On the action principle
in quantum field theory

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Short title: Action principle in QFT

Basic ideas: → July 21–22, 2001
Began: → August 8, 2001
Ended: → August 14, 2001
Initial typeset: → August 15–August 20, 2001
Last update: → March 30, 2002
Produced: → January 2, 2022

LANL xxx archive server E-print No.: hep-th/0204003

Subject Classes:
Quantum field theory

2000 MSC numbers: 81P99, 81Q99, 81T99

2001 PACS numbers: 02.90.+p, 03.70.+k, 11.10.Ef

Key-Words:
Quantum field theory, Action principle, Schwinger’s action principle
Action principles in quantum field theory, Conserved quantities
Operators of conserved quantities in quantum field theory
Energy-momentum operator, Current operator
Spin angular momentum density operator
Euler-Lagrange equations, Field equations
Derivative with respect to operator argument

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Abstract

An analysis of the Schwinger’s action principle in Lagrangian quantum field theory is presented. A solution of a problem contained in it is proposed via a suitable definition of a derivative with respect to operator variables. This results in a preservation of Euler-Lagrange equations and a change in the operator structure of conserved quantities. Besides, it entails certain relation between the field operators and their variations (which is identically valid for some fields, e.g. for the free ones). The general theory is illustrated on a number of particular examples.
1. Introduction

The paper deals with the following problems in Lagrangian quantum field theory: meaning of derivatives with respect to operator argument, order of the operators in the structure of conserved quantities, and commutation of the variations of the field operators in Schwinger’s action principle. These problems are reviewed, analyzed and their solution is proposed.

The first two of the above problems are discussed in Sect. 2. Sect. 3 reviews the Schwinger’s action principle and some its consequences. Special attention is paid to the problem with the commutativity of the fields’ variations and the field operators or/and their partial derivatives. It has been notice at first by Schwinger in his original work [1] but later, in serious books like [2], it has been forgotten. A suitable solution of that problem is proposed in Sect. 4 by giving a rigorous meaning of a derivative of operator-valued function of operator arguments with respect to such an argument. It entails preservation of (operator) Euler-Lagrange equations for the field operators and a unique definition of the operators of conserved quantities. A new moment is that the variations of the field operators cannot be completely arbitrary in the general case (e.g. for some interacting fields) as they should satisfy some conditions derived in this work. Sect 5 illustrates the general theory of Sect. 4 with particular examples (free neutral or charged scalar field, (self-)interacting scalar fields, free spinor field, and system of fields described via quadratic Lagrangian). It is presented an example of a Lagrangian, describing free (or with some self-interaction) spinor field, for which the (classical operator) Euler-Lagrange equations do not exist in a sense that they are identities, like 0 = 0. Regardless of that fact, this Lagrangian entails completely reasonable field equations. The main results of the work are summarized in Sect. 6.

In the Lagrangians we consider is not supposed normal ordering (of the products of creation and annihilation operators). Besides, no (anti)commutation (or paracommutation) relations are supposed to be fulfilled. But the results obtained are, of course, valid and if normal ordering of products is used some kind of (anti)commutation (or paracommutation) relations are taken into account.

2. Problems with the equations of motion and with conserved quantities

Suppose a system of classical fields $\varphi_i(x)$, $i = 1, \ldots, n \in \mathbb{N}$, over the Minkowski spacetime $M$, $x \in M$, is described via a Lagrangian $L$ depending on them and their first partial derivatives $\partial_\mu \varphi_i(x) = \frac{\partial \varphi_i(x)}{\partial x^\mu}$, $\{x^\mu\}$ being the (local) coordinates of $x \in M$, i.e. $L = L(\varphi_j(x), \partial_\nu \varphi_i(x))$. Here and henceforth the Greek indices $\mu, \nu, \ldots$ run from 0 to dim $M - 1 = 3$ and the Latin indices $i, j, \ldots$ run from 1 to some integer $n$. The equations of motion for $\varphi_i(x)$, known as the Euler-Lagrange equations, are

$$\frac{\partial L}{\partial \varphi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i(x))} \right) = 0 \quad (2.1)$$

and are derived from the variational principle of stationary action, known as the action principle (see, e.g. [3, § 1], [4, § 67], [2, pp. 19–20]).

The (first) Noether theorem [3, § 2] says that, if the action’s variation is invariant under a $C^1$ transformations

$$x \mapsto x^{\omega} = x^{\omega}(x) \quad x^{\omega}|_{\omega=0} = x \quad \omega = (\omega^{(1)}, \ldots, \omega^{(s)})$$

$$\varphi_i(x) \mapsto \varphi_i^{\omega}(x^{\omega}) \quad \varphi_i^{\omega}(x^{\omega})|_{\omega=0} = \varphi_i(x) \quad (2.2)$$

In this paper the Einstein’s summation convention over indices appearing twice on different levels is assumed over the whole range of their values.
depending on $s \in \mathbb{N}$ independent real parameters $\omega^{(1)}, \ldots, \omega^{(s)}$, then the quantities

$$\theta_{(a)}^{\mu}(x) := -\pi^{i\mu} \left\{ \frac{\partial \phi_{(a)}^{i\nu}(x^{(a)})}{\partial \omega^{(a)}} \bigg|_{\omega=0} - \left( \partial_{\nu} \phi_{(a)}(x) \right) \frac{\partial x^{\omega \nu}}{\partial \omega^{(a)}} \bigg|_{\omega=0} \right\} - L(x) \frac{\partial \phi_{(a)}^{\omega \mu}}{\partial \omega^{(a)}} \bigg|_{\omega=0}, \quad (2.3)$$

where $a = 1, \ldots, s$ and

$$\pi^{i\mu} := \frac{\partial L}{\partial \left( \partial_{\mu} \phi_{(a)}(x) \right)}, \quad (2.4)$$

are conserved in a sense that

$$\partial_{\mu} \theta_{(a)}^{\mu}(x) = 0. \quad (2.5)$$

In particular, the invariance with respect to spacetime translations, i.e. $x \mapsto x^b = x + b$, with $b \in M$, and $\varphi_i(x) \mapsto \varphi_i(x)$, leads to the conservation of the energy-momentum tensor:

$$T^{\mu\nu}(x) := \pi^{i\mu}(x) \partial^{i\nu} \varphi_i(x) - L(x) \eta^{\mu\nu} \quad (2.6)$$

$$\partial_{\nu} T^{\mu\nu}(x) = 0, \quad (2.7)$$

where $\eta^{\mu\nu}$ is the Lorentz metric tensor of $M$ with signature $(+ - - -)$ and the spacetime indices are raised (lowered) by $\eta^{\mu\nu}$ (by the inverse tensor $\eta_{\mu\nu}$ of $\eta^{\mu\nu}$). Analogously, the invariance relative to constant phase transformations, viz. $x \mapsto x$ and $\varphi_i(x) \mapsto e^{i\omega^{(a)} \lambda} \varphi_i(x)$ where $q = const$, $\lambda$ is a real parameter, $h$ is the Planck’s constant (divided by $2\pi$), and $c$ is the velocity of light in vacuum, implies the conservation of the corresponding current:

$$J_{\mu}(x) := \frac{q}{\hbar c} \sum_{i} \varepsilon(\varphi_i) \pi_{i\mu}^{(a)}(x) \varphi_i(x) \quad (2.8)$$

$$\partial^{\mu} J_{\mu} = 0 \quad (2.9)$$

where $\varepsilon(\varphi_i) = 0$ if $\varphi_i(x)$ is real, $\varepsilon(\varphi_i) = +1$ if $\varphi_i(x)$ is complex, and $\varepsilon(\varphi_j) = -1$ if $\varphi_j(x)$ is the complex conjugate to $\varphi_i(x)$.

Below we shall be interested in the quantum case, when the fields $\varphi_i(x)$ become linear operator depending on $x \in M$ and acting on system’s Hilbert space $F$ of states. The above scheme is repeated *mutatis mutandis* in Heisenberg picture of motion in (canonical) quantum field theory, when the fields $\varphi_i(x)$ are spacetime dependent and the state vectors are spacetime independent. Details of this procedure will be given in Sect. [3] below. However, there are three related problems which should find suitable answers:

1. How the quantum Lagrangian $L$ should be defined? For example, if a quantum system has a classical analogue, can we simply replace in the classical Lagrangian the corresponding quantum operators?
2. What is the meaning of the derivative operators $\frac{\partial}{\partial \varphi_i(x)}$ and $\frac{\partial}{\partial \left( \partial_{\nu} \varphi_i(x) \right)}$, appearing in, e.g., the Euler-Lagrange equations, when $\varphi_i(x)$ is operator-valued, not classical, functions?
3. If the previous two problems are satisfactory (well) and uniquely solved, how should be defined the conserved quantities (2.3) in the quantum case? For instance, can we write the energy-momentum operator as

$$T^{\mu\nu}(x) := \pi^{i\mu}(x) \circ \left( \partial^{\nu} \varphi_i(x) \right) - L(x) \eta^{\mu\nu}? \quad (2.10)$$

Here $\circ$ is the sign of mappings/operators composition/product and all quantities are the operator analogues of the corresponding classical ones.

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2 It is a convention whether to a complex field or to its complex conjugate to be assigned the value +1 of the function $\varepsilon$. 
Partially the nature of these problems is in the fact that, generally, the field operators \( \varphi_i(x) \) do not commute. So, the order in which the field operators or functions of them appear in some composition (product) is significant, contrary to the classical case.

The solution of the first problem for the known fields, free or not, has been found a long time ago \[ \mathbb{I} \]. The basic requirements for \( L \) being that it should be a Hermitian operator which is invariant under Lorentz/Poincaré transformations and other symmetries of the system, if any. Besides, if a quantum system has a classical analogue, the classical Lagrangian should be equal to the quantum one when in the latter the field operators are replaced with the corresponding classical fields.

A posteriori there can be different solutions of the second problem. However, a priori there is a simple rule silently followed in the literature \[ \mathbb{I} \mathbb{I} \]. According to it, one replaces in the quantum Lagrangian operator the field operators \( \varphi_i(x) \) (and their partial derivatives) with classical fields \( \varphi^c_i(x) \) and the composition of mappings sign with multiplication sign in such a way that the order of the (quantum) fields to be preserved. Then, from the so-obtained Lagrangian function \( L^c \) are calculated the derivatives \( \frac{\partial L^c}{\partial \varphi^c_i(x)} \) and \( \frac{\partial L^c}{\partial (\partial_\mu \varphi^c_i(x))} \) by preserving the order of all fields and their derivatives. At the end, one replaces in \( \frac{\partial L^c}{\partial \varphi^c_i(x)} \) and \( \frac{\partial L^c}{\partial (\partial_\mu \varphi^c_i(x))} \) the fields \( \varphi^c_i(x) \) with the field operators \( \varphi_i(x) \) and the multiplication of fields with compositions of the corresponding operators. In short, all this means that we differentiate a quantum Lagrangian with respect to its operator arguments by the same rules as in the classical case with the only addition that one should always retain the initial order of all operators \[ \mathbb{I} \mathbb{II} \ § 2 \].

