On divisorial filtrations on sheaves.

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Abstract

A notion of Poincaré series was introduced in [1]. It was developed in [2] for a multi-index filtration corresponding to the sequence of blow-ups. The present paper suggests the way to generalize the notion of Poincaré series to the case of arbitrary locally free sheaf on the modification of complex plane \(\mathbb{C}^2\). This series is expressed through the topological invariants of the sheaf. For the sheaf of holomorphic functions the answer coincides with the Poincaré series from [2].

1 Introduction

In [2] F. Delgado and S. M. Gusein-Zade have computed the Poincaré series of the multi-index filtration defined by a finite collection of divisorial valuations on the ring \(O_{\mathbb{C}^2,0}\) of germs of functions of two variables. Similar to functions, the pull back map lifts holomorphic 1-forms to the space of modification.

Therefore one could define a filtration on the space of germs of holomorphic 1-forms on \(\mathbb{C}^2\). This filtration naturally corresponds to a filtration on the space of global sections of the sheaf of 1-forms on the plane’s modification.

Calculating the Poincaré series of this filtration seems to be much more difficult than for the functions. Hence it is suggested to substitute the space of global sections by the corresponding sheaf and to calculate Euler characteristics of the quotient sheaves, organizing them into a generating series.

It is shown below that the answer for \(O_{\mathbb{C}^2,0}\) coincides with one from [2]. Theorem 1 gives the formula for an arbitrary locally free sheaf on the

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space of modification in terms of Chern classes of its restriction onto the exception al lines.

As an example, the series for the sheaf of 1-forms is calculated in Theorem 2.

By \( \nu \) denote the element \( (v_1, \ldots, v_s) \) of the lattice \( \mathbb{Z}^s \). There is a natural partial ordering on \( \mathbb{Z}^s \): \( \nu \leq w \) if every coordinate of \( \nu \) is less or equal to the corresponding coordinate of \( w \). For the pair \( \nu, w \in \mathbb{Z}^s \) let the upper bound \( \text{sup}\{\nu, w\} \) be the smallest (according to this ordering) element of \( \mathbb{Z}^s \), which is more or equal \( \nu \) and \( w \).

**Definition:** A decreasing \( s \)-index filtration on the vector space \( L \) is the family of subspaces \( \{L(\nu) | \nu \in \mathbb{Z}^s\} \) such that the following conditions hold:

1) if \( \nu_1 \leq \nu_2 \), then \( L(\nu_1) \supset L(\nu_2) \);
2) \( L(\nu) \cap L(w) = L(\text{sup}\{\nu, w\}) \);
3) \( L(0) = L(0) = (0, \ldots, 0) \).

Let \( L(\nu) \) be a \( s \)-index filtration on the space \( L \), and all quotient spaces \( L(\nu)/L(\nu + 1) \) \( (\frac{1}{1} = (1, \ldots, 1)) \) are finite dimensional. Denote \( d(\nu) = \dim L(\nu)/L(\nu + 1) \). From (2) and (3) it follows that for all \( \nu_{i_0} < \nu'_{i_0} \leq 0 \) one has

\[
L(v_1, \ldots, v_{i_0}', \ldots, v_s) = L(v_1, \ldots, v_{i_0}'', \ldots, v_s),
\]

hence the filtration is defined by the set of subspaces \( L(\nu) \) with \( \nu \) such that all its components are nonnegative.

Let \( \mathcal{L} = \mathbb{Z}[[t_1, \ldots, t_s, t_1^{-1}, \ldots, t_s^{-1}]] \) be the space of the formal Laurent series of \( s \) variables. Elements \( \mathcal{L} \) are of the form \( \sum_{\nu \in \mathbb{Z}^s} k(\nu) \cdot t^\nu \), generally speaking, infinite in all directions. \( \mathcal{L} \) is not a ring, but a module over the space of polynomials. Let

\[
Q(t_1, \ldots, t_s) = \sum_{\nu \in \mathbb{Z}^s} d(\nu) \cdot t^\nu.
\]

Since for \( v_{i_0}' < v_{i_0}'' \leq 0 \) one has

\[
d(v_1, \ldots, v_{i_0}', \ldots, v_s) = d(v_1, \ldots, v_{i_0}'', \ldots, v_s),
\]

the expression

\[
P'(t_1, \ldots, t_s) = Q(t_1, \ldots, t_s) \cdot \prod_{i=1}^{s} (t_i - 1)
\]

is a power series, i.e. an element of the subset \( \mathbb{Z}[[t_1, \ldots, t_r]] \subset \mathcal{L} \).
Definition: We call the series

\[ P_L(t_1, \ldots, t_s) = \frac{P'(t_1, \ldots, t_r)}{t_1 \cdots t_s - 1} \]

the Poincaré series of the multi-index filtration \( \{L(v)\} \) on the space \( L \).

This definition of Poincaré series was introduced in [1].

