Dynamical Linked Cluster Expansions for Spin Glasses

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Abstract

Dynamical linked cluster expansions are linked cluster expansions with hopping parameter terms endowed with their own dynamics. This amounts to a generalization from 2-point to point-link-point interactions. An associated graph theory with a generalized notion of connectivity is reviewed. We discuss physical applications to disordered systems, in particular to spin glasses, such as the bond-diluted Ising model and the Sherrington-Kirkpatrick spin glass. We derive the rules and identify the full set of graphs that contribute to the series in the quenched case. This way it becomes possible to avoid the vague extrapolation from positive integer \(n\) to \(n = 0\), that usually goes along with an application of the replica trick.

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1 Introduction

Linked cluster expansions (LCEs) have a long tradition in statistical physics. Originally applied to classical fluids, later to magnetic systems ([1],[2],[3] and references therein), they were generalized to applications in particle physics in the eighties [4]. There they have been used to study the continuum limit of a lattice $\Phi^4$ field theory in 4 dimensions at zero temperature. In [5, 6] they were further generalized to field theories at finite temperature, simultaneously the highest order in the expansion parameter was increased to 18. Usually the analytic expansions are obtained as graphical expansions. Because of the progress in computer facilities and the development of efficient algorithms for generating the graphs, it is nowadays possible to handle the order of billions of graphs. The whole range from high temperatures down to the critical region becomes available, and thermodynamic quantities like critical indices and critical temperatures are determined with high precision (the precision is comparable or even better than in corresponding high quality Monte Carlo results) [6]-[9]. An extension of LCEs to a finite volume in combination with a high order in the expansion parameter turned out to be a particularly powerful tool for investigating the phase structure of systems with first and second order transitions by means of a finite size scaling analysis [10].

Linked cluster expansions are series expansions of the free energy and connected correlation functions about an ultralocal, decoupled theory in terms of a hopping parameter $K$. The corresponding graphical representation is a sum in terms of connected graphs. The value of $K$ parametrizes the strength of interactions between fields at different lattice sites. Usually they are chosen as nearest neighbours. In contrast to the ultralocal terms of a generic interaction we will sometimes refer to hopping terms as non-ultralocal.

In this paper we develop dynamical linked cluster expansions (DLCEs). These are linked cluster expansions with hopping parameter terms that are endowed with their own dynamics. Such systems are realized in spin glasses with (fast) spins and (slow) interactions [11]-[13]. They also occur in variational estimates for SU(N)-gauge-Higgs systems, cf. [14]. Like LCEs they are expected to converge for a large class of interactions.

Formally DLCEs amount to a generalization of an expansion scheme from 2-point to point-link-point-interactions. These are interactions between fields associated with two points and with one pair of points called link. The points and links are not necessarily embedded on a lattice, and the links need not be restricted to nearest neighbours. We have developed a new multiple-line graph theory in which a generalized notion of connectivity plays a central role. Standard notions of equivalence classes of graphs like 1-line irreducible and 1-vertex irreducible graphs have been generalized, and new notions like 1-multiple-line irreducible graphs were defined in order to give a systematic classification.

The paper is organized as follows. In Sec. 2 we specify the models that admit a DLCE. We introduce multiple-line graphs and explain the idea behind the abstract
notions of multiple-line graph theory. Detailed definitions of multiple-line graphs, related notions and the computation of weights are given in Sect. 3. Sect. 4 treats the issue of renormalization in the sense of suitable resummations of graphs. Applications to spin glasses are presented in Sect. 5. There it is of particular interest that DLCEs allow for the possibility of avoiding the replica trick. In the quenched limit we derive that DLCEs must be restricted to a subclass of the corresponding graphical expansion, so-called quenched DLCEs (QDLCEs). We also list some examples for models whose phase structure is accessible to QDLCEs. To these belongs in particular the bond diluted Ising model. Sect. 6 contains the summary and conclusions.

2 A Short Primer to DLCEs

In this section we first specify the class of models for which we develop dynamical linked cluster expansions. Next we illustrate some basic notions of multiple-line graph theory, in particular the need for a new notion of connectivity.

By $\Lambda_0$ we denote a finite or infinite set of points. One of its realizations is a hypercubic lattice in $D$ dimensions, infinite or finite in some directions with the topology of a torus. $\Lambda_1$ denotes the set of unordered pairs $(x, y)$ of sites $x, y \in \Lambda_0$, $x \neq y$, also called unoriented links, and $\overline{\Lambda}_1$ a subset of $\Lambda_1$.

We consider physical systems with a partition function of the generic form

$$Z(H, I, v) \equiv \exp W(H, I, v)$$

$$= \mathcal{N} \int \mathcal{D}\phi \mathcal{D}U \exp \left(-S(\phi, U, v)\right) \exp \left(\sum_{x \in \Lambda_0} H(x)\phi(x) + \sum_{l \in \overline{\Lambda}_1} I(l)U(l)\right),$$

(1)

with measures

$$\mathcal{D}\phi = \prod_{x \in \Lambda_0} d\phi(x), \quad \mathcal{D}U = \prod_{l \in \overline{\Lambda}_1} dU(l)$$

(2)

and action

$$S(\phi, U, v) = \sum_{x \in \Lambda_0} S^0(\phi(x)) + \sum_{l \in \overline{\Lambda}_1} S^1(U(l)) - \frac{1}{2} \sum_{x, y \in \Lambda_0} v(x, y)\phi(x)U(x, y)\phi(y),$$

(3)

with non-ultralocal couplings

$$v(x, y) = v(y, x) \neq 0 \quad \text{only for } (x, y) \in \overline{\Lambda}_1,$$

in particular $v(x, x) = 0$. (4)

For later convenience the normalization via $\mathcal{N}$ is chosen such that $W[0, 0, 0] = 0$.

The field $\phi(x)$ is associated with the sites $x \in \Lambda_0$ and the field $U(l)$ lives on the links $l \in \overline{\Lambda}_1$, and we write $U(x, y) = U(l)$ for $l = (x, y)$. For definiteness and for simplicity of the notation here we assume $\phi(x) \in \mathbb{R}$ and $U(l) \in \mathbb{R}$. In our actual
applications to spin glasses the $\phi$s are the (fast) Ising spins and the $U$s $\in \mathbb{R}$ are the (slow) interactions. The action is split into two ultralocal parts, $S^0$ depending on fields on single sites, and $S^1$ depending on fields on single links $l \in \Lambda_1$. For simplicity we choose $S^1$ as the same function for all links $l \in \Lambda_1$. We may identify $\Lambda_1$ with the support of $v$,
$$\Lambda_1 = \{l = (x, y) \mid v(x, y) \neq 0\}. \tag{5}$$

The support of $v(x, y)$ need not be restricted to nearest neighbours, also the precise form of $S^0$ and $S^1$ does not matter for the generic description of DLCEs, $S^0$ and $S^1$ can be any polynomials in $\phi$ and $U$, respectively. The only restriction is the existence of the partition function.

Dynamical linked cluster expansions are induced from a Taylor expansion of $W(H, I, v) = \ln Z(H, I, v)$ about $v = 0$, the limit of a completely decoupled system. We want to express the series for $W$ in terms of connected graphs. Let us consider the generating equation
$$\frac{\partial W}{\partial v(x,y)} = \frac{1}{2} < \phi(x)U(x,y)\phi(y) >$$
$$= \frac{1}{2}(W_{H(x)I(x,y)H(y)} + W_{H(x)H(y)}W_{I(x,y)}) + W_{H(x)}W_{I(x,y)}W_{H(x)}$$
$$+ W_{H(x)}W_{H(y)}W_{I(x,y)} \tag{6}.$$

Here $\langle \cdot \rangle$ denotes the normalized expectation value w.r.t. the partition function of Eq. (1). Subscripts $H(x)$ and $I(x,y) = I(y,x) = I(l)$ denote the derivatives of $W$ w.r.t. $H(x)$ and $I(x,y)$, respectively.

Next we would like to represent the right hand side of Eq. (6) in terms of connected graphs. Once we have such a representation for the first derivative of $W$ w.r.t. $v$, graphical expansions for the higher derivatives can be traced back to the first one.

For each $W$ in Eq. (6) we draw a shaded bubble, for each derivative w.r.t. $H$ a solid line, called a $\phi$-line, with endpoint vertex $x$, and for each derivative w.r.t. $I$ a dashed line, called a $U$-line, with link label $l = (x, y)$. The main graphical constituents are shown in Fig. 1. Two $\phi$-lines with endpoints $x$ and $y$ are then joined by means of a dashed $U$-line with label $l$, if the link $l$ has $x$ and $y$ as its endpoints, i.e. $l = (x, y)$. According to these rules Eq. (6), multiplied by $v(x,y)$ and summed over $x$ and $y$, is represented by Fig. 2. Note that, because of the Taylor operation, each solid line from $x$ to $y$ carries a factor $v(x,y)$.

