Analytical and numerical studying of the perturbed Burgers equation

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Abstract

The perturbed Burgers equation is considered. This equation is used for the description of one–dimensional viscous gas dynamics, nonlinear waves in a liquid with gas bubbles and nonlinear acoustic waves. The integrability of this equation is investigated using the Painleve approach. The condition for parameters for the integrability of the perturbed Burgers equation is established. New classical and nonclassical symmetries admitted by this equation are found. All corresponding symmetry reductions are obtained. New exact solutions of these reductions are constructed. They are expressed via trigonometric and Airy functions. Stability of the exact solutions of the perturbed Burgers equation is investigated numerically.

1 Introduction

Nonlinear evolution equations play an important role in the modern mathematics and physics. There are several “universal” equations that are widely physical applicable and have remarkable mathematical properties (see e.g., Refs. [5,22]). In this set of equations one can include the Korteweg–de Vries equation, the Burgers equation, the nonlinear Schrodinger equation. Such “universal” equations are very often obtained with the help of some asymptotic approach, for example, with the help of the reductive perturbation method (see e.g., Refs. [13,22,39]). If we consider the first order approximation in the asymptotic expansion we can obtain famous nonlinear evolution equations such as the Burgers and the Korteweg–de Vries equations. On
the other hand, taking into account high order corrections in the reductive perturbation method one can obtain generalizations of the above mentioned equations. These equations can be considered as “universal” equations with high order corrections.

In this work we study the Burgers equation with high order corrections: the perturbed Burgers equation. This equation was obtained for the description of viscous gas dynamics and nonlinear acoustic waves in Refs. [8,14]. In Refs. [19,20] it was shown that the perturbed Burgers equation can be used for the description of nonlinear waves in a liquid with gas bubbles. This equation has the form (see Refs. [19,20]):

\[
 u_t + \alpha uu_x - \mu u_{xx} = \varepsilon \left( (2\mu\alpha_2 + \mu\alpha + \nu)uu_{xx} + (2\mu\alpha_1 + \mu\alpha + \nu)u_x^2 - \frac{\alpha(\alpha_2 + 2\alpha_1)}{2} u^2u_x - (\beta + \mu^2)u_{xxx} \right).
\]

(1)

where \(t\) is non–dimensional time, \(x\) is a non–dimensional Cartesian coordinate, \(u\) is non–dimensional perturbation of gas–liquid mixture density, \(\alpha, \beta, \nu, \mu\) are non–dimensional physical parameters, \(\alpha_1, \alpha_2\) are arbitrary parameters introduced by the near–identity transformations (see, Ref. [20]). Let us note that in fact Eq.(1) is a two–parametric family of equations parameterized by \(\alpha_1\) and \(\alpha_2\).

It is known that at the derivation of the perturbed Burgers equation appears an obstacle to the integrability (see Refs. [8,14,19,20,32]). The perturbed Burgers equation is integrable under a certain condition on the physical parameters in contrast to the Burgers equation. In the integrable case the perturbed Burgers equation coincides with the Sarmo–Tasso–Olver equation which was investigated (see Refs. [7,16,18,26,31]). However, the general case of the perturbed Burgers equation was not studied previously. Thus it is an interesting problem to perform an analytical and numerical study of the perturbed Burgers equation without imposing any conditions on the physical parameters.

Below we show that the perturbed Burgers equation can be transformed to a canonical form with a single parameter. This parameter can be considered as the bifurcation parameter, because there is only one value of this parameter corresponding to the integrable case of the perturbed Burgers equation. The aim of this work is to analytically and numerically study the perturbed Burgers equation at all values of the bifurcation parameter.

Using the Painlevé approach for partial differential equations we illustrate that the perturbed Burgers equation is integrable only at a certain value of
the bifurcation parameter. We construct classical and nonclassical symmetries of the perturbed Burgers equations. We show that in the general case the perturbed Burgers equation admits three classical Lie groups. In the integrable case the perturbed Burgers equations admits an infinite dimensional Lie group. Both regular and singular cases of nonclassical symmetries are considered. The regular nonclassical symmetries coincide with the classical Lie symmetries. In the singular case we find symmetries which do not correspond to classical Lie symmetries. We construct symmetry reductions of the perturbed Burgers equation and obtain their exact solutions. We numerically study the stability of a certain periodic solution of the integrable case of the perturbed Burgers equation under the variations of the bifurcation parameter. We show that this solution is stable under variations of the bifurcation parameter. We also find that initial solitary waves governed by the perturbed Burgers equation evolve to a weak shock wave with oscillating structure. We construct analytical solution of the integrable case of the perturbed Burgers equation that is similar to this weak shock wave.

To the best of our knowledge classical and nonclassical symmetries of the perturbed Burgers equation were not obtained previously. We believe that all exact solutions constructed in this work are new. The stability of the exact solutions of the perturbed Burgers equation under the variations of the bifurcation parameter is studied for the first time as well.

This work is organized as follows. In Sec. 2 we transform the perturbed Burgers equation to the canonical form. The Painleve analysis of this equation is carried out as well. Symmetries of the perturbed Burgers equation are studied in Sec. 3. We investigate both classical and nonclassical symmetries of this equation. Symmetry reductions of the perturbed Burgers equation are obtained in Sec. 4. We construct exact solutions for the symmetry reductions of the perturbed Burgers equation in Sec. 4 as well. Sec. 5 is devoted to the numerical investigation of the perturbed Burgers equation. We give final remarks and conclusions in Sec. 6.