Let \( \pi : (\mathcal{X}, D) \to (\mathbb{C}^2, 0) \) be a proper analytic map which is an isomorphism outside of the origin in \( \mathbb{C}^2 \) such that \( \pi \) is obtained by a sequence of \( s \) point blow-ups. Therefore the exceptional divisor \( D \) is the union of \( s \) irreducible components \( E_i \), each of them is isomorphic to the complex projective line.

**Lemma 1** The pull back map \( \pi^* \) is an isomorphism between \( H^0(\mathbb{C}^2, \Omega^k_{\mathbb{C}^2}) \) and \( H^0(\mathcal{X}, \Omega^k_{\mathcal{X}}) \).

**Proof.** Let \( w \) be a holomorphic \( k \)-form on \( \mathbb{C}^2 \) such that \( \pi^* w = 0 \). Then \( w = 0 \) outside the origin, so \( w = 0 \), and \( \ker \pi^* = 0 \).

Let \( \tilde{w} \) be a holomorphic \( k \)-form on the variety \( \mathcal{X} \). Since \( \pi^* \) is an isomorphism outside the exceptional divisor, there is well-defined holomorphic \( k \)-form \( w = (\pi^*)^{-1} \tilde{w} \) on \( \mathbb{C}^2 \setminus \{0\} \). By the Hartogs’ theorem it could be continued to an holomorphic \( k \)-form \( w \) on \( \mathbb{C}^2 \). Then \( \pi^* w \) coincides with \( \tilde{w} \) outside \( D \), so they coincide in every point of \( \mathcal{X} \). Therefore \( \text{Im} \pi^* = H^0(\mathcal{X}, \Omega^k_{\mathcal{X}}) \), so \( \pi^* \) is an isomorphism. \( \square \)

Let \( \mathcal{J}_i \) be the sheaf of ideals of a subscheme \( E_i \) in \( \mathcal{X} \) and

\[ d_i^j = -(E_i \circ E_j), \quad M = (m_i^j) = D^{-1}, \]

where \((\circ)\) denotes an intersection number. Let \( \tilde{E}_i \) be the smooth part of line \( E_i \), i.e., the component \( E_i \) without intersection points with all other components of the exceptional divisor.

Let \( \mathcal{G} \) be an arbitrary locally free sheaf on \( \mathcal{X} \). For every collection \( \underline{k} = (k_1, \ldots, k_s) \) of nonnegative numbers multiplication induces a natural embedding \( \mathcal{G} \otimes \prod_{i=1}^s \mathcal{J}_i^{k_i} \hookrightarrow \mathcal{G} \). Therefore a multi-index filtration with sub-sheaves is defined on \( \mathcal{G} \):

\[ \mathcal{G}(k_1, \ldots, k_s) = \mathcal{G} \otimes \prod_{i=1}^s \mathcal{J}_i^{k_i}. \]

This filtration defines a multi-index filtration \( H^0(\mathcal{X}, \mathcal{G}(k_1, \ldots, k_s)) \) on the space of the global sections \( H^0(\mathcal{X}, \mathcal{G}) \). Poincaré series of the last filtration
for $G = \mathcal{O}_X$ was computed in [1]:

$$P_O = \prod_i (1 - t^m_i)^{-\chi(E_i)}.$$ 

Following lemma 1, $H^0(X, \mathcal{O}_X(k_1, \ldots, k_s))$ could be considered as the space of functions $f$ on $\mathbb{C}^2$ such that $\pi^*f$ vanishes on component $E_i$ of the exceptional divisor with multiplicity higher or equal to $k_i$. Consider a filtration on the space of 1-forms over $\mathbb{C}^2$ where subspace with label $k$ consists of forms $w$ such that $\pi^*w$ vanishes on $E_i$ with multiplicity higher or equal $k_i$. From lemma 1 it follows that Poincaré series of this filtration coincides with Poincaré series of the filtration which is defined above in case $G = \Omega^1_X$.

Suppose $\pi$ is the unique blow-up. Let $\omega = f(x, y)dx + g(x, y)dy$ be an 1-form, $k$ is the smallest of orders of functions $f$ and $g$ at the origin. Decompose $\omega$ in the sum

$$\omega = \omega_k + \omega_{k+1} + \ldots$$

where

$$\omega_m = f_m dx + g_m dy,$$

$f_m$ and $g_m$ are the homogeneous components of $f$ and $g$ with degree $m$. Let $\theta$ be the affine coordinate in chart $\{x \neq 0\}$ of the exceptional divisor. Then $y = \theta \cdot x$, so

$$\pi^*\omega = (f_k(x, y) + \theta \cdot g_k(x, y) + f_{k+1}(x, y) + \theta g_{k+1}(x, y))dx + xg_k(x, y)dy + \ldots = x^k(f_k(1, \theta) + \theta \cdot g_k(1, \theta))dx + x^{k+1}(f_{k+1}(1, \theta) + \theta g_{k+1}(1, \theta))dx + g_k(1, \theta)dy + \ldots$$

Hence $\pi^*\omega$ has multiplicity on $D$ bigger than $k$ if and only if

$$f_k(x, y) + \theta \cdot g_k(x, y) = 0,$$

i.e., $xf_k + yg_k = 0$, so $xf_k = -yg_k$. Therefore

$$f_k = -y\varphi, \quad g_k = x\varphi$$

for a homogeneous polynomial $\varphi(x, y)$ with degree $k - 1$, thus

$$\omega_k = \varphi(x dy - ydx).$$

Then

$$\pi^*\omega = x^{k+1}(f_{k+1}(1, \theta) + \theta g_{k+1}(1, \theta))dx + g_k(1, \theta)dy + \ldots.$$
and since $g_k \neq 0$, the form $\pi^* \omega$ has a multiplicity equal to $k + 1$ on $D$.

