On degenerate blow-up profiles for the subcritical semilinear heat equation

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Abstract

We consider the semilinear heat equation with a superlinear power nonlinearity in the Sobolev subcritical range. We construct a solution which blows up in finite time only at the origin, with a completely new blow-up profile, which is cross-shaped. Our method is general and extends to the construction of other solutions blowing up only at the origin, with a large variety of blow-up profiles, degenerate or not.

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1 Introduction

We consider the following subcritical semilinear heat equation

$$\partial_t u = \Delta u + |u|^{p-1} u,$$

(1.1)

where \( u : (x, t) \in \mathbb{R}^N \times [0, T) \rightarrow \mathbb{R}, \, T > 0, \)

\( p > 1 \) and \( (N - 2)p < N + 2. \) (1.2)

We consider a solution \( u(x, t) \) blowing up in finite time \( T > 0: \)

$$\|u(t)\|_{L^\infty} \rightarrow \infty \text{ as } t \rightarrow T,$$

and \( a \in \mathbb{R}^N \) a blow-up point of \( u(x, t) : |u(a, t)| \rightarrow \infty \text{ as } t \rightarrow T. \)

From Giga and Kohn [9] and Giga, Matsui and Sasayama [11], we know that all blow-up solutions are Type 1 in the subcritical case:

$$\forall t \in [0, T), \quad \|u(t)\|_{L^\infty} \leq C(T - t)^{-\frac{1}{p-1}} \text{ for some } C > 0. \quad (1.3)$$

In order to simplify the exposition, we assume

\( N = 2 \)

1
and focus on the simplest case of the new profiles we handle in this paper (see (1.14) below). Other examples for \( N = 2 \) and some extensions to the case \( N \geq 3 \) are given in Section 1.3. Following Giga and Kohn in [8], we introduce

\[
w_a(y, s) = (T - t)^{\frac{1}{p-1}} u(x, t) \text{ where } y = \frac{x - a}{\sqrt{T - t}} \text{ and } s = -\log(T - t).
\]

(1.4)

From Giga and Kohn [10], we know that up to replacing \( u \) by \(-u\), we have

\[
w_a(y, s) \to \kappa \equiv (p - 1)^{-\frac{1}{p-1}} \text{ as } s \to \infty,
\]

(1.5)

uniformly on compact sets. According to Velázquez [25] (see also Filippas and Kohn [5] together with Filippas and Liu [6]), we may refine that convergence and obtain the following “blow-up profile” \( Q(y) \) such that

\[
w_a(y, s) - \kappa \sim Q(y, s) \text{ as } s \to \infty
\]

(1.6)

uniformly on compact sets, with:

- either

\[
Q(y, s) = -\frac{\kappa}{4ps} \sum_{i=1}^{l} h_2(y_i),
\]

(1.7)

where \( l = 1 \) or \( 2 \), after a rotation of coordinates; keeping only the leading terms in the polynomials involved in \( Q(y, s) \), we obtain the following quadratic form

\[
B(y) = \frac{\kappa}{4p} \sum_{i=1}^{l} y_i^2
\]

(1.8)

which is non zero and nonnegative;

- or

\[
Q(y, s) = -e^{-\left(\frac{m^2}{4} - 1\right)s} \sum_{j=0}^{m} C_{m,j} h_{m-j}(y_1) h_j(y_2)
\]

(1.9)

as \( s \to \infty \), for some even integer \( m = m(a) \geq 4 \), where \( y = (y_1, y_2) \), \( h_j(\xi) \) is the rescaled Hermite polynomial defined by

\[
h_j(\xi) = \sum_{i=0}^{\lfloor j/2 \rfloor} \frac{j!}{i!(j - 2i)!} (-1)^i \xi^{j-2i},
\]

(1.10)

and the multilinear form (obtained by keeping only the leading terms of the polynomials of \( Q(y) \))

\[
B(y) = \sum_{j=0}^{m} C_{m,j} y_1^{m-j} y_2^j
\]

(1.11)

is also non zero and nonnegative.

If the origin is the only zero for the multilinear form \( B(y) \) defined in (1.8) and (1.11), we are in the non-degenerate case. If not, we are in the degenerate case. Accordingly, the corresponding blow-up profile \( Q(y) \) given in (1.7) or (1.9) will be said to be non-degenerate or degenerate.
Following the classification in (1.6), (1.7) and (1.9), a natural question arises: is \(a\) an isolated blow-up point or not?

In the non-degenerate case, we know from Theorem 2 in Velázquez [24] that \(a\) is isolated. In particular, for any \(b\) in a small ball centered at \(a\) with \(b \neq a\), \(u(b, t)\) has a finite limit denoted by \(u(b, T)\), as \(t \to T\), with the following equivalents as \(b \to a\):

\[
\begin{align*}
    u(b, T) &\sim \left(\frac{(p - 1)^2}{8p} \frac{|x - a|^2}{\log |x - a|}\right)^{-\frac{1}{p-1}} & \text{if (1.7) holds with } l = 2; \\
    u(b, T) &\sim \left(\frac{(p - 1)^2}{\kappa} B(x - a)\right)^{-\frac{1}{p-1}} & \text{if (1.9) holds with } B(y) > 0 \text{ for } y \neq 0.
\end{align*}
\]

In the degenerate case, the situation is less clear. In fact, the only known examples are “artificial”, in the sense that the solution depends only on one space variable, say \(\omega \cdot x\) where \(\omega \in S^1\). A radial solution blowing-up outside the origin in 2 dimensions gives rise to a degenerate situation too. In these two examples, the blow-up point is not isolated. Apart from these trivial examples, no more solutions with a degenerate profile are known.

Following this, we wonder whether there exists a solution obeying (1.9) such that (1.9) holds for some \(m \geq 4\) with a degenerate multilinear form in (1.11) and an isolated blow-up point. In this paper, we provide such an example, which is the first ever in the subcritical range (see Theorem 1 below). Note that in the supercritical case, Merle, Raphaël and Szeftel have already provided in [17] an example of a single-point blow-up solution with a degenerate anisotropic profile, strongly relying on the existence of a stationary solution to equation (1.12) below which decays to zero at infinity (such a solution doesn’t exist in the subcritical range).

### 1.1 The existence question: state of the art and difficulties in the degenerate case

We review here the question of the existence of blow-up solutions obeying (1.7) and (1.9). Let us first mention that in the one dimensional case, the question was positively answered by Bricmont and Kupiainen in [1] (see also Herrero and Velázquez [13] for the case (1.9) with \(m = 4\)).

Let us go back to the two dimensional case and first focus on the non-degenerate case. The only examples we know concern the case (1.7) with \(l = 2\), thanks to Bricmont and Kupiainen [11] together with Merle and Zaag [15]. Such a behavior is known to be stable with respect to perturbations in initial data from [18] together with Fermanian, Merle and Zaag [29] and [4]. Note that Herrero and Velázquez showed the genericity of such a behavior in [13] and [12] dedicated to the one dimensional case, and in a non published document in higher space dimensions. We would like to mention also the solutions constructed by Nguyen and Zaag in [23], showing a refinement of (1.7) with \(l = 2\). As for the case (1.9) with a non degenerate multilinear form in (1.11), no example is available, up to our knowledge, not even for the symmetric cases with \(B(x) = |x|^m\) or \(B(x) = x_1^m + x_2^m\) with an even \(m \geq 4\).

Concerning the degenerate case in (1.9) with a degenerate multilinear form in (1.11), the only cases we know are the one-dimensional trivial cases we have just mentioned.
above, showing a non-isolated blow-up point. Apart from these trivial examples, no more solutions with a degenerate profile are known.

As stated earlier, the main goal of the paper is to provide an example of a blow-up solution obeying (1.9) in the degenerate case with an isolated blow-up point.

Let us mention that the question of having non-trivial solutions with degenerate profiles was mentioned by Hiroshi Matano, because of the failure of the formal computation. As a matter of fact, the strategy used in the non-degenerate case is ineffective in the degenerate case, as we will explain below.

Indeed, that strategy consists in working in the similarity variables setting (1.4), where equation (1.1) is transformed into the following equation satisfied by \( w \) (or \( w \) for short): for all \( y \in \mathbb{R}^2 \) and \( s \geq -\log T \),
\[
\partial_s w = \Delta w - \frac{1}{2} y \cdot \nabla w - \frac{w}{p-1} + |w|^{p-1} w. \tag{1.12}
\]

For example, the idea used by Bricmont and Kupiainen in [1] to construct their example in one space dimension with (1.9) which holds with \( m = 4 \) consists in linearizing equation (1.12) around the following profile:
\[
(p - 1 + e^{-s}|y|^4)^{-\frac{1}{p-1}}. \tag{1.13}
\]

Accordingly, if we intend to construct a solution obeying the following degenerate estimate:
\[
w_0(y, s) - \kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \text{ as } s \to \infty, \tag{1.14}
\]
uniformly on compact sets, a naive idea would be to linearize equation (1.12) around the following profile:
\[
\left( p - 1 + \frac{(p-1)^2}{\kappa} e^{-s}y_1^2y_2^2 \right)^{-\frac{1}{p-1}}, \tag{1.15}
\]
which already has the same expansion (1.14) as the solution we intend to construct.

Unfortunately, a big problem arises with this profile, since it doesn’t decay to 0 as \( |y| \to \infty \), unlike the profile in (1.13). In fact, this smallness of the profile (1.13) at infinity combined with the stability of the zero solution of (1.12) is essential in the control of the solution at infinity in space. In other words, with the profile (1.15), we can’t get such a control, and the naive idea collapses, unless we can manage to get this decaying property. Note that with the naive profile (1.15), the corresponding (approximate) solution is given by \( u(x, t) = (T-t)^{-\frac{1}{p-1}} \left( p - 1 + \frac{(p-1)^2}{\kappa} e^{-s}2y_1^2 \right)^{-\frac{1}{p-1}} \), which blows up everywhere on the axes \( x_1 = 0 \) and \( x_2 = 0 \).

### 1.2 Main result of the paper

In order to construct a solution obeying (1.14), following the previous subsection, a natural idea would be to refine the expansion (1.13) in order to get higher order terms ensuring some decaying along the axes \( y_i = 0 \), which are the degenerate directions of the naive profile (1.15). Such a refinement is given below in Lemma 2.1. It shows that the term \( C_{6,0}(h_0(y_1) + h_6(y_2)) \) may ensure that decaying property, provided that we take \( C_{6,0} = -\delta \) with \( \delta > 0 \) and large (note that the parameter \( C_{6,0} \) is free in the expansion...
of Lemma 2.1). Keeping only the leading terms of the polynomials, this leads to the following refinement of the profile (1.15):

\[ \Phi(y, s) = \left[ p - 1 + \frac{(p - 1)^2}{\kappa} \left( e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6) \right) \right]^{-\frac{1}{p-1}}, \quad (1.16) \]

as we will explain more in detail in Section 2 below. This refined version clearly has the decaying property at infinity in space, which enables us to rigorously prove the existence of a solution obeying the degenerate profile (1.14) and blowing up only at the origin, answering Matano’s question:

**Theorem 1** (A blow-up solution for equation (1.1) with a cross-shaped blow-up profile). When \( N = 2 \), there exists \( \delta_0 > 0 \) such that for any \( \delta \geq \delta_0 \), there exists a solution \( u(x, t) \) to equation (1.14) which blows up in finite time \( T \) only at the origin, with:

(i) (Inner profile)

\[ w_0(y, s) - \kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \text{ as } s \to \infty, \quad (1.17) \]

in \( L^2_\rho(\mathbb{R}^2) \) with

\[ \rho(y) = \exp\left(-\frac{|y|^2}{4}\right)/(4\pi). \quad (1.18) \]

and uniformly on compact sets.

(ii) (Intermediate profile): For any \( K > 0 \), it holds that

\[ \sup_{e^{-s}y_1^2y_2^2 + \delta e^{-2s}(y_1^6 + y_2^6) < K} |w(y, s) - \Phi(y, s)| \to 0 \text{ as } t \to T, \]

where \( \Phi(y, s) \) is defined in (1.16).

(iii) (Final profile): For any \( x \neq 0 \), \( u(x, t) \) converges to a finite limit \( u(x, T) \) uniformly on compact sets of \( \mathbb{R}^2 \setminus \{0\} \) as \( t \to T \), with

\[ u(x, T) \sim \left[ \frac{(p - 1)^2}{\kappa} (x_1^2 + x_2^2 + \delta(x_1^6 + x_2^6)) \right]^{-\frac{1}{p-1}} \text{ as } x \to 0. \quad (1.19) \]

**Remark.** One may convince himself that the profiles in (1.16) and (1.19) are cross-shaped, which justifies the title of the theorem.

**Remark.** This is the first example of a blow-up solution in the subcritical range showing a degenerate multilinear form in (1.11) with \( m \geq 4 \) and an isolated blow-up point. Note that in the literature, many examples of blow-up solutions with only one blow-up point are available (see Weissler [26], Bricmont and Kupiainen [1], Merle and Zaag [18], Nguyen and Zaag [23]). Strikingly enough, almost all the known examples are either radial or asymptotically radial, in the sense that they approach a radially symmetric blow-up profile. Up to our knowledge, the only exceptions hold in the Sobolev supercritical case, with the Type 1 solution constructed by Merle, Raphaël and Szeftel [17] (with a \( \frac{1}{5} \) rate in one direction and an exponentially decaying rate in the other), and also the Type 2 (i.e. non Type 1 (1.3)) blow-up solution due to Collot, Merle and Raphaël [2], where the profile is anisotropic for both examples. No example is available in the subcritical range. In Theorem 1, we provide such an example, with a non-radial solution blowing up only at the origin, obeying the behavior (1.9) with a degenerate multilinear form in (1.11), and an anisotropic blow-up profile.
Remark. Note that our method allows also to derive new solutions for $N = 2$ and also $N \geq 3$, blowing up only at the origin, both in the degenerate and the non-degenerate cases, showing a non necessarily radial profile in (1.9) and (1.11) (see below in Section 1.3).

Remark. As we have just written before the statement of Theorem 1, our strategy relies on the refinement of the goal (1.14) given in Lemma 2.1 below. Taking $C_{6,0} = -\delta$ where $\delta > 0$ (and large) is crucial in our argument. As a matter of fact, using our analysis and the blow-up criterion we proved in [19] for solutions of equation (1.12), we show that the construction is impossible in the case $C_{6,0} > 0$. More precisely, if $u(x,t)$ is a symmetric solution (with respect to the axes and the bissectrices) of equation (1.1) blowing up at some time $T > 0$ such that $u(0) \in L^\infty(\mathbb{R}^2)$ and estimate (1.17) holds, then, estimate (2.1) holds with $C_{6,0} \leq 0$ (note that a similar statement holds without the symmetry assumption).

1.3 Extensions of the main result

Following Theorem 1, we would like to mention that our strategy extends with no difficulty to the construction of solutions to equation (1.1) blowing up only at the origin with other types of profiles (as defined in (1.6), (1.7) and (1.9)) summarized in the following tables, respectively dedicated to the non-degenerate and the degenerate cases, where $C_0 = \frac{\(p-1\)^2}{\pi}$ and $\delta_0 > 0$ is large.

Note that in the degenerate case, the idea behind the design of the profiles we are presenting below is simple:

- First, considering the classification given in (1.6) (and its natural extension in higher dimensions), we choose the case (1.9) with some $Q(y)$ and a degenerate multilinear form $B(y)$ in (1.11); in Theorem 1 we choose $Q(y) = e^{-s}h_2(y_1)h_2(y_2)$ and $B(y) = y_1^2y_2^2$.

- Then, we refine estimate (1.6) by exhibiting higher order terms, and select among them the polynomials which live on the degenerate directions of $B(y)$; in Theorem 1, the degenerate directions are $y_1 = 0$ and $y_2 = 0$, the refinement of (1.6) is given below in (2.1) and the polynomials we select in that refinement of (1.6) are $e^{-2s}h_6(y_1)$ and $e^{-2s}h_6(y_2)$.

- Finally, we design a profile similar to (1.16), with 2 terms, one corresponding to the degenerate multilinear form $B(y)$ defined in (1.11) (which is $e^{-s}y_1^2y_2^2$ in Theorem 1), and the other to the polynomials we selected on the degenerate directions of $B(y)$ (which is $e^{-2s}(y_1^6 + y_2^6)$ in Theorem 1). The key property of that profile is that it is decaying to zero in all directions, though with different scales according to whether we are in the degenerate or the non-degenerate directions of the multilinear form $B(y)$.

Here are the 2 tables:
\[ Q(y, s) = -e^{(1 - \frac{k}{2})s} \sum_{i=0}^{k} C_{k,i} h_{k-i}(y_1) h_i(y_2) \]

\[ \text{Equivalent of } u(x, T) \text{ as } x \to 0 = [C_0 B(x)]^{-\frac{1}{p-1}} \text{ with } B(x) = \sum_{i=0}^{k} C_{k,i} x_1^{k-i} x_2^i \]

\[ Q(y, s) = -e^{-(\frac{N}{2} - 1)s} \sum_{j_1 + \cdots + j_N = k} C_{k,j_1,\ldots,j_N} h_{j_1}(y_1) \ldots h_{j_N}(y_N) \]

\[ \text{Equivalent of } u(x, T) \text{ as } x \to 0 = [C_0 B(x)]^{-\frac{1}{p-1}} \text{ with } B(x) = \sum_{j_1 + \cdots + j_N = k} C_{k,j_1,\ldots,j_N} x_1^{j_1} \ldots x_N^{j_N} \]

| Conditions |
|-------------|
| \( N = 2, \delta \geq \delta_0, \) \( k \geq 4 \text{ is even and } B(x) > 0 \text{ for } x \neq 0 \) |
| \( N \geq 3, \delta \geq \delta_0, \) \( k \geq 4 \text{ is even and } B(x) > 0 \text{ for } x \neq 0 \) |

Table 1: Extensions in the non-degenerate case.
| $Q(y,s)$                                      | Equivalent of $u(x,T)$ as $x \to 0$ | Conditions                                                                 |
|---------------------------------------------|-------------------------------------|-----------------------------------------------------------------------------|
| $-e^{-s} h_2(y_1)h_2(y_2 + a_0y_1)$         | $[C_0 (x_1^2(x_1 + a_0x_2)^2 + \delta(x_1^6 + x_2^6))]^{-\frac{1}{p-1}}$ | $N = 2, \delta \geq \delta_0, a_0 \in \mathbb{R}.$                       |
| $-e^{(1-k) s} h_{k_1}(y_1) \ldots h_{k_l}(a_l y_1 + b_l y_2)$ | $[C_0 \left( x_1^{k_1} \ldots (a_l x_1 + b_l x_2)^{k_l} + \delta(x_1^{k+2} + x_2^{k+2}) \right)]^{-\frac{1}{p-1}}$ | $N = 2, \delta \geq \delta_0, k = k_1 + \ldots + k_l, k_i \geq 2 \text{ is even, } (a_i, b_i) \neq (0,0),$ the straight lines $\{y_1 = 0\}, \ldots, \{a_i y_1 + b_i y_2 = 0\}$ are distinct. |
| $-e^{(1-k) s} \prod_{i=1}^m \left( \sum_{j \in I_i} h_{2\theta_i}(y_j) \right)$ | $\left[ C_0 \left( \prod_{i=1}^m \left( \sum_{j \in I_i} |x_j|^{2\theta_i} \right) + \delta(x_1^{k+2} + \ldots + x_N^{k+2}) \right) \right]^{-\frac{1}{p-1}}$ | $N \geq 3, \delta \geq \delta_0, k = \sum_{i=1}^m 2\theta_i,$ and the set $I_i$ make a partition of $\{1, \ldots, N\}.$ |

Table 2: Extensions in the degenerate case.
Following these examples, we would to make some comments:
- Up to our knowledge, the only known examples such that (1.9) holds with some even $m \geq 4$ and a non-degenerate form in (1.11) hold only in one space dimension. Those examples are due to Bricmont and Kupiainen [1] (see also Herrero and Velázquez [13] when $m = 4$). Strikingly enough, we couldn’t find a proof in the literature for the existence of a solution with a multilinear form $B(x) = |x|^4$ or $B(x) = x_1^4 + \cdots + x_N^4$ when $N \geq 2$. We are happy to say that our strategy provides such examples, as stated in Table 1. By the way, the non-degenerate multilinear forms $B(x)$ shown in that table need not be symmetric.
- The first example in Table 2 shows that the degeneracy directions of the multilinear form $B(x)$ need not be orthogonal, unlike one may assume from the constructed example in Theorem [1].

We proceed in several sections to prove Theorem [1]:
- first, in Section 2, we formally derive the profile in similarity variables;
- then, in Section 3, we explain our strategy for the proof;
- in Section 4, we explain the dynamics of the equation and suggest the general form of initial data;
- in Section 5, we prove a crucial $L^\infty$ bound on the solution in similarity variables;
- in Section 6, we conclude the proof of Theorem [1];
- finally, we prove the technical details in Section 7.

2 Formal determination of the profile in similarity variables

In this paper, we ask whether we can have a solution $u(x, t)$ to equation (1.1) which blows up in finite time $T > 0$ at the origin, such that estimate (1.14) holds, namely with

\begin{equation}
 w_0(y, s) - \kappa \sim -e^{-s}h_2(y_1)h_2(y_2) \quad \text{as } s \to \infty, \nonumber
\end{equation}

uniformly on compact sets. In order to simplify the calculations, we will assume that $u(x, t)$ is symmetric with respect to the axes and the bisectrices, for any $t \in [0, T)$.

As we have already discussed in Section 1.1, applying the strategy of Bricmont and Kupiainen [1] is ineffective, because the naive profile given in (1.15) is not decaying to zero along the axes $y_i = 0$. Following what we wrote in Section 1.2, the idea to refine the naive guess in (1.15) goes through the refinement of the target behavior in (1.14), which we give in the following:

Lemma 2.1 (Second order Taylor expansion). Following (1.14) and assuming the solution is symmetric with respect to the axes and bissectrices, it holds that

\begin{align}
 w_0(y, s) &= \kappa - e^{-s}h_2h_2 + e^{-2s} \left\{ -\frac{32p}{3}\kappa h_0h_0 - \frac{16p}{\kappa}(h_2h_0 + h_0h_2) \\
 &\quad - \frac{4p}{\kappa}(h_4h_0 + h_0h_4) - \frac{32p}{\kappa}h_2h_2 + C_6,0(h_6h_0 + h_0h_6) \\
 &\quad + \left(\frac{4p}{\kappa} + C_6,2\right)(h_4h_2 + h_2h_4) + \frac{p}{2\kappa}h_4h_4 \right\} + O\left(s^2e^{-3s}\right) \tag{2.1}
\end{align}

as $s \to \infty$, uniformly on compact sets, for some constants $C_6,0$ and $C_6,2$, where the notation $h_ih_j$ stands for $h_i(y_1)h_j(y_2)$. 

9
**Proof.** The proof is omitted since it is straightforward from the method we explained in Proposition 1 of our paper [22].

From this expansion, we remark that the term $C_{6,0} e^{-2s} (h_6(y_1) + h_6(y_2))$ will ensure the good decaying property, if $C_{6,0} < 0$, especially on the axes, where the main term $e^{-s} y_1^2 y_2^2$ is 0. More precisely, taking $C_{6,0} = -\delta$ for some $\delta > 0$ to be fixed large enough, then, keeping only the leading term in the polynomials, we naturally suggest the following modified version of (1.15)

$$\varphi(y, s) = \left[ \frac{E}{D} \right]^{\frac{1}{s}}$$

with

$$E = 1 + e^{-s} P(y) + e^{-2s} Q(y),$$

$$D = p - 1 + \frac{(p-1)^2}{\kappa} (e^{-s} y_1^2 y_2^2 + \delta e^{-2s} (y_1^6 + y_2^6)),$$

where $P(y)$ and $Q(y)$ are polynomials which will be chosen so that the numerator in (2.3) is positive (hence, $\varphi$ is well defined), and

$$\varphi(y, s) = \kappa - e^{-s} h_2 h_6$$

$$+ e^{-2s} \left\{ -\delta(h_6) + \gamma h_2 + h_4 h_4 \right\} + O(e^{-3s})$$

as $s \to \infty$, uniformly on compact sets, for some $\gamma$ to be fixed later. Note that the aimed expansion agrees with the prediction in Lemma 2.1 at the order $e^{-s}$ and most of the order $e^{-2s}$. Most importantly, we will also require that $\varphi(\cdot, s) \in L^\infty(\mathbb{R}^2)$ and $\varphi(y, s) \to 0$ as $|y| \to \infty$, for any $s \geq 0$, and the choice of $\gamma$ will be crucial for that.

Let us explicit the choice of $\delta, \gamma, P(y)$ and $Q(y)$. Since (2.2) directly implies that

$$\varphi(y, s) = \kappa + e^{-s} \left( \frac{\kappa}{p - 1} P(y) - y_1^2 y_2^2 \right)$$

$$+ e^{-2s} \left( \frac{\kappa}{p - 1} Q(y) + \frac{\kappa(2-p)}{2(p-1)^2} P(y)^2 - \frac{P(y)}{p - 1} y_1^2 y_2^2 - \delta y_1^6 - \delta y_2^6 + \frac{p}{2\kappa} y_1^4 y_2^4 \right) + O(e^{-3s})$$

as $s \to \infty$, uniformly on compact sets, we see by identification with (2.5) that we must have

$$P(y) = p - 1 \left( y_1^2 y_2^2 - h_2 h_2 \right),$$

$$Q(y) = p - 1 \left( \frac{P(y)}{p - 1} y_1^2 y_2^2 + \frac{\kappa(p-2)}{2(p-1)^2} P(y)^2 + \delta(y_1^6 - h_6(y_1)) + \delta(y_2^6 - h_6(y_2)) \right) + \frac{p}{2\kappa} (h_4 h_4 - y_1^4 y_2^4) + \gamma (h_4 h_4 + h_2 h_4).$$

Now, we claim the following:

**Lemma 2.2** (Good definition, boundedness and decaying at infinity of $\varphi(y, s)$ (2.2) - (2.7) - (2.8)). Take $\gamma = \frac{6\varphi_{\kappa}^2}{\kappa}$ and consider $\delta \geq \delta_{10}$ and $s \geq s_{10}(\delta)$ for some large enough $\delta_{10} \geq 1$ and $s_{10}(\delta) \geq 0$. Then:
(i) \(E \geq \frac{1}{2}\), hence \(\varphi(y, s)\) is well defined and positive.
(ii) \(\|\varphi(\cdot, s)\|_{L^\infty} \leq \kappa + C_0 e^{-\frac{\delta}{2}}\), for some \(C_0 > 0\).
(iii) \(\|\nabla \varphi(\cdot, s)\|_{L^\infty} \leq C_0 e^{-\frac{\delta}{2}}\).
(iv) \(\varphi(y, s) \to 0\) as \(|y| \to \infty\).

**Proof.** Take \(\gamma = \frac{6p-2}{\kappa}\) and consider \(\delta \geq 1\) and \(s \geq 0\) to be taken large enough.

