The mKdV and NLS hierarchies revisited

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Abstract

The purpose of this paper is to express the entire hierarchy of mKdV vector fields as restrictions of vector fields in the NLS hierarchy. The result is proved using the normal form theory of the two equations.

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1 Introduction

We consider the defocusing mKdV equation

\[ u_t = -u_{xxx} + 6u^2u_x, \]

on the circle \( T = \mathbb{R}/\mathbb{Z} \) with \( u \) real-valued. The mKdV equation can be viewed as a Hamiltonian PDE with Hamiltonian

\[ K(u) := \frac{1}{2} \int_T (u_x^2 + u^4) \, dx, \]

on the standard Sobolev space \( H^m_u := H^m(T, \mathbb{R}) \), \( m \geq 0 \), as phase space endowed with the Poisson bracket proposed by Gardner

\[ \{ F, G \}_{\partial_u} := \int_T \nabla_u F \cdot \partial_u \nabla_u G \, dx. \] (1)

Here, \( \nabla_u F \) denotes the \( L^2 \)-gradient of a \( C^1 \)-functional on \( H^m_u \). The mean value \( [u] := \int_T u \, dx \) is a Casimir for the Gardner bracket and hence is preserved by the mKdV flow. The defocusing mKdV equation then takes the form \( u_t = \{ u, K \}_{\partial_u} \). This equation admits an infinite sequence of recursively defined pairwise Poisson commuting integrals referred to as mKdV hierarchy,

\[ K_1(u) = \frac{1}{2} \int_T u^2 \, dx, \quad K_2(u) = K(u) = \frac{1}{2} \int_T (u_x^2 + u^4) \, dx, \quad \ldots. \]

Each of these Hamiltonians leads to a Hamiltonian PDE.

It is well known that the mKdV equation, being closely related to the KdV equation via the Miura map [11], is also closely related to the NLS system

\[ \begin{align*}
  i \partial_t \varphi_1 &= \partial_{\varphi_2} S = -\partial_{x\varphi_1} \varphi_1 + 2\varphi_2 \varphi_1^2, \\
  i \partial_t \varphi_2 &= -\partial_{\varphi_1} S = \partial_{x\varphi_2} \varphi_2 - 2\varphi_1 \varphi_2^2.
\end{align*} \] (2)
This system can be viewed as a Hamiltonian PDE with Hamiltonian
\[
S(\phi) = \int_T \left( \partial_x \phi_2 \partial_x \phi_1 + \phi_2^2 \phi_1^2 \right) \, dx,
\]
on the phase space \( H^m_c \times H^m_c, \) \( m \geq 0, \) with Poisson bracket
\[
\{F, G\} := -i \int_T \left( \partial_\phi F \partial_\phi G - \partial_\phi F \partial_\phi G \right) \, dx.
\]

Here \( H^m_c \) denotes the standard Sobolev space \( H^m(T, \mathbb{C}), \) \( \phi_1, \phi_2 \) denote the two components of \( \phi \in H^m_c, \) and \( \partial_\phi F, \partial_\phi G \) denote the two components of the \( L^2 \)-gradient \( \partial F \) of a \( C^1 \)-functional \( F \) on \( H^0_c. \)

The Hamiltonian \( S(\phi) \) admits for any \( m \geq 0 \) the invariant real subspaces
\[
H^m_r := \{ \phi \in H^m_c : \phi_2 = \overline{\phi_1} \}, \quad H^m_i := \{ \phi \in H^m_c : \phi_2 = -\overline{\phi_1} \}.
\]
When (2) is restricted to \( H^m_r, \) with \( \phi = (v, \overline{v}), \) one obtains the defocusing NLS (dNLS) equation
\[
i \partial_t v = i \{ v, S \} = -\partial_{xx} v + |v|^2 v, \quad S(v, \overline{v}) = \int_T (|v|_x^2 + |v|^4) \, dx.
\]

Similarly, when (2) is restricted to \( H^m_i, \) with \( \phi = (iv, i\overline{v}), \) one obtains the focusing NLS equation
\[
i \partial_t v = i \{ v, S \} = -\partial_{xx} v - |v|^2 v, \quad S(iv, i\overline{v}) = -\int_T (|v|_x^2 - |v|^4) \, dx.
\]

The NLS system (2) also admits an infinite sequence of recursively defined pairwise Poisson commuting integrals referred to as \( \text{NLS hierarchy}^1, \) \( S_1(\phi) = \int_T \phi_1 \phi_2 \, dx, \)
\[
S_2(\phi) = \frac{1}{2} \int_T (\phi_2 \partial_x \phi_1 - \phi_1 \partial_x \phi_2) \, dx,
\]
\[
S(\phi) = S_3(\phi) = \int_T (\partial_x \phi_1 \partial_x \phi_2 + \phi_1^2 \phi_2^2) \, dx,
\]
\[
S_4(\phi) = i \int_T (\partial_x \phi_1 \partial_{xxx} \phi_2 - 3\phi_1^2 \phi_2 \partial_x \phi_2) \, dx, \quad \ldots
\]
The Hamiltonian \( S_4 \) gives rise to the system
\[
\partial_t \phi_1 = \{ \phi_1, S_4 \} = -i \partial_{\phi_2} S_4 = -\partial_{xx} \phi_1 + 6\phi_1 \phi_2 \partial_x \phi_1,
\]
\[
\partial_t \phi_2 = \{ \phi_2, S_4 \} = i \partial_{\phi_1} S_4 = -\partial_{xx} \phi_2 + 6\phi_2 \phi_1 \partial_x \phi_2.
\]

This system admits, for any \( m \geq 1, \) the real invariant subspaces
\[
E^m_r := \{ \phi \in H^m_r : \phi_2 = \phi_1 \}, \quad E^m_i := \{ \phi \in H^m_i : \phi_2 = \phi_1 \}.
\]
When (5) is restricted to \( E^m_r, \) with \( \phi = (u, u) \) and \( u \) real-valued, one obtains the defocusing mKdV equation
\[
\partial_t u = -i \partial_{\phi_2} S_4 \big|_{(u, u)} = \partial_x \nabla K_2(u) = -\partial_{xx} u + 6u^2 \partial_x u.
\]

Similarly, when (5) is restricted to \( E^m_i, \) with \( \phi = (iu, iu) \) and \( u \) real-valued, one obtains the focusing mKdV equation
\[
i \partial_t u = -i \partial_{\phi_2} S_4 \big|_{(iu, iu)} = \partial_x \nabla K_2(iu) = i(-\partial_{xx} u - 6u^2 \partial_x u).
\]

\[^1\text{In comparison with [4], the } n \text{th Hamiltonian of the NLS hierarchy (for } n \geq 2 \text{) is multiplied by } (-i)^n+1 \text{ to make the corresponding Hamiltonian flow real-valued for real-valued } \phi.\]
The main purpose of this paper is to show that in the same way the entire mKdV hierarchy is contained in the NLS hierarchy. By a slight abuse of notation, we identify and its restriction to by 

\[ H^m \]

and denote its Hamiltonian vector field with respect to the Gardner bracket by 

\[ X_F = -iJ\partial F, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \]

and its restriction to is denoted by \( X^c_F = -iJ(\partial F)^4 \). Similarly, for a function \( G \) on \( H^m \) we denote its Hamiltonian vector field with respect to the Gardner bracket by 

\[ Y_G = \partial_z \nabla G. \]

**Theorem 1** For every \( m \geq 1 \) the Hamiltonian vector field \( Y_{\kappa_m} \) of the mKdV hierarchy and the Hamiltonian vector field \( X_{S_{2m}} \) of the NLS hierarchy satisfy 

\[ X^c_{S_{2m}} = (Y_{\kappa_m}, Y_{\kappa_m}) \text{ on } H^m. \]

In addition, \( S_{2m}^2 = 0 \) on \( H^m \) for every \( m \geq 1 \). \( \star \)

Loosely speaking, the above theorem says that each PDE of the mKdV hierarchy can be viewed as a subsystem of the corresponding PDE in the NLS hierarchy. The defocusing and focusing cases of the PDEs are obtained by restriction to the subspaces \( E^m_u = \{ u \in H^m_c : u \text{ real-valued} \} \) and \( E^m_i = \{ iu \in H^m_c : u \text{ real-valued} \} \), respectively. Since the defocusing mKdV equation on the circle can be identified with the KdV equation (c.f. e.g. \([7, 8]\)) and the KdV equation on the circle as well can be viewed as a subsystem of the dNLS equation.

A key ingredient into the proof of Theorem 1 are the following symmetries of the gradients of the Hamiltonian vector fields in Birkhoff coordinates. The defocusing mKdV equation admits global Birkhoff coordinates \((\varphi, \eta)_n\) \( n \geq 1 \) constructed in terms of global action-angle coordinates \((J_n, \theta_n)_{n \geq 1}\) – see [6]. To give a precise definition, we introduce for any \( m \geq 0 \) the model space \( h^m = l^2_{m+1/2}(\mathbb{N}) \times l^2_{m+1/2}(\mathbb{N}) \) with elements \((\varphi, \eta) = (\varphi_n, \eta_n)_{n \geq 1}\), where for any \( \mathfrak{h} \subset \mathbb{Z} \)

\[ \ell^2_{m+1/2}(\mathbb{N}) := \{ z \in \ell^2(\mathbb{A}, \mathbb{R}) : \sum_{n \in \mathfrak{h}} (1 + |n|^{2\alpha}) z_n^2 < \infty \}, \]

and endow this space with the Poisson structure \( \{(\varphi_n, \eta_n) = (\eta_n, \varphi_n) = \{ \} \) while all other brackets vanish. The mKdV Birkhoff map

\[ \Psi : H^m_1 \to h^1_1 \times \mathbb{R}, \quad u \mapsto ((\varphi_n, \eta_n)_{n \geq 1}, [u]) \]

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defines a bi-real-analytic, canonical diffeomorphism, which transforms every Hamiltonian of the mKdV hierarchy, on Sobolev spaces of the appropriate order, into Birkhoff normal form, that is \( K_m \circ \Psi^{-1} \) is a real analytic function of the actions \( J_n = (\xi_n^2 + \eta_n^2)/2 \) and the average alone. In these coordinates, the Hamiltonian system with Hamiltonian \( K_m \) takes the particularly simple form

\[
\dot{x}_n = \omega_{n,m} y_n, \quad \dot{y}_n = -\omega_{n,m} x_n, \quad \omega_{n,m} := \partial J_n K_m = \{K_m, \varphi_n\}_{\partial x},
\]

where \( \omega_{n,m} \) is called the \( n \)th frequency of the Hamiltonian \( K_m \).

