Spectral Analysis of Universal Conductance Fluctuations

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Abstract—Universal conductance fluctuations are usually observed in the form of aperiodic oscillations in the magnetoresistance of thin wires under variation of the magnetic field $B$. A Fourier analysis of aperiodic oscillations observed in the classical experiments by Webb and Washburn reveals a practically discrete spectrum in agreement with the scenario based on the analogy with one-dimensional systems, according to which conductance fluctuations are due to the superposition of incommensurate harmonics. A more detailed analysis reveals the existence of a continuous component, whose smallness is explained theoretically. A lot of qualitative results are obtained that confirm the presented picture: the distribution of phases, frequency differences, and the growth exponents is consistent with theoretical predictions; discrete frequencies weakly depend on the processing procedure; and the discovered shift oscillations confirm the analogy with one-dimensional systems. Microscopic estimates show that the results obtained are consistent with the geometrical dimensions of the sample.

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1. INTRODUCTION

Universal conductance fluctuations [1–4] are usually observed in the form of aperiodic oscillations in the magnetoresistance of thin wires as a function of the magnetic field $B$ [5] (Fig. 1) (see the reviews [6, 7]). The fluctuation pattern looks random, but it is fully reproduced under repeated cycling of the magnetic field. This pattern characterizes a specific realization of the random potential and is completely changed when the sample is heated to a sufficiently high temperature at which the impurities become mobile and a new impurity configuration appears (“magnetic fingerprints”).

According to the theory [1–4], for a fixed value of the field $B$, the conductance $G(B)$ experiences fluctuations on the order of $e^2/h$ under the variation of the impurity configuration; fluctuations in $G(B)$ and $G(B + \Delta B)$ are statistically independent if $\Delta B$ exceeds a certain characteristic scale $\xi_B$. It is natural to expect that, on scales larger than $\xi_B$, the oscillations in $G(B)$ in Fig. 1 are completely random. Then, their Fourier analysis should reveal a flat white noise spectrum at frequencies below $\xi_B^{-1}$.

An alternative point of view arises when comparing with the results for one-dimensional systems [8]. A magnetic field perpendicular to a thin wire creates a quadratic potential along it [9], which effectively constrains the length $L$ of the system; therefore, the variation of the magnetic field is equivalent to the variation of $L$. The resistance $\rho$ of a one-dimensional system is a strongly fluctuating quantity, and the shape of its distribution function $P(\rho)$ depends significantly on the first several moments. Indeed, the Fourier transform of $P(\rho)$ defines the characteristic function

$$F(t) = \langle e^{it\rho} \rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \langle \rho^n \rangle,$$  (1)

which is the generating function of the moments $\langle \rho^n \rangle$. If all moments of the distribution are known, then $F(t)$ can be constructed from them, after which $P(\rho)$ is determined by the inverse Fourier transform. If $\langle \rho^n \rangle$ do

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig1.png}
\caption{Conductance of a thin Au wire as a function of the magnetic field [5].}
\end{figure}
not grow too fast with \( n \), then the contribution of higher moments is suppressed by the factor \( 1/n! \), while the first few moments turn out to be significant. These moments are oscillating functions of \( L \):

\[
\langle \rho \rangle = a_1(L) + b_1(L) \cos(\omega_1 L + \varphi_1),
\]

\[
\langle \rho^2 \rangle = a_2(L) + b_2(L) \cos(\omega_2 L + \varphi_2) + b_1(L) \cos(\omega_1 L + \varphi_1) + \text{etc.}
\]

where \( a_1(L) \) and \( b_1(L) \) are monotonic functions. The point is that the growth exponent for \( \langle \rho^n \rangle \) is determined by a \((2n + 1)\)th order algebraic equation [8] one of whose roots is always real, while the rest are complex for energies within the allowed band; therefore, there are \( n \) pairs of complex conjugate roots. The expression for \( \langle \rho^n \rangle \) contains a linear combination of the corresponding exponential functions, and the complex roots provide \( n \) oscillating terms. In the general case, the frequencies \( \omega \) are incommensurate, but their incommensurability disappears deep in the allowed band under weak disorder (Section 9). According to this picture, the oscillations in \( G(B) \) in Fig. 1 are determined by the superposition of incommensurate harmonics, and their Fourier spectrum should contain discrete frequencies. This picture is indirectly confirmed by the experimental data obtained in [10] and cited in [8], according to which the distribution function \( P(\rho) \) is not stationary, but undergoes systematic aperiodic variations.

