DKMQ24 shell element with improved membrane behaviour

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Abstract

An improvement of membrane behaviour of the four-node shell element with 24 degrees of freedom DKMQ24 proposed by Katili et.al (2015) is presented. This improvement is based on the different approximation of drilling rotations based on Allman’s shape functions. This element, called DKMQ24+, was tested on eight standard benchmark problems for the case of linear elasticity. The element is free of shear and membrane locking and exhibits lower numerical error for the membrane and the membrane-dominated problems.

Keywords: DKMQ24, quadrangle shell element, 4-node finite element, warping effect, rotational degrees of freedom, drilling rotations

Nomenclature

Abbreviations

c.s. coordinate system

Latin symbols

$i$-th basis function at point $(r, s)$
penalty constants
Young’s modulus
gravitational acceleration
shear modulus
shell thickness
normal vector at node $i$
shear internal forces
vector of unknown rotations and displacements
element side in-plane normal vectors
internal moments
in-plane internal forces
element side normal vectors
coordinates in the element reference c.s.
element side directional vectors

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The transformation matrix from the global c.s. to the local c.s. at point \((r, s)\) is denoted as \(T(r, s)\). Displacements in the local c.s. are \(u_x, u_y, u_z\), and in the global c.s. they are \(u_X, u_Y, u_Z\). The local orthonormal basis at point \((r, s)\) is given by \(v_1(r, s), v_2(r, s), v_3(r, s)\). Coordinates in the local c.s. are \(x, y, z\), and in the global c.s. they are \(X, Y, Z\). Covariant basis vectors at point \((r, s)\) are \(X_r, X_s\). The Greek symbols include:

- \(\beta_x\): rotation of the \(z\)-axis towards the \(x\)-axis
- \(\beta_y\): rotation of the \(z\)-axis towards the \(y\)-axis
- \(\gamma_{xz}, \gamma_{yz}\): transversal shear strains
- \(\psi = \frac{1}{2} \left( \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} \right)\): in-plane rotation of the displacement field
- \(\varepsilon_x, \varepsilon_y, \gamma_{xy}\): membrane strains
- \(\varphi_x, \varphi_y, \varphi_z\): rotations in the local c.s.
- \(\varphi_X, \varphi_Y, \varphi_Z\): rotations in the global c.s.
- \(\kappa_x, \kappa_y, \kappa_{xy}\): bending strains

1. Introduction

Finite element modeling of shell structures is of great practical importance in engineering practice. Therefore there is a need for an efficient shell element, which would have an optimal convergence rate for different kinds of problems (membrane dominated, bending dominated and mixed shell problems), which would be at the same time free of shear locking, free of membrane locking, without zero energy modes and which would keep the optimal convergence rate also on distorted meshes. Especially, it is desired to find an optimal low-order shell element, which would provide the highest benefit-cost ratio when used in practical engineering applications. To find such an element is a tremendous task, not solved until today.

Let us restrict ourselves in this paper to 4-nodal quadrangle shell elements, based on the linear approximation of the transversal shear (the Reissner-Mindlin concept), which use both translational and rotational degrees of freedom in a node. Moreover, let us concentrate on the case of small deformations, however, an extension to the case of large deformations, keeping the small strain assumption, can be easily made in the corotational approach [1, 2, 3], in which no change of the element formulation is needed.

In order to construct a quadrangle shell element, it is possible to either consider a flat element geometry, by which however it is necessary to use the warp correction term [4] in the element formulation, or a warped element geometry. The formulations of flat quadrangles can be found in [5, 3, 6, 7, 8]. The formulations using a non-flat geometry include the QUAD4 element suggested by MacNeal [9], the famous MITC4 (Mixed Interpolation Tensorial Components) element proposed by Dvorkin and Bathe [10], which was further extended to the so called MITC4+ element [11, 12]. Another extension of the MITC4 element called MITC4S was introduced by Niemi [13]. Other recent formulations can be found in Gruttmann [14], Kim [15] or Katili [16]. The last mentioned citation introduces the DKMQ24 shell element, based
on the Naghdi/Reissner/Mindlin shell theory with six degrees of freedom per node. This shell element is an extension of the DKMQ (Discrete Kirchhoff-Mindlin Quadrangle) plate element suggested by the same author, which proved to have good numerical properties on a wide set of problems [17]. The DKMQ24 shell element has full coupling of membrane-bending energy, is free of shear locking, free of membrane locking and proved to converge on a wide set of standard benchmark problems [16]. It was successfully applied to orthotropic multilayered shells [18] and to a beam analysis [19].

The DKMQ24 shell element can be, due to its good numerical properties, considered as a promising candidate for the efficient low order quadrangle shell element. However, despite its numerical efficiency, it suffers from one drawback – the drilling rotations are independent of the remaining degrees of freedom. The stiffness matrix of the DKMQ24 shell element has a full rank due to stabilization, however, the drilling rotations do not enrich the displacement interpolation space while increasing computation costs at the same time. One possibility how to avoid this inefficiency would be to use five degrees of freedom per node only, which was proposed by Irpanni [20]. However, from a practical point of view it is of benefit to have six degrees of freedom in a node in order to easily connect more arbitrary oriented shell (or beam) elements to a single node.

The aim of our contribution is to modify the formulation of the DKMQ24 element in order to overcome this imperfection. Our proposal is to incorporate drilling rotations with the help of Allman’s shape functions [21] in the element formulation. This enriches the in-place displacement field with incomplete quadratic functions and improves the membrane behaviour of the element. The modified element formulation is named DKMQ24+. It was tested on eight well-established benchmark problems, which showed that it is free of shear locking, free of membrane locking, converges in all considered test cases, and provides the reduced numerical error in the membrane-dominated problems.

The practical implementation of the DKMQ24 and DKMQ24+ shell elements was done in the C# programming language using the Microsoft .NET Framework technology as a part of the finite element program femCalc [22].

In Section 2 we give a full and detailed derivation of the DMKQ24 shell element introduced in [16]. In Section 3 modifications of the DKMQ24 element, which construct the DKMQ24+ element, are introduced. In Section 4, a comment to practical implementation is given. In Section 5, convergence behaviour of the DKMQ24 and DKMQ24+ elements on pure membrane, pure bending and on mixed membrane-bending benchmark problems is tested and discussed. Closing remarks are given in Section 6.

2. Formulation of the DKMQ24 shell element

In this Section a detailed derivation of the DKMQ24 shell element, introduced in [16], is given. The reason for this decision is that in the reference [16] some parts of the derivation are not present.
2.1. Element characterization

The presented shell element is four-nodal with six unknowns at each node (three displacements $[u_X, u_Y, u_Z]$ and three rotations $[\varphi_X, \varphi_Y, \varphi_Z]$ in the global c.s.). The element is considered to be curved and considers full coupling between membrane and bending terms. Linear elasticity theory is assumed in the whole article.

2.2. Definition of rotations

Rotations $\varphi_x, \varphi_y, \varphi_z$ are defined using the standard definition, i.e. as rotations about axes $x, y, z$ using the right-hand-rule. Rotations $\varphi_x, \varphi_y$ can be equivalently defined as rotations of axis $z$ towards the axis $x$ or $y$ respectively, denoted by $\beta_x, \beta_y$. The relation between these definitions is the following (see Figure 1):

\begin{align*}
\beta_x &= \varphi_y \quad (1) \\
\beta_y &= -\varphi_x \quad (2)
\end{align*}

We use rotations $\beta_x, \beta_y$ during the derivation of the element, because it is more convenient. However, the final formulation of the DKMQ24 shell elements is given in the standard nodal rotations $\varphi_x, \varphi_y, \varphi_z$.

2.3. Coordinate system

The geometry of an isoparametric curvilinear shell element in the reference configuration is described by

$$X(r, s, t) = \sum_{i=1}^{4} a_i(r, s) \left[ X_i + t \frac{h}{2} n_i \right],$$

where $X_i = [X_i, Y_i, Z_i]$ is the location of the $i$-th node in the reference (undeformed) configuration, $h$ is a shell thickness and $n_i$ is a given normal vector at node $i$, see Figure 2. The shell thickness $h$ in Eq. (3) is assumed to be constant in derivation of the strain matrices (similar to the approach used in [23]). Actual thickness and its variation within the element is introduced into the constitutive relations only (Eq. (39, 40, 41)). This approximation is tested in the tapered cantilever benchmark problem in Section 4.8.
The basis functions are summarized in Table 1. At this point we can calculate a covariant vector basis with respect to all parameters

\[
X_{,r}(r,s,t) = \sum_{i=1}^{4} a_{i,r}(r,s) \left[ X_i + \frac{h}{2} n_i \right], \tag{4}
\]

\[
X_{,s}(r,s,t) = \sum_{i=1}^{4} a_{i,s}(r,s) \left[ X_i + \frac{h}{2} n_i \right], \tag{5}
\]

\[
X_{,t}(r,s,t) = \sum_{i=1}^{4} a_{i}(r,s) \frac{h}{2} n_i, \tag{6}
\]

where we have used notation \( \bullet_{,r} := \frac{\partial \bullet}{\partial r} \), \( \bullet_{,s} := \frac{\partial \bullet}{\partial s} \). These vectors are in general neither orthogonal nor orthonormal. Let us denote

\[
X_{r}(r,s) = X_{,r}(r,s,0) = \sum_{i=1}^{N} a_{i,r}(r,s) X_i, \tag{7}
\]

\[
X_{s}(r,s) = X_{,s}(r,s,0) = \sum_{i=1}^{N} a_{i,s}(r,s) X_i. \tag{8}
\]

Let us define a normal vector at any point \((r,s)\) by

\[
v_3(r,s) = \frac{X_{r}(r,s) \times X_{s}(r,s)}{||X_{r}(r,s) \times X_{s}(r,s)||}, \tag{9}
\]

with the help of which we define normal vectors at nodes \( n_i = v_3(r_i, s_i), \ i = 1, \ldots, 4 \). These vectors are used in Eq. \(3\). The remaining orthonormal vectors \( v_1(r,s) \) and \( v_2(r,s) \) can be obtained simply by

\[
v_1(r,s) = \frac{X_{r}(r,s)}{||X_{r}(r,s)||}, \tag{10}
\]

\[
v_2(r,s) = \frac{v_3(r,s) \times v_1(r,s)}{||v_3(r,s) \times v_1(r,s)||}, \tag{11}
\]

where the vectors \( v_1, v_2 \) and \( v_3 \) comprise the local coordinate system. If an orthotropic material of the shell is considered, the vector \( v_1 \) is usually chosen to be collinear with the direction of fibers.
2.4. Transformation between global and local coordinate systems

