TIME-FRACTIONAL EQUATIONS WITH REACTION TERMS: 
FUNDAMENTAL SOLUTIONS AND ASYMPTOTICS

SERENA DIPIERRO
Department of Mathematics and Statistics, University of Western Australia
35 Stirling Highway, Crawley WA 6009, Australia

BENEDETTA PELLACCI
Dipartimento di Matematica e Fisica, Università della Campania “Luigi Vanvitelli”
Viale Lincoln 5, 81100 Caserta, Italy

ENRICO VALDINOCI*
Department of Mathematics and Statistics, University of Western Australia
35 Stirling Highway, Crawley WA 6009, Australia

GIANMARIA VERZINI
Dipartimento di Matematica, Politecnico di Milano
Piazza Leonardo da Vinci 32, 20133 Milano, Italy

Abstract. We analyze the fundamental solution of a time-fractional problem, establishing existence and uniqueness in an appropriate functional space.

We also focus on the one-dimensional spatial setting in the case in which the time-fractional exponent is equal to, or larger than, $\frac{1}{2}$. In this situation, we prove that the speed of invasion of the fundamental solution is at least “almost of square root type”, namely it is larger than $ct^\beta$ for any given $c > 0$ and $\beta \in (0, \frac{1}{2})$.

1. Introduction. In this note we will consider a parabolic problem with time-fractional diffusion. The spatial diffusion will be modeled by the Laplacian operator, while the time derivative is of fractional type. In particular, we consider the so-called Caputo derivative of order $\alpha \in (0, 1)$, with initial time $t = 0$, given by

$$\partial_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\dot{u}(\tau)}{(t-\tau)^\alpha} \, d\tau. \quad (1)$$

We consider the evolution problem

$$\begin{cases}
\partial_t^\alpha u - d\Delta u = au, & x \in \mathbb{R}^N, \ t > 0 \\
u(x, 0) = \delta,
\end{cases} \quad (2)$$

where the constants $d, a$ satisfy $d > 0$, $a \geq 0$, and the $N$-dimensional Dirac delta distribution $\delta$ is centered at $x = 0$.

Time-fractional equations such as the one in (2) are very intriguing from the pure mathematical point of view, as they offer a great technical challenge to the

2010 Mathematics Subject Classification. 35R11, 35C15, 35B40, 35K57, 35K08, 26A33.

Key words and phrases. Time fractional diffusion, heat conduction with memory, Caputo fractional derivatives, fundamental solution, asymptotic behavior of solutions.

* Corresponding author: Enrico Valdinoci.
classical methods of ordinary and partial differential equations, since many of the standard techniques based on explicit barriers, maximum principle and bootstrap differentiations simply do not work in this setting. Furthermore, time-fractional diffusion naturally arises in a number of real-world phenomena in which “anomalous” diffusion takes place in view of a “memory effect” which is mathematically described by the polynomial kernel in (1).

As a matter of fact, the original motivation for the time-fractional operator in (1) arose in [6] with the goal of describing in mathematical terms an initial value problem arising in geophysics, to be confronted with experimental data.

Other powerful applications of the operator in (1) occur in the theory of viscoelastic fluids: roughly speaking, in this context one models the viscoelastic effects as an ideal superposition of “purely elastic” terms, governed by Hooke’s Law (relating the force to the displacement), and “purely viscous” terms, governed by Newton’s Law (in which forces are related instead to velocities). Such a superposition of zero-order effects (due to Hooke’s Law) and first-order ones (due to Newton’s Law) are often conveniently modeled in terms of fractional derivatives, see e.g. Section 10.2 in [26] and the references therein.

Interestingly, this viscoelastic effect and its relation with fractional calculus can be also explicitly understood in terms of mechanical systems of springs and dampers, producing the operator in (1) in a rigorous asymptotic way from fundamental physics considerations, see [35] for additional details on these models and on related problems.

Furthermore, fractional diffusive operators as in (1) and time-fractional heat equations as in (2) also arise in classical models when a complicated or fractal geometry of the environment comes into play. In particular, the classical heat equation on a very ramified comb structure naturally leads to a time-fractional equation, see [3] and [33]. The diffusive properties on highly ramified networks have also fundamental consequences in neurology, since this type of anomalous diffusion has been experimentally measured in neurons, see e.g. [34] and the references therein. See [16] for a recent review on this and related topics, and also [10] for a concrete mathematical model.

A number of natural applications of fractional derivatives also occur in statistics, mechanics, engineering and finance, see e.g. [1] and [7] for a series of concrete examples and further motivations. We also refer to [28], [29], [17], [30], and [14] for historical introductions to the fractional calculus.

The results that we obtain in this paper are the following. First of all, we prove an existence and uniqueness theory for the fundamental solution in (2) in the appropriate functional spaces, in any dimension. Then, we specialize to the case of dimension 1, with fractional exponent greater or equal than $\frac{1}{2}$, and we establish precise asymptotics for the fundamental solution, in particular by determining that the “rate of invasion” is faster than $\{ x = ct^\beta \}$ for any given $c > 0$ and $\beta \in (0, \frac{1}{2})$.

To obtain these results, the precise mathematical framework in which we work is the following. We consider the Sobolev space $H^m(\mathbb{R}^N)$ and denote by $H^{-m}(\mathbb{R}^N)$ the dual of $H^m(\mathbb{R}^N)$. For concreteness, we will take $N \geq m > N/2$ (in this way, functions in $H^m(\mathbb{R}^N)$ are necessarily continuous, and accordingly the Dirac delta belongs to $H^{-m}(\mathbb{R}^N)$; other functional spaces can be taken into account as well, provided the action of the Dirac delta is well defined).

Given $v \in H^{-m}(\mathbb{R}^N)$, we let
\[ \|v\|_{H^{-m}(\mathbb{R}^N)} := \sup_{\varphi \in H^{-m}(\mathbb{R}^N) \setminus \{0\}} \frac{|\langle v, \varphi \rangle|}{\|\varphi\|_{H^{-m}(\mathbb{R}^N)}}. \]

Then, we say that \( v \in L^\infty_\alpha ([0, T], H^{-m}(\mathbb{R}^N)) \) if, for all \( t \in [0, T] \), we have that \( v(t) \in H^{-m}(\mathbb{R}^N) \) and
\[
\sup_{t \in [0, T]} t^\alpha \|v(t)\|_{H^{-m}(\mathbb{R}^N)} < +\infty. \tag{3}\]

We also recall that, for all \( \alpha \in (0, 1) \), the time-fractional equation
\[ \partial_t^\alpha u(t) = f(t) \]
can be reduced to the Volterra integral equation
\[ u(t) = u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t f(\tau) \frac{1}{(t-\tau)^{1-\alpha}} \, d\tau, \tag{4} \]
see e.g. [9, Lemma 6.2]. Therefore, we define \( u \) to be a distributional solution (or, briefly, a solution) of (2) if
\[ u \in L^\infty_\alpha ([0, T], H^{-m}(\mathbb{R}^N)) \]
and
\[ \langle u(t), \varphi \rangle = \varphi(0) + \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\langle u(\tau), \Delta \varphi \rangle + a\langle u(\tau), \varphi \rangle}{(t-\tau)^{1-\alpha}} \, d\tau, \tag{5} \]
for all \( \varphi \in C^\infty_0(\mathbb{R}^N) \).

In this setting, we have that

**Theorem 1.1.** There exists a unique solution of (2).

Without the reaction term, i.e., when \( a = 0 \) in (2), the fundamental solution of the fractional heat equation has been introduced in [24] and some properties are presented as a consequence of the theory of special functions of Wright type. We also refer to Chapter 7, Section 5 in [25] for an approach to fundamental solutions based on transition probabilities. For initial data in suitable Lebesgue spaces a deep analysis of these types of problems has been performed in [12], [40], [41].

We remark that our situation, although related to the previous articles, presents several important structural differences. Indeed, first of all, we focus our study on the fundamental solution, i.e. our initial datum does not belong to any Lebesgue space. Furthermore, our approach is to avoid a massive use of special functions, since this may preclude a more intuitive approach to the matter, and we will in fact rely only on some very basic properties of the Mittag-Leffler function.

