A straightforward moment method to estimate the load and resistance factors

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Abstract. The load and resistance factor design is widely applied in engineering, where the load and resistance factors are introduced to account for the uncertainties in practice. Accurate estimation of the load and resistance factors is important for an appropriate design, which is generally conducted by the First Order Reliability Method (FORM). However, since the design point must be determined and derivative-based iterations have to be used, FORM is not practical for engineers. Furthermore, distributions of the load and resistance are required in FORM, which are usually unknown in practice. To overcome these deficiencies, moment methods are proposed, which estimate the load and resistance factors based on their moments. Although, the existing moment methods significantly reduce the iteration numbers, iterations are still necessary when higher accuracy is required. Therefore, a straightforward moment method is proposed in the present paper to estimate the load and resistance factors without iteration. The procedure of the proposed method is summarized in a flowchart with explicit expressions given. The accuracy of the proposed method is examined by comparison study among existing methods. It is shown that the proposed method is both accurate and efficient in estimating the load and resistance factors with no iteration required.

1. Introduction
In designing a structure, the insurance of the safety is the most important task that should be accomplished. To achieve this, the Load and Resistance Factor Design (LRFD) [1, 2] format is proposed and has been widely applied in practical engineering, where the safety of a structure is assured by making the nominal design resistances reduced by the resistance factors no less than the nominal design loads amplified by the load factors. The load and resistance factors are introduced to account for the uncertainties inherent in the determination of the nominal strength and the load effects. Accurate predetermination of the load and resistance factors based on specified reliability-based requirement is important for an appropriate design.

Generally, the load and resistance factors can be obtained by using the first order reliability method (FORM) [3], in which the “design point” must be determined, derivative-based iterations have to be used, and the problem of multiple design points has to be dealt with [4, 5]. The complexity of FORM would prevent the practicing engineers in general to perform it in engineering designs. AIJ [2] recommendation has provided a simple method based on the proposal of Mori [6], in which all the random variables are assumed to have known probability density function (PDF) and required to be transferred into lognormal random variables. However, in reality, the PDFs of some of the basic random variables are often unknown due to the lack of statistical data. ASCE [7] standard proposes simple equations to determine the load and resistance factors, where the sensitivity coefficients in the...
formulation uses approximate values. To obtain suitable load and resistance factors including random variables with unknown PDFs, Lu et al [8] proposed a method based on the moment method, where the load and resistance factors can be obtained including random variables with unknown PDFs. This moment method expands the application of LRFD into areas where the PDFs of random variables are unknown. However, it still needs iteration when high accuracy is required.

The objective of the present paper is to propose a straightforward moment method to determine the load and resistance factors with no iteration required. This paper is organized as follows. Firstly, the existing methods are reviewed. Then the proposed method is deduced with explicit formulas presented and procedure summarized. The accuracy of the proposed method is investigated with numerical examples. Finally, the findings of the present paper are concluded.

2. Review of existing methods

The LRFD format is expressed as follows

\[ \phi R \geq \sum \gamma_i S_{\text{si}} \] 

where \( \phi \) is the resistance factor; \( \gamma \) is the partial load factor to be applied to load effect \( S_i \); \( R_n \) and \( S_n \) are the nominal value of the resistance \( R \) and the load effect \( S_i \), respectively.

The appropriate expressions of \( \phi \) and \( \gamma \) are determined for reliability-based requirement:

\[ P_f \leq P_{fr}, \beta \leq \beta_f \] 

where \( P_f \) and \( \beta_f \) are the target probability of failure and target reliability index, respectively; \( P_f \) and \( \beta \) are the probability of failure and reliability index corresponding to the performance function:

\[ G(X) = R - \sum S_i \] 

where \( R \) and \( S_i \) are the random variables representing the uncertainty in the resistance and load effects.

2.1. FORM for LRFD

By performing FORM, the LRFD format can be expressed as

\[ R^* \geq \sum S^*_i \] 

where \( R^* \) and \( S^*_i \) are the values of resistance \( R \) and load \( S_i \) at the design point of FORM, respectively.

