Abstract. We study a family of gradient obstacle problems on a compact Riemannian manifold. We prove that the solutions of these free boundary problems are uniformly semiconcave and, as a consequence, we obtain some fine convergence results for the solutions and their free boundaries. Precisely, we show that the elastic and the \( \lambda \)-elastic sets of the solutions Hausdorff converge to the cut locus and the \( \lambda \)-cut locus of the manifold.

1. Introduction

Let \( M \) be a smooth \( n \)-dimensional compact Riemannian manifold without boundary. Let \( b \in M \) be a fixed point. We denote by \( d_b : M \to \mathbb{R} \) the distance function to \( b \), and by \( \text{Cut}_b(M) \) the cut locus, that is the set of points (cut points) \( p \in M \) for which there exists a geodesic \( \gamma \), starting from \( b \) and passing through \( p \), which is length minimizing between \( b \) and \( p \), but not after \( p \). The cut locus inherits much of the topology of \( M \). It is a deformation retract of \( M \setminus \{b\} \) and has the same homotopy type (see for instance [25], Chapter III, Section 4). Moreover, it is also related to the global geometry of \( M \), for instance, to the geodesic spectrum (every close geodesics starting from \( b \) crosses \( \text{Cut}_b(M) \)) and the Ambrose’s problem (see [17]).

The local structure of the cut locus can be very rich and at the same time complicated, as it seems to be closely related to the regularity of \( g \). A stratification theorem is available only when the metric \( g \) is analytic (see [22] and [5]), while in general, it is known that \( \text{Cut}_b(M) \) must have an integer Hausdorff dimension (when \( g \) is \( C^\infty \)) that might even become fractional when \( g \) is \( C^k \) (see [18] and the references therein). The sensitivity with respect to the regularity of the manifold \( (M,g) \) makes the cut locus difficult to recover by numerical methods involving discrete structures. A more stable object from this point of view is the so-called \( \lambda \)-cut locus \( \text{Cut}^\lambda_b(M) \), which we introduce in this paper in analogy with the \( \lambda \)-medial axis of Chazal and Lieutier, which is a widely studied object in Computational Geometry (see Section 1.1). We refer to [12] for a detailed account on the impact of our study to the numerical methods for the computation of the cut locus.
For any $\lambda > 0$, the $\lambda$-cut locus is defined as

$$\text{Cut}_\lambda^b(M) := \left\{ p \in M \setminus \{b\} : |\nabla d_b(p)|^2 \leq 1 - \frac{\lambda^2}{d_b^2(p)} \right\},$$

the norm of the generalized gradient $|\nabla d_b|$ being defined at every point $p \in M \setminus \{b\}$ as

$$|\nabla d_b|(p) := \max \left\{ 0, \sup_{v \in T_p M, \|v\| = 1} \partial^+_v d_b(p) \right\},$$

where $\partial^+_v d_b(p)$ is the derivative of $d_b$ in the direction $v$ (see Section 2). The $\lambda$-cut locus approximates the cut locus in the following sense: for every $\lambda > 0$, we have $\text{Cut}_\lambda^b(M) \subset \text{Cut}^b_b(M)$, while the closure of the union of $\text{Cut}_\lambda^b(M)$ over $\lambda > 0$ is precisely $\text{Cut}_b(M)$ (see Proposition 2.9). In particular, just as the cut locus, the $\lambda$-cut locus is a non-smooth set, with potentially very wild structure, even when $M$ is smooth.

In this paper we study the asymptotic behavior of a family of gradient obstacle problems on the manifold $M$ and we prove that both $\text{Cut}^b_b(M)$ and $\text{Cut}_\lambda^b(M)$ can be recovered from the solutions of these problems. Moreover, even if our study is purely theoretical, it leads to a new method for the numerical approximation of the cut locus and the $\lambda$-cut locus on a compact manifold (see Remark 1.2).

For any $m > 0$, we consider the variational minimization problem

$$\min \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), \ |\nabla u| \leq 1, \ u(b) = 0 \right\}.$$  

This problem has a unique minimizer, which we will denote by $u_m$. We consider the sets

$$E_m := \left\{ p \in M \setminus \{b\} : |\nabla u_m(p)| < 1 \right\},$$

and

$$E_{m, \lambda} := \left\{ p \in M \setminus \{b\} : |\nabla u_m(p)|^2 \leq 1 - \frac{\lambda^2}{u_m^2(p)} \right\}.$$  

Our main result is the following.

**Theorem 1.1 (Approximation of Cut$_b(M)$ and Cut$_\lambda^b(M)$).** Let $M$ be a compact Riemannian manifold of dimension $n$ and let $b \in M$ and $\lambda > 0$ be fixed. Then,

$$E_m \xrightarrow{m \to +\infty} \text{Cut}_b(M) \quad \text{in the Hausdorff sense.}$$

Moreover, for any fixed $\varepsilon > 0$, we have that

$$\sup_{p \in E_{m, \lambda}} d(p, \text{Cut}_b^\varepsilon(M)) \xrightarrow{m \to +\infty} 0, \quad \text{and} \quad \sup_{p \in \text{Cut}_b^\lambda + \varepsilon(M)} d(p, E_m) \xrightarrow{m \to +\infty} 0.$$  

**Remark 2.** (About the numerical computation of the cut locus). We notice that the direct numerical approximation of the cut locus and the $\lambda$-cut locus is difficult and requires significant computational resources. Conversely, the variational problem (1.3) consists in minimizing a convex functional under a convex constraint, which considerably simplifies this task. The numerical approach based on solving (1.3) will be the object of the forthcoming paper [12].

In order to prove Theorem 1.1, we have to study the regularity of the solutions $u_m$ and the convergence of the asymptotic behavior (as $m \to \infty$) of the sequence $(u_m)$. We gather our results about the solutions of (1.3) in the following theorem and we notice that Theorem 1.1 is in fact an immediate consequence of the claims (T5) and (T6) of Theorem 1.3 below (see Section 1.3).

**Theorem 1.3 (Regularity and convergence of $u_m$).** Let $M$ be a compact Riemannian manifold of dimension $n$ and let $b \in M$ be fixed. Then, the following holds.

1. **(T1) Regularity of $u_m$.** There exists a constant $m_0 > 0$, depending only on the manifold $M$, such that for every $m > m_0$, the minimizer $u_m$ of (1.3) is locally $C^{1,1}$ on $M \setminus \{b\}$.

2. **(T2) Properties of $E_m$.** For every $m \geq m_0$, $E_m$ is an open subset of $M$ and coincides with the set $\{u_m < d_b\}$. Moreover, $E_m$ contains $\text{Cut}_b(M)$ and is at positive distance from $b$, that is $u_m = d_b$ in a neighborhood of $b$.

3. **(T3) Monotonicity of $u_m$ and $E_m$.** For every $m \geq m' \geq m_0$, we have $u_m \geq u_{m'}$. In particular, $E_m \subset E_{m'}$.

4. **(T4) Semiconcavity of $u_m$.** For every $\rho > 0$, there are constants $C > 0$ and $m_1 > 0$, depending on $\rho$ and on the manifold $M$, such that

$$u_m \text{ is } C\text{-semiconcave on } M \setminus B_\rho(b),$$

for every $m \geq m_1$.  


(T5) Convergence of $u_m$. The sequence $u_m$ converges uniformly on $M$ to the distance function $d_b$.

(T6) Convergence of the gradients. Let $p_\infty \in M \setminus \{b\}$. Then

- for every sequence $p_m \to p_\infty$, we have
  \[
  |\nabla d_b|(p_\infty) \leq \liminf_{m \to \infty} |\nabla u_m|(p_m);
  \]

- there exists a sequence $p_m \to p_\infty$ such that
  \[
  |\nabla d_b|(p_\infty) = \lim_{m \to \infty} |\nabla u_m|(p_m).
  \]

Remark 1.4. The semiconcavity of $u_m$ (T4) and the convergence of the gradients (T6) are the most technical part of the proof and are precisely the properties that allow to approximate the $\lambda$-cut locus with the sets $E_{m,\lambda}$.

Remark 1.5. If we replace the manifold $M$ with a smooth domain $\Omega \subset \mathbb{R}^n$ and $d_b$ with the distance to the boundary of $\Omega$, the problem (1.3) becomes the classical elastic-plastic torsion problem, which we discuss in detail in Section 1.1. We notice that, for this problem, the claims (T1), (T2), (T3) and (T5) are well-known. The elastic-plastic torsion problem has a long history and inspired the study of numerous problems involving more general (even fully nonlinear) operators. The crucial point in all these problems is that the gradient constraint in (1.3) can be transformed into an obstacle constraint on the function (see Section 1.1). Until now, this property was exclusive for the Euclidean setting and for operators depending only on $\nabla u$ and $u$, but not on the points $x \in \Omega$ (in fact, for operators with variable coefficients, this equivalence is known to be false). A consequence of our analysis is that this crucial equivalence is not exclusively Euclidean but is a property of the underlying Riemannian structure of the manifold (see Proposition 1.7).

The rest of the introduction is organized as follows. In the next Section 1.1 we will discuss the relation of the $\lambda$-cut locus and the problem (1.3) to the $\lambda$-medial axis of Chazal-Lieutier and the classical elastic-plastic torsion problem. In Section 1.2 we will discuss the key points in the proof of Theorem 1.3 and the plan of the paper.

1.1. Medial axis and $\lambda$-medial axis in a domain $\Omega$. This section is dedicated to the Euclidean counterpart of Theorem 1.1. We go through the definitions of the medial axis and the $\lambda$-medial axis of a domain in the euclidean space. Then, we discuss the approximation theorem of Caffarelli and Friedman and its relation to Theorem 1.1. Throughout this section, we will use the following notation: $\Omega$ is a bounded open set with $C^2$ regular boundary in $\mathbb{R}^n$ and $d_{\partial \Omega} : \Omega \to \mathbb{R}$ is the distance function to the boundary of $\Omega$,

\[
d_{\partial \Omega}(x) := \min \{|x - y| : y \in \partial \Omega\}.
\]

1.1.1. Definition of medial axis and $\lambda$-medial axis. The medial axis $\mathcal{M}(\Omega)$ is defined as the set of points of $\Omega$ with at least two different projections on the boundary $\partial \Omega$,

\[
\mathcal{M}(\Omega) := \{x \in \Omega : \exists y, z \in \partial \Omega, \text{ such that } y \neq z \text{ and } d_{\partial \Omega}(x) = |x - y| = |x - z|\}.
\]

One crucial geometric property of the medial axis $\mathcal{M}(\Omega)$ is that it is unstable with respect to small perturbations of the boundary of $\Omega$. For instance, the medial axis of the circle consists of its center only, while the medial axis of a polygonal approximation (the regularity of the approximating sets can be improved to $C^\infty$ by rounding the corners) is the star-shaped set on Figure 1. We refer to [2] for a detailed account on medial axis, stability and computability. This instability makes computing numerically $\mathcal{M}(\Omega)$ quite tricky. Indeed, any numerical approximation of $\Omega$ (for instance, with polygons) might introduce an artificial (and large) medial set. In order to deal with this problem, in [11], Chazal and Lieutier defined the so called $\lambda$-medial axis of $\Omega$ by setting, for any $\lambda > 0$,

\[
\mathcal{M}_\lambda(\Omega) := \{x \in \Omega : r(x) \geq \lambda\},
\]

where $r(x)$ is the radius of the smallest ball containing all the projections of $x$ on the boundary $\partial \Omega$, i.e. the set $\{z \in \partial \Omega : |x - z| = d_{\partial \Omega}(x)\}$. It is known that, for $\lambda$ small enough, $\mathcal{M}_\lambda(\Omega)$ has the same homotopy type as $\mathcal{M}(\Omega)$ (see [11, section 3, theorem 2]) and that

\[
\mathcal{M}(\Omega) = \bigcup_{\lambda > 0} \mathcal{M}_\lambda(\Omega).
\]

These facts justify that $\mathcal{M}_\lambda(\Omega)$ is a good approximation of $\mathcal{M}(\Omega)$, for $\lambda$ small enough. The crucial difference though is that $\mathcal{M}_\lambda(\Omega)$ is stable with respect to small variations of $\Omega$, whereas $\mathcal{M}(\Omega)$ is not (we refer to [11,}

\[\text{Figure 1. A polygonal approximation of a circle, with its medial axis.}\]
section 4] for precise statements and proofs). Finally, we notice that the $\lambda$-medial axis can be equivalently defined (see [11, section 2.1]) as

$$\mathcal{M}_\lambda(\Omega) = \left\{ x \in \Omega : |\nabla d_{\partial \Omega}(x)|^2 \leq 1 - \frac{\lambda^2}{d_{\partial \Omega}(x)} \right\},$$

(1.11)

where $\nabla d_{\partial \Omega}$ denotes the generalized gradient wherever $d_{\partial \Omega}$ is not differentiable.

1.1.2. Approximation of the medial axis. Given a constant $m > 0$ and a domain $\Omega$, as above, we consider the following elastic-plastic torsion problem

$$\min \left\{ \int_{\Omega} \left( |\nabla v|^2 - mv \right) \, dx : v \in H^1_0(\Omega), \quad |\nabla v| \leq 1 \right\}. \quad (1.12)$$

As in the case of (1.3), the problem (1.12) has a unique minimizer, which we will denote by $v_m$. Physically speaking, $v_m$ represents the stress function of a long bar of cross section $\Omega$, twisted with an angle $m$. The elastic-plastic torsion problem and the properties of its minimizer $v_m$ have been studied by various authors in the 60’s and 70’s (see for instance [26], [4], [3], [8], [27], [9], [14] and [7]). In particular, in [4], Brezis and Sibony proved that the gradient constraint in (1.12) can be replaced with an obstacle-type constraint on the function. Precisely, the minimizer $v_m$ of (1.12) is also the (unique) minimizer of

$$\min \left\{ \int_{\Omega} \left( |\nabla v|^2 - mv \right) \, dx : v \in H^1_0(\Omega), \quad v \leq d_{\partial \Omega} \right\}. \quad (1.13)$$

Notice that this result was later generalized to a broader class of variational problems with convex constraints on the gradient (see [28], [21] and [24]). However, none of these will apply to our variant of the problem on manifolds, for which the equivalence of constraints fails in general (see Section Appendix B).

Finally, using the equivalence of (1.12) and (1.13), Caffarelli and Friedman (see [6]) proved that the sequence of elastic sets $\{ |\nabla v_m| < 1 \}$ Hausdorff converges, as $m \to +\infty$, to the medial axis $\mathcal{M}(\Omega)$. To be precise, in [6], it was showed that the elastic sets converge to the so-called ridge $\mathcal{R}(\Omega)$ which coincides with the closure of $\mathcal{M}(\Omega)$, when $\Omega$ has a $C^2$ regular boundary. This result from [6] is the euclidean counterpart of the first part of Theorem 1.1. Nevertheless, the strategies from [3] and [6] cannot be reproduced on a manifold and do not imply the convergence of the $\lambda$-medial axis. In the proof of our Theorem 1.1, we still aim at replacing the constraint on the gradient with a constraint on the function, but our approach is different and allows us to deal with the presence of the manifold and to treat both the cut locus and the $\lambda$-cut locus. In particular, we obtain the following approximation result for the $\lambda$-medial axis.

