We consider matrix orthogonal polynomials related to Jacobi type matrices of weights that can be defined in terms of a given matrix Pearson equation. Stating a Riemann–Hilbert problem we can derive first and second order differential relations that these matrix orthogonal polynomials and the second kind functions associated to them verify. For the corresponding matrix recurrence coefficients, non-Abelian extensions of a family of discrete Painlevé dP$_{IV}$ equations are obtained for the three term recurrence relation coefficients.

1. Introduction

In this paper we deal with regular matrix of weights $W(z)$, i.e., their moments

$$W_n := \frac{1}{2\pi i} \int_{\gamma} z^n W(z) \, dz, \quad n \in \mathbb{N},$$

are such that $\det \left[ W_{j+k} \right]_{j,k=0,\ldots,n} \neq 0$, $n \in \mathbb{N} := \{0, 1, \ldots\}$, and where the support of $W$, is a non self-intersecting smooth curve, $\gamma$, on the complex plane with two end points at $a, b \in \mathbb{C}$, and such that it intersects the circles $|z| = R, R \in \mathbb{R}^+$, once and only once (i.e., it can be taken as a determination curve for arg $z$).

A Riemann–Hilbert approach was used to discuss matrix biorthogonal polynomials of Hermite \cite{7,9,10,25} and Laguerre type \cite{6}. In this paper, we focus on Jacobi type

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2020 Mathematics Subject Classification. 33C45, 33C47, 42C05, 47A56.

Key words and phrases. Riemann–Hilbert problems; matrix Pearson equations; discrete integrable systems; non-Abelian discrete Painlevé IV equation.

\textsuperscript{1} Acknowledges Centro de Matemática da Universidade de Coimbra (CMUC) – UID/MAT/00324/2020, funded by the Portuguese Government through FCT/MEC and co-funded by the European Regional Development Fund through the Partnership Agreement PT2020.

\textsuperscript{2} and \textsuperscript{3} Acknowledges CIDMA Center for Research and Development in Mathematics and Applications (University of Aveiro) and the Portuguese Foundation for Science and Technology (FCT) within project UID/MAT/04106/2020.

\textsuperscript{4} Thanks financial support from the Spanish “Agencia Estatal de Investigación” research project [PGC2018-096504-B-C33], Ortogonalidad y Aproximación: Teoría y Aplicaciones en Física Matemática and research project [PID2021-122154NB-I00], Ortogonalidad y aproximación con aplicaciones en machine learning y teoría de la probabilidad.
examples: We say that a matrix of weights $W = \begin{bmatrix} W^{(1,1)} & \cdots & W^{(1,N)} \\ \vdots & \ddots & \vdots \\ W^{(N,1)} & \cdots & W^{(N,N)} \end{bmatrix} \in \mathbb{C}^{N \times N}$ with support $\gamma$ is of Jacobi type if the entries $W^{(j,k)}$ of the matrix measure $W$ can be written as

$$W^{(j,k)}(z) = \sum_{m \in I_{j,k}} \varphi_m(z) (a + z)^{\alpha_m} (b - z)^{\beta_m} \log^p(a + z) \log^q(b - z), \quad z \in \gamma,$$

where $I_{j,k}$ denotes a finite set of indexes, $\Re(\alpha_m), \Re(\beta_m) > -1$, $p_m, q_m \in \mathbb{N}$, $a \neq b$ real numbers and $\varphi_m$ is Hölder continuous, bounded and non-vanishing on $\gamma$. We assume that the determination of logarithm and the powers are taken along $\gamma$. We will request, in the development of the theory, that the functions $\varphi_m$ have a holomorphic extension to the whole complex plane.

This definition includes the non scalar examples of Jacobi type weights given in the literature [1, 8, 11, 12, 26, 32], and as far as we know it was not been studied in all its generality.

In this work, for the sake of simplicity, the finite end points of the curve $\gamma$ is taken at the origin, $a = 0$ and $b = 1$ with no loss of generality, as a similar arguments apply for $a \neq 0$ or $b \neq 1$. In [18] different examples of Laguerre matrix weights for the matrix orthogonal polynomials on the real line are studied.

The subject of orthogonal polynomials covers a wide range of topics within mathematics, as well as its applications. Particularly when dealing with Jacobi polynomials, they have played an important role in mathematical analysis. Their origins can be traced back to classical problems such as electromagnetism, potential theory and many other fields. Particularly, the Legendre and Chebyshev polynomials played a significant role in the development of spectral methods for partial differential equations [13, 27].

It was Krein [28, 29] who first introduced the matrix extensions of scalar orthogonal polynomials. On this topic, there have been some relevant papers published afterward [14, 15, 17, 21, 22, 24, 30, 32], as well as more recent contributions, including for instance [3, 24]. Numerous findings have been made such as scattering problem resolution and matrix Favard theorem [2]. Later, it was proven that matrix orthogonal polynomials sometimes satisfy some properties as do the classical orthogonal polynomials, such as the scalar type Rodrigues’ formula [19, 20]. Moreover, the last few years have seen the discovery of a plethora of families of orthogonal matrix polynomials that are eigenfunctions of certain fixed second order linear differential operators with a matrix coefficients independent of the orthogonal polynomial degree [5, 16, 18].

In this work we apply the Riemann–Hilbert analysis to the study of families of polynomials and its second kind functions that are orthogonal with respect to Jacobi type matrices of weights coming from a matrix Pearson equation. We are able to derive first and second order differential relations that these matrix orthogonal polynomials and the second kind functions associated to them verify. For the corresponding matrix recurrence coefficients, non-Abelian extensions of a family of discrete Painlevé d-$P_{IV}$ equations are obtained for the three term recurrence relation coefficients.

The structure of this work is the following: In section 2 we present the basic theory about matrix biorthogonal polynomials and state the left and right Riemann–Hilbert problems. Using orthogonal polynomials and second kind functions, the unique solution for the Riemann–Hilbert problem is given. In section 3, we discuss the analytic properties of the constant jump fundamental matrix associated with a matrix of weight solution of a Pearson type equation of Jacobi type. In section 4, a Riemann–Hilbert
approach is taken to derive the first and second order differential equation from the structure matrix. We show that these equations reduce to the scalar case when the commutativity is imposed. In section 3 we find a discrete nonlinear relation for the recursion coefficients that can be considered an extension of discrete Painlevé IV. This is accomplished by computing the explicit expression for the structure matrix.

2. BIORTHOGONALITY AND RIEMANN–HILBERT PROBLEM

Given a regular matrix of weights \( W \), we define sequences of matrix monic polynomials, \( \{p_n^L(z)\}_{n \in \mathbb{Z}} \) and \( \{p_n^R(z)\}_{n \in \mathbb{N}} \), respectively left orthogonal and right orthogonal, were deg \( p_n^L(z) = n \) and deg \( p_n^R(z) = n, n \in \mathbb{N} \), by the conditions,

\[
\frac{1}{2\pi i} \int_y p_n^L(z)W(z)z^k \, dz = \delta_{n,k} C_n^{-1}, \quad \frac{1}{2\pi i} \int_y z^k W(z) p_n^R(z) \, dz = \delta_{n,k} C_n^{-1},
\]

for \( k = 0, 1, \ldots, n \) and \( n \in \mathbb{N} \), where \( C_n \) is a nonsingular matrix.

