Theory of Caroli–de Gennes–Matricon analogs in full-shell nanowires

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Full-shell nanowires are hybrid nanostructures consisting of a semiconducting core encapsulated in an epitaxial superconducting shell. When subject to an external magnetic flux, they exhibit the Little-Parks (LP) phenomenon of flux-modulated superconductivity, an effect connected to the physics of Abrikosov vortex lines in type-II superconductors. We show that full-shell nanowires can host subgap states that are a variant of the Caroli–de Gennes–Matricon (CdGM) states in vortices. These CdGM analogs are in fact shell-induced Van Hove singularities in propagating core subbands. We elucidate their structure, parameter dependence and behavior in tunneling spectroscopy through a series of models of growing complexity. We show through microscopic numerical simulations that they exhibit a characteristic skewness towards high magnetic fields inside non-zero LP lobes resulting from the interplay of three ingredients. First, the orbital coupling to the field shifts the energy of the CdGM analogs proportionally to the flux and to their generalized angular momentum. Second, CdGM analogs coalesce into degeneracy points at flux values for which their corresponding radial wavefunctions are threaded by an integer multiple of the flux quantum. And last, the average radii of all CdGM-analog wavefunctions inside the core are approximately equal and controlled by the electrostatic band bending at the core/shell interface. As the average radius moves away from the interface, the degeneracy points shift towards larger fluxes from the center of the LP lobes, causing the skewness. This analysis provides a transparent interpretation of the nanowire spectrum that allows to extract microscopic information by measuring the number and skewness of CdGM analogs. Moreover, it allows to derive an efficient Hamiltonian of the full-shell nanowire in terms of a modified hollow-core model at the average radius.

I. INTRODUCTION

Full-shell nanowires comprised of semiconducting nanowires fully encapsulated in a thin superconducting layer, or shell, have been recently introduced in the context of topological superconductivity [11,12]. These wires could present several advantages for the generation and detection of Majorana bound states (MBSs) as compared to partial-shell ones, where the superconducting coating is limited to some facets of the nanowire [3–7]. In the full-shell case the trigger of the topological phase transition is the magnetic flux threading the nanowire produced by an external axial magnetic field, whereas in the partial-shell devices following the original proposal [8,9], it is the Zeeman effect. Partial-shell nanowires, sometimes dubbed Majorana nanowires, have been exhaustively analyzed since 2010, whereas the full-shell variant has only more recently begun being explored [10,11].

The interest of full-shell hybrid nanowires, however, extends beyond their possible relevance for topological superconductivity. The doubly-connected geometry of the superconducting shell introduces very rich physics [11,12]. In the presence of a magnetic flux $\Phi$ through the section of the hybrid nanowire, the system exhibits the so-called Little-Parks (LP) effect [20,21]. In the LP effect, the flux causes the superconducting phase in the shell to acquire a quantized winding around the nanowire axis. The winding number $n$ is an integer, also known as fluxoid number [22,23], that increases in jumps as $\Phi$ grows continuously. Winding jumps are accompanied by a repeated suppression and recovery of the superconductor gap, forming LP lobes associated with each $n$. The LP effect has been demonstrated experimentally in various regimes [11,24–26] and has been shown to be accurately described by theory based on the Ginzburg-Landau formalism [23,24,27].

Furthermore, the superconducting boundary condition imposed by the shell gives rise to a special type of fermionic subgap states through a combination of normal and Andreev reflection at the core/shell interface. These states are hybrid-nanowire analogs of the celebrated Caroli-de Gennes-Matricon (CdGM) states in Abrikosov vortex lines of type-II superconductors [23,28]. We call them analogs because both are subgap states within superconducting boundaries, bound to a region with suppressed pairing and threaded by a magnetic flux. However, several important differences exist between them. Some of these were analyzed recently in Refs. [12,24] and [35], although these states have remained relatively unexplored.

In bulk type-II superconductors, see Fig. 1(a,b), CdGM states are low energy excitations bound to the center of each vortex core, i.e., to the region of radius $r \lesssim \xi$ (with $\xi$ the bulk superconducting coherence length) [21]. Each vortex is threaded by a single flux quantum, which produces a localized suppression of the superconducting order parameter $\Delta(r)$ at the vortex core $\vec{r} = 0$, and a quantized winding of its phase in the polar angle $\varphi$ around the vortex

$$\Delta(\vec{r}) = \Delta(r) e^{i n \varphi}. \quad (1)$$
CdGM states in Type-II SCs

CdGM analogs in full-shell NWs

a

b

Δ(r)

ψ(r)²

U(r)

ψ(r)²

Δ(r)

U(r)

FIG. 1. (a) Schematic of Abrikosov vortex lines in a bulk type-II superconductor. The magnetic flux of the external magnetic field $\vec{B}$ inside each vortex is quantized to the superconducting flux quantum $Φ₀$. (b) Pairing amplitude $Δ(r)$ (blue), electrostatic potential energy $U(r)$ (green) and lowest-energy CdGM wavefunction density $|\psi(r)|²$ (yellow) as a function of radial coordinate $r$ with respect to the vortex center. (c) Schematic of a full-shell hybrid nanowire in a cylindrical approximation. The semiconducting core of radius $R_{\text{core}}$ (yellow) is fully covered by an $s$-wave superconducting shell of thickness $d_{\text{shell}} = R_{\text{shell}} - R_{\text{core}}$ (blue). The magnetic flux $Φ$ due to the field $\vec{B}$ threading the wire is not quantized and the pairing amplitude inside the shell is modulated with $Φ$ following the Little-Parks (LP) effect. (d) Same as (b) but for the full-shell wire. The conduction-band bottom inside the semiconductor exhibits a dome-like radial profile with maximum value at the center, $U_{\text{max}}$, and minimum value at the superconductor/semiconductor interface, $U_{\text{min}}$. The electrostatic potential of the metallic shell is $|U_{\text{shell}}| \gg |U_{\text{min}}|$.

Here $Δ(r) = |Δ(\vec{r})|$ denotes the pairing amplitude, with $Δ(0) = 0$, and the winding is $n = 1$. In full-shell nanowires, on the contrary, the total flux through the core is not quantized due to the thickness of the shell, see Fig. 1(c), and both the pairing amplitude inside the superconducting gap, dubbed $Ω$ from here on, are modulated with $Φ$. This is due to an orbital coupling inside the core of the form $\sim m_L Φ$. In a simplified hollow-core model for the hybrid wire, where the semiconductor wavefunction is assumed to be confined to an infinitesimal layer at the core/shell interface, the CdGM-Van Hove singularities disperse with flux symmetrically with respect to the lobe centers, where they coalesce into degeneracy points. Meanwhile, realistic solid-core wires, characterized by an angular momentum quantum number, $m_L$, much like the original CdGM states. They are also characterized by a good radial quantum number for typical nanowire radii only one or a few radial modes are occupied, in contrast to the many modes occupied in vortices.

We study the energy dispersion of CdGM analogs with magnetic flux inside each $n$-lobe. We compute both local density of states (LDOS) at the end of a semi-infinite wire, and differential conductance ($dI/dV$) through a normal/superconducting junction. The states with $m_L > 0$ ($m_L < 0$) disperse with positive (negative) slope versus $Φ$. This is due to an orbital coupling inside the core of the form $\sim m_L Φ$. In a simplified hollow-core model for the hybrid wire, where the semiconductor wavefunction is assumed to be confined to an infinitesimal layer at the core/shell interface, the CdGM-Van Hove singularities disperse with flux symmetrically with respect to the lobe centers, where they coalesce into degeneracy points. Meanwhile, realistic solid-core wires, characterized by an angular momentum quantum number, $m_L$, much like the original CdGM states. They are also characterized by a good radial quantum number for typical nanowire radii only one or a few radial modes are occupied, in contrast to the many modes occupied in vortices.