2 Transformation to the canonical form and Painleve analysis

In this work we will use the perturbed Burgers equation in the form obtained in Refs. [19, 20]. Eq. (1) can be simplified with the help of the scaling transformations and using arbitrariness of $\alpha_1$, $\alpha_2$. Indeed, let $\alpha_1$ and $\alpha_2$ have the form

$$\alpha_1 = \alpha_2 = \frac{\mu \alpha + \nu}{\mu}.$$  

(2)
Substituting the transformations
\[ x' = x + \frac{\alpha \mu}{6 \varepsilon (\mu \alpha + \nu)} t, \quad t' = (\beta + \mu^2) \varepsilon t, \quad u = \frac{\beta + \mu^2}{\mu \alpha + \nu} u' - \frac{\mu}{3 \varepsilon (\mu \alpha + \nu)}, \tag{3} \]
into equation (1) we have (primes are omitted)
\[ u_t - 3(uu_x)_x + 3\bar{\alpha} u^2 u_x + u_{xxx} = 0, \tag{4} \]
where
\[ \bar{\alpha} = \frac{\alpha (\beta + \mu^2)}{2 \mu (\mu \alpha + \nu)}. \tag{5} \]

Further we study Eq. (4).

Let us investigate whether Eq. (4) possesses the Painlevé property for partial differential equations. In accordance with the algorithm by Weiss, Tabor and Carnevale [33] we present a solution of Eq. (4) in the form
\[ u(x,t) = \phi^p \sum_{j=0}^{\infty} u_j \phi^j, \quad u_j \equiv u_j(x,t), \quad \phi \equiv \phi(x,t). \tag{6} \]

The necessary condition for Eq. (4) to possess the Painlevé property is that expansion (6) contains three arbitrary functions.

Substituting (6) into the leading order terms of Eq. (4) we get
\[ p = -1, \quad u_0^{(1,2)} = \frac{-3 \pm \sqrt{9 - 8\bar{\alpha}}}{2\bar{\alpha}} \phi_x. \tag{7} \]
Thus Eq. (4) admits two expansions in the form (6). Substituting the expression
\[ u(x,t) = u_0^{(1,2)} \phi^{-1} + u_j \phi^{j-1}, \tag{8} \]
into the leading order terms of Eq. (4) we find the Fuchs indices
\[ j_1^{(1)} = -1, \quad j_2^{(1)} = 3, \quad j_3^{(1)} = 4 + \frac{3 \sqrt{9 - 8\bar{\alpha}} - 9}{2\bar{\alpha}}, \]
\[ j_1^{(2)} = -1, \quad j_2^{(2)} = 3, \quad j_3^{(2)} = 4 - \frac{3 \sqrt{9 - 8\bar{\alpha}} + 9}{2\bar{\alpha}}. \tag{9} \]

At this step the necessary condition for Eq. (4) to possess the Painlevé property is that all Fuchs indices are real integers. Consequently \( j_3^{(1)} \) and \( j_3^{(2)} \) have to be real integers. And the expression
\[ j_3^{(1)} - 4 = \frac{3 \sqrt{9 - 8\bar{\alpha}} - 9}{\bar{\alpha}} \equiv - \frac{72}{3 \sqrt{9 - 8\bar{\alpha}} + 9}. \tag{10} \]
has to be an integer as well.
Expression in the right–hand side of (10) is an integer only at the following values of \( \bar{\alpha} \)
\[
\bar{\alpha}_1 = \frac{9}{8}, \quad \bar{\alpha}_2 = \frac{54}{49}, \quad \bar{\alpha}_3 = 1, \quad \bar{\alpha}_4 = \frac{18}{25}, \quad \bar{\alpha}_5 = 0, \quad \bar{\alpha}_6 = -2, \quad \bar{\alpha}_7 = -9, \quad \bar{\alpha}_8 = -54.
\]
(11)
We recall that both of indices \( j_3^{(1)}, j_3^{(2)} \) must be integers. This is true only for three values of \( \bar{\alpha} \): \( \bar{\alpha}_1, \bar{\alpha}_3, \bar{\alpha}_7 \).
At \( \bar{\alpha}_1 = \frac{9}{8} \) from (9) we obtain
\[
j_1^{(1)} = -1, \quad j_2^{(1)} = 3, \quad j_3^{(1)} = 0, \quad j_1^{(2)} = -1, \quad j_2^{(2)} = 3, \quad j_3^{(2)} = 0.
\]
(12)
In this case Eq. (11) does not possess the Painlevé property since the coefficient \( u_0 \) is not arbitrary function.
In the case of \( \bar{\alpha}_7 = -9 \) Fuchs indices (9) have the form
\[
j_1^{(1)} = -1, \quad j_2^{(1)} = 3, \quad j_3^{(1)} = 3, \quad j_1^{(2)} = -1, \quad j_2^{(2)} = 3, \quad j_3^{(2)} = 6.
\]
(13)
We see that there are multiple Fuchs indices. It is indicated that expansion (6) contains logarithmic terms. Thus Eq. (11) does not possess the Painlevé property at \( \bar{\alpha}_7 = -9 \).
At \( \bar{\alpha}_3 = 1 \) Fuchs indices (9) have the form
\[
j_1^{(1)} = -1, \quad j_2^{(1)} = 3, \quad j_3^{(1)} = 1, \quad j_1^{(2)} = -1, \quad j_2^{(2)} = 3, \quad j_3^{(2)} = -2.
\]
(14)
So in this case the necessary condition of the Painlevé test is held. Indeed, at \( \bar{\alpha}_3 = 1 \) Eq. (11) is the Sharmo–Tasso–Olver equation (see Refs. [26, 31]) which is linearizable by the Cole–Hopf transformation.

