We describe the non-perturbative trans-series, at both weak- and strong-coupling, of the large $N$ approximation to the beta function of the Gross-Witten-Wadia unitary matrix model. This system models a running coupling, and the structure of the trans-series changes as one crosses the large $N$ phase transition. The perturbative beta function acquires a non-perturbative trans-series completion at large but finite $N$ in the ’t Hooft limit, as does the running coupling.

I. INTRODUCTION

One of the big puzzles concerning resurgent asymptotics in QFT [1] is how it applies to the situation where the coupling is not fixed, but runs with the scale. In this short note, we explore this phenomenon in a simple solvable model, the Gross-Witten-Wadia (GWW) unitary matrix model [2, 3], which mimics a running coupling through the dependence on the lattice plaquette scale. The form of the resurgent structure changes as one crosses the large $N$ phase transition. The GWW unitary matrix model is a one-plaquette model of $2d$ Yang-Mills theory, and is defined by the partition function [2, 3]:

$$Z(t, N) = \int \frac{DU}{U(N)} \exp \left[ \frac{N}{2t} \text{tr} (U + U^†) \right]$$  \hspace{1cm} (1)

Here $t \equiv N g^2 / 2$ is the ’t Hooft coupling. The GWW model has a third-order phase transition at infinite $N$, as the specific heat develops a cusp at $t = 1$. This large $N$ third order phase transition occurs in many related examples in physics and mathematics [4–14].

For any $N$, the partition function in (1) can be compactly expressed as a Toeplitz determinant [5]:

$$Z(t, N) = \det \left[ I_{j-k} \left( \frac{N}{t} \right) \right]_{j,k=1,...,N}$$  \hspace{1cm} (2)

where $I_j$ is the modified Bessel function. While this formula is explicit, the determinant structure makes it of limited use for studying the large $N$ limit. Many alternative techniques have been developed to analyze the large $N$ limit [4–10], including the double-scaling limit described by the universal Tracy-Widom form [11]. Resurgent asymptotics for the large $N$ limit in matrix models was introduced in [15], using the pre-string difference equation. To study the analytic continuation of the large $N$ trans-series structure, where $N$ becomes complex, one can alternatively map the GWW model to a Painlevé III equation (in terms of the ’t Hooft coupling $t$), in which $N$ appears as a parameter [16].

The familiar double-scaling limit of the GWW model arises as the well-known coalescence limit reducing Painlevé III to Painlevé II [17]. In this paper, we extend this Painlevé-based approach to the analysis of the beta function of the GWW model, explaining the form of the large $N$ trans-series, at both weak and strong coupling.

A. Running Coupling and Beta Function

The running coupling is defined [2] by reintroducing a length scale (the lattice spacing $a$) into the Wilson loop via the definition

$$W(t, N) \equiv \exp \left[ -a^2 \Sigma \right]$$  \hspace{1cm} (3)

Keeping the string tension $\Sigma$ fixed therefore defines $t \equiv t(a, N)$ as a function of the scale $a$. This running coupling $t(a, N)$ can be obtained by inversion of the expression

$$a^2 \equiv - \frac{1}{\Sigma} \ln W(t, N)$$  \hspace{1cm} (4)

The beta function is then defined [2]:

$$\beta(t, N) = - \frac{\partial t(a, N)}{\partial \ln a}$$  \hspace{1cm} (5)

1 Note that for any finite $N$, the relation between the ’t Hooft coupling $t$ and the lattice scale $a$ is monotonic.
From now on, we set the string tension $\Sigma = 1$, absorbing it into the units of $a$.

