ON THE IRREDUCIBILITY OF THE SEVERI VARIETY OF NODAL CURVES IN A SMOOTH SURFACE

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Abstract. Let \( X \) be a smooth projective surface and \( L \in \text{Pic}(X) \). We prove that if \( L \) is \((2k - 1)\)-spanned, then the set \( \tilde{V}_k(L) \) of all nodal and irreducible \( D \in |L| \) with exactly \( k \) nodes is irreducible. The set \( \tilde{V}_k(L) \) is an open subset of a Severi variety of \( |L| \), the full Severi variety parametrizing all integral \( D \in |L| \) with geometric genus \( g(L) - k \).

1. Introduction

Let \( X \) be a smooth projective variety and \( L \in \text{Pic}(X) \). We recall that a zero-dimensional scheme \( Z \subset X \) is said to be curvilinear if for each \( p \in Z_{\text{red}} \) the Zariski tangent space of \( Z \) at \( p \) has at most dimension 1 (this is equivalent to the existence of a smooth curve \( C \subset X \) such that \( Z \subset C \)). We recall that the pair \((X, L)\) is said to be \( k \)-spanned (resp. \( k \)-very ample) if for each curvilinear zero-dimensional scheme (resp. each zero-dimensional scheme) \( Z \subset X \) the restriction map \( H^0(X, L) \to H^0(Z, L|_Z) \) is surjective ([1, pages 225 and 278]). Obviously \( k \)-very ampleness implies \( k \)-spannedness, but in a very particular case (but a very important case: smooth K3 surfaces and smooth Enriques surfaces) the two notions coincide ([15, Theorems 1.1 and 1.2]). Now assume that \( S \) is a smooth surface and that \( L \) is very ample. The genus \( g(L) \) of \( L \) is the arithmetic genus of any \( D \in |L| \), i.e. \( g(L) = 1 + (L^2 - \omega_X \cdot L)/2 \) (adjunction formula). Let \( V_k(L) \) denote the set of all integral \( D \in |L| \) with geometric genus \( g(L) - k \). Let \( \tilde{V}_k(L) \) be the set of all integral \( D \in |L| \) with exactly \( k \) ordinary nodes as its only singularities. Obviously \( \tilde{V}_k(X) \) is an open subset of \( V_k(L) \). In the terminology of [5] we have \( V^{L, g(L) - k}_k = V_k(L) \) and \( V^{L, k}_k = \tilde{V}_k(L) \). In many cases to check the irreducibility of \( V_k(L) \) (which of course only holds under certain assumptions) one first proves that \( \tilde{V}_k(X) \) is irreducible and then one proves that \( \tilde{V}_k(X) \) is dense in \( V_k(L) \).

In this note we prove the following result.

Theorem 1. Let \( X \) be a smooth projective surface. Fix a very ample \( L \in \text{Pic}(X) \) and a positive integer \( k \). If \( L \) is \((2k - 1)\)-spanned, then \( \tilde{V}_k(L) \) is either irreducible of dimension \( h^0(X, L) - 1 - k \) or \( \tilde{V}_k(L) = \emptyset \).

Theorem 1 improves [14, Theorem 9.1], which says that \( \tilde{V}_k(L) \) is irreducible if \( L \) is \((3k - 1)\)-very ample (so from \((3k - 1)\) to \((2k - 1)\), a huge improvement). Then M. Kemeny apply [14, Theorem 9.1] to the case in which \( X \) is a K3 surface with \( \text{Pic}(X) \cong \mathbb{Z} \), but he also remark that other methods give stronger results (in

2010 Mathematics Subject Classification. 14H10;14J99;14C20.

Key words and phrases. K3 surface; Severi variety; nodal curve; Hilbert scheme of nodal curves.

The author was partially supported by MIUR and GNSAGA of INdAM (Italy).
local ring

Let

Proof.

The “only if” part is trivial, because if

Then does not cover [5, Theorem 1], because (with the notation of [5]) it would require

We thank the referee for help in the organization of the paper.

2. The proof

We recall the following lemma, often called the curvilinear lemma, which is due to K. Chandler (\cite[Cor. 2.4]{2}, \cite[Lemma 4]{3}) and often used to compute the dimensions of secant varieties of varieties embedded in a projective space.

