A Potential Definition of Weak ω-Category

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Abstract
We give an alternative definition (weak folded category) of a weak infinite dimensional category, in an unbiased fashion, using one one dimensional quiver with composition and extra structure.

1 Weak folded categories

By a quiver we mean a diagram $M \xleftarrow{s} P \xrightarrow{t} P$ of sets $M, P$ and functions $s, t$. Elements of $M$ are referred as arrows, and are denoted as $f : A \to B$ expressing that $f \in M, s(f) = A$ and $t(f) = B$.

Let $n$ be a natural number. A path $A \rightsquigarrow B$ of length $n$ is a sequence $⟨ A, f_1, f_2, \ldots, f_n ⟩$ where $P \ni A = s(f_1)$ and $t(f_i) = s(f_{i+1})$ for all $1 \leq i < n$, and $t(f_n) = q$. Paths of length 0 are identified with vertices, and paths of length 1 are identified with the arrows.

Definition 1.1. A pre-(folded category) $C$ is a quiver $\text{Mor}_C \xrightarrow{\text{dom}} \text{Ob}_C$ equipped with an operation called ‘composition’ which associates an arrow $p \to q$ to each path $p \rightsquigarrow q$, and with an injective switchback function $J : \text{Mor}_C \to \text{Ob}_C$.

For convenience, it is also assumed that the composition of a path $⟨ A, f ⟩$ of length 1 is just $f$. We denote the composition of the empty path on object $A$ as $1_A$. Later, (the unbiased version of) associativity will imply that $1_A$ indeed behaves like an identity, at least up to equivalence. The composition of a longer path $⟨ p, f_1, \ldots, f_n ⟩$ is simply written by juxtaposition as $f_1 \cdot f_2 \cdot \ldots \cdot f_n$ or, several cases with dots: $f_1 \cdot f_2 \cdot \ldots \cdot f_n$.

We note that in this brief note, composition of arrows is intended from left to right, though function application is written on the left of the argument.

We write $C^-\cdot$ for the ‘arrow category’ of $C$, i.e. which has $\text{Ob}C^-\cdot := \text{Mor}C$ and $C^-\cdot(f, g) := \{ \alpha \in \text{Mor}C | \alpha : Jf \to Jg \}$.

(Identically, we could define it as the full subcategory of $C$ on the range of $J$.) This $C^-\cdot$ is again a pre-(folded category), with $J\alpha := \alpha$ for $\alpha : f \to g$.

Therefore, in notation we may identify the arrow $f$ with its switched-back object $Jf$, writing e.g. $\alpha : f \to g$ for $\alpha : Jf \to Jg$, especially when dealing with $C^-\cdot$.

A functor between pre-(folded categories) $\mathcal{A} \to \mathcal{B}$ is a quiver morphism which strictly preserves the composition and the switchback function in the obvious way.
For the notion of equivalence, we need a whole binary tree of pairs of morphisms which are ‘inverses’ to each other. Let \( \{0,1\}^* \) denote the set of all finite nonempty sequences of \( \{0,1\} \). Concatenation of such sequences is written by juxtaposition, so that if \( \{0,1\}^* \ni t = (t_1, t_2, \ldots, t_n) \), then e.g. \( 01 \) denotes the sequence \( (t_1, \ldots, t_n, 0, 1) \).

**Definition 1.2.** An equivalence between objects \( A, B \) is defined to be a sequence \( (f_t)_{t \in \{0,1\}^*} \) of arrows such that, letting \( X_0 := A \) and \( X_1 := B \), for any sequence \( t \) of bits, we require

\[
f_{t0} : X_{t0} \to X_{t1} \quad \text{and} \quad f_{t1} : X_{t1} \to X_{t0},
\]

where \( X_{t0} := J(f_{t0} f_{t1}) \), \( X_{t1} := J(1_{X_{t0}}) \), \( X_{110} := J(f_{t1} f_{t0}) \), and \( X_{111} := J(1_{X_{t1}}) \).

If such sequence exists, we write \( A \simeq B \), and we call \( f_0 \) an equivalence arrow with inverse \( f_1 \) (with respect to the equivalence \( (f_t)_{t \in \{0,1\}^*} \)).

