CLASSIFICATION OF EXTREMAL FUNCTIONS TO LOGARITHMIC HARDY-LITTLEWOOD-SOBOLEV INEQUALITY ON THE UPPER HALF SPACE

JINGBO DOU*
School of Mathematics and Information Science
Shaanxi Normal University
Xi’an, Shaanxi 710119, China
YE LI
Department of Mathematics
Central Michigan University
Mount Pleasant, MI 48859, USA

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Abstract. In this paper we mainly classify the extremal functions of logarithmic Hardy-Littlewood-Sobolev inequality on the upper half space \( \mathbb{R}^n_+ \), and also present some remarks on the extremal functions of logarithmic Hardy-Littlewood-Sobolev inequality on the whole space \( \mathbb{R}^n \). Our main techniques are Kelvin transformation and the method of moving spheres in integral forms.

1. Introduction. The logarithmic Hardy-Littlewood-Sobolev (HLS) inequality is an endpoint case of the classic HLS inequality. It plays an important role in analysis, statistical mechanics, conformal geometry and spectral theory.

The dual form of the logarithmic HLS inequality on \( \mathbb{S}^n \), also called Onofri inequality, was first derived on \( \mathbb{S}^2 \) by Onofri [14] in 1982. Later, the sharp logarithmic HLS inequality was established on the whole space \( \mathbb{R}^n \) by Carlen and Loss [3] and Beckner [1] in the early 1990s. Morpurgo [11] showed that the logarithmic HLS inequality on \( \mathbb{S}^n \) was the analytic expression of an extremal problem for the regularized zeta function of the Paneitz operators. Later, Branson, Fontana and Morpurgo [2] proved the sharp logarithmic HLS inequality and the corresponding Onofri inequality on the CR sphere (Heisenberg group). On Euclidean spaces and Riemannian manifolds, the best constants of inequalities are often the critical elements to show the existence of solutions to PDEs, to identify extremal geometries, to solve curvature problems and etc.. Hence, classification of extremal functions is crucial in the study of elliptic equations with critical exponents.

Recently, Dou and Zhu [6] established HLS inequality with boundary terms on the upper half space

\[
\int_{\mathbb{R}^n_+} \int_{\mathbb{R}^n_+} \frac{g(x)f(y)}{|x-y|^{n-\alpha}} \, dy \, dx \leq C_c(n, \alpha, p) \|f\|_{L^p(\partial \mathbb{R}^n_+)} \|g\|_{L^1(\mathbb{R}^n_+)},
\]

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* Corresponding author: Jingbo Dou.
for \( f \in L^p(\partial \mathbb{R}_+^n) \), \( g \in L^t(\mathbb{R}_+^n) \), where \( 1 < \alpha < n \), \( 1 < t \), \( p < \infty \) satisfy \( \frac{n-1}{p} + \frac{1}{t} + \frac{n-\alpha+1}{n} = 2 \), and \( C_n(n, \alpha, p) > 0 \) is the best constant. Inequality (1) is an extension of the classic HLS inequality. They discussed the extremal functions of inequality (1) and determined the best constant. Moreover, they also considered the limiting case of inequality (1). That is, as \( \alpha \to n^- \), the following logarithmic HLS inequality holds.

**Theorem 1.1.** (Corollary 5.3 in [6]) Assume that \( F \) and \( G \) are nonnegative \( LnL \) functions on \( \partial B_1 \) and \( B_1 \), respectively, with \( \int_{\partial B_1} F(\eta)dS_\eta = \int_{B_1} G(\xi)d\xi = 1 \). Then

\[
-2 \int_{B_1} \int_{\partial B_1} G(\xi) \ln |\xi - \eta| F(\eta)dS_\eta d\xi 
\leq \frac{1}{n} \int_{B_1} G(\xi) \ln(G(\xi))d\xi + \frac{1}{n-1} \int_{\partial B_1} F(\eta) \ln(F(\eta))dS_\eta + C_n, \tag{2}
\]

where \( C_n = \frac{\ln(n\omega_n)}{n-1} + \frac{1}{n} \int_{\partial B_1} e^{\ln(\xi)|}d\xi \) and \( I_n(\xi) = -2\omega_n^{-1} \int_{\partial B_1} \ln |\xi - \eta|dS_\eta \).

Equivalently, assume that \( f(x) \) and \( g(y) \) are nonnegative \( L\ln L \) functions on \( \partial \mathbb{R}_+^n \) and \( \mathbb{R}_+^n \), respectively, with \( \int_{\partial \mathbb{R}_+^n} f(x)dx = \int_{\mathbb{R}_+^n} g(y)dy = 1 \). Then

\[
-2 \int_{\mathbb{R}_+^n} \int_{\partial \mathbb{R}_+^n} g(y) \ln |x - y| f(x) dxdy 
\leq \frac{1}{n} \int_{\mathbb{R}_+^n} g(y) \ln(g(y))dy + \frac{1}{n-1} \int_{\partial \mathbb{R}_+^n} f(x) \ln(f(x))dx + C_n. \tag{3}
\]

Although Theorem 1.1 was stated in [6], the details of proof were not given there. We will provide the proof in appendix.

To find the extremal functions of (3) (or (2)), one can maximize the functional

\[
J(g, f) = 2 \int_{\mathbb{R}_+^n} \int_{\partial \mathbb{R}_+^n} g(y) \ln |x - y|^{-1} f(x) dxdy
- \left( \frac{1}{n} \int_{\mathbb{R}_+^n} g(y) \ln(g(y)) dy + \frac{1}{n - 1} \int_{\partial \mathbb{R}_+^n} f(x) \ln(f(x))dx \right),
\]

under the constraints \( \int_{\partial \mathbb{R}_+^n} f(x)dx = \int_{\mathbb{R}_+^n} g(y)dy = 1 \). Then we obtain the corresponding system of Euler-Lagrange equations

\[
\begin{cases}
2 \int_{\mathbb{R}_+^n} g(y) \ln |x - y|^{-1}dy - \frac{1}{n} (\ln(f(x)) + 1) = \lambda_1, \\
2 \int_{\partial \mathbb{R}_+^n} f(x) \ln |x - y|^{-1}dx - \frac{1}{n} (\ln(g(y)) + 1) = \lambda_2,
\end{cases}
\]

where \( \lambda_1 \), \( \lambda_2 \) are some constants. Let \( f(x) = e^{(n-1)u(x)-1} \) and \( g(y) = e^{nv(y)-1} \). Then we obtain the following system

\[
\begin{cases}
u(x) = \frac{2}{\tau} \int_{\mathbb{R}_+^n} e^{nv(y)} \ln |x - y|^{-1}dy + m_1, \\
u(y) = \frac{2}{\tau} \int_{\partial \mathbb{R}_+^n} e^{(n-1)u(x)} \ln |x - y|^{-1}dx + m_2.
\end{cases} \tag{4}
\]

where \( m_1 \) and \( m_2 \) are constants. The extremal functions are classified by the following result.

