QUASI-POLYNOMIAL MIXING OF CRITICAL
2D RANDOM CLUSTER MODELS

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Abstract. We study the Glauber dynamics for the random cluster (FK) model on the torus \((\mathbb{Z}/n\mathbb{Z})^2\) with parameters \((p, q)\), for \(q \in (1, 4]\) and \(p\) the critical point \(p_c\). The dynamics is believed to undergo a critical slowdown, with its continuous-time mixing time transitioning from \(O(\log n)\) for \(p \neq p_c\) to a power-law in \(n\) at \(p = p_c\). This was verified at \(p \neq p_c\) by Blanca and Sinclair, whereas at the critical \(p = p_c\), with the exception of the special integer points \(q = 2, 3, 4\) (where the model corresponds to the Ising/Potts models) the best-known upper bound on mixing was exponential in \(n\). Here we prove an upper bound of \(n^{O(\log n)}\) at \(p = p_c\) for all \(q \in (1, 4]\), where a key ingredient is bounding the number of nested long-range crossings at criticality.

1. Introduction

The random cluster (FK) model is an extensively studied model in statistical physics, generalizing electrical networks, percolation, and the Ising and Potts models, to name a few, under a single unifying framework. It is defined on a graph \(G = (V, E)\) with parameters \(0 < p < 1\) and \(q > 0\) as the probability measure over subsets \(\omega \subset E\) (or equivalently, configurations \(\omega \in \{0, 1\}^E\)), given by

\[\pi_{G, p, q}(\omega) \propto p^{\omega}(1 - p)^{|E| - |\omega|} q^{k(\omega)},\]

where \(k(\omega)\) is the number of connected components (clusters) in the graph \((V, \omega)\).

At \(q = 1\), the FK model reduces to independent bond percolation on \(G = (V, E)\), and for integer \(q \geq 2\) it corresponds via the Edwards–Sokal coupling [10] to the Ising \((q = 2)\) and Potts \((q \geq 3)\) models on \(V\). Since its introduction around 1970, the model has been well-studied both in its own right and as a means of analyzing the Ising and Potts models, with an emphasis on \(\mathbb{Z}^d\) as the underlying graph. There, for every \(q \in [1, \infty)\), the model enjoys monotonicity, and exhibits a phase transition at a critical \(p_c(q)\) w.r.t. the existence (almost surely) of an infinite cluster (see, e.g., [12] and references therein).

Significant progress has been made on the model in \(d = 2\), in particular for \(1 \leq q \leq 4\) where the model is expected to be conformally invariant (see [22, Problem 2.6]). It is known [1] that \(p_c(q) = \frac{\sqrt{q}}{1 + \sqrt{q}}\) on \(\mathbb{Z}^2\) for all \(q \geq 1\). Moreover, while the phase transition at this \(p_c\) is discontinuous if \(q > 4\) (as confirmed for all \(q > 25\) in [14] and very recently, all \(q > 4\) in [7]), it is continuous for \(1 \leq q \leq 4\) (as established in [9]). There, the probability that \(x\) belongs to the cluster of the origin decays as \(\exp(-c|x|)\) at \(p < p_c\), as a power-law \(|x|^{-\eta}\) at the critical \(p_c\), and is bounded away from 0 at \(p > p_c\).

Here we study heat-bath Glauber dynamics for the two-dimensional FK model, where the following critical slowdown phenomenon is expected: on an \(n \times n\) torus, for all \(p \neq p_c\) the mixing time of the dynamics should have order \(\log n\) (recently shown by [2]), yet at \(p = p_c\) it should behave as \(n^z\) for some universal \(z > 0\) in the presence of a continuous phase transition, and \(\exp(cn)\) in the presence of a discontinuous phase transition. The critical behavior in the former case (all \(q > 4\)) was established in a companion paper [11], as was a critical power-law in the cases \(q = 2\) ([17]) and \(q = 3\) ([11]). In this work we obtain a quasi-polynomial upper bound for non-integer \(1 < q \leq 4\) at criticality.
More precisely, Glauber dynamics for the FK measure \( \pi_{G,p,q} \) is the continuous-time Markov chain \( (X_t)_{t \geq 0} \) that assigns each edge \( e \in E \) an i.i.d. rate-1 Poisson clock, where upon ringing, \( X_t(e) \) is resampled via \( \pi_{G,p,q} \) conditioned on the values of \( X_t \) on \( E - \{e\} \). This Markov chain is reversible by construction w.r.t. \( \pi_{G,p,q} \), and may hence be viewed both as a natural model for the dynamical evolution of this interacting particle system, and as a simple protocol for sampling from its equilibrium measure. A central question is then to estimate the time it takes this chain to converge to stationarity, measured in terms of the total variation mixing time \( t_{\text{mix}} \) (see §2.2 for the related definitions).

For \( p \neq p_c \), the fact that \( t_{\text{mix}} \asymp \log n \) was established in [2] using the aforementioned exponential decay of cluster diameters in the high-temperature regime \( p < p_c \): on finite boxes with certain boundary conditions, this translates to a property known as strong spatial mixing, implying that the number of disagreements between the states of two chains started at different initial states decreases exponentially fast, thus \( t_{\text{mix}} \asymp \log n \); this result readily extends to \( p > p_c \) by the duality of the two-dimensional FK model. At \( p = p_c \), where there is no longer an exponential decay of correlations, polynomial upper bounds on \( t_{\text{mix}} \) were obtained for the Ising model in [17] and the 3-state Potts model—along with a quasi-polynomial bound for the 4-state model—in [11], using a multiscale approach that reduced the side length of the box by a constant factor in each step via a coupling argument; these carry over to the FK model for \( q = 2, 3, 4 \) by the comparison estimates of [23]. However, for non-integer \( q \), FK configurations may form macroscopic connections along the boundary of smaller-scale boxes, destroying the coupling—this is prevented for integer \( q \) thanks to the special relation between FK/Potts models. To control this effect, we prove upper and lower bounds on the total number of disjoint macroscopic connections along the boundaries of the smaller-scale boxes at \( p = p_c \) which may be of independent interest (see §1.1 as well as Fig. 1–3).

It was recently shown [13] that for \( q = 2 \) the FK Glauber dynamics on any graph \( G = (V,E) \) has \( t_{\text{mix}} \leq |E|^{O(1)} \); the technique there, however, is highly specific to the case of \( q = 2 \). Indeed, this bound does not hold on \( \mathbb{Z}^d \), for any \( d \geq 2 \), at \( p = p_c \) and \( q \) large, as follows from the exponential lower bounds of [3, 4] (see, e.g., [6, 11] for further details). The best prior upper bound on non-integer \( 1 < q < 4 \) was \( t_{\text{mix}} \leq \exp(O(n)) \).

In the present paper, we prove that for periodic boundary conditions (as well as a wide class of others, including wired and free; see Remark 1.1), the following holds:

**Theorem 1.** Let \( q \in (1,4] \) and consider the Glauber dynamics for the critical FK model on \( (\mathbb{Z}/n\mathbb{Z})^d \). There exists \( c = c(q) > 0 \) such that \( t_{\text{mix}} \lesssim n^{c \log n} \).
Remark 1.1. Theorem 1 holds for rectangles with uniformly bounded aspect ratio, under any set of boundary conditions with the following property: for every edge $e$ on the boundary of the box, there are $O(\log n)$ distinct boundary components connecting vertices on either side of $e$ (see Definitions 5.1–5.2 and Theorem 5.4). This includes, in particular, the wired and free boundary conditions, as well as, with high probability, “typical” boundary conditions: those that are sampled from $\pi_{\mathbb{Z}^2, p_c, q}$ (see Lemma 5.7).

Remark 1.2. For $q \in \{2, 3, 4\}$, the comparison estimates of [23] carry the upper bounds on the mixing time of the Potts model to the FK model, yet only for a limited class of boundary conditions (e.g., the partition of boundary vertices can have at most one cluster of size larger than $n^\epsilon$, in contrast to “typical” ones as above). The above theorem thus extends the class of FK boundary conditions for which $t_{\text{mix}}$ is quasi-polynomial.

Remark 1.3. Theorem 1 implies analogous bounds for other single-site dynamics (e.g., Metropolis), as well as global cluster dynamics (e.g., Chayes–Machta [5]) via [23].

1.1. Main techniques.

1.1.1. Disjoint long-range connections. As pointed out in [2] and later in [11], disjoint long-range clusters along the boundary of rectangles, called bridges (see Fig. 2), are a major obstacle to mixing time upper bounds for the FK Glauber dynamics.

Definition 1.4. Let $\Lambda_{n,m} = [0,n] \times [0,m] \cap \mathbb{Z}^2$. Given an FK configuration $\omega$ on $\mathbb{Z}^2 - \Lambda_{n,m}$ and a boundary edge $e \in \partial \Lambda_{n,m}$, say without loss of generality $e \in \partial_n \Lambda_{n,m}$, a bridge over $e$ is an open FK cluster in $\omega$ that contains at least one vertex in $\partial_n \Lambda_{n,m}$ to the left of $\{e\}$ and one to the right. Let $\Gamma^e(\omega)$ denote the set of all bridges of $e$.

(For a more detailed definition, we also refer the reader to Section 3.2.) In critical bond percolation ($q = 1$ and $p = 1/2$), the Berg–Kesten (BK) inequality would suggest that $\pi(|\Gamma^e| > K \log n + a) \leq \exp(-ca)$ for universal $K, c > 0$ and all $a$. In our setting of FK percolation for $1 < q < 4$ at $p = p_c$, the classical BK inequality does not hold; nevertheless, Proposition 3.9 and Lemmas 3.11–3.12 establish such a bound for $|\Gamma^e|$.

It then follows (see Corollary 3.10) that if we sample from the FK measure on a $2n \times 2n$ box $\Lambda$ with arbitrary boundary conditions, the boundary conditions this induces on the concentric inner $n \times n$ box will have order $\log n$ distinct bridges over a given edge with probability $1 - O(n^{-c})$. A simpler formulation of that is as follows.
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Figure 3. Macroscopic disjoint boundary bridges prevent the coupling of FK configurations sampled under two different boundary conditions on $\partial_s \Lambda$ from being coupled past a common horizontal dual-crossing.

Theorem 1.5. For every $q \in (1, 4]$, there exist $K' > K > 0$ and $c(q) > 0$ such that, the critical FK model has, for every $e \in \partial \Lambda_{n,n}$, say, $\frac{n}{10}$ distance from a corner of $\Lambda_{n,n}$,

$$
\pi_{\mathbb{Z}^2, p_c, q} \left( K \log n \leq |\Gamma^e(\omega)| \leq K' \log n \right) \geq 1 - O(n^{-cK'}).$
$$

The lower bound on $|\Gamma^e|$ demonstrates that the behavior of the number of bridges at $p = p_c$ is truly different than at $p \neq p_c$; there, by the exponential decay of correlations at $p < p_c$ (dual connections at $p > p_c$), the typical number of bridges over an edge is $O(1)$. The upper bound in the above theorem arises in two crucial ways in the proof of Theorem 1: (a) the typical number of bridges over an edge $f$ being $O(\log n)$ is used to disconnect all potentially destructive bridges at a cost of $e^{\Gamma_f} \lesssim n^c$ (Eq. (2.2)), and (b) the exponential upper tail beyond that is used to sustain a union bound over $n^{O(1)}$ many attempts at coupling (see §1.1.2 for more details).

To be more precise about the obstacle posed by having multiple bridges over an edge, recall the following. In [17] and then [11], the upper bounds on the mixing time of the Potts models at $\beta = \beta_c$ for $q \in \{2, 3, 4\}$ relied on RSW bounds [8, 9] to expose dual-interfaces in the FK representation, beyond which block dynamics chains could be coupled. However, the fact that chains, started from any two initial configurations, could be coupled past a dual-interface, relied on a certain conditional event, implicit in the relation between the FK and Potts models at integer $q$ (that no distinct boundary components were connected in the interior configuration). Without this conditioning, connections between two components on one side of a rectangle alter the boundary conditions elsewhere via bridges over the dual-interface, preventing coupling (see Fig. 3).

Similar difficulties were pointed out in [2] at $p < p_c$, and later in [11] at criticality when $q > 4$. In both of these cases, the exponential decay of correlations under $\pi_{\mathbb{Z}^2}^q$ ensured that all such bridges would be, with high probability, microscopic: in [2] these bridges were negligible after restricting attention to side-homogenous (wired or free on sides) boundary conditions, while in [11], relevant boundary segments could be disconnected from one another by brute-force modifications. In contrast, in the present setting at the critical point of a continuous phase transition, the power-law decay of correlations precludes such techniques; thus, at $p = p_c(q)$ obtaining sharp bounds on the number of bridges becomes not only necessary, but also substantially more delicate.
To convert the upper bounds on bridges to an upper bound on the mixing time, our dynamical analysis restricts its attention to boundary conditions with $|\Gamma^e| = O(\log n)$ for every $e \in \partial \Lambda$; we call such boundary conditions “typical” (see Definitions 5.1–5.2) and observe that wired and free boundary conditions are both typical.

1.1.2. Refined dynamical scheme. To maintain “typical” (as opposed to worst-case) boundary conditions throughout the multi-scale analysis, we turn to the Peres–Winkler censoring inequalities [20] for monotone spin systems, that were used in [19] (then later in [16]) for the Ising model under “plus” boundary, a class of boundary conditions that dominate the plus phase (observe that, in contrast, “typicality” is not monotone).

A major issue when attempting to carry out this approach—adapting the analysis of the low temperature Ising model to the critical FK model—is the stark difference between the nature of the corresponding equilibrium estimates needed to drive the multi-scale analysis. In the former, crucial to maintaining “plus” boundary conditions throughout the induction of [19] was that in the presence of favorable boundary conditions, the multiscale analysis could be controlled except with super-polynomially small probability. This yielded a bound on coupling the dynamics started at the extremal (plus and minus) initial configurations, which a standard union bound over the $O(n^2)$ sites of the box (see Fact 2.4) then transformed to a bound on $t_{\text{mix}}$.

We wish to couple the dynamics from the extremal (wired and free) initial configurations, since arbitrary starting states may induce boundary conditions on the smaller scales that are not “typical.” Even in the ideal scenario where the induced boundary conditions have no bridges, though, the probability that we fail to couple the dynamics from wired and free initial states is at least $1 - \varepsilon$ (as per the RSW estimates). In particular, even in this ideal setting, we could not afford the $O(n^2)$ factor of translating this to a bound on $t_{\text{mix}}$—the actual setting is far worse, replacing the failure probability by $1 - n^{-c}$ (Proposition 3.1). An approach based on the classical block dynamics recursion on the spectral gap (see [18] and the proofs in [11, 17]) would force one to analyze the dynamics under worst-case boundary, whereas we would like to restrict attention to the typical boundary conditions encountered throughout the dynamical process.

Therefore, in Definition 4.9 we construct a censored dynamics that mimics a block dynamics chain, and bound the total variation distance between their distributions in terms of the probability that we encounter unfavorable boundary conditions on the sub-blocks before mixing (Proposition 4.10). By doing so, we compare the censored dynamics to the block dynamics with boundary conditions modified to eliminate all $O(\log n)$ bridges over certain edges, paying a cost of $n^c$ rounds in the mixing time. We then let the block dynamics run some $n^c$ rounds (paying a union bound for the probability of ever encountering atypical boundary conditions, bounded in Corollary 5.8). The block dynamics would be making $n^c$ many independent attempts at coupling (possible due to the absence of bridges over a particular edge) beyond a dual-crossing whose existence has probability $n^{-c/2}$: see Lemma 5.6. This polynomial bound on the block dynamics coupling time translates to a quasi-polynomial bound for the censored dynamics via $O(\log n)$ recursions onto smaller scale blocks, yielding the bound on $t_{\text{mix}}$.

Finally, since periodic boundary conditions do not fall in our class of “typical” boundaries, in Section 5.3 we extend this bound first to cylinders, and then to the torus.
In this section we define the random cluster (FK) model and the FK dynamics that will be the object of study in this paper. We also recall various important results from the equilibrium theory of the FK model in §2.1, and the general theory of Markov chain mixing times (§2.2), including, in particular, that of monotone chains. Throughout the paper, for sequences $f(N), g(N)$ we will write $f \lesssim g$ if there exists a constant $c > 0$ such that $f(N) \leq cg(N)$ for all $N$ and $f \asymp g$ if $f(N) \lesssim g(N) \lesssim f(N)$.

For a more detailed exposition of much of §2.1 see [12], and for a more detailed exposition of the main ideas in §2.2 see [15].

2.1. **The FK model.** Throughout the paper, we identify an FK configuration $\omega \subset E$ with an assignment $E \to \{0, 1\}$, referring to an edge $e$ with $\omega(e) = 1$ as **open** and to an edge $e$ with $\omega(e) = 0$ as **closed**. We will drop the subscripts $p,q$ from $\pi_{G,p,q}$ whenever their value is clear from the context.

**Boundary conditions.** For a graph $G$, one can fix an arbitrary subset of the vertices to be the boundary $\partial G \subset V(G)$ so that we can define **boundary conditions** $\xi$ on $\partial G$ as follows. First augment $G$ to $G'$ by adding edges between any pair of vertices in $\partial G$ that do not share an edge in $E$; then letting $E'(\partial G)$ be the set of all edges between pairs of vertices in $\partial G$, a boundary condition $\xi$ is just an FK configuration in $\{0,1\}^{E'(\partial G)}$. Every boundary condition $\xi$ can therefore alternatively be thought of as a partition of $\partial G$ given by the clusters of $\xi$. The random cluster measure with these boundary conditions is then given by counting the number of clusters in a configuration as the number of clusters in the configuration on $G'$, fixing the restriction to $E'(\partial G)$ to be $\xi$.

The **wired** boundary condition consists of just one component consisting of all $v \in \partial G$, and the **free** boundary condition consists of only singletons, each corresponding to one vertex in $\partial G$. For ease of notation, in the former case we say $\xi = 1$ and in the latter case we say $\xi = 0$. Denote the interior of $G$ as the subgraph of $G$ given by $G^0 = (V - \partial G, E - E'(\partial G))$.

For a graph $G = (V, E)$ and a subgraph $(R, E(R))$ where $R \subset V$, denote by $\omega|_R$ or $\omega|_{E(R)}$ the restriction of the configuration $\omega \in \{0,1\}^E$ to $E(R)$.

For two domains $R_1 \subset R_2$, we say that a configuration $\omega$ on $R_2$ with boundary condition $\xi$ induces a boundary condition $\zeta$ on $R_1$ if $\zeta$ is the boundary condition induced by $\omega|_{R_2 - R_1^0 \cup \xi}$: here the union of two boundary conditions denotes the partition arising from all connections through $\omega|_{R_2 - R_1^0}$ and $\xi$ together. In such situations, when we write $\omega|_{\partial R_1}$ we mean the boundary condition induced on $R_1$ by $\omega$ on $R_2 - R_1^0$ and $\xi$. If two sites $x,y$ are in the same component of a boundary condition $\xi$, we write $x \leftrightarrow^\xi y$.

