Interval colorings of edges of a multigraph

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Let \( G = (V(G), E(G)) \) be a multigraph. The degree of a vertex \( x \) in \( G \) is denoted by \( d(x) \), the greatest degree of a vertex – by \( \Delta(G) \), the chromatic index of \( G \) – by \( \chi'(G) \). Let \( R \subseteq V(G) \).

An interval (respectively, continuous) on \( R \) \( t \)-coloring of a multigraph \( G \) is a proper coloring of edges of \( G \) with the colors 1, 2, \ldots, \( t \), in which each color is used at least for one edge, and the edges incident with each vertex \( x \in R \) are colored by \( d(x) \) consecutive colors (respectively, by the colors 1, 2, \ldots, \( d(x) \)).

In this paper the problems of existence and construction of interval or continuous on \( R \) colorings of \( G \) are investigated. Problems of such kind appear in construction of timetablings without "windows". Some properties of interval or continuous on \( V(G) \) colorings were obtained in [1,2]. Necessary and sufficient conditions of the existence of a continuous on \( V(G) \) \( \Delta(G) \)-coloring in the case when \( G \) is a tree are obtained in [3]. All non-defined concepts can be found in [4,5].

Let \( \mathfrak{N}_t \) be the set of multigraphs \( G \), for which there exists an interval on \( V(G) \) \( t \)-coloring, and \( \mathfrak{N} = \bigcup_{t \geq 1} \mathfrak{N}_t \). For every \( G \in \mathfrak{N} \), let us denote by \( w(G) \) and \( W(G) \), respectively, the least and the greatest \( t \), for which there exists an interval on \( V(G) \) \( t \)-coloring of \( G \). Evidently, \( \Delta(G) \leq \chi'(G) \leq w(G) \leq W(G) \).

**Proposition 1.** If \( G \in \mathfrak{N} \) then \( \chi'(G) = \Delta(G) \).

**Proof.** Let us consider an interval on \( V(G) \) \( w(G) \)-coloring of the multigraph \( G \). If \( w(G) = \Delta(G) \) then \( \chi'(G) = \Delta(G) \). Assume that \( w(G) > \Delta(G) \). Let us define the sets \( T(1), \ldots, T(\Delta(G)) \), where \( T(j) = \{ i / i \equiv j(\text{mod}(\Delta(G))), 1 \leq i \leq w(G) \} \), \( j = 1, \ldots, \Delta(G) \). Let \( E_j \) be the subset of edges of \( G \) which are colored by colors from the set \( T(j) \), \( j = 1, \ldots, \Delta(G) \). Clearly, \( E_j \) is a matching. For each \( j \in \{ 1, \ldots, \Delta(G) \} \), let us color the edges of \( E_j \) by the color \( j \). We shall obtain a proper coloring of edges of \( G \) with \( \Delta(G) \) colors. Hence, \( \chi'(G) = \Delta(G) \).

The Proposition is proved.

**Proposition 2.** Let \( G \) be a regular multigraph.

\( a) \) \( G \in \mathfrak{N} \) iff \( \chi'(G) = \Delta(G) \).

\( b) \) If \( G \in \mathfrak{N} \) and \( \Delta(G) \leq t \leq W(G) \) then \( G \in \mathfrak{N}_t \).

**Proof.** The proposition \( a) \) follows from the proposition 1. The proposition \( b) \) holds, since if \( t > w(G) \) then an interval on \( V(G) \) \( (t - 1) \)-coloring can be obtained from an interval on \( V(G) \) \( t \)-coloring by recoloring with the color \( t - \Delta(G) \) all edges colored by \( t \).

The Proposition is proved.

It is proved in [6] that for a regular graph \( G \), the problem of deciding whether \( \chi'(G) = \Delta(G) \) or \( \chi'(G) \neq \Delta(G) \) is \( NP \)-complete by R. Karp [7]. It follows from here and from the proposition 2
that for a regular graph $G$, the problem of determining whether $G \in \mathfrak{N}$ or $G \notin \mathfrak{N}$ is NP-complete by R. Karp.

**Lemma 1.** Let $G$ be a connected multigraph with a proper edge coloring with the colors $1, \ldots, t$, and the edges incident with each vertex $x \in V(G)$ are colored by $d(x)$ consecutive colors. Then $G \in \mathfrak{N}$.