(i) It’s enough to show that both \(P(y)\) and \(Q(y)\) are bounded from below. Since

\[ h_2(\xi) = \xi^2 - 2, \quad h_4(\xi) = \xi^4 - 12\xi^2 + 12 \quad \text{and} \quad h_6(\xi) = \xi^6 - 30\xi^4 + 180\xi^2 - 120 \]

from (1.10), it follows from (2.7) and (2.8) that

\[ P(y) = \frac{2(p-1)}{\kappa} (y_1^2 + y_2^2 - 2) \geq -\frac{4(p-1)}{\kappa} \]

and

\[ Q(y) = \frac{p-1}{\kappa} \left( \frac{2}{\kappa} \left[ y_1^2 + y_2^2 \right] + 30\delta \left[ y_1^2 + y_2^2 \right] + \frac{6p}{\kappa} \left[ y_1^2 - y_1^2 y_2^2 - y_1^2 y_2^2 + \gamma \right] \right) \]

\[ + \frac{p-1}{\kappa} \left( \frac{2}{\kappa} - \frac{6p}{\kappa} + \gamma \right) \left[ y_1^2 y_2^2 + y_2^4 \right] + 30\delta (y_1^4 + y_2^4) \]

\[ + O \left( \delta [1 + |y|^2] + [1 + |\gamma|] [1 + |y|^4] \right) \]

Choosing \(\gamma = \frac{6p-2}{\kappa}\), we see that

\[ Q(y) = \frac{30\delta(p-1)}{\kappa} [y_1^4 + y_2^4] + O \left( \delta [1 + |y|^2] + [1 + |\gamma|^4] \right) \quad \text{as} \quad |y| \to \infty, \quad (2.10) \]

Taking \(\delta\) large enough, then \(|y|\) large enough, we see that

\[ Q(y) \geq \frac{15\delta(p-1)}{\kappa} [y_1^4 + y_2^4], \quad (2.11) \]

which means that \(Q(y)\) is bounded from below. Since (2.9) implies that \(P(y)\) is bounded from below too, taking \(s\) large enough, we see from (2.2) that \(E \geq \frac{1}{2}\), hence, \(\varphi\) defined in (2.2) is well defined.

(ii) Using (2.9) and (2.10), we see from (2.2) that

\[ |\varphi(y, s)|^{p-1} \leq C \frac{(N_1 + N_2 + N_3)}{D} \]

where

\[ N_1 = 1 + e^{-s}, \quad N_2 = \delta e^{-2s} |y|^4, \quad N_3 = e^{-s} |y|^2 \]

and \(D\) is introduced in (2.4). Since \(D \geq p-1\), it follows that

\[ \frac{N_1}{D} \leq \frac{1 + e^{-s}}{p-1} \]

Introducing \(z = e^{-s} y\), we write

\[ \frac{N_2}{D} \leq \frac{C \delta e^{-s} |z|^4}{1 + z_1^2 z_2^2 + e^{-\frac{s}{2}} |z|^6} \leq \frac{C \delta e^{-s} |z|^4}{e^{-\frac{s}{2}} + e^{-\frac{s}{2}} |z|^6} = \frac{C \delta e^{-\frac{s}{2}} |z|^4}{1 + |z|^6} \leq C \delta e^{-\frac{s}{2}}. \]

(1.15)
Introducing \( X = |z|^2 \), we write
\[
\frac{N_3}{D} \leq \frac{Ce^{-\frac{4}{3}}|z|^2}{1 + e^{-\frac{2}{3}}|z|^6} = Cg_\epsilon(D), \tag{2.16}
\]
where
\[
\epsilon = e^{-\frac{s}{2}}, \ X = |z|^2 \text{ and } g_\epsilon(X) = \frac{\epsilon X}{1 + \epsilon X^\frac{3}{2}}. \tag{2.17}
\]
Since \( g_\epsilon'(X) = \frac{\epsilon(1 - 2\epsilon X^3)}{(1 + \epsilon X^3)^2} \) which changes from negative to positive at \( X = X_\epsilon \equiv (2\epsilon)^{-\frac{1}{3}} \), it follows that
\[
g_\epsilon(X) \leq g_\epsilon(X_\epsilon) = \frac{2\epsilon X_\epsilon}{3} = \frac{(2\epsilon)^{\frac{2}{3}}}{3}. \tag{2.18}
\]
Using (2.16) and (2.17), we see that
\[
\frac{N_3}{D} \leq Ce^{-\frac{4}{3}}. \tag{2.18}
\]
Collecting the estimates in (2.12), (2.14), (2.15) and (2.18), then recalling the definition (1.5) of \( \kappa \), we get the desired conclusion.

(iii) By definition (2.2) of \( \varphi(y,s) \), we write for all \( s \geq 1 \) and \( y \in \mathbb{R}^2 \),
\[
(p - 1) \log \varphi = \log E - \log D
\]
where \( E \) and \( D \) are defined in (2.3) and (2.4) (note that \( D > 0 \) by definition, and that \( E > 0 \) from item (i) of this lemma). Taking the gradient, we see that
\[
\nabla \varphi = \frac{\varphi}{p - 1} \left( \frac{\nabla E}{E} - \frac{\nabla D}{D} \right). \tag{2.19}
\]
Since
\[
|\nabla D| \leq C\delta e^{-2s}|y|^5 + Ce^{-s}|y_1|y_2 + Ce^{-s}|y_2|y_1^2 \equiv D_1 + D_2 + D_3 \tag{2.20}
\]
from (2.9) and (2.10), we write from item (ii) of this lemma
\[
|\nabla \varphi| \leq C \left[ \frac{|\nabla E|}{E} + \frac{D_1}{D} + \frac{D_2}{D} + \frac{D_3}{D} \right].
\]
Noting that
\[
E \geq \frac{E_0}{C}, \text{ where } E_0 = 1 + e^{-s}(y_1^2 + y_2^2) + e^{-2s}(y_1^4 + y_2^4), \tag{2.21}
\]
\[
|\nabla E| \leq Ce^{-s}|y| + C\delta e^{-2s}(1 + |y|^3),
\]
from (2.9), (2.11) and (2.10), using the definitions (2.4) and (2.20) of \( D \) and \( D_1 \), then proceeding as for the proof of item (ii) of this lemma, we show that
\[
\frac{|\nabla E|}{E} \leq \frac{|\nabla E|}{E_0} \leq C(\delta)e^{-\frac{4}{3}} \text{ and } \frac{D_1}{D} \leq C(\delta)e^{-\frac{4}{3}}. \tag{2.22}
\]
By symmetry, it remains only to bound the term \( \frac{D_2}{D} \), when \( y_2 \neq 0 \). Since \( \delta \geq 1 \), using the definitions (2.4) and (2.20) of \( D \) and \( D_2 \), we write
\[
\frac{D_2}{D} \leq C \frac{e^{-s}|y_1|y_2^2}{1 + e^{-s}y_1^2y_2^2 + e^{-2s}y_2^6}.
\]
As a function in the variable $|y_1|$, the left-hand side realizes its maximum for

$$|y_1| = \sqrt{\frac{1 + e^{-2s_y^6}}{e^{-s_y^2}}}. $$

Therefore, it follows that

$$\frac{D_2}{D} \leq C \sqrt{\frac{e^{-s_y^2}}{1 + e^{-2s_y^6}}} \leq Ce^{-\frac{s}{2}},$$

in particular

$$\frac{|\nabla D|}{D} \leq Ce^{-\frac{s}{2}},$$

and the estimate on $\nabla \varphi$ in item (iii) follows.

(iv) From (2.9), (2.8) and (2.10), we see that the numerator of the fraction in (2.2) is a polynomial of degree 4, whereas the denominator is of degree 6. Thus, the conclusion follows.

This concludes the proof of Lemma 2.2.

3 Strategy of the proof

From Section 2, we recall that our goal is to construct $u(x,t)$, a solution of equation (1.1) defined for all $(x,t) \in \mathbb{R}^2 \times [0,T)$ for some small enough $T > 0$, such that

$$w_0(y,s) - \kappa \sim -e^{-s} h_2(y_1) h_2(y_2) \text{ as } s \to \infty,$$

uniformly on compact sets, where $w_0(y,s)$ is the similarity variables’ version defined in (1.4). We also recall our wish to construct a solution which is symmetric with respect to the axes and the bisectrices.

Consider $\varphi(y,s)$ our candidate for the profile, defined in (2.2), (2.7) and (2.8), with $\gamma = \frac{6p-2}{\kappa}$ and $\delta \geq 1$ fixed large enough, so that Lemma 2.2 holds. From the target expansion of $w_0$ given in Lemma 2.1 (with $C_{6,0} = -\delta$ and $C_{6,2} = \gamma$) and the expansion of the profile $\varphi$ given in (2.5), we further specify our goal by requiring that $q(y,s)$ is small in some sense that will shortly given in (3.21), where

$$q(y,s) = w_0(y,s) - \varphi(y,s).$$

In order to achieve this goal, we need to write then understand the dynamics of the equation satisfied by $q(y,s)$ near 0. This is done in the following section.

3.1 Dynamics for $q(y,s)$ defined in (3.2)

Roughly speaking, our first task is to perform local estimates for bounded $y$, through a spectral analysis in $L^2_\rho$ near $\kappa$ (1.5), the constant solution of equation (1.12), assuming a uniform bound on the solution.

More precisely, from (1.12), we see that $q(y,s)$ satisfies the following equation, for all $(y,s) \in \mathbb{R}^2 \times [s_0, \infty)$, where $s_0 = -\log T$:

$$\partial_s q = (\mathcal{L} + V(y,s))q + B(y,s,q) + R(y,s),$$

(3.3)
\[
\mathcal{L}q = \Delta q - \frac{1}{2} y \cdot \nabla q + q, \quad V(y, s) = p\varphi(y, s)^{p-1} - \frac{p}{p-1}.
\]  
(3.4)

\[B(y, s, q) = |\varphi(y, s) + q|^{p-1}(\varphi(y, s) + q) - \varphi(y, s)^p - p\varphi(y, s)^{p-1}q,\]

\[R(y, s) = -\partial_s \varphi(y, s) + (\mathcal{L} - 1) \varphi(y, s) - \frac{\varphi(y, s)}{p-1} + \varphi(y, s)^p.
\]

Let us give in the following some useful properties of the terms involved in equation (3.3):

- **The linear term**: Note that $\mathcal{L}$ is a self-adjoint operator in $L^2_\rho$, the $L^2$ space with respect to the measure $\rho dy$ defined in (1.18). The spectrum of $\mathcal{L}$ consists only in eigenvalues:

\[
\text{Spec } \mathcal{L} = \{1 - \frac{m^2}{2} | m \in \mathbb{N}\}
\]

with the following set of eigenfunctions:

\[
\{h_i h_j \equiv h_i(y_1) h_j(y_2) \mid i, j \in \mathbb{N}\},
\]

which spans the space $L^2_\rho$, where we see the rescaled Hermite polynomials (1.10). Note also that

\[
\mathcal{L}(h_i h_j) = \left(1 - \frac{i + j}{2}\right) h_i h_j.
\]

- **The potential**: It is bounded in $L^\infty$ and small in $L^r_\rho$ for any $r \geq 1$, in the sense that

\[
\|V(\cdot, s)\|_{L^\infty} \leq \frac{p}{p-1} \quad \text{and} \quad \|V(\cdot, s)\|_{L^r_\rho} \leq C(r)e^{-s},
\]

for $s$ large enough, thanks to Lemma 2.2 and a small Taylor expansion.

- **The nonlinear term**: Since $\varphi$ is uniformly bounded in space and time, thanks to item (iii) of Lemma 2.2, assuming the following a priori estimate,

\[
\|q\|_{L^\infty(\mathbb{R}^2 \times [s_0, \infty))} \leq M,
\]

we easily see that $B(y, s, q)$ is superlinear, in the sense that

\[
|B(y, s, q)| \leq C(M)|q|^{\bar{p}} \quad \text{where} \quad \bar{p} = \min(p, 2) > 1,
\]

for all $y \in \mathbb{R}^2$, $q \in \mathbb{R}$ and large $s$.

- **The remainder term**: By comparing the expression (3.4) of $R(y, s)$, with the expression of equation (1.12), we see that $R(y, s)$ measures the quality of $\varphi(y, s)$ as an approximate solution of (1.12). In fact, through a straightforward calculation (see below in Section 7.1), one can show that

\[
\forall r \geq 2, \quad \forall s \geq 0, \quad \|R(s)\|_{L^r_\rho} \leq C(r)e^{-2s},
\]

which is consistent with our approach in Section 2 (particularly Lemma 2.1 and estimate (2.5)), where we constructed $\varphi$ as an approximate solution for equation (1.12).

Following these properties, constructing a small solution $q(y, s)$ to equation (3.3) seems to be reasonable, except for a serious issue: the control of the nonlinear term $B(y, s, q)$, since the power function is not continuous in $L^2_\rho$. The control of the potential term $Vq$ is delicate too. In the following section, we explain our idea to gain those controls.
3.2 Control of the potential and the nonlinear terms in equation (3.3)

In this section, we explain how we will achieve the control of the potential and the nonlinear terms $Vq$ and $B$ in equation (3.3). For simplicity, we present the argument only for the nonlinear term $B$, and restrict to the case when $p \geq 2$, hence $\bar{p} = 2$ (the actual proof for $Vq$ and $B$ will be done for any $p > 1$).

Assuming the a priori estimate (3.9), we may reduce equation (3.3) to the following linear equation with a source term

$$\partial_s q = (\mathcal{L} + \bar{V})q + R(y, s)$$

(3.12)

where

$$|\bar{V}(y, s)| = \left| V(y, s) + \frac{B(y, s, q)}{q} \right| \leq C + C(M)|q|^{\bar{p} - 1} \leq C + C(M)M^{\bar{p} - 1} \equiv \bar{C}(M). \tag{3.13}$$

Thanks to the regularizing effect of the operator $\mathcal{L}$ (see Lemma 7.1 below), we may use equation (3.12) together with (3.13) and (3.11) to control the $L^p_\rho$ norm of the solution with the $L^2_\rho$ of the solution, up to some time delay:

$$\|q(s)\|_{L^p_\rho} \leq e^{(1+\bar{C})s^*} \|q(s - s^*)\|_{L^p_\rho} + C e^{-2s}. \tag{3.14}$$

Clearly, this is equivalent to writing that

$$\|q(s)^2\|_{L^2_\rho} \leq 2e^{2(1+\bar{C})s^*} \|q(s - s^*)\|_{L^2_\rho}^2 + 2C e^{-4s},$$

which allows to control the nonlinear term $B(y, s, q)$ in $L^2_\rho$, thanks to (3.10).

Note that this control is possible thanks to the a priori estimate (3.9), which needs to be checked. We will explain that in the following section.

3.3 Proof of the a priori bound (3.9)

This is the key part of the argument, dedicated to the proof of the a priori bound (3.9), under which the local estimates presented in the previous subsections hold. In fact, as we will shortly see from the geometrical transform (3.15), we reduce the uniform bound for $w_0$ to a local estimate for $w_a$, where $a$ is arbitrary. Later in Section 5, we will see that the control of $w_a$ will follow from the spectral analysis of equation (1.12), in particular the stability of its zero solution and its heteroclinic orbit connecting $\kappa$ (1.3) and 0 and given below in (5.8).

More precisely, since $\varphi$ is uniformly bounded by Lemma 2.2, using the definition (3.2) of $q$, it is enough to control $w_0(a, s)$, for any $a \in \mathbb{R}^2$ and $s \geq s_0$, in order to prove (3.9). Using the similarity variables’ definition (1.4), we remark that

$$w_b(y, s) = w_0(y + be^\frac{s}{N}, s). \tag{3.15}$$

This way, we reduce the question to the control of $w_b(0, s)$, for any $b \in \mathbb{R}^2$ and $s \geq s_0$. Since $w_b$ satisfies equation (1.12), which is parabolic, we further reduce the question to the control of $\|w_b(s)\|_{L^2}$, for any $b \in \mathbb{R}^2$ and $s \geq s_0$. This will be done through a careful choice of initial data (at $s = s_0$) for $w_0$, which completely determines initial data for $w_b$. Then, starting from these initial data, and integrating equation (1.12) for $s \geq s_0$, we will show that $w_b$ will decrease, providing the control on $\|w_b(s)\|_{L^2}$, hence on $w_b(0, s) = w_0(be^\frac{s}{N}, s)$ (thanks to (3.15)), and finally on $\|w_0(s)\|_{L^\infty}$.
3.4 Definition of a shrinking set to make \( q(s) \to 0 \)

From our formal analysis in Section 2, in particular the target expansion of \( w_0 \) in Lemma 2.1 (with \( C_{6,0} = -\delta \) and \( C_{6,2} = \gamma \)) and the expansion of \( \varphi \) in (2.5), we may write the following target for \( q(y,s) \) defined in (3.2): \[
q(y,s) = \frac{2}{\kappa} e^{-2s}\left\{-32h_0 h_0 - 16(h_2 h_0 + h_0 h_2) - 4(h_4 h_0 + h_0 h_4) - 32h_2 h_2 \\
+ 4s(h_4 h_2 + h_2 h_4)\right\} + O(s^2 e^{-3s}) \text{ as } s \to \infty. \tag{3.16}
\]

In fact, we will not require such a sharp expansion. We will instead require that the components of \( q(y,s) \) are bounded by the same rate as the one in front of the corresponding eigenfunctions shown on the right-hand side of (3.16). More precisely, expanding any function \( v \in L_2^{\rho} \) on the eigenfunctions of \( L \) given in (3.6): \[
v = \sum_{i=0}^{7} \sum_{j=0}^{i} v_{i,j} h_{i-j} h_j + v_- , \tag{3.17}
\]
where \[
v_{i,j} = \int v(y) k_{i-j}(y_1) k_j(y_2) \rho(y) dy \tag{3.18}
\]
is the coordinate of \( v \) along the eigenfunction \( h_{i-j} h_j \) (which corresponds to the eigenvalue \( \lambda = 1 - \frac{i}{2} \) by (3.7)), and \[
k_n = h_n / \| h_n \|_{L_2^{\rho}}, \tag{3.19}
\]
our target in this paper is to construct a solution to equation (3.3) defined on some interval \([s_0, \infty)\) where \( s_0 = -\log T \) such that \[
\forall s \geq s_0, \quad q(s) \in V_A(s), \tag{3.20}
\]
where \( V_A(s) \) is defined as follows:

**Definition 3.1** (A shrinking set to trap the solution). For any \( A > 0 \) and \( s \geq 1 \), \( V_A(s) \) is the set of all \( v \in L^\infty \) such that \( \|v + \varphi(s)\|_{L^\infty} \leq 2\kappa \), \( v_{i,j} = 0 \) if \( i \) or \( j \) is odd, \( |v_{i,j}| \leq Ae^{-2s} \) if \( i \leq 4 \), \( |v_{6,0}| = |v_{6,6}| \leq As^2 e^{-3s} \), \( |v_{6,2}| = |v_{6,4}| \leq Ase^{-2s} \) and \( \|v_-\|_{L_2^{\rho}} \leq A^2 s^2 e^{-3s} \), where \( v \) is decomposed as in (3.17), and the profile \( \varphi \) defined in (3.3) and Lemma 2.2.

**Remark.** The \( L^\infty \) bound in this definition is crucial to control the nonlinear term (see (3.9) and Section 3.2).

Clearly, we see that

if \( v \in V_A(s) \), then \( \|v\|_{L_2^{\rho}} \leq CAse^{-2s} \), \tag{3.21}

for \( s \) large enough, which shows that our goal in (3.20) implies indeed that \( q(s) \to 0 \) as \( s \to \infty \) in \( L_2^{\rho}(\mathbb{R}^2) \). The next part of the paper is devoted to the rigorous proof of our goal in (3.20), and to the fact that it implies our main result stated in Theorem 1.
4 Dynamics of the equation and general form of initial data

In this section, we study the dynamics of equation \(3.3\) and suggest a general form of initial data, depending on some parameters, which will be fine-tuned in some further step to provide a suitable solution so that Theorem 1 holds.

We proceed in 3 steps, each presented in a subsection, starting by the projections of equation \(3.3\) on the different components of the decomposition \(3.17\), then, we study the behavior of the flow on the boundary of the shrinking set \(V_A(s)\) defined in \(3.1\). Finally, we suggest a general form for initial data. Note that technical details will be postponed to Section 7 below, hoping to make our exposition clearer.

4.1 Dynamics of equation \(3.3\) in the shrinking set \(V_A(s)\)

Aiming at proving \(3.20\), we assume \(q(s) \in V_A(s)\) for all \(s \in [s_0, s_1]\) for some \(s_1 \geq s_0\) and derive differential equations satisfied by \(q_i,j\) and \(q_-\) in this section. In fact, we will first derive the size of the components of the remainder term \(R(y,s)\) \(3.31\) in the following:

**Lemma 4.1** (Expansion of \(R(y,s)\) \(3.31\)). For all \(r \geq 2\), it follows that

\[
R(y,s) = \frac{p e^{-2s}}{\kappa} \{32h_0h_0 + 32(h_2h_0 + h_0h_2) + 4(h_4h_0 + h_0h_4) + 32h_2h_2 + 4(h_4h_2 + h_2h_4)\} + O(e^{-3s})
\]

in \(L^r_p(\mathbb{R}^2)\), as \(s \to \infty\).

**Proof.** Se Section 7.1 below.

Now, we project equation \(3.3\) in the following:

**Proposition 4.2** (Dynamics of equation \(3.3\) in \(V_A(s)\)). For any \(A \geq 1\), there exists \(s_{11}(A) \geq 1\) such that the following holds. Assume that

\[
q(s_0) \in L^\infty, \; \nabla q(s_0) \in L^\infty \text{ and } \forall r \geq 2, \; \|q(s_0)\|_{L^r_p} \leq C(r)A s_0 e^{-2s_0}, \tag{4.1}
\]

and that \(q(s) \in V_A(s)\) satisfies equation \(3.3\), for all \(s \in [s_0, s_1]\), for some \(s_1 \geq s_0 \geq s_{11}(A)\). Then, for all \(s \in [s_0, s_1]:\)

(i) For all \(i \in \mathbb{N}\) and 0 \(\leq j \leq i, \; |q_{i,j}(s) - (1 - \frac{1}{7})q_{i,j}(s)| \leq C_i A s e^{-3s} + R_{i,j}(s), \tag{3.17}\)

(ii) \(\frac{d}{ds}\|q_-(s)\|_{L^2_p} \leq -3\|q_-(s)\|_{L^2_p} + C_A s e^{-3s} + \|R_-(s)\|_{L^2_p}, \tag{3.17}\)

where \(q(\cdot, s)\) and \(R(\cdot, s)\) are decomposed as in \(3.17\).

**Proof.** The proof is straightforward, except for the control of the potential term \(Vq\) and the superlinear term \(B\) in equation \(3.3\), which both need a regularizing delay estimate, already mentioned informally in \(3.14\). For that reason, we only state that delay estimate here, and postpone the proof to Section 7.2 below.

**Remark.** The fact that \(q\) satisfies the PDE \(3.3\) on the interval \([s_0, s]\) and not just at a particular time \(s\) is important for the delay estimate \(3.14\).

As we have just mentioned, let us precisely state that delay estimate, which is crucial for the proof of Proposition 4.2.
Proposition 4.3 (A delay regularizing estimate for equation (3.3)). Under the hypothesis of Proposition 4.2 and for any \( r \geq 2 \) and \( s \in [s_0, s_1] \), it holds that
\[
\|q(s)\|_{L^r_\rho} \leq C(r)Ase^{-2s}.
\]

Proof. See below in Section 7.3.

4.2 Behavior of the flow on the boundary of \( V_A(s) \)

Recall that our aim is to suitably choose \( q(s_0) \) so that \( q(s) \in V_A(s) \) for all \( s \geq s_0 \).

As in the previous section, we assume that \( q(s) \in V_A(s) \) for all \( s \in [s_0, s_1] \) for some \( s_1 \geq s_0 \). This time, we will assume in addition that at \( s = s_1 \), one component of \( q(s_1) \) (as defined in (3.17)) “touches” the corresponding part of the boundary of its bound defined in Definition 4.1. We will then derive the position of the flow of that component with respect to the boundary, and see whether it is inward (which leads to a contradiction) or outward. This way, only the components with an outward flow may touch. Hopefully, those components will be in a finite dimensional space, leading the way to the application of a consequence of Brouwer’s lemma linked to some fine-tuning in initial data, in order to guarantee that the solution will stay in \( V_A(s) \) for all \( s \geq s_0 \).

More precisely, this is our statement:

Proposition 4.4 (Position of the flow on the boundary of \( V_A(s) \)). There exists \( A_{12} \geq 1 \) such that for any \( A \geq A_{12} \), there exists \( s_{12}(A) \geq 1 \) such that if the hypotheses of Proposition 4.2 hold with \( s_0 \geq s_{12} \), then:
\[ (i) \] If \( q_{i,j}(s_1) = \theta A e^{-2s_1} \) with \( i \leq 4 \) and \( j \) even, and \( \theta = \pm 1 \), then, \( \theta q_{i,j}'(s_1) > \frac{d}{ds} A e^{-2s} |_{s=s_1} \).
\[ (ii) \] If \( q_{6,j}(s_1) = \theta A s^2 e^{-3s} \) with \( j = 0 \) or \( j = 6 \), and \( \theta = \pm 1 \), then, \( \theta q_{6,j}'(s_1) > \frac{d}{ds} A s^2 e^{-3s} |_{s=s_1} \).
\[ (iii) \] If \( q_{6,j}(s_1) = \theta A s e^{-2s} \) with \( j = 2 \) or \( j = 4 \), and \( \theta = \pm 1 \), then, \( \theta q_{6,j}'(s_1) < \frac{d}{ds} A s e^{-2s} |_{s=s_1} \).
\[ (iv) \] If \( \|q_-(s_1)\|_{L^2_T} = A^2 s^2 e^{-3s_1} \), then \( \frac{d}{ds} \|q_-(s_1)\|_{L^2_T} < \frac{d}{ds} A^2 s^2 e^{-3s} |_{s=s_1} \).

Remark. The flow in items (i) and (ii) is outward, while in items (iii) and (iv), it is inward.

Proof. See Section 7.4 below.

Following this statement, we immediately see that the constraints on \( q_{0,0} = q_{6,6} \) and \( q_- \) in Definition 3.1 of \( V_A(s) \) can never achieve equality:

Corollary 4.5 (\( q_{0,2} = q_{6,4} \) and \( q_- \) never quit). Following Proposition 4.4 and assuming that \( s_1 > s_0 \), it holds that \( |q_{0,2}(s_1)| = |q_{6,4}(s_1)| < A s_1 e^{-2s_1} \) and \( \|q_-(s_1)\|_{L^2_T} < A^2 s_1^2 e^{-3s_1} \).

