ANDERSON LOCALIZATION FOR TIME QUASI PERIODIC RANDOM SCHÖDINGER AND WAVE OPERATORS

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ABSTRACT. We prove that at large disorder, with large probability and for a set of Diophantine frequencies of large measure, Anderson localization in $\mathbb{Z}^d$ is stable under localized time-quasi-periodic perturbations by proving that the associated quasi-energy operator has pure point spectrum. The main tools are the Fröhlich-Spencer mechanism for the random component and the Bourgain-Goldstein-Schlag mechanism for the quasi-periodic component. The formulation of this problem is motivated by questions of Anderson localization for non-linear Schrödinger equations.

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I. Introduction

We prove persistence of Anderson localization for random Schrödinger and random wave operators under localized time-quasi-periodic perturbations. Given an initially localized wave packet, Anderson localization is, roughly speaking, the phenomenon that the wave packet remains localized for all time. Schrödinger equation is the following:

\[ i \frac{\partial}{\partial t} \psi = (\epsilon \Delta + V) \psi, \tag{1.1} \]
on \( \mathbb{R}^d \times [0, \infty) \) or \( \mathbb{Z}^d \times [0, \infty) \), where \( \epsilon > 0 \) is a parameter, \( \Delta \) is the Laplacian (continuum or discrete), \( V \) the potential is a multiplication operator; wave equation is

\[ \frac{\partial^2}{\partial t^2} \psi = (\epsilon \Delta + V) \psi, \tag{1.2} \]
on \( \mathbb{R}^d \times [0, \infty) \) or \( \mathbb{Z}^d \times [0, \infty) \), where the right hand side (RHS) is the same as in (1.1).

In this paper, we consider \( V \) random, to be defined shortly.

When \( V \) is independent of time, Anderson localization reduces to prove that the time independent Schrödinger operator:

\[ H_0 = \epsilon \Delta + V, \tag{1.3} \]
on \( L^2(\mathbb{R}^d) \) or \( \ell^2(\mathbb{Z}^d) \) has pure point spectrum with exponentially localized (or sufficiently fast decaying) eigenfunctions. \( 0 < \epsilon \ll 1 \) is the large disorder case.

Anderson localization for time independent random Schrödinger (or wave operator) at large disorder has been well known since the seminal work of Fröhlich-Spencer [FS]. It is a topic with an extensive literature [GMP, FMSS, vDK, AM, AFHS, AENSS], to name a few.

(Time independent) quasi-periodic Schrödinger operators in one dimension are now well understood following the works in [BG, FSWi, J, Sa, Sin] and the related works [HS1,2]. Recently in their fundamental paper [BGS], Bourgain-Goldstein-Schlag proved Anderson localization in two dimensions at large disorder under appropriate arithmetic conditions on the frequency vector. (See [Bo] for an excellent review and also overview of the subject and related things.) The papers [BG, BGS] play a central role in the construction here.

Below we specialize to discrete random Schrödinger operator. \( H_0 \) is then defined as the operator:

\[ H_0 = \epsilon \Delta + V, \text{ on } \ell^2(\mathbb{Z}^d), \tag{1.4} \]

where the matrix element \( \Delta_{ij} \), for \( i, j \in \mathbb{Z}^d \) verify

\[
\begin{align*}
\Delta_{ij} &= 1, & |i - j|_{\ell^1} &= 1 \\
&= 0, & \text{otherwise}; \\
&= 2
\end{align*}
\tag{1.5}
\]
the potential function $V$ is a diagonal matrix: $V = \text{diag}(v_j), j \in \mathbb{Z}^d$, where $\{v_j\}$ is a family of independently identically distributed (iid) real random variables with distribution $g$. From now on, we write $||\cdot||$ for the $\ell^1$ norm: $||\cdot||_{\ell^1}$ on $\mathbb{Z}^d$. We denote $\ell^2$ norms by $\|\|$. The probability space $\Omega$ is taken to be $\mathbb{R}^{\mathbb{Z}^d}$ and the measure $P$ is $\prod_{j \in \mathbb{Z}^d} g(dv_j)$.

As is well known, $\sigma(\Delta) = [-2d, 2d]$. Let supp $g$ be the support of $g$, we know further (see e.g., [CFKS,PF]) that

$$\sigma(H) = [-2d, 2d] + \gamma \text{supp } g \ a.s. \quad (1.6)$$

The basic result proven in the references mentioned earlier is that under certain regularity conditions on $g$, for $0 < \epsilon \ll 1$, and in any dimension $d$, the spectrum of $H_0$ is almost surely pure point with exponentially localized eigenfunctions, i.e., Anderson localization, after the physicist P. W. Anderson [An]. Physically this manifests as a lack of conductivity due to the localization of electrons. Anderson was the first one to explain this phenomenon on theoretical physics ground.

The study of electron conduction is a many body problem. One needs to take into account the interactions among electrons. This is a hard problem. The operator $H_0$ defined in (1.4) corresponds to the so called 1-body approximation, where the interaction is approximated by the potential $V$. The equation governing the system is (1.1) on $\mathbb{Z}^d \times [0, \infty)$.

As an approximation to the many body problem, when the interaction among electrons are weak, one studies the following non-linear Schrödinger equation (cf [DS, FSWa]):

$$i\frac{\partial}{\partial t}\psi = (\epsilon \Delta + V)\psi + \delta|\psi|^p\psi, \quad (0 < \delta \ll 1, \ p > 0) \quad (1.7)$$

on $\mathbb{R}^d \times [0, \infty)$ or $\mathbb{Z}^d \times [0, \infty)$. In [AF, AFS], solutions to the non-linear eigenvalue problem corresponding to (1.7) were found, which could be used to construct time periodic solutions where higher harmonics are absent. (See [BFG], for a Nekhoroshev type theorem in a related classical setting.) But in order to obtain time quasi periodic solutions or more general time periodic solutions to (1.7), one needs to study the corresponding time dependent random Schrödinger operators.

We remark here that the non-linear Schrödinger equation in (1.7) is distinct from other more commonly studied non-linear Schrödinger equations in that the linear equation itself already has small-divisor problems. When $p = 2$, (1.7) is also called the Gross-Pitaevskii equation, which arises in the theory of vortices in boson systems [Gr, Pi].

In [SW], time-periodic, spatially localized perturbations of random Schrödinger operators were considered. It is proven that Anderson localization is stable under such perturbations.
In this paper, we prove that Anderson localization is also stable under time-quasi-periodic, spatially localized perturbations with large probability and for a set of Diophantine frequencies of large measure. The techniques here are more involved than that in [SW] as one needs to take care of the small divisor problem coming from the random component and the quasi-periodic component simultaneously.

To be precise, we study the following time-quasi-periodic random Schrödinger equation:

$$i \frac{\partial}{\partial t} \psi = (\epsilon \Delta + V + W) \psi$$

(1.8)

and the time-quasi-periodic random wave equation

$$\frac{\partial^2}{\partial t^2} \psi = (\epsilon \Delta + V + W) \psi$$

(1.9)

on $\mathbb{Z}^d \times [0, \infty)$, where as in (1.4), $V = \{v_j\}$ is a family of (time-independent) i.i.d. random variables;

$$W = W(t, j) = \sum_{k=1}^{\nu} W_k(j) \cos 2\pi (\omega_k t + \theta_k)$$

(1.10)

where

$$\omega = (\omega_1, \ldots, \omega_\nu) \in (0, 1)^\nu$$
$$\theta = (\theta_1, \ldots, \theta_\nu) \in (0, 1)^\nu.$$

To proceed further, we assume

(H1) $W_k(j)$ is such that,

$$\sum_{k=1}^{\nu} |W(j)| \leq 2\nu \delta e^{-b|j|} (0 < \delta \ll 1, b > 0)$$

(H2) $\omega$ satisfies a Diophantine condition,

$$||n \cdot \omega||_T^\nu \geq \frac{c}{|n|^A} \quad (n \neq 0, c > 0, A > 0).$$

We write $\omega \in \text{DC}_{A,c}$.  

(H3) The probability distribution $g$ has bounded support, without loss we assume $\text{supp} \ g \subset [-1,1]$  

(H4) $g$ is absolutely continuous with a bounded density $\tilde{g}$:

$$g(dv) = \tilde{g}(v) dv, \quad \|\tilde{g}\|_\infty < \infty$$
Remark. Some fast enough polynomial decay for $\sum_{k=1}^{\nu} |W(j)|$ suffices. We assume (H1) for ease in writing.

It is well known (see e.g., [Be, Ho1, 2, JL, Ya]) that the study of time-quasi-periodic equations like (1.8, 1.9) can be reduced to the spectral study of their corresponding quasi-energy operators; here

$$K = \sum_{k=1}^{\nu} i \omega_k \frac{\partial}{\partial \theta_k} + \epsilon \Delta + V + \sum_{k=1}^{\nu} W_k(j) \cos 2\pi \theta_k$$

(1.11)

for the Schrödinger equation in (1.8); and

$$K_w = (\sum_{k=1}^{\nu} \omega_k \frac{\partial}{\partial \theta_k})^2 + \epsilon \Delta + V + \sum_{k=1}^{\nu} W_k(j) \cos 2\pi \theta_k$$

(1.12)

for the wave equation in (1.9), on $\ell^2(\mathbb{Z}^d) \times L^2(T^\nu)$, (cf. [SW]).

We prove that with large probability and for a set of Diophantine frequencies $\omega \in (0, 1]^\nu$ with large measure, both $K$ and $K_w$ have pure point spectrum with exponentially decaying (in the $\mathbb{Z}^d$ direction) eigenfunctions. (For a precise statement, see the Theorem in sect. 5 on $H$ and $H_w$ which are unitary equivalents of $K$ and $K_w$ respectively.) This implies in particular (after some standard gymnastics) that with large probability, for a set of Diophantine frequencies of large measure, and initial conditions $\psi(0)$ which are localized in space, the time evolutions $\psi(t)$ of (1.8) and (1.9) are almost periodic, a.e. $\theta$, (cf. e.g., [SW, JL]).

We spare a few lines on the proof of Anderson localization for the unitary equivalents $H, H_w$, which are obtained from $K, K_w$ by a partial Fourier transform in $\theta \in T^\nu$, (see (2.2, 6.2) for the precise expressions). Let $n$ be the dual variable of $\theta, n \in \mathbb{Z}^\nu$. We know that for $0 < \epsilon \ll 1$, roughly speaking, the Green’s function decays exponentially in the $j$ directions, $j \in \mathbb{Z}^d$, due to Anderson localization of the original unperturbed operator $H_0$ defined in (1.4). To prove Anderson localization for the perturbed operators $H, H_w$ on $\ell^2(\mathbb{Z}^{d+\nu})$, we also need to prove exponential decay in the $n$ directions using quasi-periodicity. This is however the “classical” picture, as the quasi-periodic perturbation does not commute with $H_0$.

To prove Anderson localization for $H, H_w$ on $\ell^2(\mathbb{Z}^{d+\nu})$, we put the small-divisor problems originating from the random and quasi-periodic components on equal footing and deal with them concurrently. For the random component, we use the Fröhlich-Spencer (FS) approach. The version which is well adapted to our purpose is the one in [vDK], summarized in the appendix. For the quasi-periodic component, we rely on semi-algebraic considerations, Cartan type of theorems for analytic matrix valued functions developed in the series of papers by Bourgain, Goldstein and Schlag (BGS) [BG, BGS], (see also [Bo]). (The dynamics here is simpler than that in [BGS] due to
the special quasi-periodic structure of $H$, $H_w$.) The Diophantine frequencies which are excluded result from a Melnikov type of non-resonant conditions, (see Lemmas 2.3, 6.1 (2.26-2.28, 6.10)).

Finally, for the experts, we wish to add that the constructive aspect of the BGS mechanism is a more robust version of the FS mechanism. In BGS, at each scale, the number of resonant sub-regions of the previous scale can grow sub-linearly; while in FS, at each scale, the number of resonant sub-regions of the previous scale is fixed (see[vDK]). In the quasi-periodic setting, one typically falls into the BGS scenario.

2. Exponential Decay of the Green’s function of Schrödinger operator at fixed $E$ and $x$

Recall from sect. 1, the quasi-energy operator $K$:

$$K = \sum_{k=1}^{\nu} \omega_k \frac{1}{i} \frac{\partial}{\partial \theta_k} + \varepsilon \Delta + V + \sum_{k=1}^{\nu} W_k \cos 2\pi \theta_k$$ (2.1)

on $\ell^2(\mathbb{Z}^d) \times \ell^2(\mathbb{T}^\nu)$, where $\omega = (\omega_1, \omega_2 \cdots \omega_\nu) \in (0, 1]^\nu$. $V$ is the random potential on $\mathbb{Z}^d$, $0 < \varepsilon \ll 1$ and $W_k$ satisfies the decay properties specified in (H1).

