Bounding and dominating number
of families of functions on \( \mathbb{N} \)

Claude Laflamme
Department of Mathematics and Statistics
University of Calgary
Calgary, Alberta
Canada T2N 1N4

Abstract
We pursue the study of families of functions on the natural numbers, with emphasis here on the bounded families. The situation being more complicated than the unbounded case, we attack the problem by classifying the families according to their bounding and dominating numbers, the traditional scheme for gaps. Many open questions remain.

1 Introduction
Over the years, the notion of gaps of functions (or sets) of natural numbers has played an important role in the application of Set Theory to different branches of mathematics, see for example [13] for a survey. It should thus not come as a surprise that families of functions in general might have an impact. It was in fact shown recently in [2] that the structure of directed unbounded families of functions had an influence on several problems and in [3] an application of non-directed unbounded families was made. In the papers [2, 4, 7, 8], a sufficiently precise description of these families was given to address these questions.

The next step is to consider bounded families and provide a similar description. Such families are no more than generalizations of the classical notion of (linearly ordered) gaps and the situation appears quite complex; indeed, not only all partial orders of size \( \leq \aleph_1 \) embed as bounded families of functions but unbounded families themselves reflect as bounded ones. We have tried in this paper to classify the bounded families according to their bounding and dominating numbers, a criterion much weaker than cofinal equivalence for example, but which seemed a good starting point but in fact many open questions remain.

*This research was partially supported by NSERC of Canada.

1980 AMS Subj. Class. (1985 Revision) Primary 03E35; Secondary 04A20.

Key words and phrases. Gaps, ultrafilter, forcing
The ultimate results we hope to achieve are to describe the families that can be built from ZFC alone with enough details to be useful for applications; forcing is used to verify that ZFC has been exhausted, i.e. that no other families can be built from ZFC.

We express our warm thanks to Stevo Todorcevic for discussions on the topic.

2 Notation and Preliminaries

We write $\omega$ for the set of natural numbers, $\omega^\omega$ for the set of all functions on $\omega$ and $\omega^{\omega^*}$ for the set of non-decreasing (monotone) functions. We often use $f \leq g$ to abbreviate $(\forall n) f(n) \leq g(n)$, and $f \leq^* g$ for $(\forall^* n) f(n) \leq g(n)$ and similarly for $<$ and $<^*$; here "$\forall^* n$" means "for all but finitely many $n$" and similarly "$\exists^* n$" means "there exists infinitely many $n$". Also important is the ordering $f \prec g$ defined by $\lim_n g(n) - f(n) = +\infty$. As $\langle \omega^\omega, \prec \rangle$ embeds in $\langle \omega^{\omega^*}, \prec \rangle$, we shall be interested essentially in the latter structure and we shall assume for the remaining of this paper that we deal with monotone functions only.

We use $F, G, H$ to denote families of functions. Further, we shall assume that the families considered are closed downward under $\leq^*$; this simplifies greatly the discussion without interfering with the results. In particular the bounding and dominating numbers that we define are the same for a given family or its downward closure, and further it suffices to present the generators to describe a family.

Given two families of functions $H \subseteq F$, we say that $H$ is unbounded in $F$ if:

$$(\forall f \in F)(\exists h \in H) h \not\preceq^* f,$$

and $H$ is said to dominate $F$ if

$$(\forall f \in F)(\exists h \in H) f \leq^* h.$$ 

We shall in practice prove slightly more, for example that $f \prec g$ instead of only $f \leq^* g$ or that $\limsup_n f(n) - g(n) = +\infty$ instead of just $f \not\preceq^* g$; the reason is that often for applications two functions are identified if they differ only by a fixed natural number. Indeed we shall work mostly with the $\prec$ ordering. A family dominating $\omega^\omega$ is usually called a dominating family, and one unbounded in $\omega^\omega$ is simply called an unbounded family.

We define the bounding number $b(F)$ of a family $F$ as

$$b(F) = \min\{ |H| : H \subseteq F \text{ is unbounded in } F \}$$

and the dominating number $d(F)$ as

$$d(F) = \min\{ |H| : H \subseteq F \text{ dominates } F \}.$$
\( b = \mathfrak{b}(\omega^\omega) \) is the usual bounding number and \( \mathfrak{d} = \mathfrak{d}(\omega^\omega) \) the dominating number. The infinite subsets of \( \omega \) are denoted by \([\omega]^{\omega}\), the standard ordering is \( A \subseteq^* B \) if \( A \setminus B \) is finite. We shall be interested in almost disjoint families, that is families of infinite sets with pairwise finite intersections. By fixing a bijection from the rationals and \( \omega \) and considering for each irrational number a sequence of rationals converging to it, we see that there is an almost disjoint family of subsets of \( \omega \) of size \( \mathfrak{c} \), the continuum. Typical functions that will interest us are of the form \( \text{next}(\cdot, X) \) for some \( X \in [\omega]^\omega \), where

\[
\text{next}(n, X) = \text{the smallest element of } X \text{ greater than or equal to } n,
\]

and similarly for the function \( \text{last}(\cdot, X) \).

An ultrafilter is a proper family of infinite sets closed under finite intersections, supersets, and maximal with respect to those properties; in particular it must contain \( X \) or \( \omega \setminus X \) for any \( X \subseteq \omega \), and must be nonprincipal. We use \( \mathcal{U}, \mathcal{V} \) to denote ultrafilters. We write \( \chi(\mathcal{U}) \) for the minimal cardinality of a collection generating the ultrafilter \( \mathcal{U} \), and \( u \) for the least cardinality of a family of sets generating any ultrafilter. A \( P_\kappa \)-point is an ultrafilter \( \mathcal{U} \) with the property that any \( \kappa \) decreasing sequence from \( \mathcal{U} \) has a lower bound in \( \mathcal{U} \).

## 3 Unbounded Families

We shall consider in this section three sorts of unbounded (closed downward) families (of monotone functions).

**Definition 3.1**

1. The \( D \)-class (the dominating class): \( F \in D \) iff \( F \) is dominating.

2. The \( S \)-class (the superperfect class): \( F \in S \) iff
   
   i) \( (\exists h)(\forall f \in F)(\exists^\infty n)[f(n) \leq h(n)] \)
   
   ii) \( (\exists g)(\forall f)(\exists^\infty n)f(n) \leq g(n) \rightarrow f \in F. \)

3. The \( U \)-class (for an ultrafilter \( \mathcal{U} \)): \( F \in \mathcal{U} \) iff
   
   i) \( (\exists h)(\forall f \in F)[\{ n : f(n) \leq h(n) \} \in \mathcal{U}] \).
   
   ii) \( (\exists g)(\forall f)[\{ n : f(n) \leq g(n) \} \in \mathcal{U} \rightarrow f \in F]. \)

These three classes are easily seen to be distinct and we have shown in [7, 8] the relative consistency of any unbounded family of functions falling into one of these classes; that is in ZFC alone, no other unbounded family of functions can be obtained and even a single ultrafilter of your choice may be used for all members of the 3\textsuperscript{rd} class. This fulfills our original motivation for unbounded
families, in other words these descriptions are sufficiently detailed to provide answers to many general mathematical problems (see [3,4,5,6]).