Since the actual need for a new type of connectivity is not quite obvious from Fig. 2, because Eq. (6) does not contain higher than first order derivatives w.r.t. $I$,
let us consider a term

\[ W_{H(x)}W_{H(y)}W_{H(r)}W_{H(s)}W_{I(x,y)}W_{I(r,s)} \]  

(7)

occurring in the second derivative of \( W \) w.r.t. \( v(x, y), v(r, s) \). According to the above rules this term would be represented as shown in Fig. 3a. While the 2 vertices in the last term of Fig. 2 are connected in the usual sense via a common (solid) line (the dashed line with an attached bubble could be omitted in this case), the graph in Fig. 3a would be disconnected in the old sense, since neither \( x \) nor \( y \) are line-connected with \( r \) and \( s \), but -as a new feature of DLCE graphs- the lines from \( x \) to \( y \) and from \( r \) to \( s \) are connected via the dashed lines emerging from a common bubble shown in the middle of the graph. As we see from Fig. 3a, we need an additional notion of connectivity referring to the possibility of multiple-line connectivity. While the analytic expression is fixed, it is a matter of convenience to further simplify the graphical notation of Fig. 3a at \( v = 0 \). Two possibilities are shown in Fig. 3b and Fig. 3c. To Fig. 3b we later refer in the formal definition.
of the new type of multiple-line connectivity. In the familiar standard notion of connectivity two vertices of a graph are connected via lines. The vertices are line-connected. Already there, in a dual language, one could call two lines connected via vertices. The second formulation is just appropriate for our need to define when two lines are connected. The corresponding vertices mediating the connectivity of lines are visualized by tubes, in Fig. 3b we have just one of them. The tubes should be distinguished from the former type of vertices represented as full dots which are connected via bare $\phi$-lines. In Fig. 3c we show a simplified representation of Fig. 3b that we actually use in graphical expansions.

Figure 3: Representation of $W_{H(x)}W_{H(y)}W_{H(r)}W_{H(s)}W_{I(x,y)I(r,s)}$. (a) according to the rules of Fig. 1 and 2, (b) same as (a), but at $v = 0$ and simplified for a formal definition of multiple-line connectivity, cf. section 3, (c) same as (b), but for use in the actual graphical representations.

The derivative terms have to be evaluated at $v = 0$. For $v = 0$ we have a decomposition of $W$ according to

$$W(H, I, v = 0) = \sum_{x \in \Lambda_0} W^\circ(H(x)) + \sum_{l \in \Lambda_1} W^1(I(l))$$

with

$$\exp W^\circ(H) \equiv Z^\circ(H) = \frac{\int_{-\infty}^{\infty} d\phi \exp (-S^\circ(\phi) + H\phi)}{\int_{-\infty}^{\infty} d\phi \exp (-S^\circ(\phi))}$$

and

$$\exp W^1(l) \equiv Z^1(I) = \frac{\int_{-\infty}^{\infty} dU \exp (-S^1(U) + IU)}{\int_{-\infty}^{\infty} dU \exp (-S^1(U))}.$$ 

In eqs (9,10) we have omitted any single site or single link dependence, because we assume that $S^\circ$ and $S^1$ are the same for all $x \in \Lambda_0$ and $l \in \Lambda_1$, respectively.
Therefore, at \( v = 0 \), the only non-vanishing derivatives of \( W \) are

\[
W_{H(x_1)H(x_2)\ldots H(x_n)}|_{v=0} = \frac{\partial^n W(\circ (H(x_1)))}{\partial H(x_1)^n} \cdot \delta_{x_1,x_2,\ldots,x_n} \tag{11}
\]

and

\[
W_{I(l_1)I(l_2)\ldots I(l_m)}|_{v=0} = \frac{\partial^m W(\circ (I(l_1)))}{\partial I(l_1)^m} \cdot \delta_{l_1,l_2,\ldots,l_m}, \tag{12}
\]

but mixed derivatives w.r.t. \( H \) and \( I \) vanish. As anticipated in Fig.s 3b and 3c, for \( v = 0 \) we replace the dashed bubbles and graphically distinguish between bubbles with \( \phi \)-lines and bubbles with \( U \)-lines. We define

\[
\begin{align*}
\left\langle \begin{array}{c}
\vdots \\
\end{array} \right\rangle & = \nu_{\circ}^c = \left( \frac{\partial^n W(\circ (H))}{\partial H^n} \right)_{H=0} \tag{13} \\
\left\langle \begin{array}{c}
\vdots \\
\end{array} \right\rangle & = \nu_{\circ}^c = \left( \frac{\partial^m W(\circ (I))}{\partial I^m} \right)_{I=0} \tag{14}
\end{align*}
\]

for a connected \( n \)-point vertex with \( n \geq 1 \) bare \( \phi \)-lines emerging from it and

\[
\begin{align*}
\left\langle \begin{array}{c}
\vdots \\
\end{array} \right\rangle & = \nu_{\circ}^c = \left( \frac{\partial^n W(\circ (H))}{\partial H^n} \right)_{H=0} \tag{13} \\
\left\langle \begin{array}{c}
\vdots \\
\end{array} \right\rangle & = \nu_{\circ}^c = \left( \frac{\partial^m W(\circ (I))}{\partial I^m} \right)_{I=0} \tag{14}
\end{align*}
\]

for a connected \( \nu \)-line consisting of \( \nu \) bare lines. If \( \nu = 1 \), we often omit the dashed line. If the bare lines of a \( \nu \)-line are internal \( \phi \)-lines, they get vertices attached to their endpoints, if they are external \( U \)-lines, no vertices will be attached.

Let \( V \) denote the lattice volume in \( D \) dimensions. The Taylor expansion of \( W \) about \( v = 0 \) to second order in \( v \) then reads

\[
W(H, I, v) = W(H, I, v = 0) + \sum_{x,y \in \Lambda_0} v(x, y) \frac{1}{2} W_{H(x)} W_{H(y)} W_{I(x,y)} \\
+ \frac{1}{2} \sum_{x,y,r,s \in \Lambda_0} \frac{1}{4} v(x, y) v(r, s) \cdot \\
\left( 4 W_{H(y)} W_{H(s)} W_{H(r)} W_{I(x,y)} W_{I(x,y)} W_{I(r,s)} \\
+ 2 W_{H(x)} W_{H(r)} W_{H(y)} W_{H(x)} W_{I(x,y)} W_{I(r,s)} \\
+ 4 W_{H(y)} W_{H(s)} W_{H(r)} W_{H(x)} W_{I(x,y)} W_{I(r,s)} \\
+ 2 W_{H(r)} W_{H(y)} W_{H(s)} W_{H(x)} W_{I(x,y)} W_{I(r,s)} \\
+ W_{H(x)} W_{H(y)} W_{H(r)} W_{H(s)} W_{I(x,y)} W_{I(r,s)} \right)_{v=0} + \mathcal{O}(v^3), \tag{15}
\]

where we have used that \( v(x, x) = 0 \). For each \( W \) in the products of \( W \)'s we now insert Eq.s (11),(12).
If we choose \( v \) in a standard way as next-neighbour couplings

\[
v(x, y) = 2K \sum_{\mu=0}^{D-1} (\delta_{x+\hat{\mu}, y} + \delta_{x-\hat{\mu}, y})
\]

with \( \hat{\mu} \) denoting the unit vector in \( \mu \)-direction, Eq. (15) becomes in a graphical representation at \( H = I = 0 \)

\[
\frac{W(0, 0, v)}{V} = (2K) \frac{1}{2} (2D) + (2K)^2 \left\{ \frac{1}{2} (2D)^2 + \frac{1}{4} (2D) \right\} + \frac{1}{8} 2(2D) + O(K^3).
\]

For clarity, here we have written explicitly the topological symmetry factors and the lattice embedding numbers. (Usually graphs represent their full weights including these factors.) Note that the first two graphs of the second order contribution also occur in a usual LCE with frozen \( U \)-dynamics, the next two differ by an additional dashed 2-line and the last one becomes even disconnected without the dashed line.

As usual, graphical expansions for correlation functions, in particular susceptibilities, are generated from \( W(H, I, v) \) by taking derivatives w.r.t. the external fields \( H \) and \( I \). Graphically this amounts to attaching external \( \phi \)-lines and \( U \)-lines with \( (1 \text{ endpoint}) \) attached to vertices, e.g.

\[
\frac{\partial W^1(I)}{\partial I} = -S_1(U_0) + IU_0, \\
\frac{\partial^2 W^1(I)}{\partial I^2} = U_0, \\
\frac{\partial^n W^1(I)}{\partial I^n} = 0 \quad \text{for all } n > 1,
\]

i.e., no \( n \)-lines do occur with \( n > 1 \). In this case it becomes redundant to attach dashed lines to bare lines. As mentioned above, in an LCE only the first three contributions would be left in Eq. (17).
3 Graphical expansion

3.1 Multiple-line graph theory

The definition of a multiple-line graph as it will be given here is adapted to the computation of susceptibilities and the free energy, where the sum is taken over all possible locations of the fields. The definition easily generalizes to correlation functions.

