A Two-Step Quasi-Newton Identification Algorithm for Stochastic Systems with Saturated Observations

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Abstract—This paper investigates the adaptive identification and prediction problems for stochastic dynamical systems with saturated observations, which arise from various fields in engineering and social systems, but up to now still lack comprehensive theoretical studies including performance guarantees needed in practical applications. With this impetus, the paper has made the following main contributions: (i) To introduce a two-step Quasi-Newton (TSQN) algorithm to improve the performance of the identification, which is applicable to a typical class of nonlinear stochastic systems with outputs observed under possibly varying saturation. (ii) To establish the global convergence of both the parameter estimators and adaptive predictors and to prove the asymptotic normality, under the weakest possible non-persistent excitation (PE) condition, which can be applied to stochastic feedback systems with general non-stationary and correlated system signals or data. (iii) To establish useful probabilistic estimation error bounds for any given finite length of data, using either martingale inequalities or Monte Carlo experiments. A numerical example is also provided to illustrate the performance of the proposed identification algorithm.

Index Terms—Asymptotic normality, convergence, non-PE condition, stochastic systems, saturated observations.

I. INTRODUCTION

Identifying the input-output relationship and predicting the future behavior of dynamical systems based on observation data are fundamental problems in various fields including control systems, signal processes and machine learning, etc. This paper considers identification and prediction problems for stochastic dynamical systems with saturated observations. At each time, the noise-corrupted output can be observed only if its value lies in a certain range, while those observations outside this range are blind. The relationship between the system output and observation is illustrated in Fig.1, where \( y_{k+1} \) and \( u_k \) represent the system output and observation respectively, the interval \([l_k, u_k]\) is the observation range, when the system output exceed this range, the only possible observation is a constant, either \( L_k \) or \( U_k \). We call the observations produced by such a mechanism as saturated observations. Note that if we take \( L_k = l_k = 0, u_k = U_k = \infty \), then the saturation function will become the ReLu function widely used in machine learning; and if we take \( L_k = l_k = u_k = 0, U_k = 1 \), the saturation function will turn to be a binary-valued function widely used in classification problems\([1],[2]\).

![Fig. 1. Saturated observations.](image)

Saturated observations in stochastic dynamical systems exist widely in various fields including engineering (\([3],[5]\)), economics (\([6],[7],[17],[25]\)) and social systems \([8]\). We only mention several examples in three different application areas. The first example is in sensor networks (\([3]\)), where the observations of each sensor are saturated observations due to power and bandwidth limitations; The second example is in economics (\([6]\)), where \( v_k \) is interpreted as an index of consumer's intensity of desire to purchase a durable, \( y_k \) is the true purchase which can be regarded as an saturated observation, since the intensity \( v_k \) can be observed only if it exceeds a certain threshold where the true purchase takes place; The third example is in sentencing (\([8]\)), where \( y_k \) is the pronounced penalty which can also be regarded as a saturated observation, since it is constrained within the statutory range of penalty according to the related basic criminal facts.

Compared with the uncensored or unsaturated case of observations, the investigation of the saturated case turns out to be more complicated, due to the inherent nonlinearity in the observations of the underlying stochastic dynamical systems. Recently, various identification methods with saturated observations have been studied intensively with both asymptotic and non-asymptotic results, on which we give a brief review separately in the following:

First, most of the existing theoretical results are asymptotic
in nature, where the number of observations need to increase unboundedly or at least to be sufficiently large. For example, the least absolute deviation methods were considered in [15], and the strong consistency and asymptotic normality of the estimators were proven for independent signals satisfying the usual persistent excitation (PE) condition where the condition number of the information matrix is bounded. Besides, the maximum likelihood (ML) method was considered in [16], where the consistency and asymptotic efficiency were established for independent or non-random signals satisfying a stronger PE condition. Moreover, a two-stage procedure was proposed in [17], called Heckman two-step estimator, which first calculates the ML estimates of Probit model parameters ([6]) and then implements the least squares method on the ML estimates obtained in the first procedure. Furthermore, empirical measurement approach was employed in [18]–[20], where the strong consistency and asymptotic efficiency were established under periodic signals with binary-valued observations. Such observations were also considered in [21], where a strongly consistent recursive projection algorithm was given under a condition stronger than the usual PE condition.

Second, there are also a number of non-asymptotic estimation results in the literature. Despite the importance of the asymptotic estimation results as mentioned above, non-asymptotic results appear to be more practical, because one usually only has finite number of data available for identification in practice. However, obtaining non-asymptotic identification results, which are usually given under high probability, is quite challenging especially when the structure comes to nonlinear. Most of the existing results are established under assumptions that the system data are independently and identically distributed (i.i.d), e.g., the analysis of the stochastic gradient descent methods in [23]–[24], and the study of the ML method in [25]. Moreover, an online Newton method was proposed in [26], where a probabilistic error-bound was given for linear stochastic dynamical systems where the usual PE condition is satisfied.

In summary, almost all of the existing identification results for stochastic dynamical systems with saturated observations need at least the usual PE condition on the system data, and actually, most need i.i.d assumptions. Though these idealized conditions are convenient for theoretical investigation, they are hardly satisfied or verified for general stochastic dynamical systems with feedback signals (see, e.g. [14]). This inevitably brings challenges for establishing an identification theory on either asymptotic or non-asymptotic results with saturated observations under more general (non-PE) signal conditions.

Fortunately, there is a great deal of research on adaptive identification for linear or nonlinear stochastic dynamical systems with uncensored or unsaturated observations in the area of adaptive control ([9]–[13]), where the system data used can include those generated from stochastic feedback control systems. By adaptive identification, we mean that the identification algorithm is constructed recursively, where the parameter estimates are updated online based on both the current estimate and the new observation. In comparison with offline algorithms such as those widely used in statistics and machine learning, the adaptive algorithm has at least two advantages: one is that the algorithm can be updated conveniently when new data come in without restoring the old data, another is that general non-stationary and correlated data can be handled conveniently due to the structure of the adaptive algorithm. In fact, extensive investigations have been conducted in adaptive identification in the area of control systems for the design of adaptive control laws, where the system data are generated from feedback systems which are far from stationary and hard to be analyzed [28]. A remarkable analytic method for general adaptive algorithms is the well-known Ljung’s ODE method [12] which can be applied to a wide class of nonlinear recursive algorithms, and a remarkable convergence result for the classical least squares is given by Lai and Wei [13], who established the strong consistency under a weakest possible excitation condition. Of course, these results are established for traditional non-saturated observations case.

The first paper that established the strong consistency of estimators for general stochastic regression models with saturated (binary-valued) observations under non-PE condition appears to be [22], where a recursive projection Quasi-Newton type algorithm was proposed and analyzed. The non-PE condition used in [22] is similar to the weakest possible signal condition for stochastic linear regression model with uncensored observations (see [13]), which can be applied to non-stationary stochastic dynamical systems with feedback control. However, there are still some unresolved fundamental problems, for instances, a) How should a globally convergent estimation algorithm be designed for stochastic systems with general saturated observations? b) What is the asymptotic distribution of the estimation error under non-PE condition? c) How to get a useful and computable probabilistic estimation error bound under non-PE condition when the length of data is finite?