**Theorem 1.2.** Let \((u, v)\) be a pair of \( C^1 \) solutions to system (4). If \( u, v \) satisfy \( \int_{\partial \mathbb{R}_+^n} e^{(n-1)u(x)-1}dx = 1 \) and \( \int_{\mathbb{R}_+^n} e^{nv(y)-1}dy = 1 \), then, \( u, v \) must be of the following
forms on $\partial \mathbb{R}^n_+$:

$$u(\xi) = \ln \left( \frac{a_1}{|\xi - \xi_0|^2 + d^2} \right), \quad v(\xi, 0) = \ln \left( \frac{a_2}{|\xi - \xi_0|^2 + d^2} \right),$$

where $a_1, a_2 > 0, d > 0, \xi_0 \in \partial \mathbb{R}^n_+$.

According to Theorem 1.2, we get the extremal functions of inequality (3), i.e., for $x, x_0 \in \partial \mathbb{R}^n_+$,

$$f(x) = \left( \frac{a_1}{|x - x_0|^2 + d^2} \right)^{n-1}, \quad g(x, 0) = \left( \frac{a_2}{|x - x_0|^2 + d^2} \right)^n. \quad (5)$$

We note that the classic logarithmic HLS inequality studied by Beckner in [1] is of the form:

$$-2n \int_{\mathbb{R}^n} \frac{g(x)}{|x|} \ln |x - y| f(y) dy dx \leq \int_{\mathbb{R}^n} g(x) \ln(g(x)) dx + \int_{\mathbb{R}^n} f(y) \ln(f(y)) dy + K_n, \quad (6)$$

where $f(x)$ and $g(x)$ are nonnegative $L\ln L$ functions on $\mathbb{R}^n$, with $\int_{\mathbb{R}^n} f(x) dx = \int_{\mathbb{R}^n} g(x) dx = 1$ and $K_n$ is the best constant. Based on the rearrangement technique and stereographic projection, the corresponding extremal functions are classified (up to a conformal automorphism) by

$$f(x) = g(x) = \left( \frac{A}{|x|^2 + 1} \right)^n, \quad x \in \mathbb{R}^n. \quad (7)$$

Carlen and Loss [3] also classified the extremal functions of logarithmic HLS inequality (6) by competing symmetries technique.

Noticing the similarity between (3) and (6), we can also write the Euler-Lagrange equations of the extremal functions of inequality (6) as follows,

$$\begin{cases} 
  u(x) = \frac{2}{x} \int_{\mathbb{R}^n} e^{nu(x)} \ln |x - y|^{-1} dy + m_1, \\
  v(y) = \frac{2}{y} \int_{\mathbb{R}^n} e^{nv(y)} \ln |x - y|^{-1} dx + m_2.
\end{cases} \quad (8)$$

Similar to Theorem 1.2, we have the following result.

**Theorem 1.3.** Let $(u, v)$ be a pair of $C^1$ solutions to system (8). If $u, v$ satisfy $\int_{\mathbb{R}^n} e^{nu(x)} dx = \int_{\mathbb{R}^n} e^{nv(y)} dy = 1$, then, $u, v$ must be of the following forms on $\mathbb{R}^n$:

$$u(x) = \ln \left( \frac{a_1}{|x - x_0|^2 + d^2} \right), \quad v(x) = \ln \left( \frac{a_2}{|x - x_0|^2 + d^2} \right),$$

where $x, x_0 \in \mathbb{R}^n, a_1, a_2$ are some constants.

Theorem 1.3 yields another proof to the classifications of extremal functions to the logarithmic HLS inequality.

Furthermore, based on the proof similar to that in [16], it can be verified that under some suitable assumptions, the integral system (8) is equivalent to

$$\begin{cases} 
  (-\Delta)^{\frac{n}{2}} u(x) = e^{nu(x)}, \\
  (-\Delta)^{\frac{n}{2}} v(x) = e^{nu(x)},
\end{cases} \quad \text{in } \mathbb{R}^n. \quad (9)$$

That is,

**Lemma 1.4.** Let $(u, v)$ be a pair of $C^1$ strong solutions to (9) with $u = o(|x|^2), v = o(|x|^2)$ at infinity and $\int_{\mathbb{R}^n} e^{nu(x)} dx < \infty, \int_{\mathbb{R}^n} e^{nv(y)} dy < \infty$. Then $(u, v)$ satisfies (8) and vice versa.
Combining Lemma 1.4 with Theorem 1.3, we can obtain

**Theorem 1.5.** Let \((u, v)\) be a pair of \(C^1\) solutions to system (9) such that \(u = o(|x|^2)\), \(v = o(|x|^2)\) at infinity and

\[
\int_{\mathbb{R}^n} e^{nu(x)} dx < \infty, \quad \int_{\mathbb{R}^n} e^{nv(y)} dy < \infty.
\]

Then, \(u, v\) must be of the following forms on \(\mathbb{R}^n\):

\[
u(x) = \ln \left( \frac{a_1}{|x-x_0|^2 + d^2} \right), \quad v(x) = \ln \left( \frac{a_2}{|x-x_0|^2 + d^2} \right),
\]

where \(x, x_0 \in \mathbb{R}^n\), \(a_1, a_2\) are some constants.

If \(u = v\) in (9) and \(n = 2\), the above theorem was first proved by Chen and Li [4] more than 20 years ago. As an extension to (9), the higher-order equation

\[
(-\Delta)^2 u(x) = (n - 1)! e^{nu(x)} \quad \text{in} \quad \mathbb{R}^n,
\]

has been received a great deal of attention. For some specific values of \(n\), via the method of moving planes, equation (10) is classified by many authors, see e.g. Ni [13], Chou and Wan [5], Lin [10], Wei and Xu [15, 16] and the reference therein. Xu [17] considered the following conformal invariant integral equation

\[
u(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^n} \ln \left( \frac{|y|}{|x-y|} \right) K(y) e^{nu(y)} dy + C_0.
\]

For \(K(y) = (n - 1)!\), Xu proved that the integral equation (11) is equivalent to equation (10), and classified the \(C^1\) solutions of equation (11) via the method of moving spheres introduced by Li and Zhu in [9].

To classify solutions to elliptic equations on the upper half space \(\mathbb{R}^n_+\):

\[
\begin{cases}
-\Delta u = e^u, & \text{in} \quad \mathbb{R}^n_+, \\
\frac{\partial u}{\partial t} = ce^\frac{u}{2}, & \text{on} \quad \partial \mathbb{R}^n_+,
\end{cases}
\]

Li and Zhu [9] introduced a method of moving spheres, which directly yields the form of solutions:

\[
u(x, t) = \ln \frac{8\lambda^2}{[\lambda^2 + (x-x_0)^2 + (t-t_0)^2]^2},
\]

where \(\lambda > 0, (x, t) \in \mathbb{R}^n_+, x_0 \in \mathbb{R}, t_0 = c\lambda \sqrt{2}\). In this project, we employ the method of moving spheres to prove Theorem 1.2. As Li and Nirenberg improved the proof of their key calculus lemmas in [7] by assuming that the function \(f\) is only continuous, we show the three calculus lemmas for exponential functions via applying the improved proof of Li and Nirenberg.