**Theorem 1.6** (Approximation of $\mathcal{M}_\lambda(\Omega)$). Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with $C^2$ regular boundary. Then, setting

$$E^\Omega_m = \left\{ x \in \Omega : |\nabla v_m(x)| < 1 \right\} \quad \text{and} \quad E^\Omega_{m, \lambda} = \left\{ x \in \Omega : |\nabla v_m(x)|^2 \leq 1 - \frac{\lambda^2}{v_m^2(x)} \right\},$$

we have that, for any fixed $\varepsilon > 0$,

$$\sup_{x \in E^\Omega_{m, \lambda}} d(x, \mathcal{M}_\lambda(\Omega)) \rightarrow 0, \quad \text{and} \quad \sup_{x \in \mathcal{M}_{\lambda^+}(\Omega)} d(x, E^\Omega_{m, \lambda}) \rightarrow 0.$$  

1.2. Proof of Theorem 1.3 and plan of the paper. We consider the variational problem

$$\min \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), \quad u \leq d_b \right\}. \quad (1.14)$$

It is immediate to check that (1.14) admits a minimizer and that this minimizer is unique (this follows by the convexity of the functional and the constraint). We will denote by $u^d_m : M \to \mathbb{R}$ (‘$d$’ stands for the ‘distance’ constraint) the unique minimizer of (1.14).

1.2.1. Part I. Equivalence of (1.3) and (1.14). Our first aim is to show that the problems (1.3) and (1.14) are equivalent, that is the minimizers $u_m$ and $u^d_m$ are the same. Now, since every function which is 1-Lipschitz and is zero in $b$ stands below the distance function $b$, it is clear that $u_m$ can be used to test the optimality of $u^d_m$, that is, we have

$$\int_M (|\nabla u_m|^2 - mu_m) \leq \int_M (|\nabla u_m|^2 - mu_m).$$
Notice that, if we are able to prove that the minimizer $u^d_m$ is 1-Lipschitz, then we can use $u^d_m$ to test the minimality of $u_m$, i.e.
\[
\int_M (|\nabla u^d_m|^2 - mu^d_m) \geq \int_M (|\nabla u_m|^2 - mu_m).
\]
This gives that both $u_m$ and $u^d_m$ are solutions of (1.3) (and also of (1.14)), which means that they have to coincide. Thus, in order to prove that (1.3) and (1.14) are equivalent, we have to prove that
\[
|\nabla u^d_m| \leq 1 \quad \text{on} \quad M. \quad (1.15)
\]
In order to prove this, we proceed as follows:
\begin{itemize}
\item First, we prove that $u^d_m$ is $C^1$-regular locally in $M \setminus \{b\}$ (see Proposition 3.4).
\item Then, from Lemma 3.3 and Lemma 3.1, we deduce that
\[
\text{Cut}_b(M) \subset \{u^d_m < d_b\} \subset M \setminus \{b\}.
\]
In particular, since $d_b$ is smooth away from $\{b\}$ and $\text{Cut}_b(M)$, we get that on the boundary $\partial \{u^d_m < d_b\}$ both the distance function $d_b$ and the solution $u^d_m$ are differentiable and have the same gradient, which entails that $|\nabla u^d_m| = 1$ on $\partial \{u^d_m < d_b\}$.
\item Finally, we use the fact that $u^d_m$ solves the PDE
\[
\Delta u^d_m = m \quad \text{in} \quad \{u^d_m < d_b\}, \quad |\nabla u^d_m| = 1 \quad \text{on} \quad \{u^d_m = d_b\}
\]
to deduce that $|\nabla u^d_m| \leq 1$ also in the set $\{u^d_m < d_b\}$. Now, in the flat (Euclidean) case, this inequality is an immediate consequence of the fact that $|\nabla u^d_m|^2$ is subharmonic. On a general manifold $M$ the situation is more complicated as the curvature comes into play in the computation of $\Delta(|\nabla u^d_m|^2)$. For this reason we are able to prove the bound $|\nabla u^d_m| \leq 1$ on $M$ (and so the equivalence of the two problems) only in the case when $m$ is large enough. Before we give the precise statement of this result (see Proposition 1.7), let us emphasize that this is not a mere technical assumption, but a consequence of the geometry of the manifold. In fact, in the appendix (Theorem B.1), we give an example of a 2-manifold $M$ for which the bound on the gradient fails when $m$ is small.

The following is the key result for the analysis of the solution of problem (1.3). The proof is given in Section 4.

**Proposition 1.7 (Equivalence of (1.3) and (1.14)).** Let $M$ be an $n$-dimensional compact Riemannian manifold and let the constant $K \geq 0$ be a lower bound for the Ricci curvature:
\[
\text{Ric} \geq -K, \quad (1.16)
\]
where $\text{Ric}$ denotes the Ricci curvature tensor of $M$. Then, for every
\[
m \geq \frac{1}{2} \max \left\{ \sqrt{nK(1 + K \text{diam}(M)^2)}, \quad nK \text{diam}(M) \right\}, \quad (1.17)
\]
we have that
\[
|\nabla u^d_m| = 1 \quad \text{on} \quad \{d_b = u^d_m\}, \quad \text{and} \quad |\nabla u^d_m| < 1 \quad \text{in} \quad E^d_m := \{u^d_m < d_b\}. \quad (1.18)
\]
In particular, for $m$ as in (1.17), we have that $u^d_m = u_m$, where $u_m$ is the minimizer of (1.3).

Finally, as a corollary of Proposition 1.7, we obtain the first two claims of Theorem 1.3.

**Proof of Theorem 1.3 (T1) and (T2).** By Proposition 1.7 we have that $u_m = u^d_m$. From the regularity of $u^d_m$ (Proposition 3.4, Lemma 3.3 and Lemma 3.1), we obtain (T1) and (T2). □

Moreover, as in the classical case of the elastic-plastic torsion problem (see [6]), we can now use the structure of (1.14) to obtain information about the monotonicity of $E^d_m$ and the uniform convergence of $u_m$.

**Proof of Theorem 1.3 (T3) and (T5).** The uniform convergence $u^d_m \to d_b$ on $M$, as $m \to \infty$, is proved in Lemma 5.1. The monotonicity of $u_m$ and $E^d_m$, and the Hausdorff convergence of $E^d_m$ to $\text{Cut}_b(M)$, now follow from Proposition 5.2. □
1.2.2. Part II. Uniform semiconcavity and convergence of the gradients. We recall that our final objective is to prove the convergence of the sets $E_{m,\lambda}$ (Theorem 1.1) and $E_{m,\lambda}^\Omega$ (Theorem 1.6). Now, from the definition of $E_{m,\lambda}$, it is clear that this boils down to proving a convergence result for the gradients $|\nabla u_m|$. On the other hand, we cannot expect any uniform estimate on the modulus of continuity of $|\nabla u_m|$; in fact, the sequence $u_m$ converges (uniformly) to the distance function $d_\lambda$, which is not even differentiable at all points. Thus, we adopt a different strategy and we prove that the solutions are uniformly semiconcave, where our definition of semiconcavity is the following.

**Definition 1.8 (C-semiconcavity).** Given a constant $C > 0$, a function $u$ is said to be $C$-semiconcave on $M$ if and only if for any unit speed geodesic $\gamma : [a, b] \to M$, the function $t \mapsto Ct^2 - u(\gamma(t))$ is convex. Moreover,

- we say that $u$ is semiconcave if it is $C$-semiconcave for some constant $C > 0$;
- we say that $u$ is locally semiconcave if for any $p \in M$, $u$ is semiconcave in a neighborhood of $p$.

The most technical result of the paper is Theorem 1.3 (T4), which we prove in Section 6. The key result is Proposition 6.1 and applies to both Theorem 1.3 and Theorem 1.6. Let us briefly give the idea of the proof of this proposition here, directly in the setting of Theorem 1.3 (T4).

**Sketch of the proof of Theorem 1.3 (T4).** First, we fix a constant $C_d$ such that the distance function $d_b$ is $C_d$-semiconcave on $M \setminus B_\rho(b)$. Then, for every unit speed geodesic $\gamma : [a, b] \to M$, and every $\lambda \in [0, 1]$, we define the function

$$c(\gamma, \lambda) := \lambda(1-\lambda)(C_d + 1)(b-a)^2 - \left((1-\lambda)u_m(\gamma(a)) + \lambda u_m(\gamma(b)) - u_m(\gamma(\lambda a + (1-\lambda)b))\right),$$

where $\lambda a b = (1-\lambda)a + \lambda b$. We will show that the minimum of this function over all geodesics $\gamma$ and all $\lambda$ is positive, which will give that $u$ is $(C_d + 1)$-semiconcave. First, we show that for any unit speed geodesic $\gamma$ and $\lambda \in (0, 1)$, we can build a unit speed geodesic $\tilde{\gamma} : [a, b] \to M$ and $\lambda \in (0, 1)$, such that

$$c(\tilde{\gamma}, \lambda, u_m) \leq c(\gamma, \lambda, u_m) \quad \text{and} \quad \tilde{\gamma}(a, b) \subset E_m = \{u_m < d_b\}.$$

This follows from the semiconcavity of $d_b$ and the inequality $u_m \leq d_b$ (this is explained in detail in the proof of Proposition 6.1). Thus, we only need to show the semiconcavity of $u_m$ in the non-contact region $E_m$. Since $u_m$ is smooth in $E_m$, we need to prove that (see Proposition 2.2)

$$D^2 u_m \leq (C_d + 1)Id \quad \text{in} \quad E_m.$$ 

In order to prove this inequality, for every $p \in E_m$ and $X \in S^{n-1}(T_pM)$ we consider an auxiliary function of the form

$$f_\varepsilon(p, X) := D^2 u_m(X, X) + \varepsilon \left(C_1 |\nabla u_m|^2(p) + C_2 a_m^2(p) - C_3 u_m(p)\right),$$

and we show that for $\varepsilon > 0$ small enough and $m$ large enough, we have $f_\varepsilon \leq C_d + 1/2$. We suppose that the maximum of $f_\varepsilon$ is achieved for some $q \in E_m$ and some $Y \in S^{n-1}(T_qM)$ (the case when the minimum is achieved for $q \in \partial E_m$ is a consequence of known estimates for the solutions of the obstacle problem with variable coefficients, see Section 7). Then, we construct, locally around $q$, a function of the form

$$p \mapsto f_\varepsilon(p, X(p)) \quad \text{where} \quad X(p) \in S^{n-1}(T_pM),$$

and we compute its Laplacian in the variable $p$ (notice that in the flat euclidean case we can simply take the section $p \mapsto X(p)$ to be constant). Finally, we obtain that for an appropriate choice of $\varepsilon$ and $m$, the Laplacian of this function has to be positive, which contradicts the minimality of $q$ and concludes the proof. \hfill $\square$

The main part of the proof of Theorem 1.3 (T4) is contained in Proposition 6.1, which applies to both Theorem 1.3 and Theorem 1.6. In the proof of Proposition 6.1, the function $c$ is the Riemannian counterpart of the Korevaar’s convexity function (see [19]); in computing the Laplacian of $f_\varepsilon(p, X(p))$ we use some of Guan’s second order estimates for Hessian equations in Riemannian manifolds (see [15]).

At this point, the convergence of the gradients $|\nabla u_m|$ (Theorem 1.3 (T6)) follows from the uniform semiconcavity of $u_m$ by a general argument (we give the proof of this fact in Section 7). We are now in position to prove Theorem 1.1.
1.3. **Proof of Theorem 1.1.** The Hausdorff convergence of the elastic sets $E_m$ to $\text{Cut}_{b}(M)$ is a consequence from the uniform convergence (Theorem 1.3 (T5)) of the solutions $u_m$ to the distance function $d_b$, as explained in Proposition 5.2. Let us now prove the first claim in (1.6). Suppose by contradiction that there are a constant $\delta > 0$, a sequence $m_k \to \infty$ and a sequence of points $p_k$ such that

$$p_k \in E_{m_k,\lambda} \quad \text{and} \quad d(p_k, \text{Cut}_{b}(M)) > \delta. \quad (1.19)$$

By the facts that $M$ is compact and that $u_{m_k}$ coincides with the distance function $d_b$ in a neighborhood of $b$ (that does not depend on $k$), we may suppose that $p_k$ converges to some $p_{\infty} \in M \setminus \{b\}$. Now, from the uniform convergence of $u_{m_k}$ and Theorem 1.6 (T6), we get that

$$|\nabla d_b|(p_{\infty}) \leq \liminf_{k \to \infty} |\nabla u_{m_k}||(p_k) \leq \lim_{k \to \infty} \left(1 - \frac{\lambda^2}{u_{m_k}^2(p_k)}\right) = 1 - \frac{\lambda^2}{d_b^2(p_{\infty})},$$

which means that $p_{\infty} \in \text{Cut}_{b}^{\lambda}(M)$, in contradiction with (1.19).

Suppose now that the second claim in (1.6) does not hold. Then, there are a constant $\delta > 0$, a sequence $m_k \to \infty$ and a sequence of points $p_k \in \{b\}$ such that

$$p_k \in \text{Cut}_{b}^{\lambda+\varepsilon}(M) \quad \text{and} \quad d(p_k, E_{m_k,\lambda}) > \delta \quad \text{for every} \quad k \geq 0.$$  

Up to extracting a subsequence, we may suppose that $p_k$ converges to a point $p_{\infty}$ such that

$$p_{\infty} \in \text{Cut}_{b}^{\lambda+\varepsilon}(M) \quad \text{and} \quad d(p_{\infty}, E_{m_k,\lambda}) > \frac{\delta}{2} \quad \text{for every} \quad k \geq 0. \quad (1.20)$$

Now, by Theorem 1.3 (T6), there is a sequence $q_k \to p_{\infty}$ such that

$$|\nabla d_b|(p_{\infty}) = \lim_{k \to \infty} |\nabla u_{m_k}||(q_k).$$

In particular, since $p_{\infty} \in \text{Cut}_{b}^{\lambda+\varepsilon}(M)$, we have

$$\lim_{k \to \infty} \left(|\nabla u_{m_k}||(q_k) - 1 + \frac{\lambda^2}{u_{m_k}^2(q_k)}\right) = |\nabla d_b|(p_{\infty}) - 1 + \frac{\lambda^2}{d_b^2(p_{\infty})} \leq -\frac{2\varepsilon\lambda + \varepsilon^2}{d_b^2(p_{\infty})},$$

Thus, the left-hand side is negative for $k$ large enough and so, we have $q_k \in E_{m_k,\lambda}$, which is a contradiction with (1.20). This concludes the proof of Theorem 1.1. \qed

1.4. **Proof of Theorem 1.6.** As shown in Section 6, we may apply Proposition 6.1 to get that the functions $v_m$ are uniformly semiconcave on $\Omega$. It is already known that the solution $v_m$ of (1.12) and (1.13) is locally $C^{1,1}$ on $\Omega$. It is also well-known that $v_m$ converges uniformly to $d_{\partial \Omega}$ as $m \to \infty$. As a consequence, reasoning as in Section 7, we get that for every $x \in \Omega$, the following holds:

- if $x_m \to x_{\infty}$, then $|\nabla d_{\partial \Omega}|(x_{\infty}) \leq \liminf_{m \to \infty} |\nabla v_m|(x_m);$  
- there exists a sequence $x_m \to x_{\infty}$ such that $|\nabla d_{\partial \Omega}|(x_{\infty}) = \lim_{m \to \infty} |\nabla v_m|(x_m).$

Now, the conclusion follows as in the proof of Theorem 1.1. \qed

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2. **Notation, Definitions and Preliminary Results**

2.1. **General notation.** We will denote by $g$ the metric on $M$. $TM$ denotes the tangent bundle of $M$ and $T_pM$ the tangent space of $M$ at $p$. By $\mathbb{S}^{n-1}(T_pM)$ we will denote the unit sphere in $T_pM$, that is

$$\mathbb{S}^{n-1}(T_pM) := \{X \in T_pM : \ g(X, X) = 1\}.$$  

Exp : $TM \to M$ is the global exponential map, while $\exp_p$ is its restriction to $T_pM$. Finally, given a function $u$ on $M$, $Du$ is the differential of $u$, $\nabla u$ is the gradient, and $D^k u$ is the $k$-th covariant derivative (in particular, by $D$ we denote also the Riemannian connection on $M$). Thus, for smooth vector fields $X, Y : M \to TM$, we have

$$g(\nabla u, X) := Du(X) = DXu = Xu \quad \text{and} \quad D^2u(X, Y) = g(D_X(\nabla u), Y).$$
We will also use the notation $|\nabla u|^2$ for $g(\nabla u, \nabla u)$, and $\Delta u$ for the Laplace-Beltrami operator on $M$. We notice that $-\Delta$ is positive, that is, we have the integration by parts formula
\[
\int_M g(\nabla u, \nabla v) = \int_M (-\Delta u)v,
\]
for every $u, v \in C^2(M)$. Unless otherwise specified, all the integrals will be taken with respect to the volume form associated to the Riemannian metric $g$. Finally, we recall that $H^1(M)$ denotes the usual space of Sobolev functions on $M$, which is the closure of $C^1(M)$ with respect to the $H^1$-norm defined as
\[
||u||_{H^1} = \int_M |\nabla u|^2 + \int_M u^2.
\]

2.2. Semiconcave functions. In this section, we gather some of the main properties of semiconcave functions on smooth Riemannian manifolds, which we will need in the proof of Theorem 1.3. Some of these results can be found in [23], in the context of Alexandrov spaces, while for a more detailed introduction to semiconcave functions in the framework of euclidean spaces we refer to [10].