Lemma 1. Fix an integral projective variety, a very ample line bundle $L \in \text{Pic}(X)$ and a finite set $S \subset X_{\text{reg}}$. Set $n := \dim X$ and $k := |S|$. For each $p \in S$ let $2p$ be the closed subscheme of $X$ with $(I_p, X)^2$ as its ideal sheaf. Set $Z := \cup_{p \in S} 2p$. We have $h^0(X, L) - h^0(X, I_Z \otimes L) = (n+1)k$ if and only if for each choice of a degree 2 zero-dimensional scheme $Z_p \subset 2p$ we have $h^0(X, I_{\cup_{p \in S} Z_p} \otimes L) = h^0(X, L) - 2k$.

Proof. The “only if” part is trivial, because if $W \subset A \subset X$ with $A$ a zero-dimensional scheme, then $h^0(I_A \otimes L) \geq h^0(I_W \otimes L) + \deg(W) - \deg(A)$.

Now assume $h^0(X, L) - h^0(X, I_Z \otimes L) < (n+1)k$. We use $|L|$ to see $X$ as a subscheme of $\mathbb{P}^r$, $r = h^0(X, L) - 1$. For each $p \in S$ call $\mu_p$ the maximal ideal of the local ring $\mathcal{O}_{X, p}$. Since $X$ is smooth and of dimension $n$ at $p$, we have $\dim \mu_p/\mu_p^2 = n$, $\dim \mathcal{O}_{X, p}/\mu_p^2 = n + 1$ and the vector space $\mu_p/\mu_p^2$ is generated by the images of a system of $n$ regular parameters of the local ring $\mathcal{O}_{X, p}$. By the definition of 2 the sheaf $(I_p, X)^2$ is the ideal sheaf of 2p as a closed subscheme of $X$. Thus the linear span $(2p)$ in $\mathbb{P}^r$ of 2p is the Zariski tangent space of $X$ at $p$ and $\dim(2p) = n$. Thus the assumption “$h^0(X, L) - h^0(X, I_Z \otimes L) < (n+1)k$” is equivalent to assume that the linear spaces $(2p)$, $p \in S$, are not $k$ distinct and linearly independent linear subspaces of $\mathbb{P}^r$. Fix $p \in S$ and take a system of homogeneous coordinates $x_0, \ldots, x_r$ of $\mathbb{P}^r$, i.e. a basis of $H^0(X, L)$, such that $p = (1:0: \cdots:0)$ and $T_pX$ has $x_i = 0$ for all $i > n$ as its equations. Note that $x_1/x_0, \ldots, x_n/x_0$ induce a regular system of parameters, $y_1, \ldots, y_n$, of the regular local ring $\mathcal{O}_{X, p}$. Fix any line $D \subset T_pX$ containing $p$. $D$ is induced by the equations $x_i = 0$ for all $i > n$ of $T_pX$ and by $n-1$ linearly independent equations in the variables $x_1, \ldots, x_n$, say $\sum_{i=1}^{n} a_{ij} x_i = 0$, $1 \leq j \leq n-1$. Let $J \subset \mu_p$ be the ideal generated by the union of all $\sum_{ij} a_{ij} y_i \in \mu_p$ and $\mu_p^2$. $J$ gives a degree 2 scheme $Z_D = \text{Spec}(\mathcal{O}_{X, p}/J) \subset \text{Spec}(\mathcal{O}_{X, p}/\mu_p^2) = 2p$, which is the only degree 2 connected subscheme of $\mathbb{P}^r$ containing $p$ and contained in $D$ (and thus spanning $D$). Moreover $Z_D$ is the scheme-theoretic intersection of $2p$ and $D$. It is also the degree 2 connected subscheme of $D$ with $p$ as its reduction.

