Research Article

Weighted Pluricomplex Energy II

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We continue our study of the complex Monge-Ampère operator on the weighted pluricomplex energy classes. We give more characterizations of the range of the classes $\mathcal{E}_p^\varphi(\Omega)$ by the complex Monge-Ampère operator. In particular, we prove that a nonnegative Borel measure $\mu$ is the Monge-Ampère of a unique function $\varphi \in \mathcal{E}_p^\varphi(\Omega)$ if and only if $\chi(\mathcal{E}_p^\varphi(\Omega)) \subset L^1(d\mu)$. Then we show that if $\mu = (dd^c \varphi)^n$ for some $\varphi \in \mathcal{E}_p^\varphi(\Omega)$, then $\mu$ is the Monge-Ampère of a unique function $\varphi \in \mathcal{E}_p^\varphi(\Omega)$, where $f$ is given boundary data. If moreover the nonnegative Borel measure $\mu$ is suitably dominated by the Monge-Ampère capacity, we establish a priori estimates on the capacity of sublevel sets of the solutions.

As a consequence, we give a priori bounds of the solution of the Dirichlet problem in the case when the measure has a density in some Orlicz space.

1. Introduction

Let $\Omega \subset \mathbb{C}^n$ be a bounded hyperconvex domain, that is, a connected, bounded open subset of $\mathbb{C}^n$, such that there exists a negative plurisubharmonic function $\rho$ such that $\{ z \in \Omega ; \rho(z) < -c \} \subset \Omega$, for all $c > 0$. Such a function $\rho$ is called an exhaustion function. We let $\text{Psh}(\Omega)$ denote the cone of plurisubharmonic functions (psh for short) on $\Omega$ and let $\text{Psh}^{-}(\Omega)$ denote the subclass of negative functions.

As known (see [1, 2]), the complex Monge-Ampère operator $(dd^c \cdot)^n$ is well defined, as a nonnegative measure, on the set of locally bounded plurisubharmonic functions. Therefore the question of describing the measures which are the Monge-Ampère of bounded psh functions is very important for pluripotential theory, complex dynamic, and complex geometry. This problem has been studied extensively by various authors; see, for example, [2–6] and reference therein. In [7], Cegrell introduced the pluricomplex energy classes $\mathcal{E}_p(\Omega)$ and $\mathcal{E}_p^\varphi(\Omega)$ ($p \geq 1$) on which the complex Monge-Ampère operator is well defined. He proved that a measure $\mu$ is the Monge-Ampère of some function $u \in \mathcal{E}_p^\varphi(\Omega)$ if and only if it satisfies

$$\int_\Omega (-v)^p \, d\mu \leq \text{Const} \left( \int_\Omega (-v)^p (dd^c v)^n \right)^{p/(n+p)}, \quad \forall v \in \mathcal{E}_0(\Omega),$$

where $\mathcal{E}_0(\Omega)$ is the cone of all bounded psh functions $\varphi$ defined on the domain $\Omega$ with finite total Monge-Ampère mass and $\lim_{z \to \partial \Omega} \varphi(z) = 0$, for every $\zeta \in \partial \Omega$. Recently, Åhag et al. in [8] proved that, in the case $p = 1$, inequality (1) is equivalent to $\mathcal{E}_1(\Omega) \subset L^1(d\mu)$. In this note, our first objective is to extend this result by showing that, for all positive number $p$, inequality (1) is equivalent to $\mathcal{E}_p^\varphi(\Omega) \subset L^p(d\mu)$. In fact, we prove some more general result. Given a nondecreasing function $\chi : \mathbb{R}^- \to \mathbb{R}^+$, we consider the set $\mathcal{E}_p^\chi(\Omega)$ of plurisubharmonic functions of finite $\chi$-weighted Monge-Ampère energy and, in some sense, has boundary values zero. These are the functions $u \in \text{Psh}(\Omega)$ for which there exists a decreasing sequence $u_j \in \mathcal{E}_0(\Omega)$ with limit $u$ and

$$\sup_{j \in \mathbb{N}} \int_\Omega -\chi(u_j) (dd^c u_j)^n < \infty.$$

Then we have the following characterization of the image of the complex Monge-Ampère acting in the class $\mathcal{E}_p^\chi(\Omega)$.

Theorem 1. Let $\chi : \mathbb{R}^- \to \mathbb{R}^+$ be an increasing convex or homogeneous function such that $\chi(-\infty) = -\infty$. The following assertions are equivalent:

1. there exists a unique function $\varphi \in \mathcal{E}_1^\chi(\Omega)$ such that $\mu = (dd^c \varphi)^n$;
2. $\chi(\mathcal{E}_1^\chi(\Omega)) \subset L^1(d\mu)$.
Next, we extend our previous result to families of functions having prescribed boundary data. Let \( f \in \text{PSH}(\Omega) \) be a maximal psh function. We define the class \( \mathcal{C}_\chi(f) \) to be the class of psh functions \( u \) such that there exists a function \( \varphi \in \mathcal{C}_\chi(\Omega) \) such that
\[
\varphi(z) + f(z) \leq u(z) \leq f(z), \quad \forall z \in \Omega. \tag{3}
\]
Some particular cases of the classes \( \mathcal{C}_\chi(f) \) have been studied in [6, 7, 9–16].