Let $M$ be a Riemannian manifold, $u : M \to \mathbb{R}$ a given function and $\gamma : [a, b] \to M$ be a curve in $M$. It is immediate to check that the function
\[
t \mapsto Ct^2 - u(\gamma(t))
\]
is convex on $[a, b]$ if and only if
\[
(1 - \lambda)u(\gamma(a)) + \lambda u(\gamma(b)) - u(\gamma(\lambda a)) \leq C\lambda(1 - \lambda)(b - a)^2 \quad \text{for any } \lambda \in [0, 1],
\]
where here and throughout the paper, we use the notation
\[
\lambda_{ab} := (1 - \lambda)a + \lambda b \quad \text{for any } a, b, \lambda \in \mathbb{R}.
\]
In particular, this means that the function $u$ is $C$-semiconcave on $M$ if and only if (2.1) holds for any unit speed geodesic $\gamma : [a, b] \to M$. Analogously, $u$ is locally semiconcave if for every $p \in M$ there is a geodesic ball $B_p(p)$ and a constant $C_p > 0$ such that (2.1) holds (with $C = C_p$) for every unit speed geodesic $\gamma : [a, b] \to B_p(p)$.

Remark 2.1. On a compact Riemannian manifold, semiconcavity and local semiconcavity are the same.

Proposition 2.2 (Semiconcavity in terms of $D^2u$). Let $u : M \to \mathbb{R}$ be $C^2$-regular. Then
\[
D^2u \leq 2C \quad \text{on } M \quad \text{if and only if} \quad u \text{ is } C\text{-semiconcave on } M.
\]
Proof. Let $\gamma : [a, b] \to M$ be a unit speed geodesic. Then the function $t \mapsto Ct^2 - u(\gamma(t))$ is convex if and only if
\[
0 \leq 2C - \frac{d^2}{dt^2}u(\gamma(t)) = 2C - \frac{d}{dt}Du(\dot{\gamma}(t)) = 2C - \left(D^2u(\dot{\gamma}(t), \dot{\gamma}(t)) + Du(D\dot{\gamma}(t)\dot{\gamma}(t))\right) = 2C - D^2u(\dot{\gamma}(t), \dot{\gamma}(t)).
\]
The claim follows. $\square$

The semiconcavity can also be read in local coordinates as follows.

Proposition 2.3 (Semiconcavity in local coordinates). Let $u : M \to \mathbb{R}$ be a locally Lipschitz function on a Riemannian manifold $M$. Then, $u$ is locally semiconcave if and only if for any chart $\psi$ of $M$, $u \circ \psi^{-1}$ is locally semiconcave as a function on $\mathbb{R}^n$.

We postpone the proof of this proposition to Appendix A. We next show that we can define the gradient of a semiconcave function at every point.

Proposition 2.4 (The generalized gradient of a semiconcave function). Let $u : M \to \mathbb{R}$ be a locally Lipschitz and semiconcave function. Then, at every point $p \in M$, $u$ admits a directional derivative $\partial_+^u u(p)$ in any direction $v \in T_pM \setminus \{0\}$. It is defined by
\[
\partial_+^u u(p) := \frac{d}{dt}[u(\gamma(t))]_{t=0} = \lim_{t \to 0^+} \frac{u(\gamma(t)) - u(p)}{t},
\]
where $\gamma : [0, 1] \to M$ is any curve such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. Moreover, the map $v \mapsto \partial_+^u u(p)$ is 1-homogeneous and concave on $T_pM$. Thus, it attains a unique maximum in the closed unit ball of $T_pM$ at a unique vector $v_p$.

Proof. By Proposition 2.3, we can suppose that $M = \mathbb{R}^n$, $p = 0$ and that $\gamma(t) = tv$. Then the function $w(x) = C|x|^2 - u(x)$ is convex for $C$ large enough and so, the function $t \mapsto \frac{w(tv) - w(0)}{t}$ is non-decreasing in $t$, so the limit $\partial_+^u u(p) = -\partial_+^u w(0)$ exists and is finite. The convexity of the function $v \mapsto \partial_+^u w(0)$ is a consequence from the convexity of $w$. The existence of a maximum of $v \mapsto \partial_+^u w(0)$ follows. $\square$
If $\partial^+_v u(p) > 0$, then the 1-homogeneity implies that $v_p$ has norm one, and we define

$$\nabla u(p) := \partial^+_v u(p)v_p \quad \text{and} \quad |\nabla u(p)| = \partial^+_v u(p).$$

If $\partial^+_v u(p) = 0$, then we set $\nabla u(p) = 0$. Thus, the norm of $\nabla u(p)$ is given by the following formula:

$$|\nabla u(p)| = \max \{0, \max_{v \in T_p M, |v|=1} \partial^+_v u(p)\}. \quad (2.3)$$

### 2.3. Distance function, cut locus and cut points.

Let $M$ be a compact Riemannian manifold, $b \in M$ and $d_b : M \to \mathbb{R}$ be the distance function to $b$. Here we recall the definition and some of the main properties of the cut locus.

**Definition 2.5** (Cut points). Let $T > 0$ and $\gamma : [0, T] \to M$ be a unit speed geodesic such that $\gamma(0) = b$, $t_0 \in (0, T)$ and $p = \gamma(t_0)$. We say that $p$ is a cut point of $b$ along $\gamma$ if $\gamma$ is length minimizing between $b$ and $p$, but not after $p$, i.e $d_b(\gamma(t)) = t$ for $t \leq t_0$, and $d_b(\gamma(t)) < t$ for $t > t_0$.

**Definition 2.6** (Cut locus). The cut locus of $b$ in $M$, $\text{Cut}_b(M)$, is defined as the set of all cut points of $b$.

The following well-known facts about the cut locus can all be found in [25, Chapter III, Section 4]:

- $\text{Cut}_b(M)$ is the closure of the set of points $p$ in $M$, for which there are at least two minimizing geodesics connecting $b$ and $p$;
- the distance function $d_b$ is smooth outside $\text{Cut}_b(M) \cup \{b\}$ and $|\nabla d_b| = 1$ in $M \setminus (\text{Cut}_b(M) \cup \{b\})$;
- $d_b$ is differentiable at $p \in M$ if and only if there is a unique minimizing geodesics between $b$ and $p$;
- in particular, $\text{Cut}_b(M) \cup \{b\}$ is the closure of the set of points of non differentiability of $d_b$;
- the exponential map $\exp_b : T_b M \to M$ is a diffeomorphism from an open set of $T_b M$ onto $M \setminus \text{Cut}_b(M)$;
- $\text{Cut}_0(M)$ is a deformation retract of $M \setminus \{b\}$. In particular, these two sets have the same homotopy type, and so $\text{Cut}_b(M)$ inherits much of the topology of $M$ (like homology groups, for instance). See [25, Chapter III, Section 4, Proposition 4.5] for a precise statement.

We next recall that in [20, Proposition 3.4], it was proved that, for any chart $\psi$ on $M \setminus \{b\}$, the function $d_b \circ \psi^{-1}$ is locally semiconcave on $\mathbb{R}^n$. Thus, by Proposition 2.3, $d_b$ is locally semiconcave on $M \setminus \{b\}$ in the sense of Definition 1.8. Precisely, we have the following proposition

**Proposition 2.7** (Semiconcavity of the distance function). Let $M$ be a compact Riemannian manifold of dimension $n$ and $b \in M$ be a given point. Then, for every $\rho > 0$, there is a constant $C > 0$ such that the distance function $d_b$ is $C$-semiconcave on $M \setminus B_\rho(b)$.

In particular, by Proposition 2.4, for any point $p \in M \setminus \{b\}$ and any direction $v \in T_p M$, $d_b$ admits the directional derivative $\partial^+_v d_b(p)$ and so we can define $\nabla d_b$ and $|\nabla d_b|$ at every point as in (2.3). In Lemma 2.8 we give a geometric interpretation of $|\nabla d_b|(p)$ in terms of the geodesics connecting $p$ to $b$. We notice that similar results holds also in the more general framework of Alexandrov spaces, but with some additional restrictions on the curvature of the ambient space (see [1, Theorem 4.5.6] and also [1, Lemma 3.2] for the statement in the Riemannian context). We give the proof directly for the distance function to a compact subset $K$ of $M$.

**Lemma 2.8** (Geometric interpretation of the generalized gradient). Let $M$ be a smooth Riemannian manifold without boundary, $K$ a compact subset of $M$, and $d_K$ the distance function to $K$. Let $p$ be a point of $M$ such that there exist several minimizing geodesics from $p$ to $K$. We denote the set of unit speed geodesics from $p$ to $K$ that are minimizing between $p$ and $K$ by $\text{geod}(p, K)$. For any $v \in T_p M$, we have

$$\partial^+_v d_K(p) = \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v. \quad (2.4)$$

In particular,

$$|\nabla d_K|(p) = \max_{v \in T_p M, |v|=1} \min_{\gamma \in \text{geod}(p, K)} -\dot{\gamma}(0) \cdot v. \quad (2.5)$$

In particular, if $\gamma_1 : [0, d_K(p)] \to M$ and $\gamma_2 : [0, d_K(p)] \to M$ are two minimizing geodesics from $p$ to $K$, then

$$|\nabla d_K|(p) \leq \sqrt{\frac{1 + \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)}{2}}. \quad (2.6)$$
Proof. Let \( \gamma : [0, d_K(p)] \to M \) be a geodesic of \( \text{geod}(p, K) \). Let \( a = \gamma(d_K(p)/2) \). As \( \gamma \) is minimizing between \( p \) and \( \gamma(d_K(p)) \), we have \( a \notin \text{Cut}_p(M) \), and so \( p \notin \text{Cut}_a(M) \). In particular, the function \( d_a \) is differentiable at \( p \), and \( \nabla d_a(p) = -\gamma(0) \). Thus, for every \( t > 0 \), we have 
\[
\frac{d_K(\exp_p(tv)) - d_K(p)}{t} \leq \frac{d_a(\exp_p(tv)) + d_K(a) - d_K(p)}{t} = \frac{d_a(\exp_p(tv)) - d_a(p)}{t}
\]
Passing to the limit as \( t \to 0 \), we get 
\[
\partial_+^v d_K(p) \leq \min_{\gamma \in \text{geod}(p, K)} -\gamma(0) \cdot v.
\tag{2.7}
\]
Now, for every \( t > 0 \), let \( \gamma_t \in \text{geod}(\exp_p(tv), K) \). For \( t \) small enough, the length of \( \gamma_t \) is bounded by \( d_K(p) + 1 \).

By compactness of the set of geodesics of length bounded by a given constant, there exists a sequence of positive numbers \( (t_n)_{n \geq 0} \) that converges to 0, such that \( \gamma_n := \gamma_{t_n} \) converges to a unit speed geodesic \( \gamma \) as \( n \to +\infty \). As \( K \) is closed, \( \gamma \) is a geodesic from \( p \) to \( K \). What is more, we have 
\[
\text{length}(\gamma) = \lim_{n \to \infty} \text{length}(\gamma_n) = \lim_{n \to \infty} d_K(\exp_p(t_n v)) = d_K(p),
\]
so \( \gamma \in \text{geod}(p, K) \). Let \( R = \min(\text{inj}(M), d_K(p)/2) \), where \( \text{inj}(M) \) is the injectivity radius of \( M \). In particular for any \( (x, y) \) such that \( d(x, y) < R \) and \( x \neq y \), the distance function \( d(\cdot, \cdot) \) is smooth on a neighborhood of \( (x, y) \) in \( M \times M \). For \( n \in \mathbb{N} \), let \( b_n := \gamma_n(R) \), and \( b_\infty = \gamma(R) \). Let \( U, V \subset M \) be precompact neighborhoods of \( p \) and \( b_\infty \), respectively such that \( d(\cdot, \cdot) \) is smooth on \( \overline{U} \times \overline{V} \). For \( n \) big enough, we have \( \exp_p(t_n v) \in U \) and \( b_n \in V \), and so 
\[
d_K(p) \leq d_K(b_n) + d(b_n, p)
= d_K(\exp_p(t_n v)) - d(b_n, \exp_p(t_n v)) + d(b_n, p) = d_K(\exp_p(t_n v)) - \nabla_2 d(b_n, p) \cdot v + o(t_n),
\tag{2.8}
\]
where \( \nabla_2 \) is the gradient with respect to the second variable. We have 
\[
\nabla_2 d(b_n, p) \xrightarrow[n \to \infty]{} \nabla_2 d(b_\infty, p) = -\gamma(0)
\]
because \( d(\cdot, \cdot) \) is smooth on \( U \times V \). So (2.8) yields 
\[
\liminf_{n \to \infty} \frac{d_K(\exp_p(t_n v)) - d_K(p)}{t_n} \geq -\gamma(0) \cdot v.
\]
In particular, 
\[
\partial_+^v d_K(p) \geq \min_{\gamma \in \text{geod}(p, K)} -\gamma(0) \cdot v.
\]
With (2.7), this concludes the proof of (2.4). Now, (2.5) follows from (2.4) and the definition of the generalized gradient (2.3) of semiconcave functions. Finally, in order to prove (2.6), we consider the vector \( v \) that realizes the maximum in (2.5) and we write it as \( v = -\alpha\gamma_1(0) - \beta\gamma_2(0) + v_\perp \), where \( v_\perp \) is orthogonal to \( \gamma_1(0) \) and \( \gamma_2(0) \).

Then, we have 
\[
-v \cdot \dot{\gamma}_1(0) = \alpha + \beta\gamma_1(0) \cdot \dot{\gamma}_2(0) \quad \text{and} \quad -v \cdot \dot{\gamma}_2(0) = \beta + \alpha\gamma_1(0) \cdot \dot{\gamma}_2(0).
\]
In particular, 
\[
\min_{\gamma \in \text{geod}(p, K)} -\gamma(0) \cdot v \leq \frac{1}{2} \left( -v \cdot \dot{\gamma}_1(0) - v \cdot \dot{\gamma}_2(0) \right) \leq \frac{1}{2}(\alpha + \beta) \left( 1 + \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0) \right).
\tag{2.9}
\]
Now, using the fact that 
\[
\alpha^2 + \beta^2 + 2\alpha\beta\gamma_1(0) \cdot \dot{\gamma}_2(0) \leq \|v\|^2 = 1,
\]
we get that 
\[
(\alpha + \beta)^2 \leq 1 + 2\alpha\beta \left( 1 - \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0) \right) \leq 1 + \frac{1}{2}(\alpha + \beta)^2 \left( 1 - \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0) \right),
\]
which implies that 
\[
(\alpha + \beta)^2 \leq \frac{2}{1 + \dot{\gamma}_1(0) \cdot \dot{\gamma}_2(0)},
\]
which, together with (2.9), gives (2.6).

