A Piecewise Parabolic Method for Estimating Key Parameters in a Three-dimensional Internal Tidal Adjoint Model

Xianglong Cao¹, Wang Gao², Mingxuan Deng², Ning Li³ and Dong Jiang⁴,*

¹School of Mechanical Engineering, Dalian Jiaotong University, Dalian, China
²Civil Engineering College, Dalian Jiaotong University, Dalian, China
³School of Science, Dalian Jiaotong University, Dalian, China
⁴School of basic sciences for aviation, Naval Aviation university, Yantai, China

*Corresponding author email: 396846560@qq.com

Abstract. A higher-order and conservative Piecewise Parabolic Method was proposed to estimate the open boundary conditions (OBCs) in a 3D internal tidal adjoint model. The interpolation quality was verified by twin experiments. The results showed that the PPM can obtain smoother OBCs than the results obtained by Cressman interpolation. In real experiments, compared with the CI results, the PPM results can reduce the magnitude of misfit vector (m) between the simulated results and the observations of the tidal stations by 5 percent. Meanwhile, the PPM results can simulate the $M_2$ tidal constituent in the northern South China Sea (SCS) more accurate than the Cressman results.

Keywords: Piecewise Parabolic Method; Tinternal tidal adjoint model; Open boundary conditions.

1. Introduction

For the research of internal tides, open boundary conditions (OBCs) are the key parameters and absolutely crucial to the simulation results of the internal tides. Among numerous internal tides researches, a three-dimensional internal tidal adjoint model used in [1] has been proved to be more effective in obtaining reasonable OBCs estimation in the case of insufficient observation [2-4]. In the use of this adjoint assimilation model, researchers have been trying to introduce high precision interpolation to improve the parameter estimations. Jiang et al. introduced a three-dimensional Spline interpolation, which obtained much smoother estimated OBCs than the Cressman interpolation and the improved results were closer to the reality [1]. A higher-order and conservative Piecewise Parabolic Method (PPM) interpolation was adopted to optimize independent point scheme (IPS) in a PM2.5 adjoint model, and the results showed that the high precision PPM can effectively improve the model simulation accuracy and obtain a PM2.5 spatial distribution which was closer to the reality [5]. However, the PPM has not studied in the field of tides. Due to the high precision, the PPM has a certain application value for the inversion of the OBCs in the tidal adjoint model. We will discuss the interpolation effect of the PPM in the following paragraphs.

In this paper, PPM is adopted in the three-dimensional internal tidal adjoint model and the interpolation effect will be explored. This paper is organized as follows. Section 2 describes the Piecewise Parabolic interpolation Method. Section 3 shows the OBCs estimation and the simulation results. Section 4 presents the summary.
2. The Piecewise Parabolic Method

In this paper, the piecewise parabolic interpolation distribution adopted in the IPS in the adjoint model is followed the descriptions in [5]. Detailed fourth-order approximation 3D PPM scheme are described as follows.

We divide the three dimensional computing domain $\Omega$ into $I \times J \times K$ regular Eulerian cells. Let $(x_i, y_j, z_k)$ be the independent point and $C_{i,j,k}$ be the TN concentration value at the independent point.

On the cell $\Omega_{i,j,k} = \left( x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}} \right) \times \left( y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}} \right) \times \left( z_{k-\frac{1}{2}}, z_{k+\frac{1}{2}} \right)$ at time $t = t_n$, where $x_{i-\frac{1}{2}} = \frac{1}{2}(x_{i-1} + x_i)$, $y_{j-\frac{1}{2}} = \frac{1}{2}(y_{j-1} + y_j)$, $z_{k-\frac{1}{2}} = \frac{1}{2}(z_{k-1} + z_k)$, the Piecewise Parabolic interpolation distribution $\left[ \mathcal{R}C^n \right]_{i,j}(x)$ along $x$-direction is defined as follows.

\[
\left[ \mathcal{R}C^n \right]_{i,j}(x) = C^n_{i-\frac{1}{2}} + \frac{x - x_{i-\frac{1}{2}}}{h} \left( \Delta C^n_i + C^n_{i+\frac{1}{2}} \frac{x_{i+\frac{1}{2}} - x}{h} \right)
\]  

