Reciprocity and orthogonality

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Abstract. Let $p$ be a prime and let $K$ be a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers such that the group $\mathbb{Q}_p^\times$ has order $p$. The $\mathbb{F}_p$-space $K^\times/K^{\times p}$ carries a natural filtration coming from the valuation on $K$, and a natural bilinear pairing coming from the reciprocity isomorphism for the exponent $p$. We determine the orthogonal filtration for this pairing. We also prove the analogous result for $p$-fields of characteristic $p$.

(1) Let $p$ be a prime number and let $K$ be a $p$-field — a field complete for a discrete valuation with finite residue field of characteristic $p$. Let $M$ be the maximal abelian extension of $K$ of exponent $p$, $G = \text{Gal}(M|K)$, $\overline{K}^\times = K^\times/K^{\times p}$, and $\rho : \overline{K}^\times \to G$ the reciprocity isomorphism (for the exponent $p$). If $K$ has characteristic $0$ and $\mathbb{Q}_p^\times$ has order $p$, then there is a natural pairing

$$G \times \overline{K}^\times \to \mathbb{Q}_p^\times, \ (\sigma, \bar{x}) \mapsto \sigma(y)y^{-1} \ (y \in M, y^p = x).$$

Similarly, if $K$ has characteristic $p$, then there is a natural pairing

$$G \times \overline{K}^+ \to \mathbb{F}_p, \ (\sigma, \bar{x}) \mapsto \sigma(y) - y \ (y \in M, \varphi(y) = x),$$

where $\overline{K}^+ = K^+/\varphi(K^+)$ and $\varphi(z) = z^p - z$ for $z$ in any $\mathbb{F}_p$-algebra.

(2) When combined with the reciprocity isomorphism $\rho : \overline{K}^\times \to G$, we get the hilbertian pairing

$$\overline{K}^\times \times \overline{K}^\times \to \mathbb{Q}_p^\times \ (\text{resp. } \overline{K}^\times \times \overline{K}^+ \to \mathbb{F}_p).$$

The main aim of this Note is to show that the filtrations on the two factors, coming from the filtration(s) on $K^\times$ (resp. on $K^\times$ and $K^+$), are orthogonal to each other for this pairing. This will be soon made precise.

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1. Notation

(3) \((p, K, o, p, k)\). Throughout, \(p\) is a prime number and \(K\) is a \(p\)-field — a field complete for a discrete valuation with finite residue field of characteristic \(p\). The ring of integers of \(K\) is \(o\), the unique maximal ideal of \(o\) is \(p\), and the residue field is \(k = o/p\).

(4) \((e, f)\). The (absolute) residual degree of \(K\) is \(f = [K : F_p]\). If \(K\) is an extension of \(Q_p\), then \(e = [K : Q_p]f^{-1}\) is the (absolute) ramification index; if \(K\) has characteristic \(p\), then \(e = +\infty\). We often say “\(e < +\infty\)” to mean that \(K\) has characteristic 0.

(5) \((\overline{K^x}, \overline{U}_i)\). For every \(i > 0\), let \(U_i = 1 + p^i\) be the kernel of \(o^x \to (o/p^i)^x\) and denote the image of \(U_i\) in the \(F_p\)-space \(\overline{K^x} = K^x/K^{x_p}\) by \(\overline{U}_i\). Note that the image \(o^x/o^{x_p}\) of \(o^x\) in \(\overline{K^x}\) is equal to \(U_1\) because \(o^x = U_1, k^x\) and \(k^{x_p} = k^x\). The image of \(x \in K^x\) in \(\overline{K^x}\) is denoted \(\bar{x}\).

(6) \((\varphi, \overline{K^+}, \overline{p^i}, \overline{o})\). For any \(F_p\)-algebra \(A\), denote by \(\varphi : A \to A\) the endomorphism \(\varphi(x) = x^p - x\) of the additive group \(A^+\). If \(e = +\infty\) (so that \(K\) is an \(F_p\)-algebra), we put \(\overline{K^+} = K^+ / \varphi(K^+)\) and, for every \(i \in \mathbb{Z}\), denote the image of \(p^i\) in the \(F_p\)-space \(\overline{K^+}\) by \(\overline{p^i}\). Sometimes we also denote \(\overline{p^i}\) by \(\overline{o}\); it equals \(\overline{o^+}/\varphi(\overline{o^+})\). The image of \(x \in K^+\) in \(\overline{K^+}\) is denoted \(\bar{x}\).