At infinite $N$, the Wilson loop at strong and weak coupling is \[ W(t, N) \xrightarrow{N \to \infty} \begin{cases} \frac{1}{2t} & \text{strong coupling, } t \geq 1 \\ 1 - \frac{t}{2} & \text{weak coupling, } t \leq 1 \end{cases} \]

Therefore, at infinite $N$ the running coupling $t(a)$ is:

\[ t(a, N) \xrightarrow{N \to \infty} \begin{cases} \frac{1}{2} \exp \left[ a^2 \right] & \text{strong coupling, } t \geq 1 \\ \frac{1}{2} \left( 1 - \exp \left[ -a^2 \right] \right) & \text{weak coupling, } t \leq 1 \end{cases} \]

and the beta function is:

\[ \beta(t, N) \xrightarrow{N \to \infty} \begin{cases} -\frac{2t \ln(2t)}{4 \left( 1 - \frac{t}{2} \right) \ln \left( 1 - \frac{t}{2} \right)} & \text{strong coupling, } t \geq 1 \\ -\frac{2t}{4 \left( 1 - \frac{t}{2} \right) \ln \left( 1 - \frac{t}{2} \right)} & \text{weak coupling, } t \leq 1 \end{cases} \]

Gross and Witten observed that if one only had the infinite $N$ expressions at either weak or strong coupling, one might erroneously deduce the existence of spurious zeros of the beta function. See Figures 1 and 2. Similarly for the running coupling, one might deduce the incorrect behavior at small or large $a$, starting from the other limit at $N = \infty$. See Figure 3. The resolution of course is that infinite $N$ should be approached from finite $N$, with suitable large $N$ corrections included. In the next Sections we show that these finite $N$ corrections yield non-perturbative trans-series expressions both for the beta function and for the running coupling, and when these are included, the weak coupling expressions match consistently to the strong-coupling expressions. The kink in the beta function, indicating the third order phase transition, develops at $N = \infty$. See Figures 1 and 2.

![Beta Function Graph](image.png)

**FIG. 1.** Plot of the GWW beta function $\beta(t, N)$ in (9). The red solid curve shows the exact beta function for $N = 20$. The dashed and dotted lines show the strong-coupling and weak-coupling approximations, respectively, at infinite $N$, from (8). The infinite $N$ approximations show spurious zeros at $t = 1/2$ and $t = 2$, but in fact the true beta function has a single zero at $t = 0$. As $N \to \infty$, the jump at $t = 1$ shown in the red curve becomes a cusp, indicating the $N = \infty$ third-order phase transition [2, 3], and the beta function curve jumps from the infinite $N$ strong-coupling form to the infinite $N$ weak-coupling form as $t$ decreases through the phase transition. See Fig. 2 for a close-up of the cusp at $t = 1$. The finite $N$ corrections, which produce this jump, are described in Sec. II in the form of a large $N$ trans-series.
FIG. 2. Plot of the ratio of the beta function $\beta(t, N)$ to the $N = 1$ beta function, which shows the development, as $N$ increases, of the kink at the $N = \infty$ phase transition point $t = 1$.

FIG. 3. The running coupling $t(a, N)$ [red solid curve] as a function of the lattice scale $a$, for $N = 20$. The dashed and dotted curves show the strong coupling and weak coupling behavior, respectively, at infinite $N$. The true behavior of the running coupling jumps from one asymptotic curve to the other, near the $N = \infty$ phase transition point: $t = 1$. The finite $N$ corrections, in the form of a large $N$ trans-series, are described in Section III.
II. LARGE $N$ TRANS-SERIES FOR THE BETA FUNCTION

From the definition \[ \beta(t, N) = -2 \frac{\partial_t \ln W(t, N)}{W(t, N)} = -2 \frac{\partial_t (\ln \ln W(t, N))}{\partial_t W(t, N)} \] (9)

This implies that the beta function $\beta(t, N)$ inherits its non-perturbative trans-series structure directly from the trans-series structure of the Wilson loop $W(t, N)$. The large $N$ trans-series for $W(t, N)$ was studied in [16], showing how the form of the trans-series changes across the phase transition at $t = 1$. Related changes therefore occur for the beta function. For other discussions of non-perturbative effects for the GWW Wilson loop, see [18, 19].

