$(g - 2)_\mu$ from Noncommutative Geometry

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Abstract

This brief Letter demonstrates that effects from a noncommutative space-time geometry will measurably affect the value of $(g - 2)_\mu$ inferred from the decay of the muon to an electron plus two neutrinos. If the scale of noncommutivity is $O(\text{TeV})$, the alteration of the $V - A$ structure of the lepton-lepton-W vertex is sufficient to shift the inferred value of $(g - 2)_\mu$ to one part in $10^8$. This may account for the recently reported $2.6\sigma$ discrepancy between the BNL measurement $a_{\text{exp}} = 11659202(14)(6) \times 10^{-10}$ and the Standard Model prediction $a_{\text{SM}} = 11659159.6(6.7) \times 10^{-10}$.

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Introduction

The measurement of the anomalous magnetic moment of the muon, \( a_\mu \equiv (g - 2)_\mu \), has undergone continual refinement (for history and experimental details, see [1], [2]) to the point where \( a_\mu \) is now very precisely known [3]:

\[
a_{\text{expt}}^\mu = 11659202(14) \cdot 10^{-10}
\]

The experimental technique employs muons trapped in a storage ring. A uniform magnetic field \( B \) is applied perpendicular to the orbit of the muons; hence the muon spin will precess. The signal is a discrepancy between the observed precession and cyclotron frequencies. Precession of the muon spin is determined indirectly from the decay \( \mu \rightarrow e \bar{\nu}_e \nu_\mu \). Electrons emerge from the decay vertex with a characteristic angular distribution which in the Standard Model (SM) has the following form in the rest frame of the muon:

\[
dP(y, \phi) = n(y)(1 + A(y)\cos(\phi))dyd(\cos(\phi))
\]

where \( \phi \) is the angle between the momentum of the electron \( e \) and the spin of the muon, \( y = 2p_e/m_\mu \) measures the fraction of the maximum available energy which the electron carries, and \( n(y), A(y) \) are particular functions which peak at \( y = 1 \). The detectors (positioned along the perimeter of the ring) accept the passage of only the highest energy electrons in order to maximize the angular asymmetry in (2). In this way, the electron count rate is modulated at the frequency \( a_\mu eB/(2\pi mc) \).

The leading theoretical prediction of \( a_\mu \) in the SM is \( a_{\text{SM}}^\mu = 11659159.6(6.7) \cdot 10^{-10} \) which leads to a 2.6 \( \sigma \) deviation from the data:

\[
a_{\text{expt}}^\mu - a_{\text{SM}}^\mu = 43(16) \cdot 10^{-10}
\]

If this discrepancy persists as more data arrives and theoretical uncertainties improve, then there is a clear signal of new physics. Many proposals to account for this discrepancy have already appeared in the literature.

This letter is a consideration of a novel effect on the measurement of \( a_\mu \) from noncommutative geometry, a theory in which the coordinates of spacetime become noncommuting operators: \( [\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \). There is an extensive collection of papers devoted to both the theoretical foundations of noncommutative geometry [3, 4, 5, 6, 7, 8, 9, 10, 11] and its phenomenology [12, 13, 14, 15]. The reader may consult the above references for a more thorough understanding of the noncommutative quantum field theory underlying the present calculation. We will employ perturbation theory in leading powers of the dimensionful matrix of parameters \( \theta_{\mu\nu} \) in accord with the work done in [15].

Preliminaries

Although \( a_\mu \) does receive a sizable contribution from noncommutative geometry, it is a constant contribution [4], i.e., the interaction with the external magnetic field \( \Delta E \sim B_\mu \theta_{jk}e^{ijk} \) is independent of the muon spin, and therefore the experiment described above is not sensitive to this perturbation of \( a_\mu \).