It is clear from what has been said that the Fourier analysis of the function \( G(B) \) (Fig. 1) makes it possible to establish which of the two scenarios is more adequate. Below, we will show that such an analysis leads to a spectrum that looks purely discrete (Section 2): this indicates the validity of the second conception. However, there is no contradiction with the diagrammatic results [1–4] since, on the whole, the shape of the spectrum is close to that of discrete white noise, which is similar in properties to continuous noise. A more detailed analysis (Section 4) reveals a continuous component, the smallness of which is theoretically explained in Section 5. The dependence of the results on the processing procedure is discussed in Section 3: despite the obvious problems arising from the deviation from the optimal regime, the discrete frequencies of the spectral lines exhibit surprising stability, which proves their objective origin. Analysis of the real and imaginary parts of the Fourier transform \( F(\omega) \) of the function \( G(B) \) (Fig. 1) reveals the existence of fast oscillations associated with the shift of its argument from the “natural” origin, the nature of which is discussed in Section 6. After eliminating fast oscillations, we study the distribution of the phase shifts of discrete harmonics, which does not contradict their expected stochasticity (Section 7). The positions of the extrema of \( \text{Re} F(\omega) \) and \( \text{Im} F(\omega) \) differ from those of \( |F(\omega)| \), which indicates the exponential growth of harmonics expected from the analogy with one-dimensional systems (Section 8). The distribution of the growth exponents and frequency differences corresponds to the theoretical expectations for the metallic regime (Section 9). Microscopic estimates confirm the consistency of the results with the geometrical dimensions of the sample (Section 10). The results obtained in Section 2 were published previously in the short communication [11].

2. FOURIER SPECTRUM OF APERIODIC OSCILLATIONS

A Fourier analysis of the function \( G(B) \) (Fig. 1) cannot be carried out directly, since the abrupt cutoff of the experimental data gives rise to slowly decaying oscillations in its spectrum and the chaotization of the latter;\(^1\) to obtain clear results, it is necessary to use a proper smoothing function. Let us explain the situation in more detail.

Let a function \( f(x) \) be defined by the superposition of discrete harmonics and be real; then

\[
f(x) = \sum_s A_s e^{i\omega_s x} = \frac{1}{2} \sum_s (A_s e^{i\omega_s x} + A_s^* e^{-i\omega_s x}),
\]

where the frequencies \( \omega_s \) can be assumed positive without loss of generality. Then the Fourier transform of \( f(x) \) has the form

\[
F(\omega) = \pi \sum_s [A_s \delta(\omega + \omega_s) + A_s^* \delta(\omega - \omega_s)],
\]

and its modulus

\[
|F(\omega)| = \pi \sum_s |A_s| \delta(\omega + \omega_s) + \delta(\omega - \omega_s)
\]

depends only on the intensities of the spectral lines and does not contain information on the phase shifts in the corresponding discrete harmonics. Since \( |F(\omega)| \) is an even function, we can restrict ourselves to positive values of \( \omega \) and omit the first delta function in (5).

Since the function \( f(x) \) can be measured experimentally only in a certain finite range of \( x \), we actually have

\[
f(x) = \frac{1}{2} \sum_s (A_s e^{i\omega_s x} + A_s^* e^{-i\omega_s x}) G(x),
\]

where the function \( G(x) \) is equal to one within the operating range and zero outside it; this function will

\(^1\)In Fig. 14 in [5], Webb and Washburn compared the Fourier spectrum of a thin wire with the spectrum of a small ring; the latter exhibits additional oscillations associated with the Aharonov–Bohm effect. In this case, aperiodic oscillations were not the subject of discussion, and their spectrum (which was of a chaotic nature due to the abrupt cutoff) was roughly approximated by the authors and represented as the envelope of oscillations. The latter was found by comparing with Figs. 12 and 13 in [5], where chaotic oscillations are shown explicitly.
be smoothed in what follows. Then, instead of (4), we obtain
\[ F(\omega) = \frac{1}{2} \sum_{s} [A_s \sinh(\omega + \omega_s) + A_s^* \sinh(\omega - \omega_s)], \] (7)

where \( g(\omega) \) is the Fourier transform of the function \( G(x) \), which is real if \( G(x) \) is even. Thus, the restriction of the operating range leads to the replacement of delta functions by spectral lines of finite width. If the discrete frequencies are well separated and \( g(\omega) \) is strongly localized near zero, then we can neglect the overlapping of the functions \( g(\omega \pm \omega_s) \) and write
\[ |F(\omega)|^2 = \frac{1}{4} \sum_s |A_s|^2 \sinh^2(\omega - \omega_s) \] (8)

for positive frequencies. The function \( |F(\omega)|^2 \) (the so-called spectral power density [12]) more objectively characterizes the relative contribution of different harmonics since its integral over all frequencies is equal to the integral of \( |f(x)|^2 \) over all \( x \); therefore, the variation of the spectrum of \( f(x) \) for a fixed rms fluctuation leads to the intensity redistribution between different frequencies while maintaining the total power of the spectrum.