The following cases of parameter \( \bar{\alpha} \) are also interesting: \( \bar{\alpha}_4, \bar{\alpha}_5, \bar{\alpha}_6 \). For \( \bar{\alpha}_4, \bar{\alpha}_6 \) the Fuchs indices in the second expansion (6) are integers. In the case of \( \bar{\alpha} = \bar{\alpha}_5 \) equation (6) admits one expansion of form (6) with Fuchs indices that are integers. However, this expansion does not correspond to the general solution. In our opinion the presence of the Fuchs indices that are integers indicates that Eq. (11) can admit meromorphic solutions.

In this section we have found that Eq. (11) passes the Painleve test only in one case: \( \bar{\alpha} = 1 \). At any other value of \( \bar{\alpha} \) the equation does not have the Painleve property. This fact indicates that Eq. (11) is integrable only at \( \bar{\alpha} = 1 \) and this is really the case.
3 Symmetry analysis

Let us study symmetries and symmetry reductions of Eq. (4). Here we repeat Eq. (4) for the convenience

\[ E(u) = u_t - 3(uu_x)_x + 3\tilde{\alpha}u^2u_x + u_{xxx} = 0. \]  

(15)

Below we consider the application of classical and nonclassical methods for finding symmetries of (15).

3.1 Classical Lie method

First we apply the classical Lie method (see e.g., Refs. [11,27,28]) to Eq. (15). In accordance with this method Eq. (15) is invariant under transformations

\[ \tilde{x} = x + a\xi(x,t,u), \quad \tilde{t} = t + a\tau(x,t,u), \quad \tilde{u} = u + a\eta(x,t,u), \]  

(16)

if the determining equations are satisfied on the solutions of (15)

\[ X^{(3)}E|_{E=0} = 0. \]  

(17)

Here \( X^{(3)} \) is the third prolongation of the infinitesimal generator

\[ X = \xi \frac{\partial}{\partial x} + \tau \frac{\partial}{\partial t} + \eta \frac{\partial}{\partial u} \]  

(18)

that has the form

\[ X^3 = X + \eta^t \frac{\partial}{\partial u_t} + \eta^x \frac{\partial}{\partial u_x} + \eta^{xx} \frac{\partial}{\partial u_{xx}} + \eta^{xxx} \frac{\partial}{\partial u_{xxx}}. \]  

(19)

Substituting expressions for \( \eta^t, \eta^x, \eta^{xx}, \eta^{xxx} \) into (17) we obtain an over-determined system of partial differential equations for \( \xi, \tau, \eta \). This system of equations has the general solution

\[ \xi = c_1x + c_3, \quad \tau = 3c_1t + c_2, \quad \eta = -c_1u, \]  

(20)

where \( c_1, c_2, c_3 \) are arbitrary constants.

Consequently Eq. (15) is invariant under three one–parameter Lie groups with the following generators

\[ X_1 = \frac{\partial}{\partial x}, \quad X_2 = \frac{\partial}{\partial t}, \quad X_3 = x \frac{\partial}{\partial x} + 3t \frac{\partial}{\partial t} - u \frac{\partial}{\partial u}. \]  

(21)
The first generator corresponds to space translation invariance, the second generator corresponds to time translation invariance and the third one corresponds to scaling invariance. Thus there are two classical symmetry reductions of Eq. (15). The first one is the traveling wave reduction corresponding to $X_1 + X_2$ and the second one is the self–similar reduction corresponding to $X_3$. In the next section we will study these symmetry reductions.

Let us note that we cannot indicate the integrable case of (15) using symmetry analysis presented above. One can find the integrable case of (15) studying classical symmetries for the potential form of this equation. Indeed, using the variable $u = v_x$ in (15) we find that at $\bar{\alpha} = 1$ this equation admits the generator

$$X_\infty = e^\nu \psi \partial_\nu,$$

(22)

where $\psi$ is any solution of the third order linear evolution equation

$$\psi_t + \psi_{xxx} = 0.$$  

(23)

Thus one can find the integrable case of (15) using the classical Lie symmetry analysis for the potential form of (15).

### 3.2 Nonclassical method

It is known that there exist symmetries which do not coincide with the classical symmetries. These symmetries are called nonclassical symmetries [1, 21, 25, 35]. The method for finding nonclassical symmetries for partial differential equations was proposed by Bluman and Cole [1]. This method was applied for finding symmetry reductions and exact solutions of various partial differential equations (see e.g., Refs. [2–4, 9, 10, 23, 29]). Here we apply the method by Bluman and Cole to study nonclassical symmetries of Eq. (15).

According to the method by Bluman and Cole [1] we considered the additional auxiliary equation

$$\xi u_x + \tau u_t - \eta = 0.$$  

(24)

Eq. (24) is the invariant surface condition associated with operator (18).