**Domain Markov property.** For any $q$, the FK model satisfies the Domain Markov property: that is to say, for any graph $G$ and any boundary conditions $\xi$ on $\partial G$, for every subgraph $F \subset G$ and FK configuration $\eta$ on $E(G) - E(F)$,

$$
\pi^\xi_{\eta|_{E(F)}}(\omega \in \cdot) = \pi^\xi_G(\omega|_{E(F)} \in \cdot | \omega|_{E(G) - E(F)} = \eta).
$$
Monotonicity and FKG inequalities. There is a natural partial ordering to configurations and boundary conditions in the FK model: for two configurations $\omega, \omega' \in \Omega$ we say $\omega \geq \omega'$ if $\omega(e) \geq \omega'(e)$ for every edge $e \in E$, and for any two boundary conditions $\xi, \xi'$ we say that $\xi \geq \xi'$ if $x \leftrightarrow y \implies x \leftrightarrow y$ for every pair of sites $x,y \in V(\partial G)$, which is to say that $\xi'$ corresponds to a finer partition than $\xi$ of the vertices $V(\partial G)$.

An event $A$ is increasing if it is closed under addition of edges so that if $\omega \leq \omega'$, then $\omega \in A$ implies $\omega' \in A$; analogously, it is decreasing if it is closed under removal of edges. The FK model satisfies FKG inequalities for all $q \geq 1$ (i.e., it is positively correlated) so that for any two increasing events $A, B$,

$$\pi_G^\xi(A \cap B) \geq \pi_G^\xi(A) \pi_G^\xi(B).$$

This leads to monotonicity in boundary conditions for all $q \geq 1$. For any pair of boundary conditions $\xi, \xi'$ with $\xi' \leq \xi$, and any increasing event $A$,

$$\pi_G^{\xi'}(A) \leq \pi_G^{\xi}(A),$$

whence we say that $\pi_G^{\xi}$ stochastically dominates ($\geq$) $\pi_G^{\xi'}$.

Planar duality. For the purposes of this paper, we now restrict our attention to graphs that are subsets of $\mathbb{Z}^2$, the graph with vertices at $\mathbb{Z}^2$ and edges between nearest-neighbors in Euclidean distance. For a connected graph $G \subset \mathbb{Z}^2$, let $\partial G$ consist of all $v \in V$ having a $\mathbb{Z}^2$-neighbor in $\mathbb{Z}^2 - G$.

For a graph $G \subset \mathbb{Z}^2$ (in fact for any planar graph), there is a powerful duality between the FK model on $G$ and the FK model on the planar dual graph of $G$, denoted $G^\ast$. Given a planar graph $G$, we can identify to any configuration $\omega$ a dual configuration $\omega^\ast$ on $G^\ast$ where (identifying to each $e \in E(G)$, the unique dual edge $e^\ast$ passing through $e$), $\omega^\ast(e^\ast) = 1$ if and only if $\omega(e) = 0$. We sometimes identify edges with their midpoints.

For any boundary condition $\xi$ on a planar graph $G$, for all $q \geq 1$, the map $p \mapsto p^*$ where $pp^* = q(1-p)(1-p')$ can be seen to satisfy

$$\pi_{G,p,q}^\xi \overset{d}{=} \pi_{G^\ast,p^*,q}^{\xi^*},$$

where the boundary condition $\xi^*$ is determined on a case by case basis so that $(\xi^*)^* = \xi$ (in particular, the wired and free boundary conditions are dual to each other).

Planar notation. The graphs we consider will be rectangular subsets of $\mathbb{Z}^2$, denoted,

$$\Lambda_{n,m} = [0,n] \times [0,m],$$

where throughout the paper, $[0,n] := \{k \in \mathbb{Z} : 0 \leq k \leq n\}$. When $n, m$ are fixed and understood from context, we drop them from the notation. Then we denote the sides of $\partial \Lambda$ by $\partial_N \Lambda = \{0\} \times [0,m]$ and the analogously defined $\partial_S \Lambda, \partial_E \Lambda, \partial_W \Lambda$. We collect multiple sides into their union by including both subscripts, e.g., $\partial_{S,E} \Lambda = \partial_S \Lambda \cup \partial_E \Lambda$.

Consider the FK model on a rectangular graph $\Lambda$. For any $x, y \in \Lambda$, we write $x \leftrightarrow y$ if $x$ and $y$ are part of the same component of $\omega$ on $\Lambda - \partial \Lambda$ (there exists a connected set of open edges with one edge adjacent $x$ and one adjacent $y$). For a subset $R \subset \Lambda$, we write $x \leftrightarrow^R y$ to denote the existence of such a crossing within $R - \partial R$, and for two sets $A, B \subset \Lambda$ we write $A \leftrightarrow B$ if there exists $a \in A, b \in B$ such that $a \leftrightarrow b$. 
We now define the vertical crossing event for a rectangle $\Lambda$ as 
\[
C_v(\Lambda) = \partial_h \Lambda \leftrightarrow \partial_n \Lambda ,
\]
and analogously define the horizontal crossing event $C_h(\Lambda)$. One can similarly define the dual-crossing events $C_v^*(\Lambda), C_h^*(\Lambda)$ (where abusing notation, the fact that the crossings occur on $\Lambda^+$ is understood) and more generally, writing $x^\leftrightarrow y^\leftrightarrow$ denotes the existence of a connection in the dual graph. Then, crucially, planarity and self-duality of $\mathbb{Z}^2$ imply that for a rectangle $\Lambda$, we have $C_v(\Lambda) = (C_h^*(\Lambda))^c$.

Finally for two rectangles $\Lambda' \subset \Lambda$, an annulus $A = \Lambda - \Lambda'$, denote the existence of an open circuit (connected set of open edges with nontrivial homology w.r.t. $A$) by $C_o(A)$.

Gibbs measures and the FK phase transition. Infinite-volume Gibbs measures can be derived by taking limits of $\pi_{\Lambda_n}^{\xi_n}$ as $n \to \infty$ for a prescribed sequence of boundary conditions $\xi_n$: natural choices of such boundary conditions are $\xi_n = 1, 0$ or periodic so that the graph is $(\mathbb{Z}/n\mathbb{Z})^2$. If such limits exist weakly, we denote them by $\pi_{\mathbb{Z}^2}^{\xi}$, and they satisfy the DLR conditions (see, e.g., [12]).

By the self-duality of $\mathbb{Z}^2$ (up to translation), one sees that at the fixed point of $p \mapsto p^*$, $(p_{sd} = \frac{\sqrt{2}}{1 + \sqrt{2}})$, one has $\pi_{\mathbb{Z}^2}^{1, \varepsilon} \equiv \pi_{\mathbb{Z}^2}^0$, and we say the model is self-dual. The FK model for $q \geq 1$ exhibits a sharp phase transition between a high temperature phase ($p$ small) where there is no infinite component, and a low temperature phase ($p$ large) where there is almost surely an infinite component, through a critical point $p_c(q) = \inf \{ p \in [0, 1] : \pi_{\mathbb{Z}^2, p, q}(0 \leftrightarrow \infty) > 0 \}$. It was proved in [1] that for all $q \geq 1$, $p_c(q) = p_{sd}(q)$, and later in [9] that for all $q \in [1, 4]$, we have that $\pi_{\mathbb{Z}^2, p_c, q}(0 \leftrightarrow \infty) = 0$, implying $\pi_{\mathbb{Z}^2, p_c, q}^1 = \pi_{\mathbb{Z}^2, p_c, q}^0$ and continuity of the phase transition (these were established much earlier for the cases of bond percolation $q = 1$ and the Ising model $q = 2$).

Russo–Seymour–Welsh estimates. A key ingredient in the proof of the continuity of the phase transition for all $q \in [1, 4]$ was the following set of Russo–Seymour–Welsh (RSW) type estimates on crossing probabilities of rectangles uniform in the boundary conditions (such results were obtained for $q = 1$ in [21] and for $q = 2$ in [8]), which were central to all available mixing time upper bounds at $p_c$ on $\mathbb{Z}^2$ (see [11,17]):

**Theorem 2.1** ([9, Theorem 3]). Let $q \in (1, 4]$ and consider the critical FK model on $\Lambda_{n,n'}$ where $n' = [\alpha n]$ for some $\alpha > 0$. For every $\varepsilon > 0$, if $R_\varepsilon = [\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon n', (1 - \varepsilon)n']$, there exists a $p(\alpha, \varepsilon, q) > 0$ such that,
\[
\pi_\Lambda^0(\mathcal{C}_R(R_\varepsilon)) \geq p .
\]

**Corollary 2.2.** Let $q \in (1, 4]$ and consider the critical FK model on $\Lambda_{n,n'}$. Let $R_\varepsilon$ be as in Theorem 2.1; then there exists a $p(\alpha, \varepsilon, q) > 0$ such that
\[
\pi_\Lambda^0(\mathcal{C}_o(\Lambda - R_\varepsilon)) \geq p .
\]

When $1 < q < 4$, we have the a stronger bound uniform in boundary conditions:

**Proposition 2.3** ([9, Theorem 7]). Let $q \in (1, 4]$ and consider the critical FK model on $\Lambda_{n,n'}$ where $n' = [\alpha n]$ for $\alpha > 0$. There exists $p(\alpha, q) > 0$ such that,
\[
\pi_\Lambda^0(\mathcal{C}_R(\Lambda)) \geq p .
\]
Such a bound is in fact not expected to hold for \( q = 4 \), where, for instance, it is believed (see [9]) that under free boundary conditions the crossing probability goes to 0 as \( N \to \infty \).

2.2. Markov chain mixing times. In this section we introduce the dynamical notation we will be using along with several important results in the theory of Markov chain mixing times, and in particular the theory of Markov chains on monotone spin systems, that we will use in the proof of Theorem 1.

**Mixing times.** Consider a Markov chain \( (X_t)_{t \geq 0} \) with finite state space \( \Omega \), and (in discrete time) transition kernel \( P \) with invariant measure \( \pi \). In the continuous-time setup, instead of \( P^t \) we consider, for \( \omega_0, \omega \in \Omega \), the heat kernel

\[
H_t(\omega_0, \omega) = \mathbb{P}_{\omega_0}(X_t = \omega) = e^{tL}(\omega_0, \omega),
\]

where \( \mathbb{P}_{\omega_0} \) is the probability w.r.t. the law of the chain \( (X_t)_{t \geq 0} \) given \( X_0 = \omega_0 \), and \( L \) is the infinitesimal generator for the Markov process.

For two measures \( \mu, \nu \) on \( \Omega \), define the total variation distance

\[
\| \mu - \nu \|_{TV} = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)| = \inf\{ \mathbb{P}(X \neq Y) \mid X \sim \mu, Y \sim \nu \},
\]

where the infimum is over all couplings \( (\mu, \nu) \). The worst-case total variation distance of \( X_t \) from \( \pi \) is denoted

\[
d_{TV}^\epsilon(t) = \max_{\omega_0 \in \Omega} \| \mathbb{P}_{\omega_0}(X_t \in \cdot) - \pi \|_{TV},
\]

and the **total variation mixing time** of the Markov chain is given by (for \( \epsilon \in (0, 1) \)),

\[
t_{\text{mix}}(\epsilon) = \inf\{ t \geq 0 : d_{TV}^\epsilon(t) \leq \epsilon \}.
\]

For any \( \epsilon \leq \frac{1}{4} \), \( t_{\text{mix}}(\epsilon) \) is submultiplicative and the convergence to \( \pi \) in total variation distance is thenceforth exponentially fast. As such, we write \( t_{\text{mix}}, \) omitting the parameter \( \epsilon \) to refer to the standard choice \( \epsilon = 1/(2e) \).

The FK dynamics. The present paper is almost exclusively concerned with continuous-time heat-bath Glauber dynamics \( (X_t)_{t \geq 0} \) for the random cluster model on \( \Lambda \) with boundary conditions \( \xi \): this is a reversible Markov chain w.r.t. \( \pi_{\Lambda,\xi} \) defined as follows: assign i.i.d. rate-1 Poisson clocks to every edge in \( \Lambda - \partial \Lambda \); whenever the clock at an edge rings, resample its edge value according to \( \pi_{\Lambda,\xi}(\omega|_e \in \cdot | \omega|_{\Lambda-\{e\}} = X_t|_{\Lambda-\{e\}}) \). In particular, for \( e = (v,w) \in \Lambda - \partial \Lambda \), the transition rate from \( \omega \) to \( \omega \cup \{ e \} \) is

\[
\begin{cases}
    p & \text{if } v \leftrightarrow w \text{ in } \Lambda - \{ e \} \cup \xi, \\
    p/[p + q(1-p)] & \text{otherwise}.
\end{cases}
\]

An alternative view of the heat-bath dynamics is the **random mapping representation** of this dynamics: the edge updates correspond to a sequence \( (J_i, U_i, T_i)_{i \geq 1} \), in which \( T_1 < T_2 < \ldots \) are the clock ring times, the \( J_i \)'s are i.i.d. uniformly selected edges in \( \Lambda - \partial \Lambda \), and the \( U_i \)'s are i.i.d. uniform random variables on \( [0,1] \): at time \( T_i \), for \( J_i = (v,w) \), the dynamics replaces the value of \( \omega(J_i) \) by \( 1\{ U_i \leq p \} \) if \( v \leftrightarrow w \) in \( \Lambda - \{ J_i \} \cup \xi \) and by \( 1\{ U_i \leq p/[p + q(1-p)] \} \) otherwise.
**Monotonicity.** As a result of the monotonicity of the FK model for \( q \geq 1 \), the heat-bath Glauber dynamics for the FK model is monotone: for two initial configurations \( \omega' \geq \omega \), we have that for all times \( t \geq 0 \),

\[
H_t(\omega', \cdot) \succeq H_t(\omega, \cdot).
\]

Using the random mapping representation, we define the grand coupling of the set of Markov chains with all possible initial configurations, which corresponds to the identity coupling of all three random variables \( (J_i, U_i, T_i)_{i \geq 1} \) amongst all the chains; for \( q \geq 1 \), this coupling preserves the partial ordering on initial states for all subsequent times.

The following standard fact is obtained via the grand coupling (see, e.g., [19, Eq. 2.10] in the context of the Ising model, as well as [11] in the context of the FK model).

**Fact 2.4.** Consider a set \( E \) and a monotone Markov chain \( (X_t)_{t \geq 0} \) on \( \Omega = \{0,1\}^E \) with extremal configurations \( \{0,1\} \). For every \( t \geq 0 \),

\[
d_{tv}(t) \leq |E|\|\mathbb{P}_1(X_t \in \cdot) - \mathbb{P}_0(X_t \in \cdot)\|_{tv}.
\]

Combined with the triangle inequality one obtains for the sub-multiplicative quantity

\[
d_{tv}(t) = \max_{\omega_1,\omega_2 \in \Omega} \| \mathbb{P}_{\omega_1}(X_t \in \cdot) - \mathbb{P}_{\omega_2}(X_t \in \cdot)\|_{tv},
\]

that, in the FK setting,

\[
d_{tv}(t) \leq d_{tv}(t) \leq 2|E(G)|\|\mathbb{P}_1(X_t \in \cdot) - \mathbb{P}_0(X_t \in \cdot)\|_{tv}.
\]

**Censoring.** Key to our proof will be the Peres–Winkler [20] censoring inequality for monotone systems. While the theorem of [20] and its subsequent applications in e.g., [16, 19] are stated for spin systems whose sites are the vertices of the underlying graph, one can view the edges as the sites by considering the appropriate line graph; it is then easy to verify that the FK Glauber dynamics satisfies the conditions of [20, Theorem 1.1]. Further, while Theorem 1.1 of [20] is stated for the discrete-time dynamics, its formulation in continuous-time follows from the same proof: see also [19, Theorem 2.5].

**Theorem 2.5 ([20]).** Let \( \mu_T \) be the law of continuous-time Glauber dynamics at time \( T \) of a monotone system on \( \Lambda \) with invariant measure \( \pi \), whose initial distribution \( \mu_0 / \pi \) is increasing. Set \( 0 = t_0 < t_1 < \ldots < t_k = T \) for some \( k \), let \( (B_i)_{i=1}^k \) be subsets of \( \Lambda \), and let \( \mu_T \) be the law at time \( T \) of the censored dynamics, started at \( \mu_0 \), where only updates within \( B_i \) are kept in the time interval \( [t_{i-1}, t_i) \). Then

\[
\|\mu_T - \pi\|_{tv} \leq \|\mu_T - \pi\|_{tv} \quad \text{and} \quad \mu_T \leq \mu_T; \quad \text{moreover,} \quad \mu_T / \pi \quad \text{and} \quad \mu_T / \pi \quad \text{are both increasing.}
\]

**Boundary modifications.** Let \( \xi, \xi' \) be a pair of boundary conditions on \( \Lambda \) with corresponding mixing times \( t_{\text{mix}}, t_{\text{mix}}' \); define

\[
M_{\xi,\xi'} = \|\pi_{\Lambda}^\xi / \pi_{\Lambda}^\xi'\|_\infty \lor \|\pi_{\Lambda}^{\xi'} / \pi_{\Lambda}^\xi\|_\infty.
\]

It is well-known (see, e.g., [19, Lemma 2.8]) that for some \( c \) independent of \( n, \xi, \xi' \),

\[
t_{\text{mix}} \leq cM_{\xi,\xi'}^2|E(\Lambda)|t_{\text{mix}}'(2.2)
\]

(this follows from first bounding \( t_{\text{mix}} \) via its spectral gap, then using the variational characterization of the spectral gap: the Dirichlet form, expressed in terms of local variances, gives a factor of \( M_{\xi,\xi'}^2 \), and the variance produces another factor of \( M_{\xi,\xi'} \)).
3. Equilibrium estimates

In what follows, fix \( q \in (1, 4] \), let \( p = p_c(q) \) and drop \( p, q \) from the notation henceforth.

3.1. Crossing probabilities. In this subsection we present estimates on crossing probabilities that will be used to prove the desired mixing time bounds. The following is a slight extension of [11, Theorem 3.4].

**Proposition 3.1.** Let \( q \in (1, 4] \) and fix \( \alpha \in (0, 1] \). Consider the critical FK model on \( \Lambda = \Lambda_{n,n'} \) with \( [\alpha n] \leq n' \leq [\alpha^{-1} n] \). For every \( \varepsilon > 0 \), there exists \( c_4(\alpha, \varepsilon, q) > 0 \) such that for every \( x \in [\varepsilon n, 1 - \varepsilon n] \), and every boundary condition \( \xi \) on \( \partial \Lambda \), one has
\[
\pi^\xi_\Lambda((x, 0) \leftrightarrow (x, [n'])) \gtrsim n^{-c_4}.
\]

**Proof.** The proposition was proved in the case \( n' = \lfloor \alpha n \rfloor \) in [11, Theorem 3.4] by stitching together crossings of rectangles and using the RSW estimates of Theorem 2.1. Since the crossing probabilities of Theorem 2.1 are monotone in the aspect ratio, each is bounded away from zero for aspect ratios in \([\alpha, \alpha^{-1}]\), yielding the desired extension. ■

The next two results are for \( q = 4 \) (Proposition 2.3 implies both for \( 1 < q < 4 \)).