Indeed, one can give an explicit representation of the solution of (2) in terms of special functions and, to this end, it is convenient to exploit the Mittag-Leffler function
\[ E_\alpha(r) := \sum_{k=0}^{\infty} \frac{r^k}{\Gamma(1 + k\alpha)}. \tag{6} \]

Although the above series converges in more general situations, we will mainly deal with the cases \( \alpha \in (0, 1], r \in \mathbb{R} \). In this setting, the fundamental solution provided by Theorem 1.1 can be written in the form
\[ u(x,t) := \mathcal{F}^{-1} \left( E_\alpha \left( (a - 4\pi^2 d|\xi|^2) t^\alpha \right) \right), \tag{7} \]
where \( \mathcal{F} \) denotes the Fourier Transform and \( \mathcal{F}^{-1} \) its inverse.

We observe that while it is somehow straightforward to “guess” that (7) is “the” solution of (2) at a “formal” level of linear calculations, some care is needed to
establish a coherent existence and uniqueness theory in the appropriate functional spaces, and this is indeed the core of Theorem 1.1. In particular, a powerful tool which can not be applied in a direct way here is the abstract theory of uniqueness and correctness classes for Cauchy problems in topological vector spaces, as developed by Gel’fand and Shilov [15]. Indeed, such theory strongly relies on the validity of the Leibniz rule for the differentiation of a product (or, better, of a duality pairing), which fails in the time-fractional case.

Furthermore, the representation formula in (7) reveals an important structural difference with respect to the classical case. Indeed, for the standard heat equation, the Fourier Transform of the fundamental solution is a Gaussian function, thus possessing nice smoothness and decaying properties. Instead, formula (7) highlights the memory effect introduced by the time-fractional operator, which produces in this setting a significant loss of regularity in terms of functional spaces. As a matter of fact, from (7) and some well-established properties of the Mittag-Leffler function (see e.g. formula (20) here below), it follows that

\[ F \mu(\xi, t) \approx \frac{1}{\Gamma(1-\alpha)} \left( \frac{4\pi^2d}{|\xi|^2-\alpha} \right)^{\frac{N}{2}} t^\alpha \text{ as } |\xi| \to +\infty. \]  

(8)

Accordingly, for a given \( t > 0 \), we have that \( F \mu(\cdot, t) \in L^p(\mathbb{R}^N) \) if and only if \( p \in \left( \frac{N}{2}, +\infty \right) \),

(9)

which is an important difference with the classical case in which the fast decay at infinity implies integrability of any order.

Furthermore, from (9) it follows that when \( N = 1 \) we have that \( F \mu(\cdot, t) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}) \). Instead, when \( N \in \{2, 3\} \), we have that \( F \mu(\cdot, t) \in L^2(\mathbb{R}^N) \setminus L^1(\mathbb{R}^N) \), and, when \( N \geq 4 \), we have that \( F \mu(\cdot, t) \not\in L^2(\mathbb{R}^N) \). In particular, if \( N \geq 4 \), it is not even clear whether \( u \) belongs to some Lebesgue space, or it is merely a tempered distribution. From (8), one also sees that

if \( N \in \{2, 3\} \) then \( u(0, t) = +\infty \) for all \( t > 0 \),

which is also an important difference with respect to the classical case: namely, differently from the standard diffusion equation, in higher dimension the memory effect of equation (2) persists at the origin for all times, preventing any decay whatsoever from the initial singularity. Notably, this is true also when the reaction term is not present, i.e. \( a = 0 \).

The next question that we want to address is related to the “speed of invasion” of the fundamental solution. Namely, if \( u \) is as given by Theorem 1.1, one can compute the values of \( u \) on invading spheres depending on time and determine the asymptotics of these values for large time. Concretely, one can consider an increasing function \( \Theta(t) \) and look at the fundamental solution \( u \) at \( |x| = \Theta(t) \).

Roughly speaking, one expects that if \( \Theta \) is “small enough” (that is, one moves along “sufficiently slow” spheres) then the values of \( u \) will be largely affected by the “infinite” initial datum at the origin, and hence one expects that, in this case,

\[ \lim_{t \to +\infty} u(\Theta(t)e, t) = +\infty, \]  

(10)

for a given direction \( e \in S^{N-1} \). Viceversa, if \( \Theta \) is “large enough” (that is, one moves along “sufficiently fast” spheres) then the values of \( u \) will rapidly drift away from the initial datum at the origin and quickly approach the datum at infinity: in this
case, one expects that
\[ \lim_{t \to +\infty} u(\Theta(t)e, t) = 0. \]
(11)

Understanding the “optimal” velocity \( \Theta \) which produces the switch between the diverging behavior in (10) and the vanishing one in (11) is clearly an important question both in view of the development of the theory and for potential applications. Since the general situation of equation (2) is likely to be extremely rich in complications, we focus here on the special case of spatial dimension equal to 1 and fractional parameter \( \alpha \) bigger or equal than \( \frac{1}{2} \). In this case, we establish that the divergent behavior in (10) occurs for all functions \( \Theta \) that evolve slower than the square root of time. The precise result that we obtain is the following:

**Theorem 1.2.** Let \( N = 1, \alpha \in \left[ \frac{1}{2}, 1 \right) \) and \( a > 0 \). Let \( u \) be as in Theorem 1.1. Let also \( \beta \in (0, \frac{1}{2}) \) and \( c > 0 \). Then, for large \( t \),

\[ u(ct^{\beta}, t) \geq c \alpha t^{\beta - 2\alpha} E_{\alpha}(t^{\alpha})[1 - o(1)]. \]

In particular,

\[ u(ct^{\beta}, t) \to +\infty \quad \text{as} \quad t \to +\infty. \]
(12)

The proof of Theorem 1.2 is delicate and will be based on a suitable analysis exploiting some fine properties of the Mittag-Leffler function and a number of ad-hoc simplifications to take care of some wildly oscillatory behaviors of the terms describing the fundamental solution in terms of space-time power series.

Theorem 1.2 is of clear interest in itself, since it reveals an important physical feature of the “subdiffusion” provided by the time-fractional operator: namely, if the memory effect of the fractional derivative is expected to “slow down” the invasion with respect to the classical case, in which the diffusion occurs with a linear speed, our result establishes that such an invasion still occurs, in a power-like time as close as we wish to the square root function. Furthermore, Theorem 1.2 discloses a number of very interesting research directions. A few natural ones are concerned with the ranges of \( \alpha \) and \( \beta \) and with the spatial dimensions \( N \geq 2 \); we describe them as follows:

**Open Problem 1.** It would be desirable to find the “optimal speed” distinguishing the diverging from the vanishing behavior in Theorem 1.2, and in general to *determine the asymptotics in the case \( \beta \geq \frac{1}{2} \).* With respect to this, we mention that the estimate found in Theorem 1.2 is valid somehow independently on \( \alpha \), and in a way which treats the parameters \( \alpha \) and \( \beta \) in an essentially uncorrelated way. Nevertheless, the classical case corresponding to \( \alpha = 1 \) would correspond to a linear velocity of invasion (that is, \( \beta = 1 \)), and Theorem 1.2 does not capture this limit feature. In this sense, it would be desirable to investigate whether it is possible to obtain results such as in Theorem 1.2 with parameters \( \alpha \) and \( \beta \) in a clear and explicit correlation that, on the one hand, underlines the memory effect of the subdiffusion process, and, on the other hand, *recovers the classical linear speed of invasion* in the limit as \( \alpha \not\to 1 \). In general, when \( \beta \geq \frac{1}{2} \), the asymptotic and cancellation properties of the fundamental solution are likely to be different than the ones discussed in this paper and a new approach has probably to be taken into account.

**Open Problem 2.** It would be interesting to obtain a result in the spirit of Theorem 1.2 for the *highly nonlocal regime* \( \alpha \in (0, \frac{1}{2}) \). In this regime, the asymptotics of the Mittag-Leffler function are different than those considered in this paper and thus new ingredients have to be taken into consideration.
Open Problem 3. It would be desirable to understand the asymptotics of the fundamental solution in every spatial dimension \( N \geq 2 \). In this case, additional terms related to the rotational behavior of the Fourier Transform have to be taken into account. It is possible that in this case the additional factors have to be understood in terms of special functions, such as the the zeroth order Bessel function \( J_0 \), and analyzed in view of analytic tools such as the Hankel Transform. A number of technical difficulties are expected to surface in this case, since several useful identities involving the Mittag-Leffler function and its derivatives would be affected by a possible sign change in their arguments, leading to different types of cancellations.

Other interesting questions concern the optimality of the bounds obtained in Theorem 1.2, and, more generally, the possibility of obtaining sharp bounds from both sides. In particular, we mention:

Open Problem 4. It would be desirable to complement Theorem 1.2 with explicit upper bounds.

