Parabolic models for chemotaxis on weighted networks

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Abstract

In this work we consider the Keller-Segel model for chemotaxis on networks, both in the doubly parabolic case and in the parabolic-elliptic one. Introducing appropriate transition conditions at vertices, we prove the existence of a time global and spatially continuous solution for each of the two systems. The main tool is the use of the explicit formula for the fundamental solution of the heat equation on a weighted graph and of the corresponding sharp estimates.

Key words: Chemotaxis; network; transmission conditions; heat kernel.

AMS subject classification: 92C17; 92C42; 35R02; 35Q92; 35A01.

1 Introduction

We consider the classical Keller-Segel system for chemotaxis

$$
\begin{align*}
    u_t &= \Delta u - \nabla \cdot (u \nabla c) \\
    \varepsilon c_t &= \Delta c + u - \alpha c
\end{align*}
$$

(1.1)
on a finite weighted network $\Gamma$, where $\varepsilon, \alpha \geq 0$.

System (1.1) has been introduced in the early seventies in [12, 13] in order to model the aggregation phenomenon undergone by the slime mold Dictyostelium discoideum. In this biological context, $u$ represents the cell concentration of the organism and satisfies the continuity equation in (1.1), while $c$ is the chemo-attractant concentration and solves the diffusion equation in (1.1). In the Euclidean case, i.e. when (1.1) is considered on a domain of $\mathbb{R}^d$, there is a vast literature on system (1.1). Depending on the space dimension $d$, $\varepsilon > 0$ (double parabolic case) or $\varepsilon = 0$ (parabolic-elliptic case) and the initial data $u^0$, $c^0$, different phenomena can occur: global existence, finite or infinite time blow-up, peaks formation, threshold phenomena, etc. We refer to [4, 10, 22] and the references therein for more details on that problem.

Even if the study of differential equations on networks goes back to the seminal papers by Lumer [15, 16] (see also the references in [17]), in the recent time there is an increasing interest in this type of problems in connection with applications such as data transmission, traffic management, crowd motion and various problems in biology and neurobiology ([7, 8, 9, 18, 24]).
The parabolic model for chemotaxis (1.1) on networks has been recently considered in [3] to describe the evolution of the ameboid organism *Physarum polycephalum*. Indeed, during its evolution this slime mold arranges a network of thin tubes where nutrients and chemical signals are transmitted to the different parts of the organism (see [19, 25]). More precisely, in [3] system (1.1) has been analyzed numerically and the well posedness of a discrete system obtained via its finite differences approximation has been obtained. In [9] the authors consider a mathematical model in connection with the dermal wound healing process, where fibroblasts move along a polymeric scaffold to fill the wound, driven by chemotaxis. But in contrast with (1.1), in [9] the equations satisfied by the cells density $u$ are of hyperbolic type.

The goal of this paper is to consider the parabolic system (1.1) on a finite weighted network $\Gamma$ composed of $m$ edges and to prove the existence of a time global and spatially continuous solution $(u, c)$, in both cases $\varepsilon > 0$ and $\varepsilon = 0$. Therefore system (1.1) shall appear to be formally equivalent to $m$ Keller-Segel systems, one on each of the $m$ edges, coupled via the transition conditions at the vertices of the network. As usual when dealing with differential equations on networks, the transition conditions at the vertices shall play a crucial role. Coherently with the parabolic nature of the problem (see [18]), we look for a global continuous solution on the whole network and consequently we prescribe the continuity of $u$ and $c$ at the vertices (see (3.6)-(3.7)). Moreover, we require for $u$ the flux conservation at the vertices, while for $c$ a Kirchhoff type condition which guarantees the validity of the maximum principle for diffusion equations on networks (see (3.4) and (3.5) respectively). On the other hand, for simplicity reason, the network we consider has no boundary nodes. Our results can be however extended to the case of Dirichlet or mixed boundary conditions.

More specifically, we consider solution of (1.1) in the following integral sense

$$u(t, y) = P_t u^0(y) - \int_0^t P_{(t-s)} \partial_x (u(s) \partial_x c(s))(y) ds, \quad y \in \Gamma,$$

$$c(t, y) = e^{-(\alpha/\varepsilon)t} P_{(t/\varepsilon)} c^0(y) + \frac{1}{\varepsilon} \int_0^t e^{-(\alpha/\varepsilon)(t-s)} P_{((t-s)/\varepsilon)} u(s)(y) ds, \quad y \in \Gamma,$$

where $(P_t)_{t \geq 0}$ is the semigroup generated by the laplacian $-\Delta_\Gamma$ on $\Gamma$ with domain the space of continuous functions on $\Gamma$, belonging to $H^2$ on every edges of $\Gamma$ and satisfying weighted Kirchhoff conditions at the nodes (see (2.3)). The interest in considering the formulation (1.2)-(1.3) lies in the fact that $(P_t)_{t \geq 0}$ is given explicitly through the fundamental solution $H = H(t, x, y)$ (see formula (2.6) and (2.13)). Then, with this integral formula at hand, the proofs for local and global existence follow the corresponding arguments in the Euclidean case, with however specific modifications due to the network structure. In particular, it is not possible to use known results about the existence of solutions of (1.1) on a bounded interval $[0, L]$ with homogeneous Neumann boundary conditions (see [11] for instance). More than this, in order to get appropriate time bounds on the norm of $u$ and $c$ we need to prove optimal $L^p$ bounds for the heat kernel $H$ on $\Gamma$, which improve earlier results in [23, 5, 6].

We remark that the integral formulation for solutions of the heat equation on networks and the corresponding $L^p$ estimates can be the starting point to derive qualitative information about the
behavior of the solutions with respect to the structural elements of the network. We are pursuing this
analysis in a forthcoming paper. Moreover they can be useful not only for the Keller-Segel model, but
also to transpose other problems of parabolic nature from the Euclidean case to the networks.

The paper is organized as follows. In Section 2 we define the network $\Gamma$, we recall the fundamental
solution of the heat equation on $\Gamma$ and we deduce the optimal $L^p$-estimates for the heat kernel. Section 3
is devoted to the study of the parabolic-parabolic chemotaxis system on $\Gamma$, while Section 4 to the
parabolic-elliptic one. In the appendices we give the proofs of some technical results.

2 The heat equation on the network

This section is devoted to the definition of the network and to the properties of the fundamental
solution of the heat equation on that network. The fundamental solution has been computed by Roth
\[23\] for a finite network and generalized to the case of an infinite homogeneous tree and a countable
graph by Cattaneo \[5, 6\]. Therefore, notations are coherent with the ones in these papers (see also
\[20\]). However, some properties, such as the optimal $L^1$ and $L^\infty$ time decay, are not contained in the
cited papers. Their proofs are sketched in the Appendix B.

2.1 The network

Let consider a finite, connected and non-oriented (or undirected) network $\Gamma$. This means that the
underlying graph $\mathcal{G} = (V, E)$ is defined through a non empty finite set of $n$ vertices or nodes, $V :=
\{v_1, \ldots, v_n\}$, a non empty finite set of $m$ non-oriented open edges (or links), $E := \{e_1, \ldots, e_m\}$, and
that between every pair of nodes $v_i, v_j \in V$ there exists a path with edges in $E$. Furthermore, we
assume that the graph has no self-loops (no edge connecting a vertex to itself). On the other hand,
the graph can contain multiple links, i.e. the map $E \mapsto V \times V$ associating to each non-oriented edge
its endpoints can be not injective.

Every edge may have a different length. However, we parametrize and normalize each $e_j \in E$ so that to identify $\mathfrak{e}_j$ with the interval $[0, 1]$. Since the network is undirected, every edge $e_j \in E$ can be parametrized in two different ways giving rise to two oriented edges $e_j^\pm$, i.e. there exist two homeomorphism $\Pi_j^\pm : [0, 1] \mapsto (\mathfrak{e}_j)^\pm$, such that $\Pi_j^+(0) = \Pi_j^-(1)$ and $\Pi_j^-(1) = \Pi_j^+(0)$. We shall call an oriented edge an arc, and we shall denote by $a_j$ any of the two edges $e_j^\pm$. Moreover, we shall denote by $-a_j$ the arc opposite to $a_j$ and the initial and terminal endpoints of $a_j$ by $I(a_j)$ and $T(a_j)$, respectively.

We also denote by $E(v_i)$ the set of the index $j$ such that the edge $e_j$ has an endpoint at the vertex $v_i \in V$ and by $d(v_i)$ the degree of $v_i$, i.e. the cardinality of $E(v_i)$.

Next, we define a path $C$ on the network $\Gamma$ as a finite sequence of (at least two) arcs, $(a_{j_1}, \ldots, a_{j_k})$, $k \geq 2$, such that $T(a_{j_l}) = I(a_{j_{l+1}})$, $l = 1, \ldots, k - 1$. Thus a path is always oriented. We associate to each path $C = (a_{j_1}, \ldots, a_{j_k})$ its length $|C|$ as the number of the arcs composing $C$. Then, given two points $x$ and $y$ on $\Gamma$, we shall note $C_k(x, y)$ the set of the paths of length $k$ such that $x$ belongs to the
first arc of the path and $y$ belongs to the last arc of the path, i.e.

$$C_k(x, y) := \{ C = (a_{j_1}, \ldots, a_{j_k}) : x \in a_{j_1} \text{ and } y \in a_{j_k} \}, \quad k = 2, 3, \ldots .$$

A geodesic path joining $x$ to $y$ on $\Gamma$ is any path of minimum length in $\bigcup_{k \geq 2} C_k(x, y)$. We shall denote $\mathcal{L}(x, y)$ the common length of any geodesic path joining $x$ to $y$ and we also define

$$\rho(x, y) := \mathcal{L}(x, y) - 2.$$

For every $x$ and $y$ belonging to the same $\mathcal{P}_j$, we define the distance $d(x, y)$ as

$$d(x, y) := |(\Pi_j^+)^{-1}(x) - (\Pi_j^+)^{-1}(y)|, \quad x, y \in \mathcal{P}_j.$$

Then, for every $x$ and $y$ on $\Gamma$, we define the distance of $x$ to $y$ along $C = (a_{j_1}, \ldots, a_{j_k}) \in C_k(x, y)$ as

$$d_C(x, y) := d(x, T(a_{j_1})) + d(y, I(a_{j_k})) + |C| - 2.$$

Obviously, $d_C(x, y)$ is symmetric with respect to $x$ and $y$, i.e. $d_C(x, y) = d_C(y, x)$.

