A class of extended high-dimensional nonisospectral KdV hierarchies and symmetry

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Abstract: We construct a new class of $N$-dimensional Lie algebras and apply them to integrable systems. In this paper, we obtain a nonisospectral KdV integrable hierarchy by introducing a nonisospectral spectral problem. Then, a coupled nonisospectral KdV hierarchy is deduced by means of the corresponding higher-dimensional loop algebra. It follows that the $K$ symmetries, $\tau$ symmetries and their Lie algebra of the coupled nonisospectral KdV hierarchy are investigated. The bi-Hamiltonian structures of the both resulting hierarchies are derived by using the trace identity. Finally, we derive a multi-component nonisospectral KdV hierarchy related to the $N$-dimensional loop algebra, which generalizes the coupled results to an arbitrary number of components.

Key words: Multi-component nonisospectral KdV hierarchies; High-dimensional Lie algebras; Bi-Hamiltonian structures; Symmetries

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1. Introduction

A most active field of recent research involved in applied mathematics and theoretical physics is concerned with nonlinear partial differential equation [1]. The Korteweg-de Vries (KdV) equation [2], arises in the description of many phenomena in physics [3, 4], is one of the most famous nonlinear partial differential equations. In recent years, many work has been done on the research of isospectral KdV hierarchies [5–7]. However, there are few results on the extended high-dimensional nonisospectral KdV hierarchies. Therefore, it is meaningful for us to consider the generation and application of the extended high-dimensional nonisospectral KdV hierarchies in this paper. The derivation of integrable hierarchies of evolution equations involves a variety of powerful methods, including the Lax pair method proposed by Magri [8], the method that Qiao and Ma came up with to generate isospectral and nonisospectral integrable hierarchies by applying the generalized Lax representations [9–12], the approach put forward by Tu [13] and later called Tu-scheme [14]. Some integrable systems and the corresponding Hamiltonian structures as well as other properties were obtained by using the Tu scheme, such as the works in [15–17]. Based on it, many extended coupled integrable hierarchies were deduced by means of the knowledge of integrable coupling [18–21]. The nonisospectral hierarchy is a special case of generalized structure of Lax representations [22]. In [23], the L-A-B representation of nonlinear evolution equations had been discussed and its range can be determined by an approach proposed by Qiao and Strampp. Then, the authors dealt with the category of nonlinear evolution equations in [24]. Meanwhile, they proposed an approach.
for constructing the algebraic structure and $r$-matrix of nonlinear evolution equations. However, the integrable systems generated by the Tu scheme were usually presented under the case of isospectral problems. Also, to the best of our knowledge, there is very little work on generating multi-component integrable hierarchies because it is extremely dense. Recently, we constructed a multi-component non-semisimple Lie algebra for generating higher-dimensional isospectral and nonisospectral integrable hierarchies, and then derived the $\mathbb{Z}_N$ isospectral MKdV integrable coupling hierarchy and the $\mathbb{Z}_N$ nonisospectral ANKS integrable coupling hierarchy [25].

Inspired by the corresponding research results related to the Frobenius algebra [26, 27] and non-semisimple Lie algebra [28–30], we construct a class of higher-dimensional Lie algebras to generate multi-component hierarchy of soliton equations. As an application, we consider the nonisospectral KdV spatial spectral problem under the assumption case where $\lambda_t = \sum_{j=0}^n k_j(t)\lambda^{-j}$ [31, 32]. It follows that many isospectral and nonisospectral integrable systems can be obtained by reducing these resulting hierarchies. Actually, these nonisospectral integrable systems that we obtained can enrich the existing integrable models and possibly describe new nonlinear phenomena [33–38].

It is known that many integrable evolution equations possess a new set of symmetries, usually called $\tau$ symmetries, and these symmetries often constitute a Lie algebra together with the original symmetries, called $K$ symmetries [39–42]. It is known that $\tau$ symmetries are full of deep necessary [43–46]. However, to the best of our knowledge, there are very few results on the symmetries of coupled nonisospectral integrable hierarchies. Therefore, it is significant for us to investigate the $K$ symmetries, $\tau$ symmetries and their Lie algebra of the coupled nonisospectral KdV hierarchy.

2. A few expanding higher-dimensional Lie algebras

The Lie algebra $A_1$ admits two basic subalgebras [13]. One is that

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

(1)

which satisfies the commutative relations

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$  

The other one is that

$$\bar{h} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \bar{e} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \bar{f} = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(2)

which is equipped with

$$[\bar{h}, \bar{e}] = \bar{f}, \quad [\bar{h}, \bar{f}] = \bar{e}, \quad [\bar{e}, \bar{f}] = \bar{h}.$$ 

In [47], the authors presented several finite-dimensional Lie algebras. Now, we construct a few new higher-dimensional Lie algebras and generalize them to infinite dimensions. For the first subalgebra [11], we introduce the extended Lie algebras as follows:

Case 1

$$A_{12} = \text{span}\{h_i\}_{i=1}^6,$$

(3)
where
\[
\begin{align*}
\text{where} \\
\h_1 = \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad \h_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad \h_3 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, \\
\h_4 = \begin{pmatrix} 0 & \varepsilon h \\ \varepsilon h & 0 \end{pmatrix}, \quad \h_5 = \begin{pmatrix} 0 & \varepsilon e \\ \varepsilon e & 0 \end{pmatrix}, \quad \h_6 = \begin{pmatrix} 0 & \varepsilon f \\ \varepsilon f & 0 \end{pmatrix},
\end{align*}
\]

\[
\begin{align*}
[h_1, h_2] &= 2h_2, \quad [h_1, h_3] = -2h_3, \quad [h_1, h_4] = 0, \quad [h_1, h_5] = 2h_5, \quad [h_1, h_6] = -2h_6, \\
[h_2, h_3] &= h_1, \quad [h_2, h_4] = -2h_5, \quad [h_2, h_5] = 0, \quad [h_2, h_6] = h_4, \quad [h_3, h_4] = 2h_6, \\
[h_3, h_5] &= -h_4, \quad [h_3, h_6] = 0, \quad [h_4, h_5] = 2\varepsilon h_2, \quad [h_4, h_6] = -2\varepsilon h_3, \quad [h_5, h_6] = \varepsilon h_1,
\end{align*}
\]

with \( \varepsilon \in \mathbb{R} \).

Let \( G_1 = \text{span}\{h_1, h_2, h_3\}, \ G_2 = \text{span}\{h_4, h_5, h_6\} \), then \( A_{12} = G_1 \bigoplus G_2 \). Denoting
\[
\begin{align*}
[G_i, G_j] &= \{[A, B]| A \in G_i, B \in G_j\},
\end{align*}
\]

we find that the closure properties between \( G_1 \) and \( G_2 \) are as follows:

\[
\begin{align*}
[G_1, G_1] &\subseteq G_1, \quad [G_1, G_2] \subseteq G_2, \quad [G_2, G_2] \subseteq G_1.
\end{align*}
\]

**Case 2**

\[
A_{13} = \text{span}\{\overline{h}_i\}_{i=1}^9,
\]

where
\[
\begin{align*}
\overline{h}_1 &= \begin{pmatrix} h & 0 \\ 0 & h \end{pmatrix}, \quad \overline{h}_2 = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix}, \quad \overline{h}_3 = \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix}, \\
\overline{h}_4 &= \begin{pmatrix} 0 & \varepsilon h \\ \varepsilon h & 0 \end{pmatrix}, \quad \overline{h}_5 = \begin{pmatrix} 0 & \varepsilon e \\ \varepsilon e & 0 \end{pmatrix}, \quad \overline{h}_6 = \begin{pmatrix} 0 & \varepsilon f \\ \varepsilon f & 0 \end{pmatrix}, \\
\overline{h}_7 &= \begin{pmatrix} 0 & \varepsilon h \\ \varepsilon h & 0 \end{pmatrix}, \quad \overline{h}_8 = \begin{pmatrix} 0 & \varepsilon e \\ \varepsilon e & 0 \end{pmatrix}, \quad \overline{h}_9 = \begin{pmatrix} 0 & \varepsilon f \\ \varepsilon f & 0 \end{pmatrix},
\end{align*}
\]
\[ [\overline{h}_1, \overline{h}_2] = 2\overline{h}_2, \quad [\overline{h}_1, \overline{h}_3] = -2\overline{h}_3, \quad [\overline{h}_1, \overline{h}_5] = 2\overline{h}_5, \quad [\overline{h}_1, \overline{h}_6] = -2\overline{h}_6, \quad [\overline{h}_1, \overline{h}_8] = 2\overline{h}_8, \]
\[ [\overline{h}_1, \overline{h}_9] = -2\overline{h}_9, \quad [\overline{h}_2, \overline{h}_3] = \overline{h}_1, \quad [\overline{h}_2, \overline{h}_4] = -2\overline{h}_5, \quad [\overline{h}_2, \overline{h}_6] = \overline{h}_4, \quad [\overline{h}_2, \overline{h}_7] = -2\overline{h}_8, \]
\[ [\overline{h}_3, \overline{h}_4] = 2\overline{h}_4, \quad [\overline{h}_3, \overline{h}_5] = -\overline{h}_4, \quad [\overline{h}_3, \overline{h}_7] = 2\overline{h}_9, \quad [\overline{h}_3, \overline{h}_8] = -\overline{h}_7, \]
\[ [\overline{h}_4, \overline{h}_5] = 2\overline{h}_8, \quad [\overline{h}_4, \overline{h}_6] = -2\overline{h}_9, \quad [\overline{h}_4, \overline{h}_9] = 2\varepsilon\overline{h}_2, \quad [\overline{h}_4, \overline{h}_9] = -2\varepsilon\overline{h}_3, \quad [\overline{h}_5, \overline{h}_6] = \overline{h}_7, \]
\[ [\overline{h}_5, \overline{h}_7] = -2\varepsilon\overline{h}_2, \quad [\overline{h}_5, \overline{h}_9] = \varepsilon\overline{h}_1, \quad [\overline{h}_7, \overline{h}_8] = 2\varepsilon\overline{h}_5, \quad [\overline{h}_7, \overline{h}_9] = -2\varepsilon\overline{h}_6, \quad [\overline{h}_8, \overline{h}_9] = \varepsilon\overline{h}_4, \]
\[ [\overline{h}_1, \overline{h}_4] = [\overline{h}_2, \overline{h}_5] = [\overline{h}_3, \overline{h}_6] = [\overline{h}_1, \overline{h}_7] = [\overline{h}_2, \overline{h}_8] = [\overline{h}_3, \overline{h}_9] = [\overline{h}_4, \overline{h}_7] = [\overline{h}_5, \overline{h}_8] = [\overline{h}_6, \overline{h}_9] = 0. \]

Let \( \overline{C}_1 = \text{span}\{\overline{h}_1, \overline{h}_2, \overline{h}_3\}, \overline{C}_2 = \text{span}\{\overline{h}_4, \overline{h}_5, \overline{h}_6\}, \overline{C}_3 = \text{span}\{\overline{h}_7, \overline{h}_8, \overline{h}_9\} \), then \( A_{13} = \overline{C}_1 \oplus \overline{C}_2 \oplus \overline{C}_3 \). It follows that one has
\[
[\overline{C}_1, \overline{C}_1] \subseteq \overline{C}_1, \quad [\overline{C}_1, \overline{C}_2] \subseteq \overline{C}_2, \quad [\overline{C}_1, \overline{C}_3] \subseteq \overline{C}_3, \\
[\overline{C}_2, \overline{C}_2] \subseteq \overline{C}_3, \quad [\overline{C}_2, \overline{C}_3] \subseteq \overline{C}_1, \quad [\overline{C}_3, \overline{C}_3] \subseteq \overline{C}_2.
\]

Introducing a \( N \times N \) square matrix of the following form:
\[
M(A_1, A_2, \cdots, A_N) = \begin{bmatrix}
A_1 & \varepsilon A_N & \varepsilon A_{N-1} & \cdots & \varepsilon A_4 & \varepsilon A_3 & \varepsilon A_2 \\
A_2 & A_1 & \varepsilon A_N & \cdots & \varepsilon A_5 & \varepsilon A_4 & \varepsilon A_3 \\
A_3 & A_2 & A_1 & \cdots & \varepsilon A_6 & \varepsilon A_5 & \varepsilon A_4 \\
& & & \ddots & & & \\
A_{N-2} & A_{N-3} & A_{N-4} & \cdots & A_1 & \varepsilon A_N & \varepsilon A_{N-1} \\
A_{N-1} & A_{N-2} & A_{N-3} & \cdots & A_2 & A_1 & \varepsilon A_N \\
A_N & A_{N-1} & A_{N-2} & \cdots & A_3 & A_2 & A_1
\end{bmatrix} =: \begin{bmatrix} A_1, A_2, \cdots, A_N \end{bmatrix}^T, \tag{5}
\]

where \( A_m \) (1 \( \leq \) \( m \) \( \leq \) \( N \)) represent \( N \) arbitrary square matrices of the same order. Here we use the vector \( \begin{bmatrix} A_1, A_2, \cdots, A_N \end{bmatrix}^T \) to represent the corresponding \( N \times N \) matrix for convenience.