And the load and resistance factors can be obtained as [1]

\[ \varphi = R^* / R_n, \gamma_i = S^*_i / S_n \] 

Since \( R^* \) and \( S^*_i \) are obtained using derivative-based iterations, explicit expressions of \( R^* \) and \( S^*_i \) are not available. Some simplifications have been proposed to avoid iterative computations [6].

2.2. Existing moment method

By using the second moment method, the reliability index can be obtained as

\[ \beta = \beta_{2M} = \mu_c / \sigma_G, \mu_G = \mu_R - \sum \mu_{S_i}, \sigma_G^2 = (\mu_G V_R)^2 + \sum \sigma_{S_i}^2 \] 

where \( \beta_{2M} \) is the second moment reliability; \( \mu_G \) and \( \sigma_G \) are the mean and standard deviation of \( G(X) \), respectively; \( \mu_R \) and \( \mu_{S_i} \) are the mean value of the \( R \) and \( S_i \), respectively; \( V_R \) and \( \sigma_{S_i} \) are the coefficient of variation (COV) of \( R \) and standard deviation of \( S_i \), respectively. To make a balance between the economical and the reliability-based requirements, the second moment index \( \beta_{2M} \) is set to be equal to the targeted reliability index \( \beta_f \):

\[ \beta_{2M} = \beta_f \] 

Substituting Eq. \( \mu_c = \mu_R - \sum \mu_{S_i}, \sigma_G^2 = (\mu_G V_R)^2 + \sum \sigma_{S_i}^2 \) (6) into Eq. (7), Eq. (7) is rearranged as follows

\[ \mu_R (1 - \alpha_{R,M} V_R \beta_f) = \sum \mu_{S_i} (1 + \alpha_{S_i,M} V_S \beta_f) \] 

where \( \alpha_{R,M} \) and \( \alpha_{S_i,M} \) are the direction cosines of \( R \) and \( S_i \), respectively, and are given as
\[ \alpha_{R,M} = \mu_R V_R / \sigma_G, \quad \alpha_{S_i,M} = \sigma_{S_i} / \sigma_G \]  

Comparing Eq. (8) with Eq. (1), the resistance and partial load factors can be expressed as

\[ \varphi = (1 - \alpha_{R,M} V_R \beta_r) \mu_R / R_n, \quad \gamma_i = (1 + \alpha_{S_i,M} V_i \beta_i) \mu_{S_i} / S_n \]  

If \( R \) and \( S_i \) are mutually independent normal random variables, \( \beta_{2M} \) is the exact reliability index, and thus the load and resistance factors can be obtained exactly by using Eq. (10). However, in practical engineering, the \( R \) and \( S_i \) are usually non-normal variables and thus higher order reliability index are proposed. To take the advantage of the simple forms of Eq. (10), \( \beta_T \) is transformed into equivalent second-moment target reliability index \( \beta_{2T} \), which is in the same space as \( \beta_{2M} \), by using the higher order moment method. And then the reliability-based requirement given in Eq. (7) is transformed into

\[ \beta_{2M} = \beta_{2T} \]  

### Table 1. The expressions of \( \beta_{2T} \) and \( \mu_{R0} \) of existing methods

| Name           | Expression of \( \beta_{2T} \)                                                                 | Expression of \( \mu_{R0} \)                                                                 |
|----------------|-----------------------------------------------------------------------------------------------|-----------------------------------------------------------------------------------------------|
| Zhao 2000 [9]  | \[3(1 - \exp(\alpha_{3G}(-\beta_T - \alpha_{3G}/6)/3))/\alpha_{3G}\]                          | \[\sum \mu_u + \sqrt{\beta_T^2 \sum \sigma_u^2}\]                                          |
| Zhao 2001 [10] | \[\beta_T - \alpha_{3G}(\beta_T^2 - 1)/6\]                                                   | \[\sum \mu_u + \sqrt{\beta_T^2 \sum \sigma_u^2}\]                                          |
| Wang [11]      | \[3(1 - \exp(-\alpha_{3G} \beta_T^2/3))/\alpha_{3G}\]                                        | \[\sum \mu_u + \sqrt{\beta_T^2 \sum \sigma_u^2}\]                                          |
| Zhao 2007 [12] | \[l_i = 1 - \frac{\alpha_{3G}}{6(1 + 6l_i)}, \quad l_i = \frac{1}{36}(6\alpha_{3G} - 8\alpha_{3G}^2 - 14 - 2), \quad k_i = \frac{1 - 3l_i}{1 + l_i - l_i^2}, \quad \sum \mu_u + \sqrt{\beta_T^2 \sum \sigma_u^2}\] | \[l_i = \frac{1}{1 + l_i + 12l_i}\]                                                     |