The local orthonormal coordinate system \( v_1(r,s), v_2(r,s), v_3(r,s) \) forms the transformation matrix \( T(r,s) \)

\[
T(r,s) = \begin{bmatrix} v_1^T(r,s) \\ v_2^T(r,s) \\ v_3^T(r,s) \end{bmatrix}, \quad T^{-1}(r,s) = T^T(r,s) = [v_1(r,s) \ v_2(r,s) \ v_3(r,s)].
\] (12)

The same transformation rules hold for coordinates, displacements and rotations

\[
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = T \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \quad \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} = T \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix}, \quad \begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix} = T \begin{bmatrix} \varphi_X \\ \varphi_Y \\ \varphi_Z \end{bmatrix},
\] (13)

where upper-case letters refer to the reference coordinate system and lower-case letters refer to the local coordinate system. The matrix which comprises the orthonormal basis is orthogonal, therefore inversion of above relations is trivial:

\[
\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = T^T \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \begin{bmatrix} u_X \\ u_Y \\ u_Z \end{bmatrix} = T^T \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix}, \quad \begin{bmatrix} \varphi_X \\ \varphi_Y \\ \varphi_Z \end{bmatrix} = T^T \begin{bmatrix} \varphi_x \\ \varphi_y \\ \varphi_z \end{bmatrix}.
\] (14)
2.5. Displacements

Let us denote the orthonormal basis vectors \( \mathbf{v}_1(r,s), \mathbf{v}_2(r,s), \mathbf{v}_3(r,s) \) evaluated at \( i \)-th node simply by \( \mathbf{v}_i^1, \mathbf{v}_i^2, \mathbf{v}_i^3 \). The displacement field of the DKMQ24 shell element is then defined by

\[
\begin{bmatrix}
  u_X \\
  u_Y \\
  u_Z 
\end{bmatrix}
(r,s,t) = \sum_{i=1}^{4} a_i(r,s) \begin{bmatrix}
  u_{i,X} \\
  u_{i,Y} \\
  u_{i,Z} 
\end{bmatrix} + \frac{h}{2} E_i \begin{bmatrix}
  \varphi_{i,X} \\
  \varphi_{i,Y} \\
  \varphi_{i,Z} 
\end{bmatrix} + \sum_{k=5}^{8} a_k(r,s) t^2 \triangle \beta_{t,k}, \ t \in [-1,1],
\]

where

\[
E_i = -\text{Spin}(\mathbf{v}_i^3) = \begin{bmatrix}
  0 & -v_{3,Y} & v_{3,Z} \\
  v_{3,Z} & 0 & -v_{3,X} \\
  -v_{3,Y} & v_{3,X} & 0 
\end{bmatrix}.
\]

We have used the definition of the spin matrix \( \text{Spin}(\mathbf{v}) \) taken from [1]. The supplementary quadratic basis functions \( a_5, a_6, a_7, a_8 \), which enrich the bending behaviour of the element, are summarized in Table 2.

The second term in equation (15) represents the displacement increment caused by nodal rotations. Let us derive this term for the \( i \)-th node:

\[
\begin{bmatrix}
  \triangle u_{i,X} \\
  \triangle u_{i,Y} \\
  \triangle u_{i,Z}
\end{bmatrix} = z \begin{bmatrix}
  \mathbf{v}_1^i & \mathbf{v}_2^i & \mathbf{v}_3^i
\end{bmatrix} \begin{bmatrix}
  \beta_{i,x} \\
  \beta_{i,y}
\end{bmatrix} = z \begin{bmatrix}
  -\mathbf{v}_2^i & \mathbf{v}_1^i & 0
\end{bmatrix} \begin{bmatrix}
  \varphi_{i,x} \\
  \varphi_{i,y} \\
  \varphi_{i,z}
\end{bmatrix} = z \begin{bmatrix}
  -v_{3,Z} & v_{3,Y} & 0 \\
  v_{3,Z} & 0 & -v_{3,X} \\
  -v_{3,Y} & v_{3,X} & 0
\end{bmatrix} \begin{bmatrix}
  \varphi_{i,X} \\
  \varphi_{i,Y} \\
  \varphi_{i,Z}
\end{bmatrix} = z \begin{bmatrix}
  0 & v_{3,Z} & 0 \\
  -v_{3,Z} & 0 & v_{3,X} \\
  v_{3,Y} & -v_{3,X} & 0
\end{bmatrix} \begin{bmatrix}
  \varphi_{i,X} \\
  \varphi_{i,Y} \\
  \varphi_{i,Z}
\end{bmatrix}, \ z = \frac{h}{2}, \ t \in [-1,1], \ i = 1,2,3,4.
\]

\( \text{Table 2: The supplementary basis functions for midside rotations } \triangle \beta_{t,k} \text{ on side } k = 5, \ldots, 8. \)
Figure 3: Approximation of rotations $\beta_t, \beta_m$ on the $k$-th element edge.

where we have used the definition of the vector multiplication $v_3 = v_1 \times v_2$. The last term in equation (15) contains supplementary degrees of freedom $\Delta \beta_{t,k}$ (see Figure 3). The supplementary degrees of freedom enrich the rotation approximation in the tangential $t$-direction, where the quadratic approximation is considered. In the perpendicular $m$-direction to side $k$ only linear approximation of rotation $\beta_m$ is used:

$$\beta_t(t) = \left(1 - \frac{t}{L_k}\right) \beta_{t,i} + \frac{t}{L_k} \beta_{t,j} + 4 \frac{t}{L_k} \left(1 - \frac{t}{L_k}\right) \Delta \beta_{t,k},$$

$$\beta_m(t) = \left(1 - \frac{t}{L_k}\right) \beta_{m,i} + \frac{t}{L_k} \beta_{m,j}. \hspace{1cm} (18)$$

The supplementary degrees of freedom $\Delta \beta_{t,k}$ will be eliminated later on the element level, so that the only element unknowns remain three rotations and three displacements at each node (see Figure 4).
2.6. Jacobi matrix

The derivative of the nodal location in the global coordinate system \([3]\) with respect to parameters \(r, s, t\) yields

\[
J = \frac{\partial (X, Y, Z)}{\partial (r, s, t)} = [X_r, X_s, m] + t [m_r + d_r, m_s + d_s, 0],
\]

where

\[
X_r = \sum_{i=1}^{4} a_{i,r}(r, s) X_i,
\]

\[
X_s = \sum_{i=1}^{4} a_{i,s}(r, s) X_i,
\]

\[
m := \frac{h}{2} \sum_{i=1}^{4} a_i(r, s) n_i,
\]

\[
m_r := \frac{h}{2} \sum_{i=1}^{4} a_{i,r}(r, s) n_i,
\]

\[
m_s := \frac{h}{2} \sum_{i=1}^{4} a_{i,s}(r, s) n_i.
\]

The Jacobi matrix needed for the calculation of the membrane stiffness term yields

\[
J := \lim_{t \to 0} \frac{\partial (r, s, t)}{\partial (X, Y, Z)} = \lim_{t \to 0} J^{-1} = \left[ \lim_{t \to 0} J \right]^{-1} = [X_r, X_s, m]^{-1} = \begin{bmatrix}
(X_r)^T \\
(X_s)^T \\
m^T/||m||^2
\end{bmatrix}.
\]

For the calculation of the bending stiffness term the following term is also needed: \(\lim_{t \to 0} \frac{d}{dt} \frac{\partial (r, s, t)}{\partial (X, Y, Z)}\). By recalling the matrix identity

\[
\frac{dA^{-1}}{dt} = -A^{-1} \frac{dA}{dt} A^{-1},
\]

which follows from the derivation of the identity matrix \(I = AA^{-1}\), we get

\[
J' := \lim_{t \to 0} \frac{d}{dt} \frac{\partial (r, s, t)}{\partial (X, Y, Z)} = \lim_{t \to 0} \frac{dJ^{-1}}{dt} = \lim_{t \to 0} \left[ -J^{-1} \frac{dJ}{dt} J^{-1} \right] = -J [m_r, m_s, 0] J,
\]

where upper indices denote dual vectors\(^1\). Rows of the \(J'\) matrix are explicitly denoted by vectors \(o_r^T, o_s^T, o_t^T\):

\[
J' = \begin{bmatrix} o_r^T \\ o_s^T \\ o_t^T \end{bmatrix}.
\]

\(^1\)For the set of linearly independent vectors \(u_1, \ldots, u_N\) one can uniquely define a set of dual vectors \(u_1^T, \ldots, u_N^T\), which satisfy \(u_i^T u_j = \delta_{ij}\), \(i, j = 1, \ldots, N\), where \(\delta_{ij}\) is the Kronecker delta symbol.
2.7. Displacement derivatives

The derivative of the displacement field \([15]\) with respect to parameters \(r, s, t\) yields

\[
\frac{\partial (u_X, u_Y, u_Z)}{\partial (r, s, t)} = [u_r, u_s, \Phi + \beta] + t [\Phi_r + \lambda_r + \beta_r + \xi_r, \Phi_s + \lambda_s + \beta_s + \xi_s, 0],
\]

(25)

where

\[
\begin{align*}
    u_r &:= \frac{h}{2} \sum_{i=1}^{4} a_i(r, s) E_i \begin{bmatrix} \varphi_{i,X} \\ \varphi_{i,Y} \\ \varphi_{i,Z} \end{bmatrix}, \\
    u_s &:= \frac{h}{2} \sum_{i=1}^{4} a_i(r, s) E_i \begin{bmatrix} u_{i,X} \\ u_{i,Y} \\ u_{i,Z} \end{bmatrix}, \\
    \Phi &:= \frac{h}{2} \sum_{i=1}^{4} a_i(r, s) E_i \begin{bmatrix} \varphi_{i,X} \\ \varphi_{i,Y} \\ \varphi_{i,Z} \end{bmatrix}, \\
    \Phi_r &:= \frac{h}{2} \sum_{i=1}^{4} a_i(r, s) E_i \begin{bmatrix} \varphi_{i,X} \\ \varphi_{i,Y} \\ \varphi_{i,Z} \end{bmatrix}, \\
    \Phi_s &:= \frac{h}{2} \sum_{i=1}^{4} a_i(s, r) E_i \begin{bmatrix} \varphi_{i,X} \\ \varphi_{i,Y} \\ \varphi_{i,Z} \end{bmatrix}, \\
    \beta &:= \frac{h}{2} \sum_{k=5}^{8} a_k(r, s) \Delta \beta_{t,k} t_k, \\
    \beta_r &:= \frac{h}{2} \sum_{k=5}^{8} a_{k,r}(r, s) \Delta \beta_{t,k} t_k, \\
    \beta_s &:= \frac{h}{2} \sum_{k=5}^{8} a_{k,s}(r, s) \Delta \beta_{t,k} t_k.
\end{align*}
\]

