A Multigrid Method Based On Shifted-Inverse Power Technique for Eigenvalue Problem*

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Abstract

A multigrid method is proposed in this paper to solve the eigenvalue problem by the finite element method based on the shifted-inverse power iteration technique. With this scheme, solving eigenvalue problem is transformed to a series of nonsingular solutions of boundary value problems on multilevel meshes. Since the difficult eigenvalue solving is replaced by the easier solution of boundary value problems, the multigrid way can improve the overall efficiency of the eigenvalue problem solving. Some numerical experiments are presented to validate the efficiency of the new method.

Keywords. Eigenvalue problem, multigrid, shifted-inverse power iteration, finite element method.

AMS subject classifications. 65N30, 65N25, 65L15, 65B99.

1 Introduction

Solving large scale eigenvalue problems becomes a fundamental problem in modern science and engineering society. However, it is always a very difficult task to solve high-dimensional eigenvalue problems which come from physical and chemical sciences. About the solution of eigenvalue problems, [3, 7, 8, 9, 12, 17] and the references cited therein give some types of multigrid schemes which couple the multigrid method and the Rayleigh quotient iteration technique. The involved almost singular

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linear problems in these methods lead to the numerical unstability. So it is required to design some special solver for these almost singular linear problems [8, 12].

The aim of this paper is to present a type of shifted-inverse power iteration method to solve the eigenvalue problem based on the multigrid technique. The standard Galerkin finite element method for eigenvalue problems has been extensively investigated, e.g. Babuška and Osborn [1, 2], Chatelin [5] and references cited therein. Here we adopt some basic results in these papers for our analysis. The corresponding error and computational work discussion of the proposed iteration scheme for the eigenvalue problem will be analyzed. Based on the analysis, the new method can obtain optimal errors with an optimal computational work when we can solve the associated linear problems with the optimal complexity.

In order to describe our method clearly, we give the following simple Laplace eigenvalue problem to illustrate the main idea in this paper (see sections 3 and 4).

Find \((\lambda, u) \in \mathbb{R} \times H^1_0(\Omega)\) such that

\[
\begin{cases}
-\Delta u = \lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, \\
\int_{\Omega} u^2 d\Omega = 1,
\end{cases}
\]

(1.1)

where \(\Omega \subset \mathbb{R}^2\) is a bounded domain with Lipschitz boundary \(\partial \Omega\) and \(\Delta\) denotes the Laplace operator.

First, we construct a series of finite element spaces \(V_{h_1}, V_{h_2}, \cdots, V_{h_n}\), which are subspaces of \(H^1_0(\Omega)\) and defined on the corresponding series of multilevel meshes \(T_{h_k} (k = 1, 2, \cdots, n)\) such that \(V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n}\) with \(h_k = h_{k-1}/\beta\) (see, e.g., [4, 6]). Our multigrid algorithm to obtain the approximation of the eigenpairs can be defined as follows (see section 3 and section 4):

1. Solve an eigenvalue problem in the coarsest space \(V_{h_1}^:\)

   Find \((\lambda_{h_1}, u_{h_1}) \in \mathbb{R} \times V_{h_1}\) such that \(\|\nabla u_{h_1}\|_0 = 1\) and

   \[
   \int_{\Omega} \nabla u_{h_1} \cdot \nabla v_{h_1} d\Omega = \lambda_{h_1} \int_{\Omega} u_{h_1} v_{h_1} d\Omega, \quad \forall v_{h_1} \in V_{h_1}.
   \]

2. Do \(k = 1, \cdots, n - 1\)

   - Solve the following auxiliary boundary value problem:
     Find \(\tilde{u}_{h_{k+1}} \in V_{h_{k+1}}\) such that for any \(v_{h_{k+1}} \in V_{h_{k+1}}\)
     \[
     \int_{\Omega} (\nabla \tilde{u}_{h_{k+1}} \nabla v_{h_{k+1}} - \alpha_{k+1} \tilde{u}_{h_{k+1}} v_{h_{k+1}}) d\Omega = \int_{\Omega} u_{h_k} v_{h_{k+1}} d\Omega.
     \]

   - Do the normalization
     \[
u_{h_{k+1}} = \frac{\tilde{u}_{h_{k+1}}}{\|\nabla \tilde{u}_{h_{k+1}}\|_0}.
     \]
and compute the Rayleigh quotient for $u_{hk+1}$

$$\lambda_{hk+1} = \frac{\|\nabla u_{hk+1}\|_0^2}{\|u_{hk+1}\|_0^2}.$$

end Do

If, for example, $\lambda_{h1}$ is the approximation of the first eigenvalue of the problem (1.1) at the first step and $\Omega$ is a convex domain, then we can establish the following results by taking a suitable choice of $\alpha_{k+1}$ (see section 3 and section 4 for details)

$$\|\nabla (u - u_{hn})\|_0 = O(h_n), \quad \text{and} \quad |\lambda - \lambda_{hn}| = O(h_n^2).$$

These two estimates mean that we obtain asymptotic optimal errors.

In this method, we replace solving eigenvalue problem on the finest finite element space by solving a series of boundary value problems in the corresponding series of finite element spaces and an eigenvalue problems in the initial finite element space.

An outline of the paper goes as follows. In Section 2, we introduce the finite element method for the eigenvalue problem and give the corresponding basic error estimates. A type of one shifted-inverse power iteration step is given in Section 3. In Section 4, we propose a type of multigrid algorithm for solving the eigenvalue problem based on the shifted-inverse power iteration step. The computational work estimate of the eigenvalue multigrid method is given in Section 5. In Section 6, two numerical examples are presented to validate our theoretical analysis. Some concluding remarks are given in the last section.

