Cooperative Extended State Observer Based Control of Vehicle Platoons With Arbitrarily Small Time Headway

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Abstract

We study platoon control of homogeneous vehicles with linear third-order longitudinal dynamics under the constant time headway policy. The controller of each follower vehicle is only based on its own velocity, acceleration, inter-vehicle distance and velocity difference with respect to its immediate predecessor, which is all obtained by on-board sensors. We establish a dynamic model based on velocity differences between adjacent vehicles with unmodeled dynamics, for which distributed cooperative extended state observers of followers are designed to estimate the acceleration differences between adjacent vehicles. Based on estimates of the acceleration differences, distributed cooperative controllers are designed. By using the stability theory of perturbed linear systems, we show that the control parameters can be properly designed to ensure the closed-loop and string stabilities for any given positive time headway. We further show that the proposed control law based on the ideal vehicle model can still guarantee the closed-loop and string stabilities when there are small model parameter uncertainties. Also, simulation results demonstrate the robustness of the proposed control law against sensing noises, input delays and parameter uncertainties.

Key words: Vehicle platoon; Constant time headway; Extended state observer; String stability.

1 Introduction

Vehicle platoon can improve road utilization rate and reduce fuel consumption effectively (Alam, 2011). Therefore, it has attracted worldwide attention (Shladover et al., 1991; Coelingh & Solvoni, 2012). From networked control perspective, Li et al. (2015) divided a vehicle platoon system into four basic modules: node dynamics, information flow topology, formation geometry, and distributed controller, among which the formation geometry greatly affects the stability of the vehicle platoon system. Formation geometry is determined by the spacing policy. The constant time headway policy is commonly used in the literature (Klinge & Middleton, 2009; Naus et al., 2010; Xiao & Gad, 2011; Ploeg et al., 2014; Darbha et al., 2017) and it is well known that large time headway is conducive to the string stability of the vehicle platoon system (Rajamani & Zhu, 2002; Naus et al., 2010), however, the larger the time headway is, the greater the inter-vehicle distance becomes, which leads to a lower road utilization rate.

It is of interest to guarantee the stability of the vehicle platoon with small time headways. Many researches showed that using the accelerations of the preceding vehicles can reduce the lower bound of the time headway required (Rajamani & Zhu, 2002; Zhou & Peng, 2004; Naus et al., 2010; Darbha et al., 2017; Al-Jhayyish & Schmidt, 2018). All the above works assumed that the accelerations of the preceding vehicles can be obtained by the wireless communication network accurately, nevertheless, accurate communication doesn’t exist in practical applications. In addition, a control law which relies on communication data runs the risk of failure when the inter-vehicle communication network breaks down under attack. Therefore, a cooperative control law with a small time headway, which can ensure both closed-loop and string stabilities without relying on the inter-vehicle wireless communication network, is of especially significance for practical applications. Ploeg et al. (2015) and Wen & Guo (2019) proposed methods to estimate the acceleration differences between adjacent vehicles, respectively. The
control laws in [Ploeg et al. (2015) and Wen & Guo (2019)] are indeed independent of wireless communication networks. The closed-loop and string stabilities are analyzed by numerical simulations in [Ploeg et al. (2015) and Wen & Guo (2019)], especially, [Ploeg et al. (2015)] considered a third-order linear model with input delays, in which the simulation results show that the string stability is not guaranteed if the time headway is small enough.

In this paper, we consider a vehicle model with third-order linear dynamics and use the constant time headway spacing policy. We design a distributed cooperative control law for each follower vehicle only using its own velocity, acceleration, inter-vehicle distance and velocity difference with respect to its immediate predecessor, all of which can be obtained by on-board sensors. Firstly, the model of velocity difference between adjacent vehicles is established, based on which a distributed cooperative extended state observer is designed to estimate the acceleration difference between adjacent vehicles. Then based on this estimate, a distributed cooperative controller is designed for each follower. The controller for each follower consists of two parts, where the first part is a feedback term consisting of inter-vehicle distance error and its differential, the other is a feedforward term consisting of the estimate of the acceleration of the preceding vehicle. Thus, the whole cooperative control law only uses the data obtained by on-board sensors without wireless communication networks.

We analyze both closed-loop and string stabilities of the vehicle platoon system. The closed-loop system matrix is decomposed into two matrices, one of which is related to feedback parameters of the distributed controllers, and the other is regarded as the perturbation matrix, which is related to feedforward parameters. Then, we give the range of the control parameters to ensure the closed-loop stability by using the stability theory of perturbed linear systems. In the frequency domain, we analyze the transfer functions which describe the inter-vehicle distance error propagation, and give the range of the control parameters to ensure the string stability. We show that one can design control parameters properly to ensure both closed-loop and string stabilities for any given positive time headway. It should be pointed out that our method for estimating the acceleration differences between adjacent vehicles is based on distributed cooperative extended state observers, which is totally different from those in [Ploeg et al. (2015) and Wen & Guo (2019)]. Besides, we give the explicit range of control parameters quantitatively related to the system parameters to ensure both closed-loop and string stabilities for any given positive time headway.

We analyze the robustness of the proposed control law by theoretical study and numerical simulations. (i) Since the proposed control law guarantee the exponential closed-loop stability, it is naturally robust against bounded sensing noises. (ii) We show that the proposed cooperative control law based on the ideal vehicle model can still ensure the closed-loop and string stabilities provided the parameter uncertainties in the vehicle model are sufficiently small. (iii) In the numerical simulations, we consider the same vehicle model with input delays and the same time headway as in [Ploeg et al. (2015)], with additional sensing noises and uncertain model parameters. Simulations show that the closed-loop and string stabilities can be guaranteed by the proposed cooperative control law based on the ideal vehicle model without parameter uncertainties and input delays.

The rest of this paper is organized as follows. The vehicle platoon model and the control objectives are presented in Section II. In Section III, we first establish dynamic models based on velocity differences between every adjacent vehicles, then distributed cooperative extended state observers are designed to estimate the acceleration differences between adjacent vehicles. Finally, distributed cooperative controllers for follower vehicles are designed. In Section IV, we give the range of control parameters for the closed-loop and string stabilities. In Section V, we analyze the robustness of the proposed control law against model parameter uncertainties. Numerical simulations are carried out in Section VI. In Section VII, we give some conclusions.

The following notation will be used throughout this paper. For a given matrix $A$, its 2-norm and minimum singular value are denoted by $\|A\|$ and $S_{\min}(A)$, respectively; $\text{diag}(A)$ denotes a block diagonal matrix whose diagonal blocks are all matrix $A$; $\mathbb{C}$ denotes the complex domain; $\mathbb{R}$ denotes the real domain; $O$ and $I$ denote the zero matrix and the identity matrix with an appropriate size, respectively.

2 Problem formulation

The longitudinal dynamics of the vehicle is a complex system with many factors. Taking both accuracy and complexity of modeling into account and using some reasonable assumptions (Zheng et al., 2016), one can get a simplified nonlinear model of the vehicle longitudinal dynamics

\[
\begin{aligned}
\dot{p}_i(t) &= v_i(t), \\
\dot{v}_i(t) &= a_i(t), \\
\dot{a}_i(t) &= \eta_i T_{i,\text{des}}(t)/m_i R_i \tau - 2 C_i v_i(t) a_i(t)/m_i \\
&\quad - (m_i a_i(t) + C_i v_i^2(t) + m_i g f_i)/(m_i R_i \tau)
\end{aligned}
\]

where $p_i(t)$, $v_i(t)$, $a_i(t)$, $T_{i,\text{des}}(t)$ are the position, velocity, acceleration, expected driving or braking torque of the $i$th vehicle at time $t$, respectively. The constant $m_i$, $f_i$, $R_i$, $C_i$ and $\eta_i$ are the mass, the rolling resistance coefficient, the tire radius, the total air resistance coefficient and the mechanical efficiency of the drive train of
the \( i \)th vehicle, respectively. The constant \( g \) and \( \tau \) are the gravitational acceleration and the inertial delay of vehicle longitudinal dynamics, respectively.

The nonlinear model (1) can be linearized by feedback linearization, and the feedback linearization law in Zheng et al. (2016) is given by

\[
T_{i,\text{des}}(t) = R_i [\dot{C}v_i(t)(2\tau a_i(t) + v_i(t)) + m_i g f_i + m_i u_i(t)]/\eta_i, \quad i = 0, 1, \ldots, N. \tag{2}
\]

By (1) and (2), we get the following third-order linear vehicle model which is commonly used in the vehicle platoon control (Rajamani & Zhu, 2002; Ploeg et al., 2015; Zheng et al., 2016; Wen & Guo, 2019).

\[
\begin{cases}
\dot{p}_i(t) = v_i(t), \\
\dot{v}_i(t) = a_i(t), \\
\dot{a}_i(t) = -a_i(t)/\tau + u_i(t)/\tau,
\end{cases}
\tag{3}
\]

where \( u_0(t) \) is the control input of the leader vehicle, i.e. the expected value of the acceleration, and \( u_i(t) \) is the control input of the \( i \)th follower vehicle to be designed, \( i = 1, 2, \ldots, N \).

We consider the constant time headway spacing policy. The expected inter-vehicle distance is denoted by

\[
d_{r,i}(t) = r + hv_i(t), \quad i = 1, 2, \ldots, N, \tag{4}
\]

where the constant \( r \) and \( h \) are the standstill distance and the time headway, respectively.

The inter-vehicle distance error is denoted by \( e_i(t) \),

\[
e_i(t) = p_{i-1}(t) - p_i(t) - d_{r,i}(t), \quad i = 1, 2, \ldots, N, \tag{5}
\]

which is the difference between the actual inter-vehicle distance and the expected inter-vehicle distance.

The control objectives are to design \( u_i(t) \), \( i = 1, 2, \ldots, N \), for follower vehicles so that the following two objectives are satisfied.

A. closed-loop stability: all the follower vehicles tend to move at the same velocity as the leader vehicle and the inter-vehicle distance errors converge to zero, i.e.

\[
\lim_{t \to \infty} [v_i(t) - v_0(t)] = 0, \quad i = 1, 2, \ldots, N,
\]

\[
\lim_{t \to \infty} e_i(t) = 0, \quad i = 1, 2, \ldots, N.
\]

B. \( L_2 \) string stability: the inter-vehicle distance errors are not amplified during the backward propagation along the platoon, i.e.

\[
\sup_{\omega \in \mathbb{R}} \left| \frac{\mathcal{F}_i(j\omega)}{\mathcal{F}_{i-1}(j\omega)} \right| \leq 1, \quad i = 2, \ldots, N,
\]

where \( \mathcal{F}_i(s) \) is the Laplace transform of \( e_i(t) \).

