Non-uniform Dependence on Initial Data for the Camassa–Holm Equation in the Critical Besov Space

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Abstract. Whether or not the data-to-solution map of the Cauchy problem for the Camassa–Holm equation and Novikov equation in the critical Besov space $B^{3/2}_{2,1}(\mathbb{R})$ is uniformly continuous remains open. In the paper, we aim at solving the open question left in the previous works (Li et al. in J Differ Equ 269:8686–8700, 2020a; J Math Fluid Mech 22:50, 2020b) and giving a negative answer to this problem.

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Keywords. Camassa–Holm (Novikov) equation, Non-uniform continuous dependence, Critical Besov spaces.

1. Introduction

In this paper, we consider the Cauchy problem for the well-known Camassa–Holm equation

\[
\begin{aligned}
&\left\{ 
\begin{array}{l}
  u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \\
  u(x, t = 0) = u_0,
\end{array}
\right. \\
&\quad (x, t) \in \mathbb{R} \times \mathbb{R}^+,
\end{aligned}
\]

(ch)

Here the scalar function $u = u(t, x)$ stands for the fluid velocity at time $t \geq 0$ in the $x$ direction. (CH) was firstly proposed in the context of hereditary symmetries studied in [18] and then was derived explicitly as a water wave equation by Camassa–Holm [4]. (CH) is completely integrable [4,7] with a bi-Hamiltonian structure [6,18] and infinitely many conservation laws [4,18]. Also, it admits exact peaked soliton solutions (peakons) of the form $ce^{-|x-ct|}$ with $c > 0$, which are orbitally stable [15] and models wave breaking (i.e., the solution remains bounded, while its slope becomes unbounded in finite time [5,10,11]). It is worth mentioning that the peaked solitons present the characteristic for the travelling water waves of greatest height and largest amplitude and arise as solutions to the free-boundary problem for incompressible Euler equations over a flat bed, see Refs. [8,12,13,29] for the details. Because of the mentioned interesting and remarkable features, the CH equation has attracted much attention as a class of integrable shallow water wave equations in recent twenty years. Its systematic mathematical study was initiated in a series of papers by Constantin and Escher, see [9–13]. As mentioned above, the most obvious ones of infinitely many conservation laws are the conservation of the average over $\mathbb{R}$ and of the $H^1$ norm for smooth solutions with sufficient decay at infinity. Based on the latter property, Xin and Zhang [30] established that (CH) has global weak solutions for any data in $H^1(\mathbb{R})$. Later, Xin and Zhang [31] also proved the uniqueness of weak solution to (CH). We can refer the readers to see the global strong solutions in [5,9,10] and finite time blow-up strong solutions in [5,9–11] to (CH), the existence and uniqueness of global weak solutions in [14], the global conservative solutions [2] and global dissipative solutions [3] in $H^1(\mathbb{R})$.

After the phenomenon of non-uniform continuity for some dispersive equations was studied by Kenig et al. [25], the issue of non-uniform dependence on the initial data has been a fascinating object of research in the recent past. Naturally, we may wonder which regularity assumptions are relevant for
the initial data $u_0$ such that the Cauchy problem to (CH) is not uniform dependent on initial data, namely, the dependence of solution on the initial data associated with this equation is not uniformly continuous. Himonas–Misiolek [24] obtained the first result on the non-uniform dependence for (CH) in $H^s(\mathbb{T})$ with $s \geq 2$ using explicitly constructed travelling wave solutions, which was sharpened to $s > \frac{3}{2}$ by Himonas–Kenig [22] on the real-line and Himonas–Kenig–Misiolek [23] on the circle. Danchin [16,17] proved the local existence and uniqueness of strong solutions to (CH) with initial data in $B^s(CH)$ in the critical Besov spaces $B^s(CH)$ and $B^3_{2,1}$. Li and Yin [26] proved the continuity of the solution map of (CH) with respect to the initial data. Guo et al. [19] established the ill-posedness of (CH) in $H^3/2$ and in $B^3_{2,r}$ with $r \in (1,\infty)$ by proving the norm inflation. In our recent paper [27], we proved the non-uniform dependence on initial data for (CH) under the framework of Besov spaces $B^s_{p,r}$ for $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. However, whether or not the data-to-solution map of the Cauchy problem for (CH) in the critical Besov spaces $B^3_{2,1}(\mathbb{R})$ is uniformly continuous remains open. We aim at giving a negative answer to this question in this paper.