The matrix of weights \( W \) induces a sesquilinear form in the set of matrix polynomials \( \mathbb{C}^{N \times N} [z] \) given by

\[
\langle P, Q \rangle_W := \frac{1}{2\pi i} \int_y P(z) W(z) Q(z) \, dz,
\]

for which \( \{p_n^L(z)\}_{n \in \mathbb{N}} \) and \( \{p_n^R(z)\}_{n \in \mathbb{N}} \) are biorthogonal

\[
\langle p_n^L, p_m^R \rangle_W = \delta_{n,m} C_n^{-1}, \quad n, m \in \mathbb{N}.
\]

As the polynomials are chosen to be monic, we can write

\[
p_n^L(z) = I_N z^n + p_{1,n}^L z^{n-1} + p_{2,n}^L z^{n-2} + \cdots + p_{n,n}^L,
\]

\[
p_n^R(z) = I_N z^n + p_{1,n}^R z^{n-1} + p_{2,n}^R z^{n-2} + \cdots + p_{n,n}^R,
\]

with matrix coefficients \( p_{k,n}^L, p_{k,n}^R \in \mathbb{C}^{N \times N}, k = 0, \ldots, n \) and \( n \in \mathbb{N} \) (imposing that \( p_{0,n}^L = p_{0,n}^R = I_N, n \in \mathbb{N} \)). Here \( I_N \in \mathbb{C}^{N \times N} \) denotes the identity matrix.

We define, for all \( n \in \mathbb{N} \), the sequence of second kind matrix functions by

\[
Q_n^L(z) := \frac{1}{2\pi i} \int_y \frac{p_n^L(z')}{z' - z} W(z') \, dz', \quad Q_n^R(z) := \frac{1}{2\pi i} \int_y W(z') \frac{p_n^R(z')}{z' - z} \, dz'.
\]

From the orthogonality conditions (2) we have, for \( n \in \mathbb{N} \), the following asymptotic expansions, as \( |z| \to \infty \)

\[
Q_n^L(z) = -C_n^{-1} (I_N z^{-n-1} + q_{1,n} z^{-n-2} + \cdots),
\]

\[
Q_n^R(z) = -(I_N z^{-n-1} + q_{1,n} z^{-n-2} + \cdots) C_n^{-1}.
\]

We gather together these objects in the matrix

\[
Y_n^L(z) := \begin{bmatrix} p_n^L(z) & Q_n^L(z) \\ -C_{n-1} p_{n-1}^L(z) & -C_{n-1} Q_{n-1}^L(z) \end{bmatrix}, \quad Y_n^R(z) := \begin{bmatrix} p_n^R(z) & -p_{n-1}^R(z) C_{n-1} \\ Q_n^R(z) & -Q_{n-1}^R(z) C_{n-1} \end{bmatrix}.
\]

In terms of the transfer matrices, the three term recurrence relations for \( p_n^L, p_n^R \) and \( Q_n^L, Q_n^R \) read,

\[
y_{n+1}^L(z) = T_n^L(z) y_n^L(z), \quad y_{n+1}^R(z) = T_n^R(z) y_n^R(z), \quad n \in \mathbb{N},
\]
where,
\[ T_n^L = \begin{bmatrix} zI_N - \beta_n^L & C_n^{-1} \\ -C_n & 0_N \end{bmatrix}, \quad T_n^R = \begin{bmatrix} zI_N - \beta_n^R & -C_n \\ C_n^{-1} & 0_N \end{bmatrix}, \]
with initial conditions, \( P_{-1}^L = P_{-1}^R = 0_N, \ P_0^L = P_0^R = I_N, \ Q_{-1}^L(z) = Q_{-1}^R(z) = -C_n^{-1}, \)
\( Q_0^L(z) = Q_0^R(z) = S_W(z) := \frac{1}{2\pi i} \int_{\gamma} \frac{W(z')}{z-z'} \, dz', \) where \( S_W(z) \) is the Stieltjes–Markov transformation and \( \beta_n^R := C_n\beta_n^L C_n^{-1}. \) We also know, cf. for example [7], that
\[ (y_n^L(z))^{-1} = \begin{bmatrix} 0_N & I_N \\ -I_N & 0_N \end{bmatrix} y_n^R(z) \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix}. \]

Now, we state a theorem on Riemann–Hilbert problem for the Jacobi type weights.

**Theorem 1.** Given a regular Jacobi type matrix of weights \( W \) with support on \( \gamma \) we have the matrix function \( Y_n^L \) and \( Y_n^R \), defined by (3), is, for each \( n \in \mathbb{N} \), the unique solution of the following Riemann–Hilbert problems, which consists, respectively, in the determination of the following Riemann–Hilbert problems, which consists, respectively, in the determination of

(RH1): \( Y_n^L \) and \( Y_n^R \) is holomorphic in \( C \setminus \gamma \).

(RH2): Satisfies the jump condition
\[ (y_n^L(z))_+ = (y_n^L(z))_- \begin{bmatrix} I_N & W(z) \\ 0_N & I_N \end{bmatrix}, \quad (y_n^R(z))_+ = (y_n^R(z))_- \begin{bmatrix} I_N & 0_N \\ W(z) & I_N \end{bmatrix}. \]

(RH3): Has the following asymptotic behavior, as \( |z| \to \infty \)
\[ y_n^L(z) = (I_{2N} + O(1/z)) \begin{bmatrix} z^nI_N \\ 0_N \end{bmatrix}, \quad y_n^R(z) = \begin{bmatrix} I_N & 0_N \\ 0_N & I_N \end{bmatrix} (I_N + O(1/z)). \]

(RH4): \( y_n^L(z) = \begin{bmatrix} O(1) & s_1(z) \\ O(1) & s_2(z) \end{bmatrix}, \quad y_n^R(z) = \begin{bmatrix} O(1) & O(1) \\ s_1(z) & s_2(z) \end{bmatrix} \), as \( z \to 0 \), with \( \lim_{z \to 0} z s_j(z) = 0_N \) and \( \lim_{z \to \infty} z s_j^R(z) = 0_N, \) \( j = 1, 2. \)

(RH5): \( y_n^L(z) = \begin{bmatrix} O(1) & r_1(z) \\ O(1) & r_2(z) \end{bmatrix}, \quad y_n^R(z) = \begin{bmatrix} O(1) & O(1) \\ r_1(z) & r_2(z) \end{bmatrix} \), as \( z \to 1 \), with \( \lim_{z \to 1} (1-z)r_j(z) = 0_N \) and \( \lim_{z \to \infty} (1-z)r_j^R(z) = 0_N, \) \( j = 1, 2. \) The \( s_j^L, s_j^R \) (respectively, \( r_j^L, r_j^R \)) could be replaced by \( O(1/z), \) as \( z \to 0 \) (respectively, \( O(1/(1-z)), \) as \( z \to 1 \)). The \( O \) and \( o \) conditions are understood entry-wise.

