We describe the realizations of finite-dimensional Lie algebras of smooth tangential vector fields on a circle and construct “canonical” realizations of the two-dimensional noncommutative algebra, as well as the algebra $\mathfrak{sl}(2, \mathbb{R})$. It is shown that any realization of these algebras by smooth vector fields can be reduced to one of “canonical” realizations with the help of piecewise-smooth global transformations of a circle onto itself. We also deduce the formulas for the number of nonequivalent realizations.

1. Introduction

The problem of description of the realizations of Lie algebras by vector fields is extensively used, e.g., for the construction of partial differential equations with the corresponding invariance algebra and for the determination of exact solutions. However, this problem remains insufficiently well studied from the systematic point of view. For the first time, realizations of the Lie algebra on a straight line and in a plane were considered by Lie [1, pp. 1–121]. However, only after about one hundred years, the investigations of this problem became regular. Various directions in the development of this problem were described, e.g., in [2] (including the realizations of differential operators of the first order of a special form) and in [3–5] devoted to the analysis of realizations of the physical Galilei, Poincaré, and Euclidean algebras. The systematic investigations of nonequivalent realizations of real Lie algebras whose dimension does not exceed four by vector fields in the space with an arbitrary number of variables were carried out in [6], where one can find a more complete survey of the problem and a list of references.

In the cited works, realizations of the Lie algebras of the vector fields were investigated to within equivalence transformations. The problem of classification of the realizations of algebras on a certain manifold to within global equivalence transformations (on the entire manifold) is more complicated and, unlike the local theory, requires other methods of investigation. Attempts to classify the realizations of the Lie algebra of vector fields on a certain manifold (“as a whole”) were made only in few works (see [7–9]). In these works, it was proved that there exist three types of algebras, namely, one-dimensional algebras, noncommutative two-dimensional algebras, and the three-dimensional algebra $\mathfrak{sl}(2, \mathbb{R})$ that can be realized by using analytic vector fields on a circle.

The aim of the present paper is to construct all nonequivalent realizations of finite-dimensional algebras with nonzero Levi factor on a circle in the explicit form and to describe the realizations of known soluble algebras (of dimension not greater than five) in the class of vector fields without using the condition of analyticity, as was done in [7–9]. Some definitions and preliminary results were obtained in [10], where one can also find the classification of algebras whose dimension does not exceed two.

On a circle $S^1$, we introduce a parameter $\theta \in \mathbb{R}$, $0 \leq \theta < 2\pi$. Assume that the corresponding point on the circle moves clockwise as the parameter $\theta$ increases. The vector field on $S^1$ can be represented as a vector field $v(\theta)\partial_\theta$, where $v(\theta)$ is a smooth real function on a circle [11]. We can also assume that $\theta \in \mathbb{R}$ and $v(\theta)$ is a smooth $2\pi$-periodic function on the real line.

S. V. Spichak

UDC 517.986.5

Institute of Mathematics, National Academy of Sciences of Ukraine, Kyiv, Ukraine; e-mail: stas.math@gmail.com.

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In what follows, a point \( \theta_0 \) is called a *singular point* of the vector field \( v(\theta)\partial_\theta \) if \( v(\theta_0) = 0 \). By \( C \) we denote a class of vector fields \( v(\theta)\partial_\theta \) such that:

- the intervals where \( v(\theta) = 0 \), i.e., the intervals formed by singular points, do not exist;
- the functions \( v(\theta) \) are continuously differentiable, which is a natural requirement if we consider the commutators of two vector fields.

We also introduce a class of transformations \( f : S^1 \to S^1 \) of the circle onto itself with the following properties:

- these transformations are one-to-one mappings of the circle onto itself;
- \( f(\theta) \) is continuous at any point \( \theta \in S^1 \);
- these transformations are continuously differentiable at all points (except finitely many points);
- the derivative \( f'(\theta) \) tends either to \(-\infty\) or to \(+\infty\) at all discontinuity points;
- as the coordinate changes \( \bar{\theta} = f(\theta) \), the vector field from the class \( C^1 \) turns into a vector field from the same class.

A class of transformations with the indicated properties specifies *equivalence transformations* of the vector fields. We denote this class by \( \mathcal{F} \). Two realizations of an algebra of vector fields are called *nonequivalent* if it is impossible to transform one of these realizations into the other by a composition of equivalence transformations from the class \( \mathcal{F} \). Thus, we are interested in all realizations of the finite-dimensional Lie algebras of vector fields from the class \( C \) nonequivalent under the transformations \( \mathcal{F} \).

The degree of mapping corresponding to the function \( f \) is \( \deg f = \pm 1 \) (see [12]). We define the following subclasses of transformations from \( \mathcal{F} \):

- \( \mathcal{F}_0^+ \) are transformations with fixed point zero \( f(0) = 0 \) and \( \deg f = 1 \);
- \( O_\varphi \) is a clockwise rotation of the circle by an angle \( \varphi \);
- \( T \) is the reflection of the circle with respect to the axis passing through the center of the circle and point 0 on this circle.

It is easy to see that any transformation from the class \( \mathcal{F} \) is a composition of transformations from the indicated three classes. It is clear that if a transformation \( f(\theta) \in \mathcal{F}_0^+ \), then it is monotone in the entire open interval \( 0 \leq \theta < 2\pi \). If \( f(\theta) \in T \), then \( \deg f = -1 \). At the same time, if \( f(\theta) \in O_\varphi \), then \( \deg f = 1 \).

In Sec. 2, we consider two-dimensional algebras and prove auxiliary lemmas necessary for the classification of realizations of a noncommutative algebra. In Secs. 3 and 4, we perform the classification of realizations nonequivalent with respect to transformations from the class \( \mathcal{F}_0^+ \). In Sec. 5, we complete the classification of realizations of the two-dimensional noncommutative algebra and the algebra \( sl(2, \mathbb{R}) \) relative to the transformations \( O_\varphi \) and \( T \) and also present combinatorial relations for finding the number of the corresponding nonequivalent realizations.

