Improved Algorithms for Convex-Concave Minimax Optimization

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Abstract

This paper studies minimax optimization problems \( \min_x \max_y f(x, y) \), where \( f(x, y) \) is \( m_x \)-strongly convex with respect to \( x \), \( m_y \)-strongly concave with respect to \( y \) and \( (L_x, L_{xy}, L_y) \)-smooth. Zhang et al. [2019] provided the following lower bound of the gradient complexity for any first-order method:

\[
\Omega \left( \sqrt{\frac{L_x}{m_x} + \frac{L_{xy}^2}{m_x m_y} + \frac{L_y}{m_y} \ln \left( \frac{1}{\epsilon} \right)} \right)
\]

This paper proposes a new algorithm with gradient complexity upper bound \( \tilde{O} \left( \sqrt{\frac{L_x}{m_x} + \frac{L_{xy}^2}{m_x m_y} + \frac{L_y}{m_y} \ln \left( \frac{1}{\epsilon} \right)} \right) \), where \( L = \max\{L_x, L_{xy}, L_y\} \). This improves over the best known upper bound \( \tilde{O} \left( \sqrt{\frac{L_{xy}^2}{m_x m_y} \ln^3 \left( \frac{1}{\epsilon} \right)} \right) \) by Lin et al. [2020]. Our bound achieves linear convergence rate and tighter dependency on condition numbers, especially when \( L_{xy} \ll L \) (i.e., when the interaction between \( x \) and \( y \) is weak). Via reduction, our new bound also implies improved bounds for strongly convex-concave and convex-concave minimax optimization problems. When \( f \) is quadratic, we can further improve the upper bound, which matches the lower bound up to a small sub-polynomial factor.

1 Introduction

In this paper, we study the following minimax optimization problem

\[
\min_{x \in \mathbb{R}^n} \max_{y \in \mathbb{R}^m} f(x, y).
\] (1)

This problem can be thought as finding the equilibrium in a zero-sum two-player game, and has been studied extensively in game theory [Von Neumann and Morgenstern, 2007, Basar and Olsder, 1999]. This formulation also arises in many machine learning applications, including adversarial training [Madry et al., 2018, Sinha et al., 2017], prediction and regression problems [Xu et al., 2005, Taskar et al., 2006], reinforcement learning [Du et al., 2017, Dai et al., 2018, Nachum et al., 2019] and generative adversarial networks [Goodfellow et al., 2014, Arjovsky et al., 2017]. Apart from machine learning, minimax optimization has also found applications in imaging [Chambolle and Pock, 2011, Haber and Modersitzki, 2004], control [Hast et al., 2013] and economics [Nagurney, 2013].

We study the fundamental setting where \( f \) is smooth, strongly convex w.r.t. \( x \) and strongly concave w.r.t. \( y \). In particular, we consider the function class \( \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y) \), where \( m_x \) is the strong convexity modulus, \( m_y \) is the strong concavity modulus, \( L_x \) and \( L_y \) characterize the smoothness w.r.t. \( x \) and \( y \) respectively, and \( L_{xy} \) characterizes the interaction between \( x \) and \( y \) (see Definition 2). The reason to consider such a function class is twofold. First, the strongly convex-strongly concave setting is fundamental. Via reduction [Lin et al., 2020], an efficient algorithm for this setting implies efficient algorithms for other settings, including strongly convex-concave, convex-concave, and non-convex-concave settings. Second, Zhang et al. [2019] recently proved a gradient complexity lower bound \( \Omega \left( \sqrt{\frac{L_x}{m_x} + \frac{L_{xy}^2}{m_x m_y} + \frac{L_y}{m_y} \cdot \ln \left( \frac{1}{\epsilon} \right)} \right) \), which naturally depends on the above parameters.

In this setting, classic algorithms such as Gradient Descent-Ascent and ExtraGradient [Korpelevich, 1976] can achieve linear convergence [Tseng, 1995, Zhang et al., 2019]; however, their dependence on the condition
number is far from optimal. Recently, Lin et al. [2020] showed an upper bound of $\tilde{O}\left(\sqrt{L^2 / m_x m_y} \ln^3 (1/\epsilon)\right)$, which has a much tighter dependence on the condition number. In particular, when $L_{xy} > \max\{L_x, L_y\}$, the dependence on the condition number matches the lower bound. However, when $L_{xy} \ll \max\{L_x, L_y\}$, this dependence would no longer be tight (see Fig 1 for illustration). In particular, we note that, when $x$ and $y$ are completely decoupled (i.e., $L_{xy} = 0$), the optimal gradient complexity bound is $\Theta\left(\sqrt{L_x / m_x + L_y / m_y} \cdot \ln (1/\epsilon)\right)$ (the upper bound can be obtained by simply optimizing $x$ and $y$ separately). Moreover, Lin et al.’s result does not enjoy a linear rate, which may be undesirable if a high precision solution is needed.

In this work, we propose new algorithms in order to address these two issues. Our contribution can be summarized as follows.

1. For general functions in $F(m_x, m_y, L_x, L_{xy}, L_y)$, we design an algorithm called Proximal Best Response (Algorithm 4), and prove a convergence rate of

$$\tilde{O}\left(\sqrt{L_x / m_x + L_{xy} \cdot L / m_x m_y + L_y / m_y} \ln(1/\epsilon)\right).$$

It achieves linear convergence, and has a better dependence on condition numbers when $L_{xy}$ is small (see Theorem 3 and the red line in Fig. 1).

2. We obtain tighter upper bounds for the strongly-convex concave problem and the general convex-concave problem, by reducing them to the strongly convex-strongly concave problem (See Corollary 1 and 2).

3. We also study the special case where $f$ is a quadratic function. We propose an algorithm called Recursive Hermitian-Skew-Hermitian Split (RHSS($k$)), and show that it achieves an upper bound of

$$O\left(\sqrt{L_x / m_x + L_{xy}^2 / m_x m_y + L_y / m_y} \left(\frac{L^2}{m_x m_y}\right)^{\alpha(1)} \ln(1/\epsilon)\right).$$

Details can be found in Theorem 4 and Corollary 3. We note that the lower bound by Zhang et al. [2019] holds for quadratic functions as well. Hence, our upper bound matches the gradient complexity lower bound up to a sub-polynomial factor.
2 Preliminaries

In this work we are interested in strongly-convex strongly-concave smooth problems. We first review some standard definitions of strong convexity and smoothness. A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is \( L \)-Lipschitz if \( \forall x, x' \in \mathbb{R}^n \), \( \| f(x) - f(x') \| \leq L \| x - x' \| \). A function \( f : \mathbb{R}^n \to \mathbb{R}^m \) is \( L \)-smooth if \( \nabla f \) is \( L \)-Lipschitz. A differentiable function \( \phi : \mathbb{R}^n \to \mathbb{R} \) is said to be \( m \)-strongly convex if for any \( x, x' \in \mathbb{R}^n \), \( \phi(x') \geq \phi(x) + \langle \nabla \phi(x), x' - x \rangle + \frac{m}{2} \| x' - x \|^2 \). If \( m = 0 \), we recover the definition of convexity. If \( \phi \) is \( m \)-strongly convex, \( \phi \) is said to be \( m \)-concave, then \( f \) is said to be \( (L, m) \)-smooth.

\[ f \] is \( (L, m) \)-smooth if
\[ \frac{1}{L} \nabla f(x) \leq \frac{1}{m} \nabla \phi(x) \]

for all \( x \in \mathbb{R}^n \) and \( \phi \) is \( m \)-strongly convex.

**Definition 2.** The function class \( \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y) \) contains differentiable functions from \( \mathbb{R}^n \times \mathbb{R}^m \) to \( \mathbb{R} \) such that:
1. \( \forall y, f(\cdot, y) \) is \( m_x \)-smooth;
2. \( \forall x, f(x, \cdot) \) is \( m_y \)-smooth;
3. \( f \) is \( (L_x, L_{xy}, L_y) \)-smooth.

In the case where \( f(x, y) \) is twice continuously differentiable, denote the Hessian of \( f \) at \((x, y)\) by
\[ H := \begin{bmatrix} H_{xx} & H_{xy} \\ H_{yx} & H_{yy} \end{bmatrix}. \]
Then \( \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y) \) can be characterized with the Hessian; in particular we require \( m_x I \leq H_{xx} \leq L_x I, \quad m_y I \leq H_{yy} \leq L_y I \) and \( \| H_{xy} \|_2 \leq L_{xy} \).

For notational simplicity, we assume that \( L_x = L_y \) when considering algorithms and upper bounds. This is without loss of generality, since one can define \( g(x, y) := f((L_y/L_x)^{1/4} x, (L_x/L_y)^{1/4} y) \) in order to make the two smoothness constants equal. It is not hard to show that this rescaling does not change \( L_x/m_x, L_y/m_y, L_{xy} \) and \( m_x m_y \), and that \( L = \max \{ L_x, L_{xy}, L_y \} \) does not increase. Hence, we can make the following assumption without loss of generality. \(^1\)

**Assumption 1.** \( f \in \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y) \), and \( L_x = L_y \).

The optimal solution of the convex-concave minimax optimization problem \( \min_x \max_y f(x, y) \) is the saddle point \((x^*, y^*)\) defined as follows.

**Definition 3.** \((x^*, y^*)\) is a saddle point of \( f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) if \( \forall x \in \mathbb{R}^n, y \in \mathbb{R}^m \)
\[ f(x, y^*) \geq f(x^*, y^*) \geq f(x^*, y). \]

For strongly convex-strongly concave functions, it is well known that such a saddle point exists and is unique. Meanwhile, the saddle point is a stationary point, i.e. \( \nabla f(x^*, y^*) = 0 \), and is the minimizer of \( \phi(x) := \max_y f(x, y) \). For the design of numerical algorithms, we are satisfied with an close enough approximation of the saddle point, called \( \epsilon \)-saddle points.

**Definition 4.** \((\hat{x}, \hat{y})\) is an \( \epsilon \)-saddle point of \( f \) if \( \max_y f(\hat{x}, y) - \min_x f(x, \hat{y}) \leq \epsilon \).

Alternatively, we can also characterize optimality with the distance to the saddle point. In particular, let \( z^* := [x^*; y^*], \quad \hat{z} := [x; y] \), then one may require \( \| \hat{z} - z^* \| \leq \epsilon \). This implies that \(^2\)
\[ \max_y f(x, y) - \min_x f(x, \hat{y}) \leq \frac{L^2}{\min \{ m_x, m_y \} \epsilon^2}. \]

Thus, it is sufficient to find a point close enough to the saddle point.

In this work we focus on first-order methods, that is, algorithms that only access \( f \) through gradient evaluations. The complexity of algorithms is measured through the gradient complexity: the number of gradient evaluations required to find an \( \epsilon \)-saddle point (or get to \( \| \hat{z} - z^* \| \leq \epsilon \)).

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\(^1\) Note that this rescaling also does not change the lower bound.

\(^2\) See Fact 4 in the appendix for proof.
2.1 Accelerated Gradient Descent

Nesterov’s Accelerated Gradient Descent [Nesterov, 1983] is an optimal first-order algorithm for smooth and convex functions. Here we present a version of AGD for minimizing an $l$-smooth and $m$-strongly convex functions $g(\cdot)$. It is a crucial building block for the algorithms in this work.

Algorithm AGD($g, x_0, T$) [Nesterov, 2013]

Require: Initial point $x_0$, smoothness constant $l$, strongly-convex modulus $m$, number of iterations $T$

$x_0 \leftarrow x_0$, $\eta \leftarrow 1/l$, $\kappa \leftarrow l/m$, $\theta \leftarrow \left(\sqrt{\kappa} - 1\right)/\left(\sqrt{\kappa} + 1\right)$

for $t = 1, \cdots, T$ do

$x_t \leftarrow x_{t-1} - \eta \nabla g(x_{t-1})$

$x_t \leftarrow x_t + \theta(x_t - x_{t-1})$

end for

The following convergence theorem holds for AGD. It implies that the complexity is $O(\sqrt{\kappa} \ln (1/\epsilon))$, which greatly improves over the $O(\kappa \ln (1/\epsilon))$ bound for gradient descent.

Lemma 1. (Nesterov [2013, Theorem 2.2.3]) In the AGD algorithm,

$\|x_T - x^*\|^2 \leq (\kappa + 1)\|x_0 - x^*\|^2 \left(1 - \frac{1}{\sqrt{\kappa}}\right)^T$.

3 Related Work

There has been a long line of work on the convex-concave saddle point problem. Apart from GDA and ExtraGradient [Korpelevich, 1976, Tseng, 1995, Nemirovski, 2004, Gidel et al., 2019], other algorithms with theoretical guarantees include OGDA [Rakhlin and Sridharan, 2013, Daskalakis et al., 2018, Mokhtari et al., 2019, Azizian et al., 2019], Hamiltonian Gradient Descent [Abernethy et al., 2019] and Consensus Optimization [Mescheder et al., 2017, Abernethy et al., 2019, Azizian et al., 2019]. For the convex-concave case and strongly-convex-concave case, lower bounds have been provided by Ouyang and Xu [2019].

Some authors have studied variance reduction algorithms for minimax optimization [Carmon et al., 2019, Palaniappan and Bach, 2016], which is beyond the scope of this work. One special case of convex-concave functions is the so-called bilinear case, where $f(x, y) = h_1(x) + x^T Ay - g(y)$. It has been studied by Chambolle and Pock [2011], Chen et al. [2014], Ouyang and Xu [2019] and Du and Hu [2019].

Another special case is the quadratic case, where $f(x, y)$ is a quadratic function, and solving the saddle point problem amounts to solving a structured linear system. The quadratic saddle point problem has been studied extensively in the numerical analysis community [Benzi et al., 2005, Bai et al., 2003, Bai, 2009]. One of the most notable algorithms for quadratic saddle point problems is Hermitian-ske-Hermitian Split (HSS) [Bai et al., 2003]. However, most existing work do not provide a bound on the overall number of matrix-vector products.

Beyond the convex-concave setting, some researchers have also studied the nonconvex-concave case recently [Lin et al., 2019, Thekumparampil et al., 2019, Rafique et al., 2018, Lin et al., 2020, Lu et al., 2019, Nouiehed et al., 2019], with the goal being finding a stationary point of the nonconvex function $\phi(x) := \max_y f(x, y)$. By reducing to the strongly convex-strongly concave setting, Lin et al. [2020] has achieved state-of-the-art results for nonconvex-concave problems.

4 Linear Convergence and Refined Dependence on $L_{xy}$ in General Cases

4.1 Alternating Best Response

Consider the extreme case where $L_{xy} = 0$. In this case, there is no interaction between $x$ and $y$, and $f(x, y)$ can be simply written as $h_1(x) - h_2(y)$, where $h_1$ and $h_2$ are strongly convex functions. Thus, in this case,
Algorithm 1 Alternating Best Response (ABR)

Require: $g(\cdot, \cdot)$, Initial point $z_0 = [x_0; y_0]$, precision $\epsilon$, parameters $m_x, m_y, L_x, L_y$

$k_x := L_x/m_x$, $k_y := L_y/m_y$, $T \leftarrow \lceil \log_2 (4\sqrt{k_x + k_y}/\epsilon) \rceil$

for $t = 0, \cdots, T$ do
    $x_{t+1} \leftarrow \text{AGD}(f(\cdot, y_t), x_t, 2\sqrt{k_x} \ln(24k_x))$
    $y_{t+1} \leftarrow \text{AGD}(-f(x_{t+1}, \cdot), y_t, 2\sqrt{k_y} \ln(24k_y))$
end for

the following trivial algorithm solves the problem

$$x^* \leftarrow \arg\min_x f(x, y_0), \quad y^* \leftarrow \arg\max_y f(x^*, y).$$

In other words, the equilibrium can be found by directly playing the best response to each other once.

