The multiplicative attribute graph (MAG) model was introduced by Kim and Leskovec as a mathematically tractable model of certain classes of real-world networks. It is an instance of hidden graph models, and implements the plausible idea that network structure is collectively shaped by attributes individually associated with nodes. These authors have studied several aspects of this model, including its connectivity, the existence of a giant component, its diameter and the degree distribution. This was done in the asymptotic regime when the number of nodes and the number of node attributes both grow unboundedly large, the latter scaling with the former under a natural admissibility condition. In the same setting, we explore the existence (or equivalently, absence) of isolated nodes, a property not discussed in the original paper. The main result of the paper is a zero-one law for the absence of isolated nodes; this zero-one law coincides with that obtained by Kim and Leskovec for graph connectivity (although under slightly weaker assumptions). We prove these results by applying the method of first and second moments in a non-standard way to multiple sets of counting random variables associated with the number of isolated nodes.
1 Introduction

The multiplicative attribute graph (MAG) model is a mathematically tractable network model recently introduced by Kim and Leskovec [11, 12]; it implements the plausible idea that network structure is collectively shaped by attributes individually associated with nodes. MAG models are a special case of hidden variable models discussed in earlier literature where each node is endowed with a set of intrinsic (“hidden”) attributes, e.g., authority, social success, wealth, etc., and the creation of a link between two nodes expresses a mutual “benefit” based on their attributes, e.g., see references [1, 2, 8, 16, 17, 20] for examples. Here we consider the homogeneous binary MAG model where the basic idea is implemented as follows: With \( n \) nodes in the network, the attributes are modeled as \( \{0, 1\}^L \)-valued random variables (rvs) \( A(1), \ldots, A(n) \) which are assumed to be independent and identically distributed (i.i.d.). Conditionally on \( A(1), \ldots, A(n) \), edges are then created in a mutually independent manner with

\[
\Pr \left[ \text{An edge exists between node } u \text{ and node } v \mid A(1), \ldots, A(n) \right] = Q_L(A(u), A(v)), \quad u \neq v, \quad u, v = 1, \ldots, n
\]

for some Borel symmetric mapping \( Q_L : \{0, 1\}^L \times \{0, 1\}^L \to [0, 1] \) (whose form is to be specified shortly).

For each \( u = 1, \ldots, n \), the \( L \) components \( A_1(u), \ldots, A_L(u) \) of the attribute vector \( A(u) \) for node \( u \) are assumed to be i.i.d. \( \{0, 1\} \)-valued Bernoulli rvs with

\[
\Pr[A_\ell(u) = 1] = \mu(1) \quad \text{and} \quad \Pr[A_\ell(u) = 0] = \mu(0), \quad \ell = 1, \ldots, L
\]

for some \( 0 < \mu(1), \mu(0) < 1 \) such that \( \mu(0) + \mu(1) = 1 \). The homogeneous binary MAG model is then specified by taking

\[
Q_L(A(u), A(v)) = \prod_{\ell=1}^L q(A_\ell(u), A_\ell(v)), \quad u \neq v, \quad u, v = 1, \ldots, n
\]

for some symmetric \( 2 \times 2 \) matrix \( Q \equiv (q(a, b)) \) (with \( 0 < q(a, b) < 1 \), \( a, b = 0, 1 \) and \( q(0, 1) = q(1, 0) \)). Formal definitions and a complete construction are provided in Section 2. A useful way of thinking about this MAG model, especially relevant in the context of social networks, is to imagine that each network participant or node, answers a set of \( L \) binary (YES/NO) questions, e.g., Does node \( u \) exercise regularly? Does node \( u \) belong to a book club? etc. Then, \( A_\ell(u) = 1 \) (resp. \( A_\ell(u) = 0 \)) can be interpreted as a YES (resp. NO) answer to the \( \ell \)-th question answered by node \( u \).

In [11] Kim and Leskovec studied several aspects of this model, including its connectivity, the existence of a giant component, its diameter and the degree distribution. This was done in the asymptotic regime when the number \( n \) of nodes and the number \( L \) of attributes both grow unboundedly large, the latter scaling with the former under the condition \( L_n \sim \rho \ln n \) for some \( \rho > 0 \) (in which case the scaling \( n \to L_n \) is said to be \( \rho \)-admissible). In the same setting we explore the existence (or equivalently, absence) of isolated nodes in the MAG model, a property which was not discussed in the original paper [11]. The main result is a zero-one law for the absence of isolated nodes; it takes a different form depending on whether \( 1 + \rho \ln \mu(0) > 0 \) or \( 1 + \rho \ln \mu(0) < 0 \), the appropriate version being recorded in Theorem 3.1 and Theorem 3.2 respectively. These results have the same structure as the zero-one law for graph connectivity obtained by Kim and Leskovec [11, Thm. 4.2, p. 126] but are given here under weaker conditions. See Section 3 for details.

An undirected graph contains no isolated nodes if it is connected, but the converse is clearly not true in general. However, in many random graph models these two graph properties obey identical
We now give a summary of how this tailor-made approach is implemented:

zero-one laws; this is known to occur for Erdős-Rényi graphs [5,6], random geometric graphs [9,14], random key graphs [15,18] and random threshold graphs [13], to mention a few examples. While this is not universally valid as can be seen from $k$-out-$n$ random graphs [7] (also called pairwise graphs in [19]), our results establish its validity for the MAG model in the limiting regime considered here.

To prove Theorem 3.1 and Theorem 3.2 we apply the method of first and second moments to various count variables associated with the number of isolated nodes in MAG graphs: Traditionally this well-worn approach is carried out in terms of the rv $I_n(L)$ which counts the number of isolated nodes in the MAG model with $n$ nodes and $L$ attributes per node. It relies on the basic observation that $\mathbb{P}[I_n(L) = 0]$ coincides with the probability that there are no isolated nodes in the graph, and leverages the elementary inequalities

$$1 - \mathbb{E}[I_n(L)] \leq \mathbb{P}[I_n(L) = 0] \leq 1 - \frac{(\mathbb{E}[I_n(L)])^2}{\mathbb{E}[I_n(L)^2]},$$

(3)

See Section 4 for details. In principle a successful completion of this program requires exploring the limiting behavior of the sequences of moments $\{\mathbb{E}[I_n(L)]\}, n = 2, 3, \ldots$ and $\{\mathbb{E}[I_n(L)^2]\}, n = 2, 3, \ldots$ under the appropriate conditions.

For MAG models this is easier said than done, and we must resort to an indirect (and much finer) analysis: While the method of first moment can be successfully used on the rv $I_n(L)$ in a rather straightforward manner, applying the second moment method to the same rv $I_n(L)$ is problematic due to the complicated expressions for the quantities involved. Instead we introduce additional count variables, namely the rv $I_n^{(\ell)}(L)$ which tallies the number of isolated nodes (amongst the $I_n(L)$ isolated nodes) who have answered YES to exactly $\ell$ of the $L$ questions with $\ell = 0, 1, \ldots, L$. Key here is the observation that

$$\mathbb{P}[I_n(L) = 0] \leq \mathbb{P}[I_n^{(\ell)}(L) = 0], \quad \ell = 0, 1, \ldots, L.$$  

(4)

We now give a summary of how this tailor-made approach is implemented:

(i) We start the analysis in Section 5 by evaluating the first two moments of these count variables; expressions are given for the first moments in Lemma 5.1 and for the second moments in Lemma 5.2 (with the evaluation being completed in Appendix 12).

(ii) Theorem 3.1 is established in Section 7 and its proof is rather short: We begin with auxiliary “zero-infinity” laws for the first moments under conditions that mirror the ones of Theorem 3.1. Proposition 7.1 deals with the first moments of $\{I_n(L_n), n = 1, 2, \ldots\}$ and allows us to show $\lim_{n \to \infty} \mathbb{P}[I_n(L_n) = 0] = 1$ by the method of first moment under the conditions for the one-law. Proposition 7.2 captures the behavior of the first moments $\{\mathbb{E}[I_n^{(0)}(L_n)]\}, n = 1, 2, \ldots,$ and leads to the desired zero-law follows via (4) (with $\ell = 0$) upon showing that $\lim_{n \to \infty} \mathbb{P}[I_n^{(0)}(L_n) = 0] = 0$ by the method of second moment.

(iii) The proof of Theorem 3.2 is in the same vein but is a lot more involved; its major components are presented in Section 9. Here, two auxiliary “zero-infinity” laws for the first moments are needed that parallel Theorem 3.2. Proposition 8.1 deals with the first moments of the rvs $\{I_n(L_n), n = 1, 2, \ldots\}$ (as did Proposition 7.1 under the assumptions of Theorem 3.1). The first moment behavior of the rvs $\{I_n^{(\ell_n)}(L_n), n = 1, 2, \ldots\}$ is obtained for certain integer-valued sequences $n \to \ell_n$ associated with the scaling $n \to L_n$ under certain conditions. This result, which is reported in Proposition 8.2, is established in Section 10.
(iv) We give two different proofs to Proposition 8.1. The first one is presented in Section 11, and makes use of Stirling's approximation to evaluate the asymptotic behavior of various combinatorial quantities. The second proof of Proposition 8.1 is given in Section 14 and Section 15, and relies on a change of measure argument introduced in Section 13. While this second proof may be construed as less intuitive than the one provided in Section 11, it has the advantage of giving a probabilistic interpretation to the conditions appearing in Theorem 3.2.

A word on the notation and conventions in use: Unless specified otherwise, all limiting statements, including asymptotic equivalences, are understood with \( n \) going to infinity. The rvs under consideration are all defined on the same probability triple \((\Omega, \mathcal{F}, \mathbb{P})\). The construction of a probability triple sufficiently large to carry all required rvs is standard, and omitted in the interest of brevity. All probabilistic statements are made with respect to the probability measure \( \mathbb{P} \), and we denote the corresponding expectation operator by \( \mathbb{E} \). We abbreviate almost sure(ly) (under \( \mathbb{P} \)) by a.s. If \( E \) is a subset of \( \Omega \), then \( 1_E \) is the indicator rv of the set \( E \) with the usual understanding that \( 1_E(\omega) = 1 \) (resp. \( 0 \)) if \( \omega \in E \) (resp. \( \omega /\in E \)). The symbol \( \mathbb{N} \) (resp. \( \mathbb{N}_0 \)) denotes the set of non-negative (resp. positive) integers. We view sequences as mappings defined on \( \mathbb{N}_0 \); the mapping itself is denoted by bolding the symbol used for the generic element of the corresponding sequence. Unless otherwise specified, all logarithms are natural logarithms with \( \ln x \) denoting the natural logarithm of \( x > 0 \).

2 Homogeneous (binary) MAG models

The MAG model is parametrized by a number of quantities, chief amongst them the number \( n \) of nodes present in the network and the number \( L \) of attributes associated with each node – Both \( n \) and \( L \) are positive integers. Nodes are labeled \( u = 1, 2, \ldots \), while attributes are labeled \( \ell = 1, 2, \ldots \). Each of the \( L \) attributes associated with a node is assumed to be binary in nature with \( 1 \) (resp. \( 0 \)) signifying that the attribute is present (resp. absent). We conveniently organize these \( L \) attributes into a vector element \( a_L = (a_1, \ldots, a_L) \) of \( \{0, 1\}^L \).

2.1 The underlying rvs

The propensity of nodes to attach to each other is governed by their attributes in a way to be clarified shortly. The probability triple \((\Omega, \mathcal{F}, \mathbb{P})\) is assumed to carry two collections of rvs, namely the collection
\[
\{A, A_\ell, A_\ell(u), \ \ell = 1, 2, \ldots; \ u = 1, 2, \ldots\}
\]
and the triangular array
\[
\{U(u, v), \ u = 1, 2, \ldots; \ v = u + 1, u + 2, \ldots\}.
\]

The following assumptions are enforced throughout:

(i) The collection \( \{A, A_\ell, A_\ell(u), \ \ell = 1, 2, \ldots; \ u = 1, 2, \ldots\} \) and the triangular array \( \{U(u, v), \ u = 1, 2, \ldots; \ v = u + 1, u + 2, \ldots\} \) are mutually independent;

(ii) The rvs \( \{U(u, v), \ u = 1, 2, \ldots; \ v = u + 1, u + 2, \ldots\} \) are i.i.d. rvs, each of which is uniformly distributed on the interval \((0, 1)\); and
iii) The rvs \( \{A, A_\ell, A_\ell(u), \ell = 1, 2, \ldots; u = 1, 2, \ldots \} \) form a collection of i.i.d. \( \{0, 1\}\)-valued rvs with pmf \( \mu = (\mu(0), \mu(1)) \) where \( \mathbb{P}[A = 0] = \mu(0) \) and \( \mathbb{P}[A = 1] = \mu(1) \). To avoid trivial situations of limited interest, we assume that both \( \mu(0) \) and \( \mu(1) \) are elements of the open interval \((0, 1)\) such that \( \mu(0) + \mu(1) = 1 \).