2. Two-Dimensional Algebras: Auxiliary Lemmas

(A) Commutative Algebra \( A_{2,1} = A_1 \oplus A_1 \) (here and in what follows, we use the notation of algebras from [13]).

We denote vector fields \( v(\theta)\partial_\theta \) and \( w(\theta)\partial_\theta \) by \( V \in \mathcal{C} \) and \( W \in \mathcal{C} \), respectively. Assume that they are commuting. A singular point \( \theta_0 \) of the vector field \( V \) is called *degenerate* if \( v'(\theta_0) = 0 \). It is easy to see that
if \(0 \leq \theta_0 < \theta_1 < 2\pi\) are two degenerate points such that the interval \((\theta_0, \theta_1)\) does not contain degenerate points, then \(w(\theta) = \lambda v(\theta)\) on \((\theta_0, \theta_1)\), where \(\lambda \neq 0\) is an arbitrary constant. This follows from the fact that \(W \in C\) (the continuity of the derivatives of the functions \(v(\theta)\) and \(w(\theta)\)). In particular, if degenerate points are absolutely absent or there exists only one degenerate point, then \(w(\theta) = \lambda v(\theta)\) on \(S^1\). In this case, the vector fields \(V\) and \(W\) are linearly dependent and, hence, there are no realizations of the two-dimensional commutative Lie algebra (see [7–9]).

Suppose that the function \(v(\theta)\) has more than one degenerate point. Without loss of generality, we can assume that the point 0 (and, hence, \(2\pi\)) is degenerate. Then the function \(w(\theta)\) can be described as follows:

We take an arbitrary point \(\theta \in S^1\). If it is nondegenerate for the function \(v(\theta)\), then it is clear that one can find the maximal segment \([\theta_0, \theta_1]\) with two degenerate endpoints such that

\[0 \leq \theta_0 < \theta < \theta_1 < 2\pi.\]

Thus, on this segment, we have \(w(\theta) = \lambda v(\theta)\).

Further, consider the point \(\theta' \notin [\theta_0, \theta_1]\). If it is not degenerate, then we repeat the procedure. We again arrive at the relation \(w(\theta) = \lambda' v(\theta)\) on a certain segment (not on \([\theta_0, \theta_1]\)). Moreover, the coefficients \(\lambda\) and \(\lambda'\) can be different. However, if the point \(\theta'\) is degenerate, then, by virtue of the fact that \(V \in C\), there exists a nondegenerate point \(\theta'' \notin [\theta_0, \theta_1]\) arbitrarily close to this point. Thus, we repeat the above-mentioned procedure for the point \(\theta''\). As a result, the entire segment \([0, 2\pi]\) is split into (possibly, infinitely many) intervals with degenerate endpoints. On these intervals, the function \(w(\theta)\) is proportional to the function \(v(\theta)\) with nonzero different coefficients of proportionality.

(B) Noncommutative Algebra \(A_{2.2}\). Suppose that the vector fields \(V\) and \(W\) generate a noncommutative algebra. It is possible to assume that (to within their linear combination) they satisfy the commutation relation \([V, W] = W\), which is equivalent to the following condition for the functions \(v(\theta)\) and \(w(\theta)\):

\[v(\theta)w'(\theta) - v'(\theta)w(\theta) = w(\theta).\]  (1)

**Lemma 1.** There exists a singular point for the vector field \(W\).

**Proof.** Assume that the vector field \(W\) does not have singular points, i.e., \(w(\theta) > 0\) (or \(w(\theta) < 0\)) for all values \(0 \leq \theta < 2\pi\). By using (1), we get the following solution for the function \(v(\theta)\) in the entire open interval \([0, 2\pi]\):

\[v(\theta) = \left(-\int_0^\theta \frac{d\theta}{w(\theta)} + \lambda\right) w(\theta),\]  (2)

where \(\lambda\) is a constant. The function \(w(\theta)\) is \(2\pi\)-periodic on the real line. Since the integrand in (2) is positive, we conclude that \(v(0) \neq v(2\pi)\), which contradicts the periodicity of the function \(v(\theta)\).

**Lemma 2.** Singular points of the vector field \(W\) are also singular points of the vector field \(V\).

**Proof.** Assume that \(w(\theta_0) = 0\). Let \(v(\theta_0) \neq 0\). Then there exists a neighborhood \(U_{\theta_0}\) of this point in which \(v(\theta) \neq 0\). In this neighborhood, Eq. (1) can be rewritten in the form

\[w'(\theta) = \frac{1 + v'(\theta)}{v(\theta)} w(\theta).\]  (3)
Since $w(\theta_0) = 0$ and the right-hand side of Eq. (3) satisfies the Lipschitz conditions for the function $w$ uniformly in $\theta$, by virtue of the Picard theorem [14], the differential equation (3) possesses a unique solution in the neighborhood $U_{\theta_0}$. It is clear that this solution is $w \equiv 0$, which contradicts $W \in C$.

**Lemma 3.** The number of singular points of the vector field $W$ is finite.

**Proof.** Assume that the number of singular points is infinite. Since $S^1$ is a compact manifold, there exists a monotonically increasing (decreasing) sequence $\{\theta_n\}$ convergent to a point $\theta_0$ and such that $w(\theta_n) = 0$. It is easy to see that, for any $n$, there exists a regular (nonsingular) point $\tilde{\theta}_n \in (\theta_n, \theta_{n+1})$ satisfying the condition $w'(\tilde{\theta}_n) = 0$. Hence, it follows from Eq. (1) that \(v'(\tilde{\theta}_n) = -1\). Since $\lim_{n \to \infty} \tilde{\theta}_n = \theta_0$, in view of the continuous differentiability of the function $v(\theta)$, we conclude that

$$\lim_{n \to \infty} v'(\tilde{\theta}_n) = v'(\theta_0) = -1.$$ 

On the other hand, by virtue of Lemma 2, we conclude that $v(\theta_n) = v(\theta_{n+1}) = 0$. This implies that there exists a point $\tilde{\theta}_n \in (\theta_n, \theta_{n+1})$ such that $v'(\tilde{\theta}_n) = 0$. Since $\lim_{n \to \infty} \tilde{\theta}_n = \theta_0$, we find

$$\lim_{n \to \infty} v'(\tilde{\theta}_n) = v'(\theta_0) = 0.$$ 

We arrive at a contradiction.

