Quantum phase transitions in electronic systems

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Zero–temperature or quantum phase transitions in itinerant electronic systems both with and without quenched disordered are discussed. Phase transitions considered include, the ferromagnetic transition, the antiferromagnetic transition, the superconductor–metal transition, and various metal–insulator transitions. Emphasis is placed on how to determine the universal properties that characterize these quantum phase transitions. For the first three of the phase transitions listed above, one of the main physical ideas established is that in zero–temperature systems there are soft or slow modes that exist in addition to the soft order parameter fluctuations, and that these modes can couple to the critical modes. These extra soft modes are shown to have a profound effect on the quantum critical properties. For quantum phase transitions involving zero wavenumber order parameters, i.e., the ferromagnetic and superconductor–metal transitions, these extra modes effectively lead to long–ranged effective interactions between order parameter fluctuations, which in turn lead to exactly soluble critical behaviors. For the antiferromagnetic case, we argue that while in low enough dimensions disorder fluctuation effects tend to destroy long–range order, quantum fluctuations counteract this effect and in some parameter regions manage to re–establish antiferromagnetic long–range order. For the metal–insulator transition, some recent new ideas are reviewed. In particular, it is pointed out that for interacting disordered electrons, one expects that in high dimensions the metal–insulator transition is related to the phase transition that occurs in random–field magnets in high dimensions. If the analogy also holds in three dimensions this suggests that the metal–insulator transition might have glassy characteristics.

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Abbreviations

AFM Antiferromagnetism

LGW Landau–Ginzburg–Wilson

MIT Metal–insulator transition

QPT Quantum Phase Transition
I. INTRODUCTION

The study of continuous phase transitions \(^1\) has led to great advances in condensed matter physics. The insights gained from these studies \(^2\) have had remarkably broad effects, which is somewhat surprising given that phase transitions occur only at very special points in the phase diagram. In particular, phase transition theory spawned the application of renormalization techniques in condensed matter physics \(^3\), which have proved to be very powerful in a more general context. Wilson’s breakthrough paved the way for the understanding of thermal or classical phase transitions, which occur at non–zero temperature and are driven by thermal fluctuations. Let the critical temperature be \(T_c > 0\), and denote the distance from the critical point by \(t = 1 - T/T_c\). Then it turns out that the critical behavior asymptotically close to the critical point is entirely determined by classical physics. This can be seen as follows. Upon approaching the critical point, there is a diverging characteristic length in the system, viz. the correlation length \(\xi \sim |t|^{-\nu}\), with \(\nu\) an exponent that is characteristic of the class of phase transitions under consideration. Also, the characteristic time \(\xi_t\) for relaxation towards equilibrium diverges like \(\xi_t \sim \xi^z\), with \(z\) another exponent. This slow relaxation is caused by slow fluctuations of some thermodynamic quantity, the order parameter, that is characteristic of the transition. These fluctuations relax on microscopic time scales far from the transition, but become infinitely slow as \(t \to 0\). This means there is a characteristic frequency, \(\omega_c \sim 1/\xi\), that vanishes at criticality. Now quantum effects will be unimportant as long as \(\hbar \omega_c << k_B T\), which is always the case sufficiently close to a critical point with a non–zero \(T_c\). As a result, the critical behavior at, say, the Curie point in Iron, the \(\lambda\)–transition in liquid Helium, or the superconducting transition in Mercury, are determined entirely by classical physics, notwithstanding the fact that quantum mechanics is crucial for the microscopic mechanism underlying the transition and, in the last two examples, for the properties of the ordered phase.

These considerations raise the interesting question of what happens when we follow a phase separation line in a phase diagram down to zero temperature. For definiteness, let us consider a ferromagnet, or a superconductor, where we decrease \(T_c\) by diluting it with a non–magnetic or non–superconducting material. Then we obtain a schematic phase diagram like the one shown in Fig. 1, with \(J\) some parameter that triggers the transition. Normally, \(J\) is fixed for a given material, and one observes the transition by lowering the temperature. However, if we imagine crossing the phase separation line at \(T = 0\) by varying \(J\), then the above arguments suggest that quantum effects should be important for the transition, since \(T_c = 0\). We therefore expect a critical behavior in this case that is different from the one observed when crossing the phase separation line at any non–zero \(T\). This question is not academic, as was first pointed out by Suzuki \(^4\) and Hertz \(^5\). If one triggers the transition at low but non–zero \(T\), then one observes a ‘crossover’ from a region that is characterized by quantum critical behavior to one showing classical critical behavior, as indicated in the figure. Moreover, there are phase transitions that occur only at \(T = 0\), the most important examples being various metal–to–insulator transitions, which are studied experimentally by observing a smeared transition at very low temperatures.

Such phase transitions that occur in quantum mechanical systems at \(T = 0\) as a function of some non–thermal control parameter are called quantum phase transitions (QPTs). Like their finite–temperature counterparts, which for distinction are often referred to as thermal or classical phase transitions, they are characterized by a diverging correlation length \(\xi\), and a diverging relaxation time \(\xi_t\). However, the critical fluctuations that lead to these diverging length and time scales are quantum fluctuations rather than thermal ones. Apart from the origin of the relevant fluctuations being different, quantum phase transitions also differ from their classical counterparts in that their dynamic and static critical behaviors are coupled together. Technically this implies that one cannot solve for the thermodynamic properties of the phase transition without also solving for the dynamics.

There are a number of ways to see the coupling between statics and dynamics at QPTs. From a general scaling viewpoint one argument is as follows. It is obvious from Fig. 1 and the accompanying discussion that increasing the temperature from zero takes one away from the quantum critical point, much like increasing the frequency from zero takes one away from either a quantum or a classical one. Now the fact that Planck’s constant \(\hbar\) and Boltzmann’s constant \(k_B\) are chosen arbitrarily implies that in fundamental units, frequency, energy, and temperature all have the same dimensions. \(^6\) From a scaling viewpoint, this implies, for example, that if a system is at a quantum critical point, either going to a finite frequency or increasing the temperature will affect the system in the same way, and be characterized by the dynamical scaling exponent that describes the divergence of \(\xi_t\) and is usually denoted by \(z\). Since the thermodynamic observables certainly depend on the temperature, this further implies that the scaling relations or homogeneity laws for these quantities will also depend on \(z\). That is, the static and dynamic critical behaviors are

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\(^1\)We will deal only with continuous transitions, and will simply refer to them as ‘phase transitions’ or ‘critical points’.

\(^2\)This has recently been emphasized in Ref. \(^3\).

\(^3\)Not to be confused with quantum mechanics, which is concerned with small length and time scales.

\(^4\)Suzuki, M. (1973). Phys. Rev. Lett., 30, 798.

\(^5\)Hertz, J. (1976). J. Phys. C, 9, 17.

\(^6\)In fundamental units, \(\hbar = 1\) and \(k_B = 1\).
FIG. 1. Schematic phase diagram with $J$ some parameter that is responsible for the ordering. The quantum critical point is at $(T = 0, J_c)$. The dashed lines denote the boundaries of the critical region, and the dotted lines mark the crossover from quantum to classical critical behavior. The regions labeled by $c$ and $q$ show classical and quantum critical behavior, respectively.

coupled together. A more mathematical argument is that the standard Matsubara frequency technique implies that frequency and temperature are intimately related and should scale in the same way.

From a more technical point of view, a classical partition function involves separate integrals over momenta and positions that are decoupled since the kinetic energy depends only on the momenta, and the potential energy only on the positions. One can therefore solve for the thermodynamics without having to deal with the dynamics explicitly. In quantum statistical mechanics, on the other hand, the partition function of a system with Hamiltonian $H$ is given as

$$Z = \text{Tr} e^{-\mathcal{H}/k_B T}.$$  

(1.1)

All methods for calculating the above trace make use of the imaginary time technique in one form or another. The latter is based on the observation that $e^{-\beta H}$, with $\beta = 1/k_B T$, is formally identical with the time evolution operator $e^{i\mathcal{H}t/\hbar}$, with the time $t$ replaced by the imaginary quantity $i\hbar \beta$. One then proceeds to divide the imaginary time interval $[0, \beta]$ into many infinitesimal subintervals for which the time evolution can be calculated. The coupling between statics and dynamics then arises from the fact that the Hamiltonian taken at some imaginary time, $H(\tau)$, does not commute with $H$ taken at another imaginary time $\tau'$.

Although QPTs are currently a topic of great interest in condensed matter physics, they are actually a subject with a long history. The earliest work all dealt with quantum spin systems, and especially one–dimensional chain models. Important work was done by Bethe, Bloch, Tomonaga, and Lieb. In the mid–1970’s the QPT problem was cast into a more general theoretical framework. Suzuki used the so–called Trotter formula, which is one way to rewrite the trace in Eq. (1.1) by dividing $[0, \beta]$ into sub–intervals, to map some $d$–dimensional quantum spin problems onto $(d+1)$–dimensional classical statistical mechanical problems. The extra dimension was for imaginary time, which is of infinite extent in the zero–temperature limit ($\beta \to \infty$), effectively increasing the dimension of the system. Beal–Monod noted that this mapping made it obvious how to generalize the renormalization group techniques that were being used to describe classical phase transitions for an application to quantum transitions. Motivated by these works, Hertz made a number of important advances. First he showed how to derive Landau–Ginzburg–Wilson (LGW) order–parameter functionals, starting from a fermionic description of itinerant electron systems that writes the trace

\[^3\text{We will denote by Tr traces over all degrees of freedom, including continuous ones, and by tr traces over discrete degrees of freedom that are not shown explicitly.}\]
in Eq. (1.1) as a functional integral. As expected, he showed that to describe QPTs one needs to consistently describe both space and (imaginary) time fluctuations together. He further showed that Suzuki’s mapping of a \(d\)-dimensional quantum system on a \((d+1)\)-dimensional classical was too restrictive for generic systems. He found that, more generally, a \(d\)-dimensional quantum system is related to a \((d+z)\)-dimensional classical one, with \(z\) the dynamical scaling exponent for the quantum phase transition. The need for \(z \neq 1\) arises since in general in a condensed matter system, space and time do not scale in the same way. For instance, in a system with diffusive dynamics, one expects frequency to scale like the wavenumber squared, which suggests \(z = 2\). The special case \(z = 1\) occurs when space and time happen to scale in the same way. Hertz then showed that the same renormalization group ideas that had been used to characterize classical phase transitions could be applied to the quantum case for arbitrary values of \(z\).

Hertz concluded that most QPTs in three–dimensional (3–d) systems are trivial from a phase transition viewpoint in the sense that they are characterized by Landau or mean–field critical exponents. This conclusion was based on the observation that the mapping \(d \rightarrow d + z\), with \(z > 1\), lowers the upper critical dimension \(d^*_c\), above which the transition is correctly described by a mean–field/Gaussian theory, from \(d^*_c = 4\) for most classical phase transitions to \(d^*_c = 4 - z < 3\) for the quantum transition. For systems with \(z = 1\) the critical behavior would still be mean–field like up to logarithmic corrections to scaling. The only exceptions would be low–dimensional (\(d \leq 2\)) systems.

One of the main points of the current review is to emphasize that most QPTs are more complicated than suggested by the above arguments. These complications arise because in many quantum systems there are soft or slow modes that exist in addition to the critical order parameter fluctuations. These “extra” modes arise either from conservation laws (recall that statics and dynamics are coupled at QPTs), or from Goldstone modes that result from spontaneous broken continuous symmetry. These modes will in general couple to the critical modes and influence the critical behavior. If one insists on integrating out these degrees of freedom in order to obtain a standard LGW theory that is formulated in terms of the order parameter only, then this LGW theory becomes non–local. Physically, the extra modes lead to long–range interactions between the order parameter fluctuations, which in turn leads to a non–local LGW functional. This breakdown of Hertz’s theory becomes particularly obvious in systems with quenched disorder. In the cases considered by Hertz, he again found that the correlation length exponent had its mean–field value, \(\nu = 1/2\), for all physical values of the dimensionality. However, it is now known (though it was not known when Hertz derived his theory) that \(\nu\) must obey the inequality \(\nu \geq 2/d\) in generic disordered systems. This implies that the mean–field result cannot be correct for \(d < 4\).

The LGW theories mentioned above are one example of what is called effective field theories for condensed matter systems. The main motivation for deriving and studying such effective field theories is that often one is not interested in all of the detailed behavior that arises from the microscopic properties of a system. For instance, one might ask about universal effects that are the same for large classes of systems, even though the members of these classes are different with respect to their composition, their band structure, and other microscopic details. Such universal properties arise from long–wavelength and low–frequency fluctuations, i.e. processes that occur on length and time scales that are large compared to the microscopic lengths and times that determine, say, the band structure. This in turn implies that only the dominant fluctuations in this limit need to be retained explicitly and treated carefully in order to describe the universal properties. Of particular interest are soft modes, i.e. excitations that are massless in the limit of zero frequencies and wavenumbers. All other degrees of freedom can be integrated out, often in a crude approximation, without affecting the description of the universal behavior. The resulting theory for the most important fluctuations in a specified system is called an effective field theory. This is quite analogous to the situation in high–energy physics, where one also deals with effective field theories that describe the low–energy or long–distance behavior of some underlying microscopic theory. The main difference is that in condensed matter physics the microscopic theory is known, but too complicated to solve in general, and the problem consists of deriving an effective theory that is simple enough to extract the behavior of interest, while in high–energy physics the effective theory is known (i.e., it has been guessed) and the problem consists of inferring from it a more microscopic theory. In high–energy physics one thus proceeds from low to high energies, while the opposite is true in condensed matter physics. The most well–known examples of effective field theories in a condensed matter context are those that are used to describe the universal behavior near critical points. The concept is much more general, however. Even far from any phase transitions, soft modes will arise from either the conservation laws, or from the existence of Goldstone modes in a broken symmetry phase. Both of these mechanisms are common in quantum statistical mechanics, and the resulting soft modes determine the physics at large distances and low energies. In addition to these soft modes, near a phase transition the critical order parameter fluctuations are also soft. One of our objectives is to review how to derive effective field theories, both near and far from phase transition lines or points.

The current chapter is organized as follows. In Section I§ we give a brief discussion of scaling theory near (continuous) quantum phase transitions. We start Section II with a general discussion of the functional field theoretic description of itinerant electronic systems. We then discuss how to derive effective field theories for the relevant degrees of freedom, and show how to describe soft modes in general disordered electronic systems. We begin Section IV by reviewing the ferromagnetic phase transition in both disordered and clean itinerant electron systems. We will see that
We then examine the antiferromagnetic transition in disordered itinerant systems. The most important result is that quantum fluctuation effects stabilize antiferromagnetic long-range order in certain parameter regions. In Section V, we discuss some recent work on the superconductor–metal transition in bulk ($d = 3$) disordered electron systems. In Section VI we consider the disordered Fermi liquid phase, and its instability against metal–insulator transitions of the Anderson–Mott type. First we briefly review the theory based on Finkel’stein’s generalized nonlinear sigma–model, and then we discuss a recent order–parameter approach to the Anderson–Mott transition. We discuss similarities to the classical random–field magnet problem and discuss possible implications for the Anderson–Mott transition in three dimensions.

There are other interesting and important examples of quantum phase transitions that we will not discuss, for instance quantum spin chains, quantum spin glasses, and phase transitions in quantum Hall systems. A brief review of the latter subject has recently been given by Sondhi et al. [9].

II. SCALING AT QUANTUM CRITICAL POINTS

Scaling ideas have been used with great success in the description of continuous thermal phase transitions since the mid–1960’s [1]. These same ideas, properly reinterpreted, can be applied to the QPT problem. Since the scaling description of quantum phase transition has been recently reviewed in a number of places [2,9,10], we will be brief here.

In general, the mapping $d \rightarrow d + z$, discussed in Sec. I above, suggests that the classical scaling relations are valid at the QPT, with the only change being the above substitution. Although this is the standard lore, and it is true in a formal sense, we will repeatedly encounter situations where it is not valid. When such a breakdown of general scaling happens, we will mention it explicitly. More specifically, our main conclusions will be as follows. (1) The classical scaling relations that are independent of dimensionality are also valid at QPTs. (2) The classical scaling relations that do depend on $d$ (‘hyperscaling relations’) are formally valid with the substitution of $d \rightarrow d + z$. However, since many of the important QPTs are above their upper critical dimensionalities, these scaling relations in general break down because of the presence of dangerous irrelevant variables. Further, the situation is much more complicated than in classical systems above their upper critical dimensionality because, as mentioned in Sec. I, the LGW functional for describing QPT’s is often a non–local one. This feature has been shown to lead to nontrivial, dimensionality dependent, exponents and scaling relations, even above the upper critical dimension for the QPT. (3) There are scaling relationships at QPTs that have no classical analog. These identities follow from the scaling equivalence of energy, inverse time, and temperature. They depend on the dimensionality and can also be invalidated by dangerous irrelevant variables.

All scaling theories for continuous phase transitions start with the idea of a divergent correlation length scale, $\xi$. If $t$ is the dimensionless distance from the critical point, then the divergence of $\xi$ is characterized by the critical exponent $\nu$,

$$\xi \sim \frac{1}{|t|^\nu}. \quad (2.1a)$$

Typically $t$ is defined to be greater than zero in the disordered phase, and less than zero in the ordered one. At thermal phase transitions, $t$ is given by the reduced temperature, and it is very easy to control experimentally. For QPTs $t$ might represent the distance from a critical interaction strength, or from a critical impurity concentration. In the quantum case, $t$ is therefore not so easily controlled, and to vary $t$ different samples are usually used. In general this means that experimentally probing the quantum critical region, with the precision necessary for measuring critical exponents, is very difficult.

As we have mentioned in Sec. I, in addition to the diverging correlation length we need to deal with a diverging time scale. As the critical point is approached from, for example, the disordered side, the order that is being built up decays on a longer and longer time scale. This correlation time $\xi_r$ diverges at the transition, and the singularity is characterized by the dynamical scaling exponent $z$,

$$\xi_r \sim \xi^z. \quad (2.1b)$$

4The concept of dangerous irrelevant variables is defined at the end of Sec. I.

5An exception is the stress tuning technique, where uniaxial or hydrostatic stress is applied to a sample that is very close to the transition. The resulting change in the microscopic structure drives the system across the phase separation line. See Ref. [11] for an application.
Equations (2.1) represent an assumption, namely that both $\xi$ and $\xi_t$ diverge as some power of $t$. There is no principal reason why the singularity might not be stronger or weaker than that, and we will indeed encounter examples of exponential divergencies in Secs. 3 and 4 below. For now we will continue, however, with the discussion of the more conventional power–law scaling assumption.

Before formalizing the scaling ideas, we will first give two examples of how scaling (or hyperscaling) is used. First consider the singular part of the free energy density, $f_s$. The dimension of $f_s$ is that of an inverse volume times an energy. Assuming that $\xi$ and $\xi_t$ are the only relevant length and energy scales in the problem, this suggests

$$f_s \sim \xi^{-(d+z)} . \tag{2.2a}$$

Note that for the classical problem, Eq. (2.2a) is obtained with $d + z$ replaced by $d$, since in that case the relevant energy scale as $\xi \to \infty$ is just the critical temperature, $T_c \neq 0$. Using Eq. (2.1a) with $\xi_t \sim 1/t$ yields for the temperature dependence of $f_s$ at $t = 0$

$$f_s(t = 0, T) \sim T^{1+d/z} . \tag{2.2b}$$

The singular part of the specific heat thus behaves as

$$C \sim T^{d/z} . \tag{2.3}$$

That is, by using scaling ideas one can express the critical behavior of $C$ in terms of the dynamical critical exponent.

As another example, consider the conductivity, $\sigma$. In fundamental units, setting $\hbar = k_B = e^2 = 1$ (with $e$ the electron charge), the dimension of $\sigma$ is an inverse length to the power $(d-2)$. At a phase transition where $\sigma$ vanishes in the ‘disordered’ phase and is nonzero in the ‘ordered’ phase, i.e., at a metal–insulator transition, this suggests that $\sigma$ vanishes as

$$\sigma(T = 0, t) \sim \xi^{-(d-2)} \sim t^{\nu(d-2)} . \tag{2.4a}$$

We see that the conductivity exponent $s$, defined by $\sigma \sim t^s$, is related to the correlation length exponent $\nu$ by

$$s = \nu(d-2) . \tag{2.4b}$$

Equation (2.4b) is called Wegner’s scaling relation [12].