Proof. We will only prove the estimate on \( q_- \), since the other follows in the same way. Proceeding by contradiction, we assume that \( \|q_-(s_1)\|_{L^2_T} \geq A^2 s_1^2 e^{-3s_1} \). Since \( q(s) \in V_A(s) \) for all \( s \in [s_0, s_1] \) by hypothesis, it follows that
\[
\forall s \in (s_0, s_1], \quad \|q_-(s)\|_{L^2_T} \leq A^2 s^2 e^{-3s} \quad \text{and} \quad \|q_-(s_1)\|_{L^2_T} = A^2 s_1^2 e^{-3s_1}.
\]

In particular, this translates into the following estimate between the derivatives of both curves:
\[
\frac{d}{ds} \|q_-(s)\|_{L^2_T} \geq \frac{d}{ds} A^2 s^2 e^{-3s},
\]
which is a contradiction by item (iv) of Proposition 4.4. \( \square \)
4.3 Choice of initial data

Following Corollary 4.3, we see that the control of \(q(s)\) in \(V_A(s)\) reduces to the control of 5 components: \(q_{0,0}, q_{2,0}, q_{4,0}, q_{4,2}\) and \(q_{6,0}\) (together with the control of \(\|q(s) + \varphi(s)\|_{L^\infty}\), which is a major novelty of our paper). It happens that item (i) in Proposition 4.4 indicates that the flow of those components is “transverse outgoing” on the boundary of the constraint introduced in Definition 3.1. As in our various papers where we construct solutions to PDEs with a prescribed behavior (see for example [18]), the control of this finite-dimensional part will be done through a Brouwer-type lemma, involving some fine-tuning of parameters in initial data. Namely, the idea would be to introduce the following family of initial data (for \(w_0(y,s_0)\) linked to \(q(y,s_0)\) by definition (3.2)) depending on parameters \(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}\) and \(d_{6,0}\):

\[
w_0(y,s_0) = \varphi(y,s_0) + \left[ Ae^{-2s_0} S(y) + A s^2 e^{-3s} \bar{S}(y) \right] \chi(y) \]

where

\[
S(y) = d_{0,0} + d_{2,0} [h_2(y_1) + h_2(y_2)] + d_{4,0} [h_4(y_1) + h_4(y_2)] + d_{4,2} h_2(y_1) h_2(y_2),
\]

\[
\bar{S}(y) = d_{6,0} (h_6(y_1) + h_6(y_2)),
\]

(4.2)

for some sufficiently decaying function \(\chi\). In fact, in order to stick to the shape of the profile \(\varphi(2.2)\), we will in fact take the following initial data for equation (1.12):

\[
w_0(y,s_0) = \left[ \frac{E}{D} + \frac{p-1}{\kappa D^2} \left( Ae^{-2s_0} S(y) + A s^2 e^{-3s} \bar{S}(y) \right) \right]^{1/p},
\]

(4.3)

where \(E, D, S\) and \(\bar{S}\) are introduced in (2.3), (2.4) and (4.2), which yields the following initial data for equation (3.3):

\[
q(y,s_0) = w_0(y,s_0) - \varphi(y,s_0).
\]

(4.4)

In the following, we exhibit a set for the parameters so that \(q(s_0)\) is well-defined, \(q(s_0) \in V_A(s_0)\) with other smallness and decay properties, inherited from the profile \(\varphi(y,s_0)\) given in (2.2). More precisely, this is our statement:

**Proposition 4.6** (Initialization). For any \(A \geq 1\), there exists \(s_{13}(A) \geq 1\) such that for all \(s_0 \geq s_{13}(A)\), there exists a set \(D(A,s_0) \subset \mathbb{Z}^5\) such that for all parameter \(d \equiv (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in D\), we have the following 2 properties:

(i) \(E + \frac{p-1}{\kappa} \left( Ae^{-2s_0} S(y) + A s^2 e^{-3s} \bar{S}(y) \right) \geq \frac{1}{4}\), hence \(w_0(y,s_0) (4.3)\) and \(q(y,s_0) (4.4)\) are well-defined.

(ii) \(q(s_0) \in V_A(s_0)\) introduced in Definition 3.1, estimate (4.1) holds true, \(\|q_{6,2}(s_0)\| = |q_{6,4}(s_0)| \leq CAe^{-3s_0} < As_0 e^{-2s_0}, \|q_{2,0}(s_0)\|_L^2 \leq CA^{-3s_0} < A^2 s_0^2 e^{-3s_0}, \|w_0(s_0)\|_{L^\infty} \leq 1\) such that \(\kappa + Ce^{-\frac{\nu}{2}}\) and \(\|\nabla w_0(s_0)\|_{L^\infty} \leq Ce^{-\frac{\nu}{4}}\). Moreover, the function

\[
D \rightarrow \left[ -Ae^{-2s_0}, Ae^{-2s_0} \right] \times \left[ -A s_0^2 e^{-3s_0}, A s_0^2 e^{-3s_0} \right],
\]

(4.5)

is one-to-one.

**Proof.** See Section 4.3.
5 Control of $w_a(y,s)$ for any $a$

Starting from this section, our goal is to show that the $L^\infty$ bound on $q(y,s) + \varphi(y,s)$ in Definition 3.1 of $V_A(s)$ never enjoys an equality case. In other words, if $q(s_0)$ is given by (1.2) for some parameter $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6 and $q(y,s)$ satisfies equation (3.3) with $q(s) \in V_A(s)$ given in Definition 3.1 for all $s \in [s_0, s_1]$, for some $s_1 \geq s_0$, with $s_0$ large enough, then

$$\forall s \in [s_0, s_1], \quad \forall y \in \mathbb{R}^2, \quad |q(y,s) + \varphi(y,s)| = |w_0(y,s)| < 2\kappa, \quad (5.1)$$

where we have used the definition (3.2) of $q(y,s)$. Using the relation (3.15), this is equivalent to showing that

$$\forall a \in \mathbb{R}^2, \quad \forall s \in [s_0, s_1], \quad |w_a(0,s)| < 2\kappa. \quad (5.2)$$

We will proceed in several steps in order to control $w_a(y,s)$ for any $a \in \mathbb{R}^2$: we first give a uniform control of the gradient, then we explain our strategy, depending on the cases, it holds that $\|\nabla q(s)\|_{L^\infty}$ is small, provided that $s_0$ is large enough. Let us first recall our version of the Liouville theorem, stated for equation (1.12):

**Proposition 5.1** (A Liouville theorem for equation (1.12)). Under condition (1.2), consider $W(y,s)$ a solution of equation (1.12) defined and uniformly bounded for all $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s})$ for some $\bar{s} \leq +\infty$. Then, either $W \equiv 0$, or $W \equiv \pm \kappa$ or $W(y,s) = \pm \kappa (1 \pm e^{s-s^*})^{-\frac{1}{p-1}}$ for all $(y,s) \in \mathbb{R}^N \times (-\infty, \bar{s})$ and for some $s^* \in \mathbb{R}$. In all cases, it holds that $\nabla W \equiv 0$.

**Remark.** If $\bar{s} = +\infty$, then the unbounded solution $W(y,s) = \pm \kappa (1 - e^{s-s^*})^{-\frac{1}{p-1}}$ never occurs. If $\bar{s} < +\infty$, then that solution may occur with some $s^*$ satisfying $e^{\bar{s}-s^*} < 1$.

**Proof.** If $\bar{s} = +\infty$, see Theorem 1.4 page 143 in [19] for the nonnegative case and Theorem 1 page 106 in [21] for the unsigned case.

If $\bar{s} < +\infty$, then the statement follows from a small adaptation of the previous case. See Corollary 1.5 page 144 in [19] where a similar adaptation is carried out. \hfill $\square$

Let us now state our result for the gradient:

**Proposition 5.2** (Smallness of the gradient). For all $A \geq 1$ and $\delta_0 > 0$, there exists $s_{14}(A, \delta_0) \geq 1$ such that for all $s_0 \geq s_{14}(A, \delta_0)$, if $q(s_0)$ is given by (1.2) for some $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6 and $q(s) \in V_A(s)$ for all $s \in [s_0, s_1]$ for some $s_1 \geq s_0$ satisfies equation (3.3), then, for all $s \in [s_0, s_1]$, $\|\nabla q(s) + \nabla \varphi(s)\|_{L^\infty} \leq \delta_0$.

**Proof.** See Section 7.6. \hfill $\square$
5.2 Strategy for the control of \( w_a(y,s) \)

With this estimate, we can make a reduction of our goal (5.2) in the following:

Claim 5.3 (Reduction). Under the hypotheses of Proposition 5.2, assuming that

\[
\delta_0 \leq \frac{\kappa}{4} \text{ and } s_0 \geq s_{14}(A, \delta_0),
\]

we see that estimate (5.2) follows from the following:

\[
\forall a \in \mathbb{R}^2, \forall s \in [s_0, s_1], \|w_a(s)\|_{L^2_\rho} \leq \frac{3}{2}\kappa.
\]

Proof. Under the hypotheses of Proposition 5.2, assume that (5.4) holds. Noting that \( \|\nabla w_a(s)\|_{L^\infty} \leq \delta_0 \) from Proposition 5.2 together with the relations (3.2) and (3.15), we use a Taylor expansion to write

\[
|w_a(y,s) - w_a(0,s)| \leq |y| \cdot \|\nabla w_a(s)\|_{L^\infty} \leq \delta_0 |y|.
\]

Therefore, since

\[
\int \rho(y)dy = 1 \text{ and } \int |y|\rho(y)dy = 1
\]

by definition (1.18) of \( \rho \), it follows that

\[
|w_a(0,s)| \leq \int |w_a(y,s)|\rho(y)dy + \delta_0 \leq \|w_a(s)\|_{L^2_\rho} + \delta_0 \leq \frac{3}{2}\kappa + \frac{\kappa}{4} < 2\kappa,
\]

and the claim follows. \( \square \)

In the following subsections, assuming that (5.3) holds, we will prove either (5.1), (5.2) or (5.4), according to the context. Using the sharper gradient estimate at initial time \( s_0 \) given in item (ii) of Proposition 4.6, let us remark that at \( s = s_0 \), \( w_a(y,s_0) \) is “flat” in \( L^2_\rho \), in the sense that it is close to some constant independent from space, as we prove in the following:

Lemma 5.4 (Flatness of \( w_a(y,s_0) \)). For any \( A \geq 1 \), \( s_0 \geq s_{13}(A) \) and parameter \( d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in D(A, s_0) \), for any \( a \in \mathbb{R}^2 \),

\[
\|w_a(\cdot, s_0) - w_0(ae^{\frac{s}{T}} , s_0)\|_{L^2_\rho} \leq Ce^{-\frac{s}{T}},
\]

where \( s_{13}(A) \) and \( D(A, s_0) \) are defined in Proposition 4.6.

Proof. The proof follows by the same argument as Claim 5.3 in particular, estimate (5.5). \( \square \)

Following Lemma 5.4, let us remark that equation (1.12) satisfied by \( w_a(y,s) \) has 3 bounded and nonnegative explicit “flat” solutions: 0, \( \kappa \), and

\[
\psi(s) = \kappa(1 + e^s)^{-\frac{1}{\tau-1}}
\]

(note that \( \psi \) is a heteroclinic orbit connecting \( \kappa \) to 0, and that all its time shifts are also solutions). In fact, it happens that \( w_a(0, s_0) = w_0(ae^{\frac{s}{T}} , s_0) \) belongs to the interval \( [0, \kappa + Ce^{-\frac{s}{T}}] \), from (3.15) and Proposition 4.6. In other words, from Lemma 5.4 we are
in the vicinity of one of those 3 explicit solutions, whose stability properties are known! Indeed, if 0 and \( \psi \) are stable, as we will show below (see Propositions 5.6 and 5.8), \( \kappa \) has both stable and unstable directions, as one may see from the linearization of equation (1.12) around \( \kappa \) (see below in (7.31)), where the linearized operator appears to be \( \mathcal{L} \) introduced in (3.4), whose spectrum is given in (3.5). In particular, the existence of the heteroclinic orbit \( \psi \) (5.8) shows the instability of \( \kappa \). In other words, if \( w_0(\alpha e^{\frac{2p}{a}}, s_0) \) is close to \( \kappa \), we need to refine estimate (5.7) in order to better understand the dynamics of \( w_a(y,s) \) for \( s \geq s_0 \).

Accordingly, we will decompose the space into 3 regions, where \( w_a(y,s) \) will be in the vicinity of one of the 3 above-mentioned explicit solutions, leading to 3 different scenarios for the behavior of \( w_a(y,s) \) for \( s \geq s_0 \). More precisely, given \( m < M \), we introduce 3 regions \( \mathcal{R}_i(m, M, s_0) \) for \( i = 1,2,3 \) as follows:

\[
\mathcal{R}_1 = \{ a \in \mathbb{R}^2 \mid M e^{-s_0} \leq G_0(a) \}, \\
\mathcal{R}_2 = \{ a \in \mathbb{R}^2 \mid m e^{-s_0} \leq G_0(a) \leq M e^{-s_0} \}, \\
\mathcal{R}_3 = \{ a \in \mathbb{R}^2 \mid G_0(a) \leq m e^{-s_0} \},
\]

(5.9)

where

\[
G_0(a) = \frac{p-1}{\kappa} \left[ a_1^2 a_2^2 + \delta(a_1^6 + a_2^6) \right].
\]

(5.10)

In the following lemma, we will see that \( G_0(a) \) is a kind of norm which measures the size of \( w_a(0,s_0) = w_0(\alpha e^{-\frac{2p}{a}}, s_0) \):

Lemma 5.5 (Size of initial data in the three regions). For any \( M \geq 1 \), there exists \( C_{15}(M) > 0 \) such that for any \( A \geq 1 \), there exists \( s_{15}(A,M) \) such that for any \( s_0 \geq s_{15}(A,M) \) and \( d \in \mathcal{D}(A,s_0) \) defined in Proposition 4.6, for any \( m \in (0,1) \), the following holds:

- If \( a \in \mathcal{R}_1 \), then \( 0 \leq w_0(\alpha e^{\frac{2p}{a}}, s_0) \leq \kappa (1 + M)^{-\frac{1}{m-1}} + C_{15}(M) e^{-\frac{2p}{a}} \).
- If \( a \in \mathcal{R}_2 \), then \( \kappa (1+M)^{-\frac{1}{m-1}} - C_{15}(M) e^{-\frac{2p}{a}} \leq w_0(\alpha e^{\frac{2p}{a}}, s_0) \leq \kappa (1+m)^{-\frac{1}{m-1}} + C e^{-\frac{2p}{a}} \).
- If \( a \in \mathcal{R}_3 \), then \( \kappa (1+m)^{-\frac{1}{m-1}} - C e^{-\frac{2p}{a}} \leq w_0(\alpha e^{\frac{2p}{a}}, s_0) \leq \kappa + C e^{-\frac{2p}{a}} \).

Proof. See Section 7.7.

Following the 2 previous lemmas and what we have mentioned concerning the stability of the 3 explicit solutions of equation (1.12) mentioned in (6.5) and the line before, 3 scenarios become clear for the proof of estimate (5.4):

- **Scenario 1**: if \( a \in \mathcal{R}_1 \), provided that \( M \) is large enough, we are in the vicinity of the zero solution. Thanks to its stability, \( w_a(s) \) will remain small and (6.4) will follow.
- **Scenario 2**: if \( a \in \mathcal{R}_2 \), we are in the vicinity of \( \psi(s) \) (5.8). Thanks to its stability, \( w_a(s) \) will remain close to \( \psi(s) \). Since \( \psi(s) \leq \kappa \), estimate (5.4) holds.
- **Scenario 3**: if \( a \in \mathcal{R}_3 \), we are in the vicinity of \( \kappa \), which has both stable and unstable directions as mentioned earlier. For that reason, the bounds on \( w_a(0,s_0) = w_0(\alpha e^{\frac{2p}{a}}, s_0) \) given in Lemma 5.5 are not enough, and we need a more refined expansion of \( w_a(y,s_0) \), followed by an integration of PDE (1.12) satisfied by \( w_a \). This step is the key point of our argument. It is inspired by our techniques in [22] for the control of \( w_a \) where \( a \) is a blow-up point located near some given non-isolated blow-up point. In fact, from our careful design of initial data in (4.3), the integration of the PDE will show that for \( a \) not “very small” (in a sense that will naturally appear in the proof), \( w_a(s) \) will
be attracted to the vicinity of the heteroclinic orbit \( \psi \), leading us to \textit{Scenario 2}, where the stability of \( \psi \) will imply estimate (5.4). If \( a \) is "very small", then, thanks to the gradient estimate of Proposition 5.2, the boundedness of \( \|w_a(s)\|_{L^p} \) will follow from the boundedness of \( \|w_0(s)\|_{L^p} \), itself a consequence of the fact that \( q(s) \in V_A(s) \) given in Definition 3.1 (see (3.21)).

In the following, we give details for those 3 scenarios in 3 subsections.

### 5.3 Control of \( w_a(y, s) \) in Region \( \mathcal{R}_1 \)

The stability of the zero solution for equation (1.12) (under some \( L^\infty \) a priori bound) is crucial for the argument, as we wrote above in \textit{Scenario 1}. Let us state it in the following:

**Proposition 5.6** (Stability of the zero solution for equation (1.12) under an \( L^\infty \) a priori bound). There exists \( \epsilon_0 > 0 \) and \( M_0 \geq 1 \) such that if \( w \) solves equation (1.12) with \( |w(y, s)| \leq 2\kappa \) for all \( (y, s) \in \mathbb{R}^2 \times [0, \sigma_1] \) for some \( \sigma_1 > 0 \), with \( \|w(0)\|_{L^p_y} \leq \epsilon_0 \) and \( \nabla w(0)(1 + |y|)^{-k} \in L^\infty \) for some \( k \in \mathbb{N} \), then

\[
\forall s \in [0, \sigma_1], \quad \|w(s)\|_{L^p_y} \leq M_0 \|w(0)\|_{L^p_y} e^{-\frac{\epsilon_0^2}{16}}.
\]

**Proof.** The proof is somehow classical, apart from a delay estimate to control the non-linear estimate we have already presented in Section 3.2. In order to focus only on the main arguments, we postpone the proof to Section 13 in the appendix. \( \square \)

Fixing \( M \geq 1 \) such that

\[
\kappa(1 + M)^{-\frac{1}{\rho - 1}} \leq \max \left( \frac{\epsilon_0}{2\frac{\kappa}{2M_0}} \right)
\]

where \( M_0 \) and \( \epsilon_0 \) are given in Proposition 5.6, then taking \( s_0 \) large enough, we see from Proposition 4.6 that \( \nabla w(s_0) \in L^\infty \), and from Lemmas 5.5 and 5.4 that the smallness condition required in Proposition 5.6 holds in Region \( \mathcal{R}_1 \), leading to the trapping of \( w_a \) near 0, proving the bound (5.4). More precisely, this is our statement:

**Corollary 5.7** (Exponential decay of \( w_a(s) \) in Region \( \mathcal{R}_1 \)). For all \( A \geq 1 \), there exists \( s_{16}(A) \geq 1 \) such that if \( s_0 \geq s_{16}(A) \), \( d \in D(A, s_0) \) and \( a \in \mathcal{R}_1 \) defined in (5.9), then \( \nabla w(s_0) \in L^\infty \) and \( \|w_a(s_0)\|_{L^p_y} \leq 2\kappa(1 + M)^{-\frac{1}{\rho - 1}} \leq \epsilon_0 \) introduced in Proposition 5.6. If in addition we have \( |w_a(y, s)| \leq 2\kappa \), for all \( (y, s) \in \mathbb{R}^2 \times [s_0, s_2] \), for some \( s_2 \geq s_0 \), then

\[
\forall s \in [s_0, s_2], \quad \|w_a(s)\|_{L^p_y} \leq M_0 \|w_a(s_0)\|_{L^p_y} e^{-\frac{s - s_0}{p - 1}} \leq 2\kappa(1 + M)^{-\frac{1}{\rho - 1}} M_0 e^{-\frac{s - s_0}{p - 1}} \leq \kappa,
\]

where \( M_0 \) is also introduced in Proposition 5.6. In particular, (5.4) holds.

### 5.4 Control of \( w_a(y, s) \) in Region \( \mathcal{R}_2 \)

As we explained above in \textit{Scenario 2}, the stability of the solution \( \psi \) (5.8) is the key argument. Let us first state that stability result:
Proposition 5.8 (Stability of the heteroclinic orbit for equation \((1.12)\) under an \(L^\infty\) a priori bound). There exists \(M_1 \geq 1\) such that if \(w\) solves equation \((1.12)\) with \(|w(y,s)| \leq 2\kappa\) for all \((y,s) \in \mathbb{R}^2 \times [0,\sigma_1]\) for some \(\sigma_1 \geq 0\), and

\[
\nabla w(0)(1 + |y|)^{-k} \in L^\infty, \quad \|w(0) - \psi(\sigma^*)\|_{L^2_p} \leq \frac{|\psi'(\sigma^*)|}{M_1}
\]

(5.12)

for some \(k \in \mathbb{N}\) and \(\sigma^* \in \mathbb{R}\), where \(\psi\) is defined in (5.8), then

\[
\forall s \in [0,\sigma_1], \quad \|w(s) - \psi(s + \sigma^*)\|_{L^2_p} \leq M_1\|w(0) - \psi(\sigma^*)\|_{L^2_p} \frac{|\psi'(s + \sigma^*)|}{|\psi'(\sigma^*)|}.
\]

(5.13)

Remark. By definition (5.8) of \(\psi\), we have exponential decay in (5.13). Moreover, since \(\kappa\) and \(\psi(s)\) are both solutions to equation (1.12), \(\kappa\) should never satisfy condition (5.12). This is clear, except when \(\sigma^* \rightarrow -\infty\), since in that case \(\psi(\sigma^*) \rightarrow \kappa\). More precisely, we see from (5.8) that \(\kappa - \psi(\sigma^*) \sim \frac{\kappa\sigma^*}{M_1} \sim |\psi'(\sigma^*)|\), which shows that (5.12) is indeed sharp, up to a multiplying (small) factor, \(1/M_1\).

Proof. The proof is much more involved than the proof of Proposition 5.6 since by definition (5.8), \(\psi(\sigma^*)\) may be close to \(\kappa\), the unstable equilibrium of equation (1.12). As for the previous proposition, we postpone the proof to Section B in the appendix.

In the following corollary, using Proposition 4.6, Lemmas 5.4 and 5.5, we show that \(w_a(s)\) is trapped near the heteroclinic orbit \(\psi\) whenever \(a\) is in Region \(R_2\) (5.9).

More precisely, this is our statement:

Corollary 5.9 (Trapping of \(w_a(s)\) near \(\psi\) in Region \(R_2\)). There exists \(\bar{s} \in \mathbb{R}\) such that for all \(m \in (0,1)\), there exists \(\underline{c}(m) \leq \bar{s}\) such that for all \(A \geq 1\), there exists \(s_1(A,m) \geq 1\) such that for all \(s_0 \geq s_1(A,m)\), \(d \in \mathcal{D}(A,s_0)\) and \(a \in R_2\) defined in (5.9) (with \(M\) defined in (5.11)), \(\nabla w_a(s_0) \in L^\infty\) and \(\|w_a(s_0) - \psi(\sigma^*)\|_{L^2_p} \leq Ce^{-\frac{\kappa}{M_1}} \leq \frac{|\psi'(\sigma^*)|}{M_1}\)

for some \(\sigma^* \in [\underline{c}(m),\bar{s}]\), where \(M_1\) was introduced in Proposition 5.8. If in addition we have \(|w_a(y,s)| \leq 2\kappa\), for all \((y,s) \in \mathbb{R}^2 \times [s_0,s_2]\), for some \(s_2 \geq s_0\), then,

\[
\forall s \in [s_0,s_2], \quad \|w_a(s)\|_{L^2_p} \leq \psi(s + \sigma^* - s_0) + \frac{CM_1}{C(m)}e^{-\frac{\kappa}{M_1}} \leq \kappa + \frac{\kappa}{4} \leq \frac{3}{2}\kappa.
\]

In particular, (5.14) holds.

Proof. The proof is omitted, since it is a direct consequence of Proposition 5.8 thanks to Proposition 4.6, Lemmas 5.4 and 5.5.

5.5 Control of \(w_a(y,s)\) in Region \(R_3\)

Given an initial time \(s_0\), some \(m \in (0,1)\) and \(a \in R_3\) defined in (5.9), we write \(a\) as follows:

\[
a = (Ke^{-\frac{\kappa}{M_1}, Le^{-\frac{\kappa}{M_1}}}),
\]

(5.14)

for some real numbers \(K\) and \(L\). Since \(w_0(\cdot,s_0)\) is symmetric with respect to the axes and the bissectrices (see (1.3)), so is \(w_0(\cdot,s)\) for any later time \(s\). For that reason, we only consider the case where

\[
0 \leq K \leq L.
\]

(5.15)
Since
\[ w_a(y, s_0) = w_0(y + ae^{\frac{2\rho}{A}}, s_0) = w_0(y_1 + K, y_2 + L, s_0) \]  \tag{5.16}
from [3.13], we will use the explicit expression (4.3) of \( w_0 \) in order to estimate the components of \( w_a(s_0) \) in \( L^2_p \). Taking this as initial data, we will integrate equation (4.3) (which is satisfied by \( w_a \)) in order to estimate \( w_a(s) \) for later times \( s \geq s_0 \).

It happens that the outcome of the integration depends on the size of \( a \), which can be measured in terms of the position of \( K + L \) and \( A \). For that reason, we distinguish two cases in the following.

### 5.5.1 Case where \( K + L \geq A \)

From the decomposition (5.14) of \( a = (a_1, a_2) \), this is the case of “large” \( a \), where \( a_1 + a_2 \geq Ae^{\frac{2\rho}{A}} \). Let us first estimate \( w_a(s_0) \). In order to be consistent with the definition (5.2) of our profile \( \varphi \) and the decomposition in regions we suggest in (5.9), we will give the expansion of \( w_a(s_0) \) in \( L^2_p \), uniformly with respect to the small variable
\[ \iota = e^{-s_0}K^2L^2 + \delta e^{-2s_0}[K^6 + L^6] \]  \tag{5.17}
and the large parameter \( A \). This is our statement:

**Lemma 5.10** (Initial value of \( w_a(y, s) \) for large \( a \)). For any \( A \geq 1 \), there exists \( s_{18}(A) \geq 1 \) such that for any \( s_0 \geq s_{18}(A) \) and any parameter \( (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in D \) defined in Proposition 4.7, if \( w_0(y, s_0) \) is given by (4.3), then, for any \( m \in (0, 1) \) and \( a \in R_3 \) defined in (5.9) with \( a \) decomposed as in (5.14) for some \( L \geq K \geq 0 \), with \( K + L \geq A \), the following expansion holds in \( L^r_p(\mathbb{R}^2) \) for any \( r \geq 2 \):
\[ w_a(y, s_0) = \kappa - \iota - e^{-s_0} \{ 2KL^2h_1h_0 + 2K^2Lh_1h_0 + L^2h_2h_0 + 4KLh_1h_1 + K^2h_0h_2 + 2Lh_2h_1 + 2Kh_1h_2 + h_2h_2 \} + O \left( \frac{L^4}{A} \right) + O(\iota^2), \]
where \( \iota \) is defined in (5.17).