Following this procedure, one should keep in mind that a change of the order of the operators in the initial Lagrangian may result in different derivatives of it even if the Lagrangian is not changed as an operator.\[ 4 \]

When one analyzes the third of the afore presented problems, there are two guiding principles: the conserved operators \( \theta^\mu_{(a)} \) must be Hermitian and, if a system has a classical analogue, these operators should reduce to the corresponding classical conserved fields \[ \mathbb{I} \mathbb{II} \] when the field operators are replace with the corresponding to them classical fields and the composition of operators is replaced by the multiplication of (classical) fields. However, these guidelines are not enough for the explicit determination of the conserved operators and one should ‘guess’ their functional form; the result can be justified or rejected a posteriori by examining the consequences of the model hypothesis.\[ 4 \]

For instance, the straightforward transferring of (2.6) into the quantum region results in \[ \mathbb{II} \mathbb{II} \] but this operator is, generally non-Hermitian. As a working hypothesis, one may assume a ‘Hermitian symmetrization’ of (2.10), viz.

\[
T_{\mu\nu} = \frac{1}{2} \{ \pi^i_\mu(x) \circ (\partial_\nu \varphi_i(x)) + (\partial_\nu \varphi^+_i(x)) \circ (\pi^i_\mu(x)) \} - \eta_{\mu\nu} L(x) \tag{2.11}
\]

where the dagger, “\( \dagger \)”, denotes Hermitian conjugation of operators. As \( L^\dagger = L \), the last expression for energy-momentum operator operator satisfies the above-written requirements.

Similar is the situation with the current operator. Prima facie one may write (cf. (2.8))

\[
J_\mu = \frac{q}{i\hbar c} \sum_i \varepsilon(\varphi_i) \pi^+_i \circ \varphi_i \tag{2.12}
\]

\[ 3 \] If \( A \) and \( B \) are operators, the above rule implies \( \frac{\partial}{\partial x}(A \circ B) = \frac{\partial}{\partial x}(B \circ A) = B \). So, if \( A \) and \( B \) anticommute, i.e. \( A \circ B = -B \circ A \), we have \( \frac{\partial}{\partial x}(A \circ B) = B \) and \( \frac{\partial}{\partial x}(-B \circ A) = -B \). Consequently, the derivative relative to non-commuting operator argument has a ‘memory’ for the place (left or right in our example) where the arguments have been situated before the differentiation. This phenomenon will find natural explanation in Sect. \[ 4 \] in particular, see remark \[ 4 \].

\[ 4 \] Since in quantum field theory are important the constant operators \( C_{(a)} := \int_\sigma \theta^\mu_{(a)} \ d\sigma_\mu \), the integration being along some 3-dimensional spacelike surface \( \sigma \), sometimes different definitions of \( \theta^\mu_{(a)} \) may result in identical operators \( C_{(a)} \), even if different Lagrangians are employed.
but, generally, this expression is not Hermitian, \( J^\dagger_\mu \neq J_\mu \). As a working hypothesis, a ‘Hermitian symmetrization’ may be assumed (note, \( \epsilon(\varphi^\dagger_i) = -\epsilon(\varphi_i) \)):

\[
J_\mu = \frac{q}{i\hbar c} \sum_i \epsilon(\varphi_i) \{ \pi^i_\mu \circ \varphi_i - \varphi^i_\mu \circ (\pi^i_\mu)^\dagger \}. \quad (2.13)
\]

A partial discussion of the above problems with (energy-)momentum and (current or) charge operator can be found in [6].

3. Schwinger’s action principle (review and problems)

The particular variant of the variation action principle, adapted to the needs of quantum field theory, is known as the Schwinger’s action principle. Its description can be found, for instance, in [2, sec. 2.1] or in the original paper [1] (see also [7]). The purpose of the present section is a concise summary of this method, some of its consequences and problems it contains. For details, the reader is referred to [2, sec. 2.1], from where the below-presented resumé of Schwinger’s action principle is extracted.

Let there be given a system of quantum fields represented via linear field operators \( \varphi_i(x) \). Let \( L = L(x) = L(\varphi_i(x), \partial_\mu \varphi_i(x)) \) be the Lagrangian (density) operator of the system. It is supposed to depend only on the field operators and their first partial derivatives. Let \( \sigma_1 \) and \( \sigma_2 \) be two 3-dimensional spacelike surfaces and \( R \) be the 4-dimensional region (submanifold) bounded by them. The action operator is then defined by

\[
W := \frac{1}{c} \int_R L(x) \, d^4x =: \frac{1}{c} \int_{\sigma_2}^{\sigma_1} L(x) \, d^4x. \quad (3.1)
\]

Consider the (infinitesimal) transformations

\[
\begin{align*}
x^\mu &\mapsto x'^\mu = x^\mu + \delta x^\mu \\
\varphi_i(x) &\mapsto \varphi'_i(x) + \delta_0 \varphi_i(x) \quad (3.2b)
\end{align*}
\]

as a result of a change of a spacetime point \( x \) and field operator \( \varphi_i(x) \) when a transition to a new reference frame is made; in (3.2a) the symbol \( x \) in \( \varphi_i(x) \) refers to a point in the new frame. If \( x \in \sigma \) for some spacelike surface \( \sigma \), it is supposed that the infinitesimal change \( \delta_0 \varphi_i(x) \) to be generated by some generator \( F[\sigma] \) which is operator-valued functional of \( \sigma \), i.e.

\[
\delta_0 \varphi_i(x) = i\hbar [F[\sigma], \varphi_i(x)]_{\pm} \quad (3.3)
\]

where \([A, B]_{\pm} := A \circ B \pm B \circ A\) for operators \( A \) and \( B \).

The Schwinger’s action principle postulates that, if (3.2) induces the change \( W \mapsto W + \delta W \) of the action integral (3.1), then the infinitesimal change \( \delta W \) of the action integral is a difference of two surface integrals and

\[
\delta W = F[\sigma_2] - F[\sigma_1]. \quad (3.4)
\]

To work out consequences of (3.4), we notice that the variation \( \delta W \) is due to independent effects of the variations \( \delta_0 \varphi_i(x) \) of the field operators and the change \( R \mapsto R' \) of the integration region as a result of the change (3.2a) of the points of its boundary. So, neglecting second and higher order terms in the variations and applying (3.1), we can write

\[
\delta W = \frac{1}{c} \int_R \left\{ \delta_0 L + L \frac{\partial (\delta x^\mu)}{\partial x^\mu} \right\} \, d^4x \quad (3.5)
\]
where
\[ \delta_0 L := L(\varphi_i(x) + \delta_0 \varphi_i(x), \partial_\mu \varphi_i(x) + \partial_\mu (\delta_0 \varphi_i(x))) - L(\varphi_i(x), \partial_\mu \varphi_i(x)) \] (3.6)

is the variation of the Lagrangian (operator). Expanding the first term in (3.4) into Taylor series and neglecting second and higher order terms, we get
\[ \delta_0 L = \frac{\partial L}{\partial \varphi_i(x)} \circ \delta_0 \varphi_i(x) + \frac{\partial L}{\partial (\partial_\mu \varphi_i(x))} \circ \delta_0 (\partial_\mu \varphi_i(x)). \] (3.7)

Here the derivatives are understood as described in Sect. 2.

**Remark 3.1.** We have emphasized the last phrase because the transition from (3.6) to (3.7) is generally incorrect and wrong, if ‘arbitrary’ variations \(\delta_0 \varphi_i(x)\) are considered. This will be explained at length in Sect. 4. At this point, we recall only the Schwinger’s remark [1, the comments after eq. (2.17)] that the expression (3.7) for \(\delta_0 L\) should be such that terms with different positions of \(\delta_0 \varphi_i(x)\) must lead to equal portions in the variation \(\delta W\). As we shall see in Sect. 4, this is a severe restriction which, generally, cuts off part of the information, including the (anti)commutation properties of \(\delta_0 \varphi_i(x)\), the Schwinger’s action principle contains. The above problem, in other form, is mentioned in [3, p. 149] too.

**Remark 3.2.** Alternatively, one can put the variations in (3.7) to the left of the Lagrangian’s derivatives. Such a modification does not change anything, except the order of some operators, in this following. This problem is partially mentioned in [7].

Substituting (3.7) into (3.5), noting that, by its definition, \(\delta_0 (\partial_\mu \varphi_i(x)) = \partial_\mu (\delta_0 \varphi_i(x))\) and integrating by parts the term originating from the second term in (3.7), we obtain
\[ \delta W = \frac{1}{c} \int_R \left\{ \left( \frac{\partial L}{\partial \varphi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i(x))} \right) \right) \circ \delta_0 \varphi_i(x) \right. \]
\[ \left. \quad + \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i(x))} \circ \delta_0 \varphi_i(x) + L \delta x^\mu \right) \right\} d^4 x. \] (3.8)

Introducing the local (functional) variation
\[ \delta \varphi_i(x) := \varphi_i'(x') - \varphi_i(x) = \delta_0 \varphi_i(x) + (\partial_\nu \varphi_i(x)) \delta x^\nu, \] (3.9)

where \(x\) and \(x'\) refer to the coordinates of one and the same geometric point with respect to the ‘old’ and ‘new’ frames, applying the Stokes’ (Gauss’) theorem, and repeating the steps in [2, p. 63], from (3.3) we get
\[ \delta W = \frac{1}{c} \int_R \left\{ \left( \frac{\partial L}{\partial \varphi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi_i(x))} \right) \right) \circ \delta_0 \varphi_i(x) \right\} d^4 x + F[\sigma_2] - F[\sigma_1] \] (3.10)

where
\[ F[\sigma] := \frac{1}{c} \int_\sigma \{ \pi^{\mu}(x) \circ \delta_0 \varphi_i(x) + L(x) \delta x^\mu \} d\sigma_\mu \]
\[ = \frac{1}{c} \int_\sigma \{ \pi^{\mu}(x) \circ \delta \varphi_i(x) - (\pi^{\mu}(x) \circ (\partial_\nu \varphi_i(x))) - \partial_\nu^R L(x) \delta x^\nu \} d\sigma_\mu. \] (3.11)
Here $\sigma$ is a spacelike surface with surface element $d\sigma_\mu$, the notation (2.4) has been used, and $\delta^\nu_\mu$ is the (mixed) Kronecker $\delta$-symbol, i.e. $\delta^\nu_\mu = 1$ for $\mu = \nu$ and $\delta^\nu_\mu = 0$ for $\mu \neq \nu$.

Following the known argumentation (see [3, sec. 2.1] and [1, sec. 2]), from (3.10) and (3.11) a number of fundamental consequences can be derived. For example, we mention three of them.

Since (3.4) demands $\delta W$ to be a difference of two surface integrals and $R$ and $\delta_0 \varphi_i(x)$ are completely arbitrary, from (3.10) the Euler-Lagrange equations (2.1) for the field operators $\varphi_i(x)$ follow.

Identifying (3.11) with $\delta x^\nu = 0$ with the generator of the corresponding transformation (3.2) (with $\delta x^\nu = 0$), equation (3.3) implies

$$\delta \varphi_i(x) = i \hbar \left[ \int_{\sigma} \pi^{j\nu}(x') \circ \delta \varphi_j(x') \, d\sigma_\mu(x'), \varphi_i(x) \right]$$

(3.12)

as, by (3.9), $\delta \varphi_i(x) = \delta_0 \varphi_i(x)$ for $\delta x^\mu = 0$. Similarly

$$\delta \pi^{j\mu}(x) = i \hbar \left[ \int_{\sigma} \pi^{j\nu}(x') \circ \delta \varphi_j(x') \, d\sigma_\mu(x'), \pi^{j\mu}(x) \right].$$

(3.13)

The last two equalities should be identities for arbitrary $\delta \varphi_i(x)$ as long as $(x' - x)^2 = (x'^\mu - x^\mu)(x'^\nu - x^\nu)\eta_{\mu\nu} < 0$. They can be satisfied if one assumes (as additional postulate) the famous equal-time (anti)commutation relations, as it is proved in [2, pp. 65–67].

