SHIFT OPERATORS AND TORIC MIRROR THEOREM

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Abstract. We give a new proof of Givental’s mirror theorem for toric manifolds using the shift operator of equivariant parameters. The proof is almost tautological: it gives an A-model construction of the $I$-function and the mirror map. It also works for non-compact or non-semipositive toric manifolds.

1. Introduction

In 1995, Seidel [Sei97] introduced an invertible element of quantum cohomology associated to a Hamiltonian circle action. This has had many applications in symplectic topology. Seidel himself used it to construct non-trivial elements of $\pi_1$ of the group of Hamiltonian diffeomorphisms. McDuff-Tolman [MT06] calculated Seidel’s elements in a more general setting and obtained Batyrev’s ring presentation of quantum cohomology of toric manifolds. Their method, however, does not yield explicit structure constants of quantum cohomology, i.e. genus-zero Gromov-Witten invariants.

Recently, Braverman, Maulik, Okounkov and Pandharipande [OP10, BMO11, MO12] introduced a shift operator of equivariant parameters for equivariant quantum cohomology. Their shift operators reduce to Seidel’s invertible elements under the non-equivariant limit. In this paper, we show that equivariant genus-zero Gromov-Witten invariants of toric manifolds are reconstructed only from formal properties of shift operators; more precisely we recover Givental’s mirror theorem for toric manifolds. In other words, the equivariant quantum topology of toric manifolds is determined by its classical counterpart.

Recall that a Givental-style mirror theorem for toric manifolds is stated as follows:

Theorem 1.1 (Giv98b, LLY99, Iri08, Bro09, see §4.2 for more details). Let $X_\Sigma$ be a semi-projective toric manifold having a torus fixed point. Let $I(y, z)$ be the cohomology-valued hypergeometric series defined by

$$I(y, z) = z \sum_{i=1}^{m} u_i \log y / z \sum_{d \in \text{Eff}(X_\Sigma)} \left( \prod_{i=1}^{m} \frac{\prod_{c=-\infty}^{0} (u_i + cz)}{\prod_{c=-\infty}^{u_i -d} (u_i + cz)} \right) Q^d y_1^{u_1-d} \cdots y_m^{u_m-d}$$

where $u_i$, $i = 1, \ldots, m$ is the class of a prime toric divisor. Then $I(y, -z)$ lies in Givental’s Lagrangian cone $L_{X_\Sigma}$ associated to $X_\Sigma$.

We prove this theorem in the following way. The Givental cone $L_X$ [Giv04] for a smooth $T$-variety $X$ is an infinite dimensional Lagrangian submanifold of the symplectic vector space (the so-called Givental space)

$$\mathcal{H}_X = H_T^*(X) \otimes_{H_T^*(pt)} \text{Frac}(H_T^*(pt)[z])$$
and encodes all genus-zero Gromov-Witten invariants. By general theory, the shift operator associated to a $\mathbb{C}^\times$-subgroup $k: \mathbb{C}^\times \to T$ defines a vector field on $\mathcal{L}_X$:

$$L_X \ni f \mapsto z^{-1} S_k f \in T_f L_X$$

where $S_k$ denotes the shift operator acting on the Givental space $H_X$. The operator $S_k$ is determined only by classical equivariant topology of $X$ (see Definition 3.13). For toric manifolds, we have a shift operator $S_i$ for each torus-invariant prime divisor. Then we identify the $I$-function $I(y, z)$ with an integral curve of the vector fields $f \mapsto z^{-1} S_i f$.

**Theorem 1.2.** Givental’s $I$-function $I(y, z)$ is a unique integral curve which satisfies the differential equation:

$$\frac{\partial I(y, z)}{\partial y_i} = z^{-1} S_i I(y, z) \quad i = 1, \ldots, m$$

and is of the form $I(y, z) = ze^{\sum_{i=1}^m u_i \log y_i / z} (1 + \sum_{d \in \text{Eff}(X, \Sigma) \setminus \{0\}} I_d Q^d y^d)$, where we set $y^d = \prod_{i=1}^m y_i^{u_i \cdot d}$.

The $I$-function associates a mirror map $y \mapsto \tau(y) \in H_T^2(X)$ via Birkhoff factorization [CG07, Iri08]. As a corollary to our proof, we obtain the following relationship between the equivariant Seidel elements $S_i(\tau)$ and the mirror map. This generalizes a previous result [GI12] in the semipositive case obtained in joint work with Gonzalez.

**Theorem 1.3.** The mirror map $\tau(y)$ associated to the $I$-function is a unique integral curve which satisfies the differential equation

$$\frac{\partial \tau(y)}{\partial y_i} = S_i(\tau(y)) \quad i = 1, \ldots, m$$

and is of the form $\tau(y) = \sum_{i=1}^m u_i \log y_i + \sum_{d \in \text{Eff}(X, \Sigma) \setminus \{0\}} \tau_d Q^d y^d$.

The mirror map and the $I$-function are related by the formula

$$I(y, z) = z M(\tau(y), z) \Upsilon(y, z)$$

where $M(\tau, z)$ is a fundamental solution for the quantum differential equation (Proposition 2.2) and $\Upsilon(y, z)$ is an $H_T^2(X)[z]$-valued function. We can also characterize $\Upsilon(y, z)$ by the shift operator $S_i(\tau)$ acting on quantum cohomology via the differential equation:

$$\frac{\partial \Upsilon(y, z)}{\partial y_i} = [z^{-1} S_i(\tau(y)) \Upsilon(y, z)]_+.$$
This paper is structured as follows. In §2, we review equivariant quantum cohomology and in §3, we study shift operators for big quantum cohomology. In §4, we prove a mirror theorem for toric manifolds.

1.1. Notation. Unless otherwise stated, we consider cohomology groups with complex coefficients. We use the following notation throughout the paper.

- \( T \cong (\mathbb{C}^\times)^m \): an algebraic torus;
- \( X \): a smooth \( T \)-variety; \( X_\Sigma \): a smooth toric variety associated to a fan \( \Sigma \);
- \( \hat{T} = T \times \mathbb{C}^\times \);
- \( \lambda \in \text{Lie}(T) \), \( z \in \text{Lie}(\mathbb{C}^\times) \): equivariant parameters for \( \hat{T} \);
- \( H^*_T(X)_{\text{loc}} := H^*_{\hat{T}}(X) \otimes_{H^*_T(\text{pt})} \text{Frac}(H^*_T(\text{pt})) = H^*_{\hat{T}}(X) \otimes_{H^*_T(\text{pt})} \text{Frac}(H^*_T(\text{pt})[z]) \): the Givental space.

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2. Equivariant Quantum Cohomology

2.1. Hypotheses on a \( T \)-Space. Let \( T \cong (\mathbb{C}^\times)^m \) be an algebraic torus. Let \( X \) be a smooth variety over \( \mathbb{C} \) equipped with an algebraic \( T \)-action. We assume the following conditions:

1. \( X \) is semi-projective, i.e. the natural map \( X \to X_0 := \text{Spec} \, H^0(X, \mathcal{O}) \) is projective;
2. all \( T \)-weights appearing in the \( T \)-representation \( H^0(X, \mathcal{O}) \) are contained in a strictly convex cone in \( \text{Hom}(T, \mathbb{C}^\times) \otimes \mathbb{R} \) and \( H^0(X, \mathcal{O})^T = \mathbb{C} \).

A \( T \)-space \( X \) with these assumptions has nice cohomological properties, see, e.g. [HRV13]. These conditions ensure that the \( T \)-fixed set \( X^T \) is projective. We also note the following:

Proposition 2.1. A smooth \( T \)-variety \( X \) satisfying the conditions \([1], (2)\) is equivariantly formal, i.e. \( H^*_T(X) \) is a free module over \( H^*_T(\text{pt}) \) and there is a non-canonical isomorphism \( H^*_T(X) \cong H^*(X) \otimes H^*_T(\text{pt}) \) as an \( H^*_T(\text{pt}) \)-module.

Proof. We use the argument of Kirwan [Kir84, Proposition 5.8] (see also [Nak99, §5.1]). Choose a one-parameter subgroup \( k : \mathbb{C}^\times \to T \) such that \( k \) is negative on every non-zero weight of \( H^0(X, \mathcal{O}) \). This defines a \( \mathbb{C}^\times \)-action on \( X \). Let \( L \to X \) be a very ample line bundle. The \( \mathbb{C}^\times \)-action on \( X \) lifts to a \( \mathbb{C}^\times \)-linearization on \( L \), possibly after replacing \( L \) with its power \( L \otimes 1 \) [Dol03, Corollary 7.2]. Then \( L \) defines a \( \mathbb{C}^\times \)-equivariant closed embedding \( X \hookrightarrow X_0 \times \mathbb{P}^n \), where \( \mathbb{P}^n \) is equipped with a linear \( \mathbb{C}^\times \)-action. By assumption, we can embed the affine variety \( X_0 = \text{Spec}(H^0(X, \mathcal{O})) \) equivariantly into a \( \mathbb{C}^\times \)-representation \( V \) which has only positive weights. Thus we have a \( \mathbb{C}^\times \)-equivariant closed embedding \( X \hookrightarrow V \times \mathbb{P}^n \). The associated \( S^1 \)-action on \( V \times \mathbb{P}^n \) admits, with respect to the standard Kähler metric, a moment map \( \mu \) which is proper and bounded

\footnote{We use the (usual) convention that \( t \in \mathbb{C}^\times \) acts on functions by \( f(x) \mapsto f(t^{-1}x) \).}
from below. These properties allow us to use Morse theory for the moment map \( \mu|_X \). The argument in [Kir84, Nak99] shows that \( \mu|_X \) is a perfect Bott-Morse function and \( X \) is equivariantly formal. \( \square \)

2.2. Gromov-Witten Invariants. For a second homology class \( d \in H_2(X, \mathbb{Z}) \) and a non-negative integer \( n \geq 0 \), we denote by \( X_{0,n,d} \) the moduli stack of genus-zero stable maps to \( X \) of degree \( d \) with \( n \) marked points. The \( T \)-action on \( X \) induces a \( T \)-action on \( X_{0,n,d} \). It has a virtual fundamental class \( [X_{0,n,d}]_{\text{vir}} \in H_* (X_{0,n,d}, \mathbb{Q}) \) of dimension \( D = \dim X + n - 3 + c_1(X) \cdot d \). For equivariant cohomology classes \( \alpha_1, \ldots, \alpha_n \in H^*_T (X, \mathbb{Q}) \) and non-negative integers \( k_1, \ldots, k_n \), the genus-zero \( T \)-equivariant Gromov-Witten invariant is defined by

\[
\langle \alpha_1 \psi^{k_1}, \ldots, \alpha_n \psi^{k_n} \rangle^{X_T}_{0,n,d} = \int_{[X_{0,n,d}]_{\text{vir}}} \prod_{i=1}^n \text{ev}_i^* (\alpha_i) \psi_i^{k_i}.
\]

Here \( \text{ev}_i: X_{0,n,d} \to X \) is the evaluation map at the \( i \)th marked point and \( \psi_i \) denotes the equivariant first Chern class of the universal cotangent line bundle \( L_i \) over \( X_{0,n,d} \). When the moduli space \( X_{0,n,d} \) is not compact, the right-hand side is defined via the Atiyah-Bott localization formula [AB84, GPP99] and belongs to the fraction field \( \text{Frac} (H^*_T (pt)) \).