The reason to pursue their studies here is their influence on bounded families as we will see in the next section. So we now discuss the bounding and dominating number of families in these three classes. If \( \mathcal{F} \) is dominating, then \( b(\mathcal{F}) = b, \) the usual bounding number and \( d(\mathcal{F}) = d, \) the usual dominating number. Although it is possible to make structural distinctions between dominating families, applications have not made them yet necessary to analyze. If we demand that our families be closed downward under \( \leq, \) then there is only one dominating family, namely \( \omega^\omega. \)

We now turn to the \( S \)-class. In [10], Kechris showed that any unbounded Borel family must contain a superperfect tree, and we showed in [3] that any non-dominating family containing a superperfect tree belongs to the \( S \)-class.

**Proposition 3.2** If \( \mathcal{F} \) is in the \( S \)-class, then \( b(\mathcal{F}) = 1 \) or \( 2 \) and \( d = c. \) Further these two values of \( b \) are attainable.

**Proof:**

3.2.1: We first show that \( b(\mathcal{F}) \leq 2 \) if \( \mathcal{F} \) belongs to the \( S \)-class.

The point is that a bounding number of at least 3 means that the family is directed; it thus suffices to show that a directed family satisfying 2ii) in the \( S \)-class is dominating, a contradiction.

So let \( \mathcal{F} \) belong to the \( S \)-class and witnessed by \( g \) and \( h \) as in definition 3.1. Fixing any \( p \in \omega^\omega, \) define a sequence of integers by \( p_0 = 0 \) and more generally such that \( g(p_{i+1}) > p(p_i). \) If we now let \( X_i = \{\pi_{2i+n} : n \in \omega\} \) for \( i = 0, 1 \) and define \( f_i(n) = g(\text{next}(n, X_i)) \in \mathcal{F}, \) then \( \max\{f_0(n), f_1(n)\} \geq p(n) \) for each \( n. \) Since \( p \) was arbitrary, we see that \( \mathcal{F} \) must be dominating if directed, i.e. if \( b(\mathcal{F}) \geq 3. \)

3.2.2: We now show that \( d(\mathcal{F}) = c \) if \( \mathcal{F} \) belongs to the \( S \)-class.

Again, let \( \mathcal{F} \) belong to the \( S \)-class and witnessed by \( g \) and \( h. \) Choose also an increasing sequence of integers \( X = \{\pi_n : n \in \omega\} \) such that \( g(p_{i+1}) > h(p_n) \) for each \( n. \) Now for any infinite \( Y \subseteq X, \) consider the function \( f_Y(n) = g(\text{next}(n, Y)) ; \) this function must belong to \( \mathcal{F} \) as it is equal to \( g \) infinitely often. Observe however that for \( p \in \mathcal{F} \) such that \( f_Y \leq^* p, \)

\[
S_p = \{n : p(n) \leq h(n)\} \leq^* S(Y) = \bigcup\{(\pi_{n-1}, \pi_n) : \pi_n \in Y\}.
\]

Since moreover \( Y \cap Y' =^* \emptyset \to S(Y) \cap S(Y') =^* \emptyset, \) choosing an almost disjoint family of infinite subsets of \( X \) of size \( c \) shows that \( c \) functions are necessary to dominate \( \mathcal{F}. \)

3.2.3: An example of \( \mathcal{F} \) in the \( S \)-class with \( b(\mathcal{F}) = 1 \) (and \( d(\mathcal{F}) = c). \)

Fix any unbounded function \( g \in \omega^\omega \) and let \( \mathcal{F} = \{f \in \omega^\omega : (\exists^\infty n) f(n) \leq g(n)\} = \{g(\text{next}(\cdot, X)) : X \in [\omega]^{\omega}\}. \) Since \( g \) itself is unbounded in \( \mathcal{F} \) (and belongs to \( \mathcal{F} \), we have \( b(\mathcal{F}) = 1. \)
3.2.4: An example of $F$ in the $S$-class with $b(F) = 2$ (and $d(F) = \kappa$).

Consider the identity function $id(n) = n$ and for $X \in [\omega]^{\omega}$ let

$$h_X(n) = \text{next}(n, X) + |X^c \cap n|$$

and finally put $F = \{h_X : X \in [\omega]^{\omega}\}$.

The fact that $F$ belongs to the $S$-class is witnessed by the functions $g = id$ and $h(n) = 2n$. Then $b(F) \leq 2$ and it thus suffices to show that no single member of $F$ is unbounded. But given $h_X \in F$, choose an infinite $Y \subseteq X$ such that $X \setminus Y$ is also infinite. Then for each $N$ and $n$ large enough

$$h_Y(n) = \text{next}(n, Y) + |Y^c \cap n|$$

$$\geq \text{next}(n, X) + |Y^c \cap n|$$

$$\geq \text{next}(n, X) + |X^c \cap n| + N$$

$$= h_X(n) + N.$$

Thus $h_X < h_Y$ and hence $b(F) = 2$. The proof of 3.2.2 will actually give you two specific functions unbounded in $F$. This completes the proof of Proposition 3.2.

We now turn our attention to members of the $U$-class.

**Proposition 3.3** For any ultrafilter $U$, $\lambda \in \{1, 2, \omega\}$ and $\chi(U) \leq \kappa \leq c$, there is a family $F$ in the $U$-class such that $b(F) = \lambda$ and $d(F) = \kappa$.

**Proof:** First choose two functions $g, h \in \omega^{\omega}$ and an increasing sequence of integers $\langle \pi_n : n \in \omega \rangle$ such that:

1. $g \prec h$
2. $\lim_n h(n + 1) - [h(n) + n] = +\infty$
3. $(\forall n) \ h(\pi_n) \geq g(\pi_{n+2})$
4. $(\forall n) \ \pi_{n+1} - \pi_n \geq 2^n$.

Also fix an almost disjoint family $A = \langle A_\alpha : \alpha < \kappa \rangle$ such that

$$|A_\alpha \cap [\pi_n, \pi_{n+1})| = 1$$

for each $n$ and $\alpha$.

Without loss of generality we may assume that $E = \bigcup_n [\pi_{2n}, \pi_{2n+1}) \in U$ as the other case is analogous. For $X \in U$, let $P(X) = \cup\{[\pi_{2n-1}, \pi_{2n}) : X \cap [\pi_{2n}, \pi_{2n+1}) \neq \emptyset\}$.

With these preliminaries we are ready to build the desired families.

**3.3.1:** We build an $F$ in the $U$-class such that $b(F) = 1$ and $d(F) = \kappa$ where $\chi(U) \leq \kappa \leq c$.

For $X \in U$ such that $X \subseteq E$ and any $\alpha < \kappa$, let
\[ f^X_\alpha(n) = h(\text{next}(n, [A^c_\alpha \cap P(X)] \cup X)) \]

and put \( \mathcal{F} = \{ f^X_\alpha : X \subseteq E, X \in \mathcal{U}, \alpha < \kappa \} \). Observe that \( g(\text{next}(\alpha, X)) \leq f^X_\alpha \) for any \( \alpha \) and that if \( X, \alpha \) are given, then \( f^X_\alpha(n) = h(x) \) for any \( x \in X \) and therefore \( g, h \) witness that \( \mathcal{F} \) belongs to the \( \mathcal{U} \)-class.