Performing a partial Fourier series transform, in the $\mathbb{T}^\nu$ variables only, we are led to study the following unitarily equivalent operator:

$$H = \delta_j \tilde{\Delta}_n + n \cdot \omega + \varepsilon \Delta_j + V_j$$ (2.2)

on $\ell^2(\mathbb{Z}^{d+\nu})$, where

- $n \in \mathbb{Z}^\nu, j \in \mathbb{Z}^d$
- $\delta_j \tilde{\Delta}_n \overset{\text{def}}{=} \sum_{k=1}^{\nu} W_k(j) \Delta_k$ is an operator on $\ell^2(\mathbb{Z}^\nu)$, $\Delta_k$ is the standard discrete Laplacian on the $k$th copy of $\mathbb{Z}$.

$$\|\delta_j \tilde{\Delta}_n\|_{\ell^2(\mathbb{Z}^\nu)} \leq 2\nu \delta e^{-b|j|} \quad (b > 0)$$ (2.3)

by using (H1). (Some fast enough polynomial decay suffices. We assume (2.3) for ease in writing.)

- $\varepsilon \Delta_j + V_j \overset{\text{def}}{=} \varepsilon \Delta + V$, we put in the subscript $j$ to stress that it came from an operator on $\ell^2(\mathbb{Z}^d)$
- For simplicity, we now drop the tilde on $\Delta_n : \tilde{\Delta}_n \overset{\text{def}}{=} \Delta_n$. 

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• For $\Lambda \subset \mathbb{Z}^{d+\nu}$, $H_\Lambda$ is the restriction of $H$ to $\Lambda$:

$$H_\Lambda(j,n,j',n') \overset{\text{def}}{=} \begin{cases} H(j,n,j',n') & \text{if } j,n \in \Lambda \text{ and } j',n' \in \Lambda \\ 0 & \text{otherwise} \end{cases}$$

(2.4)

Let $X \subset \mathbb{R}^{\mathbb{Z}^d}$ be the set where the random Schrödinger operator $H_j = \varepsilon \Delta_j + V_j$ exhibits Anderson localization in a sense to be made precise in (2.14) where theorem 2.2 of [vDK], restated here as Theorem A is applicable. We note here only that since we require finite scale information,

$$\text{mes } X < 1, \quad \text{mes } X \gtrsim 1 - \frac{1}{L^a} (a > 0)$$

where $L$ is the initial scale.

2.1 The initial estimate (0th step)

Fix an energy $E$, fix $x \in X$, so that $H_j$ has Anderson localization. Let $\theta \in \mathbb{R}$ and define

$$H(\theta) = \delta_j \Delta_n + (n \cdot \omega + \theta) + \varepsilon \Delta_j + V_j$$

(2.5)

on $\ell^2(\mathbb{Z}^{d+\nu})$. We study the Green’s function

$$G_{\Lambda_0}(\theta, E) = (H_{\Lambda_0}(\theta) - E)^{-1}$$

(2.6)

for some $\Lambda_0 = [-N_0, N_0]^{d+\nu}$, where $N_0$ is to be determined. We call $\Lambda_0$, an $N_0$-box. We do perturbation theory and for the 0th step, we drop $\delta_j \Delta_n$. We have after diagonalization

$$H_{\Lambda_0,0} \overset{\text{def}}{=} n \cdot \omega + \theta + \mu_j$$

(2.7)

where $\mu_j$ are the eigenvalues of $H_j$. Since $\|\delta_j \Delta_n\|_{\ell^2(\mathbb{Z}^\nu)} \leq 2\nu \delta e^{-b|j|}$, from (2.3) we require that

$$|n \cdot \omega + \theta + \mu_j - E| > 2c\delta$$

(2.8)

for some $c > 2\nu$ and all $(n,j) \in \Lambda$. So we estimate the measure of the set of $\theta$ such that

$$|n \cdot \omega + \theta + \mu_j - E| \leq 2c\delta$$

(2.9)

for some $(n,j) \in \Lambda$. In this particularly simple case, we obtain

$$\text{mes } \{\theta \| (H_{\Lambda_0,0} - E)^{-1} \| \leq (c\delta)^{-1} \} \leq 4c\delta |\Lambda_0|^2$$

$$= 4c\delta (2N_0 + 1)^2(d+\nu)$$

(2.10)

Let

$$\sigma \in (0,1), N_0 = \lceil \log c\delta \rceil^{1/\sigma} + 1$$

(2.11)

($\lceil \cdot \rceil$ is the integer part) and $B_x(\Lambda_0, E)$ be the set defined in the left hand side of (2.10).

We note that

$$\text{mes } B_x(\Lambda_0, E) \leq e^{-\frac{N_0^2}{2}}$$

(2.12)

for $N_0$ satisfying (2.11) and $0 < \delta \ll 1$. 7
Lemma 2.1. There exists $\gamma' > 0$, such that for $\delta \ll 1$, on $\mathbb{R} \setminus B_x(\Lambda_0, E)$
\begin{align}
\|G_{\Lambda_0}(\theta, E)\| &< e^{N_0^\sigma} \quad \sigma \in (0, 1) \\
|G_{\Lambda_0}(\theta, E)(m, m')| &< e^{-\gamma|m-m'|}
\end{align}
(2.13)
for all $m, m' \in \Lambda$, $|m - m'| > N_0/4$.

Proof. The first inequality of (2.13) is a restatement of (2.10), (2.11). To obtain the second inequality, we use the conclusion of Theorem 2.2 of [vDK] restated here as Theorem A for the scale $N' = [N_0^{1/\alpha}] + 1$ ($1 < \alpha < 2$). Theorem A states that the set $X_{N_0}$ where there is only one pairwise disjoint bad $N'$ box contained in the $N_0$ box $[-N_0, N_0]^d$ has measure:
\begin{equation}
\text{mes} \ X_{N_0} \geq 1 - \frac{N_0^{2d}}{N^2} \geq 1 - \frac{1}{N_0^{(2p'/\alpha-2d)}} \quad (p' > 0, 1 < \alpha < 2).
\end{equation}
(2.14)
Fix $x \in X_{N_0}$, (assuming $2p'/\alpha - 2d \gg 1$), using the resolvent expansion and the first equation of (2.13) for the bad $N'$ box, we obtain that $\exists \gamma' > 0$, such that on $\mathbb{R} \setminus B_x(\Lambda, E)$,
\begin{equation}
|\langle H_{j,\Lambda}(\theta) - E \rangle^{-1}(i, i')| \leq e^{-\gamma|i-i'|}
\end{equation}
(2.15)
for all $i, i' \in [-N_0, N_0]^d$ and $|i - i'| > N/4$.

The second equation of (2.13) follows from Neumann series (in the $n$-direction), (2.15) and the decay condition on $\delta_j$ in (2.3). \hfill \Box

2.2 A Wegner estimate (in $\theta$) for all scales

We now prove an apriori estimate on $\|\langle H_{\Lambda}(\theta) - E \rangle^{-1}\|$ for all finite subsets $\Lambda \subset \mathbb{Z}^{d+\nu}$. This estimate uses the special structure of (2.5) and hence holds only for Schrödinger and not for wave equations e.g. For those more general situations, we need to resort to Cartan-type of theorem for analytic matrix valued functions a la [BGS]. (For the experts, this saves us one subroutine and moreover we only need to work with cubes in $\mathbb{Z}^{d+\nu}$.) Wave equation will be treated in sect 7.

Lemma 2.2. Let $E \in I$, an interval of length $O(1)$. Let $\Lambda$ be a finite set in $\mathbb{Z}^{d+\nu}$.
\begin{equation}
\text{mes} \ \{ \theta | \text{dist} (E, H_{\Lambda}(\theta)) \leq \kappa \} \leq C |\Lambda| \kappa
\end{equation}
(2.16)

Proof. Let $N(\theta, \lambda)$ be the # of eigenvalues of $H_{\Lambda}(\theta) \leq \lambda$
\begin{align}
\text{mes} \ \{ \theta | \text{dist} (E, H_{\Lambda}(\theta)) \leq \kappa \}
\leq & \int (N(\theta, E + \kappa) - N(\theta, E - \kappa)) d\theta \\
= & \int_{|\theta| \leq O(1)N} (N(\theta, E + \kappa) - N(\theta, E - \kappa)) d\theta,
\end{align}
(2.17)
since $\mathcal{N}(\theta, E + \kappa) = \mathcal{N}(\theta, E - \kappa)$ for $|\theta| > \mathcal{O}(1)N$. In view of (2.5)

\[ \mathcal{N}(\theta, E \pm \kappa) = \mathcal{N}(\theta \mp \kappa, E). \]  

(2.18)

Substituting (2.18) into (2.17) we obtain

\[ (2.17) = \int (\mathcal{N}(\theta - \kappa, E) - \mathcal{N}(\theta + \kappa, E))d\theta \]

\[ = \int_{\theta(1)N-\kappa}^{\theta(1)N+\kappa} \mathcal{N}(\theta, E)d\theta - \int_{-\theta(1)N+\kappa}^{\theta(1)N+\kappa} \mathcal{N}(\theta, E)d\theta \]

\[ \leq C|\mathcal{N}(\theta, E)|_{\infty} \cdot \kappa \]

(2.19)

where we used the fact that the $|\Lambda| \times |\Lambda|$ matrix $H_\Lambda(\theta)$ has $|\Lambda|$ eigenvalues. □

2.3 The first iteration (1st step)

Let

\[ N = [N^C_0] + 1, \quad (C > 1). \]  

(2.20)

Let $\Lambda = [-N, N]^{d+\nu}$. $N$ is the next scale, recall that the previous scale $N_0$ is determined by $\delta$ in (2.11). The aim of this section is to derive the analogue of Lemma 2.1 for $G_\Lambda$.

To do that we use the estimates on $G_{\Lambda_0}$ at scale $N_0$ in Lemma 2.1 and also Lemma 2.2. Let $\Lambda_0 = [-N_0, N_0]^{d+\nu} + i \subset \Lambda, i \in \Lambda$. For a fixed $\theta$, we say that $\Lambda_0$ is good if (2.13) holds, otherwise $\Lambda_0$ is bad. Recall from (2.9), (2.11) that for fixed $\theta$, at scale $N_0$, $\Lambda_0$ is bad if

\[ |n \cdot \omega + \theta + \mu_j - E| < 2e^{-N_0^\sigma} \]  

(2.21)

for some $(n, j) \in \Lambda_0$, where $\mu_j$ is an eigenvalue of $H_j$.

Let $X_N$ be the set where all $\Lambda(k) = [-N_0, N_0]^{d+k} + k \in [-N, N]^d$, have the property (2.14). Note that $X_N \subset X_{N_0}$. So

\[ \text{mes } X_N \geq 1 - \frac{(2N_0 + 1)^{2d}}{N^{2d}p'}(2N + 1)^d \quad (p' > 0) \]

(2.22)

where

\[ N' = [N_0^{1/\alpha}] + 1, 1 < \alpha < 2 \]

\[ N = [N^C_0] + 1, C > 1. \]

(2.23)

Fix $x \in X_N$. Assuming $2p'/\alpha - (2 + C)d \gg 1$, we prove
Lemma 2.3. There exists a set
\[ \Omega_N \subset (0, 1]^\nu, \quad \operatorname{mes} \Omega_N \geq 1 - e^{-N\frac{\sigma}{\nu}} \tag{2.24} \]
where \( \sigma \in (0, 1) \) is as in (2.11) and \( C > 1 \) is as in (2.20), such that if \( \omega \in \Omega_N \), then for any fixed \( \theta, E \), there is only one (pair-wise disjoint) bad \( N_0 \)-box in \( \Lambda = [-N, N]^{d+\nu} \). Moreover \( (0, 1]^\nu \setminus \Omega_N \) is contained in the union of at most \( \mathcal{O}(1)N^{4d+\nu} \) components.

Remark. It is crucial that \( \Omega_N \) is independent of \( \theta, E \), and only depends on \( x \in X_N \).

Proof. Let
\[
\Lambda_0 = [-N_0, N_0]^{d+\nu} + i \subset \Lambda
\]
\[
\Lambda'_0 = [-N_0, N_0]^{d+\nu} + i' \subset \Lambda \quad (i \neq i')
\tag{2.25}
\]
be such that \( \Lambda_0 \cap \Lambda'_0 = \emptyset \).

