}} dw, \]

that is the probability law of the r.v. \( B(W_t) \), where

\[ \mathbb{P}\{W_t \in dz\}/ds = \frac{\sqrt{2} \Gamma(3/4)e^{-x^2/\pi \sqrt{w}}}{\pi \sqrt{\pi}}, \quad z > 0, \]

and \( B \) is a standard Brownian motion.

**Remark 4.2.** For \( \nu = 1 \) we show that the solution \( u(x, t) \) in equation (4.5) coincides with the density function of the fractional Brownian motion for a fixed time \( t \). Indeed, for \( \nu = 1 \), formula (4.7) yields

\[ U(\beta, t) = \exp \left\{ -\frac{\beta^2 t^{2\nu}}{2} \right\}, \]  \hspace{1cm} (4.13)

which is the characteristic function of \( B_H(t) \).

**Remark 4.3.** For \( 1 + \frac{2\nu-1}{\nu} = 0 \), the characteristic function (4.7) becomes

\[ U(\beta, t) = \frac{1}{1 + H\beta^2 \Gamma(1 - \nu)}, \quad \left| H\beta^2 \Gamma(1 - \nu) \right| < 1, \quad H \in \left( 0, \frac{1}{\nu} \right), \]  \hspace{1cm} (4.14)

and is independent from \( t \). Indeed, in this case we have that

\[ U(\beta, t) = E_{\nu, 0, -1} \left( -H \beta^2 \right) = 1 + \sum_{k=1}^{\infty} \left( -H \beta^2 \right)^k \prod_{j=0}^{k-1} \frac{\Gamma(1 - \nu)}{\Gamma(1)} \]

\[ = \sum_{k=0}^{\infty} \left( -H \beta^2 \Gamma(1 - \nu) \right)^k = \frac{1}{1 + H\beta^2 \Gamma(1 - \nu)}. \]

when \( \left| H\beta^2 \Gamma(1 - \nu) \right| < 1 \).

We observe that, even if the function (4.14) is independent from \( t \), it satisfies the time-fractional differential equation

\[ t^\nu D^{\nu}_t f = \frac{f}{\Gamma(1 - \nu)} \]  \hspace{1cm} (4.15)

involving the Riemann–Liouville derivative. We recall that the Riemann–Liouville derivative of a constant function does not vanish and in more detail

\[ D^{\nu}_0 \text{const.} = \frac{\text{const.}}{\Gamma(1 - \nu)}. \]  \hspace{1cm} (4.16)
Remark 4.4. For \( \nu = 2H - 1 = \frac{1}{2} \) and therefore \( H = \frac{3}{4} \), the characteristic function \((4.7)\) becomes

\[
U(\beta, t) = 1 + \sum_{k=1}^{\infty} (-H^2 \beta^2 t)^k \prod_{j=0}^{2j} \left( \frac{2j+1}{2j+1} \right) = 1 + \sum_{k=1}^{\infty} (-H \sqrt{\pi} \beta^2 t)^k \prod_{j=1}^{k} \frac{1}{2j} \left( \frac{2j}{j} \right).
\]

(4.17)

We observe that the coefficients in \((4.17)\) have the following probabilistic interpretation

\[
\prod_{j=1}^{k} \frac{1}{2j} \left( \frac{2j}{j} \right) = \mathbb{P}\left( \bigcap_{j=1}^{k} \mathcal{B} \left( 2j, \frac{1}{2} \right) = j \right).
\]

(4.18)

In \((4.18)\) by \(\mathcal{B}\) we denote independent binomial r.v.’s with parameters \(2j\) and \(1/2\). For large values of \(j\), we know that \(\mathbb{P}\left( \mathcal{B} \left( 2j, \frac{1}{2} \right) = j \right) \approx 1/\sqrt{j}\) and thus the coefficients of \((4.17)\) vanish as \(1/\sqrt{k}\).

4.1 Higher order extension

Here we consider the higher-order time-fractional equation

\[
\begin{align*}
C D^\nu_0 f(x, t) &= c_k 2H t^{2H-1} \frac{d^k}{dx^k} f(x, t), \quad \nu \in (0, 1], \ 0 < H \leq 1, \\
f(x, 0) &= \delta(x),
\end{align*}
\]

(4.19)

and \(k\) is an integer greater than 2. The coefficient \(c_k = (-1)^{1+k/2}\) if \(k\) is even and \(c_k = \pm 1\) otherwise. The fractional derivative appearing in \((4.19)\) is the Caputo fractional derivative.

