SHA vs. SCHA for Modelling Secular Variation in a Small Region Such as Italy

A. De Santis¹, C. Falcone¹, and J. M. Torta²

¹Istituto Nazionale di Geofisica, Via di Vigna Murata 605, 00143, Rome, Italy
²Observatori de l'Ebre, C.S.I.C., 43520 Roquetes (Tarragona), Spain

(Received January 5, 1996; Revised April 30, 1996; Accepted May 2, 1996)

The possibility of obtaining a reasonable analytical model of the geomagnetic field secular variation (SV) in a space-time window limited to the measurements taken in Italy from 1965 onwards is discussed. The main purpose of the work is to provide a mathematical representation better for the small Italian region than the International Geomagnetic Reference Field (IGRF) which is formulated on a global scale. With this aim regional models constructed in polynomial form already exist, but they do not account for the physical constraints imposed by Laplace's equation. Spherical cap harmonic analysis (SCHA) is shown not to be suitable for SV modelling over such a small region and the final adopted model is essentially a conventional spherical expansion. It is demonstrated that it is possible to find a solution to our inversion problem that fits the data by using a limited number of harmonics, selected on the basis of significance using a regression procedure.

1. Introduction

It is well known that if a sufficiently long series of geomagnetic observatory annual mean values is examined, one notices that the values vary in a slow fashion. This is the so-called secular variation (SV), which affects all the geomagnetic elements and takes this name because it is only appreciable over a long interval of time. Once all variations of external origin (including the 11-year solar cycle period) are eliminated, SV does not manifest a clear periodicity or, better, it is intrinsically aperiodic, or else, its main period seems to be longer than the last 200 years in which the geomagnetic elements have been more or less trustworthily measured. Considerations of this kind are frequently found when dealing with the unpredictability of SV (e.g. Malin, 1985).

The global International Geomagnetic Reference Field (IGRF) model of the field (and its SV) is usually given as a definitive model (DGRF) up to approximately 5–10 years before the present epoch (depending on whether we are close to or far from the last revision) and thereafter is given as provisional (e.g. IAGA, 1992). In Italy, the SV of the IGRF in the period considered in this work (1965–1990) follows only approximately (and non-continuously) the one observed at L’Aquila Observatory. This is because it is really a global model (so that it may display a significant bias over a limited region) and is considered constant between two successive revisions (thus it is a discontinuous stepped function that changes every 5 years). The main field at a given epoch is thus computed from the linear interpolation between the sets of coefficients with epochs bracketing the desired date. The lack of fit is not only found at a particular observatory. If we compare the IGRF with the bulk of available measurements of SV in Italy (which we will describe in the next section), we find that there is a significant mean difference (especially in $X$ and $Z$, Fig. 1). Moreover the IGRF is a little impractical because, being a series of snapshot main-field models at five year intervals, to calculate the SV in those 25 years requires 720 (120 x 6) coefficients. Despite all this, the IGRF is still probably the best tool to describe the field and its SV all around the world. There remains, however, the need to produce a regional model able to represent its own measurements as well as possible and eventually better than the global models.
The present paper proposes a SV model for the Italian region for the period 1965–1990. After an introduction on some previous polynomial models for Italy, other techniques that preserve the Laplace’s equation are described, with particular attention to the spherical cap harmonic analysis (SCHA). Finally, some comparative tests between SCHA and a form of spherical harmonic analysis (SHA) support the model based on SHA.

2. Previous SV Regional Models for Italy

2.1 INGRF

The INGRF (Geomagnetic Reference Field of the Istituto Nazionale di Geofisica, I.N.G.) is a field model taken as a second degree (latitude-longitude) polynomial, that was born in the 70’s under the “Progetto Finalizzato Geodinamica” with the I.N.G. as the leading research institution. It is determined
after each magnetic survey (approximately every five years) from the measurements on an array of around a hundred repeat stations (1st order network). In addition, a 2nd order network was established with a survey of 2252 stations (with a mean density of one station per 138 km$^2$) between 1977 and 1981, the measurements being reduced to 1981.0 using the records of L'Aquila Observatory and the closest Observatory to the measuring point chosen from five other temporary observatories appropriately running at that epoch over the rest of Italy. The magnetic charts, based on this 2nd order network, are updated from time to time by using the SV estimated from the 1st order network.

