Approximation of pressure perturbations by FEM

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Abstract

In the mathematical problem of linear hydrodynamic stability for shear flows against Tollmien-Schlichting perturbations, the continuity equation for the perturbation of the velocity is replaced by a Poisson equation for the pressure perturbation. The resulting eigenvalue problem, an alternative form for the two-point eigenvalue problem for the Orr-Sommerfeld equation, is formulated in a variational form and this one is approximated by finite element method (FEM). Possible applications to concrete cases are revealed.

Key words: linear hydrodynamic stability, pressure perturbation, Poisson equation, finite element method, eigenvalue problem.

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1 The classical model and the model based on a Poisson equation for the pressure perturbation

The mathematical problem of linear hydrodynamic stability is [1]

\[
\frac{\partial u'}{\partial t} + (U \cdot \text{grad})u' + (u' \cdot \text{grad})U = -\text{grad} p' + Re^{-1} \Delta u', \quad (t, x) \in \mathbb{R}_+ \times \Omega,
\]
\[
div u' = 0, \quad (t, x) \in \mathbb{R}_+ \times \Omega,
\]
\[
u', (t, x) = 0, \quad t \in \mathbb{R}_+, \quad x \in \partial \Omega,
\]
\[
u'(0, x) = u'_0(x), \quad x \in \Omega,
\]

and its solution is the perturbation \((u', p')\), for \(t > 0\). Here all variables are non dimensional, \((U, P)\) is the basic motion in the domain \(\Omega \subseteq \mathbb{R}^n, n = 2\) or 3, \(Re = U_\infty L/\nu\) is the Reynolds number, \(L\) is a length scale, \(U_\infty\) is a velocity scale, \(\nu\) is the coefficient of kinematic viscosity.
Applying the divergence operator to \((1a)\) and using the divergence-free condition on \(u'\), we obtain
\[
\Delta p' = -\text{div} \cdot (\mathbf{U} \cdot \text{grad} u') - \text{div} \cdot ((u' \cdot \text{grad} U),
\]
\((t, x) \in \mathbb{R}_+ \times \Omega,
\]
and projecting \((1a)\) on \(n_*\), we have
\[
n_*(x) \cdot \text{grad} p'(t, x) = n_*(x) \cdot F(Re, U(x), u'(t, x)),
\]
\(t \in \mathbb{R}_+, x \in \partial \Omega,
\]
where \(n_*\) is the unit outward normal to \(\partial \Omega\) and \(F(Re, U, u') = Re^{-1} \Delta u' - (\mathbf{U} \cdot \text{grad})u' - (u' \cdot \text{grad})U\). Further form of the right hand side of the Poisson equation for \(p'\) can be obtained by using the divergence-free condition on \(U\).

Thus the mathematical problem of the linear hydrodynamic stability becomes \((1a), (2), (1c), (3), (1d)\). Of course, it is assumed that equation \((1a)\) can be continued on the \(\partial \Omega\).

2 An alternative approximation of the two - point eigenvalue problem for Orr - Sommerfeld equation

Let \(\Omega = \{x = (x, y, z) \in \mathbb{R}^3 | -\infty < x, z < \infty, 0 < y < a\}\) \((a \geq 1)\) and assume that the basic flow is of the form \(U(x, y, z) = (U(y), 0, 0)\). Let us choose Tollmien-Schlichting waves - like perturbations \(u'_0(y) = u_0(y) = (u(y), v(y), 0), \mathbf{u}'(t, x, y, z) = (u'(t, x, y), v'(t, x, y), 0),
\]
\(p'(t, x, y, z) = p(y) \exp[\alpha (x-ct)],\) where \(i = \sqrt{-1}\) \(\alpha\) is the streamwise wave number, \(c = c_r + ic_i, c_r\) is the wave speed and \(c_i\) is the amplification rate.

In this case, model \((1)\) leads to the classical two - point eigenvalue problem for Orr - Sommerfeld equation in \((c, \varphi)\), where \(\varphi\) is the nonexponential factor of the stream function, while \((1a), (2), (1c), (3), (1d)\) becomes the two - point eigenvalue problem in \((c, u, v, p)\)
\[
-\alpha Re^{-1} iu + (\alpha Re)^{-1} iu'' + Uu - \alpha^{-1} iU'v + p = cu, y \in (0, a), (4a)
\]
\[
-\alpha Re^{-1} iv + (\alpha Re)^{-1} iv'' + Uv - \alpha^{-1} ip' = cv, y \in (0, a),
\]
\(4b\)
\[
-\alpha^2 p + p'' = -2\alpha U'v, y \in (0, a),
\]
\(4c\)
\[
u(y) = 0 \text{ for } y = 0, y = a,
\]
\(4d\)
\[
v(y) = 0 \text{ for } y = 0, y = a,
\]
\(4e\)
\[
p'(y) = Re^{-1} v''(y) \text{ for } y = 0, y = a.
\]
\(4f\)