For each \( L = 1, 2, \ldots, \) we write

\[
A_L = (A_1, \ldots, A_L) \quad \text{and} \quad A_L(u) = (A_1(u), \ldots, A_L(u)), \quad u = 1, 2, \ldots.
\]

Under the enforced assumptions, the \( \{0, 1\}^L\)-valued rvs \( \{A_L, A_L(u), u = 1, 2, \ldots\} \) are i.i.d. rvs, each with i.i.d. components distributed like the generic rv \( A \). We shall also have use for the partial sum rvs

\[
S_L(u) = A_1(u) + \ldots + A_L(u), \quad u = 1, 2, \ldots
\]

and

\[
S_L = A_1 + \ldots + A_L.
\]

For each \( \ell = 1, \ldots, \) we shall say that node \( u \) exhibits (resp. does not exhibit) the \( \ell^{th} \) attribute if \( A_\ell(u) = 1 \) (resp. \( A_\ell(u) = 0 \)). In that terminology, the rv \( S_L(u) \) then counts the number of attributes exhibited by node \( u \) amongst the first \( L \) attributes. Under the enforced assumptions, the rvs \( \{S_L(u), u = 1, 2, \ldots\} \) form a sequence of i.i.d. rvs, each being distributed according to the rv \( S_L \) which is itself a Binomial rv Bin\((L, \mu(1))\).

For notational reasons we find it convenient to augment the triangular array of uniform rvs into the larger collection \( \{U(u, v), u, v = 1, 2, \ldots\} \) through the definitions

\[
U(u, u) = 1 \quad \text{and} \quad U(v, u) = U(u, v), \quad v = u + 1, \ldots \quad u = 1, 2, \ldots
\]

### 2.2 Adjacency

On the way to defining homogeneous binary MAGs, we introduce notions of adjacency between nodes based on their attributes. To do so we start with an \( 2 \times 2 \) matrix \( Q \) given by

\[
Q \equiv (q(a, b)) = \begin{pmatrix} q(1, 1) & q(1, 0) \\ q(0, 1) & q(0, 0) \end{pmatrix}.
\]

Throughout we assume the symmetry condition

\[
q(1, 0) = q(0, 1),
\]

together with the non-degeneracy conditions

\[
0 < q(a, b) < 1, \quad a, b \in \{0, 1\}.
\]

Fix \( L = 1, 2, \ldots \). With this symmetric \( 2 \times 2 \) matrix \( Q \) we associate a mapping \( Q_L : \{0, 1\}^L \times \{0, 1\}^L \rightarrow [0, 1] \) given by

\[
Q_L(a_L, b_L) = \prod_{\ell=1}^{L} q(a_\ell, b_\ell), \quad a_L, b_L \in \{0, 1\}^L.
\]

\footnote{In terms of YES/NO answers to binary questions, \( S_L(u) \) then counts the number of YES answers given by node \( u \) to the \( L \) first questions.}
Interpretations for these quantities will be given shortly. The enforced assumptions (7)-(8) on $Q$ readily imply

$$Q_L(b_L, a_L) = Q_L(a_L, b_L), \quad a_L, b_L \in \{0, 1\}^L$$

with

$$0 < Q_L(a_L, b_L) < 1, \quad a_L, b_L \in \{0, 1\}^L.$$  

Pick two nodes $u, v = 1, 2, \ldots$. We say that node $u$ is $L$-adjacent to node $v$, written $u \sim_L v$, if the condition

$$U(u, v) \leq Q_L(A_L(u), A_L(v))$$

holds, in which case an (undirected) edge from node $u$ to node $v$ is said to exist. Obviously, $L$-adjacency is a binary relation on the set of all nodes. Since $U(u, v) = U(v, u)$, it is plain from (10) that node $u$ is $L$-adjacent to node $v$ if and only if node $v$ is $L$-adjacent to node $u$ – This allows us to say that nodes $u$ and $v$ are $L$-adjacent without any risk of confusion. Node $u$ cannot be $L$-adjacent to itself because $U(u, u) = 1$ (by convention) and $Q_L(A_L(u), A_L(u)) < 1$ by (11) – In other words, $L$-adjacency will not give rise to self-loops.

We encode $L$-adjacency through the $\{0, 1\}$-valued rvs $\{\chi_L(u, v), u, v = 1, 2, \ldots\}$ given by

$$\chi_L(u, v) = 1[U(u, v) \leq Q_L(A_L(u), A_L(v))], \quad u, v = 1, 2, \ldots$$

with $\chi_L(u, v) = 1$ (resp. $\chi_L(u, v) = 0$) corresponding to the existence (resp. absence) of an (undirected) edge between node $u$ and node $v$. In view of earlier remarks, the conditions

$$\chi_L(u, u) = 0 \quad \text{and} \quad \chi_L(v, u) = \chi_L(u, v), \quad u, v = 1, 2, \ldots$$

are all satisfied.

### 2.3 Defining the homogeneous binary MAG models

Fix $n = 1, 2, \ldots$ and $L = 1, 2, \ldots$. The homogeneous binary MAG over a set of $n$ nodes, labelled $1, \ldots, n$, with each node having $L$ attributes, labelled $1, \ldots, L$, is defined as the random graph $M(n; L)$ whose edge set is determined through the rvs $\{\chi_L(u, v), u, v = 1, 2, \ldots, n\}$. From (14) it follows that edges in $M(n; L)$ are undirected and that there are no self-loops, hence any realization of $M(n; L)$ is a simple graph. For simplicity we shall refer to this model as the MAG model.

This definition is equivalent to the one given by Kim and Leskovec [11]. Indeed, with the help of Assumptions (i) and (ii), it is a simple matter to check from (15) that the rvs forming the triangular array

$$\left\{ \chi_L(u, v), \quad u = 1, \ldots, n \right\}$$

are conditionally independent given the i.i.d. attribute random vectors $\{A_L(u), u = 1, 2, \ldots, n\}$ with

$$\mathbb{P} [\chi_L(u, v) = 1 | A_L(w), w = 1, 2, \ldots, n] = \mathbb{P} [U(u, v) \leq Q_L(A_L(u), A_L(v))] A_L(w), \quad w = 1, 2, \ldots, n] = Q_L(A_L(u), A_L(v)) = \prod_{\ell=1}^{L} q(A_{\ell}(u), A_{\ell}(v)), \quad u \neq v, \quad u, v = 1, \ldots, n$$

---

2Strictly speaking, the definition given above is slightly more restrictive than the one proposed in [11] as we have eliminated by construction the possibility of self-loops, whereas such links are neglected by Kim and Leskovec as making no contributions in the limiting regime. See the discussion after Theorem 3.1 in [11].
where the symmetric mapping \( Q_L : \{0, 1\}^L \times \{0, 1\}^L \rightarrow [0, 1] \) was introduced earlier at (9). Thus, the probabilistic characteristics of \( \mathcal{M}(n, L) \) are completely determined by the matrix \( Q \) and by the pmf \( \mu \). These building blocks are assumed given and held fixed during the discussion – They will not be explicitly displayed in the notation.

Throughout we write
\[
\Gamma(a) = \mathbb{E} [q(a, A)], \quad a = 0, 1
\]
with results all given under the compact condition \( \Gamma(0) < \Gamma(1) \). When \( \Gamma(1) < \Gamma(0) \), the results can be obtained \emph{mutatis mutandis} by exchanging the roles of the attributes 0 and 1, i.e., the roles of \( \mu(0) \) (resp. \( \Gamma(0) \)) and \( \mu(1) \) (resp. \( \Gamma(1) \)) need to be interchanged in various statements. Details are left to the interested reader.

3 The main results

Fix \( n = 2, 3, \ldots \) and \( L = 1, 2, \ldots \). For each \( u = 1, \ldots, n \), node \( u \) is isolated in \( \mathcal{M}(n; L) \) if there is no other node (in \( \{1, \ldots, n\} \)) distinct from \( u \) which is \( L \)-adjacent to node \( u \). The \( \{0, 1\} \)-valued rv \( \xi_{n,L}(u) \) given by
\[
\xi_{n,L}(u) \equiv \prod_{w=1, w\neq u}^{n} (1 - \chi_L(u, w))
\]
encodes the fact that node \( u \) is isolated in \( \mathcal{M}(n; L) \).

We are interested in establishing a zero-one law for the absence of isolated nodes in MAG models when the number \( n \) of nodes and the number \( L \) of nodal attributes grow unboundedly large, the latter quantity scaling with the former. The following terminology, used repeatedly in what follows, should help simplify the presentation: A \emph{scaling} (for the number of attributes) is any mapping \( L : \mathbb{N}_0 \rightarrow \mathbb{N}_0 : n \rightarrow L_n \). With \( \rho > 0 \), the scaling \( L : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \) is said to be \( \rho \)-admissible if
\[
L_n \sim \rho \ln n,
\]
in which case it holds that
\[
L_n = \rho_n \ln n, \quad n = 1, 2, \ldots
\]
for some sequence \( \rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+ : n \rightarrow \rho_n \) such that \( \lim_{n \rightarrow \infty} \rho_n = \rho \). The sequence \( \rho : \mathbb{N}_0 \rightarrow \mathbb{R}_+ \) defined by (19) is uniquely determined by the \( \rho \)-scaling \( L : \mathbb{N}_0 \rightarrow \mathbb{N}_0 \), and is said to be \emph{associated} with it.

Interest in admissible scalings is discussed in [11]. The definition of admissibility given by Kim and Leskovec [11] uses logarithms in base two; results given here are easily reconciled with the ones in [11] through the well-known fact that \( \ln x = \ln 2 \cdot \log_2 x \) with \( \log_2 x \) denoting the logarithm of \( x \) in base 2 for \( x > 0 \). In particular, a \( \rho \)-admissible scaling as defined here at (19) is a \( \rho \ln 2 \)-scaling in the sense of Kim and Leskovec.

The zero-one law for the absence of isolated nodes takes a different form depending on the sign of \( 1 + \rho \ln \mu(0) \). The boundary case \( 1 + \rho \ln \mu(0) = 0 \) will not be considered in what follows.

3.1 The case \( 1 + \rho \ln \mu(0) > 0 \)

The result given next contains the zero-one law under the condition \( 1 + \rho \ln \mu(0) > 0 \), and is established in Section [7]
Theorem 3.1. Assume \( \Gamma(0) < \Gamma(1) \). With \( \rho > 0 \), we further assume that
\[
1 + \rho \ln \mu(0) > 0.
\] (20)

Then, for any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we have the zero-one law
\[
\lim_{n \to \infty} \mathbb{P} \left[ \mathbb{M}(n; L_n) \text{ contains no isolated nodes} \right] = \begin{cases} 
0 & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\
1 & \text{if } 1 + \rho \ln \Gamma(0) > 0.
\end{cases}
\] (21)

3.2 The case \( 1 + \rho \ln \mu(0) < 0 \)

Theorem 3.1 takes a very different form when (20) does not hold. To state the results, we introduce the quantity
\[
G(\nu, \mu) = \left( \frac{\mu}{\rho} \right)^{\nu} \left( \frac{1 - \mu}{1 - \nu} \right)^{1-\nu}, \quad 0 < \nu, \mu < 1.
\] (22)

For each \( \mu \) in \((0, 1)\) the mapping \((0, 1) \to \mathbb{R}_+ : \nu \to G(\nu, \mu)\) is well defined and continuous. By continuity we can extend it into into a continuous mapping defined on the closed interval \([0, 1]\) so that \(G(0, \mu) = \lim_{\nu \downarrow 0} G(\nu, \mu) = 1 - \mu\) and \(G(1, \mu) = \lim_{\nu \uparrow 1} G(\nu, \mu) = \mu\). This corresponds to using the convention \(0^0 = 1\) in the expression (22). In a similar way, for each \( \mu \) in \((0, 1)\) the mapping \((0, 1) \to \mathbb{R} : \nu \to \ln G(\nu, \mu)\) is well defined and continuous with
\[
\ln G(\nu, \mu) = -\nu \ln \left( \frac{\nu}{\mu} \right) - (1 - \nu) \ln \left( \frac{1 - \nu}{1 - \mu} \right), \quad 0 < \nu < 1.
\] (23)

We can also extend this second mapping into a continuous mapping defined on the closed interval \([0, 1]\) with \(\ln G(0, \mu) = \lim_{\nu \downarrow 0} \ln G(\nu, \mu) = \ln(1 - \mu)\) and \(\ln G(1, \mu) = \lim_{\nu \uparrow 1} \ln G(\nu, \mu) = \ln \mu\). This is consistent with applying the usual convention \(\ln 0 = 0\) in the expression (23). Elementary calculus shows that the mapping \([0, 1] \to \mathbb{R} : \nu \to \ln G(\nu, \mu)\) is concave, and that its maximum is achieved at \(\nu = \mu\) with \(\ln G(\mu, \mu) = 0\). Thus, the mapping \([0, 1] \to \mathbb{R} : \nu \to \ln G(\nu, \mu)\) increases on \((0, \mu)\), reaches its maximum at \(\nu = \mu\) and then decreases on \((\mu, 1)\).

With these preliminaries in place, for each \( \mu \) in \((0, 1)\) and \( \rho > 0 \), consider the non-linear equation
\[
1 + \rho \ln G(\nu, \mu) = 0, \quad \nu \in [0, 1].
\] (24)

If the condition \( 1 + \rho \ln(1 - \mu) < 0 \) holds, then the equation (24) has a non-empty set of solutions. More precisely, there always exists a root, denoted \(\nu_*(\rho)\), in the interval \((0, \mu)\) since \(1 + \rho \ln G(0, \mu) = 1 + \rho \ln(1 - \mu) < 0\) while \(1 + \rho \ln G(\mu, \mu) = 1\). Only when
\[
1 + \rho \ln G(1, \mu) = 1 + \rho \ln \mu \leq 0,
\]
does there exist a second root located in the interval \((\mu, 1)\). In what follows \(\mu(1)\) plays the role of \(\mu\).

