Switching speed distribution of spin-torque-induced magnetic
reversal

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Abstract

The switching probability of a single-domain ferromagnet under spin-current excitation is evaluated using the Fokker-Planck equation (FPE). In the case of uniaxial anisotropy, the FPE reduces to an ordinary differential equation in which the lowest eigenvalue $\lambda_1$ determines the slowest switching events. We have calculated $\lambda_1$ by using both analytical and numerical methods. It is found that the previous model based on thermally distributed initial magnetization states \cite{1} can be accurately justified in some useful limiting conditions.
I. INTRODUCTION

Fast and reliable nanosecond level writing is essential for spin-torque-induced switching in memory and recording technologies. One intrinsic source for write threshold distribution is thermal fluctuation, two aspects of which could affect reversal time $\tau_s$. First, the initial position of the magnetic moment is thermally distributed at the time the reversal field or current is applied, causing a variation in switching time. Secondly, during reversal, thermal fluctuation would modify the orbit, causing additional fluctuation for $\tau_s$ even for identical initial conditions. A reversal time distribution due to a thermalized initial condition was recently estimated [1]. Here we present a Fokker-Planck formulation for the full problem including both thermalized initial condition and reversal orbit with estimates for the reversal time and its distribution.

Fokker-Planck equations have been successfully applied to macro-spin systems with a static thermal distribution for the extraction of a small thermal escape probability. For example, the distribution for a thermally activated reversal has been analytically and numerically obtained when the applied magnetic field is smaller than the coercive field $B_c$ [2, 3, 4, 5, 6]. There the switching is through thermally assisted transition from one metastable energy minimum to equilibrium. The ensemble-averaged $\tau_s$ depends exponentially on the ratio between the reversal energy barrier and thermal energy $k_B T$. These subthreshold results have recently been extended to include the effect of spin-torque, which modifies the effective temperature $T_{\text{eff}}$ [7, 8, 9].

For fast, dynamically driven switching, the applied spin-torque is above the zero-temperature reversal threshold. For finite temperature, this leads to a time-dependent Fokker-Planck equation in which the time-evolution of the probability represents the thermalized reversal of the macro-spin. Here we solve this problem in the colinear geometry with a uniaxial anisotropy energy landscape and the presence of a spin torque.

II. FOKKER-PLANCK EQUATION

Consider an ensemble time-dependent magnetization probability density $P(n_m, t)$, where $n_m$ is the unit vector describing the direction of the macro-spin moment $m$; in spherical coordinates $n_m$ is characterized by $(\theta, \phi)$. Before one turns on the magnetic field and the
current, the probability density takes its equilibrium value. For a uniaxial anisotropy-only situation, \( P(n_m, 0) = P_0 \exp(-\xi \sin^2 \theta) \) (1) for \( 0 \leq \theta \leq \pi/2 \) and \( P(n_m, 0) = 0 \) for \( \pi/2 < \theta \leq \pi \), where \( P_0 \) is the normalization factor \( (\int_0^\pi P \sin \theta d\theta = 1) \). If we consider \( \xi \equiv KV/k_B T \gg 1 \) where \( K \) is the anisotropy constant and \( V \) is the volume of nanomagnet. To determine \( P(n_m, t) \) after one turns on the field and the current at \( t = 0 \), we need to solve the Fokker-Planck equation given below

\[
\frac{\partial P}{\partial t} + \nabla \cdot \mathbf{J} - D \nabla^2 P = 0
\]

(2)

where \( \mathbf{J} \) is the probability current and \( D = \alpha \gamma k_B T/m \) is the diffusion constant, \( \alpha \) is the damping parameter, \( \gamma \) is the gyromagnetic coefficient, and \( m = |\mathbf{m}| \). The probability current is

\[
\mathbf{J} = \frac{d\mathbf{n}_m}{dt} = -\gamma P \mathbf{n}_m \times [\mathbf{H}_e + \alpha \mathbf{n}_m \times (\mathbf{H}_e - \mathbf{H}_s)]
\]

