ASYMPTOTIC OBSERVABLES, PROPAGATION ESTIMATES AND THE
PROBLEM OF ASYMPTOTIC COMPLETENESS IN ALGEBRAIC QFT

WOJCIECH DYBALSKI

Abstract. We review recent results on the existence of asymptotic observables in algebraic
QFT. The problem of asymptotic completeness is discussed from this perspective.

1. Introduction

The problem of existence of asymptotic observables has a long tradition in algebraic quan-
tum field theory (AQFT), starting with a seminal work of Araki and Haag [1]. These authors
introduced certain time-dependent families of observables, and showed that their limits as time
tends to infinity (i.e. certain asymptotic observables), behave as idealized particle detectors.
However, in [1] the convergence was only shown on certain domains of scattering states. The
existence of such asymptotic observables on arbitrary vectors of bounded energy has remained an
open problem for over four decades. It can be expected from quantum mechanics that a solution
of this problem is a key to asymptotic completeness in AQFT.

In an ongoing project with C. Gérard we gave a solution of this problem for a certain class of
detectors [4, 5]. Moreover, we demonstrated that the linear span of the ranges of these detectors
coincides with the subspace of scattering states. This weak variant of asymptotic completeness
has a simple physical interpretation: given any initial state (think of a container with hydrogen
gas used as a proton source at the LHC), after a typical particle physics experiment one obtains a
configuration of independent particles in terms of which the measurement results are interpreted.
We show that in the context of massive theories this empirical fact can be derived from the basic
assumptions of locality and positivity of energy in a model-independent manner.

2. Framework

We work in the standard framework of AQFT, which is given by:

(1) A net of local von Neumann algebras

(2) The global algebra of this net

(3) A strongly continuous unitary representation of translations

(4) (positivity of energy)

(5) (uniqueness and cyclicity of the vacuum) 1 l

We adopt a restrictive variant of the positivity of energy assumption (4), suitable for massive
theories, which requires that

We consider only the case of outgoing asymptotic observables, since the incoming case is analogous.

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exist interacting quantum field theories which satisfy the above assumptions, for example \(\lambda \phi_1^4\) at small \(\lambda\) [3].

3. Generalized creation/annihilation operators

In concrete models, the conventional building blocks of physical observables are creation and annihilation operators. To construct their counterparts in our abstract setting, we need to control the energy-momentum transfer of elements of the algebra of observables \(\mathcal{A}\). The energy-momentum transfer (or the Arveson spectrum) of an observable \(B \in \mathcal{A}\) is denoted \(\text{supp} \hat{B}\) and defined as the support of the operator-valued distribution:

\[
\hat{B}(E, p) := (2\pi)^{-\frac{d+1}{2}} \int dt dx e^{-iEt+i px} B(t, x),
\]

where \(B(t, x) := U(t, x)BU(t, x)^*\). It has the expected properties, in particular

\[
\text{supp} \hat{B}^* = -\text{supp} \hat{B},
\]

\[
B \mathbb{1}_{\Delta}(U) \mathcal{H} \subset \mathbb{1}_{\Delta + \text{supp} \hat{B}(U)} \mathcal{H}.
\]

To use \(B^*\) as a creation operator of a single-particle state, we would like \(\text{supp} \hat{B}\) to be a small neighbourhood of a point on the mass hyperboloid \(H_m\). An operator with such an energy-momentum transfer cannot be strictly local, i.e. contained in \(\mathcal{A}(\mathcal{O})\) for some open bounded region \(\mathcal{O}\). However, it can be almost local i.e. it can be approximated in norm by operators localized in double-cones, centred at zero, with increasing radius, up to an error vanishing faster than any inverse power of this radius. A typical example of an almost local operator is

\[
B = \int dt dx A(t, x)f(t, x), \quad A \in \mathcal{A}(\mathcal{O}), f \in \mathcal{S}(\mathbb{R}^{d+1})
\]

for some open bounded \(\mathcal{O}\). Clearly, \(\text{supp} \hat{B} \subset \text{supp} \hat{f}\), so the energy-momentum transfer of \(B^*\) can be a compact set. Also, for such operators the function \((t, x) \mapsto B(t, x)\) is smooth in norm and all its derivatives are again almost local. We denote by \(\mathcal{L}_0 \subset \mathcal{A}\) the subspace of operators of the form \(3.4\) whose energy-momentum transfers are compact sets supported outside of the future lightcone \(\mathcal{O}\). The elements of \(\mathcal{L}_0\) are ‘annihilation operators’ in the sense that they annihilate the vacuum, but not in the sense of canonical commutation relations.