INGRF and its SV show the typical pitfalls of polynomial models, i.e. they do not guarantee both the geometric consistency between components and total field and the physical condition of non-existence of atmospheric currents. Some tests of physical and geometrical consistency of the field components and of their gradients have confirmed these drawbacks, especially in the region of Sicily (Meloni et al., 1994).

2.2 ITGRF

With the aim of eliminating or, at least, reducing the typical problems of the IGRF when used over Italy (as seen in Fig. 1) the ITGRF (Italian Geomagnetic Reference Field; e.g. Molina and De Santis, 1987) was introduced which was valid for 1965–1984 (although it has been extended onwards with the same philosophy, with the new revisions of the IGRF). It is a model that tries to follow the temporal variation of the field as measured at L'Aquila Observatory but uses the IGRF for the spatial variations. The idea is to consider the spatial gradients as computed from the IGRF but constraining the absolute field level to the annual mean values of L'Aquila. The magnetic elements given by the IGRF every 5 years are fitted with a second degree polynomial in latitude and longitude, with the constant term varying such that the model always gives the same difference with the field observed at L'Aquila in 1979 (which is taken as the crustal anomaly and is thus constant in time; this year was chosen because there is a good model available from Magsat satellite data).

Both these models remove the previously mentioned bias shown by the IGRF, but still have the limitations of regional polynomial models such as being only 2D (there is no dependence in altitude); INGRF has constant SV in 5-year intervals (the same trouble as the IGRF); ITGRF follows L'Aquila but there are some inconsistencies for points far from the Observatory because, in some areas, the SV spatial gradients of the IGRF may significantly depart from the actual values; and, finally, neither model satisfies the physics of the magnetic field in current-free regions (Laplace's equation).

3. Regional Analyses Using the Constraints Imposed by Laplace’s Equation

As explained above, the polynomial models do not impose any physical constraint on the geomagnetic field. It is possible, in a simple way, to take account of the absence of vertical electrical currents by making the vertical component of the curl of $\mathbf{B}$ zero (e.g. Chapman and Bartels, 1940; Stratton, 1941):

$$(\nabla \times \mathbf{B})_r = 0 \Rightarrow \frac{\partial X}{\partial \lambda} + \frac{\partial (Y \sin \theta)}{\partial \theta} = 0,$$

where $\lambda$ and $\theta$ are the longitude and the colatitude, respectively and $X$ and $Y$ are the northward and the eastward components, respectively. It has to be noted that, besides imposing a reasonable constraint, the use of this condition allows the number of coefficients to be reduced (e.g. Tsubokawa, 1952, Eqs. (24)–(26)).

Another method consists in synthesizing field values at given epochs on a latitude/longitude grid from the IGRF for the area outside the region and using the actual observations inside the region and then performing a conventional space-time global spherical harmonic analysis. The validity of the model, however, is considered restricted to the region under study.

Rectangular harmonic analysis (e.g. Alldredge, 1981) considers Laplace’s equation in cartesian coordinates:
\[ \nabla^2 V(x,y,z) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \] (2)

A solution of (2) has the form:

\[ V = Ax + By + Cz + \left[ a_{m_1}^{m_2} \cos m_1 x \cos m_2 y + b_{m_1}^{m_2} \sin m_1 x \cos m_2 y + c_{m_1}^{m_2} \sin m_1 x \sin m_2 y \right] e^{inz}; \] (3)

with the three coupling constants related by the expression: \( n^2 = m_1^2 + m_2^2 \). \( A, B, C, a_{m_1}^{m_2}, b_{m_1}^{m_2}, c_{m_1}^{m_2} \) and \( d_{m_1}^{m_2} \) represent the set of coefficients which characterise the model. The integer index \( n \) which appears in the \( z \) dependence is positive when dealing with internal sources (\( z \) is positive downwards, therefore the exponential goes to zero for increasing distances from the Earth’s surface), otherwise \( n \) is negative. The model is appropriate for small areas (say, up to a few millions of km\(^2\)) but cannot be accurately extrapolated in altitude: the radial dependence should be of the type \( 1/r^n \), \( n = 2, 3, \ldots \), while this model tends to zero more rapidly (exponentially). For a good review of these and other regional alternatives see Haines (1990).