(3)

where we have used the Landau-Lifshitz-Gilbert equation including the spin torque term \( \mathbf{H}_s = (I p)(\hbar/2q)(1/\alpha m) \mathbf{n}_s \) (\( I \) is current density, \( p \) is the spin polarization coefficient, \( q \) is the electron charge, and \( \mathbf{n}_s \) is a unit vector in the direction of the magnetization of the pinned layer). To make the solution mathematically tractable, we consider the case where the magnetic field \( \mathbf{H} \) and \( \mathbf{n}_s \) are parallel to the anisotropy axis, i.e., \( \mathbf{H}_e = (H + H_k \cos \theta) \mathbf{e}_z \) and \( \mathbf{n}_s = \mathbf{e}_z \) where \( H_k = 2K/M_s \). With these simplifications, Eq. (3) reduces to

\[
\frac{\partial P}{\partial t} = \frac{\alpha \gamma m}{\alpha \gamma k_B T} \left[ (1 - x^2) \left( \frac{dU_e}{dx} P + k_B T \frac{dP}{dx} \right) \right]
\]

(4)

where \( x = \cos \theta \), and we have defined the effective potential \( U_e = (H_s - H)mx - (H_k/2)mx^2 \). Equation (4) can be solved via the method of separation of variables. Let’s assume \( P(x, t) = f(t)u(x) \) and it is easy to see that \( f(t) = \exp(-\lambda t) \) and \( u(x) \) satisfies

\[
\frac{\alpha \gamma}{m} \frac{d}{dx} \left[ (1 - x^2) \left( \frac{dU_e}{dx} u(x) + k_B T \frac{du(x)}{dx} \right) \right] = -\lambda u(x)
\]

(5)

The above equation can be converted into a Sturm-Liouville equation,

\[
\frac{d}{dx} \left[ (1 - x^2)e^{-\beta U_e(x)} \frac{dF}{dx} \right] + ce^{-\beta U_e} F = 0
\]

(6)

where \( F(x) = e^{\beta U_e(x)} u(x) \) and \( c = \lambda m/\alpha \gamma k_B T \). Equation (6) is the same equation introduced by Brown except that \( U_e \) includes the spin-current term \( H_s \). The original FPE (4) is
now reduced to the standard eigenvalue problem. Namely, we determine eigen-function
\[ F(x) = F_n(x) \]
and eigenvalue \( \lambda = \lambda_n \) from Eq. (6) for \( n = 0, 1, 2, \ldots \), and the general solution of Eq. (4) is
\[
P(x, t) = \sum_{n=0}^{\infty} A_n e^{-\beta U_e(x)} F_n(x) e^{-\lambda_n t} \tag{7}
\]
where the coefficients in Eq. (7) are determined by the initial condition Eq. (1): \( A_n = \int_{-1}^{1} dx P(x, 0) F_n(x) \), where we have chosen the eigenfunction \( F_n(x) \) to be weighted normalized:
\[
\int_{-1}^{1} dx e^{-\beta U_e(x)} F_n^2(x) = 1.
\]

The remaining task is to solve eigenvalue \( \lambda_n \) (or \( c_n \)) and eigenfunction \( F_n \). Obviously, \( \lambda_0 = 0 \) is the lowest eigenvalue with the corresponding eigenfunction \( F_0 \) being a constant. Thus, \( u_0(x) \propto \exp(-\beta U_e) \) describes the thermal equilibrium state. The smallest nonzero eigenvalue \( \lambda_1 \) then determines the slowest decaying or switching speed.

The early theoretical attempts toward solving Eq. (6) \([2, 3, 4, 5, 6]\) had been focused on the cases where there are two potential wells separated by an energy barrier and \( H < H_k \). These solutions do not apply to the fast switching case where \( H > H_k \). In this case, there is no energy barrier and the only stable solution is at \( x = -1 \) or \( \theta = \pi \). In the following, we develop two methods to solve the Eq. (6): one is the approximate minimization which provides an upper bound for \( \lambda_1 \) and the other is to numerically determine the coefficients of polynomial series when Eq. (6) is expanded into polynomials.