It is easy to see that, to obtain a clear picture in the case of a discrete spectrum, it is necessary to have a possibly narrower shape of the spectral lines of \( g^2(\omega) \), which is ensured by an appropriate choice of the function \( G(x) \). The general strategy is determined by the properties of the integrals of rapidly oscillating functions [13]. If \( f(x) \) is discontinuous, then its Fourier transform falls off as \( 1/\omega \) at high frequencies; if the \( n \)th derivative has a discontinuity, then \( F(\omega) \sim \omega^{-n-1} \). In the case of smooth \( f(x) \), the Fourier integral is calculated by shifting the contour to the complex plane and is determined by the nearest singularity or a saddle point, which leads to the dependence \( F(\omega) \sim \omega^{-n-1} \). If a regular function is obtained by a weak smoothing of the singularity, then the exponent \( \alpha \) is small, and the exponential function manifests itself only for very large \( \omega \), while, in the rest of the region, it exhibits behavior corresponding to a singular function. In our case, we have to smooth the discontinuity of \( G(x) \), which corresponds to the behavior \( g(\omega) \sim \sinh(\omega - \omega_0)/\omega \). In this case, weak smoothing is not effective, while strong smoothing leads to a decrease in \( G(x) \) near the boundaries of the operating range and to a loss of experimental information; therefore, some reasonable compromise is needed.

We take \( G(x) \) in the form of a Fermi function symmetrized with respect to \( x, \)
\[ G(x) = \frac{1}{1 + e^{(x-\mu)/T} + e^{-(x-\mu)/T}} = \frac{1}{1 + 2 e^{-\mu/T} \cosh(x/T)}, \] (9)

for which the Fourier integral is calculated exactly:
\[ g(\omega) = \int_{-\infty}^{\infty} e^{-\omega x} f(x) dx = \frac{2\pi}{b} \sinh(\omega_0/\beta) \]
\[ x_0 = \arccosh(c/b). \] (10)

In our case, for the choice of \( x = B - \mu_0 \), the experimental data correspond to the interval \( |x| \leq \mu_0 \) with \( \mu_0 = 4 \) (in units of tesla). As a rule, we accept \( \mu = \mu_0 - 4T \), which provides a small value of \( G(\mu_0) \approx 0.02 \) at the boundary of the interval. As is clear from Fig. 2, at low \( T \), the behavior \( g(\omega) = 2\sinh(\omega_0/\omega) \) mainly persists, which is characteristic of an abrupt cutoff. The choice of \( \mu = 2 \) and \( T = 0.5 \), which was used in [11], seems reasonable; in this case, 50% of the experimental data are effectively used, and the linewidth is approximately the same as in the case of the extremal smoothing corresponding to \( \mu = T\mu_0 \) and \( x_0 = 0 \), when
\[ g(\omega) = \frac{2\pi T^2 \omega}{\sinh(\pi T \omega)} \] (11)

and the oscillations disappear completely. In the present work, it is convenient to choose (11), which uses slightly less information, but the shape of the spectrum is practically the same as in [11].

The spectral analysis of the experimental data (Fig. 1) is carried out by calculating the Fourier integral in the region \( |x| \leq \mu_0 \) with the use of the above-mentioned smoothing function: the results obtained are shown in Fig. 3. The spectrum obviously consists of discrete lines, which confirms the concept of [8]. However, in the interval \( \omega \ll 2\pi/\xi_B \) (where \( \xi_B \) is estimated as the average distance between adjacent maxima or minima in Fig. 1), the shape of the spectrum resembles dis-

\[ \text{Fig. 2. (a) Function } G(x) \text{ defined by formula (9) and (b) its Fourier transform } g(\omega) \text{ for various values of } \mu \text{ and } T: (1) \mu = 3.5, T = 0.125; (2) \mu = 3, T = 0.25; (3) \mu = 2, T = 0.5; \text{ and (4) } \mu = T\mu_0, T = 0.8. \]
As an example, consider the discrete white noise model defined by the equation [11]

\[ F(\omega) = \pi \sum_s [A_s \delta(\omega + \omega_s) + A_s^* \delta(\omega - \omega_s)]H(\omega), \quad (12) \]

where the frequencies \( \omega_s \) are equidistant (\( \omega_s = s \Delta \)), the moduli of \( A_s \) are equal (\( |A_s| = A \)), and the phases of \( A_s \) are completely random; the function \( H(\omega) \) restricts the spectrum to the region \( |\omega| \leq \Omega \) and is assumed to be even. Then, determining \( f(x) \) by the inverse Fourier transform, we have the following expression for the correlation function:

\[
\langle f(x)f(x') \rangle = \frac{1}{2} \sum_s A_s^2 H^2(\omega_s)e^{i\omega_s(x-x')} \\
= \frac{1}{2} A^2 \Delta^{-1} h(x-x'), \quad (13)
\]

where \( h(x) \) is the Fourier transform of \( H^2(\omega) \). If the function \( H(\omega) \) is smooth, then \( h(x) \) exponentially decays on the scale \( \Omega^{-1} \), which is in agreement with the diagrammatic results [1–4].