Theorem 3.2. Assume \(\Gamma(0) < \Gamma(1)\). With \(\rho > 0\), we further assume that
\[
1 + \rho \ln \mu(0) < 0.
\] (25)

Then, for any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we have the zero-one law
\[
\lim_{n \to \infty} \mathbb{P} \left[ \mathbb{M}(n; L_n) \text{ contains no isolated nodes} \right] = \begin{cases} 
0 & \text{if } 1 + \rho \ln (\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}) < 0 \\
1 & \text{if } 1 + \rho \ln (\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}) > 0.
\end{cases}
\] (26)
where $\nu_*(\rho)$ is the unique solution in the interval $(0, \mu(1))$ to the equation

$$1 + \rho \ln G(\nu, \mu(1)) = 0, \quad \nu \in [0, 1],$$

(27)

Theorem 3.2 is established in Section 9 with the help of auxiliary results discussed in Section 10 and Section 11.

3.3 On the conditions at (26)

For future reference, in order to avoid repetitions, we discuss the constraints on the sign of $1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right)$ which appear in the statement of Theorem 3.2. As we will discover shortly in subsequent sections, forthcoming arguments will require the existence of a value $\nu$ either in the range $(0, \nu_*(\rho))$ such that

$$1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right) < 0,$$

(28)
or in the range $(\nu_*(\rho), \mu(1))$ such that

$$1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right) > 0.$$

(29)

As we now argue, the existence of a value $\nu$ in the requisite intervals is indeed guaranteed by the conditions

$$1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right) < 0$$

(30)

and

$$1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right) > 0,$$

(31)

respectively: The elementary fact

$$1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right) = 1 + \rho \left(\nu \ln \Gamma(1) + (1 - \nu) \ln \Gamma(0)\right), \quad \nu \in [0, 1],$$

shows that the mapping $\nu \to 1 + \rho \ln \left(\Gamma(1)^{\nu_*(\rho)} \Gamma(0)^{1-\nu_*(\rho)}\right)$ is affine (thus continuous) on $[0, 1]$ and strictly increasing (since $\Gamma(0) < \Gamma(1)$) with intercepts at $\nu = 0$ and $\nu = 1$ given by $1 + \rho \ln \Gamma(0)$ and $1 + \rho \ln \Gamma(1)$, respectively. This elementary observation has the following implications: If (30) holds, then by continuity and monotonicity there exists a non-trivial interval $I_-(\rho) = (\alpha_-(\rho), \beta_- (\rho))$ contained in $(0, \mu(1))$ with the following properties: The interval $I_-(\rho)$ contains $\nu_*(\rho)$ and (28) holds on it. On the other hand, if (31) holds, then again by continuity and monotonicity there now exists a non-trivial interval $I_+(\rho) = (\alpha_+ (\rho), \beta_+ (\rho))$ contained in $(0, \mu(1))$ such that $\nu_*(\rho)$ belongs to $I_+(\rho)$ and (29) holds on it.

Finally, we close by noting that Kim and Leskovec couch their analysis in terms of the counts

$$\sum_{u \in V_n} 1[L - S_L(u) = j], \quad j = 0, \ldots, L$$

while here we have used instead the counts

$$\sum_{u \in V_n} 1[S_L(u) = j], \quad j = 0, \ldots, L.$$

In other words, Kim and Leskovec count the NO answers while we count the YES answers. This is why the parameters $\mu(0)$ and $\mu(1)$ need to be exchanged to go from the conditions appearing in their paper to the ones appearing here. However, Leskovec and Kim do impose additional conditions on the entries of the symmetric matrix $Q$, namely that $q(1, 1) < q(0, 1) = q(1, 0) < q(0, 0)$ (so that $\Gamma(1) < \Gamma(0)$ with their convention). Here we ask only for $\Gamma(0) < \Gamma(1)$ (with our conventions) with no additional conditions.
4 A roadmap to the proofs

4.1 Counting isolated nodes

Fix \( n = 2, 3, \ldots \) and \( L = 1, 2, \ldots \). To count the number of isolated nodes in \( \mathbb{M}(n; L) \) we introduce the rv \( I_n(L) \) given by

\[
I_n(L) = \sum_{u=1}^{n} \xi_{n,L}(u). \tag{32}
\]

Interest in these count variables stems from the observation that \( \mathbb{M}(n; L) \) contains no isolated nodes if and only if \( I_n(L) = 0 \), and that the key relation

\[
\mathbb{P} \left[ \text{\( \mathbb{M}(n; L) \) contains no isolated nodes} \right] = \mathbb{P} \left[ I_n(L) = 0 \right] \tag{33}
\]

holds. This fact will be used to establish Theorems 3.1 and 3.2 by leveraging easy bounds on the probability \( \mathbb{P} \left[ I_n(L) = 0 \right] \) in terms of the first and second moments of the rv \( I_n(L) \) (as discussed next in Section 4.2).

However, some of the forthcoming arguments will require a finer accounting which we now introduce. Recall that for each node \( u = 1, \ldots, n \), the number of attributes exhibited by node \( u \) amongst the first \( L \) attributes is captured by the rv \( S_L(u) \) introduced at (5). For each \( \ell = 0, 1, \ldots, L \), the \( \{0, 1\} \)-valued rv \( \xi_{n,L}^{(\ell)}(u) \) given by

\[
\xi_{n,L}^{(\ell)}(u) = \xi_{n,L}(u) \cdot 1 \left[ S_L(u) = \ell \right]. \tag{34}
\]

indicates whether node \( u \) is isolated in \( \mathbb{M}(n; L) \) while \( \ell \) attributes are present amongst its first \( L \) attributes.

The total number of isolated nodes in \( \mathbb{M}(n; L) \) which have \( \ell \) attributes amongst the first \( L \) attributes is then given by

\[
I_n^{(\ell)}(L) = \sum_{u=1}^{n} \xi_{n,L}^{(\ell)}(u) = \sum_{u=1}^{n} \xi_{n,L}(u) 1 \left[ S_L(u) = \ell \right]. \tag{35}
\]

Simple accounting readily yields the relations

\[
\xi_{n,L}(u) = \sum_{\ell=0}^{L} \xi_{n,L}^{(\ell)}(u) \tag{36}
\]

and

\[
I_n(L) = \sum_{\ell=0}^{L} I_n^{(\ell)}(L), \tag{37}
\]

the last one yielding the elementary bounds

\[
I_n^{(\ell)}(L) \leq I_n(L), \quad \ell = 0, 1, \ldots, L. \tag{38}
\]
4.2 The method of first and second moments

The basic strategy for proving Theorems 3.1 and 3.2 relies on the method of first and second moments applied to the number \( n \) of isolated nodes and to the related count variables \( m \). In this section we provide the main ingredients of this approach as we will need it in its various applications.

Let \( \{ Z_n, n = 1, 2, \ldots \} \) denote a collection of \( N \)-valued rvs such that \( \mathbb{E} [Z_n^2] < \infty \) for each \( n = 1, 2, \ldots \). The method of first moment \([10, Eqn (3.10), p. 55]\) relies on the well-known bound

\[
1 - \mathbb{E} [Z_n] \leq \mathbb{P} [Z_n = 0] \tag{39}
\]

while the method of second moment \([10, Remark 3.1, p. 55]\) has its starting point in the inequality

\[
\mathbb{P} [Z_n = 0] \leq 1 - \frac{\mathbb{E} [Z_n]^2}{\mathbb{E} [Z_n^2]} \tag{40}
\]

Letting \( n \) go to infinity in the resulting inequalities, we conclude from (39) that

\[
\lim_{n \to \infty} \mathbb{P} [Z_n = 0] = 1 \tag{41}
\]

if

\[
\lim_{n \to \infty} \mathbb{E} [Z_n] = 0 \tag{42}
\]

while the bound (40) implies

\[
\lim_{n \to \infty} \mathbb{P} [Z_n = 0] = 0 \tag{43}
\]

whenever

\[
\limsup_{n \to \infty} \frac{\mathbb{E} [Z_n^2]}{\mathbb{E} [Z_n]^2} \leq 1. \tag{44}
\]

Here we use this strategy when the rvs \( \{ Z_n, n = 1, 2, \ldots \} \) are count variables with the following structure: For each \( n = 1, 2, \ldots \), the rv \( Z_n \) is of the form

\[
Z_n = \sum_{u=1}^{n} \zeta_n(u)
\]

where the rvs \( \zeta_n(1), \ldots, \zeta_n(n) \) are \{0, 1\}-valued rvs. If in addition, the rvs \( \zeta_n(1), \ldots, \zeta_n(n) \) are exchangeable (as they will be here), then we easily arrive at the expressions

\[
\mathbb{E} [Z_n] = \mathbb{E} \left[ \sum_{u=1}^{n} \zeta_n(u) \right] = n \mathbb{E} [\zeta_n(1)] \tag{45}
\]

and

\[
\mathbb{E} [Z_n^2] = \mathbb{E} \left[ \left( \sum_{u=1}^{n} \zeta_n(u) \right)^2 \right] = n \mathbb{E} [\zeta_n, 1] + n(n - 1) \mathbb{E} [\zeta_n(1) \cdot \zeta_n(2)] \tag{46}
\]

by virtue of the binary nature of the rvs involved, whence

\[
\frac{\mathbb{E} [Z_n^2]}{\mathbb{E} [Z_n]^2} = \frac{1}{n} + \frac{n - 1}{n} \cdot \frac{\mathbb{E} [\zeta_n(1) \cdot \zeta_n(2)]}{\mathbb{E} [\zeta_n(1)]^2}. \tag{47}
\]
It is now plain that (44) can be achieved if we show that
\[
\lim_{n \to \infty} E[Z_n] = \infty, \tag{48}
\]
and
\[
\limsup_{n \to \infty} \frac{E[\zeta_n(1) \cdot \zeta_n(2)]}{(E[\zeta_n(1)])^2} \leq 1. \tag{49}
\]
For the problem at hand, we shall proceed as follows: With a \(\rho\)-scaling \(L : \mathbb{N}_0 \to \mathbb{N}_0\) for some \(\rho > 0\), we seek to establish the desired zero-one laws through the convergence \(\lim_{n \to \infty} P[I_n(L_n) = 0] = 0\) and \(\lim_{n \to \infty} P[I_n(L_n) = 0] = 1\). In principle this could be achieved by applying the method of first and second moments to the rvs \(\{Z_n, n = 1, 2, \ldots\}\) given by
\[
Z_n = I_n(L_n), \quad n = 1, 2, \ldots \tag{50}
\]
However, while this approach will work quite easily for the one-law, we will encounter some difficulty in applying the method of second moment to the rvs (50) and a somewhat indirect approach (based on (38)) will be adopted.

5 Evaluating the first two moments

5.1 Evaluating the first moments

We begin with an easy calculation of the first moments.

Lemma 5.1. Consider arbitrary \(n = 2, 3, \ldots\) and \(L = 1, 2, \ldots\). For each \(u = 1, \ldots, n\), with \(S_L(u)\) given by (5), it holds that
\[
E[\xi_{n,L}(u)] = \left(1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell}\right)^{n-1} \cdot P[S_L(u) = \ell], \quad \ell = 0, 1, \ldots, L \tag{51}
\]
and
\[
E[\xi_{n,L}(u)] = E\left[\left(1 - \Gamma(1) S_L(u)^{\ell} \Gamma(0)^{L-S_L(u)}\right)^{n-1}\right]. \tag{52}
\]

Recall that the rvs \(\{A, A_\ell, \ell = 1, 2, \ldots\}\) are i.i.d. \(\{0, 1\}\)-valued rvs with pmf \(\mu\), and corresponding sequence of partial sums \(\{S_L, L = 1, 2, \ldots\}\) given by (5). Under the enforced Assumptions (i)-(iii) it is plain that for each \(L = 1, 2, \ldots\), the rvs \(S_L(1), S_L(2), \ldots, S_L(n)\) are i.i.d., each distributed according to the rv \(S_L\). The two relations
\[
E[I_n^{(\ell)}(L)] = n \left(1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell}\right)^{n-1} \cdot P[S_L = \ell], \quad \ell = 0, 1, \ldots, L \tag{53}
\]
and
\[
E[I_n(L)] = n E\left[\left(1 - \Gamma(1)^{S_L} \Gamma(0)^{L-S_L}\right)^{n-1}\right] \tag{54}
\]
are now immediate consequences of the relations (55) and (57), respectively.

In what follows, for each \(L = 1, 2, \ldots\), we shall have use for the moments
\[
Q_L^*(a_L) = E[Q_L(a_L, A_L)], \quad a_L \in \{0, 1\}^L. \tag{55}
\]
Note that
\[
Q_L^*(a_L) = \mathbb{E} \left[ \prod_{\ell=1}^{L} q(a_\ell, A_\ell) \right] = \mathbb{E} \left[ q(1, A) \right]^{\sum_{\ell=1}^{L} a_\ell} \cdot \mathbb{E} \left[ q(0, A) \right]^{\sum_{\ell=1}^{L} (1 - a_\ell)} \tag{56}
\]
as we use the fact that the \(\{0, 1\}\)-valued rv \(A\) is a generic representative of the i.i.d. rvs \(A_1, \ldots, A_L\). In particular it follows that
\[
Q_L^*(A_L(u)) = \Gamma(1)^{S_L(u)} \Gamma(0)^{L-S_L(u)}, \quad u = 1, \ldots, n. \tag{57}
\]

**Proof.** It suffices to show that (51) holds since (52) follows as an easy consequence of the expression (56). Pick positive \(n = 2, 3, \ldots\) and \(L = 1, 2, \ldots\) and consider node \(u = 1, \ldots, n\). For each \(\ell = 0, 1, \ldots, L\), with the relation \((54)\) holding, a standard preconditioning argument yields
\[
\mathbb{E} \left[ \xi^{(\ell)}_{n,L}(u) \right] = \mathbb{E} \left[ 1 \left[ S_L(u) = \ell \right] \cdot \mathbb{E} \left[ \xi_{n,L}(u) \left| A_L(u) \right. \right] \right] \tag{58}
\]
as we note that the rv \(S_L(u)\) is determined by the attribute vector \(A_L(u)\).

