On twisting solutions to the Yang-Baxter equation

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Abstract

Sufficient conditions for an invertible two-tensor $F$ to relate two solutions to the Yang-Baxter equation via the transformation $R \to F_{21}^{-1}RF$ are formulated. Those conditions include relations arising from twisting of certain quasitriangular bialgebras.

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1 Introduction

The twist procedure for (quasi)-Hopf algebras developed by Drinfeld \cite{1, 2} (see also \cite{3, 4}) allows to deform the coproduct via a similarity transformation with the multiplication unchanged. Twist finds its applications in solvable models and noncommutative geometry because it appears to be very friendly to all the algebraic properties of a given Hopf algebra, including the quasitriangular structure and the structure of modules. However, to find the explicit form of an element $F$ realizing interrelations between twisted and untwisted objects is as difficult problem as that of evaluating universal $R$-matrices $R$. On the other hand, most applications of quantum groups employ their particular matrix representations, and in practice one deals with matrix solutions to the Yang-Baxter equation (YBE) rather than the universal ones. The FRT algorithm \cite{5} yields the recipe how, starting from finite-dimensional solutions to YBE, to build both quantum groups and universal $R$-matrices $R$. The latter is possible due to the factorization property of $R$ with respect to the coproduct and the famous fusion procedure \cite{7}. It is worth to note that the factorization property is virtually the only tool known for building universal twisting elements as well as their matrix realizations \cite{3, 4}, \cite{8}–\cite{13}. The finite-dimensional (matrix) version of twisting procedure was considered in the general setting in \cite{12}, but up to now there are no general criteria, except already mentioned factorization properties, for a matrix two-tensor $F$ to define twist of a given Hopf algebra. On the other hand, the problem of transforming solutions to the Yang-Baxter equation $R \rightarrow F_{21}^{-1}RF$ makes sense by itself, regardless of the possibility of expanding $F$ to the universal element $F$. In the present paper we formulate sufficient conditions guaranteeing that the transformation of concern should provide a new solution to YBE. Quite amazingly, they involve an invertible three-tensor which itself drops from YBE but ensures its fulfillment.

The article is organized as follows. Section II is devoted to the transformation
of solutions to YBE which we, by the analogy with that arising within Drinfeld’s theory, call twist. The relation between twist of bialgebras and twist of R-matrices is discussed in Section III. Section IV demonstrates some examples when twist of a matrix solution to YBE can be extended to the global twist of the quantum algebra dual to the corresponding FRT quantum semi-group. In Conclusion we discuss possible applications of the results obtained.

2 Twists of R-matrices

Throughout the paper $R$ will denote an associative algebra with unit over a field $k$. The main result of the present communication is given by the following assertion.

**Theorem 1** Let $R \in R^{\otimes 2}$ be a solution to the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12},$$

and invertible elements $F \in R^{\otimes 2}$, $\Phi \in R^{\otimes 3}$, and $\Psi \in R^{\otimes 3}$ fulfill the following conditions

$$\Phi_{123}F_{12} = \Psi_{123}F_{23}, \quad \text{(1)}$$

$$R_{12}\Phi_{123} = \Phi_{213}R_{12}, \quad \text{(2)}$$

$$R_{23}\Psi_{123} = \Psi_{132}R_{23}, \quad \text{(3)}$$

Then

$$R = \tau(F^{-1})RF, \quad \text{(4)}$$

is a solution to the Yang-Baxter equation too ($\tau$ is the permutation of the tensor components).
Transformation (4) is called *twist* of a solution to YBE. For a matrix ring $R$ this turns into a similarity transformation of the braid matrix $\hat{R} = PR$, with $P$ representing the permutation operation in the corresponding vector space. In fact, twist is determined by a pair $(F, G), \quad G \in R \otimes^3$, such that $\Phi = \bar{G} \bar{F}^{12}$ and $\Psi = \bar{G} \bar{F}_{23}$ obey the conditions of the theorem (to make formulas more readable we denote the inverse by bar).