The main purpose of this paper is to solve these problems, by introducing a two-step Quasi-Newton type adaptive identification algorithm, by refining the stochastic Lyapunov function approach, and by applying some martingale inequalities and convergence theorems. Besides, Monte Carlo method is also found quite useful in computing the estimation error bound. As we mentioned in the introduction, binary-valued observations are only special cases of saturated observations, and in this sense, the current investigation may be regarded as a continuation of the authors’ work [22], but with considerably different contents in the proposed algorithms, the main theorems and the theoretical analyses. To be specific, the main contributions of this paper can be summarized as follows:

- A new two-step Quasi-Newton (TSQN) adaptive identification algorithm is proposed for stochastic dynamical systems with saturated observations. The first step is to produce consistent parameter estimates based on the available “worst case” information, which are then used to construct the adaptation gains in the second step for improving the performance of the adaptive identification.
- Asymptotic results on the proposed new identification algorithm, including strong consistency and asymptotic normality, are established for stochastic dynamical systems with saturated observations under quite general non-
by Assumption 1, we can find an almost surely bounded sequence \( \{M_k, k \geq 0\} \) such that
\[
\sup_{x \in D} |\phi_k^T x| \leq M_k, \text{ a.s.} \tag{3}
\]

Assumption 2: The thresholds \( \{l_k, \mathcal{F}_k\}, \{u_k, \mathcal{F}_k\}, \{L_k, \mathcal{F}_k\} \) and \( \{U_k, \mathcal{F}_k\} \) are known adapted stochastic sequences, satisfying for any \( k \geq 0 \),
\[
l_k - c \leq L_k \leq l_k \leq u_k \leq U_k \leq u_k + c, \text{ a.s.,} \tag{4}
\]
and
\[
\sup_{k \geq 0} \{l_k^+\} \leq l, \sup_{k \geq 0} \{u_k\} \leq u, \text{ a.s.}, \tag{5}
\]
where \( c, l \) and \( u \) are \( \mathcal{L}_2 \) non-negative random variable, \( l_k^+ = \max(l_k, 0), u_k^- = \max(-u_k, 0) \).

Remark 1: We note that the inequalities \( L_k \leq l_k \leq u_k \leq U_k \) are natural requirements for the saturation functions to be well-defined as illustrated in Fig.1, and Assumption 2 will be automatically satisfied if \( \{L_k\} \) and \( \{U_k\} \) are \( \mathcal{L}_2 \) and bounded stochastic sequences. The conditions (4) and (5) are general assumptions that are used to guarantee the boundedness of the variances of the output prediction errors in the paper.

Assumption 3: The noise \( \{e_k, \mathcal{F}_k\} \) is an \( \mathcal{L}_2 \) martingale difference sequence and there exists a constant \( \eta > 0 \), such that
\[
\inf_{k \geq 0} \mathbb{E}\{e_{k+1}^2\} > 0, \sup_{k \geq 0} \mathbb{E}\{|e_{k+1}|^{2+\eta}\} < \infty, \text{ a.s.} \tag{6}
\]
Besides, the conditional expectation function \( G_k(x) \), defined by \( G_k(x) = \mathbb{E}\{S_k(x + e_{k+1})\} \), is known and differentiable with derivative denoted by \( G_k'(\cdot) \). Further, there exist a random variable \( M > \sup\{M_k\} \) such that
\[
0 < \inf_{|x| \leq M, k \geq 0} \{G_k'(x)\} < \sup_{|x| \leq M, k \geq 0} \{G_k'(x)\} < \infty, \text{ a.s.} \tag{7}
\]
\[
|G_k'(x) - G_k'(y)| \leq \rho |x - y|, \text{ a.s., } \forall |x|, |y| \leq M, \tag{8}
\]
where \( \rho \) is a non-negative variable, \( M_k \) is defined in (3).

Remark 2: It is worth to mention that under condition (6), the function \( G_k(\cdot) \) in Assumption 3 is well-defined for any \( k \geq 0 \), and can be calculated given the conditional probability distribution of the noise \( e_{k+1} \). In Appendix I, we have provided three typical examples to illustrate how to concretely calculate the function \( G_k(\cdot) \), which includes the classical linear stochastic regression models, censored regression models, and models with binary-valued sensors. Moreover, Assumption 3 can be easily verified if \( \{e_{k+1}, k \geq 0\} \) is i.i.d Gaussian noise and if \( \inf \{U_k - L_k\} > 0, \text{ a.s.} \). Besides, when \( l_k = -\infty \) and \( u_k = \infty \), the system (1)-(2) will degenerate to linear stochastic regression models, and Assumption 3 will degenerate to the standard noise assumption for the strong consistency of the classical least squares (13)) since \( G_k(x) \equiv x \).

For simplicity of notation, denote
\[
\inf_{|x| \leq M_k} \{G_k'(x)\} = \underline{g}_k, \sup_{|x| \leq M_k} \{G_k'(x)\} = \overline{g}_k. \tag{9}
\]
Under Assumption 3, \( \{\underline{g}_k, k \geq 0\} \) and \( \{\overline{g}_k, k \geq 0\} \) have upper bound and positive lower bound respectively, i.e.
\[
\inf_{k \geq 0} \{\overline{g}_k\} > 0, \sup_{k \geq 0} \{\overline{g}_k\} < \infty, \text{ a.s.} \tag{10}
\]
B. Algorithm

Because of its “optimality” and fast convergence rate, the classical LS algorithm is one of the most basic and widely used ones in the adaptive estimation and adaptive control of linear stochastic systems. Inspired by the analysis of the LS recursive algorithm, we have introduced a Quasi-Newton type algorithm to estimate the parameters in linear stochastic regression models with binary-valued observations in [22]. However, we find that a direct extension of the Quasi-Newton algorithm introduced in [22] from binary-valued observations to saturated observations does not give satisfactory performance, which motivates us to introduce a two-step Quasi-Newton identification algorithm as described shortly.

At first, we introduce a suitable projection operator, to ensure the boundedness of the estimates while keeping other nice properties. For the linear space \( \mathbb{R}^m \), we define a norm \( \| \cdot \|_Q \) associated with a positive definite matrix \( Q \) as \( \| x \|_Q^2 = x^T Q x \). We then define a projection operator based on \( \| \cdot \|_Q \).

Definition 1: For the convex compact set \( D \) defined in Assumption 1, the projection operator \( \Pi^k_Q(\cdot) \) is defined as

\[
\Pi^k_Q(x) = \arg \min_{y \in D} \| x - y \|_Q, \quad \forall x \in \mathbb{R}^m. \tag{11}
\]

The new two-step Quasi-Newton (TSQN) identification algorithm is defined as follows:

**Algorithm 1 Two-Step Quasi-Newton (TSQN) Identification Algorithm**

**Step 1.** Recursively calculate the preliminary estimate \( \hat{\theta}_{k+1} \) for \( k \geq 0 \):

\[
\begin{align*}
\hat{\theta}_{k+1} &= \Pi^k_{P_{k+1}} \{ \theta_k + a_k \beta_k P_k \phi_k [y_{k+1} - G_k (\phi_k^T \hat{\theta}_k)] \}, \\
P_{k+1} &= P_k - a_k \beta_k^2 P_k \phi_k \phi_k^T P_k, \\
\beta_k &= \min \left\{ \frac{P_k}{\bar{g}_k}, \frac{1}{\bar{g}_k + 1} \right\}, \\
a_k &= \frac{1}{1 + \beta_k^2 P_k \phi_k},
\end{align*}
\tag{12}
\]

where \( g_k \) and \( \bar{g}_k \) are defined as in (9), \( \Pi^k_{P_{k+1}} \) is the projection operator defined as in Definition 1, \( G_k(\cdot) \) is defined in Assumption 3, the initial values \( \theta_0 \) and \( P_0 \) can be chosen arbitrarily in \( D \) and with \( P_0 > 0 \), respectively.

**Step 2.** Recursively define the accelerated estimate \( \tilde{\theta}_{k+1} \) based on \( \hat{\theta}_{k+1} \) for \( k \geq 0 \):

\[
\begin{align*}
\tilde{\theta}_{k+1} &= \Pi^k_{P_{k+1}} \{ \theta_k + a_k \beta_k P_k \phi_k [y_{k+1} - G_k (\phi_k^T \hat{\theta}_k)] \}, \\
P_{k+1} &= P_k - a_k \beta_k^2 P_k \phi_k \phi_k^T P_k, \\
\beta_k &= \min \left\{ \frac{P_k}{\bar{g}_k}, \frac{1}{\bar{g}_k + 1} \right\}, \\
a_k &= \frac{1}{1 + \beta_k^2 P_k \phi_k},
\end{align*}
\tag{13}
\]

where \( \{ \mu_k \} \) can be any positive random process adapted to \( \mathcal{F}_k \) with \( 0 < \inf_{k \geq 0} \mu_k \leq \sup_{k \geq 0} \mu_k < \infty \), the initial values \( \hat{\theta}_0 \) and \( P_0 \) can be chosen arbitrarily in \( D \) and with \( P_0 > 0 \), respectively.

