ON ALMOST RATIONAL FINSLER METRICS

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Abstract. We study a special class of Finsler metrics which we refer to as Almost Rational Finsler metrics (shortly, AR-Finsler metrics). We give necessary and sufficient conditions for an AR-Finsler manifold \((M, F)\) to be Riemannian. The rationality of the associated geometric objects such as Cartan torsion, geodesic spray, Landsberg curvature, \(S\)-curvature, etc. is investigated. We prove for a particular subset of AR-Finsler metrics that if \(F\) has isotropic \(S\)-curvature, then its \(S\)-curvature identically vanishes. Further, if \(F\) has isotropic mean Landsberg curvature, then it is weakly Landsberg. Also, if \(F\) is an Einstein metric, then it is Ricci-flat. Moreover, we show that Randers metric cannot be AR-Finsler metric. Finally, we provide some examples of AR-Finsler metrics and introduce a new Finsler metric which is called an extended \(m\)-th root metric. We show under what conditions an extended \(m\)-th root metric is AR-Finsler metric and study its generalized Kropina change.

Keywords: Sprays; Finsler geometry; \(m\)-th root metrics, \((\alpha, \beta)\)-metrics; Einstein metrics; Generalized Kropina change.

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1. Introduction

Finsler geometry is a natural generalization of Riemannian geometry. It is wider in scope and richer in content than Riemannian geometry. A Riemannian metric is quadratic in the fiber coordinates \(y\) while a Finsler metric is not necessary be quadratic in \(y\) cf. [6]. In the literature of Finsler geometry there are some Finsler structures for which the components of its metric tensor are rational functions in \(y\), for example, Kropina metrics. The rationality in the fiber coordinates \(y\) of some geometric objects on Finsler manifold plays a vital role in its characterization. For example, Randers metrics with quadratic Riemann curvature have been investigated in [3]. Finsler spaces with rational spray coefficients have been studied in [9]. Further, in [11], it was proved that Einstein \(m\)-th root metrics are Ricci flat using the rationality of the Riemann curvature of an \(m\)-th root metric.

In this work, we study a class of Finsler metrics, for which the components of its metric tensor \(g_{ij}(x, y)\) can be written as a product of a fixed function \(\eta(x, y)\) and rational functions \(a_{ij}(x, y)\) on the tangent bundle \(TM\) (see Definition 2.1), and call it as Almost Rational Finsler metric, in short AR-Finsler metric. The rationality of the metric tensor forces several geometric objects (like spray coefficients, Riemann curvature, Ricci curvature, Berwald curvature, etc.) to have rational functions in \(y\). We explicitly calculate some geometric objects of an AR-Finsler metric, viz., Cartan torsion, Cartan mean torsion, spray coefficients, Barthel connection and \(S\)-curvature. Further, we investigate the conditions for some Finslerian geometric quantities to be rational functions in \(y\). We also observe that some geometric objects of an AR-Finsler manifold are always rational and independent of the rationality of the metric tensor.

It is interesting to note that there are some famous Finsler metrics which are examples of AR-Finsler metrics. Namely, the generalized Kropina metric, the \(m\)-th root metric
cf. [11], the Kropina change of the $m$-th root metric and the generalized Kropina change of the $m$-th root metric [7, 8]. Besides, we introduce a new Finsler metric looks like the $m$-th root metric but its coefficients are functions on $TM$ rather than functions on $M$, as in the case of $m$-th root metric, which we refer to as extended $m$-th root metric. It turns to be an AR-Finsler metric, when its coefficients are rational functions in $y$. We show that its Kropina change as well as generalized Kropina change turn also to be AR-Finsler metric. We further prove that there is no AR-Finsler metric of Randers type. The polynomial $(\alpha, \beta)$-metric has been studied by Z. Shen in [6]. In this paper, we study a special polynomial $(\alpha, \beta)$-metric and find the condition to be an AR-Finsler metric.

Some AR-Finsler metrics $F$ are rational in $y$ while others not. We prove the following two results for irrational $F$ that i) if $(M, F)$ is an AR-Finsler space of isotropic $S$-curvature, then its $S$-curvature identically vanishes; ii) an AR-Finsler space which has isotropic mean Landsberg curvature reduces to weakly Landsberg space. It is an extension of [8, Theorem 1.4]. Further, we show for certain $\eta$ that if $(M, F)$ is an AR-Finsler space of Einstein type, then it has vanishing Ricci curvature. This result represents a generalization of [11, Theorem 1.1] and [9, Theorem 1.1].

In what follows, we give the structure of this paper. Section 2 deals with basic preliminaries required for the rest of this work. In section 3, we define Almost Rational Finsler metric and study some of its characterizations and associated geometric quantities. We give necessary and sufficient for an AR-Finsler manifold to be Riemannian. We extend the results [8, Theorem 1.4], [11, Theorem 1.1] and [9, Theorem 1.1]. Further, we show that no nontrivial Randers metrics is AR-Finsler. In section 4, we give some examples AR-Finsler metrics such as Kropina metric, generalized Kropina metric and $m$-th root metric. Also, we introduce the so called extended $m$-th root metric and figure out its generalized Kropina change.

2. Preliminaries

Let $M$ be a smooth $n$-dimensional manifold and $TM$ be the corresponding tangent bundle. Let $(x^i)$ be the coordinates of any point of the base manifold $M$ and $(y^i)$ be a supporting element at the same point. The partial differentiation with respect to $x^i$ is denoted by $\partial_i$, while the partial differentiation with respect to $y^i$ (basis vector fields of the vertical bundle) is denoted by $\dot{\partial}_i$. Let us recall some basics of Finsler geometry. Most of the material presented here with further details may be found in [6]. Hereafter, the Einstein summation convention is in place.

**Definition 2.1.** A Finsler structure $F$ on a smooth manifold $M$ is a mapping

$$F : TM \to [0, \infty)$$

with the following properties:

(a) $F$ is $C^\infty$ on the slit tangent bundle $TM\{0\}$.

(b) $F(x, y)$ is positively homogeneous of first degree in $y$: $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in TM$ and $\lambda > 0$.

(c) The Hessian matrix $g_{ij}(x, y) := \frac{1}{2} \dot{\partial}_i \dot{\partial}_j F^2$ is positive-definite at each point of $TM\{0\}$.

The bilinear symmetric form $g = g_{ij}(x, y) \, dx^i \otimes dx^j$ is called the Finsler metric tensor or fundamental tensor of the Finsler manifold $(M, F)$.