**Proof.** See Section 7.8 □

Noting that
\[ \iota \leq \frac{m\kappa}{p - 1} \leq \frac{\kappa}{p - 1} \]  \tag{5.18}
from (5.17) and (5.9), it follows from this lemma that \( \| w_a(s_0) - \kappa \|_L^2 \leq C\iota + CJ \) where
\[ J \equiv e^{-s_0}(K^2 + L^2) \leq 2e^{-s_0}(K^6 + L^6)^{\frac{1}{4}} \leq 2e^{-s_0} \left( \frac{e^{2s_0}}{\delta} \right)^{\frac{1}{4}} = 2e^{-\frac{s_0}{\delta}} \left( \frac{\iota}{\delta} \right)^{\frac{1}{4}} \leq 2\iota^{\frac{1}{4}} \]  \tag{5.19}
from (5.17) and the fact that \( s_0 \geq 0 \) and \( \delta \geq 1 \) (see the beginning of Section 3). Using again (5.18), it follows that \( w_a - \kappa \) is small at \( s = s_0 \), whenever \( m \) is small. It is natural then to make a linear approximation for equation (1.12) around \( \kappa \) in order to obtain the expansion of \( w_a(y, s) \) for later times, as long as \( w_a(s) - \kappa \) remains small. In fact, we will see that the projection on \( h_0h_0 = 1 \) will dominate in \( w_a(y, s) \). For that reason, given some small \( \eta^* \) such that
\[ \eta^* \geq \frac{m\kappa}{p - 1}, \]  \tag{5.20}
where \( m \in (0, 1) \) is the constant appearing in the definition (5.9) of \( R_3 \), our integration will be valid only on the interval \([s_0, s^*]\) where \( s^* = s^*(s_0, a, \eta^*) \) is defined by

\[
e^{s^*-s_0\ell} = \eta^*
\]

and \( \ell \) is given in (5.17). Note that \( s^* \) is well defined since \( L + K \geq A > 0 \), hence \( \ell > 0 \). Note also that \( s^* \geq s_0 \) thanks to condition (5.20), as we will see in item (i) of Lemma 5.11 below. More precisely, this is our statement:

**Lemma 5.11** (Decreasing from \( \kappa \) to \( \kappa - \eta^* \) for “large” \( a \) under some a priori \( L^\infty \) bound). There exists \( M_{19} > 0 \), \( A_{19} \geq 1 \) and \( \eta_{19} > 0 \) such that for all \( A \geq A_{19} \), there exists \( s_{19}(A) \) such that for any \( s_0 \geq s_{19}(A) \) and any parameter \((d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}\) defined in Proposition 4.7, if \( w_0(y, s_0) \) is given by (4.3), then, for any \( \eta^* \in (0, \eta_{19}) \) and \( m \in (0, \min(1, \frac{\eta^*}{\kappa})) \), for any \( a \in R_3 \) defined in (5.9) where \( a \) is given by (5.14) for some \( L \geq K \geq 0 \), with \( K + L \geq A \), the following holds:

(i) \( s^* \geq s_0 \), where \( s^* \) is introduced in (5.22).

(ii) If we assume in addition that \( \|w_a(s)\|_{L^\infty} \leq 2\kappa \) for all \( s \in [s_0, s_1] \) for some \( s_1 \geq s_0 \), then, for all \( s \in [s_0, \min(s^*, s_1)] \),

\[
\|w_a(\cdot, s) - (\kappa - e^{s-s_0\ell})\|_{L^2} \leq M_{19} (\eta^* + A^{-1}) e^{s-s_0\ell} + M_{19} e^{-\frac{\eta^*}{3}}.
\]

**Proof.** See Section 7.8

With this result, we are able to prove estimate (5.4):

**Corollary 5.12** (Proof of the \( L^2_a \) bound when \( a \in R_3 \) is “large” under some \( L^\infty \) a priori bound). There exists \( A_{20} > 0 \), \( \eta_{20} > 0 \) and \( s_{20}(\eta^*) \geq 1 \), such that under the hypotheses of Lemma 5.11 if in addition \( A \geq A_{20} \), \( \eta^* \leq \eta_{20} \) and \( s_0 \geq s_{20}(\eta^*) \), then, for all \( s \in [s_0, s_1] \), \( \|w_a(s)\|_{L^2} \leq \frac{3}{2} \kappa \).

**Proof.** Following Lemma 5.11 we further assume that \( s_0 \geq s_{14}(A, 1) \) defined in Proposition 5.2 so we can apply that proposition. Consider then \( s \in [s_0, s_1] \). We distinguish 2 cases:

*Case 1: \( s \leq s^* \).* We write from Lemma 5.11 and (5.6), together with the definitions (5.17) and (5.21) of \( \ell \) and \( s^* \)

\[
\|w_a(s)\|_{L^2_a} \leq \kappa + e^{s^*-s_0\ell} + M_{19}(\eta^* + A^{-1})e^{s^*-s_0\ell} + M_{19} e^{-\frac{\eta^*}{3}}
\]

\[
\leq \kappa + \eta^* + M_{19}(\eta^* + A^{-1})\eta^* + M_{19} e^{-\frac{\eta^*}{3}}.
\]

Since \( A \geq 1 \), taking \( \eta^* \) small enough and \( s_0 \) large enough, we get the result.

*Case 2: \( s \geq s^* \).* In this case, we have \( s_0 \leq s^* \leq s \leq s_1 \). We will show that starting at time \( s^* \), \( w \) will be trapped near the heteroclinic orbit \( \psi \) (5.8), thanks to Proposition 5.8. In particular, its \( L^2_a \) will remain bounded by \( \frac{3}{2} \kappa \). More precisely, from item (ii) of Lemma 5.11 and (5.21), we see that

\[
\|w_a(s^*) - (\kappa - \eta^*)\|_{L^2_a} \leq M_{19}(\eta^* + A^{-1})\eta^* + M_{19} e^{-\frac{\eta^*}{3}}.
\]

Assuming that \( \eta^* < \kappa \), we may introduce \( \sigma^* \in \mathbb{R} \) such that \( \psi(\sigma^*) = \kappa - \eta^* \), where \( \psi \) is defined in (5.8). Noting that

\[
|\psi'(\sigma^*)| \sim \frac{\kappa e^{\sigma^*}}{p-1} \sim \kappa - \psi(\sigma^*) = \eta^* \text{ as } \eta^* \to 0,
\]

(5.22)
we see that taking \( \eta^* \) small enough, then \( A \) and \( s_0 \) large enough, we have

\[
\| w_a(s^*) - \psi (\sigma^*) \|_{L_2} \leq \frac{|\psi'(\sigma^*)|}{M_1} \left[ 1 + \frac{2}{\kappa} \| \psi' \|_{L_\infty} \right],
\]

where \( M_1 \) is introduced in Proposition 5.8. Since \( \nabla w_a(s^*) \in L^\infty \), thanks to Proposition 5.2 together with the definition (3.2) of \( q \) and the transformation (3.15), Proposition 5.8 applies and we see that at time \( s \), we have

\[
\| w_a(s) - \psi (s + \sigma^* - s^*) \|_{L_2} \leq M_1 \| w_a(s^*) - \psi (\sigma^*) \|_{L_2} \frac{|\psi'(s + \sigma^*)|}{|\psi'(\sigma^*)|} \leq \frac{3}{2} \kappa.
\]

Since \( \psi \leq \kappa \) by definition (5.8), using (5.6) we see that

\[
\| w_a(s) \|_{L_2} \leq \psi(s + \sigma^* - s^*) + \frac{\kappa}{2} \leq \frac{3}{2} \kappa.
\]

This concludes the proof of Corollary 5.12. \( \square \)

### 5.5.2 Case where \( K + L \leq A \)

Given \( m \in (0, 1) \) and \( a \in \mathcal{R}_3 \) (5.9), we aim in this section to handle the case where \( a \) is “small”, namely when it belongs to the triangle \( T_0 \) defined by

\[
T_0 = \{ (K e^{-\frac{s_0}{2}}, L e^{-\frac{s_0}{2}}) \mid 0 \leq K \leq L \text{ and } K + L \leq A \}.
\]

Let us recall from the beginning of Section 5 that \( q(s) \in V_A(s) \) given in definition 5.11 for all \( s \in [s_0, s_1] \) for some \( s_1 \geq s_0 \), hence,

\[
\forall s \in [s_0, s_1], \quad \| w_0(s) \|_{L_\infty} \leq 2\kappa.
\]

Introducing the segment

\[
G_\sigma = \{ (K' e^{-\frac{s_0}{2}}, L' e^{-\frac{s_0}{2}}) \mid 0 \leq K' \leq L' \text{ and } K' + L' = A \}
\]

and the triangle

\[
T_1 = \{ (K' e^{-\frac{s_0}{2}}, L' e^{-\frac{s_0}{2}}) \mid 0 \leq K' \leq L' \text{ and } K' + L' \leq A \},
\]

we see that

\[
T_0 = \cup_{s_0 \leq \sigma < s_1} G_\sigma \cup T_1.
\]

We then proceed in 2 steps:

- In Step 1, we handle the case of “small” \( s \), namely when \( a \in G_\sigma \) with \( s_0 \leq \sigma \leq s_1 \), and also the case where \( a \in T_1 \) with \( s_0 \leq \sigma \leq s_1 \).

- In Step 2, we handle the case of “large” \( s \), namely when \( a \in G_\sigma \) with \( s_0 \leq \sigma \leq s \leq s_1 \).

**Step 1:** Case of “small” \( a \) and “small” \( s \)
Consider $a \in \mathcal{G}_\rho$ with $s_0 \leq s \leq s_1$, or $a \in \mathcal{T}_1$ with $s_0 \leq s \leq s_1$. The conclusion will follow from the $L^p_\rho$ estimate on $w_0$ together with the gradient estimate of Proposition 5.2. Indeed, using a Taylor expansion together with (5.16) and (5.6), we write

$$w_a(0, s) = w_0(\sigma \tilde{z}, s) = w_0(0, s) + O \left( \sigma \|w_0(s)\|_{L^\infty} \right),$$

$$\int w_0(z, s) \rho(z) dz = w_0(0, s) + O \left( \|\nabla w_0(s)\|_{L^\infty} \right),$$

on the one hand. On the other hand, since $w_0 = \varphi + q$ by definition (5.22), recalling that $q(s) \in V_A(s)$, we write by definitions (2.2) and (3.18) of $\varphi$ and $q_{0,0}(s)$, together with Definition 3.1 of $V_A(s)$,

$$\int w_0(z, s) \rho(z) dz = \int \varphi(z, s) \rho(z) dz + \int q(z, s) \rho(z) dz = \int \varphi(z, s) \rho(z) dz + q_{0,0}(s)$$

$$= \kappa + O \left( e^{-s} \right) + O \left( A e^{-2s} \right).$$

Introducing $(K', L')$ such that

$$a = (K'e^{-\frac{a}{\rho}}, L'e^{-\frac{a}{\rho}}) \quad (5.24)$$

with $\sigma = s_1$ if $a \in \mathcal{T}_1$, we see that

$$0 \leq K' \leq L' \text{ and } K' + L' \leq A.$$

Recalling that $s \leq \sigma$, we write

$$|ae^{\frac{z}{\rho}}| = \sqrt{K'^2 + L'^2} e^{\frac{z}{\rho}} \leq \sqrt{2} A.$$

Taking $A \geq 1$ and recalling that $\|\nabla w_0(s)\|_{L^\infty} \leq \delta_0$ from Proposition 5.2 provided that $s_0 \geq s_{14}(A, \delta_0)$, we write from the previous estimates that

$$|w_a(0, s)| \leq \kappa + C\delta_0 + CA\delta_0 + Ce^{-s} + C Ae^{-2s} \leq \frac{3}{2} \kappa,$$

whenever $\delta_0 \leq \delta_{21}(A)$ and $s_0 \geq s_{21}(A, \delta_0)$ for some $s_{21}(A, \delta_0) \geq 1$ and $\delta_{21}(A) > 0$. In particular, estimate (5.2) holds.

**Step 2: Case of “small” $a$ and “large” $s$**

Now, we consider $a \in \mathcal{G}_\rho$ with $s_0 \leq \sigma \leq s_1$. As we will shortly see, the conclusion follows here from the case $K + L \geq A$ treated in Section 5.5.1 if one replaces there $K$, $L$ and $s_0$ by $K'$, $L'$ and $\sigma$. Indeed, by definition (5.22) of $\mathcal{G}_\rho$, we have

$$K' + L' = A,$$

where $K'$ and $L'$ are defined in (5.24). Following our strategy in Section 5.5.1, we first start by expanding $w_a(y, \sigma)$ as in Lemma 5.10.

**Lemma 5.13** (Expansion of $w_a(y, \sigma)$). For any $A \geq 1$, there exists $s_{22}(A) \geq 1$ such that for any $s_0 \geq s_{22}(A)$, the following holds: Assume that $q(s) \in V_A(s)$ satisfies equation (3.3) for any $s \in [s_0, \sigma]$ for some $\sigma \geq s_0$,
such that (4.11) holds and $\nabla q(s_0) \in L^\infty(\mathbb{R}^2)$. Assume in addition that $a = (K', L')e^{-\frac{s}{3}}$ such that (5.24) holds. Then,

$$w_a(y, \sigma) = \kappa - \ell' + O\left(\frac{\ell'}{A}\right) + O(\ell'^2) \quad (5.26)$$

- $e^{-\sigma}\{2K'L^2h_1h_0 + 2K^2L'h_1h_0 + L^2h_2h_0 + 4K'L'h_1h_1 + K'^2h_0h_2 + 2L'h_2h_1 + 2K'h_1h_2 + h_2h_2\}$

$+ q_6,2(\sigma)\{L'^3h_0h_0 + K'^4h_0h_0 + 4L'^3h_1h_1 + 4K'^3h_1h_2 + 6(K'^2 + L'^2)h_2h_2 + 4K'h_3h_2 + 4L'h_2h_3 + h_4h_2 + h_2h_4\}$

in $L^r_\rho$ for any $r \geq 2$, with $|q_6,2(\sigma)| \leq A\sigma e^{-2\sigma}$, where

$$\ell' = e^{-\sigma}K'^2L'^2 + \delta e^{-2\sigma}[K'^6 + L'^6]. \quad (5.27)$$

**Proof.** See Section 7.8.

Now, arguing as for Lemma [5.11], we see $w_a(\sigma)$ as initial data then integrate equation (1.12) to get an expansion of $w_a(s)$ for later times:

**Lemma 5.14** (Decreasing $w_a(\sigma)$ from $\kappa$ to $\kappa - \eta^*$). There exists $M_{23} > 0$, $A_{23} \geq 1$ and $\eta_{23} > 0$ such that for all $A \geq A_{23}$ and $\eta^* \in (0, \eta_{23})$, there exists $s_{23}(A, \eta^*)$ such that for any $\sigma \geq s_{23}$ and $s_1 \geq \sigma$, if $q(s) \in V_A(s)$ given in Definition [3.1] satisfies equation (3.3) on $[\sigma, s_1]$ and $\nabla q(\sigma) \in L^\infty$, if $a = (K'e^{-\frac{s}{3}}, L'e^{-\frac{s}{3}})$ with $K' + L' = A$, then:

(i) $s^* \geq \sigma$, where $s^*$ is such that $e^{s^*-\sigma}\ell' = \eta^*$ where $\ell'$ is defined in (5.27).

(ii) For all $s \in [\sigma, \min(s^*, s_1)]$,

$$\|w_a(s) - (\kappa - e^{s-\sigma}\ell')\|_{L^\beta_\rho} \leq M_{23} (\eta^* + A^{-1}) e^{s-\sigma}\ell' + M_{23} e^{-s/3}.$$

**Proof.** The proof follows from a straightforward adaptation of the proof of Lemma 5.11 given in Section 7.8. For that reason, the proof is omitted.

As in the case $K + L \geq A$, we derive from this result the following corollary where we prove estimate (5.4):

**Corollary 5.15** (Proof of the $L^\beta_\rho$ bound when $a \in \mathcal{R}_3$ is “small” and $s$ is “large”). There exists $A_{24} > 0$, $\eta_{24} > 0$ and $s_{24}(\eta^*) \geq 1$, such that under the hypotheses of Lemma 5.14, if in addition $A \geq A_{24}$, $\eta^* \leq \eta_{24}$ and $s_0 \geq s_{24}(\eta^*)$, then, for all $s \in [\sigma, s_1],

$$\|w_a(s)\|_{L^\beta_\rho} \leq \frac{3}{2}K.$$

**Proof.** The proof is omitted since this statement follows from Lemma 5.14 exactly in the same way Corollary 5.12 follows from Lemma 5.11.

### 5.5.3 Concluding statement for the control of $w_a(s)$ when $a \in \mathcal{R}_3$

Combining the previous statements given when $a \in \mathcal{R}_3$ defined in (5.9), we obtain the following statement:
Finally, if bound follows from (5.13) and (5.8), we derive the following:

3 regions in (3.4) are properly defined. Then, combining the previous statements given Lemma 5.16 (Control of \( \bar{w}(s) \) when \( a \in \mathcal{R}_3 \) if \( w(s_0) \) is given by (4.3) and \( q(s) \in V_A(s) \)).

There exist \( A_{25} \geq 1 \) and \( m_{25} \in (0, 1) \) such that for all \( A \geq A_{25} \), there exists \( \delta_{25}(A) > 0 \) such that for all \( \delta \in (0, \delta_{25}(A)) \), there exists \( s_{25}(A, \delta) \geq 1 \) such that for all \( s_0 \geq s_{25}(A, \delta) \), \( d \in \mathcal{D}(A, s_0) \) defined in Proposition (4.6) and \( s_1 \geq s_0 \), the following holds:

Assume that \( w \) is the solution of equation (1.12) with initial data \( w_0(y, s_0) \) defined in (4.3), such that for all \( s \in [s_0, s_1] \), \( q(s) \in V_A(s) \) given in Definition (3.4) where \( q(s) \) is defined in (5.22). Then, for all \( s \in [s_0, s_1] \):

(i) \( \|w(s)\|_{L^\infty} \leq \delta_0 \).

(ii) For all \( m \in (0, m_{25}) \) and \( a \in \mathcal{R}_3 \) defined in (5.9), \( |a_w(0, s)| < 2\kappa \).

Proof. The proof is omitted since it is straightforward from Claim 5.3, Corollary 5.12.

5.6 Concluding statement for the control of \( w_a(s) \) for any \( a \in \mathbb{R}^2 \)

Fixing \( M \) as in (5.11) and taking \( m = m_{25} \) introduced in Lemma 5.16, we see that the 3 regions in (3.2) are properly defined. Then, combining the previous statements given for the different regions (namely, Corollaries 5.7 and 5.9 together with Lemma 5.16), we derive the following:

Proposition 5.17 (Control of \( \|w(s)\|_{L^\infty} \) if \( w(s_0) \) is given by (4.3) and \( q(s) \in V_A(s) \)).

There exist \( A_{26} \geq 1 \) such that for all \( A \geq A_{26} \), there exists \( s_{26}(A) \geq 1 \) such that for all \( s_0 \geq s_{26}(A) \), \( d \in \mathcal{D}(A, s_0) \) defined in Proposition (4.6) and \( s_1 \geq s_0 \), the following holds:

Assume that \( w_0 \) is the solution of equation (1.12) with initial data \( w_0(y, s_0) \) defined in (4.3), such that for all \( s \in [s_0, s_1] \), \( q(s) \in V_A(s) \) given in Definition (3.4) where \( q(s) \) is defined in (5.22). Then:

(i) For all \( s \in [s_0, s_1] \), \( \|w_0(s)\|_{L^\infty} < 2\kappa \).

(ii) For all \( a \in \mathbb{R}^2 \) such that

\[
|a_1| + |a_2| \geq A e^{-\frac{\delta}{\kappa^2}},
\]

there exist \( \bar{s}(a) \geq s_0 \) and \( \bar{M}(a) \geq 0 \) such that if \( \bar{s}(a) \geq s_1 \), then for all \( s \in [\bar{s}(a), s_1] \),

\[
\|w_a(s)\|_{L^\infty} \leq \bar{M}(a)e^{-\frac{s_0}{\kappa^2}}.
\]

Proof.

(i) The proof is omitted since it is straightforward from the above-mentioned statements.

(ii) Take \( a \in \mathbb{R}^2 \) such that (5.28) holds. The conclusion follows according to the position of \( a \) in the 3 regions \( \mathcal{R}_i \) defined in in (5.9), with the constants \( M \) and \( m \) fixed in the beginning of the current subsection.

If \( a \in \mathcal{R}_1 \), then the conclusion follows from Corollary 5.10.

If \( a \in \mathcal{R}_2 \), then we see from Corollary 5.9 and its proof that Proposition 5.8 applies. In particular, \( w_a(s) \) is trapped near the heteroclinic orbit \( \psi(5.8) \), and the exponential bound follows from (5.13) and (5.14).

Finally, if \( a \in \mathcal{R}_3 \), then, following Section 5.5, we write

\[
a = (K, L)e^{-\frac{a_0}{\kappa}}
\]

as in (5.14) and reduce to the case where \( 0 \leq K \leq L \) thanks to the symmetries of initial data \( w_0(\cdot, s_0) \) defined in (4.3).

If \( K + L \geq A \), then we see from Corollary 5.12 and its proof that \( w_a(s) \) is trapped near the heteroclinic orbit \( \psi(s) \) (5.8) and the exponential bound follows from estimate (5.13).
given in Proposition 5.8. If \(K + L \leq A\), using (5.28) and (5.29), we may write \(a = (K', L')e^{-\frac{s}{2}}\) for some \(\sigma \in [s_0, s_1]\) with \(K' + L' = A\), as we did in (5.24) and (5.25). Using Corollary 5.15, we see that \(w_0(s)\) is trapped near the heteroclinic orbit \(\psi(s)\), and the conclusion follows from estimate (5.13) again. This concludes the proof of Proposition 5.17.

6 Proof of the main result

In this section, we collect all the arguments to derive Theorem 1. We proceed in 3 subsections:

- We first prove the existence of a solution of equation (3.3) trapped in the set \(V_A(s)\) defined in Definition 3.1.
- Then, we derive a better description on larger sets, yielding the so-called intermediate profile.
- Finally, we prove that the origin is the only blow-up point and derive the final profile given in (1.19).

6.1 Existence of a solution to equation (3.3) trapped in the set \(V_A(s)\)

This is our statement in this section:

**Proposition 6.1** (Existence of a solution \(q(y,s)\) in the shrinking set \(V_A(s)\)).

(i) There exist \(C_{27} > 0\), \(A \geq 1\), \(s_0 \geq 1\) and a parameter \(d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in D(A, s_0)\) defined in Proposition 4.6 such that the solution \(w_0(y,s)\) of equation (1.12) with initial data at \(s = s_0\) given by (4.3) satisfies \(q(s) \in V_A(s)\) and \(\|\nabla q(s)\|_{L^\infty} \leq C_{27}\) for any \(s \geq s_0\), where \(q\) is defined in (3.2) and the set \(V_A(s)\) is given in Definition 3.1.

(ii) If \(u(x,t) = (T-t)^{-\frac{1}{p-1}} w_0\left(\frac{x}{\sqrt{T-t}}, -\log(T-t)\right)\) with \(T = e^{-s_0}\), then \(u\) is a solution of equation (1.1) which blows up only at the origin.

**Remark.** Since initial data in (4.3) is symmetric with respect to the axes and the bissectrices, the same holds for \(w_0(y,s)\).

Before proving Proposition 6.1, let us recall the following consequence of the Liouville Theorem stated in Proposition 5.1:

**Corollary 6.2** (Behavior of the gradient). Following Proposition 6.1 it holds that \(\|\nabla w_0(s)\|_{L^\infty} \to 0\) as \(s \to \infty\).

**Proof.** This is a classical consequence of the Liouville Theorem stated in Proposition 5.1 which follows similarly as does Proposition 5.2. For a statement and a proof, see Theorem 1.1 page 141 in Merle and Zaag [19].

Let us now prove Proposition 6.1:

**Proof of Proposition 6.1.**

(i) This is a classical combination of our previous analysis, namely Lemma 2.2, Proposition 4.4, Propositions 4.6, 5.2 and 5.17. The argument relies on a reduction of the problem in finite dimensions, then the proof of the reduced problem thanks to the degree theory. We give the argument only for the sake of completeness. Expert readers may skip this proof.
In order to apply the above-mentioned statements, let us fix \( A = \max(A_{12}, 1, A_{26}) \) and \( s_0 = \max(s_{10}(\delta), s_{12}(A), s_{13}(A), s_{14}(A, 1), s_{26}(A)) \), where \( \delta \) is the constant appearing in the definition of the profile \( \varphi(y, s) \) and fixed at the beginning of Section 3.

We proceed by contradiction and assume that for all \( d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in D(A, s_0) \) defined in Proposition 4.6, \( q(d, y, s) \) doesn’t remain in the set \( V_A(s) \) for all \( s \geq s_0 \), where \( q(d, y, s) = w_0(d, y, s) - \varphi(y, s) \), \( \varphi(y, s) \) is defined in (2.2), with the constants \( \gamma \) and \( \delta \) are fixed in Lemma 2.2 and the beginning of Section 3, and \( w_0(d, y, s) \) is the solution of equation (1.12) with initial data at \( s = s_0 \) given by (4.3).

Since the function given in (4.3) is one-to-one from item (ii) in Proposition 4.6, we may take \( \tilde{d} = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in [-1, 1]^5 \) as a new parameter, where

\[
\tilde{d}_{i,j} = \frac{e^{2s_0}}{A} q_{i,j}(d, s_0) \quad \text{if} \quad (i, j) \in I_0, \quad \tilde{d}_{6,0} = \frac{e^{3s_0}}{A s_0^2} q_{6,0}(d, s_0)
\]

and \( I_0 = \{(0, 0), (2, 0), (4, 0), (4, 2)\} \). Accordingly, we introduce the notation

\[
\tilde{q}(d, y, s) = q(d, y, s), \quad \tilde{w}_0(d, y, s) = w_0(d, y, s)
\]

and so on. Since \( q(d, y, s_0) \in V_A(s_0) \) by Proposition 4.6, from continuity, we may introduce \( s^*(d) = \tilde{s}^*(d) \) as the minimal time such that \( q(d, y, s) \in V_A(s) \) for all \( s \in [s_0, s^*] \) and an equality case occurs at time \( s = s^* \) in one of the \( \leq \) defining \( V_A(s^*) \) in Definition 3.1.

Applying Proposition 5.17 we see that no equality case occurs for \( \|w_0(d, s^*)\|_{L^\infty} \).

We claim that no equality case occurs for \( |q_{6,2}(d, s^*)| = |q_{6,4}(d, s^*)| \) neither \( \|q_-(d, s_0)\|_{L^p} \).

Indeed, if \( s^*(d) = s_0 \), this follows from Proposition 4.6. If \( s^*(d) > s_0 \), then, this follows from Corollary 4.3 which can be applied thanks to Proposition 4.6. In other words, it holds that

\[
\text{either } |q_{i,j}(d, s^*)| = Ae^{-2s^*} \text{ for some } (i, j) \in I_0 \text{ or } |q_{6,0}(d, s^*)| = As^{2}e^{-3s^*}.
\]

This way, the following rescaled flow is well defined:

\[
\Phi : [-1, 1]^5 \rightarrow \partial[-1, 1]^5 \setminus \{0\} \quad d \mapsto \frac{e^{2s^*}}{A} (q_{0,0}, q_{2,0}, q_{4,0}, q_{4,2}, \frac{e^{s^*}}{s^*} q_{6,0})(d, s^*(d)).
\]

We claim that:

(a) \( \Phi \) is continuous;
(b) \( \Phi|_{[-1, 1]^5} = \text{Id} \), which will imply a contradiction, by the degree theory and finish the proof. Let us prove (a) and (b).

