Path Integration in Two-Dimensional Topological Quantum Field Theory

Stephen Sawin
Dept. of Math
MIT
Cambridge, MA 02139-4307
sawin@math.mit.edu

October 13, 2018

This research supported in part by NSF postdoctoral Fellowship #23068

Abstract

A positive, diffeomorphism-invariant generalized measure on the space of metrics of a two-dimensional smooth manifold is constructed. We use the term generalized measure analogously with the generalized measures of Ashtekar and Lewandowski and of Baez. A family of actions is presented which, when integrated against this measure, give the two-dimensional axiomatic topological quantum field theories, or TQFT's, in terms of which Durhuus and Jonsson decompose every two-dimensional unitary TQFT as a direct sum.

1 Introduction

This paper arose out of a question which the author first heard I. M. Singer propose in a graduate course: Does every two-dimensional axiomatic topological quantum field theory (TQFT) arise from a Lagrangian? That is, given Dijkgraaf's [?] classification of two-dimensional TQFT's [?, ?], can all such theories be formulated as the path integral of some action?
Using the refinement of Dijkgraaf’s work due to Durhuus and Jonsson [?], this question is easy to answer on the physical level of rigor. They prove that each two-dimensional unitary TQFT is a direct sum in an appropriate sense of theories arising from Euler number. Specifically, each such theory is a direct sum of theories \( Z_\alpha \), for \( \alpha \in \mathbb{R} \), where \( Z_\alpha \) assigns a one-dimensional Hilbert space to the circle, and when this is identified with \( \mathbb{C} \) correctly, assigns the number \( \exp(\alpha \chi(M)) \) to every 2-manifold \( M \) with boundary, where \( \chi(M) \) is the Euler number of \( M \). Durhuus and Jonsson use the parameter \( \lambda = \exp(-\alpha/2) \) to parametrize their theories.

Such a theory admits a straightforward Lagrangian formulation. The fields are metrics on \( M \), and the Lagrangian is \( \alpha/2\pi \) times the curvature two-form \( \Omega_g \) of the metric \( g \). Of course, if \( M \) is closed, then \( \int_M \Omega_g/2\pi = \chi(M) \) by Gauss-Bonnet, and in particular is a constant function on the space of metrics. Thus

\[
Z_\alpha(M) = e^{\alpha \chi(M)} = \int e^{\alpha \int_M \Omega_g/2\pi} \mathcal{D}g.
\]

If all we wanted to compute was the partition function of closed manifolds, this would be perfectly rigorous and satisfactory. However, it is clear one has not captured the notion of a path integral satisfactorily if one cannot at least reproduce the cutting and pasting arguments that justify Atiyah’s axioms. For this one needs a space of fields and a measure on this space for manifolds with boundary.

Here some difficulty occurs. As soon as one asks to be able to integrate nontrivial functions, finding a diffeomorphism-invariant measure on an infinite-dimensional space like the space of all metrics becomes quite difficult, and perhaps impossible. We work instead with ‘generalized measures’, as developed by Ashtekar and Lewandowski [?, ?] and Baez [?, ?]. They work on the space of all connections of a principal bundle, and take as their observables products of Wilson loops. They define a generalized measure on this space to be something which “knows the expectation values of these observables;” i.e., a bounded linear functional on this \( C^* \)-algebra. It turns out that a generalized measure corresponds to an honest finite Borel measure on an extension of the space of connections, called the space of generalized connections. Thus the whole theory of measure spaces can be applied to such generalized measures.

This approach works well in our context. A 2-manifold has naturally associated to it the principal \( S^1 \)-bundle coming from the tangent bundle,
and a metric gives the Levi-Civita connection on this bundle. A Wilson loop would just be $e^i$ raised to a multiple of the total geodesic curvature of the loop. It is more convenient to deviate slightly from their formulation and take the actual geodesic curvature, rather than the geodesic curvature mod $2\pi$. Thus our measurable functions will be limits of functions of the total geodesic curvature of finitely many curves. This fortunately includes the action.

We construct a diffeomorphism-invariant generalized measure on the space of metrics, with the property that, for any countable set of curves, the total geodesic curvature is almost always zero. We will see that the TQFT’s coming from Euler number can be made into rigorous path integrals against this measure.