Now, let us consider the case where $L_{xy}$ is nonzero but small. In this case, would the best response dynamics converge to the saddle point? Specifically, consider the following iterative procedure:

$$\begin{cases}
x_{t+1} \leftarrow \arg\min_x \{f(x, y_t)\} \\
y_{t+1} \leftarrow \arg\max_y \{f(x_{t+1}, y)\}
\end{cases} \quad (2)
$$

Let us define $y^*(x) := \arg\max_y f(x, y)$ and $x^*(y) := \arg\min_x f(x, y)$. Because $y^*(x)$ is $L_{xy}/m_y$-Lipschitz and $x^*(y)$ is $L_{xy}/m_x$-Lipschitz (see Fact 1 in Appendix A),

$$\|x_{t+1} - x^*\| = \|x^*(y_t) - x^*(y^*)\| \leq \frac{L_{xy}}{m_x} \|y_t - y^*\|$$

$$= \frac{L_{xy}}{m_x} \|y^*(x_t) - y^*(x^*)\| \leq \frac{L_{xy}^2}{m_x m_y} \|x_t - x^*\|.$$

Thus, when $L_{xy}^2 < m_x m_y$, (2) is indeed a contraction. In fact, we can replace the exact solution of the inner optimization problems with Nesterov’s Accelerated Gradient Descent (AGD) for constant number of steps, as described in Algorithm 1.

The following theorem holds for the Alternating Best Response algorithm. The proof of the theorem can be found in Appendix B.

Theorem 1. If $g \in F(m_x, m_y, L_x, L_{xy}, L_y)$ and $L_{xy} < \frac{1}{2}\sqrt{m_x m_y}$, Alternating Best Response returns $(x_T, y_T)$ such that

$$\|x_T - x^*\| + \|y_T - y^*\| \leq \epsilon (\|x_0 - x^*\| + \|y_0 - y^*\|),$$

and the number of gradient evaluations is bounded by (with $\kappa_x = L_x/m_x$, $\kappa_y = L_y/m_y$)

$$O \left( \sqrt{\kappa_x + \kappa_y} \cdot \ln (\kappa_x \kappa_y / \epsilon) \right).$$

Note that when $L_{xy}$ is small, the lower bound of Zhang et al. [2019] can be written as $\Omega \left( \sqrt{\kappa_x + \kappa_y} \ln(1/\epsilon) \right)$. Thus Alternating Best Response matches this lower bound up to logarithmic factors.

4.2 Accelerated Proximal Point for Minimax Optimization

In the previous subsection, we showed that Alternating Best Response matches the lower bound when the interaction term $L_{xy}$ is sufficiently small. However, in order to apply the algorithm to functions with $L_{xy} > \frac{1}{2}\sqrt{m_x m_y}$, we need another algorithmic component, namely the accelerated proximal point algorithm [Güler, 1992, Lin et al., 2020].

For a minimax optimization problem $\min_x \max_y f(x, y)$, define $\phi(x) := \max_y f(x, y)$. Suppose that we run the accelerated proximal point algorithm on $\phi(x)$ with proximal parameter $\beta$: then the number of iterations can be easily bounded, while in each iteration one needs to solve a proximal problem.
min_x \{ \phi(x) + \beta \|x - \hat{x}_t\|^2 \}. The key observation is that, this is equivalent to solving a minimax optimization problem min_x max_y \{ f(x, y) + \beta \|x - \hat{x}_t\|^2 \}. Thus, via accelerated proximal point, we are able to reduce solving min_x max_y f(x, y) to solving min_x max_y \{ f(x, y) + \beta \|x - \hat{x}_t\|^2 \}. Intuitively, the regularizer on x makes the new problem easier to solve.

This is exactly the idea behind Algorithm 2 (the idea was also used in Lin et al. [2020]). In the algorithm, M is a positive constant characterizing the precision of solving the subproblem, where we require 0 < M \leq \min \left( \frac{L_{xy}}{m_x}, \frac{L_y}{m_y} \right). If M \to \infty, the algorithm becomes an instance of accelerated proximal point on \phi(x) = max_y f(x, y).

### Algorithm 2 Accelerated Proximal Point Algorithm for Minimax Optimization

**Require:** Initial point z_0 = [x_0, y_0], proximal parameter \beta, strongly-convex modulus \mu_x

\begin{align*}
\hat{x}_0 &\leftarrow x_0, \ \kappa \leftarrow \beta / m_x, \ \theta \leftarrow \frac{2\sqrt{\kappa} - 1}{2\sqrt{\kappa + 1}}, \ \tau \leftarrow \frac{1}{2\sqrt{\kappa + \kappa}}.
\end{align*}

for t = 1, \ldots, T do

Suppose (\hat{x}, \hat{y}) = min_x max_y f(x, y) + \beta \|x - \hat{x}_{t-1}\|^2. Find (x_t, y_t) such that

\begin{align*}
\|x_t - \hat{x}\| + \|y_t - \hat{y}\| &\leq \frac{1}{M} \left( \|x_{t-1} - \hat{x}\| + \|y_{t-1} - \hat{y}\| \right) \\
\hat{x}_t &\leftarrow x_t + \theta(x_t - x_{t-1}) + \tau(x_t - \hat{x}_{t-1})
\end{align*}

end for

The following theorem can be shown for Algorithm 2. The proof can be found in Appendix C, and is based on the proof of Theorem 4.1 in [Lin et al., 2020].

**Theorem 2.** The number of iterations needed by Algorithm 2 to produce (x_T, y_T) such that

\[ \|x_T - x^*\| + \|y_T - y^*\| \leq \epsilon (\|x_0 - x^*\| + \|y_0 - y^*\|) \]

is at most (\kappa = \beta / m_x)

\[ T = 8 \sqrt{\kappa} \cdot \ln \left( \frac{28 \kappa^2 L_{xy}}{m_y} \sqrt{\frac{L_{xy}^2}{m_x m_y}} \cdot \frac{1}{\epsilon} \right). \]  

### 4.3 Proximal Alternating Best Response

With the two algorithmic components, namely Alternating Best Response and Accelerated Proximal Point in place, we can now combine them and design an efficient algorithms for general strongly convex-strongly concave functions. The high-level idea is to exploit the accelerated proximal point algorithm twice to reduce a general problem into one solvable by Alternating Best Response.

To start with, let us consider a strongly-convex-strongly-concave function f(x, y), and apply Algorithm 2 for f with proximal parameter \beta = L_{xy}. By Theorem 2, the algorithm can converge in \tilde{O} \left( \sqrt{\frac{L_{xy}}{m_x}} \right) iterations, while in each iteration we need to solve a regularized minimax problem

\[ \min_x \max_y \{ f(x, y) + \beta \|x - \hat{x}_{t-1}\|^2 \}. \]

This is equivalent to \min_y \max_x \{ -f(x, y) - \beta \|x - \hat{x}_{t-1}\|^2 \}, so we can apply Algorithm 2 once more to this problem with parameter \beta = L_{xy}. This procedure would require \tilde{O} \left( \sqrt{\frac{L_{xy}}{m_y}} \right) iterations, and in each iteration, one need to solve a minimax problem of the form

\begin{align*}
\min_x \max_y \{ -f(x, y) - \beta \|x - \hat{x}_{t-1}\|^2 + \beta \|y - \hat{y}_{t-1}\|^2 \} \\
= \min_x \max_y \{ f(x, y) + \beta \|x - \hat{x}_{t-1}\|^2 - \beta \|y - \hat{y}_{t-1}\|^2 \}.
\end{align*}

Hence, we reduced the original problem to a problem that is 2\beta-strongly convex with respect to x and 2\beta-strongly concave with respect to y. Now the interaction between x and y is (relatively) much weaker...
and one can easily see that $L_{xy} \leq \frac{1}{2} \sqrt{2\beta_3 \cdot 2\beta}$. Consequently the final problem can be solved in $\hat{O} \left( \frac{Lz}{L_{xy}} \right)$ gradient evaluations using the Alternating Best Response algorithm. We first consider the case where $L_{xy} > \max\{m_x, m_y\}$. The total gradient complexity would thus be

$$\hat{O} \left( \sqrt{\frac{L_{xy}}{m_x}} \right) \cdot \hat{O} \left( \sqrt{\frac{L_{xy}}{m_y}} \right) \cdot \hat{O} \left( \sqrt{\frac{L}{L_{xy}}} \right) = \hat{O} \left( \sqrt{\frac{L \cdot L_{xy}}{m_x m_y}} \right).$$

In order to deal with the case where $L_{xy} < \max\{m_x, m_y\}$, we shall choose $\beta_1 = \max\{L_{xy}, m_x\}$ for the first level of proximal point, and $\beta_2 = \max\{L_{xy}, m_y\}$ for the second level of proximal point. In this case, the total gradient complexity bound can be shown to be

$$\hat{O} \left( \sqrt{\frac{\beta_1}{m_x}} \right) \cdot \hat{O} \left( \sqrt{\frac{\beta_2}{m_y}} \right) \cdot \hat{O} \left( \sqrt{\frac{L + L_{xy}}{\beta_1 \beta_2}} \right) = \hat{O} \left( \sqrt{\frac{L + L_{xy}}{m_x m_y} + \frac{L_y}{m_y}} \right).$$

A formal description of the algorithm is provided in Algorithm 4, and a formal statement of the complexity upper bound is provided in Theorem 3. The proof can be found in Appendix D.

**Theorem 3.** Assume that $f \in F(m_x, m_y, L_x, L_{xy}, L_y)$. In Algorithm 4, the gradient complexity to produce $(x_T, y_T)$ such that $\|z_T - z^*\| \leq \epsilon$ is

$$O \left( \frac{L_x + L \cdot L_{xy} + L_y}{L_{xy}} \cdot \frac{1}{m_x m_y} \cdot \frac{\ln^3 \left( \frac{L^2}{m_x m_y} \right)}{\ln \left( \frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon} \right)} \right).$$

**Algorithm 3** APPA-ABR

Require: $g(\cdot, \cdot)$, Initial point $z_0 = [x_0; y_0]$, precision parameter $M_1$

1. $\beta_2 \leftarrow \max\{m_y, L_{xy}\}$, $M_2 \leftarrow \frac{36L^2}{m_x m_y}$
2. $y_0 \leftarrow y_0$, $\kappa \leftarrow \frac{\beta_2}{m_y}$, $\theta \leftarrow \frac{2\sqrt{\kappa - 1}}{2\sqrt{\kappa + 1}}$, $\tau \leftarrow \frac{1}{2\sqrt{\kappa + 4}}$, $t \leftarrow 0$
3. repeat
4. $t \leftarrow t + 1$
5. $(x_t, y_t) \leftarrow$ ABR$(g(x, y) - \beta_2\|y - y_t\|^2, [x_{t-1}; y_{t-1}], 1/M_2, 2\beta_1, 2\beta_2, 3L, 3L)$
6. $\tilde{y}_t \leftarrow y_t + \theta(y_t - y_{t-1}) + \tau(y_{t-1} - y_t)$
7. until $\|\nabla g(x_t, y_t)\| \leq \frac{\min\{m_x, m_y\}}{9L M_1} \|\nabla g(x_0, y_0)\|$

**Algorithm 4** Proximal Best Response

Require: Initial point $z_0 = [x_0; y_0]$

1. $\beta_1 \leftarrow \max\{m_x, L_{xy}\}$, $M_1 \leftarrow \frac{80L^3}{m_x m_y}$
2. $x_0 \leftarrow x_0$, $\kappa \leftarrow \frac{\beta_1}{m_x}$, $\theta \leftarrow \frac{2\sqrt{\kappa - 1}}{2\sqrt{\kappa + 1}}$, $\tau \leftarrow \frac{1}{2\sqrt{\kappa + 4}}$
3. for $t = 1, \cdots, T$ do
4. $(x_t, y_t) \leftarrow$ APPA-ABR($f(x, y) + \beta_1\|x - x_{t-1}\|^2, [x_{t-1}; y_{t-1}], M_1$)
5. $x_t \leftarrow x_t + \theta(x_t - x_{t-1}) + \tau(x_{t-1} - y_{t-1})$
6. end for

### 4.4 Implications of Theorem 3

Theorem 3 improves over the results of Lin et al. in two ways. First, Lin et al.’s upper bound has a $\ln^3(1/\epsilon)$ factor, while our algorithm enjoys linear convergence. Second, our result has a better dependence on $L_{xy}$.

To see this, note that when $L_{xy} \ll L$, $\frac{L_x}{m_x} + \frac{L \cdot L_{xy}}{m_x m_y} + \frac{L_y}{m_y} \ll \frac{L_x}{m_x} + \frac{L^2}{m_x m_y} + \frac{L_y}{m_y} \leq \frac{3L^2}{m_x m_y}$. This is also illustrated by Fig. 1, where Proximal Best Response (the red line) significantly outperforms Lin et al.’s result (the blue
We now focus on how to solve the linear system $Jz = b$. In particular, Proximal Best Response matches the lower bound when $L_{xy} > L_x$ or when $L_{xy} < \max\{m_x, m_y\}$; in between, it is able to gracefully interpolate the two cases.

As shown by Lin et al. [2020], convex-concave problems and strongly convex-concave problems can be reduced to strongly convex-strongly concave problems. Hence, Theorem 3 naturally implies improved algorithms for convex-concave and strongly convex-concave problems.

**Corollary 1.** If $f(x, y)$ is $(L_x, L_{xy}, L_y)$-smooth and $m_x$-strongly convex w.r.t. $x$, via reduction to Theorem 3, the gradient complexity of finding an $\epsilon$-saddle point is $\tilde{O}\left(\frac{m_x L_{xy} + L - L_{xy}}{m_x \epsilon}\right)$.

**Corollary 2.** If $f(x, y)$ is $(L_x, L_{xy}, L_y)$-smooth and convex-concave, via reduction to Theorem 3, the gradient complexity to produce an $\epsilon$-saddle point is $\tilde{O}\left(\sqrt{\frac{L_x + L_y}{\epsilon}} + \sqrt{L - L_{xy}}\right)$.

The precise statement as well as the proofs can be found in Appendix F. We remark that for the reduction is for constrained minimax optimization, and Theorem 3 holds for constrained problems after simple modifications to the algorithm.

# 5 Near Optimal Dependence on $L_{xy}$ in Quadratic Cases

We can see that proximal best response has near optimal dependence on condition numbers when $L_{xy} > L_x$ or when $L_{xy} < \max\{m_x, m_y\}$. However, when $L_{xy}$ falls in between, there is still a significant gap between the upper bound and the lower bound. In this section, we try to close this gap for quadratic functions, i.e. we assume that

$$f(x, y) = \frac{1}{2}x^TAx + x^TB^Ty - \frac{1}{2}y^TCy + u^Tx + v^Ty.$$ \hspace{1cm} (4)

The reason to consider quadratic functions is threefold. First, the lower bound instance by Zhang et al. [2019] is a quadratic function; thus, this lower bound applies to quadratic functions as well, so it would be interesting to match the lower bound for quadratic functions first. Second, quadratic functions are considerably easier to analyze. Third, finding the saddle point of quadratic functions is an important problem on its own, and has many applications (see Benzi et al. [2005] and references therein).

For quadratic functions of the form (4), our assumption that $f \in F(m_x, m_y, L_x, L_{xy}, L_y)$ now becomes assumptions on the singular values of matrices: $m_x I \ll A \ll L_x I$, $m_y I \ll C \ll L_y I$, $\|B\|_2 \leq L_{xy}$. In this case, the unique saddle point is given by the solution to a linear system

$$[x^*; y^*] = J^{-1}b = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}^{-1} \begin{bmatrix} -u \\ v \end{bmatrix}.$$ 

To see this, note that

$$J^{-1}b = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}^{-1} \begin{bmatrix} -u \\ v \end{bmatrix} = -\begin{bmatrix} A & B \\ B^T & -C \end{bmatrix}^{-1} \begin{bmatrix} u \\ v \end{bmatrix}.$$

Thus, $[x^*; y^*]$ is the unique stationary point of $f(x, y)$. The question then becomes: how can we solve such a linear system $J^{-1}b$ efficiently using first-order methods?

Throughout this section we assume that $L_x = L_y$ and $m_x < m_y$, which are without loss of generality, and that $m_y < L_{xy}$, as otherwise proximal best response is already near-optimal.

## 5.1 Hermitian-Skew-Hermitian-Split

We now focus on how to solve the linear system $Jz = b$, where $J := \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}$ is positive definite but not symmetric. We utilize the Hermitian-Skew-Hermitian Split (HSS) algorithm [Bai et al., 2003], which is designed to solve positive definite asymmetric systems. Define

$$G := \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix}, \quad S := \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}, \quad P := \alpha I + \beta A.$$
where $\alpha$ and $\beta$ are constants to be determined. Let $z_t := [x_t; y_t]$. Then HSS runs as
\[
\begin{align*}
(\eta P + G)z_{t+1/2} &= (\eta P - S)z_t + b, \\
(\eta P + S)z_{t+1} &= (\eta P - G)z_{t+1/2} + b.
\end{align*}
\]
\tag{5}

Here $\eta > 0$ is another constant. In this procedure, it can be shown that
\[z_{t+1} - z^* = (\eta P + S)^{-1} (\eta P - G) (\eta P + G)^{-1} (\eta P - S) (z_t - z^*).\]

The key observation of HSS is that the equation above is a contraction.

