1. Introduction

Let $I \subset k[x_0, \ldots, x_N]$ be a homogeneous ideal. For $r \geq 0$, the $r$-th symbolic power of $I$ is defined to be

$$I^{(r)} = \bigcap_{p \in \text{Ass}(R/I)} (I^r R_p \cap R).$$

Symbolic powers of ideals are interesting for a number of reasons, not least of which is that, for a radical ideal $I$, the $r$-th symbolic power $I^{(r)}$ is the ideal of all polynomials vanishing to order at least $r$ on $V(I)$ (by the Zariski-Nagata theorem).

Containment relationships between symbolic and ordinary powers are a source of great interest. As an immediate consequence of the definition, $I^{(r)} \subseteq I^r$ for all $r$. However, the other type of containment, namely that of a symbolic power in an ordinary power is much harder to pin down. It has been proved by Ein-Lazarsfeld-Smith [ELS] and Hochster-Huneke [HH] that $I^{(m)} \subseteq I^r$ for all $m \geq Nr$, but as of yet there are no examples in which this bound is sharp.

It was conjectured by Harbourne in [BDHKSS, Conjecture 8.4.3] (and later in [HaHu, Conjecture 4.1.1] in the case $e = N - 1$) that $I^{(m)} \subseteq I^r$ for all $m \geq er - (e - 1)$, where $e$ is the codimension of $V(I)$. While this conjecture holds in a number of important cases, some counterexamples have also been found. Notably, the main counterexamples come from singular points of hyperplane arrangements [BNAL]. One particular family is known in the literature under the name of Fermat configurations of points cf. $\mathbb{P}^2$ [BNAL, MS]. These have been recently generalized to Fermat-like configurations of lines in $\mathbb{P}^3$ in [MS]. The Ceva(n) arrangement of hyperplanes in $\mathbb{P}^N$ is defined by the linear factors of

$$F_{N,n} = \prod_{0 \leq i < j \leq N} (x_i^n - x_j^n),$$

where $n \geq 3$ is an integer.

In [DST] Dunnicki, Szemberg, and Tutaj-Gasinska showed that, for the ideal $I_{2,3}$ corresponding to all triple intersection points of the lines defined by linear factors of $F_{2,3}$ in $\mathbb{P}^2$, $F_{2,3} \notin I_{2,3}^2$, but $F_{2,3} \in I_{2,3}^{(3)}$. This was the first counterexample to the above mentioned conjecture. Later, in [MS] Malara and Szpond generalized this construction to $\mathbb{P}^3$, by showing that for the ideal $I_{3,n}$, corresponding to all triple intersection lines of the planes defined by the linear factors of $F_{3,n}$, $F_{3,n} \notin I_{3,n}^2$, but $F_{3,n} \in I_{3,n}^{(3)}$. In the following, the construction of counterexamples to $I^{(3)} \subseteq I_2$ from Fermat arrangements is generalized to $\mathbb{P}^N$ for all $N \geq 2$. 


2. Main result

Let \( n \in \mathbb{N} \). Let \( k \) be a field which contains a primitive \( n \)-th root of unity, \( \varepsilon \). For each \( N \in \mathbb{N} \), let \( S_N := k[x_0, x_1, \ldots, x_N] \), and define

\[
F_{N,n} := \prod_{0 \leq i < j \leq N} (x_i^n - x_j^n).
\]

Let

\[
C_{N,n} := \bigcap_{0 \leq i < j \leq N} (x_i, x_j)
\]

\[
J_{N,n} := \bigcap_{0 \leq i < j < l \leq N} (x_i - \varepsilon^a x_j, x_i - \varepsilon^b x_l, x_j - \varepsilon^{b-a} x_l),
\]

and let

\[
I_{N,n} := J_{N,n} \cap C_{N,n}.
\]

We show in Lemma 2.2 that \( I_{N,n} \) is the ideal of the \( N - 2 \) dimensional flats arising from triple intersection of hyperplanes corresponding to linear factors of \( F_{N,n} \).

**Theorem 2.1.** For all \( N \geq 2 \), \( I_{N,n}^{(3)} \not\subseteq I_{N,n}^2 \).

Before we can prove this, we must introduce a few lemmas.

**Lemma 2.2.** The ideal \( I_{N,n} \) defined above defines the union of all the \( N - 2 \) dimensional linear spaces that are intersections of at least three hyperplanes corresponding to linear factors of \( F_{N,n} \).