2.8. Membrane and bending strain matrices

The vector of element unknowns \(q\) of size \(24 \times 1\) is composed in the following way

\[
q = \begin{bmatrix} \varphi \\ u \end{bmatrix} = \begin{bmatrix} \varphi_{1,X} \\ \vdots \\ \varphi_{1,Y} \\ \vdots \\ \varphi_{1,Z} \\ \vdots \\ u_{1,X} \\ \vdots \\ u_{1,Y} \\ \vdots \\ u_{1,Z} \end{bmatrix}.
\]

(26)
The membrane and bending strain matrices can be calculated in the local coordinate system using the strain definition

\[ \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \lim_{t \to 0} \begin{bmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \\ \frac{\partial u_y}{\partial y} \end{bmatrix}, \]

\[ \kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = \lim_{t \to 0} \begin{bmatrix} \frac{d}{dz} \begin{bmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} \end{bmatrix} \end{bmatrix}, \]

\[ z = \frac{h}{2}. \]

Using the chain rule we get

\[ \frac{\partial u_x}{\partial x} = \frac{\partial u_x}{\partial (u_X, u_Y, u_Z)} \frac{\partial (u_X, u_Y, u_Z)}{\partial (r, s, t)} \frac{\partial (r, s, t)}{\partial (X, Y, Z)} \frac{\partial (X, Y, Z)}{\partial \frac{\partial u_x}{\partial x}}, \]

\[ \frac{\partial u_x}{\partial y} = \frac{\partial u_x}{\partial (u_X, u_Y, u_Z)} \frac{\partial (u_X, u_Y, u_Z)}{\partial (r, s, t)} \frac{\partial (r, s, t)}{\partial (X, Y, Z)} \frac{\partial (X, Y, Z)}{\partial \frac{\partial u_x}{\partial y}}, \]

and analogously for the remaining derivatives \( \frac{\partial u_x}{\partial y} \) and \( \frac{\partial u_x}{\partial y} \). The membrane strain \( \varepsilon \) yields

\[ \varepsilon = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = B_{m,3x24} q, \]

where

\[ B_{m,3x24} = \begin{bmatrix} E_{1,1} \\ E_{2,2} \\ E_{1,2} + E_{2,1} \end{bmatrix}, \]

\[ E_{I,J} = \begin{bmatrix} v_I^T u_r, & v_I^T u_s, & v_I^T (\Phi + \beta) \end{bmatrix} \begin{bmatrix} (X^r)^T v_J \\ (X^s)^T v_J \\ 0 \end{bmatrix} = [0^T, 0^T, 0^T], \]

\[ v_{I,X} \left[ \left( (X^r)^T v_J \right) a_{j,r} + \left( (X^s)^T v_J \right) a_{j,s} \right], \]

\[ J = 1,2,3,4 \]

where the columns are indexed by index \( j \). The bending strain \( \kappa \) yields

\[ \kappa = \begin{bmatrix} \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{bmatrix} = B_{b,3x24} q + B_{b,3x4} \Delta \beta, \]
where
\[
B_{b,\varphi,3\times24} = \begin{bmatrix}
  K_{1,1} \\
  K_{2,2} \\
  K_{1,2} + K_{2,1}
\end{bmatrix},
\]

\[
K_{I,J} = \frac{2}{h} \begin{bmatrix} v_T^T u_r, v_T^T u_s, v_T^T \Phi \end{bmatrix}
\begin{bmatrix} (o_r)^T v_J \\
(o_s)^T v_J \\
(o_t)^T v_J \end{bmatrix}
\]

\[= 0 \text{ for planar elements}\]

\[
2 \frac{h}{h} \begin{bmatrix} v_T^T \Phi_r, v_T^T \Phi_s, 0 \end{bmatrix}
\begin{bmatrix} (X^r)_T v_J \\
(X^s)_T v_J \\
(X^t)_T v_J \\
m^T v_J/||m||^2 \end{bmatrix}
\]

\[
= 0 \text{ for planar elements}\]

\[
\begin{bmatrix} (v_I)^T E_{j,1} \left[ \left( (X^r)^T v_J \right) a_{j,r} + \left( (X^s)^T v_J \right) a_{j,s} + \left( (o_t)^T v_J \right) a_j \right] \\
(v_I)^T E_{j,2} \left[ \sim \right] \\
(v_I)^T E_{j,3} \left[ \sim \right] \\
\end{bmatrix}
\]

\[
v_{I,X} \begin{bmatrix} 2 \left( (o_r)^T v_J \right) a_{j,r} + 2 \left( (o_s)^T v_J \right) a_{j,s} \end{bmatrix},
\]

\[
v_{I,Y} \begin{bmatrix} \sim \end{bmatrix},
\]

\[
v_{I,Z} \begin{bmatrix} \sim \end{bmatrix}
\]

\[I, J = 1, 2, 3, 4, \]

\[j = 1, 2, 3, 4\]

\[k = 5, 6, 7, 8\]

\[\]
where the vector of unknowns $q$ is given in formula (26). The stiffness matrix of an element $e$ is given by

$$K_e = \int_A \left[ B^T DB + B^T_s D_s B_s \right] \, dA + K_{stab},$$

(37)

where

$$B = \begin{bmatrix} B_b \\ B_m \end{bmatrix}, \quad D = \begin{bmatrix} D_b & 0 \\ 0 & D_m \end{bmatrix},$$

(38)

and $A$ is an area of the element $e$, $B_b$ is the bending strain matrix defined in equation (33), $B_m$ is the membrane strain matrix defined in equation (31) and $K_{stab}$ is a matrix stabilizing drilling rotations specified later in Eq. (67). The elasticity matrices in equation (38) equal in case of single-layered shells made of isotropic material to

$$D_b = \frac{h^3(r,s)}{12} \begin{bmatrix} E & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix},$$

(39)

$$D_m = h(r,s) \begin{bmatrix} E & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix},$$

(40)

$$D_s = \frac{5}{6} h(r,s) \begin{bmatrix} G & 0 \\ 0 & G \end{bmatrix}.$$  

(41)

Actual thickness and its variation within the element is accounted for in the calculation of $D_b$, $D_m$, $D_s$ matrices. The strain resultants evaluated in the Gauss quadrature points are

$$\begin{bmatrix} \kappa \\ \varepsilon \end{bmatrix} = Bq, \quad \gamma = \begin{bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{bmatrix} = B_s q,$$

(42)

and analogously the stress resultants evaluated in the Gauss quadrature points are

$$\begin{bmatrix} m \\ n \end{bmatrix} = DBq, \quad q = \begin{bmatrix} q_x \\ q_y \end{bmatrix} = D_s B_s q.$$  

(43)
The stress resultants \( \mathbf{m}, \mathbf{n} \) are defined in the local coordinate system defined by vectors \( \mathbf{v}_1, \mathbf{v}_2 \). If it is desired to get the stress resultants in other coordinate system \( \mathbf{v}_1', \mathbf{v}_2' \), we use the transformation

\[
\begin{bmatrix}
  m_x' \\
  m_y' \\
  m_{x'y'} \\
  n_x' \\
  n_y' \\
  n_{x'y'}
\end{bmatrix}
= \begin{bmatrix}
  \mathbf{T} & \mathbf{I} & \mathbf{T} \\
  \mathbf{I} & \mathbf{T} & \mathbf{I} \\
  \mathbf{I} & \mathbf{T} & \mathbf{I}
\end{bmatrix}
\begin{bmatrix}
  m_x \\
  m_y \\
  m_{xy} \\
  n_x \\
  n_y \\
  n_{xy}
\end{bmatrix},
\]

where \( T_{ij} = (\mathbf{v}_i')^T \mathbf{v}_j, \ i = 1, 2 \). The bending strain matrix \( \mathbf{B}_b \) and the shear strain matrix \( \mathbf{B}_s \) are derived to depend also on supplementary degrees of freedom \( \Delta \beta_{t,k} \). These supplementary degrees of freedom are then eliminated with the help of the matrix \( \mathbf{A}_{n,4\times24} \), which relates them to nodal degrees of freedom as follows:

\[
\Delta \beta = \mathbf{A}_{n,4\times24} \mathbf{q}.
\] (44)

The assembly of the bending stiffness matrix is then given by

\[
\kappa = \mathbf{B}_{b,3\times24} \mathbf{q},
\] (45)

where

\[
\mathbf{B}_{b,3\times24} = \mathbf{B}_{b,\phi,3\times24} + \mathbf{B}_{b,\Delta \beta,3\times4} \mathbf{A}_{n,4\times24}.
\] (46)

The shear stiffness matrix takes the form

\[
\mathbf{B}_{s,2\times24} = \mathbf{B}_{s,\Delta \beta,2\times4} \mathbf{A}_{n,4\times24}.
\] (47)

2.10. Evaluation of shear matrix

Matrix \( \mathbf{B}_{s,\Delta \beta,2\times4} \), which relates the transversal shear strains to the supplementary rotations \( \Delta \beta_{t,k} \), is identical to the same matrix used in the DKMQ plate element [17]. We recap the derivation for completeness.