We briefly review some relevant results from [16]. The non-perturbative trans-series form of $W(t, N)$ at any $N$ is efficiently expressed in terms of a solution to a Painlevé III equation. Define $\Delta(t, N)$ as the expectation value of the determinant in the Gross-Witten-Wadia model:

\[ \Delta(t, N) \equiv \langle \det U \rangle \] (10)

Then $W(t, N)$ is related to $\Delta(t, N)$, for any $N$, as:

\[ W(t, N) = \frac{1}{2t} \left[ 1 - \Delta^2 - \frac{t^2}{1 - \Delta^2} \left( \frac{t^2 (\partial_t \Delta)^2}{N^2} - 1 \right) \right] - \frac{t}{2} \] (11)

The expectation value $\Delta(t, N)$ satisfies the following nonlinear ordinary differential equation, as a function of the ’t Hooft coupling $t$, for any value of $N$ [5, 16, 20]:

\[ t^2 \partial_t^2 \Delta + \frac{N^2 \Delta}{t^2} (1 - \Delta^2) = \frac{\Delta}{1 - \Delta^2} \left( N^2 - t^2 (\partial_t \Delta)^2 \right) \] (12)

Notice that $N$ appears as a parameter in this equation, thereby enabling a simple analysis of the large $N$ limit, including analytic continuation in $N$. The equation [12] is directly related to the Painlevé III equation, and standard resurgent asymptotic techniques [21] permit the development of explicit trans-series expansions in various limits: for example, weak or strong ’t Hooft coupling [19].

Combining [5] and [11], the GWW beta function can also be expressed in terms of $\Delta(t, N)$:

\[ \beta(t, N) = -2t \left( 1 - \frac{2 (\Delta^2 - 1)^2}{\Delta^4 - (t^2 + 2) \Delta^2 + 1 + t^4 (\partial_t \Delta)^2 / N^2} \right) \times \ln \left[ \frac{1}{2t} \left( 1 - \Delta^2 - \frac{t^2}{1 - \Delta^2} \left( \frac{t^2 (\partial_t \Delta)^2}{N^2} - 1 \right) \right) - \frac{t}{2} \right] \] (13)

For example, from [12] we see that at infinite $N$

\[ \Delta(t, N) \overset{N \to \infty}{\rightarrow} \begin{cases} 0 & \text{strong coupling, } t \geq 1 \\ \sqrt{1 - t} & \text{weak coupling, } t \leq 1 \end{cases} \] (14)

from which follows the infinite $N$ beta function in [8].

The correspondence [13] means that we can use the trans-series structure of $\Delta(t, N)$ to study the trans-series structure of $\beta(t, N)$. And since the trans-series expansions of $\Delta(t, N)$ were shown in [16] to display concrete resurgence relations between different non-perturbative sectors in the trans-series, it follows that the same is true for the beta function $\beta(t, N)$.

We can also use the relation [13] to plot the beta function as a function of coupling, for various values of $N$: see Figures [1] and [2]. These figures illustrate the fact that for any given $N$, the weak coupling dependence merges consistently with the strong coupling dependence, with a cusp developing at the critical ’t Hooft coupling only at $N = \infty$. In particular, it is clear that the zeros of the infinite $N$ beta function at $t = 1/2$ and $t = 2$ (see Fig. [1]) are indeed spurious.

It is instructive to study the leading trans-series corrections to the infinite $N$ beta functions in [8]. The form of the trans-series changes across the phase transition, so we illustrate this change of structure by considering the leading contributions at large but finite $N$. Express the Wilson loop for any finite $N$ as

\[ W = W_{\text{pert}} + W_{\text{non-pert}} \] (15)
Keeping the leading power of the non-perturbative term, we obtain the following expression for the beta function:

\[
\beta(t, N) = -\frac{2 \ln W_{\text{pert}}(t, N)}{\partial_t \ln W_{\text{pert}}(t, N)} + \frac{2}{\partial_t W_{\text{pert}}} \left( \frac{\ln W_{\text{pert}}}{\partial_t \ln W_{\text{pert}}} \partial_t W_{\text{non-pert}} - \ln (e W_{\text{pert}}) W_{\text{non-pert}} \right) + \ldots
\]

\[
\equiv \beta_{\text{pert}}(t, N) + \beta_{\text{non-pert}}(t, N)
\]  

(16)

where the dots refer to higher powers of \( W_{\text{non-pert}} \).