Before stating our main result, we transform (CH) equivalently into the following nonlinear transport type equation

$$\begin{align*}
\begin{cases}
\partial_t u + u\partial_x u = P(u), & (x,t) \in \mathbb{R} \times \mathbb{R}^+,
\quad u(x,t = 0) = u_0,
\quad x \in \mathbb{R},
\end{cases}
\end{align*}$$

(1.1)

where

$$P(u) = P(D)\left(u^2 + \frac{1}{2}(\partial_x u)^2\right) \quad \text{with} \quad P(D) = -\partial_x (1 - \partial_x^2)^{-1}.$$  \hspace{1cm} (1.2)

Our main result is stated as follows.

**Theorem 1.1.** The data-to-solution map $u_0 \mapsto S_t(u_0)$ of the Cauchy problem (1.1)–(1.2) is not uniformly continuous from any bounded subset in $B^3_{2,1}$ into $C([0,T]; B^3_{2,1})$. More precisely, there exists two sequences of solutions $S_t(f_n + g_n)$ and $S_t(f_n)$ such that

$$\|f_n\|_{B^3_{2,1}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|g_n\|_{B^3_{2,1}} = 0$$

but

$$\liminf_{n \to \infty} \|S_t(f_n + g_n) - S_t(f_n)\|_{B^3_{2,1}} \gtrsim t, \quad \forall \ t \in [0,T_0],$$

with small time $T_0$.

**Remark 1.1.** Obviously, we generalize the previous result [27] to the critical case. The method we used in [27] does not work for the critical index $s = \frac{3}{2}$ due to technical difficulty. Precisely speaking, the transport equation theory prevents us to obtaining the estimate of solution in $B^{1/2}_{2,1}$. Our approach for proving Theorem 1.1 is new and different from that in [27].

**Remark 1.2.** The method we used in proving the Theorem 1.1 can be applied equally well to other related systems, such as the following Novikov equation (see, e.g., [20,21,28])

$$\begin{align*}
\begin{cases}
u_t + u^2 u_x = Q(u),
\quad u(x,t = 0) = u_0,
\end{cases}
\end{align*}$$

(1.3)

where

$$Q(u) = -(1 - \partial_x^2)^{-1}\left(\frac{1}{2}u_x^2 + \partial_x\left(\frac{3}{2}u^2_x + u^3\right)\right).$$

(1.4)

Then we have the following
Theorem 1.2. The data-to-solution map \( u_0 \mapsto S_t(u_0) \) of the Cauchy problem (1.3)–(1.4) is not uniformly continuous from any bounded subset in \( B_{2,1}^{3/2} \) into \( C([0,T];B_{2,1}^{3/2}) \). More precisely, there exists two sequences of solutions \( S_t(f_n + h_n) \) and \( S_t(f_n) \) such that
\[
\|f_n\|_{B_{2,1}^{3/2}} \lesssim 1 \quad \text{and} \quad \lim_{n \to \infty} \|h_n\|_{B_{2,1}^{3/2}} = 0
\]
but
\[
\liminf_{n \to \infty} \|S_t(f_n + h_n) - S_t(f_n)\|_{B_{2,1}^{3/2}} \gtrsim t, \quad \forall t \in [0,T_0],
\]
with small time \( T_0 \).

Remark 1.3. It should be mentioned that the previous result [28] was generalized to the critical case.

Remark 1.4. Based on the special construction for initial data, Guo et al. [19] obtained the ill-posedness of both (CH) and (N) in \( B_{2,r}^{3/2} \) with \( r \in (1,\infty) \) by proving the norm inflation, which implies that the data-to-solution map is not continuous. However, their construction is invalid for the case when \( r = 1 \).

Organization of our paper. In Sect. 2, we list some notations and known results which will be used in the sequel. In Sect. 3, we present the local well-posedness result and establish some technical Propositions. In Sect. 4, we prove our main theorem by adopting the strategies used in [27]. Here we should point out that the new difficulty when dealing the critical case consists in the lack of the estimate of solution in \( B_{2,1}^{3/2} \). To overcome this, we decompose the solution map as
\[
S_t(u_0^n) = S_t(u_0^n) - u_0^n - t\nu_0(u_0^n) + f_n + g_n + t(P(u_0^n) - u_0^n\partial_x u_0^n).
\]
On one hand, \( u_0^n\partial_x u_0^n \) brings us the term \( g_n\partial_x f_n \) which plays an essential role since it would not small when \( n \) is large enough; On the other hand, \( S_t(u_0^n) - u_0^n - t\nu_0(u_0^n) \) promotes us to estimate the crucial quantity \( \|S_t(u_0^n)\|_{L^\infty} \) which can be controlled by \( t\|u_0^n\|_{C^{0,1}} + \|u_0^n\|_{L^\infty} \) instead of the norm \( \|S_t(u_0^n)\|_{B_{2,1}^{3/2}} \).

