On the spectral and homological dimension of \( \kappa \)-Minkowski space

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Abstract

We extend the construction of a spectral triple for \( \kappa \)-Minkowski space, previously given for the two-dimensional case, to the general \( n \)-dimensional case. This takes into account the modular group naturally arising from the symmetries of the geometry, and requires the use of notions that have been recently developed in the frameworks of twisted and modular spectral triples. First we compute the spectral dimension, using an appropriate weight, and show that in general it coincides with the classical one. We also study the classical limit and the analytic continuation of the associated zeta function. Then we compare this notion of dimension with the one coming from homology. To this end, we compute the twisted Hochschild dimension of the universal enveloping algebra underlying \( \kappa \)-Minkowski space. The result is that twisting avoids the dimension drop, similarly to other examples coming from quantum groups. In particular, the simplest such twist is given by the inverse of the modular group mentioned above.

1 Introduction

The \( \kappa \)-Poincaré Hopf algebra \([1, 2]\) and the related non-commutative spacetime \( \kappa \)-Minkowski \([3]\) are interesting toy models to study features arising from quantum gravity. They have been much investigated in the Hopf algebraic framework, but so far there have been only a few studies \([4, 5, 6]\) from the point of view of (Euclidean) non-commutative geometry in the sense of Alain Connes \([7]\). Recently we argued in \([8]\) that a natural starting point for such a construction is a KMS weight which is invariant under the \( \kappa \)-Poincaré symmetries.

This introduces difficulties for the construction of a spectral triple, since the necessary modifications to the axioms, due to the presence of the modular group associated to a KMS weight, are not yet well understood. In the literature two different approaches have been advocated to deal, at least partly, with these issues: the first one is that of twisted spectral triples \([9]\), where one requires the boundedness of a twisted commutator, where the twist is given by an automorphism of the algebra; the second one is that of modular spectral triples \([10, 11, 12]\) where one, roughly speaking, replaces the operator trace with a weight having a non-trivial modular group. We used ideas from both of these approaches in \([8]\), where we constructed a “modular spectral triple” for the two dimensional \( \kappa \)-Minkowski space. The

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quotes are necessary because, even though the construction follows the spirit of this approach, strictly speaking it does not satisfy its axioms (but see also [13][14]). Indeed they arise from the study of geometries with a periodic modular group, while in our case we do not have this periodicity. Another general source of difficulties comes from the fact that we are dealing with a non-compact geometry: this is an aspect which has not been developed too much in this approach to non-commutative geometry (but see [15]), and we feel that it should be important to make some progress in this respect to make more contact with physics.

After this general discussion let us recall briefly the construction given in [8]. Starting from an algebra \( \mathcal{A} \), naturally associated to the commutation relations of \( \kappa \)-Minkowski space, we introduce a Hilbert space \( \mathcal{H} \) via the GNS-construction for the KMS weight \( \omega \) we mentioned above. We emphasize that the main difference with respect to other approaches is in the choice of this weight, which is motivated by the symmetries. The use of the twisted commutator turns out to be necessary if one wants to satisfy a boundedness condition and have a good classical limit. Moreover these requests, together with some symmetry conditions, single out a unique operator \( D \) and unique twist \( \sigma \) such that the twisted commutator is bounded.

However the triple \((\mathcal{A}, \mathcal{H}, D)\) is not finitely summable, an outcome which is hinted by the mismatch in the modular properties of the weight \( \omega \) and the non-commutative integral. We will repeat and emphasize this argument later in the paper, so we do not elaborate further here. This mismatch also hints at the possibility that, by choosing an appropriate weight in the sense of modular spectral triples, we can obtain a finite spectral dimension. This is indeed the case and we find that, in this setting, the spectral dimension is finite and coincides with the classical one. Moreover, by computing the residue at the spectral dimension of the corresponding zeta function, we recover the weight \( \omega \) up to a constant.

Having summarized the construction, which is essential for our investigations here, we now come to the contents of the paper. First we give an extension of the results mentioned above to the \( n \)-dimensional case. This turns out to be fairly easy, and in doing so we will skip most the details of the construction, which can be easily filled using the detailed arguments provided in [8]. Secondly we provide further evidence that, although this construction is still not well understood as part of a general framework, it should be relevant for the description of the non-commutative geometry of \( \kappa \)-Minkowski space. Using the same ingredients of the two-dimensional case, we find that the spectral dimension according to our definition is in general equal to the classical one. Moreover, by computing the residue at the spectral dimension of the associated zeta function, we recover the weight \( \omega \) as in the two-dimensional case. These results confirm the intuition that, in passing from the two-dimensional case to the general one, little changes. Next we analyze some properties of the zeta function that we introduced. We show that, by taking the limit of the deformation parameter to zero, it reduces as it should to the classical setting. Also, as in the commutative setting, this zeta function can be analytically continued to a meromorphic function on the complex plane, with only simple poles. The poles of the commutative case still remain, but additional ones appear due to the presence of the deformation parameter. The significance of these poles remains to be investigated.

Another important issue we analyze is the homological dimension of this geometry. In the framework of non-commutative geometry this notion is given by the dimension of the Hochschild homology, which in the commutative case coincides with the spectral dimension. However in many examples, coming in particular from quantum groups, one finds that the
homological dimension is lower than the spectral dimension, a phenomenon known as \textit{dimension drop}. In many cases it is possible to avoid this drop by introducing a twist in the homology theory, as seen for example in [22, 23]. Here we compute the \textit{twisted Hochschild homology} \cite{24} of the universal enveloping algebra associated to \( \kappa \)-Minkowski space. Similarly to the examples we mentioned above, we show that the dimension drop occurs at the level of Hochschild homology, but can be avoided by introducing a twist. More interestingly, the simplest twist which avoids the drop is the inverse of the modular group of the weight \( \omega \), while the other possible twists are given by its positive powers. This should be compared to the case of [22, 23] and other examples, where the twist is the inverse of the modular group of the Haar state, and therefore seems to be a general feature of these non-commutative geometries.

2 The spectral triple

2.1 The \( \kappa \)-Poincaré and \( \kappa \)-Minkowski algebras

In this subsection we summarize the algebraic properties of the \( \kappa \)-Poincaré and \( \kappa \)-Minkowski Hopf algebras, which we denote respectively by \( \mathcal{P}_\kappa \) and \( \mathcal{M}_\kappa \). Actually we can restrict our attention to the translations sector of the \( \kappa \)-Poincaré algebra, which is all that is needed to define the \( \kappa \)-Minkowski space, while the complete algebra can be obtained by the bicrossproduct construction [3]. The construction in an arbitrary number of dimensions was given in [16]. We have the generators \( P_\mu \), with \( \mu \) ranging from 0 to \( n-1 \), which satisfy \( [P_\mu, P_\nu] = 0 \). The rest of the Hopf algebra structure is contained in the coproduct \( \Delta : \mathcal{P}_\kappa \to \mathcal{P}_\kappa \otimes \mathcal{P}_\kappa \), the counit \( \varepsilon : \mathcal{P}_\kappa \to \mathbb{C} \) and antipode \( S : \mathcal{P}_\kappa \to \mathcal{P}_\kappa \) which are given by

\[
\Delta(P_0) = P_0 \otimes 1 + 1 \otimes P_0 , \quad \Delta(P_j) = P_j \otimes 1 + e^{-P_0/\kappa} \otimes P_j ,
\]

\[
\varepsilon(P_\mu) = 0 , \quad S(P_0) = -P_0 , \quad S(P_j) = -e^{P_0/\kappa} P_j .
\]

We adopt the usual general relativistic convention of greek indices going from 0 to \( n-1 \), while latin indices go from 1 to \( n-1 \). From the defining relations we see that the translations sector is an Hopf subalgebra of \( \mathcal{P}_\kappa \), and we denote it by \( \mathcal{T}_\kappa \). We define the \( \kappa \)-Minkowski space \( \mathcal{M}_\kappa \) as the dual Hopf algebra to this subalgebra [3]. If we denote the pairing by \( \langle \cdot, \cdot \rangle : \mathcal{T}_\kappa \times \mathcal{M}_\kappa \to \mathbb{C} \), then the structure of \( \mathcal{M}_\kappa \) is determined by the duality relations

\[
\langle t, xy \rangle = \langle t^{(1)}, x \rangle \langle t^{(2)}, y \rangle ,
\]

\[
\langle ts, x \rangle = \langle t, x^{(1)} \rangle \langle s, x^{(2)} \rangle .
\]