Case 3
\[
A_{1N} = \text{span}\{\overline{h}_1\}_{i=1}^{3N}, \tag{6}
\]

with
\[
\overline{h}_1 = M(h, 0, \cdots, 0), \quad \overline{h}_2 = M(e, 0, \cdots, 0), \quad \overline{h}_3 = M(f, 0, \cdots, 0), \\
\overline{h}_4 = M(0, h, \cdots, 0), \quad \overline{h}_5 = M(0, e, \cdots, 0), \quad \overline{h}_6 = M(0, f, \cdots, 0), \\
\cdots
\]
\[
\overline{h}_{3N-2} = M(0, 0, \cdots, h), \quad \overline{h}_{3N-1} = M(0, 0, \cdots, e), \quad \overline{h}_{3N} = M(0, 0, \cdots, f), \quad N = 1, 2, \cdots,
\]
\[
[\overline{h}_3k, -2, \overline{h}_3k, -1] = [\overline{h}_3k, -1, \overline{h}_3k, 0] = 0, \quad i, k = 1, 2, \cdots, N,
\]
\[
[\overline{h}_3k, -2, \overline{h}_3k, -1] = \begin{cases}
2\overline{h}_3(k+i-1), & 1 \leq k \leq N - i + 1, \\
2\varepsilon\overline{h}_3(k+i-1-N), & N - i + 2 \leq k \leq N,
\end{cases}
\]
\[
[\overline{h}_3k, -2, \overline{h}_3k] = \begin{cases}
-2\overline{h}_3(k+i-1), & 1 \leq k \leq N - i + 1, \\
-2\varepsilon\overline{h}_3(k+i-1-N), & N - i + 2 \leq k \leq N,
\end{cases}
\]
\[
[\overline{h}_3k, -1, \overline{h}_3k] = \begin{cases}
\overline{h}_3(k+i-2), & 1 \leq k \leq N - i + 1, \\
\varepsilon\overline{h}_3(k+i-1-N), & N - i + 2 \leq k \leq N,
\end{cases}
\]
where $M$ is $N \times N$ matrix given by (5) and $h, e, f$ are the second-order matrices given by (1), Let $\tilde{G}_k = \text{span} \{ \tilde{h}_{3k-2}, \tilde{h}_{3k-1}, \tilde{h}_{3k} \}$, $k = 1, 2, \cdots, N$, then $A_{1N} = \tilde{G}_1 \bigoplus \tilde{G}_2 \bigoplus \cdots \bigoplus \tilde{G}_N$. Thus, we obtain

$$[\tilde{G}_i, \tilde{G}_j] \subseteq \tilde{G}_{i+j-1-\delta N}, \quad \delta = \begin{cases} 0, & 2 \leq i + j \leq N + 1, \\ 1, & N + 2 \leq i + j \leq 2N, \end{cases} \quad i, j = 1, 2, \cdots, N.$$ 

Similarly, we introduce the following extended Lie algebras related to the subalgebra (2):

**Case 4**

$$A_{22} = \text{span} \{ \tilde{e}_i \}_{i=1}^{6} \quad \text{(7)}$$

where

$$\tilde{e}_1 = \begin{pmatrix} \tilde{h} & 0 \\ 0 & \tilde{h} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \tilde{e}_2 = \begin{pmatrix} \tilde{e} & 0 \\ 0 & \tilde{e} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\tilde{e}_3 = \begin{pmatrix} \tilde{f} & 0 \\ 0 & \tilde{f} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \tilde{e}_4 = \begin{pmatrix} \tilde{e} \tilde{f} & 0 \\ 0 & \tilde{e} \tilde{f} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\tilde{e}_5 = \begin{pmatrix} \varepsilon \tilde{e} & 0 \\ \tilde{e} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{e}_6 = \begin{pmatrix} \varepsilon \tilde{f} & 0 \\ \tilde{f} & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & -\varepsilon \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$[\tilde{e}_1, \tilde{e}_2] = \tilde{e}_3, \quad [\tilde{e}_1, \tilde{e}_3] = \tilde{e}_2, \quad [\tilde{e}_1, \tilde{e}_4] = 0, \quad [\tilde{e}_1, \tilde{e}_5] = \tilde{e}_6, \quad [\tilde{e}_1, \tilde{e}_6] = -\tilde{e}_5,$$

$$[\tilde{e}_2, \tilde{e}_3] = -\tilde{e}_1, \quad [\tilde{e}_2, \tilde{e}_4] = -\tilde{e}_6, \quad [\tilde{e}_2, \tilde{e}_5] = 0, \quad [\tilde{e}_2, \tilde{e}_6] = -\tilde{e}_4, \quad [\tilde{e}_3, \tilde{e}_4] = -\tilde{e}_5, \quad [\tilde{e}_3, \tilde{e}_5] = \tilde{e}_4, \quad [\tilde{e}_3, \tilde{e}_6] = 0, \quad [\tilde{e}_4, \tilde{e}_5] = \varepsilon \tilde{e}_3, \quad [\tilde{e}_4, \tilde{e}_6] = \varepsilon \tilde{e}_2, \quad [\tilde{e}_5, \tilde{e}_6] = \varepsilon \tilde{e}_1.$$ 

Let $\bar{G}_1 = \text{span} \{ \tilde{e}_1, \tilde{e}_2, \tilde{e}_3 \}, \quad \bar{G}_2 = \text{span} \{ \tilde{e}_4, \tilde{e}_5, \tilde{e}_6 \}$, then $A_{22} = \bar{G}_1 \bigoplus \bar{G}_2$. It follows that one has

$$[\bar{G}_1, \bar{G}_1] \subseteq \bar{G}_1, \quad [\bar{G}_1, \bar{G}_2] \subseteq \bar{G}_2, \quad [\bar{G}_2, \bar{G}_2] \subseteq \bar{G}_1.$$ 

**Case 5**

$$A_{2N} = \text{span} \{ \tilde{e}_i \}_{i=1}^{3N} \quad \text{(8)}$$

with

$$\tilde{e}_1 = M(\tilde{h}, 0, \cdots, 0), \quad \tilde{e}_2 = M(\tilde{e}, 0, \cdots, 0), \quad \tilde{e}_3 = M(\tilde{f}, 0, \cdots, 0),$$

$$\tilde{e}_4 = M(0, \tilde{h}, \cdots, 0), \quad \tilde{e}_5 = M(0, \tilde{e}, \cdots, 0), \quad \tilde{e}_6 = M(0, \tilde{f}, \cdots, 0),$$

$$\cdots$$

$$\tilde{e}_{3N-2} = M(0, 0, \cdots, \tilde{h}), \quad \tilde{e}_{3N-1} = M(0, 0, \cdots, \tilde{e}), \quad \tilde{e}_{3N} = M(0, 0, \cdots, \tilde{f}), \quad N = 1, 2, \cdots,$$
\[ [\tilde{e}_{3i-2}, \tilde{e}_{3k-2}] = [\tilde{e}_{3i-1}, \tilde{e}_{3k-1}] = [\tilde{e}_{3i}, \tilde{e}_{3k}] = 0, \quad i, k = 1, 2, \ldots, N, \]

\[ [\tilde{e}_{3i-2}, \tilde{e}_{3k-1}] = \begin{cases} \tilde{e}_{3(k+i-1)}, & 1 \leq k \leq N - i + 1, \\ \tilde{e}_{3(k+i-1-N)}, & N - i + 2 \leq k \leq N, \end{cases} \]

\[ [\tilde{e}_{3i-2}, \tilde{e}_{3k}] = \begin{cases} \tilde{e}_{3(k+i-1)-1}, & 1 \leq k \leq N - i + 1, \\ \tilde{e}_{3(k+i-1-N)-1}, & N - i + 2 \leq k \leq N, \end{cases} \]

\[ [\tilde{e}_{3i-1}, \tilde{e}_{3k}] = \begin{cases} -\tilde{e}_{3(k+i)-2}, & 1 \leq k \leq N - i + 1, \\ -\tilde{e}_{3(k+i-1-N)-2}, & N - i + 2 \leq k \leq N, \end{cases} \]

where \( M \) is \( N \times N \) matrix given by (5) and \( \overline{h}, \overline{r}, \overline{f} \) are given by (2).

Let \( \overline{g}_k = \text{span}\{\tilde{e}_{3k-2}, \tilde{e}_{3k-1}, \tilde{e}_{3k}\} \), \( k = 1, 2, \ldots, N \), then \( A_{2N} = \overline{g}_1 \oplus \overline{g}_2 \oplus \cdots \oplus \overline{g}_N \). Thus, we obtain

\[ [\overline{g}_i, \overline{g}_j] \subseteq \overline{g}_{i+j-1-\delta N}, \quad \delta = \begin{cases} 0, & 2 \leq i + j \leq N + 1, \\ 1, & N + 2 \leq i + j \leq 2N, \end{cases} \quad i, j = 1, 2, \ldots, N. \]

We believe that other Lie algebras could also be extended to the similar higher-dimensional forms. For example, the Lie algebra \( \text{so}(3) \) admits a set of bases \( f_1, f_2, f_3 \),

\[ f_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (9) \]

whose commutators are

\[ [f_1, f_2] = f_3, \quad [f_1, f_3] = -f_2, \quad [f_2, f_3] = f_1. \]

The Lie algebra \( \text{so}(3) \) can be extended into:

**Case 6**

\[ A_{32} = \text{span}\{\overline{f}_i\}_{i=1}^6, \quad (10) \]

where

\[ \overline{f}_1 = \begin{pmatrix} f_1 \\ 0 \end{pmatrix}, \quad \overline{f}_2 = \begin{pmatrix} f_2 \\ 0 \end{pmatrix}, \quad \overline{f}_3 = \begin{pmatrix} f_3 \\ 0 \end{pmatrix}, \quad \overline{f}_4 = \begin{pmatrix} 0 & \varepsilon f_1 \\ f_1 & 0 \end{pmatrix}, \quad \overline{f}_5 = \begin{pmatrix} 0 & \varepsilon f_2 \\ f_2 & 0 \end{pmatrix}, \quad \overline{f}_6 = \begin{pmatrix} 0 & \varepsilon f_3 \\ f_3 & 0 \end{pmatrix}, \]

\[ [\overline{f}_1, \overline{f}_2] = \overline{f}_3, \quad [\overline{f}_1, \overline{f}_3] = -\overline{f}_2, \quad [\overline{f}_1, \overline{f}_4] = 0, \quad [\overline{f}_1, \overline{f}_5] = \overline{f}_6, \quad [\overline{f}_1, \overline{f}_6] = -\overline{f}_5, \]

\[ [\overline{f}_2, \overline{f}_3] = \overline{f}_1, \quad [\overline{f}_2, \overline{f}_4] = -\overline{f}_6, \quad [\overline{f}_2, \overline{f}_5] = 0, \quad [\overline{f}_2, \overline{f}_6] = \overline{f}_4, \quad [\overline{f}_3, \overline{f}_4] = \overline{f}_5, \]

**Case 7**

\[ A_{3N} = \text{span}\{\overline{f}_i\}_{i=1}^{3N}, \quad (11) \]
with
\[ f_1 = M(f_1, 0, \ldots, 0), \quad f_2 = M(f_2, 0, \ldots, 0), \quad f_3 = M(f_3, 0, \ldots, 0), \]
\[ f_4 = M(0, f_1, \ldots, 0), \quad f_5 = M(0, f_2, \ldots, 0), \quad f_6 = M(0, f_3, \ldots, 0), \]
\[ \ldots \]
\[ f_{3N-2} = M(0, 0, \ldots, f_1), \quad f_{3N-1} = M(0, 0, \ldots, f_2), \quad f_{3N} = M(0, 0, \ldots, f_3), \quad N = 1, 2, \ldots, \]
where \( M \) is given by (5) and \( f_1, f_2, f_3 \) are given by (9).