Remark 3: As described above, our identification algorithm is actually defined by two successive steps, between which the main difference is the construction of the adaptation gains. In the first step, the scalar adaptation gain \( \beta_k \) is constructed by using the bounds \( g_k \) and \( \bar{g}_k \) defined in (9), in a similar way as that constructed in the identification algorithm of [22]. Though the strong consistency of the preliminary estimate \( \hat{\theta}_k \) in the first step may be established following a similar arguments as in [22], its convergence speed appears to be not good enough, and its asymptotic normality also appears to be hard to establish, because the scalar adaptation gain \( \beta_k \) is simply constructed by using the “worst case” information \( g_k \) and \( \bar{g}_k \).

To overcome these shortcomings, the second step estimation is introduced with the following two features: (i) To improve the performance of the estimation algorithm, the scalar adaptation gain \( \beta_k \) is defined in an adaptive way by using the preliminary estimates \( \hat{\theta}_k \) generated in the first step, and (ii) To ensure the asymptotic normality of the estimation errors under non-PE condition, the regularization factor \( \mu_k \) is taken as a “noise variance” estimate constructed by using the online estimates (see Theorem 3). Simulations in Section 5 also demonstrate that the convergence speed of the parameter estimates given in the second step outperforms that of the first step.

III. MAIN RESULTS

In this section, we give some asymptotic results of the TSQN identification algorithm. To be specific, we will establish asymptotic upper bounds for both the parameter estimation errors and the adaptive prediction errors in Subsection III-A, study the asymptotic normality of the TSQN algorithm in Subsection III-B and give high probabilistic error bounds for any given finite number of data in Subsection III-C.

A. Asymptotic error bounds

**Theorem 1:** Under Assumptions 1-3, the estimate \( \hat{\theta}_k \) given by the TSQN Algorithm has the following upper bound almost surely as \( k \to \infty \):

\[
\| \hat{\theta}_{k+1} \|^2 = O \left( \frac{\log \lambda_{\max}(k)}{\lambda_{\min}(k)} \right), \quad a.s. \tag{14}
\]

where \( \hat{\theta}_{k+1} = \theta - \hat{\theta}_{k+1}, \lambda_{\min}(k) \) and \( \lambda_{\max}(k) \) are the minimum and maximum eigenvalues of the matrix \( \sum_{i=0}^k \phi_i \phi_i^T + \lambda_0 I \) respectively, and the initial value satisfies \( \lambda_0 > \| P_0^{-1} \| \).

**Remark 4:** From Theorem 1, it is easy to see that the algorithm will converge to the true parameter almost surely if

\[
\frac{\log \lambda_{\max}(k)}{\lambda_{\min}(k)} = o(1), \quad a.s. \tag{15}
\]

This condition does not need the independence and stationarity conditions on the system regressors and hence applicable to stochastic feedback control systems, and is known to be the weakest possible convergence condition of the classical least squares in the linear case (see [13]), which is much weaker than the traditional PE condition, i.e. \( \lambda_{\max}(k) = O(\lambda_{\min}(k)) \).

Moreover, since the true parameter is the interior of \( D \), by
the strong consistency of the parameter estimates, it is easy to see that after some finite time, the projection operator will become not necessary in the computation process.

**Corollary 1:** Under conditions of Theorem 1, if the excitation condition (15) is strengthened to $k = O(\lambda_{\text{min}}(k))$, then the convergence rate of the TSQN algorithm can be improved to the following iterated logarithm rate:

$$\|\hat{\theta}_{k+1}\|^2 = O\left(\frac{\log \log k}{k}\right), \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (16)

We note that the convergence rate (16) is known to be the best rate of convergence of the classical least squares in the linear case. It is worth noting that we do not know how to establish such a best possible convergence rate for the preliminary estimate given in the first step.

Given the parameter estimate $\hat{\theta}_k$ by the above TSQN algorithm, one can define the adaptive predictor for the output as follows:

$$\hat{y}_{k+1} = G_k(\phi_k^T \hat{\theta}_k).$$

Usually, the difference between the best predictor and the adaptive predictor along the sample path, can be measured by the regret defined as follows:

$$R_k = (E_k \{y_{k+1}\} - \hat{y}_{k+1})^2.$$  \hspace{1cm} (17)

The following theorem gives an asymptotic result for the regret $R_k$.

**Theorem 2:** Let Assumptions 1-3 hold. Then the sample paths of the accumulated regrets will have the following upper bound:

$$\sum_{i=1}^{k} R_i = O(\log \lambda_{\text{max}}(k)), \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (18)

The convergence of the accumulated regrets in Theorem 2 does not require any excitation condition to hold, and thus can be easily applied to closed-loop control systems. We remark that the order $\log k$ is known to be the best possible order in the linear case (see [31]).

**B. Asymptotic normality**

In this subsection, we study the asymptotic distribution properties of the estimation under a general non-PE condition, and show that our algorithm is asymptotically efficient in some typical cases.

For this, let us now take the regulation factor sequence $\{\mu_k\}$ in the second step of the TSQN algorithm as

$$\mu_k = \sigma_k(\phi_k^T \hat{\theta}_k),$$

and the function $\sigma_k(\cdot)$ is defined by

$$\sigma_k(x) = E_k\{[S_k(x + e_{k+1}) - G_k(x)]^2\}, \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (19)

Under Assumption 1-3, it is not difficult to obtain that the function $\sigma_k(\cdot)$ has the following properties:

$$\sup_{k \geq 0} \{\mu_k\} \leq \sup_{|x| \leq M_k, k \geq 0} |\sigma_k(x)| < \infty, \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (20)

and

$$\inf_{k \geq 0} \{\mu_k\} \geq \inf_{|x| \leq M_k, k \geq 0} |\sigma_k(x)| > 0, \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (21)

We are now in a position to present a theorem on asymptotic normality of the parameter estimate $\hat{\theta}_k$ under a general non-PE condition.

**Theorem 3:** Let the Assumptions 1-3 be satisfied. Assume that $\{\phi_k, k \geq 0\}$ is an $L_2$ sequence and satisfies as $k \to \infty$,

$$\frac{\log k}{\sqrt{\lambda_{\text{min}}(k)}} = o(1), \hspace{1cm} \text{a.s.}$$  \hspace{1cm} (22)

where $\lambda_{\text{min}}(k)$ is the same as that in Theorem 1. Besides, assume that for each $k \geq 0$, there exists a non-random positive definite matrix $\Delta_k$ such that as $k \to \infty$,

$$\Delta_{k+1}^{-1} Q_{k+1}^{-1} \overset{p}{\to} I,$$  \hspace{1cm} (23)

where $Q_{k+1}^{-1} = \sum_{i=0}^{k} \frac{(G_i(\phi_i^T \theta))}{\sigma_i(\phi_i^T \theta)} (\phi_i(\phi_i^T \theta))^{-1} + P_0^{-1}$. Then the estimate $\hat{\theta}_k$ given by the TSQN algorithm has the following asymptotically normal property as $k \to \infty$:

$$Q_{k+1}^{-1/2} \hat{\theta}_{k+1} \overset{d}{\to} N(0, I),$$  \hspace{1cm} (24)

where $\hat{\theta}_{k+1} = \theta - \Delta_{k+1}^{-1} Q_{k+1}^{-1} \hat{\theta}_k$, $\Delta_{k+1}$ and $\Delta_{k+1}^{-1}$ mean the convergence in probability and in distribution, respectively.

**Remark 4:** Notice that if $\{\phi_k\}$ is a determined sequence, then $\Delta_k$ can be simply chosen as $Q_k^{-1/2}$. Moreover, if $\{\phi_i(\phi_i^T \theta)\}$ is a stationary and ergodic random sequence with positive covariance matrix, then $\Delta_k$ can be taken as $\sqrt{k} [E\{(G_i(\phi_i^T \theta)) \sigma_i(\phi_i^T \theta)^{-1} (\phi_i(\phi_i^T \theta))^{-1}\}]^{1/2}$.