First let us prove the equality

$$F_{12} \bar{\Psi}_{312} R_{13} R_{23} \Phi_{123} F_{12} = \hat{R}_{13} \hat{R}_{23}. \quad (5)$$

Indeed, using conditions (1–3) along with the definition (4) we find

$$F_{12} \bar{\Psi}_{312} R_{13} R_{23} \Phi_{123} F_{12} = \bar{F}_{21} \bar{R}_{12} \bar{F}_{12} \bar{F}_{312} \bar{R}_{13} \bar{R}_{23} \Phi_{123} F_{12} \bar{R}_{12} F_{21}$$

$$= \bar{F}_{21} \bar{R}_{12} \bar{F}_{12} \bar{F}_{312} \bar{R}_{13} \bar{R}_{23} \Phi_{123} F_{12} \bar{R}_{12} F_{21}$$

$$= \bar{F}_{21} \bar{F}_{321} \bar{R}_{12} \bar{R}_{13} \bar{R}_{23} \Phi_{213} F_{21} \bar{R}_{12} F_{21}$$

$$= \bar{F}_{21} \bar{F}_{321} \bar{R}_{23} \bar{R}_{13} \Phi_{213} F_{21}$$

$$= \tau_{12}(\bar{F}_{12} \bar{F}_{312} \bar{R}_{13} \bar{R}_{23} \Phi_{123} F_{12})$$

$$= \tau_{12}(\bar{R}_{13} \bar{R}_{23}) = \hat{R}_{23} \hat{R}_{13}, \quad (6)$$
Theorem 1 can be understood within the framework of the bialgebra twist theory in its matrix formulation rendered in some detail in the next section. Although twist of an R-matrix might not be extended to the global bialgebra twist as discussed later on, it possesses many familiar features, for example, the composition property. For a pair \((F, G)\) the inverse is \((\bar{F}, \bar{G})\). For two pairs \((F, G)\) and \((F', G')\), where the latter is defined through the new R-matrix \(\tau(\bar{F})RF\), there exists their composition \((FF', GG')\). And, finally, \((e^2, e^3)\) (the units in the tensor square and cube of \(R\)) realizes the identical transformation. Another similarity with the bialgebra twist is the existence of the set of gauge transformations of the twisting pair as will be shown in the next section.

3 On the global twist and braid group representation equivalence

Although the most evident applications of the observation made in the previous section can be relevant to finite-dimensional matrix rings \(R\), usually those of fundamental representations of Hopf algebras of interest, Theorem 1 holds for any \(R\). So, one can consider \(R = \text{Mat}(N)[[\lambda, \lambda^{-1}]]\) and R-matrices depending on the spectral parameter \(\lambda\). As another example, let us take a quasitriangular bialgebra \(H\) as \(R\) and an element \(F \in H^{\otimes 2}\) satisfying the twist equation \[ (\Delta \otimes \text{id})(F)_{12} = (\text{id} \otimes \Delta)(F)_{23}. \] Then, one can put \(F = \mathcal{F}\) and \(G\) to be the expression on either side of (7).

To explain the result obtained we shall use the formalism dual to the FRT algorithm of constructing quantum semi-groups. Recall that the tensor bialgebra \(T(R)\) over \(R\) is introduced as the direct sum of ideals \(T(R) = \sum_{m=0}^{\infty} R^{\otimes n}\), where \(R^0\) is isomorphic...
to $k$, the field of scalars. The unit in $T(\mathcal{R})$ is represented by the sum of idempotents $1 = \sum_{n \geq 0} e^n$, the units in $\mathcal{R} \otimes_n$, respectively. Multiplication by $e^n$ realizes the projection homomorphism $T(\mathcal{R}) \rightarrow \mathcal{R} \otimes_n$, and for $n = 0$ this coincides with the counit mapping to $k$, the coproduct being introduced on the basis elements $x^{i_1..i_n} \in \mathcal{R} \otimes_n$ as

$$\Delta(x^{i_1..i_n}) = e^0 \otimes x^{i_1..i_n} + \ldots + x^{i_1..i_k \otimes x^{i_{k+1}..i_n} + \ldots + x^{i_1..i_n} \otimes e^0}.$$ 