**Lemma 2** ([Bai et al., 2003]). Define $M(\eta) := (\eta P + S)^{-1} (\eta P - G) (\eta P + G)^{-1} (\eta P - S)$. Then\(^3\)
\[\rho(M(\eta)) \leq \|M(\eta)\|_2 \leq \max_{\lambda_i \in sp(P^{-1}G)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right| < 1.\]

Lemma 2 provides an upper bound on the iteration complexity of HSS, as in the original analysis of HSS [Bai et al., 2003]. However, it does not consider the computational cost per iteration. In particular, the matrix $\eta P + S$ is also asymmetric, and in fact corresponds to another quadratic minimax optimization problem. The original HSS paper did not consider how to solve this subproblem for general $P$. Our idea is to solve the subproblem recursively, as in the next subsection.

## 5.2 Recursive HSS

In this subsection, we describe our algorithm Recursive Hermitian-skew-Hermitian Split, or RHSS($k$), which uses HSS in $k - 1$ levels of recursion. Specifically, RHSS($k$) calls HSS with parameters $\alpha = m_x/m_y$, $\beta = L_{xy}^{-\frac{1}{2}} - \frac{x^2}{\|x\|^2}$, $\eta = L_{xy}^{\frac{1}{2}} - \frac{1}{x}$. In each iteration, it solves two linear systems. The first one, which is associated with $\eta P + G$, can be solved with Conjugate Gradient [Hestenes et al., 1952] as $\eta P + G$ is symmetric positive definite. The second one is associated with
\[\eta P + S = \begin{bmatrix} \eta (\alpha I + \beta A) & B \\ -B^T & \eta (I + \beta C) \end{bmatrix},\]

which is a quadratic minimax optimization problem. RHSS($k$) then makes a recursive call RHSS($k - 1$) to solve this subproblem. When $k = 1$, we simply run the Proximal Best Response algorithm (Algorithm 4).

A detailed description of RHSS($k$) for $k \geq 2$ is given in Algorithm 5.

Our main result for RHSS($k$) is the following theorem, whose proof is deferred to Appendix G. Note that for an algorithm on quadratic functions, the number of matrix-vector products is the same as the gradient complexity.

**Theorem 4.** There exists constants $C_1, C_2$, such that the number of matrix-vector products needed to find $(x_T, y_T)$ such that $\|z_T - z^*\| \leq \epsilon$ is at most
\[
\sqrt{\frac{L_{xy}^2}{m_x m_y} + \left(\frac{L_x}{m_x} + \frac{L_y}{m_y}\right) \left(1 + \frac{L_{xy}}{\max\{m_x, m_y\}}\right)^{\frac{1}{2}}} \cdot C_1 \ln \left(\frac{C_2 L_x^2}{m_x m_y} \right)^{k+3} \sqrt{\frac{1}{\epsilon}}. \tag{6}
\]

If $k$ is chosen as a fixed constant, the comparison of (6) and the lower bound [Zhang et al., 2019] is illustrated in Fig. 1. One can see that as $k$ increases, the upper bound of RHSS($k$) gradually fits the lower bound. One may also try to choose the optimal $k$. In particular, we can show the following corollary.

**Corollary 3.** When $k = \Theta \left(\frac{\ln \left(\frac{L_{xy}^2}{m_x m_y}\right)}{\ln \left(\frac{L_x^2}{m_x m_y}\right)}\right)$, the number of matrix vector products that RHSS($k$) need to find $z_T$ such that $\|z_T - z^*\| \leq \epsilon$ is
\[
\sqrt{\frac{L_{xy}^2}{m_x m_y} + \left(\frac{L_x}{m_x} + \frac{L_y}{m_y}\right) \ln \left(\frac{L_x^2}{m_x m_y} \right)^{\frac{1}{2}}} \cdot \left(\frac{L_x^2}{m_x m_y}\right)^{o(1)}. \tag{6}
\]

In other words, for the quadratic saddle point problem, RHSS($k$) with the optimal choice of $k$ matches the lower bound up to a sub-polynomial factor.

\(^3\)Here $\rho(\cdot)$ stands for the spectral radius of a matrix, and $sp(\cdot)$ stands for its spectrum.
Algorithm 5 RHSS($k$) (Recursive Hermitian-skew-Hermitian Split)

**Require:** Initial point $[x_0; y_0]$, precision $\epsilon$, parameters $m_x$, $m_y$, $L_{xy}$

$t \leftarrow 0$, $M_1 \leftarrow \frac{192L_{xy}^2}{m_y^2}$, $M_2 \leftarrow \frac{16L_{xy}^2}{m_y}$, $\alpha \leftarrow \frac{m_x}{m_y}$, $\beta \leftarrow \frac{L_{xy}^2}{m_y}$, $\eta \leftarrow \frac{L_{xy}^2}{m_y}$, $\bar{\epsilon} \leftarrow \frac{m_x\epsilon}{L_{xy}^2 + m_y}$

repeat

\[
\begin{bmatrix}
  r_1 \\
  r_2
\end{bmatrix} \leftarrow \begin{bmatrix}
  \eta (\alpha I + \beta A) & -B \\
  B^T & \eta (I + \beta C)
\end{bmatrix} \begin{bmatrix}
  x_t \\
  y_t
\end{bmatrix} + \begin{bmatrix}
  -u \\
  v
\end{bmatrix}.
\]

Call conjugate gradient to compute

\[
\begin{bmatrix}
  x_{t+1/2} \\
  y_{t+1/2}
\end{bmatrix} \leftarrow \text{CG} \left( \begin{bmatrix}
  \eta (\alpha I + \beta A) + A \\
  \eta (I + \beta C) + C
\end{bmatrix}, \begin{bmatrix}
  r_1 \\
  r_2 \\
  x_t \\
  y_t \\
  1/M_1
\end{bmatrix} \right).
\]

Compute

\[
\begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix} \leftarrow \begin{bmatrix}
  \eta I + \eta \beta A - A \\
  0
\end{bmatrix} \begin{bmatrix}
  x_{t+1/2} \\
  y_{t+1/2}
\end{bmatrix} + \begin{bmatrix}
  -u \\
  v
\end{bmatrix}.
\]

Call RHSS($k-1$) with initial point $[x_t; y_t]$ and precision $1/M_2$ to solve

\[
\begin{bmatrix}
  x_{t+1} \\
  y_{t+1}
\end{bmatrix} \leftarrow \begin{bmatrix}
  \eta (\alpha I + \beta A) & B \\
  -B^T & \eta (I + \beta C)
\end{bmatrix}^{-1} \begin{bmatrix}
  w_1 \\
  w_2
\end{bmatrix}.
\]

$t \leftarrow t + 1$

until $\|Jz_t - b\| \leq \bar{\epsilon}\|Jz_0 - b\|$

Algorithm 6 The Conjugate Gradient Algorithm: CG$(A, b, x_0, \epsilon)$ [Allaire and Kaber, 2008]

$r_0 \leftarrow b - Ax_0$, $p_0 \leftarrow r_0$, $k \leftarrow 0$

repeat

$\alpha_k \leftarrow \frac{r_k^T r_k}{p_k^T A p_k}$

$x_{k+1} \leftarrow x_k + \alpha_k p_k$

$r_{k+1} \leftarrow r_k - \alpha_k A p_k$

$\beta_k \leftarrow \frac{r_{k+1}^T r_{k+1}}{r_k^T r_k}$

$p_{k+1} \leftarrow r_{k+1} + \beta_k p_k$

$k \leftarrow k + 1$

until $\|r_k\| \leq \epsilon\|b - Ax_0\|$

Return $x$

6 Conclusion

In this work, we studied convex-concave minimax optimization problems. For general strongly convex-strongly concave problems, our Proximal Best Response algorithm achieves linear convergence and better dependence on $L_{xy}$, the interaction parameter. Via known reductions [Lin et al., 2020], this result implies better upper bounds for strongly convex-concave functions. For quadratic functions, our algorithm RHSS($k$) is able to match the lower bound up to a sub-polynomial factor.

In future research, one interesting direction is to extend RHSS($k$) to general strongly convex-strongly concave functions. Another important direction would be to shave the remaining sub-polynomial factor from the upper bound for quadratic functions.

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A Some Useful Properties

In this section, we review some useful properties of functions in $\mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y)$. Some of the facts are known (e.g., Lin et al. [2020] and Zhang et al. [2019]) and we provide the proofs for completeness.

**Fact 1.** Suppose $f \in \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y)$. Let us define $y^*(x) := \arg \max_y f(x, y)$, $x^*(y) := \arg \min_x f(x, y)$, $\phi(x) := \max_y f(x, y)$ and $\psi(y) := \min_x f(x, y)$. Then, we have that

1. $y^*$ is $L_{xy}/m_y$-Lipschitz, $x^*$ is $L_{xy}/m_x$-Lipschitz;
2. $\phi(x)$ is $m_x$-strongly convex and $L_x + L_{xy}^2/m_y$-smooth; $\psi(y)$ is $m_y$-strongly concave and $L_y + L_{xy}^2/m_x$-smooth.

**Proof.**

1. Consider arbitrary $x$ and $x'$. By definition, $\nabla_y f(x, y^*(x)) = \nabla_y f(x', y^*(x')) = 0$. By the definition of $(L_x, L_{xy}, L_y)$-smoothness, $\|\nabla_y f(x', y^*(x))\| \leq L_{xy} \|x - x'\|$. Thus

$$m_y \|y^*(x) - y^*(x')\| \leq \|\nabla_y f(x', y^*(x))\| \leq L_{xy} \|x - x'\|.$$  
This proves that $y^*(\cdot)$ is $L_{xy}/m_y$-Lipschitz. Similarly $x^*(\cdot)$ is $L_{xy}/m_x$-Lipschitz.

2. By Danskin’s Theorem, $\nabla \phi(x) = \nabla_x f(x, y^*(x))$. Thus, $\forall x, x'$

$$\|\nabla \phi(x) - \nabla \phi(x')\| = \|\nabla_x f(x, y^*(x)) - \nabla_x f(x', y^*(x'))\|$$

$$\leq \|\nabla_x f(x, y^*(x)) - \nabla_x f(x, y^*(x'))\| + \|\nabla_x f(x, y^*(x')) - \nabla_x f(x', y^*(x'))\|$$

$$\leq L_{xy} \|y^*(x) - y^*(x')\| + L_x \|x - x'\|$$

$$\leq \left( L_x + \frac{L_{xy}^2}{m_y} \right) \|x - x'\|.$$  
On the other hand, $\forall x, x'$,

$$\phi(x') - \phi(x) - (x' - x)^T \nabla \phi(x) = f(x', y^*(x')) - f(x, y^*(x)) - (x' - x)^T \nabla_x f(x, y^*(x))$$

$$\geq f(x', y^*(x)) - f(x, y^*(x)) - (x' - x)^T \nabla_x f(x, y^*(x))$$

$$\geq \frac{m_x}{2} \|x' - x\|^2.$$  
Thus $\phi(x)$ is $m_x$-strongly convex and $\left( L_x + \frac{L_{xy}^2}{m_y} \right)$-smooth. By symmetric arguments, one can show that $\psi(y)$ is $m_y$-strongly concave and $\left( L_y + \frac{L_{xy}^2}{m_x} \right)$-smooth.

**Fact 2.** Let $z := [x; y]$ and $z^* := [x^*; y^*]$. Then

$$\frac{1}{\sqrt{2}} (\|x - x^*\| + \|y - y^*\|) \leq \|z - z^*\| \leq \|x - x^*\| + \|y - y^*\|.$$  

**Proof.** This can be easily proven using the AM-GM inequality.

**Fact 3.** Let $z := [x; y] \in \mathbb{R}^{m+n}$, $z^* := [x^*; y^*]$. Then

$$\min\{m_x, m_y\} \|z - z^*\| \leq \|\nabla f(x, y)\| \leq 2L \|z - z^*\|.$$  

**Proof.** By properties of strong convexity [Nesterov, 2013], $\forall x, y$

$$f(x, y^*(x)) - f(x, y) \leq \frac{1}{2m_y} \|\nabla_y f(x, y)\|^2.$$  

Similarly,

$$f(x, y) - f(x^*(y), y) \leq \frac{1}{2m_x} \|\nabla_x f(x, y)\|^2.$$
Thus,

$$\| \nabla f(x, y) \|^2 = \| \nabla_x f(x, y) \|^2 + \| \nabla_y f(x, y) \|^2 \geq 2 \min \{ m_x, m_y \} (\phi(x) - \psi(y)).$$

Here \( \phi(\cdot) = \max_y f(\cdot, y), \psi(\cdot) = \min_x f(x, \cdot). \) By Proposition 1, \( \phi \) is \( m_x \)-strongly concave while \( \psi \) is \( m_y \)-strongly concave. Hence

$$\phi(x) - \psi(y) \geq \frac{\min \{ m_x, m_y \}}{2} (\| x - x^* \|^2 + \| y - y^* \|^2) = \frac{\min \{ m_x, m_y \}}{2} \| z - z^* \|^2.$$

It follows that \( \| \nabla f(x, y) \| \geq \min \{ m_x, m_y \} \| z - z^* \|. \) On the other hand,

$$\| \nabla_x f(x, y) \| \leq L_{xy} \| y - y^* \| + L_x \| x - x^* \|,$$

$$\| \nabla_y f(x, y) \| \leq L_{xy} \| x - x^* \| + L_y \| y - y^* \|.$$

As a result \( \| \nabla f(x, y) \|^2 \leq L (\| x - x^* \| + \| y - y^* \|)^2 \leq 4L^2 \| z - z^* \|^2. \)

**Fact 4.** Let \( \hat{z} = [\hat{x}; \hat{y}] \). Then \( \| \hat{z} - z^* \| \leq \epsilon \) implies

$$\max_y f(\hat{x}, y) - \min_x f(x, \hat{y}) \leq \frac{L^2}{\min \{ m_x, m_y \}} \epsilon^2.$$

**Proof.** Define \( \phi(x) = \max_y f(x, y) \) and \( \psi(y) = \min_x f(x, y). \) Then

$$\max_y f(\hat{x}, y) - \min_x f(x, \hat{y}) = \phi(\hat{x}) - \psi(\hat{y}).$$

By Fact 1, \( \phi \) is \( (L_x + L_{xy}^2/m_x) \)-smooth while \( \psi \) is \( (L_y + L_{xy}^2/m_x) \)-smooth. Since \( \phi(x^*) = \psi(y^*), \nabla \phi(x^*) = 0, \nabla \psi(y^*) = 0, \)

$$\phi(\hat{x}) - \psi(\hat{y}) \leq \frac{1}{2} \left( L_x + \frac{L_{xy}^2}{m_x} \right) \| \hat{x} - x^* \|^2 + \frac{1}{2} \left( L_y + \frac{L_{xy}^2}{m_y} \right) \| \hat{y} - y^* \|^2$$

$$\leq \frac{1}{2} \left( L + \min \{ L_x, L_y \} \right) \left( \| \hat{x} - x^* \|^2 + \| \hat{y} - y^* \|^2 \right)$$

$$\leq \frac{L^2}{\min \{ m_x, m_y \}} \epsilon^2.$$

**B Proof of Theorem 1**

**Theorem 1.** If \( g \in F(m_x, m_y, L_x, L_{xy}, L_y) \) and \( L_{xy} < \frac{1}{2} \sqrt{m_x m_y} \), Alternating Best Response returns \((x_T, y_T)\) such that

$$\| x_T - x^* \| + \| y_T - y^* \| \leq \epsilon \left( \| x_0 - x^* \| + \| y_0 - y^* \| \right),$$

using \( (\kappa_x = L_x/m_x, \kappa_y = L_y/m_y) \)

$$O \left( \left( \sqrt{\kappa_x + \kappa_y} \right) \cdot \ln \left( \kappa_x \kappa_y \right) \ln \left( \frac{\kappa_x \kappa_y}{\epsilon} \right) \right).$$

