Efficient Quantum Voting with Information-Theoretic Security

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Ensuring security and integrity of elections constitutes an important challenge with wide-ranging societal implications. Classically, security guarantees can be ensured based on computational complexity, which may be challenged by quantum computers. We show that the use of quantum networks can enable information-theoretic security for the desirable aspects of a distributed voting scheme in a resource-efficient manner. In our approach, ballot information is encoded in quantum states that enable an exponential reduction in communication complexity compared to classical communication. In addition, we provide an efficient and secure anonymous queuing protocol. As a result, our scheme only requires modest quantum memories with size scaling logarithmically with the number of voters. This intrinsic efficiency together with certain noise-robustness of our protocol paves the way for its physical implementation in realistic quantum networks.

I. INTRODUCTION

Voting protocols have been studied extensively in classical cryptography [1] and numerous classical voting protocols have been developed [2–5]. To have a fair and secure voting outcome, an ideal protocol should meet the following requirements:

1. **Correctness**: With no adversary interference, the protocol correctly counts all voters.
2. **Accountability**: Only eligible voters are allowed to vote with only one vote per voter.
3. **Verifiability**: Voters should be able to verify their vote and the outcome of the election.
4. **Anonymity**: The identity of the voter cannot be compromised.

At present, the security of advanced voting protocols relies on assumptions about the computational complexity of certain mathematical problems [6], such as integer factorization [7]. However, some of these problems can be potentially solved efficiently on future large-scale quantum computers [8]. At the same time, quantum systems may allow the sharing of quantum information between voters that can, in principle, avoid computational assumptions in favor of information-theoretic security [9]. Even if the encrypted information is stored, no future advances in computational power will enable its decoding. This observation has spurred active development of quantum voting protocols, see, e.g., [10] for a recent review. However, existing approaches are either inefficient or do not satisfy all desirable security criteria. While early contributions [11, 12] explored the use of distributed entangled states to compute the voting function, their security is not rigorously established, especially in practical settings. For instance, quantum communication that is exponential in the number of voters is required to prevent double-voting [10, 13].

In this Article, we describe a novel approach to this
problem that exploits an exponential separation between quantum and classical communication complexity to authenticate voters and prevent forgery. Our scheme (Figure 1) features a (generally untrusted) tallyman, who distributes ballots encoded in quantum states to the voters. We make use of a special encoding that has been shown to provide an exponential reduction in communication complexity [14] compared to classical encodings for solving a specific matching problem [15]. This feature is related to a threshold [16] for locally decodable codes [17] and serves as the basis for unforgeable quantum money [18]. We use the communication advantage to ensure efficient scaling of resources for a large number of voters. Specifically, the total number of qubits communicated in our scheme scales polynomially with the number of voters and the size of local qubit memories only logarithmically with the number of voters. Importantly, we prove that this specific encoding also prevents forgery of votes. More precisely, we show that the voter can only learn enough information to output one vote, via projective measurement that collapses the quantum state encoding ballot information.

This ballot state is distributed to voters via quantum teleportation using preshared entanglement. In the case of an untrusted tallyman, anonymous state teleportation schemes [19, 20] can be employed to ensure voter anonymity [21]. Distributed computation of the voting function is enabled by anonymous broadcast of the voters’ ballots and votes [22]. To enable secure collision detection for both of these subroutines, we generalize a previous single-bit scheme [22] to allow computation of a distributed sum efficiently and prove its information security. It improves the efficiency of the anonymous state transfer and broadcast by queuing the senders in advance. In contrast, commonly used classical collision detection [23] relies on pairwise authenticated channels (consuming more resources) and has not been proven secure against quantum adversaries [24]. Note that compared to prior quantum secure summation protocols [25], our approach shares the output with all parties while retaining efficiency.

In our approach, the tallyman is required to create and teleport the ballot state to each voter using photonic channels. The voters should be able to do permutations of the computational basis states allowing them to measure the ballot state in different bases. We outline below how this can be achieved using one quantum node per voter containing a number of memory qubits that only scales logarithmically with the total number of voters. Consequently, quantum nodes with only tens of qubits are required for thousands of voters.

Our main result can be summarized as (formalized later in Theorem 5):

The desirable cryptographic properties (Item 1 – Item 4) of a voting scheme can be satisfied for N voters with information-theoretic security with efficient scaling involving total quantum communication of $O(N^4 \log N)$ qubits and $O(\log N)$ memory qubits per voter.

## II. THE VOTING SCHEME

We first describe the specific steps of the voting scheme assuming a trusted tallyman. We will then discuss the amendments to the scheme that allow for an untrusted tallyman.

**Creating the quantum ballots.** The protocol begins with the tallyman sampling a random bit string $x \in \{0, 1\}^n$ of $n$ bits. As discussed later, $n = O(N^2)$, where $N$ is the number of voters, to ensure that the protocol runs correctly. For instance, $n = 6$ and the sampled bit string is $x = 101100$.

The tallyman then prepares $N$ copies of the ballot state

$$|\psi_x\rangle = \frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} (-1)^{x_i} |i\rangle.$$  (1)

Here $x_i$ is the $i$th bit of $x$. The quantum state can be encoded across $\lceil \log n \rceil$ qubits using a binary encoding of $i$. Following our example of $x = 101100$, the state $|\psi_x\rangle$ is

$$\frac{1}{\sqrt{6}} (|000\rangle + |001\rangle - |010\rangle - |011\rangle + |100\rangle + |101\rangle),$$  (2)

where $|000\rangle \equiv |0\rangle \otimes |0\rangle \otimes |0\rangle$ denotes a three-qubit state.

After creating the ballot states, the tallyman distributes one ballot state to each voter. Using quantum teleportation, only $\lceil \log n \rceil$ Bell pairs need to be shared between each voter and the tallyman. We make use of a special encoding that allows quantum money encoding of $x$.

**Casting a vote.** Having received a quantum ballot state, a voter can, by means of a projective measurement, learn the parity of two random bits from the encoded bit string $x$. To illustrate this idea, we can view each of the basis states $|i\rangle$ as nodes. Measuring in the basis $\{|i\rangle\}$ would correspond to projecting at random on one of the nodes. Instead, a voter randomly chooses another complete measurement basis consisting of projectors on superpositions of two states $\frac{1}{\sqrt{2}} (|i\rangle \pm |j\rangle)$ (where $i \neq j$).

This can be viewed as picking a pairwise matching of all the nodes and projecting on a single random edge of the corresponding matching (see Figure 2). Following the above example, a measurement basis could, e.g., consist of projectors on states

$$\frac{1}{\sqrt{2}} \{|000\rangle \pm |011\rangle, |001\rangle \pm |100\rangle, |010\rangle \pm |101\rangle\}. $$  (3)

By this measurement, the voter learns the parity between bit $x_i$ and $x_j$ since obtaining measurement outcome $|i\rangle \pm |j\rangle$ corresponds to the parity of $x_i$ and $x_j$ being even (+) or odd (−).
The voter classically encodes their vote \( v \in \{0, 1\} \) by calculating the vote bit \( a = p \oplus v \), where \( \oplus \) denotes addition modulo 2. Here, we use the notion that even (odd) parity is assigned bit value \( p = 0 \) (\( p = 1 \)). To compute the voting function, the voters broadcast their measured edge \((i, j)\) and their vote bit \( a \) as described below. Once all voters have broadcasted their votes, the bit string \( x \) will be revealed by the tallyman. This will enable the voters to decode each other’s votes having the complete information of \( x \).

Importantly, \( x \) is not to be revealed until after all voters have cast their votes (the case of a dishonest tallyman is treated below). In this way, a correct encoding and subsequent decoding of a vote can only be guaranteed if the voter can sample the correct parity of an edge from measurement of the ballot state. In what follows, we prove that given \( k \) copies of the ballot state, a voter will only be able to obtain the parity of \( k \) distinct edges (meaning that the edges do not share a node) in a deterministic fashion. Obtaining more than \( k \) parities can only be achieved by randomly guessing a parity with a success probability of \( 1/2 \).

We use this property to rule out double voting by repeating the voting protocol \( O(N) \) times and averaging the votes over the rounds. If any collection of \( k \) voters (having access to \( k \) ballot states) tries to output more than \( k \) votes, they can only do so by randomly guessing parities for the extra votes, which will average to an even number of 0 and 1 votes over the repetitions and will cancel upon calculating the margin of the election.

The above arguments necessitate that only votes from distinct edges are counted. Otherwise, a vote can be forged, since \( x_i \oplus x_j = (x_i \oplus x_k) \oplus (x_k \oplus x_j) \). In the case that two voters share a node, only the first one broadcasted will be counted. The choice not to abort the protocol if overlapping edges appear prevents denial-of-service attacks, as detailed below, but poses the problem that two honest voters might sample overlapping edges by chance, which would result in one of the votes not being counted correctly. However, by increasing the length of the bit string \( n \), the probability of this event can be made arbitrarily small. Importantly, the number of qubits only scales as \( \log n \), implying efficient use of quantum resources.

To retain privacy, the broadcast of the votes should be anonymous. For this, the one-bit anonymous broadcast scheme of Ref. [22] is used repeatedly to broadcast both the edge and the agreement of a voter. The protocol assumes that the voters are connected in a ring structure by Bell pairs as illustrated in Figure 1. Each ring of Bell pairs can be used to anonymously broadcast one bit of classical information. For a specific ring, a voter can choose (with probability \( O(1/N) \)) to test the Bell pair correlations in the ring by measuring their two qubits each in the X basis. Otherwise, the voter performs a Bell measurement on the two qubits. If all voters perform a Bell measurement, the correlations between their measurement outcomes can be used as a shared key that exactly one voter can use to encode and anonymously broadcast one bit of classical information. If one or more voters choose to test, all voters that did not test announce the outcomes of their Bell measurement, which will allow the testers to verify that the correlations in the ring are correct.

Anonymous queuing. One complication of the above protocol is that one ring of Bell pairs only allows a single voter to broadcast one bit. However, to cast a vote, the voters need to broadcast both the edge \((i, j)\) of their ballot and their vote bit \( a \), corresponding to \( 2\log n + 1 \) bits of classical information. Therefore, voters need to be queued such that the order in which the bits are broadcasted is known. At the same time, collisions where two voters try to broadcast simultaneously are prevented. Here, a position in the queue corresponds to a collection of \( O(N\log n) \) Bell pairs used to broadcast the vote information of one voter. Previous works [20, 24] have used a classical collision detection algorithm, which relies on pairwise authenticated channels between the voters and has not been proven secure against quantum adversaries [24].

We solve this problem by introducing an anonymous queuing algorithm. It uses a multiparty secure sum that can be viewed as a generalization of the one-bit protocol of Ref. [22]. Let us assume that the voters share entangled states of the form

\[
\frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} |i\rangle |i\rangle
\]

in a ring; i.e., each voter shares an entangled state with each of their two neighbors. Here \( M = 2^{\lfloor \log N \rfloor} \) and the above state therefore corresponds to sharing \( \lfloor \log N \rfloor \) Bell pairs between the two voters. With probability \( 1/N \), voters test the fidelity of the entanglement by measur-
Computing their halves of the entangled states in a generalized text.