2.3. Quantum Cohomology. Let \( \text{Eff}(X) \subset H_2(X, \mathbb{Z}) \) denote the semigroup of homology classes of effective curves. We write \( Q \) for the Novikov variable and define \( M[Q] \) to be the space of formal power series:

\[
M[Q] = \left\{ \sum_{d \in \text{Eff}(X)} a_d Q^d : a_d \in M \right\}
\]

with coefficients in a module \( M \). When \( M \) is a ring, \( M[Q] \) is also a ring. Let \( (\cdot, \cdot) \) denote the \( T \)-equivariant Poincaré pairing on \( H^*_T (X) \):

\[
(\alpha, \beta) = \int_X \alpha \cup \beta.
\]

If \( X \) is not compact, we define the right-hand side via the localization formula. Therefore \( (\cdot, \cdot) \) takes values in \( \text{Frac} (H^*_T (pt)) \) in general. Let \( \{ \phi_i \}_{i=0}^N \) be a basis of \( H^*_T (X) \) over \( H^*_T (pt) \). We write \( \{ \tau^i \}_{i=0}^N \) for the dual co-ordinates on \( H^*_T (X) \) and \( \tau = \sum_{i=0}^N \tau^i \phi_i \) for a general point on \( H^*_T (X) \). The (big) quantum product \( \ast \) is defined by the formula

\[
(\phi_i \ast \phi_j, \phi_k) = \sum_{d \in \text{Eff}(X)} \sum_{n=0}^\infty \frac{Q^d}{n!} \langle \phi_i, \phi_j, \phi_k, \tau, \ldots, \tau \rangle^{X_T}_{0,n+3,d}.
\]

We note that the quantum product \( \phi_i \ast \phi_j \) is defined without localization:

\[
\phi_i \ast \phi_j \in H^*_T (X)[Q][\tau^0, \ldots, \tau^N].
\]

In fact, \( \phi_i \ast \phi_j \) can be written as the push-forward

\[
\sum_{d \in \text{Eff}(X)} \sum_{n=0}^\infty \frac{Q^d}{n!} \text{PD} \text{ev}_3^* \left( \text{ev}_1^*(\phi_i) \text{ev}_2^*(\phi_j) \prod_{l=1}^{n+3} \text{ev}_l^*(\tau) \cap [X_{0,n+3,d}]_{\text{vir}} \right).
\]
along the proper evaluation map $ev_3$, and hence the localization is not necessary. The properness of $ev_3$ follows from the assumption that $X$ is semi-projective.

2.4. Quantum Connection and Fundamental Solution. The quantum connection is the operator

$$\nabla_i : H^*_T(X)[z][\tau^0, \ldots, \tau^N] \to z^{-1} H^*_T(X)[z][\tau^0, \ldots, \tau^N]$$

defined by

$$\nabla_i = \frac{\partial}{\partial \tau^i} + \frac{1}{z}(\phi_i \ast).$$

The quantum connection has a parameter $z$: we identify it with the equivariant parameter for an additional $\mathbb{C}^\times$-action. Set $\hat{T} = T \times \mathbb{C}^\times$ and consider the $\hat{T}$-action on $X$ induced by the projection $\hat{T} \to T$. Then we have $H^*_\hat{T}(X) \cong H^*_T(X)[z]$. The quantum connection is known to be flat, and admits a fundamental solution:

$$M(\tau) : H^*_\hat{T}(X)[Q][\tau^0, \ldots, \tau^N] \to H^*_\hat{T}(X)_{loc}[Q][\tau^0, \ldots, \tau^N]$$

satisfying the quantum differential equation:

$$z \frac{\partial}{\partial \tau^i} M(\tau) = M(\tau)(\phi_i \ast)$$

or equivalently $(\partial/\partial \tau^i) \circ M(\tau) = M(\tau) \circ \nabla_i$, where $H^*_\hat{T}(X)_{loc} := H^*_\hat{T}(X) \otimes H^*_\hat{T}(pt)$ is the localized equivariant cohomology. The following proposition is well-known, see [Giv98a, §1], [Pan98, Proposition 2].

**Proposition 2.2.** A fundamental solution is given by

$$(M(\tau)\phi_i, \phi_j) = (\phi_i, \phi_j) + \sum_{d \in \text{Eff}(X), n \geq 0 \atop (d,n) \neq (0,0)} \frac{Q^d}{n!} \left< \phi_i, \tau, \ldots, \tau, \frac{\phi_j}{z - \psi} \right>_{0,n+2,d}^{X,T}.$$

**Remark 2.3.** Expanding $1/(z - \psi) = \sum_{n=0}^{\infty} \psi^n/z^{n+1}$, we find that $M(\tau)\phi_i$ takes values in $H^*_\hat{T}(X)[z^{-1}]$. By the localization calculation, it also follows that $M(\tau)\phi_i$ takes values in $H^*_\hat{T}(X)_{loc}$. The localized $\hat{T}$-equivariant cohomology $H^*_\hat{T}(X)_{loc}$ is also called the Givental space [Giv04].

3. Shift Operator

The shift operator for equivariant quantum cohomology has been introduced by Okounkov-Pandharipande [OP10], Braverman-Maulik-Okounkov [BMO11] and Maulik-Okounkov [MO12]. We discuss its (straightforward) extension to the big quantum cohomology.
3.1. Twisted Homomorphism. We write \( \hat{T} = T \times \mathbb{C}^x \). For a group homomorphism \( k: \mathbb{C}^x \to T \), we consider the \( \hat{T} \)-action \( \rho_k \) on \( X \) defined by
\[
\rho_k(t, u)x = tu^k \cdot x
\]
where \((t, u) \in \hat{T}, x \in X \) and \( u^k \in T \) denotes the image of \( u \in \mathbb{C}^x \) under \( k \). Let \( \lambda \in \text{Lie}(T) \) denote the equivariant parameter for \( T \) and let \( z \in \text{Lie}(\mathbb{C}^x) \) denote the equivariant parameter for \( \mathbb{C}^x \). The identity map \( \text{id}: (X, \rho_0) \to (X, \rho_k) \) is equivariant with respect to the group automorphism
\[
\phi_k: \hat{T} \to \hat{T}, \quad \phi_k(t, u) = (tu^{-k}, u).
\]
Therefore the identity map induces an isomorphism
\[
\Phi_k: H^*_x(T, \rho_0)(X) \cong H^*_x(T, \rho_k)(X)
\]
such that
\[
(3.1) \quad \Phi_k(f(\lambda, z)\alpha) = f(\lambda + kz, z)\Phi_k(\alpha)
\]
where \( \alpha \in H^*_x(T, \rho_0)(X) \) and \( f(\lambda, z) \in H^*_T(\text{pt}) \) is a polynomial function on \( \text{Lie}(\hat{T}) \). Referring to the property (3.1), we say that \( \Phi_k \) is a \( k \)-twisted homomorphism.

**Notation 3.1.** We write \( H^*_x(T, \rho)(X) \) for the \( \hat{T} \)-equivariant cohomology of \( X \) with respect to the \( \hat{T} \)-action \( \rho \) on \( X \). When \( \rho \) is omitted, \( H^*_x(X) \) means \( H^*_x(T, \rho_0)(X) \).

3.2. Bundle Associated to a \( \mathbb{C}^x \)-Subgroup.

**Definition 3.2** (associated bundle). Let \( k: \mathbb{C}^x \to T \) be a group homomorphism. Consider the \( \mathbb{C}^x \)-action on \( X \times (\mathbb{C}^2 \setminus \{0\}) \) given by \( s \cdot (x, (v_1, v_2)) = (s^k \cdot x, (s^{-1}v_1, s^{-1}v_2)) \). Let \( E_k \) denote the quotient space:
\[
E_k := X \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^x.
\]
We have a natural projection \( \pi: E_k \to \mathbb{P}^1 \) given by \( \pi([x, (v_1, v_2)]) = [v_1, v_2] \) and \( E_k \) is a fiber bundle over \( \mathbb{P}^1 \) with fiber \( X \). We consider the \( \hat{T} \)-action on \( E_k \) given by \( (t, u) \cdot [x, (v_1, v_2)] = [t \cdot x, (v_1, uv_2)] \). Let \( X_0 \) denote the fiber of \( E_k \to \mathbb{P}^1 \) at \( [1, 0] \) and let \( X_\infty \) denote the fiber at \([0, 1]\). Note that we have
\[
X_0 \cong (X, \rho_0) \quad \text{and} \quad X_\infty \cong (X, \rho_k)
\]
as \( \hat{T} \)-spaces.

**Definition 3.3.** A group homomorphism \( k: \mathbb{C}^x \to T \) is said to be semi-negative if \( k \) is non-positive on each \( T \)-weight of \( H^0(X, \mathcal{O}) \). We say that \( k \) is negative if \( k \) is negative on each non-zero \( T \)-weight of \( H^0(X, \mathcal{O}) \).

**Remark 3.4.** When \( X \) is complete, every \( \mathbb{C}^x \)-subgroup is negative.

Suppose that \( k: \mathbb{C}^x \to T \) is semi-negative and consider the \( \mathbb{C}^x \)-action on \( X \) induced by \( k \). Let \( L \) be a very ample line bundle on \( X \). As discussed in the proof of Proposition [2.1], we may assume that \( L \) admits a \( \mathbb{C}^x \)-linearization. By tensoring \( L \) with a \( \mathbb{C}^x \)-character, we may assume that all the \( \mathbb{C}^x \)-weights on \( H^0(X, L^{\otimes n}) \) are negative for \( n > 0 \). Let \( p: X \times \mathbb{C}^2 \to X \) be the natural projection. Then \( p^*L \) is a \( \mathbb{C}^x \)-equivariant line bundle.
on $X \times \mathbb{C}^2$, where $\mathbb{C}^\times$ acts on the base by $s \cdot (x, (v_1, v_2)) = (s^k \cdot x, (s^{-1}v_1, s^{-1}v_2))$. We can see that

$$H^0(X \times \mathbb{C}^2, (p^*L)^{\otimes n}) = \bigoplus_{i=0}^{\infty} H^0(X, L^{\otimes n}(-i) \otimes \mathbb{C}[v_1, v_2]^{(i)}$$

where the superscript $(l)$ means the component of $\mathbb{C}^\times$-weight $l$. The unstable locus for the $\mathbb{C}^\times$-action on $(X \times \mathbb{C}^2, p^*L)$, in the sense of Geometric Invariant Theory (GIT), is $X \times \{0\}$ and therefore we find that $E_k$ is the GIT quotient of $X \times \mathbb{C}^2$, i.e. $E_k = \text{Proj}(\bigoplus_{n=0}^{\infty} H^0(X \times \mathbb{C}^2, (p^*L)^{\otimes n}))$. This proves:

**Lemma 3.5.** If $k$ is semi-negative, $E_k$ is semi-projective.

Let $k: \mathbb{C}^\times \to T$ be a semi-negative subgroup and consider the $\mathbb{C}^\times$-action on $X$ induced by $k$. A $\mathbb{C}^\times$-fixed point $x \in X$ defines a section of $E_k \to \mathbb{P}^1$:

$$\sigma_x = (\{x\} \times \mathbb{P}^1) \subset E_k.$$

We now define a minimal section among all such sections associated to fixed points. Using the argument in the proof of Proposition [21], we obtain a $\mathbb{C}^\times$-equivariant closed embedding $X \hookrightarrow \mathbb{P}^n \times \mathbb{C}^l$ where $\mathbb{C}^l$ is a $\mathbb{C}^\times$-representation with only non-negative weights. In particular, for every point $x \in X$, the limit $\lim_{s \to 0} s^k \cdot x$ exists. This implies the existence of the Bialynicki-Birula decomposition [BB73, Theorem 4.1] for $X$: if $X^{C^\times} = \bigsqcup_i F_i$ is the decomposition of the $\mathbb{C}^\times$-fixed locus $X^{C^\times}$ into connected components, we have the induced decomposition of $X$

$$X = \bigsqcup_i U_i, \quad U_i = \{x \in X : \lim_{s \to 0} s^k \cdot x \in F_i\}$$

into locally closed smooth subvarieties $U_i$. In particular there exists a unique $\mathbb{C}^\times$-fixed component $F_{\text{min}} \subset X$ such that all the $\mathbb{C}^\times$-weights on the normal bundle to $F_{\text{min}}$ are positive. The moment map $\mu$ for the associated $S^1$-action attains a global minimum on $F_{\text{min}}$. We call the class of a section $\sigma_{\text{min}}$ of $E_k$ associated to a point in $F_{\text{min}}$ the minimal section class. We write

$$H^2_{\text{sec}}(E_k, \mathbb{Z}) = \left\{d \in H^2(E_k, \mathbb{Z}) : \pi_*(d) = [\mathbb{P}^1] \right\},$$

$$\text{Eff}(E_k)_{\text{sec}} = \text{Eff}(E_k) \cap H^2_{\text{sec}}(E_k, \mathbb{Z})$$