Then we look for classical symmetries admitted by Eqs. (15) and (24) simultaneously. One can show that the invariance condition for Eq. (24) satisfies without imposing any conditions on $\xi, \tau$ and $\eta$ (see Refs. [21, 35]).

Let us note that if $X$ is a nonclassical symmetry generator then $\lambda X$ is a nonclassical symmetry generator as well for any function $\lambda(x, t, u) \neq 0$ (see Refs. [21, 35]). Thus further we have to consider two cases of nonclassical
operators $X$. The first one is the case of $\tau \neq 0$, where without loss of generality we can assume that $\tau = 1$. The other one is the case of $\tau = 0$, where without loss of generality we can assume that $\xi = 1$. The case of $\tau \neq 0$ is called the regular case where we have an overdetermined system of equations for infinitesimals. In the singular case $\tau = 0$ we obtain a single partial differential equation for infinitesimal $\eta$. As it was shown by Kunzinger and Popovych [21] every solution of this equation generates family of solutions of a considered equation.

Let us study the case of $\tau \neq 0$ (we assume that $\tau = 1$). Applying the classical Lie method to (15) and (24) and using (24) and its differential consequences for eliminating derivatives involving time and using (15) for eliminating higher order derivative with respect to $x$ we obtain the system of equations for $\xi$ and $\eta$:

\[
\begin{align*}
\xi_u &= 0, \quad \eta_{uu} = 0, \quad \eta_u + \xi_x = 0, \\
\xi_{xx} + \eta + u \xi_x - \eta_{xx} &= 0, \\
\eta_t + \eta_{xxx} + 3\eta \xi_x - 3\eta_{xx} + 3\alpha u^2 \eta_x &= 0, \\
\xi_{xxx} + \xi_t - 3\eta_{xxu} + 3\xi_x + 6\eta_t - 6\alpha u \eta - 3u \xi_{xx} + 6u \eta_{xu} - 6\alpha u^2 \xi_x &= 0.
\end{align*}
\]

Solving this system of equations we find

\[
\xi = \frac{x + c_4}{3(t - c_5)^3}, \quad \eta = \frac{u}{3(t - c_5)},
\]

where $c_4, c_5$ are arbitrary constants.

Similarity variables connected with (26) have the form

\[
u = \frac{1}{(t - c_5)^{1/3}} f(\theta), \quad \theta = \frac{x + c_4}{(t - c_5)^{1/3}}.
\]

We can see that reduction corresponding to (27) coincides with the reduction generated by the linear combination of classical generators $X_1, X_2$ and $X_3$.

Let us consider the case of $\tau = 0$. In this case without loss of generality we assume $\xi = 1$. Applying the classical Lie approach and excluding space derivatives using (24) and its differential consequence and (15) we obtain the determining equation for infinitesimal $\eta$:

\[
\begin{align*}
\eta^3 \eta_{uuu} + 3 \left[2\alpha u - u \eta_{uu} + (\eta_{uu} - 2) \eta_u + \eta_{xu} \right] \eta^2 + \\
+ 3 \left[\eta_x (\eta_{uu} - 3) + (\eta_u - 2u) \eta_{xu} + 3\eta_{xxx} \right] \eta + \\
+ 3\alpha u^2 \eta_x + 3\alpha \eta \eta_{xx} - 3u \eta_{xx} + \eta_t + \eta_{xxx} &= 0.
\end{align*}
\]

We cannot find the general solution of Eq. (28). Kunzinger and Popovych [21] showed that the problem for construction all solutions of (28) is completely equivalent to the problem of the construction of all one-parametric
Table 1: Nonclassical infinitesimals admitted by (15)

| n | $\bar{\alpha}$ | $\eta$ |
|---|---|---|
| 1 | $\bar{\alpha} = \frac{18}{25}$ | $\eta^1 = \frac{u}{2x+c_6} + \frac{u^2}{(2x+c_6)^2}$ |
| 2 | $\bar{\alpha} \neq 0$ | $\eta^2 = \frac{1}{4} (3 \pm \sqrt{9 - 8\bar{\alpha}}) u^2 + \frac{c_7 \pm 2x}{6\sqrt{9 - 8\bar{\alpha}} + \sqrt{9 - 8\bar{\alpha}} \pm 2c_8}$ |
| 3 | $\bar{\alpha} \neq 0$ | $\eta^3 = \frac{1}{4} (3 \pm \sqrt{9 - 8\bar{\alpha}}) u^2 + c_9$ |
| 4 | $\bar{\alpha} = 0$ | $\eta^4 = \frac{1}{4} u^2 + c_{10}x + c_{11}$ |
| 5 | $\bar{\alpha} = 0$ | $\eta^5 = \frac{2x+c_{12}}{c_{11}-18t}$ |

solutions families of Eq. (15). However every particular solution of (28) generates nonclassical symmetry reduction of (15). Using this fact we can find exact solutions families of (15) that are not invariant under classical Lie groups admitted by (15).

We seek for solution of (28) assuming that $\eta$ is polynomial in $u$. In this way we find infinitesimals which are presented in Table 1. Parameters $c_i$, $i = 6, \ldots, 13$ from Table 1 are arbitrary.