Finally, to each non-oriented edge $e_j \in E$ we associate a positive weight $\kappa(e_j)$ and we assume that

$$0 < \kappa_0 \leq \kappa(e_j) \leq \kappa_1, \quad \forall j = 1, \ldots, m.$$

The weights $\kappa(e_j)$ shall influence the transmission or the reflection of $u$ through the nodes. Indeed, for each couple of arcs $(a_i, a_j)$, we introduce the transfer/reflection coefficient from $a_i$ to $a_j$ as

$$\epsilon_{(a_i \rightarrow a_j)} := \begin{cases} 2\kappa(e_j)(\sum_{l \in E(T(a_i))} \kappa(e_l))^{-1} & \text{if } T(a_i) = I(a_j) \text{ and } a_j \neq -a_i \quad \text{(transmission)} \\ 2\kappa(e_j)(\sum_{l \in E(T(a_i))} \kappa(e_l))^{-1} - 1 & \text{if } a_j = -a_i \quad \text{(reflection)} \\ 0 & \text{otherwise} \end{cases}$$

(2.1)

The weight $\epsilon(C)$ of a path $C = (a_{j_1}, \ldots, a_{j_k})$ is then the product of the transfer/reflection coefficients of all the pairs of consecutive arcs composing $C$, i.e.

$$\epsilon(C) := \prod_{l=1}^{k-1} \epsilon_{(a_{j_l} \rightarrow a_{j_{l+1}})}.$$  

(2.2)

It is worth noticing that in case of reflection, the coefficient $\epsilon_{(a_i \rightarrow -a_i)}$ may be negative, and so also the weight $\epsilon(C)$ of all path $C$ passing through $a_i$ and $-a_i$ consecutively. Moreover, $\epsilon_{(a_i \rightarrow a_j)} \neq \epsilon_{(a_j \rightarrow a_i)}$ and $\epsilon(C) \neq \epsilon(-C)$, in general.

The definitions (2.1) of the transfer/reflection coefficients and (2.2) of the paths weights come naturally in the construction of the fundamental solution of the heat equation on the network given in Appendix A.
2.2 The heat equation on the network

A function $u$ defined on the network $\Gamma$ is a collection of $m$ functions $(u_j)_{j=1}^m$ such that $u_j := u_{|e_j}$. To every function $u$ on $\Gamma$ we associate the vector valued function $\tilde{u} = (\tilde{u}_1, \ldots, \tilde{u}_m)$ defined on $[0,1]$ such that $\tilde{u}_j := u \circ \Pi_j^\pm$. Then, we denote $u_j'(x)$ and $u_j''(x)$, $x \in e_j$, the derivatives $\tilde{u}_j'(\xi)$ and $\tilde{u}_j''(\xi)$ with respect to $\xi \in (0,1)$, $\xi = (\Pi_j^\pm)^{-1}(x)$. We also define the exterior normal derivative of $u_j$ at the endpoints of the arc $a_j$ as

$$\frac{\partial u_j(I(a_j))}{\partial n} = -\lim_{h \to 0^+} \frac{\tilde{u}_j(h) - \tilde{u}_j(0)}{h} \quad \text{and} \quad \frac{\partial u_j(T(a_j))}{\partial n} = \lim_{h \to 0^-} \frac{\tilde{u}_j(1 + h) - \tilde{u}_j(1)}{h}.$$

Next, we define the space of continuous functions on $\Gamma$

$$C^0(\Gamma) := \{u = (u_j)_{j=1}^m : u_j(v_i) = u_k(v_i) \text{ if } j,k \in E(v_i), \ i = 1,\ldots,n\},$$

the integral of a function $u$ over $\Gamma$

$$\int_\Gamma u(x)dx := \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{u}_j(\xi) \, d\xi,$$

the Lebesgue spaces

$$L^p(\Gamma) := \{u = (u_j)_{j=1}^m : \|u\|_{L^p(\Gamma)}^p := \sum_{j=1}^m \kappa(e_j) \|\tilde{u}_j\|_{L^p(0,1)}^p < \infty\}, \quad p \in [1,\infty),$$

$$L^{\infty}(\Gamma) := \{u = (u_j)_{j=1}^m : \|u\|_{L^{\infty}(\Gamma)} := \max_{1 \leq j \leq m} \kappa(e_j) \|\tilde{u}_j\|_{L^{\infty}(0,1)} \},$$

and the Sobolev spaces

$$H^{1,\infty}(\Gamma) := \{u \in C^0(\Gamma) : u' \in L^{\infty}(\Gamma)\},$$

$$H^r(\Gamma) := \{u \in C^0(\Gamma) : \|u\|_{H^r(\Gamma)}^2 := \sum_{j=1}^m \kappa(e_j) \|\tilde{u}_j\|_{H^r(0,1)}^2 < \infty\}.$$

With these notations, we can now introduce the operator $(D(-\Delta_\Gamma), -\Delta_\Gamma)$, where the domain $D(-\Delta_\Gamma)$ is the set of the function $u$ in $H^2(\Gamma)$ satisfying the transmission conditions of Kirchhoff type at every vertex $v_i \in V$,

$$D(-\Delta_\Gamma) := \{u \in H^2(\Gamma) : \sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial u_j}{\partial n}(v_i) = 0, \ i = 1,\ldots,n\}, \quad (2.3)$$

and, for all $u \in D(-\Delta_\Gamma)$, the laplacian $\Delta_\Gamma$ on $\Gamma$ is naturally defined as $\Delta_\Gamma u = u''$. Then, $-\Delta_\Gamma$ is densely defined in the Hilbert space $L^2(\Gamma)$ endowed with the scalar product

$$(u,v)_{L^2(\Gamma)} = \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{u}_j'(\xi)\tilde{v}_j'(\xi) \, d\xi.$$

It is also symmetric and positive and consequently accretive. Thanks to the transmission conditions, it can be also proved that $-\Delta_\Gamma$ is $m$-accretive and therefore self-adjoint (see for instance [14]). Hence, we
can associate to $-\Delta \Gamma$ a semigroup of contractions on $L^2(\Gamma)$, say $(T(t))_{t \geq 0}$, whose generator is $\Delta \Gamma$. To conclude, given any $f \in L^2(\Gamma)$, the function $u(t) = T(t)f$ is the unique solution of the heat equation

$$\begin{cases}
    \partial_t u = \Delta \Gamma u & \text{on } (0, \infty) \times \Gamma, \\
    u(0) = f & \text{on } \Gamma.
\end{cases}$$

(2.4)

in the space $C([0, \infty), L^2(\Gamma)) \cap C((0, \infty), D(-\Delta \Gamma)) \cap C^1((0, \infty), L^2(\Gamma))$.

Problem (2.4) can be also written as a system of $m$ heat equations coupled through the continuity and transmission conditions at the vertex, i.e.

$$\begin{cases}
    \partial_t u_j = \partial_{xx} u_j & \text{on } (0, \infty) \times e_j, j = 1, \ldots, m \\
    u_j(0) = f_j & \text{on } e_j, j = 1, \ldots, m \\
    u_j(t, v_i) = u_k(t, v_i) \text{ if } j, k \in E(v_i), i = 1, \ldots, n \quad t > 0 \quad \text{(continuity)} \\
    \sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial u_j}{\partial n} (t, v_i) = 0, i = 1, \ldots, n \quad t > 0 \quad \text{(transmission condition)}
\end{cases}$$

(2.5)

These transmission conditions together provide, for each node $v_i$, a system of $d(v_i)$ equations for the $d(v_i)$ components $u_j$ of the solution $u$ such that $j \in E(v_i)$. Moreover, they reduces at the vertex $v_i$ of degree 1, $d(v_i) = 1$, to the homogeneous Neumann boundary condition.

Finally, it is worth noticing that the choice of the orientation of the edges $e_j$ has no consequences, since the heat equation (2.4)-(2.5) and the problem (1.1) are invariant under the transformation $\xi \rightarrow (1 - \xi)$ that commute $\Pi_j^+ \rightarrow \Pi_j^-$ and vice versa, as well as all the definitions given above. On the other hand, orientation appears to be necessary for the construction of the fundamental solution of the heat equation on $\Gamma$ below, that we shall use for the resolution of (1.1). For the analysis of (2.4)-(2.5) through the abstract semigroup method see [14] and the reference therein.

2.3 The fundamental solution of the heat equation on the network

Let $G(t, z) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{z^2}{4t}}$ denote the heat kernel on $(0, \infty) \times \mathbb{R}$, and consider the function defined on $(0, \infty) \times \Gamma \times \Gamma$ as

$$H(t, x, y) = \delta_{i,j} \kappa^{-1}(e_i) G(t, d(x, y)) + L(t, x, y), \quad x \in \Gamma_i, y \in \Gamma_j, i, j \in \{1, \ldots, m\},$$

(2.6)

where $\delta_{ij}$ is the usual Kronecker’s delta function and

$$L(t, x, y) = \sum_{k \geq \rho(x,y)} \sum_{C \in C_{k+2}(x,y)} \kappa^{-1}(e_i) \epsilon(C) G(t, d_C(x, y)), \quad x \in \Gamma_i, y \in \Gamma_j, i, j \in \{1, \ldots, m\}.$$

(2.7)

The first term in (2.6) is simply the restriction of the fundamental solution of the heat equation on each edge of the network. The second term is determined in such a way that the function $H$ satisfies the continuity and transmission conditions in (2.5) with respect to $y$, for any fixed $x \in \Gamma$ (see Appendix A).