In this work we study the structure and properties of CdGM analogs in realistic full-shell wires, and the information these states can provide about key nanowire aspects through local measurements. We identify the CdGM analogs as Van Hove singularities of the $n$-dependent, quasi-one dimensional, traverse subbands propagating along the axis of the proximitized nanowire core. We use a cylindrical model to describe the hybrid wire, although this approximation is not critical to our findings. Subbands are thus characterized by an angular momentum quantum number, $m_L$, much like the original CdGM states. They are also characterized by a good radial quantum number (for typical nanowire radii only one or a few radial modes are occupied, in contrast to the many modes occupied in vortices).
The flux displacement of the degeneracy points is a crucial quantity to understand the subgap spectrum of full-shell nanowires. It is directly related to the spatial wavefunction distribution of the CdGM analogs, or more specifically, to their average radius \( R_{av} \) inside the core. As we will see, the degeneracy points occur at magnetic fields such that the flux inside \( R_{av} \) is an integer multiple of the superconducting flux quantum. We perform numerical simulations for Al/InAs full-shell models to show how \( R_{av} \), and hence the CdGM spectrum within each lobe, are affected by the different materials of the Al shell and the InAs core and the appearance of a charge accumulation layer close to the interface. By direct inspection of the number of subgap states and their skewness, it is possible to ascertain the wire’s effective doping and wave function distribution, which in turn are an approximate measure of the potential profile \( U(r) \) inside the semiconductor, see Fig. 1(d) (shaded in green). This provides an indirect but powerful tool to characterize the screened and otherwise inaccessible interior of full-shell devices. Finally, we discuss a modified hollow-core model for the full-shell wire, conveniently tailored to account for the degeneracy-point skewness. We show that it provides very similar results to the full-solid simulations in the presence of \( U(r) \), but at a considerably reduced computational cost.

In most of this work we neglect Zeeman and spin-orbit coupling (SOC) inside the semiconductor since they have a minor effect on the CdGM analogs. Spin is therefore dropped as an inert degree of freedom. The Zeeman effect merely produces small splittings inside small-\( n \) LP lobes in the otherwise spin-degenerate subgap spectrum. The SOC, on the other hand, is of course essential for the existence of the topological phase and the emergence of MBs. As we will show, however, it leaves the rest of the subgap spectrum practically unaffected.

This paper is organized as follows. In Sec. II we summarize the physics of the LP effect of the shell, deferring the technical details to Apps. A and B. In Sec. III we characterize the bandstructure, LDOS and quantum numbers of the nanowire modes using the simplified, hollow-core approximation to the full-shell nanowire. We show how the proximity effect induced by the shell gives rise to Van Hove singularities that become degenerate at special points. In Sec. IV we generalize the model to a finite semiconducting layer thickness, dubbed the tubular-nanowire model, which results in a reduction of the average wavefunction radius of each mode. This leads to a shift of the degeneracy points towards higher fields and the skewness of the CdGM analogs. In Sec. V we connect to the experimentally-relevant model of a solid-core nanowire with a finite band-bending electrostatic potential profile in the core. We compute the LDOS at one end of the wire and show how it is related to the tunnelling differential conductance of current experiment.11,12 We also develop the modified hollow-core description and compare it to the solid core model. In Sec. VI we discuss the effect of including Zeeman and SOC to our models. Finally, in Sec. VII we summarize our main findings and conclude.

II. THE LITTLE-PARKS EFFECT OF THE SHELL

We start by describing the effect of the threading flux \( \Phi \) on the superconducting shell alone, i.e., the blue region in Fig. 1(c). Consider a hollow superconducting cylinder along the \( \hat{z} \) direction, of thickness \( d_{shell} \), outer radius \( R_{shell} \) and inner radius \( R_{core} = R_{shell} - d_{shell} \). A magnetic field \( \vec{B} = B_{\hat{z}} \hat{z} \) is applied along its axis. In the symmetric gauge, the vector potential for \( \vec{B} \) reads \( \vec{A} = \frac{1}{2}(\vec{B} \times \vec{r}) = (−y, x, 0)B_{z}/2 = A_{\varphi}\hat{\varphi}, \) where \( A_{\varphi} = Br/2 \). Here \( r \) is the radial coordinate and \( \varphi \) denotes the polar angle around \( \hat{z} \). The magnetic field produces a flux into the cylinder, defined as

\[
\Phi = \pi R_{LP}^2 B_{z},
\]

\[
R_{LP} = \frac{R_{shell} + R_{core}}{2}.
\]

Note that \( \Phi \) is taken at the mean radius \( R_{LP} \) of the shell.

In superconductors under magnetic fields, a useful quantity related to \( \Phi \) is the fluxoid \( \Phi' \), which is quantized in units of \( \Phi_0 = h/2e \), the superconducting flux quantum.\textsuperscript{10,11} This was established by F. London,\textsuperscript{22} who defined the fluxoid \( \Phi' \) as the sum of the magnetic flux \( \Phi \) and an extra term involving the superconducting order parameter \( \Delta(\vec{r}) \), and representing the circulation of persistent supercurrents that arise in response to the magnetic flux. In our hollow cylinder under an axial flux, these supercurrents flow in the \( \varphi \) direction, around the cylinder. If \( d_{shell} \) is much greater than the London penetration depth,\textsuperscript{22,13} \( \lambda_{L} \), the persistent supercurrent term vanishes deep inside the superconductor, and thus the magnetic flux \( \Phi \) is also quantized at large distances, \( \Phi = \Phi' \). This is the case also in vortices inside bulk superconductors. However, for thin superconducting shells as the ones considered here, the Meissner,\textsuperscript{14,15} expulsion of the magnetic flux is negligible and thus the magnetic field in the superconductor, as well as in its interior, is essentially the same as the applied one (and is hence not quantized). In this case, the second term involving the screening supercurrents oscillates with flux as the fluxoid increases in units of \( \Phi_0 \), which in turn leads to a modulation of the pairing amplitude \( \Delta \), superconducting gap \( \Omega \) and critical temperature \( T_c \), with a period \( \Phi_0 \). This is known as the LP effect.\textsuperscript{10,13,14,15}

From a complementary point of view, the flux modulation of the superconducting properties is a consequence of the pair-breaking effect of the magnetic field on the Cooper pairs in the superconductor. This pair-breaking effect is minimal at integer values of \( \Phi/\Phi_0 \), where \( \Delta \) and \( \Omega \) reach a maximum, and strongest at half-integer values, where they are minimized. Shells with a finite (zero) \( \Omega \) at this point are said to be in the non-destructive (destructive) regime, see Fig. 2. In the non-destructive case the
fluxoid number $n$ and other physical observables perform abrupt, first-order transitions at lobe boundaries.

Alternatively to the London theory, the quantization of the fluxoid can also be established within the Ginzburg-Landau formalism for the complex superconducting order parameter $\Delta(\vec{r}) = \Delta(r, \varphi)$ (we ignore any $z$ dependence). Since this is a single-valued complex quantity, its phase must change by an integer multiple of $2\pi$, $n \in \mathbb{Z}$, when completing a closed path around the cylinder, $\varphi \rightarrow \varphi + 2\pi$. This winding number is in fact the fluxoid number $n = \Phi/\Phi_0$. One can thus write $\Delta(\vec{r})$ as in Eq. [1] for arbitrary $n$.