Infinitesimal generator $\eta^1$ corresponds to the rational solution of Eq. (15) at $\bar{\alpha} = 18/25$. Using generator $\eta^2$ we can obtain exact solution of Eq. (15) expressed via the Airy functions. Below we show that this solution is the generalization of the self–similar solution of Eq. (15). The third generator $\eta^3$ corresponds to rational and solitary wave solutions of Eq. (15). Using generators $\eta^1$, $\eta^3$ we can obtain rational, solitary wave and Airy function solutions of Eq. (15) at $\bar{\alpha} = 0$. We believe that at least generators $\eta^1$, $\eta^2$, $\eta^3$ correspond to the solutions of Eq. (15) that cannot be found with the help of the classical Lie approach. Below we construct exact solutions of Eq. (15) corresponding to generators $\eta^1$, $\eta^2$, $\eta^3$.

Also one can find additional infinitesimal generator at $\bar{\alpha} = 1$:

$$\eta^6 = u^2 + v u, \quad \bar{\alpha} = 1, \quad (29)$$

where function $v$ satisfies Eq. (15) at $\bar{\alpha} = 1$. Using this infinitesimal generator we can construct auto–Backlund transformations for the integrable case of (15). Indeed, because $v$ satisfies Eq. (15) at $\bar{\alpha} = 1$ and using the invariant surface condition (24) at $\tau = 0, \xi = 1$ we obtain

$$u_x = u^2 - v u, \quad u_t = [(v u)_x - v u^2]_x, \quad (30)$$

where $u$ and $v$ both satisfies Eq. (15) at $\bar{\alpha} = 1$. Using these auto–Backlund transformations we can generate infinite series of solutions of the Sharma–Tasso–Olver equation. Let us note that auto–Backlund transformations (30)
were found in Ref. [18]. Here we recover these transformations using the symmetry approach. It is worth to note that for the first time the nonclassical symmetries were used for constructing auto–Backlund transformations by Nucci [24].

In this section we have found the classical symmetry generators admitted by Eq. (4). Further we use these generators for constructing traveling wave and self–similar solutions of Eq. (4). We also obtain several nonclassical generators admitted by Eq. (4). Below we use some of these generators to construct solutions of Eq. (4) which are not invariant under classical Lie groups.

4 Reductions and exact solutions

In this section we study symmetry reductions of Eq. (4). We construct several families of exact solutions of this equation. Both classical and non-classical symmetry reductions are considered.

4.1 Traveling wave solutions

Let us consider reduction of Eq. (4) corresponding to the linear combination of generators \(X_1\) and \(X_2\). Using the traveling wave variable \(u(x,t) = y(z)\), \(z = x - C_0 t\) and integrating with respect to \(z\) from Eq. (4) we get

\[
C_1 - C_0 y + \bar{\alpha} y^3 - 3y y_z + y_{zz} = 0,
\]

where \(C_1\) is an integration constant. Introducing new variable \(v = y_z\) we obtain the Abel equation of the second kind

\[
v v_y = 3yv - (\bar{\alpha} y^3 - C_0 y + C_1).
\]

(32)

Eq. (32) can be transformed to the canonical form with the help of the substitution \(\zeta = 3/2 y^2\):

\[
v v_{\zeta} - v = -\frac{2}{9} \bar{\alpha} \zeta + \tilde{C}_0 - \tilde{C}_1 \zeta^{-1/2},
\]

(33)

where \(\tilde{C}_0 = C_0/3\), \(\tilde{C}_1 = C_1/\sqrt{6}\).

List of the Abel equations which can be solved is presented in handbook by Zaitsev and Polyanin [34]. General solution of (33) at \(\tilde{C}_0, \tilde{C}_1 \neq 0\) can be found only in the case of \(\bar{\alpha} = 1\) (see Ref. [34]).
For constructing particular solutions of (31) we use the simplest equation method proposed in Ref. [15]. We find that Eq. (31) at \( C_1 = 0 \) has the following solution

\[
y = -\frac{3}{2\bar{\alpha}} \pm \sqrt{\frac{9 - 8\bar{\alpha}}{4\bar{\alpha}}} \sqrt{B \tanh\left\{ \sqrt{B} (z - z_0) \right\}}, \quad B = \frac{2C_0\bar{\alpha}}{9 - 3\sqrt{9 + 8\bar{\alpha} - 4\bar{\alpha}}}. \tag{34}
\]

One can see that solution (34) cannot be used at \( \bar{\alpha} = 0 \). In this case Eq. (31) admits stationary solution \( (C_0 = 0) \) at \( C_1 = 0 \)

\[
y = -\frac{2}{3} \sqrt{B} \tanh\left\{ \sqrt{B} (z - z_0) \right\}, \tag{35}
\]

where \( B \) is an arbitrary constant. Let us note that solutions (34), (35) can be obtained by the direct integration of (31).

The plots of solution (34) for several values of \( \bar{\alpha} \) are presented in Fig. 1. We can see that solution (34) is a weak shock wave.