**Lemma 3.6.** If $k$ is semi-negative, we have $\text{Eff}(E_k)_{\text{sec}} = \sigma_{\text{min}} + \text{Eff}(X)$.

**Proof.** The compact case was discussed in [GIT2, Lemma 2.2]. Take a negative one-parameter subgroup $l: \mathbb{C}^\times \to T$ and consider the $\mathbb{C}^\times$-action on $E_k$ induced by $\mathbb{C}^\times \xrightarrow{l} T \times \{1\} \subset \hat{T}$. Observe that all non-zero $\mathbb{C}^\times$-weights on $H^0(E_k, \mathcal{O})$ are negative. This means that $E_{k,0} := \text{Spec} H^0(E_k, \mathcal{O})$ has a unique $\mathbb{C}^\times$-fixed point 0 and $\lim_{s \to 0} s \cdot x = 0$ for all $x \in E_{k,0}$. Therefore every curve can be deformed, via the $\mathbb{C}^\times$-action, to a stable curve in the fiber $K$ of $E_k \to E_{k,0}$ at $0 \in E_{k,0}$ in the same homology class. Since $\hat{T}$-action on $E_k$ preserves $K$ and $K$ is compact, we may further deform a curve in $K$ to a $\hat{T}$-invariant stable curve. A $\hat{T}$-invariant stable curve in $E_k$ is a union of a section class $\sigma_x$ associated to a $T$-fixed point $x \in X$ and effective curves in $X_0 \cup X_\infty$. Suppose that two different fixed points $x, y \in X^T$ are connected by a $k(\mathbb{C}^\times)$-orbit, i.e. $\exists p \in X$, then...
\[
x = \lim_{s \to \infty} s^k \cdot p \quad \text{and} \quad y = \lim_{s \to 0} s^k \cdot p.
\]
The closure \( C = k(\mathbb{C}^\times) \cdot p \) is isomorphic to \( \mathbb{P}^1 \) and \( \sigma_x, \sigma_y \) are contained in a Hirzebruch surface
\[
C \times (\mathbb{C}^2 \setminus \{0\}) / \mathbb{C}^\times \subset E_k.
\]

Then one finds \( \sigma_x = \sigma_y + a[C] \) for some \( a > 0 \). Using the Bialynicki-Birula decomposition for the \( k(\mathbb{C}^\times) \)-action on \( X \), we find that every \( T \)-fixed point is connected to a \( T \)-fixed point on \( F_{\min} \) by a chain of \( k(\mathbb{C}^\times) \)-orbits. The conclusion follows.

**Lemma 3.7.** We have an isomorphism
\[
H^*_T (E_k) \cong \left\{ (\alpha, \beta) \in H^*_T (X) \oplus H^*_{T, \rho_k} (X) : \alpha - \Phi_k^{-1}(\beta) \equiv 0 \mod z \right\}
\]
which sends \( \tau \) to \( \tau|_{X_0}, \tau|_{X_\infty} \). Recall that \( z \) is the equivariant parameter for \( \mathbb{C}^\times \) and we have a canonical isomorphism \( H^*_T (X) \cong H^*_T (X)[z] \).

**Proof.** Consider the Mayer-Vietoris exact sequence associated to the covering \( E_k = U_0 \cup U_\infty \) with \( U_0 = \pi^{-1}(\mathbb{C}) \) and \( U_\infty = \pi^{-1}(\mathbb{P}^1 \setminus \{0\}) \). We have \( H^*_T (U_0) \cong H^*_T (X), \) \( H^*_T (U_\infty) \cong H^*_T (X) \) and \( H^*_T (U_0 \cap U_\infty) \cong H^*_T (X). \) The map \( H^*_T (U_0) \oplus H^*_T (U_\infty) \to H^*_T (U_0 \cap U_\infty) \) is surjective and is given by \( (\alpha, \beta) \mapsto (\alpha - \Phi_k^{-1}(\beta))|_{z=0} \).

**Notation 3.8.** By Lemma [3.7], for \( \tau \in H^*_T (X) \), there exists \( \hat{\tau} \in H^*_T (E_k) \) such that \( \hat{\tau}|_{X_0} = \tau \) and \( \hat{\tau}|_{X_\infty} = \Phi_k(\tau) \). This defines a map \( \hat{\cdot} : H^*_T (X) \to H^*_T (E_k) \). This is not \( H^*_T (pt) \)-linear.

### 3.3. Shift Operator.

**Definition 3.9 (shift operator).** Let \( k : \mathbb{C}^\times \to T \) be a semi-negative group homomorphism. For \( \tau \in H^*_T (X) \), we define \( S_k(\tau) : H^*_T,\rho_0 (X)[Q] \to H^*_T,\rho_k (X)[Q] \) by
\[
\left( \widetilde{S}_k(\tau) \right) \alpha, \beta = \sum_{d \in \text{Eff}(E_k)^{\text{sec}}} \frac{Q^{d - \sigma_{\min}}}{n!} \langle \iota_0 \alpha, \iota_\infty \beta, \hat{\tau}, \ldots, \hat{\tau} \rangle_{E_k, \hat{T}}^{E_k, \hat{T}}
\]
where \((\cdot, \cdot)\) in the left-hand side is the \( \hat{T} \)-equivariant Poincaré pairing on \( H^*_T,\rho_k (X), \alpha \in H^*_T,\rho_0 (X), \beta \in H^*_T,\rho_k (X), \sigma_{\min} \) is the minimal section class for \( E_k \), and \( \iota_0 : X_0 \to E_k \), \( \iota_\infty : X_\infty \to E_k \) are the natural inclusions. We also define
\[
S_k(\tau) = \Phi_k^{-1} \circ \widetilde{S}_k(\tau) : H^*_T (X)[Q] \to H^*_T (X)[Q].
\]

Note that \( \widetilde{S}_k \) is untwisted but \( S_k \) is \((-k)\)-twisted (see (3.11)).

**Remark 3.10.** When \( k \) is semi-negative, \( E_k \) is semi-projective by Lemma [3.5] and thus the shift operator \( S_k \) is defined without localization: we may rewrite \( \widetilde{S}_k \) as the push-forward along an evaluation map (see (3.1)). When \( k \) is not semi-negative, we can still define \( S_k \) over \( \text{Frac}(H^*_T (pt)) \) after choosing a suitable section class \( \sigma_{\min} \).

**Remark 3.11.** Since the map \( \tau \mapsto \hat{\tau} \) is not \( H^*_T (pt) \)-linear, \( S(\tau) \) cannot be written as formal power series in the \( H^*_T (pt) \)-valued variables \( \tau^0, \ldots, \tau^N \). For \( \alpha_1, \ldots, \alpha_l \in H^*_T (X) \) and \( \mathbb{C} \)-valued variables \( t^1, \ldots, t^l \), the shift operator \( S(\tau) \) with \( \tau = \sum_{i=1}^l t^i \alpha_i \) is a formal power series in \( t^1, \ldots, t^l \).
Remark 3.12 (divisor equation). Suppose that \( \tau = h + \tau' \) with \( h \in H^2_T(X) \). Using the divisor equation, we have:

\[
\left( \tilde{S}_k(\tau)\alpha, \beta \right) = e^{-h(k)} \sum_{d \in \text{Eff}(X)} \frac{Q^d e^{h-d}}{n!} (t_{\alpha}, t_{\infty}, \beta, \tau', \ldots \tau')_{0,n+2,\sigma_{\min}+d}
\]

where \( h(k) \) is the pairing between \( k \) and the restriction \( h|_x \in H^2_T(pt) \cong \text{Lie}(T)^* \) of \( h \) to a fixed point \( x \) in the minimal fixed component \( F_{\min} \) (with respect to \( k \)). Note that \( \hat{h} \cdot \sigma_{\min} = -h(k) \).

By the localization theorem of equivariant cohomology \[AB84\], the restriction to the \( T \)-fixed subspace \( X^T \) induces an isomorphism

\[
i^* : H^*_T(X)_{\text{loc}} \xrightarrow{\cong} H^*_T(X^T)_{\text{loc}} = H^*(X^T) \otimes \text{Frac}(H^*_T(pt)).
\]

We use this to define the shift operator on the Givental space \( H^*_T(X)_{\text{loc}} \).

Definition 3.13 (shift operator on the Givental space). Let \( X^T = \bigsqcup_i F_i \) be the decomposition of \( X^T \) into connected components. Let \( N_i \) be the normal bundle to \( F_i \) in \( X \). Let \( N_i = \bigoplus_{\alpha} N_{i,\alpha} \) denote the \( T \)-eigenbundle decomposition, where \( T \) acts on \( N_{i,\alpha} \) by the character \( \alpha \in \text{Hom}(T, \mathbb{C}^\times) \). Let \( \rho_{i,\alpha,j} \), \( j = 1, \ldots, \text{rank}(N_{i,\alpha}) \) denote the Chern roots of \( N_{i,\alpha} \). For a semi-negative \( k \in \text{Hom}(\mathbb{C}^\times, T) \), we define:

\[
\Delta_i(k) = Q^\sigma_{i-\sigma_{\min}} \prod_{\alpha} \prod_{j=1}^{\text{rank}(N_{i,\alpha})} \prod_{c=-\infty}^{0} \frac{Q^c}{(e^{\rho_{i,\alpha,j} + \alpha + cz}) (e^{-\rho_{i,\alpha,j} + \alpha + cz})} \in H^*_T(F_i)_{\text{loc}}[Q]
\]

where \( \alpha \) is regarded as an element of \( H^2_T(pt, \mathbb{Z}) \), \( \sigma_i \) is the section class of \( E_k \) associated to a fixed point in \( F_i \) and \( \sigma_{\min} \) is the minimal section class of \( E_k \). Note that all but finite factors in the infinite product cancel. We define the operator \( S_k : H^*_T(X)_{\text{loc}} \to H^*_T(X)_{\text{loc}} \) by the following commutative diagram:

\[
\begin{array}{ccc}
H^*_T(X)_{\text{loc}} & \xrightarrow{S_k} & H^*_T(X)_{\text{loc}} \\
i^* \downarrow & & \downarrow i^* \\
H^*_T(X^T)_{\text{loc}} & \xrightarrow{\bigoplus \Delta_i(k)e^{-zk\delta_\lambda}} & H^*_T(X^T)_{\text{loc}}
\end{array}
\]

where we use the decomposition \( H^*_T(X^T)_{\text{loc}} \cong \bigoplus_i H^*(F_i) \otimes \text{Frac}(H^*_T(pt)) \) in the bottom arrow and \( e^{-zk\delta_\lambda} \) acts on \( \text{Frac}(H^*_T(pt)) \) by \( f(\lambda, z) \mapsto f(\lambda - k z, z) \). The operator \( S_k \) is a \((-k)\)-twisted homomorphism.

The following is a key property of the shift operator.

Theorem 3.14. We have \( M(\tau) \circ S_k(\tau) = S_k \circ M(\tau) \), where \( M(\tau) \) is the fundamental solution in Proposition 2.3.