Here we have \( t,s \in \mathcal{T}_\kappa \), \( x,y \in \mathcal{M}_\kappa \) and we use the Sweedler notation for the coproduct

\[
\Delta x = \sum_i x^{(1)}_{(i)} \otimes x^{(2)}_{(i)} = x^{(1)} \otimes x^{(2)} .
\]

From the pairing we deduce that \( \mathcal{M}_\kappa \) is non-commutative, since \( \mathcal{T}_\kappa \) is not cocommutative, that is the coproduct in \( \mathcal{T}_\kappa \) is not trivial. On the other hand, since \( \mathcal{T}_\kappa \) is commutative we have that \( \mathcal{M}_\kappa \) is cocommutative. The algebraic relations for the \( \kappa \)-Minkowski Hopf algebra \( \mathcal{M}_\kappa \) are
\[ [X^0, X^j] = -\kappa^{-1} X^j, \quad \Delta X^\mu = X^\mu \otimes 1 + 1 \otimes X^\mu. \]

This concludes the usual presentation of the \(\kappa\)-Poincaré and \(\kappa\)-Minkowski Hopf algebras. In this paper we are going to use a slightly different presentation, as done in \[17\], to avoid issues of convergence in the algebra. Instead of considering the exponential \(e^{-P_0/\kappa}\) as a power series in \(P_0\), we consider it as an invertible element \(E\) and rewrite the defining relations as

\[
\begin{align*}
[P_\mu, P_\nu] &= 0, \quad [P_\mu, E] = 0, \\
\Delta(P_0) &= P_0 \otimes 1 + 1 \otimes P_0, \quad \Delta(P_j) = P_j \otimes 1 + E \otimes P_j, \quad \Delta(E) = E \otimes E, \\
\varepsilon(P_\mu) &= 0, \quad \varepsilon(E) = 1, \\
S(P_0) &= -P_0, \quad S(P_j) = -E^{-1} P_j, \quad S(E) = E^{-1}.
\end{align*}
\]

In this form, the role of the subalgebra \(T_\kappa\) is played by the one generated by \(P_\mu\) and \(E\), which we call the extended momentum algebra and denote again by \(T_\kappa\). An appropriate pairing defining \(\kappa\)-Minkowski space can be easily written in terms of these generators. More importantly it can be made into a Hopf \(*\)-algebra by defining the involution as \(P_\mu^* = P_\mu\) and \(E^* = E\).

In the following we are going to consider the case of Euclidean signature and so, strictly speaking, we should refer to the Euclidean counterpart of the \(\kappa\)-Poincaré algebra, which is known as the quantum Euclidean group. However the boost generator \(N\) is not going to play a central role in our discussion, which is going to be based on the extended momentum algebra, and therefore most of our relations do not depend on the signature. Henceforth we only make reference to the \(\kappa\)-Poincaré algebra and make some remarks when needed.

One more remark on the notation: we are going to write all formulae in terms of the parameter \(\lambda := \kappa^{-1}\), instead of \(\kappa\). The motivation comes from the fact that the Poincaré algebra is obtained in the “classical limit” \(\lambda \to 0\), in a similar fashion to the classical limit \(\hbar \to 0\) of quantum mechanics. This makes more transparent checking that some formulae reduce, in this limit, to their respective undeformed counterparts.

### 2.2 The algebraic construction

We begin by generalizing to \(n\) dimensions the construction of the \(*\)-algebra given in \[17\]. We will skip most of the computations, since they are completely analogous to the two-dimensional case, but we will provide some details regarding the modular aspects of the construction.

The underlying algebra of the \(n\)-dimensional \(\kappa\)-Minkowski space is the enveloping algebra of the Lie algebra with generators \(ix^0\) and \(ix^k\) (with \(k = 1, \ldots, n - 1\)), fulfilling the commutation relations \([x^0, x^k] = i\lambda x^k\). It has a faithful \(n \times n\) matrix representation \(\varphi\) given by

\[
\begin{align*}
\varphi(ix^0) &= \begin{pmatrix}
-\lambda & \cdots & 0 \\
\vdots & \ddots & 0 \\
0 & \cdots & 0
\end{pmatrix}, & \varphi(ix^k) &= \begin{pmatrix}
0 & \cdots & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 0 & \cdots & 0
\end{pmatrix}.
\end{align*}
\]

The matrix \(\varphi(ix^k)\) has non-zero values only in the \((k + 1)\)-th column. An element of the
associated group $G$ can be presented in the form

$$S(a) = \begin{pmatrix} e^{-\lambda a_0} & a_1 & \cdots & a_{n-1} \\ 0 & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \quad (2.1)$$

Here we use the notation $a = (a_0, \vec{a})$, where $\vec{a} = (a_1, \cdots, a_{n-1})$. The group operations written in the $(a_0, \cdots, a_{n-1})$ coordinates are given by

$$S(a)S(b) = S(a_0 + b_0, \vec{a} + e^{-\lambda a_0} \vec{b}), \quad S(a)^{-1} = S(-a_0, -e^{\lambda a_0} \vec{a}). \quad (2.2)$$

**Proposition 1.** The left and right invariant measures on $G$ are given respectively by $d\mu_L(a) = e^{\lambda(n-1)a_0} d^m a$ and $d\mu_R(a) = d^m a$, where $d^m a$ is the Lebesgue measure on $\mathbb{R}^n$.

**Proof.** We do the computation for the left invariant case. Using the $(a_0, \cdots, a_{n-1})$ coordinates and the group operations given in (2.2) we easily find

$$\int f(a \cdot b) d\mu_L(b) = \int f(a_0 + b_0, \vec{a} + e^{-\lambda a_0} \vec{b}) e^{\lambda(n-1)b_0} d^m b$$

$$= \int f(a_0 + b_0, \vec{a} + \vec{b}) e^{\lambda(n-1)(a_0 + b_0)} d^m b$$

$$= \int f(b_0, \vec{b}) e^{\lambda(n-1)b_0} d^m b = \int f(b) d\mu_L(b).$$

The right invariant case is treated similarly. \hfill \square

We have that $G$ is not a unimodular group, with the modular function $e^{-\lambda(n-1)a_0}$ playing a central role in the following. We consider the convolution algebra of $G$ with respect to the right invariant measure, and we identify functions on $G$ with functions on $\mathbb{R}^n$ by the parametrization (2.1). The convolution algebra is an involutive Banach algebra consisting of integrable functions on $\mathbb{R}^n$ with product $\ast$ and involution $\hat{\ast}$ given by

$$(f \hat{\ast} g)(a) = \int f(a_0 - a'_0, \vec{a} - e^{-\lambda(a_0-a'_0)} \vec{a}') g(a'_0, \vec{a}') d^m a,'$$

$$f^\ast(a) = e^{\lambda(n-1)a_0} \mathcal{F}^{-1}(-a_0, -e^{\lambda a_0} \vec{a}).$$

We pass from momentum space to configuration space via the Fourier transform. The star-product and involution are defined, in terms of the convolution algebra operations, as

$$f \ast g := \mathcal{F}^{-1} (\mathcal{F}(f) \hat{\ast} \mathcal{F}(g)), \quad f^\ast := \mathcal{F}^{-1} (\mathcal{F}(f)^\ast).$$

These formulae are written for clarity using the unitary convention for the Fourier transform, but in the following we will use the physicists convention with the $(2\pi)^n$ in momentum space. We restrict our attention to the following space of functions.
Definition 2. Denote by $S_c$ the space of Schwartz functions on $\mathbb{R}^n$ with compact support in the first variable, that is for $f \in S_c$ we have $\text{supp}(f) \subseteq K \times \mathbb{R}^{n-1}$ for some compact $K \subset \mathbb{R}$. We define $\mathcal{A} = \mathcal{F}(S_c)$, where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^n$.

On this space we can safely perform all the operations we need. The next proposition shows that $\mathcal{A}$ is a $*$-algebra and gives explicit formulae for the star product and the involution. We use the notation $x = (x_0, \vec{x})$ and $\vec{x} = (x_1, \ldots, x_{n-1})$ for the coordinates on $\mathcal{A}$.