The above scaling arguments have used only what in the literature is called a naive or engineering (with apologies to engineers) dimensional analysis. For instance, we have assumed that since the conductivity is measured in units of centimeters to the power $2-d$, it will scale like $\xi^{2-d}$. In general this type of argument is not correct. The conductivity, for example, could acquire an ‘anomalous dimension’ $\theta$ by scaling like $\sigma \sim \xi^{2-d+\theta} a^{-\theta}$, where the ‘wrong’ dependence on the length scale $\xi$ is compensated for by a dependence on some microscopic length scale $a$. However, for some quantities, such as $f$ and $\sigma$ above, there are additional arguments that show that they have zero anomalous dimension so that the above argument is correct. There is, however, yet another way for scaling relations like Eq. (2.4b) to break down, which we will discuss at the end of the current Section.

Let us now formalize the scaling approach a bit. Suppose we are interested in an observable $Q$ as a function of $t, T$, frequency $\Omega$, wavenumber $k$, and possibly an external field $h$. The power law singularities that characterize continuous phase transitions imply that the singular or scaling part $Q_s$ of the observable is a generalized homogeneous function of the appropriate variables. That is, the usual scaling ansatz, which typically can be derived by using renormalization group methods, is

$$Q_s(t, T, h; k, \Omega) = b^{-[Q]} Q_s(b^{1/\nu} t, b^{2} T, b^{\nu_s} h; bk, \Omega) . \tag{2.5}$$

Here $b$ is an arbitrary length rescaling factor, and the number $[Q]$ is called the scale dimension of the quantity $Q$. The remaining exponents in Eq. (2.5) have been chosen so that $b$ really is the length rescaling factor. For example, if $Q = \xi$ is the correlation length, then at $T = h = k = \Omega = 0$, and using $[\xi] = -1$ (as one should if $b$ is a length rescaling factor), we have $\xi(t) = b^1 \xi(b^{1/\nu t})$. Choosing $b = 1/t'$ gives $\xi(t) = \xi(1)/t'$, i.e., $\xi$ correctly diverges as $t^{-1}$ as $t \to 0$, and $\xi(1)$ is the value of the correlation length far away from the critical point.

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$^6$We will use units such that $\hbar = 1$ in what follows.

$^7$We will see in Sec. 3 in which sense the metallic phase can be considered the ordered one.
To illustrate the usefulness of these homogeneity laws, let us consider the conductivity $\sigma$ again, as a function of $t$, $T$, and $\Omega$,

$$\sigma(t, T, \Omega) = b^{-(d-2)} \sigma(b^{1/\nu} t, b^{z} T, b^{\nu} \Omega) .$$  \hspace{1cm} (2.6a)$$

By choosing $b = 1/t^\nu$, Eq. (2.6a) gives,

$$\sigma(t, T, \Omega) = t^{\nu(d-2)} \sigma(1, T/t^\nu, \Omega/t^\nu)$$

$$= t^{\nu(d-2)} F_\sigma(T/t^\nu, \Omega/t^\nu) ,$$  \hspace{1cm} (2.6b)

and by choosing $b = 1/T^{1/z}$ we obtain,

$$\sigma(t, T, \Omega) = T^{(d-2)/z} \sigma(t/T^{1/\nu}, 1, \Omega/T)$$

$$= T^{(d-2)/z} G_\sigma(t/T^{1/\nu}, \Omega/T) .$$  \hspace{1cm} (2.6c)

Here we have defined the functions $F_\sigma(x, y) = \sigma(1, x, y)$ and $G_\sigma(x, y) = \sigma(x, y)$. The important point is that, say, at $\Omega = 0$ the quantity $\sigma/t^{\nu(d-2)}$ is a function of $T/t^\nu$ alone, rather then of $t$ and $T$ separately. The resulting collapse of data on a single scaling curve has been historically important for the confirmation of scaling at thermal phase transitions \cite{1}, and it is equally useful at QPTs. We will consider an example in Sec. VII B 3 below.

Equations (2.6) reflect the fact that $t$ is not the only relevant distance from the critical point. At zero frequency and wavenumber there are two relevant variables: $t$ and $T$. For either $t \neq 0$ or $T \neq 0$ the system is away from the critical surface, and the exponents $\nu$ and $z$ are measures of the relevance of $t$ and $T$, respectively. Of course, in any system there are numerous other variables that are irrelevant. Let us denote a generic irrelevant variable by $u$. In a homogeneity law such as Eqs. (2.6), these irrelevant variables also occur, but they have negative scale dimensions, $[u] < 0$. We generalize Eq. (2.6a) at $\Omega = 0$ to

$$\sigma(t, T, u) = b^{-(d-2)} \sigma(b^{1/\nu} t, b^{z} T, b^{\nu} u) .$$  \hspace{1cm} (2.7a)

Choosing $b = 1/t^\nu$ now gives

$$\sigma(t, T, u) = t^{\nu(d-2)} \sigma(1, T/t^\nu, t^{-\nu[u]} u) ,$$  \hspace{1cm} (2.7b)

which for $t \to 0$ yields (remember $[u] < 0$)

$$\sigma(t \to 0, T, u) = t^{\nu(d-2)} \sigma(1, T/t^\nu, 0)$$

$$= t^{\nu(d-2)} F_\sigma(T/t^\nu) .$$  \hspace{1cm} (2.7c)

We see that close to the critical point, $t \to 0$, the dependence on the irrelevant variable drops out. This is a very important result and is one of the reasons for the universal behavior near continuous phase transitions.

We conclude this subsection by introducing the concept of a dangerous irrelevant variable \cite{14}. Let us use Eq. (2.7a) as an example. Implicit in the last equality in that equation is the assumption that $\sigma(1, T/t^\nu, u = 0)$ is neither zero nor infinity. This is a nontrivial assumption which can break down. If $t$ does, then the irrelevant variable $u$ is called a dangerous irrelevant variable and actually modifies the scaling behavior, and scaling equalities, even though it is irrelevant in the technical sense of the renormalization group. Whether or not a given irrelevant variable is dangerous with respect to a certain observable can usually not be determined on general grounds, but requires explicit calculations. In the context of thermal phase transitions, it is dangerous irrelevant variables that lead to a breakdown of hyperscaling above the upper critical dimensionality \cite{13}. In the following sections we will repeatedly see that dangerous irrelevant variables play an even more important role in QPTs, mostly due to the small value of the upper critical dimensionality of the latter. In particular, we will see in Sec. VII B 2 that Wegner’s scaling law, Eqs. (2.4b), (2.7c), is violated at certain metal–insulator transitions due to a dangerous irrelevant variable.

\section*{III. FERMIONIC FIELD THEORY}

The many–fermion problem has a long history, due to its importance with respect to electrons in condensed matter systems. In order to describe nonperturbative effects in these systems, such as phase transitions involving spontaneous symmetry breaking, it is convenient to go to a field theoretic description. In such a formalism the important underlying symmetries are most apparent, and, at least formally, it is relatively easy to integrate out the degrees of freedom that are irrelevant for describing the physics one wants to focus on. Here we will use standard field theoretic techniques \cite{13} to sketch the development of a theory for disordered interacting electrons, following along the lines of Ref. \cite{14}.
A. Grassmannian field theory

Since the description of fermions involves anticommuting variables, any field theory or functional Feynman path integral description for electrons must be formulated in terms of anticommuting or Grassmann variables. For the sake of simplicity we consider a model for a homogeneous electron fluid subject to a random potential that models the quenched disorder. For most of the effects we want to study, the inclusion of microscopic details like the underlying crystal structure and the resulting band structure effects, or the explicit inclusion of phonons, is not necessary. The partition function can then be written

\[ Z = \int D[\bar{\psi}, \psi] \exp[S] \]  

Here the functional integration is with respect to Grassmann valued fields, \( \bar{\psi} \) and \( \psi \), and the action \( S \) is given by

\[ S = -\int dx \sum_\sigma \bar{\psi}_\sigma(x) \partial_\tau \psi_\sigma(x) + S_0 + S_{\text{dis}} + S_{\text{int}} \]  

We use a \((d+1)\)-vector notation, with \( x = (x, \tau) \), and \( \beta = 1/T \) is the inverse temperature, and \( \sigma \) is the spin label. \( S_0 \) describes free electrons with chemical potential \( \mu \),

\[ S_0 = \int dx \sum_\sigma \bar{\psi}_\sigma(x) \left( \frac{\nabla^2}{2m} + \mu \right) \psi_\sigma(x) \]  

with \( m \) the fermion mass. We will find it useful to go to a Fourier representation with wavevectors \( k \) and fermionic Matsubara frequencies \( \omega_n = 2\pi T(n+1/2) \), and a \((d+1)\)-vector notation, \( k = (k, \omega_n) \). Then \( S_0 \) reads,

\[ S_0 = \sum_{k, \sigma} \bar{\psi}_\sigma(k) \left[ i\omega_n - \frac{k^2}{2m} + \mu \right] \psi_\sigma(k) \]  

Here we have redefined \( S_0 \) to include the first term on the right–hand side of Eq. (3.2a). \( S_{\text{dis}} \) describes a static random potential, \( u(x) \), that couples to the fermionic number density;

\[ S_{\text{dis}} = -\int dx \sum_\sigma \bar{\psi}_\sigma(x) \psi_\sigma(x) \]  

and \( S_{\text{int}} \) describes a spin–independent two–particle interaction,

\[ S_{\text{int}} = -\frac{1}{2} \int dx_1 dx_2 \sum_{\sigma_1, \sigma_2} \bar{\psi}_{\sigma_1}(x_1) \bar{\psi}_{\sigma_2}(x_2) \psi_{\sigma_2}(x_2) \psi_{\sigma_1}(x_1) \]  

The interaction potential \( v(x) \) will be discussed further below.

For simplicity, the random potential in Eq. (3.2c) is taken to be delta–correlated with a Gaussian distribution. Its second moment is,

\[ \{ u(x) u(y) \}_{\text{dis}} = \frac{1}{\pi N_F \tau_{\text{el}}} \delta(x - y) \]  

where \( \{ \ldots \}_{\text{dis}} \) denotes the disorder average. Here \( \tau_{\text{el}} \) is the elastic–scattering mean–free time, and \( N_F \) denotes the bare single–particle density of states (for both spin projections). More general random potential models can be used, but any differences between such generalizations and Eq. (3.3) are irrelevant for the long–wavelength effects we are concerned with in this review.

For disordered systems, single–particle momentum excitations are not long lived, but decay exponentially on a time scale given by the elastic mean–free time, \( \tau_{\text{el}} \). The important physics on the longest length and time scales is

\[ \text{We will denote fermionic frequencies by } \omega_n, \text{ and bosonic ones by } \Omega_n \text{ throughout.} \]
controlled by two–particle excitations that are soft either because they are related to conserved quantities or because of a mechanism related to Goldstone’s theorem. For a detailed discussion of the soft modes in a disordered, interacting electronic system we refer the reader to Ref. [15]. The main conclusion is that of all the modes that appear in $S_{\text{int}}$, the dominant soft modes are those that involve fluctuations of either the particle number density $n_n$, or the spin density $n_s$, or density fluctuations, $n_c$, in the particle–particle or Cooper channel. In Fourier space, these densities are given in terms of fermion fields by

$$n_n(q) = \sqrt{T/V} \sum_{k,\sigma} \bar{\psi}_\sigma(k) \psi_\sigma(k+q)$$ \hspace{1cm} (3.4a)

$$n_s(q) = \sqrt{T/V} \sum_{k,\sigma,\sigma'} \bar{\psi}_\sigma(k) \bar{\sigma}_{\sigma\sigma'} \psi_{\sigma'}(k+q)$$ \hspace{1cm} (3.4b)

$$n_c(q) = \sqrt{T/V} \sum_k \psi_\uparrow(k) \psi_\downarrow(-k+q)$$ \hspace{1cm} (3.4c)

with $\bar{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ the Pauli matrices, and

This means that in Eq. (3.2d) we should project onto these fluctuations. The net result is that $S_{\text{int}}$ can be split into three terms, each with a different interaction amplitude,

$$S_{\text{int}} = S_{\text{int}}^{(s)} + S_{\text{int}}^{(t)} + S_{\text{int}}^{(c)}$$ \hspace{1cm} (3.5)

with the singlet, triplet, and Cooper channel terms simply describing interactions between the densities given in Eqs. (3.4),

$$S_{\text{int}}^{(s)} = -\frac{\Gamma^{(s)}}{2} \sum_q n_n(q) n_n(-q)$$ \hspace{1cm} (3.6a)

$$S_{\text{int}}^{(t)} = \frac{\Gamma^{(t)}}{2} \sum_q n_s(q) \cdot n_s(-q)$$ \hspace{1cm} (3.6b)

$$S_{\text{int}}^{(c)} = -\frac{\Gamma^{(c)}}{2} \sum_q n_c(q) n_c(-q)$$ \hspace{1cm} (3.6c)

Here $\Gamma^{(i)}$, $i = s, t, c$, are singlet, triplet and Cooper channel interaction amplitudes, which are in turn given by angular averages of the interaction potential at specified momenta.

Since the system contains quenched disorder, it is necessary to average the free energy or $\ln Z$. This is accomplished by means of the replica trick \[13\], which is based on the formal identity

$$\ln Z = \lim_{n \to 0} (Z^n - 1)/n$$ \hspace{1cm} (3.7)

Introducing $n$ identical replicas of the system (with $n$ an integer), labeled by the index $\alpha$, and carrying out the disorder average, we obtain

$$\tilde{Z} \equiv \{Z^n\}_\text{dis} = \int D[\bar{\psi}^\alpha, \psi^\alpha] e^S$$ \hspace{1cm} (3.8)

We again separate $S$ into free, disordered and interacting parts,

\[\text{For short–ranged model interactions, this can be done in a straightforward way} \] [15]. A Coulomb interaction requires a somewhat more elaborate procedure \[17\]. However, the final result is given by Eqs. (3.6) in either case.
reference ensemble'.

where the

integral, the replicated partition function, Eq. (3.8), can be written as

\[ S = \sum_{\alpha=1}^{n} (\tilde{S}_0^\alpha + \tilde{S}_{dis}^\alpha + \tilde{S}_{int}^\alpha) \quad . \]

\[ \tilde{S}_0^\alpha \] and \( \tilde{S}_{int}^\alpha \) are given by Eqs. (3.2b) and (3.3) and (3.4), respectively with \( \psi \rightarrow \psi^\alpha \) and \( S_{dis}^\alpha \) is given by,

\[ \tilde{S}_{dis}^\alpha = \frac{1}{2\pi N_F \tau_0} \sum_{\beta=1}^{n} \sum_{\{k\}} \sum_{n,m,\sigma,\sigma'} \delta_{k_1+k_3, k_2+k_4} \psi^\alpha_{n\sigma}(k_1) \psi^\alpha_{m\sigma}(k_3) \psi^{\bar{\alpha}}_{m\bar{\sigma}}(k_4) \psi^{\bar{\alpha}}_{n\bar{\sigma}}(k_4) \quad . \]

At the end of all calculations, one takes the limit \( n \rightarrow 0 \) (assuming that both the analytic continuation from integer to real \( n \) and the limit exist) to obtain the desired averages for quenched disorder.

**B. Order parameter field theories**

Starting with the pioneering work of Landau [19], it emerged over the years that an expedient way to describe any phase transition is to formulate an effective field theory, the LGW functional, entirely in terms of an appropriate order parameter, and to bury the information about all other degrees of freedom in the vertices of that order parameter field theory. As we will see, this approach is more problematic for quantum phase transitions than for thermal ones. The reason is that the LGW approach hinges in part on an implicit assumption, namely that the order parameter fluctuations are the only soft modes in the system. Although this is not true for most of the quantum phase transitions we will be interested in, it turns out that the LGW approach nevertheless provides an economical way of solving the problem, provided one adequately deals with the ensuing complications. In the current subsection we therefore show how to derive LGW functionals of ferromagnetic, antiferromagnetic, and superconducting order parameters, respectively; for the time being without worrying about any additional soft modes. The question of an order parameter description of metal–insulator transitions is more intricate, and does require a thorough examination of the symmetry and soft mode properties of the underlying fermionic field theory, Sec. III A, as a prerequisite. This analysis will be given in Sec. III C below.

1. Magnetic order parameters

It is well known that the model defined in Sec. III A contains a ferromagnetic phase. Large values of the coupling constant \( \Gamma \) and small values of \( T \) favor ferromagnetism, while small values of \( \Gamma \) and large values of \( T \) favor paramagnetism. The appropriate order parameter for the transition between these two phases is the magnetization \( m \) or the expectation value of the spin density defined in Eq. (3.4), \( \mathbf{m} = \langle \mathbf{n}_s(x) \rangle \). Technically, the derivation of a LGW functional is easily achieved by means of a trick due to Stratonovich and Hubbard [20], which is based on the identity

\[ \int D[M] \ e^{-(\Gamma \langle t \rangle/2) \int dx \ M(x) \cdot M(x) + \Gamma \langle t \rangle \int dx \ M(x) \cdot n_s(x)} = \text{const} \times e^{S_{int}^{(t)}} \quad . \]

Here \( M(x) \) is a classical (i.e., complex number valued) auxiliary field that couples linearly to the spin density. The strategy is now to apply this Gaussian decoupling procedure, which is referred to as a Hubbard–Stratonovich decoupling or as ‘uncompleting the square’, to \( S_{int}^{(t)} \) while leaving all other terms in the action unchanged. We thus write the action as

\[ S = S_{ref} + \ln \int D[M] \ e^{-(\Gamma \langle t \rangle/2) \int dx \ M(x) \cdot M(x) + \Gamma \langle t \rangle \int dx \ M(x) \cdot n_s(x)} \quad , \]

where the \( S_{ref} \) contains all pieces of the action other than \( S_{int}^{(t)} \). We will refer to the system described by \( S_{ref} \) as the ‘reference ensemble’.

By applying the Hubbard–Stratonovich transformation explained above, and formally performing the fermionic integral, the replicated partition function, Eq. (3.8), can be written as

\[ \tilde{Z} = e^{-nF_0/T} \prod_{\alpha} D[M^\alpha] \exp\left(-\Phi[M]\right) \quad , \]

\[ \Phi[M] = \sum_{\alpha=1}^{n} \left(\tilde{S}_0^\alpha + \tilde{S}_{dis}^\alpha + \tilde{S}_{int}^\alpha\right) \].
with $F_0$ the noncritical part of the free energy. The LGW functional $\Phi$ reads

$$\Phi[M] = \frac{F(t)}{2} \sum_\alpha \int dx \ M^\alpha(x) \cdot M^\alpha(x)$$

$$- \ln \left\{ \exp \left( -\Gamma(t) \sum_\alpha \int dx \ M^\alpha(x) \cdot n_\alpha(x) \right) \right\}_{\text{ref}} \ .$$  (3.12b)

Here $\langle \ldots \rangle_{\text{ref}}$ denotes an average taken with respect to the reference ensemble. A useful interpretation of the second contribution to the LGW functional is obtained by realizing that it represents the free energy of the reference ensemble in the presence of an external magnetic field that is proportional to the order parameter field $M$. Also notice that the average of $M$ is proportional to the magnetization $m$, by virtue of the linear coupling between $M$ and $n_\alpha$.