We order the points $p_1, \ldots, p_k$ of $S$ and set $S_i := \{p_1, \ldots, p_i\}$. Set $S_0 := \emptyset$. Let $h$ be the last non-negative integer $< k$ such that the linear space $L_h$ spanned by $T_{p_1}X, \ldots, T_{p_h}X$ has dimension $h(n+1) - 1$. By the definition of $h$ we have $T_{p_{h+1}}X \cap L_h \neq \emptyset$. Fix $o \in T_{p_{h+1}}X \cap L_h$. There is $E \subset \{1, \ldots, h\}$ and $a_i \in T_{p_i}X$ such that $o$ is in the linear span of the set $\{a_i\}_{i \in E}$. For each $p \in S$ the $k$-dimensional
linear subspace $T_pX$ is the union of all lines of $\mathbb{P}^r$ passing through $p$ and contained in $T_pX$. We saw the each such line is spanned by a degree 2 subscheme of $2p$. Thus for each $a \in T_pX$ there is a degree 2 connected zero-dimensional scheme $W < 2p$ such that $a$ is contained in the line $(W)$ spanned by $W$; $W$ is unique if and only if $a \neq p$. Thus there are degree 2 connected zero-dimensional schemes $Z_i \subset 2p_i$, $i \in E \cup \{h + 1\}$, such that $o$ is contained in the line $(Z_{h+1})$ and $a_i, i \in E$, is contained in the line $(Z_i)$. If $E \subseteq \{1, \ldots, h\}$ for all $i \in \{1, \ldots, h\} \setminus E$ fix any degree 2 connected scheme $Z_i \subset 2p_i$. Thus $o$ is in the linear span of the union of the $h$ lines $\cup_{i=1}^h(Z_i)$. Thus $\cup_{i=1}^{h+1}(Z_i)$ spans a linear space of dimension $< (h + 1)n - 1$. If $h \leq k - 2$ call $Z_i, i = h + 2, \ldots, k$, an arbitrary connected degree 2 zero-dimensional scheme $Z_i \subset 2p_i$. By construction the lines $(Z_i), 1 \leq i \leq k$, are either not distinct or not linearly independent. Thus $h^0(X, L) - h^0(X, L_{\cup Z_i} \otimes L) > 2k$. □

Note that the scheme $Z$ in the statement of Lemma[I] have degree $k(n + 1)$, while the scheme $\cup_{p \in S}Z_p$ has degree $2k$.

Remark 1. Take $(X, L)$ as in Lemma[I]. An obvious consequence of Lemma[I] is that $h^0(L) \geq k(n + 1)$. In the set-up of Theorem[I] (hence with $n = 2$) we want to have $h^0(L) > 3k$ (not just $h^0(L) \geq 3k$, which would follows from the $k$-spannedness of $L$ and Lemma[I]). If $L$ is $k$-spanned and $h^0(L) = 3k$, then $V(L) = \emptyset$ (see the proof of Theorem[I]). This is also explicitly stated in the set-up of K3-surfaces in [I]. Remark at page 173).

Proof of Theorem[I]: Fix any finite set $S \subset X$ such that $|S| = k$. Set $V(S, L) := \{D \in \hat{V}_k(L) \mid S = \text{Sing}(D)\}$. Set $Z := \cup_{p \in S}2p$. By Lemma[I] we have $h^0(I_Z \otimes L) = h^0(L) - 3k$. Note that $|I_Z \otimes L|$ parametrizes all curves $D \subset |L|$ which are singular at each point of $S$. Thus $V(S, L) = \emptyset$ if $h^0(L) = 3k$, while $V(S, L)$ is a Zariski open subset of a projective space of dimension $h^0(L) - 3k - 1$ if $h^0(L) > 3k$. Since this is true for all subsets of $X$ with cardinality $k$, we get $V_k(L) = \emptyset$ if $h^0(L) = 3k$. Now assume $h^0(L) > 3k$. Since the set of all subsets of $X$ with cardinality $k$ forms an irreducible variety of dimension $2k$, we would get that $V_k(L)$ is irreducible of dimension $h^0(L) - k - 1$ if $V(S, L) \neq \emptyset$ for a general $S$. Note that if $V(S, L) = \emptyset$ for a general $S$ with cardinality $k$, then the same holds for all $S$ with cardinality $k$.

Remark 2. There is a list of pairs $(X, L)$ with $L$ a $k$-very ample line bundle with $L^2 \leq 4k + 4$ ([II] Table 2] and [I] Remark 1.4 and note 1]).

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