GEVREY SOLUTIONS OF SINGULARLY PERTURBED DIFFERENTIAL EQUATIONS, AN EXTENSION TO THE NON-AUTONOMOUS CASE

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Abstract. We generalize the results on the existence of an over-stable solution of singularly perturbed differential equations to the equations of the form\[ \varepsilon \ddot{x} - F(x, t, \dot{x}, k(t), \varepsilon) = 0. \]In this equation, the time dependence prevents from returning to the well known case of an equation of the form \[ \varepsilon \frac{dy}{dx} = F(x, y, a, \varepsilon) \]where \( a \) is a parameter. This can have important physiological applications. Indeed, the coupling between the cardiac and the respiratory activity can be modeled with two coupled van der Pol equations. But this coupling vanishes during the sleep or the anesthesia. Thus, in a perspective of an application to optimal awake, we are led to consider a non autonomous differential equation.

1. Introduction. We consider a singular equation of the form
\[ \varepsilon \ddot{x} - F_1(x, t, \dot{x}, k(t), \varepsilon) = 0 \]where \( k \) is a vector of unknown functions whom the dimension will be specified later. By posing \( y = \dot{x} \) we get
\[ \begin{align*}
\dot{x} &= y \\
\varepsilon \dot{y} &= F_1(x, t, y, k(t), \varepsilon).
\end{align*} \]In the case where \( F_1(x, t, y, k, \varepsilon) = -x + y(1 - x^2) + k(t)g(t) \), this equation models the cardiac system forced by the respiratory system, as presented in [9]. In this situation, by posing \( k(t) = a/g(t) \), the equation is a classic forced van der Pol system. This equation has been extensively studied, non exhaustive examples can be found in [13, 2, 12, 21, 4]. Matzinger in [14] has proved the existence of a uniform asymptotic development of the canard solutions. Canards are special solutions of slow-fast systems, that is, systems of ODEs where the variables evolve on different timescales, due to the presence of a small parameter \( 0 < \varepsilon \ll 1 \). Canard trajectories follow the attracting part of the fast nullcline, pass close to a bifurcation point of

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the fast subsystem and then follow the repelling part of fast nullcline for a large amount of time. The canard phenomenon was first discovered by Benoît, Callot, Diener & Diener in [1] in the van der Pol oscillator, and studied by means of non-standard analysis. Later on, similar results were established with standard techniques, in particular by matching asymptotic [11] and geometrical approaches to singular perturbation theory (the so-called blow-up method) [10]. Apart from the fact that they follow repelling objects for long intervals of time, canards (in 2D-systems) are special in that they exist for a very narrow parameter interval, which can be proved to be exponentially small in $\varepsilon$. The transition through canard solutions in parameter space is usually referred to as an explosion [3].

More generally, if the function $F_1$ is time independent and $k$ is a vector of parameters, the situation is similar to the one studied by M. Canalis-Durand, J.P. Ramis, R. Schäfke and Y. Sibuya in [8]. In this article, the authors proved the existence of an over-stable solution that tends, as $\varepsilon \to 0$, to some given slow curve $\phi_0(x)$ satisfying $F_1(x, \phi_0(x), k_0, 0) = 0$. They proved the Gevrey character of a formal solution, then they used this formal solution to construct a quasi-solution. A quasi-solution is a function which satisfies the equation except for an exponentially small term. Finally, they proved the existence of a true solution, exponentially close to the quasi-solution. The Gevrey character of formal solutions of differential equations has been extensively studied, see for example [16, 17, 18, 20].

The notion of Gevrey series also have applications in the context of the theory of Fliess operators (see [23, 24]). These operators, as input-output map of a non-linear control system, can be applied in biology, for example to the modeling of homeostasis.

Here, we generalize the results obtained by the authors in [8] to any analytic (the precise hypotheses are specified below) functions $F_1(x, t, y, k(t), \varepsilon)$.

We eliminate the time variable in the equations (2) by posing $\theta = \omega t$. We obtain the following system:

$$
\begin{align*}
\dot{x} &= y \\
\varepsilon \dot{y} &= F_1(x, \theta, y, k(\theta), \varepsilon) \\
\theta' &= \omega.
\end{align*}
$$

Then, we pose:

$$
O_1(y) = y \partial_x y + \omega \partial_\theta y.
$$

(3)

and we will consider the singular Burgers-like equation

$$
\varepsilon O_1(y) = F_1(x, \theta, y, k, \varepsilon),
$$

(4)

where $0 < \varepsilon \ll 1$, $(x, \theta) \in \mathbb{C}^2$, $k(\theta)$ is a vector function of $\theta$. The “time” variable $\theta$ is taken in $\mathbb{C}$ in order to make the computation easier. We want to study the over-stable solutions that tends, as $\varepsilon \to 0$ to some given slow curve of (3), noted $\phi_0(x, \theta)$ satisfying $F_1(x, \theta, \phi_0(x, \theta), k_0(\theta), 0) = 0$.

We consider the case where the derivative

$$
A_0(x, \theta) = \frac{\partial F_1}{\partial y}(x, \theta, \phi_0(x, \theta), k_0(\theta), 0)
$$

is null at $x = x_0$. We suppose that $x_0$ is a zero of order $m$ of $A_0(x, \theta)$ for all $\theta \in D_{r_2}(\theta_0)$ for a certain value $\theta_0$, where $D_{r_2}(\theta_0)$ means the disc centered in $\theta_0$ of radius $r_2$. We have $k(\theta) = (k_1(\theta), ..., k_m(\theta))$. We will generalize the results provided in [8]; namely, we will prove the existence of the over-stable solutions and provide some asymptotic properties.

We suppose that
Hypothesis 1. a) The function $F_1$ is analytic in $x, \theta, y$ and $k$ on $D = D_{r_1}(x_0) \times D_{r_2}(\theta_0) \times D_{r_3}(y_0) \times D_{r_4}(k_0)$, where $D_{r_i}(x_0)$ means the disc centered in $x_0$ of radius $r_i$.

b) The function $F_1$ is analytic in $\varepsilon$ on an open sector $V$ whose vertex is at the origin and $F_1$ is Gevrey asymptotic of order 1 to the formal series $\sum f_k(x, \theta, y, k)\varepsilon^k$ on $V$. The functions $f_k$ are holomorphic on $D$.

c) There exists a point $(x_0, \theta_0, y_0, k_0) \in \mathcal{D}$ such that $F_1(x_0, \theta_0, y_0, k_0, 0) = 0$.

We provide here the definition of Gevrey asymptotic of order $1/\sigma$:

**Definition 1.1.** $F$ is Gevrey asymptotic of order $1/\sigma$ to the formal series $\sum f_k(x, \theta, y, k)\varepsilon^k$ if there exist positive constants $A$ and $C$ such that for all $\varepsilon \in V$, $(x, \theta, y, k) \in \mathcal{D}$ and all $n \in \mathbb{N}^*$

$$\left| F(x, \theta, y, k, \varepsilon) - \sum_{0}^{n-1} f_k(x, \theta, y, k)\varepsilon^k \right| \leq CA^{n/\sigma} \Gamma(n/\sigma + 1)|\varepsilon|^n,$$

where $\Gamma$ is the usual special function.

In addition, in our case, the formal series is Gevrey of order 1. That means that there exist two constants $A$ and $C$ such that for all $n \in \mathbb{N}^*$ we have $|f_k(x, \theta, y, k)| \leq CA^n(n - 1)!$. Let us recall two useful definitions:

**Definition 1.2.**
- The set $\mathcal{L}_0 := \{(x, \theta, y) \in \mathbb{C}^3/F_1(x, \theta, y, k, 0) = 0 \}$ is called the slow set of equation (3).
- A slow curve $\mathcal{C}_0$ of the equation (3) is a smooth subset of the slow set $\mathcal{L}_0$. It is the graph of a function $\phi_0$ holomorphic on $D_{r_1} \times D_{r_2} \subset D_{r_1} \times D_{r_2}$. In this case, we have $F_1(x, \theta, \phi_0(x, \theta), k_0(\theta), 0) = 0$ and $\phi_0(x_0, \theta_0) = y_0$.