Let
\[ \Lambda_{0,j} \]
be the projection of \( \Lambda_0 \) onto \( \mathbb{Z}^d \)
\[ \Lambda_{0,n} \]
be the procection of \( \Lambda_0 \) onto \( \mathbb{Z}^\nu \)

and similarly for \( \Lambda' \).

Assume both \( \Lambda_0 \) and \( \Lambda'_0 \) are bad, then there exist \( (n, j) \in \Lambda, (n', j') \in \Lambda' \), such that
\[
|n \cdot \omega + \theta + \mu_j - E| < 2e^{-N_0^\sigma} \tag{2.26}
\]
\[
|n' \cdot \omega + \theta + \mu_{j'} - E| < 2e^{-N_0^\sigma} \tag{2.27}
\]
Subtracting (2.27) from (2.26), we obtain
\[
|(n - n') \cdot \omega + (\mu_j - \mu_{j'})| < 4e^{-N_0^\sigma}. \tag{2.28}
\]
Since \( \Lambda_0 \cap \Lambda'_0 = \emptyset \).

\[ (n, j) \neq (n', j') \tag{2.29} \]

There are 2 possibilities:

- \( n = n' \)

In this case \( \Lambda_{0,n} \cap \Lambda_{0,n'} \neq \emptyset \), so \( \Lambda_{0,j} \cap \Lambda_{0,j'} = \emptyset \). Anderson localization for \( H_j \), Theorem A then implies that on \( X_N \), \( |\mu_j - \mu_{j'}| \geq e^{-N_0^\beta} \) for some
\[
\beta \in (0, \sigma) \tag{2.30}
\]
for all \( \Lambda_{0,j}, \Lambda_{0,j'} \subset \Lambda_j, \Lambda_{0,j} \cap \Lambda_{0,j'} = \emptyset \) and any pair of eigenvalues \( \mu_j \in \sigma(H_j), \mu_{j'} \in \sigma(H_{j'}) \) (2.30) is in contradiction with (2.28). So there can be only 1 (pairwise disjoint) bad \( N_0 \)-box.

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Let 
\[
m = n - n', \\
\lambda = \mu_j - \mu_{j'} ,
\]
(2.31)
then
\[
m \in [-2N, 2N]\nu \setminus \{0\}
\]
(2.32)
\(\lambda\) can take on at most \((2N + 1)^{2d}(2N_0 + 1)^{2d}\) different values.

So (2.28) corresponds to at most \(O(1)N^{4d+\nu}\) inequalities in \(\omega \in (0, 1)^\nu\) of the form
\[
|m \cdot \omega + \lambda| \leq 4e^{-N_0^\sigma} .
\]
(2.33)
For each equation in (2.33), it is simple to see that the set of \(\omega \in (0, 1)^\nu\) such that (2.33) is satisfied, has one single component of measure \(\leq O(1)e^{-N_0^\sigma}\). We hence obtain the lemma for \(N\) large enough.

Let
\[
\Lambda = [-N, N]^{d+\nu}, \quad B_x(\Lambda, E) = \{\theta | \|G_{\Lambda}(\theta, E)\| \geq e^{N^\sigma}\}.
\]
(2.34)
From (2.16),
\[
\text{mes } B_x(\Lambda, E) \leq e^{-\frac{N^\sigma}{C}} \text{ for } N \gg 1.
\]
(2.35)
For any \(x \in X_N, X_N\) defined in (2.22, 2.14), using (2.14), (2.34) Lemma 2.3 and resolvent expansion a la Fröhlich-Spencer, we obtain our main estimate,

**Lemma 2.4.** For all \(\theta \in \mathbb{R}\setminus B_x(\Lambda_0, E)\)
\[
\|G_{\Lambda}(\omega, \theta, E)\| < e^{N^\sigma}, \\
|G_{\Lambda}(\omega, \theta, E)(m, m')| < e^{-\gamma|m-m'|}
\]
(2.36)
for all \(m, m' \in \Lambda_0, |m - m'| > N/4, \omega \in \Omega_N \subset (0, 1)^\nu, \text{ mes } \Omega_N \geq 1 - e^{N(\sigma/2C)}\) where \(\gamma'/2 < \gamma < \gamma', \gamma, \sigma\) are the same as in Lemma 2.1, \(N_0\) as defined in (2.20).

### 2.4 A large deviation estimate (in \(\theta\)) for the Green’s functions at all scales

We now build upon the estimates in Lemmas 2.1 and 2.4 to obtain estimates for Green’s functions at all scales. In order that the bad set in \(\theta\) be of small measure at larger scales, we need to let in more bad boxes at the smaller scales. (In Lemma 2.1, there is no bad box, while in Lemma 2.4, there is one.) The number of bad boxes is controlled by using semi-algebraic sets as in [BGS] and Lemma 2.3.
Assume $0 < \delta \ll 1$ is sufficiently small so that Lemma 2.1 holds for all $N \in [N_0, N_0']$, $\alpha' > 1$, $N_0 \gg 1$ determined by (2.11). From Lemma 2.4, both equations in (2.36) hold for all $N \in [N_0', N_C']$ on $\mathbb{R} \setminus \mathcal{B}_{x,N}$ and $\omega \in \Omega_N$, where $C > 1$ is to be determined shortly. The probability subspace is then further restricted to be

$$X = \bigcap_{N \in [N_0, N_C]} X_N$$

(2.37)

where $X_N$ is defined similarly to (2.22).

For what is to follow, it is more convenient to slightly modify the definition and let

$$G_x^{\gamma,\sigma}(\Lambda, E) \overset{\text{def}}{=} \{ \theta \in \mathbb{R} \mid \| G_A(\theta, E) \| < e^{N\sigma},$$

$$| G_A(\theta, E)(m, m') | < e^{-\gamma|m-m'|}$$

$$\forall m, m' \in \Lambda, |m-m'| > N/4 \}$$

$$B_x^{\gamma,\sigma}(\Lambda, E) \overset{\text{def}}{=} \mathbb{R} \setminus G_x^{\gamma,\sigma},$$

(2.38)

where $\gamma > 0$, $0 < \sigma < 1$, $\Lambda \subset \mathbb{Z}^{d+\nu}$ is a cube of side length $2N + 1$.

Lemma 2.1 and Lemma 2.4 can be summarized as

**Proposition 2.5.** There exist $\gamma > 0$, $0 < \sigma < 1$, such that for $0 < \delta \ll 1$, $\delta_j$ satisfying (2.3), $0 < \varepsilon \ll 1$, there exists $N_0$, such that for all $N \in [N_0, N_0']$ ($C > 1$), $\Lambda = [-N, N]^{d+\nu} \subset \mathbb{Z}^{d+\nu}$, cubes of side length $2N + 1$,

$$\sup_{\omega \in \Omega} \text{mes} (\Omega_x \cap \Omega_{x,N} \subset (0, 1]^\nu)$$

(2.39)

if

$$\omega \in \Omega_x = \bigcap_{N \in [N_0, N_0']} \Omega_{x,N} \subset (0, 1]^\nu$$

and $\Omega_{x,N}$ is as in Lemma 2.3, $\text{mes} \Omega_x \geq 1 - e^{-N_0^{\sigma/2}}$.

Let $X_{N,i}$ be defined as in (2.22) with $[-N, N]^{d+\nu} + i$ replacing $[-N, N]^{d+\nu}$; $\Omega_{N,i}$ defined as in Lemma 2.3 with $\Lambda(i) = [-N, N]^{d+\nu} + i$ in place of $\Lambda = [-N, N]^{d+\nu}$. Denote by $DC_{A,c}(M)$, the set of $\omega \in (0, 1]^\nu$, such that (H2) is verified for $n \in [-M, M]^\nu$. We now prove

**Lemma 2.6.** Suppose all the assumptions of Proposition 2.5 is valid. Let $C > 10(d + \nu), 0 < \sigma < 1/2, N_1 = N_0^{C'}$. Then for all $N \in [N_1, N_1^2]$, $\Lambda$ cubes of side length $2N + 1$, let

$$X = \bigcap_{N \in [N_0, N_1^2]} X_N \bigcap_{i \in [-2N_0, 2N_0]^d} X_{N_0,i}.$$
For any \( x \in X \), let
\[
\Omega_x = \bigcap_{N \in [N_0, N_1^2]} \Omega_{x,N} \cap \bigcap_{i \in [-2N_0, 2N_0]^d} \Omega_{x,N_0,i}
\]
mes \( \Omega_x \geq 1 - e^{-N_0^a/2} \). If \( \omega \in \Omega_x \cap DC_{A,c}(N_1^2) \) then
\[
\sup_{x \in X, E} \text{mes} (B_{x}^{\gamma', \sigma}(A, E)) \leq e^{-N_0^a/2}
\]
(2.40)
where \( \gamma' = \gamma - N^{-\kappa}, \kappa = \kappa(\sigma, \gamma) > 0 \).

**Proof.** Fix \( N \in [N_1, N_1^2] \) and let
\[
\Lambda = [-N, N]^d + \nu,
\]
\[
T = [-N_0, N_0]^d \times [-N, N]\nu \subset \Lambda
\]
(2.41)

Let
\[
\Lambda_0 = [-N_0, N_0]^d + \nu
\]
\[
\Lambda_0(i) = \Lambda_0 + i.
\]
(2.42)

Define
\[
\mathcal{A} \overset{\text{def}}{=} \bigcup_{i \in [-2N_0, 2N_0]^d} B_{x}^{\gamma', \sigma}(\Lambda_0(i), E).
\]
(2.43)

Since the conditions on the Green’s function in (2.38) can be rewritten as polynomial inequalities by using Cramer’s rule (ratio of determinants) as in [BG, BGS], \( \mathcal{A} \) is semi-algebraic of total degree less than
\[
(2N_0 + 1)^{2(d + \nu)} \cdot (2N_0 + 1)^{2(d + \nu)} \cdot (4N_0 + 1)^d
\]
\[
= O_{d,\nu}(1) N_0^{5(d + \nu)},
\]
(2.44)
where the first factor corresponds to the degree of each polynomial for each pair of points in a \( N_0 \)-box, the second is an upperbound on the \( \# \) of pairs in each \( N_0 \)-box plus the one for the Hilbert-Schmidt norm, the third is the \( \# \) of such \( N_0 \)-boxes. \( \mathcal{A} \) is therefore the union of at most \( O_{d,\nu}(1) N_0^{5(d + \nu)} \) intervals in \( \mathbb{R} \) by using Theorem 1 in [Ba] (see also [BGS], where the special case we need is restated as Theorem 7.3.)

For any fixed \( \theta \in \mathbb{R} \), let
\[
I = \{ n \in [-N, N]^{\nu} \mid n \cdot \omega + \theta \in \mathcal{A} \}.
\]
(2.45)
Then for $\omega \in \Omega_N \cap DC_{A,c}$

$$|I| \leq O_{d,\nu}(1)N_0^{5(d+\nu)}$$  \hspace{1cm} (2.46)

by using (2.39). This is because if there exist $n, n' \in [-N, N]^\nu$, $n \neq n'$, then for $\omega \in DC_{A,c}$

$$|(n - n') \cdot \omega| \geq \frac{c}{|N|^A} \gg e^{-N^{\sigma}/2} (c > 0, A > 0).$$  \hspace{1cm} (2.47)

Hence each interval can contain at most 1 integer $n \in [-N, N]^\nu$.

We therefore conclude that for any fixed $\theta \in \mathbb{R}$,

$$\#\{i \in \Lambda | \Lambda_0(i) \cap T \neq \emptyset, \Lambda_0(i) \text{ is a bad } N_0\text{-box} \} \leq O_{d,\nu}(1)N_0^{5(d+\nu)}$$  \hspace{1cm} (2.48)

where $\Lambda, T, \Lambda_0$ as defined in (2.41, 2.42).

For the $N_0$-boxes $\Lambda_0(i)(i \in \Lambda)$, such that $\Lambda_0(i) \cap T \neq \emptyset$, we use Lemma 2.3 and (2.3) to conclude that for $\omega \in \Omega_N, (N \in [N_1, N_2])$, $\exists i_0 \in \Lambda \setminus T$, such that $\forall i \in \Lambda$, such that $\Lambda_0(i) \cap T = \emptyset$, if $\Lambda_0(i) \cap \Lambda_0(i_0) = \emptyset$, then

$\Lambda_0(i)$ is a good $N_0$-box.  \hspace{1cm} (2.49)

We now introduce an intermediate scale $\bar{N}$:

$$\log N_0 < \log \bar{N} < \log N.$$  \hspace{1cm} (2.50)

Let $\bar{C} \in (10(d + \nu), C), \bar{N} = [N_0^{\bar{C}}]$

$$\bar{\Lambda} = [-\bar{N}, \bar{N}]^{d+\nu}.$$  \hspace{1cm} (2.51)

Let $\mathcal{I}$ be the set defined in (2.48).