For the dNLS equation global Birkhoff coordinates \( (x_n, y_n)_{n \in \mathbb{Z}} \) can be constructed in terms of global action-angle coordinates \( (I_n, \theta_n)_{n \in \mathbb{Z}} \) -- see [10, 4] and references therein. As the model space we choose the Hilbert space \( H_r^m = \ell^2_r(\mathbb{Z}) \times \ell^2_r(\mathbb{Z}) \), \( m \geq 0 \), with elements \( (x, y) = (x_n, y_n)_{n \in \mathbb{Z}} \), which is endowed with the Poisson structure \( \{x_n, y_n\} = -\{y_n, x_n\} = -1^2 \). The dNLS Birkhoff map

\[
\Omega: H_r^0 \to h_r^0, \quad \varphi \mapsto (x_n, y_n)_{n \in \mathbb{Z}},
\]
defines a bi-real-analytic, canonical diffeomorphism, which transforms every Hamiltonian of the NLS hierarchy, on Sobolev spaces of the appropriate order, into Birkhoff normal form, that is \( S_m \circ \Omega^{-1} \) is a real analytic function of the actions \( I_n = (x_n^2 + y_n^2)/2 \) alone. In these coordinates, the Hamiltonian system with Hamiltonian \( S_m \) is given by

\[
\dot{x}_n = -\omega_{n,m} y_n, \quad \dot{y}_n = \omega_{n,m} x_n, \quad \omega_{n,m} := \partial I_n S_m = \{\theta_n, S_m\},
\]

where \( \omega_{n,m} \) is called the \( n \)th frequency of \( S_m \).

We obtain the following relation of the frequencies of the two hierarchies in Birkhoff coordinates.

**Theorem 3** For any \( n \geq 1 \) and \( m \geq 1 \), we have on \( H_r^{m-1} \),

\[
(-1)^m \omega_{n,2m}^2 = \eta_{n,m}. \quad \checkmark
\]

**Remark.** Note that \( \eta_{n,m} = \omega_{n,m-1}^\text{KdV} \circ B \) for any \( n \geq 1 \) and \( m \geq 1 \). Moreover, the symmetry \( \omega_{n,2m}^2 = -\omega_{n,2m}^4 \) for any \( n \geq 1 \) and \( m \geq 1 \) was obtained in [3].

Theorem 3 follows from Theorem 1 and the following relation of the Birkhoff coordinates of the mKdV and dNLS equations.

**Theorem 4** On \( H_r^1 \),

(i) \( I_n^1 \) vanishes if and only if the average \( [v] \) is zero, and for any \( n \geq 1 \), \( I_n^1 \) vanishes if and only if \( J_n \) is zero. (Note that \( I_{-n}^1 = I_n^1 \) for any \( n \geq 1 \).)

(ii) Each \( I_n^1, n \in \mathbb{Z} \), is a real analytic function of the actions \( (J_m)_{m \geq 1} \) and the average alone.

Conversely, the average and each \( J_n, n \geq 1 \), are real analytic functions of the actions \( (I_n)_{n \in \mathbb{Z}} \) alone.

(iii) For any \( n, m \geq 1 \), one has \( \{\theta_{m,n}, J_n\}_{\partial x} = \delta_{m,n} \). In particular, since \( \{\varphi_m, J_n\}_{\partial x} = -\delta_{m,n} \), it follows that \( \varphi_m + \theta_{m,n}^1 \) is a function of the actions \( (J_m)_{m \geq 1} \) and the average alone. \( \checkmark \)

**Method of proof.** The \( m \)th Hamiltonian \( S_m \) of the NLS hierarchy and its gradient \( \partial S_m \) satisfy on \( H_r^{m-1} \), \( m \geq 1 \), the trace formulas

\[
\frac{1}{2m-1} S_m = \sum_{m \in \mathbb{Z}} J_{n,m}, \quad \frac{1}{2m-1} \partial S_m = \sum_{m \in \mathbb{Z}} \partial I_{n,m},
\]

\( ^2 \)Since we closely follow [6] for the mKdV and [3] for the NLS normal form, respectively, we did not change the signs of the Poisson brackets on the model spaces, hence they are opposite.
The actions are defined in terms of spectral data of the Zakharov-Shabat operator

\[ L(\varphi) = \begin{pmatrix} i & 0 \\ 0 & -1 \end{pmatrix} \partial_x + \begin{pmatrix} 0 & \varphi_2 \\ \varphi_1 & 0 \end{pmatrix}, \]

which arises in the Lax-pair formulation of NLS. More to the point, they are defined as functions of the discriminant \( \Delta(\lambda, \varphi) \) of the fundamental solution associated to \( L(\varphi) \). We prove several symmetries of \( \Delta \) and its gradient \( \partial \Delta \) under the transformations \( \varphi \mapsto P\varphi \) and \( \varphi \mapsto R_\alpha \varphi, \alpha \in \mathbb{R} \), from which we obtain corresponding symmetries of the actions \( I_{n,m}, n \in \mathbb{Z}, m \geq 1 \), and their gradient \( \partial I_{n,m} \). This establishes Theorem 2.

In the Lax-pair formulation of mKdV there arises the Hill operator

\[ L_{\text{mKdV}}(u) = -\partial_x^2 + B(u), \]

with the potential \( B(u) \) given by the Miura map \( B(u) = u_x + u^2 \). Theorem 1 now follows from an identification of the spectra of the operators \( L(u, u) \) and \( L_{\text{mKdV}}(u) \). It turns out that for any solution \( f = (f_1, f_2) \) of \( L(u, u)f = \lambda f \), the function \( g = f_1 + if_2 \) is a solution of \( L_{\text{mKdV}}(u)g = \lambda^2 g \). This implies that the Floquet matrix \( \hat{M}(\lambda) \) of \( L(u, u) \) is conjugated to the Floquet matrix \( \hat{M}_{\text{mKdV}}(\mu) \) with \( \mu = \lambda^2 \) – see [1]. Hence the discriminants \( \Delta(\lambda) \) and \( \Delta_{\text{mKdV}}(\mu) \) coincide at \( \mu = \lambda^2 \). The Hamiltonian hierarchies can be obtained from the asymptotic expansions of the corresponding discriminants, which gives for any \( m \geq 1 \) the identities

\[ \frac{1}{2} S_{2m-1}^2 = K_m, \quad S_{2m}^2 = 0. \]

Since by Theorem 2 (applied for \( \varphi = (u, u) \)) we have

\[ X_{S_{2m}}^2 = (Y_{S_{2m-1}}^2, Y_{S_{2m-1}}^2), \]

Theorem 1 follows immediately.

The construction of the Birkhoff coordinates is based on the one of action-angle variables. Since the action variables can be obtained from spectral data of the operators \( L_{\text{mKdV}}(u) \) and \( L(\varphi) \), respectively, the observed relation of the discriminants for \( \varphi = (u, u) \) allow us to derive Theorem 4.

**Related work.** Chodos observed in [1] that the Floquet matrix \( \hat{M}(\lambda) \) of \( L(u, u) \) is conjugated to the Floquet matrix \( M_{\text{mKdV}}(\mu) \) of \( L_{\text{mKdV}}(u) \) with \( \mu = \lambda^2 \). He uses this to obtain the identity

\[ K_m = \frac{1}{2} S_{2m-1}^2, \quad m \geq 0, \quad (6) \]

on the Sobolev spaces of the appropriate order, by realizing the NLS and mKdV Hamiltonians as traces of certain powers of the operators \( L(u, u) \) and \( L_{\text{mKdV}}(u) \), respectively. His approach, however, seems not to be suited to compare the Hamiltonian vector fields of the NLS and the mKdV hierarchies, which is necessary to identify the PDEs in the mKdV hierarchy as subsystems of the NLS hierarchy. Note that Theorem 1 does not follow immediately from (6) by differentiation. Indeed, the indices of the identity \( K_m = \frac{1}{2} S_{2m-1}^2 \) for the Hamiltonians themselves are different from the indices of the identity \( Y_{K_m} = X_{S_{2m}}^2 \) for the Hamiltonian vector fields. This is due to the fact that the Poisson structure (1) of mKdV involves an additional derivative \( \partial_x \) in comparison to the Poisson structure (4) of NLS.

Dickey [2] shows several algebraic relations of the NLS hierarchy and derives the mKdV hierarchy from the former by the method of Drinfeld-Sokolov reduction. However, the obtained relations are implicit in contrast to the explicit formulas given in Theorem 1 & 2.
Item (ii) of Theorem 2 has been obtained by Magri [9] in the case of a $C_0^\infty$-potential on $[0, 1]$. In this case, the NLS system (2) can be written in Bi-Hamiltonian form
\[\partial_t \varphi = K \partial S_3, \quad \text{and} \quad \partial_t \varphi = K_2 \partial S_2.\] (7)

Here, $K = -iJ$ denotes the standard Poisson structure and $K_2$ denotes the second Poisson structure
\[K_2 f = \partial_x Pf - iR \varphi \left( \int_0^x (f_1 \varphi_1 - f_2 \varphi_2) \, dx + \int_1^x (f_1 \varphi_1 - f_2 \varphi_2) \, dx \right),\]
with Poisson bracket $\{F, G\}_2 = \int_x \partial_x F \partial_x G \, dx$. Both Poisson structures are compatible on $C_0^\infty$ in the sense that
\[\{F, G\}_\lambda := \{F, G\} - \lambda \{F, G\}_2\]
is a Poisson bracket for any real $\lambda$, which due to the nonlinear nature of the Jacobi identity is a nontrivial constraint. The second Poisson structure $K_2$ is non-constant. Furthermore, one obtains the following more general version of Theorem 2 (ii)
\[K \partial S_{m+1} = K_2 \partial S_m, \quad m \geq 1.\] (8)

Note that one has $K_2|_{E_c} = \partial_x$, hence item (ii) indeed follows from (8).