**Remark 5:** If we take $l_k = -\infty$ and $u_k = \infty$, then the system (1)-(2) will degenerate to the standard linear stochastic regression model. In this case, the matrix $Q_{k+1}$ is equal to $\sum_{i=1}^{k} \frac{1}{\sigma_i} \phi_i^T \phi_i^T$, where $\sigma_i$ is the variance of the noise $e_{i+1}$. Thus, our TSQN algorithm is asymptotically efficient [31], provided that $\{\phi_k\}$ is a deterministic sequence and $\{e_k\}$ is independent with Gaussian distribution. Moreover, the next corollary shows that in the typical binary-valued observation case, our algorithm is also asymptotically efficient.

**Corollary 2:** Let the conditions of Theorem 3 hold and $L_k = l_k = u_k$, where the nonlinear models degenerate to linear regression models with binary-valued observations. If $\{\phi_k\}$ is a deterministic sequence and $\{e_k\}$ is an independent sequence, then our TSQN algorithm is asymptotically efficient in the sense of

$$I_k^{-1/2} \hat{\theta}_k \overset{d}{\to} N(0, I),$$  \hspace{1cm} (25)

where $I_k$ is the fisher information matrix $I_k$ given data $\{(y_{k+1}, \phi_k), \hspace{1cm} 0 \leq k \leq n\}$.

In fact, the fisher information matrix $I_k$ in this case can be calculated as follows:

$$I_{k+1} = -E\left[\frac{\partial^2 \log p(y_1, y_2, \cdots, y_{k+1})}{\partial \theta^2}\right] = Q_{k+1}^{-1}.$$  \hspace{1cm} (26)

**Remark 7:** Asymptotic confidence interval. When the sample size $n$ tends to $\infty$, the $1 - \alpha$ asymptotically correct confidence sets usually take the following form ([33], [32]):

$$C = \{\theta : \|\hat{Q}_{n+1}^{-1/2} \hat{\theta}_{n+1}\|^2 \leq \lambda_{m, \alpha}^2\},$$  \hspace{1cm} (27)

where $\hat{Q}_{n+1}$ is an estimate of $Q_{n+1}$ and can be taken as

$$\left(\sum_{i=1}^{n} \frac{(G_i(\phi_i^T \theta_n))}{\sigma_i(\phi_i^T \theta_n)} (\phi_i(\phi_i^T \theta_n))^{-1}\right) \overset{d}{\to} \lambda_{m, \alpha}^2$$  \hspace{1cm} (28)

in our current case, $\lambda_{m, \alpha}^2$ is the
$\alpha-$quantile of the standard $X^2$-distribution with degrees of freedom $m$, where $m$ is the dimension of the parameter $\theta$. Based on Theorem 3, the confidence set $C$ may be computed approximately when $n$ is large enough. Moreover, since

$$
\|\tau_j^T\hat{\theta}_{n+1}\| = \|\tau_j^T\hat{\theta}_{n+1} - \hat{\theta}_{n+1}\| \\
\leq \|\tau_j^T\hat{\theta}_{n+1}\| \cdot \|\hat{\theta}_{n+1} - \bar{\theta}_{n+1}\|,
$$

(29)

where $\tau_j$ is the $j^{th}$ column of the identity matrix, from which we can get a more detailed $1-\alpha$ asymptotically correct confidence intervals for the components of the estimation error vector as follows:

$$
C(j) = \left(\hat{\theta}_{n+1}^j - \sqrt{\frac{\min_{k \leq N} \hat{\theta}_{n+1}^j} \frac{\hat{\theta}_{n+1}^j}} \hat{\theta}_{n+1}^j + \sqrt{\frac{\min_{k \leq N} \hat{\theta}_{n+1}^j} \frac{\hat{\theta}_{n+1}^j}}\right),
$$

(30)

where $\hat{\theta}_{n+1}^j$ is the $j^{th}$ diagonal element of $\hat{\theta}_{n+1}$.

C. Non-asymptotic analysis

Though an asymptotic bound can be given based on Theorem 3 as discussed in Remark 7 which, however, requires that the number of data samples is sufficiently large. As a result, the asymptotic bound is hard to apply in real situations where one only has finite number of data samples. In this subsection, we will provide some upper bounds for the estimation errors with high probability when the number of data samples is given and finite.

1) Lyapunov function-based confidence interval: In this subsection, we give a Lyapunov function-based confidence interval based on the analysis of a Lyapunov function used in the proof of Lemma 1. For convenience, we assume the initial values in the TSQN algorithm satisfy $\log|P_0^{-1}| > 1, \log|P_0^{-1}| > 1$, and introduce the following notations to be used throughout the sequel:

$$
w_{k+1} = y_{k+1} - K_k(\phi_k^T \theta).
$$

(31)

Theorem 4: Under Assumptions 1-3, assume that $\{w_k^2\}$ is an $L_2$ sequence. Then for any given $N \geq 1$ and any $0 < \alpha < \frac{1}{2}$, each component $\hat{\theta}_{n+1}^j$ of $\hat{\theta}_{n+1}$ satisfies the following inequality with probability at least $1-2\alpha$:

$$
|\hat{\theta}_{n+1}^j|^2 \leq P_{n+1}^j(\sigma_b \log|P_{n+1}^{-1}| + C \lambda_N + \Gamma), j = 1, \cdots, m,
$$

(32)

and with probability at least $1-2\alpha$, we have

$$
R_n \leq 2\delta_0(\sigma_b \log|P_{n+1}^{-1}| + C \lambda_N + \Gamma),
$$

(33)

where

$$
C = 4\Psi(\sigma_a + \Gamma + 1)^{2+\tau} + 2\Psi(\sigma_a + \Gamma + 6\gamma + 1)^{2+\tau},
$$

$$
\Gamma = V_0 + \sigma_b \log|P_0^{-1}| + \frac{\Phi tr(P_0)\sigma_b}{2\sigma_1} + 18\sigma_1 \frac{1-\alpha}{\alpha},
$$

(34)

Besides, $P_{n+1}^j$ is the $j^{th}$ diagonal component of $P_{n+1}$, and $\delta_0 = \sup_{0 \leq k \leq N} \{k + \beta^2 \theta_k^T P_k \theta_k\}, \Phi = \sup_{0 \leq k \leq N} \{k + \beta^2 \theta_k^T P_k \theta_k\}, \Phi = \sup_{0 \leq k \leq N} \beta^2 \theta_k^T \theta_k , \sigma_b = \sup_{0 \leq k \leq N} \{k + \beta^2 \theta_k^T P_k \theta_k\}$.

Let the empirical distribution function of the generated samples for the $j^{th}$ component be

$$
F_{K}^{(j)}(x) = \frac{1}{K} \sum_{i=1}^{K} I_{(\hat{H}_j(X_i) \leq x)}, \forall x \in R, j = 1, \cdots, m.
$$

(35)

Under the above mentioned assumptions and notations, we have the following proposition on the confidence interval with finite data length:

Proposition 1: For any positive $\alpha$ and $t$ with $\alpha + t < 1$, and any $j = 1, \cdots, m$, the $j^{th}$ component of the estimation error $\hat{\theta}_{n+1}^j$ generated by the TSQN algorithm belongs to the following confidence interval with probability at least $1-\alpha-t$:

$$
\hat{\theta}_{n+1}^j \in [z_{K}^{(j)}(\alpha - \frac{1}{2} - \sqrt{\frac{\ln \frac{2 - \ln t}{2K}}}), z_{K}^{(j)}(1 - \alpha - \frac{1}{2} + \sqrt{\frac{\ln \frac{2 - \ln t}{2K}}})],
$$

(36)

where $z_{K}^{(j)}(\alpha)$ and $z_{K}^{(j)}(\alpha)$ are the $\alpha$ quantiles of the distribution $F_{K}^{(j)}$ and $F_{K}^{(j)}$ respectively.
It is obvious that the confidence interval of the estimation error given in Proposition 1 will decrease asymptotically, as the number of random samplings $K$ used in Proposition 1 increases. We remark that there are at least two advantages of Proposition 1, one is that it is applicable to the case where the data length $n$ is given and finite, another is that the confidence interval may be better than that given in Theorem 4 in some applications.