This measure is an extremely simple one, and the theories that arise from it certainly deserve the name toy models. But, the geometric work which must be done is probably part of the work necessary to approach interesting generally covariant theories mathematically. In particular, this is a truly diffeomorphism-invariant theory. That is, one can integrate functions of the curvatures of any countable collection of smooth curves, even if the curves are not analytic. It is hoped that this will be a step towards constructing fully diffeomorphism-invariant generalized measures on the space of connections with a nonabelian group.

This paper is not the first to give rigorous path-integral formulations of TQFT’s. [?] gives a state-sum on a triangulation for theories $Z_\alpha$, when $\exp(\alpha) \in \mathbb{N}$, and [?] describes theories based on finite-groups, which are a subset of these, as a finite sum. In both cases the sums may be taken as discrete path integrals, and can be fit into the framework of Section III.

I would like to thank Isadore Singer, John Baez, Scott Axelrod, Eric Weinstein, Dan Stroock and Richard Dudley for comments, suggestions and conversations.

## 2 Constructing the Generalized Measure

Let $M$ be a smooth, compact 2-manifold, possibly with boundary. By a curve in $M$ we will always mean a smooth immersion of the circle into $M$, considered up to positive reparametrization.

If $c : S^1 \to M$ is a smooth immersion, the geodesic curvature of $c$ with
respect to a metric $g$ at a point $t \in S^1$, $\kappa_{c,g}(t)$, is the value of the Levi-Civita connection one-form of $g$ on the tangent to the lift of the curve to $TM$, at $t$. It depends on the parametrization, but the total curvature, $k_c(g) = \int_{S^1} \kappa_{c,g}(t) dt$, does not, and thus is a function of the curve $c$.

If $c_1, \ldots, c_n$ are curves in $M$, and $f$ is any bounded continuous function on $n$ real variables, we get a bounded continuous function $F_{f,c_1,\ldots,c_n}$ on $\mathcal{G}$, the set of all metrics on $M$, sending $g \in \mathcal{G}$ to $f(k_{c_1}(g), \ldots, k_{c_n}(g))$. Continuous bounded complex-valued functions on $\mathcal{G}$ form a commutative $C^*$-algebra with complex conjugation as involution and the sup norm as norm, and the closure $C$ in this norm of the algebra of all functions $F_{f,c_1,\ldots,c_n}$ is a $C^*$-subalgebra. We will call a bounded linear functional on $C$ a \textit{generalized measure} on $\mathcal{G}$.

The group of diffeomorphisms of $M$ acts as a group of homeomorphisms on $\mathcal{G}$, and thus as a group of $C^*$-isomorphisms on the bounded continuous functions on $\mathcal{G}$. These isomorphisms take $C$ to itself, so they also act on $C$, where a diffeomorphism $D$ sends $F_{f,c_1,\ldots,c_n}$ to $F_{f,D(c_1),\ldots,D(c_n)}$. Call a generalized measure \textit{diffeomorphism-invariant} if its value on an element of $C$ is unchanged by the action of a diffeomorphism. Also, call a generalized measure \textit{positive} if its value on any nonnegative-valued function in $C$ is nonnegative.

The terminology warrants an explanation. By a fundamental theorem of $C^*$-algebras [?, Thm. 1.4.1], $C$ is the algebra of continuous functions on some compact Hausdorff space $\mathfrak{G}$ into which $\mathcal{G}$ maps continuously and densely. In particular, a bounded linear functional on $C$ corresponds by the Riesz representation theorem to a finite Borel measure on $\mathfrak{G}$. What’s more the action of diffeomorphisms of $M$ on $\mathcal{G}$ extends to a continuous action on $\mathfrak{G}$, and thus acts on the finite Borel measures of $\mathfrak{G}$, the action being the one already described for linear functionals on $C$. Thus diffeomorphism-invariant generalized measures on $\mathcal{G}$ are exactly diffeomorphism-invariant finite Borel measures on $\mathfrak{G}$. Likewise, an element of $C$ is nonnegative if and only if it is nonnegative as a function on $\mathfrak{G}$, so positive generalized measures are exactly positive, finite, Borel measures on $\mathfrak{G}$.