As \( h \) is unbounded in \( \mathcal{F} \), we conclude readily that \( \mathfrak{b}(\mathcal{F}) = 1 \). We must now show that the dominating number is \( \kappa \). If \( \mathcal{B} \) is a base for the ultrafilter \( \mathcal{U} \), then

\[ \mathcal{H} = \{ f^X_\alpha : X \in \mathcal{B}, \alpha < \kappa \} \]

clearly dominates the family \( \mathcal{F} \) and therefore \( \mathfrak{d}(\mathcal{F}) \leq \chi(\mathcal{U}) \cdot \kappa = \kappa \). On the other hand, fix a family \( \mathcal{H} \subseteq \mathcal{F} \) of size less than \( \kappa \), and fix some ordinal \( \beta \in \kappa \) not mentioned in any indexing of the functions from \( \mathcal{H} \). But if \( X \in \mathcal{U}, X \subseteq E \) and \( \alpha \neq \beta \), then \( f^X_\alpha \) does not dominate \( f^E_\beta \). Indeed the set \( A^c_\alpha \cap P(X) \setminus A^c_\beta \) is infinite as otherwise \( P(X) \cap A_\beta \subseteq^* A_\alpha \), and since \( P(X) \) is an infinite union of intervals of the form \( [\pi_n, \pi_{n+1}] \), \( P(X) \cap A_\beta \) is infinite and thus \( A_\beta \cap A_\alpha \) is infinite contradicting that \( \mathcal{A} \) is an almost disjoint family.

But now for any \( x \in A^c_\alpha \cap P(X) \setminus A^c_\beta \), \( f^X_\alpha(x) = h(x) \) and \( f^E_\beta(x) \geq h(x+1) \) and as \( \lim_n h(n+1) - h(n) = +\infty \) we get \( \limsup_n f^E_\beta(n) - f^X_\alpha(n) = +\infty \) as well. Therefore, no member of \( \mathcal{H} \) dominates the function \( f^E_\beta \in \mathcal{F} \) and we conclude that \( \mathfrak{d}(\mathcal{F}) \geq \kappa \) and thus \( \mathfrak{d}(\mathcal{F}) = \kappa \).\[ \text{3.32: We build an } \mathcal{F} \text{ in the } \mathcal{U} \text{-class such that } \mathfrak{b}(\mathcal{F}) = 2 \text{ and } \mathfrak{d}(\mathcal{F}) = \kappa \text{ for any } \chi(\mathcal{U}) \leq \kappa \leq \kappa. \]

For \( X \in \mathcal{U} \) such that \( X \subseteq E \) and any \( \alpha < \kappa \), let

\[ f^X_\alpha(n) = h(\text{next}(n, [A^c_\alpha \cap P(X)] \cup X)) + |X^c \cap n| \]

and put \( \mathcal{F} = \{ f^X_\alpha : X \subseteq E, X \in \mathcal{U}, \alpha < \kappa \} \). For any \( f^X_\alpha \in \mathcal{F} \) and \( x \in X \), we have \( f^X_\alpha(x) \leq h(x) + x \) and therefore \( g \) and \( h(n) = h(n) + n \) witness that \( \mathcal{F} \) belongs to the \( \mathcal{U} \)-class.

We first show that the bounding number is 2. No \( f^X_\alpha \) itself is unbounded in \( \mathcal{F} \) since choosing \( Y \in \mathcal{U} \) such that \( X \setminus Y \) is infinite, we get for each \( N \) and \( n \) large enough

\[
 f^X_\alpha(n) = h(\text{next}(n, [A^c_\alpha \cap P(X)] \cup X)) + |X^c \cap n| \\
 \leq h(\text{next}(n, [A^c_\alpha \cap P(Y)] \cup Y)) + |X^c \cap n| \\
 \leq h(\text{next}(n, [A^c_\alpha \cap P(Y)] \cup Y)) + |Y^c \cap n| - N \\
 = f^Y_\alpha(n) - N
\]

and therefore \( f^X_\alpha \prec f^Y_\alpha \).

However, we claim that for any \( \alpha \neq \beta \), the pair \( \{ f^E_\alpha, f^E_\beta \} \) is unbounded in \( \mathcal{F} \). To verify this, we consider any \( f^X_\gamma \in \mathcal{F} \), without loss of generality \( \alpha \neq \gamma \). For any \( x \in A^c_\alpha \cap P(X) \setminus A^c_\gamma \), which is infinite, we have \( f^X_\gamma(x) \leq h(x) + x \) and \( f^E_\alpha(x) \geq h(x+1) \). But as \( \lim \sup_n h(n+1) - [h(n) + n] = +\infty \) by assumption
we get \( \limsup_n f_\alpha^X(n) - f_\alpha^X(n) = +\infty \) as well. The fact that \( d(F) = \kappa \) is proved as in the previous example.

\[3.3: \text{We build an } F \text{ in the } U\text{-class such that } b(F) = \omega \text{ and } d(F) = \kappa \text{ where } \chi(U) \leq \kappa \leq \epsilon. \]

For any \( a \in [\kappa]^{<\omega} \) and \( X \in U \) such that \( X \subseteq E \) we let

\[
f_\alpha^X(n) = h(\text{next}(n, \left[ \bigcap_{a \in A} A_a^\alpha \cap P(X) \cup X \right]) + |X \cap n|
\]

and put \( F = \{ f^X_a \subseteq E, X \in U, a \in [\kappa]^{<\omega} \} \). Observe that for any such \( f^X_a \) and \( x \in X \subseteq E \), \( f^X_a(x) = h(x) + |X \cap x| \leq h(x) + x \) and therefore \( g \) and \( h'(n) = h(n) + n \) again witness that \( F \) belongs to the \( U \)-class.

Our first task is to show that \( F \) is directed and therefore \( b(F) = \omega \). But given \( f^X_a \) and \( f^Y_b \), put \( c = a \cup b \) and choose \( Z \in U \) such that \( Z \subseteq X \cap Y \) and \( X \cap Y \setminus Z \) is infinite; then \( f^Z_c > f^X_a \cup f^Y_b \). To show now that \( b(F) \leq \omega \), choose \( A \in [\kappa]^{<\omega} \) and we prove that the collection \( H = \{ f^X_a : a \in A \} \subseteq F \) is unbounded in \( F \). So let us fix \( f^X_a \in F \) and choose \( \beta \in A \setminus a \). Now the set \( \bigcap_{a \in A} A_a^\alpha \cap P(X) \setminus A_\beta^\alpha \) is infinite as otherwise we would obtain \( P(X) \cap A_\beta \leq \omega \), and as \( P(X) \cap A_\beta \) is infinite \( A_\beta \) would have infinite intersection with some \( A_a \) contradicting that \( A \) is an almost disjoint family. But now for \( x \in \bigcap_{a \in A} A_a^\alpha \cap P(X) \setminus A_\beta^\alpha \),

\[
f^X_a(x) = h(\text{next}(x, \left[ \bigcap_{a \in A} A_a^\alpha \cap P(X) \cup X \right]) + |X \cap x| \leq h(x) + x,
\]

\[
f^\beta_\alpha(x) = h(\text{next}(x, A_\beta^\alpha \cap P(E) \cup E)) + |E \cap x| \geq h(x + 1).
\]

As \( \limsup_n h(n+1) - [h(n) + n] = +\infty \), we get that \( \limsup_n f^\alpha_\beta(n) - f^X_a(n) = +\infty \) as well and \( H \) is indeed unbounded in \( F \).

The verification the the dominating number is \( \kappa \) is again very similar to the first example.

This completes the proof of Proposition \( 3.3 \). \( \square \)

**Corollary 3.4** For any \( \lambda \in \{1, 2, \omega \} \) and \( \nu \leq \kappa \leq \epsilon \), there is a family \( F \) in the \( U \)-class such that \( b(F) = \lambda \) and \( d(F) = \kappa \).

