A new gradient elasticity model for the elastic boreholes

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Abstract. A new gradient elasticity model is employed to discuss strain gradient effects and its ability in predicting size effects on an elastic rock mass with microstructure, around an axisymmetric borehole under internal pressure and remote isotropic compressive stress. The constitutive equation of the model involves the Laplacian of the strain tensor multiplied by the gradient coefficient. The formulated boundary value problem is solved analytically to derive stress, strain, and displacement distributions and discuss respective gradient effects on the mechanical behavior of the rock mass and the corresponding borehole stability. The paper concludes with the employment of the Rankine failure criterion to investigate size effects on the stress concentration factor at the perimeter of the borehole, and the comparison with another gradient elasticity model which involves the Laplacian of the hydrostatic part of the strain tensor.

1. Introduction
Nowadays, the demands of technology are very high for the thorough study of the mechanical behavior of materials. Although generalized theories of classical elasticity had been proposed in the 1960s they did not apply extensively to engineering problems because they provided a large number of phenomenological constants and the technology of that time did not require a detailed study of the effect of microstructure on elastic behavior and the corresponding stress analysis. The size effects (i.e. the strength of the specimen depends on its size), which related to the material microstructure and cannot be predicted by classical elasticity (CE), require the use of a generalized theory (or a non-local theory) such as gradient elasticity (GE) which introduces higher-order strain (or stress) gradients in the constitutive law. The GE can adequately describe the effect of the microstructure on the macro scale \([1-4]\). Since deformation is always non-uniform at the microstructural level, the influence of strain gradients in the macroscopic material behavior becomes increasingly important as the dimensions of the specimen decrease down to the micro and nanoscale. This, in turn, results in phenomena that usually are not observed in the macroscale including size-dependent strength and stress-strain responses. The size effects relate to the question of the transferability of mechanical test results of geometrically similar scaled-down structural models to the full-scale structures using similitude laws. They also concern about the validity of small-scale laboratory-type test results and their use as a basis for the computational modeling of large-scale components \([5]\).
The problem addressed in the present work calculates stresses, strains, and displacements around boreholes in elastic intact rocks under internal pressure and external compressive stress introducing a new constitutive equation of gradient elasticity. The solution is used to describe the failure mechanism around the borehole and to discuss the occurrence of size effects. The borehole radius enters with its own dimensional identity normalized by the characteristic length which is associated with the gradient coefficient and assumed a material constant. The presence of this ratio between the geometrical length and the characteristic length represents the ability of gradient elasticity in interpreting size effects for otherwise geometrically similar specimens. From the analytical expression of the stress concentration factor (SCF), the size effect also arises, which turns out to be a function of the borehole radius. Finally, the comparison of the present SCF expression with the corresponding one resulting from an earlier gradient elasticity model (where the Laplacian of the hydrostatic part of the strain tensor was taken into account in the constitutive law), is made.

2. Analysis and results
A new gradient elasticity model is used, incorporating the Laplacian of the strain tensor in Hooke’s law. The appropriate gradient constitutive equation (gradient model I) reads:

\[
\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij} - 2\mu c \nabla^2 \varepsilon_{ij}
\]  

where \(\lambda, \mu\) are Lamé constants, \(\delta_{ij}\) is the unit tensor, \(\varepsilon_{ij}\) is the strain tensor, \(\sigma_{ij}\) is the stress tensor, and \(\nabla^2\) is the Laplacian operator.