Often, a function $F$ satisfies the above mentioned conditions is called a regular Finsler metric. Instead of positive definite, if the metric tensor $g$ is non-degenerate at each point of $TM\{0\}$, the pair $(M, F)$ is called a pseudo-Finsler manifold. If $F$ satisfies the conditions
On a Finsler manifold

Definition 2.4. The Cartan torsion associated with a Finsler metric $F$ is given by $C = C_{ijk} dx^i \otimes dx^j \otimes dx^k$, where $C_{ijk} = \frac{1}{4} \partial_k g_{ij} = \frac{1}{4} \partial_i \partial_j \partial_k F^2$. The mean Cartan torsion is denoted by $I = I_k dx^k$, where $I_k = g^{ij} C_{ijk}$.

An immediate consequence of the definition of the Cartan tensor is that a Finsler manifold is Riemannian if and only if $C_{ijk} = 0$. Deicke proved, in [2], that a regular Finsler metric is Riemannian if and only if $I_k$ vanishes.

In 1941, Randers metrics were first studied by the physicist G. Randers, from the standpoint of general relativity [5]. Further, in 1957, R. S. Ingarden applied Randers metrics to the theory of the electron microscope and named them Randers metrics. A Finsler manifold $(M, F)$ is of Randers type if $F = \alpha + \beta$, where $\alpha = \sqrt{\alpha_{ij}(x) y^i y^j}$ is a Riemannian metric and $\beta = b_i(x) dx^i$ is a 1-form on $M$ with $||\beta||_\alpha = \sqrt{\alpha^{ij}(x) b_i(x) b_j(x)} < 1$. The metric $F$ is then said to be a Randers metric. As a generalization of Randers metric, M. Matsumoto introduced $(\alpha, \beta)$-metrics in [4].

Definition 2.3. A Finsler metric $F = \alpha \phi \left( \frac{y}{s} \right) = \alpha \phi(s)$ is called $(\alpha, \beta)$-metric if $\phi$ is smooth positive function defined on the interval $(-b_0, b_0)$ such that

$$\phi(s) - \phi(s) \phi'(s) + (b^2 - s^2) \phi''(s) > 0, \quad |s| \leq ||\beta||_\alpha < b_0.$$

It is known that, the metric tensor associated to $F$ is given by

$$g_{ij} = (\phi^2 - s^2 \phi') \alpha_{ij} + (\phi \phi'' + (\phi')^2) b_i b_j + \frac{1}{\alpha} \left( \phi \phi' - s \left[ \phi \phi'' + (\phi')^2 \right] \right) \left( b_i y_j + b_j y_i - \frac{2}{\alpha} y_i y_j \right),$$

where $y_j := \alpha_{ij} y^i$.

Definition 2.4. On a Finsler manifold $(M, F)$, the geodesics are characterized by

$$\frac{d^2 x^i}{dt^2} + G^i(x, \frac{dx}{dt}) = 0,$$

where $G^i$ are called spray coefficients of $F$ which defined by

$$G^i = \frac{1}{4} g^{ir} \left( y^k \partial_r \partial_k F^2 - \partial_r F^2 \right) = \frac{1}{2} g^{ir} \left( y^k y^s \partial_k g_{rs} - 2 y^l y^s \partial_r g_{ls} \right).$$

It is worth noting that for a Riemannian metric $F$, the spray coefficients $G^i(x, y)$ are quadratic in $y$, whereas for Finsler metric they are highly nonlinear cf. [1, 6]. The geodesic spray induces a nonlinear connection $N^i_j := \partial_j G^i$ which is called the Barthel (or Cartan nonlinear) connection associated with $(M, F)$. Thereby, we have the direct sum decomposition

$$T_u(TM) = H_u(TM) \oplus V_u(TM), \quad \forall u \in TM.$$

The basis vector fields of the horizontal bundle are denoted by $\delta_i := \partial_i - N^i_j \partial_j$.

In addition, $G^i_{jh} := \partial_i N^j_h = \partial_h \partial_j G^i$ are the coefficients of Berwald connection. Consequently, $G^i_{jk} := \partial_k G^i_{jh} = \partial_k \partial_h \partial_j G^i$ are the coefficients of Berwald curvature. It is known that, a Finsler manifold with $G^i_{jk} = 0$ is said to be Berwaldian.

Definition 2.5. A Finsler manifold $(M, F)$ is called Landsbergian if Landsberg curvature $L = L_{ijk} dx^i \otimes dx^j \otimes dx^k$, where $L_{ijk} = \frac{1}{2} y^m g_{ms} G^s_{ijk}$, vanishes. $(M, F)$ is said to be weakly Landsberg manifold if the mean Landsberg curvature $J = J_k dx^k$ vanishes, where

$$J_k = g^{ij} L_{ijk} = y^s I_{k|s} = y^s \partial_s I_k - 2 G^s \partial_s I_k - N^i_k I_s.$$
A Finsler manifold \((M, F)\) has isotropic mean Landsberg curvature (or relatively isotropic \(J\)-curvature) if \(J\) can be written in the form
\[
J(x, y) = A(x) F(x, y) I(x, y), \text{ where } A \in C^\infty(M).
\]

**Definition 2.6.** A Finsler metric is called Douglas metric if the Douglas curvature \(D = D^i_{jkl} \partial_i \otimes dx^j \otimes dx^k \otimes dx^l\) vanishes, where
\[
D^i_{jkl} = G^i_{jkl} - \frac{1}{n+1} \partial_j \partial_k \partial_l \left( y^i N^n \right).
\]

**Definition 2.7.** The Riemann curvature \(R = R^i_k \partial_i \otimes dx^k\) of a Finsler space is defined by
\[
R^i_k = 2 \partial_k G^i - y^j \partial_j \partial_k G^i + 2G^i \partial_k \partial_j G^j - \partial_j G^i \partial_k G^j.
\]

The Weyl projective curvature \(W = W^i_j \partial_i \otimes dx^j\) is given by
\[
W^i_j = Q^i_j - \frac{1}{n+1} y^j \partial_i Q^i_j,
\]
where \(Q^i_j = R^i_j - \frac{Ric}{n+1} \delta^i_j\) and \(Ric = R^m_m\). Further, the \(\chi\)-curvature is defined by
\[
\chi_l = -\frac{1}{6} \left\{ 2 \partial_l R^i_i + \partial_l Ric \right\}.
\]

**Definition 2.8.** Given a volume form \(dV := \sigma(x) dx^1 \wedge \ldots \wedge dx^n\) on \((M, F)\), for any \(x \in M\) and \(y \in T_x M \setminus \{0\}\), the distortion \(\tau\) of the Finsler metric \(F\) is defined by \(\tau(x, y) = \ln \left( \frac{\sqrt{\det(g_{ij}(x, y))}}{\sigma(x)} \right)\). The derivative of \(\tau\) along a geodesic \(\gamma\) with \(\gamma(0) = x\) and \(\gamma'(0) = y\) is called the \(S\)-curvature at \(x\) along \(y\). More explicitly, the \(S\)-curvature is defined by
\[
S(x, y) = \frac{d}{dt} [\tau(\gamma(t), \gamma'(t))]_{t=0}.
\]