**Definition 1.3.** An over-stable solution of the equation (4) is a couple $(y^*(x, \theta, \varepsilon), k^*(\theta, \varepsilon))$ of functions such that:
- $y^*$ is holomorphic in $x$ and $\theta$ in a well chosen domain $D_1(x, \theta) \subset \mathbb{C}^2$ and $k^*$ is holomorphic in a well chosen domain $D_2(\theta) \subset \mathbb{C}$
- $y^*$ and $k^*$ are holomorphic in $\varepsilon$ in a well chosen sector $W$ and satisfy
  $$\varepsilon \mathcal{O}_1(y^*) = F_1(x, \theta, y^*, k^*, \varepsilon)$$
  for $(x, \theta) \in D_1(x, \theta)$ and $\varepsilon \in W$.
- $\lim_{\varepsilon \to 0} y^* = \phi_0$ uniformly in $x$ and $\theta$ in $W$.
- $\lim_{\varepsilon \to 0} k^* = k_0$ uniformly in $\theta$.

The hypothesis of transversality: This hypothesis is defined exactly in the same manner as in [8]. We recall it here for the comfort of the reader.

We denote

$$A_0(x, \theta) = \frac{\partial F_1}{\partial y}(x, \theta, \phi_0, k_0, 0) = (x - x_0)^m K(x, \theta)$$

where $K$ is analytic in both variables on a neighborhood of $(x_0, \theta_0)$ and $K(x_0, \theta) \neq 0$ for $\theta \in D_{r_2}(\theta_0)$. We rewrite the right hand side of equation (4) in the form

$$F_1(x, \theta, y, \varepsilon) = A_0(x, \theta)(y - \phi_0) + B_0(x, \theta)(k - k_0) + \mathcal{F}(x, \theta, y, k, \varepsilon)$$

where

$$B_0(x, \theta) = \frac{\partial F_1}{\partial k}(x, \theta, \phi_0, k_0, 0).$$
Then, we have
\[ A_0(x, \theta)(y - \phi_0) + B_0(x, \theta)(k - k_0) = \varepsilon O_1(y) - F(x, \theta, y, k, \varepsilon). \]
We consider the equation of the form
\[ A_0(x, \theta)(y - \phi_0) + B_0(x, \theta)(k - k_0) = g(x, \theta) \]
which is equivalent to
\[ (x - x_0)^m K(x, \theta)(y(x, \theta) - \phi_0(x, \theta)) + B_0(x, \theta)(k(\theta) - k_0(\theta)) = g(x, \theta). \]
We obtain
\[ K(x, \theta)(y - \phi_0) = \frac{g(x, \theta) - B_0(x, \theta)(k - k_0)}{(x - x_0)^m}. \]
Let us define the shift operator \( S \) by
\[ S[f(x, \theta)] = f(x - x_0) \quad \text{if } x \neq x_0 \]
\[ S[f(x_0, \theta)] = \partial_x f(x_0, \theta). \]
It comes from the equation (5) that the function \( y \) is analytic in \( x_0 \) if and only if,
\[ S^j[g(x_0, \theta)] = S^j[B_0(x_0, \theta)](k - k_0) \quad \forall j \in [0, m - 1], \]
where \( S^j \) means \( S \) applied \( j \) times. We obtain a system of \( m \) equations with \( m \) unknown \( (k_1(\theta), \ldots, k_m(\theta)) \). In order to always have a unique solution we need the \( m \times m \) matrix
\[ S^m = [S^j[B_0(x_0, \theta)]]_{0 \leq j \leq m - 1} = [S^j[\partial_k F_1(x_0, \theta, \phi_0, k_0, 0)]]_{1 \leq t \leq m, 0 \leq j \leq m - 1} \]
to be invertible on \( D_{r_2}(\theta_0) \).
In this case, the function \( k \) is uniquely determined by equation (6). In addition, a holomorphic solution \( y \) is given by
\[ y(x, \theta) = \phi_0 + \frac{1}{K(x, \theta)} S^m[(g - B_0)(x, \theta)](k - k_0)(\theta). \]

Then, the invertibility of \( S^m[B_0(x_0, \theta)] \) is equivalent to the transversality hypothesis which is given by

**Hypothesis 2.** We denote \( A_0(x, \theta) = \partial_y F_1(x, \theta, 0, 0, 0) = xG(x) \), with \( G(0) \neq 0 \) and we denote \( B_0(x, \theta) = \partial_k F_1(x, \theta, 0, 0, 0) \). The mapping
\[ \mathcal{H}(D_{r_1} \times D_{r_2}) \times \mathcal{H}(D(r_2)) \rightarrow \mathcal{H}(D_{r_1} \times D_{r_2}) \]
\[ (y, k) \rightarrow A_0 y + B_0 k \]
where \( \mathcal{H}(D) \) is the set of holomorphic functions on \( D \), is bijective.

For all \( \theta \in D_{r_2} \), we pose
\[ N(\theta) = \sup_{|X|_m = 1} \left| \left| \left[ \left[ S^j[B_0(x, \theta)] \right] \right]^{-1} X \right| \right|. \]
If \( \sup_{\theta \in D_{r_2}} N(\theta) = +\infty \), it is sufficient to choose a smaller \( r_2 \) to ensure that \( N(\theta) \) is bounded on \( D_{r_2} \). Similarly, if \( \inf_{(x, \theta) \in D_{r_1} \times D_{r_2}} |K(x, \theta)| = 0 \), it suffices to consider a smaller \( r_1 \) or a smaller \( r_2 \) to ensure that it exists a constant \( \alpha > 0 \) such that \( \forall (x, \theta) \in D_{r_1} \times D_{r_2}, |K(x, \theta)| \geq \alpha \). Thus without loss of generality, we make the following assumption :

**Hypothesis 3.** There exist a constant \( C < +\infty \) and a constant \( \alpha > 0 \) such that \( \sup_{\theta \in D_{r_2}} N(\theta) < C \) and \( \forall (x, \theta) \in D_{r_1} \times D_{r_2}, |K(x, \theta)| \geq \alpha \).

Under the hypothesis 1, 2 and 3, we have the following theorem
Main Theorem. Let us consider the singular partial differential equation (4) and suppose that the hypotheses 1 and 2 are satisfied. Let us assume that solution satisfy the hypothesis of transversality. Then, the equation (4) has a unique formal slow curve corresponding to the time function $k_0(\theta)$ with $\phi_0(x_0, \theta_0) = y_0$ and that

$$A_0(x, \theta) = \frac{\partial F_1}{\partial y}(x, \theta, \phi_0(x), k_0, 0) \quad \text{and} \quad B_0(x, \theta) = \frac{\partial F_1}{\partial k}(x, \theta, \phi_0(x), k_0, 0)$$

satisfy the hypothesis of transversality. Then, the equation (4) has a unique formal solution

$$\dot{y} = \sum_{j=1}^{\infty} b_j(x, \theta) \varepsilon^j$$

$$\dot{k} = \sum_{j=1}^{\infty} c_j(\theta) \varepsilon^j.$$

where $b_j$'s are analytic in $D_{r_1} \times D_{r_2}$, $c_j$'s are analytic in $D_{r_2}$ and $b_0 = \phi_0$, $c_0 = k_0$.

In addition, these formal series are Gevrey of order 1. It means that

$$\sup_{(x, \theta) \in D_{r_1} \times D_{r_2}} |b_j(x, \theta)| \leq MN^j \Gamma(j + 1),$$

$$\sup_{\theta \in D_{r_2}} |c_j(\theta)| \leq MN^j \Gamma(j + 1),$$

where $M$ and $N$ are constant.

Moreover, for $\beta \in \mathbb{R}$ and sufficiently small $\tilde{r}_1$, $\tilde{r}_2$, $\varepsilon_0$, $\delta_0 > 0$, such that $W = \{\varepsilon, \arg(\varepsilon - \beta)\} < \delta_0, |\varepsilon| < \varepsilon_0\}$ defines a proper sub-sector $W$ of $V$, there is an over-stable solution $(y^*, k^*) : W \times D_{r_2}(0) \to \mathbb{C}$, $y^* : W \times D_{r_1}(x_0) \times D_{r_2}(0) \to \mathbb{C}$ of (3) having $(\hat{y}, \hat{k})$ as asymptotic expansion of Gevrey of order 1 uniformly for $(x, \theta) \in D_{r_1}(x_0) \times D_{r_2}(0)$.

This theorem asserts that the formal solution is unique and provides a good approximation of a true solution. To prove this theorem, the sketch of proof is identical as the one used by the authors in [8].

The outline of the paper is as follows: the second and the third section provide some intermediate results needed to prove the theorem 1. The fourth section is dedicated to the proof of the Gevrey character of the formal solution. In the fifth section, we show the existence of a quasi-solution, obtained as the Laplace transform of the Borel transform of the formal solution. Finally, the sixth section proves the existence of a true solution, approximated by the quasi solution which ends the proof of the theorem 1.