The principal feature of $T(\mathcal{R})$ is that for any bialgebra $\mathcal{H}$ and a representation $\rho: \mathcal{H} \rightarrow \mathcal{R}$ there is the unique extension to the homomorphism of bialgebras $\mathcal{H} \rightarrow T(\mathcal{R})$. It is built by means of the multiple coproduct $\Delta^n: \mathcal{H} \rightarrow \mathcal{H} \otimes^n$ defined for $n = 0$ as the counit, for $n = 1$ as the identical mapping, and for higher $n$ it is $\Delta^2 \equiv \Delta$, $\Delta^3 \equiv (\Delta \otimes id) \circ \Delta$, and so on. Then the homomorphism of concern is specified by the mappings $(\rho \otimes^n \circ \Delta^n): \mathcal{H} \rightarrow \mathcal{R} \otimes^n$.

For a given solution to the Yang-Baxter equation $R \in \mathcal{R} \otimes^2$, one defines a subalgebra $\mathcal{U} = \sum_{n=0}^\infty \mathcal{U}^n$, where $\mathcal{U}^0 = k$, $\mathcal{U}^1 = \mathcal{R}$, and $\mathcal{U}^n = \{z | z \in \mathcal{R} \otimes^n, R_{ii+1}z = \tau_{ii+1}(z)R_{ii+1}, 0 < i < n\}$ ($\tau_{ii+1}$ is the permutation between $i$-th and $i+1$-th sites). Such tensors are called $R$-symmetric, and in the case of matrix rings they just commute with the braid matrix $\hat{R}$.

**Theorem 2** $\mathcal{U}$ is a quasitriangular sub-bialgebra in $T(\mathcal{R})$.

It follows immediately from the definition that $\mathcal{U}$ is a sub-bialgebra indeed. Its universal $R$-matrix is decomposed into the sum of its $\mathcal{R} \otimes^m \otimes \mathcal{R} \otimes^n$-components $R^{m,n}$; for $mn = 0$ it is just $e^m \otimes e^n$, the unit of $\mathcal{R} \otimes^m \otimes \mathcal{R} \otimes^n$, and if $n = n'$ and $m$ are both non-zero, one has

$$R^{m,n'} = (R_{11'} \ldots R_{1n'}) (R_{2n'} \ldots R_{21'}) \ldots (R_{mn'} \ldots R_{m1'}),$$

where primes mark indices numbering $R$-factors in the second tensor component. Note that the bialgebra $\mathcal{U}$ is dual to the quantum semi-group $\mathcal{A}_R$ generated by basis elements of the linear space $\mathcal{R}^*$ with the RTT relations imposed.
Usually, one is interested in Hopf algebras, which require additional relations of the quantum determinant type imposed on the generators of the quantum semi-group. Such relations eliminate just few degrees of freedom while dramatically complicate algebraic structure mixing the homogeneous components. So, we prefer to work with bialgebras, in the dual sector represented by the direct sum of their ideals \( \mathcal{U} = \sum_{n \geq 0} \mathcal{U} \cap R^\otimes n \).

The component representation of the twist equation (7) in \( T(R) \) reads

\[
F_{m+n,k}^m (F_{m}^n \otimes e^k) = F_{m,n+k}^m (e^m \otimes F_{n,k}^n),
\]

with \( F_{i,j} \) being the images of the universal twisting element \( \mathcal{F} \) in \( R^{\otimes i} \otimes R^{\otimes j} \subset T(R)^{\otimes 2} \). It is equal to \( F_{m,n} = (\rho^\otimes m \circ \Delta^m \otimes \rho^\otimes n \circ \Delta^n)(\mathcal{F}) \), where \( \rho \) is the representation of \( \mathcal{H} \) in \( R \). Given a solution to (7), for any quasitriangular bialgebra one can construct twisted quasitriangular bialgebra with the new universal \( R \)-matrix \( R = \tau(\mathcal{F})^{-1}R\mathcal{F} \).