For a local coordinate system in \((M, F)\), the \(S\)-curvature has the following expression
\[
S(x, y) = N^m_m(x, y) - y^m \partial_m \log(\sigma(x)).
\]

Another non-Riemannian object, manifested from the differentiation of \(S\)-curvature, is called Berwald mean curvature (or \(E\)-curvature) \(E = E_{ij} dx^i \otimes dx^j\), where
\[
E_{ij} = \frac{1}{2} \partial_i \partial_j S = \frac{1}{2} \partial_i \partial_j N^m_m.
\]

### 3. Almost rational Finsler metrics

**Definition 3.1.** A Finsler metric \(F\) on a manifold \(M\) is called Almost Rational Finsler (simply, AR-Finsler) metric if the coefficients \(g_{ij}(x, y)\) of its metric tensor can be expressed as follows:
\[
g_{ij}(x, y) = \eta(x, y) a_{ij}(x, y),
\]
where

(a) \(\eta : TM \rightarrow [0, \infty)\) is a smooth function,
(b) the matrix \((a_{ij}(x, y))_{1 \leq i, j \leq n}\) is symmetric positive definite with each \(a_{ij}(x, y)\) be a rational function in the fiber coordinate \(y\),
(c) \(\{\eta(x, y) a_{ij}(x, y)\}\) is positive homogeneous of degree zero in \(y\) for all \(1 \leq i, j \leq n\).

The pair \((M, F)\) is said to be AR-Finsler manifold. In addition if \(\eta\) is a rational function in \(y\), we call \(F\) as rational Finsler metric and the pair \((M, F)\) as rational Finsler structure.
If we relax condition (b) as the matrix \((a_{ij}(x,y))_{1 \leq i,j \leq n}\) is nondegenerate (respectively, degenerate) then, we deal with pseudo or nondegenerate (respectively, degenerate) AR-Finsler metric.

**Remark 3.2.** It is clear that from expression \((11)\), the metric tensor \(g_{ij}(x,y)\) are rational functions in \(y\) if and only if \(\eta\) is a rational function in \(y\). For example, in \([10, \text{Examples 3, 4}]\) the function \(\eta\) is a rational function in \(y\).

**Lemma 3.3.** Let \((M, F)\) be an AR-Finsler manifold. Then we have the following:

1. The function \(\frac{F^2}{\eta}(x,y)\) is rational in \(y\),
2. The inverse metric tensor of \(F\) is given by \(g^{ij}(x,y) = \frac{a^{ij}(x,y)}{\eta(x,y)}\),
3. The expression \(\{\eta \partial_\i a_{jk} + a_{jk} \partial_\i \eta\}\) is symmetric in the indices \(i, j, k\).

**Proof.** (1) follows directly from the fact that the multiplication of rational functions always results a rational function and \(\frac{F^2}{\eta}(x,y) = a_{ij}(x,y) y^i y^j\). (2) is straightforward from the fact \(g_{ij} g^{ik} = \delta^k_i\). (3) follows by plugging (11) into the Cartan torsion formula \(C_{ijk} = \frac{1}{2} \partial_k g_{ij}\), which is totally symmetric in its indices.

**Proposition 3.4.** Let \((M, F)\) be an \(n\)-dimensional AR-Finsler manifold, \(n \geq 2\). Then, the function \(\partial_k \log(\eta(x,y))\) is a rational function in \(y\) for all \(i = 1, \ldots, n\).

**Proof.** It is clear that \(2C_{ijk} = 2C_{kij}\). That is, in the view of Lemma 3.3 (3),

\[
\eta \partial_\i a_{jk} + a_{jk} \partial_\i \eta = \eta \partial_k a_{ij} + a_{ij} \partial_k \eta \iff \\
\eta \left(\partial_k a_{ij} - \partial_\i a_{ij}\right) = -a_{jk} \partial_\i \eta + a_{ij} \partial_k \eta.
\]

Multiplying both sides of the previous relation by \(a^{ji}\), we get

\[
\eta a^{ji} \left(\partial_k a_{ij} - \partial_\i a_{ij}\right) = -\delta^i_k \partial_\i \eta + n \partial_k \eta = (n - 1) \partial_k \eta.
\]

Thus, we have

\[
(12) \quad \partial_k \log(\eta(x,y)) = \frac{1}{n-1} a^{ji} \left(\partial_k a_{ij} - \partial_\i a_{ij}\right).
\]

Therefore, \(\partial_k \log(\eta(x,y))\) are rational functions in \(y\).

It is known that the partial derivatives \(\partial_k\) and \(\partial_\i\) commutes. Also, the differential of any rational function \(f(x,y)\) in \(y\) with respect to \(x\) remains rational in \(y\), thereby, we have:

**Corollary 3.5.** The function \(\partial_k \partial_{k_1} \partial_{k_2} \ldots \partial_{k_r} \log(\eta(x,y))\) is a rational function in \(y\) for all \(1 \leq k, k_1, \ldots, k_r \leq n, r \in \mathbb{N}\).

**Remark 3.6.** Given an AR-Finsler manifold, the rationality of the associated Cartan torsion \(C_{ijk}(x,y)\) in \(y\) depends on the rationality of the function \(\eta\) in \(y\). This results from the expression

\[
C_{ijk} = \eta \partial_\i a_{jk} + a_{jk} \partial_\i \eta.
\]

In other words, \(C_{ijk}(x,y)\) are rational functions in \(y\) if and only if \(\eta\) is rational function in \(y\). However, the following proposition shows that rationality of the mean Cartan torsion of an AR-Finsler manifold is independent of the rationality of the function \(\eta\).

**Proposition 3.7.** Let \((M, F)\) be an AR-Finsler manifold. Then, the mean Cartan torsion is rational in \(y\).
Proof. By Lemma 3.3 (2), we get

\[ I_i = g^{jk} C_{ijk} = \frac{a^{jk}}{\eta} \left( \eta \dot{\partial}_{a_{jk}} + a_{jk} \dot{\partial}_{\eta} \right) = a^{jk} \dot{\partial}_{a_{jk}} + n \dot{\partial}_i \log(\eta). \]

Therefore, the proof is completed by the use of Proposition 3.4. \(\square\)

In the view of formulae (12) and (13), we have the following result.

**Corollary 3.8.** Let \((M, F)\) be an \(n\)-dimensional AR-Finsler manifold, \(n \geq 2\). Then, the mean Cartan torsion \(I_k\) can be expressed as follows:

\[ I_k = \frac{1}{n-1} a^{rs} \left( n \dot{\partial}_r a_{sk} - \dot{\partial}_k a_{rs} \right). \]

**Proposition 3.9.** Let \((M, F)\) be an \(n\)-dimensional AR-Finsler manifold. Then, we have

\[ \delta_k \log(\eta) = -\frac{1}{n} a^{ij} a_{ijk}, \]

where \(\delta\) is the horizontal covariant derivative with respect to the Berwald connection.

