Automating Renormalization of Quantum Field Theories

A. D. Kennedy
adk@ph.ed.ac.uk

Thomas Binoth
Thomas.Binoth@ed.ac.uk

Tom Rippon
t.o.rippon@sms.ed.ac.uk

School of Physics, University of Edinburgh
King’s Buildings, Mayfield Road, Edinburgh, EH9 3JZ, Scotland

ABSTRACT

We give an overview of state-of-the-art multi-loop Feynman diagram computations, and explain how we use symbolic manipulation to generate renormalized integrals that are then evaluated numerically. We explain how we automate BPHZ renormalization using “henges” and “sectors”, and give a brief description of the symbolic tensor and Dirac $\gamma$-matrix manipulation that is required. We shall compare the use of general computer algebra systems such as Maple with domain-specific languages such as FORM, highlighting in particular memory management issues.

Categories and Subject Descriptors
J.2 [Computer Applications]: Physical sciences and engineering—Physics; I.1.2,4 [Computing Methodologies]: Symbolic and algebraic manipulation—Algorithms, Applications

General Terms
Algorithms, Languages.

Keywords
Quantum Field Theory, Renormalization Theory, Feynman Diagrams.

1. INTRODUCTION

It has been said that “in the thirties, under the demoralizing influence of quantum-theoretic perturbation theory, the mathematics required of a theoretical physicist was reduced to a rudimentary knowledge of the Latin and Greek alphabets.” Likewise, the ability to evaluate multi-loop Feynman diagram can be reduced to an ability to do some simple tensor algebra, some graphical manipulations to purge the calculations of divergences, a knowledge of some basic properties of $\Gamma$ functions, and the ability to evaluate integrals numerically. Since this is obviously damaging to the delicate egos of theoretical physicists we advocate that the entire procedure can — and should — be automated and handed over to computers which do not have egos and, as is well-known, never make mistakes.

Precision measurements related to properties of elementary particles are nowadays on a level which makes the inclusion of quantum corrections to theoretical pre- and postdictions mandatory. As will be explained below these quantum corrections are evaluated in the context of quantum field theoretic perturbation theory, the loop expansion. This allows for a systematic evaluation of quantum corrections to — in principle — any order in the coupling parameters of the theory under consideration.

1.1 Loops and Legs: the State of the Art

The complexity of perturbative calculations grows with the number of free internal momenta which are integrated over (loops), and the number of external particles of the process under consideration (legs). Only a limited number of observables have actually been calculated beyond the one-loop level. For one-loop scattering amplitudes on the other hand only one complete scattering process with more that five external legs has been evaluated up to now: the electroweak corrections to four fermion production in electron positron collisions which is highly relevant for $e^+e^-$-collider phenomenology [23]. For hadron colliders a lot of progress has been made very recently concerning the evaluation of six-point amplitudes. Different methods have been designed to tackle this problem and benchmark amplitudes for six-photon [37, 10, 38] and six-gluon scattering [13, 6, 24, 4, 50] are now available. In the one-loop case so-called unitarity based methods have played a prominent role in these developments (see [6] for a review and references therein). The unitarity approach is not based on Feynman diagrams but rather on bootstraping tree level amplitudes. Although it has a big potential its applicability to amplitudes with internal and external masses needs further developments before it may replace the Feynman diagrammatic approach.

It should be noted that the closed-form structure of one-loop amplitudes is completely understood. One can show that each one-loop Feynman integral is expressible in terms of dilogarithms. Even at two loops the situation is less clear: only a limited number of four-point amplitudes, mainly for massless particles, are known. The corresponding Feynman diagrams are reduced first to an independent set of basis functions, the so-called Master integrals, by using integration by parts identities [17] and relations from Lorentz invariance [26]. The analytic knowledge of the amplitude re-
lies then on the successful evaluation of the Master integrals. This is a formidable task and typically only successful if the number of scales is very restricted. For example a breakthrough in that direction was the analytical evaluation based on Mellin-Barnes representations of the massless planar [41] and non-planar [45] four-point two loop integrals in 2000. Although there has been some progress for more complicated cases it has been very slow. Unfortunately, even direct numerical evaluation is highly non-trivial as the presence of infrared (IR) and ultraviolet (UV) divergences within the graphs necessitates regularization; even in their absence integrable singularities due to internal thresholds requires new numerical methods [34, 40, 5, 22].

1.2 Beyond Two Loops

Beyond the two loop level essentially only one-scale problems have been evaluated so far. A very remarkable result here is the recent analytical evaluation of the three-loop splitting functions in QCD [19, 14]. By mapping the splitting functions to Mellin moment integrals one can generate a hierarchy of difference equations which have to be solved recursively. The project took many person-years and needed to accumulate a database of integrals which occupied about 3.5 GBytes of disk space. The algebraic manipulations relied on the domain-specific language FORM, or more precisely on constant improvements of this code. Indeed, the successful completion of this important calculation relied on the fact that the author of the algebraic manipulation code was a member of the collaboration.

The FORM code was also the basis of the few known four-loop computations in perturbative QCD. All renormalization constants are known at this level [16]. An important milestone was in this respect the evaluation of the four-loop β-function in QCD [16] in the so-called \( \overline{\text{MS}} \) renormalization scheme. The problem was mapped to the evaluation of three loop propagator functions which were computed with the FORM package Mincer. The project amounted to the evaluation of about 50,000 Feynman diagrams and its confirmation by an independent group was an important issue [21]. The latter mapped the problem to the evaluation of the divergent part of some four-loop tadpole master integrals which were evaluated by solving integration by parts identities using a dedicated C++ code. To give a prominent example of a five-loop contribution to a very precisely measured observable acting on Hilbert spaces and satisfying various commutation and anticommutation relations. Nowadays we prefer to think of them in terms of functional (or path) integrals, which give us greater geometrical insight (for example for the introduction of Fade’yev-Popov ghosts for gauge-fixing non-abelian gauge theories) with just as high a level of mathematical rigour (namely not very much).