Let us evaluate an average transversal shear strain on each side. Taking an arbitrary side \( k \), the directional parameter is denoted by \( t \). The transversal shear strain in direction \( t \) has the form

\[
\tau_{tz} = \frac{q_t}{D_s},
\] (48)

where \( q_t \) is a shear internal force and \( D_s \) is the shear stiffness, which in case of single-layered isotropic shells equals to \( \frac{G}{2}Gh \), where \( G \) is the shear modulus and \( h \) is an average shell thickness on side \( k \). The stress equilibrium yields

\[
q_t = m_{t,t} + m_{t,n,n},
\] (49)
and Hooke’s law has the form
\[
\begin{bmatrix}
  m_t \\
  m_n \\
  m_{tn}
\end{bmatrix} =
\begin{bmatrix}
  D_b & * & * \\
  * & & * \\
  * & * & *
\end{bmatrix}
\begin{bmatrix}
  \beta_{t,t} \\
  \beta_{n,n} \\
  \beta_{t,n} + \beta_{n,t}
\end{bmatrix},
\] (50)

where \( D_b \) is an average bending stiffness on the side \( k \), which in case of single-layered isotropic shell equals to \( \frac{Eh^3}{12(1-\nu^2)} \), where \( E \) is Young’s modulus and \( h \) the shell thickness. We now consider a linear approximation of angle \( \beta \) in normal \( m \)-direction and quadratic in tangential \( t \)-direction according to Eq. \( (19) \). Both approximations do not depend on the \( m \)-direction, which immediately yields
\[
\gamma_{tz} = D_b \beta_{t,tt} = -\frac{2}{3} \Phi_k \Delta \beta_{t,k}, \quad \Phi_k = \frac{12}{L_k^2} D_b.
\] (51)

We see that the transversal shear approximation \( \gamma_{tz} \) is constant on the side. If we would like to transform these shear strains back to the reference plane, we have to incorporate the determinant of the transformation and also the parametrization sign change, see Figure 5:
\[
\begin{align}
\gamma_{xz,5} &= (\det J)_5 \gamma_{tz,5} = + \frac{L_5}{2} \left( -\frac{2}{3} \Phi_5 \Delta \beta_{t,5} \right), \\
\gamma_{xz,6} &= (\det J)_6 \gamma_{tz,6} = + \frac{L_6}{2} \left( -\frac{2}{3} \Phi_6 \Delta \beta_{t,6} \right), \\
\gamma_{xz,7} &= (\det J)_7 \gamma_{tz,7} = - \frac{L_7}{2} \left( -\frac{2}{3} \Phi_7 \Delta \beta_{t,7} \right), \\
\gamma_{xz,8} &= (\det J)_8 \gamma_{tz,8} = - \frac{L_8}{2} \left( -\frac{2}{3} \Phi_8 \Delta \beta_{t,8} \right).
\end{align}
\] (52-55)

As in the MITC4 element \([10]\) transverse shear strains are approximated with
\[
\begin{align}
\gamma_{rz} &= \frac{1}{2} (1-s) \gamma_{tz,5} + \frac{1}{2} (1+s) \gamma_{tz,7}, \\
\gamma_{sz} &= \frac{1}{2} (1-r) \gamma_{tz,6} + \frac{1}{2} (1+r) \gamma_{tz,8},
\end{align}
\] (56-57)

where
\[
\begin{bmatrix}
  \gamma_{xz} \\
  \gamma_{yz}
\end{bmatrix} = J^{-1} \begin{bmatrix}
  \gamma_{rz} \\
  \gamma_{sz}
\end{bmatrix},
\] (58)

so we finally get
\[
\begin{bmatrix}
  \gamma_{xz} \\
  \gamma_{yz}
\end{bmatrix} = B_{s,\Delta \beta,2\times4} \begin{bmatrix}
  \Delta \beta_{t,5} \\
  \Delta \beta_{t,6} \\
  \Delta \beta_{t,7} \\
  \Delta \beta_{t,8}
\end{bmatrix},
\] (59)

\[
B_{s,\Delta \beta,2\times4} = \frac{1}{6} \begin{bmatrix}
- J_{11}^{-1}(1-s)L_5 \Phi_5 & - J_{12}^{-1}(1+r)L_6 \Phi_6 & J_{11}^{-1}(1+s)L_7 \Phi_7 & J_{12}^{-1}(1-r)L_8 \Phi_8 \\
- J_{21}^{-1}(1-s)L_5 \Phi_5 & - J_{22}^{-1}(1+r)L_6 \Phi_6 & J_{21}^{-1}(1+s)L_7 \Phi_7 & J_{22}^{-1}(1-r)L_8 \Phi_8
\end{bmatrix}.
\]
2.11. Evaluation of matrix $A_n$

We use the Hu-Washizu functional according to which

$$\int_{0}^{L_k} (\gamma_{t_z} - \bar{\gamma}_{t_z}) \, dt = 0,$$

where $\gamma_{t_z}$ is a shear approximated from displacement and $\bar{\gamma}_{t_z}$ is an average shear approximated by Eq. [51]. We have

$$\gamma_{t_z} = \frac{\partial u_z}{\partial t} + \beta_t =$$
$$u_{z,j} - u_{z,i} + \left(1 - \frac{t}{L_k}\right) \beta_{t,i} + \frac{t}{L_k} \beta_{t,j} + 4 \frac{t}{L_k} \left(1 - \frac{t}{L_k}\right) \triangle \beta_{t,k},$$

(61)

$$\bar{\gamma}_{t_z} = -\frac{2}{3} \Phi_k \triangle \beta_{t,k},$$

(62)

$$0 = \int_{0}^{L_k} (\gamma_{t_z} - \bar{\gamma}_{t_z}) \, dt = u_{z,j} - u_{z,i} + \frac{L_k}{2} \left(\beta_{t,i} + \beta_{t,j}\right) + \frac{2}{3} L_k (1 + \Phi_k) \triangle \beta_{t,k}.$$  

By rewriting the scalar unknowns to vector ones we get

$$n_k^T [u_i - u_j] - \frac{L_k}{2} t_k^T (\varphi_i + \varphi_j) = \frac{2}{3} L_k (1 + \Phi_k) \triangle \beta_{t,k}.$$  

(63)
In the matrix form the equation (63) takes the form

\[
\begin{bmatrix}
A^I_w & A^{II}_w & A^{III}_w & A^{IV}_w & A^V_w & A^VI_w
\end{bmatrix}
\begin{bmatrix}
q
\end{bmatrix} =
\begin{bmatrix}
\frac{2}{3}L_5(1+\Phi_5) & \frac{2}{3}L_6(1+\Phi_6) & \frac{2}{3}L_7(1+\Phi_7) & \frac{2}{3}L_8(1+\Phi_8)
\end{bmatrix}
\begin{bmatrix}
\Delta \beta_{1,5} \\ \Delta \beta_{1,6} \\ \Delta \beta_{1,7} \\ \Delta \beta_{1,8}
\end{bmatrix},
\]

(64)

\[
d_k = -\frac{L_k}{2} l_k, \ k = 5, 6, 7, 8,
\]

\[
n_k = \frac{n_i + n_j}{2}, \ k = 5, 6, 7, 8,
\]

\[
A^I_w = \begin{bmatrix}
d_5,X & d_5,X & 0 & 0 \\
0 & d_6,X & d_6,X & 0 \\
0 & 0 & d_7,X & d_7,X \\
d_8,X & 0 & 0 & d_8,X
\end{bmatrix},
\]

\[
A^{II}_w = \begin{bmatrix}
d_5,Y & d_5,Y & 0 & 0 \\
0 & d_6,Y & d_6,Y & 0 \\
0 & 0 & d_7,Y & d_7,Y \\
d_8,Y & 0 & 0 & d_8,Y
\end{bmatrix},
\]

\[
A^{III}_w = \begin{bmatrix}
d_5,Z & d_5,Z & 0 & 0 \\
0 & d_6,Z & d_6,Z & 0 \\
0 & 0 & d_7,Z & d_7,Z \\
d_8,Z & 0 & 0 & d_8,Z
\end{bmatrix},
\]

\[
A^{IV}_w = \begin{bmatrix}
n_5,X & -n_5,X & 0 & 0 \\
0 & n_6,X & -n_6,X & 0 \\
0 & 0 & n_7,X & -n_7,X \\
-n_8,X & 0 & 0 & n_8,X
\end{bmatrix},
\]

\[
A^V_w = \begin{bmatrix}
n_5,Y & -n_5,Y & 0 & 0 \\
0 & n_6,Y & -n_6,Y & 0 \\
0 & 0 & n_7,Y & -n_7,Y \\
-n_8,Y & 0 & 0 & n_8,Y
\end{bmatrix},
\]

\[
A^{VI}_w = \begin{bmatrix}
n_5,Z & -n_5,Z & 0 & 0 \\
0 & n_6,Z & -n_6,Z & 0 \\
0 & 0 & n_7,Z & -n_7,Z \\
-n_8,Z & 0 & 0 & n_8,Z
\end{bmatrix},
\]

where \( n_k \) is the normal vector corresponding to the side \( k \) starting with node \( i \) and ending with node \( j \).

Finally, the supplementary rotations \( \Delta \beta_{i,k} \) can be evaluated by the following equation

\[
\Delta \beta = A_{n,4 \times 24} q, \quad A_{n,4 \times 24} = A^{-1}_{\Delta \beta,4 \times 4} A_{w,4 \times 24}.
\]

(65)

2.12. Drilling rotations stabilization

The shell element must be stabilized with respect to drilling rotations \( \varphi_z \) in order to avoid the singular stiffness matrix, which appears if a node connects coplanar elements. We follow the stabilization procedure given in [10]. The added energy, which stabilizes drilling rotation has the form

\[
\Pi_{stab} = \frac{c_l}{2} \left[ G \int_A h \varphi_z^2(r,s) \, dA + \frac{E}{12} \int_A h^3 \left[ \varphi_{z,x}^2(r,s) + \varphi_{z,y}^2(r,s) \right] \, dA \right],
\]

(66)
where \( c_1 \) is the penalty constant. The value \( c_1 = 0.001 \) is considered in our implementation of the DKMQ24 element as well as in [16]. The corresponding stiffness matrix in the global coordinate system has the form:

\[
K_{\text{stab};8+i,8+j} = c_1 \int_{-1}^{1} \int_{-1}^{1} \left\{ G h \, a_i(r,s) a_j(r,s) + \frac{E h^3}{12} \left[ a_{i,x}(r,s) a_{j,x}(r,s) + a_{i,y}(r,s) a_{j,y}(r,s) \right] \right\} \left( n_i \right)^T n_j \det J \, dr ds, \quad i,j = 1, \ldots, 4.
\]

The first term is called the MacNeal stabilization and has rank 1, the second has rank 3. It can be proven that such a matrix removes all zero energy modes connected with drilling rotations [16].