Proposition 3. For $f, g \in \mathcal{A}$ we have

$$
(f \star g)(x) = \int e^{ip_0x_0}(\mathcal{F}_0 f)(p_0, \vec{x})g(x_0, e^{-\lambda p_0} \vec{x})\frac{dp_0}{2\pi},
$$

$$
f^*(x) = \int e^{ip_0x_0}(\mathcal{F}_0 \bar{f})(p_0, e^{-\lambda p_0} \vec{x})\frac{dp_0}{2\pi}.
$$

We have that $f \star g \in \mathcal{A}$ and $f^* \in \mathcal{A}$, so that $\mathcal{A}$ is a $*$-algebra.

Proof. Let us show how this works for the involution. From the definitions we get

$$
f^*(x) = \mathcal{F}^{-1}(\mathcal{F}(f)^*)(x) = \int e^{ipx}(\mathcal{F}(f)^*)(p)\frac{dp}{(2\pi)^n} = \int e^{ipx} e^{(n-1)\lambda p_0} (\mathcal{F}\bar{f})(-p_0, -e^{\lambda p_0} \vec{p})\frac{dp}{(2\pi)^n}.
$$

Now using the change of variables $\vec{p} \to e^{-\lambda p_0} \vec{p}$ we find

$$
f^*(x) = \int e^{ip_0x_0} e^{i\lambda p_0 \vec{p} \cdot \vec{x}} (\mathcal{F}\bar{f})(-p)\frac{dp}{(2\pi)^n} = \int e^{i p_0 x_0} e^{i\lambda p_0 \vec{p} \cdot \vec{x}} (\mathcal{F}\bar{f})(p)\frac{dp}{(2\pi)^n}.
$$

Finally performing the Fourier transform in the $\vec{p}$ variables we find

$$
f^*(x) = \int e^{i p_0 x_0} (\mathcal{F}_0 \bar{f})(p_0, e^{-\lambda p_0} \vec{x})\frac{dp_0}{2\pi}.
$$

The product can be computed in a similar way. For more details see [17].

A nice property of this algebra is that it comes naturally with an action of the $\kappa$-Poincaré algebra $\mathcal{P}_\kappa$ on it. In [17] it was proven that $\mathcal{A}$ is a left $\mathcal{P}_\kappa$-module $*$-algebra, which means that the action of the $\kappa$-Poincaré symmetries on $\mathcal{A}$ preserves the Hopf algebraic structure. In particular the action of the translations sector is elementary, with $(P_\mu \triangleright f)(x) = -i(\partial_\mu f)(x)$ and $(E \triangleright f)(x) = f(x_0 + i\lambda, \vec{x})$. This remains true for the $n$-dimensional case.

Now we can introduce a Hilbert space by using the GNS-construction for $\mathcal{A}$, after the choice of some weight $\omega$. There is a natural choice which respects the symmetries of the $\kappa$-Poincaré Hopf algebra, see [17, 18]: it is simply given by the integral of a function $f \in \mathcal{A}$ with respect to the Lebesgue measure over $\mathbb{R}^n$, and we denote it by $\omega$. Contrarily to the commutative case it does not satisfy the trace property.
Proposition 4. For \( f, g \in \mathcal{A} \) we have the twisted trace property

\[
\int (f \ast g)(x) d^n x = \int (\sigma^{-1}(g) \ast f)(x) d^n x,
\]

where we define \( \sigma(g)(x) := g(x_0 + i\lambda, \bar{x}) \).

Proof. First we use the change of variables \( \bar{x} \to e^{\lambda p_0} \bar{x} \) and obtain

\[
\int (f \ast g)(x) d^n x = \int \int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, \bar{x}) g(x_0, e^{-\lambda p_0} \bar{x}) \frac{dp_0}{2\pi} d^n x
\]

\[
= \int \int e^{ip_0 x_0} e^{(n-1)\lambda p_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0} \bar{x}) g(x_0, \bar{x}) d^n x \frac{dp_0}{2\pi}.
\]

Now, using the analyticity of the functions of \( \mathcal{A} \) in the first variable, we can shift \( x_0 \to x_0 + i(n-1)\lambda \) to obtain the action of \( \sigma^{-1} \) on \( g \), that is

\[
\int (f \ast g)(x) d^n x = \int \int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0} \bar{x}) g(x_0 + i(n-1)\lambda, \bar{x}) d^n x \frac{dp_0}{2\pi}
\]

\[
= \int \int e^{ip_0 x_0} (\mathcal{F}_0 f)(p_0, e^{\lambda p_0} \bar{x}) \sigma^{n-1}(g)(x_0, \bar{x}) d^n x \frac{dp_0}{2\pi}.
\]

It only remains to rewrite this expression in terms of the \( \ast \)-product. Writing explicitly the Fourier transform \( \mathcal{F}_0 f \) we have

\[
\int (f \ast g)(x) d^n x = \int \int e^{ip_0 x_0} \int e^{-ip_0 y_0} f(y_0, e^{\lambda p_0} \bar{x}) \sigma^{n-1}(g)(x_0, \bar{x}) dy_0 d^n x \frac{dp_0}{2\pi}.
\]

We need to do some rearranging: change \( p_0 \to -p_0 \), relabel \( y_0 \leftrightarrow x_0 \) and exchange the order of the \( x_0 \) and \( y_0 \) integral. The result of these operations is

\[
\int (f \ast g)(x) d^n x = \int \int e^{ip_0 x_0} f(x_0, e^{-\lambda p_0} \bar{x}) \int e^{-ip_0 y_0} \sigma^{n-1}(g)(y_0, \bar{x}) dy_0 d^n x \frac{dp_0}{2\pi}.
\]

But now the last integral is just the Fourier transform of \( \sigma^{n-1}(g) \) in the \( y_0 \) variable, so

\[
\int (f \ast g)(x) d^n x = \int \int e^{ip_0 x_0} (\mathcal{F}_0 \sigma^{n-1}(g))(p_0, \bar{x}) f(x_0, e^{-\lambda p_0} \bar{x}) \frac{dp_0}{2\pi} d^n x.
\]

Finally we observe that the right hand side is just the integral of the function \( (\sigma^{n-1}(g) \ast f)(x) \), which proves the result. \( \square \)

As observed in [S] this property can be rephrased as a KMS condition for \( \omega \).

Proposition 5. The weight \( \omega \) satisfies the KMS condition with respect to the modular group \( \sigma^\omega \), defined by \( (\sigma^\omega_t f)(x_0, \bar{x}) := f(x_0 - t(n-1)\lambda, \bar{x}) \). The associated modular operator is \( \Delta_\omega = e^{-(n-1)\lambda P_0} \), where \( P_0 = -i\partial_0 \).
On the Hilbert space $\mathcal{H}$, obtained by the GNS-construction for $\omega$, the algebra $\mathcal{A}$ acts via left multiplication, that is $\pi(f)\psi := f \star \psi$. In the following we omit the representation symbol $\pi$ and just write $f$ for the operator of left multiplication by this function.

It is important to point out that the Hilbert space $\mathcal{H}$ is not $L^2(\mathbb{R}^n)$. On the other hand, using the fact that $\mathcal{A}$ is dense in both Hilbert spaces, one can easily find a unitary operator between the two. One can also find, using this unitary operator, the Schwartz kernel of a certain class of operators which will be of interest to us in the following. These results are the content of the next proposition, for details about the derivation see [8].

**Proposition 6.** The Hilbert space $\mathcal{H}$ obtained by the GNS-construction for $\omega$ is unitarily equivalent to $L^2(\mathbb{R}^n)$, via the unitary operator given by

$$
(Uf)(x) = \int e^{ipx_0} (\mathcal{F}_0 f)(p_0, e^{-\lambda p_0} \hat{x}) \frac{dp_0}{2\pi}.
$$

Consider now the operator $U\pi(f)g(P)U^{-1}$ acting on $L^2(\mathbb{R}^n)$, where $f \in \mathcal{A}$ and $P_\mu = -i\partial_\mu$. Then its Schwartz kernel is given by

$$
K(x, y) = \int e^{ip(x-y)} (Uf)(x_0, e^{\lambda p_0} \hat{x}) g(p_0, e^{-\lambda p_0} \hat{p}) \frac{dp_0}{2\pi}.
$$

### 2.3 Dirac operator and differential calculus

The next step is the introduction of a self-adjoint operator $D$ satisfying certain conditions, the so-called Dirac operator. From the analysis given in [8] we know that to obtain a boundedness condition for $D$ we need to use a twisted commutator [9]. This amounts to introducing an automorphism $\sigma$ of the algebra $\mathcal{A}$, the twist. Then for each $f \in \mathcal{A}$ the operator $[D, f]_\sigma = Df - \sigma(f)D$ should be bounded. One needs to find what are the possible choices for $D$ and the automorphism $\sigma$ such that this condition is fulfilled. We consider some additional assumptions which are related to the symmetries and the classical limit, which we state precisely below.