However, we point out that the condition $\varphi \in C_0^\infty(0, 1]$ is neither dynamically invariant for the NLS system (2) nor for the mKdV system (5). Moreover, in the case of periodic boundary conditions, one verifies using the case that $\varphi$ is a nontrivial constant, that the NLS system (2) is not Bi-Hamiltonian in the sense of Magri. Indeed, the identity
\[K \partial S_3 = K_2 \partial S_2\]
does not hold for any linear operator $K_*$, since for this choice of the potential $\partial S_2$ vanishes, $\partial S_3$ does not, and $K$ is invertible. In fact, one infers that (8) generically does not hold for $m$ even. However, one can restrict $K \partial S_{m+1}$ and $K_2 \partial S_m$ to the invariant subspaces
\[\mathcal{M}_- = \{ \varphi(1-x) = -\varphi(x) \}, \quad \mathcal{M}_+ = \{ \varphi(1-x) = \varphi(x) \},\]
where all odd, respectively even, derivatives of $\varphi$ vanish on the boundary of $[0, 1]$. On these spaces (8) holds for $m$ odd – see Section 5.

Furthermore, Theorem 2 and Theorem 3 are related to [3]. In particular, the identities $I^1_n = I^2_n$ and $\theta_{n-1}^+= -\theta_n^+$, for any $n \geq 1$ are proved there implying $\omega_{n, 2m}^1 = -\omega_{n, 2m}^1$.

Finally, we mention the work of Zakharov & Kuznetsov [14], where multiscale expansions are proposed to discover relations between various integrable PDEs.

Organization of this paper. In Section 2 the mKdV and NLS action variables as well as the spectral data needed to define them are introduced. In Section 3 the discriminants of mKdV and NLS are compared and Theorem 4 (i)-(ii) are proven. In Section 4 the symmetries of the Hamiltonians in the NLS hierarchy under the transformations $\varphi \mapsto P \varphi$ and $\varphi \mapsto R_{\alpha} \varphi$, $\alpha \in \mathbb{R}$, are obtained and subsequently used in Section 5 to prove Theorem 1, Theorem 2, Theorem 3, and Theorem 4 (iii).

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2 Setup

We begin by briefly recalling the definition of the mKdV and NLS action variables as well as the properties of the spectral data needed to define them – see e.g. [5, 6, 4].

**mKdV action variables.** The Miura transform [11]

\[ H^3_e \to H^0_e, \quad u \mapsto B(u) = u_x + u^2, \]

when restricted to \( H^1_e \) where mKdV is well-posed, maps solution of the defocusing mKdV equation onto solutions of the KdV equation. This allows us to use the setup for KdV as in [5] and pull back all defined objects using the Miura transform – see also [6]. For a potential \( u \in H^1_e \) consider the Hill operator

\[ L_{mKdV}(u) = -\partial_x^2 + B(u) \]
on the interval \([0, 2]\) of twice the length of the period of \( u \) with periodic boundary conditions. By a slight abuse of notation, the spectrum of \( L_{mKdV}(u) \) is called the periodic spectrum of \( u \) and is denoted by \( \text{spec}(u) \). It is known to be discrete and to consist of a sequence \( \mu_0^+, \mu_1^+, \mu_2^+ \ldots \) of periodic eigenvalues, which, when counted with their multiplicities, can be ordered lexicographically – first by their real part and second by their imaginary part – such that

\[ \mu_0^+ \leq \mu_1^- \leq \mu_1^+ \leq \ldots \leq \mu_{n-1}^- \leq \mu_n^+ \leq \ldots, \quad \mu_0^\pm = n^2 \pi^2 + \ell_n. \]

Here \( \ell_n \) denotes a generic \( \ell^2 \)-sequence. For any \( n \geq 1 \) we define the gap length

\[ \delta_n(u) := \mu_n^+(u) - \mu_n^-(u). \]

When \( u \) is real-valued, the periodic spectrum and the gap lengths are real-valued.

To obtain a suitable characterization of the periodic spectrum, let \( y_1(x, \mu, u) \) and \( y_2(x, \mu, u) \) be the two standard fundamental solutions of \( L_{mKdV} = \mu y \) and denote by \( \Delta_{mKdV}(\mu, u) = y_1(1, \mu, u) + y_2(1, \mu, u) \) the associated discriminant. The periodic spectrum of \( u \) is precisely the zero set of the entire function \( \Delta^2_{mKdV}(\mu) - 4 \), and we have the product representation

\[ \Delta^2_{mKdV}(\mu) - 4 = 4(\mu_0^+ - \mu) \prod_{m \geq 1} \frac{(\mu_m^+ - \mu)(\mu_m^- - \mu)}{\pi^4}. \]  

(9)

The \( \mu \)-derivative is denoted by \( \Delta^\bullet_{mKdV}(\mu) := \partial_\mu \Delta_{mKdV}(\mu) \).

For each potential \( u \in H^1_e \) there exists an open neighborhood \( V_{mKdV} \) within \( H^1_e \) such that the straight lines

\[ G_0 = \{ \mu_0^+ - t \mid t \geq 0 \}, \quad G_n^{mKdV} = [\mu_n^-, \mu_n^+], \quad n \geq 1, \]

are disjoint from each other for every potential in \( V_{mKdV} \). Actually, for \( V_{mKdV} \) sufficiently small, there exist mutually disjoint neighborhoods \( (U_n^{mKdV})_{n \geq 0} \subset \mathbb{C} \), called isolating neighborhoods, such that \( G_n^{mKdV} \) is contained in \( U_n^{mKdV} \) for every \( n \geq 0 \) and every potential in \( V_{mKdV} \), and \( U_n^{mKdV} = \{ |\mu - n^2 \pi^2| \leq \pi/4 \} \) for \( n \) sufficiently large. The union of all \( V_{mKdV} \) with \( u \in H^1_e \) is denoted by \( W_{mKdV} \).

To define the action variables in terms of contour integrals in the complex plane, we introduce the canonical branch of the square root of \( \Delta^2_{mKdV}(\mu) - 4 \) by stipulating on \( H^1_e \) that

\[ i \sqrt{\Delta^2_{mKdV}(\mu) - 4} > 0 \quad \text{for} \quad \mu \in (\mu_0^+, \mu_1^-). \]  

(10)

This root admits an analytic extension onto \((\mathbb{C} \setminus \bigcup_{n \geq 0} U_n^{mKdV}) \times V_{mKdV} \) for any \( u \in W_{mKdV} \) – see also [5].
To proceed, we define for any $u \in W^{mKdV}$ on $(C \setminus \bigcup_{n \geq 0} U_{n, u}^{mKdV}) \times V_{u}^{mKdV}$ the mapping

$$F_{mKdV}(\mu) = \int_{\mu}^{\infty} \frac{\Delta_{mKdV}^{*}}{\sqrt{\Delta_{mKdV} - 4}} \, dz,$$

(11)

where the path of integration is chosen to not intersect any open gap except possibly at its endpoints. This mapping is analytic on $(C \setminus \bigcup_{n \geq 0} U_{n, u}^{mKdV}) \times V_{u}^{mKdV}$, and locally around $G_{n}^{mKdV}$

$$F_{mKdV}(\mu) + in\pi = \cosh^{-1} \left( \frac{\Delta_{mKdV}(\mu)}{2} \right)$$

$$:= \log \left( \frac{-1}{2} \left( \Delta_{mKdV}(\mu) + \sqrt{\Delta_{mKdV}^{2}(\mu) - 4} \right) \right),$$

with log denoting the principal branch of the logarithm. Moreover, for a finite gap potential $u$, the mapping $F_{mKdV}$ has the following asymptotic expansion along $\mu = a_{n}^{2}$ with $a_{n} = (n + 1/2)\pi$,

$$F_{mKdV}\big|_{\mu = a_{n}^{2}} = -i a_{n} + i \sum_{1 \leq k \leq N} \frac{2k_{k}}{(2a_{n})^{2k-1}} + O(a^{-2N-1}), \quad n \to +\infty. \quad (12)$$

The $n$th $mKdV$ action, $n \geq 1$, of $u \in W^{mKdV}$ is then given by

$$J_{n} := \frac{-1}{4\pi} \int_{\Sigma_{n}} \mu^{-1} F_{mKdV}(\mu) \, d\mu,$$

(13)

where $\Sigma_{n}$ denotes any sufficiently close counter clockwise oriented circuit around $G_{n}^{mKdV}$ which does not enclose the origin. The action $J_{n}$ vanishes if and only if the gap length $\delta_{n}$ is zero – see [6, 13] for details.

**NLS action variables.** For a potential $\varphi = (\varphi_{1}, \varphi_{2}) \in H_{c}^{0}$, consider the Zakharov-Shabat operator

$$L(\varphi) := \begin{pmatrix} i & 1 \\ -i & 0 \end{pmatrix} \frac{d}{dx} + \begin{pmatrix} \varphi_{2} \\ \varphi_{1} \end{pmatrix}$$

on the interval $[0, 2]$ with periodic boundary conditions. By a slight abuse of notation, the spectrum of $L(\varphi)$ is called the periodic spectrum of $\varphi$ and is denoted by $\text{spec}(\varphi)$. It is known to be discrete and to consist of a sequence of pairs of complex eigenvalues $\lambda_{n}^{\pm}(\varphi)$ and $\lambda_{n}^{-}(\varphi)$, $n \in \mathbb{Z}$, listed with algebraic multiplicities, such that when ordered lexicographically

$$\cdots \leq \lambda_{n-1}^{+} \leq \lambda_{n}^{-} \leq \lambda_{n}^{+} \leq \lambda_{n+1}^{-} \leq \cdots, \quad \lambda_{n}^{\pm} = n\pi + \delta_{n}^{\pm}.$$  

We also define the gap lengths

$$\gamma_{n}(\varphi) := \lambda_{n}^{+}(\varphi) - \lambda_{n}^{-}(\varphi).$$

Denote by $M(x, \lambda, \varphi)$ the standard fundamental solution of $L(\varphi)M = \lambda M$, and introduce the discriminant $\Delta(\lambda, \varphi) := \text{tr} M(1, \lambda, \varphi)$. The periodic spectrum of $\varphi$ is precisely the zero set of the entire function $\Delta^{2}(\lambda) - 4$, and we have the product representation

$$\Delta^{2}(\lambda) - 4 = -4 \prod_{n \in \mathbb{Z}} \frac{(\lambda_{n}^{+} - \lambda)(\lambda_{n}^{-} - \lambda)}{\pi_{n}^{2}}, \quad \pi_{n} := \begin{cases} n\pi, & n \neq 0, \\ 1, & n = 0. \end{cases}\quad (14)$$

We also need the $\lambda$-derivative $\Delta^{*} := \partial_{\lambda} \Delta$.