**gradient evaluations.**

**Proof.** Define \( \hat{x}_{t+1} := \arg \min_x f(x, y_t) \). Let us define \( y^*(x) := \arg \max_y f(x, y), \ x^*(y) := \arg \min_x f(x, y) \) and \( \phi(x) := \max_y f(x, y) \). Also define \( \hat{x}_{t+1} := \arg \min_x f(x, y^*(x_t)) \) and \( x_{t+1} := \arg \min_x f(x, y_t). \)
Similarly, It follows that

\[ \|x^*(y_t) - x^*\| = \|x^*(y_t) - x^*(y^*)\| \leq \frac{L_{xy}}{m_x} \|y_t - y^*\|, \]

\[ \|y^*(x_{t+1}) - y^*\| = \|y^*(x_{t+1}) - y^*(x^*)\| \leq \frac{L_{xy}}{m_y} \|x_{t+1} - x^*\|. \]

By a standard analysis of accelerated gradient descent (Lemma 1), since \( \hat{x}_{t+1} = x^*(y_t) \) is the minimum of \( f(\cdot, y_t) \) and \( x_t \) is the initial point,

\[
\|x_{t+1} - \hat{x}_{t+1}\|^2 \leq (\kappa_x + 1) \|x_t - \hat{x}_{t+1}\|^2 \cdot \left(1 - \frac{1}{\sqrt{\kappa_x}}\right)^{2\sqrt{\kappa_x} \ln(24\kappa_x)}
\leq \|x_t - \hat{x}_{t+1}\|^2 \cdot (\kappa_x + 1) \cdot \exp \{-2 \ln(24\kappa_x)\}
\leq \frac{1}{256} \|x_t - \hat{x}_{t+1}\|^2.
\]

That is,

\[
\|x_{t+1} - x^*(y_t)\| \leq \frac{1}{16} \|x_t - x^*(y_t)\| \leq \frac{1}{16} (\|x_t - x^*\| + \|x^*(y_t) - x^*\|).
\]

Thus

\[
\|x_{t+1} - x^*\| \leq \|x_{t+1} - x^*(y_t)\| + \|x^*(y_t) - x^*\| \leq \frac{17}{16} \cdot \frac{L_{xy}}{m_x} \|y_t - y^*\| + \frac{1}{16} \|x_t - x^*\|. \tag{7}
\]

Similarly,

\[
\|y_{t+1} - y^*(x_{t+1})\| \leq \frac{1}{16} \|y_t - y^*(x_{t+1})\| \leq \frac{1}{16} (\|y_t - y^*\| + \|y^*(x_{t+1}) - y^*\|).
\]

Thus

\[
\|y_{t+1} - y^*\| \leq \|y_{t+1} - y^*(x_{t+1})\| + \|y^*(x_{t+1}) - y^*\|
\leq \frac{17}{16} \cdot \frac{L_{xy}}{m_y} \|x_{t+1} - x^*\| + \frac{1}{16} \|y_t - y^*\|
\leq \left(\frac{17^2}{16^2} \cdot \frac{L_{xy}^2}{m_x m_y} + \frac{1}{16}\right) \|y_t - y^*\| + \frac{17L_{xy}}{256m_y} \|x_t - x^*\|
\leq 0.35 \|y_t - y^*\| + \frac{17L_{xy}}{256m_y} \|x_t - x^*\|. \tag{8}
\]

Define \( C := 4\sqrt{m_y/m_x} \). By adding (7) and \( C \) times (8), one gets

\[
\|x_{t+1} - x^*\| + C\|y_{t+1} - y^*\| \leq \left(\frac{1}{16} + \frac{17L_{xy}}{64\sqrt{m_x m_y}}\right) \|x_t - x^*\| + \left(0.35C + \frac{17}{16} \cdot \frac{L_{xy}}{m_x}\right) \|y_t - y^*\|
\leq \frac{1}{2} \|x_t - x^*\| + \left(0.35 + \frac{17L_{xy}}{64\sqrt{m_x m_y}}\right) C \|y_t - y^*\|
\leq \frac{1}{2} (\|x_t - x^*\| + C\|y_t - y^*\|).
\]

It follows that

\[
\|x_T - x^*\| + C\|y_T - y^*\| \leq 2^{-T} (\|x_0 - x^*\| + C\|y_0 - y^*\|).
\]

If \( C \geq 1 \), then

\[
\|x_T - x^*\| + \|y_T - y^*\| \leq 4 \sqrt{m_y/m_x} \cdot 2^{-T} \cdot (\|x_0 - x^*\| + \|y_0 - y^*\|).
\]

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On the other hand, if \( C < 1 \), then
\[
\|x_T - x^*\| + \|y_T - y^*\| \leq \frac{2^{-T}}{C} (\|x_0 - x^*\| + \|y_0 - y^*\|) = \sqrt{\frac{m_x}{m_y}} \cdot 2^{-T-1} (\|x_0 - x^*\| + \|y_0 - y^*\|).
\]
Since \( \max\{m_x/m_y, m_y/m_x\} \leq L \max\{m_x, m_y\} \),
\[
\|x_T - x^*\| + \|y_T - y^*\| \leq 4 \sqrt{\frac{L}{\min\{m_x, m_y\}}} \cdot 2^{-T} (\|x_0 - x^*\| + \|y_0 - y^*\|).
\]
(9)

The theorem follows from this inequality.

\[\square\]

C Proof of Theorem 2

**Theorem 2.** Assume that \( M \geq 20\sqrt{2} + \frac{L}{m_x} + \frac{L^2 m_x}{m_y} \left(1 + \frac{L}{m_y}\right) \). The number of iterations needed by Algorithm 2 to produce \((x_T, y_T)\) such that
\[
\|x_T - x^*\| + \|y_T - y^*\| \leq \epsilon (\|x_0 - x^*\| + \|y_0 - y^*\|)
\]
is at most (\( \kappa = \beta/m_x \))
\[
\hat{T} = 8\sqrt{\kappa} \cdot \ln \left( \frac{28\kappa^2 L}{m_y} \frac{L^2}{m_x m_y} \cdot \frac{1}{\epsilon} \right).
\]
(10)

Before proving the theorem, we would first state the inexact accelerated proximal point algorithm [Lin et al., 2020], which is the basis of Algorithm 2.

**Algorithm 7** Inexact Accelerated Proximal Point Algorithm (Inexact APPA)

**Require:** Initial point \( x_0 \), proximal parameter \( \beta \), strongly convex module \( m \)
\[
x_0 \leftarrow x_0, \ \kappa \leftarrow \beta/m, \ \theta \leftarrow \frac{2^{-\kappa - 1}}{2\sqrt{\kappa} + 1}, \ \tau \leftarrow \frac{1}{2\sqrt{\kappa} + 1}
\]
for \( t = 1, \cdots, T \) do

- Find \( x_t \) such that \( g(x_t) + \beta\|x_t - \hat{x}_{t-1}\|^2 \leq \min_x \{ g(x) + \beta\|x - x_{t-1}\|^2 \} + \delta_t \)
- \( \hat{x}_t \leftarrow x_t + \theta(x_t - x_{t-1}) + \tau(x_t - x_{t-1}) \)

end for

The following lemma about the inexact APPA algorithm follows directly from the proof of Theorem 4.1 [Lin et al., 2020]. We state it without proving it.

**Lemma 3.** Suppose that \( \{x_t\}_{t \geq 0} \) is generated by running the inexact APPA algorithm on \( g(\cdot) \). There exists a sequence \( \{\Lambda_t\}_{t \geq 0} \) such that

1. \( \Lambda_t \geq g(x_t) \)
2. \( \Lambda_0 - g(x^*) \leq 2(g(x_0) - g(x^*)) \)
3. \( \Lambda_{t+1} - g(x^*) \leq \left(1 - \frac{1}{2\sqrt{\kappa}}\right) (\Lambda_t - g(x^*)) + 11\kappa \delta_{t+1} \)

Here \( \Lambda_t \) can be recursively defined as follows (see also Lin et al. [2020]),
\[
\Lambda_0 := g(x_0) + \frac{m\|x^* - x_0\|^2}{4},
\]
\[
\Lambda_{t+1} := \frac{1}{2\sqrt{\kappa}} \left( g(x_{t+1}) + 2\beta(x_t - x_{t+1})^T (x^* - x_{t+1}) + \frac{m\|x^* - x_{t+1}\|^2}{4} + 14\kappa^3/\delta_{t+1} \right) + \left(1 - \frac{1}{2\sqrt{\kappa}}\right) \Lambda_t.
\]
However, we do not need to make use of the explicit definition of \( \Lambda_t \).

Now we are ready to prove Theorem 2.
Proof. Define $\phi(x) := \max_y f(x,y)$ and $\hat{L} := L + L^2_{xy}/m_y$. Then $\phi(x)$ is $m_x$-strongly convex and $\hat{L}$-smooth. Observe that

$$x^*_t = \arg\min_x \left[ \phi(x) + \beta \|x - x_{t-1}\|^2 \right],$$

$$y^*_t = \arg\max_y [f(x^*_t, y)].$$

Thus Algorithm 2 is an instance of the inexact APPA algorithm on $\phi(x)$ with proximal parameter $\beta$ and strongly convex module $m_x$, and with

$$\delta_t = \phi(x_t) + \beta \|x_t - x_{t-1}\|^2 - \min_x \left\{ \phi(x) + \beta \|x - x_{t-1}\|^2 \right\} \leq \frac{\hat{L} + 2\beta}{2} \|x_t - x^*_t\|^2. \tag{11}$$

Here we used the fact that, for a $L$-smooth function $g(\cdot)$ whose minimum is $x^*$, $g(x) - g(x^*) \leq \frac{L}{2} \|x - x^*\|^2$.

Define $C_1 := \|x_0 - x^*\| + \|y_0 - y^*\|$ and $C_0 := 44\kappa\sqrt{\kappa}L + 2\beta C_1^2$. Let us state the following induction hypothesis

$$\Delta_t := A_t - \phi(x^*) \leq C_0 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^t, \tag{12}$$

$$\epsilon_t := \|x_t - x_t^*\| + \|y_t - y_t^*\| \leq C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^\frac{t}{2}. \tag{13}$$

It is easy to verify that with our choice of $C_0$ and $C_1$, both (12) and (13) hold for $t = 0$.

Now, assume that (12) and (13) hold for $t = 1, 2, \cdots, t$. Define $y^*(\cdot) := \arg\max_y f(\cdot, y)$. By Fact 1, $y^*(\cdot)$ is $(L/m_y)$-Lipschitz. Thus

$$\|y_t - y_{t+1}^*\| \leq \|y_t^* - y_{t+1}^*\| + \|y_t - y_t^*\|$$

$$\leq \|y^*(x_t^*) - y^*(x_{t+1}^*)\| + \epsilon_t$$

$$\leq \frac{L}{m_y} \cdot (\|x_t^* - x_t\| + \|x_t - x_{t+1}^*\|) + \epsilon_t$$

$$\leq \left( \frac{L}{m_y} + 1 \right) \epsilon_t + \frac{L}{m_y} \|x_t - x_{t+1}^*\|.$$

It follows that

$$\epsilon_{t+1} \leq \frac{1}{M} \left( \|x_t - x_{t+1}^*\| + \|y_t - y_{t+1}^*\| \right) \leq \frac{1 + \frac{L}{m_y}}{M} \cdot \left( \|x_t - x_{t+1}^*\| + \epsilon_t \right). \tag{14}$$

Note that by Lemma 3 and the induction hypothesis (12)

$$\phi(x_{t+1}^*) - \phi(x^*) \leq \left( 1 - \frac{1}{2\sqrt{\kappa}} \right) \Delta_t \leq C_0 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^t.$$

By the $m_x$-strong convexity of $\phi(\cdot)$ (Fact 1),

$$\|x_{t+1}^* - x^*\| \leq \sqrt{\frac{2}{m_x} (\phi(x_{t+1}^*) - \phi(x^*))} \leq \sqrt{\frac{2C_0}{m_x} \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^\frac{t}{2}}.$$

Meanwhile

$$\|x_t - x^*\| \leq \sqrt{\frac{2}{m_x} (\phi(x_t) - \phi(x^*))} \leq \sqrt{\frac{2C_0}{m_x} \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^\frac{t}{2}}.$$

Therefore

$$\|x_t - x_{t+1}^*\| \leq \|x_t - x^*\| + \|x_{t+1}^* - x^*\| \leq 2 \sqrt{\frac{2C_0}{m_x} \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^\frac{t}{2}}. \tag{15}$$
By (14), (13) and the fact that $M \geq 20\kappa \sqrt{2\kappa + \frac{L}{m_x}}(1 + L/m_y)$,

$$\epsilon_{t+1} \leq \frac{1 + \frac{L}{m_y}}{M} \left( 2\sqrt{\frac{2C_0}{m_x}} + C_1 \right) \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}}$$

$$\leq \frac{1 + \frac{L}{m_y}}{M} \cdot \left( 1 + 2\sqrt{\frac{44\kappa^2(\hat{L} + 2\beta)}{m_x}} \right) C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \quad (C_0 = 44\kappa^1.5 \frac{L + 2\beta}{2} C_1^2)$$

$$\leq \frac{1 + 2\sqrt{44\kappa \frac{L + 2\beta}{m_x}}}{20\kappa \sqrt{2\kappa + \frac{L}{m_x}}} \cdot C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \quad (2\sqrt{44} + 1 < 15)$$

$$\leq \frac{3}{4} C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \leq C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}}.$$  

Therefore (13) holds for $t + 1$. Meanwhile, by (11) and Lemma 3,

$$\Delta_{t+1} \leq \left( 1 - \frac{1}{2\sqrt{\kappa}} \right) \Delta_t + \frac{\hat{L} + 2\beta}{2} \cdot \frac{1}{4\sqrt{\kappa}} \cdot \epsilon_{t+1}$$

$$\leq \left( 1 - \frac{1}{2\sqrt{\kappa}} \right) C_0 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} + \frac{11\kappa \cdot \hat{L} + 2\beta}{2} \cdot \frac{1}{4\sqrt{\kappa}} \cdot C_1^2 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}}$$

$$= C_0 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} + \frac{11\kappa \cdot \hat{L} + 2\beta}{2} \cdot \frac{1}{4\sqrt{\kappa}} \cdot C_1^2 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}}.$$

where we used the fact that

$$11\kappa \cdot \frac{L + 2\beta}{2} \cdot C_1^2 = \frac{1}{4\sqrt{\kappa}} \cdot 44\kappa^1.5 \frac{L + 2\beta}{2} C_1^2 = \frac{C_0}{4\sqrt{\kappa}}.$$

Thus (12) also holds for $t + 1$. By induction on $t$, we can see that (12) and (13) both hold for all $t \geq 0$.

As a result,

$$\|x_T - x^*\| \leq \sqrt{\frac{2}{m_x} \left| \phi(x_T) - \phi(x^*) \right|} \leq \sqrt{\frac{2}{m_x} \cdot 44\kappa \sqrt{\kappa} \frac{L + 2\beta}{2} C_1^2 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}}}$$

$$\leq C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \sqrt{88\kappa \sqrt{\kappa} \cdot \left( \frac{L^2}{m_x m_y + \kappa} \right)}.$$

Meanwhile,

$$\|y_T - y^*\| \leq \|y_T - y^*(x_T)\| + \|y^* - y^*(x_T)\| \leq \epsilon_T + \frac{L_{xy}}{m_y} \|x_T - x^*\|.$$

Therefore

$$\|x_T - x^*\| + \|y_T - y^*\| \leq \epsilon_T + \left( \frac{L_{xy}}{m_y} + 1 \right) \|x_T - x^*\|$$

$$\leq C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} + \frac{2L}{m_y} \cdot C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \cdot \sqrt{88\kappa \sqrt{\kappa} \cdot \left( \frac{L^2}{m_x m_y} + \kappa \right)}$$

$$\leq C_1 \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \cdot \left[ 1 + \frac{27\kappa^2 L}{m_y} \cdot \sqrt{\frac{L^2}{m_x m_y}} \right]$$

$$\leq \frac{28\kappa^2 L}{m_y} \sqrt{\frac{L^2}{m_x m_y}} \cdot \left( 1 - \frac{1}{4\sqrt{\kappa}} \right)^{\frac{1}{2}} \cdot (\|x_0 - x^*\| + \|y_0 - y^*\|),$$

which proves the theorem.
D Proof of Theorem 3

Theorem 3. Assume that $f \in F(m_x, m_y, L_x, L_{xy}, L_y)$. In Algorithm 4, the gradient complexity to produce $(x_T, y_T)$ such that $\|x_T - z^*\| \leq \epsilon$ is

$$O \left( \sqrt{\frac{L_x}{m_x} + \frac{L \cdot L_{xy}}{m_x m_y}} \cdot \frac{L_y}{m_y} \cdot \ln^3 \left( \frac{L^2}{m_x m_y} \right) \cdot \ln \left( \frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon} \right) \right).$$

Proof. We start the proof by verifying $f(x, y) + \beta_1 \|x - \hat{x}\|^2 - \beta_2 \|y - \hat{y}\|^2$ can indeed be solved by calling ABR($\cdot$, $[x_0, y_0], 1/M_2, 2\beta_1, 2\beta_2, 3L, 3L$). Observe that $L_{xy} \leq \beta_1, \beta_2 \leq L$. Since $f(x, y) + \beta_1 \|x - \hat{x}\|^2 - \beta_2 \|y - \hat{y}\|^2$ is $2\beta_1$-strongly convex w.r.t. $x$ and $2\beta_2$-strongly concave w.r.t. $y$, we can see that $L \geq \frac{1}{2} \sqrt{2\beta_1 \cdot 2\beta_2} \geq L_{xy}$. We can also verify that $f(x, y) + \beta_1 \|x - \hat{x}\|^2 - \beta_2 \|y - \hat{y}\|^2$ is $3L$-smooth, which follows from the fact that $L + \max\{2\beta_1, 2\beta_2\} \leq 3L$.