**Lemma 4.** If $\theta_0$ is a singular point of the vector field $W$, then it is degenerate for this field (i.e., $w'(\theta_0) = 0$).

**Proof.** Since $v'(\theta_0) = 0$ (see Lemma 2), we get

$$v(\theta) = v'(\theta_0)(\theta - \theta_0) + h(\theta) \quad \text{and} \quad w(\theta) = w'(\theta_0)(\theta - \theta_0) + g(\theta),$$

where $h$ and $g$ are continuously differentiable functions and

$$h(\theta_0) = g(\theta_0) = h'(\theta_0) = g'(\theta_0) = 0. \quad (4)$$

In addition, by using Eq. (1), we find

$$[v(\theta) \partial_{\theta}, w(\theta) \partial_{\theta}] = [(\theta - \theta_0)(v'(\theta_0)g'(\theta) - w'(\theta_0)h'(\theta))$$

$$+ h(\theta)(w'(\theta_0) + g'(\theta)) - g(\theta)(v'(\theta_0) + h'(\theta))] \partial_{\theta}$$

$$= [w'(\theta_0)(\theta - \theta_0) + g(\theta)] \partial_{\theta}.$$

We divide both sides of this equation by $\theta - \theta_0$ and pass to the limit as $\theta \to \theta_0$. By using relation (4) and the L’Hospital theorem, we conclude that $w'(\theta_0) = 0$.

**Lemma 5.** Under the equivalence transformation $\tilde{\theta} = f(\theta)$ from the class $\mathcal{F}$, a singular (resp., regular) point of the vector field $W$ is mapped into a singular (resp., regular) point of the vector field $\tilde{W} = \tilde{w}(\tilde{\theta}) \partial_{\tilde{\theta}}$.

**Proof.** (a) Let $w(\theta_0) \neq 0$. Then $f'(\theta)$ is continuous at the point $\theta_0$. Assume that this is not true. Then there exists a neighborhood of this point in which $f'(\theta)$ is continuous (except $\theta_0$) and $\lim_{\theta \to \theta_0} f'(\theta) = \pm \infty$
(the sign depends on \( \deg f \)). For the transformed vector field \( \tilde{W} \), we have the relation \( \tilde{w}(f(\theta)) = w(\theta)f'(\theta) \). Since \( w(\theta) \neq 0 \) in the above-mentioned neighborhood, we conclude that

\[
f'(\theta) = \frac{\tilde{w}(f(\theta))}{w(\theta)},
\]

and the continuity of the functions \( f \) and \( \tilde{w} \) implies that \( \lim_{\theta \to \theta_0} f'(\theta) \) is finite, i.e., we arrive at a contradiction. Recall that the continuity of the function \( \tilde{w} \) follows from the property that any equivalence transformation \( f \in F \) maps a vector field from the class \( C^1 \) into a vector field from the same class.

We now assume that \( \tilde{w}(f(\theta_0)) \neq 0 \) (i.e., \( f(\theta_0) \) is singular). Since the right-hand side of the differential equation (5) satisfies the Lipschitz conditions for the argument \( f(\theta) \) uniformly in \( \theta \) the function \( \tilde{w} \) is continuously differentiable), the unique solution \( f(\theta) \) exists in a neighborhood of the point \( \theta_0 \). A constant function \( f(\theta) = f(\theta_0) = \text{const} \) satisfies Eq. (5). Hence, it is a solution in the indicated neighborhood. However, in this case, the mapping \( f(\theta) \) is not bijective, which contradicts the property of equivalence transformation. Therefore, \( \bar{\theta}_0 = f(\theta_0) \) is a regular point.

(b) Let \( \theta_0 \) be a regular point. Assume that \( \tilde{w}(f(\theta_0)) \neq 0 \) (i.e., \( f(\theta_0) \) is regular). By Lemma 3, there exists a neighborhood of the point \( \theta_0 \) in which all points are regular (except \( \theta_0 \)). From the previous part of the proof, we get [see (5)] the following relation for the regular points of the neighborhood: \( f'(\theta) > 0 \) (or \( f'(\theta) < 0 \)) if \( \deg f > 0 \) (or \( \deg f < 0 \)), and \( \lim_{\theta \to \theta_0} f'(\theta) = \pm \infty \). Thus, we can easily show that the inverse transformation \( \theta = f^{-1}(\tilde{\theta}) \) is continuously differentiable in the entire neighborhood of the point \( \bar{\theta}_0 = f(\theta_0) \) \( (f^{-1}(\bar{\theta}_0) = \theta_0) \). Therefore, by using the arguments from the previous part of the proof [item (a)], we conclude that the regular point \( \bar{\theta}_0 \) is mapped into the singular point \( \theta_0 \) under the mapping \( f^{-1} \). Thus, we arrive at a contradiction.

**Corollary 1.** The number of singular points of the vector field is invariant under any equivalence transformation.

3. Realizations of Soluble Algebras

The realizations of the algebra \( A_{2,1} = V \oplus W \) have been described in the previous section. Namely, if \( V = \langle e_1 \rangle = v(\theta)\partial_0 \) and \( W = \langle e_2 \rangle = w(\theta)\partial_0 \), then the realization of the algebra can be described as follows: \( w(\theta) = \lambda v(\theta) \) on the segments \( [\theta', \theta''] \), where \( \lambda \) are constants, \( \theta' \) and \( \theta'' \) are degenerate points, and all points of the interval are nondegenerate. In what follows, we also denote the relationship between the realizations of two one-dimensional commutative algebras by \( \langle e_1 \rangle \sim \langle e_2 \rangle \).