Consider transformations (3.2) leaving the action operator unchanged, i.e. such that

$$\delta W = 0.$$  

(3.14)

Then, by (3.10) and (2.1),

$$F[\sigma_1] = F[\sigma_2]$$

(3.15)

for any 3-dimensional spacetime surfaces $\sigma_1$ and $\sigma_2$, i.e. the operators (3.11) are surface-independent:

$$\frac{\delta F[\sigma]}{\delta \sigma(y)} = 0 \quad y \in \sigma$$

(3.16)

where $\frac{\delta}{\delta \sigma(y)}$ means the derivative of a functional of $\sigma$ relative to $\sigma$ at $y \in \sigma$ [2, p. 10]. From here follows that the ‘current (density)’

$$f^\mu(x) := \pi^{j\mu}(x) \circ \delta \varphi_j(x) - \left( \pi^{j\mu}(x) \circ (\partial_\nu \varphi_j(x)) - \delta^\mu_\nu L(x) \right) \delta x^\nu$$

is conserved, viz.

$$\partial_\mu f^\mu(x) = 0.$$  

(3.17)

It is a simple exercise to be verified, if we take (2.2) as a particular realization of (3.2), then (3.16) and (3.17) will respectively read

$$f^\mu(x) = - \sum_{a=1}^{s} \theta^\mu_{(a)}(x) \delta \omega^{(a)}$$

(3.18)

$$\partial_\mu \theta^\mu_{(a)}(x) = 0,$$

(3.19)
where
\[
\theta_{(a)}^\mu(x) := -\pi^\mu \circ \left\{ \frac{\partial \varphi_i(x)}{\partial \omega^a(x)} \right\}_{\omega=0} - \left( \partial_\nu \varphi_i(x) \right) \frac{\partial x^\nu}{\partial \omega^a(x)} \right\}_{\omega=0} - L(x) \frac{\partial x^\mu}{\partial \omega^a(x)} \right\}_{\omega=0}
\]
(3.20)
is the quantum version of (2.3). In this way, we arrive to the quantum variant of the (first) Noether theorem saying that, if the action (3.1) is invariant under finite parameter transformation (2.2), the current operators (3.20) are conserved in a sense of (3.19) or, equivalently, in a sense that the integrals
\[
C_{(a)}(\sigma) := \int_\sigma \theta_{(a)}^\mu(x) \, d\sigma
\]
are surface-independent, i.e.
\[
\frac{\delta C_{(a)}(\sigma)}{\delta \sigma(y)} = 0 \quad y \in \sigma.
\]
(3.22)

In particular, the choice \( \sigma = \{ x : x^0 = ct = \text{const} \} \) results in
\[
C_{(a)}(t) := \int_{x^0=ct} \theta_{(a)}^\mu(x) \, d^3x
\]
(3.21)
\[
\frac{dC_{(a)}(t)}{dt} = 0.
\]
(3.23)

We shall end the review of the Schwinger’s action principle and its consequences with the remark that the particular transformations (3.2b) with the choices
\[
\delta x^\mu = a^\mu \quad \delta_0 \varphi_i(x) = 0
\]
(3.23)
\[
\delta x^\mu = 0 \quad \delta_0 \varphi_i(x) = \varepsilon(\varphi_i) \frac{q}{\hbar c} \lambda \varphi_i(x),
\]
(3.24)
where \( a^\mu \) and \( \lambda \) are real parameters, lead to the canonical energy-momentum operator (2.10) and current operator (2.12), respectively, and, consequently, to the accompanying them problems, as discussed in Sect. 2.

In remark 3.1, we mentioned that the representation (3.7) for the r.h.s of (3.6) is, in the general case, incorrect and wrong. Since the Lagrangian \( L \) is supposed to be polynomial or convergent power series in \( \varphi_i(x) \) and \( \partial_\mu \varphi_i(x) \), it must be a sum of terms like
\[
\alpha \psi_1(x) \circ \cdots \circ \psi_a(x),
\]
where \( \alpha \) is real or complex number, \( a \in \mathbb{N} \) and \( \psi_b(x), b = 1, \ldots, a, \) is a field operator of a partial derivative of a field operator relative to some coordinate. The variation of such a term, under the transformation (3.2b), is
\[
\sum_{b=1}^a \psi_1(x) \circ \cdots \circ \psi_{b-1}(x) \circ \delta_0 \psi_b(x) \circ \psi_{b+1}(x) \circ \cdots \circ \psi_a(x)
\]
and can be put in the form (3.7), i.e.
\[
\left( \sum_{b=1}^a \psi_1(x) \circ \cdots \circ \psi_{b-1}(x) \circ \psi_{b+1}(x) \circ \cdots \circ \psi_a(x) \right) \circ \delta_0 \psi_b(x),
\]
if and only if
\[
[\delta_0 \psi_b(x), \psi_{b+1}(x) \circ \cdots \circ \psi_a(x)]_\pm = 0.
\]
(3.25)
These are the conditions Schwinger assumed to hold in \[1\] the comments after eq. (2.17) (see also \[7\]). The particular form of these conditions depends, of course, on the concrete Lagrangian under consideration. Generally, they say that the Lagrangian and variations of the field operators cannot be completely independent and arbitrary.

Since, usually, the Lagrangian is considered as a basic object in the theory, one may ask: can the conditions (3.25) hold for and arbitrary Lagrangian? The answer is positive. For example, if the variations \(\delta_0 \phi_i(x)\) are chosen as multiples of the identity mapping \(\text{id}_F\) of the system’s Hilbert space \(F\) of states, i.e.

\[
\delta_0 \phi_i(x) = f_i(x) \text{id}_F
\]

for completely arbitrary functions \(f_i: M \to C\), then

\[
\delta_0 (\partial_\mu \phi_i(x)) = (\partial_\mu f_i(x)) \text{id}_F
\]

and, hence, the conditions (3.25) are identically satisfied. It is a simple verification to be proved that the choices (3.26) are sufficient for a rigorous derivation of all the results concerning Schwinger’s action principle reviewed above, as well as the ones in the literature \[1,2,7–9\].

However, the choices (3.26) entail also the problems with the conserved quantities mentioned in Sect. 2.

As the purpose of the present paper is not the investigation of the conditions under which (3.7) holds, we shall end this section with a simple example illustrating the problem with (3.7). Consider a free neutral scalar field \(\phi(x)\) with mass parameter \(m\). Its Lagrangian is \[2, 4\]

\[
L = -\frac{1}{2} m^2 c^4 \phi \circ \phi + \frac{1}{2} c^2 \hbar^2 (\partial_\mu \phi) \circ (\partial^\mu \phi)
\]

so that

\[
\delta L = -\frac{1}{2} m^2 c^4 \{\phi \circ (\delta_0 \phi) + (\delta_0 \phi) \circ \phi\} + \frac{1}{2} c^2 \hbar^2 \{(\partial_\mu \phi) \circ (\delta_0 (\partial^\mu \phi)) + (\delta_0 (\partial_\mu \phi)) \circ (\partial^\mu \phi)\}.
\]

(3.29)

Obviously, we can write the last expression in the form (3.7), i.e. as

\[
\delta L = \frac{\partial L}{\partial \phi} \circ \delta_0 \phi + \frac{\partial L}{\partial (\partial_\mu \phi)} \circ \delta_0 (\partial_\mu \phi),
\]

(3.30)

where

\[
\frac{\partial L}{\partial \phi} = -m^2 c^4 \phi, \quad \frac{\partial L}{\partial (\partial_\mu \phi)} = c^2 \hbar^2 (\partial^\mu \phi),
\]

(3.31)

if and only if

\[
(\delta_0 \phi) \circ \phi = \phi \circ (\delta_0 \phi), \quad (\delta_0 (\partial_\mu \phi)) \circ (\partial^\mu \phi) = (\partial_\mu \phi) \circ (\delta_0 (\partial^\mu \phi)).
\]

If one assumes these equalities (or (3.31)) to hold for completely arbitrary \(\delta_0 \phi\), as it is done everywhere in the literature, (the Schur’s lemma (see, e.g., \[10\], sec. 8.2) or \[11\], ch. 5, sec. 3) implies that the field \(\phi\) must be proportional to the identity mapping of system’s Hilbert space \(F\), \(\phi(x) = g(x) \text{id}_F\) with \(g: M \to C\) of class \(C^2\) such that \(m^2 c^2 g + \hbar^2 (\partial_\mu \circ \partial^\mu) g = 0\).

---

5 The choices (3.26) explain also the ‘usual’ meaning of the derivatives with respect to operators, as the ones in (2.4) and (3.8).
due to (2.1) (i.e. due to the equation (1.2') below). This is equivalent to a consideration of a classical free real scalar field \( g \). However, if we restrict the variety of variations \( \delta_0 \varphi \) to the choice (3.26), i.e.

\[
\delta_0 \varphi(x) = f(x) \text{id}_\mathcal{F}
\]  

with completely arbitrary \( f: M \to \mathbb{C} \), we recover the standard quantum field theory of a free neutral scalar field \( \varphi \) [2–4], accompanied with the mentioned problems concerning the conserved quantities, i.e. the energy-momentum tensorial operator in this particular case.

### 4. A solution of the problems

As we pointed in Sect. 3, a possible solution of the problem with the representation (3.7) is to restrict the field variations to multiples of the identity mapping \( \text{id}_\mathcal{F} \) (see (3.26)) which rigorously reproduces the known results and problems following from Schwinger’s action principle. However, it is our opinion, the representation (3.7), as well as all efforts to ensure its validity, is not inherent to the Schwinger’s variational principle of quantum field theory. By imposing it, one restricts the possible Lagrangians and/or the variety of possible variations of the field operators by a purely technical reason, which is not in harmony with the other principles of quantum field theory. Moreover, by demanding the validity of (3.7), one ‘changes the rules of the game’ after it has been started, viz. after the variational principle is formulated and the extraction of consequences of it has began, one suddenly imposes the equality (3.7) only because it is valid in the classical case. We find such a situation unsatisfactory and propose the below-described solution of all problems mentions until now. But, to illustrate the method we intend to apply, we first consider the example Lagrangian (3.28).