More specifically, since the network is composed of $m$ edges, it holds, for all $k \in \mathbb{N}$, that

$$\text{card}(C_{k+2}(x,y)) \leq 2 \left( \max_{i=1,\ldots,m} d(v_i) \right)^{k+1} \leq 2m^{k+1}.$$
Hence, the sum with respect to the paths \( C \in C_{k+2}(x,y) \) in (2.7) is finite. We also have that the coefficients (2.1) are uniformly bounded: \( |\epsilon_{(a_i,\rightarrow a_j)}| \leq 2\kappa_1 \kappa_0^{-1} := \bar{\epsilon}, \ i,j = 1, \ldots, m \). Therefore, by (2.8), we get

\[
|L(t, x, y)| \leq \kappa_0^{-1} \sum_{k=0}^{+\infty} (\tau m)^{k+1} \frac{e^{-k^2/4t}}{\sqrt{\pi t}} < \infty. \tag{2.9}
\]

The latter estimate implies that the series giving \( L(t, x, y) \) is normally convergent over \([t_1, t_2] \times \Gamma \times \Gamma\), for any fixed \( t_1, t_2 > 0 \). Therefore, the associated vector valued function \( \tilde{H} = \tilde{H}(t, \xi, \eta) \) is continuous with respect to \((t, \xi, \eta) \in (0, \infty) \times [0, 1] \times [0, 1]\), component by component. Similarly, for any fixed \( \xi \in (0, 1) \), the derivatives \( \partial_t \tilde{H}, \partial_\eta \tilde{H} \) and \( \partial_{\eta \eta} \tilde{H} \) exist and are continuous with respect to \((t, \xi, \eta) \in (0, \infty) \times (0, 1) \times (0, 1)\). They can be computed differentiating under the sum sign and \( \tilde{H} \) satisfies the heat equation \( \partial_t \tilde{H} = \partial_{\eta \eta} \tilde{H} \), component by component. These and other properties of the function \( H \) are resumed below.

**Theorem 2.1** (23). Let \( H \) be the function defined in (2.6). Then,

(i) \( H \) is continuous on \((0, \infty) \times \Gamma \times \Gamma\);

(ii) \( \partial_t H(t, x, y) \) exists for all \((t, x, y) \in (0, \infty) \times \Gamma \times \Gamma\) and it is continuous on \((0, \infty) \times \Gamma \times \Gamma\);

(iii) the derivatives \( \partial_\eta H(t, \xi, \eta) \) and \( \partial_{\eta \eta} H(t, \xi, \eta) \) exist for all \((t, \xi, \eta) \in (0, \infty) \times (0, 1) \times (0, 1)\) and are continuous on \((0, \infty) \times (0, 1) \times (0, 1)\);

(iv) \( H(t, x, \cdot) \in D(-\Delta_G) \) for all \((t, x) \in (0, \infty) \times \Gamma\);

(v) \( \partial_t H(t, x, y) = \partial_{yy} H(t, x, y) \) for all \((t, x, y) \in (0, \infty) \times \Gamma \times \Gamma\);

(vi) for all \( f \in C^0(\Gamma) \), \( \int_\Gamma H(t, x, y) f(x) dx \to f(y) \) for \( t \to 0^+ \), uniformly with respect to \( y \in \Gamma \);

(vii) for all \( f \in C^0(\Gamma) \), the function

\[
P_t f(y) := \int_\Gamma H(t, x, y) f(x) dx, \quad (t, y) \in (0, \infty) \times \Gamma \tag{2.10}
\]

with \( P_0 f = f \) is the unique continuous solution of the initial valued problem (2.1).

Moreover, \( H \) is symmetric with respect to \( x, y \in \Gamma \), i.e. \( H(t, x, y) = H(t, y, x) \) for all \( t \in (0, \infty) \) and the properties above hold true with respect to \( x \), for any fixed \( y \).

The function \( H \) is also the unique function satisfying properties (i)-(vii) in Theorem (2.1). As observed in (23), \( (P_t)_{t \geq 0} \) is a strongly continuous semigroup on \( L^2(\Gamma) \), whose infinitesimal generator is the closure of \( -\Delta_G \) in \( L^2(\Gamma) \). It is obviously the same semigroup determined in (14) by variational methods.

It is worth noticing that \( H \) is not a priori positive since the weights \( \epsilon(C) \) could be negative. Furthermore, the spatial symmetry of \( H \) is due to the symmetry of \( G \) and to fact that changing \( x \) with \( y \) in (2.6)-(2.7), the path \( C \) changes into \(-C\) and \( \kappa^{-1}(e_i) \epsilon(C) = \kappa^{-1}(e_j) \epsilon(-C) \), (see (2.10)-(2.2)). The
construction of $H$ has been done in [23] in the case $\kappa(e_j) = 1, \forall j$. The generalization to the case of a weighted graph has been considered in [5, 6], where however the construction is not detailed. Therefore, we give the general construction in the Appendix A for the reader convenience.

We close this section showing the optimal decay in time of $H$ and its derivatives. The proofs are sketched in the Appendix B.

**Proposition 2.2.** Let $H$ be defined as in (2.6). Then,

$$\int_{\Gamma} H(t, x, y) dy = 1, \quad \forall (t, x) \in (0, \infty) \times \Gamma,$$

(2.11)

and there exist constants $C_i > 0$, $i = 1, \ldots, 4$, such that for all $t > 0$ it holds

$$\sup_{x \in \Gamma} \|H(t, x, \cdot)\|_{L^1(\Gamma)} \leq C_1,$$

(2.12)

$$\|H(t)\|_{L^\infty(\Gamma \times \Gamma)} \leq C_2(1 + t^{-1/2}),$$

(2.13)

$$\sup_{x \in \Gamma} \|\partial_y H(t, x, \cdot)\|_{L^1(\Gamma)} + \sup_{y \in \Gamma} \|\partial_y H(t, \cdot, y)\|_{L^1(\Gamma)} \leq C_3(1 + t^{-1/2}),$$

(2.14)

$$\|\partial_y H(t)\|_{L^\infty(\Gamma \times \Gamma)} \leq C_4(1 + t^{-1}).$$

(2.15)

Moreover, since $H$ is symmetric with respect to $x$ and $y$, all the above properties hold true changing $x$ with $y$.

3 The parabolic-parabolic Keller-Segel system on the network

According to the notations and definitions of the previous section, system (1.1), endowed with the natural continuity and transmission conditions, can be written on the network $\Gamma$ as following

$$\partial_t u_j = \partial_{yy} u_j - \partial_y (u_j \partial_y c_j) \quad \text{on } (0, \infty) \times e_j, \quad j = 1, \ldots, m,$$

(3.1)

$$\varepsilon \partial_t c_j = \partial_{yy} c_j + u_j - \alpha c_j \quad \text{on } (0, \infty) \times e_j, \quad j = 1, \ldots, m,$$

(3.2)

$$u_j(0, y) = u^0_j(y) \quad \text{and} \quad c_j(0, y) = c^0_j(y), \quad y \in \Gamma, \quad j = 1, \ldots, m,$$

(3.3)

$$\sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial u_j}{\partial n}(t, v_i) = 0, \quad t > 0, \quad i = 1, \ldots, n,$$

(3.4)

$$\sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial c_j}{\partial n}(t, v_i) = 0, \quad t > 0, \quad i = 1, \ldots, n,$$

(3.5)

$$u_j(t, v_i) = u_k(t, v_i) \text{ if } j, k \in E(v_i), \quad t > 0, \quad i = 1, \ldots, n,$$

(3.6)

$$c_j(t, v_i) = c_k(t, v_i) \text{ if } j, k \in E(v_i), \quad t > 0, \quad i = 1, \ldots, n.$$

(3.7)

As for problem (2.5), there is no coupling among the $m$ systems (3.1)-(3.2)-(3.3) on each edge $e_j$. The systems are coupled only through the transmission conditions (3.4) and (3.5), which express the conservation of the flux at the vertices for $u$ and $c$ (Kirchhoff condition), and through the continuity
conditions (3.6) and (3.7). Again, conditions (3.4)-(3.7) give exactly $2d(v_i)$ equations for the $2d(v_i)$ functions $u_j, c_j$ such that $j \in E(v_i)$. Furthermore, conditions (3.4) and (3.5), together with the continuity of $u$, guarantee the conservation of the initial mass

$$\int_{\Gamma} u(t,y) \, dy = \int_{\Gamma} u^0(y) \, dy =: M, \quad t > 0. \quad (3.8)$$

Indeed

$$\frac{d}{dt} \int_{\Gamma} u(t,y) \, dy = \sum_{j=1}^{m} \kappa(e_j) \frac{d}{dt} \int_{0}^{1} \tilde{u}_j(t,\eta) \, d\eta = \sum_{j=1}^{m} \kappa(e_j) \int_{0}^{1} (\partial_{\eta \eta} \tilde{u}_j(t,\eta) - \partial_{\eta} (\tilde{u}_j(t,\eta) \partial_{\eta} \tilde{c}_j(t,\eta))) \, d\eta$$

$$= \sum_{j=1}^{m} \kappa(e_j) \partial_{\eta} \tilde{u}_j(t,\eta) - \tilde{c}_j(t,\eta) \partial_{\eta} \tilde{c}_j(t,\eta)) = \sum_{j=1}^{n} \sum_{j \in E(v_i)} \kappa(e_j) \partial_{u_j} \partial_{c_j}^* (t,v_i) = 0.$$

Remark 3.1 A different but again natural condition that one can impose at the vertices instead of (3.4) is the conservation of the total flux

$$\sum_{j \in E(v_i)} \kappa(e_j) \partial_{u_j} \partial_{c_j}^* (t,v_i) = 0, \quad t > 0, \ i = 1, \ldots, n.$$

However, the latter together with the continuity of $u$ at the vertices and the Kirchhoff condition (3.5) imply (3.4).

