Testable constraint on near-tribimaximal neutrino mixing

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Abstract

General lowest order perturbations to hermitian squared mass matrices of leptons are considered away from the tribimaximal (TBM) limit in which a weak flavor basis with mass diagonal charged leptons is chosen. The three measurable TBM-deviants are expressed linearly in terms of perturbation induced dimensionless coefficients appearing in the charged lepton and neutrino flavor eigenstates. With unnatural cancellations assumed to be absent and the charged lepton perturbation contributions to their flavor eigenstates argued to be small, we analytically derive that a deviation from maximal atmospheric neutrino mixing and CP violation in neutrino oscillations cannot both be observably large, posing the challenge of verification to forthcoming experiments at the intensity frontier.

1 Introduction

The phenomenon of mixing between different generations of quarks/leptons has now been experimentally studied fairly well [1]. The three quark mixing angles are at present quite well-measured. Though the leptonic mixing angles are not known as well, one has credible nonzero $3\sigma$ upper and lower bounds on them. CP-violation has been investigated quite thoroughly in the quark sector, but as yet there is no reliable observation of CP-violation involving only leptons. Quark mixing angles are known to become progressively smaller in order of magnitude as one moves from 1-2 to 2-3 and 1-3 generation mixing. This fact can be understood qualitatively in terms of a hierarchical quark mass matrix. The mixing angles, that emerge from such a mass matrix, are small and turn out to be given roughly by the mass ratios of relevant generations of quarks. Since the masses of both up- and down-type quarks are strongly hierarchical with respect to generations, this ties in with observation. In complete contrast, the leptonic mixing angles have been found to be much larger and show a different pattern. The qualitative difference between quark and lepton mixing patterns is made starkly evident by a quantitative comparison of the
Consequently, we take a complex symmetric mass matrix for them. In the mass basis, that is, we put forth the following viewpoint. While each of \( U_u, U_d, U_{\ell} \) shows a hierarchical structure, this is not true of \( U_{\nu} \) which is governed by a different principle. The way to gain new insights into this principle is through more precise measurements of the leptonic mixing angles and of the associated CP-violating phase \( \delta_{CP} \) as well as of the concerned neutrino masses. These can test mixing constraints from specific theoretical ideas. Our aim in this paper is to derive some such constraint which is experimentally testable.

We follow the procedure of Ref. [2] and take neutrinos to be light Majorana particles occurring in three generations. More definitely, the family symmetry controlling their mixing can be taken to be independent of the neutrino mass hierarchy. Though one need not make any specific assumption on the neutrino mass hierarchy, such considerations are most natural for quasi-degenerate neutrinos. Even if there is any mass hierarchy among neutrinos, it can be presumed to be quite mild. Thus we separate the issue of the mixing of neutrinos from that of their mass hierarchy.

For fermions of type \( t = u, d, \ell, \nu \), we can define the mass basis as the one in which the corresponding mass matrix \( M_t \) is diagonal. We can also consider the flavor basis in which the fermions \( | \chi^f \rangle \) are flavor eigenstates but the mass matrix \( M_{\ell f} \) is not necessarily diagonal. The hermitian squared mass matrix \( M_{\ell}^T M_{\ell} \) in each basis is related by a unitary transformation \( U_{\ell} \):

\[
U_{\ell}^T M_{\ell f}^T M_{\ell f} U_{\ell} = M_{\ell}^T U_{\ell}.
\]

We henceforth use the superscript zero to denote the TBM limit. In this limit we choose to work in the weak flavor basis in which the charged leptons have a diagonal Dirac mass matrix:

\[
M_{\ell}^0 = \text{diag.} \left( m_{\ell_1}^0, m_{\ell_2}^0, m_{\ell_3}^0 \right). \tag{3}
\]

Though the masses of the charged leptons \( l = e, \mu, \tau \) show a pronounced hierarchical pattern with respect to generations, one suspects that such may not be the case with neutrinos. What operates for the mixing latter, possibly related their presumed Majorana nature\(^\text{\footnote{We follow the procedure of Ref. [2] and take neutrinos to be light Majorana particles occurring in three generations. Consequently, we take a complex symmetric mass matrix for them. In the mass basis, that is \( M_{\ell} = \text{diag.} \left( m_{\nu_1}, m_{\nu_2}, m_{\nu_3} \right) \) with \( m_{\nu_1} = |m_{\nu_1}|, m_{\nu_2} = |m_{\nu_2}|e^{-i\alpha_2}, m_{\nu_3} = |m_{\nu_3}|e^{-i\alpha_3} \) and \( \alpha_{21}, \alpha_{31} \) as Majorana phases.}}\) originating, say from some kind of a seesaw mechanism [2], is perhaps some underlying family symmetry. Though one need not approximate magnitudes [2] of the elements of the respective unitary matrices \( V_{CKM} \) and \( U_{PMNS} \):

\[
|V_{CKM}| \sim \begin{pmatrix} 0.9 & 0.2 & 0.004 \\ 0.2 & 0.9 & 0.01 \\ 0.008 & 0.04 & 0.9 \end{pmatrix}, |U_{PMNS}| \sim \begin{pmatrix} 0.8 & 0.5 & 0.2 \\ 0.4 & 0.6 & 0.7 \\ 0.4 & 0.6 & 0.7 \end{pmatrix}. \tag{1}
\]

The rest of the paper is organized as follows. Section 2 is devoted to a discussion of tribimaximal mixing and its breaking. In section 3 we set up our basic lowest order perturbative formalism which is meant to compute the deviations away from tribimaximality. Section 4 contains the derivation of the theoretical consequences of the said formalism. In Section 5 we discuss the experimentally testable constraint arising therefrom. The final section 6 summarizes our conclusions.