**IV. PROOFS OF THE MAIN RESULTS**

For convenience, we denote

\[ \tilde{\psi}_k = G_k(\phi_k^T \theta) - G_k(\phi_k^T \tilde{\theta}_k), \] (37)
\[ \psi_k = G_k(\phi_k^T \theta) - G_k(\phi_k^T \hat{\theta}_k). \] (38)

To prove the main theorem, we need to establish the following lemma first.

**Lemma 1:** Let Assumptions 1-3 be satisfied. Then the parameter estimate $\hat{\theta}_k$ given by TSQN Algorithm has the following property as $k \to \infty$:

\[ \tilde{\theta}_{k+1}^{-1} P_{k+1} \tilde{\theta}_{k+1} + \sum_{i=0}^{k} a_i \psi_i^2 = O(\log \lambda_{\max}(k)) . \] (39)

where $\tilde{\theta}_k$ is defined as $\theta - \hat{\theta}_k$, $\psi_k$ is defined as in (38).

**Proof:**

Following the analysis ideas of the classical least-squares for linear stochastic regression models (see e.g., [11], [13], [28]), we consider the following stochastic Lyapunov function:

\[ V_{k+1} = \tilde{\theta}_{k+1}^{-1} P_{k+1} \tilde{\theta}_{k+1} . \] (40)

By (13), we know that

\[ P_{k+1}^{-1} = P_k^{-1} + \mu_k^{-1} \beta_k \phi_k \phi_k^T . \] (41)

Hence, multiplying $a_k \phi_k^T P_k$ from the left hand side and noticing the definition of $a_k$, we know that

\[ a_k \phi_k^T P_k \tilde{\theta}_k^{-1} = a_k \phi_k^T (I + \mu_k^{-1} \beta_k \phi_k \phi_k^T) = \mu_k^{-1} \phi_k^T . \] (42)

Also by (38) and the definition of $\beta_k$ in (13), we know that

\[ G_k(\phi_k^T \theta) - G_k(\phi_k^T \tilde{\theta}_k) = G_k(\phi_k^T \theta) - G_k(\phi_k^T \hat{\theta}_k) + G_k(\phi_k^T \hat{\theta}_k) - G_k(\phi_k^T \tilde{\theta}_k) \] (43)
\[ = \psi_k + \beta_k \phi_k^T (\hat{\theta}_k - \tilde{\theta}_k) = \psi_k + \beta_k \phi_k^T (\theta - \hat{\theta}_k - \tilde{\theta}_k) . \]

Hence,

\[ \psi_k = \beta_k \phi_k^T \hat{\theta}_k = \tilde{\psi}_k - \beta_k \phi_k^T \tilde{\theta}_k . \] (44)

Moreover, by Lemma 2 in Appendix II, (40), (42), and (44), we know that

\[ V_{k+1} \leq [\tilde{\theta}_k - a_k \beta_k \phi_k \phi_k^T (\psi_k + w_{k+1})] P_{k+1}^{-1} \tilde{\theta}_k . \]
\[ = V_k + \mu_k^{-1} \beta_k^2 \phi_k^T P_k \tilde{\theta}_k \psi_k \]
\[ + a_k \beta_k^2 \phi_k^T \psi_k P_k^{-1} (\psi_k - \psi_k) P_k \phi_k \psi_k^T \phi_k \psi_k^T \phi_k \]
\[ + 2a_k \beta_k^2 \phi_k^T \phi_k \psi_k w_{k+1} \]
\[ + \beta_k^2 a_k^2 \phi_k^T \phi_k \psi_k w_{k+1} \]
\[ = V_k - \psi_k^2 \]
\[ + a_k \beta_k^2 \phi_k^T \psi_k \]
\[ + 2\mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k) w_{k+1} \]
\[ - \psi_k \psi_k \]
\[ + \mu_k^{-1} a_k \beta_k^2 \phi_k^T \phi_k \psi_k w_{k+1} \]
\[ + \mu_k^{-1} a_k \beta_k^2 \phi_k^T \phi_k \phi_k \psi_k w_{k+1} \]
\[ = V_k - \psi_k^2 + \mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k)^2 \]
\[ + 2\mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k) w_{k+1} + \mu_k^{-1} a_k \beta_k^2 \phi_k^T \phi_k \psi_k w_{k+1} \]
\[ \leq V_k + \mu_k^{-1} \beta_k^2 \phi_k^T P_k \tilde{\theta}_k \psi_k \]
\[ + 2\mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k) w_{k+1} + \mu_k^{-1} a_k \beta_k^2 \phi_k^T \phi_k \psi_k w_{k+1} . \] (45)

where $w_{k+1}$ is defined in (31). Summing up both sides of (45) from 0 to $n$ and using (44), we have

\[ V_{n+1} \leq V_0 - \sum_{k=0}^{n} a_k \psi_k^2 + \sum_{k=0}^{n} \mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k)^2 \]
\[ - \sum_{k=0}^{n} 2a_k \psi_k w_{k+1} \]
\[ + \sum_{k=0}^{n} 2\mu_k^{-1} (\psi_k - \beta_k \phi_k^T \tilde{\theta}_k) w_{k+1} \]
\[ + \sum_{k=0}^{n} \mu_k^{-1} a_k \beta_k^2 \phi_k^T \phi_k \psi_k w_{k+1} , \quad \text{a.s.} \] (46)

We now analyze the RHS of (46) term by term. First, we have

\[ \sup_{|x| \leq M_k} E_k \{|S_k(x + e_k)|^{2+\gamma} \} < \infty , \quad \text{a.s.} \] (47)

Since $\phi_k^T \theta$ is $F_k$-measurable and $|\phi_k^T \theta| \leq M_k$, a.s. by (3), we can easily have $\sup E_k \{|w_{k+1}|^{2+\gamma} \} < \infty$, a.s. Thus, by Lemma 3 in Appendix II, we know that

\[ \sum_{k=1}^{n} 2a_k \psi_k w_{k+1} = o \left( \sum_{k=1}^{n} a_k \psi_k^2 \right)^{\frac{1}{2}+\gamma} \]
\[ = o \left( \sum_{k=1}^{n} a_k \psi_k^2 \right) + O(1) , \quad \text{a.s., } \forall \gamma > 0 , \] (48)

where we have used the fact that $\sup_{k \geq 0} \{a_k\} \leq \sup_{k \geq 0} \{\mu_k^{-1}\} < \infty$, a.s. Similarly, we have

\[ \sum_{k=1}^{n} 2\mu_k^{-1} (\psi_k - \beta_k \phi_k^T \hat{\theta}_k) w_{k+1} \]
\[ = o \left( \sum_{k=1}^{n} \mu_k^{-1} (\psi_k - \beta_k \phi_k^T \hat{\theta}_k)^2 \right)^{\frac{1}{2}+\gamma} \]
\[ = o \left( \sum_{k=1}^{n} \mu_k^{-1} (\psi_k - \beta_k \phi_k^T \hat{\theta}_k)^2 \right) + O(1) , \quad \text{a.s.} \] (49)