For details of the standard definition of graphs in the framework of linked cluster expansions and related notions we refer e.g. to [4, 5]. Here, for convenience, we briefly recall the very definition of a graph to point out the new properties of multiple-line graphs as defined below in this section.

A (standard LCE) graph or diagram is a structure

\[ \tilde{\Gamma} = (\tilde{L}_{\Gamma}, \tilde{B}_{\Gamma}, \tilde{E}_{\Gamma}, \tilde{\Phi}_{\Gamma}), \]  

(20)

where \( \tilde{L}_{\Gamma} \) and \( \tilde{B}_{\Gamma} \neq \emptyset \) are disjoint sets of internal lines and vertices of \( \tilde{\Gamma} \), respectively. \( \tilde{E}_{\Gamma} \) is a map

\[ \tilde{E}_{\Gamma} : \tilde{B}_{\Gamma} \rightarrow \{0, 1, 2, \ldots\}, \]

\[ v \rightarrow \tilde{E}_{\Gamma}(v) \]  

(21)

that assigns to every vertex \( v \) the number of external lines \( \tilde{E}_{\Gamma}(v) \) attached to it. Finally, \( \tilde{\Phi}_{\Gamma} \) is the incidence relation that assigns internal lines to their two endpoints.

A multiple-line graph or multiple-line diagram is a structure

\[ \Gamma = (L_{\Gamma}, M_{\Gamma}, B_{\Gamma}, E_{\Gamma}^{(\phi)}, E_{\Gamma}^{(U)}, \Phi_{\Gamma}, \Psi_{\Gamma}). \]  

(22)

\( L_{\Gamma}, M_{\Gamma} \) and \( B_{\Gamma} \) are three mutually disjoint sets,

\[ L_{\Gamma} = \text{set of bare internal lines of } \Gamma, \]  

(23)

\[ M_{\Gamma} = \text{set of multiple lines of } \Gamma, \]  

(24)

\[ B_{\Gamma} = \text{set of vertices of } \Gamma. \]  

(25)

\( E_{\Gamma}^{(\phi)} \) is a map

\[ E_{\Gamma}^{(\phi)} : B_{\Gamma} \rightarrow \{0, 1, 2, \ldots\}, \]

\[ v \rightarrow E_{\Gamma}^{(\phi)}(v) \]  

(26)

that assigns to every vertex \( v \) the number of bare external \( \phi \)-lines \( E_{\Gamma}^{(\phi)}(v) \) attached to \( v \). Every such \( \phi \)-line represents a field \( \phi \). The number of external \( \phi \)-lines of \( \Gamma \) is denoted by \( E_{\Gamma} = \sum_{v \in B_{\Gamma}} E_{\Gamma}^{(\phi)}(v) \). Similarly, \( E_{\Gamma}^{(U)} \) is a map

\[ E_{\Gamma}^{(U)} : M_{\Gamma} \rightarrow \{0, 1, 2, \ldots\}, \]

\[ m \rightarrow E_{\Gamma}^{(U)}(m) \]  

(27)
that assigns to every multiple line \( m \) the number of external \( U \)-lines \( E_{\Gamma}^{(U)}(m) \) attached to \( m \). Every such \( U \)-line represents a field \( U \) associated with a lattice link. The number of external \( U \)-lines of \( \Gamma \) is given by \( E_{\Gamma}^{(U)}(m) = \sum_{m \in \mathcal{M}_{\Gamma}} E_{\Gamma}^{(U)}(m) \).

Furthermore, \( \Phi_{\Gamma} \) and \( \Psi_{\Gamma} \) are incidence relations that assign bare internal lines to their endpoint vertices and to their multiple lines, respectively. We treat lines as unoriented. The generalization to oriented lines is easily done. More precisely, let \( (\mathcal{B}_{\Gamma} \times \mathcal{B}_{\Gamma})' \) be the set of unordered pairs of vertices \((v, w)\) with \( v, w \in \mathcal{B}_{\Gamma}, v \neq w \). (The bar implies unordered pairs, the prime the exclusions of \((v, v)\), \(v \in \mathcal{B}_{\Gamma}\).) As in the standard linked cluster expansion, self-lines are excluded. Every bare internal line is then mapped onto its pair of endpoints via

\[
\Phi_{\Gamma} : \mathcal{L}_{\Gamma} \rightarrow (\mathcal{B}_{\Gamma} \times \mathcal{B}_{\Gamma})'.
\]  

(28)

We say that \( v \) and \( w \) are the endpoint vertices of \( l \in \mathcal{L}_{\Gamma} \) if \( \Phi_{\Gamma}(l) = (v, w) \). If there is such an \( l \in \mathcal{L}_{\Gamma} \), \( v \) and \( w \) are called neighbours. Similarly, \( \Psi_{\Gamma} \) is a map

\[
\Psi_{\Gamma} : \mathcal{L}_{\Gamma} \rightarrow \mathcal{M}_{\Gamma},
\]

\[
l \rightarrow \Psi_{\Gamma}(l)
\]

(29)

that maps every bare internal line to a multiple line. A multiple line \( m \in \mathcal{M}_{\Gamma} \) is composed of bare internal lines \( l \in \mathcal{L}_{\Gamma} \) which belong to \( m \) in the sense that \( \Psi_{\Gamma}(l) = m \). \( l_{\mathcal{M}_{\Gamma}}(m) \) is the total number of bare internal lines belonging to \( m \). With \( \nu = l_{\mathcal{M}_{\Gamma}}(m) + E_{\Gamma}^{(U)}(m) \), \( m \) is called a \( \nu \)-line. We always require that \( \nu \geq 1 \). On the other hand, every bare internal line belongs to one and only one multiple line. For simplicity we often identify a 1-line with the only one bare line that belongs to it.

Next we introduce some further notions that will be used later. External vertices are vertices having external \( \phi \)-lines attached,

\[
\mathcal{B}_{\Gamma, \text{ext}} = \{ v \in \mathcal{B}_{\Gamma} \mid E_{\Gamma}^{(\phi)}(v) \neq 0 \},
\]

(30)

whereas internal vertices do not, \( \mathcal{B}_{\Gamma, \text{int}} = \mathcal{B}_{\Gamma} \setminus \mathcal{B}_{\Gamma, \text{ext}} \). Similarly, external multiple lines have external \( U \)-lines attached,

\[
\mathcal{M}_{\Gamma, \text{ext}} = \{ m \in \mathcal{M}_{\Gamma} \mid E_{\Gamma}^{(U)}(m) \neq 0 \},
\]

(31)

and the complement in \( \mathcal{M}_{\Gamma} \) are the internal multiple lines, \( \mathcal{M}_{\Gamma, \text{int}} = \mathcal{M}_{\Gamma} \setminus \mathcal{M}_{\Gamma, \text{ext}} \).

For every pair of vertices \( v, w \in \mathcal{B}_{\Gamma}, v \neq w \), let \( \mathcal{L}^\Gamma(v, w) \) be the set of lines with endpoint vertices \( v \) and \( w \), and \( |\mathcal{L}^\Gamma(v, w)| \) the number of these lines. Thus \( \mathcal{L}^\Gamma(v, w) \) is the set of lines \( v \) and \( w \) have in common. With \( E_{\Gamma}^{(\phi)}(v) \) denoting the number of external \( \phi \)-lines attached to \( v \in \mathcal{B}_{\Gamma} \),

\[
t_{\mathcal{B}_{\Gamma}}(v) = \sum_{w \in \mathcal{B}_{\Gamma}} |\mathcal{L}^\Gamma(v, w)| + E_{\Gamma}^{(\phi)}(v)
\]

(32)

is the total number of bare lines attached to \( v \).

10
Some topological notions and global properties of graphs will be of major interest in the following. A central notion is the connectivity of a multiple-line graph. Recall that we want to consider the DLCE expansion of the free energy and of truncated correlation functions as an expansion in connected graphs. As indicated in section 2, the main generalization compared to the common notion of connectivity of a graph which is required here is that an additional type of connectivity is provided by multiple-lines. To define the connectivity of a multiple-line graph $\Gamma$, $\Gamma$ first is mapped to a (standard) LCE graph $\overline{\Gamma}$ to which the standard notion of connectivity applies. There are various equivalent ways to define such a map. We choose the following one.

- For every multiple-line $m \in \mathcal{M}_\Gamma$ define a new vertex $w(m)$. Let $\widetilde{\mathcal{B}}_\Gamma = \{ w(m) | m \in \mathcal{M}_\Gamma \}$ and define $\overline{\mathcal{B}} = \mathcal{B}_\Gamma \cup \widetilde{\mathcal{B}}_\Gamma$ as the union of the vertices of $\Gamma$ and the new set of vertices originating from the multiple-lines.