**A. Large N expansion at strong ‘t Hooft coupling**

In the strong coupling limit, \( \Delta_{\text{pert}} \) is identically zero, so \( \Delta(t, N) \) is purely non-perturbative \([16]\). Consequently, from \([11]\) we deduce that the Wilson loop \( W(t, N) \) has only one perturbative term, \( W_{\text{pert}} = \frac{1}{2t} \), which is independent of \( N \), and equal to the familiar infinite \( N \) Wilson loop in \([8]\). At finite \( N \), the further corrections are all non-perturbative. Keeping the leading such non-perturbative correction \([16, 18, 19]\),

\[
W_{\text{strong}}(t, N) = \frac{1}{2t} - \frac{t e^{-2NS_{\text{strong}}(t)}}{4\pi N^2 (t^2 - 1)} \left( 1 - \frac{t (3 + 14t^2)}{12N(t^2 - 1)^{3/2}} + \frac{t^2 (81 + 804t^2 + 340t^4)}{288N^2(t^2 - 1)^3} + \ldots \right) + \ldots
\]

(17)

where the large \( N \) instanton action at strong coupling is

\[
S_{\text{strong}}(t) = \text{arccosh}(t) - \sqrt{1 - 1/t^2}
\]

(18)

This translates into a non-perturbative large \( N \) instanton correction to the infinite \( N \) beta function in \([8]\):

\[
\beta_{\text{strong}}(t, N) = -2t \ln(2t)
\]

\[
-\frac{1}{N\pi} \frac{2t^2 \ln(2t)}{\sqrt{t^2 - 1}} e^{-2NS_{\text{strong}}(t)} \left( 1 + \frac{t (6t^2 - 6 - (14t^2 - 9) \ln(2t))}{12N(t^2 - 1)^{3/2} \ln(2t)} + \ldots \right) + \ldots
\]

(19)

Note the appearance of further terms involving \( \ln(t) \) in the fluctuations about the leading large \( N \) instanton term, consistent with general trans-series structure \([21][23]\).

At any finite \( N \), the expression \([19]\) has an unphysical divergence at \( t = 1 \), arising from use of the Debye expansion for the Bessel functions \([24]\). In \([16]\), the leading large \( N \) correction for the Wilson loop at strong coupling was calculated more precisely to be:

\[
W_{\text{strong}}(t, N) \approx \frac{1}{2t} - \frac{1}{2t} \left[ (J_N(N/t))^2 - J_{N-1}(N/t) J_{N+1}(N/t) \right] + \ldots
\]

(20)

This leading correction, in terms of Bessel J functions, is exponentially small at large \( N \), and represents a resummation of all fluctuations about the leading large \( N \) instanton exponential factor in \([17]\). At finite \( N \), expression \([20]\) is therefore much more accurate than the conventional large \( N \) expression \([17]\) in the vicinity of the large \( N \) phase transition, at \( t = 1 \), where instantons and their fluctuations condense \([16][23]\).

A uniform large \( N \) instanton expression is obtained by using the uniform large \( N \) approximation \([26]\) for the Bessel functions appearing in \([20]\). This is a nonlinear analogue of the uniform WKB approximation, smooth through the transition point for any finite \( N \), and expressed in terms of an Airy function rather than an exponential \([16][26]\). Physically, this uniform large \( N \) approximation arises from the merging of two saddles at the large \( N \) phase transition. A similar expression, along with a corresponding uniform approximation, can be deduced for the beta function at large \( N \), in the strong coupling regime:

\[
\beta_{\text{strong}}(t, N) \approx -2t \ln(2t) - 2t \left( (J_N(N/t))^2 + (2 \ln(2t) - 1)J_{N-1}(N/t) J_{N+1}(N/t) \right) + \ldots
\]

(21)

**B. Large N expansion at weak ‘t Hooft coupling**

In the weak coupling regime, the infinite \( N \) expression in \([14]\), \( \Delta \sim \sqrt{1 - t} \), receives both perturbative and non-perturbative corrections at finite \( N \):

\[
\Delta(t, N) = \Delta_{\text{pert}}(t, N) + \Delta_{\text{nonpert}}(t, N)
\]

(22)
This structure flows through to the Wilson loop and to the beta function.

