THE $p$-TORSION SUBGROUP SCHEME OF AN ELLIPTIC CURVE

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ABSTRACT. Let $k$ be a field of positive characteristic $p$.

Question: Does every twisted form of $\mu_p$ over $k$ occur as subgroup scheme of an elliptic curve over $k$?

We show that this is true for most finite fields, for local fields and for fields of characteristic $p \leq 11$. However, it is false in general for fields of characteristic $p \geq 13$, which is related to the fact that the Igusa curves are not rational in these characteristics.

Moreover, we also analyse twisted forms of $p$-torsion subgroup schemes of ordinary elliptic curves and the analogous problems for supersingular curves.

INTRODUCTION

Let $k$ be a field of positive characteristic $p$ and let $E$ be an ordinary elliptic curve over $k$. Then $\ker(F)$, the kernel of the Frobenius morphism $F : E \to E^{(p)}$, is an infinitesimal group scheme of length $p$ over $k$, which is a twisted form of $\mu_p$. As usual, $\mu_p$ denotes the subgroup scheme of $p$th roots of unity of $\mathbb{G}_m$.

Since twisted forms of $\mu_p$ and elliptic curves are such fundamental objects in algebraic geometry it is natural to ask

Question (A). Given a field $k$ of positive characteristic $p$, does every twisted form of $\mu_p$ over $k$ occur as subgroup scheme of an elliptic curve over $k$?

We will prove that this question has a positive answer in the following cases:

Theorem. If a field of positive characteristic $p$ is

(i) a finite field with $p^n$ elements such that $p \leq 17$ or $n \geq 2$, or
(ii) a field of the form $k((t))$, or
(iii) of characteristic $p \leq 11$,

then Question (A) has a positive answer for this field.

Conversely, we will prove that these results are rather sharp:

Theorem. If a field of positive characteristic $p$ is

(i) a prime field $\mathbb{F}_p$ with $p \geq 19$, or
(ii) of the form $k((t))$ with $p \geq 13$

then Question (A) has a negative answer for this field.

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We will see that we cannot improve our results by also allowing twisted forms of \(\mu_p\) on nonproper or singular curves carrying group structures.

Although we will prove a Hasse principle for twisted forms of \(\mu_p\) over global fields, we will show that there is no Hasse principle for realising twisted forms of \(\mu_p\) as subgroup schemes of elliptic curves.

Moreover, the \(p\)-torsion subgroup scheme \(E[p]\) of an elliptic curve sits inside an extension

\[
\begin{align*}
0 & \to \ker(F) \to E[p] \to \ker(V) \to 0 
\end{align*}
\]

(here, \(V : E(p) \to E\) denotes Verschiebung), and is endowed with a bilinear, alternating and nondegenerate pairing, the Weil pairing. Due to this pairing, kernel and cokernel of (1) become Cartier dual group schemes. Hence we may ask

**Question (B).** Given a field \(k\) of positive characteristic \(p\), does every twisted form of \(\mu_p \oplus (\mathbb{Z}/p\mathbb{Z})\) over \(k\) that is endowed with a bilinear, alternating and nondegenerate pairing occur as subgroup scheme of an elliptic curve over \(k\)?

For perfect fields, the sequence (1) is split from which it follows that Question (A) and Question (B) are equivalent for these fields. Also, if already Question (A) has a negative answer for a field, Question (B) cannot have a positive answer. In view of our positive results we can only hope for local fields and fields of characteristic \(p \leq 11\) to have a positive answer to Question (B):

**Theorem.** If a field of positive characteristic \(p\) is

(i) a field of the form \(k((t))\) where \(k\) is perfect, or
(ii) if \(p \leq 11\),

then Question (B) has a positive answer for this field.

Finally, we answer the analogous questions for supersingular curves: if \(E\) is supersingular then \(\ker(F)\) is a twisted form of \(\alpha_p\), and \(E[p]\) is a twisted form of \(M_2\). We refer to [O, Section II.15.5] for the definition of this latter group scheme, which is the unique non-split extension of \(\alpha_p\) by itself that is autodual. On the other hand, we will see that neither \(\alpha_p\) nor \(M_2\) possess twisted forms over fields, whence \(\ker(F) \cong \alpha_p\) and \(E[p] \cong M_2\) for every supersingular elliptic curve.

This article is organised as follows:

In Section 1 we recall a couple of facts from [K–M] and [L–S] about twisted forms of \(\mu_p\), Hasse invariants and \(p\)-torsion subgroup schemes of ordinary elliptic curves.

In Section 2 we use Weil’s results on elliptic curves over finite fields and Honda–Tate theory [Ta] to answer Questions (A) and (B) for finite fields.

In Section 3 we use universal formal deformations of supersingular elliptic curves and a result of Igusa on the Hasse invariants of such deformations to get a positive answer to Question (A) for all fields of the form \(k((t))\). In particular, this gives a positive answer to Question (A) for local fields. Finally, we prove a Hasse principle for twisted forms over global fields using class field theory.
In Section 4 we first answer Question (A) positively for all fields of characteristic $p \leq 11$ using explicit computations with Hasse invariants. Then we prove that Question (A) has a negative answer for the function field of $\mathbb{P}^1$ in characteristic $p \geq 13$ and for function fields of elliptic curves in characteristic $p \geq 17$. This result is closely related to Igusa’s result [Ig] that the Igusa curves $Ig(p)$ are not rational for $p \geq 13$ and of general type for $p \geq 17$.

In Section 5 we show that Question (B) also has a positive answer for local fields and for fields of characteristic $p \leq 11$.

Finally, in Section 6 we discuss kernel of Frobenius and the $p$-torsion subgroup scheme of supersingular elliptic curves over fields.