Finally, $F(v)$ consists of forms

$$\varphi(xdy - ydx) + f dx + gdy,$$

where $\varphi$ is a homogeneous polynomial with degree $v - 2$ and $f$ and $g$ have orders, bigger or equal to $v$. Let us compute $P(t)$. Denote by $d_k = k + 1$ the dimension of the space $D_k$ of homogeneous polynomials of two variables with degree $k$, $R_k$ be the space of 1-forms $\varphi(xdy - ydx)$ where $\varphi$ is a homogeneous polynomial with degree $k - 1$. Then $\dim R_k = d_{k-1},$

$$F(k)/F(k + 1) \cong [(D_k \oplus D_k)/R_{k-1}] \oplus R_{k-2},$$

hence $\dim(F(k)/F(k + 1)) = 2d_k - d_{k-1} + d_{k-2},$

$$P(t) = \frac{2 - t + t^2}{(1 - t)^2} = \frac{1 + t}{(1 - t)^2} + 1.$$

## Geometrical Poincaré series

The space of global sections and its dimension are quite sophisticated invariants of a sheaf. It is simpler to calculate its Euler characteristic. Let

$$h(\underline{v}) = \chi\left(\mathcal{X}, \mathcal{G}(\underline{v})/\mathcal{G}(\underline{v} + 1)\right).$$

Since the sheaf $\mathcal{G}(\underline{v})/\mathcal{G}(\underline{v} + 1)$ is supported on $D$, one could see that

$$h(\underline{v}) = \chi\left(D, \mathcal{G}(\underline{v})/\mathcal{G}(\underline{v} + 1)\right).$$

Consider a formal Laurent series $\tilde{Q}(t_1, \ldots, t_s) = \sum_{\underline{v} \in \mathbb{Z}^s} h(\underline{v}) \cdot \underline{v}$ and denote

$$\tilde{P}'(t_1, \ldots, t_s) = \tilde{Q}(t_1, \ldots, t_s) \cdot \prod_{i=1}^{s} (t_i - 1).$$

As above, $\tilde{P}'$ is a power series, and the series

$$\tilde{P}_G(t_1, \ldots, t_s) = \frac{\tilde{P}'(t_1, \ldots, t_s)}{t_1 \cdots t_s - 1}$$

is well-defined.
For $I \subset I_0 = \{1, 2, \ldots, s\}$ by $|I|$ denote the number of elements in $I$, and let $\underline{1}_I$ be the element of $\mathbb{Z}^r$ such that its components labeled by numbers from $I$ are equal to 1, and other components are equal to zero. Denote

$$k(v) = -\sum_{I \subset I_0} (-1)^{|I|} \chi\left(D, \mathcal{G}(v)/\mathcal{G}(v + \underline{1}_I)\right).$$

**Lemma 2**

$$\tilde{P}_G(t) = \sum_{v \in \mathbb{Z}_{\geq 0}} k(v) \cdot t^v.$$

**Proof.** The coefficient at the monomial $t^v$ in the series

$$\left(\sum_{I \subset I_0} k(v) \cdot t^v\right)(t_1 \cdots t_s - 1)$$

is equal to

$$-\sum_{I \subset I_0} (-1)^{|I|} \chi\left(D, \mathcal{G}(v-\underline{1})/\mathcal{G}(v-1+\underline{1}_I)\right) + \sum_{I \subset I_0} (-1)^{|I|} \chi\left(D, \mathcal{G}(v)/\mathcal{G}(v+1_\underline{I})\right) =$$

$$= \sum_{I \subset I_0} (-1)^{|I|}(h(v - 1 + \underline{1}_I) + h(v)) = \sum_{I \subset I_0} (-1)^{|I|}h(v - 1 + \underline{1}_I).$$

Otherwise, the coefficient at the monomial $t^v$ in the series

$$\tilde{P}'(t) = \left(\sum_{v \in \mathbb{Z}^r} h(v) \cdot t^v\right) \cdot \prod_{i=1}^s (t_i - 1)$$

is also equal to $\sum_{I \subset I_0} (-1)^{|I|}h(v - 1 + \underline{1}_I)$. □

Denote $\mathcal{G}|_{E_i}$ by $\mathcal{G}_i$, and let $r$ be the rank of $\mathcal{G}$.