In this work we are interested in the regime $d_{\text{shell}} \ll \lambda_s$, so that the pairing amplitude is constant, $\Delta(\vec{r}) = \Delta$. In a ballistic model for the shell, the pairing amplitude $\Delta$ turns out to be equal to the superconducting gap $\Omega$. This is also the case for a time-reversal-symmetric superconductor in the dirty limit according to Anderson’s theory. The experimental shells we consider here are diffusive, owing in particular to unavoidable oxidation of their outer surface. In the presence of pair-breaking perturbations, like magnetic impurities or (as in our case) magnetic fields in a diffusive superconductor, $\Omega$ is different from $\Delta$. This was originally described by Abrikosov and Gor’kov whose theory was later applied to the LP effect. The technical details and relevant equations are given in App. A.

The magnetic flux $\Phi$ produces the LP modulation of the shell gap $\Omega(\Phi)$ [Eq. (A5)] and pairing amplitude $\Delta(\Phi)$ [Eq. (A4)], which exhibit re-emergent lobes centered around integer $n = \Phi/\Phi_0$ as pointed out in the introduction, each of them is characterized by a different fluxoid number $n$. The precise $\Omega(\Phi)$ and $\Delta(\Phi)$ profiles depend on the geometric parameters of the shell, and on the superconducting diffusive coherence length $\xi$, see Eq. (A6) in App. A. Figure 2 shows these results for typical nanowire shell parameters.

### III. HOLLOW-CORE NANOWIRE

Following Ref. [1] we consider a basic model of a cylindrically symmetric full-shell wire that combines (i) the effect of the magnetic flux on the superconducting shell (the LP effect), (ii) the proximity effect on the core subbands with well-defined angular momentum, and (iii) the effect of the magnetic flux on the core subbands.

Point (i) is summarized in the preceding section. Regarding (ii), the hybrid wire consists of a semiconducting core with effective mass $m^*$ and radius $R_{\text{core}}$ covered by a superconducting shell of thickness $d_{\text{shell}} = R_{\text{shell}} - R_{\text{core}}$. Given a Hamiltonian $H_{\text{core}}$ for the normal core electrons, we wish to write an effective Hamiltonian $H$ in the presence of the shell by integrating out the shell degrees of freedom. This procedure introduces a self-energy $\Sigma_{\text{shell}}$ into the Green’s function $G(\omega) = [\omega - H_{\text{core}} - \Sigma_{\text{shell}}(\omega)]^{-1}$. The effective Hamiltonian for the system is then defined as $H \equiv \omega - G^{-1}(\omega) = H_{\text{core}} + \Sigma_{\text{shell}}(\omega)$, which is in general frequency dependent. Note that we use $\hbar = 1$ throughout, so that $\omega$ has units of energy. In the following we neglect non-local self-energy components (a valid approximation for disordered shell[1]) and also any non-uniformity of the self-energy along the wire length, so that $\Sigma_{\text{shell}}$ depends only on frequency and the angle $\varphi$ around the cylinder axis, $\Sigma_{\text{shell}}(\omega, \varphi)$. As discussed in App. [3] the form of $\Sigma_{\text{shell}}$ for a diffusive shell is expressed in terms of a normal decay rate $\Gamma_N$ from the core into the shell and a function $u(\omega)$ given by Eq. [2],

$$
\Sigma_{\text{shell}}(\omega, \varphi) = -\Gamma_N \frac{u(\omega) \tau_0 - e^{i n \varphi} \tau_x}{\sqrt{1 - u(\omega)^2}}.
$$

In the following subsection we define the minimal model that captures also (iii). We call it the hollow-core approximation.

#### A. Model

In the hollow-core approximation all charge carriers in the core are localized at the interface with the superconductor and thus their radial coordinate is fixed to
its eigenstates can be classified in terms of the integer angular momentum $l$ and $m$, so its projection $m$ is a good quantum number of the eigenstates of the Hamiltonian. Since both values are degenerate we neglect spin until Sec. IV. Note that both $A_{r}$ and $p_{r}$ should be evaluated on the hollow-core surface at $r = R_{core}$.

1. Quantum numbers

The hollow-core model in Eq. (4) exhibits three symmetries that can be used to classify its eigenstates:\[ H = \left[ \frac{(p_{r} + eA_{r} \tau_{z})^{2} + \frac{p_{z}^{2}}{2m^{*}}}{} - \mu \right] \tau_{z} + \Sigma_{shell}(\omega, \varphi), \]

where $p_{r} = -\frac{1}{r}i\partial_{r}$, $p_{z} = -i\partial_{z}$ are the momentum operators for electrons, $\mu$ is the semiconductor chemical potential, $e > 0$ is the unitary charge and $\tau_{z}$ are the Pauli matrices for the electron/hole degree of freedom. The nanowire is subject to a magnetic field as described in Eq. (2). Thin lines are for $\Gamma_{N} = 0$ Nambu subbands [i.e., normal electron (solid) and hole (dashed) bands]. These are coloured according to their corresponding $m_{L}$ quantum number. Thick lines correspond to finite $\Gamma_{N}$, and are labeled by the corresponding $m_{L}$ quantum number.

The key feature to note in the finite $\Gamma_{N}$ bands is the appearance of avoided crossings between normal electron and hole subbands with $m_{L}$ differing by the fluxoid number $n$. The avoided crossings are due to Andreev reflection at the core/shell interface, and result in Van Hove singularities (black dots) in the LDOS that move in energy with $\Phi$. These Van Hove singularities are the full-shell analogs of CdGM states in type-II superconductors.

The energies of the Van Hove singularities for different $m_{L}$’s become degenerate at certain values of magnetic flux, see Figs. 3a,d. These special values of $\Phi$ correspond to integer multiples of $\Phi_{0}$, when the flux $\Phi$ is computed as $\Phi = \pi R_{core}^{2} B_{z}$, which in the thin-shell limit ($R_{LP} = R_{core}$) coincides with Eq. (2). Thus, for thin shells and hollow cores, the proximity-induced Van Hove singularities become degenerate at the center of each lobe.

C. Local density of states

The Van Hove singularities and their degeneracies can also be visualized in terms of the LDOS at the end of a semi-infinite hollow-core nanowire, given by

\[ \rho(\omega) = -\frac{1}{\pi} \sum_{m_{L}} \text{Im} \text{Tr} G^{0}_{m_{L}}(\omega). \]

Here, the retarded Green’s function $G^{0}_{m_{L}}(\omega)$ may be computed by discretizing the rotated Hamiltonian $\tilde{H}$ in a one-dimensional lattice along $z$, with lattice constant $a_{0}$, and using standard methods of scattering theory to obtain $\rho_{m_{L}}(\omega)$ in the first unit cell. The superindex $0$ here stands for the first site of the semi-infinite chain. The trace $\text{Tr}$ is taken over the remaining electron/hole degree of freedom.