(7) \((b_p^{(i)})\). We denote by \(b_p^{(i)}\) \((i > 0)\) be the sequence of positive integers \(\equiv 0 \pmod{p}\), namely \(b_p^{(i)} = i + [(i - 1)/(p - 1)]\), where, for every \(x \in \mathbb{R}\), \([x]\) is the largest integer in the interval \(] - \infty, x]\).

(8) \((c)\). If \(e < +\infty\) and \(e \equiv 0 \pmod{p - 1}\) (for example when \(pK^X\) has order \(p\)), then we abbreviate \(c = (p - 1)^{-1}e\). Note that in this case \(pc = b_p^{(e)} + 1 = e + c\).

(9) \((v, \bar{v})\). The surjective valuation \(K^x \to \mathbb{Z}\) is denoted by \(v\) (so that \(e = v(p)\)); it induces an isomorphism \(\bar{v} : \overline{K^x}/\overline{U}_1 \to \mathbb{Z}/p\mathbb{Z}\). For this reason, in order to determine the structure the the filtered \(F_p\)-space \(\overline{K^x}\), it is enough to study the filtered \(F_p\)-space \(\overline{U}_1\).

2. The filtered \(F_p\)-space \(\overline{U}_1\)

(10) Let us determine the filtration on the \(F_p\)-space \(\overline{U}_1\). We will see that the dimension \(d = \dim_{F_p} \overline{U}_1\) is finite or infinite according as \(e < +\infty\) or \(e = +\infty\). When \(e < +\infty\), one has \(d = [K : Q_p] + \dim_{F_p}(pK^x)\), so there are two subcases according as \(pK^x\) is trivial or has order \(p\). More precisely, by studying what the endomorphism \((\ )^p : U_1 \to U_1\) does to the filtration on \(U_1\), one determines the filtration on \(\overline{U}_1\) as follows.
Suppose that \( e = +\infty \). For \( i > 0 \), the inclusion \( \bar{U}_{i+1} \subset \bar{U}_i \) is an equality if \( i \equiv 0 \) (mod. \( p \)), and has codimension \( f \) if \( i \not\equiv 0 \) (mod. \( p \)).

Suppose that \( e < +\infty \) and \( pK^\times \) is trivial. Then \( \bar{U}_i \) is trivial for \( i > \beta(p) \). For \( i \in [1, \beta(p)] \), the inclusion \( \bar{U}_{i+1} \subset \bar{U}_i \) is an equality if \( i \equiv 0 \) (mod. \( p \)), and has codimension \( f \) if \( i \not\equiv 0 \) (mod. \( p \)).

Suppose that \( e < +\infty \) and \( pK^\times \) has order \( p \). Then \( \bar{U}_i \) is trivial for \( i > \beta(p) \) and \( \bar{U}_pc \) has order \( p \). For \( i \in [1, \beta(p)] \), the inclusion \( \bar{U}_{i+1} \subset \bar{U}_i \) is an equality if \( i \equiv 0 \) (mod. \( p \)), and has codimension \( f \) if \( i \not\equiv 0 \) (mod. \( p \)).

Remark. Whenever \( \bar{U}_{i+1} \) has codimension \( f \) in \( \bar{U}_i \), the quotient \( \bar{U}_i/\bar{U}_{i+1} \) is canonically isomorphic to \( U_i/U_{i+1} \).

Denote the reduction map \( \sigma \rightarrow k \) by \( a \mapsto \hat{a} \) and let \( S : k \rightarrow F_p \) be the trace map. If \( e < +\infty \) and if \( \zeta \in pK^\times \) has order \( p \), then \( v(p\pi) = \beta(p) \), where \( \pi = 1 - \zeta \), and the map \( 1 + a_p\pi \mapsto \zeta^a (a \in \sigma) \) is an isomorphism \( \bar{U}_pc \rightarrow pK^\times \), independent of \( \zeta \). For proofs and more information, see [2, Prop. 42], for example.

3. The filtered \( F_p \)-space \( \overline{K^+} \)

Suppose that \( e = +\infty \). In analogy with the foregoing, by studying what the endomorphism \( \varphi : K^+ \rightarrow K^+ \) does to the filtration on \( K^+ \) (by the powers \( p^i \) (\( i \in \mathbb{Z} \)) of \( p \)), one determines the filtration on \( \overline{K^+} \).

For every \( i < 0 \), one has \( \overline{p^i} = \{0\} \). The group \( \overline{p^i} = \bar{\sigma} \) has order \( p \). For every \( i < 0 \), the inclusion \( \overline{p^{i+1}} \subset \overline{p^i} \) is an equality if \( i \equiv 0 \) (mod. \( p \)), and has codimension \( f \) if \( i \not\equiv 0 \) (mod. \( p \)).