The description of the field by means of an expansion of its magnetic potential in terms of Legendre polynomials and trigonometric functions is possible in a global analysis of the phenomenon using data from all around the world. This in fact implies an analysis over the complete range of the two angular variables, the longitude \([0, 2\pi]\) and the colatitude \([0, \pi]\). When observations are available over a limited region of the Earth’s surface or a regional analysis only is required, the approach used in a global analysis is now unsuitable. A solution to this problem was proposed by Haines (1985a). His method stems from the choice of the region: if the analysis is limited to the measurements taken over a spherical cap, the longitude \( \lambda \) still varies between 0 and \( 2\pi \), but the colatitude is now limited to the interval \([0, \theta_0]\), \( \theta_0 \) being the cap half-angle. The Legendre polynomials \( P_n^m(\cos \theta) \), which constitute a complete basis in the space \( L^2[-1,1] \), are not appropriate any more in the new space \( L^2[\cos \theta_0, 1] \).

Having to work with a set of data limited to a region, the idea of Haines (1985a) consists of choosing adequate basis functions for such a region. SCHA proposes the use of those solutions of Laplace’s equation which constitute an orthogonal basis within the sub-space defined by the spherical cap. In particular, the \( \theta \)-functions form two sets of orthogonal functions \( P_n^m \) within the interval \([0, \theta_0]\), chosen in order to satisfy either

\[ P_n^m(\cos \theta_0) = 0, \] (4)

or

\[ \frac{dP_n^m(\cos \theta_0)}{d\theta} = 0. \] (5)

### 3.1 The fractional spherical harmonics

Equations (4) and (5) must be solved for \( n \), thus defining the particular solutions of Laplace’s equation. In general, the boundary conditions are now completely different from those in the global case, which required that \( n \) was integer. The SCHA solutions are termed as “fractional” (Schmitz, 1989), because usually the corresponding values of degree \( n \) are not integers. In particular, the so-called \( P_n^m \) possesses a new index \( k \): seeking the (non-integer) value of \( n \) that satisfy the new boundary conditions,
alternate solutions of (4) and (5) are found which, for a particular \( m \), are ordered according to \( k \). The expression for the potential can be then written in the conventional form, substituting \( n \) for \( n_k \):

\[
V(r, \theta, \lambda) = \sum_{k=0}^{K} \sum_{m=0}^{k} (a / r)^{n_k(m)+1} P_{n_k}^m(m) \left\{ g_k \cos m\lambda + h_k \sin m\lambda \right\}.
\]

Notice that the \( n_k \) depend on \( m \) and it is therefore necessary that they appear inside both summations. The angles \( \theta \) and \( \lambda \) refer to colatitude and longitude relative to the pole of the spherical cap.

Having assumed that the arrangement of this analysis is exclusively mathematical, one wonders what these \( P_{n_k}^m \) functions really are and whether they represent something physical. On one hand, it is to be noted that the non-integer degree of these functions forces them to go to infinity at \( \theta = 180^\circ \), but this, in practice, is not really important because the \( P_{n_k}^m \) are only considered within the interval \([0,0^\circ]\) and completely lose their significance outside the cap. In the next section it will be shown how this fact becomes significant when one tries to extrapolate a model created by a SCHA beyond the region covered by the data. Notice, on the other hand, that the summation includes the root \( n = 0 \), since the monopole (expressed by \( P_0^0 \)) is a constant function and, therefore, implicitly satisfies condition (5). This does not mean, however, that SCHA postulates the presence of monopoles, on the contrary the \( P_0^0 \) term is just proportional to the average vertical field over the region, which is obviously zero in SHA but not, in general, in SCHA. This confirms that in this mathematical “game” the meaning of a spherical harmonic degree is completely different from the conventional one, which assigns it a precise role in the physics of geomagnetism. Remember, in fact, that the term \( V_{nm} \sim 1/r^{n+1} \) is related to the potential due to a \((n+1)\)pole and in the case of \( n \) being non-integer, one wonders what is meant by a “non-integer multipole”.

4. The Secular Variation in Italy from 1965 to 1990

The aim of an analysis of Italian observatory and repeat station data for the period 1965 and onwards is to provide a practical and easy-to-use SV model for reducing data from magnetic surveys to a base epoch and to update Italian magnetic charts.

4.1 Data processing and model construction criteria

The three-component annual mean values from 5 geomagnetic observatories (two Italian and 3 from neighbouring countries; stars in Figs. 2 and 5) and the data from the magnetic surveys developed in Italy from 1965 to 1992 (crosses in Figs. 2 and 5) on the 2nd order network make up the data set used in the present analysis. For the survey data we used the values provided by the I.N.G. and thus already reduced to fixed epochs, roughly centred at the middle of each magnetic survey period. For instance, the measurements of the last survey 1989–1992 were reduced to 1990.0 (Meloni et al., 1994); the same procedure was applied in the former surveys lasting from 3 to 7 years. This results in a mean interval of approximately 5 years between two successive data from the repeat stations.