These expanded higher-dimensional Lie algebras can be applied to different spectral problems, and then generate multi-component integrable hierarchies. Below we consider a specific application, that is the KdV spectral problem.

### 3. A nonisospectral KdV hierarchy

For the first set of basis (1), we consider the corresponding loop algebra\[ \widetilde{A}_{11} = \text{span}\{h(n), e(n), f(n)\} \]
which satisfies the commutators
\[ [h(n), e(m)] = 2e(m+n), \quad [h(n), f(m)] = -2f(m+n), \quad [e(n), f(m)] = h(m+n), \quad m, n \in \mathbb{Z}, \]
where
\[ h(n) = h\lambda^n, \quad e(n) = e\lambda^n, \quad f(n) = f\lambda^n. \]

Based on the KdV spatial spectral problem (see [4]), we introduce the following nonisospectral problem based on \( \widetilde{A}_{11} \)
\[ \begin{cases} 
\psi_x = M\psi, \quad M = \frac{1}{4}e(1) - uc(0) + f(0) = \begin{pmatrix} 0 & -u + \frac{1}{4} \\ 1 & 0 \end{pmatrix}, \\
\psi_t = N\psi, \quad N = ah(0) + be(0) + cf(0) = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \\
\lambda_t = \sum_{m \geq 0} k_m(t)\lambda^{-m}, 
\end{cases} \quad (12) \]
where \( a = \sum_{m \geq 0} a_m\lambda^{-m}, \quad b = \sum_{m \geq 0} b_m\lambda^{-m}, \quad c = \sum_{m \geq 0} c_m\lambda^{-m}. \)

A direct calculation gives \( \frac{\partial M}{\partial \lambda} \lambda_t = \frac{1}{4} \sum_{m \geq 0} k_m(t)e(-m). \) By solving the following nonisospectral stationary zero-curvature equation of (12)
\[ N_x = \frac{\partial M}{\partial \lambda} \lambda_t + [M, N], \quad (13) \]
we obtain the recursion equations:
\[ \begin{cases} 
a_{mx} = -uc_m + \frac{1}{4}c_{m+1} - b_m, \\
b_{mx} = \frac{1}{4}k_m(t) - \frac{1}{4}a_{m+1} + 2ua_m, \\
c_{mx} = 2a_m, \end{cases} \quad (14) \]
which have the equivalent forms

\[
\begin{align*}
    a_m &= \frac{1}{2} c_{mx}, \\
    b_m &= -\frac{1}{4} c_{m+1} + \partial^{-1} u c_{mx} + \frac{1}{4} k_m(t) x, \\
    c_{m+1} &= (\partial^2 + 2u + 2\partial^{-1} u \partial)c_m + \frac{1}{2} k_m(t) x.
\end{align*}
\]

(15)

To the recursion equations (15), we take the initial values \(a_0 = 0\), then

\[
\begin{align*}
    a_0 &= 0, \quad c_0 = \alpha, \quad c_1 = 2\alpha u + \frac{1}{2} k_0(t) x, \quad b_0 = -\frac{1}{2} \alpha u + \frac{1}{8} k_0(t) x, \\
    a_1 &= \alpha u_x + \frac{1}{4} k_0(t), \quad c_2 = 2\alpha u_{xx} + 6\alpha u^2 + k_0(t)(xu + \partial^{-1} u) + \frac{1}{2} k_1(t) x, \\
    b_1 &= -\frac{\alpha}{2} u_{xx} - \frac{\alpha}{2} u^2 - \frac{1}{4} k_0(t)(xu + \partial^{-1} u) - \frac{1}{8} k_1(t) x + \frac{1}{4} k_2(t) x, \\
    \ldots
\end{align*}
\]

Denoting that

\[
N^{(n)} = N^{(n)}_+ + N^{(n)}_- = N\lambda^n, \quad N^{(n)}_+ = \sum_{i=0}^{n} (a_i h(n - i) + b_i e(n - i) + c_i f(n - i)),
\]

\[
deg h(n) = \deg(h\lambda^n) = n, \quad \lambda^{(n)}_+ = \lambda^{(n)}_t + \lambda^{(n)}_t = \sum_{i=0}^{n} k_i(t) \lambda^{n-i} + \sum_{i=n}^{\infty} k_i(t) \lambda^{n-i}.
\]

(13) can be broken down into

\[
-N^{(n)}_{+x} + \frac{\partial M}{\partial \lambda} \lambda^{(n)}_t + [M, N^{(n)}_+] = N^{(n)}_{-x} - \frac{\partial M}{\partial \lambda} \lambda^{(n)}_t - [M, N^{(n)}_-].
\]

(17)

It follows that the gradations of the left-hand side of (17) are obtained as follows:

\[
\deg N^{(n)}_+ =: (0, 0, 0) \geq 0, \quad \deg \left( \frac{\partial M}{\partial \lambda} \lambda^{(n)}_t \right) =: (0, 0, 0) \geq 0, \quad \deg([M, N^{(n)}_+]) =: (0, 1, 0; 0, 0, 0) \geq 0,
\]

which indicate the minimum gradation of the left-hand side of (17) is zero. Similarly, the maximum gradation of the right-hand side of (17) is also 0. Thus, we obtain the following equation by taking these terms which have the gradations 0:

\[
-N^{(n)}_{+x} + \frac{\partial M}{\partial \lambda} \lambda^{(n)}_t + [M, N^{(n)}_+] = \frac{1}{2} a_{n+1} e(0) - \frac{1}{4} c_{n+1} h(0).
\]

(18)

Thus, we take the modified term \(\Delta_n = -\frac{1}{4} c_{n+1} e(0)\) so that for \(N^{(n)} = N^{(n)}_+ + \Delta_n\). By solving the nonisospectral zero curvature equation

\[
\frac{\partial M}{\partial u} u_t + \frac{\partial M}{\partial \lambda} \lambda^{(n)}_t - N^{(n)}_+ + [M, N^{(n)}] = 0,
\]

(19)

we obtain the nonisospectral KdV hierarchy as follows:

\[
u_{t_n} = \frac{1}{2} c_{n+1,x} = \frac{1}{2} \left[ (\partial^2 + 2u + 2\partial^{-1} u \partial)c_n + \frac{1}{2} k_n(t) x \right]_x = \frac{1}{2} \partial L c_n + \frac{1}{4} k_n(t),
\]

where \(L = \partial^2 + 2u + 2\partial^{-1} u \partial\). The first two equations in the above hierarchy of soliton equations are

\[
u_{t_0} = c_0 + \frac{1}{4} k_0(t),
\]

(21)
The equation (22) becomes the famous KdV equation when $\alpha = 1$, $k_0(t) = k_1(t) = 0$. Actually, the nonisospectral KdV hierarchy (20) can be reduced to the isospectral KdV hierarchy by taking $k_m(t) = 0$, $m = 0, 1, 2, \cdots$.

3.1. Bi-Hamiltonian structures

In this section, we discuss the Hamiltonian structure of the hierarchy (20) by means of the trace identity (see [13]). Denoting the trace of the square matrices A and B by $<A, B> = \text{tr}(AB)$. From the spectral problem (12), we can directly calculate

$$<N, \frac{\partial M}{\partial u}> = -2c, \quad <N, \frac{\partial M}{\partial \lambda}> = \frac{1}{4}c.$$  

It follows that the trace identity

$$\frac{\delta}{\delta u}(<N, \frac{\partial M}{\partial \lambda}>) = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma}(<N, \frac{\partial M}{\partial u}>)$$

admits that

$$\frac{\delta c_m}{\delta u} = -4(\gamma - m + 1)c_{m-1}, \quad m \geq 1.$$  

One can find that $\gamma = -\frac{1}{2}$ via substituting $m = 1$ into the above equation by means of (16). Therefore, we obtain

$$c_m = \frac{\delta H_{m+1}}{\delta u}, \quad H_{m+1} = \frac{c_{m+1}}{2(m+1)}, \quad m \geq 0. \quad (23)$$

Furthermore, we obtain the bi-Hamiltonian structures of the hierarchy (20) as follows:

$$u_t = K_n = \frac{1}{2} \frac{\partial c_{n+1}}{\partial u} =: Jc_{n+1} = J \frac{\delta H_{n+2}}{\delta u}$$

$$= J(L \frac{\delta H_{n+1}}{\delta u} + \frac{1}{2} k_n(t) x) =: M \frac{\delta H_{n+1}}{\delta u} + \frac{1}{4} k_n(t)$$

$$= J(L^{n+1} c_0 + \frac{1}{2} \sum_{m=0}^{n} k_m(t) L^{n-m} x) = \Phi^{n+1} Jc_0 + \frac{1}{2} \sum_{m=0}^{n} k_m(t) \Phi^{n-m} Jx,$$

where $J$ and $M$ are two Hamiltonian operators and $\Phi$ is a hereditary symmetry operator as follows:

$$J = \frac{1}{2} \partial, \quad M = JL = \frac{1}{2}(\partial^3 + 2\partial u + 2u \partial), \quad \Phi = JLJ^{-1} = \partial^2 + 2ux \partial^{-1} + 4u. \quad (25)$$

We can conclude that the integrable hierarchy (20) is integrable in the sense of Liouville (see [48]), and we obtain the Abelian algebra of symmetries

$$[K_i, K_j] = K'_i(u)[K_j] - K'_j(u)[K_i] = 0, \quad i, j \geq 0, \quad (26)$$

and the Abelian algebras of conserved functionals,

$$\{H_i, H_j\}_J = \int \left( \frac{\delta H_i}{\delta u} \right)^T J \frac{\delta H_j}{\delta u} dx = 0, \quad \{H_i, H_j\}_M = \int \left( \frac{\delta H_i}{\delta u} \right)^T J \frac{\delta H_j}{\delta u} dx = 0, \quad i, j \geq 0. \quad (27)$$
4. A coupled nonisospectral KdV hierarchy

For the Lie algebra \( A_{12} \) (see eq. 30), we consider the corresponding loop algebra

\[ \tilde{A}_{12} = \text{span}\{h_1(n), h_2(n), h_3(n), h_4(n), h_5(n), h_6(n)\} \]

where

\[ h_i(n) = h_i^n, \quad i = 1, 2, \ldots, 6. \]

Introducing the following nonisospectral problem based on \( \tilde{A}_{11} \)

\[
\begin{aligned}
\psi_x &= U \psi, \quad U = \frac{1}{4} h_2(t) - u_1 h_2(0) + h_3(0) - u_2 h_5(0) = \begin{bmatrix} U_1 & \varepsilon U_2 \\ U_2 & U_1 \end{bmatrix}, \\
\psi_t &= W \psi, \quad W = a h_1(0) + b h_2(0) + c h_3(0) + e h_4(0) + f h_5(0) + g h_6(0) = \begin{bmatrix} W_1 & \varepsilon W_2 \\ W_2 & W_1 \end{bmatrix}, \\
\lambda &= \sum_{m \geq 0} k_m(t) \lambda^{-m},
\end{aligned}
\]

where

\[
U_1 = \begin{pmatrix} 0 & -u_1 + \frac{4}{3} \\ 1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -u_2 \\ 1 & 0 \end{pmatrix}, \quad W_1 = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad W_2 = \begin{pmatrix} e & f \\ g & -e \end{pmatrix}
\]

\[ a = \sum_{m \geq 0} a_m \lambda^{-m}, \quad b = \sum_{m \geq 0} b_m \lambda^{-m}, \quad c = \sum_{m \geq 0} c_m \lambda^{-m}, \quad e = \sum_{m \geq 0} e_m \lambda^{-m}, \quad f = \sum_{m \geq 0} f_m \lambda^{-m}, \quad g = \sum_{m \geq 0} g_m \lambda^{-m}. \]