The expressions of \( \beta_{2T} \) of different methods are listed in Table 1, where \( \alpha_{3G} \) and \( \alpha_{4G} \) are the skewness and kurtosis of the performance \( G(X) \), respectively, and expressed as

\[ \alpha_{3G} = [\alpha_{3R}(\mu_R V_R)^3 - \sum \alpha_{3S_i} \sigma_{S_i}^3] / \sigma_G^3 \]  

\[ \alpha_{4G} = [\alpha_{4R}(\mu_R V_R)^4 + 6\sigma_R^2 \sum \alpha_{3S_i} \sigma_{S_i}^2 + \sum \alpha_{4S_i} \sigma_{S_i}^4 + 6 \sum \alpha_{3S_i} \sigma_{S_i}^2 \sigma_{S_j}^2] / \sigma_G^4 \]  

where \( \alpha_{3R} \) and \( \alpha_{3S_i} \) are the skewness of the resistance \( R \) and load \( S_i \), respectively; \( \alpha_{4R} \) and \( \alpha_{4S_i} \) are the kurtosis of the resistance \( R \) and load \( S_i \), respectively. With \( \alpha_{3G} \) and \( \alpha_{4G} \) given, \( \beta_{2T} \) can be easily determined with the aid of Table 1, and then the resistance factor \( \phi \) and partial load factor \( \gamma_i \) are obtained by solving Eq. (11) and are expressed as

\[ \varphi = (1 - \alpha_{R,M} V_R \beta_{2T}) \mu_R / R_n, \quad \gamma_i = (1 + \alpha_{S_i,M} V_i \beta_{2T}) \mu_{S_i} / S_n \]  

Eq. (13) in the same form as Eq. (10), while \( \beta_{2T} \) is applied in Eq. (13) instead of \( \beta_T \) in Eq. (10). When \( \alpha_{3G} = 0 \) and \( \alpha_{4G} = 3 \), \( \beta_{2T} = \beta_T \) and Eq. (13) reduce to Eq. (10). Thus, Eq. (13) is used to represent the expressions applied to estimate load and resistance factors by moment methods.

According to Eqs. (9)-(13) and Table 1, the values of \( \phi \) and \( \gamma_i \) depend on the value of \( \mu_{R0} \), which is unknown in the design process. Therefore, Eq. (13) has to be evaluated iteratively, and the initial mean resistance, \( \mu_{R0} \), is listed in Table 1. The accurate determination of \( \mu_{R0} \) is necessary for an efficient method. However, according to Table 1, \( \mu_{R0} \) are defined by experienced formula in all the existing moment methods, which are not precise enough and may cause the iterations. Thus a new formula is proposed in this paper to avoid the iterations required in the existing moment methods.

3. Proposed model
To simplify the procedure of estimating the load and resistance factors, a suitable formula to predetermine the $\mu_R$ is proposed. Once $\mu_R$ is determined, the load and resistance factors can be directly obtained with the aid of Eqs. (9), (13), and thus the iterations are avoided.