In this section we assume $\varepsilon > 0$ and we consider solutions of the Keller-Segel system in the integral form (1.2)-(1.3). Then, for $f$ integrable over $\Gamma$, we introduce the notation

$$(H(t) * f)(y) := \int_{\Gamma} H(t,x,y) f(x) \, dx, \quad y \in \Gamma.$$ 

Thanks to the continuity of the heat kernel $H$ on $\Gamma$, if $u$ is continuous on $\Gamma$ and $c$ satisfies the Kirchhoff condition (3.5), equations (1.2)-(1.3) read equivalently as

$$u(t,y) = (H(t) * u^0)(y) + \int_{0}^{t} (\partial_x H(t - s) * (u(s) \partial_x c(s)))(y) \, ds, \quad y \in \Gamma; \quad (3.9)$$

$$c(t,y) = e^{-(\alpha/\varepsilon) t} (H(t \varepsilon^{-1}) * c^0)(y) + \frac{1}{\varepsilon} \int_{0}^{t} e^{-(\alpha/\varepsilon)(t-s)} (H((t-s) \varepsilon^{-1}) * u(s))(y) \, ds, \quad y \in \Gamma. \quad (3.10)$$

It is worth noticing that thanks to property (2.11), any integral solution (3.9)-(3.10) satisfies the mass conservation (3.8).

Theorem 3.2 (Local existence). Let $\varepsilon > 0$, $\alpha \geq 0$ and assume $u^0 \in L^\infty(\Gamma)$, $c^0 \in W^{1,\infty}(\Gamma)$. Then, there exist $T = T(||u^0||_{L^\infty(\Gamma)}, ||\partial_x c^0||_{L^\infty(\Gamma)}, \varepsilon) > 0$ and a unique integral solution (3.9)-(3.10) of the Keller-Segel system with

$$u \in L^\infty((0,T); C^0(\Gamma)), \quad c \in L^\infty(0,T; W^{1,\infty}(\Gamma)),$$

satisfying the transmission conditions (3.4) and (3.5) and the mass conservation (3.8).
Proof. For $u_0$ given, $A := \|u^0\|_{L^\infty(\Gamma)}$, $K := \sup_{t>0,y \in \Gamma} \|H(t, \cdot, y)\|_{L^1(\Gamma)} > 1$ and $T > 0$ to be chosen later, let

$$B := \{ u \in L^\infty((0,T) \times \Gamma) : u(0, y) = u^0(y) \text{ and } \sup_{0 \leq t < T} \|u(t)\|_{L^\infty(\Gamma)} \leq K A + 1 \}$$

and $d(u_1, u_2) := \sup_{0 \leq t < T} \|u_1(t) - u_2(t)\|_{L^\infty(\Gamma)}$. Next, for $u \in B$ fixed, $c_0$ given and $c$ defined through $u$ by (3.10), we define on $B$ the map

$$\Psi(u)(t, y) := (H(t) \ast u^0)(y) + \int_0^t (\partial_s H(t - s) \ast (u(s) \partial_x c(s)))(y) ds, \quad (t, y) \in (0, T) \times \Gamma. \quad (3.11)$$

Since $(B, d)$ is a non empty complete metric space, we shall prove the claimed local existence using the Banach fixed point theorem.

First step: $\Psi(B) \subset B$. From (3.11) we have

$$|\Psi(u)(t, y)| \leq K A + \kappa_0^{-2} \sup_{0 \leq s < T} \|u(s)\|_{L^\infty(\Gamma)} \int_0^t \|\partial_s H(t - s, \cdot, y)\|_{L^1(\Gamma)} \|\partial_x c(s)\|_{L^\infty(\Gamma)} ds, \quad y \in \Gamma. \quad (3.12)$$

Next, owing to the property $\partial_y H(t, x, y) = -\partial_x H(t, x, y)$, by (3.10) we get for any $y \in \Gamma$

$$\partial_y c(t, y) = e^{-((\alpha/\varepsilon)t) \partial_y H(t \varepsilon^{-1} \ast c^0)(y)} + \frac{1}{\varepsilon} \int_0^t e^{-((\alpha/\varepsilon)(t-s)) \partial_y H((t-s)\varepsilon^{-1}) \ast u(s))(y)} ds \quad (3.13)$$

Furthermore, we observe that embedding the non-oriented network $\Gamma$ into the oriented one $(V, \{e_j^\pm ; j = 1, \ldots, m\})$, denoted by $2\Gamma$, by construction the fundamental solution $H$ of the heat equation (2.5) on $\Gamma$ is also solution on $2\Gamma$ and $\int_{2\Gamma} H(t, x, y) f(x) dx = 2 \int_{\Gamma} H(t, x, y) f(x) dx$, for any $f$ integrable on $\Gamma$ (and so on $2\Gamma$). Therefore, for any $y \in \Gamma$,

$$\int_{\Gamma} \partial_x H(t \varepsilon^{-1}, x, y) c^0(x) dx = \frac{1}{2} \int_{2\Gamma} \partial_x H(t \varepsilon^{-1}, x, y) c^0(x) dx \quad \begin{aligned}
= \frac{1}{2} \sum_{j=1}^m \kappa(e_j) \int_0^1 \partial_{\xi} \tilde{H}_j(t \varepsilon^{-1}, \xi, \eta) c^0_j(\xi) d\xi + \frac{1}{2} \sum_{j=1}^m \kappa(e_j) \int_0^1 \partial_{\eta} \tilde{H}_j(t \varepsilon^{-1}, 1 - \xi, \eta) c^0_j(1 - \xi) d\eta \\
= \frac{1}{2} \sum_{j=1}^m \kappa(e_j) [\tilde{H}_j(t \varepsilon^{-1}, \xi, \eta) c^0_j(\xi)]_0^1 - \frac{1}{2} \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{H}_j(t \varepsilon^{-1}, \xi, \eta) \partial_{\xi} c^0_j(\xi) d\xi \\
+ \frac{1}{2} \sum_{j=1}^m \kappa(e_j) [\tilde{H}_j(t \varepsilon^{-1}, 1 - \xi, \eta) c^0_j(1 - \xi)]_0^1 + \frac{1}{2} \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{H}_j(t \varepsilon^{-1}, 1 - \xi, \eta) \partial_{\xi} c^0_j(1 - \xi) d\xi
= - (H(t \varepsilon^{-1}) \ast \partial_x c^0)(y),
\end{aligned}$$

and (3.13) becomes

$$\partial_y c(t, y) = -e^{-((\alpha/\varepsilon)t) \partial_x c^0}(y) + \frac{1}{\varepsilon} \int_0^t e^{-((\alpha/\varepsilon)(t-s)) \partial_y H((t-s)\varepsilon^{-1}) \ast u(s))(y)} ds.$$
Using (2.12) we arrive at the following estimate for the spatial derivative of $c$

$$|\partial_y c(t,y)| \leq \kappa_0^{-1} K \|\partial_x c^0\|_{L^\infty(\Gamma)} + (\varepsilon \kappa_0)^{-1} \sup_{0 \leq s < T} \|u(s)\|_{L^\infty(\Gamma)} \int_0^t \|\partial_y H((t-s)\varepsilon^{-1},\cdot,y)\|_{L^1(\Gamma)} ds, \quad y \in \Gamma,$$

and by (2.14)

$$\|\partial_y c(t)\|_{L^\infty(\Gamma)} \leq \kappa_0^{-1} K \|\partial_x c^0\|_{L^\infty(\Gamma)} + C(K A + 1)(\varepsilon^{-1} t + \varepsilon^{-\frac{1}{2}} t^\frac{1}{2}), \quad \text{(3.14)}$$

where $C > 0$ does not depend on $\varepsilon$. Finally, plugging (3.14) into (3.12) and using the decaying properties of $H$ again, we get for $t \in (0,T)$

$$\|\Psi(u)(t)\|_{L^\infty(\Gamma)} \leq K A + C(K A + 1) \int_0^t (1 + (t-s)^{-1/2}) \|\partial_x c(s)\|_{L^\infty(\Gamma)} ds$$

$$\leq K A + \tilde{C}(t + t^{1/2})(1 + \varepsilon^{-1} t + \varepsilon^{-\frac{1}{2}} t^\frac{1}{2}),$$

where $\tilde{C} = \tilde{C}(K, A, \Gamma, \|\partial_x c^0\|_{L^\infty(\Gamma)})$. Therefore, for $T = T(\|u^0\|_{L^\infty(\Gamma)}, \|\partial_x c^0\|_{L^\infty(\Gamma)}, \varepsilon) > 0$ sufficiently small, it holds

$$\sup_{0 \leq t < T} \|\Psi(u)(t)\|_{L^\infty(\Gamma)} \leq K A + 1.$$

To obtain the claim, we also observe that $\Psi(u)(0,y) = u^0(y)$ since by definition $H(0)*u^0 = u^0$.

**Second step:** $\Psi$ is a contraction map on $B$. Let $u_1, u_2 \in B$. By (3.11) and arguing as in the previous step, we get for all $(t,y) \in (0,T) \times \Gamma$

$$|\Psi(u_1) - \Psi(u_2)|(t,y) \leq \int_0^t \|\partial_x H(t-s) * [(u_1 - u_2)\partial_x c_1 + u_2(\partial_x c_1 - \partial_x c_2)](s)(y)\| ds$$

$$\leq d(u_1, u_2)\kappa_0^{-2} \int_0^t \|\partial_x H(t-s,\cdot,y)\|_{L^1(\Gamma)} \|\partial_x c_1(s)\|_{L^\infty(\Gamma)} ds$$

$$+ (K A + 1)\kappa_0^{-2} \int_0^t \|\partial_x H(t-s,\cdot,y)\|_{L^1(\Gamma)} \|\partial_x c_1 - \partial_x c_2\|_{L^\infty(\Gamma)} ds, \quad \text{(3.15)}$$

and for all $t \in (0,T)$

$$\|\partial_y c_1 - \partial_y c_2\|_{L^\infty(\Gamma)} \leq C d(u_1, u_2)(\varepsilon^{-1} t + \varepsilon^{-\frac{1}{2}} t^\frac{1}{2}). \quad \text{(3.16)}$$

Plugging (3.14) and (3.16) into (3.15) and using (2.14), we arrive at

$$\|\Psi(u_1) - \Psi(u_2)(t)\|_{L^\infty(\Gamma)} \leq \tilde{C} d(u_1, u_2)(t + t^{1/2})(1 + \varepsilon^{-1} t + \varepsilon^{-\frac{1}{2}} t^\frac{1}{2}).$$

Hence, for $T$ sufficiently small again, $\Psi$ is a contraction on $B$.