## 2 Broken tribimaximal mixing and its effects

There is a vast literature [2, 3, 4, 5] covering theoretical ideas on the principle governing \( M_{\ell f} \) and \( U_{\nu} \). Our focus, however, is on tribimaximal (TBM) mixing [6, 7, 8, 9] which is elegant, predictive and can be given a solid theoretical foundation [10, 11] from specific realizations of discrete family symmetries such as \( A_4, S_3 \) and \( \Delta_{27} \). Some of the latter have also suggested a few neutrino mixing sum-rules [12, 13, 14].

The broken TBM paradigm is explained in detail below. From our consideration, we obtain two alternative experimentally testable possibilities, at least one of which is obligatory.

### 2.1 Broken tribimaximal mixing

For fermions of type \( t = u, d, \ell, \nu \), we can define the mass basis as the one in which the corresponding mass matrix \( M_t \) is diagonal. We can also consider the flavor basis in which the fermions \( | \chi^f \rangle \) are flavor eigenstates but the mass matrix \( M_{\ell f} \) is not necessarily diagonal. The hermitian squared mass matrix \( M_{\ell}^T M_{\ell} \) in each basis is related by a unitary transformation \( U_{\ell} \):

\[
U_{\ell}^T M_{\ell f}^T M_{\ell f} U_{\ell} = M_{\ell}^T U_{\ell}.
\]

We henceforth use the superscript zero to denote the TBM limit. In this limit we choose to work in the weak flavor basis in which the charged leptons have a diagonal Dirac mass matrix:

\[
M_{\ell}^0 = \text{diag.} \left( m_{\ell_1}^0, m_{\ell_2}^0, m_{\ell_3}^0 \right). \tag{3}
\]
On the other hand, in the charged lepton sector, some kind of symmetry breaking terms, which characterize their contributions to expected to be of the same order of magnitude. Moreover, our results on neutrino mixing do not need to dent analysis with the most general TBM violating perturbation matrices whose nonzero elements are low scale, or any specific discrete family symmetry. In fact, we perform a lowest order model independent of perturbations to be of the same order of magnitude and treat them to the lowest order. Much effort has already been expended in this direction. However, we do have something new and interesting to say. We bring out a novel feature of the near-TBM mixing of neutrinos in terms of an effort has already been expended in this direction. However, we do have something new and interesting to say. We bring out a novel feature of the near-TBM mixing of neutrinos in terms of an effort has already been expended in this direction. 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Of course, the neglected $O(\epsilon^2)$ terms are estimated to be only at a couple of percent level which is much below the present and expected future accuracy of proposed neutrino oscillation experiments.

3 Lowest order perturbation away from tribimaximality

For charged leptons $\ell$ the normalized eigenvectors in the mass basis and the flavor basis are identical in the TBM limit. Thus we can take

$$\begin{align*}
|\chi_1^{\ell0}\rangle &= |\chi_1^{\ell0}\rangle_{\ell f} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
|\chi_2^{\ell0}\rangle &= |\chi_2^{\ell0}\rangle_{\ell f} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
|\chi_3^{\ell0}\rangle &= |\chi_3^{\ell0}\rangle_{\ell f} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}$$

(8)

Moreover, the charged lepton mass matrix is identical in each basis in the same limit, namely

$$M^0_{\ell f} = M^0_{\ell}.$$  

(9)

Adding a perturbation $M^\prime_{\ell f} (= \lambda_{ij})$ to $M^0_{\ell f}$ so that $M_{\ell f} = M^0_{\ell f} + M^\prime_{\ell f}$, we can construct the corresponding matrix $M'_{\ell f}$ in the mass basis as

$$M'_{\ell f} = U_{\ell f}^\dagger M^0_{\ell f} U_{\ell f}.$$  

(10)

Moreover,

$$M'^\dagger_{\ell f} M_{\ell f} = U_{\ell f}^\dagger M^0_{\ell f} U_{\ell f}.$$  

(11)

Turning to neutrinos in the TBM limit, we can write

$$U^\nu_\nu^0 M^0_{\nu f} M'^0_{\nu f} U^\nu_\nu = \text{diag.} (|m_{\nu 1}|^2, |m_{\nu 2}|^2, |m_{\nu 3}|^2)$$

(12)

with

$$U^\nu_\nu = \begin{pmatrix} \sqrt{2/3} & \sqrt{1/3} & 0 \\ -\sqrt{1/6} & \sqrt{1/3} & \sqrt{1/2} \\ \sqrt{1/6} & -\sqrt{1/3} & \sqrt{1/2} \end{pmatrix}.$$  

(13)

The normalized flavor eigenvectors $|\chi_i^{\nu0}\rangle$ of $M^\nu_{\nu f} M'^0_{\nu f}$ for $i = 1, 2, 3$ are the columns of $U^\nu_\nu$ while those in the mass basis are identical to the charged lepton ones. Thus

$$\begin{align*}
|\chi_1^{\nu0}\rangle &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},
|\chi_2^{\nu0}\rangle &= \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
|\chi_3^{\nu0}\rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.
\end{align*}$$