Thus, the results obtained in this section reconcile the two alternative scenarios indicated at the beginning of the paper. On the one hand, the spectrum is discrete, which confirms the concept of [8]. On the other hand, its shape roughly corresponds to discrete white noise, whose properties are close to those of continuous white noise.

3. DEPENDENCE OF THE RESULTS ON THE PROCESSING PROCEDURE

Let us discuss the dependence of the results on the choice of the smoothing function (9), which, for \( \mu = 7 \ln 2 \) and \( T = 0.8 \), on this and subsequent figures, the values of \( F(\omega) \) are multiplied by 10.

![Fig. 3. Fourier analysis of the experimental data of Fig. 1 for the smoothing function (9) with \( \mu = 7 \ln 2 \) and \( T = 0.8 \). On this and subsequent figures, the values of \( F(\omega) \) are multiplied by 10.](image)

Despite the obvious problems arising when deviating from the optimal processing regime, the frequencies of discrete harmonics exhibit surprising stability when \( \beta \) varies by more than an order of magnitude (Fig. 5), and there is no doubt about their objective origin. The slight dependence on \( \beta \) is associated with a change in the shape of the lines and their mutual influence on each other. In fact, Fig. 5 demonstrates how well the independent harmonic approximation works.

If the experimental range of variation of the field \( B \) were smaller, then the optimal line resolution mode (Fig. 4a) could be missing, and a transition from Fig. 4b with merged lines to Fig. 4c with spurious oscillations would occur. We can assume that, even under the present experimental conditions, the line resolution is incomplete, and a partial merging of lines occurs.

4. CONTINUOUS COMPONENT OF THE SPECTRUM

The Fourier spectrum in Fig. 3 looks purely discrete, which does not seem quite natural. The analogy with one-dimensional systems leads to the conclusion that the distribution function \( P(\rho) \) undergoes systematic variations of deterministic nature, which lead to oscillations in \( \rho \) in a particular sample. However, similar oscillations (of a random nature) should also occur for a stationary distribution \( P(\rho) \) due to the finite width of the latter. It would be more natural if Fig. 3 contained a continuous component, while the discrete lines were observed against its background. In fact, the
continuous component is indeed present, and below we will try to evaluate it.

According to (8), in the neglect of the interaction of harmonics, the spectrum $|F(\omega)|^2$ is represented as a linear combination of functions $g^2(\omega - \omega_\alpha)$ whose shape is known beforehand. The verification of relation (8) is shown in Fig. 6a, where the frequencies $\omega_\alpha$ were determined from the positions of the maxima in Fig. 3 and the amplitudes $|A_\alpha|$, by the height of these maxima; as for the shape of $g^2(\omega - \omega_\alpha)$, no fitting was made. The agreement looks satisfactory but is not complete: the linewidth differs from the theoretical one either upwards or downwards, and the observed line shape is not always symmetric.

It is natural to assume that this is due to the presence of a continuous component of the spectrum. Assuming that the latter component is a slowly varying function and measuring $\omega_0$ from the center of the line, we can set

$$ F(\omega) = Ag(\omega) + B $$

(14)

and consider $B$ to be constant within the linewidth. Then

$$ |F(\omega)|^2 = c_1 g^2(\omega) + c_2 g(\omega) + c_3, $$

(15)

where

$$ c_1 = |A|^2, \quad c_2 = 2|A||B|\cos \chi, \quad c_3 = |B|^2 $$

(16)

and $\chi$ is determined by the difference of $A$ and $B$ phases. Assuming that $g(\omega)$ is normalized to unity at $\omega = 0$, we can easily understand that the function $g^2(\omega)$ corresponds to a narrower maximum than $g(\omega)$; therefore, the line broadens for $c_2 > 0$ and narrows for $c_2 < 0$, turning out to be asymmetric in the case of a significant change in $B$ near the maximum. According to (15), $|F(\omega)|^2$ is determined by the superposition of three basis functions $g^2(\omega)$, $g(\omega)$, and 1, whose coefficients can be found from the minimization of the rms deviation. This is a standard processing procedure [12], which is linear and unique. However, in practice, it leads to unphysical results due to the violation of the condition

$$ |c_2| \leq 2|c_1||c_3|. $$

(17)
which follows from (16). Apparently, the optimal fit corresponds to the limiting values of $\pm 1$ for $\cos \chi$; in this case, the parameters $A$ and $B$ can be considered real, assuming that

$$F_0 = |F(0)|.$$ Here we take into account that the procedure used is justified near the maximum of $|F(\omega)|^2$, and its position is naturally kept fixed. Fitting by formula (18) separately for each line leads to Fig. 6b: for most of the lines, the agreement is almost perfect, and the absence of such agreement for some of the $m$ is apparently attributed to the presence of secondary harmonics, which “are hidden” against the background of the main spectral lines.