**Proof.** It is known that \(g_{ijk} = 0\). Thus, by (11), we obtain

\[ 0 = (\eta a_{ij})_{ik} = \eta_{ik} a_{ij} + \eta a_{ijk} = (\delta_k \eta) a_{ij} + \eta a_{ijk} \iff (\delta_k \eta) a_{ij} = -\eta a_{ijk}. \]

Then, we have

\[ a_{ij} \delta_k \log(\eta) = -a_{ijk}. \]

Hence, the proof is completed by multiplying both sides of the last relation by \(a^{ij}\) and taking into account \(a^{ij} a_{ij} = n\). \(\square\)

**Theorem 3.10.** Let \((M, F)\) be a regular AR-Finsler manifold. Then, \((M, F)\) is Riemannian if and only if the functions \(\dot{\partial}_i \log(\eta)\) have the following form:

\[ \dot{\partial}_i \log(\eta) = -\frac{1}{n} a^{jk} \dot{\partial}_a_{jk}. \]

**Proof.** Suppose \((M, F)\) is Riemannian. Thus, \(C_{ijk} = 0\), this means that,

\[ 0 = \eta \dot{\partial}_a_{jk} + a_{jk} \dot{\partial}_\eta \Rightarrow a_{jk} \dot{\partial}_i \log(\eta) = -\dot{\partial}_a_{jk} \Rightarrow n \dot{\partial}_i \log(\eta) = -a^{jk} \dot{\partial}_a_{jk}. \]

For the converse, assume that \(\dot{\partial}_i \log(\eta) = -\frac{1}{n} a^{jk} \dot{\partial}_a_{jk}\). Then, we get

\[ n \dot{\partial}_i \log(\eta) = -a^{jk} \dot{\partial}_a_{jk} \Rightarrow n \dot{\partial}_\eta = -\eta a^{jk} \dot{\partial}_a_{jk} \Rightarrow a^{jk} a_{jk} \dot{\partial}_i \eta = -\eta a^{jk} \dot{\partial}_a_{jk} \iff a^{jk} \left( \eta \dot{\partial}_a_{jk} + a_{jk} \dot{\partial}_\eta \right) = 0 \iff a^{jk} C_{ijk} = 0. \]

In view of Lemma 3.3 (2),

\[ a^{jk} C_{ijk} = 0 \iff \frac{1}{\eta} a^{jk} C_{ijk} = 0 \iff g^{jk} C_{ijk} = 0 \iff I_i = 0. \]

Therefore, \((M, F)\) is Riemannian space by Deicke’s theorem [2]. \(\square\)

**Corollary 3.11.** Let \((M, F)\) be a regular AR-Finsler manifold, \(n \geq 2\). Then, the necessary and sufficient condition for \((M, F)\) to be Riemannian is

\[ a^{jk} \left( n \dot{\partial}_a_{ji} - \dot{\partial}_a_{jk} \right) = 0. \]

**Proof.** By making use of Theorem 3.10 and formula (12), we get

\[ \dot{\partial}_i \log(\eta) = -\frac{1}{n} a^{jk} \dot{\partial}_a_{ijk} = \frac{1}{n-1} a^{jk} \left( \dot{\partial}_a_{ijk} - \dot{\partial}_a_{ijk} \right), \]

which is equivalent to

\[ \frac{1}{n-1} a^{jk} \dot{\partial}_a_{ji} + \left( \frac{1}{n} - \frac{1}{n-1} \right) a^{jk} \dot{\partial}_a_{jk} = 0 \iff \frac{1}{n(n-1)} a^{jk} \left( n \dot{\partial}_a_{ji} - \dot{\partial}_a_{jk} \right) = 0. \]
Proposition 3.12. Let \((M,F)\) be an AR-Finsler manifold. Then, the geodesic spray coefficients \(G^i(x,y)\) of \(F\) are rational functions in \(y\) if and only if \(\partial_k \log(\eta(x,y))\) are rational functions in \(y\).

Proof. Plug expression (11) into formula (2), we obtain
\[
G^i = \frac{1}{2} \left( y^j y^l - \frac{1}{2} y^r y^s a_{rs} a^{jl} \right) \partial_l \log(\eta) + \frac{1}{2} y^k y^s a^{kl} \left( \partial_k a_{ls} - \frac{1}{2} \partial_l a_{ks} \right).
\]
Assume that \(\partial_k \log(\eta(x,y))\) are rational functions in \(y\). The proof is completed by noting that the multiplication of rational functions is rational and the partial derivative \(\partial_k\) the rational functions \(a_{ij}(x,y)\) with respect to \(x\) remains rational in \(y\). The converse follows directly from (17).

Proposition 3.13. Let \((M,F)\) be an AR-Finsler manifold. Then, the coefficients of Barthel connection \(N^i_j\) are given by
\[
N^i_j = \frac{1}{2} \left[ y^j \partial_i \log(\eta) + \delta^j_i y^k \partial_k \log(\eta) \right]
+ \frac{1}{4} y^r \left( 2 a_{ir} a^{jl} + y^r [a^{jl} \partial_l a_{rs} + a_{rs} \partial_l a^{jl}] \right) \partial_i \log(\eta)
+ \frac{1}{2} \left( y^i y^j - \frac{1}{2} y^r y^s a_{rs} a^{jl} \right) \partial_i \partial_j \log(\eta)
+ \frac{1}{2} \partial_i \left( y^k y^s a^{lj} \left[ \partial_k a_{ls} - \frac{1}{2} \partial_l a_{ks} \right] \right).
\]
Consequently, \(N^i_j(x,y)\) are rational functions in \(y\) if and only if the functions \(\partial_k \log(\eta(x,y))\) are rational in \(y\).

Proof. The expression of \(N^i_j(x,y)\) immediately follows by differentiating formula (17) with respect to \(y^j\). The necessary and sufficient condition for \(N^i_j(x,y)\) to be rational functions in \(y\) is a direct consequence of Corollary 3.5 and Proposition 3.12 along with using the fact that the multiplication of rational functions is rational and the partial derivative of a rational function in \(y\) with respect to \(x\) remains rational in \(y\).

Corollary 3.14. Let \((M,F)\) be an \(n\)-dimensional AR-Finsler manifold, \(n \geq 2\). Then, the functions \(\delta_k \log(\eta(x,y))\) are rational in \(y\) if and only if \(\partial_i \log(\eta(x,y))\) are rational functions in \(y\).

Proof. Since \(\delta_k \log(\eta) = \delta_k \log(\eta) - N^k_j \partial_j \log(\eta)\), thus, the proof is completed by making use of Propositions 3.4 and 3.13 together with using the fact the multiplication of rational functions is rational.