For the last term on the RHS of (46), let us take $X_k = \beta_k \phi_k$
in Lemma 4 in Appendix II, we get
\[
\sum_{k=0}^{n} a_k \beta_k^2 \phi_k^T P_k \phi_k = O(\log \lambda_{\max}(n)), \quad \text{a.s.} \tag{50}
\]
Moreover, from Lyapunov inequality, we have for any \( \delta \in (2, \min(\eta, 4)) \)
\[
\sup_{k \geq 0} E_k \{w_{k+1}^2 - E_k \{w_{k+1}^2\}\} < \infty, \quad \text{a.s.} \tag{51}
\]
Denote \( \Lambda_n = \sum_{k=0}^{n} (a_k \beta_k^2 \phi_k^T P_k \phi_k)^{\frac{1}{2}} \), by Lemma 3 in Appendix II with \( \alpha = \frac{\delta}{2} \), we get
\[
\sum_{k=0}^{n} \mu_k^{-1} a_k \beta_k^2 \phi_k^T P_k \phi_k (w_{k+1}^2 - E_k \{w_{k+1}^2\}) = O_{\gamma}(\Lambda_n \log^{\frac{\gamma}{2}}(\Lambda_n) + \gamma) \tag{52}
\]
Hence, from (50) and (52)
\[
\sum_{k=0}^{n} \mu_k^{-1} a_k \beta_k^2 \phi_k^T P_k \phi_k w_{k+1}^2 \\
\leq \sum_{k=0}^{n} \mu_k^{-1} a_k \beta_k^2 \phi_k^T P_k \phi_k (w_{k+1}^2 - E_k \{w_{k+1}^2\}) + \sup_{k \geq 0} \{E_k \{w_{k+1}^2\}\} \left( \sum_{k=0}^{n} \mu_k^{-1} a_k \beta_k^2 \phi_k^T P_k \phi_k \right) \\
= O(\log \lambda_{\max}(n)) \quad \text{a.s.}
\]
For the third term on RHS of (46), let
\[
\zeta_k = \frac{\tilde{\psi}_k}{\phi_k^T \theta_k} I(\phi_k^T \theta_k \neq 0) + \theta_k^T I(\phi_k^T \theta_k = 0), \tag{54}
\]
by (9), we then have
\[
0 < \zeta_k \leq \bar{\gamma}_k, \quad 0 < \beta_k \leq \bar{\gamma}_k, \quad \text{a.s.} \tag{55}
\]
Hence, we can obtain that
\[
\sum_{k=1}^{n} \mu_k^{-1}(\tilde{\psi}_k - \beta_k \phi_k^T \theta_k)^2 \\
= \sum_{k=1}^{n} \mu_k^{-1}(\zeta_k - \beta_k)^2(\phi_k^T \theta_k)^2 = O(\sum_{k=1}^{n}(\phi_k^T \theta_k)^2), \tag{56}
\]
where we have used the fact that \( |\zeta_k - \beta_k| \leq \sup_{k \geq 0} \{\bar{\gamma}_k\} < \infty \).
We now prove
\[
\sum_{k=1}^{n}(\phi_k^T \tilde{\theta}_k)^2 = O(\log \lambda_{\max}(n)). \tag{57}
\]
For this, we consider the Lyapunov function
\[
\tilde{V}_{k+1} = \tilde{\theta}_k^T P_k \tilde{\theta}_{k+1}, \tag{58}
\]
Similarly to (45), we have the following property:
\[
\tilde{V}_{n+1} \leq V_0 + \sum_{k=0}^{n} (\beta_k \phi_k^T \tilde{\theta}_k - \bar{\alpha}_k \phi_k^T P_k \phi_k \tilde{\theta}_k^2) \\
+ \sum_{k=0}^{n} \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k \phi_k \tilde{\theta}_k^2 \\
- \sum_{k=0}^{n} \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k \phi_k \tilde{\theta}_k^2 w_{k+1} \\
+ \sum_{k=0}^{n} \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k (w_{k+1}^2 - E_k \{w_{k+1}^2\}), \quad \text{a.s.} \tag{59}
\]
By the definition of \( \beta_k \), we have \( |\beta_k \phi_k^T \tilde{\theta}_k - \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k| \leq |\tilde{\theta}_k| \). Besides, following the similar analysis for the noise term as in (48)-(53), we will have
\[
\tilde{V}_{n+1} + \sum_{k=0}^{n} (\beta_k \phi_k^T \tilde{\theta}_k - \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k^2) \\
= O(\log \lambda_{\max}(n)), \quad \text{a.s.}
\]
Also by the definition of \( \beta_k \) in (12) and \( \tilde{\psi}_k \) in (37), we have
\[
\tilde{\psi}_k^2 \geq g_k^2 (\phi_k^T \tilde{\theta}_k)^2 \geq \beta_k^2 (\phi_k^T \tilde{\theta}_k)^2, \quad \text{a.s.} \tag{61}
\]
and
\[
\sum_{k=0}^{n} (\beta_k \phi_k^T \tilde{\theta}_k - \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k^2) \\
\geq \frac{1}{2} \bar{\alpha}_k \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k \phi_k \tilde{\theta}_k^2, \quad \text{a.s.} \tag{62}
\]
Moreover, Since \( \{\beta_k\} \) and \( \{\phi_k\} \) are bounded, we obtain that
\[
\inf_{k \geq 0} \{\bar{\alpha}_k\} = \inf_{k \geq 0} \left\{ \frac{1}{1 + \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k^2} \right\} \geq \inf_{k \geq 0} \left\{ \frac{1}{1 + \beta_k^2 \phi_k^T \tilde{P}_0 \phi_k \tilde{\theta}_k^2} \right\} > 0, \quad \text{a.s.} \tag{63}
\]
Note that \( \{\beta_k\} \) has a positive lower bounded almost surely, (57) can be obtained by (60) (62) and (63).
Finally, combining (46), (48), (49), (53), (56) and (57), we get the desired result
\[
V_{k+1} + \sum_{i=0}^{n} \mu_i \psi_i^2 = O(\log \lambda_{\max}(k)), \quad \text{a.s.} \tag{64}
\]

**Proof of Theorem 1 and Theorem 2:** For the proof of Theorem 1, note that
\[
V_{n+1} \geq \inf_{k \geq 0} \{\mu_k^{-1}\} \inf_{k \geq 0} \{g_k^2\} (\lambda_{\min}(n) - ||F_0^{-1}|| ||\tilde{\theta}_{n+1}||)^2, \quad \text{a.s.}
\]
Theorem 1 follows immediately from Lemma 1, because \( \lambda_{\min}(n) \geq \lambda_0 > ||F_0^{-1}|| \) and \( \inf_{k \geq 0} \{\mu_k\} > 0, \inf_{k \geq 0} \{g_k\} > 0 \).
For the proof of Theorem 2, from the definition of \( \psi_k \) in (38), we have
\[
\psi_k^2 \geq g_k^2 (\phi_k^T \tilde{\theta}_k)^2, \tag{65}
\]
where \( \inf_{k \geq 0} \{g_k\} > 0 \) by (10). Besides, Since \( \{\mu_k\}, \{\beta_k\} \) and \( \{\phi_k\} \) are bounded, we obtain that
\[
\inf_{k \geq 0} \{\bar{\alpha}_k\} = \inf_{k \geq 0} \left\{ \frac{1}{\mu_k + \beta_k^2 \phi_k^T \tilde{P}_k \phi_k \tilde{\theta}_k^2} \right\} > 0, \quad \text{a.s.} \tag{66}
\]
Thus Theorem 2 also follows from Lemma 1.

Proof of Theorem 3:

Similar result has been proven in [13] for linear stochastic regression models, where some key ideas may also be used here to prove Theorem 3 for our current nonlinear stochastic regression models, and the details will be presented elsewhere.