Looking over (3.29) and (3.30), we see that, for a free neutral scalar field, the derivative \( \frac{\partial L}{\partial \varphi} \) should be regarded as a mapping acting on the operators on \( \mathcal{F} \), not on vectors in \( \mathcal{F} \), such that

\[
\frac{\partial L}{\partial \varphi} : v \mapsto \frac{\partial L}{\partial \varphi} (v) = -\frac{1}{2}m^2 c^4 \varphi \circ v + v \circ \varphi
\]

for any \( v: \mathcal{F} \to \mathcal{F} \). Defining similarly \( \frac{\partial L}{\partial (\partial_\mu \varphi)} \) by

\[
\frac{\partial L}{\partial (\partial_\mu \varphi)} : v \mapsto \frac{1}{2} c^2 \hbar^2 ((\partial^\mu \varphi) \circ v + v \circ (\partial^\mu \varphi))
\]

we can replace (3.30) with the equality

\[
\delta L = \frac{\partial L}{\partial \varphi} (\delta_0 \varphi) + \frac{\partial L}{\partial (\partial_\mu \varphi)} (\delta_0 (\partial_\mu \varphi))
\]

(4.1)

where no additional conditions, like (3.32), on \( \varphi \) and \( \delta_0 \varphi \) have been imposed. Further, repeating the derivation of Euler-Lagrange equations, with (4.1) for (3.7), we get

\[
\left( \frac{\partial L}{\partial \varphi} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \varphi)} \right) \right) (\delta_0 \varphi) = 0.
\]

(4.2)

Substituting here the just-obtained derivatives, we find

\[
\frac{1}{2} \left( -m^2 c^4 \varphi - c^2 \hbar^2 \Box (\varphi) \right) \circ \delta_0 \varphi + \delta_0 \varphi \circ \left( -m^2 c^4 \varphi - c^2 \hbar^2 \Box (\varphi) \right) = 0
\]

(4.2′)

where \( \Box := \partial_\mu \partial^\mu \) is the D’Alembert operator and no additional condition have been imposed. The choice \( \delta_0 \varphi = \text{id}_\mathcal{F} \) reduces the last equality to the standard Klein-Gordon equation for \( \varphi \),

\[
m^2 c^2 \varphi + \hbar^2 \Box (\varphi) = 0
\]

(4.2″)

which, in turn, converts (4.2′) into identity. So, without the additional conditions (3.32), we derived from the action principle the ‘right’ Klein-Gordon equation describing free neutral scalar field. For the energy-momentum operator \( T_{\mu\nu} \) of such a field, one can repeat its
derivation, as described in Sect. 3, but with the new definition of the Lagrangian’s derivatives. The result reads (cf. (2.10))

\[ T_{\mu\nu} = \pi_\mu (\partial_\nu \varphi) - L \eta_{\mu\nu} \]  

(4.3)

where \( \pi^\mu := \frac{\partial L}{\partial (\partial_\mu \varphi)} : v \mapsto \frac{1}{2} c^2 h^2 ( (\partial^\mu \varphi) \circ v + v \circ (\partial^\mu \varphi) ) \) for \( v : \mathcal{F} \to \mathcal{F} \), or, equivalently

\[ T_{\mu\nu} = \frac{1}{2} c^2 h^2 \{ (\partial_\mu \varphi) \circ (\partial_\nu \varphi) + (\partial_\nu \varphi) \circ (\partial_\mu \varphi) \} + \frac{1}{2} ( m^2 c^4 \varphi - c^2 h^2 (\partial_\mu \varphi) \circ (\partial_\mu \varphi) ) \eta_{\mu\nu}. \]  

(4.3)

Thus, without any additional hypotheses, the action principle (3.4) leads to the ‘Hermitian symmetrized’ energy-momentum tensor (2.11) (with the ‘usual’ meaning of the \( \pi^\mu \), i.e. \( \pi^\mu = c^2 h^2 \partial^\mu \varphi \)). It can be proved that, after imposing the commutation relations and normal ordering, the quantum field \( \varphi \) described via (4.2) and \( T_{\mu\nu} = c^2 h^2 (\partial_\mu \varphi) \circ (\partial_\nu \varphi) - \eta_{\mu\nu} L \), corresponding to (2.10) and usually considered in the literature [3, 4].

Having in mind the above example, we turn now our attention to the general case of arbitrary Lagrangian, which is supposed to be polynomial or convergent power series in the field operators and their first partial derivatives.

The main idea of the following is the derivatives of the Lagrangian with respect to the field operators, usually, satisfy some (anti)commutation relations and, hence, in it \( \omega \) is a \( \mathbb{C} \)-vector space, \( \omega \subseteq \{ \mathcal{F} \to \mathcal{F} \} \) be a subset of the space of operators acting on \( \mathcal{F} \), \( n \in \mathbb{N} \), \( u_1, \ldots, u_n \in \omega \), \( u \in \{ u_1, \ldots, u_n \} \), and \( f : (u_1, \ldots, u_n) \mapsto f(u_1, \ldots, u_n) : \mathcal{F} \to \mathcal{F} \) be operator-valued function of \( u_1, \ldots, u_n \) which is polynomial (or convergent power series) in its operator arguments. The derivative of \( f \) with respect to \( u \) is an \( n \)-argument mapping with domain \( \omega \times \cdots \times \omega \) (\( n \)-times), denoted by \( \frac{\partial f}{\partial u} \)(\( u \)), such that:

(i) Its value at \( (u_1, \ldots, u_n) \), denoted by \( \frac{\partial f}{\partial u} (u_1, \ldots, u_n) := \frac{\partial f}{\partial u} \mid _{(u_1,\ldots,u_n)} \) and \( \omega \to \{ \mathcal{F} \to \mathcal{F} \} \) is a mapping \( \omega \to \{ \mathcal{F} \to \mathcal{F} \} \) from the subset \( \omega \) on the space of operators on \( \mathcal{F} \).

(ii) The mapping \( \frac{\partial f}{\partial u} : f \mapsto \frac{\partial f}{\partial u} \) is linear relative to complex-valued functions on \( M \). In particular, it is \( \mathbb{C} \)-linear.

(iii) Let \( v : \mathcal{F} \to \mathcal{F} \) be such that \( u + v \in \omega \), \( a \in \mathbb{N} \), \( i_1, \ldots, i_a \in \{ 1, \ldots, n \} \) and \( I := \{ i \in \{ i_1, \ldots, i_a \} : u_i = u \} \) be the set of indices which label all operators among \( u_{i_1}, \ldots, u_{i_a} \) equal to \( u \). Then

\[ \left( \frac{\partial}{\partial u} (u_{i_1} \circ \cdots \circ u_{i_a}) \right) (v) := \sum_{i \in I} \left( u_{i_1} \circ \cdots \circ u_{i_a} \right) \mid _{u_i = v}. \]  

(4.5)

In particular, if \( I \) is empty, \( I = \emptyset \), the r.h.s. of (4.5) is set equal to the zero operator of \( \mathcal{F} \).

Remark 4.1. The restriction \( u + v \in \omega \) is essential one in quantum field theory, in which the field operators, usually, satisfy some (anti)commutation relations and, hence, in it \( \omega \neq \{ \mathcal{F} \to \mathcal{F} \} \); for some general remarks on that item, see [12] sec. 21.1. For instance, let us find the derivative of \( A \circ B \) with respect to \( A \), where \( A, B : \mathcal{F} \to \mathcal{F} \) are anticommuting
operators, $A \circ B = -B \circ A$. In this particular case $\frac{\partial}{\partial x}(A \circ B)$ is defined only on those $v: \mathcal{F} \to \mathcal{F}$ for which $(A + v) \circ B = -B \circ (A + v)$, i.e. such that $v \circ B = -B \circ v$, and hence $\omega = \{z: \mathcal{F} \to \mathcal{F}: z \circ B = -B \circ z\}$. In accord with (4.3), we have $\frac{\partial (A \circ B)}{\partial A}(v) = v \circ B = -B \circ v$, the evaluation of the derivative on element $w \in \{\mathcal{F} \to \mathcal{F}\}\setminus \omega$ leads to a contradiction, $\frac{\partial (A \circ B)}{\partial A}(w) \neq \frac{\partial (-B \circ A)}{\partial A}(w)$.

Remark 4.2. From a viewpoint of functional analysis, the definition 4.1 defines the notion of partial Fréchet derivative of particular kind of functionals employed in quantum field theory. From this position, the r.h.s. of (4.4) is nothing else, but the Fréchet differential of the partial Fréchet derivative of particular kind of functionals employed in quantum field theory.

In short, definition 4.1 means that a derivative of operator-valued function, polynomial or convergent power series, of operator arguments with respect to some of its operator arguments is calculated by differentiating each its term according to (4.5). In particular, this is valid for Lagrangians of the type we consider in this work.

It is a trivial checking to show that the derivatives introduce via definition 4.1 possess all ‘standard’ derivative properties; in particular, they satisfy the Leibnitz rule for differentiation of compositions (products) of functions and the rule for differentiation of composite functions.

If $c: M \to \mathbb{C}$ and $u + c(x)\text{id}_\mathcal{F} \in \omega$, $x \in M$, it is a trivial corollary of definition 4.1 that

$$\frac{\partial f(u_1, \ldots, u_n)}{\partial u}(c(x)\text{id}_\mathcal{F}) = c(x)\frac{\partial^2 f(u_1, \ldots, u_n)}{\partial u},$$

(4.6)

where all operators are supposed to be linear and $\frac{\partial^2}{\partial u}$ means the ‘classical’ derivative with respect to $u$ as it was defined in Sect. 4 i.e. the derivative in the r.h.s. of (4.6) should be calculated as if $u_1, \ldots, u_n$ were classical fields over $M$ with a preservation of the relative order of all operators. This is exactly the definition of a derivative of a function of non-commuting arguments accepted, e.g., in [3, § 2]. For example, we have $\frac{\partial^2}{\partial x}(v) = \varphi^2 \circ v + \varphi \circ v \circ \varphi + v \circ \varphi^2$ and $\frac{\partial^3}{\partial x}(c(x)\text{id}_\mathcal{F}) = 3c(x)\varphi^2$ where $\varphi^a := \varphi \circ \ldots \varphi$ ($a$-times) for $a \in \mathbb{N}$ and $\varphi: \mathcal{F} \to \mathcal{F}$.

Equipped with the new definition of a derivative of a Lagrangian with respect to a field operator or its partial derivative, it is trivial to verify that (4.4) is an equivalent version of (3.6), up to second and higher order terms, without making any additional hypotheses with respect to the Lagrangian or/and variations of the field operators. Moreover, if we consider the variations (3.26), then, in view of (1.6) and (4.4), we get

$$\delta_0 L|_{\delta_0 \varphi(x)=f_i(x)\text{id}_\mathcal{F}} = \left\{ \frac{\partial L}{\partial \varphi_i(x)} \circ \delta_0 \varphi_i(x) + \frac{\partial L}{\partial (\varphi_i(x))} \circ \delta_0 (\varphi_i(x)) \right\} \bigg|_{\delta_0 \varphi(x)=f_i(x)\text{id}_\mathcal{F}}$$

$$= f_i(x)\frac{\partial L}{\partial \varphi_i(x)} + \left( \varphi_i(x) \frac{\partial L}{\partial (\varphi_i(x))} \right)$$

(4.7)

from where all standard consequences (and problems) of the Schwinger’s action principle can be derived (recovered). It is natural to be expected that the restriction of the field variations to ones given via (3.26) should lead to a broadening of the consequences of the variational principle (3.4). Below we shall examine them without accepting any additional conditions, like (3.26) or, in the general case, (3.25). In this case, one can expect new consequences of the Schwinger’s action principle, which otherwise are ‘swallowed’ by (3.26) (or (3.25)) in its standard presentations.