The effect of noncommutative geometry on this measurement does however enter in the manner in which the muon spin is measured in its decay. Each of the W-boson vertices in the decay diagram Fig.1(a) receives corrections from noncommutative geometry at the one loop level, as

\[ \text{for a partial list, see } [3] \]
Figure 1: (a) The muon decay to an electron plus two neutrinos. Each vertex receives a noncommutative loop correction (b) (with \( l = e, \mu \)) which upsets the electron’s angular distribution.

shown in Fig. 1(b). One might expect such corrections to be negligible, but in fact the loop integral in Fig. 1(b) involves \( \theta \)-dependent vertices which lead to integrals of the form

\[
\int \frac{d^4k}{16\pi^2} e^{ip\cdot q} \prod_{i=1}^{4} \frac{1}{k_i^4}
\]

for loop momenta much larger than the external momenta \( p, q \). In the limit \( |\theta| \to 0 \) the integral formally diverges so one has to renormalize carefully (see [15] for a discussion of this point). The generic size of the noncommutative contribution will be \( \alpha \prod_{i=1}^{4} \frac{1}{k_i^4} \) which for fast muons \( (p_{\mu} \approx 3 \, \text{GeV at BNL}) \) and low scales of noncommutivity \( (|\theta| \approx (1 \, \text{TeV})^{-2}) \) gives a suppression factor of \( O(10^{-8}) \) relative to the tree level decay diagram. Since the current deviation of the SM prediction from experiment in (3) is of this size, we see that noncommutative effects cannot be neglected on the basis of their magnitude.

More importantly, the appearance of the antisymmetric object \( \theta_{\mu\nu} \) in the decay amplitude leads to combinations of the muon and electron spins and momenta which alter the modulation frequency of the decay rate (2). Specifically, one anticipates factors of \( (\vec{p}_e \cdot \vec{p}_\mu)(\vec{p}_e \cdot \theta \cdot \vec{p}_\mu) \) which for electron momenta close to their kinematic limit \( (i.e., y = 1) \) behaves like \( \cos(\phi)\sin(\phi) \). In what follows we explicitly demonstrate these terms exist in the decay rate.

The Calculation

Define the muon decay amplitude

\[
\mathcal{M} = \frac{G_F}{\sqrt{2}} \overline{\nu}_\mu (C_i \mathcal{O}_i^{\alpha}) \nu_1 \overline{\nu}_2 (C_j^{\alpha} \mathcal{O}_j^{\alpha}) u_\mu
\]

involving the electron, muon, and neutrino \((1, 2)\) spinors and the most general set of operators at the interaction vertices, \( \mathcal{O}_i \) \((i \subset \{ S, P, A, V, T \}) \) which may depend on momenta. The muon decay rate is proportional to the squared matrix element

\[
|\mathcal{M}|^2 = \frac{G_F^2}{4} T_e T_\mu
\]

\[
T_e \equiv tr \left( \overline{\nu}_e (C_i \mathcal{O}_i^{\alpha}) \nu_1 \overline{\nu}_1 (C_j^{\alpha} \mathcal{O}_j^{\alpha}) u_e \right)
\]

\[
T_\mu \equiv tr \left( \overline{\nu}_2 (C_k \mathcal{O}_{k,\alpha}) u_\mu \overline{\nu}_2 (C_l^{\beta} \mathcal{O}_{l,\beta}) u_2 \right)
\]

\[^{‡}\text{for an excellent treatment of the corresponding SM calculation, see [13]}\]
This is a product of two terms: the electron trace \( T_e \) and the muon trace \( T_\mu \). If \( \theta \) were zero, all operators would be of the standard \( V - A \) form, and the traces would be:

\[
T_e(SM) = 4 \left( q_1^\alpha p_1^\beta + q_1^\beta p_1^\alpha - (q_1 \cdot p_1) g^{\alpha \beta} + i q_1^\gamma p_\delta e^{\alpha \beta \gamma \delta} \right)
\]