**Lemma 3** Suppose that $0 \to \mathcal{E} \to \mathcal{F} \to \mathcal{G} \to 0$ is an exact sequence of sheaves on $\mathbb{P}^1$, $\mathcal{K}$ is a locally free sheaf on $\mathbb{P}^1$. Then the sequence $0 \to \mathcal{K} \otimes \mathcal{E} \to \mathcal{K} \otimes \mathcal{F} \to \mathcal{K} \otimes \mathcal{G} \to 0$ is also exact.

**Proof.** By the Birkhoff-Grothendieck splitting theorem every locally free sheaf on $\mathbb{P}^1$ is isomorphic to a direct sum of pervasive: $\mathcal{K} = \sum_{i=1}^r \mathcal{K}_i$. Since the multiplication of the exact triple on the pervasive sheaf does not disturb it exactness, sequences

$$0 \to \mathcal{K}_i \otimes \mathcal{E} \to \mathcal{K}_i \otimes \mathcal{F} \to \mathcal{K}_i \otimes \mathcal{G} \to 0$$

are exact. Therefore the desired sequence is exact as the direct sum of exact ones. □
Lemma 4 Let \( D = \sum_i k_i p_i \) be an effective divisor on \( \mathbb{P}^1 \). If 
\[
0 \to E \to F \to G \to 0
\]
is an exact sequence of locally free sheaves, then
\[
\chi(E - D) - \chi(F - D) + \chi(G - D) = 0.
\]

**Proof.** \( \chi(F - D) = \chi(F) - \text{deg}(D) \cdot \text{rk}(F) = \chi(E) + \chi(G) - \text{deg}(D) \cdot (\text{rk}(E) + \text{rk}(G)) = \chi(E - D) + \chi(G - D). \) □

Lemma 5 By \( c_{(i)} \) denote the value of the first Chern class of \( G_i \) on the fundamental homological class \([E_i]\). Then
\[
\chi\left( \frac{E_i G(k)}{G(k + 1_{(i)})|_{E_i}} \right) = r + c_{(i)} + r \cdot k \cdot d^i.
\]

**Proof.** Let \( \nu_i \) be the normal sheaf to the \( E_i \) in the variety \( X \). One has an exact triple 
\[
0 \to J_i^{2i} \to J_i \to \nu_i^* \to 0
\]
is also exact. Since \( G_i \) is locally free, one has an exact sequence
\[
0 \to G_i \otimes J_i^{ki} \to G_i \otimes J_i \to G_i \otimes \nu_i^{-ki} \to 0,
\]
therefore by lemma 4
\[
\chi\left( \frac{E_i G(k)}{G(k + 1_{(i)})|_{E_i}} \right) = \chi\left( G_i \otimes \nu_i^{-ki} \otimes \prod_{b \neq i} J_b^{kb} \right).
\]

Let \( a \) be a canonical generator in \( H^2(\mathbb{P}^1, \mathbb{Z}) \), then Chern character of \( G_i \) is equal to \( ch(G_i) = r + c_{(i)} a \). Otherwise, \( ch(\nu_i^*) = 1 + d^ia \), then
\[
ch(G_i \otimes \nu_i^{-ki}) = (r + c_{(i)} a)(1 + d^i a)^{ki} = r + (c_{(i)} + rd^i k_i) a,
\]
so \( \chi(G_i \otimes \nu_i^{-ki}) = r + c_{(i)} + rd^i k_i \).

Hence
\[
\chi\left( \frac{E_i G(k)}{G(k + 1_{(i)})} \right) = r + c_{(i)} + rd^i k_i - \sum_{E_j \cap E_i \neq \emptyset} r k_j = r + c_{(i)} + r \sum_j d^i k_j.
\]
□

For \( I \subset I_0 \) by \( \mu(I) \) denote a number of intersection points of different lines with labels from the set \( I \).
Lemma 6 Let $\mathcal{H} = \mathcal{G}(k)/\mathcal{G}(k + 1)$. Then
\[
\chi(\mathcal{D}, \mathcal{H}) = r\mu(I) + \sum_{i \in I} \left( r + c_{(i)} + r(k + 1_{I \setminus \{i\}})d_i \right).
\]

Proof. Consider $i \in I$. One could see that
\[
\chi\left( E_i, \mathcal{G}(k)/\mathcal{G}(k + 1) \right) = \chi\left( E_i, \mathcal{G}(k)/\mathcal{G}(k + 1_{I \{i\}}) \right) + \chi\left( E_i, \mathcal{G}(k + 1_{I \{i\}})/\mathcal{G}(k + 1_i) \right) = -r \sum_{b \in I \setminus \{i\}} d_{ib} + r + c_{(i)} + r(k + 1_{I \{i\}})d_i.
\]
Since $\mathcal{H}$ is supported on $\bigcup_{i \in I} E_i$, from the Mayer-Vietoris exact sequence it follows that
\[
\chi(\mathcal{D}, \mathcal{H}) = \sum_{i \in I} \left( r + c_{(i)} + r(k + 1_{I \{i\}})d_i \right) - r \mu(I) = \sum_{i \in I} \left( r + c_{(i)} + r(k + 1_{I \{i\}})d_i \right) + 2r \mu(I) - r \mu(I).
\]
\[
\square
\]
Using lemma 6, one could calculate the series $\tilde{P}_G(t)$ for $G = \mathcal{O}_X$. Let
\[
m_i^l = -1_{I \{i\}}d_i
\]
be the number of intersection points $E_i$ with other components of exceptional divisor $\mathcal{D}$ with labels from the set $I$.