### 4.2 Self–similar solutions

Let us consider the self–similar reduction of Eq. (4). Using the variables

\[
u = C_2 t^{-1/3} f(\theta), \quad \theta = C_3 x t^{-1/3}, \tag{36}
\]

from Eq. (1) at \( C_2 = -C_3 = -(3)^{-1/3} \) we get

\[
(\theta f)_\theta = 3(ff_\theta)_\theta + 3\bar{\alpha} f^2 f_\theta + f_{\theta\theta\theta}. \tag{37}
\]
Integrating (38) once we have
\[ \theta f = 3f f_\theta + \bar{\alpha} f^3 + f_{\theta\theta} + C_4, \] (38)
where \( C_4 \) is an integration constant. It is known that general solution of Eq. (38) can be obtained in the case of \( \bar{\alpha} = 1 \) (see Ref. [34]). However at the arbitrary value of \( \bar{\alpha} \) Eq. (38) admits particular solution which is expressed via the Airy functions.

Indeed, if the following relation holds
\[ C_4 = -\frac{\sqrt{9 - 8\bar{\alpha}} \pm 3}{3\sqrt{9 - 8\bar{\alpha}} \mp 4\bar{\alpha} \pm 9}, \] (39)
Eq. (38) has the solution in the form
\[ f = \frac{3 \pm \sqrt{9 - 8\bar{\alpha}}}{2\bar{\alpha}} \times \frac{(-B)^{1/3}(C_5 A_i \{\frac{1}{-B} (\theta - \theta_0)\} + Bi' \{-(-B)^{1/3}(\theta)\})}{C_5 A_i\{\frac{1}{-B} (\theta - \theta_0)\} + Bi\{-(-B)^{1/3}(\theta)\}}, \quad \bar{\alpha} \neq 0 \] (40)
where
\[ B = \frac{2\bar{\alpha}}{9 \pm 3\sqrt{9 - 8\bar{\alpha}} - 4\bar{\alpha}} \] (41)
and ' denotes the derivative with respect to the function argument, \( C_5 \) is an arbitrary constant.

Figure 2: Self–similar solution (42) of Eq. (4) for \( C_6 = 0.1 \) and \( C_7 = 1.1 \).
In the case of $\bar{\alpha} = 0$ we cannot find self-similar solution of Eq. (4). However, one can obtain stationary solution of Eq. (4) at $\bar{\alpha} = 0$ that is expressed via the Airy functions. This solution can be obtained using generator $\eta^4$ or by the direct integration of the stationary case of Eq. (4) at $\bar{\alpha} = 0$.

Let us note that using the Cole–Hopf transformation one can obtain more general self-similar solution of Eq. (4) at $\bar{\alpha} = 1$. This solution has the form

$$f = \frac{\Psi_\theta}{\Psi}, \quad \Psi = \int (C_6 \text{Ai}(\theta) + \text{Bi}(\theta)) d\theta + C_7,$$  \hspace{1cm} (42)

where $C_6$ and $C_7$ are arbitrary constants.

Using (36) and (42) we can find dependence of (42) on $x$ and $t$. The plot of this solution at various values of $t$ is presented in Fig. 2. From Fig. 2 we see that this solution is a weak shock wave with oscillations on a wave crest. When time $t$ increases the number of oscillations and the amplitude of waves decrease and the width of wave front increases. Waves of this type were observed experimentally in a liquid with gas bubbles.

### 4.3 Nonclassical reductions and their exact solutions

Let us study nonclassical symmetry reductions of Eq. (4). First we consider reduction corresponding to generator $\eta^1$ from Table 1.

Using $\eta^1$ we find similarity variables for Eq. (4) at $\bar{\alpha} = \frac{18}{25}$

$$u(x, t) = \sqrt{2x + c_6} h(t) - \frac{5}{3(2x + c_6)},$$  \hspace{1cm} (43)

where function $h(t)$ satisfies the equation

$$h_t + \frac{54}{25} h^3 = 0$$  \hspace{1cm} (44)

Solving this equation and using (43) we obtain the rational solution of Eq. (4) at $\bar{\alpha} = \frac{18}{25}$

$$u(x, t) = \frac{5}{3} \left( \frac{2x + c_6}{12t + t_0} - \frac{1}{2x + c_6} \right),$$  \hspace{1cm} (45)

where $t_0$ is an arbitrary constant. We see that we obtain the exact solution of Eq. (4) that is not invariant under classical Lie symmetries of this equation.

Let us consider the generator $\eta^5$ from Table 1. In this case symmetry reduction has the form

$$u(x, t) = \frac{x^2 - c_{12}x}{2c_{13} - 18t} + h(t),$$  \hspace{1cm} (46)
where \( h(t) \) satisfies the equation

\[
h_t = \frac{1}{4(c_{13} - 9t)^2} \left( 12(c_{13} - 9t)h + 3c_{12}^2 \right). \tag{47}\]

Using (46), (47) we find the following solution of Eq. (4) at \( \bar{\alpha} = 0 \)

\[
u = \frac{(2x - c_{12})^2}{8(c_{13} - 9t)} + \frac{C_8}{(c_{13} - 9t)^{1/3}}, \tag{48}\]

where \( C_8 \) is an integration constant.

Symmetry generators \( \eta^3 \) and \( \eta^4 \) correspond to the traveling wave and stationary solutions that were obtained above. It is interesting to consider generator \( \eta^2 \), generator \( \eta^3 \) at \( c_9 = 0 \) and generator \( \eta^4 \) at \( c_{10} = c_{11} = 0 \).