**Lemma 3.2.** Let \( q = 4 \) and fix \( \alpha \in (0, 1] \). Consider the critical FK model on \( \Lambda = \Lambda_{n,n'} \) with \( [\alpha n] \leq n' \leq [\alpha^{-1} n] \) and \((1, 0)\) boundary conditions denoting wired on \( \partial_h \Lambda \) and free elsewhere. For every \( \varepsilon > 0 \), there exists \( p(\alpha, \varepsilon) > 0 \) such that
\[
\pi^1_\Lambda(C_v([0, n] \times [0, (1 - \varepsilon)n'])) \geq p(\varepsilon).
\]

**Proof.** Note that for an \( n \times n \) square with wired boundary conditions on the N,S sides, and free boundary conditions elsewhere, the probability of a vertical crossing is, by self-duality, \( 1/2 \). By bounding the Radon–Nikodym derivative, it is easy to see that under the same boundary conditions but with the north and south sides disconnected from each other, the same probability is bounded below by some \( p_0(q) > 0 \).

Moreover, by Theorem 2.1 and monotonicity in boundary conditions, there exists \( p_1(\varepsilon) > 0 \) such that
\[
\pi^1_\Lambda(C_h([\varepsilon n, (1 - \varepsilon)n] \times [(1 - \varepsilon)\alpha n, (1 - \varepsilon)\alpha n])) \geq p_1.
\]

The measure on \([[(\varepsilon/4)n, (1 - \varepsilon/4)n] \times [0, (1 - \varepsilon)n]\) conditioned on the above crossing event stochastically dominates the measure induced on it by wired on the N,S sides and free on the E,W sides of \([((1 - \alpha + \varepsilon/4)^n/2, (1 + \alpha - \varepsilon/4)^n/2] \times [0, (1 - \varepsilon)\alpha n]\). By monotonicity in boundary conditions inequality the probability of a vertical crossing in \([0, n] \times [0, (1 - \varepsilon)\alpha n]\) is thus bigger than \( p_0p_1 \). Finally, by Corollary 2.2 and monotonicity of crossing probabilities in aspect ratio, there exists \( p_2(\alpha, q) > 0 \) such that
\[
\pi^{1,0}_\Lambda(C_v(\Lambda - [((1 - \alpha)^n/2, (1 + \alpha)^n/2] \times [(1 - \varepsilon)\alpha n, (1 - \varepsilon)\alpha n'])) \geq p_2
\]
holds for every \( [\alpha n] \leq n' \leq [\alpha^{-1} n] \). By the FKG inequality, stitching the three crossings together implies the desired lower bound for \( p = p_0p_1p_2 \). ■

**Corollary 3.3.** Let \( q = 4 \) and fix \( \alpha \in (0, 1] \). Consider the critical FK model on \( \Lambda = \Lambda_{n,n'} \) with \( [\alpha n] \leq n' \leq [\alpha^{-1} n] \) and boundary conditions, denoted by \((1, 0, 1, 0)\), that are wired on \( \partial_{N,S} \Lambda \) and free on \( \partial_{E,W} \Lambda \). Then there exists \( p(\alpha) > 0 \) such that
\[
\pi^{1,0,1,0}_\Lambda(C_v(\Lambda)) \geq p.
\]
Proof. For all \( n' \leq n \) this follows immediately from self-duality and monotonicity in boundary conditions. For \( n \leq n' \leq \lceil \alpha^{-1}n \rceil \), by monotonicity in boundary conditions and Lemma 3.2, for any \( \varepsilon \in (0, 1) \), there is a \( p(1, \varepsilon) > 0 \) such that,
\[
\pi^{1,0,1,0}_\Lambda (C_v([0, n] \times [0, \varepsilon n])) \geq p,
\]
and by reflection symmetry, \( \pi^{1,0,1,0}_\Lambda (C_v([0, n] \times [n' - \varepsilon n, n'])) \geq p \). Let
\[
A_\varepsilon = \Lambda - [\varepsilon n, (1 - \varepsilon)n] \times [\varepsilon n, n' - \varepsilon n].
\]
Since \( C_o(A_\varepsilon) \) can be lower bounded by four crossings of rectangles, each of whose probabilities is monotone in the aspect ratio and thus bounded away from 0 uniformly over \( n \leq n' \leq \lceil \alpha^{-1}n \rceil \), we have that \( \pi^{1,0,1,0}_\Lambda (C_v(A_\varepsilon)) \geq p' \) uniformly over \( n \leq n' \leq \lceil \alpha^{-1}n \rceil \) for some \( p'(\alpha, \varepsilon) \).

Now observe that
\[
(C_v([0, n] \times [0, \varepsilon n]) \cap C_v([0, n] \times [n' - \varepsilon n, n'])) \cap C_o(A_\varepsilon) \subset C_v(\Lambda).
\]
After fixing any small \( \varepsilon > 0 \), by the FKG inequality, there exists some \( p(\alpha) > 0 \) such that for every \( n \leq n' \leq \lceil \alpha^{-1}n \rceil \), one has \( \pi^{1,0,1,0}_\Lambda (C_v(\Lambda)) \geq p \), as required.

3.2. Boundary bridges. In this subsection we define boundary bridges of the FK model and related notation. As explained in detail in §1.1, the presence of boundary bridges will be the key obstacle to coupling and, in turn, to mixing time bounds.

Definition 3.4. Consider a rectangle \( \Lambda = \Lambda_{n,n'} \) with boundary conditions \( \xi \), and a connected segment \( L = [a, b] \times \{n'\} \subset \partial \Lambda \). A component \( \gamma \subset \partial \Lambda \) of \( \xi \) is a bridge over \( L \) if there exist \( v = (v_1, v_2), w = (w_1, w_2) \in \gamma \) such that \( v \xleftarrow{\xi} w \) and
\[
v_1 < a \quad \text{and} \quad w_1 > b.
\]
Note that every two distinct bridges \( \gamma_1 \neq \gamma_2 \) over \( L \) are disjoint in \( \xi \). Denote by \( \Gamma^L = \Gamma^L(\xi) \) the set of all bridges over the segment \( L \). Define bridges on subsets of \( \partial \Lambda_s, \partial \Lambda_e, \partial \Lambda_w \) analogously.

Figure 4. A pair of boundary bridges, \( \gamma_i, \gamma_{i+1} \), over \( e \in \partial \Lambda \) induced by a configuration on \( \Lambda - R \) and separated by a dual-bridge over \( e \).
Definition 3.5 (hull and length of a bridge). The west and east hulls of a bridge $\gamma$ over $L = [a, b] \times \{n'\}$ are defined as

$$
\text{hull}_w(\gamma) = \left[ \max\{x \leq a : (x, n') \in \gamma\}, a \right] \times \{n'\},
$$

$$
\text{hull}_e(\gamma) = \left[ b, \min\{x \geq b : (x, n') \in \gamma\} \right] \times \{n'\},
$$

so that the hulls of a bridge $\gamma$ are connected subsets of $\partial_n \Lambda$ (see Fig. 4). The west and east lengths of $\gamma$ are defined to be

$$
\ell_w(\gamma) = \left| \text{hull}_w(\gamma) \right|, \quad \ell_e(\gamma) = \left| \text{hull}_e(\gamma) \right|.
$$

Given the above convention, for any $L$ and $\xi$ we can define an east-ordering of $\Gamma^L(\xi)$ as $(\gamma_1, \gamma_2, \ldots, \gamma_{|\Gamma^L|})$ where, for all $i < j$,

$$
\ell_e(\gamma_i) < \ell_e(\gamma_j).
$$

Note that, in this ordering of the bridges, $\text{hull}_e(\gamma_i) \subseteq \text{hull}_e(\gamma_j)$ for all $i < j$. Define a west-ordering of $\Gamma^L$ analogously.

Definition 3.6. For a subset $R \subset \Lambda$, an induced boundary condition on $\partial R$ is one that can be identified with the component structure of an edge configuration $\omega|_{\Lambda-R}$ along with the boundary condition on $\Lambda$.

Using the above definitions, and planarity, one can check the following useful facts (depicted in Fig. 4). For concreteness we use the east-ordering of $\Gamma^L = \{\gamma_1, \ldots, \gamma_{|\Gamma^L|}\}$.

Fact 3.7. Let $\Lambda \supset R$ with boundary conditions $\xi$, and let $L \subset \partial_n R$. If $\gamma_i$, for $i < |\Gamma^L|$, is the $i$-th bridge in the east-ordering of $\Gamma^L$, then either the two connected components of $\partial_n R - (\text{hull}_w(\gamma_i) \cup L \cup \text{hull}_e(\gamma_i))$ are connected in $\Lambda - R$, or each of these components is connected to $\partial \Lambda$ in $\Lambda - R$.

Fact 3.8. Let $\Lambda \supset R$ with boundary conditions $\xi$, and let $L \subset \partial_n R$. For every two induced bridges $\gamma_1 \neq \gamma_2$ over a segment $L$ such that $\text{hull}_e(\gamma_1) \subseteq \text{hull}_e(\gamma_2)$, either the two sets $(\text{hull}_w(\gamma_2) \triangle \text{hull}_w(\gamma_1))$ and $(\text{hull}_e(\gamma_2) \triangle \text{hull}_e(\gamma_1))$ are dual-connected in $\Lambda - R$, or each of these sets is dual-connected to $\partial \Lambda$ in $\Lambda - R$.

3.3. Estimating the number of boundary bridges. In this section, we bound the number of distinct induced boundary bridges over a segment of $\partial R$.

When sampling boundary conditions on $R \subset \Lambda$ under $\pi^L_\Lambda$, the induced bridges over $e$ and all properties of them, are measurable w.r.t. $\omega|_{\Lambda-R^e}$. For any configuration $\omega$, we denote by $\Gamma^e = \Gamma^e(\omega|_{\Lambda-R^e}, \xi)$ the set of all bridges over $e$ corresponding to that configuration on $\Lambda$, with the above defined west and east orderings.

The main estimate on $|\Gamma^e|$, that will be key to the proof of Theorem 1, is the following.

Proposition 3.9. Let $q \in (1, 4]$ and fix $\alpha \in (0, 1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $n' \geq \lfloor \alpha n \rfloor$, along with the subset $R = \Lambda_{n,n'/2}$. There exists $c(\alpha, q) > 0$ such that for every $\gamma \in \partial_n R$, every boundary condition $\xi$, and every $K > 0$,

$$
\pi^L_\Lambda(\omega : |\Gamma^e| \geq K \log n) \lesssim n^{-cK}.
$$

(3.1)
Moreover, there exists \( c'(\alpha, q) > 0 \), and for every \( \varepsilon > 0 \) there is some \( K_0(\varepsilon) \), such that for every \( e \in [n^\varepsilon, n - n^\varepsilon] \times \{ \left\lfloor \frac{n}{2} \right\rfloor \} \), every boundary condition \( \xi \), and every \( K < K_0 \),

\[
\pi^\varepsilon_{\Lambda} (\omega : |\Gamma^e| \geq K \log n) \gtrsim n^{-cK}.
\]

In addition to the tail behavior of \( |\Gamma^e| \), we can also classify its typical behavior, showing that a fixed edge indeed has order \( \log n \) bridges over it with high probability (cf. the case of \( p \neq p_c(q) \) where this quantity is typically \( O(1) \)).

**Corollary 3.10.** Let \( q \in (1, 4] \) and \( \alpha \in (0, 1] \). Consider a rectangle \( \Lambda = \Lambda_{n,n'} \) with \( n' \geq [\alpha n] \), along with \( R = \Lambda_{n,n'}/2 \). There exists \( c(\alpha, q) > 0 \), and for every \( \varepsilon > 0 \), there exist \( K' > K(\varepsilon) > 0 \), such that for every \( e \in [n^\varepsilon, n - n^\varepsilon] \times \{ \left\lfloor \frac{n}{2} \right\rfloor \} \) and every \( \xi \),

\[
\pi^\varepsilon_{\Lambda} (|\Gamma^e| \notin [K \log n, K' \log n]) \lesssim n^{-c}.
\]

The model at \( p = p_c(q) \), \( q \in (1, 4] \) is believed to be scale-invariant; in line with this, having nested bridges whose length grow exponentially induces \( c \log n \) clusters ranging in scale between \( O(1) \) and \( O(n^\varepsilon) \). Indeed, this is how the lower bound of Corollary 3.10 is obtained. It will, therefore, be important for us to split the set \( \Gamma^e \) into those bridges according to their proximity to their interior bridges, as well as the boundary.

For the rest of this subsection, since \( e \) is fixed, if \( e \) is in the western half of \( \partial \Lambda R \) then we will use the east-ordering of \( \Gamma^e \) and otherwise we will use the west-ordering of \( \Gamma^e \). If \( e \) is in the western half of \( \partial \Lambda R \) define the following subsets of \( \Gamma^e \):

\[
\Gamma_1^e = \Gamma_1^e(\omega_{|\Lambda_{-R}}, \xi) = \{ \gamma_i \in \Gamma^e : \ell_E(\gamma_{i-1}) \leq \frac{n}{6}, \ell_E(\gamma_i) \leq 2\ell_E(\gamma_{i-1}) \},
\]
\[
\Gamma_2^e = \Gamma_2^e(\omega_{|\Lambda_{-R}}, \xi) = \{ \gamma_i \in \Gamma^e : \ell_E(\gamma_{i-1}) \geq \frac{n}{6}, n - x - \ell_E(\gamma_i) \geq \frac{1}{2} (n - x - \ell_E(\gamma_{i-1})) \}.
\]

For \( e \) in the eastern half of \( \partial \Lambda R \), define \( \Gamma_1^e \) and \( \Gamma_2^e \) analogously, by replacing \( \ell_E \) with \( \ell_W \) and \( n - x \) with \( x \). For convenience, let \( \gamma_0 \) be the possibly nonexistent bridge given by the two vertices incident to the edge \( e \), which will allow us to treat \( \gamma_1 \) as we would treat the other \( \gamma_i \)’s.

Before proving Proposition 3.9, we present the two lemmas central to the upper bound (3.1) of Proposition 3.9, proving exponential tails on each of \( |\Gamma_1^e| \) and \( |\Gamma_2^e| \) beyond \( O(\log n) \). Together, these will imply the \( O(\log n) \) upper bound on \( |\Gamma^e| \), so we defer the proofs of the two lemmas until after completing the proof of Proposition 3.9 using the lemmas. We conclude this section with a proof of Corollary 3.10.

**Lemma 3.11.** There exists \( c_1(\alpha, q) > 0 \) such that for every \( e \in \partial \Lambda R \), \( \xi \) and \( K > 0 \),

\[
\pi^\varepsilon_{\Lambda} (\omega : |\Gamma_1^e| \geq K \log n) \lesssim n^{-c_1K}.
\]

**Lemma 3.12.** There exists \( c_2(\alpha, q) > 0 \) such that for every \( e \in \partial \Lambda R \), \( \xi \) and \( K > 0 \),

\[
\pi^\varepsilon_{\Lambda} (\omega : |\Gamma_2^e| \geq K \log n) \lesssim n^{-c_2K}.
\]

With these two lemmas in hand the proof of Proposition 3.9 is greatly simplified.

**Proof of Proposition 3.9.** We begin with the upper bound. Fix an edge \( e \in \partial \Lambda R \) and a boundary condition \( \xi \) on \( \partial \Lambda \). Without loss of generality suppose that \( e \) is in the western half of \( \partial \Lambda R \) and use the east-ordering of \( \Gamma^e = \{ \gamma_1, \gamma_2, \ldots, \gamma_{|\Gamma^e|} \} \). Observe that violating the second condition in \( \Gamma_1^e \) means that \( \ell_E(\gamma_i) \) has at least doubled the length of
its predecessor, whereas violating the second condition in $\Gamma_0^\varepsilon$ means that $n - x - \ell_k(\gamma_i)$ is at most half the corresponding quantity of its predecessor. Noting that violating the length condition of $\ell_k(\gamma_i-1)$ (compared to $n/6$) is disjoint between $\Gamma_1^\varepsilon$ and $\Gamma_2^\varepsilon$, and since $\ell_k(\gamma_i) \leq n$ and $n - x - \ell_k(\gamma_i) \geq 1$ for all $i$, we deterministically have

$$|\Gamma^\varepsilon - (\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon)| \leq 2 \log_2 n \leq 3 \log n.$$ 

Using a union bound,

$$\pi_A^\varepsilon (|\Gamma_1^\varepsilon \cup \Gamma_2^\varepsilon| \geq (K - 3) \log n) \leq \pi_A^\varepsilon (|\Gamma_1^\varepsilon| \geq \frac{K-3}{2} \log n) + \pi_A^\varepsilon (|\Gamma_2^\varepsilon| \geq \frac{K-3}{2} \log n).$$

The bounds on the two terms on the right-hand side are given by Lemmas 3.11–3.12, respectively. Taking the minimum of $c_1, c_2$ in those lemmas then implies that there exists $c(\alpha, q) > 0$ such that

$$\pi_A^\varepsilon (|\Gamma^\varepsilon| \geq K \log n) \leq n^{-c(K-3)/2}.$$ 

In order to prove the lower bound, for any $\varepsilon > 0$, fix any edge $e = (x, \lfloor \frac{n'}{2} \rfloor)$ with $x \in [n^\varepsilon, n - n^\varepsilon]$. For $i \geq 1$, suppressing the dependence on $e$, define the sets

$$\tilde{R}_i^N = [x - 2i + 1, x + 2i + 1] \times \left[ \left[ \frac{n'}{2}, \frac{n'}{2} \right] + 2i + 2^i + 2^{-i+1} \right],$$

$$\tilde{R}_i^W = [x + 2i, x + 2i + 1] \times \left[ \left[ \frac{n'}{2} - 2i, \frac{n'}{2} + 2i + 2^i + 2^{-i+1} \right] \right],$$

$$\tilde{R}_i^W = [x - 2i + 1, x - 2i] \times \left[ \left[ \frac{n'}{2}, \frac{n'}{2} \right] - 2i + 2^i + 2^{-i+1} \right].$$

When $K < K_0 := \frac{\varepsilon \log 4}{4}$, for every $i \leq 2K \log n$, we have $\tilde{R}_i^W, \tilde{R}_i^N, \tilde{R}_i^W \subset \Lambda$. Also define the following crossing events.

$$\mathcal{A}_i = C_v(R_i^W) \cap C_h(R_i^N) \cap C_v(R_i^W),$$

$$\mathcal{A}_i^* = C_v^*(R_i^W) \cap C_h^*(R_i^N) \cap C_v^*(R_i^W).$$

Then by definition of distinct bridges in $\Lambda - R_i^W$, we observe that for each $k$,

$$\{ |\Gamma^\varepsilon| \geq K \log n \} \supset \bigcap_{i=1}^{K \log n} \mathcal{A}_{2i-1} \cap \mathcal{A}_{2i}^*.$$ 

By monotonicity in boundary conditions, the FKG inequality, and Theorem 2.1, there exists $p(\alpha, q) > 0$ such that for every $i \leq 2K \log n$,

$$\pi_A^\varepsilon (\mathcal{A}_i) \geq \pi^0_{R_i^W}(C_v(R_i^W)) \pi^0_{R_i^N}(C_h(R_i^N)) \pi^0_{R_i^W}(C_v(R_i^W)) \geq p,$$

$$\pi_A^\varepsilon (\mathcal{A}_i^*) \geq \pi^1_{R_i^W}(C_v^*(R_i^W)) \pi^1_{R_i^N}(C_h^*(R_i^N)) \pi^1_{R_i^W}(C_v^*(R_i^W)) \geq p.$$ 

Thus, if $K < K_0$, we have $\pi_A^\varepsilon (|\Gamma^\varepsilon| \geq K \log n) \geq p^{2K \log n}$, as required. 