This is the analogue of the notion of generalized measures on the space of connections developed by Ashtekar & Lewandowski and Baez. It is helpful to think of the generalized measure as an honest measure on the enlarged space, and to rely on our knowledge and intuition about measures. This is what we do in reconstructing the TQFT from the generalized measure we construct. It would be possible to do everything without reference to $\mathfrak{G}$, simply relying on
the definition of a generalized measure as a bounded linear functional. This certainly has an appealing concreteness, and might dispel a slightly mystical feel associated to these highly abstract spaces and structures. The language of path integrals is so compelling, however, that the exposition seems greatly aided by the ability to use explicit measures and integrals.

The measure we construct is defined by the very simple property that for any finite set of curves, almost every generalized metric assigns these curves total curvature zero.

**Theorem 1** There exists a positive, diffeomorphism-invariant generalized measure sending each $F_{f,c_1,...,c_n}$ to $f(0,...,0)$.

To prove the functional is bounded, in fact of norm one, the key geometric step is to see that its value on a function is contained in the range of the function: that for any finite set of curves, there is a metric which makes their total curvatures all simultaneously zero. This straightforward sounding fact is quite subtle, because of the complicated manner in which the curves might intersect.

**Lemma 1** Let $\{c_i\}_{i=1}^n$ be a set of curves in a smooth 2-manifold $M$. There is a metric on $M$ which makes the total curvature on each $c_i$ equal to zero.

**Proof:** First, recall that no curve can touch the same point $x$ in $M$ infinitely many times. If it did, the preimage of $x$ in $S^1$ would have an accumulation point, and the curve would not be an immersion at that point.

Consider each $x \in K$, the union of the ranges of the $c_i$’s. Identify the tangent plane to $M$ at $x$ with $\mathbb{R}^2$ in such a fashion that no tangent vector to any $c_i$ is on the $y$-axis. Extend this to a coordinate patch, and consider a neighborhood $N_x$ small enough that no tangent to $c_i$ is parallel to the $y$-axis. Since $K$ is compact, it is covered by finitely many such $N_i$’s, $i = 1,\ldots,k$. Choose a triangulation of $M$, and subdivide as necessary until every triangle intersecting $K$ lies in some $N_i$. Perturb this triangulation slightly so that the vertices do not lie on $K$ and no edge is ever horizontal.
with respect to any coordinate patch. Notice each $c_i$ intersects each edge transversely, and hence finitely many times.

Now choose nonintersecting neighborhoods of each edge $e$ minus a neighborhood of the vertices, and identify them with a region of $\mathbb{R}^2$ via a map $\phi_e$, so that the edge is identified with a straight line and each intersection with a $c_i$ is perpendicular. Finally, identify each open face of the triangulation minus a disk around each vertex with a region of $\mathbb{R}^2$, so that the overlap maps with $\phi_e$ are isometries, and all three edges are vertical, pointing up or down according to whether they pointed up or down in the original coordinate patch. See Figure 1 for an example of this process.

The metric induced by these maps is a flat metric on $M$ with finitely many disks removed. Extend it to all of $M$. The total curvature of $c_i$ with respect to this metric is the sum of the total curvature of $c_i$ in each triangle. However, within one triangle $c_i$ goes without self-intersection from being horizontal to being horizontal with the same orientation. Since the metric is flat, the total curvature of this piece is zero. Thus the total curvature of each $c_i$ in this metric is zero.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Putting a flat metric on a piece of $K$}
\end{figure}

**Proof:** *(Of Theorem 1)* It suffices to show this gives a well-defined norm-one linear functional on all functions of the form $F_{f,c_1,\ldots,c_n}$, since these are dense in $C$. If $F_{f,c_1,\ldots,c_n}$ and $F_{g,d_1,\ldots,d_n}$ represent the same
function, find by Lemma 1 a metric such that all \( c_i \) and \( d_i \) have zero total curvature. In each case the value of the functional is just the value of the function on this metric, and therefore is the same. Since the value of the functional is always gotten by evaluation of the function at some point, it is clearly smaller than the sup norm of that function, and therefore is of norm one.

Call the measure we have constructed \( D_g \).

3 Constructing the TQFT

We must say what it means for a TQFT to ‘arise from a path integral.’