For clean systems, and a more restricted reference ensemble, the above derivation of an LGW functional for quantum ferromagnets is due to Hertz \[4\]. Our derivation \[21\] has generalized his approach to include disorder, and electron interaction in multiple channels. The standard way to proceed with a discussion of the paramagnet–to–ferromagnet transition is a Landau expansion or expansion of the LGW functional $\Phi$ in powers of the order parameter field $M$. The structure of this expansion is

$$\Phi[M] = \sum_{l=2}^\infty \Phi_l[M]$$

$$= \frac{1}{2} \sum_\alpha \int dx_1 dx_2 \ X^{(2)}_{ab}(x_1, x_2) M^\alpha_a(x_1) M^\alpha_b(x_2)$$

$$+ \frac{1}{3!} \sum_\alpha \int dx_1 dx_2 dx_3 \ X^{(3)}_{abc}(x_1, x_2, x_3) M^\alpha_a(x_1) M^\alpha_b(x_2) M^\alpha_c(x_3)$$

$$- \frac{1}{4!} \sum_{\alpha, \beta} \int dx_1 dx_2 dx_3 dx_4 \ X^{(4)}_{abcd}(x_1, x_2, x_3, x_4) M^\alpha_a(x_1) M^\alpha_b(x_2) M^\alpha_c(x_3) M^\alpha_d(x_4) + O(M^5) \ ,$$  (3.12c)

The vertex functions $X^{(l)}$ are given in terms of the spin density correlations of the reference ensemble, and can be calculated within any theory of interacting disordered electrons. This we will come back to in Sec. \[IVB\] below, where we will see that this Landau expansion is ill–behaved in the sense that the vertex functions cannot be localized in space and time.

For an antiferromagnet the interesting fluctuations are those of the staggered magnetization. Our simple continuum model, Sec. \[IIA\] actually has no antiferromagnetic phase. However, order parameter theories for antiferromagnets can be derived from more complicated microscopic theories \[22\]. The net result is again a LGW functional of the form of Eq. (3.12), with the fluctuating magnetization $M$ replaced by a different vector order parameter $N$, whose average is proportional to the staggered magnetization. The structure of the vertex functions in this case is different from the ferromagnetic one. We will model these vertex functions in Sec. \[IVB\].

\[2. \text{Superconducting order parameter}\]

For superconductors, the procedure is completely analogous to that in the ferromagnetic case, except that one uses a Hubbard–Stratonovich decoupling on the Cooper channel interaction, $S^{(c)}_{\text{int}}$. We thus write

$$\tilde{S} = \tilde{S}_{\text{ref}} + \tilde{S}^{(c)}_{\text{int}} \ ,$$  (3.13)

\[10\] $F_0$ is noncritical as long as the reference ensemble does not undergo a phase transition of its own. This is not an entirely trivial point, as perturbation theory within the reference ensemble will in general create a nonvanishing spin–triplet interaction, even though there was none in the bare system. For all of our order parameter theories we assume that the reference ensemble is far from any phase transitions, since else the mode separation that is implicit in Eq. (3.11) breaks down, and nothing is gained by the Hubbard–Stratonovich transformation.

\[11\] We denote the microscopic fermionic action by $S$, LGW functionals that depend only on the classical order parameter field by $\Phi$, and effective fermion actions by $A$, respectively.
with \( S_{\text{lat}}^{(c)} \) the replicated generalization of Eq. (3.6c), and proceed as in Sec. III B 1, with the only difference being that we now have a complex valued order parameter that we denote by \( \Psi \). The LGW functional reads
\[
\Phi[\Psi] = -\Gamma^{(c)} \sum_{\alpha} \int dq \, |\Psi^\alpha(q)|^2 - \ln \left( e^{-\Gamma^{(c)} \sum_{\alpha} \int dq \left( \Psi^{\alpha*}(q) n_\alpha^c(q) + \Psi^{\alpha}(q) \bar{n}_\alpha^c(q) \right)} \right)_{\text{ref}}.
\]
(3.14)

The Landau expansion in powers of the order parameter is again analogous to the ferromagnetic case, Eq. (3.12c), except that due to gauge invariance only even powers of \( \Psi \) appear, and due to the complex valuedness of \( \Psi \) the vertex functions are scalars rather than tensors. This order parameter field theory will be discussed in Sec. V below.

C. The nonlinear sigma–model

Before we continue the discussion of the effective field theories derived in the previous subsection, it is useful to discuss the properties of the underlying fermionic field theory in some detail, in particular with respect to its symmetry properties and soft mode structure. This is very helpful for an understanding of the properties of the order parameter theories, and it is necessary in order to derive an order parameter theory for metal–insulator transitions.

The symmetry analysis of the fermionic theory is very technical and involved, and the details can be found in Ref. [15]. Here our goal is to explain the general structure of the theory, and the logic of the analysis, by exploiting a remarkable analogy between classical spin models for Heisenberg ferromagnets and disordered electrons that was first noted and used by Wegner [23].

1. Digression: The nonlinear sigma–model for classical Heisenberg ferromagnets

Let us consider, as a model for a classical Heisenberg ferromagnet, an \( O(N) \) symmetric \( \phi^4 \)–theory with a magnetic field \( h \) in the 1–direction. The action
\[
S[\phi, h] = \int d\mathbf{x} \left[ r(\phi(\mathbf{x}))^2 + c(\nabla \phi(\mathbf{x}))^2 \right] + u \int d\mathbf{x} \left( \phi(\mathbf{x}) \cdot \phi(\mathbf{x}) \right)^2 - h \int d\mathbf{x} \phi_1(\mathbf{x}) ,
\]
(3.15a)
determines the partition function
\[
Z[h] = \int D[\phi] e^{-S[\phi, h]} ,
\]
(3.15b)
and \( r, c, \) and \( u \) are real–valued coefficients that span the parameter space of the theory. The following discussion of this action, along standard lines [24], will turn out to be very useful as an analogy for the fermionic theory we are interested in.

The crucial property of the action, Eq. (3.15a), that we want to exploit is its symmetry: For zero external magnetic field, \( h = 0 \), the action is invariant under \( O(N) \) rotations of the field \( \phi \). Let us consider a particular rotation, between \( \phi_1 \) and \( \phi_i \) with \( 2 \leq i \leq N \), through an infinitesimal angle \( \theta \),
\[
\phi_1 \rightarrow \phi'_1 = \phi_1 + \delta \phi_1 = \phi_1 + \theta \phi_i ,
\]
\[
\phi_i \rightarrow \phi'_i = \phi_i + \delta \phi_i = \phi_i - \theta \phi_1 .
\]
(3.16)

We also introduce a source \( \mathbf{J}(\mathbf{x}) \) for the \( \phi \)–field, and consider the generating functional
\[
Z[h, \mathbf{J}] = \int D[\phi] e^{-S[\phi, h] + \int d\mathbf{x} \mathbf{J}(\mathbf{x}) \cdot \phi(\mathbf{x})} .
\]
(3.17)
If we change integration variables from \( \phi \) to \( \phi' \), then the Jacobian of that transformation is equal to one. To linear order in the infinitesimal group parameter \( \theta \) we thus obtain

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12The fact that classical ferromagnets are useful here for pedagogical purposes, while in Secs. III B 1 and IV A we discuss quantum ferromagnets, is accidental.

13This source can be thought of as a spatial modulation of the magnetic field \( h \).
\[
\int D[\vec{\phi}] \left( -\delta S[\vec{\phi}, h] + \int d\mathbf{x} \, \vec{J}(\mathbf{x}) \cdot \delta \vec{\phi}(\mathbf{x}) \right) e^{-S[\vec{\phi}, h]} = 0 ,
\]  

(3.18)

where \(\delta S = S[\vec{\phi}'] - S[\vec{\phi}]\). By differentiating with respect to \(J_i(\mathbf{x})\) and setting \(\vec{J} = 0\) after the differentiation we obtain \(N - 1\) Ward identities,

\[
\langle \delta S \cdot \phi_i(\mathbf{x}) \rangle + \langle \delta \phi_i(\mathbf{x}) \rangle = 0 , \quad (i = 2, \ldots, N) ,
\]

(3.19)

where the brackets denote an average with respect to the action \(S\). The only part of \(S\) that is not invariant under the rotation is the magnetic field term. The Ward identities, Eq. (3.19), thus take the form of a relation between the transverse (with respect to the magnetic field) two-point correlation functions and the longitudinal one-point function,

\[
h \int d\mathbf{y} \, \langle \phi_i(\mathbf{x}) \phi_i(\mathbf{y}) \rangle = \langle \phi_1(\mathbf{x}) \rangle .
\]

(3.20a)

Since the left-hand side of this identity is \(h\) times the homogeneous transverse susceptibility \(\chi_t\), while the right-hand side is equal to the magnetization \(m\), this can be rewritten as

\[
\chi_t = m/h .
\]

(3.20b)

We thus obtain the well-known result that the transverse zero-field susceptibility diverges everywhere in the ordered phase, i.e. whenever \(m > 0\). This is a particular example of Goldstone’s theorem: Since there is an \((N - 1)\)-parameter continuous symmetry \((O(N)\) in our case) that is spontaneously broken \((m = \langle \phi_1 \rangle_{h \to 0} \neq 0\) in the ordered phase), there are \(N - 1\) massless excitations (the transverse \(\phi-\phi\) correlation functions in the limit \(h \to 0\)).

In ordinary perturbation theory the masslessness of the \(N - 1\) Goldstone modes is not manifest, but rather has to be established by means of explicit calculations order by order \([24]\). This is clearly undesirable, and raises the question of whether one can reformulate the theory so that this qualitative feature is explicitly displayed. This can indeed be done, and the result takes the form of a so-called nonlinear sigma-model \([26]\). The basic idea is the realization that the massless Goldstone modes correspond to purely transverse fluctuations of the vector fields, as the above derivation of the Ward identity demonstrates.\(^{14}\) Let us therefore decompose the vector field \(\vec{\phi}\) into its modulus \(\rho\) and a unit vector field \(\hat{\phi}\),

\[
\vec{\phi}(\mathbf{x}) = \rho(\mathbf{x}) \hat{\phi}(\mathbf{x}) , \quad \hat{\phi}^2(\mathbf{x}) \equiv 1 ,
\]

(3.21)

Let \(\hat{\psi}\) be a particular fixed unit \(N\)-vector. Then any \(\hat{\phi}\) can be generated from \(\hat{\psi}\) by means of an \((O(N)\)-rotation). The set of the \(\hat{\phi}\) is thus isomorphic to \((O(N)\) modulo the subgroup that leaves \(\hat{\psi}\) invariant, which is \((O(N - 1)\). The \(\hat{\phi}\) therefore provide a representation of the homogeneous space \((O(N)/O(N - 1)\). In terms of \(\rho\) and \(\hat{\phi}\) the action reads,

\[
S[\rho, \hat{\phi}] = \int d\mathbf{x} \left[ c \rho^2(\mathbf{x}) \left( \nabla \hat{\phi}(\mathbf{x}) \right)^2 + c (\nabla \rho(\mathbf{x}))^2 + r \rho^2(\mathbf{x}) \right] + u \int d\mathbf{x} \rho^4(\mathbf{x}) - h \int d\mathbf{x} \rho(\mathbf{x}) \hat{\phi}_1(\mathbf{x}) ,
\]

(3.22a)

and switching from the functional integration variables \(\vec{\phi}\) to \((\rho, \hat{\phi})\) leads to a Jacobian or invariant measure

\[
I[\rho] = \prod_x \rho^{N - 1}(x) .
\]

(3.22b)

The important point is that in this formulation of the theory, the field \(\hat{\phi}\) appears only in conjunction with two gradient operators. \(\hat{\phi}\) represents the \(N - 1\) soft Goldstone modes of the problem, while \(\rho\) represents the massive modes. Now we parametrize \(\phi\),

\[
\hat{\phi}(\mathbf{x}) = (\sigma(\mathbf{x}), \vec{\sigma}(\mathbf{x})) ,
\]

(3.23a)

where

\([^14]\)Notice that the \((O(N)\) transformation leaves the modulus of the vector field invariant.
We split off the expectation value of the massive $\rho$–field, $\rho(x) = R + \Delta \rho(x)$, with $R = \langle \rho(x) \rangle$, and expand in powers of $\pi$ and $\Delta \rho$. Rescaling the coupling constants with appropriate powers of $R$, the action can be written

$$S[\rho, \pi] = S_{NL \sigma M}[\pi] + \Delta S[\rho, \pi] .$$

Here

$$S_{NL \sigma M}[\pi] = \frac{1}{t} \int dx \left[ (\nabla \pi(x))^2 + (\nabla \sigma(x))^2 \right] - h \int dx \sigma(x) ,$$

is the action of the $O(N)/O(N - 1)$ nonlinear sigma–model, which derives its name from the notation used in Eqs. (3.23), and

$$\Delta S[\rho, \pi] = r \int dx \left( \Delta \rho(x) \right)^2 + c \int dx \left( \nabla \Delta \rho(x) \right)^2 + 4Ru \int dx \left( \Delta \rho(x) \right)^3 + u \int dx \left( \Delta \rho(x) \right)^4$$

$$+ O \left( \Delta \rho \sigma, \Delta \rho \left( \nabla \phi \right)^2 \right) ,$$

contains the corrections to it, which are all massive. If we neglect all fluctuations of the massive $\rho$–field, then we are left with the $O(N)/O(N - 1)$ nonlinear sigma–model in the usual parametrization.

The separation of soft and massive modes is now complete, and will be preserved order by order in perturbation theory. We will explain in the next subsection how to perform an analogous separation of modes for the fermionic action of Sec. III A. Before we do that, however, let us briefly discuss a way to rewrite Eq. (3.24b) that will be very useful later. We enforce the constraint $\hat{\phi}^2 = 1$ explicitly by means of a Lagrange multiplier field $\lambda$,

$$S_{NL \sigma M}[\hat{\phi}, \lambda] = \frac{1}{t} \int dx \left[ (\nabla \hat{\phi}(x))^2 + \lambda(x) \left( \hat{\phi}^2(x) - 1 \right) \right] .$$

Usually one integrates out $\lambda$, which eliminates $\sigma$ in terms of $\pi$, and expands in powers of $\pi$. This leads to a $(2 + \epsilon)$–expansion for the Heisenberg transition. Alternatively, however, one can integrate out the $\pi$ fields to obtain an effective action $A_{eff}$ in terms of $\sigma$ and $\lambda$,

$$A_{eff}[\sigma, \lambda] = \frac{1}{t} \int dx \left[ (\nabla \sigma(x))^2 + (\sigma^2(x) - 1) \lambda(x) \right] + \frac{1}{2} (N - 1) \text{Tr} \ln \left( -\nabla^2 + \lambda(x) \right) .$$

Now look for a homogeneous saddle–point solution. Denoting the saddle–point values of the fields again by $\sigma$ and $\lambda$, respectively, we find

$$\lambda \sigma = 0 ,$$

$$\sigma^2 = 1 - (N - 1) t \frac{1}{V} \frac{1}{p^2 + \lambda} .$$

These saddle–point equations correctly reproduce the exact critical behavior for $d > 4$, which is mean–field like. They also yield the exact critical behavior for all $d > 2$ in the limit $N \to \infty$ (with $tN$ held fixed) [24].

2. Symmetry properties of the fermion model

It is possible to rewrite the fermionic action of Sec. III A in complete analogy to the treatment of the much simpler $\phi^4$–theory in the previous subsection. The first question that arises in this context is which symmetry group takes the place of $O(N)$. For this purpose, it is convenient to rewrite the fermionic theory in terms of four–component bispinors $\eta$ and $\eta^+$ that are defined by [27]

$$\eta_{\alpha}^a(x) = \frac{1}{\sqrt{2}} \left( \begin{array}{c} \phi_{\alpha n}^a(x) \\ \bar{\psi}_{\alpha n}^a(x) \\ \psi_{\alpha n}^a(x) \\ -\phi_{\alpha n}^a(x) \end{array} \right) ,$$

(3.27a)
and

\[ (\eta^\alpha)^n (x) = i(C\eta)^n (x) = \frac{i}{\sqrt{2}} \left( \begin{array}{c} -\psi^\alpha_n (x) \\ -\psi^\alpha_{n1} (x) \\ \psi^\alpha_{n(2)} (x) \\ -\psi^\alpha_{n(3)} (x) \end{array} \right), \]  

(3.27b)

where

\[ i\gamma^{\alpha\beta} = \delta_{nm} \delta^{\alpha\beta} c_{ij} \]

(3.27c)

with \( c \) the charge–conjugation matrix

\[ c = \begin{pmatrix} 0 & s_2 \\ s_2 & 0 \end{pmatrix} = i\tau_1 \otimes s_2 \]  

(3.27d)

The four degrees of freedom represented by the bispinor are the particle–hole or number density degrees of freedom, and the two spin degrees of freedom. We have also defined a basis in spin–quaternion space as \( s \otimes \bar{s} \) \((r, i = 0, 1, 2, 3)\), with \( \tau_0 = s_0 \) the 2 \( \times \) 2 identity matrix, and \( \tau_j = -s_j = \pm i\sigma_j \) \((j = 1, 2, 3)\), with \( \sigma_j \) the Pauli matrices. In this basis, the channels \( r = 0, 3 \) and \( r = 1, 2 \) describe the particle–hole and particle–particle degrees of freedom, and the channels \( i = 0 \) and \( i = 1, 2, 3 \) describe the spin–singlet and spin–triplet, respectively. In the space of bispinors we further define a scalar product,

\[ \sum_n \sum_\alpha \text{tr} \left( (\eta^\alpha)^n \otimes \eta^\alpha_n \right) \equiv \langle \eta, \eta \rangle \]  

(3.28)

Now let \( \eta \) be transformed by means of an operator \( \hat{T} \): \( \eta \to \hat{T}\eta \). By rewriting the action in terms of the bispinors, it is easy to see that the free fermion action \( S_\phi \), Eq. (3.22b), is invariant under any transformations of the bispinors that leave the metric \( \langle \eta, \eta \rangle \) invariant, except for the part that is proportional to the external frequency, \( i\omega \). The disorder part of the action \( S_{\Delta \phi} \), Eq. (3.29b), is also invariant under such transformations. Remembering that \( \eta^+ \) is related to \( \eta \) by the charge conjugation matrix \( C \), Eq. (3.27d), and using \( CT = C^{-1} \), we find that in order to leave \( \langle \eta, \eta \rangle \) invariant, \( \hat{T} \) must obey

\[ \hat{T}^T C \hat{T} = C \]  

(3.29)

For a system with \( 2N \) frequency labels (\( N \) positive ones, including 0, and \( N \) negative ones), and \( n \) replicas, Eq. (3.29) defines a representation of the symplectic group \( \text{Sp}(8Nn, C) \) over the complex numbers \( C \). \( \text{Sp}(8Nn, C) \) thus plays a role analogous to that of \( O(N) \) in \( \phi^4 \)-theory, and the external frequency in \( S_\phi \) can be thought of as being analogous to the external magnetic field in the latter.\(^{[28]} \) The electron–electron interaction, which also breaks the symmetry, turns out to be very similar to \( i\omega \) in its effect, see below.

The next question is whether there is a phase where the symplectic symmetry at zero frequency is spontaneously broken, and if so, what plays the role of the order parameter and the Goldstone modes. To see this it is convenient to define a Grassmannian matrix field \( B_{12} \), with \( 1 \equiv (n_1, \alpha_1) \), etc, by

\[ B_{12} (x) = \eta_1^+(x) \otimes \eta_2(x) \]  

(3.30a)

or, in Fourier space and with all components written out,

\[ i^j B^{\alpha\beta}_{nm} (q) = \sum_k i^j \eta_n^\alpha (k)^j \eta_m^\beta (k + q) \]  

(3.30b)

where \( i^\eta \) denotes the elements of \( \eta \), and \( i\eta \) those of \( \eta^+ \). It is often convenient to expand the \( B \) into the spin–quaternion basis defined above,

\[ B_{12} = \sum_r \sum_i i^r B_{12r} (\tau_r \otimes s_i) \]  

(3.30c)

---

\(^{15}\)This is true even though \( i\omega \) couples to a term that is bilinear in \( \psi \) or \( \eta \), while \( h \) in Sec. III C 1 couples linearly to \( \phi \).
The set of $B$ is isomorphic to a set of classical, i.e. complex number valued, matrix fields. One can thus introduce a classical matrix field $Q$, and constrain $B$ to $Q$ by means of a Langrange multiplier field, $\Lambda$. The fermionic degrees of freedom can then be integrated out, and one obtains a theory in terms of the matrix fields $Q$ and $\Lambda$. In terms of these objects, the density of states, $N$, as a function of energy or frequency $\omega$ measured from the Fermi surface is

$$N(\epsilon_F + \omega) = \frac{4}{\pi} \text{Re} \left. \langle 0 | Q_{nn}(x) | 0 \rangle \right|_{\omega_n \to \omega + i0}.$$  

(3.31)

Similarly, the number density susceptibility $\chi_n$, and the spin density susceptibility, $\chi_s$, are given by

$$\chi^{(i)}(q, \Omega_n) = 16T \sum_{m_1, m_2} \sum_{r=0,3} \left\langle i r (\Delta Q)_{m_1-n,m_1}^\alpha(q) i r (\Delta Q)_{m_2,m_2+n}^\alpha(-q) \right\rangle,$$

(3.32)

with $\Delta Q = Q - \langle Q \rangle$, and $\chi^{(0)} = \chi_n$ and $\chi^{(1,2,3)} = \chi_s$. $\Omega_n = 2\pi Tn$ is a bosonic Matsubara frequency.