We say $\bar{\Lambda}(i)(i \in \Lambda)$ is good if

$$\bar{\Lambda}(i) \cap (\mathcal{I} \cup \Lambda_0(i_0)) = \emptyset,$$  \hspace{1cm} (2.52)

where $\Lambda_0(i_0)$ is as in (2.49), otherwise it is bad.

Let $\mathcal{F}$ be a family of pairwise disjoint bad $\bar{N}$-boxes in $\bar{\Lambda} = [-\bar{N}, \bar{N}]^{d+\nu}$. (2.48, 2.49) imply that

$$\#\mathcal{F} \leq O_{d,\nu}(1)N_0^{5(d+\nu)} + 2^{d+\nu}$$

$$= O_{d,\nu}(1)N_0^{5(d+\nu)}.$$  \hspace{1cm} (2.53)
If $\bar{\Lambda}(i)$ is good, then
\[ \forall j \in \bar{\Lambda}(i), \exists j' \in \bar{\Lambda}(i) \text{ such that } \Lambda_0(j') \subset \bar{\Lambda}(i) \]
and
\[ \text{dist} \left( j, \partial_\ast \Lambda_0(j') \right) \geq N_0, \tag{2.54} \]
where $\partial_\ast \Lambda_0(j')$ is the interior boundary of $\Lambda_0(j')$ relative to $\bar{\Lambda}(i)$:
\[ \partial_\ast \Lambda_0(j') = \{ z \in \Lambda_0(j') | \exists z' \in \bar{\Lambda}(i) \setminus \Lambda_0(j'), |z' - z| = 1 \}. \tag{2.55} \]

An easy resolvent expansion (see e.g. Lemma 2.2 and proof of Corollary 4.5 in [BGS]) then shows that
\[ |G_{\bar{\Lambda}(i)}(m, m')| < e^{-\gamma|m - m'| + CN_0} \tag{2.56} \]
for all $m, m' \in \bar{\Lambda}(i), |m - m'| > N_0/4$.

For $N \in [N_1, N_1^2] = [N_0^C, N_0^2C]$, we have
\[ \#F < N^\sigma \tag{2.57} \]
with
\[ \frac{5(d + \nu)}{2C} < \sigma < \frac{5(d + \nu)}{C} \tag{2.58} \]
for
\[ C > 10(d + \nu), \sigma < 1/2. \tag{2.59} \]

For all $C > 10(d + \nu)$, we can choose $\bar{C}$ satisfying (2.51), so that for all $N \in [N_1, N_1^2]$, we obtain (2.40) from the estimates at scale $\bar{N}$ by applying Lemma 2.1 of [BGS] with a single step iteration. (This is possible because $\sigma < 1/2$, so $\exists \alpha > 2$ such that $\alpha \sigma < 1$, where $\alpha$ is the geometric expansion factor. See the first inequality of (2.12) of [BGS].)

The measure estimate in $\theta$ is supplied by Lemma 2.2.

We do not repeat the details of this iteration, except noting the following small variations:

- Because of the apriori estimate in Lemma 2.2, which holds at all scales, we only need to estimate Green’s functions for cubes in $\mathbb{Z}^{d+\nu}$.
- To eliminate $G_{\Lambda}(i, i'), i, i' \in \Lambda, |i - i'| > N/4$, we make an exhaustion $\{S_j(i)\}_{j=0}^{\ell}$ of $\Lambda$ of width $2\bar{N}$ centered at $i$ (as in [BGS]):
\[
\begin{align*}
S_{-1}(i) & \overset{\text{def}}{=} \emptyset \\
S_0(i) & \overset{\text{def}}{=} \bar{\Lambda}(i) \cap \Lambda \\
S_j(i) & \overset{\text{def}}{=} \bigcup_{k \in \delta_{j-1}(i)} \bar{\Lambda}(k) \cap \Lambda \text{ for } 1 \leq j \leq \ell 
\end{align*} \tag{2.60}
\]
where \( \ell \) is maximal such that \( S_\ell(i) \neq \Lambda \).

- In the iteration, we need to estimate \( G_A(m', m) \), where \( A = S_j \setminus S_{j'} \), for some \( j' < j \leq \ell \), is an annulus. \( m' \in \partial_{ss} S_{j'}, \partial_{ss} S_j \), is the exterior boundary of \( S_{j'} \), relative to \( \Lambda \):

\[
\partial_{ss} S_{j'} = \{ z \mid z \in \Lambda \setminus S_{j'}, \exists z' \in S_{j'}, \ |z - z'| = 1 \} \tag{2.61}
\]

is concave; \( m \in \partial_{s} S_j \) is the interior boundary of \( S_j \) as defined in (2.55) \( \partial_{s} S_j \) is convex.

- When \( A \) is good (for the precise definition, see Lemma 2.2 of [BGS]). We estimate \( G_A(m', m) \) using \( \bar{\Lambda} \) cubes, which are all "good". We always start the resolvent expansion from \( m \in \partial_{s} S_j \), for the property that \( \forall m \in \partial_{s} S_j, \exists m'' \in A, \) such that \( \bar{\Lambda}(m'') \in A \) and \( \text{dist}(m, \partial_{s} \bar{\Lambda}(m'')) \geq N_0 \). For the last term in the expansion, we use the apriori estimate in Lemma 2.2. This way we avoid having to estimate Green’s functions in regions of the form \( \bar{\Lambda}(z) \cap \Lambda \). We obtain exponential decay as in (2.56).

- When \( A \) is bad, we resort to Lemma 2.2. From (2.60), we need (2.16) to hold for at most

\[
\mathcal{O}_{d, \nu}(1) \frac{N}{N} \cdot N^{d+\nu} \leq \mathcal{O}_{d, \nu}(1) N^{d+\nu+1} \tag{2.62}
\]

number of annuli. Combining (2.16) with (2.62) we obtain the estimate in measure in (2.40).

Using Proposition 2.5 and Lemma 2.6, we obtain the main estimate of this section by induction:

**Proposition 2.7.** There exist \( \gamma > 0, 0 < \sigma < 1/2, N_0 \in \mathbb{N}, X \subset \mathbb{R}^d \),

\[
\text{mes } X \geq 1 - N_0^{-1} \tag{2.63}
\]

\[
\Omega_x \subset (0, 1]^\nu, \text{mes } \Omega_x \geq 1 - e^{-N_0^\sigma/2}, \tag{2.64}
\]

such that for \( x \in X, \omega \in \Omega_x \cap DC_{A,c}, \)

\[
0 < \delta \ll 1, \delta_j \text{ satisfying } (2.3), \text{ for all } N \geq N_0,
\]

\[
\Lambda = [-N, N]^{d+\nu} + i, \text{ all } i \in [-2N, 2N]^d,
\]

\[
\sup_{x \in X, E} \text{mes } (B_x^{\nu, \sigma}(\Lambda, E)) \leq e^{-N_0^\sigma/2}. \tag{2.65}
\]

**Proof.** In view of the proof of Lemma 2.1, in particular (2.22) and the proof of Lemma 2.6:

\[
\text{mes } (\mathbb{R}^d \setminus X) \leq \sum_{L_i = N_0}^{\infty} \sum_{L'_i \leq L_{i+1} \leq L'^{2c}_i} L_i^{-2p'} (2L_i + 1)^{2d} (6L_{i+1} + 1)^d \tag{2.66}
\]
where the factor 6 in the last factor of (2.66) comes from the fact that at each scale \( L \), we need estimates for all cubes \([-L, L]^d + i, i \in [-2L, 2L]^d \), see (2.43).

(2.63) is satisfied if

\[
\bar{N}_0^{2p'/\alpha} N_0^{2d} N_0^{2C} N_0^{2Cd} \leq N_0^{-2},
\]

which leads to

\[
p' > \alpha (C + 1)(d + 1).
\]

In view of (2.67)

\[
p' > 42(d + \nu)d.
\]

(2.64) is verified by removing a set \( \Omega_L \) at each scale \( L \) similar to Lemma 2.3. Because of the decay property of \( \delta_j \) in (2.3), the RHS of (2.26, 2.27) are replaced by \( \mathcal{O}(1)e^{-L'} \) (\( \mathcal{O}(1) \) is the same for all scales), where \( L' \) is the previous scale, see proof of Lemma 2.1. Summing over the scales, we obtain (2.64).

3. **Exponential Decay of the Green’s function of Schrödinger operator at fixed \( E \) and \( \theta \)**

This section is in some sense a minor image of the previous section. Here we study the operator \( H(\theta) \) defined in (2.5) for fixed \( \theta \), but we are allowed to “move” the random variables \( x \in \mathbb{R}^d \). Without loss of generality, we set \( \theta = 0 \) and study

\[
H = \delta_j \Delta_n + n \cdot \omega + \epsilon \Delta_j + V_j
\]
on \( \ell^2(\mathbb{Z}^{d+\nu}) \). We first prove the analogue of Lemma 2.2.

**Lemma 3.1.** Let \( E \in I \), an interval of length \( \mathcal{O}(1) \). Let \( \Lambda \subset \mathbb{Z}^{d+\nu} \) be a finite set. Then

\[
\text{mes} \{ x | \text{dist} (E, H_{\Lambda}(x)) \leq \kappa \} \leq C \kappa |\Lambda| \| \tilde{g} \|_{\infty}.
\]

**Proof.** Let \( N(E, x) \) be the \# of eigenvalues of \( H_{\Lambda} \leq \lambda \)

\[
\text{mes} \{ x | \text{dist} (E, H_{\Lambda}(x)) \leq \kappa \}
\]

\[
\leq \int (N(E + \kappa, x) - N(E - \kappa, x)) \prod_{i \in \Lambda_j} \tilde{g}(v_i) dv_i
\]

\[
= \int_{E-\kappa}^{E+\kappa} \frac{d}{d\lambda} N(\lambda, x) d\lambda \prod_{i \in \Lambda_j} \tilde{g}(v_i) dv_i
\]

\[
= \int_{E-\kappa}^{E+\kappa} \sum_{i \in \Lambda_j} \frac{\partial}{\partial v_i} N(\Lambda, x) \prod_{i \in \Lambda_j} \tilde{g}(v_i) dv_i
\]

\[
\leq C \kappa |\Lambda| \| \tilde{g} \|_{\infty}
\]
where $\Lambda_j = \Lambda \cap \mathbb{Z}^d$ and each $\partial/\partial v_i$ is seen as a rank $|\Lambda \cap (\mathbb{Z}^\nu + i)|$ perturbation as usual.

We define the good and bad sets in analogy with (2.38):

$$G_\theta^{\gamma,\sigma}(\Lambda, E) \overset{\text{def}}{=} \{ x \in \mathbb{R}^{d} \ | \ |G_\Lambda(x, E)| < e^{N\sigma}, \ G_\Lambda(x, E)(m, m') < e^{-\gamma|m-m'|} \ \forall m, m' \in \Lambda, |m - m'| > N/4 \}$$

$$B_\theta^{\gamma,\sigma}(\Lambda, E) \overset{\text{def}}{=} \mathbb{R}^{d} \setminus G_\theta^{\gamma,\sigma}(\Lambda, E)$$

where $\gamma > 0, 0 < \sigma < 1, \Lambda = [-N, N]^{d+\nu} + i$ for some $i \in \mathbb{Z}^{d+\nu}$.

**Lemma 3.2.** There exist $\gamma > 0, 0 < \sigma < 1, 0 < \epsilon_0 < 1, N_0(\sigma), p > 0$, such that for all $0 < \epsilon < \epsilon_0, N \geq N_0$, there exists $0 < \delta(N) \ll 1$, such that

$$\sup_{\theta, E} \text{mes}(B_\theta^{\gamma,\sigma}(\Lambda, E)) \leq N^{-p}.$$  \hspace{1cm} (3.5)

**Proof.** We prove (3.5) by perturbation. Let

$$H_{\Lambda, 0} \overset{\text{def}}{=} n \cdot \omega + \epsilon \Delta_j + V_j$$

on $\ell^2(\Lambda)$ with Dirichlet boundary conditions as defined in (2.4),

$$H_j \overset{\text{def}}{=} \epsilon \Delta_j + V_j$$

on $\ell^2(\Lambda \cap \mathbb{Z}^d)$.