Consider a contravariant functor from the tensor category of smooth, compact \( n \)-manifolds with boundary and smooth immersions to the tensor category of probability spaces and measurable functions. Specifically, this is an assignment of a probability space \( \mathcal{F}_M \) to every manifold \( M \), and a measurable map \( \mathcal{F}_f : \mathcal{F}_N \to \mathcal{F}_M \) to each immersion \( f : M \to N \), such that \( \mathcal{F}_g \mathcal{F}_f = \mathcal{F}_{fg} \), \( \mathcal{F}_1 = 1 \), \( \mathcal{F}_{M \cup N} = \mathcal{F}_M \times \mathcal{F}_N \), \( \mathcal{F}_{f \cup g} = \mathcal{F}_f \times \mathcal{F}_g \), the empty \( n \)-manifold gets sent to a one-point probability space, and the obvious map from \( M \cup N \) to \( N \cup M \) gets sent to the obvious map from \( \mathcal{F}_M \times \mathcal{F}_N \) to \( \mathcal{F}_N \times \mathcal{F}_M \). We also ask that if \( i \) is an embedding then \( \mathcal{F}_i \) is measure-preserving, in the sense that the preimage of measurable sets have the same measure. Thus in particular diffeomorphisms correspond to measure-space isomorphisms.

If \( i : N \to M \) is an embedding, so that \( \mathcal{F}_i : \mathcal{F}_M \to \mathcal{F}_N \) is measure-preserving, consider an integrable function \( f \) from \( \mathcal{F}_M \) to \( \mathbb{C} \). It defines a measure on \( \mathcal{F}_N \), by assigning to each measurable set the integral of \( f \) on the pullback of that set. This measure is absolutely continuous with respect to the probability measure on \( \mathcal{F}_N \), and thus arises from some function \( E^M_N(f) \) defined a.e., by the Radon-Nikodym theorem. The map \( E \) is called a conditional expectation, and satisfies that, for \( f, f' \) functions on \( \mathcal{F}_M \) and \( \mathcal{F}_N \) respectively,

\[
\int_{\mathcal{F}_M} E^M_N(f)f'Dg = \int_{\mathcal{F}_M} ff'Dg,
\]

where all functions are pulled back to \( \mathcal{F}_M \).

Our final condition on this functor is that if \( M_1 \cap M_2 \subset N \) are all submanifolds of some \( M \), \( f_1 \) and \( f_2 \) are integrable functions on \( \mathcal{F}_{M_1} \) and \( \mathcal{F}_{M_2} \),
respectively, and the product of their pullbacks to $\mathfrak{S}_M$ is also integrable, then $E_N^M(f_1 f_2) = E_N^M(f_1) E_N^M(f_2)$, where by $f_1$ and $f_2$ we really mean their pullbacks to $\mathfrak{S}_M$. This odd condition expresses the intuitive notion that having specified a field on $N$, the possible ways of extending it to $M_1$ and $M_2$ are independent.

Suppose a portion of the boundary of $M$ has been identified with a $d-1$-manifold $\Sigma$ in some way. This identification can be extended to an embedding $i$ of $\Sigma \times I$ into $M$. The embedding is certainly not unique, but two different such embeddings can be related by a boundary-fixing automorphism of $M$, so that the measure-preserving map $\mathfrak{F}_i : \mathfrak{F}_M \to \mathfrak{F}_{\Sigma \times I}$ is unique up to a measure-isomorphism of $\mathfrak{F}_M$ (again, having picked an identification of the boundary with $\Sigma$). If we define $\mathfrak{F}_\Sigma$ to be $\mathfrak{F}_{\Sigma \times I}$, $\mathfrak{F}$ is a contravariant functor with the same properties for $n-1$-manifolds, and there is a measure preserving natural transformation $\mathfrak{F}_M \to \mathfrak{F}_{\partial M}$.

Now for each compact, closed, oriented $n$-manifold $M$, let $S_M$ be an integrable function on $\mathfrak{F}_M$. Suppose $S_{M \cup N} = S_M \times S_N$, $S_M = \bar{S}_M^*$, the bar indicating complex conjugate and $M^*$ indicating $M$ with the reverse orientation, and if $f : M \to N$ is an onto immersion, then $S_N = S_M \mathfrak{F}_{f}$. Call such an $S$ an action. Define $Z(M) = E_{\partial M}^M(S_M)$, an integrable function on $\mathfrak{F}_{\partial M}$. Notice that $Z(M) = \bar{Z}(M^*)$.