### 3 Cooperative control law based on extended state observer

Suppose that there is no wireless communication between vehicles. We consider predecessor following topology, and each follower vehicle in the platoon relies on the on-broad sensors to measure its own velocity, acceleration, the inter-vehicle distance and the velocity difference with respect to its immediate predecessor. However, the sensor of each follower vehicle cannot measure the acceleration of its immediate predecessor. Instead, we can design an observer to estimate the acceleration difference between adjacent vehicles. The extended state observer (ESO) was first put forward by Han (1995), whose core idea is to expand the unmodeled dynamics into new state and then according to the new state equation, an extended state observer is designed to estimate all states of the system. However, the extended state observer proposed by Han (1995) is nonlinear, which has difficulty in parameter tuning and stability analysis. In order to overcome the above problems, Gao (2003) put forward a linear extended state observer, which simplifies parameter tuning and is also beneficial for stability analysis. In this paper, we design a linear extended state observer to estimate the acceleration difference between adjacent vehicles.

According to (3), the models of the velocity difference between adjacent vehicles are given by

\[
\begin{cases}
\dot{v}_{d,i}(t) = a_{d,i}(t), \\
\dot{a}_{d,i}(t) = q_i(t) - u_i(t)/\tau, \\
\dot{q}_i(t) = w_i(t),
\end{cases}
\tag{6}
\]

where

\[
\begin{align*}
v_{d,i}(t) &= v_{i-1}(t) - v_i(t), \\
a_{d,i}(t) &= a_{i-1}(t) - a_i(t), \\
q_i(t) &= (-a_{i-1}(t) + u_{i-1}(t) + a_i(t))/\tau, \\
w_i(t) &= (a_{i-1}(t) - u_{i-1}(t) - a_i(t) + u_i(t)) \\
&\quad + \tau\dot{u}_{i-1}(t)/\tau^2.
\end{align*}
\tag{7}
\]

Here, \( q_i(t) \) is the unmodeled dynamics, which contains the control input and the acceleration of \( i-1 \)th vehicle that cannot be measured directly by the \( i \)th vehicle. We design a linear extended state observer for (6)

\[
\begin{cases}
\dot{z}_{1,i}(t) = z_{2,i}(t) + \beta_1(v_{d,i}(t) - z_{1,i}(t)), \\
\dot{z}_{2,i}(t) = z_{3,i}(t) + \beta_2(v_{d,i}(t) - z_{1,i}(t)) - u_i(t)/\tau, \\
\dot{z}_{3,i}(t) = \beta_3(v_{d,i}(t) - z_{1,i}(t)), \quad i = 1, 2, \ldots, N,
\end{cases}
\tag{11}
\]

where \( z_{2,i}(t) \) is the estimate of \( a_{d,i}(t) \). The constants \( \beta_1 > 0, \beta_2 > 0 \) and \( \beta_3 > 0 \) are the observer gains to be designed.
The output $z_{2,i}(t)$ of the extended state observer (11) is the estimate of the acceleration difference between the $i$th vehicle and the $i$th vehicle. Then the estimate of the acceleration of $i$th vehicle is given by $z_{2,i}(t) + a_i(t)$. Combining with the velocity and the acceleration of the $i$th vehicle, and the inter-vehicle distance and the velocity difference between the $i$th vehicle and the $i$th vehicle, the controller of $i$th follower vehicle is designed as

$$u_i(t) = k_p e_i(t) + k_v (v_{d,i}(t) - h a_i(t)) + k_a (z_{2,i}(t) + a_i(t)), \quad i = 1, 2, ..., N,$$

where $k_p > 0$, $k_v > 0$, $k_a > 0$ are the control parameters to be designed. The controller (12) consists of two parts. The first part $k_p e_i(t) + k_v (v_{d,i}(t) - h a_i(t))$ is the feedback item, which consists of the inter-vehicle distance error between the adjacent vehicles and its differential; while the second part $k_a (z_{2,i}(t) + a_i(t))$ is the feedforward item, which consists of the estimate of the acceleration of the $i$th vehicle. It is worth mentioning that the design of the extended state observer (11) and the controller (12) only uses the information obtained by on-board sensors of followers.

4 Stability analysis of vehicle platoon

In reality, the leader vehicle in the platoon cannot always perform the shifting motion. The velocity of the leader vehicle will reach at a steady state within a finite time $t_f$, that is, there exists $t_f$, such that $u_0(t) = 0$, $t \geq t_f$. So we make the following reasonable assumption.

**Assumption 1** $\lim_{t \to \infty} u_0(t) = 0$, $\lim_{t \to \infty} \dot{u}_0(t) = 0$.

Denote $\Psi(k_p, k_v) = \begin{bmatrix} \Psi_{11} & O \\ \Psi_{21} & \Psi_{22} \end{bmatrix} \in \mathbb{R}^{6N \times 6N}$, where $\Psi_{11}$, $\Psi_{21}$ and $\Psi_{22}$ are $3N$ order square matrices with $\Psi_{22} = diag(\mathcal{H})$,

$$A = \begin{bmatrix} 0 & 1 & -h \\ 0 & 0 & -1 \\ \frac{k_p}{\tau} & k_v & -1 - k_v h \end{bmatrix}, \quad \Psi_{21} = \begin{bmatrix} \mathcal{C} & \mathcal{D} & \mathcal{C} \\ \mathcal{D} & \mathcal{C} & \mathcal{C} \\ \vdots & \vdots & \vdots \\ \mathcal{E} & \mathcal{D} & \mathcal{C} \end{bmatrix},$$

$$\Psi_{11} = \begin{bmatrix} A & B & A \\ \vdots & \vdots & \vdots \\ B & A \end{bmatrix}, \quad \mathcal{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{k_p}{\tau^2} & -\frac{k_v}{\tau^2} & 1 + k_v h \end{bmatrix} \tau^2,$$

$$\mathcal{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{k_p}{\tau^2} & -\frac{k_v}{\tau^2} & 1 + k_v h \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} -\beta_1 & 1 & 0 \\ -\beta_2 & 0 & 1 \\ -\beta_3 & 0 & 0 \end{bmatrix}, \quad \mathcal{H} = \begin{bmatrix} -\beta_1 & 1 & 0 \\ -\beta_2 & 0 & 1 \\ -\beta_3 & 0 & 0 \end{bmatrix}.$$

For the closed-loop stability, we have the following theorem.

**Theorem 1** Suppose that Assumption 1 holds. Consider the system (3) under the control law (11) and (12). For any given $h > 0$, if $\beta_1 > 0$, $\beta_2 > 0$, $\beta_1 \beta_2 - \beta_3 > 0$ and $k_p > 0$, $k_v > 0$,

$$k_v > \left\{ \begin{array}{ll} \frac{((1 - k_p h^2)^2 + 4 k_p h \tau)^{1/2} - (1 + k_p h^2)}{2 h}, & \text{if } h < \tau, \\ 0, & \text{if } h \geq \tau, \end{array} \right.$$

$$0 < k_a < \left\{ \begin{array}{ll} \tau^2 \min_{\omega \in \mathbb{R}} S_n(i \omega I - \Psi)/(\tau + 1), & \text{if } N = 1, \\ \tau^2 \min_{\omega \in \mathbb{R}} S_n(i \omega I - \Psi)/(k_a h + \tau \beta_2 + 4 \tau + 4), & \text{if } N = 2, \\ \tau^2 \min_{\omega \in \mathbb{R}} S_n(i \omega I - \Psi)/(2N - 5) \min_{\omega \in \mathbb{R}} S_n(i \omega I - \Psi)^1/2, & \text{if } N \geq 3, \end{array} \right.$$

then $v_i(t) - v_0(t)$ and $e_i(t)$ both converge to zero exponentially, $i = 1, 2, ..., N$, where $\Theta = (N(1 + \tau) + (N - 1)(\tau \beta_2 + 2 \tau + 2 + k_v h) + (N - 2)(k_p + k_v + 2 k_v h + 2))/\tau^2$.

The proof of Theorem 1 is given in Appendix A.

**Remark 1** Theorem 1 shows that the control law (11) and (12) can be properly designed such that the closed-loop system is exponentially stable, so the proposed control law is naturally robust against sensing noises, i.e. under the control law (11) and (12) with bounded sensing noises in the measurements of $v_{d,i}(t)$, $e_i(t)$ and $a_i(t)$, the inter-vehicle distance errors will converge to a neighborhood of zero whose size is proportional to the bound of noise intensities.

For the $L_2$ string stability of the platoon, we have the following theorem.

**Theorem 2** Consider the system (3) under the control law (11) and (12). Let $k_p = \mu_k k$, $k_v = \mu_k k$, $k_a = \mu_k k$, $\beta_1 = 3 \omega_a$ and $\beta_2 = 3 \omega_a$, $\beta_3 = \omega_a$, where $k$, $\mu_k$, $\mu_v$, $\mu_a$, $\omega_a$ are positive parameters to be designed. For any given
Suppose that the real vehicles have the following third-order linear longitudinal dynamics

\[
\begin{aligned}
    \dot{v}_i(t) &= v_i(t), \\
    a_i(t) &= a_i(t), \\
    \dot{a}_i(t) &= -(1/\tau + \epsilon_i) a_i(t) + (1/\tau + \epsilon_i) u_i(t),
\end{aligned}
\]

where the definitions of \( u_i(t) \), \( i = 0, 1, ..., N \), are the same as in (3). The constant \( \tau \) is the known nominal inertial delay of vehicle longitudinal dynamics. The constant \( \epsilon_i \) is the parameter uncertainty caused by linearization or heterogeneity. We assume that the parameter uncertainties are bounded, i.e. there is 0 ≤ \( \tau < 1/\tau \) such that \(|\epsilon_i| \leq \tau, i = 0, 1, ..., N\).

To ensure the closed-loop stability by applying the control law (11) and (12) to (18), we have the following theorem.