2.13. Numerical integration

Elements are integrated by means of the standard 2 \( \times \) 2 Gauss quadrature except of the stabilization term (67), for which a reduced one-point integration rule is used.

3. Improvements to the DKMQ24 shell element

In this Section we introduce two modifications of the DKMQ24 shell element, derived in Section 2.

3.1. The treatment of drilling rotations

In the original derivation drilling rotations are stabilized according to formula (66), which causes the local element stiffness matrix to have a full rank even for the coplanar element configuration. However, this treatment keeps drilling rotations independent from the remaining degrees of freedom and as a consequence the element behaviour is not improved although we pay for these additional degrees of freedom by an increase in computational costs. Therefore, we use the approach proposed in [25], where the drilling rotations are considered as derivatives of the in-plane displacement field. The space of basis functions is enriched by incomplete quadratic polynomials in this case.

Let us start with calculation of a rotation around a normal vector \( \mathbf{n} \) at an arbitrary point \((r,s)\)

\[
\varphi_z(r,s) = \mathbf{n} \cdot \begin{bmatrix} \varphi_X(r,s) \\ \varphi_Y(r,s) \\ \varphi_Z(r,s) \end{bmatrix}.
\]

The displacement field \( \mathbf{u} = \begin{bmatrix} u_X(r,s) \\ u_Y(r,s) \\ u_Z(r,s) \end{bmatrix} \) given by formula (15) is then enriched by the following term

\[
\Delta \mathbf{u} = \sum_{k=5}^{8} a_k(r,s) L_k \frac{L_k}{8} \left( \varphi_{I(k)\,z} - \varphi_{J(k)\,z} \right) (-l_k) = \sum_{k=5}^{8} a_k(r,s) L_k L_k \frac{L_k}{8} \mathbf{n} \cdot \begin{bmatrix} \varphi_{I(k)\,X} \\ \varphi_{I(k)\,Y} \\ \varphi_{I(k)\,Z} \end{bmatrix} - \begin{bmatrix} \varphi_{J(k)\,X} \\ \varphi_{J(k)\,Y} \\ \varphi_{J(k)\,Z} \end{bmatrix},
\]

(69)
where start node $I(k)$ and end node $J(k)$ for the side $k = 5, 6, 7, 8$ is given by

| $k$ | $I(k)$ | $J(k)$ |
|-----|--------|--------|
| 5   | 1      | 2      |
| 6   | 2      | 3      |
| 7   | 3      | 4      |
| 8   | 4      | 1      |

and $l_k$ is the in-plane normal vector corresponding to the side $k$, see Figure 4, given by $l_k = \frac{n_k \times t_k}{||n_k \times t_k||}$, where $n_k = \frac{n_{I(k)} + n_{J(k)}}{2}$, $t_k = \frac{P_{J(k)} - P_{I(k)}}{||P_{J(k)} - P_{I(k)}||}$, see Figure 6. The membrane strain field $\varepsilon = \begin{bmatrix} \varepsilon_x(r, s) \\ \varepsilon_y(r, s) \\ \gamma_{xy}(r, s) \end{bmatrix}$ given by formula (32) is enriched by the term $\begin{bmatrix} F_{1,1} \\ F_{1,2} + F_{2,1} \end{bmatrix}$, where

$$F_{K,L} = v_K^T(r, s) \frac{\partial(\triangle u_X, \triangle u_Y, \triangle u_Z)}{\partial(r, s, t)} \frac{\partial(r, s, t)}{\partial(X, Y, Z)} v_L(r, s) = \sum_{k=5}^{8} \frac{L_k}{8} (v_K^T l_k) \left[ ((X^r)^T v_L) a_{k,r}(r, s) + ((X^s)^T v_L) a_{k,s}(r, s) \right] N_k, \quad K, L = 1, 2, \quad (70)$$

where

$$N_k = n \cdot \begin{bmatrix} \varphi I(k), X \\ \varphi I(k), Y \\ \varphi I(k), Z \end{bmatrix} - \begin{bmatrix} \varphi J(k), X \\ \varphi J(k), Y \\ \varphi J(k), Z \end{bmatrix} = n_X e_{I(k)} - n_X e_{J(k)} + n_Y e_{4+I(k)} - n_Y e_{4+J(k)} + n_Z e_{8+I(k)} - n_Z e_{8+J(k)}. \quad (71)$$

where the vector $e_i$ is a zero vector with 1 at index $i$. The bending strain field $\kappa$ given by Eq. (35) stays unchanged. To conclude this section, let us remark that the DKMQ24 shell element has improved bending behaviour, due to quadratic enrichment of the displacement field with the help of out-of-plane rotations $\varphi_x, \varphi_y$ and in the same manner, the DKMQ24+ shell element has improved membrane behaviour due to quadratic enrichment of the displacement field with the help of in-plane (drilling) rotations $\varphi_z$. 
3.2. The static condensation

We enrich the displacement field by a bubble mode according to the approach given in [25]. Two new degrees of freedom \( u_1, u_2 \) corresponding to the mid-element node 9 are considered:

\[
\mathbf{u}_n(r, s) = a_0(r, s) \left[ u_1 \mathbf{v}_1(0, 0) + u_2 \mathbf{v}_2(0, 0) \right], \quad a_0(r, s) = (1 - r^2)(1 - s^2).
\]  

(72)

The corresponding strain vector has the form

\[
\varepsilon_n = B_{n,3\times2} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.
\]  

(73)

These additional degrees of freedom are statically condensed out on the element level, therefore the shell element has again 24 degrees of freedom. The static condensation is based on standard procedure applied on the membrane part only (Appendix B). The statically condensed strain matrix \( B \) takes the form

\[
B = \begin{bmatrix} B_b \\ B_m - B_{n,3\times2} K_{n,n}^{-1} K_{m,n}^{T} \end{bmatrix}
\]  

(74)

where

\[
K_{m,n} = \int_e B_{m,3\times24}^{T} D_m B_{n,3\times2} \, dA,
\]  

(75)

\[
K_{n,n} = \int_e B_{n,3\times2}^{T} D_m B_{n,3\times2} \, dA.
\]  

(76)

3.3. Penalty term

The drilling rotations stabilization term [66], used in the case of the DKMQ24 shell element, is not used in case of the DKMQ24+ shell element. However, a penalty term which forces the drilling rotations \( \varphi_z(r, s) \) to be equal to a rotation of the displacement field \( \psi \) at the same point, derived by Hughes and Brezzi [25], is considered. The energy term takes the form

\[
\Pi_{stab} = \frac{c_2 G h}{2} \int_A \left[ \psi(r, s) - \varphi_z(r, s) \right]^2 \, dA,
\]  

(77)

where

\[
\psi(r, s) = \frac{1}{2} \left( \frac{\partial u_y}{\partial x}(r, s) - \frac{\partial u_x}{\partial y}(r, s) \right),
\]  

(78)

\[
\frac{\partial u_y}{\partial x}(r, s) = \sum_{i=1}^{4} a_{i,x}(r, s) \mathbf{v}_2(r, s) \cdot \begin{bmatrix} u_{i,X} \\ u_{i,Y} \\ u_{i,Z} \end{bmatrix},
\]  

(79)

\[
\frac{\partial u_x}{\partial y}(r, s) = \sum_{i=1}^{4} a_{i,y}(r, s) \mathbf{v}_1(r, s) \cdot \begin{bmatrix} u_{i,X} \\ u_{i,Y} \\ u_{i,Z} \end{bmatrix},
\]  

(80)

\[
a_{i,x}(r, s) = a_{i,r}(r, s) (X^r(r, s))^T \mathbf{v}_1(r, s) + a_{i,s}(r, s) (X^s(r, s))^T \mathbf{v}_1(r, s),
\]  

(81)

\[
a_{i,y}(r, s) = a_{i,r}(r, s) (X^r(r, s))^T \mathbf{v}_2(r, s) + a_{i,s}(r, s) (X^s(r, s))^T \mathbf{v}_2(r, s),
\]  

(82)

\[
\varphi_z(r, s) = \sum_{i=1}^{4} a_{i}(r, s) \mathbf{v}_3(r, s) \cdot \begin{bmatrix} \varphi_{i,X} \\ \varphi_{i,Y} \\ \varphi_{i,Z} \end{bmatrix},
\]  

(83)
and $c_2$ is the penalty constant and is taken as $c_2 = 10^{-6}$ in our implementation. The corresponding stiffness matrix in the global coordinate system has the form:

$$K_{\text{stab};i,j} = c_2 G h \int_{-1}^{1} \int_{-1}^{1} g_{24,i}(r,s) g_{24,j}(r,s) \det J \, dr \, ds, \quad i,j = 1, \ldots, 24, \quad (84)$$

where

$$g_{24}(r,s) = \begin{bmatrix}
- a_1(r,s)v_{3,X}(r,s), & \ldots, & - a_1(r,s)v_{3,Y}(r,s), & \ldots, & - a_1(r,s)v_{3,Z}(r,s),
- a_1(r,s)v_{3,Y}(r,s), & \ldots, & - a_4(r,s)v_{3,Y}(r,s), & \ldots, & - a_4(r,s)v_{3,Z}(r,s),
- a_1(r,s)v_{3,Z}(r,s), & \ldots, & - a_4(r,s)v_{3,Z}(r,s), & \ldots, & - a_4(r,s)v_{3,Z}(r,s),
a_{1,x}(r,s)v_{2,X}(r,s) - a_{1,y}(r,s)v_{1,X}(r,s), & \ldots, & a_{4,x}(r,s)v_{2,X}(r,s) - a_{4,y}(r,s)v_{1,X}(r,s),
a_{1,y}(r,s)v_{2,Y}(r,s) - a_{1,x}(r,s)v_{1,Y}(r,s), & \ldots, & a_{4,y}(r,s)v_{2,Y}(r,s) - a_{4,x}(r,s)v_{1,Y}(r,s),
a_{1,z}(r,s)v_{2,Z}(r,s) - a_{1,y}(r,s)v_{1,Z}(r,s), & \ldots, & a_{4,z}(r,s)v_{2,Z}(r,s) - a_{4,y}(r,s)v_{1,Z}(r,s),
\end{bmatrix}. \quad (85)$$

The one point integration rule is used to integrate the stiffness matrix \[84\] in order to relax the prescribed condition [77].