**Proof.** A very similar proof can be found in [7] and [6]. \( \square \)

3. **Fundamental matrices**

3.1. **Pearson equation.** Here we consider matrix of weights, \( W \), satisfying a matrix Pearson type equation
\[ z(1-z)W'(z) = h^L(z)W(z) + W(z)h^R(z), \]
with entire matrix functions \( h^L, h^R. \) If we take a matrix function \( W^L \) such that
\[ z(1-z)(W^L)'(z) = h^L(z)W^L(z), \]
then there exists a matrix function \( W^R(z) \) such that \( W(z) = W^L(z)W^R(z) \) with
\[ z(1-z)(W^R)'(z) = W^R(z)h^R(z). \]
The reciprocal is also true.
The solution of (5) and (6) will have possibly branch points at 0 and 1, cf. [34]. This means that the exists constant matrices, $C^R_j, C^L_j$, with $j = 0, 1$, such that

\begin{align}
(7) \quad (W^L(z)_- = (W^L(z))_+ C^L_0, \quad (W^R(z)_- = C^R_0 (W^R(z))_+, \quad \text{in } (0, 1), \\
(8) \quad (W^L(z)_- = (W^L(z))_+ C^L_1, \quad (W^R(z)_- = C^R_1 (W^R(z))_+, \quad \text{in } (1, +\infty).
\end{align}

We introduce, the constant jump fundamental matrices, for $n \in \mathbb{N}$,

\begin{align}
(9) \quad Z^L_n(z) := y^L_n(z) \begin{bmatrix}
W^L(z) & 0_N \\
0_N & (W^R(z))^{-1}_-
\end{bmatrix}, \quad Z^R_n(z) := \begin{bmatrix}
W^R(z) & 0_N \\
0_N & (W^L(z))^{-1}_-
\end{bmatrix} y^R_n(z).
\end{align}

The constant jump fundamental matrices $Z^L_n$ and $Z^R_n$ satisfy, for each $n \in \mathbb{N}$, the following properties:

i) Are holomorphic on $\mathbb{C} \setminus [0, +\infty)$.

ii) Present the following constant jump condition on $(0, 1)$

\begin{align}
(Z^L_n(z))_+ = (Z^L_n(z))_- \begin{bmatrix}
C^L_0 & C^L_1 \\
0_N & I_N
\end{bmatrix}, \quad (Z^R_n(z))_+ = \begin{bmatrix}
I_N & 0_N \\
C^R_0 & C^R_1
\end{bmatrix} (Z^R_n(z))_-.
\end{align}

iii) Present the following constant jump condition on $(1, +\infty)$

\begin{align}
(Z^L_n(z))_+ = (Z^L_n(z))_- \begin{bmatrix}
C^L_1 & 0_N \\
0_N & C^R_1
\end{bmatrix}, \quad (Z^R_n(z))_+ = \begin{bmatrix}
C^R_1 & 0_N \\
0_N & C^L_1
\end{bmatrix} (Z^R_n(z))_-.
\end{align}

Now, we will explicit the constant jump matrix in the special case when we have the following decompositions for the matrix of weights, $W(z) = W^L(z)W^R(z)$, with:

\begin{align}
(10) \quad z \left( W^L \right)'(z) = \tilde{h}^L(z) W^L(z), \quad (1 - z) \left( W^R \right)'(z) = W^R(z) \tilde{h}^R(z),
\end{align}

where $h^L$ and $\tilde{h}^R$ are entire functions. Therefore, the matrix $W(z) = W^L(z)W^R(z)$ is such that,

\begin{align}
z(1 - z)W'(z) = h^L(z)W(z) + W(z)\tilde{h}^R(z),
\end{align}

where $h^L(z) = (1 - z)h^L(z)$ and $\tilde{h}^R(z) = z\tilde{h}^R(z)$.

General solutions $W^L$ and $W^R$ of (10) are given explicitly (cf. for example [34]) by

\begin{align}
(11) \quad W^L(z) = H^L(z)e^{\alpha z} W^L_0, \quad W^R(z) = W^R_0 (1 - z)^\beta H^R(z),
\end{align}

where $H^L(z)$ and $H^R(z)$ are entire and nonsingular matrix functions, and $\alpha, \beta$ are constant matrices, as well as $W^L_0$ and $W^R_0$ are constant nonsingular matrices.

It is easy to see that $W$, within this decomposition, is a Jacobi type weight matrix defined by (1). From (11), the constant jump fundamental matrices $Z^L_n(z)$ and $Z^R_n(z)$ have the following constant jump condition on $(0, 1)$

\begin{align}
(Z^L_n(z))_+ = (Z^L_n(z))_- \begin{bmatrix}
(W^L_0)^{-1} e^{-2\pi i \alpha} W^L_0 & (W^L_0)^{-1} e^{-2\pi i \alpha} W^L_0 \\
0_N & I_N
\end{bmatrix},
\end{align}

\begin{align}
(Z^R_n(z))_+ = \begin{bmatrix}
I_N & 0_N \\
I_N & (W^R_0)^{-1} e^{2\pi i \alpha} W^L_0
\end{bmatrix} (Z^R_n(z))_-,
\end{align}

as well as, the constant jump condition on $(1, +\infty)$

\begin{align}
(Z^L_n(z))_+ = (Z^L_n(z))_- \begin{bmatrix}
(W^L_0)^{-1} e^{-2\pi i \alpha} W^L_0 & 0_N \\
0_N & W^R_0 e^{2\pi i \beta (W^R_0)^{-1}}
\end{bmatrix},
\end{align}

\begin{align}
(Z^R_n(z))_+ = \begin{bmatrix}
I_N & 0_N \\
I_N & (W^R_0)^{-1} e^{2\pi i \alpha} W^L_0
\end{bmatrix} (Z^R_n(z))_-,
\end{align}

\begin{align}
(Z^R_n(z))_+ = \begin{bmatrix}
I_N & 0_N \\
I_N & (W^R_0)^{-1} e^{2\pi i \alpha} W^L_0
\end{bmatrix} (Z^R_n(z))_-.
\end{align}
As well as the following properties hold (cf. [7]):

\[
(Z_n^R(z))_+ = \begin{bmatrix}
W_0^R e^{-2\pi i \beta (W_0^R)^{-1}} & 0_N \\
0_N & (W_0^L)^{-1} e^{2\pi i \alpha} W_0^L
\end{bmatrix} (Z_n^R(z))_-.
\]