2. Transformation of the equation. We do the following transformation

$$y(x, \theta) \leftarrow y(x, \theta) - \phi_0(x, \theta)$$

$$k(\theta) \leftarrow k(\theta) - k_0(\theta)$$

$$x \leftarrow x - x_0$$

$$\theta \leftarrow \theta - \theta_0$$

The new initial values are $y_0 = 0$, $x_0 = 0$ and $k_0 = 0$. We obtain

$$\varepsilon [(y + \phi_0) \partial_x y + \omega \partial_\theta y] = F_1(x + 1, \theta, y + \phi_0, k + \psi_0, \varepsilon) - \varepsilon (y + \phi_0) \partial_x \phi_0$$

We then have the prepared equation

$$\varepsilon \mathcal{O}(y) = F(x, \theta, y, k, \varepsilon)$$

where

- $\mathcal{O}(y) = (y + \phi_0) \partial_x y + \omega \partial_\theta y$
- $F$ is holomorphic on $D_{r_1}(0) \times D_{r_2}(0) \times D_{r_3}(0) \times D_{r_4}(0) \subset \mathbb{C}^4$,
We define \( 0 < \rho < r_1 \) and \( 0 < \gamma < r_2 \). We consider the function \( d_1 \) on the open disk \( D_{r_1} = D_{r_1}(0) \)
\[
d_1(x) = \begin{cases} 
  r_1 - |x| & \text{if } |x| \geq \rho \\
  r_1 - \rho & \text{if } |x| < \rho,
\end{cases}
\]
and the function \( d_2 \) on the open disk \( D_{r_2} = D_{r_2}(0) \)
\[
d_2(\theta) = \begin{cases} 
  r_2 - |\theta| & \text{if } |\theta| \geq \gamma \\
  r_2 - \gamma & \text{if } |\theta| < \gamma.
\end{cases}
\]

Remark 1. The transversality hypothesis remains unchanged.

3. The adapted Nagumo norm. In order to prove the Gevrey property of the formal solution, we will need the adapted Nagumo norm.

Definition 3.1. We define \( 0 < \rho < r_1 \) and \( 0 < \gamma < r_2 \). We consider the function \( d_1 \) on the open disk \( D_{r_1} = D_{r_1}(0) \)
\[
d_1(x) = \begin{cases} 
  r_1 - |x| & \text{if } |x| \geq \rho \\
  r_1 - \rho & \text{if } |x| < \rho,
\end{cases}
\]
and the function \( d_2 \) on the open disk \( D_{r_2} = D_{r_2}(0) \)
\[
d_2(\theta) = \begin{cases} 
  r_2 - |\theta| & \text{if } |\theta| \geq \gamma \\
  r_2 - \gamma & \text{if } |\theta| < \gamma.
\end{cases}
\]

Remark 2. \( \forall x, d_1(x) \leq r_1 - \rho \) and \( \forall \theta, d_2(\theta) \leq r_2 - \gamma \)

Proposition 1. If \( x \) and \( y \in D_{r_1} \), then \( |x - y| \geq d_1(x) \) and if \( \theta \) and \( \delta \in D_{r_2} \), then \( |\theta - \delta| \geq d_2(\theta) \).

Definition 3.2. Let us consider \( p \in \mathbb{N}^* \) and \( f \) a holomorphic function on \( D_{r_1} \times D_{r_2} \). Then, the adapted Nagumo norm is given by
\[
\|f\|_p = \sup_{(x,\theta) \in D_{r_1} \times D_{r_2}} [\|f(x,\theta)|d_1(x)|d_2(\theta)|^p].
\]

Of course, the Nagumo norm depends on the choice of \( \rho \) and \( \gamma \). In addition, \( \|f\|_p \) is infinite for some \( f \in D_{r_1} \times D_{r_2} \). If \( f \) is smooth on the closure of \( D_{r_1} \times D_{r_2} \), then:
\[
\|f\|_p \leq (r_1 - \rho)^p(r_2 - \gamma)^p \sup_{(x,\theta) \in D_{r_1} \times D_{r_2}} |f(x)|.
\]

In all cases,
\[
|f(x,\theta)| \leq \|f\|_p d_1(x)^{-p} d_2(\theta)^{-p} \quad \forall |x| < r_1, |\theta| < r_2,
|f(x,\theta)| \leq \|f\|_p \delta_1^{-p} \delta_2^{-p} \quad \text{if } |x| \leq r_1 - \delta_1, 0 < \delta_1 < r_1 - \rho, |\theta| \leq r_2 - \delta_2, 0 < \delta_2 < r_2 - \gamma.
\]

The modified Nagumo norm (see [15, 22]), was first used in the context of singularly perturbed equations by R. Schäfke in [19] and by Canalis-Durand in [5, 7, 6].

The following inequalities follow immediately from the definition of the adapted Nagumo norm:

Proposition 2.
\[
|f(x,\theta)| \leq \|f\|_p d_1(x)^{-p} d_2(\theta)^{-p}
\]
\[
\|f + g\|_p \leq \|f\|_p + \|g\|_p
\]
\[
\|\alpha f\|_p \leq \|\alpha\|_p \|f\|_p
\]
\[
\|fg\|_{p+q} \leq \|f\|_p \|g\|_q.
\]

We can bound the Nagumo norm of the partial derivatives with the following lemma:
Lemma 3.3.
\[ ||\partial_x f||_{p+1} \leq e(p+1)||f||_p \]
and
\[ ||\partial_y f||_{p+1} \leq e(p+1)||f||_p \]
where \( e = \exp(1) \).

Proof. By using the Cauchy formula, we get:
\[
|\partial_x f(y, \theta)| = \left| \frac{1}{2\pi} \int_{|x-y|=R} \frac{f(x, \theta)}{|x-y|^p} dx \right| \leq \frac{1}{R} \max_{|x-y|=R} |f(x, \theta)|
\]
with \( R \) arbitrary between 0 and \( r_1 - y \).

But,
\[
\forall x, \theta, \ |f(x, \theta)| \leq ||f||_p d_1(x)^{-p} d_2(\theta)^{-p},
\]
then,
\[
\max_{|x-y|=R} |f(x, \theta)| \leq ||f||_p d_2(\theta)^{-p} \max_{|x-y|=R} d_1(x)^{-p}.
\]
In addition, \( \forall x/|x-y| = R, d_1(y) \leq d_1(x) + R \). It follows that
\[
|\partial_x f(y, \theta)| \leq \frac{1}{R} ||f||_p d_2(\theta)^{-p} (d_1(y) - R)^{-p}.
\]
With the choice \( R = d_1(y)/(p+1) \), we obtain
\[
|\partial_x f(y, \theta)| (p+1) ||f||_p d_2(\theta)^{-p} d_1(y)^{(p+1)} \leq e(p+1)||f||_p
\]
then
\[
|\partial_x f(y, \theta)| d_2(\theta)^{p+1} d_1(y)^{(p+1)} \leq e(p+1)||f||_p.
\]
The result follows.

The proof of the second point follows the same scheme.

Corollary 1.
\[ ||(f + \phi_0)\partial_x f||_{p+1} \leq e(p+1)||f||_p ||f + \phi_0||_0. \]

Finally, the following lemma holds:

Lemma 3.4.
\[ ||Sf||_p \leq \frac{2}{\rho} ||f||_p \]

Proof. For all \( x \in D_{r_1} \), we have:
\[
|f(x, \theta)| \leq d_1(x)^{-p} d_2(\theta)^{-p} ||f||_p \quad \text{and} \quad |f(0, \theta)| \leq d_1(0)^{-p} d_2(\theta)^{-p} ||f||_p \leq d_1(x)^{-p} d_2(\theta)^{-p} ||f||_p.
\]

- If \( |x| \geq \rho \), then, we have
\[
\left| \frac{f(x, \theta) - f(0, \theta)}{x} \right| \leq \frac{2}{\rho} d_1(x)^{-p} d_2(\theta)^{-p} ||f||_p,
\]
- If \( |x| < \rho \), by the maximum principle,
\[
|Sf(x, \theta)| \leq \max_{|y|=\rho} |Sf(y, \theta)| \leq \frac{2}{\rho} (r_1 - \rho)^{-p} d_2(\theta)^{-p} ||f||_p = \frac{2}{\rho} d_1(x)^{-p} d_2(\theta)^{-p} ||f||_p.
\]
\[
\square
\]
4. Formal solution. Let us consider the transformed equation (8)
$$\varepsilon \mathcal{O}(y) = F(x, \theta, y, k, \varepsilon).$$

With the hypothesis 2, The following proposition holds:

**Proposition 3.** The equation (4) has a unique formal solution

$$y = \sum_{j=1}^{\infty} b_j(x, \theta)\varepsilon^j, \quad k = \sum_{j=1}^{\infty} c_j(\theta)\varepsilon^j,$$

with $b_j$ holomorphic on $D_{r_1} \times D_{r_2}$ and $c_j$ holomorphic on $D_{r_2}$.