The above construction of generalized creation and annihilation operators is well known since early days of AQFT. We conclude this section with a more recent concept of ‘improper’ annihilation operators, introduced in our recent work [4]. They are defined as maps

\[
a_B : \mathcal{H}_c \to \mathcal{H} \otimes L^2(\mathbb{R}^d),
\]

\[
a_B \Psi)(x) = B(x)\Psi,
\]

where \(B \in \mathcal{L}_0\) and \(\mathcal{H}_c\) is the domain of vectors \(\Psi \in \mathcal{H}\) s.t. \(\Psi = \mathbb{1}_{\Delta}(U)\Psi\) for some compact set \(\Delta\). The fact that \(x \mapsto B(x)\Psi\) is square-integrable is non-trivial, but it follows from Lemma 2.2 of [2]. Similarly, for a family \(B_1, \ldots, B_n\) of operators from \(\mathcal{L}_0\), we define

\[
a_{B_1, \ldots, B_n} : \mathcal{H}_c \to \mathcal{H} \otimes L^2(\mathbb{R}^{nd}),
\]

\[
(a_{B_1, \ldots, B_n} \Psi)(x_1, \ldots, x_n) = B_1(x_1) \ldots B_n(x_n)\Psi.
\]

4. Propagation observables and asymptotic observables

We recall that in non-relativistic scattering theory, time-dependent estimates on the propagation properties of solutions of evolution equations are usually called propagation estimates, and time-dependent observables used to derive them are called propagation observables. Asymptotic observables are limits (as time \(t \to \infty\)) of propagation observables evolved in the Heisenberg picture.

Let us first consider the classical phase space \(T^* \mathbb{R}^d = \mathbb{R}^d \times (\mathbb{R}^d)^*\), \(h \in \mathcal{S}(T^* \mathbb{R}^d)\) and a classical propagation observable

\[
t \mapsto h_t \in \mathcal{S}(T^* \mathbb{R}^d),
\]

\(2\)This definition is slightly more restrictive than the one from [4, 5].
where $h_t(x, \xi) := h(x/t, \xi)$. We can elevate it to a quantum-mechanical propagation observable with the help of the Weyl quantization:

\begin{equation}
(4.2) \quad t \mapsto h^w_t \in B(L^2(\mathbb{R}^d)),
\end{equation}

where

\begin{equation}
(4.3) \quad (h^w_t u)(x) = (2\pi)^{-d} \int e^{i(x-y) \cdot \xi} h_t \left( \frac{x+y}{2} \right) u(y) dy d\xi, \quad u \in L^2(\mathbb{R}^d).
\end{equation}

Next, using the map $a_B$, $B \in \mathcal{L}_0$, we define a quantum-field-theoretical propagation observable:

\begin{equation}
(4.4) \quad t \mapsto a_B^* (\mathbb{1}_H \otimes h^w_t) a_B,
\end{equation}

which is a family of (unbounded) operators on $\mathcal{H}_c$.

Given the propagation observable (4.4), the corresponding asymptotic observable is approximated (as $t \to \infty$) by:

\begin{equation}
(4.5) \quad C_t := e^{itH} a_B^* (\mathbb{1}_H \otimes h^w_t) e^{-itH}.
\end{equation}

We mention as an aside that in the case of $h$ independent of momentum $\xi$ the above formula gives

\begin{equation}
(4.6) \quad C_t = \int dx h(x/t)(B^* B)(t, x),
\end{equation}

which is the standard Araki–Haag detector [1, 2].