With (17) as a point of departure, we have
\[
\xi_{n,L}(u) = \prod_{w=1, w \neq u}^{n} (1 - \chi_L(u, w)) = \prod_{w=1, w \neq u}^{n} 1 \left[ U(u, w) > Q_L(A_L(u), A_L(w)) \right].
\]

Under the enforced independence assumptions, we readily conclude to
\[
\mathbb{E} \left[ \xi_{n,L}(u) \left| A_L(1), \ldots, A_L(n) \right. \right] = \prod_{w=1, w \neq u}^{n} (1 - Q_L(A_L(u), A_L(w))).
\]
The smoothing property of conditional expectations readily gives
\[
\mathbb{E} \left[ \xi_{n,L}(u) \left| A_L(u) \right. \right] = \mathbb{E} \left[ \mathbb{E} \left[ \xi_{n,L}(u) \left| A_L(1), \ldots, A_L(n) \right. \right| A_L(u) \right] \right.
\]
\[
= \mathbb{E} \left[ \prod_{w=1, w \neq u}^{n} (1 - Q_L(A_L(u), A_L(w))) \left| A_L(u) \right. \right] \right]
\]
\[
= \mathbb{E} \left[ \prod_{w=1, w \neq u}^{n} (1 - Q_L(a_L, A_L(w))) \right]_{a_L = A_L(u)}
\]
\[
= (1 - Q_L^*(A_L(u)))^{n-1}
\]
where the last two steps made use of the fact that the rvs \(A_L(1), \ldots, A_L(n)\) are i.i.d. rvs. Using (58) we obtain
\[
\mathbb{E} \left[ \xi^{(\ell)}_{n,L}(u) \right] = \mathbb{E} \left[ 1 \left[ S_L(u) = \ell \right] \cdot (1 - Q_L^*(A_L(u)))^{n-1} \right]
\]
\[
= \mathbb{E} \left[ 1 \left[ S_L(u) = \ell \right] \cdot \left( 1 - \Gamma(1)^{S_L(u)} \Gamma(0)^{L-S_L(u)} \right)^{n-1} \right]
\]
by virtue of (57), and the desired conclusion (51) follows in a straightforward manner. \(\square\)
5.2 Evaluating the second moments

The expressions for the second order quantities are much more involved as the next intermediary result already shows.

**Lemma 5.2.** Consider arbitrary \( n = 2, 3, \ldots \) and \( L = 1, 2, \ldots \). For distinct \( u, v = 1, \ldots, n \), it holds that

\[
\mathbb{E} \left[ \xi_{n,L}(u) \xi_{n,L}(v) \middle| A_L(u), A_L(v) \right] = (1 - Q_L(A_L(u), A_L(v))) \cdot \left( 1 - \tilde{Q}_L(A_L(u), A_L(v)) \right)^{n-2}
\]

where for arbitrary \( a_L \) and \( b_L \) in \( \{0, 1\}^L \), we have set

\[
\tilde{Q}_L(a_L, b_L) = Q_L^*(a_L) + Q_L^*(b_L) - Q_L^{**}(a_L, b_L)
\]

with

\[
Q_L^{**}(a_L, b_L) = \mathbb{E} [Q_L(a_L, A_L(u))Q_L(b_L, A_L(v))].
\]

The proof of this result can be found in Appendix 12. In principle, it is now possible to evaluate the expressions

\[
\mathbb{E} \left[ \xi_{n,L}^{(k)}(u) \xi_{n,L}^{(\ell)}(v) \right], \quad k, \ell = 0, \ldots, L
\]

for distinct \( u, v = 1, \ldots, n \). Indeed, for \( k, \ell = 0, 1, \ldots, L \), not necessarily distinct, the relation yields

\[
\xi_{n,L}^{(k)}(u) \xi_{n,L}^{(\ell)}(v) = 1[S_L(u) = k] \cdot 1[S_L(v) = \ell] \cdot \xi_{n,L}(u) \xi_{n,L}(v)
\]

and an easy preconditioning argument leads to

\[
\mathbb{E} \left[ \xi_{n,L}^{(k)}(u) \cdot \xi_{n,L}^{(\ell)}(v) \right] = \mathbb{E} \left[ 1[S_L(u) = k] 1[S_L(v) = \ell] \cdot \mathbb{E} \left[ \xi_{n,L}(u) \xi_{n,L}(v) \middle| A_L(u), A_L(v) \right] \right]
\]

because the rvs \( S_L(u) \) and \( S_L(v) \) are determined by the attribute vectors \( A_L(u) \) and \( A_L(v) \), respectively. Using (63) we also readily obtain

\[
\mathbb{E} \left[ \xi_{n,L}(u) \xi_{n,L}(v) \right] = \sum_{k=0}^L \sum_{\ell=0}^L \mathbb{E} \left[ \xi_{n,u}^{(k)}(L) \xi_{n,v}^{(\ell)}(v) \right].
\]
With arbitrary $a_L$ and $b_L$ in $\{0, 1\}^L$, we note from (61) that

$$Q_\star^+ (a_L, b_L) = \mathbb{E}(Q_L(a_L, A_L)Q_L(b_L, A_L))$$

$$= \mathbb{E}\left[\prod_{\ell=1}^L q(a_\ell, A_\ell)q(b_\ell, A_\ell)\right]$$

$$= \prod_{\ell=1}^L \mathbb{E}[q(a_\ell, A_\ell)q(b_\ell, A_\ell)]$$

$$= \prod_{\ell=1}^L \mathbb{E}\left[q(1, A)^2\sum_{\ell=1}^L a_\ell b_\ell \mathbb{E}[q(1, A)q(0, A)]^{a_\ell(1-b_\ell)+b_\ell(1-a_\ell)}\mathbb{E}\left[q(0, A)^2\right]^{(1-a_\ell)(1-b_\ell)}\right]$$

by arguments similar to the ones used for reaching the expression (63). Here lies the rub: The quantities $Q_\star^+(A_L(u))$ and $Q_\star^+(A_L(v))$ depend on $A_L(u)$ and $A_L(v)$ only through the sums $S_L(u)$ and $S_L(v)$, respectively. On the other hand, $Q_\star^+(A_L(u), A_L(v))$ does not depend on $A_L(u)$ and $A_L(v)$ only through the sums $S_L(u)$ and $S_L(v)$, but instead through the three sums $\sum_{\ell=1}^L A_\ell(u)A_\ell(v)$, $\sum_{\ell=1}^L (A_\ell(u) - A_\ell(v)) + A_\ell(v) (1 - A_\ell(u)))$ and $\sum_{\ell=1}^L (1 - A_\ell(u)) (1 - A_\ell(v))$.

Fortunately, the exact expression (63) will not be needed as only the following crude bounds will suffice: For $k, \ell = 0, 1, \ldots, L$, not necessarily distinct, the expression (63) yields the bound

$$\mathbb{E}\left[\xi_{n,L}(u) \cdot \xi_{n,L}(v)\right] \leq \mathbb{P}[S_L(u) = k, S_L(v) = \ell]$$

$$= \mathbb{P}[S_L(u) = k] \mathbb{P}[S_L(v) = \ell]$$

since

$$\mathbb{E}\left[\xi_{n,L}(u) \xi_{n,L}(v) \mid A_L(u), A_L(v)\right] \leq 1 \quad a.s.$$

6 Two useful technical results

The next two technical lemmas will be useful in a number of places. We present them here, with their proofs, for easy reference. The first one relies on the following well-known fact [4, Prop. 3.1.1, p. 116]: For any sequence $a : \mathbb{N}_0 \to \mathbb{R}_+$, we have

$$\lim_{n \to \infty} (1 - a_n)^n = e^{-c}$$

(65)

for some $c$ in $[0, \infty]$ if and only if

$$\lim_{n \to \infty} na_n = c.$$  

(66)

**Lemma 6.1.** Consider a $\rho$-admissible scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$ for some $\rho > 0$. For any sequence $\nu : \mathbb{N}_0 \to [0, 1]$ such that $\lim_{n \to \infty} \nu_n = \nu$ for some $\nu$, it holds that

$$\lim_{n \to \infty} \left(1 - \left(\Gamma(1)\nu \Gamma(0)^{1-\nu_n}\right)^{L_n}\right)^{n-1} = \begin{cases} 0 & \text{if } 1 + \rho \ln (\Gamma(1)^{\nu} \Gamma(0)^{1-\nu}) > 0 \\ 1 & \text{if } 1 + \rho \ln (\Gamma(1)^{\nu} \Gamma(0)^{1-\nu}) < 0. \end{cases}$$

(67)
Proof. It follows from the equivalence (65)-(66) (with $a_n = (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L$) that the convergence
\[
\lim_{n \to \infty} \left(1 - (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L\right)^{n-1} = e^{-c}
\] (68)
takes place for some $c$ in $[0, \infty]$ if and only if
\[
\lim_{n \to \infty} (n - 1) (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L = c.
\] (69)

For each $n = 1, 2, \ldots$, the $\rho$-admissibility of the scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$ yields
\[
(n - 1) (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L = (n - 1) (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^{\rho_n\ln n}
\]
\[
= \frac{n - 1}{n} e^{(1 + \rho_n \ln (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n}))\ln n}
\]
\[
= \frac{n - 1}{n} e^{1 + \rho_n \ln (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})}
\] (70)
where the sequence $\rho : \mathbb{N}_0 \to \mathbb{R}_+$ is the unique sequence associated with the $\rho$-admissible scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$.

The conclusion (67) readily follows from the equivalence of (68) and (69) once we note that
\[
\lim_{n \to \infty} (1 + \rho_n \ln (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})) = 1 + \rho \ln (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}).
\]
Indeed $1 + \rho \ln (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}) < 0$ (resp. $1 + \rho \ln (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}) > 0$) yields $c = 0$ (resp. $c = \infty$) in (69), whence $e^{-c} = 1$ (resp. $e^{-c} = 0$) in (68).

A little more can be extracted from the arguments given above: The usual exponentiation argument and (70) readily yield
\[
n \left(1 - (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L\right)^{n-1} \leq e^{n - \frac{n - 1}{n} e^{1 + \rho_n \ln (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})}}
\]
for $n = 1, 2, \ldots$. Therefore, when $1 + \rho \ln (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}) > 0$, the stronger result
\[
\lim_{n \to \infty} n \left(1 - (\Gamma(1)^{\nu_n}\Gamma(0)^{1-\nu_n})^L\right)^{n-1} = 0
\] (71)
also holds.

Lemma 6.2. Consider a $\rho$-admissible scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$ for some $\rho > 0$. For any sequence $C : \mathbb{N}_0 \to (0, \infty)$ such that $\lim_{n \to \infty} C_n = C$ for some $C > 0$, it holds that
\[
\lim_{n \to \infty} nC_n^{L_n} = \begin{cases} 
\infty & \text{if } 1 + \rho \ln C > 0 \\
0 & \text{if } 1 + \rho \ln C < 0.
\end{cases}
\] (72)
Proof. The \( \rho \)-admissibility of the scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) yields
\[
nC_n^{L_n} = ne^{L_n \ln C_n} = ne^{\rho_n \ln C_n - \ln n} = n^{1 + \rho_n \ln C_n}, \quad n = 2, 3, \ldots
\]
where the sequence \( \rho : \mathbb{N}_0 \to \mathbb{R}_+ \) is the unique sequence associated with the \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \). Letting \( n \) go to infinity readily yields the desired conclusion (72) since
\[
\lim_{n \to \infty} (1 + \rho_n \ln C_n) = 1 + \rho \ln C.
\]

7 A proof of Theorem 3.1

The proof of Theorem 3.1 proceeds in two steps. The first step deals with the first moment conditions (42) and (48), and is contained in the following “zero-infinity” law for the first moment – Note the analogy with Theorem 3.1.

Proposition 7.1. Assume \( \Gamma(0) < \Gamma(1) \). With \( \rho > 0 \), we further assume that (20) holds. For any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we have
\[
\lim_{n \to \infty} \mathbb{E}[I_n(L_n)] = \begin{cases} 
\infty & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\
0 & \text{if } 1 + \rho \ln \Gamma(0) > 0.
\end{cases}
\]

Proof. Fix \( n = 2, 3, \ldots \). Under the assumed inequality \( \Gamma(0) < \Gamma(1) \), the expression (54) implies
\[
\mathbb{E}[I_n(L)] \leq n (1 - \Gamma(0)^L)^{n-1} \leq ne^{-(n-1)\Gamma(0)^L} = e^{\ln n - (n-1)\Gamma(0)^L}, \quad L = 1, 2, \ldots
\]
Now, for any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) we have
\[
\mathbb{E}[I_n(L_n)] \leq e^{\ln n - (n-1)\Gamma(0)^L_n}
\]
with
\[
\ln n - (n - 1)\Gamma(0)^{L_n} = \ln n - (n - 1)\Gamma(0)^{\rho_n \ln n} = \ln n - \frac{n - 1}{n} n^{1 + \rho_n \ln \Gamma(0)}
\]
where the sequence \( \rho : \mathbb{N}_0 \to \mathbb{R}_+ \) is the unique sequence associated with the \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \). Under the condition \( 1 + \rho \ln \Gamma(0) > 0 \), we have
\[
\lim_{n \to \infty} (\ln n - (n - 1)\Gamma(0)^{L_n}) = -\infty
\]
and the conclusion \( \lim_{n \to \infty} \mathbb{E}[I_n(L_n)] = 0 \) follows upon letting \( n \) go to infinity in (76).