**Proof.** In order to prove this proposition, we write the equation (8) in the following manner:

$$A_0(x, \theta)y + B_0(x, \theta)k = \varepsilon \mathcal{O}(y) - \varepsilon \psi_1(x, \theta) - \sum_{p+|\vec{q}|+r \geq 2} f(x, \theta)y^p k^{\vec{q}} \varepsilon^r,$$

where $\vec{q} = (q_1, \ldots, q_m) \in (\mathbb{N}^+)^m$ and $|\vec{q}| = q_1 + \ldots + q_m$. All these functions are holomorphic. The $b_j$ and $c_j$ are obtained recursively.

**Remark 3.** The right hand side of the equation 16 has coefficient 0 for $\varepsilon^0$.

In order to prove that $k$ and $y$ are Gevrey of order 1, we will consider the inverse of the function introduced by the transversality condition. We note

$$P_1 : \mathcal{H}(D_{r_1} \times D_{r_2}) \rightarrow \mathcal{H}(D_{r_1} \times D_{r_2}),$$

$$P_2 : \mathcal{H}(D_{r_1} \times D_{r_2}) \rightarrow \mathcal{H}(D_{r_2}).$$

This mapping is such that $\forall(x, \theta) \in D_{r_1} \times D_{r_2}, f \in \mathcal{H}(D_{r_1} \times D_{r_2}),$

$$A_0(x, \theta)(P_1f)(x, \theta) + B_0(x, \theta)(P_2f)(x, \theta) = f.$$

We pose $g = P_1(f)$, $a = P_2(f)$.

$$x^m K(x, \theta)g(x, \theta) + B_0(x, \theta)a(\theta) = f(x, \theta).$$

It follows that

$$B_0(0, \theta)a(\theta) = f(0, \theta).$$

**Lemma 4.1.** There exist two constants $K_1$ and $K_2$ such that

$$\|P_1f\|_p \leq K_1\|f\|_p,$$

$$\|P_2f\|_p \leq K_2\|f\|_p.$$

**Proof.** Let $f \in \mathcal{H}(D_{r_1} \times D_{r_2})$ be a holomorphic function. Let us pose $g = P_1(f)$, $a = P_2(f)$. As in the equation (6), we have that for all $j = 0, \ldots, m - 1,$

$$S^j[f(0, \theta)] = S^j[B_0(0, \theta)a(\theta)].$$

Via the transversality hypothesis, the matrix $(S^j[B_0(0, \theta)])_{i,j}$, with $i = 1, \ldots, m$ and $j = 0, \ldots, m - 1$, is invertible. For all $\theta \in D_{r_2}$, via the first inequality of the proposition 2 and the hypothesis 3, we obtain

$$|a(\theta)| \leq C'(S^j[f(0, \theta)])_{j=0,\ldots,m-1} \leq C\|S^j[f]\|_{p} d_1(0)^{-p} d_2(\theta)^{-p} \leq K_1\|f\|_p d_1(0)^{-p} d_2(\theta)^{-p}.$$
where $K_1$ may be chosen independent of $f$. By taking the $\|\cdot\|_p$ norm of this inequality, we get
\[
\|a\|_p \leq \| (K_1 \|f\|_p d_1(0)^{-p} d_2(\theta)^{-p}) \|_p \\
\leq K_1 \|f\|_p d_1(0)^{-p} \|d_2(\theta)^{-p}\|_p.
\]
We can compute $\|d_2^p\|_p$:
\[
\|d_2^p\|_p = \sup_{(x,\theta) \in D_{r_1} \times D_{r_2}} |d_1(x)^p d_2(\theta)^p d_2(\theta)^{-p}| = d_1(0)^p.
\]
It follows that
\[
\|a\|_p \leq K_1 \|f\|_p.
\]
Furthermore, we have:
\[
g(x, \theta) = \frac{1}{K(x, \theta)} f(x, \theta) - B_0(x, \theta) a(\theta) = K(x, \theta) S^m [f(x, \theta) - B_0(x, \theta) a(\theta)],
\]
where $S^m$ means $S$ applied $m$ times. From the lemma 3.4 we have
\[
\|g\|_p \leq \|K^{-1}\|_0 \|S^m f - B_0 a\|_p \\
\leq \|K^{-1}\|_0 \left(\frac{2}{\rho}\right)^m \|f - B_0 a\|_p \\
\leq \left(\frac{2}{\rho}\right)^m \|K^{-1}\|_0 \|f\|_p + \|B_0\|_0 \|a\|_p
\]
where $K_2$ is independent of $f$. \hfill \Box

Let us state the first intermediary theorem:

**Theorem 4.2.** Let us consider the equation (8) and let us note the unique formal solution as the formal series
\[
y = \sum_{j=1}^{\infty} b_j(x, \theta) \varepsilon^j \\
k = \sum_{j=1}^{\infty} c_j(\theta) \varepsilon^j.
\]
Then, $y$ is Gevrey of order 1 on $D_{r_1} \times D_{r_2}$ and $k$ is Gevrey of order 1 on $D_{r_2}$. It means that
\[
\sup_{(x, \theta) \in D_{r_1} \times D_{r_2}} |b_j(x, \theta)| \leq MN^2 \Gamma(j + 1),
\]
\[
\sup_{\theta \in D_{r_2}} |c_j(\theta)| \leq MN^2 \Gamma(j + 1),
\]
where $M$ and $N$ are two constants and $\Gamma$ is the classical special function.

**Proof.** We will use the series $u = \sum u(x, \theta) \varepsilon^j = A_0 y + B_0 x$. We recover $y = P_1 u$ and $k = P_2 u$ with the convention that $P_i$ is applied to each factor of the series. We get
\[
u = A_0 P_1(u) + B_0 P_2(u) \\
= \varepsilon \mathcal{O}(P_1 u) - \varepsilon \psi_1(x, \theta) - \sum_{p+q+r \leq 2} f_{p, q, r}(x, \theta)(P_1 u)^p (P_2 u)^q \varepsilon^r \\
= \varepsilon \mathcal{O}(P_1 u) - \sum_{r=1}^{\infty} \psi_k(x, \theta) \varepsilon^r - \sum_{r=1}^{\infty} A_r(x, \theta) \varepsilon^r P_1 u - \sum_{r=1}^{\infty} B_r(x, \theta) \varepsilon^r P_2 u \\
- \sum_{p+q \geq 2} \sum_{r \geq 0} f_{p, q, r}(x, \theta) \varepsilon^r (P_1 u)^p (P_2 u)^q \varepsilon^r.
\]
which is equivalent to
\[ u = \varepsilon O(P_1 u) - A(x, \theta, \varepsilon)P_1 u - B(x, \theta, \varepsilon)P_2 u - \psi(x, \theta, \varepsilon) \]
\[ - \sum_{p+q \geq 2} f_{p,q}(x, \theta, \varepsilon)(P_1 u)^p (P_2 u)^q. \]  
(22)

with
\[ A(x, \theta, \varepsilon) = \sum_{i=1}^{\infty} A_i(x, \theta)\varepsilon^i \]
\[ B(x, \theta, \varepsilon) = \sum_{i=1}^{\infty} B_i(x, \theta)\varepsilon^i \]
\[ \psi(x, \theta, \varepsilon) = \sum_{i=1}^{\infty} \psi_i(x, \theta)\varepsilon^i \]
\[ f_{p,q}(x, \theta, \varepsilon) = \sum_{i=1}^{\infty} f_{p,q,i}(x, \theta)\varepsilon^i. \]

**Definition 4.3.** We say that a series \( g(\varepsilon): (x, \theta) \rightarrow \sum_{l} g_l(x, \theta)\varepsilon^l \) is majorized (noted \( \ll \)) by \( h(z) = \sum_{l} h_l z^l \) if and only if
\[ \|g_j\|_j \leq |h_j|_j! \]
for all \( j \).