Since the constant \(C_c(n, \alpha, p)\) in inequality (1) could not be immediately applied to derive the logarithmic HLS inequality which is different from that on the whole space, we first need to give an estimate to \(C_c(n, \alpha, p)\), then deduce the logarithmic HLS inequality by the differentiation argument used in [1].

Theorem 1.3 tells us that we can compute the explicit form of the best constant \(K_n\) of inequality (6) from the explicit form of \(u\) and \(v\). However, according to the conclusion of Theorem 1.2, we could not show the explicit form of the best constant \(C_n\) of inequality (3) (or (2)) since we don’t know the form of the extremal function \(v\) on \(\mathbb{R}^n_+\).

The paper is organized as follows. Kelvin transformation and some key lemmas are presented in Section 2. In Section 3, by employing Li and Nirenberg’s standard
2. Some lemmas. In this section, we show Kelvin transformation to system (4), and present some key calculus lemmas which ensure that the moving sphere procedure could be carried out.

For $R > 0$, we give some notations as follows.

$$B_R(x) = \{ y \in \mathbb{R}^n \mid |y - x| < R, x \in \mathbb{R}^n \}, \Sigma_{x,R}^n = \mathbb{R}^n \setminus B_R^+(x),$$

$$B_R^{-1}(x) = \{ y \in \partial \mathbb{R}^n_+ \mid |y - x| < R, x \in \partial \mathbb{R}^n_+ \}, \Sigma_{x,R}^{n-1} = \partial \mathbb{R}^n_+ \setminus B_R^{-1}(x),$$

$$B_R^+(x) = \{ y = (y_1, y_2, \ldots, y_n) \in B_R(x) \mid y_n > 0, x \in \partial \mathbb{R}^n_+ \}.$$

Let $x \in \partial \mathbb{R}^n_+$ and $\lambda > 0$. We define the following transformation:

$$\omega_{x,\lambda} (\xi) = \omega (\xi^{x,\lambda}) - 2 \ln \left( \frac{|\xi - x|}{\lambda} \right), \quad \xi \in \mathbb{R}^n_+ \setminus \{ x \},$$

where

$$\xi^{x,\lambda} = x + \frac{\lambda^2 (\xi - x)}{|\xi - x|^2}$$

is the Kelvin transformation of $\xi$ with respect to $B_\lambda(x)$.

**Lemma 2.1.** Let $(u, v)$ be a pair of positive solutions to system (4). Then for $x \in \partial \mathbb{R}^n_+$,

$$u_{x,\lambda}(\xi) = \frac{2}{e} \int_{\mathbb{R}^n_+} e^{nu_{x,\lambda}(\eta)} \ln |\xi^{x,\lambda} - \eta^{x,\lambda}|^{-1} d\eta - 2 \ln \left( \frac{|\xi - x|}{\lambda} \right) - m_1, \quad (13)$$

$$v_{x,\lambda}(\eta) = \frac{2}{e} \int_{\partial \mathbb{R}^n_+} e^{(n-1)v_{x,\lambda}(\xi)} \ln |\xi^{x,\lambda} - \eta^{x,\lambda}|^{-1} d\xi - 2 \ln \left( \frac{|\eta - x|}{\lambda} \right) - m_2. \quad (14)$$

Moreover,

$$u_{x,\lambda}(\xi) - u(\xi) = \frac{2}{e} \int_{\Sigma_{x,\lambda}^n} P(x, \lambda; \xi, \eta) \left( e^{nu_{x,\lambda}(\eta)} - e^{nv(\eta)} \right) d\eta,$$

$$v_{x,\lambda}(\eta) - v(\eta) = \frac{2}{e} \int_{\Sigma_{x,\lambda}^{n-1}} P(x, \lambda; \xi, \eta) \left( e^{(n-1)v_{x,\lambda}(\xi)} - e^{(n-1)v(\xi)} \right) d\xi,$$

where

$$P(x, \lambda; \xi, \eta) = \ln \left( \frac{|\xi - x||\eta - \xi^{x,\lambda}|}{\lambda |\xi - \eta|} \right).$$

Moreover,

$$P(x, \lambda; \xi, \eta) > 0, \text{ for } \forall \xi \in \Sigma_{x,\lambda}^{n-1}, \eta \in \Sigma_{x,\lambda}^n, \lambda > 0.$$

**Proof.** For any $x \in \partial \mathbb{R}^n_+$ and $\lambda > 0$, let

$$y = \eta^{x,\lambda} = x + \frac{\lambda^2 (\eta - x)}{|\eta - x|^2}.$$

The $n$ space forms in $y$ variable and $\eta$ variable are related by

$$dy = \left( \frac{\lambda}{|\eta - x|} \right)^{2n} d\eta.$$
To simplify the calculations, we define
\[ A^+ (\xi, \lambda) = \int_{B_\lambda^+ (x)} e^{nv(y)} \ln |\xi^{x, \lambda} - y|^{-1} dy, \]
\[ A^- (\xi, \lambda) = \int_{\Sigma_{n, \lambda}} e^{nv(y)} \ln |\xi^{x, \lambda} - y|^{-1} dy. \]
That is, \( u \) can be written as follows:
\[ u(\xi^{x, \lambda}) = \frac{2}{e} \left( A^+ (\xi^{x, \lambda}) + A^- (\xi^{x, \lambda}) \right) - m_1. \]
By a direct calculation we have
\[ A^+ (\xi, \lambda) = \int_{B_\lambda^+ (x)} e^{nv(y)} \ln |\xi^{x, \lambda} - y|^{-1} dy = \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta, \]
and
\[ A^- (\xi, \lambda) = \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta. \]
Combining the above, we have
\[ u(\xi^{x, \lambda}) = \frac{2}{e} \left( \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta \right) - m_1 \]
\[ = \frac{2}{e} \left( \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta \right) - m_1 \]
\[ + \frac{2}{e} \left( \int_{B_\lambda^+ (x)} e^{nv(y)} \ln |\xi^{x, \lambda} - y|^{-1} dy - m_1 \right) \]
Therefore,
\[ u_{x, \lambda}(x) = \frac{2}{e} \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta - 2 \ln \left( \frac{|\xi - x|}{\lambda} \right) - m_1. \]
Similarly,
\[ u_{x, \lambda}(\eta) = \frac{2}{e} \int_{\Sigma_{n, \lambda}} e^{(n-1)u_{x, \lambda}(\xi)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta - 2 \ln \left( \frac{|\eta - x|}{\lambda} \right) - m_2. \]
On the other hand,
\[ u_{x, \lambda}(\xi) - u(\xi) = \frac{2}{e} \left( A^+ (\xi^{x, \lambda}) + A^- (\xi^{x, \lambda}) \right) - 2 \ln \left( \frac{|\xi - x|}{\lambda} \right) - \frac{2}{e} \left( A^+ (\xi) + A^- (\xi) \right) \]
\[ = \frac{2}{e} \left( (A^+ (\xi^{x, \lambda}) - A^+ (\xi)) + (A^- (\xi^{x, \lambda}) - A^- (\xi)) \right) - 2 \ln \left( \frac{|\xi - x|}{\lambda} \right), \]
where
\[ A^+ (\xi^{x, \lambda}) - A^+(\xi) \]
\[ = \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi^{x, \lambda} - \eta^{x, \lambda}|^{-1} d\eta - \int_{\Sigma_{n, \lambda}} e^{nv_{x, \lambda}(\eta)} \ln |\xi - \eta^{x, \lambda}|^{-1} d\eta \]
In view of and