4. Convergence tests

The behaviour of the DKMQ24 and DKMQ24+ elements has been tested on eight well-established benchmark problems, including one pure membrane benchmark problem suggested by Cook [26], six shell and plate benchmark problems suggested by MacNeal and Harder [27] and Belytscho [28], which were reused by Katili [16] and one plate benchmark problem which incorporates a varying shell thickness. We compare the results of the DKMQ24 element from [16] and of the DKMQ24 and DKMQ24+ elements from our implementation. In all cases, the geometric and material linearity is considered and there is no initial deformation.
4.1. Cook’s membrane

The standard Cook’s membrane problem tests an element behaviour for the case of in-plane shear loading. The problem also tests the effect of mesh distortion. The membrane is fully fixed at $X = 0$ and loaded by a distributed force at $X = L_1$ (see Figure 7). The $Z$-component of the displacement is evaluated at the test point $A$. The analytical solution for this example is not available, therefore the reference solution is obtained numerically using the DKMQ24+ element on the refined mesh $192 \times 128$. The numerical results are summarized in Table 3. We observe massive reduction of the relative error in case of the DKMQ24+ shell element in comparison to the DKMQ24 shell element.

![Figure 7: Cook’s membrane with mesh $2 \times 2$.](image)

| Mesh | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|------|-------------|--------|---------|
|      | $u_Z(A)$ rel.err. | $u_Z(A)$ rel.err. | $u_Z(A)$ rel.err. |
|      | [m] | [%] | [m] | [%] | [m] | [%] |
| $2 \times 2$ | - | - | 11.845 | 50.57 | 22.399 | 6.54 |
| $4 \times 4$ | - | - | 18.299 | 23.64 | 23.417 | 2.29 |
| $8 \times 8$ | - | - | 22.079 | 7.87 | 23.780 | 0.77 |
| $16 \times 16$ | - | - | 23.430 | 2.23 | 23.902 | 0.26 |
| $32 \times 32$ | - | - | 23.818 | 0.61 | 23.943 | 0.09 |

Reference solution: $u_Z(A) = 23.965$ m (DKMQ24+ element on mesh $192 \times 128$)

Table 3: Deflection $u_Z(A)$ for Cook’s membrane problem
4.2. Hemispherical shell

The hemispherical shell problem, consisting of a hemisphere loaded by four opposite single forces, is depicted in Figure 8. From symmetric reasons only one quarter of a hemisphere is calculated, due to which the following symmetry conditions are applied: \( u_Y = \varphi_X = \varphi_Z = 0 \) on the side AC, \( u_X = \varphi_Y = \varphi_Z = 0 \) on the side BC. Additionally, the following boundary condition is applied at point C: \( u_Z = 0 \). The analytical solution to this problem is given by [27] as \( u_X(A) = -u_Y(B) = 0.0924 \text{ m} \). As we can see in Table 4, the DKMQ24+ shell element exhibits superior performance if compared to the DKMQ24 shell element at all test cases. Note that this example is sensitive to quality of the mesh and also it is the only benchmark in our set, for which the mesh is not exactly specified by its depiction in Figure 8. In our case we have used the Laplacian smoothing of internal nodes [29] until we have converged to a unique mesh for the calculation.
Table 4: Deflection $u_X(A)$ for the hemispherical shell problem. Note that the increase of the relative error in the last row to 0.02 is only caused by insufficient number of digits of the reference value.
4.3. Scordelis-Lo roof

The Scordelis-Lo roof is a cylindrical surface with radius $R$, length $L$ and thickness $h$, reinforced at positions $Y = 0$ and $Y = L$ by two rigid diaphragms (see Figure 9). Because of symmetry only one quarter is calculated. The shell is loaded by a self-weight, where gravitational acceleration is considered to be $g = 10 \text{ m s}^{-2}$. The following boundary and symmetric conditions are applied: $u_X = u_Z = \varphi_Y = 0$ on the side AD, $u_X = \varphi_Y = \varphi_Z = 0$ on the side CD and $u_Y = \varphi_X = \varphi_Z = 0$ on the side CB. The analytical solution is given in [30].

With this benchmark problem we test not only nodal displacements, but also values of the nodal stress-resultants. We point out that these values are extrapolated from values given in the Gauss quadrature points. The comparison of displacements in Tables 5 and 6 reveals that the DKMQ24+ element has less precise results on the coarsest mesh $4 \times 4$, and more precise results on all other meshes if compared to the DKMQ24 element. The comparison of the stress resultants in Tables 7 and 8 shows, that the DKMQ24+ element exhibits significantly better approximation of the stress-resultants if compared to the DKMQ24 element.

Figure 9: Scordelis-Lo roof with mesh $4 \times 4$.

$L = 6 \text{ m}$
$R = 3 \text{ m}$
$h = 0.03 \text{ m}$
$\varphi = 40^\circ$
$E = 30000 \text{ MPa}$
$\nu = 0$
$h\rho = 625 \text{ kg m}^{-2}$
| Mesh | DKMQ24 | DKMQ24 | DKMQ24+ |
|------|--------|--------|---------|
|      | $u_Z(B)$ | rel.err. | $u_Z(B)$ | rel.err. | $u_Z(B)$ | rel.err. |
|      | [mm]    | [%]    | [mm]    | [%]    | [mm]    | [%]    |
| $4 \times 4$ | $-0.03425$ | 5.12 | $-0.034258$ | 5.10 | $-0.038230$ | 5.90 |
| $8 \times 8$ | $-0.03528$ | 2.26 | $-0.035284$ | 2.26 | $-0.036529$ | 1.19 |
| $16 \times 16$ | $-0.03585$ | 0.70 | $-0.035858$ | 0.67 | $-0.036180$ | 0.22 |
| $32 \times 32$ | - | - | $-0.036070$ | 0.08 | $-0.036120$ | 0.06 |

Reference solution: $u_Z(B) = -0.0361$ m (theory of deep shell, [30])

### Table 5: Deflection $u_Z(B)$ for the Scordelis-Lo roof problem

| Mesh | DKMQ24 | DKMQ24 | DKMQ24+ |
|------|--------|--------|---------|
|      | $u_Z(C)$ | rel.err. | $u_Z(C)$ | rel.err. | $u_Z(C)$ | rel.err. |
|      | [mm]    | [%]    | [mm]    | [%]    | [mm]    | [%]    |
| $4 \times 4$ | $0.00513$ | 5.18 | $0.005130$ | 5.17 | $0.005775$ | 6.75 |
| $8 \times 8$ | $0.00529$ | 2.21 | $0.005294$ | 2.15 | $0.005492$ | 1.52 |
| $16 \times 16$ | $0.00538$ | 0.55 | $0.005378$ | 0.58 | $0.005431$ | 0.38 |
| $32 \times 32$ | - | - | $0.005407$ | 0.05 | $0.005420$ | 0.18 |

Reference solution: $u_Z(C) = 0.00541$ m (theory of deep shell, [30])

### Table 6: Deflection $u_Z(C)$ for the Scordelis-Lo roof problem

Deflection $u_Z(B)$ for the Scordelis-Lo roof problem

![Graph showing deflection $u_Z(B)$ for the Scordelis-Lo roof problem](image)
| Mesh  | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|-------|-------------|---------|---------|
|       | \(n_y(B)\) | \(n_y(B)\) | \(n_y(B)\) |
|       | rel.err. | rel.err. | rel.err. |
|       | [mm] | [%] | [mm] | [%] | [mm] | [%] |
| 4 \(\times\) 4 | 1.874 | 8.85 | 1.7401 | 15.36 | 1.9661 | 4.37 |
| 8 \(\times\) 8 | 1.998 | 2.82 | 1.9478 | 5.26 | 2.0290 | 1.31 |
| 16 \(\times\) 16 | 2.041 | 0.73 | 2.0273 | 1.40 | 2.0493 | 0.33 |
| 32 \(\times\) 32 | 2.054 | 0.10 | 2.0504 | 0.27 | 2.0555 | 0.02 |

Reference solution: \(n_y(B) = 2.056 \text{kNm}^{-1}\) (theory of shallow shell, [30])

Table 7: Internal force \(n_y(B)\) for the Scordelis-Lo roof problem

Deflection \(u_z(C)\) for the Scordelis-Lo roof problem

Internal force \(n_y(B)\) for the Scordelis-Lo roof problem
| Mesh   | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|--------|-------------|--------|---------|
|        | $m_x(C)$ rel.err. | $m_x(C)$ rel.err. | $m_x(C)$ rel.err. |
|        | [mm] | [%] | [mm] | [%] | [mm] | [%] |
| 4 × 4  | 498.430 | 22.24 | 498.741 | 22.19 | 555.367 | 13.36 |
| 8 × 8  | 592.680 | 7.54  | 592.745 | 7.53  | 609.390 | 4.93  |
| 16 × 16| 620.750 | 3.16  | 620.909 | 3.13  | 624.956 | 2.50  |
| 32 × 32| 628.280 | 1.98  | 628.908 | 1.89  | 629.314 | 1.82  |

Reference solution: $m_x(C) = 641.000 \text{kN}$ (theory of shallow shell, [30])

Table 8: Internal moment $m_x(C)$ for the Scordelis-Lo roof problem

Internal moment $m_x(C)$ for the Scordelis-Lo roof problem

![Graph showing the internal moment $m_x(C)$ for different meshes and solutions.](image-url)
4.4. Pinched cylinder shell

The pinched cylinder, depicted in Figure 10, is loaded by two opposite pinching forces \( F \). The cylinder has a rigid diaphragm at both ends. Only one eighth of the model is calculated due to symmetries (to which the force \( F/4 \) is applied only). The boundary conditions are as follows: \( u_X = u_Z = \varphi_Y = 0 \) on the side AD, on the remaining sides symmetric conditions are prescribed: \( u_Z = \varphi_X = \varphi_Y = 0 \) on side AB, \( u_Y = \varphi_X = \varphi_Z = 0 \) on the side BC and \( u_X = \varphi_Y = \varphi_Z = 0 \) on side CD. Two different shell thicknesses are considered in our calculation, \( h = 0.03 \) m and \( h = 0.3 \) m. We use Young’s modulus \( E = 3 \) MPa, which is a usual value for this standard benchmark example [28] instead of the value 30000 MPa used in [16].