Theorem 1.2 is also related to a number of results in the recent literature dealing with the asymptotics of nonlocal heat equation and of possibly nonlinear fractional equations. With respect to this point, we observe that the nonlinear setting in the nonlocal case happens to be better understood in the case of space-fractional, rather than time-fractional, equations: for instance, in [4], [5] and [27] a very detailed long time asymptotics is given for nonlinear space-fractional equations. Conversely, the case of time-fractional equations seems to be more difficult to consider, since the memory effect given by the Caputo derivative does not often permit a direct use of barriers and maximum principles in nonlinear scenarios. For time-fractional equations with large time decay, several results have been recently obtained in [12], [40], [41], [37], [22], [21], [32], [31], [38], [18], [36], [19], [11], [2] and [20]. Differently from the previous literature, in this paper we consider a reaction term, namely, the term \( au \) on the right hand side of (2). This term deeply affects the analysis of the problem, since when \( a > 0 \) the arguments of the Mittag-Leffler functions involved in the computations can become positive, thus exhibiting an exponential behavior for large times.

Of course, this sign change is not only a merely technical occurrence, but it is indeed responsible of the diverging structure described in formula (12). Indeed, while in the existing literature the time-fractional heat equations were studied without a reaction term, obtaining decay results for large times, Theorem 1.2 here provides the first result of divergence for large times, in view of the positive reaction term when \( a > 0 \).

We also point out that, after this paper has been completed and posted in ArXiv, the very interesting article [42] has appeared, which also considered, from a different point of view, suitable functional settings for time-fractional equations with reaction terms. In general, however, the previous literature in [42] and in the references therein mainly focus on the case in which the initial datum belongs to either \( L^2 \) or \( L^1 \cap L^2 \), hence the functional setting that we consider here is structurally different.

The rest of this article is organized as follows. Section 2 is devoted to the proof of Theorem 1.1. First, we will focus on the uniqueness result in Theorem 1.1, which will be the consequence of a series of general integral estimates. For this, we will combine suitable methodologies of ordinary differential equations, integral equations, and distribution theory. Then, the existence result of Theorem 1.1 will be proved by explicitly checking that the setting in (7) provides a solution of this
problem. This part is also not straightforward, since one has to check that the 
definition of (7) is compatible with the notion of solution and that some of the 
“formal” computations that one would like to perform are indeed well-justified in a 
functional analysis framework.

In Section 3 we recall some useful properties of the Mittag-Leffler function. They 
will be exploited in Section 4, where we prove Theorem 1.2. For this, we will need to 
carefully analyze the cancellations occurring in the Fourier Transform representation 
of the fundamental solution. These cancellations are dictated by the oscillatory 
kernel of the Fourier Transform and one of the key ingredients of our analysis will 
be to discover an alternate cancellation structure similar to that of the “Leibniz 
Criterion” for numerical series.

2. The fundamental solution of the reaction-diffusion equation. In this 
section, we discuss the existence and uniqueness of the fundamental solution of (2).

The main idea is that such a fundamental solution can be written explicitly in 
terms of the Mittag-Leffler function (recall (7)). This explicit representation will be 
valid in any dimension. Moreover, the fundamental solution turns out to be unique 
in the sense of distributions. The technical computations needed to check uniqueness 
are contained in Subsection 2.1, while the existence and explicit representation of 
the solution are discussed in Subsection 2.2. In our setting, Theorem 1.1 will follow 
from the subsequent results in Corollary 1 and Lemma 2.3.

2.1. The fundamental solution of the reaction-diffusion equation: Unique-
ness of the solution. We discuss now the uniqueness result claimed in Theo-
rem 1.1. To this end, we establish first a pivotal result, which will then be applied 
to the difference of two possible solutions.

Lemma 2.1. Let \( w \in C^\infty(\mathbb{R}^n) \). Let \( T > 0 \) and

\[
v \in L^\infty_\alpha([0, T], H^{-m}(\mathbb{R}^N))
\]

be such that, for all \( \varphi \in C^\infty_0(\mathbb{R}^N) \) and \( t \in [0, T] \),

\[
\langle S(t), \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^t \langle S(\tau), w\varphi \rangle \frac{1}{(t - \tau)^{1-\alpha}} \, d\tau,
\]

where \( S \) is the Fourier Transform of \( v \). Then \( v(t) = 0 \) for all \( t \in [0, T] \).

Proof. We fix \( R \geq 1 \), and we take \( \phi \in C^\infty_0(B_R) \). We see that (up to normalizing 
constants that we omit for the sake of simplicity)

\[
\| \mathcal{F}(w) \|_{H^m(\mathbb{R}^N)} = \sum_{|\beta| \leq m} \| D^\beta(\mathcal{F}(w\phi)) \|_{L^2(\mathbb{R}^N)} = \sum_{|\beta| \leq m} \| \mathcal{F}(\xi^\beta w\phi) \|_{L^2(\mathbb{R}^N)}
\]

\[
= \sum_{|\beta| \leq m} \| \xi^\beta w\phi \|_{L^2(\mathbb{R}^N)} \leq \| w \|_{L^\infty(B_R)} \sum_{|\beta| \leq m} \| \xi^\beta \phi \|_{L^2(\mathbb{R}^N)}
\]

\[
= \| w \|_{L^\infty(B_R)} \sum_{|\beta| \leq m} \| \mathcal{F}(\xi^\beta \phi) \|_{L^2(\mathbb{R}^N)}
\]

\[
= \| w \|_{L^\infty(B_R)} \sum_{|\beta| \leq m} \| D^\beta(\mathcal{F}(\phi)) \|_{L^2(\mathbb{R}^N)}
\]

\[
= \| w \|_{L^\infty(B_R)} \| \mathcal{F}(\phi) \|_{H^m(\mathbb{R}^N)}.
\]
We define
\[
\sigma(t) := \sup_{\tau \in [0,t]} \sup_{\psi \in C^\infty_0(B_R) \setminus \{0\}} \frac{t^\alpha}{\|\psi\|_{H^m(\mathbb{R}^N)}^2} \big| \langle S(t), \psi \rangle \big|,
\]
where \( \hat{\psi} = \mathcal{F}(\psi) \) denotes the Fourier Transform of \( \psi \).

We point out that
\[
\|\langle S(t), \psi \rangle\|_{H^m(\mathbb{R}^N)} = \frac{\|\langle \hat{\psi}, S(t) \rangle\|_{H^m(\mathbb{R}^N)}}{\|\hat{\psi}\|_{H^m(\mathbb{R}^N)}}
\]
and thus, by (13) and (3), we see that \( \sigma(t) \leq C \), for some \( C > 0 \), for all \( t \in [0,T] \) (in particular, \( \sigma(t) \) is finite). Moreover, for all \( t \in [0,T] \) and all \( \psi \in C^\infty_0(B_R) \),
\[
\|\langle S(t), \psi \rangle\|_{H^m(\mathbb{R}^N)} \leq \frac{\sigma(t) \|\hat{\psi}\|_{H^m(\mathbb{R}^N)}}{t^\alpha}.
\]

In this way, for all \( T_0 \geq 0 \) and all \( t \in [0,T_0] \), we have that
\[
\|\langle S(t), \phi \rangle\| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\|\langle S(\tau), w\phi \rangle\|_{\tau^\alpha(t-\tau)^{1-\alpha}}}{\Delta(\alpha)} d\tau
\]
\[
\leq \frac{1}{\Gamma(\alpha)} \int_0^t \sigma(\tau) \frac{\|\hat{\phi}\|_{H^m(\mathbb{R}^N)}}{\tau^\alpha(t-\tau)^{1-\alpha}} d\tau
\]
\[
\leq \frac{\|w\|_{L^\infty(B_R)}}{\Gamma(\alpha)} \|\hat{\psi}\|_{H^m(\mathbb{R}^N)} \frac{\sigma(T_0) \Xi(T_0)}{\Gamma(\alpha) \alpha}
\]
for all \( t \in [0,T_0] \), where
\[
\Xi(t) := \int_0^t \frac{d\tau}{\tau^\alpha(t-\tau)^{1-\alpha}} = \frac{4^{\alpha-1} \sqrt{\pi \Gamma(1-\alpha) t^{2-2\alpha}}}{\Gamma(3/2-\alpha)}.
\]

As a consequence, we obtain that
\[
\sigma(T_0) \leq \frac{\|w\|_{L^\infty(B_R)}}{\Gamma(\alpha)} \frac{\sigma(T_0) \Xi(T_0)}{\Gamma(\alpha) \alpha}
\]
In particular, if \( T_0 \) is sufficiently small, it follows that \( \sigma(T_0) \leq \sigma(T_0)/2 \), and therefore \( \sigma(T_0) = 0 \).