1. Generalities

In this section we recall a couple of facts concerning the kernel of Frobenius, Hasse invariants and $p$-torsion subgroup schemes of ordinary elliptic curves. For generalities about torsion subgroup schemes of elliptic curves we refer to [K–M, Chapter 8.7]. For supersingular elliptic curves, see Section 6 below.

Let $S$ be a base scheme of characteristic $p > 0$ and $E$ be an elliptic curve over $S$. Then $\ker(F)$, the kernel of Frobenius $F : E \to E^{(p)}$, is a finite and flat group scheme of length $p$ over $S$. The $p$-Lie algebra of $\ker(F)$ is a projective $O_S$-module of rank 1 and thus coincides with the $p$-Lie algebra of $E$.

We will now assume that $S = \text{Spec } T$ is affine with $\text{Pic}(T) = 0$, e.g., $T$ could be a field, a local ring or a polynomial ring over a field. In this case a sheaf of projective $O_S$-modules of rank 1 admits a global basis $t \in O_S$ identifying this module with $T$. Moreover, giving $S$ the structure of a sheaf of $p$-Lie algebras is equivalent to prescribing the action of the $p$-operator on this basis, i.e., $t[p] = \lambda t$ for some $\lambda \in T$. Note however, that we have chosen a basis for $O_S$, which makes $\lambda$ only unique up to multiplication by elements of $T \times (p-1)$. Under our assumptions on $S$, it follows from the Tate–Oort classification [T–O] of group schemes of prime order that the $p$-Lie algebra of $\ker(F)$ determines $\ker(F)$ uniquely.

We will now assume that $E$ is an ordinary elliptic curve over $S$, which is equivalent to $\lambda \in T^\times$ in our setup. Then $\ker(F)$ is a twisted form of $\mu_p$ over $S$. Since the automorphism group scheme $\text{Aut}(\mu_p)$ of $\mu_p$ is isomorphic to $\mu_{p-1}$, twisted forms of $\mu_p$ are classified by $H^1_{\text{ét}}(S, \mu_{p-1})$. Using the Kummer sequence $0 \to \mu_{p-1} \to \mathbb{G}_m \to \mathbb{G}_m \to 0$ and our assumption $\text{Pic}(T) = 0$ we deduce that the coboundary map in cohomology induces an isomorphism of groups

$$T^\times / T^\times (p-1) \to H^1_{\text{ét}}(S, \mu_{p-1}).$$

Here we see directly that the $p$-Lie algebra determines the twisted form of $\mu_p$ uniquely. See also [L–S, Section 1] for the preceding discussion.

We recall from [K–M, Section 12.4] that the Hasse invariant of an elliptic curve is defined to be the linear mapping induced by the Verschiebung $V$ on $p$-Lie algebras: $h(E) = \text{Lie}(V) : \text{Lie}(E^{(p)}) \to \text{Lie}(E)$. Using the identification $\text{Lie}(E^{(p)}) = \text{Lie}(E) \otimes_p$, we may regard the Hasse invariant as an element
of the projective $\mathcal{O}_S$-module $\text{Lie}(E)^{(1-p)}$, which is of rank one. Choosing a basis $u \in \text{Lie}(E)$, we can identify $h(E)$ with $\lambda u^{(1-p)}$ for some $\lambda' \in T$, which is unique up to $(p-1)$-st powers. As carried out in [L–S, Section 3] the Hasse invariant determines the $p$-Lie algebra of $E$ up to isomorphism and conversely. More precisely, we have $\lambda' = \lambda^{-1}$ by [L–S, Proposition 3.2]. In particular, Question (A) of the introduction is equivalent to

**Question (A′).** Given a field $k$ of positive characteristic $p$, does every element of $k^\times /k^\times (p-1) \cong H^1_{\text{ét}}(\text{Spec } k, \mu_{p-1})$ occur as Hasse invariant of an ordinary elliptic curve over $k$?

To describe the $p$-torsion subgroup scheme $E[p]$ of an ordinary elliptic curve $E$ over $S$, let us recall the setup of [L–S, Section 3]: this group scheme is a twisted form of $G := \mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$. On $E[p]$ there exists a a bilinear, alternating and non-degenerate pairing, the Weil pairing. This pairing makes the identity component $\ker(F)$ of $E[p]$ the Cartier dual of the étale quotient $\ker(V)$.

$$0 \to \ker(F) \to E[p] \to \ker(V) \to 0.$$  

(1)

In particular, not every twisted form of $G$ can occur as $p$-torsion subgroup scheme of an ordinary elliptic curve. However, on $G$ we may define a bilinear, alternating and non-degenerate pairing by

$$\Phi : G \times G \to \mu_p \quad (\mu, i, (\nu, j)) \mapsto \mu^j/\nu^i,$$

and denote by $\mathcal{A} := \text{Aut}(G, \Phi)$ the automorphism scheme of those automorphisms of $G$ that respect $\Phi$. Then $E[p]$ defines a cohomology class in $H^1_{\text{ét}}(S, \mathcal{A})$. We can now reformulate Question (B) of the introduction:

**Question (B′).** Given a field $k$ of positive characteristic $p$, can every cohomology class in $H^1_{\text{ét}}(\text{Spec } k, \mathcal{A})$ be realised by the $p$-torsion subgroup scheme of an ordinary elliptic curve over $k$?

In [L–S] Section 2 we described $\mathcal{A}$ explicitly: it sits inside a split short exact sequence

$$1 \to \mu_{p-1} \to \mathcal{A} \to \mu_p \to 1.$$  

(2)

It turns out that $\mathcal{A}$ is not commutative for $p \geq 5$ and then nonabelian group cohomology is needed to describe twisted forms of $(G, \Phi)$. Taking cohomology in (2) induces a surjective homomorphism of pointed sets

$$H^1_{\text{ét}}(S, \mathcal{A}) \to H^1_{\text{ét}}(S, \mu_{p-1}) \to 1$$  

(3)

mapping the class of $E[p]$ to the class of $\ker(F)$. Since $\mathcal{A}$ is nonabelian for $p \geq 5$, the description of the fibres of (3) is a little bit tricky. Roughly speaking, these fibres describe extension classes like the short exact sequence (1). We refer to [L–S, Theorem 3.4] for details.