To circumvent extra votes outputted by the tallyman, the election is run in parallel with different tallymen with independent, private randomness. Each of these elections should yield the same result if the tallymen are honest, and we only need to assume the presence of an honest majority. Any dishonest tallymen will be identified through the discrepancy in the outcome of their elections and there is thus a risk of exposure if a tallyman chooses to behave dishonestly. The independence in the randomness implies that dishonest tallymen cannot initiate attacks based on correlated randomness with honest tallymen.

To ensure the anonymity of the voters, the ballot states are teleported anonymously to them. This way, the tallyman might tamper with a ballot state but will have no information about which voter received the tampered.
ballot. Consequently, anonymity will not be compromised.

To anonymously teleport the ballot states, we make use of anonymous quantum state transfer [20]. This protocol relies on shared GHZ states among the voters and the tallyman. A GHZ state can be distilled from a ring of Bell pairs connecting the voters and the tallyman (see Figure 1): performing a CNOT gate followed by a measurement on the target qubit combines Bell pairs into the nodes of a GHZ state, up to local single-qubit rotations [27]. From the GHZ state, a Bell pair is distilled between the tallyman and an anonymous voter, through which a qubit can be transferred by means of quantum teleportation.

To distill a Bell pair, all but the tallyman and the receiving voter measure their qubits in the X basis and broadcast the result (the receiver broadcasts a dummy bit to stay anonymous). The anonymity is linked with the fidelity of the GHZ state (see Theorem 4), which can be estimated by the voters through random checks of the correlations in the state, similar to the one-bit anonymous broadcast protocol. In other words, the voters either choose to use a GHZ state for transfer or for testing. In this way, they can detect if an adversary tries to compromise anonymity, since the attempt will corrupt the correlations in the GHZ state.

One GHZ state enables only a single voter to receive one qubit anonymously. The anonymous queuing of the voters (as described above) allows the voters to know in which position (comprising \(O(N^2 \log n)\) Bell states) they are to receive a ballot state and to avoid collisions.

**Resource requirement.** As detailed above, all subroutines of the voting protocol can be realized using shared Bell pairs between pairs of voters and the tallyman in a ring structure. The total resource consumption of one round (out of \(O(N)\) rounds) of the voting protocol is \(O(N^2 \log N)\) qubits of communication in the case when the tallyman is trusted and \(O(N^3 \log N)\) otherwise, due to the anonymous state transfer [26].

Importantly, the size of the local qubit memories of the voters and the tallyman only scales as \(O(\log N)\) assuming a sequential operation of the voting scheme where Bell pairs are distributed at the start of each subroutine (e.g., ballot distribution, anonymous queuing) instead of in advance. In this way, the size of the memories will be determined by the ballot state and the anonymous queuing algorithm, which both require \(O(\log N)\) memory qubits.

In the rest of this paper, we outline the elements of a physical implementation and describe the cryptographic properties of the scheme. The precise statements of the algorithms and technical material completing the proofs are relegated to the Supplemental Material [26].

### III. IMPLEMENTATION

The execution of our voting scheme requires an underlying quantum network capable of distributing high-fidelity Bell pairs between adjacent voters in a ring structure. In summary, the scheme consists of the steps:

1. First, anonymous queuing is run to establish the order of ballot distribution and vote casting. The tallyman repeatedly distributes \(O(\log N)\) Bell pairs to the voters until the queuing is finished.

2. After the queuing, the tallyman again distributes \(O(\log N)\) Bell pairs for anonymous state teleportation. Once an anonymous channel of \(O(\log N)\) Bell pairs between the tallyman and a voter has been established, the tallyman creates a ballot state and teleports it to the voter.

3. Having received the ballot state, the voter measures in a random basis to calculate a (classical) bit used for encrypting votes.

4. Steps 2 and 3 are repeated until all voters have their vote bits. The tallyman now repeatedly distributes \(O(\log N)\) Bell pairs to allow the voters to anonymously broadcast their encrypted votes one-by-one following the order dictated by the queue. Finally, the tallyman classically broadcasts the bit string \(x\) used for decryption.

Note that the number of Bell pairs that a voter needs to simultaneously store and operate on scales as \(O(\log N)\) due to the efficient encoding of the quantum ballot. Only modestly-sized local quantum processors are thus required even for a large number of voters.

The most demanding operations that have to be performed are the creation and measurements of the ballot states. As we now discuss, these operations can, nonetheless, be implemented using single quantum emitters strongly coupled to photonic resonators, which function as efficient spin-photon interfaces [28, 29] (see Figure 4).

**Ballot creation.** The key idea to simplify ballot creation involves mapping between unary and binary encodings (see Figure 4(a)). The ballot state is first encoded in unary using a qudit time-bin encoding where, ideally, a single photon is distributed equally in \(n\) time bins. Such a state can either be created using a Raman-type single-emitter scheme [30] or approximated using weak coherent pulses. The bit string \(x\) is encoded by applying a phase of \(\pi\) to photons in select time bins, e.g., by passing them through a phase shifter.

The photonic qudit is subsequently compressed into \(\log n\) qubits using spin-dependent reflection from a single-sided cavity [29, 31]. A resonant photon will be reflected with (without) a \(\pi\) phase shift if the emitter qubit is in a state that is uncoupled (coupled) to an excited state by the cavity field [32], which realizes a controlled-phase gate between the qubit and photon. Reflecting the photon off a logarithmic number of
such emitter-cavity systems, routing time bins to cavities via the binary representation of the time bin number [33, 34], will entangle the spin registers with the photonic qudit. Subsequently erasing the time-bin information with an interferometric (X-basis) measurement will teleport the ballot state to the registers up to a phase correction determined by the measurement outcome. The operation is heralded on photon detection, which mitigates the detrimental effects of loss in the system to significantly boost the fidelity. Alternatively, the erasing of the time-bin information can be achieved with atomic detection as outlined in Refs. [33, 34].

The coherence time of the emitter qubits will limit the size of the ballot state that can be created using the unary to binary encoding outlined above. In particular, the coherence time should be long compared to the duration of the single photon qudit pulse. Furthermore, the complexity of the final interferometric measurement also increases with the size of the qudit state since more time bins need to be interfered. The length of the qudit pulse will be \( n\Delta t \) where \( n \) is the length of the bit string \( x \) and \( \Delta t \) is the duration of one time-bin. The latter will be determined by the photon emission time of the emitters and the optical switching rate for routing the time bins to the cavities. For diamond defect centers, Purcell-enhanced photon emission times of about 100 ps [29] would be compatible with time bins on the order of 1 ns, requiring GHz-rate optical switching. For coherence times on the order of 10 ms [28], this would give a limit of \( n \lesssim 10^7 \), compatible with thousands of voters. We note that the size of the required quantum memory would only be around 24 qubits.

Once the ballot state has been encoded in the emitter-cavity registers, it is transferred to a voter via quantum teleportation using the anonymously distributed Bell pairs. For emitters based on color centers in diamond, the Bell pair qubit may be stored in a nearby nuclear spin [28, 35]. A Bell measurement is performed using the direct electronic-nuclear spin coupling. The correction bit strings necessary for teleportation can be broadcasted to the voter since they do not contain any information about \( x \).

**Ballot measurement.** Voters may sample \( \log n \) random matchings by measuring one qubit of the ballot state in the \( X \) basis and the rest in the \( Z \) basis. The other matchings can be obtained by permuting the basis states. The \( X \), CNOT, and Toffoli gates are universal in this sense, and the circuit size is bounded by \( O(n \log n) \) [36]. In practice, sampling over all matchings may not be necessary, so smaller circuits may suffice, but then the tallyman has more influence on which edge is obtained.

Such operations can be performed with the same type of spin-cavity register as described above. To realize a CNOT gate, Bell states are first established between the spin-cavity systems using spin-photon controlled gates, similar to the ballot state creation except with a photonic time-bin qubit instead of qudit (Figure 4(b)). Since the ballot state has been transferred to nuclear mem-

![FIG. 4. Implementation of the hidden matching communication problem. (a) To create the ballot state, the tallyman first generates a single photon in a superposition of time bins. A time-dependent phase is applied to encode the tallyman's randomness. Teleporting to quantum memories, via cavity-assisted photonic gates followed by interferometric measurement, compresses the photonic qudit into a binary qubit code. (b) Similarly, Bell pairs are created by reflecting a time-bin photonic qubit off two cavities, followed by measurement. This entanglement enables teleported CNOT gates, which permute basis states to sample different matchings.](image)

**Noise-robustness.** The realistic imperfect fidelity of the Bell states underlying the voting scheme reduces anonymity [20, 22], but is testable so the amount of loss is known. Furthermore, our unforgeability result of the quantum ballots remains unchanged for noisy channels. Errors on the ballot state are detected by the voters at the end of the protocol since they verify whether their vote was counted correctly. While the vote-verification of a group of voters may fail, an election can still be carried out given that this group is small enough compared to the margin of the election.

Running the election in parallel by splitting voters in communities of size \( O(\log N) \), each having separate tallymen, allows for an error rate that diminishes polynomially in \( N \) and, therefore, requires only \( \text{poly}(N) \) rounds of repetition (on average) across all communities to have an effectively error-free election. Once a tally is conducted in a community, the tallymen communicate \( O(\log N) \) bits to share the margin in their local election. Each tallyman can compute the outcome of the full election locally with this information. This tree structure, combined with network topologies other than
IV. SECURITY ANALYSIS

Threat model. We assume that the voters are partitioned into two sets: honest and dishonest. In our threat model, we trust the set of honest voters to follow the protocol. The same voting protocol is run by multiple tallymen, each having independent randomness, a majority of whom are assumed to be honest. No further assumptions are made and the dishonest tallymen can collude with the set of malicious voters. Additionally, no limits are placed on the computational power of the malicious voters.

We provide definitions for the security properties of a voting protocol, accounting for information-theoretic security. Compared to prior work [10] in the setting of computationally-bounded adversaries, our properties account for the more general setting of information-theoretic security and therefore provide a stronger security guarantee. We then prove that our protocol satisfies these properties. Our ideal protocol does not leak any information as a function of the number of voters, in contrast to Ref. [10] where the distinguishing probability is allowed to be 1/2 + negl(N) — a weaker requirement where a negligible (inverse superpolynomial) amount of information can be learned. We begin by formally defining the desirable properties of a voting protocol introduced informally in the beginning of this article.

Definition 1 (Correctness). A voting protocol \( \Pi_{\text{vote}} \) is said to satisfy the correctness property if for every set of honest voters \( \{V_i\}_{i=1}^{N} \), given a set of votes \( \{v_i\}_{i=1}^{N} \),

\[
\Pr[\Pi_{\text{vote}}(\{v_i\}_{i=1}^{N}) = \text{MAJ}_N(\{v_i\}_{i=1}^{N})] \geq 1 - \epsilon .
\]

Equivalently, Definition 1 asserts that the protocol outputs the correct answer in the absence of adversarial voters with probability close to unity. Theorem 1 demonstrates that our voting protocol satisfies this property.