As a consequence of Lemma 2.8 and in particular of (2.6), we obtain the \( \lambda \)-cut locus approximates the cut locus in the following sense.
Proposition 2.9. Suppose that $M$ is a compact Riemannian manifold, the point $b \in M$ is fixed and that $d_b$ is the distance function to $b$. Then, for every $\lambda > 0$, $\text{Cut}^\lambda_b(M) \subset \text{Cut}_b(M)$. Moreover, the cut locus $\text{Cut}_b(M)$ is the closure of the union $\bigcup_{\lambda > 0} \text{Cut}^\lambda_b(M)$.

Proof. The inclusion $\text{Cut}^\lambda_b(M) \subset \text{Cut}_b(M)$ follows from the fact that $d_b$ is differentiable and $|\nabla d_b| = 1$ outside $\text{Cut}_b(M) \cup \{b\}$. In order to prove the second claim, we fix a point $p \in \text{Cut}_b(M)$. Then, there is a sequence of points $p_n \in \text{Cut}_b(M)$ for each of which there are at least to different minimizing geodesics from $p_n$ to $b$. Now, from (2.6), we have that $p_n \in \text{Cut}^\lambda_b(M)$ for some $\lambda_n > 0$. This concludes the proof. \qed

3. Regularity of $u^d_m$

This section is dedicated to the $C^{1,1}$ regularity of the minimizer $u^d_m$ of (1.14). We recall the following result.

Lemma 3.1 (Regularization of the obstacle, [16]). For any $m > 0$, there exists a function $\tilde{d}_b$ which is smooth on $M \setminus \{b\}$, such that

$$u^d_m \leq \tilde{d}_b \leq d_b \text{ on } M, \quad \text{and } \tilde{d}_b < d_b \text{ on } \text{Cut}_b(M).$$

In particular, $u^d_m$ is also the solution of the obstacle problem

$$\min \left\{ \int_M |\nabla u|^2 - mu : u \in H^1(M), \ u \leq \tilde{d}_b \right\}. \quad (3.1)$$

One could adapt to the manifold framework the regularity theorems for the classical obstacle problem on a euclidean domain and, with the preceding lemma, deduce the regularity of $u^d_m$. Rather than doing that, we will use Lemma 3.1 to reduce our problem to a classical obstacle problem on a euclidean domain. Let us start with the following regularity lemma.

Lemma 3.2 (Continuity of $u^d_m$). For any $m > 0$, the function $u^d_m$ is continuous on $M$.

Proof. We will reduce our problem to a classical obstacle type variational problem on an open subset of $\mathbb{R}^n$, by a series of elementary modifications, and apply a classical $W^{2,p}$ regularity theorem.

From Lemma 3.1, we know that there exists an open set $U \subset M$ and $\varepsilon > 0$ such that $\text{Cut}_b(M) \subset U$ and $u^d_m \leq d_b - \varepsilon$ on $U$. As a consequence, on the set $U$, $u^d_m$ verifies the Euler-Lagrange equation of (1.14), i.e $\Delta u^d_m = -2m$. In particular, it is $C^\infty$ smooth on $U$. Let $\Omega \subset M$ be a smooth open set such that $\overline{U} \subset \Omega$, $\partial\Omega \subset U$ and $\text{Cut}_b(M) \cap \overline{\Omega} = \emptyset$.

As $U^c \subset \Omega$, it suffices to show that $u^d_m$ is continuous on $\Omega$. As $\partial\Omega \subset U$, $u^d_m$ is smooth on $\partial\Omega$, so there exists a smooth function $v_m$ on $\overline{\Omega}$ such that $v_m = u^d_m$ on $\partial\Omega$. Then, one can check that $u^d_m$ is a solution of the following variational problem:

$$\min \left\{ \int_{\Omega} |\nabla u|^2 - mu : u \in H^1(\Omega), \ u \leq \tilde{d}_b \text{ in } \Omega, \ u = v_m \text{ on } \partial\Omega \right\}.$$ 

As a consequence, $u^d_m - v_m$ is a solution of the following variational problem:

$$\min \left\{ \int_{\Omega} |\nabla v|^2 - (m + \Delta v_m)v : v \in H^1_0(\Omega), \ v \leq d_b - v_m \text{ in } \Omega \right\}. \quad (3.2)$$

Because we have $\text{Cut}_b(M) \cap \overline{\Omega} = \emptyset$, the exponential map at $b$ is a diffeomorphism onto $\Omega$. Let $\phi : \Omega \to \overline{\Omega} \subset \mathbb{R}^n$ be a normal coordinates chart centered at $b$. Let $g = (g^{ij})$ denotes the metric of $M$ in the coordinates defined by $\phi$, and $\det g$ its determinant. We recall that the Riemannian volume measure is given in coordinates by $\sqrt{\det g} d\nu$. So we have

$$\int_{\Omega} |\nabla v|^2 - (m + \Delta v_m)v = \int_{\overline{\Omega}} \left( g^{ij} \partial_i (v \circ \phi^{-1}) \partial_j (v \circ \phi^{-1}) \sqrt{\det g} - (m + \Delta v_m \circ \phi^{-1}) (v \circ \phi^{-1}) \right) \sqrt{\det g} \, dx,$$

so $(u^d_m - v_m) \circ \phi^{-1}$ is a minimizer of

$$\min \left\{ \int_{\overline{\Omega}} \left( g^{ij} \sqrt{\det g} \partial_i w \partial_j w - Fw \right) dx : w \in H^1_0(\overline{\Omega}), \ w \leq \psi \right\}, \quad (3.3)$$

where we have set $\psi := (d_b - v_m) \circ \phi^{-1}$ and $F := (m + \Delta v_m) \circ \phi^{-1} \sqrt{\det g}$. We want to apply $[29, \text{Theorem 4.32}]$. For this we need to write the above variational problem into a variational inequality. Let $w$ be a competitor
Lemma 3.5 below, from which we deduce that we have

which yields a contradiction by the maximum principle. Then, the maximum of

is the elliptic operator defined on \( H^1_0(\Omega) \) by \( A w := -\partial_j (g^{ij} \sqrt{\det g} \partial_i w) \). From there, we can apply [29, Theorem 4.32] to deduce that, for any \( p < n \), if \( A \psi \wedge F \in L^p(\Omega) \), then \( A w \in L^p(\Omega) \). To check that \( A \psi \wedge F \in L^p(\Omega) \), it is enough to check that \( A (d_b \circ \phi^{-1}) \) is integrable at \( 0 \). But this is a consequence of the fact that \(-\Delta d_b \circ \phi^{-1} = \sqrt{\det g} A (d_b \circ \phi^{-1}) \), and Lemma 3.5 below, from which we deduce that \( A (d_b \circ \phi^{-1}) \) is equivalent to \( \frac{n-1}{|x|} \) when \( x \) goes to \( 0 \). 

Therefore, for \( p < n \), \( A (d_b \circ \phi^{-1}) \) is integrable at \( 0 \), and so \( A w \in L^p(\Omega) \). By elliptic regularity, this implies \( w \in W^{2,p}(\Omega) \), for any \( p < n \). By the Sobolev embeddings, \( w \) is then continuous on \( \Omega \), and so \( u_m^d \) is continuous on \( \Omega \). This concludes the proof.

We can now define the set \( E_m^d := \{ u_m^d < d_b \} \), for any \( m > 0 \). It is an open subset of \( M \), on which \( u_m^d \) solves the equation \( \Delta u_m^d = -2m \). We can now prove the following lemma.

**Lemma 3.3.** For any \( m > 0 \), we have \( u_m^d = d_b \) in a neighborhood of \( b \).

**Proof.** Let us assume that we have constructed a \( C^1 \) function \( v \) on \( \overline{B}_R(b) \) for some \( R > 0 \), such that

\[
\begin{align*}
\{ & v \leq d_b & \text{in } B_R(b), & & (3.4) \\
& v = d_b & \text{in } B_\varepsilon(b) & \text{for some } \varepsilon \in (0, R), & & (3.5) \\
& v < 0 & \text{in } \partial B_R(b), & & (3.6) \\
& \Delta v \geq -m & \text{in } B_R(b) & \text{in the distributional sense.} & & (3.7)
\end{align*}
\]

We will then show that we have \( u_m^d \geq v \). The construction of \( v \) is postponed to the end of the proof. From Lemma 3.2, we know that the function \( v - u_m^d \) is continuous. Let us first assume that \( v - u_m^d \) attains a positive maximum at a point \( x \in \overline{B}_R(b) \). We have

\[
0 < v(x) - u_m^d(x) \leq d_b(x) - u_m^d(x),
\]

so \( x \in E_m^d \). Moreover, we have \( u_m^d \geq 0 \) since \( \max(u_m^d, 0) \) is a better competitor than \( u_m^d \) in (1.14), so

\[
v - u_m^d \leq v < 0 \quad \text{on} \quad \partial B_R(b),
\]

and so \( x \in B_R(b) \). Hence the function \( v - u_m^d \) attains a positive maximum inside the open set \( E_m^d \cap B_R(b) \), but its Laplacian verifies in the distributional sense:

\[
\Delta (v - u_m^d) = \Delta v + m \geq 0,
\]

which yields a contradiction by the maximum principle. Then, the maximum of \( v - u_m^d \) on \( \overline{B}_R(b) \) is non-positive, and we get

\[
u_m^d \geq v = d_b \quad \text{in} \quad B_\varepsilon(b),
\]

which concludes the proof.

Let us now construct the function \( v \) that was used above. Let \( R > 0 \) be small enough so that \( \overline{B}_R(b) \) is contained in a normal neighborhood of \( b \). In polar coordinates around \( b \), we define \( v \) as a radial function. For \( \varepsilon > 0 \) to be chosen small enough later, let \( f : [0, R] \to [0, \infty) \) be the \( C^1 \) function such that

\[
\begin{align*}
\{ & f(r) = r & \text{if } r \leq \varepsilon, & & (3.9) \\
& f''(r) + \frac{n-1}{r} f'(r) = -\frac{m}{2} & \text{if } r > \varepsilon.
\end{align*}
\]

If \( n = 2 \), the unique \( C^1 \) solution to this system is given by:

\[
\begin{align*}
\{ & f(r) = r & \text{if } r \leq \varepsilon, & & (3.10) \\
& f(r) = \varepsilon + \frac{m}{8} (\varepsilon^2 - r^2) + \left( \varepsilon + \frac{m}{4} \varepsilon^2 \right) \ln \left( \frac{r}{\varepsilon} \right) & \text{if } r > \varepsilon.
\end{align*}
\]

If \( n \geq 3 \), then the solution is

\[
\begin{align*}
\{ & f(r) = r & \text{if } r \leq \varepsilon, & & (3.11) \\
& f(r) = \varepsilon + \frac{m}{4n} (\varepsilon^2 - r^2) & \text{if } r \leq \varepsilon, \\
& \left( \varepsilon^{n-1} + \frac{m}{2n} \varepsilon^n \right) \frac{1}{n-2} \left( \frac{1}{\varepsilon^{n-2}} + \frac{1}{r^{n-2}} \right) & \text{if } r > \varepsilon.
\end{align*}
\]
Then, we set in standard polar coordinates \( v(x) = f(r) \) for \( x \in B_R(b) \). For \( r \leq \varepsilon \), the constraint (3.5) is verified by definition. (For \( r > \varepsilon \), we chose \( f \) so that \( \Delta v \) is small, but still bigger than \(-m.\))

Let us show that (3.4) holds. Let us set \( g(r) := f(r) - r \) and prove that \( g \leq 0 \). We have \( g(r) = 0 \) for \( r \leq \varepsilon \) so it is sufficient to prove that \( g'(r) \leq 0 \) for \( r \geq \varepsilon \). But, as \( f \) verifies (3.9), \( g \) verifies

\[
g'' + \frac{n-1}{r} g' = -m - \frac{n-1}{r} \quad \text{for} \quad r \geq \varepsilon.
\]

In particular, whenever \( g'(r) = 0 \), we have \( g''(r) < 0 \). This implies \( g'(r) \leq 0 \) for \( r \geq \varepsilon \), and so (3.4) is verified.

Now let us show that (3.7) holds if \( R \) has been taken small enough. We use the following expression of the Laplacian in coordinates:

\[
\Delta v = \frac{1}{\sqrt{\det g}} \partial_r \left( \sqrt{\det g} \partial_r v \right),
\]

where \( g = (g^{ij}) \) is the metric of the manifold \( M \), and \( \det g \) its determinant. We apply this formula to polar coordinates to find that, on \( B_R(b) \setminus \partial B_{\varepsilon}(b) \), we have in the classical sense

\[
\Delta v = \frac{1}{\sqrt{\det g}} \partial_r \left( \sqrt{\det g} f'(r) \right) = f'' + \frac{\partial_r \det g}{2 \det g} f' = -\frac{m}{2} + \frac{(\partial_r \det g}{2 \det g} - \frac{n-1}{r}) f'.
\]

(3.12)

Note that by applying the Laplacian formula in polar coordinates to the distance function \( d_b(x) = r \), we find that

\[
\Delta d_b = \frac{\partial_r \det g}{2 \det g}.
\]

(3.13)

Because of Lemma 3.5, we also have

\[
\Delta d_b(x) = \frac{n-1}{r} + o(1).
\]

With (3.12) and (3.13), this last equation yields in the classical sense

\[
\Delta v = -\frac{m}{2} + o(1) f'(r) \quad \text{on} \quad B_R(b) \setminus \partial B_{\varepsilon}(b).
\]

(3.14)

Moreover, it is clear from the following expression that \( f' \) is bounded on \([\varepsilon, R]\), by a constant independent of \( R \), as long as we choose \( R \leq 1 \):

\[
f'(r) = \frac{m}{n} + \left( \varepsilon^{n-1} + \frac{m}{n} \varepsilon^n \right) \frac{1}{r^{n-1}} \quad \text{for} \quad \varepsilon \leq r \leq R.
\]

Hence from (3.14) we see that by taking \( R \) small enough (independently of \( \varepsilon \)), we can ensure that

\[
\Delta v \geq -m \quad \text{on} \quad B_R(b) \setminus \partial B_{\varepsilon}(b).
\]

But from (3.14), we see that the above is also true on \( B_{\varepsilon}(b) \) if \( \varepsilon \) is small enough. Thus the function \( v \) is \( C^1 \) on \( B_R(b) \) and verifies \( \Delta v(x) \geq -m \) when \( x \notin \partial B_{\varepsilon}(b) \), hence (3.7) holds. It is also clear from (3.10) and (3.11) that the constraint (3.6) is verified if \( \varepsilon \) is taken small enough. This concludes the proof.

We can now prove the \( C^{1,1} \) regularity of \( u_m^d. \)

**Proposition 3.4.** For any \( \varepsilon > 0 \), the function \( u_m^d \) belongs to \( C^{1,1}(M \setminus B_{\varepsilon}(b)) \).