**Theorem 3** Suppose that Assumption 1 holds. Consider the system (18) under the control law (11) and (12). For any given \( h > 0 \), if \( \beta_1 > 0, \beta_3 > 0, \beta_1\beta_2 - \beta_3 > 0, k_0 > 0 \),

\[
k_v > \left\{ \begin{array}{ll}
    (((1 - k_p h^2)^2 + 4 k_p h \tau)^{1/2} - (1 + k_p h^2)) / (2h), & \text{if } h < \tau, \\
    0, & \text{if } h \geq \tau,
\end{array} \right.
\]

\[
    Z_1(k_p, k_v)^2 > Z_2(k_p, k_v) - c_r(\Psi(k_p, k_v)) < 0,
\]

\[
    Z_3(k_p, k_v)^2 + Z_4(k_p, k_v) - c_r(\Psi(k_p, k_v)) < 0,
\]

\[
    Z_5(k_p, k_v)^2 < Z_6(k_p, k_v) - c_r(\Psi(k_p, k_v)) < 0,
\]

\[
0 < k_a < \left\{ \begin{array}{ll}
    \tau^2 (r_c(\Psi) - Z_1)^2 - Z_2^2 \tau^2 + Z_2 \tau^2 + 7\tau + 1, & \text{if } N = 1, \\
    \tau^2 (r_c(\Psi) - Z_3)^2 - Z_4^2 \tau^2 + Z_4 \tau^2 + k_v h + 4 \tau + 4, & \text{if } N = 2, \\
    (\tau^2(\Theta + Z_5)^2 + Z_6^2 + 4(2N - 5)Z_6^2 \tau^2 + 2\tau + 1) (\Psi - Z_5)^2 - Z_6^2 \tau^2 + 2\tau + 1, & \text{if } N \geq 3,
\end{array} \right.
\]

The proof of Theorem 2 is given in Appendix B.

### 5 Robustness against parameter uncertainties

The control law (11) and (12) is designed based on the ideal vehicle model (3) with completely known parameters. In this section, we will give the range of control parameters to ensure the closed-loop and string stabilities when there are parameter uncertainties in the vehicle model. It is shown that the control law (11) and (12) based on the ideal vehicle model can still ensure the closed-loop and string stabilities with small model parameter uncertainties.

Suppose that the real vehicles have the following third-order linear longitudinal dynamics

\[
\begin{aligned}
    \dot{v}_i(t) &= v_i(t), \\
    a_i(t) &= a_i(t), \\
    \dot{a}_i(t) &= -(1/\tau + \epsilon_i) a_i(t) + (1/\tau + \epsilon_i) u_i(t),
\end{aligned}
\]

then \( v_i(t) - v_0(t) \) and \( e_i(t) \) both converge to zero exponentially, \( i = 1, 2, ..., N \), where \( r_c(\Psi(k_p, k_v)) = \min S_n(\omega I - \Psi(k_p, k_v), Y_1 = k_h + 2, Y_2 = (k_h + 2)/\tau + \beta_2 + 2, Y_3 = k_v + 5 k_v h + 4, Y_4 = 10 + k_v + 6 k_v h + 3 \beta_2 + 5, Y_5 = N(k_h + 1) + (N - 1)(k_v + 3 k_v h + 4) + (N - 2)(k_v + k_v + 2 k_v h + 3), Y_6 = N(2 + k_v h + \beta_2 + 2) + (N - 1)(6 + k_v + 4 k_v h + \beta_2 + 2) + (N - 2)(5 + 2 k_v + 2 k_v h + 4 k_v h)/\tau \).
The proof of Theorem 3 is given in Appendix C.

Remark 2 From Theorem 3, we know that the control law (11) and (12) is robust against model parameter uncertainties. If \( \tau = 0 \), then (20) naturally holds and (21) degenerates into (14). By the continuity of (20) and (21) with respect to \( \tau \), we know that if the control law (11) and (12) is designed according to Theorem 1 based on the ideal vehicle model (3), then the closed-loop and string stabilities can still be ensured provided \( \tau \) is sufficiently small.

The control parameters can also be properly designed to ensure the \( L_2 \) string stability by applying the control law (11) and (12) to (18), we have the following theorem.

Theorem 4 Consider the system (18) under the control law (11) and (12). Let \( k_p = \mu_k k, k_v = \mu_v k, k_a = \mu_a k, \mu_k, \mu_v, \mu_a, \omega_o \) are positive parameters to be designed. For any fixed \( h > 0 \), if \( \mu_a > 0, \mu_v > 0, \mu_p > 0 \),

\[
\mu_v > \max \left\{ 4 \mu_a b^2 / (h b^2), 2 \mu_b (\tau h b^2), \mu_p b^2, \tau \mu_p \right\},
\]

\[
\omega_o > \max \left\{ -1 + \sqrt{1 - 4 \lambda_3 / (2 \lambda_1)}, 16 b^2 \mu_a \right\},
\]

\[
k \geq \max \{ \theta_1, \theta_2, \theta_3, \theta_4, \gamma_5 / \gamma_3 \},
\]

then \( \sup_{\omega \in \mathbb{R}} \left| \frac{\beta_i(\omega)}{\beta_{i-1}(\omega)} \right| \leq 1 \), where \( \beta = \frac{1}{2} + \tau, \mu = \frac{1}{2} - \tau, \rho = 1 - 4 \lambda_3 / (2 \lambda_1), 16 b^2 \mu_a, \).

The proof of Theorem 4 is given in Appendix D.

6 NUMERICAL SIMULATIONS

Suppose there are 1 leader vehicle and 5 follower vehicles with the following third-order linear longitudinal dynamics with input delays

\[
\begin{align*}
\dot{p}_i(t) & = p_i(t), \\
v_i(t) & = v_i(t), \\
\dot{a}_i(t) & = - (1/\tau + \epsilon_i) a_i(t) + (1/\tau + \epsilon_i) u_i(t - \phi),
\end{align*}
\]

where the nominal inertial delay \( \tau = 0.1 \) and the input delay \( \phi = 0.2 \) as in [Ploeg et al. (2015)]. The parameter uncertainties are given by \( \epsilon_0 = -0.8, \epsilon_1 = 0.1, \epsilon_2 = 0.5, \epsilon_3 = -0.2, \epsilon_4 = 0.65, \epsilon_5 = -0.3 \).

The control input of the leader vehicle is given by

\[
\begin{align*}
\rho_0(t) = \begin{cases} 
0.5, & 0 < t \leq 2, \\
0, & t > 2.
\end{cases}
\end{align*}
\]
The initial velocities are given by \( v_i(0) = 10 \, m/s \), \( i = 0, 1, ..., 5 \). The initial accelerations are given by \( a_i(0) = 0 \, m/s^2 \), \( i = 0, 1, ..., 5 \). The initial positions are taken as \( p_0(0) = 30 \, m \), \( p_1(0) = 24 \, m \), \( p_2(0) = 18 \, m \), \( p_3(0) = 12 \, m \), \( p_4(0) = 6 \, m \), \( p_5(0) = 0 \, m \). The standstill distance is given by \( r = 3 \, m \). In practical applications, the velocity differences between adjacent vehicles \( v_{d,i}(t) \), \( i = 1, 2, ..., N \), measured by on-board sensors are usually corrupted by random noises. In the numerical simulations, we implement the sampled-data version of the control law (11) and (12) with \( \xi(k) \) replaced by \( \xi_i(k) \), where \( \sigma = 0.005 \) is the sampling period and \( \{\xi_i(k), k = 0, 1, \ldots\} \) is a sequence of random variables with the uniform distribution \( U(-0.005, 0.005) \). Let \( h = 0.3 \, s \). We choose \( k_p = 0.2 \), \( k_v = 1.5 \), \( k_a = 0.6 \) and \( \omega_0 = 20 \), \( \beta_1 = 60 \), \( \beta_2 = 1200 \) and \( \beta_3 = 8000 \). The actual and the estimated acceleration differences between the 3rd and 4th follower vehicles are shown in Fig. 1(a). The evolution of vehicles’ accelerations and velocities are shown in Fig. 1(b) and Fig. 1(c), respectively. The evolution of inter-vehicle distance errors are shown in Fig. 1(d).

![Fig. 1. (a)](image1.png)
![Fig. 1. (b)](image2.png)
![Fig. 1. (c)](image3.png)
![Fig. 1. (d)](image4.png)

**Fig. 1.** Vehicle platoon under the control law (11) and (12) with \( h = 0.3 \, s \). (a) The actual and the estimated acceleration differences between the 3rd and 4th follower vehicles. (b) Accelerations of the vehicles. (c) Velocities of the vehicles. (d) Inter-vehicle distance errors.

From Fig. 1(a), it is shown that although the ESO (11) are designed based on the nominal \( \tau \) and the measurements of \( v_{d,i}(t) \) are corrupted by random noises, the output of the ESO \( z_{2,i}(t) \) can still track the acceleration differences between adjacent vehicles \( a_{d,i}(t) \). Due to the relatively small observer gain, the sensing noises are not significantly amplified. Fig. 1(b) shows that the accelerations of follower vehicles converge to a small neighborhood of zero as the acceleration of leader vehicle goes to zero. Fig. 1(c) shows that velocity differences between adjacent vehicles converge to a small neighborhood of zero. By Fig. 1(d), we know that the inter-vehicle distance errors converge to a small neighborhood of zero and they are not amplified in the backward propagation along the platoon.

Then let the time headway \( h \) change to 0.01s. By Fig. 2(a), we can see that the control law (11) and (12) with the previous parameters, i.e. \( k_p = 0.2 \), \( k_v = 1.5 \), \( k_a = 0.6 \), \( \beta_1 = 60 \), \( \beta_2 = 1200 \) and \( \beta_3 = 8000 \), cannot ensure the string stability any longer, but the closed-loop stability is still guaranteed. We reselect \( k_p = 0.01 \), \( k_v = 0.2 \), \( k_a = 0.8 \), \( \beta_1 = 60 \), \( \beta_2 = 1200 \) and \( \beta_3 = 8000 \). It can be seen from Fig. 2(b) that the closed-loop and string stabilities are both guaranteed by reselecting the control parameters. It is shown that for a smaller time headway \( h \), a smaller \( k_p \) can be chosen to ensure string stability at the cost of slower convergence.