If $R$ and $S_i$ ($i=1, \ldots, n$) are independent normal random variables, the exact value of $\mu_R$ can be obtained by solving Eqs. (6) and (7), and is expressed as

$$\mu_R = \frac{1 + \sqrt{1 - \omega_R^2 \omega_S^2}}{\omega_R} \sum \mu_{S_i}, \omega_R = 1 - \beta^2 V_R^2, \omega_S = 1 - \beta^2 V_S^2$$

where $V_S = \sum \sigma_i^2 / \sum \mu_{S_i}$ is the COV of $S$, considering all $S_i$ ($i=1, \ldots, n$) as a whole load effect $S$.

In general, the resistance $R$ and partial loads $S_i$ ($i=1, \ldots, n$) are not normally distributed, and then the exact value of $\mu_R$ is obtained by solving Eqs. (6) and (11) with the aid of Table 1 and given as

$$\mu_R = \frac{1 + \sqrt{1 - \omega_R^2 \omega_S^2}}{\omega_R} \sum \mu_{S_i}$$

$$\omega_R' = 1 - \beta_{2T_{check}}^2 V_R^2, \omega_S = 1 - \beta_{2T_{check}}^2 V_S^2$$

where $\beta_{2T_{check}}$ is the a checking value of $\beta_{2T}$, obtained at the checking value of $\mu_R$ with the aid of Table 1. To predetermine the value of $\beta_{2T}$, a checking value of $\mu_R$, named as $\mu_{R_{check}}$, is given as

$$\mu_{R_{check}} = 2(V_R^2 + 1)\sum \mu_{S_i}$$

The procedure of the proposed method is depicted in Figure 1, where the procedure circled by bold dotted line is the first step of the proposed method to predetermine $\mu_R$. As illustrated in Figure 1, with $\mu_R$ predetermined, the iterations required in the existing moment methods are avoided in the proposed method. And since the step to predetermine $\mu_R$ includes only explicit expressions, the proposed method is easy to conduct in practical engineering.

4. Application in structural reliability

4.1. Example 1

The first example considers the following performance function [7]

$$G(X) = R - (D + L + S)$$

where $R$, $D$, $L$ and $S$ are the resistance, dead load, live load and snow load, respectively. The distribution and the first four moments of the random variables are listed in Table 2.

| Variables | Distribution | $\mu/D_n$ | $V_i$ | $\sigma_i$ | $\alpha_{S_i}$ | $\alpha_{S_i}$ | $\mu_S/R_n$ or $\mu_S/S_n$ | $S_n/D_n$ |
|-----------|--------------|-----------|-------|------------|----------------|----------------|---------------------------|----------|
| $R$       | Lognormal    | 0.09      | 0.271 | 3.131      | 1.06           | -              |                           | -        |
| $D$       | Normal       | 1         | 0.25  | 0.250      | 0              | 3              | 1.00                      | 1        |
| $L$       | Gamma        | 0.175     | 0.59  | 0.103      | 1.180          | 5.089          | 0.35                      | 0.5      |
| $S$       | Gumbel       | 0.6874    | 0.21  | 0.144      | 1.140          | 5.4            | 0.982                     | 0.7      |
According to Figure 1, four steps are conducted to obtain the resistance factor $\phi$ and partial load factor $\gamma_i$. Different formulas in Table 1 are used, and the results are listed in Table 3. As can be observed from Table 3, the biggest relative differences among $\beta_{T,\text{check}}$ and $\beta_{ST}$ obtained by different methods are relatively small. This indicates there is no big difference in the accuracy of these expressions of $\beta_{T,\text{check}}$ and $\beta_{ST}$ obtained by different formulas listed in Table 1. To make the proposed method more practical, Zhao 2001’s formula is applied in the following example.