In fact, from the definition of \(Z_n^L(z)\) we have

\[
(Z_n^L(z))_+ = (Y_n^L(z))_+ \begin{bmatrix}
(W^L(z))_+ & 0_N \\
0_N & (W^R(z))_+^{-1}
\end{bmatrix},
\]

and taking into account Theorem 1 we successively get

\[
(Z_n^L(z))_+ = (Y_n^L(z))_+ \begin{bmatrix}
(W^L(z))_+ & 0_N \\
0_N & (W^R(z))_+^{-1}
\end{bmatrix} = (Y_n^L(z))_+ \begin{bmatrix}
(W^L(z))_+^{-1} & 0_N \\
0_N & (W^R(z))_+^{-1}
\end{bmatrix}.
\]

Similarly over \((1, +\infty)\) we have

\[
(Z_n^L(z))_+ = (Y_n^L(z))_+ \begin{bmatrix}
(W^L(z))_+ & 0_N \\
0_N & (W^R(z))_+^{-1}
\end{bmatrix} = (Y_n^L(z))_+ \begin{bmatrix}
(W^L(z))_+^{-1} & 0_N \\
0_N & (W^R(z))_+^{-1}
\end{bmatrix}.
\]

To complete the proof we only have to see that

\[
\begin{bmatrix}
W^L \\
W^R
\end{bmatrix} = H^L e^{2\pi i \alpha z} \beta W_0^L,
\begin{bmatrix}
W^L \\
W^R
\end{bmatrix} = W_0^R e^{2\pi i \alpha (1 - z)^2} H^R,
\]

and then check that

\[
Z_n^R(z) = \begin{bmatrix}
0 & -I_N \\
I_N & 0
\end{bmatrix} (Z_n^L(z))^{-1} \begin{bmatrix}
0 & I_N \\
-I_N & 0
\end{bmatrix},
\]

which is a consequence of (9) within the definition of \(Y_n^L\) and \(Y_n^R\), cf. [3].

3.2. Structure matrix and zero curvature formula. In parallel to the matrices \(Z_n^L(z)\) and \(Z_n^R(z)\), for each factorization we introduce what we call structure matrices given in terms of the left, respectively right, logarithmic derivatives by,

\[
M_n^L(z) := (Z_n^L(z))' (Z_n^L(z))^{-1}, \quad M_n^R(z) := (Z_n^R(z))^{-1} (Z_n^R(z))' (z).
\]

It is not difficult to see that

\[
M_n^R(z) = -\begin{bmatrix}
0 & -I_N \\
I_N & 0
\end{bmatrix} M_n^L(z) \begin{bmatrix}
0 & I_N \\
-I_N & 0
\end{bmatrix}, \quad n \in \mathbb{N},
\]

as well as the following properties hold (cf. [7]):

i) The transfer matrices satisfy

\[
T_n^L(z) Z_n^L(z) = Z_{n+1}^L(z), \quad Z_n^R(z) T_n^R(z) = Z_{n+1}^R(z), \quad n \in \mathbb{N}.
\]

ii) The zero curvature formulas holds for all \(n \in \mathbb{N},

\[
\begin{bmatrix}
I_N & 0_N \\
0_N & 0_N
\end{bmatrix} = M_{n+1}^L(z) T_n^L(z) - T_n^R(z) M_n^L(z),
\]
\[ \begin{bmatrix} I_N & 0_N \\ 0_N & 0_N \end{bmatrix} = T_n^R(z) M_{n+1}^R(z) - M_n^R(z) T_n^R(z). \]

Now, we discuss the holomorphic properties of the structure matrices just introduced.

**Theorem 2.** Let \( W \) be a regular Jacobi matrix weight that satisfies a Pearson type equation \( \langle 1 \rangle \) that admits a factorization \( W(z) = W^L(z) W^R(z) \), where \( W^L \) and \( W^R \) satisfy \( \langle 3 \rangle \) and \( \langle 5 \rangle \). Then, the structure matrices \( M_n^L(z) \) and \( M_n^R(z) \) are, for each \( n \in \mathbb{N} \), meromorphic on \( \mathbb{C} \), with singularities located at \( z = 0 \) and \( z = 1 \), which happens to be a removable singularity or a simple pole.

**Proof.** Let us prove the statement for \( M_n^L(z) \). The matrix function \( M_n^L(z) \) is holomorphic in \( \mathbb{C} \setminus \{0, +\infty\} \) by definition, cf. \( \langle 12 \rangle \). Due to the fact that \( Z_n^L(z) \) has a constant jump on \( (0, 1) \cup (1, +\infty) \), cf. \( \langle 7 \rangle \) and \( \langle 9 \rangle \), the matrix function \( (Z_n^L)’ \) has the same constant jump on \( (0, 1) \cup (1, +\infty) \), so that the matrix \( M_n^L(z) \) has no jump on \( (0, 1) \cup (1, +\infty) \), and it follows that at \( z = 0 \) and \( z = 1 \), \( M_n^L(z) \) has an isolated singularity.

From \( \langle 9 \rangle \) and \( \langle 12 \rangle \) it holds
\[
M_n^L(z) = (Z_n^L)'(z) (Z_n^L(z))^{-1} = (Y_n^L)'(z) (Y_n^L(z))^{-1} + \frac{1}{z(z-1)} Y_n^L(z) \begin{bmatrix} h_n^L(z) & 0_N \\ 0_N & -h_n^R(z) \end{bmatrix} (Y_n^L(z))^{-1},
\]
where \( Y_n^L \) is given in \( \langle 3 \rangle \). Each entry of the matrix \( Q_n^L(z) \) is the Cauchy transform of certain function, \( f \), of type
\[
f(z) = \sum_{j \in \mathcal{I}} \varphi_j(z) z^{\alpha_j} (1 - z)^{\beta_j} \log^{q_j} (z) \log^{q_j} (1 - z),
\]
where \( \varphi_j(z) \) is, for each \( j \in \mathcal{I} \), an entire function with \( \text{Re}(\alpha_j), \text{Re}(\beta_j) > -1, p_j, q_j \in \mathbb{N} \), and \( \mathcal{I} \) is a finite set of indices. It’s clear that
\[
\lim_{z \to 0} zf(z) = 0_N \quad \text{and} \quad \lim_{z \to 1} (1-z)f(z) = 0_N.
\]

By \( \langle 23 \rangle \ \S 8.3-8.6 \) and \( \langle 31 \rangle \), we deduce that the Cauchy transform of \( f \) have the same properties:
\[
\lim_{z \to 0} z \int_0^1 \frac{f(t)}{t-z} \, dt = 0_N \quad \text{and} \quad \lim_{z \to 1} (1-z) \int_0^1 \frac{f(t)}{t-z} \, dt = 0_N.
\]

Now, we will prove that
\[
\lim_{z \to 0} z^2 \left( \int_0^1 \frac{f(t)}{t-z} \, dt \right)' = 0_N \quad \text{and} \quad \lim_{z \to 1} (1-z)^2 \left( \int_0^1 \frac{f(t)}{t-z} \, dt \right)' = 0_N.
\]