Definition 2 (Accountability). A voting protocol \( \Pi_{\text{vote}} \) is said to satisfy the accountability property if for \( \forall \delta > 0 \),

\[
\Pr[\text{honest votes are counted}] = 1 ,
\]

and,

\[
\Pr[\text{malicious votes are counted}] \leq \delta .
\]

Definition 2 asserts that the voting protocol always counts votes output by honest voters but does not count, with high probability, votes output by any voter that is not honest. Theorem 2 demonstrates that our voting protocol satisfies this property.

Definition 3 (Verifiability). A voting protocol \( \Pi_{\text{vote}} \) is said to satisfy the verifiability property if every voter \( V_i, i \in [N] \) can check that their vote \( v_i \) is tallied and any auditor can verify the outcome of the election by looking at the transcript of \( \Pi_{\text{vote}} \).

Definition 3 asserts that the protocol is transparent in that it allows voters to check the tallying process. Theorem 3 demonstrates that our voting protocol satisfies this property.

Definition 4 (Anonymity). A voting protocol \( \Pi_{\text{vote}} \) satisfies the anonymity property with parameter \( \gamma \in (0, 1) \) if the following holds: Given an honest set of voters \( B \subset \{V_i\}_{i=1}^{N} \) and a malicious set of voters \( A = \{V_i\}_{i=1}^{N} \setminus B \) along with their choice of permutation \( \pi : B \rightarrow B \), such that \( |A| \leq N, \forall \) strategies \( S \) used by \( A \),

\[
\Pr_{b \in \{0,1\}}[S(\Pi_{\text{vote}}) = b] \leq \frac{1}{2} + \gamma ,
\]

where \( b \) is a bit that is chosen uniformly at random, such that \( \Pi_{\text{vote}} \) runs for the configuration of votes chosen by \( A \) for \( b = 1 \) and the permuted version for \( b = 0 \).

Definition 4 asserts that malicious voters cannot associate a voter’s identity to the vote cast by them. More precisely, anonymity is maintained when malicious voters cannot distinguish between any two voting configurations output by honest voters on observing the announcement of the honest voters regardless of the strategy that they use. Theorem 4 demonstrates that our voting protocol satisfies this property.

We now describe how the voting protocol introduced in this article satisfies the requirements of an ideal voting scheme formally introduced above. First, in the presence of only honest voters, the protocol computes the correct outcome of the voting function.

Theorem 1 (Correctness). The voting protocol described in this article satisfies Definition 1.

Proof Sketch. By construction, assuming that the ballots do not have overlapping edges, the voting protocol outputs the majority winner of the election. The probability that all voters sample the parity of independent bits can be shown to be

\[
1 - O\left(\frac{N^2}{n}\right),
\]

from simple combinatorial considerations. To avoid collisions between voters, \( n = O(N^2) \), where \( n \) is the number of random bits generated by the tallyman, which are encoded in \( \log n \) qubits.

The second requirement says that no voter can cast more than one vote. Recall that votes are computed as \( v = p \oplus a \), where a ballot \( ((i,j), p) \) is acquired by solving the communication problem and the agreement \( a \in \{0,1\} \) is determined by the desired vote. We prove that no voter can output more than one vote that aligns
with their intended candidate deterministically, and that extra votes do not bias the election. Casting more than one vote results in all but the first vote being distributed in an unbiased binomial distribution. The influence of the other votes is suppressed by repeating the protocol.

**Lemma 1.** Given $N$ copies of the ballot state $|\psi_x\rangle$, a string of $N' \geq N$ votes may be outputted with probability

$$\Pr[(N' - N) \text{ votes}] \leq \frac{1}{2^{N-N}}.$$  \hspace{1cm} (13)

**Proof Sketch.** Votes are only counted if they do not have intersecting vertices. We can therefore consider the task of outputting the overall parity of more than $2N$ bits given access to $N$ ballot states, which bounds the success probability of outputting more than $N$ votes by reduction. Following a similar approach as in Ref. [16], we bound the success probability of any measurement strategy correctly distinguishing between states encoding different parities of the random bits. In particular, we show that all extra votes are outputted with $\Pr[v_i = 0] = \Pr[v_i = 1] = 1/2$, i.e., with zero bias. \hfill \square

With an appropriate number of repetitions, using the lemma above, the protocol provides a strong degree of accountability.

**Theorem 2 (Accountability).** The voting protocol described in this article, on repeating $r = O(N, 1/\gamma)$ times with averaging of outputs, satisfies Definition 2.

**Proof Sketch.** The adversary can technically cast any number of votes. However, outputting extra votes will eventually lead to edge collisions, causing all but the first of their votes to be dropped. As a result, malicious voters can output no more than $O(N)$ vote attempts in a round of the protocol. Repeating $\Pi_{\text{vote}}$ using fresh randomness over $O(N)$ rounds and averaging the outcomes suppresses the noise introduced by the extra votes such that they cancel in the computation of the margin. The first vote, conversely, is biased and contributes to the tally. \hfill \square

Extending the notion of accountability, the result of the election should be verifiable by both the voters and any external auditor.

**Theorem 3 (Verifiability).** The voting protocol described in this article satisfies Definition 3.

**Proof.** After running $\Pi_{\text{vote}}$, a transcript containing the edges $(i, j)$ along with the agreement bit $a_i$ of every voter is produced. Since $x$ is revealed after every run, a voter can verify that the vote they cast was correctly counted by checking that $(x_i \oplus x_j) \oplus a_i = v_i$. An external verifier simply runs the computation in Theorem 2 to obtain the margin, by which the election is decided. \hfill \square

Another critical notion is anonymity: no subset of voters can conspire to identify the votes of the remaining set of voters. Perfect anonymity amounts to only being able to randomly guess the identity of a voter; i.e., the success probability of linking a vote to a voter is $1/N$. In practice, however, anonymity is reduced by a parameter $\gamma$, due to the deviation from perfect fidelity of the entangled states used in the underlying protocols for distributing the ballot states [20] and the collection of votes [22]. This notion of information-theoretic security allows for noisy quantum states [20].

**Theorem 4 (Anonymity).** The voting protocol described in this article satisfies Definition 4 with any parameter $\gamma > 0$.

**Proof Sketch.** The security of the anonymous broadcast in our protocol relies on the standard reduction of quantum key distribution to the error-corrected case [22, 38]. The mutual information between eavesdroppers and the broadcasted messages is suppressed. The agreement bit itself is masked with the randomness of the uncompromised key. The underlying ballot state can be anonymously distributed to the voters up to a certain tolerance of fidelity given by $\gamma$ [20]. The total loss of information by broadcast and transfer can then be bounded by a union bound. \hfill \square

The voting protocol therefore satisfies all the desirable cryptographic properties outlined in Section I and has efficient consumption of communication resources, as summarized in Theorem 5.

**Theorem 5 (Main Theorem).** Given $e, \delta, \gamma > 0$, voters $\{V_i\}_{i=1}^N$ with votes $\{v_i\}_{i=1}^N$, and the majority voting function $\text{MAJ}_N : \{0, 1\}^N \rightarrow \{0, 1\}$, where $e$ is the tolerance to tag collisions, $\delta$ is the tolerance to failure during repetition, and $\gamma$ is the deviation from perfect anonymity, $\exists n = O(N^2, 1/e)$ such that

$$\Pr [\Pi_{\text{vote}}(x, \{V_i\}_{i=1}^N) = \text{MAJ}_N(\{v_i\}_{i=1}^N)] \geq 1 - e - \delta,$$

where $r = O(N, 1/\gamma)$ is the number of rounds of $\Pi_{\text{vote}}$. The scheme satisfies the security properties specified in Definitions 1 – 4 with $\gamma$-anonymity using no more than $O(N^3 \log N)$ qubits of communication in $\Pi_{\text{vote}}$.

**Proof Sketch.** The main theorem follows from the proofs of the component theorems and the analysis of the algorithms, included in the Supplemental Material [26]. \hfill \square

**Noisy votes.** Malicious voters are free to attempt to output as many extra votes as possible. However, there is a limit before losing anonymity. More specifically, in order for the anonymous broadcast to go through, the voters must use the shared Bell pairs [22]. As described earlier, the influence of extra vote attempts is suppressed by repeating the protocol a modest number of times. While a malicious voter can input noise into the broadcast channel, this attack can be detected by the honest voter at the end of the protocol. Our protocol does not guarantee counting any vote that a malicious voter may cast. In particular, any voter that attempts to deterministically output more than one vote may have
none of their votes counted in the election.

**DoS attacks.** Using the fact that a voter does not want to compromise their identity in conjunction with Theorem 1, we can rule out certain families of denial-of-service (DoS) attacks during any phase other than one that uses the anonymous broadcast. Any attempt will either compromise their identity, lead to extra votes that are suppressed during repetitions as shown in Theorem 2, or result in their votes being not counted. As mentioned prior, while noise injected by malicious voters during anonymous broadcast can be detected, the anonymity in the protocol disallows identifying them, leading to a vulnerability to a noise-based DoS attack. This problem is foreseen in Ref. [10] and left open for future work.

When running $\Pi_{\text{vote}}$, any malicious voter $V_i \in A_i$ attempting to guess an edge $e = (i', j')$ that intersects with one cast by an honest voter $V_j$ will succeed with vanishing probability (Theorem 1). On the other hand, if $V_i$ observes the broadcasted edge in the tag of an honest voter $V_j$, then $V_i$ can only cast their tag afterwards (as in the order instantiated by the anonymous queue), in which case any vote with overlapping edge is discarded during the execution of $\Pi_{\text{vote}}$. If $V_i$ has output edges $\{e'\}$ that do not intersect with those of any other voters and are not obtained from the ballot state, then these votes will be suppressed during repetition (Theorem 2). Lastly, note that if some subset of malicious voters attempts to choose an intersecting set of edges on purpose to compute parities of other edges deterministically, then those extra edges broadcasted in a tag with a vote will be dropped, since they intersect with at least one of the prior edges. In such a case, only the first vote will be counted and subsequent votes will be dropped, which removes incentive for the malicious adversaries to try this attack. These guarantees show that any DoS attack which relies on the protocol aborting due to intersecting edges can be avoided.