(1)

\[
\Delta C^n_i = \langle C^n \rangle_{i+\frac{1}{2}} - \langle C^n \rangle_{i-\frac{1}{2}}
\]  

(2)

\[
C^n_{i+\frac{1}{2}} = 6\left( C^n_i - \left( \langle C^n \rangle_{i+\frac{1}{2}} + \langle C^n \rangle_{i-\frac{1}{2}} \right) / 2 \right)
\]  

(3)

\[
\langle C^n \rangle_{i+\frac{1}{2}} = \frac{7}{12} \left( C^n_i + C^n_{i+1} \right) - \frac{1}{12} \left( C^n_{i+2} + C^n_{i-1} \right).
\]  

(4)

Where $\langle C^n \rangle_{i+\frac{1}{2}}$ is the value at the point of $x_{i+\frac{1}{2}}$. Meanwhile, Taylor expansion is adopted in the boundary region. This interpolation distribution can obtain fourth-order approximation Scheme of the PPM and preserve mass conservation. The computations along $y$-direction and $z$-direction are as similar as the equations 1 through 4.

3. Twin Experiments

The credibility of the PPM in the 3D internal tidal model is evaluated by twin experiments. The model region is fixed in the SCS, from 17°30' N to 24°10' N and from 116° E to 124° E (see Figure 1). The horizontal resolution is 5'×5', and there are total 97×81 horizontal grids in the discussion area. The horizontal eddy viscosity coefficient $A_h$ is fixed as 1000 m²/s. We fix coefficients of bottom $\kappa$ and interface friction $A_v$ as constant, $\kappa = 2.0 \times 10^{-3}$ and $A_v = 3 \times 10^{-2}$ m²/s, respectively. The angular frequency of $M_2$ tide is $1.4050789025 \times 10^{-4}$ m/s and the whole-time step is 496.863 s (1/90 of the period of $M_2$ constituent). We adopt the same vertical profile of initial potential density shown in [1], the undisturbed interface of the adjoint model is placed at the depth of 500m. Meanwhile, we will carry out a targeted estimation study on the eastern open boundary of the computing area, because it has been proved to be the important resource for tidal forces in the SCS [1-4].

We use the same 3D Internal Tidal adjoint model as in Jiang and Lv [1]. To solve the ill-posed caused by the large dimension of the OBCs, the independent point (IP) scheme was adopted in all the experiments. The positions of the independent points and observation sites are proposed in Figure 1. The experimental procedure is shown in Figure 2.
Figure 1. Map of the computing area. The red ‘*’ presents independent points. Blue stars present positions of tidal observation stations. Black dots are observation positions along the T/P ground tracks [Jiang].

Figure 2. Experimental flow chart.

3.1. Twin Experiments and Numerical Results
In this part, performances of the PPM are compared with the Cressman Interpolation (CI) results under the same experimental settings.

For the open boundary grid point \((i, j)\), the height of water level \(\zeta\) of \(M_2\) tidal force at the \(n\)-th time step is subject to

\[
\zeta_{i,j}^n = a_{i,j} \cos(\omega_n \Delta t) + b_{i,j} \sin(\omega_n \Delta t)
\]

Where \(\omega\) is the frequency of \(M_2\) constituent, \(\Delta t\) denotes the time step length. The Fourier coefficients \((a_{i,j}, b_{i,j})\) are used as the adjustable parameters of the model in this study. And only the estimation of the Fourier coefficient \(a\) is carried out, for simplicity. \(b\) is set to be 0.

The Fourier coefficient \(a\) is defined by trigonometric function \(a_j = 0.5 \cdot \sin \frac{2k\pi(j - 3)}{n_y - 1}\), where \(j\) represents the meridional indexes of the open boundary point, \(n_y\) shows the total number of the open boundary points. \(k\) is used to represent the complexity of the spatial distribution.
Table 1. K values of the prescribed spatial distributions and the corresponding IPS.

| Distribution number | 1 | 2 | 3 | 4 |
|---------------------|---|---|---|---|
| k                   | 0.5 | 1.0 | 1.5 | 2.0 |
| Independent points  | 5 | 5 | 7 | 9 |