We shall now work out the explicit form of the action variation (3.3) with (3.7) replaced by (1.4) with the new meaning of the derivatives in it. Inserting (4.4) into (3.5) and inte-

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\[ \text{See the remark in } [3, \text{p. 149}] \text{ on the above topic.} \]
grating by parts the term coming from the second one in (4.4), we get (cf (3.8))

\[
\delta W = \frac{1}{c} \int_R \left\{ \sum_i \left( \frac{\partial L}{\partial \phi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} \right) \right) \left( \delta_0 \phi_i(x) \right) + \sum_\mu \frac{\partial}{\partial x^\mu} \left( \sum_i \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} \right) \delta_0 \phi_i(x) \right\} \mathrm{d}^4x. \quad (4.8)
\]

From here, repeating *mutatis mutandis* the derivation of (3.10), as given in [2], we obtain

\[
\delta W = \frac{1}{c} \int_R \left\{ \sum_i \left( \frac{\partial L}{\partial \phi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} \right) \right) \left( \delta_0 \phi_i(x) \right) \right\} \mathrm{d}^4x + F[\sigma_2] - F[\sigma_1] \quad (4.9)
\]

where

\[
F[\sigma] := \frac{1}{c} \int \sum_\mu \left\{ \sum_i \pi^{\mu i}(x) (\delta \phi_i(x)) - \sum_\nu \left( \sum_i \pi^{\nu i}(x) (\partial_\nu \phi_i(x)) - \delta_\nu^\mu L(x) \right) \delta x^\nu \right\} \mathrm{d}\sigma_\mu \quad (4.10)
\]

with

\[
\pi^{\mu i}(x) := \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} : \{F \to F\} \to \{F \to F\} \quad (4.11)
\]

being the derivative of $L$ relative to $\partial_\mu \phi_i(x)$ according to definition [4.1]. Formally, (4.9) and (4.10) can be obtained from (3.10) and (3.11) via the next replacements:

\[
\frac{\partial L}{\partial \phi_i(x)} \circ v \mapsto \frac{\partial L}{\partial \phi_i(x)}(v) \quad \left\{ \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial \phi_i(x)} \right) \right\} \circ v \mapsto \left\{ \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial \phi_i(x)} \right) \right\}(v)
\]

\[
\pi^{\mu i} \circ v = \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} \circ v \mapsto \pi^{\mu i}(v) = \frac{\partial L}{\partial (\partial_\mu \phi_i(x))}(v) \quad (4.12)
\]

where $v: F \to F$ (is some variation, i.e. $\delta_0 \phi_i(x)$ or $\delta_0 \phi_i$). As we shall see further, these changes can be used for ‘repairing’ the standard consequences of Schwinger’s action principle. Evidently, the replacements opposite to (4.12) transform (4.9) and (4.10) to (3.10) and (3.11), respectively.

Evidently, in view of (1.1), it is clear that the ‘old’ variation (3.10) (with the standard meaning of the derivatives in it) and the ‘new’ variation (4.9) (with the derivatives in it given via definition [4.1]) are identical if variations like (3.26), i.e. multipliers of the identity mapping, are employed.

Let us proceed with extraction of consequences of the variational principle (3.4) on a base of the representation (4.4).

Since (3.4) states that $\delta W$ must be a difference of two surface integrals, (4.4) implies the vanishment of the volume integral in it, which, due to the arbitrariness of the integration region $R$, is equivalent to

\[
\sum_i \left\{ \left( \frac{\partial L}{\partial \phi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial L}{\partial (\partial_\mu \phi_i(x))} \right) \right)(v_i) \right\} = 0 \quad (4.13)
\]

where, for brevity, we have denoted by $v_i$ the variation $\delta_0 \phi_i(x)$ of $\phi_i(x)$, $v_i := \delta_0 \phi_i(x)$. This is the prototype of the *operator Euler-Lagrange equations for the field operators $\phi_i(x)$*.

---

\[\text{By virtue of properties (ii) and (iii) in definition 4.1, the integration by parts is a rigorous operation; in general, we have: } \frac{\partial}{\partial x^\mu} (\partial_\mu v) = \partial_\mu (\frac{\partial}{\partial x^\mu}(v)) - (\partial_\mu (\frac{\partial}{\partial x^\mu})(v)).\]
and their variations $v_i := \delta_0 \varphi_i(x)$. We should emphasize, now this is an equation both for $\varphi_i(x)$ and $v_i$, contrary to the standard procedure where one gets, due to the arbitrariness of $v_i$, equations only for $\varphi_i(x)$. So, the new moment with respect to the ‘old’ variational principle is that (4.13) puts, in general, restrictions both on the field operators and on their variations, i.e. the variations cannot be considered as completely arbitrary (if one does not want to deal with trivial fields in some cases). In Sect. 5, examples will be considered when completely arbitrary and not such variations are admissible; the particular situation depends on the concrete Lagrangian employed. However, from the derivation of (4.9) (and therefore of (4.13)), it is clear that the variations (3.20), i.e. ones proportional to the identity mapping $\text{id}_F$ of the system’s Hilbert space $F$ of states, are always admissible. For them, in view of (4.6) and the complete arbitrariness of the functions $f_i: M \to \mathbb{C}$ in (3.24), we derive from (1.13) the ‘classical’ Euler-Lagrange equations for the field operators as

$$
\frac{\partial^\mu L}{\partial \varphi_i(x)} - \frac{\partial}{\partial x^\mu} \left( \frac{\partial^\mu L}{\partial (\partial_\mu \varphi_i(x))} \right) = 0
$$

which, due to the meaning of the operator derivatives in it, coincide with the ones obtained form the ‘old’ variations (1.10). But it should clearly be understood, if one requires (4.13) to be valid for completely arbitrary variations $v_i$, other restrictions on the field operators may arise. The alternative point of view is to look on (4.14) as on field equations for the field operators and on the remaining consequences of (4.13), if any, as on restrictions on the admissible variations $v_i$. We shall discuss this topic in Sect. 5 on concrete examples; in particular, in Subsect. 5.6 it will be presented an example of a Lagrangian which leads to completely reasonable field equations which are not the Euler-Lagrange equations for it (the latter being simply identities with respect to the fields and their variations).

Let us turn now to the problem with conserved quantities. Consider a transformation (3.2), in which the field operators $\varphi_i(x)$ and their variations $v_i = \delta_0 \varphi_i(x)$ satisfy (4.13), leaving the action (3.1) unchanged. For these operators equations (3.1)–(3.15) with (4.13) and (4.14)) with $F[\sigma]$ defined, now, by (4.10), not by (3.11). Therefore, the ‘current(density)’ (cf. (3.16))

$$
f^\mu(x) := \sum_i \pi^\mu_i(x)(\delta \varphi_i(x)) - \sum_\nu \left( \sum_i \pi^\mu_i(x)(\partial_\nu \varphi_i(x)) - \delta_\nu^\mu L(x) \right) \delta x^\nu
$$

is conserved, i.e. satisfies the continuity equation (3.17). In particular, if an $s$-parameter transformations (2.3) satisfy the above conditions, the equalities (4.13) and (3.17) will reduce respectively to (3.18) and (3.19) with the ‘Noether currents’ $\theta^\mu_{(a)}(x)$, $a = 1, \ldots, s$, given by (cf. (3.20))

$$
\theta^\mu_{(a)}(x) := - \sum_i \pi^\mu_i(x) \left( \frac{\partial \varphi_i^\omega(x^\omega)}{\partial \omega^{(a)}} \right) \Big|_{\omega=0} + \sum_{i \nu} \pi^\mu_i(x) \left( \partial_\nu \varphi_i(x) \right) \left( \frac{\partial x^\omega_{\nu}}{\partial \omega^{(a)}} \right) \Big|_{\omega=0} - L(x) \frac{\partial x^\omega_{\mu}}{\partial \omega^{(a)}} \Big|_{\omega=0}
$$

where $\pi^\mu_i(x)$ is defined via (4.11). The equations (3.21)–(3.22), of course, remain valid with the new definition (4.16) of $\theta^\mu_{(a)}(x)$.

To feel better the difference between (4.13) and (3.16) (or between (4.16) and (3.20)), let us consider the transformations (3.23) and (3.24), generating in the classical case the

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8 This does not exclude the coincidence of the final equations for $\varphi_i$; in particular, such is the case with the free fields — see Sect. 5.

9 As a rule, the Lagrangians describing free fields admit arbitrary variations, while those describing (self-)interacting fields require some restrictions on the variety of field variations.

10 In some cases, these new restrictions imply the field operators to be proportional to $\text{id}_F$, i.e., in a sense, leading to classical, not quantum, fields.
energy-momentum tensor (2.6) and current vector (2.8), respectively. For them the operator (4.15) reduces respectively to \( T^\mu(x) = -\sum_i \pi^\mu_i(x) (\partial^i \varphi_i(x)) - \eta^\mu\nu L(x) \) (4.17) and \( J^\mu(x) = \frac{q_i \hbar}{c} \sum_i \varepsilon(\varphi_i) \pi^\mu_i(x) (\varphi_i(x)) \) (4.18) are the energy-momentum and (charge) current operators, respectively. In some cases, these expressions may differ significantly from (2.10) and (2.12), respectively, which will be illustrated on concrete examples in Sect. 5.

As a last example of a conserved operator quantity, we consider the angular momentum operator \( M_\lambda^{\mu\nu} \). Suppose the action operator is invariant under 4-rotations, i.e. under the changes \( x^\mu \mapsto x^\epsilon^\mu = x^\mu + \varepsilon^\mu\nu x^\nu \), with \( x^\nu := \eta_{\nu\mu} x^\mu \) and antisymmetric real parameters \( \varepsilon^\mu\nu = -\varepsilon^\nu\mu \), and \( \varphi_i(x) \mapsto \varphi_i^\epsilon(x^\epsilon) \) with \( \varphi_i^\epsilon(x^\epsilon) = \varphi_i(x) + \sum_{\mu<\nu} I^j_{i\mu\nu} \varphi_j(x) \varepsilon^\mu\nu + \cdots \), where the dots stand for second and higher order terms in \( \varepsilon^\mu\nu \) and \( I^j_{i\mu\nu} = -I^j_{i\nu\mu} \) are numbers characterizing the behaviour of the field operators under 4-rotations. Since \( x^\epsilon^\rho = x^\rho + \sum_{\mu<\nu} (\partial^\rho_{\mu} x^\nu - \partial^\nu_{\mu} x^\rho) \varepsilon^\mu\nu \), from (4.16) (with changed sign), in view of (4.17), we obtain

\[
M_\lambda^{\mu\nu} = (x^\mu T^\lambda_\nu - x^\nu T^\lambda_\mu) + S_\mu^\lambda_{\nu}, \tag{4.19}
\]

where

\[
S_\mu^\lambda_{\nu} := \sum_{i,j} \pi^\lambda_i(\varphi_j) I^j_{i\mu\nu} \tag{4.20}
\]

is the spin angular momentum operator.

As for the quantization rules, equation (3.12) should now be replaced by

\[
\delta \varphi_i(x) = i\hbar \int_\sigma \left[ \sum_j \pi^{\mu\nu}(\varphi_j(x')) (\delta \varphi_j(x')) , \varphi_i(x) \right] d\sigma \nu(x') \tag{4.21}
\]

and similarly for (3.13). These equations and the variations (3.26), combined with the known argumentation [2, sect 2.1 (ii)], produce the canonical (anti)commutation relations. However, a different choice of the field variations, if such ones are admissible, may result in new restrictions on the field operators.

5. Examples

The main purpose of this section is an illustration of the general theory of Sect. 4 for particular Lagrangians. As we shall see, known results are reproduce with some corrections. The ‘quadratic’ Lagrangians will be pointed as the ‘best’ ones selected by the Schwinger’s action principle.