For any potential $\varphi \in H_{c}^{0}$, there exists an open neighborhood $V_{\varphi}$ within $H_{c}^{0}$ for which there exist disjoint closed discs $(U_{n})_{n \in \mathbb{Z}}$ centered on the real axis such that $G_{n} := [\lambda_{n}^{-}, \lambda_{n}^{+}]$ is contained in the interior of $U_{n}$ for any potential in $V_{\varphi}$ and any $n \in \mathbb{Z}$, and $U_{n} = \{ |\lambda - n\pi| \leq \pi/4 \}$ for $|n|$
sufficiently large. Such discs are called *isolating neighborhoods*, and we denote the union of all $V_\varphi$ with $\varphi \in \mathcal{H}^0_c$ by $W$.

To define the action variables in terms of contour integrals in the complex plane, we introduce the *canonical root* $\sqrt{\Delta^2(\lambda) - 4}$ by stipulating on $\mathcal{H}^0_c$ that
\begin{equation}
i \sqrt{\Delta^2(\lambda) - 4} > 0, \quad \lambda_0^+ < \lambda < \lambda_1^-.
\end{equation}

This root admits an analytic continuation onto $(\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} U_n) \times V_\varphi$.

To proceed, we define for any $\varphi \in W$ on $(\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} U_n) \times V_\varphi$ the mapping
\begin{equation}F(\lambda) := \frac{1}{2} \left( \int_{\lambda_0^-}^\lambda \frac{\Delta^*(z)}{\sqrt{\Delta^2(z) - 4}} \, dz + \int_{\lambda_0^+}^\lambda \frac{\Delta^*(z)}{\sqrt{\Delta^2(z) - 4}} \, dz \right).
\end{equation}

This map is analytic on $(\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} U_n) \times V_\varphi$ with gradient
\begin{equation}\partial F = \frac{\partial \Delta}{\sqrt{\Delta^2 - 4}}.
\end{equation}

Furthermore, $F(\lambda_0^+) = F(\lambda_0^-) = 0$, and
\begin{equation}F(\lambda) = \int_{\lambda_0^-}^\lambda \frac{\Delta^*(z)}{\sqrt{\Delta^2(z) - 4}} \, dz = \int_{\lambda_0^+}^\lambda \frac{\Delta^*(z)}{\sqrt{\Delta^2(z) - 4}} \, dz.
\end{equation}

If $\varphi \in \mathcal{H}^0_c$, then locally around $G_n$
\begin{equation}F(\lambda) + i n \pi = \cosh^{-1} \left( \frac{\Delta(\lambda)}{2} \right) = \log \left( \frac{1}{2} \right) \left( \Delta(\lambda) + \sqrt{\Delta^2(\lambda) - 4} \right).
\end{equation}

Moreover, if $\varphi$ is a finite gap potential, then there exists $\Lambda > 0$ such that
\begin{equation}F(\lambda, \varphi) = -i \lambda + \frac{1}{2} \sum_{n \geq 1} S_n(\varphi) (2\lambda)^n, \quad |\lambda| > \Lambda.
\end{equation}

For $\varphi \in W$ the $n$th *NLS action variable*, $n \in \mathbb{Z}$, is given by
\begin{equation}I_n = -\frac{1}{\pi} \int_{\Gamma_n} F(\lambda) \, d\lambda,
\end{equation}

with $\Gamma_n$ being a sufficiently close counter clockwise oriented circuit around $G_n$. The action $I_n$ vanishes if and only if the gap length $\gamma_n$ is zero – see [4, 12] for details.

## 3 Identity for the discriminants

In this section we establish the following identity relating the discriminants of the Zakharov-Shabat operator with the one of a corresponding Hill operator and discuss several applications.

**Theorem 5** For all $\lambda \in \mathbb{C}$ and $u \in H^1_c$,
\begin{equation}\Delta_{mKdV}(\lambda^2, u) = \Delta(\lambda, \varphi_u),
\end{equation}
where $\varphi_u := (u, u)$. \hfill $\times$

The proof of this theorem is based on an observation by Chodos [1] relating the fundamental solutions of the Hill operator $L_{mKdV}(u)$ and the Zakharov-Shabat operator $L(\varphi_u)$. Given $u \in H^1_c$ and $\lambda \in \mathbb{C}$, define
\begin{equation}A(x, \lambda, u) := \left( \begin{array}{cc} 1 & i \lambda \\ u - i \lambda & iu - \lambda \end{array} \right).
\end{equation}

Note that $\det A(x, \lambda, u) = -2\lambda$, hence $A(x, \lambda, u)$ is invertible for any $\lambda \neq 0, 0 \leq x \leq 1$, and $u \in H^1_c$. Furthermore, $A(x, \lambda, u)$ is 1-periodic in $x$. Finally, denote by $M_{mKdV}$ the fundamental solution of $L_{mKdV}$ and by $M$ the one of $L$. 

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Lemma 2  Suppose \( u \in H^1_\ell \). If for some \( f = (f_1, f_2) \in H^2(\mathbb{R}) \) and \( \lambda \in \mathbb{C} \),
\[
L(\varphi_u)f = \lambda f,
\]
then
\[
L_{mKdV}(u)(f_1 + if_2) = \lambda^2(f_1 + if_2).
\]
Moreover, if \( \lambda \neq 0 \), then
\[
M_{mKdV}(x, \lambda^2, u) = A(x, \lambda, u)M(x, \lambda, \varphi_u)A(0, \lambda, u)^{-1},
\]
In particular, at \( x = 1 \),
\[
M_{mKdV}(1, \lambda^2, u) = A(0, \lambda, u)M(1, \lambda, \varphi_u)A(0, \lambda, u)^{-1}. \quad \times
\]
Proof. If \( f \) is a solution of \( L(\varphi_u)f = \lambda f \), then
\[
\partial_x f = \begin{pmatrix} -i\lambda & iu \\ -iu & i\lambda \end{pmatrix} f, \quad \partial_x^2 f = i(\partial_x u)J f + (-\lambda^2 + u^2)f, \quad J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
\]
and hence
\[
(-\partial_x^2 + u^2 + \partial_x u)(f_1 + if_2) = \lambda^2(f_1 + if_2).
\]
Let \( M = (m_1, m_2) \) denote the entries of the fundamental solution of \( L \), and define
\[
y_1 = m_1 + im_3, \quad y_2 = m_2 + im_4,
\]
then by the preceding calculation
\[
y_1' = (u - i\lambda)m_1 + (iu - \lambda)m_3, \quad y_2' = (u - i\lambda)m_2 + (iu - \lambda)m_4.
\]
Thus \( Y = \begin{pmatrix} y_1 \\ y_1' \\ y_2 \\ y_2' \end{pmatrix} \) is a fundamental solution of \(-y'' + B(u)y = \lambda y\) with
\[
Y = AM = \begin{pmatrix} 1 \\ u - i\lambda \\ iu - \lambda \end{pmatrix} \begin{pmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{pmatrix}.
\]
As \( A(0, \lambda, u) \) is invertible if and only if \( \lambda \neq 0 \), and \( u \) is 1-periodic in \( x \), the claim follows. \[\Box\]

Proof of Theorem 5. By Lemma 2, the fundamental solutions \( M_{mKdV}(1, \lambda^2, u) \) and \( M(1, \lambda, \varphi_u) \), evaluated at \( x = 1 \), are conjugated for \( \lambda \in \mathbb{C} \setminus \{0\} \). Thus their discriminants coincide, \( \Delta_{mKdV}(\lambda^2, u) = \Delta(\lambda, \varphi_u) \). By continuity this identity also holds for \( \lambda = 0 \). \[\Box\]

Consequently, as already noted in [3], the discriminant \( \Delta(\lambda, \varphi_u) \) for \( u \in H^1_\ell \) is an even function of \( \lambda \). Recall from (14) that the periodic spectrum of \( \varphi_u \) is precisely the zero set of \( \Delta(\lambda, \varphi_u) - 4 \). Thus, it follows from the asymptotic behavior \( \lambda_n^\pm(n) = n\pi + \ell_n^2 \) and the lexicographical ordering that
\[
\lambda_n^+(\varphi_u) = -\lambda_n^-(\varphi_u), \quad n \geq 0.
\]
(20)

Further symmetries of the discriminant will be obtained in Section 4.

Lemma 3  For every \( u \in H^1_\ell \),
\[
\mu_n^+(u) = (\lambda_n^+(\varphi_u))^2 = (\lambda_n^-(\varphi_u))^2, \quad \mu_n^-(u) = (\lambda_n^-(\varphi_u))^2, \quad n \geq 1,
\]
and \( \mu_n^\pm(u) \) has the same geometric multiplicity as \( \lambda_n^\pm(\varphi_u) \). In particular, \( G_n^2(\varphi_u) = G_n^{mKdV}(u) \) for any \( n \geq 1 \), and \( \delta_n(u) = 0 \) iff \( \gamma_n(\varphi_u) = 0 \). \[\times\]
Proof. By Theorem 5, \( \mu = \lambda^2 \) is an eigenvalue of \( L_{mKdV}(u) \) if and only if \( \lambda \) is an eigenvalue of \( L(\varphi_u) \). If \( u \) is real-valued, then the periodic spectra of \( u \) and \( \varphi_u \) are real, and due to the symmetry and the lexicographical ordering

\[
0 \leq \mu_0^2 < \mu_1 \leq \mu_1^+ \leq \cdots, \quad \cdots \leq \lambda_0^- \leq 0 < \lambda_1^+ \leq \lambda_1^- \leq \cdots.
\]

Consequently, \( \mu_0^+ = (\lambda_0^+)^2 = (\lambda_0^-)^2 \), and \( \mu_0^- = (\lambda_0^-)^2 \) as well as \( \mu_n^+ = (\lambda_n^+)^2 \) for any \( n \geq 1 \). Thus \( G_n^2 = [\lambda_n^-, (\lambda_n^+)^2] = G_n^{mKdV} \) for any \( n \geq 1 \).

Finally, \( \mu_n^+ = \mu_n^- \) if and only if \( \lambda_n^+ = \lambda_n^- \), and the fundamental solutions \( M_{mKdV}(1, \mu_n^+, u) \) and \( M(1, \lambda_n^+, \varphi_u) \) are conjugated by Lemma 2. Thus there exist two linear independent eigenfunctions for \( \mu_n = \mu_n^+ = \mu_n^- \) if and only if they exist for \( \lambda_n = \lambda_n^+ = \lambda_n^- \).