In fact,
\[
z(1-z) \left( \int_0^1 \frac{f(t)}{t-z} \, dt \right)' = \int_0^1 \frac{z(1-z)f(t)}{(t-z)^2} \, dt
\]
\[
= \int_0^1 \frac{(t-z)(t+z-1)f(t)}{(t-z)^2} \, dt + \int_0^1 \frac{t(1-t)f(t)}{(t-z)^2} \, dt,
\]
\[
= \int_0^1 \frac{t+z-1}{t-z} f(t) \, dt - \frac{(1-t)f(t)^1}{t-z} \bigg|_0^1 + \int_0^1 [t(1-t)f(t)]' \, dt.
\]
From the boundary conditions, the first term is zero and we get
\[ z(1 - z) \left( \int_0^1 \frac{f(t)}{t - z} \, dt \right)' = - \int_0^1 f(t) \, dt + \int_0^1 \frac{t(1 - t)f'(t)}{t - z} \, dt. \]

We return back to (16), and see that this is equivalent to prove that
\[ \lim_{z \to 0} z^2 (1 - z) \left( \int_0^1 \frac{f(t)}{t - z} \, dt \right)' = 0_N. \]

This follows from the fact that the Stieltjes transform of \( z(1 - z)f'(z) \), i.e.
\[ \int_0^1 \frac{t(1 - t)f'(t)}{t - z} \, dt, \]
is of the same type of \( f \). Then,
\[ \lim_{z \to 0} \left( \int_0^1 \frac{t(1 - t)f'(t)}{t - z} \, dt \right) = 0_N, \quad \lim_{z \to 1} (1 - z) \left( \int_0^1 \frac{t(1 - t)f'(t)}{t - z} \, dt \right) = 0_N, \]
and (16) follows.

Now, as each entry of the matrix \( Q_n^f(z) \) is a Cauchy transform of certain function \( f \) described previously, by using (15) and (16) we have that,
\[ \left( Y_n^L \right)'(z) = \begin{bmatrix} O(1) & o\left( \frac{1}{z} \right) \\ O(1) & o\left( \frac{1}{z^2} \right) \end{bmatrix}, \quad \left( Y_n^L(z) \right)^{-1} = \begin{bmatrix} o\left( \frac{1}{z} \right) & o\left( \frac{1}{z^2} \right) \\ O(1) & O(1) \end{bmatrix}, \quad z \to 0 \]
and
\[ \left( Y_n^L \right)'(z) = \begin{bmatrix} O(1) & o\left( \frac{1}{1 - z^2} \right) \\ O(1) & o\left( \frac{1}{(1 - z)^2} \right) \end{bmatrix}, \quad \left( Y_n^L(z) \right)^{-1} = \begin{bmatrix} o\left( \frac{1}{1 - z^2} \right) & o\left( \frac{1}{1 - z^2} \right) \\ O(1) & O(1) \end{bmatrix}, \quad z \to 1. \]

This implies that
\[ \lim_{z \to 0} z^2 \left( Y_n^L(z) \right)'(z) \left( Y_n^L(z) \right)^{-1} = \lim_{z \to 0} z^2 \begin{bmatrix} o\left( \frac{1}{z} \right) + o\left( \frac{1}{z^2} \right) & o\left( \frac{1}{z} \right) + o\left( \frac{1}{z} \right) \\ o\left( \frac{1}{z} \right) + o\left( \frac{1}{z} \right) & o\left( \frac{1}{z} \right) + o\left( \frac{1}{z} \right) \end{bmatrix} = \lim_{z \to 0} z^2 \begin{bmatrix} o\left( \frac{1}{z} \right) & o\left( \frac{1}{z} \right) \\ o\left( \frac{1}{z} \right) & o\left( \frac{1}{z} \right) \end{bmatrix} = 0_{2N}, \]
and
\[ \lim_{z \to 1} (1 - z)^2 \left( Y_n^L(z) \right)'(z) \left( Y_n^L(z) \right)^{-1} = \lim_{z \to 1} (1 - z)^2 \begin{bmatrix} o\left( \frac{1}{1-z^2} \right) & o\left( \frac{1}{1-z^2} \right) \\ o\left( \frac{1}{1-z^2} \right) & o\left( \frac{1}{1-z^2} \right) \end{bmatrix} = 0_{2N}. \]

Straightforward calculation and similar considerations lead us to
\[ \lim_{z \to 0} Y_n^L(z) \begin{bmatrix} h^L(z) & 0_N \\ 0_N & -h_R(z) \end{bmatrix} \left( Y_n^L(z) \right)^{-1} = 0_{2N}, \]
\[ \lim_{z \to 1} (1 - z) Y_n^L(z) \begin{bmatrix} h^L(z) & 0_N \\ 0_N & -h_R(z) \end{bmatrix} \left( Y_n^L(z) \right)^{-1} = 0_{2N}. \]

Finally we arrive to
\[ \lim_{z \to 0} z^2 M_n^L(z) = 0_{2N} \quad \text{and} \quad \lim_{z \to 1} (1 - z)^2 M_n^L(z) = 0_{2N}. \]

By analogous arguments we get the results for \( M_n^R \). \( \square \)
4. Differential relations from the Riemann–Hilbert problem

Our objective is to derive differential equations satisfied by the biorthogonal matrix polynomials associated to regular Jacobi type matrices of weights. Here we use the Riemann–Hilbert problem approach in order to derive these differential relations.

Let us define a new matrix functions,

\[
\begin{align*}
\tilde{M}_n^L(z) &= z(1-z)M_n^L(z), \\
\tilde{M}_n^R(z) &= z(1-z)M_n^R(z),
\end{align*}
\]

then \(\tilde{M}_n^L(z)\) and \(\tilde{M}_n^R(z)\) are matrices of entire functions, cf. Theorem 2

**Proposition 1** (First order differential equation for the fundamental matrices). In the conditions of Theorem 2 we have that

\[
\begin{align*}
z(1-z)(Y_n^L)'(z) + Y_n^L(z) &\begin{bmatrix} h_n^L(z) & 0_N \\
0_N & -h_n^R(z) \end{bmatrix} = \tilde{M}_n^L(z)Y_n^L(z) \\
z(1-z)(Y_n^R)'(z) + \begin{bmatrix} h_n^R(z) & 0_N \\
0_N & -h_n^L(z) \end{bmatrix} Y_n^R(z) &= Y_n^R(z)\tilde{M}_n^R(z).
\end{align*}
\]

**Proof.** Equations (17) and (18) follows immediately from the definition of the matrices \(\tilde{M}_n^L(z)\) and \(\tilde{M}_n^R(z)\) in (12). \(\square\)