Let now
\[
T_* := \sup \left\{ t \in [0,T] \mid \text{s.t. } \sigma(t) = 0 \right\}.
\]

We have just proved that \( T_* > 0 \), and we now claim that
\[
T_* = T.
\]

For this, suppose by contradiction that \( T_* \in (0,T) \). Then, for all \( \epsilon \in [0,T-T_*] \), and all \( t \in [0,T_*+\epsilon] \), we set \( \hat{t} := \max\{t,T_*\} \) and we have that
\[
\|\langle S(t), \phi \rangle\| \leq \frac{1}{\Gamma(\alpha)} \int_{T_*}^{\hat{t}} \frac{\|\langle S(\tau), w\phi \rangle\|_{(t-\tau)^{1-\alpha}}}{\Delta(\alpha)} d\tau
\]
\begin{align*}
\leq \frac{1}{\Gamma(\alpha)} \int_{T_*}^{t} \sigma(\tau) \frac{\|F(w\phi)\|_{L^m(\mathbb{R}^N)}}{\tau^\alpha(t-\tau)^{1-\alpha}} d\tau \\
\leq \frac{\sigma(T_* + \epsilon)}{\Gamma(\alpha)} \frac{\|w\|_{L^\infty(B_R)}}{\|\hat{\phi}\|_{L^m(\mathbb{R}^N)}} \int_{T_*}^{t} \frac{d\tau}{\tau^\alpha(t-\tau)^{1-\alpha}} \\
= \frac{\sigma(T_* + \epsilon)}{\Gamma(\alpha)} \frac{\|w\|_{L^\infty(B_R)}}{\|\hat{\phi}\|_{L^m(\mathbb{R}^N)}} \left(\Xi(t) - \Xi(T_*)\right) \\cdot \frac{\|\Xi(T_* + \epsilon) - \Xi(T_*)\|}{\|\Xi(T_*)\|}.
\end{align*}

As a consequence,
\[\sigma(T_* + \epsilon) \leq \frac{\sigma(T_* + \epsilon)}{\Gamma(\alpha)} \frac{\|w\|_{L^\infty(B_R)}}{\|\hat{\phi}\|_{L^m(\mathbb{R}^N)}} \left(\Xi(T_* + \epsilon) - \Xi(T_*)\right) \cdot \frac{\|\Xi(T_*)\|}{\|\Xi(T_*)\|}.
\]
In particular, for a suitably small \(\epsilon > 0\),
\[\sigma(T_* + \epsilon) \leq \frac{\sigma(T_* + \epsilon)}{2},\]
which gives that \(\sigma(T_* + \epsilon) = 0\). This contradicts the maximality of \(T_*\) and so it proves (14).

In particular, we obtain that \(\langle S(t), \phi \rangle = 0\) for all \(t \in [0, T]\), for all \(\phi \in C_0^\infty(B_R)\), and for all \(R > 0\). Hence, we find that \(\langle S(t), \phi \rangle = 0\) for all \(t \in [0, T]\) and \(\phi \in C_0^\infty(\mathbb{R}^n)\). Since \(C_0^\infty(\mathbb{R}^n)\) is dense in \(H^m(\mathbb{R}^N)\), we conclude that \(\langle S(t), \phi \rangle = 0\) for all \(t \in [0, T]\) and \(\phi \in H^m(\mathbb{R}^N)\). Hence \(v(t) = \hat{S}(t) = 0\), as desired. \(\Box\)

From this, we can establish the uniqueness claim in Theorem 1.1:

**Corollary 1.** There exists at most one solution of (2).

**Proof.** Suppose that \(u_1\) and \(u_2\) are both solutions of (2), and let \(v := u_1 - u_2\). Since \(v(t) \in H^{-m}(\mathbb{R}^N)\), we can consider its Fourier Transform in the distributional sense, that we denote by \(\hat{S}(t)\). In view of (5), we know that
\[\langle v(t), \psi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\langle v(\tau), \Delta \phi \rangle + a\langle v(\tau), \phi \rangle}{(t-\tau)^{1-\alpha}} d\tau,
\]
for all \(\psi \in H^m(\mathbb{R}^N)\).

Hence, given any \(\varphi \in C_0^\infty(\mathbb{R}^N)\), we can apply the previous identity to the Fourier Transform of \(\varphi\), that we denote by \(\hat{\varphi} := \hat{\phi}\), and find that
\[\langle S(t), \varphi \rangle = \langle v(t), \hat{\varphi} \rangle = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{d\langle v(\tau), \Delta \hat{\phi} \rangle + a\langle v(\tau), \hat{\phi} \rangle}{(t-\tau)^{1-\alpha}} d\tau,
\]
for all \(\varphi \in C_0^\infty(\mathbb{R}^N)\).

We also set \(\eta(x) := |x|^2 \varphi(x) \in C_0^\infty(\mathbb{R}^N)\). We observe that
\[\Delta \hat{\phi}(\xi) = \sum_{j=1}^N \partial^2_{\xi_j} \int_{\mathbb{R}^N} \varphi(x) e^{-2\pi i x \cdot \xi} dx = -4\pi^2 \sum_{j=1}^N \int_{\mathbb{R}^N} x_j^2 \varphi(x) e^{-2\pi i x \cdot \xi} dx
\]
\[= -4\pi^2 \int_{\mathbb{R}^N} \eta(x) e^{-2\pi i x \cdot \xi} dx = -4\pi^2 \hat{\eta}(\xi).
\]
Notice also that
\[-4\pi^2 d\eta(x) + a\varphi(x) = (-4\pi^2 d|x|^2 + a)\varphi(x) = w(x)\varphi(x),
\]
where
\[w(x) := -4\pi^2 d|x|^2 + a.
\]
Consequently,
\[
d(v(\tau), \Delta \hat{\phi}) + a(v(\tau), \hat{\phi}) = -4\pi^2 d(v(\tau), \hat{\eta}) + a(v(\tau), \hat{\phi}) = -4\pi^2 d(S(\tau), \eta) + a(S(\tau), \varphi) = \langle S(\tau), w \varphi \rangle.
\]

We plug this information into (15) and we find that
\[
\langle S(t), \varphi \rangle = \frac{1}{\Gamma(\alpha)} \int_0^t \langle S(\tau), w \varphi \rangle \left( \frac{t}{t-\tau} \right)^{1-\alpha} d\tau,
\]
for all \( \varphi \in C_0^\infty(\mathbb{R}^N) \).

Then, in light of Lemma 2.1, we conclude that \( v(t) = 0 \), and thus \( u_1(t) = u_2(t) \), for all \( t \in [0, T] \). \( \square \)

2.2. The fundamental solution of the reaction-diffusion equation: Existence of the solution. Here we focus on the existence statement in Theorem 1.1, which will be obtained via the representation formula in (7) that exploits the Mittag-Leffler function \( E_\alpha \). In our framework, the crucial role played by \( E_\alpha \) consists in the fact that
\[
\partial_\alpha t^\alpha (E_\alpha(\lambda t^\alpha)) = \lambda E_\alpha(\lambda t^\alpha),
\]
for all \( \lambda \in \mathbb{R} \), see e.g. [9, Theorem 4.3].

Moreover, it is convenient to extend the setting in (6) by also defining
\[
E_{\alpha, \alpha}(r) := \sum_{k=0}^\infty \frac{r^k}{\Gamma(\alpha + k\alpha)}.
\]

For later use, we recall some well-known properties of the Mittag-Leffler functions in the following lemma.

**Lemma 2.2.** Let \( 0 < \alpha < 1 \). Then
\[
E'_{\alpha}(r) = \frac{1}{\alpha} E_{\alpha, \alpha}(r) \quad \text{for every } r \in \mathbb{R},
\]
\[
E_{\alpha}(r) > 0, \ E_{\alpha, \alpha}(r) > 0 \quad \text{for every } r \in \mathbb{R},
\]
\[
E_{\alpha}(r) = \frac{1}{\alpha} \exp r^{1/\alpha} + O \left( \frac{1}{r} \right), \quad r \to +\infty,
\]
\[
E_{\alpha}(r) = -\frac{1}{\Gamma(1-\alpha)} \frac{1}{r} + O \left( \frac{1}{r^2} \right), \quad r \to -\infty.
\]

**Proof.** All the properties are well-known: (17) is immediate, recalling that \( \Gamma(1 + (k+1)\alpha) = \alpha(k+1)\Gamma((k+1)\alpha) \); (18) is trivial for \( r \geq 0 \), while for \( r < 0 \) it descends from (17), and from the fact that \( E_\alpha(-z) \) is completely monotonic for \( z > 0 \) and \( 0 \leq \alpha \leq 1 \) (see e.g. [13], eq. (6) on p. 207); finally, (19) and (20) are equations (10) and (7) on p. 208 and p. 207 in [13], respectively. \( \square \)

**Remark 1.** Since \( E_1(r) = e^r \), the claims in Lemma 2.2 hold true also when \( \alpha = 1 \), except of course (20).