A direct consequence of Propositions 3.12 and 3.13 is the following result.

Corollary 3.15. Let \((M,F)\) be an AR-Finsler manifold. Then, the Berwald connection \(G^i_{jk}\) and Berwald curvature \(G^i_{jkl}\) are rational functions in \(y\) if and only if the functions \(\partial_i \log(\eta(x,y))\) are rational in \(y\).

Proposition 3.16. Let \((M,F,dV)\) be an AR-Finsler manifold equipped with an arbitrary volume form. Then, the \(S\)-curvature is given by
\[
S = \frac{1}{2} \left( y^r y^s a_{rs} + a_{rs} \partial_r a^m \right) \partial_i \log(\eta)
+ \frac{1}{2} \left( y^m y^j - \frac{1}{2} y^r y^s a_{rs} a^{ml} \right) \partial_m \partial_i \log(\eta) + \frac{1}{2} y^k a^{ml} \left( \partial_m a_{lk} + a_{lm} \partial_l a_m - \frac{1}{2} \partial_l a_{mk} \right)
+ \frac{1}{2} y^k y^s \left[ \partial_m a^{ml} \left( \partial_k a_{ls} - \frac{1}{2} \partial_l a_{ks} \right) \right] - y^m \partial_m \log(\sigma(x)).
\]
Thus, $S$-curvature is rational function in $y$ if and only if the functions $\partial_i \log (\eta(x, y))$ are rational in $y$. Consequently, the $E$-curvature is rational in $y$ if and only if the functions $\partial_i \log (\eta(x, y))$ are rational in $y$.

Proof. It follows from Propositions 3.13 together with formulae (9) and (10) along with using the fact that the multiplication of rational functions is rational.

It is worth mentioning that some Finsler structures $F$ are rational in $y$ while others are not. For example, Kropina metric is rational in $y$, generalized Kropina metric $F(x, y) = \frac{\alpha^{k+1}}{\beta^2}$, where $k$ is an even number, is not rational function in $y$ (see section 4, for details). The following result holds only when a Finsler metric $F$ is not rational function in $y$.

Theorem 3.17. Let $(M, F, dV)$ be an AR-Finsler manifold equipped with an arbitrary volume form such that $F(x, y)$ is not rational in $y$ and the functions $\partial_i \log (\eta(x, y))$ are rational in $y$. If $(M, F, dV)$ has isotropic $S$-curvature, then $(M, F)$ has vanishing $S$-curvature.

Proof. A Finsler metric $F$ is said to have isotopic $S$-curvature if there exists a function $A$ in $M$ such that

$$S(x, y) = (n + 1) A(x) F(x, y).$$

Assume that the functions $\partial_i \log (\eta(x, y))$ are rational in $y$. Then, by Proposition 3.7, the $S$-curvature is rational in $y$, on the other hand $F(x, y)$ is not rational in $y$ by hypothesis. Therefore, $S$ identically vanishes.

Proposition 3.18. Let $(M, F)$ be an AR-Finsler manifold. Then, the Douglas curvature components $D_{jkl}$ are rational functions in $y$ if and only if the functions $\partial_i \log (\eta(x, y))$ are rational in $y$.

Proof. It results from Proposition 3.13 and Corollary 3.15 together with formula (5).

Proposition 3.19. Let $(M, F)$ be an AR-Finsler manifold, $n \geq 2$. Then, the Landsberg curvature components $L_{ijk}$ are rational functions in $y$ if and only if $\partial_i \log (\eta(x, y))$ are rational functions in $y$.

Proof. It results by making use of Corollary 3.15 along with the formula

$$L_{ijk} = \frac{1}{2} y^m g_{ms} G^s_{ijk}.$$

Lemma 3.20. Let $(M, F)$ be an AR-Finsler manifold with $n \geq 2$. Then, the mean Landsberg curvature $J_k(x, y)$ are rational functions in $y$ if and only if $\partial_k \log (\eta(x, y))$ are rational functions in $y$.

Proof. Proposition 3.7 reads that the functions $I_k(x, y)$ are rational in $y$. Thereby, $\partial_i I_k(x, y)$ and $\partial_i I_k(x, y)$ are rational functions in $y$. Propositions 3.12 and 3.13 say that the rationally of the quantities $G^s(x, y)$ and $N^s_k(x, y)$ in $y$ is equivalent to the rationally of the functions $\partial_k \log (\eta(x, y))$ in $y$. Thus, by formula (3), the proof is completed.

Theorem 3.21. Let $(M, F)$ be an AR-Finsler manifold with $n \geq 2$ such that $F(x, y)$ is not a rational function in $y$ and the functions $\partial_i \log (\eta(x, y))$ are rational in $y$. If $(M, F)$ has isotropic mean Landsberg curvature, then $(M, F)$ is weakly Landsberg manifold.

Proof. It is clear from Lemma 3.20 that $J_k(x, y)$ are rational functions in $y$. Suppose $(M, F)$ is an isotropic mean Landsberg manifold, that is $J$ can be written as in (4). But $F$ is not rational function in $y$, by our assumption, and $I$ is rational in $y$ by Proposition 3.7. Therefore, by formula (4), $J$ must identically vanishes. Thus, $(M, F)$ is a weakly Landsberg manifold.
**Proposition 3.22.** Let \((M, F)\) be an AR-Finsler manifold, \(n \geq 2\). Then, the Riemannian curvature is a rational function in \(y\) if and only if the functions \(\partial_i \log (\eta(x, y))\) are rational in \(y\) for. Consequently, the Weyl curvature and \(\chi\)-curvature are rational in \(y\) if and only if \(\partial_i \log (\eta(x, y))\) are rational functions in \(y\).

**Proof.** Let \(F\) be an AR-Finsler metric such that \(\partial_i \log (\eta(x, y))\) are rational functions in \(y\). It is clear from Propositions 3.12 that \(G^i(x, y)\) are rational functions in \(y\). Thus, \(\partial_k G^i(x, y)\) are rational functions in \(y\) as the differentiation with respect to the manifold coordinates \(\partial_k\) is not affecting the rationality of \(G^i(x, y)\) in \(y\). Also the product of rational functions is a rational function, considering (6) leads to the rationality of \(R^i_k(x, y)\) in \(y\). Similarly, by formulae (7) and (8) the proof is completed. \(\square\)

The following result holds only in case of the AR-Finsler metrics with \(\eta\) is not a rational function in \(y\). It may be considered as a generalization of [11, Theorem 1.1] and [9, Theorem 1.1].

**Theorem 3.23.** Let \((M, F)\) be an AR-Finsler manifold \(n \geq 2\) such that the function \(\eta\) is not rational in \(y\) and the functions \(\partial_i \log (\eta(x, y))\) are rational in \(y\). If \(F\) is an Einstein metric then it is Ricci-flat.