V. CONCLUSION

In this paper, we present an efficient quantum voting protocol satisfying the desirable security criteria with information-theoretic security (Theorem 5). It is a distributed ballot scheme where the solution to a communication complexity problem authenticates voters and encodes their votes. We prove that the quantum encoding prevents forgery of votes, in that only one vote can be recovered from each state. The election outcome can then be computed in a secure multiparty setting using anonymous broadcast. We provide a generalized quantum key distribution protocol that enables secure sum, which we use for unconditionally secure collision detection and queuing of voters. An election is run with $O(N^3 \log N)$ qubits of communication, repeated $O(N)$ times to achieve security. Importantly, the poly($N$) scaling is efficient in the number of voters $N$, in contrast to previous schemes. Furthermore, the protocol requires local memories of size $O(\log N)$ qubits for each voter, ensuring that only modestly-sized quantum processors are necessary even for a large number of voters. The reduction in resources, for both communication and computation, removes a fundamental obstacle toward practicality. While beyond the reach of current quantum networks, the amount of quantum processing is modest and many of the key elements of the scheme have already been demonstrated in experiments. Consequently, this work advances the state-of-the-art in cryptography and provides a practical application for quantum networks. The quantum scheme enables capabilities beyond classical voting protocols and may provide hardware-level security.

The $O(N)$ overhead introduced by averaging to suppress adversarial votes can potentially be avoided in a tag-based scheme [26]. The tallyman $T$ sends $s = O(\log N)$ copies of the ballot to every voter, and unless the voter can output $s$ parities of the string $x$ at uniformly random locations, the vote is rejected. Then, the probability that an adversary can output votes correctly is bounded by $1/N^c$, for some $c \geq 1$. Communication resources decrease to $\tilde{O}(N^3)$ qubits at the expense of a marginal increase in randomness to $\tilde{O}(N^2)$ bits, suppressing logarithmic factors. However, since a voter makes multiple announcements to cast a single vote, the anonymity of such a protocol becomes challenging to prove in the presence of adaptive adversaries.

Finally, we remark that the voting protocol does not satisfy a property called **Receipt-Freeness**, which asserts that no voter should be able to prove that they voted in a certain way. Malicious adversaries can use the underlying algebraic structure of the secret key in the anonymous broadcast to learn information about how the rest of the voters voted. We believe that “vote selling” may still be prevented, and that it is a modest criterion given the voting protocol’s other highly desirable guarantees.

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Supplemental Material: Efficient Quantum Voting with Information-Theoretic Security

We specify the algorithms in the quantum voting protocol and provide technical analysis. Furthermore, we complete the proofs of the protocol’s cryptographic properties. Finally, we discuss the tag-based variant of the voting scheme.

Section S.I elaborates on the algorithms described in the main text. In particular, we state the overarching voting protocol for calculating the election outcome (Algorithm S.1) and the subroutines of anonymous state transfer (Algorithm S.2) and key generation (Algorithm S.3). We then state the protocols for computing a distributed secure sum (Algorithm S.4), anonymously queueing the voters (Algorithm S.5), and anonymously broadcasting and computing the votes of every voter (Algorithm S.6). Then, we analyze aspects of how the voting protocol runs, bounding the probability of voters sampling intersecting pairs of bits from the quantum ballot state (Lemma S.2) and the number of rounds required in the queuing algorithm (Lemma S.3). Finally, we consider noise in the distribution of the quantum ballot state (Lemma S.4) and bound the probability of a successful run under independent bit flip errors.

Section S.II provides the full security proofs for the voting protocol. First, we prove the unforgeability of the quantum ballot, beginning with the result for one copy of the state (Lemma S.5) and afterward in general for multiple copies (Theorem S.1). Then, we bound the influence of adversarial votes when averaging over rounds of the protocol (Theorem S.2). Finally, we show anonymity of the voters (Theorem S.3).

We introduce a variant of the scheme in Section S.III. It enables savings in quantum communication (Theorem S.4), but requires a more involved security analysis of anonymity.

S.I. ALGORITHMS

We present the pseudocode for all the algorithms along with brief informal explanations for each algorithm.

Algorithm S.1: DECISION

Result: $\text{MAJ}_N(\{v_i\}_{i=1}^N)$
1 #0=#1=0 ;
2 for $r' \in [r], r = O(N)$ do
3 \hspace{1em} Run $\text{VOTING}$ (Alg. S.6) ;
4 \hspace{1em} for $l \in [N]$ do
5 \hspace{2em} if $(i_l,j_l) \cap (i_{l'},j_{l'}) = \varnothing, \forall l' < l$ then
6 \hspace{3em} if $v_l = 0$ then
7 \hspace{4em} #0 = #0 + 1 ;
8 \hspace{3em} else
9 \hspace{4em} #1 = #1 + 1 ;
10 \hspace{2em} end
11 \hspace{1em} end
12 \hspace{1em} if $p_l \neq x_i \oplus x_{i_l}$ then
13 \hspace{2em} $b_l = 1$ ;
14 \hspace{1em} else
15 \hspace{2em} $b_l = 0$ ;
16 \hspace{1em} end
17 \hspace{1em} Compute $B_{r'} = \sum_{l=1}^N b_l \mod N$ (SUM);
18 end
19 Compute averages $\overline{0} = \#0/r, \overline{1} = \#1/r$ and margin $\overline{0} - \overline{1}$ ;
20 Compute average number of complaints $\sum_{r'} B_{r'}/r$ ;
21 if $|\overline{0} - \overline{1}| > 2 \sum_{r'} B_{r'}/r$ then
22 \hspace{1em} Output winner $(1 - \text{sgn}(\overline{0} - \overline{1}))/2$ ;
23 end
Algorithm S.2: TRANSFER

Result: Qubit transmitted from $T$ to $V_f$ with $\epsilon$-anonymity

1. $T$ shares a GHZ state with voters $\{V_l\}_{l \in [N]}$:
2. for $V_l, l \in [N]$ do
   3. Sample $x \in \{0, 1\}^S, S = O(\log N)$;
   4. Anonymously broadcasts $X = \prod_{i=1}^S x_i$;
5. end
6. if $X = 0$ then
   7. Random $V_{j'}, j \in [N]$ runs VERIFICATION [20];
8. else
   9. $\{V_l\}_{l \in [N]}$ run ANONYMOUS ENTANGLEMENT [20], establishing Bell pair $|\phi^+\rangle$ between $V_{j'}, j' \in Q([N])$ and $T$ with $\epsilon$-anonymity;
10. Quantum teleportation from $T$ to $V_{j'}$ using $|\phi^+\rangle$;
11. end

Algorithm S.3: KEY GENERATION

Result: $K = \sum_{i=1}^N k_i \mod N = 0$

1. for $V_l, l \in [N]$ do
2.   $V_l$ and $V_{l+1}$ share $\tilde{\sum}_{i=1}^N |i\rangle_l |i\rangle_{l+1}$;
3.   $V_l$ samples $r_l \in \{0\} \cup \{1\}^{N-1}$;
4.   if $r_l = 0$ then
5.     Measure $X_l \equiv |i\rangle_l \otimes |i\rangle_{l+1} \implies |i\rangle_l |j\rangle_{l+1}$, measure $X_{l,1} \otimes Z_{l,2}$, with outcome $x_{l,1}, z_{l,2}$;
6.     Broadcast $x_{l,1}$;
7.   else
8.     Bell measurement: unitary $|i\rangle_l |j\rangle_{l+1} (\text{C}_{l}X_{l,2})^{N-1} |i\rangle_l |j + (N-1)i\rangle_{l+1}$, measure $X_{l,1} \otimes Z_{l,2}$, with outcome $x_{l,1}, z_{l,2}$;
9.     Broadcast $z_{l,2}$;
10. end
11. Broadcast $p_l$;
12. end
13. if $\exists l' \in [N]: r_{l'} = 0$ then
14. for $V_{j'}, l \in [N]$ do
15.   Broadcast $x_{l,1}$;
16. end
17. Minimum fidelity $F_{\text{min}}$ of $X_{l,1} \otimes X_{j',2}$ measurements, for $V_i, V_j$ connected by Bell measurements;
18. else
19.   $K = \sum_{i=1}^N z_{l,2} \mod N = 0$;
20. end

Algorithm S.4: SUM

Result: $B^N, B = \sum_{i=1}^N b_i \mod N$

1. Key generation (Alg. S.3), obtaining $K = \sum_{i=1}^N k_i \mod N = 0$;
2. for $V_l, l \in [N]$ do
3.   Broadcast $(k_l + b_l) \mod N$;
4. end
5. for $V_{j'}, l \in [N]$ do
6.   Compute $\sum_{i=1}^N (k_i + b_i) \mod N = \sum_{i=1}^N b_i \mod N$;
7. end
Algorithm S.5: QUEUE

Result: $Q = \sigma([N]), \sigma \in S_N$

1 for $i \in [N]$ do
2 while $\sigma(i) = \emptyset$ do
3 for $V_l, l \in [N]$ do
4 $V_l$ samples $r_l \overset{R}{\leftarrow} \{0\}^{N-i} \cup \{1\}$;
5 if $i \not\in Q(i-1)$ and $r_l = 1$ then
6 $b_l = 1$;
7 else
8 $b_l = 0$;
9 end
10 end
11 Compute $B_N, B = \sum_{i=1}^{N} b_i \mod N$ (Alg. S.4);
12 if $B = 1$ and $r_l = 1$ then
13 $\sigma(i) = l$;
14 end
15 end
16 end

Algorithm S.6: VOTING

Result: $\{(i_l, j_l, v_l)\}_{l=1}^{N}$, for each $V_l$

1 Anonymous queue (Alg. S.5) $Q$;
2 $T$ distributes $|\psi_x\rangle \otimes N$, where $x \overset{R}{\leftarrow} \{0, 1\}^n$, anonymously [20] to $\{V_l\}_{l \in Q}$;
3 for $V_l, l \in Q$ do
4 Solve $(i, j), p$ from $x$ [15];
5 Set agreement bit $a = p \oplus v$;
6 Anonymous broadcast [22] of $(i_l, j_l), a_l$;
7 end
8 $T$ broadcasts $x$;
9 for $V_l, l \in [N] \equiv \{1, \ldots, N\}$ do
10 Compute $v_l \equiv p_l \oplus a_l, i \in [N]$;
11 end

A. Decision Function

The calculation of the election winner in majority vote relies on averaging over $O(N)$ rounds in order to suppress adversarial votes. The margin of the election is also computed. In each round of the election, voters can anonymously complain if their vote was incorrectly counted. If the number of such complaints is less than half the margin, then the election outcome is unchanged and valid. This process is made explicit in Algorithm S.1.

B. Anonymous State Transfer

The algorithm used for anonymous state transfer (Algorithm S.2) is adapted from [20]. A successful run of the protocol establishes one Bell pair between a voter $V_l$ and and tallyman $T$ with $\epsilon$-anonymity (Lemma S.1). Testing of the entanglement limits the success probability to $O(1/N)$. Since a GHZ state can be shared with $N$ Bell pairs, anonymous teleportation of one qubit consumes $O(N^2)$ entangled pairs.