The existence of systematic errors and the approximation of computing the first temporal derivative by finite differentiation present some problems. The simplest way to define a SV datum is to consider the difference between two successive field data divided by the corresponding interval of time between the two observations, assigning the value to the central epoch of the interval. However there exists the risk that an error on a field datum (for instance \( B(t_2) \)) will be propagated to the two successive SV data \((B(t_1,2)\) and \(B(t_2,3))\). This means that the SV data are correlated and this should be taken into account in the construction of the data weight matrix. It is rather tedious to solve this problem analytically, but it can be overcome by resorting to data smoothing. Each component of each station is first fitted by a time function and then its derivative computed analytically. This allows equal weights to be given to all data, smoothing the errors and removing the correlation between successive SV data. These errors could, moreover, be
uniformly re-distributed along the whole time period. Another way to avoid numerical differentiation and to overcome the difficulty of the estimation of SV when the interval between consecutive data values is many years consists of modelling main-field differences (Haines, 1993).

In brief, the repeat station data have been interpolated in time with a fourth degree polynomial and we have computed the time derivative value close to the intermediate point between two successive data. A total of 416 (repeat stations + observatories) vectorial SV data were finally available. We assumed the same weight for all them. This is quite reasonable because in any case the contribution of the information given by the observatories is weighted the most, considering the amount of available data (one per year during the 25-year span plus those of 1991 and 1992 added to have some control on the trend of the SV after 1990) in contrast with the 5 or 6 data from each repeat station.

We then expressed the SV in spherical harmonics in space and expanded the corresponding Gauss coefficients in power series in time up to the third degree. This is coherent with the choice of a fourth degree polynomial for the interpolation of B at individual stations:

\[
\begin{align*}
\dot{X} &= \sum_{k=1}^{K} \sum_{m=0}^{k} (a/r)^{n_k(m)+2} \cos \theta \left\{ \frac{dP_{m}(\cos \theta)}{d\theta} \right\} \left\{ g^{m}_{k}(t) \cos m\lambda + h^{m}_{k}(t) \sin m\lambda \right\}, \\
\dot{Y} &= \sum_{k=1}^{K} \sum_{m=1}^{k} (a/r)^{n_k(m)+2} \sin \theta \left\{ \frac{dP_{m}(\cos \theta)}{d\theta} \right\} \left\{ g^{m}_{k}(t) \sin m\lambda - h^{m}_{k}(t) \cos m\lambda \right\}, \\
\dot{Z} &= -\sum_{k=0}^{K} \sum_{m=0}^{n_k(m)+1} (a/r)^{n_k(m)+2} \left\{ g^{m}_{k}(t) \cos m\lambda + h^{m}_{k}(t) \sin m\lambda \right\};
\end{align*}
\]

with,

\[
\begin{align*}
\left\{ g^{m}_{k}(t) \right\} &= \sum_{p=0}^{3} \left\{ g_{k,p}^{m} \right\} \left[ \frac{t - t_0}{f_T} \right]^p, \\
\left\{ h^{m}_{k}(t) \right\} &= \sum_{p=0}^{3} \left\{ h_{k,p}^{m} \right\} \left[ \frac{t - t_0}{f_T} \right]^p;
\end{align*}
\]

and with the convention that in case of SHA \( n_k(m) \) becomes \( n \) and \( k \) is replaced by \( n \). In SHA, moreover, the \( n = 0 \) term is usually taken \textit{a priori} to be zero, and is not considered any further. \( t_0 \) is conveniently chosen within the time interval 1965–1990. We include a factor \( f_T \) to normalize the time interval to approximately \([-1,1]\).

4.2 Remarks on the model selection

The degree of an (integer or fractional) spherical harmonic is related to the spatial wavelength that it can represent. The minimum degree of the harmonics corresponding to a spherical cap increases by diminishing \( \theta_0 \) (the degree zero term does not enter in this argument because it simply corresponds to a constant level). Thus for instance, for a spherical cap of 7° (which roughly corresponds to the area of concern) the smallest value of \( n_k \) is 14.6 which corresponds to a wavelength of about 2700 km (comparable with the cap arc). This means that the fractional spherical harmonics introduced for a small cap are actually fairly inappropriate to represent a phenomenon originating in the Earth’s core such as the SV. In fact, this does not mean that SCHA fails but merely that, to have a reasonable SV model, it would be necessary to consider all the harmonics, and so an infinite number of coefficients.