\[ A_0^-(\xi, \lambda) - A_0^- (\xi) = \int_{\Sigma_{x, \lambda}^n} e^{nu_{x, \lambda}(\eta)} \ln \left( \frac{|\xi - x|}{|\xi - \xi, \lambda|} \right) d\eta, \]

and

\[
A_0^- (\xi, \lambda) - A_0^- (\xi) = \int_{\Sigma_{x, \lambda}^n} e^{nu(y)} \ln |\xi - \eta|^{-1} d\eta - \int_{\Sigma_{x, \lambda}^n} e^{nu(y)} \ln |\xi - \eta|^{-1} d\eta
\]

\[
= \int_{\Sigma_{x, \lambda}^n} e^{nu(y)} \ln \left( \frac{|\xi - \eta|}{|\xi - \xi, \lambda|} \right) d\eta.
\]

Since

\[
\left| \frac{\xi - \eta, \lambda}{\xi - \xi, \lambda} \right| = \frac{|\xi - x|^2 |\eta - \xi, \lambda|}{\lambda^2 |\tau - \eta|},
\]

we have

\[
u_{x, \lambda}(\xi) - u(\xi) = \frac{2}{e} \int_{\Sigma_{x, \lambda}^n} \ln \left( \frac{|\xi - x|^2 |\eta - \xi, \lambda|}{\lambda |\xi - \eta|} \right) (e^{nu_{x, \lambda}(\eta)} - e^{nu(\eta)}) d\eta
\]

\[
+ \int_{\Sigma_{x, \lambda}^n} \ln \left( \frac{|\xi - x|^2}{\lambda^2} \right) e^{nu(\eta)} d\eta - \ln \left( \frac{|\xi - x|^2}{\lambda^2} \right).
\]

In view of \( \int_{\mathbb{R}^n_+} e^{nu(y)}^{-1} dy = 1 \), we find that

\[
\int_{\mathbb{R}^n_+} e^{nu(y)} dy = \int_{\Sigma_{x, \lambda}^n} e^{nu_{x, \lambda}(\eta)} d\eta + \int_{\Sigma_{x, \lambda}^n} e^{nu(\eta)} d\eta = e.
\]

Therefore,

\[
u_{x, \lambda}(\xi) - u(\xi) = \frac{2}{e} \int_{\Sigma_{x, \lambda}^n} \ln \left( \frac{|\xi - x|^2 |\eta - \xi, \lambda|}{\lambda |\xi - \eta|} \right) (e^{nu_{x, \lambda}(\eta)} - e^{nu(\eta)}) d\eta.
\]

Using the same method, we obtain

\[
v_{x, \lambda}(\eta) - v(\eta) = \frac{2}{e} \int_{\Sigma_{x, \lambda}^{n-1}} \ln \left( \frac{|\eta - x|^2 |\xi - \eta, \lambda|}{\lambda |\xi - \eta|} \right) (e^{nu_{x, \lambda}(\xi)} - e^{nu(\xi)}) d\xi.
\]

Finally, we discuss the sign of \( P(x, \lambda; \xi, \eta) \). For \( \xi \in \Sigma_{x, \lambda}^n, n \in \Sigma_{x, \lambda}^n \) and \( \lambda > 0 \), it is easy to verify that

\[
\left( |\xi - x| |\eta - \xi, \lambda| \right)^2 - \left( \lambda |\xi - \eta| \right)^2
\]

\[
= |\xi - x|^2 |\eta - x| - 2 |\xi - x| \left( |\xi - x| \left( \frac{\lambda^2 |\xi - x|}{|\xi - x|^2} + \frac{\lambda^4}{|\xi - x|^2} \right) - \lambda^2 (|\xi - x|^2 - 2 |\xi - x, \eta - x| + |\eta - x|^2) + |\xi - x|^2 \right)
\]

\[
= |\xi - x|^2 |\eta - x| + \lambda^4 - \lambda^2 |\xi - x|^2 - \lambda^2 |\eta - x|^2
\]

\[
= (|\xi - x|^2 - \lambda^2)(|\eta - x|^2 - \lambda^2) > 0.
\]

Hence, we conclude that \( P(x, \lambda; \xi, \eta) = \ln \left( \frac{|\xi - x| |\eta - \xi, \lambda|}{\lambda |\xi - \eta|} \right) > 0. \)

To prove the calculus key lemmas for exponential functions, we need to recall the following key lemmas introduced by Li and Nirenberg in [7]. The earlier visions with stronger assumptions were first proved by Li and Zhu [9], and Li and Zhang [8].
Lemma 2.2. (Lemma 5.7 in [7]) For \( n \geq 1 \) and \( \mu \in \mathbb{R} \), if \( f \) is a function defined on \( \mathbb{R}^n \) and valued in \((\infty, +\infty)\) satisfying
\[
\left(\frac{\lambda}{|y-x|}\right)^\mu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) \leq f(y), \quad \forall \lambda > 0, \ |y-x| \geq \lambda, x, y \in \mathbb{R}^n,
\]
then \( f(x) = \text{constant} \).

Lemma 2.3. (Lemma 5.8 in [7]) Let \( n \geq 1 \) and \( \mu \in \mathbb{R} \), and \( f \in C^0(\mathbb{R}^n) \). Suppose that for every \( x \in \mathbb{R}^n \), there exists \( \lambda > 0 \) such that
\[
\left(\frac{\lambda}{|y-x|}\right)^\mu f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) = f(y), \quad \forall y \in \mathbb{R}^n \setminus \{x\}.
\]
Then there are \( a \geq 0, d > 0 \) and \( \bar{x} \in \mathbb{R}^n \), such that
\[
f(x) = \pm a\left(\frac{1}{d + |x - \bar{x}|^2}\right)^\frac{d}{2}.
\]

The following three calculus key lemmas are essential to carry out the moving sphere procedure.