Lemma 7 If $s > 2$ then
\[
\tilde{P}_G(t) \equiv 0.
\]

Proof. From lemma 2 it follows that the coefficient at $t^k$ is equal to
\[
- \sum_{I \subset I_0} (-1)^{|I|} \chi\left( \mathcal{D}, \mathcal{G}(k)/\mathcal{G}(k + 1) \right) = - \sum_{I \subset I_0} (-1)^{|I|} \left( \sum_{i \in I} \left( 1 + (k + 1_{I \{i\}})d_i \right) \right).
\]
Denote \( v_i = 1 + kd_i \), then the expression for the coefficient could be represented in the form

\[
- \sum_{I \subseteq I_0} (-1)^{|I|}(\mu(I) + \sum_{i \in I} (v_i - m_i^I)).
\]

Since in the sum \( \sum_{i \in I} m_i^I \) every intersection point is taken into account twice, it is clear that \( \sum_{i \in I} m_i^I = 2\mu(I) \). Hence the expression is equal to

\[
- \sum_{I \subseteq I_0} (-1)^{|I|} \sum_{i \in I} (v_i - \frac{m_i^I}{2}) = - \sum_{i} (v_i \sum_{I \ni i} (-1)^{|I|} + \sum_{I \ni i} (-1)^{|I|} \frac{m_i^I}{2}) =
\]

\[
= 0 + \frac{1}{2} \sum_{i} \sum_{I: i \in I, \forall j \in I d_i < 0} (-1)^{|I|} m_i^I \sum_{J: \forall j \in J d_i^J = 0} (-1)^{|J|}.
\]

If there are divisors, which does not intersect \( E_i \), one has

\[
\sum_{J: \forall j \in J d_i^J = 0} (-1)^{|J|} = 0.
\]

In the opposite case \( E_i \) is intersected by \( s - 1 \) divisors and the \( i \)-th term is equal to \(-\frac{1}{2} \sum_{m=0}^{s-1} (-1)^m m \binom{s-1}{m} \) which vanishes if \( s - 1 > 1 \). \( \Box \)

This result is unnatural, but it shows that \( P_G \) could not be interpreted as generalization of the notion of Poincaré series in any sense.

Let \( \psi(x) \) be equal to \( x \) if \( x \geq 0 \) and equal to zero in the opposite case.

**Lemma 8** Let \( \mathcal{G} = \mathcal{O}_X \). Then

\[
\dim H^0 \left( \mathcal{D}, \mathcal{G}(k)/\mathcal{G}(k + 1) \right) = \mu(I) + \sum_{i \in I} \psi\left( \chi(E_i, \mathcal{G}(k)/\mathcal{G}(k + 1)) \right).
\]

**Proof.** Compute \( \dim H^0 \left( E_i, \mathcal{G}(k + 1\setminus \{i\})/\mathcal{G}(k + 1\setminus \{i\}) \right) \). Let \( f \) be the function on \( X \) representing the section of the corresponding sheaf on \( E_i \). Since \( \mathcal{I}_i^{k_i}/\mathcal{I}_i^{k_i+1} \simeq \nu_i^{-k_i} \), the class of \( f \) is a well-defined section of \( \nu_i^{-k_i} \simeq \mathcal{O}(k_i d_i) \).

The space of global sections of this bundle which vanish at intersection points with other divisors \( E_j, j \neq i \) with order bigger or equal to \( k_i d_i \) is canonically isomorphic to the

\[
H^0 \left( \mathbb{P}^1, \mathcal{O}(k_i d_i^i + \sum_{j \neq i} \sum_{k_i d_i^j} + \sum_{b \in I \setminus \{i\}} d_i^b) \right)
\]
and its dimension is equal to \( \psi((\bar{k} + 1_{\mathbb{I} \setminus \{i\}})d^i + 1) \). Furthermore,

\[
\dim H^0\left( \mathbb{P}^1, \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_{\mathbb{I} \setminus \{i\}}) \right) = -1_{\mathbb{I} \setminus \{i\}}d^i
\]

and \( H^1\left( \mathbb{P}^1, \mathcal{G}(\bar{k} + 1_{\mathbb{I} \setminus \{i\}})/\mathcal{G}(\bar{k} + 1_I) \right) = 0 \), hence from the exact sequence

\[
0 \to \mathcal{G}(\bar{k} + 1_{\mathbb{I} \setminus \{i\}})/\mathcal{G}(\bar{k} + 1_I) \to \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_I) \to \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_{\mathbb{I} \setminus \{i\}}) \to 0
\]

\[
\dim H^0\left( E_i, \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_I) \right) = \psi\left( \chi(E_i, \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_I)) \right) - 1_{\mathbb{I} \setminus \{i\}}d^i.
\]

Since \( \mathcal{G}(\bar{k})/\mathcal{G}(\bar{k} + 1_I) \) is supported on the union of lines with numbers from \( I \), it is sufficient to consider the space of its global sections over this union. For such global section we could construct the collection of sections over \( E_i \) which are uniquely determined by sets of their zeros up to the multiplication on a constant.