2.1 QFT as a Functional Integral

The quantities we want to calculate are matrix elements of the scattering \((S)\) matrix, and these can be written as integrals over all possible field configurations with a measure \(\exp(iS(\phi))\) where \(S = \int d^nx L(\phi, \partial \phi)\) is the action which is the space-time integral of the Lagrangian density \(L\) defining the theory. We take the dimension of space-time to be \(n\) rather than four as this is a convenient way of “regulating” the divergences that occur; when we have removed all the divergences by some “renormalization” procedure that we shall discuss later then we will take the limit that \(n \to 4\) (or \(n \to 6\) which is convenient for the scalar \(\phi^4\) theory we are using as an example). For a scalar field theory, which for simplicity is all we shall consider here, the Lagrangian is \(L = \frac{1}{2} \phi (\partial^2 + m^2) \phi + \lambda \phi^4\). A typical S-matrix element is then

\[
\langle \phi(x) \phi(y) | S | \phi(z) \rangle = \frac{1}{Z} \int d\phi \phi(x) \phi(y) \phi(z) e^{iS(\phi)},
\]

where the points \(x, y\) and \(z\) are at time \(t = \infty\) and \(z\) is at \(t = -\infty\). The integral is over all fields \(\phi\) that is \(d\phi = \prod_x d\phi(x)\) where the product is taken over all space-time points. We shall not dwell on the precise definition of such integrals but we will consider the situation where the coupling constant \(\lambda\) is small and we can apply perturbation theory. Formally one can consider that the weight factor \(\exp(S)\) should be written as \(\exp(iS/\hbar)\) in order to get the dimensions correct, and that as Planck’s constant \(\hbar\) is very small the integral will be dominated by field values near to the minimum of the action which occurs at \(\phi = 0\). In reality this is completely bogus, firstly because we are interested in phenomena where the natural scale is set by working in units where \(\hbar = c = 1\), and secondly because our \(\phi^4\) action is not bounded below as \(\phi \to -\infty\). To justify our argument we need to point out that we are really expanding in the powers of \(\lambda\) rather than \(\hbar\), and that for small perturbations about \(\phi = 0\) the theory does not notice the “non-perturbative” instability of the vacuum. We could try to be a little less cavalier and consider a \(\phi^4\) interaction instead of or as well as the \(\phi^3\) one, but it turns out that this theory is also sick, and so we will keep things simple and just ignore these problems.

2.2 Perturbation Theory

The perturbative expansion we use is just an (unjustified) generalization of the method of steepest descents to an infinite dimensional integral. We add a linear source term \(J\) to the action so that we can write

\[
\int d\phi \phi(x) \phi(y) \phi(z) e^{iS(\phi)} = \frac{(-i)^3}{Z} \left. \frac{\partial^3 Z(J; \lambda)}{\partial J(x) \partial J(y) \partial J(z)} \right|_{J=0}
\]

where

\[
Z(J; \lambda) \equiv \int d\phi e^{i\lambda \phi + iJ \phi} \sim \exp \left[ i \int d^nx \lambda \phi \frac{\partial^3}{\partial J(x)^3} \right] Z_0(J; 0)
\]

where we have written \(J \phi\) for \(\int d^nx J(x) \phi(x)\), and \(Z_0(J) \equiv \int d\phi e^{(i\lambda \phi + iJ \phi)} \propto e^{-\frac{1}{2} \lambda J} \), where we have introduced the 2

The reader should trust us that realistic theories with gauge fields, fermions, ghosts and so forth just add some technical complications but do not essentially alter the strategy and methods we propose.
kernel $K = \partial^2 + m^2$ and its Green’s function $\Delta \equiv K^{-1}$ with appropriate boundary conditions. We obtain the required asymptotic expansion by expanding $\exp(i \lambda \partial^2 / \partial J^2)$ as a Taylor series, and then each resulting term can be drawn as a Feynman diagram with the rules that there is a factor of $i \lambda$ associated with each vertex whose position is integrated over all space-time locations, and a propagator $\Delta$ with each edge. If we Fourier transform to momentum space then we obtain the equivalent diagrams but with the rules that momentum is conserved at each vertex and there is an integral over the momentum flowing round each loop, and the propagators become $\Delta(k) = (k^2 - m^2 + i0^+)^{-1}$.

### 3. DIVERGENCES AND RENORMALIZATION

The infamous UV divergence disease of QFT is now immediately apparent, it manifests itself even if we work in Euclidean rather than Minkowski space. Even the simplest one-loop diagram corresponding to the Euclidean integral

$$I(p) \equiv \int \frac{d^n k}{[k^2 + m^2][(k+p)^2 + m^2]} < c \int_0^\infty dk \frac{||k||^{n-1}}{\max(||k||, m)^4}$$

for some constant $c$, and this clearly diverges in $n \geq 4$ dimensions.

What saves the day is that the divergence is local (a Dirac $\delta$-function or its derivatives in space-time, or equivalently a polynomial in the external momentum $p$ in momentum space). We may see this easily by differentiating the Feynman integral with respect to $p$:

$$\frac{\partial I(p)}{\partial p_\mu} = -\int \frac{d^n k}{[k^2 + m^2][(k+p)^2 + m^2]} 2(k+p)_\mu.$$ 

For large $||k||$ the integrand now behaves as $||k||^{-5}$ rather than $||k||^{-4}$ as before; differentiating $d = n - 3$ times suffices to render the Feynman convergent. We may therefore express the Feynman integral as

$$I(p) = T^d I(p) + \int_0^p d^n p_1 \cdots \int_0^{p^d-1} d^n p_d \partial^d I(p_d)$$

by iterating the fundamental theorem of calculus $d$ times (otherwise known as Taylor’s theorem) where $T^d I(p)$ is a polynomial of degree $d$ with divergent coefficients. We have simplified the notation by writing $\partial$ for $\partial / \partial p_\mu$, suppressing all the tensor indices (such as $\mu$), and lumping together all the external momenta into one term.

We can remove the offending divergent polynomial by adding it to the action as a new term giving rise to a vertex that is formally of one-loop order. The number of loops in a Feynman diagram is formally equal to the corresponding power of $\hbar$ in the perturbation expansion, so we may choose to imagine that all the couplings in the action, $\lambda$, $m$, and the coefficient $Z$ of the kinetic term $\phi \partial^2 \phi$ that we forgot to include originally, may be expanded in powers of $\hbar$ where all the terms but the first just serve to remove the unwanted divergences. Clearly there are two requirements that must be satisfied for this to work: (i) all the divergences must be of the form of terms that occur in the action, and (ii) all the divergences must add up in just the right way to correspond to such counterterms. We usually rephrase (i) by saying that we must include all monomials in the action that are allowed (i.e., which do not violate physically necessary conditions such as locality and unitarity), as if we do not we will have to come back later and put them in so as to cancel the divergences that arise. Of course, we can only allow there to be a finite number of parameters (and thus monomials) in the action if we are to have a theory with any predictive power, but happily we can get away with including only monomials of dimension less than $n$, as this suffices to cancel all possible divergences by a simple power-counting argument (q.v., [32]).

#### 3.1 Regulators

Of course this is all rather messy and embarrassing, so we reformulate the procedure in the following language: in order to define a QFT we need to introduce a regulator of some kind which makes our manipulations mathematically well-defined. We then renormalize the theory by adjusting the “bare” coefficients in the action to absorb all the would-be divergences, and finally we take the regulator away to obtain a well-defined finite theory as the limit of a renormalized regularized functional integral. There are many choices of regulator, working in $n$ dimensions (“dimensional regularization” [44, 45, 12, 18]) is just one rather convenient possibility, and they are all supposed to give the same answer. What turns out to be crucial is that the regulator preserves as many of the symmetries of the original theory as possible, as we can then exclude counterterms that do not also have these symmetries.