The sequence (1) is just the connected-étale exact sequence for $E[p]$ and it is well-known that such sequences split over perfect fields. In our case this can also be seen from (3), since its kernel $H^1_{\text{ét}}(\text{Spec } k, \mu_p)$ is trivial for perfect fields. More
generally, (1) splits if and only if $j(E) \in k^p$ by [L-S, Proposition 3.3]. In any case we note

**Remark 1.1.** For a perfect field, Questions (A) and (B) are equivalent.

2. **Finite Fields**

In this section we answer Questions (A) and (B) for finite fields. After classifying twisted forms over finite fields we use Weil’s results about counting points of elliptic curves over finite fields and Honda–Tate theory [Ta] to decide which twisted forms of $\mu_p$ over a finite field can be realised as subgroup schemes of elliptic curves over this finite field.

**Proposition 2.1.** Let $k = \mathbb{F}_q$ be the finite field with $q = p^n$ elements. Then there exists a bijection of sets and an isomorphism $\varphi$ of abelian groups

$$\{ \text{twisted forms of } \mu_p \text{ over } k \} \to k^\times / k^\times (p-1) \xrightarrow{\varphi} \mathbb{F}_p^\times .$$

In particular, there are precisely $(p-1)$ twisted forms of $\mu_p$ over $k$.

**Proof.** We have seen in Section 1 that twisted forms of $\mu_p$ over $k$ correspond bijectively to one-dimensional $p$-Lie algebras over $k$ whose $p$-operator is non-trivial. There we have also seen that these $p$-Lie algebras correspond bijectively to the set $k^\times / k^\times (p-1)$. Finally, we leave it to the reader to check that

$$\varphi : k^\times / k^\times (p-1) \to \mathbb{F}_p^\times , \quad x \mapsto x^m \quad \text{with} \quad m = \frac{q-1}{p-1}$$

defines an isomorphism of abelian groups. □

**Theorem 2.2.** Let $k = \mathbb{F}_q$ be the finite field with $q = p^n$ elements and assume that $p \leq 17$ or $n \geq 2$. Then every twisted form of $\mu_p$ over $k$ occurs as subgroup scheme of an elliptic curve over $k$. In particular, Questions (A) and (B) have a positive answer for $k$.

**Theorem 2.3.** Let $\mathbb{F}_p$ be a prime field with $p \geq 19$ elements. Then the set of twisted forms that occur as subgroup schemes of elliptic curves over $\mathbb{F}_p$ corresponds via the map $\varphi$ of Proposition 2.1 to the set

$$\{ [\beta] \in \mathbb{F}_p^\times \mid \beta \in \mathbb{Z} - \{0\}, \beta^2 < 4p \} \subset \mathbb{F}_p^\times ,$$

which is a proper subset. In particular, Questions (A) and (B) have a negative answer for $\mathbb{F}_p$.

**Proof (of both theorems).** Since finite fields are perfect, Questions (A) and (B) are equivalent by Remark 1.1. By Proposition 2.1 there are no twisted forms of $\mu_p$ in characteristic $p = 2$ and we may thus assume $p \geq 3$.

Let $E$ be an elliptic curve over the finite field $k = \mathbb{F}_q$ with $q = p^n$ elements. Then the characteristic polynomial on the first $\ell$-adic cohomology group is of the form $x^2 - \beta x + q$, where $\beta$ is an integer satisfying $\beta^2 < 4q$, confer [Si, Chapter V, Section 2]. By loc.cit. we also know that the number of $k$-rational points on $E$ is equal to

$$\# E(k) = 1 - \beta + q.$$
Since \( p \geq 3 \) we may assume that \( E \) is given by an equation of the form \( y^2 = f(x) \). Denote by \( A_p^m \) the coefficient of \( x^{p^m-1} \) in \( f(x)^{(p^m-1)/2} \). Then \( A_p \) is the Hasse invariant of \( E \) and its class in \( k^\times /k^\times(p-1) \) corresponds to the twisted form \( \ker(F) \) of \( \mu_p \) as in Proposition 2.1 and as explained in Section 1. From the proof of [Si, Theorem V.4.1] we get
\[
A_q = A_p^{1+p+\ldots+p^{n-1}} = A_p^{(p-1)/2} \varphi(A_p),
\]
where \( \varphi \) is as in (4). In particular, \( A_q \) lies in \( F_p \). From loc.cit. we also get
\[
A_q \equiv 1 - \#E(k) \mod p.
\]
Putting these results together, we infer
\[
\beta = 1 - \#E(k) + q \equiv A_q = \varphi(A_p) \mod p,
\]
i.e., \( \beta \mod p \) as an element of \( F_p \) determines \( \ker(F) \) via the correspondence of Proposition 2.1.

Theorem 2.3 now follows: in fact, \( \beta \) is an integer fulfilling \( \beta^2 < 4q \), and if \( q = p^2 \geq 19 \) there are strictly less than \( p - 1 \) possibilities for \( \beta \). On the other hand, there are precisely \( p - 1 \) twisted forms of \( \mu_p \) over \( F_p \) and hence not every one of them can be realised as kernel of Frobenius on an elliptic curve.

Conversely, for \( h \in F_p^\times \) and \( q = p^n \) with \( p \leq 17 \) or \( n \geq 2 \) there exists an integer \( \beta \) with \( \beta^2 < 4q \) and \( \beta \equiv h \mod p \). Now, Honda–Tate theory tells us that there exists an elliptic curve over \( k = F_q \) with characteristic polynomial of Frobenius equal to \( x^2 - \beta x + q \), confer [Ta]. This proves Theorem 2.2.