## 2 Discretization by finite element method

In this section, we introduce some notations and error estimates of the finite element approximation for the eigenvalue problem. The letter $C$ (with or without subscripts) denotes a generic positive constant which may be different at its different occurrences through the paper. For convenience, the symbols $\lesssim, \gtrsim$ and $\approx$ will be used in this paper. That $x_1 \lesssim y_1, x_2 \gtrsim y_2$ and $x_3 \approx y_3$, mean that $x_1 \leq C_1 y_1, x_2 \geq c_2 y_2$ and $c_3 x_3 \leq y_3 \leq C_3 x_3$ for some constants $C_1, c_2, c_3$ and $C_3$ that are independent of mesh sizes (see, e.g., [14]).

Let $(V, \| \cdot \|)$ be a real Hilbert space with inner product $(\cdot, \cdot)$ and norm $\| \cdot \|$, respectively. Let $a(\cdot, \cdot), b(\cdot, \cdot)$ be two symmetric bilinear forms on $V \times V$ satisfying

$$a(w, v) \lesssim \|w\|\|v\|, \quad \forall w \in V \text{ and } \forall v \in V, \quad \text{(2.1)}$$

$$\|w\|^2 \lesssim a(w, w), \quad \forall w \in V \text{ and } 0 < b(w, w), \quad \forall w \in V \text{ and } w \neq 0. \quad \text{(2.2)}$$

From (2.1) and (2.2), we know that $\| \cdot \|_a := a(\cdot, \cdot)^{1/2}$ and $\| \cdot \|$ are two equivalent norms on $V$. We assume that the norm $\| \cdot \|$ is relatively compact with respect to
the norm \( \| \cdot \|_b := b(\cdot, \cdot)^{1/2} \) in the sense that any sequence which is bounded in \( \| \cdot \| \), one can extract a subsequence which is Cauchy with respect to \( \| \cdot \|_b \). We shall use \( a(\cdot, \cdot) \) and \( \| \cdot \|_a \) as the inner product and norm on \( V \) in the rest of this paper.

We assume that \( V_h \subset V \) is a family of finite-dimensional spaces that satisfy the following assumption:

For any \( w \in V \)

\[
\lim_{h \to 0} \inf_{v_h \in V_h} \| w - v_h \|_a = 0. \tag{2.3}
\]

Let \( P_h \) be the finite element projection operator of \( V \) onto \( V_h \) defined by

\[
a(w - P_h w, v_h) = 0, \quad \forall w \in V \text{ and } \forall v_h \in V_h. \tag{2.4}
\]

Obviously

\[
\| P_h w \|_a \leq \| w \|_a, \quad \forall w \in V. \tag{2.5}
\]

For any \( w \in V \), by (2.3) we have

\[
\| w - P_h w \|_a = o(1), \quad \text{as } h \to 0. \tag{2.6}
\]

Define \( \eta_a(h) \) as

\[
\eta_a(h) = \sup_{f \in V, \| f \|_b = 1} \inf_{v_h \in V_h} \| T f - v_h \|_a, \tag{2.7}
\]

where the operator \( T : V' \mapsto V \) is defined as

\[
a(T f, v) = b(f, v), \quad \forall f \in V' \text{ and } \forall v \in V. \tag{2.8}
\]

In order to derive the error estimate of eigenpair approximation in the weak norm \( \| \cdot \|_b \), we need the following weak norm error estimate of the finite element projection operator \( P_h \).

**Lemma 2.1.** ([2, Lemma 3.3 and Lemma 3.4])

\[
\eta_a(h) = o(1), \quad \text{as } h \to 0, \tag{2.9}
\]

and

\[
\| w - P_h w \|_b \lesssim \eta_a(h) \| w - P_h w \|_a, \quad \forall w \in V. \tag{2.10}
\]

In our methodology description, we are concerned with the following general eigenvalue problem:

Find \( (\lambda, u) \in \mathcal{R} \times V \) such that \( b(u, u) = 1 \) and

\[
a(u, v) = \lambda b(u, v), \quad \forall v \in V. \tag{2.11}
\]
For the eigenvalue $\lambda$, there exists the following Rayleigh quotient expression (see, e.g., [1, 2, 16])

$$\lambda = \frac{a(u, u)}{b(u, u)}. \quad (2.12)$$

From [2, 5], we know the eigenvalue problem (2.11) has an eigenvalue sequence $\{\lambda_j\}$:

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots, \quad \lim_{k \to \infty} \lambda_k = \infty,$$

and the associated eigenfunctions

$$u_1, u_2, \cdots, u_k, \cdots,$$

where $b(u_i, u_j) = \delta_{ij}$. In the sequence $\{\lambda_j\}$, the $\lambda_j$ are repeated according to their geometric multiplicity.

Let $M(\lambda_i)$ denote the unit ball in the eigenfunction set corresponding to the eigenvalue $\lambda_i$ which is defined by

$$M(\lambda_i) = \{ w \in V : w \text{ is an eigenfunction of (2.11) corresponding to } \lambda_i \text{ and } \|w\|_a = 1 \}.$$

From [1, 2], each eigenvalue $\lambda_i$ can be defined as follows

$$\lambda_i = \inf_{v \in V, v \perp M(\lambda_j) \text{ for } \lambda_j < \lambda_i} \frac{a(v, v)}{b(v, v)}.$$

Now, let us define the finite element approximations of the problem (2.11). First we generate a shape-regular decomposition of the computing domain $\Omega \subset \mathcal{R}^d$ ($d = 2, 3$) into triangles or rectangles for $d = 2$ (tetrahedrons or hexahedrons for $d = 3$). The diameter of a cell $K \in \mathcal{T}_h$ is denoted by $h_K$. The mesh diameter $h$ describes the maximum diameter of all cells $K \in \mathcal{T}_h$. Based on the mesh $\mathcal{T}_h$, we can construct a finite element space denoted by $V_h \subset V$.