More precisely, we prove the following result.

**Theorem 2.** Let \( \mu \) be a nonnegative measure in \( \Omega \), let \( \chi : \mathbb{R}^- \rightarrow \mathbb{R}^- \) be an increasing convex or homogeneous function such that \( \chi(-\infty) = -\infty \), and let \( f \) be a maximal function. Then, if \( \mu = (dd^c u)^n \) for some \( u \in \mathcal{C}_\chi(\Omega) \) then there exists a unique function \( \varphi \in \mathcal{C}_\chi(f) \) such that \( \mu = (dd^c \varphi)^n \).

Moreover, when the nonnegative measure \( \mu \) is dominated by the Monge-Ampère capacity, we give an estimate of the growth of solutions of the equation \( (dd^c \varphi)^n = \mu \). As in [12], let us consider the function
\[
F_{\mu}(t) := \sup \{ \mu(K), K \in \Omega; \text{cap}_n(K) \leq t \}, \quad \forall t \geq 0. \tag{4}
\]
Then \( F := F_{\mu} \) is a nondecreasing function on \( \mathbb{R}^+ \) and satisfies
\[
\mu(K) \leq F_{\mu} \left( \text{cap}_n(K) \right), \quad \forall \text{Borel subsets } K \subset \Omega. \tag{5}
\]
Write \( F(x) = F_{\mu}(x) = x(e(-\ln(x/n)))^n \) where \( e : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) is nondecreasing.

Such measures dominated by the Monge-Ampère capacity have been extensively studied by Kolodziej in [3–5]. He proved that if \( \phi : \partial \Omega \rightarrow \mathbb{R} \) is a continuous function and \( \int_{\Omega} \phi(t) dt < +\infty \), then \( \mu \) is the Monge-Ampère of a unique function \( \varphi \in \text{PSH}(\Omega) \) with \( \varphi|_{\partial \Omega} = \phi \).

When \( \int_{\Omega} \phi(s) ds = +\infty \), we have the following estimate.

**Theorem 3.** Let \( \mu \) be a positive finite measure. Assume, for all compact subsets \( K \subset \Omega \),
\[
\mu(K) \leq F_{\mu} \left( \text{cap}_n(K) \right). \tag{6}
\]
Then there exists a unique function \( \varphi \in \mathcal{C}(f) \) such that \( \mu = (dd^c \varphi)^n \), and
\[
\text{cap}_n \left( \{ \varphi < f - s \} \right) \leq \exp \left( -nH^{-1}(s) \right), \quad \forall s > 0. \tag{7}
\]
Here \( H^{-1} \) is the reciprocal function of \( H(x) = e \int_0^x \phi(t) dt + s_0(\mu) \).

In particular if \( \int_{\Omega} \phi(t) dt < +\infty \) then
\[
0 \leq f - \varphi \leq e \int_{\Omega} \phi(t) dt + e(e(0) + \mu(\mu)^{1/n}). \tag{8}
\]

The paper is organised as follows. In Section 2, we recall the definitions of the energy classes \( \mathcal{C}_\chi(\Omega) \) and some classes of psh functions introduced by Cegrell [7, 13, 14] and we prove Theorem 1. In Section 3, we prove Theorem 2. As a consequence, we generalize the main result in the paper [9]. In Section 4, we prove Theorem 3. As application, we give a priori bound of the solution of Dirichlet problem in the case when the measure \( \mu = gd\lambda \), where \( g \) belongs to some Orlicz space \( L \log^a L \).

**2. Energy Classes with Zero Boundary Data \( \mathcal{C}_\chi \)**

Let us recall some Cegrell’s classes (Cf. [7, 13, 14]). The class \( \mathcal{C}(\Omega) \) is the set of plurisubharmonic functions \( u \) such that, for all \( z_0 \in \Omega \), there exists a neighbourhood \( V_{z_0} \) of \( z_0 \) and \( u_j \in \mathcal{C}(\Omega) \), a decreasing sequence which converges towards \( u \) in \( V_{z_0} \) and satisfies \( \sup_j \int_{V_{z_0}} (dd^c u^j)^n < +\infty \). Cegrell [13] has shown that the operator \( (dd^c \varphi)^n \) is well defined on \( \mathcal{C}(\Omega) \), continuous under decreasing limits, and the class \( \mathcal{C}(\Omega) \) is stable under taking maximum; that is, if \( u \in \mathcal{C}(\Omega) \) and \( v \in \text{PSH}(\Omega) \) then \( \max(u, v) \in \mathcal{C}(\Omega) \). This class is the largest class with these properties (Theorem 4.5 in [13]). The class \( \mathcal{C}(\Omega) \) has been further characterized by Blocki [17, 18] and Le Mau et al. in [19].