**Proof of (a):** Using items (i) and (ii) in Proposition 4.4, we see that the flow of the five components \( |q_{i,j}| \) with \( (i, j) \in I_0 \cup \{(6, 0)\} \) is transverse outgoing on the corresponding boundary (i.e. \( Ae^{-2s} \) if \( (i, j) \in I_0 \) or \( As^{2}e^{-3s} \) if \( (i, j) = (6, 0) \)). This implies the continuity of \( \Phi \).

**Proof of (b):** If \( \tilde{d} \in \partial[-1, 1]^5 \) and \( d \) is the corresponding original parameter in \( D \), then we see by definition (6.1) that

\[
\text{either } |q_{i,j}(d, s_0)| = Ae^{-2s_0} \text{ for some } (i, j) \in I_0 \text{ or } |q_{6,0}(d, s_0)| = As^{2}e^{-3s_0}.
\]

Since \( q(d, s_0) \in V_A(s_0) \) by Definition 3.1 this implies that \( s^*(\tilde{d}) = s^*(d) = s_0 \). From (6.2) and (6.1), we see that \( \Phi(\tilde{d}) = \tilde{d} \) and (b) holds.
Since we know from the degree theory that there is no continuous function from $[-1, 1]^2$ to its boundary which is equal to the identity on the boundary, a contradiction follows. Thus, there exists a parameter $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{6,0}) \in \mathcal{D}(A, s_0)$ defined in Proposition 4.4 such that $q(d, y, s) \in \mathcal{V}_k(s)$ for all $s \geq s_0$. Applying Proposition 5.2 and Lemma 2.2, we see that $\|\nabla q(d, s)\|_{L^\infty} \leq 1 + C_0$, for all $s \geq s_0$.

(ii) From the similarity variables definition (i.1), $u$ is indeed a solution of equation (1.1) defined for all $(x, t) \in \mathbb{R}^2 \times [0, T)$. From the classical theory by Giga and Kohn [10], we see that when $b \to \infty$, $w_b(s) \to \kappa$ when $b$ is a blow-up point, and $w_b(s) \to 0$ if not, in $L^2_{\rho}$.

Therefore, it’s enough to prove that $w_0(s) \to \kappa$ and $w_a(s) \to 0$ as $s \to \infty$ in order to conclude.

First, since $q(s) \in \mathcal{V}_k(s)$ for all $s \geq s_0$ from item (i), we see from (3.21), the relation (3.2) and the definition (2.2) of $\varphi$ that $w_0(s) \to \kappa$ as $s \to \infty$.

Second, if $a \neq 0$, we see that for any $s_1 \geq 2 \log \frac{A}{|a_1| + |a_2|}$, (5.23) holds. Since Proposition 5.17 holds here, applying its item (ii), we see that for some $\bar{s}(a) \geq s_0$ and $\bar{M}(a) > 0$,

any $s_1 \geq \max(2 \log \frac{A}{|a_1| + |a_2|}, \bar{s}(a))$, for any $s \in [\bar{s}(a), s_1]$, $\|w_a(s)\|_{L^2_{\rho}} \leq \bar{M}(a) e^{-\frac{A}{2s}}$. Making $s_1 \to \infty$, we see that $w_a(s) \to 0$. This concludes the proof of Proposition 6.1.

\[6.2 \text{ Intermediate profile}\]

From the construction given in Proposition 6.1 together with (3.21) and the expansion (2.5) of the profile $\varphi$ defined in (2.2), we have a solution $w_0(y, s)$ of equation (1.12) such that our goal in (1.14) holds, namely

$w_0(y, s) = \kappa - e^{-s} h_2(y_1) h_2(y_2) + o(e^{-s})$ as $s \to \infty$,

in $L^2_{\rho}$ and uniformly on compact sets, by parabolic regularity.

For large $s$, the second term of this expansion will be small with respect to the constant $\kappa$, which means that we only see a flat shape for $w_0$. Extending the convergence beyond compact sets, say, for $|y| \sim e^{\frac{s}{2}}$, would allow to see $w$ escape that constant. This is indeed what Velázquez did in Theorem 1 page 1570 of [24], proving that $w_0$ has the degenerate profile (1.15), in the sense that

$\sup_{|y| < K e^{\frac{s}{2}}} \left| w_0(y, s) - \left( p - 1 + \frac{(p - 1)^2}{\kappa} e^{-s} y_1^2 y_2^2 \right)^{-\frac{1}{p-1}} \right| \to 0$ as $s \to \infty$,

for any $K > 0$.

If we see in this expansion that $w_0$ departs from the constant $\kappa$ (1.5) for $y_1 = K_1 e^{\frac{s}{2}}$ and $y_2 = K_2 e^{\frac{s}{2}}$ for some $K_1 > 0$ and $K_2 > 0$, this is not the case on the axes, namely when $y_1 = 0$ or $y_2 = 0$. In accordance with our idea in deriving the profile $\varphi$ in (2.2), our techniques in this paper provide us with the following sharper profile, valid on a larger region:

**Proposition 6.3 (Intermediate profile).** Consider $w_0(y, s)$ the solution of equation (1.12) constructed in Proposition 6.1. For any $K > 0$, it holds that

$\sup_{e^{-s} y_1^2 y_2^2 + \delta e^{-2s} (y_1^2 + y_2^2) < K} |w_0(y, s) - \Phi(y, s)| \to 0$ as $t \to T$,

where $\Phi(y, s)$ is defined in (1.10).
Remark. Φ(y,s) is referred to as the “intermediate profile”, since it holds in the region \{s \geq s_0 = -\log T\}, which corresponds to the region \{0 \leq t < T\} in the original variables defined by (1.4). Later in Proposition 6.3, we will define the “final profile”, as a limit of \( u(a,t) \) as \( t \to T \) when \( a \neq 0 \), and holds in some sense for \( t = T \), justifying its character as “final”.

Proof. Consider \( w_0 \) the solution of equation (1.12) constructed in Proposition 6.1 and defined for all \((y,s) \in \mathbb{R}^2 \times [s_0, \infty)\). Given \( K_0 > 0 \) and \( \epsilon > 0 \), we look for \( S_0 = S_0(K_0, \epsilon) \) such that for any \( S \geq S_0 \) and \( Y \in \mathbb{R}^2 \) such that

\[
e^{-S}Y^2_1Y^2_2 + \delta e^{-2S}(Y^6_1 + Y^6_2) < K_0, \tag{6.3}\]

it holds that

\[
|w_0(Y,s) - \Phi(Y,s)| \leq C\epsilon, \tag{6.4}\]

for some universal \( C > 0 \), where \( \Phi(y,s) \) is defined in (1.16). Consider then \( S \geq s_0 \) and \( Y \in \mathbb{R}^2 \) such that (6.3) holds. Since \( w_0(y,s) \) is symmetric with respect to the axes and the bissectrices (see the remark following Proposition 6.1), we may assume that

\[
0 \leq Y_1 \leq Y_2. \tag{6.5}\]

Following our strategy in Section 6.5.1, we introduce \( a = a(Y,S) \) such that

\[
ae^{-\frac{S}{2}} = Y. \tag{6.6}\]

Since

\[
w_0(Y,S) = w_a(0,S) \tag{6.7}\]

by (6.15), our conclusion will follow from the study of \( w_a(y,s) \) for \( s \in [s_0,S] \). From (6.3) and (6.6), we see that

\[
G_0(a) \leq \frac{K_0(p - 1)}{\kappa}e^{-S}, \tag{5.10}\]

where \( G_0 \) is defined in (5.10). In particular,

\[
0 \leq a_1 \leq a_2 \leq \left( \frac{K_0(p - 1)}{\kappa \delta} \right)^{\frac{1}{2}} e^{-\frac{S}{2}}. \tag{6.8}\]

Consider \( \bar{A} > 0 \) to be fixed large enough later. Taking \( S \geq S_1 \) for some \( S_1(K_0, \bar{A}) \geq s_0 \), we see that

\[
a_1 + a_2 \leq \bar{A}e^{-\frac{S}{2}}. \tag{6.9}\]

Therefore, we may introduce \( \sigma \geq s_0, K' \) and \( L' \) such that

\[
a = (K', L')e^{-\frac{S}{2}} \text{ with } 0 \leq K' \leq L' \text{ and } K' + L' = \bar{A}. \tag{6.9}\]

The conclusion will follow by applying Lemma 5.14 and Proposition 5.8 exactly as we did in Corollaries 5.15 and 5.12. In order to apply those two statements, let us assume that \( \bar{A} \geq \max(A, A_{23}) \) and consider \( \eta^* \in (0, \eta_{23}] \), where \( A \) is fixed in Proposition 6.1 and \( A_{23} \) together with \( \eta_{23} \) are introduced in Lemma 5.14.

From (6.8) and (6.9), we see that whenever \( S \geq S_2 \) for some \( S_2(K_0, \bar{A}, \eta^*) \geq s_0 \), it follows that \( \sigma \geq s_{23}(\bar{A}, \eta^*) \) defined in Lemma 5.14. Let us then assume that \( S \geq S_2 \).
Since $\tilde{A} \geq A$, we see from Proposition 6.1 and Definition 3.1 that $q(s) \in V_A(s) \subset V_{\tilde{A}}(s)$ and $\nabla q(s) \in L^\infty$, for all $s \geq s_0$.

Using (6.9), we see that Lemma 5.14 applies with any $s_1 \geq \sigma$. In particular, if we introduce $s^*$ such that

$$e^{s^*-\sigma} \psi = \eta^*$$

(6.10)

where $\psi$ is defined in (5.27), then we see that $s^* \geq \sigma$ and for all $s \in [\sigma, s^*]$,

$$\|w_a(s) - (\kappa - e^{s^*-\sigma}) \psi \|_{L^\rho} \leq M_{23} \left( \eta^* + \frac{1}{A} \right) \eta^* + M_{23} e^{-\frac{\eta}{\eta^*}},$$

(6.11)

where the constant $M_{23}$ is defined in Lemma 5.14.

Assuming that $\eta^* < \kappa$, we may introduce $\sigma^* \in \mathbb{R}$ such that

$$\psi(\sigma^*) = \kappa - \eta^*,$$

(6.12)

where $\psi$ is defined in (5.8). Using (5.22), (6.9) and (6.8), we see that whenever $\tilde{A} \geq A_3$, $\eta^* \leq \eta_3$ and $S \geq S_3$, for some $A_3 \geq 1$, $\eta_3 > 0$ and $S_3(K_0, \tilde{A}, \eta^*)$, we have

$$\|w_a(s^*) - \psi(s^*)\|_{L^\rho} \leq M_{23} \left( \eta^* + \frac{1}{A} \right) \eta^* + M_{23} e^{-\frac{\eta}{\eta^*}} \leq \frac{\|\psi'(\sigma^*)\|_{L^\rho}}{M_1},$$

(6.13)

where $M_1$ is introduced in Proposition 5.8. Since $\nabla w_a(s^*) \in L^\infty$ and $\|w_a(s)\|_{L^\infty} \leq 2\kappa$ for any $s \geq s^*$, thanks to Proposition 6.11 together with the definition (3.2) of $q$ and the transformation (3.15), Proposition 5.8 applies to the shifted function $w_a(y, s + s^*)$ (with any $\sigma_1 \geq 0$), and we see that for all $s \geq s^*$, we have

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^\rho} \leq M_1 \|w_a(s^*) - \psi(\sigma^*)\|_{L^\rho} \frac{\|\psi'(s + \sigma^*)\|_{L^\rho}}{\|\psi'(\sigma^*)\|_{L^\rho}}.$$

Since $\psi' \in L^\infty$ by definition (5.8) of $\psi$, using again (6.13) and (5.22), we see that

$$\forall s \geq s^*, \|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^\rho} \leq M_1 M_{23} \|\psi'\|_{L^\infty} \left( \eta^* + \frac{1}{A} + \frac{e^{-\eta}}{\eta^*} \right),$$

(6.14)

whenever $\eta^* \leq \eta_4$, for some $\eta_4 > 0$.

Now, if $s \in [\sigma, s^*]$, using (6.6), we write

$$\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^\rho} \leq \|w_a(s) - (\kappa - e^{s^*-\sigma}) \psi \|_{L^\rho} + \|\psi(s + \sigma^* - s^*) - \kappa + e^{s^*-\sigma} \psi' \|_{L^\rho}.$$ (6.15)

Since $s \leq s^*$, hence $s + \sigma^* - s^* \leq \sigma^*$ and $\psi$ is decreasing by definition (5.8), we write from (6.12) and (6.10)

$$|\psi(s + \sigma^* - s^*) - \kappa| \leq |\psi(s^*) - \kappa| = \eta^*$$

and $e^{s^*-\sigma} \psi' \leq e^{s^*-\sigma} \psi' = \eta^*$. (6.16)

Therefore, using (6.15) together with (6.11), we see that

$$\forall s \in [\sigma, s^*], \|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^\rho} \leq M_{23} \left( \eta^* + \frac{1}{A} \right) \eta^* + M_{23} e^{-\frac{\eta}{\eta^*}} + 2\eta^*.$$ (6.17)
Finally, if \( s \in [s_0, \sigma] \), noting again that \( s + \sigma^* - s^* \leq \sigma + \sigma^* - s^* \leq \sigma^* \), we may use (6.16) and write similarly
\[
\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_p} \leq \|w_a(s) - \kappa\|_{L^2_p} + \eta^*.
\]
Recalling that \( \nabla w(s) \in L^\infty \) from the construction in Proposition 6.1, we may use a Taylor expansion as in the proof of Claim 5.3 and write
\[
\|w_a(s) - \kappa\|_{L^2_p} \leq C\|w_a(0, s) - \kappa\|_{L^2_p} + C\|\nabla w_0(s)\|_{L^\infty}.
\] (6.18)
Since \( w_a(0, s) = w_0(ae^{\frac{\chi}{2}}, s) \) by (6.15) and
\[
|ae^{\frac{\chi}{2}}| = |(K', L')|e^{\frac{\mu \rho}{2}} \leq \sqrt{K'' + L''} e^{\frac{\mu \rho}{2}} \leq A\sqrt{2}
\]
by (6.9) and the fact that \( s \leq \sigma \), we may use again a Taylor expansion for \( w_0 \) and write
\[
|w_a(0, s) - \kappa| = |w_0(ae^{\frac{\chi}{2}}, s) - \kappa| \leq C\|w_0(s) - \kappa\|_{L^2_p} + C(1 + \bar{A})\|\nabla w_0(s)\|_{L^\infty}.
\]
Furthermore, using the construction in Proposition 6.1 together with the definitions (6.2) and (6.2) of \( q(y, s) \) and \( \varphi(y, s) \), and estimate (6.21), we write
\[
\|w_0(0, s) - \kappa\|_{L^2_p} \leq \|q_0(s)\|_{L^2_p} + \|\varphi(s) - \kappa\|_{L^2_p} \leq CAs e^{-2s} + Ce^{-s},
\]
where \( A \) is given in Proposition 6.1. Collecting the previous estimates, we write
\[
\forall s \in [s_0, \sigma], \quad \|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_p} \leq CAs e^{-2s} + Ce^{-s} + C(1 + \bar{A})\|\nabla w(s)\|_{L^\infty} + \eta^*.
\] (6.19)
Using (6.9) and (6.8) and the gradient estimate in Corollary 6.2, we see from (6.14), (6.17) and (6.19) that taking \( \bar{A} \geq A_5, \eta^* \leq \eta_5 \) and \( S \geq S_5 \) for some \( A_5(\epsilon) \geq 1, \eta_5(\epsilon) > 0 \) and \( S_5(\epsilon, \eta^*, K_0, \bar{A}) \geq s_0 \), we get for all \( s \geq s_0 \),
\[
\|w_a(s) - \psi(s + \sigma^* - s^*)\|_{L^2_p} \leq 4\epsilon.
\]
Taking \( s = S \), using (6.6) and (6.7), then proceeding as with (6.18), we write
\[
|w_0(Y, S) - \psi(S + \sigma^* - s^*)| = |w_0(0, S) - \psi(S + \sigma^* - s^*)| \leq C\|w_0(0, S) - \psi(S + \sigma^* - s^*)\|_{L^2_p} + C\|\nabla w_0(0, S)\|_{L^\infty} \leq C\epsilon + C\|\nabla w_0(0, S)\|_{L^\infty}.
\]
Taking \( S \) larger and using again the gradient estimate of Corollary 6.2, we see that
\[
|w_0(Y, S) - \psi(S + \sigma^* - s^*)| \leq 2C\epsilon.
\] (6.20)
Since \( e^{s^* - \sigma'} = \eta^* \) from (6.10), \( e^{s^* - \sigma'} \sim \eta^* \) as \( \eta^* \) goes to 0 from (5.22), and \( e^{S^* - \sigma'} = e^{-\delta Y^1 Y^2 + \delta e^{-2S}(Y^1 + Y^2)} \) by definitions (6.6), (6.9) and (6.27) of \( a, (K', L') \) and \( \sigma' \), we write
\[
e^{S + \sigma^* - s^*} = e^{s^* - \sigma'} e^{S - \sigma} = \left( \frac{\rho - 1}{\kappa} + \bar{\eta} \right) \eta^* e^{s^* - \sigma} e^{S - \sigma} = \left( \frac{\rho - 1}{\kappa} + \bar{\eta} \right) \sigma' e^{S - \sigma}
\]
where \( \bar{\eta} \) goes to 0 as \( \eta^* \) goes to 0. Recalling condition (6.3), then making an expansion of \( \psi \) defined in (6.8), we see that for \( \eta^* \leq \eta_6 \) for some \( a_6(K_0, \epsilon) > 0 \), we have
\[
\left| \psi(S + \sigma^* - s^*) - \left[ p - 1 + \frac{(p - 1)^2}{\kappa} \left( e^{-\delta Y^1 Y^2 + \delta e^{-2S}(Y^1 + Y^2)} \right) \right] \right| \leq \epsilon.
\] (6.21)
Combining (6.20) and (6.21) yields (6.4). This concludes the proof of Proposition 6.3.
6.3 Derivation of the final profile

Throughout this section, we consider \( u(x,t) \) the solution of equation (1.1) constructed in Proposition 6.1 which blows up at time \( T > 0 \) only at the origin.

Using Propositions 6.1 and 6.3, together with the similarity variables’ transformation (1.3), we derive the following:

**Corollary 6.4** (Intermediate profile for \( u(x,t) \)). For any \( K > 0 \), it holds that

\[
\sup_{x_1^2 x_2^2 + \delta(x_1^6 + x_2^6) \leq K (T-t)} \left| (T-t)^{\frac{1}{p-1}} u(x,t) - \left[ p - 1 \right. \right. \left. + \frac{(p-1)^2}{\kappa} \left( a_1^2 a_2^2 + \delta (a_1^6 + a_2^6) \right) \right] \right| \rightarrow 0 \text{ as } t \rightarrow T.
\]

Using this result, we derive the following estimate for the final profile:

**Proposition 6.5** (Final profile). For any \( a \neq 0 \), \( u(a,t) \) has a limit as \( t \rightarrow T \), denoted by \( u(a,T) \). Moreover,

\[
u(a,T) \sim u^*(a) \text{ as } a \rightarrow 0, \text{ where } u^*(a) = \left( \frac{(p-1)^2}{\kappa} \left( a_1^2 a_2^2 + \delta (a_1^6 + a_2^6) \right) \right)^{-\frac{1}{p-1}}. \quad (6.22)
\]

**Remark.** Following the remark given after Proposition 6.3, we justify the name of \( u^* \) as “final profile”.

This result is in fact a consequence of the following ODE localization property of the PDE, proved in our earlier paper [19], and which is a direct consequence of the Liouville Theorem stated in Proposition 5.4.

**Proposition 6.6** (ODE localization for \( u(x,t) \)). For any \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that for all \( (x,t) \in \mathbb{R}^2 \times [0,T) \),

\[(1 - \epsilon) u(x,t)^p - C_\epsilon \leq \partial_t u(x,t) \leq (1 + \epsilon) u(x,t)^p + C_\epsilon.\]

**Remark.** Since initial data given in (4.3) are nonnegative, the same holds for \( w_0(y,s) \), solution of equation (1.12), and \( u(x,t) \), solution of equation (1.1), both constructed in Proposition 6.7. This justifies the notation \( u(x,t)^p \) without absolute value.

**Proof.** See Theorem 1.7 page 144 in [19]. The fact that initial data given in (4.3) is in \( W^{2,\infty} (\mathbb{R}^2) \) is necessary to have the estimate in \( \mathbb{R}^2 \times [0,T) \), otherwise, if initial data in only in \( L^\infty (\mathbb{R}^2) \), we will have \( u(t_0) \in W^{2,\infty} (\mathbb{R}^2) \) for any \( t_0 > 0 \) by parabolic regularity, and the statement will hold uniformly in \( \mathbb{R}^2 \times [t_0,T) \) from any \( t_0 \in (0,T) \).

Now, we are ready to prove Proposition 6.5 thanks to Corollary 6.4 and Proposition 6.6.

**Proof of Proposition 6.5.** The existence of the limiting profile \( u(a,T) \) follows by compactness, exactly as in Proposition 2.2 page 269 in Merle [16]. It remains to prove (6.22).

Consider any \( K_0 > 0 \). Given any small enough \( a \neq 0 \), we introduce the time \( t^*(a) \in [0,T) \) such that

\[
a_1^2 a_2^2 + \delta (a_1^6 + a_2^6) = K_0 \frac{\kappa}{p-1} (T - t^*(a)). \quad (6.23)
\]
Note that $t^*(a) \to T$ as $a \to 0$. The conclusion will follow from the study of $u(a, t)$ on the time interval $[t^*(a), T]$. Consider first some arbitrary $\epsilon > 0$.

**Step 1: Initialization.** From Corollary 6.4, we see that
\[
\left| (T - t^*(a))^{\frac{1}{1 - p}} u(a, t^*(a)) - \kappa (1 + K_0)^{-\frac{1}{p - 1}} \right| \leq \epsilon \kappa (1 + K_0)^{-\frac{1}{p - 1}},
\]
provided that $a$ is small enough.

**Step 2: Dynamics for $t \in [t^*(a), T]$.** From Proposition 6.6, we see that
\[
\forall t \in [t^*(a), T), \quad (1 - \epsilon) u(a, t)^p - C_\epsilon \leq \partial_t u(x, t) \leq (1 + \epsilon) u(a, t)^p + C_\epsilon
\]
for some $C_\epsilon > 0$. Using (6.24) and (6.25), one easily shows that for $a$ small enough,
\[
\forall t \in [t^*(a), T), \quad C_\epsilon \leq \epsilon u(a, t)^p.
\]
Therefore, we see from (6.25) that
\[
\forall t \in [t^*(a), T), \quad (1 - 2\epsilon) u(a, t)^p \leq \partial_t u(x, t) \leq (1 + 2\epsilon) u(a, t)^p.
\]
Using again (6.24), we may explicitly integrate the 2 differential inequalities we have just derived and write for all $t \in [t^*(a), T)$,
\[
\kappa \left[ (T - t^*(a))(1 + K_0)(1 - \epsilon)^{1 - p} - (1 - 2\epsilon)(t - t^*(a)) \right]^{\frac{1}{p - 1}} \leq u(a, t)
\]
\[
\leq \kappa \left[ (T - t^*(a))(1 + K_0)(1 + \epsilon)^{1 - p} - (1 + 2\epsilon)(t - t^*(a)) \right]^{\frac{1}{p - 1}}.
\]
Making $t \to T$, we see that
\[
\kappa (T - t^*(a))^{\frac{1}{p - 1}} \left[ (1 + K_0)(1 - \epsilon)^{1 - p} - (1 - 2\epsilon) \right]^{\frac{1}{p - 1}} \leq u(a, T)
\]
\[
\leq \kappa (T - t^*(a))^{\frac{1}{p - 1}} \left[ (1 + K_0)(1 + \epsilon)^{1 - p} - (1 + 2\epsilon) \right]^{\frac{1}{p - 1}}.
\]
Taking $\epsilon$ small enough, we see that
\[
\left| u(a, T) - \kappa (T - t^*(a))^{\frac{1}{p - 1}} K_0^{\frac{1}{p - 1}} \right| \leq C(K_0) \epsilon (T - t^*(a))^{\frac{1}{p - 1}}.
\]
Using the definitions (6.23) and (6.22) of $t^*(a)$ and $u^*(a)$, we see that
\[
|u(a, T) - u^*(a)| \leq C'(K_0) \epsilon u^*(a).
\]
Since $\epsilon > 0$ was arbitrarily chosen, this concludes the proof of Proposition 6.6.

**6.4 Conclusion of the proof of the main statement**

In this section, we gather the previous statements to derive Theorem 1.

**Proof of Theorem 1.** Let us first note that $\delta > 0$ was chosen large enough in the beginning of Section 3. In fact, as one can easily see from the proof, our analysis holds for any $\delta \geq \delta_0$, for some large enough $\delta_0$.

Using Proposition 6.4, we have the existence of a solution $u(x, t)$ of equation (1.1) blowing up at some time $T > 0$ only at the origin, such that $q(s) \in V_A(s)$ given in Definition 4.1, for any $s \geq s_0 = -\log T$, where $q$ is defined in (3.2). In particular, $\|q(s)\|_{L^2} \leq A e^{-q s}$ for all $s \geq s_0$, by (3.21). From the expansion (2.5) of the profile $\varphi$ defined in (2.2), we see that item (i) of Theorem 1 holds in $L^2(\mathbb{R}^2)$. Using parabolic regularity, the convergence holds also uniformly in any compact set of $\mathbb{R}^2$.

As for items (ii) and (iii) of Theorem 1 they directly follow from Propositions 6.3 and 6.5. This concludes the proof of Theorem 1.
7 Proof of the technical details

In this section, we proceed in several subsections to justify all the technical ingredients, which were stated without proofs in the previous sections.

7.1 Estimates of the remainder term $R(y,s)$ defined in (3.4)

In this section, we prove estimate (3.11) and Lemma 4.1. Since the former is a direct consequence of the latter, thanks to the decomposition (3.17), we only prove the lemma.