**Third step:** conclusion. By the previous steps, it follows that there exists a unique fixed point $u \in B$ of $\Psi$ and that $(u, c)$ satisfies the integral system (3.12) + (3.10). Furthermore, $c$ is continuous on $\Gamma$ and satisfies the transmission condition (3.5) because $H$ is continuous on $\Gamma$ and satisfies the same condition. Again because of the regularity of $H$, $u$ is differentiable on $\Gamma$ and $c$ is twice differentiable.
Consequently, performing an integration by part on each edge in the second term of the r.h.s. of (3.9), $u$ can be also written as

$$u(t, y) = (H(t) * u^0)(y) + \int_0^t \left\{ \sum_{i=1}^{n} H(t - s, v_i, y) \sum_{j \in E(v_i)} \kappa(e_j) u_j(s, v_i) \frac{\partial c_j}{\partial n}(s, v_i) \, ds \right\} - \int_0^t (H(t - s) * \partial_x(u(s)\partial_x c(s)))(y) \, ds,$$

implying that $u(t) \in C^0(\Gamma)$ holds true for all $t \in (0, T)$. Finally, the continuity of $u$ together with (3.5) gives that

$$u(t, y) = (H(t) * u^0)(y) - \int_0^t (H(t - s) * \partial_x(u(s)\partial_x c(s)))(y) \, ds.$$

So that $u$ satisfies (3.4) and the proof is complete. \qed

We conclude this section showing the existence of a classical solution of system (3.1) - (3.7) in $(0, T)$ for any $T > 0$, i.e. we do not exclude that the solution blow-up for $T \to +\infty$. More specifically, we shall prove the following.

**Theorem 3.3** (Global existence and positivity). Under the hypothesis of Theorem 3.2, for all $T > 0$ there exists a solution $(u, c)$ of the Keller-Segel system on the time interval $[0, T]$. Moreover, if the initial data $u^0$ and $c^0$ are nonnegative, the solution $(u, c)$ is nonnegative.

**Proof.** The global existence result is obtained by extending the local in time solution obtained in Theorem 3.2. Indeed, let $T_{max}$ be the maximal time of existence of the obtained local solution. Then, the limits as $t \to T_{max}$ of $u$ and $c$ exist and depend only on $\|u^0\|_{L^\infty(\Gamma)}$, $\|\partial_x c^0\|_{L^\infty(\Gamma)}$ and $\varepsilon$. Therefore, it is possible to extend $(u(t), c(t))$ behind $T_{max}$, iteratively as many time as it is necessary to reach $T > 0$.

In order obtain the positivity of the solution $(u, c)$ when the initial data are positive, we analyze the time evolution of $\int_{\Gamma} \phi(u(t, y)) \, dy$, where $\phi$ is a smooth function on $\mathbb{R}$ such that $\phi(z) > 0$ if $z < 0$, $\phi(z) = 0$ if $z \geq 0$ and there exists $C > 0$ such that $0 \leq \phi''(z)z^2 \leq C\phi(z)$, for all $z \in \mathbb{R}$. Owing to (3.1) and to the Kirchhoff conditions (3.4) and (3.5), we have for any $\delta > 0$

$$\frac{d}{dt} \int_{\Gamma} \phi(u(t, y)) \, dy = \sum_{j=1}^{m} \kappa(e_j) \int_0^1 \phi'(\tilde{u}_j(t, \eta))(\partial_{\eta\eta}\tilde{u}_j - \partial_{\eta}(\tilde{u}_j\partial_{\eta}\tilde{c}_j))(t, \eta) \, d\eta$$

$$\leq -\sum_{j=1}^{m} \kappa(e_j) \int_0^1 \phi''(\tilde{u}_j)(\partial_{\eta}\tilde{u}_j)^2(t, \eta) \, d\eta$$

$$+ \sum_{j=1}^{m} \kappa(e_j)\|\partial_{\eta}\tilde{c}_j(t)\|_{L^\infty(0,1)} \int_0^1 \phi''(\tilde{u}_j)|\tilde{u}_j|\|\partial_{\eta}\tilde{u}_j(t, \eta)\| \, d\eta$$

$$\leq (\frac{\delta}{2} - 1) \int_{\Gamma} \phi''(u)(\partial_{\eta}u)^2(t, y) \, dy + \frac{\kappa_0}{2\delta} \|\partial_{\eta}c(t)\|_{L^\infty(\Gamma)}^2 \int_{\Gamma} \phi''(u)u^2(t, y) \, dy.$$
Choosing \( \delta < 2 \), by the properties of \( \phi \) we get the differential inequality
\[
\frac{d}{dt} \int_{\Gamma} \phi(u(t,y)) \, dy \leq \frac{C \kappa^{-1}}{2 \delta} \| \partial_y c(t) \|_{L^\infty(\Gamma)} \int_{\Gamma} \phi(u(t,y)) \, dy.
\]
Applying the Gronwall lemma, we obtain that \( \phi(u(t,y)) = 0 \), so that \( u(t,y) \geq 0 \).

The positivity of \( c \) does not follow from (3.10), since \( H \) is not a priori positive, as observed before. Instead, it follows from the maximum principle for parabolic equations on network [2], taking also into account that \( u \) is positive.

Remark 3.4 (Energy) As for the euclidian case, solutions of the Keller-Segel system (1.1) on \( \Gamma \) that satisfy the continuity and transmission conditions (3.4)-(3.7), satisfy also the energy dissipation equation
\[
\frac{d}{dt} \mathcal{E}(u(t), c(t)) = - \int_{\Gamma} u(t,x) |\partial_x (\log u - c)|^2(t,x) \, dx - \varepsilon \int_{\Gamma} (\partial_t c(t,x))^2 \, dx,
\]
where \( \mathcal{E} \) is the usual free energy associated to the Keller-Segel system, i.e.
\[
\mathcal{E}(u,v) := \int_{\Gamma} u \log u \, dx - \int_{\Gamma} u c \, dx + \frac{1}{2} \int_{\Gamma} |\partial_x c|^2 \, dx + \frac{\alpha}{2} \int_{\Gamma} c^2 \, dx.
\]
In particular, the global solution of Theorem 3.3 satisfies (3.17).

4 The parabolic-elliptic Keller-Segel system on the network

We shall consider in this section the parabolic-elliptic system (1.1), i.e. \( \epsilon = 0 \), not only for the sake of completeness, but also to put in evidence the different behaviour of the two systems on a network: in contrast with the parabolic-parabolic case of the previous section, here positives solutions can not exhibit blow-up as \( t \to +\infty \).

The integral solution \((u,c)\) in (3.9)-(3.10), reads now as
\[
u(t,y) = (H(t) * u^0)(y) + \int_0^t (\partial_x H(t-s) * (u(s)\partial_x c(s)))(y) \, ds,
\]
where \( c \) is the weak solution of the elliptic equation on \( \Gamma \)
\[
-\partial_{yy} c_j = u_j - \alpha c_j \quad y \in e_j, \ j = 1, \ldots, m,
\]
\[
c_j(t,v_i) = c_k(t,v_i) \quad j, k \in E(v_i), \ i = 1, \ldots, n,
\]
\[
\sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial c_j}{\partial n}(t,v_i) = 0, \quad i = 1, \ldots, n.
\]

Elliptic equations on networks are studied in [21]. Following [21], we can state the existence result below. We sketch some ideas of the proof in the Appendix for the reader’s convenience.
Proposition 4.1. Let $\alpha > 0$. Given $z \in L^\infty(\Gamma)$, the elliptic problem

\[-w''_j = z_j - \alpha w_j \quad x \in e_j, \ j = 1, \ldots, m,\]
\[w_j(v_i) = w_k(v_i) \quad j, k \in E(v_i), \ i = 1, \ldots, n, \tag{4.3}\]
\[\sum_{j \in E(v_i)} \kappa(e_j) \frac{\partial w_j}{\partial n}(v_i) = 0, \quad i = 1, \ldots, n,\]

admits a unique weak solution $w \in H^1(\Gamma)$. Moreover $w \in W^{1,\infty}(\Gamma)$ and there exists $c = c(\Gamma, \alpha) > 0$ such that

\[c(\Gamma, \alpha) \inf \{z\} \leq w(x) \leq c(\Gamma, \alpha) \sup \{z\}, \quad \forall x \in \Gamma, \tag{4.4}\]
\[\|w'\|_{L^\infty(\Gamma)} \leq \|z\|_{L^\infty(\Gamma)}. \tag{4.5}\]

Finally if $z \in C^0(\Gamma)$, then $w \in C^2(\Gamma) := \{u \in C^0(\Gamma) : \tilde{u}_j \in C^2((0,1)), \ j = 1, \ldots, m\}$.

Our main result of this section is the following global existence theorem.

Theorem 4.2 (Global existence). Let $\varepsilon = 0, \alpha > 0$ and assume $u^0 \in L^\infty(\Gamma), u^0 \geq 0$. Then, system (4.1)-(4.2) has a global in time integral-weak solution $(u, c)$ such that $u \geq 0, c \geq 0$ and

\[u \in L^\infty((0,\infty); C^0(\Gamma)), \quad c \in L^\infty((0,\infty); C^2(\Gamma)).\]

Moreover, the conservation of mass (3.8) and the transmission conditions (3.4) and (3.5) are also satisfied.

Proof. The local existence result can be proved as in Theorem 3.2 simply replacing the estimate (3.14) with the estimate (4.5).