The class \( \mathcal{C}(\Omega) \) is the global version of \( \mathcal{C}(\Omega) \): a function \( u \) belongs to \( \mathcal{C}(\Omega) \) if and only if there exists a decreasing sequence \( u_j \in \mathcal{C}(\Omega) \) converging towards \( u \) in all of \( \Omega \), which satisfies \( \sup_j \int_{\Omega} (dd^c u^j)^n < +\infty \). The class \( \mathcal{C}(\Omega) \) has been further characterized in [12, 17].

Let \( \Omega_j \subset \Omega \) be an increasing sequence of strictly pseudoconvex domains such that \( \Omega = \bigcup_j \Omega_j \). Let \( u \in \mathcal{C}(\Omega) \) be a given psh function and put
\[
u_{\Omega_j} := \sup \{ \varphi \in \text{PSH}(\Omega); \varphi \leq u \text{ on } \Omega \setminus \Omega_j \}. \tag{9}
\]
Then we have \( \nu_{\Omega_j} \in \mathcal{C}(\Omega) \) and \( \nu_{\Omega_j} \) is an increasing sequence. Let \( \overline{u} := (\lim_j \nu_{\Omega_j})^* \). It follows from the properties of \( \mathcal{C}(\Omega) \) that \( \overline{u} \in \mathcal{C}(\Omega) \). Note that the definition of \( \overline{u} \) is independent of the choice of the sequence \( \Omega_j \) and is maximal; that is, \((dd^c \overline{u})^n = 0 \). \( \overline{u} \) is the smallest maximal psh function above \( u \). Define \( \mathcal{N}(\Omega) := \{ u \in \mathcal{C}(\Omega); \overline{u} = 0 \} \). In fact, this class is the analogues of potentials for subharmonic functions.

**Definition 4.** Let \( \chi : \mathbb{R}^- \rightarrow \mathbb{R}^- \) be a nondecreasing function. We let \( \mathcal{C}_{\chi}(\Omega) \) denote the set of all functions \( u \in \text{PSH}(\Omega) \) for which there exists a sequence \( u_j \in \mathcal{C}(\Omega) \) decreasing to \( u \) in \( \Omega \) and satisfying
\[
\sup_{j \in \mathbb{N}} \int_{\Omega} -\chi(u_j) (dd^c u_j)^n < +\infty. \tag{10}
\]
It was proved in [15, 20] that if \( \chi \neq 0 \) then \( \mathcal{C}_{\chi}(\Omega) \subset \mathcal{C}(\Omega) \). \tag{11}

In particular, for any function \( u \in \mathcal{C}_{\chi}(\Omega) \), the complex Monge-Ampère operator \( (dd^c u)^n \) is well defined as nonnegative measure. Furthermore, if \( \chi(t) < 0 \) for all \( t > 0 \), then
\[
\mathcal{C}_{\chi}(\Omega) = \left\{ u \in \mathcal{N}(\Omega); \int_{\Omega} -\chi(u) (dd^c u)^n < +\infty \right\}. \tag{12}
\]
The class $\mathcal{E}_\chi(\Omega)$ has been characterized by the speed of
decrease of the capacity of sublevel sets [11, 12].
Recall that the Monge-Ampère capacity has been introduced
and studied by Bedford and Taylor in [1]. Given $K \subset \Omega$,
a compact subset, its Monge-Ampère capacity relatively to $\Omega$
is defined by
\[
\text{cap}_\Omega(K) := \sup \left\{ \int_K (dd^c u)^n; \ u \in \text{PSH}(\Omega), \ -1 \leq u \leq 0 \right\}.
\] (13)
The following estimates (cf [12]) will be useful later on. For
any $\varphi \in \mathcal{E}_0$
\[
t^n \text{cap}_\Omega(\varphi < -s - t) \leq \int \chi(\varphi < -s) \leq s^n \text{cap}_\Omega(\varphi < -s), \ \forall s, t > 0.
\] (14)
Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be a nondecreasing function. Without
loss of generality, from now on, we assume that $\chi(0) = 0$. We
define the class $\mathcal{E}_\chi(\Omega)$
\[
\mathcal{E}_\chi(\Omega) := \left\{ \varphi \in \text{PSH}^- (\Omega); \ \int_{-t}^{\infty} t^n \chi'(t) \text{cap}_\Omega(\varphi < -t) dt < +\infty \right\}.
\] (15)
Proposition 5. One has $\mathcal{E}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$, while
$\mathcal{E}_\chi(\Omega) \subset \mathcal{E}_\chi(\Omega)$, where $\chi(t) = \chi\left(\frac{t}{2}\right)$.
(16)
Moreover, if $\chi : (-\infty, -t_\chi] \to \mathbb{R}^-$ is convex, then
$\mathcal{E}_\chi(\Omega) = \mathcal{E}_\chi(\Omega)$.
(17)
Here $t_\chi$ denote the real number satisfying $\chi(t) < 0$, for all $t < -t_\chi$,
and $\chi(t) = 0$, for all $t \geq -t_\chi$.
Proof. Compare with [12, 20].