Based on the suitable choice of \( f_n \) and \( g_n \), we prove that the data-to-solution map is not uniformly continuous.

2. Littlewood–Paley Analysis

We firstly introduce some notations which will be used throughout this paper.

The symbol \( A \lesssim \) (resp. \( \gtrsim \)) \( B \) means that there is a uniform positive constant \( c \) independent of \( A \) and \( B \) such that \( A \leq (\text{resp.} \geq) cb \). Given a Banach space \( X \), we denote its norm by \( \| \cdot \|_X \). We use the simplified notation \( \|f_1,\ldots,f_n\|_X = \|f_1\|_X + \cdots + \|f_n\|_X \) if without confusion. For all \( f \in S' \), the Fourier transform \( \mathcal{F}f \) (also denoted by \( \hat{f} \)) is defined by
\[
\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)dx \quad \text{for any } \xi \in \mathbb{R}.
\]
The inverse Fourier transform allows us to recover \( u \) from \( \hat{f} \):
\[
f(x) = \mathcal{F}^{-1}\hat{f}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \hat{f}(\xi)d\xi.
\]
Next, we will recall some facts about the Littlewood–Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties (see [1] for more details).

There exists a couple of smooth functions \( (\chi, \varphi) \) valued in \( [0,1] \), such that \( \chi \) is supported in the ball \( B \triangleq \{ \xi \in \mathbb{R} : |\xi| \leq \frac{4}{3} \} \), and \( \varphi \) is supported in the ring \( C \triangleq \{ \xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{5}{3} \} \). Moreover,
\[
\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}.
\]
For every $f \in \mathcal{S}'(\mathbb{R})$, the inhomogeneous dyadic blocks $\Delta_j$ are defined as follows

$$
\Delta_j f = \begin{cases}
0, & \text{if } j \leq -2; \\
\chi(D)f = \mathcal{F}^{-1}(\chi \mathcal{F}f), & \text{if } j = -1; \\
\varphi(2^{-j}D)f = \mathcal{F}^{-1}(\varphi(2^{-j}\mathcal{F}f)), & \text{if } j \geq 0.
\end{cases}
$$

In the inhomogeneous case, the following Littlewood–Paley decomposition makes sense

$$
f = \sum_{j \geq -1} \Delta_j f \quad \text{for any } f \in \mathcal{S}'(\mathbb{R}).
$$

**Definition 2.1** (See [1]). Let $s \in \mathbb{R}$ and $(p, r) \in [1, \infty]^2$. The nonhomogeneous Besov space $B^s_{p, r}(\mathbb{R})$ is defined by

$$
B^s_{p, r}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p, r}(\mathbb{R})} < \infty \right\},
$$

where

$$
\|f\|_{B^s_{p, r}(\mathbb{R})} = \begin{cases}
\left( \sum_{j \geq -1} 2^{sjr} \|\Delta_j f\|_{L^p(\mathbb{R})} \right)^{\frac{1}{r}}, & \text{if } 1 \leq r < \infty, \\
\sup_{j \geq -1} 2^{sj} \|\Delta_j f\|_{L^p(\mathbb{R})}, & \text{if } r = \infty.
\end{cases}
$$

Finally, we give some important properties which will be also often used throughout the paper.

**Lemma 2.1** (See [1]). For $s > 0$, then for any $u, v \in B^s_{2, 1}(\mathbb{R}) \cap L^\infty(\mathbb{R})$, we have

$$
\|uv\|_{B^s_{2, 1}(\mathbb{R})} \leq C\left( \|u\|_{B^s_{2, 1}(\mathbb{R})} \|v\|_{L^\infty(\mathbb{R})} + \|v\|_{B^s_{2, 1}(\mathbb{R})} \|u\|_{L^\infty(\mathbb{R})} \right).
$$

In particular, we have the embedding $B^{1/2}_{2, 1}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and

$$
B^s_{p, q}(\mathbb{R}) \hookrightarrow B^t_{p, r}(\mathbb{R}) \quad \text{for } s > t \quad \text{or} \quad s = t, 1 \leq q \leq r \leq \infty.
$$

**Lemma 2.2** (Lemma 3.26 in [1]). Let $(p, r) \in [1, \infty]^2$, $s > 1$ and $u_0 \in B^s_{p, r}(\mathbb{R})$. Assume that $u \in L^\infty([0, T]; B^s_{p, r}(\mathbb{R}))$ solves (1.1)–(1.2). Then there exists a constant $C = C(s, p)$ and a universal constant $C'$ such that for all $t \in [0, T]$, we have

$$
\|u(t)\|_{B^s_{p, r}(\mathbb{R})} \leq \|u_0\|_{B^s_{p, r}(\mathbb{R})} \exp \left( C \int_0^t \|u(\tau)\|_{C^{0, 1}(\mathbb{R})} \d \tau \right),
$$

$$
\|u(t)\|_{C^{0, 1}(\mathbb{R})} \leq \|u_0\|_{C^{0, 1}(\mathbb{R})} \exp \left( C' \int_0^t \|\partial_x u(\tau)\|_{L^\infty(\mathbb{R})} \d \tau \right).
$$

Let us recall the local well-posedness result for (CH) in the critical Besov spaces.