In Fig. 4 we show the calculated LDOS $\rho(\omega)$ as a function of energy $\omega$ and normalized flux $\Phi/\Phi_{0}$, for different values of core/shell coupling $\Gamma_{N}$. Said coupling controls together with $k_{z}$ and $s_{z}$. In the normal case, then, the canonical transformation that diagonalizes the Hamiltonian is simply $U = e^{-im_{L}\varphi - ik_{z}z}$.
FIG. 3. (a) Bogoliubov-de Gennes bandstructure of an infinitely long hollow-core nanowire of radius $R_{\text{core}} = R_{\text{shell}}$, as a function of longitudinal momentum $k_z$ at the center of the $n = 0$ LP lobe (zero flux). Thin solid (dashed) lines correspond to the normal electron (hole) subbands, with shell/core coupling $\Gamma_N = 0$, whereas thick lines correspond to the superconducting state, with $\Gamma_N \neq 0$. The number of occupied subbands depends on the filling $\mu$. Different colors signal different (generalized) angular momentum number $m_l$ ($m_L$) for the normal (superconducting) subbands. Both $m_l$ and $m_L$ are integers in the $n = 0$ lobe, see legend. The superconducting pairing turns the finite-momentum electron-hole crossings with equal $m_L$ into anticrossings, with Van Hove singularities arising at the edges of the corresponding gaps, see black dots. In the absence of applied flux, $\Phi = 0$, all anticrossings are equal in magnitude and centered at zero energy. As a result, all Van Hove singularities are degenerate. (b) Same as (a) but for $\Phi/\Phi_0 = 0.49$, close to the edge of the $n = 0$ lobe. The previously degenerate Van Hove singularities split in energy due to the different dispersion with flux of electron and hole $m_L$ subbands. (c) Same as (b) but for $\Phi$ close to the edge of the $n = 0$ lobe. The normal bands are very similar to (b), but the anticrossing pattern has changed, as the pairing only couples electron and holes with $m_L$ differing by $n$. The superconducting-subband colors represent half-integer $m_L$ quantum numbers. (d) Same as (c) but at the center of the $n = 1$ lobe where the hollow core is threaded by one flux quantum. The Van Hove-singularity degeneracies of different subbands are recovered.

FIG. 4. Local density of states (LDOS) at the end of a semi-infinite hollow-core nanowire (in arbitrary units) vs energy $\omega$ and applied normalized flux $\Phi/\Phi_0$, displaying half of the zeroth lobe, first and second lobes. CdGM analogs are shown as subgap features below the LP shell gap $\Omega(\Phi)$, together with their generalized angular momentum $m_L$. From top to bottom: (a) weak superconductor/semiconductor coupling, $\Gamma_N = 0$.1$\Omega(0)$, (b) intermediate coupling, $\Gamma_N = 0.8\Omega(0)$ and (c) strong coupling, $\Gamma_N = 3\Omega(0)$. Degeneracy points (where all CdGM-Van Hove states cross) happen at the center of each lobe (where the normalized flux $\Phi/\Phi_0$ is an integer), see arrows in (c). The lowest subgap level is dubbed the induced gap $\Omega^*(\Phi)$. Parameters: $R_{\text{core}} = R_{\text{shell}} = 70$ nm, $\Omega(0) = \Delta(0) = 0.23 \text{ meV}$, $\mu \approx 1 \text{ meV}$, $\xi = 70$nm, $m^* = 0.023 m_e$ and $a_0 = 5$ nm.

the magnitude of the induced gap $\Omega^*(\Phi)$, which is smaller than the gap in the shell $\Omega(\Phi)$. The energy of the coalescing van Hove singularities at zero flux is $\Omega^*(0)$.

In the LDOS simulation of Fig. 4 we have focused on the non-destructive LP regime with $d_{\text{shell}} \approx 0$, so that all lobes have an identical shape. The (spin-degenerate) Van Hove singularities, visible as sharp, flux-dependent subgap features in each lobe, are labeled with their corresponding $m_L$ quantum numbers. Note that each CdGM analog at any given $\Phi$ consists of both the Van Hove singularity itself (seen with a bright orange color in Fig. 4) and a tail extending above or below it in $\omega$ till the par-
The model of singularities depends on $\mu$ and is different in even and odd lobes. Since $m_L$ is integer in even lobes (including 0), these contain an odd number of Van Hove pairs. Odd lobes, in contrast, have half-integer $m_L$ [see Eq. (5)], so they contain an even number of Van Hove pairs, see Fig. 4.

We saw in the bandstructures of Fig. 3 how Van Hove singularities become degenerate at the center of each lobe. In the LDOS this is visible as coalescing singularities, forming a characteristic fountain-like pattern around degeneracy points, and symmetrically around $\omega = 0$. The slope with which the singularities disperse with flux away from the degeneracy points is proportional to $m_L$. This is ultimately due to the orbital coupling term

$$\frac{1}{2m^2R_E^2}\Phi_i m_L \tau_z = \omega_L m_L \tau_z,$$

(9)

that is present in Eq. (6) after expanding the square. Here, $\omega_L = eB_z/2m^2$ is the Larmor frequency. The slope is not constant, however, as the spectral density at the band edge in $\Sigma_{\text{shell}}$ repels the CdGM analogs as they approach $\Omega(\Phi)$.

IV. TUBULAR-CORE NANOWIRE

To continue building towards the realistic nanowire model in Sec. VI, we next generalize the hollow-core model by giving a finite thickness $d_{\text{core}} = R_{\text{core}} - R_{\text{inner}}$ to the semiconductor, so that it spans a finite range of radii $r \in [R_{\text{inner}}, R_{\text{core}}]$, see Fig. 5(a), while keeping the potential in the core $r$-independent. We dub this the tubular-core nanowire model.

A. Model

The tubular-core generalization introduces radial kinetic energy into the model, and consequently radially quantized modes. The corresponding BdG effective Hamiltonian reads

$$H = \left[\frac{\left(p_r + eA_r(r)\tau_z\right)^2 + p^2}{2m^2} - \mu\right] \tau_z + \Sigma_{\text{shell}}(\omega, \phi),$$

(10)

where $p_r^2 = -\frac{1}{r^2}\partial_r(r\partial_r)$. Note that we have also restored the radial dependence of $A_r(r) = B_z r/2$. The same canonical transformation $U$ as for Eq. (6) reduces the above to

$$\tilde{H} = \left[\frac{(m_L - \frac{1}{2}n\tau_z + \frac{1}{2} \frac{\Phi}{\Phi_0} r^2 \tau_z)^2}{2m^2r^2} + \frac{k^2 + p^2}{2m^2} - \mu\right] \tau_z + \Sigma_{\text{shell}}(\omega, \phi).$$

(11)

To find the eigenstates $\tilde{\Psi}(r)$ of $\tilde{H}$ we follow the DLL-FDM scheme of Ref. 56. We first discretize the radial coordinate $r$ with a lattice spacing $a_0$, replacing derivatives with finite differences in the differential eigenvalue equation $\tilde{H}\psi(r) = \epsilon\Psi(r)$. We then absorb the Jacobian $J = r$ of the cylindrical coordinates into modified discrete eigenstates $F(r_i) = \tilde{\Psi}(r_i)/\sqrt{r_i}$ and into the corresponding Hamiltonian $H' = r^{1/2} \tilde{H} r^{-1/2}$. With this we arrive at a discrete eigenvalue problem $\sum_i H'_{ii'} F(r_i) = \epsilon F(r_i)$ with a Hermitian Hamiltonian matrix $H'_{ii'}$, whose discrete eigenstates are, by virtue of their definition, trivially orthonormal without $J$. $\sum_i F^*_i(r_i) F_i(r_i) = \delta_{ii'}$. The kinetic energy $\tau_z p^2/m$ in $H'$ transforms, in the discrete $H'_{ii'}$, into an onsite term $t_i = 2t_0 \tau_z$ plus a radial hopping $t_{ii'} = -t_0 \tau_z \sqrt{r_i r_{i'}}$ between the nearest neighbors, where $t_0 = 1/(2m^2 a_0^2)$. Note that the $r/\sqrt{r_i r_{i'}}$ factor directly stems from the cylindrical Jacobian, but does not break the symmetry $t_{ii'} = t_{i'i}$. Also, when applying the above DLL-FDM scheme to systems including the origin $r = 0$, the correct boundary condition must be implemented there. This is done by excluding the $r = 0$ site and multiplying $t_i$ at the $r = a_0$ site by $3/4$. 55

B. Degeneracy point shifts and skewness

The evolution of the subgap Van Hove singularities as we go from the hollow-core to the tubular-core nanowire of decreasing $R_{\text{inner}}$ is shown in Fig. 5 all the way to $R_{\text{inner}} = 0$. Note that in this model the electrostatic potential across the tubular core is kept uniform for simplicity. The most immediate effect of gradually reducing $R_{\text{inner}}$ is a shift of the degeneracy points (originally located at the center of each LP lobe in the hollow-core nanowire) towards higher fields for $n > 0$, see e.g. the difference between Fig. 5(b) and Fig. 5(c). The shift can cause the degeneracy points, together with the diamond-shaped gap below them, to exit the $n \neq 0$ lobes altogether, see Figs. 5(e-g). In the process, the CdGM-Van Hove singularities become skewed towards higher fields for $n > 1$.