Nevertheless, after some trials with different values of maximum value \( K \), we found no significant differences in terms of standard error for models with \( K > 4 \). For example, the SCH model with \( K = 5 \)
SHA vs. SCHA for Modelling Secular Variation in a Small Region

provided 58 significant space-time coefficients with standard error of $\sigma = 4.3$ nT/year. The selection of the most significant coefficients was done with a stepwise regression procedure based on Efroymson's algorithm (Draper and Smith, 1966) which has been widely used when dealing with this kind of analysis (e.g. Haines, 1985a; Haines and Torta, 1994). The rather large value of $\sigma$ is not surprising: the experimental data, as discussed, are affected by systematic error and by some external contribution due to the quasi 11-year period solar activity. Nor are they well filtered either by the operation of annual averaging or by the temporal approximation by a polynomial fitting within the 25-year span considered. The results are certainly discouraging, as can be appreciated in Fig. 2 (top), which shows the contours for the epoch 1985.0. The contours are characterized by the small wavelengths necessary for an analysis over a 7° cap, and their sinuosity is due to the lack of information between the data points, thus enabling the small wavelengths to oscillate in the regions not filled by data.

Torta et al. (1992) have shown that by simply enlarging the cap (by using harmonics associated with larger wavelengths) it is possible to give a good field representation. In these conditions, however, there is no longer a relationship between the studied region and the spherical cap: the necessity of imposing specific boundary conditions on the Legendre functions on the remote border of a cap loses its meaning.

The extreme case is the analysis with conventional spherical harmonics, or SHA. In case of using the same number of harmonics as the IGRF SV model, up to maximum degree 8 ($80 \times 4$ space-time coefficients), the problem would obviously be improperly posed because the functions used are orthogonal over an area much larger than that being investigated (Haines, 1990). Such a model would have nonsensical coefficients, with orders of magnitude much greater than the typical values of the SV phenomenon, which are normally under $10^2$ nT/year (the Schmidt semi-normalization enables us to work with coefficients of the same order of magnitude as that of the represented phenomenon). If, on the contrary, we select the most significant harmonics (such as to provide a standard deviation comparable to that supplied by the SCH

---

Fig. 2. Contours of the three cartesian components ($\hat{X}$, $\hat{Y}$ and $\hat{Z}$, from left to right) of SV (in nT/year) at 1985.0 from: (top) a spatial-temporal SCHA on a 7° spherical cap (boundary shown); (bottom) as top but from a SHA. Locations of the observatories (stars) and repeat stations (crosses) used in the analysis are also shown.
model) the results of Fig. 2 (bottom) are obtained. The number of space-time coefficients is 16 and the standard error is again \( \sigma = 4.3 \) nT/year.

The misfits of the SCHA and the "selected" SHA models with respect to the data are practically the same, so comparison must be based on other criteria. The comparison of the number of coefficients is not decisive for the choice of a model, but in this context, where we want an easy-to-use model, the less complex model with fewer coefficients is better. As indicated by Haines (1990), the minimum wavelength of the regional SH model is even larger than that of global models; but in our opinion this is not a problem as SV is characterized by large wavelengths.

4.3 Subsequent tests and final model

From the analysis of the contour maps it can be concluded that the set of fractional spherical harmonics associated with a cap of half-angle of 7° is not able to represent a phenomenon, like SV, with wavelengths typically larger than those shown in Fig. 2. Since the numerical comparison with the original data does not justify these conclusions, we attempted to confirm them with a study of the SV synthesized from the IGRF coefficients for 1987.5 (just dealing with a spatial analysis, since our emphasis is on the spatial aspect of SV modelling). We constructed two different sets of synthetic data, either synthesizing the data at the locations of the Italian stations or on a regular grid within the cap. The results were again similar to those obtained from SCHA applied to real data; they improved a little when the number of data were increased, but were always worse than those obtained with "selected" SHA (see Figs. 3 and 4).