Otherwise, if we have a collection of points with multiplicities on \( E_i \), let us draw through every point a germ of analytical curve intersecting \( E_i \) with the corresponding multiplicity. Under projection on \( \mathbb{C}^2 \) we will have a germ of a reducible curve. Let us define this germ by equation \( \{ g = 0 \} \) and consider a function \( \pi^*g \) (\( g \) is determined uniquely up to the multiplication on the function, which does not vanish at the origin). Since on every \( E_i \) section corresponding to \( \pi^*g \) has as much zeros as the section, from which we have started, it follows that \( \pi^*g \) matches the same sections of the same powers of conormal bundles.

Therefore Mayer-Vietoris sequence is also exact in the part with global sections. This note proves the lemma. □

Analogous to lemma 2 it is easy to prove that the coefficient at \( t^k \) in \( P_L \) is equal to \( -\sum_{I \subset I_0} (-1)^{|I|} \dim(L(\bar{k})/L(\bar{k} + 1_I)) \).

**Definition:** The series

\[
P^g_{\mathfrak{g}}(\mathfrak{z}) = -\sum_{k \in \mathbb{Z}_{\geq 0}} t^k \sum_{I \subset I_0} (-1)^{|I|} \left( r\mu(I) + \sum_{i \in I} \psi \left( \chi(E_i, \mathcal{G}(\bar{k} + 1_{\mathbb{I} \setminus \{i\}})/\mathcal{G}(\bar{k} + 1_I)) \right) \right)
\]

is said to be geometrical Poincaré series of the filtration on the sheaf \( \mathcal{G} \).

It is clear from lemma 7 that geometrical Poincaré series for \( \mathcal{O}_{\mathbb{C}^2,0} \) coincides with the series \( P_{H^0} \) computed in \( [2] \), so it is reasonable to consider the geometrical Poincaré series as a proper generalization of the notion of Poicaré series for the space of global sections.
Lemma 9 Let $a_i$ be nonnegative integers. Then

$$- \sum_{I \subseteq I_0} (-1)^{|I|} \left( \mu(I) + \sum_{i \in I, a_i + 1 - m_i^I > 0} (a_i + 1 - m_i^I) \right) = \chi(\prod_i S^{a_i} \tilde{E}_i).$$

Proof. Denote $f(\underline{a}, \mathcal{D}) = \chi(\prod_i S^{a_i} \tilde{E}_i)$. Let us prove the proposition of lemma by induction on the number of exceptional lines. If the line is unique, then $a_1 + 1 = (-1)^{a_1} (-1)^2 = \chi(S^{a_1} E_1)$. If there are two lines, it’s easy to check the lemma’s proposition: $(a_1 + 1)+(a_2 + 1) - ((a_1 + 1) + (a_2 + 1) + 1) = 1$

if $a_1 > 0, a_2 > 0$; other cases are analogous. Let us prove the inductive transition. Since the dual graph of resolution is a tree, it has a vertex with degree 1, i. e., a divisor $E_j$ which intersects the unique line $E_l$. Suppose that $a_i \neq 0$, then $\chi(\tilde{E}_j) = 1$, hence $\chi(S^{a_i} \tilde{E}_j) = 1$; otherwise, if $\tilde{E}_l = \tilde{E}_l \cup (E_j \cap E_l)$, then

$$S^{a_i} \tilde{E}_l = S^{a_i} \tilde{E}_l \cap (E_j \cap E_l) \times S^{a_i-1} \tilde{E}_l,$$

and $\chi(S^{a_i} \tilde{E}_l) = \chi(S^{a_i} \tilde{E}_l) - \chi(S^{a_i-1} \tilde{E}_l)$. Denote $\hat{\mathcal{D}} = \cup_{i \neq j} E_i, \hat{m}_i^I = \hat{m}_i^{I \cup \{j\}}$

Then $\prod_i \chi(S^{a_i} \tilde{E}_i) = \prod_{i \neq j} \chi(S^{a_i} \tilde{E}_i) = f(\underline{a}, \mathcal{D}) - f(\underline{a} - \underline{1}_{\{l\}}, \hat{\mathcal{D}})$. Otherwise, for $A, B, C \subseteq I_0$ denote

$$\mathcal{M}_{A,B,C} = \{(i, I) : i \in I \subseteq I_0, A \cap I = \emptyset, B \subseteq I, i \notin C, a_i + 1 - m_i^I > 0\},$$

and $\hat{\mathcal{M}}_{A,B,C} = \{(i, I) : i \in I \subseteq I_0, A \cap I = \emptyset, B \subseteq I, i \notin C, a_i + 1 - \hat{m}_i^I > 0\}$.