Lemma S.1 ([20]). \(\forall \epsilon > 0\), given that the distributed GHZ states have fidelity $F \geq \sqrt{1 - \epsilon^2}$, even in the presence of malicious adversaries $B$,

\[ \Pr[B \text{ guesses } V_l] \leq \frac{1}{N} + \epsilon, \]

where a voter $V_l$ and tallyman $T$ establish a Bell pair.
C. Key Generation

The general key generation algorithm for \( N \) levels is outlined in Algorithm S.3. It is used in the anonymous broadcast [22] and to compute the secure sum. When all voters perform Bell measurements on their shared Bell pairs, they obtain a secret key. With probability \( 1/N \), a voter performs a test measurement on the fidelity of the entanglement. Key generation then succeeds with probability \( 1/e \) asymptotically [22], which results in constant overhead.

D. Anonymous Queuing & Voting

Algorithm S.4 computes the distributed secure sum of the votes cast by voters. It relies on the key generation algorithm (Algorithm S.3) to “mask” the vote with perfect randomness. It outputs the sum of the inputs of the voters (modulo \( N \)).

Algorithm S.5 instantiates the anonymous queue that delegates the order in which the voters cast their vote. Every unequipped voter samples randomly to enter the queue with equal probability. If the bit is one, the voter casts a desire to queue themselves in the next position in the queue. This request is granted if the distributed secure sum of the bits cast by all voters is 1. The protocol need only be repeated \( O(N) \) times (Lemma S.3) to queue every voter.

Algorithm S.6 is the actual protocol used by the voters to cast their votes. First, the tallyman anonymously distributes (Algorithm S.2) the hidden-matching state to every voter. Then, the voters anonymously broadcast [22] their tags, which consist of an agreement bit as well as the indices of the measured edge, in the order instantiated by the anonymous queue (Algorithm S.5). The tallyman then broadcasts the random string and the votes are computed for every voter.

E. Resource Consumption

One round of the voting protocol has total quantum communication of \( O(N^3 \log N) \) qubits, enumerated below.

- The distributed secure sum consumes \( O(N \log N) \) bits of entanglement, performed \( O(N) \) times to construct the queue.
- The quantum superposition state compresses the tallyman’s \( n \) bits of randomness in \( \log n \) qubits. To avoid intersecting edges between voters, \( n = O(N^2) \) (Lemma S.2). Copies of the state are communicated to \( N \) voters, consuming a GHZ state per qubit [19] (with an overhead of \( N \) for verifying the entanglement), which may be distributed with consumption of \( N \) Bell pairs [39].
- The anonymous broadcast protocol requires a ring of maximally entangled states connecting the voters, consuming \( O(N) \) Bell pairs per broadcasted bit, including test measurements verifying the entanglement. A single ballot and vote constitute \( 2 \log n + 1 \) bits.

The number of rounds, used to amplify security by removing the influence of adversarial votes, is \( O(N) \) in our scheme, in contrast to comparable protocols that are susceptible to double-voting if the number of rounds is not exponential in the number of voters [10, 13].

F. Edge Collision

The following lemma bounds the number of edges that can share an intersecting vertex between two voters given a ballot state. As a result, for \( n = O(N^2/e) \), adversarial voters can guess \( O(N) \) ballot configurations before colliding with another voter. More specifically, Lemma S.2 implies that, for any \( \alpha > 0 \), if a single voter outputs \( O(N^1+\alpha) \) votes, the probability of a shared vertex between some pair of voters becomes \( O(N^\alpha) \) which is much greater than \( \epsilon \) for sufficiently large \( N \), and causes the protocol to almost certainly fail.

**Lemma S.2.** Given \( n = O(N^2/e) \), \( x \stackrel{R}{\leftarrow} \{0,1\}^n \) and two voters \( V, V' \) with perfect matchings on \( n \) vertices \( M \stackrel{R}{\leftarrow} \mathcal{M}_2, \mathcal{M}' \stackrel{R}{\leftarrow} \mathcal{M}_2 \) and edges \( e \stackrel{R}{\leftarrow} M, e' \stackrel{R}{\leftarrow} M' \), the probability that they share a vertex is,

\[
\Pr[e' \cap e = \emptyset \mid e, e'] \leq O \left( \frac{N^2}{n} \right) \leq \epsilon.
\]  

(S.1)
Proof. Since each edge $e$ is drawn by a voter i.i.d.,

$$ \Pr[e \leftarrow V, e' \leftarrow V'] = \Pr[e \leftarrow V] \Pr[e' \leftarrow V'] .$$  \hfill (S.2)

The probability that two voters sample an edge with intersecting vertices is equivalent to the number of edges with a shared vertex given two matchings $M, M'$ that are sampled independently normalized by the number of matchings. We denote the event of a shared vertex between two voters $V, V'$ that have sampled edges $e, e'$ as $E$. Since we want to bound it for every pair of voters, we begin by first applying a union bound over the number of distinct voter pairs to the event $E$,

$$ \Pr \left[ \bigcup_{V \neq V'} E \right] \leq \sum_{V \neq V'} \Pr[E] $$  \hfill (S.3)

$$ = \sum_{V \neq V'} \Pr[e, e', E] $$  \hfill (S.4)

$$ = \sum_{V \neq V'} \Pr[e] \Pr[e', E|e] $$  \hfill (S.5)

$$ = \sum_{V \neq V'} \Pr[e] \Pr[e', E|e] $$  \hfill (S.6)

We now use conditional probabilities to split up the joint distribution $\Pr[e, e', E]$ and note that,

$$ \Pr_{e \in M} [e] = \Pr_{e \in M} [e] = \sum_{M \in M} \Pr[e|M] \Pr[M] .$$

Using the above, we expand as,

$$ \Pr \left[ \bigcup_{V \neq V'} E \right] \leq \sum_{V \neq V'} \sum_{M \in M} \Pr[e|M] \Pr_{e \in M} [e] \Pr_{e \in M} [e', E|e] $$  \hfill (S.7)

$$ = \sum_{V \neq V'} \sum_{M \in M} \Pr[e|M] \Pr_{e \in M} [e] \Pr_{e \in M} [e', E|e, M'] \Pr_{e \in M} [M'] .$$  \hfill (S.8)

After applying the union bound over the $n \choose 2$ pairs, marginalizing over the sums of choosing a matching $M$ uniformly at random, and choosing the first edge $e$ uniformly at random from the $n \choose 2$ edges in a matching $M$, we compute the probability that the second edge $e'$ chosen uniformly at random from another random matching $M'$ has an intersecting vertex.

$$ \Pr \left[ \bigcup_{V \neq V'} E \right] \leq \frac{2(n)}{n} \Pr_{e \in M'} [e', E|e, M'] $$  \hfill (S.9)

$$ \leq \frac{2(n)}{n} (2n - 3) \Pr [e' \in M'] .$$  \hfill (S.10)

In the above equation, we note that the number of intersecting edges $e'$ with some fixed $e$ are $2n - 3$, and compute the probability of the intersecting edge as the fraction of matchings over $n - 2$ vertices to those over $n$ vertices (fixing
where we proceed by explicitly counting the number of matchings in $M_{\frac{n}{2}}$ and $M_{\frac{n}{2}}$, and setting $n = O(N^2/\epsilon)$ to be the choice for the number of random bits that the tallyman uses to draw $x$.

### G. Queue

We show that the adaptive strategy in the anonymous queuing algorithm requires $O(N)$ rounds, resulting in the consumption of $O(N^2 \log N)$ Bell pairs overall.

**Lemma S.3.**

\[
\Pr[\text{QUEUE terminates in } O(N) \text{ rounds}] = 1 - 1/c, \tag{S.16}
\]

for any constant $c$.

**Proof.** Let $p_i$ be the probability of successfully filling the $i$th spot in the queue. To succeed, out of the $N - i + 1$ unqueued voters, one voter attempts to enter the queue with probability $1/(N - i + 1)$ while the rest choose not to. Then, combinatorially,

\[
p_i = (N - i + 1)\left(\frac{1}{N - i + 1}\right)^1(1 - \frac{1}{N - i + 1})^{(N-i+1)-1} = (1 - \frac{1}{N - i + 1})^{N-i} \tag{S.17}
\]

\[
\xrightarrow{N \to \infty} 1/e. \tag{S.18}
\]

The total time $T = t_1 + \ldots + t_N$ is the sum of the times needed to fill the individual spots. Having a geometric distribution, the expected value is $\mathbb{E}(t_i) = 1/p_i$. By linearity,

\[
\mathbb{E}(T) = \sum_{i=1}^{N} \mathbb{E}(t_i) = \sum_{i=1}^{N} 1/p_i \tag{S.20}
\]

\[
= \sum_{i=1}^{N-1} \left(\frac{N - i + 1}{N - i}\right)^{N-i} + 1 \tag{S.21}
\]

\[
\xrightarrow{N \to \infty} N \cdot e. \tag{S.22}
\]

Markov’s inequality [40] bounds the probability of a long run time, for any $c > 0$:

\[
\Pr[T \geq ceN] \leq 1/c. \tag{S.24}
\]
H. Bit-Flip Errors

We demonstrate that the effect of independent bit-flip errors on the ballot state is independent of the amount of random bits \( n \) (Lemma S.4). The expected number of repetitions so that the voters can cast valid votes is \( \exp(N) \) with probability \( \geq 1 - \exp(-e^2n) \). However, note that the expected number of voters with corrupted parities when \( N \) ballot states \( |\psi_x\rangle \) are distributed is \( \leq (\frac{1}{2} + e')n \). If the margin of victory is greater by some constant amount, then it suffices to have far fewer rounds of repetition since the probability that the same voter has errors in their parity computation also decreases exponentially, and so every round will likely have different voters with errors. By running the election in \( O(N/\log N) \) communities of size \( O(\log N) \) each, we only incur \( 1/poly(N) \) probability of single-qubit Pauli X errors (in every community). Therefore, with \( poly(N) \) repetitions on expectation, an error-free run is conducted.

**Lemma S.4.** Given \( N \) copies of the state \( |\psi_x\rangle \) transferred under a bit-flip channel \( X_p \) independently for every qubit and some arbitrary \( e' > 0 \), the probability that \( N \) votes are cast is

\[
\Pr_p[\text{N votes}] \geq O(2^{-N}),
\]

provided that \( x \) is \( e' \)-balanced (has roughly equivalent 1s and 0s). The error channel is

\[
X_p: \rho \rightarrow (1-p)\rho + p(Xp\rho),
\]

for single-qubit density matrices \( \rho \).

**Proof.** Choose some arbitrarily small \( e' > 0 \). We define \( \#0(x) \) and \( \#1(x) \) as the number of zeroes and ones in a bit string \( x \). Notice that a Chernoff bound [40] implies that

\[
\#1(x) \in \left(\left\lfloor \frac{1}{2} - e'\right\rfloor n, \left\lceil \frac{1}{2} + e'\right\rceil n\right],
\]

\[
\#0(x) \in \left(\left\lfloor \frac{1}{2} - e'\right\rfloor n, \left\lceil \frac{1}{2} + e'\right\rceil n\right),
\]

with probability \( \geq 1 - \exp(-e^2n) \), which is the precise characterization of \( x \in \{0,1\}^n \) being \( e' \)-balanced.