(A) **Realizations of the Algebra \( A_{2,2} \).** By virtue of Lemmas 1 and 3, we can assume that there exists a vector field \( W \) with \( n \geq 1 \) singular points \( \theta_k \). According to Lemma 4, they are also degenerate. It is easy to see that, in view of the equivalence transformations and Lemma 5, we can set \( \theta_k = \frac{2\pi k}{n}, \ k = 0, 1, \ldots, n - 1 \). We now consider the interval \( \triangle_k = (\theta_k, \theta_{k+1}) \) and denote

\[
\bar{\theta}_k = \frac{\theta_k + \theta_{k+1}}{2} = \frac{\pi(2k + 1)}{n}.
\]

Further, we construct the following continuously differentiable transformation \( f \) on \( \triangle_k \) satisfying the following conditions:

\[
f(\theta_k) = \theta_k, \quad f(\theta_{k+1}) = \theta_{k+1}, \quad f(\bar{\theta}_k) = \bar{\theta}_k.
\]
Assume that \( w(\theta) > 0, \quad \theta \in \triangle_k \). Consider the Cauchy problem for this interval:

\[
  w(\theta) f'(\theta) = 1 - \cos(n f(\theta)), \quad f(\tilde{\theta}_k) = \tilde{\theta}_k. \tag{7}
\]

It has the following solution:

\[
f(\theta) = \frac{2}{n} \arctan \left( -nI(\theta) \right) + \tilde{\theta}_k, \quad \text{where} \quad I(\theta) = \int_{\theta_k}^{\theta} \frac{d\theta}{w(\theta)}, \quad \theta \in \triangle_k. \tag{8}
\]

It is clear that the integral \( I(\theta) \) converges for any point from the interval \( \triangle_k \). By virtue of Lemma 4, the integral does not converge at the ends of this interval:

\[
  \lim_{\theta \to \theta_k + 0} I(\theta) = -\infty \quad \text{and} \quad \lim_{\theta \to \theta_k + 1 - 0} I(\theta) = +\infty.
\]

By using these relations, we can easily show that transformation (8) satisfies conditions (6) and transforms the vector field \( w(\theta) \partial_\theta \) into the vector field \( (1 - \cos(n\tilde{\theta})) \partial_\theta \).

Similarly, if \( w(\theta) < 0 \) for \( \theta \in \triangle_k \), then we can find the equivalence transformation mapping the vector field \( w(\theta) \partial_\theta \) onto the vector field \( (\cos(n\tilde{\theta}) - 1) \partial_\theta \).

Thus, in each interval \( \triangle_k \), we get the vector field

\[
  W = \pm \left( 1 - \cos(n\theta) \right) \partial_\theta
\]

(the "tilde" sign is omitted). Substituting the function \( w(\theta) = \pm \left( \cos(n\theta) - 1 \right) \) in Eq. (1), we can easily get the solution for the function \( v(\theta) \) in the indicated interval \( \triangle_k \):

\[
v(\theta) = \frac{1}{n} \sin(n\theta) + \lambda_k \left( 1 - \cos(n\tilde{\theta}) \right), \quad \lambda_k \in \mathbb{R}. \tag{9}
\]

Further, in (9), we can set the constants \( \lambda_k \) equal to zero. To this end, we consider the transformation \( f \) of a circle onto itself of the following form:

\[
f(\theta) = \frac{2}{n} \arctan \left( \cot \frac{n\theta}{2} + \lambda_k n \right) + \frac{2\pi k}{n} \quad \text{for} \quad \theta \in \triangle_k.
\]

It is easy to see that the indicated transformation \( \tilde{\theta} = f(\theta) \) belongs to the above-mentioned class \( \mathcal{F}_0^+ \), has fixed points \( \theta_k \), does not change the form of the vector field: \( \overline{W} = \pm \left( 1 - \cos(n\tilde{\theta}) \right) \partial_\theta \), and changes the form of the vector field so that \( \overline{V} = \overline{v}(\overline{\theta}) \partial_\overline{\theta} \), where the function \( \overline{v}(\overline{\theta}) \) has the form (9) and all \( \lambda_k = 0 \).

As a result, we arrive at the following statement:

**Theorem 1.** Any realization of the two-dimensional commutative algebra \( A_{2,2} \) of vector fields on a circle is equivalent to

\[
  A_{2,2}^n = \left\langle \sigma^n(\theta)(1 - \cos(n\theta)) \partial_\theta, \frac{1}{n} \sin(n\theta) \partial_\theta \right\rangle, \tag{10}
\]

where \( n \in \mathbb{N} \), and the function \( \sigma^n(\theta) \) is equal either to 1 or to \(-1\) on the segments \( \left[ \frac{2\pi k}{n}, \frac{2\pi (k + 1)}{n} \right] \), \( k = 0, \ldots, n - 1 \).
(B) Realizations of the Algebras $A_{k,i}$ Whose Dimensions Are Greater than Two. It is known that the problem of classification of nonisomorphic soluble Lie algebras is solved only for real Lie algebras up to the sixth order, inclusively (see, e.g., [15–18]). In our work, we present the realizations of soluble Lie algebras on a circle over the field $\mathbb{R}$ whose dimension does not exceed five. We use the notation introduced in [13], where one can find commutative relations both for the irreducible algebras $A_{2,i}$ of dimension $k$ ($i$ is the ordinal number) and for the algebras reducible into direct sums. Assume that the set of basis elements of the algebra contains elements $e_l$, $e_m$, and $e_n$ such that

$$[e_l, e_m] \neq 0 \quad \text{and} \quad [e_l, e_n] = [e_m, e_n] = 0.$$ 

It is clear that the realizations of these algebras are absent on a circle. Indeed, it follows from the last conditions that $e_l \sim e_n$ and $e_m \sim e_n$. Therefore, $e_l \sim e_m$ and $[e_l, e_m] = 0$, which contradicts the first condition. This statement excludes the possibility of realization of numerous soluble algebras on a circle. In particular, this is true for all reducible algebras.