Under transformations of the bispinors, the $Q$ and $\Lambda$ transform accordingly. One can then derive Ward identities in analogy to Eqs. (3.14) - (3.20a), with the vector field $\vec{\phi}$ replaced by the matrix field $Q$, and the source $J$ also being a matrix. The detailed calculation shows that the role of the transverse fields $\phi_i$ ($i = 2, \ldots, N$) is played by the matrix elements $Q_{nm}$ with $nm < 0$, i.e. by products of fermion fields whose frequency indices have different signs, while the analogs of the longitudinal field are matrix elements with $nm > 0$. The Goldstone mode equation analogous to Eq. (3.20) takes the form,

$$\left\langle 0 Q_{n_1n_2}^\alpha(k) 0 Q_{n_1n_2}^\beta(-k) \right\rangle_{k=0} = \frac{\pi N(\epsilon_F)}{16|\Omega_{n_1-n_2}|}, \quad (n_1n_2 < 0).$$

(3.33)

Here $N(\epsilon_F)$ is the exact density of states at the Fermi level. The salient point is that, as long as $N(\epsilon_F) > 0$, the $Q$-$Q$ correlation function at zero momentum diverges like $1/|\Omega_{n_1-n_2}|$. We have therefore identified $0 Q_{n_1n_2}^\alpha$ for $n_1n_2 < 0$ as a soft mode. As written, Eq. (3.33) is valid only for noninteracting electrons. However, an analysis of the interaction term shows that it does not spoil the property of the $Q$-$Q$ correlation function on the left–hand side of Eq. (3.33) being massless as long as the density of states at the Fermi level is nonzero. Further, one can invoke additional symmetries of the action to show that Eq. (3.33) also holds for $\frac{i}{2} Q$ with $i, r \neq 0$. The soft modes in the particle–hole channel ($r = 0, 3$) and particle–particle channel ($r = 1, 2$) are often called ‘diffusons’, and ‘cooperons’, respectively. They are all soft in the absence of physical processes that break the additional symmetries which link the general $\frac{i}{2} Q$ to the $0 Q$. Breaking these symmetries reduces the number of soft modes, see Sec. VII below.

It is important to realize that the above $Q$-$Q$ correlation function corresponds to a very general four-fermion correlation, with no restrictions on the frequencies $n_1$ and $n_2$ other than that they must have opposite signs. It is thus not related to the fermion number density, or some other conserved quantity, although the density–density correlation function can be obtained as a linear combination of these more general propagators, see Eq. (3.33). The physical reason for the softness of these modes is thus not a conservation law, but rather a spontaneously broken symmetry, viz. the rotation symmetry in $\psi$ or $\eta$–space between fermion fields with positive and negative frequency indices, respectively, or the symmetry between retarded and advanced degrees of freedom. This symmetry is broken whenever one has a nonzero density of states at the Fermi level (which is just the difference between the spectra of the retarded and advanced Green functions, respectively), and the soft $Q$ excitations are the corresponding Goldstone modes. They are analogous to the transverse spin fluctuations in the classical Heisenberg model.

3. Separation of soft and massive modes, and the nonlinear sigma–model for fermions

We now know that the correlation functions of the $Q_{nm}$ with $nm < 0$ are soft, while those with $nm > 0$ are massive. Our next goal is to separate these degrees of freedom in such a way that the soft modes remain manifestly soft to all orders in perturbation theory, in analogy to the treatment of the $O(N)$–symmetric Heisenberg model in Sec. II C. Such a separation was first achieved by Schäfer and Wegner 29 for non–interacting electrons. The generalization of their treatment for the case of interacting electron was given in Ref. 18. Here we sketch the logic of the reasoning, for details we refer to that reference.

The matrices $Q$ under consideration are complex $8 N n \times 8 N n$ matrices, or, alternatively, quaternion–valued $4 N n \times 4 N n$ matrices. However, all of their matrix elements are not independent; the definition of the $Q$ in terms of the fermionic bispinors implies symmetry properties that reduce the number of independent matrix elements. Using these
symmetry properties, one can show that the $Q$ can be diagonalized by means of elements of the unitary symplectic group $\text{USp}(8Nn,\mathbb{C})$. The most general $Q$ can thus be written

$$Q = \tilde{S} D \tilde{S}^{-1},$$

(3.34)

where $D$ is diagonal, and $\tilde{S} \in \text{USp}(8Nn,\mathbb{C})$.

However, diagonalization is more than we want. Since we know that the $Q_{nm}$ with $nm < 0$ are soft, while those with $nm > 0$ are massive, we are interested in generating the most general $Q$ from a matrix $P$ that is block–diagonal in Matsubara frequency space,

$$P = \begin{pmatrix} P^> & 0 \\ 0 & P^< \end{pmatrix},$$

(3.35)

where $P^>$ and $P^<$ are matrices with elements $P_{nm}$ where $n, m > 0$ and $n, m < 0$, respectively. This can easily be achieved. Due to the symmetry properties of $P^>$ and $P^<$, the most general $P$ can be obtained from $D$ by an element $U$ of $\text{USp}(4Nn,\mathbb{C}) \times \text{USp}(4Nn,\mathbb{C})$. The most general $Q$ can therefore be written

$$Q = S P S^{-1},$$

(3.36)

with $S = \tilde{S}U^{-1}$. The set of transformations $S$ is the set of all cosets of $\text{USp}(8Nn,\mathbb{C})$ with respect to $\text{USp}(4Nn,\mathbb{C}) \times \text{USp}(4Nn,\mathbb{C})$, i.e. the $S$ form the homogeneous space $\text{USp}(8Nn,\mathbb{C})/\text{USp}(4Nn,\mathbb{C}) \times \text{USp}(4Nn,\mathbb{C})$.

This achieves the desired separation of our degrees of freedom into soft and massive ones. The massive degrees of freedom are represented by the matrix $P$, while the soft ones are represented by the transformations $S \in \text{USp}(8Nn,\mathbb{C})/\text{USp}(4Nn,\mathbb{C}) \times \text{USp}(4Nn,\mathbb{C})$. The analogy with the $O(N)$–Heisenberg model is now obvious: The unitary–symplectic coset space, identified above as the space whose elements comprise the soft modes of the theory, is a matrix generalization of the homogeneous space $O(N)/O(N-1) \times O(1)$, which represents the soft modes $\phi$ in the $O(N)$ vector model.

In order to formulate the field theory in terms of the soft and massive modes, one also needs the invariant measure $I[P]$, or the Jacobian of the transformation from the $Q$ to the $P$ and the $S$, defined by

$$\int D[Q] \ldots = \int D[P] I[P] \int D[S] \ldots .$$

(3.37)

$I[P]$ has been constructed explicitly \[13\]. However, it is not necessary to know its explicit form to derive the nonlinear sigma–model.

The further development of the theory is analogous to the derivation of Eqs. (3.24), although the technical details are more intricate \[13\]. One splits the expectation value $\langle P \rangle$ off the massive field $P$, and defines a new matrix field \[17\]

$$\hat{Q}(x) = S(x) \langle P \rangle S^{-1}(x) ,$$

(3.38)

$\hat{Q}$ has the properties $\hat{Q}^2 = \text{const}$, and $\text{tr} \hat{Q} = 0$. The action can then be written as a generalized matrix nonlinear sigma–model that contains only the soft modes $\hat{Q}$, and corrections that are all massive. The nonlinear sigma–model part of the action reads

$$A_{\text{NL}q_M} = -\frac{1}{2G} \int d^4x \, \text{tr} \left( \nabla \hat{Q}(x) \right)^2 + 2H \int d^4x \, \text{tr} \left( \Omega \hat{Q}(x) \right) + A_{\text{int}}[\hat{Q}] ,$$

(3.39a)

with the matrix field $\hat{Q} = \hat{Q} - \langle P \rangle$, and $A_{\text{int}}$ the interaction part of the action, Eq. (3.39), replicated and rewritten in terms of the $Q$–matrices. \[24\] The explicit derivation yields the coupling constants as $G = 8/\pi \sigma_0$, with $\sigma_0$ the conductivity in self–consistent Born approximation, and $H = \pi N_F/8$ which can be interpreted as the bare quasi–particle or specific heat density of states \[20,31,32\]. $G$ serves as a measure of the disorder in the system. $\Omega$ is a diagonal matrix whose diagonal elements are the fermionic Matsubara frequencies $\omega_n$. $\hat{Q}$ is subject to the constraints,

\[16\] This group is defined as the intersection of a unitary and a symplectic group, $\text{USp}(2n,\mathbb{C}) \equiv U(2n,\mathbb{C}) \cap \text{Sp}(2n,\mathbb{C})$ \[25\]. It can be represented by unitary matrices that are also symplectic.

\[17\] Strictly speaking, $\hat{Q}$ contains only the real part of $\langle P \rangle$, and we also drop a normalization factor.

\[18\] One can write either $\hat{Q}$ and $\hat{\tilde{Q}}$ in the action, Eq. (3.39a). The only resulting difference is an uninteresting constant contribution to $A_{\text{NL}q_M}$. 

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where the factor of $1/\pi$ in the ordered (i.e., metallic) phase, and to integrate out the soft modes, i.e., in analogy to Eqs. (3.25). We separate $\hat{Q}$ into blocks in frequency space,

$$\hat{Q}_{nm} = \begin{cases} Q_{nm}, & \text{if } n \geq 0, m \geq 0 \text{ or } n < 0, m < 0, \\ q_{nm}, & \text{if } n \geq 0, m < 0, \\ q_{nm}^\dagger, & \text{if } n < 0, m \geq 0, \\ 0, & \text{if } n \geq 0, m < 0, m < 0 \end{cases} ,$$

and enforce the constraint $\hat{Q}^2 = 1$, Eq. (3.39b), by means of a matrix Lagrange multiplier field $\Lambda$,

$$\prod_x \delta[\hat{Q}^2(x) - 1] = \int D[\Lambda] \exp\left\{ -\frac{1}{2G} \int dx \, tr \left( \Lambda(x) [\hat{Q}^2(x) - 1] \right) \right\} ,$$

where the factor of $1/2G$ has been inserted for convenience. By decomposing $\Lambda$ into blocks like $\hat{Q}$ one sees that the elements $\Lambda_{nm}$ with $nm > 0$, together with the tracelessness condition, Eq. (3.39b), are sufficient for enforcing the constraint $\hat{Q}^2 = 1$. We can therefore restrict ourselves to $\Lambda_{nm}$ with $nm > 0$. The action, Eq. (3.39a), can then be written

$$A_{NL\sigma M}[\hat{Q}, \Lambda] = -\frac{1}{2G} \int dx \, tr \left[ \Lambda(x) [\hat{Q}^2(x) - 1] + (\nabla \hat{Q}(x))^2 \right] + 2H \int dx \, tr \left( \hat{Q}(x) \right) + A_{int}[\hat{Q}] .$$

The soft modes $q$ can now be integrated out to obtain an effective action entirely in terms of $Q$ and $\Lambda$. We obtain

$$A_{eff} = \ln \int D[q, q^\dagger] \exp(A_{NL\sigma M}[\hat{Q}, \Lambda])$$

$$= A_{NL\sigma M}[Q, \Lambda] - \frac{1}{2} Tr \ln M[\Lambda] .$$

Here $M$ is a complicated matrix that depends linearly on $\Lambda$. By constructing a saddle–point solution of $A_{eff}$ one can now construct a mean–field theory of the metal–insulator transition in analogy to Eqs. (3.26). This will be discussed in Sec. VI B 2 below.

IV. MAGNETIC TRANSITIONS AT ZERO TEMPERATURE

A. Itinerant ferromagnets

Perhaps the most obvious example of a quantum phase transition is the paramagnet–to–ferromagnet transition of itinerant electrons at $T = 0$ as a function of the exchange interaction between the electron spins. Early theoretical work on this transition suggested that the critical behavior in the physical dimensions $d = 2$ and $d = 3$ was mean–field
like, as it is in the thermal ferromagnetic transition in \( d > 4 \). As discussed in Sec. III, the reason for this suggestion was that the phase transition in a quantum mechanical system in \( d \) dimensions is related to a classical phase transition in \( d + 2 \) dimensions. Formally it was shown \[ \] that \( z = 3 \) and \( z = 4 \) in clean and disordered itinerant quantum ferromagnets, respectively. This appeared to reduce the upper critical dimension \( d_{c}^+ \), above which fluctuations are unimportant and simple mean-field theory yields the correct critical behavior, from \( d_{c}^+ = 4 \) in the classical case to \( d_{c}^+ = 1 \) and \( d_{c}^+ = 0 \) in the clean and disordered quantum cases, respectively. If this were true, then this QPT would be uninteresting from a critical phenomena point of view.

It is now known that this is not the case \[ \] \cite{21,22,23}. In the early work there was an implicit assumption that the only modes or excitations that are important for describing the phase transition were the critical modes. For the finite temperature phase transitions this is generally correct. \cite{4} However, at zero temperature the statics and dynamics are coupled together, which tends to increases the number of soft modes. Further, at \( T = 0 \) there are soft excitations that exist in the entire metallic phase due to a spontaneously broken symmetry that is unrelated to magnetism. These particle–hole excitations (‘diffusons’) and particle–particle excitations (‘cooperons’) were discussed in Sec. III. These modes cause what are known as weak localization effects, \cite{24} and they are soft due to their diffusive nature. It turns out that these additional soft modes couple to the critical modes and influence the critical behavior.

1. Disordered ferromagnets

We now proceed with the discussion of the LGW functional derived in Sec. III, along the lines of Ref. \cite{21}. As discussed in Sec. III, formally expanding the LGW functional \( \Phi[M] \), Eq. (3.124), in powers of \( M \) yields Eq. (3.12c).

Of particular interest to us is the quadratic or Gaussian term, which in Fourier space is given by

\[
\Phi_2(M) = \frac{1}{2} \sum_{q,\Omega_n} \sum_{\alpha} \left( \frac{1}{\Gamma(t)} - \chi_s(q, \Omega_n) \right) |M^\alpha(q, \Omega_n)|^2.
\]  

(4.1)

Here we have scaled \( M \) with \( 1/\Gamma(t) \). \( \chi_s \) is the Fourier transform of the dynamical spin susceptibility in the reference ensemble,

\[
\chi_s(x_1 - x_2) = \langle n_{s,a}(x_1)n_{s,a}(x_2) \rangle_{\text{ref}},
\]

(4.2a)

where \( n_{s,a} \) \((a = 1, 2, 3)\) is one component of the spin density vector \( n_s \), Eq. (3.41). Spin density conservation implies that at small frequency and wavenumber, the Fourier transform of \( \chi_s \) has a diffusive structure \[ \] \cite{45}.

\[
\chi_s(q, \Omega_n) = \chi_0(q) \frac{Dq^2}{|\Omega_n| + Dq^2},
\]

(4.2b)

where \( D \) is the spin diffusion coefficient and \( \chi_0(q) \) is the static spin susceptibility, both in the reference ensemble. In order to reach criticality, the frequency must be taken to zero before the wavenumber. \cite{21} In the critical limit, we can thus expand,

\[
\chi_s(q, \Omega_n) = \chi_0(q) \left[ 1 - |\Omega_n|/Dq^2 + \ldots \right].
\]

(4.2c)

It can be easily verified that the corrections to the leading terms in Eq. (4.2c) are irrelevant for the critical behavior.

\[^{19}\text{However, complicated soft mode structures can occur in classical systems as well, for instance in liquid crystals. See Ref. [14] for an example.}\]

\[^{20}\text{The term ‘weak localization’ is loosely defined, and often misunderstood. We use it to refer to the nonanalytic behavior of electronic correlation functions in the limit of zero momentum and/or frequency that is induced by quenched disorder, or by a combination of interactions and quenched disorder, and that occurs even if the system is far from any kind of phase transition. The physical reason for these nonanalyticities is the diffusive motion of the electrons in the presence of quenched disorder. For a detailed discussion see, e.g., Refs. [12, 13, and 14].}\]

\[^{21}\text{This can be seen as follows. Since the magnetization is conserved, ordering on a length scale } L \text{ requires some spin density to be transported over that length, which takes a time } t \sim L^2/D, \text{ with } D \text{ the spin diffusion coefficient. Now suppose the coherence length is } \xi, \text{ and we look at the system at a momentum scale } q \text{ or a length scale } L \sim 1/q < \xi. \text{ Because of the time it takes the system to order over that scale, the condition for criticality is } L^2 < \text{Min}(Dt, \xi^2). \text{ In particular, one must have } L^2 < Dt, \text{ or } \Omega \sim 1/t < Dq^2.\]
Next comes the crucial step of calculating $\chi_0(q)$ in the reference ensemble. For a long time it was assumed that $\chi_0(q)$ was an analytic function of $q^2$. This might seem plausible, given that the reference ensemble describes a physical system that is far away from any critical point. However, it is now known that this reasoning is incorrect, and that any itinerant electron system at zero temperature has long–range correlations everywhere in its phase diagram, even far away from any critical point \[44\]. These long–range correlations are due to the extra soft modes that generically exist in quantum systems. In disordered systems, these effects are known collectively as weak localization effects. For small values of $|q|$, and for $d > 2$, the leading behavior of $\chi_0(q)$ is

$$\chi_0(q) = c_0 - c_{d-2}|q|^{d-2} - c_2q^2,$$  

(4.3)

where the $c_i$ are positive constants. Notice that the susceptibility decreases with increasing wavenumber. The negative sign of the $|q|^{d-2}$ term, whose prefactor $c_{d-2}$ vanishes for noninteracting electrons, is due to the fact that the interaction increases $\chi_0$, and its effect gets weaker at larger values of $|q|$. For $d < 2$, the electrons are localized and a different theory is needed.

Using Eqs. (4.2) and (4.3), the Gaussian part of the LGW functional can be written more explicitly,

$$\Phi_2(M) = \frac{1}{2} \sum_{q,\Omega_n} \sum_{\alpha} \left[ t_0 + q \cdot \mathbf{M}_n(q,\Omega_n) \right]^2,$$  

(4.4a)

$$\Phi_2(M) = \frac{1}{2} \sum_{q,\Omega_n} \sum_{\alpha} ^{2} \left( t_0 + q \cdot \mathbf{M}_n(q,\Omega_n) \right)^2,$$  

(4.4b)

where we have omitted the prefactors of the various terms, and

$$t_0 = \frac{1}{\Gamma(t)} - \chi_0(q = 0),$$  

(4.4b)

is the bare distance from the critical point. We recognize Eq. (4.4b) as a generalized Stoner criterion for ferromagnetism. Physically, the most important term in Eq. (4.4a) is the $|q|^{d-2}$, which means that in real space the spin density fluctuations interact through a potential that decays with distance $r$ as $r^{2-d}$. It is well known from the theory of classical phase transitions that long–range interactions suppress fluctuation effects, and the critical behavior in systems with such interactions can therefore be determined exactly \[46\]. For example, using renormalization group methods it can be established that all terms higher than $l = 2$ in the Landau expansion, Eq. (3.12c), are renormalization group irrelevant for $d > 2$, so that the upper critical dimension for fluctuations is $d^u_c = 2$. From this it follows that many of the critical exponents can be determined exactly from $\Phi_2$.