**Lemma 2.3** (See [17]). For any initial data $u_0$ which belongs to

$$
B_R = \left\{ \psi \in B^\frac{3}{2}_{2, 1} : \|\psi\|_{B^\frac{3}{2}_{2, 1}} \leq R \right\} \quad \text{for any } R > 0.
$$

Then there exists some $T = T(\|u_0\|_{B^\frac{3}{2}_{2, 1}}) > 0$ such that (CH) has a unique solution $S_t(u_0) \in \mathcal{C}([0, T]; B^\frac{3}{2}_{2, 1})$. Moreover, we have

$$
\|S_t(u_0)\|_{B^\frac{3}{2}_{2, 1}} \leq C\|u_0\|_{B^\frac{3}{2}_{2, 1}}.
$$
3. The Key Estimations

Firstly, we need to introduce smooth, radial cut-off functions to localize the frequency region. Precisely, let \( \hat{\phi} \in C_0^\infty(\mathbb{R}) \) be an even, real-valued and non-negative function on \( \mathbb{R} \) and satisfy

\[
\hat{\phi}(\xi) = \begin{cases} 
1, & \text{if } |\xi| \leq \frac{1}{4}, \\
0, & \text{if } |\xi| \geq \frac{1}{2}.
\end{cases}
\]

Next, we need to establish the following crucial lemmas which will be used later on.

**Lemma 3.1.** Let \((p, r) \in [1, \infty] \times [1, \infty)\). We define the high frequency function \(f_n\) and the low frequency functions \(g_n, h_n\) as follows

\[
f_n = 2^{-\frac{2}{3}n} \phi(x) \sin \left( \frac{17}{12} 2^n x \right), \quad g_n = \frac{12}{17} 2^{-n} \phi(x) \quad \text{and} \quad h_n = \frac{12}{17} 2^{-\frac{2}{3}} \phi(x), \quad n \gg 1.
\]

Then for any \(\sigma \in \mathbb{R}\), we have

\[
\|f_n\|_{L^\infty} \leq C 2^{-\frac{2}{3}n} \phi(0) \quad \text{and} \quad \|\partial_x f_n\|_{L^\infty} \leq C 2^{-\frac{2}{3}} \phi(0), \tag{3.1}
\]

\[
\|g_n, \partial_x g_n\|_{L^\infty} \leq C 2^{-n} \phi(0) \quad \text{and} \quad \|h_n, \partial_x h_n\|_{L^\infty} \leq C 2^{-\frac{2}{3}} \phi(0), \tag{3.2}
\]

\[
\|g_n\|_{B^{\sigma, p} r} \leq C 2^{(-n/2) + \sigma} \|\phi\|_{L^p} \quad \text{and} \quad \|h_n\|_{B^{\sigma, p} r} \leq C 2^{(-\frac{2}{3} + \sigma)} \|\phi\|_{L^p}, \tag{3.3}
\]

\[
\|f_n\|_{B^{\sigma, p} r} \leq C 2^{(\sigma - \frac{2}{3})n} \|\phi\|_{L^p}, \tag{3.4}
\]

\[
\liminf_{n \to \infty} \|g_n \partial_x f_n\|_{B^{\sigma, p} r, \infty} \geq M_1, \tag{3.5}
\]

\[
\liminf_{n \to \infty} \|h_n^2 \partial_x f_n\|_{B^{\sigma, p} r, \infty} \geq M_2, \tag{3.6}
\]

for some positive constants \(C, M_1, M_2\).