From the trials with synthetic data (comparison with a well known field) and from the results obtained with real data, it is deduced that the primary goal of producing a model to represent the SV field in our space-time window (and where the geometrical and rotational consistencies are respected) can be achieved with a compact model using the first few integer harmonics (SHA) for the spatial representation and a third degree polynomial for the temporal representation. SCHA unfortunately has the above mentioned limits when using SV data within a small cap. By the way, the same problem would also have been met using other truly regional techniques (e.g. RHA) because the corresponding first (longer) harmonic would be of the size of the region of study. The definitive model proposed here (in Fig. 5 some "snapshots" in successive years are shown) is an SH model comprising only 16 spatial-temporal coefficients (Table 1), making it possible to be used with a simple calculator.

It is important to bear in mind that we do not want to claim that SCHA is either an exaggeratedly complex or wrong technique: the above mentioned conclusions are limited to the particular application with which we have dealt. It has to be noted that this technique is very useful for studying crustal magnetic anomalies (with many examples in the recent literature), variation fields of external origin (Haines and Torta, 1994), or the SV itself when the regions are rather vast (for example, it has successfully been applied over Canada by Haines, 1985b).

Figure 6 shows the original data from a repeat station (Contrada Misteci, 37.44°N, 14.05°E) with the corresponding model values computed for the interval 1965 to 1990. Finally, we estimated the goodness of the final model by computing its misfit with respect to the unsmoothed SV data and then comparing it with the one computed for the IGRF. The final model has an r.m.s. difference of 6.3 nT/year, while the IGRF has an r.m.s. difference of 13.5 nT/year.

A question still remains as to whether it is possible to use the information restricted to a region for gaining some conclusions on the morphology of the originating sources. Vertical extrapolation (up or down) from this regional analysis has been studied and in particular the upward continuation at an altitude of 400 km using the SH model generated from the synthetic IGRF data for 1987.5 compares perfectly with that of the IGRF, while for the downward continuation the model maintains the correspondence and only begins to fail in the proximity of the core boundary. This means that this kind of regional SHA works reasonably well for fields like SV of internal origin. We do not however recommend the use of such improperly posed inversion to separate fields into their external and internal parts, for example with variation fields of either ionospheric or magnetospheric origin and their corresponding internally induced fields.
Fig. 3. $\vec{x}$, $\vec{y}$ and $\vec{z}$ (from top to bottom), in nT/year, from the IGRF (dashed lines) and the SCH model determined from synthetic IGRF SV data (solid lines), along the 12°E meridian (horizontal axes refer to degrees of latitude). At the top the northern hemisphere, at the bottom the region containing the Italian region: the oscillations are due to the use of very small wavelengths.
Fig. 4. The same as Fig. 3, but solid lines correspond to an SH model (selecting the most significant coefficients). Notice the change of scale at the top, and the almost perfect agreement at the bottom.
Fig. 5. Behaviour of the $\dot{X}$, $\dot{Y}$ and $\dot{Z}$ (from left to right) SV components (in nT/year) from 1965 to 1990, from the model in spherical harmonics (16 spatial-temporal coefficients, Table 1). Locations of the observatories (stars) and repeat stations (crosses) used in the analysis are shown.

Table 1. Significant coefficients of the spherical harmonic expansion, with $t_0 = 1985$ and $f_T = 10$.

| $n$ | $m$ | $g_{0,0}^m$ | $h_{0,0}^m$ | $g_{1,1}^m$ | $h_{1,1}^m$ | $g_{2,2}^m$ | $h_{2,2}^m$ | $g_{0,3}^m$ | $h_{0,3}^m$ | $g_{1,3}^m$ | $h_{1,3}^m$ |
|-----|-----|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 1   | 0   | 24.80       | -30.93      | -29.52      | -3.23       |             |             |             |             |             |             |
| 2   | 0   | -34.23      | 27.73       |             | 0.00        | -6.98       |             |             |             |             |             |
| 2   | 1   | 0.00        | -67.07      |             |             |             |             |             |             |             |             |
| 3   | 1   | 25.81       | 28.47       |             |             |             |             |             |             |             |             |
| 4   | 2   | -24.86      | 0.00        |             |             |             |             |             |             | -2.28       | 3.95        |
On the other hand, a problem still remains as to whether it is possible to impose some kind of constraint which allows the limits due to the lack of uniqueness of the solutions to be overcome. Sometimes this is solved by "filling-in" the remainder of the Earth with data synthesized from the IGRF, which can certainly limit the number of possible solutions and, above all, enable a conventional analysis with all the coefficients commonly used in global analyses ($n_{\text{max}} = 8$). This forces the model to follow the IGRF outside the region. Another possibility is to impose some kind of regularization. With this aim a further step can be to add some extra constraints to the model coefficients, for instance by adding information on the spatial gradients (e.g. De Santis et al., 1996).