Then we define the approximation for the eigenpair $(\lambda, u)$ of (2.11) by the finite element method as:

Find $(\bar{\lambda}_h, \bar{u}_h) \in \mathcal{R} \times V_h$ such that $a(\bar{u}_h, \bar{u}_h) = 1$ and

$$a(\bar{u}_h, v_h) = \bar{\lambda}_h b(\bar{u}_h, v_h), \quad \forall v_h \in V_h. \quad (2.14)$$

From (2.14), we know the following Rayleigh quotient expression for $\bar{\lambda}_h$ holds (see, e.g., [1, 2, 16])

$$\bar{\lambda}_h = \frac{a(\bar{u}_h, \bar{u}_h)}{b(\bar{u}_h, \bar{u}_h)}. \quad (2.15)$$
Similarly, we know from \[2, 5\] the eigenvalue problem (2.14) has eigenvalues

\[0 < \bar{\lambda}_{1,h} \leq \bar{\lambda}_{2,h} \leq \cdots \leq \bar{\lambda}_{k,h} \leq \cdots \leq \bar{\lambda}_{N_h,h},\]

and the corresponding eigenfunctions

\[\bar{u}_{1,h}, \bar{u}_{2,h}, \cdots, \bar{u}_{k,h}, \cdots, \bar{u}_{N_h,h},\]

where \(b(\bar{u}_{i,h}, \bar{u}_{j,h}) = \delta_{ij}, 1 \leq i, j \leq N_h\) (\(N_h\) is the dimension of the finite element space \(V_h\)).

From the minimum-maximum principle (see, e.g., [1, 2]), the following upper bound result holds

\[\lambda_i \leq \bar{\lambda}_{i,h}, \quad i = 1, 2, \cdots, N_h.\]

Similarly, let \(M_h(\lambda_i)\) denote the unit ball in the eigenfunction set corresponding to the eigenvalue \(\lambda_i\) which is defined by

\[M_h(\lambda_i) = \{ w_h \in V_h : w_h \text{ is an eigenfunction of (2.14) corresponding to } \lambda_i \text{ and } \| w_h \|_a = 1 \}. \quad (2.16)\]

From [1, 2], each eigenvalue \(\lambda_{i,h}\) can be defined as follows

\[\lambda_{i,h} = \inf_{v_h \in V_h, v_h \perp M_h(\lambda_j) \text{ for } \lambda_j < \lambda_i} \frac{a(v_h, v_h)}{b(v_h, v_h)}. \quad (2.17)\]

Then we define

\[\delta_h(\lambda_i) = \sup_{w \in M(\lambda_i)} \inf_{v_h \in V_h} \| w - v_h \|_a. \quad (2.18)\]

For the eigenpair approximations by finite element method, there exist the following error estimates.

**Proposition 2.1.** ([1, Lemma 3.7, (3.29b)], [2, P. 699] and [5])

(i) For any eigenfunction approximation \(\bar{u}_{i,h}\) of (2.14) \((i = 1, 2, \cdots, N_h)\), there is an eigenfunction \(u_i\) of (2.11) corresponding to \(\lambda_i\) such that \(\| u_i \|_b = 1\) and

\[\| u_i - \bar{u}_{i,h} \|_a \leq C_i \delta_h(\lambda_i). \quad (2.19)\]

Furthermore,

\[\| u_i - \bar{u}_{i,h} \|_b \leq C_i \eta_a(h) \| u_i - \bar{u}_{i,h} \|_a. \quad (2.20)\]

(ii) For each eigenvalue, we have

\[\lambda_i \leq \bar{\lambda}_{i,h} \leq \lambda_i + C_i \delta_h^2(\lambda_i). \quad (2.21)\]

Here and hereafter \(C_i\) is some constant depending on \(i\) but independent of the mesh size \(h\).
3 One shifted-inverse power iteration step with multigrid method

In this section, we present a type of one shifted-inverse power iteration step to improve the accuracy of the given eigenvalue and eigenfunction approximations. This iteration method only contains solving auxiliary boundary value problems in the finer finite element space.

To analyze our method, we introduce the error expansion of the eigenvalue by the Rayleigh quotient formula which comes from [1, 2, 11, 16].

Lemma 3.1. ([1, 2, 11, 16]) Assume \((\lambda, u)\) is a true solution of the eigenvalue problem (2.11) and \(0 \neq \psi \in V\). Let us define

\[ \hat{\lambda} = \frac{a(\psi, \psi)}{b(\psi, \psi)}. \] (3.1)

Then we have

\[ \hat{\lambda} - \lambda = \frac{a(u - \psi, u - \psi)}{b(\psi, \psi)} - \lambda \frac{b(u - \psi, u - \psi)}{b(\psi, \psi)}. \] (3.2)

For simplicity, here we only state the numerical method for the first eigenvalue. Assume we have obtained an eigenpair approximation \((\lambda_{1,h_k}, u_{1,h_k}) \in \mathcal{R} \times V_{h_k}\). Now we introduce a type of iteration step to improve the accuracy of the current eigenpair approximation \((\lambda_{1,h_k}, u_{1,h_k})\). Let \(V_{h_{k+1}} \subset V\) be a finer finite element space such that \(V_{h_k} \subset V_{h_{k+1}}\). Based on this finer finite element space, we define the following one shifted-inverse power iteration step.

Algorithm 3.1. One Shifted-inverse Power Iteration Step

1. Solve the following boundary value problem:

   Find \(u_{1,h_{k+1}} \in V_{h_{k+1}}\) such that for any \(v_{h_{k+1}} \in V_{h_{k+1}}\)

   \[ a(u_{1,h_{k+1}}, v_{h_{k+1}}) - \alpha_{1,k+1} b(u_{1,h_{k+1}}, v_{h_{k+1}}) = b(u_{1,h_k}, v_{h_{k+1}}). \] (3.3)

2. Do the normalization for \(u_{1,h_{k+1}}\) such that \(\|u_{1,h_{k+1}}\|_a = 1\) and compute the Rayleigh quotient for \(u_{1,h_{k+1}}\)

   \[ \lambda_{1,h_{k+1}} = \frac{a(u_{1,h_{k+1}}, u_{1,h_{k+1}})}{b(u_{1,h_{k+1}}, u_{1,h_{k+1}})}. \] (3.4)