We can easily show that this is also true for almost all irreducible algebras. Thus, assume that there exist a realization of the algebra $A_{3,9}$ on a circle with the following nonzero commutative relations:

$$[e_1, e_3] = qe_1 - e_2, \quad [e_2, e_3] = e_1 + qe_2, \quad q > 0.$$ 

It follows from these relations that $e_1 \sim e_2$, i.e., the conditions

$$e_2 = \lambda e_1, \quad [e_1, e_3] = (q - \lambda)e_1, \quad \text{and} \quad [\lambda e_1, e_3] = (1 + q\lambda)e_1$$

are locally satisfied on a certain interval, which implies that $\lambda(q - \lambda) = (1 + q\lambda)$ but this is impossible for $\lambda \in \mathbb{R}$. Similar analyses performed for the other soluble algebras yield the following assertion:

**Theorem 2.** There exist the following realizations of the soluble Lie algebras $A_{k,i} = \langle e_1, \ldots, e_k \rangle$ of vector polynomials on a circle ($k \leq 5$):

- $A_{3,1} = 3A_1$, where $e_1 \sim e_2 \sim e_3$;
- $A_{3,5}$, where $e_2 \sim e_1$, $\langle e_1, e_3 \rangle$ form realizations $A_{2,2}^n$ of the algebra $A_{2,2}$ in Theorem 1;
- $4A_1$, where $e_1 \sim e_2 \sim e_3 \sim e_4$;
- $A_{4,5}$ ($q = p = 1$), where $e_2 \sim e_1$, $e_3 \sim e_1$, $\langle e_1, e_4 \rangle = A_{2,2}^n$;
- $5A_1$, where $e_1 \sim e_2 \sim e_3 \sim e_4 \sim e_5$;
- $A_{5,7}$ ($q = p = r = 1$), where $e_2 \sim e_1$, $e_3 \sim e_1$, $e_4 \sim e_1$, $\langle e_1, e_5 \rangle = A_{2,2}^n$.

4. **Algebras with Nonzero Levi Factor**

It is know that any Lie algebra can be represented in the form of a semidirect sum of soluble and semisimple subalgebras (see, e.g., [19, pp. 170–172]). If this semisimple subalgebra (Levi factor) is nonzero, then the algebra contains a simple subalgebra. We now prove the following theorem:

**Theorem 3.** If the dimension of a simple algebra of vector fields is greater than three, then the realizations of this algebra on a circle do not exist.
Proof. Assume that there exist a realization of the algebra
\[ L = \langle e_1, e_2, \ldots, e_s \rangle = \langle \varphi_1 \partial_\theta, \varphi_2 \partial_\theta, \ldots, \varphi_s \partial_\theta \rangle, \]
where \( s > 3 \), \( \varphi_i(\theta) \partial_\theta \in \mathcal{C} \). It is clear that one can find an interval \( I \) where \( \varphi_1(\theta) \neq 0 \) and an equivalence transformation \( \mathcal{F} \) such that, in the realization of the algebra we can assume that \( \varphi_1(\theta) = 1 \) on the indicated interval and that 0 also belongs to \( I \). By the definition of Lie algebra, we get the relation \( [e_1, e_i] = C^k_i \varphi_k \) (where \( C^k_i = C^k_{i1} \) are structural constants of the algebra). This implies that \( \varphi'_i = C^k_i \varphi_k, \ i = 2, \ldots, s \) (here and in what follows, the operation of summation is carried out over the repeated indices). Hence, the functions \( \varphi_i \) are infinitely differentiable. We also can assume that \( \varphi_i(0) = 0, \ i = 2, \ldots, s \) (by changing the basis of the algebra).

We say that \( l \in \mathbb{N} \) is the order of the function \( \varphi(\theta) \) if \( \varphi^{(k)}(0) = 0 \) for
\[
0 \leq k a \varphi^{(l)}(0) \neq 0.
\]

If this \( l \) does not exist, then we assume that the function \( \varphi(\theta) \) has the infinite order. Further, if all functions \( \varphi_i(\theta), \ i = 2, \ldots, s, \) have the infinite order, then we can easily show that \( \langle e_2, \ldots, e_s \rangle \) is an ideal of the algebra \( L \), which contradicts the assumption that the algebra is simple. Then there exists \( l \in \mathbb{N} \), which is the minimal order among these functions. Without loss of generality, we can assume that \( l \) is the order of the function \( \varphi_2(\theta) \) and the orders of the other functions are greater than \( l \) (by changing the basis). Let \( l > 1 \). Since the order of the function \( \varphi'_2(\theta) \) is equal to \( l - 1 > 0 \), it is clear that the commutator
\[
[e_1, e_2] = \varphi'_2(\theta) \partial_\theta
\]
cannot be a linear combination of basis operators of the algebra. Hence, \( l = 1 \).

Similarly, if we consider the basis elements \( \langle e_3, \ldots, e_s \rangle \), then we can prove that the order of the function \( \varphi_3(\theta) \) is equal to 2, while the orders of the other functions \( \varphi_i(\theta) \ (i > 3) \) are greater than 2 and finite. Moreover, we can change the basis to guarantee that the corresponding orders \( m_i, \ i > 3, \) can be ordered: 2, where the function \( \chi(\theta) \) has the order \( m_3 + 1 \); thus, taking into account the last inequalities, we can easily show that this commutator cannot be a linear combination of basis operators of the algebra. Therefore, \( s \) can be equal only to 3.

It is known that there exist only two simple algebras whose dimensions do not exceed 3: \( \mathfrak{so}(3) \) and \( \mathfrak{sl}(2, \mathbb{R}) \). The realizations of the algebra \( \mathfrak{so}(3) \) of vector fields on the straight line (and, hence, also on the circle) do not exist (see, e.g., [1, 8]). Consider the algebra \( \mathfrak{sl}(2, \mathbb{R}) \). We can easily prove the following theorem:

**Theorem 4.** Any realization of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \) of vector fields on the circle is equivalent to
\[
\left( \sigma^n(\theta)(1 - \cos(n\theta)) \partial_\theta, \frac{1}{n} \sin(n\theta) \partial_\theta, \frac{\sigma^n(\theta)}{n^2}(1 + \cos(n\theta)) \partial_\theta \right), \tag{11}
\]
where \( n \in \mathbb{N} \) and \( \sigma^n(\theta) = \pm 1 \) on the segments \( \left[ \frac{2\pi k}{n}, \frac{2\pi(k + 1)}{n} \right], \ k = 0, \ldots, n - 1 \).