![Figure 1. The borehole problem: geometry and loadings.](image_url)

The gradient coefficient \(c\) multiplies the Laplacian of the strain tensor; its value is related to a characteristic scale of the material. It is considered that the physical meaning of \(c\) is connected to the region over which non-locality acts, smoothing high variations of the elastic stress field. From the present gradient constitutive law that applies to the borehole problem and solving the relevant boundary value problem, a set of fourth-order equations arises with the displacements as the sole unknowns. Consequently, the higher order of the governing equations means that higher-order
boundary conditions must be formulated. A successful practice for achieving the general solution adopts, for the boundary conditions, second-order displacement derivatives \[4\], \[6\]. The borehole problem (figure 1) describes the mechanical behaviour of a circular hole with radius \(R\) under internal pressure \(p\) and remote biaxial (isotropic) compression \(\sigma\) under plane strain conditions. For the axisymmetric configuration at hand, shearing stresses and tangential displacement are zero everywhere; the only non-zero displacement component is the radial one, \(u_r = u\). Thus, in polar coordinates \((r, \theta)\), the (classical) kinematic equations relating the radial and tangential strain to the radial displacement are:

\[ e_r = \frac{du}{dr}, \quad e_\theta = \frac{u}{r}. \]  

\[ \text{(2)} \]

Derivatives with respect to the angular coordinate \(\theta\) vanish, so that equation (1) yields:

\[ \sigma_r = (\lambda + 2\mu)e_r + \lambda e_\theta - 2\mu c \left[ \frac{d^2 e_r}{dr^2} + \frac{1}{r} \frac{de_r}{dr} \right], \]

\[ \sigma_\theta = \lambda e_r + (\lambda + 2\mu)e_\theta - 2\mu c \left[ \frac{d^2 e_\theta}{dr^2} + \frac{1}{r} \frac{de_\theta}{dr} \right]. \]

\[ \text{(3)} \]

\[ \text{(4)} \]

Upon substitution of equations (2) into equations (3) and (4), we get:

\[ \sigma_r = (\lambda + 2\mu) \left( \frac{du}{dr} \right) + \lambda \left( \frac{u}{r} \right) - 2\mu c \left[ \frac{d^2 u}{dr^2} + \frac{1}{r} \frac{du}{dr} \right], \]

\[ \sigma_\theta = \lambda \left( \frac{du}{dr} \right) + (\lambda + 2\mu) \left( \frac{u}{r} \right) - 2\mu c \left[ \frac{1}{r^2} \frac{d^2 u}{dr} + \frac{1}{r} \frac{du}{dr} \right]. \]

\[ \text{(5)} \]

\[ \text{(6)} \]

The only non-trivial equilibrium equation is the one in the radial direction, i.e. (neglecting the body forces):

\[ \frac{d\sigma_r}{dr} - \frac{\sigma_r}{r} = 0. \]

\[ \text{(7)} \]

Upon substitution of equations (5) and (6) into equilibrium equation (7), a fourth-order ordinary differential equation for \(u\) is obtained. Its general solution is easily derived as

\[ u = A \frac{r}{\ell_G} + B \left( \frac{\ell_G}{r} \right) + C \cdot I_1 \left( \frac{r}{\ell_G} \right) + D \cdot K_1 \left( \frac{r}{\ell_G} \right), \]

\[ \text{(8)} \]

where \(A, B, C, D\) are integration constants, \(I_1\) and \(K_1\) are the first-order modified Bessel functions of first and second kind, respectively, and \(\ell_G\) is a characteristic (material) length related to the gradient coefficient \(c\) by:

\[ \ell_G = \left[ \frac{(1-2\nu)c}{1-\nu} \right]^{1/2}, \]

\[ \text{(9)} \]

where \(\nu\) is the Poisson ratio. In order to determine the integration constants in equation (8), four boundary conditions are required: two classical boundary conditions, i.e. \(\sigma_r = p\) at \(r = R\) and \(\sigma_r = \sigma\) at \(r \to \infty\), and two non-standard, i.e. \(d^2 u / dr^2 = 0\) at \(r = R\) and at \(r \to \infty\). Hence, the integration constants are found as follows:

\[ A = \sigma \cdot \ell_G \left[ 2(\lambda + \mu) \right]^{-1}, \]