Proof. A similar intertwining property has been discussed in \[OP10, BM01, MO12\]. We calculate \( S_k(\tau) \) using \( \hat{T} \)-equivariant localization. We refer the reader to \[GP99, CK99\] for localization arguments in Gromov–Witten theory. Fix a section class \( d \in \text{Eff}(E_k) \). A \( \hat{T} \)-fixed stable map \( f : (C, x_1, \ldots, x_{n+2}) \to E_k \) of degree \( d \) is of the form:
• $C = C_0 \cup C_{\sec} \cup C_{\infty}$ with $C_{\sec} \cong \mathbb{P}^1$;
• $f_0 = f|_{C_0}$ is a $T$-fixed stable map to $X_0$;
• $f_{\infty} = f|_{C_{\infty}}$ is a $T$-fixed stable map to $X_{\infty}$;
• $f_{\sec} = f|_{C_{\sec}}$ is a section of $E_k$ associated to a $T$-fixed point in $X$ (see (3.2)).

Recall that the tangent space $T^1$ and the obstruction space $T^2$ at the stable map $f$ fit into the exact sequence

$$
0 \longrightarrow \text{Ext}^0(\Omega^1_C(x), \mathcal{O}_C) \longrightarrow H^0(C, f^*T_{E_k}) \longrightarrow T^1 \\
\longrightarrow \text{Ext}^1(\Omega^1_C(x), \mathcal{O}_C) \longrightarrow H^1(C, f^*T_{E_k}) \longrightarrow T^2 \longrightarrow 0
$$

where $x = x_1 + \cdots + x_{n+2}$. The virtual normal bundle at $f$ is:

$$
\mathcal{N}^\text{vir} = T^1,\text{mov} - T^2,\text{mov} = \chi(f^*T_{E_k})^{\text{mov}} - \chi(\Omega^1_C(x), \mathcal{O}_C)^{\text{mov}}
$$

where "mov" means the moving part with respect to the $\hat{T}$-action and $\chi(\mathcal{E}) = H^0(C, \mathcal{E}) - H^1(C, \mathcal{E})$, $\chi(\mathcal{E}, \mathcal{F}) = \text{Ext}^0(\mathcal{E}, \mathcal{F}) - \text{Ext}^1(\mathcal{E}, \mathcal{F})$ denotes the Euler characteristics. Let $p, q$ denote the nodal intersection points $C_0 \cap C_{\sec}, C_{\infty} \cap C_{\sec}$ respectively. Using the normalization exact sequence $0 \to \mathcal{O}_C \to \mathcal{O}_{C_0} \oplus \mathcal{O}_{C_{\sec}} \oplus \mathcal{O}_{C_{\infty}} \to \mathcal{O}_p \oplus \mathcal{O}_q \to 0$, we find:

$$
\chi(f^*T_{E_k})^{\text{mov}} = \chi(f^*T_{X_0})^{\text{mov}} + \chi(f^*T_{X_{\infty}})^{\text{mov}} + \chi(f_{\sec}^*T_{E_k}) \\
+ \xi + \xi^{-1} - (T_{f(p)}E)^{\text{mov}} - (T_{f(q)}E)^{\text{mov}}
$$

where $\xi$ is the one-dimensional $\mathbb{C}^\times$-representation of weight one. We write $x_0 = x_0 + x_\infty$ where $x_0, x_{\infty}$ are divisors on $C_0, C_{\infty}$ respectively. Then we have

$$
\chi(\Omega^1_C(x), \mathcal{O}_C)^{\text{mov}} = T_pC_0 \otimes T_pC_{\sec} + T_qC_{\infty} \otimes T_qC_{\sec} \\
- \chi(\Omega^1_{C_0}(x_0), \mathcal{O}_{C_0})^{\text{mov}} - \chi(\Omega^1_{C_{\infty}}(x_{\infty} + q), \mathcal{O}_{C_{\infty}})^{\text{mov}}.
$$

The $\hat{T}$-fixed locus in the moduli space $(E_k)_{0, n+2,d}$ is given by

$$
\bigcup_{1 \leq i \leq \{1, \ldots, n+2\}, \sigma_i = d} \left( (X_0)_{0,1_i \cup \mathbb{P}^1, d_0} \times_{F_i} ((X_{\infty})_{0,1_2 \cup d_0, \infty})^T \right)
$$

where $F_i, \sigma_i$ are as in Definition 3.13. Combining (3.4), (3.5), we find that the virtual normal bundle $\mathcal{N}_{i}^{\text{vir}}$ on the component $((X_0)_{0,1_1 \cup \mathbb{P}^1, d_0})^T \times_{F_i} ((X_{\infty})_{0,1_2 \cup d_0, \infty})^T$ is:

$$
\mathcal{N}_{i}^{\text{vir}} = \mathcal{N}_0^{\text{vir}} + \mathcal{N}_{\sec, i} + N_{F_i/X_0} - N_{F_i/X_{\infty}} + L_{-1}^{-1} \otimes \xi + L_{-1}^{-1} \otimes \xi^{-1}
$$

where $\mathcal{N}_0^{\text{vir}}$ is the virtual normal bundle of $(X_0)_{0,1_1 \cup \mathbb{P}^1, d_0}$ in $(X_0)_{0,1_2 \cup \mathbb{P}^1, d_0}$, $\mathcal{N}_{\sec, i}$ is the vector bundle with fiber $\chi(f_{\sec}^*T_{E_k})^{\text{mov}}$. Let $N_{F_i/X} = N_i = \bigoplus_{\alpha} N_{i, \alpha}$ be decomposition as in Definition 3.13. The normal bundle of $F_i \times \mathbb{P}^1$ in $E_k$ is

$$
\bigoplus_{\alpha} N_{i, \alpha} \otimes \mathcal{O}_{\mathbb{P}^1}(-\alpha \cdot k).
$$

Thus we find:

$$
\mathcal{N}_{\sec, i} = \xi \oplus \xi^{-1} \oplus \bigoplus_{\alpha} N_{i, \alpha} \otimes \left( \bigoplus_{c \leq 0} \xi^c - \bigoplus_{c < 0} \xi^c \right).
$$

(3.6)
The virtual localization formula gives:

\[
\left( \bar{S}_k(\tau) \alpha, \beta \right) = \sum_{i,k,l,a,b} \sum_{d_0+d_\infty+\sigma_i = d} \left\langle z\alpha, \tau, \ldots, \tau, \frac{(t_{0,i} \star \phi_{i,a})}{z - \psi} \right\rangle_{X_0,1} \frac{Q_{d_0}^k}{k!} x_{0,k+2,d_0} \times \left( \int_{F_i} e_{\tilde{\mathcal{T}}}(N_{\text{sec},i}) \phi_i^a \phi_b^b \right) \left\langle \frac{(t_{\infty,i} \star \phi_{i,b})}{-z - \psi}, \tau', \ldots, \tau', \tau', \ldots, \tau', \ldots, -z\beta \right\rangle_{X_\infty,1} \frac{Q_{d_\infty}^l}{l!}
\]

where \( \alpha \in H^*_F(X_0), \beta \in H^*_F(X_\infty), \tau' = \Phi_k(\tau) \), the maps \( \iota_{0,i} : F_i \to X_0, \iota_{\infty,i} : F_i \to X_\infty \) are the natural inclusions, \( \{ \phi_{i,a} \} \subset H^*(F_i) \) is a basis, \( \{ \phi_i^a \} \) is the dual basis such that \( \int_{F_i} \phi_{i,a} \otimes \phi_b^b = \delta_{ab} \). Note that we have by \[3.15\],

\[
\frac{Q_{\sigma_i - \sigma_{\min}}^{\sigma_{\min}}}{e_{\tilde{\mathcal{T}}}(N_{\text{vir},i})} = \frac{1}{z(-z)} e_{\tilde{\mathcal{T}}}(N_{F_i/X_\infty}) \left( e^{kz\partial_k} \Delta_i(k) \right).
\]

Combining these equations, we conclude

\[
\left( \bar{S}_k(\tau) \alpha, \beta \right) = \left( \bar{S}_k M(\tau, z) \alpha, M'(\tau', -z) \beta \right)
\]

where we write the argument \( z \) in the fundamental solution explicitly and

- \( \bar{S}_k : H^*_F(X_0)_{\text{loc}} \to H^*_F(X_\infty)_{\text{loc}} \) is a map defined similarly to \( S_k \) by replacing \( \bigoplus_i \Delta_i(k) e^{-kz\partial_k} \) in the diagram \[3.13\] with \( \bigoplus_i (e^{kz\partial_k} \Delta_i(k)) \);

- \( M'(\tau', z) \) is defined similarly to Proposition \[2.2\] by replacing \( T \)-equivariant Gromov-Witten invariants there with \( (\tilde{T}, \rho_k) \)-equivariant invariants.

Note that \( M'(\tau', z) = \Phi_k \circ M(\tau, z) \circ \Phi_k^{-1} \) and \( \bar{S} = \Phi_k \circ S \). The conclusion follows from the so-called “unitarity” \( M(\tau, -z)^* = M(\tau, z)^{-1} \) of the fundamental solution (see [Giv88, 1]).

Theorem \[3.14\] and the differential equation \( \partial_i \circ M(\tau) = M(\tau) \circ \nabla_i \) show:

**Corollary 3.15.** The shift operator commutes with the quantum connection, i.e. \([\nabla_i, S_k(\tau)] = 0 \) for \( i = 0, \ldots, N \).

This corollary is shown in [MO12, 8.8] in the case where \( \tau = 0 \). We also remark that the shift operators commute each other.

**Corollary 3.16.** We have \( S_k \circ S_l = Q^{d(k,l)}_{d(k,l)} S_k S_l \) for some \( d(k,l) \in H_2(X, \mathbb{Z}) \) which is symmetric in \( k \) and \( l \). In particular, \( S_k \circ S_l = Q^{d(k,l)}_{d(k,l)} S_k S_l \), \( [S_k, S_l] = [S_k, S_l] = 0 \).

**Proof.** Consider the \( X \)-bundle \( E_{k,l} \) over \( \mathbb{P}^1 \times \mathbb{P}^1 \) given by

\[
E_{k,l} = X \times (\mathbb{C}^2 \setminus \{0\}) \times (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^\times \times \mathbb{C}^\times
\]

where \( (s_1, s_2) \subset \mathbb{C}^\times \times \mathbb{C}^\times \) acts on \( X \times \mathbb{C}^2 \times \mathbb{C}^2 \) by \( (s_1, s_2) \cdot (x, (a_1, a_2), (b_1, b_2)) = (s_1^k s_2^l, (s_1^{-1} a_1, s_1^{-1} a_2), (s_2^{-1} b_1, s_2^{-1} b_2)) \). Note that \( E_{k,l}|_{\mathbb{P}^1 \times \{1:0\}} \cong E_k \) and \( E_{k,l}|_{\{1:0\} \times \mathbb{P}^1} \cong E_l \) and \( E_{k,l}|_{\Delta(\mathbb{P}^1)} \cong E_{k+l} \), where \( \Delta(\mathbb{P}^1) \subset \mathbb{P}^1 \times \mathbb{P}^1 \) denotes the diagonal. The addition in \( H_2(E_{k,l}, \mathbb{Z}) \) defines a map \( \# : H^2_{\text{sec}}(E_l, \mathbb{Z}) \times H^2_{\text{sec}}(E_k, \mathbb{Z}) \to H^2_{\text{sec}}(E_{k+l}, \mathbb{Z}) \). For any \( T \)-fixed point \( x \), the section class \( \sigma_x \) (see \[3.2\]) associated to \( x \) satisfies \( \sigma_x \# \sigma_x = \sigma_x \).

A straightforward computation now shows that \( S_k \circ S_l = Q^{\sigma_{\min}(k+l) - \sigma_{\min}(k) \# \sigma_{\min}(l)}_{\sigma_{\min}(k+l) \# \sigma_{\min}(l)} S_{k+l} \).
where $\sigma_{\min}(k)$ denotes the minimal section class of $E_k$. The conclusion follows by setting $d(k, l) = \sigma_{\min}(k + l) - \sigma_{\min}(k)\#\sigma_{\min}(l)$ and the commutativity of $\#$.