**Proposition 2.** In the conditions of Theorem 2 If \(h_n^L(z) = A^L z + B^L\) and \(h_n^R(z) = A^R z + B^R\), then the left and right fundamental matrices are given respectively by,

\[
\begin{align*}
\tilde{M}_n^L(z) &= \begin{bmatrix} (A^L - nl_N z + [p_{1,n}^L A^L^\ast + p_{2,n}^L + nl_N + B^L) \quad (A^L C_n^1 + C_n^1 A^L - (2n + 1) C_n^1) \\
-C_n^1 - A^L C_n^1 + (2n - 1) C_n^1 & (nl_N - A^R) z + [p_{1,n}^R A^R^\ast - p_{1,n}^R - nl_N - B^R \end{bmatrix} \\
\tilde{M}_n^R(z) &= \begin{bmatrix} (A^R - nl_N) z - [p_{1,n}^R A^R^\ast + p_{1,n}^R + nl_N + B^R) \quad -C_n^1 - A^R C_n^1 + (2n - 1) C_n^1 \\
A^R C_n^1 - C_n^1 A^R - (2n + 1) C_n^1 & (nl_N - A^R) z - [p_{1,n}^R A^R^\ast - p_{1,n}^R - nl_N - B^R \end{bmatrix}
\end{align*}
\]

**Proof.** Taking \(|z| \to +\infty\) in (14) we have that

\[
Y_n^L = \begin{bmatrix} I_N z^{n+1} + p_{1,n}^L z^{n-1} + \cdots & -C_n^1 (I_N z^{n-1} + q_{1,n}^L z^{n-2} + \cdots) \\
-C_n^1 (I_N z^{n-1} + p_{1,n}^L z^{n-1} + \cdots) & I_N z^{n-2} + q_{1,n-1}^L z^{n-3} + \cdots \end{bmatrix}
\]

\[
(Y_n^L)^{-1} = \begin{bmatrix} I_N z^{n-2} + q_{1,n-1}^L z^{n-3} + \cdots & -C_n^1 (I_N z^{n-1} + q_{1,n}^L z^{n-2} + \cdots) \\
C_n^1 (I_N z^{n-1} + q_{1,n-1}^L z^{n-2} + \cdots) & I_N z^{n+1} + p_{1,n}^L z^{n-1} + p_{2,n-1}^L z^{n-3} + \cdots \end{bmatrix} C_n^1
\]

Hence, as \(|z| \to +\infty\)

\[
z(1-z) \begin{bmatrix} (Y_n^L)'(z) \\
(Y_n^L)^{-1} \end{bmatrix} = \begin{bmatrix} -nl_N z + nl_N -(nq_{1,n-1}^L + (n-1)p_{1,n}^L) & -(2n + 1) C_n^1 \\
(2n-1) C_n^1 & nl_N z - nl_N + np_{1,n}^L + (n+1) q_{1,n-1}^L \end{bmatrix} + O(1/z).
\]

Since \(z \mapsto z(1-z) (Y_n^L)'(z) (Y_n^L(z))^{-1}\) is holomorphic over \(\mathbb{C}\), by Liouville theorem we deduce that,

\[
z(1-z) \begin{bmatrix} (Y_n^L)'(z) \\
(Y_n^L(z))^{-1} \end{bmatrix} = \begin{bmatrix} -nl_N z + nl_N -(nq_{1,n-1}^L + (n-1)p_{1,n}^L) & -(2n + 1) C_n^1 \\
(2n-1) C_n^1 & nl_N z - nl_N + np_{1,n}^L + (n+1) q_{1,n-1}^L \end{bmatrix}.
\]

Using again Liouville’s theorem we get,
we get the stated result (21). The equation (22) follows in a similar way from definition
Since
Proof. Differentiating in (12) we get
\( (22) \)
\( (21) \)
\( p_{L,n} = q_{L,n-1} \) and \( p_{L,n} = q_{L,n} \). Then \( (19) \) follows.
By considering the identities \( p_{R,n} = -q_{L,n-1} \) and \( p_{R,n} = -q_{L,n} \), then \( (19) \) follows.
\( \square \)

Now, we introduce the \( N \) map, \( N(F(z)) = F'(z) + \frac{F^2(z)}{z} \).

Proposition 3 (Second order differential equation for the fundamental matrices). In
the conditions of Theorem 2 we have that

\( (21) \)
\( z(1-z) (Y_n^L)' + (Y_n^L) [2h^k + (1-2z)I_N -2h^R + (1-2z)I_N] + Y_n^L \left[ N(h^k) 0_N \right] = N(M^L_n)Y_n^L, \)
\( (22) \)
\( z(1-z) (Y_n^R)' + [2h^k + (1-2z)I_N -2h^R + (1-2z)I_N] (Y_n^R)' + \left[ N(h^R) 0_N \right] = Y_n^R N(M^R_n). \)

Proof: Differentiating in (12), we get
\[ (Z_n^L)' (Z_n^L)^{-1} = \frac{\left( M_n^L \right)'}{z(1-z)} - (1-2z) + \frac{\left( M_n^L \right)^2}{z^2(1-z)^2}, \]
so that
\[ z(1-z) (Z_n^L)' (Z_n^L)^{-1} + (1-2z) = \left( M_n^L \right)' + \frac{\left( M_n^L \right)^2}{z(1-z)} = N(M_n^L). \]

Now let us see that
\[ (1-2z)M_n^L = z(1-z) (Y_n^L)' (Y_n^L)^{-1} + Y_n^L \left[ h^k 0_N \right] (Y_n^L)^{-1}. \]

From (5) we have
\[ z(1-z) (W^L)' (W^L)^{-1} = \frac{(h^l)^2}{z(1-z)} - \frac{1-2z}{z(1-z)} h^l + (h^l)', \]
and
\[ z(1-z) (W^R)' (W^R)^{-1} = \frac{(h^R)^2}{z(1-z)} + \frac{1-2z}{z(1-z)} h^R - (h^R)'. \]

Since
\[ z(1-z) (Z_n^L)' (Z_n^L)^{-1} = z(1-z) (Y_n^L)' (Y_n^L)^{-1} + Y_n^L \left[ 2h^k 0_N \right] (Y_n^L)^{-1} + Y_n^L \left[ z(1-z) (W^L)' (W^L)^{-1} + \frac{1}{z(1-z)} (W^R)' (W^R)^{-1} \right] (Y_n^L)^{-1}, \]
we get the stated result (21). The equation (22) follows in a similar way from definition
of \( M_n^R \) in (12).