**Lemma 4.4.** If \( g \ll h(z) \) and \( \tilde{g} \ll \tilde{h}(z) \), then,
\[ g(\varepsilon)\tilde{g}(\varepsilon) \ll h(z)\tilde{h}(z), \]
\[ \varepsilon \partial_x g(\varepsilon) \ll \varepsilon z h(z), \]
\[ \varepsilon \partial_y g(\varepsilon) \ll \varepsilon z h(z). \]  
(23)

**Proof.**
- The first part follows from the definition of the product series and the fact that \( \forall k \in [0, j], k!(j-k) \leq j! \).
- For the second part, we compute
\[ \varepsilon z h(z) = \sum_{j=1}^{\infty} e h_{j-1} z^j, \]
\[ \varepsilon \partial_x g = \varepsilon \sum_{j=0}^{\infty} \partial_x g_j \varepsilon^j \]
\[ = \varepsilon \sum_{j=1}^{\infty} \partial_x g_{j-1} \varepsilon^j. \]

The result follows from the lemma 3.3.
- The proof of the third part is identical.

By applying \( \|\|_{j^!} \) to the coefficient of \( \varepsilon^j \) in the equation (22) we obtain the majorant series \( \hat{A}(z), \hat{B}(z), \hat{f}_{pq}(z) \) of \( A(x, \theta, \varepsilon), B(x, \theta, \varepsilon) \) and \( f_{pq}(x, \theta, \varepsilon) \). From the hypothesis 1-a) and the Cauchy estimate, all these series have the same non-zero radius of convergence. In addition, the series
\[ \sum_{p+q \geq 2} f_{p,q}(z) g^{p+q} \]
converge if \( z \) and \( g \) are small enough. We can now consider the majorant series
\[ v(z) = \hat{\psi}(z) + \varepsilon z K v(z) (\omega + \|\phi_0\|_0 + K v(z)) + (\hat{A}(z) + \hat{B}(z)) K v(z) \]
\[ + \sum_{r=2}^{\infty} \left( \sum_{p+q=r} \hat{f}_{p,q}(z) \right) K^r v(z)^r, \]  
(24)
where \( K = \max(K_1, K_2) \), the constants from the lemma 4.1. Because we have 
\[ \psi(0) = A(0) = B(0) = 0 \], it is clear that the equation (24) has a unique formal
solution \( v(z) = \sum_{j=1}^{\infty} v_j z^j \), such that the \( v_j \)'s are non negative.

Moreover, the equation (24) is an implicit equation for \( v(z) \). Therefore, the series
\( v(z) \) converges. There exist two constants \( M \) and \( N \) such that \( v_j \leq MN^j \).

We will now demonstrate that \( y \approx v(z) \) majorizes the solution \( u \) of the equation (22).
In order to make things clearer, let us note \( Ru \) the right hand part of the equation (22) and note \( \hat{Ru}(z) \) the right hand part of the equation (24). Then, from the
lemmas 4.1 and 4.4, it is true that
\[ y \ll v(z) \Rightarrow Ry \ll \hat{R}v(z). \tag{25} \]

Let us begin with \( u_0 = \sum_{j=1}^{\infty} 0 \varepsilon^j \), \( v_0(z) = \sum_{j=1}^{\infty} 0z^j \). We have \( u_0 \ll v_0(z) \). We define
recursively \( u_k = Ru_{k-1} \) and \( v_k(z) = \hat{R}v_{k-1}(z) \). From the equation (25), we get
\( u_k \ll v_k(z) \).

Because the coefficients of \( \varepsilon^1, \varepsilon^2, ..., \varepsilon^n \) in \( u \) (or the same coefficient of \( z^j \) for \( \hat{v} \))
determine the coefficients of \( \varepsilon^1, \varepsilon^2, ..., \varepsilon^{n+1} \) in \( Ru \) (or \( \hat{R}v \)), it implies that these
coefficients correspond to the coefficients of the formal solution \( u \) (respectively \( v \))
of the equation (22) (respectively equation (24)). It follows that the relation (12)
implies that, for all \( \delta_1 \in [0, r_1 - \rho], \delta_2 \in [0, r_2 - \gamma] \) we have
\[ |u_k(x, \theta)| \leq \|u\|_k \delta_1^{-k} \delta_2^{-k} \leq k! v_k \delta_1^{-k} \delta_2^{-k} \]
for \( |x| \leq r_1 - \delta_1, |\theta| \leq r_2 - \delta_2 \). It follows that \( u \) is Gevrey of order 1 on the each
product of sub-disk \( D_{\tilde{r}_1} \times D_{\tilde{r}_2} \) included in \( D_{r_1} \times D_{r_2} \). Via the applications \( P_1 \) and
\( P_2 \) and the lemma 4.1, we obtain that \( y = P_1 u \) and \( a = P_2 u \) are Gevrey of order 1
too. It ends the proof of theorem 4.2.

5. Quasi solutions. Let us consider the equation (4) and its formal solution \( \hat{y} = \sum b_j(x, \theta) \varepsilon^j, k = \sum c_j(\theta) \varepsilon^j \). In the following, we work on a sub-domain \( D_{\tilde{r}_1} \times D_{\tilde{r}_2} \subset\]
(26)
\[ \tilde{D}_{r_1} \times \tilde{D}_{r_2} \) such that the series are Gevrey of order 1. To simplify, we suppose that
the real part of \( \varepsilon \) is positive and that the sector \( V \) from the hypothesis 1-b) is given by
\[ V = \{ \varepsilon; |\arg(\varepsilon)| < \delta_0, 0 < |\varepsilon| < \varepsilon_0 \} \]
with \( \delta_0 < \pi/2 \).

We define the Borel transform of \( \hat{y}, \tilde{k} \) and \( F \) as follows:

**Definition 5.1.** The Borel transform of \( \hat{y}, \tilde{k} \) and \( F \) are given by

\[
\hat{b}(x, \theta, t) = \sum_{j=1}^{\infty} b_j(x, \theta) \frac{t^{j-1}}{(j-1)!} \text{ for } (x, \theta) \in \tilde{\mathcal{C}}, |t| < N, \\
\tilde{c}(\theta, t) = \sum_{j=1}^{\infty} c_j(\theta) \frac{t^{j-1}}{(j-1)!} \text{ for } \theta \in \tilde{\mathcal{D}}_{r_2}, |t| < N, \\
\tilde{G}(x, \theta, y, k, t) = \sum_{j=1}^{\infty} f_j(x, \theta, y, k) \frac{t^{j-1}}{(j-1)!} \text{ for } (x, \theta, y, k) \in \mathcal{D}, |t| < 1/A,
\]

where \( N > 0 \) is given by the theorem 4.2, \( A > 0 \) and \( \mathcal{D} \) is as in the hypothesis 1-a).
We choose $T$ such that $0 < T < \min(1/N, 1/A)$ and we pose

\[
\tilde{y}(x, \theta, \varepsilon) = \int_0^T e^{-t/\varepsilon} \tilde{b}(x, \theta, t) dt + \varepsilon \tilde{b}(x, \theta, T) e^{-T/\varepsilon},
\]

\[
\tilde{k}(\theta, \varepsilon) = \int_0^T e^{-t/\varepsilon} \tilde{c}(\theta, t) dt + \varepsilon \tilde{c}(\theta, T) e^{-T/\varepsilon},
\]

\[
\tilde{F}(x, \theta, y, k, \varepsilon) = f_0(x, \theta, y, k) + \int_0^T e^{-t/\varepsilon} \tilde{G}(x, \theta, y, k, t) dt + \varepsilon \tilde{G}(x, \theta, y, k, T) e^{-T/\varepsilon},
\]

(27)

with $\text{Re}(1/\varepsilon) > 0$ and $(x, \theta, y, k) \in D$. The functions $\tilde{y}$, $\tilde{k}$ and $\tilde{F}$ are close to the truncated Laplace transforms of the series $\tilde{b}$, $\tilde{c}$ and $\tilde{G}$. We add a corrective term in their definitions so that they are Laplace transform of continuous functions.

As in [8], we can apply the Watson lemma, thus we have

\[
\tilde{y}(x, \theta, \varepsilon) \sim \sum_{j=1}^{\infty} b_j(x, \theta) \varepsilon^j,
\]

\[
\tilde{k}(\theta, \varepsilon) \sim \sum_{j=1}^{\infty} c_j(\theta) \varepsilon^j,
\]

\[
\tilde{F}(x, \theta, y, k, \varepsilon) \sim \sum_{j=1}^{\infty} f_j(x, \theta, y, k) \varepsilon^j,
\]

(28)

as $\varepsilon \to 0$ in all the sector \{\varepsilon, |\arg(\varepsilon)| < $\pi/2 - \delta$\} with $\delta > 0$. As $(\tilde{F} - F)(x, \theta, y, k, \varepsilon)$ is flat when $\varepsilon \to 0$ in $V$, $(\tilde{F} - F)$ is exponentially small. More precisely, there exists $K_F > 0$ such that

\[
|(\tilde{F} - F)(x, \theta, y, k, \varepsilon)| \leq K_F |e^{-T/\varepsilon}|.
\]

The functions $(\tilde{y}, \tilde{k})$ are expected to be a good approximation of the solution. Then, we define the corrective term as

\[
\tilde{R}(x, \theta, \varepsilon) = \varepsilon O(\tilde{y})(x, \theta, \varepsilon) + F(x, \theta, \tilde{y}(x, \theta, \varepsilon), \tilde{k}(\theta, \varepsilon), \varepsilon)
\]

(29)

for $(x, \theta) \in \tilde{C}$, $\varepsilon$ in $V$.