Lemma 2.4. For \( n \geq 1 \), if \( f \) is a function defined on \( \mathbb{R}^n \) and valued in \((\infty, +\infty)\) satisfying
\[
f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 2 \ln\left(\frac{|y-x|}{\lambda}\right) \leq f(y), \quad \forall \lambda > 0, \ |y-x| \geq \lambda, x, y \in \mathbb{R}^n,
\]
then \( f(x) = \text{constant} \).

Proof. Write \( h(x) = e^{f(x)} \). It follows from (15) that
\[
h(y) = e^{f(y)} \geq e^{f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 2 \ln\left(\frac{|y-x|}{\lambda}\right)} = \left(\frac{\lambda}{|y-x|}\right)^2 e^{f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right)}
\]
\[
= \left(\frac{\lambda}{|y-x|}\right)^2 h\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right).
\]
By Lemma 2.2, we obtain \( f(x) = \text{constant} \). \( \square \)

Lemma 2.5. Let \( n \geq 1 \) and \( \mu \in \mathbb{R} \), and \( f \in C^0(\mathbb{R}^n) \). Suppose that for every \( x \in \mathbb{R}^n \), there exists \( \lambda > 0 \) such that
\[
f(y) = f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 2 \ln\left(\frac{|y-x|}{\lambda}\right), \quad \forall y \in \mathbb{R}^n \setminus \{x\}.
\]
Then there are \( a \geq 0, d > 0 \) and \( \bar{x} \in \mathbb{R}^n \), such that
\[
f(x) = \ln\left(\frac{a}{d + |x - \bar{x}|^2}\right).
\]

Proof. Set \( h(x) = e^{f(x)} \). Since \( f \in C^0(\mathbb{R}^n) \), we get \( h \in C^0(\mathbb{R}^n) \). It follows from (16) that
\[
h(y) = e^{f(y)} = e^{f\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right) - 2 \ln\left(\frac{|y-x|}{\lambda}\right)} = \left(\frac{\lambda}{|y-x|}\right)^2 h\left(x + \frac{\lambda^2(y-x)}{|y-x|^2}\right),
\]
By Lemma 2.3, we have
\[
h(x) = \frac{a}{|x - \bar{x}_0|^2 + d}.
\]
The lemma is proved. \( \square \)
Lemma 2.6. For \( n \geq 1 \), if \( f \) is a function defined on \( \mathbb{R}^n_+ \) and valued in \( (-\infty, +\infty) \) satisfying
\[
 f(x + \frac{\lambda^2(y - x)}{|y - x|^2}) - 2\ln \left( \frac{|y - x|}{\lambda} \right) \leq f(y), \quad \forall \lambda > 0, \ |y - x| \geq \lambda, y \in \mathbb{R}^n_+, x \in \partial \mathbb{R}^n_+,
\]
then
\[
 f(z) = f(z', t) = f(0, t), \quad \forall z = (z', t) \in \mathbb{R}^n_+.
\]

Proof. The proof of this lemma is similar to that of Lemma 3.7 in [6] which extended Lemma 5.7 in [7] to the upper half space. For any \( z = (z', z_n) \in \mathbb{R}^n_+ \), choose \( y^i = (y^i, y^i_n) \) with \( y^i \neq z' \), \( y^i_n > z_n \) and \( y^i_n \to z_n \) as \( i \to \infty \). Choose \( b^i > 1 \), so that
\[
x^i := (x^i, 0) = x^i(b^i) = y^i + b^i(z - y^i) \in \partial \mathbb{R}^n_+.
\]
Also define
\[
 \lambda_i := \lambda_i(b^i) = \sqrt{|z - x^i||y^i - x^i|}.
\]
Then,
\[
 z = x^i + \frac{\lambda_i^2(y^i - x^i)}{|y^i - x^i|^2},
\]
and by the assumption of Lemma 2.6,
\[
 f(z) + 2\ln \left( \frac{\lambda_i}{|y^i - x^i|} \right) \leq f(y^i).
\]
Since
\[
 \lim_{i \to \infty} \frac{\lambda_i}{|y^i - x^i|} = \lim_{i \to \infty} \sqrt{\frac{|z - x^i|}{|y^i - x^i|}} = 1,
\]
we obtain \( f(z', z_n) \leq f(y^i, z_n) \). Since \( y^i \) and \( z' \) are arbitrary, we prove the lemma. \( \square \)

3. Proof of Theorem 1.2. We devote this section to completing Theorem 1.2 by the method of moving spheres in integral form.

Lemma 3.1. Under the same assumptions as in Theorem 1.2, if \((u, v)\) is a pair of solution to system \((4)\), then for any \( x \in \partial \mathbb{R}^n_+ \), there exists a small positive number \( \lambda_0(x) \) such that for all \( 0 < \lambda < \lambda_0(x) \)
\[
 u_{x, \lambda}(\xi) \leq u(\xi), \quad \xi \in \Sigma_{x, \lambda}^{-1},
\]
\[
 v_{x, \lambda}(\eta) \leq v(\eta), \quad \eta \in \Sigma_{x, \lambda}.
\]
Proof. Without loss of generality we may assume \( x = 0 \), and we use the notation \( u_\lambda = u_{0, \lambda}, \xi^\lambda = \xi^{x, \lambda} \). By the definition,
\[
 \frac{\partial}{\partial \xi_i} u(\xi) = -\frac{2}{e} \int_{\mathbb{R}^n_+} \frac{\xi_i - \eta_i}{|\xi - \eta|^2} e^{\nu(\eta)} \, d\eta.
\]
Then
\[
 \nabla_\xi (u(\xi) + \ln |\xi|) : \xi = -\frac{2}{e} \int_{\mathbb{R}^n_+} \frac{|\xi|^2 - (\xi, \eta)}{|\xi - \eta|^2} e^{\nu(\eta)} \, d\eta + 1.
\]
According to \( \int_{\mathbb{R}^n_+} e^{\nu(\eta) - 1} \, d\eta = 1 \) and the triangle inequality, one obtains
\[
 \nabla_\xi (u(\xi) + \ln |\xi|) : \xi \geq 1 - \left( \frac{2}{e} \int_{\mathbb{R}^n_+} \frac{1}{|\xi - \eta|} e^{\nu(\eta)} \, d\eta \right) |\xi|.
\]
It is easy to check that \( \int_{\mathbb{R}^n_+} \frac{1}{\sqrt{2\pi}} e^{-\nu(n)\eta} d\eta \leq C \). Then there exists \( r_0 > 0 \) such that
\[
\nabla \xi(u(\xi) + \ln |\xi|) \cdot \xi > 0, \quad \forall 0 < |\xi| < r_0.
\]