Proof of Theorem 4 (i). For \( u \in H^1 \) it follows from (13), (19), and Lemma 3 that for any \( n \geq 1 \)

\[
J_n(u) = 0 \Leftrightarrow \delta_n(u) = 0 \Leftrightarrow \gamma_n(\varphi_u) = 0 \Leftrightarrow I_n(\varphi_u) = 0.
\]

One concludes from (14) – see also [4] – that \( \Delta(\lambda, \varphi) - 2 \) vanishes at \( \lambda_0^\pm \) and is strictly positive on the interior of \( G_0 \). By (20), \( G_0(\varphi_u) \) is a symmetric interval around zero, hence \( I_0(\varphi_u) \) is strictly positive if any only if

\[
\Delta(0, \varphi_u) = \Delta_{mKdV}(0, u) \neq 2.
\]

It was observed in [5] that

\[
\Delta_{mKdV}(0, u) = 2 \cosh([u]), \quad (21)
\]

hence \( I_0(\varphi_u) \) vanishes if and only if the average \([u] \) vanishes.

Lemma 4 For \( u \in H^1 \) and \( \lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_n \) with \( \Re \lambda > 0 \),

\[
\sqrt{\Delta_{mKdV}^2(\lambda^2, u)} - 4 = \sqrt{\Delta^2(\lambda, \varphi_u)} - 4. \times
\]

Proof. On \( D = (\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_n) \cap \{ \Re \lambda > 0 \} \) the canonical NLS root of \( \varphi_u \) is analytic, and, by Lemma 3, \( D \) is mapped by \( \lambda \mapsto \lambda^2 \) onto the domain \( \mathbb{C} \setminus \bigcup_{n \geq 0} G_n^{mKdV} \) where the canonical mKdV root of \( u \) is analytic. Moreover, \( \Delta_{mKdV}^2(\lambda^2, u) - 4 = \Delta^2(\lambda, \varphi_u) - 4 \), by Theorem 5, hence these roots differ at most by a sign. Recall from (10) that the canonical mKdV root is chosen such that

\[
i \sqrt{\Delta_{mKdV}^2(\mu, u)} - 4 > 0, \quad \mu_n^+ < \mu < \mu_1^-,
\]

and from (15) that the canonical NLS root is chosen such that

\[
i \sqrt{\Delta^2(\lambda, \varphi_u)} - 4 > 0, \quad \lambda_0^+ < \lambda < \lambda_1^-.
\]

Since \( \mu_0^+ = (\lambda_0^+)^2 \) and \( \mu_1^- = (\lambda_1^-)^2 \), both roots have the same sign provided that \( \Re \lambda > 0 \).

Lemma 5 Suppose \( u \in H^1 \), then on \( \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_n \) provided \( \Re \lambda > 0 \),

\[
F_{mKdV}(\lambda^2, u) = F(\lambda, \varphi_u). \times
\]

Proof. Since \( \Delta(\lambda, \varphi_u) = 2\Delta_{mKdV}(\lambda^2, u) \) by Theorem 5, we conclude with (16) and Lemma 4,

\[
F(\lambda, \varphi_u) = \int_{\lambda_0^+}^{\lambda} \frac{\Delta(\lambda, \varphi_u)}{\sqrt{\Delta^2(\lambda, \varphi_u)} - 4} \, d\lambda = \int_{\lambda_0^+}^{\lambda} \frac{\Delta_{mKdV}(\lambda^2, u)}{\sqrt{\Delta_{mKdV}^2(\lambda^2, u)} - 4} 2\lambda \, d\lambda.
\]

Now substituting \( w = \lambda^2 \), and using that \( \mu_n^+ = (\lambda_0^+)^2 \geq 0 \) and \( \Re \lambda > 0 \), yields in view of (11)

\[
F(\lambda, \varphi_u) = \int_{(\lambda_0^+)^2}^{\lambda^2} \frac{\Delta_{mKdV}(u, w)}{\sqrt{\Delta_{mKdV}^2(u, w)} - 4} \, dw = F_{mKdV}(\lambda^2, u). \]

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At this point, we may recover Chodos’ observation in the framework of the NLS hierarchy, which in addition allows us to prove that the Hamiltonians $S_{2m}$ for any $m \geq 1$ vanish at any point $\varphi_u$. To simplify notation, for any functional $f$ we set $f^2(u) := f(\varphi_u)$.

**Proposition 6** The Hamiltonians of the mKdV and NLS hierarchies satisfy for every $m \geq 1$

$$K_m = \frac{1}{2} S_{2m-1}^2 \text{ on } H^{m-1}_r, \quad S_{2m}^2 \text{ on } H^m_r. \quad \blacktriangle$$

**Proof.** It immediately follows from the preceding lemma and the expansions (12) and (18) that for each $m \geq 1$,

$$K_m(u) = \frac{1}{2} S_{2m-1}(\varphi_u) \text{ on } H^{m-1}_r, \quad S_{2m}(\varphi_u) = 0 \text{ on } H^m_r.$$

Since both hand sides are analytic in $u$, these identities extend to all of $H^{m-1}_r$ and $H^m_r$, respectively, by Lemma 14. \hfill \blacktriangle

To be able to compare the actions of mKdV and dNLS, it is convenient to introduce actions defined on integer levels $k$ – see [10, 4]. More precisely, for $u \in W^{mKdV}$ the $n$th mKdV action, $n \geq 1$, on level $k \in \mathbb{Z}$ is defined by

$$J_{n,k}(u) := -\frac{1}{4\pi} \int_{\Sigma_n} \mu^{k-2} F_{mKdV}(\mu, u) \, d\mu, \quad (22)$$

where $\Sigma_n$ is a sufficiently close circuit around $G^m_{mKdV}$ which does not enclose the origin. Note that $J_n = J_{n,1}$. Similarly, for $\varphi_u \in W$, the $n$th NLS action, $n \in \mathbb{Z}$, on level $k \in \mathbb{Z}$ is defined by

$$I_{n,k}(\varphi_u) := -\frac{1}{\pi} \int_{\Gamma_n} \lambda^{k-1} F(\lambda, \varphi_u) \, d\lambda, \quad (23)$$

where $\Gamma_n$ is a sufficiently close circuit around $G_n$ which in the case $n \neq 0$ does not enclose the origin, while $\Gamma_0$ is a circuit around the origin. Note that $I_n = I_{n,1}$.

**Lemma 7** On $H^1_r$ for any $n \geq 1$ and any $k \in \mathbb{Z}$,

$$2 J_{n,k}(u) = -\frac{1}{\pi} \int_{\Gamma_n} \lambda^{2k-3} F(\lambda, \varphi_u) \, d\lambda = I_{n,2k-2}(\varphi_u).$$

In particular, $I_{n,2k-2}$ is an analytic extension of $(u, u) \mapsto 2J_{n,k}(u)$ onto an open neighborhood of $\mathcal{E}^0$ within $H^0_r$. \hfill \blacktriangle

**Proof.** For any $n \geq 1$, the map $\lambda \mapsto \lambda^2$ maps any sufficiently close circuit $\Gamma_n$ around $G_n$ bijectively onto a circuit $\Sigma_n$ around $G^m_{mKdV}$. Consequently, by the transformation formula and the previous lemma

$$I_{n,2k-2} = -\frac{1}{\pi} \int_{\Gamma_n} \lambda^{2k-3} F(\lambda) \, d\lambda = -\frac{1}{2\pi} \int_{\Gamma_n} \lambda^{2k-5} F_{mKdV}(\lambda^2) 2\lambda \, d\lambda = -\frac{1}{2\pi} \int_{\Sigma_n} \mu^{k-2} F_{mKdV}(\mu) \, d\mu = 2 J_{n,k}. \quad \blacktriangle$$

**Proof of Theorem 4 (ii).** After possibly shrinking $W^{mKdV}$, we may assume that $u \in W^{mKdV}$ implies $\varphi_u \in W$. As a result, $I_n^u(u) = I_n(\varphi_u)$, $n \in \mathbb{Z}$, defines an analytic function on $W^{mKdV}$. Moreover, $\nabla_u \Delta^2(\lambda) = \nabla_u \Delta_{mKdV}(\lambda^2)$ by Theorem 5, hence

$$\nabla I_n^u(u) = -\frac{1}{\pi} \int_{\Gamma_n} \frac{\nabla_u \Delta^4(\lambda)}{\sqrt{\Delta^2(\lambda, \varphi_u) - 4}} \, d\lambda = -\frac{1}{\pi} \int_{\Gamma_n} \frac{\nabla_u \Delta_{mKdV}(\lambda^2)}{\sqrt{\Delta_{mKdV}(\lambda^2, u) - 4}} \, d\lambda.$$
As shown in [5, Lemma 10.2], we have \( \{ \Delta_{\text{mKdV}}(\mu_1), \Delta_{\text{mKdV}}(\mu_2) \} \theta_e = 0 \) for any \( \mu_1, \mu_2 \in \mathbb{C} \). Hence, for any \( n \in \mathbb{Z} \) and \( m \geq 1 \) on \( W_{\text{mKdV}} \),

\[
\{ I_n^2, J_m \} \theta_e = \frac{1}{4 \pi^2} \int_{\Gamma_n} \int_{\mathbb{S}_m} \frac{\{ \Delta_{\text{mKdV}}(\lambda^2), \Delta_{\text{mKdV}}(\mu) \} \theta_e}{\sqrt{\Delta_{\text{mKdV}}(\lambda^2) - 4 \sqrt{\Delta_{\text{mKdV}}(\mu) - 4}}} \, d\lambda \, d\mu = 0.
\]

Consequently, each \( I_n^2 \circ \Psi^{-1} \), where \( \Psi \) denotes the mKdV Birkhoff map, is a real analytic function of the actions \( (J_m)_{m \geq 1} \) and the average \([u]\) alone.