Our main result, stated in Theorem 4.1 below, concerns the strong convergence as $t \to \infty$ of approximating sequences of the form

\begin{equation}
(4.7) \quad t \mapsto C_{1,t} \ldots C_{n,t} \mathbb{1}_\Delta(U),
\end{equation}

where $\Delta \subset G_{2m}$ is an open set, whose extension is small compared to $m$, and $C_{i,t}$, $i = 1, \ldots, n$ are defined as in (4.5). So far we can handle detectors which satisfy the following admissibility conditions:

(a) The energy-momentum transfers of $B^*_i$ are small neighbourhoods of distinct points $p_i$ on the mass hyperboloid $H_m$ s.t. $p_1 + \cdots + p_n \in \Delta$.

(b) The functions $h_{1,t} \in S(T^* \mathbb{R}^d)$ have the form

\[ h_{i,t}(x, \xi) = h_{0,i}(x/t) \chi(x/t - \nabla \omega(\xi)), \]

where $h_{0,i} \in C^\infty_c(\mathbb{R}^d)$ have mutually disjoint supports, $\chi \in C^\infty_c(\mathbb{R}^d)$ is supported in a small neighbourhood of zero and $\omega(\xi) := \sqrt{\xi^2 + m^2}$.

The first assumption above says that $B^*_i$ are ‘creation operators’ of single-particle states with energy-momentum vectors centered around $p_i$. The disjointness of supports of $h_{0,i}$ in the second assumption ensures that the corresponding detectors are localized in spacelike separated regions for large $t$, which helps to exploit locality. The function $\chi$ keeps the average velocity $x/t$ close to the instantaneous velocity $\nabla \omega(\xi)$ in accordance with the expected ballistic motion of a particle at asymptotic times.

**Theorem 4.1.** Let $\Delta \subset G_{2m}$ be a small open subset, $B_i, h_i$, $i = 1, \ldots, n$, be admissible as specified in (a), (b) above and let $C_{i,t}$ be given by (4.5). Then there exists the limit

\begin{equation}
(4.8) \quad Q^+_{n}(\Delta) := \lim_{t \to \infty} C_{1,t} \ldots C_{n,t} \mathbb{1}_\Delta(U).
\end{equation}

5. **OUTLINE OF THE PROOF OF THEOREM 4.1**

The proof of Theorem 4.1, given in [3], relies on the method of propagation estimates combined with the Haag–Ruelle scattering theory. Here we outline a different argument (also obtained jointly with C. Gérard) which does not use the Haag–Ruelle theory.

We note that for $n = 1$ condition (a) is not compatible with $\Delta \subset G_{2m}$, hence the theorem does not provide any information on the convergence of one detector. Let us consider the first interesting case which is $n = 2$: Making use of locality and disjointness of supports of $h_{0,1}$, $h_{0,2}$ (condition (b)) we can write

\begin{equation}
(5.1) \quad C_{1,t} C_{2,t} \mathbb{1}_\Delta(U) = e^{itH} a^*_{B_1, B_2} (\mathbb{1}_H \otimes h^w_{1,t} h^w_{2,t}) a_{B_1, B_2} e^{-itH} \mathbb{1}_\Delta(U) + O(t^{-\infty}),
\end{equation}

where $h^w_{1,t}$, $h^w_{2,t}$ are defined as in (4.5).
where $O(t^{-\infty})$ denotes a term which vanishes in norm faster than any inverse power of $t$. Thus it is enough to show strong convergence of

$$f(t) := e^{itH}a_{B_1,B_2}(\mathbb{1}_H \otimes h_{1,t}^w h_{2,t}^w)a_{B_1,B_2}e^{-itH}\mathbb{1}_\Delta(U)$$

as $t \to \infty$. The key observation, which allows us to transport methods from quantum-mechanical scattering theory to the present context, is that the time-derivative of $f$ can be expressed by the Heisenberg derivative of the quantum-mechanical propagation observable $t \mapsto h_{1,t}^w h_{2,t}^w$. More precisely