**Remark 2.4.** The class \( [\beta] = 1 \in F_p^\times \) corresponds to \( \mu_p \) and thus Honda–Tate theory shows moreover that there always exists an elliptic curve \( E \) over \( F_p \) with \( \ker(F) \cong \mu_p \) and thus \( E[p] \cong \mu_p \oplus (\mathbb{Z}/p\mathbb{Z}) \).

**Remark 2.5.** From the description in Theorem 2.3 of those twisted forms of \( \mu_p \) that are realisable as subgroup schemes of elliptic curves it is not difficult to see that this subset usually does not form a subgroup.

### 3. Local Fields

In this section we show that Question (A) has a positive answer for all fields of the form \( k((t)) \) using deformation theory of elliptic curves and a result of Igusa. Also, we prove a Hasse principle for twisted forms of \( \mu_p \) and classify these forms over local fields.

**Theorem 3.1.** Let \( k \) be a field of positive characteristic \( p \). Then every twisted form of \( \mu_p \) over \( k((t)) \) occurs as subgroup scheme of an elliptic curve over \( k((t)) \). In particular, Question (A) has a positive answer for \( k((t)) \).

**Proof.** Choose a supersingular elliptic curve over \( F_p \) and let \( E \rightarrow F_p[[t]] \) be its universal formal deformation. By a theorem of Igusa ([K–M Corollary 12.4.4.]), the Hasse invariant of \( E \) has simple zeroes, i.e., the Hasse invariant can be lifted to an element of \( F_p[[t]] \) of the form
\[
c \cdot t \cdot t^{(p-1)} \cdot (1 + tg(t))
\]
for some $c \in \mathbb{F}_p^\times$, some $n \geq 0$ and $g(t) \in \mathbb{F}_p[[t]]$. Using the fact that the equation $x^{p-1} - 1$ has $p - 1$ different zeroes in $\mathbb{F}_p$ and using Hensel’s lemma, it is easy to see that this Hasse invariant is congruent to $c \cdot t$ modulo $\mathbb{F}_p((t))^{(p-1)}$.

Base changing from $\mathbb{F}_p$ to $k$ and using then base changes of the form $t \mapsto a \cdot t^m$ we can realise every class in $k((t))^{\times}/k((t))^{(p-1)}$ as Hasse invariant of an elliptic curve over $k((t))$. □

Now, let $K$ a global field of positive characteristic $p$, i.e., a finite extension of $\mathbb{F}_p(t)$. For a place $v$ of $K$ we denote by $K_v$ the completion with respect to $v$. The next result shows that twisted forms of $\mu_p$ over global fields obey a Hasse principle.

**Proposition 3.2** (Hasse principle). Let $K$ be global field of positive characteristic $p$. Then the natural homomorphism

$$H^1_{\text{ét}}(\text{Spec } K, \mu_{p-1}) \to \prod_v H^1_{\text{ét}}(\text{Spec } K_v, \mu_{p-1}),$$

where the product is taken over all places of $K$, is injective. In particular, a twisted form of $\mu_p$ over $K$ is trivial if and only if it is trivial over every completion.

**Proof.** The total space of a $\mu_{p-1}$-torsor over $\text{Spec } K$ is the spectrum of an Artin algebra over $K$, hence a product of fields $L \supset K$. If this torsor is non-trivial then $L/K$ is a non-trivial extension and due to the $\mu_{p-1}$-action this extension is a Galois extension with abelian Galois group. By class field theory there exists a place $v$ of $K$ such that the induced extension $L_v/K_v$ on completions is non-trivial. This implies that the induced $\mu_{p-1}$-torsor over $K_v$ is non-trivial and our injectivity statement follows. □

**Remark 3.3.** Let $K = k((t))$ for some field $k$ of positive characteristic $p$. Using Hensel’s lemma it is easy to see that the valuation $\nu : K^\times \to \mathbb{Z}$ induces a short exact sequence

$$0 \to H^1_{\text{ét}}(\text{Spec } k, \mu_{p-1}) \to H^1_{\text{ét}}(\text{Spec } K, \mu_{p-1}) \xrightarrow{\nu} \mathbb{Z}/(p-1)\mathbb{Z} \to 0,$$

which can be split using the uniformiser $t \in K$. In particular, if $K$ is a local field, i.e., if $k$ is a finite field $\mathbb{F}_q$, then applying Proposition 2.1 we obtain an isomorphism

$$H^1_{\text{ét}}(\text{Spec } \mathbb{F}_q((t)), \mu_{p-1}) \cong (\mathbb{Z}/(p-1)\mathbb{Z})^2.$$

In particular, there are only finitely many twisted forms of $\mu_p$ over local fields.

## 4. IGUSA CURVES

In this section we will first prove that Question (A) has a positive answer for all fields of positive characteristic $p \leq 11$. Then, however, we will see that Question (A) has a negative answer for the fields $k(t)$ in characteristic $p \geq 13$ and for function fields of elliptic curves in characteristic $p \geq 17$. This result is closely related to the geometry of the Igusa curves.

**Theorem 4.1.** Let $k$ be a field of characteristic $p \leq 11$. Then every twisted form of $\mu_p$ over $k$ occurs as subgroup scheme of an elliptic curve over $k$. In particular, Question (A) has a positive answer for $k$. 
Since there are no twisted forms of \( \mu_p \) in characteristic 2, the result trivially holds true for \( p = 2 \). From the discussion in Section 1 we know that Question (A) is equivalent to Question (A'), i.e., we have to realise every class in \( k^\times/k^\times(p-1) \) as Hasse invariant of an elliptic curve over \( k \).