Theorem 4.2. The solution \(f(x, t), x \in \mathbb{R}, t \geq 0, \) to \((4.19)\) is

\[
f(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\beta x} E_{\nu,1+\frac{2H-1}{\nu}} \left( c_k 2H(-i\beta)^k t^{\nu+2H-1} \right) d\beta.
\]

(4.20)

Proof. Applying the Fourier transform

\[
F(\beta, t) = \int_{\mathbb{R}} e^{i\beta x} f(x, t) dx,
\]

(4.21)

to \((4.19)\), we have

\[
\begin{align*}
C D^\nu_0 F (\beta, t) &= c_k 2H t^{2H-1} (-i\beta)^k F (\beta, t), \\
F(\beta, 0) &= 1.
\end{align*}
\]

(4.22)

The solution to \((4.22)\) is immediately obtained:

\[
F(\beta, t) = E_{\nu,1+\frac{2H-1}{\nu}} \left( c_k 2H(-i\beta)^k t^{\nu+2H-1} \right),
\]

(4.23)

Recall that the solution is proved unique for \(1/2 \leq H \leq 1\). From \((4.23)\) it is immediate to obtain \((4.20)\).

In the special case \(H = 1/4, \nu = 3/4\) and \(k = 2n\), the Fourier transform of the solution of \((4.19)\) reads

\[
F(\beta, t) = \int_{0}^{\infty} e^{-\beta^2 y} e^{-\frac{\pi^2}{\nu^2}} dy.
\]

(4.24)

Thus the inverse Fourier transform coincides with the law of a pseudoprocess of order \(2n\) evaluated at the random time \(W_t\). In a similar way the odd-order case can be treated. For a wide discussion about pseudoprocesses we refer for example to \([17]\) and the references therein.
4.2 Space-time bifractional equations

Theorem 4.1 can be modified by introducing the Riesz fractional derivative, in the right hand side of (4.11), thus obtaining

$$\left\{ \begin{array}{ll}
\mathcal{D}_0^H f(x, t) = H t^{2H-1} \frac{\partial^x}{\partial |x|^H} f(x, t), & \nu \in (0, 1], \ 0 < \alpha < 2, \ 0 < H < 1, \\
\frac{\partial^x}{\partial |x|^H} f(x, 0) = \delta(x).
\end{array} \right. \tag{4.25}$$

This equation has been object of recent interest in the framework of anomalous diffusion processes \cite{2,3}. Similarly to Theorem 4.4, we apply the Fourier transform to (4.1), arriving at

$$\left\{ \begin{array}{ll}
\mathcal{D}_0^H \mathcal{F}(\beta, t) = -H t^{2H-1} |\beta|^\alpha \mathcal{F}(\beta, t), & \nu \in (0, 1], \ 0 < \alpha < 2, \\
\mathcal{F}(\beta, 0) = 1.
\end{array} \right. \tag{4.26}$$

The solution to (4.26) is thus

$$f(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\beta x} e_{\nu, \frac{\alpha \beta}{\nu}} (\nu t)^{\nu + 2H - 1} d\beta. \tag{4.27}$$

We observe that (4.27) coincides with equation (63) of \cite{3} with a suitably adaptation of the notation.

**Remark 4.5.** When $H = 1/4$, $\nu = 3/4$, on the basis of the results discussed in Remark 4.7, see equation (4.11), we have that

$$F(\beta, t) = \frac{2\sqrt{2}\Gamma(3/4)}{\pi \sqrt{t}} \int_0^\infty e^{-x/\nu} e^{-|\beta|\nu x} dz, \tag{4.28}$$

that coincides with the probability law of the r.v. $Y_a(\mathcal{M}_1)$, where

$$\mathbb{P}\left( \mathcal{M}_1 \in dz \right) = \frac{2\sqrt{2}\Gamma(3/4)e^{-z}}{\pi \sqrt{t}}, \quad z > 0 \tag{4.29}$$

and $Y_a$ is a symmetric stable r.v. of order $\alpha$.