\[ \text{(10)} \]
In figures 3 and 4, the radial and the tangential stress component are plotted vs. the radial coordinate for three different characteristic lengths (\( \ell_g \)). It is observed that, for a radial distance of about four times the radius of the borehole, the gradient elasticity coincides with the classical elasticity. In figures 3 and 4, the radial and the tangential stress component are plotted vs. the radial coordinate for three different characteristic lengths (\( \ell_g \)). The radial stress distributions do not show a significant difference while for the tangential stress distributions a remarkable difference is observing for a radial distance smaller than about four times the radius of the borehole. The bigger the characteristic length (\( \ell_g \)) value the lower line of the tangential stress is shown in figure 4.
Figure 2. Radial displacement distributions vs. radial distance in gradient elasticity (GE) (model I) and classical elasticity (CE) for:
\[ \sigma = 90\,\text{MPa}, \quad p = 30\,\text{MPa}, \quad \nu = 0.3, \]
\[ E = 80\,\text{GPa}, \quad R = 0.1\,\text{m}, \]
\[ \ell_G = 0.03\,\text{m} / 0.05\,\text{m} / 0.1\,\text{m}. \]

Figure 3. Radial stress distributions vs. radial distance in gradient elasticity (GE) (model I) and classical elasticity (CE) for:
\[ \sigma = 90\,\text{MPa}, \quad p = 30\,\text{MPa}, \quad \nu = 0.3, \]
\[ R = 0.1\,\text{m}, \quad \ell_G = 0.07\,\text{m} / 0.1\,\text{m} / 0.2\,\text{m}. \]

Figure 4. Tangential stress distributions vs. radial distance in gradient elasticity (GE) (model I) and classical elasticity (CE) for:
\[ \sigma = 90\,\text{MPa}, \quad p = 30\,\text{MPa}, \quad \nu = 0.3, \]
\[ R = 0.1\,\text{m}, \quad \ell_G = 0.07\,\text{m} / 0.1\,\text{m} / 0.2\,\text{m}. \]
3. Stress concentration factor

The maximum stress over the whole domain is provided by the tangential stress at the hole perimeter (i.e. at \( r = R \)):

\[
\sigma_{\theta_{|r=R}} = 2\sigma - p - (\sigma - p) \left[ \frac{2\nu}{1-2\nu} \frac{\ell_G}{R} \right] \left[ \frac{2K_1(\frac{R}{\ell_G}) - R_{\sigma}K_0(\frac{R}{\ell_G})}{R_{\sigma}K_0(\frac{R}{\ell_G}) + 2(1-\nu)\frac{\ell_G}{R} + R_{\sigma}K_1(\frac{R}{\ell_G})} \right]. \tag{17}
\]

In the present gradient approach (gradient model I), the critical conditions are supposed to be achieved simply when the maximum normal stress \( \sigma_{\text{max}} \) attains a threshold value, that is the material compressive strength \( \sigma_c \). (Rankine failure criterion). If the external compressive stress \( \sigma \) is written as a multiple of the applied internal pressure \( (p) \), i.e. \( \sigma = k \cdot p \) (where \( k \geq 1 \)), then the following form for the stress concentration factor \( \text{SCF} = \sigma_{\theta_{|r=R}} / p \) is obtained:

\[
\text{SCF}_I = (2k - 1) - (k - 1) \left[ \frac{2\nu}{1-2\nu} \frac{1}{R_{\sigma}} \left[ 2K_1(\frac{R_{\sigma}}{\ell_G}) - R_{\sigma}K_0(\frac{R_{\sigma}}{\ell_G}) \right] \right]
\]

\[
K_0(\frac{R_{\sigma}}{\ell_G}) + 2(1-\nu)\frac{1}{1-2\nu} \frac{1}{R_{\sigma}} + R_{\sigma}K_1(\frac{R_{\sigma}}{\ell_G}) \right]
\]