\[ W_{\text{weak}}(t, N) = \left( 1 - \frac{t}{2} - \frac{t^2}{8N^2(1-t)} + \ldots \right) \]

\[ - \frac{i}{2\sqrt{2\pi N^{3/2}}} \frac{t}{(1-t)^{1/4}} e^{-NS_{\text{weak}}(t)} \left( 1 + \frac{8 + 12t + 9t^2}{96N(1-t)^{3/2}} + \ldots \right) \]

(23)

where the large \( N \) instanton action at weak coupling is

\[ S_{\text{weak}}(t) = \frac{2\sqrt{1-t}}{t} - 2 \arctanh \left( \sqrt{1-t} \right) \]

(24)

The corresponding large \( N \) trans-series expansion for the beta function has the form

\[ \beta_{\text{weak}}(t, N) = 2(2-t) \log \left( 1 - \frac{t}{2} \right) \left[ 1 - \frac{t}{2} \left( t - t^2 + \frac{4 - 3t}{4N^2(2-t)(1-t)^2} \log \left( 1 - \frac{t}{2} \right) + \ldots \right) \right] \]

\[ - i \sqrt{\frac{2}{\pi N}} \frac{(1-t)^{1/4}}{t} e^{-NS_{\text{weak}}(t)} (1 + \ldots) + \ldots \]

(25)

C. Large N Double-scaling Limit

It is well known that the double-scaling limit is described by the Painlevé II equation [2][3][15]. In our approach this can be seen as follows. In the double-scaling limit, zoomed in to the immediate vicinity of the GWW phase transition at \( t = 1 \), the Rossi equation (12) reduces to a Painlevé II equation in terms of the scaled variable \( \kappa \) which measures the scaled deviation from \( t = 1 \):

\[ t = 1 + \frac{\kappa}{N^{2/3}}, \quad \Delta(t, N) = \frac{t^{1/3}}{N^{2/3}} V(\kappa) \]

(26)

Here \( V(\kappa) \) is the real Hastings-McLeod solution of the Painlevé II equation [15][16]. In this double-scaling limit, the Wilson loop behaves as

\[ W_{\text{double-scaling}}(\kappa) \approx \frac{1}{2} - \frac{\kappa}{2N^{2/3}} + \frac{(\kappa + V^2(\kappa))^2 - (V'(\kappa))^2}{2N^{4/3}} + O \left( \frac{1}{N^2} \right) \]

(27)

and the beta function as

\[ \beta_{\text{double-scaling}}(\kappa) \approx -2 \ln(2) - \frac{4 \ln 2}{N^{2/3}} \left( \kappa \ln(2e) + V^2(\kappa) + 2\kappa V(\kappa)V'(\kappa) + 2V^3(\kappa)V'(\kappa) - V'(\kappa)V''(\kappa) \right) \]

\[ + O \left( \frac{1}{N^{4/3}} \right) \]

(28)

This matches smoothly to the strong- and weak-coupling sides of the phase transition, as shown for the double-scaling limit of \( \Delta(t, N) \) in [16].

III. LARGE N TRANS-SERIES FOR THE RUNNING COUPLING

At infinite \( N \), the running coupling has the form in [7]. The finite \( N \) corrections, described in the previous section for the beta function, lead also to trans-series structures for \( t(a, N) \). At strong coupling, where the scale \( a \) is large, the corrections are naturally expressed in terms of the Wilson loop, \( W = \exp[-a^2] \); while at weak coupling, where the scale \( a \) is small, the corrections are naturally expressed in terms of \( 1 - W = 1 - \exp[-a^2] \). The infinite \( N \) phase transition occurs at \( W = 1/2 \). At any finite \( N \), the running coupling, \( t(a, N) \) solves the scaling equation

\[ \frac{\partial t}{\partial a} = 2 \frac{W(t, N) \ln W(t, N)}{\partial_t W(t, N)} \]

(29)
which is both non-linear and non-perturbative. It is convenient to consider the coupling as a function of the Wilson loop $W$. At $N = \infty$ we have:

$$t(W, N) \xrightarrow{N \to \infty} \begin{cases} \frac{1}{2W} & \text{strong coupling, } W \leq \frac{1}{2} \\ 2(1 - W) & \text{weak coupling, } W \geq \frac{1}{2} \end{cases}$$  \hspace{1cm} (30)