(14)

whereas

$$\begin{align*}
|\chi_1^{\nu0}\rangle_{\ell f} &= \begin{pmatrix} \sqrt{2/3} \\ -\sqrt{1/6} \\ \sqrt{1/6} \end{pmatrix},
|\chi_2^{\nu0}\rangle_{\ell f} &= \begin{pmatrix} \sqrt{1/3} \\ \sqrt{1/3} \\ \sqrt{1/2} \end{pmatrix},
|\chi_3^{\nu0}\rangle_{\ell f} &= \begin{pmatrix} 0 \\ \sqrt{1/3} \\ \sqrt{1/3} \end{pmatrix}.
\end{align*}$$

(15)

Once the perturbation is introduced, we have $M_{\ell f} = M^0_{\ell f} + M'^{\prime}_{\ell f}$, whereas $M'^0_{\nu f}$ obey the TBM conditions $\text{[14]}$ while $(M'^{\prime}_{\nu f})_{ij} \equiv \mu_{ij}$ violate them. Note that, unlike the real diagonal $M^0_{\nu f}$ and the general $M'^{\prime}_{\nu f}$, both $M^0_{\nu f}$ and $M'^{\prime}_{\nu f}$ have to be complex symmetric matrices in order to make the corresponding neutrinos Majorana particles.

We now expand the perturbed eigenstates for both charged leptons and neutrinos at the lowest order. We choose to use a compact notation covering both cases by introducing perturbation parameters $\epsilon_{\ell k}^{\nu \nu}$
(for $i, k = 1, 2, 3$). Thus we can write the $i$th first order perturbed eigenvectors of $M_{ij}^\dagger M_{ij}$ on one hand and of $M_{i\ell}^\dagger M_{i\ell}$ on the other as

$$|\chi_{i}^{\nu,\ell,\epsilon}_{f}\rangle = |\chi_{i}^{0,\nu,\ell,\epsilon}_{f}\rangle + \sum_{k \neq i} \epsilon_{ik}^{\nu,\ell,\epsilon} |\chi_{k}^{0,\nu,\ell,\epsilon}_{f}\rangle + O(\epsilon^2).$$

Two new quantities have been introduced in (16). They are defined by

$$\epsilon_{ik}^{\nu,\ell,\epsilon} = -\epsilon_{ik}^{\nu,\ell,\epsilon^*} = (|m_{\nu,\ell}\rangle^2 - |m_{\nu,ik}\rangle^2)^{-1} P_{ik},$$

(17)

$$P_{ik}^{\nu,\ell,\epsilon} = \langle \chi_{i}^{0,\nu,\ell,\epsilon} | M_{\nu,\ell}^{\dagger} M_{\nu,\ell}^{\dagger} + M_{\nu,\ell}^{\dagger} M_{\nu,\ell}^{\dagger} | \chi_{k}^{0,\nu,\ell,\epsilon}\rangle.$$

(18)

Note that (17) and (18) have been written in the mass basis utilizing the fact that $\epsilon_{ik}$, as well as $P_{ik}$, do not change from one basis to the other.

Turning to (16), we see that its LHS for $i = 1, 2, 3$ can be identified with three corresponding columns of $U_{\nu,\ell}$. Thus

$$U_{\nu,\ell} = (|\chi_{1}^{\nu,\ell,\epsilon}_{f}\rangle |\chi_{2}^{\nu,\ell,\epsilon}_{f}\rangle |\chi_{3}^{\nu,\ell,\epsilon}_{f}\rangle).$$

Neglecting $O(\epsilon^2)$ terms, it follows from (19) that

$$U_{\ell} = \begin{pmatrix} 1 & -\epsilon_{12} & -\epsilon_{13} \\ \epsilon_{12} & 1 & -\epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & 1 \end{pmatrix}$$

(20)

and

$$U_{\nu} = \begin{pmatrix} \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \epsilon_{12} & \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} \epsilon_{12} & -\sqrt{\frac{2}{3}} \epsilon_{13} - \sqrt{\frac{1}{3}} \epsilon_{23} \\ \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} \epsilon_{12} + \sqrt{\frac{1}{3}} \epsilon_{13} & \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \epsilon_{12} + \sqrt{\frac{1}{3}} \epsilon_{23} & \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \epsilon_{13} + \sqrt{\frac{1}{3}} \epsilon_{23} \\ \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \epsilon_{12} - \sqrt{\frac{1}{3}} \epsilon_{13} - \sqrt{\frac{1}{3}} \epsilon_{23} & \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} \epsilon_{12} - \sqrt{\frac{1}{3}} \epsilon_{13} + \sqrt{\frac{1}{3}} \epsilon_{23} & \sqrt{\frac{2}{3}} - \sqrt{\frac{1}{3}} \epsilon_{13} + \sqrt{\frac{1}{3}} \epsilon_{23} \end{pmatrix}.$$

(21)

Let us define the Majorana phase matrix

$$K \equiv \text{diag.} (1, e^{i\frac{\theta_{12}}{2}}, e^{i\frac{\theta_{13}}{2}}).$$

(22)

Then $U_{PMNS}K^{-1}$ can be written in the PDG convention [1] with $c_{ij} \equiv \cos \theta_{ij}$ and $s_{ij} \equiv \sin \theta_{ij}$ as

$$U_{PMNS}K^{-1} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{13}c_{23} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}s_{13} \end{pmatrix}.$$

(23)

We can now make the identification

$$U_{PMNS}K^{-1} = U_{\ell}^\dagger U_{\nu}$$

(24)

and work out the consequences from (20), (21) and (23).