Proof of Proposition 1: For any given positive $\alpha$ and $t$ with $\alpha + t < 1$, denote $v = \sqrt{\frac{\ln 2 - \ln t}{2K}}$ and let

$$E_1 = \{\hat{\theta}_n^j \in [z_K^j (\frac{\alpha}{2} - v), z_K^j (1 - \frac{\alpha}{2} + v)]\},$$

$$E_2 = \{\hat{\theta}_n^j \in [z_K^j (\frac{\alpha}{2})], z_K^j (1 - \frac{\alpha}{2})]\},$$

$$E_3 = \{F_K^j (z_K^j (\frac{\alpha}{2} - v) - F_j (z_K^j (1 - \frac{\alpha}{2} + v)) < v)\} \cap \{F_K^j (z_K^j (1 - \frac{\alpha}{2} + v) - F_j (z_K^j (1 - \frac{\alpha}{2} + v)) < v)\}.$$ (67)

For every $\omega \in E_k$, we have

$$F_j (z_K^j (\frac{\alpha}{2} - v)) < F_j (z_K^j (\frac{\alpha}{2} - v)) + v = \frac{\alpha}{t},$$

$$F_j (z_K^j (1 - \frac{\alpha}{2} + v)) > F_j (z_K^j (1 - \frac{\alpha}{2} + v)) - v = 1 - \frac{\alpha}{2},$$

which means

$$z_K^j (\frac{\alpha}{2}) \geq z_K^j (\frac{\alpha}{2} - v),$$

$$z_K^j (1 - \frac{\alpha}{2}) \leq z_K^j (1 - \frac{\alpha}{2} + v).$$ (69)

Hence, we have

$$(E_2 \cap E_3) \subset E_1.$$ (70)

From the definition of $E_2$, we have

$$P(E_2) < \alpha.$$ (71)

We now prove that

$$P(E_3) < t.$$ (72)

Since $E(F_j (x)) = E(\frac{1}{K} \sum_{i=1}^{K} I_{\{|H(x_i)\leq x\}}) = F_j (x)$ for any $x \in R$, by Hoffding’s inequality, we have

$$P\{F_K^j (z_K^j (\frac{\alpha}{2} - v)) - F_j (z_K^j (\frac{\alpha}{2} - v)) \leq \frac{t}{2}\}$$

$$P\{F_K^j (z_K^j (1 - \frac{\alpha}{2} + v) - F_j (z_K^j (1 - \frac{\alpha}{2} + v)) \geq \frac{t}{2}\}$$

Thus (72) holds true. From (70), (71) and (72), we finally have $P(E_1) \geq 1 - \alpha - t$, which proves Proposition 1.

V. NUMERICAL SIMULATIONS

In this section, we compare the convergence speeds of the preliminary parameter estimate $\hat{\theta}_k$ in the first step with the accelerated parameter estimate in the second step $\hat{\theta}_k$ of TSQN algorithm by a simulation example. Let the saturation function $S_k(\cdot)$ in the model (1)-(2) be time-invariant with

$$S_k(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 15 \\ 15 & x > 15 \end{cases}.$$ (74)

The regressors $\{\phi_j\}_{m=0}$ and observations $\{y_{i+1}\}$ are generated by the following dynamical system model:

$$\begin{cases} \phi_{k+1} = A \phi_k + u_k \\ y_{k+1} = S_k(\phi_k^T \theta + e_{k+1}) \end{cases}.$$ (75)

The diminishingly excited input $u_k = (u_k^{(1)}, \cdots, u_k^{(10)})$, where $u_k^{(j)}$ are independent with the distribution $u_k^{(j)} \sim N(0, 1)$ and $u_k^{(i)} \sim \frac{1}{\sqrt{10}} N(0, 1)$ for any $k \geq 0$ and $i = 2, \cdots, m$. The state matrix $A = \text{diag}\{0.3, 0.5, 0.1, 0.01, 0.9, 0.95, 0.5, 0.4, 0.6, 0.1\}$, the parameter $\theta = [-1.2, 0.5, 1, -0.5, 1.5, -1, 1.8, 0.8, -1.8, 0.4]^T$, and the noise sequence $\{e_{k+1}\}$ is i.i.d with normal distribution $N(0, 1)$. Let $\phi_0 = 0$, it can be verified that the PE condition is not satisfied in this case. Let the convex compact parameter set be $D = \{x \in \mathbb{R}^{10} : |x^{(i)}| \leq 2\}$, on which the estimate is projected. The preliminary estimate $\hat{\theta}_k$ and accelerated estimate $\hat{\theta}_k$ will be generated by our TSQN algorithm, respectively.

Fig 2 shows the trajectory of the estimation error of the preliminary estimate $\hat{\theta}_k$ and the accelerated estimate $\hat{\theta}_k$, where we can see that the accelerated estimate $\hat{\theta}_k$ in the second step does indeed outperform the performance of the preliminary estimate $\hat{\theta}_k$ in the first step, in terms of the convergence of the parameter estimation error.

Fig 3 shows the convergence result of the averaged regrets of adaptive prediction under TSQN algorithm, where one can see again that the averaged regrets in the second step outperforms that in the first step.

VI. CONCLUSION

Motivated by various application backgrounds, we have in this paper studied the problem of adaptive identification
and prediction problems of stochastic dynamical systems with saturated observations. To improve the performance of the estimation algorithm designed previously by using a single step nonlinear Quasi-Newton method, we have proposed a new two-step TSQN algorithm to estimate the unknown parameters. It is shown that the strong consistency and the asymptotic normality of the estimate can be established under general non-PE conditions as the data length increases to infinity. When the data length is given and finite, it is also shown that the estimation performance can also be guaranteed with high probability by using either Lyapunov function-based method or Monte Carlo-based method, which appears to be more suitable for application problems where only finite length of data are available. Simulation example also demonstrates that the performance of the proposed TSQN algorithm is better than the single Quasi-Newton algorithm even under non-PE conditions of the data. The proposed new TSQN algorithm has also been used successfully in sentencing computation problems with real finite data set in [8]. For future investigation, there are still a number of interesting problems need to be solved in theory, for example, how to establish global convergence or estimation error bounds for adaptive estimation algorithms of more complicated stochastic regression models including multi-layer neural networks, and how to solve adaptive control problems with saturated observations for stochastic dynamical control systems, etc.

**APPENDIX I**

In this appendix, we give three examples for the calculation of the functions $G_k(\cdot)$ and $\sigma_k(\cdot)$.

**Example 1:** Let us consider the case where $l_k = -\infty$ and $u_k = \infty$ for any $k \geq 1$, then the model degenerate to the classical linear regression model, and in this case we have

$$ G(x) = \mathbb{E}[S(x + \epsilon_{k+1}) \mid F_k] \equiv x, $$

$$ \sigma(x) = \mathbb{E}[|S(x + \epsilon_{k+1}) - G(x)|^2] \mid F_k] = \mathbb{E}[\epsilon_{k+1}^2 \mid F_k]. $$

**Example 2:** Let us consider the case where $L_k = l_k = u_k = 0, U_k = 1$, then the saturated function will turn to be a binary-valued function, which is widely used in classification problem. Let the noise $\epsilon_k$ is $F_k-$measurable with the conditional probability distribution function $F_k(\cdot)$, then we have

$$ G_k(x) = 1 - F_k(-x), $$

$$ \sigma_k(x) = F_k(-x)[1 - F_k(-x)]. $$

**Example 3:** Let us consider the case where $L_k = l_k < u_k = U_k$ for any $k \geq 1$, the noise $\epsilon_k$ is $F_k-$measurable with the conditional probability distribution function $F_k(\cdot)$ and the conditional probability density function $f_k(\cdot)$, then the function $G_k(\cdot)$ and $\sigma_k(\cdot)$ can be calculated as follows:

$$ G_k(x) = \int_{l_k}^{u_k} t f_k(t-x) dt + l_k F_k(l_k-x), $$

$$ \sigma_k(x) = \int_{l_k}^{u_k} [t - G_k(x)]^2 f_k(t-x) dt $$

$$ + (l_k - G_k(x))^2 F_k(l_k-x) + (u_k - G_k(x))^2 (1 - F_k(u_k-x)). $$

(76)

In particular, when the noise sequence $\{\epsilon_k\}$ is independent and normally distributed with $\epsilon_k \sim N(0, \sigma^2)$, $F_k(\cdot) \equiv F(\cdot)$ and $f_k(\cdot) \equiv f(\cdot)$ will correspond to the distribution function and density function of $N(0, \sigma^2)$ respectively, and $G_k(x)$ and $\sigma_k(x)$ can be calculated further as follows:

$$ G_k(x) = u_k + (l_k - x) F(l_k-x) - (u_k - x) F(u_k-x) + \sigma^2 [f(l_k-x) - f(u_k-x)], $$

$$ \sigma_k(x) = (u_k - G_k(x))^2 + \sigma^2 F(u_k-x) $$

$$ + [(G_k(x) - x)^2 - (u_k - G_k(x))^2] F(u_k-x) $$

$$ - [(l_k - G_k(x))^2 - (G_k(x) - x)^2 - \sigma^2]^2 F(l_k-x) $$

$$ + \sigma^2 [(l_k + x - 2G_k(x)) f(l_k-x) $$

$$ - (u_k + x - 2G_k(x)) f(u_k-x)]. $$

(77)