Four given spatial distributions of \( a \) are shown in Figure 3. Estimated results obtained by PPM and CI are also presented in Figure 3. As can be seen clearly from the figures, although both of the two interpolation methods can better invert the given distributions, the open boundary curve obtained by the PPM is more smooth. Combined with Figure 4, it can be found that, for the four kinds of given open boundary curves, the larger errors of the open boundary curves retrieved by linear interpolation scheme are mainly concentrated in the middle and two end points of the curve, mainly because the given distribution has strong nonlinear changes in these places. However, the traditional CI is unable to describe the nonlinear change accurately. In addition, it can be seen from Figure 4 that, with the increase of the number of independent points, the CI will produce a "bull's eye" phenomenon, that is, the inversion error around the independent points is large, while the PPM results do not present the "bull's eye" phenomenon. On the contrary, the overall error of the open boundary condition obtained by PPM is very small, so smooth open boundary curves can be obtained, which is more in line with the actual physical significance compared with CI.

Figure 3. Given and inverted OBCs of Fourier coefficients in twin experiments.

Figure 4. Inversion errors of Fourier coefficients \( a \) in twin experiments.
3.2. Practical Experiment

According to the conclusion of Sections 3.1 and 3.2, the OBCs are estimated and co-tidal charts for the $M_2$ constituent in the SCS are simulated by the two interpolations, respectively. The co-tidal charts for the $M_2$ constituent in the SCS are shown in Figure 5. The T/P data are used to test the simulations. Magnitude of misfit vector between the simulated results and the observations of the tidal stations are shown in Table 2.

![Figure 5](image-url)  
*Figure 5. Co-tidal charts for the $M_2$ constituent in the SCS. (a) is obtained by CI and (b) is by PPM.*

| Tidal Station Index | Exp | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 |
|---------------------|-----|----|----|----|----|----|----|----|----|----|----|
| CI                  |     | 0.027 | 0.114 | 0.058 | 0.043 | 0.040 | 0.09 | 0.032 | 0.073 | 0.053 | 0.093 |
| PPM                 |     | 0.032 | 0.047 | 0.052 | 0.041 | 0.021 | 0.081 | 0.037 | 0.053 | 0.037 | 0.043 |

The co-tidal charts for the $M_2$ constituent obtained by the two interpolation methods are both presented the characteristics of the SCS. The magnitude of misfit vector for PPM is smaller than that for the CI. Therefore, the PPM can effectively improve the simulations in realistic domain to some extent.

4. Summary

A high order, Piecewise Parabolic interpolation is presented for the OBCs reconstruction in the 3D internal tidal adjoint model. In this paper, PPM presents its effectiveness of solving the ill-posedness in all the experiments. And the numerical results show that PPM is more effective than the CI in parameter reconstructions and model simulations. In the practical cases, compared with the CI results, the PPM gains the rational co-tidal charts for the $M_2$ constituent in the SSC, with smaller magnitude of misfit vector. Future research will focus on the applications of the Piecewise Parabolic interpolation in the estimation of other important parameters in the 3D internal tidal adjoint model.
Acknowledgments
This research was financially supported by the National Science Foundation of Liaoning Province (20180550292).

References
[1] Dong J, Haibo C, Guangzhen J and Xianqing L 2018 Estimating smoothly varying open boundary conditions for a 3D internal tidal model with an improved independent point scheme Journal of Atmospheric and Oceanic Technology (Electronic Materials vol 35) chapter 6 pp 173–214
[2] C Zhang and X.Q. Lv 2008 Parameter estimation for a three-dimensional numerical barotropic tidal model with adjoint method International Journal for Numerical Methods in Fluids chapter 57 pp 47–92
[3] J.C Zhang and X.Q. Lv 2010 Inversion of three-dimensional tidal currents in marginal seas by assimilating satellite altimetry Computer Methods in Applied Mechanics & Engineering chapter 199, pp 3125–3136.
[4] H.B.Chen, A.Z. Cao J.C. Zhang C.B. Miao and X.Q. Lv 2014 Estimation of spatially varying open boundary conditions for a numerical internal tidal model with adjoint method Mathematics & Computers in Simulation, chapter 97, pp 14-38.
[5] Ning Li Yongzhi Liu Xianqing LV Jicai Zhang and Kai FU 2017 The High Order Conservative Method for the Parameters Estimation in a PM2.5 Transport Adjoint Model Advances in Meteorology pp 1-13