Therefore, we can apply Theorem 1 and conclude that at line 5 of Algorithm 3

$$\|x_t - x_t^*\| + \|y_t - y_t^*\| \leq \frac{1}{M_2} \left( \|x_{t-1} - x_t^*\| + \|y_{t-1} - y_t^*\| \right),$$

where $(x_t^*, y_t^*) := \min_x \max_y \{g(x, y) - \beta_2 \|y - y_{t-1}\|^2\}$, and such $(x_t, y_t)$ is found in a gradient complexity of

$$O \left( \frac{L'}{\beta_1} + \frac{L^2}{\beta_1 \beta_2} \cdot \ln \left( \frac{L^2}{\beta_1 \beta_2} \cdot M_2 \right) \right) = O \left( \frac{L'}{\beta_1} + \frac{L^2}{\beta_2} \cdot \ln^2 \left( \frac{L^2}{m_x m_y} \right) \right).$$

Next, we verify that Algorithm 3 is an instance of Algorithm 2 on the function $-g(x, y)$. Notice that

$$\min_{x, y} \{ -g(x, y) + \beta \|y - \hat{y}\|^2 \} = -\min_{x, y} \{ g(x, y) - \beta \|y - \hat{y}\|^2 \}.$$ 

That is, $\min_{x, y} \{ g(x, y) - \|y - \hat{y}\|^2 \}$ has the same saddle point as $-g(x, y) + \beta \|y - \hat{y}\|^2$. Thus, we only need to verify that

$$M_2 \geq 20 \cdot \frac{\beta_2}{m_y} \left( 1 + \frac{L'}{m_x} + \frac{L^2}{m_x m_y} \right),$$

where $(m'_x, m'_y, L'_x, L_{xy}', L'_y)$ are parameters for $f(x, y) + \beta_1 \|x\|^2$, and $L' = \max\{L_{xy}, L'_x, L'_y\}$. Note that $m'_x \geq m_x + 2\beta_1$, $m'_y = m_y$, $L'_x = L'_y \leq L + 2\beta_1$, $L_{xy} \leq \beta_1, \beta_2 \leq L$. Thus

$$\text{RHS of (16)} \leq 20 \cdot \frac{\beta_2}{m_y} \sqrt{\frac{2\beta_2}{m_y} + \frac{L + 2\beta_1}{m_y} + \frac{L^2}{m_y (m_x + 2\beta_1)} \cdot \left( 1 + \frac{L + 2\beta_1}{m_x + 2\beta_1} \right)} \leq 20 \cdot \frac{L}{m_y} \sqrt{\frac{2L}{m_y} + \frac{3L}{m_y} + \frac{L_{xy}}{2m_y} \cdot \left( 1 + \frac{L}{m_x} \right)} \leq \frac{96L^{2.5}}{m_x m_y^{1.5}} = M_2.$$ 

Therefore, Algorithm 3 is indeed an instance of Inexact APPA Algorithm 7. Notice that by the stopping condition of Algorithm 3,

$$\left( \|x_t - x^*\| + \|y_t - y^*\| \right) \leq \frac{\sqrt{2}}{\min\{m_x, m_y\}} \|\nabla g(x_t, y_t)\| \quad \text{(Fact 3 and 2)}$$

$$\leq \frac{\sqrt{2}}{\min\{m_x, m_y\}} \cdot \frac{\min\{m_x, m_y\}}{9LM_1} \|\nabla g(x_0, y_0)\| \leq \frac{\sqrt{2}}{\min\{m_x, m_y\}} \cdot \frac{\min\{m_x, m_y\}}{9LM_1} \cdot 6L \left( \|x_0 - x^*\| + \|y_0 - y^*\| \right) \leq \frac{1}{M_1} \left( \|x_0 - x^*\| + \|y_0 - y^*\| \right).$$

\footnote{Here $g(x, y)$ refers to the argument passed to Algorithm 3, which in our case has the form $f(x, y) + \beta \|x - \hat{x}_{t-1}\|^2.$}
Thus when Algorithm 3 returns,
\[ \|x_t - x^*\| + \|y_t - y^*\| \leq \frac{1}{M_1} (\|x_0 - x^*\| + \|y_0 - y^*\|) \]  
(17)

On the other hand, suppose that
\[ \|x_t - x^*\| + \|y_t - y^*\| \leq \frac{1}{M_1} \min\{m_x, m_y\} \cdot (\|x_0 - x^*\| + \|y_0 - y^*\|), \]
we can show that
\[ \|\nabla g(x_t, y_t)\| \leq 6L (\|x_t - x^*\| + \|y_t - y^*\|) \]
\[ \leq \frac{\min\{m_x, m_y\}}{2M_1} (\|x_0 - x^*\| + \|y_0 - y^*\|) \]
\[ \leq \frac{1}{M_1} \|\nabla g(x_0, y_0)\|. \]

Thus in this case Algorithm 3 must return. By Theorem 2, we can see that Algorithm 3 always returns in at most
\[ O\left(\sqrt{\frac{\beta_2}{m_y}} \cdot \ln\left(\frac{L^2}{m_x m_y} \cdot \frac{12L}{\min\{m_x, m_y\} M_1}\right)\right) = O\left(\sqrt{\frac{\beta_1}{m_y}} \cdot \ln\left(\frac{L^2}{m_x m_y}\right)\right) \]  
(18)

iterations.

Finally, we verify that Algorithm 4 is an instance of Algorithm 2 on \( f(x, y) \) with parameter \( \beta_1 \). Note that by (17), we only need to verify that
\[ M_1 = \frac{80L^3}{m_x^5 m_y^5} \geq 20 \cdot \frac{\beta_1}{m_x} \sqrt{\frac{2\beta_1}{m_x} + \frac{L}{m_x} + \frac{L^2}{m_x m_y} (1 + \frac{L}{m_y})}. \]

Observe that
\[ 20 \cdot \frac{\beta_1}{m_x} \sqrt{\frac{2\beta_1}{m_x} + \frac{L}{m_x} + \frac{L^2}{m_x m_y}} (1 + \frac{L}{m_y}) \leq 20 \cdot \frac{L}{m_x} \sqrt{\frac{2L}{m_x} + \frac{L}{m_x} + \frac{L^2}{m_x m_y}} \]
\[ \leq 20 \cdot \frac{L}{m_x} \cdot \sqrt{\frac{4L^2}{m_x m_y} \cdot \frac{2L}{m_y} = M_1}. \]

Therefore Algorithm 4 is indeed an instance of Algorithm 2 on \( f(x, y) \). As a result, by Theorem 2, the number of iterations needed such that \( \|z_T - z^*\| \leq \epsilon \) is
\[ O\left(\sqrt{\frac{\beta_1}{m_x}} \cdot \ln\left(\frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon}\right)\right). \]
(19)

We now compute the total gradient complexity. Recall that \( \beta_1 = \max\{m_x, L_{xy}\} \), while \( \beta_2 = \max\{m_y, L_{xy}\} \). By (19), (18) and (D), the total gradient complexity of Algorithm 4 to reach \( \|z_T - z^*\| \leq \epsilon \) is
\[ O\left(\sqrt{\frac{\beta_1}{m_x}} \cdot \ln\left(\frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon}\right)\right) \cdot \sqrt{\frac{\beta_2}{m_y}} \cdot \ln\left(\frac{L^2}{m_x m_y}\right) \cdot \sqrt{\frac{L}{\beta_1} + \frac{L}{\beta_2} \cdot \ln^2\left(\frac{L^2}{m_x m_y}\right)} \]
\[ = O\left(\frac{L (\beta_1 + \beta_2)}{m_x m_y} \cdot \ln\left(\frac{L^2}{m_x m_y}\right) \ln\left(\frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon}\right)\right). \]

If \( L_{xy} \geq \max\{m_x, m_y\} \), then \( \beta_1 = \beta_2 = L_{xy} \), so
\[ \sqrt{\frac{L (\beta_1 + \beta_2)}{m_x m_y}} = \sqrt{\frac{2L L_{xy}}{m_x m_y}} \leq 2 \sqrt{\frac{L_x}{m_x} + \frac{L_{xy}}{m_x m_y} + \frac{L_y}{m_y}}. \]
Now consider the case where $L_{xy} < \max\{m_x, m_y\}$. Without loss of generality, assume that $m_x \leq m_y$. Suppose that $L_{xy} < m_y$, then $L = L_x, \beta_2 = m_y$, while $\beta_1 \leq m_y$. Hence

$$\sqrt{\frac{L(\beta_1 + \beta_2)}{m_x m_y}} \leq \sqrt{\frac{L_x \cdot 2m_y}{m_x m_y}} = \sqrt{\frac{2L_x}{m_x}} \leq 2\sqrt{\frac{L_x + L_{xy} + L_y}{m_x m_y}}.$$ 

Thus, in either case, $\sqrt{\frac{L(\beta_1 + \beta_2)}{m_x m_y}} = O\left(\sqrt{\frac{L_x + L_{xy} + L_y}{m_x m_y}}\right)$. We conclude that the total gradient complexity of Algorithm 4 to find a point $z_T = [x_T; y_T]$ such that $\|z_T - z^*\| \leq \epsilon$ is

$$O\left(\sqrt{\frac{L_x + L_{xy} + L_y}{m_x m_y}} \cdot \ln^3 \left(\frac{L^2}{m_x m_y} \cdot \frac{\|z_0 - z^*\|}{\epsilon}\right)\right).$$

\square

### E Application to Constrained Problems

In the constrained minimax optimization problem, $x$ is constrained to a compact convex set $\mathcal{X} \subseteq \mathbb{R}^n$ while $y$ is constrained to a compact convex set $\mathcal{Y} \subseteq \mathbb{R}^m$. For constrained minimax optimization problems, saddle points are defined as follows.

**Definition 9.** $(x^*, y^*)$ is a saddle point of $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ if $\forall x \in \mathcal{X}, y \in \mathcal{Y},$

$$f(x, y^*) \geq f(x^*, y) \geq f(x^*, y^*).$$

**Definition 10.** $(\hat{x}, \hat{y})$ is an $\epsilon$-saddle point of $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ if

$$\max_{y \in \mathcal{Y}} f(\hat{x}, y) - \min_{x \in \mathcal{X}} f(x, \hat{y}) \leq \epsilon.$$ 

We will use $P_\mathcal{X}[: \]$ to denote the projection onto convex set $\mathcal{X}$. Assuming efficient projection oracles, our algorithms can all be easily adapted to the constrained case. In particular, for Algorithm 1, we only need to replace AGD with the constrained version; that is, set $x_t \leftarrow P_\mathcal{X} [x_{t-1} - \eta \nabla f(\hat{x}_{t-1})]$.

For Algorithm 3 and 4, the modified versions are presented below. The only significant change is the addition of a projected gradient descent-ascent step in line 5-6 of Algorithm 3 and line 5-6 and 9-10 of Algorithm 4.

#### E.1 Algorithmic Modifications

**Algorithm 8** AGD($g, x_0, T$) with Projections [Nesterov, 2013, (2.2.63)]

**Require:** Initial point $x_0$, smoothness constant $l$, strongly-convex modulus $m$, number of iterations $T$

1. $\eta \leftarrow 1/l$, $\kappa \leftarrow l/m$, $\theta \leftarrow (\sqrt{\kappa} - 1)/\sqrt{\kappa} + 1$
2. $x_1 \leftarrow P_\mathcal{X} [x_0 - \eta \nabla g(x_0)], \hat{x}_1 \leftarrow x_1$
3. for $t = 2, \ldots, T + 1$ do
   4. $x_t \leftarrow P_\mathcal{X} [x_{t-1} - \eta \nabla g(\hat{x}_{t-1})]$
   5. $\hat{x}_t \leftarrow x_t + \theta (x_t - x_{t-1})$
4. end for

For Algorithm 1, the only necessary modification is to add projection steps to the Accelerated Gradient Descent Procedure. The reason for the extra gradient step on line 2 is technical. From the original analysis [Nesterov, 2013, Theorem 2.2.3], it only follows that

$$\|x_{T+1} - x^*\|^2 \leq \left[\|x_1 - x^*\|^2 + \frac{2}{m} (f(x_1) - f(x^*))\right] \cdot \left(1 - \frac{1}{\sqrt{\kappa}}\right)^T.$$
For constrained problems, \( f(x_1) - f(x^*) \leq \frac{L}{2}\|x_1 - x^*\|^2 \) does not hold. However, with the initial projected gradient step, it can be shown that \( \|x_1 - x^*\| \leq \|x_0 - x^*\| \) and that \( f(x_1) - f(x^*) \leq \frac{L}{2}\|x_0 - x^*\|^2 \) (see Lemma 5). Thus
\[
\|x_{T+1} - x^*\|^2 \leq (\kappa + 1)\|x_0 - x^*\|^2 \left( 1 - \frac{1}{\sqrt{\kappa}} \right)^T.
\]

For Algorithm 3 and 4, the modified versions are presented below.

**Algorithm 9** APPA-ABR (for Constrained Optimization)

**Require:** \( g(\cdot, \cdot) \), Initial point \( z_0 = [x_0; y_0] \), precision parameter \( M_1 \)

1. \( \beta_2 \leftarrow \max\{m_y, L_{xy}\}, M_2 \leftarrow \frac{200L^3}{m_xm_y^2} \)
2. \( \hat{y}_0 \leftarrow y_0, \kappa \leftarrow \beta_2/m_y, \theta \leftarrow \frac{2\sqrt{\kappa} - 1}{2\sqrt{\kappa} + 1}, \tau \leftarrow \frac{1}{2\sqrt{\kappa} + \kappa}, T \leftarrow 8\sqrt{\kappa} \ln \left( \frac{400e^2 L^2 M_1}{m_x \sqrt{m_y^2 \kappa}} \right) \)
3. for \( t = 1, \ldots, T \) do
   4. \( \left( x_t', y_t' \right) \leftarrow \text{ABR}(g(x, y) - \beta_2\|y - \hat{y}_{t-1}\|^2, [x_{t-1}; y_{t-1}], 1/M_2, 2\beta_1, 2\beta_2, 3L, 3L) \)
   5. \( x_t \leftarrow P_X \left( x_t' - \theta \nabla_x g(x_t', y_t') \right) \)
   6. \( y_t \leftarrow P_Y \left( y_t' + \frac{1}{\theta} \nabla_y g(x_t', y_t') \right) \)
   7. \( \hat{y}_t \leftarrow y_t + \theta (y_t - y_{t-1}) + \tau (y_t - y_{t-1}) \)
4. end for

**Algorithm 10** Proximal Best Response (for Constrained Optimization)

**Require:** Initial point \( z_0 = [x_0; y_0] \), precision parameter \( M_1 \)

1. \( \beta_1 \leftarrow \max\{m_x, L_{xy}\}, M_1 \leftarrow \frac{120L^3}{m_x^2 m_y} \)
2. \( x_0 \leftarrow x_0, \kappa \leftarrow \beta_1/m_x, \theta \leftarrow \frac{2\sqrt{\kappa} - 1}{2\sqrt{\kappa} + 1}, \tau \leftarrow \frac{1}{2\sqrt{\kappa} + \kappa}, T \leftarrow 8\sqrt{\kappa} \ln \left( \frac{400e^2 L^2 M_1}{m_x \sqrt{m_y^2 \kappa}} \right) \)
3. for \( t = 1, \ldots, T \) do
   4. \( \left( x_t', y_t' \right) \leftarrow \text{APPA-ABR}(f(x, y) + \beta_1\|x - \hat{x}_{t-1}\|^2, [x_{t-1}, y_{t-1}], M_1) \)
   5. \( x_t \leftarrow P_X \left( x_t' - \theta \nabla_x f(x_t', y_t') \right) \)
   6. \( y_t \leftarrow P_Y \left( y_t' + \frac{1}{\theta} \nabla_y f(x_t', y_t') \right) \)
   7. \( \hat{x}_t \leftarrow x_t + \theta (x_t - x_{t-1}) + \tau (x_t - x_{t-1}) \)
4. end for

The most significant change is the addition of a projected gradient descent-ascent step in line 5-6 of Algorithm 3 and line 5-6 and 9-10 of Algorithm 4. The reason for this modification is very similar to that of the initial projected gradient descent step for AGD. For unconstrained problems, a small distance to the saddle point implies a small duality gap (Fact 4); however this may not be true for constrained problems, since the saddle point may no longer be a stationary point. This is also true for minimization: if \( x^* = \arg\min_{x \in X} g(x) \) where \( g(x) \) is a \( L \)-smooth function \( g(x) - g(x^*) \leq \frac{L}{2}\|x - x^*\|^2 \) may not hold.