**APPENDIX II**

Lemma 2: ( [34]). The projection operator given by Definition 1 satisfies

$$ \|\Pi^k_Q(x) - \Pi^k_Q(y)\|_Q \leq \|x - y\|_Q \quad \forall x, y \in \mathbb{R}^m $$

(78)

Lemma 3: ( [9]). Let $\{w_n, F_n\}$ be a martingale difference sequence and $\{f_n, F_n\}$ an adapted sequence. If

$$ \sup_n \mathbb{E}[\|w_{n+1}\|^\alpha \mid F_n] < \infty \quad a.s. $$

(79)

for some $\alpha \in (0, 2]$, then as $n \to \infty$:

$$ \sum_{i=0}^n f_i w_{i+1} = O(s_n(\alpha) \log^{\frac{1}{2} + \eta}(s_n(\alpha) + \varepsilon)) \quad a.s., \forall \eta > 0, $$

(80)

where

$$ s_n(\alpha) = \left( \sum_{i=0}^n |f_i|^{\alpha} \right)^{\frac{1}{\alpha}}. $$

(81)

Lemma 4: ( [13]). Let $X_1, X_2, \cdots$ be a sequence of vectors in $\mathbb{R}^m (m \geq 1)$ and let $A_n = A_0 + \sum_{i=1}^n X_i X_i^T$. Let $|A_n|$ denote the determinant of $A_n$. Assume that $A_0$ is nonsingular, then as $n \to \infty$

$$ \sum_{k=0}^n X_k A_k^{-1} X_k \leq \log(|A_n|) + \log(|A_0|). $$

(82)

Lemma 5: ( [28]). Let $X_1, X_2, \cdots$ be any bounded sequence of vectors in $\mathbb{R}^m (m \geq 1)$. Denote $A_n = A_0 + \sum_{i=1}^n X_i X_i^T$ with $A_0 > 0$, then we have

$$ \sum_{k=0}^\infty (X_k A_k^{-1} X_k)^2 < \infty. $$

(83)

Lemma 6: ( [35]). For each $n \geq 1$, let $\{S_{n,i} = \sum_{j=1}^n X_{n,i}, F_j, 1 \leq j \leq n < \infty\}$ be an $L_2$ stochastic sequence on $(\Omega, F, p)$ satisfying

$$ \sum_{k=1}^n \mathbb{E}(X_{n,k} \mid F_{k-1}) \overset{p}{\to} 0, $$

(84)

$$ \sum_{k=1}^n \mathbb{E}(X_{n,k}^2 \mid F_{k-1}) - (\mathbb{E}(X_{n,k} \mid F_{k-1}))^2 \overset{p}{\to} \eta^2, $$

(85)

and

$$ \sum_{k=1}^n \mathbb{E}(X_{n,k}^2 I(\|X_{n,k}\| > \varepsilon) \mid F_{n,k-1}) \overset{p}{\to} 0, \quad \varepsilon > 0. $$

(86)
for some non-negative constant $\eta^2$, then we have $S_{n,n} \xrightarrow{d} N(0, \eta^2)$.

**Lemma 7:** If $\{S_n = \sum_{k=1}^{n} X_k, F_n, n \geq 1\}$ is an $\mathcal{L}_2$ martingale with $\mathbb{E}[S_1] = 0$ and $\mathcal{F}_0 = (\emptyset, w)$, then for any positive constants $x, y$,

$$P\{S_n \geq x \sum_{k=1}^{n} \mathbb{E}[X_k^2 | \mathcal{F}_{k-1}] + y, \text{some } n \geq 1\} \leq \frac{1}{1 + xy} \tag{87}$$

### References

[1] W. S. McChlloch and W. Pitts, “A logical calculus of the ideas immanent in nervous activity,” The Bulletin of Mathematical Biophysics, vol. 5, no. 10, pp. 115-133, Dec. 1943.

[2] S. I. Gallant, “Perceptron-based learning algorithms,” IEEE Transactions on Neural Networks, vol. 1, no. 2, pp. 179-191, Jun. 1990.

[3] C. Rago, P. Willett and Y. Bar-Shalom, “Censoring sensors: a low-communication-rate scheme for distributed detection,” IEEE Transactions on Aerospace and Electronic Systems, vol. 32, no. 2, pp. 554-568, Apr. 1996.

[4] S. Appadwedula, V. V. Veeravalli, and D. L. Jones, “Decentralized detection with censoring sensors,” IEEE Transactions on Signal Processing, vol. 56, no. 4, pp. 1362-1373, Mar. 2008.

[5] S. Tan, J. Guo, Y. Zhao and J. Zhang, “Adaptive control with saturation-constrained observations for drag-free satellites—a set-valued identification approach,” Science China Information Sciences, vol. 64, no. 10, pp. 1-12, Oct. 2021.

[6] J. Tobin, “Estimation of relationships for limited dependent variables,” Econometrica: Journal of the Econometric Society, vol. 26, no. 1, pp. 24-36, Jan. 1958.

[7] S. B. Caudill and J. D. Jackson, “Heteroscedasticity and grouped data regression,” Southern Economic Journal, vol. 60, no. 1, pp. 128-135, Jul. 1993.

[8] F. Wang, L. Zhang and L. Guo, “Sentencing computation: nonlinear models and reliability analysis,” to be published, Jul. 2022.

[9] H. F. Chen and L. Guo, Identification and Stochastic Adaptive Control, Birkhäuser, Boston, 1991.

[10] L. Ljung, “Consistency of the least-squares identification method,” IEEE Transactions on Automatic Control, vol. 21, no. 15, pp. 779-781, Oct. 1976.

[11] J. B. Moore, “On strong consistency of least squares identification algorithms,” Automatica, vol. 14, no. 5, pp. 505-509, Sep. 1978.

[12] L. Ljung, “Analysis of recursive stochastic algorithms,” IEEE Transactions on Automatic Control, vol. 22, no. 4, pp. 551-575, Aug. 1977.

[13] T. L. Lai and C. Z. Wei, “Least squares estimates in stochastic regression models with applications to identification and control of dynamic systems,” The Annals of Statistics, vol. 10, no. 1, pp. 154-166, Mar. 1982.

[14] L. Guo, “Feedback and uncertainty: Some basic problems and results,” Annual Reviews in Control, vol. 24, pp. 27-36, May. 2020.

[15] J. L. Powell, “Least absolute deviations estimation for the censored regression model,” Journal of Econometrics, vol. 25, no. 3, pp. 303-325, Jul. 1984.

[16] T. L. Lai and Z. Ying, “Asymptotically efficient estimation in censored and truncated regression models,” Statistica Sinica, vol. 2, no. 1, pp. 17-46, Jan. 1992.

[17] J. J. Heckman, “The common structure of statistical models of truncation, sample selection and limited dependent variables and a simple estimator for such models,” Annals of Economic and Social Measurement, vol. 5, no. 4, pp. 475-492, Oct. 1976.

[18] L. Y. Wang, J. F. Zhang and G. G. Yin, “System identification using binary sensors,” IEEE Transactions on Automatic Control, vol. 48, no. 11, pp. 1892-1907, Nov. 2003.

[19] L. Y. Wang, G. G. Yin and J. F. Zhang, “Joint identification of plant rational models and noise distribution functions using binary-valued observation,” Automatica, vol. 42, no. 4, pp. 535-547, Apr. 2006.

[20] L. Y. Wang and G. G. Yin, “Asymptotically efficient parameter estimation using quantized output observations,” Automatica, vol. 43, no. 7, pp. 1178-1191, Jul. 2007.

[21] J. Guo and Y. Zhao, “Recursive projection algorithm on FIR system identification with binary-valued observations,” Automatica, vol. 49, no. 11, pp. 3396-3401, Nov. 2013.