Generator \( \eta^2 \) corresponds to the exact solution of Eq. (4) that is expressed via the Airy functions. This solution does not coincide with self–similar solution (40) and has the form

\[
u = \left( \frac{3t(-3 + \sqrt{9 - 8\bar{\alpha}}) + 2c_8}{c_8(3 + \sqrt{9 - 8\bar{\alpha}}) - 12\bar{\alpha}t} \right)^{1/3} \frac{C_9 \text{Ai}'(\theta) + \text{Bi}'(\theta)}{C_9 \text{Ai}(\theta) + \text{Bi}(\theta)}, \tag{49}\]

where \( ' \) denotes the derivative with respect to function argument, \( C_9 \) is an arbitrary constant. Solution (49) is the generalization of self–similar solution (40) and coincides with this solution at \( c_7 = c_8 = 0 \).

Using generator \( \eta^3 \) at \( c_9 = 0 \) and generator \( \eta^4 \) at \( c_{10} = c_{11} = 0 \) we can obtain the following rational solutions of Eq. (4)

\[
u = \left( \frac{3t(-3 + \sqrt{9 - 8\bar{\alpha}}) + 2c_8}{c_8(3 + \sqrt{9 - 8\bar{\alpha}}) - 12\bar{\alpha}t} \right)^{-\frac{2}{3}}, \tag{50}\]

where \( x_0 \) is an arbitrary constant. Thus we find one–parametric families of stationary rational solutions of Eq. (4).

In this section we have found several families of exact solutions of Eq. (4). Two families of exact solutions correspond to the classical similarity reductions of Eq. (4). The other families of solutions correspond to the non-classical similarity reductions of Eq. (4). We believe that all exact solutions found in this section are new.
5 Numerical investigation

As it was mentioned above the perturbed Burgers equation is integrable only at one value of the bifurcation parameter ($\bar{\alpha} = 1$). So at this value of $\bar{\alpha}$ one can obtain a plenty of exact solutions. There is an interesting question. What happens with the solutions of the integrable case of Eq. (4) when the value of $\bar{\alpha}$ is being varied? In other words, it is interesting to study the stability of solutions for Eq. (4) with respect to perturbations of the parameter $\bar{\alpha}$. For this purpose we will use the numerical approach.

Let us consider the boundary value problem for Eq. (4) with periodic boundary conditions and some initial condition. We use the integrating factor with the fourth–order Runge–Kutta approximation method for the numerical calculations (see Refs. [6,12,17]).

We can present the perturbed Burgers equation in the form

$$u_t = L(u) + N(u),$$  \hspace{1cm} (51)

where $L(u) = -u_{xxx}$ and $N(u) = 3(uu_x)_x - 3\bar{\alpha}u^2u_x$. The boundary conditions and the initial condition have the form

$$u_{ix}(x,t) = u_{ix}(x+H,t), \quad i = 0, 1, 2,$$

$$u(x,0) = u_0(x),$$  \hspace{1cm} (52)

where $u_{ix} = \partial^i u / \partial x^i$ and $H$ is a period.

We discretize the spatial part of (51) using the Fourier transform. As a result we obtain the system of ordinary differential equations

$$\hat{u}_t = \hat{L}(\hat{u}) + \hat{N}(\hat{u}), \quad \hat{u} \big|_{t=0} = \hat{u}_0,$$  \hspace{1cm} (53)

where $\hat{u}$, $\hat{L}$ and $\hat{N}$ are the Fourier transforms of $u$, $L$ and $N$ respectively.

Applying the integrating factor in (53)

$$\hat{u} = e^{\hat{L}t}v,$$  \hspace{1cm} (54)

we have

$$v_t = e^{-\hat{L}t}N(e^{\hat{L}t}v).$$  \hspace{1cm} (55)

We solve this system of ordinary differential equations using the fourth-order Runge–Kutta method.

To verify our numerical algorithm we use one of the simplest periodic exact solutions of Eq. (4) at $\bar{\alpha} = 1$. This solution has the form

$$u = \frac{k \sin(kx + k^3 t - \phi_0)}{C_{10} + \cos(kx + k^3 t - \phi_0)},$$  \hspace{1cm} (56)
where $k$, $C_{10}$ and $\phi_0$ are arbitrary constants.

We calculate the average error of our numerical algorithm by the formula

$$Er = \frac{1}{N} \sum_{i=1}^{N} |u_{\text{exact}}^i - u_{\text{num}}^i|,$$

(57)

where $N$ is the number of approximation points on the $x$ axis.

We use solution (56) at $t = 0$ as the initial condition for problem (52). We calculated the average error by formula (57) for $N = 256$ and $N = 512$. Total time of calculations was chosen equal to 10. We obtain that the magnitude of the average error in both cases is very small and equals to $10^{-10}$ and to
10^{-14}$ respectively. Thus we can use $N = 256$ in all other calculations. We have also tested our numerical algorithm for the long time of calculations. The dependence of average error (57) on time $t$ is presented in Fig. 3.

Let us investigate the stability of solution (56) under the perturbations of parameter $\bar{\alpha}$. We again use this solution as the initial condition and perform calculations for $\bar{\alpha} > 1$. The calculations are performed until numerical solution does not break up. In this way for each value of $\bar{\alpha}$ we can obtain the value of time $t = t^*$ at which the solution breaks up. The values of $\bar{\alpha}$ versus $t^*$ is presented in Fig. 4. From Fig. 4 we see that there is exist a critical value of parameter $\bar{\alpha}^*$ at which time $t^*$ becomes very small. This critical value $\bar{\alpha}^*$ is located between 4.3 and 4.4.