The stationary zero-curvature equation

\[ W_x = \frac{\partial U}{\partial \lambda} \lambda + [U, W], \]

gives rise to the recursion equations:

\[
\begin{aligned}
a_{mx} &= -u_1 c_m + \frac{1}{4} c_{m+1} - b_m - \varepsilon u_2 g_m, \\
b_{mx} &= \frac{1}{4} k_m(t) - \frac{1}{2} a_{m+1} + 2 u_1 a_m + 2 \varepsilon u_2 e_m, \\
c_{mx} &= 2 a_m, \\
e_{mx} &= -u_1 g_m + \frac{1}{4} g_{m+1} - f_m - u_2 c_m, \\
f_{mx} &= -\frac{1}{2} e_{m+1} + 2 u_1 e_m + 2 u_2 a_m, \\
g_{mx} &= 2 e_m.
\end{aligned}
\]

which have the equivalent forms

\[
\begin{aligned}
a_m &= \frac{1}{2} c_{mx}, \\
b_m &= -\frac{1}{4} c_{m+1} + \partial^{-1}(u_1 c_{mx} + \varepsilon u_2 g_{mx}) + \frac{1}{4} k_m(t)x, \\
c_{m+1} &= (\partial^2 + 2 u_1 + 2 \partial^{-1} u_1 \partial)c_m + (2 \varepsilon u_2 + 2 \varepsilon \partial^{-1} u_2 \partial)g_m + \frac{1}{2} k_m(t)x, \\
e_m &= \frac{1}{2} g_{mx}, \\
f_m &= -\frac{1}{2} g_{m+1} + \partial^{-1}(u_1 g_{mx} + u_2 c_{mx}), \\
g_{m+1} &= (2 u_2 + 2 \partial^{-1} u_2 \partial)c_m + (\partial^2 + 2 u_1 + 2 \partial^{-1} u_1 \partial)g_m.
\end{aligned}
\]
To the recursion equations \(31\), we take the initial values \(a_0 = e_0 = 0\), it follows that one has

\[
\begin{align*}
    a_0 &= e_0 = 0, \quad g_0 = \alpha_2, \quad g_1 = 2\alpha_1u_1 + 2\alpha_2u_2, \\
    b_0 &= -\frac{1}{2}\alpha_1u_1 - \frac{\varepsilon}{2}\alpha_2u_2 + \frac{1}{8}k_0(t)x, \quad f_0 = -\frac{1}{2}\alpha_2u_1 - \frac{1}{2}\alpha_1u_2, \\
    a_1 &= \alpha_1u_{1x} + \varepsilon\alpha_2u_{2x} + \frac{1}{4}k_0(t), \quad e_1 = \alpha_2u_{1x} + \alpha_1u_{2x}, \\
    c_2 &= 2\alpha_1u_{1xx} + 2\alpha_2u_{2xx} + 6\alpha_1u_1^2 + 6\varepsilon\alpha_1u_2^2 + 12\varepsilon\alpha_2u_1u_2 + k_0(t)xu_1 + k_0(t)\partial^{-1}u_1 + \frac{1}{2}k_1(t)x, \\
    g_2 &= 2\alpha_2u_{1xx} + 2\alpha_1u_{2xx} + 6\alpha_2u_1^2 + 6\varepsilon\alpha_2u_2^2 + 12\alpha_1u_1u_2 + k_0(t)xu_2 + k_0(t)\partial^{-1}u_2, \\
    \ldots
\end{align*}
\]

Denoting that

\[
W^{(n)} = W\lambda^n, \quad W_+^{(n)} = \sum_{i=0}^{n} (a_ih_1(n-i) + b_ih_2(n-i) + c_ih_3(n-i) + e_ih_4(n-i) + f_ih_5(n-i) + g_ih_6(n-i)),
\]

\(29\) can be broken down into

\[
-W_+^{(n)} + \frac{\partial U}{\partial \lambda} \lambda_+^{(n)} + [U, W_+^{(n)}] = W_-^{(n)} - \frac{\partial U}{\partial \lambda} \lambda_-^{(n)} - [U, W_-^{(n)}].
\]  \(33\)

The minimum gradation of the left-hand side and the maximum gradation of the right-hand side of \(35\) are both zero. It follows that one has:

\[
-W_+^{(n)} + \frac{\partial U}{\partial \lambda} \lambda_+^{(n)} + [U, W_+^{(n)}] = \frac{1}{2}a_{n+1}c(0) - \frac{1}{4}c_{n+1}h(0).
\]  \(34\)

Thus, we take the modified term \(\Delta_n = -\frac{1}{4}c_{n+1}h_2(0) - \frac{3}{4}g_{n+1}h_5(0)\) so that for \(V^{(n)} = W_+^{(n)} + \Delta_n\). The nonisospectral zero curvature equation

\[
\frac{\partial U}{\partial t_{t_1}} + \frac{\partial U}{\partial \lambda} \lambda^{(n)} + V^{(n)}(n) + [U, V^{(n)}] = 0,
\]  \(35\)

leads to the coupled nonisospectral KdV hierarchy as follows:

\[
u_{t_n} = \mathcal{K}_n = \begin{pmatrix} \frac{1}{2}c_{n+1,x} \\ \frac{1}{2}g_{n+1,x} \end{pmatrix} = J \begin{pmatrix} c_{n+1} \\ g_{n+1} \end{pmatrix} = J[\mathcal{L} \begin{pmatrix} c_n \\ g_n \end{pmatrix} + \frac{1}{2}k_n(t) \begin{pmatrix} x \\ 0 \end{pmatrix}],
\]  \(36\)

where \(J\) is given by \(25\) and

\[
\mathcal{L} = \begin{pmatrix} \partial^2 + 4u_1 - 2\partial^{-1}u_{1x} & \varepsilon(4u_2 - 2\partial^{-1}u_{2x}) \\ 4u_2 - 2\partial^{-1}u_{2x} & \partial^2 + 4u_1 - 2\partial^{-1}u_{1x} \end{pmatrix} = \begin{pmatrix} \mathcal{L}_{11} & \varepsilon\mathcal{L}_{21} \\ \varepsilon\mathcal{L}_{12} & \mathcal{L}_{22} \end{pmatrix},
\]

which is determined by the recursion equations \(31\). The first two examples in the above hierarchy of soliton equations are

\[
u_{t_0} = \frac{1}{2} \begin{pmatrix} c_{1,x} \\ g_{1,x} \end{pmatrix} = \alpha_1 \begin{pmatrix} u_{1x} \\ u_{2x} \end{pmatrix} + \alpha_2 \begin{pmatrix} \varepsilon u_{2x} \\ u_{1x} \end{pmatrix} + \frac{1}{4}k_0(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix},
\]  \(37\)
Substituting from (28), the integrable system (39) has the following Lax pairs:

\[
\begin{align*}
\lambda & = \alpha_1 \begin{pmatrix}
(6u_1u_{1x} + 6\varepsilon u_2u_{2x}) \\
(6u_1u_{2x} + 6u_2u_{1x} + 6\varepsilon u_2u_{2x})
\end{pmatrix} + \alpha_2 \begin{pmatrix}
(6\varepsilon u_{2xx} + 6\varepsilon u_2u_{1x}) \\
(u_{1xx} + 6u_1u_{1x} + 6\varepsilon u_2u_{2x})
\end{pmatrix}, \\
\psi & = \begin{pmatrix}
2u_1 + xu_{1x} \\
2u_2 + xu_{2x}
\end{pmatrix} + \frac{1}{2}k_0(t) \begin{pmatrix} 2 \alpha_1 \alpha_2 \\ 1 \end{pmatrix}.
\end{align*}
\]

which is a coupled nonisospectral KdV equation.

Taking \(\alpha_1 = 1, \alpha_2 = 0, k_0(t) = k_1(t) = 0\), and (38) becomes the Frobenius KdV equation

\[
\begin{align*}
&\begin{cases}
\frac{\partial u_1}{\partial t} = u_{1xx} + 6u_1u_{1x} + 6\varepsilon u_2u_{2x}, \\
\frac{\partial u_2}{\partial t} = u_{2xx} + 6u_1u_{2x} + 6u_2u_{1x},
\end{cases}
\end{align*}
\]

which is derived by the Frobenius-Virasoro algebra (see [26, 27]). (39) can be reduced to the coupled KdV equation (see [49, 50]) and the complexification of the KdV equation by taking \(\varepsilon = 0\) and \(\varepsilon = -1\) respectively.

From [28], the integrable system (39) has the following Lax pairs:

\[
\begin{align*}
\psi_x &= U\psi = \begin{bmatrix}
U_1 & \varepsilon U_2 \\
U_2 & U_1
\end{bmatrix} \psi, \quad U_1 = \begin{pmatrix} 0 & -u_1 + \frac{1}{4} \\ -1 & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & -u_2 \\ 0 & 0 \end{pmatrix}, \\
\psi_t &= V\psi = \begin{bmatrix}
V_1 & \varepsilon V_2 \\
V_2 & V_1
\end{bmatrix} \psi, \quad V_1 = \begin{pmatrix} u_{1x} & u_1 - 2u_1^2 - \varepsilon u_2 - \frac{1}{2}\lambda u_1 + \frac{1}{4}\lambda^2 \\ 2u_1 + \lambda & -u_{1x} \end{pmatrix}, \\
& \quad V_2 = \begin{pmatrix} u_{2x} & u_2 - 4u_1u_2 - \frac{1}{2}\lambda u_2 \\ 2u_2 & -u_{2x} \end{pmatrix}.
\end{align*}
\]

4.1. Bi-Hamiltonian structures

In order to establish the bi-Hamiltonian structures of the coupled nonisospectral KdV hierarchy [30], we use the following two sets of component-trace identity (see [30]):

\[
\begin{align*}
&\left( \frac{\partial}{\partial u} < W_1, \frac{\partial U_1}{\partial x} > + < W_2, \frac{\partial U_1}{\partial u} > \right) = \lambda^{-\gamma} \frac{\partial}{\partial x} \lambda^{\gamma} \left( < W_1, \frac{\partial U_1}{\partial u} > + < W_2, \frac{\partial U_1}{\partial x} > \right), \\
&\left( \frac{\partial}{\partial u} < W_1, \frac{\partial U_1}{\partial x} > + \varepsilon < W_2, \frac{\partial U_1}{\partial x} > \right) = \lambda^{-\gamma} \frac{\partial}{\partial x} \lambda^{\gamma} \left( < W_1, \frac{\partial U_1}{\partial u} > + \varepsilon < W_2, \frac{\partial U_1}{\partial x} > \right).
\end{align*}
\]

Substituting

\[
\begin{align*}
&W_1, \frac{\partial U_1}{\partial u} > + < W_2, \frac{\partial U_1}{\partial u} > = \begin{pmatrix} -g \\ -c \end{pmatrix}, \quad < W_1, \frac{\partial U_1}{\partial u} > + \varepsilon < W_2, \frac{\partial U_1}{\partial u} > = \begin{pmatrix} -c \end{pmatrix}, \\
&W_1, \frac{\partial U_2}{\partial \lambda} > + < W_2, \frac{\partial U_2}{\partial \lambda} > = \begin{pmatrix} 1 \end{pmatrix} g, \quad < W_1, \frac{\partial U_1}{\partial \lambda} > + \varepsilon < W_2, \frac{\partial U_2}{\partial \lambda} > = \frac{1}{4} c,
\end{align*}
\]

into the above component-trace identity and comparing powers of \(\lambda\), we obtain

\[
\begin{align*}
\frac{\delta c_m}{\delta u} &= -4(\gamma - m + 1) g_m, \quad \frac{\delta g_m}{\delta u} = -4(\gamma - m + 1) c_m, \quad m \geq 1.
\end{align*}
\]

where \(W_1, W_2, U_1, U_2\) is given by [28]. One can find that \(\gamma = -\frac{1}{2}\) via substituting \(m = 1\) into [31] by means of \([16]\). Therefore, we obtain

\[
\begin{align*}
\left( \begin{array}{c}
c_m \\
g_m
\end{array} \right) &= \frac{\delta H_{1,m+1}}{\delta u}, \quad \left( \begin{array}{c}
g_m \\
c_m
\end{array} \right) = \frac{\delta H_{2,m+1}}{\delta u}, \quad m \geq 0.
\end{align*}
\]

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where $H_{1,m+1} = \frac{\delta_{m+1}}{2m+1}$ and $H_{2,m+1} = \frac{\delta_{m+1}}{2m+1}$ are the Hamiltonian functionals. From (42), one can find that the Hamiltonian structure of the hierarchy (36) consists of two component and let us discuss the two components separately.