3.4. **Relation to the Seidel Representation.** Taking the $z \to 0$ limit of shift operators, we obtain a big quantum cohomology version of the Seidel representation [Sei97]. The author learned the idea of big Seidel elements from Eduardo Gonzalez during joint work [GI12] with him.

**Definition 3.17** (Seidel elements). Let $k \in \text{Hom}(\mathbb{C}^x, T)$ be a semi-negative homomorphism. The element $S_k(\tau) := \lim_{z \to 0} S_k(\tau)1$ of $H^*_T(X)[Q][\tau^0, \ldots, \tau^m]$ is called the **Seidel element**.

By Corollary 3.13, the $z \to 0$ limit of the operator $S_k(\tau)$ commutes with the quantum multiplication, and therefore coincides with the quantum multiplication by $S_k(\tau)$ (see also [MO12, §8]). By Corollary 3.16, we have

$$S_k(\tau) \ast S_l(\tau) = Q^{d(k,l)} S_{k+l}(\tau).$$

This is called the **Seidel representation**.

3.5. **Relation to the $\hat{\Gamma}$-Integral Structure.** We remark a relationship between the shift operator and the $\hat{\Gamma}$-integral structure introduced in [Iri09, KKP08, CIJ14]. For quantum cohomology of the Hilbert scheme of points on $\mathbb{C}^2$, it has been observed in [OPT10] that certain $\Gamma$-factors play an important role in the difference equation associated to the shift operators.

We recall the $\hat{\Gamma}$-class of $X$. Let $\delta_1, \ldots, \delta_D$ denote the $T$-equivariant Chern roots of the tangent bundle $TX$ such that $c^T(TX) = (1 + \delta_1) \cdots (1 + \delta_D)$. The $T$-equivariant $\hat{\Gamma}$-class of $X$ is the class

$$\hat{\Gamma}_X = \hat{\Gamma}(TX) = \prod_{i=1}^D \Gamma(1 + \delta_i)$$

in $H^*_T(X) = \prod_{p=0}^{\infty} H^p_T(X)$. Here $\Gamma(z) = \int_0^{\infty} e^{-t z} t^{-1} dt$ is Euler’s $\Gamma$-function. By Taylor expansion, the right-hand side becomes a symmetric formal power series in $\delta_1, \ldots, \delta_n$ and thus can be expressed in terms of the equivariant Chern classes of $TX$.

The $\hat{\Gamma}$-integral structure assigns the following homogeneous flat section $s(E)$ of the quantum connection to a $T$-equivariant vector bundle $E \to X$:

$$s(E) = (2\pi)^{-D/2} M(\tau)^{-1} z^{-\mu} e^{c_1(X)}\hat{\Gamma}_X(2\pi i)^{\text{deg}/2} \text{ch}^T(E)$$

where $D = \dim_{\mathbb{C}} X$, $M(\tau)$ is the fundamental solution in Proposition 2.2, $\mu \in \text{End}_{\mathbb{C}}(H^*_T(X))$ is the Hodge grading operator defined by $\mu(\phi_i) = \left(\frac{\deg \phi_i}{2} - \frac{D}{2}\right)\phi_i$, $e^{c_1(X)} = e^{c_1(X) \log z}$ and $(2\pi i)^{\text{deg}/2} \text{ch}^T(E) = \sum_{p=0}(2\pi i)^p \text{ch}^T_p(E)$. The section $s(E)$ is flat, i.e. $\nabla_i s(E) = 0$ and is homogeneous in the sense that

$$\left[ z \frac{\partial}{\partial z} + \mu + \sum_{i=0}^N \left(1 - \frac{1}{2} \deg \phi_i\right) \tau^i \frac{\partial}{\partial \tau^i} + \sum_{i=0}^N \rho^i \frac{\partial}{\partial \rho^i} \right] s(E) = 0$$

where we set $c_1(X) = \sum_{i=0}^N \rho^i \phi_i$. A key property of $s(E)$ is that the pairing

$$\langle s(E)(\tau, e^{-\tau_1} z), s(F)(\tau, z) \rangle$$

is flat, i.e. $\nabla_i s(E) = 0$ and is homogeneous in the sense that
equals the $T$-equivariant Euler pairing $z^{-\deg/2} (2\pi i)^{\deg/2} \chi(E, F)$, where $\chi(E, F) = \sum_{i=0}^D (-1)^i \chi^T(\text{Ext}^i(E, F)) \in H_T^*(pt)$. This follows from an appropriate equivariant Hirzebruch-Riemann-Roch formula. See [CIJ14, §2-3] for more details.

The $T$-equivariant $K$-group is a module over $K_T^0(pt) = \mathbb{C}[T]$ and the Chern character $\exp^T: K_T^0(pt) \to H_T^*(pt)$ can be viewed as the pull-back by the universal covering $\exp: \text{Lie}(T) = \mathbb{C}^m \to T = (\mathbb{C}^*)^m$. A deck-transformation of this covering is given by the shift of equivariant parameters $\lambda_j \to \lambda_j + 2\pi i$. This suggests that $s(E)$ should be “invariant” under integral shifts of equivariant parameters.

**Proposition 3.18.** When the Novikov variable $Q$ is set to be one, the flat section $s(E)$ is invariant under the shift operator:

$$S_k s(E) = s(E)$$

for every semi-negative $k \in \text{Hom}(\mathbb{C}^*, T)$.

**Proof.** As is discussed in [CIJ14, §3], the divisor equation shows that the specialization $Q = 1$ of the Novikov variable is well-defined for $s(E)$. In view of the intertwining property in Theorem 3.14, it suffices to show that

$$S_k \left( z^{-\mu_e \cdot c_1(X)} \tilde{\chi}_X(2\pi i)^{\deg/2} \, \text{ch}(E) \right) = z^{-\mu_e \cdot c_1(X)} \tilde{\chi}_X(2\pi i)^{\deg/2} \, \text{ch}(E).$$

The restriction to the $T$-fixed component $F_i$ gives

$$\left[ z^{-\mu_e \cdot c_1(X)} \tilde{\chi}_X(2\pi i)^{\deg/2} \, \text{ch}(E) \right]_{F_i} = z^{D/2} \cdot \gamma_{(F_i)}/z \left( z^{-\deg/2} \tilde{\gamma}_{(F_i)} \right) \times \prod_{\alpha} \prod_{j=1}^{\text{rank } N_{\alpha,i}} z^{(\rho_{i,\alpha,j} + \alpha)/z} \Gamma \left( 1 + \frac{\rho_{i,\alpha,j}}{z} + \frac{\alpha}{z} \right) \sum_{\epsilon} e^{2\pi i \epsilon/z}$$

where $\epsilon$ ranges over $T$-equivariant Chern roots of $E$ and we use the notation from Definition 3.13. The conclusion easily follows from the identity $\Gamma(1 + z) = z\Gamma(z)$. □

4. Toric Mirror Theorem

In this section we give a new proof of a mirror theorem [Giv98b] for toric manifolds.

4.1. Toric Manifolds. We fix notation for toric manifolds. For background materials on toric manifolds, we refer the reader to [Oda88, Aud04, CLS11]. Let $N \cong \mathbb{Z}^D$ denote a lattice. A toric manifold is given by a rational simplicial fan $\Sigma$ in the vector space $N_R = N \otimes \mathbb{R}$. We assume that

- each cone $\sigma$ of $\Sigma$ is generated by part of a $\mathbb{Z}$-basis of $N$;
- the support $|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma$ of $\Sigma$ is convex and full-dimensional;
- $\Sigma$ admits a strictly convex piecewise linear function $\eta: |\Sigma| \to \mathbb{R}$.

These assumptions ensure that the corresponding toric variety $X_\Sigma$ is smooth and satisfies the hypotheses in §2.1. We do not require that $X$ is compact, or $c_1(X)$ is semipositive. Let $b_1, \ldots, b_m \in N$ be primitive integral generators of one-dimensional cones of $\Sigma$. The shift by $2\pi i$ is superseded by the shift by $z$ because of the operators $z^{-\mu}$ and $(2\pi i)^{\deg/2}$. 

\[\text{the shift by } 2\pi i \text{ is superseded by the shift by } z \text{ because of the operators } z^{-\mu} \text{ and } (2\pi i)^{\deg/2}.\]
Let $\beta : \mathbb{Z}^m \to \mathbb{N}$ be the homomorphism sending the standard basis vector $e_i \in \mathbb{Z}^m$ to $b_i$. The fan sequence is the exact sequence

$$0 \longrightarrow \mathbb{L} \longrightarrow \mathbb{Z}^m \overset{\beta}{\longrightarrow} \mathbb{N} \longrightarrow 0$$

with $\mathbb{L} = \text{Ker}(\beta)$. Set $K = \mathbb{L} \otimes \mathbb{C}^\times$. The inclusion $\mathbb{L} \hookrightarrow \mathbb{Z}^m$ induces the inclusion $K \hookrightarrow (\mathbb{C}^\times)^m$ of tori and defines a linear $K$-action on $\mathbb{C}^m$. The toric variety associated to $\Sigma$ is given by the GIT quotient

$$X_{\Sigma} = U/K, \quad U = \mathbb{C}^m \setminus Z$$

where $Z \subset \mathbb{C}^m$ is the common zero set of monomials $z^I = z_{i_1} \cdots z_{i_k}$ with $I = \{i_1, \ldots, i_k\}$ such that $\{b_i : 1 \leq i \leq m, i \notin I\}$ spans a cone in $\Sigma$. We consider the $T$-action on $X_{\Sigma}$ induced by the $T = (\mathbb{C}^\times)^m$-action on $\mathbb{C}^m$.

Let $\lambda_i \in H^*_T(pt) \cong \text{Lie}(T)^\ast$ denote the class corresponding to the $i$-th projection $T \to \mathbb{C}^\times$. We have

$$H^*_T(pt) = \mathbb{C}[\lambda_1, \ldots, \lambda_m].$$

All the $T$-weights of $H^0(X_{\Sigma}, \mathcal{O})$ are contained in the cone $\sum_{i=1}^m \mathbb{R}_{\geq 0}(\lambda_i)$ and therefore the condition (2) in §2.1 is satisfied. A cocharacter $k : \mathbb{C}^\times \to T$ is semi-negative in the sense of Definition 3.3 if $\lambda_i \cdot k \geq 0$ for all $i = 1, \ldots, m$.

Let $u_i \in H^*_T(X_{\Sigma})$ denote the class of the torus-invariant divisor $\{z_i = 0\}$ defined as the vanishing set of the $i$th co-ordinate $z_i$ on $\mathbb{C}^m$. The $T$-equivariant cohomology ring of $X_{\Sigma}$ is generated by these classes:

$$H^*_T(X_{\Sigma}) \cong H^*_T(pt)[u_1, \ldots, u_m]/(\mathfrak{I}_1 + \mathfrak{I}_2)$$

where $\mathfrak{I}_1$ is the ideal generated by $\prod_{i \in I} u_i$ such that $\{b_i : i \in I\}$ does not span a cone in $\Sigma$ and $\mathfrak{I}_2$ is the ideal generated by $\sum_{i=1}^m \chi(b_i)(u_i - \lambda_i)$ with $\chi \in \text{Hom}(\mathbb{N}, \mathbb{Z})$.

4.2. Mirror Theorem. Define a cohomology-valued hypergeometric series $I(y, z)$ by the formula:

$$I(y, z) = z e^{\sum_{i=1}^m u_i \log y_i / z} \sum_{d \in \text{Eff}(X_{\Sigma})} \left( \prod_{i=1}^m \int_{c=-\infty}^{0} (u_i + cz) \right)^d y_1^{u_1 - d} \cdots y_m^{u_m - d}.$$

This formula defines an element of $H^*_T(X_{\Sigma})_{\text{loc}}[Q][\log y]$. We may write $I(y, z)$ as a sum over $H_2(X_{\Sigma}, \mathbb{Z})$ since the summand automatically vanishes if $d \notin \text{Eff}(X_{\Sigma})$.