We now prove Lemmas 3.11–3.12, whose proofs constitute the majority of the work in obtaining Proposition 3.9.
Proof of Lemma 3.11. Assume without loss of generality that \( e = [x-1, x] \times [n'] \) is such that \( x \leq \frac{n}{2} \) and use the east-ordering of \( \Gamma^e = \{\gamma_1, \gamma_2, \ldots, \gamma_{|\Gamma^e|}\} \). In order to obtain an upper tail on \( |\Gamma^e| \), let us describe a revealing procedure for the FK configuration \( \omega \) on \( \Lambda - R^e \) under \( \pi^e_{\Lambda} \).

Let \( F = [0, n] \times [\lfloor \frac{n'}{2} \rfloor, n'] = \Lambda - R^e \) (so that \( \omega|_F, \xi \) is the set of connections with respect to which the existence/properties of bridges are measurable). We can sequentially reveal \( \gamma_1, \ldots, \gamma_{|\Gamma^e|} \) by exposing the open clusters of \( \partial \) one at a time, starting from those adjacent to the right-vertex of \( e \). Such procedures for exposing the clusters have been used in related settings (see, e.g., [2, 11, 17]); we formally describe the procedure here since in our case it involves long-range interactions imposed by the FK boundary conditions. One can reveal the open cluster \( C_v \) containing a vertex \( v \) in the set \( F \) by

1. initializing the set \( C = \{v\} \);
2. exposing the values of \( \omega \) on all edges in \( E(F) \) that contain vertices in \( C \);
3. adding to \( C \) any vertices that are now connected by a path of open edges to \( \{v\} \) (including possibly the connections imposed by the boundary condition \( \xi \));
4. repeating the process from step (1) with the new set \( C \).

Any open cluster containing vertices in \( \partial R \) on both the right and left sides of \( e \) is a bridge over \( e \). In order to reveal the first \( m \) bridges over the edge \( e \), we can iteratively reveal the open clusters of \( \partial R \) in \( F \), starting initially with the cluster of \( (x, \lfloor \frac{n'}{2} \rfloor) \), and continuing to the right along \( \partial R \), until \( m \) distinct bridges have been exposed. Using this revealing procedure, the edges which are revealed in order to expose the first \( m \) bridges over \( e \) are either enclosed by \( \gamma_m \) and \( \partial R \), or belong to the outer boundary of \( \gamma_m \) and are closed, thus forming a bounding dual-path.

Let \( (\mathcal{F}_m) \) be the filtration associated with the above revealing process for the bridges \( (\mathcal{F}_m \text{ reveals } \gamma_1, \ldots, \gamma_m) \) over the edge \( e \). Our aim is to prove that for every \( m \geq 1 \),

\[
\pi^e_{\Lambda}(\gamma_m \in \Gamma^e \mid \mathcal{F}_{m-1}) \leq p, \tag{3.5}
\]

(if \( \gamma_m \) doesn’t exist, we vacuously say \( \gamma_m \notin \Gamma^e \)) for the choice of

\[
p = 1 - p_1p_2p_3 < 1, \tag{3.6}
\]

where \( p_1(\alpha, q), p_2(\alpha, q), p_3(\alpha, q) > 0 \) are defined as follows:

- \( p_1 \) is given by Proposition 2.3 with aspect ratio 1 for \( 1 < q < 4 \) and by Lemma 3.2 with the choice \( \varepsilon = 1/2 \) and aspect ratio 1/2 for \( q = 4 \),
- \( p_2 \) is the probability given by Theorem 2.1 for \( \varepsilon = 1/4 \) and aspect ratio \( 6 \lor \alpha^{-1} \),
- \( p_3 \) is the probability given by Theorem 2.1 for \( \varepsilon = 1/3 \) and aspect ratio 1.

Let us first conclude the proof of Lemma 3.11 given (3.5). By iteratively conditioning on \( (\mathcal{F}_i)_{i \geq 1} \) we see that the sequence of indicators \( \{1\{\gamma_i \in \Gamma^e\}\}_{i \geq 1} \) is stochastically dominated by the i.i.d. sequence \( (Z_i)_{i \geq 1} \) where \( Z_i \sim \text{Bernoulli}(p) \). At the same time, by definition of the set \( \Gamma^e \), through this revealing process, as soon as \( \lceil \log_2 n \rceil \) many of the indicators \( \{1\{\gamma_i \in \Gamma^e\}\} \) are zero, all subsequent ones are deterministically zero (note that once \( \ell_n(\gamma_i) > n/6 \), every subsequent bridge will also have this property).
After conditioning on ζ (via the configuration in the blue shaded region), the probability of the purple and green dual-crossings is greater than \( p_1 p_2 p_3 \), bounding the probability of \( \{ \gamma_m \in \Gamma_1^e \} \). Therefore, we can bound

\[
\pi^e_\Lambda(|\Gamma_1^e| \geq r) \leq \mathbb{P}(\text{Bin}(r + \lceil \log_2 n \rceil - 1, p) \geq r)
\]

which, upon taking \( r = K \log n \) and using, say, Hoeffding’s inequality once \( K \geq 2p^{-1} \), yields the desired estimate.

We now turn to proving the conditional estimate of (3.5). First observe that by Fact 3.8 and the definition of hull \( e(\gamma_m - 1) \), if \( L_W, L_E \) are the two connected subsets of \( \partial_n R - \text{hull}_e(\gamma_m - 1) \), the event

\[
E_m = \{ L_W \xrightarrow{F^*} L_E \text{ or } L_E \xrightarrow{F^*} \partial \Lambda \}
\]

satisfies \( E_m \supset \{|\Gamma^e| \geq m \} \supset \{ \gamma_m \in \Gamma_1^e \} \). In fact, the revealing process of \( F_{m-1} \) reveals precisely the dual-path that bounds the open cluster of \( \gamma_m - 1 \), and that dual-path is either a dual connection from \( L_W \) to \( L_E \) in \( F^* \), or it is the west-most dual crossing from \( L_E \) to \( \partial \Lambda \) that is to the right of \( \gamma_m - 1 \). Either way, denote by \( \zeta \) the dual-bridge/crossing revealed as such by \( F_{m-1} \) (see Fig. 5), and let \( (z, \lfloor n'^2 \rceil) \) be the west-most point of \( \zeta \cap \partial_n R \).

Let \( k = \ell_e(\gamma_m - 1) \); in order for \( \gamma_m \in \Gamma_1^e \), necessarily \( \ell_e(\gamma_m) \leq 2k \) and

\[
(z, \lfloor n'^2 \rceil) \in [x + k, x + 2k] \times \{ \lfloor n'^2 \rceil \} =: I.
\]

We will establish the desired upper bound of (3.5) uniformly over \( F_{m-1}, \zeta \) and \( k \). It suffices to only consider \( k \leq \frac{n}{6} \) because otherwise, \( \ell_e(\gamma_m) > \frac{n}{6} \) and therefore \( \gamma_m \notin \Gamma_1^e \) deterministically.

Note that conditional on \( F_{m-1} \) (which contains the \( \sigma \)-algebras generated by \( \zeta \) and \( k \)), by Fact 3.7, the event \( \{ \gamma_m \in \Gamma_1^e \} \) implies the event \( S \), stating that either \( \zeta \) is a dual-bridge and \( I \) is primal-connected in \( F \cup \xi \) to the left component of \( \partial_n R - \text{hull}_e(\zeta) \), or alternatively \( \zeta \) is a dual-crossing to \( \partial \Lambda \) and \( I \) is primal-connected to \( \partial \Lambda \) in \( F \). Thus,
in this conditional space,

\[
\{ \gamma_m \notin \Gamma^*_i \} \supset \left\{ \zeta \overset{E^*}{\longrightarrow} [x + 2k, x + 3k] \times \{|n/4|\} \right\}, \tag{3.7}
\]

since the right-hand side of Eq. (3.7) implies \( S^c \) which implies the left-hand side.

In order to lower bound the probability of the last event in Eq. (3.7), let \( D^* \) be the outer (if \( \zeta \) is a dual-crossing in \( F \), then eastern) connected component of \( F^* - \zeta \), and let \( D \) be its dual. Define also the following subsets of \( \Lambda \):

\[
R_1 = [x, x + k] \times \left[ 0, \left\lfloor \frac{n}{4} \right\rfloor \right] + \min \{k, \frac{an}{4} \},
\]

\[
R_2 = [x, x + 3k] \times \left[ \left\lfloor \frac{n}{4} \right\rfloor + \min \{\frac{k}{2}, \frac{an}{8} \}, \left\lfloor \frac{n}{4} \right\rfloor + \min \{k, \frac{an}{4} \} \right],
\]

\[
R_3 = [x + 2k, x + 3k] \times \left[ \left\lfloor \frac{n}{4} \right\rfloor, \frac{n}{4} \right] + \min \{k, \frac{an}{4} \},
\]

whereby, the event in the right-hand side of Eq. (3.7) can be written as \( \{ \zeta \overset{E^*}{\longrightarrow} \partial_k R_3 \} \).

For any \( i = 1, 2, 3 \), define the following crossing events (see Fig. 5):

\[
C^*_i(R_i \cap D) = \left\{ \partial_k R_i \overset{R_i \cap D}{\leftrightarrow} \partial_k \omega R_i \right\},
\]

\[
C^*_i(R_i \cap D) = \left\{ \partial_k R_i \overset{R_i \cap D}{\leftrightarrow} \partial_k \omega R_i \right\}. \tag{3.8}
\]

(observe that implicit in \( (C^*_i(R_i \cap D))^c \) is the event \( \{ \partial_k R_i \cap \emptyset \} \cap \{ \partial_k R_i \cap \emptyset \} \), and similarly, implicit in \( (C^*_i(R_i \cap D))^c \) is the event \( \{ \partial_k R_i \cap \emptyset \} \cap \{ \partial_k R_i \cap \emptyset \} \).)

**Claim 3.13.** Conditional on \( F_{m-1} \) (and in particular also \( \zeta \) and \( k \)),

\[
\{ \gamma_m \notin \Gamma^*_i \} \supset \left( C^*_i(R_1 \cap D) \cap C^*_i(R_2 \cap D) \cap C^*_i(R_3 \cap D) \right).
\]

**Proof.** Suppose that \( \omega \) satisfies the events on the right-hand. Recall that \( \zeta \) is such that \( \partial_k R_3 \cap D \neq \emptyset \) and \( \partial_k R_3 \cap D \neq \emptyset \), and \( \partial_k R_3 \leftrightarrow \partial_k R_3 \) in \( R_3 \cap D \) since \( \omega \in C^*_i(R_3 \cap D) \). Consider \( R_3 \cap D \) with boundary conditions wired on \( \partial_k R_3 \cap D \) and free on \( \zeta \) and \( \partial_k R_3 \cap D \); then the boundary conditions on \( R_3 \cap D \) alternate between free and wired on boundary curves ordered clockwise as \( L^w_1, L^w_2, L^w_3, \ldots \); by planarity and the choice of generalized Dobrushin boundary conditions, for any two wired boundary curves \( L^w_i, L^w_{i+1} \), either \( L^w_i \leftrightarrow L^w_{i+1} \) or \( L^w_i \leftrightarrow L^w_{i+1} \) for some \( j \neq i \). Picking the two wired boundary segments of \( \partial_k R_3 \cap D \) closest to \( \partial_k R_3 \), the aforementioned fact that \( \partial_k R_3 \leftrightarrow \partial_k R_3 \) in \( R_3 \cap D \) implies that either \( \partial_k R_3 \leftrightarrow \partial_k R_3 \) or \( \partial_k R_3 \leftrightarrow \partial_k R_3 \). In the former, \( \{ \gamma_m \notin \Gamma^*_i \} \) holds by Eq. (3.7), so suppose only the latter holds and call the dual-crossing \( \zeta_3 \).

Since \( \partial_k R_3 \leftrightarrow \partial_k R_3 \), both \( \partial_k R_2 \cap D \) and \( \partial_k R_2 \cap D \) are nonempty. Clearly, \( \zeta_3 \) splits \( R_2 \cap D \) into the subset to its east, \( U_E \), and that to its west, \( U_W \). Consider the set to its east, \( U_E \), with boundary conditions that are wired on \( \partial_k R_2 \cap D \) and free on \( \zeta \) and \( \partial_k R_2 \cap D \). Since \( \zeta_3 \) and \( \zeta \) are vertex-disjoint (by our assumption that \( \partial_k R_3 \leftrightarrow \zeta \) in \( R_3 \cap D \)), and the wired boundary segments adjacent to \( \zeta_3 \) are disconnected in \( U_E \), it must be that either \( \zeta_3 \leftrightarrow \zeta \) or \( \zeta_3 \leftrightarrow \partial_k R_2 \) in \( U_E \). Using the same reasoning on \( U_W \), either \( \zeta_3 \leftrightarrow \zeta \) or \( \zeta_3 \leftrightarrow \partial_k R_2 \) in \( U_W \). Combining these, either \( \zeta_3 \leftrightarrow \zeta_3 \), in which case \( \zeta_3 \leftrightarrow \partial_k R_3 \), or alternatively \( \partial_k R_2 \leftrightarrow \zeta_3 \leftrightarrow \partial_k R_2 \) in \( R_2 \cap D \). In the former case, by Eq. (3.7), \( \{ \gamma_m \notin \Gamma^*_i \} \); assume therefore that only the latter case holds, and let \( \zeta_2 \) be a dual-crossing between \( \partial_k R_2 \) to \( \partial_k R_2 \) that intersects \( \zeta_3 \).
Finally, we can deduce that \( \partial_y R_1 \cap D \) and \( \partial_y R_1 \cap D \) are nonempty as \( \zeta_2 \) and \( \zeta \) are vertex-disjoint (by our assumptions \( \zeta \leftrightarrow \zeta_3 \) and \( \zeta_2 \leftrightarrow \zeta_3 \)). Considering now \( U_8 \), the subset of \( R_1 \cap D \) south of \( \zeta_2 \) with wired boundary conditions on \( \partial_{e,w} R_1 \cap D \) and free elsewhere, as before we deduce that either \( \zeta_2 \leftarrow \zeta \) or \( \zeta_2 \leftarrow \partial_y R_1 \) in \( U_8 \). Since, by definition of \( \zeta \), deterministically \( \partial_y R_1 \cap D = \emptyset \), the former must hold, and \( \zeta \leftrightarrow \partial_y R_3 \) through \( \zeta_2 \) and \( \zeta_3 \), and Eq. (3.7) concludes the proof. ■

We will next bound the probability of each of the events \( C_v(R_1 \cap D) \), \( C_h(R_2 \cap D) \) and \( C_v(R_3 \cap D) \), which, using the above claim, will translate to a bound on \( \{ \gamma_m \notin \Gamma \} \).

To see this, first note that by planarity, for all \( i = 1, 2, 3 \) and every subset \( D \),

\[
C_v(R_i \cap D) > (C_h(R_i))^c = C_v(R_i),
\]

and likewise for horizontal crossing events. Define the rectangle \( \tilde{R}_1 \supseteq R_1 \) by

\[
\tilde{R}_1 = [x, x + k] \times [\lceil \frac{w}{2} \rceil, n'] \subset \Lambda.
\]

Let the boundary conditions (1, 0) on \( \tilde{R}_1 \) be free on \( \partial_y \tilde{R}_1 \) and wired on \( \partial_{n,e,w} \tilde{R}_1 \). Combining Eq. (3.9), monotonicity in boundary conditions, and the domain Markov property, we get for \( p_1(\alpha, q) > 0 \) given by Eq. (3.6),

\[
\pi_\Lambda^\xi (C_v^*(R_1 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \geq \pi_{R_1}^1 (C_v^*(R_1 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \\
\geq \pi_{R_1}^{1,0} (C_v^*(R_1)) \geq p_1,
\]

where the last inequality follows from Proposition 2.3, Lemma 3.2 and self-duality. We stress that wiring of \( \partial_{n,e,w} \tilde{R}_1 \) allowed us to ignore the information revealed on \( \tilde{R}_1 \) as far as the configuration in \( R_1 \cap D \) is concerned, and the fact that \( \omega|_{\zeta} \) is closed allowed us to place a free boundary on \( \partial_y \tilde{R}_1 \), supporting Lemma 3.2.

Next, consider the rectangle \( \tilde{R}_2 \supseteq R_2 \) defined by

\[
\tilde{R}_2 = [x - k, x + 4k] \times [\lceil \frac{w}{2} \rceil, n'] \supset \Lambda,
\]

so that \( \tilde{R}_2 \subset \Lambda \) since \( k = \ell_\Lambda(\gamma_{m-1}) \leq n/6 \). By monotonicity in boundary conditions and Eq. (3.9), we get that for the choice of \( p_2(\alpha, q) > 0 \) given by Eq. (3.6),

\[
\pi_\Lambda^\xi (C_h^*(R_2 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \geq \pi_{R_2}^1 (C_h^*(R_2 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \\
\geq \pi_{R_2}^{1,0} (C_h^*(R_2)) \geq p_2.
\]

Similarly, applying the exact same treatment of \( \tilde{R}_2 \) to

\[
\tilde{R}_3 = [x + k, x + 4k] \times [\lceil \frac{n'}{4} \rceil, n'] \subset \Lambda,
\]

(it is possible to encapsulate \( R_3 \) by a rectangle with wired boundary conditions since \( \zeta \) does not intersect \( \partial_y R_3 \) in our conditional space) shows that

\[
\pi_\Lambda^\xi (C_v^*(R_3 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \geq \pi_{R_3}^1 (C_v^*(R_3 \cap D) \mid \ell_\Lambda(\gamma_{m-1}) = k, \mathcal{F}_{m-1}, \zeta) \\
\geq \pi_{R_3}^{1,0} (C_v^*(R_3)) \geq p_3,
\]

for \( p_3(\alpha, q) > 0 \) as defined in Eq. (3.6).
Putting these all together, by the FKG inequality and Claim 3.13,

\[ \pi_\Lambda^\varepsilon(\gamma_m \notin \Gamma_e^i | \mathcal{F}_{m-1}) \geq p_1 p_2 p_3, \]

implying the desired (3.5), and concluding the proof.

Proof of Lemma 3.12. Without loss of generality suppose \( e \) is in the western half of \( \partial_n R \) and use the east-ordering of bridges so that \( \Gamma_e = \{ \gamma_1, ..., \gamma_{|\Gamma_e^\varepsilon|} \} \).