Remark. Denote the passage to the quotient \( \sigma \rightarrow \bar{\sigma} \) (resp. the reduction map \( \sigma \rightarrow k \)) by \( x \mapsto \bar{x} \) (resp. \( x \mapsto \hat{x} \)), and the trace map \( k \rightarrow F_p \) by \( S \). Then the map \( \bar{a} \mapsto S(\hat{a}) (a \in \sigma) \) is an isomorphism \( \sigma \rightarrow F_p \). For proofs and more information, see [3, Prop. 11], for example.

4. Breaks and levels

Let \( E \) be a cyclic extension of \( K \) of degree \( p \). The ramification filtration on the group \( G = \text{Gal}(E|K) \) has a unique break \( \varepsilon(E) \) (the integer \( j \) such that \( G^j = G, G^{j+1} = \{1\} \)). We have \( \varepsilon(E) = -1 \) if and only if \( E \) is unramified over \( K \); otherwise, \( \varepsilon(E) > 0 \). We recall what the possibilities for \( \varepsilon(E) \) are, and how it is related to another invariant of \( E \) in some cases.

Suppose that \( e < +\infty \) and \( pK^\times \) has order \( p \), and let \( D \subset \overline{K^\times} \) be a line (a 1-dimensional subspace). There is a unique integer \( j \) such that
D \subset \bar{U}_j$ but $D \not\subset \bar{U}_{j+1}$, with the convention that $\bar{U}_0 = \overline{K^\times}$. We define the level $\delta(D)$ of $D$ to be $pc - j$. We have seen that $\delta(D) \in [0, pc]$, and if $\delta(D) \equiv 0 \pmod{p}$, then $\delta(D) = 0$ or $\delta(D) = pc$.

21 Suppose that $e = +\infty$ and let $D \subset \overline{K^+}$ be a line. There is a unique integer $j$ such that $D \subset \overline{p^j}$ but $D \not\subset \overline{p^{j+1}}$, and we define the level $\delta(D)$ of $D$ to be $pc - j$. We have $\delta(D) \in [0, +\infty]$, and if $\delta(D) \equiv 0 \pmod{p}$, then $\delta(D) = 0$, as we have seen.

22 Suppose that $e < +\infty$ and $pK^\times$ has order $p$ (resp. $e = +\infty$). Let $E$ be a cyclic extension of $K$ of degree $p$, and let $D \subset \overline{K^\times}$ (resp. $D \subset \overline{K^+}$) be the line such that $E = K(\sqrt[p]{D})$ (resp. $E = K(\sqrt[p]{-1}(D))$). If $E$ is unramified over $K$, then $D = \bar{U}_{pc}$ (resp. $D = \bar{o}$). If $E$ is ramified over $K$, then $\varepsilon(E) = \delta(D)$.

23 It follows that if $E$ is ramified over $K$, then $\varepsilon(E) = b_p^{(i)}$ for some $i \in [1, e]$ or $\varepsilon(E) = pc$ (resp. $\varepsilon(E) = b_p^{(i)}$ for some $i > 0$), and all these possibilities do occur.

24 Remark. The only case not covered by this proposition is when $e < +\infty$ and $pK^\times$ is trivial. One can compute $\varepsilon(E)$ in this case by replacing $K$ by $K' = K(\sqrt[p]{T})$ and $E$ by $E' = EK'$. If $E$ is ramified over $K$, then $\varepsilon(E) = b_p^{(i)}$ for some $i \in [1, e]$, and all these possibilities do occur. In particular, $\varepsilon(E) \not\equiv 0 \pmod{p}$, as in the case $e = +\infty$. See [2, Prop. 63], for example. In all three cases, there are only finitely many $E$ with a given $\varepsilon(E)$, and their number can be easily computed.

5. Orthogonality

25 Recall that for every galoisian extension $M$ of $K$, the profinite group $G = \text{Gal}(M|K)$ comes equipped with a decreasing filtration $(G^t)_{t \in [-1, +\infty]}$ — the ramification filtration in the upper numbering — by closed normal subgroups which is separated ($\bigcap_t G^t = \{1\}$) and exhaustive ($G^{-1} = G$). We put $G^{t+} = \bigcup_{s > t} G^s$, and call $t$ a ramification break for $G$ if $G^{t+} \neq G^t$, as in the case of degree-$p$ cyclic extensions above. In general, $G^0$ is the inertia subgroup of $G$ and $G^{0+}$ is the (wild) ramification subgroup.