**Proof.** Let $\alpha(e)$ be the color of the edge $e \in E(G)$. Without loss of generality, we assume that $\min_e \alpha(e) = 1$, $\max_e \alpha(e) = t$. For the proof of the lemma it is suffice to show, that if $t \geq 3$, then each color $r$, $1 < r < t$, is used for at least one edge. Since $G$ is connected, then there exists a simple path $P = (x_0, e_1, x_1, \ldots, x_{k-1}, e_k, x_k)$ in it, where $e_i = (x_{i-1}, x_i)$, $i = 1, \ldots, k$ and $\alpha(e_1) = t$, $\alpha(e_k) = 1$. If $\alpha(e_i) \neq r$, $i = 1, \ldots, k$, let us consider in $P$ the vertex $x_{i_0}$ with the greatest index, satisfying the inequality $\alpha(e_{i_0}) > r$. Then $\alpha(e_{i_0}) < r$. It follows from the condition of the lemma that there is an edge incident with the vertex $x_{i_0}$ colored by the color $r$.

The Lemma is proved.

**Theorem 1.** Let $G$ be a connected graph without triangles. If $G \in \mathfrak{N}$ then $W(G) \leq |V(G)| - 1$.

**Proof** by contrary. Assume that there exist connected graphs $H$ in $\mathfrak{N}$ without triangles with $W(H) \geq |V(H)|$. Let us choose among them a graph $G$ with the least number of edges. Clearly, $|E(G)| > 1$. Consider an interval on $V(G)$ coloring of $G$. The color of an edge $e$ is denoted by $\alpha(e)$, the set $\{v \in V(G)/ (u, v) \in E(G)\}$ by $I(u)$. Let $\mathfrak{M}$ be the set of all simple paths with the initial edge colored by the color $W(G)$ and the final edge colored by the color 1. For each $P \in \mathfrak{M}$ with the sequence $e_1, \ldots, e_t$ of edges, $t \geq 2$, let us set in correspondence the sequence $\alpha(P) = (\alpha(e_1), \ldots, \alpha(e_t))$ of colors. Let us show that there is a path $P_0$ in $\mathfrak{M}$ for which $\alpha(P_0)$ is decreasing.

Let $\alpha(e') = W(G)$, $e' = (x_0, x_1)$, and $d(x_1) \geq d(x_0)$. Since $|E(G)| > 1$, then $d(x_1) \geq 2$. Let us construct the sequence $X$ of vertices as follows:

Step 1. $X := \{x_0, x_1\}$.

Step 2. Let $x_i$ be the last vertex in the sequence $X$. If $I(x_i) \setminus X = \emptyset$ or $\alpha(x_i, y) > \alpha(x_{i-1}, x_i)$ for each $y \in I(x_i) \setminus X$ then the construction of $X$ is completed. Otherwise let us choose from $I(x_i) \setminus X$ the vertex $x_{i+1}$, for which $\alpha(x_i, x_{i+1}) = \min \alpha(x_i, y)$, where the minimum is taken on all $y \in I(x_i) \setminus X$. Let us bring $x_{i+1}$ in $X$ and repeat the step 2.

Suppose that $X$ is constructed and $X = \{x_0, x_1, \ldots, x_k\}$. Clearly, $X$ defines a simple path $P_0 = (x_0, e_1, x_1, \ldots, e_k, x_k)$, where $e_i = (x_{i-1}, x_i)$, $i = 1, \ldots, k$. Let us show that $\alpha(e_k) = 1$.

Suppose $1 < \alpha(e_k) < W(G)$. Let us define a graph $H$ as follows:

$H = \{ \begin{array}{ll} G - x_k, & \text{if } d(x_k) = 1 \\ G - e_k, & \text{if } d(x_k) \geq 2. \end{array}$

Let us show that $H$ is connected. Assume the contrary. Then $H = G - e_k$. Let $H_1, H_2$ be the connected components of $H$, $x_{k-1} \in V(H_1)$, $x_k \in V(H_2)$, and $G_1, G_2$ be the subgraphs of $G$ induced, respectively, by the subsets $V(H_1) \cup \{x_k\}$ and $V(H_2) \cup \{x_{k-1}\}$. The coloring of the graph $G$ induces the coloring of $G_1$ satisfying the conditions of the lemma. Therefore $G_1 \in \mathfrak{N}$, and, since $|E(G_1)| < |E(G)|$ then $W(G_1) \leq |V(G_1)| - 1$, $i = 1, 2$. It is not difficult to check that $W(G) \leq W(G_1) + W(G_2) - 1$. From here we obtain the inequality $W(G) \leq |V(G)| - 1$, which contradicts the choice of $G$. Therefore $H$ is connected.

It follows from the lemma that the coloring of edges of $H$ induced by the coloring of $G$ is an interval on $V(H)$ coloring of $G$. Then $W(H) \geq W(G) \geq |V(G)| \geq |V(H)|$. The obtained inequality contradicts the choice of $G$, because $|E(H)| < |E(G)|$. Consequently, $\alpha(e_k) = 1$. 