For sufficiently small $R_{\text{inner}}$, however, the skewness is inverted, see Figs. 5(g-i). This inversion happens sooner at higher LP lobes. The skewness of CdGM analogs is thus found to be a direct consequence of the shift of the degeneracy points, which becomes the central concept in understanding the tubular-core nanowire.

The shift of degeneracy points can be readily understood in terms of the radial wavefunction profile of modes inside the core. In the hollow-core case we showed that degeneracy points appeared at integer normalized flux,
as experienced by core states. The fact that this condition matched the integer normalized flux as experienced by the shell (center of LP lobes) was a consequence of the simplifying assumption that $R_{\text{core}} = R_{\text{shell}} = R_{\text{LP}}$, since then the area spanned by the superconductor and core states coincided. Now, in the tubular-core model, as the core wavefunctions are allowed to spread inwards within the interval $r \in [R_{\text{inner}}, R_{\text{core}}]$, see Fig. 6(c), the flux they experience at a given magnetic field decreases with respect to the LP flux $\Phi$ through the shell. This shifts the degeneracy points towards higher magnetic fields.

An approximate analytical expression for the shift can be derived by considering that the flux experienced by the spread-out wavefunction is the same as if it were concentrated at its average radius $R_{\text{av}} = \langle r \rangle$. The flux at which the degeneracy point happens in the $n = 1$ lobe, $\Phi_{\text{dp}} = \pi R_{\text{LP}}^2 B_{\text{clos}}^2$, then becomes

$$\Phi_{\text{dp}} / \Phi_0 = (1 + \delta n_{\text{dp}}) = \frac{R_{\text{LP}}^2}{R_{\text{av}}^2}. \quad (12)$$

We analyze the validity of this approximation in Fig. 6. Taking Fig. 6(d) as a starting point, which corresponds to a tubular nanowire with $R_{\text{inner}} = 50 \text{ nm}$ and $R_{\text{core}} = 70 \text{ nm}$, we show in Fig. 6(a,b) the change in the dimensionless shift $\delta n_{\text{dp}}$ for two different values of $R_{\text{shell}}$ (and thus of $R_{\text{LP}}$). For clarity, we have fixed the shell gap $\Omega(\Phi)$ to a constant $\Omega(0) = 0.23 \text{ meV}$, so that the LP modulation does not obscure the degeneracy point shift [the corresponding LDOS with a flux-dependent $\Omega(\Phi)$ in the destructive regime is shown in panels Fig. 6(c,d) for comparison]. It is clear from Eq. 12 that for $R_{\text{av}} < \sqrt{2/3} R_{\text{LP}}$ we have $\delta n_{\text{dp}} > 1/2$, which pushes the degeneracy point out of the $n = 1$ LP lobe. This is the case of Fig. 6(b), where $R_{\text{LP}}$ is increased by setting $R_{\text{shell}} = 90 \text{ nm}$. In this situation, the $n = 1$ degeneracy point does not correspond any more to a stable configuration for any value of magnetic field, since the fluxoid number in the ground state changes from $n = 1$ to $n = 2$ already at $\Phi / \Phi_0 = 3/2 < 1 + \delta n_{\text{dp}}$. Instead, the degeneracy point can be found in a metastable configuration of the $n > 1$ LP lobe, represented inside the dashed white square of Fig. 6(b). Note, however, that the $n = 1$ CdGM skewness does not depend on whether the degeneracy point is in a stable or metastable configuration.

FIG. 5. (a) Schematics of the full-shell nanowire cross section for a tubular semiconducting core with inner $R_{\text{inner}}$ and outer $R_{\text{core}}$ radii. The outer shell radius is $R_{\text{shell}}$. (b-i) LDOS in arbitrary units versus energy $\omega$ and normalized flux $\Phi / \Phi_0$ for increasing tubular-core thickness $d_{\text{core}} = R_{\text{core}} - R_{\text{inner}}$, from the hollow-core approximation in (b) to the solid-core case in (i). The corresponding values of $R_{\text{inner}}$ are displayed in (a). As the core thickness increases, the degeneracy points shift to larger flux within each $n \neq 0$ LP lobe, skewing the CdGM analogs and shifting the gap $\Omega^*$ below them. The electrostatic potential inside the semiconductor is uniform, and is adjusted in each panel to yield 13 (spin-degenerate) subbands occupied at zero flux. The superconductor/semiconductor coupling $\Gamma_N$ is adjusted to have $\Omega^*(0) \approx 0.2 \text{ meV}$. Other parameters: $a_0 = 5 \text{ nm}$, $R_{\text{core}} = R_{\text{shell}} = 70 \text{ nm}$, $\Omega(0) = 0.23 \text{ meV}$, $\xi = 70 \text{ nm}$.
the lowest excitation at $k_a$ in a solid-core semiconducting nanowire with a dome-like potential profile $U_m$ at the bottom. Different (normal) angular momentum subbands $R$ with $\Phi_m$ modulation with flux that corresponds to a shell coherence length $\xi = 140$ nm. (e) Wavefunction modulus ($|\Psi(r)|$) of the lowest excitation at $k_z = 0$ in the normal state ($\Gamma_N = 0$) vs radial coordinate $r$ for a tubular-core semiconducting nanowire of external radius $R_{core} = 70$ nm and different thicknesses, from $d_{core} = R_{core} - R_{inner} = 10$ nm at the top to 70 nm (solid core) at the bottom. Different (normal) angular momentum subbands $m_l$ are indicated with different colors. (f) Same as (e) but for a solid-core semiconducting nanowire with a dome-like potential profile $U(r)$, see Eq. $13$. Different values of $U_{max} - U_{min}$ from top to bottom are indicated (with $U_{max} = 0$ and exponent $\nu = 2$). Other parameters like in Fig. 5.

V. SOLID-CORE NANOWIRE

A. Model

The development of the nanowire model culminates in this section, in which we consider a more accurate approximation to the actual full-shell wires studied in recent experiments.\cite{high12}. These are all solid-core nanowires with $R_{inner} = 0$. We keep the cylindrical approximation, since we find that a more complicated hexagonal wire cross-section, which is computationally much more expensive, does not significantly affect the skewness and overall properties of the CdGM analogs (not shown here).