The next problem is whether we can construct a family \( F \) in the \( U \)-class with an uncountable bounding number. We show that this requires a \( P \)-point and therefore, in view of Shelah’s consistency result \( [14] \) that there might be no such \( P \)-points, we cannot construct such families in \( ZFC \) alone.

**Proposition 3.5** If \( F \) in the \( U \)-class has an uncountable bounding number, then there is a finite-to-one map \( m \) such that \( m(U) \) is a \( P \)-point.

**Proof:** Fix functions \( g \) and \( h \) witnessing that \( F \) belongs to the \( U \)-class and define a sequence of integers such that \( \pi_0 = 0 \) and more generally such
that \( g(\pi_{n+1}) > h(\pi_n) \). We may assume without loss of generality that \( E = \bigcup_n [\pi_{2n}, \pi_{2n+1}] \in \mathcal{U} \) as the other case is analogous. Now for any \( X \subseteq E \), if \( X \subseteq E \), any \( f \in \mathcal{F} \) with \( g(\text{next}(-, X)) \leq^* f \) must satisfy

\[
S(f) = \{ n : f(n) \leq h(n) \} \subseteq^* T(X) = \bigcup \{ [\pi_{2n-1}, \pi_{2n+1}] : X \cap [\pi_{2n}, \pi_{2n+1}] \neq \emptyset \}
\]

Now define a map \( m \in \omega^\omega \) by \( m^{-1}(Y_n) = n \) and consider \( \mathcal{V} = m(\mathcal{U}) \). Certainly \( \mathcal{V} \) is a (non principal) ultrafilter as \( m \) is finite-to-one. To show it is actually a \( P \)-point, let \( \{ Y_n : n \in \omega \} \subseteq \mathcal{V} \) be given and consider the sets \( X_n = m^{-1}(Y_n) \cap \mathcal{U} \). Since we are assuming that the bounding number of \( \mathcal{F} \) is uncountable, fix a function \( f \in \mathcal{F} \) such that \( g(\text{next}(-, X_n)) \leq^* f \) for each \( n \). Therefore \( S(f) \subseteq^* T(X_n) \) for each \( n \) and thus \( m(S(f)) \subseteq^* m(T(X_n)) \subseteq Y_n \). Since moreover \( S(f) \in \mathcal{U} \) and therefore \( m(S(f)) \in m(\mathcal{V}) \), the proof is complete.

Under the existence of \( P \)-points or more generally \( P_\kappa \) points, one can easily construct members of the \( \mathcal{U} \)-class with bounding number \( \kappa \) by fixing some \( g \in \omega^\omega \) and defining \( \mathcal{F} = \{ g(\text{next}(-, X)) : X \in \mathcal{U} \} \). Thus in general we have:

**Proposition 3.6** There is a \( P_\kappa \) ultrafilter if and only if there if a family \( \mathcal{F} \) in the \( \mathcal{U} \)-class with bounding and dominating number \( \kappa \).

We can also deduce from the proof of Proposition 3.5 that \( d(\mathcal{F}) \geq u \) for any family \( \mathcal{F} \) in the ultrafilter class, but I do not know if \( d(\mathcal{F}) \geq \chi(\mathcal{U}) \) whenever \( \mathcal{F} \) belongs to the \( \mathcal{U} \)-class with witness \( \mathcal{U} \).

## 4 Bounded Families

Let \( \mathcal{F} \) be a bounded family and let

\[
\mathcal{F}_\downarrow = \{ g \in \omega^\omega : ( \forall f \in \mathcal{F} ) f \prec g \}
\]

Certainly \( \mathcal{F}_\downarrow \) is nonempty as \( \mathcal{F} \) is bounded and the pair \( (\mathcal{F}, \mathcal{F}_\downarrow) \) forms a gap in the sense that there is no \( h \in \omega^\omega \) such that

\[
( \forall f \in \mathcal{F} )( \forall g \in \mathcal{F}_\downarrow ) f \prec h \prec g.
\]

To make a first distinction between bounded families, we make the following definition.

**Definition 4.1**

\[
b(\mathcal{F}) = \min \{ |\mathcal{H}| : \mathcal{H} \subseteq \mathcal{F}_\downarrow \text{ and } \mathcal{H} \text{ is unbounded in } \mathcal{F}_\downarrow \text{ in the reverse order } \}
\]

\[
= \min \{ |\mathcal{H}| : \neg( \exists g \in \mathcal{F}_\downarrow ) ( \forall h \in \mathcal{H} ) g \prec h \}
\]
We loosely call $b_{\downarrow}(F)$ and for that matter $F_{\downarrow}$ depending on the context the upper bound of $F$ and we will classify the families according to this cardinal $b_{\downarrow}(F)$ which takes either the value 1 or an infinite regular cardinal; notice that the value 2 cannot occur here. Observe also that if $H \subseteq F_{\downarrow}$ is unbounded in $F_{\downarrow}$ in the reverse order as above, then the pair $(F, H)$ is also a gap. Much work has been done on gaps $(F, H)$ for which both $F$ and $H$ are linearly ordered by $\prec$; in particular gaps $(F, G)$ for which $b(F) = d(F)$. Such gaps are usually qualified as $(b(F), b_{\downarrow}(F))$ gaps. Here we will work in a more general situation.

4.1 Bounded families with a countable upper bound

Unbounded families have much influence on the bounded ones; we can use the results of §3 to construct families with countable upper bounds and various bounding and dominating numbers.

Proposition 4.2 There are families $F$ with countable upper bounds, that is $b_{\downarrow}(F) = 1$ or $\omega$, such that:

1. $b(F) = \mathfrak{b}$ and $d(F) = \mathfrak{d}$.
2. $b(F) = 1$ or $2$ and $d(F) = \aleph$.
3. $b(F) = 1$, $2$ or $\omega$ and $u \leq d(F) \leq c$.

Proof: The goal of the proof is to build families $F$ with the same bounding and dominating number as the families from §3; we fix for our constructions the functions $g(n) = n^2$ and more generally for $\ell \in \omega$

$$g_{\ell}(n) = n^2 - \ell \log n \text{ or } g_{\ell}(n) = n^2 - \ell$$

depending on the context.

We shall build gaps $(F, \{g_{\ell}: \ell \in \omega\})$ giving us families $F$ with upper bounds 1 or $\omega$, depending on which collection $\{g_{\ell}: \ell \in \omega\}$ one chooses, and with the appropriate bounding and dominating numbers.

Observe first that irrespective of the collection we choose, we have

$$g_{\ell+1}(n + 1) \geq g_{\ell}(n)$$

for each $n$ and $\ell$; this will make our verifications easier. Now if $H$ is any unbounded family and $h \in H$, let

$$f_h(n) = g_m(n) \text{ if } h(m - 1) < n \leq h(m)$$

and put $F(H) = \{f_h: h \in H\}$. In this case we have :

Claim 4.3 $b(H) = b(F(H))$ and $d(H) = d(F(H))$. 

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**Proof:** It suffices to prove that for all $h_1, h_2 \in H$, we have

$$h_1 \preceq^* h_2 \text{ iff } f_{h_1} \preceq^* f_{h_2}.$$ 

To verify this, suppose first that $h_1(m) \leq h_2(m)$ for all $m \geq M$ and fix $n \geq h_2(M)$.