For the first component, we have:

$$ u_{1n} = \left( \frac{1}{2} c_{n+1,x} \right) = \frac{1}{2} \left( \begin{array}{ccc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{array} \right) \left( \begin{array}{c} c_{n+1} \\ \varepsilon g_{n+1} \end{array} \right) = J_1 \left( \begin{array}{c} c_{n+1} \\ \varepsilon g_{n+1} \end{array} \right) = J_1 \frac{\delta H_{1,n+2}}{\delta u}, $$

\begin{equation}
= J_1 \left( L^T \frac{\delta H_{1,n+1}}{\delta u} + k_n(t) \left( \begin{array}{c} 0 \\ \frac{\varepsilon}{\delta} \end{array} \right) \right) = \bar{M}_1 \frac{\delta H_{1,n+1}}{\delta u} + k_n(t) J_1 \left( \begin{array}{c} 0 \\ \frac{\varepsilon}{\delta} \end{array} \right), \quad n \geq 1,
\end{equation}

(43)

where $L^T$ is the device matrix of matrix $L$ in (36), $J_1$ and $\bar{M}_1$ are two Hamiltonian operators and $\bar{\Phi}_1$ is a hereditary symmetry operator as follows:

$$ J_1 = \frac{1}{2} \left( \begin{array}{ccc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{array} \right), \quad \bar{M}_1 = J_1 L^T = \frac{1}{2} \left( \begin{array}{ccc} \partial_1^3 + 2 \partial u_1 + 2 u_1 \partial_2 & 2 \partial u_2 + 2 u_2 \partial_1 \\ 2 \partial u_2 + 2 u_2 \partial_1 & \frac{\varepsilon}{\delta} (\partial_1^3 + 2 \partial u_1 + 2 u_1 \partial_1) \end{array} \right), $$

$$ \bar{\Phi}_1 = J_1 L^T J_1^{-1} = \left( \begin{array}{ccc} \partial_1^2 + 4 u_1 + 2 u_1 \partial_1 & \varepsilon (4 u_2 + 2 u_2 \partial_1) \\ 4 u_2 + 2 u_2 \partial_1 & \partial_1^2 + 4 u_1 + 2 u_1 \partial_1 \end{array} \right). $$

(44)

For the second component, one has:

$$ u_{2n} = \left( \frac{1}{2} c_{n+1,x} \right) = \frac{1}{2} \left( \begin{array}{ccc} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \end{array} \right) \left( \begin{array}{c} g_{n+1} \\ c_{n+1} \end{array} \right) = J_2 \left( \begin{array}{c} g_{n+1} \\ c_{n+1} \end{array} \right) = J_2 \frac{\delta H_{2,n+2}}{\delta u}, $$

\begin{equation}
= J_2 \left( L^T \frac{\delta H_{2,n+1}}{\delta u} + k_n(t) \left( \begin{array}{c} 0 \\ \frac{\varepsilon}{\delta} \end{array} \right) \right) = \bar{M}_2 \frac{\delta H_{2,n+1}}{\delta u} + k_n(t) J_2 \left( \begin{array}{c} 0 \\ \frac{\varepsilon}{\delta} \end{array} \right), \quad n \geq 1,
\end{equation}

(45)

where $J_2$ and $\bar{M}_2$ are two Hamiltonian operators and $\bar{\Phi}_2$ is a hereditary symmetry operator as follows:

$$ J_2 = \frac{1}{2} \left( \begin{array}{ccc} 0 & \partial_1 \\ \partial_1 & 0 \end{array} \right), \quad \bar{M}_2 = J_2 L^T = \frac{1}{2} \left( \begin{array}{ccc} \varepsilon (2 \partial u_2 + 2 u_2 \partial_1) & \partial_1^3 + 2 \partial u_1 + 2 u_1 \partial_2 \\ \partial_1^3 + 2 \partial u_1 + 2 u_1 \partial_2 & 2 \partial u_2 + 2 u_2 \partial_1 \end{array} \right), $$

$$ \bar{\Phi}_2 = J_2 L^T J_2^{-1} = \left( \begin{array}{ccc} \partial_1^2 + 4 u_1 + 2 u_1 \partial_1 & \varepsilon (4 u_2 + 2 u_2 \partial_1) \\ 4 u_2 + 2 u_2 \partial_1 & \partial_1^2 + 4 u_1 + 2 u_1 \partial_1 \end{array} \right) = \bar{\Phi}_1 =: \bar{\Phi}. $$

(46)

After comparing and analyzing the results of these two components, we obtain the bi-Hamiltonian structures...
of the coupled nonisospectral KdV hierarchy \[36\] as follows:

\[
\begin{align*}
    u_{t_n} &= J_1 \frac{\delta H_{1,n+2}}{\delta u} = M_1 \frac{\delta H_{1,n+2}}{\delta u} + k_n(t) J_1 \left( \frac{x}{2}, 0 \right) + \frac{c_1}{\varepsilon g_1} J_1 \left( c_1, 0 \right) + \frac{1}{2} \sum_{m=1}^{n} k_m(t) \Phi^{-m} J_1 \left( \frac{x}{2}, 0 \right), \\
    u_{t_n} &= J_2 \frac{\delta H_{2,n+2}}{\delta u} = M_2 \frac{\delta H_{2,n+2}}{\delta u} + k_n(t) J_2 \left( 0, \frac{x}{2} \right) + \frac{g_1}{c_1} J_2 \left( 0, \frac{x}{2} \right) + \frac{1}{2} \sum_{m=1}^{n} k_m(t) \Phi^{-m} J_2 \left( 0, \frac{x}{2} \right),
\end{align*}
\]

with the recursion operator

\[
\bar{\Phi} = \begin{pmatrix}
    \partial^2 + 4u_1 + 2u_{1x} \partial^{-1} & \varepsilon (4u_2 + 2u_{2x} \partial^{-1}) \\
    4u_2 + 2u_{2x} \partial^{-1} & \partial^2 + 4u_1 + 2u_{1x} \partial^{-1}
\end{pmatrix}.
\]

After a simple calculation, one can find that the two equations of \[47\] can be combined into one equation as follows:

\[
    u_{t_n} = K_n = \alpha_1 \Phi^n \left( \frac{u_{1x}}{u_{2x}} \right) + \alpha_2 \Phi^n \left( \frac{\varepsilon u_{2x}}{u_{1x}} \right) + \frac{1}{2} \sum_{m=0}^{n} k_m(t) \Phi^{-m} \left( \frac{1}{2}, 0 \right).
\]

which is equivalent to the coupled nonisospectral KdV hierarchy \[36\]. Selecting \( \alpha_1 = 1, \alpha_2 = 0 \), and then \[48\] becomes

\[
    u_{t_n} = K_n = \Phi^n \left( \frac{u_{1x}}{u_{2x}} \right) + \frac{1}{2} \sum_{m=0}^{n} k_m(t) \Phi^{-m} \left( \frac{1}{2}, 0 \right),
\]

where \( \Phi \) is given by \[17\].

Obviously, both \( M_1 \) and \( M_2 \) of \[17\] are antisymmetric operators, that is \( M_1 = -M_1, M_2 = -M_2 \), and thus we have

\[
M_1 = \Phi J_1 = J_1 \Phi, \quad M_2 = \Phi J_2 = J_2 \Phi,
\]

where \( \Phi^* \) is the complex conjugate of \( \Phi \). We can conclude that the integrable hierarchy \[36\] is integrable in the sense of Liouville, and then we obtain the Abelian algebra of symmetries

\[
[K_i, K_j] = K_i(u)[K_j] - K_j(u)[K_i] = 0, \quad i, j \geq 0,
\]

and the component Abelian algebras of conserved functionals,

\[
\{H_{1,i}, H_{1,j}\} = \int \frac{\delta H_{1,i}}{\delta u} \frac{\delta H_{1,j}}{\delta u} dx = 0, \quad \{H_{1,i}, H_{1,j}\}_M = \int \frac{\delta H_{1,i}}{\delta u} J_1 \frac{\delta H_{1,j}}{\delta u} dx = 0,
\]

\[
\{H_{2,i}, H_{2,j}\} = \int \frac{\delta H_{2,i}}{\delta u} \frac{\delta H_{2,j}}{\delta u} dx = 0, \quad \{H_{2,i}, H_{2,j}\}_M = \int \frac{\delta H_{2,i}}{\delta u} J_2 \frac{\delta H_{2,j}}{\delta u} dx = 0.
\]

5. Symmetries and their Lie algebra of the coupled nonisospectral KdV hierarchy

In this section, we will discuss the \( K \) symmetries, \( \tau \) symmetries and their Lie algebra of the coupled nonisospectral KdV hierarchy \[49\]. Let us first recall some basic notions and properties related to symmetries. Assume \( G(u) = U(u, u_x, \ldots) \) is a function, then the Gateaux derivative can be defined as: \( G'(u)[\sigma] = \)
lim_{n \to 0} \frac{d}{d\eta} G(u + \eta \sigma). \, \sigma_t = K'_n \sigma$ is a linearised equation associated with a given soliton hierarchy $u_t = K_n$, and then we call $\sigma$ a symmetry of the given soliton hierarchy. In integrable hierarchy \[17\], one has:

\[
\Phi = \begin{pmatrix}
\partial^2 + 4u_1 + 2u_{1x}\partial^{-1} \\
4u_2 + 2u_{2x}\partial^{-1}
\end{pmatrix} \varepsilon (4u_2 + 2u_{2x}\partial^{-1}) =: \phi, \quad K_0 = \begin{pmatrix} u_{1x} \\ u_{2x} \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},
\]

\[K_1 = \phi K_0 = \begin{pmatrix} u_{1xx} + 6u_1 u_{1x} + 6\varepsilon u_2 u_{2x} \\ u_{2xx} + 6u_1 u_{2x} + 6u_2 u_{1x} \end{pmatrix}.
\]

Here we use $\phi$ to represent $\Phi$ for convenience.

**Lemma 1.** $\phi$ is a hereditary symmetry of the hierarchy \[19\], that is,

\[
\phi'[\phi f]g - \phi'[\phi g]f = \phi\{\phi'[f]g - \phi'[g]f\}. \tag{53}
\]

**Proof.** This theorem can be directly calculated via the definition of Gateaux derivative.

For any $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, one has

\[
\phi'[f] = \begin{pmatrix} 4f_1 + 2f_{1x}\partial^{-1} \\ 4f_2 + 2f_{2x}\partial^{-1} \end{pmatrix} \varepsilon (4f_2 + 2f_{2x}\partial^{-1}) = \begin{pmatrix} 4f_1 + 2f_{1x}\partial^{-1} \\ 4f_2 + 2f_{2x}\partial^{-1} \end{pmatrix}.
\]

It follows that we obtain

\[
\phi'[f]g - \phi'[g]f = \begin{pmatrix} 2f_{1x}\partial^{-1}g_1 - 2g_{1x}\partial^{-1}f_1 + 2\varepsilon f_{2x}\partial^{-1}g_2 - 2\varepsilon g_{2x}\partial^{-1}f_2 \\ 2f_{2x}\partial^{-1}g_1 - 2g_{2x}\partial^{-1}f_1 + 2\varepsilon f_{1x}\partial^{-1}g_2 - 2\varepsilon g_{1x}\partial^{-1}f_2 \end{pmatrix}.
\]

Since

\[
\phi f = \begin{pmatrix} f_{1xx} + 4u_1 f_1 + 2u_{1x}\partial^{-1}f_1 + \varepsilon (4u_2 f_2 + 2u_{2x}\partial^{-1}f_2) \\ f_{2xx} + 4u_1 f_2 + 2u_{1x}\partial^{-1}f_2 + 4u_2 f_1 + 2u_{2x}\partial^{-1}f_1 \end{pmatrix} =: \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},
\]

\[
\phi g = \begin{pmatrix} g_{1xx} + 4u_1 g_1 + 2u_{1x}\partial^{-1}g_1 + \varepsilon (4u_2 g_2 + 2u_{2x}\partial^{-1}g_2) \\ g_{2xx} + 4u_1 g_2 + 2u_{1x}\partial^{-1}g_2 + 4u_2 g_1 + 2u_{2x}\partial^{-1}g_1 \end{pmatrix} =: \begin{pmatrix} s_3 \\ s_4 \end{pmatrix},
\]

then

\[
\phi'[\phi f]g - \phi'[\phi g]f = 2\begin{pmatrix} s_{1x}\partial^{-1}g_1 - s_{3x}\partial^{-1}g_2 + 2s_{1x}g_1 - 2s_{3x}g_2 \\ s_{2x}\partial^{-1}g_1 - s_{4x}\partial^{-1}g_2 + 2s_{2x}g_1 - 2s_{4x}g_2 \end{pmatrix}.
\]

After a direct calculation, one can find that \[53\] holds.