The analysis for the general $n$-dimensional case is essentially identical to the two-dimensional case, so we skip the computations and refer to [8] for details.

As in the classical case we enlarge the Hilbert space to accommodate for spinors. Therefore we consider $\mathcal{H} = \mathcal{H}_r \otimes \mathbb{C}^{[n/2]}$, where $\mathcal{H}_r$ is the Hilbert space previously introduced. Here $[n/2]$ is the dimension of the spinor bundle on $\mathbb{R}^n$, and we use the notation $\Gamma^\mu$ for the matrix representation of the Clifford algebra, which satisfy $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$. Then we can write $D$ in the form $D = \Gamma^\mu \hat{D}_\mu$, where $\hat{D}_\mu$ are self-adjoint operators on $\mathcal{H}_r$.

Now we state our assumptions for the Dirac operator $D$ and the automorphism $\sigma$. We denote by $\rho$ the map from the extended momentum algebra $\mathcal{T}_\kappa$ to (possibly unbounded) operators on $\mathcal{H}_r$, see [8]. Since $D$ should be determined by the symmetries, we assume that $\hat{D}_\mu = \rho(D_\mu)$ for some $D_\mu \in \mathcal{T}_\kappa$, which is basically the requirement of equivariance. Similarly we assume that $\sigma$ is given by $\sigma(f) = \sigma \triangleright f$ for some $\sigma \in \mathcal{T}_\kappa$, which to be an automorphism must have a coproduct of the form $\Delta(\sigma) = \sigma \otimes \sigma$. Since the parameter $\lambda$ is a physical quantity of the model, which has the dimension of a length, the Dirac operator must have the dimension of an inverse length. Moreover we require that $D$ reduces to the classical Dirac operator in the limit $\lambda \to 0$, by which we mean that for all $\psi \in \mathcal{A}$ we should have $\lim D_\mu \psi = \tilde{P}_\mu \psi$. 

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Proposition 7. Under the assumptions given above, we have that there is a unique operator $D$ and a unique automorphism $\sigma$ such that $[D, f]_\sigma$ is bounded for every $f \in \mathcal{A}$. They are given by $D = \Gamma^\mu D_\mu$, with $D_0 = \lambda^{-1}(1 - e^{-\lambda P_0})$ and $D_j = P_j$, while $\sigma = e^{-\lambda P_0}$.

Notice that formally for $\lambda \to 0$ we obtain the usual Dirac operator on $\mathbb{R}^n$. We have the interesting relation $\Delta_\omega D^2 = C$, where $\Delta_\omega$ is the modular operator of the weight $\omega$ and $C$ is the first Casimir of the $\kappa$-Poincaré algebra, which is given by

$$C = \frac{4}{\lambda^2} \sinh^2 \left( \frac{\lambda P_0}{2} \right) + \sum_{j=1}^{n-1} e^{\lambda P_0} P_j^2. $$

Now we discuss some aspects of the differential calculus associated with the operator $D$. For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ one defines the $\mathcal{A}$-bimodule $\Omega_D^1$ of one-forms as the linear span of operators of the form $a[D, b]$, with $a, b \in \mathcal{A}$. Then $d(a) = [D, a]$ is a derivation of $\mathcal{A}$ with values in $\Omega_D^1$, that is $d(ab) = d(a)b + ad(b)$, which immediately follows from the properties of the commutator. In the twisted case this definition must be modified, since we have

$$[D, ab]_\sigma = [D, a]_\sigma b + \sigma(a)[D, b]_\sigma. $$

The necessary modification is very simple [9]. One simply defines $\Omega_D^1$ to be the linear span of operators of the form $a[D, b]_\sigma$, with the bimodule structure given by $a \cdot [D, b]_\sigma \cdot c = \sigma(a)[D, b]_\sigma c$. Then it is obvious that $d_\sigma(a) = [D, a]_\sigma$ is a derivation of $\mathcal{A}$ with values in $\Omega_D^1$.

In the non-compact case, already at the untwisted level, it is not completely clear how one should generalize this notion. One can replace the algebra $\mathcal{A}$ with some unitization, as done in [15]. However in this case there is no analogue of the one-form $dx^\mu$, since the function $x^\mu$ does not belong to $\mathcal{A}$ or some unitization of it. Nevertheless, it is clear that in the commutative case $[D, x^\mu]$ extends to a bounded operator, in particular it is equal to $-i\Gamma^\mu$. This is also the case in this non-commutative setting; indeed notice that for any $f \in \mathcal{A}$ we have the equality

$$[D, f]_\sigma = \Gamma^\mu (D_\mu \triangleright f), $$

where $D_0 = \lambda^{-1}(1 - e^{-\lambda P_0})$ and $D_j = P_j$. Now it is easy to see that the twisted commutator $[D, x^\mu]_\sigma$ extends to a bounded operator and in particular $[D, x^\mu]_\sigma = -i\Gamma^\mu$, as in the commutative case. Adopting the natural notation $df = [D, f]_\sigma$, we can write $df = dx^\mu (iD_\mu \triangleright f)$. Then from the bimodule structure on $\Omega_D^1$ it then follows that

$$df = dx^\mu \cdot (iD_\mu \triangleright f) = \sigma^{-1}(iD_\mu \triangleright f) \cdot dx^\mu. $$

In [19] the introduction of bicovariant differential calculi on $\kappa$-Minkowski space was investigated. It follows from our construction that the differential calculus defined by the operator $D$ is an example of such a bicovariant differential calculus. We have the relation

$$x^\mu \cdot dx^\nu - dx^\nu \cdot x^\mu = \sigma(x^\mu) dx^\nu - dx^\nu x^\mu = i\lambda \delta_0^\mu dx^\nu. $$

Therefore in the notation of [19] we obtain $[x^\mu, dx^\nu] = iA^\mu_\rho dx^\rho$ with $A^\mu_\rho = \lambda \delta_0^\mu \delta_\rho^\nu$. 

9
3 Spectral dimension

3.1 The spectral dimension

In [8] it was shown, for the two-dimensional case, that the ingredients introduced in the previous section do not give a finitely summable (twisted) spectral triple. We can try to interpret this result with the following heuristic argument: suppose we did find a spectral dimension \( n \), coinciding with the classical dimension. Then, from the general properties of twisted spectral triples, it would follow that \( \varphi(ab) = \varphi(\sigma^n(b)a) \), where \( \sigma(a) = e^{-\lambda P_0}a e^{\lambda P_0} \) and \( \varphi \) is the non-commutative integral (defined, for example, in terms of the Dixmier trace). The weight \( \omega \), on the other hand, satisfies \( \omega(f \ast g) = \omega(\sigma^n(g) \ast f) \), where \( \sigma^n(a) = e^{-\lambda P_0}a e^{\lambda P_0} \). Therefore we have a mismatch between the modular properties of the weight \( \omega \) and the integral \( \varphi \), which shows that we can not recover the weight \( \omega \) from the non-commutative integral.

This argument leaves open the possibility that this could happen if the spectral dimension were equal to \( n - 1 \), but this is shown not to be the case by the explicit calculation. In [8] it was argued that one has to use a weight to obtain finite summability, as in the framework of modular spectral triples [10, 11, 12]. The relevant definition for us is the following.

**Definition 8.** Let \((\mathcal{A}, \mathcal{H}, D)\) be a non-compact modular spectral triple with weight \( \Phi \). We say that it is finitely summable and call \( p \) the spectral dimension if the following quantity exists

\[
p := \inf\{ s > 0 : \forall a \in \mathcal{A}, a \geq 0, \ |\Phi(a(D^2 + 1)^{-s/2})| < \infty \}.
\]

We can choose our weight to be of the form \( \Phi(\cdot) = \text{Tr}(\Delta^{\cdot}) \), where \( \Delta \) is a positive and invertible operator. We call it the modular operator associated to the weight \( \Phi \), and denote the corresponding modular group by \( \sigma^{\cdot} \). As we discussed above, the mismatch of one power of \( e^{-\lambda P_0} \) suggests setting \( \Delta = e^{-\lambda P_0} \) as the modular element. It is instructive to consider a slightly more general situation, which we discuss in the following proposition.