4 Consequences of lowest order perturbation

Let us define $L \equiv U_{\ell}^\dagger U_{\nu}$ and $N \equiv U_{PMNS}K^{-1}$. The identification $L_{ij} = N_{ij}$ as per (24) leads to nine equations which are detailed in convenient combinations in the Appendix. Not all of these are independent, but they lead to four independent constraint conditions and three equations for the three

$$\begin{align*}
\text{(20)} & \quad \text{for two new quantities defined in (16).} \\
\text{(19)} & \quad \text{for the perturbed eigenvectors.} \\
\text{(20)} & \quad \text{for the Majorana phase matrix.} \\
\text{(21)} & \quad \text{for the matrix elements.} \\
\end{align*}$$
TBM-deviants $c_{12} - \sqrt{2/3}, c_{23} - s_{23}$ and $s_{13} e^{i\delta_{CP}}$. The constraint conditions follow from the fact that four of the elements of $N$ are real. They are given by

\[
\begin{align*}
\text{Im } \epsilon_{12}' &= O(\epsilon^2), \\
\text{Im } (\epsilon_{13}' - \sqrt{2} \epsilon_{23}') &= O(\epsilon^2), \\
\text{Im } \epsilon_{23}' &= O(\epsilon^2), \\
\text{Im } (\epsilon_{12}' - \epsilon_{13}') &= O(\epsilon^2).
\end{align*}
\]

Neglecting $O(\epsilon^2)$ terms, the three measurable TBM-deviants are linear in $\epsilon$ parameters and may be given as

\[
\begin{align*}
c_{12} - \sqrt{2/3} &= \sqrt{1/2} \left( \sqrt{1/3} - s_{12} \right) = \sqrt{1/3} \left( \epsilon_{12}' - \sqrt{1/6} \left( \epsilon_{12}' - \epsilon_{13}' \right) \right), \\
c_{23} - s_{23} &= -\sqrt{2/3} \left( \epsilon_{13}' - \sqrt{2} \epsilon_{23}' \right) = -\sqrt{2} \epsilon_{23}', \\
s_{13} e^{i\delta_{CP}} &= -\sqrt{1/3} \left( \sqrt{2} \epsilon_{13}' + \epsilon_{23}' \right) + \sqrt{1/2} \left( \epsilon_{12}' + \epsilon_{13}' \right).
\end{align*}
\]

The derivation of Eqs. (23)–(31) appears in the Appendix.

Because of (20), the real and imaginary parts of (31) enable us to write, modulo $O(\epsilon^2)$ terms, that

\[
\tan \delta_{CP} = \frac{3 \text{Im } \epsilon_{23}' - \sqrt{3/2} \text{Re } (\epsilon_{12}' + \epsilon_{13}')} {\text{Re } (\sqrt{2} \epsilon_{13}' + \epsilon_{23}') - \sqrt{3/2} \text{Re } (\epsilon_{12}' + \epsilon_{13}')}.
\]

The above equation may be recast in terms of the basis independent Jarlskog invariant $J$ which equals

\[
\text{Im } \left[ (U_t^\dagger U_\nu)_{e1}(U_t^\dagger U_\nu)_{\mu2}(U_t^\dagger U_\nu)_{\tau3} (U_t^\dagger U_\nu)_{\mu1} \right].
\]

We then have

\[
J = -\frac{1}{\sqrt{6}} \text{Im } [\epsilon_{23}' - \frac{1}{\sqrt{6}} (\epsilon_{12}' + \epsilon_{13}')] + O(\epsilon^2).
\]

Let us now explore, to the lowest order in $\epsilon$, the consequences of (17) and (18) by explicitly taking elements of the respective perturbing mass matrices for neutrinos and charged leptons. We take

\[
(M'_{ij})_{ij} = \mu_{ij}
\]

and

\[
(M'_{ij})_{ij} = (M'_{ij})_{ij} + O(\epsilon^2) = \lambda_{ij}
\]

with $\lambda_{ij}$ and $\mu_{ij} = \mu_{ji}$ as complex mass dimensional parameters naturally expected to be of the same order of magnitude. The identity of the charged lepton mass basis and flavor basis in the TBM limit makes the calculations in this case quite straightforward. From (17) and (18), we can easily derive

\[
\begin{align*}
\epsilon_{12}' &= (m_\mu^2 - m_\mu^2)^{-1} (m_\mu^2 \lambda_{21} + m_\mu^2 \lambda_{12}), \\
\epsilon_{23}' &= (m_\tau^2 - m_\mu^2)^{-1} (m_\tau^2 \lambda_{32} + m_\mu^2 \lambda_{23}), \\
\epsilon_{13}' &= (m_\mu^2 - m_\tau^2)^{-1} (m_\tau \lambda_{31} + m_\mu^2 \lambda_{13}).
\end{align*}
\]

We want to comment on the magnitudes of $\epsilon_{23}'$ and $\epsilon_{13}'$. In order for them to be large, the relevant $\lambda$ parameters would need to be of order $m_\tau$. That is in contradiction with our premise that nonzero charged lepton perturbation matrix elements cannot be very different in order of magnitude from those for neutrinos. Thus we expect that $|\epsilon_{23}'|$ and $|\epsilon_{13}'|$ to be quite small. Because of the strongly hierarchical
nature of charged lepton masses, (27) and (28) can be satisfied without unnatural cancellations only by
\(\lambda_{12}, \lambda_{21}, \lambda_{13}, \lambda_{31}, \lambda_{23}, \lambda_{32}\) all being real to order \(\epsilon\). One then automatically obtains that
\[
\text{Im } \epsilon'_{12} = O(\epsilon^2) = \text{Im } \epsilon'_{13}.
\] (39)