**Lemma 2.** $\phi$ is a strong symmetry of the hierarchy \[19\], that is,

\[
\phi'[K_m] = [K'_m, \phi], \quad m \geq 0. \tag{54}
\]

**Proof.** A direct calculation gives rise to

\[
\phi'[K_0] = \begin{pmatrix} 2u_{1xx}\partial^{-1} + 4u_{1x} \\ 2u_{2xx}\partial^{-1} + 4u_{2x} \end{pmatrix} \varepsilon (2u_{2xx}\partial^{-1} + 4u_{2x}).
\]
For a test function \( \sigma = (\sigma_1, \sigma_2)^T \), we have

\[
(K_0)'[\sigma] = \frac{d}{d\eta} |_{\eta=0} \left( \begin{array}{c} (u_1 + \eta \sigma_1)_x \\ (u_2 + \eta \sigma_2)_x \end{array} \right) = \partial \left( \begin{array}{c} \sigma_1 \\ \sigma_2 \end{array} \right) \implies (K_0)' = \partial.
\]

Hence, we obtain

\[
(K_0)'[\phi] = (\partial_3 + 2u_1 x + 4u_1 \partial + 2u_1 x \partial) (\partial_3 + 2u_2 x + 4u_2 \partial) - 1\varepsilon (2u_2 x + 4u_2 \partial).
\]

It follows that we have \( \phi'[K_0] = [K_0', \phi] = K_0' \phi - \Phi K_0' \). Therefore, we verify that (54) holds since the hereditary symmetry property of \( \phi \).

**Theorem 1.**

\[
[K_m, K_n] = K'_m[K_n] - K'_n[K_m] = 0, \quad m, n = 0, 1, 2, \cdots
\]

where \( K_m = \phi^m \left( \begin{array}{c} u_{1x} \\ u_{2x} \end{array} \right) \), \( K_n = \phi^n \left( \begin{array}{c} u_{1x} \\ u_{2x} \end{array} \right) \).

**Proof.** Based on lemma 1, lemma 2, the formula

\[
(\phi G)'[\sigma] = \phi'[\sigma] G + \Phi G'[\sigma],
\]

and the operator \( \phi \), we can verify that (55) holds (see [42]).

**Lemma 3.**

\[
[K_m, \sigma_0] = K'_m \left[ \begin{array}{c} 1 \\ \frac{1}{2} \varepsilon \\ \frac{1}{2} \end{array} \right] = (2m + 1) H K_{m-1}, \quad m = 1, 2, \cdots
\]

where \( H = \left( \begin{array}{cc} 1 & \varepsilon \\ 0 & 1 \end{array} \right) \).

**Proof.** Based on (52), one has

\[
[K_1, \sigma_0] = \left( \begin{array}{c} u_{1xxx} + 6u_1 u_{1x} + 6\varepsilon u_2 u_{2x} \\ u_{2xxx} + 6u_1 u_{2x} + 6\varepsilon u_2 u_{1x} \end{array} \right)' \left[ \begin{array}{c} 1 \\ \frac{1}{2} \varepsilon \\ \frac{1}{2} \end{array} \right] = 3 \left( \begin{array}{c} u_{1x} + \varepsilon u_{2x} \\ u_{1x} + u_{2x} \end{array} \right) = 3 H K_0.
\]

Assume (57) holds for \( m = l - 1 \), that is

\[
[K_{l-1}, \sigma_0] = (2k - 1) H K_{l-2}, \quad l \geq 3.
\]

Since

\[
\phi' \sigma_0 = \phi' \left[ \begin{array}{c} 1 \\ \frac{1}{2} \varepsilon \\ \frac{1}{2} \end{array} \right] = 2H.
\]

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It follows that one has
\[
[K_l, \sigma_0] = (\phi^l \sigma_0)(u_{1x} u_{2x})' \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right] = \phi^l \phi^{l-1} (u_{1x} u_{2x})' + \phi(\phi^{l-1} (u_{1x} u_{2x}))' \left[ \begin{array}{c} 1 \\ 1 \\ 1 \end{array} \right]
\]
\[= 2H \phi^{l-1} (u_{1x} u_{2x}) + \phi[K_{l-1}, \sigma_0] = 2HK_{l-1} + \phi(2l-1)HK_{l-2} = (2l+1)HK_{l-1}.
\]
Therefore, (59) holds by induction.

Lemma 4.
\[
[K_m, \phi^n \sigma_0] = (2m+1)HK_{m+n-1}, \quad m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots.
\]

Proof. From Lemma 3, one can find that (59) holds when \(n=1\). Assume (59) holds for \(n = l-1\), that is
\[
[K_m, \phi^{l-1} \sigma_0] = (2m+1)HK_{m+l-2}, \quad l \geq 3.
\]
Then, we have
\[
[K_m, \phi^l \sigma_0] = K_m'[\phi^l \sigma_0] - (\phi^l \sigma_0)'[K_m]
\]
\[= K_m'[\phi^l \sigma_0] - \phi'[K_m] \phi^{l-1} \sigma_0 - \phi(\phi^{l-1} \sigma_0)'[K_m]
\]
\[= K_m'[\phi^l \sigma_0] - K_m' \phi - \phi K_m' \phi^{l-1} \sigma_0 - \phi(\phi^{l-1} \sigma_0)'[K_m]
\]
\[= \phi[K_m, \phi^{l-1} \sigma_0] = (2m+1)HK_{m+l-1}.
\]
Hence, we conclude that (59) holds by induction.

Lemma 5.
\[
[\phi^m \sigma_0, \sigma_0] = 2m \phi^{m-1} H \sigma_0, \quad m = 1, 2, \ldots.
\]

Proof. When \(m = 1\), one has
\[
[\phi \sigma_0, \sigma_0] = (\phi \sigma_0)' \sigma_0 = \left( xu_{1x} + 2u_1 + \varepsilon xu_{2x} + 2 \varepsilon u_2 \right)' \left[ \begin{array}{c} 1 \\ 2 \end{array} \right] = \left( 1 + \varepsilon \right) 2 = 2H \sigma_0.
\]
Assume
\[
[\phi^{l-1} \sigma_0, \sigma_0] = 2(l-1) \phi^{l-2} H \sigma_0, \quad l \geq 3.
\]
It follows that
\[
[\phi^l \sigma_0, \sigma_0] = (\phi^l \sigma_0)' \sigma_0 = \phi'[\sigma_0] \phi^{l-1} \sigma_0 + \phi(\phi^{l-1} \sigma_0)' \sigma_0
\]
\[= 2H \phi^{l-1} \sigma_0 + \phi(2(l-1) \phi^{l-2} H \sigma_0
\]
\[= 2l \phi^{l-1} H \sigma_0.
\]
Therefore, (60) holds.

Lemma 6. Introducing the notation \(F_{n,m} = (\phi^{n-1} \sigma_0)'[\phi^m \sigma_0]\), then
\[
\phi'[\phi^{n-1} \sigma_0] \phi^{m-1} \sigma_0 - F_{n,m} = \phi^{n-1} \phi'[\sigma_0] \phi^{m-1} \sigma_0 - \phi F_{n,m-1}, \quad n, m = 1, 2, \ldots.
\]

(61)
Proof. Using (56) repeatedly, one has

\[ F_{n,m} = \phi'[\phi^m \sigma_0] \phi^{n-2} \sigma_0 + \phi F_{n-1,m} \]
\[ = \phi'[\phi^m \sigma_0] \phi^{n-2} \sigma_0 + \phi \phi'[\phi^m \sigma_0] \phi^{n-3} \sigma_0 + \phi^2 F_{n-2,m} \]
\[ = \ldots \]
\[ = \sum_{j=2}^{n} \phi^{j-2} \phi'[\phi^m \sigma_0] \phi^{n-j} \sigma_0. \]

It follows that we obtain the following equation by using (53) repeatedly

\[ \phi'[\phi^{n-1} \sigma_0] \phi^{m-1} \sigma_0 - F_{n,m} = \phi'[\phi^{n-1} \sigma_0] \phi^{m-1} \sigma_0 - \phi'[\phi^m \sigma_0] \phi^{n-2} \sigma_0 - \sum_{j=3}^{n} \phi^{j-2} \phi'[\phi^m \sigma_0] \phi^{n-j} \sigma_0 \]
\[ = \phi \phi'[\phi^{n-2} \sigma_0] \phi^{m-1} \sigma_0 - \phi \phi'[\phi^{n-1} \sigma_0] \phi^{m-2} \sigma_0 - \phi \phi'[\phi^m \sigma_0] \phi^{n-3} \sigma_0 - \sum_{j=4}^{n} \phi^{j-2} \phi'[\phi^m \sigma_0] \phi^{n-j} \sigma_0 \]
\[ = \ldots \]
\[ = -\phi F_{n,m-1} + \phi^{n-1} \phi'[\sigma_0] \phi^{m-1} \sigma_0. \]

\[ \square \]

Lemma 7.

\[ [\phi^m \sigma_0, \phi^n \sigma_0] = 2(m-n) \phi^{m+n-1} H \sigma_0, \; m = 1, 2, \ldots, \; n = 0, 1, 2, \ldots. \] (62)

Proof. From Lemma 5, one can find that (62) holds when \( m = 0, \; n = 0, 1, 2, \ldots \). Assume (62) holds for \( m = l - 1 \), that is

\[ [\phi^{l-1} \sigma_0, \phi^n \sigma_0] = 2(l-n-1) \phi^{l+n-2} H \sigma_0, \; l \geq 2. \]

In order to prove that (62) holds, we need to prove that \( [\phi \sigma_0, \phi^n \sigma_0] = 2(l-n) \phi^{l+n-1} H \sigma_0 \), which is equivalent to proving that the following recurrence relationship holds

\[ [\phi^l \sigma_0, \phi^n \sigma_0] = \phi^n \phi'[\sigma_0] \phi^{l-1} \sigma_0 + \phi \phi^{l-1} \sigma_0, \phi^n \sigma_0] = 2(l-n) \phi^{l+n-1} H \sigma_0. \] (63)

Actually, the recurrence relationship (63) can be obtained by means of (53), (56) and (61).

\[ \square \]

According to the coupled nonisospectral KdV hierarchy (49), (52) and the above lemmas, we let

\[ \tau_0^m = (2m+1)tH K_{m-1} + \sigma_0, \; m = 1, 2, \ldots, \] (64)
\[ \tau_n^m = \phi^n \tau_0^m = (2m+1)tH K_{m+n-1} + \phi^n \sigma_0, \; n = 0, 1, 2, \ldots. \] (65)

Then, the \( \tau \) symmetries of the coupled nonisospectral KdV hierarchy can be deduced.

**Theorem 2.** \( \tau_n^m \) are symmetries of the coupled nonisospectral KdV hierarchy (49), that means

\[ (\tau_n^m)_t = K_m' [\tau_n^m], \; m = 1, 2, \ldots, \; n = 0, 1, 2, \ldots. \] (66)
Proof. From (55) and (57), one has
\[ K_m' \tau_0^m = (2m + 1)tHK_{m-1} + (2m + 1)HK_m - \sigma_0 \]
\[ = (2m + 1)tHK_m - (2m + 1)HK_m - (2m + 1)tHK_m - (2m + 1)HK_m = 0. \]

Hence, (66) holds for \( n = 0 \). We conclude that (66) holds because \( \phi \) is a strong symmetry of (49).

**Theorem 3.**
\[ [K_m, \tau_n^l] = (2m + 1)HK_{m+n-1}, \quad l, m = 1, 2, \ldots, \quad n = 0, 1, 2, \ldots. \]

**Proof.** By means of (55), (59) and (65), we have
\[ [K_m, \tau_n^l] = [K_m, (2l + 1)tHK_{l+n-1} + \phi^n \sigma_0] = [K_m, \phi^n \sigma_0] = (2m + 1)HK_{m+n-1}. \]

**Theorem 4.**
\[ [\tau_m^l, \tau_n^m] = 2(l - n)H\tau_{l+n-1}^m, \quad m = 1, 2, \ldots, \quad l, n = 0, 1, 2, \ldots. \]

**Proof.** From (55), (59), (62) and (65), we obtain
\[ [\tau_m^l, \tau_n^m] = [(2m + 1)tHK_{m+n-1} + \phi^n \sigma_0] = (2m + 1)tHK_{m+n-1} + \phi^n \sigma_0 \]
\[ = (2m + 1)tHK_{m+n-1} + \phi^n \sigma_0 = (2m + 1)tHK_{m+n-1} + \phi^n \sigma_0 = 2(l - n)H\tau_{l+n-1}^m. \]

**Theorem 5.** From theorem 1, theorem 3 and theorem 4, we find that the \( K \) symmetries and \( \tau \) symmetries of (49) constitute a set of infinite dimensional Lie algebras of the following structure:
\[ [K_m, K_n] = 0, \quad [K_m, \tau_n^l] = (2m + 1)HK_{m+n-1}, \quad [\tau_m^l, \tau_n^m] = 2(l - n)H\tau_{l+n-1}^m. \]

In the following, we consider some conserved quantities of the coupled nonisospectral KdV hierarchy (49).