Let $M_1$ have a portion of its boundary isomorphic to an $n-1$-manifold $\Sigma$, and let $M_2$ have a portion of its boundary isomorphic to $\Sigma^*$, and consider the manifold $M$ formed by gluing them together by identifying points along this boundary. We may just as well identify a neighborhood of both boundaries with $\Sigma \times I$, and identify them along these submanifolds. Thus if $\Sigma_1$ and $\Sigma_2$ are the rest of the boundary of $M_1$ and $M_2$ respectively, we have

\[
E_{\Sigma_1 \cup \Sigma_2}^M(S_M) = E_{\Sigma_1 \cup \Sigma_2}^M(S_{M_1} S_{M_2})
\]

\[
= E_{\Sigma_1 \cup \Sigma_2}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma} E_{\Sigma_1 \cup \Sigma_2 \cup \Sigma}^M(S_{M_1} S_{M_2})
\]

\[
= E_{\Sigma_1 \cup \Sigma_2 \cup \Sigma}^M E_{\Sigma_1 \cup \Sigma_2 \cup \Sigma}^M(S_{M_1}) E_{\Sigma_1 \cup \Sigma_2 \cup \Sigma}^M(S_{M_2})
\]

\[
= E_{\Sigma_1 \cup \Sigma_2}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma} E_{\partial M_1}^{\Sigma_1 \cup \Sigma_2 \cup \Sigma}(S_{M_1}) E_{\partial M_2}^{M_2}(S_{M_2})
\]

with $d-1$-manifolds such as $\Sigma_1$ used to denote the associated subspaces of the $d$-manifolds diffeomorphic to $\Sigma_1 \times I$, for convenience. The last step is because $\Sigma_2$ is disjoint from $M_1$ and $\Sigma \cup \Sigma_1$, so the probability spaces of the unions are products, and thus $E_{\Sigma_1 \cup \Sigma_2}^M = E_{\Sigma_1 \cup \Sigma_2}^{M_1 \cup M_2} = E_{\Sigma_1 \cup \Sigma}^{M_1 \times id}$ for
functions of $\mathfrak{F}_{M_1} \times \mathfrak{F}_{\Sigma}$. Now $\mathfrak{F}_{\Sigma_1 \cup \Sigma_2}$ is a product probability space, so denoting elements of $\mathfrak{F}_{\Sigma_1}$, $\mathfrak{F}_{\Sigma_2}$, and $\mathfrak{F}_{\Sigma}$ by $a$, $b$, and $c$ respectively, this says

$$\mathcal{Z}(M)(a, b) = \int \mathcal{Z}(M_1)(a, c)\mathcal{Z}(M_2)(b, c)\mathcal{D}c.$$ 

In particular, gluing $M$ to $M^*$ along their common boundary and taking $\mathcal{Z}$ of the result yields $\int \mathcal{Z}(M)\mathcal{Z}(M)$, and we see $\mathcal{Z}(M)$ is in $L_2(\mathfrak{F}_{\partial M})$ for all $M$. We may thus interpret $\mathcal{Z}(M)$, $\mathcal{Z}(M_1)$ and $\mathcal{Z}(M_2)$ above as the kernels of Hilbert-Schmidt operators $\mathfrak{F}_{\Sigma_1} \to \mathfrak{F}_{\Sigma_2}$, $\mathfrak{F}_{\Sigma_1} \to \mathfrak{F}_{\Sigma}$, and $\mathfrak{F}_{\Sigma} \to \mathfrak{F}_{\Sigma_2}$ respectively, and then the above gluing law becomes

$$\mathcal{Z}(M) = \mathcal{Z}(M_2)\mathcal{Z}(M_1)$$

the product being composition of Hilbert-Schmidt operators.

Now consider $\mathcal{Z}(\Sigma \times I) \in L_2(\mathfrak{F}_\Sigma \times \mathfrak{F}_\Sigma)$ as the kernel of a Hilbert-Schmidt operator $P_\Sigma$ on $L_2(\mathfrak{F}_\Sigma)$. Let $\mathcal{Z}(\Sigma)$ be the range of $P_\Sigma$.

**Theorem 2** Let $\mathfrak{F}$ be a contravariant functor as above, and let $S$ be an action for it. Then $\mathcal{Z}$ is a unitary TQFT.

**Proof:** We first argue that $\mathcal{Z}(M)$ lies in $\mathcal{Z}(\partial M)$. Notice that $M$ is equal to $M$ glued to $\partial M \times I$. In particular, $\mathcal{Z}(M) = P_\partial M \mathcal{Z}(M)$ and thus is in $\mathcal{Z}(\Sigma)$.