5.1. Free neutral scalar field

The Lagrangian of a free neutral scalar field \( \varphi = \varphi^\dagger \) with mass parameter \( m \) is (3.28). In accord with definition 4.1, the action of its operator derivatives on an operator \( v \) are (cf. (3.31))

\[
\frac{\partial L}{\partial \varphi}(v) = -\frac{1}{2} m^2 c^4 (\varphi \circ v + v \circ \varphi).
\]

\[
\pi^\mu(v) = \frac{\partial L}{\partial(\partial^\mu \varphi)}(v) = \frac{1}{2} c^2 \hbar^2 ((\partial^\mu \varphi) \circ v + v \circ (\partial^\mu \varphi)). \tag{5.1}
\]
Hence the Euler-Lagrange relation (4.13) for it reads

\[
\frac{1}{2}(-m^2 c^4 \varphi - c^2 \hbar^2 \Box(\varphi)) \circ v + v \circ \frac{1}{2}(-m^2 c^4 \varphi - c^2 \hbar^2 \Box(\varphi)) = 0
\]

(5.2)

where \( \Box = \partial_\mu \partial^\mu \) is the D’Alembert operator. The choice \( v = \text{id}_\mathcal{F} \), \( \mathcal{F} \) being the field’s (system’s) Hilbert space of states, results in the Klein-Gordon equation

\[
m^2 c^2 \varphi + \hbar^2 \Box(\varphi) = 0
\]

(5.3)

which corresponds to (4.14) with Lagrangian (3.28). Evidently, (5.3) converts (5.3) into identity relative to \( v \). Thus the equations (4.14) do not impose any restrictions for the variations \( v \) in a case of the Lagrangian (3.28). In view of (5.1), the energy-momentum operator (4.17) now reads

\[
T_{\mu \nu} = \frac{1}{2} c^2 \hbar^2 \{ (\partial_\mu \varphi) \circ (\partial_\nu \varphi) + (\partial_\nu \varphi) \circ (\partial_\mu \varphi) \} - \eta_{\mu \nu} \{ -\frac{1}{2} m^2 c^4 \varphi \circ \varphi + \frac{1}{2} c^2 \hbar^2 (\partial_\mu \varphi) \circ (\partial^\mu \varphi) \},
\]

(5.4)

which corresponds to (2.11) with Lagrangian (3.28), not to (2.10), when the additional conditions (1.29), i.e. (3.32) in the particular case, in Schwinger’s action principle are imposed.

In almost the same way, the reader may wish to consider an electromagnetic field with 4-potential operators \( A_\mu \) and, e.g., gauge invariant Lagrangian \( L = -\frac{1}{4} c^2 \hbar^2 F_{\mu \nu} F^{\mu \nu} \) with \( F_{\mu \nu} := \partial_\mu A_\nu - \partial_\nu A_\mu \). In this case, we have \( \frac{\partial L}{\partial \varphi^\dagger} = 0 \) and (do not sum over \( \mu \))

\[
\pi^\mu_{\nu}(v_\mu) = \frac{\partial L}{\partial (\partial_\nu \varphi^\dagger)}(v_\mu) = \frac{1}{2} c^2 \hbar^2 \{ v_\mu \circ F_{\nu \mu} + F_{\nu \mu} \circ v_\mu \}
\]

for a variation \( v_\mu \) of \( A_\mu \).

### 5.2. Free charged scalar field

The standard choice of a Lagrangian of a free charged scalar field \( \varphi \neq \varphi^\dagger \) with mass parameter \( m \) is 

\[
L = -m^2 c^4 \varphi \circ \varphi + c^2 \hbar^2 (\partial_\mu \varphi^\dagger) \circ (\partial^\mu \varphi).
\]

(5.5)

So, we have:

\[
\frac{\partial L}{\partial \varphi}(v) = -m^2 c^4 \varphi^\dagger \circ v \quad \pi^\mu(v) = \frac{\partial L}{\partial (\partial_\mu \varphi)}(v) = c^2 \hbar^2 (\partial^\mu \varphi^\dagger) \circ v
\]

\[
\frac{\partial L}{\partial \varphi^\dagger}(w) = -m^2 c^4 \varphi \circ w \quad \pi^\mu_{\nu}(w) = \frac{\partial L}{\partial (\partial_\mu \varphi^\dagger)}(w) = c^2 \hbar^2 w \circ (\partial^\mu \varphi)
\]

(5.6)

for operators \( v \) and \( w \) having a meaning of variations of \( \varphi \) and \( \varphi^\dagger \), respectively. Therefore the relation (4.13) now reads:

\[
(-m^2 c^4 \varphi^\dagger - c^2 \hbar^2 \Box(\varphi^\dagger)) \circ v + v \circ (-m^2 c^4 \varphi - c^2 \hbar^2 \Box(\varphi)) = 0.
\]

(5.7)

From here, using the standard choices \((v, w) = (0, \text{id}_\mathcal{F}), (\text{id}_\mathcal{F}, 0)\), we derive the Klein-Gordon equations

\[
m^2 c^2 \varphi + \hbar^2 \Box(\varphi) = 0 \quad m^2 c^2 \varphi^\dagger + \hbar^2 \Box(\varphi^\dagger) = 0,
\]

(5.8)

\[\text{11}\] In some sense, the Lagrangian \( L = -\frac{1}{4} m^2 c^4 (\varphi \circ \varphi + \varphi \circ \varphi^\dagger) + \frac{1}{2} c^2 \hbar^2 (\partial_\mu \varphi^\dagger) \circ (\partial^\mu \varphi) + (\partial_\mu \varphi) \circ (\partial^\mu \varphi^\dagger) \) is better than (5.3). But, after imposing the commutation relations and normal ordering, the quantum field theory arising from both Lagrangians turns to be one and the same.
which convert \(5.7\) into identity relative to \(v\) and \(w\). Thus, the variations \(v\) and \(w\) in \(5.7\) can be completely arbitrary. Substituting \(5.6\) into \(4.17\) and \(4.18\), we get the energy-momentum and current operators respectively as

\[
T_{\mu\nu} = c^2 h^2 \{(\partial_\mu \varphi^\dagger) \circ (\partial_\nu \varphi) + (\partial_\nu \varphi^\dagger) \circ (\partial_\mu \varphi)\} - \eta_{\mu\nu} \{-m^2 c^4 \varphi \circ \varphi + c^2 h^2 (\partial_\mu \varphi^\dagger) \circ (\partial^\mu \varphi)\}
\]

\(5.9\)

\[
J_\mu = -i\hbar q \{ (\partial_\mu \varphi^\dagger) \circ \varphi - \varphi^\dagger \circ (\partial_\mu \varphi)\},
\]

\(5.10\)

where, for definiteness, we have chosen \(\varepsilon(\varphi) = +1\) and \(\varepsilon(\varphi^\dagger) = -1\). These expressions correspond to \(2.11\) and \(2.13\), not to \(2.10\) and \(2.12\), respectively. Evidently, \(T_{\mu\nu} = T_{\nu\mu}, T_{\mu\nu}^\dagger = T_{\nu\mu}^\dagger,\) and \(J_\mu^\dagger = J_\mu\). Consequently, our formalism produces the known results form the literature \([3, 4]\), where expressions, like \(5.9\) and \(5.10\), are more a matter of convention/postulate than a one of rigorous derivation.

### 5.3. Self-interacting neutral scalar field

Consider a neutral scalar field \(\varphi = \varphi^\dagger\) with Lagrangian

\[
L = -\frac{1}{2} m^2 c^4 \varphi \circ \varphi + \frac{1}{2} c^2 h^2 (\partial_\mu \varphi) \circ (\partial^\mu \varphi) + a \varphi^k
\]

\(5.11\)

where \(a\) is a real non-zero parameter, \(k \in \mathbb{N}\), and \(\varphi^k := \varphi \circ \ldots \varphi\) \((k\text{-times})\). Applying definition \(4.1\), we get:

\[
\frac{\partial L}{\partial \varphi}(v) = -\frac{1}{2} m^2 c^4 \varphi \circ (v \circ v \circ \varphi)
\]

\[
+ a(v \circ \varphi^{k-1} \circ v \circ v \circ \varphi^{k-2} + \cdots + v \circ v \circ \varphi^{k-2} \circ v \circ \varphi + \varphi^{k-1} \circ v)
\]

\(5.12\)

\[
\pi^\mu(v) = \frac{\partial L}{\partial (\partial_\mu \varphi)}(v) = \frac{1}{2} c^2 h^2 ((\partial_\mu \varphi) \circ v + v \circ (\partial_\mu \varphi)).
\]

So, the energy-momentum operator is given by \(5.4\), with the term in the braces replaced by the r.h.s. of \(5.11\), and the relation \(4.13\) reads (cf. \(5.2\))

\[
\frac{1}{2}(-m^2 c^4 \varphi - c^2 h^2 \Box(\varphi)) \circ v + v \circ \frac{1}{2}(-m^2 c^4 \varphi - c^2 h^2 \Box(\varphi))
\]

\[
+ a(v \circ \varphi^{k-1} \circ v \circ v \circ \varphi^{k-2} + \cdots + v \circ v \circ \varphi^{k-2} \circ v \circ \varphi + \varphi^{k-1} \circ v) = 0.
\]

\(5.13\)

Choosing \(v = \text{id}_\mathcal{F}\), we obtain a ‘classical’ Euler-Lagrange equation for \(\varphi\):

\[
m^2 c^4 \varphi + c^2 h^2 \Box(\varphi) = ak \varphi^{k-1}.
\]

\(5.14\)

Combining \(5.14\) with \(5.13\), we see that the variation \(v = \delta_0 \varphi\) cannot be arbitrary (for \(a \neq 0\)) as it must satisfy the equation

\[
(1-k/2)(\varphi^{k-1} \circ v + v \circ \varphi^{k-1}) + \varphi \circ v \circ \varphi^{k-2} + \varphi^2 \circ v \circ \varphi^{k-3} + \cdots + \varphi^{k-2} \circ v \circ \varphi = 0
\]

\(5.15\)

which always has solutions of the form \(v = f(x) \text{id}_\mathcal{F}\) with \(f : M \to \mathbb{C}\). In particular, for \(k = 1, 2, 3, 4\) this equation respectively reads:

\[
0 = 0 \quad (k = 1) \quad 0 = 0 \quad (k = 2)
\]

\[
[\varphi, [\varphi, v]] = 0 \quad (k = 3) \quad [\varphi^2, [\varphi, v]] = 0 \quad (k = 4).
\]

\(5.16\)

Consequently, for \(k \geq 3\) either the set of the variation \(v\) should be restricted to the ones satisfying \(5.15\) with \(\varphi\) being a solution \(5.14\), or to the field equation \(5.14\) should be added the condition \(5.15\) with arbitrary operator/variation \(v\).
5.4. Interacting neutral scalar fields