It is worth emphasizing that the nonnegativity of the initial data $u_0$ is not necessary for the local existence. On the other hand, if $u^0 \geq 0$, then the positivity of the solution can be proved as for the corresponding result in Theorem 3.3.

To prove the global existence of a nonnegative solution $(u, c)$, we extend to the network some classical estimates on the $L^p$ norms of the solutions of the Keller-Segel system on $\mathbb{R}^d$. By multiplying (3.1) for $w^{p-1}, p > 1$, and integrating over $\Gamma$ we get

\[\frac{d}{dt} \int_{\Gamma} u^p dy = \sum_{j=1}^m \kappa(e_j) \frac{d}{dt} \int_0^1 \tilde{u}_j^p d\eta = p \sum_{j=1}^m \kappa(e_j) \int_0^1 \tilde{u}_j^{p-1} \left[ \partial_\eta \tilde{u}_j - \partial_\eta (\tilde{u}_j \partial_\eta \tilde{c}_j) \right] d\eta\]
\[= p \sum_{i=1}^n \sum_{j \in E(v_i)} \kappa(e_j) u_j^{p-1}(t, v_i) \frac{\partial u_j}{\partial n} - u_j \frac{\partial c_j}{\partial n}(t, v_i)\]
\[-4 \frac{(p-1)}{p} \int_{\Gamma} |\partial_\gamma u^{p/2}|^2 dy + 2(p-1) \int_{\Gamma} u^{p/2} \partial_\gamma u^{p/2} \partial_\gamma c dy.\]

Using the continuity of $u$, the transmission conditions (3.4) and (3.5) and the Hölder inequality, we obtain

\[\frac{d}{dt} \int_{\Gamma} u^p dy \leq -2 \frac{(p-1)}{p} \int_{\Gamma} |\partial_\gamma u^{p/2}|^2 dy + \frac{1}{2} p(p-1) \|\partial_\gamma c(t)\|_{L^\infty(\Gamma)}^2 \int_{\Gamma} u^p dy. \tag{4.6}\]
Next, we shall estimate $\|\partial_y c(t)\|_{L^\infty(\Gamma)}$ uniformly in time. By (4.13), we immediately get that $\bar{c} \in C^2(0,1)$, for $j = 1, \ldots, m$. Integrating over $\Gamma$ the equation on $c$ and observing that

$$
\int_{\Gamma} \partial_y c(t,y) dy = \sum_{j=1}^{m} \kappa (e_j) \int_{0}^{1} \partial_{\eta} \bar{c}_j(t,\eta) d\eta = \sum_{i=1}^{n} \sum_{j \in E(v_i)} \kappa (e_j) \frac{\partial c_i}{\partial n}(t,v_i) = 0,
$$

we get, by the mass conservation property for $u$, i.e.

$$\alpha \int_{\Gamma} c(t,y) dy = \int_{\Gamma} u(t,y) dy = \int_{\Gamma} u_0(y) dy = M. \tag{4.7}$$

Integrating now the equation on $c_j$ over $[0,\eta], \eta \in [0,1]$, we have for all $j = 1, \ldots, m$,

$$
\partial_\eta \bar{c}_j(t,\eta) = \partial_\eta c_j(t,0) + \int_{0}^{\eta}[\alpha \bar{c}_j(t,\sigma) - \bar{u}_j(t,\sigma)]d\sigma,
$$

where $\partial_\eta \bar{c}_j(t,0)$ is the internal derivative. Thus, since $u$ and $c$ are nonnegative, using (4.7), we obtain for $y = \Pi_j^\pm(\eta)$ and $j = 1, \ldots, m$,

$$\partial_\eta c_j(t,\Pi_j^\pm(0)) - \kappa_0^{-1} M \leq \partial_\eta c_j(t,y) \leq \partial_\eta c_j(t,\Pi_j^\pm(0)) + \kappa_0^{-1} M. \tag{4.8}$$

Hence, in order to bound $\|\partial_y c(t)\|_{L^\infty(\Gamma)}$ it is sufficient to bound $\partial_\eta c(t)$ at the vertices $v_i$ of $\Gamma$.

Integrating again the l.h.s. inequality in (4.8) over $[0,y], y \in e_j$, and using the positivity of $c$, we get

$$y \partial_\eta c_j(t,\Pi_j^\pm(0)) - y \kappa_0^{-1} M \leq c_j(t,y),$$

and after one more integration over $[0,1]$ we arrive at

$$
\frac{1}{2} \partial_\eta c_j(t,\Pi_j^\pm(0)) - \frac{1}{2} \kappa_0^{-1} M \leq \int_{0}^{1} \bar{c}_j(t,\eta) d\eta \leq (\alpha \kappa_0)^{-1} M,
$$

i.e.

$$\max_{i=1,\ldots,n} \max_{j \in E(v_i)} \partial_\eta c_j(t,v_i) \leq C_1(\alpha,\kappa_0) M. \tag{4.9}$$

Plugging (4.9) into the r.h.s. of (4.8), we arrive at the uniform in time upper bound for $\partial_\eta c(t)$

$$\partial_\eta c(t) \leq C_2(\alpha,\kappa_0) M. \tag{4.10}$$

On the other hand, integrating twice as before the r.h.s. inequality in (4.8), we obtain

$$- c_j(t,\Pi_j^\pm(0)) - \frac{1}{2} \kappa_0^{-1} M \leq - \frac{1}{2} \partial_\eta c_j(t,\Pi_j^\pm(0)). \tag{4.11}$$

Furthermore, thanks to (4.7) and (4.10), we observe that for all $j = 1, \ldots, m$ and $y = \Pi_j^\pm(\eta) \in \bar{e}_j$, it holds

$$0 \leq c_j(t,y) = \int_{0}^{1} \bar{c}_j(t,\sigma) d\sigma + \int_{0}^{\eta} \int_{\sigma}^{\eta} \partial_\sigma \bar{c}_j(t,z) dz d\sigma \leq C_3(\alpha,\kappa_0) M. \tag{4.12}$$
Plugging (4.12) into (4.11), we arrive at a uniform in time lower bound for \( \partial_y c_j(t, \Pi_j^\pm(0)) \), \( j = 1, \ldots, m \), giving a uniform in time lower bound for \( \partial_y c(t) \) from the l.h.s. of (4.8). Resuming, we have obtained the existence of a constant \( C(\alpha, \kappa_0) > 0 \) such that, in the maximal time interval of existence of the solution, it holds

\[
\| c(t) \|_{W^{1, \infty}(\Gamma)} \leq C(\alpha, \kappa_0) M. \tag{4.13}
\]

To conclude, we replace (4.13) into (4.6) and we apply the iterative method introduced in [1] to get an uniform in time estimate of \( \| u(t) \|_{L^\infty(\Gamma)} \). The global existence of the nonnegative solution \( (u, c) \) follows by the classical continuation in time argument.

## A Appendix A: the heat kernel on networks

This Appendix is devoted to the construction of the fundamental solution (2.6)-(2.7) of the heat equation (2.4) on a weighted graph.

Let \( \Gamma \) be the considered weighted network. We recall that the edges \( e_j \) are open. Let \( x \) be a fixed point say of \( e_1 \). Let \( y \) be an arbitrary point of \( \Gamma \setminus V \) and \( \pm a_j, j = 1, \ldots, m \), the two arcs containing \( y \). We consider the following function

\[
H(t, x, y) = \delta_{1,j} \kappa^{-1}(e_1) G(t, d(x, y)) + G(t, d(y, I(a_j))) \ast \phi_{a_j}(t, x) + G(t, d(y, I(-a_j))) \ast \phi_{-a_j}(t, x), \tag{A.1}
\]

where \( \phi_{a_j} \) and \( \phi_{-a_j} \) are continuous functions with respect to \( t \in [0, \infty) \), associated to the arc \( \pm a_j \) respectively and to be determined for all \( j = 1, \ldots, m \). Moreover, for functions \( f \) and \( g \) defined on \( \mathbb{R}_+ \), the \( \ast \) operator is defined as following

\[
(g \ast f)(t) = \int_0^t g(s)f(t-s)ds, \quad t \geq 0.
\]

The following technical lemma will be useful in the sequel, (see [23 Lemmas 1 and 2]).

**Lemma A.1.** The following identities hold true

\[
\begin{align*}
(i) & \quad \frac{\partial}{\partial z}(G(t, z) \ast f) \big|_{z=0^+} = -\frac{1}{2} f(t), \quad \text{if } f \in C^0([0, \infty)); \\
(ii) & \quad \frac{\partial}{\partial t} G(t, z_1) \ast G(t, z_2) = G(t, z_1), \quad \text{for all } z_1 \neq 0 \text{ and } z_2 \in \mathbb{R}; \\
(iii) & \quad \frac{z_1}{t} G(t, z_1) \ast \frac{z_2}{t} G(t, z_2) = \frac{z_1+z_2}{t} G(t, z_1+z_2), \quad \text{for all } z_1, z_2 \neq 0.
\end{align*}
\]

It is easily seen that \( H(\cdot, x, \cdot) \) is a solution of the heat equation on \( (0, \infty) \times \Gamma \setminus V \). Moreover, denoting \( \theta_{\pm j}(t, x) \) the function \( G(t, 0) \ast \phi_{\pm a_j}(t, x) \) and using Lemma [A.1](ii), it is easy to see that, for all \( y \in \Gamma \setminus V \), the function \( H \) can be written also as

\[
H(t, x, y) = \delta_{1,j} \kappa^{-1}(e_1) G(t, d(x, y)) + t^{-1} d(y, I(a_j)) G(t, d(y, I(a_j))) \ast \theta_{+j}(t, x) + t^{-1} d(y, I(-a_j)) G(t, d(y, I(-a_j))) \ast \theta_{-j}(t, x). \tag{A.2}
\]
Introducing the vector function \( \Theta(t, x) = (\theta_{-1}, \theta_{+1}, \ldots, \theta_{-m}, \theta_{+m})^\top (t, x) \) and the \( 1 \times 2m \) matrix function \( \Psi_j(t, y) = (\psi_{-j, 1}, \psi_{+j, 1}, \ldots, \psi_{-j, m}, \psi_{+j, m}) (t, y) \) such that \( \psi_{\pm j} = 0 \) if \( l \neq j \) and

\[
\psi_{\pm j}(t, y) = t^{-1} d(y, I(\pm a_j)) G(t, d(y, I(\pm a_j))) ,
\]

\( (A.2) \) reads as

\[
H(t, x, y) = \delta_{1, j} \kappa^{-1}(e_1) G(t, d(x, y)) + \Psi_j(t, y) \ast \Theta(t, x). \tag{A.3}
\]

Next, for any fixed \( x \in e_1 \), we shall determine \( \Theta(\cdot, x) \) in such a way that \( H \) satisfies the following conditions on each ramifications nodes \( v \in V \),

1. **continuity condition**: there exists \( b(t, x, v) \in \mathbb{R} \) such that \( \lim_{y \to v} H(t, x, y) = b(t, x, v), \ j \in E(v) \);

2. **transmission condition**: \( \sum_{j \in E(v)} \kappa(e_j) \frac{\partial}{\partial n} H(t, x, v) = 0 \).