#### 3.2 Renormalization Conditions and the Renormalization Group

In order to fix the finite parts of the counterterms in the action we need to specify a set of renormalization conditions, i.e., a set of experimentally determined values for some processes that can be solved for the parameters in the action. This is no different from what happens in classical theories, except that the parameters themselves have a less obvious physical meaning. Indeed, we can choose the renormalization conditions in many different ways; for example we can specify them as a function of some energy scale $\mu$, and the fact that they all fix the same physical theory tells us the physical quantities must not depend upon $\mu$. The invariance under this group of reparameterizations is known as renormalization group invariance, and is very useful because it is not in general respected by the perturbative expansion to any given order.

The parameters in the action that are just constants on the classical level thus become in general scale dependent if radiative corrections are included. In the context of Quantum Chromodynamics (QCD), our theory of the strong interactions, this leads to asymptotic freedom, the logarithmic decrease of the coupling strength with increasing interaction energy. For QCD the perturbative approach is limited to the high energy regime where the gauge coupling parameter, $\alpha_s$, is sufficiently small.

In the case of gauge theories (which are the theories we really use to describe nature) we must choose renormalization conditions that preserve the gauge symmetries or the whole theory falls apart. On the other hand there are certain symmetries, such as some forms of chiral symmetry (a peculiar symmetry that occurs because our theories involve fields that transform as spinor representations of the covering group $\text{Spin}(3, 1)$ of the Lorentz group $\text{SO}(3, 1)$) and scale invariance symmetry cannot be maintained by any regulator and thus are not symmetries of the quantum theory even
though they look like perfectly good symmetries of the underlying classical action. There are physical implications of these so-called anomalies, such as the decay of a pion into two photons $\pi \rightarrow \gamma \gamma$, and this is evidence that renormalization is necessary and not just an ugly contrivance that we could avoid by being more clever.

What will concern us for the rest of this paper is a more detailed investigation of (ii), namely showing that all the divergences are local and appear in the right way to be cancelled by counterterms in the action. For instance, if we go beyond one-loop Feynman diagrams all are the divergences still local? The trouble is that multi-loop Feynman diagrams are very complicated integrals, and it is hard to see what is going on without getting lost. Nevertheless, with a little effort one can see that a typical two loop diagram \( \includegraphics{2loop_diagram} \) has non-local divergences; however, before abandoning all hope we notice that the one-loop counterterms we introduced to cancel the one-loop divergences have to appear in loop diagrams themselves. Their divergent coefficients thus multiply the non-local finite parts of the corresponding Feynman integrals also leading to non-local divergences. What we must show is that these one loop countergrapths, which are formally of two loop order \( (O(h^2)) \) exactly cancel the non-local part of our two loop graphs, leaving only a local divergence that can be absorbed by adding local two loop counterterms to the action.

### 3.3 Henges and the $R$ and $\bar{R}$ Operations

This is most easily done in two stages. The first is we define a procedure that removes all divergences by local Taylor series subtractions, and then we show that the subtractions thus made add up to give local counterterms. We will discuss the first stage here, as it is what needs to be implemented to automate the renormalization process; the second is a purely combinatorial proof which the interested reader can find in the literature \cite{14, 2, 19}.

The two-loop diagram \( \includegraphics{2loop_diagram} \) illustrates the difficulty we face. Clearly when all the loop momenta get large simultaneously the diagram diverges as \( \|k\|^{2n-10} \), and this overall divergence is local. However, what happens when one of the two three-line loop momenta gets large while the other stays small? How do we disentangling these overlapping sub-divergences? Our approach is a systematic decomposition of the space of all loop momenta based on the structure of the graph itself.

We need only consider graphs which are one-particle irreducible (1PI), that is ones which remain connected when any line is cut. For any line \( \ell \) in a 1PI Feynman diagram \( \mathcal{G} \) we can decompose the graph uniquely into a single loop containing \( \ell \) stringing together a set of 1PI subgraphs that we will call a Henge \( \mathcal{H}(\mathcal{H}, \ell) \) \cite{15, 32}. We will discuss efficient ways of representing and computing Henges in \cite{3}.

Clearly at any point the space of all loop momenta some line must carry the smallest (Euclidean) momentum. Hence we may decompose an arbitrary Feynman diagram \( \mathcal{G} \) into a sum of contributions each from a region where a particular line \( \ell \in \mathcal{G} \) carries the smallest momentum, and in each such region we can use our Henge decomposition to write the Feynman integral as

\[
I_\lambda(\mathcal{G}) = \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty d^4 k_\ell \, i_{\lambda \ell} (\mathcal{G}/\mathcal{H}(\mathcal{G}, \ell)) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} I_\lambda(\Theta),
\]

where \( I_\lambda(\Theta) \) is the Feynman integral for the graph \( \Theta \) restricted to the region of momentum space where all the lines carry momenta of magnitude greater than \( \lambda \), and \( i_{\lambda \ell} (\mathcal{G}/\mathcal{H}) \) is the product of all the lines in the single loop graph \( \mathcal{G}/\mathcal{H} \) obtained by shrinking all the graphs in \( \mathcal{H} \) to points, again restricted to have momentum of magnitude greater than \( \lambda \). The "small momentum cutoff" \( \lambda \) serves as a convenient parameter for our recursive definition.

As an example let us consider the overlapping two-loop diagram we considered before. Its concomitant Feynman integral is

\[
I(p) = \int d^4 k \, d^4 k' \, \Delta(k) \Delta(k+p) \Delta(k-k') \Delta(k'+p) \Delta(k').
\]

The Henges that arise are \( \includegraphics{henge_diagram} \), \( \includegraphics{henge_diagram} \), \( \includegraphics{henge_diagram} \), where the heavy lines show the Henges corresponding to each of the light lines. For example, the contribution from the region where the light line in the first diagram is smallest is

\[
I_\lambda \left( \includegraphics{henge_diagram} \right) = \int_\lambda^\infty d^4 k'' \, \Delta(k'') \, I_{k''} \left( \includegraphics{henge_diagram} \right)
\]

where \( \Delta(k) \equiv \Delta(k) \theta(\|k\| - \lambda) \) and the inner integral (corresponding to the blob in the diagram on the left and the heavy loop in that on the right) is

\[
\int_{\|k''\|}^\infty d^4 k \, \Delta(k) \Delta(k+p) \Delta(k''(k+k''+p)) \Delta(k''(k+k')).
\]

Observe how the graphical structure dictated the change of variable \( k'' = k - k' \) to the loop momentum of the shrunken graph.