Then we obtain a new eigenpair approximation \((\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}\). Summarize the above two steps into

\[ (\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \text{Correction}(\alpha_{1,k+1}, u_{1,h_k}, V_{h_{k+1}}). \]
Theorem 3.1. After one correction step, there exists the exact solution \((\bar{\lambda}_{1,h_{k+1}}, \bar{u}_{1,h_{k+1}})\) of the eigenvalue problem \((2.14)\) in the finite space \(V_{h_{k+1}}\) such that the resultant approximation \((\bar{\lambda}_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}\) has the following error estimates

\[
\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\| \leq \frac{|\bar{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}|}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} \|u_{1,h_{k}} - \bar{u}_{1,h_{k+1}}\|_a; 
\]

and

\[
\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\| \leq \frac{|\bar{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}|}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} \left( \sqrt{\frac{\lambda_{2,h_{k+1}}}{\lambda_{2,h_k}}} \|u_{1,h_k} - \bar{u}_{1,h_k}\|_a \right)
\]

\[+ \sqrt{\lambda_{2,h_{k+1}} \|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a} \].

(3.6)

Proof. Let us define \(\gamma = \bar{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}\), \(u_{1,h_{k+1}} = \gamma u_{1,h_{k+1}}\) and \(w_{h_{k+1}} = u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\). Then we know

\[a(u_{1,h_{k+1}}, v_{h_{k+1}}) - \alpha_{1,k+1}b(u_{1,h_{k+1}}, v_{h_{k+1}}) = \gamma b(u_{1,h_{k}}, v_{h_{k+1}}), \ \forall v_{h_{k+1}} \in V_{h_{k+1}}. \]

(3.7)

Choosing \(\bar{u}_{1,h_{k+1}}\) such that \(u_{1,h_k} - \bar{u}_{1,h_{k+1}} \perp M_{h_{k+1}}(\lambda_1)\), from problems (3.3) and

\[a(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) - \alpha_{1,k+1}b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}) = \gamma b(\bar{u}_{1,h_{k+1}}, v_{h_{k+1}}), \ \forall v_{h_{k+1}} \in V_{h_{k+1}}. \]

(3.8)

we have \(w_{h_{k+1}} \perp M_{h_{k+1}}(\lambda_1)\).

From (3.3), (3.8) and (2.17), the following estimates hold

\[
\|w_{h_{k+1}}\|_a^2 = \frac{\alpha_{1,k+1}}{\lambda_{2,h_{k+1}}} \|w_{h_{k+1}}\|_a^2 + \frac{|\gamma|}{\lambda_{2,h_{k+1}}} \|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a \|w_{h_{k+1}}\|_a 
\]

Then we have

\[
\|w_{h_{k+1}}\|_a \leq \frac{|\bar{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}|}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} \|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a, 
\]

(3.9)

which means we have obtained the first desired result (3.5). Similarly, we can chose \(\bar{u}_{1,h_k}\) such that \(u_{1,h_k} - \bar{u}_{1,h_k} \perp M_{h_k}(\lambda_1)\). Then from (3.3), (3.8) and (2.17), the following estimates hold

\[
\|w_{h_{k+1}}\|_a^2 = \frac{\alpha_{1,k+1}}{\lambda_{2,h_{k+1}}} \|w_{h_{k+1}}\|_a^2 + \frac{|\gamma|}{\lambda_{2,h_{k+1}}} \|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a \|w_{h_{k+1}}\|_a 
\]

(3.9)
onto the space $V_{1,h_k}$. It means that
\[
\|w_{h_{k+1}}\|_a \leq |\gamma| b(\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}, w_{h_{k+1}}) + \frac{\alpha_{1,k+1}}{\lambda_{2,h_{k+1}}} \|w_{h_{k+1}}\|_a 
+ \frac{|\gamma|}{\sqrt{\lambda_{2,h_{k+1}}}} \|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\| b \|w_{h_{k+1}}\|_a,
\]
onto the space $V_{1,h_k}$. It means that
\[
\|w_{h_{k+1}}\|_a \leq \frac{\bar{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} \left( \sqrt{\frac{\lambda_{2,h_{k+1}}}{\lambda_{2,h_k}}} \|u_{1,h_k} - \bar{u}_{1,h_k}\|_a 
+ \sqrt{\lambda_{2,h_{k+1}}} \|\bar{u}_{1,h_k} - \bar{u}_{1,h_{k+1}}\| b \right). \tag{3.10}
\]
This is the desired result (3.6) and the proof is complete. \hfill \Box

**Remark 3.1.** We discuss the different choice of $\alpha_{1,k+1}$.

1. If $\alpha_{1,k+1} = 0$, i.e. we don’t use shift, actually Algorithm 3.1 is the well-known two grid method [16]. From (3.5), we have
\[
\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\|_a \leq \frac{\bar{\lambda}_{1,h_{k+1}}}{\lambda_{2,h_{k+1}}} \|u_{1,h_k} - \bar{u}_{1,h_k}\|_a \tag{3.11}
\]
This indicates that the two grid step has linear convergence speed rate. But if the gap between $\bar{\lambda}_{1,h_{k+1}}$ and $\lambda_{2,h_{k+1}}$ is small, the convergence is slow.

2. From Lemma 3.1, the following estimate holds
\[
\lambda_{1,h_k} - \bar{\lambda}_{1,h_k} = \frac{a(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, \bar{u}_{1,h_{k+1}} - u_{1,h_k})}{b(u_{1,h_k}, u_{1,h_k})} 
- \bar{\lambda}_{1,h_k} \frac{b(\bar{u}_{1,h_{k+1}} - u_{1,h_k}, \bar{u}_{1,h_{k+1}} - u_{1,h_k})}{b(u_{1,h_k}, u_{1,h_k})} 
\lesssim \|u_{1,h_k} - \bar{u}_{1,h_k}\|_a^2.
\]
If $\alpha_{1,h_{k+1}} = \lambda_{1,h_k}$, we have
\[
\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\|_a \lesssim \|u_{1,h_k} - \bar{u}_{1,h_{k+1}}\|_a^3 \tag{3.12}
\]
which means that the Rayleigh quotient iteration has the cubic convergence rate.
The good choice for $\alpha_{1,k+1}$ sometimes is not so easy to obtain since it depends on $\bar{\lambda}_{1,k+1}$ and $\bar{\lambda}_{2,k+1}$ which are unknown. But from (3.5), if

$$\frac{|\bar{\lambda}_{1,k+1} - \alpha_{1,k+1}|}{\bar{\lambda}_{2,k+1} - \alpha_{1,k+1}} < 1,$$

the accuracy of the solution of the (3.3) can be improved through more times iteration. Then we can design the following modified one multi shifted-inverse power iteration step.