As a consequence, the resulting function \( H \) shall satisfy all the properties (i)-(vii) in Theorem 2.1.

Let \( v \in V \) be an arbitrary fixed node. Because of the invariance of \( (A.1) \) with respect to the parametrizations of \( \pm a_j \), we can assume that \( I(a_j) = v = T(-a_j) \), for all \( j \in E(v) \), without loss of generality. Then, computing with respect to the parametrization of \( a_j \) and passing to the limit \( y \to v \) in \( (A.1) \), the continuity condition reads as

\[
H(t, x, I(a_j)) = \delta_{1, j} \kappa^{-1}(e_1) G(t, d(x, I(a_j))) + G(t, 0) \ast \phi_{a_j}(t, x) + G(t, 1) \ast \phi_{-a_j}(t, x) = b(t, x, v). \tag{A.4}
\]

By Lemma \( (A.1) \) (ii), it holds

\[
G(t, 1) \ast \phi_{-a_j}(t, x) = \frac{1}{t} G(t, 1) \ast \theta_{-j}(t, x),
\]

and \( (A.4) \) can be written in term of the thetas functions as

\[
\delta_{1, j} \kappa^{-1}(e_1) G(t, d(x, I(a_j))) + \theta_{+j}(t, x) + \frac{1}{t} G(t, 1) \ast \theta_{-j}(t, x) = b(t, x, v), \quad j \in E(v). \tag{A.5}
\]

Multiplying \( (A.5) \) by \( \kappa(e_j) \) and summing over all \( j \in E(v) \), we get

\[
\sum_{j \in E(v)} \delta_{1, j} G(t, d(x, I(a_j))) + \sum_{j \in E(v)} \kappa(e_j) \theta_{+j}(t, x) + \sum_{j \in E(v)} \kappa(e_j) \frac{1}{t} G(t, 1) \ast \theta_{-j}(t, x) = b(t, x, v) \sum_{j \in E(v)} \kappa(e_j). \tag{A.6}
\]

On the other hand, according to the definition of the exterior normal derivatives and to Lemma \( (A.1) \) (i), it holds true that

\[
\frac{\partial}{\partial n} H(t, x, I(a_j)) = -\delta_{1, j} \kappa^{-1}(e_1) \frac{1}{2t} d(x, I(a_j)) G(t, d(x, I(a_j)))
\]

\[
+ \frac{1}{2} \phi_{a_j}(t, x) - \frac{1}{2t} G(t, 1) \ast \phi_{-a_j}(t, x), \quad j \in E(v).
\]

Again, by Lemma \( (A.1) \) (ii), we easily obtain the identity

\[
\frac{\partial}{\partial n} H(t, x, I(a_j)) \ast \frac{1}{\sqrt{\pi t}} = -\delta_{1, j} \kappa^{-1}(e_1) G(t, d(x, I(a_j))) + \theta_{+j}(t, x) - \frac{1}{t} G(t, 1) \ast \theta_{-j}(t, x), \quad j \in E(v). \tag{A.7}
\]

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Since the transmission condition at the node \( v \) is equivalent to the condition
\[
\sum_{j \in E(v)} \kappa(e_j) \frac{\partial}{\partial n} H(t, x, v) \ast \frac{1}{\sqrt{\pi t}} = 0,
\]
multiplying (A.7) by \( \kappa(e_j) \) and summing over all \( j \in E(v) \), we get
\[
\sum_{j \in E(v)} \kappa(e_j) \theta_+(t, x) = \sum_{j \in E(v)} \delta_{1,j} G(t, d(x, I(a_j))) + \sum_{j \in E(v)} \kappa(e_j) \frac{1}{t} G(t, 1) \ast \theta_-(t, x). \tag{A.8}
\]
Plugging (A.8) into (A.6), we have
\[
2 \sum_{j \in E(v)} \delta_{1,j} G(t, d(x, I(a_j))) + 2 \sum_{j \in E(v)} \kappa(e_j) \frac{1}{t} G(t, 1) \ast \theta_-(t, x) = b(t, x, v) \sum_{j \in E(v)} \kappa(e_j).
\]
The latter identity gives us the value \( b(t, x, v) \) in term of the thetas functions, for any node \( v \in V \).
Defining \( \kappa(v) := \sum_{j \in E(v)} \kappa(e_j) \) and plugging the obtained expression for \( b(t, x, v) \) into (A.5), we arrive at the identity
\[
\delta_{1,j} \kappa^{-1}(e_1) G(t, d(x, I(a_j))) + \theta_+ j(t, x) + \frac{1}{t} G(t, 1) \ast \theta_-(t, x) = \frac{2}{\kappa(v)} \sum_{i \in E(v)} \delta_{1,i} G(t, d(x, I(a_i))) + \frac{2}{\kappa(v)} \sum_{i \in E(v)} \kappa(e_i) \frac{1}{t} G(t, 1) \ast \theta_-(t, x),
\]
i.e. for any \( j \in E(v) \) it holds
\[
\theta_+(t, x) = \left(\frac{2 \kappa(e_1)}{\kappa(v)} - 1\right) \delta_{1,j} \kappa^{-1}(e_1) G(t, d(x, I(a_j))) + \sum_{i \in E(v), i \neq j} \frac{2 \kappa(e_i)}{\kappa(v)} \delta_{1,i} \kappa^{-1}(e_i) G(t, d(x, I(a_i)))
+ \left(\frac{2 \kappa(e_j)}{\kappa(v)} - 1\right) \frac{1}{t} G(t, 1) \ast \theta_-(t, x) + \sum_{i \in E(v), i \neq j} \frac{2 \kappa(e_i)}{\kappa(v)} \frac{1}{t} G(t, 1) \ast \theta_-(t, x). \tag{A.9}
\]
Taking into account that the sum of the first two terms in (A.9) is not identically zero iff \( 1 \in E(v) \) and using the transfrt/reflection coefficients (2.1), (A.9) can be written as
\[
\theta_+(t, x) = \epsilon_{(-a_1 \rightarrow a_j)} \kappa^{-1}(e_1) G(t, d(x, I(a_1))) + \sum_{i \in E(v)} \epsilon_{(-a_i \rightarrow a_j)} \frac{1}{t} G(t, 1) \ast \theta_-(t, x), \quad j \in E(v). \tag{A.10}
\]
Now it is worth noticing that changing the roles between \( a_j \) and \(-a_j \) and making all the above computations with respect to the parametrization of \(-a_j \), gives us that formula (A.10) also holds true for \( \theta_-(t, x) \), i.e.
\[
\theta_-(t, x) = \epsilon_{(a_1 \rightarrow -a_j)} \kappa^{-1}(e_1) G(t, d(x, I(-a_1))) + \sum_{i \in E(v)} \epsilon_{(a_i \rightarrow -a_j)} \frac{1}{t} G(t, 1) \ast \theta_+(t, x), \quad j \in E(v). \tag{A.11}
\]
Let denote the two arcs \( \pm a_i \), supported by \( e_i \) as \( \pm i \) and the transfer/reflection coefficients \( \epsilon_{(\pm a_i, \pm a_j)} \) as \( \epsilon_{\pm i, \pm j} \). We recall that \( \epsilon_{\pm i, \pm j} \neq 0 \) iff \( T(\pm i) = I(\pm j) \). Let denote \( I \) the ordered set \( \{-1, 1, \ldots, -m, m\} \). Then, (A.10)-(A.11) reads both as
\[
\theta_l(t, x) = \kappa^{-1}(e_1)[\epsilon_{-1,l}G(t, d(x, I(1))) + \epsilon_{1,l}G(t, d(x, I(-1)))] + \sum_{k \in I} \frac{1}{t}G(t, 1) * \theta_k(t, x), \quad l \in I,
\]
i.e. the function \( \Theta \) satisfies the \( 2m \) system
\[
\Theta(t, x) = \Lambda_1(t, x) + T(t) * \Theta(t, x), \quad (A.12)
\]
where \( \Lambda_1(t, x) \) is the \( 2m \) vector of components
\[
\lambda_l(t, x) = \kappa^{-1}(e_1)[\epsilon_{-1,l}G(t, d(x, I(1))) + \epsilon_{1,l}G(t, d(x, I(-1)))] , \quad l \in I,
\]
and \( T(t) \) is the \( 2m \) squared matrix \( T(t) = \frac{1}{t}G(t, 1)((\epsilon_{k,l})_{k,l \in I})^\top \).