- For every bare internal line $l \in \mathcal{L}_\Gamma$ define two new internal lines $l_1, l_2$ and incidence relations

$$
\overline{\Phi}(l_1) = (v_1, w(\Psi_\Gamma(l))),
\overline{\Phi}(l_2) = (v_2, w(\Psi_\Gamma(l))),
$$

(33)

where $v_1$ and $v_2$ are the two endpoint vertices of $l$. The set of all lines $l_1, l_2$, for all $l \in \mathcal{L}_\Gamma$, is denoted by $\overline{\mathcal{L}}$.

- Define the external incidence relations

$$
\overline{E} : \overline{\mathcal{B}} \rightarrow \{0, 1, 2, \ldots \},
\overline{E}(v) = E_\Gamma^{(o)}(v), \text{ for } v \in \mathcal{B}_\Gamma,
\overline{E}(v) = E_\Gamma^{(u)}(m), \text{ for } v = w(m) \in \widetilde{\mathcal{B}}_\Gamma.
$$

(34)

Now, $\overline{\Gamma}$ is defined by

$$
\overline{\Gamma} = (\overline{\mathcal{L}}, \overline{\mathcal{B}}, \overline{E}, \overline{\Phi}).
$$

(35)

Having defined the standard LCE graph $\overline{\Gamma}$ for any multiple-line graph $\Gamma$, we call $\Gamma$ multiple-line connected or just connected if $\overline{\Gamma}$ is connected (in the usual sense). In Fig. 4 we have given two examples for a connected (upper graph) and a disconnected (lower graph) multiple-line graph.

The next important notion is the topological equivalence of two multiple-line graphs. Two multiple-line graphs

$$
\Gamma_i = (\mathcal{L}_i, \mathcal{M}_i, \mathcal{B}_i, E_i^{(o)}, E_i^{(u)}, \Phi_i, \Psi_i), \quad i = 1, 2
$$

(36)

are called (topologically) equivalent if there are three invertible maps

$$
\phi_1 : \mathcal{B}_1 \rightarrow \mathcal{B}_2,
\phi_2 : \mathcal{L}_1 \rightarrow \mathcal{L}_2,
\phi_3 : \mathcal{M}_1 \rightarrow \mathcal{M}_2,
$$

(37)
Figure 4: Example of multiple-line connectivity. The upper multiple-line graph $\Gamma_1$ is connected because the graph $\overline{\Gamma}_1$ is connected in the conventional sense. The lower multiple-line graph $\Gamma_2$ is disconnected because $\overline{\Gamma}_2$ is so.

such that

$$\Phi_2 \circ \phi_2 = \Phi_1 \circ \phi_1,$$
$$\Psi_2 \circ \phi_2 = \phi_3 \circ \Psi_1, \quad (38)$$

and

$$E^{(\phi)}_2 \circ \phi_1 = E^{(\phi)}_1,$$
$$E^{(U)}_2 \circ \phi_3 = E^{(U)}_1. \quad (39)$$

Here $\circ$ means decomposition of maps, and

$$\overline{\phi}_1 : B_1 \times B_1' \rightarrow B_2 \times B_2',$$
$$\overline{\phi}_1(v, w) = (\phi_1(v), \phi_1(w)). \quad (40)$$

A symmetry of a multiple-line graph $\Gamma = (\mathcal{L}, \mathcal{M}, \mathcal{B}, E^{(\phi)}, E^{(U)}, \Phi, \Psi)$ is a triple of maps $\phi_1 : B \rightarrow \mathcal{B}, \phi_2 : \mathcal{L} \rightarrow \mathcal{L}$ and $\phi_3 : \mathcal{M} \rightarrow \mathcal{M}$ such that

$$\Phi \circ \phi_2 = \overline{\phi}_1 \circ \Phi,$$
$$\Psi \circ \phi_2 = \phi_3 \circ \Psi, \quad (41)$$

and

$$E^{(\phi)} \circ \phi_1 = E^{(\phi)},$$
$$E^{(U)} \circ \phi_3 = E^{(U)}. \quad (42)$$

The number of these maps is called the symmetry number of $\Gamma$. 

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We denote by $G_{E_1,E_2}(L)$ the set of equivalence classes of connected multiple-line graphs with $L$ bare internal lines, $E_1$ external $\phi$-lines and $E_2$ external $U$-lines. Furthermore we set

$$G_{E_1,E_2} := \bigcup_{L \geq 0} G_{E_1,E_2}(L).$$

(43)

A multiple line graph $\Gamma$ does not need to have a vertex. If $b_\Gamma = 0$, we have $L_\Gamma = 0$ as well. If in addition $\Gamma$ is connected, $M_\Gamma$ consists of only one element, with all external $U$-lines attached to it. (We anticipate that $\Gamma$ is 1-multiple-line irreducible (1MLI) by definition. For the definition of 1MLI cf. section 4 below.) The only graph of $G_{0,E}(L = 0)$ is given by

$$\Gamma = \begin{cases} E. \end{cases}$$

(44)

It represents the leading term of the susceptibility

$$\chi_{0,E} = \frac{1}{V D} \sum_{l_1,\ldots,l_E \in \Lambda_1} <U(l_1) \cdots U(l_E)>^c$$

(45)

and is given by $\partial^E W^1(I)/\partial I^E|_{I=0}$. The index $c$ in (43) stands for truncated (connected) correlation.

By removal of a $\nu$-line $m \in M_\Gamma$ we mean that $m$ is dropped together with all bare internal lines and all external $U$-lines that belong to $m$. This notion is explained in Fig. 5a. (It is used in section 4 for 1-lines to define 1-particle irreducible (1PI) and 1-line irreducible (1LI) multiple-line graphs.)

On the other hand, by decomposition of a $\nu$-line $m \in M_\Gamma$ we mean that $m$ is dropped together with the external $U$-lines of $m$, but all bare internal lines that belong to $m$ are kept in the graph, being identified now with 1-lines. This notion will be used below to define 1MLI and renormalized multiple-line moments. It is illustrated in Fig. 5b.

Similarly, decomposition of a vertex $v \in B_\Gamma$ means to remove the vertex $v$ and to attach the free end of every line that entered $v$ before to a new vertex, a separate one for each line. This notion is used to define 1-vertex-irreducible (1VI) and renormalized vertex moments for multiple-line graphs. For an example see Fig. 5c.

### 3.2 Susceptibilities and weights

In the last section we have defined multiple-line graphs and the notions of connectivity and equivalence of such graphs. The definition is chosen in such a way that the series expansions of the free energy and of truncated correlation functions are obtained as a sum over equivalence classes of connected multiple-line graphs. The
number $L$ of bare internal lines of a multiple-line graph $\Gamma$ counts the order in the expansion parameter $v(x, y)$ to which $\Gamma$ contributes. If $v(x, y)$ is of the form
\[ v(x, y) = 2K \sum_{z \in \mathcal{N}(x)} \delta_{y,z} \] (46)
with $\mathcal{N}(x)$ any finite $x$-dependent set of lattice sites, the contribution of $\Gamma$ is a multiple of $(2K)^L$. Often used special cases are the nearest neighbour interactions
\[ v(x, y) = 2K \sum_{\mu=0}^{D-1} (\delta_{x,y+\mu} + \delta_{x,y-\mu}) \] (47)
and the uniform interaction
\[ v(x, y) = 2K \left( 1 - \delta_{x,y} \right) \] (48)
which is used in models of spin glasses and partially annealed neural networks.

Susceptibilities of the $\phi$ and $U$ fields will be represented as
\[
\chi_{E_1,E_2} = \frac{1}{VD} \sum_{x_1,\ldots,x_{E_1}\in\Lambda_0, l_1,\ldots,l_{E_2}\in\Lambda_1} \langle \phi(x_1) \cdots \phi(x_{E_1}) U(l_1) \cdots U(l_{E_2}) \rangle^c
\]
\[ \equiv \frac{1}{VD} \sum_{x_1,\ldots,x_{E_1}\in\Lambda_0, l_1,\ldots,l_{E_2}\in\Lambda_1} \frac{\partial^{E_1+E_2} W(H,I,v)}{\partial H(x_1) \cdots \partial H(x_{E_1}) \partial I(l_1) \cdots \partial I(l_{E_2})} \bigg|_{H=I=0} \]
\[ = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \Theta_{E_1,E_2}(L)} w(\Gamma) \] (49)
with lattice volume $V$ and dimension $D$. Similar representations hold for higher moments $\mu$. 

Figure 5: Removal (a) and decomposition (b) of a 5-line. Decomposition (c) of a vertex.
The weight \( w(\Gamma) \) of a multiple-line graph \( \Gamma \in \mathcal{G}_{E_1,E_2}(L) \) is given as the product of the following factors

- for every vertex \( v \in \mathcal{B}_\Gamma \) a factor
  \[
  v_n^{oc} = \left( \frac{\partial^n W(H)^\circ}{\partial H^n} \right)_{H=0} ,
  \]  
  where \( n = t_{B_\Gamma}(v) \) is the total number of bare lines attached to \( v \).