With this decomposition it is obvious that the divergences can be put into two classes, overall divergences that occur when all the momenta simultaneously get large, and sub-divergences that are isolated to the subgraphs occurring in the Henges. Let us introduce an operator \( R \) that removes all the divergences from a given Feynman diagram: it first removes all the sub-divergences (an operation that is called \( \bar{R} \)) by recursively applying \( R \) to the elements of the Henges

\[
\bar{R} I_\lambda(\mathcal{G}) \equiv \sum_{\ell \in \mathcal{G}} \int_\lambda^\infty d^4 k_\ell \, i_{\lambda \ell} (\mathcal{G}/\mathcal{H}(\mathcal{G}, \ell)) \prod_{\Theta \in \mathcal{H}(\mathcal{G}, \ell)} R I_\lambda(\Theta),
\]

and then it removes the overall divergence (if necessary) by subtracting the leading terms of the Taylor series expansion in the external momenta as described before for the one-loop case, \( R I_\lambda(\mathcal{G}) \equiv I_\lambda(\mathcal{G}) - T^{\text{leading}(\mathcal{G})} \bar{R} I_\lambda(\mathcal{G}) \). The number of terms removed by the Taylor subtraction operation \( T \) is fixed by simple power counting rules (q.v. \cite{14, 19}). Note the subtle but vital fact that the subtraction term has its small momentum cutoff set to zero.

### 3.4 BPH and Zimmermann’s Forests

In order for this to be a valid renormalization procedure we need to prove two things: first that all the overall divergences are local — that is polynomial in the external momenta —
so that they get eaten by the Taylor series subtraction, and second that all the subtractions that are made add up in such a way as to correspond to counterterms in the action. The former condition is equivalent to showing that the derivatives $\frac{d^{k}I_{\lambda}(\bar{G})}{d\phi^{k}}$ with respect to the external momenta are finite, and follows by induction from (a) the inductive assumption that $|I_{\lambda}(\bar{G})| < c \cdot \max(\lambda, m^{d_{\text{deg}}(\bar{G})})$ for some constant $c$, (b) the requirement that differentiation lowers the power-counting degree, $\deg(\bar{G}) \leq \deg(G) - 1$, and (c) the fact that differentiation commutes with the $R$ operation, $[\partial, R] = 0$.

To prove (c), as well as the condition (ii) above, we rewrite the $R$ operation in purely graphical form (as it was originally defined by Bogoliubov and Parasiul in [8]). Namely,

$$\tilde{R}I(\bar{G}) = \sum_{S \in \mathcal{V}(\bar{G})} I(\bar{G}/S) \cdot \prod_{\Gamma \in \mathcal{S}} \left( -\tau^{\deg(\Gamma)} \tilde{R}I(\Gamma) \right).$$

Here $\mathcal{S}$ is a spinney, that is a covering of $G$ by a set of 1PI subgraphs (possibly including single vertices), and the proper wood $\bar{G}$ is the set of all such spinneys (excluding the one consisting of just $G$ itself). The notation $I(G/\Gamma) + f(\Gamma)$ means that the 1PI subgraph $\Gamma$ in $G$ is to be shrunk to a point and replaced with the value $f(\Gamma)$.

Proving the equivalence of this definition of $R$ to the previous one for $\lambda = 0$ is left as an exercise for the dedicated reader who need just show that in the recursive Henge decomposition the sequence of subtractions made always correspond to some spinney, and that for each spinney $S$ the corresponding subtractions occur in the Henge decomposition for every ordering of the line momenta in $G/S$.

One can expand out the recursion even more and obtain an explicit formula for all the subtractions that are made in a graph in terms of nested non-overlapping Taylor subtractions; the necessary graphical apparatus for this was given by Zimmermann in [51] in his forest formula. Zimmermann introduced his forest formula because it explicitly defines a convergent integral corresponding to a Feynman diagram without the need for any intermediate regularization scheme.

This property is the reason for our practical use of the $R$ operation: we use it to generate the integrand of a manifestly convergent integral, although in practice this is more easily implemented using Henge recursion. From a formal point of view the utility of Zimmermann’s approach is limited by the fact that although it defines finite integrals these are not directly related to the original functional integral in a straightforward way, and therefore the various identities between Feynman integrals that follow because of this that are needed to guarantee desirable properties such as gauge invariance (Ward or BRS identities) or unitarity (cutting relations) are no longer manifest. Indeed the former are still untrue in the case of anomalies despite the fact that no regulator is introduced.

The fact that Bogoliubov’s definition is purely graphical makes it obvious that $[\partial, R] = 0$, and it is also possible to give a combinatorial argument that shows that all the subtractions made, summed over all Feynman graphs, formally corresponds to adding $-\tau^{\deg} R e^{\lambda \phi^{n}}$ to the action [11, 2].

### 3.5 Power Counting

The BPH theorem tells us how to remove all the divergences from a Feynman diagram by local subtractions, but it does not guarantee that the number of counterterms required is finite. If it is not then the resulting theory has little if any predictive power, as an infinite number of renormalization conditions will be needed to fix the finite parts of all these new vertices in the theory. The class of theories that can be renormalized with a finite number of counterterms, called renormalizable theories, is therefore of central interest. We can give a simple power counting rule that tells us when a theory is renormalizable (but not the converse).

The degree function introduced in [33] must be chosen such that $|\Delta(k)| \leq c \cdot \max(\lambda, m^{d_{\text{deg}}(S)})$ for all vertices, propagators (lines), and their derivatives in the Feynman rules associating integrals with graphs; this is the starting point for the inductive proof that $I_{\lambda}(\bar{G}) \leq c \cdot \max(\lambda, m^{d_{\text{deg}}(G)})$ that is used to establish the BPH theorem in our approach. For our working example of $\phi^{4}$ theory in $n$ dimensions the degree of each propagator is $d = -2$ and the vertices have degree $d' = 0$. The degree of a graph with $I$ internal lines, $V$ vertices, and $L$ loops is thus $deg = 2d + d'L + Ln$, recalling that there is an $n$-dimensional momentum integration associated with each loop. Now, any connected graph with $V$ vertices has a spanning tree containing exactly $V - 1$ edges, and the remaining $I - V + 1$ edges each give rise to an independent loop; furthermore each internal line has two ends and, in our theory, each vertex connects three lines, so “conservation of line ends” implies that $3V = 2I + E$ where $E$ is the number of external lines. Eliminating $L$ and $I$ from these equations we find that $deg = n - E_{f}^{2}(d + n) + V_{f}^{2}(2d' + 3d + n) \equiv n - E \dim(\phi) + V \dim(V)$ where the dimension of the field $\dim(\phi) = \frac{1}{2}(n - 2)$ and of the vertex is $\dim(V) = \frac{1}{2}(n - 6)$.