Choose first $m$ such that

$$h_1(m - 1) < n \leq h_1(m)$$

and $\ell$ such that

$$h_2(\ell - 1) < n \leq h_2(\ell).$$

Observe that we must have $\ell \leq M$ and thus

$$f_{h_1}(n) = g_m(n) \leq g_\ell(n) = f_{h_2}(n).$$

Suppose now for the other direction that $f_{h_1}(n) \leq f_{h_2}(n)$ for all $n \geq h_2(N)$ and fix $n \geq N$; we show that $h_1(n) \leq h_2(n)$. But if for the sake of a contradiction we have $h_2(n) < h_1(n)$, pick $\ell \geq n + 1$ such that $h_2(\ell - 1) < h_1(n) \leq h_2(\ell)$.

Then

$$f_{h_1}(h_1(n)) = g_n(h_1(n))$$

and

$$f_{h_2}(h_1(n)) = g_\ell(h_1(n)) \leq g_{n+1}(h_1(n)) < g_n(h_1(n))$$

and we obtain the desired contradiction. This proves the claim.

The Proposition is now proved by replacing $H$ by the appropriate families of §3. Actually, to obtain $F(H) \subseteq \omega^{\omega}$, we should first replace the families $H$ by $H' = \{h(n) + n : h \in H\}$ for example to ensure that we have strictly increasing functions; observe that this does not affect the bounding and dominating number.

There is however more than just reflecting unbounded families to bounded ones, indeed let us see how close we are. Let $F$ be a family of functions and $\{g_n : n \in \omega\}$ a collection such that $F \prec g_{n+1} \preceq^* g_n$ for each $n$, and assume without loss of generality that $g_{n+1}(k) + 1 \leq g_n(k)$ for each $k$ and $n$. For $f \prec \{g_n : n \in \omega\}$, we define

$$h_f(n) = \max\{k : g_n(k) \leq f(k)\}$$

and put $H(F) = \{h_f : f \in F\}$. The following proposition, due to Rothberger, shows that unbounded families are always involved somehow.

**Proposition 4.4** (*Rothberger*) The pair $(F, \{g_n : n \in \omega\})$ is a gap if and only if $H(F)$ is an unbounded family.
Proof: Suppose first that the family $\mathcal{H}(\mathcal{F})$ is bounded, say by $h$; we might as well assume that $n < h(n) < h(n + 1)$ for each $n$. Define a function $p$ by:

$$p(j) = g_m(j)$$

where $m$ is the smallest integer such that $h(m + 1) > j$.

As $j$ increases, $m$ increases as well and therefore $p < g_m$ for each $m$. Now for any $f \in \mathcal{F}$, and therefore for $h_f \in \mathcal{H}$, choose $N$ large enough so that

$$(\forall n \geq N) \ h_f(n) < h(n).$$

Hence for all $m \geq h(N)$, if we let $\ell \geq N$ be as large as possible such that $m \geq h(\ell)$, we obtain:

$$m \geq h(\ell) > h_f(\ell)$$

and therefore

$$f(m) < g_\ell(m) = p(m).$$

We conclude that $\mathcal{F} \prec p \prec \{g_n : n \in \omega\}$ and thus the pair $(\mathcal{F}, \{g_n : n \in \omega\})$ is not a gap.

For the other direction, since we have the implication

$$f \leq^* f' \rightarrow h_f \leq^* h_{f'},$$

we conclude readily that $\mathcal{H}(\mathcal{F})$ is bounded if the pair $(\mathcal{F}, \{g_n : n \in \omega\})$ is not a gap.

Corollary 4.5 $\mathcal{D}(\mathcal{F}) \geq b$ for any $\mathcal{F}$ with countable upper bound.

Since $f \leq^* f' \rightarrow h_f \leq^* h_{f'}$, we also obtain

Corollary 4.6 $\mathcal{D}(\mathcal{H}(\mathcal{F})) \leq \mathcal{D}(\mathcal{F})$ and $b(\mathcal{F}) \leq b(\mathcal{H}(\mathcal{F}))$ unless $b(\mathcal{H}(\mathcal{F})) = 1$ in which case $b(\mathcal{F}) \leq 2$.

This allows us to extend Proposition 4.2 as follows.

Proposition 4.7 Let $\mathcal{H}$ be any unbounded family and $\lambda \leq b$ a regular (infinite) cardinal. Then there is a family $\mathcal{F}$ with countable upper bound such that

$$b(\mathcal{F}) = \min\{\lambda, b(\mathcal{H})\} \text{ and } \mathcal{D}(\mathcal{F}) = \mathcal{D}(\mathcal{H}).$$

Proof: To simplify the calculations, we fix the functions $g_k(n) = n^k - kn$ for $k \in \omega$ and an increasing sequence of sets $(X_\alpha : \alpha < \lambda)$ such that $X_\beta \setminus X_\alpha$ is infinite whenever $\alpha < \beta$; this is guaranteed by $\lambda \leq b$.

Without loss of generality, we may assume that each $h \in \mathcal{H}$ is strictly increasing, that $h(n) > n$ for each $n$ and that the range is included in $X_\alpha$. Now for $h \in \mathcal{H}$ and $\alpha < \lambda$, define

$$f_{h, \alpha}(n) = g_m(\text{last}(n, X_\alpha)) + |X_\alpha \cap n| \text{ where } h(m - 1) < n \leq h(m)$$

and put $\mathcal{F} = \{f_{h, \alpha} : h \in \mathcal{H}, \alpha < \lambda\}$. As $\mathcal{H}(\mathcal{F}) = \mathcal{H}$, we conclude from Corollary 4.6 that $(\mathcal{F}, \{g_k : k \in \omega\})$ is a gap and that $b(\mathcal{F}) \leq b(\mathcal{H}) + 1$ and $\mathcal{D}(\mathcal{F}) \geq \mathcal{D}(\mathcal{H})$. 

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Claim 4.8 $b(F) \leq \lambda$.

**Proof:** Fix $h \in H$ and let $S = \{f_{h,\alpha} : \alpha < \lambda\}$. We show that $S (\subseteq F)$ is unbounded in $F$. Indeed, fix any $h' \in H$ and any $\alpha < \lambda$ and consider any $\beta$, $\alpha < \beta < \lambda$; we claim that $f_{h,\beta}(n) \geq f_{h',\alpha}(n)$ for infinitely many $n$, indeed on almost all $x \in X_\beta \setminus X_\alpha$. For fix such an $x$, if $h'(m-1) < x \leq h'(m)$, then

$$f_{h',\alpha}(x) = g_m(last(x, X_\alpha)) + |X_\alpha \cap x|$$
$$\leq g_m(x-1) + x$$
$$= (x-1)^{x-1} - m(x-1) + x$$

and if $h(\ell-1) < x \leq h(\ell)$ then

$$f_{h,\beta}(x) = g_\ell(last(x, X_\beta)) + |X_\beta \cap x|$$
$$\geq g_\ell(x) = x^x - \ell x.$$  

As $\ell, m \leq x$, we get $f_{h,\beta}(x) \geq f_{h',\alpha}(x)$ for almost all such $x$'s. This proves the claim.

Claim 4.9 $b(F) \geq \min\{\lambda, b(H)\}$.

**Proof:** Let $S \subseteq F$, $|S| < \min\{\lambda, b(H)\}$, and fix $\zeta < \lambda$, $T \subseteq H$ such that

$$S \subseteq \{f_{h,\alpha} : h \in T, \alpha < \zeta\} \quad \text{and} \quad |T| < b(H).$$

Therefore choose an $h' \in H$ such that $h <^* h'$ for any $h \in T$ and we show that $f_{h,\alpha} <^* f_{h',\zeta}$ for all $h \in T$ and $\alpha < \zeta$, and thus $S$ is bounded in $CF$.