**Proof.** We reproduce the proof of Lemma 3.2, but we replace the open set \( \Omega \) with \( \tilde{\Omega} := \Omega \setminus B_{\varepsilon}(b) \), and the function \( u_m \) with a function \( \tilde{u}_m \) that is smooth and such that \( u_m = \tilde{u}_m \) on \( \partial \tilde{\Omega} \). We know that such a function exists because \( u_m \) is smooth on \( \partial B_{\varepsilon}(b) \) for \( \varepsilon \) small enough, as it can be seen from Lemma 3.3. This way, we can apply the stronger \( W^{2,\infty} \) regularity result for the obstacle problem [29, Theorem 4.38], since \( d_b \) is smooth on \( \tilde{\Omega} \). We get that \( \tilde{u}_m^d \) belongs to \( W^{2,\infty} = C^{1,1}(\tilde{\Omega}) \). As \( u_m \) is smooth on \( E_m^d \) and \( \partial \tilde{\Omega} \subset E_m^d \), then \( u_m^d \) is \( C^{1,1} \) on \( \tilde{\Omega} \cup E_m^d = M \setminus B_{\varepsilon}(b) \).

We end this section with the following computational lemma, which we used in the proof of Lemma 3.3.

**Lemma 3.5.** We have

\[
\Delta d_b(p) = \frac{n-1}{d_b(p)} + o(1).
\]

(3.15)
Proof. We compute $\Delta d_b$ in normal coordinates centered at $b$. Let $g = (g^{ij})$ be the metric of $M$ in these coordinates. We have

$$\Delta d_b(x) = \frac{1}{\sqrt{\det g}} \partial_i \left( \sqrt{\det g} g^{ij} \partial_j d_b \right)(x).$$

In normal coordinates, the metric is euclidean up to order $1$ as $x$ goes to $0$. So we have

$$g^{ij}(x) = \delta^{ij} + o(x), \quad \partial_i \left( \sqrt{\det g} g^{ij} \right)(x) = o(1) \quad \text{and} \quad \frac{1}{\sqrt{\det g}} = 1 + o(x).$$

Moreover, in normal coordinates, we have $d_b(x) = |x|$, and so

$$\delta^{ij} \partial_i d_b(x) = \frac{n-1}{|x|},$$

which gives precisely (3.15). \hfill \square

4. Equivalence of the two constraints

Proof of Proposition 1.7. As above, we denote by $u_m^d$ the minimizer of (1.14). In order to show that $u_m^d$ solves (1.3), it is sufficient to show that $u_m^d$ is an admissible competitor in (1.3), that is, $|u_m^d| \leq 1$ on $M$. Recall that the function $u_m^d$ is $C^1$ except at $b$, by Proposition 3.4.

First, suppose that $x \neq b$ is in the contact set $E_m^d := \{ u_m^d = d_b \}$. By Lemma 3.1, we have $x \notin \text{Cut}_b(M)$, and so the distance function $d_b$ is differentiable at $x$. It is a simple consequence of the constraint $u_m^d \leq d_b$ and the equality $u_m^d(x) = d_b(x)$ that we have $\nabla u_m^d(x) = \nabla d_b(x)$. The desired inequality $|\nabla u_m^d(x)| \leq 1$ follows.

In the non-contact set $E_m^d = \{ u_m^d < d_b \}$, the function $u_m^d$ solves the PDE

$$\Delta u_m^d = -2m. \quad (4.1)$$

In particular it is smooth, and we may apply the Bochner-Weitzenböck formula:

$$\Delta \left( |\nabla u_m^d|^2 \right) = 2 \text{Ric}(\nabla u_m^d, \nabla u_m^d) + 2 |D^2 u_m^d|^2 + 2(\Delta u_m^d, \nabla u_m^d), \quad (4.2)$$

where Ric denotes the Ricci curvature tensor on the manifold $M$ and $D^2 u_m^d$ is the second covariant derivative of $u_m^d$. The last term is $0$ because of (4.1). As for the second term, we have:

$$|D^2 u_m^d|^2 \leq \frac{1}{n} (\text{Trace}(D^2 u_m^d))^2 = \frac{m^2}{n}, \quad (4.3)$$

where the last inequality is due to (4.1). As the manifold $M$ is compact, there exists a constant $K > 0$ (depending on $M$ only) such that the Ricci curvature is bounded from below by $-K$. In the end, (4.2) yields

$$\Delta \left( |\nabla u_m^d|^2 \right) + 2K |\nabla u_m^d|^2 \geq \frac{8}{n} m^2. \quad (4.4)$$

Now notice that by (4.1),

$$\Delta \left( (u_m^d)^2 \right) = 2 u_m^d \Delta u_m^d + 2 |\nabla u_m^d|^2 = -4 m u_m^d + 2 |\nabla u_m^d|^2,$$

so (4.4) gives

$$\Delta \left( |\nabla u_m^d|^2 + K(u_m^d)^2 \right) \geq \frac{8}{n} m^2 - 4 K m u_m^d \geq \frac{8}{n} m^2 - 4 K m d_b \geq \frac{8}{n} m^2 - 4 K m \text{diam}(M).$$

Thus, if $m \geq \frac{4}{K} \text{diam}(M)$, the function $|\nabla u_m^d|^2 + K(u_m^d)^2$ is subharmonic in the non-contact set $E_m^d$. From Lemma 3.3, we have $E_m^d \subset M \setminus \{ b \}$, and with Proposition 3.4, we get that the function $|\nabla u_m^d|^2 + K(u_m^d)^2$ is continuous on $E_m^d \subset M \setminus \{ b \}$. Therefore we may apply the maximum principle to get

$$|\nabla u_m^d|^2 \leq |\nabla u_m^d|^2 + K(u_m^d)^2 \leq \sup_{\partial E_m^d} \left( |\nabla u_m^d|^2 + K(u_m^d)^2 \right) = 1 + K \sup_{\partial E_m^d} (u_m^d)^2 \leq 1 + K \text{diam}(M)^2 \leq 1 + K \text{diam}(M)^2 \leq 1 + K \text{diam}(M)^2.$$

With (4.4), this last inequality gives

$$\Delta \left( |\nabla u_m^d|^2 \right) \geq \frac{8}{n} m^2 - 2K(1 + K \text{diam}(M)^2)$$

Thus, whenever the right-hand side is nonnegative, the maximum principle applied to the function $|\nabla u_m^d|^2$ on the open set $E_m^d$ implies that $|\nabla u_m^d|^2 < 1$ on this set. This concludes the proof. \hfill \square
5. Convergence of the non-contact set

In this section we show that the non-contact set $E^d_m = \{ u_m^d < d_b \}$ (which coincides with $E_m$, for $m$ large enough, as we showed in the previous section) Hausdorff-converges to $\text{Cut}_b(M)$.

**Lemma 5.1.** We have $\|d_b - u_m^d\|_{L^\infty(M)} \leq \frac{C}{m}$, for some positive constant $C$ depending on $M$ only.

**Proof.** We only need to prove the proposition for $m$ large enough. Therefore, thanks to Proposition 1.7, we will assume that $m$ is large enough so that $|\nabla u_m^d| \leq 1$. We only need to show the estimate on $E^d_m$ since outside this set, $u_m^d$ and $d_b$ are the same. We will show that for $m$ large enough, we have

$$\forall \overline{p} \in E^d_m, \exists p \in (E^d_m)^c \text{ such that } d(p, \overline{p}) < 5n/m. \tag{5.1}$$

This will conclude the proof since by the 1-Lipschitzianity of $u_m^d$ and $d_b$, we then have

$$|d_b(\overline{p}) - u_m^d(\overline{p})| \leq |d_b(p) - u_m^d(p)| + 2d(\overline{p}, p) = 0 + 2d(\overline{p}, p) \leq \frac{10n}{m},$$

which is what we need. In order to prove (5.1), we argue by contradiction and assume that $B_{5n/m}(\overline{p}) \subset E^d_m$. We want to apply the maximum principle to the function $v$ defined on $B_{5n/m}(\overline{p})$ by the following formula

$$v(p) := u_m^d(p) - \inf_{\partial B_{5n/m}(\overline{p})} u_m^d + \frac{m}{2n} \left( d(p) - \frac{5n}{m} \right)^2.$$  

For any $p \in B_{5n/m}(\overline{p})$, we have $\Delta u_m^d(p) = -2m$ because we have assumed $B_{5n/m}(\overline{p}) \subset E^d_m$. To estimate the Laplacian of $\partial_b$, we use some normal coordinates $(x')$ centered at $\overline{p}$. In these coordinates, the metric is euclidean up to order 1, uniformly in $\overline{p}$ since $M$ is compact, and $\partial(\overline{x}) = |x|$ (see Lemma 3.5). We get that for $m$ large enough, independently of $\overline{p}$,

$$\forall p \in B_{5n/m}(\overline{p}), \quad \Delta d_b^2(p) \leq 2(2n).$$

All in all, we obtain on $B_{5n/m}(\overline{p}) \subset E^d_m$,

$$\Delta v \leq -2m + \frac{m}{2n}2(2n) = 0.$$  

So we can apply the maximum principle to $v$ to get

$$v(\overline{p}) \geq \inf_{\partial B_{5n/m}(\overline{p})} v,$$

i.e.

$$u_m^d(\overline{p}) - \inf_{\partial B_{5n/m}(\overline{p})} u_m^d + \frac{m}{4n} \left( \frac{5n}{m} \right)^2 \geq 0. \tag{5.2}$$

As we have taken $m$ large enough so that $|\nabla u_m^d| \leq 1$, we also have

$$u_m^d(\overline{p}) - \inf_{\partial B_{5n/m}(\overline{p})} u_m^d \leq \frac{5n}{m} \leq \frac{m}{4n} \left( \frac{5n}{m} \right)^2,$$

which contradicts the estimate (5.2). This concludes the proof. \hfill \Box

**Proposition 5.2** (Monotonicity of $u_m^d$ and $E^d_m$, and convergence of $E^d_m$). For any $m > m' > 0$, we have

$$u_{m'}^d \leq u_m^d \leq d_b \quad \text{and} \quad \text{Cut}_b(M) \subset E^d_m \subset E^d_{m'}.$$

Moreover,

$$E^d_m \rightarrow \text{Cut}_b(M) \text{ in the Hausdorff sense.}$$

**Proof.** The fact that, for any $m > 0$, $\text{Cut}_b(M) \subset E^d_m$, is a direct consequence of Lemma 3.1. Let us prove the second inclusion. For $m > m' > 0$, note that by the respective minimality of $u_m^d$ and $u_{m'}^d$, we have

$$\int_M |\nabla \max(u_{m'}^d, u_m^d)|^2 - m \int_M \max(u_{m'}^d, u_m^d) \geq \int_M |\nabla u_m^d|^2 - m \int_M u_m^d,$$

and

$$\int_M |\nabla \min(u_{m'}^d, u_m^d)|^2 - m' \int_M \min(u_{m'}^d, u_m^d) \geq \int_M |\nabla u_m^d|^2 - m' \int_M u_m^d.$$
Using the formulas
\[
\nabla \max(u_m^d, u_m^d) = \nabla u_m^d 1_{\{u_m^d > u_m^d\}} + \nabla u_m^d 1_{\{u_m^d \leq u_m^d\}},
\]
\[
\nabla \min(u_m^d, u_m^d) = \nabla u_m^d 1_{\{u_m^d > u_m^d\}} + \nabla u_m^d 1_{\{u_m^d \leq u_m^d\}},
\]
we obtain
\[
\int_{\{u_m^d > u_m^d\}} \left( |\nabla u_m^d|^2 - |\nabla u_m^d|^2 \right) \geq -m \int_{\{u_m^d > u_m^d\}} \left( u_m^d - u_m^d \right),
\]
and
\[
\int_{\{u_m^d > u_m^d\}} \left( |\nabla u_m^d|^2 - |\nabla u_m^d|^2 \right) \geq -m \int_{\{u_m^d > u_m^d\}} \left( u_m^d - u_m^d \right).
\]
Summing these two inequalities, we get
\[
0 \geq (m - m') \int_{\{u_m^d > u_m^d\}} \left( u_m^d - u_m^d \right),
\]
and so \(u_m^d \geq u_m^d\). In particular, \(E_m^d \subset E_m^d\).

We are left to show the Hausdorff convergence in \(E_m^d\) to \(\text{Cut}_b(M)\). Given \(\varepsilon > 0\), let us set
\[
\Omega_{\varepsilon} := \{ x \in M : d(x, \text{Cut}_b(M)) > \varepsilon \}.
\]
We will show that for \(m\) large enough we have \(E_m^d \subset (\Omega_{2\varepsilon})^c\), which will conclude the proof. Let \(\phi : M \to \mathbb{R}\) be a function such that \(\phi \leq d_b\) on \(M\), \(\phi = d_b\) on \(\Omega_{2\varepsilon}\), \(\phi < d_b\) on \(\partial \Omega_{\varepsilon}\), and \(\phi\) is smooth on \(M\) except at \(b\). We want to apply the maximum principle to the function \(\phi - u_m^d\) on \(E_m^d \cap \Omega_{\varepsilon}\). We have
\[
\Delta (\phi - u_m^d) = \Delta \phi + 2m \quad \text{on} \quad E_m^d \cap \Omega_{\varepsilon},
\]
so for \(m\) large enough the function \(\phi - u_m^d\) is subharmonic on \(E_m^d \cap \Omega_{\varepsilon}\). On \(\partial \Omega_{\varepsilon}\), we have \(\phi < d_b\) and \(u_m^d\) converges uniformly to \(d_b\) as \(m\) tends to \(+\infty\) (Lemma 5.1) so \(\phi - u_m^d \leq 0\), for \(m\) large enough. On \(\partial E_m^d\), we have \(\phi - u_m^d = \phi - d_b \leq 0\). Thus the maximum principle implies that for \(m\) large enough, we have \(\phi - u_m^d \leq 0\) on \(E_m^d \cap \Omega_{\varepsilon}\). As \(\phi = d_b\) on \(\Omega_{2\varepsilon}\), we get \(u_m^d \geq d_b\) on \(E_m^d \cap \Omega_{2\varepsilon}\). Since by definition we have \(u_m^d < d_b\) on \(E_m^d\), we get \(E_m^d \subset (\Omega_{2\varepsilon})^c\), which concludes the proof. \(\square\)

6. SEMICONVEXITY

This section is dedicated to the semiconvexity of the solutions to the obstacle problems (1.14) and (1.13). The key result is Proposition 6.1, which applies to both Theorem 1.3 and Theorem 1.6.

In the case of Theorem 1.6, we have \(\bar{M} = \Omega\) and \(\partial M = \partial \Omega\).

**Proposition 6.1.** Let \(M = \bar{M} \cup \partial M\) be a smooth compact Riemannian manifold, with (possibly empty) boundary \(\partial M\). Suppose that for some constants \(L > 0\) and \(C > 0\), we are given the following:

(a) a function \(d : \bar{M} \to \mathbb{R}\), which is bounded and \(C\)-semiconvex on \(\bar{M}\);

(b) a family of functions \(u_m : \bar{M} \to \mathbb{R}^n\), for \(m > 0\), such that:

\(\text{(b.1)}\) for every \(m > 0\), \(u_m \leq d\) on \(M\);

\(\text{(b.2)}\) for every \(m > 0\), \(u_m\) is \(L\)-Lipschitz on \(\bar{M}\);

\(\text{(b.3)}\) on the set \(E_m := \{ u_m < d \}\), \(u_m\) is \(C^\infty\) smooth and

\[-\Delta u_m = 2m \quad \text{in} \quad E_m;\]

\(\text{(b.4)}\) \(E_m\) is precompact in \(\bar{M}\);

\(\text{(b.5)}\) for every \(\eta > 0\), for every \(m > 0\), there is a neighborhood \(N_{\eta,m}\) of \(\partial E_m\) in \(\bar{M}\) such that

\[D^2 u_m \leq (C + \eta) I\]

in \(E_m \cap N_{\eta,m}\).