**Lemma 2.3.** Let \( T > 0 \). Problem (2) is solved in \( t \in [0, T] \) by
\[
u(x,t) := \mathcal{F}^{-1} \left( E_\alpha \left( (a - 4\pi^2 d |\xi|^2)t^\alpha \right) \right),
\]
being \( \mathcal{F} \) the Fourier Transform.
Proof. First of all, we show that, if \( u \) is as in (21), for every \( t \in [0,T] \),
\[
    u(t) \in H^{-m}(\mathbb{R}^N), \quad \text{and} \quad \sup_{t \in [0,T]} \langle u(t), \phi \rangle_{H^m(\mathbb{R}^N)} < +\infty,
\]  
(22)
as long as
\[
    N \ni m > \frac{N - 4}{2}.
\]
(23)

To check this, we first observe that, for every smooth and compactly supported function \( \phi \) and any \( \alpha \in \mathbb{N}^N \), considering the Fourier Transform \( \hat{\phi} = \mathcal{F}\phi \), we have that
\[
    \|D^\alpha \phi\|_{L^2(\mathbb{R}^N)}^2 = \|\mathcal{F}(D^\alpha \phi)\|_{L^2(\mathbb{R}^N)}^2 = \|\xi^\alpha \hat{\phi}\|_{L^2(\mathbb{R}^N)}^2.
\]

Also, if \( \rho := \frac{1}{4} \sqrt{\frac{t}{2\pi}} \) and \( \xi \in \mathbb{R}^N \setminus B_\rho \), we have that
\[
    2\pi^2 d|\xi|^2 - \alpha \geq 0
\]
and therefore
\[
    \frac{1}{4\pi^2 d|\xi|^2 - \alpha} \leq \frac{1}{2\pi^2 d|\xi|^2}.
\]

From this and (20), we see that
\[
    \int_{\mathbb{R}^N \setminus B_\rho} E_\alpha ((a - 4\pi^2 d|\xi|^2) t^\alpha) \hat{\phi}(\xi) \, d\xi 
\leq C \int_{\mathbb{R}^N \setminus B_\rho} \frac{\hat{\phi}(\xi)}{(4\pi^2 d|\xi|^2 - \alpha) t^\alpha} \, d\xi
\leq \frac{C}{t^\alpha} \sqrt{\int_{\mathbb{R}^N \setminus B_\rho} |\xi|^{2m}|\hat{\phi}(\xi)|^2 \, d\xi \int_{\mathbb{R}^N \setminus B_\rho} \frac{d\xi}{|\xi|^{4+2m}}}
\leq \frac{C}{t^\alpha} \sqrt{\int_{\mathbb{R}^N} |\xi|^{2m}|\hat{\phi}(\xi)|^2 \, d\xi} \leq \frac{C}{t^\alpha} \|D^m \phi\|_{L^2(\mathbb{R}^N)},
\]
(24)
for some \( C > 0 \) varying from line to line and where (23) has been exploited.

On the other hand,
\[
    \int_{B_\rho} E_\alpha ((a - 4\pi^2 d|\xi|^2) t^\alpha) \hat{\phi}(\xi) \, d\xi 
\leq C \int_{B_\rho} |\hat{\phi}(\xi)| \, d\xi \leq C \sqrt{\int_{B_\rho} |\hat{\phi}(\xi)|^2 \, d\xi}
\]
\[
\leq C \|\hat{\phi}\|_{L^2(\mathbb{R}^N)} = C \|\phi\|_{L^2(\mathbb{R}^N)}.
\]

This and (24), up to renaming \( C \), yield that
\[
    \frac{C \|\phi\|_{H^m(\mathbb{R}^N)}}{t^\alpha} \geq \int_{\mathbb{R}^N} E_\alpha ((a - 4\pi^2 d|\xi|^2) t^\alpha) \hat{\phi}(\xi) \, d\xi
= \int_{\mathbb{R}^N} \mathcal{F}^{-1} \left( E_\alpha ((a - 4\pi^2 d\cdot|\cdot|^2) t^\alpha) \right) \phi(x) \, dx
= \|\langle u(t), \phi \rangle\|.
\]

This proves (22).

From this, we know that \( u \in L^\infty_\alpha ([0,T], H^{-m}(\mathbb{R}^N)) \). Hence, it remains to show that (5) is satisfied. To this end, we recall (4) and (16), and we write that
\[
    E_\alpha (t^\alpha) = 1 + \frac{\lambda}{\Gamma(\alpha)} \int_0^t E_\alpha (\lambda \tau^\alpha) \frac{d\tau}{(t - \tau)^{1-\alpha}} d\tau.
\]
Using this formula with \( \lambda := a - 4\pi^2d|\xi|^2 \), and recalling (21), we obtain that
\[
\hat{u}(\xi, t) = E_\alpha((a - 4\pi^2d|\xi|^2)t^\alpha) = 1 + \frac{a - 4\pi^2d|\xi|^2}{\Gamma(\alpha)} \int_0^t E_\alpha((a - 4\pi^2d|\xi|^2)t^\alpha) \frac{d\tau}{(t - \tau)^{1-\alpha}}
\]
As a consequence, for every \( \varphi \in C_0^{\infty}(\mathbb{R}^N) \),
\[
(u(t), \varphi) - \varphi(0) = \langle (\hat{u}(t) - 1, \hat{\varphi}) \rangle = \int_{\mathbb{R}^N} \left( (a - 4\pi^2d|\xi|^2)\hat{\varphi}(\xi) \int_0^t \frac{E_\alpha((a - 4\pi^2d|\xi|^2)t^\alpha)}{(t - \tau)^{1-\alpha}} \, d\tau \right) d\xi = \int_{\mathbb{R}^N} \left( a\hat{\varphi}(\xi) + d\mathcal{F}(\Delta \varphi)(\xi) \frac{E_\alpha((a - 4\pi^2d|\xi|^2)t^\alpha)}{(t - \tau)^{1-\alpha}} \right) d\xi = \frac{1}{\Gamma(\alpha)} \int_0^t \int_{\mathbb{R}^N} \frac{a\hat{\varphi}(\xi) + d\mathcal{F}(\Delta \varphi)(\xi)}{(t - \tau)^{1-\alpha}} \, d\xi \, d\tau = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{a\hat{\varphi}(\xi) + d\mathcal{F}(\Delta \varphi)(\xi)}{(t - \tau)^{1-\alpha}} \, d\tau \]
and this establishes (5).

**Remark 2.** In case \( \alpha = 1 \), then (2) reduces to the standard heat equation, and recalling that \( E_1(r) = e^r \) Lemma 2.3 provides the well-known expression of the solution:
\[
u(x, t) = \frac{1}{(4\pi dt)^{N/2}} \exp \left( at - \frac{|x|^2}{4dt} \right).
\]

3. **Some estimates about Mittag-Leffler functions.** This section presents some useful facts on the Mittag-Leffler function and on similar special functions. These properties will be utilized in the next section to prove Theorem 1.2. For other interesting properties of the Mittag-Leffler function see e.g. [39], [23], and the references therein.