To see this we define the order parameter correlation function by

$$G(q,\Omega_n) = \langle |M_n(q,\Omega_n)|^2 \rangle,$$  

(4.5a)

$$G(q,\Omega_n) = \langle |M_n(q,\Omega_n)|^2 \rangle,$$  

(4.5b)

with $M_n$ an arbitrary component of the vector field $\mathbf{M}$. For small $|q|$ and $\Omega_n$ we find

$$G(q,\Omega_n) = \frac{1}{t + |q|^{d-2} + q^2 + \Omega_n/q^2}.$$  

(4.5b)

Here we have again omitted all prefactors of the various terms in the denominator in Eq. (4.51), since they are irrelevant for our purposes. In this equation we have also anticipated that irrelevant variables will renormalize the $t_0$ in Eq. (4.40) to $t$, the actual distance from the critical point, in Eq. (4.51). For $q = \Omega_n = 0$, the correlation function $G$ determines the magnetic susceptibility $\chi_m \sim G(q = 0,\Omega = 0)$ in zero field. Hence we have $\chi_m(t) \sim t^{-1} = t^{-\gamma}$, where the last relation defines the critical exponent $\gamma$. In our case,

$$\gamma = 1,$$  

(4.6)

which is valid for all $d > 2$. $\gamma$ thus has its usual mean–field value. However, for non–zero wavenumbers the anomalous $|q|^{d-2}$ term dominates the usual $q^2$ dependence for all $d < 4$. By inspection, the correlation length exponent, defined by $\xi \sim t^{-\nu}$, is given by

$$\nu = \begin{cases} 1/(d-2), & \text{for } 2 < d < 4, \\ 1/2, & \text{for } d > 4. \end{cases}$$  

(4.7)

Note that $\nu \geq 2/d$, as it must be according to the discussion in Sec. I. The wavenumber dependence of $G$ at $t = 0$ is characterized by the exponent $\eta$, which is defined as $G(q,\Omega_n = 0) \sim |q|^{-2+\eta}$. From Eq. (4.5b) we obtain

$$\eta = \begin{cases} 4-d, & \text{for } 2 < d < 4, \\ 0, & \text{for } d > 4. \end{cases}$$  

(4.8)
Finally, the dynamical scaling exponent $z$ is read off the relation $\xi^z \sim \xi^2/t$ to be

$$z = \begin{cases} 
  d &, \text{for } 2 < d < 4 \\
  4 &, \text{for } d > 4 
\end{cases}, \quad (4.9)$$

Note that all of these exponents ‘lock into’ their mean-field values at a dimensionality $d_{c+}^+ = 4$.

To determine the critical exponents $\beta$ and $\delta$ we need the equation of state in the ordered phase. An important point is that although the quartic ($l = 4$) term is irrelevant in the technical sense of the renormalization group, it is nevertheless needed to determine the critical behavior of the magnetization. The reason is that the coefficient of the $M^4$-term is an example of a dangerous irrelevant variable. Further, the same mechanism that causes the nonanalyticity in Eq. (4.3) leads to a divergence of that coefficient in the long-wavelength, low-frequency limit, i.e. the function $X^{(4)}$ in Eq. (3.12) cannot be localized in space and time. The terms with $l > 4$ in Eq. (3.12) are even more singular in this limit. Despite this unpleasant behavior of the field theory, it is easy to see by power counting in constructing the equation of state in the ordered phase. An important point is that although the quartic ($l = 4$) term is irrelevant in the technical sense of the renormalization group, it is nevertheless needed to determine the critical behavior of the magnetization. The reason is that the coefficient of the $M^4$-term is an example of a dangerous irrelevant variable. Further, the same mechanism that causes the nonanalyticity in Eq. (4.3) leads to a divergence of that coefficient in the long-wavelength, low-frequency limit, i.e. the function $X^{(4)}$ in Eq. (3.12) cannot be localized in space and time. The terms with $l > 4$ in Eq. (3.12) are even more singular in this limit.

The equation of state, Eq. (4.10), gives

$$tm + m^{d/2} + m^3 = h,$$  \quad (4.10)

where we again have left off the irrelevant prefactors. Here $m$ is the modulus of the magnetization vector, and $h$ is the external magnetic field. Notice the term $m^{d/2}$, which occurs in addition to what is otherwise an ordinary mean-field equation of state. It occurs because of the singularities mentioned above. For $d < 6$ it dominates the $m^3$ term, and hence determines the critical exponents $\beta$ and $\delta$. The zero-field magnetization near criticality behaves by definition of the exponent $\beta$ as $m(t,h=0) \sim t^\beta$. The equation of state, Eq. (4.10), gives

$$\beta = \begin{cases} 
  2/(d - 2) &, \text{for } 2 < d < 6 \\
  1/2 &, \text{for } d > 6 
\end{cases}, \quad (4.11a)$$

Similarly, the exponent $\delta$, defined by $m(t=0,h) \sim h^{1/\delta}$, is

$$\delta = \begin{cases} 
  d/2 &, \text{for } 2 < d < 6 \\
  3 &, \text{for } d > 6 
\end{cases}. \quad (4.11b)$$

Note that these relations imply yet another upper critical dimension, $d_{c+}^+ = 6$, defined as the dimension where $\beta$ and $\delta$ ‘lock into’ their mean-field values of $1/2$ and $3$, respectively.

The critical behavior of the specific heat, $c_V$, has also been calculated. It is most convenient to discuss the specific heat coefficient, $\gamma_V = \lim_{T \to 0} c_V/T$, which in a Fermi liquid would be a constant. Its behavior at criticality, $t = 0$, is adequately represented by the following integral,

$$\gamma_V = \int_0^\Lambda dq \frac{q^{d-1}}{T + q^d + q^4 + h^{1-1/\delta}/q^2}.$$

Remarkably, in zero magnetic field, $\gamma_V$ diverges logarithmically as $T \to 0$ for all dimensions $2 < d < 4$. From a scaling viewpoint this is a consequence of the dynamical scaling exponent $z$ being exactly equal to the spatial dimensionality, $d$, in this dimensionality range. In terms of a scaling function, Eq. (4.12) implies

$$\gamma_V(t,T,h) = \Theta(4 - d) \ln b + F_\gamma(\theta h^{1/\nu}, T \beta^z, h b^{\delta/\nu}) \quad (4.13)$$

Here $\Theta(x)$ denotes the step function, $b$ is an arbitrary scale factor, and $F_\gamma$ is a scaling function that cannot be determined by general arguments alone.

The complexity of the critical behavior displayed above is astonishing, given that it is controlled, after all, by a simple Gaussian fixed point. For instance, the appearance of several different ‘upper critical dimensionalities’ is surprising at first sight. As the scaling analysis makes clear, this is a direct consequence of the nonanalytic terms in the LGW functional, which in turn results from the ‘additional’ soft modes. Furthermore, the critical behavior of different quantities can be affected by different dangerous irrelevant variables. For instance, the critical behavior of the magnetization is governed by the coefficient of the (irrelevant) quartic term in the LGW functional, while the specific heat coefficient is not affected by this dangerous irrelevant variable. As a result, the Wilson ratio, $W = (m/h)/\gamma_V$, diverges at the criticality, while general scaling [47] predicts that it should be a universal finite number.

Another interesting example for the breakdown of general scaling is provided by the temperature dependence of the magnetization $m$, and of the magnetic susceptibility $\chi_m$. A detailed analysis [21,8] reveals that these quantities...
do not depend on the critical temperature scale, and therefore their temperature scaling behavior is not determined by the dynamical critical exponent \( z \), but rather by the scale dimensions of subleading temperature scales. One finds for \( m \) and \( \chi_m \) the following homogeneity laws,

\[
m(t, T, H) = b^{-\beta/\nu} m(t^1/\nu, T^{\nu/\beta}, H^{b\beta/\nu}) ,
\]

\[
\chi_m(t, T, H) = b^{\gamma/\nu} \chi_m(t^{1/\nu}, T^{\nu/\beta}, H^{b\beta/\nu}) .
\]

Here the crossover exponent \( \phi \) turns out to be given not by \( \nu z \), but rather by [39],

\[
\phi = \begin{cases} 
2 \nu = 2/(d - 2) , & \text{for } 2 < d < \sqrt{5} + 1 , \\
2/(d - 2) , & \text{for } \sqrt{5} + 1 < d < 4 , \\
2 \nu = 1 , & \text{for } d > 6 .
\end{cases}
\]

The correct physical interpretation of the temperature dependence of \( m \) and \( \chi_m \) is not dynamical scaling, but rather the crossover from the quantum critical region to a regime whose behavior is dominated by the classical Gaussian fixed point [48].

2. Clean ferromagnets

In the previous subsection, the problem of disordered ferromagnets was considered. The only point in that discussion where the disorder was important was the diffusive dispersion relation of the ‘extra’ soft modes. This raises the question whether similar effects might exist in clean itinerant ferromagnets. The first question that arises in the context is what, if anything, will replace the \( q^{d-2} \)–term in the static spin susceptibility, Eq. (4.3), in clean systems. To answer this, let us consider the perturbation theory for \( \chi_0(q) \). What leads to the nonanalyticity in Eq. (4.3) is the coupling of two diffusive modes, which mathematically takes the form of a mode–mode coupling integral of the type

\[
\int d\mathbf{k} \int d\omega \, \frac{1}{\omega + |\mathbf{k}|} \frac{1}{\omega + (|\mathbf{q}| + \mathbf{k})^2} ,
\]

with \( \mathbf{q} \) the external momentum, and we have set the diffusion coefficient equal to unity. Renormalization group techniques have been used to show that this is indeed the leading small–\( q \) behavior of \( \chi_0 \).

What changes in a clean system? The soft modes are still the density and spin density fluctuations, and in addition more general particle–hole excitations. All of these have a linear dispersion relation, i.e., \( \omega \sim |\mathbf{q}| \). One might thus expect \( \chi_0(q) \) in a clean system to have a mode–mode coupling contribution analogous to that given by Eq. (4.15), but with ballistic modes instead of diffusive ones,

\[
\int d\mathbf{k} \int d\omega \, \frac{1}{\omega + |\mathbf{k}|} \frac{1}{\omega + |\mathbf{q}|} .
\]

In generic dimensions, expanding in \( |\mathbf{q}| \) leads to [39],

\[
\chi_0(q) \sim \text{const} + d_{d-1} |\mathbf{q}|^{d-1} + d_2 |\mathbf{q}|^2 ,
\]

at \( T = 0 \). Here \( d_{d-1} \) and \( d_2 \) are constant prefactors. For \( d \leq 3 \), the nonanalytic term in Eq. (4.17) represents the leading \( \mathbf{q} \)–dependence of \( \chi_0 \). In \( d = 3 \), one finds a \( q^2 \ln 1/|\mathbf{q}| \) term, and in \( d > 3 \) the analytic contribution is the leading one.

If \( d_{d-1} < 0 \) in Eq. (4.17), then the same arguments used in the previous section apply here. For \( d < 3 \) one obtains the critical exponents

\[
\beta = \nu = 1/(d - 1) , \quad \eta = 3 - d , \quad \delta = z = d , \quad \gamma = 1 ,
\]

22Refs. [23,44] contained a mistake in the result for \( \phi \), which was corrected in Ref. [39].
and Eq. (4.13) is valid with $4 - d$ replaced by $3 - d$. For $d > 3$ all of the exponents have their mean–field values, and in $d = 3$ there are logarithmic corrections to mean–field like critical behavior.

As in the disordered case discussed in the previous subsection, the temperature scaling of the magnetization is complicated and violates general scaling. The homogeneity laws for $m$ and $\chi_m$ are given by Eqs. (4.14a), (4.14b), with a crossover exponent $\phi$ given by

$$
\phi = \begin{cases} 
\nu = 1/(d - 1) & , \text{ for } 1 < d < 2 , \\
(d/2(d - 1))^{1/(d - 1)} & , \text{ for } 2 < d < 3 , \\
3/(d + 1) & , \text{ for } 3 < d < 5 , \\
\nu = 1/2 & , \text{ for } d > 5 . 
\end{cases}
$$

(4.19)

These results for the quantum critical behavior of clean itinerant ferromagnets disagree with those of Ref. [4]. It should be emphasized that these discrepancies cannot be related to differences in the underlying models. If the model of Ref. [4] is properly renormalized, then it shows the same critical behavior as discussed above, since all features of the action that are not included in the bare model will be generated by the renormalization group. This was shown in Ref. [39]. It highlights the important point that a simple power counting analysis yields the right answer only if no qualitatively new terms are generated under renormalization.

Detailed calculations have confirmed the existence of the $|q|^{d-1}$–term in Eq. (4.17) [40]. However, to lowest order in perturbation theory the coefficient $d_{q-1}$ of that term is positive. For $d \leq 3$ this implies that $\chi_0$ increases with increasing $q$, like $|q|^{d-1}$. For any physical system for which this were the true asymptotic behavior at small $q$, this would have remarkable consequences for the zero–temperature phase transition from the paramagnetic to the ferromagnetic state as a function of the exchange coupling. One possibility is that the ground state of the system will not be ferromagnetic, irrespective of the strength of the spin triplet interaction, since the functional form of $\chi_0$ leads to the instability of any homogeneously magnetized ground state. Instead, with increasing interaction strength, the system would undergo a transition from a paramagnetic Fermi liquid to some other type of magnetically ordered state, most likely a spin density wave. While there seems to be no observational evidence for this, let us point out that in $d = 3$ the effect is only logarithmic, and would hence manifest itself only as a phase transition at exponentially small temperatures, and exponentially large length scales, that might well be unobservable. For $d \leq 2$, on the other hand, there is no long–range Heisenberg ferromagnetic order at finite temperatures, and the suggestion seems less exotic. Furthermore, any finite concentration of quenched impurities will reverse the sign of the leading nonanalyticity, and thus make a ferromagnetic ground state possible again.

Another possibility is that the zero–temperature paramagnet–to–ferromagnet transition is of first order. It has been shown in Ref. [39] that the nonanalyticity in $\chi_0(q)$ leads to a similar nonanalyticity in the magnetic equation of state, which takes the form

$$
tm - v_d m^d + u m^3 = h .
$$

(4.20)

with $m$ the magnetization, $h$ the external magnetic field, and $u > 0$ a positive coefficient. If the soft mode mechanism discussed above is the only mechanism that leads to nonanalyticities, then the sign of the remaining coefficient $v_d$ in Eq. (4.20) should be the same as that of $d_{q-1}$ in Eq. (4.17), i.e. $v_d > 0$. This would imply a first order transition for $2 < d < 3$. In this case the length scale that the scenario of the previous paragraph would have been attributed to a spin density wave would instead be related to the critical radius for nucleation at the first order phase transition.

Finally, it is possible that terms of higher order in the interaction amplitude could change the sign of $d_{q-1}$ in Eq. (4.17) for interaction strengths sufficient to cause ferromagnetism. In this case one obtains a continuous quantum phase transition with critical exponents as given by Eqs. (4.18) and (4.19). Further work is clearly needed on the clean itinerant quantum ferromagnetism problem.

B. Disordered antiferromagnets

Quantum antiferromagnetism (AFM) has experienced a surge of interest in recent years, mainly in efforts to explain the magnetic properties of high–$T_c$ materials. Much of the theoretical work has focused on one–dimensional spin chains, both with and without quenched disorder. In $d = 3$, the QPT in clean itinerant systems is mean–field like, since $z = 2$ and hence $d + z$ is greater than the classical upper critical dimension $d_c^+ = 4$ for all $d > 2$. In this case Hertz’s LGW theory [4] does not break down, since the order parameter (the staggered magnetization) is not a homogeneous quantity. Therefore the additional soft modes that make the ferromagnetic problem so interesting do not couple to the order parameter fluctuations, and the vertices in Eq. (4.12c) can be localized in space and time. There has been relatively little work, on the other hand, on the disordered AFM problem in $d > 1$. Even though in
this case as well the soft noncritical modes of the previous section do not influence the AFM transition, the inclusion of quenched disorder leads to a very interesting and difficult problem. Following earlier work by Das Gupta and Ma \cite{50}, Bhatt and Lee \cite{51} have studied some aspects of the interplay between AFM and strong quenched disorder. They considered a model for the insulating phase of a doped semiconductor consisting of an ensemble of randomly distributed, AFM coupled Heisenberg spins with a very broad distribution of coupling constants, \( J \). Using a numerical renormalization procedure they found that with decreasing temperature, an increasing number of spin pairs freeze into inert singlets, but that some unpaired spins remain and give an essentially free spin contribution, \( \chi_m \sim 1/T \), to the magnetic susceptibility as \( T \to 0 \). The net result was a ‘random–singlet’ phase, with a sub–Curie power law \( T \) dependence of the magnetic susceptibility as \( T \to 0 \). The quantum nature of the spins thus prevents the classically expected long–range order of either AFM or spin glass type.

Bhatt and Fisher \cite{52} applied similar ideas to the disordered metallic phase. They argued that rare fluctuations in the random potential always lead to randomly distributed local moments to which the methods of Bhatt and Lee can be applied, but now in the metallic phase. Their conclusion was that the local moments cannot be quenched by either the Kondo effect or the RKKY interaction induced by the conduction electrons. This again leads to a random–singlet phase with a magnetic susceptibility that diverges as \( T \to 0 \), albeit slower than any power.

These results seem to imply the unlikely conclusion that antiferromagnetic long–range order can never exist in the ground state of disordered systems. In \( d = 1 \) this conclusion is known to be wrong \cite{53}. In higher dimensions it clearly warranted further investigation. In Ref. \cite{53} it was established that in some parts of the phase diagram, long–range order can exist. The basic idea is that quantum fluctuations weaken the metallic random–singlet phase since they enhance the interaction of the isolated local moment electrons with their environment. These interactions in turn restore long–range order by suppressing the random–singlet phase that would otherwise pre–empt an AFM transition.

This result was established by studying a model of an itinerant AFM with a spatially random Néel temperature, or mass term. This model can be derived from a fermionic description, \cite{23} just like the ferromagnetic one was derived in Sec. \cite{22}. The only difference is that the antiferromagnetic order parameter, \( N(x, \tau) \), is multiplied by a phase function that represents perfect AFM order. Because of this phase factor the singular soft modes responsible for the nonlocalities in the ferromagnetic LGW theory do not couple to the order parameter. The resulting replicated local field theory for \( N \) has an action

\[
S[N^\alpha] = \frac{1}{2} \int dx \, dy \int_0^{1/T} dt \, d\tau' \sum_\alpha N^\alpha(x, \tau) \Gamma(x - y, \tau - \tau') N^\alpha(y, \tau') + u \int dx \int_0^{1/T} dt \sum_\alpha (N^\alpha(x, \tau) \cdot N^\alpha(x, \tau))^2
\]

\[
- \Delta \int dx \int_0^{1/T} dt \, d\tau' \sum_{\alpha, \beta} (N^\alpha(x, \tau) \cdot N^\beta(x, \tau')) \left( N^\beta(x, \tau') \cdot N^\beta(x, \tau') \right) .
\]

The Fourier transform of the bare two–point vertex function \( \Gamma(x, \tau) \) is

\[
\Gamma(q, \Omega_n) = t + q^2 + |\Omega_n| .
\]

\[
(4.22)
\]

\( u \) is the coupling constant for the usual quartic term, and \( \Delta \) is the strength of the random mass term. Physically, \( u \) is a measure of the strength of quantum fluctuation effects, while \( \Delta \) is a measure of disorder fluctuations. Note that \( u, \Delta \geq 0 \), so the presence of disorder (i.e., \( \Delta \neq 0 \)) has a destabilizing effect on the field theory. To initiate the analysis of Eq. (4.21) we put \( T = 0 \), and first consider the case \( \Delta = 0 \). We again define the scale dimension of a length \( L \) to be \( [L] = -1 \) (see Eq. (2.22)), and that of time to be \( [\tau] = -z \), and look for a Gaussian fixed point where \( z = 2 \) and \( \eta = 0 \). Power counting in \( d \) dimensions shows that \( [u] = 4 - (d + z) \), so that \( u \) is irrelevant, and the Gaussian fixed point is stable, for all \( d > 2 \). The other exponents can then be readily determined; for instance, \( \nu = 1/2 \). In contrast, the term proportional to \( \Delta \) carries an extra time integral, so with respect to the Gaussian fixed point we have \( [\Delta] = 4 - d \). Hence the disorder is relevant for \( d < 4 \), and the Gaussian fixed point is no longer stable in the presence of disorder. This instability of the Gaussian fixed point can also be inferred from the Harris criterion \cite{3}.