Fix some edge \( (i,j) \) that a voter \( V_l \) chooses (with probability \( 2/\log_2(n) \)) and an \( e' \)-balanced \( x \). Note that, even under the presence of errors, the probability that the measurement is correct is if \( |i\rangle \rightarrow |i'\rangle \) and \( |j\rangle \rightarrow |j'\rangle \), such that

\[
\left( (x_P = x_i) \land (x_P = x_j) \right) \lor \left( (x_P = 1 \oplus x_i) \land (x_P = 1 \oplus x_j) \right) = 1.
\]

(S.27)

Note that the probability of the first event in Equation S.27 is simply:

\[
\left( \sum_{l=0}^{\log_2 n} \Pr[x_P(x_i) = x_l] \Pr[d_H(x_P, x_i) = t] \right) \left( \sum_{l=0}^{\log_2 n} \Pr[x_P = x_j] \Pr[d_H(x_P, x_j) = t] \right)
\]

\[
\geq 4 \left( \frac{1}{2} - e' \right)^2 \left( \frac{1}{2} - e' - \frac{1}{n} \right)^2,
\]

(S.28)

where \( d_H(x, y) \) denotes the Hamming distance between strings \( x \) and \( y \). We used the fact that the \( X_P \) channel acts on \( |i\rangle \) and \( |j\rangle \) independently and that \( x \) is drawn uniformly at random, since we are in the regime of the Chernoff bound set in Equation S.26. By symmetry,

\[
\left( \sum_{l=0}^{\log_2 n} \Pr[x_P = 1 \oplus x_i] \Pr[d_H(x_P, x_i) = t] \right) \left( \sum_{l=0}^{\log_2 n} \Pr[x_P = 1 \oplus x_j] \Pr[d_H(x_P, x_j) = t] \right)
\]

\[
\geq 4 \left( \frac{1}{2} - e' \right)^2 \left( \frac{1}{2} - e' - \frac{1}{n} \right)^2.
\]

(S.29)

The total probability that a single voter \( V_l \) measures the correct parity of an edge \( (i,j) \) chosen uniformly at random from \( M_2 \) given

\[
|\psi_x\rangle \xrightarrow{X_P^{\log_2 n}} |\psi'_x\rangle
\]

(S.30)
is

\[ \Pr[p_{ij} = p'_{ij}] \geq 8 \left( \frac{1}{2} - \epsilon' \right)^2 \left( \frac{1}{2} - \epsilon' - \frac{1}{n} \right)^2. \] (S.31)

Since there are \( N \) independent copies of \(|\psi_x\rangle\), the probability of valid votes being cast correctly by all voters is

\[ \Pr[\text{N valid votes}] \geq 2^{3N} \left( \frac{1}{2} - \epsilon' \right)^{2N} \left( \frac{1}{2} - \epsilon' - \frac{1}{n} \right)^{2N} \]
\[ = 2^{-N} (1 - 2\epsilon')^{2N} (1 - 2(\epsilon' + \frac{1}{n}))^{2N} \]
\[ = 2^{-N} (1 - 4\epsilon' + o(1))(1 - 4(\epsilon' + \frac{1}{n})N + o(1)) \]
\[ = 2^{-N} \cdot O(1), \] (S.32)

if we choose \( \epsilon' = \frac{1}{cN} \), for some arbitrarily large \( c > 0 \). \( \square \)

### S.II. SECURITY PROOFS

Verifiability is proven directly in the main text. Correctness follows from Lemma S.2. The proof of accountability relies on the unforgeability of the quantum ballot. This result is demonstrated for one copy of the state in Lemma S.5, and then we generalize the proof strategy for multiple copies in Theorem S.1. Then, Theorem S.2 shows the suppression of adversarial votes via repetition, completing the proof of accountability. To prove anonymity, we show that the protocol does not leak information about the voters, so that guessing a particular identity cannot be done better than random (Theorem S.3).

#### A. Unforgeability

**Lemma S.5.** Given one copy of \(|\psi_x\rangle\), a computationally unbounded adversary \( \text{Adv} \) can output more than one parity \( p \) of an edge \((i, j)\) with probability

\[ \Pr[\text{Adv} \rightarrow (p, (i, j))] \leq 1/2. \] (S.33)

**Proof.** We bound the success probability of outputting the parity of a subset \( t \) bits and show that it becomes 1/2 for \( t > 2 \). That is, the adversary cannot forge a vote from one copy of the state better than a coin flip. A similar approach bounds the quantum communication complexity of the problem [16].

Here, the tallyman has \( x \in \{0, 1\}^n \), which is encoded in \( \rho(x) = |\psi_x\rangle \langle \psi_x| \). The adversary outputs a subset \( G \) of \( [n] \) of size \( t \) and the parity of the bits in that subset. Without loss of generality, the adversary’s protocol consists of a measurement \( \Pi_G \) to output a subset, followed by a measurement of its parity (Fact S.1). \( G_+ \) (or \( G_- \)) denote the subsets of \( \{0, 1\}^n \) where that parity is + (or −). The probability of distinguishing between two states \( \rho_+, \rho_- \) that occur with respective probabilities \( p_+, p_- \) is at most, under a positive-operator valued measure (POVM),

\[ \frac{1}{2} + \frac{1}{2} ||p_+\rho_+ - p_-\rho_-||_{tr}, \] (S.34)

where the deviation from 1/2 is denoted \( \epsilon_{bias} \). Here, \( ||\cdot||_{tr} \) denotes the trace norm. In our case,

\[ p_+ = p_- = \frac{1}{2^n} \langle \rho, \Pi_G \rangle, \] (S.35)

\[ \rho_+ = \frac{1}{\langle \rho, \Pi_G \rangle} \sum_{x \in G_+} \sqrt{\Pi_G} \rho(x) \sqrt{\Pi_G}, \] (S.36)

\[ \rho_- = \frac{1}{\langle \rho, \Pi_G \rangle} \sum_{x \in G_-} \sqrt{\Pi_G} \rho(x) \sqrt{\Pi_G}. \] (S.37)
The bias conditioned on projecting onto $G$ is

$$
e_{\text{bias}|G} \leq \frac{1}{2} \frac{1}{2^n} \cdot \sum_{x \in G^+} \sqrt{\Pi_G} \frac{\rho(x)}{\langle \rho, \Pi_G \rangle} \sqrt{\Pi_G} = \frac{1}{2} \frac{1}{2^n} \cdot \sum_{x \in G^+} \sqrt{\Pi_G} \frac{\rho(x)}{\langle \rho, \Pi_G \rangle} \sqrt{\Pi_G} \|_\text{tr},$$

(S.38)

$$= \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} \rho(x|G) - \sum_{x \in G^-} \rho(x|G) \|_\text{tr},$$

(S.39)

where $x|G$ is the projection of $x$ onto the $t$ elements of $G$. The overall bias is

$$e_{\text{bias}} = \sum_G \langle \rho, \Pi_G \rangle \cdot e_{\text{bias}|G}.$$  

(S.40)

To cancel most of the terms for each $x$, pair elements in $G_+$ and $G_-$ that have the minimum Hamming distance of one. Here, $\delta_{ij}$ is the Kronecker delta function.

$$e_{\text{bias}|G} \leq \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} [\rho(x|G) - \rho(x|G \oplus 0^{t-1}1)] \|_\text{tr}$$

(S.41)

$$= \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} \frac{1}{t} \sum_{i,j=1}^{t-1} (-1)^{x_i+x_j} |i\rangle \langle j| - \frac{1}{t} \sum_{i,j=1}^{t-1} (-1)^{x_i+x_j+\delta_{ij}+\delta_{ij}} |i\rangle \langle j| \|_\text{tr}$$

(S.42)

$$= \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} \frac{1}{t} \sum_{i=1}^{t-1} (-1)^{x_i} |i\rangle \langle i| - \sum_{i=1}^{t-1} (-1)^{x_i+\delta_{ij}+\delta_{ij}} |i\rangle \langle j| \|_\text{tr}$$

(S.43)

$$= \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} \frac{2}{t} \sum_{i=1}^{t-1} (-1)^{x_i} |i\rangle \langle i| + \sum_{j=1}^{t-1} (-1)^{x_j} |j\rangle \langle j| \|_\text{tr}$$

(S.44)

$$= \frac{1}{2} \frac{1}{2^n} \| \sum_{x \in G^+} \frac{2}{t} \sum_{i=1}^{t-1} (-1)^{x_i+\delta_{ij}} |i\rangle \langle i| + |t\rangle \langle t| \|_\text{tr}.$$  

(S.45)

Before proceeding, note that $\rho(x|G) = \rho(\bar{x}|G)$, where $\bar{x}$ is the conjugate string with all bits flipped. For $t$ odd, the parity flips upon conjugation, so the bias is zero in this case. Thus, restrict to the case where $t$ is even. Now, invoke cancellations between elements of $G_+$, observing that even and odd-parity substrings of length $t - 2$ contribute equally, for $t > 2$:

$$e_{\text{bias}|G} \leq \frac{1}{2} \frac{1}{2^n} \frac{1}{t} \sum_{i=1}^{t-1} \sum_{x \in G^+} (-1)^{x_i+\delta_{ij}} \| \langle i| + |t\rangle \langle i| \|_\text{tr}$$

(S.46)

$$= \frac{1}{2} \frac{1}{2^n} \frac{1}{t} \begin{cases} \| \sum_{i=1}^{t-1} [2^{n-t}(2)](i| \langle i| + |t\rangle \langle i|) \|_\text{tr} & \text{if } t = 2 \\ \| \sum_{i=1}^{t-1} [2^{n-t}(2) \cdot 1 \cdot 2^{t-2} - 2 \cdot 1 \cdot 2^{t-2}])(i| \langle i| + |t\rangle \langle i|) \|_\text{tr} & \text{if } t > 2 \\ \| \sum_{i=1}^{t-1} [2^{n-t}(2)](i| \langle i| + |t\rangle \langle i|) \|_\text{tr} & \text{if } t = 2 \\ 0 & \text{if } t > 2 \end{cases}$$

(S.47)

$$= \begin{cases} \frac{1}{2} & \text{if } t = 2 \\ 0 & \text{if } t > 2 \end{cases}.$$  

(S.48)

Thus, the optimal strategy is to choose $t = 2$ for all $G$ in the POVM, resulting in overall bias

$$e_{\text{bias}} = \sum_G \langle \rho, \Pi_G \rangle \cdot e_{\text{bias}|G}$$

(S.50)

$$= \frac{1}{2} \sum_G \langle \rho, \Pi_G \rangle$$

(S.51)

$$= \frac{1}{2}.$$  

(S.52)
since \( \sum_G \Pi_G = 1 \) and \( \rho \) is a normalized density matrix. That is, only one vote (parity and edge) can be outputted deterministically. Otherwise, success is due to guessing:

\[
\Pr[\text{Adv} \rightarrow (p, (i,j))] = \frac{1}{2} + \epsilon_{\text{bias}} = \frac{1}{2}.
\] (S.53)

\[\square\]

**Fact S.1 ([16]).** For an arbitrary POVM \( \Pi = \{\Pi_{a,b}\} \) indexed by \( a \in \mathcal{A}, b \in \mathcal{B} \), one can construct POVMs

\[
P = \{P_a\}_{a \in \mathcal{A}} \quad \text{and} \quad Q_a = \{Q_{a,b}\}_{b \in \mathcal{B}} \quad \text{for} \ a \in \mathcal{A},
\] (S.54)
such that for any state \( \rho \) that \( \Pi \) acts on, applying \( P \), followed by \( Q_a \), where \( a \) is the outcome of \( P \), gives the same output distribution. In particular,

\[
P_a = \sum_{b \in \mathcal{B}} \Pi_{a,b} \quad \text{and} \quad Q_{a,b} = \sqrt{P_a^+ \Pi_{a,b} \sqrt{P_a^+}},
\] (S.55)

where \( P_a^+ \) is the Moore-Penrose pseudoinverse of \( P_a \).