Theorem 6. Let $\chi : \mathbb{R}^- \to \mathbb{R}^-$ be an increasing convex
function such that $\chi(-\infty) = -\infty$. The following conditions are equivalent:

1. there exists a unique function $\varphi \in \mathcal{E}_\chi(\Omega)$ such that

\[
\mu = (dd^c \varphi)^n;
\]
2. $\chi(\mathcal{E}_\chi(\Omega)) \subset L^1(\Omega, d\mu)$;
3. there exists a constant $C > 0$ such that

\[
\int \chi(u) d\mu \leq C, \ \forall u \in \mathcal{E}_0(\Omega);
\] (18)
4. there exists a constant $A > 0$ such that

\[
\int \chi(u) d\mu \leq C_2 \max \left(1, \left(\int_0^\infty s^n \chi'(s) \text{cap}_\Omega(u < -s) ds\right)^{1/n}\right),
\] (19)
\[
\forall u \in \mathcal{E}_0(\Omega);
\]
5. there exists a locally bounded function $F : \mathbb{R}^+ \to \mathbb{R}^+$
such that $\limsup_{t \to +\infty} F(t)/t < 1$ and

\[
\int \chi(u) F(u) d\mu \leq \int \chi(u) d\mu \leq F(C_\chi(u)), \ \forall u \in \mathcal{E}_0(\Omega).
\] (20)

The equivalences (1) $\leftrightarrow$ (3) $\leftrightarrow$ (4) are proved in [11]
(Theorem 5.1) and the implication (5) $\Rightarrow$ (1) is proved in
[12] (Theorem 5.2). For the sake of completeness we include a complete proof.

Proof. We start by the implication (1) $\Rightarrow$ (2). Let $u, \varphi \in \mathcal{E}_\chi(\Omega)$. It follows from Proposition 5 that $u + \varphi \in \mathcal{E}_\chi(W)$.
Hence

\[
\int \chi(u + \varphi) (dd^c (u + \varphi))^n < \infty.
\] (21)

Now, for the implication (2) $\Rightarrow$ (3), assume that (3) is not satisfied. Then for each $j \in \mathbb{N}$ we can find a function $u_j \in \mathcal{E}_0(\Omega)$ such that

\[
\int \chi(u_j) d\mu \geq 2^{3j}.
\] (22)

Consider the function

\[
u := \sum_{j=1}^\infty \frac{1}{2^{3j}} u_j;
\] (23)

Observe that

\[
(\nu < 0) \subset \left(\bigcup_{j=1}^\infty (u_j < -2^j)\right).
\] (24)

Hence

\[
\text{cap}_\Omega(\nu < 0) \leq \sum_{j=1}^\infty \text{cap}_\Omega(u_j < -2^j);
\] (25)

Now, since the weight $\chi$ is convex or homogeneous and using the estimates (14), we get

\[
\int_0^\infty s^n \chi'(s) \text{cap}_\Omega(u < -s) ds
\]
\[
\leq \sum_{j=1}^\infty \int_0^\infty (2^{1/2s})^n \chi'(s) \text{cap}_\Omega(u < -2^j) ds
\]
\[
\leq \sum_{j=1}^\infty \int_0^\infty (2^{1/2s})^n \chi'(s) \text{cap}_\Omega(u < -2^j) ds
\]
\[
\leq 2^n \int_0^\infty \frac{1}{2^{3j}} < \infty.
\] (26)
Hence \( u \in E_\chi(\Omega) \). On the other hand, from (22) we have
\[
\int_\Omega -\chi(u) \, d\mu \geq \frac{1}{2n} \int_\Omega -\chi(u_j) \, d\mu \geq 2^j, \quad \forall j \in \mathbb{N},
\]
which yields a contradiction.

Now, we prove that (3) \( \Rightarrow \) (4). Let \( \psi \in \mathcal{E}_0(\Omega) \), denote\( E_j(\psi) := \int_\Omega -\chi(\psi) (dd^c \psi)^n \). If \( \psi \in \mathcal{E}_\chi(\Omega) \), that is, \( C_\chi(\psi) \leq 1 \), then
\[
\int_\Omega -\chi(\psi) \, d\mu \leq 2^n = C.
\]
If \( C_\chi(\psi) > 1 \) the function \( \bar{\psi} \) defined by
\[
\bar{\psi} := \frac{\psi}{1 + C_\chi(\psi)^{1/n}} \in \mathcal{E}_\chi(\Omega).
\]
Indeed, from the monotonicity of \( \chi \), we have
\[
\int_0^\infty \chi'(s) s^n \text{cap}_\Omega \left( \frac{\psi}{1 + C_\chi(\psi)^{1/n}} < -s \right) \, ds = \frac{1}{C_\chi(\psi)} \int_0^\infty \chi'(s) (sC_\chi(\psi)^{1/n})^n \cdot \text{cap}_\Omega \left( \psi < -s - sC_\chi(\psi)^{1/n} \right) \, ds
\leq \frac{1}{C_\chi(\psi)} \int_0^\infty \chi'(s) s^n \text{cap}(\psi < -s) \, ds = 1.
\]
It follows from (18) and the convexity of \( \chi \) that
\[
\int_\Omega -\chi(\psi) \, d\mu \leq 2C_\chi(\psi)^{1/n} \int_\Omega -\chi \left( \frac{\psi}{1 + C_\chi(\psi)^{1/n}} \right) \, d\mu
\leq AC_\chi(\psi)^{1/n}.
\]
Hence we get (19).