Fortunately, there is a simple fix to this problem. By applying projected gradient descent-ascent once, we can assure that a small distance implies small duality gap. This is specified by the following lemma, which is the key reason why our result can be adapted to the constrained problem.

**Lemma 4.** Suppose that \( f \in F(m_x, m_y, L_x, L_{xy}, L_y), (x^*, y^*) \) is a saddle point of \( f \), \( z_0 = (x_0, y_0) \) satisfies \( \|z_0 - z^*\| \leq \epsilon \). Let \( \hat{z} = (\hat{x}, \hat{y}) \) be the result of one projected GDA update, i.e.
\[
\hat{x} \leftarrow P_X \left( x_0 - \frac{1}{2L} \nabla_x f(x_0, y_0) \right), \\
\hat{y} \leftarrow P_Y \left( y_0 + \frac{1}{2L} \nabla_y f(x_0, y_0) \right).
\]
Then \( \| \hat{z} - z^* \| \leq \epsilon \), and
\[
\max_{\mathbf{y} \in \mathcal{Y}} f(\hat{\mathbf{x}}, \mathbf{y}) - \min_{\mathbf{x} \in \mathcal{X}} f(\mathbf{x}, \hat{\mathbf{y}}) \leq 2 \left( 1 + \frac{L^2_{xy}}{\min\{m_x, m_y\}^2} \right) L\epsilon^2.
\]

The proof of Lemma 4 is deferred to Sec. E.3.

Because we would use Lemma 4 to replace (11) in the analysis of Algorithm 3 and 4, we would need to accordingly increase \( M_1 \) to \( \frac{120L^3}{m_x^2 m_y^2} \) and \( M_2 \) to \( \frac{200L^3}{m_x m_y^2} \). Apart from this, another minor change in Algorithm 3 is that it would terminate after a fixed number of iterations instead of based on a termination criterion. The number of iterations is chosen such that \( \| x_T - x^* \| + \| y_T - y^* \| \leq \frac{1}{\sqrt{T}} \| x_0 - x^* \| + \| y_0 - y^* \| \) is guaranteed.

### E.2 Modification of Analysis

We now claim that after modifications to the algorithms, Theorem 3 holds for constrained cases.

**Theorem 3.** (Modified) Assume that \( f \in \mathcal{F}(m_x, m_y, L_x, L_{xy}, L_y) \). In Algorithm 4, the gradient complexity to find an \( \epsilon \)-saddle point
\[
O \left( \sqrt{\frac{L_x}{m_x}} + \frac{L \cdot L_{xy}}{m_x m_y} + \frac{L_y}{m_y} \right) \ln^3 \left( \frac{L^2}{m_x m_y} \ln \left( \frac{L^2}{m_x m_y} \frac{L \| x_0 - x^* \|^2}{\epsilon} \right) \right).
\]

The proof of this theorem is, for the most part, the same as the unconstrained version. Hence, we only need to point out parts of the original proof that need to be modified for the constrained case.

To start with, Theorem 1 holds in the constrained case. The proof of Theorem 1 only relies on the analysis of AGD and the Lipschitz properties in Fact 1, and both still hold for constrained problems. (See Lemma B.2 Lin et al. [2020] for the proof of Fact 1 in constrained problems.)

As for Theorem 2, the key modification is about (11). As argued above, (11) uses the property \( g(x) - g(x^*) \leq \frac{L}{2} \| x - x^* \|^2 \), which does not hold in constrained problems, since the optimum may not be a stationary point. Here, we would use Lemma 4 to derive a similar bound to replace (11). Note that originally (11) is only used to derive \( \delta_t \leq \frac{L + 2\beta_1}{2} \epsilon_t^2 \). Using Lemma 4, we can replace this with
\[
\delta_t \leq \max_{y \in \mathcal{Y}} \{ f(x_t, y) + \beta \| x_t - \hat{x}_{t-1} \|^2 \} - \min_{x \in \mathcal{X}} \{ f(x, y_t) + \beta \| x - \hat{x}_{t-1} \|^2 \}
\leq 2 \left( 1 + \frac{L^2_{xy}}{m_x m_y} \right) L\epsilon_t^2.
\]

Accordingly, we can change \( C_0 \) to \( 44k\sqrt{\kappa} \cdot 2L \left( 1 + \frac{L^2_{xy}}{m_x m_y} \right) C^2_1 \), and the assumption on \( M \) to
\[
M \geq 20k \sqrt{\frac{4L}{m_x} \left( 1 + \frac{L^2_{xy}}{m_x m_y} \right) \left( 1 + \frac{L}{m_y} \right)}.
\]

Then Theorem 2 would hold for the constrained case as well.

Finally, as for Theorem 3, we need to re-verify that \( M_1 \) and \( M_2 \) satisfy the new assumptions of \( M \) in order to apply Theorem 2. Observe that
\[
20 \cdot \frac{\beta_2}{m_y} \cdot \sqrt{\frac{4L^2 + 2\beta_1}{m_y^2}} \cdot \left( 1 + \frac{L^2_{xy}}{2\beta_1 \cdot m_y} \right) \cdot \left( 1 + \frac{L}{m_x} \right)
\leq 20 \cdot \frac{L}{m_y} \cdot \sqrt{\frac{18L^3}{m_x m_y^2} \cdot \frac{2L}{m_x} \cdot \frac{200L^3}{m_x m_y^2} = M_2,}
\]

and that
\[
20 \cdot \frac{\beta_1}{m_x} \cdot \sqrt{\frac{4L}{m_x} \cdot \frac{2L^2_{xy}}{m_x m_y} \cdot \frac{2L}{m_y} \cdot \frac{80\sqrt{2}L^{3.5}}{m_x m_y^{1.5}} \leq M_1.}
\]
It follows that the number of iterations needed to find \( \|z_f - z^*\| \leq \epsilon \) is
\[
O\left(\sqrt{\frac{F_{\text{X}}}{m_X} + L \cdot F_{\text{XY}} + \frac{L_y}{m_Y}} \ln^3 \left( \frac{L^2}{m_X m_Y} \right) \ln \left( \frac{L^2}{m_X m_Y} \cdot \|z_0 - z^*\| / \epsilon \right) \right).
\]

It follows from Lemma 4 that the duality gap of \((\hat{x}, \hat{y})\) is at most
\[
\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) \leq 2 \left( 1 + \frac{L_{xy}}{\min\{m_x, m_y\}}^2 \right) Le.
\]

Resetting \( \epsilon \) to \( \sqrt{\frac{e \min\{m_x, m_y\}^2}{4L^2}} \) proves the theorem.

### E.3 Properties of Projected Gradient

**Lemma 5.** If \( g : X \to \mathbb{R} \) is \( L \)-smooth, \( x^* = \arg\min_{x \in X} g(x) \), \( \hat{x} = P_X \left[ x_0 - \frac{1}{T} \nabla g(x_0) \right] \), then \( \|\hat{x} - x^*\| \leq \|x_0 - x^*\| \), and \( g(\hat{x}) - g(x^*) \leq \frac{L}{2} \|x_0 - x^*\|^2 \).

**Proof.** By Corollary 2.2.1 [Nesterov, 2013], \((x_0 - \hat{x})^T(x_0 - x^*) \geq \frac{1}{2}\|\hat{x} - x_0\|^2\). Therefore
\[
\|\hat{x} - x^*\|^2 = \|(x_0 - x^*) + (\hat{x} - x_0)\|^2 \\
= \|x_0 - x^*\|^2 + 2(x_0 - x^*)^T(\hat{x} - x_0) + \|\hat{x} - x_0\|^2 \\
\leq \|x_0 - x^*\|^2.
\]

Meanwhile, note that \( \hat{x} = \arg\min_{x \in X} \left\{ \nabla g(x_0)^T x + \frac{L}{2} \|x - x_0\|^2 \right\} \). By the optimality condition and the \( L \)-strong convexity of \( \nabla g(x_0)^T x + \frac{L}{2} \|x - x_0\|^2 \), we have
\[
\nabla g(x_0)^T \hat{x} + \frac{L}{2} \|\hat{x} - x_0\|^2 + \frac{L}{2} \|x_1 - x^*\|^2 \leq \nabla g(x_0)^T x^* + \frac{L}{2} \|x^* - x_0\|^2.
\]

Thus
\[
\nabla g(x_0)^T(\hat{x} - x^*) \leq \frac{L}{2} \left[ \|x^* - x_0\|^2 - \|\hat{x} - x_0\|^2 - \|\hat{x} - x^*\|^2 \right] .
\]

It follows that
\[
g(\hat{x}) - g(x^*) \leq \nabla g(\hat{x})^T(\hat{x} - x^*) \\
= \nabla g(x_0)^T(\hat{x} - x^*) + (\nabla g(\hat{x}) - \nabla g(x_0))^T(\hat{x} - x^*) \\
\leq \frac{L}{2} \|x^* - x_0\|^2 - \frac{L}{2} \|\hat{x} - x_0\|^2 - \frac{L}{2} \|\hat{x} - x^*\|^2 + L \|\hat{x} - x_0\| \cdot \|\hat{x} - x^*\| \\
\leq \frac{L}{2} \|x^* - x_0\|^2.
\]

We then prove Lemma 4.

**Proof of Lemma 4.** This can be seen as a special case of Proposition 2.2 [Nemirovski, 2004]. Define the gradient descent-ascent field to be \( F(z) := \left[ \nabla_x f(x, y) \ - \nabla_y f(x, y) \right] \). Note that the \( z \) can also be written as
\[
\hat{z} = \arg\min_{z \in \mathcal{X} \times \mathcal{Y}} \left\{ L \|z - z_0\|^2 + F(z)^T z \right\} .
\]
Now, define $z' = (x', y')$ to be
\[
x' \leftarrow P_x \left[ x_0 - \frac{1}{2L} \nabla_x f(\hat{x}, \hat{y}) \right],
\]
\[
y' \leftarrow P_y \left[ y_0 + \frac{1}{2L} \nabla_y f(\hat{x}, \hat{y}) \right].
\]

In other words, $z' = \arg \min_{z \in \mathcal{X} \times \mathcal{Y}} \{ L\|z - z_0\|^2 + F(\hat{z})^T z \}$. By the optimality condition and $2L$-strong convexity of $L\|z - z_0\|^2 + F(\hat{z})^T z$, for any $z \in \mathcal{X} \times \mathcal{Y}$,
\[
L\|z' - z_0\|^2 + F(\hat{z})^T z' + L\|z' - z\|^2 \leq L\|z - z_0\|^2 + F(\hat{z})^T z.
\]

Similarly, by optimality of $\hat{z}$,
\[
L\|\hat{z} - z_0\|^2 + F(z_0)^T \hat{z} + L\|z' - \hat{z}\|^2 \leq L\|z' - z_0\|^2 + F(z_0)^T z'.
\]

Thus
\[
F(\hat{z})^T (\hat{z} - z) = F(\hat{z})^T (z' - z) + F(z_0)^T (\hat{z} - z')
\]
\[
\leq L \left( \|z - z_0\|^2 - \|z' - z_0\|^2 - \|z' - z\|^2 \right) + (F(\hat{z}) - F(z_0))^T (\hat{z} - z')
\]
\[
\leq L \left( \|z' - z_0\|^2 - \|z' - z\|^2 \right) + 2L\|\hat{z} - z_0\| \cdot \|\hat{z} - z'\| - L \|\hat{z} - z'\|^2 - L\|\hat{z} - z_0\|^2
\]

Here we used the fact that for any $z_1, z_2, \|F(z_1) - F(z_2)\| \leq 2L\|z_1 - z_2\|$. Note that (by convexity and concavity)
\[
F(\hat{z})^T (\hat{z} - z) = \nabla_x f(\hat{x}, \hat{y})^T (\hat{x} - x) - \nabla_y f(\hat{x}, \hat{y})^T (\hat{y} - y)
\]
\[
\geq [f(\hat{x}, \hat{y}) - f(x, \bar{y})] + [f(\bar{x}, y) - f(\hat{x}, \bar{y})]
\]
\[
\geq f(\bar{x}, y) - f(x, \bar{y}).
\]

If we choose $x$ and $y$ to be $x^*(\bar{y})$ and $y^*(\bar{x})$, we can see that
\[
\max_{y \in \mathcal{Y}} f(\bar{x}, y) - \min_{x \in \mathcal{X}} f(x, \bar{y}) \leq L\|z - z_0\|^2
\]
\[
\leq 2L\|z - z^*\|^2 + 2L\|x^* - z_0\|^2
\]
\[
\leq 2L\|x^*(\bar{y}) - x^*\|^2 + 2L\|y^*(\bar{x}) - y^*\|^2 + 2L\|z^* - z_0\|^2
\]
\[
\leq 2L^2 \frac{x_0}{\min\{m_x, m_y\}} \cdot L\|\hat{z} - z^*\|^2 + 2L\|z^* - z_0\|^2.
\]

By Corollary 2.2.1 [Nesterov, 2013], $(x_0 - \hat{x})^T (x_0 - x^*) \geq \frac{1}{2}\|\hat{x} - x_0\|^2$. Therefore
\[
\|\hat{x} - x^*\|^2 = \|(x_0 - x^*) + (\hat{x} - x_0)\|^2
\]
\[
= \|x_0 - x^*\|^2 + 2(x_0 - x^*)^T (\hat{x} - x_0) + \|\hat{x} - x_0\|^2
\]
\[
\leq \|x_0 - x^*\|^2.
\]

Similarly, $\|\hat{y} - y^*\| \leq \|y_0 - y^*\|$. Thus
\[
\|\hat{z} - z^*\|^2 = \|\hat{x} - x^*\|^2 + \|\hat{y} - y^*\|^2 \leq \|z_0 - z^*\|^2 \leq \epsilon^2.
\]

It follows that
\[
\max_{y \in \mathcal{Y}} f(\bar{x}, y) - \min_{x \in \mathcal{X}} f(x, \bar{y}) \leq 2L \cdot \left( \frac{L^2 \frac{x_0}{\min\{m_x, m_y\}} + 1}{L^2 \frac{x_0}{\min\{m_x, m_y\}} + 1} \right) \epsilon^2.
\]

\[\square\]
F Implications of Theorem 3

In this section, we discuss how Theorem 3 implies improved bounds for strongly convex-concave problems and convex-concave problems via reductions established in Lin et al. [2020].

Let us consider minimax optimization problem \(\min_{x \in X} \max_{y \in Y} f(x, y)\), where \(f(x, y)\) is \(m_x\)-strongly convex with respect to \(x\), concave with respect to \(y\), and \((L_x, L_{xy}, L_y)\)-smooth. Here, we assume that \(X\) and \(Y\) are bounded sets, with diameters \(D_x = \max_{x, x' \in X} \|x - x'\|\) and \(D_y = \max_{y, y' \in Y} \|y - y'\|\).