Choose first $N \in X_\alpha$ such that $X_\alpha \setminus N \subseteq X_\zeta$ and fix $n \geq N$; if $m$ is such that

$$h'(m-1) < n \leq h'(m),$$

and $\ell$ such that

$$h(\ell-1) < n \leq k(\ell)$$

we obtain, with $x = last(n, X_\zeta)$,

$$f_{h,\alpha}(n) = g_\ell(last(n, X_\alpha)) + |X_\alpha \cap n|$$
$$\leq g_\ell(last(n, X_\zeta)) + |X_\alpha \cap n|$$
$$= g_\ell(x) + |X_\alpha \cap n|$$
$$= x^x - \ell x + |X_\alpha \cap n|$$

and

$$f_{h',\zeta}(n) = g_m(last(n, X_\zeta)) + |X_\zeta \cap n|$$
$$= g_m(x) + |X_\zeta \cap n|$$
$$= x^x - mx + |X_\zeta \cap n|.$$  

But $m \leq \ell$ (for $n$ large enough) and as $X_\zeta \setminus X_\alpha$ is infinite, we get $f_{h,\alpha}(n) < f_{h',\zeta}(n)$ for almost all $n$, in fact $f_{h,\alpha} \leq f_{h',\zeta}$. This proves the claim.

Finally, as we already know that $\delta(F) \geq \delta(H)$, we must show the reverse inequality. But $F$ is generated by $\lambda \times \delta(H) = \delta(H)$ functions, and the proof is complete. \qed
This gives an idea of what can be done in terms of bounded families with countable upper bound, they all involve unbounded families by proposition 4.4, but this is only very partial information and a lot of freedom remains.

4.2 Families with upper bound $\omega_1$

One of the surprising construction in ZFC is a gap build by Hausdorff which has bounding and dominating number $\omega_1$. Lusin build one with bounding number 1 and dominating number $\omega_1$; it is this construction that we will adapt to produce gaps with various bounding and dominating numbers. Although in both Hausdorff’s and Lusin’s construction the upper bound is at most $\omega_1$, I do not know if could be $\omega_2$; if $\beta > \omega_1$, they certainly cannot.

**Proposition 4.10** For each $\lambda \in \{1, 2, \omega\}$ and $\omega_1 \leq \kappa \leq \mathfrak{c}$, there is a family $F$ with upper bound at most $\omega_1$ such that $b(F) = \lambda$ and $d(F) = \kappa$.

**Proof:**
We first build $\{f_\alpha : \alpha < \omega_1\}$ and $\{g_\alpha : \alpha < \omega_1\}$ such that:

1: $(\forall \alpha < \beta) f_\alpha + id \prec g_\beta \prec g_\alpha$, where $id$ is the identity function $id(n) = n$.

2: $(\forall \alpha)(\forall a \in [\omega_1 \setminus \{\alpha\}]^\omega) \limsup_n f_\alpha(n) - \max_{\gamma \in a}\{f_\alpha(n) + n\} = +\infty$.

3: $(\forall \alpha)(\forall n) f_\alpha(n) \leq g_\alpha(n)$.

4: $(\forall \alpha < \beta)(\exists n) f_\alpha(n) > g_\beta(n)$.

Let us first observe that this construction, essentially due to Lusin, will give us a gap.

**Claim 4.11** The collection $\langle\{f_\alpha : \alpha < \omega_1\}, \{g_\alpha : \alpha < \omega_1\}\rangle$ is a gap.

**Proof:** Suppose on the contrary that $\{f_\alpha : \alpha < \omega_1\} \prec h \prec \{g_\alpha : \alpha < \omega_1\}$ for some function $h$. Choose $X \in [\omega_1]^\omega$ and $n$ so that

a: $(\forall \alpha, \beta \in X) f_\alpha \upharpoonright n = f_\beta \upharpoonright n$ and $g_\alpha \upharpoonright n = g_\beta \upharpoonright n$.

b: $(\forall m \geq n) f_\alpha(m) \leq g_\beta(m)$.

Thus $(\forall \alpha < \beta \in X)(\forall k)$

$$f_\alpha(k) = f_\beta(k) \leq g_\beta(k) \quad \text{if } k < n$$

$$f_\alpha(k) \leq g_\beta(k) \quad \text{if } k \geq n$$

But this contradicts requirement 4. $\square$

If we can accomplish this construction, we put $F_1 = \{f_\alpha : \alpha < \omega_1\}, G = \{g_\alpha : \alpha < \omega_1\}$ and we get a gap $(F, G)$ with $b(F_1) = 1$ and $d(F_1) = \omega_1$. Choosing
functions $0 < h_n < h_{n+1} < id$ and using $F_2 = \{f_\alpha + h_n : \alpha < \omega_1, n \in \omega\}$, we obtain a family with $b(F_2) = 2$ and $d(F_2) = \omega_1$. Finally, we let $F_\omega = \{\max_{n \in \omega} \{f_n\} + h_n : a \in [\omega_1]^{<\omega}, n \in \omega\}$ we obtain a family with $b(F) = \omega$ and $d(F) = \omega_1$. To obtain families with various dominating number, fix for example $f_0$ and choose a set $X = \{x_n : n \in \omega\}$ such that $f_0(x_{n+1}) > f_0(x_n) + x_n$ and let $A = \{A_\alpha : \alpha < \kappa\} \subseteq \mathcal{P}(X)$ an almost disjoint family. Assume further that we actually have $0 < 2h_n < 2h_{n+1} < id$. Now for $\beta < \kappa$, define

$$f_0^\beta(n) = \max\{f_0(n), f_0(last(n, A_\beta)) + \frac{1}{2}last(n, A_\beta)\}.$$

Notice that for $\beta \neq \beta'$, if $x_{n+1} \in A_\beta \setminus A_{\beta'}$,

$$f_0^\beta(x_{n+1}) = f_0(x_{n+1}) + \frac{1}{2}x_{n+1}$$

$$f_0^\beta(x_{n+1}) \leq \max\{f_0(x_{n+1}), f_0(x_n) + x_n\} = f_0(x_{n+1})$$

and therefore $f_0^\beta(x_{n+1}) - f_0^\beta(x_{n+1}) \geq x_{n+1}$ and hence for each $m$

$$\limsup_k f_0^\beta(k) - [f_0^\beta(k) + h_m(k)] = +\infty.$$

We can then replace $f_0$ in the above families by $\{f_0^\beta : \beta < \kappa\}$ to obtain families with dominating number $\kappa$.

The construction proceeds by induction on $\alpha$, that is we start with

$$f_0(n) = n \text{ and } g_0(n) = n^2$$

Now assume that we have already defined the functions $\{f_\xi : \xi \in \alpha\}$ and $\{g_\xi : \xi \in \alpha\}$ such that:

**2.1:** $(\forall \beta < \gamma < \alpha) f_\beta + id < g_\gamma < g_\beta$.

**2.2:** $(\forall \beta < \alpha)(\forall a \in [\alpha \setminus \beta])^{<\omega} \limsup_n f_\beta(n) - \max_{\gamma \in a} \{f_\gamma(n) + n\} = +\infty$.

**2.3:** $(\forall \beta < \alpha)(\forall n) f_\beta(n) \leq g_\beta(n)$.

**2.4:** $(\forall \beta < \gamma < \alpha)(\exists n) f_\beta(n) > g_\gamma(n)$.

and we proceed to build $f_\alpha$ and $g_\alpha$ in countably many steps. As $\alpha$ is countable, we list $\alpha \times [\alpha]^{<\omega}$ as $\{(a_k, a_k) : k \in \omega\}$, $\{f_\beta : \beta < \alpha\}$ as $\{f_k : k \in \omega\}$ and $\{g_\beta : \beta < \alpha\}$ as $\{g_k : k \in \omega\}$.