**Algorithm 3.2. Multi Shifted-inverse Power Iteration Step**

1. Set $u^{0}_{1,h_{k+1}} = u_{1,h_{k}}$.
2. Do $j = 0, \cdots, \ell - 1$
   - Solve the following boundary value problems:
     Find $u^{j+1}_{1,h_{k+1}} \in V_{h_{k+1}}$ such that for any $v_{h_{k+1}} \in V_{h_{k+1}}$
     $a(u^{j+1}_{1,h_{k+1}}, v_{h_{k+1}}) - \alpha^{j+1}_{1,k+1} b(u^{j+1}_{1,h_{k+1}}, v_{h_{k+1}}) = b(u^{j}_{1,h_{k+1}}, v_{h_{k+1}})$. (3.13)
   - Normalize $u^{j+1}_{1,h_{k+1}}$ and compute the Rayleigh quotient for $u^{j+1}_{1,h_{k+1}}$
     $$\lambda^{j+1}_{1,h_{k+1}} = \frac{a(u^{j+1}_{1,h_{k+1}}, u^{j+1}_{1,h_{k+1}})}{b(u^{j+1}_{1,h_{k+1}}, u^{j+1}_{1,h_{k+1}})}. \quad (3.14)$$
   end Do
3. Set $u_{1,h_{k+1}} = u^{\ell}_{1,h_{k+1}}$ and $\lambda_{1,h_{k+1}} = \lambda^{\ell}_{1,h_{k+1}}$.

Then we obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$. Summarize the above two steps into

$$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \text{Correction}((\{\alpha^{j}_{1,k+1}\}_{j=1}^{\ell}, u_{1,h_{k}}, V_{h_{k+1}})).$$

In Algorithm 3.2, we can adjust $\ell$ such that the following estimate satisfies

$$\|u_{1,h_{k+1}} - \bar{u}_{1,h_{k+1}}\|_a < \frac{1}{\beta} \left( \sqrt{\frac{\bar{\lambda}_{2,h_{k+1}}}{\bar{\lambda}_{2,h_{k}}} \|u_{1,h_{k}} - \bar{u}_{1,h_{k}}\|_a} \right. \left. + \sqrt{\bar{\lambda}_{2,h_{k+1}} \|\bar{u}_{1,h_{k}} - \bar{u}_{1,h_{k+1}}\|_b} \right), \quad (3.15)$$

where $\beta$ is a constant defined in the following section. In fact, $\ell$ can be very small ($\ell = 2$ or 3) and the modified iteration step makes the choice of $\alpha_{1,k+1}$ become not so sharp as in Algorithm 3.1 and it improves the stability of the iteration step.
4 Multigrid scheme for the eigenvalue problem

In this section, we introduce a type of multigrid scheme based on the One Shifted-inverse Power Iteration Step defined in Algorithm 3.1. This type of multigrid method can obtain the optimal error estimate as same as solving the eigenvalue problem directly in the finest finite element space.

In order to do multigrid scheme, we define a sequence of triangulations $T_{h_k}$ of $\Omega$ determined as follows. Suppose $T_{h_1}$ is given and let $T_{h_k}$ be obtained from $T_{h_{k-1}}$ via regular refinement (produce $\beta^d$ subelements) such that

$$h_k = \frac{1}{\beta} h_{k-1}.$$ 

Based on this sequence of meshes, we construct the corresponding linear finite element spaces such that

$$V_{h_1} \subset V_{h_2} \subset \cdots \subset V_{h_n},\quad (4.1)$$

and the following relation of approximation errors hold

$$\eta_a(h_k) \approx \frac{1}{\beta} \eta_a(h_{k-1}), \quad \delta_{h_k}(\lambda) \approx \frac{1}{\beta} \delta_{h_{k-1}}(\lambda), \quad k = 2, \ldots, n.\quad (4.2)$$

We also define

$$\alpha_{1,k+1} = \max \left\{ 0, \frac{2\beta \lambda_{1,h_k} - \lambda_{2,h_k}}{2\beta - 1} \right\}, \quad k = 1, \ldots, n - 1.\quad (4.3)$$

Algorithm 4.1. Eigenvalue Multigrid Scheme

1. Construct a series of nested finite element spaces $V_{h_1}, V_{h_2}, \cdots, V_{h_n}$ such that (4.1) and (4.2) holds.

2. Solve the following eigenvalue problem:

   Find $(\lambda_{1,h_1}, u_{1,h_1}) \in \mathcal{R} \times V_{h_1}$ such that $a(u_{1,h_1}, u_{1,h_1}) = 1$ and

   $$a(u_{1,h_1}, v_{h_1}) = \lambda_{1,h_1} b(u_{1,h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.\quad (4.4)$$

3. Do $k = 1, \cdots, n - 1$

   Obtain a new eigenpair approximation $(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) \in \mathcal{R} \times V_{h_{k+1}}$ by a correction step

   $$(\lambda_{1,h_{k+1}}, u_{1,h_{k+1}}) = \text{Correction}(\alpha_{1,k+1}, u_{1,h_k}, V_{h_{k+1}}).\quad (4.5)$$

   end Do
Finally, we obtain an eigenpair approximation \((\lambda_{1,h_n}, u_{1,h_n}) \in \mathcal{R} \times V_{h_n}\).