The uniform and the distorted mesh are analyzed. For the distorted mesh a bias factor, i.e. the ratio of the largest and smallest element edge, of 10 is used. The analytical solution for the case \( h = 0.03 \) m is based on the Kirchhoff theory proposed by Lindberg, et. al. [31], which follows the solution of Flügge [32], and equals to \( u_Z(C) = -1.825 \) mm. The analytical solution for the case \( h = 0.3 \) m, taken from [33], is based on the Mindlin theory and equals to \( u_Z(C) = -0.01261 \) mm. However, as reported in [16] and according to our numerical results, the numerical results converge to slightly larger values. Therefore, numerical values results for DKMQ24+ shell elements on mesh 128×128 are taken as reference values.

The numerical results presented in Tables 9-12 reveal slow convergence of the all considered elements, which is in concordance with other authors [28][34] and is a consequence of a single point loading. At this
example our modifications from Section 3 do not bring a clear advantage, in some cases the DKMQ24+ element, in comparison with the DKMQ24 element, exhibits lower error and in some cases higher error.

Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.03$ m, uniform mesh

| Mesh     | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|----------|-------------|--------|---------|
|          | $u_Z(C)$    | rel.err. | $u_Z(C)$ | rel.err. | $u_Z(C)$ | rel.err. |
|          | [mm]   | [%]     | [mm]   | [%]     | [mm]   | [%]     |
| 4 × 4    | −1.1255 | 39.26   | −1.1242 | 39.33   | −1.1847 | 36.07   |
| 8 × 8    | −1.7246 | 6.93    | −1.7238 | 6.97    | −1.7462 | 5.76    |
| 16 × 16  | −1.8599 | 0.37    | −1.8595 | 0.35    | −1.8647 | 0.63    |
| 32 × 32  | -       | -       | −1.8569 | 0.21    | −1.8583 | 0.29    |

Reference solution: $u_Z(C) = −1.853$ mm (DKMQ24+ element on mesh 128 × 128)

Table 9: Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.03$ m, uniform mesh
Table 10: Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.03\, \text{m}$, distorted mesh

| Mesh   | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|--------|-------------|--------|---------|
|        | $u_Z(C)$    | rel.err. | $u_Z(C)$ | rel.err. | $u_Z(C)$ | rel.err. |
|        | [mm]  | [%] | [mm] | [%] | [mm] | [%] |
| 4 × 4  | −0.2397  | 87.06 | −0.2425 | 86.91 | −0.2540 | 86.29 |
| 8 × 8  | −1.2326  | 33.46 | −1.2310 | 33.57 | −1.1948 | 35.52 |
| 16 × 16 | −1.7144 | 7.48 | −1.7136 | 7.52 | −1.7073 | 7.86 |
| 32 × 32 | - | - | −1.8206 | 1.75 | −1.8196 | 1.80 |

Reference solution: $u_Z(C) = -1.853\, \text{mm}$ (DKMQ24+ element on mesh 128 × 128)

Table 11: Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.3\, \text{m}$, uniform mesh

| Mesh   | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|--------|-------------|--------|---------|
|        | $u_Z(C)$    | rel.err. | $u_Z(C)$ | rel.err. | $u_Z(C)$ | rel.err. |
|        | [mm]  | [%] | [mm] | [%] | [mm] | [%] |
| 4 × 4  | −0.011445  | 19.97 | −0.011346 | 20.28 | −0.011521 | 19.58 |
| 8 × 8  | −0.012180  | 14.83 | −0.012125 | 15.38 | −0.012026 | 16.08 |
| 16 × 16 | −0.012822 | 10.34 | −0.012638 | 11.89 | −0.012393 | 13.29 |
| 32 × 32 | - | - | −0.013099 | 8.39 | −0.012910 | 9.79 |

Reference solution: $u_Z(C) = -0.0143\, \text{mm}$ (DKMQ24+ element on mesh 128 × 128)

Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.03\, \text{m}$, distorted mesh
### Table 12: Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.3$ m, distorted mesh

| Mesh   | DKMQ24 ($^\text{16}$) | DKMQ24 | DKMQ24+ |
|--------|-----------------------|--------|---------|
|        | $u_Z(C)$ rel.err. | $u_Z(C)$ rel.err. | $u_Z(C)$ rel.err. |
|        | [mm] [\%] | [mm] [\%] | [mm] [\%] |
| $4 \times 4$ | $-0.008617$ 39.74 | $-0.008665$ 39.16 | $-0.008656$ 39.16 |
| $8 \times 8$ | $-0.011443$ 19.98 | $-0.011439$ 20.28 | $-0.011396$ 20.28 |
| $16 \times 16$ | $-0.012508$ 12.53 | $-0.012426$ 13.29 | $-0.012211$ 14.69 |
| $32 \times 32$ | - - | $-0.013038$ 9.09 | $-0.012771$ 10.49 |

Reference solution: $u_Z(C) = -0.0143$ mm (DKMQ24+ element on mesh $128 \times 128$)

Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.3$ m, uniform mesh

Deflection $u_Z(C)$ for the pinched cylinder problem, thickness $h = 0.3$ m, distorted mesh
4.5. Hyperbolic paraboloid shell

The hyperbolic paraboloid shell problem is depicted in Figure 11. The shell middle surface is described by \( Z = \frac{b}{a} XY \). The shell is loaded by a self-weight, where gravitational acceleration is considered to be \( g = 10 \text{ m s}^{-2} \). The shell is fully clamped on all sides ABCD, i.e. \( u_X = u_Y = u_Z = \varphi_X = \varphi_Y = \varphi_Z = 0 \).

The analytical solution is taken from [35]. Our implementation shows slightly better results on coarser meshes. The results are otherwise almost identical and we can observe that our modification from Section 3 does not play an important role in case of this benchmark.

![Hyperbolic paraboloid shell](image)

Figure 11: Hyperbolic paraboloid shell with mesh 4 \( \times \) 4.

| Mesh | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|------|----------|--------|--------|
| \( u_Z(O) \) | \( \text{rel.err.} \) | \( u_Z(O) \) | \( \text{rel.err.} \) | \( u_Z(O) \) | \( \text{rel.err.} \) |
| [mm] | [%] | [mm] | [%] | [mm] | [%] |
| 2 \( \times \) 2 | 35.90 | 45.93 | 35.642 | 44.88 | 35.645 | 44.90 |
| 4 \( \times \) 4 | 27.07 | 10.06 | 26.657 | 8.36 | 26.772 | 8.83 |
| 8 \( \times \) 8 | 25.02 | 1.72 | 24.720 | 0.49 | 24.709 | 0.44 |
| 16 \( \times \) 16 | 24.86 | 1.07 | 24.546 | 0.22 | 24.549 | 0.21 |
| 32 \( \times \) 32 | - | - | 24.524 | 0.31 | 24.525 | 0.31 |

Reference solution: \( u_Z(O) = 24.600 \text{ mm} \) [35]

Table 13: Deflection \( u_Z(O) \) for the hyperbolic paraboloid shell problem
Deflection \( u_Z(O) \) for the hyperbolic paraboloid shell problem

![Graph showing deflection vs. number of elements]

4.6. Twisted beam

A shell strip of length \( L \), width \( b \) and thickness \( h \) is twisted in the undeformed configuration by angle \( \frac{\pi}{2} \) between points \( O \) and \( A \) (see Figure 12). The strip is clamped at \( X = 0 \) and loaded by a single force \([0, F_Y, F_Z]\) at point \( A \). The shell middle surface is described by: \([X(\varphi, s), Y(\varphi, s), Z(\varphi, s)] = \left[ \frac{2\varphi L}{\pi}, \ s \sin \varphi, \ s \cos \varphi \right]\), \( 0 \leq \varphi \leq \frac{\pi}{2}, -\frac{b}{2} \leq s \leq \frac{b}{2} \). The example was introduced by MacNeal and Harder \[27\] and represents an important test example at which many flat quadrilateral shell elements fail to converge. This is with accordance with our numerical experience, which shows that the DKMQ24/DKMQ24+ elements fail to converge if the coupling term, including derivatives of the element normal with respect to reference coordinates \[34\], which is identicaly zero for flat elements, is missing. The analytical solution based on the beam theory (which ignores the deformation in the transverse direction) is taken from \[16\]. Four different settings of loading forces and shell thicknesses were tested, see Tables 14-17. The new element DKMQ24+ exhibits significant improvement in all considered test cases if compared to the original DKMQ24 element.
| Mesh | DKMQ24 | DKMQ24 | DKMQ24+ |
|------|--------|--------|---------|
|      | $u_Y(A)$ | rel.err. | $u_Y(A)$ | rel.err. | $u_Y(A)$ | rel.err. |
| $2 \times 12$ | 5.200 | 1.07 | 5.1721 | 1.60 | 5.2162 | 0.76 |
| $4 \times 24$ | 5.217 | 0.74 | 5.2142 | 0.79 | 5.2435 | 0.24 |
| $8 \times 48$ | 5.238 | 0.34 | 5.2404 | 0.30 | 5.2510 | 0.09 |
| $16 \times 96$ | - | - | 5.2478 | 0.16 | 5.2505 | 0.10 |

Reference solution: $u_Y(A) = 5.256$ m

Table 14: Deflection $u_Y(A)$ for the twisted beam problem with $h = 0.0032$ m, $F_Y = 0.001$ N, $F_Z = 0$.

| Mesh | DKMQ24 | DKMQ24 | DKMQ24+ |
|------|--------|--------|---------|
|      | $u_Z(A)$ | rel.err. | $u_Z(A)$ | rel.err. | $u_Z(A)$ | rel.err. |
| $2 \times 12$ | 1.274 | 1.55 | 1.2673 | 2.06 | 1.2767 | 1.34 |
| $4 \times 24$ | 1.285 | 0.70 | 1.2855 | 0.66 | 1.2903 | 0.29 |
| $8 \times 48$ | 1.290 | 0.31 | 1.2911 | 0.23 | 1.2926 | 0.11 |
| $16 \times 96$ | - | - | 1.2926 | 0.11 | 1.2929 | 0.08 |

Reference solution: $u_Z(A) = 1.294$ m

Table 15: Deflection $u_Z(A)$ for the twisted beam problem with $h = 0.0032$ m, $F_Y = 0$, $F_Z = 0.001$ N