**Proposition 9.** Let \( \Phi_t(\cdot) = \text{Tr}(\Delta^{\cdot}_t) \) be the weight with modular operator \( \Delta^t = e^{-t\lambda P_0} \). Then, for any \( f \in \mathcal{A} \), we have \( \Phi_t(f(D^2 + \mu^2)^{-s/2}) < \infty \) if and only if \( t > 0 \) and \( s > n - 1 + t \). In other words, the spectral dimension exists for \( t > 0 \) and is given by \( p = n - 1 + t \).

**Proof.** First of all notice that we have \( \Delta^t f = \sigma^t(f) \Delta^t \), so that we can consider without loss of generality the operator \( A := f\Delta^t(D^2 + \mu^2)^{-s/2} \). Using the unitary operator \( U \) we can consider \( A \) as an operator on \( L^2(\mathbb{R}^n \otimes \mathbb{C}^{2^{n/2}}) \) whose symbol, thanks to Proposition [10] is given by \( a(x, \xi) := (Uf)(x_0, e^{i\xi_0} \xi) G^\Delta_{s,t}(\xi_0, e^{-\lambda\xi_0} \xi) \), where we have defined

\[
G^\Delta_{s,t}(\xi) = e^{-t\xi_0} \left( \lambda^{-2} \left( 1 - e^{-\lambda\xi_0} \right)^2 + \xi^2 + \mu^2 \right)^{-s/2}.
\]

To prove that the operator \( A \) is trace-class it suffices to show that its symbol and certain number of its derivatives are integrable [23]. We now show that the symbol \( a(x, \xi) \) is integrable. With a simple change of variables we can factorize the integral as

\[
\int |a(x, \xi)| d^n x d^n \xi = \int G^\Delta_{s,t}(\xi) d^n \xi \int |(Uf)(x)| d^n x.
\]
The integral of \((Uf)(x)\) is clearly finite. We can now perform the integral in the variables \((\xi_1, \cdots, \xi_{n-1})\) using the well known formula
\[
\int (\xi^2 + a^2)^{-z/2} d^N \xi = \pi^{N/2} \frac{\Gamma \left( \frac{z-N}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} a^{-\left( z-N \right)},
\]
which is valid for \(\text{Re}(z) > N\). Then we have
\[
\int G^\Delta_{s,t}(\xi) d^p \xi = \pi^{(n-1)/2} \frac{\Gamma \left( \frac{s-(n-1)}{2} \right)}{\Gamma \left( \frac{s}{2} \right)} \int e^{-t\lambda\xi_0} \left( \lambda^{-2} \left( 1 - e^{-\lambda\xi_0} \right)^2 + \mu^2 \right)^{-\frac{s-(n-1)}{2}} d\xi_0,
\]
provided that \(s > n - 1\). To proceed further we consider the asymptotics of the integrand
\[
\tilde{I}_t(s) := e^{-t\lambda\xi_0} \left( \lambda^{-2} \left( 1 - e^{-\lambda\xi_0} \right)^2 + \mu^2 \right)^{-\frac{s-(n-1)}{2}}.
\]
For \(\xi_0 \to +\infty\) we have \(\tilde{I}_t \sim e^{-t\lambda|\xi_0|}\), so it integrable provided that \(t > 0\), independently of \(s\). In the other regime \(\xi_0 \to -\infty\) we have instead
\[
\tilde{I}_t \sim e^{t\lambda|\xi_0|} e^{-\left( s-(n-1) \right)\lambda|\xi_0|} = e^{-\left( s-(n-1)-t \right)\lambda|\xi_0|},
\]
which is integrable when \(s > n - 1 + t\). It is easy to see that the various derivatives of the symbol \(a(x, \xi)\) are integrable under these conditions. Finally taking the infimum over \(s\) we obtain that the spectral dimension is \(p = n - 1 + t\).

We see that by introducing the weight \(\Phi_t\) we are able to obtain a finite spectral dimension. But what about the free parameter \(t\)? A natural choice is to fix \(t = 1\), in such a way that the spectral dimension coincides with the classical dimension \(n\). To understand this ambiguity let us consider for a moment a generic situation, where we have a weight \(\omega\).

Let us consider now our specific case: the twist of the commutator is\(\Phi\). We see that by introducing the weight \(\Phi_t\) we are able to recover the KMS condition for the weight \(\omega\). This provides strength to the argument that recovering the weight \(\omega\) from the non-commutative integral provides the right guidance in this setting. At the same time it shows that this is not enough to fix the free parameter \(t\), since it disappears in the combination \(\sigma^\Phi_t \circ \sigma^p\). We do not know at the moment what kind of condition could select the value \(t = 1\) uniquely (apart from recovering the classical dimension, of course).
3.2 Poles of the zeta function

In the following we fix $t = 1$. Then the function $\Phi(f(D^2 + \mu^2)^{-s/2})$ has a singularity at $s = 1$, whose nature we now want to investigate, along with its analytic continuation to the complex plane. The singularities of this kind of “zeta function” play an important role in the local index formula of Connes and Moscovici [26]. Before starting the analysis, let us briefly review the commutative case of $\mathbb{R}^n$. The Dirac operator is given by $D = -i\Gamma^\mu \partial_\mu$, where $\Gamma^\mu$ are the gamma matrices satisfying the relations $\{\Gamma^\mu, \Gamma^\nu\} = 2\delta^{\mu\nu}$ and the dimension of the spinor bundle is $2^{[n/2]}$. We consider the zeta function defined by

$$\zeta_f(z) = \text{Tr}(f(D^2 + \mu^2)^{-z/2}) .$$

Here $\mu$ is, as usual, a non-zero real number needed to compensate for the lack of invertibility of $D$. An immediate computation shows that

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} \int (\xi^2 + \mu^2)^{-z/2} d^n \xi \int f(x)d^n x ,$$

where the coefficient $2^{[n/2]}$ comes from the trace over the spinor bundle. The integral over $\xi$ is finite for $\text{Re}(z) > n$ and we get

$$I_c(z) := \int (\xi^2 + \mu^2)^{-z/2} d^n \xi = \pi^{n/2} \mu^{n-z} \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{z}{2}\right)} . \quad (3.1)$$

We obtain an analytic continuation using well-known properties of the gamma function, and we find that the only singularities of $\zeta_f(z)$ are simple poles. Indeed $\Gamma(z)$ has poles on the negative real axis at $z = 0, -1, -2, \cdots$, so that the function $\Gamma\left(\frac{z-n}{2}\right)$ has poles at $z = n - 2m$, where $m \in \mathbb{N}_0$. When $n$ is even the poles at $z = 0, -2, -4, \cdots$ are canceled by the zeroes of $\Gamma\left(\frac{z}{2}\right)$. Then the result is that $\zeta_f(z)$ has simple poles at $z = n, n - 2, \cdots$ when $n$ is even, and has simple poles at $z = n, n - 2, \cdots, 1, -1, -3, \cdots$ when $n$ is odd.

For compact Riemannian manifolds this kind of zeta function has been studied by Minakshisundaram and Pleijel [20], and here we have the analogous result for $\mathbb{R}^n$. We can easily compute the residue at $z = n$ of $\zeta_f(z)$, which is given by

$$\text{Res}_{z=n}\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} \frac{2\pi^{n/2}}{\Gamma\left(\frac{z}{2}\right)} \int f(x)d^n x .$$

Now we are ready to study the singularities and the analytic continuation in the case of $\kappa$-Minkowski space, where the relevant zeta function is defined by

$$\zeta_f(z) := \Phi\left(f(D^2 + \mu^2)^{-z/2}\right) ,$$

where we recall that $\Phi(\cdot) = \text{Tr}(\Delta \Phi \cdot)$ and we omit the representation symbol $\pi$.

**Proposition 10.** Let $f \in \mathcal{A}$ and $\text{Re}(z) > n$. Then we have

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} I(z) \int f(x)d^n x ,$$

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where \( I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z)) \), with the function \( I_c(z) \) being the classical result given in (3.1), which is independent of \( \lambda \), and the function \( I_\lambda(z) \) being given by

\[
I_\lambda(z) = \pi^{(n-1)/2} \mu^{(n-1)-z} \frac{\Gamma \left( \frac{z-(n-1)}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} \lambda^{-1} \, _2F_1 \left( \frac{1}{2}, z - \frac{(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda \mu)^2} \right).
\]

The function \( I(z) \) reduces to the classical one \( I_c(z) \) in the limit \( \lambda \to 0 \).