Then

$$\sum_{I \subseteq I_0} (-1)^{|I|} \mu(I) + \sum_{i \in I, a_i + 1 - m_i^I > 0} (a_i + 1 - m_i^I) =$$

$$\sum_{I} (-1)^{|I|} \mu(I) + \sum_{(i,l) \in \mathcal{M}_{\emptyset, \emptyset}} (-1)^{|I|} (a_i + 1 - m_i^I) + \sum_{(j,l) \in \mathcal{M}_{\emptyset, \{i\}, \emptyset}} (-1)^{|I|} (a_j + 1 - m_j^I) +$$

$$\sum_{(j,l) \in \mathcal{M}_{\emptyset, \{i\}, \emptyset}} (-1)^{|I|} (a_j + 1 - m_j^I) + \sum_{(i,l) \in \mathcal{M}_{\emptyset, \{j\}, \emptyset}} (-1)^{|I|} (a_i + m_i^I) +$$

$$\sum_{i \in I, k \notin I} (-1)^{|I|} \mu(I) + \sum_{i \in I, k \notin I} (-1)^{|I|} \mu(I) - f(\underline{a}, \mathcal{D}) =$$

$$\sum_{i \in I, l \neq i} (-1)^{|I|} (a_{i} + 1) + \sum_{i \in I, l \notin I} (-1)^{|I|} a_{i} +$$

$$\sum_{(i,l) \in \mathcal{M}_{\emptyset, \{i\}, \emptyset}} (-1)^{|I|} (a_i + 1 + m_i^I) +$$

$$\sum_{(i,l) \in \mathcal{M}_{\emptyset, \{j\}, \emptyset}} (-1)^{|I|} (a_j + 1 + m_j^I) + \sum_{i \in I, l \notin I} (-1)^{|I|} (\mu(I) + 1) -$$

$$\sum_{i \in I, l \notin I} (-1)^{|I|} \mu(I) - f(\underline{a}, \mathcal{D}) = 0 + f(\underline{a} - \underline{1}_{\{l\}}, \hat{\mathcal{D}}) - f(\underline{a}, \hat{\mathcal{D}}) = - f(\underline{a}, \hat{\mathcal{D}}).$$

If $a_l = 0$, then $f(\underline{a}, \mathcal{D}) = f(\underline{a}, \hat{\mathcal{D}})$. Otherwise,
\[
\sum_{I} (-1)^{|I|} (\mu(I) + \sum_{i \in I, a_i + 1 - m_i} (-1)^{|I|}(a_i + 1 - m_i)) = \\
\sum_{I \cup \{i\} \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} (a_j + 1 - m_j) + \sum_{I \cup \{i\} \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|}(a_j + 1 - m_j) + \\
\sum_{(i, j) \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} (a_j + 1 - m_j) + \sum_{I \cup \{i\} \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} \mu(I) + \\
\sum_{I \cup \{i\} \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} \mu(I) - f(\alpha, \tilde{D}) = \sum_{I \cup \{i\} \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} (a_j + 1) + \\
\sum_{(i, j) \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} (a_i + 1 - \hat{m}_i) + \sum_{(i, j) \in \mathcal{M}_{I, \emptyset}} (-1)^{|I|} (-d_i) - f(\alpha, \tilde{D}) = -f(\alpha, \tilde{D}).
\]

Let \(\zeta_i = 1\) if every exceptional line intersects \(E_i\) and \(\zeta_i = 0\) in the opposite case.

**Lemma 10** Suppose that \(a_i\) are nonnegative integers and \(u_i\) are arbitrary numbers. Then

\[
- \sum_{i, I : i \in I, a_i + 1 - m_i > 0} (-1)^{|I|} u_i = \sum_i \zeta_i u_i (-1)^{a_i} \left(1 - \chi(\tilde{E}_i)\right).
\]

**Proof.** - \(\sum_{i, I : i \in I, a_i + 1 - m_i > 0} (-1)^{|I|} u_i = \)

\[
= - \sum_i u_i \sum_{I : i \in I, a_i + 1 - m_i > 0} (-1)^{|I|} = \\
= \sum_i u_i \sum_{I : i \in I, a_i + 1 - m_i > 0} (-1)^{|I|} \sum_{J : \forall j \in J | I| > 0} (-1)^{|J|}.
\]

The \(i\)-th term is equal to 0 if there exists a line, which does not intersect \(E_i\), and it is equal to \(u_i \sum_{s=0}^{a_i} (-1)^s \binom{2-\chi(\tilde{E}_i)}{s} = u_i (-1)^{a_i} \binom{1-\chi(\tilde{E}_i)}{a_i}\), if all lines intersect \(E_i\). \(\Box\)