Let us first recall some basic notions and definitions (see \([40–42]\)).

**Definition 1.** For a given integrable hierarchy \( u_t = K_n(u) \), \( \nu \) is called the conserved covariance when it satisfies
\[ \frac{d\nu}{dt} + K'\nu = 0 \]
where \( K' \) denotes the linearized operator of \( K \), and \( K'^\ast \) is a conjugate operator of \( K' \).
Proposition 1. For a given integrable hierarchy \( u_t = K_n(u) \), if \( \sigma \) is its symmetry and \( \nu \) is the conserved covariance, then
\[
\int_{-\infty}^{\infty} \nu \sigma dx = < \nu, \sigma >.
\]
Actually, one has \( \frac{d}{dt} < \nu, \sigma > = 0 \) since \( < \nu, \sigma > \) is independent of time \( t \).

Definition 2. If \( F'f = < \nu, f > \), for \( \forall f \in S \), then \( \nu \) is called the gradient of the functional \( F \), that means \( \nu = \frac{\delta F}{\delta u} \).

Proposition 2. If \( \nu' = \nu'^* \), then \( \nu \) is the gradient of the following functional
\[
F = \int_{0}^{1} < \nu(\lambda u), u > d\lambda.
\]

Proposition 3. If \( I \) is a conserved quantity of the hierarchy \( u_t = K_n(u) \), and the conserved covariance \( \nu \) satisfies
\[
I'K_n = < \nu, K_n >,
\]
then
\[
\frac{\partial I}{\partial t} + < \nu, K_n > = 0,
\]
that is
\[
\frac{\partial \nu}{\partial t} + K_n'^* \nu + \nu'K_n = 0.
\]
Therefore, the conserved quantities associated with the integrable hierarchy \( u_t = K_n(u) \) are derived as follows:
\[
I_m = \int_{0}^{1} < \partial_x^{-1} K_m(\lambda u), u > d\lambda.
\]

Based on the above definitions and propositions, we obtain a few conserved qualities of the hierarchy as follows:

\[
I_0 = \int_{0}^{1} < [\partial_x^{-1} K_0(\lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix})] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} > d\lambda = \int_{0}^{1} \int_{-\infty}^{\infty} \lambda (u_1^2 + u_2^2) dx d\lambda = \frac{1}{2} \int_{-\infty}^{\infty} (u_1^2 + u_2^2) dx,
\]
\[
I_1 = \int_{0}^{1} < [\partial_x^{-1} K_1(\lambda \begin{pmatrix} u_1 \\ u_2 \end{pmatrix})] \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} > d\lambda = \int_{-\infty}^{\infty} \left( \frac{1}{2} u_1 u_{1,xx} + \frac{1}{2} u_2 u_{2,xx} + u_1^3 + \varepsilon u_1 u_2^2 + 2u_1 u_2^3 \right) dx
\]
\[
= \int_{-\infty}^{\infty} (-\frac{1}{2} u_1^2 - \frac{1}{2} u_2^2 + \varepsilon u_1 u_2^2 + 2u_1 u_2^3) dx,
\]

where \( K_0 \) and \( K_1 \) are given by [12].
6. A multi-component nonisospectral KdV hierarchy

For the Lie algebra $A_{1N}$ (see eq. (74)), we consider the corresponding loop algebra

$$\bar{A}_{1N} = \text{span}\{\bar{h}_1(n), \bar{h}_2(n), \bar{h}_3(n), \ldots, \bar{h}_{3N-2}(n), \bar{h}_{3N-1}(n), \bar{h}_N(n)\}$$

where

$$\bar{h}_i(n) = \bar{h}_i\lambda^n, \ i = 1, 2, \ldots, 3N.$$

Introducing the following $N$-dimensional nonisospectral problem based on $\bar{A}_{1N}$

$$\begin{aligned}
\psi_x &= U\psi, \quad U = \begin{pmatrix} \bar{h}_2(1) + \bar{h}_3(0) - \sum_{i=1}^{N} u_i \bar{h}_{3i-1}(0) \end{pmatrix}^T, \\
\psi_t &= W\psi, \quad W = \sum_{i=1}^{N} (a_i \bar{h}_{3i-2}(0) + b_i \bar{h}_{3i-1}(0) + c_i \bar{h}_{3i}(0)) = \begin{pmatrix} \bar{W}_1, \bar{W}_2, \ldots, \bar{W}_N \end{pmatrix}^T, \\
\lambda_t &= \sum_{m \geq 0} k_m(t)\lambda^{-m},
\end{aligned}$$

where

$$\begin{aligned}
U_1 &= \begin{pmatrix} 0 & -u_1 + \frac{1}{4}h_2(1) \\ 1 & 0 \end{pmatrix}, \\
U_l &= \begin{pmatrix} 0 & -u_l \\ 0 & 0 \end{pmatrix}, \quad l = 2, 3, \ldots, N, \\
W_k &= \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix},
\end{aligned}$$

$$a_k = \sum_{m \geq 0} a_{km}\lambda^{-m}, \quad b_k = \sum_{m \geq 0} b_{km}\lambda^{-m}, \quad c_k = \sum_{m \geq 0} c_{km}\lambda^{-m}, \quad k = 1, 2, \ldots, N.$$

The stationary zero-curvature equation

$$W_x = \frac{\partial U}{\partial \lambda} \lambda_t + [U, W],$$

(74)

gives rise to the recursion equations:

$$\begin{aligned}
a_{kl,x} &= \frac{1}{4}c_{k,l+1} - b_{k,l} - \sum_{i+j=k+1 \atop 1 \leq i,j \leq k} c_{il}u_j + \sigma \varepsilon \sum_{m+n=k+1 \atop k+1 \leq m,n \leq N} c_{ml}u_m, \\
b_{kl,x} &= \frac{1}{4}k_l(t) - \frac{1}{2}a_{k,l+1} + 2\sum_{i+j=k+1 \atop 1 \leq i,j \leq k} a_{il}u_j + 2\sigma \varepsilon \sum_{m+n=k+1 \atop k+1 \leq m,n \leq N} a_{ml}u_m, \\
c_{kl,x} &= 2a_{kl}, \quad l \geq 0, \quad k = 1, 2, \ldots, N,
\end{aligned}$$

(75)

which are equivalent to

$$\begin{aligned}
a_{kl} &= \frac{1}{2}c_{kl}, \quad l \geq 0, \quad k = 1, 2, \ldots, N, \\
b_{kl} &= -\frac{1}{4}c_{k,l+1} + \theta^{-1}\left( \sum_{i+j=k+1 \atop 1 \leq i,j \leq k} c_{il}u_j + \sigma \varepsilon \sum_{m+n=k+1 \atop k+1 \leq m,n \leq N} c_{ml}u_m \right) + \frac{1}{4}k_l(t)x, \\
c_{k,l+1} &= c_{kl} + 2\sum_{i+j=k+1 \atop 1 \leq i,j \leq k} c_{il}u_j + 2\sigma \varepsilon \sum_{m+n=k+1 \atop k+1 \leq m,n \leq N} c_{ml}u_m \\
&\quad + 2\theta^{-1}\left( \sum_{i+j=k+1 \atop 1 \leq i,j \leq k} c_{il}u_j + \sigma \varepsilon \sum_{m+n=k+1 \atop k+1 \leq m,n \leq N} c_{ml}u_m \right) + \frac{1}{2}k_l(t)x.
\end{aligned}$$

(76)
To the recursion equations (76), we take the initial values $a_{k0} = 0$, it follows that one has

$$ a_{k0} = 0, \quad c_{k0} = \beta_1, \quad c_{k1} = 2\beta_1 \sum_{j=1}^{k} u_j + 2\beta_1 \sigma \varepsilon \sum_{n=k+1}^{N} u_n + \frac{1}{2} k_0(t)x, $$

$$ b_{k0} = -\beta_1 \frac{k}{2} \sum_{j=1}^{k} u_j - \beta_1 \frac{k}{2} \sigma \varepsilon \sum_{n=k+1}^{N} u_n + \frac{1}{8} k_0(t)x, \quad a_{k1} = \beta_1 \sum_{j=1}^{k} u_{j,x} + \beta_1 \sigma \varepsilon \sum_{n=k+1}^{N} u_{n,x} + \frac{1}{4} k_0(t), $$

$$ c_{k2} = 2\beta_1 u_{k,xx} + 6\beta_1 \left( \sum_{j=1}^{k} u_j u_j + \sigma \varepsilon \sum_{n=k+1}^{N} u_m u_n \right) + k_0(t)(x + \partial^{-1}) \left( \sum_{j=1}^{k} u_j + \sigma \varepsilon \sum_{n=k+1}^{N} u_n \right) + \frac{1}{2} k_1(t)x, $$

Denoting that

$$ \overline{W}^{(n)} = \overline{W}^{(n)} - \sum_{i=1}^{N} (a_i \overline{h}_{3i-2}(n) + b_i \overline{h}_{3i-1}(n) + c_i \overline{h}_{3i}(n)) = \overline{W}^{(n)} + \overline{W}^{(n)}, $$

$$ \overline{W}^{(n)} = \sum_{i=1}^{N} \sum_{n=0}^{N} (a_{im} \overline{h}_{3i-2}(0) + b_{im} \overline{h}_{3i-1}(0) + c_{im} \overline{h}_{3i}(0)) \lambda^{n-m}. $$

Taking the modified term $\Delta_n = -\frac{1}{4} \sum_{i=1}^{N} c_{i,n+1} \overline{h}_{3i-1}(0)$ so that for $\overline{V}^{(n)} = \overline{W}^{(n)} - \Delta_n$. It follows that the nonisospectral zero curvature equation

$$ \frac{\partial \overline{U}}{\partial u} u_t + \frac{\partial \overline{U}}{\partial \lambda} \lambda^{(n)} - \overline{V}^{(n)} + [\overline{U}, \overline{V}^{(n)}] = 0, $$

gives rise to the multi-component nonisospectral KdV hierarchy as follows:

$$ u_{t,n} = \overline{R}_n = \frac{1}{2} \partial \left( \begin{array}{c} c_{1,n+1} \\ \vdots \\ c_{N,n+1} \end{array} \right) = J \left( \begin{array}{c} c_{1,n+1} \\ \vdots \\ c_{N,n+1} \end{array} \right), $$

$$ =: \overline{M} \left( \begin{array}{c} \frac{1}{4} k_n(t) \\ 1 \\ \vdots \end{array} \right) - \Phi^n J \left( \begin{array}{c} c_{1,1} \\ \vdots \\ c_{N,1} \end{array} \right) + \frac{1}{2} k_m(t) \Phi^{n-m} \left( \begin{array}{c} 1 \\ \vdots \end{array} \right), $$

where $J$ is given by (77) and $\overline{L}$ is determined by the recursion equations (70).
The first two examples in the above hierarchy of soliton equations are

\[
 u_{t_0} = \begin{pmatrix}
 u_1 \\
 u_2 \\
 \vdots \\
 u_N
\end{pmatrix}_{t_0} = \frac{1}{2} \partial \begin{pmatrix}
 c_{1,1} \\
 c_{2,1} \\
 \vdots \\
 c_{N,1}
\end{pmatrix},
\]

\[
 u_{k,t_0} = \frac{1}{2} (c_{k,1})_x = \beta_1 \sum_{j=1}^{k} u_{j,x} + \beta_1 \sigma \varepsilon \sum_{n=k+1}^{N} u_{n,x} + \frac{1}{4} k_0(t), \quad k = 1, \cdots, N.
\]

\[
 u_{t_1} = \begin{pmatrix}
 u_1 \\
 u_2 \\
 \vdots \\
 u_N
\end{pmatrix}_{t_1} = \frac{1}{2} \partial \begin{pmatrix}
 c_{1,2} \\
 c_{2,2} \\
 \vdots \\
 c_{N,2}
\end{pmatrix},
\]

\[
 u_{k,t_1} = \frac{1}{2} (c_{k,2})_x = \beta_1 u_{k,xxx} + 3 \beta_1 \sigma \varepsilon \sum_{i+j=k+1, 1 \leq i, j \leq k} u_i u_j + \sigma \varepsilon \sum_{m+n=k+1, k+1 \leq m, n \leq N} u_m u_n \right)_x
\]

\[
 + \frac{1}{2} k_0(t) [2 \sum_{j=1}^{k} u_j + \sigma \varepsilon \sum_{n=k+1}^{N} u_n] + x(\sum_{j=1}^{k} u_{j,x} + \sigma \varepsilon \sum_{n=k+1}^{N} u_{n,x}) + \frac{1}{4} k_1(t), \quad k = 1, \cdots, N,
\]

which is a multi-component nonisospectral KdV equation.