Proof. Assume that the algebra \( \mathfrak{sl}(2, \mathbb{R}) = \langle e_1, e_2, e_3 \rangle \). As a subalgebra, it contains the algebra \( A_{2,2} = \langle e_1, e_2 \rangle \) all nonequivalent realizations \( A^i_{2,2} \) of which are described in Theorem 1. Hence, we can assume that, for the nonequivalent realizations of the algebra \( \mathfrak{sl}(2, \mathbb{R}) \), the basis vector fields \( e_1 \) and \( e_2 \) have the form (10). By using the commutation relations between all operators \( e_1, e_2, \) and \( e_3 \), we can easily show that the operator \( e_3 \) has the form \( \frac{\sigma^n(\theta)}{n^2}(1 + \cos(n\theta)) \partial_\theta \).
Further, if the Levi factor is a semisimple algebra, i.e., the direct sum of simple algebras, then the realizations of these algebras do not exist, as shown in the previous section prior to Theorem 2.

Finally, if we consider the semidirect sums of the algebra $\mathfrak{sl}(2, \mathbb{R})$ (as the Levi factor) and known (at present) soluble algebras, then we can show that the corresponding algebras do not exist in the one-dimensional case and, in particular, on the circle. To this end, it suffices to consider all cases of these algebras in which a soluble subalgebra (radical) coincides with one of the algebras specified in Theorem 2. In view of the commutation relations between the operators of the algebra $\mathfrak{sl}(2, \mathbb{R})$ and the corresponding operators of soluble algebras (radicals), we arrive at a contradiction.

Thus, Theorems 1, 2, and 4 give the full list of all possible realizations of the noncommutative Lie algebras of vector fields on the circle.

5. Some Combinatorial Relations

In Theorem 1, we determine all nonequivalent realizations (10) of the two-dimensional noncommutative algebra $A_{2,2}$ on the circle with respect to the transformations $F^+_0$. However, the set of these realizations contain realizations that are equivalent under the transformations of the algebra $O_\varphi$ (rotations) and $T$ (inversions) (see the definitions of these transformations in the introduction). If we describe the number of realizations of this algebra that are nonequivalent under all three indicated transformations, i.e., the number of collections of the corresponding functions $\sigma^n(\varphi)$, then, by virtue of Theorems 1 and 4, we get the same number of nonequivalent realizations for the algebra $\mathfrak{sl}(2, \mathbb{R})$.

(A) Nonequivalent Realizations on the Circle with Respect to Rotations. We first describe realizations equivalent under the transformations $O_{\varphi}$ and find their number. It is clear that if a certain couple of realizations is equivalent under rotations of the circle by an angle $\varphi$, then the numbers $n$ of singular points for the corresponding fields $W$ and $\overline{W}$ coincide and, moreover, $O_{\varphi} = O_{\varphi k} \equiv O_k$, where $\varphi_k = \frac{2\pi k}{n}$, $k = 0, 1, \ldots, n - 1$. In realization (10), we define a signature of length $n$ as the vector $\sigma^n = (\sigma_0, \sigma_1, \ldots, \sigma_{n-1})$, where $\sigma_k = \pm 1$ and coincide with the corresponding values of the function $\sigma^n(\varphi)$ on the segments $\left[\frac{2\pi k}{n}, \frac{2\pi (k + 1)}{n}\right]$. Then, with the help of rotations $O_k$, this realization transforms into realization (10) with the signature $\tilde{\sigma} = O_k(\sigma) = (\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{n-1})$, where

$$\tilde{\sigma}_i = \sigma_{n-k+i(\text{mod} n)}, \quad i = 0, 1, \ldots, n - 1.$$  

These signatures $\sigma$ and $\tilde{\sigma}$ are called equivalent under the rotations $O_k$. Thus, the entire set of signatures of the same length [i.e., realizations (10)] decomposes into the equivalence classes with respect to rotations. It is clear that the realizations with signatures of different lengths are nonequivalent.

Let $d \in \mathbb{N}$, $1 \leq d \leq n$, be the minimum number such that $O_d(\sigma) = \sigma$ (this number exists because $O_n(\sigma) \equiv \sigma$).

This signature is called $d$-periodic. Then we can easily show that:

- $d$ is the divisor of $n$: $d|n$;
- if $d \neq d'$, then the $d$- and $d'$-periodic signatures are nonequivalent.

We introduce the notation:

- $L^n$ is the number of equivalence classes of signatures of length $n$;
- $M^n_d$ is the number of equivalence classes of $d$-periodic signatures of length $n$;
- $N^n_d$ is the number of different (possibly, equivalent) $d$-periodic signatures of length $n$. 

Since under rotations, the \(d\)-periodic signature \(\sigma\) transforms into the \(d\)-periodic signature \(\tilde{\sigma}\) and the signatures \(\sigma\) and \(O_k(\sigma)\) are different, for \(k < d\), then each class \(M_d^n\) contains \(d\) different signatures and, hence, \(N_d^n = dM_d^n\). Further, if \(d|n\), then it is clear that \(N_d^n = N_d^d\) and we get

\[
L^n = \sum_{d|n} M_d^n = \sum_{d|n} \frac{N_d^n}{d} = \sum_{d|n} \frac{P_d}{d}, \quad \text{where} \quad P_d = N_d^d
\]

(12)

(here and in what follows, the sum is taken over all divisors \(d\) of the number \(n\)). Since the total number of all different signatures is equal to \(2^n\), we get

\[
\sum_{d|n} N_d^n = \sum_{d|n} P_d = 2^n.
\]

(13)

It is clear that \(P_1 = 2\) (the signature \(\sigma_0\) is equal either to 1 or to –1). Thus, we get the recurrence relation (13) for the evaluation of \(P_d\). Further, by using relation (12), we can find the number \(L^n\) of equivalence classes of signatures.