26 We take $M$ to be the maximal abelian extension of $K$ of exponent $p$ and determine the ramification breaks of $G$. When $e < +\infty$ and $pK^\times$ has order $p$ (resp. $e = +\infty$), so that $M = K(\sqrt[p]{K^\times})$ (resp. $M = K(\sqrt[p]{-1}(K))$), we have the pairing

$$G \times \overline{K^\times} \to pK^\times \quad \text{(resp.} \ G \times \overline{K^+} \to F_p),$$

as recalled in the Introduction, and we show that the filtration on $\overline{K^\times}$ (resp. $\overline{K^+}$) is orthogonal to the filtration on $G$ in a certain precise sense.
The maximal tamely ramified extension $M_1$ of $K$ in $M$ is the unramified degree-$p$ extension $M_1 = K(\sqrt[p]{U_{pc}})$ (resp. $M_1 = K(\varphi^{-1}(\mathfrak{c}))$), as we have recalled.

Suppose that $e < +\infty$ and $pK^\times$ has order $p$. We have $G^t = G^1$ for $t \in [-1, 1]$, and, for $t \in [0, pc + 1]$,

$$G^{t\perp} = \bar{U}_{pc-\lceil t \rceil + 1}$$

under $G \times \overline{K^\times} \to \overline{pK^\times}$. The positive ramification breaks in the filtration on $G$ occur precisely at the $b^{(i)}_p (i \in \mathbb{[1, e]})$ and at $pc$. $\square$

Suppose that $e = +\infty$. We have $G^t = G^1$ for $t \in [-1, 1]$, and, for $t > 0$,

$$G^{t\perp} = \mathbb{F}_p^{-\lceil t \rceil + 1}$$

under $G \times \overline{K^\times} \to \mathbb{F}_p$. The positive ramification breaks in the filtration on $G$ occur precisely at the $b^{(i)}_p (i > 0)$. $\square$

Remark. For the proofs, see [2] and [3] respectively. Now suppose that $e < +\infty$ and $pK^\times$ is trivial, and put $K' = K(\sqrt[p]{\mathbb{T}})$, $\Gamma = \text{Gal}(K'|K)$ and $M' = MK'$. Let $\psi : \Gamma \to \mathbb{F}_p^\times$ be the cyclotomic character giving the action of $\Gamma$ on $pK^\times$. It can be checked that the subspace $D \subset K'^\times/\psi^{e}\mathbb{N}$ such that $M' = K'(\sqrt[p]{D})$ is precisely the $\omega$-eigenspace for the action of $\Gamma$. Hence or otherwise, one shows that the positive ramification breaks in the filtration on $G$ occur precisely at the $b^{(i)}_p (i \in \mathbb{[1, e]})$.

Let $L$ be an abelian extension of $K$ of exponent $p$. It follows from the foregoing and Herbrand’s theorem — the ramification filtration in the upper numbering is compatible with the passage to the quotient — that the ramification breaks of $\text{Gal}(L|K)$ are integers. This is a special case of the Hasse-Arf theorem, valid for all abelian extensions of local fields. The advantage of the direct proof in this special case is that one can specify which integers occur.

6. Norms

Suppose that $e < +\infty$ and $pK^\times$ has order $p$ (resp. $e = +\infty$), and let $i \in [0, pc + 1]$ (resp. $i \in \mathbb{N}$) be an integer. Let $L_i = K(\sqrt[p]{U_{pc-i+1}})$ (resp. $L_i = K(\varphi^{-1}(p^{-i+1}))$). Recall that $L_0 = K$, and that $L_1$ is the unramified degree-$p$ extension of $K$. The inductive limit of the $L_i$ (which is nothing but $L_{pc+1}$ if $e < +\infty$ and $pK^\times$ has order $p$) is equal to the maximal abelian extension $M$ of $K$ of exponent $p$.

We have $N_{L_i|K}(L_i^\times) = U_iK^{x,p}$ for every $i \in [0, pc + 1]$ (resp. $i \in \mathbb{N}$), with the convention that $U_0 = K^\times$. $\square$
(34) All this can presumably be proved by studying, as in [4, Chapter V], what the norm maps \( N_{L_i|K} : L_i^\times \to K^\times \) do to the filtrations. It follows that \( K^\times /N_{L_i|K}(L_i^\times ) = \overline{K^\times }/\overline{U}_i \) for every \( i \in [0, pc + 1] \) (rep. \( i \in \mathbb{N} \)). When \( i = 1 \), the surjective valuation on \( K \) induces an isomorphism \( \overline{v} : K^\times /N_{L_1|K}(L_1^\times ) \to \mathbb{Z}/p\mathbb{Z} \).