Consider a system of two neutral scalar fields $\varphi_1$ and $\varphi_2$ with Lagrangian

$$ L = -\frac{1}{2}m^2c^4 \varphi_1 \circ \varphi_1 + \frac{1}{2}c^2\hbar^2(\partial_\mu \varphi_1) \circ (\partial^\mu \varphi_1) $$

$$ -\frac{1}{2}m^2c^4 \varphi_2 \circ \varphi_2 + \frac{1}{2}c^2\hbar^2(\partial_\mu \varphi_2) \circ (\partial^\mu \varphi_2) + a\varphi_1 \varphi_2 \quad (5.17) $$

where $a$ is a non-vanishing real parameter. Performing a procedure similar to the ones in the previous subsections, we obtain the field equations

$$ m^2c^4 \varphi_1 + c^2\hbar^2\Box(\varphi_1) = a\varphi_2 \quad m^2c^4 \varphi_2 + c^2\hbar^2\Box(\varphi_2) = a\varphi_1 \quad (5.18) $$

and the conditions

$$ [\varphi_1, v_2] = 0 \quad [\varphi_2, v_1] = 0 \quad (5.19) $$

which must satisfy the solutions of (5.18) and the variations $v_1$ and $v_2$ of $\varphi_1$ and $\varphi_2$, respectively. The last conditions are quite natural from physical viewpoint because they mean that we can make a variation of $\varphi_1$ independently of $\varphi_2$ and vice versa. The energy-momentum operator of the system under consideration turns to be

$$ T_{\mu\nu} = \frac{1}{2}c^2\hbar^2 \sum_{i=1,2} \left\{ (\partial_\mu \varphi_i) \circ (\partial_\nu \varphi_i) + (\partial_\nu \varphi_i) \circ (\partial_\mu \varphi_i) \right\} - \eta_{\mu\nu}L. \quad (5.20) $$

5.5. Free Dirac (spinor) field 1. Standard Lagrangian

As a standard Lagrangian of a spin $\frac{1}{2}$ Dirac field $\psi$, we take [3]

$$ L = \frac{1}{2}i\hbar c\{\overline{\psi}_\gamma^\mu \circ (\partial_\mu \psi) - (\partial_\mu \overline{\psi}) \gamma^\mu \circ \psi\} - mc^2\overline{\psi} \circ \psi. \quad (5.21) $$

Here: $\gamma^\mu$ are the Dirac’s $\gamma$-matrices, $\overline{\psi} := \psi^\dagger \gamma^0$ is the Dirac conjugate of $\psi = (\psi_0, \psi_1, \psi_2, \psi_3)^T$ with $^T$ being the matrix transposition sign, in products like $\overline{\psi}_\gamma^\mu$ the matrix multiplication sign is dropped, and $\circ$ denotes a composition combined with matrix multiplication, e.g., $\overline{\psi} \circ \psi = \sum_\mu \overline{\psi}_\mu \circ \psi_\mu = (\gamma^0)_{\mu\nu} \psi_\mu \circ \psi_\nu$. If $v$ and $\overline{\psi}$ denote variations of $\psi$ and $\overline{\psi}$, respectively, we, in view of definition (4.13), obtain

$$ \frac{\partial L}{\partial \psi}(v) = -\frac{1}{2}i\hbar c(\partial_\mu \overline{\psi}) \gamma^\mu \circ v - mc^2\overline{\psi} \circ v \quad \frac{\partial L}{\partial (\partial_\mu \psi)}(v) = \frac{1}{2}i\hbar c\gamma^\mu \circ v $$

$$ \frac{\partial L}{\partial \overline{\psi}}(\overline{\psi}) = \frac{1}{2}i\hbar c\gamma^\mu (\partial_\mu \psi) - mc^2\overline{\psi} \circ \psi \quad \frac{\partial L}{\partial (\partial_\mu \overline{\psi})}(\overline{\psi}) = -\frac{1}{2}i\hbar c\gamma^\mu \psi. \quad (5.22) $$

Thus, the basic relation (4.13) takes the form

$$ \{i\hbar c(\partial_\mu \overline{\psi}) \gamma^\mu + mc^2\overline{\psi}\} \circ v + \overline{\psi} \circ \{i\hbar c(\partial_\mu \psi) - mc^2\psi\} = 0. \quad (5.23) $$

The choices when $v$ or $\overline{\psi}$ is the zero operator and $\overline{\psi}$ or $v$, respectively, is arbitrary, result in the Dirac equation and its conjugate, viz.

$$ i\hbar c \gamma^\mu (\partial_\mu \psi) - mc\psi = 0 \quad i\hbar c (\partial_\mu \overline{\psi}) \gamma^\mu + mc\overline{\psi} = 0. \quad (5.24) $$

\[12\] More precisely, the choice when all but one of the components $v_\alpha$ and $\overline{\psi}_\alpha$ vanish, results in the (conjugate) Dirac equation for this component; the substitution of these results into (5.24) converts it into identity relative to $v_\alpha$ and $\overline{\psi}_\alpha$. 

These equations convert (5.23) into identity relative to \( v \) and \( \overline{\psi} \). So, the variations of a free spinor field are completely arbitrary. Combining (5.22) with (4.17)–(4.20), we get the energy-momentum, current and spin angular momentum operators as:

\[
T_{\mu\nu} = \frac{i}{\hbar} \{ \overline{\psi} \gamma_\mu \odot (\partial_\nu \psi) - (\partial_\nu \overline{\psi}) \gamma_\mu \odot \psi \} \tag{5.25}
\]

\[
J_\mu = q c \overline{\psi} \gamma_\mu \odot \psi \tag{5.26}
\]

\[
S^\lambda_{\mu\nu} = -\frac{1}{4} \hbar c \overline{\psi} (\gamma^\lambda \sigma_{\mu\nu} + \sigma_{\mu\nu} \gamma^\lambda) \odot \psi \quad \sigma_{\mu\nu} := \frac{i}{2} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu), \tag{5.27}
\]

where we have used that a 4-rotation \( x^\mu \mapsto x^\mu + \varepsilon^{\mu\nu} x_\nu \) implies \( \psi \mapsto e^{-\frac{i}{2} \sigma_{\mu\nu} \varepsilon^{\mu\nu}} \psi \) and \( \overline{\psi} \mapsto e^{+\frac{i}{2} \sigma_{\mu\nu} \varepsilon^{\mu\nu}} \overline{\psi} \). These expressions are identical with the ones in [3].

5.6. Free Dirac (spinor) field
2. Charge symmetric Lagrangian

In the present subsection, we shall present an example of a Lagrangian for which the Euler-Lagrange equations are identities, like \( 0 = 0 \), but which, regardless of this ‘strange’ fact, entails completely reasonable field equations. In a classical sense, this Lagrangian is a ‘completely singular’ one, as all its classical derivatives are zero, but it is not a constant operator.

Let \( \psi \) be a Dirac 4-spinor and (see, e.g., [3,3,13])

\[
\dot{\psi} := C \overline{\psi} \mathsf{T} = (\overline{\psi} C^\mathsf{T}) \mathsf{T} \tag{5.28}
\]

be its charge conjugate one, where the matrix \( C \) satisfies the conditions

\[
C^{-1} \gamma^\mu C = -\gamma^\mu \mathsf{T} := -(\gamma^\mu)^\mathsf{T} \quad C^\mathsf{T} = -C. \tag{5.29}
\]

Let us consider \( \psi \) and \( \dot{\psi} \) as independent field variables. In their terms, the Lagrangian (5.21) reads

\[
L' = -\frac{1}{2} \hbar c \{ (\dot{\psi}^\mathsf{T} (x) C^{-1} \gamma^\mu (x) \odot \partial_\mu \psi (x) - (\partial_\mu \dot{\psi}^\mathsf{T} (x)) C^{-1} \gamma^\mu \odot \psi (x)) \} + mc^2 \dot{\psi}^\mathsf{T} (x) C^{-1} \odot \psi (x). \tag{5.30}
\]

We would like to emphasize on the change of the signs and the appearance of the matrix \( C \) in (5.30) with respect to (5.21). An alternative to this Lagrangian is a one with changed positions of \( \psi \) and \( \dot{\psi} \), viz.

\[
L'' = -\frac{1}{2} \hbar c \{ \psi^\mathsf{T} (x) C^{-1} \gamma^\mu (x) \odot (\partial_\mu \dot{\psi} (x)) - (\partial_\mu \psi^\mathsf{T} (x)) C^{-1} \gamma^\mu \odot \dot{\psi} (x) \} + mc^2 \psi^\mathsf{T} (x) C^{-1} \odot \dot{\psi} (x). \tag{5.31}
\]

Evidently, the variables \( \psi \) and \( \dot{\psi} \) do not enter in (5.30) and (5.31) on equal footing. We shall try to ‘symmetrize’ the situation by considering a Lagrangian which is the half sum of the last two ones, i.e.

\[
L''' = \frac{1}{4} \hbar c \{ -\dot{\psi}^\mathsf{T} (x) C^{-1} \gamma^\mu (x) \odot (\partial_\mu \psi (x)) + (\partial_\mu \dot{\psi}^\mathsf{T} (x)) C^{-1} \gamma^\mu \odot \psi (x) - \psi^\mathsf{T} (x) C^{-1} \gamma^\mu \odot (\partial_\mu \dot{\psi} (x)) + (\partial_\mu \psi^\mathsf{T} (x)) C^{-1} \gamma^\mu \odot \dot{\psi} (x) \}
\]

\[
+ \frac{1}{2} mc^2 \{ \psi^\mathsf{T} (x) C^{-1} \odot \psi (x) + \dot{\psi}^\mathsf{T} (x) C^{-1} \odot \dot{\psi} (x) \}. \tag{5.32}
\]
Let $v$ and $\bar{v}$ denote variations of $\psi$ and $\bar{\psi}$, respectively. Applying definition 4.1, we can calculate the derivatives of the Lagrangians (5.30)-(5.32). They are as follows:

$$
\frac{\partial L'}{\partial \psi}(v) = +\frac{1}{2} \text{i} \hbar c \{ C^{-1} \gamma^\mu (\partial_\mu \bar{\psi}) \}^\top \circ v - mc^2 \{ C^{-1} \bar{\psi} \}^\top \circ v
$$

$$
\frac{\partial L'}{\partial \bar{\psi}}(\bar{v}) = -\frac{1}{2} \text{i} \hbar c \bar{v}^\top \circ C^{-1} \gamma^\mu (\partial_\mu \psi) + mc^2 v^\top \circ C^{-1} \bar{\psi}
$$

$$
\frac{\partial L'}{\partial (\partial_\mu \psi)}(v) = -\frac{1}{2} \text{i} \hbar c \{ C^{-1} \gamma^\mu \bar{\psi} \}^\top \circ v \quad \frac{\partial L'}{\partial (\partial_\mu \bar{\psi})}(\bar{v}) = +\frac{1}{2} \text{i} \hbar c \bar{v}^\top \circ C^{-1} \gamma^\mu \psi
$$

$$
\frac{\partial L''}{\partial \psi}(v) = -\frac{1}{2} \text{i} \hbar c \bar{v}^\top \circ C^{-1} \gamma^\mu (\partial_\mu \bar{\psi}) + mc^2 v^\top \circ C^{-1} \bar{\psi}
$$

$$
\frac{\partial L''}{\partial \bar{\psi}}(\bar{v}) = +\frac{1}{2} \text{i} \hbar c \{ C^{-1} \gamma^\mu (\partial_\mu \psi) \}^\top \circ \bar{v} - mc^2 \{ C^{-1} \psi \}^\top \circ \bar{v}
$$

$$
\frac{\partial L''}{\partial (\partial_\mu \psi)}(v) = +\frac{1}{2} \text{i} \hbar c \bar{v}^\top \circ C^{-1} \gamma^\mu \psi - \frac{1}{2} \text{i} \hbar c \{ C^{-1} \gamma^\mu \bar{\psi} \}^\top \circ \bar{v}
$$