We now consider the case \( 1 + \rho \ln \Gamma(0) < 0 \): Fix \( n = 2, 3, \ldots \). For each \( L = 1, 2, \ldots \), the bound (38) (with \( \ell = 0 \)) yields
\[
\mathbb{E}[I_n(0)^L] = n (1 - \Gamma(0)^L)^{n-1} \cdot \mathbb{P}[S_L = 0] \leq \mathbb{E}[I_n(L)]
\]
as we make use of (53) (with \( \ell = 0 \)). Recall that \( \mathbb{P}[S_L = 0] = \mu(0)^L \) since \( S_L \) is a binomial rv \( \text{Bin}(L, \mu(1)) \). Now, for any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) we can write

\[
\mathbb{E} \left[ I_n^{(0)}(L_n) \right] = n\mu(0)^{L_n} \left( 1 - \Gamma(0)^{L_n} \right)^{n-1} \leq \mathbb{E} \left[ I_n(L_n) \right].
\]  

(78)

Let \( n \) go to infinity in (78): Lemma 6.1 (with \( \nu_n = 0 \) for all \( n = 1, 2, \ldots \)) gives 
\[
\lim_{n \to \infty} \left( 1 - \Gamma(0)^{L_n} \right)^{n-1} = 1
\]
under the condition \( 1 + \rho \ln \Gamma(0) < 0 \), while Lemma 6.2 (with \( C_n = \mu(0) \) for all \( n = 1, 2, \ldots \)) yields 
\[
\lim_{n \to \infty} n\mu(0)^{L_n} = \infty
\]
under (20). Thus, 
\[
\lim_{n \to \infty} \mathbb{E} \left[ I_n^{(0)}(L_n) \right] = \infty,
\]
and the desired conclusion \( \lim_{n \to \infty} \mathbb{E} \left[ I_n(L_n) \right] = \infty \) follows.

Upon inspecting the proof of Proposition 7.1 we see (with the help of (78)) that we have also shown the following result to be used shortly.

**Proposition 7.2.** Assume \( \Gamma(0) < \Gamma(1) \). With \( \rho > 0 \) further assume that (20) holds. For any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we have

\[
\lim_{n \to \infty} \mathbb{E} \left[ I_n^{(0)}(L_n) \right] = \begin{cases} \infty & \text{if } 1 + \rho \ln \Gamma(0) < 0 \\ 0 & \text{if } 1 + \rho \ln \Gamma(0) > 0. \end{cases}
\]  

(79)

The reason for this additional “infinity-zero” law will soon become apparent as we turn next to the proof of Theorem 3.1.

Let \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) denote a \( \rho \)-admissible scaling. Under the condition \( 1 + \rho \ln \Gamma(0) > 0 \), Proposition 7.1 yields 
\[
\lim_{n \to \infty} \mathbb{E} \left[ I_n(L_n) \right] = 0,
\]
whence \( \lim_{n \to \infty} \mathbb{P} \left[ I_n(L_n) = 0 \right] = 1 \) by the method of first moment, and this establishes the one-law part of Theorem 3.1.

In view of the second moment results of Section 5.2, a straightforward application of the method of second moments to the count rvs (50) appears problematic; instead we focus on the related count variables

\[
Z_n = I_n^{(0)}(L_n), \quad n = 1, 2, \ldots
\]  

(80)

Under the condition \( 1 + \rho \ln \Gamma(0) < 0 \), Proposition 7.2 already gives the convergence 
\[
\lim_{n \to \infty} \mathbb{E} \left[ I_n^{(0)}(L_n) \right] = \infty.
\]

If we were able to establish the appropriate version of (49), namely

\[
\lim_{n \to \infty} \sup \frac{\mathbb{E} \left[ I_n^{(0)}(L_n) \cdot I_n^{(0)}(L_n,2) \right]}{\left( \mathbb{E} \left[ I_n^{(0)}(L_n) \right] \right)^2} \leq 1,
\]  

(81)

we would then be in a position to conclude \( \lim_{n \to \infty} \mathbb{P} \left[ I_n(L_n) = 0 \right] = 0 \) by the method of second moment applied to the rvs (80). Using the bound (33) (with \( \ell = 0 \)) we would immediately obtain 
\[
\lim_{n \to \infty} \mathbb{P} \left[ I_n(L_n) = 0 \right] = 0,
\]
and the proof of the zero-law part of Theorem 3.1 would be completed.

To establish (81) we proceed as follows: Fix \( n = 2, 3, \ldots \) and \( L = 1, \ldots \). Applying (51) (with \( \ell = 0 \)) gives

\[
\mathbb{E} \left[ I_n^{(0)}(L_n) \right] = \left( 1 - \Gamma(0)^{L} \right)^{n-1} \cdot \mathbb{P} [S_L(1) = 0] = \left( 1 - \Gamma(0)^{L} \right)^{n-1} \cdot \mu(0)^L.
\]

On the other hand, specializing (64) to \( k = \ell = 0 \) we obtain the bound

\[
\mathbb{E} \left[ I_n^{(0)}(L_n) \cdot I_n^{(0)}(L_n,2) \right] \leq \mathbb{P} [S_L(1) = 0] \mathbb{P} [S_L(2) = 0] = \mu(0)^{2L},
\]
whence
\[
\mathbb{E} \left[ \xi_{n,L}^{(0)}(1) \cdot \xi_{n,L}^{(0)}(2) \right] \leq \frac{\mu(0)^{2L}}{\left( (1 - \Gamma(0)L)^{n-1} \cdot \mu(0)L \right)^2} = \frac{1}{(1 - \Gamma(0)L)^{2(n-1)}}.
\]

As we substitute according to the \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) in this last inequality we obtain
\[
\mathbb{E} \left[ \xi_{n,L_n}^{(0)}(1) \cdot \xi_{n,L_n}^{(0)}(2) \right] \leq \frac{1}{(1 - \Gamma(0)L_n)^{2(n-1)}}, \quad n = 2, 3, \ldots
\]

Let \( n \) go infinity in this last inequality: Under the condition \( 1 + \rho \ln \Gamma(0) < 0 \) we readily get (81) as desired since \( \lim_{n \to \infty} (1 - \Gamma(0)L_n)^n = 1 \) by virtue of Lemma 6.1 (with \( \nu_n = 0 \) for all \( n = 1, 2, \ldots \)).

The remainder of the paper deals with the proof of Theorem 3.2.

**8 Auxiliary zero-infinity laws associated with Theorem 3.2**

Although the arguments for proving Theorem 3.2 are similar to the ones used in the proof of Theorem 3.1 there are major differences in some of the technical details. This should already be apparent from Proposition 8.2 below which will act as the appropriate analog to Proposition 7.2.

Again we begin by investigating the appropriate first moment conditions (42) and (48). This is contained in the following “zero-infinity” law for the first moment – Note the analogy with Theorem 3.2.

**Proposition 8.1.** Assume \( \Gamma(0) < \Gamma(1) \). With \( \rho > 0 \) further assume that (25) holds. For any \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we have
\[
\lim_{n \to \infty} \mathbb{E} \left[ I_n(L_n) \right] = \begin{cases} 
\infty & \text{if } 1 + \rho \ln \left( (1)^{\nu*(\rho)} \Gamma(0)^{1-\nu*(\rho)} \right) < 0 \\
0 & \text{if } 1 + \rho \ln \left( (1)^{\nu*(\rho)} \Gamma(0)^{1-\nu*(\rho)} \right) > 0
\end{cases}
\] (82)

where \( \nu*(\rho) \) is the unique solution in the interval \((0, \mu(1))\) to the equation (27).

We give two proofs of Proposition 8.1. The first one is given in Section 11 and uses Stirling’s approximation to obtain the asymptotic of various quantities. The second proof is given in Appendix (Section 14 and Section 15), and relies on a change of measure argument introduced in Section 13. While this second proof might be less intuitive than the one provided in this section, it has the advantage of giving a probabilistic interpretation to the quantity (22).

As in the proof Theorem 3.1 we need to complement the “zero-infinity” law of Proposition 8.1. This time, however, the needed result assumes a more complicated form than the one taken in Proposition 7.2. First we need to set the stage: Our starting point is a scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) with the property \( \lim_{n \to \infty} L_n = \infty \), a condition automatically satisfied by \( \rho \)-admissible scalings. Pick \( \nu \) in \((0, 1)\), and consider any sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) such that
\[
\ell_n \leq L_n, \quad n = 1, 2, \ldots
\] (83)
under the additional property
\[ \lim_{n \to \infty} \frac{\ell_n}{L_n} = \nu. \tag{84} \]
We refer to any sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) satisfying the conditions \( 83 \)-\( 84 \) as a sequence \( \nu \)-associated with the scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \). A \( \nu \)-associated sequence can be easily generated through the formula
\[ \ell_n = \lfloor \nu L_n \rfloor \text{ for all } n = 1, 2, \ldots. \]
Any \( \nu \)-associated sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) induces the sequence \( \nu : \mathbb{N}_0 \to [0, 1] \) defined by
\[ \nu_n = \frac{\ell_n}{L_n}, \quad n = 1, 2, \ldots \]
In this notation the constraints \( 83 \) and \( 84 \) can now be expressed as
\[ \ell_n = \nu_n L_n, \quad n = 1, 2, \ldots \tag{85} \]
and
\[ \lim_{n \to \infty} \nu_n = \nu. \tag{86} \]
The next result is established in Section 10.

**Proposition 8.2.** Assume \( \Gamma(0) < \Gamma(1) \). With \( \rho > 0 \), we further assume that \( 25 \) holds. Consider an \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), and any \( \nu \)-associated sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) with \( \nu \) in \( (0, 1) \). Under the condition \( 30 \), the parameter \( \nu \) can be selected in the interval \( (\nu_*(\rho), \mu(1)) \) so that
\[ \lim_{n \to \infty} \mathbb{E} \left[ I_n(\ell_n)(L_n) \right] = \infty. \tag{87} \]

In Section 11 and Section 10 we will have the opportunity to use Stirling’s approximation for factorials given by
\[ p! \sim \sqrt{2\pi p} \left( \frac{p}{e} \right)^p \quad (p \to \infty). \tag{88} \]

## 9 A proof of Theorem 3.2

Consider a \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) for some \( \rho > 0 \). Under the condition \( 1 + \rho \ln \Gamma(1)\nu_*(\rho) \Gamma(0) \Gamma(0)^{1-\nu_*(\rho)} > 0 \), Proposition 8.1 yields
\[ \lim_{n \to \infty} \mathbb{E} [I_n(L_n)] = 0, \]
whence \( \lim_{n \to \infty} \mathbb{P} [I_n(L_n) = 0] = 1 \) by the method of first moments, and this establishes the one-law part of Theorem 3.2.

Assume now that \( 1 + \rho \ln \Gamma(1)\nu_*(\rho) \Gamma(0) \Gamma(0)^{1-\nu_*(\rho)} < 0 \). Here as well, we will not attempt to apply the method of second moment directly to the count variables \( 50 \) in order to establish the zero-law part of Theorem 3.2. Under the enforced assumptions, we shall show instead that the parameter \( \nu \) can be selected in \( (\nu_*(\rho), \mu(1)) \) in such a manner that the method of second moment applies to the count variables
\[ Z_n = I_n(\ell_n)(L_n), \quad n = 1, 2, \ldots \tag{89} \]
where the sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) is \( \nu \)-associated with the scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) for the selected value of \( \nu \).
This will require showing the validity of both

\[
\lim_{n \to \infty} \mathbb{E} \left[ I_{n}^{(\ell_n)}(L_n) \right] = \infty
\]  

and

\[
\limsup_{n \to \infty} \mathbb{E} \left[ \left( \xi_{n,L}^{(\ell_n)}(1) \cdot \xi_{n,L}^{(\ell_n)}(2) \right) \right] \leq 1.
\]  

Once this is done, it will follow from the method of second moment applied to the rvs (89) that \( \lim_{n \to \infty} \mathbb{P} \left[ I_{n}^{(\ell_n)}(L_n) = 0 \right] = 0 \). Using the bound (38) (with \( L = L_n \) and \( \ell = \ell_n \) for each \( n = 2, 3, \ldots \)) we immediately obtain \( \lim_{n \to \infty} \mathbb{P} \left[ I_{n}(L_n) = 0 \right] = 0 \), and the zero-law part of Theorem 3.2 will then be established.