2
Hence, we have constructed the path $P_0 \in \mathfrak{M}$ for which the sequence $\alpha(P_0)$ is decreasing.

Let us denote by $\vartheta$ the set of all shortest paths $P$ from $\mathfrak{M}$, for which the sequence $\alpha(P)$ is decreasing. Let $k$ be the length of paths in $\vartheta$.

Let us define the sets $\vartheta_1, \ldots, \vartheta_k$: $\vartheta_1 = \vartheta$, and $\vartheta_i$ is the subset of paths from $\vartheta_{i-1}$ with the greatest color of the $i$-th edge, $i = 2, \ldots, k$.

Let us choose from $\vartheta_k$ some path $P_1 = (x_0, e_1, x_1, \ldots, x_{k-1}, e_k, x_k)$. Let $A(i) = \{y \in I(x_i) / \alpha(e_{i+1}) < \alpha(x_i, y) < \alpha(e_i)\}$. Clearly, $|A(i)| = \alpha(e_i) - \alpha(e_{i+1}) - 1$, $i = 1, \ldots, k - 1$.

Let us show that $A(i) \cap \{x_0, x_1, \ldots, x_k\} = \emptyset$, $i = 1, \ldots, k - 1$. Suppose that there exist such $i_0$, $j_0$, that either $x_{i_0} \in A(j_0)$ or $x_{j_0} \in A(i_0)$. Let us define the path $P$ as follows. If $i_0 \neq 0$, $j_0 \neq k$, then $P = (x_0, e_1, x_1, \ldots, x_{i_0}, \alpha(x_{i_0}, x_{j_0}), x_{j_0}, \ldots, x_k)$. If $i_0 = 0$, then $P = (x_1, e_1, x_0, (x_0, x_{j_0}), x_{j_0}, \ldots, x_k)$. If $j_0 = k$, then $P = (x_0, e_1, x_1, \ldots, x_{i_0}, x_{i_0}, x_k, e_k, x_{k-1})$. In all three cases the sequence $\alpha(P)$ is decreasing, and the length of $P$ is less than the length of $P_1$, which contradicts the choice of $P_1$. Therefore

$$A(i) \cap \{x_0, x_1, \ldots, x_k\} = \emptyset, \quad i = 1, \ldots, k - 1 \quad (1)$$

Let us show that $A(i) \cap A(j) = \emptyset$, $1 \leq i < j \leq k - 1$. Suppose that there exist such $i_0$, $j_0$, for which $1 \leq i_0 < j_0 \leq k - 1$ and $A(i_0) \cap A(j_0) \neq \emptyset$.

Since there is no triangle in $G$ then $j_0 - i_0 \geq 2$. Let $v \in A(i_0) \cap A(j_0)$. Let us consider the path $P = (x_0, e_1, x_1, \ldots, x_{i_0}, (x_{i_0}, v), (v, x_{j_0}), x_{j_0}, \ldots, x_{k-1}, e_k, x_k)$. Clearly, the sequence $\alpha(P)$ is decreasing. If $j_0 - i_0 \geq 3$ then the length of $P$ is less than the length of $P_1$, and if $j_0 - i_0 = 2$ then $\alpha(x_{i_0}, v) > \alpha(e_{i+1})$. In both cases it contradicts the choice of $P_1$. Therefore

$$A(i) \cap A(j) = \emptyset, \quad 1 \leq i < j \leq k - 1 \quad (2)$$

From (1) and (2), it follows that

$$|V(G)| \geq \left| \bigcup_{i=1}^{k-1} A(i) \right| + k + 1 = k + 1 + \sum_{i=1}^{k-1} |A(i)| = k + 1 + \sum_{i=1}^{k-1} (\alpha(e_i) - \alpha(e_{i+1}) - 1) =$$

$$= k + 1 + W(G) - 1 - (k - 1) = 1 + W(G).$$

It contradicts the choice of $G$.

The Theorem is proved.

**Corollary 1.** If $G$ is a connected bipartite graph, and $G \in \mathfrak{M}$, then $W(G) \leq |V(G)| - 1$.

Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph. Let us denote by $w_1(G)$ and $W_1(G)$, respectively, the least and the greatest $t$, for which there exists an interval on $V_1(G)$ $t$-coloring of $G$. Evidently, $W_1(G) = |E(G)|$. 

**Theorem 2.** For any $t$, $w_1(G) \leq t \leq W_1(G)$, there exists an interval on $V_1(G)$ $t$-coloring of the multigraph $G$.