The proof follows the same argument used to prove Lemma 3.11. In what follows we describe the necessary modifications that are needed here. Recall the prescribed revealing process for the configuration on \( F = [0, n] \times [n'_{\varepsilon}] \) described in the proof of Lemma 3.11; recall also that \( (\mathcal{F}_m) \) is the filtration corresponding to the process of sequentially revealing the distinct bridges over the edge \( e \). Our goal is to prove the following analogue of (3.5), that for every \( m \geq 1, \)

\[ \pi_\Lambda^\varepsilon(\gamma_m \in \Gamma_e^2 | \mathcal{F}_{m-1}) \leq p, \]  

for the choice of \( p = 1 - p_1 p_2 p_3 < 1 \) where,

- \( p_1 \) is given by Proposition 2.3 with aspect ratio \( \alpha \) for \( 1 < q < 4 \) and by Lemma 3.2 with the choice \( \varepsilon = 1/2 \) and aspect ratio \( \alpha/2 \) for \( q = 4 \),
- \( p_2 \) is the probability given by Theorem 2.1 for \( \varepsilon = 1/8 \) and aspect ratio \( 6/\alpha \),
- \( p_3 \) is the probability given by Theorem 2.1 for \( \varepsilon = 1/3 \) and aspect ratio \( \alpha \).

Indeed, by iteratively conditioning on \( (\mathcal{F}_i)_{i \geq 1} \), the bound (3.10) allows us to stochastically dominate the sequence of indicators \( (1\{\gamma_i \in \Gamma_e^i\})_{i \geq 1} \) by the i.i.d. sequence \( (Z_i)_{i \geq 1} \) where \( Z_i \sim \text{Bernoulli}(p) \), and moreover by definition of \( \Gamma_e^2 \), as soon as \( \lfloor \log_2 n \rfloor \) of the indicators are zero, all subsequent ones are deterministically zero. The desired inequality then follows by comparison to \( \mathbb{P}(\text{Bin}(r + \lfloor \log_2 n \rfloor - 1, p) \geq r) \) for \( r = K \log n \).

As before, we consider a fixed \( m \), and let \( L_w, L_e \) be the left and right connected components of \( \partial_n R - \text{hull}_e(\gamma_{m-1}) \). As in the proof of Lemma 3.11, reveal \( \gamma_{m-1} \), in...
which case we reveal the enclosing dual-path $\zeta$ attaining
\[
E_m = \left\{ L_E \xleftarrow{F^*} L_w \text{ or } L_E \xrightarrow{F^*} \partial A \right\},
\]
whose west-most vertex of intersection with $\partial A R$ is marked by $(z, |n^w/2|)$. Conditionally
on $F_{m-1}$, which contains the $\sigma$-algebras of $\gamma_{m-1}, \zeta$ and $\ell_E(\gamma_{m-1}) = k$,

Also for any instance of the configuration revealed by $F_{m-1}$, we can set $k = \ell_E(\gamma_{m-1})$
as before, and let
\[
l := n - (x + \ell_E(\gamma_{m-1})) .
\]
If $n - z < l/2$, deterministically $\gamma_m \notin \Gamma_2^\varepsilon$ (as argued in the proof of Proposition 3.9),
hence we may assume that $n - z \geq l/2$; moreover, since $k \geq \frac{n}{6}$, it must be that $\frac{l}{6} \leq k$.
Define the following subsets of $\Lambda$:
\[
R_1 = [n - l - \frac{l}{6}, n - l] \times [\left\lfloor \frac{n'}{2} \right\rfloor, |\frac{n'}{2}| + \frac{\alpha l}{6}],
R_2 = [n - l - \frac{l}{3}, n - \frac{l}{6}] \times [\left\lfloor \frac{n'}{2} \right\rfloor, |\frac{n'}{2}| + \frac{\alpha l}{6}],
R_3 = [n - \frac{l}{2}, n - \frac{l}{3}] \times [\left\lfloor \frac{n'}{2} \right\rfloor, |\frac{n'}{2}| + \frac{\alpha l}{6}].
\]
Define $C^*_v(R_1 \cap D), C^*_h(R_2 \cap D), C^*_v(R_3 \cap D)$ as in Eq. (3.8). As in Claim 3.13,
\[
\{\gamma_m \notin \Gamma_2^\varepsilon\} \supset \left( C^*_v(R_1 \cap D) \cap C^*_h(R_2 \cap D) \cap C^*_v(R_3 \cap D) \right).
\]
Finally, for $\tilde{R}_i$, $i = 1, 2, 3$ given by
\[
\tilde{R}_1 = [n - l - \frac{l}{6}, n - l] \times [\left\lfloor \frac{n'}{2} \right\rfloor, |\frac{n'}{2}| + \frac{\alpha l}{3}],
\tilde{R}_2 = [n - l - \frac{l}{3}, n] \times \left[ \left\lfloor \frac{n'}{2} \right\rfloor - \frac{\alpha l}{6}, |\frac{n'}{2}| + \frac{\alpha l}{6} \right],
\tilde{R}_3 = [n - \frac{l}{2}, n] \times \left[ \left\lfloor \frac{n'}{2} \right\rfloor - \frac{\alpha l}{6}, |\frac{n'}{2}| + \frac{\alpha l}{6} \right],
\]
(note that all three are subsets of $\Lambda$, by the fact that $l \leq n$ and $n' \geq |\alpha n|$), the same
monotonicity argument used in the proof of Lemma 3.11 now implies (see Fig. 6) that
\[
\pi^\xi_\Lambda (\gamma_{m-1} \notin \Gamma_2^\varepsilon \mid F_{m-1}) \geq p_1 p_2 p_3,
\]
implying (3.10) and concluding the proof.

By matching the tail estimate of Proposition 3.9 with a lower bound, we can straight-
forwardly see that an order $\log n$ bridges over a fixed edge is indeed typical.

**Proof of Corollary 3.10.** Fix an arbitrary $\varepsilon > 0$, any boundary condition $\xi$, and any $e \in [n^\varepsilon, n - n^\varepsilon] \times \left\{ \left\lfloor \frac{n'}{2} \right\rfloor \right\}$. For this $e$, recall the definitions of the rectangles $R_i^{E}, R_i^{F}, R_i^{W}$
as well as their subsets $R_i^{E}, R_i^{F}, R_i^{W}$ from (3.2)–(3.3). As before, when $M < M_0 := \frac{\varepsilon \log 4}{4}$,
for every $i \leq 2 M \log n$, all these are subsets of $\Lambda$ and we can define the crossing events
\[
\mathcal{A}_i = C_v(R_i^{W}) \cap C_h(R_i^{E}) \cap C_v(R_i^{F}) , \quad \text{and} \quad \mathcal{A}^*_i = C_v^*(R_i^{W}) \cap C_h^*(R_i^{E}) \cap C_v^*(R_i^{F}) .
\]
Now for each $i \leq M \log n$, we can define the event $\chi_i := \mathcal{A}_{2i-1} \cap \mathcal{A}^*_{2i}$, and notice that
\[
|\Gamma_e| \geq \sum_{i=1}^{M \log n} 1\{\chi_i\}.
\]
Observe that for each $i$, the event $\mathcal{A}_i$ is measurable with respect to the configuration $\omega$ on the half-annulus $R_{i}^N \cup R_{i}^S \cup R_{i}^E$. By a similar reasoning as before, there exists some $p = p(\alpha, q) > 0$ such that for every $i = 1, \ldots, 2M \log n$, and every configuration $\eta$,

\[
\pi^\xi_A(\mathcal{A}_i \mid \omega |_{A - \tilde{R}_{i}^{N,E,W}} = \eta) \geq \pi^0_{\tilde{R}_{i}^{N}}(C_v(R_{i}^N))\pi^0_{\tilde{R}_{i}^{E}}(C_v(R_{i}^E))\pi^0_{\tilde{R}_{i}^{S}}(C_h(R_{i}^S)) \geq p, \\
\pi^\xi_A(\mathcal{A}_i^* \mid \omega |_{A - \tilde{R}_{i}^{N,E,W}} = \eta) \geq \pi^1_{\tilde{R}_{i}^{N}}(C_v^*(R_{i}^N))\pi^1_{\tilde{R}_{i}^{E}}(C_v^*(R_{i}^E))\pi^1_{\tilde{R}_{i}^{S}}(C_h^*(R_{i}^S)) \geq p.
\]

Observe that for $i \neq j$ the interiors of $\tilde{R}_{i}^{N,E,W}$ and $\tilde{R}_{j}^{N,E,W}$ are disjoint. As a consequence, we can also deduce by the domain Markov property and monotonicity, that for every configuration $\eta$, for every $i = 1, \ldots, M \log n$,

\[
\pi^\xi_A(\chi_i \mid \omega |_{A - \tilde{R}_{i}^{N,E,W} - \tilde{R}_{j}^{N,E,W}} = \eta) \geq p^2.
\]

In particular, the sequence of indicators $(1\{\chi_i\})_{i = 1, \ldots, M \log n}$ stochastically dominates a sequence of i.i.d. Bernoulli($p^2$) random variables. We therefore deduce that

\[
|\Gamma^e| \geq \sum_{i = 1}^{M \log n} 1\{\chi_i\} \geq \text{Bin}(M \log n, p^2)
\]

Choosing $K < \frac{p^2}{2} M$, and using Hoeffding’s inequality to bound the probability that the binomial random variable on the right-hand side is at most $K \log n$, we see that

\[
\pi^\xi_A(\mid\Gamma^e\mid \leq K \log n) \leq \exp\left[ -\frac{1}{4} Mp^4 \log n \right].
\]

Combining this via a union bound with the upper bound from (3.1) in Proposition 3.9 implies the desired.

### 3.4. Disjoint crossings.

To extend our mixing time bound from favorable boundary conditions (see §5.1) to periodic boundary conditions (which are not in that class) in §5.3, we need an analogous bound on the number of disjoint crossings of a rectangle.

For a rectangle $R$ and a configuration $\omega |_{R}$, let $\Psi_R = \Psi_R(\omega |_{R})$ be the set containing every component $A \subset V(R)$ (connected via the edges of $\omega |_{R}$) that intersects both $\partial_s R$ and $\partial_n R$. We will need the following equilibrium estimate similar to Proposition 3.9.

**Proposition 3.14.** Let $q \in (1, 4]$ and $\alpha \in (0, 1]$. Consider the critical FK model on $\Lambda = \Lambda_{n,n'}$ with $n' \geq \lfloor \alpha n \rfloor$, and the subset $R = [0, n] \times [n' \cdot \frac{2n'}{3}, 2n' \cdot \frac{2n'}{3}]$. There exists $c(\alpha, q) > 0$ such that for every boundary condition $\xi$ and every $m \geq 3$,

\[
\pi^\xi_A(\mid\Psi_R\mid \geq m) \leq e^{-cm}.
\]

**Proof.** We will prove by induction that, for all $m \geq 1$,

\[
\pi^\xi_A(\mid\Psi_R\mid \geq m) \leq (1 - p)^{m-2}, \tag{3.11}
\]

where $p > 0$ is as given by Proposition 2.3 with aspect ratio $3/\alpha$ when $1 < q < 4$, and is as given by Corollary 3.3 with aspect ratio $\alpha/3$ when $q = 4$.

The cases $m = 1, 2$ are trivially satisfied for any $0 < p < 1$. Now let $m \geq 3$, and suppose that Eq. (3.11) holds for $m - 1$; the proof will be concluded once we show that

\[
\pi^\xi_A(\mid\Psi_R\mid \geq m \mid \mid\Psi_R\mid \geq m - 1) \leq 1 - p.
\]
Conditioned on the existence of at least \( m - 1 \) distinct components in \( \Psi_R \), we can condition on the west-most component in \( \Psi_R \) (by revealing all dual-components of \( \omega |_R \) incident to \( \partial_w R \), then revealing the primal-component of the adjacent primal-crossing). We can also condition on the \( m - 2 \) east-most components in \( \Psi_R \) (by successively repeating the aforementioned procedure from east to west, i.e., replacing \( \partial_w R \) above by \( \partial_e R \) to reveal some component \( C \in \Psi_R \), then by its western boundary \( \partial_w C \), etc.).

Through this process, we can find two disjoint vertical dual-crossings \( \zeta_1, \zeta_2 \) of \( R \), each one a simple dual-path; the set \((R^* - \zeta_1 - \zeta_2)^*\) consists of three connected subsets of \( R \); let \( D \) denote the middle one. There are exactly \( m - 1 \) elements of \( \Psi_R \) in \( R - D \), thus its \( m \)-th element, if one exists, must belong to \( D \). Since every edge in \( \zeta_1 \cup \zeta_2 \) is dual-open, for any such choice of \( \zeta_1, \zeta_2 \), we then have

\[
\pi^\xi_{\Lambda}(|\Psi_R| \geq m \mid |\Psi_R| \geq m - 1, \zeta_1, \zeta_2) = \pi^\xi_{\Lambda}(\mathcal{C}_v(D) \mid \zeta_1, \zeta_2),
\]

Using the domain Markov property and monotonicity of boundary conditions,

\[
\pi^\xi_{\Lambda}(\mathcal{C}_v(D) \mid \zeta_1, \zeta_2) \leq \pi^{1,0,1,0}_{D}(\mathcal{C}_v(D)),
\]

where \((1,0,1,0)\) boundary conditions on \( D \) denote those that are free on \( \zeta_1, \zeta_2 \) and wired on \( \partial R \cap \partial D \). Again by monotonicity (in boundary conditions and crossing events),

\[
1 - \pi^{1,0,1,0}_{R}(\mathcal{C}_v(D)) \leq \pi^{1,0,1,0}_{R}(\mathcal{C}_v(D) \mid \omega_{\zeta_1} = 0, \omega_{\zeta_2} = 0) \leq 1 - \pi^{1,0,1,0}_{R}(\mathcal{C}_v^*(R)),
\]

where, following the notation of Corollary 3.3, \((1,0,1,0)\) boundary conditions on a rectangle \( R \) are wired on \( \partial_n R \) and free on \( \partial_w R \). By monotonicity in boundary conditions and the definition of \( p \), the right-hand side is bounded above by

\[
1 - \pi^{1,0,1,0}_{R}(\mathcal{C}_h^n([0, n] \times [\frac{n'}{3}, \frac{2n'}{3} + \frac{an}{3}])) \leq 1 - p.
\]

4. Dynamical tools

In this section, we introduce the main techniques we use to control the total variation distance from stationarity for the random cluster heat-bath Glauber dynamics.

4.1. Modifications of boundary conditions. Crucial to the proof of Theorem 1 is the modification of boundary bridges so that we can couple beyond FK interfaces as done in [17]; in this subsection we define boundary condition modifications and control the effect such modifications can have on the mixing time.

**Definition 4.1** (segment modification). Let \( \xi \) be a boundary condition on a rectangle \( \Lambda \) which corresponds to a partition \( \{P_1, ..., P_k\} \) of \( \partial \Lambda \), and let \( \Delta \subset \partial \Lambda \). The **segment modification on** \( \Delta \), denoted by \( \xi_{\Delta^c} \), is the boundary condition that corresponds to the partition \( \{P_1 - V(\Delta), ..., P_k - V(\Delta)\} \cup \bigcup_{v \in V(\Delta)} \{v\} \) of \( \partial \Lambda \).

**Definition 4.2** (bridge modification). Let \( \xi \) be a boundary condition on \( \partial \Lambda \), corresponding to a partition \( \{P_1, ..., P_k\} \) of \( \partial \Lambda \). Let \( \Gamma^e \) be the set of disjoint bridges in \( \xi|_{\partial \Lambda} \) over the edge \( e = (x, y) \in \partial_\Lambda \), corresponding to the components \( \{P_{ij}\}_{j=1} \), as per Definition 3.4. The **bridge modification of** \( \xi \) over \( e \), denoted \( \xi^e \), is the boundary condition associated to the partition where every \( P_{ij} \) is split into two components,

\[
P^e_{ij} = \{(v_1, v_2) \in P_{ij} : v_1 - x < 0\} \quad \text{and} \quad P^w_{ij} = \{(v_1, v_2) \in P_{ij} : v_1 - x > 0\}.
\]
Lemma 4.6. Let $P$ be a boundary condition on $\partial \Lambda$, corresponding to a partition $\{P_1, \ldots, P_k\}$ of $\partial \Lambda$. The side modification $\xi^*$ is defined as follows. Split every $P_j$ into its four sides, that is, for $i = N, S, E, W$, let
\[ P_j^i = \{ v \in P_j : v \in \partial_i \Lambda \} , \]
where for the corner vertices the choice is arbitrary (for concreteness, associate the corner with the side that follows it clockwise). Then for every $\xi$, the modified $\xi^*$ has no components that contain vertices in more than one side of $\partial \Lambda$.

It will be useful to have a notion of distance between boundary conditions.

**Definition 4.4.** For any pair of boundary conditions, $\xi, \xi'$ define the symmetric distance function $d(\xi, \xi')$ as follows: if $\xi''$ is the unique smallest (in the previously defined partial ordering) boundary condition with $\xi'' \geq \xi$ and $\xi'' \geq \xi'$, define $d(\xi, \xi') = k(\xi'') - k(\xi) + k(\xi'') - k(\xi')$, where $k(\xi)$ is the number of distinct components in the partition induced by $\xi$.

If $\xi$ is a boundary condition on $\Lambda$ and $\xi'$ is a any of the above boundary modifications of $\xi$, then $\xi' \leq \xi$ and the partition associated to it is a refinement of $\xi$; this implies that $d(\xi, \xi') = k(\xi) - k(\xi')$. One can easily verify the following.

**Fact 4.5.** For a segment $\Delta$, we have $d(\xi, \xi^\Delta) \leq |V(\Delta)|$; for an edge $e$, we have $d(\xi, \xi^e) = |E^e|$; for the side modification $\xi^*$, we have that $d(\xi, \xi^*)$ is bounded above by three times the number of components in $\xi$ with vertices in multiple sides of $\partial \Lambda$.

We now present a lemma bounding the effect on total variation mixing from modifying the boundary conditions. Recall that for two boundary conditions $\xi, \xi'$ on $\Lambda$, we defined in the preliminaries the quantity $M_{\xi, \xi'} = \| \pi^\xi_\Lambda \|_{\infty} + \| \pi^\xi_\Lambda \|_{\infty}$, and we have from Eq. (2.2), that $t_{\text{mix}} \lesssim M_{\xi, \xi'} |E(\Lambda)| t_{\text{mix}}$. Moreover, using the notation of [19] and [16], for an initial configuration $\omega_0$, and boundary condition $\xi$, let
\[ d_{TV}^{(\omega_0, \xi)}(t) = \| \mathbb{P}_{\omega_0}(X_t \in \cdot) - \pi^\xi_\Lambda \|_{TV} , \]
where here and throughout the paper, for any Markov chain $(X_t)_{t \geq 0}$, $\mathbb{P}_\omega(X_t \in \cdot) = \mathbb{P}(X_t \in \cdot \mid X_0 = \omega)$ with boundary conditions $\xi$; when clear from the context we may drop the boundary condition superscript from the notation.