When \( N = 1, 2 \), the system (81) is reduced to the nonisospectral KdV equation (22) and the coupled non-isospectral KdV equation (38) respectively.

7. Conclusions and discussions

The higher-dimensional Lie algebras (3)-(11) were constructed. As the applications, we considered the nonisospectral problems (12), (28), (73) respectively, and thus deduced the nonisospectral KdV hierarchy (20), the coupled nonisospectral KdV hierarchy (30) and the multi-component nonisospectral KdV hierarchy (79). By reducing these hierarchies, we obtained the famous KdV equation (22), coupled nonisospectral KdV equation (38) and multi-component nonisospectral KdV equation (81) respectively. It follows that the bi-Hamiltonian structures of these resulting hierarchies were derived by means of the Tu scheme and thus showing their Liouville integrability. Additionally, we found that the \( K \) symmetries and \( \tau \) symmetries of the coupled nonisospectral KdV hierarchy constitute a set of infinite dimensional Lie algebras.

This paper only considers the KdV space spectral problem. In fact, the idea and the method used here are universal for other isospectral and nonisospectral problems. The Riemann-Hilbert method for multi-component systems based on higher-order matrix spectral problems had been discussed (see [51–53]). It is quite intriguing for us to consider the application of the Riemann-Hilbert approach to the multi-component KdV systems. Also, the application of \( \partial \)-dressing method for the construction of solutions to the multi-component KdV systems (see [54]), which is worthy of further work.

Declarations

Ethics approval and consent to participate All authors approve ethics and consent to participate.
Consent for publication All authors consent for publication.

Availability of data and materials In this paper, we have no data and materials used.

Competing interests The authors declare that they have no conflicts of interest.

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References
[1] A.C. Newell, Solitons in Mathematics and Physics. Philadelphia, PA:SIAM, 1985.
[2] D.J. Korteweg and G. de Vries, On the change of form of long waves advancing in a rectangular canal and a new type of long stationary waves. Phi. Mag., 39(1895)422-423.
[3] P.G. Drazin and R.S. Johnson, Solitons: an Introduction. Cambridge University Press, 2002.
[4] M.J. Ablowitz and H. Segur, Solitons and the Inverse Scattering Transform. Philadelphia, PA: SIAM, 1981.
[5] J.P. Zubelli and F. Magri, Differential Equations in the Spectral Parameter, Darboux Transformations and a Hierarchy of Master Symmetries for KdV. Commun. Math. Phys., 141(1991)329-351.
[6] N.A. Kudryashov, Lax pair and first integrals of the traveling wave reduction for the KdV hierarchy. Appl. Math. Comput., 350(2019)323-330.
[7] Y.F. Zhang and H. Tam, Three kinds of coupling integrable couplings of the Korteweg-de Vries hierarchy of evolution equations. J. Math. Phys., 51(2010)043510.
[8] F. Magri, Nonlinear Evolution Equations and Dynamical Systems. Springer Lecture Notes in Physics 120, Berlin: Springer, 1980, P. 233.
[9] W.X. Ma, An approach for constructing non-isospectral hierarchies of evolution equations. J. Phys. A: Math. Gen., 25(1992)L719-L726.
[10] W.X. Ma, A simple scheme for generating nonisospectral flows from the zero curvature representation. Phys. Lett. A, 179(1993)179-185.
[11] Z.J. Qiao, New hierarchies of isospectral and non-isospectral integrable NLEEs derived from the Harry-Dym spectral problem. Physica A, 252(1998)377-387.
[12] Z.J. Qiao, Generation of soliton hierarchy and general structure of its commutator representations. Acta Math. Appl. Sin-E, 18(2)(1995)287-301.
[13] G.Z. Tu, The trace identity, a powerful tool for constructing the Hamiltonian structure of integrable systems. J. Math. Phys., 30(1989)330-338.
[14] W.X. Ma, A new hierarchy of Liouville integrable generalized Hamiltonian equations and its reduction. Chin. J. Contemp. Math., 13(1)(1992)79.
[15] X.X. Xu, An integrable coupling hierarchy of the Mkdv-integrable systems, its Hamiltonian structure and corresponding nonisospectral integrable hierarchy. Appl. Math. Comput., 216(1)(2010)344-353.
[16] Y.F. Zhang and H. Tam, A few integrable systems and spatial spectral transformations. Commun. Nonlinear Sci., 14(11)(2009)3770-3783.
[17] Y.F. Zhang and W.J. Rui, A few continuous and discrete dynamical systems. Rep. Math. Phys., 78(1)(2016)19-32.
[18] Y.F. Zhang, H.Q. Zhang and Q.Y. Yan, Integrable couplings of Botie-Pempinelli-Tu (BPT) hierarchy. Phys. Lett. A, 299(5-6)(2002)543-548.
[19] W.X. Ma, Enlarging spectral problems to construct integrable couplings of soliton equations. Phys. Lett. A, 316(1-2)(2003)72-76.
[20] E.G. Fan and Y.F. Zhang. A simple method for generating integrable hierarchies with multi-potential functions. Chaos, Soliton Fract., 25(2)(2005)425-439.
[21] L. Luo, W.X. Ma and E.G. Fan, The algebraic structure of zero curvature representations associated with integrable couplings. Int. J. Mod. Phys. A, 23(2008)1309-1325.
[22] Z.J. Qiao, Generalized structure of Lax representations for nonlinear evolution equation. Appl. Math. Mech., 18(1997)625-630.
[23] Z.J. Qiao and W. Strampp, L-A-B representation for nonlinear evolution equations and its applications. Physica A, 293(2001)143-156.
[24] Z.J. Qiao, C.W. Cao and W. Strampp, Category of nonlinear evolution equations, algebraic structure, and $\tau$-matrix. J. Math. Phys., 44(2003)701-722.
[25] H.F. Wang and Y.F. Zhang, A new multi-component integrable coupling and its application to isospectral and nonisospectral problems. Commun. Nonlinear Sci., 105(2022)106075.
[26] B. Dubrovin, I.A.B. Strachan, Y.J. Zhang and D.F. Zuo, Extended affine Weyl groups of BCD-type: Their Frobenius manifolds and Landau-Ginzburg superpotentials. Adv. Math., 351(2019)897-946.
[27] D.F. Zuo, The Frobenius-Virasoro algebra and Euler equations. J. Geom. Phys., 86(2014)203-210.
[28] W.X. Ma, X.X. Xu and Y.F. Zhang, Semi-direct sums of Lie algebras and discrete integrable couplings. J. Math. Phys., 47(2006)053501.
[29] W.X. Ma and M. Chen, Hamiltonian and quasi-Hamiltonian structures associated with semidirect sums of Lie algebras. J. Phys. A: Math. Gen., 39(2006)10787-10801.
[30] H.F. Wang and Y.F. Zhang, A kind of nonisospectral and isospectral integrable couplings and their Hamiltonian systems. Commun. Nonlinear Sci., 99(2021)105822.
[31] Y.F. Zhang, J.Q. Mei and H.Y. Guan, A method for generating isospectral and nonisospectral hierarchies of equations as well as symmetries. J. Geom. Phys., 147(2020)103538:1-15.
[32] Y.F. Zhang and H.F. Wang. Schemes for generating different nonlinear Schrödinger integrable equations and their some properties. Acta Math. Appl. Sin-E, 38(3)(2022)579-600.
[33] D. Levi, Hierarchies of integrable equations obtained as nonisospectral (in $x$ and $t$) deformations of the Schrödinger spectral problem. Phys. Lett. A, 119(9)(1987)453-456.
[34] D. Levi and O. Ragnisco, Non-isospectral deformations and Darboux transformations for the third-order spectral problem. Inverse Probl., 4(3)(1988)815.
[35] P.R. Gordoa, A. Pickering and Z.N. Zhu, New 2+1 dimensional nonisospectral Toda lattice hierarchy. J. Math. Phys., 48(2)(2007)023515.
[36] P.A. Clarkson, P.R. Gordoa and A. Pickering, Multicomponent equations associated to non-isospectral scattering problems. Inverse Probl., 13(6)(1997)1463-1476.
[37] X.K. Chang, X.M. Chen and X.B. Hu. A generalized nonisospectral Camassa-Holm equation and its multipeakon solutions. Adv. Math., 263(2014)154-177.
[38] H.F. Wang and Y.F. Zhang. Application of Riemann-Hilbert method to an extended coupled nonlinear Schrödinger equations. J. Comput. Appl. Math., 420(2023)114812.
[39] W.X. Ma, K-symmetries and $\tau$-symmetries of evolution equations and their Lie algebras. J. Phys. A: Math. Gen., 23(1990)2707-2716.
[40] Y.S. Li, A kind of evolution equations and the deform of spectral. Sci. Sin. A, 25(1982)385-390.
[41] Y.S. Li and G.C. Zhu, New set of symmetries of the integrable equations, Lie algebras and non-isospectral evolution equations (I). Sci. Sin. A, 17(1987)235-241.
[42] Y.S. Li and G.C. Zhu, New set of symmetries of the integrable equations, Lie algebras and non-isospectral evolution equationsII: AKNS suystem. J. Phys. A: Math. Gen., 19(1986)3713-3725.
[43] P.J. Olver, On the Hamiltonian structure of evolution equations. Math. Proc. Cambridge, 88(1980)71-88.
[44] H.H. Chen, Y.C. Lee and J.E. Lin, On the direct construction of the inverse scattering operators of integrable nonlinear Hamiltonian systems. Physica D, 26(1987)165-170.
[45] X.Y. Zhu and D.J. Zhang, Symmetries and Their Lie Algebra of a Variable Coefficient Korteweg-de Vries Hierarchy. Chin. Ann. Math., 37B(2016)543-552.
[46] D.Y. Chen, H.W. Xin and D.J. Zhang, Lie algebraic structures of some (1+2)-dimensional Lax integrable systems. Chaos, Soliton Fract., 15(2003)761-770.
[47] Y.F. Zhang, E.G. Fan and H.W. Tam, A few expanding Lie algebras of the Lie algebra $A_1$ and applications. Phys. Lett. A, 359(2006)471-480.
[48] G.Z. Tu, On Liouville integrability of zero-curvature equations and the Yang hierarchy. J. Phys. A: Math. Gen. 22(1989)2375-2392.
[49] P. Casati and G. Ortenzi, New integrable hierarchies from vertex operator representations of polynomial Lie algebras. J. Geom. Phys., 56(2006)418-449.
[50] A.P. Fordy, A.G. Reyman and M.A. Semenov-Tian-Shansky, Classical r-matrices and compatible Poisson brackets for coupled KdV systems. Lett. Math. Phys., 17(1989)25-29.
[51] X.G. Geng, K.D. Wang and M.M. Chen, Long-Time Asymptotics for the Spin-1 Gross-Pitaevskii Equation. Commun. Math. Phys., 382(2021)585-611.
[52] W.X. Ma, Y.H. Huang and F.D. Wang, Inverse scattering transforms and soliton solutions of nonlocal reverse-space nonlinear Schrödinger hierarchies. Stud. Appl. Math., 145(2020)1-23.
[53] Y. Zhang, H.H. Dong and D.S. Wang, Riemann-Hilbert problems and soliton solutions for a multi-component cubic-quintic nonlinear Schrödinger equation. J. Geom. Phys., 149(2020)103569.
[54] H.F. Wang and Y.F. Zhang, $\bar{\partial}$-dressing method for a few (2+1)-dimensional integrable coupling systems. Theor. Math. Phys., 208(2021)452-470.