Since the right hand side of the equation 28 are formal solutions of equation 8, it is clear that, for sufficiently small $\varepsilon$ in $V$,

\[
\tilde{R}(x, \theta, \varepsilon) \sim 0 + 0 \varepsilon + ...
\]

as $\varepsilon \to 0$.

We have the following theorem:

**Theorem 5.2.** The functions $\tilde{y}$ and $\tilde{k}$ previously defined are quasi-solutions of the equation (8). It means that there exists a constant $K > 0$ such that the corrective term $\tilde{R}$ defined in the equation (29) satisfies

\[
|\tilde{R}(x, \theta, \varepsilon)| \leq |\exp(-T/\varepsilon)| (\varepsilon \in V, \varepsilon \to 0, (x, \theta) \in D_{\varepsilon_1} \times D_{\varepsilon_2}).
\]

**Proof.** Since $|\tilde{F} - F| \leq K_F |e^{-T/\varepsilon}|$, we just need to prove that

\[
|\tilde{S}(x, \theta, \varepsilon)| \leq K |e^{-T/\varepsilon}|
\]

with

\[
\tilde{S}(x, \theta, \varepsilon) = \varepsilon O(\tilde{y}) - \tilde{F}(x, \theta, \tilde{y}, \tilde{k}, \varepsilon).
\]

(30)
As in [8], we will demonstrate that $\tilde{S}$ is the Laplace transform of a continuous function $\Phi(x, \theta, t)$ of, at most, exponential growth. It means that

$$\tilde{S}(x, \theta, \varepsilon) = \int_0^\infty e^{-t/\varepsilon} \Phi(x, \theta, t) dt$$  \hspace{1cm} (31)

where, for each $(x, \theta) \in D_{\tilde{r}_1} \times D_{\tilde{r}_2}$, $\Phi(x, \theta, t)$ has a power series in powers of $t$ of radius of convergence $T$.

In this case, if $\Phi(x, \theta, t) \sim \sum_0^\infty \phi_n(x, \theta) t^n$ then, by Watson’s lemma: $\tilde{S}(x, \theta, \varepsilon) \sim \sum_0^\infty \phi_n(x, \theta) \varepsilon^n$. But, we know that $\tilde{S}$ is flat (since $\tilde{R}$ is flat) then, all $\phi_j$ vanish in $[0, T]$. This proves the theorem.

To prove the equality (31), we demonstrate that both terms on the right-hand side of the equation (30) have this property.

By definition, $\tilde{y}$ is the Laplace transform of

$$b(x, \theta, t) := \begin{cases} \tilde{b}(x, \theta, t) & \text{if } 0 \leq t \leq T \\ \tilde{b}(x, \theta, T) & \text{if } T \leq t. \end{cases}$$  \hspace{1cm} (32)

We define $k(\theta, t)$ from $\tilde{c}$ and $\tilde{F}$ from $\tilde{G}$ in the same way.

We note $\mathcal{L}(f(X, t))(\varepsilon)$ the Laplace transform of a function $f$ in $t$. Then, we have $\partial_x \tilde{y} = \mathcal{L}(\partial_x b)$ and $\partial_\theta \tilde{y} = \mathcal{L}(\partial_\theta b)$. In addition, $\phi_0(x) = \mathcal{L}[\phi_0(x) \delta_0(t)]$ where $\delta_0$ is the Dirac function in $0$. Thus,

$$(\tilde{y} + \phi_0) \partial_x \tilde{y} = \int_0^\infty e^{-t/\varepsilon} (\partial_x b \ast_t [b(x, \theta, .) + \phi_0(x) \delta_0(.)](t)) dt$$

where $\ast_t$ denotes the convolution with respect to $t$, furthermore, we have

$$\omega \partial_\theta \tilde{y} = \omega \int_0^\infty e^{-t/\varepsilon} \partial_\theta b dt.$$  \hspace{1cm} (33)

Finally,

$$\varepsilon \mathcal{O}(\tilde{y}) = \int_0^\infty e^{-t/\varepsilon} (\partial_x b \ast_t [b(x, \theta, .) + \phi_0(x) \delta_0(.)](t) + \omega \partial_\theta b) dt$$

$$= \int_0^\infty e^{-t/\varepsilon} B(x, \theta, t) dt.$$  \hspace{1cm} (33)

For the second term, we write

$$\tilde{F}(x, \theta, \tilde{y}, \tilde{k}, \varepsilon) = \sum_{p+|\overrightarrow{q}| \geq 0} \hat{f}_{p \overrightarrow{q}}(x, \theta, \varepsilon) \tilde{y}^p \tilde{k}^\overrightarrow{q}.$$  

Using the definition of $\tilde{F}$, we write

$$\sum_{p+|\overrightarrow{q}| \geq 0} \hat{f}_{p \overrightarrow{q}}(x, \theta, \varepsilon) \tilde{y}^p \tilde{k}^\overrightarrow{q} - f_0(x, \theta, \tilde{y}, \tilde{k}) = \int_0^T e^{-t/\varepsilon} G(x, \theta, \tilde{y}, \tilde{k}, t) dt$$

which is equivalent to

$$\sum_{p+|\overrightarrow{q}| \geq 0} \hat{f}_{pq}(x, \theta, \varepsilon) \tilde{y}^p \tilde{k}^\overrightarrow{q} - \sum_{p+|\overrightarrow{q}| \geq 0} \hat{f}_{pq}(x, \theta, 0) \tilde{y}^p \tilde{k}^\overrightarrow{q}$$

$$= \int_0^T e^{-t/\varepsilon} \sum_{p+|\overrightarrow{q}| \geq 0} \tilde{g}_{pq}(x, \theta, t) \tilde{y}^p \tilde{k}^\overrightarrow{q} dt.$$
The Borel transform of a Gevrey series is convergent, thus, we can exchange sum and integral. We obtain
\[
\sum_{p+1 \, | \, \bar{q} \geq 0} \tilde{f}_{pq}(x, \theta, \varepsilon) \tilde{y}^p \tilde{k}^\bar{q} - \sum_{p+1 \, | \, \bar{q} \geq 0} \tilde{f}_{pq}(x, \theta, 0) \tilde{y}^p \tilde{k}^\bar{q} \\
= \sum_{p+1 \, | \, \bar{q} \geq 0} \left( \int_0^T e^{-t/\varepsilon} \tilde{g}_{pq}(x, \theta, t) dt \right) \tilde{y}^p \tilde{k}^\bar{q}.
\]
The sum \( \sum_{p+1 \, | \, \bar{q} \geq 0} \tilde{g}_{pq} \tilde{y}^p \tilde{k}^\bar{q} \) is convergent, thus it exists a constant \( M \) such that
\[
|\tilde{g}_{pq}| \leq Mr^{-(p+|\bar{q}|)}
\]
with \( r < \min(r_3, r_4) \) being as in the hypothesis 1-a). The same holds for \( \tilde{f}_{pq}(x, \theta, 0) \).
By the definition of \( b \) and \( k \), we can suppose that \( |b(x, \theta, t)| \leq M \) and \( |k(\theta, t)| \leq M \) for \( (x, \theta) \in D_{\tilde{f}_1} \times D_{\tilde{f}_2}, t \geq 0 \). Since a Laplace transform product is the Laplace transform of a convolution product, we have
\[
\tilde{y}^p \tilde{k}^\bar{q} = \int_0^\infty e^{-t/\varepsilon} b^p \ast_t k^\bar{q} dt
\]
where \( k^\bar{q} = k_1^{q_1} \ast_t \ldots \ast_t k_m^{q_m} \) and \( k_i^{q_i} = k_1 \ast_t k_2 \ast_t \ldots \ast_t k_i, q_i \) times. Of course, \( b^{*p} \)

is defined in the same manner. By induction, we verify that
\[
|b^{*p} \ast_t k^\bar{q}| \leq M \frac{(Mt)^{l-1}}{(l-1)!}
\]
with \( l = p + |\bar{q}| \). We develop the sum:
\[
\sum_{p+1 \, | \, \bar{q} \geq 1} \tilde{f}_{pq}(x, \theta, \varepsilon) \tilde{y}^p \tilde{k}^\bar{q} = \sum_{p+1 \, | \, \bar{q} \geq 1} \left( \tilde{f}_{pq}(x, \theta, 0) \tilde{y}^p \tilde{k}^\bar{q} + \int_0^\infty e^{-t/\varepsilon} \tilde{g}_{pq} \tilde{y}^p \tilde{k}^\bar{q} \right)
\]
\[
= \sum_{p+1 \, | \, \bar{q} \geq 1} \left( \int_0^\infty e^{-t/\varepsilon} \tilde{f}_{pq}(x, \theta, 0) b^{*p} \ast_t k^\bar{q} \right.
\]
\[
+ \int_0^\infty e^{-t/\varepsilon} \tilde{g}_{pq} \tilde{y}^p \tilde{k}^\bar{q} \ast_t b^{*p} \ast_t k^\bar{q} \bigg) \right).
\]