**Proof** by induction on $|V_1(G)|$.

If $|V_1(G)| = 1$ then the proposition of the theorem is true. Suppose that the proposition of the theorem is true for all $G'$ with $|V_1(G')| = p$. Suppose that $|V_1(G)| = p + 1$, and assume there exists an interval on $V_1(G)$ $t$-coloring of $G$, $w_1(G) \leq t < W_1(G)$. Among vertices of $V_1(G)$ which are incident with edges colored by the color $t$, let us choose a vertex $x_1$ with the smallest degree. There is an edge $e_1$ colored by the color $t + 1 - d(x_1)$ which is incident with the vertex $x_1$.

1) If there exists an edge different from $e_1$ and colored by the color $t + 1 - d(x_1)$, then, by recoloring $e_1$ with the color $t + 1$ we shall obtain an interval on $V_1(G)$ $(t + 1)$-coloring of $G$. 

3
2) Let $e_1$ be the unique edge colored by the color $t + 1 - d(x_1)$, and $s$ be the maximum color which is used for more than one edge. Clearly, $1 \leq s < t < |E(G)|$.

2a) Let $t + 1 - d(x_1) < s < t$. Let us recolor each edge with the color $i$, where $i = t + 1 - d(x_1), \ldots, s$, by the color $i + t - s$, and let us recolor each edge with the color $i$, where $i = s + 1, \ldots, t$, by the color $(i + t - s) \bmod t + t - d(x_1)$. In the obtained interval on $V_1(G)$ $t$-coloring, among that vertices from $V_1(G)$ which are incident with edges colored by $t$ (there are more than 1 such edges), we shall choose a vertex $x_2$ with the smallest degree and recolor the incident with it edge with the color $i + 1 - d(x_2)$ by the color $t + 1$. We shall obtain an interval on $V_1(G)$ $(t + 1)$-coloring of $G$.

2b) Let $1 \leq s < t + 1 - d(x_1)$. Removing $x_1$ from $G$, we shall obtain a multigraph $G'$ with an interval on $V_1(G')$ $(t - d(x_1))$-coloring. Clearly, $|E(G')| = |E(G)| - d(x_1)$ and $t - d(x_1) < |E(G')| = W_1(G')$. By the assumption of induction there exists an interval on $V_1(G')$ $(t + 1 - d(x_1))$-coloring of $G'$. We shall color the edges incident with the vertex $x_1$ by the colors $t + 2 - d(x_1), \ldots, t + 1$ and obtain an interval on $V_1(G)$ $(t + 1)$-coloring of $G$.

The Theorem is proved.

In the work [8] in terms of timetables the NP-completeness was proved for the problem of a 3-coloring of a bipartite graph with preassignments in one part. A bipartite graph $H = (V_1(H), V_2(H), E(H))$ with $\Delta(H) = 3$ is given, where the set $V_1(H)$ contains no pendent vertex, and, for each $x \in V_1(H)$, a set $T(x)$ is preassigned, $T(x) \subseteq \{1, 2, 3\}$, $|T(x)| = d_H(x)$. The required is to determine does there exist a proper coloring of edges of $H$ with the colors 1, 2, 3, at which the edges incident with each vertex $x \in V_1(H)$ are colored by colors from the set $T(x)$.

**Theorem 3.** For a bipartite multigraph with the greatest degree 3 of a vertex, the problem of deciding whether a 3-coloring, continuous on one part, exists or not, is NP-complete.

**Proof.** Let $H'$ be a graph isomorphic to the graph $H$, $V(H) \cap V(H') = \emptyset$, and to each vertex $y \in V(H)$ a vertex $y' \in V(H')$ corresponds. Let us construct a bipartite multigraph $G_1$ as follows.

For each $y \in V_2(H)$, connect the vertices $y$ and $y'$ with one edge if $d_H(y) = 2$, and with two parallel edges if $d_H(y) = 1$. Clearly, $\Delta(G_1) = 3$.

Set
$$T(y') := T(y), T(y) := T(y) \text{ for each } y \in V_1(H),$$
$$T(y') := \{1, 2, 3\}, T(y) := \{1, 2, 3\} \text{ for each } y \in V_2(H).$$

Let $V_{ij} = \{y \in V(G_1)/T(y) = \{i, j\}\}, 1 \leq i < j \leq 3$.