**Proof.** By Propositions 3.22, \(\text{Ric} = R^i_i\) is a rational function in \(y\). By our postulates, \(F\) is an Einstein metric, that is,

\[
\text{Ric} = (n - 1) K F^2, \quad \text{where } K \in C^\infty(M).
\]

The function \(\eta\) is not rational in \(y\) thereby, \(F^2 = \eta a_{ij} y^i y^j\) is not a rational function in \(y\). Hence, \(K = 0\) which means \(\text{Ric} = 0\). \(\square\)

**Remark 3.24.** It may be remarked here that we can not expect all irrational Finsler metrics \(g_{ij}\) in the form of (11). For example, Shen’s circles with radius 1 and centered at \((0,0)\) cf. [3]. Let \(M = \mathbb{R}^2\) equipped with the following Finsler metric of Randers type

\[
F(x, y) = F(x^1, x^2; y^1, y^2) = \sqrt{(y^1)^2 + (y^2)^2 + A(x^1, x^2) (y^1 + y^2)},
\]

where \(A(x^1, x^2) \in C^\infty(M)\). It is easy to see that, the associated metric tensor has the following components:

\[
g_{11} = 1 + A^2(x^1, x^2) + A(x^1, x^2) \frac{(y^1)^3 + (y^2)^3 + y^1 (y^2)^2}{[(y^1)^2 + (y^2)^2]^{\frac{3}{2}}},
\]

\[
g_{22} = 1 + A^2(x^1, x^2) + A(x^1, x^2) \frac{(y^1)^3 + (y^2)^3 + y^2 (y^1)^2}{[(y^1)^2 + (y^2)^2]^{\frac{3}{2}}},
\]

\[
g_{12} = A^2(x^1, x^2) + A(x^1, x^2) \frac{(y^1)^3 + (y^2)^3}{[(y^1)^2 + (y^2)^2]^{\frac{3}{2}}}.
\]

This metric has irrational spray coefficients

\[
G^1 = \frac{y^2}{2} \sqrt{(y^1)^2 + (y^2)^2}, \quad G^2 = \frac{y^1}{2} \sqrt{(y^1)^2 + (y^2)^2}.
\]

Further, its Cartan torsion is irrational in \(y\) while its Riemannian curvature is quadratic.

The Shen’s circles example motivates us to study the case of Randers metrics which leads to the following result.

**Theorem 3.25.** There is no AR-Finsler manifold of Randers type.
Proof. Suppose that \( F = \alpha + \beta \). Thus, its metric tensor is given by
\[
g_{ij}(x, y) = \frac{1}{2} \partial_i \partial_j F^2(x, y) = \frac{1}{2} \partial_i \partial_j (\alpha(x, y) + \beta(x, y))^2
\]
\[
= \frac{1}{2} \partial_i \partial_j \alpha^2(x, y) + \frac{1}{2} \partial_i \partial_j \beta^2(x, y) + \partial_i \partial_j (\alpha(x, y) \beta(x, y))
\]
\[
= \alpha_{ij}(x) + b_i(x) b_j(x) + \frac{1}{\alpha} \left\{ \left( \alpha_{ij}(x) - \frac{y_i y_j}{\alpha^2(x, y)} \right) \beta(x, y) + b_i(x) y_j + b_j(x) y_i \right\},
\]
where \( y_i = \alpha_{ij}(x) y^j \). It is clear that \( \alpha_{ij} + b_i b_j \) is a function of \( x \) only, thus
\[
\left\{ \left( \alpha_{ij}(x) - \frac{y_i y_j}{\alpha^2(x, y)} \right) \beta(x, y) + b_i(x) y_j + b_j(x) y_i \right\}
\]
is a rational function in \( y \) and \( \frac{1}{\alpha(x, y)} \) is not rational function in \( y \). Therefore, \( g_{ij} \) can be written in the form of (11) with
\[
\eta(x, y) = \frac{1}{\alpha(x, y)} \quad \text{and} \quad a_{ij}(x, y) = \left\{ \left( \alpha_{ij}(x) - \frac{y_i y_j}{\alpha^2(x, y)} \right) \beta(x, y) + b_i(x) y_j + b_j(x) y_i \right\}
\]
if and only if \( \{ \alpha_{ij}(x) + b_i(x) b_j(x) \} \) vanish identically, which is impossible. Hence, there is no Randers structure whose fundamental metric can be written in the form of (11). \( \square \)

Remark 3.26. It should be noted that, Randers metrics of Berwald type cf. [6] (in which \( \beta \) is parallel with respect to \( \alpha \)) are examples of non AR-Finsler metrics with rational, more precisely quadratic, spray coefficients.

4. EXAMPLES OF AR-FINSLER METRICS

In this section we study some examples of AR-Finsler metrics.

The generalized Kropina metric.

The Finsler function of generalized Kropina metric is defined by \( F = \alpha \phi(s) \) where \( \phi(s) = \frac{1}{s^m}, \ m \neq 0, -1 \). By substituting in formula (1), the metric tensor is given by
\[
g_{ij} = \frac{m+1}{s^{2m}} \alpha_{ij} + \frac{m(2m+1)}{s^{2m+2}} b_i b_j - \frac{m(m+2)}{\alpha s^{2m+1}} \left( b_i y_j + b_j y_i - \frac{s}{\alpha} y_i y_j \right).
\]

It is clear that, the function \( \frac{1}{s^{2m}} = \left( \frac{1}{s} \right)^{m+1} = \left( \frac{\alpha_{ij}(x) y^i y^j}{b_i(x) b_j(x) y^i y^j} \right)^{m+1} \) is a rational function in \( y \) when \( m \) is integer and irrational function in \( y \) when \( m \) not integer. Similarly, the functions \( \frac{1}{s^{2m+2}} = \left( \frac{\alpha_{ij}(x) y^i y^j}{b_i(x) b_j(x) y^i y^j} \right)^m \) and \( \frac{1}{s^{2m+1}} = \frac{1}{s b_i(x) y^i} \left( \frac{\alpha_{ij}(x) y^i y^j}{b_i(x) b_j(x) y^i y^j} \right)^m \) are rational functions in \( y \) when \( m \) is integer and irrational functions in \( y \) when \( m \) not integer. Which leads to the metric (18) is rational in \( y \) when \( m \) is integer. Further, expression (18) can be written as follows:
\[
g_{ij} = \frac{1}{s^{2m}} \left[ (m+1) \alpha_{ij} + \frac{m(2m+1)}{s^2} b_i b_j - \frac{m(m+2)}{\alpha s} \left( b_i y_j + b_j y_i - \frac{s}{\alpha} y_i y_j \right) \right].
\]

Thereby, expression (18) has the form of (11) with
\[
\eta = \frac{1}{s^{2m}} \quad \text{and} \quad a_{ij} = (m+1) \alpha_{ij} + \frac{m(2m+1)}{s^2} b_i b_j - \frac{m(m+2)}{\alpha s} \left( b_i y_j + b_j y_i - \frac{s}{\alpha} y_i y_j \right).
\]

It should be noted that, the functions \( a_{ij}(x, y) \) are rational in \( y \). Hence, the generalized Kropina metric is an AR-Finsler metric. By direct calculations, we get
\[
\hat{\partial}_t \log(\eta(x, y)) = -2m \frac{b_i(x)}{b_i(x) y^i} + 2m \frac{\alpha_{ij}(x) y^i y^j}{\alpha_{ij}(x) y^i y^j}.
\]
Remark 4.1. In particular for \( k = 1 \), \( F \) is the well known Kropina metric, which has been studied in details in [4, 6].