Following Lin et al. [2020], let us consider the function

\[
f_{\epsilon, y}(x, y) := f(x, y) - \frac{\epsilon \|y - y_0\|^2}{2D_y^2}.
\]

Recall that \((\hat{x}, \hat{y})\) is an \(\epsilon\)-saddle point of \(f\) if \(\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) \leq \epsilon\). We now show that a \((\epsilon/2)\)-saddle point of \(f_{\epsilon, y}\) would be an \(\epsilon\)-saddle point of \(f\). Let \(x^*() := \arg \min_{x \in X} f(x, \cdot)\) and \(y^*() := \arg \max_{y \in Y} f(\cdot, y)\). Obviously, for any \(x \in X, y \in Y\),

\[
f(x, y) - \frac{\epsilon}{2} \leq f_{\epsilon, y}(x, y) \leq f(x, y).
\]

Thus, if \((\hat{x}, \hat{y})\) is a \((\epsilon/2)\)-saddle point of \(f_{\epsilon, y}\), then

\[
\begin{align*}
&f(\hat{x}, y^*(\hat{x})) \leq f_{\epsilon, y}(\hat{x}, y^*(\hat{x})) + \frac{\epsilon}{2} \leq \max_{y \in Y} f_{\epsilon, y}(\hat{x}, y) + \frac{\epsilon}{2}, \\
&f(x^*(\hat{y}), \hat{y}) \geq f_{\epsilon, y}(x^*(\hat{y}), \hat{y}) \geq \min_{x \in X} f_{\epsilon, y}(x, \hat{y}).
\end{align*}
\]

It immediately follows that

\[
\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) \leq \frac{\epsilon}{2} + \max_{y \in Y} f_{\epsilon, y}(\hat{x}, y) - \min_{x \in X} f_{\epsilon, y}(x, \hat{y}) \leq \epsilon.
\]

Thus, to find an \(\epsilon\)-saddle point of \(f\), we only need to find an \((\epsilon/2)\)-saddle point of \(f_{\epsilon, y}\). We can now prove Corollary 1 by reducing to (the constrained version of) Theorem 3.

Observe that \(f_{\epsilon, y}\) belongs to \(F(m_x, \frac{D_x}{m_x} L_x, L_{xy}, L_y + \frac{D_y}{D_y} L_y)\). Thus, by Theorem 3, the gradient complexity of finding a \((\epsilon/2)\)-saddle point in \(f_{\epsilon, y}\) is

\[
\tilde{O} \left( \sqrt{\frac{L_x}{m_x} + \left(\frac{L \cdot L_{xy}}{m_x} + L_y\right)} \cdot \frac{D_y^2}{\epsilon} \cdot \ln^4 \left( \frac{(D_{xy} + D_y)^2 L_x^2}{m_x \epsilon} \right) \right) = \tilde{O} \left( \sqrt{\frac{m_x \cdot L_y + L \cdot L_{xy}}{m_x \epsilon}} \right),
\]

which proves Corollary 1.

**Corollary 1.** If \(f(x, y)\) is \((L_x, L_{xy}, L_y)\)-smooth and \(m_x\)-strongly convex w.r.t. \(x\), via reduction to Theorem 3, the gradient complexity of finding an \(\epsilon\)-saddle point is \(\tilde{O} \left( \sqrt{\frac{L^2}{m_x \epsilon}} \right)\).

In comparison, Lin et al.’s result in this setting is \(\tilde{O} \left( \sqrt{\frac{L}{m_x \epsilon}} \right)\). Meanwhile a lower bound for this problem has been shown to be \(\Omega \left( \sqrt{\frac{L^2}{m_x \epsilon}} \right)\) [Ouyang and Xu, 2019]. It can be seen that when \(L_{xy} \ll L\), our bound is a significant improvement over Lin et al.’s result, as \(m_x \cdot L_y + L \cdot L_{xy} \ll L^2\).

Similarly, if \(f : X \times Y \rightarrow \mathbb{R}\) is convex with respect to \(x\), concave with respect to \(y\) and \((L_x, L_{xy}, L_y)\)-smooth, we can consider the function

\[
f_{\epsilon}(x, y) := f(x, y) + \frac{\epsilon \|x - x_0\|^2}{4D_x^2} - \frac{\epsilon \|y - y_0\|^2}{4D_y^2}.
\]

It can be shown that for any \(\hat{x} \in X\),

\[
\max_{y \in Y} \left\{ f(\hat{x}, y) + \frac{\epsilon \|\hat{x} - x_0\|^2}{4D_x^2} - \frac{\epsilon \|y - y_0\|^2}{4D_y^2} \right\} \geq \max_{y \in Y} f(\hat{x}, y) - \frac{\epsilon}{4}.
\]

\(^5\)Here it is assumed that \(\epsilon\) is sufficiently small, i.e. \(\epsilon \leq \max\{L_{xy}, m_x\} D_y^2\).
Similarly, for any $\hat{y} \in Y$,
\[
\min_{x \in X} \left\{ f(x, \hat{y}) + \frac{\epsilon \|x - x_0\|^2}{4D_X^2} - \frac{\epsilon \|\hat{y} - y_0\|^2}{4D_Y^2} \right\} \leq \min_{x \in X} f(x, \hat{y}) + \frac{\epsilon}{4}.
\]
Therefore, if $(\hat{x}, \hat{y})$ is an $(\epsilon/2)$-saddle point of $f$, it is an $\epsilon$-saddle point of $f$, as
\[
\max_{y \in Y} f(\hat{x}, y) - \min_{x \in X} f(x, \hat{y}) \leq \frac{\epsilon}{2} + \max_{y \in Y} f_e(\hat{x}, y) - \min_{x \in X} f_e(x, \hat{y}) \leq \epsilon.
\]
Observe that $f_e$ belongs to $\mathcal{F}(\frac{x}{\sqrt{L_x}}, \frac{y}{\sqrt{L_y}}, L_x + \frac{\sqrt{L_{xy}}}{\sqrt{L_y}})$, $L_y$. Thus, by Theorem 3, the gradient complexity of finding an $(\epsilon/2)$-saddle point of $f_e$ is
\[
O \left( \left( \sqrt{\frac{L_x D_X^2 + L_y D_Y^2}{\epsilon}} + \frac{D_X D_Y \sqrt{L_y L_{xy}}}{\epsilon} \right) \cdot \ln^4 \left( \frac{L(D_x + D_y)^2}{\epsilon} \right) \right),
\]
which proves Corollary 2.

**Corollary 2.** If $f(x, y)$ is $(L_x, L_{xy}, L_y)$-smooth and convex-concave, via reduction to Theorem 3, the gradient complexity to produce an $\epsilon$-saddle point is $\tilde{O} \left( \frac{1}{\epsilon} \right)$ [Nemirovski, 2004]. Meanwhile, a lower bound for this setting has shown to be $\Omega \left( \frac{L_x}{\sqrt{\epsilon}} + \frac{L_{xy}}{\epsilon} \right)$ [Ouyang and Xu, 2019]. Again, our result can be a significant improvement over Lin et al.’s result if $L_{xy} \ll L$, and is closer to the lower bound.

**G Proof of Theorem 4**

We will start by proving several useful lemmas.

**Lemma 2.** ([Bai et al., 2003]) Define $M(\eta) := (\eta P + S)^{-1} (\eta P - G) (\eta P + G)^{-1} (\eta P - S)$. Then
\[
\rho(M(\eta)) \leq \|M(\eta)\|_2 \leq \max_{\lambda_i \in sp(P^{-1}G)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right| < 1.
\]

**Proof of Lemma 1.** We provide a proof for completeness. First, observe that
\[
M(\eta) = (\eta P + S)^{-1} \frac{\eta P - G}{(\eta P + G)^{-1} (\eta P - S)}
= P^{-\frac{1}{2}} (\eta I + P^{-\frac{1}{2}} GP^{-\frac{1}{2}})^{-1} (\eta I - P^{-\frac{1}{2}} GP^{-\frac{1}{2}}) (\eta I + P^{-\frac{1}{2}} GP^{-\frac{1}{2}})^{-1} (\eta I - P^{-\frac{1}{2}} SP^{-\frac{1}{2}}) P^{\frac{1}{2}}.
\]
Let $G := P^{-\frac{1}{2}} GP^{-\frac{1}{2}}, \hat{S} := P^{-\frac{1}{2}} SP^{-\frac{1}{2}}$. Then $M(\eta)$ is similar to
\[
(\eta I + \hat{S})^{-1} (\eta I - G)(\eta I + \hat{G})^{-1} (\eta I - \hat{S}),
\]
which is then similar to
\[
(\eta I - \hat{G})(\eta I + \hat{G})^{-1} (\eta I - \hat{S})(\eta I + \hat{S})^{-1}.
\]
The key observation is that $(\eta I - \hat{S})(\eta I + \hat{S})^{-1}$ is orthogonal, since
\[
\left( (\eta I + \hat{S})^{-1} \right)^T (\eta I - \hat{S})(\eta I - \hat{S}) (\eta I + \hat{S})^{-1}
=(\eta I - \hat{S})^{-1} (\eta I + \hat{S})(\eta I - \hat{S})(\eta I + \hat{S})^{-1}
=(\eta I - \hat{S})^{-1} (\eta I + \hat{S})(\eta I + \hat{S})^{-1} = I.
\]
Therefore
\[
\rho(M(\eta)) \leq \| (\eta I - \hat{G})(\eta I + \hat{G})^{-1}(\eta I - \hat{S})(\eta I + \hat{S})^{-1} \|_2 \\
\leq \| (\eta I - \hat{G})(\eta I + \hat{G})^{-1} \|_2 \cdot \| (\eta I - \hat{S})(\eta I + \hat{S})^{-1} \|_2 \\
= \max_{\lambda_i \in sp(\eta)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right| = \max_{\lambda_i \in sp(\eta)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right|.
\]

We now proceed to state some useful lemmas for the proof of Theorem 4.

**Lemma 6.** The following statements about the eigenvalues and singular values of matrices hold:

1. The singular values of $J$ fall in $[m_x, L_{xy} + L_x]$;
2. The condition number of $\eta P + G$ is at most $\frac{3m_x}{m_x} \left( \frac{m_x}{r_{xy}} \right)^\frac{1}{2}$;
3. The condition number of $\eta P + G$ is at most $L_{x} / m_{x}$.
4. The eigenvalues of $\eta (\alpha I + \beta A)$ fall in $[\eta \alpha, 2\eta \beta L_x]$. The eigenvalues of $\eta (I + \beta C)$ fall in $[\eta, 2\eta L_x]$.

**Proof of Lemma 6.**

1. Consider an arbitrary $x \in \mathbb{R}^{n+m}$ with $\|x\|_2 = 1$. Construct a set of orthonormal vectors $\{x_1, \cdots, x_{n+m}\}$ with $x_1 = x$. Then

\[
x^T J^T J x = \sum_{i=1}^{n+m} x^T J^T x_i x_i^T J x = \sum_{i=1}^{n+m} (x^T J^T x_i)^2 \geq (x^T J^T x)^2.
\]

Since $J = G + S = \begin{bmatrix} A & 0 \\ 0 & C \end{bmatrix} + \begin{bmatrix} 0 & B \\ -B^T & 0 \end{bmatrix}$, where $S$ is skew-symmetric, $x^T J^T x = x^T G x \geq m_x$. Thus

\[
\sigma_{\min}(J) = \sqrt{\lambda_{\min}(J^T J)} \geq m_x.
\]

Meanwhile,

\[
\lambda_{\max}(J^T J) \leq \|G\|_2 + \|S\|_2 \leq L_{xy} + L_x.
\]

2. Note that

\[
\eta P + G = \begin{bmatrix} \eta(\alpha I + \beta A) + A \\ \eta(I + \beta C) + C \end{bmatrix}.
\]

Thus

\[
\|\eta P + G\|_2 \leq \max\{\eta(\alpha + \beta L_x) + L_x, \eta(1 + \beta L_x) + L_x\} = \eta(1 + \beta L_x) + L_x.
\]

On the other hand

\[
\lambda_{\min}(\eta P + G) \geq \min\{\eta \alpha + \eta \beta m_x + m_x, \eta + \eta \beta m_x + m_x\} = \eta \alpha + \eta \beta m_x + m_x.
\]

Thus the condition number of $\eta P + G$ is at most

\[
\frac{\eta(1 + \beta L_x) + L_x}{\eta \alpha + \eta \beta m_x + m_x} \leq \frac{L_x}{\eta \alpha} + \frac{1 + \beta L_x}{\eta \alpha} \leq \frac{L_x}{\eta \alpha} + \frac{2\beta L_x}{\eta \alpha} = \frac{L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2} + \frac{2L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2} \leq \frac{3L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}.
\]

3. On the other hand,

\[
\frac{\eta(1 + \beta L_x) + L_x}{\eta \alpha + \eta \beta m_x + m_x} = \frac{\eta + \eta \beta L_x + L_x}{\eta \alpha + \eta \beta m_x + m_x} \leq \max\left\{ \frac{1}{\alpha}, \frac{L_x}{m_x} \right\} = \frac{L_x}{m_x}.
\]
4. Finally let us consider matrices $\eta(\alpha I + \beta A)$ and $\eta(I + \beta C)$. Obviously
\[
\eta(\alpha I + \beta A) \succ \eta \alpha I, \quad \eta(I + \beta C) \succ \eta I.
\]
Meanwhile
\[
||\eta(\alpha I + \beta A)|| \leq \eta \cdot (\alpha + \beta L_x) \\
\leq \eta (1 + \beta L_x) \\
\leq 2\eta \beta L_x. \quad (\alpha < 1)
\]
Similarly $||\eta(I + \beta C)|| \leq 2\eta \beta L_x$.

**Lemma 7.** With our choice of $\eta$, $\alpha$ and $\beta$,
\[
\rho(M(\eta)) \leq ||M(\eta)||_2 \leq 1 - \frac{1}{2} \left( \frac{m_y}{L_{xy}} \right)^{\frac{1}{k}}.
\]

**Proof of Lemma 7.** By Lemma 1,
\[
\rho(M(\eta)) \leq ||M(\eta)||_2 \leq \max_{\lambda_i \in sp(P^{-1}G)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right|.
\]
Observe that
\[
P^{-1}G = \left[ (\alpha I + \beta A)^{-1} A \quad (I + \beta C)^{-1} C \right].
\]
The eigenvalues of $(\alpha I + \beta A)^{-1} A$ are contained in
\[
\left[ \frac{m_x}{\alpha + \beta m_x}, \frac{L_x}{\alpha + \beta L_x} \right] \subseteq \left[ \frac{m_y}{2}, \frac{1}{\beta} \right]. \quad (\beta m_x \leq \alpha)
\]
Similarly the eigenvalues of $(I + \beta C)^{-1} C$ are contained in
\[
\left[ \frac{m_y}{1 + \beta m_y}, \frac{L_x}{1 + \beta L_x} \right] \subseteq \left[ \frac{m_y}{2}, \frac{1}{\beta} \right]. \quad (\beta m_y \leq 1)
\]
Recall that $\eta = L_{xy}^{1/k} m_y^{1-1/k} = \sqrt{m_y/\beta}$. As a result,
\[
\max_{\lambda_i \in sp(P^{-1}G)} \left| \frac{\lambda_i - \eta}{\lambda_i + \eta} \right| \leq \max \left\{ \frac{1}{\beta + \sqrt{m_y/\beta}}, \frac{\sqrt{m_y/\beta} - \frac{m_y}{2}}{\frac{1}{\beta} + \sqrt{m_y/\beta}} \right\} \leq 1 - \frac{\sqrt{m_y/\beta}}{2} = 1 - \frac{1}{2} \left( \frac{m_y}{L_{xy}} \right)^{\frac{1}{k}}.
\]

**Lemma 8.** When RHSS($k$) terminates $\|z_t - z^*\| \leq \epsilon \|z_0 - z^*\|$.

**Proof of Lemma 8.**
\[
\sigma_{\min}(J) \|z_t - z^*\| \leq \|Jz_t - b\| \leq \epsilon \|Jz_0 - b\| \leq \sigma_{\max}(J) \|z_0 - z^*\|.
\]
We know that $\sigma_{\min}(J) \geq m_x$ and that $\sigma_{\max}(J) \leq L_x + L_{xy}$. Thus
\[
\|z_t - z^*\| \leq \frac{\epsilon \cdot (L_x + L_{xy})}{m_x} \|z_0 - z^*\| = \epsilon \|z_0 - z^*\|.
\]
Lemma 10. In RHSS($k$),
\[ \|z_{t+1} - z^*\| \leq \left( 1 - \frac{1}{4} \left( \frac{m_y}{L_{xy}} \right) \right) \|z_t - z^*\|. \]  
(20)