Let us define a bipartite multigraph $G$ as follows:

$$V(G) = V(G_1) \cup \{x_1/x \in V_{23}\} \cup \{x_1, x_2/x \in V_{13}\},$$
$$E(G) = E(G_1) \cup \{(x, x_1)/x \in V_{23}\} \cup \{(x, x_1), (x_1, x_2)/x \in V_{13}\}.$$

Clearly, a 3-coloring of $H$ with preassignments in $V_1(H)$ exists if and only if a 3-coloring of edges of $G_1$ exists at which the edges incident with each vertex $x \in V(G_1)$ are colored by colors from the set $T(x)$. Such coloring of edges of $G_1$ exists if and only if a 3-coloring of $G$ continuous on $V(G)$ exists. It is not difficult to check that the collection of degrees of vertices of $V_1(G)$ coincides with the collection of degrees of vertices of $V_2(G)$. Therefore a continuous on $V(G)$ 3-coloring of $G$ exists if and only if a continuous on $V_1(G)$ 3-coloring of $G$ exists.

The Theorem is proved.

**Theorem 4.** Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph. If for each edge $(x, y)$, where $x \in V_1(G)$, the condition $d(x) \geq d(y)$ holds, then $G$ has a continuous on $V_1(G)$ $\Delta(G)$-coloring.
Proof. Let $V_1(G) = \{x_1, \ldots, x_p\}$, $d(x_1) \geq \ldots \geq d(x_p)$, and already a proper coloring of edges incident with the vertices $x_1, \ldots, x_n$ ($n \geq 1$) is constructed so that the edges incident with the vertex $x_i$ are colored by the colors $1, \ldots, d(x_i)$, $i = 1, \ldots, n$. If $n < p$ and with the vertex $x_{n+1}$ the edges $(x_{n+1}, y(1)), \ldots, (x_{n+1}, y(d(x_{n+1}))$ are incident, then, sequentially for each $j = 1, \ldots, d(x_{n+1})$ do as follows. If the color $j$ is absent in the vertex $y(j)$, then we shall color the edge $(x_{n+1}, y(j))$ by the color $j$. Otherwise a color $k$ is absent in $y(j)$, $1 \leq k \leq d(x_{n+1})$. We shall recolor the longest path consisting of edges colored by $j$ and $k$ with the initial vertex $y(j)$ and we shall color the edge $(x_{n+1}, y(j))$ by the color $j$.

The Theorem is proved.

Corollary 2. If

$$\min_{x \in V_1(G)} d(x) \geq \max_{y \in V_2(G)} d(y),$$

then $G$ has a continuous on $V_1(G)$ $\Delta(G)$-coloring.

Proposition 3. The problem of deciding whether a proper coloring of edges of a bipartite multigraph with the fixed number of edges of each color exists is $NP$-complete.

Proof. Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph with $\Delta(G) = 3$. Set $n_i = |\{x/ x \in V_1(G), d(x) \geq i\}|$, $i = 1, 2, 3$. Clearly, a continuous on $V_1(G)$ $3$-coloring of $G$ exists if and only if there exists a proper coloring of edges of $G$ with the colors $1, 2, 3$, at which by each color $i$ $n_i$ edges are colored, $i = 1, 2, 3$. Therefore, the proposition follows from the theorem.

The Proposition is proved.

Some sufficient conditions for the existence of a proper coloring of edges of a bipartite multigraph with the fixed number of edges colored by each color are found in [9,12].

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Resume (in Armenian)

Let $G = (V_1(G), V_2(G), E(G))$ be a bipartite multigraph, and $R \subseteq V_1(G) \cup V_2(G)$. A proper coloring of edges of $G$ with the colors $1, \ldots, t$ is called interval (respectively, continuous) on $R$, if each color is used for at least one edge and the edges incident with each vertex $x \in R$ are colored by $d(x)$ consecutive colors (respectively, by the colors $1, \ldots, d(x)$), where $d(x)$ is a degree of the vertex $x$. We denote by $w_1(G)$ and $W_1(G)$, respectively, the least and the greatest values of $t$, for which there exists an interval on $V_1(G)$ coloring of the multigraph $G$ with the colors $1, \ldots, t$.

In the paper the following basic results are obtained.

Theorem 2. For an arbitrary $k$, $w_1(G) \leq k \leq W_1(G)$, there is an interval on $V_1(G)$ coloring of the multigraph $G$ with the colors $1, \ldots, k$.

Theorem 3. The problem of recognition of the existence of a continuous on $V_1(G)$ coloring of the multigraph $G$ is $NP$-complete.

Theorem 4. If for any edge $(x, y) \in E(G)$, where $x \in V_1(G)$, the inequality $d(x) \geq d(y)$ holds then there is a continuous on $V_1(G)$ coloring of the multigraph $G$.

Theorem 1. If $G$ has no multiple edges and triangles, and there is an interval on $V(G)$ coloring of the graph $G$ with the colors $1, \ldots, k$, then $k \leq |V(G)| - 1$.  

5
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