After determining the coefficients $c_1$, $c_2$, and $c_3$, we can eliminate from $|F(\omega)|^2$ the contributions proportional to $g^2(\omega)$ and $g(\omega)$. Carrying out this procedure for all lines, we obtain the “residual” spectrum shown in Fig. 7. The sharp peaks in this spectrum are apparently associated with secondary discrete harmonics, everything else being naturally attributed to the continuous component, which amounts to $10–15\%$ of the intensity of the main lines; the reason for the smallness of this component is discussed in Section 5. The spectrum of the continuous component approximately corresponds to white noise for $\omega \approx 2\pi/\xi_B$, but the assumed slowness of its variation is not conformed, so the result obtained should be considered only as a rough estimate.\(^4\)

Strictly speaking, the quantity $B$ in (15) represents not only the continuous component, but also the contributions of adjacent discrete lines. This does not affect the correctness of the determination of the amplitude $A$, which is confirmed by a similar calculation for the smoothing function with $\mu = 2$ and $T = 0.5$: the “tails” of $g(\omega)$ for the latter function are significantly different from those of function (11), but the result is not too different from that illustrated in Fig. 7.

Note that the spectrum $|F(\omega)|^2$ in Fig. 3 looks “more discrete” than it really is: the point is that the quantity $B$, being approximately constant in the neighborhoods of the maxima, changes its sign in the intervals between some lines. The residual spectrum in Fig. 7 does not reduce to $B$ but contains processing errors and interference effects between adjacent lines.

5. WHY IS THE SPECTRUM PRACTICALLY DISCRETE?

Let us discuss the reasons for the smallness of the continuous component of the spectrum. According to [14–19], the evolution of the distribution $P(\rho)$ in one-dimensional systems is described by the diffusion-type equation

$$4$$ For asymmetric lines, the processing is ambiguous due to the possibility of fitting their right or left slopes. This ambiguity was used to avoid the appearance of unphysical negative values of $|F(\omega)|^2_{\text{res}}$.,
\[ \frac{\partial P(\rho)}{\partial L} = \alpha \frac{\partial}{\partial \rho} \left[ \rho(1 + \rho) \frac{\partial P(\rho)}{\partial \rho} \right], \]  \hspace{1cm} (19)

in which \( \alpha L \) plays the role of time. This equation was obtained in the random phase approximation, which works well in the quasi-metallic regime, i.e., deep in the allowed band for weak disorder [8]. The natural initial condition for (19) is

\[ P(\rho) = \delta(\rho) \quad \text{for} \quad L = 0, \hspace{1cm} (20) \]

since, for zero length of the system, its resistance is zero regardless of the realization of the random potential. This initial condition leads to the distribution

\[ P(\rho) = (\alpha L)^{-1} \exp[-\rho/\alpha L] \hspace{1cm} (21) \]

for small \( L \) (when typical \( \rho \) are small) and to the log-normal distribution for large \( L \) (when typical \( \rho \) are large). For distribution (21), the mean value \( \langle \rho \rangle \) coincides with the standard deviation \( \sigma \), while, in the log-normal regime, \( \sigma \) grows faster than \( \langle \rho \rangle \).

Consider a more general initial condition, the meaning of which is discussed below:

\[ P(\rho) = \delta(\rho - \rho_0) \quad \text{for} \quad L = L_0. \hspace{1cm} (22) \]

The solution of Eq. (19) with the initial condition (22) is close to the Gaussian distribution for large \( L \) (when typical \( \rho \) are large). For distribution (21), the mean value \( \langle \rho \rangle \) coincides with the standard deviation \( \sigma \), while, in the log-normal regime, \( \sigma \) grows faster than \( \langle \rho \rangle \).

Let us discuss the meaning of the initial condition (22). Suppose we are measuring the resistance \( \rho \) of the system on the length \( L_0 \), creating various impurity configurations; for a sufficiently large number of configurations, we reproduce distribution (21). Let now change the procedure and choose only those configurations for which \( \rho \) falls into a small interval around \( \rho_0 \); thus, an ensemble with a narrow distribution of the type (22) is artificially created whose evolution leads to the picture shown in Fig. 8. Now we take a single sample whose resistance is \( \rho_0 \) on the length \( L_0 \). The function \( \rho(L) \) for such a sample can be obtained theoretically if all the details of the impurity configuration are known. Usually, such information is missing, and only general statistical properties of the random potential are known. In this case, only an approximate corridor of possibilities for the function \( \rho(L) \) can be established, which is illustrated in Fig. 8.