Thus, if \(n = p\) is a prime number, then \(P_1 + P_p = 2^p\), i.e., \(P_p = 2^p - 2\).

If \(n = p_1p_2\), \(p_i\) are prime numbers, then \(P_1 + P_{p_1} + P_{p_2} + P_{p_1p_2} = 2^n\), i.e., \(P_n = 2^n - 2^{p_1} - 2^{p_2} + 2\).

Moreover,

\[
L_n = \frac{P_1}{1} + \frac{P_{p_1}}{p_1} + \frac{P_{p_2}}{p_2} + \frac{P_{p_1p_2}}{p_1p_2} = \frac{2^{p_1} - 2}{p_1} + \frac{2^{p_2} - 2}{p_2} + \frac{2^n - 2^{p_1} - 2^{p_2} + 2}{p_1p_2}.
\]

We now deduce the formula for the quantities \(L^n\) and \(P_d\). To this end, we introduce the required notation. Assume that \(n\) can be factorized into prime numbers as follows: \(n = p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_s^{\alpha_s}\), \(\alpha_i > 0\). Further, we assume that \(n\) and, hence, the numbers \(s\), \(p_i\), and \(\alpha_i\) are fixed. Moreover, we introduce the following three vectors: \(\tilde{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_s)\), \(\tilde{\beta} = (\beta_1, \beta_2, \ldots, \beta_s)\), where \(0 \leq \beta_i \leq \alpha_i\) are arbitrary numbers, and \(\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_s)\), where \(\varepsilon_i = \{0; 1\}\).

It is possible to prove the following theorem:

**Theorem 5.** Suppose that \(d = p_1^{\beta_1}p_2^{\beta_2} \cdots p_s^{\beta_s}\) is a divisor of the number \(n\), i.e., \(\beta_i \leq \alpha_i\) (and denote \(\tilde{\beta} \leq \tilde{\alpha}\)). Also let \(\{\tilde{\gamma}\} = p_1^{\gamma_1}p_2^{\gamma_2} \cdots p_s^{\gamma_s}\) and \(|\tilde{\varepsilon}| = \varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_s\). Then the following relation is true:

\[
P_d = P_{\{\tilde{\beta}\}} = \sum_{\tilde{\varepsilon} \leq \tilde{\beta}} (-1)^{\tilde{\varepsilon}} 2^{(|\tilde{\varepsilon}| - \tilde{\beta})}.
\]

(14)

In view of relation (12), the number of equivalence classes of signatures of length \(n\) is

\[
L_n = L_{\{\tilde{\alpha}\}} = \sum_{\tilde{\beta} \leq \tilde{\alpha}} \frac{1}{\{\tilde{\beta}\}} \sum_{\tilde{\varepsilon} \leq \tilde{\beta}} (-1)^{\tilde{\varepsilon}} 2^{(|\tilde{\varepsilon}| - \tilde{\beta})}.
\]

(15)

**Proof.** We prove relation (14) by induction on \(k = |\tilde{\beta}| = \beta_1 + \beta_2 + \ldots + \beta_s\) (\(s\) and \(p_i\) are fixed). If \(k = 0\), then \(\{\tilde{\beta}\} = \{0\} = 1\) and \(P_1 = 2\) (see above). On the other hand,

\[
P_1 = P_{\{\tilde{\beta}\}} = (-1)^0 2^{(0)} = 2,
\]
i.e., the required relation is true. Assume that it is true for all \( l = |\beta| \leq k. \) Let \( |\beta| = k + 1. \) By using relation (13), we get

\[
P_{\{\beta\}} = 2^{\langle \beta \rangle} - \sum_{\tilde{\delta} < \gamma \leq \beta} P_{\{\beta - \gamma\}} \quad (\text{summation over } \gamma).
\]

Since \( |\beta - \gamma| \) (14) and the right-hand side of relation (16) under the sign of summation is formed by the powers of two \( 2^{\langle \beta - \gamma \rangle}, \) where \( |\tilde{\delta}| > 0. \)

We find the coefficient of \( 2^{\langle \beta - \gamma \rangle} \) on the right-hand side of (16). Let \( |\varepsilon| = l > 0. \) Then it follows from relation (14) that this power of two is contained in the composition of the terms \( P_{\{\beta - \gamma + \delta\}} \) for which \( 0 \leq |\delta| < |\varepsilon| \) and the coefficient of this power in the corresponding term is equal to \( (-1)^r, \) where \( 0 \leq |\delta| = r. \) By analyzing all possible vectors \( \tilde{\delta}, \) we determine the following general coefficient from relation (16):

\[
- \sum_{0 \leq r < l} (-1)^r C_l^r = - \sum_{0 \leq r < l} (-1)^r C_l^r + (-1)^l = -(1 - 1)^l + (-1)^l = (1)^{|\varepsilon|}.
\]

We compute the coefficient of \( 2^{\langle \beta - \gamma \rangle}, \) where some component for the vector \( \gamma \) is such that \( \gamma_i > 1 \) and there are \( l \) nonzero components of this vector. Then this power of two is contained in the composition of the terms \( P_{\{\beta - \gamma + \varepsilon\}} \) for which \( 0 \leq |\varepsilon| < |\gamma| \) and the coefficient of this power in this term is equal to \( (-1)^r, \) where \( 0 \leq |\varepsilon| = r \leq l. \) Sorting out all possible vectors \( \varepsilon, \) we determine the following general coefficient from relation (16):

\[
- \sum_{0 \leq r \leq l} (-1)^r C_l^r = -(1 - 1)^l = 0.
\]