In principle one needs to know the wavefunction profile to compute its $R_{av} = \langle r \rangle$ in order to use Eq. $12$. However, in the case of a uniform electrostatic potential, the wavefunction is approximately symmetric around the geometric mean radius $(R_{inner} + R_{core})/2$ of the core for all $m_l$, as long as $R_{inner}/R_{core} \gtrsim 0.5$, so that the effect of the Jacobian is small. This is shown in Fig. 6(e). Hence, we can use the approximation $R_{av} \approx (R_{inner} + R_{core})/2$ in Eq. $12$. This yields $\delta n_{dp} = 0.36$ for the parameters of Fig. 6(a), which is very close the the numerical result of 0.40 observed in that figure. The same happens for Fig. 6(b), where the analytical solution is $\delta n_{dp} = 0.78$ and the numerical one is 0.83. Therefore, we find that taking $R_{av}$ as the average core radius and using it for the purpose of determining the flux that threads the wavefunction is a good approximation in the tubular-core model. Deviations are expected only when the wavefunction spreads substantially away from the core/shell interface. In this case different $m_l$ exhibit different $\langle r \rangle$, and the (metastable) degeneracy point becomes blurred and is no longer well defined (not shown).
FIG. 7. (a) Schematics of a solid-core, full-shell, semi-infinite nanowire with semiconductor electrostatic potential $U(r)$ in its interior. (b-c) LDOS in arbitrary units as a function of energy $\omega$ and normalized flux $\Phi/\Phi_0$ for a $R_{\text{core}} = 70$ nm, $R_{\text{shell}} = 80$ nm nanowire with shell coherence length $\xi = 70$ nm, core potential maximum $U_{\text{max}} = 0$ and different values of $U_{\text{min}}$, from a shallow (b) to a deep (d) dome-like profile. (e) Schematics of a full-shell nanowire-based normal-superconductor tunnel junction. The potential-barrier profile $U_b(z)$ in the uncovered semiconductor region between the normal metal (N) and the full-shell wire (S) only depends on $z$. (f-h) Differential conductance $dI/dV$ (in units of the conductance quantum $G_0$) versus normalized flux for the same full-shell nanowires as in (b-d), and for a sharp tunnel barrier of width 50 nm and height 60 meV (f), 110 meV (g) and 170 meV (h). (i-l) Same as (e-l) but for a longer tunnel junction; width 150 nm and heights 25 meV (j), 50 meV (k) and 110 meV (l). Parameters: Column (b,f,j) has $\Gamma_N = 90\Omega(0)$; column (c,g,k) has $\Gamma_N = 30\Omega(0)$; and column (d,h,l) has $\Gamma_N = 20\Omega(0)$. Other parameters like in Fig. 5.

$$U(r) = U_{\text{min}} + (U_{\text{max}} - U_{\text{min}}) \left( \frac{r}{R_{\text{core}}} \right)^{\nu},$$

(13)

see illustrations in Fig. 1(d) and Fig. 7(a). We specialize our simulations for $\nu = 2$, $U_{\text{max}} = 0$, and $U_{\text{min}}$ ranging from $-140$ meV to $-40$ meV, as suggested by microscopic calculations.\(^{13,17}\) The BdG effective Hamiltonian for the solid-core nanowire in the $m_L, k_z$-rotated basis then reads

$$\hat{H} = \left[ \begin{array}{c}
\left( m_L - \frac{1}{2} m \tau_z + \frac{1}{2} \Phi \frac{r^2}{R_{\text{core}}} \tau_z \right)^2 + \frac{k^2}{2m^*} + p_z^2 + U(r) \\
+ \Sigma_{\text{shell}}(\omega, 0) \end{array} \right] \tau_z,$$

\hspace{1cm} (14)

B. LDOS and transport

Even though $R_{\text{inner}} = 0$ in the solid-core model, we find that the LDOS dependence with $\Phi$ for $R_{\text{core}} = 70$ nm, shown in Figs. 7(b-d) for three values of $U_{\text{min}}$, is rather similar to that of tubular-core nanowires with $R_{\text{core}} = 70$ nm and $R_{\text{inner}} = 30-50$ nm, Figs. 5(d-f). The reason is that the potential $U(r)$ concentrates the wavefunction of the various modes to a region close to the core/shell interface. Indeed, we find that the wavefunction profiles for the different $m_l$ normal modes, depicted in Fig. 6(f) for various values of $U_{\text{min}}$, are very similar to those found for the tubular-core wire with varying semiconducting tube thicknesses, Fig. 6(e). Furthermore, the shape of the wavefunction in the solid-core model depends only weakly on the potential $U(r)$ for realistic values $U_{\text{min}} < -40$ meV. The main effect of increasing the band bending (taking more negative $U_{\text{min}}$ values) is increasing the number of occupied $m_l$ modes, and hence the number of CdGM-Van Hoves visible within each lobe [compare Fig. 7(b) to Fig. 7(d)].

In a typical experiment, the local subgap spectral structure in hybrid wires is measured via tunneling-transport spectroscopy. The technique measures the differential conductance $dI/dV$ between a normal probe at bias $V$ and a grounded wire, across a gate-tuneable barrier. For high and sharp barriers, it was shown that the $dI/dV$ becomes proportional to the BdG LDOS at the contact,\(^{57,58}\) hence the name tunneling spectroscopy. We confirm this by computing $dI/dV$ in the non-interacting Green’s function formalism across a sharp $L_b = 50$ nm-long Gaussian $U_b(z)$ barrier, with a normal probe defined using the same model as the solid-core nanowire but with-
out the shell-induced $\Sigma_{\text{shell}}$. The resulting $dI/dV$ indeed matches the LDOS closely, see Fig. 7(f-h), with three notable differences that can be attributed to the small but finite $L_0$. First, Van Hove singularities appear much sharper in the $dI/dV$ than in the LDOS, to the point that they could be easily mistaken for discrete subgap levels. Second, a small particle-hole \( V \rightarrow -V \) asymmetry is visible in $dI/dV'$, whereas the BdG LDOS is symmetric by definition. Last, and most significant, the small but finite $L_0$ makes the barrier more sensitive to modes with smaller $|m_L|$ values [see $m_L$ labels in Fig. 8(b), showing a case similar to Fig. 7(c)]. For smaller (larger) values of $|m_L|$, different subbands are deeper (shallower) in $k_z$-space, which translates into a slower (faster) evanescent decay inside the barrier. This makes the transmission probability through the barrier to acquire a strong $m_L$ dependence as barrier length increases. As a result, Van Hove singularities with larger $|m_L|$ appear much fainter in the $dI/dV$, or they may even become undetectable.

Finally, Fig. 8(b) shows the radially-resolved LDOS in the middle of the $n = 1$ lobe. This is closely related to the profile of the core wavefunctions that were shown in Fig. 6(f), but this time including their coupling to the shell. We confirm that most of the subgap states remain concentrated around a certain $R_{av}$ inside the core [see dashed line in panel 8(b)]. We can see that, except for states close to the gap edge, there is a rather small leakage of core states into the shell, even for strong proximity effect (\( \Omega(0) \approx \Omega(0) \), perfect epitaxial contact). Note also the existence of other fainter modes, particularly at low energy, with maxima away from $R_{av}$ and closer to the nanowire axis. These correspond Van Hove singularities in higher radial-momentum subbands, which may become populated in the lowest $|m_L|$ sector for dense enough and/or thick enough nanowires.