Due to the definition, we have that \( A \simeq B \) if and only if there are arrows \( f : A \to B \) and \( g : B \to A \) such that \( J(fg) \simeq J(1_A) \) and \( J(gf) \simeq J(1_B) \).

It also directly follows that functors preserve equivalences.

**Definition 1.3.** A globular pre-(folded category) is a pre-(folded category) \( C \) which has only ‘globular cells’, in the sense that whenever there exists any \( \alpha : Jf \to Jg \), it implies that \( \text{dom } f = \text{dom } g \) and \( \text{cod } f = \text{cod } g \).

Due to globularity, we can introduce the following partial functions: \( \text{dom}^n \) and \( \text{cod}^n \) by setting

\[
\begin{align*}
\text{dom}^{n+1}(\alpha) & := \text{dom}(J^{-1}(\text{dom}^n(\alpha))) \\
\text{cod}^{n+1}(\alpha) & := \text{cod}(J^{-1}(\text{cod}^n(\alpha))).
\end{align*}
\]

Furthermore, let \( C^n \) denote the smallest full substructure of the product of \( n \) copies of \( C \), which contains all paths \( (f_1, f_2, \ldots, f_n) \) of \( C \) as objects, for \( n \geq 1 \). As it is supposed to be closed under the switchback function \( J \), we have that

\[
C^n \supseteq \{ (\alpha_1, \alpha_2, \ldots, \alpha_n) \mid \exists k : (\text{cod}^k(\alpha_i) = \text{dom}^k(\alpha_{i+1}) \text{ for all } i < n) \}.
\]

Consequently, e.g. for \( n = 2 \), if \( (\alpha_0) : f_0 \simeq f_1 \) and \( (\beta_0) : g_0 \simeq g_1 \) with \( \text{cod } f_1 = \text{cod } f_0 = \text{dom } g_0 = \text{dom } g_1 \), then in \( C^2 \) we have that \( (f_0, g_0) \simeq (f_1, g_1) \), as all pairs of arrows \( (\alpha_0, \beta_0) \) and their switched-back objects are present in \( C^2 \).

In the globular setting, we extend the series of \( C^n \) by introducing \( C^0 \) to be the smallest substructure of \( C \) that contains all points of \( C \), and hence all identities \( 1_A \), all possible compositions of these, but basically nothing else. (Note that \( C^0 \) is not required to be a full substructure.)
Definition 1.4. A (globular) weak folded category is a globular pre-(folded category) \( \mathcal{C} \) which has the following additional properties and structures:

- A functor \( \mu_n : \mathcal{C}^{[n]} \to \mathcal{C}^\sim \) for each \( n \in \mathbb{N} \) which altogether play the role of ‘horizontal composition’, such that it extends the originally given composition, i.e. it extends the object map \( \langle f_1, f_2, \ldots, f_n \rangle \mapsto f_1 f_2 \cdots f_n \).

- We require that \( \mu_1 \) be the identity map, and \( \mu_0(1_A) = 1_{f_1 A} \).

- **Weak associativity with natural coherence equivalences:**

  For each path \( f = \langle A_0, f_1, f_2, \ldots, f_n \rangle \) on objects \( A_0, A_1, \ldots, A_n \) and for each pair of indices \( i, j \) such that \( 1 \leq i + 1 \leq j \leq n \) we require a fixed equivalence \( \theta_{f,i,j} : J(f_1 f_2 \cdots f_n) \simeq J(f_1 \cdots (f_i \cdots (f_j \cdots (f_n)) \cdots)), \) called coherence equivalences which altogether satisfy the conditions below.

  (The case \( i + 1 = j \) corresponds to placing the ‘empty parenthesis’ at \( A_j = \text{dom} f_j \), which is interpreted as the composition \( f_1 \cdots f_i A_j f_j \cdots f_n \).)

  \( \mu \) is composition, i.e. it extends the object map \( \langle i \rangle \mapsto i \).