where the prime stands for the differentiation with respect to \(y\), \(c\) is an eigenvalue and \((u, v, p)\) is an eigenvector.
Let \( L^2(0, a) \) be the Hilbert space of all measurable complex-valued functions \( u \), defined on \((0, a)\), for which \( \int_0^a |u(x)|^2 dx < \infty \) and the inner product is \( (u, v) = \int_0^a u(x)v(x) dx \). Denote \( Du \) the generalized derivative and consider the spaces \( H^1_0(0, a) = \{ u \in L^2(0, a) \mid Du \in L^2(0, a), u(0) = u(a) = 0 \} \), \( L^2_0(0, a) = \{ p \in L^2(0, a) \mid \int_0^a p(x) dx = 0 \} \).

Multiply (4a), (4b) and (4c) by arbitrary test functions \( f_1, f_2 \) and \( g \) respectively, integrate over \((0, a)\), apply partial integration if necessary and take into account the two point conditions (4d) - (4f) to obtain the following weak formulation of problem (4) in \((c, u, v, p) \in \mathbb{C} \times H^1_0(0, a)^2 \times L^2_0(0, a)\):

\[
-\alpha Re^{-1}i \int_0^a u f_1 dy - (\alpha Re)^{-1}i \int_0^a u f_1' dy + \int_0^a U u f_1 dy - \alpha Re^{-1}i \int_0^a v f_2 dy - (\alpha Re)^{-1}i \int_0^a v f_2' dy + \int_0^a U v f_2 dy
\]

\[
-\alpha Re^{-1}i \int_0^a v f_2 dy - (\alpha Re)^{-1}i \int_0^a v f_2' dy + \int_0^a U v f_2 dy
\]

\[
\alpha^{-1} \int_0^a f_1 f_1' dy = c \cdot \int_0^a f_1 f_1 dy, \forall f_1 \in H^1_0(0, a),
\]

\[
\alpha^{-1} \int_0^a f_2 f_2' dy = c \cdot \int_0^a f_2 f_2 dy, \forall f_2 \in H^1_0(0, a),
\]

\[
-\alpha^2 \int_0^a p g dy - \int_0^a p g' dy = -Re^{-1} v''(a) g(a)
\]

\[
+ Re^{-1} v''(0) g(0) - 2\alpha i \int_0^a U' v g dy, \forall g \in L^2_0(0, a).
\]

3 Approximation of problem (5) - (7) by FEM

In order to perform this approximation, let us divide the interval \([0, a]\) in \(N + 1\) subintervals \( K = K_j = [y_j, y_{j+1}], 0 \leq j \leq N, \) where \(0 = y_0 < y_1 < \ldots < y_{N+1} = a\). The sets \( K \) represent a triangulation \( \mathcal{T}_h \) of \([0, a]\).

The approximate basic shear flow \( U_h(x, y, z) = (U_h(y), 0, 0) \) is defined by \( U_h(y) = U(y), y \in [0, a] \). The approximate amplitudes \( u_h(y), v_h(y) \) and \( p_h(y) \) correspond to the exact ones, \( u(y), v(y) \) and \( p(y) \), respectively.
Introduce the spaces \( V_h = \{ v : [0, a] \to \mathbb{C} \mid v \in C[0, a], v_{\text{is on}} K_j \}
\) an one-dimensional polynomial, having complex coefficients, of degree 2, 0 \( \leq j \leq N, v(0) = v(a) = 0 \) and \( M_h = \{ v : [0, a] \to \mathbb{C} \mid v \in C[0, a], v_{\text{is on}} K_j \}
\) an one-dimensional polynomial, having complex coefficients, of degree 1, 0 \( \leq j \leq N \).

Correspondingly, variational problem (3) - (7) is approximated by the following problem in \((c_h, u_h, v_h, p_h) \in \mathbb{C} \times V_h^2 \times M_h,\)

\[
- \alpha Re^{-1}i \int_{0}^{a} u_h^j 1dy - (\alpha Re)^{-1}i \int_{0}^{a} u_h^j 1'dy + \int_{0}^{a} U u_h^j 1dy - \\
- \alpha^{-1}i \int_{0}^{a} U'v_h^j 1dy + \int_{0}^{a} p_h^j 1dy = c_h \cdot \int_{0}^{a} u_h^j 1dy, \forall f_1 \in V_h, \\
- \alpha Re^{-1}i \int_{0}^{a} v_h^j 2dy - (\alpha Re)^{-1}i \int_{0}^{a} v_h^j 2'dy + \int_{0}^{a} U v_h^j 2dy \\
+ \alpha^{-1}i \int_{0}^{a} p_h^j 2'dy = c_h \cdot \int_{0}^{a} v_h^j 2dy, \forall f_2 \in V_h, \\
- \alpha^2 \int_{0}^{a} p_h^j gdy - \int_{0}^{a} p_h^j g'dy = -Re^{-1}v_h^j(a)g(a) \\
+ Re^{-1}v_h^j(0)g(0) - 2\alpha i \int_{0}^{a} U'v_h^j gd, \forall g \in M_h.
\]