(B) Nonequivalent Realizations on the Circle with Respect to Inversions. We now consider the inversions \( T \) of a circle about the “vertical” axis, i.e., the transformations for which the degree of mapping \( \text{deg} \ T = -1, \) such that \( \theta \rightarrow \theta' = 2\pi - \theta, \) \( 0 \leq \theta < 2\pi. \) It is easy to see that, under the transformation \( T, \) the realization with signature \( \sigma \) turns into the realization with signature

\[
\tilde{\sigma} = T(\sigma) = (-\sigma_{n-1}, -\sigma_{n-2}, \ldots, -\sigma_0).
\]

Then some signatures nonequivalent under rotations \( O_k \) (and, hence, the realizations of algebras) that belong to one of the equivalence classes \( L_n \) (it is called \( O \)-equivalence) can be equivalent under inversion. Since the relation \( O_k T = T O_{-k} \) is true, we can easily show that, under the action of inversion, a set of elements from any \( O \)-equivalence class is bijectively mapped into a set of elements from a certain \( O \)-equivalence class. Thus, the number of equivalence classes under the transformations \( O_k \) and \( T \) is smaller than \( L_n. \) By \( K_n \) we denote the number of these classes. The following theorem is true:

**Theorem 6.** The set of realizations of the algebras \( A_{2,2} \) (or \( sl(2, \mathbb{R}) \supset A_{2,2} \)) of vector fields on a circle for which the vector field \( W \) [basis element of the algebra; see (1)] has \( n \) singular points, decomposes into equivalence classes with respect to the set of transformations \( \mathcal{F} \) of the circle onto itself. The number \( K_n \) of these classes is as follows:

(a) if \( n \) is an odd number, then

\[
K_n = \frac{L_n}{2};
\]
(b) if \( n \) is an even number, then
\[
K_n = \frac{L_n + I_n}{2},
\]
where \( I_n \) is the number of equivalence classes under the transformations \( O_k \) whose elements have signatures with the same number of positive and negative components \( \sigma_i \) and are invariant under the inversion \( T \). If \( n = 2k \), then \( I_n = 2^{k-1} \).

**Proof.** (a) Let \( n = 2k - 1 \). The entire set of representatives of the equivalence classes under the rotations \( O_k \) can be split into two subsets: with signatures in which the number of positive components is greater or smaller than the number of negative components, respectively. The inversion transformation \( T \) establishes a one-to-one correspondence between these subsets: every element of one subset is transformed by inversion into an element of the second subset and, moreover, two different elements are transformed into two different elements, which proves assertion (a).

(b) Let \( n = 2k \). Since the inversion establishes a mapping of the set of \( O \)-equivalence classes (under rotations) onto itself, every \( L_n \) class turns either into another class or into itself. Assume that the number of these \( T \)-invariant classes is equal to \( I_n \). Then, in view of the fact that \( T^2 = I \) (identity mapping), the number of equivalence classes under the compositions of rotations and inversions is equal to
\[
K_n = \frac{L_n - I_n}{2} + I_n = \frac{L_n + I_n}{2}.
\]

We determine the number \( I_n \).

Assume that a certain class is mapped onto itself, i.e., each element \( \sigma \) satisfies the relation \( T \sigma = O_1 \sigma \). Then \( l = 2s \) is even. Indeed, let \( l = 2s + 1 \). Thus, \( T \sigma = O_{2s+1} \sigma \) and the relation \( O_k T = T O_{-k} \) implies that either \( T O_k \sigma = O_1 O_s \sigma \) or \( T \tilde{\sigma} = O_1 \tilde{\sigma} \), where
\[
\tilde{\sigma} = O_s \sigma = (\tilde{\sigma}_0, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_{2k-1})
\]
is a representative of the same class. Since \( T \tilde{\sigma} = (-\tilde{\sigma}_{2k-1}, \ldots) \) and \( O_1 \tilde{\sigma} = (\tilde{\sigma}_{2k-1}, \ldots) \), we arrive at a contradiction. Hence, \( T \sigma = O_{2s} \sigma \) and \( T \tilde{\sigma} = \tilde{\sigma} \), where \( \tilde{\sigma} = O_k \sigma \). These signatures are called \( T \)-invariant. Therefore, a \( T \)-invariant signature can be chosen as a representative of each \( T \)-invariant class. It is clear that this signature has the form \( \sigma = (\alpha, T \alpha) \), where \( \alpha \) is a signature of length \( k = \frac{n}{2} \). Sorting out all possible \( \alpha \), we obtain the entire set \( M \) of \( T \)-invariant signatures. It is clear that their number is equal to \( 2^k \). What signatures of this kind are equivalent under rotations?

The set \( M \) can be described in a different way. Let \( \sigma \) be \( d \)-periodic. Then \( d = 2l \), where \( l \) is a divisor of \( k \). Indeed, \( \sigma = T \sigma = O_d \sigma \) and, as shown above, \( d \) is even. The entire set \( M \) can be split into subsets of signatures of different periodicity that are not equivalent. Let the signatures \( \sigma = (\beta, T \beta, \ldots, \beta, T \beta) \) and \( \delta = (\gamma, T \gamma, \ldots, \gamma, T \gamma) \) be \( 2l \)-periodic, where the signatures \( \beta \) and \( \gamma \) of length \( l \) are not \( T \)-invariant (otherwise, the period is smaller than \( 2l \)). Assume that \( \sigma \) and \( \delta \) do not coincide and are equivalent under rotations: \( \sigma = O_{-m} \delta \). Then \( m \) is a multiple of \( l \). Indeed, let \( m \). Then \( O_{2l(-m)} \delta = \delta \), which contradicts the \( 2l \)-periodicity of \( \delta \). Similarly, \( m \) cannot belong to the interval \( l \). This yields \( \beta = T \gamma \) and, hence, the entire set \( M \) is split into “couples” of signatures equivalent under rotations. Hence, the number of signatures nonequivalent under \( O_\cdot \) and \( T \)-transformations is equal to \( 2^k / 2 = 2^{k-1} \).

Theorems 5 and 6 give the complete classification of realizations of the two-dimensional noncommutative algebra \( A_{2,2} \) and algebra \( sl(2, \mathbb{R}) \) on a circle.
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