For the implication (4) \( \Rightarrow \) (5), we consider \( F(t) = A \text{max}(1, t^{1/n}) \).

(5) \( \Rightarrow \) (1). It follows from [12] (Theorem 4.5) that the class \( E_\chi(\Omega) \) characterizes pluripolar sets in the sense that if \( P \) is a locally pluripolar subset of \( \Omega \) then \( P \subset \{ v = -\infty \} \), for some \( v \in E_\chi(\Omega) \). Then assumption (20) on \( \mu \) implies that it vanishes on pluripolar sets. It follows from [13] that there exists a function \( u \in \mathcal{E}_\chi(\Omega) \) and \( f \in L^1_{\text{loc}}((dd^c u)^n) \) such that \( \mu = f(dd^c u)^n \).

Consider \( \mu_j := \text{min}(f, j)(dd^c u)^n \). This is a finite measure which is bounded from above by the complex Monge-Ampère measure of a bounded function. It follows therefore from [3] that there exist \( \varphi_j \in E_\chi(\Omega) \) such that
\[
(dd^c \varphi_j)^n = \text{min}(f, j)(dd^c u)^n.
\]
The comparison principle shows that \( \varphi_j \) is a decreasing sequence. Set \( \varphi = \lim_{j \to \infty} \varphi_j \). It follows from (20) that
\[
\int_\Omega -\chi(\varphi_j) (dd^c \varphi_j)^n \leq F \left( \int_0^\infty \chi'(s) s^n \text{cap}_\Omega(\varphi_j < -s) \, ds \right).
\]

Hence
\[
\sup_j \int_0^\infty \chi'(s) s^n \text{cap}_\Omega(\varphi_j < -s) \, ds < \infty.
\]
This implies that
\[
\int_0^\infty \text{cap}_\Omega(\{ \varphi < -t \}) \, dt < +\infty.
\]
Then \( \varphi \neq -\infty \) and therefore \( \varphi \in E_\chi(\Omega) \).

We conclude now by continuity of the complex Monge-Ampère operator along decreasing sequences that \( (dd^c \varphi)^n = \mu \). The uniqueness of \( \varphi \) follows from the comparison principle.

\[ \square \]

### 3. The Weighted Energy Class with Boundary Values

Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be a nondecreasing function and let \( f \in \mathcal{M}(\Omega) \) be a maximal psh function. We define the class \( E_\chi(\Omega) \) (resp., \( \mathcal{N}(f) \), \( \mathcal{F}(f) \), \( \mathcal{F}^n(f) \), and \( \mathcal{F}^a(f) \)) to be the class of psh functions \( u \) such that there exists a function \( \varphi \in E_\chi(\Omega) \) (resp., \( \mathcal{N}(f) \), \( \mathcal{F}(f) \), \( \mathcal{F}^n(f) \), and \( \mathcal{F}^a(f) \)) such that
\[
\varphi(z) + f(z) \leq u(z) \leq f(z), \quad \forall z \in \Omega.
\]

Later on, we will use repeatedly the following well known comparison principle from [1] as well as its generalizations to the class \( \mathcal{N}(f) \) (cf. [10, 14]).

**Theorem 7** (see [1, 10, 14]). Let \( f \in E_\chi(\Omega) \) be a maximal function and \( u, v \in \mathcal{N}(f) \) be such that \( (dd^c u)^n \) vanishes on all pluripolar sets in \( \Omega \). Then
\[
\int_{\{u < v\}} (dd^c u)^n \leq \int_{\{u < v\}} (dd^c v)^n.
\]

Furthermore if \( (dd^c u)^n = (dd^c v)^n \) then \( u = v \).

The following lemma, which gives an estimate of the size of sublevel set in terms of the mass of Monge-Ampère measure, will be useful shortly.

**Lemma 8.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be a nondecreasing function such that \( \chi(t) < 0 \), for all \( t < 0 \), and \( f \in \mathcal{E}_\chi(\Omega) \) be a maximal function. Then for all \( \varphi \in E_\chi(f) \)
\[
t^n \text{cap}_\Omega(\varphi < -s - t + f)
\leq \int_{\{\varphi < -s - t + f\}} (dd^c \varphi)^n, \quad \forall s > 0, \ t > 0.
\]