An interesting implication of the equivalent system (8) is that there are no closed conditions on \( F_{1,1} \) directly involved in deformation of \( R = R^{1,1} = (\rho \otimes \rho)(\mathcal{R}) \), which is a matrix solution to YBE. Actually, to obtain new, twisted solutions to the Yang-Baxter equation there is no need to satisfy the whole set of equations (8) recovering the universal element \( \mathcal{F} \), it is sufficient to restrict the study by only the small part of them. This is the observation which underlines Theorem 1.

Let us investigate the question when a twist of an \( R \)-matrix can be extended to the twist of the entire bialgebra \( \mathcal{U} \). Having introduced tensors \( \Omega^2 = F, \Omega^3 = G \), one can see that they satisfy the equalities

\[
R_{ii+1} \Omega^n = \tau_{ii+1}(\Omega^n) \tilde{R}_{ii+1}, \quad 0 < i < n \tag{10}
\]

for \( n = 2, 3 \). If \( R \) is a matrix ring, this establishes a local isomorphism of the braid group \( B_3 \) local representations specified by the matrices \( R \) and \( \tilde{R} \).

**Theorem 3** The pair \( (\Omega^2, \Omega^3) \) is extended to the twist of the bialgebra \( \mathcal{U} \) if and only if for each \( n > 3 \) there exists an invertible element \( \Omega^n \in R^{\otimes n} \) fulfilling (10). Twisting
element is uniquely defined up to an isomorphism via the formula

\[ F^{m,n} = \Omega^{m+n}(\tilde{\Omega}^m \otimes \tilde{\Omega}^n), \quad m, n \geq 0, \]  

where for \( i = 0, 1 \) we set \( \Omega^i = e^i \), the units of \( R^\otimes i \).

It is easy to see that \( F^{m,n} \) introduced according to (11) satisfy (9) and indeed lie in \( U \otimes 2 \) (their each component is \( R \)-symmetric). Let us prove the converse. Given the universal twisting element, define \( \Omega^n \) for \( n > 2 \) as the product

\[ \Omega^n = F^{1,n-1}(e^1 \otimes F^{1,n-2}) \cdots (e^{n-2} \otimes F^{1,1}) = F^{1,n-1}(\Omega^1 \otimes \Omega^{n-1}). \]  

(12)

We are going to state (11) and that would evidently be enough because then we can employ the induction method and the \( R \)-symmetry of the elements \( F^{m,n} \). Conditions (11) hold if one of the numbers \( m \) and \( n \) are zero. They are also true by construction for \( m = 1 \) and any \( n \). Then, for \( m \geq 1 \) one has

\[ \Omega^{1+m+n} = F^{1,m+n}(\Omega^1 \otimes \Omega^{m+n}) = F^{1,m+n}(e^1 \otimes (F^{m,n}(\Omega^m \otimes \Omega^n))) = \]

\[ = F^{1,m+n}(e^1 \otimes F^{m,n})(e^1 \otimes \Omega^m \otimes \Omega^n) = \]

\[ = F^{1+m,n}(F^{1,m} \otimes e^n)(e^1 \otimes \Omega^m \otimes \Omega^n) = \]

\[ = F^{1+m,n}(e^1 \otimes \Omega^m) \otimes \Omega^n = F^{1+m,n}(\Omega^{1+m} \otimes \Omega^n) \]

by induction. Thus, as subalgebras in \( T(R) \), \( U \) and its twisted counterpart \( \tilde{U} \) are related by the similarity transformation with the element \( \Omega = \sum_{n \geq 0} \Omega^n \) and \textit{vice versa}. This immediately implies the uniqueness of the global twist because two different \( \Omega \)'s are linked via an \( R \)-symmetric element \( u \) which, by definition, belongs to \( U \) (that is also a manifestation of the twist composition property). It realizes the inner automorphism \( h \rightarrow u^{-1}hu \) leading to the transformation \( F \rightarrow \Delta(u^{-1})F(u \otimes u) \). If it happens so that given \( \Omega^i, i = 2, 3 \), cannot be expanded to a universal element \( F \), yet there are gauge transformations of the element \( F \) leading to trivial or isomorphic deformations.
of the R-matrix. For every invertible $R$-symmetric $u^i \in R^i$, $i = 1, 2, 3$, substitution $(\Omega^2, \Omega^3) \rightarrow \left(u^2 \Omega^2(u^1 \otimes u^1), u^3 \Omega^3(u^1 \otimes u^1 \otimes u^1)\right)$, results in the similarity transformation $\tilde{R} \rightarrow (\tilde{u}^1 \otimes \tilde{u}^1)\tilde{R}(u^1 \otimes u^1)$.