**Proof.** Direct computations give (3.1) and (3.2). Notice that

\[
\text{supp } \hat{g}_n \subset \left\{ \xi \in \mathbb{R} : 0 \leq |\xi| \leq \frac{1}{2} \right\},
\]

then, we have

\[
\hat{\Delta_j g}_n = \varphi(2^{-j} \xi) \hat{g}_n(\xi) \equiv 0 \quad \text{for} \quad j \geq 0,
\]

which implies

\[
\Delta_j g_n \equiv 0 \quad \text{for} \quad j \geq 0.
\]

By the definitions of \(g_n\) and the Besov space, we deduce that

\[
\|g_n\|_{B^{\sigma, p} r} = \frac{12}{17} 2^{-(n/2) + \sigma} \|\Delta_{-1} \phi\|_{L^p} \leq C 2^{-(n/2 + \sigma)} \|\phi\|_{L^p}.
\]

Notice that

\[
\text{supp } \hat{h}_n \subset \left\{ \xi \in \mathbb{R} : 0 \leq |\xi| \leq \frac{1}{2} \right\} \Rightarrow \text{supp } \hat{h}_n^2 \subset \left\{ \xi \in \mathbb{R} : 0 \leq |\xi| \leq 1 \right\},
\]

then, we have

\[
\text{supp } h_n^2 \partial_x f_n \subset \left\{ \xi \in \mathbb{R} : \frac{17}{12} 2^n - \frac{3}{2} \leq |\xi| \leq \frac{17}{12} 2^n + \frac{3}{2} \right\},
\]
which implies
\[
\Delta_j(h_n^2 \partial_x f_n) = \begin{cases} 
 h_n^2 \partial_x f_n, & \text{if } j = n, \\
 0, & \text{if } j \neq n.
\end{cases}
\]

By the definitions of \( f_n \) and \( h_n \), we obtain for some \( \delta > 0 \)
\[
\|h_n^2 \partial_x f_n\|_{B^\frac{3}{2}}^{\frac{3}{2}} = 2^{\frac{3}{2}} \| \Delta_n(h_n^2 \partial_x f_n) \|_{L^2} = 2^{\frac{3}{2}} \|h_n^2 \partial_x f_n\|_{L^2}
\]
\[
= \left\| \frac{12}{17} \phi^3(x) \cos \left( \frac{17}{12} 2^n x \right) + \left( \frac{12}{17} \right)^2 2^{-n} \phi^2(x) \partial_x \phi(x) \sin \left( \frac{17}{12} 2^n x \right) \right\|_{L^2}
\]
\[
\geq \frac{12}{17} \| \phi^3(x) \cos \left( \frac{17}{12} 2^n x \right) \|_{L^2} - C2^{-n}
\]
\[
\geq \frac{1}{8} \cdot \frac{12}{17} \delta \phi^3(0) \left( \int_0^\delta \left| \cos \left( \frac{17}{12} 2^n x \right) \right|^2 dx \right)^{1/2} - C2^{-n}.
\]

We thus deduce that (3.6) (see Lemma 3.2 in [27] for more details).

Following the same procedure of the Proof of Lemmas 3.3 and 3.4 in [27], we can prove (3.4) and (3.5) with suitable modification. Here we omit the details.

Now, we establish the estimate involving \( S_t(u_0) - u_0 - t v_0(u_0) \) which is crucial in proving Theorem 1.1.

**Proposition 3.1.** Assume that \( \|u_0\|_{B^\frac{3}{2}} \leq 1 \). Under the assumptions of Theorem 1.1, we have
\[
\|S_t(u_0) - u_0 - t v_0(u_0)\|_{B^\frac{3}{2}} \leq Ct^2 E(u_0),
\]
where we denote \( v_0(u_0) := P(u_0) - u_0 \partial_x u_0 \) and
\[
E(u_0) := 1 + \|u_0\|_{L^2}^2 \|u_0\|_{B^\frac{3}{2}}^\frac{3}{2} + \|u_0\|_{L^\infty} \left( \|u_0\|_{B^\frac{3}{2}}^\frac{3}{2} + \left( \|u_0\|_{L^\infty} + \|u_0\|_{C^1} \right) \|u_0\|_{B^\frac{3}{2}}^\frac{7}{2} \right).
\]

**Proof.** For simplicity, we denote \( u(t) = S_t(u_0) \). Firstly, according to Lemma 2.3, there exists a small time \( T = T(\|u_0\|_{B^\frac{3}{2}}) \) such that the solution \( u(t) \in C([0, T]; B^\frac{3}{2}) \), namely,
\[
\|u(t)\|_{L^\infty B^\frac{3}{2}} \leq C\|u_0\|_{B^\frac{3}{2}}^\frac{3}{2} \leq C.
\]

Applying Lemma 2.2 to Eq.(1.1), we have for all \( t \in [0, T] \) and \( \gamma \geq \frac{3}{2} \)
\[
\|u(t)\|_{L^\gamma B^\frac{3}{2}} \leq \|u_0\|_{B^\frac{3}{2}}^\gamma \exp \left( C \int_0^T \|u\|_{B^\frac{3}{2}}^\gamma \, d\tau \right) \leq C\|u_0\|_{B^\frac{3}{2}}.
\]