**Lemma 3.1.** Let \( \alpha \geq 1/2 \) and \( r \geq 0 \). Then
\[
E_\alpha(r) \leq \alpha E_\alpha'(r) + 1 - \frac{1}{\Gamma(\alpha)} + \left( \frac{1}{\Gamma(1 + \alpha)} - \frac{1}{\Gamma(2\alpha)} \right) r.
\]

**Proof.** It follows by (6), (17), and by the fact that
\[
s \geq \frac{3}{2} \Rightarrow \Gamma'(s) > 0, \quad \text{i.e.} \quad k \geq 2 \Rightarrow \frac{1}{\Gamma(1 + k\alpha)} \leq \frac{1}{\Gamma(\alpha + k\alpha)}
\]
(this is well known, see e.g. [8]). Indeed,
\[
E_\alpha(r) - 1 - \frac{r}{\Gamma(1 + \alpha)} = \sum_{k=2}^{\infty} \frac{r^k}{\Gamma(1 + k\alpha)} \leq \sum_{k=2}^{\infty} \frac{r^k}{\Gamma(\alpha + k\alpha)} = E_{\alpha,\alpha}(r) - \frac{1}{\Gamma(\alpha)} - \frac{r}{\Gamma(2\alpha)},
\]
as desired. \( \square \)
Lemma 3.2. Let $\alpha \geq 1/2$ and $r > 0$. Then

$$E_\alpha(r) \geq \frac{\alpha}{r} E'_\alpha(r) - \frac{1}{\Gamma(\alpha)} + 1 - \frac{1}{\Gamma(2\alpha)}.$$

Proof. We have

$$rE_\alpha(r) = \sum_{k=0}^{\infty} \frac{r^{k+1}}{\Gamma(1 + k\alpha)} = \sum_{k=1}^{\infty} \frac{r^k}{\Gamma(1 - \alpha + k\alpha)}.$$

Since $\alpha \geq 1/2$ we have $1 - \alpha \leq \alpha$. Using again the monotonicity of $\Gamma$ we infer

$$k \geq 2 \implies \frac{1}{\Gamma(1 - \alpha + k\alpha)} \geq \frac{1}{\Gamma(\alpha + k\alpha)},$$

and

$$rE_\alpha(r) - r = \sum_{k=2}^{\infty} \frac{r^k}{\Gamma(1 - \alpha + k\alpha)} \geq \sum_{k=2}^{\infty} \frac{r^k}{\Gamma(\alpha + k\alpha)} = E_{\alpha,\alpha}(r) - \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(2\alpha)} r,$$

as desired. \qed

Remark 3. Since

$$E_1(r) = e^r, \quad E_{1/2}(r) = \left(1 + \frac{2}{\sqrt{\pi}} \int_0^r e^{-s^2} ds\right) e^{r^2},$$

we obtain

$$E'_1(r) = E_1(r), \quad E'_{1/2}(r) = 2rE_{1/2}(r) + \frac{2}{\sqrt{\pi}}.$$

In particular, up to the lower order terms, Lemma 3.1 is optimal for $\alpha = 1$, while Lemma 3.2 is optimal for $\alpha = 1/2$.

4. Estimates of the fundamental solution. The goal of this section is to prove Theorem 1.2. To this end, in the following we study the fundamental solution $u$ defined in Theorem 1.1, restricting to dimension $N = 1$. From now on, we take $a := 1$ and $d := 1$, the general case following by suitable change of variables in the integral defining $u$. In this way, formula (7) becomes

$$u(x, t) = \int_{\mathbb{R}} E_\alpha((1 - 4\pi^2|\xi|^2)t^\alpha) \cos(2\pi x \xi) d\xi$$

$$= 2 \int_0^{+\infty} E_\alpha((1 - 4\pi^2|\xi|^2)t^\alpha) \cos(2\pi x \xi) d\xi$$

$$= \frac{1}{\pi} \int_0^{+\infty} E_\alpha((1 - |\xi|^2)t^\alpha) \cos(x \xi) d\xi.$$

We will now prove Theorem 1.2 through a sequence of lemmas. The first of this lemmas splits the integral into (25) into an infinite sum. This will be convenient in order to detect the alternate cancellations arising from the oscillatory kernel of the Fourier Transform.
Lemma 4.1. We have that
\[ u(x,t) = \frac{1}{\pi} \sum_{k=0}^{\infty} (-1)^k a_k(x,t), \]
with \[ a_0 := a_0(x,t) := \int_0^{\pi} E_\alpha((1 - \xi^2)t^\alpha) \cos(x \xi) \, d\xi \]
\[ a_k := a_k(x,t) := (-1)^k \int_{\pi/2}^{\pi} E_\alpha((1 - \xi^2)t^\alpha) \cos(x \xi) \, d\xi, \quad \text{for } k \geq 1. \]
\[ (26) \]
Moreover, for every \( t > 0, \, x \geq 0, \, k \in \mathbb{N}, \) we have that
\[ a_k(x,t) > 0, \]
and, furthermore,
\[ a_k(x,t) \to 0 \quad \text{as} \quad k \to +\infty. \]
\[ (27) \]
In addition,
\[ a_0 > \frac{a_1}{2}, \]
and, if \( k \geq 1, \)
\[ a_{k+1}(x,t) < a_k(x,t). \]
\[ (29) \]
\[ (30) \]
Proof. The claim in \( (26) \) plainly follows from \( (25) \).

Then, since the integral defining \( u \) converges, then also the series in \( (26) \) does, and consequently \( a_k \to 0 \) pointwise as \( k \to +\infty \), which establishes \( (28) \).

Moreover, the positivity of \( a_k \) claimed in \( (27) \) easily follows from \( (18) \) and the fact that
\[ \frac{\pi}{2x}(2k - 1) < \xi < \frac{\pi}{2x}(2k + 1) \implies (-1)^k \cos(x \xi) > 0. \]

Finally, if \( k \geq 1, \) we perform the change of variable \( \xi = \eta + \pi/x \) in \( a_{k+1} \). In this way, we infer that
\[ a_{k+1}(x,t) = (-1)^{k+1} \int_{\pi/2}^{3\pi/2} E_\alpha \left( \left( 1 - \left( \eta + \frac{\pi}{x} \right)^2 \right) t^\alpha \right) \cos(x \eta + \pi) \, d\eta \]
\[ = (-1)^k 2 \int_{\pi/2}^{3\pi/2} E_\alpha \left( \left( 1 - \left( \eta + \frac{\pi}{x} \right)^2 \right) t^\alpha \right) \cos(x \eta) \, d\eta. \]

Hence, by the strict monotonicity of \( E_\alpha \), see \( (17) \) and \( (18) \), we conclude that
\[ a_{k+1}(x,t) < (-1)^k 2 \int_{\pi/2}^{3\pi/2} E_\alpha \left( (1 - \eta^2) t^\alpha \right) \cos(x \eta) \, d\eta = a_k(x,t), \]
which concludes the proof of \( (30) \).

Similarly, one sees that \( (29) \) holds true, since the integral defining \( a_0 \) lies in \( (0, \pi/2x) \). \( \square \)

Next result states the quantities \( a_k \) in an analytically convenient way.
Lemma 4.2. Let $a_k$ be as in (26). Then,

$$a_0(x,t) = \int_{1-(\frac{x}{\pi})^2}^1 E_{\alpha}(t^\alpha \rho) \frac{\cos(x\sqrt{1-\rho})}{\sqrt{1-\rho}} \, d\rho$$

$$a_k(x,t) = (-1)^k \int_{1-(\frac{x}{\pi})^2}^1 E_{\alpha}(t^\alpha \rho) \frac{\cos(x\sqrt{1-\rho})}{\sqrt{1-\rho}} \, d\rho, \quad k \geq 1.$$ 

Proof. The desired claim follows by a direct computation, using the (monotone decreasing) change of variables

$$\rho := 1 - \xi^2 \in (-\infty, 1], \quad \xi = \sqrt{1-\rho}, \quad d\xi = -\frac{d\rho}{2\sqrt{1-\rho}}.$$ 

The next result analyzes the “first coefficient” $a_0$, which in the end will turn out to be the dominant one to understand the long time behavior of the fundamental solution.