$$\partial_t f(t) = e^{itH}a_{B_1,B_2}(\mathbb{1}_H \otimes \mathcal{D}(h_{1,t}^w h_{2,t}^w))a_{B_1,B_2}e^{-itH}\mathbb{1}_\Delta(U) + O(t^{-\infty}),$$

where $\mathcal{D} = \partial_t + i[\omega, \cdot]$ and $\omega = \omega(-i\nabla_{x_1}) + \omega(-i\nabla_{x_2})$. Thus the asymptotic time-evolution of a relativistic QFT is governed by a quantum-mechanical Hamiltonian $\omega$. In the case of $n = 2$, using relation (5.3) and the standard phase space propagation estimates (see e.g. [3]) one can show the convergence of $t \mapsto f(t)$ [4]. In this case functions $\chi$ appearing in condition (b) above are not needed and one can prove the convergence of a product of two conventional Araki–Haag detectors [4.6]. For $n > 2$ things become more complicated due to our limited understanding of quantum mechanical dispersive systems, that is systems of particles with non-quadric dispersion relations. In this case the phase space propagation estimate is not available and we had to derive a new propagation estimate to control the convergence of $t \mapsto f(t)$. As explained in more detail in [5], it requires the presence of the functions $\chi$ in the propagation observables.

Let us conclude this section with a few remarks about the key property (5.3): The term involving $\partial_t h_{1,t}^w h_{2,t}^w$ is self-explanatory. Let us indicate how $\partial_t e^{itH}$ and $\partial_t e^{-itH}$ give rise to the commutator with $\omega$: We note the identities

$$B_1(x_1)B_2(x_2)\mathbb{1}_\Delta(U) = \mathbb{1}_{\{0\}}(U)B_1(x_1)\mathbb{1}_{H_m}(U)B_2(x_2)\mathbb{1}_\Delta(U),$$

(5.4)

$$B_1(x_1)B_2(x_2)H\mathbb{1}_\Delta(U) = HB_1(x_1)B_2(x_2)\mathbb{1}_\Delta(U) + [B_1(x_1), B_2(x_2), H]\mathbb{1}_\Delta(U) + [B_1(x_1), H]B_2(x_2)\mathbb{1}_\Delta(U) + [B_2(x_2), H]B_1(x_1)\mathbb{1}_\Delta(U),$$

(5.5)

where (5.4) follows from condition (a) and (5.5) is a simple computation. The first term on the r.h.s. of (5.5) vanishes due to translation invariance of the vacuum. The term involving the double commutator contributes to $O(t^{-\infty})$ on the r.h.s. of (5.3) due to locality and the disjointness of supports of $h_{1,0}$ and $h_{2,0}$. The last two terms on the r.h.s. give rise to the quantum-mechanical Hamiltonian $\omega$. In fact, keeping (5.3) in mind, we note that

$$\mathbb{1}_{\{0\}}(U)[B_1(x_1), H]\mathbb{1}_{H_m}(U) = \mathbb{1}_{\{0\}}(U)B_1(x_1)\omega(P)\mathbb{1}_{H_m}(U) = \omega(-i\nabla_{x_1})\mathbb{1}_{\{0\}}(U)B_1(x_1)\mathbb{1}_{H_m}(U),$$

(5.6)

where the last step follows by taking the Fourier transform of $\omega$ and exploiting the invariance of the vacuum.