In order to obtain \( u_h, v_h \), we use a basis of real functions of \( V_h \). Let \( J = J_K = \{1, 2, 3\} \) be the local numeration for the nodes of \( K \), where 1, 3 correspond to \( y_j, y_{j+1} \) respectively and 2 corresponds to a node between \( y_j \) and \( y_{j+1} \). Let \( \{\phi_n, n \in J\} \) be the local quadratic basis of functions on \( K \) corresponding to the local nodes. Let \( J_s = \{1, \ldots, 2N + 1\} \) be the global numeration for the nodes of \([0, a]\), where the nodes corresponding to \( y_0 \) and \( y_{N+1} \) are not taken into account, and let \( L_1 \) be a matrix whose elements are the elements of \( J_s \). Its rows are indexed by the elements \( K \in T_h \) and its columns, by the local numeration \( n \in J \). We take the value 0 at the locations of \( L_1 \) which we do not consider in the computations, i.e. the locations where \( K = K_0, n = 1 \) and \( K = K_N, n = 3 \). Write \( n_s = L_1(K, n) \) or, simply, \( n_s \) for the element \( n_s \) of \( L_1 \) which depends on \( K \) and \( n \). Let \( \{ \Phi_{n_s}, 0 \}, \{0, \Phi_{n_s}\}; n_s \in J_s \) be a basis of functions of \( V_h^2 \). If \( n_s \) corresponds
to $y_j$, then $\Phi_{n_\ast}$ is a real quadratic function on $K_{j-1}$ and $K_j$ and its value is zero on $[0,a] \backslash (K_{j-1} \cup K_j)$. If $n_\ast$ lies between $y_j$ and $y_{j+1}$, then $\Phi_{n_\ast}$ is a real quadratic function on $K_j$ and its value is zero on $[0,a] \backslash K_j$. We have $\Phi_{n_\ast}(y) = \phi_n(y)$, where $y \in K$, $n_\ast = L_1(K,n)$. Let $u_{n_\ast}, v_{n_\ast}$ be the values of $u_h, v_h$ at the nodes $n_\ast, n_\ast \in J_\ast$. Retaining our convention about $n_\ast = L_1(K,n)$, we do not write in the sequel the conditions $n \neq 1$ for $K = K_0$ and $n \neq 3$ for $K = K_N$. We have

$$u_h(y) = \sum_{n_\ast \in J_\ast} u_{n_\ast} \Phi_{n_\ast}(y) = \sum_{K \in T_h} \sum_{y \in K} u_{n_\ast} \phi_n(y)$$

and a similar expression for $v_h(y)$.

In order to obtain $p_h$, we use a basis of real functions of $M_h$. Let $I = I_K = \{1,2\}$ be the local numeration for the nodes of $K$, where 1, 2 correspond to $y_j, y_{j+1}$ respectively. Let $\{\psi_m, m \in I\}$ be the local affine basis of functions on $K$ corresponding to the local nodes. Let $I_\ast = \{0,1,\ldots, N, N + 1\}$ be the global numeration for the nodes of $[0,a]$ and let $L_2$ be a matrix whose elements are the elements of $I_\ast$. Its rows are indexed by the elements $K \in T_h$ and its columns, by the local numeration $m \in I$. Write $m_\ast = L_2(K,m)$ or, simply, $m_\ast$ for the element $m_\ast$ of $L_2$ which depends on $K$ and $m$. Let $\{\Psi_{m_\ast}, m_\ast \in I_\ast\}$ be a basis of functions of $M_h$. If $m_\ast$ corresponds to $y_j$, then $\Psi_{m_\ast}$ is a real affine function on $K_{j-1}$ and $K_j$ and its value is zero on $[0,a] \backslash (K_{j-1} \cup K_j)$. We have $\Psi_{m_\ast}(y) = \psi_m(y)$, where $y \in K$, $m_\ast = L_2(K,m)$. Let $p_{m_\ast}$ be the values of $p_h$ at the nodes $m_\ast, m_\ast \in I_\ast$. We have

$$p_h(y) = \sum_{m_\ast \in I_\ast} p_{m_\ast} \Psi_{m_\ast}(y) = \sum_{K \in T_h} \sum_{y \in K} p_{m_\ast} \psi_m(y).$$