Set \( \tilde{u} = S_t(u_0) - u_0 \), then we deduce from Eq.(1.1) that
\[
\left\{ \begin{array}{l}
\partial_t \tilde{u} + S_t(u_0) \partial_x S_t(u_0) = P(S_t(u_0)), \\
\tilde{u}_0 = 0.
\end{array} \right.
\]

Then, we have
\[
\|\tilde{u}(t)\|_{L^\infty} \leq \int_0^t \|\partial_x \tilde{u}\|_{L^\infty} \, d\tau
\]
\[
\leq \int_0^t \|S_t(u_0) \partial_x S_t(u_0)\|_{L^\infty} \, d\tau + \int_0^t \|P(S_t(u_0))\|_{L^\infty} \, d\tau
\]
\[
\leq C \int_0^t \|S_t(u_0)\|_{L^2}^\gamma \, d\tau \quad \text{by Lemma 2.2}
\]
\[
\leq C t \|u_0\|_{L^2}^\gamma
\]
\[
\leq C t \|u_0\|_{L^2}^\gamma
\]
where we have used the estimate
\[ \|P(S_t(0))\|_{L^\infty} \leq C\|S_t(0)\|_{C^{0,1}} \]
from the fact that \((1 - \partial_x^2)^{-1} f = G * f \) with \(G(x) = \frac{1}{2} e^{-|x|} \).

Note that \(P(D)\) is a multiplier of degree \(-1\), by Lemma 2.1, we have for \(\gamma \geq \frac{3}{2}\)
\[
\|P(u)\|_{B^{\gamma,1}_{2,1}} \leq C \left\| u^2 + \frac{1}{2} u_x^2 \right\|_{B^{-1}_{2,1}} \\
\leq C \left\| u, u_x \right\|_{L^\infty} \left\| u, u_x \right\|_{B^{-1}_{2,1}} \\
\leq C \left\| u \right\|_{B^{\gamma,1}_{2,1}} \left\| u \right\|_{B^{\gamma,1}_{2,1}}.
\]
(3.12)

By Lemma 2.1, we obtain from (3.8) and (3.12) that
\[
\|u(t) - u_0\|_{B^{\gamma,1}_{2,1}} \leq \int_0^t \|\partial_\tau u\|_{B^{\gamma,1}_{2,1}} \, d\tau \\
\leq \int_0^t \|P(u)\|_{B^{\gamma,1}_{2,1}} d\tau + \int_0^t \|u\|_{B^{\gamma,1}_{2,1}} d\tau \\
\leq C t \left( \|u\|_{B^{\gamma,1}_{2,1}}^2 + \|u\|_{L^\infty} \|u_x\|_{B^{\gamma,1}_{2,1}} \right) \\
\leq C t \left( \|u_0\|_{B^{\gamma,1}_{2,1}}^2 + \|u_0\|_{L^\infty} + \|u_0\|_{C^{0,1}} \right) \|u_0\|_{B^{\gamma,1}_{2,1}}.
\]
(3.13)

Similarly, we obtain from (3.8), (3.9) and (3.12) that
\[
\|u(t) - u_0\|_{B^{\gamma,1}_{2,1}} \leq \int_0^t \|\partial_\tau u\|_{B^{\gamma,1}_{2,1}} \, d\tau \\
\leq \int_0^t \|P(u)\|_{B^{\gamma,1}_{2,1}} d\tau + \int_0^t \|u\|_{B^{\gamma,1}_{2,1}} d\tau \\
\leq C t \left( \|u\|_{B^{\gamma,1}_{2,1}} + \|u\|_{L^\infty} \|u\|_{B^{\gamma,1}_{2,1}} \right) \\
\leq C t \left( \|u_0\|_{B^{\gamma,1}_{2,1}}^2 + \|u_0\|_{L^\infty} + \|u_0\|_{C^{0,1}} \right) \|u_0\|_{B^{\gamma,1}_{2,1}}.
\]
(3.14)

Using Lemma 2.1 again, we obtain that
\[
\|u(t) - u_0 - t v_0(u_0)\|_{B^{\gamma,1}_{2,1}} \leq \int_0^t \|\partial_\tau u - v_0(0)\|_{B^{\gamma,1}_{2,1}} \, d\tau \\
\leq \int_0^t \|P(u) - P(u_0)\|_{B^{\gamma,1}_{2,1}} d\tau + \int_0^t \|u\|_{B^{\gamma,1}_{2,1}} d\tau \\
\leq \int_0^t \|u(\tau) - u_0\|_{B^{\gamma,1}_{2,1}} d\tau + \int_0^t \|u(\tau) - u_0\|_{L^\infty} \, d\tau \\
+ \int_0^t \|u(\tau) - u_0\|_{B^{\gamma,1}_{2,1}} \, d\tau \\
\leq \int_0^t \|u(\tau) - u_0\|_{B^{\gamma,1}_{2,1}} d\tau + \|u_0\|_{L^\infty} \int_0^t \|u(\tau) - u_0\|_{L^\infty} d\tau \\
+ \|u_0\|_{L^\infty} \int_0^t \|u(\tau) - u_0\|_{B^{\gamma,1}_{2,1}} d\tau,
\]
(3.15)