Then, for every \(\eta > 0\), there exists \(m_0 > 0\) such that

\[u_m \text{ is } (C + \eta)-\text{semiconvex on } \bar{M}, \text{ for every } m \geq m_0.\]

**Application to Theorem 1.6.** In order to apply Proposition 6.1 to Theorem 1.6, we take \(\bar{M} = \Omega\) and \(\partial M = \partial \Omega\). The function \(d\) is the distance function \(d_{\partial \Omega}\) to the boundary of \(\Omega\), while \(u_m\) is the solution \(v_m\) of (1.13) (thus, the Lipschitz constant from (b.2) is \(L = 1\), which means that the conditions (b.1), (b.2) and (b.3) are fulfilled. When \(\Omega\) is \(C^2\) regular, the set \(M(\Omega)\) is contained in \(\Omega\). Now, as the elastic sets \(\{u_m < d\}\) Hausdorff-converge to \(M(\Omega)\) (see [6]) we get that, for large \(m\), \(u_m\) coincides with \(d\) in a neighborhood of \(\partial \Omega\). Thus, (b.4) is fulfilled. As \(\Omega\) is \(C^2\), the function \(d_{\partial \Omega}\) is known to be \(C\)-semiconvex in \(\Omega\) for some \(C > 0\) (see
[10, (iii) of Proposition 2.2.2]), so (a) is fulfilled. Finally, condition (b.5) is a consequence of [13, Chapter 2, Theorem 3.8]. Thus, there exists a constant $C > 0$ such that for $m$ big enough, $v_m$ is $C$-semiconcave in $\Omega$.

**Application to Theorem 1.3 (T4).** In the case of Theorem 1.3, we take $\hat{M} = M \setminus \overline{B}_\rho(x_0)$ and $\partial M = \partial B_p(b)$, where $B_p(b)$ is a small geodesic ball centered at the base point $b$. The function $d$ is the distance function $d_b$ to the base point, while $u_m$ is the solution of (1.14). The semiconcavity of the distance function $d$ in $M \setminus B_p(b)$ was proved in [20], see Proposition 2.7. By Proposition 1.7, for large $m$, the problems (1.14) and (1.3) are equivalent and so we can take $L = 1$ in (b.2), and we also have that (b.1) are (b.3) are fulfilled. Next, we notice that by Lemma 3.3 we have that $u_m = d$ in a neighborhood of $b$, which proves (b.4) by choosing the radius $\rho$ small enough. Finally, in Lemma 6.2 we will prove that also the condition (b.5) is fulfilled.

**Proof of Proposition 6.1.** First, we notice that by dividing all the functions by $L$, we can assume that $L = 1$. Let $\eta > 0$. As in Definition 1.8, for $a, b \in \mathbb{R}$ and $\lambda \in (0, 1)$, we will use the notation

$$\lambda_{ab} := (1 - \lambda)a + \lambda b.$$  

For any unit speed geodesic $\gamma : [a, b] \to \hat{M}$, $\lambda \in (0, 1)$ and $v$ a function on $\hat{M}$, let us define

$$c(\gamma, \lambda, v) := \lambda(1 - \lambda)(C + \eta)(b - a)^2 - \left((1 - \lambda)v(\gamma(a)) + \lambda v(\gamma(b)) - v(\gamma(\lambda_{ab}))\right).$$

We aim to show the following:

$$\inf_{\gamma, \lambda, u_m} c(\gamma, \lambda, u_m) \geq 0,$$  

(6.1)  

where the infimum is taken over unit speed geodesics defined over finite intervals. Let us argue by contradiction and assume that (6.1) does not hold.

Let us show that we may assume that the infimum is actually taken over unit speed geodesics $\gamma : [a, b] \to \hat{M}$ such that

$$\gamma((a, b)) \subset E_m = \{u_m < d\}.  

(6.2)$$

Let $\gamma : [a, b] \to \hat{M}$ be a unit speed geodesic, and $\lambda \in (0, 1)$, such that $c(\gamma, \lambda, u_m) < 0$. Let us assume that $\gamma$ does not verify (6.2). We will build a geodesic $\hat{\gamma}$ that does verify (6.2), and $\hat{\lambda} \in (0, 1)$, such that

$$c(\hat{\gamma}, \hat{\lambda}, u_m) < c(\gamma, \lambda, u_m).$$

First, notice that if $\gamma(\lambda_{ab}) \notin E_m$, then we have $u_m(\gamma(\lambda_{ab})) = d(\gamma(\lambda_{ab})), u_m(\gamma(a)) \leq d(\gamma(a))$ and $u_m(\gamma(b)) \leq d(\gamma(b))$, and so

$$c(\gamma, \lambda, u_m) \geq c(\gamma, \lambda, d) > 0,$$  

(6.3)  

where the last inequality comes from the $C$-semiconcavity of $d$. This is contradictory, so $\gamma(\lambda_{ab}) \in E_m$. As $\gamma$ does not verify (6.2), there exists $t \in (0, \lambda_{ab}) \cup (\lambda_{ab}, 1)$, such that $\gamma(t_{ab}) \notin E_m$. Up to reparametrization of $\gamma$, we may assume that $t \in (0, \lambda_{ab})$. We can define

$$\mu := \min \{s \in (0, \lambda) : \forall r \in (s, \lambda), \gamma(r_{ab}) \in E_m\}.$$  

We have $\gamma(\mu_{ab}) \notin E_m$, and $\gamma((\mu_{ab}, \lambda_{ab})) \subset E_m$. Figure 2 may help justify intuitively the following construction. Let $\hat{\lambda} \in (0, 1)$ be such that

$$\tilde{\lambda}_{\mu_{ab}} = \lambda_{ab}.$$  

(6.4)

**Figure 2. Construction of $\tilde{\gamma}$ and $\tilde{\lambda}$.**

Let $\tilde{\gamma}$ be the unit speed geodesic defined by $\tilde{\gamma} := \gamma([\mu_{ab}, b])$. Let us set $f(t) := (C + \eta)(b - a)^2 - u_m(\gamma(t))$. Then

$$c(\tilde{\gamma}, \tilde{\lambda}, u_m) = (1 - \tilde{\lambda})f(\mu_{ab}) + \tilde{\lambda}f(b) - f(\lambda_{\mu_{ab}})$$

$$= (1 - \tilde{\lambda})f(\mu_{ab}) + \tilde{\lambda}f(b) - f(\lambda_{ab})$$

$$= c(\gamma, \lambda, u_m) - (1 - \lambda)f(a) + (\tilde{\lambda} - \lambda)f(b) + (1 - \tilde{\lambda})f(\mu_{ab}).$$  

(6.5)
Now after some elementary calculations, (6.4) translates into
\[
\begin{cases}
1 - \lambda &= (1 - \tilde{\lambda})(1 - \mu), \\
\tilde{\lambda} - \lambda &= -(1 - \tilde{\lambda})\mu,
\end{cases}
\]
so (6.5) becomes
\[
c(\tilde{\gamma}, \tilde{\lambda}, u_m) = c(\gamma, \lambda, u_m) - (1 - \tilde{\lambda})(1 - \mu)f(a) + \mu f(b) - f(\mu_{ab})
\]
\[= c(\gamma, \lambda, u_m) - (1 - \tilde{\lambda})c(\gamma, \mu, u_m).
\]
Using the fact that $\gamma(\mu_{ab}) \notin \mathcal{E}_m$, we deduce, as in (6.3), that
\[
c(\gamma, \mu, u_m) \geq c(\gamma, \mu, d) > 0.
\]
This yields
\[
c(\tilde{\gamma}, \tilde{\lambda}, u_m) < c(\gamma, \lambda, u_m).
\]
Moreover the unit speed geodesic $\tilde{\gamma} : [\mu_{ab}, b] \rightarrow \hat{M}$ verifies $\tilde{\gamma}(\mu_{ab}, \tilde{\mu}_{ab}) \subset \mathcal{E}_m$. Now, arguing as above, if there exists $t \in (\tilde{\lambda}, 1)$ such that $\tilde{\gamma}(t_{\mu_{ab}}) \notin \mathcal{E}_m$, then we may build two numbers $\nu \in (\tilde{\lambda}, 1)$ and $\tilde{\lambda} \in (0, 1)$ such that the unit speed geodesic $\tilde{\gamma} := \tilde{\gamma}_{[\nu_{ab}, \nu_{ab}]}$ verifies
\[
c(\tilde{\gamma}, \tilde{\lambda}, u_m) < c(\tilde{\gamma}, \tilde{\lambda}, u_m),
\]
and
\[
c(\mu_{ab}, \nu_{ab}) \in \mathcal{E}_m.
\]
So we now need to show that
\[
\inf_{\gamma, \lambda} c(\gamma, \lambda, u_m) \geq 0,
\]
where the infimum is taken over unit speed geodesics $\gamma : [a, b] \rightarrow \hat{M}$ such that $\gamma((a, b)) \subset \mathcal{E}_m$.

By continuity of $u_m$, (6.6) is equivalent to simply saying that $u_m$ is $(C + \eta)$-semiconcave on $\mathcal{E}_m$. Therefore, as $u_m$ is smooth on $\mathcal{E}_m$, by Proposition 2.2, we only need to show the pointwise condition
\[
D^2 u_m \leq (C + \eta) Id \quad \text{on} \quad \mathcal{E}_m.
\]
Now, let $C_1, C_2, C_3 > 0$ be some constants to be determined later, and $\varepsilon > 0$ to be chosen small enough later. For $p \in \mathcal{E}_m$ and $X \in \mathbb{S}^{n-1}(T_pM)$, we define
\[
f_\varepsilon(p, X) := D^2 u_m(X, X) + \varepsilon \left(C_1 |\nabla u_m|^2(p) + C_2 u_m^2(p) - C_3 u_m(p)\right).
\]
We will show that for a good choice of constants $C_1, C_2, C_3$, depending only of $M$ and $|d|_{L^\infty}$, for any $\varepsilon > 0$ small enough, depending only on $M, |d|_{L^\infty}$ and $\eta$, we have for any $m$ large enough,
\[
f_\varepsilon(p, X) \leq C + \frac{2\eta}{3} \quad \text{for every} \quad p \in \mathcal{E}_m \quad \text{and every} \quad X \in \mathbb{S}^{n-1}(T_pM).
\]
This will conclude the proof since, as $u_m$ is bounded by $|d|_{L^\infty}$ and 1-Lipschitz, we will then get
\[
D^2 u_m(X, X) \leq C + \frac{2\eta}{3} + \varepsilon C(M, |d|_{L^\infty}),
\]
where $C(M, |d|_{L^\infty}) > 0$ is a constant depending on $M$ and $|d|_{L^\infty}$ only. But this implies (6.7) if $\varepsilon$ has been taken small enough.

Suppose by contradiction that
\[
\sup_{X \in \mathbb{S}^{n-1}(T_qM)} f_\varepsilon(p, X) > C + \frac{2\eta}{3}.
\]
Let us assume that $m$ is large enough so that $D^2 u_m \leq (C + \eta/3) Id$ in a neighborhood of $\partial \mathcal{E}_m$. In particular, we get that for $\varepsilon$ small enough, depending only on $M, |d|_{L^\infty}$ and $\eta$, $f_\varepsilon < C + \frac{2\eta}{3}$ in a neighborhood of $\partial \mathcal{E}_m$.

Thus, by (6.10) and the precompactness of $\mathcal{E}_m$, there exist $q \in \mathcal{E}_m$ and $Y \in \mathbb{S}^{n-1}(T_qM)$ such that
\[
f_\varepsilon(q, Y) = \sup_{X \in \mathbb{S}^{n-1}(T_qM)} f_\varepsilon(p, X).
\]
In the following, $C^M$ will denote any constant that depends only on $M$. Let us pick some normal coordinates at $q$ such that $\partial_i(q) = Y$. We then extend the vector $Y$ into a vector field (still denoted by $Y$) in a neighborhood of $q$, by setting $Y := \partial_i / |\partial_i|$. As $D\partial_i(q) = 0$, we also have $DY(q) = 0$. Moreover, as the manifold $M$ is compact, $D^2Y(q)$ is bounded by a constant that depends on $M$ only: we have

$$|D^2Y(q)| \leq C^M. \quad (6.12)$$

We will show that the Laplacian of $p \mapsto f_q(p, Y(p))$ is positive at $q$, which contradicts the maximality of $(q, Y(q))$ in (6.11). Let us estimate $\Delta(D^2u_m(Y, Y))$ at the point $q$, using the abstract index notation.

$$ \Delta(D^2u_m(Y, Y)) = g^{ab}D_aD_b(D^2u_mY^cY^d) = g^{ab} \left( D^4_{abcd}u_mY^cY^d + D^3_{ac}D_bu_mD_b(Y^cY^d) + D^2_{ac}D_bu_mD^2_{ab}(Y^cY^d) \right). \quad (6.13)$$

We may divide the right-hand side into four terms and estimate them at the point $q$ individually. The second term is null because it contains $D_b(Y^cY^d) = (D_bY^c)Y^d + Y^cD_bY^d$, and $DY = 0$. The third term is also null, for the same reason. By (6.12), we can estimate the fourth term as follows:

$$ g^{ab}D^2_{cd}u_mD^2_{ab}(Y^cY^d) \geq -C^M |D^2u_m| \geq -\frac{C^M}{\varepsilon} - \varepsilon |D^2u_m|^2. \quad (6.14)$$

It now remains to estimate the first term of (6.13). Using the notation

$$ D_{[ab]} := D_aD_b - D_bD_a, $$

we compute

$$ D_aD_bD_cD_du_m = D_aD_bD_cD_du_m + D_{[ac]}D_bD_du_m + D_cD_aD_{[bd]}u_m + D_cD_aD_{[bd]}u_m. $$

By definition of the Riemann tensor we have

$$ D_aD_{[bc]}D_du_m = D_a(R_{bcde}D^e_wu_m) = (D_aR_{bcde})D^e_wu_m + R_{bcde}D_aD^e_wu_m, $$

and so

$$ |D_aD_{[bc]}D_du_m| \geq -C^M |D^2u_m|. \quad (6.15) \quad \text{Likewise,} \quad |D_cD_{[ad]}D_du_m| \geq -C^M |D^2u_m|. \quad (6.16) $$

To compute the term $D_{[ac]}D_bD_du_m$, let us pick some coordinates $(x^i)$ and write $D_bD_du_m = D^2_{ij}u_mdx^idx^j$. Then, we have

$$ D_{[ac]}D_bD_du_m = D_{[ac]}(D^2_{ij}u_mdx^idx^j) = \left( D_{[ac]}D^2_{ij}u_mdx^idx^j + D^2_{ij}u_mD_{[ac]}(dx^idx^j) + D^2_{ij}u_mdx^idx^jD_{[ac]}(dx^j) \right) $$

$$ = 0 + D^2_{ij}u_mR_{acde}(dx^e)dx^idx^j + D^2_{ij}u_mdx^idx^jR_{acde}(dx^e) $$

$$ = R_{acde}D^e_wu_m + R_{acde}D_aD^e_wu_m. $$

and so

$$ |D_{[ac]}D_bD_du_m| \geq -C^M |D^2u_m|. \quad (6.17)$$

By symmetry of the tensor $D^2u_m$, we have

$$ D_cD_aD_{[bd]}u_m = 0. \quad (6.18)$$

Putting (6.15), (6.16), (6.17) and (6.18) together, we find

$$ |g^{ab}D_aD_bD_cD_du_m| \geq -C^M |D^2u_m| - C^M |D^2u_m| - |g^{ab}D_cD_aD_bD_du_m|. $$