In our case, \( \mathcal{A} = \{G \mid G \subset [n], |G| = t\} \) and \( \mathcal{B} = \{+, -\} \).

**Theorem S.1.** Given \( N \) copies of the ballot state \( |\psi_x\rangle \), a computationally unbounded adversary \( \text{Adv} \) can output a string of \( N' \geq N \) parities \( p_i \) of edges \( (i_j, j_i) \), with probability

\[
\Pr[\text{Adv} \rightarrow \{p_i, (i_j, j_i)\}_{i=1}^{N'-N}] \leq \frac{1}{2^{N'-N}}.
\] (S.56)

**Proof.** We now want to show that given \( N \) copies of the state \( |\psi_x\rangle \), the adversary cannot output successful forgeries, with probability more than coin flips. To do so, we generalize the calculation of the bias in Lemma S.5. Here, the projection is onto a subset \( G \) of \([n]\). Exploiting a data-processing inequality (Fact S.2), let \( G' \) be the union of the \( N \) subsets.

\[
\epsilon_{\text{bias}}|G \leq \frac{1}{2} \left[ \frac{1}{2^{N}} \sum_{x \in G'_+} \rho(x|G')^\otimes N - \frac{1}{2^{N}} \sum_{x \in G'_-} \rho(x|G')^\otimes N \right] \text{tr}
\] (S.57)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \sum_{x \in G'_+} \rho(x|G')^\otimes N - \rho(x|G' \oplus 0^{t-1}1)^\otimes N \right] \text{tr}
\] (S.58)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \sum_{x \in G'_+} \frac{2}{N} \sum_{\alpha} (\sum_{i_{\alpha} \neq \beta} (-1)^{\sum_{x_{\alpha} \neq x_{\beta}}+x_t} |i_{\alpha} \rangle \langle j_{\alpha}|) \right] \text{tr}
\] (S.59)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \frac{1}{N} \sum_{x \in G'_+} \sum_{\alpha} \sum_{i_{\alpha} \neq \beta} (-1)^{\sum_{x_{\alpha} \neq x_{\beta}}+x_t} |i_{\alpha} \rangle \langle j_{\alpha}|) \right] \text{tr}
\] (S.60)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \frac{1}{N} \sum_{x \in G'_+} \sum_{\beta} \sum_{i_{\beta} \neq \beta} (-1)^{\sum_{x_{\beta} \neq x_{\beta}}+x_t} |i_{\beta} \rangle \langle j_{\beta}|) \right] \text{tr}
\] (S.61)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \frac{1}{N} \sum_{x \in G'_+} \sum_{\beta} \sum_{i_{\beta} \neq \beta} |i_{\beta} \rangle \langle j_{\beta}|) \right] \text{tr}
\] (S.62)

\[
= \frac{1}{2} \left[ \frac{1}{2^{N}} \frac{1}{N} \sum_{x \in G'_+} \sum_{\beta} |i_{\beta} \rangle \langle j_{\beta}|) \right] \text{tr}.
\] (S.63)

In Equation S.59 is the difference for each \( x \in G'_+ \) flipping \( x_t \), due to complete interference of phases: \( |i_{\alpha} \rangle \langle j_{\alpha}| \) denote the tensor product of basis states, with an odd number of \( t \) substitutions; \( \alpha \) indexes all such tuples. Crucially, \( c \cdot x_t \mod 2 = x_t \) for any odd \( c \). The bit is flipped back in Equation S.60. Inducting the procedure over all bits \( x_t \) yields
Equation S.61: $|i_\alpha \rangle \langle j_\beta|$ is composed of a tensor product of an odd number of each basis vector; $\beta$ indexes all such tuples. For clarity,

\[
|i_\alpha \rangle \langle j_\alpha| \equiv |i_1\rangle \ldots |i_N\rangle \langle j_1\rangle \ldots \langle j_N| \quad \text{s.t.} \quad |\{i_\alpha = t\}| + |\{j_\alpha = t\}| = c \text{ for } c|2 = 1, \quad (S.64)
\]

\[
|i_\beta \rangle \langle j_\beta| \equiv |i_1\rangle \ldots |i_N\rangle \langle j_1\rangle \ldots \langle j_N| \quad \text{s.t.} \quad |\{i_\beta = i\}| + |\{j_\beta = i\}| = c_i \text{ for } c_i|2 = 1, \quad (S.65)
\]

\[
\forall i_\alpha, j_\alpha, l_0, l_1, i, \in \{1, \ldots, t\}. \quad (S.66)
\]

The constraint dictates which terms appear in Equation S.63, imposing

\[
\epsilon_{\text{bias}}|_G = 0 \quad \text{if} \quad N < t/2. \quad (S.67)
\]

In the voting scheme, a valid vote consists of an edge $(i, j)$ and the parity of its bits $p_{i,j}$. The overall bias is

\[
\epsilon_{\text{bias}} = \sum_G \langle \rho, \Pi_G \rangle \cdot \epsilon_{\text{bias}}|_G \leq \frac{1}{2}, \quad (S.68)
\]

achieved by the ideal protocol. By reduction (Lemma S.6), a maximum of $N$ votes can be outputted with unity success probability; outputting any more (i.e., a forgery) can only be done through guessing (Equation S.67).

\[\square\]

**Fact S.2 ([41]).** For any Hermitian operator $R$ and any subnormalized POVM $\{P_a\}_a$,

\[
\sum_a \|\sqrt{P_a} R \sqrt{P_a}\|_{\text{tr}} \leq \|R\|_{\text{tr}}. \quad (S.69)
\]

**Lemma S.6.** Evaluating the parity of $t$ bits reduces to the evaluation of $t/2$ pair-wise parities (assuming $t$ even).

**Proof.** Exploiting associativity of the parity function,

\[
\bigoplus_{i=1}^t x_i = \bigoplus_{j=1}^{t/2} (x_j \oplus x_{j+1}), \quad (S.70)
\]

where $x_i \mapsto x_j$ is any permutation of the bits. \[\square\]

### B. Security Amplification

**Theorem S.2.** $\forall \delta' > 0$, given $r = O(N)$ repetitions of $\Pi_{\text{vote}}$, $\Pr[N' > N \text{ votes are counted}] \leq \delta'$.

**Proof.** Generic ballots cast have $N' > N$ votes, where $N$ copies of $|\psi_x\rangle$ are distributed. The combined ballots look like

\[
\left\{(i_1, j_1), a_{i_1 j_1}\right\}, \ldots, \left\{(i_{N'}, j_{N'}), a_{i_{N'} j_{N'}}\right\} \subset [n]^2 \times \{0, 1\}. \quad (S.71)
\]

Theorem S.1 implies that the $N' - N$ adversarial votes align in the desired direction with probability $1/2$ each. To avoid collisions between voters and proceed with the protocol, $N' - N \leq O(N)$, by Lemma S.2. The following tail bound applies to the votes.

**Lemma S.7 (Additive Chernoff Bound [40]).** For $X = \sum_{i=1}^n X_i$, where $X_i$ are independent, identically distributed Bernoulli random variables, taking the value 1 with probability $p$ and 0 otherwise,

\[
\Pr[|X - E[X]| \geq \sqrt{n} \delta] \leq 2e^{-2\delta^2}. \quad (S.72)
\]

The influence of the adversarial votes is suppressed by averaging. All votes are counted in one round of the protocol, and the average over $r$ rounds is taken for both the 0 and 1 votes (Algorithm S.1). By linearity, consider the adversarial votes separately. Denote by $X$ the total number of adversarial 0 votes (or 1, by symmetry, since the adversary cannot bias the output):

\[
X = \sum_{i=1}^r \sum_{j=1}^{\eta_i} x_{ij}, \quad (S.73)
\]
where \( x_{ij} \) corresponds to round \( i \) and vote \( j \) in that round, out of \( \eta_i \) votes. By Equation S.72, the influence of the adversary is bounded by

\[
\Pr(|X - r\eta/2| \geq \sqrt{n}\eta \delta) \leq 2e^{-2c^2},
\]

where \( \eta = \max_i \eta_i \). Dividing by \( r \) to take the average over the rounds yields

\[
\Pr \left[ \frac{|\bar{X} - \eta/2|}{\sqrt{\frac{r}{\eta}}} \geq \frac{\sqrt{\frac{r}{\eta}}}{2}\delta \right] \leq 2e^{-2c^2},
\]

where \( \bar{X} = X/r \). To limit the absolute deviation to be less than 1 with high probability, set \( r = \eta = O(N) \), since a maximum of \( O(N) \) adversarial votes can be inputted into one round. More explicitly, for a deviation of \( c \), the number of rounds of averaging is \( r = \eta/c^2 \).

To compute the margin of the election, subtract the average number of 0 votes from the average number of 1 votes. Since the adversarial 0 or 1 votes are symmetric and both represented by \( X \), their average values cancel within an additive factor \( 2c = O(1) \). In majority rule, the election is decided by the sign of the margin. Note further, that in order to estimate the number of 0 and 1 votes, which is necessary to compute a transitive voting function \( f_{\text{vote}} \) (Definition S.1), this technique only works under the assumption that every voter casts a vote.

C. Anonymity

**Theorem S.3.** The voting protocol is \( \gamma \)-anonymous with fidelity \( \sqrt{1 - \gamma^2} \) of the GHZ states used in Algorithm S.2.

**Proof.** In the key generation (Algorithm S.3), the \( X \) basis measurements do not leak information about the key values \( i, j \). Specifically, \( X = U_{\text{QFT}}^\dagger ZU_{\text{QFT}} \), where \( Z \) is the computational basis and the quantum Fourier transform matrix is specified by

\[
(U_{\text{QFT}})_{ab} = (e^{2\pi i/N})^{ab}, \quad 0 \leq a, b \leq N - 1.
\]

\( X \) and \( Z \) are mutually unbiased bases, so they satisfy the following entropic uncertainty relation \([42, 43]\)

\[
H(X) + H(Z) \geq \log N.
\]

The Shannon entropy \( H(Z) \) is 0 after \( Z \) measurement, so \( H(X) = \log N \) is maximal; i.e., the measurement outcomes \( X \) are uniformly distributed and uninformative.