![Fig. 2. (a)](image5.png)
![Fig. 2. (b)](image6.png)

**Fig. 2.** Vehicle platoon under the control law (11) and (12) with \( h = 0.01 \, s \). (a) \( k_p = 0.2 \), \( k_v = 1.5 \), \( k_a = 0.6 \), \( \beta_1 = 60 \), \( \beta_2 = 1200 \) and \( \beta_3 = 8000 \). (b) \( k_p = 0.01 \), \( k_v = 0.2 \), \( k_a = 0.8 \), \( \beta_1 = 60 \), \( \beta_2 = 1200 \) and \( \beta_3 = 8000 \).

The transient performance of the closed-loop system can be investigated from the structure of the compound controller (12) containing proportional differential feedback and feedforward terms. The convergence rate of the closed-loop system is mainly determined by the proportional differential term. The feedforward term reduces the influence of disturbances.

![Fig. 3](image7.png)

**Fig. 3.** The evolution of inter-vehicle distance errors between the leader vehicle and the first follower vehicle with different control parameters.
Let any two of the control parameters \(k_p, k_v\) and \(k_a\) be fixed and the other one be changed. The evolution of inter-vehicle distance errors between the leader and the first follower under different control parameters are shown in Fig. 3. By Fig. 3, we know that a larger \(k_p\) leads to a faster convergence and a larger \(k_v\) leads to a slower convergence with smaller inter-vehicle distance error. The parameter \(k_a\) has little effect on the convergence rate, and a larger \(k_a\) brings a smaller inter-vehicle distance error.

7 Conclusion

We have considered the platoon control for homogeneous vehicles with third-order linear dynamics model. The constant time headway spacing policy is adopted. Firstly, the distributed cooperative extended state observers are designed to estimate the acceleration differences between adjacent vehicles. Then the controller of each follower vehicle is designed by its own velocity, acceleration, velocity difference and estimated acceleration difference with respect to its immediate predecessor. The information required by the control law can be obtained by on-board sensors. The closed-loop stability of the vehicle platoon system is analyzed by the stability theory of perturbed linear systems, and the sufficient conditions to ensure the closed-loop stability are given. Also the string stability is analyzed in the frequency domain and the range of the control parameters that guarantee the string stability is presented. It is shown that for any given positive time headway, control parameters can be properly designed to guarantee both closed-loop and the string stabilities of the vehicle platoon system. In addition, it has been shown that the closed-loop and string stabilities of the vehicle platoon system can be guaranteed with small model parameter uncertainties.

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Appendix A Proof of Theorem 1

The proof of Theorem 1 needs the following lemmas.

Lemma 1 (Hinrichsen & Pritchard 1986; Gniadkowski 1993)
Suppose \(\dot{x}(t) = Ax(t)\) is exponentially stable, where \(A \in \mathbb{C}^{n \times n}\). Denote \(r_c(A) = \min_{\omega \in \mathbb{R}} S_n(\omega I - A)\). If \(\|B\| < r_c(A)\), then \(\dot{x}(t) = (A + B)x(t)\) is exponentially stable. Further, there exists \(B \in \mathbb{C}^{n \times n}\) with \(\|B\| = r_c(A)\), such that \(\dot{x}(t) = (A + B)x(t)\) is not asymptotically stable.

Lemma 2 For any \(A \in \mathbb{R}^{n \times n}\), \(\|A\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\), where \(a_{ij}\) is the element of the \(i\)th row and \(j\)th column of \(A\).

Proof. Denote

\[
\begin{aligned}
    b_{ij}^{pq} &= \begin{cases} 
        a_{ij}, & p = i, q = j, \\
        0, & \text{otherwise}.
    \end{cases} 
\end{aligned}
\]

Define \(A_{ij} = [b_{pq}^{ij}]_{n \times n}\), where \(b_{pq}^{ij}\) is the element of the \(p\)th row and \(q\)th column of \(A_{ij}\).

By (25) and the definition of \(A_{ij}\), we have

\[
A = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij}.
\]

This together with the triangle inequality of matrix norm leads to

\[
\|A\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} \|A_{ij}\|. 
\]

By the definition of the 2-norm of matrix, we have

\[
\|A_{ij}\| = \sqrt{\lambda_{\max}(A_{ij}^T A_{ij})},
\]

where \(\lambda_{\max}(A_{ij}^T A_{ij})\) is the maximum eigenvalue of \(A_{ij}^T A_{ij}\).

By (25) and the definition of \(A_{ij}\), we get

\[
(A_{ij}^T A_{ij})_{pq} = \sum_{k=1}^{n} b_{kp}^{ij} b_{kq}^{ij} = \left\{ \begin{array}{ll}
    a_{ij}^2, & p = j, q = j, \\
    0, & \text{otherwise}.
\end{array} \right.
\]

This together with (27) gives \(\|A_{ij}\| = \|a_{ij}\|\). Then by (26), we get \(\|A\| \leq \sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|\). □

Proof of Theorem 1. Denote

\[
X_i(t) = [p_i(t), v_i(t), a_i(t)]^T, \quad i = 0, 1, 2, ..., N, 
\]

\[
E_i(t) = [e_{1,i}(t), e_{2,i}(t), e_{3,i}(t)]^T, \quad i = 1, 2, ..., N, 
\]

\[
F_i(t) = [e_i(t), v_d,i(t), a_d,i(t)]^T, \quad i = 1, 2, ..., N, 
\]

\[
W(t) = [F_1^T(t), F_2^T(t), \ldots, F_N^T(t), E_1^T(t), E_2^T(t), \ldots, E_N^T(t)]^T,
\]

\[
\Delta(t) = [\xi_1(t), 0, \ldots, 0, \xi_2(t), \xi_2(t), 0, \ldots, 0]^T,
\]

where

\[
e_{1,i}(t) = z_{1,i}(t) - v_{d,i}(t),
\]

\[
e_{2,i}(t) = z_{2,i}(t) - a_{d,i}(t),
\]

\[
e_{3,i}(t) = z_{3,i}(t) - q_i(t),
\]

\[
\delta_i(t) = [0, a_0(t) - k_a a_0(t)/\tau]^T,
\]

\[
\xi_1(t) = e_{1,i}(t), \quad \xi_2(t) = e_{2,i}(t), \quad \xi_3(t) = e_{3,i}(t).
\]
\(\zeta_1(t) = \left[0, 0, (u_0(t) - (1 + k_a) a_0(t) - \tau \dot{a}_0(t))/\tau^2\right]^T,\)  
(37)

\(\zeta_2(t) = \left[0, 0, (k_a a_0(t) - \tau k_a a_0(t) - k_a k_v h a_0(t) - \tau k_a \dot{a}_0(t))/\tau^2\right]^T,\)  
(38)

\(\zeta_3(t) = \left[0, 0, k_a^2 a_0(t)/\tau^2\right]^T.\)  
(39)

From (8), (12) and (34), we know

\[u_i(t) = k_p c_i(t) + k_v (v_{d,i}(t) - h a_i(t)) + k_a (a_i(t) - c_2(t)), \quad i = 1, 2, \ldots, N.\]  
(40)

This together with (3), (4), (5), (7), (28) and (29) leads to

\[
\dot{X}_i(t) = \begin{cases} 
A_0 X_i(t) + B_0 u_i(t), & i = 0, \\
A X_i(t) + B X_{i-1}(t) + C E_i(t) + L r, & i = 1, \ldots, N,
\end{cases}
\]  
(41)

where

\[
A_0 = \begin{bmatrix} 0 & 1 & 0 \\
0 & 0 & 1 \\
0 & -\frac{1}{\tau} \end{bmatrix}, B_0 = \begin{bmatrix} 0 \\
0 \\
\frac{1}{\tau} \end{bmatrix}, L = \begin{bmatrix} 0 \\
0 \\
-k_p \end{bmatrix},
\]

\[
A = \begin{bmatrix} 0 & 1 & 0 \\
k_p & -k_v & 1 \\
-k_v \tau & -k_v & 1 \\
\frac{1}{\tau} \end{bmatrix}, B = \begin{bmatrix} 0 \\
0 \\
k_v \\
\frac{k_a \tau}{\tau} \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
k_a \\
\frac{k_a}{\tau} \end{bmatrix},
\]

From (4), (5), (28) and (30), we get

\[
F_i(t) = P X_{i-1}(t) - Q X_i(t) - L_1 r, \quad i = 1, 2, \ldots, N,\]  
(42)

where

\[
P = \begin{bmatrix} 1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \end{bmatrix}, Q = \begin{bmatrix} 1 & h & 0 \\
0 & 1 & 0 \\
0 & 0 & -1 \end{bmatrix}, L_1 = \begin{bmatrix} 1 \\
0 \\
0 \end{bmatrix}.
\]

From (36), (41) and (42), we obtain

\[
\dot{E}_i(t) = \begin{cases} 
AF_i(t) + GE_i(t) + \delta_i(t), & i = 1, \\
AF_i(t) + B_1 F_{i-1}(t) + GE_i(t), & i = 2, \ldots, N,
\end{cases}
\]  
(43)

where

\[
B_1 = \begin{bmatrix} 0 & 0 & 1 \\
0 & \frac{k_a}{\tau} \end{bmatrix}, G = \begin{bmatrix} 0 & 0 & 0 \\
0 & \frac{k_a}{\tau} & 0 \end{bmatrix}.
\]