The special polynomial \((\alpha, \beta)\)-metric.

Let \( F := \alpha \phi(s) \) where \( \phi(s) = a s^k + b s^m \), \( m = k(\text{mod}2) \) such that
\[
a (m-1) s^m + b (k-1) s^k > 0, \quad |s| < b_\alpha.
\]
By straight forward calculations, we get
\[
g_{ij} = -\left( \frac{O}{2m} s^{2m} + \frac{Z}{k} s^{2k} + \frac{U}{(m+k)} s^{m+k} \right) \alpha_{ij} + A s^{2m} + B s^{2k} + H s^{m+k} \frac{b_i b_j}{s^2} - \frac{O s^{2m} + Z s^{2k} + U s^{m+k}}{s \alpha} \left( b_i y_j + y_i b_j - \frac{s}{\alpha} y_i y_j \right),
\]
Thereby, \( g_{ij} \) can be written in the form:
\[
g_{ij} = -\left( \frac{O}{2m} s^{2m} + \frac{Z}{k} s^{2k} + \frac{U}{(m+k)} s^{m+k} \right) \alpha_{ij} + A s^{2m} + B s^{2k} + H s^{m+k} \frac{b_i b_j}{s^2} - \frac{O s^{2m} + Z s^{2k} + U s^{m+k}}{s \alpha} \left( b_i y_j + y_i b_j - \frac{s}{\alpha} y_i y_j \right),
\]
where \( A := a^2 m (2m-1) \), \( B := b^2 k (2k-1) \), \( H := a b (m+k-1) (m+k) \),
\( O := 2 a^2 m (m-1) \), \( Z := b^2 k (k-1) \), \( U := a b (m+k-2) (m+k) \) are real constants.

Therefore, \( g_{ij} \) are rational functions in \( y \). Indeed, the coefficients of \( \alpha_{ij}(x) \) are linear combination of the terms \( s^{2m}, s^{2k} \) and \( s^{m+k} \) which are rational functions in \( y \). Similarly, the coefficients of \( b_i(x) b_j(x) \) are the linear combination of \( s^{2m-2}, s^{2k-2} \) and \( s^{k+m-2} \), which are also rational functions in \( y \). The last term in \( g_{ij} \) is the product of rational functions in \( y \), namely \( \frac{s}{\alpha} = \frac{s}{\beta} \), with the linear combination of the functions \( s^{2m}, s^{2k}, s^{k+m} \). It may be noted here that the term \( \frac{s}{\alpha} = \frac{s}{\beta} \) itself is rational in \( y \). Since \( g_{ij} \) is a rational function in \( y \), we get
\[
\dot{b}_i \log(\eta(x,y)) = 0 = \partial_i \log(\eta(x,y)).
\]

The \( m \)-th root metric.

The Finsler mapping of the \( m \)-th root type is defined by, cf. [11],
\[
F(x,y) := \left( a_{i_1i_2...i_m}(x) y^{i_1} y^{i_2} ... y^{i_m} \right)^{\frac{1}{m}}.
\]
In fact, \( F(x,y) \) is a rational function in \( y \) if and only if it is quadratic in \( y \), that is \( F \) is Riemannian. In addition, \( F \) is positive definite when \( m \) is even. It is clear that for \( m \geq 3 \) and \( n \geq 2 \), the \( m \)-th root metric is an irrational function in \( y \).
Let the polynomial
\[(20)\]
\[A := a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m} = F^m.\]

Thereby, its fundamental metric tensor has the following components,

\[g_{ij}(x, y) = \frac{1}{m} A^{\frac{2}{m} - 2} \left[ A \partial_i \partial_j A + \frac{2 - m}{m} \partial_i A \partial_j A \right].\]

Thus, \(F\) is an AR-Finsler metric. Indeed, \(g_{ij}(x, y)\) satisfies (11) where

\[\eta(x, y) = \frac{1}{m} A^{\frac{2}{m} - 2} = \frac{1}{m} F^{2-2m} \quad \text{and} \quad a_{ij} = A \partial_i \partial_j A - \frac{2-m}{m} \partial_i A \partial_j A.\]

Obviously, \(\eta(x, y)\) is an irrational positive homogeneous function of degree \((2 - m)\) in \(y\) and \(a_{ij}(x, y)\) are rational positive homogeneous functions of degree \((m - 2)\) in \(y\). By direct calculations, we get

\[\partial_i \log(\eta(x, y)) = \frac{2(1-m)}{m^2} \partial_i A(x, y) \frac{A(x, y)}{a_{ij}} = \frac{2(1-m)}{m} a_{i_1 i_2 \ldots i_m}(x) y^{i_2} \ldots y^{i_m}
\]

which are rational functions in \(y\). Similarly,

\[\partial_i \log(\eta(x, y)) = \frac{2(1-m)}{m^2} \partial_a a_{i_1 i_2 \ldots i_m}(x) y^{i_1} y^{i_2} \ldots y^{i_m}
\]

are rational functions in \(y\).

**Remark 4.2.** A special case of (19) is the Finsler metric \(F(x, y) = f(x)(y^1 y^2 y^3)^{\frac{1}{3}}\) which defined in the conic domain \(D = T^* \mathbb{R}^3 - \{(x', y') \in T^* \mathbb{R}^3 | y^i \neq 0\}\). It is an AR-Finsler manifold which does not admit a semi-concurrent vector field [10, Remark 3.13].

The generalized Kropina change \(\tilde{F} = \frac{F^{k+1}}{\beta^2}\) of an \(m\)-th root metric \(F\). Here, \(F\) is defined in (19) and \(k\) is an arbitrary positive real number cf. [7].