Feeding this information, we can simplify (32) and (33) to
\[
\tan \delta_{\text{CP}} = \frac{3 \text{ Im } \epsilon''_{13}}{\text{Re } (\sqrt{2} \epsilon'_{13} + \epsilon''_{23}) - \sqrt{3/2} \text{ Re } (\epsilon'_{12} + \epsilon'_{13})},
\] (40)
\[
J = -\frac{1}{\sqrt{6}} \text{ Im } \epsilon''_{23} + O(\epsilon^2)
\] (41)
respectively.

Turning to neutrinos next, the relevant off-diagonal elements of \(M'_\nu = U_0^T M'_\nu' U_0\) are
\[
(M'_\nu)_{12} = \frac{1}{3\sqrt{2}} (2\mu_{11} + \mu_{12} - \mu_{13} - \mu_{22} + 2\mu_{23} - \mu_{33}),
\] (42)
\[
(M'_\nu)_{23} = \frac{1}{\sqrt{6}} (\mu_{12} + \mu_{13} + \mu_{22} - \mu_{33}),
\] (43)
\[
(M'_\nu)_{13} = \frac{1}{\sqrt{3}} (\mu_{12} + \mu_{13} - \frac{1}{2}\mu_{22} + \frac{1}{2}\mu_{33}).
\] (44)

It is now convenient to define
\[
\Delta^0_{ij} \equiv |m^0_{\nu_i}|^2 - |m^0_{\nu_j}|^2,
\] (45)
\[
a^0_{ij} \equiv m^0_{\nu_i} \mp m^0_{\nu_j}.
\] (46)

Then we take (17) and (18) and successively consider the index combinations \(i = 1, k = 2\) and \(i = 2, k = 3\) as well as \(i = 1, k = 3\). Separating the real and imaginary parts and using (45) and (46), we obtain the following six equations
\[
2 \Delta^0_{12} \left( \frac{i \text{ Im } \epsilon''_{12}}{\text{Re } \epsilon'_{12}} \right) = a^+_{21} (M'_\nu)_{12} \mp \text{ c.c.},
\] (47)
\[
2 \Delta^0_{23} \left( \frac{i \text{ Im } \epsilon''_{23}}{\text{Re } \epsilon'_{23}} \right) = a^+_{32} (M'_\nu)_{23} \mp \text{ c.c.},
\] (48)
\[
2 \Delta^0_{13} \left( \frac{i \text{ Im } \epsilon''_{13}}{\text{Re } \epsilon'_{13}} \right) = a^+_{31} (M'_\nu)_{13} \mp \text{ c.c.}
\] (49)

Needless to add, order \(\epsilon^2\) terms have been neglected in deriving the above results.

5 Results and discussion

Eq. (49) has a simple consequence if we exclude unnatural cancellations. In conjunction with (29), it forces the combination of \(\mu_{ij}\), occurring in \((M'_\nu)_{12}\), i.e. \(2\mu_{11} + \mu_{12} - \mu_{13} - \mu_{22} + 2\mu_{23} - \mu_{33}\), to be real. It also implies that \(m^0_{\nu_2} - m^0_{\nu_3}\) is real, the latter forcing \(\alpha^0_{21}\) to be 0 or \(\pi\). However, our key observation follows from combining (48) and (49) with (26). That yields the equality
\[
\text{Im } \left[ (m^0_{\nu_3} - m^0_{\nu_2}) (\mu_{12} + \mu_{13} + \mu_{22} - \mu_{33}) \right] = \text{Im } \left[ (m^0_{\nu_3} - m^0_{\nu_1}) (\mu_{12} + \mu_{13} + \frac{1}{2}\mu_{22} - \frac{1}{2}\mu_{33}) \right].
\] (50)
There are two ways to satisfy (50) without any unnatural cancellation, at least one of which is obligatory. Either we must have option (1), namely that \( m_{\nu_1}^0 = m_{\nu_2}^0 \) and \( \mu_{22} = \mu_{33} \) or there must be option (2), namely that \( m_{\nu_3}^0, m_{\nu_2}^0, m_{\nu_1}^0, \mu_{12} + \mu_{13} \) and \( \mu_{22} = \mu_{33} \) are all real so that each side of (50) vanishes. Take (1) first. Since \( m_{\nu_1}^0 | m_{\nu_2}^0 | m_{\nu_3}^0 | m_{\nu_4}^0 \) by choice, we now have \( |m_{\nu_1}^0| = |m_{\nu_2}^0| \) and \( \alpha_{21} = 0 \). Further, with \( \mu_{22} = \mu_{33} \), the implication from from (18) and (19) is that \( \sqrt{2} \Re \epsilon_{\nu_3}^{\alpha} = \Re \epsilon_{\nu_3}^{\alpha} + O(\epsilon^2) \). Consequently, it follows from (27) and (30) that \( s_{23} - s_{23} = -\sqrt{2} \epsilon_{23}^\alpha + O(\epsilon^2) \) which leads to the result \( |s_{23} - s_{23}| = |\epsilon_{23}^\alpha| + |O(\epsilon^2)| << |O(\epsilon^2)| \). The strong inequality in the last step has been based on the discussion which followed (38). Thus option (1) says that the magnitude of any deviation from maximal atmospheric mixing, being of order \( |\epsilon_{23}^\alpha| \) and small, will be below the sensitivity of the forthcoming experiments. Let us turn to alternative (2). Now we have \( \alpha_{23}^\alpha = \alpha_{33}^\alpha = 0 \). Further, by use of (35), we derive that \( \Im \epsilon_{23}^\alpha = O(\epsilon^2) \). As a result, by virtue of (10) as well as (11), one concludes that \( s_{13} \sin \delta_{CP} = O(\epsilon^2) \) and \( J = O(\epsilon^2) \), so that both would be unobservably small in experiments planned for the near future. The consequence of option (2) is that CP-violation in neutrino oscillations would not be seen in those experiments.