\[
T_\mu(SM) = 4 \left( q_2^\alpha p_2^\beta + q_2^\beta p_2^\alpha - (q_2 \cdot p_2) g^{\alpha \beta} + i q_2^\gamma p_\delta e^{\alpha \beta \gamma \delta} \right) - 4m \left( q_2^\beta s_\mu^\alpha + q_2^\alpha s_\mu^\beta - (q_2 \cdot s_\mu) g^{\alpha \beta} + i q_2^\gamma s_\mu^\delta e^{\alpha \beta \gamma \delta} \right)
\]

where \( m \) is the muon mass and we neglect the mass of the electron in this and all that follows. The lowest order contribution from noncommutative geometry will be proportional to one power of \( \theta \), so to extract it one calculates the contribution to \( |M|^2 \) from each way it is possible to change one \( V - A \) operator into a noncommutative one, giving altogether twenty \( \mathcal{O}(\theta) \) terms in \( |M|^2 \). To find the precise form of these operators, we next calculate the loop. In Fig. 2 we show the loop with incoming charged lepton momentum \( p \) and outgoing neutrino momentum \( q \). The loop amplitude is

\[
\mathcal{M}_{\text{loop}} = \int \frac{d^4 k}{(2\pi)^4} \bar{\psi}_e(q) \left[ -ig\gamma^\gamma(1 - \gamma_5) \right] \left[ \frac{1}{(k-q)^2 - m_e^2} \eta \frac{i}{k-m} \left[ -ie\gamma^\alpha \right] \right] u(p) \times g^\beta g^{[q + k - 2p]d + g^{[q + p - 2k]d} + g^{[k + p - 2q]d}] \exp[i k \cdot \theta \cdot (p - q)]
\]

which becomes

\[
\bar{\psi}_e(q) \gamma^\beta e \int \left( \frac{d^4 k}{(2\pi)^4} \frac{N_1^\eta + N_2^\eta + N_3^\eta}{(k^2 - m_e^2)^2(k-q)^2 - m_e^2} \right) u(p)
\]

\[
N_1^\eta = (\bar{q} + k - 2\bar{p})(1 - \gamma_5)(k + m) \gamma^\eta
\]

\[
N_2^\eta = \gamma^\beta (1 - \gamma_5)(\bar{k} + m) \gamma^\gamma (q + k - 2p)^\eta
\]

\[
N_3^\eta = \gamma^\eta (1 - \gamma_5)(\bar{p} + k - 2\bar{q})
\]

Now using the on-shell condition \( \bar{\psi}_e(p) \bar{p} = m \bar{\psi}_e(p) \) and only retaining terms which couple \( \theta_{\mu \nu} \) to the overall Dirac structure \( \bar{\psi}_e(p) \) we arrive at

\[
N_1^\eta \rightarrow 2m \bar{\eta}(1 + \gamma_5) \gamma^\eta - 2\bar{\eta} p^\rho (1 + \gamma_5)
\]

\[
N_2^\eta \rightarrow m \bar{k} \gamma^\eta (1 + \gamma_5) - 2k^\eta \bar{k}(1 - \gamma_5)
\]

\[
N_3^\eta \rightarrow m \gamma^\eta (1 - \gamma_5) \bar{k}
\]

Of the above terms in the numerator, the dominant one is the tensor piece of \( N_2^\eta \), i.e. the one proportional to \( k^\eta \bar{k} \), since it has the most powers of \( k \). To compute its effect, we consider first the alteration of the electron trace, keeping the \( V - A \) vertices of the muon trace intact. This tensor

\textit{i.e. terms containing} \( k^\eta \) or \( \bar{k} \), since \( \theta \) needs to be contracted with the electron or muon spin.
part of the electron trace $T_e$ is

$$T_e = \text{tr} \left( p_e (1 - \gamma_5 s_e) \gamma^\mu \gamma^\alpha \theta_{\mu\nu} (p_e - q_1)^\rho q_1 \gamma^\beta (1 - \gamma_5) \right)$$