Deflection $u_Y(A)$ for the twisted beam problem with $h = 0.0032$ m, $F_Y = 0.001$ N, $F_Z = 0$
Table 16: Deflection $u_Y(A)$ for the twisted beam problem with $h = 0.32\, \text{m}$, $F_Y = 1000\, \text{N}$, $F_Z = 0$

| Mesh   | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|--------|-------------|--------|----------|
|        | $u_Y(A)$ rel.err. | $u_Y(A)$ rel.err. | $u_Y(A)$ rel.err. |
| 2 × 12 | 5.393 [m] [57] | 5.370 [m] [100] | 5.435 [m] [22] |
| 4 × 24 | 5.403 [m] [39] | 5.399 [m] [45]  | 5.418 [m] [11] |
| 8 × 48 | 5.410 [m] [24] | 5.412 [m] [24]  | 5.416 [m] [15] |
| 16 × 96| - [m] [-]     | 5.415 [m] [17]  | 5.416 [m] [14] |

Reference solution: $u_Y(A) = 5.424\, \text{m}$ [27]

Deflection $u_Z(A)$ for the twisted beam problem with $h = 0.0032\, \text{m}$, $F_Y = 0$, $F_Z = 0.001\, \text{N}$

Deflection $u_Y(A)$ for the twisted beam problem with $u_Y(A), h = 0.32\, \text{m}$, $F_Y = 1000\, \text{N}$, $F_Z = 0$
Table 17: Deflection $u_Z(A)$ for the twisted beam problem with $h = 0.32\,\text{m}$, $F_Y = 0$, $F_Z = 1000\,\text{N}$

\[
\begin{array}{cccccc}
\text{Mesh} & \text{DKMQ24} & \text{DKMQ24} & \text{DKMQ24+} \\
\text{m} & \text{rel.err.} & \text{m} & \text{rel.err.} & \text{m} & \text{rel.err.} \\
\hline
2 \times 12 & 1.624 & 7.41 & 1.6182 & 7.74 & 1.7473 & 0.38 \\
4 \times 24 & 1.711 & 2.45 & 1.7105 & 2.48 & 1.7501 & 0.22 \\
8 \times 48 & 1.740 & 0.80 & 1.7408 & 0.75 & 1.7513 & 0.15 \\
16 \times 96 & - & - & 1.7491 & 0.28 & 1.7518 & 0.13 \\
\end{array}
\]

Reference solution: $u_Z(A) = 1.754\,\text{m}$ [27]

Deflection $u_Z(A)$ for the twisted beam problem with $h = 0.32\,\text{m}$, $F_Y = 0$, $F_Z = 1000\,\text{N}$

4.7. Square clamped plate under uniform loading with distorted mesh

A fully clamped pressure loaded square plate with distorted mesh is considered (see Figure 13). Only one quarter is computed due to symmetries. Boundary and symmetry conditions are as follows:

$u_X = u_Y = u_Z = \varphi_X = \varphi_Y = \varphi_Z = 0$ on sides AB and AD, $u_X = \varphi_Y = \varphi_Z = 0$ on side BC and
| Mesh | DKMQ24 [16] | DKMQ24 | DKMQ24+ |
|------|-------------|--------|---------|
|      | \( u_Z(C) \) | rel.err. | \( u_Z(C) \) | rel.err. | \( u_Z(C) \) | rel.err. |
|      | [mm] | [%] | [mm] | [%] | [mm] | [%] |
| 2 x 2 | -1929.0 | 28.37 | -1910.8 | 27.73 | -1910.8 | 27.73 |
| 4 x 4 | -1616.1 | 8.03 | -1906.6 | 7.39 | -1906.6 | 7.39 |
| 8 x 8 | -1538.0 | 2.81 | -1529.2 | 2.22 | -1529.2 | 2.22 |
| 16 x 16 | -1519.0 | 1.54 | -1510.6 | 0.98 | -1510.6 | 0.98 |
| 32 x 32 | - | - | -1506.1 | 0.68 | -1506.1 | 0.68 |

Reference solution: \( u_Z(C) = -1495.95 \text{ mm} \) [37]

Table 18: Deflection \( u_Z(C) \) for the plate with distorted mesh

\( u_Y = \varphi_X = \varphi_Z = 0 \) on side CD. This benchmark problem tests sensitivity to mesh distortion. The reference solution is taken from [37].

This benchmark is taken as an example of pure bending, where our modifications of the membrane part of the DKMQ24 element does not play any role. Table 18 reflects this fact, where results of our implementation of both DMKQ24 and DKMQ24+ elements are identical. However, these results are slightly better than results taken from [16].

\[ \text{Deflection } u_Z(C) \text{ for the plate with distorted mesh} \]

\[ \text{Reference solution } u_Z(C) = -1495.95 \text{ mm} \] [37]

---

\[ ^2 \text{The value 62.83 given in [16] is not the deformation } u_Z \text{ as stated, but } \frac{G_{uZ}}{h_p}. \]
4.8. Tapered cantilever

The tapered cantilever benchmark problem tests quality of numerical approximation in case of varying shell thickness. The cantilever beam of length $L$, depth $d$ and the beam height described by $h(X) = h_1 + \frac{X}{L}(h_2 - h_1)$ is fully fixed at $X = 0$ and loaded by a force $F$ at $X = L$ in direction $-Z$ (see Figure 14). Our modification plays no role in this example due to pure bending loading. We consider two cases, the constant element thickness and varying element thickness, evaluated independently in each Gauss point, in calculation of the bending stiffness matrix (39). The numerical results show, that consideration of the varying shell thickness in the constitutive relation (39) only, is fully satisfactory.

\[
\begin{array}{cccccc}
\text{Mesh} & \text{DKMQ24+ (const.thickness)} & \text{DKMQ24+ (varying thickness)} \\
\hline
& u_Z(C) & \text{rel.err.} & u_Z(C) & \text{rel.err.} \\
1 \times 1 & 70.547 & 61.44 & 45.086 & 3.17 \\
2 \times 1 & 55.804 & 27.70 & 43.498 & 0.46 \\
4 \times 1 & 47.425 & 8.53 & 43.592 & 0.24 \\
8 \times 1 & 44.640 & 2.15 & 43.684 & 0.03 \\
16 \times 1 & 43.930 & 0.53 & 43.698 & 0.00 \\
\end{array}
\]

Reference solution: $u_Z(C) = 43.699$ mm (see Appendix A)

Table 19: Deflection $u_Z(C)$ for the tapered cantilever problem
5. Conclusions

The new quadrangle shell element DKMQ24+ with four nodes and six degrees of freedom per node, based on the DKMQ24 quadrangle shell element introduced by Katili et al. [16], with improved membrane behaviour was introduced, fully derived and tested on a set of standard benchmark problems.

The new shell element DKMQ24+ exhibits superior behaviour on pure membrane problems and on shell problems, in which the in-plane loading is significantly present. Moreover, accuracy of the stress-resultant values is significantly improved as shown on the Scordeli-Lo roof problem. On shell problems with prevailing bending loading the elements DKMQ24+ and DKMQ24 deliver similar results. On pure bending problems both elements are identical.

The DKMQ24+ shell element passes all our convergence tests, without presence of neither shear locking nor membrane locking and converges in all considered cases to the reference solution. Moreover, our modification, based on the quadratic enrichment of the in-plane approximations with the help of drilling rotations, does not introduce any additional computational costs when compared to the DKMQ24 shell element. To conclude, the DKMQ24+ shell element could be considered as a replacement of the DKMQ24 shell element.

5.1. Appendix A – Analytical solution to tapered cantilever beam benchmark

The beam deflection without consideration of shear is described by the Euler-Bernoulli equation

\[ u''(X) = -\frac{M(X)}{EI(X)}, \]  

(86)
where for the rectangular tapered cross-section we have \( I(X) = \frac{w^3(X)}{12} \), \( h(X) = h_1 + \frac{X}{L} (h_2 - h_1) \). The loading moment equals to \( M(X) = -F(L - X) \). We get the following problem

\[
\frac{d^2 u}{dX^2} = \frac{12FL}{Eh_1^3} \left(1 - a \frac{X}{L}\right) \left[ \frac{aX}{L}\right]^k, \quad a = 1 - \frac{h_2}{h_1}, \quad u(0) = u'(0) = 0, \tag{87}
\]

where \( 0 < a \leq 1 \). Using the Taylor’s expansion of the right hand side at \( a = 0 \)

\[
\frac{1}{(1 - a \frac{X}{L})^3} = 1 + \sum_{k=1}^{\infty} \frac{(k+1)(k+2)}{2} \left[ \frac{aX}{L}\right]^k, \tag{88}
\]

and double integration of Eq. \( \text{(87)} \) we get the maximum deflection at the tip

\[
u(L) = \frac{4FL^3}{Eh_1^3} \left[ 1 + \sum_{k=1}^{\infty} \frac{3a^k}{3 + k} \right]. \tag{89}
\]

5.2. Appendix B – Static condensation

We consider the following system of equations defined on a single element \( e \)

\[
\begin{bmatrix}
K_{m,m} & K_{m,n} \\
K_{m,n} & K_{n,n}
\end{bmatrix}
\begin{bmatrix}
q_m \\
q_n
\end{bmatrix}
=
\begin{bmatrix}
f_m \\
0
\end{bmatrix}, \tag{90}
\]

where

\[
K_{m,m} = \int_e B_{m,3 \times 24}^T D_m B_{m,3 \times 24} \, dA, \tag{91}
\]

\[
K_{m,n} = \int_e B_{m,3 \times 24}^T D_m B_{n,3 \times 2} \, dA, \tag{92}
\]

\[
K_{n,n} = \int_e B_{n,3 \times 2}^T D_m B_{n,3 \times 2} \, dA. \tag{93}
\]

If we eliminate the unknown \( q_n \), we get the eliminated system of equations of the form

\[
\tilde{K}_{m,m} q_m = f_m, \tag{94}
\]

where

\[
\tilde{K}_{m,m} = K_{m,m} - K_{m,n} K_{n,n}^{-1} K_{m,n}^T, \tag{95}
\]

From the programming point of view the following equivalent form can be advantageous

\[
\tilde{K}_{m,m} = \int_e \tilde{B}^T D_m \tilde{B} \, dA, \quad \tilde{B} = B_m - B_n K_{n,n}^{-1} K_{m,n}^T, \tag{96}
\]

because the condensation is implemented only on the strain matrix level.

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