with \( R_{\sigma} = R / \ell_G \). The first term in the right-hand part of the equation (18) is the classical one while the second term is the gradient “correction” which is a function of \( k, R_{\sigma} \) and \( \nu \). This is how it appears the size-dependence of the stress concentration factor of the gradient model I. In figure 5 the stress concentration factor \( \text{SCF}_I \) is plotted vs. the dimensionless borehole radius \( R_{\sigma} \) for \( k=3 \). It is observed that when \( R_{\sigma} \) is ranging between 1 and 2.38 then \( \text{SCF}_I < \text{SCF}_{\text{classic}} \) = 5 while for \( R_{\sigma} > 2.38 \), then \( \text{SCF}_I > \text{SCF}_{\text{classic}} \). In case that \( R_{\sigma} = 6.05 \), the \( \text{SCF}_I \) takes its maximum value (=5.237). The minimum value of \( \text{SCF}_I \) is 4.25. For \( R_{\sigma} \rightarrow \infty \), we have \( \text{SCF}_I \rightarrow \text{SCF}_{\text{classic}} \). In fact it can be considered that for \( R_{\sigma} > 6.05 \) there is no significant difference between \( SCF_I \) and \( SCF_{\text{classic}} \).

In case that a different version of gradient elasticity is used (we call it here as gradient model II) [6], i.e.

\[
\sigma_{ij} = \lambda \varepsilon_{ik} \delta_{ij} + 2\mu \varepsilon_{ij} - c\lambda \nabla^2 \varepsilon_{ik} \delta_{ij}, \tag{19}
\]

the stress concentration factor takes the following form for the same problem configuration [7]:

\[
\text{SCF}_{II} = (2k - 1) - (k - 1) \left[ \frac{2K_0(\frac{R_{\sigma}}{\ell_G})}{R_{\sigma}K_1(\frac{R_{\sigma}}{\ell_G}) + K_0(\frac{R_{\sigma}}{\ell_G})} \right]. \tag{20}
\]

Here the gradient “correction” is a function of \( k \) and \( R_{\sigma} \). From equation (20) it turns out that for \( R_{\sigma} > 1 \) then \( \text{SCF}_{II} < \text{SCF}_{\text{classic}} \) while for \( R_{\sigma} \rightarrow \infty \), we have \( \text{SCF}_{II} \rightarrow \text{SCF}_{\text{classic}} = 5 \). The minimum value of \( \text{SCF}_{II} \) is 3.354. For small values of dimensionless borehole radius the stress concentration factor derived from gradient model I shows a smaller deviation (15%) from the classical solution than the gradient model II (33%). In figure 5, the stress concentration factors distributions versus the dimensionless borehole radius are illustrated, in gradient elasticity (model I and model II) and classical elasticity for \( k = 3 \).
Figure 5. Stress concentration factor (SCF) distributions versus the dimensionless borehole radius in Gradient Elasticity (GE) (model I and II) and in Classical Elasticity (CE) [for $k = 3$].

It is also noted that the predictions of the gradient models are in qualitative agreement with the most common experimental trend, i.e. smaller specimens are stronger. Moreover, for relatively large specimens gradient solutions approach the size independent predictions of classical elasticity.

4. Conclusions
The present work provides an analytical solution for the borehole problem in an elastic rock under internal pressure and external biaxial isotropic compression. The solution is achieved by using a new constitutive equation of gradient elasticity, which takes into account the Laplacian of the strain tensor. The corresponding boundary value problem was solved and the solution is given in a closed-form. Explicit expressions of stresses, strains, and displacements are provided, where in addition to the classic part there is the "correction" due to the gradient terms. Regarding the stress field, there is a remarkable difference in the tangential stress component (the gradient tensile stress component takes lower values than the classical one). The higher the value of the characteristic length $\ell_G$ is taking, the greater the deviation from the classical solution occurs. This variation applies to radial distances approximately five times the radius of the borehole. The same observation applies to radial displacement. As for the radial stress component, the differences between gradient and classical elasticity, are negligible. Concerning the stress concentration factor (SCF) it is shown its dependency on the radius of the borehole, i.e. the occurrence of size effects. For dimensionless borehole radius $R_G < 5$ a usual size effect is observed, while for $R_G > 5$ we have an inverse effect. It further follows that the lower the value of the borehole radius is, the stronger the dependence of the SCF on the size of the hole. The SCFII (gradient model II) presents a more intense size effect than the SCFI (gradient model I).
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