+ \text{tr} \left( p_e (1 - \gamma_5 s_e) \gamma^\alpha (1 - \gamma_5) q_1 \gamma^\mu \gamma^\beta \theta_{\mu\rho} (p_e - q_1)^\rho \right)$$

$$\times \frac{g_2^2}{16\pi^2} \ln \left| m_\mu^2 \theta \right|$$

(11)

which, after some Dirac algebra, dotting into the SM muon trace (7), and integration over the neutrino momenta $q_{1,2}$ (since these are not observed) gives

$$|\mathcal{M}|^2 \supset \frac{G_F^2 g_2^2 e m_\mu^6}{64\pi} \ln \left| m_\mu^2 \theta \right| (s_e \cdot \hat{p}_e)(s_\mu \cdot \hat{\theta} \hat{p}_e)$$

(12)

The other half of the calculation, keeping the electron trace fixed and inserting $\theta$-dependent operators into the muon trace, yields a very similar result. For high electron momenta, the muon neutrino and electron antineutrino momenta are approximately opposite that of the electron, forcing the spin of the electron to match the spin of the muon. In this case the product $(s_e \cdot \hat{p}_e)(s_\mu \cdot \hat{\theta} \hat{p}_e)$ becomes approximately $\cos(\phi)\sin(\phi)$ since $\vec{s}_e \approx -\vec{s}_\mu$ and $\theta_{\mu\nu}$ is antisymmetric. This upsets the $\cos(\phi)$ angular dependence that the SM predicts in (2) potentially at the level of 1 part in $10^8$.

1 Concluding Remarks

It is interesting not only that noncommutative geometry can account for the recent measurement of $a_\mu$ if the scale of noncommutivity is of the order of 1 TeV, but also that a noncommutative spacetime at this energy can account for $\epsilon_K$ and possibly some of the CP violating observables in $B$-meson physics [13]. The caveat however is that $\theta_{\mu\nu}$, being an intrinsically directional object, is subject to being averaged away if experiments collect and average data over time scales of days or longer due to the rotation of the Earth. In a storage ring such as the one at BNL, the circulation of the muons at their cyclotron frequency introduces an additional averaging of the components of $\theta$, so some of the effects of noncommutative geometry are bound to be projected away. Nonetheless, it is hoped that experimenters will look for a time-varying effect in the data for $a_\mu$ which would be a definite positive signal of noncommutative geometry.

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References

[1] F. J. M Farley and E. Picasso, Quantum Electrodynamics, ed. T. Kinoshita. Singapore: World Scientific, 1990

[2] B. Lee Roberts, Int. J. Mod. Phys. A 15S1, 386 (2000)

[3] H. N. Brown, et al., Phys. Rev. Lett 86, 2227 (2001)

[4] A. Czarnecki and W. J. Marciano, Nucl. Phys. (Proc. Suppl.) B 76, 245 (1999)

[5] http://phyppro1.phy.bnl.gov/g2muon/new_theory.html

[6] A. Connes, Noncommutative Geometry. New York: Academic Press, 1994

[7] E. Witten, Nuc.Phys B 268, 253 (1986); N. Seiberg and E. Witten, JHEP 9909:032 (1999)

[8] J. Madore, An introduction to noncommutative differential geometry and its physical applications. New York: Cambridge University Press, 1999

[9] M. Chaichian, A. Demichev, P. Presnajder, Nuc. Phys. B 567, 360 (2000)

[10] S. Minwalla, M. Van Raamsdonk, N. Seiberg, JHEP 0002:020 (2000)

[11] M. Hayakawa, hep-th/9912167

[12] J. J. L. Hewett, F. J. Petriello, T. G. Rizzo, hep-ph/0010354

[13] M. Chaichian, M. M. Sheikh-Jabbari, A. Tureanu, Phys. Rev. Lett 86, 2716 (2001)

[14] I. F. Riad and M. M. Sheikh-Jabbari, JHEP 0008:045 (2000)

[15] I. Hinchliffe and N. Kersting, hep-ph/0104137

[16] E. D. Commins, Weak Interactions, New York: McGraw-Hill, 1973