5. Conclusions

A model of the geomagnetic secular variation has been determined valid in the Italian area from 1965 to 1990. The data have been processed in such a way so as to reduce part of the systematic error associated
with conventional processing (systematic since it is due to the effect of well-defined geomagnetic phenomena which depend on both the time of measurement and the geographical region containing the station).

Some models were already available but none of them (excluding the global IGRF model) satisfied the physics of the problem. The use of (conventional or fractional) spherical harmonics has produced models perfectly valid in terms of minimum deviation with respect to the original data and of coherence with Laplace’s equation. Nevertheless, the SCH model is not appropriate in terms of spectral content (too many short wavelength features of SV) and is not stable in the inversion. On the other hand, the selected SH model is more reasonable, being in agreement with the gross features of a global SV model such as that of the IGRF. Processing and analyzing the data in search of the best model has required the use of a powerful computer but, on the contrary, once determined, the SV model that we recommend consists of a SHA with a selection of the most significant harmonics which, considering its final simplicity and compactness, may be implemented in any computing medium.

We thank two anonymous referees for their comments and suggestions. This is a contribution under the Project “Mathematical models of the geomagnetic field in Europe” funded by the Italian-Spanish Joint Commission.

REFERENCES

Alldredge, L. R., Rectangular harmonic analysis applied to the geomagnetic field, *J. Geophys. Res.*, 86, 3021–3026, 1981.
Chapman, S. and J. Bartels, *Geomagnetism*, 542 pp., Clarendon Press, Oxford, 1940.
De Santis, A., C. Falcone, and J. M. Torta, Simple additional constraints on regional models of the geomagnetic secular variation field, *Phys. Earth Planet. Inter.*, 97, 15–21, 1996.
Draper, N. R. and H. Smith, *Applied Regression Analysis*, 709 pp., John Wiley, New York, 1966.
Haines, G. V., Spherical cap harmonic analysis, *J. Geophys. Res.*, 90, 2583–2591, 1985a.
Haines, G. V., Spherical cap harmonic analysis of geomagnetic secular variation over Canada 1960–1983, *J. Geophys. Res.*, 90, 12563–12574, 1985b.
Haines, G. V., Regional magnetic field modelling: a review, *J. Geomag. Geoelectr.*, 42, 1001–1018, 1990.
Haines, G. V., Modelling geomagnetic secular variation by main-field differences, *Geophys. J. Int.*, 114, 490–500, 1993.
Haines, G. V. and J. M. Torta, Determination of equivalent current sources from spherical cap harmonic models of geomagnetic field variation, *Geophys. J. Int.*, 118, 499–514, 1994.
IAGA Division V, Working Group 8, International Geomagnetic Reference Field, 1991 Revision, *Geophysics*, 57, 956–959, 1992.
Malin, S. R. C., On the unpredictability of geomagnetic secular variation, *Phys. Earth Planet. Inter.*, 39, 293–296, 1985.
Meloni, A., O. Battelli, A. De Santis, and G. Dominici, The 1990.0 magnetic repeat station survey and normal reference fields for Italy, *Annali di Geofisica*, XXXVII, 949–967, 1994.
Molina, F. and A. De Santis, Consideration and proposal for the best utilization of IGRF over areas including a geomagnetic observatory, *Phys. Earth Planet. Inter.*, 48, 379–385, 1987.
Schmitz, D., Spherical harmonic analysis, in *The Encyclopedia of Solid Earth Geophysics*, edited by D. E. James, pp. 1217–1221, Van Nostrand Reinhold Co., New York, 1989.
Stratton, J. A., *Electromagnetic Theory*, 615 pp., McGraw-Hill, New York, 1941.
Torta, J. M., A. Garcia, J. J. Curto, and A. De Santis, New representation of geomagnetic secular variation over restricted regions by means of Spherical Cap Harmonic Analysis: application to the case of Spain, *Phys. Earth Planet. Inter.*, 74, 209–217, 1992.
Tsubokawa, I., Reduction of the results obtained by the magnetic survey of Japan (1948–51) to the epoch 1950.0 and reduction of the empirical formulae expressing the magnetic elements, *Bull. Geogr. Surv. Inst.*, 3, 1–29, 1952.