Conversely, by the preceding lemma each \( J_m, m \geq 1 \), extends to an analytic function \( \tilde{J}_m = J_m,0 \) on an open neighborhood of \( \mathcal{E}^1_r \) within \( \mathcal{H}^0_r \). Moreover,

\[
\{ \tilde{J}_m, I_n \} = \{ I_m,0, I_n \} = \frac{1}{4 \pi^2} \int_{\Gamma_n} \int_{\mathbb{S}_m} \frac{\{ \Delta(z), \Delta(w) \}}{\sqrt{\Delta^2(z) - 4 \sqrt{\Delta^2(w) - 4}}} \, dz \, dw = 0,
\]

using that \( \{ \Delta(z), \Delta(w) \} = 0 \) for any \( z, w \in \mathbb{C} \) by [4, Lemma 8.3]. So, with \( \Omega \) denoting the NLS Birkhoff mapping, \( \tilde{J}_m \circ \Omega^{-1} \) is a real analytic function of the actions \( (I_n)_{n \in \mathbb{Z}} \) alone. For the average we have for \( u \in H^1_r \) by (21)

\[
[u] = \cosh^{-1} \left( \frac{\Delta(0, \varphi_u)}{2} \right) = - \int_{\lambda_0}^{\tau_0} \frac{\Delta^\ast(\lambda, \varphi_u)}{\sqrt{\Delta^2(\lambda, \varphi_u) - 4}} \, d\lambda,
\]

where \( \tau_0 = (\lambda_0^+ + \lambda_0^-)/2 = 0 \) and the path of integration is chosen to run on the right hand side of the straight line connecting \( \lambda_0^- \) and \( \tau_0 \) in the complex plane. One shows by exactly the same arguments as in the proof of [12, Proposition A2] that the latter defines a real analytic function on all of \( H^0_r \). Since it only depends on the periodic spectrum, it is a real analytic function of the actions alone. \( \blacksquare \)

4 Symmetries of the Zakharov-Shabat discriminant

In this section we obtain several symmetries of the Zakharov-Shabat discriminant under the transformations

\[
\varphi \mapsto P \varphi, \quad \varphi \mapsto R_\alpha \varphi, \quad \varphi \mapsto T \varphi, \quad P := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad R_\alpha := \begin{pmatrix} e^{i \alpha} & 0 \\ 0 & e^{-i \alpha} \end{pmatrix}, \quad \alpha \in \mathbb{R},
\]

and \( T \varphi(x) = \varphi(1 - x) \). By Theorem 5 those symmetries translate into corresponding symmetries of the Hill discriminant. To simplify notation, let

\[
J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad R := R_{\pi/2} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\]

Theorem 6 The discriminant \( \Delta \) has the following symmetries

(i) \( \Delta(\lambda, \varphi) = \Delta(-\lambda, P \varphi) = \Delta(-\lambda, T \varphi) = \Delta(\lambda, R_\alpha \varphi) \) for all \( \lambda \in \mathbb{C} \);

In particular, for all \( n \in \mathbb{Z} \),

\[
\lambda_n^\pm(\varphi) = -\lambda_n^\pm(P \varphi) = -\lambda_n^\pm(T \varphi) = \lambda_n^\pm(R_\alpha \varphi), \quad G_n(\varphi) = -G_n(P \varphi) = -G_n(T \varphi) = G_n(R_\alpha \varphi),
\]

hence one can choose \( W \) to be invariant under \( P, T, \) and \( R_\alpha \) for any \( \alpha \in \mathbb{R} \).

(ii) \( \Delta^\ast(\lambda, \varphi) = -\Delta^\ast(-\lambda, P \varphi) = -\Delta^\ast(-\lambda, T \varphi) = \Delta^\ast(\lambda, R_\alpha \varphi) \) for all \( \lambda \in \mathbb{C} \) and all \( \alpha \in \mathbb{R} \).

(iii) \( \partial \Delta(\lambda, \varphi) = P \partial \Delta(-\lambda, P \varphi) = T \partial \Delta(-\lambda, T \varphi) = R_\alpha \partial \Delta(\lambda, R_\alpha \varphi) \) for all \( \lambda \in \mathbb{C} \) and \( \alpha \in \mathbb{R} \).
(iv) If $\varphi \in W$, then for all $\lambda \in \mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_n$ and all $\alpha \in \mathbb{R}$,

$$\sqrt{\Delta^2(\lambda, \varphi) - 4} = -\sqrt{\Delta^2(-\lambda, P\varphi) - 4} = -\sqrt{\Delta^2(-\lambda, T\varphi) - 4} = \sqrt{\Delta^2(\lambda, R_\alpha \varphi) - 4}.$$  

(v) $\partial_\alpha \Delta(\lambda, \varphi) - 2\lambda R \partial \Delta(\lambda, \varphi) = \xi(\lambda, \varphi) J \varphi$ where the function

$$\xi(x, \lambda, \varphi) := \left( (n_1(\lambda, \varphi) - n_4(\lambda, \varphi)) + 2i \int_0^1 (R \varphi \cdot \partial \Delta(\lambda, \varphi)) \, dy \right)$$

is 1-periodic in $x$ and satisfies $\xi(x, \lambda, \varphi) = -\xi(x, -\lambda, P\varphi) = -\xi(1-x, -\lambda, T\varphi) = \xi(x, \lambda, R_\alpha \varphi)$ for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, and $\alpha \in \mathbb{R}$.  

**Fundamental solution.** Since $\Delta(\lambda, \varphi)$ is the trace of the fundamental solution $\hat{M}(\lambda, \varphi) = M(x, \lambda, \varphi)|_{x=1}$, it follows from Lemma 13 that $\Delta(\lambda, \varphi) = \Delta(-\lambda, P\varphi)$, $\Delta(\lambda, R_\alpha \varphi) = \Delta(\lambda, \varphi)$, and $\Delta(-\lambda, T\varphi) = \Delta(\lambda, \varphi)$. Differentiating these identities with respect to $\lambda$ and $\varphi$ gives items (ii) and (iii) of Theorem 6.

Recalling from (14) that the periodic spectrum is the zero set of $\Delta^2 - 4$, we conclude that

$$\lambda \in \text{spec}(\varphi) \iff -\lambda \in \text{spec}(P\varphi) \iff -\lambda \in \text{spec}(T\varphi) \iff \lambda \in \text{spec}(R_\alpha \varphi).$$

From the lexicographical ordering and the asymptotic behavior $\lambda^\pm_n = n\pi + \ell^2_n$, we further infer that $\lambda^\pm_n(P\varphi) = -\lambda^\mp_n(T\varphi) = \lambda^\pm_n(R_\alpha \varphi)$ for any $n \in \mathbb{Z}$. This proves item (i) of Theorem 6.

**Canonical root.** Clearly, $\Delta^2(\lambda, \varphi) - 4 = \Delta^2(-\lambda, P\varphi) - 4 = \Delta^2(-\lambda, T\varphi) - 4 = \Delta^2(\lambda, R_\alpha \varphi) - 4$ on $W$, hence

$$\sqrt{\Delta^2(\lambda, \varphi) - 4} = \sqrt{\Delta^2(-\lambda, P\varphi) - 4} = \sqrt{\Delta^2(-\lambda, T\varphi) - 4} = \sqrt{\Delta^2(\lambda, R_\alpha \varphi) - 4},$$

where the signs $\zeta_P$, $\zeta_T$, and $\zeta_{R_\alpha}$ have modulus one, are locally constant in $\varphi$, and independent of $\lambda$ as $\Delta^2(\lambda) - 4$ does not vanish on $\mathbb{C} \setminus \bigcup_{n \in \mathbb{Z}} G_n$. The straight line connecting $\varphi$ and the origin is compact in $W$, and further $\sqrt{\Delta^2(\lambda) - 4}|_{\varphi=0} = -2i \sin(\lambda)$, hence $\zeta_P = \zeta_T \equiv -1$ and $\zeta_{R_\alpha} = 1$. This proves (iv) of Theorem 6.

**Gradient Symmetry.** The gradient of $\Delta$ can be represented by the components of $M$ — see [4, Section 4]. To further simplify notation we denote $\hat{M} := M|_{x=1}$. Let $\xi_\pm$ denote the eigenvalues of $\hat{M}$, then for all $\lambda$ with $\hat{n}_2 \neq 0$,

$$i \partial \Delta = \hat{n}_2 f_+ \ast f_-,$$

where $\hat{n}_2 = \left( \begin{smallmatrix} h_1 & k_1 \\ h_2 & k_2 \end{smallmatrix} \right)$. The function $\lambda \mapsto \hat{n}_2(\lambda, \varphi)$ vanishes identically if and only if $\varphi$ is the zero potential, thus for $\varphi \neq 0$ we have $\hat{n}_2(\lambda, \varphi) \neq 0$ for generic $\lambda$. Denote $\Phi = (\varphi_2 \varphi_1)$, then we can write $\partial_x f_\pm = R(\Phi - \lambda) f_\pm$ as $f_\pm$ is a solution of $L f_\pm = \lambda f_\pm$. A straightforward computation shows $(Ra) \ast b = a \ast (Rb) = -R(a \ast b)$ and

$$(R\Phi a) \ast b + a \ast (R\Phi b) = i P \Phi ((Ja) \ast b + a \ast (Jb))$$

for any two vectors $a, b$. Consequently,

$$\partial_x (f_+ \ast f_-) = (\partial_x f_+ \ast f_-) + (f_+ \ast \partial_x f_-)$$

$$= (R(\Phi - \lambda) f_+ \ast f_-) + (f_+ \ast R(\Phi - \lambda) f_-)$$

$$= 2 \lambda R(f_+ \ast f_-) + i P \Phi ((J f_+ \ast f_-) + (f_+ \ast J f_-)) \cdot$$

We conclude from (24) that for generic $\lambda$

$$\partial_x \partial \Delta(\lambda, \varphi) - 2 \lambda R \partial \Delta(\lambda, \varphi) = P \Phi, \quad \Pi = \hat{n}_2 ((J f_+ \ast f_-) + (f_+ \ast J f_-)).$$
Note that the function $\Pi$ is one periodic since $f(0) = 0$ and $0, 0, 1$. We proceed by computing the $x$-derivative of $\Pi$. To this end, we compute

$$
\partial_x(Jf_\pm * f_\pm) = (JR(\Phi - \lambda)f_\pm * f_\pm) + (JRf_\pm * R(\Phi - \lambda)f_\pm)
$$

$$
= (JR^2f_\pm * f_\pm) + (JRf_\pm * R^2f_\pm) - \lambda(JR^2f_\pm * f_\pm) + (JRf_\pm * Rf_\pm)
$$

$$
= (JR^2f_\pm * f_\pm) + (JRf_\pm * Rf_\pm),
$$

where we used that $JR^2a * b + Ja * Rb = 0$. Furthermore, note that $(JR^2a * b) + (Ja * R^2b) = (JR^2a * a) + (Ja * R^2a) = -((\Phi) \cdot (a * b))e_0$ where $a \cdot b = a_1b_1 + a_2b_2$ and $e_0 = (1, -1)$. As a consequence,

$$
\partial_x(Jf_\pm * f_\pm + f_\pm * Jf_\pm) = -2((\Phi) \cdot (f_\pm \ast f_\pm))e_0,
$$

so that

$$
\partial_x \Pi = 2(P^2 \Phi - \Phi P) \partial \Delta = 2i(\Phi \cdot \partial \Delta)e_0, \quad e_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
$$

Here $a \cdot b = a_1b_1 + a_2b_2$. Since $\Pi(0) = (m_1 - \hat{m}_4)e_0$ we conclude

$$
\Pi = (m_1 - \hat{m}_4) + 2i \int_0^x (\Phi \cdot \partial \Delta) \, dy e_0.
$$

Note that $P\Phi e_0 = J\varphi$ so finally

$$
\partial_x \Delta - 2\lambda R \partial \Delta = \left((m_1 - \hat{m}_4) + 2i \int_0^x (\Phi \cdot \partial \Delta) \, dy \right) J\varphi.
$$

The properties of $\xi$ follow immediately from the properties of $\Delta$. This completes the proof of Theorem 6.