- for every multiple line \( m \in \mathcal{M}_\Gamma \) a factor
  \[
  m_\nu^{1c} = \left( \frac{\partial^{\nu} W^{1}(I)}{\partial I^{\nu}} \right)_{I=0} ,
  \]  
  where \( \nu = l_{\mathcal{M}_\Gamma}(m) + E^{(U)}_\Gamma(m) \), that is \( m \) is a \( \nu \)-line,

- a factor \( 1/S_\Gamma \), where \( S_\Gamma \) is the topological symmetry number of \( \Gamma \),

- a factor counting the permutation symmetry of external \( \phi \)-lines,
  \[
  \frac{E^{(\phi)}_\Gamma!}{\prod_{v \in \mathcal{B}_\Gamma} E^{(\phi)}_\Gamma(v)!} ,
  \]  

- a factor counting the permutation symmetry of external \( U \)-lines,
  \[
  \frac{E^{(U)}_\Gamma!}{\prod_{m \in \mathcal{M}_\Gamma} E^{(U)}_\Gamma(m)!} ,
  \]  

- the lattice embedding number of \( \Gamma \), which is the number of ways \( \Gamma \) can be embedded on a lattice of given geometry, e.g. on a hypercubic lattice. To this end, the vertices of \( \Gamma \) (if any) are placed onto lattice sites. One arbitrary vertex is placed at a fixed lattice site, in order to account for the volume factor \( 1/V \) in (19). A priori there is no exclusion principle. This means that any number of vertices can be placed at the same lattice site. (This is sometimes called free embedding.) Two restrictions apply to the embeddings. The first constraint results from the fact that a bare internal line represents a hopping propagator \( v(x, y) \), with lattice sites \( x \) and \( y \) at which the two endpoint vertices of the line are placed at. A reasonable computation of the embedding number takes into account the particular form of \( v(x, y) \) from the very beginning. The second constraint is that bare lines of the same multiple-line have to be mapped on the same pair of sites.

For example, if \( v(x, y) \) is the nearest neighbour interaction (17), two vertices which have at least one line in common are to be placed at nearest neighbour lattice sites. On the other hand, a propagator \( v(x, y) \) of the form (18) implies a rather weak constraint in that \( x \) and \( y \) must be different, but otherwise can be freely placed over the lattice.
We remark that in case of a non-trivial internal symmetry (such as considered in section 7) the expressions of Eq.s (49)-(51) must be modified appropriately. In particular, the weight (51) of a multiple-line does no longer take such a simple form.

4 Renormalization

Truncated correlation functions, susceptibilities and other moments are obtained as sums over multiple-line graphs that are connected. Their number rapidly grows with increasing order, that is with increasing number of bare internal lines. The procedure of ”renormalization” means that the connected moments are represented in terms of reduced ones. The reduced moments are obtained by summation over multiple-line graph classes which are more restricted than just by their property of being connected. Of course the number of graphs of such classes is smaller. Only the most restricted multiple-line graph classes must be constructed. The subsequent steps towards the moment computation are most conveniently done by operating analytically with the reduced moments. In particular, it is no longer necessary to generate all connected and the corresponding intermediate multiple-line graph classes.

A connected multiple-line graph \( \Gamma \) is called 1-particle irreducible (1PI) if it satisfies the following condition. Remove an arbitrary 1-line of \( \Gamma \). There is at most one connected component left that has external lines attached. (This notion is the same as the one used in the context of Feynman graphs.) On the other hand, if in addition the remaining graph is still connected, then \( \Gamma \) is called 1-line irreducible (1LI). In many cases it is sufficient to use only the second notion. It is for instance sufficient that all vertices are constrained to have only an even number of lines attached, or more generally, if graphs and subgraphs with one external line are forbidden. For notational simplicity we assume in the following that this is the case and henceforth refer only to the notion 1LI \(^4\). The generalization to the case in which 1LI and 1PI graphs must be distinguished goes along the same lines as for LCEs, which was discussed in [13].

By \( \mathcal{G}^{1LI}_{E_1,E_2}(L) \) we denote the subset of multiple-line graphs \( \Gamma \in \mathcal{G}_{E_1,E_2}(L) \) that are 1LI. 1LI-susceptibilities are defined as series in the hopping parameter similarly as in (43) by restricting the summation to 1LI graphs,

\[
\chi^{1LI}_{E_1,E_2} = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \mathcal{G}^{1LI}_{E_1,E_2}(L)} w(\Gamma).
\]

Susceptibilities are easily obtained in a closed form in terms of 1LI-susceptibilities \( \chi^{1LI} \). It can be shown that the \( \chi^{1LI} \)s can be obtained by an appropriate Legendre transform. For instance

\[
\chi_{2,0} = \frac{\chi^{1LI}_{2,0}}{1 - \tilde{v}(0)\chi^{1LI}_{2,0}},
\]

\(^4\)In ref. [4, 5] the term 1PI was used instead.
\[ \chi_{2,1} = \frac{\chi_{2,1}^{LL}}{(1 - \tilde{v}(0)\chi_{2,0}^{LL})^2}, \]  
\[ (55) \]

where \( \tilde{v}(k) \) is the Fourier transform of the hopping propagator \( v(x, y) \),

\[ v(x, y) = \int_{-\pi}^{\pi} \frac{d^Dk}{(2\pi)^D} e^{-ik \cdot (x-y)} \tilde{v}(k). \]  
\[ (56) \]

In LCEs the second important resummation comes from so called vertex renormalizations. This means partial resummation of graphs with specific properties such as having one external vertex only. These sums then are considered as "renormalized vertices" replacing the vertices of graphs with complementary properties. The procedure naturally leads to the notion of 1-vertex irreducibility (1VI) and renormalized moments.

In DLCE we follow this procedure. The very definition of 1VI has to be modified slightly for multiple-line graphs because of the enhanced connectivity properties due to multiple-lines. In addition, as a natural generalization, we supplement vertex renormalization by multiple-line renormalization.

A multiple-line graph \( \Gamma \) is called 1-vertex irreducible (1VI) if it satisfies the following condition. Decompose an arbitrary vertex \( v \in \mathcal{B}_\Gamma \). Every remaining connected component has then at least one external line attached. It can be a \( \phi \)-line or a \( U \)-line. We write

\[ G_{E_1, E_2}^{1VI}(L) = \{ \Gamma \in G_{E_1, E_2}^{1LI}(L) \mid \Gamma \text{ is } 1VI \} \]  
\[ (57) \]

for the set of equivalence classes of graphs that are both 1LI and 1VI, with \( E_1 \) external \( \phi \)-lines, \( E_2 \) external \( U \)-lines and \( L \) bare internal lines.

The renormalized vertex moment graphs are 1LI graphs that have precisely one external vertex and no external multiple line,

\[ Q_k(L) = \{ \Gamma \in G_{k,0}^{1LI}(L) \mid \text{there is } v \in \mathcal{B}_\Gamma \text{ with } E^{(\phi)}_\Gamma(v) = k \}. \]  
\[ (58) \]

A multiple-line graph \( \Gamma \) is called 1-multiple-line irreducible (1MLI) if it satisfies the following criterion. Decompose an arbitrary multiple-line \( m \in \mathcal{M}_\Gamma \). Every remaining connected component has then at least one external line attached. It can be a \( \phi \)-line or a \( U \)-line. We write

\[ G_{E_1, E_2}^{1MLI}(L) = \{ \Gamma \in G_{E_1, E_2}^{1LI}(L) \mid \Gamma \text{ is } 1MLI \}. \]  
\[ (59) \]

The renormalized multiple-line moment graphs are graphs that are 1LI and have precisely one external multiple-line, but no external vertex,

\[ R_k(L) = \{ \Gamma \in G_{0,k}^{1LI}(L) \mid \text{there is } m \in \mathcal{M}_\Gamma \text{ with } E^{(U)}_\Gamma(m) = k \}. \]  
\[ (60) \]

The equivalence classes of graphs that are both 1VI and 1MLI are denoted by

\[ S_{E_1, E_2}(L) = G_{E_1, E_2}^{1VI}(L) \cap G_{E_1, E_2}^{1MLI}(L). \]  
\[ (61) \]
With the renormalized moment graphs as defined above, the 1LI-susceptibilities are now obtained in the form
\[ \chi_{E_1,E_2}^{1LI} = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in S_{E_1,E_2}(L)} \tilde{w}(\Gamma). \] (62)

The weights \( \tilde{w}(\Gamma) \) are given as a product of factors as described in the last subsection, with the following two exceptions.

- The vertex coupling constants \( v_{n}^{oc} \) are replaced by the renormalized vertex moments
  \[ v_{n}^{oc} \rightarrow v_{n}^{c} = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in Q_n(L)} w(\Gamma). \] (63)

- The multiple line coupling constants \( m_{\nu}^{1c} \) are replaced by the renormalized multiple line moments
  \[ m_{\nu}^{1c} \rightarrow m_{\nu}^{c} = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in R_{\nu}(L)} w(\Gamma). \] (64)

In the series representations above, the \( w(\Gamma) \) are computed according to the rules of subsection 3.2.