Notice, in these equalities the matrix transposition serves only to ensure proper matrix multiplication. Substituting the just calculated derivatives in the basic equation (4.13), we see that for the Lagrangians (5.30)-(5.32) it reduces respectively to

$$
\left( A(\bar{\psi}) \right)^\top \circ v - \bar{v}^\top \circ A(\psi) = 0 \quad (5.33)
$$

$$
-v^\top \circ A(\bar{\psi}) + (A(\psi))^\top \circ \bar{v} = 0 \quad (5.34)
$$

$$
\left( A(\bar{\psi}) \right)^\top \circ v - v^\top \circ A(\bar{\psi}) + (A(\psi))^\top \circ \bar{v} - \bar{v}^\top \circ A(\psi) = 0 \quad (5.35)
$$

where

$$
A: \psi \mapsto A(\psi) := \text{i} \hbar c C^{-1} \gamma^\mu \partial_\mu \psi - mc^2 C^{-1} \bar{\psi} \quad (5.36)
$$

Let $\alpha \in \{0, 1, 2, 3\}$ be arbitrarily fixed. Choosing all but the $\alpha$th component of $v = 0$ (resp. $\bar{v} = 0$) to vanish and setting this component to be equal to the identity mapping $id_F$, we see that (5.33) or (5.34) is equivalent to the equations $A(\psi_\alpha) = 0$ and $A(\bar{\psi}_\alpha) = 0$ for all $\alpha = 0, 1, 2, 3$, which, by virtue of (5.36), are equivalent to the system of Dirac equations

$$
\text{i} \hbar c \gamma^\mu \partial_\mu \psi - mc^2 \bar{\psi} = 0 \quad \text{i} \hbar c \gamma^\mu \partial_\mu \bar{\psi} - mc^2 \psi = 0. \quad (5.37)
$$

However, if one substitutes in (5.33) the just-described choices of $v$ and $\bar{v}$, one will get the identity $0 = 0$ for any one of them, instead of some equations for the components of $\psi$ and $\bar{\psi}$. In this sense, classical Euler-Lagrange equations for the Lagrangian (5.32) do not exist.
In this case, which cannot be handled by the standard methods, we shall proceed as follows. Let us make the same as the above selection of the variations \( v \) and \( \bar{v} \) without setting the non-vanishing components to be equal to \( \text{id}_\mathcal{F} \). Such choices reduces (5.33) to the next system of relations (do not sum over \( \alpha \))

\[
[A(\bar{\psi}_\alpha), v_\alpha] = 0 \quad [A(\psi_\alpha), \bar{v}_\alpha] = 0
\]  

(5.38)

for the variations \( v_\alpha \) and \( \bar{v}_\alpha \). If we now choose the operators \( v_\alpha \) and \( \bar{v}_\alpha \) to range in a unitary representation in \( \mathcal{F} \) of some group, then the Schur’s lemma\( ^[3] \) implies the existence of classical functions, not operators, \( f_{\psi,\alpha}(x) \) and \( f_{\bar{\psi},\alpha}(x) \), such that

\[
A(\psi_\alpha(x)) = f_{\psi,\alpha}(x) \text{id}_\mathcal{F} \quad A(\bar{\psi}_\alpha(x)) = f_{\bar{\psi},\alpha}(x) \text{id}_\mathcal{F}.
\]  

(5.39)

In view of (5.36), these equations are equivalent to the system

\[
\text{i}h\gamma^\mu \partial_\mu \psi - mc^2 \psi = \chi_\psi \text{id}_\mathcal{F} \quad \text{i}h\gamma^\mu \partial_\mu \bar{\psi} - mc^2 \bar{\psi} = \bar{\chi}_\psi \text{id}_\mathcal{F}.
\]  

(5.40)

where \( \chi_\psi \) and \( \bar{\chi}_\psi \) are some classical, not operator-valued, spinors. It is trivial to be checked that (5.40) converts (5.33) into identity with respect to \( v \) and \( \bar{v} \). Therefore the system of (Dirac equations with, generally, non-vanishing r.h.s.) (5.40) plays a role of a system of field equations for the Lagrangian (5.32).

Ending this example, we note that if one wants the Lagrangian (5.32) to describe a free spinor field, the choices

\[
\chi_\psi = 0 \quad \bar{\chi}_\psi = 0
\]  

(5.41)

should be made; otherwise, the equations (5.40) describe a spin \( \frac{1}{2} \) quantum field with some selfinteraction. Elsewhere we shall demonstrate that the Lagrangians (5.30)–(5.32) and the additional conditions (5.41) lead to one and the same quantum field theory of free spinor fields.

5.7. General quadratic Lagrangian

Let us consider a system of quantum fields \( \varphi_i \) with a Lagrangian

\[
L = a^i \varphi_i + m^{ij} \varphi_i \circ \varphi_j + b^{ij} \partial_\mu \varphi_i + g^{ij\mu}(\partial_\mu \varphi_i) \circ (\partial_\nu \varphi_j) + c^{ij\mu} \varphi_i \circ (\partial_\mu \varphi_j) + d^{ij\mu}(\partial_\mu \varphi_j) \circ \varphi_i,
\]  

(5.42)

where \( a^i, m^{ij}, b^{ij}, g^{ij\mu}, c^{ij\mu}, \) and \( d^{ij\mu} \) are some (dimensional) constants. Calculating \( \frac{\partial L}{\partial \varphi_i} \) and \( \pi^{\mu i} = \frac{\partial L}{\partial (\partial_\mu \varphi_i)} \), according to definition 4.4, and substituting them into (1.13), we get the relation

\[
\alpha^i \circ v_i + v_i \circ \beta^i + a^i v_i = 0
\]  

(5.43)

where

\[
\alpha^i := m^{ij} \varphi_j + (c^{ij\mu} - d^{ij\mu}) \partial_\mu \varphi_j - g^{ij\mu} \partial_\mu \partial_\nu \varphi_j
\]

(5.44)

\[
\beta^i := m^{ij} \varphi_j + (d^{ij\mu} - c^{ij\mu}) \partial_\mu \varphi_j - g^{ij\mu} \partial_\mu \partial_\nu \varphi_j
\]

and \( v_i = \delta_0 \varphi_i \) is a variation of \( \varphi_i \). Choosing \( v_i = f_i(x) \text{id}_\mathcal{F} \) with arbitrary \( f: M \to \mathbb{C} \), from (5.43), we obtain the Euler-Lagrange equations

\[
\alpha^i + \beta^i + a^i \text{id}_\mathcal{F} = 0.
\]  

(5.45)
The combination of (5.45) and (5.43) results in the condition
\[ \sum_i [\alpha^i, v_i] = 0 \]  
(5.46)

which is equivalent to \[ \sum_i [\beta^i, v_i] = 0 \]  
for the field operators \( \varphi_i \) and their variations \( v_i \). These conditions can be satisfied identically relative to \( v_i \) if all \( \alpha^i \) (and, hence, all \( \beta^i \)) happen to be proportional to the identity operator \( \text{id}_F \). In particular, this will be the case when
\[
m_{ij} = m_{ji} \quad c_{ij \mu} - d_{ji \mu} = d_{ij \mu} - c_{ji \mu} \quad g^{i \mu j \nu} = g^{j \nu i \mu},
\]  
(5.47)

so that \( \alpha^i = \beta^i \) (see (5.44)) and the field equations (5.45) read
\[
\alpha^i + \frac{1}{2} a^i \text{id}_F = 0.
\]  
(5.48)

As a special case of (5.47), the one of free fields should be single out. In this important case, we have (do not sum over \( i \! \! \! \! = \! \! \! \! 1 \! \! \! \! )
\[
a^i = b^i = c^{ij \mu} = d^{ij \mu} = 0 \quad m_{ij} = m_{ij}^2 c^{4 \delta_{ij}} \quad g^{i \mu j \nu} = g^{j \nu i \mu},
\]  
(5.49)

where \( m_i \) and \( g^{\mu \nu} = g^{\nu \mu} \) are some (dimensional) constants.

However, having in mind the considerations in Subsect. 5.6, one should take into account the possibility that the Euler-Lagrange equations (5.43) may turn to be identities, in which case the field equations should be derived from (5.43) in a different way.

The energy-momentum operator corresponding to the Lagrangian (5.42), in view of (4.17), is
\[
T^{\mu \nu} = b^{i \mu} \partial^\nu \varphi_i + g^{i \mu j \lambda} \partial^\nu \varphi_j \circ (\partial_\lambda \varphi_i) + g^{j \lambda i \mu} \partial_\lambda \varphi_j \circ (\partial^\nu \varphi_i)
\]
\[
+ c^{i j \mu} \varphi_j \circ (\partial^\nu \varphi_i) + d^{i j \mu} \partial_\nu \varphi_i \circ \varphi_j - \eta^{\mu \nu} L.
\]  
(5.50)

If the system possesses a charge, the corresponding current operator can be obtained formally from the r.h.s. of (5.50) by deleting the last term and replacing \( \partial^\nu \) with \( \frac{1}{i \hbar} \delta \varepsilon (\varphi_i) \) (see (4.18)).

6. Conclusion

In this paper we have given an analysis of some aspects and corollaries of the Schwinger’s action principle in (canonical) quantum field theory. As it was demonstrated, in the ‘standard’ presentation, this variational principle contains an additional hypothesis, which does not follow logically from the rest of the theory and modifies the conserved quantities so that they do not always have the required properties. We have removed the mentioned hypothesis by giving a suitable meaning of derivatives of operator-valued functions of operator arguments with respect to such an argument. After that modification is done, the following important consequences of the Schwinger’s action principle were derived:

i. The classical (standard) Euler-Lagrange equation of motion for the field operators remain the same as before the change.

ii. The conserved quantities (operators) are changed so that they have the required properties, at least in the examples considered.

iii. In the general case (of (self-)interacting fields), the variations of the field operators cannot be completely arbitrary as they should satisfy some equations in which the field operators, satisfying the Euler-Lagrange equations, enter.
iv. Any variations of the field operators proportional to the identity mapping/operator, like (3.26), are always admissible.

v. The variations if item iv above are sufficient for the derivation of the Euler-Lagrange equations and all spacetime conserved quantities. However, for other variations, such as the ones connected with internal symmetries (e.g. like (constant) phase transformations leading to charge conservation), one should always check whether they are admissible in a sense that they must satisfy the equations mentioned in point iii above.

vi. If one insists on keeping the field variations completely arbitrary, it is quite likely that, for interacting fields, field operators which are multiples of the identity operator will be the only solutions of the variational problem determining them.\footnote{For instance, such is the case considered in Subsect. 5.4: if (5.19) holds for any \( v_1 \) and \( v_2 \), then \( \varphi_a(x) = f_a(x) \text{id}_\mathbb{C} \), \( a = 1, 2 \), for some \( f_a: M \to \mathbb{C} \) (as a result of, e.g., Schur’s lemma — see \([10\), sec. 8.2\] or \([11\), ch. 5, sec. 3\]).}

At the end, since the Euler-Lagrange equations are not changed after the described correction of Schwinger’s action principle, the modification in the structure of conserved quantities (operators), and, possibly, other equations for the fields (and their variations) should be regarded as the main outcome of the present investigation.

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