In $E_m$, $u_m$ has constant Laplacian, so

$$ g^{ab}D_cD_dD_aD_bu_m = D_cD_dg^{ab}D_aD_bu_m = D_cD_d\Delta u_m = 0. $$

So we get

$$ |g^{ab}D_aD_bD_cD_du_m| \geq -C^M |D^2u_m| - C^M |D^2u_m|. $$

From this and the fact $Y$ has norm 1, we deduce

$$ |g^{ab}D_aD_bD_cD_du_mY^cY^d| \geq -C^M \varepsilon^{-1} - \varepsilon |D^2u_m|^2 - \varepsilon |D^2u_m|^2. $$

Combining this equation with (6.13) and (6.14), we obtain at the point $q$,

$$ \Delta(D^2u_m(Y, Y)) \geq -C^M \varepsilon^{-1} - 2\varepsilon |D^2u_m|^2 - \varepsilon |D^2u_m|^2. \quad (6.19) $$
We recall the Bochner-Weitzenböck formula:
\[
\Delta \left( |\nabla u_m|^2 \right) = 2 \text{Ric}(\nabla u_m, \nabla u_m) + 2 |D^2 u_m|^2 + 2(\Delta u_m, \nabla u_m).
\]
As \( M \) is compact, there exists a constant \( K > 0 \) such that \( \text{Ric} \geq -K \). Using the fact that \( u_m \) has constant Laplacian in \( E_m \), we get
\[
\Delta \left( |\nabla u_m|^2 \right) \geq 2 |D^2 u_m|^2 - 2K |\nabla u_m|^2.
\]
Furthermore, using the fact that \( \Delta u_m = -2m \) in \( E_m \) again, we find
\[
\Delta (u_m^2) = 2 |\nabla u_m|^2 - 2mu_m \\
\geq 2 |\nabla u_m|^2 - 2m |d|_{L^\infty}^2,
\]
\[
\Delta u_m = -2m.
\]
(6.20)
(6.21)
(6.22)
Using (6.20), (6.21) and (6.22), we get
\[
\Delta \left( \epsilon |\nabla u_m|^2 + (K + 1)\epsilon u_m^2 - ((K + 1)|d|_{L^\infty} + 1)u_m \right) \geq 2\epsilon |D^2 u_m|^2 + \epsilon |\nabla u_m|^2 + \epsilon m.
\]
Setting \((C_1,C_2,C_3) = (1,K+1,(K+1)|d|_{L^\infty} + 1)\), and recalling the definition of \( f_\epsilon \) (6.8), we obtain thanks to (6.19):
\[
\Delta (f_\epsilon(p,Y(p)))_{p=q} \geq -C^M \epsilon^{-1} + \epsilon \eta.
\]
In particular, if \( m \) is large enough, depending on \( \eta \) and \( \epsilon \), this contradicts the maximality of \((q,Y(q))\) in (6.11).
This concludes the proof of (6.9) and Proposition 6.1.

In order to apply Proposition 6.1 to problem (1.14), we will need the following lemma.

**Lemma 6.2** (Bound of \( D^2 u_m \) near \( \partial E_m \)). Let \( u_m \) be the solution of (1.3), as in Theorem 1.3. Let \( \epsilon > 0 \) be smaller than the distance from \( b \) to \( \text{Cut}_b(M) \). Let \( E_m := \{u_m < d_b\} \). From Proposition 5.2, we know that for \( m \) large enough, we have \( E_m \subset M \setminus B(b,\epsilon) \). Let \( C > 0 \) be such that \( d_b \) is \( C \)-semiconcave on \( M \setminus B(b,\epsilon) \). Then, for any \( m \) large enough, for any \( \eta > 0 \), there is a neighborhood \( \mathcal{N}_{n,m} \) of \( \partial E_m \) in \( M \setminus B(b,\epsilon) \) such that
\[
D^2 u_m \leq (C + \eta) I_d \text{ in } E_m \cap \mathcal{N}_{n,m}.
\]
(6.23)

**Proof.** We will use a theorem for obstacle problems on \( \mathbb{R}^n \). Let us show that \( u_m \) is the solution of an obstacle problem on an open subset of \( \mathbb{R}^n \). Then, we will apply [13, Chapter 2, Theorem 3.8] to conclude that (6.23) holds.

The minimality of \( u_m \) in (1.14) implies
\[
-\Delta u_m - 2m \geq 0, \quad u_m \leq d_b \quad \text{and} \quad (-\Delta u_m - 2m)(u_m - d_b) = 0.
\]
(6.24)
Let \( \tilde{\Omega} \) be defined as in the proof of Proposition 3.4. Let \( \phi : \tilde{\Omega} \to \tilde{U} \) be a normal coordinates chart. Writing down (6.24) in these coordinates, we find
\[
A\tilde{u}_m - 2m \geq 0, \quad \tilde{u}_m \leq \psi \quad \text{and} \quad (A\tilde{u}_m - 2m)(\tilde{u}_m - \psi) = 0,
\]
where \( A \) is the Laplacian of \( M \) in the coordinates defined by \( \phi, \tilde{u}_m = u_m \circ \phi^{-1} \) and \( \psi = d_b \circ \phi^{-1} \). This is the form of [13, Chapter 2, equation (3.16)], so we can apply [13, Chapter 2, Theorem 3.8], to deduce that
\[
\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{q \to p} \left\{ D^2 \tilde{u}_m(\phi(q))(X,X) \right\} \leq D^2 \psi(\phi(p))(X,X).
\]
(6.25)
Moreover, we have
\[
D^2 \tilde{u}_m = D^2 u_m \circ (D \phi^{-1}, D \phi^{-1}) + Du_m \circ D^2 \phi^{-1},
\]
\[
D^2 \psi = D^2 d_b \circ (D \phi^{-1}, D \phi^{-1}) + D d_b \circ D^2 \phi^{-1},
\]
and \( Du_m = D d_b \) on \( \partial E_m \) because \( u_m \) is \( C^1 \). Thus, (6.25) yields:
\[
\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{q \to p} \left\{ D^2 u_m(q)(X_q,X_q) \right\} \leq D^2 d_b(p)(X_p,X_p),
\]
where we have set \( X_q := D \phi^{-1}(\phi(q))X \). As \( d_b \) is \( C \)-semiconcave, with Proposition 2.2, we get
\[
\forall p \in \partial E_m, \forall X \in \mathbb{R}^n \lim_{q \to p} \left\{ D^2 u_m(q)(X_q,X_q) \right\} \leq C |X_p|^2.
\]
(6.26)
From there, we deduce that
\[
\text{for } q \in E_m \text{ close enough to } \partial E_m, \text{ we have } D^2 u_m(q) \leq C + \eta.
\]
(6.27)
Indeed, if not, there exist a sequence \((q_k)\) of points of \(E_m\) whose distance to \(\partial E_m\) goes to 0, and a sequence \((X_k)\) of unit vectors of \(\mathbb{R}^2\) such that for any \(k \in \mathbb{N}\),

\[
D^2u_m(q_k)((X_k)_{q_k}, (X_k)_{q_k}) > C + \eta.
\]  

(6.28)

As \(E_m\) is precompact, up to extracting a subsequence, we can assume that \((q_k)\) converges to a point \(p \in \partial E_m\), and \((X_k)\) converges to a vector \(Y \in \mathbb{R}^n\). Because of (6.26), we have

\[
\lim_{k \to \infty} D^2u_m(q_k)(Y_{q_k}, Y_{q_k}) \leq C.
\]  

(6.29)

Furthermore, we know from Proposition 3.4 that \(D^2u_m\) is locally bounded. As \((X_k)_{q_k} - Y_{q_k}\) converges to 0 when \(k\) goes to \(\infty\), this implies

\[
\lim_{k \to \infty} D^2u_m(q_k)((X_k)_{q_k}, (X_k)_{q_k}) - D^2u_m(q_k)(Y_{q_k}, Y_{q_k}) = 0.
\]  

(6.30)

Inequalities (6.28), (6.29) and (6.30) yield a contradiction. So (6.27) is true. This concludes the proof. □

7. Convergence of the gradients

In this section, we show that the uniform semiconcavity of \(u_m\) implies the convergence of the gradients in the sense of Theorem 1.3 (T6). We notice that the results from this section also apply to more general sequences of semiconcave functions.

7.1. Lower semicontinuity. In this section, we prove the first inequality in Theorem 1.3 (T6) (see Proposition 7.2). We start by the following lemma.

Lemma 7.1. Let \(u : M \to \mathbb{R}\) be a \(C\)-semiconcave function. Let \(p, q \in M\) be such that there exists a geodesic from \(p\) to \(q\). Then,

\[
u(q) \leq u(p) + |\nabla u(p)| d(p, q) + \frac{C}{2} d(p, q)^2,
\]

where \(|\nabla u(p)|\) is the norm of the generalized gradient, defined in (2.3).

Proof. Let \(\gamma : [0, d(p, q)] \to M\) be a geodesic from \(p\) to \(q\). Consider the function \(f(t) = \frac{1}{2} C t^2 - u(\gamma(t))\). By the semiconcavity of \(u\), we know that \(f\) is convex. Thus, we have

\[
f(d(p, q)) \geq f(0) + f'(0) d(p, q).
\]

On the other hand, setting \(\dot{\gamma}(0) := v \in T_p(M)\), by construction, we have

\[
f(0) = -u(p), \quad f(d(p, q)) = \frac{C}{2} d(p, q)^2 - u(q), \quad \text{and} \quad f'(0) = -\partial^+_v u(p).
\]

Thus, we obtain

\[
u(q) \leq u(p) + d(p, q) \partial^+_v u(p) + \frac{C}{2} d(p, q)^2 \leq u(p) + |\nabla u(p)| d(p, q) + \frac{C}{2} d(p, q)^2.
\]  

□

Proposition 7.2. Let \(M\) be a Riemannian manifold and let \(C > 0\) be a fixed constant. Let \(u_k : M \to \mathbb{R}\) be a sequence of \(C\)-semiconcave continuous functions that converges locally uniformly to a continuous function \(u_\infty : M \to \mathbb{R}\). Then, \(u_\infty\) is also \(C\)-semiconcave, and for any sequence of points \(p_k \to p_\infty \in M\), we have

\[
|\nabla u_\infty|(p_\infty) \leq \liminf_{k \to \infty} |\nabla u_k|(p_k).
\]  

(7.1)

Proof. First, notice that the \(C\)-semiconcavity of \(u_\infty\) is an immediate consequence of the pointwise convergence and the \(C\)-semiconcavity of \(u_k\). In particular, the generalized gradients \(|\nabla u_m|(p_k)\) and \(|\nabla u_\infty|(p_\infty)\) are well-defined by Proposition 2.4. Thus, we only need to prove (7.1). We notice that (7.1) is trivial if \(|\nabla u_\infty|(p_\infty) = 0\). Thus, we suppose that \(|\nabla u_\infty|(p_\infty) > 0\). In particular, there are a vector \(v \in \mathbb{S}^{n-1}(T_{p_\infty} M)\) and a unit speed geodesic \(\gamma\) with \(\gamma(0) = p_\infty\) and \(\dot{\gamma}(0) = v\) such that

\[
|\nabla u_\infty|(p_\infty) = \lim_{t \to 0^+} \frac{u_\infty(\gamma(t)) - u_\infty(\gamma(0))}{t}.
\]

In particular, for any \(\varepsilon > 0\), we can find \(q \in M\) such that \(d(p_\infty, q) \leq \varepsilon\) and

\[
|\nabla u_\infty|(p_\infty) \leq \frac{u_\infty(q) - u_\infty(p_\infty)}{d(q, p_\infty)} + \varepsilon.
\]
Then, by the uniform convergence of \( \eta_k \) and Lemma 7.1, we get
\[
|\nabla u_k(p_\infty)| \leq \liminf_{k \to \infty} \frac{u_k(q) - u_k(p_k)}{d(q, p_k)} + \varepsilon \leq \liminf_{k \to \infty} |\nabla u_k(p_k)| + \frac{C}{2} d(q, p_k) + \varepsilon \\
\leq \liminf_{k \to \infty} |\nabla u_k(p_k)| + (C + 1)\varepsilon,
\]
which concludes the proof, as the inequality holds for any \( \varepsilon \).

\[ \square \]

### 7.2. Proof of Theorem 1.3 (T6)

The claim (1.8) follows from Proposition 7.2. Thus, we only need to prove (1.9). First, notice that, if \( |\nabla d_b(p, p_\infty)| = 1 \), then (1.9) follows from (1.8) and the fact that \( \eta_m \) is 1-Lipschitz. Let now \( |\nabla d_b(p, p_\infty)| < 1 \). Suppose by contradiction that there are a subsequence \( \eta_k \to +\infty \) and constants \( \varepsilon > 0 \) and \( \eta_0 > 0 \) such that
\[
|\nabla d_b(p, p_\infty) + \varepsilon| \leq |\nabla \eta_k(p)| \quad \text{for every} \quad p \in B_{\eta_0}(p_\infty) \quad \text{and every} \quad k \geq 0.
\]
We now fix \( \eta \leq \eta_0 \), which will be chosen later in the proof. Let \( (q_t) \) be the curve defined by
\[
q_0 = p_\infty \quad \text{and} \quad \frac{dq_t}{dt} = \nabla u_m(q_t).
\]
Let \( T > 0 \) be such that for any \( t \in [0, T] \), \( d(q_t, p_\infty) \leq \eta \), and in particular
\[
|\nabla d_b(p, p_\infty) + \varepsilon| \leq |\nabla \eta_k(t)| \quad \text{for every} \quad t \in [0, T].
\]
We have
\[
\eta_k(q_T) - \eta_k(p_\infty) = \int_0^T |\nabla \eta_k(q_t)|^2 dt \geq \int_0^T (|\nabla d_b(p, p_\infty) + \varepsilon|^2 dt = T(|\nabla d_b(p, p_\infty) + \varepsilon|^2).
\]
As \( \eta_k \) is bounded by the diameter of \( M \), this estimate implies that there exists a finite biggest time \( T > 0 \) such that for any \( t \in [0, T] \), \( d(q_t, p_\infty) \leq \eta \). In particular, \( d(p_\infty, q_T) = \eta \). Let \( \gamma \) be a unit speed minimizing geodesic between \( p_\infty \) and \( q_T \). By Proposition 2.7, there is a constant \( C_d > 0 \) such that \( d_t \) is \( C_d \)-semiconcave in \( B_{\eta_0}(p_\infty) \). In particular, by Lemma 7.1, we have that
\[
d_b(q_T) - d_b(p, p_\infty) \leq |\nabla d_b(p, p_\infty)| d(p, q_T) + C_d d(p, q_T)^2 = |\nabla d_b(p, p_\infty)| \eta + C_d \eta^2.
\]
On the other hand,
\[
\eta_k(q_T) - \eta_k(p_\infty) = \int_0^T |\nabla \eta_k(q_t)| \left| \frac{dq_t}{dt} \right| dt \geq \int_0^T (|\nabla d_b(p, p_\infty) + \varepsilon| \left| \frac{dq_t}{dt} \right| dt
\]
\[
= (|\nabla d_b(p, p_\infty) + \varepsilon| \int_0^T \left| \frac{dq_t}{dt} \right| dt \geq (|\nabla d_b(p, p_\infty) + \varepsilon| d(q_0, q_T)
\]
\[
= (|\nabla d_b(p, p_\infty) + \varepsilon| \eta).
\]
Combining (7.2) and (7.3), we get that
\[
\varepsilon \eta - C_d \eta^2 \leq \left( \eta_k(q_T) - \eta_k(p_\infty) \right) - \left( d_b(q_T) - d_b(p_\infty) \right) \leq 2\norm{\eta_k - d_b}_{L^\infty(M)}.
\]
Now, taking \( \eta \) small enough, we get that
\[
\frac{1}{2} \varepsilon \eta \leq 2\norm{\eta_k - d_b}_{L^\infty(M)} \quad \text{for every} \quad k \geq 0,
\]
but this is in contradiction with the uniform convergence of \( \eta_k \) to \( d_b \).

\[ \square \]

### Appendix A. Appendix about Semiconcavity

In this section we prove that defining local semiconcavity through charts (as in [20]), or through geodesics, is the same (see Proposition 2.3). We recall the notation \( \lambda_{ab} = (1 - \lambda)a + \lambda b \), for \( a, b \in \mathbb{R} \) and \( \lambda \in [0, 1] \) and we notice that the \( C \)-semiconcavity of \( u : M \to \mathbb{R} \) (in the sense of Definition 1.8) can be rewritten as
\[
\lambda_{u(\gamma(a)), u(\gamma(b))} - u(\gamma(\lambda_{ab})) \leq C \lambda(1 - \lambda)(b - a)^2,
\]
for every unit speed geodesic \( \gamma : [a, b] \to M \) and any \( \lambda \in [0, 1] \).