Lemma 4.3. If $1/2 \leq \alpha < 1$, $t > 0$, and

$$0 < \ell < \frac{\pi}{2} < x,$$

then

$$a_0(x,t) \geq \frac{\alpha \cos \ell}{\ell} \frac{x}{t^{2\alpha}} \left[ E_{\alpha}(t^\alpha) - E_{\alpha}\left(t^\alpha \left(1 - \frac{\ell^2}{x^2}\right)\right) + c_0(x,t) \right]$$

where $c_0(x,t)$ is given by

$$c_0(x,t) := \frac{t^{\alpha}}{\alpha \Gamma(\alpha)} \ln \left(1 - \frac{\ell^2}{x^2}\right) + \frac{\ell^2 t^{2\alpha}}{\alpha x^2} \left(1 - \frac{1}{\Gamma(2\alpha)}\right).$$

Proof. By Lemma 4.2, for any $0 < \ell(x) < \pi/2$ we have

$$a_0(x,t) \geq \int_{1-(\frac{x}{\pi})^2}^1 E_{\alpha}(t^\alpha \rho) \frac{\cos(x\sqrt{1-\rho})}{\sqrt{1-\rho}} \, d\rho \geq \frac{x \cos \ell}{\ell} \int_{1-(\frac{x}{\pi})^2}^1 E_{\alpha}(t^\alpha \rho) \, d\rho.$$ 

The assumption on $x$ insures that $\rho > 0$ in the integration interval. As a consequence, using Lemma 3.2, we infer that

$$a_0(x,t) \geq \frac{x \cos \ell}{\ell} \int_{1-(\frac{x}{\pi})^2}^1 \left[ \frac{\alpha}{t^{\alpha} \rho} E_{\alpha}'(t^\alpha \rho) - \frac{1}{\Gamma(\alpha)t^{\alpha} \rho} + 1 - \frac{1}{\Gamma(2\alpha)} \right] \, d\rho$$

$$\geq \frac{\alpha x \cos \ell}{t^{2\alpha} \ell} \left[ E_{\alpha}(t^\alpha) - E_{\alpha}\left(t^\alpha \left(1 - \frac{\ell^2}{x^2}\right)\right) \right]$$

$$+ \frac{x \cos \ell}{\ell} \left[ \frac{1}{t^{\alpha} \Gamma(\alpha)} \ln \left(1 - \frac{\ell^2}{x^2}\right) + \left(1 - \frac{1}{\Gamma(2\alpha)}\right) \frac{\ell^2}{x^2} \right]$$

yielding the conclusion. □

We need now to compare the coefficient $a_0$ with the other terms. For this, we have the following result, estimating the next terms from above.

Lemma 4.4. If $1/2 \leq \alpha < 1$, $k \geq 1$, $t > 0$ and

$$x > \frac{\pi}{2}(2k+1),$$
then
\[ a_k(x, t) \leq \frac{2\alpha}{(2k-1)\pi} \frac{x}{t^\alpha} \left[ E_\alpha \left( t^\alpha \left( 1 - \frac{\pi^2}{4x^2(2k-1)^2} \right) \right) - E_\alpha \left( t^\alpha \left( 1 - \frac{\pi^2}{4x^2(2k+1)^2} \right) \right) + c_k(x, t) \right] \]

where \( c_k(x, t) \) is given by

\[ c_k(x, t) := \frac{4k\pi}{x(2k-1)} \left( 1 - \frac{1}{\Gamma(\alpha)} \right) + \frac{t^\alpha k\pi}{x(2k-1)} \left( \frac{1}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(2\alpha)} \right) \left( 4 - \left( \frac{\pi}{x} \right)^2 (1 + 4k^2) \right). \]

**Proof.** If \( k \geq 1 \) then, using Lemma 3.1,

\[ a_k(x, t) = (-1)^k \int_{1-(\frac{\pi}{x})^2(2k+1)^2}^{1-(\frac{\pi}{x})^2(2k-1)^2} E_\alpha(t^\alpha \rho) \cos \left( \frac{x(1-\rho)}{\sqrt{1-\rho}} \right) \rho d\rho \]

\[ \leq \frac{2x}{(2k-1)\pi} \int_{1-(\frac{\pi}{x})^2(2k+1)^2}^{1-(\frac{\pi}{x})^2(2k-1)^2} \alpha E'_\alpha(t^\alpha \rho) d\rho \]

\[ + \frac{4k\pi}{x(2k-1)} \left( 1 - \frac{1}{\Gamma(\alpha)} \right) + \frac{t^\alpha k\pi}{x(2k-1)} \left( \frac{1}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(2\alpha)} \right) \left( 4 - \left( \frac{\pi}{x} \right)^2 (1 + 4k^2) \right), \]

concluding the proof.

With the previous results, we are now in the position of completing the proof of Theorem 1.2 by detecting an infinite number of alternate cancellations.

**Proof of Theorem 1.2.** By the elementary Leibniz criterion for series with terms having alternate sign, we have that

\[ u(x, t) > a_0(x, t) - a_1(x, t). \]

For \( x > 3\pi/2 \) we can apply Lemma 4.3, with \( \cos \ell = \ell \), and Lemma 4.4, with \( k = 1 \). We obtain

\[ u(x, t) \geq \frac{\alpha x}{t^{2\alpha}} \left[ E_\alpha(t^\alpha) - E_\alpha \left( t^\alpha \left( 1 - \frac{\ell^2}{x^2} \right) \right) - t^\alpha E_\alpha \left( t^\alpha \left( 1 - \frac{A^2}{x^2} \right) \right) \right. \]

\[ \left. + t^\alpha E_\alpha \left( t^\alpha \left( 1 - \frac{A^2}{x^2} \right) \right) + c_0(x, t) - c_1(x, t) \right], \]

where

\[ 0 < \ell < A_- := \frac{\pi}{2} < A_+ := \frac{3\pi}{2}. \]

Let \( m > 0 \) and \( \beta > 0 \). Substituting \( x = t^\beta/m \) we have, for \( t \) large,

\[ u \left( \frac{t^\beta}{m}, t \right) \geq \frac{\alpha}{m} t^{\beta-2\alpha} E_\alpha(t^\alpha) \left[ 1 - \frac{E_\alpha(t^\alpha(1-\ell^2m^2t^{-2\beta}))}{E_\alpha(t^\alpha)} \right. \]

\[ \left. - 2t^\alpha \frac{E_\alpha(t^\alpha(1-A^2m^2t^{-2\beta}))}{E_\alpha(t^\alpha)} + 2t^\alpha \frac{E_\alpha(t^\alpha(1-A^2m^2t^{-2\beta}))}{E_\alpha(t^\alpha)} \right] \]
\[
\begin{align*}
&+ c_0 \left( \frac{t^\beta}{m} , t \right) - c_1 \left( \frac{t^\beta}{m} , t \right) \Bigg] \Bigg). \\
\end{align*}
\]

Since \( \beta > 0 \), we infer that, as \( t \to +\infty \), all the arguments of the Mittag-Leffler functions diverge to \(+\infty\). Let us first study the asymptotic behavior of \( c_0 \left( \frac{t^\beta}{m} , t \right) \) and \( c_1 \left( \frac{t^\beta}{m} , t \right) \). From (19) we easily deduce that
\[
c_0 \left( \frac{t^\beta}{m} , t \right) = \frac{\mu^\alpha}{\Gamma(\alpha)} \ln(1 - \ell^2 m^2 t^{-2\beta}) + \frac{1 - \frac{1}{\Gamma(2\alpha)}}{\Gamma(1 + \alpha)} m^2 \ell^2 t^{2(\alpha - \beta)} = o(1)
\]
and
\[
c_1 \left( \frac{t^\beta}{m} , t \right) = \frac{4\pi m t^{-\beta}}{\exp(t) + o(1)}
\]

Applying again (19) we obtain
\[
E_\alpha \left( t^\beta (1 - \ell^2 m^2 t^{-2\beta}) \right) = \exp \left( t(1 - \ell^2 m^2 t^{-2\beta})^{1/\alpha} + o(1) \right) = o(1) \quad \text{as} \quad t \to +\infty.
\]

Taking \( \beta < 1/2 \) we obtain
\[
E_\alpha \left( t^\alpha (1 - A_m^2 m^2 t^{-2\beta}) \right) = o(1) \quad \text{as} \quad t \to +\infty.
\]

Analogously,
\[
l^\alpha \frac{E_\alpha \left( t^\alpha (1 - A_m^2 m^2 t^{-2\beta}) \right)}{E_\alpha (t^\alpha)} = l^\alpha \exp \left( - \frac{1}{\alpha} A_m^2 m^2 t^{1-2\beta} \right) + l^\alpha o(\exp(-t)) = o(1) \quad \text{as} \quad t \to +\infty,
\]
and this establishes Theorem 1.2.

\[\square\]

**Acknowledgments.** S. D. and E. V. are supported by the Australian Research Council Discovery Project 170104880 NEW - Nonlocal Equations at Work.

S. D. is supported by the DECRA Project DE180100957 PDEs, free boundaries and applications and by the Fulbright Foundation.

G. V. is partially supported by the project ERC Advanced Grant 2013-339958 Complex Patterns for Strongly Interacting Dynamical Systems - COMPAT.

G. V. and B. P. are also partially supported by the PRIN Grant 2015KB9WPT Variational methods, with applications to problems in mathematical physics and geometry.