In addition, the sum
\[
\sum_{p+1 \, | \, \bar{q} \geq 1} \tilde{f}_{pq}(x, \theta, 0) b^{*p} \ast_t k^\bar{q} + \tilde{g}_{pq}(x, \theta, t) \ast_t b^{*p} \ast_t k^\bar{q}
\]
is dominated by
\[
\sum_{l=1}^\infty \left( \sum_{p+1 \, | \, \bar{q} = l} M \frac{(Mt)^{l-1}}{(l-1)!} + \frac{(Mt)^l}{(l+1)!} \right)
\]
which is equal to
\[
Me^{Mt/r} \left( t - \frac{r}{M} \right) (1 + t) - r(1 + t).
\]
Thus, we can exchange sum and integral. We obtain
\[
\sum_{p+1 \, | \, \bar{q} \geq 1} \tilde{f}_{pq}(x, \theta, \varepsilon) \tilde{y}^p (x, \theta, \varepsilon) \tilde{k}^\bar{q} (\theta, \varepsilon) = \int_0^\infty e^{-t/\varepsilon} v(x, \theta, t) dt
\]
where
\[
v(x, \theta, t) = \sum_{p+1 \, | \, \bar{q} \geq 1} \tilde{f}_{pq}(x, \theta, 0) b^{*p} \ast_t k^\bar{q} + \tilde{g}_{pq}(x, \theta, t) \ast_t b^{*p} \ast_t k^\bar{q}.
\]
Finally,

\[ \tilde{S}(x, \theta, \varepsilon) = \int_0^\infty e^{-t/\varepsilon} (B(x, \theta, t) - \tilde{g}_{00}(x, \theta, t) - v(x, \theta, t)) \, dt \]

\[ = \int_0^\infty e^{-t/\varepsilon} \Phi(x, \theta, t) \, dt. \]

The sum

\[ \sum_{l=1}^{\infty} \left( \sum_{p+q=|l|} f_{p,q} b^p * f_{l-q} + \tilde{g}_{p,q} b^p * f_{l-q} \right) \]

is normally convergent, therefore the functions \( t \to v(x, \theta, t) \), \( t \to B(x, \theta, t) \) and \( \Phi \) are continuous. The same estimate holds on the complex disk \( |t| \leq T \) (where \( e^t \) is replaced by \( e^{|t|} \)), then \( t \to \Phi(x, \theta, t) \) is analytic. It ends the proof of the theorem.

6. **Over-stable solution.** We have shown in the previous section that the equation (8) has quasi-solution \( \tilde{y} \), \( \tilde{k} \) that satisfies the equation (8) except for an exponentially small error \( \tilde{R} \). So, we will choose \( k = \tilde{k} \) and demonstrate that, with this choice, there exists an exact over-stable solution \( y(x, \theta) \) of (8). To that end, let us pose

\[ \varepsilon \mathcal{O}(y) = F(x, \theta, \tilde{k}, \varepsilon) = G(x, \theta, \varepsilon) \]

the equation with this choice of \( k \). We make the transformation

\[ y(x, \theta) = \tilde{y}(x, \theta, \varepsilon) + \tilde{\Delta} \]

and we obtain a differential equation for \( \tilde{\Delta} \):

\[ H_\varepsilon(x, \theta, \tilde{\Delta}, \partial_x \tilde{\Delta}, \partial_\theta \tilde{\Delta}) = 0 \]

where

\[ H_\varepsilon(x, \theta, z, p_x, p_\theta) = \varepsilon[\mathcal{O}(\tilde{y}) + u \partial_x \tilde{y} + z p_x + \tilde{y} p_x + \omega p_\theta] - \tilde{G}(x, \theta, \varepsilon, z), \]

and \( \tilde{G}(x, \theta, \varepsilon, z) = G(x, \theta, \varepsilon, \tilde{y} + z) \). With this notation, we have the following lemma:

**Lemma 6.1.** Equation (35) has a solution \( \tilde{\Delta} \) analytic for small \( x, \theta \) and \( \varepsilon \) with small \( |\arg(\varepsilon)| \) and there exist two constants \( K \) and \( \alpha \) such that

\[ |\tilde{\Delta}| \leq K e^{-\alpha/|\varepsilon|} \]

in the definition domain of \( \tilde{\Delta} \).

This lemma proves the theorem 1.

To prove the lemma, we will demonstrate the existence and uniqueness of the solution with the method of characteristics, and then prove the inequality.

The characteristic equation of (35) is

\[ \dot{x} = z + \tilde{y}(x, \theta, \varepsilon) = f_1(x, \theta, z) \]

\[ \dot{z} = \frac{1}{\varepsilon} \left( \tilde{G}(x, \theta, \varepsilon, z) - \varepsilon \mathcal{O}(\tilde{y}) \right) - z \partial_x \tilde{y} = f_2(x, \theta, z) \]

\[ \dot{\theta} = \omega = f_3(x, \theta, z). \]

All the functions implied are analytic in \( x, \theta \) and \( z \), thus the function \( (x, \theta, z) \to (f_1(x, \theta, z), f_2(x, \theta, z), f_3(x, \theta, z)) \) is locally Lipschitz. We choose as initial condition \( z(0) = 0 \), \( x(0) = x_0 \) with \( |x_0| \leq \tilde{r}_1/8 \) (we will justify this choice below) and \( \theta(0) = \theta_0 \). We will now prove that the solution of (37) exists on a segment \( |0, \tau| \)
independent of $\varepsilon$ and verifies the inequality $|z(t)| \leq K|e^{-\alpha/\varepsilon}|$ for some $\alpha > 0$. In addition, we will prove that we can cover a sub-domain $D_{\tilde{r}_1} \times D_{\tilde{r}_2} \subset D_{\tilde{r}_1} \times D_{\tilde{r}_2}$ which doesn’t vanish when $\varepsilon \to 0$ with $(x(t), \theta(t))$.

We choose a constant $D$ such that $\tilde{G}(x, \theta, \varepsilon, w)$ is well defined for $(x, \theta) \in D_{\tilde{r}_1} \times D_{\tilde{r}_2}, \varepsilon \in V, |w| \leq D$. If $\tilde{r}_1, \tilde{r}_2, D$ and the opening of the sector $V$ are sufficiently small, there exists a constant $L_1$ such that

$$\left| \frac{\partial \tilde{G}(x, \theta, \varepsilon, z)}{\partial z} \right| \leq L_1 \text{ for } (x, \theta) \in D_{\tilde{r}_1} \times D_{\tilde{r}_2}, \varepsilon \in V, |z| \leq D.$$  

Therefore,

$$|\tilde{G}(x, \theta, \varepsilon, z) - \tilde{G}(x, \theta, \varepsilon, 0)| \leq L_1 |z|.$$  

In addition, there exist two constants $L_2 > 0$ and $L_3 > 0$ such that

$$\forall (x, \theta) \in D_{\tilde{r}_1} \times D_{\tilde{r}_2}, \varepsilon \in V, |\partial \tilde{G} y| \leq L_2,$$

$$\forall (x, \theta) \in D_{\tilde{r}_1} \times D_{\tilde{r}_2}, \varepsilon \in V, |\partial \tilde{G} \theta y| \leq L_3.$$

For $\varepsilon$ fixed, let us pose

$$\tau_1(\varepsilon) = \sup \{ \tau > 0 \mid |t| \in [0, \tau], |z(t)| \leq D \},$$

$$\tau_2(\varepsilon) = \sup \{ \tau > 0 \mid |t| \in [0, \tau], |x(t)| \leq \tilde{r}_1 \},$$

$$\tau_3(\varepsilon) = \sup \{ \tau > 0 \mid |t| \in [0, \tau], |\theta(t)| \leq \tilde{r}_2 \},$$

and

$$\tau(\varepsilon) = \min(\tau_1, \tau_2, \tau_3).$$

For a certain $\alpha > 0$ (because $\tilde{G}(x, \theta, \varepsilon, 0) = \tilde{R}$ defined in (29)), we have the inequality

$$|\dot{z}(t)| \leq \frac{1}{\varepsilon} Ke^{-2\alpha/|\varepsilon|} + \left( \frac{L_1}{\varepsilon} + L_2 \right) |z(t)|$$

for all $t \in [0, \tau(\varepsilon)]$. For all $\varepsilon$, it is clear that $\tau(\varepsilon) > 0$, let us prove that $\tau(\varepsilon)$ does not converge to 0 when $\varepsilon \to 0$.