\( \square \)

We introduce the following \( C^{2N\times 2N} \) valued functions
\[ H_n^L = \begin{bmatrix} H_{1,1,n}^L & H_{1,2,n}^L \\ H_{2,1,n}^L & H_{2,2,n}^L \end{bmatrix} := N(M_n^L), \quad H_n^R = \begin{bmatrix} H_{1,1,n}^R & H_{1,2,n}^R \\ H_{2,1,n}^R & H_{2,2,n}^R \end{bmatrix} := N(M_n^R). \]
It holds that the second order matrix differential equations (21) and (22) split in the following differential relations:

\[ z(1 - z)(P_n^L)' + (P_n^L)'(2h^L + (1 - 2z)I_N) + p_n^L N(h^L) = H_{1,1,n}^L - H_{1,2,n}^L C_n - I_n P_{n-1}^L, \]

\[ z(1 - z)(Q_n^L)' - (Q_n^L)'(2h^R + (1 - 2z)I_N) + Q_n^L N(-h^R) = H_{1,1,n}^L - H_{1,2,n}^L C_n - Q_{n-1}^L, \]

\[ z(1 - z)(P_n^R)' + (P_n^R)'(2h^R + (1 - 2z)I_N) + Q_n^R N(h^R)P_{n-1}^R = P_{n+1}^R - P_{n-1}^R C_n - h^R, \]

\[ z(1 - z)(Q_n^R)' - (Q_n^R)'(2h^R + (1 - 2z)I_N) - Q_n^R N(-h^R) = Q_{n+1}^R - Q_{n-1}^R C_n - h^R. \]

Using the calculation made in (19), we want to recover here some known formulas in the scalar case.

**Example.** Let us consider the weight \( W(z) = z^\alpha (1 - z)^\beta, \) with \( \alpha, \beta \) scalars in \((-1, \infty)\). Then, the scalar second order equation for \( \{P_n^L\}_{n \in \mathbb{N}} \) and \( \{Q_n^L\}_{n \in \mathbb{N}} \) (cf. for example [33]) is given by

\[ z(1 - z)P_n''(z) + (1 + \alpha - (\alpha + \beta + 2)z)P_n'(z) + n(\alpha + \beta + n + 1)P_n(z) = 0, \]

\[ z(1 - z)Q_n''(z) + (1 + \alpha - (\alpha + \beta - 2)z)Q_n'(z) + (n + 1)(\alpha + \beta + n)Q_n(z) = 0. \]

In fact, from (19),

\[ \tilde{M}_n^L(z) = \begin{pmatrix} \frac{\alpha + \beta + n}{2} + n & -C_n^{-1}(\alpha + \beta + 2n + 1) \\ C_n^{-1}(\alpha + \beta + 2n - 1) & \frac{(\alpha + \beta + n)z - p_n^L - n - \frac{\alpha}{2}}{} \end{pmatrix}, \]

it is easy to see that

\[ \left( \tilde{M}_n^L(z) \right)^2 = \left( -\frac{\alpha + \beta + n}{2}z + p_n^L + n + \frac{\alpha}{2} \right)^2 - \alpha_n((\alpha + \beta + 2n)^2 - 1)I_2, \]

and also, \( \left( \tilde{M}_n^L(z) \right)' = \begin{pmatrix} -\frac{\alpha + \beta + n}{2} & 0 \\ 0 & \frac{\alpha + \beta + n}{2} - n \end{pmatrix}. \) Using now proposition 3 we get

\[ z(1 - z)P_n''(z) + (1 + \alpha - (\alpha + \beta + 2)z)P_n'(z) - n(\alpha + \beta + n + 1)P_n(z) = \frac{n(\alpha + n) + p_n^L(\alpha + \beta + 2n)}{1 - z} + \frac{\alpha^2 + (p_n^L + \frac{\alpha}{2} + n)^2 + \gamma_n((\alpha + \beta + 2n)^2 - 1)}{z(1 - z)}P_n(z). \]

By equalizing poles between left and right hand side on 0 then on 1 we have

\[ \left( \frac{\alpha^2}{4} + \left( p_n^L + \frac{\alpha}{2} + n \right)^2 + \gamma_n((\alpha + \beta + 2n)^2 - 1) \right)P_n(0) = 0 \]

\[ (n(\alpha + n) + p_n^L(\alpha + \beta + 2n))P_n(1) = 0 \]

which, taking into account \( P_n(0), P_n(1) \neq 0, \) leads to the representation of \( p_n^L \) and \( \gamma_n, \) as well as (23). The equation (24) for the \( \{Q_n\}_{n \in \mathbb{N}} \) follows from the above considerations.

5. **Matrix discrete Painlevé IV**

We can consider, using the notation introduced before, the matrix weight measure \( W(z) = W_L(z)W_R(z) \) such that

\[ z(1 - z)(W^L)'(z) = (h_0^L + h_1^L z + h_2^L z^2)W^L(z), \]

\[ z(1 - z)(W^R)'(z) = W^R(z)(h_0^R + h_1^R z + h_2^R z^2). \]
From Theorem 2, we get the matrix $\tilde{M}_n = z(1 - z)M_n^L$ is given explicitly by

\[
(\tilde{M}_n)_{11} = C_n^{-1} h_n^R C_{n-1} + (h_n^L + h_n^L z + h_n^R z^2) + h_n^L q_{n,R,n-1} + p_{L,n}^1 h_n^L + z(h_n^L q_{R,n-1} + p_{R,n}^1 h_n^L + h_n^L q_{R,n-1}^2 + p_{L,n}^1 h_n^L + p_{L,n}^1 h_n^L q_{R,n-1}^2 + n I_N - z n I_N + p_{L,n}^1),
\]

\[
(\tilde{M}_n)_{12} = (h_n^1 + h_n^2 z + h_n^2 q_{R,n-1} + p_{L,n}^1 h_n^2 C_{n-1} + C_{n}^{-1} (h_n^1 R^1 + h_n^2 p_{R,n-1} + q_{L,n}^1 h_n^R) - (2n + 1) C_{n-1}^{-1},
\]

\[
(\tilde{M}_n)_{21} = -C_{n-1} (h_n^1 + h_n^2 z + h_n^2 q_{R,n-1} + p_{L,n}^1 h_n^2)
\]

\[
- (h_n^1 R^2 z + h_n^2 p_{R,n-1} + q_{L,n-1}^1 h_n^R) C_{n-1} + (2n - 1) C_{n-1},
\]

\[
(\tilde{M}_n)_{22} = -C_{n-1} h_n^2 C_{n-1} - (h_n^1 R^2 + h_n^2 C_{n-1}) - h_n^1 p_{R,n-1} - q_{L,n-1} h_n^1
\]

\[- z(h_n^1 R^2 p_{R,n-1} + q_{L,n-1} R^2 h_n^1) - R^2 p_{R,n-1} - h_n^2 (2 p_{R,n-1} - q_{L,n-1} R^1 p_{R,n-1} - n I_N + z n I_N) - p_{L,n}^1.
\]

Using the three term recurrence relation for $\{p_{L,n}^1\}_{n \in \mathbb{N}}$ we get that $p_{L,n}^1 - p_{L,n+1}^L = \beta_n^L$.

and $p_{L,n}^2 - p_{L,n+1}^2 = \beta_n^L p_{L,n} + \gamma_n^L$ where $\gamma_n^L = C_{n-1}^{-1} C_{n-1}$. Consequently,

\[
p_{L,n}^1 = -\sum_{k=0}^{n-1} \beta_k^L, \quad p_{L,n}^2 = \sum_{i,j=0}^{n-1} \beta_k^L \beta_j^L - \sum_{k=0}^{n-1} \gamma_k^L.
\]

In the same manner, from the three term recurrence relation for $\{Q_{L,n}^2\}_{n \in \mathbb{N}}$ we deduce that $q_{L,n-1}^1 = \beta_n^L := C_n^2 \beta_n C_{n-1}$ and $q_{L,n-1}^2 = \beta_n^L q_{L,n-1}^1 + \gamma_n^L$, where $\gamma_n^L = C_n^{-1} C_{n-1}$.