At stage $N$, suppose that we have $f_\alpha \restriction n$ and $g_\alpha \restriction n$ for some $n$, such that:

**3.1:** $(\forall k < N)(\exists m < n) f_{\alpha_k}(m) - \max_{\gamma \in (a_k \cup \alpha_\beta) \setminus \{a_k\}} \{f_\gamma(m) + m\} \geq N$,
3.2: \( \forall k < N)(\exists m < n) \ f_\alpha(m) - \max_{\gamma \in a_k} \{f_\gamma(m) + m\} \geq N, \)

3.3: \( \forall m < n \ f_\alpha(m) \leq g_\alpha(m), \)

3.4: \( \forall k < N)(\exists m < n) \ f^k(m) > g_\alpha(m). \)

We will also ensure that for \( m \geq n \)

3.5: \( \max_{k < N} \{f^k(m) + m\} + 2N \leq g_\alpha(m) + N \leq \min_{k < N} \{g^k(m)\}. \)

This will help satisfy 2.1. Requirements 3.1 and 3.2 will ensure 2.2, 3.3 will give 2.3 and 3.4 gives 2.4.

For the construction at stage \( N \), first choose \( m_0 > n \) such that:

1. \( f^N(m_0) - \max_{k < N} \{f^k(m_0) + m_0\} > N. \)
2. \( f^N(m_0) > g_\alpha(n - 1). \)

Then we define, for \( n \leq m \leq m_0, \)

\[
\begin{align*}
g_\alpha(m) &= \max_{k < N} \{g_\alpha(n - 1), f^k(m) + m + N\} \\
f_\alpha(m) &= f_\alpha(n - 1)
\end{align*}
\]

This fulfills 3.4 as \( g_\alpha(m_0) < f^N(m_0). \)

Now given \( m_i \) for \( i < N + 1, \) choose \( p_i > m_i \) such that:

1. \( f_\alpha(p_i) - \max_{\gamma \in a_i \setminus \{\alpha_i\}} \{f_\gamma(p_i) + p_i\} \geq N + 1. \)
2. \( \max_{\gamma \in a_i \setminus \{\alpha_i\}} \{f_\gamma(p_i)\} \geq f_\alpha(m_i). \)

Then we define, for \( m_i \leq m \leq p_i, \)

\[
\begin{align*}
f_\alpha(m) &= f_\alpha(m_i) \\
g_\alpha(m) &= \max_{k < N} \{g_\alpha(m_i), f_\alpha(m) + N, f^k(m) + m + N\}.
\end{align*}
\]

This handles 3.1.

Finally choose \( n' > p_N \) large enough so that for all \( m \geq n', \) we have

\[
\begin{align*}
g_\alpha(p_N) &\leq \max \left\{ \max_{\gamma \in \cup_{k < N + 1} a_k} \{f_\gamma(m) + m\}, \max_{k < N + 1} \{f^k(m) + m\} \right\} \\
&\leq \min_{k < N} \{g^k(m)\} - 3(N + 1)
\end{align*}
\]

and define

\[
\begin{align*}
f_\alpha(m) &= f_\alpha(p_N) \text{ for } p_N < m < n' \\
&= \max_{\gamma \in \cup_{k < N + 1} a_k} \{f_\gamma(n') + n'\} + N + 1 \\
&\text{ for } m = n' \\
g_\alpha(m) &= \max_{k < N} \{f_\alpha(m) + N, f^k(m) + N\} \text{ for } p_N < m \leq n'
\end{align*}
\]

This satisfies 3.2 and observe that we are able to keep our promise 3.5. This completes the construction and proves the Theorem 4.1.

As far as uncountable bounding number is concerned, a Hausdorff gap provides a family \( F \) with upper bound at most \( \omega_1, \) \( b(F) = d(F) = \omega_1. \) I do not know if there is always such a family \( F \) with large dominating number, say \( d(F) = \omega \) for example.
4.3 Families with upper bound $b$

In view of Rothberger’s result and the fact that the smallest size of an unbounded family in $\omega^\omega$ is $b$, it is not at all surprising that this cardinal has a role to play in bounded families. We have the following result.

**Proposition 4.12** For any $\lambda \in \{1, 2, \omega\}$, and $\lambda \leq \kappa \leq \omega$, there is a family $F$ with upper bound $b$ such that $b(F) = \lambda$ and $d(F) = \kappa$.

**Proof:** We provide a general construction which will work for all values of $\lambda$ and $\kappa$.

Fix an increasing unbounded family $\langle h_\alpha : \alpha < b \rangle$ and let $f(n) = n^2$. Now for $\ell \in \omega$ define

$$f_\ell(n) = n^2 + \ell \log(n),$$

and thus $f_\ell < f_{\ell+1}$

and for $\alpha < b$ put

$$g_\alpha(n) = f_m(n) = n^2 + m \log(n)$$

if $h_\alpha(m - 1) < n \leq h_\alpha(m)$.

These functions are technically not in $\omega^\omega$ because of the log function, but one could easily take instead the smallest integer greater than or equal to these values. Notice that

$$f_\ell < f_{\ell+1} < g_\beta < g_\alpha$$

for all $\alpha < \beta$ and $\ell$.

**Claim 4.13** For all $X \in [\omega]^\omega$, the pair $\langle \{f_\ell : \ell \in \omega\} \cup \{g_\alpha : \alpha < b\} \rangle$ is a gap.

**Proof of the claim:** Suppose otherwise that there is a function $h$ such that

$$(\forall \ell)(\forall \alpha) f_\ell | X < h < g_\alpha | X.$$

Then we define

$$p(n) = \min \{x \in X : (\forall y \in X \setminus x) h(y) > f_n(y)\}$$

It now suffices to show that $h_\alpha \leq^* p$ for each $\alpha$ to obtain a contradiction. But fix $N$ large enough so that

$$\forall x \in X, x \geq N \rightarrow h(x) < g_\alpha(x)$$

So for each $n$ with $p(n) \geq N$ we must have $p(n) \geq h_\alpha(n)$ as well; indeed, if $x = p(n) < h_\alpha(n)$, we get

$$h(x) = h(p(n)) > f_\alpha(x).$$
Now choose \( m \leq n \) such that \( h_\alpha(m - 1) < x \leq h_\alpha(m) \), then
\[
g_\alpha(x) = f_m(x) \leq f_n(x)
\]
and therefore \( g_\alpha(x) < h(x) \), a contradiction. This proves the claim. \( \square \)

Now let \( \mathcal{A} = \{ A_\alpha : \alpha < \kappa \} \) be an almost disjoint family and for each \( \alpha < \kappa \) and \( \ell \in \omega \), let
\[
f'_\alpha,\ell(k) = f_\ell(\text{last}(n, A_\alpha))
\]
Then certainly \( f'_{\alpha,\ell} < f'_{\alpha,\ell+1} \). Further, if \( \alpha \neq \beta \) and \( \ell, k \) are given, pick \( n \in A_\beta \setminus A_\alpha \), and thus
\[
f'_{\beta,\ell}(n) = f_\ell(n) = n^2 + \ell \log(n)
\]
but as \( \text{last}(n, A_\alpha) \leq n - 1 \) we obtain
\[
f'_{\alpha,k}(n) \leq (n - 1)^2 + k \log(n - 1) = n^2 - 2n + 1 + k \log(n - 1)
\]
and hence \( \limsup_n f'_\alpha,\ell(n) - f'_{\alpha,k}(n) = +\infty. \)

If we now let \( F = \{ f'_\alpha,\ell : \alpha < \kappa, \ell \in \omega \} \), we obtain a family with upper bound \( b \), bounding number \( 2 \) and dominating number \( \kappa \times \omega = \kappa \).