Proof of Lemma 4.1. Take $r \geq 2$. In the following, all the expansions are valid in $L^r_p$ for $s \to \infty$. This is the case in particular for (2.5). As one may convince himself by differentiating the expression (2.2) then making an expansion as $s \to \infty$, we will find the same result as if we have directly differentiated the expression (2.5). Using (3.7), it follows that

$$
\partial_s \varphi(y,s) = e^{-s}h_2 h_2 - 2e^{-2s} \left\{ -\delta (h_6 h_0 + h_0 h_6) + \gamma (h_4 h_2 + h_2 h_4) + \frac{p}{2\kappa} h_4 h_4 \right\} + O(e^{-3s}) \text{,}
$$

$$(\mathcal{L} - 1) \varphi(y,s) = 2e^{-s}h_2 h_2 + e^{-2s} \left\{ 3\delta (h_6 h_0 + h_0 h_6) - 3\gamma (h_4 h_2 + h_2 h_4) - \frac{2p}{\kappa} h_4 h_4 \right\} + O(e^{-3s}) \text{.}
$$

Since $\|\varphi(\cdot,s)\|_{L^\infty} \leq \kappa + C_0$ for all $s \geq 0$ (see item (ii) of Lemma 2.2), using a Taylor expansion, we write

$$
\varphi^p = \kappa^p + p \kappa^{p-1} (\varphi - \kappa) + \frac{p(p-1)}{2} \kappa^{p-2} (\varphi - \kappa)^2 + O((\varphi - \kappa)^3) \text{.}
$$

Since

$$
\varphi - \kappa = O(e^{-s}) \text{ and } (\varphi - \kappa)^2 = e^{-2s} h_2(y_1)^2 h_2(y_2)^2 + O(e^{-3s})
$$

from (2.3), noting that

$$
h_2(\xi)^2 = h_4(\xi) + 8 h_2(\xi) + 8 h_0(\xi)
$$

by definition (1.10), the result follows from the above estimates, thanks to the definitions (3.21) and (1.5) of $R(y,s)$ and $\kappa$. This concludes the proof of Lemma 4.1.

$\square$

7.2 Dynamics of equation (3.3)

We prove Proposition 4.2 in this section.

Proof of Proposition 4.2. Consider $q(s) \in V_A(s)$ a solution of equation (3.3) on some time interval $[s_0, s_1]$ for some $A > 0$. Note that $\|q(s)\|_{L^\infty}$ is uniformly bounded, hence, (3.10) holds. Assume that the initial condition (4.1) holds. Consider then $s \in [s_0, s_1]$.

(i) Consider $i \in \mathbb{N}$ and $0 \leq j \leq i$. If we multiply equation (3.3) by $k_{i-j}(y_1) k_j(y_2)$ where the polynomial $k_n$ is introduced in (3.19), we get the conclusion, by definition (3.18) of $q_{i,j}$ together with (3.7), (3.8), (3.21), (3.10) and Hölder’s inequality.

(ii) If $P_-$ is the $L^2_p$ projector on

$$
E_- \equiv \text{span}\{h_{i-j}h_j \mid i \geq 8 \text{ and } 0 \leq j \leq i\} \text{,}
$$

(7.1)
then \( q_-(s) = P_-(q(s)) \) and \( R_-(s) = P_-(R(s)) \). Applying this projector to equation (3.3), we get

\[
\partial_s q_- = \mathcal{L} q_- + P_-(Vq + B) + R_-. 
\]

Multiplying by \( q_- \) and then integrating in space, we write

\[
\frac{1}{2} \frac{d}{ds} \| q_- \|_{L^2_{\rho}}^2 = \int_{\mathbb{R}^2} q_- \mathcal{L} q_- \rho dy + \int_{\mathbb{R}^2} q_- [P_-(Vq + B) + R_-] \rho dy. 
\]

Since the highest eigenvalue of \( \mathcal{L} \) on \( E_- \) is \( \lambda = -3 \) (see (3.7)), it follows that

\[
\int_{\mathbb{R}^2} q_- \mathcal{L} q_- \rho dy \leq -3 \| q_- \|_{L^2_{\rho}}^2.
\]

Recalling that \( |B| \leq C|q|^\bar{p} \) where \( \bar{p} = \min(p, 2) > 1 \) thanks to (3.10), then using the Cauchy-Schwarz inequality, we write

\[
\left| \int_{\mathbb{R}^2} q_- [P_-(Vq + B) + R_-] \rho dy \right| \leq \| q_- \|_{L^2_{\rho}} \left[ \| P_-(Vq + B) \|_{L^2_{\rho}} + \| R_- \|_{L^2_{\rho}} \right] 
\]

\[
\leq \| q_- \|_{L^2_{\rho}} \left[ \| Vq \|_{L^2_{\rho}} + C \| B \|_{L^2_{\rho}} + \| R_- \|_{L^2_{\rho}} \right] 
\]

\[
\leq \| q_- \|_{L^2_{\rho}} \left[ \| V \|_{L^{4}_{\rho}} \| q \|_{L^{4}_{\rho}} + C \| q \|_{L^{\bar{p}}_{\rho}}^{\bar{p}} + \| R_- \|_{L^2_{\rho}} \right].
\]

Using the delay regularizing estimate given in Proposition 4.3 together with the estimate (3.8) satisfied by \( V \), we conclude the proof of Proposition 4.2.

### 7.3 Parabolic regularity

In this section, we prove Proposition 4.3 which gives a parabolic regularity estimate for equation (3.3). Through a Duhamel formulation for equation (3.3), we reduce the question to the linear level, where the estimate was already proved by Herrero and Velázquez [15], as we recall in the following, along with another classical regularity estimate:

**Lemma 7.1.** (Regularizing effect of the operator \( \mathcal{L} \))

(i) (**Herrero and Velázquez [15]**) For any \( r > 1 \), \( \bar{r} > 1 \), \( v_0 \in L^r_{\rho}(\mathbb{R}^N) \) and \( s > \max \left( 0, -\log \left( \frac{-1}{r-1} \right) \right) \), it holds that

\[
\| e^{s \mathcal{L}} v_0 \|_{L^r_{\rho}(\mathbb{R}^N)} \leq \frac{C(r, \bar{r}) e^s}{(1 - e^{-s}) \frac{r}{\bar{r}} (r - 1 - e^{-s}(\bar{r} - 1)) \frac{N}{2N}} \| v_0 \|_{L^\bar{r}_{\rho}(\mathbb{R}^N)}.
\]

(ii) Consider \( r \geq 2 \) and \( v_0 \in L^r_{\rho}(\mathbb{R}^N) \) such that \( |v_0(y)| + |\nabla v_0(y)| \leq C(1 + |y|^k) \) for some \( k \in \mathbb{N} \). Then, for all \( s \geq 0 \), we have \( \| e^{s \mathcal{L}} v_0 \|_{L^r_{\rho}(\mathbb{R}^N)} \leq C e^s \| v_0 \|_{L^r_{\rho}(\mathbb{R}^N)} \).

(iii) For any \( v_0 \in L^\infty(\mathbb{R}^N) \) and \( s \geq 0 \), it holds that \( e^{s \mathcal{L}} v_0 \in L^\infty \) with \( \| e^{s \mathcal{L}} v_0 \|_{L^\infty} \leq C e^s \| v_0 \|_{L^\infty} \).

(iv) For any \( v_0 \in W^{1,\infty}(\mathbb{R}^N) \) and \( s > 0 \), it holds that \( e^{s \mathcal{L}} \nabla v_0 \in L^\infty \) with \( \| e^{s \mathcal{L}} \nabla v_0 \|_{L^\infty} \leq \frac{C e^s}{\sqrt{1 - e^{-s}}} \| v_0 \|_{L^\infty} \).

**Remark.** When \( r = \bar{r} \) in item (i), then constant in the right-hand side blows up as \( s \to 0 \), which may seem surprising, since one expects some continuity of the norm in \( L^r_{\rho} \). In fact, such a continuity is obtained in item (ii), thanks to an additional growth control of the gradient.
Proof.
(i) See Section 2 page 139 in [15] where the one-dimensional case is proved. The adaptation to higher dimensions is straightforward.
(ii) See Lemma 2.1 in [22].
(iii) and (iv) These estimates are straightforward from the definition of the kernel:

\[ e^{sL}(y, x) = \frac{e^s}{|4\pi(1 - e^{-s})|^{N/2}} \exp \left[ \frac{|ye^{-\frac{s}{2}} - x|^2}{4(1 - e^{-s})} \right]. \tag{7.2} \]

Let us now give the proof of Proposition 4.3.

Proof of Proposition 4.3. Let us consider \( A > 0 \) and \( s_1 \geq s_0 \geq 0 \), and assume that \( q(y, s) \) satisfies equation (3.3) for all \( (y, s) \in \mathbb{R}^2 \times [s_0, s_1] \), with \( q(s) \in V_A(s) \) defined in (3.11). Assume also that the initial condition (4.1) holds. Note in particular that \( \|q(s)\|_{L^\infty} \) is uniformly bounded. Using (3.12) together with (3.13), we see that for some universal constant \( C^* > 0 \) and for almost every \((y, s) \in \mathbb{R}^2 \times [s_0, s_1]\), we have

\[ \partial_s|q| \leq (\mathcal{L} + C^*)|q| + |R|. \tag{7.3} \]

Consider now some \( r \geq 2 \) and introduce \( s^* = -\log\left(\frac{2}{e} r^2\right) \geq 0 \), which is involved in the statement of Lemma 7.1. Consider then some \( s \in [s_0, s_1] \), and introduce

\[ s' = \max[s_0, s - s^*]. \tag{7.4} \]

We then introduce the following Duhamel formulation of (7.3) on the interval \([s', s]\):

\[ |q(s)| \leq e^{(\mathcal{L} + C^*)(s-s')}|q(s')| + \int_{s'}^{s} e^{(\mathcal{L} + C^*)(s-\tau)}|R(\tau)|d\tau, \]

which implies that

\[ \|q(s)\|_{L^r_\mathbb{R}(\mathbb{R}^2)} \leq I + J \tag{7.5} \]

where

\[ I = \|e^{(\mathcal{L} + C^*)(s-s')}|q(s')|\|_{L^r_\mathbb{R}(\mathbb{R}^2)} \text{ and } J = \int_{s'}^{s} \|e^{(\mathcal{L} + C^*)(s-\tau)}|R(\tau)|\|_{L^r_\mathbb{R}(\mathbb{R}^2)}d\tau. \]

Let us first bound \( J \) then \( I \).

Since \( R(y, s) \) and \( \nabla R(y, s) \) are clearly bounded by a polynomial in \( y \) by definition (3.4), we write from item (ii) of Lemma 7.1 and the bound (3.11) on \( R \):

\[
J \leq C \int_{s'}^{s} e^{(1+C^*)(s-\tau)}\|R(\tau)\|_{L^r_\mathbb{R}(\mathbb{R}^2)}d\tau \\
\leq C \int_{s'}^{s} e^{(1+C^*)(s-\tau)}e^{-2\tau}d\tau \leq e^{(1+C^*)(s-s')}e^{-2s'}d\tau.
\]

Since \( s' \geq s - s^* \) from (7.4), it follows that \( s - s' \leq s^* \) and \( e^{-2s'} \leq e^{2s^*-2s} \), hence

\[ J \leq C s^* e^{(3+C^*)s^*} e^{-2s}. \tag{7.6} \]
In order to bound $I$, we consider 2 cases:
- If $s - s^* \geq s_0$, then $s' = s - s^*$ and $s - s' = s^*$. Using item (i) in Lemma 4.1, we write
  \[ I \leq C(r)\|q(s - s^*)\|_{L^2_{\rho}}. \]
Since $q(s - s^*) \in V_A(s - s^*)$ by hypothesis, it follows from (3.21) that
\[ \|q(s - s^*)\|_{L^2_{\rho}} \leq CA(s - s^*)e^{-2(s - s^*)} \leq Ce^{2s^*}Ase^{-2s}. \]
Therefore, we conclude that
\[ I \leq C(r)As e^{-2s}. \] (7.7)
- If $s - s^* < s_0$, then $s' = s_0$. This time, from hypothesis (4.1), we see that we can apply item (ii) of Lemma 4.1 to write
\[ I \leq Ce^{(1+C^*)(s-s_0)}\|q(s_0)\|_{L^2_{\rho}} \leq C(r)e^{(1+C^*)s^*}Ase^{-2s_0}. \]
Since $s_0 \leq s < s_0 + s^*$, it follows that
\[ I \leq C(r)As e^{-2s}. \] (7.8)
Combining (7.5), (7.6), (7.7) and (7.8) concludes the proof of Proposition 4.3. \hfill \square

7.4 Position of the flow of (3.3) on the boundary of $V_A(s)$

We prove Proposition 4.3 in this section.

Proof of Proposition 4.3. Consider $q(s) \in V_A(s)$ a solution of equation (3.3) on some time interval $[s_0, s_1]$ for some $A > 0$, where $s_0 \geq s_{11}(A)$ is large enough so that both Proposition 4.2 and Lemma 4.1 apply. Assume that the initial condition (4.1) holds. We only prove items (i) and (iv), since items (ii) and (iii) follow in the same way.

(i) Consider $i \leq 4$, with $i$ and $j$ both even, and assume that
\[ q_{i,j}(s_1) = \theta Ae^{-2s_1}, \]
for some $\theta = \pm 1$. Since Lemma 4.1 implies that $|R_{i,j}(s)| \leq Ce^{-2s}$, by definition (3.18), we write from item (i) of Proposition 4.2
\[ \theta q_{i,j}'(s_1) \geq (1 - \frac{i}{2})Ae^{-2s_1} - C_iAs_1e^{-3s_1} - Ce^{-2s_1} \geq \frac{1 - i}{2}Ae^{-2s_1}, \] (7.9)
on the one hand, taking $A$ large enough, then $s_0$ large enough. On the other hand, we have
\[ \frac{d}{ds} Ae^{-2s}\big|_{s=s_1} = -2Ae^{-2s_1}. \] (7.10)
Since $\frac{1 - i}{2} \geq \frac{1 - 4}{2} = -\frac{3}{2} \geq -2$, the conclusion follows.

(iv) Let us assume that $\|q_-(s_1)\|_{L^2_{\rho}} = A^2s^2_1e^{-3s_1}$. Using Lemma 4.1, we see that $\|R_-(s)\|_{L^2_{\rho}} \leq Ce^{-3s}$, by definition (3.17). Using item (ii) in Proposition 4.2, it follows that
\[ \frac{d}{ds} \|q_-(s_1)\|_{L^2_{\rho}} \leq 3A^2s^2_1e^{-3s_1} + CAs_1e^{-3s_1} + Ce^{-3s_1} \leq 3A^2s^2_1e^{-3s_1} + A^2s_1e^{-3s_1}, \] (7.11)
on the one hand, taking $A$ large enough, then $s_0$ large enough. On the other hand, we compute
\[ \frac{d}{ds} A^2s^2e^{-3s}\big|_{s=s_1} = A^2e^{-3s_1}(2s_1 - 3s^2_1), \] (7.12)
and the conclusion follows. This concludes the proof of Proposition 4.3. \hfill \square
7.5 Details for the initialization

We prove Proposition 4.6 here.

Proof of Proposition 4.6. We will be using the notation $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0})$ for simplicity. Let us consider $A \geq 1$ and $s_0 \geq 1$. The first item (i) will be proved for any $d \in [-2, 2]^5$. The set $D$ will be introduced while proving item (ii). Since the set $D$ we intend to construct will be in $[-2, 2]^5$, there will be no need to revisit the proof of item (i) afterwards. Note that all the expansions given below are valid in $L^r_p$ for any $r \geq 2$.

(i) By Definition 4.2 of $S(y)$ and $S'(y)$ and Definition 2.1 of $D$, together with (2.13) and (2.15), we write

\[
\frac{A e^{-2s_0} |S(y)|}{D} \leq \frac{C A e^{-2s_0} (1 + |y|^4)}{D} \leq C A e^{-\frac{2}{3} s_0},
\]

\[
\frac{A s_0^2 e^{-3s_0} |S(y)|}{D} \leq C A s_0^2 e^{-3s_0} \frac{1 + |y|^6}{e^{-2s_0} + e^{-2s_0}|y|^6} \leq C A s_0^2 e^{-s_0}. \tag{7.13}
\]

Using item (i) of Lemma 2.2, the conclusion follows for $s_0$ large enough.

(ii) Since $D \geq p - 1 > 0$ by definition 2.1 of $D$, using the expressions 4.3 and 2.2 of $w_0(y, s_0)$ and $\varphi$, together with item (ii) of Lemma 2.2 and (7.13), we write

\[
|w_0(y, s_0)|^{p-1} \leq |\varphi(y, s_0)|^{p-1} + \frac{p - 1}{\kappa D} (A e^{-2s_0} S(y) + A s_0^2 e^{-3s_0} S'(y)) \leq \frac{1}{p - 1} + C e^{-\frac{2}{3} s_0}, \tag{7.14}
\]

for $s_0$ large enough, and the bound on $\|w_0(s_0)\|_{L^\infty}$ follows.

Taking the logarithm then the gradient of $w_0(y, s_0)$ (4.3), we write

\[
|\nabla w_0(y, s_0)| \leq \frac{|w_0(y, s_0)|}{p - 1} \left[ \frac{|\nabla E|}{E} + \frac{|\nabla D|}{D} \right] \tag{7.15}
\]

where

\[
E = E + \frac{p - 1}{\kappa D} (A e^{-2s_0} S(y) + A s_0^2 e^{-3s_0} S'(y)).
\]

Using (7.13), (2.3), (2.9) and (2.11), we write

\[
E \geq \frac{E_0}{C} \text{ where } E_0 = 1 + e^{-s_0}(y_1^2 + y_2^2) + e^{-2s_0}(y_1^4 + y_2^4). \tag{7.16}
\]

Then, we write

\[
\nabla \left[ \frac{S(y)}{D} \right] = \frac{\nabla S(y)}{D} - \frac{S(y) \nabla D}{D^2},
\]

hence, by definition 4.2 of $S(y)$, we have

\[
A e^{-2s_0} \left| \nabla \left[ \frac{S(y)}{D} \right] \right| \leq C A e^{-2s_0} \frac{(1 + |y|^3)}{D} + A e^{-2s_0} \frac{|S(y)|}{D} \leq C A e^{-\frac{2}{3} s_0},
\]

thanks to (7.13) and (2.23), together with the technique we used for (2.15). Similarly, we derive that

\[
A s_0^2 e^{-3s_0} \left| \nabla \left[ \frac{S(y)}{D} \right] \right| \leq A s_0^2 e^{-\frac{5}{3} s_0}.
\]

Therefore, using (7.15), (7.16) together with (2.22), we write

\[
\frac{|\nabla E|}{E} \leq C \frac{|\nabla E|}{E_0} + C A e^{-\frac{2}{3} s_0} + C A s_0^2 e^{-\frac{5}{3} s_0} \leq C e^{-\frac{2}{3} s_0}. \tag{7.17}
\]
We claim that it is enough to prove that
\[ |\nabla w_0(y, s_0)| \leq Ce^{-\frac{s_0}{p}}. \]

Arguing as for (2.40), we show the following:
\[
w_0(y, s_0) = \kappa + e^{-s_0} \left( \frac{\kappa}{p - 1} P(y) - \frac{\kappa(2 - p)}{2(p - 1)} y_1^2 y_2^2 \right) + e^{-2s_0} \left( \frac{\kappa}{p - 1} Q(y) + AS(y) + \frac{\kappa(2 - p)}{2(p - 1)} P(y)^2 - \frac{P(y)}{p - 1} y_1^2 y_2^2 - \delta y_1^6 - \delta y_2^6 + \frac{p}{2\kappa} y_1^4 y_2^4 \right) + A_0^2 e^{-3s_0} \bar{S}(y) + O(Ae^{-3s_0}),
\]
uniformly for \( d \in [-2, 2]^5 \). Using again the expansion (2.6) of \( \varphi \) together with definition (3.18) of (7.17), we obtain the following bound
\[ \|q_-(s_0)\|_{L^2} = O(Ae^{-3s_0}) \]
and
\[ q_{i,j}(s_0) = O(Ae^{-3s_0}) \]
whenever \((i, j)\) and \((i, i - j)\) are not in \( I_0 \cup \{(6, 0)\} \).

Recalling estimate (7.14) and the definition (4.1) of \( q(y, s_0) \), we see that this clearly gives the existence of \( D \subset [-2, 2]^5 \) such that \( q(s_0) \in V_A(s_0) \), (3.11) holds and \( |q_{i,2}(s_0)| + \|q_-(s_0)\|_{L^2} \leq CAe^{-3s_0} \), whenever \( d \in D \), with the function in (4.5) one-to-one, provided that \( s_0 \) is large enough. This concludes the proof of Proposition 4.6.

7.6 Gradient estimate in the shrinking set

This section is devoted to the proof of Proposition 5.2 thanks to the Liouville theorem recalled in Proposition 5.1.

**Proof of Proposition 7.6** Consider \( A \geq 1 \) and \( \delta_0 > 0 \). Proceeding by contradiction, we may exhibit a sequence \( q_n \) of solutions to equation (3.3) defined for all \( (y, s) \in \mathbb{R}^2 \times [s_{0,n}, s_{1,n}] \) for some \( s_{1,n} \geq s_{0,n} \geq n \) such that \( q_n(s_0, n) \) is given by (4.1) for some parameter \((d_{0,0,n}, d_{2,0,n}, d_{4,0,n}, d_{4,2,n}, d_{6,0,n}) \in \Delta_n \) where \( \Delta_n = \Delta(A, s_{0,n}) \) is defined in Proposition (4.6), with \( q_n(s) \in V_A(s) \) for all \( s \in [s_{0,n}, s_{1,n}] \), and
\[
\|\nabla q_n(s_{2,n}) + \nabla \varphi(s_{2,n})\|_{L^\infty} > \delta_0 \text{ for some } s_{2,n} \in [s_{0,n}, s_{1,n}].
\]
Note that
\[ s_{2,n} \to \infty \text{ as } n \to \infty. \]

We claim that it is enough to prove that
\[ s_{2,n} - s_{0,n} \to \infty \text{ as } n \to \infty \]
in order to conclude. Indeed, if (7.21) holds, then, introducing
\[ w_n(y, s) = q_n(y, s + s_{2,n}) + \varphi(y, s + s_{2,n}), \]  
we see from (7.22) that \( w_n \) is a solution of equation (1.12) defined for all \((y, s) \in [s_{0,n} - s_{2,n}, 0] \). Since \( q_n(s) \in V_A(s) \) for all \( s \in [s_{0,n}, s_{2,n}] \), it follows by Definition 3.1 that \( \| q_n(s) + \varphi(s) \|_{L^\infty} \leq 2\kappa \). Using this together with (7.19) we see that for \( n \) large enough, we have:
\[ \forall s \in [s_{0,n} - s_{2,n}, 0], \| w_n(s) \|_{L^\infty} \leq 2\kappa, \quad (7.23) \]
\[ \| \nabla w_n(0) \|_{L^\infty} \geq \delta_0. \quad (7.24) \]
Applying a classical parabolic regularity technique to equation (1.12), we obtain the following estimate:

**Lemma 7.2** (Parabolic regularity for equation (1.12)). For all \( n \in \mathbb{N} \), it holds that
\[ \| w_n \|_{C^{2,1,\alpha}(\mathbb{R}^2 \times [s_{0,n}-s_{2,n}+1, 0])} \leq C_0 \quad (7.25) \]
for some \( C_0 > 0 \), where \( C^{2,1,\alpha} \) stands for the set of functions of space and time, with 2 space derivatives and 1 time derivative which are \( \alpha \)-Hölder continuous in both variables.

**Proof.** Since this estimate is classical, we leave its justification to Section A in the Appendix.

Recalling that \( s_{0,n} - s_{2,n} \to -\infty \) from (7.21), we may use the compactness provided by this lemma, combined to a diagonal process, in order to extract a subsequence (still denoted by \( w_n \)) such that \( w_n \to w \) in \( C^{2,1} \) of any compact subset of \( \mathbb{R}^2 \times (-\infty, 0] \). Since \( w_n \) is a solution of equation (1.12), the same holds for \( w \). From Properties (7.23) and (7.24), it follows that
\[ \forall s \leq 0, \| w(s) \|_{L^\infty} \leq 2\kappa, \]
\[ \| \nabla w(0) \|_{L^\infty} \geq \delta_0. \]
Using Proposition 5.1 we see that this is a contradiction. Thus, it remains to prove (7.21) in order to conclude.

**Proof of (7.21):** Since \( w_n \) satisfies equation (1.12), differentiating this equation in space, we obtain the following vector-valued equation on \( \nabla w_n \), for all \((y, s) \in \mathbb{R}^2 \times [s_{0,n} - s_{2,n}, 0] \):
\[ \partial_s \nabla w_n = (\mathcal{L} - \frac{3}{2} - \frac{1}{p - 1}) \nabla w_n + p|w_n|^{p-1} \nabla w_n, \]
where the operator \( \mathcal{L} \) is defined in (3.4). Using a Duhamel formulation based on the kernel (7.2), together with the \( L^\infty \) bound in (7.23), we see that for all \( s \in [s_{0,n} - s_{2,n}, 0] \):
\[ \| \nabla w_n(s) \|_{L^\infty} \leq e^{-\frac{1}{2} + \frac{1}{p - 1}} \| \nabla w_n(s_{0,n} - s_{2,n}) \|_{L^\infty} \]
\[ + p(2\kappa)^{p-1} \int_{s_{0,n} - s_{2,n}}^0 \| \nabla w_n(s') \|_{L^\infty} ds'. \]
Since $\kappa^{p-1} = \frac{1}{p-1}$ by definition (1.5) using Gronwall’s lemma, we see that
\begin{equation}
\|\nabla w_n(0)\|_{L^\infty} \leq e^{(\kappa^{p-1}-\frac{1}{2})(s_2, n-s_0, n)}\|\nabla w_n(s_0, n - s_2, n)\|_{L^\infty},
\end{equation}
on the one hand. On the other hand, recalling that by hypothesis, $q_n(s_0, n)$ is given by (4.4) for some parameters $(d_0, 0, n, d_0, 2, n, d_4, 2, n, d_6, 0, n) \in \mathcal{D}_n$ where $\mathcal{D}_n = \mathcal{D}(A, s_0, n)$ is defined in Proposition (4.6), we see from that proposition and definition (7.22) of $w_n$
that
\begin{equation}
\|\nabla w_n(s_0, n - s_2, n)\|_{L^\infty} = \|\nabla q_n(s_0, n) + \nabla \varphi(s_0, n)\|_{L^\infty} \leq Ce^{-\frac{s_n}{6}}.
\end{equation}
Using this together with (7.22) and (7.26), we write
\begin{equation}
0 < \delta_0 \leq \|\nabla w_n(0)\|_{L^\infty} \leq e^{(\kappa^{p-1}-\frac{1}{2})(s_2, n-s_0, n)}\|\nabla w_n(s_0, n - s_2, n)\|_{L^\infty}
\leq Ce^{(\kappa^{p-1}-\frac{1}{2})(s_2, n-s_0, n)} e^{-\frac{s_0}{6}}.
\end{equation}
Since $s_0, n \geq n$ and \( \frac{\kappa^{p-1}-\frac{1}{2}}{p-1} = \frac{3}{2} > \frac{1}{2} > 0 \), it follows that $s_2, n - s_0, n \to \infty$, and (7.21) holds. Since we have already showed that (7.21) yields a contradiction, this concludes the proof of Proposition 5.2

7.7 Size of the solution in the 3 regions

In this section, we prove Lemma 5.5.