In order to see if there are other fixed points that might be stable, a one–loop renormalization group calculation for the model given by Eq. (4.21) was performed in Ref. \cite{54}. The results are shown in a schematic flow diagram on the critical surface in Fig. \cite{22}. One finds indeed a stable, critical fixed point that we call the random fixed point. In

\[23\] Of course it is necessary to consider fermions on a lattice, as a continuum model does not display antiferromagnetism.
FIG. 2. Schematic flow diagram on the critical surface. The unstable Gaussian (G) and the stable random (R) fixed points are shown. The dotted line denotes the boundary of the basin of attraction for the random fixed point. (From Ref. [54].)

Figure 2, both the Gaussian and the random fixed points are indicated. The linear renormalization group eigenvalues on the critical surface are complex, so that the corrections to scaling at the transition corresponding to the random fixed point are oscillatory in nature. More importantly, the random fixed point has only a limited range of attraction. If the initial values of $\Delta$ and $u$ are inside the basin of attraction for that fixed point, then the system undergoes a continuous phase transition, and standard techniques can be used to calculate the critical exponents. If, on the other hand, the initial values of $\Delta$ and $u$ place the system outside of that basin of attraction, then one finds runaway flows. The correct interpretation of this result is not obvious. One possible interpretation is a fluctuation induced first order phase transition. However, for a state with simple AFM order this possibility is not realized [55]. Another obvious possibility is that for some parameter values, long-range order is not possible. The latter possibility is consistent with the phase diagram shown in Fig. 3, in which the random–singlet phase (no long–range order) and the AFM phase compete with one another. The basic idea is that although disorder destabilizes the field theory, and hence works against ordering, quantum fluctuations as measured by $u$ can counteract this and lead to long–range order. For this to happen, however, the initial values at $u$ and $\Delta$ must not be too small since $u$ is a renormalization group irrelevant

FIG. 3. Schematic phase diagram, showing the antiferromagnetic (AFM) and random–singlet (RS) phases in the $J – \Delta$ plane, with $J$ the effective AFM coupling constant, and $\Delta$ from Eq. (4.21). For $\Delta = 0$, one has an AFM for $J > J_c$, and a Fermi liquid for $J < J_c$. The dependence of the phase boundary on the fluctuation parameter $u$ is not shown. (From Ref. [54].)
variable for $d = 4 - \epsilon$. In other regions of parameter space where no critical fixed point exists, the disorder wins out and scales to large values. The resulting field theory is unstable, but has local instanton or local magnetic moment solutions. Presumably these local solutions represent the random singlet phase. Qualitatively, the most striking prediction is that for small disorder, there is no long–range order, but that for intermediate values of $\Delta$, long–range order does exist. For further discussions we refer to the original publication [74].

V. SUPERCONDUCTOR–METAL TRANSITION AT ZERO TEMPERATURE

The transition from a normal metal to a superconductor at $T = 0$ provides an interesting contrast to the magnetic transitions discussed in Sec. [5, 6]. Again there are soft modes in addition to the order parameter fluctuations, in this case particle–hole excitations in the Cooper channel or ‘cooperons’, that are integrated out in deriving a LGW functional, and that lead to long–range effective interactions between the order parameter fluctuations. For superconductors, the additional soft modes are stronger than in the magnetic case, and they completely dominate the physics. As a result, the quantum critical behavior of superconductors is BCS–like. There are, however, very strong corrections to scaling that lead to a broad transition region as $d \to 2$.

Let us concentrate on the realistic case of a superconductor with a nonzero density of elastic scatterers, so that the normal state conductivity is finite. The transition can then be triggered by varying either the attractive electron–electron interaction or the disorder. We proceed by performing a Landau expansion or expansion of the LGW functional $\Phi$, Eq. (3.14), in powers of the order parameter. Since gauge invariance is not broken in the reference ensemble, only even powers of $\Psi$ appear. The coefficients in this expansion, i.e. the vertex functions of the effective field theory, are connected correlation functions of the anomalous or Cooper channel density $n_c$, Eq. (3.4c), in the reference ensemble. In particular, the Gaussian vertex is determined by the pair propagator or anomalous density–density correlation function in the reference ensemble. Denoting the latter by $C(q)$, the Gaussian term in the LGW functional reads

$$\Phi^{(2)}[\Psi] = \sum_\alpha \int dq \Psi^{*\alpha}(q) \left[ 1/|\Gamma^{(c)}| - C(q) \right] \Psi^{\alpha}(q), \quad (5.1a)$$

where we have scaled $\Psi$ with $1/\Gamma^{(c)}$. $C(q)$ is the connected propagator $C(q) = \{ (n_c(q) n_c(q)) \}_{dis}$, where $\{ \ldots \}_{dis}$ denotes the disorder average. $C(q)$ is a complicated correlation function. However, since the reference ensemble is by construction a Fermi liquid, the structure of this correlation function is known. Renormalization group arguments show that the structure of $C$ at low frequencies and long wavelengths in the limit $T \to 0$ is [22]

$$C(q) = \frac{Z}{\hbar} \frac{\ln(\Omega_0/(Dq^2 + |\Omega_n|))}{1 + (\delta k_c/\hbar) \ln(\Omega_0/(Dq^2 + |\Omega_n|))}, \quad (5.1b)$$

Here $\Omega_0 = k_BT_0/\hbar$ is a frequency cutoff on the order of the Debye frequency (for phonon–mediated superconductivity), and $\delta k_c$ is the repulsive interaction in the Cooper channel that is generated within perturbation theory even though the bare $\Gamma^{(c)}$ vanishes. $D$ is the diffusion coefficient of the electrons, $h$ is the renormalized frequency renormalization constant whose bare value is $H = \pi N_F/8$, see Eq. (3.39a), and $Z$ is the wavefunction renormalization. All of these parameters characterize the reference ensemble, and it is known that they provide a complete characterization of $C(q)$ [57]. Using Eq. (5.1b) in Eq. (5.1a) we obtain, after again rescaling $\Psi$,

$$\Phi^{(2)}[\Psi] = \sum_\alpha \int dq \Psi^{*\alpha}(q) \left[ t + \frac{1}{\ln(\Omega_0/(Dq^2 + |\Omega_n|))} \right] \Psi^{\alpha}(q), \quad (5.1c)$$

where $t = -(Z |K_c| - \delta k_c)/\hbar$, with $K_c = \pi N_F^2 \Gamma^{(c)}/4$.

Let us now discuss this result for the Gaussian LGW theory. From the structure of the Gaussian vertex we immediately read off the values of the exponents $\eta$, $\gamma$, and $z$, defined as $\Gamma^{(2)}(q, \Omega = 0) \sim |q|^{2-\eta}$, $\Gamma^{(2)}(q, \Omega = 0) \sim t^\gamma$, and $\xi_T \sim \xi^z$, with $\xi_T$ the relaxation time, see Eq. (2.11). They are

$$\eta = 2, \quad \gamma = 1, \quad z = 2, \quad (5.2)$$

By scaling $|q|$ with the correlation length $\xi$, we also obtain the behavior of the latter,

$$\xi \sim e^{1/2} t. \quad (5.3)$$
The exponent $\nu$ therefore does not exist, $\nu = \infty$.

In order to determine the behavior of the order parameter and the free energy, we need to consider the higher order terms in the Landau expansion. The coefficient of the quartic term is a nonlinear anomalous density susceptibility in the reference ensemble which we denote by $C^{(4)}$. Due to the cumulant expansion with respect to the disorder average there are two different contributions to this coefficient, $C^{(4)} = C_1^{(4)} + C_2^{(4)}$, where $C_1^{(4)}$ is the disorder average of the four–point correlation function for a given disorder realization, while $C_2^{(4)}$ is the disorder average of the two–point function squared. 

Given that the constant contribution to the Gaussian coefficient, $C(q)$, is barely the leading term in the limit $q \to 0$, and that the quartic coefficient in the magnetic case was singular (see the discussion after Eq. (4.9)), one does not expect $C^{(4)}$ to be finite in the limit of vanishing frequencies and wavenumbers. Indeed, a calculation shows that both $C_1^{(4)}$ and $C_2^{(4)}$ are singular in this limit. Cutting off the singularity by means of a wave number $|p|$, one finds for the leading contributions $C_1^{(4)} \sim u_4/|p|^4 \ln^4 |p|$ and $C_2^{(4)} \sim v_4/|p|^{4-d}$, respectively, with $u_4$ and $v_4$ finite coefficients. 

The same method shows that the most divergent contribution to the coefficient of the term of order $|\Psi|^{2n}$ diverges like

$$C^{(2n)} \sim \frac{u_{2n}}{|p|^{4(n-1)} \ln^{2n} |p|},$$

with $u_{2n}$ a finite coefficient. This implies that the Landau expansion of the cutoff regularized LGW theory is an expansion in powers of $\Psi/|p|^2 \ln(1/|p|)$. The order parameter field theory, rather than having a simple LGW form, is thus again strongly non–local. This is qualitatively the same effect as in the magnetic case, but here the singularities one encounters are even stronger.

As in the magnetic case, the functional $\Phi$ can now be analyzed by using standard techniques. We are looking for a fixed point where the functional dependence of the 2–point vertex on $q$ and $\Omega_n$, Eq. (5.1a), is not renormalized. This fixes the exponents $\eta$ and $z$. As in Secs. II and IV, we define the scale dimension of the correlation length to be $[\xi] = -1$. Power counting then shows that the coefficients $u_{2n}$ (n ≥ 2) of the non–Gaussian terms have scale dimensions $u_{2n} = (n-1)(2-d)$, and hence are irrelevant operators with respect to the fixed point for all dimensions $d > 2$. $|v_4| = -2(2-d)$ for $2 < d < 4$, and the higher cumulants are even more irrelevant. The upper critical dimension is therefore $d^c_+ = 2$, and for $d > 2$ the critical behavior obtained from the Gaussian theory is exact.

The scaling behavior of the order parameter, and of the free energy, is determined by the term of $O(\Psi^4)$ which is a dangerous irrelevant variable with respect to these quantities. For scaling purposes, the cutoff wavenumber $|p|$ can be replaced by the inverse correlation length, $|p| \sim \xi^{-1}$. The scaling behavior $\Psi \sim |p|^2 \ln(1/|p|)$ observed above then immediately leads to

$$\Psi \sim \frac{\Theta(-t)}{|t|} \xi^{-1/|t|},$$

One must notice, however, that the order parameter field $\Psi$ is distinct from the physical gap function $\Delta$. The latter determines the gap in the single–particle excitation spectrum, and hence scales like the frequency or like $\xi^2$; $\Delta \sim |t| \Psi \sim e^{-1/|t|}$. From Eqs. (6.3) and (6.6) we see that even though the exponents $\nu$ and $\beta$ do not exist, we can assign a value to their ratio,

$$\beta/\nu = 2,$$

in the sense that $\Delta \xi^2 \sim \text{const}$.

We next consider the critical behavior of the penetration depth $\lambda$ and the upper critical field $H_{c2}$. Since we have shown that the mean–field/Gaussian theory yields the exact critical behavior at $T = 0$, all relations between observables that are derived within BCS theory are valid. In particular, we have $\lambda \sim 1/\sqrt{\Delta}$, and $H_{c2} \sim \Delta$, which in conjunction with $\Delta \sim \xi^2$ yields

24 The same structure appears in the ferromagnetic LGW functional. Here, however, $C_2^{(4)}$ is more important than its analog in the magnetic case, as we will see below.

25 This result for $C_2^{(4)}$ holds for $2 < d < 4$, and we have neglected possible logarithmic corrections to the power law. For $d > 4$, $C_2^{(4)}$ is finite.
\[ \lambda \sim \xi \quad , \quad H_{c2} \sim \xi^{-2} \]  
Similarly, we can determine the scale dimension of the conductivity \( \sigma \) or the resistivity \( \rho \). The real part of the frequency dependent conductivity in the superconducting phase has a singular contribution, which within BCS theory is given by

\[ \text{Re} \, \sigma_s(\Omega) = \frac{\pi^2}{2} \sigma_n \Delta \delta(\Omega) \]

with \( \sigma_n \) the conductivity in the normal state. For scaling purposes, \( \delta(\Omega) \sim 1/\Omega \sim 1/\Delta \). \( \sigma_n \) is determined entirely by properties of the reference ensemble, and hence it does not show any critical behavior and its scale dimension is zero. We conclude that the scale dimension of \( \sigma_s \) vanishes. If we assume that the conductivity has only one scaling part, then the same is true for the conductivity or resistivity in general, and we obtain

\[ \rho(t, T) = \rho(t \ln b, Tb^2) \]

The resistivity thus shows a step discontinuity from a finite value to zero as one crosses the phase boundary at \( T = 0 \).

We now turn to the free energy density \( f \). Hyperscaling suggests that \( f \) scales like \( f \sim T/V \sim \xi^{-(d+2)} \), which leads to a homogeneity law

\[ f(t, T, u_4) = b^{-(d+2)} f(t \ln b, Tb^2, u_4 b^{2-d}) \]

where of the irrelevant operators we have written only \( u_4 \) explicitly. \( f \) is proportional to \( u_4 \Delta^4 \sim 1/u_4 \), and hence the effective scale dimension of \( f \) is \([f] = 4\), which yields

\[ f(t, T) = b^{-4} f(t \ln b, Tb^2) \]

Hyperscaling is violated by the usual mechanism that is operative above an upper critical dimension, viz. by means of the quartic coefficient being dangerously irrelevant with respect to the free energy. The more exotic violation mechanisms that we encountered in the magnetic case are not realized here. By differentiating twice with respect to \( T \) one obtains the specific heat coefficient \( \gamma(t, T) \),

\[ \gamma(t, T) = \gamma(t \ln b, Tb^2) \]

This implies, among other things, that the specific heat coefficient approaches a constant as the temperature is lowered to zero at the critical coupling strength. More detailed calculations show a step discontinuity in \( \gamma(T = 0) \) at \( t = 0 \).

The quartic term whose coefficient is \( C_2^{(4)} \) yields corrections to scaling. \( C_2^{(4)} \) represents fluctuations in the position of the critical point: By making the coupling constant \( \Gamma^{(c)} \) a random variable, and integrating out that randomness, one obtains a term of that structure. By repeating the arguments of Ref. [59], one finds that the relative fluctuations of the position of the critical point decay anomalously slowly, viz. \( \Delta t/t \sim \xi^{(d-2)} \). Translating that into the corresponding fluctuations of \( T_c \) via \( T_c = T_0 \exp(-1/|t|) \) we obtain

\[ \Delta T_c/T_c \sim T_0^{(d-2)/2} F(\ln(T_0/T_c)) \]

with \( F(x) \) a function that depends parametrically on the disorder. Since it depends on \( T_c \) only logarithmically, the latter dependence is weak. Equation (5.14) predicts very strong disorder fluctuations in thin superconducting films. This prediction is in semi–quantitative agreement with recent experiments by Hsu et al. [10].

**VI. METAL–INSULATOR TRANSITIONS**

**A. Disordered Fermi liquids**

In recent years there has been a considerable amount of work on using renormalization group ideas to derive and justify Landau’s Fermi liquid theory [6]. The main motivation was the idea that non–Fermi liquid notions might be important for understanding, for example, the normal state of the high \( T_c \) materials, and that in order to derive and discuss non–Fermi liquid behavior, one must first have a deeper understanding of why most systems are Fermi liquids. In this context, renormalization group ideas are used to understand an entire phase and not just a single (although important) critical point in the phase diagram. Technically this implies that one must look for completely stable renormalization group fixed points, as opposed to the more usual critical fixed points that are associated with phase transitions.

In this subsection we briefly review how renormalization group ideas can be applied to derive and justify the existence of a disordered Fermi liquid phase for \( d > 2 \), and, in addition, can be used to derive what are commonly known as weak localization effects [6].
1. The disordered Fermi liquid fixed point

To proceed we parametrize the $\hat{Q}$ in Eq. (3.39b) in analogy with the parameterization of $\hat{\phi}$ given by Eqs. (3.23). We write $\hat{Q}$ in a block matrix form as,

$$\hat{Q} = \begin{pmatrix} \sqrt{1-qq^\dagger} & \frac{q}{q^\dagger} \\ \frac{q^\dagger}{q} & -\sqrt{1-q^\dagger q} \end{pmatrix}, \quad (6.1)$$

where the matrix $q$ has elements $q_{nm}$ whose frequency labels are restricted to $n \geq 0, m < 0$. Now, while the sigma–model, Eq. (3.39a), can be expressed entirely in terms of $q$, the corrections to the sigma–model action also depend on

$$\Delta P = P - \langle P \rangle. \quad (6.2)$$

Considering the total action,

$$A = A_{NL\sigma M} + \Delta A, \quad (6.3)$$

we perform a momentum shell renormalization group procedure. For the rescaling part of this transformation, we need to assign scale dimensions to the soft field $q$, and to the massive field $\Delta P$ as well. If the scale dimension of a length $L$ is $[L] = -1$, we write, in analogy to scaling near a critical point,

$$[q(x)] = \frac{1}{2} (d - 2 + \eta') \quad ,$$  

$$[\Delta P(x)] = \frac{1}{2} (d - 2 + \eta) \quad ,$$  

which defines the exponents $\eta$ and $\eta'$. The stable Fermi–liquid fixed point of the theory is characterized by the choice

$$\eta = 2 \quad , \quad \eta' = 0. \quad (6.5a)$$

Physically, $\eta' = 0$ corresponds to diffusive correlations of $q$, and $\eta = 2$ means that the correlations of the $\Delta P$ are of short range. This is indeed what one expects in a disordered Fermi liquid. In addition, we must specify the scale dimension of frequency and temperature, i.e., the dynamical scaling exponent $z = [\omega] = [T]$. In order for the fixed point to be consistent with diffusion, that is with frequencies that scale like the squares of wavenumbers, we must choose

$$z = 2. \quad (6.5b)$$

Now we expand the sigma–model action, Eq. (3.39a), in powers of $q$. In a symbolic notation that leaves out everything not needed for power counting purposes, we write,

$$A_{NL\sigma M} = \frac{1}{G} \int dx (\nabla q)^2 + H \int dx \omega q^2 + \Gamma T \int dx q^2 + O(\nabla^2 q^4, \omega q^4, T q^3) \quad ,$$  

with the bare coupling constants $G \sim 1/\sigma_0$ and $H \sim N_F$. $\Gamma$ can stand for any of the three interaction constants $\Gamma^{(s)}$, $\Gamma^{(t)}$, or $\Gamma^{(c)}$. Power counting gives

$$[G] = [H] = [\Gamma] = 0. \quad (6.6)$$

These terms therefore make up part of the fixed point action.

Next consider the corrections that arise within the sigma–model. The leading ones are indicated in Eq. (1.6). We denote the corresponding coupling constants by $C_{\nabla^2 q^4}$, etc., with a subscript that identifies the structure of the respective contribution to the action. One finds,

$$[C_{\nabla^2 q^4}] = [C_{\omega q^4}] = [(C_{T q^3})^2] = -(d - 2). \quad (6.7)$$

In the last equality in Eq. (6.8) we have considered the square of $C_{T q^3}$, since any contributions to physical correlation functions contain the corresponding term squared. We see that all of these operators are irrelevant with respect to
the disordered Fermi–liquid fixed point for all \( d > 2 \), and that they become marginal in \( d = 2 \) and relevant in \( d < 2 \). All other terms in the sigma–model are at least as irrelevant as those considered above.