**Remark 4.6.** When $H = 1/2$, the solution (4.27) reduces to

$$f(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\beta x} E_{\nu, 1} \left( -\frac{1}{2} |\beta|^\alpha t^\nu \right) d\beta. \tag{4.30}$$

The expansion of the Mittag--Leffler function yields

$$f(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\beta x} \frac{\sin \frac{\nu \pi}{\nu_0}}{\nu_0} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \frac{\nu_0 \pi}{\nu_0} + 1} e^{-\left(\frac{1}{2}\nu_0 t^\nu\right) r} dr d\beta \tag{4.31}$$

$$= \frac{\sin \frac{\nu \pi}{\nu_0}}{2\pi^2} \int_0^\infty \frac{r^{\nu-1}}{r^{2\nu} + 2r^\nu \cos \frac{\nu_0 \pi}{\nu_0} + 1} \int_{\mathbb{R}} e^{-i\beta x} e^{-\left(\frac{1}{2}\nu_0 t^\nu\right) r} dr d\beta.$$

Then $f(x, t)$ is the distribution of the time-changed r.v. $Y_{a/\nu} \left( \mathcal{M}_1, t \right)$ where $\mathcal{M}_1$ is the Lamperti r.v. and $Y_{a/\nu}$ is a symmetric stable r.v. of order $0 < \alpha/\nu < 2$.

**Remark 4.7.** If $\nu = \gamma \eta$, we can expand the Mittag--Leffler function appearing in (4.30) with an alternative formula (see \cite{2} page 291), i.e.

$$E_{\gamma, 1}(-\theta t^\nu) = \frac{\sin \gamma \pi}{\pi} \int_0^\infty \frac{r^{\gamma-1}}{r^{2\gamma} + 2r^\gamma \cos \gamma \pi + 1} E_{\eta, 1}(-r \theta^{1/\gamma} t^\eta) dr, \tag{4.32}$$
where $\gamma \in (0,1]$, $\eta \in (0,1]$. In our case, for $H = 1/2$ we obtain that

$$f(x, t) = \frac{1}{2\pi} \int_R \frac{e^{-\beta \sin \gamma \pi \frac{t}{\beta}}}{\pi} \int_0^\infty \int_0^\infty \frac{r^{\gamma-1}}{r^{2\gamma} + 2r^\gamma \cos \gamma \pi + 1} \frac{E_{\gamma, 1}(-H^{1/2}|\beta|^{1/\gamma}rt^\gamma)}{dr} \frac{d\beta}{d\gamma}$$

$$= \sin \gamma \pi \frac{\gamma \pi}{2\pi^3} \int_0^\infty \int_0^\infty \int_0^\infty \frac{r^{\gamma-1}}{r^{2\gamma} + 2r^\gamma \cos \gamma \pi + 1} \frac{E_{\gamma, 1}(-H^{1/2}|\beta|^{1/\gamma}rt^\gamma)}{dr} \frac{d\beta}{d\gamma}$$

$$= \sin \gamma \pi \sin \eta \pi \int_0^\infty \frac{\gamma \pi}{2\pi^3} \int_0^\infty \int_0^\infty \frac{r^{\gamma-1}}{r^{2\gamma} + 2r^\gamma \cos \gamma \pi + 1} \frac{E_{\gamma, 1}(-H^{1/2}|\beta|^{1/\gamma}rt^\gamma)}{dr} \frac{d\beta}{d\gamma}$$

Therefore, for $H = 1/2$ and $0 < \alpha/\gamma \eta < 2$, $f(x, t)$ is the distribution of the time-changed random variable $Y_{\alpha/\gamma \eta}(tW, t\Psi_{\gamma}(t))$, where $W(t)$ is the Lamperti r.v. and $Y_{\alpha/\gamma \eta}$ is a symmetric stable law of order $\frac{\alpha}{\gamma \eta}$.

**Remark 4.8.** We observe that for $\nu = 1$, $\alpha \in (0,2)$ and $H \in (0,1)$, from (4.27) we obtain the law of a symmetric stable process with a deterministic time-change, whose characteristic function is given by

$$F(\beta, t) = \exp \left\{ -\frac{t^{2H}}{2} |\beta|^\alpha \right\}.$$  

(4.34)

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