where we have used
\[
\|P(u) - P(u_0)\|_{B^{\gamma,1}_{2,1}} \leq C \left\| (u + u_0)(u - u_0) + \frac{1}{2} \partial_\tau (u + u_0) \partial_x (u - u_0) \right\|_{B^{\gamma,1}_{2,1}}.
\]
In this section, we prove Theorems 1.1 and 1.2 by using Propositions 3.1 and 3.2, respectively.

\[ \leq C\|u - u_0\|_{B^2_{2,1}}^{\frac{3}{2}} \|u + u_0\|_{B^2_{2,1}}^{\frac{1}{2}} \]

Plugging (3.11)–(3.13) into (3.14) yields the desired result (3.7). Thus, we complete the proof of Proposition 3.1.

Also, the following estimate involving \( S_t(u_0) - u_0 - tw_0(u_0) \) is crucial in proving Theorem 1.2. \( \square \)

**Proposition 3.2.** Assume that \( \|u_0\|_{B^2_{2,1}}^{\frac{3}{2}} \leq 1 \). Under the assumptions of Theorem 1.2, we have

\[
\|S_t(u_0) - u_0\|_{L^\infty} \leq Ct\|u_0\|_{C^{0,1}}, \\
\|S_t(u_0) - u_0\|_{B^2_{2,1}}^{\frac{3}{2}} \leq Ct\left(\|u_0\|_{B^2_{2,1}}^{\frac{3}{2}} + \|u_0\|_{C^{0,1}}\right), \\
\|S_t(u_0) - u_0\|_{B^2_{2,1}}^{\frac{5}{2}} \leq Ct\left(\|u_0\|_{B^2_{2,1}}^{\frac{5}{2}} + \|u_0\|_{C^{0,1}}\right), \\
\|S_t(u_0) - u_0 - tw_0\|_{B^2_{2,1}}^{\frac{3}{2}} \leq Ct^2F(u_0),
\]

where we denote \( w_0(u_0) := Q(u_0) - u_0^2\partial_xu_0 \) and

\[ F(u_0) := 1 + \|u_0\|_{C^{0,1}}^2 \|u_0\|_{B^2_{2,1}}^{\frac{5}{2}} + \|u_0\|_{C^{0,1}}^4 \|u_0\|_{B^2_{2,1}}^{\frac{5}{2}}. \]

**Proof.** The proof follows the same manner from Proposition 3.1, we omit the details. \( \square \)

4. **Non-uniform Continuous Dependence**

In this section, we prove Theorems 1.1 and 1.2 by using Propositions 3.1 and 3.2, respectively.

**Proof of Theorem 1.1.** We set \( u^n_0 = f_n + g_n \) and compare the solution \( S_t(u^n_0) \) and \( S_t(f_n) \). We obviously have

\[ \|u^n_0 - f_n\|_{B^2_{2,1}}^{\frac{3}{2}} = \|g_n\|_{B^2_{2,1}}^{\frac{3}{2}} \leq C2^{-n}, \]

which means that

\[ \lim_{n \to \infty} \|u^n_0 - f_n\|_{B^2_{2,1}}^{\frac{3}{2}} = 0. \]

From Lemma 3.1, one has

\[ \|u^n_0, f_n\|_{B^2_{2,1}} \leq C2^{(\sigma - \frac{3}{2})n} \text{ for } \sigma \geq \frac{3}{2}, \]

\[ \|u^n_0, f_n\|_{L^\infty} \leq C2^{-n} \text{ and } \|\partial_xu^n_0, \partial_xf_n\|_{L^\infty} \leq C2^{-\frac{3}{2}}, \]

which implies

\[ \text{E}(u^n_0) + \text{E}(f_n) \leq C. \]