These dimensions can be read directly from the Lagrangian describing the theory by some simple rules. We see that if $\dim(V) > 0$ then we can find arbitrarily overall divergent graphs just by increasing the number of vertices sufficiently, whereas if $\dim(V) < 0$ then only a finite number of graphs can have overall divergences (such theories are called superrenormalizable). The case $\dim(V) = 0$ (renormalizable) is particularly interesting as there can be an infinitude of overall divergent graphs, but as their degree is bounded by $E \dim(\phi)$ (provided this is positive) only a finite number of Green’s functions are overall divergent. In four dimensions $n = 4$ we have $\dim(\phi) = 1$ and $\dim(V) = -1$ for $\phi^{4}$ theory, so it is superrenormalizable, whereas in six dimensions $\dim(\phi) = 2$ and $\dim(V) = 0$ it is renormalizable.

### 4. NUMERICAL VERSUS ANALYTIC EVALUATION OF INTEGRALS

Heretofore people have usually calculated Feynman integrals by using some regulator to make all the manipulations well-defined and then arranged the resulting expressions to cancel the would-be divergences between graphs and countergraphs, leaving answers with a finite limit as the regulator is removed. This procedure is effective if all the integrals can be evaluated in closed form [13] but it is very difficult to completely automate the procedure as the closed-form evaluation of the integrals is a new challenge at each order of the loop expansion.

\footnote{In fact, what is really needed is that the would-be divergent parts can be evaluated in closed form. The usual folklore is that the work required to evaluate the divergent parts of diagrams with $\ell + 1$ loops is about the same as that to evaluate the finite parts with $\ell$ loops.}
We wish to use the $R$ operation to generate manifestly finite Feynman integrals, which we can then evaluate numerically. The advantages of this approach are obvious, but there is a price to pay. First, we need to evaluate the albeit finite integrals for every value of the external momenta and parameters (renormalization conditions) separately. Second, the integrals are over a unit cube in $I - 1$ dimensions (q.v., where $I$ is the number of edges in the graph, and thus we have to resort to Monte Carlo methods in all but the most trivial cases, so we are only able to achieve limited precision. Third, in most cases where perturbation theory is applicable we are expanding in a small parameter (the fine structure constant $\alpha \approx 1/137$ in QED for example), so there is no point in computing the $\ell + 1$ loop contributions unless the errors at $\ell$ loops are sufficiently small. Our window of opportunity is to compute one loop higher order that the current best closed-form solutions.

5. GRAPHICAL ALGORITHMS

We now consider how we can implement the $R$ operation in practice in our symbolic–numeric scheme. It is much easier to implement the more recursive Henge approach than Bogoliubov’s definition or Zimmermann’s explicit forest formula, as the complexity of the problem is then naturally handled by a correspondingly recursive program. To do this we need some reasonably efficient graph manipulation algorithms, as the text-book methods using adjacency matrices and suchlike are slow in practice. Fortunately such algorithms are well-known, such as the Galler–Fisher algorithm or finding equivalence classes given pairwise equivalence relations, which may be equivalently viewed as a means of splitting a graph into its connected components.

The basic idea is to represent a graph as a set of edges each represented as a pair of vertices, and to label the vertices with small integers; for instance the graph \( egin{array}{c} \uparrow \downarrow \end{array} \) could be represented as \( \{12, 23, 34, 41, 24\} \). We create an array \( F \) indexed by vertices and initialised to some special value (such as zero), and then run once through the edges \( (ij) \), computing the ancestor of each of its vertices, where the ancestor \( A(i) \) is \( i \) if \( F[i] = 0 \) or \( A(F[i]) \) otherwise. If the ancestors are unequal \( A(i) \neq A(j) \) then we set \( F[A(i)] \leftarrow A(j) \). When we have exhausted the supply of edges the ancestor of any vertex labels the connected component that the vertex belongs to.

This algorithm is pleasantly easy to extend to carry out other graphical tasks efficiently. To find all the irreducible edges in a connected graph (those that may be cut without disconnecting it) we just keep track of the route from a vertex to its ancestor by introducing an \( \mathbb{Z}_2 \) chain-valued array \( P \) indexed by vertices that contains the chain of edges connecting \( i \) to \( F[i] \), and defining the route \( R(i) \) from \( i \) to \( A(i) \) to be zero if \( i = A(i) \) or the chain \( R(i) = P[i] \cup R(F[i]) \) otherwise. Upon examining the edge \( (ij) \) we note that the chain \( C_{ij} \equiv R(j) \cup (ij) \cup R(i) \) connects \( A(i) \) to \( A(j) \), so if \( A(i) \neq A(j) \) as well as updating \( F[i] \) as above we also set \( P[i] \leftarrow C_{ij} \). If, on the other hand, \( A(i) = A(j) \) then we have discovered that \( C_{ij} \) is a closed loop, and all the edges occurring in this loop are marked as irreducible. All the edges not so marked by the end of the loop over edges are reducible.

One use of this algorithm is to route momenta through a graph. We multiply each loop by a symbol for the corresponding loop momentum, and for each vertex \( i \) at which external momentum \( p \) enters the graph we multiply the route \( R(i) \) by \( p \). The coefficient of any edge in the sum of all these \( p \)-chains is the momentum flowing through that edge.

Indeed, it is but a small step further to devise an efficient Henge-finding algorithm. Start with a 1PI graph \( G \) and remove any edge \( \ell \); as \( G \) is just a set of edges this graph is just the set of edges \( G = \{\ell\} \). Now apply the previous algorithm to this graph, partitioning the lines into two sets \( G = I \cup R \) of irreducible lines and reducible lines, and apply the first algorithm to partition \( I \) into disconnected pieces \( I = \{\emptyset\} \). The required Henge \( \mathcal{H}(G, \ell) \) is just this set, and the single loop \( \mathcal{G}/\mathcal{H}(G, \ell) = R \cup \{\emptyset\} \).

Using these algorithms we have implemented the subtraction terms specified by the \( R \) operation on an arbitrary diagram in \( \phi^3 \) theory, and the generalization to more interesting theories with more complicated actions is both straightforward and in progress. It is perhaps worth noting for the benefit of the dedicated reader who has thought about the equivalence of the Henge and Bogoliubov definitions of \( R \) that too naive an implementation of the former will produce the same spinnery multiple times, corresponding to different orderings of lines in \( \mathcal{G} - \mathcal{S} \).