6. The Problem of Asymptotic Completeness

We recall that under the assumptions from Section 2 the Haag–Ruelle scattering theory [8] [10] gives a canonical wave operator whose range will be denoted by $\mathcal{H}^+$. $\mathcal{H}^+$ can equivalently be seen as a subspace of $\mathcal{H}$ spanned by (outgoing) scattering states, including the vacuum (0-particle state) and $\mathcal{H}_1 := \mathbb{1}_{H_m}(U)\mathcal{H}$ (single-particle states). The property of asymptotic completeness requires that $\mathcal{H}^+ = \mathcal{H}$ i.e. that all states in the physical Hilbert space can be interpreted as configurations of particles. It is well known that this property does not follow from the Haag–Kastler postulates: Firstly, it could happen that not all the superselection sectors of the theory are accommodated in $\mathcal{H}$. Single-particle states of charged particles from the omitted sectors would then be missing in $\mathcal{H}$ and consequently their scattering states would not belong to $\mathcal{H}^+$. But scattering configurations of such particles with total charge zero would clearly belong to $\mathcal{H}$ thus violating asymptotic completeness. Secondly, even if all the superselection sectors are accommodated in $\mathcal{H}$, the physical Hilbert space may still contain pathological states with too many local degrees of freedom which do not admit any particle interpretation. Such states appear e.g. in certain generalized free fields [7].

One approach to the problem of asymptotic completeness in AQFT is to amend the Haag–Kastler postulates with some physically motivated a priori conditions which should characterize theories with a reasonable particle interpretation. This approach, pioneered in [8], resulted in a family of phase space conditions which brought many important insights in AQFT, but did not
have much impact on its scattering theory. Our strategy is different: we try to get as close as possible to proving complete particle interpretation without adopting additional assumptions. A posteriori our results can be reformulated as a condition for asymptotic completeness.

To illustrate this strategy, let us discuss the following result from [1] which complements Theorem 4.1.

**Theorem 6.1.** Under the assumptions of Theorem 4.1, the range of each asymptotic observable \( Q_n^*(\Delta) \) belongs to the subspace of scattering states \( \mathcal{H}_\Delta(U)\mathcal{H}^+ \). Moreover, \( \mathcal{H}_\Delta(U)\mathcal{H}^+ \) is spanned by the ranges of \( Q_n^*(\Delta) \), \( n \in \mathbb{N} \).

We note that the second part of this result ensures that ‘sufficiently many’ asymptotic observables, constructed in Theorem 4.1 are non-zero. This part is relatively easy to prove proceeding similarly as in [1]. The essential part of Theorem 6.1 is the first statement, which says that a certain subspace of \( \mathcal{H} \) (namely the span of the ranges of the asymptotic observables \( Q_n^*(\Delta) \)), which \textit{a priori} has nothing to do with scattering states, is in fact contained in the subspace of scattering states \( \mathcal{H}^+ \). Thus the requirement that the ranges of \( Q_n^*(\Delta) \) (together with the vacuum vector and single-particle states) span the entire Hilbert space \( \mathcal{H} \), is a condition for asymptotic completeness.

If we are ready to interpret \( Q_n^*(\Delta) \) as particle detectors, Theorem 6.1 has a simple physical interpretation. It says that for any initial state \( \Psi \in \mathcal{H} \), the state resulting from the measurement \( Q_n^*(\Delta) \) will be a vector in \( \mathcal{H}^+ \), that is a configuration of particles. Theorem 6.1 also says that if \( \Psi \) belongs to the orthogonal complement of \( \mathcal{H}^+ \) (as a neutral configuration of particles from superselection sectors omitted in \( \mathcal{H} \), a state with too many local degrees of freedom or any other pathology Haag–Kastler postulates may admit) then \( \Psi \) will be annihilated by \( Q_n^*(\Delta) \). This insight cannot be obtained by methods from [1], where only the existence of asymptotic observables on \( \mathcal{H}^+ \) is considered.

In the language of many-body scattering theory our results concern the problem of asymptotic completeness in the free region (i.e. outside of the collision planes \( \{x_1 = x_2\}, \{x_1 = x_3\} \), etc.) In fact, condition (b) above ensures that the classical propagation observables, corresponding to \( H_n \), vanish near the collision planes. A natural future direction is to drop condition (b) and understand the asymptotic dynamics of AQFT close to the collision planes. As this condition enters already in the first step of our analysis (see (5.1)), this goal surely requires new ideas.

Taking quantum mechanics as a guide, some relativistic counterpart of the Mourre theory may be needed here. One can speculate that the covariance under Lorentz boosts, which we did not exploit in our analysis so far, will provide this missing ingredient.

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