If \( p \geq 5 \) then every elliptic curve can be given by a Weierstraß equation of the form \( y^2 = x^3 + ax + b \). Depending on \( p \) we obtain the following Hasse invariants:

| \( p \) | \( 2a \) | \( 3b \) | \( 9ab \) |
|-------|-------|-------|-------|
| 5     |       |       |       |
| 7     |       |       |       |
| 11    |       |       |       |

Hence if \( 5 \leq p \leq 11 \) we can easily find for every class \( [h] \in k^\times/k^\times(p-1) \) an elliptic curve with Hasse invariant \( [h] \).

We leave the case \( p = 3 \) to the reader. □

**Remark 4.2.** Of course, one can ask what freedom one has choosing the elliptic curve containing a given twisted form of \( \mu_p \) as subgroup scheme. Using the automorphism groups of (special) elliptic curves to twist a given curve it follows from Lemma 4.4 below that

- \( p = 2 \) There are no twisted forms of \( \mu_2 \) over fields and every ordinary elliptic curve contains \( \mu_2 \) as subgroup scheme.
- \( p = 3 \) Given a twisted form \( \bar{\mu}_3 \) of \( \mu_3 \) and an arbitrary ordinary elliptic curve \( E \) there exists a quadratic twist of \( E \) containing \( \bar{\mu}_3 \) as subgroup scheme.
- \( p = 5 \) Every twisted form of \( \mu_5 \) can be realised as subgroup scheme of an elliptic curve with \( j = 1728 \).
- \( p = 7 \) Every twisted form of \( \mu_7 \) can be realised as subgroup scheme of an elliptic curve with \( j = 0 \).

In characteristic \( p = 11 \) there usually does not exist one single elliptic curve such that all twisted forms of \( \mu_p \) occur as subgroup schemes of twists of this particular curve.

We now come to the main result of this section, namely that in characteristic \( p \geq 13 \) Question (A) has a negative answer in general. As the proof will show this is closely related to the Igusa curves not being rational if \( p \geq 13 \). We recall from [K–M, Chapter 12.3] that the Igusa moduli problem classifies ordinary elliptic curves \( E \) over base schemes \( S \) of positive characteristic \( p \) such that the Frobenius pullback \( E^{(p)} \) contains an \( S \)-rational \( p \)-division point. For \( p \geq 3 \) this moduli problem is representable by a smooth affine curve \( \text{Ig}(p)^\text{ord} \) over \( \mathbb{F}_p \), whose smooth compactification is denoted by \( \overline{\text{Ig}(p)^\text{ord}} \). The geometry of these curves has been analysed in [IG], and in particular their genera have been determined there:

**Proposition 4.3** (Igusa). Let \( \overline{\text{Ig}(p)^\text{ord}} \) be the smooth compactification of the Igusa curve in positive characteristic \( p \geq 3 \). Then this curve is

(i) rational if \( p \leq 11 \), is  
(ii) elliptic if \( p = 13 \), and is  
(iii) of general type if \( p \geq 17 \). □

Next, we determine the effect that twisting an elliptic curve has on its Hasse invariant.
Lemma 4.4. Let $E$ be an ordinary elliptic curve over a field $k$ of characteristic $p \geq 3$ and let $[h] \in k^\times/k^\times(p-1)$ be its Hasse invariant.

(i) If $E^D$ is the quadratic twist of $E$ with respect to $D \in k^\times/k^\times2$ then the Hasse invariant of $E^D$ is $[hD^{(p-1)/2}]$.

(ii) If $j(E) = 0$ (and thus $p \equiv 1 \mod 3$) and $E^D$ is the sextic twist of $E$ with respect to $D \in k^\times/k^\times6$ then the Hasse invariant of $E^D$ is $[hD^{(p-1)/6}]$.

(iii) If $j(E) = 1728$ (and thus $p \equiv 1 \mod 4$) and $E^D$ is the quartic twist of $E$ with respect to $D \in k^\times/k^\times4$ then the Hasse invariant of $E^D$ is $[hD^{(p-1)/4}]$.

PROOF. In characteristic $p \geq 5$ this can be seen from the explicit computation of twists in [S1, Chapter X, Proposition 5.4] together with the description of the Hasse invariant as the coefficient of $x^{p-1}$ in $f(x)^{(p-1)/2}$ if the elliptic curve is given by $y^2 = f(x)$, see [S1, Chapter V, Theorem 4.1]. We leave the case $p = 3$ to the reader. □

Theorem 4.5. Let $k$ be a field of characteristic $p \geq 13$. Then there exists a twisted form of $\mu_p$ over $k(t)$ that does not occur as subgroup scheme of an elliptic curve over $k(t)$. In particular, Question (A) has a negative answer for $k(t)$.

PROOF. Consider the field extension $k(t) \subset L := k(t)[\root{(p-1)}\of{t}]$, which is Galois with cyclic Galois group of order $p - 1$. Hence $\text{Spec } L \rightarrow \text{Spec } k(t)$ is a $\mu_{p-1}$-torsor and we denote by $\tilde{\mu}_p$ be twisted form of $\mu_p$, which arises by twisting $\mu_p$ with this torsor. Note that $L = k(u)$ for $u := \root{(p-1)}\of{t}$, i.e., $L$ is the function field of $\mathbb{P}^1_k$.

By way of contradiction, we assume that there exists an elliptic curve $E$ over $k(t)$ containing $\tilde{\mu}_p$ as subgroup scheme.