**Theorem 1** Geometrical Poincaré series of the filtration on the sheaf \(\mathcal{G}\) is equal to the regular part of the Laurent series

\[
r \prod_i t_i^{-\frac{a_i}{r}} (1-\frac{m_i}{r})^{-\chi(\tilde{E}_i)} - r \prod_i t_i^{-\frac{a_i}{r}} (1-\frac{m_i}{r})^{-1} \sum_i \zeta_i \left(-\frac{1}{r}\right) (1-\frac{m_i}{r})^{2-\chi(\tilde{E}_i)},
\]

where \(\{x\}\) is a fractional part of \(x\), and \(\lfloor x\rfloor\) is the smallest integer bigger or equal to \(x\).
Proof. From lemma 5 it follows that the coefficient at $t^k$ in the geometrical Poincaré series for sheaf $G$ is equal to

$$-\sum_{I \subseteq I_0} (-1)^{|I|} \left( r \mu(I) + \sum_{i \in I} \psi((r + c_{(i)} + r(k + 1_{(i)})d^i)) \right).$$

Denote $v_i = r + c_{(i)} + r \cdot k \cdot d^i$. It is clear that if there exists a term in the sum which does not vanish, then $v_i > 0$. Hence the coefficient is equal to

$$-\sum_{I \subseteq I_0} (-1)^{|I|} \left( r \mu(I) + \sum_{i \in I, v_i - rm_i > 0} (v_i - rm_i) \right) =$$

$$-r \sum_{I \subseteq I_0} (-1)^{|I|} \left( \mu(I) + \sum_{i \in I, \frac{v_i}{r} - m_i > 0} \left( \frac{v_i}{r} - \left\lfloor \frac{v_i}{r} \right\rfloor - m_i \right) \right) + r \sum_{i, I_i \subseteq I, \frac{v_i}{r} - m_i > 0} (-1)^{|I_i|} \left\{ -\frac{v_i}{r} \right\} =$$

$$r \chi \left( \prod_i S^{\left\lfloor \frac{v_i}{r} \right\rfloor - 1} \tilde{E}_i \right) - \sum_i \zeta_i \left\{ -\frac{v_i}{r} \right\} \left( 1 - \chi(\tilde{E}_i) \right)^{\left\lfloor \frac{v_i}{r} \right\rfloor - 1} \left( 1 - \chi(\tilde{E}_i) \right)^{-1}.$$

Furthermore, $\left\lfloor \frac{w_i}{r} \right\rfloor = 1 + \left\lfloor \frac{c_{(i)}}{r} \right\rfloor + kd_i > 0$, i.e., $w_i = \left\lfloor \frac{c_{(i)}}{r} \right\rfloor + kd_i$ can be an arbitrary nonnegative integer and

$$k = (w - \left\lfloor \frac{w}{r} \right\rfloor) M.$$

Therefore the Poincaré series is equal to

$$\sum_{w \in \mathbb{Z}_{\geq 0}} \left( r \chi \left( \prod_i S^{w_i} \tilde{E}_i \right) - r \sum_i \zeta_i \left( \frac{c_{(i)}}{r} \right) (-1)^{w_i} \left( 1 - \chi(\tilde{E}_i) \right) \right) t^{(w - \left\lfloor \frac{w}{r} \right\rfloor) M} =$$

$$= r \prod_i t_i^{-\left\lfloor \frac{c_{(i)}}{r} \right\rfloor} \prod_i (1 - t^{m_i})^{-\chi(\tilde{E}_i)} -$$

$$r \sum_i \zeta_i \left\{ \frac{c_{(i)}}{r} \right\} \prod_j t_j^{-\left\lfloor \frac{c_{(j)}}{r} \right\rfloor} (1 - t^{m_j})^{-1} - \chi(\tilde{E}_i) \prod_{j \neq i} (1 - t^{m_j})^{-1}.$$

Generally speaking, this series contains also finite amount of terms with negative powers $k$. We’ll make the desired expression by throwing them away. □
Theorem 2  Geometrical Poincaré series for the sheaf of 1-forms is equal to the regular part of the series

\[ 2 \prod_{i} t_i^{\left\lfloor \frac{d_i}{2} \right\rfloor} (1 - t^{m_i}) \chi(E_i) - 2 \prod_{i} t_i^{\left\lfloor \frac{d_i}{2} \right\rfloor} (1 - t^{m_i})^{-1} \sum_{i} \zeta_i \{ -\frac{d_i^2}{2} \} (1 - t^{m_i})^2 - \chi(E_i). \]

Proof. In this case \( \mathcal{G} = \Omega^1_X \). Then \( \mathcal{G}_i = T^* \mathcal{X}|_{E_i} \). Consider an exact sequence

\[ 0 \to T E_i \to T \mathcal{X}|_{E_i} \to \nu_i \to 0. \]

Dual complex is:

\[ 0 \to \nu_i^* \to T^* \mathcal{X}|_{E_i} \to T^* E_i \to 0. \]

Therefore \( c_{(i)} = < 1 (T^* \mathcal{X}|_{E_i}), [E_i] > = d_i - 2. \) Now the proposition of the theorem follows from the theorem 1. \( \square \)

If the line is unique, then series is equal to \( \frac{1+t}{(1-t)^2} \). It differs from the Poincaré series on the space of global sections by 1.

If there are two divisors, the answer is equal to \( \frac{1+t_1 t_2}{(1-t_1 t_2)(1-t_1 t_2)} \).

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