By (4)-(11), (29), (30), (33)-(40) and (43), we have

\[
\dot{E}_i(t) = \begin{cases} 
CF_i(t) + H_1 E_i(t) + \zeta_i(t), & i = 1, \\
CF_i(t) + D_1 F_{i-1}(t) + H_1 E_i(t) + J E_{i-1}(t) + \zeta_i(t), & i = 2, \\
CF_i(t) + D_1 F_{i-1}(t) + \varepsilon_1 F_{i-2}(t) + H_1 E_i(t) + J E_{i-2}(t) + \zeta_i(t), & i = 3, \\
CF_i(t) + D_1 F_{i-1}(t) + \varepsilon_1 F_{i-2}(t) + J F_{i-3}(t) + H_1 E_i(t) + J E_{i-1}(t) + J E_{i-2}(t), & i = 4, \ldots, N,
\end{cases}
\]  
(44)

where

\[
H_1 = \begin{bmatrix} -\beta_1 & 1 & 0 \\
-\beta_2 & 0 & 1 \\
-\beta_3 & -k_a & 0 \end{bmatrix}, F = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & k_a^2 \end{bmatrix},
\]

\[
J = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
-\frac{k_a^2}{\tau} & 0 \end{bmatrix}, I = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
\beta_2 k_a & (1 + k_v) k_a & -k_a \end{bmatrix}.
\]

\[
D_1 = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
\frac{k_v k_a}{\tau^2} & \frac{k_v^2}{\tau^2} & -2 k_v h - \frac{k_v^2 h^2}{\tau^2} - \frac{k_a}{\tau} \end{bmatrix},
\]

\[
E_1 = \begin{bmatrix} 0 & 0 & 0 \\
0 & 0 & 0 \\
-k_v k_a & -k_v & 2(1 + k_v) k_a \end{bmatrix}.
\]
From (31), (32), (43) and (44), we know
\[ W(t) = (\Psi + \hat{\Psi})W(t) + \Delta(t), \quad (45) \]
where \( \Psi = \begin{bmatrix} \hat{\Psi}_{11} & \hat{\Psi}_{12} \\ \hat{\Psi}_{21} & \hat{\Psi}_{22} \end{bmatrix} \) and
\[ \hat{\Psi}_{11} = \begin{bmatrix} 0 & B_2 \\ \vdots & \vdots \\ B_2 & 0 \end{bmatrix}, \hat{\Psi}_{22} = \begin{bmatrix} H_2 & I & H_2 \\ J & I & H_2 \\ J & I & H_2 \end{bmatrix}, \]
\[ \hat{\Psi}_{21} = \begin{bmatrix} 0 & D_2 \\ \vdots & \vdots \\ D_2 & 0 \end{bmatrix}, \hat{\Psi}_{12} = \begin{bmatrix} \mathcal{H}_2 \\ J \mathcal{I} \mathcal{H}_2 \\ J \mathcal{I} \mathcal{H}_2 \end{bmatrix}, \]

\[ B_2 = B_1 - B, D_2 = D_1 - D, E_2 = E_1 - E, \mathcal{H}_2 = \mathcal{H}_1 - \mathcal{H}. \]

Firstly, we analyze the stability of \( \Psi \). The eigenvalues of \( \Psi \) are only related to \( A \) and \( \mathcal{H} \). Calculating the characteristic polynomial of \( A \), we obtain
\[ |sI - A| = s^3 + \left( \frac{1 + k_v h}{\tau} \right) s^2 + \left( \frac{k_v + k_p h}{\tau} \right) s + \frac{k_p}{\tau}. \quad (46) \]

The Rouse table corresponding to (46) is given by
\[ s^3 \quad 1 \quad \beta_2 \\ s^2 \quad \beta_1 \quad \beta_3 \\ s^1 \quad \beta_1 \beta_2 - \beta_3 \\ s^0 \quad \beta_1 \quad \beta_3 \]

By \( \beta_1 > 0, \beta_3 > 0, \beta_1 \beta_2 - \beta_3 > 0 \), we know that the elements of the first column of the Rouse table corresponding to (47) are all greater than zero. From Rouse criterion, \( \mathcal{A} \) is stable. Then \( \Psi \) is stable. From the definition of \( \|\hat{\Psi}\| \) and Lemma 2, we know
\[ \|\hat{\Psi}\| \leq \begin{cases} k_n(1 + \tau)/\tau^2, & \text{if } N = 1, \\ k_n(k_v h + \tau \beta_2 + 4\tau + 4)/\tau^2, & \text{if } N = 2, \\ (2N - 5)k_n^2/\tau^2 + \Theta k_n, & \text{if } N \geq 3. \end{cases} \quad (48) \]

From (14) and (48), we get \( \|\hat{\Psi}\| < \min_{\omega \in \mathbb{R}} S_n(i\omega I - \Psi) \). It is known from the definition of \( \Delta(t) \) and Assumption 1 that \( \lim_{t \to \infty} \Delta(t) = 0 \). By Lemma 1 and (45), we know \( W(t) \) converges to zero exponentially. Then \( v_{i-1}(t) - v_i(t) \) and \( \dot{v}_i(t) \) both converge to zero exponentially, \( i = 1, 2, ..., N \), which implies \( v_{i+1}(t) - v_i(t) \) converges to zero exponentially, \( i = 1, 2, ..., N \). \( \square \)

Appendix B   Proof of Theorem 2

**Proof of Theorem 2.** By (4), (5) and (7), we get
\[ a_i(t) = \frac{v_{d,i}(t) - \dot{e}_i(t)}{h}. \quad (49) \]

Taking the Laplace transform of (49), we have
\[ \mathcal{A}_i(s) = \frac{\mathcal{F}_{d,i}(s) - se_i(s)}{h}, \quad (50) \]

where \( \mathcal{A}_i(s) \) and \( \mathcal{F}_{d,i}(s) \) are the Laplace transform of \( a_i(t), v_{d,i}(t) \), respectively. From (3), we know
\[ u_i(t) = \tau \dot{a}_i(t) + a_i(t). \quad (51) \]

This together with (49) leads to
\[ u_i(t) = \tau \frac{\dot{v}_{d,i}(t) - \dot{e}_i(t)}{h} + \frac{v_{d,i}(t) - \dot{e}_i(t)}{h}. \quad (52) \]

Taking the Laplace transform of (52), we get
\[ \mathcal{U}_i(s) = \tau \frac{s\mathcal{F}_{d,i}(s) - s^2\dot{e}_i(s)}{h} + \frac{\mathcal{F}_{d,i}(s) - se_i(s)}{h}, \quad (53) \]
where \( \mathcal{L}(s) \) is the Laplace transform of \( u(t) \). Taking the Laplace transform of (11), we have

\[
\begin{align*}
\mathcal{L}_1(s) &= \mathcal{L}_2(s) + \beta_1(\mathcal{L}_d(s) - \mathcal{L}_1(s)), \\
\mathcal{L}_2(s) &= \mathcal{L}_3(s) + \beta_2(\mathcal{L}_d(s) - \mathcal{L}_1(s)) - \frac{1}{\tau} \mathcal{L}_1(s), \\
\mathcal{L}_3(s) &= \beta_3(\mathcal{L}_d(s) - \mathcal{L}_1(s)),
\end{align*}
\]

(54)

where \( \mathcal{L}_1(s) \), \( \mathcal{L}_2(s) \) and \( \mathcal{L}_3(s) \) are the Laplace transform of \( z_1(t) \), \( z_2(t) \) and \( z_3(t) \), respectively. Substituting (53) into (54), we obtain

\[
\mathcal{L}_2(s) = \left( (-s^3 + (h \beta_2 \tau - \beta_3 \tau - 1)s^2/\tau + (h \beta_3 \tau - \beta_1)s/\tau) \mathcal{L}_d(s) + (s^4 + (\beta_1 + 1)s^3/\tau + \beta_4s^2/\tau) \mathcal{L}_1(s) / (h(s^2 + \beta_1 s^2 + \beta_2 s + \beta_3)) \right)
\]

(55)

By (12), (49) and (51), we get

\[
\tau \bar{a}_i(t) + a_i(t) = k_p \bar{c}_i(t) + k_c \bar{e}_i(t) + k_a (z_{2,i}(t) + a_i(t)).
\]

(56)

Taking the Laplace transform of (56), we have

\[
\tau s \mathcal{L}(s) + \mathcal{L}(s) = k_p \mathcal{L}(s) + k_c \mathcal{L}_1(s) + k_a \mathcal{L}_2(s)
\]

(57)

Denote \( H(s) = \frac{\mathcal{L}(s)}{\mathcal{L}_1(s)} \). By (50), (102) and (57), we get

\[
H(s) = \frac{-\tau s^5 + n_4s^4 + n_3s^3 + n_2s^2 + n_1 + \alpha_0}{-\tau s^5 + n_4s^4 + n_3s^3 + n_2s^2 + n_1 + \alpha_0},
\]

(58)

where

\[
\begin{align*}
\alpha_0 &= -k_p h \beta_3, \\
n_0 &= -k_p h \beta_2, \\
n_1 &= -k_p h \beta_2 - (1 - \alpha_k + n_k h) \beta_3, \\
n_2 &= -k_p h + \alpha_k / \tau - (1 - \alpha_k + n_k h) \beta_2 - \tau \beta_3, \\
n_3 &= -k_p h - \alpha_k / \tau - (1 - \alpha_k + n_k h) \beta_1 - \tau \beta_2, \\
n_4 &= -1 - k_v h - \tau \beta_1, \\
\beta_0 &= (k_a - 1) \beta_3, \\
\beta_1 &= -k_a \beta_1 / \tau - (1 - \alpha_k + n_k h) \beta_2 - (1 - \alpha_k + n_k h) \beta_3, \\
\beta_2 &= -k_p h - \alpha_k / \tau - \tau \beta_3 - \alpha_k / \tau, \\
\beta_3 &= -\tau \beta_1 - 1.
\end{align*}
\]

By (4), (5) and (7), we get

\[
\bar{e}_{i-1}(t) - \bar{e}_i(t) = v_{d,i-1}(t) - v_{d,i}(t) - h \bar{v}_{d,i}(t).
\]

(59)

Denote \( G_c(s) = \frac{\mathcal{L}(s)}{\mathcal{L}_1(s)} \). Taking the Laplace transform of (59), we have

\[
s \mathcal{L}_1(s) - s \mathcal{L}_1(s) = \mathcal{L}_d(s) - \mathcal{L}_1(s) - hs \mathcal{L}_d(s).
\]

This together with \( \mathcal{L}_d(s) = H(s) \mathcal{L}_1(s) \) leads to

\[
G_c(s) = \frac{s - H(s)}{s - (hs + 1)H(s)}.
\]

This together with (58) leads to

\[
G_c(s) = \frac{(k_v s^4 + \omega_0^2 s^2 + \omega_0^4 s + k_p \beta_3) / (\tau s^6)}{\tau s^5 + \tau s^4 + \tau s^3 + \tau s^2 + \tau s + k_p \beta_3},
\]

(60)

where

\[
\begin{align*}
\omega_0 &= k_p h \beta_3, \\
\omega_1 &= k_p h \beta_2, \\
\omega_2 &= k_p h + k_v h, \\
\omega_3 &= k_v h + k_a \beta_2 + k_p, \\
\omega_4 &= k_v h + k_a \beta_2 + k_p h + k_v h + k_a / \tau + k_v,
\end{align*}
\]

\[
\omega_5 = \tau \beta_1 + k_v h + 1.
\]