Suppose first that $j(E) \in k$. Then there exists a twist $E'$ of $E$, which is already defined over $k$. The corresponding twist $\tilde{\mu}'_p$ is then also defined over $k$. If $j(E) \notin \{0, 1728\}$ then $E'$ is a quadratic twist of $E$, see [S1, Chapter X, Proposition 5.4]. Now, $\tilde{\mu}_p$ corresponds to the class $[t] \in k(t)^\times/k(t)^\times(p-1)$. Since $\tilde{\mu}'_p$ is obtained by a quadratic twist (still assuming $j(E) \notin \{0, 1728\}$ for the moment), say with twisting parameter $D$, the twisted form of $\tilde{\mu}'_p$ corresponds to the class $[tD^{(p-1)/2}]$ by Lemma 4.2. Using $k[t]$ with its standard valuation $v$ such that $v(t) = 1$, we see that for $p \geq 5$ and for all $D \in k(t)^\times$ we have $v(tD^{(p-1)/2}) \neq 0$. In particular, the class of $\tilde{\mu}'_p$ in $k(t)^\times/k(t)^\times(p-1)$ cannot be represented by an element of $k$, which implies that $\tilde{\mu}'_p$ cannot be defined over $k$. If $j(E) = 0$ or $j(E) = 1728$ we have to consider also sextic and quartic twists, but again, for $p \geq 11$, no sextic or quartic twist of $\tilde{\mu}_p$ can be defined over $k$. We conclude that $j(E) \notin k$, i.e., $j(E)$ is transcendental over $k$.

Since $\tilde{\mu}_p$ becomes isomorphic to $\mu_p$ over $L$, we see that $\mu_p$ is a subgroup scheme of $E_L := E \times_{\text{Spec } k(t)} \text{Spec } L$. This implies that $E_L^{(p)}$ contains an $L$-rational $p$-division point and we obtain a classifying morphism $\varphi : \text{Spec } L \rightarrow \text{Ig}(p)^{\text{ord}}$ to the Igusa curve, which yields a morphism $\varphi_k : \text{Spec } L \rightarrow \text{Ig}(p)^{\text{ord}} \times_{\text{Spec } k} \text{Spec } k$. There is a dominant morphism $\text{Ig}(p) \rightarrow \mathbb{P}^1$ induced by the $j$-invariant and using that $j(E)$ is transcendental over $k$ we infer that $\varphi_k$ is dominant. Since $L$ is the
function field of $\mathbb{P}^1_k$, the existence of $\varphi_k$ implies that $\overline{\text{Ig}(p)}^\text{ord}$ is a rational curve. This contradicts Proposition 4.3 and we conclude that the curve $E$ we started with does not exist.

**Theorem 4.6.** Let $k$ be a field of characteristic $p \geq 17$ and $E$ be an elliptic curve over $k$. Then there exists a twisted form of $\mu_p$ over the function field $k(E)$ that does not occur as subgroup scheme of an elliptic curve over $k(E)$. In particular, Question (A) has a negative answer for $k$.

**Proof.** We consider the morphism $E \to E$, which is given by multiplication with $p - 1$. This induces a field extension $k(E) \subset k(E)$, which is Galois with group $(\mathbb{Z}/(p - 1)\mathbb{Z})^2$. In particular, there exists a subfield $k(E) \subset L \subset k(E)$, such that $L/k(E)$ is Galois with group $\mathbb{Z}/(p - 1)\mathbb{Z}$. Twisting $\mu_p$ with the $\mu_{p-1}$-torsor $\text{Spec } L \to \text{Spec } k(E)$, we obtain a twisted form $\tilde{\mu}_p$ of $\mu_p$. We note that $L$ is the function field of an elliptic curve over $k$, which is in particular a geometrically irreducible curve over $k$.

Assume there exists an elliptic curve $X$ over $k(E)$ containing $\tilde{\mu}_p$ as subgroup scheme.

As in the proof of Theorem 4.5, one first shows that $j(X) \notin k$. Otherwise there exists a quadratic (resp. quartic, resp. sextic) twist $\tilde{\mu}_p'$ of $\tilde{\mu}_p$, which is defined over $k$, which implies that the extension $L/k(E)$ is given by taking the $(p - 1)$,st root of an element of the form $aD^m$ with $m = (p - 1)/2$ (resp. $m = (p - 1)/4$, resp. $m = (p - 1)/6$) and $a \in k$. However, this implies that $L$ is the function field of a curve over $k$, which is not geometrically irreducible, a contradiction.

From $j(X) \notin k$, we infer again that $\text{Spec } L$ maps dominantly to $\text{Ig}(p)$, and using similar arguments as in the proof of Theorem 4.5, we conclude that $\overline{\text{Ig}(p)}^\text{ord}$ has genus at most one, which contradicts Proposition 4.3.

If we choose $k$ to be a finite field in the previous two theorems, then we obtain examples of global fields $K$ over which there exist a twisted form $\tilde{\mu}_p$ of $\mu_p$ that cannot be realised as subgroup scheme of an elliptic curve over $K$. On the other hand, Theorem 3.1 tells us that for every place $v$ of $K$ the group scheme $\tilde{\mu}_p \times_{\text{Spec } K} \text{Spec } K_v$ over the completion $K_v$ can be realised as subgroup of an elliptic curve over $K_v$. Thus, although there exists a Hasse principle for twisted forms over global fields by Proposition 3.2, we note:

**Corollary 4.7** (no Hasse principle). Let $K$ be the function field of $\mathbb{P}^1$ over a finite field of characteristic $p \geq 13$ or the function field of an elliptic curve over a finite field of characteristic $p \geq 17$. Then there is no Hasse principle for realising twisted forms of $\mu_p$ over $K$ as subgroup schemes of elliptic curves.

In view of the negative results Theorem 4.5 and Theorem 4.6 one can ask whether one can realise more twisted forms of $\mu_p$ by also allowing singular or nonproper curves carrying group structures rather than only elliptic curves. This turns out not to help:

If we do not insist on properness (but on geometric integrality), we are led to considering twisted forms of $G_m$. If we do not insist on smoothness (but on
geometric integrality) then we are led to considering projective curves $C$ with $h^1(C, \mathcal{O}_C) = 1$ and having one singularity. The smooth locus of $C$ is a twisted form of $G_m$ if the singularity is a node and a twisted form of $G_a$ if the singularity is a cusp. Thus, we find twisted forms of $\mu_p$ also on nodal curves.