**Proof.** Let \( 0 \leq a, b < n \), and let \( 0 \leq i < j \leq l \leq N \). Then

\[
(x_i - \varepsilon^a x_j, x_i - \varepsilon^b x_l) = (x_i - \varepsilon^a x_j, x_i - \varepsilon^b x_l, x_j - \varepsilon^{b-a} x_l)
\]

defines the intersection of the three hyperplanes corresponding to \( (x_i - \varepsilon^a x_j), (x_i - \varepsilon^b x_l), \) and \( (x_j - \varepsilon^{b-a} x_l) \). Furthermore

\[
(x_i, x_j) = (x_i - \varepsilon^a x_j : a = 0, 1, \ldots, n),
\]

so \((x_i, x_j)\) defines the intersection of \( n \) hyperplanes corresponding to linear factors of \( x_i^n - x_j^n \).

It remains to be seen that all \( N - 2 \) dimensional linear spaces that arise as intersections of at least three hyperplanes corresponding to linear factors of \( F_{N,n} \) are accounted for above. Let \( L \) be the ideal defining such a linear space, then \( L \) contains three linearly dependent binomials of the form \( x_i - \varepsilon^a x_j, x_k - \varepsilon^b x_l, x_u - \varepsilon^c x_v \). Without loss of generality (after multiplication by appropriate powers of \( \varepsilon \)) this yields \( i = k \) and \( \{j, l\} = \{u, v\} \). If \( j \neq l \) then \( L = (x_i - \varepsilon^a x_j, x_i - \varepsilon^b x_l) \) is one of the primes appearing in the decomposition of \( J_{N,n} \) and if \( j = l \) then \( L = (x_i, x_j) \) is one of the primes appearing in the decomposition of \( C_{N,n} \). \( \square \)

**Lemma 2.3.** Let \( R \) and \( S \) be finitely generated graded-local Noetherian rings. Let \( m \) be the homogeneous maximal ideal of \( R \). Let \( I \subset R \) be a homogeneous ideal, and suppose \( F \not\subseteq I^r \) for some \( r \in \mathbb{N} \). Let \( J \subset S \) be an ideal, and let \( \pi : S \to R \) be a (not necessarily homogeneous) ring homomorphism such that \( \pi(J) \subseteq I \). If \( G \in R \) is such that \( \pi(G) = Fg \), where \( g \not\in m \), then \( G \not\subseteq J^r \).
Thus \( F g = \pi(G) \subseteq (\pi(J))^r \subseteq \Gamma' \). Then

\[
F g = \pi(G) \subseteq (\pi(J))^r \subseteq \Gamma'.
\]

This lemma allows us to construct an inductive argument for the main theorem.

**Proof of Theorem 2.1**. By Lemma 2.2 \( F_{N,n} \) must vanish to order 3 or greater on each of the linear spaces whose union is \( V(I_{N,n}) \), thus \( F_{N,n} \in I^{(3)}_{N,n} \). To finish the proof, it suffices to show that for all \( N \geq 2 \), \( F_{N,n} \notin I^2_{N,n} \).

We argue by induction on \( N \). For \( N = 2 \), this is proved in the paper of Dumnicki, Szemberg, and Tutaj-Gasinska [DST].

For \( N > 3 \), assume that \( F_{N-1,n} \notin I^2_{N-1,n} \) and consider the evaluation homomorphism \( \pi : S_N \to S_{N-1} \) defined by \( \pi(x_N) = 1 \) and \( \pi(x_i) = x_i \) for \( i \leq N - 1 \). Then:

\[
\pi(I_{N,n}) \subseteq C_{N-1,n} \cap \left( \bigcap_{0 \leq i < N} (x_i, 1) \right) \subseteq I_{N-1,n}.
\]

We note that \( \pi(F_{N,n}) = F_{N-1,n}g \) where \( g = \prod_{0 \leq i < N} (x_i^n - 1) \). Since \( g \notin (x_0, \ldots, x_{N-1}) \), we conclude by Lemma 2.3 that \( F_{N,n} \notin I^2_{N,n} \). □

**3. Concluding Remarks**

Another proof for the noncontainment noncontainment \( I^{(3)}_{N,n} \subseteq I^2_{N,n} \) has been found by Grzegorz Malara and Justyna Szpond and can bee seen in their upcoming paper [MS2].

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