Givental’s mirror theorem [Giv98b] (generalized later in LLY99, Iri08, Bro09) states the following:

**Theorem 4.1.** The function $I(y, -z)$ lies on the Givental cone associated to genus-zero Gromov-Witten theory of $X_{\Sigma}$.

We explain the meaning of the statement. The Givental cone $\mathcal{L}$ [Giv04] is a subset of $H^*_T(X_{\Sigma})_{\text{loc}}[Q]$ consisting of points of the form:

$$(1.1) \quad -z + \mathbf{t}(z) + \sum_{i=0}^N \sum_{n=0}^{\infty} \sum_{d \in \text{Eff}(X_{\Sigma})} \frac{Q^d}{n!} \left( \phi_i^j \bigg|_{-z - \psi} , \mathbf{t}(\psi), \ldots, \mathbf{t}(\psi) \right)_{0, n+1, d} \phi_i$$

with $\mathbf{t}(z) \in H^*_T(X_{\Sigma})[Q] = H^*_T(X_{\Sigma})[z][Q]$. The Givental cone $\mathcal{L}$ can be written as the graph of the differential of the genus-zero descendant Gromov-Witten potential, and
encodes all genus-zero descendant Gromov-Witten invariants. Theorem 4.1 says that $I(y, z)$ is of the form (4.1), for some $t(z) \in H^*_T(X_\Sigma)[z][Q][\log y]$ with $t(z)|_{Q=\log y=0} = 0$. For toric manifolds, the above $I$-function determines the Givental cone and hence all the genus-zero Gromov-Witten invariants completely.

In this paper, we use an alternative description [Giv04] of the Givental cone $\mathcal{L}$. We can write $\mathcal{L}$ as the union

$$\mathcal{L} = \bigcup_{\tau \in H^*_T(X_\Sigma)[Q]} zT_{\tau}$$

of the semi-infinite subspaces $T_{\tau} = M(\tau, -z)H_T(X_\Sigma)[z][Q]$, where $M(\tau, -z)$ denotes the fundamental solution from Proposition 2.2 with the sign of $z$ flipped. The subspace $T_{\tau}$ is a (common) tangent space to $\mathcal{L}$ along $zT_{\tau} \subset \mathcal{L}$. Therefore, it suffices to show that $I(y, z)$ can be written in the form

$$I(y, z) = zM(\tau(y), z)\Upsilon(y, z)$$

for some $\tau(y) \in H^*_T(X_\Sigma)[Q][\log y]$ and $\Upsilon(y, z) \in H^*_T(X_\Sigma)[z][Q][\log y]$.

4.3. Proof. The idea of the proof is as follows. Let $e_i$ denote the cocharacter $\mathbb{C}^\times \to T = (\mathbb{C}^\times)^m$ given by the inclusion of the $i$th factor. Let $S_i = S_{e_i}$, $S_i = S_{e_i}$ denote the corresponding shift operators. In view of Theorem 3.14, the shift operator $S_i$ defines a vector field on the Givental cone:

$$(4.2) \quad \mathcal{L} \ni f \mapsto z^{-1} S_i f \in T_f \mathcal{L}.$$ 

These vector fields define commuting flows by Corollary 3.16. We will identify the $I$-function with an integral submanifold of these vector fields.

Consider the $\mathbb{C}^\times$-action on $X_\Sigma$ induced by the cocharacter $e_i \in \text{Hom}(\mathbb{C}^\times, T)$. The minimal fixed component $F_{\min}$ for this $\mathbb{C}^\times$-action is the toric divisor $\{z_i = 0\}$. Let $E_i = E_{e_i}$ denote the associated bundle. For a fixed point $x \in X_\Sigma$, we set $d_i(x) = \sigma_x - \sigma_{\min} \in H_2(X_\Sigma, \mathbb{Z})$, where $\sigma_x \in H_2^{\text{ev}}(E_k)$ is the section (1.12) of $E_i$ associated to $x$ and $\sigma_{\min} \in H_2^{\text{ev}}(E_k)$ is the minimal section class of $E_i$. We write $u_j(x) \in H_2^T(\text{pt})$ for the restriction of $u_j$ to $x$.

**Lemma 4.2.** With the notation as above, we have

$$u_j(x) \cdot e_i = \delta_{ij} - u_j \cdot d_i(x).$$

**Proof.** Consider the $\widehat{T}$-invariant divisor $\{z_j = 0\} \times \mathbb{P}^1$ in $E_i$ and let $\hat{u}_j$ denote the $\widehat{T}$-equivariant Poincaré dual of the divisor. Then we have $\hat{u}_j|_{(x,[1,0])} = u_j(x)$ and $\hat{u}_j|_{(x,[0,1])} = u_j(x) + (u_j(x) \cdot e_i)z$. The localization formula gives

$$\hat{u}_j \cdot \sigma_x = u_j|_{(x,[1,0])} + u_j|_{(x,[0,1])} z = -u_j(x) \cdot e_i.$$ 

Similarly we have $\hat{u}_j \cdot \sigma_{\min} = -u_j(y) \cdot e_i$ for any $T$-fixed point $y$ in the divisor $F_{\min} = \{z_i = 0\}$. If $i \neq j$, taking $y$ away from $\{z_j = 0\}$, we get $u_j(y) = 0$. If $i = j$, $u_j(y) \cdot e_i = 1$. Therefore $\hat{u}_j \cdot \sigma_{\min} = -\delta_{ij}$. The conclusion follows. \qed
Lemma 4.3. The $I$-function is an integral curve of the vector field \((\ref{eq:vector_field})\), that is, for $i \in \{1, \ldots, m\}$, we have

$$z \frac{\partial}{\partial y_i} I(y, z) = S_i I(y, z).$$

Proof. Note that all the $T$-fixed points on $X_\Sigma$ are isolated. Let $x \in X^T$ be a fixed point. It suffices to show that

$$z \frac{\partial}{\partial y_i} I_x(y, z) = \Delta_x(e_i) e^{-z \partial I} I_x(y, z)$$

where $I_x(y, z)$ is the restriction of the $I$-function to $x$ and

$$\Delta_x(e_i) = Q_{d_i(x)} \prod_{j=1}^{m} \frac{\prod_{c=-\infty}^{0}(u_j(x) + cz)}{\prod_{c=-\infty}^{0-\sum_{j=1}^{m} (u_j)d_i(x))} (u_j(x) + cz).$$

Using Lemma \((\ref{eq:vector_field})\), we have

$$\Delta_x(e_i) e^{-z \partial I} I_x(y, z) = ze^{\sum_{j=1}^{m} u_j(x) \log y_j / z} e^{-z \log y_i + \sum_{j=1}^{m} (u_j d_i(x)) \log y_j}$$

$$\times Q_{d_i(x)} \sum_{d \in H_2(X_\Sigma, \mathbb{Z})} \left( \prod_{j=1}^{m} \frac{\prod_{c=-\infty}^{0}(u_j(x) + cz)}{\prod_{c=-\infty}^{0-\sum_{j=1}^{m} (u_j)d_i(x))} (u_j(x) + cz) \right) Q^d y^d$$

where $y^d = \prod_{j=1}^{m} y_j^{u_j - d_i}$. Changing variables $d \to d - d_i(x)$ and using again Lemma \((\ref{eq:vector_field})\), we find that this equals $z \frac{\partial}{\partial y_i} I(y, z).$ \qed

We identify the classical shift operators:

**Notation 4.4.** We set $v_i := u_i - \lambda_i \in H^2_T(X_\Sigma)$ and write $v_i(x) \in H^2_T(pt)$ for the restriction of $v_i$ to a $T$-fixed point $x$.

**Lemma 4.5.** Let $f(v, \lambda)$ be a cohomology class in $H^*_T(X_\Sigma)$ expressed as a polynomial in $v_1, \ldots, v_m$ and $\lambda_1, \ldots, \lambda_m$. When we write $\tau \in H^*_T(X_\Sigma)$ as a polynomial $\tau(v, \lambda)$ in $v_1, \ldots, v_m$ and $\lambda_1, \ldots, \lambda_m$, we have

$$\lim_{Q \to 0} S_i(\tau) f(v, \lambda) = u_i e^{(\tau(v, \lambda - e_i) - \tau(v, \lambda)) / z} f(v, \lambda - z e_i)$$

where $\lambda - ze_i = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_i - z, \lambda_{i+1}, \ldots, \lambda_m)$. In particular the classical Seidel elements are given by:

$$\lim_{Q \to 0} S_i(\tau) = u_i e^{-\frac{\partial\tau(\cdot, \lambda)}{\partial \lambda_i}}.$$

Proof. Recall from Theorem \((\ref{thm:seidel})\) that we have $S_i \circ M(\tau) = M(\tau) \circ S_i(\tau)$. Since $\lim_{Q \to 0} M(\tau) = e^{\tau / z}$, we have

$$\lim_{Q \to 0} S_i(\tau) f(v, \lambda) = e^{-\tau / z} \left( \lim_{Q \to 0} S_i(\tau) \right) e^{\tau / z} f(v, \lambda).$$

By definition of $S_i$, this vanishes when restricted to a fixed point outside of the minimal fixed component \{$z_i = 0$\} with respect to $e_i$. On the other hand, for any $T$-fixed point
\[ \lim_{Q \to 0} S_i(t) f(v, \lambda) \bigg|_{x} = e^{-\tau(v(x), \lambda)/z} u_i(x) e^{-z\partial x_i} \left[ e^{\tau(v(x), \lambda)/z} f(v(x), \lambda) \right] \\
= u_i(x) e^{(\tau(v(x), \lambda - e_i z) - \tau(v, \lambda))/z} f(v(x), \lambda - e_i z) \]

where we set \( v(x) = (v_1(x), \ldots, v_m(x)) \). The conclusion follows. \( \square \)

**Lemma 4.6.** Let \( x \) be a \( T \)-fixed point on \( X_{\Sigma} \). The restriction \( u_j(x) \) is a linear combination of \( \lambda_i \) such that \( x \) does not lie on the divisor \( \{ z_i = 0 \} \).

**Proof.** Note that if \( x \) does not lie on the divisor \( \{ z_i = 0 \} \), we have \( u_i(x) = 0 \) and thus \( v_i(x) = -\lambda_i \). This together with the linear relation \( \sum_{i=1}^m \chi(b_i)v_i = 0, \chi \in \text{Hom}(N, Z) \) determines \( v_1(x), \ldots, v_m(x) \) uniquely. This implies the conclusion. \( \square \)

Let \( \mathcal{Z} = \mathcal{L}|_{z \to -z} \) denote the Givental cone with the sign of \( z \) flipped. By the description in \( \S 4.2 \), we have a parametrization of the Givental cone (with the sign of \( z \) flipped) \( \mathcal{Z} \) by \( (\tau, \Upsilon) \in H^*_T(X) \times H^*_T(X) = H^*_T(X) \times H^*_T(X)[z] \) as:

\[ (\tau, \Upsilon) \mapsto zM(\tau, z) \Upsilon \in \mathcal{Z}. \]

The vector field \( (4.2) \) on \( \mathcal{Z} \) corresponds to the following vector field on \( H^*_T(X) \times H^*_T(X)[z] \):

\[ (V_i)_{\tau, \Upsilon} = (S_i(\tau), [z^{-1}S_i(\tau)\Upsilon]_+) \]

where \( S_i(\tau) \) is the Seidel element in Definition 3.17 and \([ \cdots ]_+ \) denotes the projection to the polynomial part in \( z \) (i.e. removing the \( z^{-1} \)-term). In fact, if we have a curve \( t \mapsto (\tau(t), \Upsilon(t)) \) with \( \tau(0) = S_i(\tau(0)), \Upsilon(0) = [S_i(\tau(0))\Upsilon(0)]_+ \), the corresponding curve \( f(t) = zM(\tau(t), z)\Upsilon(t) \) on \( \mathcal{Z} \) satisfies

\[ f'(0) = M(\tau(0), z)(S_i(\tau(0)) \ast_{\tau(0)} \Upsilon(0)) + zM(\tau(0), z)[z^{-1}S_i(\tau(0))\Upsilon(0)]_+ \]

\[ = M(\tau(0), z)S_i(\tau(0))\Upsilon(0) = z^{-1}S_i f(0) \]

where we used \( z\partial_z M(\tau, z) = M(\tau, z)(\phi_r \ast \tau) \) in the first line and Theorem 3.14 in the second line. Since the vector fields \( (4.2) \) commute each other, the corresponding vector fields \( V_i, i = 1, \ldots, m \) also commute each other. In what follows, we show the existence of an integral curve for the vector field \( V_i \) with prescribed asymptotics.