Proof of Lemma 5.5. Consider $A \geq 1, s_0 \geq s_13(A)$ and $d = (d_0, 0, d_0, 2, d_4, 2, d_6, 0) \in \mathcal{D}(A, s_0)$, where $s_13(A)$ and $\mathcal{D}(A, s_0)$ are defined in Proposition 4.6. Consider then $a \in \mathbb{R}^2$ and introduce $y = ae^{\frac{a}{\kappa}}$. By definition (1.3) and (2.2) of initial data $w_0(y, s_0)$ and the profile $\varphi$, arguing as for (2.12), we may improve that estimate and write
\begin{equation}
\left|w_0(y, s_0)^{p-1} - \frac{1}{D}\right| \leq C \left\{ e^{-s_0} + \frac{N_2 + N_3}{D} + \frac{Ae^{-2s_0}S(y)}{D^2} + \frac{As^2_0e^{-3s_0}S(y)}{D^2} \right\} \leq Ce^{-\frac{s_0}{6}},
\end{equation}
where $D$, $N_2$, $N_3$, $S(y)$ and $\bar{S}(y)$ are defined in (2.4), (2.13) and (4.2), and where we have used the bounds (2.14), (2.18) and (4.13). By definitions (2.4) and (5.10) of $D$ and $G_0(a)$, we see that
\begin{equation}
D = p - 1 + \frac{(p-1)^2}{\kappa} \left( e^{-s_0}y_1^2y_2^2 + \delta e^{-2s_0}(y_1^6 + y_2^6) \right) = (p-1)[1 + e^{s_0}G_0(a)].
\end{equation}
Consider now 2 nonnegative numbers $m$ and $M$ such that $0 < m \leq 1 \leq M$. If $a \in \mathcal{R}_1$ (resp. $\mathcal{R}_2$, resp. $\mathcal{R}_3$) defined in (5.9), we see by definition (5.9) that $(p-1)(1+M) \leq D$ (resp. $(p-1)(1+m) \leq D \leq (p-1)(1+M)$, resp. $D \leq (p-1)(1+m)$). Since $0 \leq w_0(y, s_0) \leq \kappa + Ce^{-\frac{s_0}{6}}$ by Proposition 4.6 combining this with (7.27) concludes the proof of Lemma 5.5.

7.8 Details for the control of $w_a(y, s)$ for $a$ in Region $\mathcal{R}_3$

In this section, we prove Lemmas 5.10, 5.11 and 5.13.
Proof of Lemma 5.10. Consider $A \geq 1$ and $s_0 \geq s_{13}(A)$, together with the parameter $d = (d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}(A, s_0)$, where $s_{13}$ and $\mathcal{D}$ are defined in Proposition 4.6. Consider also $w_0(y, s_0)$ defined in (4.3). Recalling that $\mathcal{D} \subset [-2, 2]^3$, we may write the following Taylor expansion:

$$w_0(y, s) - \kappa \left[ 1 - \frac{X}{p-1} - \frac{e^{-s_0}P}{p-1} \right] \leq C \left\{ I + J^2 + \mathbb{1}_{1 < p < \frac{3}{2}} \frac{1}{p-1} \right\}$$

(7.28)

where

$$X(y_1, y_2, s_0) = \frac{p-1}{\kappa} \left[ e^{-s_0}y^2_1 y^2_2 + \delta e^{-2s_0}(y^6_1 + y^6_2) \right],$$

$$I = X e^{-s_0} |P| + e^{-2s_0} |Q| \left( (1 + X) + X^2 (1 + e^{-s_0} |P| + e^{-2s_0} |Q|) \right) + A e^{-2s_0} (1 + |y|^4) + A \delta e^{-3s_0} (1 + |y|^6),$$

$$J = X + e^{-s_0} |P| + I,$$

the polynomials $P$ and $Q$ are given in (2.7) and (2.8) and have respectively 2 and 4 as degree (see (2.9) and (2.10)), and the constant $\delta \geq 1$ was already fixed large enough at the beginning of Section 3.

Consider now some $m \in (0, 1)$ and $a \in \mathcal{R}_3$ defined in (5.14), with $a$ decomposed as in (5.13), for some $L \geq K \geq 0$, with $L + K \geq A$. Given some $r \geq 2$, we may use the relation (5.10) together with (7.28) to derive an expansion for $w_a(y, s_0)$, showing error terms bounded by small terms in scales of $1/(L + K)$ and $e^{-s_0}$. In particular, the following expansions are useful:

$$P(y_1 + K, y_2 + L) = \frac{2(p-1)}{\kappa} \left[ K^2 + L^2 + 2 + 2K h_1(y_1) + 2L h_2(y_2) + 4h_1(y_2) + h_2(y_1) + h_2(y_2) \right],$$

$$X(y_1 + K, y_2 + L, s_0) = e^{-s_0} \left[ y^2_1 y^2_2 + 2L y^2_1 y^2_2 + 2K y_1 y^2_2 + L^2 y^2_1 + 2K L y_1 y_2 + K^2 y^2_2 + 2K L^2 y_1 + 2K^2 L y_2 + K^2 L^2 \right] + \delta e^{-2s_0} (K^6 + L^6) + O((K^5 + L^5) e^{-2s_0})$$

in $L^p_a(\mathbb{R}^2)$. This latter estimate can be easily written in the Hermite polynomials basis (1.10). Since by definition (5.17) of $\iota$ and (5.18), it follows that

$$\iota \leq \frac{mK}{p-1} \leq \frac{\kappa}{p-1}, \quad K + L \leq 2 \iota \frac{\eta^*}{p} \quad \text{and} \quad KL \leq \sqrt{\iota} \frac{\eta^*}{p},$$

one can easily bound all the error terms by $O\left( \frac{1}{\bar{x}} \right)$ and $O\left( \iota^2 \right)$, as required by the statement of the lemma. This concludes the proof of Lemma 5.10. \qed

Proof of Lemma 5.11. Take $A \geq 1$ and $s_0 \geq \max\{s_{13}(A), s_{18}(A)\}$ where $s_{13}$ and $s_{18}$ are defined in Proposition 4.6 and Lemma 5.10. Consider then $(d_{0,0}, d_{2,0}, d_{4,0}, d_{4,2}, d_{6,0}) \in \mathcal{D}$ defined in Proposition 4.6 and initial data $w(y, s_0)$ defined in (4.3). Consider also some $\eta^* \leq \frac{m}{p}$, $0 < m < n(p-1)$ and $a \in \mathcal{R}_3$ (5.9) given by (5.14) for some $L \geq K \geq 0$ such that $L + K \geq A$. In this case, Proposition 4.6 applies, and so does Lemma 5.10. In particular, the expansion given there holds for $w_a(y, s_0)$.

(i) Note that condition (5.20) holds from the choice of $\eta^*$ and $m$. Therefore, using (5.18), we see that $\iota \leq \frac{mK}{p-1} \leq \eta^*$. By definition (5.21) of $s^*$, it follows that $s^* = s_0 + \log \frac{\eta^*}{\iota} \geq s_0$. 

47
(ii) Assume now that
\[ \forall s \in [s_0, s_1], \quad \|w_a(s)\|_{L^\infty} \leq 2\kappa, \] (7.29)
for some \( s_1 \geq s_0 \). Introducing
\[ v_a = w_a - \kappa \text{ and } \bar{s} = \min(s^*, s_1), \] (7.30)
we work in the following in the interval \([s_0, \bar{s}]\), and proceed in 3 steps in order to give the proof:

- In Step 1, we write an equation satisfied by \( v_a \) and project it on the various components \( v_{a,i,j} \) defined in (3.18).
- In Step 2, we integrate those equations.
- In Step 3, we collect the previous information to conclude the proof.

**Step 1: Dynamics for \( v_a \)**

Since \( w_a \) satisfies equation (1.12), by definition (7.30), it follows that \( v_a \) satisfies the following equation:
\[ \forall s \in [s_0, \bar{s}], \quad \partial_s v_a = \mathcal{L}v_a + \bar{B}(v_a), \] (7.31)
where the linear operator \( \mathcal{L} \) is introduced in (3.4) and
\[ \bar{B}(v_a) = |\kappa + v_a|^{p-1}(\kappa + v_a) - \kappa^p - \kappa p^{p-1}v_a. \] (7.32)

In this step, we project equation (7.31) in order to write differential inequalities satisfied by the various components \( v_{a,i,j} \) defined in (3.18) as well as \( \bar{P}(v_a) \) where \( \bar{P} \) is the \( L^2_\rho \) orthogonal projector on
\[ \bar{E} = \text{span}\{h_i h_j \mid (i,j) \notin \{(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (1,2), (2,1), (2,2)\} \}, \] (7.33)
the orthogonal supplement of the directions appearing in the expansion of Lemma 5.10.

This is our statement:

**Lemma 7.3** (Projections of equation (7.31)). Under the hypotheses of Lemma 5.11, for all \( s \in [s_0, \bar{s}] \), for all \( i \in \mathbb{N} \) and \( j = 0, \ldots, i \), it holds that
\[ \left| v_{a,i,j}'(s) - \left(1 - \frac{i}{2}\right) v_{a,i,j}(s) \right| \leq C(i)\|v_a(s)\|_{L^2_\rho}^2. \] (7.34)

In addition,
\[ \frac{d}{ds}\|\bar{P}(v_a(s))\|_{L^2_\rho} \leq -\frac{1}{2}\|\bar{P}(v_a(s))\|_{L^2_\rho} \] (7.35)
\[ + \sum_{\{s_0 \leq s \leq s_0 + 2\}} C_{28} \|v_a(s_0)\|_{L^2_\rho}^2 + \sum_{\{s_0 + 2 \leq s \leq \bar{s}\}} C_{28} \|v_a(s - 2)\|_{L^2_\rho}^2, \]
for some universal constant \( C_{28} \), where \( \bar{s} \) is defined in (7.30) and the projector \( \bar{P} \) is defined right before (7.33).

**Proof.** Note first from (7.29), (7.30) and (7.32) that
\[ \forall s \in [s_0, \bar{s}], \quad |\bar{P}(v_a)| \leq C|v_a|^2. \] (7.36)
This way, the proof follows as for Proposition 4.2 proved in Section 7.2. More precisely, identity (7.34) follows from equation (7.31) and the quadratic estimate (7.36) exactly as for item (i) of that proposition. As for (7.35), arguing as for item (ii) of the same proposition, we write

\[
\frac{d}{ds} \| \bar{P}(v_a(s)) \|_{L^2} \leq -\frac{1}{2} \| \bar{P}(v_a(s)) \|_{L^2} + C \| v_a(s) \|^2_{L^2},
\]

since \( \lambda = -\frac{1}{2} \) is the largest eigenvalue of \( E \) corresponding to the components spanning \( \bar{E} \), the image of the projector \( \bar{P} \) (see (3.7)). Note that \( \lambda = -\frac{1}{2} \) corresponds to the eigenfunctions \( h_3h_0 \) and \( h_0h_3 \).

The question then reduces to the control of \( \| v_a(s) \|^2_{L^2} = \| v_a(s) \|^2_{L^2} \). Arguing as we did in Section 7.3 for the proof of Proposition 4.3 we can apply here items (i) and (ii) of Lemma 7.4 (with a delay time equal to 2). Indeed, \( v_a(s), \ v_a(0) \) and \( \nabla v_a(s) \) are in \( L^\infty(\mathbb{R}^2) \), thanks to (7.29) and Proposition 4.6 through the transformations (7.30) and (5.16). This concludes the proof of Lemma 7.3.

\[ \square \]

**Step 2: Integration of the equations of Lemma 7.3**

We claim the following:

**Lemma 7.4** (Integration of equation (7.31)). There exist \( M_{29}, \ A_{29} \geq 1 \) and \( \eta_{29} > 0 \) such that under the hypotheses of Lemma 5.11, if \( A \geq A_{29} \) and \( \eta^* \leq \eta_{29} \), then for all \( s \in [s_0, \bar{s}] \), it holds that

\[
\forall i = 0, \ldots, 4 \text{ and } j = 0, \ldots, i, \quad \| \bar{P}(v_{a,i,j}(s)) \|_{L^2} \leq M_{29}(\eta^* + A^{-2})e^\tau \;, \\
\| \bar{P}(v_a(s)) \|_{L^2} \leq e^{-\frac{\tau}{2}} \| \bar{P}(v_a(0)) \|_{L^2} + M_{29}(\eta^* + A^{-2})e^\tau \;,
\]

provided that \( A \) and \( s_0 \) are large enough and \( \eta^* \) is small enough, where \( \bar{s}, \ \tau \) and \( P \) are defined in (7.30), (5.17) and right before (7.33), with \( \tau = s - s_0 \).

**Proof.** This integration is at the heart of our argument, as it was already the case in our previous work [22] dedicated to the classification of the possible behaviors near non isolated blow-up points for equation (1.1) (see in particular Section 4.4 in that paper). Looking in that paper is certainly convincing for the expert reader. To be nice to all readers, we summarize the integration argument below.

Consider \( M_{29} > 0 \) to be fixed later. For \( s_0 \) large enough, we note that both identities in Lemma 7.4 are true at \( s = s_0 \). Therefore, we may proceed by contradiction and assume that one of these identities doesn’t hold at some time in the interval \( [s_0, \bar{s}] \) where \( \bar{s} \) is defined in (7.30). If \( s_{29} \) is the infimum of such times, then we see from continuity that both identities hold on the interval \( [s_0, s_{29}] \) and that one of them has an equality case at \( s = s_{29} \). In the following, we will prove that no equality case occurs, yielding a contradiction.

Since both identities hold for all \( s \in [s_0, s_{29}] \), using the estimates at initial time \( s = s_0 \) given in Lemma 5.10 and using the definition (7.30) of \( v_a \), we derive the following bounds for \( s_0 \) large enough and for all \( s \in [s_0, s_{29}] \) and \( r \in \{2, 4\} \):

\[
\| v_a(s_0) \|_{L^2} \leq M_{29}(\tau + J) \text{ where } J = e^{-s_0}(K^2 + L^2), \\
\| v_a(s) \|_{L^2} \leq M_{29}(e^\tau \tau + J) + M_{29}M_{29}(\eta^* + A^{-2})e^\tau \;,
\]

49
for some universal constant $M'_{29} > 0$, where $\tau = s - s_0$ hereafter. Taking $\eta^* > 0$ small enough and $A$ large enough, we write for all $s \in [s_0, s_29]$, 

$$\|v_a(s)\|_{L^2} \leq 2M'_{29}(e^{r\tau} + J).$$

(7.38)

Using Lemma 7.3 together with (7.38), we write for $s_0$ large enough, for all $i = 0, \ldots, 4, j = 0, \ldots, i$ and $s \in [s_0, s_29]$, 

$$\left|v'_{a,i,j}(s) - \left(1 - \frac{i}{2}\right)v_{a,i,j}(s)\right| \leq C_{29}M'_{29}^2(e^{r\tau} + J^2),$$

$$\frac{d}{ds}\|P(v_a(s))\|_{L^2} \leq -\frac{1}{2}\|P(v_a(s))\|_{L^2} + C_{29}M'_{29}^2(e^{r\tau} + J^2),$$

for some universal constant $C_{29} > 0$.

Integrating the first equation, we see that for all $s \in [s_0, s_29]$, 

$$|v_{a,i,j}(s) - e^{(1 - \frac{i}{2})r}v_{a,i,j}(s_0)| \leq C_{29}M'_{29}^2(e^{r\tau} + 2e^\tau J^2).$$

(7.39)

Integrating the second inequality, we see that for all $s \in [s_0, s_29]$, 

$$\|P(v_a(s))\|_{L^2} \leq e^{-\frac{1}{2}}\|P(v_a(s_0))\|_{L^2} + C_{29}M'_{29}^2\left(\frac{2}{5}e^{2\tau} + 2J^2\right).$$

(7.40)

Recalling that $\tau = s - s_0 \leq s_29 - s_0 \leq \bar{s} - s_0 \leq s^* - s_0$, we see by definition (5.21) that 

$$e^\tau \leq \eta^*.$$ 

(7.41)

Moreover, recalling that $K + L \geq A$ and $\delta \geq 1$ (see the beginning of Section 3), we write by definitions (7.37) and (5.17) of $J$ and $\iota$: 

$$J^2 \leq e^{-2s_0}\frac{(K + L)^2}{A^2}(K^2 + L^2)^2 \leq e^{-2s_0}\frac{16}{A^2}(K^6 + L^6) \leq \frac{16\tau}{\delta A^2} \leq \frac{16\tau}{A^2}.$$ 

Using this together with (7.41), (7.39) and (7.40), we see that for $s_0$ large enough, we have for all $s \in [s_0, s_29]$, 

$$\forall i = 0, \ldots, 4 \text{ and } j = 0, \ldots, i, \quad |v_{a,i,j}(s) - e^{(1 - \frac{i}{2})r}v_{a,i,j}(s_0)| \leq 32C_{29}M'_{29}^2(\eta^* + A^{-2})e^{r\tau},$$

$$\|P(v_a(s))\|_{L^2} \leq e^{-\frac{1}{2}}\|P(v_a(s_0))\|_{L^2} + 32C_{29}M'_{29}^2(\eta^* + A^{-2})e^{r\tau}.$$ 

Fixing 

$$M_{29} = 33C_{29}M'_{29}^2,$$

we see that no equality case occurs in both identities shown in Lemma 7.4. A contradiction follows from the beginning of the proof. This concludes the proof of Lemma 7.4.  

\[\square\]

**Step 3: Conclusion of the proof of Lemma 5.11**

Recalling the transformation (7.30), then using Lemma 7.4 together with Lemma 5.10 and the Cauchy-Schwarz inequality, we see that for all $s \in [s_0, \bar{s}]$, 

$$\|w_a(\cdot, s) - (\kappa - e^{r\tau})\|_{L^2} \leq M''_{29}\left\{M_{29}(\eta^* + A^{-2})e^{r\tau} + e^{r\tau}\left(\frac{L}{A} + \tau^2\right) + (L^2 + K^2)e^{-s_0}\right\},$$

50
for some universal constant $M_{29}^\prime > 0$. Using the bounds (5.19) and (5.18), then recalling that $\delta \geq 1$ from the beginning of Section 3 we see that

$$e^{-s_0}(L^2 + K^2) \leq 2e^{-\frac{4\omega}{p-1}} \left( \frac{\kappa}{p-1} \right)^{\frac{1}{7}}.$$  

This concludes the proof of Lemma 5.11.



Proof of Lemma 5.13. Since $w_0(y, \sigma) = \varphi(y, \sigma) + q(y, \sigma)$ by the relation (3.2), proceeding as in (5.16), we may introduce

$$\varphi_a(y, \sigma) = \varphi(y + ae^\frac{2}{7}, \sigma) = \varphi(y_1 + K', y_2 + L', \sigma),$$

$$q_a(y, \sigma) = q(y + ae^\frac{2}{7}, \sigma) = q(y_1 + K', y_2 + L', \sigma)$$ (7.42)

(use (5.21)). This way, we write $w_a(y, \sigma) = \varphi_a(y, \sigma) + q_a(y, \sigma)$, and the proof of (5.26) follows by adding the expansions of $\varphi_a(y, \sigma)$ and $q_a(y, \sigma)$, performed in 2 steps.

**Step 1: The expansion of $\varphi_a(y, \sigma)$**

We claim that the expansion of $\varphi_a(y, \sigma)$ follows from Lemma 5.10. Indeed, the input in that lemma is initial data $w_0(y, s_0)$ (4.3), and if one takes the parameter $(d_{0,0}, d_{2,0}, d_{4,0}, d_{1,2}, d_{6,0}) = (0, 0, 0, 0, 0)$ and formally replaces $s_0$ by $\sigma$ in the definition (4.3) of initial data $w_0(y, s_0)$, then, we recover $\varphi(y, \sigma)$ defined in (2.2). In addition, the point $a$ we consider is given by (5.24) with $K' + L' = A$, which falls in the framework considered in Section 5.5.1. Therefore, Lemma 5.10 applies and we see that

$$\varphi_a(y, \sigma) = \kappa - c' - e^{-\sigma} \left\{ 2K'L'^2 h_1 h_0 + 2K'^2 L'h_1 h_2 + L'^2 h_2 h_0 + 4K'L'h_1 h_1 + K'^2 h_0 h_2 
\quad + 2L'h_2 h_1 + 2K' h_1 h_2 + h_2 h_2 \right\} + O \left( \frac{c'}{A} \right) + O(c'^2),$$ (7.43)

in $L^r_q$ for any $r \geq 2$, where $c'$ is given in (5.27).

**Step 2: The expansion of $q_a(y, \sigma)$**

Take $r \geq 2$. Using the decomposition (3.17) for $q(y, \sigma)$, we write from (7.42):

$$q_a(y, \sigma) = \tilde{q}_a(y, \sigma) + \check{q}_a(y, \sigma)$$ (7.44)

where

$$\tilde{q}_a(y, \sigma) = \sum_{i=0}^{7} \sum_{j=0}^{i} q_{i,j}(y_1 + K') h_{i-j}(y_2 + L'), \quad \check{q}_a(y, \sigma) = q(y_1 + K', y_2 + L', \sigma).$$ (7.45)

Concerning $\tilde{q}_a(y, \sigma)$, recalling that $q(\sigma) \in V_A(\sigma)$ defined in Definition 3.1, then proceeding as for the proof of Lemma 5.10 given at the beginning of this subsection, we derive that

$$\tilde{q}_a(y, \sigma) = q_{0,2}(\sigma) \left\{ L'^4 h_0 h_0 + K'^4 h_0 h_2 + 4L'^3 h_2 h_1 + 4K'^3 h_1 h_2 + 6(K'^2 + L'^2) h_2 h_2 
\quad + 4K' h_3 h_2 + 4L'h_2 h_3 + h_4 h_2 + h_2 h_4 \right\} + \frac{c'}{A},$$ (7.46)

in $L^r_q$. As for $\check{q}_a(y, \sigma)$, we can bound it thanks to the following parabolic regularity estimate on $q_-(\sigma)$:
Lemma 7.5 (Parabolic regularity for \( q_-(\sigma) \)). Under the hypotheses of Lemma 5.13, it holds that for all \( r' \geq 2 \),

\[
\forall s \in [s_0, \sigma], \quad \|q_-(s)\|_{L_{r'}^\rho} \leq C(r')A^2s^2e^{-3s}.
\]

Indeed, using (7.45), we write

\[
\int |q_\alpha(y, \sigma)|^r \rho(y) dy = \int |q_-(z, \sigma)|^r \rho(z_1 - K', z_2 - L') dz.
\] (7.47)

Then, by definition (1.18), we write

\[
\rho(z_1 - K', z_2 - L') = \rho(z) e^{K'z_1 + L'z_2} e^{K'z_1 + L'/2}.
\]

Using the Cauchy-Schwarz inequality together with Lemma 7.5, we write

\[
\int |q_-(z, \sigma)|^r \rho(z) dz 
\leq e^{K'/2 + L'/2} \left( \int |q_-(z, \sigma)|^{2r} \rho(z) dz \right)^{\frac{1}{2}} \left( \int e^{K'z_1 + L'z_2} \rho(z) dz \right)^{\frac{1}{2}} 
\leq e^{\frac{r}{4}(K'z_1 + L'z_2)} \left( \int |q_-(z, \sigma)|^{2r} \rho(z) dz \right)^{\frac{1}{2}} \leq Ce^{\frac{r}{4}(K'z_1 + L'z_2)}(A^2\sigma^2 e^{-3\sigma})^{r}.
\] (7.48)

Using (5.25) and recalling the definition (5.27) of \( \iota' \), then taking \( s_0 \) large enough (remember that \( \sigma \geq s_0 \)), we see that

\[
e^{\frac{r}{4}(K'z_1 + L'z_2)}A^2\sigma^2 e^{-3\sigma} \leq e^{-2\sigma} \frac{A^6}{A} = e^{-2\sigma} \frac{(K' + L')^6}{A} \leq \frac{2^6}{A} \frac{e^{-2\sigma}(K'z_1 + L'z_2)}{A^\delta}.
\] (7.49)

Using (7.46), (7.47), (7.48) and (7.49), we obtain an expansion for \( q_\alpha(z, \sigma) \), by (7.44). Adding the expansion (7.43), we obtain the desired expansion (5.26) in Lemma 5.13. It remains to justify Lemma 7.5.

Proof of Lemma 7.5. Assume here that \( s_0 \geq s_1(A) \) defined in Proposition 4.2 so that Proposition 4.3 applies. Consider then some \( r' \geq 2 \). Proceeding as for item (ii) of Proposition 4.2, we project equation (3.3) for all \( s \in [s_0, \sigma] \) as follows

\[
\partial_s q_- = \mathcal{L}q_- + G \text{ where } G = P_-(Vq + B + R)
\]

and \( P_- \) is the \( L^2 \rho \) projector on the subspace \( E_- \) (7.11). Given some \( \sigma_0 \in [s_0, \sigma] \), we may write a Duhamel formulation based on the kernel given in (7.2):

\[
q_-(\sigma) = e^{(\sigma-\sigma_0)\mathcal{L}} q(\sigma_0) + \int_{\sigma_0}^{\sigma} e^{(\sigma-\sigma')}\mathcal{L} G(\sigma') d\sigma'.
\]

Taking the \( L_{r'}^\rho \) norm, we write

\[
\|q_-(\sigma)\|_{L_{r'}^\rho} \leq I + II \equiv e^{(\sigma-\sigma_0)\mathcal{L}} q(\sigma_0)\|_{L_{r'}^\rho} + \int_{\sigma_0}^{\sigma} e^{(\sigma-\sigma')}\mathcal{L} G(\sigma')\|_{L_{r'}^\rho} d\sigma'.
\] (7.50)

We start by bounding \( II \). We claim that for any \( \sigma' \in [s_0, \sigma] \), \( G(\sigma') \) and its gradient have polynomial growth in \( y \), allowing the application of item (ii) of Lemma 7.4. Indeed,
by definition (3.1), together with the $L^\infty$ bound on $q(s)$ from Definition 3.1 and the gradient estimate of Proposition 5.2, we see that $Vq + B(q) + R$ and its gradient have polynomial growth in $y$. By definition of the $P_-$ operator (see (7.11) and (3.17)), so has $G$. Applying item (ii) of Lemma 7.1 and write by definition (7.50) of $I$ bounded (see the hypotheses of Lemma 5.13 and Definition 3.1 of $V_A(s)$), we see that

$$
|II| \leq \int_{\sigma_0}^{\tilde{\sigma}} \|G(\sigma')\|_{L'_0} d\sigma'.
$$

(7.51)

Since

$$
\|P_-(g)\|_{L'_0} \leq C(r')\|g\|_{L'_0}
$$

(see again (7.11) and (3.17)), we write

$$
\|G(\sigma')\|_{L'_0} \leq \|Vq(\sigma')\|_{L'_0} + \|B(q(\sigma'))\|_{L'_0} + \|R_-(\sigma')\|_{L'_0}.
$$

Using Proposition 4.3 and proceeding as for item (ii) of Proposition 4.2, we see that

$$
\|G(\sigma')\|_{L'_0} \leq C'(r')A\sigma'e^{-3\sigma'}.
$$

(7.52)

As for the term $I$, introducing $\sigma^*(r') > 0$ and $C^*(r') > 0$ such that the following delay regularizing effect holds for any $v \in L_0^2$ (see item (i) of Lemma 7.1):

$$
\|e^{\sigma^*(r')L}(v)\|_{L'_0} \leq C^*(r')\|v\|_{L_0^2},
$$

(7.53)

we distinguish two cases in the following:

**Case 1:** $\sigma \geq s_0 + \sigma^*$. Fixing $\sigma_0 = \sigma - \sigma^*$, we see that $\sigma_0 \geq s_0$. Using (7.53), we see by definition (7.50) of $I$ and Definition 3.1 of $V_A(s)$ that

$$
|I| \leq C^*(r')\|q_-(\sigma - \sigma^*)\|_{L_0^2} \leq C^*(r')A^2(\sigma - \sigma^*)^2e^{-3(\sigma - \sigma^*)}.
$$

Using (7.50), (7.51), and (7.52), we see that

$$
\|q_-(\sigma)\|_{L'_0} \leq C^*(r')A^2\sigma^2e^{-3(\sigma - \sigma^*)} + \sigma^*C(r')A(\sigma - \sigma^*)e^{-3(\sigma - \sigma^*)} \leq \tilde{C}(r')A^2\sigma^2e^{-3\sigma}
$$

and the conclusion of Lemma 7.5 follows.