**Table 3. Results of the proposed method.**

| Variables | Zhao 2000 | Zhao 2001 | Wang | Zhao 2007 |
|-----------|-----------|-----------|------|-----------|
| $\beta_{T,\text{check}}$ | 2.9202 | 2.9190 | 2.9292 | 2.9667 |
| $\beta_{ST}$ | 2.9810 | 2.9810 | 2.9827 | 3.0299 |
| $\mu_R$ | 3.0665 | 3.0659 | 3.0710 | 3.0898 |
| $\phi$ | 0.8697 | 0.8697 | 0.8694 | 0.8657 |
| $\gamma_D$ | 1.4519 | 1.4519 | 1.4518 | 1.4577 |
| $\gamma_L$ | 0.5038 | 0.5038 | 0.5038 | 0.5058 |
| $\gamma_S$ | 1.1967 | 1.1967 | 1.1967 | 1.1995 |

4.2. **Example 2**

The second example considers the following performance function:

$$G(X) = R - (D + L + S + W)$$

where $W$ is the wind load. The statistical information of the random variables is listed in Table 4.

**Table 4. Information of the random variables in Eq. (18).**

| Variables | Distribution | $\mu_i/D_n$ | $V_i$ | $\sigma_i$ | $\alpha_{3i}$ | $\alpha_{4i}$ | $\mu_B/R_n$ or $\mu_S/S_{ni}$ |
|-----------|--------------|-------------|------|------------|--------------|-------------|-----------------|
| $R$       | Lognormal    | —           | 0.15 | 0.453      | 3.368        | 1.10        |                 |
| $D$       | Normal       | 0.10        | 0.1  | 0          | 3            | 1           |                 |
| $L$       | Lognormal    | 0.5         | 0.4  | 0.2        | 1.264        | 5.969       | 0.45            |
| $S$       | Gumbel       | 2.0         | 0.25 | 0.5        | 1.140        | 5.4         | 0.47            |
| $W$       | Gumbel       | 2.0         | 0.20 | 0.4        | 1.140        | 5.4         | 0.60            |
Since the distributions of the random variables are known, the FORM can be performed and the results are used herein as the benchmark. By using the moments of the random variables, the existing and proposed methods can be performed. The changes in the mean value of $R$ ($\mu_R$), the iteration numbers required and the resistance factor $\phi$ with the increase of $\beta_T$ by using different methods are depicted in Figures 2 (a-c), respectively, which reveal that:

(i) As can be seen from Figure 2 (a), $\mu_R$ obtained by the moment methods are slightly larger than that obtained by using FORM, this error may come from the approximation of $\beta_{2T}$ in the moment methods. When $\beta_{2T}$ is relative small, the results obtained by the proposed method and the existing moment methods are almost the same. When the $\beta_{2T}$ becomes larger, the difference between the proposed method and the existing method becomes larger, while the $\mu_R$ obtained by the proposed method becomes closer with that obtained by the FORM, which indicates the proposed method is suitable to conduct LRFD for $\beta_T$ changes from 1 to 3.

(ii) As can be seen from Figure 2 (b), the derivative-based iteration numbers of FORM for LRFD varied from 9 to 22 as $\beta_T$ changed from 1.0 to 3.0. The iteration numbers required in the existing method reduce significantly compared with those in FORM. However, one may have to conduct iteration steps to satisfy the tolerable error ($\varepsilon = 0.03$), except for some particular cases. There is no need of iteration in the proposed method.

(iii) As can be seen from Figures 2 (c), $\phi$ obtained by the existing methods and those by the proposed method coincide with each other, with the only exception of the resistance factor $\phi$ obtained by using Zhao 2007’s formula. However, the difference in the resistance factor $\phi$ obtained by using the proposed and Zhao 2007’s methods is not large. These again indicate the efficiency of the proposed method. The resistance factor $\phi$ obtained by the moment methods and FORM are different, this can be explained by that different combinations of load and resistances factors can result in the same design results. In design practice, if the resistance factor determined by a certain method is adopted, it is important that the corresponding load factors should be used.

![Figure 2. Comparison of the LRFD by using different method.](image)

5. Conclusion
A straightforward method to estimate the load and resistance factors for reliability-based structural design without iterations is proposed. The procedure, summarized in flowchart, shows that the proposed method avoids iterations, which are needed by the existing methods. Thus the proposed method is much easier to apply. Comparison studies among the proposed and the exiting methods are conducted. The results of the proposed method are similar with those of the FORM which indicates the accuracy of the proposed method.

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