Note that
\[
u(\xi^\lambda) + \ln |\xi^\lambda| = u_\lambda(\xi) + 2 \ln \left( \frac{|\xi|}{\lambda} \right) + \ln \left( \frac{\lambda^2}{|\xi|} \right) = u_\lambda(\xi) + \ln |\xi|.
\]

Therefore,
\[
u_\lambda(\xi) < u(\xi), \quad \forall 0 < \lambda < |\xi| < r_0.
\]

Meanwhile, by Fatou’s lemma
\[
\liminf_{|\xi| \to \infty} \frac{2}{|\xi|} \int_{\mathbb{R}^n_+} \ln \left( \frac{|\xi|}{\xi - \eta} \right) e^{\nu(\eta)} d\eta \geq 0.
\]

Then for \( \forall |\xi| \geq r_0, u(\xi) + 2 \ln |\xi| \geq 0 \). For small \( \lambda_0 \in (0, r_0) \) and for \( 0 < \lambda < \lambda_0 \),
\[
u_\lambda(\xi) = u_\lambda(\xi) - 2 \ln \frac{|\xi|}{\lambda} \leq \sup_{\eta < \lambda} + u(\xi) + 2 \ln \lambda_0 \leq u(\xi).
\]

Therefore, there exists \( \lambda_0 > 0 \) such that for all \( 0 < \lambda < \lambda_0 \), \( \nu_{x,\lambda}(\xi) \leq u(\xi) \) with \( \xi \in \Sigma_{x,\lambda}^n \). Similarly, \( v_{x,\lambda}(\eta) \leq v(\eta) \) with \( \eta \in \Sigma_{x,\lambda}^n \).

Define
\[
\bar{\lambda}(\xi) = \sup \{ \mu > 0 | u_{x,\lambda}(\xi) \leq u(\xi) \text{ and } v_{x,\lambda}(\eta) \leq v(\eta) \},
\]

\[
\forall \lambda \in (0, \mu), \forall \xi \in \Sigma_{x,\lambda}^{n-1}, \forall \eta \in \Sigma_{x,\lambda}^n.
\]

**Lemma 3.2.** For some \( x_0 \in \partial \mathbb{R}^n_+ \), if \( \bar{\lambda} = \bar{\lambda}(x_0) < \infty \), then
\[
u_{x_0,\bar{\lambda}}(\xi) = u(\xi), \quad \xi \in \partial \mathbb{R}^n_+,
\]
\[
u_{x_0,\bar{\lambda}}(\eta) = v(\eta), \quad \eta \in \mathbb{R}^n_+.
\]

**Proof.** For simplicity, we only show \( \nu_{x_0,\bar{\lambda}} = u \) for all \( \xi \in \partial \mathbb{R}^n_+ \). Without loss of generality, we may assume \( x_0 = 0 \), and we write \( \bar{\lambda} = \bar{\lambda}(0), u_\lambda = u_{0,\lambda}, \eta^\lambda = \eta^{0,\lambda}, \Sigma_\lambda = \Sigma_{0,\lambda}^n \) and \( \Sigma_\lambda = \Sigma_{0,\lambda}^{n-1} \).

By the definition of \( \bar{\lambda} \)
\[
u_{\bar{\lambda}}(\xi) \leq u(\xi), \quad \xi \in \Sigma_{\lambda}^{n-1},
\]
\[
u_{\bar{\lambda}}(\eta) \leq v(\eta), \quad \eta \in \Sigma_{\lambda}^n.
\]

Also by the positivity of the kernel \( P(0, \bar{\lambda}; \xi, \eta) \), either \( u_{\bar{\lambda}}(\xi) = u(\xi) \) and \( v_{\bar{\lambda}}(\eta) = v(\eta) \) for all \( \xi \in \Sigma_{\lambda}^{n-1} \) and \( \eta \in \Sigma_{\lambda}^n \), or \( u_{\bar{\lambda}}(\xi) < u(\xi), v_{\bar{\lambda}}(\eta) < v(\eta) \) for all \( \xi \in \Sigma_{\lambda}^{n-1} \) and \( \eta \in \Sigma_{\lambda}^n \).

If \( u_{\bar{\lambda}}(\xi) = u(\xi) \), then there is no contradiction.

If \( u_{\bar{\lambda}}(\xi) < u(\xi), v_{\bar{\lambda}}(\eta) < v(\eta) \) for all \( \xi \in \Sigma_{\lambda}^{n-1} \) and \( \eta \in \Sigma_{\lambda}^n \), we would like to show there is a contradiction.

By Fatou’s lemma,
\[
\liminf_{|\xi| \to \infty} \frac{2}{|\xi|} \int_{\Sigma_{\lambda}^n} \ln \left( \frac{|\xi| - \eta}{|\lambda^\lambda - \eta|} \right) e^{\nu_\lambda(\eta)} d\eta - e^{\nu_{\bar{\lambda}}(\eta)} d\eta
\]

\[
= \liminf_{|\xi| \to \infty} \frac{2}{|\xi|} \int_{\Sigma_{\lambda}^n} \ln \left( \frac{|\xi| - \eta}{|\lambda^\lambda - \eta|} \right) e^{\nu_\lambda(\eta)} d\eta - e^{\nu_{\bar{\lambda}}(\eta)} d\eta
\]
Similarly, there exists $\epsilon_1 \in (0,1)$ such that $u(\xi) - u_\lambda(\xi) \geq \epsilon_1$ for $\forall \xi \in \Sigma^{n-1}_{\lambda+1}$.

By the explicit formula for $u_\lambda$ and $\int_{\mathbb{R}^n_+} e^{nv(y)-1}dy = 1$, for any $\lambda > \bar{\lambda}$ and $\xi \in \Sigma^{n-1}_\lambda$

$$u_\lambda(\xi) - u_\lambda(\xi) = \frac{2}{e} \int_{\mathbb{R}^n_+} \ln \left( \frac{|y|}{\lambda} \right) e^{nv(y)} dy.$$

Then by Fatou's lemma

$$\liminf_{\|\xi\| \to \infty} (u(\xi) - u_\lambda(\xi)) \geq \frac{2}{e} \int_{\mathbb{R}^n_+} \ln \left( \frac{\lambda}{\lambda} \right) e^{nv(y)} dy = 2 \ln \left( \frac{\lambda}{\lambda} \right) > 0.$$

Then there exists $0 < \epsilon_2 < \epsilon_1$ such that

$$u(\xi) - u_\lambda(\xi) = u(\xi) - u_\lambda(\xi) + u_\lambda(\xi) - u_\lambda(\xi) \geq \frac{\epsilon_1}{2}, \quad \forall \xi \in \Sigma^{n-1}_{\lambda+1}, \bar{\lambda} \leq \lambda \leq \lambda + \epsilon_2. \quad (17)$$