6. FEYNMAN PARAMETERS

We have described in some detail the means of obtaining a finite Feynman integral, but we still have to evaluate it. There are various ways of doing this: we could, for example, directly evaluate the loop integrals in momentum space, leading to an \( nL \)-fold real integral. For some regulators, such as the lattice, where we discretize the functional integral by approximating space-time by a hypercubic grid, this is essentially the only way to proceed. In practice, however, we usual carry out perturbative QFT calculations using dimensional regularization in which case all of the propagators are essentially inverse quadratic forms in the momenta, as we used above. In this case there is a convenient \( \Gamma \) function identity that allows us to write a product of such propagators as a power of a single quadratic form

$$\frac{\Gamma(\alpha)\Gamma(\beta)}{A^\alpha B^\beta} = \Gamma(\alpha + \beta) \int_0^\infty dt \ t^{\beta-1} (A + Bt)^{\alpha + \beta}.$$  

This can be iterated to establish the formula

$$\prod_{i=1}^N \frac{1}{Q_i} = (N-1)! \int_0^1 dx_1 \cdots \int_0^1 dx_N \ \delta \left( 1 - \sum_{k=1}^N x_k \right) \left[ \sum_{k=1}^N x_k Q_k \right]^N.$$  

The quantities \( x_k \) being known as Feynman parameters.

Introducing Feynman parameters and interchanging the order of the momentum and parameter integrals (which is valid in the presence of a regulator) we can combine all \( nL \) loop momenta into a single vector and carry out the momentum integration using another \( \Gamma \) function identity,

$$\int \frac{d^\omega k}{[k^2 + F(p, m)]^\alpha} = \pi^\frac{\alpha - \omega}{\Gamma(\alpha)} F(p, m)^{\omega - \alpha}.$$  

\(^{\#} A \ p \)-chain consists of sums of edges with coefficients being ± symbolic names for momenta.
and related formulae obtained from this by differenting with respect to the external momentum $p$. We are then left with $I$ parameter integrals to evaluate, in our case numerically.

6.1 Sectors

It is interesting to see how the Henge decomposition is reflected in Feynman parameter space. To this end we define a sector of Feynman parameter space as a region where the parameters are totally ordered, for example one such sector is $x_1 > x_2 > \cdots > x_N$. If we restrict the integral in equation (1) to this sector we find

$$\int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{N-1}} dx_N \frac{\delta \left( 1 - \sum_{k=1}^N x_k \right)}{\left( \sum_{j=1}^N x_j Q_j \right)^N} = \frac{1}{(N-1)!} \prod_{i=1}^N \frac{1}{\sum_{j=1}^N Q_j},$$

so for example the contribution to the product $T^3$ represented using Feynman parameters $t$, $u$, and $v$ from the sector $t > u > v$ is $\frac{1}{(t+u)(t+u+v)}$. This means that the quadratic form denominator $T$ corresponding to parameter $t$ is guaranteed to be smaller than all the other denominators, which means that it corresponds to a line carrying the smallest momentum in the Henge decomposition.

We note in passing that especially in the context of overlapping IR divergences the iterated application of sector decomposition has proved to be a powerful way to deal numerically not only with multi-loop Feynman diagrams but also high-dimensional phase-space integrals.

7. TENSOR AND $\gamma$-MATRIX MANIPULATION

In this pedagogical presentation we have illustrated our calculations using only scalar fields; in general we need to introduce vector fields for gauge bosons and spinor fields for fermions, these lead to the additional complication of having tensor expression in the Feynman integrand numerators. Actually our symbolic manipulation code already has to deal with such numerators, as they occur when we Taylor subtract even for scalar theories. What we end up with is some complicated SO(4) tensor expressions which need to be simplified into some canonical form; these expressions can themselves be viewed as graphs, where the tensors are the vertices and the edges are pairs of contracted indices, and we use the algorithms of section 8 to carry out this simplification.

For spinors, which strictly speaking are representations of the universal covering group Spin(n) of SO(n), the numerators also become functions of spinor operators built out of the Dirac (Clifford) algebra generators $\gamma_{\mu}$ that satisfy the anticommutation relations $\{\gamma_{\mu}, \gamma_{\nu}\} = 2g_{\mu\nu}$ where $g$ is the metric tensor. Efficient algorithms are known for simplifying $\gamma$-matrix expressions and have been implemented in the REDUCE package CVIT; these algorithms are based on reducing the “spin-network” or “bird-track” diagrams in terms of sums of irreducible representations of Spin(n) and their 3j and 6j coefficients. We are also currently investigating the generalization of these methods to handle arbitrary representations of SO(n), which might lead to much more efficient algorithms for symbolic tensor manipulation than exist at present.

8. INFRARED CANCELLATIONS

The dedicated reader might have observed that we do not live in Euclidean space, but in Minkowski space. Fortunately this has absolutely no effect on the algebraic manipulations we have described: we end up with a renormalized integrand depending on a set of SO(4, C) invariants constructed out of the external momenta, such as $p^2$, $p \cdot q$, $\epsilon_{\mu\nu\rho\sigma} q^{\mu} r^{\nu} s^{\rho} t^{\sigma}$ and the difference between the real forms $SO(4, R)$ and $SO(3, 1, R)$ is just that $p^2$ can be negative, for example.

While the cancellation of UV divergences works just as well in Minkowski space as it does in Euclidean space, new IR divergences can occur. Previously we evaded all IR divergences by keeping the mass $m$ of our particles non-zero, in the real world we need to handle the fact that some real particles are massless (such as the photon) and also that there are a variety of other IR singularities possible because $k^2 = 0 \neq k = 0$ in Minkowski space.

There are two different situations to consider:

- There are true IR divergences that make the theory meaningless. For example, massless scalar field theory does not exist because it is IR sick in this manner.
- There are IR singularities that occur at special values of the external momenta. The emission of a zero-energy photon in QED exemplifies this situation.

We shall reject theories in the first case, whereas in the second we evade the problem by insisting that on physical grounds one must average the cross-section for emission of soft photons (Brehmsstrahlung) over some experimental resolution for the detection of the photon energy.

The crucial observation is that while the Green’s functions may have IR divergences the cross-sections smeared over experimental resolutions are always finite. At the diagrammatic level what happens is that there are cancellations between loop integrals (or Feynman parameter integrals) and phase space integrals, so our intention is to extend our program to treat loop and phases space integrals consistently. What we must do is to ensure that all divergences cancel locally in the integrand, just as we do for Green’s functions by our use of the $R$ operation. To this end we intend to generate the integrands for cross sections by representing them as cut bubble diagrams, where the cut propagators correspond to on-shell particles convoluted with the experimental resolution functions.

9. COMPUTATIONAL ISSUES

Our prototype program is written in Maple and generates C code: an example of its use is illustrated in Figures 1 and 2. This is unusual in that most symbolic computations of Feynman diagrams make use of domain-specific languages such as FORM, which are relatively unsophisticated.