To establish the convergence statements (90) and (91), we proceed as follows: By Proposition 8.2 we already know that there exists some \( \nu \) in the interval \( (\nu_*(\rho), \mu(1)) \) such that (87), namely (90), holds -- in fact the proof shows that it happens for \( \nu \) in the interval \( (\nu_*(\rho), \beta_-(\rho)) \). It remains only to establish (91) for any \( \nu \) selected in the interval \( (\nu_*(\rho), \beta_-(\rho)) \). To that end, fix \( n = 2, 3, \ldots \) and \( L = 1, 2, \ldots \). Using the expression (51) we obtain

\[
\mathbb{E} \left[ \xi_{n,L}^{(\ell)}(1) \right] = \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell} \right)^{n-1} \cdot \mathbb{P} \left[ S_L(1) = \ell \right] = \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell} \right)^{n-1} \cdot \left( \frac{L}{\ell} \right) \mu(1)^{\ell} \mu(0)^{L-\ell}
\]  

on the range \( \ell = 0, 1, \ldots, L \). On the other hand, specializing (64) to \( k = \ell \) yields

\[
\mathbb{E} \left[ \xi_{n,L}^{(\ell)}(1) \cdot \xi_{n,L}^{(\ell)}(2) \right] \leq \mathbb{P} \left[ S_L(1) = \ell \right] \mathbb{P} \left[ S_L(2) = \ell \right] = \left( \frac{L}{\ell} \right)^2 \mu(1)^{\ell} \mu(0)^{L-\ell}
\]  

whence

\[
\mathbb{E} \left[ \frac{\xi_{n,L}^{(\ell)}(1) \cdot \xi_{n,L}^{(\ell)}(2)}{\left( \mathbb{E} \left[ \xi_{n,L}^{(\ell)}(1) \right] \right)^2} \right] \leq \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell} \right)^{-2(n-1)}.
\]

Now, substitute in this last inequality according to the given \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) and the sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \) \( \nu \)-associated with it where \( \nu \) appearing in (86) is the one selected earlier in the interval \( (\nu_*(\rho), \beta_-(\rho)) \). This yields

\[
\mathbb{E} \left[ \xi_{n,L}^{(\ell_n)}(1) \cdot \xi_{n,L}^{(\ell_n)}(2) \right] \left( \mathbb{E} \left[ \xi_{n,L}^{(\ell_n)}(1) \right] \right)^{-2(n-1)} \leq \left( 1 - \Gamma(1)^{\ell_n} \Gamma(0)^{L_n-\ell} \right)^{-2(n-1)} = \left( 1 - \left( \Gamma(1)^{\nu_n} \Gamma(0)^{(1-\nu_n)} \right)^{L_n} \right)^{-(n-1)}
\]

Letting \( n \) go infinity in this last inequality we conclude \( \lim_{n \to \infty} \left( 1 - \left( \Gamma(1)^{\nu_n} \Gamma(0)^{(1-\nu_n)} \right)^{L_n} \right)^{n-1} = 1 \) by virtue of Lemma 6.1 since \( 1 + \rho \ln \Gamma(1)^{\nu} \Gamma(0)^{1-\nu} < 0 \) for the value \( \nu \) we selected in the interval \( (\nu_*(\rho), \beta_-(\rho)) \). This establishes (91) and the proof of Theorem 3.2 is now complete.
10 A proof of Proposition \(8.2\)

Fix \(n = 2, 3, \ldots\) and \(L = 1, 2, \ldots\). Our point of departure is the expression (83), namely

\[
\mathbb{E} \left[ I_n^{(L)}(L) \right] = n \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L-\ell} \right)^{n-1} \cdot \mathbb{P} \left[ S_L = \ell \right]
\]

on the range \(\ell = 0, 1, \ldots, L\).

Pick \(\nu\) in \((0, 1)\). Substituting \(L\) and \(\ell\) in this last relation according to the scaling \(L : \mathbb{N}_0 \rightarrow \mathbb{N}_0\) and any \(\nu\)-associated sequence \(\ell : \mathbb{N}_0 \rightarrow \mathbb{N}\) satisfying (83) (or equivalently, (85)) and (84) for the selected \(\nu\), we get

\[
\mathbb{E} \left[ I_n^{(\ell_n)}(L_n) \right] = n \left( 1 - \Gamma(1)^{\ell_n} \Gamma(0)^{L_n-\ell_n} \right)^{n-1} \cdot \left( \frac{L_n}{\ell_n} \right) \mu(1)^{\ell_n} \mu(0)^{L_n-\ell_n}
= n \left( \frac{L_n}{\nu_n L_n} \right) \left( \mu(1)^{\nu_n} \mu(0)^{1-\nu_n} \right)^{L_n} \cdot \left( 1 - \left( \Gamma(1)^{\nu_n} \Gamma(0)^{1-\nu_n} \right)^{L_n} \right)^{n-1}
\]

where \(\nu_n L_n\) and \(L_n - \nu_n L_n = (1 - \nu_n) L_n\) are integers by construction.

After standard simplifications, Stirling’s formula readily yields

\[
\left( \frac{L_n}{\nu_n L_n} \right) \sim \frac{\sqrt{2\pi L_n} (L_n)^{L_n}}{\sqrt{2\pi \nu_n L_n} (\nu_n L_n)^{\nu_n L_n} \cdot \sqrt{2\pi (1 - \nu_n) L_n} (1 - \nu_n) L_n^{(1-\nu_n)L_n}} \cdot \frac{1}{1 - \frac{\nu_n}{(1 - \nu_n)^{1-\nu_n}}}
\]

so that

\[
n \left( \frac{L_n}{\nu_n L_n} \right) \left( \mu(1)^{\nu_n} \mu(0)^{1-\nu_n} \right)^{L_n} \sim \frac{n}{\sqrt{2\pi \nu_n (1 - \nu_n) L_n}} \cdot \left( \frac{\mu(1)^{\nu_n} \mu(0)^{1-\nu_n}}{\nu_n^{\nu_n} (1 - \nu_n)^{1-\nu_n}} \right)^{L_n}
= \frac{n}{\sqrt{2\pi \nu_n (1 - \nu_n) L_n}} \cdot G(\nu_n, \mu(1))^{L_n}.
\]

Collecting we obtain

\[
\mathbb{E} \left[ I_n^{(\ell_n)}(L_n) \right] \sim \frac{n}{\sqrt{2\pi \nu_n (1 - \nu_n) L_n}} \cdot G(\nu_n, \mu(1))^{L_n} \cdot \left( 1 - \left( \Gamma(1)^{\nu_n} \Gamma(0)^{1-\nu_n} \right)^{L_n} \right)^{n-1}
\]

as we make use of (86) in the last step.

Recall now that both conditions (25) and (30) are enforced. Therefore, as discussed at the end of Section 3, condition (28) holds on some interval \(I_-(\rho) = (\alpha_-(\rho), \beta_-(\rho)) \subseteq (0, \mu(1))\), said interval containing \(\nu_*(\rho)\). As we restrict \(\nu\) to be an element of \((\nu_*(\rho), \beta_-(\rho))\), we conclude by Lemma 6.1 that

\[
\lim_{n \to \infty} \left( 1 - \left( \Gamma(1)^{\nu_n} \Gamma(0)^{1-\nu_n} \right)^{L_n} \right)^{n-1} = 1,
\]
and the desired conclusion \( \lim_{n \to \infty} \mathbb{E} \left[ I_{n}^{(\ell_n)}(L_n) \right] = \infty \) follows provided we can show that

\[
\liminf_{n \to \infty} \frac{n \cdot G(\nu_n, \mu(1))^{L_n}}{\sqrt{L_n}} > 0. \tag{96}
\]

It is always possible to find \( \varepsilon > 0 \) so that the interval \((\nu - \varepsilon, \nu + \varepsilon)\) is contained in the interval \((\nu_\star(\rho), \beta_\star(\rho))\). By virtue of (84) there exists a finite integer \( n(\varepsilon) \) such that \( \nu - \varepsilon < \nu_n < \nu + \varepsilon \) whenever \( n \geq n(\varepsilon) \), and on that range, the monotonicity of the mapping \( \nu' \to 1 + \rho \ln G(\nu', \mu(1)) \) on \((0, \mu(1))\) yields

\[
0 < 1 + \rho \ln G(\nu - \varepsilon, \mu(1)) \leq 1 + \rho \ln G(\nu_n, \mu(1))
\]

because \( 1 + \rho \ln G(\nu', \mu(1)) > 0 \) on the interval \((\nu_\star(\rho), \beta_\star(\rho))\). Returning to the proof of Lemma 6.2 (with \( C_n = G(\nu_n, \mu(1)) \) for all \( n = 1, 2, \ldots \)), we see that (73) yields the bounds

\[
n \cdot G(\nu_n, \mu(1))^{L_n} = n^{1 + \rho_n \ln G(\nu_n, \mu(1))} \geq n^{1 + \rho_n \ln G(\nu - \varepsilon, \mu(1))}, \quad n \geq n(\varepsilon)
\]

where the sequence \( \rho : \mathbb{N}_0 \to \mathbb{R}_+ \) is the unique sequence associated with the \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \). It is then plain that

\[
\liminf_{n \to \infty} \frac{n \cdot G(\nu_n, \mu(1))^{L_n}}{\sqrt{L_n}} \geq \liminf_{n \to \infty} \frac{n^{1 + \rho_n \ln G(\nu - \varepsilon, \mu(1))}}{\sqrt{n}} = \infty \tag{97}
\]

since \( 1 + \rho \ln G(\nu - \varepsilon, \mu(1)) > 0 \). This establishes (96), and the proof of Proposition 8.2 is now complete.

\[
\]

11 A proof of Proposition 8.1

Assume \( \Gamma(0) < \Gamma(1) \), and consider a \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \) for some \( \rho > 0 \).

Under the condition \( 1 + \rho \ln \Gamma(1)^{\nu_\star(\rho)} \Gamma(0)^{1 - \nu_\star(\rho)} < 0 \), Proposition 8.2 asserts the existence of \( \nu \) in \((0,1)\) such that \( \lim_{n \to \infty} \mathbb{E} \left[ I_{n}^{(\ell_n)}(L_n) \right] = \infty \) for any \( \nu \)-associated sequence \( \ell : \mathbb{N}_0 \to \mathbb{N} \). It now follows that \( \lim_{n \to \infty} \mathbb{E} \left[ I_n(L_n) \right] = \infty \), and the infinity part of Proposition 8.1 holds – This is an immediate consequence of the bound (38) (with \( L = L_n \) and \( \ell = \ell_n \) for each \( n = 2, 3, \ldots \)).

As we now turn to establishing the zero-law in (38), assume that the condition \( 1 + \rho \ln \Gamma(1)^{\nu_\star(\rho)} \Gamma(0)^{1 - \nu_\star(\rho)} > 0 \) holds: As discussed at the end of Section 3 under this condition there exists \( \varepsilon \) sufficiently small in \((0, \nu_\star(\rho))\) so that \( \alpha_\star(\rho) < \nu_\star(\rho) - \varepsilon \), hence \( 1 + \rho \ln \left( \Gamma(1)^{\nu_\star(\rho) - \varepsilon} \Gamma(0)^{1 - \nu_\star(\rho) + \varepsilon} \right) > 0 \). Select such a value of \( \varepsilon \) and keep it fixed throughout the proof.

Fix \( n = 2, 3, \ldots \). It follows from (54) that

\[
\mathbb{E} \left[ I_n(L_n) \right] = n \sum_{\ell = 0}^{L_n} \binom{L_n}{\ell} \mu(1)^{\ell} \mu(0)^{L_n - \ell} \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L_n - \ell} \right)^{n-1} = \sum_{\ell = 0}^{\lfloor (\nu_\star(\rho) - \varepsilon)L_n \rfloor} n \binom{L_n}{\ell} \mu(1)^{\ell} \mu(0)^{L_n - \ell} \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L_n - \ell} \right)^{n-1} + \sum_{\ell = \lfloor (\nu_\star(\rho) - \varepsilon)L_n \rfloor + 1}^{L_n} n \binom{L_n}{\ell} \mu(1)^{\ell} \mu(0)^{L_n - \ell} \left( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L_n - \ell} \right)^{n-1}.
\]
We will obtain the desired conclusion \( \lim_{n \to \infty} \mathbb{E} [I_n(L_n)] = 0 \) by showing that

\[
\lim_{n \to \infty} \sum_{\ell=0}^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} n \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell} \left( 1 - \Gamma(1)^\ell \Gamma(0)^{L_n-\ell} \right)^{n-1} = 0
\]  

(98)

and

\[
\lim_{n \to \infty} \sum_{\ell=\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor + 1}^{L_n} n \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell} \left( 1 - \Gamma(1)^\ell \Gamma(0)^{L_n-\ell} \right)^{n-1} = 0.
\]  

(99)

To establish (98), we proceed as follows: First, for \( \ell = 0, 1, \ldots, L_n \), note the crude bounds

\[
n \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell} \left( 1 - \Gamma(1)^\ell \Gamma(0)^{L_n-\ell} \right)^{n-1} \leq n \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell}.
\]

Since \( \nu_\star(\rho) - \varepsilon \) lies in \( (0, \mu(1)) \), the quantity \( \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell} \) increases with \( \ell \) on the range \( \ell = 0, 1, \ldots, \lfloor (\nu_\star(\rho) - \varepsilon)L_n \rfloor \), and we obtain the bound

\[
\sum_{\ell=0}^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} n \left( \frac{L_n}{\ell} \right) \mu(1)^\ell \mu(0)^{L_n-\ell} \left( 1 - \Gamma(1)^\ell \Gamma(0)^{L_n-\ell} \right)^{n-1} \leq L_n \cdot n \left( \frac{L_n}{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right) \mu(1)^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \mu(0)^{L_n-\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor}.
\]

(100)