The authors are members of INdAM-GNAMPA. Part of this work was written on the occasion of a visit of B. P. and G. V. to the University of Melbourne, and of G. V. to the University of Western Australia.
REFERENCES

[1] N. Abatangelo and E. Valdinoci, Getting acquainted with the fractional Laplacian, in Contemporary Research in Elliptic PDEs and Related Topics, vol. 33 of Springer INdAM Ser., Springer, Cham, 2019, 1–105. https://link.springer.com/chapter/10.1007/978-3-030-18921-1_1.

[2] E. Affili and E. Valdinoci, Decay estimates for evolution equations with classical and fractional time-derivatives, J. Differential Equations, 266 (2019), 4027–4060.

[3] V. E. Arkhincheev and E. M. Baskin, Anomalous diffusion and drift in a comb model of percolation clusters, J. Exp. Theor. Phys., 73 (1991), 161–165. http://www.jetp.ac.ru/cgi-bin/e/index/e/73/1/p161?a=list.

[4] X. Cabré, A.-C. Coulon and J.-M. Roquejoffre, Propagation in Fisher-KPP type equations with fractional diffusion in periodic media, C. R. Math. Acad. Sci. Paris, 350 (2012), 885–890.

[5] X. Cabré and J.-M. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equations, Comm. Math. Phys., 320 (2013), 679–722.

[6] M. Caputo, Linear models of dissipation whose $Q$ is almost frequency independent. II, Fract. Calc. Appl. Anal., 11 (2008), 4–14, Reprinted from Geophys. J. R. Astr. Soc., 13 (1967), 529–539. https://www.annalsofgeophysics.eu/index.php/annals/article/viewFile/5950/5122.

[7] A. Carbotti, S. Dipierro and E. Valdinoci, Local density of solutions to fractional equations, De Gruyter Studies in Mathematics 74. De Gruyter, Berlin, https://www.degruyter.com/view/product/534026.

[8] W. E. Deming and C. G. Colcord, The minimum in the gamma function, Nature, 135 (1935), 917.

[9] K. Diethelm, The Analysis of Fractional Differential Equations, An application-oriented exposition using differential operators of Caputo type. Vol. 2004 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010.

[10] S. Dipierro and E. Valdinoci, A simple mathematical model inspired by the Purkinje cells: from delayed travelling waves to fractional diffusion, Bull. Math. Biol., 80 (2018), 1849–1870.

[11] S. Dipierro, E. Valdinoci and V. Vespri, Decay estimates for evolution equations with fractional time-diffusion, J. Evol. Equ., 19 (2019), 435–462.

[12] S. D. Eidelman and A. N. Kochubei, Cauchy problem for fractional diffusion equations, J. Differential Equations, 199 (2004), 211–255.

[13] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, Higher Transcendental Functions. Vol. III, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1955, https://www.ams.org/books/chel/379/chel379-endmatter.pdf. Based, in part, on notes left by Harry Bateman.

[14] F. Ferrari, Weyl and Marchaud Derivatives: A forgotten history, Mathematics, 6 (2018), 6. https://www.mdpi.com/2227-7390/6/1/6.

[15] I. M. Gel’fand and G. E. Shilov, Generalized Functions. Vol. S, Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1967, Theory of differential equations, Translated from the Russian by Meinhard E. Mayer. https://www.ams.org/books/chel379/chel379-endmatter.pdf.

[16] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado and J. H. T. Bates, The role of fractional calculus in modeling biological phenomena: A review, Commun. Nonlinear Sci. Numer. Simul., 51 (2017), 141–159.

[17] S. L. Kalla and B. Ross, The development of functional relations by means of fractional operators, in Fractional Calculus (Glasgow, 1984), vol. 138 of Res. Notes in Math., Pitman, Boston, MA, 1985, 32–43.

[18] J. Kemppainen, J. Siljander, V. Vergara and R. Zacher, Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$, Math. Ann., 366 (2016), 941–979.

[19] J. Kemppainen, J. Siljander and R. Zacher, Representation of solutions and large-time behavior for fully nonlocal diffusion equations, J. Differential Equations, 263 (2017), 149–201.

[20] J. Kemppainen and R. Zacher, Long-time behavior of non-local in time Fokker–Planck equations via the entropy method, Math. Models Methods Appl. Sci., 29 (2019), 209–235.

[21] Y. Luchko, Initial-boundary-value problems for the one-dimensional time-fractional diffusion equation, Fract. Calc. Appl. Anal., 15 (2012), 141–160.
[22] Y. Luchko, Initial-boundary problems for the generalized multi-term time-fractional diffusion equation, J. Math. Anal. Appl., 374 (2011), 538–548.

[23] F. Mainardi, On some properties of the Mittag-Leffler function $E_\alpha(-t^\alpha)$, completely monotone for $t > 0$ with $0 < \alpha < 1$, Discrete Contin. Dyn. Syst. Ser. B, 19 (2014), 2267–2278.

[24] F. Mainardi, Y. Luchko and G. Pagnini, The fundamental solution of the space-time fractional diffusion equation, Fract. Calc. Appl. Anal., 4 (2001), 153–192. https://arxiv.org/pdf/cond-mat/0702419.pdf.

[25] M. M. Meerschaert and A. Sikorskii, Stochastic Models for Fractional Calculus, vol. 43 of De Gruyter Studies in Mathematics, Walter de Gruyter & Co., Berlin, 2012. https://www.degruyter.com/view/product/129781.

[26] I. Podlubny, Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. https://www.degruyter.com/view/product/129781.

[27] B. Ross, The development, Theory and Application of the Gamma-function and a Profile of Fractional-calculus, ProQuest LLC, Ann Arbor, MI, 1974. Thesis (Ph.D.)–New York University. http://gateway.proquest.com.pros.lib.unimi.it/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:7417154.

[28] B. Ross, The development of fractional calculus 1695–1900, Historia Math., 4 (1977), 75–89.

[29] B. Ross, Origins of fractional calculus and some applications, Internat. J. Math. Statist. Sci., 1 (1992), 21–34.

[30] J. Sánchez and V. Vergara, Long-time behavior of bounded global solutions to systems of nonlinear integro-differential equations, Asymptot. Anal., 85 (2013), 167–178.

[31] J. Sánchez and V. Vergara, Long-time behavior of nonlinear integro-differential evolution equations, Nonlinear Anal., 91 (2013), 20–31.

[32] T. Sandev, A. Schulz, H. Kantz and A. Iomin, Heterogeneous diffusion in comb and fractal grid structures, Chaos Solitons Fractals, 114 (2018), 551–555.

[33] F. Santamaria, S. Wils, E. D. Schutter and G. J. Augustine, The diffusional properties of dendrites depend on the density of dendritic spines, Eur. J. Neurosci., 34 (2011), 561–568. https://www.ncbi.nlm.nih.gov/pmc/articles/PMC3156966/.

[34] H. Schiessel, C. Friedrich and A. Blumen, Applications to problems in polymer physics and rheology, in Applications of Fractional Calculus in Physics, World Sci. Publ., River Edge, NJ, 2000, 331–376.

[35] E. Topp and M. Yangari, Existence and uniqueness for parabolic problems with Caputo time derivative, J. Differential Equations, 262 (2017), 6018–6046.

[36] V. Vergara and R. Zacher, A priori bounds for degenerate and singular evolutionary partial integro-differential equations, Nonlinear Anal., 73 (2010), 3572–3585.

[37] V. Vergara and R. Zacher, Optimal decay estimates for time-fractional and other nonlocal subdiffusion equations via energy methods, SIAM J. Math. Anal., 47 (2015), 210–239.

[38] R. Wong and Y.-Q. Zhao, Exponential asymptotics of the Mittag-Leffler function, Constr. Approx., 18 (2002), 355–385.

[39] R. Zacher, Maximal regularity of type $L_p$ for abstract parabolic Volterra equations, J. Evol. Equ., 5 (2005), 79–103.

[40] R. Zacher, Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces, Funkcial. Ekvac., 52 (2009), 1–18.

[41] R. Zacher, Time fractional diffusion equations: Solution concepts, regularity, and long-time behavior, in Handbook of Fractional Calculus with Applications. Vol. 2, De Gruyter, Berlin, 2019, 159–179. https://www.degruyter.com/viewbooktoc/product/497030.

Received for publication September 2019.

E-mail address: serena.dipierro@uwa.edu.au
E-mail address: benedetta.pellaccia@unicampania.it
E-mail address: enrico.valdinoci@uwa.edu.au
E-mail address: gianmaria.verzini@polimi.it