Proof of Lemma 10. Let us define
\[ z_{t+1/2} = \begin{bmatrix} \hat{x}_{t+1/2} \\ \hat{y}_{t+1/2} \end{bmatrix} = \left[ \begin{array}{c} \eta (\alpha I + \beta A) + A \\ \eta (I + \beta C) + C \end{array} \right]^{-1} \begin{bmatrix} r_t \\ r_{t} \end{bmatrix}. \]
Since \( (\eta P + G)(z_{t+1/2} - \hat{z}_{t+1/2}) \leq \frac{1}{L_{xy}} \|z_{t+1/2} - \hat{z}_{t+1/2}\| \),
\[ \|z_{t+1/2} - \hat{z}_{t+1/2}\| \leq \frac{|\lambda_{\max}(\eta P + G)|}{M_1 \lambda_{\min}(\eta P + G)} \|z_t - \hat{z}_{t+1/2}\| \]
\[ \leq \frac{L_{xy}}{M_1 m_x} \|z_t - \hat{z}_{t+1/2}\| = \frac{m_x m_y^3}{192 L^4} \|z_t - \hat{z}_{t+1/2}\|. \]
Because \( \hat{z}_{t+1/2} - z^* = (\eta P + G)^{-1} (\eta P - S) (z_t - z^*) \),
\[ \|\hat{z}_{t+1/2} - z^*\| \leq \|\eta P + G\|^{-1} \|\eta P - S\|_2 \|z_t - z^*\| \]
\[ \leq \frac{1}{\eta \alpha} \cdot (L_{xy} + \eta \alpha + \eta \beta L_{xy}) \cdot \|z_t - z^*\| \]
\[ \leq \left( 1 + \frac{2L}{m_x} \right) \|z_t - z^*\|. \]
It follows that
\[ \|z_t - \hat{z}_{t+1/2}\| \leq \|z_t - z^*\| + \|\hat{z}_{t+1/2} - z^*\| \leq \left( 2 + \frac{2L}{m_x} \right) \|z_t - z^*\|. \]
By plugging this into (21), one gets
\[ \|z_{t+1/2} - \hat{z}_{t+1/2}\| \leq \frac{m_x m_y^3}{192 L^4} \cdot \left( 2 + \frac{2L}{m_x} \right) \|z_t - z^*\| \leq \frac{m_y^3}{48 L^3} \|z_t - z^*\|. \]
(22)
Now, let us define
\[ \hat{z}_{t+1} := (\eta P + S)^{-1} [(\eta P - G)\hat{z}_{t+1/2} + b], \]
\[ \hat{z}_{t+1} := (\eta P + S)^{-1} [(\eta P - G)z_{t+1/2} + b]. \]
First let us try to bound \( \|\hat{z}_{t+1} - \hat{z}_{t+1}\|. \) Observe that \( \hat{z}_{t+1} - z^* = (\eta P + S)^{-1}(\eta P - G)(z_{t+1/2} - z^*), \) so
\[ \|\hat{z}_{t+1} - \hat{z}_{t+1}\| = \| (\hat{z}_{t+1} - z^*) - (z_{t+1} - z^*) \|
= \| (\eta P + G)^{-1} (\eta P - S) (z_{t+1/2} - z_{t+1/2}) \|
\leq \| (\eta P + G)^{-1} (\eta P - S) \|_2 \cdot \|z_{t+1/2} - z_{t+1/2}\|
\leq \frac{3L^2}{m_x^2} \cdot \frac{m_y^3}{48 L^3} \|z_t - z^*\| = \frac{m_x}{16L} \|z_t - z^*\|. \]
(23)
Next, by Lemma 8 on RHSS($k - 1$),
\[ \|z_{t+1} - \hat{z}_{t+1}\| \leq \frac{1}{M_2} \|z_t - \hat{z}_{t+1}\| \leq \frac{1}{M_2} \left( \|z_t - z^*\| + \|\hat{z}_{t+1} - z^*\| \right). \]
By Lemma 7,
\[ \|\hat{z}_{t+1} - z^*\| = \|M(\eta)(z_t - z^*)\|_2 \leq \left(1 - \frac{1}{2} \left(\frac{m_y}{L_{xy}}\right)^{\frac{1}{2}}\right) \|z_t - z^*\|. \] (24)

Thus
\[ \|z_{t+1} - \hat{z}_{t+1}\| \leq \frac{1}{M_2} \left(2\|z_t - z^*\| + \|\hat{z}_{t+1} - \hat{z}_{t+1}\|\right). \] (25)

Combining (23) and (25), one gets
\[
\|z_{t+1} - \hat{z}_{t+1}\| \leq \frac{2}{M_2} \|z_t - z^*\| + \left(1 + \frac{2}{M_2}\right) \|\hat{z}_{t+1} - \hat{z}_{t+1}\|
\leq \frac{m_y}{8L_{xy}} \|z_t - z^*\| + \frac{m_y}{8L} \|z_t - z^*\| \leq \frac{m_y}{4L_{xy}} \|z_t - z^*\|.
\]

Combining this with (24), one gets
\[
\|z_{t+1} - z^*\| \leq \|\hat{z}_{t+1} - z^*\| + \|\hat{z}_{t-1} - z_t\|
\leq \left(1 - \frac{1}{2} \left(\frac{m_y}{L_{xy}}\right)^{\frac{1}{2}}\right) \|z_t - z^*\| + \frac{m_y}{4L_{xy}} \|z_t - z^*\|
\leq \left(1 - \frac{1}{4} \left(\frac{m_y}{L_{xy}}\right)^{\frac{1}{2}}\right) \|z_t - z^*\|.
\]

Finally, we are ready to prove Theorem 4.

**Theorem 4.** There exists constants $C_1, C_2$, such that the number of matrix-vector products needed to find $(x_T, y_T)$ such that $\|z_T - z^*\| \leq \epsilon$ is at most
\[
\sqrt{\frac{L_{xy}^2}{m_x m_y} + \left(\frac{L_x}{m_x} + \frac{L_y}{m_y}\right)^2 \left(1 + \left(\frac{L_{xy}}{\max\{m_x, m_y\}}\right)^{\frac{1}{2}}\right)} \cdot \left(C_1 \ln \left(\frac{C_2 L_x^2}{m_x m_y}\right)\right)^{k+3} \ln \left(\frac{\|z_0 - z^*\|}{\epsilon}\right).
\] (26)

**Proof of Theorem 4.** By Lemma 10, when running RHSS($k$),
\[ \|z_T - z^*\| \leq \left(1 - \frac{1}{4} \left(\frac{m_y}{L_{xy}}\right)^{\frac{1}{2}}\right)^T \|z_0 - z^*\|.
\]

Thus, when $T > 4 \left(\frac{L_{xy}}{m_y}\right)^{1/k} \ln \left(\frac{\|z_0 - z^*\|}{\epsilon}\right)$, one can ensure that $\|z_T - z^*\| \leq \epsilon$. Now we can focus on the number of matrix-vector products needed per iteration, which comes in two parts: the cost of calling conjugate gradient and the cost of calling RHSS($k-1$).

**Conjugate Gradient cost** The matrix to be solved via conjugate gradient is $\eta P + G$. By Lemma 6, its condition number is upper bounded by $\frac{3L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{1/k}$. By Lemma 9, the number of matrix-vector products needed for calling CG is
\[
\left\lceil \sqrt{\frac{3L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{1/k}} \ln \left(2\sqrt{\frac{3L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{1/k}} M_1 \right)\right\rceil \leq c_1 \sqrt{\frac{L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{\frac{1}{2}}} \cdot \ln \left(\frac{c_2 L_x^2}{m_x m_y}\right),
\] (27)
for some constants $c_1, c_2 > 0$.  

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RHSS\((k-1)\) cost  By Lemma 6, the new saddle point problem involving \(\eta P + S\) has parameters \(m_x' = \eta a_s\), \(m_y' = \eta l_y\), \(L_x' = 2\eta b_x L_x\), \(L_y' = L_{xy}\). It is easy to see that \(m_x' = \eta \geq m_x\), \(m_y' = (m_x/m_y)m_y' \geq m_y\), and that \(L_x' = L_y' \leq 2L_x\). Thus \(L' = \max\{L_x', L_y', L_{xy}'\} \leq 2L\). Assuming that Theorem 4 holds for RHSS\((k-1)\), then the number of matrix-vector products needed for the new saddle point problem can be bounded by

\[
\left[\frac{L_{xy}^2}{m_x' m_y'} + \frac{L_x^2}{m_x^2} + \frac{L_y^2}{m_y^2}\right] \left(1 + \frac{L_{xy}}{\max\{m_x', m_y'\}}\right)^{1/(k-1)} \left(C_1 \ln \left(\frac{C_2 L^2}{m_x' m_y'}\right)\right)^{k+2} \ln \left(\frac{4L^2 M_2}{m_x^2}\right).
\]

(a)

Here we used Lemma 8, that when \(\|z_t - z^*\| \leq \left(\frac{m_y}{L_x+L_{xy}}\right)^2 \|z_0 - z^*\|\), RHSS\((k-1)\) returns. Assume that \(C_1 > 8\). Note that

\[
\frac{L_{xy}^2}{m_x' m_y'} = \frac{L_{xy}^2}{m_x^2 m_y^2} \frac{m_x^2}{m_y^2} \frac{2^{2/k}}{m_x^2 m_y^2} = \frac{L_{xy}^2}{m_x^2 m_y^2} \left(\frac{m_y}{L_{xy}}\right)^{2/k}.
\]

Therefore

\[
(a) \leq \sqrt{\frac{L_{xy}^2}{m_x^2 m_y^2} \left(\frac{m_y}{L_{xy}}\right)^{2/k} + \frac{2L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{2/k}} \cdot \left(1 + \left(\frac{L_{xy}}{m_y}\right)^{2/k}\right)
\]

\[
\leq 2 \sqrt{\frac{L_{xy}^2}{m_x^2 m_y^2} \left(\frac{m_y}{L_{xy}}\right)^{2/k} + \frac{L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{2/k}}.
\]

\[
\ln \left(\frac{C_2 L^2}{m_x' m_y'}\right) \leq \ln \left(\frac{4C_2 L^2}{m_x m_y}\right) \leq 2 \ln \left(\frac{C_2 L^2}{m_x m_y}\right),
\]

\[
(b) \leq \ln \left(\frac{64 L^4}{m_x^2 m_y^2}\right) \leq \ln \left(\frac{8L^2}{m_x m_y}\right).
\]

Thus the cost of calling RHSS\((k-1)\) is at most

\[
4C_1^{k+2} \ln^{k+3} \left(\frac{C_2 L^2}{m_x m_y}\right) \sqrt{\frac{L_{xy}^2}{m_x^2 m_y^2} \left(\frac{m_y}{L_{xy}}\right)^{2/k} + \frac{L_x}{m_x} \left(\frac{m_y}{L_{xy}}\right)^{2/k}}.
\]

(28)

In the case where \(k = 2\), RHSS\((k-1)\) is exactly Proximal Best Response (Algorithm 4). Hence, by Theorem 3, the number of matrix-vector products needed is at most

\[
O\left(\sqrt{\frac{L_{xy} \cdot \max\{L_{xy}, L'\}}{m_x' m_y'} + \frac{L_x}{m_x^2} + \frac{L_y}{m_y^2}} \cdot \ln^{4} \left(\frac{L_{xy}^2}{m_x' m_y'}\right) \cdot \ln \left(\frac{L'M_2}{m_x' m_y'}\right)\right)
\]

\[
=O\left(\sqrt{\frac{L_{xy}^2}{m_x^2 m_y^2} + \frac{L_x \sqrt{m_y}}{m_x \sqrt{L_{xy}}} \ln^{5} \left(\frac{L^2}{m_x m_y}\right)}\right).
\]

By this, we mean there exists constants \(c_3, c_4 > 0\) such that the number of matrix-vector products needed is

\[
c_3 \sqrt{\frac{L_{xy}^2}{m_x^2} + \frac{L_x \sqrt{m_y}}{m_x \sqrt{L_{xy}}}} \ln^{5} \left(\frac{c_4 L^2}{m_x m_y}\right).
\]

Thus, (28) also holds for \(k = 2\), provided that \(C_2 \geq c_4\) and \(C_1 \geq c_3\).
Total cost. By combining (27) and (28), we can see that the cost (i.e. number of matrix-vector products) of RHSS(k) per iteration is

\[
(4C_1^{k+2} + c_1) \ln^{k+3} \left( \frac{\max\{c_2, C_2\} L^2}{m_x m_y} \right) \sqrt{\frac{L_{xy}^2}{m_x m_y} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}} + \frac{L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}.
\]

Let us choose \( C_2 > \max\{c_2, 8\} \) and \( C_1 > \max\{c_1, 20\} \). Then, in order to ensure that \( \|z_T - z^*\| \leq \epsilon \), the number of matrix-vector products that RHSS(k) needs is

\[
4 \left( \frac{L_{xy}}{m_y} \right)^{1/k} \ln \left( \frac{\|z_0 - z^*\|}{\epsilon} \right) \cdot (4C_1^{k+2} + c_1) \ln^{k+3} \left( \frac{C_2 L^2}{m_x m_y} \right) \sqrt{\frac{L_{xy}^2}{m_x m_y} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}} + \frac{L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}
\]

\[
\leq 20C_1^{k+2} \ln \left( \frac{\|z_0 - z^*\|}{\epsilon} \right) \ln^{k+3} \left( \frac{C_2 L^2}{m_x m_y} \right) \sqrt{\frac{L_{xy}^2}{m_x m_y} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}} + \frac{L_x}{m_x} \left( \frac{m_y}{L_{xy}} \right)^\frac{1}{2}
\]

\[
\leq \sqrt{\frac{L_{xy}^2}{m_x m_y} + \frac{L_x}{m_x} + \frac{L_y}{m_y}} \ln \left( \frac{\|z_0 - z^*\|}{\epsilon} \right) \cdot \left( \frac{L_{xy}^2}{m_x m_y} \right)^\frac{1}{2} \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \ln \left( \frac{L_{xy}^2}{m_x m_y} \right) \cdot (C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \ln \left( \frac{L_{xy}^2}{m_x m_y} \right))^{k+3}.
\]

We now discuss how to choose the optimal \( k \). Observe that

\[
(6) \leq \sqrt{\frac{L_{xy}^2}{m_x m_y} + \frac{L_x}{m_x} + \frac{L_y}{m_y}} \ln \left( \frac{\|z_0 - z^*\|}{\epsilon} \right) \cdot \left( \frac{L_{xy}^2}{m_x m_y} \right)^\frac{1}{2} \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \ln \left( \frac{L_{xy}^2}{m_x m_y} \right) \cdot (C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \ln \left( \frac{L_{xy}^2}{m_x m_y} \right))^{k+3}.
\]

Compared to the lower bound, there is only one additional factor \( (a) \), whose logarithm is

\[
\ln \left( \left( \frac{L^2}{m_x m_y} \right)^\frac{1}{2} \right) C_1^{k+3} \ln^{k+3} \left( \frac{C_2 L^2}{m_x m_y} \right) = \frac{1}{2k} \ln \left( \frac{L^2}{m_x m_y} \right) + (k + 3) \ln \left( C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \right),
\]

which is minimized when \( k = \sqrt{(\ln \left( \frac{L^2}{m_x m_y} \right)) / (2 \ln (C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right)))} \), and the minimum value is

\[
3 \ln \left( C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \right) + \frac{1}{2} \ln \left( \frac{L^2}{m_x m_y} \right) \ln \left( C_1 \ln \left( \frac{C_2 L^2}{m_x m_y} \right) \right) = o \left( \ln \left( \frac{L^2}{m_x m_y} \right) \right).
\]

I.e. \((a)\) is sub-polynomial in \( \frac{L^2}{m_x m_y} \). This proves Corollary 3, which states that, when

\[
k = \Theta \left( \sqrt{\ln \left( \frac{L^2}{m_x m_y} \right) / \ln \left( \frac{L^2}{m_x m_y} \right)} \right),
\]

the number of matrix vector products that RHSS(k) needs to find \( z_T \) such that \( \|z_T - z^*\| \leq \epsilon \) is

\[
\sqrt{\frac{L_{xy}^2}{m_x m_y} + \frac{L_x}{m_x} + \frac{L_y}{m_y}} \ln \left( \frac{\|z_0 - z^*\|}{\epsilon} \right) \cdot \left( \frac{L^2}{m_x m_y} \right)^{o(1)}.
\]

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