Eavesdroppers are detected via testing. The security analysis of the anonymous broadcast protocol applies. Specifically, Boykin [22] shows

\[
I(E; F(M)|A) \leq (4 + 4\sqrt{2})H(F(K))\sqrt{1 - F_{\text{min}}},
\]

where the mutual information between the eavesdroppers \( E \) and a function of the message \( F(M) \), conditioned on the announcement \( A \), is bounded by the entropy of the function on the key \( K \). Notably, if the fidelity of the test measurements is large enough for errors to be correctable by a code, the eavesdropper’s information goes to zero.

Since voters can detect eavesdroppers and abort, we restrict ourselves to the case where a key was successfully generated and show that the distributions of every single voter under announcements in the distributed sum or broadcast are the same. Let \( A \) represent the subset of malicious voters. The malicious voters \( A \) set a voting configuration \( \{v_1, \ldots, v_N\} \) for all the voters and choose a permutation over the honest voters \( \pi : B \to B \) to permute their votes. For simplicity, assume that the voting function is transitive (Definition S.1), i.e., it only depends on the number of 0 and 1 votes. This class includes majority vote via a decision on the margin. Also assume a binary alphabet, but note that more than two inputs can still be encoded in a bit string.

**Definition S.1 (Transitive Voting Function).** The voting function \( f_{\text{vote}} : \{0, 1\}^N \to \{0, 1\} \) is said to be transitive if

\[
f_{\text{vote}}(v_1, \ldots, v_N) = f_{\text{vote}}(\pi(v_1, \ldots, v_N)), \forall \pi \in S_N,
\]

where \( v_i \in \{0, 1\}, i \in [N] \equiv \{1, \ldots, N\} \) are the votes and \( S_N \) is the symmetric group on \( N \) letters.
Consider the two cases to be distinguished in the anonymity game.

1. \( b = 0 \):

   The voters vote in the configuration set by \( A \). Voter \( V_i \) announces \( A_i = k_i + c_i \mod N \) where \( k_i \in \{0, \ldots, N - 1\} \) and \( c_i \in \{0, \ldots, N - 1\} \). By injectivity, the probability distributions are the same:
   
   \[
   P(A_i) = P(k_i). \tag{S.80}
   \]

   The distribution of the announcements of the honest voters is then uniform:
   
   \[
   (A_1, \ldots, A_{|B|}) \xrightarrow{\mathcal{R}} \mathcal{U}\{0, N - 1\}^{|B|}. \tag{S.81}
   \]

2. \( b = 1 \):

   The votes are exchanged by the chosen permutation \( \pi \). Note that since \( f_{\text{vote}} \) is transitive, applying \( \pi \) on the honest voters does not change the outcome of the election and, therefore, the adversary gains nothing by observing the outcome of the election as \( f_{\text{vote}}(v_1, \ldots, v_N) = f_{\text{vote}}(\pi(v_1, \ldots, v_N)) \).

   Note that the permutation \( \pi : i \mapsto j \) does not change the distribution of the announcement \( A_i = k_i + c_j \mod N \):
   
   \[
   P(A_i) = P(k_i). \tag{S.82}
   \]

   The announcements are then sampled uniformly at random:
   
   \[
   (A_{\pi(1)}, \ldots, A_{\pi(|B|)}) \xrightarrow{\mathcal{R}} \mathcal{U}\{0, N - 1\}^{|B|}. \tag{S.83}
   \]

   The distributions of the announcements \( A_i, A_{\pi(i)} \) of every voter \( V_i \) are the same for every permutation \( \pi \). Given that \( \Pr[b = 0] = \Pr[b = 1] \), there is no bias between the two cases, by Equation S.34. Therefore, the adversary can do no better than flipping a coin.

   Further, note that every honest voter \( V_i \) chooses an edge \( e_i \) independently and uniformly at random from a random matching \( \mathcal{M}_2 \). Therefore, every honest voter’s announcement is independent and there is no adaptive strategy that \( A \) can use to predict some information of an upcoming announcement, assuming Equation S.78 and the distributional equivalence implied by the cases. This means that the total information gained by the adversary is simply the sum of the information gained from each announcement, which is shown to be 0 above because of indistinguishability (Equation S.81 and Equation S.83). This argument also shows where the hidden matching state \( |\psi_x\rangle \) assists in the protocol \( \Pi_{\text{vote}} \), as it allows the honest voters \( V_i \) the ability to cast their vote with a tag uncorrelated to that of other voters or one that is chosen by the tallyman.

   Lastly, even with complete knowledge of \( x \), the tallyman \( T \) cannot distinguish one announcement from the other since the tallyman does not know which edge corresponds to which voter (with guessing advantage more than \( \gamma \)). This is accomplished via the use of Algorithm S.2. To show this, we bound the loss of anonymity from multiple runs of Algorithm S.2. Specifically, given the required number of GHZ states, let the minimum starting fidelity of a Bell state be
   
   \[
   F_{\text{min}}^0 = 1 - \epsilon, \tag{S.84}
   \]

   for some \( \epsilon > 0 \). With this assumption, we can apply the BBPSSW purification protocol [44] for \( r \) rounds. This will increase the fidelity to,
   
   \[
   F_{\text{min}}^r = \frac{1}{p_{\text{suc}}} \cdot \left( F_{\text{min}}^{r-1} \right)^2 + \frac{(1 - F_{\text{min}}^{r-1})^2}{3}, \tag{S.85}
   \]

   where \( F_{\text{min}}^r \) is the minimum fidelity across all Bell states after \( r \) rounds of purification, and,

   \[
   p_{\text{suc}} = (F_{\text{min}}^{r-1})^2 + \frac{2}{3} F_{\text{min}}^{r-1} (1 - F_{\text{min}}^{r-1}) + \frac{5}{9} (1 - F_{\text{min}}^{r-1})^2, \tag{S.86}
   \]
where $p_{\text{suc}}$ denotes the probability that the purification succeeds. By a Taylor expansion of Equation S.85 using Equation S.84, we obtain that

$$F_\text{min}^1 = 1 - \frac{2}{3}e - \frac{2}{3}e^2 + O(e^3) \approx 1 - \frac{2}{3}e = \frac{1}{3}(1 + 2F_\text{min}^0).$$

(S.87)

After some algebra, this yields that,

$$F_\text{min}^r = \frac{1}{3^r}(c_r + 2^r \cdot F_\text{min}^0),$$

(S.88)

where $c_1 = 1$ and,

$$c_r = 3^{r-1} + 2 \cdot c_{r-1} = 3^r - 2^r.$$  

(S.89)

This allows us to further simplify Equation S.88 to its final form as,

$$F_\text{min}^r = \frac{1}{3^r}(3^r - 2^r(1 - F_\text{min}^0)) = 1 - \left(\frac{2}{3}\right)^r e.$$  

(S.90)

Algorithm S.2 consumes $O(N)$ GHZ states to anonymously transfer 1 qubit and is repeated $O(N \log N)$ times for every run of $\Pi_{\text{vote}}$, thereby using $O(N^2 \log N)$ GHZ states in total. Additionally, $\Pi_{\text{vote}}$ itself is repeated $O(N)$ times, bringing the total number of GHZ states that need to be purified to $O(N^3 \log N)$. Note that a GHZ state can be created from $N$ Bell states with no more than $O(N)$ loss of fidelity, assuming perfect gates in the circuit. Such a circuit implementation has been proposed [27] for the aforementioned GHZ state preparation. Therefore, by a union bound on Lemma S.1, we see that our total loss of anonymity is,

$$\text{anonymity leak} \leq e' \cdot O(N^4 \log N),$$

where $e'$ is the desired fidelity to have $\gamma$-anonymity. To ensure this, the anonymity leak is upper bounded by $\gamma$,

$$\text{anonymity leak} \leq e' \cdot O(N^4 \log N) = \gamma.$$  

(S.91)

Using Equation S.90 and the fact that the desired final fidelity by Equation S.91 is,

$$1 - \left(\frac{2}{3}\right)^r e = F_\text{min}^r = \sqrt{1 - \frac{\gamma^2}{N^8 \log^2 N}},$$

we have the following bound on the number of rounds $r$ required in the purification process,

$$r = \log\left(\frac{4e}{3\gamma^2} \cdot N^8 \log^2 N\right).$$  

(S.92)

Now, because the protocol is a recurrence protocol, the number of expected Bell states required to purify one GHZ state to the desired degree of fidelity is $O(2^r)$. This yields,

$$\# \text{ Bell states} = O\left(2^r \log\left(\frac{4e}{3\gamma^2} N^8 \log^2 N\right)\right) = O\left(\frac{4e}{3\gamma^2} N^8 \log^2 N\right).$$  

(S.93)

This can be repeated for all $O(N^3 \log N)$ GHZ states that will be used in the repetitions of $\Pi_{\text{vote}}$.  

\[\square\]

**S.III. EXTENSION**

Each of $t$ tallymen may send $s$ extra copies of the ballot state to each voter, introducing a tag consisting of random pairs of bits and their parities (Fig. S.1):

$$\left(((i_1, j_1), p_{i_1j_1}), \ldots, ((i_sj_s), p_{i_sj_s})\right) \subset [n]^2 \times \{0, 1\}.$$  

(S.94)
For the vote to be counted, the tag has to be correct. Extending Eq. S.56, the probability to pass authentication without access to the ballot states is exponentially small in the length of the tag:

$$\Pr[(N' - N) \text{ votes}] \leq \frac{1}{2^{\frac{1}{sN}'}} \frac{1}{2^{N' - N}}.$$  \hspace{1cm} (S.95)

This feature prevents attacks where unverified adversaries attempt to force the protocol to abort. With tags from multiple tallymen, it forces them to cooperate to verify votes. The resource consumption becomes $O(stN \log(sN))$, times a factor of $O((stN)^2)$ for anonymous transmission. For security, the number of copies in the tag is $s = O(\log N)$ (Theorem S.4). The number of tallymen $t$ can be some constant greater than one to compare outcomes.

**Theorem S.4.** \(\forall \delta' > 0, \) given \(s = O(\log N)\) copies of \(|\psi_x\rangle\), \(\Pr[N' > N \text{ votes are counted}] \leq \delta'.\)

**Proof.** Given \(s\) additional copies of the ballot state per voting ballot, the probability of casting adversarial votes is suppressed by a factor \(1/2^s\). Extending the reasoning of Theorem S.2, each voter has \(O(sN)\) such attempts, by Lemma S.2. To limit the influence on the election to one vote across all voters,

$$\frac{sN}{2^s} \leq \frac{1}{N}.$$  \hspace{1cm} (S.96)

Then, \(2^s/s \geq N^2\), so asymptotically, a tag of length \(s = O(\log N)\) suffices.

---

**FIG. S.1.** To amplify security, voters receive several copies of the ballot state from multiple tallymen. A vote is valid if the additional parities are correct. The outcomes from multiple tallymen are compared to suppress cheating.