On the other hand if we let \( F = \{ \max\{ f'_\alpha,\ell : \alpha \in a \} : a \in [\kappa]^{<\omega}, \ell \in \omega \} \), we obtain a family with again upper bound \( b \) but bounding number \( \omega \) and dominating number \( \kappa \).

Moreover, if we had used the functions \( f_\ell(n) = n^2 + \ell \) instead, then the family \( F = \{ f'_{\alpha,0} : \alpha < \kappa \} \) would constitute a family with upper bound \( h \), bounding number \( 1 \) and dominating number \( \kappa \).

This completes the proof of Proposition 4.12. \( \square \)

The obvious question now is whether we can have a family with upper bound \( b \) and uncountable bounding number; we shall see that there is no such family in the Mathias model and hence such families cannot be constructed in ZFC alone.

## 5 Models with few families of functions

We shall be interested in two forcing notions.

**Definition 5.1**

**Mathias forcing** \( M_1 = \{ \langle a, A \rangle : a \in [\omega]^{<\omega}, A \text{ is an infinite subset of } \omega \text{ disjoint from } a \} \) equipped with the ordering
\[
\langle a, A \rangle \leq \langle b, B \rangle \text{ iff } A \subseteq B, b \subseteq a, \text{ and } a \setminus b \subseteq B.
\]
**Matet forcing** $M_2 = \{ \langle a, A \rangle : a \in [\omega]^\omega, A \text{ is an infinite set of pairwise disjoint finite subsets of } \omega \}$ equipped with the ordering

$$\langle a, A \rangle \leq \langle b, B \rangle \text{ iff } b \subseteq a, a \setminus b$$

and members of $A$ are finite unions of elements of $B$.

We use $M_1$ and $M_2$ to denote the models obtained from a model of CH by an $\aleph_2$ iteration with countable support of the (proper) partial orders $M_1$ and $M_2$ respectively.

It is known from [3] that $M_2$ satisfies $u < g$ and hence by [7, 8] that the only unbounded (downward closed) families in this model are from the 3 classes described in §3. Further, $M_2$ satisfies $b = u = \aleph_1$ and $d = c = \aleph_2$. On the other hand, the Mathias model $M_1$ satisfies $b = d = u = c = \aleph_2$.

One can modify Baumgartner’s result that Mathias forcing preserves towers to gaps with uncountable upper bound and bounding number and extend it to Matet forcing as well:

**Proposition 5.2** If $M \models ZFC$ and $G$ is either $M_1$ or $M_2$ generic, then any $(\omega_1, \omega_1)$ gap from $M$ remains a gap in $M[G]$.

The iteration lemmas of Shelah [15] give us:

**Proposition 5.3** In $M_1$ or $M_2$, the only linearly ordered $(\lambda, \kappa)$ gaps have either $\lambda \leq \omega$, $\kappa \leq \omega$ or else $\lambda = \kappa = \omega_1$.

Here are therefore the families we get in $M_1$. From Propositions 4.2 we get families $F$ with countable upper bound with $b(F) = 1, 2$ or $\omega$ and $d(F) = \omega_2$. By Proposition 4.7, we get families with countable upper bound, bounding number $\omega_1$ or $\omega_2$ and dominating number $\aleph_2$ by using $H = \omega^{\omega_1}$. There are no such families with $d(F) = \omega_1$ by Corollary 4.5. For families with upper bound $\omega_1$ or $\omega_2 = b$, Propositions 4.11 and Hausdorff’s result provide a general context, although I do not know if $M_1$ has a family with upper bound and bounding number $\omega_1$, and dominating number $\aleph_2$; there is however an unbounded family $F$ in $M_1$ with $b(F) = \omega_1$ and $d(F) = \omega_2$. The two Propositions above 5.2 and 5.3 justify our remark of §4.3 that no gaps with upper bound $b$ has uncountable bounding number in $M_1$. Indeed, a standard argument would force such a family $F$ to reflect to some $F_\alpha = F \cap M_1[G_\alpha]$ for some $\alpha < \omega_2$ where $b(F_\alpha) = b(F_\alpha) = \omega_1$ and be equivalent in this model to a linearly ordered $(\omega_1, \omega_1)$ gap. Since this gap would be preserved to $M_1$, we obtain $b(F) = \omega_1$, a contradiction.

In $M_2$ we have a little more:

**Proposition 5.4** In $M_2$ there are no $<^*$-increasing or $<^*$-decreasing chains of length $\omega_2$. 
Proof: It suffices to prove that there are no increasing chains of size $\omega_2$. Let $\mathcal{F}$ be such a chain. If $\mathcal{F}$ is unbounded, it would have to belong to one of the 3 classes described in §3. Clearly $\mathcal{F}$ cannot be dominating as $b(\mathcal{F}) = \omega_2 > 2$. If finally $\mathcal{F}$ would belong to the $\mathcal{U}$-class, then Proposition 3.6 would provide us with a $P_{\omega_2}$-point which do not exist in $M_2$ by [4]. Therefore $\mathcal{F}$ must be bounded and the above preservation results show that $b_{\downarrow}(\mathcal{F})$ is countable; then Rothberger’s result, Proposition 3.2, produces an unbounded $<_\ast$-increasing chain $\mathcal{H}(\mathcal{F})$ of size $\omega_2$ which we have just showed does not exist.

This provides an alternative model to Theorem 3.1 of Shelah and Stepans [16] showing the failure of Nyikos’ axiom 6.5. Indeed the above shows that any family has an unbounded subset of size at most $\omega_1$ and since NCF holds in $M_2$ as it follows from $u < g$, we conclude that $\text{cof}(\omega_\omega/U) = d = \aleph_2$ for all ultrafilters $U$ (see [1]).

To summarize, we have the following bounded families in $M_2$. For $\lambda = 1, 2$ or $\omega$ and $\aleph_1 \leq \kappa \leq \aleph_2$, Proposition 4.2 gives us $\mathcal{F}$ with countable upper bound such that $b(\mathcal{F}) = \lambda$ and $\mathfrak{d}(\mathcal{F}) = \kappa$. For $\lambda = \omega_1$, Proposition 4.2 again gives us $\mathcal{F}$ countable upper bound, bounding number $\omega_1$ and dominating number $\aleph_2$; as $u = \aleph_1 < d = \aleph_2$, we get a $P_{\aleph_1}$-point in $M_2$ and Proposition 3.6 together with Proposition 4.7 give us a family with countable upper bound, bounding number $\omega_1$ and dominating number $\aleph_1$. There are no such families with bounding number $\omega_2$ as remarked above. Now for families with upper bound $\omega_1$, there those with bounding number $\lambda = 1, 2$ or $\omega$ and dominating number between $\lambda$ and $\omega_2$ by Propositions 4.11 and 4.12 as $b = \omega_1$. Hausdorff’s result provides one with bounding and dominating number $\omega_2$ and again I do not know if there is one with (upper bound $\omega_2$) bounding number $\omega_1$ and dominating number $\omega_2$. There are no families with upper bound $\omega_2$.

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