**Proof.** Fix \( s, t > 0 \). Let \( K \subset \{ \varphi < f - s - t \} \) be a compact subset. Then
\[
\text{cap}_\Omega(K) = \int_\Omega (dd^c u_K)^n = \int_{\{\varphi < f - s - t\}} (dd^c u_K)^n
\leq \int_{\{\varphi < -s - t\}} (dd^c \varphi)^n.
\]
where \( u^*_K \) is the relative extremal function of the compact \( K \) and \( v := f - s + tu^*_K \). It follows from Theorem 7 that
\[
\frac{1}{t^n} \int_{(\varphi < v)} (dd^c v)^n = \frac{1}{t^n} \int_{(\varphi < \max(\varphi,v))} (dd^c \max(\varphi,v))^n \\
\leq \frac{1}{t^n} \int_{(\varphi < \max(\varphi,v))} (dd^c \varphi)^n \\
= \frac{1}{t^n} \int_{(\varphi < f - s + tu^*_K)} (dd^c \varphi)^n \\
\leq \frac{1}{t^n} \int_{(\varphi < f - s)} (dd^c \varphi)^n.
\]
Taking the supremum over all \( K \)'s yields the first inequality. \( \square \)

**Proposition 9.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function. Then one has
\[
\mathcal{E}_\chi(f) \subset \left\{ u \in \text{PSH}(\Omega) ; u \leq f, \right\}
\]
\[
\int_0^{+\infty} s^\chi'(s) \text{ cap}_\Omega (u < f - 2s) ds < +\infty \right\}. \tag{41}
\]
In particular, if \( \chi \not\equiv 0 \), then \( \text{cap}_\Omega (u < f - s) < +\infty \) for all \( s > 0 \) and \( u \in \mathcal{E}_\chi(f) \).

**Proof.** Let \( u \in \mathcal{E}_\chi(f) \). Then there exists a function \( \varphi \in \mathcal{E}_\chi(\Omega) \) such that \( \varphi + f \leq u \). Therefore \( (u < f - s) \subset (\varphi < -s) \). It follows from Lemma 8
\[
\int_0^{+\infty} s^\chi'(-s) \text{ cap}_\Omega (u < f - 2s) ds \\
\leq \int_0^{+\infty} s^\chi'(-s) \text{ cap}_\Omega (\varphi < -2s) ds \\
\leq \int_0^{+\infty} \chi'(-s) \int_{(\varphi < -s)} (dd^c \varphi)^n ds \\
= \int_\Omega -\chi(\varphi) (dd^c \varphi)^n < \infty.
\]
\( \square \)

**Theorem 10.** Let \( \chi : \mathbb{R}^- \to \mathbb{R}^- \) be an increasing function which satisfies \( \chi(-\infty) = -\infty \) and \( f \in \mathcal{E}_\chi(\Omega) \) a maximal function. Then if there exists a decreasing sequence \( u_j \in \mathcal{E}_0(f) \) such that
\[
\sup_j \int_\Omega -\chi(u_j - f) (dd^c u_j)^n < \infty \tag{43}
\]
then \( u := \lim_{j \to \infty} u_j \in \mathcal{E}_\chi(f) \) and \( \chi(u - f) \in L^1((dd^c u)^n) \).

Conversely, if \( u \in \mathcal{E}_\chi(f) \) and \( \chi(u - f) \in L^1((dd^c u)^n) \) then there exists sequence \( u_j \in \mathcal{E}_0(f) \) decreasing towards \( u \) such that
\[
\sup_j \int_\Omega -\chi(u_j - f) (dd^c u_j)^n < \infty. \tag{44}
\]

**Proof.** Assume that \( \mu = (dd^c v)^n \) for some \( v \in \mathcal{E}_\chi(\Omega) \). Let \( \Omega^j \) be a fundamental sequence of strictly pseudoconvex subsets of \( \Omega \). Choose a sequence \( f_j \in \text{PSH}(\Omega^j) \cap C(\Omega) \) decreasing towards \( f \) on \( \Omega \) and \( f_j \) is maximal on \( \Omega^j_{\rho_{j+1}} \). It follows from [13] that there exist a function \( g \in \mathcal{E}_0 \) and a function \( \theta \in L^1_{\text{loc}}(dd^c g)^n \) such that
\[
\mu = \theta(dd^c g)^n. \tag{49}
\]
Consider the measure \( \mu_j = \mathcal{P}_{\Omega_j} \min(\theta, j)(dd^c g)^n \), where \( \mathcal{P}_{\Omega_j} \) denotes the characteristic function of the set \( \Omega_j \). Now, solving the Dirichlet problem in the strictly pseudoconvex domain \( \Omega_j \), we state that there exist functions \( u_j, v_j \in \text{PSH}(\Omega_j) \cap C(\overline{\Omega_j}) \) such that

\[
(dd^c u_j)^n = (dd^c v_j)^n = \mu_j, \quad v_j = 0, \quad u_j = f_j \quad \text{on } \partial \Omega_j.
\]

(50)

By the comparison principle, we have \( u_j, v_j \) are decreasing sequences and

v + f \leq v_j + f_j \leq u_j \leq f_j \quad \text{on } \Omega_j.

(51)

Letting \( j \to +\infty \) we get that \( u := \lim_{j \to +\infty} u_j \in \mathcal{E}_c(f) \). The continuity of the complex Monge-Ampère operator under monotonic sequences yields that \((dd^c u)^n = \mu\). Uniqueness of \( u \) follows from the comparison principle.