Now, we consider that $W = W^L$ and $W^R = I_N$, and then use the representation for $\{p_{L,n}^1\}_{n \in \mathbb{N}}$ and $\{Q_{L,n}^2\}_{n \in \mathbb{N}}$ in $z$ powers, the (1, 2) and (2, 2) entries in [17] read

\[
(2n + 1)(I_N - \beta_n^L + h_n^L + h_n^L (\gamma_{n+1}^L) + (\gamma_n^L)^2) + h_n^L
\]

\[
= [p_{L,n}^1, h_n^1] + [p_{L,n}, h_n^1] - [p_{L,n}^1, h_n^1] - p_{L,n}^1 - C_{n-1}^{-1} p_{L,n+1} C_n,
\]

\[
\beta_n^L - (\beta_n^L)^2 = \gamma_n^L (h_n^1 + (\beta_n^L + \gamma_n^L) + [p_{L,n}^1, h_n^1] + h_n^1 - (2n + 1) I_N)
\]

\[
- (h_n^1 (\beta_n^L + \gamma_n^L) + [p_{L,n}^1, h_n^1] + h_n^1 - (2n + 3) I_N) \gamma_n^L + [p_{L,n}^1, p_{L,n+1}^1].
\]

We can write these equations as follows

\[
(25) \quad (2n + 1) I_N + h_n^L + h_n^L (\gamma_{n+1}^L) + h_n^L (\beta_n^L + h_n^1 - (2n + 1) I_N) \beta_n^L + \sum_{k=0}^{n-1} \beta_k^L
\]

\[
+ C_n^{-1} \sum_{k=0}^{n-1} \beta_k^L C_n = \left[ \sum_{k=0}^{n-1} \beta_k^L, h_n^1 \right] \sum_{k=0}^{n-1} \beta_k^L - \left[ \sum_{k=0}^{n-1} \beta_k^L \beta_j^L - \sum_{k=0}^{n-1} \gamma_k^L h_n^1 \right] - \left[ \sum_{k=0}^{n-1} \beta_k^L, h_n^1 \right].
\]

\[
(26) \quad \beta_n^L - (\beta_n^L)^2 - \gamma_n^L (h_n^1 (\beta_n^L + \beta_n^L) + h_n^1 - (2n + 1) I_N) + (h_n^1 (\beta_n^L + \beta_n^L) + h_n^1)
\]

\[- (2n + 3) I_N) \gamma_n^L = \gamma_n^L \left[ \sum_{k=0}^{n-1} \beta_k^L, h_n^1 \right] - \left[ \sum_{k=0}^{n-1} \beta_k^L, h_n^1 \right] \gamma_n^L + \left[ \sum_{k=0}^{n-1} \beta_k^L, \sum_{k=0}^{n-1} \beta_k^L \right].
\]

We will show now that this system contains a noncommutative version of an instance of discrete Painlevé IV equation.

We see, on the r.h.s. of the nonlinear discrete equations (25) and (26) nonlocal terms (sums) in the recursion coefficients $\beta_n^L$ and $\gamma_n^L$, all of them inside commutators. These nonlocal terms vanish whenever the three matrices $(h_n^1, h_n^1, h_n^1)$ form an Abelian set, so that $(h_n^1, h_n^1, h_n^1, \beta_n^L, \gamma_n^L)$ is also an Abelian set. In this commutative setting we have

\[
(2n + 1) I_N + h_n^L + h_n^L (\gamma_{n+1}^L + \gamma_n^L) + (h_n^L \beta_n^L + h_n^1 - (2n + 1) I_N) \beta_n^L + p_{L,n}^1 + p_{L,n+1}^1 = 0_N,
\]
\[ \beta_n^\ell - (\beta_n^\ell)^2 - \gamma_n^\ell (h_2^\ell \beta_n^\ell + \beta_n^{\ell-1}) + h_1^\ell - (2n - 1) I_N \] 
\[ + h_1^\ell - (2n + 3) I_N) \gamma_{n+1}^\ell = 0_N. \]

In terms of
\[ \xi_n := \frac{h_0^\ell}{2} + n I_N + h_2^\ell \gamma_n + p_{L_n}^1 \quad \text{and} \quad \mu_n := h_2^{\ell+1} + h_1^\ell - (2n + 1) I_N, \]
the above equations reads as
\[ -\mu_n \beta_n^\ell = \xi_n + \xi_{n+1} \quad \text{and} \quad \beta_n^\ell (\xi_n - \xi_{n+1}) = \mu_{n+1} \gamma_{n+1} - \gamma_{n} \mu_{n-1}. \]

Now, we multiply the second equation by \( \mu_n \) and taking into account the first one we arrive to
\[ - (\xi_n + \xi_{n+1}) (\xi_n - \xi_{n+1}) = -\gamma_n \mu_{n-1} \mu_n + \gamma_{n+1} \mu_n \mu_{n+1}, \]
and so
\[ \xi_{n+1}^2 - \xi_n^2 = \gamma_{n+1} \mu_n \mu_{n+1} - \gamma_n \mu_{n-1} \mu_n. \]

Hence,
\[ \xi_{n+1}^2 - \xi_0^2 = \gamma_{n+1} \mu_n \mu_{n+1} \quad \text{and} \quad \beta_n^\ell \mu_n = - (\xi_n + \xi_{n+1}) \]
coincide to the ones presented in [4] as discrete Painlevé IV (dPIV) equation. In fact, taking \( \nu_n = \mu_n^{-1} \) we finally arrive to
\begin{align*}
\nu_n \nu_{n+1} &= \frac{h_0^\ell (\xi_{n+1}^2 - \xi_0^2/2 - n I_N - p_{L_n}^1)}{\xi_{n+1}^2 - \xi_0^2}, \\
\xi_n + \xi_{n+1} &= \left( \left( h_2^\ell \right)^{-1} - \left( h_2^\ell \right)^{-1} \nu_n^{-1} - (2n + 1) \left( h_2^\ell \right)^{-1} \right) \nu_n^{-1}.
\end{align*}

Now, we are able to state that,

**Theorem 3 (Non-Abelian extension of the dPIV).** Equations (25) and (26) defines a nonlocal nonlinear non-Abelian system for the recursion coefficients.

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