It should be noted that in option (1) one needs to use degenerate perturbation theory \( \text{25, 26, 27} \) with respect to the TBM limit for the 1-2 sector of neutrinos. In the latter case, the perturbation splits the 1-2 mass degeneracy and generates the solar neutrino mass difference with \( m_{\nu_1}^0 = m_{\nu_2}^0 = m_{\nu_3}^0 \). One then obtains

\[
\Delta_{21} = \sqrt{(p_{11}^\nu - p_{22}^\nu)^2 + p_{12}^\nu^2},
\]

as calculated using (13). Additionally, to order \( \epsilon^\nu \) and \( \epsilon^\ell \), \( s_{13} \epsilon^{i\delta_{CP}} \) can be obtained in terms of \( m_{\nu_3}^0, m_{\nu_2}^0, \bar{m}_{\nu_1}^0, m_{\nu_1}^0, n_{\nu_1}^0, n_{\nu_2}^0 \) as well as the \( \mu \) and \( \lambda \) parameters by using (31) and employing the expressions for the \( \epsilon \) parameters. We choose not write that full expression here.

Some comments on the issue of unnatural cancellations are in order. The TBM breaking terms in the mass matrix of charged leptons do not leave any residual symmetry except possibly some rephasing invariants. As stated earlier, given that \( m_\ell << m_\mu << m_\tau \), the cancellations required to avoid the reality condition on all \( \lambda_{ij} \) (for \( i \neq j \)) cannot be effected by any such invariance. In the neutrino case, there generally is a residual \( Z_2 \) symmetry \( \text{7, 8, 9, 10, 11, 12, 13} \) after TBM is broken. Even such a discrete symmetry does not generally enable one to obtain the concerned complicated equality between specific combinations of TBM violating perturbation parameters, TBM invariant neutrino masses as well as Majorana phases. We feel, therefore, that our argument ruling out such cancellations is sound and our conclusions are reliable.

Let us finally remark on the relevance of our result to planned experiments at the proton beam intensity frontier. The determination of the sign of the neutrino mass ordering is one of their aims. It is noteworthy that the constraint on neutrino mixing parameters, derived by us, is independent of this issue just as the consequences of exact TBM are. Those experiments will also investigate neutrino mixing parameters. A combination \( \text{25, 29} \) of data from the ongoing and upcoming runs of T2K and NO\( \nu \)A experiments would probe \( |s_{23} - \frac{1}{\sqrt{2}}| \) from the conversion probability \( P(\nu_\mu \rightarrow \nu_e) \). Now, in case a nonzero value of that quantity is measured, being of magnitude comparable in percentage terms to (100 \( s_{13} \))\% of the maximal value of \( s_{23} \), our condition (2) would hold and predict a non-observation of CP-violation in neutrino oscillations from the above data. Contrariwise, the failure to measure any such deviation outside error bars would mean that our condition (1) would operate with \( s_{13} \sin \delta_{CP} = O(\epsilon^\nu), J = O(\epsilon^\nu) \) permitted; that would bolster the hope of detecting CP non-conservation for oscillating neutrinos from the difference in oscillation probabilities \( P(\nu_\mu \rightarrow \nu_e) - P(\nu_\mu \rightarrow \nu_\tau) \). The latter would be good news not only for a combined analysis of data from forthcoming runs of \( \text{30} \) of T2K and NO\( \nu \)A but also for future experiments with superbeams, such as LBNE \( \text{31, LBNO 32, 33} \) or a neutrino factory at 10 GeV \( \text{34} \). Current evidence from global analyses, either for a non-maximal \( \theta_{23} \) or a nonzero \( \sin \delta_{CP} / J \) is by no means robust and an experimental resolution of these two issues is urgently called for.
6 Concluding summary

In this paper we have considered general perturbations at the lowest order to hermitian squared mass matrices $M_{\ell f}^2 M_{\ell f}$ and $M_{\nu f}^2 M_{\nu f}$ respectively for charged leptons and neutrinos in the flavor basis of each and away from their TBM limits by carefully taking into account the unitary relation between the mass basis and the flavor basis. We have utilized the fact that columns of the said unitary matrix are the perturbed eigenstates. We have derived linear expressions for the three measurable TBM-deviants in terms of the dimensionless coefficients that appear in the perturbed charged lepton and neutrino eigenstates. We have further derived four independent constraints on the imaginary parts of the latter from the requirement that four of the elements of $U_{\ell f}^\dagger U_{\nu f}$ have to be real. With the plausible assumptions of the mixing caused by the strongly mass hierarchical charged leptons being significantly smaller than that due to neutrinos and no unnatural cancellations, we have derived a result, forcing one of two possibilities, which should be testable in the foreseeable future. This main result of ours can be stated succinctly in the language of mathematical logic. Proposition A: an accurate description of neutrino mixing is given by lowest order perturbed tribimaximality without unnatural cancellations and with the mixing from the strongly mass hierarchical charged leptons being significantly smaller than that from neutrinos. Proposition B: $|s_{23} - \sqrt{1/2}| = O(\epsilon^\nu)$. Proposition C: $s_{13} \sin \delta_{CP}/J = O(\epsilon^\nu)$. Then $A \cap (B \cup C) = \emptyset$.