We have already demonstrated the gluing law. In particular $P_\Sigma P_\Sigma$ is the operator associated to the manifold formed by gluing $\Sigma \times I$ to itself along one $\Sigma$. Since this is $\Sigma \times I$ again, We have $P_\Sigma P_\Sigma = P_\Sigma$, and thus it acts as the identity on its range $\mathcal{Z}(\Sigma)$.

Since $\mathcal{Z}(\Sigma_1 \cup \Sigma_2)$ is the range of $P_{\Sigma_1 \cup \Sigma_2} = P_{\Sigma_1} \otimes P_{\Sigma_2}$, it is the tensor product of the ranges of the two factors, so it is $\mathcal{Z}(\Sigma_1) \otimes \mathcal{Z}(\Sigma_2)$. Of course, if $\Sigma$ is the empty $n - 1$ manifold then $\mathfrak{F}_\Sigma$ is the one-point probability space and $P_\Sigma$ is the identity, so $\mathcal{Z}(\Sigma) = \mathbb{C}$.

Now $P_\Sigma$ is Hilbert-Schmidt and hence compact, so acting as the identity on $\mathcal{Z}(\Sigma)$ means that $\mathcal{Z}(\Sigma)$ is finite-dimensional. Thus we can write $\mathcal{Z}(\Sigma \times I)$ as $\sum_{i=1}^{n} f_i \otimes \bar{f_i}$, where $\{f_i\}$ is an orthonormal basis for $\mathcal{Z}(\Sigma)$ of functions of $\mathfrak{F}_\Sigma$. Since $\mathcal{Z}(\Sigma \times I)$ is
also an element of $Z(\Sigma) \otimes Z(\Sigma^*)$, we have that $Z(\Sigma^*)$ contains the complex conjugate of $Z(\Sigma)$ in $L_2(\mathfrak{F}_\Sigma)$. Since $(\Sigma^*)^* = \Sigma$, $Z(\Sigma^*)$ is equal to the complex conjugate of $Z(\Sigma)$, and thus $Z(\Sigma^*)$ is dual in the obvious way to $Z(\Sigma)$. $Z(\Sigma \times I)$ is the canonical element of $Z(\Sigma) \otimes Z(\Sigma^*)$.

We have already seen that any diffeomorphism of $\Sigma$ to $\Sigma'$ corresponds to a measure space isomorphism of $\mathfrak{F}_\Sigma$ to $\mathfrak{F}_{\Sigma'}$. This diffeomorphism extends to a diffeomorphism of $\Sigma \times I$ to $\Sigma' \times I$, and so the isomorphism takes $Z(\Sigma \times I)$ to $Z(\Sigma' \times I)$. We thus get an isomorphism from $Z(\Sigma)$ to $Z(\Sigma')$. More generally, if $M$ is diffeomorphic to $M'$, the diffeomorphism restricted to the boundary gives an isomorphism of $Z(\partial M)$ to $Z(\partial M')$ sending $Z(M)$ to $Z(M')$. This and the functoriality of $\mathfrak{F}$ make $Z$ functorial.

Finally, since $Z(M^*) = \overline{Z(M)}$, we have that they are the kernels of adjoint Hilbert-Schmidt operators. This is all we needed to show for $Z$ to be a unitary TQFT \cite{??}, \cite{??}.

\begin{theorem}
The assignment to each $M$ of the measure space $\mathfrak{G}$ of generalized measures on $M$ with the measure constructed in the previous section, together with the action $S_M = \exp(\alpha \int_M \Omega_g / 2\pi)$, for a fixed $\alpha \in \mathbb{R}$, satisfy the assumptions of the previous theorem, and thus give a 2-dimensional unitary TQFT. What’s more, this is exactly the TQFT $Z_\alpha$ described by Durhuus and Jonsson \cite{??}.
\end{theorem}