Using Stirling’s formula, we get the asymptotic equivalence

\[
\left( \frac{L_n}{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right) \sim \frac{\sqrt{L_n}}{\sqrt{2\pi \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor \cdot (L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor)}} \cdot A_n
\]

(101)

where for each \( n = 1, 2, \ldots \), the factor \( A_n \) is given by

\[
A_n \equiv \left( \frac{L_n}{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right)^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \cdot \left( \frac{L_n}{L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right)^{L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor}.
\]

After simplifications and rearrangements it follows that

\[
\left( \frac{L_n}{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right) \cdot \mu(1)^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \mu(0)^{L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor}
\sim \frac{\sqrt{L_n}}{\sqrt{2\pi \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor \cdot (L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor)}} \cdot A^*_n
\]

(102)

where for each \( n = 1, 2, \ldots \) we have

\[
A^*_n \equiv \left( \frac{\mu(1)L_n}{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right)^{\lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \cdot \left( \frac{\mu(0)L_n}{L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor} \right)^{L_n - \lfloor (\nu_\star(\rho)-\varepsilon)L_n \rfloor}.
\]

(103)

as we recall the definition (22) of \( G(\cdot, \cdot) \).
Noting that
\[
\lim_{n \to \infty} \frac{\sqrt{L_n}}{\sqrt{2\pi \left[ (\nu_\star - \varepsilon) L_n \right] \cdot (L_n - [(\nu_\star - \varepsilon) L_n])}} = 0,
\]
we conclude that
\[
\left( \frac{L_n}{[(\nu_\star - \varepsilon) L_n]} \right)^{\mu(1) \frac{[(\nu_\star - \varepsilon) L_n]}{L_n}} \leq G \left( \frac{[(\nu_\star - \varepsilon) L_n]}{L_n}, \mu(1) \right)^{L_n}
\]
for \( n \) sufficiently large, and the upper bound
\[
\sum_{\ell=0}^{\left\lfloor \frac{\nu_\star L_n}{\varepsilon} \right\rfloor} n \left( \frac{L_n}{\ell} \right)^{\mu(1) \ell \mu(0)^{L_n-\ell}} \left( 1 - \Gamma(1)^{\ell} \mu(0)^{L_n-\ell} \right)^{n-1}
\leq L_n \cdot n G \left( \frac{[(\nu_\star - \varepsilon) L_n]}{L_n}, \mu(1) \right)^{L_n}
\]
(104)
then follows for sufficiently large \( n \).

Next, the sequence \( \rho : \mathbb{N}_0 \to \mathbb{R}_+ \) being the unique sequence associated with the \( \rho \)-admissible scaling \( L : \mathbb{N}_0 \to \mathbb{N}_0 \), we write
\[
L_n \cdot n G \left( \frac{\nu_\star (\rho - \varepsilon) L_n}{L_n}, \mu(1) \right)^{L_n} = e^{\ln(rho \ln n) + (1 + rho \ln n) \ln n}
\]
for each \( n = 1, 2, \ldots \) where we have set
\[
C_n = G \left( \frac{\nu_\star (\rho - \varepsilon) L_n}{L_n}, \mu(1) \right).
\]

Obviously we have \( \lim_{n \to \infty} \frac{\nu_\star (\rho - \varepsilon) L_n}{L_n} = \nu_\star (\rho - \varepsilon) \), while the definition of \( \nu_\star (\rho) \) implies \( 1 + \rho \ln G (\nu_\star (\rho - \varepsilon), \mu(1)) < 0 \). Thus, letting \( n \) go to infinity in (104) yields
\[
\lim_{n \to \infty} L_n \cdot n G \left( \frac{[(\nu_\star - \varepsilon) L_n]}{L_n}, \mu(1) \right)^{L_n} = 0
\]
and (98) holds.

As we turn to showing (99) we note the successive bounds
\[
\sum_{\ell=\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor + 1}^{L_n} n \left( \frac{L_n}{\ell} \right)^{\mu(1) \ell \mu(0)^{L_n-\ell}} \left( 1 - \Gamma(1)^{\ell} \mu(0)^{L_n-\ell} \right)^{n-1}
\leq n \left( 1 - \Gamma(1) \right)^{\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor \Gamma(0)^{L_n-\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor} n-1
\leq n \left( 1 - \Gamma(1)^{\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor} \Gamma(0)^{L_n-\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor} \right)^{n-1}, \quad n = 1, 2, \ldots
\]
Indeed, the quantity \( 1 - \Gamma(1)^{\ell} \Gamma(0)^{L_n-\ell} \) is monotonically decreasing in \( \ell \) under the assumption \( \Gamma(1) > \Gamma(0) \), and a straightforward probabilistic interpretation yields
\[
\sum_{\ell=\left\lfloor \frac{\nu_\star (\rho - \varepsilon) L_n}{\varepsilon} \right\rfloor + 1}^{L_n} \left( \frac{L_n}{\ell} \right)^{\mu(1) \ell \mu(0)^{L_n-\ell} \leq 1}.
\]

25
The condition $1 + \rho \ln \left( \Gamma(1)^{\nu_*(\rho)-\varepsilon} \Gamma(0)^{1-\nu_*(\rho)+\varepsilon} \right) > 0$ implies
\[
\lim_{n \to \infty} n \left( 1 - \Gamma(1)^{\nu_*(\rho)-\varepsilon} \Gamma(0)^{L_n - \nu_*(\rho)-\varepsilon} \right)^{n-1} = 0
\]
by the remark following the proof of Lemma 6.1 and the convergence (99) holds. This completes the proof of Proposition 8.1.

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12 Appendix: A proof of Lemma \[5.2\]

The arguments are very similar to the ones given in the proof of Lemma \[5.1\] Pick positive \( n = 2, 3, \ldots \) and \( L = 1, 2, \ldots \), and consider distinct nodes \( u, v = 1, \ldots, n \). For \( k, \ell = 0, 1, \ldots, L \), not necessarily distinct, we start from the relation (63). Note that the product \( \xi_{n,L}(u) \xi_{n,L}(v) \) can be expressed as

\[
\xi_{n,L}(u) \xi_{n,L}(v) = \prod_{w=1, w \neq u}^{n} (1 - \chi_L(u, w)) \cdot \prod_{w=1, w \neq v}^{n} (1 - \chi_L(v, w))
\]

with factors represented as

\[
1 - \chi_L(u, v) = 1 \cdot [U(u, v) > Q_L(A_L(u), A_L(v))]
\]

and

\[
\prod_{w=1, w \neq u,v}^{n} (1 - \chi_L(u, w)) (1 - \chi_L(v, w)) = \prod_{w=1, w \neq u,v}^{n} 1 \cdot [U(u, w) > Q_L(A_L(u), A_L(w))] \cdot 1 \cdot [U(v, w) > Q_L(A_L(v), A_L(w))].
\]
Under the enforced independence assumptions, it is now straightforward to conclude that

\[
\mathbb{E} \left[ \xi_{n,L}(u) \xi_{n,L}(v) \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

\[
= (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))) \cdot \prod_{w=1, w \neq u, v}^n (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(w))) (1 - Q_L(\mathbf{A}_L(v), \mathbf{A}_L(w))).
\]

The smoothing property of conditional expectations is again invoked, this time to obtain

\[
\mathbb{E} \left[ \xi_{n,L}(u) \xi_{n,L}(v) \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

\[
= (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(v))) \cdot \mathbb{E} \left[ \prod_{w=1, w \neq u, v}^n \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

(105)

where

\[
\mathbb{E} \left[ \prod_{w=1, w \neq u, v}^n \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

\[
= \mathbb{E} \left[ \prod_{w=1, w \neq u, v}^n (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(w))) (1 - Q_L(\mathbf{A}_L(v), \mathbf{A}_L(w))) \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

\[
= \mathbb{E} \left[ \prod_{w=1, w \neq u, v}^n (1 - Q_L(\mathbf{a}_L, \mathbf{A}_L(w))) (1 - Q_L(\mathbf{b}_L, \mathbf{A}_L(w))) \right]_{\mathbf{a}_L=\mathbf{A}_L(u), \mathbf{b}_L=\mathbf{A}_L(v)}
\]

(106)

under the enforced i.i.d. assumptions on the rvs \( \mathbf{A}_L(1), \ldots, \mathbf{A}_L(n) \). In the notation introduced earlier at (55) and (61) we can write

\[
\mathbb{E} \left[ (1 - Q_L(\mathbf{a}_L, \mathbf{A}_L)) (1 - Q_L(\mathbf{b}_L, \mathbf{A}_L)) \right]_{\mathbf{a}_L=\mathbf{A}_L(u), \mathbf{b}_L=\mathbf{A}_L(v)}
\]

(107)

This allows us to conclude that

\[
\mathbb{E} \left[ \prod_{w=1, w \neq u, v}^n (1 - Q_L(\mathbf{A}_L(u), \mathbf{A}_L(w))) (1 - Q_L(\mathbf{A}_L(v), \mathbf{A}_L(w))) \left| \mathbf{A}_L(u), \mathbf{A}_L(v) \right. \right]
\]

\[
= (1 - Q_L^*(\mathbf{A}_L(u)) - Q_L^*(\mathbf{A}_L(v)) + Q_L^{**}(\mathbf{A}_L(u), \mathbf{A}_L(v)))^{n-2},
\]

(108)

and substituting into (105) we obtain the desired conclusion (59).
13 Appendix: A change of measure

As stated earlier, all rvs are defined on the measurable space \((\Omega, \mathcal{F})\) and their statistics computed under the given probability measure \(P\) as stipulated by Assumptions (i)-(iii). To proceed we will find it convenient to embed \(P\) on the \(\sigma\)-field \(\mathcal{F}\) with the definition (22) used in the last step and where we have set assumption:

(iii-\(\nu\)) The rvs \(\{A, A_\ell, A_\ell(u), \ \ell = 1, 2, \ldots; \ u = 1, 2, \ldots\}\) form a collection of \(i.i.d.\) \(\{0, 1\}\)-valued rvs with pmf \(\nu = (\nu, 1 - \nu)\) where

\[
P_\nu[A = 0] = 1 - \nu \quad \text{and} \quad P_\nu[A = 1] = \nu.
\]

Let \(E_\nu\) denote the expectation operator associated with \(P_\nu\).

Obviously, we have \(P \equiv P_\nu\) when selecting \(\nu = \mu(1)\). It is always possible to construct a measurable space \((\Omega, \mathcal{F})\), the appropriate collections of rvs on it and a collection \(\{P_\nu, \ \nu \in (0, 1)\}\) of probability measures defined on the \(\sigma\)-field \(\mathcal{F}\) with the requisite properties; details are well known and omitted here for the sake of brevity.

In fact, given \(\nu\) in \((0, 1)\), for each \(L = 1, \ldots\), the probability measures \(P\) and \(P_\nu\) are mutually absolutely continuous when restricted to the \(\sigma\)-field \(\sigma\{A_1, \ldots, A_L\}\) with Radon-Nikodym derivative given by

\[
\left(\frac{dP}{dP_\nu}\right)_L = \prod_{\ell=1}^L \left(\frac{\mu(1)}{\nu}\right)^{A_\ell} \left(\frac{1 - \mu(1)}{1 - \nu}\right)^{1 - A_\ell} = \left(\frac{\mu(1)}{\nu}\right)^{S_L} \left(\frac{1 - \mu(1)}{1 - \nu}\right)^{L - S_L}.
\]

However, the probability measures \(P\) and \(P_\nu\) are not mutually absolutely continuous on the entire \(\sigma\)-field \(\mathcal{F}\).

To take advantage of this change of measure we proceed as follows: Fix \(\nu\) in \((0, 1)\), \(n = 2, 3, \ldots\) and \(L = 1, 2, \ldots\). The expression (54) can be written

\[
E[I_n(L)] = nE \left[ (1 - \Gamma(1)^{S_L} \Gamma(0)^{L - S_L})^{n - 1} \right] = n \cdot E_\nu \left[ (1 - \Gamma(1)^{S_L} \Gamma(0)^{L - S_L})^{n - 1} \cdot \left(\frac{\mu(1)}{\nu}\right)^{S_L} \left(\frac{1 - \mu(1)}{1 - \nu}\right)^{L - S_L} \right] = n \left( \left(\frac{\mu(1)}{\nu}\right)^{\nu} \left(\frac{1 - \mu(1)}{1 - \nu}\right)^{1 - \nu} \right)^L \cdot E_n(\nu, L) = nG(\nu, \mu(1))^L \cdot E_n(\nu, L)
\] (109)

with the definition 22 used in the last step and where we have set

\[
E_n(\nu, L) = E_\nu \left[ (1 - \Gamma(1)^{S_L} \Gamma(0)^{L - S_L})^{n - 1} \cdot \left(\frac{\mu(1)}{\nu}\right)^{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}^{S_L - L\nu} \right].
\] (110)

For future reference we note the decomposition

\[
E_n(\nu, L) = E_n^+(\nu, L) + E_n^-(\nu, L)
\] (111)
with $E^+_n(\nu, L)$ and $E^-_n(\nu, L)$ given by

$$E^+_n(\nu, L) = \mathbb{E}_\nu \left[ (1 - \Gamma(1)^{S_L} \Gamma(0)^{L-S_L})^{n-1} \cdot \left( \frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)} \right)^{S_L-L\nu} 1_{[S_L - \nu L > 0]} \right]$$

and

$$E^-_n(\nu, L) = \mathbb{E}_\nu \left[ (1 - \Gamma(1)^{S_L} \Gamma(0)^{L-S_L})^{n-1} \cdot \left( \frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)} \right)^{S_L-L\nu} 1_{[S_L - \nu L \leq 0]} \right].$$

It is plain that

$$\frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)} > 1 \text{ if and only if } \nu < \mu(1). \quad (112)$$

We shall also use the simple fact that

$$\Gamma(1)^{S_L} \Gamma(0)^{L-S_L} = \left( \frac{\Gamma(1)}{\Gamma(0)} \right)^{L \cdot \left( \frac{\Gamma(1)}{\Gamma(0)} \right)^{S_L-L\nu}}. \quad (113)$$

These observations form the basis for the arguments given next.