For $\epsilon$ sufficiently small, $H_j$ has Anderson localization, (cf. appx.) So for any fixed $\lambda$ starting at some scale $N'$, with log $N' < \log N$, using Theorem A and Lemma 3.1,

$$|(H_j - \lambda)^{-1}(j, j')| < e^{-\gamma|j-j'|}$$

for all $j, j' \in \Lambda \cap \mathbb{Z}^d, |j - j'| > N/4$

with probability $\geq 1 - O_d(1)N^{2d/N^{12p'}} - e^{-N^{\sigma}}, 0 < \sigma < 1, p' > 0$ to be specified, $N' = N^{1/\alpha}(1 < \alpha < 2)$ and where we used Lemma 3.1.

Let

$$\lambda = E + n \cdot \omega, \ n \in \Lambda \cap \mathbb{Z}^\nu$$

and summing over the probability, we obtain that

$$|(H_{\Lambda, 0} - E)^{-1}(j, j')| < e^{-\gamma|j-j'|}$$

(3.10)
for all \( j, j' \in \Lambda \cap \mathbb{Z}^d \), \(|j - j'| > N/4\) with probability
\[
\geq 1 - \mathcal{O}_d(1) \frac{N^{2d}}{N^{2p'}} - \mathcal{O}_d(1) N^{d} e^{-N^\sigma}.
\]

We remark here that the set defined by (3.10), \( X_{N'} \subset X_{N_0}, X_{N_0} \) as in (2.14), \( \text{mes } X_{N'} \leq \text{mes } X_{N_0} \), because of the need for Lemma 3.1, as we do not have \( \theta \) at our disposal.

To obtain exponential decay of \( G_{\Lambda} \), we use the resolvent expansion:
\[
(H_{\Lambda} - E)^{-1}(n, j, n', j') = (H_{\Lambda, 0} - E)^{-1}(n, j; n', j')
\]
\[
+ \left( \sum_{k=1}^{K} [(H_{\Lambda, 0} - E)^{-1} \delta_j \Delta_n]^k (H_{\Lambda, 0} - E)^{-1} \right) (n, j; n', j')
\]
\[
+ \left( [(H_{\Lambda, 0} - E)^{-1} \delta_j \Delta_n]^{K+1} (H_{\Lambda} - E)^{-1} \right) (n, j, n', j')
\]
(3.11)
where \( K = \mathcal{O}(n - n') \). Using (3.10, 3.2, 2.3), we arrive at (3.5) for
\[
\delta = \delta(N) \ll 1, N' = [N^{1/\alpha}], 1 < \alpha < 2, p = \frac{2p'}{\alpha} - 2d
\] (assuming \( p' \) large enough), \( N \) large enough and a \( \gamma \) which is slightly smaller than that in (3.10).

Lemma 3.2 is the analogue of Proposition 2.5. We now prove

**Lemma 3.3.** Suppose \( \delta \) is such that Lemma 3.2 holds for all \( N \in [N_0, N_0^C] \) with \( C > 10(d + \nu) \). Let \( N_1 = N_0^C \). Then for all \( N \in [N_1, N_1^2] \), \( \Lambda = [-N, N]^{d+\nu} + i, i \in \mathbb{Z}^{d+\nu} \), if \( \omega \in DC_{A, c}(2N) \) then
\[
\sup_{E, \theta} \text{mes } (B_{\gamma', \sigma}^\gamma(\Lambda, E)) \leq N^{-p}
\] (3.13)
where \( \gamma' = \gamma - N^{-\nu}, \kappa = \kappa(\sigma, \gamma) > 0, p \) is the same as in Lemma 3.2, (3.12).

**Proof.** This is almost a mirror image of the proof of Lemma 2.6. Hence we will only point out the small differences.

Fix \( N \in [N_1, N_1^2] \) and let
\[
\Lambda = [-N, N]^{d+\nu}
\]
\[
T(i) = \{[-N_0, N_0]^2 + i\} \times [-N, N]^\nu \quad (i \in \Lambda \cap \mathbb{Z}^d)
\]
\[
\Lambda_0 = [-N_0, N_0]^{d+\nu}
\]
\[
\Lambda_0(i) = \Lambda_0 + i \quad (i \in \Lambda).
\] (3.14)
• For a given $T(i), i \in \Lambda \cap \mathbb{Z}^d$

$$\text{Prob}\{\Lambda_0(i + j) \text{ are good, } \forall j \in [-N, N]^\nu\} \geq 1 - \mathcal{O}_\nu(1) \frac{N^\nu}{N_0^p} \quad (3.15)$$

by using Lemma 3.2.

• For any $x \in X'_{N_0} \subset X_{N_0}$, for a given $T(i)$, from semi-algebraic considerations as in (2.43-2.48) for $\omega \in DC_{A,c}(2N)$

$$\#\{i' \in [-N, N]^{d+\nu}\: | \: \Lambda_0(i') \cap T(i) \neq \emptyset, \Lambda_0(i') \text{ is a bad } N_0\text{-box}\} \leq \mathcal{O}_{d,\nu}(1)N_0^{5(d+\nu)} \quad (3.16)$$

as in (2.48).

For a given

$$T(i) = \{[-N_0, N_0]^d + i\} \times [-N, N]^\nu,$$

$i \in \Lambda \cap \mathbb{Z}^d$, we say $T(i)$ is bad if the condition in (3.15) is violated, i.e., if $\exists j \in [-N, N]^\nu$, such that $\Lambda_0(i + j)$ is a bad $N_0\text{-box}$. Let $J$ be an even integer. From (3.15)

$$\text{Prob}\{\text{there are at least } J \text{ pairwise disjoint bad } T(i) \text{ in } \Lambda\} \leq \left(\mathcal{O}_\nu(1) \frac{N^\nu}{N_0^p}\right)^2 \cdot \mathcal{O}_d(1)N^{2dJ/2} = \mathcal{O}_{d,\nu}(1) N_0^{5(d+\nu)J} \quad (3.17)$$

assuming

$$p > 2C(d + \nu) > 20(d + \nu)^2. \quad (3.18)$$

For a given $C$, let $p$ be such that $p/2C - d - \nu = 1$ and choose $J = p+1$ or $J = p+2$, so that $J$ is even. So

$$(3.17) \leq \mathcal{O}_{d,\nu}(1)N^{-(p+1)}. \quad (3.19)$$

• Using (3.16, 3.19, 3.2), introducing another scale $N = N_0^C, C \in (10(d + \nu), C)$, we obtain Lemma 3.2 just like the way we obtained Lemma 2.6.

Iterating using Lemmas 3.2, 3.3, we arrive at the main estimate of this section.

**Proposition 3.4.** There exist $\gamma > 0, 0 < \sigma < 1/2, p > 0$, such that for $\omega \in DC_{A,c}, 0 < \epsilon \ll 1, 0 < \delta \ll 1, \delta_j$ satisfying (2.3), there exists $N_0(\epsilon, \delta)$, such that for all $N \geq N_0, \Lambda \subset \mathbb{Z}^{d+\nu}$, cubes of side length $2N + 1$

$$\sup_{\theta, E} \text{mes}\left(B_{\theta,\epsilon}^{\gamma,\sigma}(\Lambda, E)\right) \leq N^{-p}. \quad (3.20)$$
4. The elimination of $E$ and frequency estimates

The goal of this section is to transform the large deviation estimates in $\theta$ in (2.63) into estimates in $\omega$ and to eliminate the dependence of the singular set on $E$ in the process. (Recall that $\theta$ is an auxiliary variable that was not in the original problem (2.2).) This is needed to prove Anderson localization. We need two scales $N, \bar{N}$, $\log \log \bar{N} \ll \log N$. Let $\Lambda_N(i) = [-N, N]^{d+\nu} + i, i \in \mathbb{Z}^d, \Lambda_{\bar{N}} = [-\bar{N}, \bar{N}]^{d+\nu}, \Omega_N(i)$ the set of $\omega$ as defined in Lemma 2.3 for the box $\Lambda_N^{(i)}$ and $DC_{A,c}(N)$, the set of $\omega$ satisfying (H2) for $n \in [-N, N]^{\nu}$.

For a fixed $x \in X', X' \subset X$ as in (2.63), define $B_x(N, \bar{N}) \subset (0, 1]^{\nu} \times \mathbb{R}$ as

$$B_x(N, \bar{N}) = \bigcup_{\{i \in \mathbb{Z}^d | \Lambda_N(i) \cap \Lambda_{\bar{N}} \neq \emptyset\}} \{(\omega, \theta) \in (0, 1]^{\nu} \times \mathbb{R} | \exists E, \text{ such that } \| (H_{\bar{N}}(\omega, 0) - E)^{-1} \| \geq e^{\bar{C}N} \text{ and } \theta \in B_{\gamma, \sigma}^x(\Lambda_N(i), E)\} \quad (4.1)$$

and $S_x(N) \subset (0, 1]^{\nu} \times \mathbb{R}$ as

$$S_x(N) = \bigcup_{N \asymp N^C} \left\{ B_x(N, \bar{N}) \cap \left( \bigcap_{\{i \in \mathbb{Z}^d | \Lambda_N(i) \cap \Lambda_{\bar{N}} \neq \emptyset\}} \Omega_N(i) \times \mathbb{R} \right) \right\} \bigcap \left( DC_{A,c}(\bar{N} + N) \times \mathbb{R} \right) \quad (4.2)$$

where $\Omega_N(i)$ is the set of $\omega$ defined as in Lemma 2.3 for the box $[-N, N]^{d+\nu} + i$.

In view of (4.1, 4.2), at each scale $N$, we need estimates for all cubes $[-N, N]^{d+\nu}$, where $\bar{N} \asymp N^C$. Similar considerations as in (2.67-2.69) lead to $p' > 422d(d + \nu)^2$

$$\text{mes } X' \geq 1 - N_0^{-1} \quad (4.3)$$

$N_0$ as in Proposition 2.7.

Likewise $\Omega_x$ of Proposition 2.7 is reduced to $\Omega'_x \subset \Omega_x$ accordingly,

$$\text{mes } \Omega'_x \geq 1 - e^{-\frac{1}{2}N_0^{\nu/2}} \quad (4.4)$$

if $N_0 \gg 1$. For simplicity, we now drop the prime:

$$X \overset{\text{def}}{=} X', \Omega_x \overset{\text{def}}{=} \Omega'_x.$$

**Lemma 4.1.** Let $N, \bar{N} \in \mathbb{N}$ be such that $\bar{N} \asymp N^C (C > 1)$. For any $x \in X$,

$$\text{mes } S_x(N) < e^{-\frac{1}{2}N_0^{\nu/2}} \quad 0 < \sigma < 1/2; \quad (4.5)$$
Moreover for any $\theta \in \mathbb{R}$, the section

$$S_{x,\theta}(N) = \{\omega \in (0, 1)^\nu | (\omega, \theta) \in S_x(N)\}$$

is a union of at most $N^{7C(d+\nu)}$ components.

**Proof.** For a given $\bar{N}$, fix

$$\omega \in \bigcap_{i \in \mathbb{Z}^d \setminus \Lambda_N(i) \cap \Lambda_N \neq \emptyset} \Omega_N(i) \cap DC_{A,c}(\bar{N} + N). \quad (4.6)$$

Let

$$\lambda \in \sigma(H_{\bar{N}}(\omega, 0)).$$

Then

$$\text{mes} \left\{ \bigcup_{i \in \mathbb{Z}^d \setminus \Lambda_N(i) \cap \Lambda_N \neq \emptyset} \bigcup_{\lambda \in \sigma(H_{\bar{N}}(\omega, 0))} B_x^{\gamma,\sigma}(\Lambda_N(i), \lambda) \right\}$$

$$\leq O_{d,\nu}(1) \bar{N}^d \cdot \bar{N}^d + \nu \cdot e^{-N^{\sigma/2}}$$

$$\leq O_{d,\nu}(1) e^{-\frac{1}{4} N^{\sigma/2}} \quad (4.7)$$

for $\bar{N} \asymp N^C$, where we used Proposition 2.7.