**Lemma 4.6.** Let $\xi, \xi'$ be a pair of boundary conditions on $\partial \Lambda$. Then,
\[ M_{\xi, \xi'} \leq q^{d(\xi', \xi)} , \tag{4.1} \]
and consequently, there exists an absolute $c > 0$ such that for every $t > 0$,
\[ \max_{\omega_0 \in \{0, 1\}} d_{TV}^{(\omega_0, \xi)}(t) \leq 8 \max_{\omega_0 \in \{0, 1\}} d_{TV}^{(\omega_0, \xi')}\left( c |E(\Lambda)|^{-2} q^{-4d(\xi', \xi)} t \right) + \exp\left(-q^{d(\xi', \xi)}\right) . \tag{4.2} \]
Proof. Adapting an argument of [19] to the FK setting, Lemma 5.4 of [11] proves a version of this lemma for two coupled probability measures \( P, P^\Delta \) over pairs \( \xi, \xi^\Delta \). The proof for arbitrary pairs of boundary conditions, \( \xi, \xi' \), is identical; letting \( P \) be a point mass at \( \xi \) completes the proof. ■

4.2. Censored block dynamics. We next define the censored and systematic block dynamics whose coupling is the core of the dynamical analysis used to prove Theorem 1. This coupling may be of general interest in the study of mixing times of monotone Markov chains, where one only has control on mixing times in the presence of favorable boundary conditions. We therefore present it in more generality than necessary for the proof of Theorem 1: consider the heat-bath dynamics for a monotone spin or edge system on a graph \( G \) with boundary \( \partial G \) that satisfies the domain Markov property and has extremal configurations \( \{0, 1\} \) and invariant measure \( \pi^G \).

Definition 4.7 (systematic block dynamics). Let \( B_0, \ldots, B_{s-1} \) denote a finite cover of \( E(G) \) (or \( V(G) \) for a spin system) and for \( k \geq 1 \) let \( i_k := (k - 1) \mod s \).

The systematic block dynamics \( (Y_k)_{k \geq 0} \) is a discrete-time flavor of the block dynamics w.r.t. \( \{B_i\} \), with blocks that are updated in a sequential deterministic order: at time \( k \), the chain updates \( B_{i_k} \) by resampling \( \omega|_{B_{i_k}} \sim \pi^G_i(\cdot | \omega|_{G -(B_{i_k})}) \).

Remark 4.8. The systematic block dynamics as defined has unique invariant measure \( \pi^G_i \), but it is neither time-homogenous nor reversible. If one wanted a time-homogenous and reversible analogue, one could, e.g., in each time step update all \( s \) blocks sequentially in forward and then reverse order, i.e., in the order \( (B_0, ..., B_{s-1}, ..., B_0) \).

Definition 4.9 (censored block dynamics). Let \( B_0, ..., B_{s-1} \) be as before and consider a set \( \Gamma_i \) of permissible boundary conditions for \( B_i \), and fix \( \epsilon > 0 \). The censored block dynamics \( (\tilde{X}_t)_{t \geq 0} \) is the continuous-time single-bond (single-site) heat-bath dynamics that simulates \( \tilde{Y}_k \) as follows. For a given \( \epsilon > 0 \), define

\[
T = T(\epsilon) = \max_i \max_{\xi \in \Gamma_i} t^\epsilon_{\text{mix}}^{B_i}(\epsilon), \tag{4.3}
\]

where \( t^\epsilon_{\text{mix}}^{B_i} \) is the mixing time of standard heat-bath dynamics on the block \( B_i \) with boundary conditions \( \xi \). Let \( i_k := (k - 1) \mod s \) and let the chain \( \tilde{X}_t \) be obtained from the standard heat-bath dynamics by censoring, as in Theorem 2.5, for every integer \( k \geq 1 \), along the interval \( (k - 1)T, kT \), all updates except those in \( B_{i_k} \).

Proposition 4.10 (comparison of censored / systematic block dynamics). Let \( (\tilde{X}_t)_{t \geq 0} \) and \( (Y_k)_{k \geq 0} \) be the censored and systematic block dynamics, respectively, w.r.t. some blocks \( B_0, ..., B_{s-1} \) and permissible boundary conditions \( \Gamma_i \) on \( G \) with boundary conditions \( \xi \) and initial state \( \omega_0 \), as per Definitions 4.7-4.9. Let \( \rho := \max_{k \geq 1} \max_{\xi \in [0, s-1]} \mathbb{P}_{\omega_0}(Y_k|_{\partial B_i} \notin \Gamma_i) \), \( \tag{4.4} \)

where \( Y_k|_{\partial B_i} \) is the boundary condition induced on \( \partial B_i \) by the configuration \( Y_k \) on \( G \setminus B_i^c \). Then for every \( \epsilon > 0 \), every integer \( k \geq 0 \), and \( T \) as in (4.3),

\[
\|\mathbb{P}_{\omega_0}(\tilde{X}_{kT} \in \cdot) - \mathbb{P}_{\omega_0}(Y_k \in \cdot)\|_{TV} \leq k(\rho + \epsilon). \tag{4.5}
\]
We claim that (4.5) holds.

In the setting of Proposition 4.10, when the initial state is \(\omega_0\) and let \(\delta_k = \|P_{\omega_0}(X_{kT} \in \cdot) - P_{\omega_0}(Y_k \in \cdot)\|_{TV}\) denote its left-hand side; observe that \(\delta_0 = 0\) by definition, and suppose that \(\delta_k \leq k(\rho + \varepsilon)\) for some \(k\). Denote by \(i = i_{k+1}\) the block that is updated at time \(k + 1\) by the systematic block dynamics, and let \(X_{k+1}^{(i)}\) and \(Y_{k+1}^{(i)}\) be the censored and systematic chains corresponding to the block sequence \((B_{(i+1) \mod s})_{i \geq 0}\) (where the block \(B_i\) is the first to be updated). By the Markov property and the triangle inequality,

\[
\delta_{k+1} \leq \frac{1}{2} \sum_{\omega, \omega'} \left( \|P_{\omega_0}(X_{kT} = \omega') - P_{\omega_0}(Y_k = \omega')\|_{TV} \right) \|P_{\omega'}(X_{kT} = \omega) - P_{\omega'}(Y_{k+1}^{(i)} = \omega)\|_{TV}
\]

The last summand in (4.6) satisfies

\[
\sum_{\omega} \|P_{\omega_0}(Y_k = \omega)\|_{TV} \|P_{\omega}(X_T^{(i)} \in \cdot) - P_{\omega}(Y_{k+1}^{(i)} \in \cdot)\|_{TV}
\]

by the definition of \(T = T_1(\varepsilon)\) and \(\rho\); here we identified the configuration on \(G - B_i^o\) with the boundary it induces on \(\partial B_i\). Combined with Eq. (4.6), this completes the proof of Eq. (4.5).

**Remark 4.11.** Although we defined the systematic and censored block dynamics for deterministic block updates, one could easily formulate the same bound for the usual block dynamics with random updates, where the \(s\) sub-blocks are each assigned i.i.d. Poisson clocks (cf. [18]), by also randomizing the order in which the censored block dynamics updates sub-blocks, using the identity coupling on the corresponding clocks.

**Proof of Proposition 4.10.** We now prove Eq. (4.5) by induction on \(k\). Fix any \(\omega_0\) and let \(\delta_k = \|P_{\omega_0}(X_{kT} \in \cdot) - P_{\omega_0}(Y_k \in \cdot)\|_{TV}\) denote its left-hand side; observe that \(\delta_0 = 0\) by definition, and suppose that \(\delta_k \leq k(\rho + \varepsilon)\) for some \(k\). Denote by \(i = i_{k+1}\) the block that is updated at time \(k + 1\) by the systematic block dynamics, and let \(X_{k+1}^{(i)}\) and \(Y_{k+1}^{(i)}\) be the censored and systematic chains corresponding to the block sequence \((B_{(i+1) \mod s})_{i \geq 0}\) (where the block \(B_i\) is the first to be updated). By the Markov property and the triangle inequality,

\[
\delta_{k+1} \leq \frac{1}{2} \sum_{\omega, \omega'} \left( \|P_{\omega_0}(X_{kT} = \omega') - P_{\omega_0}(Y_k = \omega')\|_{TV} \right) \|P_{\omega'}(X_{kT} = \omega) - P_{\omega'}(Y_{k+1}^{(i)} = \omega)\|_{TV}.
\]

The last summand in (4.6) satisfies

\[
\sum_{\omega} \|P_{\omega_0}(Y_k = \omega)\|_{TV} \|P_{\omega}(X_T^{(i)} \in \cdot) - P_{\omega}(Y_{k+1}^{(i)} \in \cdot)\|_{TV}
\]

by the definition of \(T = T_1(\varepsilon)\) and \(\rho\); here we identified the configuration on \(G - B_i^o\) with the boundary it induces on \(\partial B_i\). Combined with Eq. (4.6), this completes the proof of Eq. (4.5).

**Remark 4.12.** In the setting of Proposition 4.10, when the initial state is \(\omega_0 \in \{0, 1\}\) (either minimal or maximal), one can obtain the following improved bound. Set

\[
T = \max_i \max_{\xi \in \Gamma_i} t_{\text{mix}}^{\xi, B_i}(\omega_0 | B_i, \varepsilon),
\]

where \(t_{\text{mix}}^{\xi, B_i}(\omega_0, \varepsilon) = \inf\{t : d_{TV}^{\omega_0, \xi}(t) \leq \varepsilon\}\), relaxing the previous definition (4.3) of \(T\) to only consider the initial state \(\omega_0\). Let \((X_t)\) be the censored block dynamics w.r.t. this new value of \(T\), and denote by \((\tilde{X}_t)\) the modification of \((X_t)\) where, for every \(k \geq 1\), the configuration of the block \(B_{\tilde{B}_k}\) (i.e., the block that is to be updated in the interval \(((k - 1)T, kT]\)) is reset at time \((k - 1)T\) to the original value of \(\omega_0\) on that block. We claim that (4.5) holds\(^1\) for the relaxed value of \(T\) in (4.7) if we replace \(X_t\) by \(\tilde{X}_t\).

\(^1\)In fact, (4.5) is valid for \(X_t\) with the relaxed \(T\) in (4.7) for every \(\omega_0\), not just for the maximal and minimal configurations; however, it is when \(\omega_0 \in \{0, 1\}\) that the modified dynamics \(\tilde{X}_t\) can easily be compared to \(X_t\), and thereafter to \(X_t\), via the censoring inequality of Theorem 2.5.
Indeed, all the steps in the above proof of Proposition 4.10 remain valid up to the final inequality, at which point the fact that we consider \( \bar{X}_t \) (as opposed to \( \bar{X}_1 \)) implies that

\[
\max_{\omega|_{\partial B_t} \in \Gamma_t} \left\| P_\omega(\bar{X}^{(i)}_T \in \cdot) - P_\omega(Y^{(i)}_1 \in \cdot) \right\|_{TV} = \max_{\xi \in \Gamma_t} \left\| P_{\omega|_{\partial B_t}}(\bar{X}^{(i)}_T \in \cdot) - \pi^{(i)}_{B_t} \right\|_{TV},
\]

which is at most \( \varepsilon \) when \( T \) is as defined in (4.7).

5. Proof of main result

In this section, we prove Theorem 1 by combining the equilibrium estimates of §3 with the dynamical tools provided in §4. We first establish an analog of Theorem 1 (Theorem 5.4) for “typical” boundary conditions (defined in §5.1 below), and then, using Proposition 3.14, derive from it the case of periodic boundary conditions in §5.3.

The effect of boundary bridges (which may foil the multiscale coupling approach, as described in §1.1) is controlled by restricting the analysis to those boundary conditions that have \( O(\log n) \) bridges, and applying Proposition 4.10 to bound the mixing time under such boundary conditions. We now define the favorable boundary conditions for which we prove a mixing time upper bound of \( n^{O(\log n)} \).

5.1. Typical boundary conditions. We first define the class of “typical” boundary conditions on a segment (e.g., \( \partial_n \Lambda \)).

**Definition 5.1** (typical boundary conditions on a segment). For \( K > 0, N \geq 1 \), and a segment \( L \), let \( \Xi_{K,N} \) be the set of boundary conditions \( \xi \) on \( L \) such that

\[
|\Gamma^e(\xi)| \leq K \log N \quad \text{for every } e \in L.
\]

We will later see (as a consequence of Lemma 5.7 below) that the boundary conditions on each of the sides of a box \( \Lambda \) induced by the infinite-volume FK measure \( \pi_{\mathbb{Z}^2} \) belong to the class of “typical” boundary conditions with high probability.

Next, we define the global property we require of typical boundary conditions.

**Definition 5.2** (typical boundary conditions on \( \partial \Lambda \)). Let \( \Upsilon_{K_1,K_2,N} = \Upsilon_{K_1,K_2,N}^{\Lambda} \) be the set of boundary conditions \( \xi \) on \( \partial \Lambda \) such that \( \xi|_{\partial \Lambda} \in \Xi_{K_1,N} \) for every \( i = n,s,e,w \), and \( \xi \) has at most \( K_2 \log N \) distinct components with vertices on different sides of \( \partial \Lambda \).

**Remark 5.3.** The wired and free boundary conditions on a side \( \partial_i \Lambda \) are always in \( \Upsilon_{K_1,K_2,N} \) whenever \( K_1 \log N \geq 1 \) and \( K_2 \log N \geq 1 \) (in the former all vertices are in just one component and in the latter no two vertices are in the same component).

5.2. Mixing under typical boundary conditions. Since periodic boundary conditions are not in \( \Upsilon_{K_1,K_2,N} \) for any \( K_2 > 0 \), we first bound the mixing time on rectangles \( \Lambda_{N,N'} \) where \( N' = \lceil \alpha N \rceil \) for \( \alpha \in (0,1] \), with boundary conditions \( \xi \in \Upsilon_{K_1,K_2,N} \).

**Theorem 5.4.** Let \( q \in (1,4) \) and fix \( \alpha \in (0,1] \) and \( K_1, K_2 > 0 \). Consider the Glauber dynamics for the critical FK model on \( \Lambda_{N,N'} \) with \( \alpha N \leq N' \leq N \) and boundary conditions \( \xi \in \Upsilon_{K_1,K_2,N} \). Then there exists \( c = c(\alpha,q,K_1,K_2) > 0 \) such that

\[
t_{\text{Mix}} \lesssim N^{c \log N}.
\]
Observe that if we define
\[ \Upsilon_{K,N} := \Upsilon_{K,2K,N}, \quad (5.1) \]
clearly \( \Upsilon_{K_1,K_2,N} \subset \Upsilon_{\max\{K_1,K_2\},N} \), so it suffices to consider \( \Upsilon_{K,N} \) for general \( K > 0 \).

The proof of Theorem 5.4 proceeds by analyzing the censored and systematic block dynamics on \( \Lambda \), obtaining good control on the systematic block dynamics using the RSW estimates of [9], then comparing it to the censored block dynamics. The choice of parameters for which we will apply Proposition 4.10 is the following.

**Definition 5.5** (block choice for censored / systematic block dynamics). Let \( q \in (1,4) \) and for any \( n' \leq n \leq N \), consider the critical FK Glauber dynamics on \( \Lambda_{n,n'} \). Let
\[ B_E = \left[ \frac{n}{4}, n \right] \times [0,n'], \]
\[ B_w = \left[ 0, \frac{3n}{4} \right] \times [0,n'], \]
ordered as \( B_0 = B_E , B_1 = B_w \) as in the setup of Proposition 4.10. For \( K = \max\{K_1,K_2\} \) given by Theorem 5.4, let \( \Gamma_i = \Upsilon_{K,N} \) be the set of permissible boundary conditions for the block \( B_i \) in \( \Lambda_{n,n'} \).

Before proving Theorem 5.4 we will prove two lemmas that will be necessary for the application of Proposition 4.10. We first introduce some preliminary notation.

For any \( n \leq N \), label the following edges in \( \partial \Lambda_{n,n'} \):
\[ e_s^c = (\lfloor \frac{n}{2} \rfloor + \frac{1}{2}, 0), \quad \text{and} \quad e_s^s = (\lfloor \frac{n}{2} \rfloor + \frac{1}{2}, n'). \]

Recall the definitions of the bridge modification \( \xi^c \) and the side modification \( \xi^s \) from Definitions 4.2–4.3. We will, throughout the proof of Theorem 5.4, for any boundary condition \( \xi \in \partial \Lambda_{n,n'} \), let the modification \( \xi' \leq \xi \) be given by
\[ \xi' := \xi^c_s \land \xi^s_s \land \xi^s, \quad (5.2) \]
i.e., the bridge modification of \( \xi \) on \( e_s^c \) and \( e_s^s \), combined with the side modification \( \xi^s \).

If \( \Xi_{K,N}, \Upsilon_{K,N} \) are the sets of boundary conditions defined in Definition 5.5, we let \( \Xi_{K,N}, \Upsilon_{K,N} \) be the sets corresponding to the modification \( \xi \mapsto \xi' \) of every element in the original sets. Observe that \( \Upsilon_{K,N} \subset \Upsilon_{K,N} \) and likewise, \( \Xi_{K,N} \subset \Xi_{K,N} \).

**Lemma 5.6.** Let \( \alpha \in (0,1) \) and consider the systematic block dynamics \( \{Y_k\}_{k \in \mathbb{N}} \) on \( \Lambda_{n,n'} \) with \( \lfloor \alpha n \rfloor \leq n' \leq n \) and blocks given by Definition 5.5. There exist \( c_Y, c_*(\alpha, q) > 0 \) such that for every two initial configurations \( \omega_1, \omega_2 \), and every boundary condition \( \xi \in \partial \Lambda_{n,n'} \), modified to \( \xi' \) by Eq. (5.2), for all \( k \geq 2 \),
\[ \| \pi^c\xi'_{\omega_1}(Y_k \in \cdot) - \pi^c\xi'_{\omega_2}(Y_k \in \cdot) \|_{TV} \leq \exp(-c_Y kn^{-c_*}). \]

In particular, for all \( k \geq 2 \),
\[ \max_{\omega_0} \| \pi^c\xi'_{\omega_0}(Y_k \in \cdot) - \pi^c_{\Lambda_{n,n'}}(Y_k \in \cdot) \|_{TV} \leq \exp(-c_Y kn^{-c_*}). \]

**Proof.** We construct a coupling between the two systematic block dynamics chains, starting from two arbitrary initial configurations \( \omega_1, \omega_2 \), as follows. The systematic block dynamics first samples a configuration on \( B_E^o \) (the interior of \( B_E \)) according to
\[ \pi_{B^{\star}_E}^{\epsilon' \omega_i} \text{ for } i = 1, 2, \] where \( (\epsilon', \omega_i) \) is the boundary condition induced by \( \omega_i \mid_{B^{\star}_E - B^{\star}_W} \cup \xi' \) on \( \partial B^{\star}_E \). By Proposition 3.1, applied to the box

\[ R^{\star} = B^{\star}_W \cap B^{\star}_E, \]

and monotonicity in boundary conditions,

\[ \pi_{B^{\star}_E}^{\epsilon^{\star}_S \rightarrow \epsilon^{\star}_N} \succ n^{-c^{\star}_s}, \]

where \( c^\star(\min\{\frac{1}{2}, \alpha\}, \varepsilon = \frac{1}{4}, q) > 0 \) is given by that proposition.