**Proof.** From the proof of Proposition 13 we have

\[
\text{Tr} \left( f \Delta (D^2 + \mu^2)^{-z/2} \right) = \frac{2^{[n/2]}}{(2\pi)^n} \int G_s^\Delta(\xi)d^n \xi \int (Uf)(x)d^n x,
\]

where we recall that we have set \( t = 1 \). Similarly to the classical case we set \( I(z) := \int G_s^\Delta(\xi)d^n \xi \). We already partially computed this integral, and the result was

\[
I(z) = \pi^{(n-1)/2} \frac{\Gamma \left( \frac{z-(n-1)}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} \int e^{-t\xi_0} \left( \lambda^{-2} \left( 1 - e^{-\xi_0} \right)^2 + \mu^2 \right) \frac{z-(n-1)}{2} d\xi_0.
\]

We need to compute the last integral. First we do the change of variable \( r = e^{-\xi_0} \) and obtain

\[
I(z) = \pi^{(n-1)/2} \frac{\Gamma \left( \frac{z-(n-1)}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} \lambda^{z-n} \int_0^\infty \left( 1 - r \right)^2 + (\lambda \mu)^2 \frac{z-(n-1)}{2} dr.
\]

This integral can be computed analytically. We use the formula

\[
\int_0^\infty (1 - r)^2 + a^2 \, dr = a^{-2z} \left[ \frac{a \sqrt{\pi} \Gamma \left( \frac{n}{2} \right)}{2} \frac{\Gamma \left( z - \frac{1}{2} \right)}{\Gamma \left( z \right)} + \frac{1}{2} \, _2F_1 \left( \frac{1}{2}, z; \frac{3}{2}; -\frac{1}{a^2} \right) \right],
\]

which is valid for \( \text{Re}(z) > 1/2 \). Here \( _2F_1(a, b; c; z) \) is the ordinary hypergeometric function. Therefore the integral in \( I(z) \) is finite for \( \text{Re}(z) > n \) and we have

\[
I(z) = \frac{1}{2} \pi^{n/2} \mu^{z-n} \frac{\Gamma \left( \frac{z-n}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} + \frac{1}{2} I_\lambda(z),
\]

where we have defined the function

\[
I_\lambda(z) := 2\pi^{(n-1)/2} \mu^{(n-1)-z} \frac{\Gamma \left( \frac{z-(n-1)}{2} \right)}{\Gamma \left( \frac{z}{2} \right)} \lambda^{-1} \, _2F_1 \left( \frac{1}{2}, z - \frac{(n-1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda \mu)^2} \right).
\]

Notice that we have \( I(z) = \frac{1}{2}(I_c(z) + I_\lambda(z)) \). Finally we have \( \int U f = \int f \), which is valid for \( f \in \mathcal{A} \), from which the first part of the proposition follows.
Now we want to consider the classical limit of $I(z)$, in the case $\text{Re}(z) > n$. Using the linear transformation formulae for the hypergeometric function $2F_1(a, b; c; z)$ it is easy to obtain an asymptotic expansion for large negative $z$, see \[21\]. This expansion takes the form

$$2F_1(a, b; c; -z) \sim \frac{\Gamma(c)\Gamma(b - a)}{\Gamma(b)\Gamma(c - a)} z^{-a} + \frac{\Gamma(c)\Gamma(a - b)}{\Gamma(a)\Gamma(c - b)} z^{-b}.$$  

With this result it is easy to compute the limit

$$\lim_{\lambda \to 0} \lambda^{-1} 2F_1 \left( \frac{1}{2}, \frac{z - (n - 1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda \mu)^2} \right) = \frac{\sqrt{\pi}}{2} \frac{\Gamma \left( \frac{z - n}{2} \right)}{\Gamma \left( \frac{z - (n - 1)}{2} \right)} \lambda.$$  

Then we see that $I(z)$ reduces to $I_c(z)$ in the classical limit $\lambda \to 0$.

**Corollary 11.** For $f \in \mathcal{A}$ we have

$$\text{Res}_{z=n} \zeta_f(z) = c_n \omega(f),$$

where the constant is defined as $c_n = \frac{2^{[n/2]} \pi^{n/2}}{(2\pi)^n \Gamma \left( \frac{n}{2} \right)}$.

**Proof.** The function $I_\lambda(z)$ is regular at $z = n$, so we get one-half of the classical residue, that is we have $\text{Res}_{z=n} I(z) = \pi^{n/2} / \Gamma \left( \frac{n}{2} \right)$. The result follows immediately.  

**Proposition 12.** Let $f \in \mathcal{A}$. Then the zeta function

$$\zeta_f(z) = \frac{2^{[n/2]}}{(2\pi)^n} I(z) \int f(x) d^n x$$

has a meromorphic extension to the whole complex plane with only simple poles.

**Proof.** Since $I(z) = \frac{1}{2} (I_c(z) + I_\lambda(z))$, where $I_c(z)$ is the integral arising in the commutative case, the zeta function $\zeta_f(z)$ has the poles of the commutative case plus additional poles coming from the function $I_\lambda(z)$. To study them consider the hypergeometric function $2F_1(a, b; c; z)$, with the assumption that $c$ does not belong to $\{0, -1, -2, \cdots \}$. The series defining $2F_1(a, b; c; z)$ is convergent in the open disk $|z| < 1$, but can be analytically continued to the entire complex plane with a branch cut from $z = 1$ to $z = \infty$. Therefore the function

$$2F_1 \left( \frac{1}{2}, \frac{z - (n - 1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda \mu)^2} \right)$$

does not have any poles in $z$. Now recall that the function $I_\lambda(z)$ is defined by

$$I_\lambda(z) = 2\pi^{(n-1)/2} \lambda^{-(n-1)-z} \frac{\Gamma \left( \frac{z - (n-1)}{2} \right)}{\Gamma \left( \frac{\hat{z}}{2} \right)} 2F_1 \left( \frac{1}{2}, \frac{z - (n - 1)}{2}; \frac{3}{2}; -\frac{1}{(\lambda \mu)^2} \right).$$

Therefore the only poles of this function come from the ratio of the two gamma functions. These are simple poles, from which the claim follows.

It is interesting to note that the poles of the function $I_c(z)$ come from the ratio $\Gamma \left( \frac{z - n}{2} \right) / \Gamma \left( \frac{\hat{z}}{2} \right)$, while the poles of the function $I_\lambda(z)$ come from the ratio $\Gamma \left( \frac{z - (n - 1)}{2} \right) / \Gamma \left( \frac{\hat{z}}{2} \right)$: the latter are therefore the poles of the $(n - 1)$-dimensional case. If we think of the Lorentzian version of $\kappa$-Minkowski space, we can relate this result to the different properties of the time direction from the space directions, evident already from the commutation relations.
4 Twisted homology

4.1 Motivation and preliminaries

In this section we want to study the homological properties of $\kappa$-Minkowski space. In the non-compact setting it is not completely clear, at least as far as we understand, which algebra should be considered in this respect. A possibility is to consider a certain unitalization of the algebra $\mathcal{A}$ in consideration, as done in [15]. On the other hand, already at the commutative level, if we consider $\mathbb{R}^n$ we would like to have an analog of the volume form $dx^1 \wedge \cdots \wedge dx^n$, but is clear that the functions $x^n$ do not belong to a unital algebra.

Our plan is to investigate the homological properties of the enveloping algebra $U(\mathfrak{g}_\kappa)$, where $\mathfrak{g}_\kappa$ is the Lie algebra underlying $\kappa$-Minkowski space. General results for the twisted homology of an enveloping algebra are given in [24], where the twist is called the Nakayama automorphism. Here we choose a more elementary approach, which involves the explicit computation using the Chevalley-Eilenberg complex for the Lie algebra $\mathfrak{g}_\kappa$. This choice also allows us to do a more detailed comparison with other examples coming from quantum groups.