**Theorem 4.1.** After implementing Algorithm 4.1, there exists an eigenpair \((\lambda_1, u_1) \in \mathcal{R} \times M(\lambda_1)\) of (2.11) such that the resultant eigenpair approximation \((\lambda_{1,h_n}, u_{1,h_n})\) has the following error estimates

\[
\|u_{1,h_n} - u_1\|_a \leq C\delta_{h_n}(\lambda_1),
\]

\[
|\lambda_{1,h_n} - \lambda_1| \leq C\delta^2_{h_n}(\lambda_1),
\]

when we chose \(\alpha_{1,k+1}\) as in (4.3) and the mesh size \(h_1\) is small enough.

**Proof.** If we chose the \(\alpha_{1,k+1}\) as in (4.3) and \(2\beta \lambda_{1,h_k} - \lambda_{2,h_1} > 0\), then \(\alpha_{1,k+1} = \frac{2\beta \lambda_{1,h_k} - \lambda_{2,h_1}}{2\beta - 1}\) and we have

\[
\frac{\tilde{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} = \frac{|\lambda_{2,h_1} - \tilde{\lambda}_{1,h_{k+1}} + 2\beta (\tilde{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k})|}{2\beta (\lambda_{2,h_{k+1}} - \lambda_{1,h_k}) + \lambda_{2,h_1} - \lambda_{2,h_{k+1}}}
= \frac{1}{2\beta} + \mathcal{O}(h_1^2) < \frac{1}{\beta},
\]

(4.8)

when \(h_1\) is small enough.

If we chose the \(\alpha_{1,k+1}\) as in (4.3) and \(2\beta \lambda_{1,h_k} - \lambda_{2,h_1} < 0\), then \(\alpha_{1,k+1} = 0\) and we have

\[
\frac{\tilde{\lambda}_{1,h_{k+1}} - \alpha_{1,k+1}}{\lambda_{2,h_{k+1}} - \alpha_{1,k+1}} = \frac{|\lambda_{1,h_k} + (\tilde{\lambda}_{1,h_{k+1}} - \lambda_{1,h_k})|}{\lambda_{2,h_1} + \lambda_{2,h_{k+1}} - \lambda_{2,h_1}}
= \frac{1}{2\beta} + \mathcal{O}(h_1^2) < \frac{1}{\beta},
\]

(4.9)

when \(h_1\) is small enough.

From Lemma 2.1, (4.8)-(4.9), Theorem 3.1 and the recursive relation (4.2), the final eigenfunction approximation \(u_{1,h_n}\) and the direct finite element approximation \(\bar{u}_{1,h_n}\) has the following estimates

\[
\|u_{1,h_n} - \bar{u}_{1,h_n}\|_a \leq \frac{1}{\beta} \|u_{1,h_{n-1}} - \bar{u}_{1,h_{n-1}}\|_a + C\eta_a(h_{n-1})\delta_{h_{n-1}}(\lambda_1)
\leq C \left( \sum_{k=1}^{n-1} \left( \frac{1}{\beta} \right)^{n-1-k} \eta_a(h_{k})\delta_{h_k}(\lambda_1) \right)
\leq C \left( \sum_{k=1}^{n-1} \left( \frac{1}{\beta} \right)^{n-1-k} \beta^{n-k} \left( \frac{1}{\beta} \right)^{k-1} \right) \eta_a(h_{1})\delta_{h_{1}}(\lambda_1)
\leq C \frac{\beta^2}{\beta - 1} \eta_a(h_{1})\delta_{h_{1}}(\lambda_1).
\]

(4.10)

Then the desired result (4.6) can be derived by combining (2.19), (4.10) and the triangle inequality. From Lemma 3.1 and (4.6), we can obtain the desired results (4.7). \qed
Remark 4.1. We also investigate the condition of the boundary value problem with different choices of $\alpha_{1,k+1}$. If we chose $\alpha_{1,k+1}$ as (4.3) and $2\beta \lambda_{1,h_k} - \lambda_{2,h_1} > 0$, the condition of linear problem (3.3) has the following estimate

$$\frac{\hat{\lambda}_{n,h_{k+1}} - \alpha_{1,k+1}}{\lambda_{1,h_{k+1}} - \alpha_{1,k+1}} \approx \frac{\lambda_1}{\lambda_1 - \alpha_{1,k+1}} \frac{\hat{\lambda}_{n,h_{k+1}}}{\lambda_1 - \lambda_2} \approx \frac{2\beta \lambda_1}{\lambda_2 - \lambda_1} \text{cond}(A),$$

where the convergence results of $\lambda_{1,h_k}$, $\lambda_{1,h_k}$ and $\lambda_{2,h_1}$ are used.

5 Work estimate of eigenvalue multigrid scheme

In this section, we turn our attention to the estimate of computational work for Algorithm 4.1. We will show that Algorithm 4.1 makes solving the eigenvalue problem need almost the same work as solving the corresponding boundary value problem if we adopt the multigrid method to solve the involved linear problems (3.3) (see, e.g., [4, 7, 8, 14, 15]).

First, we define the dimension of each level linear finite element space as $N_k := \text{dim} V_{h_k}$. Then we have

$$N_k \approx \left(\frac{1}{\beta}\right)^{d(n-k)} N_n, \quad k = 1, 2, \cdots, n. \quad (5.1)$$

Theorem 5.1. Assume the eigenvalue problem solved in the coarse space $V_{h_1}$ needs work $O(M_{h_1})$ and the work of the multigrid solver in each level space $V_{h_k}$ is $O(N_k)$ for $k = 2, 3, \cdots, n$. Then the work involved in Algorithm 4.1 is $O(N_n + M_{h_1})$. Furthermore, the complexity will be $O(N_n)$ provided $M_{h_1} \leq N_k$.