Problem (8) - (10) becomes the following algebraic eigenvalue problem

$$\sum_{K \in T_h} \sum_{n_\ast \in J} u_{n_\ast} \left[-\alpha \Re^{-1} i \int K \phi_n \phi_k dz - (\alpha \Re)^{-1} i \int K \phi_n' \phi_k' dz + \int K U \phi_n \phi_k dz\right] + \sum_{K \in T_h} \sum_{n_\ast \in J} v_{n_\ast} \left[-\alpha^{-1} i \int K U' \phi_n \phi_k dz\right] + \sum_{K \in T_h} \sum_{m \in I} p_{m_\ast} \int K \psi_m \phi_k dz = c_h \cdot \left\{ \sum_{K \in T_h} \sum_{n_\ast} u_{n_\ast} \int K \phi_n \phi_k dz \right\}$$

(11)
\[
\sum_{K \in T_h} \sum_{n \in J} v_{n_k} \left[ \alpha Re^{-1} i \int_{K} \phi_n \phi_k dz - (\alpha Re)^{-1} i \int_{K} \phi'_n \phi'_k dz \right. \\
+ \left. \int_{K} U \phi_n \phi_k dz \right] + \sum_{K \in T_h} \sum_{m \in I} p_{m_k} \left[ \alpha^{-1} i \int_{K} \psi_m \phi_k dz \right] \\
= c_h \cdot \left\{ \sum_{K \in T_h} \sum_{n \in J} v_{n_k} \int_{K} \phi_n \phi_k dz \right\} \\
\]

(12)

\[
\sum_{K \in T_h} \sum_{m \in I} p_{m_k} \left[ -\alpha^2 i \int_{K} \psi_m \psi_k dz - \int_{K} \psi'_m \psi'_k dz \right] \\
= - \sum_{K=K_N; n \in J_{K_N}; n \neq 3} v_{n_k} Re^{-1} \Phi'_n(a) \Phi_{N+1}(a) \\
+ \sum_{K=K_0; n \in J_{K_0}; n \neq 1} v_{n_k} Re^{-1} \Phi''_n(0) \Phi_1(0) \\
+ \sum_{K \in T_h} \sum_{n \in J} v_{n_k} \left( -2\alpha i \int_{K} U' \phi_n \psi_k dz \right),
\]

(13)

for all \( k \in J \), \( \ell \in I \), for all \( K \in T_h \), \( k \neq 1 \) when \( K = K_0 \), \( k \neq 3 \) when \( K = K_N \).

The eigenvalue problem in \( c_h \in \mathbb{C} \), \( A_h = (u_1, v_1, \ldots, u_{2N+1}, v_{2N+1}) \in \mathbb{C}^{2(N+1)} \), \( B_h = (p_0, p_1, \ldots, p_{N+1}) \in \mathbb{C}^{N+2} \) represented by relations (11) - (13) can be written in the following matrix form

\[
K_h A_h + L_h B_h = c_h S_h A_h, \quad G_h B_h = H_h A_h,
\]

(14)

where \( K_h, S_h \in \mathbb{C}^{2(N+1) \times 2(N+1)}, L_h \in \mathbb{C}^{2(N+1) \times (N+2)}, G_h \in \mathbb{C}^{(N+2) \times (N+2)}, H_h \in \mathbb{C}^{(N+2) \times 2(N+1)} \).

Matrix \( G_h \) is positive definite and symmetric. So problem (14) becomes

\[
(K_h + L_h G_h^{-1} H_h) A_h = c_h S_h A_h, \quad B_h = G_h^{-1} H_h A_h,
\]

(15)

The eigenvalue problem given by the first equation of (15) can be solved by using the QZ method or, after multiplication by \( S_h^{-1} \) at the left, by using the QR or LR method.

4 Possible applications to concrete fluid flows

Once (15) solved, several linear hydrodynamic stability characteristics can be determined numerically. Keeping \( \alpha \) and \( Re \) fixed, (15) yields approximations of \( c \) and of amplitude distributions \( u(y), v(y) \) and \( p(y) \). Repeating this operation with various values of \( \alpha \) and \( Re \), we can identify the
pairs \((Re, \alpha)\) where the approximate value \(c_i = 0\). Thus, we can construct the neutral curve. We also can determine the curves \(c_i(Re, \alpha) = \text{constant}\) of constant amplification factor and the wave speed \(c_r(Re, \alpha)\), for \((Re, \alpha)\) belonging to the neutral curve.

In particular, problem (15) is the basis of a computer program for the case of Prandtl’s boundary layer, which will be presented elsewhere.

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