Similarly, there exists $0 < \epsilon_3 < \epsilon_1$ such that

$$v(\eta) - v_\lambda(\eta) = v(\eta) - v_\lambda(\eta) + v_\lambda(\eta) - v_\lambda(\eta) \geq \frac{\epsilon_1}{2}, \quad \forall \eta \in \Sigma_{\lambda+1}^n, \bar{\lambda} \leq \lambda \leq \lambda + \epsilon_3. \quad (18)$$

For $\epsilon \in (0, \epsilon_2)$ which we choose below, we have for $\bar{\lambda} \leq \lambda \leq \bar{\lambda} + \epsilon$ and for $\xi \in B_{\lambda+1} \setminus B_{\lambda}$

$$u(\xi) - u_\lambda(\xi) = \frac{2}{e} \int_{\Sigma_{\lambda}} P(0, \lambda; \xi, \eta)(e^{nv(\eta)} - e^{nv_\lambda(\eta)}) d\eta$$

$$\geq \frac{2}{e} \int_{B_{\lambda+1} \setminus B_{\lambda+2}^+} P(0, \lambda; \xi, \eta)(e^{nv(\eta)} - e^{nv_\lambda(\eta)}) d\eta$$

$$+ \frac{2}{e} \int_{B_{\lambda+1} \setminus B_{\lambda+2}^+} P(0, \lambda; \xi, \eta)(e^{nv(\eta)} - e^{nv_\lambda(\eta)}) d\eta$$

$$\geq \frac{2}{e} \int_{B_{\lambda+1} \setminus B_{\lambda+2}^+} P(0, \lambda; \xi, \eta)(e^{nv(\eta)} - e^{nv_\lambda(\eta)}) d\eta$$

$$+ \frac{2}{e} \int_{B_{\lambda+1} \setminus B_{\lambda+2}^+} P(0, \lambda; \xi, \eta)(e^{nv(\eta)} - e^{nv_\lambda(\eta)}) d\eta.$$

According to (18), there exists $\delta_1 > 0$ such that

$$e^{nv(\eta)} - e^{nv_\lambda(\eta)} \geq \delta_1, \quad \forall \eta \in B_{\lambda+3}^+ \setminus B_{\lambda+2}^+.$$

Note that

$$\nabla_\xi P(0, \lambda; \xi, \eta) \cdot \xi|_{\xi| = \lambda} = \frac{|\eta|^2 - |\xi|^2}{|\xi - \eta|^2} > 0, \quad \forall \eta \in \Sigma^n_{\lambda}.$$

Since

$$P(0, \lambda; \xi, \eta) = 0, \quad \forall |\xi| = \lambda,$$

and $P(0, \lambda; \xi, \eta)$ is positive and smooth in the relevant region $B_{\lambda+3}^+ \setminus B_{\lambda+2}^+$, we have

$$P(0, \lambda; \xi, \eta) \geq \delta_2 (|\xi| - \lambda), \quad \forall \xi \in B_{\lambda+1} \setminus B_{\lambda}, \eta \in B_{\lambda+3}^+ \setminus B_{\lambda+2}^+,$$

where $\delta_2 > 0$ is some constant independent of $\epsilon$. According to the mean value theorem, there must be some constant $C > 0$ independent of $\epsilon$ such that

$$|e^{nv_\lambda(\eta)} - e^{nv_\lambda(\eta)}| \leq C(\lambda - \bar{\lambda}) \leq C\epsilon, \quad \forall \eta \in B_{\lambda+1}^+ \setminus B_{\lambda}^+,$$
for some $a\not= d$, then in view of Lemma 2.4, we conclude that $u = C_0$ is a constant. This violates the assumption $\int_{\mathbb{R}^n} e^{(n-1)u(x)} \, dx = 1$. Theorem 1.2 is established.

\begin{equation}
\int_{B^+_{x+1}\setminus B^+_{x}} P(0, \lambda; \xi, \eta) \, d\eta = \int_{B^+_{x+1}\setminus B^+_{x}} \ln \left( \frac{\left| \xi \right|}{\lambda} \right) \, d\eta + \int_{B^+_{x+1}\setminus B^+_{x}} \ln \left( \frac{\left| \xi - \eta \right|}{\left| \xi - \eta \right|} \right) \, d\eta
\end{equation}

$$\leq C(\left| \xi \right| - \lambda) + C(\left| \xi^0 - \xi \right|) \leq C(\left| \xi \right| - \lambda).$$

It follows that for small $\epsilon > 0$ and $\lambda \leq \lambda \leq \lambda + \epsilon$

$$u(\xi) - u_\lambda(\xi) \geq -Ce \int_{B^+_{x+1}\setminus B^+_{x}} P(0, \lambda; \xi, \eta) \, d\eta + \frac{2}{e} \delta_1 \delta_2 (\left| \xi \right| - \lambda) \int_{B^+_{x+1}\setminus B^+_{x+2}} d\eta$$

$$\geq \frac{2}{e} \delta_1 \delta_2 \int_{B^+_{x+1}\setminus B^+_{x+2}} d\eta - Ce (\left| \xi \right| - \lambda) \geq 0.$$
Appendix. In appendix, we present the proof of Theorem 1.1. To achieve this aim, we first need to project HLS inequality (1) to $B_1$ by Kelvin transform as follows (see Corollary 5.1 in [6]):

$$\int_{B_1} \int_{\partial B_1} \frac{G(\xi)F(\eta)}{|\xi - \eta|^{n+\alpha}} dS_\eta d\xi \leq C_F \|F\|_{L^p(\partial B_1)} \|G\|_{L^r(B_1)},$$

(21)

where $p = \frac{2(n-1)}{n+\alpha-2}$, $t = \frac{2n}{n+\alpha}$ and

$$C_F = C_F(n, \alpha) = (n\omega_n)^{-\frac{n+\alpha-2}{2(n-1)}} \left( \int_{B_1} \left( \int_{\partial B_1} \frac{1}{|\xi - \eta|^{n+\alpha}} dS_\eta \right)^{\frac{2n}{n+\alpha}} d\xi \right)^{\frac{n-\alpha}{2n}}.$$

It is not easy to calculate the exact value of $C_F$ directly. Since we are interested in the limiting case $\alpha \to n^-$, we only need to obtain an estimate of $C_F$ as $\alpha \to n^-$. The following proposition provides an estimate of $C_F$ as $\alpha \to n^-$. A similar proof of the estimate of $C_F$ can be found in [12].

**Proposition 1.** For $n \geq 2$ and $\theta = \frac{n-\alpha}{n}$, the following holds:

$$C_F(n, \alpha) = C_F(\theta) = 1 + \frac{\theta}{n-1} \ln(n\omega_n) + \frac{\theta}{n} \ln \left( \int_{B_1} e^{|I_n(\xi)|} d\xi + o(\theta) \right),$$

(22)

as $\theta \to 0$, where $I_n(\xi) = -2\omega_n^{-1} \int_{\partial B_1} \ln |\xi - \eta| dS_\eta$.