In [14] we have described an algorithmic construction of graphs that is the first step for an automatic computer aided generation. In exceptional cases DLCEs can be summed up in a closed form. Otherwise, a computer aided generation of graphs is unavoidable, because the proliferation of DLCE graphs is pronounced even compared to LCE graphs. For LCE graphs we know that billions of graphs must be included to obtain the critical exponents with the accuracy of some per mil.

5 Applications to spin glasses

In this section we consider applications of DLCEs to disordered systems, in particular to spin glasses with "slow" interactions coupled to "fast" spins. The interactions \( J \) are assumed to be in equilibrium with a thermal heat bath of inverse temperature \( \beta' \), while the spins \( \sigma \) are equilibrated according to a second inverse temperature \( \beta \). Both systems need not be mutually in equilibrium. Let \( Z_\beta(J) \) be the partition function that describes the equilibrium distribution of the spins for given \( J \)s,
\[ Z_\beta(J) = \sum_{\{\sigma_i = \pm 1\}} \exp (\beta \sum_{i<j} J_{(i,j)} \sigma_i \sigma_j). \] (65)

The sum runs over pairs \((i, j)\) that need not be restricted to nearest neighbours only.
We further assume that the dynamics of the time evolution of the slow interactions $J$ is governed by a Langevin equation

$$
N \frac{d}{dt} J_{(i,j)} = - \frac{\partial}{\partial J_{(i,j)}} \mathcal{H}(J) + \sqrt{N} \eta_{ij}(t)
$$

with

$$
\mathcal{H}(J) = - \frac{1}{\beta} \ln Z_\beta(J) + \frac{1}{2} \mu N \sum_{i<j=1}^N J_{(i,j)}^2.
$$

Here $Z_\beta(J)$ is given by (65), $N$ is the total number of spins, $\mu$ is a positive constant and $\eta_{ij}$ is a stochastic gaussian white noise of zero mean and correlation

$$
< \eta_{ij}(t) \eta_{kl}(t') > = \frac{2}{\beta'} \delta_{(ij),(kl)} \delta(t-t').
$$

Such a Langevin equation for the $J$s can be derived from an ansatz which is motivated by neural networks [11]-[13]. Moreover, since the time evolution of the $J$s is determined by a dissipative Langevin equation, the equilibrium distribution of the slow variables is again a Boltzmann distribution, governed now by the second temperature $\beta' - 1$,

$$
Z'_{\beta'} = N \int_{-\infty}^{\infty} \prod_{i<j=1}^N dJ_{(i,j)} \exp (-\beta' \mathcal{H}(J)),
$$

with $N$ some normalization that will be specified below. The effective Hamiltonian $\mathcal{H}$ of $J$ is given by (67).

It is these equilibrium aspects of coupled systems of fast spins and slow interactions that we can treat analytically with DLCEs, as we will show below.

Let us first rewrite $Z'_{\beta'}$ in the form

$$
Z'_{x\beta} = \int_{-\infty}^{\infty} \prod_{i<j=1}^N \left( \sqrt{QN \frac{2\pi}{J_{(i,j)}}} \right) \cdot \exp \left(-\frac{1}{2} Q N \sum_{i<j} J_{(i,j)}^2 \right) Z_\beta(J)^x
$$

$$
\equiv [[Z_\beta(J)^x]],
$$

where we have introduced $Q = \beta \mu$ and real $x = \beta'/\beta$ as the ratio of two temperatures. The normalization has been chosen such that $[[1]] = 1$. In the limit of $x \to 0$ for fixed $\beta$, i.e. $\beta' \to 0$, we have a quenched system. The $J$ only feel the infinitely high temperature $\beta' - 1$, but are decoupled from the spins. The time scale of fluctuations of the spins is assumed to be so short that the $J$ are only sensitive to averages of the $\sigma$. Vice versa, the spin dynamics does depend on the $J$s. Therefore the quantity of physical interest is not

$$
\ln \left[ \int_{-\infty}^{\infty} \prod_{i<j=1}^N \left( \sqrt{QN \frac{2\pi}{J_{(i,j)}}} \right) \cdot \left( \sum_{\{\sigma_i\}} \exp \left( \beta \sum_{i<j=1}^N J_{(i,j)} \sigma_i \sigma_j \right) \right)^x \cdot \exp \left(-\frac{1}{2} Q N \sum_{i<j} J_{(i,j)}^2 \right) \right]
$$

(71)
where fluctuations of the $\sigma$s do influence the $J$s, but
\[
\int_{-\infty}^{\infty} \prod_{i<j=1}^{N} \left( \sqrt{\frac{QN}{2\pi}} dJ_{(i,j)} \right) \ln \left[ \sum_{\{\sigma_i\}} \exp \left( \beta \sum_{i<j=1}^{N} J_{(i,j)} \sigma_i \sigma_j \right) \right] \cdot \exp \left( -\frac{1}{2} QN \sum_{i<j} J_{(i,j)}^2 \right),
\]
(72)

or, in a shorthand notation,
\[
\int \mathcal{D}J \ln Z_\beta(J) \equiv [[\ln Z_\beta(J)]].
\]
(73)

Usually one rewrites
\[
\int \mathcal{D}J \ln Z_\beta(J) = \int \mathcal{D}J \lim_{x \to 0} \frac{Z_\beta(J)^x - 1}{x} = \lim_{x \to 0} \frac{\ln \{1 + ([Z_\beta(J)^x] - 1)\}}{x} = \lim_{x \to 0} \ln Z'_{x\beta} \frac{x}{Z'_{x\beta}}.
\]
(74)

For the second equality sign one has assumed that $\int \mathcal{D}J$ commutes with $\lim_{x \to 0}$, in the third one that $\lim_{x \to 0}[[Z_\beta(J)^x]] = 1$. So far, $x$ as the ratio of two temperatures is real. Rewriting the left hand side of (74) according to the right hand side is called the replica trick [17]. The uncontrolled approximation that usually enters the replica trick is that now the right hand side is evaluated for positive integer $x \equiv n$ and extrapolated to $n = 0$. Clearly a function that is known only for positive integer $n$ does not have a unique extrapolation to $n = 0$ without further assumptions. Nevertheless, this approximation is made, because it is rather convenient. For integer $n$, $Z_\beta(J)^n$ is the partition function of an $n$ times replicated system of which the logarithm is taken after the integration over the $J$s. It is seen as follows. We rewrite
\[
Z_\beta(J)^n = \sum_{\{\sigma^{(a)}_i\}} \exp \left( \beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J_{(i,j)}^{(a)} \sigma^{(a)}_i \sigma^{(a)}_j \right),
\]
(75)

with $a = 1, \ldots, n$ labelling the replicated spin variables, so that
\[
Z'_{n\beta} = \int_{-\infty}^{\infty} \prod_{i<j=1}^{N} dJ_{(i,j)} \cdot \sum_{\{\sigma^{(a)}_i\} = \pm 1} \exp \left( -S(J, \sigma^{(a)}) \right),
\]
\[
S(J, \sigma^{(a)}) = -\beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J^{(a)}_{(i,j)}{\sigma^{(a)}_i}{\sigma^{(a)}_j} + \frac{1}{2} QN \sum_{i<j} J_{(i,j)}^2.
\]
(76)

Linear terms in $\sigma$ and $J$ may be included according to
\[
S_{lin} = -h \sum_{a=1}^{n} \sum_{i=1}^{N} {\sigma^{(a)}_i} + c \sum_{i<j=1}^{N} J_{(i,j)}
\]
(77)
with constant external fields \( h \) and \( c \).

Apparently, because of integer \( n \), \( Z'_{n\beta} \) has the form of models to which DLCE applies, with a hopping term

\[
S_{\text{hop}}(J, \sigma^{(a)}) = -\beta \sum_{a=1}^{n} \sum_{i<j=1}^{N} J_{(i,j)}\sigma^{(a)}_i \sigma^{(a)}_j,
\]

(78)

a single link action

\[
S^1(J_{(i,j)}) = cJ_{(i,j)} + \frac{1}{2}QN J_{(i,j)}^2,
\]

(79)

and a single site action

\[
S^o(\sigma^{(a)}_i) = -h \sum_{a=1}^{n} \sigma^{(a)}_i.
\]

(80)

Depending on \( n \) we distinguish the following cases.

- \( n = 1 \). First we note that for \( n = 1 \) we can directly apply DLCE to \( \ln Z'_{\beta' = \beta} \) and to derived quantities to obtain their series expansions in \( \beta \). But from a physical point of view, in a disordered system one is not interested in \( n = 1 \), because \( n = 1 \) corresponds to the completely annealed situation, in which the fast spins and the slow interactions are in mutual equilibrium. (In contrast, in particle physics one is interested in the \( n = 1 \) case, cf. our applications of DLCEs in the framework of variational cumulant expansions of the \( SU(2) \) Higgs model [14].)