\footnote{At least “if we keep our feet on the ground and ignore gravity” (E. Mottola).}

\footnote{In fact probably only the photon, as neutrinos seem to have masses, albeit very small ones. Gluons, the gauge bosons for QCD, are also massless; but as QCD is a strongly interacting confined theory they do not exist as physical “on-shell” particles.

In mathematical terms we say that the Green’s functions in Minkowski space are (tempered) distributions rather than functions.
but can execute their limited repertoire of operations on very large expressions with great efficiency.

9.1 Code Generation

We could try to evaluate the final numerical integrals within Maple, but other than for debugging purposes this is too inefficient even using the NAG integration routines packaged within Maple. We therefore need to generate efficient numerical code to evaluate the integrand for use within our special-purpose Monte Carlo integration programs. It is not too important which language is used, our example is in C but it would be trivial to generate Fortran if that was preferred.

9.2 Memory Management and Laziness

The computational model that FORM uses is to stream a sum of many terms from disk to disk, either applying pattern-based transformations or sorting and collecting terms as the data passes through the processor. This model is justified by the huge size of the expressions that are generated in realistic Feynman diagram computations, unlike the toy example of Figure 1. While our Maple program works well on moderate size problems it dies a horrible death when the entire problem no longer fits in memory.

We suggest that the huge intermediate expressions generated are not really an intrinsic property of the problem, but are an artefact of the way we think about and implement the programs. To be specific, our program first applies the R operation generating a sum of Taylor subtraction terms as it recurses through the Henge decomposition; it then runs through all these terms converting them into Feynman parameter integrals; it then collects all the terms with similar quadratic forms in their denominator before finally generating C code. This multi-pass approach is not necessary, it is just convenient for thinking about and debugging the program, as it separates the computation into logically independent steps.

We could avoid generating huge intermediate expressions by generating the terms lazily, and this could be done declaratively without altering the logic of the program if Maple implemented a lazy map primitive, or a yield statement for use within loops. We would then generate each term and immediately apply subsequent operations on it until we reach the “collection” phase, in which terms could be collected into elements of a hashed array.

Eventually this approach will still run out of memory if this array has to live in (virtual) memory, but this could be circumvented by writing the array elements to disk when necessary.

10. CONCLUSIONS

Current perturbative calculations in QFT are unthinkable without the use of computer algebra. Whether this requires domain-specific languages such as FORM or more careful memory choreography in general purpose systems such as Maple remains to be seen. All of the pieces of the calculation, including renormalization, can be fully automated except for evaluation of the Feynman parameter integrals where the lack of closed-form solutions forces us into the arms of numerical integration.

11. REFERENCES

[1] C. Anastasiou, S. Beerli, and A. Daleo. Evaluating multi-loop Feynman diagrams with infrared and threshold singularities numerically. arXiv:hep-ph/0703282, 2007.
[2] S. A. Anikin and O. I. Zav’yalov. Counterterms in the formalism of normal products. Teor. Mat. Fiz., 26:162, 1976. [Theor. Math. Phys., 26, 105 (1976)].
[3] S. A. Anikin, O. I. Zav’yalov, and M. K. Polivanov. A simple proof of the Bogolyubov–Parasyuk theorem. Teor. Mat. Fiz., 17:189, 1973. [Theor. Math. Phys. 17, 1082 (1973)].
```c
#include <math.h>

float term2(float x4, float x5, float x6, float x7, float x8) {
    float SMadjr1c1;
    float SMadjr2c2;
    float SmuSq;
    float SMadjr1c1 = x4*x5 + x6 + x7 + x8;
    float SMadjr2c2 = -SMadjr1c1;
    float t11 = m*m;
    float t13 = x4*x4;
    SmuSq = p3p3*x4 + t11*SMadjr1c1 - (2.0*SMadjr2c2*t13 - (-SMadjr2c2*x4=x4)*p3p3);
    float t20 = 0.3141593E1*0.3141593E1;
    float t21 = t20*t20;
    float t12 = x4+x5+x6+x7+x8;
    float t24 = logf(SmuSq);
    float t25 = sqrtf(x4*x6 + x7*x4 + x8*x4 + x5*x6 + x7*x5 + x8*x5);
    return(-t21*t20*SmuSq*t24/t25);    
}
```

Figure 2: Part of the C code generated by the example of Figure 1.

[4] J. Bedford, A. Brandhuber, B. J. Spence, and G. Travaglini. Non-supersymmetric loop amplitudes and MHV vertices. Nucl. Phys., B712:59–85, 2005. arXiv:hep-th/0412108
[5] C. F. Berger, Z. Bern, L. J. Dixon, D. Forde, and D. A. Kosower. Bootstrapping one-loop QCD amplitudes with general helicities. Phys. Rev., D74:036009, 2006. arXiv:hep-ph/0604195
[6] Z. Bern, L. J. Dixon, and D. A. Kosower. On-shell methods in perturbative QCD. arXiv:hep-ph/0704.2798, 2007.
[7] T. Binoth and G. Heinrich. An automatized algorithm to compute infrared divergent multi-loop integrals. Nucl. Phys., B585:741–759, 2000. arXiv:hep-ph/0004013
[8] T. Binoth and G. Heinrich. Numerical evaluation of multi-loop integrals by sector decomposition. Nucl. Phys., B680:375–388, 2004. arXiv:hep-ph/0305234
[9] T. Binoth and G. Heinrich. Numerical evaluation of phase space integrals by sector decomposition. Nucl. Phys., B693:134–148, 2004. arXiv:hep-ph/0402263
[10] T. Binoth, G. Heinrich, T. Gehrmann, and P. Mastrolia. Six-photon amplitudes. arXiv:hep-ph/0703311 2007.