The concentration of all the lowest radial subbands around a common $R_{av}$ supports our interpretation of the consistent skewness of the CdGM analogs with flux. It also points to an interesting simplification of our solid-core model, which we dub the modified hollow-core model, identical to the original hollow-core model in Sec. III but with $R_{\text{core}}$ replaced by $R_{av}$. The $U$-transformation $\hat{H}$ thus reads

$$
\hat{H} = \left[ \left( m_L - \frac{1}{2} n \tau_z + \frac{1}{2} \frac{\Phi}{\Phi_0} \frac{R_{av}^2}{R_{LP}^2} \tau_z \right)^2 + k_z^2 \frac{2 m^*}{2 m^* - \mu} \right] \tau_z + \Sigma_{\text{shell}}(\omega, 0).
$$

The self-energy $\Sigma_{\text{shell}}$ is kept the same as in the solid core model, with the same dependence on $d_{\text{shell}}$ and $R_{LP}$, see Eq. A6. The value of $\Gamma_N$ in Eq. 3, however, needs to be adjusted to keep $\Omega^*(0)$ as in the solid core case. The definition of $\Phi$ and the LP lobes remain unchanged. The factor $R_{av}^2/R_{LP}^2$ in the angular kinetic energy produces the required shift $\delta n_{\text{dip}}$ of the degeneracy points. The resulting modified hollow-core model above is almost trivial to solve numerically when compared to the solid core, and exhibits a similar phenomenology, as shown in the

![Fig. 8.](image)

**VI. ROLE OF SPIN-ORBIT COUPLING AND ZEEMAN SPLITTING**

Up to this point we have neglected SOC and Zeeman splitting in our models. In the context of topological superconductivity, SOC is a crucial ingredient as it controls the magnitude of the topological gap and localizes the Majorana modes in these systems. However, as we show in this section, the SOC does not have a significant effect on the rest of the subgap spectrum, whereas the Zeeman coupling produces only a small splitting of
otherwise degenerate CdGM analog states.

The generalization of the Hamiltonian in Eq. (14) to include SOC was discussed in Ref. [11]. It involves writing the canonical transformation $\mathcal{U}$ in terms of the eigenvalues $m_J$ of the total generalized angular momentum, $J_z = -i\partial_z + \frac{1}{2}\sigma_z + \frac{1}{4}n\tau_z$ (instead of the $m_L$ eigenvalues of the generalized orbital momentum $L_z$ used so far). In the presence of a radial SOC, the $m_J$’s are good quantum numbers of the Hamiltonian eigenstates even with Zeeman. Note that now $m_J$ is (half-integer) integer in the (even) odd lobes. The Zeeman effect can be included in the BdG Hamiltonian $\tilde{H}$ by adding the term

$$V_Z = \frac{1}{2}g\mu_B B_z \sigma_z,$$

(16)

where $\mu_B$ is the Bohr magneton and $g$ is the nanowire Landé $g$-factor. Following this scheme, we computed the LDOS of a solid-core nanowire with and without Zeeman and SOC. We use $g = 12$ for our simulations and a SOC of the form $\alpha(r)^2 \cdot (\sigma \times \mathbf{r})^2$. Here $\alpha(r)$ is written, using a standard 8-band model in terms of the radial potential gradient $\partial_r U(r)$,

$$\alpha(r) = \alpha_0 \partial_r U(r) = \frac{P^2}{3} \left[ \frac{1}{\Delta_g^2} - \frac{1}{(\Delta_{\text{off}}^2 + \Delta_g^2)^2} \right] \partial_r U(r).$$

(17)

Using the Kane parameter $P = 919.7$ meV nm, the semiconductor gap $\Delta_g = 417$ meV and split-off gap $\Delta_{\text{off}} = 390$ meV, relevant for InAs, we obtain $\alpha_0 = 1.19$ nm$^{-2}$ (see Ref. [60] for more elaborate approximations).

The simulations are shown in Figs. 9(a,b) for $g = \alpha = 0$ and in Figs. 9(c,d) with finite $g$ and $\alpha$. In the two rows we choose different values of $U_{\text{min}}$ that result in topologically trivial and non-trivial phases for the $m_J = 0$ sector when SOC is included, see Figs. 9(c) and 9(d), respectively. In the latter we note that the SOC indeed produces a sharp Majorana zero mode with $m_J = 0$ throughout the first lobe, as expected (recall that the nanowire is semi-infinite). More generally, SOC also introduces a few additional, very sharp subgap features, while Zeeman produces a small splitting of each Van Hove peak. Overall, however, the CdGM analog spectrum remains almost unperturbed.

VII. SUMMARY AND CONCLUSIONS

We have shown that full-shell nanowires can host analogs of the CdGM states in type-II superconductors, which result from flux quantization in the LP effect. Although the flux itself is not quantized as in an Abrikosov vortex, the integer number of superconducting-phase twists in finite-$n$ LP lobes similarly stabilizes a variety of low-energy subgap states in the nanowire core. Despite the various differences mentioned in the introduction between the original CdGM states and their full-shell analogs, their essential nature is the same: they are subgap states resulting from the two-dimensional confinement of a superconductor with a finite winding in its phase.

At a more detailed level, however, full-shell CdGM analogs are Van Hove singularities in propagating core subbands with a far richer structure. Their ring-like wavefunction within the wire cross section is maximal around a characteristic radius $R_{\text{av}}$ that depends on the dome-like potential profile within the semiconductor. This turns out to have important consequences for their evolution with flux, $\Phi$.

We found that degeneracy points appear, where all Van Hove singularities coalesce, and which shift toward larger absolute values of the threading flux $\Phi$ for $n \neq 0$ lobes, even disappearing beyond the lobe edge. This shift occurs as $R_{\text{av}}$ is reduced relative to $R_{\text{core}}$, or as $R_{L \text{P}}$ increases with respect to $R_{\text{core}}$. The shift leaves behind a bundle of CdGM-Van Hove singularities that fill the whole LP-modulated parent gap $\Omega(\Phi)$ towards the left (right) side of each $n > 0$ ($n < 0$) lobes, but that tends to leave a characteristically shifted induced gap $\Omega^*(\Phi)$ on the opposite side. The degeneracy-point shift is proportional to $n$, so that the shifted gap is more visible in the $|n| = 1$ lobe and tends to disappear faster with decreasing $R_{\text{av}}$ for $|n| \geq 2$.

Furthermore, due to their orbital coupling to the field, CdGM analogs disperse with $\Phi$ with a positive or negative slope depending on their angular momentum $m_L$. For $\omega > 0$ states, the states with negative $m_L$ are pushed toward the parent gap edge, leaving mostly $m_L > 0$ states below the LP gap edge that exhibit a systematic skew-
ness towards higher $\Phi$ values within finite-$n$ LP lobes, see e.g. Fig. 8. The $m_L$ quantum numbers of these $\omega > 0$ CdGM analogs are ordered from smaller to larger values from the LP shell gap edge towards zero energy, while the opposite is true for $\omega < 0$ states. By contrast, this is the opposite order than CdGM states in type-II vortex cores.

The number, energy and skewness of the CdGM analogs in full-shell wires can be accessed experimentally using tunneling spectroscopy with sufficiently short tunnel barriers. The measured CdGM spectrum can be used to extract a wealth of otherwise inaccessible microscopic information about the electronic structure of the encapsulated nanowire. This includes details about the electrostatic potential profile inside the core, the resulting carrier density, its spatial distribution characterized by $R_{sv}$, the angular momentum of each mode or the transparency of the core/shell interface. The value of $R_{sv}$ can furthermore be used to define a modified hollow-core model that qualitatively captures most of the spectral features of the more complex microscopic model. Our analysis remains robust when introducing additional complexity, such as SOC, Zeeman splittings or even non-cylindrical nanowire cross sections.

In devices with longer tunneling barriers, tunneling spectroscopy becomes less sensitive to CdGM analogs with higher angular momentum. Also, long barriers often carry the added problem that quantum-dot-like states, localized around the barrier, appear within the gap. These are not part of the bulk nanowire spectrum and strongly depend on the barrier details. It should be possible to verify experimentally that a measured subgap feature in the tunneling conductance is indeed a CdGM analog, and not a quantum dot state, by checking that it is largely insensitive to changes in the tunnel barrier. The CdGM analogs, as Van Hove singularities of the nanowire bandstructure, are associated to the extended bulk of the nanowire, not to the local barrier details. Two distinguishing properties of the CdGM analogs are their skewness with flux and the presence of a shifted gap. Since CdGM analogs are in fact van Hove singularities, they are only fully developed in sufficiently long nanowires. However, they should also be visible in shorter nanowires of length $L$ as a collection of longitudinally quantized levels with a level spacing $\sim 1/L$, concentrated around the energy of the $L \rightarrow \infty$ singularities.[13]

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**Appendix A: Little-Parks effect of a diffusive shell**

In this Appendix we summarize standard results for the dependence of the pairing amplitude $\Delta(\Phi)$ and superconductor energy gap $\Omega(\Phi)$ with flux $\Phi$ in a diffusive superconducting shell. We follow closely the presentation of Ref. [19].