Let $E$ be such that

$$\|(H_{\bar{N}}(\omega, 0) - E)^{-1}\| \geq e^{\bar{C}N},$$

then $\exists \lambda \in \sigma(H_{\bar{N}}(\omega, 0))$ such that

$$|E - \lambda| \leq e^{-\bar{C}N}. \quad (4.8)$$

Using the resolvent equation, we have

$$(H_{\Lambda_N(i)} - \lambda)^{-1} = (H_{\Lambda_N(i)} - E)^{-1} + (E - \lambda)(H_{\Lambda_N(i)} - \lambda)^{-1}(H_{\Lambda_N(i)} - E)^{-1}. \quad (4.9)$$

(4.9) and Lemma 2.2 then imply that

$$\mathcal{G}_{x}^{\gamma,\sigma}(E) \supset \mathcal{G}_{x}^{\gamma',\sigma'}(\lambda) \quad (4.10)$$

with $0 < \gamma' < \gamma$, $0 < \sigma' < \sigma$. Using (4.10) and taking the union over $\bar{N} \asymp N^C$, we obtain (4.5) with $\gamma, \sigma$ slightly smaller than that in (4.7).

To prove the second statement, we need to bound the degree of $S_{x,\theta}(N)$ for a fixed $\theta$. The conditions in (4.2) can be reexpressed in polynomial inequalities by using Hilbert Schmidt norm and Cramer’s rule as before. The sets $\Omega_N(i)$ are defined by polynomial
(monomial) inequalities as in (2.28, 2.33), similarly for $DC_{A,c}(\tilde{N} + N)$. So $S_\ell(N)$ is semi-algebraic.

Using a special case of Theroem 1 in [Ba] as stated in Theorem 7.3 of [BGS], we obtain that for any fixed $\theta$, $S_{x,\theta}(N)$ is the union of at most

$$O_d,\nu(1)\{N^2(d+\nu)(N^2(d+\nu) + N^2(d+\nu) + N^2(d+\nu)) \cdot N^d \cdot N^c \cdot \tilde{N}^{d+\nu}\}^\nu$$

connected components. □

We need one more lemma, before transferring the estimate in $(\omega, \theta)$ in (4.5) into an estimate in $\omega$ only.

**Lemma 4.2.** Let

$$S(N) \subset ((0, 1]^{\nu} \cap DC_{A,c}(N)) \times \mathbb{R}$$

be a set with the properties:

- For each $\theta \in \mathbb{R}$, the section

  $$S_\theta = \{\omega \in ((0, 1]^{\nu} \cap DC_{A,c}(N)) | (\omega, \theta) \in S\}$$

  is a union of at most $M$ components

- $\text{supp} S \subset (0, 1]^{\nu} \times [-N_0, N_0]$.

Let $K \gg MN_0$. Then

$$\text{mes}\{\omega \in ((0, 1]^{\nu} \cap DC_{A,c}(2K)) | (\omega, \ell \cdot \omega) \in S(N) \text{ for some } \ell, |\ell| \sim K\}$$

$$\leq O(1)(K^\nu MN_0(\text{mes} S(N))^{1/3} + MN_0K^{-1}). \quad (4.11)$$

**Proof.** We use a similar strategy as in the proof of Lemma 6.1 in [BG].

$$\text{mes}\{\omega \in ((0, 1]^{\nu} \cap DC_{A,c}(2K)) | (\omega, \ell \cdot \omega) \in S(N) \text{ for some } \ell, |\ell| \sim K\}$$

$$\leq \sum_{\ell,|\ell|\sim K} \int_{(0,1]^{\nu}} \chi_{S(N)}(\omega, \ell \cdot \omega) d\omega. \quad (4.12)$$

Let

$$\ell \cdot \omega \overset{\text{def}}{=} \theta = |\ell|\omega_\ell, \quad (4.13)$$

where $\omega_\ell$ is the projection of $\omega$ in the $\ell$ direction. Let $\omega_\ell^\perp$ be the orthogonal component. So

$$(4.12) \leq \sum_{\ell,|\ell|\sim K} \frac{1}{|\ell|} \int_{-N_0}^{N_0} d\theta \int d\omega_\ell^\perp \chi_{S(N)}\left(\left(\frac{\theta}{|\ell|}, \omega_\ell^\perp\right), \theta\right). \quad (4.14)$$
Fix $\theta$ and bound
\[ \#\{\ell \mid |\ell| \lesssim K \text{ and } \omega = \left( \frac{\theta}{|\ell|}, \omega_{\ell'} \right) \in S_\theta(N) \}. \]

We distinguish two cases:

- $|S_\theta| > \gamma$ (4.15)
- $|S_\theta| \leq \gamma$ (4.16)

where $0 < \gamma \ll 1$ is to be specified.

- If $|S_\theta| > \gamma$, then the contribution to (4.14) is bounded by
  \[ K^{\nu - 1} \{ \text{mes } \theta \in [-N_0, N_0] \mid |S_\theta| > \gamma \} \leq O(1)K^{\nu - 1}\gamma^{-1}|S|. \] (4.17)

- Assume (4.16) and $|\theta| < aK^{-A}$ ($a > 0$ to be specified), the contribution to (4.14) is bounded by
  \[ O(1)K^{\nu - A - 1}a. \] (4.18)

- Assume (4.16), $|\theta| \geq aK^{-A}$ and moreover
  \[ \exists \ell, \ell', \ell \neq \ell', \text{ such that } \exists \omega, \omega' \text{ in the same component of } S_\theta \] (4.19)

Since $\theta$ is fixed
\[ |\ell \cdot \omega - \ell' \cdot \omega'| = 0 \] (4.20)
\[ \ell \cdot \omega - \ell' \cdot \omega' = (\ell - \ell') \cdot \omega + \ell' \cdot (\omega - \omega') \]
\[ |(\ell - \ell') \cdot \omega| \geq \frac{a}{(2K)^A} \]

for $\omega \in DC_{A,a}(2K)$. So (4.19) implies
\[ |\ell' \cdot (\omega - \omega')| \geq \frac{a}{(2K)}. \]

Hence
\[ |\omega_\ell - \omega_{\ell'}| \geq \frac{a}{(2K)^A+1} \] (4.21)

(4.16, 4.21) imply that the contribution of (4.19) to (4.14) is bounded by
\[ O(1)K^{\nu - 1}N_0M\gamma K^{A+1}a^{-1} = O(1)K^{\nu + A}N_0Ma^{-1}\gamma. \] (4.22)
The contribution from the negation of (4.19) is bounded by

$$O(1)K^{-1}N_0 M. \quad (4.23)$$

Summing over (4.17, 4.18, 4.22, 4.23) and taking $\gamma = |S|^{2/3}, a = |S|^{1/3}K^A$, we obtain the lemma. \qed

Combining Lemmas 4.1 and 4.2, we arrive at the conclusion of this section. Let $\bar{N} \asymp N^C$,

$$\tilde{\Omega}_{N, \bar{N}} = \bigcap_{\{i \in \mathbb{Z}^d | \Lambda_N(i) \cap \Lambda_S \neq \emptyset\}} \Omega_N(i) \quad (4.24)$$

$\Omega_N(i)$ as defined in Lemma 2.3 for the box $[-N, N]^{d+\nu} + i$ we have

**Lemma 4.3.** Let $N \in \mathbb{N}$ be sufficiently large. Fix $x \in X$. Let

$$\tilde{\Omega}_N = \tilde{\Omega}_{N, \bar{N}} \cap DC_{A,c}(\bar{N} + N)$$

be the set such that

(4.25) There is $\bar{N} \asymp N^C, \ell \in \mathbb{Z}^\nu, |\ell| \sim N^\tau (\tau > 7C(d + \nu)\nu)$ and $E$ such that

$$\| (H_{\bar{N}}(\omega, 0) - E)^{-1} \| \geq e^{CN} \quad (\bar{C} > 0) \quad (4.26)$$

and there is $i \in \mathbb{Z}^d, \Lambda_N(i) \cap \Lambda_N \neq \emptyset$. Such that

$$|(H_{\Lambda_N}(i) - E)^{-1}(\omega, \ell \cdot \omega)(m, m')| > e^{-\gamma|m - m'|} \quad (4.27)$$

for some $m, m' \in \Lambda_N(i), |m - m'| > N/4$,

$$\text{mes } \tilde{\Omega}_N \leq N^{-\tilde{q}} \quad 0 < \tilde{q} < (\tau - 7C(d + \nu)\nu - 1). \quad (4.28)$$

**Remarks.**

- $\ell$ could be taken larger, e.g. $|\ell| \sim N^{\log N}$ as in [BG, BGS]. But in view of the probability estimate for the random part, which is only polynomial coming from [vDK]. We take $|\ell| \sim N^\tau$, assuming $p \gg 1$ ($p$ as in (3.12)).

- The probability estimate for the random part can be improved to subexponential by allowing more bad boxes. But for now, we leave it as it is.
5. Proof of Anderson Localization for the Schrödinger Operator

We now prove Anderson localization, i.e., pure point spectrum with exponentially decaying eigenfunctions, for $H$ defined in (2.2).

**Theorem.** There exists $\epsilon_0$, such that $\forall 0 < \epsilon < \epsilon_0, \eta > 0$, given a bounded interval $I \subset \mathbb{R}$, $\exists \delta_0(\eta, \epsilon)$, such that $\forall 0 < \delta < \delta_0(\eta, \epsilon)$

$$\exists \tilde{X}_{\eta, \epsilon, W} \subset \mathbb{R}^{2d}, \text{mes} \tilde{X}_{\eta, \epsilon, W} \geq 1 - \eta,$$

such that

$$\forall x \in \tilde{X}_{\eta, \epsilon, W}, \exists \Omega_{x, \eta, \epsilon, W} \subset (0, 1]^\nu,$$

$$\text{mes} \Omega_{x, \eta, \epsilon, W} \geq 1 - \eta,$$

such that

$$\forall \omega \in \Omega_{x, \eta, \epsilon, W}, \delta_j \text{ satisfying (2.3)}$$

$H$ has Anderson localization in $I$.

We need the analogue of Lemma 4.3.

**Lemma 5.1.** Let $N \in \mathbb{N}$ be sufficiently large. Fix $\theta = 0$. Let $\omega \in DC_{A, c}(N)$.

There is $\bar{N} \asymp N^C, \ell \in \mathbb{Z}^{d+\nu}, |\ell| \sim N^\tau, E$ (5.1) such that

$$\| (H_{\bar{N}}(\omega, 0) - E)^{-1} \| \geq e^{\tilde{C}N} \quad (\tilde{C} > 0)$$

(5.2)

and there is $\ell \in \mathbb{Z}^{d+\nu}, |\ell| \sim N^\tau (\tau > 7C(d + \nu)\nu), \Lambda_N(\ell)$ satisfying

$$(\Lambda_N(\ell) \cap \mathbb{Z}^d) \cap (\Lambda_{\bar{N}} \cap \mathbb{Z}^d) = \emptyset$$

(5.3)

such that

$$|(H_{\Lambda_N(\ell)} - E)^{-1}(m, m')| > e^{-\gamma|m-m'|}$$

(5.4)

for some $m, m' \in \Lambda_N(\ell), |m - m'| > N/4$.

$$\text{Prob} \leq N^{-q} \quad (q > p - \tau(d + \nu) - 1)$$

(5.5)

**Proof.** (5.3) implies that $H_{\Lambda_N(\ell)}$ is independent from $H_{\bar{N}}$ and hence its eigenvalues. Summing over the probabilistic estimates in (3.5), we obtain (5.5). (The lemma holds...
as soon as (5.3) is satisfied due to independence. We take $|\ell| \sim N^\tau$ in view of Lemma 4.3).

**Proof of the theorem.**

Let $X_\epsilon \subset \mathbb{R}^d$ be the probability subspace defined in (4.4), i.e., Proposition 2.7 with the modification $i \in [-\bar{N}, \bar{N}]^d, \bar{N} \sim N^C$, (2.63) still holds in this case.

Let $\tilde{X}_N$ be the complement of the set defined in (5.1). Let

$$\tilde{X}_{N_0} = \bigcap_{N > N_0} \tilde{X}_N \bigcap \tilde{X}_\epsilon.$$

Fix $x \in \tilde{X}_{N_0}$. Let $\Omega_{N,x}$ be the complement of the set defined in (4.26). Let

$$\Omega_{x,N_0} = \bigcap_{N > N_0} \Omega_{N,x}.$$

For any given $\eta > 0$, $\exists N_0$, such that

$$\tilde{X}_{\eta,\epsilon} \overset{\text{def}}{=} \tilde{X}_{N_0},$$

Satisfying

$$\text{mes } \tilde{X}_{\eta,\epsilon} \geq 1 - \eta$$

and

$$\Omega_{x,\eta,\epsilon} \overset{\text{def}}{=} \Omega_{x,N_0}.$$

Satisfying

$$\text{mes } \Omega_{x,\eta,\epsilon} \geq 1 - \eta,$$

if $p' > 422(d + \nu)\nu$ (cf. A1, A2, 2.69, 3.12, 3.18, 4.25, 4.28, 5.5).