Proof. Let $W_k$ denote the work of the correction step in the $k$-th finite element space $V_{h_k}$. Then with the correction definition, we have

$$W_k = O(N_k). \quad (5.2)$$

Iterating (5.2) and using the fact (5.1), we obtain

$$W_n = \sum_{k=1}^{n} W_k = O\left(M_{h_1} + \sum_{k=2}^{n} N_k\right) = O\left(\sum_{k=2}^{n} N_k + M_{h_1}\right)$$

$$= O\left(\sum_{k=2}^{n} \left(\frac{1}{\beta}\right)^{d(n-k)} N_n + M_{h_1}\right) = O(N_n + M_{h_1}). \quad (5.3)$$

This is the desired result $O(N_n + M_{h_1})$ and the one $O(N_n)$ can be obtained by the condition $M_{h_1} \leq N_n$. \qed

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6 Numerical results

In this section, two numerical examples are presented to illustrate the efficiency of the multigrid scheme proposed in this paper.

6.1 Model eigenvalue problem

Here we give the numerical results of the multigrid scheme for the Laplace eigenvalue problem on the two dimensional domain $\Omega = (0, 1) \times (0, 1)$. The sequence of finite element spaces are constructed by using linear element on the series of mesh which are produced by regular refinement with $\beta = 2$ (connecting the midpoints of each edge). In this example, we use two meshes which are generated by Delaunay method as the initial mesh $T_{h_1}$ to investigate the convergence behaviors. Figure 1 shows the corresponding initial meshes: one is coarse and the other is fine.

Algorithm 4.1 is applied to solve the eigenvalue problem. For comparison, we also solve the eigenvalue problem by the direct method.

![Figure 1: The coarse and fine initial meshes for Example 1](image)

Figure 2 gives the corresponding numerical results for the first eigenvalue $\lambda_1 = 2\pi^2$ and the corresponding eigenfunction on the two initial meshes illustrated in Figure 1.

From Figure 2, we find the multigrid scheme can obtain the optimal error estimates as same as the direct eigenvalue solving method for the eigenvalue and the corresponding eigenfunction approximations.

We also check the convergence behavior for multi eigenvalue approximations with Algorithm 4.1. Here the first six eigenvalues $\lambda = 2\pi^2, 5\pi^2, 5\pi^2, 8\pi^2, 10\pi^2, 10\pi^2$ are investigated. We adopt the right one in Figure 1 as the initial mesh and the corresponding numerical results are shown in Figure 3. Figure 3 also exhibits the optimal convergence of the multigrid scheme.
6.2 More general eigenvalue problem

Here we give the numerical results of the multigrid scheme for solving a more general eigenvalue problem on the unit square domain $\Omega = (0,1) \times (0,1)$.

Find $(\lambda, u)$ such that

$$
\begin{cases}
-\nabla \cdot A \nabla u + \phi u = \lambda \rho u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega, \\
\int_{\Omega} \rho u^2 d\Omega = 1,
\end{cases}
$$

(6.1)

where

$$
A = \begin{pmatrix}
1 + (x_1 - \frac{1}{2})^2 & (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) \\
(x_1 - \frac{1}{2})(x_2 - \frac{1}{2}) & 1 + (x_2 - \frac{1}{2})^2
\end{pmatrix},
$$

$$
\phi = e^{(x_1 - \frac{1}{2})(x_2 - \frac{1}{2})} \quad \text{and} \quad \rho = 1 + (x_1 - \frac{1}{2})(x_2 - \frac{1}{2}).
$$

We first solve the eigenvalue problem (6.1) by linear finite element space on the coarse mesh $T_{h_1}$. Then refine the mesh by the regular way to produce a series of meshes $T_{h_k} (k = 2, \cdots, n)$ with $\beta = 2$ (connecting the midpoints of each edge) and solve the auxiliary boundary value problem (3.3) in the finer linear finite element space $V_{h_k}$ defined on $T_{h_k}$.

In this example, we also use two coarse meshes which are shown in Figure 1 as the initial meshes to investigate the convergence behaviors. Since the exact solution is not known, we choose an adequately accurate eigenvalue approximations with the extrapolation method (see, e.g., [10]). Figure 4 gives the corresponding numerical results for the first six eigenvalue approximations and the first eigenfunction approximations. Here we also compare the numerical results with the direct algorithm. Figure 4 also exhibits the optimal convergence of Algorithm 4.1.
7 Concluding remarks

In this paper, we give a type of multigrid scheme to solve the eigenvalue problems. The idea here is to combine the shifted-inverse power iteration method and the multilevel meshes to transform the solution of eigenvalue problem to a series of solutions of the corresponding boundary value problems which can be solved by the multigrid method. As stated in the numerical examples, *Eigenvalue Multigrid Scheme* defined in Algorithm 4.1 for one eigenvalue can be extended to the corresponding version for multi eigenvalues. We can state the following version of *Eigenvalue Multigrid Scheme* for $m$ eigenvalues.

Similarly, we first give a type of *One Correction Step for Multi Eigenvalues* for the given eigenpair approximations \( \{\lambda_{j,h_k}, u_{j,h_k}\}_{j=1}^m \).

**Algorithm 7.1. One Correction Step for Multi Eigenvalues**

1. Do \( j = 1, \ldots, m \)
   - Find \( \tilde{u}_{j,h_{k+1}} \in V_{h_{k+1}}, \) such that \( \forall v_{h_{k+1}} \in V_{h_{k+1}} \)
   
   \[
   a(\tilde{u}_{j,h_{k+1}}, v_{h_{k+1}}) - \alpha_{j,k+1} b(\tilde{u}_{j,h_{k+1}}, v_{h_{k+1}}) = b(u_{j,h_k}, v_{h_{k+1}}). \tag{7.1}
   