Substituting \( s = j \omega \) into (60), we get

\[
G_c(j \omega) = \frac{x_n(\omega) + y_n(\omega) j}{x_d(\omega) + y_d(\omega) j},
\]

(61)

where

\[
\begin{align*}
x_n(\omega) &= k_p h \beta_3 - \tau \omega^2 + k_v h, \\
y_n(\omega) &= \tau \omega^2 - \tau \omega^2 - \tau \omega^2 - \tau \omega^2 - \tau \omega^2 + \omega_4 h, \\
x_d(\omega) &= k_p h \beta_3 - \tau \omega^2 + \tau \omega^2 + \tau \omega^2 + \tau \omega^2 + \tau \omega^2 + \tau \omega^2 + \omega_4 h.
\end{align*}
\]

By \( \mu_p > 0 \) and \( \mu_a > 0 \), we know \( \alpha_5 > 0 \) and \( \gamma_5 > 0 \). From (17), we obtain \( k \geq \gamma_5 / \alpha_5 \). This together with \( \alpha_5 > 0 \) and \( \gamma_5 > 0 \) leads to

\[
\alpha_5 k^2 - \gamma_5 k \geq 0.
\]

(62)

From (15) and \( \mu_a > 0 \), we know \( \mu_v > \mu_a / h \). By (16), we know \( \omega_v > \theta_\mu \). This together with \( \mu_v > \mu_a / h \) leads to \([h^2 \mu^2_\omega - \mu_\omega^2] \omega^2_\omega - \lambda_4 \omega^4_\omega > 0 \). By (16), we know \( \omega_v > 16 \mu_a / (3 \tau h^2 \mu_p) \). This together with \( \mu_p > 0 \) and \( \mu_a > 0 \) leads to \((3h^2 \mu^2_\omega - 16 \mu_a \tau h_p / \tau) \omega^2_\omega > 0 \). So we get \( \alpha_4 > 0 \). From (17), we know \( k \geq \theta_4 \). This together with \( \alpha_4 > 0 \) and \( \rho_4 > 0 \) leads to

\[
\alpha_4 k^2 + \gamma_4 k + \rho_4 \geq 0.
\]

(63)

By (15), we know \( \mu_v > \sqrt{3} \mu_a / h \). This together with \( \mu_a > 0 \) leads to \( \lambda_1 > 0 \). By (15), we get \( \mu_v > (\tau - 2h) \mu_p / 2 \). This leads to \( 12h \mu_p + 12 \mu_c - 6 \mu_p \tau > 0 \). This together with \( \mu_v > 0 \) leads to \( \lambda_1 > 0 \). By (16), we know \( \omega_v > \theta_\lambda \). This together with \( \alpha_1 > 0 \) and \( \alpha_3 > 0 \) leads to \( \alpha_3 > 0 \).
From (17), we know $k \geq \theta_3$. This together with $\alpha_3 > 0$ and $\rho_3 > 0$ leads to

$$\alpha_3 k^2 + \gamma_3 k + \rho_3 \geq 0. \quad (64)$$

By (15) and $\mu_a > 0$, we know $\mu_a > 0$. This together with $\mu_p > 0$ and $\mu_a = 0$ leads to $\alpha_2 > 0$. From (17), we obtain $k \geq \theta_2$. This together with $\alpha_2 > 0$ and $\rho_2 > 0$ leads to

$$\alpha_2 k^2 + \gamma_2 k + \rho_2 \geq 0. \quad (65)$$

From (17), we obtain $k \geq \theta_1$. This together with $\alpha_1 > 0$ and $\rho_1 > 0$ leads to

$$\alpha_1 k^2 + \gamma_1 k + \rho_1 \geq 0. \quad (66)$$

By (62)-(66), we know

$$(\alpha_5 k^2 - \gamma_5 k)\omega^2 + (\alpha_4 k^2 + \gamma_4 k + \rho_4)\omega^4 + (\alpha_3 k^2 + \gamma_3 k + \rho_3)\omega^6 + (\alpha_2 k^2 + \gamma_2 k + \rho_2)\omega^8 + (\alpha_1 k^2 + \gamma_1 k + \rho_1)\omega^{10} + \tau^2 \omega^{12} \geq 0, \forall \omega \in \mathbb{R}. \quad (67)$$

Through calculation, we get

$$
\begin{align*}
\alpha_5 k^2 - \gamma_5 k &= 2k_p \beta_3 \phi_2 + \phi_1^2 - 2k_p \beta_3 \phi_2 - \phi_1^2, \\
\alpha_4 k^2 + \gamma_4 k + \rho_4 &= 2\phi_2 \phi_3 + 2k_p \beta_3 \phi_4 + \phi_2^2 - \phi_1^2, \\
\alpha_3 k^2 + \gamma_3 k + \rho_3 &= 2\phi_2 \phi_4 + 2\phi_1 \phi_5 + \phi_2^2 - \phi_1^2, \\
\alpha_2 k^2 + \gamma_2 k + \rho_2 &= \phi_2^2 + 2\phi_2 \phi_5 - k_5^2 - 2\phi_3 \phi_5, \\
\alpha_1 k^2 + \gamma_1 k + \rho_1 &= \phi_3^2 - 2\phi_4 \phi_5.
\end{align*}
$$

This together with (67) leads to

$$(2k_p \beta_3 \phi_2 + \phi_1^2 - 2k_p \beta_3 \phi_2 - \phi_1^2)\omega^2 + (2\phi_2 \phi_3 + 2k_p \beta_3 \phi_4 + \phi_2^2 - \phi_1^2)\omega^4 + (2\phi_2 \phi_4 + 2\phi_1 \phi_5 + \phi_2^2 - \phi_1^2)\omega^6 + (\phi_2^2 + 2\phi_2 \phi_5 - k_5^2 - 2\phi_3 \phi_5)\omega^8 + (\phi_3^2 - 2\phi_4 \phi_5)\omega^{10} + \tau^2 \omega^{12} \geq 0, \forall \omega \in \mathbb{R}. \quad (68)$$

Through calculation, we know

$$\begin{align*}
x_d^2(\omega) + y_d^2(\omega) - x_n^2(\omega) - y_n^2(\omega) &= (2k_p \beta_3 \phi_2 + \phi_1^2 - 2k_p \beta_3 \phi_2 - \phi_1^2)\omega^2 + (2\phi_2 \phi_3 + 2k_p \beta_3 \phi_4 + \phi_2^2 - \phi_1^2)\omega^4 + (2\phi_2 \phi_4 + 2\phi_1 \phi_5 + \phi_2^2 - \phi_1^2)\omega^6 + (\phi_2^2 + 2\phi_2 \phi_5 - k_5^2 - 2\phi_3 \phi_5)\omega^8 + (\phi_3^2 - 2\phi_4 \phi_5)\omega^{10} + \tau^2 \omega^{12},
\end{align*}$$

This together with (68) leads to

$$x_d^2(\omega) + y_d^2(\omega) - x_n^2(\omega) - y_n^2(\omega) \geq 0, \forall \omega \in \mathbb{R}. \quad (69)$$

By (69), we know

$$\frac{x_d^2(\omega) + y_d^2(\omega)}{x_d^2(\omega) + y_d^2(\omega)} \leq 1, \forall \omega \in \mathbb{R}. \quad (70)$$

Through calculation, we obtain

$$\left| \frac{x_n(\omega) + y_n(\omega)j}{x_d(\omega) + y_d(\omega)j} \right| = \frac{\sqrt{x_d^2(\omega) + y_d^2(\omega)}}{\sqrt{x_d^2(\omega) + y_d^2(\omega)}},$$

This together with (70) leads to

$$\left| \frac{x_n(\omega) + y_n(\omega)j}{x_d(\omega) + y_d(\omega)j} \right| \leq 1, \forall \omega \in \mathbb{R}. \quad (71)$$

From (61) and (71), we know $|G_e(j\omega)| \leq 1$ for any $\omega \in \mathbb{R}$. That is $\left| \frac{E_i(t)}{E_i(t)} \right| \leq 1$ for any $\omega \in \mathbb{R}$. \qed

**Appendix C  Proof of Theorem 3**

**Proof of Theorem 3.** For simplicity of presentation, we denote $b_i = \frac{1}{\tau} + \epsilon_i$. According to (18), the models of the velocity difference between adjacent vehicles are given by

$$\begin{align*}
\dot{v}_{d,i}(t) &= a_{d,i}(t), \\
\dot{a}_{d,i}(t) &= q_{1,i}(t) - \frac{1}{\tau} u_i(t), \quad i = 1, 2, ..., N, \\
q_{1,i}(t) &= u_{1,i}(t),
\end{align*} \quad (72)$$

where the definitions of $a_{d,i}(t)$ and $v_{d,i}(t)$ are the same as in (6),

$$\begin{align*}
q_{1,i}(t) &= -b_{i-1} a_{i-1}(t) + b_{i-1} u_{i-1}(t) + b_i a_i(t) - \epsilon_i u_i(t), \\
w_{1,i}(t) &= b_{i-1}^2 a_{i-1}(t) - b_{i-1}^2 u_{i-1}(t) + b_{i-1} a_{i-1}(t) - b_{i}^2 a_i(t) + b_i^2 u_i(t) - \epsilon_i \hat{u}_i(t). \quad (73)
\end{align*}$$

Denote

$$\begin{align*}
X_i(t) &= [p_i(t), v_i(t), a_i(t)]^T, \quad i = 0, 1, 2, ..., N, \\
E_i(t) &= [e_{1,i}(t), e_{2,i}(t), e_{3,i}(t)]^T, \quad i = 1, 2, ..., N, \\
F_i(t) &= [e_{i}(t), v_{d,i}(t), a_i(t)]^T, \quad i = 1, 2, ..., N, \\
W(t) &= [F_1^T(t), F_2^T(t), \ldots, F_N^T(t), E_1^T(t), E_2^T(t), \ldots, E_N^T(t)]^T, \\
\Delta(t) &= [\delta_1^T(t), 0, \ldots, 0, \zeta_1^T(t), \zeta_2^T(t), \zeta_3^T(t), 0, \ldots, 0]^T. \quad (78)
\end{align*}$$
From (8), (12) and (81), we know
\[ e_{1,i}(t) = z_{1,i}(t) - v_{d,i}(t), \]
\[ e_{2,i}(t) = z_{2,i}(t) - a_{d,i}(t), \]
\[ e_{3,i}(t) = z_{3,i}(t) - q_i(t), \]
\[ \delta_i(t) = [0, a_0(t), b_k a_0(t)]^T, \]
\[ \zeta_i(t) = [0, 0, -b_0 a_0(t) + b_0 u_0(t) + b_1 k a_0(t) - k_v e_1 a_0(t) + k_v k_1 \epsilon_1 b_1 a_0(t) + e_1 k_1 a_0(t)]^T, \]
\[ \zeta_2(t) = [0, 0, b_0^2 k a_0(t) + k_0^2 e_2 b_1 a_0(t) - k_v b_1 a_0(t) + k_a h e_1 b_1 a_0(t) - b_1 k_v a_0(t)]^T, \]
\[ \zeta_3(t) = [0, 0, k_0^2 b_1 b_2 a_0(t)]^T. \]