\begin{proof}
We have not fully described the data, since we have not associated a measurable map to each immersion. Let $i : M \to N$ be such an immersion. This gives a map $i^* : \mathcal{G}_N \to \mathcal{G}_M$, taking each metric on $N$ to its pullback on $M$. If $F_{f,c_1,\ldots,c_n}$ is a typical element of $C$ for $M$, then $F_{f,c_1,\ldots,c_n} \circ i^* = F_{f,i(c_1),\ldots,i(c_n)}$, an element of $C$ for $N$. This is a $C^*$-homomorphism, and thus induces a continuous onto map $\mathfrak{G}_i : \mathfrak{G}_N \to \mathfrak{G}_M$. If $i$ is an embedding, the integral of each $F$ is unchanged by $\mathfrak{G}_i$, so it is measure-preserving. The assignment of $i^*$ to $i$ is a contravariant functor, so the assignment $\mathfrak{G}$ is too.
\end{proof}
The conditional expectation $E^M_N$ is trivial, because the space of integrable functions up to a.e. equivalence is one dimensional. In fact, if $N$ is nonempty $E^M_N(F_{f,c_1,...,c_n}) = F_{g,d_1,...,d_k}$, where $d_i$ are any curves on $N$ and $g$ is any function with $g(0,...,0) = f(0,...,0)$. If $N$ is the empty manifold, then it is the function on the one-point probability space with value $f(0,...,0)$. Either way it is always true that $E^M_N(f_1)E^M_N(f_2) = E^M_N(f_1f_2)$ a.e., for any functions $f_i$ on $\mathfrak{S}_M$.

$S_M$ is an element of $C$, because it is just $\alpha(\chi(M) + x_g/2\pi)$, where $x_g$ is the total curvature around the boundary. Thus it is integrable. It clearly pulls back through onto immersion, its value on a union of manifolds is the product of its values on the individual manifolds, and reversing orientation does not change it and thus in particular sends it to its complex conjugate for $\alpha \in \mathbb{R}$.

The TQFT is now easy to describe. An element of $\mathfrak{G}_M$ assigns a total curvature to each curve, and two will go to the same element of $\mathfrak{S}_M'$ if they assign the same value to all curves in $M'$. $L_2$ functions on $\mathfrak{G}_M'$ will be spanned by those of the form $F_{f,c_1,...,c_n}$ with the $c_i$’s in $M'$. Such a function has zero norm if $f(0,...,0) = 0$. Thus $L_2(\mathfrak{G}_\Sigma) = \mathbb{C}$ for every one-manifold $\Sigma$. For any $M$, $Z(M)$ is the function $\exp(\alpha(\chi(M) + x/2\pi))$, where $x$ is the total curvature around the boundary, and thus corresponds as an element of $\mathbb{C}$ to $\exp(\alpha \chi(M))$. Each $P_\Sigma$ is then 1, so $Z(\Sigma) = \mathbb{C}$, with $Z(M) = \exp(\alpha \chi(M))$. This is exactly the TQFT constructed by Durhuus and Jonsson.

4 Remarks

- All that we did here would work as well for piecewise smooth curves, and for curves with endpoints.

- Of course, these theories have obvious observables, namely functions of the total curvature of a given set of closed curves. Unfortunately, the expectation value of these observables are all zero.
• One might wonder whether the work of constructing a generalized measure was really necessary. If we had defined the $C^*$-algebra $C$ to be spanned only by functions of the total curvature of the boundary of $M$, we would have had no difficulty finding a measure on $\mathcal{G}$ which made everything in $C$ integrable and satisfied the restrictions necessary to make the value of the partition function $Z_\alpha$ (specifically, the integral of $\exp((\alpha + \bar{\alpha})x_B/2\pi)$ must be 1, where $x_B$ was the total curvature of the boundary). However, such a measure would not have had the functorial properties needed to apply Theorem 2. By reproducing in a rigorous fashion the heuristic arguments leading from path integrals to Atiyah’s axioms, Theorem 2 represents a reasonable definition of what it means for a TQFT to arise rigorously from a path integral.

• We began with the question: Do all two-dimensional axiomatic unitary topological quantum field theories arise as path integrals? We have only actually shown that every such is a direct sum of theories with this property. But it is straightforward to write the direct sum of two path integral theories as a path integral: for connected $M$ the measure space of fields is just the union of the two measure spaces, and the action is just the function which is each respective action on each component.

• A slight modification of the above argument gives a path-integral formulation of the nonunitary TQFT $Z_\alpha$ for $\alpha \in \mathbb{C} - \mathbb{R}$. These represent many of the irreducible nonunitary two-dimensional TQFT’s, but not all: there are nilpotent theories [?] for which no path integral formulation is known. They are simple enough that one might conceivably hope to find actions for them, but their geometry is less transparent than the $Z_\alpha$.  

12