In order to prove Proposition 2.3, we need the following lemma, which shows how to estimate the difference between two geodesics linking a pair of given points, for two different metrics.

**Lemma A.1.** Let \( g \) be a metric on the unit ball \( B_1(0) \subset \mathbb{R}^n \). There exists a constant \( B > 0 \) such that for any unit speed geodesic \( \gamma : [a, b] \to (B_1(0), g) \) and \( \lambda \in [0, 1] \), we have
\[
|\gamma(\lambda_{ab}) - \lambda_{\gamma(a)\gamma(b)}| \leq B \lambda(1 - \lambda)(b - a)^2.
\]
Proof. It suffices to prove that the estimate holds for $\lambda \leq \frac{1}{2}$, as the case $\lambda \geq \frac{1}{2}$ can be deduced by considering $\bar{\gamma} : t \mapsto \gamma(b-t)$ instead of $\gamma$. A unit speed geodesic $\gamma : [a, b] \to (B_{1}(0), g)$ satisfies the geodesic equation

$$\ddot{\gamma}^{i} + \Gamma^{i}_{jk} \dot{\gamma}^{j} \dot{\gamma}^{k} = 0,$$

where $\Gamma^{i}_{jk}$ are the Christoffel symbols of the metric $g$. As $\gamma$ is unit speed, the $(\dot{\gamma}^{i})$ are bounded, uniformly in $\gamma$. Therefore, there exists a constant $\alpha > 0$ independent of $\gamma$ such that $|\ddot{\gamma}| \leq \alpha$. By integration, we find

$$|\gamma(t) - \gamma(a) - \dot{\gamma}(a)(t-a)| \leq \alpha(t-a)^{2}.$$

Evaluating this expression at $b$ yields

$$|\gamma(b) - \gamma(a) - \dot{\gamma}(a)(b-a)| \leq \alpha(b-a)^{2}.$$

From these two estimates, we deduce

$$\left| \gamma(t) - \gamma(a) - \frac{\gamma(b) - \gamma(a)}{b-a}(t-a) \right| \leq \alpha(t-a)^{2} + \alpha(b-a)(t-a).$$

Taking $t = (1-\lambda)a + \lambda b$ in this estimate yields

$$|\gamma((1-\lambda)a + \lambda b) - ((1-\lambda)\gamma(a) + \lambda \gamma(b))| \leq \alpha \lambda (1+\lambda)(b-a)^{2} = \frac{\alpha(1+\lambda)}{1-\lambda} \lambda (1-\lambda)(b-a)^{2}.$$

Taking $B := \frac{\alpha(1+\lambda)}{1-\lambda}$, this proves the desired estimate when $\lambda \leq 1/2$. This concludes the proof. \qed

Proof of Proposition 2.3. Let us assume that $u$ is locally semiconcave. Let $\psi : U \to V$ be a chart from an open set $U$ of $M$ to an open set $V$ of $\mathbb{R}^{n}$, and $y \in V$. Let $f := u \circ \psi^{-1}$. We want to show that $f$ is semiconcave in a neighborhood of $y$, as a function of $\mathbb{R}^{n}$. We first observe that $f$ is locally semiconcave on the manifold $(V, \psi_{*}g)$. Let $V' \subset V$ be a neighborhood of $y$ that is geodesically convex for the metric $\psi_{*}g$, and such that there exists a constant $C > 0$ such that $f$ is $C$-semiconcave on $(V', \psi_{*}g)$. Let $d$ denote the distance function on $(V', \psi_{*}g)$. Up to taking $V'$ smaller, we may assume that the metric $\psi_{*}g$ is bounded on $V'$, and so there exists a constant $\beta > 0$ such that

$$\forall x, y \in V', \quad d(x, y) \leq \beta |x-y|.$$

Let $x, y \in V'$ be such that $[x, y] \subset V'$, and $\lambda \in [0,1]$. Let $\gamma : [a, b] \to V'$ be a unit speed geodesic of $(V', \psi_{*}g)$ from $x$ to $y$. By the $C$-semiconcavity of $f$ on $(V', \psi_{*}g)$, we have

$$\lambda_{f(x)f(y)} - f(\lambda_{xy}) = \lambda_{f(\gamma(a))f(\gamma(b))} - f(\lambda_{\gamma(a)\gamma(b)}) \leq C\lambda (1-\lambda)(b-a)^{2} + f(\lambda_{ab}) - f(\lambda_{\gamma(a)\gamma(b)}) \leq C\lambda (1-\lambda)(b-a)^{2} + \operatorname{Lip}(f) |\lambda_{ab} - \lambda_{\gamma(a)\gamma(b)}|.$$

Applying Lemma A.1 above, we get a constant $B > 0$ such that

$$\lambda_{f(x)f(y)} - f(\lambda_{xy}) \leq (C + \operatorname{Lip}(f) B) \lambda (1-\lambda)(b-a)^{2} = (C + \operatorname{Lip}(f) B) \lambda (1-\lambda)(d(x, y))^{2} \leq (C + \operatorname{Lip}(f) \beta^{2}) \lambda (1-\lambda) |x-y|^{2},$$

and so $f$ is semiconcave on $V'$, as a function of $\mathbb{R}^{n}$.

Reciprocally, let us assume that $u \circ \psi^{-1}$ is locally semiconcave as a function of $\mathbb{R}^{n}$ for any chart $\psi$. Then, we can show that $u \circ \psi^{-1}$ is locally semiconcave for the metric $\psi_{*}g$, for any chart $\psi$, by using the same technique. From there we deduce that $u$ is locally semiconcave. This concludes the proof. \qed

Appendix B. A counter-example to the equivalence of (1.3) and (1.14) for small $m$

Theorem B.1. There exist a surface of revolution $M$ and a parameter $m > 0$ such that $u_{m} \neq u^{d}_{m}$.

Proof of Theorem B.1. Let $r_{\theta}$ denote the rotation of $\mathbb{R}^{3}$ of angle $\theta \in [0, 2\pi]$ around the $z$-axis. Let $T := 10^{10}$ and $r, h : [0, T] \to \mathbb{R}$ be two smooth functions such that:

[Insert additional details or equations here as needed.]
\[ \gamma : t \mapsto (r(t), 0, h(t)) \] is a unit speed curve.

\[ M := \{ r_\theta(\gamma(t)) : (t, \theta) \in [0, T] \times [0, 2\pi] \} \]

is a smooth surface,

\[ r(0) = r(T) = 0, \]

\[ r \leq 2, \]

\[ r([1, 2]) \subset [1, 2], \]

\[ r([3, 4]) \subset (0, 10^{-10}), \]

\[ r([5, T - 1]) \subset [1, 2]. \]

This information is pictured in Figure B. We chose \( b = (0, 0, 0) \) as the base point on \( M \), and \( m = 10^{-10} \). Let us assume that \( u^d_m = u_m \) and build a better competitor in (1.14) to contradict the minimality of \( u^d_m \).

We will first reduce (1.14) to a one-dimensional problem. Note that the functional we are minimizing is rotation-invariant. More precisely, for any \( \theta \in (0, 2\pi) \) and \( u \in H^1(M) \), we have

\[ \int_M |\nabla(u \circ r_\theta)|^2 - m(u \circ r_\theta) = \int_M |\nabla u|^2 - mu. \tag{B.1} \]

By the uniqueness of the minimizer \( u^d_m \), we deduce that \( u^d_m \) is rotation-invariant, i.e. there exists a function \( p_m : [0, T] \rightarrow \mathbb{R} \) such that for any \( \theta \in [0, 2\pi) \) and \( t \in [0, T] \), \( u^d_m(r_\theta(\gamma(t))) = p_m(t) \). Thus \( u^d_m \) is a minimizer of (1.14) among rotation-invariant functions.

Let \( u : M \rightarrow \mathbb{R} \) be any rotation-invariant function, and \( \rho : [0, T] \rightarrow \mathbb{R} \) be such that for any \( \theta \in [0, 2\pi) \), \( u(r_\theta(\gamma(t))) = \rho(t) \). We will translate the minimization problem (1.14) on \( u \) into a problem on \( \rho \).

First, because \( M \) is a surface of revolution, all the geodesics starting from \( b = (0, 0, 0) \) have a constant angle \( \theta \). Thus, they are of the form \( t \mapsto r_\theta(\gamma(t)) \) for some \( \theta \in [0, 2\pi) \). These are actually unit speed geodesics as \( \gamma \) is unit speed. Hence, \( d_b(r_\theta(\gamma(t))) = t \), and the constraint \( u \leq d_b \) in (1.14) is equivalent to \( \rho(t) \leq t \).

Secondly, we translate the \( H^1 \) constraint. To this end, let us define some coordinates \((t, \theta)\) on \( M \) via the map

\[ \phi : (0, T) \times (0, 2\pi) \rightarrow M, \quad \phi(t, \theta) = r_\theta(\gamma(t)). \]

We have

\[ \int_M |\nabla u|^2 = \int_0^{2\pi} \int_0^T (|\nabla u|^2 \circ \phi) J \phi \, dt \, d\theta \]

\[ = \int_0^{2\pi} \int_0^T |\nabla u|^2 (r_\theta(\gamma(t))) r(t) \, dt \, d\theta = 2\pi \int_0^T |\nabla u|^2 (\gamma(t)) r(t) \, dt, \tag{B.2} \]

because \( u \) is rotation-invariant. Moreover, as \( u \) is rotation-invariant, its gradient at the point \( \gamma(t) \) is parallel to \( \gamma'(t) \), and so

\[ |\rho'(t)| = |\nabla u(\gamma(t)) \cdot \gamma'(t)| = |\nabla u(\gamma(t))| |\gamma'(t)| = |\nabla u(\gamma(t))|. \]

Hence (B.2) gives

\[ \int_M |\nabla u|^2 = 2\pi \int_0^T \rho(t)^2 r(t) \, dt \]

Thus, the constraint \( u \in H^1(M) \) in (1.14) is equivalent to \( v \in H^1((0, T), r(t) dt) \).

Thirdly, we may compute the functional likewise:

\[ \int_M |\nabla u|^2 - mu = 2\pi \int_0^T (\rho(t)^2 - m\rho(t)) r(t) \, dt. \]

All in all, as \( u^d_m \) is a minimizer in (1.14), \( \rho_m \) is a minimizer of:

\[ \inf \left\{ \int_0^T \left( \rho'(t)^2 - m \rho(t) \right) r(t) \, dt : \rho \in H^1((0, T), r(t) dt), \rho(t) \leq t \right\}. \tag{B.3} \]

The idea of the rest of the proof is the following. First, we recall the assumption \( u^d_m = u_m \), which means that \( |\nabla u^d_m| \leq 1 \), and so \( |\rho_m| \leq 1 \). Now, if \( \rho_m(4) \) is close to 4, then \( \rho'_m(t) \) is close to 1 for \( t \leq 4 \), so a competitor \( v \) such that \( \rho'(t) \) is small for \( t \leq 4 \) will contradict the minimality of \( \rho_m \) in (B.3). If on the contrary \( \rho_m(4) \) is significantly smaller than 4, then for \( t \geq 4, \rho_m(t) \) will be significantly smaller than \( t \), so a competitor \( \rho \) such that \( \rho(t) \) is closer to \( t \) for \( t \geq 4 \) will contradict the minimality of \( \rho_m \) in (B.3). Because we chose \( r \) very small on
the interval $[3, 4]$ (see Figure 3), we can define a competitor $\rho$ independently on $[0, 3]$ and $[4, T]$, without paying much for the behavior of $\rho$ on $[3, 4]$.

**Case one:** $\rho_m(4) \in [3, 4]$. Let us define a competitor $\rho$ for (B.3):

$$
\rho : [0, T] \to \mathbb{R}, \quad \rho(t) = \begin{cases} 0 & \text{if } t \in [0, 3] \\
4(t - 3) & \text{if } t \in [3, 4] \\
\rho_m(t) + 4 - \rho_m(4) & \text{if } t \geq 4
\end{cases}.
$$

Let us call $\mathcal{F}(\rho)$ the functional appearing in (B.3). We have, from the definition of $r$ and $\rho$,

$$
\mathcal{F}(\rho) = \int_0^4 (16 - 4m(t - 3))r(t)dt + \int_4^T \left( \rho_m^2(t) - m\rho_m(t) \right) r(t)dt - m(4 - \rho_m(4)) \int_4^T r(t)dt
$$

so

$$
\mathcal{F}(\rho) - \mathcal{F}(\rho_m) \leq 16 \cdot 10^{-10} - \int_0^4 (\rho_m^2(t) - m\rho_m(t)) r(t) dt.
$$

We are left to bound from below the integral term in (B.5). By the Hölder inequality we have

$$
\int_1^2 \rho_m' \leq \left( \int_1^2 \frac{1}{r} \right)^{1/2} \left( \int_1^2 \rho_m^2 r \right)^{1/2},
$$

and so

$$
\int_1^2 \rho_m^2 r \geq \frac{(\rho_m(2) - \rho_m(1))^2}{\int_1^2 \frac{1}{r}} \geq (\rho_m(2) - \rho_m(1))^2,
$$

by the construction of $r$. Now we use the fact $u_m'' = u_m$, which means that $|\nabla u_m''| \leq 1$, and so $|\rho'_m| \leq 1$. With the running assumption $\rho_m(4) \geq 3.5$, this implies $\rho_m(2) \geq 1.5$. As $\rho_m(1) \leq 1$, we get $\rho_m(2) - \rho_m(1) \geq 0.5$. Then, (B.6) and (B.5) yield

$$
\mathcal{F}(\rho) - \mathcal{F}(\rho_m) \leq 16 \cdot 10^{-10} - 0.25 + 16m.
$$

Recalling that we have chosen $m = 10^{-10}$, it contradicts the minimality of $\rho_m$ in (B.3).

**Case two:** $\rho_m(4) \leq 3.5$. We use the same competitor $\rho$ as in case one. We even perform similar estimates, the only difference being that we don’t estimate the term $-m(4 - \rho_m(4)) \int_4^T r(t)dt$ by 0 as in (B.4). Thus (B.5) becomes instead:

$$
\mathcal{F}(\rho) - \mathcal{F}(\rho_m) \leq 16 \cdot 10^{-10} - \int_1^2 \rho_m^2(t)r(t)dt + 16m - m(4 - \rho_m(4)) \int_4^T r(t)dt.
$$

Recalling that we have chosen $m = 10^{-10}$, $T = 10^{10}$ and $r \geq 1$ between 5 and $T - 1$, it contradicts the minimality of $\rho_m$ in (B.3). This concludes the proof.

\[\square\]

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