Corollary 8  On $W$ for any $\lambda \in \mathbb{C} \setminus \bigcup_{\gamma_n \neq 0} G_n(\varphi)$,

$$
F(\lambda, \varphi) = - F(-\lambda, P\varphi) = - F(-\lambda, T\varphi) = F(\lambda, R_{\alpha} \varphi),
$$

$$
\partial F(\lambda, \varphi) = - P\partial F(-\lambda, P\varphi) = - T\partial F(-\lambda, T\varphi) = R_{\alpha} \partial F(\lambda, R_{\alpha} \varphi).
$$

Proof. Suppose $\varphi \in W$. In view of Theorem 6,

$$
F(-\lambda, P\varphi) = \int_{\lambda_0}^{\lambda} \frac{\Delta^*(z, P\varphi)}{\sqrt{\Delta^2(z, P\varphi) - 4}} \, dz = \int_{\lambda_0}^{\lambda} \frac{\Delta^*(z, \varphi)}{\sqrt{\Delta^2(z, \varphi) - 4}} \, dz = - F(\lambda, \varphi).
$$

In a similar way, one verifies that $F(\lambda, R_{\alpha} \varphi) = - F(-\lambda, T\varphi) = F(\lambda, \varphi)$. The identity for the gradients follows by differentiation.

Corollary 9  (i) If $P\varphi = R_{\alpha} \varphi$ for some $\alpha \in \mathbb{R}$, then for all $\lambda \in \mathbb{C}$

$$
\partial_x \partial F(\lambda, \varphi) - 2\lambda R \partial F(\lambda, \varphi) = \partial_x \partial F(-\lambda, \varphi) + 2\lambda R \partial F(-\lambda, \varphi).
$$

(ii) If $T\varphi = \pm \varphi$, then for all $\lambda \in \mathbb{C}$,

$$
\partial_x \partial F(\lambda, \varphi) - 2\lambda R \partial F(\lambda, \varphi) - 2i \int_0^x (R\varphi \cdot \partial F(\lambda, \varphi)) \, dy
$$

$$
= \partial_x \partial F(-\lambda, \varphi) + 2\lambda R \partial F(-\lambda, \varphi) - 2i \int_0^x (R\varphi \cdot \partial F(-\lambda, \varphi)) \, dy.
$$

Proof. Since $\partial F(\lambda) = \frac{\partial \Delta(\lambda)}{\sqrt{\Delta^2(\lambda) - 4}}$, the first claim follows from Theorem 6 using that $P\varphi = R_{\alpha} \varphi$ implies $\xi(x, -\lambda, \varphi) = - \xi(x, \lambda, \varphi)$. The second claim follows in analogous fashion using that $T\varphi = \pm \varphi$ implies $\hat{m}_1(-\lambda, \varphi) - \hat{m}_4(-\lambda, \varphi) = -(\hat{m}_1(\lambda, \varphi) - \hat{m}_4(\lambda, \varphi))$. 

January 29, 2016
5 Symmetries of the NLS actions and NLS hamiltonians

In this section we obtain several symmetries of the gradients of the NLS action variables which are subsequently used to prove Theorem 4 (iii), Theorem 2, Theorem 1, and Theorem 3. Recall from (23) that for \( \varphi \in W \) the \( n \)th NLS action on level \( k \in \mathbb{Z} \) is given by

\[
I_{n,k}(\varphi) = -\frac{1}{\pi} \int_{\Gamma_{n}(\varphi)} \lambda^{k-1} F(\lambda, \varphi) \, d\lambda,
\]

where \( \Gamma_{n}(\varphi) \) denotes a sufficiently close circuit around \( G_{n}(\varphi) \).

**Lemma 10**

(i) On \( W \) we have for any \( k \geq 1 \), any \( n \in \mathbb{Z} \), and any \( \alpha \in \mathbb{R} \),

\[
\partial I_{n,k}(\varphi) = (-1)^{k-1} P\partial I_{-n,k}(P\varphi) = (-1)^{k-1} T\partial I_{-n,k}(T\varphi) = R_{\alpha} \partial I_{n,k}(R_{\alpha} \varphi).
\]

(ii) If \( P\varphi = R_{\alpha} \varphi \) for some \( \alpha \in \mathbb{R} \), then for any \( k \geq 1 \) and any \( n \in \mathbb{Z} \),

\[
\partial_{x} \partial I_{-n,k-1} - 2R\partial I_{-n,k} = (-1)^{k+1}(\partial_{x} \partial I_{n,k-1} - 2R\partial I_{n,k}).
\]

(iii) If \( T\varphi = \pm \varphi \), then for any \( k \geq 1 \) and any \( n \in \mathbb{Z} \),

\[
\partial_{x} \partial I_{-n,k-1} - 2R\partial I_{-n,k} - 2i \int_{0}^{\pi} (R\partial I_{-n,k-1})
\]

\[
= (-1)^{k+1} \left( \partial_{x} \partial I_{n,k-1} - 2R\partial I_{n,k} - 2i \int_{0}^{\pi} (R\partial I_{n,k-1}) \right). \quad \times
\]

Item (i) for the case \( k = 1 \) has been obtained in [3].

**Proof.** (i) By Theorem 6, \( G_{n}(\varphi) = -G_{-n}(P\varphi) = -G_{-n}(T\varphi) = G_{n}(R_{\alpha} \varphi) \) for any \( n \in \mathbb{Z} \). If, in addition \( \varphi \in W \), then there exists a set of isolating neighborhoods \( (U_{n})_{n \in \mathbb{Z}} \) which are mutually disjoint discs centered on the real axis such that \( G_{n} \subset U_{n} \). They can be chosen such that \( U_{n}(\varphi) = -U_{-n}(P\varphi) = -U_{-n}(T\varphi) = U_{n}(R_{\alpha} \varphi) \) for all \( n \in \mathbb{Z} \). In particular, for any circuit \( \Gamma_{n}(\varphi) \) sufficiently close around \( G_{n}(\varphi) \), its inversion at the origin, \( -\Gamma_{n}(\varphi) \), defines a circuit around \( G_{-n}(P\varphi) \) of the same orientation as \( \Gamma_{n}(\varphi) \). Thus, with the substitution \( \lambda \mapsto -\lambda \),

\[
I_{-n,k}(P\varphi) = -\frac{1}{\pi} \int_{-\Gamma_{n}(\varphi)} \lambda^{k-1} F(\lambda, P\varphi) \, d\lambda
\]

\[
= \frac{1}{\pi} \int_{\Gamma_{n}} (-\lambda)^{k-1} F(-\lambda, P\varphi) \, d\lambda = (-1)^{k-1} I_{n,k}(\varphi),
\]

where we used that \( F(-\lambda, P\varphi) = -F(\lambda, \varphi) \) by Corollary 8. Similarly, using \( F(-\lambda, T\varphi) = -F(\lambda, \varphi) \) one shows that \( I_{-n,k}(T\varphi) = I_{n,k}(\varphi) \), and using \( F(\lambda, R_{\alpha} \varphi) = F(\lambda, \varphi) \) one shows that \( I_{n,k}(R_{\alpha} \varphi) = I_{n,k}(\varphi) \). Differentiating these identities gives

\[
\partial I_{n,k}(\varphi) = (-1)^{k-1} P\partial I_{-n,k}(P\varphi) = R_{\alpha} \partial I_{n,k}(R_{\alpha} \varphi).
\]

(ii) If \( P\varphi = R_{\alpha} \varphi \), the periodic spectrum of \( \varphi \) is symmetric, and one has by Corollary 9

\[
\partial_{x} \partial I_{-n,k-1} - 2R\partial I_{-n,k} = -\left( -\frac{1}{\pi} \int_{\Gamma_{n}} (-\lambda)^{k-2} \left( \partial_{x} \partial F(-\lambda) + 2\lambda R\partial F(-\lambda) \right) \, d\lambda \right)
\]

\[
= (-1)^{k+1} \left( -\frac{1}{\pi} \int_{\Gamma_{n}} \lambda^{k-2} \left( \partial_{x} \partial F(\lambda) - 2\lambda R\partial F(\lambda) \right) \, d\lambda \right)
\]

\[
= (-1)^{k+1}(\partial_{x} \partial I_{n,k-1} - 2R\partial I_{n,k+1}).
\]
((iii) If \( T\varphi = \pm \varphi \), the periodic spectrum of \( \varphi \) is symmetric, and one has

\[
\partial_x \partial I_{n,k-1} - 2R \partial I_{n,k} - 2i \int_0^x (R\varphi \cdot \partial I_{n,k-1})
= - \left( \frac{1}{\pi} \int_\mathbb{R} (\varphi x \partial F(-\lambda) + 2\lambda R \partial F(-\lambda) - 2i \int_0^x (R\varphi \cdot \partial F(-\lambda)) \, d\lambda \right)
= (-1)^{k+1} \left( \partial_x \partial I_{n,k-1} - 2R \partial I_{n,k} - 2i \int_0^x (R\varphi \cdot \partial I_{n,k-1}) \right).
\]

**Proof of Theorem 4 (iii).** Suppose \( \varphi_u \in W \). Using that \( PR = -iJ \) we obtain from Lemma 10 (ii), applied in the case \( k = 1 \),

\[
-iJ(\partial I_n - \partial I_{-n})|_{\varphi_u} = \frac{1}{2} \partial_x P(\partial I_{n,0} + P\partial I_{n,0})|_{\varphi_u}.
\]