Let us start by recalling some notions from homological algebra, following the exposition given in [27]. Let $\mathfrak{g}$ be a Lie algebra and $M$ be a left $\mathfrak{g}$-module. The Lie algebra homology of $\mathfrak{g}$ with coefficients in $M$ is, by definition, the homology of the Chevalley-Eilenberg complex

$$M \overset{\delta}{\leftarrow} M \otimes \Lambda^1 \mathfrak{g} \overset{\delta}{\leftarrow} M \otimes \Lambda^2 \mathfrak{g} \overset{\delta}{\leftarrow} \cdots,$$

where $\Lambda^k \mathfrak{g}$ denotes the $k$-th exterior power of $\mathfrak{g}$ and the differential $\delta$ is defined by

$$\delta(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{i < j} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_n$$

$$+ \sum_{i=1}^n (-1)^i X_i(m) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n,$$

where the hat denotes omission. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. Given a $U(\mathfrak{g})$-bimodule $M$, we define the left $\mathfrak{g}$-module $M^{ad}$, where $M^{ad} = M$ as vector spaces and the left module structure is defined for all $X \in \mathfrak{g}$ and $m \in M$ by

$$X(m) = Xm - mX.$$

We can define a map

$$\varepsilon : M^{ad} \otimes \Lambda^n \mathfrak{g} \rightarrow M \otimes U(\mathfrak{g})^{\otimes n}$$

from the Lie algebra complex to the Hochschild complex by

$$\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{s \in S_n} \text{sgn}(s)m \otimes X_{s(1)} \otimes \cdots \otimes X_{s(n)}.$$

One can prove that $\varepsilon : C(\mathfrak{g}, M^{ad}) \rightarrow C(U(\mathfrak{g}), M)$ is a quasi-isomorphism, so it induces an isomorphism between the corresponding homology groups

$$H_*(\mathfrak{g}, M^{ad}) \cong H_*(U(\mathfrak{g}), M).$$
In particular if we choose \( M = \sigma U(\mathfrak{g}) \), that is \( U(\mathfrak{g}) \) with the bimodule structure \( a \cdot b \cdot c = \sigma(a)bc \), then on the right we have the \textit{twisted Hochschild homology} \( H_*(U(\mathfrak{g}), \sigma U(\mathfrak{g})) \). The \textit{twisted Hochschild dimension} is defined, according to [21], as the maximum of the homological dimension of \( H_*(U(\mathfrak{g}), \sigma U(\mathfrak{g})) \) over all the automorphisms \( \sigma \) of \( U(\mathfrak{g}) \). The case \( \sigma = \text{id} \) gives the usual Hochschild homology. Interest in this twisted homology theory comes from the fact that, in several examples coming from quantum groups, it allows to avoid the phenomenon of dimension drop. We will see that this is the case also here.

4.2 The two dimensional case

Now we can start the computation for the two dimensional case. For clarity we use the notation \( x_1, x_2 \) instead of \( x_0, x_1 \) as done in the previous sections. Since the Lie algebra is two dimensional the complex is simply given by

\[
M \xleftarrow{\delta} M \otimes \Lambda^1 g \xleftarrow{\delta} M \otimes \Lambda^2 g \xleftarrow{\delta} \Lambda^3 g.
\]

The differential \( \delta \) acting on \( M \otimes \Lambda^2 g \) takes the form

\[
\delta(m \otimes X_1 \wedge X_2) = -m \otimes [X_1, X_2] - X_1(m) \otimes X_2 + X_2(m) \otimes X_1.
\]

We write \( X_1 \) and \( X_2 \) in the \( x_1, x_2 \) basis as \( X_1 = c_1^1 x_1 + c_1^2 x_2 \) and \( X_2 = c_2^1 x_1 + c_2^2 x_2 \), for some coefficients. Their commutator is given by

\[
[X_1, X_2] = (c_1^1 c_2^2 - c_2^1 c_1^2) i \lambda x_2.
\]

Notice that for \( m \otimes X_1 \wedge X_2 \) to be non-trivial we need \( c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0 \). Indeed we have

\[
m \otimes X_1 \wedge X_2 = (c_1^1 c_2^2 - c_2^1 c_1^2) m \otimes x_1 \wedge x_2.
\]

**Proposition 13.** The twisted homological dimension of \( U(\mathfrak{g}_\kappa) \) is equal to two.

**Proof.** Since \( \Lambda^3 \mathfrak{g} \) is trivial we only have to show that there exists a non-trivial element \( m \otimes X_1 \wedge X_2 \) such that \( \delta(m \otimes X_1 \wedge X_2) = 0 \). Computing the differential we get

\[
\delta(m \otimes X_1 \wedge X_2) = -(c_1^1 c_2^2 - c_2^1 c_1^2) ((i \lambda m + x_1(m)) \otimes x_2 - x_2(m) \otimes x_1).
\]

Since \( c_1^1 c_2^2 - c_2^1 c_1^2 \neq 0 \), the condition \( \delta(m \otimes X_1 \wedge X_2) = 0 \) implies

\[
(i \lambda m + x_1(m)) \otimes x_2 - x_2(m) \otimes x_1 = 0.
\]

This in turn implies the conditions \( x_2(m) = 0 \) and \( i \lambda m + x_1(m) = 0 \).

We have \( X(m) = \sigma(X)m - mX \), where \( \sigma \) is an automorphism of the form

\[
\sigma(x_1) = x_1 + i \mu, \quad \sigma(x_2) = x_2.
\]

By the Poincaré–Birkhoff–Witt theorem, we can write \( m \in U(\mathfrak{g}_\kappa) \) as

\[
m = \sum_{a,b} f_{a,b} x_1^a x_2^b,
\]

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where the sum is finite, \( f_{a,b} \) are numerical coefficients and the exponents are non-negative integers. Since the automorphism \( \sigma \) acts trivially on \( x_2 \), the condition \( x_2(m) = 0 \) implies that \( m \) commutes with \( x_2 \), that is \( m \) should not depend on \( x_1 \).

The second condition, on the other hand, can be rewritten as

\[
i\lambda m + x_1(m) = i(\lambda + \mu)m + [x_1, m] = 0 .
\]

An easy computation then shows that

\[
[x_1, m] = \sum_{a,b} f_{0,b}[x_1, x_2^b] = i\lambda \sum_{a,b} f_{0,b}bx_2^b .
\]

Plugging this result into \( i\lambda m + x_1(m) = 0 \) we obtain

\[
i\lambda m + x_1(m) = i(\lambda + \mu)\sum_{a,b} f_{0,b}x_2^b + i\lambda \sum_{a,b} f_{0,b}bx_2^b
\]

\[
= \sum_{a,b} f_{0,b}i(\lambda(1 + b) + \mu)x_2^b = 0 .
\]

Since \( b \) is a non-negative integer, this equation is satisfied if and only if \( \mu = -\lambda(1 + b) \), for some \( b \in \mathbb{N}_0 \). In particular, the simplest choice \( b = 0 \) corresponds to the automorphism \( \sigma(x_1) = x_1 - i\lambda, \sigma(x_2) = x_2 \) which has been considered in [8].

4.3 The \( n \)-dimensional case

Let us write \( \delta = \delta_1 + \delta_2 \), where \( \delta_1 \) and \( \delta_2 \) are given respectively by the first and second line of equation (4.1). To study the case of a general dimension we start by proving two lemmata, which allows to rewrite the differential in a easier form. The first one is valid for any Lie algebra \( \mathfrak{g} \), and simply requires some gymnastics with differential forms, while the second one is related to the simple structure of the commutation relations of the Lie algebra \( \mathfrak{g}_\kappa \).

**Lemma 14.** Let \( X_i \in \mathfrak{g} \) be given by \( X_i = c^j_i x_j \), where \( c^j_i \) are numerical coefficients and \( \{x_j\} \) is a basis of the Lie algebra \( \mathfrak{g} \). Then we have

\[
\delta_2(m \otimes X_1 \wedge \cdots \wedge X_n) = \det C \sum_{j=1}^n x_j(m) \otimes x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n ,
\]

where \( C \) is the matrix formed by the coefficients \( c^j_i \).