Notice that

\[ S_t(u^n_0) = S_t(u^n_0 - u^n_0 + tw_0(u^n_0)) + f_n + g_n + t(P(u^n_0) - u^n_0\partial_xu^n_0) \]
\[ = I_1(u^n_0) \]
\[ S_t(f_n) = S_t(f_n - f_n + tw_0(f_n)) + f_n + t(P(f_n) - f_n\partial_xf_n) \]
\[ = I_2(u^n_0) \]
\[ u^n_0\partial_xu^n_0 - f_n\partial_xf_n = g_n\partial_xf_n + u^n_0\partial_xg_n, \]

using the triangle inequality and Proposition 3.1, we deduce that

\[ \|S_t(u^n_0) - S_t(f_n)\|_{B^2_{2,1}}^{\frac{3}{2}} \]
where we have performed the following easy computations
\[
\|u_0^n \partial_x g_n\|_{B_{2,1}^{3/2}} \leq C\|u_0^n\|_{B_{2,1}^{3/2}} \|g_n\|_{B_{2,1}^{3/2}} \leq C2^{-n},
\]
\[
\|P(u_0^n) - P(f_n)\|_{B_{2,1}^{3/2}} \leq C\|g_n\|_{B_{2,1}^{3/2}} \|u_0^n + f_n\|_{B_{2,1}^{3/2}} \leq C2^{-n}.
\]
Combining the fact from Lemma 3.1
\[
\liminf_{n \to \infty} \|g_n \partial_x f_n\|_{B_{2,1}^{3/2}} \geq M_1,
\]
then we deduce from (4.1) that
\[
\liminf_{n \to \infty} \|S_t(f_n + g_n) - S_t(f_n)\|_{B_{2,1}^{3/2}} \geq t \quad \text{for } t \text{ small enough.}
\]
This completes the proof of Theorem 1.1. \qed

**Proof of Theorem 1.2.** We set \( u_0^n = f_n + h_n \) and compare the solution \( S_t(u_0^n) \) and \( S_t(f_n) \). Obviously, we have
\[
\lim_{n \to \infty} \|u_0^n - f_n\|_{B_{2,1}^{3/2}} = \lim_{n \to \infty} \|h_n\|_{B_{2,1}^{3/2}} = 0.
\]
Lemma 3.1 tells us that
\[
\|u_0^n, f_n\|_{C^{1,1}} \leq C2^{-\frac{n}{2}} \quad \text{and} \quad \|u_0^n, f_n\|_{B_{2,1}^{3/2}} \leq C2^{(\sigma - \frac{3}{2})n} \quad \text{for } \sigma \geq \frac{3}{2},
\]
which implies
\[
F(u_0^n) + F(f_n) \leq C.
\]
Using the triangle inequality and Proposition 3.2, we deduce that
\[
\|S_t(u_0^n) - S_t(f_n)\|_{B_{2,1}^{3/2}} \quad = \quad \|I_1(w_0) - I_2(w_0) + g_n - t((u_0^n)^2 \partial_x u_0^n - f_n^2 \partial_x f_n - Q(u_0^n) + Q(f_n))\|_{B_{2,1}^{3/2}} \\
\geq \quad t\|h_n^2 \partial_x f_n\|_{B_{2,1}^{3/2}} \quad - \quad t\|2f_n h_n \partial_x f_n + (u_0^n)^2 \partial_x h_n + Q(f_n) - Q(u_0^n)\|_{B_{2,1}^{3/2}} \\
\geq \quad t \|h_n^2 \partial_x f_n\|_{B_{2,1}^{3/2}} \quad - \quad Ct2^{-\frac{n}{2}} \quad - \quad Ct^2,
\]
where we have used that
\[
(u_0^n)^2 \partial_x u_0^n - f_n^2 \partial_x f_n = h_n^2 \partial_x f_n + 2f_n h_n \partial_x f_n + (u_0^n)^2 \partial_x h_n
\]
and
\[
\|\partial_x h_n\|_{B_{2,1}^{3/2}} \leq C\|u_0^n\|_{B_{2,1}^{3/2}}^2 \|h_n\|_{B_{2,1}^{3/2}} \leq C2^{-\frac{n}{2}},
\]
\[
\|f_n h_n \partial_x f_n\|_{B_{2,1}^{3/2}} \leq \|f_n\|_{L\infty} \|h_n\|_{L\infty} \|f_n\|_{B_{2,1}^{3/2}} + \|\partial_x f_n\|_{L\infty} \|h_n\|_{B_{2,1}^{3/2}} \|f_n\|_{B_{2,1}^{3/2}} \|f_n\|_{B_{2,1}^{3/2}} \leq C2^{-n}.
\]
Combining the fact from Lemma 3.1
\[ \liminf_{n \to \infty} \| h_n^2 \partial_x f_n \|_{B_{2,1}^2}^2 \gtrsim M_2, \]
then we deduce from (4.2) that
\[ \liminf_{n \to \infty} \| S_t(f_n + h_n) - S_t(f_n) \|_{B_{2,1}^2}^2 \gtrsim t \]
for \( t \) small enough.
This completes the proof of Theorem 1.2. \( \square \)

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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