We conclude this section with the remark that in the case of quasitriangular bialgebra $\mathcal{H}$ admitting twist with the element $\mathcal{F}$ there is the abstract form of $F^{m,n}$ belonging to $\mathcal{H}^{\otimes m} \otimes \mathcal{H}^{\otimes n}$. It is built with the help of the multiple coproduct applied to the components of the twisting element: $\mathcal{F}^{m,n} = (\Delta^m \otimes \Delta^n)(\mathcal{F})$. Formula (11) then gives the abstract element $\Omega$ intertwining $R$- and $\tilde{R}$-symmetric tensors in $\mathcal{H}^{\otimes n}$ and in algebraically isomorphic $\tilde{\mathcal{H}}^{\otimes n}$. Element $\Omega$ appeared in [15] as the necessary condition for the global twist factorization of the unitary universal R-matrix and was applied to the $XXZ$-model of spin $\frac{1}{2}$ chain described by the trigonometric solution to the Yang-Baxter equation. The relation to the symmetric group $S_n$ representations was discussed there as well.

### 4 Factorization of twisting elements

In the present section we give some illustrations to the constructions considered above. Given a bialgebra $\mathcal{H}$ and a solution $\mathcal{F} \in \mathcal{H} \otimes \mathcal{H}$ to the pair of equations

\begin{align}
(id \otimes \Delta)(\mathcal{F}) &= \mathcal{F}_{13}\mathcal{F}_{12}, \\
(\Delta \otimes id)(\mathcal{F}) &= \mathcal{F}_{13}\mathcal{F}_{23},
\end{align}

satisfying the additional condition

\[ \mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}, \]

it is possible to twist $\mathcal{H}$ by $\mathcal{F}$. In a matrix representation, the element $\mathcal{F}$ is given by its tensor components decomposed into the products

\[ F^{m,n'} = (F_{1n'} \ldots F_{11'})(F_{2n'} \ldots F_{21'}) \ldots (F_{mn'} \ldots F_{m1'}) \]
(cf. the notation in formula (8)). This expression is exactly the same as for the universal R-matrix (8), and that is due to the factorization conditions (13), (14) similar to those held for universal R-matrices. The homogeneous components of the global intertwiner Ω are evaluated using (12) and (13):

\[ Ω^n = (F_{1n} \ldots F_{12})(F_{2n} \ldots F_{23}) \ldots (F_{n-1,n}). \]

The most natural situation for such twists appears when the bialgebra \( \mathcal{H} \) is isomorphic to the tensor product of its sub-bialgebras \( \mathcal{A} \) and \( \mathcal{B} \) and \( \mathcal{F} \) actually belongs to \( \mathcal{A} \otimes \mathcal{B} \). In the matrix language (13-15) read

\[ R_{23}F_{13}F_{12} = F_{12}F_{13}R_{23}, \quad (16) \]
\[ R_{12}F_{23}F_{13} = F_{13}F_{23}R_{12}, \quad (17) \]
\[ F_{12}F_{23} = F_{23}F_{12}. \quad (18) \]

Conversely, each solution to the system (16–18) generates a twist of the R-matrix extended to the global twist of \( \mathcal{U} \). Twisting elements fulfilling (16–18) were used for explaining Fronsdal-Galindo deformation of the standard Drinfeld-Jimbo quantum groups \( \mathcal{U}_q(sl(2N + 1)) \) [11].