14 Appendix: A proof of Proposition 8.1 – The zero-law

Consider a $\rho$-admissible scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$ such that (25) holds, or equivalently,

$$1 + \rho \ln(1 - \mu(1)) < 0. \quad (114)$$

By the discussion preceding the statement of Theorem 3.2 the non-linear equation (27) admits a single solution $\nu^* (\rho)$ in the interval $(0, \mu(1))$ and

$$1 + \rho \ln G(\nu, \mu(1)) < 0, \quad \nu \in (0, \nu^* (\rho)).$$

It follows from Lemma 6.2 (with $C_n = G(\nu, \mu(1))$ for all $n = 1, 2, \ldots$) that

$$\lim_{n \to \infty} n G(\nu, \mu(1))^{L_n} = 0, \quad \nu \in (0, \nu^* (\rho)).$$

Therefore, by virtue of (109) the desired result $\lim_{n \to \infty} \mathbb{E} \left[ I_n(L_n) \right] = 0$ will be established if we show that

$$\limsup_{n \to \infty} E_n(\nu, L_n) < \infty \quad (115)$$

for some $\nu$ in $(0, \nu^* (\rho))$.

This issue is explored with the help of the decomposition (111): Fix $n = 2, 3, \ldots$ and pick $\nu$ in the interval $(0, \nu^* (\rho))$. Thus, (112) holds, and we have

$$\left( \frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)} \right)^{S_{L_n}-L_n\nu} \leq \left( \frac{\mu(1)}{\nu} \cdot \frac{1-\nu}{1-\mu(1)} \right)^{(1-\nu)L_n}$$

since $S_{L_n} \leq L_n$. Using $\Gamma(0) < \Gamma(1)$ in (113) we then conclude that

$$\left( \frac{\Gamma(1)}{\Gamma(0)} \right)^{L_n} \leq \Gamma(1)^{S_{L_n}} \Gamma(0)^{L_n-S_{L_n}} \text{ on } [S_{L_n} - L_n\nu > 0],$$
whence
\[
(1 - \Gamma(1)^{S_{L_n}}(0)^{L_n-S_{L_n}})^{n-1} \leq \left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{L_n}\right)^{n-1}
\] on \([S_{L_n} - L_n \nu > 0]\).

Using these bounds in the definition of \(E^+_n(\nu, L_n)\), we obtain
\[
E^+_n(\nu, L_n)
\leq \left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{L_n}\right)^{n-1} \cdot \left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)^{(1-\nu)L_n} \mathbb{P}_\nu [S_{L_n} - \nu L_n > 0]
\]
\[
\leq \left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{L_n}\right)^{n-1} \cdot \left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)^{(1-\nu)L_n}. \quad (116)
\]

Next we turn to bounding \(E^-_n(\nu, L_n)\). Because \(\Gamma(0) < \Gamma(1) < 1\), we always have
\[
(1 - \Gamma(1)^{S_{L_n}}(0)^{L_n-S_{L_n}})^{n-1} \leq 1
\]
and exploiting the bound \((112)\) gives
\[
\left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)^{S_{L_n}-L_n \nu} \leq 1 \text{ on } [S_{L_n} - L_n \nu \leq 0].
\]

We readily conclude \(E^-_n(\nu, L_n) \leq \mathbb{P}_\nu [S_{L_n} - L_n \nu \leq 0] \leq 1\) by applying these two bounds to the expression of \(E^-_n(\nu, L_n)\).

Thus, in order to establish \((115)\) we need only show that
\[
\limsup_{n \to \infty} E^+_n(\nu, L_n) < \infty \quad (117)
\]
for some \(\nu\) in \((0, \nu_*(\rho))\), possibly under additional conditions which ensure that the constraint \((29)\) also holds. As per the discussion following Theorem 3.2, the condition \((31)\) guarantees \((29)\) when \(\nu\) is selected in the interval \((\alpha_*(\rho), \nu_*(\rho))\), as we do from now on.

First, let the sequence \(\rho : \mathbb{N}_0 \to \mathbb{R}_+\) be the unique sequence associated with the \(\rho\)-admissible scaling \(L : \mathbb{N}_0 \to \mathbb{N}_0\). For each \(n = 2, 3, \ldots\) consider each of the factors in the bound at \((116)\). We find that
\[
\left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{L_n}\right)^{n-1} = \left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{\rho_n \ln n}\right)^{n-1}
\]
\[
\leq e^{-(n-1)(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{\rho_n \ln n}}
\]
\[
= e^{-\frac{(n-1)}{n} \cdot \rho_n \ln(n(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}))} \quad (118)
\]
and
\[
\left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)^{(1-\nu)L_n} = \left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)^{(1-\nu)\rho_n \ln n}
\]
\[
= n^{(1-\nu)\rho_n \ln\left(\frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)}\right)} \quad (119)
\]

By the \(\rho\)-admissibility of the scaling \(L : \mathbb{N}_0 \to \mathbb{N}_0\), for every \(\varepsilon > 0\) there exists a positive integer \(n_*(\varepsilon)\) such that \(\rho - \varepsilon < \rho_n < \rho + \varepsilon\) whenever \(n \geq n_*(\varepsilon)\). On that range the bounds \((118)\) and \((119)\) imply
\[
\left(1 - (\Gamma(1)^{\nu}\Gamma(0)^{1-\nu})^{L_n}\right)^{n-1} \leq e^{-\frac{(n-1)}{n} \cdot \rho_n \ln(n(\Gamma(1)^{\nu}\Gamma(0)^{1-\nu}))} \quad (120)
\]

and
\[
\left( \frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)} \right)^{(1 - \nu)L_n} \leq n^{(1 - \nu)(\rho + \epsilon) \ln \left( \frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)} \right)}
\]
(121)
as we recall that $\Gamma(0)$ and $\Gamma(1)$ both live in $(0, 1)$ and the inequality (112) holds. Given that (29) holds for the choice of $\nu$, then it is also the case that
\[
1 + (\rho + \epsilon) \ln \left( \Gamma(1) \nu \Gamma(0)^{(1 - \nu)} \right) > 0
\]
(122)
provided $\epsilon > 0$ is selected small enough (as we do from now on).

Let $n$ go to infinity in (116). It is plain from (118) that
\[
\lim_{n \to \infty} e^{-(n-1)(\Gamma(1)^{\nu} \Gamma(0)^{(1 - \nu)})^{\rho_n} \ln n} = 0
\]
by virtue of condition (122), while (119) implies
\[
\lim_{n \to \infty} \left( \frac{\mu}{\nu} \cdot \frac{1 - \nu}{1 - \mu} \right)^{(1 - \nu)\rho_n \ln n} = \infty
\]
under (112). Nevertheless, appealing to the bounds (120) and (121), we have $\lim_{n \to \infty} E_n^+(\nu, L_n) = 0$ in view of the fact that
\[
\lim_{n \to \infty} \left( e^{-(n-1)(\Gamma(1)^{\nu} \Gamma(0)^{(1 - \nu)})^{\rho_n} \ln n} \cdot n^{(1 - \nu)(\rho + \epsilon) \ln \left( \frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)} \right)} \right) = 0.
\]
This is because the first factor goes to zero like $e^{-n^\delta}$ (with $\delta > 0$) while the second factor explodes to infinity like $n^{\beta}$ (with $\beta > 0$). Obviously, $\limsup_{n \to \infty} E_n^-(\nu, L_n) \leq 1$ and the conclusion $\limsup_{n \to \infty} E_n(\nu, L_n) \leq 1$ follows. This concludes the proof of the zero-law in Theorem 3.2.

15 Appendix: A proof of Proposition 8.1 – The infinity-law

Consider a $\rho$-admissible scaling $L : \mathbb{N}_0 \to \mathbb{N}_0$ such that (25) holds, or equivalently, (114). We already know that
\[
1 + \rho \ln G(\nu, \mu(1)) > 0, \quad \nu \in (\nu_*(\rho), \mu(1)),
\]
(123)
and the convergence
\[
\lim_{n \to \infty} nG(\nu, \mu(1))^L_n = \infty, \quad \nu \in (\nu_*(\rho), \mu(1))
\]
follows by Lemma 6.2 (with $C_n = G(\nu, \mu(1))$ for all $n = 1, 2, \ldots$). By virtue of (109) the desired result $\lim_{n \to \infty} E_I[n(L_n)] = \infty$ will be established if we show that
\[
\liminf_{n \to \infty} E_n^+(\nu, L_n) > 0
\]
(124)
for some $\nu$ in $(\nu_*(\rho), \mu(1))$ possibly constrained by some additional condition.

Pick $\nu$ still in $(\nu_*(\rho), \mu(1))$ for the time being, and fix $n = 2, 3, \ldots$ Because (112) holds here, we have
\[
\left( \frac{\mu(1)}{\nu} \cdot \frac{1 - \nu}{1 - \mu(1)} \right)^{S_{L_n} - L_n \nu} \geq 1 \text{ on } [S_{L_n} - L_n \nu > 0]
\]
(125)
so that
\[ E_n^+(\nu, L_n) \geq \mathbb{E}_\nu \left[ (1 - \Gamma(1) S_{L_n} \Gamma(0)^{L_n - S_{L_n}})^{n-1} \mathbb{1}_{[S_{L_n} - \nu L_n > 0]} \right]. \quad (126) \]

Next, we write
\[ (1 - \Gamma(1) S_{L_n} \Gamma(0)^{L_n - S_{L_n}})^{n-1} = \left( 1 - \left( \Gamma(1) \frac{S_{L_n}}{L_n} \Gamma(0)^{1 - \frac{S_{L_n}}{L_n}} \right)^{L_n} \right)^{n-1} \quad (127) \]
and note that
\[ \left| (1 - \Gamma(1) S_{L_n} \Gamma(0)^{L_n - S_{L_n}})^{n-1} \right| \leq 1. \]

Now further restrict the value of \( \nu \) to the interval \((\nu_*(\rho), \beta_-(\rho))\) discussed at the end of Section 3. Condition (30) ensures that (28) holds, and by Lemma 6.1 (with \( \nu_n = \frac{S_{L_n}}{L_n} \) for all \( n = 1, 2, \ldots \), with the help of (127)), we have the convergence
\[ \lim_{n \to \infty} \left( 1 - \Gamma(1) S_{L_n} \Gamma(0)^{L_n - S_{L_n}} \right)^{n-1} = 1. \quad \mathbb{P}_\nu - \text{a.s.}, \quad (128) \]
Indeed, the Strong Law of Large Numbers (under \( \mathbb{P}_\nu \)) yields the convergence
\[ \lim_{n \to \infty} \frac{S_{L_n}}{L_n} = \nu \quad \mathbb{P}_\nu - \text{a.s.}, \]
and this leads to the needed conclusion
\[ \lim_{n \to \infty} \left( 1 + \rho_n \ln \left( \Gamma(1) \frac{S_{L_n}}{L_n} \Gamma(0)^{1 - \frac{S_{L_n}}{L_n}} \right) \right) = 1 + \rho \ln \left( \Gamma(0)^{1 - \nu} \Gamma(1)^\nu \right) < 0 \quad \mathbb{P}_\nu - \text{a.s.} \]
under (28).

Pick \( \varepsilon \) in \((0, 1)\). It follows from the bound (126) that
\[ E_n^+(\nu, L_n) \geq (1 - \varepsilon) \mathbb{P}_\nu [A_n(\varepsilon) \cap [S_{L_n} - \nu L_n > 0]], \quad n = 2, 3, \ldots \quad (129) \]
where for notational simplicity we have introduced the event
\[ A_n(\varepsilon) = \left[ \left( 1 - \Gamma(1) S_{L_n} \Gamma(0)^{L_n - S_{L_n}} \right)^{n-1} > 1 - \varepsilon \right]. \]
Since a.s. convergence implies convergence in probability (under \( \mathbb{P}_\nu \)), it is plain from (128) that
\[ \lim_{n \to \infty} \mathbb{P}_\nu [A_n(\varepsilon)] = 1. \] On the other hand we also have \( \lim_{n \to \infty} \mathbb{P}_\nu [S_{L_n} - L_n \nu > 0] = \frac{1}{2} \) by the Central Limit Theorem (under \( \mathbb{P}_\nu \)), whence \( \lim_{n \to \infty} \mathbb{P}_\nu [A_n(\varepsilon) \cap [S_{L_n} - \nu L_n > 0]] = \frac{1}{2} \) by standard arguments. Therefore, \( \lim\inf_{n \to \infty} E_n^+(\nu, L_n) \geq (1 - \varepsilon)/2 \) and the desired conclusion \( \lim\inf_{n \to \infty} E_n^+(\nu, L_n) \geq 1 \) follows since \( \varepsilon \) is arbitrary in \((0, 1)\). This conclude the proof of the infinity-law in Proposition 8.1. \( \blacksquare \)