One can check that, the fundamental metric tensor \(\tilde{g}_{ij}(x, y)\) satisfies (11) with

\[\tilde{\eta}(x, y) = \frac{A^{2k+2-m}(x, y)}{\beta^{2k}(x, y)} = \frac{F^{2(k+1)-m}(x, y)}{\beta^{2k}(x, y)} \quad \text{and} \quad \tilde{a}_{ij} = \frac{k(2k+1)}{\beta^2} A b_i b_j + \frac{(k+1)}{m} \partial_i \partial_j A + \frac{(k+1)(2k-m+2)}{m^2 A} \partial_i A \partial_j A - \frac{2k(k+1)}{m^2} \left( b_i \partial_i A + b_j \partial_j A \right).
\]

Therefore, \(\tilde{F}\) is an AR-Finsler metric.

Thereby, \(\tilde{\eta}(x, y)\) is an irrational positive homogeneous function of degree \((2 - m)\) in \(y\) and \(\tilde{a}_{ij}(x, y)\) are positive homogeneous of degree \((m - 2)\) rational functions in \(y\). By direct calculations, one can see that \(\partial_i \log(\tilde{\eta}(x, y))\) and \(\partial_i \log(\tilde{\eta}(x, y))\) are rational functions in \(y\).

**Remark 4.3.** In particular for \(k = 1\), \(\tilde{F}\) becomes the Kropina change of an \(m\)-th root metric which has been studied in [8].

The extended \(m\)-th root metric.

Now, let us introduce the following Finsler metric that can be considered as an extension of the \(m\)-th root metric. It is given by

\[(21)\]
\[F(x, y) := \left( \mu_{i_1 i_2 \ldots i_m}(x, y) y^{i_1} y^{i_2} \ldots y^{i_m} \right)^{\frac{1}{m}},\]

where \(\mu_{i_1 i_2 \ldots i_m}(x, y)\) symmetric in all indices, positive homogeneous of degree \((0)\) in \(y\). In fact, \(F(x, y)\) is a rational function in \(y\) if and only if it is quadratic in \(y\) and \(\mu_{i_1 i_2 \ldots i_m}(x, y)\)
are rational functions in $y$. It is clear that for $m \geq 3$ and $n \geq 2$, an extended $m$-th root metric (21) is irrational function in $y$. Let

$$A(x, y) = \mu_{i_1i_2...i_m}(x, y) y^{i_1} y^{i_2}... y^{i_m} = F^m.$$ 

Thus, the supporting element is given by

$$l_i = \dot{\hat{\alpha}}_i F = \dot{\hat{\alpha}}_i A^{1/m} = \frac{1}{m} A^{1-m} \dot{\hat{\alpha}}_i A.$$ 

The associated metric tensor has the following components

$$g_{ij}(x, y) = A^{\frac{2-m}{m}} \left[ \frac{1}{m} A \dot{\hat{\alpha}}_i \dot{\hat{\alpha}}_j A + \frac{2-m}{m^2} \dot{\hat{\alpha}}_i A \dot{\hat{\alpha}}_j A \right].$$

It should be noted that, when $\mu_{i_1i_2...i_m}(x, y)$ are rational functions in $y$, then $g_{ij}(x, y)$ satisfies (11) where

$$\eta(x, y) = A^{\frac{2-m}{m}} = F^{2-2m} \quad \text{and} \quad a_{ij} = \frac{1}{m} A \dot{\hat{\alpha}}_i \dot{\hat{\alpha}}_j A - \frac{2-m}{m^2} \dot{\hat{\alpha}}_i A \dot{\hat{\alpha}}_j A.$$ 

Indeed, in case of $\mu_{i_1i_2...i_m}(x, y)$ are rational functions in $y$, the function $A(x, y)$ is rational in $y$. One can see that, $\eta(x, y)$ is an irrational positive homogeneous of degree $(2-m)$ in $y$ and $a_{ij}(x, y)$ are rational positive homogeneous functions of degree $(m-2)$ in $y$. Consequently, $F$ is an AR-Finsler metric.

By direct calculations, we get

$$\dot{\hat{\alpha}}_i \log(\eta(x, y)) = (2-2m) \frac{\dot{\hat{\alpha}}_i F(x, y)}{F(x, y)} = (2-2m) \frac{\dot{\hat{\alpha}}_i A(x, y)}{A(x, y)} = \frac{m}{m} \mu_{i_1i_2...i_m}(x, y) y^{i_1}... y^{i_m} y^{i_1} y^{i_2}... y^{i_m} \dot{\hat{\alpha}}_i \mu_{i_1i_2...i_m}(x, y)$$ 

which are rational functions in $y$. Similarly, $\partial_i \log(\eta(x, y))$ are rational functions in $y$.

The generalized Kropina change of the extended $m$-th root metric with rational coefficients.

Here, $F$ is given by (21), where $\mu_{i_1i_2...i_m}(x, y)$ are rational functions in $y$. One can check that, $\hat{F} := \frac{F^{k+1}}{F^k}$ is an AR-Finsler metric. Indeed, its fundamental tensor $\hat{g}_{ij}(x, y)$ satisfies (11) with

$$\hat{\eta}(x, y) = A^{\frac{2k+2-m}{m}}(x, y) = \frac{F^{2(k+1)-m}(x, y)}{\beta^{2k}(x, y)} \quad \text{and}$$

$$\hat{a}_{ij} = \frac{k(2k+1)}{\beta^2} A b_i b_j + \frac{(k+1)}{m} \dot{\hat{\alpha}}_i \dot{\hat{\alpha}}_j A \left( \frac{(k+1)(2k-m+2)}{m^2 A} \dot{\hat{\alpha}}_i A \dot{\hat{\alpha}}_j A - \frac{2k(k+1)}{m \beta} (b_j \dot{\hat{\alpha}}_i A + b_i \dot{\hat{\alpha}}_j A) \right).$$

Thereby, $\hat{\eta}(x, y)$ is an irrational positive homogeneous function of degree $(2-m)$ in $y$ and $\hat{a}_{ij}(x, y)$ are rational functions in $y$ and positive homogeneous of degree $(m-2)$ in $y$. Moreover, one can show that $\dot{\hat{\alpha}}_i \log(\eta(x, y))$ and $\partial_i \log(\eta(x, y))$ are rational functions in $y$.

Remark 4.4. In particular for $k = 1$, $\hat{F}$ becomes the Kropina change of an extended $m$-th root metric with rational coefficients.
Remark 4.5. In the above mentioned examples, we observe the following:

(i) The generalized Kropina change of an $m$-th root metric or extended $m$-th root metric with rational coefficients preserves the almost rationality of the Finsler metric.

(ii) The geometric objects associated to the AR-Finsler metrics, namely, $I_k$, $G^i$, $N^i_j$, $G^i_{jk}$, $G^i_{jkl}$, $D^i_{jkl}$, $L^i_{jkl}$, $J_k$, $R^i_j$, $\chi_i$, $W^i_j$ and the $S$-curvature are rational functions in $y$. It follows from the results of §3 and these AR-metrics satisfy the property that $\partial_i \log(\eta(x,y))$ are rational functions in $y$.

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