The variational method can be obtained by multiplying \( F(x) \) to Eq. (6) and then by integrating from -1 to 1,
\[
\int_{-1}^{1} dx F \frac{d}{dx} \left[ (1 - x^2) e^{-\beta U_e} \frac{dF}{dx} \right] + c \int_{-1}^{1} e^{-\beta U_e} F^2 dx = 0.
\]
Performing integration by parts of the first term, we find
\[
c = \mathcal{D} / \mathcal{H} \tag{8}
\]
where
\[
\begin{align*}
\mathcal{D} &= \int_{-1}^{1} dx e^{-\beta U_e} (1 - x^2) \left( \frac{dF}{dx} \right)^2 \\
\mathcal{H} &= \int_{-1}^{1} dx e^{-\beta U_e} F^2(x)
\end{align*}
\]
Since both \( \mathcal{D} \) and \( \mathcal{H} \) are definitely positive, the variational principle is to choose a trial function \( F(x) \) such that \( c \) is minimized, and the non-zero minimum \( c \) represents the upper
bound of the lowest eigenvalue $c_1$. Additional constriction of $F(x)$ is its orthogonality with $F_0$ which is a constant, i.e.,
\[
\int_{-1}^{1} e^{-\beta U_e} F(x) dx = 0. \tag{10}
\]
Since $\exp(-\beta U_e)$ is sharply peaked at $x = -1$ in our case, we change the variable as $x = y - 1$ and thus $\beta U_e = \eta y - \xi y^2 + \text{const}$, where we define $\eta = \frac{m}{k_B T} (H_s - H + H_k)$ and $\xi = \frac{m}{k_B T} \frac{1}{2} H_k$.

We consider a simple trial function $F(y) = 1 + b_1 y + b_2 y^2$ where $b_1$ and $b_2$ are variational parameters. By placing $F(y)$ into Eq. (9) and (10), we have
\[
\begin{align*}
\mathcal{D} &= \int_{0}^{2} y(2 - y) e^{-\eta y + \xi y^2} (b_1 + 2 b_2 y)^2 dy \\
\mathcal{H} &= \int_{0}^{2} e^{-\eta y + \xi y^2} (1 + b_1 y + b_2 y^2)^2 dy \\
\int_{0}^{2} e^{-\eta y + \xi y^2} (1 + b_1 y + b_2 y^2) dy &= 0
\end{align*}
\tag{11}
\]
After the completion of these three integrations, the eigenvalue $c_1$ is obtained by minimizing $\mathcal{D}/\mathcal{H}$ with respect to $b_2$ or $b_1$. The above integrations can be analytically obtained if we notice that $y$ is limited to $1/\eta$ due to exponential $e^{-\eta y}$ and thus the term $e^{\xi y^2}$ is small and can be expanded as: $e^{\xi y^2} = 1 + \xi y^2 + \frac{1}{2} \xi^2 y^4$. By using $\int_{0}^{2} y^n e^{-\eta y} dy \approx n! / \eta^{n+1}$, for example, the third equation in Eq. (11) is
\[
\frac{1}{\eta} + \frac{2 \xi}{\eta^3} + \frac{12 \xi^2}{\eta^5} + \left( \frac{1}{\eta^2} + \frac{6 \xi}{\eta^4} + \frac{60 \xi^2}{\eta^6} \right) b_1
+ \left( \frac{2}{\eta^3} + \frac{24 \xi}{\eta^5} + \frac{360 \xi^2}{\eta^7} \right) b_2 = 0
\tag{12}
\]
Note that the leading orders of $b_1$ and $b_2$ are $\eta$ and $\eta^2$ respectively, and the above equation is valid up to $\eta^{-3}$. If we consider the fast switching case, i.e. $\eta \gg \xi$, or $H_s - H \gg H_k$, the equation (12) can be further approximated as
\[
b_1 = -2 b_2 \eta^{-1} - \eta \tag{13}
\]
We can similarly calculate $\mathcal{H}$ and $\mathcal{D}$. Substituting $b_1$ by Eq. (13), the expression of $\mathcal{D}/\mathcal{H}$ contains only $b_2$. Then, we find $b_2$ by minimizing $\mathcal{D}/\mathcal{H}$
\[
\frac{d}{db_2} \left( \frac{\mathcal{D}}{\mathcal{H}} \right) = 0 \tag{14}
\]
The results are:

\[
\begin{aligned}
&b_2 = \frac{1}{2} \eta^2, \quad b_1 = -2 \eta \\
&\mathcal{H} = \frac{1}{\eta}, \quad D = 4, \quad c_1 = \frac{D \mathcal{H}}{\eta} = 4 \eta
\end{aligned}
\]

Rewriting in terms of original parameters, we find the slowest decaying rate is

\[
\lambda_1 = 4 \alpha \gamma (H_s - H) = 4 \left( \frac{I}{I_c} - 1 \right) \tau_0^{-1}
\]

where \( \tau_0^{-1} = \frac{n \mu B}{q m} I_c = \alpha \gamma (H + H_k) \).