It is easy to show that \( \Delta A \) in Eq. (6.3) also contributes to the disordered Fermi–liquid fixed point action. We denote this contribution by \( \Delta A^* \). An inspection shows that \( \Delta A^* \) depends only on \( \Delta P \) and does not couple to \( q \). Further, all of the corrections to \( \Delta A \) are more irrelevant for \( 2 < d < 4 \) than the operators with scale dimensions equal to \( 2 - d \) that are indicated in Eq. (6.8). The scale dimension of the least irrelevant additional corrections is equal to \(-2\).

The above renormalization group arguments show that the theory contains a disordered Fermi–liquid fixed point that is stable for all \( d > 2 \). The effective fixed point action is,

\[
\mathcal{A}_{\text{eff}}^* = \mathcal{A}_{NL,s} + \Delta A^*[\Delta P] \quad ,
\]

and the leading irrelevant terms near this fixed point have scale dimensions given by Eq. (6.8).

2. Scaling behavior of observables

We now discuss the physical meaning of the corrections to scaling induced by the irrelevant operators that we have identified above. Let us denote by the generic name \( u \) any of the least irrelevant operators whose scale dimension is \([u] = -(d-2)\), and let us discuss various observables, viz. the conductivity \( \sigma \), the specific heat coefficient \( \gamma \), the single–particle density of states \( N \), and the spin susceptibility \( \chi_\ast \). Which of the various operators with scale dimension \(-d(2-d)\) is the important one depends on the quantity under consideration.

Let us first consider the dynamical conductivity, \( \sigma(\omega) \). Its bare value is proportional to \( 1/G \), and according to Eq. (6.7) its scale dimension is zero. We therefore have the scaling law,

\[
\sigma(\omega,u) = \sigma(\omega b^2, ub^{-(d-2)}) \quad ,
\]

where \( b \) is an arbitrary renormalization group scale factor. By putting \( b = 1/\omega^{1/2} \), and using \( z = 2 \), Eq. (6.51), as well as the fact that \( \sigma(1,x) \) is an analytic function of \( x \), we find that the conductivity has a singularity at zero frequency, or a long–time tail, of the form

\[
\sigma(\omega) \sim \text{const} + \omega^{(d-2)/2} \quad .
\]

This nonanalyticity is well known from perturbation theory for both noninteracting and interacting electrons \[42,43\]. The above analysis proves that the \( \omega^{(d-2)/2} \) is the exact leading nonanalytic behavior.

The specific heat coefficient, \( \gamma_V = c_V/T \), is proportional to the quasiparticle density of states \( H \) \[30,31,62\], whose scale dimension vanishes according to Eq. (6.7). We thus have a scaling law

\[
\gamma_V(T,u) = \gamma_V(T b^2, ub^{-(d-2)}) \quad ,
\]

which leads to a low-temperature behavior

\[
\gamma_V(T) \sim \text{const} + T^{(d-2)/2} \quad .
\]

From perturbation theory it is known \[42\] that \( \gamma_V \) shows this behavior only for interacting electrons, while for non–interacting systems the prefactor of the nonanalyticity vanishes. This can not be seen by our simple counting arguments.

The single–particle density of states, \( N \), is proportional to the expectation value of \( Q \), and to study the leading correction to the finite fixed point value of \( N \) it suffices to replace \( Q \) by \( Q \). Then we have, in symbolic notation, \( N \sim 1 + \langle qq' \rangle + \ldots = 1 + \Delta N \). The scale dimension of \( \Delta N \) is \([\Delta N] = 2[q] = d - 2\). We find the scaling law

\[
\Delta N(\omega) = b^{-(d-2)} \Delta N(\omega b^2) \quad ,
\]

which leads to the so-called Coulomb anomaly \[63\],

\[
N(\omega) \sim \text{const} + \omega^{(d-2)/2} \quad .
\]

Again, this behavior is known to occur only in the presence of electron-electron interactions.

Finally, we consider the static, wave vector dependent spin susceptibility, \( \chi_0(q) \). \( \chi_0 \) is given by a \( Q-\bar{Q} \) correlation function, and the leading correction to the finite Fermi–liquid value is obtained by replacing both of the \( Q \) by \( q \). Then we have a term of the structure \( \chi_0 \sim T \int d\mathbf{x} \ q^1q \), with scale dimension \([\chi_0] = 0\). The relevant scaling law is
\[ \chi_0(q, u) = \chi_0(q, ub^{-(d-2)}) , \]  
which leads to a nonanalytic dependence on the wave number,
\[ \chi_0(q) \sim \text{const} - |q|^{(d-2)} . \]

This behavior is also known from perturbation theory \cite{21}, and holds only for interacting electrons. As we have seen in Sec. IV A above, it has interesting consequences for the theory of ferromagnetism.

To summarize, we see from the above arguments that all of the so–called weak–localization effects, i.e. nonanalytic dependencies of various observables on frequency, temperature, or wave number, in disordered electron systems that are well known from perturbation theory, emerge naturally in the present context as the leading corrections to scaling near the Fermi–liquid fixed point of a general field theory for disordered interacting electrons. Apart from providing an aesthetic, unifying, and very simple explanation for these effects, our arguments also prove that they do indeed constitute the leading nonanalytic behavior, a conclusion that cannot be drawn from perturbation theory alone.

We finally note that nonanalyticities that are very similar to those discussed above occur in classical fluids. In that context they are known as long–time tail effects, and they were first discussed theoretically by using many–body perturbation theory and mode coupling theory \cite{64,44}. Later, they were examined using renormalization group ideas, and they were shown to be related to corrections to the scaling behavior near a hydrodynamic fixed point \cite{65}.

B. The Anderson–Mott transition

It is well known that at sufficiently large disorder, the metallic disordered Fermi liquid phase discussed in the last subsection becomes unstable against an insulating phase. Such metal–insulator transitions (MITs) are observed in doped semiconductors and other disordered electron systems, and the generalized nonlinear sigma–model shown in Eq. (3.39a) is capable of describing them \cite{32}. Metal–insulator transitions whose critical behavior is determined by both the electron–electron interaction and the disorder, are commonly referred to as Anderson–Mott transitions, to distinguish them from purely disorder driven MITs (‘Anderson transitions’) and purely correlation driven ones (‘Mott transitions’), respectively. They can be grouped into two broad classes: (1) Those that are related to fixed points in the vicinity of \( d = 2 \), and (2) one related to a Gaussian fixed point in high \((d > 6)\) dimensions. Somewhat strangely, these transitions seem to be quite different in nature, and attempts to extrapolate down from \( d = 6 \) to \( d = 3 \) yields results that are incompatible with attempts to extrapolate up from \( d = 2 \) to \( d = 3 \). As a result, the theoretical description of the MITs that are observed in 3–d systems is actually an open problem, the substantial amount of effort that has gone into this subject notwithstanding. We therefore do not pretend here to be able to give a coherent theoretical picture. Rather, we present the results near \( d = 2 \) and the results in high dimensions separately, and then add some speculations about the behavior in \( d = 3 \), with emphasis on the points that are not understood.

1. Anderson–Mott transition near two dimensions

This subject has been reviewed in Ref. \cite{32}, and we refer the reader there for details and references to the original work. Here we give a brief, updated overview designed to tie this subject in with the more recent developments that are reviewed elsewhere in this chapter.

The MITs near \( d = 2 \) fall into several distinct universality classes that are related to external fields that couple to various degrees of freedom in the spin quaternion space defined in Sec. III C 2. Specifically, an external magnetic field (to be referred to as MF), magnetic impurities (MI), and spin–orbit scattering (SO) all break some of the additional symmetries mentioned after Eq. (3.33) that lead to \( Q \)-matrix sectors \( ^{i}Q \) with \( i, r \neq 0 \) being soft modes \cite{27}. The \( r = 0, i = 0 \) sector is directly controlled by the Ward identity, Eq. (3.33), and it always is soft. The universality class with no additional symmetry breakers is called the generic class, and is denoted by G. Table lists the remaining soft modes for all of these universality classes.

In addition, the critical properties turn out to depend on the type of electron–electron interaction considered; they are different depending on whether a short–ranged model interaction is used, or a long–range Coulomb interaction. For each of the four universality classes G, MF, MI, and SO one therefore distinguishes between short–ranged (SR)

---

\[26\] The short–ranged case is not entirely academic. It has been proposed to realize it by putting a grounded metallic plate behind the sample which cuts off the long–ranged Coulomb interaction by means of the image charges induced in the plate \cite{38}.
and the parametrization, Eq. (6.1), of the $Q^\beta$ exponent on which modes are soft, and therefore on the universality class. This is also indicated in Table III. By means of exponent relations that are listed in Table III. Which quantities are critical and which are not depends on the various universality classes as shown in Table II. Other critical exponents of interest are related to these three critical exponents (cf. Sec. II) we choose the correlation length exponent $\nu$, the dynamical exponent $z$, and the exponent $\beta$ that describes the vanishing of the density of states at the Fermi surface, $N(\epsilon_F) \sim t^{\beta}$. Their values for the various universality classes are shown in Table I. Other critical exponents of interest are related to these three by means of exponent relations that are listed in Table II. Which quantities are critical and which are not depends on which modes are soft, and therefore on the universality class. This is also indicated in Table II.

The theoretical analysis of these universality classes near $d = 2 \pm 1$ starts from the nonlinear sigma–model, Eq. (3.39a), and the parametrization, Eq. (6.1), of the $Q$–matrix or an equivalent parametrization. This procedure does not make use of the order parameter field theory derived in Sec. II C 4, and the fact that the density of states is the order parameter for the transition does not become apparent. Rather, one constructs a loop expansion that is equivalent to an expansion in powers of $q$, and proceeds with a renormalization group analysis analogous to that for the $O(N)$–Heisenberg model in $d = 2 + \epsilon$ dimension [24]. For the universality classes MI, MF, and SO, this procedure readily leads to fixed points corresponding to MITs, and the asymptotic critical properties at these transitions have been worked out to lowest order in a $2 + \epsilon$ expansion [35]. For the conductivity and the density of states, the results are in reasonable agreement with experiments, although the ill–behaved nature of the $2 + \epsilon$ expansion precludes quantitative comparisons [32]. Measurements of the spin susceptibility and the specific heat, on the other hand, have shown non–Fermi liquid like behavior on either side of the MIT [69], which the nonlinear sigma–model can not explain. Very similar results were obtained experimentally on systems that are believed to be in the universality class G [70]. These features are usually interpreted in terms of local magnetic moments that are believed to be ubiquitous in disordered electronic systems [71]. Currently no unified theory exists that can describe the interplay between these local moments and the critical behavior of the transport properties at an MIT.

For the universality classes MI (SR) and SO (SR) it turns out that the electron–electron interaction amplitude scales to zero under renormalization [35]. The critical behavior in these cases is therefore the same as for the corresponding universality classes of non–interacting electrons [35]. The density of states is uncritical ($\beta = 0$), and the dynamical exponent is $z = d$. For the class MI (SR) one finds a fixed point at one–loop order that corresponds to a MIT, with a correlation length exponent $\nu$ as given in Table II. The behavior for the class SO (SR) was unclear for a long time, since in this case there is no fixed point that is accessible by means of a $2 + \epsilon$ expansion. However, there is now good

| Universality class | Symmetry breaker | Soft modes |
|-------------------|-----------------|------------|
| MF                | magnetic field  | $r = 0, 3; i = 0, 3$ |
| MI                | magnetic impurities | $r = 0, 3; i = 0$ |
| SO                | spin–orbit scattering | $r = 0, 1, 2, 3; i = 0$ |
| G                 | none            | all        |

TABLE I. Universality classes and soft modes for the MIT near $d = 2$.

| Exponent | SR | MI | LR | SR | MF | LR | SR | SO | LR | SR | SO | LR | G |
|----------|----|----|----|----|----|----|----|----|----|----|----|----|---|
| $\nu$    | $\frac{1}{2} - \frac{2\epsilon}{\epsilon + 1} + O(\epsilon^2)$ | $\frac{1}{2} + O(1)$ | $\frac{1}{2} + O(1)$ | $\frac{1}{2} + O(1)$ | $\approx 1.3 \pm 0.2$ | $\frac{1}{2} + O(1)$ | $\approx 0.75$ |
| $\beta$  | 0  | $\frac{1}{2} + O(1)$ | non–universal | $\frac{1}{2} - \frac{2\epsilon}{\epsilon + 1} + O(1)$ | 0 | $\frac{1}{2} + O(1)$ | $\approx 0.50$ |
| $z$      | $d$ | $2 + \frac{1}{2} + O(\epsilon^2)$ | $d$ | $d$ | $d$ | $2 + O(\epsilon^2)$ | $\approx 5.91$ |

TABLE II. Values for the three independent exponents $\nu$, $\beta$, and $z$ for the eight universality classes in $d = 2 + \epsilon$. Values are given for $d = 2 + \epsilon$ dimensions except for class G, where approximate values for $d = 3$ based on a two–loop approximation in Ref. [6] are shown, and for class SO (SR), where the value shown for $\nu$ is the numerical result given in Ref. [57]. $\beta$ for the class MF (SR) is known to one-loop order, but depends on non–universal quantities, see Ref. [32]. The class G (SR) has never been considered.
TABLE III. Physical quantities, their critical behavior, and relevant exponent relations. Also listed are the universality classes for which the respective quantities show critical behavior. The order parameter density is the density of states. The first two columns are generally valid, while the last two pertain to the Anderson–Mott transition near \( d = 2 \) only. In high dimensions, there is only one universality class, and some exponent relations change, see Sec. VI B 3.

| Physical quantity | Scaling behavior | Exponent relation | Universality classes |
|-------------------|------------------|-------------------|---------------------|
| Correlation length | \( \xi(t) \sim t^{-\nu} \) | independent exponent | all |
| Correlation time   | \( \xi_t \sim t^z \) | independent exponent | all |
| Density of states  | \( N(\varepsilon_F, t) \sim t^\beta \) | independent exponent | MI(LR), MF, SO(LR), G |
| Electrical conductivity | \( \sigma(t, T = \omega = 0) \sim t^s \) | \( s = \nu(d - 2) \) | all |
| Specific heat      | \( c_V(t = 0, T) \sim T^{1-\kappa/z} \) | \( \kappa = z - d \) | MI(LR), SO(LR) |
| Heat diffusion coefficient | \( D_h(t, T = 0) \sim t^{s_h} \) | \( s_h = \nu(z - 2) \) | MI(LR), SO(LR), G |
| Spin diffusion coefficient | \( D_s(t, T = 0) \sim t^{s_s} \) | \( s_s = \nu(z - 2) \) | G |
| Order parameter density correlation function | \( G(k, t = T = 0) \sim |k|^{-\nu - \eta} \) | \( \eta = 2 - d + 2\beta/\nu \) | MI(LR), MF, SO(LR), G |
| Order parameter density susceptibility | \( \chi_{OP}(t, T = 0) \sim t^{-\gamma} \) | \( \gamma = \nu d - 2\beta \) | MI(LR), MF, SO(LR), G |

Evidence from both numerical analyses and high–order perturbation theory for a MIT in \( d = 3 \) allows the renormalization group analysis yields MIT fixed points where the interaction strength scales to a finite number, so that the critical behavior is different from that of noninteracting electrons. The results are listed in Table I. For the MI and MF universality classes the Cooper channel \((r = 1, 2\) in the spin– quaternion basis) does not contribute to the soft modes, see Table I. It is therefore sufficient to renormalize the particle–hole degrees of freedom. For the class SO (LR), on the other hand, one has to deal with the Cooper channel as well. How to properly do this in an interacting system has given rise to some controversy. The current state of affairs is that no satisfactory renormalization group treatment of the Cooper channel interaction amplitude \( \Gamma^{(c)} \) exists. However, there is general agreement that \( \Gamma^{(c)} \) is irrelevant at the MIT fixed point, so that the asymptotic critical behavior is unaffected by this unsolved problem. It is also likely that \( \Gamma^{(c)} \) is only marginally irrelevant, and gives rise to logarithmic corrections to scaling.

The behavior one encounters in class G is substantially more complicated than that in any of the other universality classes. Low–order applications of the perturbative renormalization group reveal a runaway behavior of the spin– triplet interaction constant, \( \Gamma^{(t)} \), and the flow equations do not allow for a fixed point corresponding to a MIT. There is a popular belief that this runaway flow is an indication of local–moment formation, which weak–order perturbation theory is inadequate to describe. We do not believe that this is the correct interpretation of the runaway flow, for the following reasons. As has been shown in Ref. [73], the leading singularities in the runaway region, i.e. for the renormalized \( \Gamma^{(t)} \to \infty \), can be resummed exactly to all orders in the loop expansion. The result is a pair of coupled integral equations for the spin and heat diffusion coefficients, the solution of which yields a phase transition where the spin diffusion coefficient vanishes. This transition has also been studied with renormalization group techniques, which corroborated the results obtained from the integral equations. The conductivity or charge diffusion coefficient was found to be uncritical, and decoupled from the spin transport in the critical region, so the transition is not a MIT. A very remarkable aspect of these results was that the critical behavior could be obtained exactly, apart from possible logarithmic corrections, in all dimensions \( d > 2 \). The reasons for this unusual feature were not clear at the time. Originally, the nature of this transition was not known, and the original papers speculated about it leading to a pseudomagnetic phase with ‘incompletely frozen spins’. While such an ‘incompletely frozen spin phase’ might have had something to do with local moments, later developments showed that these speculations were incorrect. The exact critical behavior found in Refs. turns out to be identical with that of the exactly soluble paramagnet–to–ferromagnet transition in disordered itinerant electron systems that was reviewed in Sec. V A 1 above.
FIG. 4. Schematic phase diagram for disordered itinerant electron systems at $T = 0$ close to $d = 2$, in the plane spanned by the spin–triplet interaction constant $\Gamma^{(t)}$ and the disorder $G$. The phases shown are the paramagnetic metal (PM), the ferromagnetic metal (FM), and the insulator (I). It is not known whether there is another phase transition within I from a ferromagnetic to a paramagnetic insulator.

This can hardly be a coincidence, and we therefore believe that the correct interpretation of the runaway flow found in the G universality class is a transition to a long–range ordered ferromagnetic phase, not an instability against local moment formation. The fact that the runaway flow occurs for arbitrarily small values of the disorder and the interaction strength, provided that one works close to $d = 2$, suggests a phase diagram for low–dimensional systems as shown in Fig. 4. The point is that close to two dimensions, the paramagnetic metal phase and the insulator phase are always separated by a ferromagnetic phase. This is the reason why the MIT in the G universality class is not accessible by means of an $\epsilon$–expansion about $d = 2$, and it is what is causing the runaway flow in the $2 + \epsilon$ expansion. Whether or not the paramagnetic metal phase is unstable against the formation of local moments, and if so, why this physical feature is not reflected in the sigma–model, is a separate question. We also point out that there are other competing instabilities in this universality class, in addition to the ones noted above. For example, in Ref. 79 we have argued that in $d = 2$, there is an instability to a novel type of even–parity superconductivity. All of this suggest that the actual phase diagram in and near two dimensions is very complicated and, at this point, not understood.

For $d > 2 + \epsilon_c$, with $\epsilon_c$ a critical value that was estimated in Ref. 78, there is a direct transition from the paramagnetic metal to an insulating phase, see Fig. 5. Obviously, this transition cannot be described by means of a controlled $\epsilon$–expansion about $d = 2$. The properties of this transition in $d = 3$ have been estimated by means of a two–loop approximation [66], the results of which are shown in Table II.

Finally, we mention that all existing treatments of the Anderson–Mott transition conclude that $d_c = 2$ is the lower critical dimension of the problem, in the sense that in $d = 2$ there is no metallic phase for any degree of disorder. Until recently, the experimental situation was in agreement with this. Experiments that seem to show a transition from an insulating phase to a conducting one in 2–$d$ Si MOSFET systems therefore came as a considerable surprise [80]. Several theoretical speculations have been put forward [81], but the nature of this phenomenon is currently unclear.

2. Anderson–Mott transition in high dimensions

Let us now return to the generalized sigma–model in the form given by Eq. (3.42), where the soft modes have been integrated out. This rewriting of the action one would assume to lead to a simple formulation of the problem in sufficiently high dimensions, where the soft modes do not provide the leading physical effects. This expectation will prove to be correct, and will allow us to determine the exact critical behavior at a metal–insulator transition in dimensions $d > 6$ [38].

We look for saddle–point solutions $Q_{sp}$, $\Lambda_{sp}$ that are spatially uniform and satisfy
FIG. 5. Schematic phase diagram for a 3-d disordered itinerant electron system in the plane spanned by the spin–triplet interaction constant \( \Gamma \) and the disorder \( G \). The phases shown are a paramagnetic metal (PM), a ferromagnetic metal (FM), a paramagnetic insulator (PI), and a ferromagnetic insulator (FI). M denotes a multicritical point.

\[
\begin{align*}
\Gamma^{(1)} & \quad \text{FM} \\
\text{PM} & \quad \text{M} \\
\text{PI} & \quad \text{FI} \\
G & 
\end{align*}
\]

where the superscript (0) denotes the saddle–point approximation. The replica, frequency, and spin–quaternion structures in Eqs. (6.14) are motivated by the fact that \( \langle i_r Q_{nm} \rangle \) and \( \langle i_r \Lambda_{nm} \rangle \) have these properties, and that in the mean–field approximation averages are replaced by the corresponding saddle–point values.

Taking the extremum of the action, Eq. (3.42), with respect to \( \Lambda \) or \( \ell \) one obtains

\[
\left( N_n^{(0)} \right)^2 = 1 - G \Gamma^{(s)} f_n(\ell^{(0)}) ,
\]

and taking the extremum with respect to \( Q \) or \( N_n^{(0)} \) gives

\[
\ell^{(0)} = 2GH\omega_n/N_n^{(0)} .
\]

\( f_n \) in Eq. (6.15a) is a functional of \( \ell \) that contains several frequency integrations over \( \ell_n^{(0)} \), and that in addition depends parametrically on the external frequency \( \omega_n \), on the disorder, and on the interaction constants \( \Gamma^{(s)} \). For simplicity we show only the spin–singlet interaction, \( \Gamma^{(s)} \), explicitly. It has been shown in Ref. [38] that the other interaction channels do not qualitatively change the conclusions.

\( N_n \) is simply related to the density of states through Eq. (3.31). For fixed \( \Gamma^{(s)} \), Eqs. (6.15) describe a decrease of the density of states with increasing disorder \( G \). Furthermore, \( N(\epsilon_F + \omega) \) is nonanalytic at \( \omega = 0 \). An iteration to first order in \( G \) recovers the ‘Coulomb anomaly’ that is well known from perturbation theory \([38]\), see Eq. (6.12b) above. With further increasing disorder, \( N(\epsilon_F) \) vanishes at a critical value \( G_c \) of the disorder. The mean–field approximation thus describes a phase transition with a vanishing density of states at the Fermi level, the hallmark of the Anderson–Mott transition. (Transport properties we will discuss shortly). Note that in the absence of interactions, \( N^{(0)} \equiv 1 \). This reflects the fact that the density of states does not vanish at an Anderson transition \([32]\).

The saddle point equations, Eqs. (6.15), together with an expansion to Gaussian order about the saddle point, determine the critical behavior of this Anderson–Mott transition in high dimensions. One finds standard mean–field/Gaussian values for all static exponents,

\[
\beta = \nu = 1/2 , \quad \gamma = 1 , \quad \eta = 0 , \quad \delta = 3 ,
\]

(6.16a)
and for the dynamical critical exponent,

\[ z = 3 \quad (6.16b) \]

The mean-field critical behaviors of the charge and spin diffusion coefficients have also been determined \[38\], with the result that they both are proportional to the order parameter. For the exponents \( s \) and \( s_s \) defined in Table III\[32\] this means \[27\]

\[ s = s_s = 1/2 \quad (6.16c) \]

We now must ask what the upper critical dimensionality for the problem is, i.e. the dimension \( d_c^+ \) above which the above critical behavior is exact. A superficial inspection of the action suggests \( d_c^+ = 4 \). However, it turns out that under renormalization additional contributions are created that were not in the bare action, and that lead to \( d_c^+ = 6 \) \[38\]. The structure of these additional terms is reminiscent of the structure encountered in the theory of magnets in random magnetic fields. This becomes plausible if one recalls that the random potential in the action couples to the electron density, see Eq. (3.2c). If we Fourier transform from imaginary time to Matsubara frequencies, we obtain a term proportional to

\[ \int d\mathbf{x} u(\mathbf{x}) \sum_n \bar{\psi}_n(\mathbf{x}) \psi_n(\mathbf{x}) \quad (6.17) \]

Since the expectation value \( \langle \bar{\psi}_n(\mathbf{x}) \psi_n(\mathbf{x}) \rangle \) determines the density of states, this means that the random potential couples to the order parameter field for the phase transition, just like a random magnetic field couples to the magnetic order parameter. A detailed technical analysis \[38\] does indeed confirm this analogy. Not only is \( d_c^+ = 6 \), but many structures of the theory coincide with those encountered in the theory of the random-field Ising model \[83\], including a quartic coupling constant, \( u \), that acts as a dangerous irrelevant variable even below the upper critical dimension \( d_c^+ = 6 \). An explicit \( 6 - \epsilon \) expansion of the critical exponents yields also results that, at least to one-loop order, coincide with those for the random-field Ising model.

The implications of these results for the critical behavior in \( d = 3 \) are discussed in the next subsection. For dimensions \( d > 6 \), the mean-field results reviewed above represent the first example of an Anderson–Mott transition, or metal–insulator transition of disordered interacting electrons, for which the critical behavior has been determined exactly. Remarkably, this transition has nothing obvious in common with the Anderson–Mott transitions that are described by the same model in \( d = 2 + \epsilon \) and that have been discussed in Sec. VI B 1 above. In particular, there is no sign here of the various universality classes that are present in \( d = 2 + \epsilon \). This is not just the strong universality that one expects from a mean–field theory, as it persists in the \( 6 - \epsilon \) expansion. It is therefore likely that the Anderson–Mott transition described above is of a different nature than those found in \( d = 2 + \epsilon \). Which description is closer to what actually happens in 3–d systems is currently not known.

### 3. Anderson–Mott transitions in three dimensions: Conventional scaling scenario

The similarities between the Anderson–Mott transition in high dimensionalities and random–field magnets mentioned in the last subsection gives rise to a scaling description of the former \[38\] that is quite different from the one presented in Sec. VI B 1. It is important to point out that for the case of random–field magnets, this scaling scenario almost certainly is not correct. However, there are subtle technical differences between the two systems that suggest that the situation may be different in the case of the Anderson–Mott transition. We therefore present this scenario, which is a conventional (i.e., power–law) scaling description, here, and a different, more exotic, possibility in Sec. VI B 4 below. Which of these two possibilities provides a better description of the actual behavior in \( d = 3 \) is not known.

There are two crucial ingredients to a conventional scaling description of the Anderson–Mott transition based on the order parameter formalism of Secs. III C 1 and VI B 2. The first one is the realization that the electron–electron interaction is renormalization group irrelevant at this transition. \[28\] As a result, the frequency mixing that produced

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27 The conductivity \( \sigma \) is related to the charge diffusion coefficient \( D_c \) according to the Einstein relation \( \sigma = D_c \partial n / \partial \mu \), with \( \partial n / \partial \mu \) the thermodynamic compressibility. The result quoted for \( s \) assumes that \( \partial n / \partial \mu \) has a noncritical contribution.

28 This is true despite the fact that the interaction is necessary for the transition to occur, see Eq. (6.15a).
nontrivial dynamical exponents $z$ in the $2 + \epsilon$ expansion, is absent here, the frequency plays the role of an external field that is conjugate to the order parameter, and $z$ is not independent. If we denote the scale dimension of the external field by $y_h$, we have

$$z = y_h = \frac{\delta \beta}{\nu}$$ \hspace{1cm} (6.18)

The second important point is the existence of a quartic coupling constant, $u$, that acts as a dangerous irrelevant variable with a scale dimension of $[u] = -\theta$. This adds a third independent static exponent, $\theta$, to the usual two. The order parameter obeys a scaling or homogeneity relation,

$$N(t, \Omega, u, \ldots) = b^{(2 - d - \eta)/2} N(tb^{1/\nu}, \Omega b^{z}, ub^{-\theta}, \ldots)$$ \hspace{1cm} (6.19a)

Here we have denoted by $\tilde{z}$ the scale dimension of $\Omega$ or $T$, $[\Omega] = [T] = \tilde{z}$. Upon elimination of the dangerous irrelevant variable $u$, $\tilde{z}$ turns into the effective dynamical exponent $z$,

$$N(t, \Omega) = b^{(2 + \theta - d - \eta)/2} N(tb^{1/\nu}, \Omega b^{\tilde{z}})$$ \hspace{1cm} (6.19b)

This relates the OP exponent $\beta$ to the three independent exponents $\nu$, $\eta$, and $\theta$ through the scaling law,

$$\beta = \frac{\nu}{2} (d - \theta - 2 + \eta)$$ \hspace{1cm} (6.20a)

The remaining static exponents are given by the usual scaling laws, with $d \to d - \theta$ due to the violation of hyperscaling by the dangerous irrelevant variable,

$$\delta = (d - \theta + 2 - \eta)\nu/2\beta$$ \hspace{1cm} (6.20b)

Now we consider the thermodynamic susceptibilities $\partial/n/\partial \mu$, $\gamma_V$, and $\chi_s$. From general arguments given in Ref. [38], we expect all of them to share the same critical behavior. Denoting their singular parts collectively by $\chi_{sing}$, we have

$$\chi_{sing}(t, T) = b^{-d+\theta+z} \chi_{sing}(tb^{1/\nu}, Tb^{\tilde{z}})$$ \hspace{1cm} (6.21)

This links the static critical behavior of the thermodynamic susceptibilities, characterized by the exponent $\kappa$, $\chi_{sing} \sim t^\kappa$ (see Table III), to that of the OP,

$$\kappa = \beta$$ \hspace{1cm} (6.22)

where we have used the scaling laws, Eqs. (6.18), (6.20b). The fact that all of the thermodynamic susceptibilities scale like the order parameter is a consequence of the random–field or static disorder fluctuations being dominant over the quantum fluctuations.

Let us also consider the scaling behavior of the transport coefficients. The charge, spin, and heat diffusion coefficients, which we collectively denote by $D$, all obey the same homogeneity relation,

$$D(t, \Omega) = b^{2-z} D(tb^{1/\nu}, \Omega b^{z})$$ \hspace{1cm} (6.23)

By the definition of the exponents $s_s$ and $s_h$ (see Table III), this yields

$$s_s = s_h = \beta - \nu \eta$$ \hspace{1cm} (6.24)

If the thermodynamic density susceptibility, $\partial n/\partial \mu$, has a noncritical background contribution, then the conductivity exponent $s$ is also given by Eq. (6.24). Using Eq. (6.20a), we find the following generalization of Wegner’s scaling law, Eq. (2.4b) and Table III,

$$s = \frac{\nu}{2} (d - 2 - \theta - \eta)$$ \hspace{1cm} (6.25)

With these results, we can write the homogeneity law for the static electrical conductivity as

$$\sigma(t, T) = b^{-s/\nu} \sigma(tb^{1/\nu}, Tb^{\tilde{z}}) = t^s F_\sigma(T/t^{\nu z})$$ \hspace{1cm} (6.26)

where the scaling function $F_\sigma$ is determined by $\sigma(t = 1, T)$, i.e. by the conductivity far from the transition. The salient point is that, according to this scaling scenario, $\sigma/t^s$ in the critical region is a function of $T/t^{\nu z}$, rather than of
FIG. 6. Dynamical scaling plot of the conductivity data from Fig. 1 of Ref. [85]. The plot assumes a critical stress \( S_c = 6.71 \) kbar, and exponent values \( s = 0.29, \nu z = 1.82 \). Only data in the temperature range \( T < 60 \) mK have been included in the plot, and different symbols denote different stress values, from \( S = 6.59 \) kbar to \( S = 8.03 \) kbar. We have chosen \( T_0 = 100 \) K, and the relation between \( t \) and \( S - S_c \) was taken from Ref. [86], viz. \( t = (S - S_c) \times 10^{-3} / (\text{kbar})^{-1} \). The inset shows that the data cease to scale once the temperature region \( 60 \) mK < \( T < 225 \) mK is taken into account. (From Ref. [84].)

T and \( t \) separately. This can be checked experimentally. In Eq. (6.26), the exponent \( s \) is related to the independent exponents \( \nu, \eta, \) and \( \theta \) by Eq. (6.23). An important point in this context is again the Harris criterion [8], which requires \( \nu \geq 2/d \). If Wegner scaling, Eq. (2.41), were valid (as it is, e.g., in the transitions in \( d = 2 + \epsilon \) discussed in Sec. VI B 1), then it would follow that \( \nu \geq 2/3 \) in three dimensions. This puts a severe constraint on any interpretation of experimental results, and has been a much-discussed issue in the case of the metal–insulator transition that is observed in Si:P [32]. A thorough discussion of these experiments in the light of Eq. (6.26) has been given in Ref. [84]. The net conclusion was that the data allow for reasonable dynamical scaling plots, provided that one chooses values of \( s \) that are substantially smaller than 2/3. An example of such a scaling plot is shown in Fig. 6. This indicated that an interpretation of this experiment in terms of conventional power law scaling is possible only within the framework of a theory that, like the present one, violates Wegner scaling. This seems to favor the order parameter description of the Anderson–Mott transition over the \( (2 + \epsilon) \)-dimensional approach that was reviewed in Sec. VI B 1. However, a competing experiment on the same material [57] obtained results that are incompatible with Ref. [86] and allow for scaling plots of comparable quality with a value of \( s \approx 1.3 \). This experimental controversy has not been resolved [88], and the experimental accuracy is not sufficient for a scaling plot to discriminate among them. For a more detailed discussion of this point we refer the reader to Ref. [84].

4. Anderson–Mott transitions in three dimensions: Activated scaling scenario

In the previous subsection we gave a scaling theory for the Anderson–Mott transition, assuming it was a conventional continuous phase transition, albeit with dangerous irrelevant variables playing a crucial role. There is, however, another possibility. We have seen that the Anderson–Mott transition near \( d = 6 \) has structural similarities to the transition that occurs in a classical random–field Ising model near \( d = 6 \). The latter has been predicted and observed to display glass–like features, and activated rather than power–law scaling in \( d = 3 \). If the analogy to the Anderson–Mott transition still holds in \( d = 3 \) (which at this point is merely a subject of speculation), then the remarkable conclusion is that the Anderson–Mott transition also has glassy aspects. This possibility has been explored in some detail in Ref. [84]. Here we briefly show how to modify the scaling theory for the Anderson–Mott transition in order to allow for this possibility.

The chief assumption in a scaling theory for a phase transition that shows activated scaling is that the critical time
scale grows exponentially with $\xi$,

$$\ln (\tau/\tau_0) \sim \xi^\psi,$$  \hspace{1cm} (6.27)$$

with $\tau_0$ a microscopic time scale, and $\psi$ a generalized dynamical exponent. Physically, this equation implies an exponential growth of the relaxation time as the transition is approached, i.e., behavior that is typical of a glass transition. As a result of this extreme critical slowing down, the system’s equilibrium behavior near the transition becomes inaccessible for all practical purposes, and the systems falls out of equilibrium on realizable experimental time scales. Below we will point out some of the experimental consequences of this. Before we do so, let us briefly discuss the physical ideas behind our picture of a ‘glassy’ Anderson–Mott transition. As already noted, the Anderson–Mott transition problem contains an intrinsic frustration feature: Electron–electron interactions always suppress the local density of states, i.e. the order parameter, while the random potential can cause either local increases or decreases in the order parameter. This frustration means, for example, that local insulating clusters exist inside the metallic phase. Elimination of these clusters requires a large free energy barrier to be overcome. These barriers are assumed to grow like $L^\psi$, with $L$ some length scale, which near the critical point is given by $\xi$. Via the Arrhenius law, this leads to Eq. (6.27).

To construct a scaling theory for this type of transition, the usual homogeneity laws need to be generalized to allow for activated scaling. Since the barriers are expected to be normally distributed, while the relaxation times have a much broader distribution, the natural scaling variables are $\ln \tau/\tau_0$ or $\ln(T_0/T)$, with $T_0$ a microscopic temperature scale such as the Fermi temperature. Considering a variable $Q$, which we assume to be self-averaging, we therefore expect a homogeneity law

$$Q(t, T) = b^{-[Q]} F_Q \left( \frac{tb^{1/\nu}}{\ln(T_0/T)} \right),$$  \hspace{1cm} (6.28)$$

where $F_Q$ is a scaling function. For the tunneling density of states this implies,

$$N(t, T) = b^{-\beta/\nu} F_N \left( \frac{tb^{1/\nu}}{\ln(T_0/T)} \right).$$  \hspace{1cm} (6.29a)$$

Eliminating the parameter $b$ by choosing $b^\psi = \ln(T_0/T)$ gives

$$N(t, T) = \frac{1}{[\ln(T_0/T)]^{\beta/\nu}} G_N [t^{\nu\psi} \ln(T_0/T)],$$  \hspace{1cm} (6.29b)$$

with $G_N$ another scaling function. Equation (6.29b) illustrates a general and important result: At criticality, $t = 0$, the critical singularities will be only logarithmic in $\ln(T_0/T)$. This is the quantum analog of the fact that classical static glass transitions are experimentally inaccessable.

Other quantities that are singular at the critical point have been discussed in Ref. [84]. Here we explicitly consider the magnetic susceptibility, and the specific heat coefficient. Given that the entropy and the magnetization, as observables become singular at various values of $t$, the metallic phase means that there is a Griffiths phase, or a region away from the critical point in which certain thermodynamic quantities, satisfy homogeneity laws like Eq. (6.28), it has been argued [84] that neither $\chi_m$ nor $\gamma_V$ satisfy such a law. As a consequence of this, one finds that these quantities are singular even in the metallic phase. For example, $\chi_m$ scales like

$$\chi_m(t, T) = \frac{T^{-1 + \text{const} \times t^{\nu\psi}}}{[\ln(T_0/T)]^{(d-\theta)/\psi-2\phi}}.$$  \hspace{1cm} (6.30)$$

$\phi$ is an exponent related to the difference in scaling between $T$ and a magnetic field $H$. The singular behavior in the metallic phase means that there is a Griffiths phase, or a region away from the critical point in which certain observables become singular at various values of $t$.

Finally, it has been argued in Ref. [84] that the conductivity, $\sigma$, is not self averaging and that instead one should consider $\ell_{\sigma} \equiv \{[\ln(\sigma_0/\sigma)]_{\text{dis}}\}$, with $\sigma_0$ the bare conductivity. The physically reason for this is that $\sigma$ is related to a relaxation time which, as mentioned above, is log–normally distributed. $\ell_{\sigma}$ obeys a scaling law

$$\ell_{\sigma}(t, T) = b^{\psi} F_{\sigma} \left( \frac{tb^{1/\nu}}{\ln(T_0/T)} \right) \equiv \ln(T_0/T) G_{\sigma} \left( t^{\nu\psi} \ln(T_0/T) \right).$$  \hspace{1cm} (6.31)$$
The important point is that the observable conductivity at $T = 0$ will vanish exponentially fast as $t \to 0$, but that it is impossible to observe this behavior because it requires exponentially low temperatures. The same arguments also imply that one should expect large sample–to–sample variations in the measured conductivity.

There is some experimental support for the glassy picture presented above. First, it has been known for some time that there are large sample–to–sample variations in the conductivity measurements in doped semiconductors. Already in the early 1980’s this was observed in Si:P for $t \leq 10^{-3}$ in the mK temperature range [11,86]. Later, similar effects were blamed on some type of thermal decoupling [87]. The so–called Griffiths like phase in which $\chi_m$ and $\gamma_V$ are singular already in the metallic phase has also been seen in the doped semiconductors. Conventionally this behavior has been attributed to local moment formation [89], and the vanishing of the Kondo temperature [52,90] in disordered systems. Whether, or how, these effects are related to the Griffiths singularities that arise in our glassy phase transition theory is not obvious and warrants further work.

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