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  \( a_1 \) For each path \( f = \langle A_0, f_1, f_2, \ldots, f_n \rangle \) on objects \( A_0, A_1, \ldots, A_n \) and for each pair of indices \( i, j \) such that \( 1 \leq i + 1 \leq j \leq n \) we have the following commutativity (up to equivalence).

  \[
  f_1 \cdots f_i f_j \cdots f_k f_l \cdots f_n \longrightarrow f_1 \cdots (f_i \cdots f_j) \cdots f_k f_l \cdots f_n
  \]

  \( a_2 \) For any pair of indices \( (i, j) \), \( (k, l) \) such that \( 1 \leq i + 1 \leq k + 1 \leq l \leq j \leq n \), with \( j' : = j - (l - k) \), \( n' : = n - (j - i) \), \( k' : = k - (i - 1) \) and \( l' : = l - (i - 1) \), we have

  \[
  (\theta_{f,i,j})_0 \cdot (\theta_{f(i,j),k',l'})_0 \simeq (\theta_{f,k,l})_0 \cdot (\theta_{f(i,j),l'})_0
  \]

  where \( f^{(ij)} \) denotes the path \( \langle A_0, f_1, f_2, \ldots, (f_i \cdots f_j), \ldots, f_n \rangle \) of length \( n - (j - i) \). Putting in other way, we can say that the following diagram commutes (up to equivalence).

  \[
  f_1 \cdots f_i \cdots f_k \cdots f_l \cdots f_n \longrightarrow f_1 \cdots (f_i \cdots f_k) \cdots f_l \cdots f_n
  \]

  \( b \) Whenever paths \( f = \langle A_0, f_1, f_2, \ldots, f_n \rangle \) and \( g = \langle A_0, g_1, g_2, \ldots, g_n \rangle \) are given with arrows \( \alpha_k : J f_k \to J g_k \) and a pair of indices \( 1 \leq i + 1 \leq j \leq n \), we have the following commutativity (up to equivalence), using auxiliary variables \( n' : = n - (j - i) \) and \( j' : = j - (i - 1) \).

  \[
  (\theta_{f,i,j})_0 \cdot \mu_{n'}(\alpha_1, \ldots, \alpha_j) \cdot (\theta_{g,i,j})_0 \simeq \mu_n(\alpha_1, \ldots, \alpha_n)
  \]
An object $A$ is called \textit{discrete}, if the only arrow ending or starting in $A$ is the identity $1_A$. An object $A$ is said to be \textit{hereditary discrete} if all the following objects $A_n$ are discrete ($n \in \mathbb{N}$):

$$A_0 := A, \quad A_{n+1} := J(1_{A_n}).$$

Objects might also be referred as 0\textit{-cells}, and an object $A$ is called \textit{(n+1)-cell} if $A = Jf$ for any arrow $f$ for which both $\text{dom}f$ and $\text{cod}f$ are $n$\textit{-cells}. (In this case, the arrow $f$ is also referred to as an $n+1$\textit{-cell}.)

Note that, by this, all $n$\textit{-cells} are automatically $m$\textit{-cells} for any $m < n$.

**Examples:**

1. Categories can be identified within weak folded categories as those that has the following property:
   For all morphism $f$, the object $Jf$ is discrete. Consequently, they are also hereditary discrete.
   Conversely, if a category $\mathcal{A}$ is given, let $\text{Ob}\mathcal{C}$ be the disjoint union
   $$\text{Ob}\mathcal{A} \sqcup \text{Mor}\mathcal{A} \times \mathbb{N}$$
   where $\langle f, 0 \rangle$ stands for $f$, i.e. $Jf := \langle f, 0 \rangle$, and $\langle f, n+1 \rangle$ stands for $1_{\langle f, n \rangle}$.
   It is then straightforward to find the rest of the data.

2. Similarly, a weak folded category represents an unbiased bicategory if any 2\textit{-cell} is discrete. (Consequently, as all $n$\textit{-cells} are 2\textit{-cells} if $n \geq 2$, in this case the 2\textit{-cells} are actually hereditary discrete.)

3. For a topological space $X$, consider the weak folded category $\mathcal{C}(X)$, the objects of which are all points (level 0), paths (level 1), homotopies between paths (level 2), homotopies between homotopies (level 3), and so on. There can be morphism only between objects of the same level, and these morphisms are then (endpoint fixing) homotopies from the next level.
   Note that if we pack one more parameter into the notion of path/homotopy, namely the length $t$, and define homotopy between functions $K \to X$ as a function $K \times [0, t] \to X$ for some $t \in \mathbb{R}, \quad t \geq 0$ then the straightforward composition on $\mathcal{C}(X)$ becomes strictly associative.