The Abrikosov-Gor’kov theory[50,51] describes a superconductor in the presence of paramagnetic impurities. When the mean free path is small enough so that the superconductor is in the diffusive regime, the quasi-classical retarded Green’s function is given by

$$g^S(\omega) = -\pi \nu_F \frac{u(\omega)\tau_0 - \tau_x}{\sqrt{1 - u(\omega)^2}}, \quad (A1)$$

where $\nu_F$ is the density of state at the Fermi level (in the normal state) and $\tau_x$ are Pauli matrices in Nambu space. The complex function $u(\omega)$ is obtained as the solution of

$$u(\omega) = \frac{\omega}{\Delta(\Lambda)} + \frac{\Lambda u(\omega)}{\Delta(\Lambda) - \sqrt{1 - u(\omega)^2}}. \quad (A2)$$

It depends on the depairing parameter $\Lambda$ introduced by the spin-polarized impurities and on the pairing amplitude $\Delta(\Lambda)$, which at zero temperature is given by

$$\Delta(\Lambda) = \frac{\nu_F V_{eph}}{2} \int_{-\omega_D}^{\omega_D} d\omega \text{Re} \frac{1}{\sqrt{\omega^2 - 1}}, \quad (A3)$$

where $\text{Re}$ is the real part, $V_{eph}$ is the phonon-mediated effective electron-electron interaction and $\omega_D$ the Debye frequency. These quantities are related by the BCS relation $\Delta(0) = 2\omega_D c^{-1/\nu_F} V_{eph}$. For a given $\Delta(\Lambda)$ Eq. (A2) can be expressed as a fourth-order polynomial with root $u(\omega)$ chosen so as to satisfy the appropriate continuity and asymptotic behaviors for retarded Green’s functions.

For finite pair-breaking Abrikosov-Gor’kov found a closed form solution for the pairing amplitude

$$\ln \frac{\Delta(\Lambda)}{\Delta(0)} = -P \left( \frac{\Lambda}{\Delta(\Lambda)} \right),$$

$$P(z \leq 1) = \frac{\pi}{4} z,$$

$$P(z \geq 1) = \ln \left( z + \sqrt{z^2 - 1} \right) + \frac{z}{2} \text{arctan} \left( \frac{1}{\sqrt{z^2 - 1}} \right) - \frac{\sqrt{z^2 - 1}}{2z}, \quad (A4)$$

where $\Delta(0)$ is the pairing of the pure (ballistic) superconductor, i.e., for $\Lambda = 0$. Note that $\Lambda$ has energy units and is bounded by $0 \leq \Lambda \leq \Delta(0)/2$. The equation for $\Delta(\Lambda)$ has to be solved self-consistently.

Subsequently, Skalski et al.[22] found an analytical expression for the energy gap, defined by the edge of the branch cut at $u(\Omega)^2 = 1$ in Eq. (A1), and given by

$$\Omega(\Lambda) = \left( \frac{\Delta(\Lambda)^{2/3} - \Lambda^{2/3}}{3/2} \right)^{3/2}. \quad (A5)$$
Note that the energy gap $\Omega$ is only equal to the pairing amplitude $\Delta$ in the absence of depairing effects, and is smaller otherwise. There even exists a region of $\Lambda$ close to $\Delta(0)/2$ for which the gap in the excitation spectrum is zero even though the shell is still a superconductor in the sense of having a non-zero order parameter. This is the regime of so-called gapless superconductivity.

The problem of a superconductor containing paramagnetic impurities is very similar to the problem of an ordinary diffusive superconductor in the presence of an external magnetic field\textsuperscript{\cite{Lutchyn2010}}. Thus, we can identify the depairing parameter produced by the magnetic impurities above with an analogous depairing produced by the magnetic flux, $\Lambda(\Phi)$. Assuming now cylindrical symmetry, a standard Ginzburg-Landau theory of the LP effect\textsuperscript{\cite{Furusaki1990_1,Furusaki1990_2,Kopasz2015}} provides an explicit connection between flux and depairing

\[
\Lambda(\Phi) = \frac{\xi^2 k_B T_c}{\pi R_{LP}^2} \left[ 4 \left( n - \frac{\Phi}{\Phi_0(0)} \right)^2 + \frac{d_{shell}^2}{R_{LP}^2} \left( \frac{\Phi^2}{\Phi_0^2} + n^2 - \frac{1}{3} \right) \right],
\]

\[
n(\Phi) = \Phi/\Phi_0 = 0, \pm 1, \pm 2, \ldots,
\]

where $\xi$ is the diffusional superconducting coherence length and $T_c$ is the zero-flux critical temperature. At zero field $\Delta(0) = 0$, $\Omega(0) = \Delta(0)$, and $k_B T_c \approx \Omega(0)/1.76$, where $k_B$ is the Boltzmann constant.

The solution for Eqs. (A4)-(A6) is qualitatively different depending on the ratio $R_{LP}/\xi$ and $d_{shell}/R_{LP}$. It ranges from the non-destructive regime ($\Omega$ is non-zero, satisfied for $R_{LP}/\xi \gtrsim 0.6$ if $d_{shell} \to 0$) to the destructive regime ($\Omega$ vanishes in a finite window around odd half-integer $\Phi/\Phi_0$, satisfied for smaller $R_{LP}/\xi$\textsuperscript{29}). The different regimes both for $\Delta$ and $\Omega$ are represented in Fig. 2. As a guideline, some typical values representative of recent experiment\textsuperscript{\cite{Lutchyn2010,Peña2018}} are $\xi \sim 100$ nm, $R_{core} \sim 70$ nm, $d_{shell} \sim 10$ nm and $\lambda_L = 150$ nm. These parameters correspond to a superconductor in the dirty limit, which is the regime where the above theory is applicable, and the one relevant to current experiments.

**Appendix B: Self-energy from a diffusive shell**

The proximity effect of the diffusive superconducting shell described in App. A onto the semiconducting core of the full-shell nanowire can be accounted for by means of a self-energy $\Sigma_{shell}$ acting on the core’s surface. Using a tight-binding language where $t_f$ is the hopping parameter between a surface site of the core lattice and the shell, the shell self-energy reads

\[
\Sigma_{shell}(\omega, \varphi) = t_f^2 g^S(\omega, \varphi),
\]

where $g^S(\omega, \varphi)$ is the same one of Eq. (A1) but including now the dependence of the pairing amplitude with the fluxoid number $n$ and the polar angle $\varphi$.

\[
g^S(\omega, \varphi) = -\pi \nu_F \frac{u(\omega)\tau_0 - e^{in\varphi}\tau_x}{\sqrt{1 - u(\omega)^2}}.
\]

Defining $\Gamma_N = \pi \nu_F^2 t_f^2$ as the normal decay rate from the core into the shell, we can write

\[
\Sigma_{shell}(\omega, \varphi) = -\Gamma_N \frac{u(\omega)\tau_0 - e^{in\varphi}\tau_x}{\sqrt{1 - u(\omega)^2}},
\]

where $u(\omega)$ is given in Eq. (A2).

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