Equation (19) is obtained in the random phase approximation, which eliminates all oscillatory phenomena. Fortunately, the evolution of \( \langle \rho \rangle \) can be investigated exactly without any assumptions (see Appendix B). In the quasi-metallic regime, the following result is valid for natural ideal leads [8]:

\[ \langle \rho \rangle = \rho_0 + \frac{1 + 2\rho_0 (e^{\pm i} - 1) + \frac{e^2}{\delta} \sqrt{\rho_0 (1 + \rho_0)}}{2} \times \left[ e^{x L} \sin \psi - e^{-x L} \sin(2\delta L + \psi) \right], \hspace{1cm} (23) \]

which is obtained for the discrete Anderson model; here \( \delta = k_d a_0 \), \( \epsilon^2 = W^2/4\delta^2 \), \( l = (L - L_0)/a_0 \), \( k_F \) is the Fermi momentum, \( a_0 \) is the lattice constant, \( W \) is the amplitude of the random potential, and \( \psi \) is determined by the difference of the phases appearing in the transfer matrix given on the scale \( L_0 \). It is easy to see that there exist oscillations whose period is determined by the de Broglie wavelength \( 2\delta L = 2k_F L \) for \( L_0 = 0 \); the amplitude of these oscillations can be comparable to \( \rho_0 \ll 1 \) despite the presence of the small parameter \( \epsilon^2/\delta \). In the random phase approximation, the quantity \( \psi \) is completely stochasticized, and averaging over it eliminates oscillations, restoring the result following from Eq. (19) (see (A.6) in Appendix A). Outside the metallic regime \( (\epsilon^2 \approx \delta) \), the amplitude of oscillations certainly exceeds \( \rho_0 \),

\[ \langle \rho \rangle = \rho_0 + \frac{1}{3} \left( \frac{\epsilon^2}{\delta} \right)^{2/3} \left( 1 + \frac{2\rho_0}{2} - \cos \psi \sqrt{\rho_0 (1 + \rho_0)} \right) \times \left[ e^{x L} - 2e^{-x L} \cos \left( \frac{3\pi L}{2} + \frac{\pi}{3} \right) \right], \hspace{1cm} (24) \]

and the oscillations do not disappear when averaged over \( \psi \) due to the fundamental inapplicability of the random phase approximation [8] (here \( x = \frac{\delta}{\epsilon} \)).
The oscillation amplitude in the metallic regime increases in the presence of foreign leads,

\[
\langle \rho \rangle = \rho_0 + \frac{1 + 2p_0}{2} [-1 + \Delta_0^2 e^{2i\psi} - \Delta_0^2 e^{-2i\psi} \cos 2\delta/l] \\
+ \Delta_1 \sqrt{\rho_0 (1 + p_0)} e^{2i\psi} \Delta_2 \cos \psi - e^{-2i\psi} \\
\times \{(\Delta_2 - 1) \cos \psi \cos 2\delta/l + \cos(2\delta/l + \psi)\},
\]

where the parameters \(\Delta_1\) and \(\Delta_2\) are defined in Appendix B and can be large.

Thus, under sufficiently general conditions, the oscillation amplitude of \(\langle \rho \rangle\) is comparable to \(\rho_0\) and certainly exceeds \(\sigma\) in the neighborhood of \(L_0\). For higher moments, the possibility of obtaining results of the type (23)–(25) looks almost unrealistic due to the tediousness of calculations, and we can give only general considerations. For the second moment \(\langle \rho^2 \rangle\), it is natural to expect that the oscillation amplitude is comparable to \(\rho_0^2\) and significantly exceeds the value of \(\sigma^2\). Thus, the variation of the distribution width due to systematic variations of \(P(\rho)\) significantly exceeds the distribution width in the absence of oscillations. Therefore, the oscillations of the higher moments also prove to be significant.

The applicability of the analysis performed to the situation under consideration is determined by the fact that, for the smoothing function with \(\mu = 2\) and \(\beta = 2\) used in [1], the interval \(B = 2–6\ T\) is effectively used, which corresponds to a threefold change in the field. Since \(L \propto B^{1/2}\), this corresponds to a 1.7-fold change in \(L\). And, when \(L_0\) is chosen in the middle of the interval, the deviations from \(L_0\) turn out to be at the level of 30%. For the smoothing function with \(\mu = 7\ln2\), which leads to (11), this deviation is even smaller. Note that the situation does not change with an increase in the experimental range of the fields used, since the arguments for the choice of the smoothing function remain the same: only the value of \(\mu_0\) changes, while all the proportions in Fig. 2 are maintained.

6. SHIFT OSCILLATIONS

Returning to the formulas of Section 2, we can easily notice that expression (7) is more general than (8): the former is an exact consequence of (6), while the latter suggests a weak overlap of spectral lines. It can be expected that processing based on (7) will lead to a smoother dependence for \(|F(\omega)|^2\). Unfortunately, these expectations are not confirmed: the resulting picture is not too different from Fig. 7 and does not justify a more complex processing. However, the study of the real and imaginary parts of \(\tilde{F}(\omega)\) reveals many interesting things, which are discussed in the present and subsequent sections.