However, the group scheme of automorphisms of $G_m$ fixing the neutral element is $\mathbb{Z}/2\mathbb{Z}$ generated by $t \mapsto t^{-1}$. Thus, on twisted forms of nodal projective curves or on twisted forms of $G_m$, we can only realise quadratic twists of $\mu_p$. Using quadratic twists of an elliptic curve $E$ containing $\mu_p$ as subgroup scheme, which always exists by Remark 2.4, we see that all quadratic twists of $\mu_p$ can be realised as subgroup schemes of elliptic curves. Thus we have shown

**Proposition 4.8.** Let $k$ be field of positive characteristic $p$. Then every twisted form of $\mu_p$ that occurs on a twisted form of $G_m$, or on a twisted form of the singular nodal curve of arithmetic genus one can also be realised as subgroup scheme of an elliptic curve over $k$. \hfill \Box

## 5. The $p$-Torsion Subgroup Scheme

We have seen in the previous sections that Question (A) has a positive answer for local fields (Theorem 3.1). as well as for all fields of characteristic $p \leq 11$ (Theorem 4.1). We will see in this section that for those fields also Question (B) has a positive answer.

We need two technical lemmas to start with.

**Lemma 5.1.** Let $G := \mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$, $\Phi : G \times G \to \mu_p$ and $A = \text{Aut}(G, \Phi)$ as in Section 2. Let $k$ be a field of positive characteristic $p$ and let $\text{Spec } L \to \text{Spec } k$ be an $A$-torsor. Then

(i) as a $k$-algebra $L$ is isomorphic to $k[x, y]/(x^p - a, y^{p-1} - b)$ for some elements $a, b \in k$.

(ii) as a scheme the twist $(G, \Phi) \wedge^A \text{Spec } L$ is isomorphic to the spectrum of $k \oplus k[y]/(y^{p-1} - b) \oplus L$.

Moreover, a $p^2$-dimensional $k$-algebra can carry at most one structure of a Hopf algebra making its spectrum into a twisted form of $(G, \Phi)$.

**Proof.** We have seen in Section 1 that $A \cong \mu_{p-1} \times \mu_p$. Taking $\mu_{p-1}$-invariants we can factor $\text{Spec } L \to \text{Spec } k$ as $k \subset k' := k[x]/(x^p - a)$ for some $a \in k$ and then $L = k'[z]/(z^{p-1} - c)$ for some $c \in k'$. But then $b := c^p$ lies in $k$ and $k[x, y]/(x^p - a, y^{p-1} - b)$ is contained in $L$. Comparing their dimensions as $k$-vector spaces they must be equal and we get the first assertion.

The action of $A$ on $G$ is described in [L–S], Section 2. This action has three orbits: one orbit consists of $\text{Spec } k$ (corresponding to the zero section), one orbit has length $(p - 1)$ (corresponding to $\mathbb{Z}/p\mathbb{Z}$ minus the zero section) with infinitesimal isotropy groups, and finally one orbit has length $p(p - 1)$ upon which $A$ acts without fixed scheme. From this we get the second assertion.

Finally, we note that every twisted form of $(G, \Phi)$ is autodual, i.e., isomorphic to its own Cartier dual. Now, the coalgebra structure of a commutative and cocommutative Hopf algebra is determined by its algebra structure, confer [W], Section...
2.4]. Hence a $k$-algebra can carry at most one structure of a Hopf algebra making its spectrum into a twisted form of $(G, \Phi)$.

**Lemma 5.2.** Let $E$ be an elliptic curve over a field of positive characteristic $p$ with Hasse invariant $[h] \in k^\times/k^\times(p-1)$ and $j$-invariant $j(E)$. Then, as a scheme the $p$-torsion subgroup scheme $E[p]$ is isomorphic to the spectrum of $k \oplus M \oplus L$, where $M = k[y]/(y^{p-1} - h)$ and $L = M[x]/(x^p - j(E))$.

**Proof.** As a scheme, the twisted form $\mathrm{ker}(V)$ of $\mathbb{Z}/p\mathbb{Z}$, which is nothing but the étale quotient of $E[p]$ (compare with (1)) is isomorphic to the spectrum of $k \oplus k[y]/(y^{p-1} - h)$. These two summands correspond to the two $A$-orbits of length 1 and $p - 1$ that we have seen in the proof of Lemma 5.1.

To determine the third summand, we may assume $j(E) \not\in k^p$ for otherwise $E[p]$ is the direct sum of $\ker(V)$ and its Cartier dual and the result is true in this case. We note that $E[p]$ becomes isomorphic to $\mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$ over $k[\phi y]/(y^p - j(E), y^{p-1} - h)$. From this it follows easily that $k[y]/(y^p - j(E))$ is contained in this third summand, which implies that it contains $k[x, y]/(x^p - j(E), y^{p-1} - h)$. Since this latter $k$-algebra is $p(p-1)$-dimensional like the summand we are looking for, they have to coincide.

To answer Question (B) and to avoid trivialities, we may assume by Remark [1.1] that the field we are dealing with is not perfect.

**Proposition 5.3.** Let $k$ be a nonperfect field of characteristic $p$. Then Question (B) has a positive answer for $k$ if for every $[h] \in k^\times/k^\times(p-1)$ and every purely inseparable field extension $k \subset k'$ of degree $p$, there exists an elliptic curve $E$ over $k$ with Hasse invariant $[h]$ and $j$-invariant $j(E)$ such that $k' = k[\sqrt[p]{j(E)}]$.