**Case 2:** $s_0 \leq \sigma \leq s_0 + \sigma^*$. Fixing $\sigma_0 = s_0$, and noting that $q(s_0)$ and $\nabla q(s_0)$ are bounded (see the hypotheses of Lemma 5.13 and Definition 3.1 of $V_A(s_0)$), we can apply item (ii) of Lemma 7.1 and write by definition (7.50) of $I$ and Definition 3.1

$$
|I| \leq Ce^{\sigma-s_0}\|q_-(s_0)\|_{L_0^2} \leq Ce^{\sigma-s_0}A^2s_0^2e^{-3s_0} \leq Ce^\sigma A^2\sigma^2e^{-3(\sigma - \sigma^*)},
$$

and the conclusion follows as in Case 1. This concludes the proof of Lemma 7.5.

This concludes the proof of Lemma 5.13 too.

A classical parabolic regularity estimate for equation (1.12)

We prove Lemma 7.2 here. Since the argument was extensively used in our earlier papers, we won’t give details.

53
Proof of Lemma 7.2. We proceed in 2 steps: we first justify the estimate locally in space, then we extend it to the whole space.

Step 1: Proof of a local version, where $\mathbb{R}^2$ is replaced by $B(0,1)$, the unit ball of $\mathbb{R}^2$. When restricting to $B(0,1)$, one can use the similarity variables’ transformation to translate the problem into a regularity question for equation (1.1). Using the technique of Step 2 page 1060 of [20], which relies on Theorem 3 page 406 of Friedman [7], we get the result.

Step 2: Extension to the whole space $\mathbb{R}^2$. Introducing for any $a \in \mathbb{R}^2$,

$$w_{a,n}(y,s) = w_n(y + ae^2, s),$$

we see from the similarity variables transformation (1.4) that $w_{a,n}$ is also a solution of (1.12) defined for all $s \in [s_{0,n}, s_{2,n}]$ and satisfying the uniform bound (7.23). Applying the same local regularity technique on $w_{a,n}$ as in Step 1, we show that (7.25) holds also for $w_{a,n}$ with $\mathbb{R}^2$ replaced by $B(0,1)$, uniformly in $a \in \mathbb{R}^2$. Using (A.1) and varying $a$ in the whole space $\mathbb{R}^2$, we recover the full estimate (7.25) (on $\mathbb{R}^2$) for $w_n$. This concludes the proof of Lemma 7.2.

\[\square\]

B Stability results for equation (1.12)

This section is devoted to the proof of Propositions 5.6 and 5.8.

Proof of Proposition 5.6. Consider $w$ a solution of equation (1.12) defined for all $(y,s) \in \mathbb{R}^2 \times [0,\sigma_1]$ for some $\sigma_1 \geq 0$, with

$$|w(y,s)| \leq 2\kappa \text{ and } \nabla w(0)(1 + |y|)^{-k} \in L^\infty$$

(B.1)

for some $k \in \mathbb{N}$. We aim at proving that

$$\forall s \in [0, \sigma_1], \quad \|w(s)\|_{L^2_\rho} \leq M_0\|w(0)\|_{L^2_\rho}e^{-\frac{s^p}{p-1}},$$

(B.2)

provided that

$$\|w(0)\|_{L^2_\rho} \leq \epsilon_0,$$

(B.3)

for some large $M_0 \geq 1$ and small $\epsilon_0 > 0$.

We will assume that

$$\|w(0)\|_{L^2_\rho} > 0,$$

(B.4)

otherwise $w \equiv 0$ and (B.2) is trivial.

Since $\Delta w - \frac{1}{2} y \cdot \nabla w = \frac{1}{\rho} \nabla \cdot (\rho \nabla w)$ by definition (1.18) of $\rho$, multiplying equation (1.12) by $w\rho$, integrating in space, then using an integration by parts, we write for all $s \in [0, \sigma_1]$:

$$\frac{1}{2} \frac{d}{ds} \int w(y,s)^2 \rho(y)dy \leq -\frac{1}{p-1} \int w(y,s)^2 \rho(y)dy + \int |w(y,s)|^{p+1} \rho(y)dy.$$  (B.5)

Note that the fact that $w(0) \in L^\infty$ and $\nabla w(0)(1 + |y|)^{-k} \in L^\infty$ for some $k \in \mathbb{N}$ (see (B.1)) is important to justify this integration by parts, as it is the case in item (ii) of Lemma 7.1. The conclusion will follow from 2 arguments: a rough estimate for general data, then a delicate estimate for small data. In the final step, we combine both arguments to conclude.
Step 1: A rough estimate for general data
Using (B.5) together with (B.1) and the definition (15) of \( \kappa \), we write for all \( s \in [0, \sigma_1] \),
\[
\frac{1}{2} \frac{d}{ds} \int w(y, s)^2 \rho(y) dy \leq \frac{2^{p-1} - 1}{p-1} \int w(y, s)^2 \rho(y) dy,
\]
hence,
\[
\|w(s)\|_{L^2_p}^2 \leq e^{\frac{(2^{p-1} - 1)\rho}{p-1} s} \|w(0)\|_{L^2_p}^2. \tag{B.6}
\]

Step 2: A delicate estimate for small data and large \( \sigma_1 \)
Using again equation (1.12), together with (B.1) and the definitions (15) and (84) of \( \kappa \) and \( L \), we write for almost every \((y, s) \in \mathbb{R}^2 \times [0, \sigma_1] \),
\[
\partial_s |w| \leq (L - 1 + \frac{(2^{p-1} - 1)}{p-1}) |w|.
\]
Using the regularizing effect of Lemma 7.1, we derive the existence of \( s_* > 0 \) and \( C^* > 0 \) such that if \( \sigma_1 \geq s_* \), then for all \( s \in [s_*, \sigma_1] \), we have
\[
\|w(s)\|_{L^{p+1}_p} \leq C^* \|w(s - s_*)\|_{L^2_p}. \tag{B.7}
\]
Now, if \( s \in [0, \min(s_*, \sigma_1)] \), we use the \( L^\infty \) bound (B.1) and the definition (1.5) of \( \kappa \) to derive the following rough control of the nonlinear term:
\[
\|w(s)\|_{L^{p+1}_p}^2 \leq (2\kappa)^{p-1} \|w(s)\|_{L^2_p}^2 = \frac{2^{p-1}}{p-1} \|w(s)\|_{L^2_p}^2.
\]
Using (B.5) with these 2 controls of the nonlinear term, we write for all \( s \in [0, \sigma_1] \), the following delay differential inequality:
\[
\frac{1}{2} \frac{d}{ds} \|w(s)\|_{L^2_p}^2 \leq -\frac{1}{p-1} \|w(s)\|_{L^2_p}^2 \tag{B.8}
\]
\[
+ \mathbb{1}_{\{0 \leq s \leq s_*\}} \frac{2^{p-1}}{p-1} \|w(s)\|_{L^2_p}^2 + \mathbb{1}_{\{s_* \leq s \leq \sigma_1\}} (C^*)^{p+1} \|w(s - s_*)\|_{L^{p+1}_p}^2.
\]

Step 3: Conclusion of the proof
Fixing \( M_0 > 0 \) and \( \epsilon_0 \) such that
\[
M_0 = 2 \max \left( 1, e^{\frac{2^{p-1}}{p-1} s_*} \right) \text{ and } 4 (C^* M_0)^{p+1} \epsilon_0^{p-1} e^{\frac{2^{p-1}}{p-1}} = M_0^2 \tag{B.9}
\]
we are ready to finish the proof of (B.2), if (B.3) and (B.4) hold. By continuity of the \( L^2_p \) norm of \( w(s) \) \(^3\) and noting that \( M_0 > 1 \), if we proceed by contradiction and assume that identity (B.2) fails, then we may introduce \( \bar{s} \in (0, \sigma_1] \) such that
\[
\forall s \in [0, \bar{s}], \quad \|w(s)\|_{L^2_p} \leq M_0 \|w(0)\|_{L^2_p} e^{\frac{2^{p-1}}{p-1}}, \tag{B.10}
\]
\[
\|w(\bar{s})\|_{L^2_p} = M_0 \|w(0)\|_{L^2_p} e^{\frac{2^{p-1}}{p-1}}. \tag{B.11}
\]
\(^3\)This is a consequence of the continuity in \( L^\infty \) for equation (1.13), through the transformation (1.4).
We will reach a contradiction in each of the 2 cases we consider in the following. 

**Case 1:** $\bar{s} \leq s_*$. In this case, using (B.6), the assumption (B.4) and the choice of $M_0$ in (B.9), we write

$$
\|w(\bar{s})\|_{L^p}^2 \leq e^{\frac{n-1}{p-1}}\|w(0)\|_{L^p}^2 \leq e^{\frac{n-1}{p-1}}\|w(0)\|_{L^p}^2
$$

$$
\leq M_0 \|w(0)\|_{L^p}^2 e^{-\frac{\bar{s}}{p-1}} < M_0 \|w(0)\|_{L^p}^2 e^{-\frac{\bar{s}}{p-1}},
$$

and a contradiction follows by (B.11).

**Case 2:** $s_* \leq \bar{s} \leq \sigma_1$. In this case, identity (B.8) holds for any $s \in [0, \bar{s}]$. Using (B.6) when $0 \leq s \leq s_*$ and (B.10) when $s_* \leq s \leq \bar{s}$, we write for all $s \in [0, \bar{s}]$:

$$
\frac{1}{2} \frac{d}{ds} \|w(s)\|_{L^p}^2 \leq -\frac{1}{p-1} \|w(s)\|_{L^p}^2
$$

$$
+ \begin{cases} 2p-1 e^{\frac{n-1}{p-1}} \|w(0)\|_{L^p}^2 & \text{if } s_* \leq s \leq \bar{s} \\
\|w(s)\|_{L^p}^2 - \|w(0)\|_{L^p}^2 & \text{if } s_* \leq s \leq \bar{s} \end{cases}
$$

Integrating this equation, we see that

$$
\|w(\bar{s})\|_{L^p}^2 \leq e^{-\frac{\bar{s}}{p-1}} \left( \|w(0)\|_{L^p}^2 + e^{\frac{n-1}{p-1}} \|w(0)\|_{L^p}^2 + 2 \left( C^* M_0 \|w(0)\|_{L^p}^2 \right)^{p+1} e^{\frac{\bar{s}}{p-1}} \right)
$$

$$
\leq e^{-\frac{\bar{s}}{p-1}} \left\{ \left( \frac{M_0}{2} \right)^2 \|w(0)\|_{L^p}^2 + \left( \frac{M_0}{2} \right)^2 \|w(0)\|_{L^p}^2 + \frac{M_0^2}{2} \|w(0)\|_{L^p}^2 \right\}
$$

$$
\leq \frac{3}{4} M_0^2 e^{-\frac{\bar{s}}{p-1}} \|w(0)\|_{L^p}^2
$$

thanks to the definition (B.9) of $M_0$ and $\epsilon_0$, together with (B.4). Thus, a contradiction follows from (B.11). This concludes the proof of Proposition 5.6. \qed

Now, we give the proof of Proposition 5.8.

**Proof of Proposition 5.8** Consider $w$ a solution of equation of (1.12) defined for all $(y, s) \in \mathbb{R}^2 \times [0, \sigma_1]$ for some $\sigma_1 \geq 0$, with

$$
\nabla w(0)(1 + |y|)^{-k} \in L^\infty \text{ and } |w(y, s)| \leq 2\kappa, \quad (B.12)
$$

for some $k \in \mathbb{N}$. We will prove that for some universal constant $M_1 \geq 1$, if

$$
\|w(0) - \psi(\sigma^*)\|_{L^p} \equiv \epsilon_1 \leq \frac{|\psi'(\sigma^*)|}{M_1} \quad (B.13)
$$

for some $\sigma^* \in \mathbb{R}$, where $\psi$ is defined in (5.8), then, it holds that

$$
\forall s \in [0, \sigma_1], \quad \|w(s) - \psi(s + \sigma^*)\|_{L^p} \leq M_1 \|w(0) - \psi(\sigma^*)\|_{L^{p+1}} \frac{|\psi'(s + \sigma^*)|}{|\psi'(\sigma^*)|} \quad (B.14)
$$

We may assume that

$$
\epsilon_1 > 0, \quad (B.15)
$$

56
otherwise \( w(y, s) = \psi(s + \sigma^*) \) for any \( s \geq 0 \), from the uniqueness of solutions to equation (1.12), and (B.14) is trivial.

Since \( w \) and \( \psi \) are both solutions of (1.12), introducing

\[ \tilde{\psi}(s) = \psi(s + \sigma^*) \text{ and } v(y, s) = w(y, s) - \tilde{\psi}(s), \]  

we write the following PDE satisfied by \( v \), for all \((y, s) \in \mathbb{R}^2 \times [0, \sigma_1]\):

\[ \partial_s v = \Delta v - \frac{1}{2} y \cdot \nabla v - \frac{v}{p-1} + p|\tilde{w}|^{p-1}v, \tag{B.17} \]

where

\[ \tilde{w}(y, s) \in [w(y, s), \tilde{\psi}(s)]. \tag{B.18} \]

Arguing as for (B.5), we derive the following identity from (B.17), for all \((y, s) \in [0, \sigma_1]\):

\[ \frac{1}{2} \tilde{z}'(s) \leq -\frac{z(s)}{p-1} + p \int |\tilde{w}(y, s)|^{p-1}v(y, s)^2 \rho(y)dy, \text{ where } z(s) = \int v(y, s)^2 \rho(y)dy. \tag{B.19} \]

The fact that \( \nabla w(0)(1 + |y|)^{-k} \in L^\infty \) is useful to justify the integration by parts in (B.19) elsewhere. We proceed in 2 steps, first deriving a differential inequality for \( z(s) \), then using a Gronwall argument to conclude.

**Step 1: A differential inequality on \( z(s) \).**

Since \( \tilde{w} \) and \( \tilde{\psi} \) are bounded by (5.8), (B.18) and (B.12), using the definitions (B.18) and (B.16) of \( \tilde{w} \) and \( v \), we write by continuity:

\[ ||\tilde{w}(y, s)|^{p-1} - \tilde{\psi}^{p-1}| \leq C_0|\tilde{w}(y, s) - \tilde{\psi}(s)|^{\bar{p}-1} \leq C_0|v(y, s)|^{\bar{p}-1} \]

for some \( C_0 > 0 \), where \( \bar{p} = \min(p, 2) \). Plugging this in (B.19), we write

\[ \frac{1}{2} \tilde{z}'(s) \leq -\frac{1}{p-1} + p\tilde{\psi}(s)^{p-1}z(s) + C_0 \int |v(y, s)|^{\bar{p}-1} \rho(y)dy. \tag{B.20} \]

Let us now bound \( \|v(s)\|_{L^{\bar{p}}_{p+1}} \). Using equation (B.17), the bound (B.12), the definitions (B.18) and (5.8) of \( \tilde{w} \) and \( \tilde{\psi} \), together with the definitions (1.3) and (3.1) of \( \kappa \) and \( \mathcal{L} \), we write for almost every \((y, s) \in \mathbb{R}^2 \times [0, \sigma_1]\),

\[ \partial_s|v| \leq (\mathcal{L} - 1 + \frac{(2p-1)(p-1)}{p-1})|v|. \tag{B.21} \]

Arguing as for (B.7), we see that if \( \sigma_1 \geq s_0 \), then, we have for all \( s \in [s_0, \sigma_1] \),

\[ \|v(s)\|_{L^{\bar{p}}_{p+1}} \leq \tilde{C}\|v(s - s_0)\|_{L^2_{p}} = \tilde{C}z(s - s_0)^{\frac{1}{2}}, \tag{B.22} \]

for some possibly different \( s_0(p) > 0 \) and \( \tilde{C} > 0 \). Now, if \( s \in [0, \min(s_0, \sigma_1)] \), using (B.12) and the definition (5.8) of \( \psi \), we see by definition (B.16) of \( v \) that \( |v| \leq 3\kappa \) and \( \nabla v(0)(1 + |y|)^{-k} \in L^\infty \). Therefore,

\[ \int |v(y, s)|^{\bar{p}+1} \rho(y)dy \leq (3\kappa)^{\bar{p}-1} \int v(y, s)^2 \rho(y)dy. \tag{B.23} \]
In addition, using (B.21), we see that we can apply item (ii) of Lemma 7.1 and get from (B.13)
\[ \forall s \in [0, \min(s_*, \sigma_1)], \quad \|v(s)\|_{L^2_{\rho}} \leq C^* \|v(0)\|_{L^2_{\rho}} = C^* \epsilon_1. \]

Using this together with (B.23), (B.20) and (B.22), we see that for all \( s \in [0, \sigma_1] \),
\[ \frac{1}{2} z'(s) \leq \left[-\frac{1}{p-1} + p\psi(s)^{p-1}\right] z(s) \]
\[ + C_1 \mathbf{1}_{0 \leq s \leq s_*} \epsilon_1^2 + C_1 \mathbf{1}_{s_* < s \leq \sigma_1} z(s - s_*)^{\frac{p+1}{2}}, \]
for some \( C_1 > 0 \).

**Step 2: A Gronwall estimate**

Let us define
\[ z_p(s) = \mathcal{S}(s + \sigma^*) \quad \text{where} \quad z_p(s) = \frac{e^{-\frac{2s}{p-1}}}{(1 + e^{-s})^{\frac{p}{p-1}}} = \left[ \frac{p - 1}{\kappa} \psi'(s) \right]^2 \]
and \( \psi \) is defined in (5.8). Since \( \psi(s) \) satisfies equation (1.12), it follows that \( \tilde{z}_p(s) \) is a solution of the linear part of (B.24), namely
\[ \tilde{z}_p'(s) = 2 \left( -\frac{1}{p-1} + p\psi(s)^{p-1} \right) \tilde{z}_p(s). \]

Then, we introduce the following barrier
\[ \tilde{z}(s) = \frac{M'_1 \epsilon_1^2}{z_p(\sigma^*)} \tilde{z}_p(s), \]
where \( M'_1 > 1 \) will be fixed large enough later. With this definition and recalling the definition (B.19) of \( z(s) \), we suggest to prove that
\[ \forall s \in [0, \sigma_1], \quad z(s) \leq \tilde{z}(s), \]  
if \( \epsilon_1 \) defined in (B.13) is small enough, which clearly implies (B.14), by definition (B.25) of \( z_p \). We proceed by contradiction and assume that identity (B.28) fails. Since
\[ 0 < z(0) = \epsilon_1^2 < M'_1 \epsilon_1^2 = \tilde{z}(0), \]
by (B.19), (B.16), (B.3), (B.15), (B.27) and (B.25), using the continuity in time of the \( L^2_{\rho} \) norm of \( v(s) \) solution of equation (B.17) (which is a consequence of the continuity in \( L^\infty \) for equation (1.1), through the transformations (1.4) and (B.16)), we see that (B.28) holds at least on a small interval to the right of 0. Hence, we may introduce \( \bar{s} \in (0, \sigma_1] \) such that
\[ \forall s \in [0, \bar{s}], \quad z(s) \leq \tilde{z}(s), \]
\[ z(\bar{s}) = \tilde{z}(\bar{s}). \]

Using the differential inequality (B.24) together with the auxiliary function \( \tilde{z}_p \) which satisfies equation (B.26), (B.29) and (B.30), we write the following Gronwall estimate:
\[ z(\bar{s}) \leq \tilde{z}_p(\bar{s}) \left\{ \frac{\epsilon_1^2}{\tilde{z}_p(0)} + C_1 \epsilon_1^2 J_1 + C_1 \left( \frac{M'_1 \epsilon_1^2}{z_p(\sigma^*)} \right)^{\frac{p+1}{2}} J_2 \right\} \]
58
where
\[ J_1 = \int_{0}^{s_*} \frac{d\sigma}{z_p(\sigma)} \quad \text{and} \quad J_2 = \int_{s_*}^{s} \frac{\bar{z}_p(\sigma - s_*)^{\frac{p+1}{2}}}{\bar{z}_p(\sigma)} d\sigma. \]

By definition (B.25), we see that for all \( s' \in [0, s_*] \), we have \( z_p(\sigma' + s') \geq z_p(\sigma^*) e^{-2\frac{(p+1)}{p-1}s_*} \),

hence
\[ J_1 = \int_{\sigma^* + s_*}^{\sigma' + s_*} \frac{d\sigma'}{z_p(\sigma')} \leq \frac{s_* e^{2\frac{(p+1)}{p-1}s_*}}{z_p(\sigma^*)}. \]

We also have
\[ J_2 = \int_{\sigma^* + s_*}^{\sigma' + s_*} \frac{z_p(\sigma' - s_*)^{\frac{p+1}{2}}}{z_p(\sigma')} d\sigma' \leq \int_{-\infty}^{\infty} \frac{z_p(\sigma' - s_*)^{\frac{p+1}{2}}}{z_p(\sigma')} d\sigma' \equiv C_2(s_*). \]

Imposing that
\[ \epsilon_1^2 \leq \frac{z_p(\sigma^*)}{M'_1 (C_1 C_2(s_*))^{\frac{2}{p-1}}}, \] (B.32)

we see that
\[ z(\bar{s}) \leq \frac{\epsilon_1^2}{z_p(\sigma^*)} \bar{z}_p(\bar{s}) [1 + C_1 s_* e^{2\frac{(p+1)}{p-1}s_*} + 1]. \]

Fixing
\[ M'_1 = 3 + C_1 s_* e^{2\frac{(p+1)}{p-1}s_*}, \]
we see that a contradiction follows from (B.31), (B.30) and (B.27) (remember that \( \bar{z}(s) > 0 \) by definition (B.27), together with (B.25) and (B.15)). Thus, (B.28) holds. Since \( z_p(s) = C|\psi'(s)|^2 \) from (B.25), using the definitions (B.19) and (B.27) of \( \bar{z}(s) \) and \( \bar{z}(s) \), together with the condition (B.32), we conclude the proof of Proposition 5.8. \( \Box \)

References

[1] J. Bricmont and A. Kupiainen. Universality in blow-up for nonlinear heat equations. *Nonlinearity*, 7(2):539–575, 1994.

[2] C. Collot, P. Merle, and P. Raphaël. Strongly anisotropic type II blow up at an isolated point. *J. Am. Math. Soc.*, 33(2):527–607, 2020.

[3] C. Fermanian Kammerer, F. Merle, and H. Zaag. Stability of the blow-up profile of non-linear heat equations from the dynamical system point of view. *Math. Annalen*, 317(2):195–237, 2000.

[4] C. Fermanian Kammerer and H. Zaag. Boundedness up to blow-up of the difference between two solutions to a semilinear heat equation. *Nonlinearity*, 13(4):1189–1216, 2000.

[5] S. Filippas and R. V. Kohn. Refined asymptotics for the blowup of \( u_t - \Delta u = u^p \). *Comm. Pure Appl. Math.*, 45(7):821–869, 1992.

[6] S. Filippas and W. X. Liu. On the blowup of multidimensional semilinear heat equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(3):313–344, 1993.
[7] A. Friedman. Interior estimates for parabolic systems of partial differential equations. *J. Math. Mech.*, 7:393–417, 1958.

[8] Y. Giga and R. V. Kohn. Asymptotically self-similar blow-up of semilinear heat equations. *Comm. Pure Appl. Math.*, 38(3):297–319, 1985.

[9] Y. Giga and R. V. Kohn. Characterizing blowup using similarity variables. *Indiana Univ. Math. J.*, 36(1):1–40, 1987.

[10] Y. Giga and R. V. Kohn. Nondegeneracy of blowup for semilinear heat equations. *Comm. Pure Appl. Math.*, 42(6):845–884, 1989.

[11] Y. Giga, S. Matsui, and S. Sasayama. Blow up rate for semilinear heat equations with subcritical nonlinearity. *Indiana Univ. Math. J.*, 53(2):483–514, 2004.

[12] M. A. Herrero and J. J. L. Velázquez. Comportement générique au voisinage d’un point d’explosion pour des solutions d’équations paraboliques unidimensionnelles. *C. R. Acad. Sci. Paris Sér. I Math.*, 314(3):201–203, 1992.

[13] M. A. Herrero and J. J. L. Velázquez. Flat blow-up in one-dimensional semilinear heat equations. *Differential Integral Equations*, 5(5):973–997, 1992.

[14] M. A. Herrero and J. J. L. Velázquez. Generic behaviour of one-dimensional blow up patterns. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 19(3):381–450, 1992.

[15] M. A. Herrero and J. J. L. Velázquez. Blow-up behaviour of one-dimensional semilinear parabolic equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(2):131–189, 1993.

[16] F. Merle. Solution of a nonlinear heat equation with arbitrarily given blow-up points. *Comm. Pure Appl. Math.*, 45(3):263–300, 1992.

[17] F. Merle, P. Raphaël, and J. Szeftel. On strongly anisotropic type I blowup. *Int. Math. Res. Not.*, 2020(2):541–606, 2020.

[18] F. Merle and H. Zaag. Stability of the blow-up profile for equations of the type $u_t = \Delta u + |u|^{p-1}u$. *Duke Math. J.*, 86(1):143–195, 1997.

[19] F. Merle and H. Zaag. Optimal estimates for blowup rate and behavior for nonlinear heat equations. *Comm. Pure Appl. Math.*, 51(2):139–196, 1998.

[20] F. Merle and H. Zaag. Refined uniform estimates at blow-up and applications for nonlinear heat equations. *Geom. Funct. Anal.*, 8(6):1043–1085, 1998.

[21] F. Merle and H. Zaag. A Liouville theorem for vector-valued nonlinear heat equations and applications. *Math. Annalen*, 316(1):103–137, 2000.

[22] F. Merle and H. Zaag. Behavior rigidity near non-isolated blow-up points for the semilinear heat equation. *Internat. Math. Res. Notices*, 07 2021.

[23] V. T. Nguyen and H. Zaag. Finite degrees of freedom for the refined blow-up profile of the semilinear heat equation. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(5):1241–1282, 2017.
[24] J. J. L. Velázquez. Higher-dimensional blow up for semilinear parabolic equations. *Comm. Partial Differential Equations*, 17(9-10):1567–1596, 1992.

[25] J. J. L. Velázquez. Classification of singularities for blowing up solutions in higher dimensions. *Trans. Amer. Math. Soc.*, 338(1):441–464, 1993.

[26] F. B. Weissler. Single point blow-up for a semilinear initial value problem. *J. Differential Equations*, 55(2):204–224, 1984.

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