The functions \(\text{Re} F(\omega)\) and \(\text{Im} F(\omega)\) turn out to be rapidly oscillating (see Fig. 9a). The reason for the oscillations is easy to understand: if \(F(\omega)\) is the Fourier transform of \(f(x)\), then a shift of the origin \(x\) leads to the correspondence

\[ f(x - a) \leftrightarrow e^{i\alpha a} F(\omega) \]

and, for large \(a\), fast oscillations appear. In this regard, the question arises about the “natural” choice of the origin, for which the Fourier transform changes slowly. By the average oscillation period in Fig. 9a, we can establish that \(a = 4.3\), and the elimination of the factor \(\exp(i\alpha a)\) leads to Fig. 9b, where \(\text{Re} F(\omega)\) and \(\text{Im} F(\omega)\) vary on the same scale as \(|F(\omega)|\). A natural origin is determined by the condition \(x = B - \mu_0 = a\) and corresponds to a magnetic field of \(B_0 = 8.3\ T\), which lies beyond the upper boundary of the experimental range of fields. Since large fields correspond to the small size of the system, we can propose the following interpretation.

When studying the evolution of \(P(\rho)\) as a function of the length \(L\) of a one-dimensional system, the natural origin is at \(L = 0\). However, if the evolution starts from a finite scale \(L_0\), then the distribution for large \(L\) is the same as in the case of \(L_0 = 0\). Therefore, by studying the situation for large \(L\) and extrapolating to the initial stage of evolution, we cannot establish from what scale the evolution starts:7 we can only state that this scale is small compared to the considered ones. In terms of the magnetic field, this means that evolution starts from a certain large field \(B_0\). However, due to the nonlinear character of the dependence \(L \propto B^{-1/2}\), the value of \(B_0\) obtained as a result of linear extrapolation turns out to be not too large.8

The proposed interpretation seems logical and is an indirect confirmation of the analogy with one-dimensional systems. Formal arguments for the choice of the natural origin are presented in Appendix C.

7. PHASE DISTRIBUTION

After establishing the natural origin of the argument of \(f(x)\), we can use the information on the real and
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and imaginary parts of $F(\omega)$ for further analysis. Estimating the values of $\text{Re} F(\omega)$ and $\text{Im} F(\omega)$ at the points of maximum of $|F(\omega)|^2$, we can determine the complex phases of the coefficients $A_s$ in (6)–(8). Their distribution is illustrated in Fig. 10.

As discussed in Section 2, discrete white noise is analogous to continuous one if the phases of $A_s$ are completely random. According to Fig. 10, the distribution of these phases is fairly uniform and does not contradict the expected stochastization. It would be interesting to check how the phase distribution changes when passing to other impurity configurations.

8. EVIDENCE OF EXPONENTIAL GROWTH

According to formula (7), the contributions of discrete harmonics to $F(\omega)$ are proportional to $g(\omega - \omega_s)$, and, in the neglect of their interaction, should lead to the extrema of $\text{Re} F(\omega)$ and $\text{Im} F(\omega)$ at the points $\omega_s$.

However, Fig. 9b demonstrates that the extrema of $\text{Re} F(\omega)$ and $\text{Im} F(\omega)$ are located at different points, which do not coincide with the maximum points of $|F(\omega)|^2$. This indicates that formula (7) fails and casts doubt on the original expression (6) from which (7) has been derived.

Recall that, according to (2), the oscillation amplitudes are not constant but are subjected to exponential growth. Taking this fact into account, instead of (6), we should use the expression

$$f(x) = \frac{1}{2} \sum_s \left[ A_s e^{i\omega_s x + \alpha_s x} + A_s^* e^{-i\omega_s x - \alpha_s x} \right] g(x),$$

(27)

which leads to the following result instead of (7):

$$F(\omega) = \frac{1}{2} \sum_s \left[ A_s g(\omega + \omega_s - i\alpha_s) + A_s^* g(\omega - \omega_s - i\alpha_s) \right].$$

(28)

Let us focus on the contribution of a single harmonic $\omega_s$ and, shifting the origin of $\omega_s$, set

$$A = A' + iA'' = |A| e^{i\phi},$$

$$g(\omega - i\alpha) = g_s(\omega) + ig_s'(\omega).$$

If $\omega_1$ and $\omega_2$ are the positions of the extrema of $\text{Re} F(\omega)$ and $\text{Im} F(\omega)$, then the relations

$$A'g_1'(\omega_1) - A''g_1(\omega_1) = 0,$$

$$A'g_2'(\omega_2) + A''g_2(\omega_2) = 0,$$

(30)

are valid, and the phase $\phi$ of the coefficient $A$ is determined by the condition

$$\tan \phi = \frac{g_1'(\omega_1)}{g_2'(\omega_1)} = -\frac{g_2'(\omega_2)}{g_1'(\omega_2)}.$$