7. Reciprocity

(35) Keep the previous notation and continue to suppose that \( e < +\infty \) and \( pK^\times \) has order \( p \) (resp. \( e = +\infty \)) As \( L_1 \) is the unramified degree-\( p \) extension of \( K \), the group \( \text{Gal}(L_1|K) \) has a canonical generator \( \sigma \) — the one which reduces to the \( k \)-automorphism \( x \mapsto x^q \) (\( q = \text{Card } k \)) of the residue field of \( L_1 \).

(36) There is a unique isomorphism \( \rho_1 : \overline{K^\times }/\overline{U}_1 \to \text{Gal}(L_1|K) \) such that \( \rho_1(\overline{\pi}) = \sigma \) for every uniformiser \( \pi \) of \( K \). The kernel of the resulting map \( K^\times \to \text{Gal}(L_1|K) \) is \( N_{L_i|K}(L_i^\times ) \).

(37) For \( i \in [1, pc+1] \) (resp. \( i > 0 \)), and for every intermediate extension \( K \subset E \subset L_i \) we have the galoisian projection \( \text{Gal}(L_i|K) \to \text{Gal}(E|K) \). In particular, we hav the projection \( \text{Gal}(L_i|K) \to \text{Gal}(L_1|K) \). Recall also that \( K^\times /N_{L_i|K}(L_i^\times ) = \overline{K^\times }/\overline{U}_i \).

(38) For \( i \in [1, pc + 1] \) (resp. \( i > 0 \)), there is a unique isomorphism \( \rho_i \) making the square

\[
\begin{array}{ccc}
\overline{K^\times }/\overline{U}_i & \xrightarrow{\rho_i} & \text{Gal}(L_i|K) \\
\downarrow & & \downarrow \\
\overline{K^\times }/\overline{U}_1 & \xrightarrow{\rho_1} & \text{Gal}(L_1|K)
\end{array}
\]

commute, and such that for every intermediate extension \( K \subset E \subset L_i \), the kernel of the map \( K^\times \to \text{Gal}(E|K) \) deduced from \( \rho_i \) is \( N_{E|K}(E^\times ) \).

(39) Let \( M = K(\sqrt[p]{K^\times }) \) (resp. \( M = K(\rho^{-1}(K)) \)) be the maximal abelian extension of \( K \) of exponent \( p \). It follows from the foregoing that there is a unique isomorphism \( \rho : K^\times \to \text{Gal}(M|K) \) of profinite groups such that the resulting map \( K^\times \to \text{Gal}(L_1|K) \) takes every uniformiser of \( K \) to \( \sigma \) and such that for every intermediate extension \( K \subset E \subset M \) of finite degree, the kernel of the resulting map \( K^\times \to \text{Gal}(E|K) \) is \( N_{E|K}(E^\times ) \). Moreover, \( \rho(\overline{U}_i) = \text{Gal}(M|K)^i \) for every \( i \in [1, pc + 1] \) (resp. \( i > 0 \)).
8. Orthogonality bis

(40) Suppose that $e < +\infty$ and $\mathfrak{p}K^\times$ has order $p$. Combining the kummerian pairing $G \times \bar{K}^\times \to \mathfrak{p}K^\times$ with the reciprocity isomorphism $\rho : \bar{K}^\times \to G$, we get the hilbertian pairing $\bar{K}^\times \times \bar{K}^\times \to \mathfrak{p}K^\times$.

(41) For every $i \in [0, pc + 1]$, the orthogonal complement of $\bar{U}_i$ for the hilbertian pairing is $\bar{U}_{pc-i+1}$.

(42) Suppose finally that $e = +\infty$. As before, combining the pairing $G \times \bar{K}^+ \to \mathbb{F}_p$ with the reciprocity isomorphism $\rho : \bar{K}^\times \to G$, we get the hilbertian pairing $\bar{K}^\times \times \bar{K}^+ \to \mathbb{F}_p$.

(43) For every $i \in \mathbb{N}$, the orthogonal complement of $\bar{U}_i$ for the hilbertian pairing is $\mathbb{F}_p^{-i+1}$ and vice versa.

(44) Remark. In both cases, $\bar{U}_0 = \bar{K}^\times$ by convention. This Note was written in response to a question by Hatice Boylan on MathOverflow [1].

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