Another possible factorizations of the twisting element with respect to the coproducts (in this case, twisted and non-twisted ones) are [13, 14]

\[ (id \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}, \quad (19) \]
\[ (\tilde{\Delta} \otimes id)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}. \quad (20) \]

The system of equations (19), (20), and (7) is determined by its any pair. Conditions (19, 20) are the generalization of Reshetikhin’s twist, in which (20) is substituted by
(\Delta \otimes id)(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13} and the Yang-Baxter relation \mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12} (originally there were some excessive additional conditions which were loosened later in [8]).

Matrix version of (19–20) reduces to

\[ R_{23}\mathcal{F}_{12}\mathcal{F}_{13} = \mathcal{F}_{13}\mathcal{F}_{12}R_{23}, \tag{21} \]

\[ \tilde{R}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\tilde{R}_{12}. \tag{22} \]

Indeed, as was shown in [12], any \( F \) fulfilling (21–22) defines the global twist possessing (19–20). Again, using factorization (19) we find

\[ \Omega^n = (F_{1 2} \ldots F_{1 n})(F_{2 3} \ldots F_{2 n}) \ldots (F_{n-1 n}) \]

for the global intertwiner \( \Omega \). It is interesting to note that for such a twist the element \( \mathcal{F} \) carries out an algebra homomorphism \( a \rightarrow \langle \mathcal{F}, id \otimes a \rangle \) from the quantum semi-group \( \mathcal{A}_R \sim \mathcal{H}^* \) to \( \mathcal{H} \), while the transposed mapping is a homomorphism from \( \mathcal{A}_{\tilde{R}} \) to \( \tilde{\mathcal{H}} \sim \mathcal{H} \).

Composition of these mappings with the representation \( \rho \) yields in its turn homomorphisms of the twisted and non-twisted semi-groups to \( R \) which are determined on the generators by the element \( F \). The necessary and sufficient conditions for the existence of such homomorphisms are just exactly equations (19–20). Thus, there is a tool for verification whether two solutions to YBE are related via the twist with factorization conditions (19) and (20): among all the invertible elements \( F \) intertwining \( R \) and \( \tilde{R} \) one should find those defining homomorphisms from the corresponding quantum groups into \( R \).

5 Conclusion

The present investigation shows that transformation \( \tilde{R} = \tau(F^{-1})RF \) of a solution to YBE leads to a new solution if there exists a three-tensor \( \Omega^3 \) relating \( R \)- and \( \tilde{R} \)-symmetric three-tensors. For the global twist of the bialgebra \( \mathcal{U} \) defined by \( R \), one
should require the existence of invertible elements $\Omega^n$ relating $R$- and $\bar{R}$-symmetric $n$-tensors for every $n$. This means the equivalence between the corresponding representations of the braid groups $B_n$. Since dimension $n = 3$ proves to be crucial for the Yang-Baxter equation, an interesting question is whether two representations of $B_n$ are locally isomorphic if such an isomorphism takes place for $B_3$. If so, that could reduce the problem of building twist of bialgebras, within the matrix formalism, to solving the finite set of relatively simple equations in the matrix tensor square and cube.

Although twist establishes an equivalence between monoidal categories of representations of quasitriangular Hopf algebras, the physical content of related integrable models can change significantly. So, the jordanian deformation of the $XXX$-model of spin $\frac{1}{2}$ chain leads to the non-Hermitian Hamiltonian [16]. Preservation of its spectrum under that particular transformation is occasional, rather, and does not take place in other cases, for example, in transition from the standard quantum Toda chain to the system related to the Cremmer-Gervais R-matrix [17]. On the other hand, there is a successful experience of applying twisting technique to obtain simpler expressions for correlation functions [15], and the global intertwiner $\Omega$ introduced for the special case of the $XXZ$-model in [15] and studied on a somewhat general basis in the present paper should play an essential role in that process.

Another possible application of the present consideration is finding new solutions to the matrix Yang-Baxter equation including those depending on the spectral parameter. Particular realization of this line requires essentially using computer algebra programming because in the simplest case of two dimensions all the solutions has already been listed in [14]; and that is beyond the scope of our communication being a separate and elaborate problem.
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