**Lemma 1.5.** The following statements hold for a weak folded category:

\begin{enumerate}
\item Suppose that arrows $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ are given such that $Jf_k \simeq Jg_k$ for all $k = 1, 2, \ldots, n$, then we also have $J(f_1 f_2 \cdots f_n) \simeq J(g_1 g_2 \cdots g_n)$.
\end{enumerate}
b) If \( f_1, \ldots, f_n \) are composable equivalence arrows with inverses \( g_1, \ldots, g_n \), then there are equivalence arrows \( u_1, \ldots, u_m \) with inverses \( v_1, \ldots, v_m \) such that

\[
\begin{align*}
u_1 u_2 \cdots u_m : (f_1 \cdots f_n)(g_n \cdots g_1) & \to 1_{\text{dom } f_1} \\
v_m \cdots v_2 v_1 : 1_{\text{dom } f_1} & \to (f_1 \cdots f_n)(g_n \cdots g_1)
\end{align*}
\]

Proof. a) According to the note under Def.1.3 the \( n \)-tuple \( \langle f_1, \ldots, f_n \rangle \) is already equivalent to \( \langle g_1, \ldots, g_n \rangle \) in \( C[^n] \), and \( \mu_n : C[^n] \to C \) is assumed to be a functor, so that it preserves equivalences.

b) First we have to apply adequate coherence equivalences, in order to arrive to \( f_1(f_2(\ldots(f_n g_n)\ldots)g_2)g_1 \) from \( (f_1 \cdots f_n)(g_n \cdots g_1) \), these define the first few \( u_i \)'s we are looking for. Then we apply the hypotheses that \( g_i \) is an inverse of \( f_i \), yielding an equivalence arrow \( f_i g_i \to 1_{A_i} \), so we can start to eliminate the pairs from the middle, repeatedly using the coherence equivalences for the identities \( 1_{A_i} \) placed in the composition, for \( i = n, \ldots, 2, 1 \).

**Proposition 1.6.** In a weak folded category, equivalence between objects as defined in Def.1.2 is an equivalence relation.

**Proof.** Reflexivity holds because of the presence of identities, symmetry is obvious from definition, and transitivity follows from part b) of the previous lemma.

Now, we are basically allowed to apply all the usual tools about isomorphisms to equivalences, only that the equalities are replaced by certain equivalences. For instance, we have the following.

**Proposition 1.7.** Suppose that \( u : A \to B \) is an equivalence with inverse \( v \) and that \( f : B \to C, g : A \to C \) are arrows such that \( uf \simeq g \). Then we equally have \( f \simeq vg \). Of course, the dual statement also holds.

**Proof.** We have \( f \simeq 1_B f \simeq (vu)f \simeq v(uf) \simeq vg \).

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**Remarks.**

1. Instead of a quiver \( \text{Mor} C \xrightarrow{\text{dom}} \text{Ob} C \xrightarrow{\text{cod}} \text{Ob} C \) with an injective switchback function \( J : \text{Mor} C \to \text{Ob} C \), we could have started out from only one class \( \text{Ob} C \) with two partial functions \( \text{dom}, \text{cod} : \text{Ob} C \to \text{Ob} C \), both are defined on the same subclass which we can denote by \( C'^* \). All the rest can be rephrased for this setup.

2. The unbiased approach (having \( n \)-ary compositions as part of the basic structure), unwrapped from [Leinster], can be equally well replaced by the ordinary approach of binary compositions and identities with coherence pentagon and triangles.

3. If we drop globularity, and define horizontal composition (i.e. the functors \( \mu_n \)) along some predefined functors \( \text{left}, \text{right} : C'^* \to C \) then we arrive to an infinite dimensional version of Double-like categories, with cubic cells, where horizontal and vertical arrows are not distinguished. In particular, it seems that we can describe this way the (pseudo-) double category of
quintets of an arbitrary bicategory, though such a structure is weakly associative in both horizontal and vertical directions. (Such doubly weak double categories are studied in [Verity] and [Morton].)

References

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