**Corollary 12.** Let \( \mu \) be nonnegative measure in \( \Omega \) with total finite mass \( \mu(\Omega) < +\infty \) and let \( f \) be a maximal function. Then there exists a uniquely determined function \( \varphi \in \mathcal{F}^a(f) \) such that \((dd^c \varphi)^n = \mu \) if and only if \( \mu \) vanishes on pluripolar subsets.

**Proof.** It follows from [13] that there exist a function \( \psi \in \mathcal{E}_c(\Omega) \) and a function \( \theta \in L^1_{loc}(dd^c g)^n \) such that

\[
\mu = \theta \ (dd^c \psi)^n.
\]

(52)

By [3], there exists a unique \( h \in \mathcal{E}_c(\Omega) \) such that \((dd^c h)^n = \min(\theta, j)(dd^c g)^n \). The comparison principle yields that \( h \) is a decreasing sequence. Let denote by \( h := \lim_{j \to +\infty} h_j \). It follows from Lemma 8 that \( h \neq -\infty \). Therefore \( h \in \mathcal{F}^a \). By the continuity of the complex Monge-Ampère operator under decreasing sequences, we have \((dd^c h)^n = \mu \). Now, since

\[
\mathcal{F}^a = \bigcup_{x \in \text{convex; } x(0) \leq \Omega} \mathcal{E}_c(\Omega)
\]

(53)

then there exists a convex function \( \chi : \mathbb{R} \to \mathbb{R} \) with \( \chi(0) \neq 0 \) and \( \chi(-\infty) = -\infty \) such that \( h \in \mathcal{E}_c(\chi) \). By Theorem 11, we can find a unique function \( \varphi \in \mathcal{E}_c(\Omega) \) such that \((dd^c \varphi)^n = \mu \). \( \square \)

**4. Measures Dominated by Capacity**

Throughout this section, \( \mu \) denotes a fixed nonnegative measure of finite total mass \( \mu(\Omega) < +\infty \). We want to solve the Dirichlet problem

\[
(dd^c \varphi)^n = \mu, \quad \text{with } \varphi \in \mathcal{F}^a(f), \quad \varphi|_{\partial \Omega} = \psi_f,
\]

(54)

and measure how far the distance between the solution \( \varphi \) and the given boundary data \( f \) is from being bounded, by assuming that \( \mu \) is suitable dominated by the Monge-Ampère capacity.

Measures dominated by the Monge-Ampère capacity have been extensively studied by Kołodziej in [3–5]. The main result of his study, achieved in [4], can be formulated as follows. Fix \( \varepsilon : \mathbb{R} \to [0, +\infty) \) a continuous decreasing function and set \( F_\varepsilon(x) := x[\varepsilon(-\ln x/n)]^n \). If for all compact subsets \( K \subset \Omega \)

\[
\mu(K) \leq F_\varepsilon(\text{cap}_K(K)), \quad \text{where } \int_0^{+\infty} \varepsilon(t) \, dt < +\infty,
\]

(55)

and \( l : \partial \Omega \to \mathbb{R} \) is a continuous function, then \( \mu = (dd^c \varphi)^n \) for some continuous function \( \varphi \in \text{PSH}(\Omega) \) with \( \varphi|_{\partial \Omega} = l \).

The condition \( \int_0^{+\infty} \varepsilon(t) \, dt < +\infty \) means that \( \varepsilon \) decreases fast enough towards zero at infinity. This gives a quantitative estimate on how fast \( e(-\ln \text{Cap}_K(K)/n) \), hence \( \mu \) decreases towards zero as \( \text{Cap}_K(K) \to 0 \).

When \( \int_0^{+\infty} \varepsilon(t) \, dt = +\infty \), it is still possible to show that \( \mu = (dd^c \varphi)^n \) for some function \( \varphi \in \mathcal{F}(\Omega) \), but \( \varphi \) will generally be unbounded. We now measure how far it is from being so.

**Theorem 13.** Let \( \mu \) be a nonnegative finite measure. Assume for all compact subsets \( K \subset \Omega \)

\[
\mu(K) \leq F_\varepsilon(\text{cap}_K(K)),
\]

(56)

Then there exists a unique function \( \varphi \in \mathcal{F}^a(f) \) such that \( \mu = (dd^c \varphi)^n \), and

\[
\text{cap}_K(\{\varphi < f - s\}) \leq \exp\left(-nH^{-1}(s)\right), \quad \forall s > 0.
\]

(57)

Here \( H^{-1} \) is the reciprocal function of \( H(x) = e \int_0^x \varepsilon(t) \, dt + ee(0) + \mu(\Omega)/n \).

The proof is almost the same as that of Theorem 5.1 in [12], except that we use Corollary 12 for the existence of the solution and Lemma 8 to estimate the capacity of sublevel set.