**Proof.** Denoting by \( C_{i,j} \) the \((i, j)\)-minor of the matrix \( C \) we can write

\[
X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n = \sum_{j=1}^n C_{i,j}x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n .
\]

If we expand \( X_i \) in the basis of the generators we can write

\[
X_i(m) = \sum_{k=1}^n c^k_i x_k(m) .
\]
Then the second line of the differential $\delta$ given by (4.1) becomes

$$\delta_2 = \sum_{j=1}^{n} \sum_{k=1}^{n} x_k (m) \otimes \sum_{i=1}^{n} (-1)^i c_k^i C_{i,j} x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$  

The sum over $i$ of $(-1)^i c_k^i C_{i,j}$ looks like a Laplace expansion of the determinant of some matrix. Indeed, it is the determinant of the matrix obtained from $C$ by replacing the $j$-th column, given by $c_a^k$ with $a = 1, \ldots, n$, with the column $c_j^k$. If $k \neq j$ then, after this replacement, we obviously have two linearly dependent columns, so the determinant vanishes. On the other hand if $k = j$ we obtain $\det C$, independent of $j$. So we can write

$$\sum_{i=1}^{n} (-1)^i c_k^i C_{i,j} = \det C \delta_j^k.$$

Plugging this result into the previous formula we find the result. 

\[ \square \]

**Lemma 15.** With the same notation as above, consider the Lie algebra $g_\kappa$ with commutation relations $[x_1, x_j] = i \lambda x_j$, where $j > 1$. Then we have

$$\delta_1 (m \otimes X_1 \wedge \cdots \wedge X_n) = \det C i \lambda (n - 1) m \otimes \hat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n.$$  

**Proof.** We start by computing the commutator of two elements $X_i$ and $X_j$:

$$[X_i, X_j] = i \lambda \sum_{k=1}^{n} (c_i^k c_j^k - c_i^k c_j^k) x_k = i \lambda (c_i^1 X_j - c_j^1 X_i).$$

Then the first line of the differential $\delta$ given by (4.1) becomes

$$\delta_1 = \sum_{i<j} i \lambda c_i^1 (-1)^{i+j} m \otimes X_j \wedge X_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge X_n$$

$$- \sum_{i<j} i \lambda c_j^1 (-1)^{i+j} m \otimes X_i \wedge X_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge X_n.$$  

Now we can bring $X_i$ and $X_j$ to their missing spots, picking up some signs. When we move $X_i$ we have to go across $i - 1$ terms, so we pick a $(-1)^{i-1}$, while when we move $X_j$ we have to go across $j - 2$ terms, since also $X_i$ is missing, so we pick a $(-1)^{j-2}$. Then we have

$$\delta_1 = \sum_{i<j} i \lambda c_i^0 (-1)^i m \otimes X_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge X_n$$

$$+ \sum_{i<j} i \lambda c_j^0 (-1)^j m \otimes X_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge X_n.$$  

Notice that now $j$ and $i$ do not appear anymore respectively in the first and the second sum. It is not difficult to see that we can rewrite them as

$$\delta_1 = \sum_{i=1}^{n} i \lambda (n - i) c_i^1 (1)^i m \otimes X_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge X_n$$

$$+ \sum_{j=1}^{n} i \lambda (j - 1) c_j^1 (-1)^j m \otimes X_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge X_n.$$
Summing the two contributions we get
\[ \delta_1 = i\lambda(n-1) \sum_{i=1}^{n} (-1)^{i} c_i^1 m \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_n. \]

Writing the wedge products in terms of the minors of \( C \) we obtain
\[ \delta_1 = i\lambda(n-1) \sum_{i=1}^{n} (-1)^{i} c_i^1 m \otimes x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n. \]

Finally, using the same arguments of the previous lemma, we obtain
\[ \delta_1 = \det C i\lambda(n-1) m \otimes \hat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n. \]

This concludes the proof of the lemma.

**Theorem 16.** Let \( g_\kappa \) be the Lie algebra associated with \( \kappa \)-Minkowski space in \( n \)-dimensions, which is characterized by the commutation relations \([x_1, x_j] = i\lambda x_j\), where \( j > 1 \). Then the twisted homological dimension of \( U(g_\kappa) \) is equal to \( n \).

**Proof.** As in the two dimensional case, we only need to show that there is an element \( m \otimes X_1 \wedge \cdots \wedge X_n \) such that \( \delta(m \otimes X_1 \wedge \cdots \wedge X_n) = 0 \). Putting together the two previous lemmata we have the following expression for the differential
\[ \delta(m \otimes X_1 \wedge \cdots \wedge X_n) = \det C i\lambda(n-1) m \otimes \hat{x}_1 \wedge x_2 \wedge \cdots \wedge x_n \]
\[ + \det C \sum_{j=1}^{n} x_j(m) \otimes x_1 \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n. \]

Since \( \det C \) is different from zero, we need to impose the conditions \( x_j(m) = 0 \) for \( j = 2, \cdots , n \). Again we have to consider automorphisms which are of the form
\[ \sigma(x_1) = x_1 + i\mu, \quad \sigma(x_j) = x_j. \]

Then \( x_j(m) = [x_j, m] = 0 \) implies that \( m \) does not depend on \( x_1 \). The other condition we need to impose is \( i\lambda(n-1) m + x_1(m) = 0 \), which can be rewritten in the form
\[ i(\lambda(n-1) + \mu) m + [x_1, m] = 0. \]

By the Poincaré–Birkhoff–Witt theorem we can write \( m \in U(g_\kappa) \) as
\[ m = \sum_{a_2, \cdots , a_n} f_0(a_2, \cdots , a_n) x_2^{a_2} \cdots x_n^{a_n}, \]

where the sum is finite, \( f_0(a_2, \cdots , a_n) \) are numerical coefficients and the exponents are non-negative integers. We have already imposed the condition that \( m \) does not depend on \( x_1 \). Now we
compute the commutator of \( m \) with \( x_1 \)

\[
[x_1, m] = \sum_{a_2, \ldots, a_n} f_{0,a_2,\ldots,a_n} [x_1, x_{a_2}^2 \cdots x_{a_n}^n]
\]

\[
= \sum_{a_2, \ldots, a_n} f_{0,a_2,\ldots,a_n} ( [x_1, x_{a_2}^2] x_{a_3}^3 \cdots x_{a_n}^n + \cdots + x_{a_2}^2 \cdots x_{a_{n-1}}^{a_{n-1}} [x_1, x_{a_n}^n])
\]

\[
= i\lambda \sum_{a_2, \ldots, a_n} f_{0,a_2,\ldots,a_n} (a_2 + \cdots + a_n) x_{a_2}^2 \cdots x_{a_n}^n.
\]

Using this result we finally obtain

\[
i\lambda m + x_1(m) = \sum_{a_2, \ldots, a_n} f_{0,a_2,\ldots,a_n} i(\lambda(n - 1 + a_2 + \cdots + a_n) + \mu) x_{a_2}^2 \cdots x_{a_n}^n = 0.
\]

Since \( a_2, \ldots, a_n \) are non-negative integers, this equation is satisfied if and only if \( \mu = -\lambda(n - 1 + a_2 + \cdots + a_n) \), for some \( a_2, \ldots, a_n \in \mathbb{N}_0 \).

We notice from this result that, by choosing the simplest case where \( a_2 = \cdots = a_n = 0 \), we obtain the automorphism given by \( \sigma(x_1) = x_1 - i(n - 1)\lambda \) and \( \sigma(x_j) = x_j \). This is exactly the inverse of the modular group \( \sigma_1^\omega \) of the weight \( \omega \) we introduced in the first part, which was the starting point for the construction. Other choices of the \( a_2, \ldots, a_n \) coefficients give an automorphism \( \sigma \) which is a negative power of this modular group. This is exactly the same thing that happens for the twisted homology of \( SL_q(2) \) [22], for the Podleś spheres [23] and for other examples which come from quantum groups.

There is another feature of this result which is worth mentioning. Consider again the simplest non-trivial cycle, which we obtain by setting \( f_{0,\ldots,0} = 1 \) and all other coefficients zero. Passing from the Lie algebra complex to the Hochschild complex via

\[
\varepsilon(m \otimes X_1 \wedge \cdots \wedge X_n) = \sum_{s \in S_n} \text{sgn}(s) m \otimes X_{s(1)} \otimes \cdots \otimes X_{s(n)},
\]

we see that this cycle corresponds to

\[
c = \det C \sum_{s \in S_n} \text{sgn}(s) 1 \otimes x_{s(1)} \otimes \cdots \otimes x_{s(n)}.
\]

We notice that it has the same form as in commutative case. Indeed the analogy goes further since, as we discussed in the section on the Dirac operator and the differential calculus, we have that \( [D, x^\mu]_\sigma = -i\Gamma^\mu \). Therefore, if we represent this cycle on the Hilbert space by \( a_0[D, a_1]_\sigma \cdots [D, a_n]_\sigma \), we get exactly the orientation cycle of the commutative case, which corresponds to the volume form \( dx^1 \wedge \cdots \wedge dx^n \).

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