**Proof.** For any $\xi \in B_1$, there exists a positive constant $C$ independent of $\xi$ such that

$$\int_{\partial B_1} \ln |\xi - \eta| dS_\eta \leq C.$$

According to the Taylor’s expansion of the function $f(\theta) = |\xi - \eta|^{-2\theta}$ with respect to $\theta$ at 0, we have

$$\int_{\partial B_1} |\xi - \eta|^{-2\theta} dS_\eta = \int_{\partial B_1} dS_\eta - 2\theta \int_{\partial B_1} \ln |\xi - \eta| dS_\eta + o(\theta)$$

$$= n\omega_n(1 + o(\theta))(1 + \frac{\theta}{n} I_n(\xi)).$$

Since

$$(1 + \frac{\theta}{n} I_n(\xi))^{\frac{n}{2}} = e^{I_n(\xi)}(1 + o(\theta))$$

for $\theta \to 0$, we have

$$\left( \int_{B_1} (1 + \frac{\theta}{n} I_n(\xi))^{\frac{n}{2}} d\xi \right)^{\frac{2}{n}} = (1 + o(\theta)) \left( \int_{B_1} e^{I_n(\xi)} d\xi \right)^{\frac{n}{2}}.$$

Combining the above estimates, we obtain

$$\left( \int_{B_1} (\int_{\partial B_1} |\xi - \eta|^{-2\theta} dS_\eta)^{\frac{n}{2}} d\xi \right)^{\frac{2}{n}} = n\omega_n(1 + o(\theta))(\int_{B_1} e^{I_n(\xi)} d\xi)^{\frac{n}{2}}.$$

Hence, for $\theta$ small enough,

$$C_F(n, \alpha) = C_F(\theta) = (n\omega_n)^{-\frac{2(n-1)}{2(n+\alpha-1)}} \left( \int_{B_1} (\int_{\partial B_1} |\xi - \eta|^{-2\theta} dS_\eta)^{\frac{n}{2}} d\xi \right)^{\frac{n}{2}}$$

$$= (1 + o(\theta))(n\omega_n)^{\frac{\alpha}{n+\alpha}} \left( \int_{B_1} e^{I_n(\xi)} d\xi \right)^{\frac{n}{2}}.$$

(23)
It is easy to check from (23) that
\[ C_e(0) = 1, \]
\[ \frac{d}{d\theta}(C_e(0)) = \frac{\ln(n\omega_n)}{n-1} + \frac{1}{n} \int_{B_1} e^{l_n(\xi)} d\xi. \]
Hence, using the Taylor’s expansion of \( C_e(\theta) \) with respect to \( \theta \) at 0, we conclude (22).

**Proof of Theorem 1.1.** Using a differentiation argument as Beckner [1], we can obtain “endpoint information” at \( p = 1 \). Write
\[ \Phi(F,G) = C_e\|F\|_{L^p(\partial B_1)}\|G\|_{L^p(B_1)} - \int_{\partial B_1} \int_{B_1} \frac{G(\xi)F(\eta)}{|\xi - \eta|^{n-\alpha}} d\eta d\xi. \]
It follows from (21) that \( \Phi(F,G) \geq 0 \).

Letting \( \theta = \frac{n-\alpha}{2} \), we claim \( \frac{d\Phi(F,G)}{d\theta}|_{\theta=0} \geq 0 \). Indeed, it is obvious that \( p = \frac{n-1-\theta}{n-\theta} \rightarrow 1 \) and \( t = \frac{n}{n-\theta} \rightarrow 1 \) as \( \theta \rightarrow 0 \). Meanwhile,
\[ \left. \frac{d\Phi}{d\theta} \right|_{\theta=0} = \frac{n}{(n-\theta)^2} \mid_{\theta=0} = \frac{1}{n-1}, \]
\[ \left. \frac{dt}{d\theta} \right|_{\theta=0} = \frac{n}{(n-\theta)^2} \mid_{\theta=0} = \frac{1}{n}. \]

Since \( \int_{\partial B_1} F(\eta)d\eta = \int_{B_1} G(\xi)d\xi = 1 \), it is easy to check
\[ \frac{d}{d\theta} \left( \int_{B_1} \int_{\partial B_1} \frac{G(\xi)F(\eta)}{|\xi - \eta|^{n-\alpha}} d\eta d\xi \right) \mid_{\theta=0} = -2 \int_{B_1} \int_{\partial B_1} G(\xi)F(\eta) \ln |\xi - \eta| d\eta d\xi, \]
\[ \frac{d}{d\theta} \left( \|F\|_{L^p(\partial B_1)} \right) \mid_{\theta=0} = \frac{1}{n-1} \int_{\partial B_1} F(\eta) \ln(F(\eta)) d\eta, \]
\[ \frac{d}{d\theta} \left( \|G\|_{L^p(B_1)} \right) \mid_{\theta=0} = \frac{1}{n} \int_{B_1} G(\xi) \ln(G(\xi)) d\xi. \]

From Proposition 1, \( \lim_{\theta \to 0^+} C_e(\theta) = 1 \) and
\[ \frac{d}{d\theta}(C_e(0)) = \frac{1}{n-1} \ln(n\omega_n) + \frac{1}{n} \ln \int_{B_1} e^{l_n(\xi)} d\xi. \]
Also when \( \theta = 0 \), \( \|F\|_{L^p(\partial B_1)} = \int_{\partial B_1} F(\eta)d\eta = 1 \) and \( \|G\|_{L^p(B_1)} = \int_{B_1} G(\xi)d\xi = 1 \). Hence,
\[ \left. \frac{d\Phi}{d\theta} \right|_{\theta=0} = \frac{1}{n-1} \ln(n\omega_n) + \frac{1}{n} \ln \int_{B_1} e^{l_n(\xi)} d\xi + \frac{1}{n-1} \int_{\partial B_1} F(\eta) \ln(F(\eta)) d\eta \]
\[ + \frac{1}{n} \int_{B_1} G(\xi) \ln(G(\xi)) d\xi + 2 \int_{B_1} \int_{\partial B_1} G(\xi)F(\eta) \ln |\xi - \eta| d\eta d\xi \geq 0. \]

That is,
\[ -2 \int_{B_1} \int_{\partial B_1} G(\xi) \ln |\xi - \eta| F(\eta) d\eta d\xi \]
\[ \leq \int_{B_1} G(\xi) \ln(G(\xi)) d\xi + \frac{1}{n-1} \int_{\partial B_1} F(\eta) \ln(F(\eta)) d\eta + C_n, \]
where \( C_n = \frac{1}{n-1} \ln(n\omega_n) + \frac{1}{n} \ln \int_{B_1} e^{l_n(\xi)} d\xi. \)
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E-mail address: jbdou@snnu.edu.cn
E-mail address: li4y@cmich.edu