The solution of system (A.12) is given by
\[
\Theta(t, x) = \Lambda_1(t, x) + \sum_{n=1}^{\infty} T(t)^* n * \Lambda_1(t, x), \quad (A.13)
\]
where \( T(t)^* n \) is the iterated convolution \( T(t) * \cdots * T(t) \). Plugging (A.13) into (A.3) we obtain for
\( x \in e_i \) and \( y \in e_j \)
\[
H(t, x, y) = \delta_{i,j} \kappa^{-1}(e_1)G(t, d(x, y)) + \Psi_j(t, y) * \Lambda_1(t, x) + \sum_{n=1}^{\infty} \Psi_j(t, y) * T(t)^* n * \Lambda_1(t, x). \quad (A.14)
\]
Let observe that the term \( \Psi_j(t, y) * \Lambda_1(t, x) \) in (A.14) is not equal to 0 iff \( (\pm i, \pm j) \) is a path. In the same way, the first term of the series in (A.14) is the sum of \( \psi_{\pm i, \pm j}(t, y) * T(t)^* l \) over all \( l \in I \) and that each of these terms is not equal to 0 iff \( (\pm i, \pm j, l) \) forms a path. Reasoning iteratively and using Lemma A.1(ii)-(iii) to compute the nonzero terms in the right hand side of (A.14), we arrive finally to the expression
\[
H(t, x, y) = \delta_{i,j} \kappa^{-1}(e_1)G(t, d(x, y)) + \sum_{k \geq p(x, y)} \sum_{C \in C_{k+2}(x, y)} \kappa^{-1}(e_1) \epsilon(C) G(t, d_C(x, y)).
\]
Since initially \( x \) is an arbitrarily fixed point of \( e_1 \), the previous formula holds true also for any \( x \in e_i \), \( i = 1, \ldots, m \), and by continuity for any \( x \) vertex of \( \Gamma \), giving (2.6)-(2.7).

**B Appendix B: Proof of Propositions 2.2 and 4.1**

**Proof of Proposition 2.2** We shall prove (2.11) first for any fixed \( x \in \Gamma \setminus V \). By the continuity of \( H \) on \( \Gamma \), (2.11) holds true for all \( x \in \Gamma \). Moreover, since \( \int_{\Gamma} H(t, x, y)dy \) is continuous with respect to \( t \) and \( \partial_t \int_{\Gamma} H(t, x, y)dy = \int_{\Gamma} \partial_y H(t, x, y)dy = 0 \), it is enough to prove that \( \lim_{t \to a^+} \int_{\Gamma} H(t, x, y)dy = 1 \).
To begin, we have
\[ \int_\Gamma H(t, x, y) dy = \int_0^1 G(t, |\xi - \eta|) d\eta + I(t, x) = 1 - \int_{\mathbb{R}\setminus(0,1)} G(t, |\xi - \eta|) d\eta + I(t, x), \] (B.1)
where \( \xi = (\Pi^+_i)^{-1}(x) \) and \( \eta = (\Pi^+_j)^{-1}(y) \). The limit as \( t \to 0^+ \) of the second term in the r.h.s. of (B.1) is obviously 0. Concerning the remainder \( I \), for any \( x \in e_i \) fixed, there exists \( \delta \in (0, 1) \) such that \( \xi \in (\delta, 1 - \delta) \). Then, we have
\[ |I| = \left| \sum_{j=1}^m \frac{\kappa(e_j)}{\kappa(e_i)} \int_0^1 \sum_{k \geq \rho(x, y)} \sum_{C \in C_{k+2}(x, y)} \epsilon(C) G(t, d_C(x, y)) d\eta \right| \leq m \kappa_1 \kappa_0^{-1} \sum_{k=0}^{+\infty} e^{-\delta^2 + k^2/4t} (\tau m)^{k+1}, \]
so that \( \lim_{t \to 0^+} I(t, x) = 0 \).

Next, let us notice that the function \( L(t, x, y) \) in (2.14) is not a priori positive (because of \( \epsilon(C) \), see (2.11)-(2.12)), and so \( H(t, x, y) \) too. Therefore, (2.11) does not give (2.12). However, it follows by (2.6) and (2.9) with \( t \leq 1 \) that
\[ |H(t, x, y)| \leq K t^{-\frac{3}{2}} \left[ 1 + \sum_{k=0}^{+\infty} (\tau m)^k e^{-k^2/4t} \right] = K t^{-\frac{3}{2}}, \quad \forall (x, y) \in \Gamma \times \Gamma. \] (B.2)
In particular \( |H(1, x, y)| \leq K \). Applying the maximum principle for the heat equation on networks [2], it holds
\[ |H(t, x, y)| \leq K, \quad \forall (t, x, y) \in [1, \infty) \times \Gamma \times \Gamma, \] (B.3)
and (2.12) follows for \( t \geq 1 \). On the other hand, for \( t \in (0, 1) \) and \( x = \Pi_i^+(\xi) \in \mathcal{T}_i \) arbitrarily fixed, we have
\[ \int_\Gamma |H(t, x, y)| dy \leq \int_0^1 G(t, |\xi - \eta|) d\eta + \int_\Gamma |L(t, x, y)| dy \]
\[ \leq 1 + \sum_{j=1}^m \frac{\kappa(e_j)}{\kappa(e_i)} \sum_{k=0}^{+\infty} (\tau m)^k e^{-k^2/4t} \int_0^1 \frac{1}{\sqrt{\pi t}} e^{-\eta^2/4t} d\eta \leq 1 + K. \]
The proof of (2.12) is now complete. From (B.2) and (B.3), estimate (2.13) follows too.

For the derivative of \( H \) we have
\[ \partial_y H(t, x, y) = \delta_{i,j} \kappa^{-1}(e_i) \partial_y G(t, d(x, y)) + \partial_y L(t, x, y), \quad \forall (t, x, y) \in (0, \infty) \times \Gamma \times \Gamma, \]
where
\[ \partial_y L(t, x, y) = \sum_{k \geq \rho(x, y)} \sum_{C \in C_{k+2}(x, y)} \kappa^{-1}(e_i) \epsilon(C) \partial_y G(t, d_C(x, y)). \]
Observing that for any \( C \in C_{k+2}(x, y) \) we have \( d_C(x, y) = (1 - \xi) + k + \eta \), we get the estimate
\[ |\partial_y H(t, x, y)| \leq K t^{-1} \left[ 1 + \sum_{k=0}^{+\infty} (\tau m)^k e^{-k^2/4t} + \sum_{k=0}^{+\infty} (\tau m)^k \frac{k}{2\sqrt{t}} e^{-k^2/4t} \right], \quad \forall (t, x, y) \in (0, \infty) \times \Gamma \times \Gamma. \] (B.4)
Moreover,
\[
\sum_{k=0}^{+\infty} (\tau m)^k \frac{k}{2\sqrt{t}} e^{-k^2/4t} \leq \frac{1}{2\sqrt{t}} e^{-1/4t} \sum_{k=1}^{+\infty} (\tau m)^k k e^{-(k-1)^2/4t},
\]
and (2.15) follows for \( t \leq 1 \), from (B.4) and (B.5). For \( t > 1 \), we obtain (2.15) as before, observing that \( |\partial_y H(1, x, y)| \leq K \) and applying again the maximum principle.

It remains to prove (2.14) for \( t \in (0, 1) \), since for \( t \geq 1 \) (2.14) follows by the obtained estimate \( \|\partial_y H(t)\|_{L^\infty(\Gamma \times \Gamma)} \leq K \). Let \( x = \Pi^\pm_i (\xi) \in \overline{\Omega} \) be arbitrarily fixed. We have
\[
\int_\Gamma |\partial_y H(t, x, y)| dy \leq \int_0^1 G(t, |\xi - \eta|) \frac{|\xi - \eta|}{2t} d\eta + \int_\Gamma |\partial_y L(t, x, y)| dy
\]
\[
\leq K t^{-1/2} + C \sum_{j=1}^{m} \frac{\kappa(e_j)}{\kappa(e_i)} \sum_{k=0}^{+\infty} (\tau m)^{k+1} \frac{(1 - \xi)}{t} e^{-(1-\xi)^2/4t} e^{-k^2/4t} \int_0^1 e^{-\eta^2/4t} \frac{d\eta}{2\sqrt{t}}
\]
\[
+ C \sum_{j=1}^{m} \frac{\kappa(e_j)}{\kappa(e_i)} \sum_{k=0}^{+\infty} (\tau m)^{k+1} \frac{k}{t} e^{-k^2/4t} \int_0^1 e^{-\eta^2/4t} \frac{d\eta}{2\sqrt{t}}
\]
\[
\leq K t^{-1/2} + C \sum_{k=0}^{+\infty} (\tau m)^{k} k e^{-k^2/4t} + C \sum_{k=0}^{+\infty} (\tau m)^{k} \frac{k}{2\sqrt{t}} e^{-k^2/4t}.
\]

Arguing as in (B.3), we arrive at
\[
\sup_{x \in \Gamma} \|\partial_y H(t, x, \cdot)\|_{L^1(\Gamma)} \leq K t^{-1/2}, \quad \forall \ t \in (0, 1).
\]

Finally, the estimate \( \sup_{y \in \Gamma} \|\partial_y H(t, \cdot, y)\|_{L^1(\Gamma)} \leq K t^{-1/2} \), for \( t \in (0, 1) \), can be obtained as above, changing the role between \( \xi \) and \( \eta \), and the proof is complete.

**Proof of Proposition 4.1** We say that a function \( w \in H^1(\Gamma) \) is a weak solution of (4.3) if it is a solution of the variational formulation of (4.3)
\[
a(w, \phi) = (z, \phi)_{L^2(\Gamma)}, \quad \forall \ \phi \in H^1(\Gamma),
\]
where \( a(w, \phi) := \int_\Gamma (w'(x)\phi'(x) + \alpha w(x)\phi(x)) dx \) is a continuous, coercive bilinear form on \( H^1(\Gamma) \). A standard application of the Lax-Milgram’s Theorem gives the existence and uniqueness of the solution of (B.6). The inequality (4.3) is consequence of the maximum principle (see [21]). To prove (4.5), observe that the equation and (4.4) imply that \( w_j'' \) is bounded for all \( j = 1, \ldots, m \) and
\[
\|w''\|_{L^\infty(\Gamma)} \leq \|z\|_{L^\infty(\Gamma)}.
\]
By the previous estimate and the Morrey’s inequality we get the estimate (4.5).
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