- \( n > 1 \), integer. Again we apply DLCE to \( \ln Z'_{\beta' = n\beta} \), but have to account for the permutation symmetry between the replicas. Formally, the replica symmetry plays a role similar to an internal symmetry, e.g. an \( O(N) \) symmetry in a scalar Higgs model. DLCEs with nontrivial internal symmetries have been discussed in connection with the \( SU(2) \) Higgs model [14]. Thus we can study ”‘unquenched’” equilibrium aspects of systems with two temperatures and compare the results from DLCEs adapted to ”internal” replica symmetry with Monte Carlo simulations for the same \( n \) [18, 19].

- \( n = 0 \), the quenched limit. As we will show in the next section, in order to discuss the \( x \rightarrow 0 \) limit, we need not refer to \( n \) times replicated systems \( Z_\beta(J)^n \) characterized by (78)-(80), but just to \( Z_\beta(J) \) given by (78)-(80) with \( n = 1 \). By means of special DLCEs, so-called quenched DLCEs (QDLCEs), we directly calculate the left hand side of (74). Therefore, setting \( n = 1 \) in (78)-(80) in QDLCEs does not imply the completely annealed case, because we first take the logarithm of \( Z_\beta(J) \) and then average over the \( J_s \).
5.1 Avoiding the replica trick

First we adapt the notation to section 2 to include more general cases. \( \Lambda_0 \) denotes the support of the spins, that is the set of lattice sites, with \( V = |\Lambda_0| \) denoting their total number. \( \overline{\Lambda}_1 \subseteq \Lambda_1 \) are the pairs of sites whose spins interact. In accordance with \( (70) \), we write for the normalized link-average of a function \( f(J) \)

\[
[[f(J)]] = \int \mathcal{D}J \ f(J)
\]

with

\[
\mathcal{D}J = \prod_{l \in \overline{\Lambda}_1} d\mu(J(l)) ,
\]

\[
d\mu(J) = N_1 \ dJ \ \exp (-S^1(J)) , \ \int_{-\infty}^{\infty} d\mu(J) = 1.
\]

It is convenient to introduce the single link expectation values

\[
< g(J) >_1 \equiv \int d\mu(J) \ g(J)
\]

and the generating function \( W^1(I) \) by

\[
\exp W^1(I) \equiv < \exp(IJ) >_1.
\]

The way in which the replica trick can be avoided is examplified for the free energy density \( W_{sp}/V \) of the spin system averaged over the link couplings. The partition function of the spin system for a given distribution of the link interactions \( J(x,y) \) is given by

\[
\exp W_{sp}(J) = N_{sp} \ \int \mathcal{D}\sigma \ \exp (-S_{sp}(\sigma, J)),
\]

where \( W_{sp}(0) = 0 \) and

\[
S_{sp}(\sigma, J) = -\frac{1}{2} \sum_{x,y \in \Lambda_0} v(x,y) \sigma(x) \sigma(y) J(x,y),
\]

\[
\mathcal{D}\sigma = \prod_{x \in \Lambda_0} d\sigma(x) \cdot \exp (-S^{\circ}(\sigma(x))).
\]

Without loss of generality we identify the support of the interaction \( v(x,y) = v(y,x) \) with the set \( \overline{\Lambda}_1 \) of lattice sites where \( \mathcal{D}J \) is supported,

\[
\overline{\Lambda}_1 = \{ l = (x,y) \in \overline{\Lambda}_0 \times \overline{\Lambda}_0 \mid v(x,y) \neq 0 \}.
\]

For simplicity we assume \( v(x,y) \) to be of the form \( (46) \), so that \( K \) is a measure of the strength of the interactions \( v(x,y) \).
The free energy density of the spin system allows for a series expansion in the standard LCE sense, with the link field \( J(l) \) playing the role of a "background field",

\[
\frac{1}{V} W_{sp}(J) = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} w_{sp}(\Gamma, J). \tag{88}
\]

Here \( \mathcal{G}_E^{sp}(L) \) (with \( E = 0 \)) denotes the set of equivalence classes of connected LCE graphs with \( E \) external lines and \( L \) internal lines. The spin-weights \( w_{sp}(\Gamma, J) \) are of the form

\[
w_{sp}(\Gamma, J) = R_{sp}(\Gamma) \sum_{\Lambda_1 \rightarrow \Gamma} \prod_{l \in \Lambda_1} J(l)^{m(l)}. \tag{89}
\]

The sum is taken over all non-vanishing lattice embeddings of the graph \( \Gamma \). It runs over all maps of internal lines of the graph \( \Gamma \) to pairs of lattice sites of \( \Lambda_1 \) that are consistent with the graph topology in the sense discussed in section 3. For every \( l \in \Lambda_1, m(l) \) denotes the number of lines of \( \Gamma \) that are mapped onto the link \( l \) by the embedding. All other factors that contribute to the weight are collected in the prefactor \( R_{sp}(\Gamma) \), including the inverse topological symmetry number of \( \Gamma \).

Next we want to express \( \left[ \frac{1}{V} W_{sp}(J) \right] \) as a series in \( K \) by means of DLCE. Toward this end we set \( f(J) = W_{sp}(J) \) and insert the series (88) with (89) into (81). At this stage we are not concerned with question of (uniform or dominated) convergence and obtain

\[
\left[ \frac{1}{V} W_{sp}(J) \right] = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} \int \mathcal{D}J \ w_{sp}(\Gamma, J)
= \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in \mathcal{G}_0^{sp}(L)} R_{sp}(\Gamma) \sum_{\Lambda_1 \rightarrow \Gamma} \prod_{l \in \Lambda_1} < J(l)^{m(l)} >_1 . \tag{90}
\]

The next step is to express the full single link expectation values in terms of the connected ones. They are related by

\[
< J^m >_1 = \sum_{\Pi \in \mathcal{P}(m)} \prod_{P \in \Pi} < J^{|P|} >_{1_P}^{*}, \tag{91}
\]

where \( \mathcal{P}(m) \) denotes the set of all partitions of \( m = \{1, \ldots, m\} \) into non-empty, mutually disjoint subsets of \( m \). \( |P| \) is the number of elements of the set \( P \). The relation (91) is equivalent to the partition of all lines of \( \Gamma \) that are mapped to the same lattice link into multiple lines, with every multiple line contributing a factor

\[
< J^{|P|} >_{1_P}^{*} = \frac{\partial^{|P|} W_1^c(I)}{\partial I^{|P|}} \bigg|_{I=0} = m_{|P|}^{1_c} . \tag{92}
\]

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Using (91), (92) we rewrite (90) as

$$[[ {\frac{1}{V} W_{sp}(J)} ]] = \sum_{L \geq 0} (2K)^L \sum_{\Gamma \in G_{0}^{sp}(L)} R^{sp}(\Gamma) \sum_{\Pi \in P(\mathcal{L}_{\Gamma})} \left( \prod_{P \in \Pi} m_{[P]}^{1c} \right) \left( \sum_{\Pi \rightarrow \Lambda_{1}} \prod_{i \in \Lambda_{1}} 1 \right).$$

The last summation in (93) is over all maps $\mathcal{L}_{\Gamma} \rightarrow \Lambda_{1}$ of the lines of $\Gamma$ to the lattice links of $\Lambda_{1}$ subject to the constraint that all lines that belong to the same multiple-line corresponding to some $P \in \Pi$ are mapped onto the same lattice link.

Finally we rewrite (93) as a sum over multiple-line graphs. To this end, we first observe that for every $\Gamma \in G_{0}^{sp}(L)$, every partition $\Pi \in P(\mathcal{L}_{\Gamma})$ of the lines of $\Gamma$ into multiple-lines generates a multiple-line graph $\Delta = (\Gamma, \Pi)$ in the obvious way. Let us denote by $\mathcal{G}_{0,0}(L)$ the subset of multiple-line graphs of $G_{0,0}(L)$ that stay connected after decomposition of all multiple lines. (These are the multiple-line graphs which stay connected in the usual graph theoretical sense, when the dashed lines are omitted.) For every $\Delta \in \mathcal{G}_{0,0}(L)$ there is a unique $\Gamma(\Delta) \in G_{0}^{sp}(L)$ and at least one $\Pi \in P(\mathcal{L}_{\Gamma(\Delta)})$ such that $(\Gamma(\Delta), \Pi) = \Delta$. Let $n_{\Delta}$ be the (uniquely determined) number of partitions $\Pi \in P(\mathcal{L}_{\Gamma(\Delta)})$ with $(\Gamma(\Delta), \Pi) = \Delta$, and $\Pi(\Delta)$ such an arbitrary partition. Eq. (93) then becomes

$$[[ {\frac{1}{V} W_{sp}(J)} ]] = \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \mathcal{G}_{0,0}(L)} n_{\Delta} R^{sp}(\Gamma(\Delta)) \left( \prod_{P \in \Pi(\Delta)} m_{[P]}^{1c} \right) \left( \sum_{\Pi(\Delta) \rightarrow \Lambda_{1}} \prod_{i \in \Lambda_{1}} 1 \right).$$

The last bracket of (94) is the lattice embedding factor of the multiple-line graph $\Delta$. The second bracket from the right does not depend on the choice of $\Pi(\Delta)$ and is the product of the multiple-line coupling constants as defined in section 3. Finally, $n_{\Delta} R^{sp}(\Gamma(\Delta))$ is precisely the remaining part of the weight of $\Delta$ that was described in detail in section 3, endowed with the correct inverse topological symmetry number of the multiple-line graph $\Delta$ (because of the factor $n_{\Delta}$).