Observe that if \( \int_0^{+\infty} \varepsilon(t) \, dt < +\infty \) then \( H \) is bounded by \( e \int_0^{+\infty} \varepsilon(t) \, dt + ee(0) + \mu(\Omega)/n \). Hence \( H^{-1}(t) = +\infty \), \( \forall t \geq e \int_0^{+\infty} \varepsilon(t) \, dt + ee(0) + \mu(\Omega)/n \). Therefore

\[
0 \leq f - \varphi \leq e \int_0^{+\infty} \varepsilon(t) \, dt + ee(0) + \mu(\Omega)/n.
\]

(58)

Now, we consider the case when \( \mu = f d\lambda \) is absolutely continuous with respect to Lebesgue measure.

Let \( G \subset C^n \) denote a generic subspace of \( C^n \) that is a real subspace such that \( G + fG = C^n \), where \( G \) is the usual complex structure on \( C^n \) (cf. [21] for more details). \( G \) will be endowed with the induced Euclidean structure and the corresponding Lebesgue measure which will be denoted by \( \lambda_G \).

Let \( \alpha > 0 \) be a positive real number. According to [22, 23], the Orlicz space \( \text{Log}^{a*}(d\lambda_G) \) consists of \( \lambda_G \)-measurable functions \( g \) defined on \( \Omega \cap G \) such that

\[
\int_{\Omega \cap G} \frac{|f|^s}{\lambda} \log^{a*} \left(1 + \frac{|f|}{\lambda}\right) d\lambda_G < +\infty, \quad \text{for some } \lambda > 0.
\]

(59)

On the space \( \text{Log}^{a*}(d\mu) \), we define the norm

\[
\|f\|_{\text{Log}^{a*}(d\mu)} := \inf \left\{ \lambda > 0; \int_{\Omega} \frac{|f|^s}{\lambda} \log^{a*} \left( e + \frac{|f|}{\lambda}\right) d\lambda_G < 1 \right\}.
\]

(60)
The dual space to $L^{\text{log}^{n+\alpha}}L$ is the exponential class $\text{Exp}^{1/n+\alpha}$; that is, the vector space

$$\text{Exp}^{1/n+\alpha} := \left\{ f : \Omega \to \mathbb{C} \; \exists \lambda > 0 : \int_{\Omega} \exp\left(\frac{|f|}{\lambda}\right)^{1/n+\alpha} - 1d\lambda_G < \infty \right\},$$

equipped with the norm

$$\left\| f \right\|_{\text{Exp}^{1/n+\alpha}} := \inf\left\{ \lambda > 0 ; \int_{\Omega} \left( \exp\left(\frac{|f|}{\lambda}\right)^{1/n+\alpha} - 1 \right) d\lambda_G < 1 \right\}.$$  

for $f \in L^{\text{log}^{n+\alpha}}L$ and $g \in \text{Exp}^{1/n+\alpha}$, where $C_{n,\alpha} > 0$ is a positive constant depending only on $n$ and $\alpha$. By a simple computation, we have

$$\left\| 1 \right\|_{\text{Exp}^{1/n+\alpha}(K)} = \frac{1}{\text{log}^{n+\alpha}(1 + 1/\lambda_G(K))}. \quad (61)$$

**Corollary 14.** Let $\mu = \int_D gd\lambda_G$ be a measure with nonnegative density $g \in L^{\text{log}^{n+\alpha}}L(\Omega \cap G)$. Then there exists a unique bounded function $\varphi \in \mathcal{P}(f) \cap L^{\infty}(\Omega)$ such that $(dd^c \varphi)^n = \mu$ and

$$0 \leq f - \varphi \leq C \left\| g \right\|_{L^{\text{log}^{n+\alpha}}L}, \quad (62)$$

where $C > 0$ only depends on $n, \alpha, \Omega$, and $G$.

**Proof.** We claim that there exists a constant $C > 0$ such that

$$\mu(K) \leq C \left\| g \right\|_{L^{\text{log}^{n+\alpha}}L}^{1/n} \cdot \text{cap}_\Omega^{(\alpha+\alpha)/n}(K), \quad (63)$$

for compact $K \subset \Omega$.

Indeed, Hölder’s inequality and inequality (66) yield

$$\mu(K) \leq \left\| g \right\|_{L^{\text{log}^{n+\alpha}}L} \cdot \frac{1}{\text{log}^{n+\alpha}(1 + 1/\lambda_G(K))}, \quad (64)$$

for compact $K \subset \Omega$.

By [21] we have

$$\lambda_G(K) \leq C \exp\left( -\frac{1}{\text{cap}_\Omega(K)} \right), \quad \forall \text{compact } K \subset \Omega,$$

where $C > 0$ is a constant which depends only on $\Omega$ and $G$.

Inequality (66) follows by combining (67) and (68). Then we apply Theorem 13 with

$$\varepsilon(x) = C \left\| g \right\|_{L^{\text{log}^{n+\alpha}}L} C \left\| g \right\|_{L^{\text{log}^{n+\alpha}}L} e^{-\alpha x/n}, \quad (69)$$

which yields

$$0 \leq f - \varphi \leq \varepsilon \int_0^x \varepsilon(t) dt + \varepsilon(0) + \mu(\Omega)^{1/n} \quad (70)$$

\hfill \Box

**Conflict of Interests**

The author declares that there is no conflict of interests regarding the publication of this paper.

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