The proof uses lemmas 4.3 and 5.1 and follows the same strategy as in [BG, BGS]. So we will only highlight the main points.

- **Generalized eigenfunctions of $H$**: $H \psi = E \psi$ has the apriori bound

  $$|\psi(m)| \leq 1 + |m|^c$$

  from the Schnol-Simon Theorem [Sh, Sim].

- **Let $\psi$ be a non-zero eigenfunction of $H$**: $H \psi = E \psi$. Let $\Lambda \subset \mathbb{Z}^{d+\nu}$ and assume $E \not\in \sigma(H_{\Lambda})$. Then for all $m \in \mathbb{Z}^{d+\nu}$

  $$\psi(m) = \sum_{\{m' \in \Lambda \mid \exists m'' \in \mathbb{Z}^{d+\nu} \setminus \Lambda, |m' - m''| = 1\}} (H_{\Lambda} - E)^{-1}(m, m')\psi(m'').$$

  (5.7)
• From semi-algebraic considerations and the restriction of \( \omega \in \Omega_{x,\eta,\epsilon} \) (see Lemma 2.3), \( \exists \kappa > 0, M \simeq N (\bar{N} \simeq N^C, C > 10(d + \nu), \) cf. Lemma 2.6) such that

\[
\# \{ j \in \mathbb{Z}^{d+\nu} | \Lambda_N(j) \cap \Lambda_M \neq \emptyset, \Lambda_N(j) \text{ is a bad } N \text{-box} \} \leq M^{1-\kappa}.
\]  

(5.8)

Let \( \mathcal{I} \) be the set defined in (5.8). For \( C \) large enough, there exists an annulus

\[ A = \Lambda_L \setminus \Lambda_{L'} \subset \Lambda_M \]

of width \( L' - L > 2N \) such that

\[ A \cap \mathcal{I} = \emptyset. \]

• Without loss, assume \( \psi(0) = 1. \) Using (5.6, 5.7) first with \( \Lambda = \Lambda_N(j), j \in A \) and then \( \Lambda = \Lambda_{\bar{N}} \) we obtain

\[ \| (H_{\bar{N}} - E)^{-1} \| \geq e^{\bar{C}N} \]

\[ (\bar{C} > 0). \]  

(5.9)

• Lemma 4.3, 5.1 and an application of the resolvent equation as used earlier imply that \( \forall i \in \partial \Lambda_{2K}(0), K \sim N^\tau \)

\[ |G_{\Lambda_K(i)}(E, m, m')| \leq \exp(-|m - m'|). \]  

(5.10)

\( m, m' \in \Lambda_K(i), |m - m'| \geq K/4. \)

We note that

\[ \bigcup_{i \in \partial \Lambda_{2K}(0)} \Lambda_K(i) = \Lambda_{3K}(0) \setminus \Lambda_K(0) \overset{\text{def}}{=} U \]

(5.6, 5.7, 5.10) imply that

\[ |\psi(m)| < e^{-|m|/2} \]

for \( m \) such that \( \text{dist}(m, \partial U) \geq K/4, \) provided \( N \) and thus \( K \) are large. \( \square \)

6. Proof of Anderson localization for wave operator

The quasi-energy operator \( K_w \) in the wave case is

\[ K_w = -\sum_{k=1}^\nu \sum_{k'=1}^\nu \omega_k \omega_{k'} \frac{\partial^2}{\partial \theta_k \partial \theta_{k'}} + \epsilon \Delta + V + \sum_{k=1}^\nu W_k \cos 2\pi \theta_k \]  

(6.1)
on $\ell^2(\mathbb{Z}^d) \times L^2(\mathbb{T}^\nu)$, where $\omega = (\omega_1, \omega_2, \ldots, \omega_\nu) \in (0,1]^\nu$. $V$ is the random potential on $\mathbb{Z}^d$, $0 < \epsilon \ll 1$, and $W_k$ satisfies the decay properties specified in (H1). Compared with the quasi-energy operator $K$ for Schrödinger in (2.1), the only difference is that the $\theta$ derivatives are second order.

Performing a partial Fourier series transform in the $\mathbb{T}^\nu$ variables as in sect. 2, we are led to study the following unitarily equivalent operator:

$$H_w = \delta_j \tilde{\Delta}_n + (n \cdot \omega)^2 + \epsilon \Delta_j + V_j$$

(6.2)

on $\ell^2(\mathbb{Z}^{d+\nu})$, where $n \in \mathbb{Z}^\nu$, $j \in \mathbb{Z}^d$ and $\delta_j \tilde{\Delta}_n$ is as in (2.3). We proceed as in the Schrödinger case and introduce the parameter $\theta \in \mathbb{R}$. We define

$$H_w(\theta) = \delta_j \tilde{\Delta}_n + (n \cdot \omega + \theta)^2 + \epsilon \Delta_j + V_j$$

(6.3)

on $\ell^2(\mathbb{Z}^{d+\nu})$ and study the Green’s functions

$$G_{w,\Lambda}(\theta, E) = (H_{w,\Lambda}(\theta) - E)^{-1}$$

(6.4)

for a class of finite sets $\Lambda \subset \mathbb{Z}^{d+\nu}$ and $\Lambda \n\mathbb{Z}^{d+\nu}$ to be specified shortly.

The main difference between $H_w(\theta)$ in (6.3) and $H(\theta)$ in (2.5) is that

$$\frac{\partial H(\theta)}{\partial \theta} = 1;$$

(6.5)

while

$$\frac{\partial H_w(\theta)}{\partial \theta} = 2(n \cdot \omega + \theta)$$

(6.6)

which could be 0. So the apriori estimate in Lemma 2.2 for Schrödinger does not apply here. ($\theta$ and $E$ are no longer equivalent, see (2.17).) We need to resort to Cartan type of theorem for analytic matrix-valued functions as in [BGS]. Unlike Lemma 2.2, which holds at all scales this requires a multi-scale analysis. At each scale $N (N \gg 1)$ we need measure estimates on the bad sets at two previous scales $N_0, N_1$ with

$$\log N_0 < \log N_1 \ll \log N.$$

We extend the class of finite subsets $\Lambda$ of $\mathbb{Z}^{d+\nu}$, which were previously cubes to elementary regions: (as in [BGS])

$$\Lambda \overset{\text{def}}{=} R \setminus (R + m)$$

(6.7)

where $m \in \mathbb{Z}^{d+\nu}$ is arbitrary and $R$ is a rectangle

$$R = \times_{i=1}^{d+\nu} [-M_i, M_i] + k,$$

(6.8)
$k \in \mathbb{Z}^{d+\nu}$. The size of $\Lambda$, denoted by $\ell(\Lambda)$, is simply its diameter. We denote by $\mathcal{ER}(M)$, the set of all elementary regions of size $M$.

Let

$$\sigma \in (0, 1), N = \lceil N_0^C \rceil + 1 \quad (C > 1) \quad (6.9)$$

$N_0 \gg 1$ (determined by $\delta, \sigma$ similar to (2.11), $\delta$ as in (2.31)),

$$\Lambda \subset [-N, N]^{d+\nu}, \Lambda \in \mathcal{ER}(N).$$

Let $\Lambda, \Lambda'$ be two elementary regions. Let $\tilde{\Lambda}, \tilde{\Lambda}'$ be their respective convex envelop. We say that $\Lambda$ and $\Lambda'$ are disjoint if $\tilde{\Lambda} \cap \tilde{\Lambda}' = \emptyset$. Following is the analogue of Lemma 2.3.

**Lemma 6.1.** Fix $x \in X_N, X_N$ as in Sect. 2. There exists a set $\Omega_N \subset (0, 1)^\nu$, $\text{mes } \Omega_N \geq 1 - e^{-N^2\sigma}$

such that if $\omega \in \Omega_N$, then for any fixed $\theta, E$, there are at most two pair-wise disjoint bad $\Lambda_0$'s,

$$\Lambda_0 \in \bigcup_{N_0 \leq M \leq 2N_0} \mathcal{ER}(M)$$

in $\Lambda$. Moreover, $(0, 1)^\nu \setminus \Omega_N$ is contained in the union of at most $N^{5(d+\nu)}\nu$ components.

**Proof.** We follow the same line of argument as in the proof of Lemma 2.3. Let

$$\Lambda_0, \Lambda'_0, \Lambda''_0 \in \bigcup_{N_0 \leq M \leq 2N_0} \mathcal{ER}(M)$$

$\Lambda_0, \Lambda'_0, \Lambda''_0 \subset \Lambda$, be pair-wise disjoint.

Let

$$\Lambda_{0,j} \text{ be the projection of } \Lambda_0 \text{ onto } \mathbb{Z}^d.$$ 

$$\Lambda_{0,n} \text{ be the projection of } \Lambda_0 \text{ onto } \mathbb{Z}^\nu$$

and similarly for $\Lambda'_0, \Lambda''_0$.

Assume $\Lambda_0, \Lambda'_0$ and $\Lambda''_0$ are all bad, then there exist

$$\begin{align*}
n \in \Lambda_{0,n} & \quad n' \in \Lambda'_{0,n} & \quad n'' \in \Lambda''_{0,n} \quad \text{such that} \\
j \in \Lambda_{0,j} & \quad j' \in \Lambda'_{0,j} & \quad j'' \in \Lambda''_{0,j} \\
 \cdot \omega + \theta + \mu_j - E & \leq 2e^{-N_0^a} \\
n' \cdot \omega + \theta + \mu_j' - E & \leq 2e^{-N_0^a} \\
n'' \cdot \omega + \theta + \mu_j'' - E & \leq 2e^{N_0^a} \quad (6.10)
\end{align*}$$
Assume $n, n', n''$ are distinct, otherwise say $n = n'$, then the first two inequalities in (6.10) imply that
\[(\mu_j - \mu_j') < 4e^{-N_0^\sigma}\]
(6.11)
where $\mu_j$ is an eigenvalue for $\Lambda_{0,j}$ and $\mu_j', \Lambda_{0,j}'$. Since $\Lambda_{0,j} \cap \Lambda_{0,j}' = \emptyset$, Theorem A, in particular (A3) implies that $|\mu_j - \mu_j'| \geq e^{-N_0^\beta} \gg e^{-N_0^\sigma}$ by choosing $\sigma > \beta$ as in the proof of Lemma 2.3.

Subtracting the inequalities in (6.10) pairwise, we get two inequalities with linear dependence on $\theta$ and independent of $E$. Eliminating the dependence on $\theta$, we obtain the following:
\[|\langle n - n' \rangle \cdot \omega(n - n'') \cdot \omega(n' - n'') \cdot \omega + (n - n'') \cdot \omega(\mu_j - \mu_j') - (n - n') \cdot \omega(\mu_j - \mu_j')| \leq 4e^{-N_0^\sigma} (|(n - n') \cdot \omega| + |(n - n'') \cdot \omega|).\]
(6.12)
Since $\omega \in (0, 1), n - n', n - n'' \in [-2N, 2N]$, RHS of (6.12) \[\leq 16N\nu E e^{-N_0^\sigma} \leq e^{-\frac{N_0^\sigma}{2}}\]
(6.13)
for appropriate $\sigma, C$. (Recall $N = [N_0^C] + 1$).

Let
\[\begin{cases}
  m = n - n', m' = n - n'' \\
  \lambda = \mu_j - \mu_j', \lambda' = \mu_j - \mu_j'.
\end{cases}\]
(6.14)
The solutions to (6.12) is contained in the solutions to
\[|\langle m \cdot \omega \rangle (m' \cdot \omega) (m - m') \cdot \omega + (\lambda m' - \lambda' m) \cdot \omega| \leq e^{-\frac{N_0^\sigma}{2}} \quad (m \neq m', m \neq 0, m' \neq 0).\]
(6.15)
Assume $\omega$ is a solution to
\[\langle m \cdot \omega \rangle (m' \cdot \omega) (m - m') \cdot \omega + (\lambda m' - \lambda' m) \cdot \omega = 0 \quad (m \neq m', m \neq 0, m' \neq 0)\]
(6.16)
Let $\omega \to \omega + \delta(|\delta| \ll 1)$. It is easy to see that we can always choose $\delta$ so that the third order variation
\[|\langle m \cdot \delta \rangle (m' \cdot \delta) (m - m') \cdot \delta| > c'|\delta|^3\]
(6.17)
for all $m \neq 0, m' \neq 0, m' \neq m'$, where $c' > 0$ is independent of $m, m'$. We obtain
\[\text{mes} \{\omega \in (0, 1)\}'|(6.15) \text{ is satisfied}\}
\[\leq Ce^{-\frac{N_0^\sigma}{3}}\]
(6.18)
where $C$ only depends on $\nu$.