**Proposition 4.7.** There exist unique functions

\[ \tau(y) \in H^*_T(X_{\Sigma})[Q][\log y] \quad \text{and} \quad \Upsilon(y, z) \in H^*_T(X_{\Sigma})[z][Q][\log y] \]

which are of the form

\[ \tau(y) = \sum_{i=1}^m u_i \log y_i + \sum_{d \in \text{Eff}(X_{\Sigma}), d \neq 0} Q^d y^d \tau_d \]

\[ \Upsilon(y, z) = 1 + \sum_{d \in \text{Eff}(X_{\Sigma}), d \neq 0} Q^d y^d \Upsilon_d \]
with \( y^d = \prod_{j=1}^{m} y_j^{u_j} \) and give an integral curve for the vector field \( \mathbf{V}_i \):

\[
\frac{\partial \tau(y)}{\partial y_i} = S_i(\tau(y)) \quad \text{and} \quad \frac{\partial \Upsilon(y, z)}{\partial y_i} = [(z^{-1}S_i(\tau(y))\Upsilon(y, z))]_+
\]

for all \( 1 \leq i \leq m \).

**Proof.** Write \( \tau(y) = \sum_{j=1}^{m} u_j \log y_j + \tau' \). The divisor equation in Remark 3.12 gives

\[
S_i(\tau(y)) = y_i^{-1} S_i(\tau'; Qy).
\]

where \( S_i(\sigma; Qy) \) is obtained from \( S_i(\sigma) \) by replacing \( Q^d \) with \( Q^d y^d \). Therefore we need to solve for the differential equations:

\[
\begin{align*}
(4.3) \quad y_i \frac{\partial \tau'}{\partial y_i} = S_i(\tau'; Qy) - u_i & \quad \text{and} \quad y_i \frac{\partial \Upsilon}{\partial y_i} = [(z^{-1}S_i(\tau'; Qy)\Upsilon)]_+.
\end{align*}
\]

We expand

\[
\tau' = \sum_{d \in \text{Eff}(X_\Sigma), d \neq 0} \tilde{\tau}_d(y)Q^d, \quad \Upsilon = \sum_{d \in \text{Eff}(X_\Sigma)} \tilde{\Upsilon}_d(y)Q^d
\]

with \( \tilde{\Upsilon}_0(y) = 1 \) and solve for the coefficients \( \tilde{\tau}_d(y) \), \( \tilde{\Upsilon}_d(y) \) recursively. Note that the equation (4.3) holds true mod \( Q \) by Lemma 1.3.

First we solve for \( \tau' \). Choose a Kähler class \( \omega \) such that \( \omega \cdot d_1 = \omega \cdot d_2 \) for \( d_1, d_2 \in \text{Eff}(X_\Sigma) \) if and only if \( d_1 = d_2 \). This defines a positive real grading on the Novikov ring \( \mathbb{C}[Q] \) such that \( \deg Q^d = \omega \cdot d \). Take \( d_0 \in \text{Eff}(X_\Sigma) \setminus \{0\} \). Suppose by induction that there exist \( \tilde{\tau}_d \) for all \( d \) with \( \omega \cdot d < \omega \cdot d_0 \) such that \( \tilde{\tau}_d = \tau_d y^d \) for some \( \tau_d \in H^*_{\tau}(X) \) and that \( \tau' = \sum_{\omega \cdot d < \omega \cdot d_0} \tilde{\tau}_d Q^d \) satisfies the differential equation (4.3) modulo terms of degree \( \geq \omega \cdot d_0 \). We think of \( \tau_d \) as being expressed as a polynomial in \( v_1, \ldots, v_m \) and \( \lambda_1, \ldots, \lambda_m \). Comparing the coefficients of \( Q^{d_0} \) of the differential equation, we obtain using Lemma 1.3 that:

\[
\frac{\partial \tilde{\tau}_{d_0}}{\partial y_i} + u_i \frac{\partial \tilde{\tau}_{d_0}}{\partial \lambda_i} = \left( \text{an expression in } \tilde{\tau}_d \right) \quad \text{with } \omega \cdot d < \omega \cdot d_0.
\]

Here the right-hand side is of the form \( g_i(v, \lambda) y^{d_0} \) by induction hypothesis, where \( g_i(v, \lambda) \) is a polynomial in \( v_1, \ldots, v_m \) and \( \lambda_1, \ldots, \lambda_m \). Setting \( \tilde{\tau}_{d_0} = \tau_{d_0} y^{d_0} \), we obtain

\[
(u_i \cdot d_0) \tau_{d_0} + (v_i + \lambda_i) \frac{\partial \tau_{d_0}}{\partial \lambda_i} = g_i(v, \lambda).
\]

The Kähler class can be written as a non-negative linear combination of \( u_i \), and thus there exists \( i_0 \) such that \( u_{i_0} \cdot d_0 > 0 \). Then we can solve for the polynomial \( \tau_{d_0} = \tau_{d_0}(v, \lambda) \) from the above equation with \( i = i_0 \) recursively from the highest order term in \( \lambda_{i_0} \). Setting \( \tau(y) = \sum u_i \log y_i + \sum_{\omega \cdot d \leq \omega \cdot d_0} \tau_d y^d Q^d \), we have

\[
\frac{\partial \tau(y)}{\partial y_i} \equiv S_i(\tau(y)).
\]
modulo terms of degree $\geq \omega \cdot d_0$ for $i \neq i_0$ and modulo terms of degree $> \omega \cdot d_0$ for $i = i_0$. The commutativity of the flow implies that we have for $i \neq i_0$,
\[
\frac{\partial}{\partial y_{i_0}} \left( \frac{\partial \tau}{\partial y_i} - S_i(\tau(y)) \right) = \frac{\partial^2 \tau(y)}{\partial y_i \partial y_{i_0}} - (d_{\partial \tau/\partial y_{i_0}}) S_i(\tau(y)) \equiv \frac{\partial S_{i_0}(\tau(y))}{\partial y_i} - (d_{S_{i_0}(\tau(y))} S_i(\tau(y))
\]
\[
= (d_{\partial \tau/\partial y_{i_0}} S_i)(\tau(y)) - (d_{S_{i_0}(\tau(y))} S_i)(\tau(y))
\]
modulo terms of degree $> \omega \cdot d_0$. Using the divisor equation again, we have
\[
y_i \left( \frac{\partial \tau(y)}{\partial y_i} - S_i(\tau(y)) \right) = u_i + y_i \frac{\partial \tau'}{\partial y_i} - S_i(\tau', Qy).
\]
Modulo terms of degree $> \omega \cdot d_0$, this is $\alpha(Qy)^{d_0}$ for some $\alpha = \alpha(v, \lambda) \in H^2_{\Sigma}(X)$. Now the coefficient of $Q^{d_0}$ of equation (4.4) gives (by Lemma 4.5):
\[
(u_{i_0} \cdot d_0) \alpha + u_{i_0} \frac{\partial \alpha}{\partial \lambda_{i_0}} = 0.
\]
We want to show that $\alpha = 0$ as a cohomology class. Consider the restriction $\alpha(x)$ of $\alpha$ to a $T$-fixed point $x \in X_{\Sigma}$. If $x$ lies in the divisor $\{z_{i_0} = 0\}$, $v_j(x) \in H^2_{\Sigma}(pt)$ is a linear combination of $\lambda_{j'}$ with $j' \neq i_0$ by Lemma 4.6. Thus
\[
\left. \frac{\partial \alpha}{\partial \lambda_{i_0}} \right|_x = \frac{\partial \alpha(x)}{\partial \lambda_{i_0}}.
\]
If $x$ is not in the divisor $\{z_{i_0} = 0\}$, $u_{i_0}(x) = 0$. Therefore, by restricting to $x$, we have
\[
(u_{i_0} \cdot d) \alpha(x) + u_{i_0}(x) \frac{\partial \alpha(x)}{\partial \lambda_{i_0}} = 0.
\]
This shows that $\alpha(x) = 0$ recursively from the highest order term in $\lambda_{i_0}$. Note that the same argument shows the uniqueness of $\tau_{d_0}$. This completes the induction.

Next we solve for $\Upsilon$ assuming that $\tau'$ is already solved. Let $\omega$ be a Kähler class as above and $d_0 \in \text{Eff}(X_{\Sigma})$ be a non-zero effective class. Suppose by induction that there exist $\tilde{\Upsilon}_d$ for all $d$ with $\omega \cdot d < \omega \cdot d_0$ such that $\tilde{\Upsilon}_d = \Upsilon dy^d$ and that $\Upsilon = \sum_{\omega \cdot d < \omega \cdot d_0} \tilde{\Upsilon}_d Q^d$ satisfies the differential equation (4.3) modulo terms of degree $\geq \omega \cdot d_0$. We regard $\Upsilon_d$ as a polynomial in $v_1, \ldots, v_m$ and $\lambda_1, \ldots, \lambda_m$. Comparing the coefficients of $Q^{d_0}$ of the differential equation using Lemma 4.3, we obtain
\[
y_i \frac{\partial \tilde{\Upsilon}_{d_0}(v, \lambda)}{\partial y_i} - \left[ z^{-1}(v_i + \lambda_i) \tilde{\Upsilon}_{d_0}(v, \lambda - e_i z) \right]_+ = \left( \text{an expression in } \tilde{\Upsilon}_d \right)
\]
with $\omega \cdot d < \omega \cdot d_0$.