[11] N. N. Bogoliubov and O. S. Parasiuk. Über die multiplikation der kausalfunktionen in der quantentheorie der felder. Acta Math., 97:227, 1957.
[12] C. G. Bollini and J. J. Giambiagi. Dimensional renormalization: The number of dimensions as a regularizing parameter. Nuovo Cim., B12:20–25, 1972.
[13] R. Britto, B. Feng, and P. Mastrolia. The cut-constructible part of QCD amplitudes. Phys. Rev., D73:105004, 2006. arXiv:hep-ph/0602178
[14] W. E. Caswell and A. D. Kennedy. A simple approach to renormalization theory. Phys. Rev., D25:392, 1982.
[15] W. E. Caswell and A. D. Kennedy. The asymptotic behaviour of Feynman integrals. Phys. Rev., D26:3073, 1983.
[16] K. G. Chetyrkin. Four-loop renormalization of QCD: Full set of renormalization constants and anomalous dimensions. Nucl. Phys., B710:499–510, 2005. arXiv:hep-ph/0405193
[17] K. G. Chetyrkin and F. V. Tkachov. Integration by parts: The algorithm to calculate $\beta$ functions in 4 loops. Nucl. Phys., B192:159–204, 1981.
[18] G. M. Cicuta and E. Montaldi. Narrow-resonance saturation of current algebra sum rules and evidence for the singlet nature of A. Lett. Nuovo Cimento, 4:329, 1972.
[19] J. Collins. Renormalization. Cambridge University Press, 1984.
[20] P. Cvitanović and A. D. Kennedy. Spinors in negative dimensions. Physica Scripta, 26:5–14, 1982.
[21] M. Czakon. The four-loop QCD $\beta$-function and anomalous dimensions. Nucl. Phys., B710:485–498, 2005. arXiv:hep-ph/0411261.
[22] M. Czakon. Automatized analytic continuation of Mellin–Barnes integrals. Comput. Phys. Commun., 175:559–571, 2006. arXiv:hep-ph/0511200.
[23] A. Denner, S. Dittmaier, M. Roth, and L. H. Wieders. Complete electroweak O($\alpha$) corrections to charged-current $e^+e^-\rightarrow 4$ fermion processes. Phys. Lett., B612:223–232, 2005.
[24] R. K. Ellis, W. T. Giele, and G. Zanderighi. The cut-constructible part of QCD amplitudes. Nucl. Phys., B693:59–85, 2005. arXiv:hep-ph/0406053.
[25] G. Heinrich and V. A. Smirnov. Analytical evaluation of dimensionally regularized massive on-shell double boxes. Phys. Lett., B598:55–66, 2004. arXiv:hep-ph/0406053.
[26] K. Hepp. Proof of the Bogolyubov–Parasiuk theorem on renormalization. Commun. Math. Phys., 2:301–303, May 1964.
[27] T. Gehrmann and E. Remiddi. Differential equations for two-loop four-point functions. Nucl. Phys., B580:485, 2000. arXiv:hep-ph/9912329.
[28] G. Heinrich and V. A. Smirnov. Analytical evaluation of dimensionally regularized massive on-shell double boxes. Phys. Lett., B598:55–66, 2004. arXiv:hep-ph/0406053.
[29] K. Hepp. Proof of the Bogolyubov–Parasiuk theorem on renormalization. Commun. Math. Phys., 2:301–303, 1966.
[30] V. A. Ilyin, A. P. Kryukov, A. Y. Rodionov, and A. Y. Taranov. Fast algorithm for calculation of Dirac $\gamma$-matrix traces. SIGSAM Bull., 23(4):15–24, 1989.
[31] A. D. Kennedy. Clifford algebras in 2ω dimensions. J. Math. Phys., 22:1330–1337, 1981.
[31] A. D. Kennedy. Spinography: Diagrammatic methods for spinors in Feynman diagrams. *Phys. Rev.*, D26:1936, 1982.

[32] A. D. Kennedy. A simple proof of the BPH theorem. In *High energy physics and quantum field theory*, pages 331–338, 1996. Proceedings of 11th International Workshop on High-Energy Physics and Quantum Field Theory (QFTHEP 96), St. Petersburg.

[33] T. Kinoshita and M. Nio. The tenth-order QED contribution to the lepton $g - 2$: Evaluation of dominant $\alpha^5$ terms of muon $g - 2$. *Phys. Rev.*, D73:053007, 2006. arXiv:hep-ph/0512330.

[34] D. E. Knuth. *The Art of Computer Programming*, volume 1. Addison-Wesley, 1997.

[35] S. Laporta and E. Remiddi. The analytical value of the electron $g - 2$ at order $\alpha^3$ in QED. *Phys. Lett.*, B379:283–291, 1996. arXiv:hep-ph/9602417.

[36] S. Moch, J. A. M. Vermaseren, and A. Vogt. The three-loop splitting functions in QCD: The non-singlet case. *Nucl. Phys.*, B688:101–134, 2004. arXiv:hep-ph/0403192.

[37] Z. Nagy and D. E. Soper. Numerical integration of one-loop Feynman diagrams for $n$-photon amplitudes. *Phys. Rev.*, D74:093006, 2006. arXiv:hep-ph/0610028.

[38] G. Ossola, C. G. Papadopoulos, and R. Pittau. Numerical evaluation of six-photon amplitudes. arXiv:hep-ph/0704.1271 [hep-ph], 2007.

[39] O. S. Parasiuk. К теории $R$-операции Богословска. *Ukr. Math. Z.*, 12:287, 1960.

[40] G. Passarino. An approach toward the numerical evaluation of multiloop Feynman diagrams. *Nucl. Phys.*, B619:257–312, 2001. arXiv:hep-ph/0108252.

[41] V. A. Smirnov. Analytical evaluation of double boxes. arXiv:hep-ph/0209177, 2002.

[42] H. Strubbe. Manual for SCHOONSCHIP: a CDC 6000/7000 program for symbolic evaluation of algebraic expressions. *Comp. Phys. Commun.*, 8:1–30, 1974.

[43] G. ’t Hooft. Dimensional regularization and the renormalization group. *Nucl. Phys.*, B61:455–468, 1973.

[44] G. ’t Hooft and M. J. G. Veltman. Regularization and renormalization of gauge fields. *Nucl. Phys.*, B44:189–213, 1972.

[45] J. B. Tausk. Non-planar massless two-loop Feynman diagrams with four on-shell legs. *Phys. Lett.*, B469:225–234, 1999. arXiv:hep-ph/990506.

[46] T. van Ritenberg, J. A. M. Vermaseren, and S. A. Larin. The four-loop $\beta$ function in quantum chromodynamics. *Phys. Lett.*, B400:379–384, 1997. arXiv:hep-ph/9701390.

[47] J. A. Vermaseren. Harmonic sums, Mellin transforms and integrals. *Int. J. Mod. Phys.*, A14(13):2037, 1999.

[48] J. A. Vermaseren. New features of FORM. http://arXiv:math-ph/0010025, 2000.

[49] A. Vogt, S. Moch, and J. A. M. Vermaseren. The three-loop splitting functions in QCD: The singlet case. *Nucl. Phys.*, B691:129–181, 2004. arXiv:hep-ph/0404111.

[50] Z. Xiao, G. Yang, and C.-J. Zhu. The rational part of QCD amplitude. III: The six-gluon. *Nucl. Phys.*, B758:53–89, 2006. arXiv:hep-ph/0607017.

[51] W. Zimmermann. Local operator products and renormalization in quantum field theory. In S. Deser, M. Grisaru, and H. Pendleton, editors, *Lectures on Elementary Particles and Quantum Field Theory*, volume 1, Cambridge, MA, 1971. MIT Press.