In summary, we obtain the series expansion of the link-averaged free energy density in terms of DLCE graphs,

$$[[ {\frac{1}{V} W_{sp}(J)} ]] = \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \mathcal{G}_{0,0}(L)} w(\Delta).$$

The weight $w(\Delta)$ of a multiple-line graph $\Delta$ is defined and computed according to the rules given in section 3.

Eq. (95) is the series representation of the link-averaged free energy density of the spin system, i.e. the free energy density of the $n = 0$ replica system, in terms of DLCE graphs. It looks much like the series representation of the 1-replica system, which is given by

$$\frac{1}{V} W_{1-repl} \equiv \frac{1}{V} \ln \left[ \exp W_{sp}(J) \right] = \sum_{L \geq 0} (2K)^L \sum_{\Delta \in \mathcal{G}_{0,0}(L)} w(\Delta)$$

(96)
according to the discussion of section 2. We recall that \( G_{0,0}(L) \) is the set of DLCE vacuum graphs with \( L \) bare lines that are connected in the generalized DLCE sense. Comparing (95) and (96), the transition from \( n = 1 \) to \( n = 0 \) replicas is achieved by keeping only the subset \( G_{0,0}(L) \subseteq G_{0,0}(L) \) of multiple-line graphs that are connected in the original (LCE) sense.

We emphasize that the restriction of DLCEs to QDLCEs is not an ad hoc (or intuitively motivated) assumption but a derived consequence of the fact that the logarithm is taken before the integration \( \int \mathcal{D}J \). This procedure accounts for all graphs that contribute to a given order in \( K \). Thus we do have to truncate the series unless the series can be completely summed up, as it happens in exceptional cases.

We expect that the series (95) are convergent for a large class of interactions \( S^1(J) \) and \( v(x, y) \) if the coupling constant \( K \) is sufficiently small. For special interactions most of the multiple-line graphs yield vanishing contributions so that we can further restrict the sum to a subset of \( G_{0,0}(L) \). An example is given by the mean field type of interaction of the Sherrington-Kirkpatrick model, cf. Sect. 5.2.

5.2 Applications of QDLCEs

In the following we list some examples for systems of which we can study the phase structure by means of QDLCEs. Their actions are special cases of (78)-(80) with \( n = 1 \) (as explained above) and the following choice of variables.

- Infinite range models. Choose \( J_{(i,j)} \in \mathbb{R} \) as before, \( \sigma_i \in \pm 1, i, j \in \{1, \ldots, N\} \),

\[
\begin{align*}
v(x, y) &= K (1 - \delta_{x,y}), \\
\Lambda_1 &= \Lambda_1 \text{ is the set of all pairs of sites}, \\
S^1(J) &= N \frac{1}{2} J^2.
\end{align*}
\]

Now the sum over the sites in (78) runs over arbitrary pairs \((i, j), i < j\), and we obtain the infinite range Sherrington-Kirkpatrick model. For infinite range and in the thermodynamic limit \((N \to \infty)\), the phase structure can be solved by replica mean field theory, cf. e.g. [10]. For QDLCEs the infinite range and \( N \to \infty \) limits imply that only tree graphs of 2-lines contribute to the series of the free energy density, such as

\[
(98)
\]
The reason for that is that each 2-line gets a factor of $1/N$ from the $S_1$-part of the action, but each vertex gets a factor $(N - 1)$ from the embedding onto a lattice $\Lambda_0$. The contribution of every tree graph to the free energy is proportional to $N$. If a chain of 2-lines connecting the vertices gets closed, forming a loop, there is one $(N - 1)$ less in the total embedding factor. Thus the contribution is suppressed by $1/(N - 1)$ for every loop and vanishes in the thermodynamic limit. Because of the simple tree structure there is a chance for summing up the series. This is currently under investigation.

- Finite range connectivity. The sum $\sum_{i<j}$ of the spins is now restricted to next-neighbours or, more generally, to a finite number of pairs. Rather than specifying $S_1(J_{i,j})$ of (79), it is sufficient for DLCEs and QDLCEs to choose $\exp(-S_1(J_{i,j}))$. Let

$$\exp(-S_1(J)) = (1 - p)\delta(J) + p\delta(J - 1)$$

with $p \in [0, 1]$. The variables $J_{i,j} \in \{0, 1\}$ can then be interpreted as occupation numbers of the bonds. Furthermore, if we choose $\sigma_i \in \{\pm 1\}$ we obtain a

- bond-diluted Ising model.

Choosing $\sigma_i \in \mathbb{Z}_q$ we obtain a

- bond-diluted $q$-state Potts model.

If $\sigma_i \in S_q$, we obtain a

- bond-diluted Heisenberg model.

QDLCEs provide a systematic analytic expansion for disordered systems with bond dilution in a quenched limit. Coming from the high temperature (small $\beta$) region one can study the phase structure as a function of the degree of dilution. Work in this direction is in progress.

6 Summary and Conclusions

In this paper we have introduced a new expansion scheme for 3-point interactions or, more precisely, for point-link-point interactions. This scheme generalizes linked cluster expansions for 2-point interactions by including hopping parameter terms endowed with their own dynamics. In chapters 3-4 we have developed a multiple-line graph theory with an additional new type of multiple-line connectivity. We have introduced appropriate equivalence classes of graphs and discussed the issue of renormalization. These notions are required for an algorithmic generation of graphs. Because of the fast proliferation of graphs already at low orders in the expansion, a computer aided implementation becomes unavoidable, if one is interested in higher orders of the expansion than we have computed so far.
In Sect. 5 we have shown how to avoid the replica trick for calculating the free energy of disordered systems in the quenched limit. DLCEs are a systematic expansion method to study the phase structure of disordered systems. It is systematic in the sense that we do not restrict the expansion to certain subclasses of graphs that can be summed up, but we identify and keep all graphs that contribute to a given order in the expansion parameter. DLCEs provide an analytic tool for studying systems in situations in which it has been impossible so far.

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References

[1] M. Wortis, "Linked cluster expansion", in Phase transition and critical phenomena, vol.3, eds. C. Domb and M.S. Green (Academic Press, London 1974).

[2] C. Itzykson, J.-M. Drouffe, "Statistical field theory", vol.2, Cambridge University Press, 1989.

[3] A.J. Guttmann, ”Asymptotic analysis of Power-Series Expansions”, in Phase transition and critical phenomena, vol.13, eds. C. Domb and J.L. Lebowitz (Academic Press).

[4] M. Lüscher and P. Weisz, Nucl. Phys. B300[FS22] (1988) 325.

[5] T. Reisz, Nucl. Phys. B450 (1995) 569.

[6] T. Reisz, Phys. Lett. 360B (1995) 77.

[7] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, Nucl. Phys. B459 (1996) 207.

[8] S. Zinn, S.-N. Lai and M. E. Fisher, Phys. Rev. E54 (1996) 1176.

[9] P. Butera and N. Comi, Phys. Rev. E55 (1997) 6391.

[10] H. Meyer-Ortmanns and T. Reisz, J. Stat. Phys. 87 (1997) 755.

[11] D. Sherrington and S. Kirkpatrick, Phys. Rev. Lett 35 (1975) 1972.

[12] V. Dotsenko, S. Franz and M. Mézard, J. Phys. A27 (1994) 2351.

[13] R. W. Penney, A. C. C. Coolen, and D. Sherrington, J. Phys. A: Math. Gen. 26 (1993) 3681. A. C. C. Coolen, R. W. Penney, and D. Sherrington, Phys. Rev. B48 (1993) 16116.
[14] H. Meyer-Ortmanns and T. Reisz, Int. J. Mod. Phys. A14 (1999) 947.

[15] T. Reisz, Nucl. Phys. B527 (1998) 363.

[16] K. H. Fischer and J. A. Hertz, “Spin Glasses”, Cambridge University Press, Cambridge 1991.

[17] J. Zinn-Justin, “Quantum Field Theory and Critical Phenomena”, 3rd Edition, Oxford Science Publication, Clarendon Press, Oxford 1996.

[18] D. Sherrington, J. Phys. A13 (1980) 637; I. Kondor, J. Phys. A16 (1983) L127.

[19] B. Berg and W. Janke, Phys. Rev. Lett 80 (1998) 4771