Appendix: Derivation of mixing constraints

Neglecting $O(\epsilon^2)$ terms, we may write,

$$ |\psi_{1f}^{0\nu}| = |\psi_{1f}^{0\nu}| + \epsilon_{12} |\psi_{2f}^{0\nu}| + \epsilon_{13} |\psi_{3f}^{0\nu}|, \quad (A-1) $$

$$ |\psi_{2f}^{0\nu}| = -\epsilon_{12}^{*} |\psi_{1f}^{0\nu}| + |\psi_{2f}^{0\nu}| + \epsilon_{23} |\psi_{3f}^{0\nu}|, \quad (A-2) $$

$$ |\psi_{3f}^{0\nu}| = -\epsilon_{12}^{*} |\psi_{1f}^{0\nu}| - \epsilon_{23}^{*} |\psi_{2f}^{0\nu}| + |\psi_{3f}^{0\nu}|, \quad (A-3) $$

where

$$ |\psi_{1f}^{0\nu}| = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ -\sqrt{\frac{1}{2}} \end{pmatrix}, |\psi_{2f}^{0\nu}| = \begin{pmatrix} \sqrt{\frac{1}{2}} \\ 0 \end{pmatrix}, |\psi_{3f}^{0\nu}| = \begin{pmatrix} 0 \\ \sqrt{\frac{1}{2}} \end{pmatrix}. \quad (A-4) $$

and

$$ |\psi_{1f}^{0\ell}| = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, |\psi_{2f}^{0\ell}| = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |\psi_{3f}^{0\ell}| = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (A-5) $$

By using $U_{\ell} = (|\psi_{1f}^{0\nu}| |\psi_{2f}^{0\nu}| |\psi_{3f}^{0\nu}|)$ and $U_{\nu} = (|\psi_{1f}^{0\ell}| |\psi_{2f}^{0\ell}| |\psi_{3f}^{0\ell}|)$, one is led to the respective expressions for $U_{\ell}$ and $U_{\nu}$, as given in the text. If we define $L \equiv U_{\ell}^\dagger U_{\nu}$, then neglecting $O(\epsilon^2)$ terms, the
nine elements of the $L$ matrix are

$$L_{11} = \sqrt{\frac{2}{3}} + \sqrt{\frac{1}{3}} \epsilon_{12} - \sqrt{\frac{1}{2}} \epsilon_{12} + \sqrt{\frac{1}{6}} \epsilon_{13}, \quad (A-6)$$

$$L_{12} = \sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}} \epsilon_{12} + \sqrt{\frac{1}{2}} \epsilon_{12} - \sqrt{\frac{1}{6}} \epsilon_{13}, \quad (A-7)$$

$$L_{13} = -\sqrt{\frac{2}{3}} \epsilon_{13} - \sqrt{\frac{1}{3}} \epsilon_{13} + \sqrt{\frac{1}{2}} \epsilon_{12} + \sqrt{\frac{1}{6}} \epsilon_{13}, \quad (A-8)$$

$$L_{21} = -\sqrt{\frac{1}{6}} + \sqrt{\frac{1}{3}} \epsilon_{12} + \sqrt{\frac{1}{2}} \epsilon_{13} - \sqrt{\frac{2}{3}} \epsilon_{12} + \sqrt{\frac{1}{6}} \epsilon_{23}, \quad (A-9)$$

$$L_{22} = \sqrt{\frac{1}{3}} + \sqrt{\frac{1}{6}} \epsilon_{12} + \sqrt{\frac{1}{2}} \epsilon_{23} - \sqrt{\frac{2}{3}} \epsilon_{12} - \sqrt{\frac{1}{6}} \epsilon_{23}, \quad (A-10)$$

$$L_{23} = \sqrt{\frac{1}{2}} + \sqrt{\frac{1}{6}} \epsilon_{13} - \sqrt{\frac{1}{3}} \epsilon_{23} + \sqrt{\frac{1}{2}} \epsilon_{23}, \quad (A-11)$$

$$L_{31} = \sqrt{\frac{1}{6}} - \sqrt{\frac{1}{3}} \epsilon_{12} + \sqrt{\frac{1}{2}} \epsilon_{23} - \sqrt{\frac{2}{3}} \epsilon_{13} + \sqrt{\frac{1}{6}} \epsilon_{23}, \quad (A-12)$$

$$L_{32} = -\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{6}} \epsilon_{12} + \sqrt{\frac{1}{2}} \epsilon_{23} - \sqrt{\frac{2}{3}} \epsilon_{23} - \sqrt{\frac{1}{6}} \epsilon_{23}, \quad (A-13)$$

$$L_{33} = \sqrt{\frac{1}{2}} - \sqrt{\frac{1}{6}} \epsilon_{13} + \sqrt{\frac{1}{3}} \epsilon_{23} - \sqrt{\frac{1}{2}} \epsilon_{23}. \quad (A-14)$$