We can condition on the west-most vertical dual-crossing between \( \epsilon^{\star}_S \) and \( \epsilon^{\star}_N \) (if such a dual-crossing exists) as follows: reveal the open components of \( \partial B^{\star}_E \cap [0, \lfloor \frac{n}{2} \rfloor] \times [0, n'] \) as in [17] or [11], so that no edges in other components are revealed. If the open components do not connect to the eastern half of \( \partial R^{\star} \), i.e., to \( \partial R^{\star} \cap [\lceil \frac{n}{2} \rceil + 1, n] \times [0, n'] \), then it must be the case that the desired open dual-crossing exists and can be exposed without revealing any information about edges east of it.

By monotonicity in boundary conditions, if under \( \pi_{B^{\star}_E}^{1} \) such a vertical dual-connection from \( \epsilon^{\star}_S \) to \( \epsilon^{\star}_N \) exists, the grand coupling (see §2.2) ensures that the same under \( \pi_{B^{\star}_E}^{\epsilon' \omega_i} \) for any \( \omega_i \mid_{\Lambda_{n,n'} - B^{\star}_W} \). By definition of the modification \( \epsilon' \), there are no bridges over \( \epsilon^{\star}_S \), no bridges over \( \epsilon^{\star}_N \), and no components of \( \epsilon' \) with vertices in multiple sides of \( \partial \Lambda_{n,n'} \); thus, conditional on this vertical dual-crossing, the following event holds:

\[ \bigcap \left\{ v \leftrightarrow^\epsilon' w : \begin{align*} v &\in \partial \Lambda_{n,n'} \cap [0, \frac{n}{2}] \times [0, n'] \\ w &\in \partial \Lambda_{n,n'} \cap [\frac{n}{2}, n] \times [0, n'] \end{align*} \right\}. \]
By the domain Markov property (see Fig. 7), for any pair \( \omega_1 \upharpoonright_{B_w - B_w^o} \) and \( \omega_2 \upharpoonright_{B_w - B_w^o} \),
\[
\pi_{B_w}^{\xi', \omega_1} \left( \omega \mid [\frac{3n}{4}, n] \times [0, n'] \right) \big| e_s^* \xleftarrow{R^*} e_N^* \overset{d}{=} \pi_{B_w}^{\xi', \omega_2} \left( \omega \mid [\frac{3n}{4}, n] \times [0, n'] \right) \big| e_s^* \xleftarrow{R^*} e_N^*,
\]
using that the boundary conditions to the east of the vertical dual-crossing are the same under both measures. (In the presence of bridges over \( e_s^* \) or \( e_N^* \) the above distributional equality does not hold; different configurations west of such a dual-crossing could still induce different boundary conditions east of the dual-crossing, preventing coupling (as illustrated in Fig. 3)—cf. the case of integer \( q \) where this problem does not arise.)

This implies that, on the event \( e_s^* \xleftarrow{R^*} e_N^* \), the grand coupling couples the two systematic block dynamics chains so that they agree on \( \Lambda_{n,n'} - B_w^o \) with probability 1. In this case, let \( \eta \) be the resulting configuration on \( B_E - B_w^o \), so that
\[
\eta = Y_1 \upharpoonright_{B_E - B_w^o}.
\]
If the two chains were coupled on \( B_E - B_w^o \), the boundary conditions \((\xi', \eta)\) on \( \partial B_w \) would be the same for any pair of systematic block dynamics chains with initial configurations \( \omega_1, \omega_2 \); in particular the identity coupling would couple them on all of \( \Lambda_{n,n'} \) in the next step when \( B_w \) is resampled from \( \pi_{B_w}^{\xi', \eta} \). Thus, for some \( c > 0 \),
\[
\| \mathbb{P}_{\omega_1}^{\xi'} (Y_2 \in \cdot) - \mathbb{P}_{\omega_2}^{\xi'} (Y_2 \in \cdot) \|_{TV} \leq 1 - cn^{-c\ast}.
\]

Since the systematic block dynamics is Markovian and all of the above estimates were uniform in \( \omega_1 \) and \( \omega_2 \), the probability of not having coupled in time \( k \) under the grand coupling is bounded above by
\[
(1 - cn^{-c\ast})^{\lfloor k/2 \rfloor} \leq \exp(\alpha k_2 |\mathcal{R}^e| n^{-c\ast}).
\]

The next lemma will be key to obtaining the desired upper bound on \( \rho \) as defined in (4.4); it shows that with high probability, the boundary conditions induced by the FK measure on a segment will be in \( \Xi_{K,N} \), hence the term “typical” boundary conditions.

**Lemma 5.7.** Fix \( q \in (1, 4] \). There exists \( c_T(q) > 0 \) so that, for every \( \Xi_{K,N} \) given by Definition 5.1 on \( \Lambda_{n,n'} \) with \( n' \leq n \leq N \) and \( K > 0 \), and every boundary condition \( \xi \),
\[
\pi_{B_w}^{\xi} (\omega \mid \partial_k B_w \notin \Xi_{K,N}) \lesssim N^{-c_T K},
\]
where \( \omega \upharpoonright_{\partial_k B_w} \) denotes the boundary conditions induced on \( \partial_k B_w \) by \( \omega \upharpoonright_{B_E - B_w^o} \). The same statement holds when exchanging \( E \) and \( W \).

**Proof.** By symmetry, it suffices to prove the bound for the boundary conditions on \( \partial_k B_w \). Consider the rectangle
\[
R = [\frac{n}{2}, n] \times [0, n'].
\]
By Proposition 3.9 with aspect ratio \( \frac{1}{2} \), there exists \( c(q) = c(\alpha = \frac{1}{2}, q) > 0 \) such that, for every edge \( e \in \partial_k B_w \) and every boundary condition \( \eta \) on \( \partial R \),
\[
\pi^\eta_R (|\Gamma^e| \geq K \log N) \lesssim N^{-cK},
\]
where \( \Gamma^e \) is the set of edges adjacent to \( e \).
where, for a configuration $\omega_R$ on $R$, we recall that $|\Gamma^c|$ is the number of disjoint bridges in $\omega_R|_{R-B^0_e} \cup \xi_R$ over $e$. A union bound over all $n'$ edges on $\partial_e B_w$ implies that
\[
\max_{\eta} \pi^\eta_R(\omega|_{\partial_e B_w} \notin \Xi_{K,N}) \lesssim n'^{-cK} \lesssim N^{-cK+1},
\]
using $n \leq N$. Consequently,
\[
\pi_{\partial_e B_w}^\xi(\omega|_{\partial_e B_w} \notin \Xi_{K,N}) = \mathbb{E}_{\pi_{\partial_e B_w}^\xi} \left[ \pi_R^\xi(\omega|_{\partial_e B_w} \notin \Xi_{K,N}) \right] \lesssim N^{-cK+1},
\]
where the expectation is w.r.t. $\pi_{\partial_e B_w}^\xi$ over the boundary conditions $\xi_R$ induced on $R$ by $\xi$ and the configuration on $B_e - R^0$. This concludes the proof of the lemma. \hfill \blacksquare

**Corollary 5.8.** Fix $q \in (1,4]$, and consider the systematic block dynamics on $\Lambda_{n,n'}$ for $n' \leq n \leq N$ with block choices as given in Definition 5.5. There exists $c_T(q) > 0$ so that, for every fixed $K > 0$ and every boundary condition $\xi' \in \Upsilon'_{K,N}$ on $\partial \Lambda_{n,n'}$,
\[
\rho \lesssim N^{-c_T K},
\]
where $\rho$ is as defined as in (4.4) w.r.t. the initial configuration $\omega_0 \in \{0,1\}$ and the permissible boundary conditions $\Upsilon_{K,N}$.

**Proof.** Let $Y_k$ be the systematic block dynamics on $\Lambda_{n,n'}$ where $n \leq N$. Recall the definition of $\rho$ in Eq. (4.4), so that in the present setting,
\[
\rho = \max_{\omega_0 \in \{0,1\}} \max_{k \geq 1} \max_{i \in \{e,w\}} \mathbb{P}_{\omega_0}^\xi \left( Y_k|_{\partial_e B_i} \notin \Upsilon_{K,N} \right).
\]
In the first time step, $\omega_0|_{B_e}$ induces wired or free boundary conditions on $\partial_e B_e$ and so, by Remark 5.3, the boundary condition on $\partial_e B_e$ is trivially in $\Xi_{K,N}$. Furthermore, the boundary conditions on $\partial_{n,K,s} B_e$ also belong to $\Xi_{K,N}$ by the hypothesis $\xi' \in \Upsilon_{K,N}$. Finally, there cannot be more than $2K \log N$ components in the boundary condition on $\partial B_e$ consisting of vertices on multiple sides for the following reason: as a result of the side modification on $\xi'$, such components can only arise from connections between $\partial_n B_e$ and the bridges in $\Gamma^{(n/4,0)}$ and $\Gamma^{(n/4,n')}$; however, there are at most $K \log N$ bridges in each set under any configuration on $\Lambda - B^0_e$ (summing to at most $2K \log N$ components, as claimed). Altogether, $Y_1|_{\partial_e B_e} \in \Upsilon_{K,N}$ deterministically.

To address all subsequent time steps, by reflection symmetry and the definition of the systematic block dynamics, it suffices to consider $Y_2|_{\partial B_w}$. By Lemma 5.7, the probability that a boundary condition on $\partial_e B_e$ induced by the systematic dynamics will not be in $\Xi_{K,N}$ is $O(N^{-c_T K})$, with $c_T > 0$ from that lemma. The fact that, deterministically, the boundary conditions on $\partial_{n,s,w} B_w$ are in $\Xi_{K,N}$, and there are at most $2K \log N$ components of the boundary condition on $\partial B_w$ containing vertices of multiple sides of $\partial B_w$, follows by the same reasoning argued for the first time step. \hfill \blacksquare

We are now in a position to prove Theorem 5.4.

**Proof of Theorem 5.4.** Consider $\Lambda = \Lambda_{N,N'}$ with aspect ratio $\alpha \in (0,1]$ and boundary conditions $\xi \in \Upsilon_{K,N}$ for a fixed
\[
K \geq K_0 := 6(c_4 + 1) \max\{c_T^{-1},1\}, \tag{5.3}
\]
where \( c_\ast = c_\ast \left( \min \{ \tilde{\alpha}, \frac{1}{2} \} \right) \) is the constant given by Proposition 3.1, and \( c_T = c_T(q) \) is given by Corollary 5.8. It suffices to prove the proposition for all \( K \) sufficiently large, as \( \Upsilon_{K,N} \subset \Upsilon_{K',N} \) for every \( K \leq K' \).

We prove the following inductively in \( n \in [1, N] \): for every \( K > K_0 \) as above, every \( (\tilde{\alpha} \wedge \frac{1}{2}) n \leq n' \leq n \), and every \( \xi \in \Upsilon_{K,N} \), if

\[
t_n = N^{2(c_\ast + \lambda + 1) \log 4/n} \quad \text{where} \quad \lambda := 32K \log q + 5,
\]

then Glauber dynamics for the critical FK model on \( A_{n,n'} \) has

\[
\| \mathbb{P}^\xi_1 (X_{t_n} \in \cdot) - \mathbb{P}_0^\xi (X_{t_n} \in \cdot) \|_{TV} \leq N^{-3} . \tag{5.4}
\]

To see that Eq. (5.4) implies Theorem 5.4, note that (2.1), with the choice \( n = N \), implies that \( d_{TV}(N^{c(\tilde{\alpha}, q) \log N}) = O(1/N) = o(1) \) for some \( c(\tilde{\alpha}, q) > 0 \).

For the base case, fix a large constant \( M \), where clearly \( t_{\text{mix}} = O(1) \) for all \( n \leq M \). Next, let \( m \in [M, N] \), and assume (5.4) holds for all \( n \in [1, m-1] \). Consider the censored and systematic block dynamics, \( (\bar{X}_t)_{t \geq 0} \) and \( \{Y_k\}_{k \geq 0} \), respectively, on the blocks defined in Definition 5.5 on \( \Lambda_m = \Lambda_{m,m'} \) for some \( (\tilde{\alpha} \wedge \frac{1}{2}) m \leq m' \leq m \) and boundary conditions \( \xi \in \Upsilon_{K,N} \).

Recall that \( \xi \in \Upsilon_{K,N} \) has at most \( K \log N \) bridges over any edge and at most \( 2K \log N \) components spanning multiple sides of \( \partial \Lambda_m \); thus, by Fact 4.5, the boundary modification \( \xi' \) defined in (5.2) satisfies \( d(\xi', \xi) \leq 8K \log N \). By the definition of \( \lambda \), we have \( |E|^2 q^{ld(\xi', \xi)} = o(N^\lambda) \). Hence, by Lemma 4.6 (Eq. (4.2), where we increased the time on the right-hand to \( N^\lambda \), for large enough \( N \), by the monotonicity of \( d_{TV} \) and the above bound on \( d(\xi', \xi) \), we have that for all \( k,T \geq 0 \),

\[
\| \mathbb{P}^\xi_1 (X_{N^\lambda kT} \in \cdot) - \mathbb{P}_0^\xi (X_{N^\lambda kT} \in \cdot) \|_{TV} \leq 2 \max_{\omega_0 \in \{0,1\}} \| \mathbb{P}^\xi_{\omega_0} (X_{N^\lambda kT} \in \cdot) - \pi^\xi_{\Lambda_m} \|_{TV}
\]

\[
\leq 16 \max_{\omega_0 \in \{0,1\}} \| \mathbb{P}^{\xi'}_{\omega_0} (X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m} \|_{TV} + 2e^{-N^{\lambda/4} \lambda},
\]

and subsequently, by Theorem 2.5,

\[
\| \mathbb{P}^\xi_1 (X_{N^\lambda kT} \in \cdot) - \mathbb{P}_0^\xi (X_{N^\lambda kT} \in \cdot) \|_{TV} \leq 16 \max_{\omega_0 \in \{0,1\}} \| \mathbb{P}^{\xi'}_{\omega_0} (X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m} \|_{TV} + 2e^{-N^{\lambda/4} \lambda} . \tag{5.5}
\]

We will next show that the first term in the right-hand above satisfies

\[
\max_{\omega_0 \in \{0,1\}} \| \mathbb{P}^{\xi'}_{\omega_0} (X_{kT} \in \cdot) - \pi^{\xi'}_{\Lambda_m} \|_{TV} = o(N^{-3}) , \tag{5.6}
\]

which will imply (5.4) if we choose \( k,T \) such that \( N^\lambda kT \leq t_m \). By triangle inequality,
where the last inequality is valid for every \( k \geq 2 \) by Lemma 5.6. Using \( \Upsilon_{K,N}^i \subset \Upsilon_{K,N} \) and Proposition 4.10,
\[
\max_{\omega_0 \in \{0,1\}} \| \mathbb{P}^e_{\omega_0}(X_{kT} \in \cdot) - \pi_{\Lambda_m}^e \|_{TV} \leq k(\rho + \varepsilon(T)) + e^{-c_N^2k - \varepsilon},
\]
and so, combined with (5.5),
\[
\| \mathbb{P}_i^e(X_{N^{\lambda K} T} \in \cdot) - \mathbb{P}_0^e(X_{N^{\lambda K} T} \in \cdot) \|_{TV} \leq 16k(\rho + \varepsilon(T)) + 16e^{-c_N^2k - \varepsilon} + 2e^{-N^{\lambda/4}}, \tag{5.7}
\]
where \( \rho \) and \( \varepsilon \) were given in (4.3)-(4.4), that is, in our context,
\[
\varepsilon(T) = \max_{\omega' \in \Omega} \max_{i \in \{e,w\}} \max_{\zeta \in \Upsilon_{K,N}^i} \| \mathbb{P}^e_{\omega'}(X_T \in \cdot) - \pi_{\Upsilon_{K,N}}^e \|_{TV},
\]
\[
\rho = \max_{k \geq 1} \max_{i \in \{e,w\}} \mathbb{P}_{\omega_0} \left( Y_{k \mid \partial B_i} \notin \Upsilon_{K,N}^i \right).
\]
We will bound \( \varepsilon(T) \) by the inductive assumption for the choice of
\[
T := k t_{[3m/4]} N^\lambda K \log N, \quad \text{where} \quad k := c_N^{-1}(c_* + 6) N^\lambda \log N. \tag{5.8}
\]
In order to apply the induction hypothesis for a box whose side lengths are smaller by a constant factor vs. the original dimensions of \( m \times m' \), we repeat the above analysis for the sub-block \( B_i \) (whose dimensions are \( \lfloor \frac{3}{4} m \rfloor \times \lfloor \frac{3}{4} m' \rfloor \)), and get from Fact 2.4 and the above arguments that
\[
\varepsilon(T) \lesssim N^2 \max_{i \in \{e,w\}} \max_{\zeta \in \Upsilon_{K,N}^i} \| \mathbb{P}_0^e(B_i) - \mathbb{P}_1^e(B_i) \|_{TV},
\]
which by reapplying (5.7) at the lower scale of the \( B_i \)'s implies that
\[
\varepsilon(T) \lesssim N^2 k \left( \rho' + \varepsilon' \left( \frac{T_{ENX}}{kN^2} \right) \right) + N^2 e^{-c_N^2k - \varepsilon} + N^2 e^{-N^{\lambda/4}},
\]
where \( \varepsilon' \) and \( \rho' \) are the counterparts of \( \varepsilon(T) \) and \( \rho \) w.r.t. the sub-blocks (as per Definition 5.5) of \( B_i \) rotated by \( \pi/2 \). (N.b. this rotation is crucial to ensuring that the aspect ratios of the rectangles we consider remain uniformly bounded as we recurse down in scale, and consequently the coupling probabilities satisfy the same lower bound; this rotation is also what forces us to maintain “typical” boundary conditions on all four sides of the rectangles we are considering as opposed to, say, just on \( \partial_{e,w} \Lambda \).)