By matching the expansions of $W$, we deduce the following large $N$ trans-series structures for $t$ as a function of $W$ (and hence of $a$)

$$t(W, N) = \begin{cases} \frac{1}{2W} + \sum_{k=1}^{\infty} e^{-kN\hat{S}_{\text{strong}}(W)} f_{\text{strong}}^{(k)}(W, N), & \text{strong coupling, } W \leq \frac{1}{2} \\ \sum_{k=0}^{\infty} e^{-kN\hat{S}_{\text{weak}}(W)} f_{\text{weak}}^{(k)}(W, N), & \text{weak coupling, } W \geq \frac{1}{2} \end{cases}$$  \hspace{1cm} (31)

The actions $\hat{S}_{\text{strong}}(W)$ and $\hat{S}_{\text{weak}}(W)$ are the strong and weak coupling actions $S_{\text{strong}}(t)$ and $S_{\text{weak}}(t)$, evaluated at the infinite $N$ values of $t$ as given in (30):

$$\hat{S}_{\text{strong}}(W) = S_{\text{strong}} \left( \frac{1}{2W} \right) = \arccosh \left( \frac{1}{2W} \right) - \sqrt{1 - 4W^2}$$

$$\hat{S}_{\text{weak}}(W) = S_{\text{weak}} \left( 2(1 - W) \right) = \frac{\sqrt{2W - 1}}{1 - W} - 2 \text{arctanh} \left( \sqrt{2W - 1} \right)$$  \hspace{1cm} (32)

The leading terms in the strong coupling trans-series (31) read:

$$t_{\text{strong}}(W, N) \approx \frac{1}{2W} e^{-N\hat{S}_{\text{strong}}(W)} \left( 1 - \frac{6W^2 + 7}{6N(1 - 4W^2)^{3/2}} - \frac{324W^4 + 804W^2 + 85}{72N^2(1 - 4W^2)^3} + \ldots \right) + \ldots$$  \hspace{1cm} (33)

understood as being expanded in $W = \exp[-a^2]$. At weak coupling

$$t_{\text{weak}}(W, N) \approx 2(1 - W) \left( 1 - \frac{1}{2N^2(2W - 1)} + \frac{(8W^2 + 5W - 9)(1 - W)^2}{8N^4(2W - 1)^4} + \ldots \right)$$

$$- \frac{2i}{\pi N^{3/2}} \frac{1 - W}{(2W - 1)^{1/4}} e^{-N\hat{S}_{\text{weak}}(W)} \left( 1 - \frac{9W^2 + 5}{24N(2W - 1)^{3/2}} + \ldots \right) + \ldots$$  \hspace{1cm} (34)

understood as being expanded in $(1 - W) = (1 - \exp[-a^2])$.

**IV. CONCLUSIONS**

The Gross-Witten-Wadia unitary matrix model is a one-plaquette model of 2 dimensional lattice Yang-Mills theory, which has the interesting feature of a third-order phase transition at infinite $N$, in addition to a running coupling $[2, 3]$. The perturbative beta function for this model acquires a non-perturbative trans-series completion at large but finite $N$ in the ’t Hooft limit, as does the running coupling. The ’t Hooft coupling runs with the scale $a$, and the trans-series rearranges itself across the phase transition. Physically, this transition is identified with the condensation of instantons $[24]$, with different kinds of instantons dominating at weak- and strong-coupling $[15, 27, 28]$. Technically, the beta function $\beta(t, N)$ can be expressed explicitly in terms of the expectation value $\Delta(t, N) \equiv \langle \det U \rangle$, whose resurgent trans-series structure was studied in detail in $[15]$. The beta function $\beta(t, N)$ inherits its trans-series structure from that of $\Delta(t, N)$, and therefore the beta function trans-series also has full resurgent properties, including concrete relations between different instanton sectors. It would be interesting to study further this trans-series structure directly in the renormalization group approach to matrix models $[29, 32]$.

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