From (83), (88) and (89), we obtain
\[ \dot{F}_i(t) = \begin{cases} A_{1i} F_i(t) + G_i E_i(t) + \delta_i(t), & i = 1, \\ A_{1i} F_i(t) + B_{1i} F_{i-1}(t) + G_i E_i(t), & i = 2, \ldots, N, \end{cases} \]
where
\[ B_{1i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b_i k_a & 0 \end{bmatrix}, G_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \]

By (4), (5), (7), (8), (11), (18), (73)-(77), (80)-(87) and (90), we have
\[ \dot{H}_{1i} = \begin{bmatrix} -\beta_1 & 1 & 0 \\ -\beta_2 & 0 & 1 \\ -\beta_3 - k_a \beta_2 \epsilon_i & -k_a b_i^2 - k_a h b_i \epsilon_i & k_a \epsilon_i \end{bmatrix}, \]
\[ \mathcal{F}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_0^2 b_{i-1} b_{i-2} & 0 \end{bmatrix}, \mathcal{J}_i = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_0^2 b_{i-1} b_{i-2} & 0 \end{bmatrix}, \]
\[ \mathcal{C}_{1i} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -k_p k_v h \beta_i \epsilon_i & k_p \epsilon_i - k_i b_i^2 - k_v h b_i^2 - k_a \epsilon_i \end{bmatrix}. \]
From (78), (79), (90) and (91), we know
\[
\hat{\Psi} = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & k_p h b_{i-1} + k_v b_{i-1} \\
k_p k_v b_{i-1} + k_p k_v b_{i-1} + k_v k_v b_{i-1} - k_v k_v b_{i-1} - 2 k_v b_{i-1} \\
0 & 0 & 0 \\
0 & 0 & k_v k_v b_{i-1} - k_v k_v b_{i-1} - k_v k_v b_{i-1} - k_v k_v b_{i-1} - 2 k_v b_{i-1} \\
\end{bmatrix}
\]

Then \(A_{2i} = A_{1i} - A, B_{2i} = B_{1i} - B, C_{2i} = C_{1i} - C\),
\(D_{2i} = D_{1i} - D, E_{2i} = E_{1i} - E, H_{2i} = H_{1i} - H\).

Firstly, we analyze the stability of \(\Psi\). The eigenvalues of \(\Psi\) are only related to \(A\) and \(H\). Calculating the characteristic polynomial of \(A\), we obtain
\[
|sI - A| = s^3 + \left(\frac{1 + k_v h}{\tau}\right) s^2 + \left(\frac{k_v + k_p h}{\tau}\right) s + \frac{k_p}{\tau}.
\]

The Rouse table corresponding to (93) is given by
\[
s^3 \quad 1 \quad \frac{k_v + k_p h}{\tau} \\
s^2 \quad 1 + k_v h \quad \frac{k_v}{\tau} \\
s^1 \quad h k_v^2 + (1 + h^2 k_v) k_v + (h - \tau) k_p \quad 0 \\
s^0 \quad \frac{k_v}{\tau} \\
\]

By \(k_p > 0\) and (19), we know that the elements of the first column of the Rouse table corresponding to (93) are all greater than zero. From Rouse criterion, \(A\) is stable. Calculating the characteristic polynomial of \(H\), we get
\[
|sI - H| = s^3 + \beta_1 s^2 + \beta_2 s + \beta_3.
\]

The Rouse table corresponding to (94) is given by
\[
s^3 \quad 1 \quad \beta_2 \\
s^2 \quad \beta_1 \quad \beta_3 \\
s^1 \quad \beta_2 - \beta_3 \quad 0 \\
s^0 \quad \beta_3 \\
\]

By \(\beta_1 > 0, \beta_3 > 0, \beta_4 \beta_2 - \beta_3 > 0\), we know that the elements of the first column of the Rouse table corresponding to (94) are all greater than zero. From Rouse criterion, \(H\) is stable. Then \(\Psi\) is stable.

From the definition of \(\|\hat{\Psi}\|\) and Lemma 2, we know
\[
\|\hat{\Psi}\| \leq \begin{cases}
\left((1 + \tau^2 + Y_1 \tau^2 + Y_2 \tau) k_a + Z_1 \tau^2 + Z_2 \tau, \right. \\
+ Z_3 \tau^2 + Z_4 \tau, \\
\left.\frac{((k_b h + \tau \beta_2 + 4 \tau + 4) \tau^2 + Y_5 \tau^2 + Y_4 \tau)}{\tau} k_a + \frac{Z_5 \tau^2 + Z_6 \tau}{\tau}, \right.
\end{cases}
\]

if \(N = 1, 2, 3\), respectively.

From (20), (21) and (95), we get \(\|\hat{\Psi}\| < r_c(\Psi)\). It is known from the definition of \(\Delta(t)\) and Assumption 1 that

\[14\]
\[
\lim_{t \to \infty} \Delta(t) = 0.\] By Lemma 1 and (92), we know \(W(t)\) converges to zero exponentially. Then \(v_i(t) - \nu_0(t)\) and \(e_i(t)\) both converge to zero exponentially, \(i = 1, 2, \ldots, N\). \(\Box\)

**Appendix D  Proof of Theorem 4**

**Proof of Theorem 4.** For simplicity of presentation, we denote \(b_i = \frac{1}{\tau} + \epsilon_i\). By (4), (5) and (7), we get
\[
a_i(t) = \frac{v_{d,i}(t) - \dot{e}_i(t)}{h}. \tag{96}
\]
Taking the Laplace transform of (96), we have
\[
\mathcal{A}_i(s) = \frac{\mathcal{Y}_{d,i}(s) - s \mathcal{E}_i(s)}{h}, \tag{97}
\]
where \(\mathcal{A}_i(s)\) and \(\mathcal{Y}_{d,i}(s)\) are the Laplace transform of \(a_i(t), v_{d,i}(t)\), respectively. From (18), we know
\[
u_i(t) = \frac{\dot{a}_i(t)}{b_i} + a_i(t). \tag{98}
\]
This together with (96) leads to
\[
u_i(t) = \frac{\dot{e}_i(t) + v_{d,i}(t) - \dot{e}_i(t)}{b_i h} + \frac{v_{d,i}(t) - \dot{e}_i(t)}{h}. \tag{99}
\]
Taking the Laplace transform of (99), we get
\[
\mathcal{Y}_i(s) = \frac{s \mathcal{Y}_{d,i}(s) - s^2 \mathcal{E}_i(s)}{b_i h} + \frac{\mathcal{Y}_{d,i}(s) - s \mathcal{E}_i(s)}{h}, \tag{100}
\]
where \(\mathcal{Y}_i(s)\) is the Laplace transform of \(u_i(t)\). Taking the Laplace transform of (11), we have
\[
\begin{cases}
s \mathcal{Z}_{1,i}(s) = \mathcal{Z}_{2,i}(s) + \beta_1(\mathcal{Y}_{d,i}(s) - \mathcal{Z}_{1,i}(s)), \\
s \mathcal{Z}_{2,i}(s) = \mathcal{Z}_{3,i}(s) + \beta_2(\mathcal{Y}_{d,i}(s) - \mathcal{Z}_{1,i}(s)) - \frac{1}{\tau} \mathcal{Y}_i(s), \\
s \mathcal{Z}_{3,i}(s) = \beta_3(\mathcal{Y}_{d,i}(s) - \mathcal{Z}_{1,i}(s)),
\end{cases} \tag{101}
\]
where \(\mathcal{Z}_{1,i}(s), \mathcal{Z}_{2,i}(s)\) and \(\mathcal{Z}_{3,i}(s)\) are the Laplace transform of \(z_{1,i}(t), z_{2,i}(t)\) and \(z_{3,i}(t)\), respectively. Substituting (100) into (101), we obtain
\[
\mathcal{Z}_{2,i}(s) = \frac{(-s^3 + (\tau b_i h \beta_2 - \beta_3 - b_i) s^2 + (\tau b_i h \beta_3 - b_i \beta_1) s) \mathcal{Y}_{d,i}(s) + (s^4 + (\beta_1 + b_i) s^3 + b_i \beta_1 s^2) \mathcal{E}_i(s)}{(\tau b_i h (s^3 + \beta_1 s^2 + \beta_2 s + \beta_3))}. \tag{102}
\]
By (12), (96) and (98), we get
\[
\frac{\dot{a}_i(t)}{b_i} + a_i(t) = k_p e_i(t) + k_v \dot{e}_i(t) + k_a (z_{2,i}(t) + a_i(t)). \tag{103}
\]
Taking the Laplace transform of (103), we have
\[
\frac{s \mathcal{A}_i(s)}{b_i} + \mathcal{A}_i(s) = k_p \mathcal{E}_i(s) + k_v s \mathcal{E}_i(s) + k_a \mathcal{A}_i(s) + k_a \mathcal{Z}_{2,i}(s). \tag{104}
\]
Denote \(H_i(s) = \frac{\mathcal{Y}_i(s)}{\mathcal{E}_{i-1}(s)}\). By (97), (102) and (104), we get
\[
H_i(s) = \frac{s^5 + n_{44} s^4 + n_{33} s^3 + n_{22} s^2 + n_{11} s + n_{00}}{-s^4 + d_{31} s^3 + d_{21} s^2 + d_{11} s + d_{01}}, \tag{105}
\]
where
\[
\begin{align*}
n_{00} &= -b_j k_p \beta_3, \\
n_{11} &= -b_j k_p \beta_2 - (b_j + b_k h - b_i k_a) \beta_3, \\
n_{21} &= -(b_j k_p h + b_j k_a / \tau) \beta_1 - (b_j + b_k h - b_i k_a) \beta_2 - \beta_3, \\
n_{31} &= -(b_j k_p h - b_i k_a / \tau - (b_j + b_k h) k_a / \tau - b_i k_a) \beta_1 - \beta_2, \\
n_{44} &= -b_j - b_i k_a / \tau + b_j k_a - b_k h - \beta_1, \\
&= (b_j k_a - b_i \beta_3, \\
d_{11} &= -(b_j k_a \beta_1 / \tau - (b_j - b_i k_a) \beta_2 - (1 - b_i k_a) \beta_3, \\
d_{21} &= -(b_j - b_i k_a + k_a / \tau) \beta_1 - (1 - b_i k_a) \beta_2 - b_i k_a / \tau, \\
d_{31} &= -(b_j - b_i k_a / \tau + b_j k_a) - \beta_1. \\
\end{align*}
\]
By (4), (5) and (7), we get
\[
\dot{e}_{i-1}(t) - \dot{e}_i(t) = v_{d,i-1}(t) - v_{d,i}(t) - h \dot{v}_{d,i}(t). \tag{106}
\]
Denote \(G_{ci}(s) = \frac{\mathcal{E}_{i-1}(s)}{\mathcal{E}_{i-1}(s)}\). Taking the Laplace transform of (106), we have
\[
s \mathcal{E}_{i-1}(s) - s \mathcal{E}_i(t) = \mathcal{Y}_{d,i-1}(s) - \mathcal{Y}_{d,i}(s) - h s \mathcal{Y}_{d,i}(s). \tag{107}
\]
This together with \(\mathcal{Y}_{d,i}(s) = H_i(s) \mathcal{E}_i(s)\) leads to
\[
G_{ci}(s) = \frac{s - H_i(s)}{s - (h s + 1) H_i(s)}. \tag{108}
\]
This together with (105) leads to
\[
G_{ci}(s) = \frac{(b_i k_a s^4 + \bar{p}_{31} s^3 + \bar{p}_{21} s^2 + \bar{p}_{11} s + b_j k_p \beta_3)}{(s^5 + \bar{d}_{31} s^3 + \bar{d}_{21} s^2 + \bar{d}_{11} s + \bar{d}_{01})}, \tag{109}
\]
where
\[\pi_{1i} = b_i k_p \beta_2 + b_i k_v \beta_3,\]
\[\pi_{2i} = b_i k_p \beta_7 + b_i k_v \beta_2 + b_i k_a \beta_3,\]
\[\pi_{3i} = b_i k_v \beta_7 + b_i k_a 2 + b_i k_p,\]
\[\vec{d}_{1i} = b_i k_p \beta_2 + (b_i k_v + b_i k_p) \beta_3,\]
\[\vec{d}_{2i} = b_i k_p \beta_2 + (b_i k_v + b_i k_p) \beta_2 + (b_i + b_i k_v) \beta_3,\]
\[\vec{d}_{3i} = (b_i k_v + b_i k_h + k_a / \tau) \beta_1 + (b_i + b_i k_v) \beta_2 + b_i k_v + b_i k_h + b_i k_a / \tau,\]
\[\vec{d}_{5i} = \beta_1 + b_i k_v + k_a / \tau - b_i k_a.\]