   - Do the following orthogonalization and normalization

   \[
   \hat{u}_{j,h_{k+1}} := \tilde{u}_{j,h_{k+1}} - \sum_{\ell=1}^{j-1} a(\hat{u}_{j,h_{k+1}}, \tilde{u}_{\ell,h_{k+1}}) \tilde{u}_{\ell,h_{k+1}}, \tag{7.2}
   
   u_{j,h_{k+1}} := \frac{\hat{u}_{j,h_{k+1}}}{\|\hat{u}_{j,h_{k+1}}\|_a} \tag{7.3}
   \]
Figure 4: The errors of the multigrid algorithm for the first eigenvalue $\pi^2$ and the corresponding eigenfunction, where $u_h^{\text{dir}}$ and $\lambda_h^{\text{dir}}$ denote the eigenfunction and eigenvalue approximation by direct eigenvalue solving.

end Do

2. Compute the new eigenvalue approximations

$$\lambda_{j,h}^{k+1} = \frac{a(u_{j,h}^{k+1}, u_{j,h}^{k+1})}{b(u_{j,h}^{k+1}, u_{j,h}^{k+1})}, \quad \forall j = 1, \cdots, m. \quad (7.4)$$

We summarize the above two steps into

$$\{\lambda_{j,h}^{k+1}, u_{j,h}^{k+1}\}_{j=1}^m = \text{Correction}(\{a_{j,k+1}\}_{j=1}^m, \{\lambda_{j,h}^{k}, u_{j,h}^{k}\}_{j=1}^m, V_{h,k+1}).$$

Similarly to (3.2), we can also define a modified one correction step for multi eigenvalues. Furthermore, Algorithm 7.1 can be modified to the following form where we need to solve a small dimensional eigenvalue problem.

Algorithm 7.2. One Correction Step for Multi Eigenvalues

1. Do $j = 1, \cdots, m$

   Find $\tilde{u}_{j,h,k+1} \in V_{h,k+1}$, such that $\forall v_{h,k+1} \in V_{h,k+1}$ satisfying

   $$a(\tilde{u}_{j,h,k+1}, v_{h,k+1}) - \alpha_{j,k+1} b(\tilde{u}_{j,h,k+1}, v_{h,k+1}) = b(u_{j,h,k}, v_{h,k+1}). \quad (7.5)$$

end Do

2. Build a finite dimensional space $\tilde{V}_{h,k+1} = \text{span}\{\tilde{u}_{1,h,k+1}, \cdots, \tilde{u}_{m,h,k+1}\}$ and solve the following eigenvalue problem:

   Find $(\lambda_{j,h,k+1}, u_{j,h,k+1}) \in \mathcal{R} \times \tilde{V}_{h,k+1}$, $j = 1, 2, \cdots, m$, such that $a(u_{j,h,k+1}, u_{j,h,k+1}) = 1$ and

   $$a(u_{j,h,k+1}, v_{h,k+1}) = \lambda_{j,h,k+1} b(u_{j,h,k+1}, v_{h,k+1}), \quad \forall v_{h,k+1} \in \tilde{V}_{h,k+1}. \quad (7.6)$$
We summarize the above two steps into
\[ \{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=1}^m = \text{Correction}(\{\alpha_{j,k+1}\}_{j=1}^m, \{\lambda_{j,h_k}, u_{j,h_k}\}_{j=1}^m; V_{h_{k+1}}). \]

Based on Algorithm 7.1 or 7.2, we can give the corresponding multigrid method.

**Algorithm 7.3. Eigenvalue Multigrid Scheme for Multi Eigenvalues**

1. Construct a series of nested finite element spaces \( V_{h_1}, V_{h_2}, \ldots, V_{h_n} \) such that (4.1) and (4.2) hold.

2. Solve the following eigenvalue problem:

   Find \( (\lambda_{h_1}, u_{h_1}) \in \mathcal{R} \times V_{h_1} \) such that
   \[
   a(u_{h_1}, v_{h_1}) = \lambda_{h_1} b(u_{h_1}, v_{h_1}), \quad \forall v_{h_1} \in V_{h_1}.
   \] (7.7)

   Choose \( m \) eigenpairs \( \{\lambda_{j,h_1}, u_{j,h_1}\}_{j=1}^m \) which approximate to our desired eigenvalues and their eigenspaces.

3. Do \( k = 1, \ldots, n - 1 \)

   Obtain new eigenpair approximations \( \{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=1}^m \in \mathcal{R} \times V_{h_{k+1}} \) by a correction step

   \[
   \{\lambda_{j,h_{k+1}}, u_{j,h_{k+1}}\}_{j=1}^m = \text{Correction}(\{\alpha_{j,k+1}\}_{j=1}^m, \{\lambda_{j,h_k}, u_{j,h_k}\}_{j=1}^m; V_{h_{k+1}}).
   \] end Do

Finally, we obtain \( m \) eigenpair approximations \( \{\lambda_{j,h_n}, u_{j,h_n}\}_{j=1}^m \in \mathcal{R} \times V_{h_n} \).

We can also define
\[
\alpha_{j,k+1} = \max \left\{ 0, \frac{2\beta \lambda_{j,h_k} - \lambda_{j+1,h_k}}{2\beta - 1} \right\}, \quad j = 1, \ldots, m - 1
\] (7.8)
and
\[
\alpha_{m,k+1} = \max \left\{ 0, \frac{2\beta \lambda_{m,h_k} - \lambda_{m+1,h_k}}{2\beta - 1} \right\}.
\] (7.9)

Based on the above definitions of \( \alpha_{j,k+1} \), we can also give the error analysis for this version of eigenvalue multigrid method in the similar way used in Sections 3 and 4. If we use the multigrid method to solve the boundary value problems included in Algorithm 7.3, the computational work involved in the multi eigenvalues version is \( O(m^2 N_n) \). Furthermore, the parallel computation can be used to solve (7.5) for different \( j \).

We can replace the multigrid method by other types of efficient iteration schemes such as algebraic multigrid method, the type of preconditioned schemes based on the subspace decomposition and subspace corrections (see, e.g., [4, 14]), and the domain decomposition method (see, e.g., [13]). Furthermore, the framework here can also be coupled with the parallel method and the adaptive refinement technique. The ideas can be extended to other types of linear eigenvalue problems and other types problems. These will be investigated in our future work.
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