Here the right-hand side is of the form $g_i(v, \lambda) y^{d_0}$ for some polynomial $g_i(v, \lambda)$ in $v_1, \ldots, v_m$ and $\lambda_1, \ldots, \lambda_m$. Setting $\tilde{\Upsilon}_{d_0} = \Upsilon_{d_0} y^{d_0}$, we have
\[
(u_1 \cdot d_0) \Upsilon_{d_0}(v, \lambda) - \left[ z^{-1} v_i \Upsilon_{d_0}(v, \lambda - e_i z) \right]_+ = g_i(v, \lambda).
\]
As before, we can find \( i_0 \) such that \( u_{i_0} \cdot d_0 > 0 \). The Lemma 4.3 below shows that we can solve for \( \Upsilon_{d_0}(\tau, \lambda) \) recursively from the highest order term in \((z, \lambda_{i_0})\) using this equation with \( i = i_0 \). Setting \( \Upsilon = \sum_{\omega \leq \omega_0} \Upsilon_{d_0}Q^{\omega} \), we have

\[
\frac{\partial \Upsilon(y)}{\partial y_i} \equiv \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+
\]

modulo terms of degree \( \geq \omega \cdot d_0 \), modulo terms of degree \( > \omega \cdot d_0 \) for \( i = i_0 \). We have for \( i \neq i_0 \),

\[
\frac{\partial}{\partial y_{i_0}} \left( \frac{\partial \Upsilon}{\partial y_i} - \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ \right) = \frac{\partial^2 \Upsilon}{\partial y_i \partial y_{i_0}} - \frac{\partial}{\partial y_{i_0}} \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ 
\]

\[
\equiv \frac{\partial}{\partial y_i} \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ - \frac{\partial}{\partial y_{i_0}} \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ 
\]

\[
\equiv \left[ z^{-1}(d_{S_i(\tau(y))S_{i_0}})(\tau(y))\Upsilon(y) + z^{-1}S_{i_0}(\tau(y))\frac{\partial \Upsilon(y)}{\partial y_i} \right]_+ 
\]

\[
- \left[ z^{-1}(d_{S_{i_0}(\tau(y))S_i})(\tau(y))\Upsilon(y) + z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+.
\]

modulo terms of degree \( > \omega \cdot d_0 \). The commutativity of the flows \( V_i, i = 1, \ldots, m \) implies for \( i \neq j \),

\[
\left[ z^{-1}(d_{S_j(\tau)}S_j)(\tau)\Upsilon + \left[ z^{-1}S_j(\tau)\Upsilon(y) \right]_+ \right]_+ = \left[ z^{-1}(d_{S_i(\tau)}S_i)(\tau)\Upsilon + z^{-1}S_i(\tau)\Upsilon(y) \right]_+.
\]

Therefore we have:

\[
\frac{\partial}{\partial y_{i_0}} \left( \frac{\partial \Upsilon(y)}{\partial y_i} - \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ \right) \equiv \left[ z^{-1}S_{i_0}(\tau(y))\frac{\partial \Upsilon(y)}{\partial y_i} - \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ \right]_+
\]

modulo terms of degree \( > \omega \cdot d_0 \). By the divisor equation, we have

\[
y_i \left( \frac{\partial \Upsilon(y)}{\partial y_i} - \left[ z^{-1}S_i(\tau(y))\Upsilon(y) \right]_+ \right) = y_i \frac{\partial \Upsilon(y)}{\partial y_i} - \left[ z^{-1}S_i(\tau'(y))\Upsilon(y) \right]_+.
\]

This is of the form \( \alpha(Qy)^{d_0} \) for some \( \alpha = \alpha(v, \lambda, z) \in H^2_T(X_S) \), modulo terms of degree \( > \omega \cdot d_0 \). Hence the differential equation [4.3] implies via Lemma 4.5 that:

\[
(u_{i_0} \cdot d_0)\alpha - \left[ z^{-1}u_{i_0}\alpha(v, \lambda - e_{i_0}z, z) \right]_+ = 0.
\]

We will show that \( \alpha = 0 \) in the cohomology group. By restricting this to a \( T \)-fixed point \( x \) and using a similar argument as before, we obtain

\[
(u_{i_0} \cdot d_0)\alpha(x) - \left[ z^{-1}(v_{i_0}(x) + \lambda_{i_0})e^{-z\partial_{\lambda_{i_0}}}\alpha(x) \right]_+ = 0
\]

for the restriction \( \alpha(x) \in H^2_T(pt) \) of \( \alpha \) to \( x \). Using Lemma 4.8, we can show that \( \alpha(x) = 0 \) recursively from the highest order term in \((\lambda_{i_0}, z)\). Therefore \( \alpha = 0 \). This completes the induction and the proof.

In the above proof, we used the following lemma:

**Lemma 4.8.** Let \( \mathbb{C}[\lambda, z]^{(n)} \) denote the degree \( n \) homogeneous component of the polynomial ring \( \mathbb{C}[\lambda, z] \). The map \( \mathbb{C}[\lambda, z]^{(n)} \to \mathbb{C}[\lambda, z]^{(n)} \) given by

\[
f(\lambda, z) \mapsto cf(\lambda, z) - \left[ z^{-1}\lambda f(\lambda - z, z) \right]_+
\]

for any polynomial \( f(\lambda, z) \).
is an isomorphism for $c > 0$, where $[\cdots]_+$ denotes the projection to the polynomial part in $z$.

Proof. Changing the sign of $z$, we consider the mapping $\mathbb{C}[\lambda, z]^{(n)} \to \mathbb{C}[\lambda, z]^{(n)}$ given by $f(\lambda, z) \mapsto cf(\lambda, z) + [z^{-1}\lambda f(\lambda + z, z)]_+$. We use the two bases of $\mathbb{C}[\lambda, z]^{(n)}$: 

\[
\{z^n, (\lambda - z)z^{n-1}, \ldots, (\lambda - z)^n, (\lambda - z)^n\}, \\
\{z^n, \lambda z^{n-1}, \ldots, \lambda^{-1}z, \lambda^n\}.
\]

In these two bases, the map is represented by the following matrix:

\[
\begin{pmatrix}
c & -c & c & \cdots & (-1)^{n-1}c & (-1)^n c \\
1 & c & -2c & \cdots & (-1)^{n-2}(n-1)c & (-1)^{n-1}nc \\
0 & 1 & c & \cdots & (-1)^{n-2}c & (-1)^{n-2}(\frac{n}{2}) c \\
0 & 0 & 1 & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \cdots & c & -nc \\
0 & 0 & 0 & \cdots & 1 & c
\end{pmatrix}
\]

It is easy to see that the determinant of this matrix is a polynomial in $c$ with non-negative coefficients. In particular the matrix is invertible for $c > 0$. \qed

We now come to the final step of the proof. Let $\tau(y)$, $\Upsilon(y, z)$ be as in Proposition 4.7. Then, as discussed in a paragraph preceding Proposition 4.7, $y \mapsto f(y) := zM(\tau(y), z)\Upsilon(y, z)$ defines an integral manifold for the vector fields in (4.2). We shall show that $f(y) = I(y, z)$. Using the divisor equation for $M(\tau, z)$, we find that $f(y)$ is of the form:

\[(4.6) \quad f(y) = z e^{\sum_{i=1}^m u_i \log y_i/z} \left(1 + \sum_{d \in \mathbb{E}(X_\Sigma) \setminus \{0\}} f_d Q^d y^d\right)\]

with $f_d \in H^2(X_{\Sigma})_{\text{loc}}$. In view of Lemma 4.3, the following lemma shows that $f(y) = I(y, z)$ and completes the proof of Theorem 4.1.

**Lemma 4.9.** The family of elements $y \mapsto f(y)$ of the form (4.6) satisfying $\partial_y f(y) = z^{-1} S_if(y)$, $i = 1, \ldots, m$ is unique.

**Proof.** Suppose that we have two families $f_1(y)$, $f_2(y)$ of elements of the form (4.6) satisfying $\partial_y f_j(y) = z^{-1} S_i f_j(y)$, $j = 1, 2$, $i = 1, 2, \ldots, m$. The difference $g(y) = f_1(y) - f_2(y)$ satisfies the same differential equation and is of the form

\[g(y) = z e^{\sum_{i=1}^m u_i \log y_i/z} \sum_{d \in \mathbb{E}(X_\Sigma) \setminus \{0\}} g_d Q^d y^d.\]

Choose a Kähler class $\omega$ and suppose by induction that we know $g_d = 0$ for all $d \in \mathbb{E}(X_\Sigma)$ with $\omega \cdot d < \omega \cdot d_0$ for some $d_0 \in \mathbb{E}(X_\Sigma) \setminus \{0\}$. Let $x$ be a $T$-fixed point. Let $\delta$ be the set of indices $i$ such that $x$ does not lie on the toric divisor $\{z_i = 0\}$. The Kähler class $\omega$ can be written as a positive linear combination of non-equivariant limits of $u_i$.
with \( i \in \delta \). Therefore, there exists \( i_0 \in \delta \) such that \( u_{i_0} \cdot d_0 > 0 \). The coefficient in front of \( Q^{d_0} \) of the equation \( \partial_{y_{i_0}} g(y) = z^{-1}S_{i_0} g(y) \) restricted to the fixed point \( x \) gives:
\[
(u_{i_0} \cdot d_0) g_{d_0}(x) = 0
\]
since \( x \) does not lie on the minimal fixed component \( \{ z_{i_0} = 0 \} \) with respect to \( e_{i_0} \). Therefore \( g_{d_0}(x) = 0 \). Since \( x \) is arbitrary, \( g_{d_0} = 0 \). This completes the induction and the proof.

4.4. Example. Consider the toric variety \( X_\Sigma = \mathbb{P}^{m-1} \). In this case we have \( m \) shift operators \( S_1, \ldots, S_m \) corresponding to \( m \) toric divisors. It is well-known that the mirror map \( \tau(y) \) and the function \( \Upsilon(y) \) are trivial:
\[
\tau(y) = \sum_{i=1}^{m} u_i \log y_i, \quad \Upsilon(y) = 1.
\]
Generalizing the differential equation in Lemma 4.3, we can show that
\[
S_1 \cdots S_m I(y, z) = z \partial_{y_1} \cdots z \partial_{y_m} I(y, z)
\]
when \( i_1, \ldots, i_a \) are distinct. This together with the intertwining property \( S_i \circ M(\tau, z) = M(\tau, z) \circ S_i(\tau) \) and the divisor equation \( S_i(\tau(y)) = y_i^{-1} S_i(0; Qy) \) implies:
\[
S_{i_1}(0; Qy) \cdots S_{i_a}(0; Qy) 1 = z \nabla u_{i_1} \cdots z \nabla u_{i_a} 1 \bigg|_{\tau(y)} = \begin{cases} u_{i_1} \cdots u_{i_a} & \text{if } a < m; \\ Qy_1 \cdots y_m & \text{if } a = m, \end{cases}
\]
where \( i_1, \ldots, i_a \) are distinct and \( S_i(0; Qy) \) means \( S_i(0)|_{Q \rightarrow Qy_1 \cdots y_m} \). This determines the action of \( S_i(0) \) completely. Since the one-parameter subgroup \( e_1 + \cdots + e_m \) acts on \( \mathbb{P}^{m-1} \) trivially, we have a relation \( S_1(\tau) \circ \cdots \circ S_m(\tau) = Q \) by Corollary 3.10. Writing \( u_i = v + \lambda_i \) for \( i = 1, \ldots, m \), we recover the relation:
\[
(z \nabla_v + \lambda_1) \cdots (z \nabla_v + \lambda_m) 1 \bigg|_{\tau=0} = Q
\]
in the equivariant small quantum \( D \)-module of \( \mathbb{P}^{m-1} \).

4.5. Remarks. We first remark a relation to the results in [GI12]. Let \( X_\Sigma \) be a compact toric manifold such that \( c_1(X_\Sigma) \) is nef. In this case, the mirror map \( \tau(y) \) takes values in \( H^2_c(X) \). We write
\[
\tau(y) = \sum_{i=1}^{m} (\log y_i - g^i(y)) u_i
\]
for some \( \mathbb{C} \)-valued functions \( g^i(y) \). Using the divisor equation from Remark 3.12, the differential equation in Proposition 4.7 implies:
\[
y_i \frac{\partial \tau(y)}{\partial y_i} = e^{g^i(y)} S_i(0; Q e^{\tau(y)})
\]
where we set \( S_i(0; Q e^{\tau(y)}) = S_i(0)|_{Q \rightarrow Q e^{\tau(y)}} \). The left-hand side is called the Batyrev element in [GI12] and this recover the relationship between the Seidel and the Batyrev elements in [GI12, Theorem 1.1].

We should also recover a mirror theorem for the extended \( I \)-function [CCIT13] by considering the shift operators corresponding to general semi-negative cocharacters \( k \in \mathbb{C} \).
$\mathbb{Z}_{\geq 0}^m \subset \text{Hom}(\mathbb{C}^*, T)$. It would be also interesting to see if our method can be generalized to toric orbifolds [CCIT13, CCFK14], toric fibrations [Bro09], or other $T$-varieties.

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