Substituting \(s = j \omega\) into (107), we get
\[G_e(j \omega) = \frac{x_{n1}(\omega) + y_{d1}(\omega)}{x_{d1}(\omega) + y_{d1}(\omega)},\]

where \(x_{n1}(\omega) = b_i k_p \beta_2 - \pi_{2i} \omega^2 + b_i k_v \omega^2, y_{d1}(\omega) = \pi_{3i} \omega - \pi_{3i} \omega^3, x_{d1}(\omega) = b_i k_p \beta_3 - \vec{d}_{2i} \omega^2 + \vec{d}_{4i} \omega^4 - \omega^6, y_{d1}(\omega) = \vec{d}_{4i} \omega - \vec{d}_{5i} \omega^3 + \vec{d}_{5i} \omega^5.\]

Through calculation, we get
\[\alpha_{5i} k_2 - \gamma_{5i} k_1 = 2b_i k_p \beta_3 - \pi_{2i} \omega^2 + 2b_i k_v \omega^2 - \pi_{3i} \omega^3, x_{d1}(\omega) = b_i k_p \beta_3 - \vec{d}_{2i} \omega^2 + \vec{d}_{4i} \omega^4 - \omega^6, y_{d1}(\omega) = \vec{d}_{4i} \omega - \vec{d}_{5i} \omega^3 + \vec{d}_{5i} \omega^5.\]

By (22), we know \(k_1 = \tau \omega_0\). This leads to \(\lambda_4 > 0\). From (22) and \(\omega_{\mu} > 0\), we know \(\mu_\nu > b_i k_\mu (b_i k_\mu)\). By (23), we know \(\omega_0 > \sqrt{4b_i^2 \mu_\mu (b_i^2 \mu_\mu - b_i^2 \mu_\mu)}\). This together with \(v_\nu > b_i k_\mu (b_i k_\mu)\) leads to \(b_i^2 \mu_\mu (b_i^2 \mu_\mu - b_i^2 \mu_\mu)\). This together with \(\mu_\nu > 0\) and \(\mu_\nu > 0\) leads to \(b_i^2 \mu_\mu (b_i^2 \mu_\mu - b_i^2 \mu_\mu)\). This together with \(\mu_\nu > 0\) and \(\mu_\nu > 0\) leads to
\[\alpha_{4i} k + \gamma_{4i} k + \rho_{4i} = 0.\]

By (22), we know \(\mu_\nu > \mu_\mu (b_i k_\mu)\). This leads to \(\mu_\nu > 0\) and \(\mu_\nu > 0\). This together with \(\mu_\nu > 0\) and \(\mu_\nu > 0\) leads to \(\alpha_{3i} k + \gamma_{3i} k + \rho_{3i} = 0\). This together with \(\mu_\nu > 0\) and \(\mu_\nu > 0\) leads to \(\alpha_{3i} k + \gamma_{3i} k + \rho_{3i} = 0\).
\[ k \geq \max \{ 0, (\gamma_2i + \sqrt{\gamma_2i^2 - 4\alpha_1i\beta_1i})/(2\alpha_1i) \}. \] This together with \( \alpha_2i > 0 \) and \( \rho_2i > 0 \) leads to
\[ \alpha_2i^2k^2 + \gamma_2i^2k + \rho_2i \geq 0. \] (113)

By (22), we know \( \mu_\nu > \mu_\nu^0/h(b) \). This together with \( \mu_\alpha > 0 \), we know \( \mu_\alpha/\tau - \gamma_\nu \beta_\nu + h(b) \mu_\nu > 0 \) and \( \beta_1 < \alpha_1i < \tau_1i \). From (24) and \( \tau_1i > 0 \), we obtain \( k \geq \max \{ 0, (\gamma_2i + \sqrt{\gamma_2i^2 - 4\alpha_1i\beta_1i})/(2\alpha_1i) \}. \) This together with \( \alpha_1i > 0 \) and \( \rho_1i > 0 \) leads to
\[ \alpha_1i^2k^2 + \gamma_1i^2k + \rho_1i \geq 0. \] (114)

By (110)-(114), we know
\[
\begin{align*}
(\alpha_1i^2k^2 - \gamma_1i^2k)i^2 + (\alpha_1i^2k^2 + \gamma_1i^2k + \rho_1i)i^4 \\
+ (\alpha_3i^2k^2 + \gamma_3i^2k + \rho_3i)i^6 + (\alpha_2i^2k^2 + \gamma_2i^2k + \rho_2i)i^8 \\
+ (\alpha_1i^2k^2 + \gamma_1i^2k + \rho_1i)i^{10} + i^{12} \geq 0, \quad \forall i \in \mathbb{R}.
\end{align*}
\] (115)

Through calculation, we know
\[
\begin{align*}
x_{3i}(\omega) + y_{3i}(\omega) - x_{3i}(\omega) - y_{3i}(\omega) \\
= (2b_3i\beta_3\bar{\tau}_{3i} + \bar{d}_{3i}^2 - 2b_3i\beta_3\bar{d}_{3i} - \bar{m}_{3i}^2)\omega^2 \\
+ (2\tau_{3i}\bar{m}_{3i} + 2b_3i\beta_3\bar{d}_{3i} + \bar{d}_{3i}^2 - \bar{m}_{3i}^2 - 2b_3i\beta_3\bar{d}_{3i}^2)\omega^4 \\
- 2\tau_{3i}\bar{d}_{3i}\omega^6 + (\bar{d}_{3i} + \bar{d}_{3i}^2 - b_3i\bar{k}_{3i})\omega^8 \\
- 2\bar{d}_{3i}\bar{d}_{3i}\omega^{10} + (\bar{d}_{3i} - \bar{d}_{3i}^2)\omega^{12} \geq 0, \quad \forall \omega \in \mathbb{R}.
\end{align*}
\] (116)

This together with (116) leads to
\[ x_{3i}(\omega) + y_{3i}(\omega) - x_{3i}(\omega) - y_{3i}(\omega) \geq 0, \quad \forall \omega \in \mathbb{R}. \] (117)

By (117), we know
\[ \frac{x_{3i}(\omega) + y_{3i}(\omega)}{x_{3i}(\omega) + y_{3i}(\omega)} \leq 1, \quad \forall \omega \in \mathbb{R}. \] (118)

Through calculation, we obtain
\[
\frac{x_{3i}(\omega) + y_{3i}(\omega)}{x_{3i}(\omega) + y_{3i}(\omega)} = \frac{\sqrt{x_{3i}(\omega) + y_{3i}(\omega)}}{\sqrt{x_{3i}(\omega) + y_{3i}(\omega)}}
\]

This together with (118) leads to
\[ \frac{x_{3i}(\omega) + y_{3i}(\omega)}{x_{3i}(\omega) + y_{3i}(\omega)} \leq 1, \quad \forall \omega \in \mathbb{R}. \] (119)

From (108) and (119), we know \( |G_{\omega j}(\omega)| \leq 1 \) for any \( \omega \in \mathbb{R} \). That is \( \frac{\frac{e_{\omega j}(\omega)}{G_{\omega j}(\omega)}}{\frac{\omega}{1+\omega}} \leq 1 \) for any \( \omega \in \mathbb{R} \). □

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