**Proof.** Let $\tilde{G}$ be a twisted form of $(G, \Phi)$ over $k$. By Lemma 5.1 there exist $a, b \in k$ such that the scheme underlying $\tilde{G}$ is isomorphic to the spectrum of $k \oplus M \oplus L$, where $M = k[y]/(y^{p-1} - b)$ and $L = M[x]/(x^p - a)$. By Lemma 5.2 and our assumptions there exists an elliptic curve $E$ over $k$ such that the scheme underlying $E[p]$ also is isomorphic to the spectrum of $k \oplus M \oplus L$. But then, again by Lemma 5.1, the two schemes $\tilde{G}$ and $E[p]$ are also isomorphic as group schemes.

**Theorem 5.4.** Let $k$ be a field of positive characteristic $p \leq 11$. Then every twisted form of $\mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$ endowed with a bilinear, alternating and nondegenerate pairing can be realised as subgroup scheme of an elliptic curve over $k$. In particular, Question (B) has a positive answer for $k$.

**Proof.** By Theorem [4.1] and Remark [1.1] we may assume that $k$ is not perfect. Given a purely inseparable field extension $k \subset k'$ of degree $p$, there exists an element $j \in k$ such that $k' = k[\sqrt[p]{j}]$. Then we choose an elliptic curve $E$ over $k$ with $j(E) = j$.

By Proposition 5.3 it remains to realise all possible Hasse invariants on such curves. In characteristic $p = 2$ there is only the Hasse invariant $h = [1]$ and we get our result in this characteristic. In characteristic $p = 3$ we can use quadratic
twists of this curve $E$ to realise every possible Hasse invariant (see Remark 3.2) and since twisting does not change the $j$-invariant, we also get our result in this characteristic.

In characteristic $p = 5$ we may assume that elliptic curves are given by a Weierstraß equation of the form $y^2 = x^3 + ax + b$. The Hasse invariant of such a curve is $[2a] \in k^{\times}/k^{\times(p-1)}$ (see also the proof of Theorem 4.1) and we can realise every class of $k^{\times}/k^{\times(p-1)}$ by the Hasse invariant of an elliptic curve. Replacing $a$ by $a^p$ does not change the Hasse invariant. But then a straightforward computation shows that $k[\sqrt[5]{j(E)}]$ coincides with $k[\sqrt[p]{b}]$. Choosing thus $a$ and $b$ appropriately, we see that the assumptions of Proposition 5.3 are fulfilled and our statement follows for characteristic 5.

The remaining cases $p = 7$ and $p = 11$ are similar to $p = 5$ and therefore left to the reader. □

**Theorem 5.5.** Let $k$ be a perfect field of positive characteristic $p$. Then every twisted form of $\mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$ endowed with a bilinear, alternating and nondegenerate pairing can be realised as subgroup scheme of an elliptic curve over $k((t))$. In particular, Question (B) has a positive answer for $k((t))$.

**Proof.** By Theorem 5.1 given a class in $k((t))^{\times}/k((t))^{\times(p-1)}$ there exists an elliptic curve $E$ over $k((t))$ having this class as Hasse invariant. Moreover, looking at the proof of this result we see that we may assume that this elliptic curve has a $j$-invariant that does not lie in $k$. If $j(E)$ lies in $k((t))^p$ then there exists an elliptic curve $E'$ over $k((t))$ such that $E = E'^{(p)}$. It is easy to see that the Hasse invariants of $E$ and $E'$ coincide. We may thus assume that we have realised the given Hasse invariant on an elliptic curve $E$ with $j(E) \not\in k((t))^p$. Since there is only one inseparable extension $K'$ of $k((t))$ of degree $p$ (here we use that $k$ is perfect), we must have $K' = k((t))[\sqrt[p]{j(E)}]$. Applying Proposition 5.3 our result follows. □

6. **Supersingular elliptic curves**

Finally, we describe $p$-torsion subgroup schemes of supersingular elliptic curves, which turn out to be much simpler than those of ordinary elliptic curves.

The kernel of Frobenius of a supersingular elliptic curve is a twisted form of $\alpha_p$. Since $\text{Aut}(\alpha_p) = G_a$ is a smooth group scheme, $\alpha_p$ does not possess twisted forms over fields by Hilbert 90.

Over perfect fields, there exists only one non-split extension of $\alpha_p$ by itself that is autodual, namely $M_2$ in the notation of [10, Section II.15.5]

\[(5) \quad 0 \to \alpha_p \to M_2 \to \alpha_p \to 0.\]

Hence the $p$-torsion subgroup scheme of a supersingular elliptic curve is a twisted form of $M_2$. In particular, $M_2$ plays the role that $\mu_p \oplus (\mathbb{Z}/p\mathbb{Z})$ plays for ordinary elliptic curves. Since $\alpha_p$ does not possess twisted forms over fields, twisted forms of $M_2$ correspond to twisted splittings of $\alpha_p$, which are parametrised by $H^1_{\text{f}}(\text{Spec } k, \text{Hom}(\alpha_p, \alpha_p))$, confer [D–G, Chapter III.6.3.5]. On the other hand, $\text{Hom}(\alpha_p, \alpha_p) \cong G_a$ is a smooth group scheme, and hence this cohomology group
is trivial for fields by Hilbert 90. We deduce that $M_2$ does not possess twisted forms over fields.

We have thus shown that the questions analogous to Questions (A) and (B) trivially hold true for supersingular elliptic curves:

**Theorem 6.1.** Let $k$ be a field of positive characteristic $p$.

(i) The kernel of Frobenius of a supersingular elliptic curve over $k$ is isomorphic to $\alpha_p$. The group scheme $\alpha_p$ does not possess twisted forms over $k$.

(ii) The $p$-torsion subgroup scheme of a supersingular elliptic curve over $k$ is isomorphic to $M_2$. The group scheme $M_2$ does not possess twisted forms over $k$.

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