We performed analogous calculations for $\bar{\alpha} \in [0, 1)$. We have found that solution remains stable until $t \sim 500$ for all $\bar{\alpha} \in [0, 1)$, thus there is no critical value of $\bar{\alpha}$ on this interval.

![Figure 5: Evolution of solution (56) governed by Eq. (4) for $\bar{\alpha} = 3.57$.](image)

In Fig. 5 we demonstrate evolution of solution (56) governed by Eq. (4) at $\bar{\alpha} = 3.57$. We see that the solution remains close to the exact solution until time $t = 12.5$. On the next stage of evolution the numerical solution is slightly deformed in comparison to the exact solution. At $t = 50$ the numerical solution significantly changes the form. However, it remains stable.

From Figs. 4, 5 we see that periodic exact solution (56) is stable under the variations of parameter $\bar{\alpha}$. Thus under small variations of $\bar{\alpha}$ we can use solution (56) for the description of waves governed by the nonintegrable case of Eq. (4).

Let us study the propagation of solitary waves governed by Eq. (4) at values of the parameter $\bar{\alpha}$ typical for a real liquid with gas bubbles. We consider waves propagated in water with carbon dioxide bubbles with equilibrium bubbles radius equal to $R_0 = 2.810^{-5}$m. In this case the parameter
Figure 6: Evolution of a solitary wave in water with carbon dioxide bubbles governed by Eq. (4) at $\bar{\alpha} = 1.12$.

$\bar{\alpha}$ has value $\bar{\alpha} \simeq 1.12$.

Figure 7: Evolution of a compact wave in water with carbon dioxide bubbles governed by (4) at $\bar{\alpha} = 1.12$.

In Fig. 6 we demonstrate the evolution of initial solitary wave $u_0 = 10 \cosh^{-2}\{x - 10\}$ governed by Eq. (4). We see that the solitary wave is transformed to the weak shock wave with the oscillating tail after the wave crest.
Let us consider as the initial condition the wave with compact support:

\[
u_0 = \begin{cases}
sin^2(x/3), & x \in [3\pi, 12\pi] \\
0, & x \not\in [3\pi, 12\pi]
\end{cases}
\] (58)

From Fig. 7 we see that this wave is transformed to a weak shock wave with the oscillating tail after the wave crest as well. We perform analogous calculations for initial Gaussian and rectangular pulses. We obtained that these initial pulses evolve to the same weak shock wave.

![Figure 8: Exact solution (59) of the integrable case of Eq. (4) at k = 0.4, C_{11} = 70, t = 15, \phi^1_0 = \phi^2_0 = 15.](image)

We can see that several types of solitary wave initial conditions evolve to the same asymptotic profile. We found that solutions of Eq. (4) have the same asymptotic behavior at large values of \(t\). Thus there is a stable solution of Eq. (4) that describes dynamics of solitary waves at large values of \(t\).

Let us note that the integrable case of Eq. (4) admits analytical solution that is very similar to this asymptotic profile. This solution has the form

\[
u = -\frac{\Psi_x}{\Psi}, \quad \Psi = C_{11} + \cos(kx + k^3t + \phi^{(1)}_0) + \exp(-kx + k^3t + \phi^{(2)}_0),
\] (59)

where \(k, C_{11}, \phi^{(1)}_0\) and \(\phi^{(2)}_0\) are arbitrary constants. The plot of solution (59) at \(t = 15\) is presented in Fig. 8. From Fig. 8 we see that solution (59) is very similar to numerical solutions obtained above.
6 Final remarks and conclusions

The perturbed Burgers equation was investigated. We have shown this equation passes the Painleve test only in the case of $\bar{\alpha} = 1$. This case corresponds to the integrable Sharma–Tasso–Olver equation. We have applied the classical Lie method to the perturbed Burgers equation. Three classical generators admitted by this equation have been found: translations in $x$ and $t$ and scaling in $x$, $t$ and $u$. Some nonclassical symmetries of the perturbed Burgers equation have been obtained using the method by Bluman and Cole. Exact solutions of classical and nonclassical symmetry reductions have been constructed. We have obtained traveling wave solutions, self–similar solutions and three families of solutions that are invariant under nonclassical symmetries. We believe that all found exact solutions are new.

The stability of exact solutions of the perturbed Burgers equation was investigated numerically. We have shown that periodic solution (57) is stable under the variation of parameter $\bar{\alpha}$. We have seen that dynamics of waves described by nonintegrable case of Eq. (1) remains close to the dynamics of waves described by integrable case of Eq. (1) at small variations of parameter $\bar{\alpha}$. It can be conjectured that other periodic solutions of the integrable case of Eq. (1) are also stable under variations of $\bar{\alpha}$.

We have found that the dynamics of waves governed by Eq. (2) at large values of time is the same for various solitary wave initial conditions. The initial disturbances evolve to the same weak shock wave with oscillations after wave crest. So for large values of time there is a stable solution of Eq. (1). We have shown that integrable case of Eq. (1) admits analytical solution in the form of the weak shock wave with oscillations after wave crest.

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