Of course, $\tau_3(\varepsilon)$ is independent of $\varepsilon$. It remains to prove that $\tau_1(\varepsilon)$ and $\tau_2(\varepsilon)$ do not vanish.

Let us consider the equation of $z$:

$$|\dot{z}(t)| \leq \frac{K}{\varepsilon} e^{-2\alpha/|\varepsilon|} + \frac{L}{\varepsilon} |z| \forall t \leq \tau(\varepsilon)$$

with $L = L_1 + L_2 > L_1 + \varepsilon L_2$. By applying Gronwall lemma, we get

$$|z(t)| \leq \frac{K}{L} e^{-(2\alpha - Lt)/|\varepsilon|} \forall t \leq \tau(\varepsilon)$$

therefore,

$$|z(t)| \leq \frac{K}{L} e^{-\alpha/|\varepsilon|} \forall t \leq \min(\tau(\varepsilon), \alpha/L).$$

We choose $\varepsilon_0 > 0$ such that

$$\frac{K}{L} e^{-2\alpha/\varepsilon_0} \leq D/2.$$  

Thus, we have $|z(t)| \leq D/2$ for all $t \leq \min(\tau(\varepsilon), \alpha/L), |\varepsilon| \leq \varepsilon_0$. By definition of $\tau_1(\varepsilon)$, we have that $\tau_1(\varepsilon) \geq \min(\tau(\varepsilon), \alpha/L)$.

If $\tau(\varepsilon) = \tau_1(\varepsilon)$, $\tau_2(\varepsilon) \geq \tau_1(\varepsilon) \geq \alpha/L$ for $|\varepsilon| \leq \varepsilon_1$, then $z$ is exponentially small on an interval that is independent of $\varepsilon$. Now, let us assume that $\tau(\varepsilon) = \tau_2(\varepsilon)$ and let us prove that $\tau_2(\varepsilon)$ does not vanish.
Let us consider the equation in $x$:

\[
\dot{x} = z + \tilde{y} \\
|\dot{x}| \leq |z| + L_2|x| + L_2|x_0| + |\tilde{y}(0, \theta, \varepsilon)| \\
|\dot{x}| \leq |z| + L_2|x| + L_2|x_0| + L_3|\theta - \theta_0| \\
|\dot{x}| \leq |z| + L_2|x| + L_2|x_0| + L_3\omega \tau(\varepsilon) \\
|\dot{x}| \leq \frac{K}{L} e^{-2\alpha|\varepsilon|} + L_2|x| + L_2|x_0| + L_3\omega \tau(\varepsilon), \forall t \leq \min(\tau_2(\varepsilon), \alpha/L).
\]

By applying the Gronwall lemma, and considering that $|\theta| = |\theta_0 + \omega \tau| \leq \tilde{r}_3$ and $|x_0| \leq \tilde{r}_1/8$ we obtain that, $\forall t \leq \min(\tau_2(\varepsilon), \alpha/L)$,

\[
|x| \leq \frac{K}{L} e^{-2\alpha|\varepsilon|} + L_2\tilde{r}_1/8 + L_3\tilde{r}_3 (e^{L_2t} - 1) + |x_0| e^{L_2t} \\
|x| \leq \left( \frac{K}{L} e^{-2\alpha|\varepsilon|} + \frac{\tilde{r}_1}{8} + \frac{L_2\tilde{r}_3}{L_2} \right) (e^{L_2t} - 1) + |x_0| e^{L_2t}.
\]

We choose $\varepsilon_1$ such that

\[
\frac{K}{L} L_2 e^{-\alpha/\varepsilon_1} \leq \frac{\tilde{r}_1}{16},
\]

and we suppose that

\[
\tilde{r}_3 \leq \frac{L_2\tilde{r}_1}{16L_3}.
\]

If this is not the case, we replace $\tilde{r}_3$ by $\frac{L_2\tilde{r}_1}{16L_3}$ at the beginning of this demonstration. Since $x_0 \in D_{\tilde{r}_1/8}$, then we have the following inequality

\[
|x| \leq \frac{\tilde{r}_1}{4} (e^{L_2t} - 1) + \frac{\tilde{r}_1}{8} \text{ for all } |\varepsilon| \leq \varepsilon_1.
\]

It follows that,

\[
|x| \leq \frac{\tilde{r}_1}{2} \text{ for all } t \leq \min(\tau_2(\varepsilon), 2\alpha/L, \ln(2)/L_2).
\]

This last inequality implies that $\tau_2(\varepsilon) \geq \min(2\alpha/L, \ln(2)/L_2)$ for all $|\varepsilon| \leq \varepsilon_1$.

Finally, the solution of (37) exists on an interval independent of $\varepsilon$ and on this interval, $z$ is exponentially small. The functions $f_1(x, \theta, z)$, $f_2(x, \theta, z)$ and $f_3(x, \theta, z)$ are locally Lipschitz. Thus, for each initial value, there exists a unique solution on an interval $I$. This interval contains $[0, \tau(\varepsilon)]$, therefore, the solutions do not intersect. Let us define $C$ as the domain such that, for all $t \in [-\tau, \tau]$, and $\theta_0 \in C$, $\theta_0 + t\omega \in D_{\tilde{r}_3}$ with $\tau = \inf_{\varepsilon \in V, |\varepsilon| \leq \min(\varepsilon_1, \varepsilon_0)} (\tau(\varepsilon))$. Then, by taking as initial condition $z(0) = 0$, $x(0) = x_0 \in D_{\tilde{r}_1/8}$ and $\theta(0) = \theta_0 \in C$, the solution covers a domain in $x$ and a domain in $\theta$, both independent of $\varepsilon$. The solutions obtained by the characteristic lines are solutions of the partial differential equation (35). This proves the main theorem.

7. Conclusion. In this paper, we extended the results obtained by the authors in [8]. We have proved that, in the non-autonomous case, still exist over-stable solutions, and that they can be approximated by a Gevrey series. These results may have important applications to model the physiology of the cardio-respiratory coupling in human (see From conservative to dissipative non-linear differential systems. An application to the cardio-respiratory regulation, J. Demongeot et al. in the same issue). In addition, it seems that these results could be extended to a larger class of equations of the form $\varepsilon O y(X) = F(X, y, \varepsilon)$, where $X \in \mathbb{C}^n$ and $O$ is a differential operator. The search of a minimal set of hypotheses on $O$ and $F$ may be the subject of a future work.
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Obituary.

René Thomas (1928-2017), in memoriam

The present paper is the third of a trilogy initiated by René Thomas four years ago, the two others being: 1 - C. Antonopoulos, V. Basios, J. Demongeot, P. Nardone & R. Thomas, Linear and nonlinear arabesques: a study of closed chains of negative 2-element circuits. Int. J. Bifurcation and Chaos, 23, 9, 30033 (2013).
2 - J. Demongeot, D. Istrate, H. Khlaifi, L. Mégret, C. Taramasco & R. Thomas, From conservative to dissipative non-linear differential systems. An application to the cardio-respiratory regulation (present issue). The research undertaken by René Thomas has focused on a wide variety of topics, from DNA biochemistry (notably the denaturation of nucleic acids) to genetics (in particular that of phages), theoretical biology, and system dynamics. He was passionate about the aesthetics of mathematics and often saw the beauty of a curve that he simulated on his computer before its utility in genetics or physiology. The present work is a tribute to his creative genius and his unwavering friendship with his colleagues.

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