This yields the following new bound on (5.6):
\[
\max_{\omega_0 \in \{0,1\}} \| \mathbb{P}_{\omega_0}^e(X_{kT} \in \cdot) - \pi_{\Lambda_m}^e \|_{TV} \lesssim N^2 k^2 \left( \rho + \rho' + \varepsilon' \left( \frac{T_{ENX}}{kN^2} \right) \right) + kN^2 e^{-c_N^2k - \varepsilon} + o(N^{-3}).
\]
Note that the dimensions of the sub-blocks of \( B_i \) (those under consideration in \( \varepsilon'(T) \)) are \( \lfloor \frac{3}{4} m \rfloor \times \lfloor \frac{3}{4} m' \rfloor \). Hence, by the inductive assumption at scale \( \lfloor \frac{3}{4} m \rfloor \) and Fact 2.4,
\[
\varepsilon' \left( t_{[3m/4]} \right) = O(1/N),
\]
which, along with (2.1) and the sub-multiplicativity of \( d_{TV}(t) \), yields that for \( T \) from (5.8),
\[
\varepsilon' \left( \frac{T}{K N^2} \right) = \varepsilon' \left( t_{[3m/4]} K \log N \right) \lesssim N^{-K} \leq N^{-6(c_* + 1)}.
\]
By Corollary 5.8, we have \( \rho \lesssim N^{-c_N^2K} \leq N^{-6(c_* + 1)} \) by our choice of \( K_0 \), and similarly for \( \rho' \). So, for \( k = N^{c_*+1} \) as in (5.8), \( k^2 \rho \lesssim N^{-4(c_*+6)+o(1)} = o(N^{-5}) \), and similarly, \( k^2 \rho' = o(N^{-5}) \). Finally, this choice of \( k \) guarantees that \( kN^2 \exp(-c_N^2k - \varepsilon) \) is at most.
$kN^{-c_{\ast}^{-4}} = o(N^{-3})$. Combining the last three displays with these bounds yields (5.6). The proof is concluded by noting that indeed $N^\lambda kT \leq N^{2c_{\ast} + 2\lambda + o(1)} t_{m/4} \leq t_m$. ■

5.3. Mixing on the torus. Here we extend Theorem 5.4 to the $n \times n$ torus, proving Theorem 1. Observe that the periodic FK boundary conditions identified with $(\mathbb{Z}/n\mathbb{Z})^2$ in fact have order $n$ components with vertices on multiple sides of $\partial \Lambda$. We thus have to extend the bound of Theorem 5.4 to periodic boundary conditions using the topological structure of $(\mathbb{Z}/n\mathbb{Z})^2$. The proof draws from the extension of mixing time bounds in [17] and [11] from fixed boundary conditions to $(\mathbb{Z}/n\mathbb{Z})^2$. In the present setting, having to deal with a specific class of boundary conditions forces us to reapply the bridge modification and the censored and systematic block dynamics techniques.

We first bound the mixing time on a cylinder with typical boundary conditions on its non-periodic sides. In what follows, for any $\Lambda_{n,n'}$, label the following edges:

- $e^s_{sw} = (0, \lceil \frac{n'}{2} \rceil + \frac{1}{2})$,
- $e^s_{sk} = (n, \lceil \frac{n'}{2} \rceil + \frac{1}{2})$,
- $e^s_{nw} = (0, \lceil \frac{9n'}{10} \rceil + \frac{1}{2})$,
- $e^s_{ne} = (n, \lceil \frac{9n'}{10} \rceil + \frac{1}{2})$.

Then define the modification $\xi'$ of boundary conditions $\xi$ by

$$\xi' = \xi^{e_{sw}} \wedge \xi^{e_{sk}} \wedge \xi^{e_{nw}} \wedge \xi^{e_{ne}} \wedge \xi^s$$

(5.9)

and define $\Xi'_{K,N}, \Upsilon'_{K,N}$ as before, for the new modification. We say that a boundary condition on $\partial_{n,s} \Lambda$ is in $\Upsilon'_{K,N}$ if its restriction to each side is in $\Xi'_{K,N}$ and there are fewer than $2K \log N$ distinct components with vertices in $\partial_n \Lambda$ and $\partial_s \Lambda$, and analogously for boundary conditions on $\partial_{e,w} \Lambda$.

**Theorem 5.9** (Mixing time on a cylinder). Fix $q \in (1,4]$, $\alpha \in (0,1]$ and $K > 0$. There exists some $c_\alpha(q,K) > 0$ such that the critical FK model on $\Lambda = \Lambda_{N,N'}$ with $\alpha N \leq N' \leq \alpha^{-1}N$ and boundary conditions, denoted by $(p, \xi)$, that are periodic on $\partial_{n,s} \Lambda$ and $\xi \in \Upsilon'_{K,N}$ on $\partial_{e,w} \Lambda$, satisfies $t_{mix} \lesssim N^{c \log N}$.

**Proof.** We will use a similar approach as in the proof of Theorem 5.4 to reduce the cylinder to rectangles with “typical” boundary conditions. It suffices to prove the theorem for large $K$, since $\Upsilon_{K,N} \subset \Upsilon'_{K',N}$ for $K \leq K'$. We establish it for every fixed $K \geq K_0 + K_0'$ where $K_0 = 4(c_{\ast} + 1)(c_{\ast}^{-1} \vee 1)$ and $K_0' = K_0(c_{\ast}^{-1} \vee 1)$, in which $c_{\ast} = c_{\ast}(\frac{\alpha}{q}, \frac{1}{4}, q) > 0$ is given by Proposition 3.1, the constant $c_T$ is $c(\frac{2\alpha}{q}, q) > 0$ from Proposition 3.9, and $c_\Psi = c_\Psi(\frac{2\alpha}{q}, q) > 0$ is given by Proposition 3.14.

Define, as in Definition 4.9, the censored and systematic block dynamics on

$$B_0 := \left[0, N\right] \times \left[0, \frac{N'}{2}\right] \cup \left[0, N\right] \times \left[\frac{2N'}{3}, N'\right],$$

$$B_1 := \left[0, N\right] \times \left[0, \frac{2N'}{3}\right] \cup \left[0, N\right] \times \left[\frac{4N'}{5}, N'\right].$$

The choice of boundary class on $B_i$ for $i = 0, 1$ is $\Gamma_i = \Upsilon_{3K,N}$. Observe that by translating vertically on the universal cover, the blocks $B_0$ and $B_1$ are, by construction, $N \times \frac{4}{5}N'$ rectangles with non-periodic boundary conditions. These blocks and the coupling scheme are depicted in Figure 8.
It again suffices, by Fact 2.4, to show that there exists some \( c(\alpha, q, K) > 0 \) such that
\[
\|P_{p, \xi}^1(X_{N_{\log N}} \in \cdot) - P_{0, \xi}^0(X_{N_{\log N}} \in \cdot)\|_{TV} \leq N^{-3}.
\] (5.10)

In the setting of the cylinder, the side modification \((p, \xi)\) of \((p, \xi)\) only disconnects \(\partial_{w} \Lambda\) from \(\partial_{e} \Lambda\), and so, if \(\xi'\) is as in (5.9), then \(d(\xi', \xi) \leq 6K \log N\). Thus, by (4.2), the triangle inequality and Theorem 2.5 (as explained in the derivation of (5.5)), if
\[
t_\text{N} = N^\lambda k T \quad \text{for} \quad \lambda = 24K + 5
\]
(so that \(|\mathcal{E}|^4_k = o(N^\lambda))\), then for every \(k, T \geq 0\),
\[
\|P_{p, \xi}^1(X_{t_N} \in \cdot) - P_{0, \xi}^0(X_{t_N} \in \cdot)\|_{TV} \leq 16 \max_{\omega_0 \in \{0, 1\}} \|P_{\omega_0}^{p, \xi'}(X_{kT} \in \cdot) - \pi_{\Lambda}^{p, \xi'}\|_{TV} + 2e^{-N^{\lambda/4}},
\]
which, by Proposition 4.10, is at most
\[
16 \max_{\omega_0 \in \{0, 1\}} \|P_{\omega_0}^{p, \xi'}(Y_k \in \cdot) - \pi_{\Lambda}^{p, \xi'}\|_{TV} + 16k(\rho + \varepsilon(T)) + 2e^{-N^{\lambda/4}},
\] (5.11)
where \(\varepsilon(T)\) and \(\rho\) are given by (4.3) and (4.4), respectively, w.r.t. the blocks \(B_0, B_1\), the permissible boundary conditions \(\Upsilon_{3K, N}\), and the initial configuration \(\omega_0 \in \{0, 1\}^2\):
\[
\varepsilon(T) = \max_{\omega_0} \max_{i \in \{0, 1\}} \max_{\xi \in \Upsilon_{3K, N}} \|P_{\omega_0}^{\xi; B_i}(X_T \in \cdot) - \pi_{\Lambda}^{\xi}\|_{TV},
\]
\[
\rho = \max_{k \geq 1} \max_{i \in \{0, 1\}} P_{\omega_0}(Y_k |_{\partial B_i} \notin \Upsilon_{3K, N}).
\]

We next bound the first term in the right-hand side of Eq. (5.11) by the probability of not coupling the systematic block dynamics chains started from two arbitrary initial configurations under the grand coupling (cf. Lemma 5.6). In the first time step, we try to couple the chains started from \(\omega_1, \omega_2\) on
\[
R := [0, N] \times \left[\frac{3N'}{5}, \frac{4N'}{5}\right],
\]
so that in the second step the identity coupling couples them on all of \( \Lambda \). It suffices to couple the systematic chains started from \( \omega_1 = 0 \) and \( \omega_2 = 1 \) under the grand coupling. In order to couple samples from the \((0, \xi')\) and \((1, \xi')\) boundary conditions on \( R \) (induced by \( \omega_1 = 0 \) and \( \omega_2 = 1 \) resp.), define the following two sub-blocks of \( B_0 \):

\[
R_s := [0, N] \times \left[ \frac{2N'}{5}, \frac{3N'}{5} \right], \quad R_n := [0, N] \times \left[ \frac{4N'}{5}, N' \right].
\]

By Proposition 3.1, monotonicity in boundary conditions, and the FKG inequality,

\[
\min_{\eta} \pi^{\eta, \xi'}_{B_0} \left( e^*_{SW} \leftrightarrow e^*_{SE}, e^*_{NW} \leftrightarrow e^*_{NE} \right) \geq N^{-2c}. \]

By the Domain Markov property, and the definition of the boundary modification \( \xi' \),

\[
\pi^{1, \xi'}_{B_0} \left( \omega \upharpoonright_R | e^*_{SW} \leftrightarrow e^*_{SE}, e^*_{NW} \leftrightarrow e^*_{NE} \right) = \pi^{0, \xi'}_{B_0} \left( \omega \upharpoonright_R | e^*_{SW} \leftrightarrow e^*_{SE}, e^*_{NW} \leftrightarrow e^*_{NE} \right). \]

As before, using the grand coupling and revealing edges from \( \partial_n R_s \) and \( \partial_s R_n \) until we reveal a pair of such horizontal dual-crossings, by monotonicity, we can couple \( \pi^{\omega_1, \xi'}_{B_0} \) and \( \pi^{\omega_2, \xi'}_{B_0} \) on \( R \) with probability at least \( cN^{-2c} \). On that event, the two chains are coupled in the next step (and thereafter) on all of \( \Lambda \) with probability 1. By the definition of the systematic block dynamics, we conclude that, for some \( c_Y > 0 \) and every \( k \geq 2 \),

\[
\max_{\omega_0 \in \{0, 1\}} \| \pi^{p, \xi'}_{\omega_0} (Y_k \in \cdot) - \pi^{p, \xi'}_\Lambda (\cdot) \|_{TV} \leq \exp(-c_Y k N^{-2c}).
\]

To bound \( \rho \), first note that, for \( \omega_0 \in \{0, 1\} \), the block \( B_0 \) has boundary conditions \((0, \xi')\) or \((1, \xi')\), both of which are in \( \Upsilon_{3K, N} \) by Remark 5.3. Thereafter, the uniformity of Proposition 3.9 in boundary conditions implies that for every \( \eta \),

\[
\pi^{\eta, \xi'}_{B_0} (\omega \upharpoonright_{\partial B_1} \notin \Xi_{K, N}) \leq N^{-c_T K},
\]

and likewise when exchanging \( B_0 \) and \( B_1 \). We need to also bound the number of boundary components intersecting distinct sides of \( B_0 \) or \( B_1 \). We can bound the connections between \( \partial_{n, s} B_i \) and \( \partial_{e, w} B_i \) (for \( i = 0, 1 \)) deterministically by \( 4K \log N \) as in the proof of Corollary 5.8. In the present setting there could also be (multiple) open components connecting \( \partial_{s} B_i \) to \( \partial_{s} B_i \) in \( \Lambda - B_i \). By Proposition 3.14 and monotonicity in boundary conditions, for every \( \eta \),

\[
\pi^{\eta, \xi'}_{B_0} (| \Psi_{\Lambda - B_i} | \geq K \log N) \leq N^{-c_\ell K},
\]

where, as in that proposition, \( | \Psi_{\Lambda - B_i} | \) is the number of distinct vertical crossings of \( \Lambda - B_i \). By the choices of \( K_0 \) and \( K'_0 \), a union bound yields

\[
\rho \leq \max_{\eta} \pi^{\eta, \xi'}_{B_0} (\omega \upharpoonright_{\partial B_1} \notin \Upsilon_{3K, N}) \leq N^{-4c, -4}.
\]

Observe that on their respective translates, \( B_0 \) and \( B_1 \) are \( N \times \frac{3}{5} N' \) rectangles, so we can bound \( \max_{\xi} \max_{\xi' \in \Upsilon_{3K, N}} \pi^{\xi, B_i}_{\text{mix}} \) using Theorem 5.4; by that theorem, rotational symmetry, and the sub-multiplicativity of \( d_{TV} \), we have that for some \( c_B(\alpha, q, K) > 0 \),

\[
\varepsilon(T) \lesssim \exp(-c_B^{-1} T N^{-c_B \log N}).
\]
uniformly over \( \alpha N \leq N' \leq \alpha^{-1} N \). Altogether, combining the bounds on \( \rho, \varepsilon, \) and the systematic block dynamics distance from stationarity, in Eq. (5.11), we see that for

\[
k = N^{2c_s+1} \quad \text{and} \quad T = N^{(c_B+1) \log N}
\]

one has

\[
\| \mathbb{P}_1^{p,\xi}(X_{tN} \in \cdot) - \mathbb{P}_0^{p,\xi}(X_{tN} \in \cdot) \|_{TV} = o(N^{-3}),
\]

implying Eq. (5.10) and concluding the proof.

**Proof of Theorem 1.** The theorem is obtained by reducing the mixing time on the torus to that on a cylinder and then applying Theorem 5.9. Fix \( \tilde{\alpha} \in (0, 1] \) and consider \( \Lambda = \Lambda_{n,n'} \) with \( \tilde{\alpha} n \leq n' \leq \tilde{\alpha}^{-1} n \) and periodic boundary conditions, denoted by \( (p) \), identified with \( (\mathbb{Z}/n\mathbb{Z}) \times (\mathbb{Z}/n'\mathbb{Z}) \) (the special case \( n' = n \) is formulated in Theorem 1).

Let \( c_s = c_s(\tilde{\alpha}, \frac{1}{2}, q) > 0 \) be given by Proposition 3.1 and let \( c_T, c_\Psi, K_0 \) and \( K_0' \) be given as in the proof of Theorem 5.9. Define \( K = K_0 + K_0' \). We consider the censored and systematic block dynamics with the block choices,

\[
B_0 := [0, n] \times [0, n'] \cup [0, n] \times \left[ \frac{3n'}{5}, n' \right] \quad B_1 := [0, n] \times \left[ 0, \frac{3n'}{5} \right] \cup [0, n] \times \left[ \frac{4n'}{5}, n' \right],
\]

and boundary class

\[
\Upsilon_{3K,n} := \left\{ \xi : \xi|_{\partial_s,A} = p, \xi|_{\partial_c,w,A} \in \Upsilon_{3K,n} \right\}.
\]

By Theorem 2.5, the triangle inequality and Proposition 4.10, for every \( k, T \geq 0, \)

\[
\| \mathbb{P}_1^{p}(X_{kT} \in \cdot) - \mathbb{P}_0^{p}(X_{kT} \in \cdot) \|_{TV} \leq 2 \max_{\omega_0 \in \{0,1\}} \| \mathbb{P}_{\omega_0}^{p}(Y_k \in \cdot) - \pi_\Lambda^{p}\|_{TV} + 2k(\rho + \varepsilon(T))\),
\]

where \( \rho \) and \( \varepsilon(T) \) are w.r.t. the class \( \Upsilon_{3K,n} \) of permissible boundary conditions. It suffices, as before, to prove that the right-hand side is \( o(n^{-3}) \) and then use (2.1) and the sub-multiplicativity of \( d_{TV}(t) \) to obtain the desired result.

Recall the edges \( e_{SW}^*, e_{NW}^*, e_{SE}^* \) and \( e_{NE}^* \) on \( \Lambda_{n,n'} \). As in the proof of Theorem 5.9, if

\[
R_s := [0, n] \times \left[ \frac{2n'}{5}, \frac{3n'}{5} \right], \quad R_N := [0, n] \times \left[ \frac{4n'}{5}, n' \right],
\]

then by Proposition 3.1 and the FKG inequality, we have

\[
\pi_{B_0}^{1,p} \left( e_{SW}^* \xrightarrow{R_s^*} e_{SE}^*, e_{NW}^* \xrightarrow{R_N^*} e_{NE}^* \right) \geq n^{-2c_s}.
\]

Crucially, while no boundary modification was done in this case, the periodic sides of \( B_0 \) have no bridges over the four designated edges, and the two horizontal dual-crossings, from the event above, disconnect its non-periodic sides \( \partial_s B_0 \) and \( \partial_c B_0 \) from \( \partial_s B_1 \) and \( \partial_c B_1 \). Therefore, if that event occurs for the systematic block dynamics chain started from \( \omega_0 = 1 \), the grand coupling carries it to the chains started from all other initial states, and yields a coupling of all these chains on \( \left[ \frac{3n'}{5}, \frac{4n'}{5} \right] \times [0, n'] \supset \partial B_1 \). By definition of the systematic block dynamics and sub-multiplicativity of \( d_{TV}(t) \), for \( k \geq 2, \)

\[
\max_{\omega_0 \in \{0,1\}} \| \mathbb{P}_{\omega_0}^{p}(Y_k \in \cdot) - \pi_\Lambda^{p}\|_{TV} \leq \exp(-c_\Psi k n^{-2c_s}). \quad (5.12)
\]
Observe that at every time step of the systematic block dynamics, the block $B_i (i = 0, 1)$ is an $n \times \frac{2n'}{5}$ rectangle with periodic boundary conditions on $\partial_{e,W} B_i$ and boundary conditions $\eta$ induced by the chain on $\partial_{n,S} B_i$. By Theorem 5.9, for some $c(\bar{\alpha}, q, K) > 0$,

$$\max_i \max_{(p, \eta) \in \mathcal{Y}_{p, 3K, n}^i} \tau_{\text{mix}}^{(p, \eta), B_i} \lesssim n^{c \log n},$$

and by sub-multiplicativity of $\bar{d}_{TV}(t)$, we have $\varepsilon(T) \lesssim \exp(-c^{-1}Tn^{c \log n})$. As in the proof of Theorem 5.9, since the estimate on $\rho$ was uniform in the boundary conditions, we again have $\rho \lesssim n^{-4c-4}$ (using Propositions 3.9 and 3.14). Combining the bounds on $\rho$ and $\varepsilon$ with (5.12), there exists some $c(\bar{\alpha}, q, K) > 0$ such that

$$\|P^p(X_{n^{c \log n}} \in \cdot) - P^0(X_{n^{c \log n}} \in \cdot)\|_{TV} = o(n^{-3}),$$

as required. ■

Acknowledgment. The authors thank the anonymous referees for their helpful suggestions. R.G. was supported in part by NSF grant DMS-1507019. E.L. was supported in part by NSF grant DMS-1513403.

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