There are at most $N^{2d+5\nu}$ equations of the form (6.15). Let

$$\Omega_N = \{\omega \in (0,1]^\nu | (6.15) \text{ is satisfied } \forall m, m', \lambda, \lambda'\}$$

We obtain the lemma by using (6.18) and Basu’s theorem stated as Theorem 7.3 in [BGS].

Assume $N \gg 1$, $\Lambda \in \mathcal{ER}(N)$, $\Lambda \subset [-N,N]^{d+\nu}$. Let $X \subset \mathbb{R}^{2d}$ be defined similarly as in Lemma 2.6. For any $x \in X$, define $\Omega_x$ similarly to that in Lemma 2.6. Combining Lemma 6.1 and semi-algebraic considerations, we have as in Lemma 2.6 that there exists $\delta_0 > 0$, such that for any fixed $\theta, x \in X, \omega \in \Omega_x \cap \text{DC}_{A,c}(2N)$

$$\# \{m \in [-N,N]^{d+\nu} | \exists \Lambda_0 \in \mathcal{ER}(M), N_0 \leq M \leq 2N_0, \Lambda_0 \subset m + [-M,M]^{d+\nu}, \Lambda_0 \text{ is bad}\} \leq N^{1-\delta_0}.$$  \hspace{1cm} (6.19)

where $N_0 = \lfloor N^{1/C} \rfloor$ for appropriate $C$ depending on $\delta_0$ only.

Recall the definition of good and bad regions for fixed $\theta$. $\Lambda_0$ of size $\ell(\Lambda_0)$ is good if

$$\|G_{\Lambda_0}(\theta, E)\| < e^{N_0^\sigma} \quad |G_{\Lambda_0}(\theta, E)(m, m')| < e^{-\gamma|m-m'|}$$

for all $m, m' \in \Lambda_0$, $|m - m'| > \ell(\Lambda_0)/4$ where $\sigma > 0, \gamma > 0$. Otherwise it is bad. As in (2.38), $G^{\gamma,\sigma}_x(\Lambda_0, E)$ is the set of $\theta \in \mathbb{R}$ such that (6.20) holds and $B^{\gamma,\sigma}_x(\Lambda_0, E)$ is the complement set.

The following lemma plays the role of Lemma 2.2 for $H_w$.

**Lemma 6.2.** Assume $\varepsilon, \delta \ll 1$ and (2.3). There exist $\sigma, \rho, \gamma > 0$ satisfying $0 < \sigma, \rho < 1, \sigma + \delta_0 > 1 + 3\rho$, where $\delta_0$ is as in (6.19), and $C_1(\sigma, \rho) \gg \frac{1}{\rho}$, $N_0 \leq N_1$, satisfying

$$\bar{N}_0(\gamma, \sigma, \rho) \leq 100N_0 \leq N_1^\rho,$$  \hspace{1cm} (6.21)

such that for any $N_0 \leq M \leq N$, and any $\Lambda \in \mathcal{ER}(M)$

$$\sup_{x \in X, E} \text{mes} \left( B^{\gamma,\sigma}_\theta(\Lambda, E) \right) \leq \exp(-\ell(n)^\rho).$$  \hspace{1cm} (6.22)

Assume moreover that

$$\omega \in \Omega_x \cap \text{DC}_{A,c}(2N_1^{\rho C_1}).$$

Then for all $\Lambda \in \mathcal{ER}(N)$

$$\sup_{x \in X, E} \text{mes} \{ \theta \in \mathbb{R} | \|G_{\Lambda}(\theta, E)\| > e^{N^b} \} < e^{-N^{3\rho}}$$  \hspace{1cm} (6.23)
where \( N_0^{C_1} \leq N \leq N_1^{pC_1} \).

The proof of the above lemma is very similar to the proof of Lemma 4.4 of [BGS], (see also Chap XIV of [Bo] for a more detailed exposition). So instead of replicating the proof, we only sketch the main line of arguments.

- As mentioned earlier in (6.5, 6.6), contrary to the Schrödinger case, the first order variation can vanish. So we need to resort to analytic and subharmonic function theory to control the measure of \( B_1^{\sigma}(\Lambda, E) \). To do that we need 2 scales \( M, M_1 \), with \( \log M < \log M_1 \ll \log N \).

- Fix \( \theta \), at scale \( M \) \( (N_0 \leq M \leq 2N_0) \), let \( \Lambda_* \) be, roughly speaking, the complement of the set in (6.19). For more precise definition, which requires a partition of \( \Lambda \), see the beginning of the proof of Lemma 4.4 in [BGS]. Using an elementary resolvent expansion (Lemma 2.2 of [BGS]), we obtain an upper bound on \( \|G_{\Lambda}(E, \theta)\| \) by using the decay estimate on the \( \Lambda_0 \)’s, elementary regions at scale \( M \), in \( \Lambda_* \). By definition they are all good. By standard Neumann series arguments, this bound is preserved inside the disk \( B(\theta, e^{-N_0}) \subset \mathbb{C} \).

**Remark.** We have control over the size of \( \Lambda_* \) via (6.19), but not its geometry. Typically \( \Lambda_* \) is non-convex. Hence the need for elementary regions which are more general than cubes, in particular L-shaped regions, in view of Lemma 2.2 of [BGS].

- Define a matrix-valued analytic function \( A(\theta') \) on \( B(\theta, e^{-N_0}) \) as

\[
A(\theta') = R_{\Lambda^c} H_w(\theta') R_{\Lambda^c} - R_{\Lambda^c} H_w(\theta') R_{\Lambda} G_{\Lambda}(E, \theta') R_{\Lambda} H_w(\theta') R_{\Lambda^c}
\]

(6.24)

where \( \Lambda^c = \Lambda \setminus \Lambda_* \), \( R_{\Lambda} \), \( R_{\Lambda^c} \) are projections. From (6.19), \( A(\theta') \) is a rank \( O(N^{1-\delta_0}) \times O(N^{1-\delta_0}) \) matrix. The raison d’être of introducing \( A(\theta') \) is the following inequality:

\[
\|A(\theta')^{-1}\| \lesssim \|G_{\Lambda}(\theta', E)\| \lesssim e^{2N_0}\|A(\theta')^{-1}\|,
\]

(see Lemma 4.8 of [BGS]). So to bound \( \|G_{\Lambda}(\theta', E)\| \), it is sufficient to bound \( \|A(\theta')^{-1}\| \), which is of smaller dimension.

- Toward that end, we introduce an intermediate scale \( M_1 \), \( \log M_1 > \log M \). We work in an interval \( \Theta = \{\theta'|\theta' - \theta| < e^{-N_0}\} \). Using (6.21) for the \( \Lambda_1 \)’s at scale \( M_1 \) and in \( \Lambda \), the same elementary resolvent expansion, we obtain a bound on \( \|G_{\Lambda}(\theta', E)\| \) except for a set of \( \theta' \) of measure smaller than \( e^{-O(N_1^{p})} \). So there exists \( y \in \Theta \), such that we have both a lower bound on \( \|A(\theta')\| \) at \( \theta' = y \), and an apriori upper bound on \( \|A(\theta')\| \) in the disk \( B(y, e^{-N_0}/2) \), which comes from boundedness of \( H_w \) and the bound on \( \|G_{\Lambda}(E, \theta)\| \) (see (6.24)).

- Transfering the estimates on \( \|A(\theta')\| \) into estimates on \( \log \left| \det A(\theta') \right| \), which is subharmonic and using either Cartan type of theorem (see sect. 11.2 in [Le]) or...
proceeding as in the proof of Lemma 4.4 of [BGS] or Chap XIV of [Bo], we obtain
the lemma by covering the interval \( I = (-\mathcal{O}(N^c_{\rho C_1}), \mathcal{O}(N^c_{\rho C_1})) \) with intervals of
size \( e^{-N_0} \). (Recall (6.21) and that for all \( \theta \notin I \), \( H_{w,\Lambda} - E \) is automatically
invertible.) \( \square \)

Lemma 3.1 remains valid for \( H_{w,\Lambda} \), as the first order variation in \( x \) remains to be
1. We can now proceed as in the Schrödinger case to prove Anderson localization for
the wave operator \( H_w \). We obtain the same Theorem as in sect. 5 with \( H_w \) in place
of \( H \).

APPENDIX: LOCALIZATION RESULTS FOR RANDOM SCHRODINGER OPERATORS

Random Schrödinger operator is the operator
\[
H = \epsilon \Delta + V \text{ on } \ell^2(\mathbb{Z}^d)
\]
where \( \epsilon > 0 \) is a parameter, \( \Delta(i,j) = 1 \) if \( |i - j| = 1 \) and zero otherwise, \( V = \{v_i\}_{i \in \mathbb{Z}^d} \)
is a family of independent identically distributed (iid) random variables with common
probability distribution \( g \). The spectrum of \( H \) is given by
\[
\sigma(H) = \sigma(\epsilon \Delta) + \sigma(V) = [-2\epsilon d, 2\epsilon d] + \text{supp } g.
\]

There are a few versions of Anderson localization results for \( H \), The one that is
most adapted for our purpose is proven in [vDK], which we restate below.

For any \( L \in \mathbb{N} \), let \( \Lambda_L(i) = [-L,L]^d + i, i \in \mathbb{Z}^d \). Let \( m > 0, E \in \mathbb{R} \). \( \Lambda_L(i) \) is
\( (m,E) \)-regular (for a fixed \( V \)) if \( E \notin \sigma(H_{\Lambda_L(i)}) \) and
\[
|G_{\Lambda_L(i)}(E; j, j')| \leq e^{-m|j - j'|}
\]
for all \( j, j' \in \Lambda_L(i), |j - j'| > L/4 \).

**Theorem A.** Let \( I \subset \mathbb{R} \) be a bounded interval. Suppose that for some \( L_0 > 0 \), we have

\[
\text{Prob } \{ \text{for any } E \in I \text{ either } \Lambda_{L_0}(i) \text{ or } \Lambda_{L_0}(j) \text{ is } (m_0, E)\text{-regular} \} \geq 1 - \frac{1}{L_0^{2p'}} \quad (A1)
\]

for some \( p' > d, m_0 > 0 \), and any \( i, j \in \mathbb{Z}^d, |i - j| > 2L_0 \)

\[
\text{Prob } \{ \text{dist } (E, \sigma(H_{\Lambda_L(0)})) < e^{-L^\beta} \} \leq 1/L^q \quad (A2)
\]
for some $\beta$ and $q$, $0 < \beta < 1$ and $q > 4p + 6d$ all $E$ with

$$\text{dist} (E, I) \leq \frac{1}{2} e^{-L\beta},$$

and all $L \geq L_0$. Then there exists $\alpha, 1 < \alpha < 2$, such that if we set $L_{k+1} = [L_k^\alpha] + 1$, $k = 0, 1, 2 \ldots$ and pick $m, 0 < m < m_0$, there is $Q < \infty$, such that if $L_0 > Q$, we have that for any $k = 0, 1, 2 \ldots$

$$\text{Prob} \{\text{for any } E \in I \text{ either } \Lambda_{L_k(i)} \text{ or } \Lambda_{L_k(j)} \text{ is } (m, E) \text{ regular} \} \geq 1 - \frac{1}{L_k^{2p}}$$

for any $i, j \in \mathbb{Z}^d$ with $|i - j| > 2L_k$.

Remark. On the same probability subspace,

$$\text{dist} (\sigma(H_{\Lambda_{L_k(i)}}), \sigma(H_{\Lambda_{L_k(j)}})) > e^{-L_k^\beta}, \quad \beta > 0 \quad (A3)$$

if $|i - j| > 2L_k$. This is part of the ingredient of the proof of Theorem A.

(A1) is verified if $\epsilon$ is sufficiently small. (A2) is provided by the Wegner Lemma if $g$ is absolutely continuous with a bounded density $\tilde{g}$.

$$\{\text{Prob}\{\text{dist} (E, \sigma(H_\Lambda)) \leq K\} \leq CK|\Lambda|\|\tilde{g}\|_\infty.$$

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