Recall that \( f^\sharp(u) := f(\varphi_u) \) for any \( C^1 \)-functional \( f \) on \( H^0_c \), and

\[
\nabla_u f^\sharp = (\partial f) \cdot (1,1)^\top|_{\varphi_u} = (\partial_1 f + \partial_2 f)|_{\varphi_u}.
\]

In particular,

\[
-iJ(\partial I_n - \partial I_{-n})|_{\varphi_u} = \frac{1}{2} \partial_x \nabla_u I^\sharp_{n,0}(1,1)^\top,
\]

and in general for any \( f \),

\[
\{ f, I_n - I_{-n} \}|_{\varphi_u} = -i \int_T (\partial f) \cdot J(\partial I_{n,1} - \partial I_{-n,1}) \, dx|_{\varphi_u}
= \frac{1}{2} \int_T (\nabla_u f^\sharp) \partial_x \nabla_u I^\sharp_{n,0} \, dx|_{u}
= \frac{1}{2} \left( f^\sharp, I^\sharp_{n,0} \partial_x \right)|_{u} = \{ f^\sharp, J_n \partial_x \}|_{u},
\]

where we used the identity \( I^\sharp_{n,0} = 2J_n \) from Lemma 7 in the last step. Consequently, for \( n, m \geq 1 \),

\[
\{ \theta^\sharp_m, J_n \partial_x \}|_{u} = \{ \theta_m, I_n - I_{-n} \}|_{\varphi_u} = \delta_{m,n}.
\]

The next result implies Theorem 2.

**Proposition 11**  
(i) For every \( \varphi \in H^{k-1}_c \) with \( k \geq 1 \), and any real \( \alpha \),

\[
\partial S_k(\varphi) = (-1)^{k-1} P \partial S_k(P\varphi) = (-1)^{k-1} T \partial S_k(T\varphi) = R_\alpha \partial S_k(R_\alpha \varphi).
\]

(ii) If \( \varphi \in H^{2m-1}_c \), \( m \geq 1 \), with \( P\varphi = R_\alpha \varphi \) for some real \( \alpha \), then

\[
-iJ \partial S_{2m}(\varphi) = \partial_\alpha \partial S_{2m-1}(\varphi).
\]

(iii) If \( \varphi \in H^{2m-1}_c \), \( m \geq 1 \), with \( T\varphi = \pm \varphi \), then

\[
R(\partial S_{2m}) = \partial_\alpha (\partial S_{2m-1}) - 2i \int_0^x (R\varphi \cdot \partial S_{2m-1}).
\]

**Remark.** One verifies by direct computation that generically item (ii) and (iii) do not hold when \( 2m \) is replaced by \( 2m + 1 \) if \( m \geq 2 \). \( \neg \circ \)
Proof. (i) On $W \cap \mathcal{H}_c^{k-1}$ the sum $\sum_{n \in \mathbb{Z}} I_{n,k}$ converges locally uniformly to an analytic function – see [4, Section 13] – and satisfies for $k \geq 1$

$$\sum_{n \in \mathbb{Z}} I_{n,k} = \frac{1}{2^{k-1}} S_k.$$ Since $\partial I_{n,k}(\varphi) = (-1)^{k-1} P \partial I_{-n,k}(P \varphi) = (-1)^{k-1} T \partial I_{-n,k}(T \varphi) = R_\alpha \partial I_{n,k}(R_\alpha \varphi)$ by Lemma 10 (i), the first identity of Theorem 2, $\partial S_k(\varphi) = (-1)^{k-1} P \partial S_k(P \varphi) = (-1)^{k-1} T \partial S_k(T \varphi) = R_\alpha \partial S_k(R_\alpha \varphi)$, follows for $\varphi \in \mathcal{H}^{k-1}_c$. Since the Hamiltonians are analytic on $\mathcal{H}^{k-1}_c$, the identity extends to $\mathcal{H}^{k-1}_c$ by Lemma 14.

(ii) Suppose $\varphi \in \mathcal{H}^{2k-1}_c$ with $P \varphi = R_\alpha \varphi$ for some $\alpha \in \mathbb{R}$. Summing identity (ii) of Lemma 10 over $n \in \mathbb{Z}$ yields

$$\frac{1}{2^{2m-1}} 2 R(\partial S_{2m}) = \sum_{n \in \mathbb{Z}} 2 R(\partial I_{n,2m}) = \sum_{n \in \mathbb{Z}} \partial_x (\partial I_{n,2m-1}) = \frac{1}{2^{2m-2}} \partial_x (\partial S_{2m-1}).$$

Since both sides are analytic on $\mathcal{H}^{2k-1}_c$, the identities extend by Lemma 14.

(iii) Similarly as for the previous item one obtains provided $T \varphi = \pm \varphi$,

$$\frac{1}{2^{2m-1}} 2 R(\partial S_{2m}) = \sum_{n \in \mathbb{Z}} 2 R(\partial I_{n,2m}) = \sum_{n \in \mathbb{Z}} \partial_x (\partial I_{n,2m-1}) - 2i \int_0^\varphi (R \varphi \cdot \partial I_{n,2m-1})
= \frac{1}{2^{2m-2}} \partial_x (\partial S_{2m-1}) - 2i \int_0^\varphi (R \varphi \cdot \partial S_{2m-1}).$$

\textbf{Proof of Theorem 1.} At any point $\varphi_0 \in \mathcal{H}_c^{2m-1}$ we have by Theorem 2 (ii)

$$X_{S_{2m}} = -iJ(\partial S_{2m}) = iJRR(\partial S_{2m}) = \partial_x P(\partial S_{2m-1}), \quad m \geq 1,$$

where we used that $iJR = P$. Furthermore, $(\partial S_{2m-1}) = P(\partial S_{2m-1})$ by Theorem 2 (i), hence

$$X_{S_{2m}}^\varphi = \frac{1}{2}(\partial_x (\partial S_{2m-1})^2 + \partial_x (\partial S_{2m-1})^2)(1,1)^T = \frac{1}{2}(Y^\varphi_{S_{2m-1}}, Y^\varphi_{S_{2m-1}})^T.$$ Since $\frac{1}{2} S_{2m-1} = Y_{Km}$ by Proposition 6, it now follows that

$$X_{S_{2m}}^\varphi = (Y_{Km}, Y_{Km}), \quad m \geq 1.$$ \textbf{Proof of Theorem 3.} Suppose $J_n \neq 0$ and let $\phi_t$ be a local flow for the vector field $Y_{Km}$, then

$$\eta_{n,m} = -\{\theta_n, K_m\} \eta_x = \lim_{t \to 0} \frac{d}{dt} \theta_n \circ \phi_t.$$ Since $\theta_n + \theta_n^\varphi$ is a function of the mKdV actions only, we have

$$-\frac{d}{dt} \theta_n \circ \phi_t = \lim_{t \to 0} \frac{d}{dt} \theta_n^\varphi \circ \phi_t.$$ As $X_{S_{2m}}^\varphi = Y_{Km}$ by Theorem 1, $\phi_t$ is also a local flow for the vector field $X_{S_{2m}}^\varphi$ and hence

$$\lim_{t \to 0} \frac{d}{dt} \theta_n^\varphi \circ \phi_t = \{\theta_n, S_{2m}\} = \omega_{n,2m}.$$
A Symmetries of the ZS fundamental solution

Let $M(x, \lambda, \varphi) = (m_1, m_2, m_3, m_4)$ denote the fundamental solution of the Zakharov-Shabat operator

$L(\varphi) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \frac{d^2}{dx^2} + \begin{pmatrix} \varphi & \psi \\ 0 & 0 \end{pmatrix}.$

The following symmetries under the transformations $P$, $R_\alpha$, and $T$, introduced in Section 4 have been noted in [3].

Lemma 13 For any $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$, and $\varphi \in \mathcal{H}_r$,

$$M(x, -\lambda, P\varphi) = JM(x, \lambda, \varphi)J^{-1} = \begin{pmatrix} m_4(x, \lambda, \varphi) & -m_3(x, \lambda, \varphi) \\ -m_2(x, \lambda, \varphi) & m_1(x, \lambda, \varphi) \end{pmatrix}.$$

$$M(x, \lambda, R_\alpha \varphi) = R_{\alpha/2}M(x, \lambda, \varphi)R_{\alpha/2}^{-1} = \begin{pmatrix} m_1(x, \lambda, \varphi) e^{i\alpha} & m_2(x, \lambda, \varphi) e^{-i\alpha} \\ e^{-i\alpha}m_3(x, \lambda, \varphi) & m_4(x, \lambda, \varphi) \end{pmatrix}.$$

$$M(x, -\lambda, T\varphi) = PJM(1 - x, \lambda, \varphi)\tilde{M}(\lambda, \varphi)^{-1}J^{-1}P.$$

In particular,

$$\tilde{M}(-\lambda, T\varphi) = \begin{pmatrix} m_4(\lambda, \varphi) & m_2(\lambda, \varphi) \\ m_3(\lambda, \varphi) & m_1(\lambda, \varphi) \end{pmatrix}.$$

B Analyticity

Lemma 14 Let $X_r$ be an $\mathbb{R}$-Banach space and denote by $X$ its complexification. Assume that $U \subset X$ is an open connected neighborhood of $U_r = U \cap X_r$ and that $f: U \rightarrow \mathbb{C}$ is an analytic map. If $f|_{U_r} = 0$, then $f \equiv 0$. \(\blacksquare\)

Proof. Near any $u \in U_r$ the map $f$ is represented by its Taylor series,

$$f(u + h) = \sum_{n \geq 0} \frac{1}{n!} d^n_u f(h, \ldots, h),$$

where the series converges absolutely and uniformly (cf. e.g. [1, Theorem A.3]). Since $f|_{U_r} = 0$, it follows that for any $h \in X_r$ and any $n \geq 0$, $d^n_u f(h, \ldots, h) = 0$. As $f$ is analytic, $d^n_u f$ is symmetric and $\mathbb{C}$-multilinear, hence it follows from the polarization identity that $d^n_u f(h, \ldots, h) = 0$ holds also for any $h$ in the complexification $X$ of $X_r$. This implies $f \equiv 0$ in a neighborhood $V_u$ of $u$ with $V_u \subset U$. Since $U$ is connected it follows that $f \equiv 0$ on all of $U$ by the identity theorem. \(\blacksquare\)

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