Although the variational method is able to analytically obtain the upper bound for the rate of the relaxation, its accuracy is unknown. A more rigorous method is to expand \( F(x) \) via Taylor series, i.e., \( F(x) = \sum a_n x^n \) \((n = 0, 1, 2...\)). By substituting it into Eq. (6), we find the recurrence formula for coefficient \( a_n \)

\[
(n + 2)(n + 1)a_{n+2} - \zeta (n + 1)a_{n+1} + [c_n - n(n - 1) + n(2 \xi - 2)]a_n + \zeta (n - 1)a_{n-1} - 2 \xi (n - 2)a_{n-2} = 0
\]

where \( \zeta = \eta - 2 \xi = \frac{m}{k_B T}(H_s - H) \). In the case of \( \zeta = \xi = 0 \), the above equation reduces to a two-term recursion formula, and the solutions are the Legendre polynomials with \( c_n = n(n + 1) \). This is the well-known solution for the thermal decay of a free particle \([2]\). In the case of \( \zeta = 0 \), Eq. (16) reduces to a three-term recursion formula, and the continued fraction method can be used to obtain the smallest nonzero eigenvalue \( c_1 = 2 - \frac{4}{5} \xi + O(\xi^2) \)[3].

In our case where both \( \zeta \) and \( \xi \) are non-zero, above methods can not be used directly. One way to solve the problem is to numerically calculate Eq. (16) by simply keeping enough terms (i.e. up to a large number \( N = 300 \)) to ensure the convergence of \( c_1 \). In Fig. 1, we show that the relaxation rates as a function of temperature \( (H_k \text{ is fixed}) \) for different \( \frac{I}{I_c} \)(\( \geq 1 \)). In Fig. 2, we show that the relaxation rate is linearly dependent of \( \frac{I}{I_c} - 1 \) when \( \frac{I}{I_c} > 1.5 \).

By comparing with Eq. (15) derived from the variational approach, we find that to a good approximation the relaxation rate is a factor of two smaller, i.e.,

\[
\lambda_1 = 2(\frac{I}{I_c} - 1) \tau_0^{-1}.
\]
III. COMPARISON WITH SUN’S MODEL

Sun’s model [1] assumes that thermally distributed initial magnetization states determine the distribution of switching time in case of \( I > I_c \). In this model, the switching time at zero temperature is estimated as \( \tau = \tau_0 (I/I_c - 1)^{-1} \ln(\pi/2\theta) \), where \( \theta \) is the initial angle of magnetization whose distribution is given by Eq. (1). The corresponding distribution of switching time is calculated from the definition: \( D(\tau)d\tau = -P(\theta)\sin\theta d\theta \). By defining the switching probability density \( D(\tau)d\tau = -P(n_m, 0)\sin\theta d\theta \) and utilizing the above relation between \( \tau \) and \( \theta \), the probability of not being switched is

\[
E_r(t) \equiv 1 - \int_0^t D(\tau)d\tau \simeq \frac{\pi^2}{4} \xi^4 \exp \left( -\frac{2(I/I_c - 1)}{\tau_0} t \right)
\]

where a long time limit is taken. The relaxation rate is identical with Eq. (17). This suggests the initial-condition randomization is the leading cause for switching time distribution in the limit of large \( \xi \) and \( I/I_c \gg 1 \). Deviation occurs when \( \xi \) is small, such as shown in Fig. 1, or when the current is near \( I_c \), as in Fig. 2.

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Figure Caption

FIG.1 (Color online) Numerical results of the smallest nonzero eigenvalues of Eq. (16) for increasing $\xi$, in cases of different $I/I_c$. $\tau_0^{-1} = \alpha \gamma (H + H_k)$. 

FIG.2 (Color online) Numerical results of the smallest nonzero eigenvalues of Eq. (16) for increasing $I/I_c - 1$, where $\xi = 80$. The red line is the fitting line: $\lambda_1 \tau_0 = 2(I/I_c - 1)$. 
FIG. 1:
Numerical results, $\xi = 80$

Fitting: $\lambda_1 \tau_0 = 2 (I / I_c - 1)$

FIG. 2: