IRREDUCIBILITY AND GALOIS GROUP OF HECKE POLYNOMIALS

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Abstract. Let $T_{n,k}(X)$ be the characteristic polynomial of the $n$-th Hecke operator acting on the space of cusp forms of weight $k$ for the full modular group. We show that if there exists $n \geq 2$ such that $T_{n,k}(X)$ is irreducible and has the full symmetric group as Galois group, then the same is true of $T_{p,k}(X)$ for all primes $p$.

1. Introduction

Let $k$ be an even integer, $S_k$ be the space of cusp forms of weight $k$ for the full modular group $SL(2, \mathbb{Z})$ and $d_k$ be its dimension. We denote by $T_{n,k}(X)$ the characteristic polynomial of the Hecke operator $T_n$ acting on $S_k$. It is well known that $T_{n,k}(X)$ belongs to $\mathbb{Z}[X]$ and is monic. A conjecture of Maeda asserts that the Hecke algebra of $S_k$ over $\mathbb{Q}$ is simple, and that its Galois closure over $\mathbb{Q}$ has Galois group the full symmetric group. A popular extension of this conjecture, often called Maeda’s Conjecture, states that $T_{p,k}(X)$ is irreducible in $\mathbb{Q}[X]$ and has full Galois group over $\mathbb{Q}$ for every prime $p$. This conjecture has implications for the nonvanishing of $L$-functions attached to modular forms [12], [5], for the work of Maeda and Hida [9] on base changes to totally real number fields for level 1 eigenforms, and for the Inverse Galois Problem [17].

There has been some progress toward Maeda’s Conjecture in the last 20 years with both theoretical and computational approaches. The main contributions take two different directions. The first is to verify the irreducibility of $T_{2,k}(X)$ and compute the Galois group for different weights $k$. For example, in [3] Buzzard did the verifications for $T_{2,k}(X)$ with all weights of the form $k = 12l$ with $l \leq 19$ prime. The furthest verifications up to the author’s knowledge are due to Ghitza and McAndrew [8] for $k \leq 12000$.

The second direction consists in showing irreducibility of $T_{p,k}(X)$ assuming the irreducibility of $T_{q,k}(X)$ for some $q$. One of the first results [6] uses the trace formula in characteristic $p$ to show that if some $T_{q,k}(X)$ is irreducible and has full Galois group, then the same holds for $T_{p,k}(X)$ in a set of primes of density $5/6$. Combining this with a computational approach, Farmer and James [7] show that $T_{p,k}(X)$ is irreducible and has full Galois group for $p \leq 2000$ and $k \leq 2000$. Ahlgren [1] went further in the verifications and showed that if $T_{n,k}(X)$ is irreducible and has full Galois group, then the same

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is true of $T_{p,k}(X)$ for each prime $p \leq 4000000$. Perhaps the most significant progress so far for the irreducible part of the conjecture is due to Baba and Murty’s work [2], who improved the density of the set of primes for which the $T_{p,k}$ are irreducible from $5/6$ to $1$. They study Frobenius distributions and Galois representations of Hecke eigenforms. In this article, we improve the result [2] and settle this second direction of work by showing that:

**Theorem 1.1.** Let $k$ be a positive even integer. If $T_{n,k}(X)$ is irreducible and has full Galois group for some $n$, then the same holds for all $T_{p,k}(X)$ with $p$ prime.

It follows from Theorem 1.1 together with Ghitza and McAndrew’s result [8] that:

**Corollary 1.2.** Let $k$ be a positive even integer $\leq 12000$. For every prime $p$, $T_{p,k}(X)$ is irreducible and has full Galois group.

Hence Corollary 1.2 provides the first examples of spaces $S_k$ that satisfy Maeda’s Conjecture in the version introduced previously.

The proof of Theorem 1.1 consists in showing that part (ii) of Lemma 3.1 never holds. This is done by contradiction when looking at the action of the Hecke operators on a special family of cusp forms denoted by $f_{k,D,d}$ that we define in the next section.

2. Definition and properties of $f_{k,D,d}$

Let $Q_\Delta$ be the set of all quadratic forms $Q(x,y) = ax^2 + bxy + cy^2$ with integer coefficients and discriminant $b^2 - 4ac = \Delta$. Classical genus theory associates to each discriminant $\Delta$ and fundamental discriminant divisor $d$ of $\Delta$ an $\text{SL}(2, \mathbb{Z})$-invariant character $\chi_d : Q_\Delta \to \{\pm 1, 0\}$ by setting, for $Q = [a, b, c] \in Q_\Delta$,

$$
\chi_d(Q) = \begin{cases} 
\left( \frac{d}{r} \right) & \text{if } (a, b, c, d) = 1, \\
0 & \text{otherwise},
\end{cases}
$$

where in the first case $r$ is an integer prime to $d$ represented by $Q$ and $\left( \cdot \mid \cdot \right)$ is the Kronecker symbol. Such an $r$ always exists and the value of $\left( \frac{d}{r} \right)$ is independent of the choice of $r$. The genus character $\chi_d$ is also invariant under the Fricke involution:

$$
\chi_d([a, b, c]) = \chi_d([c, -b, a]).
$$

Throughout this section we let $k \geq 2$ be an integer, $D$ be any discriminant and $d$ be a fundamental discriminant such that

$$
(1) \quad \text{sign } d = \text{sign } D = (-1)^k.
$$
We consider the functions defined on the upper complex half plane $\mathcal{H}$ by

$$f_{k,D,d}(z) := C_k(Dd)^{k-\frac{1}{2}} \sum_{a,b,c \in \mathbb{Z}, \atop b^2 - 4ac = Dd} \frac{\chi_d([a,b,c])}{(az^2 + bz + c)^k}$$

where $C_k$ is an unimportant normalising factor depending only on $k$. The functions above were introduced by Kohnen in [11] generalising Zagier’s well-known functions $f_{k,\Delta}$ introduced in [19] in connection with the Doi-Naganuma lift for Hilbert modular forms and defined only for positive discriminants $\Delta$ and even integers $k \geq 2$ by:

$$f_{k,\Delta}(z) := C_k\Delta^{k-\frac{1}{2}} \sum_{a,b,c \in \mathbb{Z}, \atop b^2 - 4ac = \Delta} \frac{1}{(az^2 + bz + c)^k} \quad (z \in \mathcal{H}).$$

The functions $f_{k,\Delta}$ and $f_{k,D,d}$ are cusp forms of weight $2k$ for $\text{SL}(2,\mathbb{Z})$ and play an important role in the theory of modular forms of half-integral weight of level 1 (cf. [12]) and higher level respectively (cf. [11]).

**Theorem 2.1.** The function

$$\Omega_{k,z,d}(\tau) = \sum_{D > 0, \atop D \equiv 0,1 \quad (\text{mod} \ 4)} f_{k,D,d}(z) e^{2\pi i |D| \tau} \quad (z, \tau \in \mathcal{H})$$

is for each fixed $z$ a cusp form of weight $k + \frac{1}{2}$ on $\Gamma_0(4)$ with respect to $\tau$.

This was proved by Kohnen in the more general context of modular forms of half-integral weight with higher level (cf. [11], Th.1). In the case where $d = 1$, it is also an immediate consequence of a result of Vignéras [16] on theta series associated to indefinite quadratic forms.

Below we give a closed formula for the action of the Hecke operators $T_p,2k$ on the functions $f_{k,D,d}$. The proof of Proposition 2.2 follows the proof of the analogous statement for non-holomorphic modular forms of weight zero given in [20] (proof of equation 36). Such a formula is given when there is no genus character by Parson in [15].

**Proposition 2.2.** For each prime $p$ not dividing $d$, the action of the $p$-th Hecke operator on $f_{k,D,d}$ can be described in the following closed form

$$T_{p,2k}f_{k,D,d} = f_{k,Dp^2,d} + \left(\frac{D}{p}\right) p^{k-1}f_{k,D,d} + p^{2k-1}f_{k,D,p^2,d}$$

(with the convention $f_{k,D,p^2,d} = 0$ if $p^2 \nmid D$).

**Proof.** We have that
\[ T_{p,2k}(f_{k,D,d})(z) = p^{2k-1}f_{k,D,d}(pz) + \frac{1}{p} \sum_{j=1}^{p} f_{k,D,d}\left(\frac{z+j}{p}\right) \]

\[ = C_k(Dd)^{k-\frac{1}{2}} \sum_{b^2-4ac=DaD} \left( \frac{p^{2k-1} \chi_d([a,b,c])}{(apz)^2 + bpz + c)^k} + \frac{p^{2k-1} \chi_d([a,b,c])}{(az + j)^2 + b(z + j)p + cp^2)^k} \right). \]

Set \( r = [a, b, c](p) = ap^2 + bp + c = [ap^2, bp, c](1) \). Then \( r \) is represented by both \([a, b, c]\) and \([ap^2, bp, c]\), and so

\[ \chi_d([a, b, c]) = \left( \frac{d}{p} \right) = \chi_d([ap^2, bp, c]) \]

(since \( p \nmid d \)). Since \( \chi_d \) is invariant under translations and the Fricke involution, we also have that

\[ \chi_d([a, b, c]) = \chi_d([a, bp, cp^2]) = \chi_d([a, bp + 2aj, cp^2 + bpj + aj^2]). \]

Thus

\[ T_{p,2k}(f_{k,D,d})(z) = C_k(Dd)^{k-\frac{1}{2}} \sum_{b^2-4ac=DaD} \frac{n(a, b, c)}{(az^2 + bz + c)^k}, \]

where

\[ n(a, b, c) = \chi_d([a, b, c]) \left( \epsilon\left(\frac{a}{p^2}, \frac{b}{p}, c\right) + \sum_{j=1}^{p} \epsilon\left(\frac{a}{p}, \frac{b - 2aj}{p}, \frac{c - bj + aj^2}{p^2}\right) \right) \]

and \( \epsilon(a, b, c) \) equals 1 if \( a, b, c \) are integral, 0 otherwise. By [20, p.201],

\[ n(a, b, c) = \chi_d([a, b, c]) \left( 1 + \left(\frac{Dd}{p}\right) \epsilon\left(\frac{a}{p}, \frac{b}{p}, \frac{c}{p}\right) + p \epsilon\left(\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right) \right). \]

If \( p \) divides \( a, b, c \), then

\[ \chi_d([a, b, c]) = \left( \frac{d}{p} \right) \chi_d\left(\left[\frac{a}{p^2}, \frac{b}{p}, \frac{c}{p}\right]\right), \]

and, if \( p^2 \) divides \( a, b, c \), then

\[ \chi_d([a, b, c]) = \chi_d\left(\left[\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right]\right). \]

Hence we have

\[ \sum_{b^2-4ac=DaD} \epsilon\left(\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right) \chi_d([a, b, c]) = \sum_{b^2-4ac=DaD} \left( \frac{d}{p} \right) \frac{\chi_d([a, b, c])}{p^k(az^2 + bz + c)^k} \]

and

\[ \sum_{b^2-4ac=DaD} \epsilon\left(\frac{a}{p^2}, \frac{b}{p^2}, \frac{c}{p^2}\right) \chi_d([a, b, c]) = \sum_{b^2-4ac=DaD} \frac{\chi_d([a, b, c])}{p^{2k}(az^2 + bz + c)^k}. \]
We recall that the periods $r_n(f)$ of a cusp form $f$ of weight $2k$ are defined by
\[ r_n(f) = \int_0^{i\infty} f(z)z^n dz \quad (0 \leq n \leq 2k-2). \]
These are collected in the period polynomial $r(f) \in \mathbb{C}[X]_{2k-2}$ defined by
\[ r(f)(X) = \int_0^{\infty} f(z)(z - X)^{2k-2} dz = \sum_{n=0}^{2k-2} (-1)^n \binom{2k-2}{n} r_n(f) X^{2k-2-n}. \]
The polynomial above splits into an odd and an even part respectively defined by
\[ r^-(f)(X) = - \sum_{0 < n < 2k-2 \atop n \text{ odd}} \binom{2k-2}{n} r_n(f) X^{2k-2-n} \]
and
\[ r^+(f)(X) = \sum_{0 \leq n < 2k-2 \atop n \text{ even}} \binom{2k-2}{n} r_n(f) X^{2k-2-n}. \]
The periods of $f_{k,\Delta}$ were calculated in [13] and were shown to be rational. In the proposition below we give explicit formulas for the periods of the function $f_{k,D,d}$. When $d = 1$ and $k$ is even, the proposition is covered by Theorem 4 in [13].

Let $\Delta$ be a positive fundamental discriminant. Denote by $\zeta(s)$ the Riemann zeta function as usual and by $\zeta_{\Delta}$ the Dedekind zeta-function of the quadratic field of discriminant $\Delta$ defined for $\Re(s) > 1$ by
\[ \zeta_{\Delta}(s) = \sum_{a \in \mathcal{O}_{\Delta}} \frac{1}{N(a)^s} \]
(where $a$ ranges over non-zero ideals of $\mathcal{O}_\Delta$) and by meromorphic continuation for other $s \in \mathbb{C}$. Let $\zeta_{\mathcal{A}}$ be the partial zeta-function restricted to the class $\mathcal{A}$. The function $\zeta_{\mathcal{A}}(s)$ satisfies the functional equation (see e.g. [14], chapter VII, §5)
\[ \Gamma \left( \frac{1-s}{2} \right)^2 \zeta_{\mathcal{A}-1}(1-s) = \Delta^{s-1/2} \pi^{1-2s} \Gamma \left( \frac{s}{2} \right)^2 \zeta_{\mathcal{A}}(s), \]
where $\mathfrak{d} = \sqrt{\Delta} \mathcal{O}_\Delta$ is the different of $\mathbb{Q}(\sqrt{\Delta})$.

For $m$ and $N$ positive integers, we denote by $H(m,N)$ Cohen’s numbers defined in [1]. We recall that if $N$ is a fundamental discriminant, then
\[ H(m,N) = L(1-m, \chi_N), \]
where $L(s, \chi_N)$ is the Dirichlet $L$-series of the Kronecker symbol $\chi_N(\cdot) = \left( \frac{N}{\cdot} \right)$. 
Finally we define
\[
P_{k,D,d}(X) := \sum_{\substack{b^2 - 4ac = Dd \ a < 0 < c}} \chi_d([a, b, c]) (aX^2 + bX + c)^{k-1}.
\]

**Proposition 2.3.** Let \(D, d\) be two discriminants satisfying (1) and such that \(d\) and \(Dd\) are fundamental. We have
\[
r^+(f_{k,D,d})(X) = P_{k,D,d}(X) + \frac{H(k, |D|)H(k, |d|)}{2\zeta(1 - 2k)} (X^{2k-2} - 1).
\]

**Proof.** For each class \(\mathcal{A}\) of binary quadratic forms of discriminant \(\Delta\), define
\[
f_{k,\Delta,A}(z) = C_k \Delta^\frac{k-1}{2} \sum_{[a,b,c] \in A} \frac{1}{(az^2 + bz + c)^k}
\]
and
\[
\tilde{f}_{k,\Delta,A} = f_{k,\Delta,A} + f_{k,\Delta,A'}
\]
where \(\mathcal{A}' = \{[a, -b, c] : [a, b, c] \in \mathcal{A}\}\).

Similarly, define
\[
P_{k,\Delta,A}(X) = \sum_{\substack{[a,b,c] \in A \ a < 0 < c}} (aX^2 + bX + c)^{k-1}
\]
and denote by \(P_{k,\Delta,A}^+\) the even part. Then it follows from [13] (Theorem 5 and the formula in line 15 of the proof) that
\[
r^+(\tilde{f}_{k,\Delta,A})(X) = P_{k,\Delta,A}^+(X) + (-1)^k P_{k,\Delta,A^*}^+(X) + \frac{\zeta(1 - k)}{\zeta(1 - 2k)} (X^{2k-2} - 1)
\]
where \(\mathcal{A}^* = \{-a, -b, -c : [a, b, c] \in \mathcal{A}\}\).

Now, since \(\chi_d(\mathcal{A}) = \chi_d(\mathcal{A}')\), we have that
\[
f_{k,D,d} = \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) f_{k,Dd,\mathcal{A}} = \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) f_{k,Dd,\mathcal{A}'}
\]
and
\[
\tilde{f}_{k,Dd} = \frac{1}{2} \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) \tilde{f}_{k,Dd,\mathcal{A}}
\]
Therefore,
\[
r^+(f_{k,D,d})(X) = \frac{1}{2} \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) r^+(\tilde{f}_{k,Dd,\mathcal{A}})(X).
\]

Since \(\chi(\mathcal{A}) = (-1)^k \chi(\mathcal{A}^*)\) and \(P_{k,D,d}(X)\) is even, we have
\[
P_{k,D,d}(X) = \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) P_{k,Dd,\mathcal{A}}^+(X) = (-1)^k \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) P_{k,Dd,\mathcal{A}^*}^+(X).
\]
Hence,
\[
\frac{1}{2} \sum_{\text{disc } \mathcal{A} = Dd} \chi_d(\mathcal{A}) \left( P_{k,Dd,\mathcal{A}}^+(X) + (-1)^k P_{k,Dd,\mathcal{A}^*}^+(X) \right) = P_{k,D,d}(X).
\]
By (6) and (8) we have that
\[ r^+ (f_{k,D,d})(X) = P_{k,D,d}(X) + \sum_{\text{disc } A = Dd} \chi_d(A) \frac{\zeta_A(1-k)}{2}\zeta(1-2k)\left(X^{2k-2} - 1\right). \]

The following lemma finishes the proof of Proposition 2.3.

**Lemma 2.4.** For any integer \( k \geq 2 \),
\[ \sum_{\text{disc } A = Dd} \chi_d(A)(1-k) = H(k,|D|)H(k,|d|). \]

**Proof.** If \( k \) is odd, then both sides in the identity of the lemma vanish. Hence we can suppose that \( k \) is even if convenient. Wong introduces in [18] the real function
\[ F_{k,D,d}(x) := \sum_{a,b,c \in \mathbb{Z}, \sqrt{b^2 - 4ac} = Dd} \chi_d([a,b,c])(ax^2 + bx + c)^{k-1} \quad (x \in \mathbb{R}) \]
defined for any integer \( k \geq 2 \), generalising Zagier’s \( F_{k,\Delta} \) introduced in [21], defined only when \( k \) is even and \( \Delta \) a positive discriminant:
\[ F_{k,\Delta}(x) := \sum_{a,b,c \in \mathbb{Z}, \sqrt{b^2 - 4ac} = \Delta} (ax^2 + bx + c)^{k-1} \quad (x \in \mathbb{R}). \]

One the one side, Wong generalises Zagier’s calculation of the average value of the \( f_{k,\Delta} \) by showing that
\[ \int_0^1 F_{k,D,d}(x) \, dx = \frac{H(k,|D|)H(k,|d|)}{2\zeta(1-2k)}. \]

On the other side, we have
\[ \int_0^1 F_{k,D,d}(x) \, dx = \sum_{Q = [a,b,c] \in Q_{D,d}/\Gamma_{\infty}} \chi_d(Q) \beta_k(Q), \]
where \( \beta_k(Q) := \int_{-\infty}^{\infty} \max(0,Q(x))^{k-1} \, dx \) and \( \Gamma_{\infty} \) is the group of translations. Now
\[ r.h.s. \text{ of (10)} = \sum_{\text{disc } A = Dd} \chi_d(A) \sum_{Q = [a,b,c] \in A/\Gamma_{\infty}} \beta_k(Q). \]

Making the substitution \( x = \frac{-b-\sqrt{b^2 - 4ac}}{2a} \), one finds that
\[ \beta_k(Q) = c_k(Dd)^{k-\frac{1}{2}}|a|^{-k} \quad \text{with} \quad c_k = \frac{1}{2^{2k-1}} \frac{\Gamma(k)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+\frac{1}{2})}. \]

Hence
\[ r.h.s. \text{ of (10)} = c_k(Dd)^{k-\frac{1}{2}} \sum_{\text{disc } A = Dd} \chi_d(A) \sum_{Q = [a,b,c] \in A/\Gamma_{\infty}} |a|^{-k}. \]
Denote by $N_A(n)$ the number of integers $b$ modulo $2n$ such that $b^2 \equiv Dd \pmod{4n}$ and $[-n, b, (Dd - b^2)/4n] \in A$. Then the number of $Q \in A/\Gamma_\infty$ with first coefficient $a = -n < 0$ is precisely $N_A(n)$, so

$$r.h.s. \text{ of (10)} = c_k(Dd)^{k - \frac{1}{2}} \sum_{\text{disc } A = Dd} \chi_d(A) \sum_{n=1}^\infty \frac{N_A(n)}{n^k}.$$ 

Now, we can rewrite $\zeta_A(s)$ as (see [21], section 8 for details):

$$\zeta_A(s) = \zeta(2s) \sum_{n=1}^\infty \frac{N_A(n)}{n^s},$$
so

$$r.h.s. \text{ of (10)} = c_k(Dd)^{k - \frac{1}{2}} \sum_{\text{disc } A = Dd} \chi_d(A) \frac{\zeta_A(k)}{\zeta(2k)}.$$

Suppose $k$ is even. From the functional equation (4) and the functional equation for $\zeta(s)$ we have the identity:

$$\frac{\zeta_{2k}^{-1}(1 - k)}{\zeta(1 - 2k)} = 2c_k(Dd)^{k - \frac{1}{2}} \frac{\zeta_A(k)}{\zeta(2k)}.$$

Therefore

$$r.h.s. \text{ of (10)} = \sum_{\text{disc } A = Dd} \chi_d(A) \frac{\zeta_{2k}^{-1}(1 - k)}{2\zeta(1 - 2k)}$$
$$= \sum_{\text{disc } A = Dd} \chi_d(A) \frac{\zeta_A(1 - k)}{2\zeta(1 - 2k)},$$

where in the last equality we used

$$\chi_d(0) = 1 \quad \text{and} \quad \chi_d(A^{-1}) = \chi_d(A).$$

Finally the equality (9) gives

$$\frac{H(k, |D|)H(k, |d|)}{2\zeta(1 - 2k)} = \sum_{\text{disc } A = Dd} \chi_d(A) \frac{\zeta_A(1 - k)}{2\zeta(1 - 2k)}.$$

\[\square\]

3. Proof of Theorem 1.1

Let $K_{2k}$ be the field generated by the Fourier coefficients of the Hecke basis of $S_{2k}$. The following Lemma due to Conrey, Farmer and Wallace [6] (see Proposition 2) is a consequence of the group $Gal(K_{2k}/\mathbb{Q})$ acting on both the individual coefficients and the Hecke basis.

**Lemma 3.1.** Let $k$ be a positive integer. Suppose $T_{n,2k}(X)$ is irreducible and has full Galois group for some $n$. Then for each $m$ either

(i) $T_{n,2k}(X)$ is irreducible and has full Galois group, or,

(ii) $T_{n,2k}(X) = (X - \lambda)^{d_{2k}}$ for some $\lambda \in \mathbb{Z}$. 


Let $k$ be a positive integer $\geq 2$ and $p$ a prime. Suppose that $T_{n,2k}(X)$ is irreducible and has full Galois group for some $n$. Then, by Lemma 3.1, either $T_{p,2k}(X)$ is irreducible and has full Galois group, either $T_{p,2k}(X) = (X - \lambda_{p,k})^{d_{2k}}$ for some $\lambda_{p,k} \in \mathbb{Z}$ depending on $p$ and $k$. Suppose that we are in this second case. Every element in $S_{2k}$ can be written in the Hecke basis, where the operator $T_{p,2k}$ acts linearly, so every element in $S_{2k}$ is an eigenvector with eigenvalue $\lambda_{p,k}$ for the operator $T_{p,2k}$. In particular, so it is the function $f_{k,D,d}$. Therefore

$$T_{p,2k} f_{k,D,d} = \lambda_{p,k} f_{k,D,d}.$$  

Throughout we choose $D$ and $d$ both fundamental satisfying (11) such that $Dd$ is also fundamental and neither $p \mid d$ nor $p^2 \mid D$. It follows from (3) and (11) that

$$f_{k,Dp^2,d} = \left( \frac{\lambda_{p,k} - (D/p)^{k-1}}{p^{k-1}} \right) f_{k,D,d}.$$  

Now, Shimura correspondence between forms of integral and half-integral weight for level 1 gives the following result due to Kohnen [10]:

**Theorem 3.2.** There is an isomorphism as modules over the Hecke algebra between the space $S_{2k}$ and the space $S_{k+1/2}$ of cusp forms of weight $k + 1/2$ on $\Gamma_0(4)$ having a Fourier development of the form

$$g(\tau) = \sum c(n)e^{2\pi i n\tau}, \quad c(n) = 0 \text{ unless } (-1)^k n \equiv 0, 1 \pmod{4}.$$  

If $f(z) = \sum a(n)e^{2\pi inz} \in S_{2k}$ is a normalised eigenform and $g$ as in (13) the corresponding form of half-integral weight, then the Fourier coefficients of $f$ and $g$ are related by

$$c(n^2|M|) = c(|M|) \sum_{m|n} \mu(m) \left( \frac{M}{m} \right) m^{k-1} a \left( \frac{n}{m} \right),$$  

where $M$ is an arbitrary fundamental discriminant with $(-1)^k M > 0$ and $\mu(m)$ is the Möbius function.

Let us apply Theorem 3.2 to the function $\Omega_{k,z,d}(\tau)$ defined by (2). We obtain for each fixed $z \in \mathcal{H}$,

$$f_{k,Dp^2,d}(z) = f_{k,D,d}(z) \left( a(p) - \left( \frac{D}{p} \right)^{p^{k-1}} \right),$$  

where $a(p)$ is the $p$-th Fourier coefficient of the normalised eigenform in $S_{2k}$ corresponding to $\Omega_{k,z,d}(\tau)$, and a priori can depend on $p, k, d$ and $z$. It follows from (12) and (15) that

$$\lambda_{p,k} = a(p).$$  

In particular $a(p)$ depends only on $p$ and $k$. Hence, if $d'$ is another fundamental discriminant satisfying (11) and such that $p \nmid d'$, the $p$-th Fourier coefficient
of the normalised eigenform in $S_{2k}$ corresponding to $\Omega_{k,z,d}(\tau) - \Omega_{k,z,d'}(\tau)$ is zero. By Theorem 3.2 we obtain

$$ f_{k,Dp^2,d'} - f_{k,Dp^2,d'} = -(f_{k,D,d} - f_{k,D,d'}) \left( \frac{D}{p} \right) p^{k-1}. \quad (17) $$

It follows from (12) and (17) that

$$ (f_{k,D,d} - f_{k,D,d'}) \left( \chi_p - \left( \frac{D}{p} \right) p^{k-1} \right) = -(f_{k,D,d} - f_{k,D,d'}) \left( \frac{D}{p} \right) p^{k-1}, $$

so

$$ f_{k,D,d,d}(z) = f_{k,D,d'}(z). $$

In particular we have that

$$ r^+(f_{k,D,d})(X) = r^+(f_{k,D,d'})(X). \quad (18) $$

Suppose $k$ is odd. The coefficient of the constant term in $r^+(f_{k,D,d})(X)$ is equal to

$$ C_{k,D,d} = \sum_{\substack{b^2 - 4ac = Dd \atop a < 0 < c}} \chi_d([a, b, c]) c^{k-1}. \quad (19) $$

We set $D = -3$ and

$$ d, d' = \begin{cases} 
-4, -7 & \text{if } p \neq 2, 7, \\
-7, -11 & \text{if } p = 2, \\
-4, -11 & \text{if } p = 7.
\end{cases} $$

There is one single class of forms of discriminant 12, 21 or 33, so the characters $\chi_d$, $\chi_d'$ are trivial for all the possible values of $D, d, d'$ above.

There are exactly four forms $[a, b, c]$ with discriminant 12 and $a < 0 < c$: $[1, \pm 2, 2]$ and $[2, \pm 2, 1]$. With discriminant 21 there are six forms $[-1, \pm 1, 5], [-3, \pm 3, 1]$ and $[-5, \pm 1, 1]$. With discriminant 33 there are 14 forms: $[-1, \pm 1, 8], [-2, \pm 1, 4], [-2, \pm 3, 3], [-3, \pm 3, 2], [-4, \pm 1, 2], [-6, \pm 3, 1], [-8, \pm 1, 1]$. Clearly:

$$ C_{k,-3,-4} < C_{k,-3,-7} < C_{k,-3,-11}, $$

which contradicts (18).

Suppose $k$ is even. We consider $D = 5$, $d = 1$, $d' = 13$ if $p \neq 13$ and $d' = 8$ if $p = 13$. The coefficient of the $X^2$ term in $r^+(f_{k,D,d})(X)$ is equal to

$$ B_{k,D,d} = (k - 1) \sum_{\substack{b^2 - 4ac = Dd \atop a < 0 < c}} \chi_d([a, b, c]) c^{k-3} \left( ac + \frac{k - 2}{2} b^2 \right). \quad (20) $$

There are exactly two forms $[a, b, c]$ with discriminant 5 and $a < 0 < c$: $[-1, \pm 1, 1]$. With discriminant 65 there are exactly 22 forms: $[-1, \pm 1, 16], [-2, \pm 1, 8], [-2, \pm 3, 7], [-4, \pm 1, 4], [-4, \pm 7, 1], [-5, \pm 5, 2], [-7, \pm 3, 2], [-8, \pm 1, 2], [-10, \pm 5, 1], [-14, \pm 3, 1], [-16, \pm 1, 1]$. With discriminant 40 there are 12: $[-1, \pm 2, 9], [-2, \pm 4, 3], [-3, \pm 2, 3], [-3, \pm 4, 2], [-6, \pm 4, 1], [-9, \pm 2, 1]$. 
We have that
\[ \frac{B_{k,5,1}}{k-1} = -4 + k, \]
\[ \frac{B_{k,5,13}}{k-1} = \begin{cases} 
-50 & \text{if } k = 2, \\
\leq -190 & \text{if } 4 \leq k \leq 34, \\
> 12^k & \text{if } k \geq 35
\end{cases} \]
and
\[ \frac{B_{k,5,8}}{k-1} = \begin{cases} 
\leq -22 & \text{if } 2 \leq k \leq 6, \\
> 2^k & \text{if } k \geq 7.
\end{cases} \]
Hence \( B_{k,5,1} \neq B_{k,5,13} \) and \( B_{k,5,1} \neq B_{k,5,8} \), which contradicts (18).

In both cases we end up with a contradiction when we suppose that \( T_{p,2k}(X) \) satisfies (ii) of Lemma 3.1. Therefore, \( T_{p,2k}(X) \) is irreducible and has full Galois group.

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