Similarly, defining $N \equiv U_{PMNS} K^{-1}$ and again neglecting $O(\epsilon^2)$ terms

$$N = \begin{pmatrix}
  c_{12} & s_{12} & s_{13} e^{-i\delta_{CP}} \\
  -s_{12} c_{23} - \sqrt{\frac{1}{3}} s_{13} e^{i\delta_{CP}} & c_{12} c_{23} - \sqrt{\frac{1}{6}} s_{13} e^{i\delta_{CP}} & s_{23} \\
  s_{12} s_{23} - \sqrt{\frac{1}{3}} s_{13} e^{i\delta_{CP}} & -c_{12} s_{23} - \sqrt{\frac{1}{6}} s_{13} e^{i\delta_{CP}} & c_{23}
\end{pmatrix}, \quad (A-15)$$

Expanding in $\epsilon$, the relations $\sqrt{2} c_{12} + s_{12} = \sqrt{3} + O(\epsilon^2)$ and $c_{23} + s_{23} = \sqrt{2} + O(\epsilon^2)$ are automatic. The equality $L = N$ leads to the mixing constraint relations. Specifically, the identification of elements or their combinations

$$L_{11} = N_{11}, \quad L_{21} - L_{31} = N_{21} - N_{31}, \quad L_{21} + L_{31} = N_{21} + N_{31},$$

$$L_{12} = N_{12}, \quad L_{22} + L_{32} = N_{22} + N_{32}, \quad L_{22} - L_{32} = N_{22} - N_{32},$$

$$L_{13} = N_{13}, \quad L_{33} + L_{23} = N_{33} + N_{23}, \quad L_{33} - L_{23} = N_{33} - N_{23},$$
neglecting $O(\epsilon^2)$ terms, lead respectively to the equations

\begin{align}
  c_{12} - \frac{\sqrt{2}}{3} = \frac{1}{\sqrt{2}} \left( \sqrt{\frac{1}{3}} - s_{12} \right) = \sqrt{\frac{1}{3}} \epsilon_{12}^\nu - \sqrt{\frac{1}{6}} (\epsilon_{12}^\nu - \epsilon_{13}^\nu), \quad & (A-16) \\
  -\sqrt{\frac{1}{3}} (c_{23} - s_{23}) - \frac{2}{\sqrt{3}} s_{13} \epsilon^{\delta CP} = \sqrt{2} \epsilon_{13}^\nu - \sqrt{\frac{1}{2}} (\epsilon_{12}^\nu + \epsilon_{13}^\nu) + \sqrt{\frac{1}{6}} (\epsilon_{23}^\nu + \epsilon_{23}^\nu), \quad & (A-17) \\
  -\sqrt{2} s_{12} = -2 \sqrt{\frac{1}{6}} + 2 \sqrt{\frac{1}{3}} \epsilon_{12}^\nu - \sqrt{\frac{1}{3}} (\epsilon_{12}^\nu - \epsilon_{13}^\nu) + \sqrt{\frac{1}{6}} (\epsilon_{12}^\nu - \epsilon_{13}^\nu), \quad & (A-18) \\
  s_{12} = \sqrt{\frac{1}{3}} - \sqrt{\frac{2}{3}} \epsilon_{12}^\nu + \sqrt{\frac{1}{3}} (\epsilon_{12}^\nu - \epsilon_{13}^\nu), \quad & (A-19) \\
  \sqrt{\frac{2}{3}} (c_{23} - s_{23} - s_{13} \epsilon^{\delta CP}) = \sqrt{2} \epsilon_{23}^\nu - \sqrt{\frac{1}{3}} (\epsilon_{12}^\nu + \epsilon_{13}^\nu + \epsilon_{23}^\nu + \epsilon_{23}^\nu), \quad & (A-20) \\
  \sqrt{2} c_{12} = 2 \sqrt{\frac{1}{3}} + 2 \sqrt{\frac{1}{6}} \epsilon_{12}^\nu - \sqrt{\frac{1}{3}} (\epsilon_{23}^\nu - \epsilon_{23}^\nu) - \sqrt{\frac{1}{3}} (\epsilon_{12}^\nu - \epsilon_{13}^\nu), \quad & (A-21) \\
  s_{13} \epsilon^{\delta CP} = -\sqrt{\frac{1}{3}} (\sqrt{2} \epsilon_{13}^\nu + \epsilon_{23}^\nu) + \sqrt{\frac{1}{2}} (\epsilon_{12}^\nu + \epsilon_{13}^\nu) \quad & (A-22) \\
  c_{23} + s_{23} = \sqrt{2} + \frac{1}{\sqrt{2}} (\epsilon_{23}^\nu - \epsilon_{23}^\nu), \quad & (A-23) \\
  c_{23} - s_{23} = -\sqrt{\frac{2}{3}} \epsilon_{13}^\nu - \sqrt{\frac{1}{2}} (\epsilon_{23}^\nu + \epsilon_{23}^\nu) + \frac{2}{\sqrt{3}} \epsilon_{23}^\nu. \quad & (A-24)
\end{align}

Eq. (27) is a direct consequence of (A-23). Eq. (26) follows from (A-20) and (A-22), while Eq. (25) follows from (A-19) and (A-21). Now Eq. (28) obtains from (A-21), whereas Eq. (29) is just a rewritten form of (A-16) with the input of Eq. (28). Eq. (30) follows from (A-17) and (A-20). Finally, Eq. (31) is nothing but (A-22).

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