(31)
If the experimental values of $w_1$ and $w_2$ are known, then the algorithm for determining $\phi$ and $\alpha$ is as follows. According to (29), $g_1(\omega)$ and $g_2(\omega)$ depend on $\alpha$, where $g_2(\omega) \to 0$ as $\alpha \to 0$; therefore, for small $\alpha$, the first fraction in (31) is large, while the second is small in absolute value. By increasing $\alpha$, we can achieve equality (31), which determines the values of $\tan \phi$ and $\alpha$. Since, by the known value of the tangent, $\phi$ can be determined up to additive contributions that are multiples of $\pi$, we can use the following relations to establish the correct quadrant for $\phi$:

$$A'g_1(\omega) - A'g_2(\omega) = F_1,$$

$$A'g_2(\omega) + A'g_1(\omega) = F_2,$$

where $F_1$ and $F_2$ are the values of $\Re F(\omega)$ and $\Im F(\omega)$ at the extremum points.

If the function $g(\omega)$ in (11) is normalized to unity at $\omega = 0$, then, setting

$$z = \pi T\omega, \quad \gamma = \pi T\alpha,$$

we have

$$g_1(\omega) = \frac{z \cos \gamma \sinh z + \gamma \sin \gamma \cosh z}{\sinh^2 z + \sin^2 \gamma},$$

$$g_2(\omega) = \frac{z \sin \gamma \cosh z - \gamma \cos \gamma \sinh z}{\sin^2 z + \sin^2 \gamma},$$

$$|g(\omega + i\delta)|^2 = \frac{z^2 + \gamma^2}{\sinh^2 z + \sin^2 \gamma}.$$

It is easy to see that the extrema of $g_1(\omega)$ and $|g(\omega + i\delta)|$ occur at $\omega = 0$, i.e., that the finiteness of $\alpha$ does not affect their position; accordingly, it is natural to measure the frequencies $w_1$ and $w_2$ from zero. The function $g_2(\omega)$ is odd, and its small addition to $g_1(\omega)$ shifts the extremum to the left or right, depending on the sign of the addition. For the functions $\Re F(\omega)$ and $\Im F(\omega)$, the additions to $g_1(\omega)$ proportional to $g_2(\omega)$, have opposite signs, which provides different signs for $\omega_1$ and $\omega_2$. For small $\alpha$, we can restrict ourselves to the first order in $\gamma$ in (34), and, taking into account the smallness of $\omega_1$ and $\omega_2$, obtain

$$\tan \phi = -\frac{\omega_1}{\alpha} \frac{\alpha}{\omega_2}, \quad \alpha^2 = -\omega_1\omega_2.$$

In this approximation, it follows from (32) that $A' \approx F_1$ and $A'' \approx F_2$, which allows us to establish the sign of $\alpha$ and choose the correct quadrant for the phase $\phi$, assuming that the latter lies in the interval $(-\pi, \pi)$:

$$\alpha = -\sqrt{-\omega_1\omega_2 \text{sign}(\omega_1 F_2)},$$

$$\phi = -\arctan(\omega_0/\alpha) + \pi \text{sign} F_2 (1 - \text{sgn} F_1)/2.$$

Note that $|\tan \phi|$ is determined only by the values of $\omega_1$ and $\omega_2$, while $F_1$ and $F_2$ are required only to choose the correct quadrant. This estimate of $\phi$ is fundamentally different from that used in the previous section and coincides with it only if the shifts of the extrema of $\Re F(\omega)$ and $\Im F(\omega)$ are indeed related to the exponential growth.

A comparison of the two estimates of $\phi$ (see Table 1) shows that they approximately coincide for all harmonics and confirms the correctness of the suggested mechanism for the shift of extrema. Small differences between the estimates can be attributed to the approximate character of formula (36) and to complicating factors such as the mutual influence of harmonics, the existence of a continuous spectrum, experimental errors, etc.

### Table 1. Estimates of phases obtained according to Section 7 (\(\phi_s\)) and by formula (36) (\(\bar{\phi}_s\))

| $s$ | $\phi_s$, deg | $\bar{\phi}_s$, deg | $s$ | $\phi_s$, deg | $\bar{\phi}_s$, deg |
|-----|----------------|----------------------|-----|----------------|----------------------|
| 1   | 170            | 161                  | 10  | 37             | 37                   |
| 2   | -88            | -84                  | 11  | 155            | 145                  |
| 3   | 125            | 120                  | 12  | -15            | -20                  |
| 4   | -172           | -162                 | 13  | 111            | 121                  |
| 5   | -78            | -67                  | 14  | -96            | -106                 |
| 6   | 47             | 40                   | 15  | -81            | -72                  |
| 7   | 144            | 153                  | 16  | 87             | 84                   |
| 8   | 59             | 49                   | 17  | -144           | -140                 |
| 9   | -124           | -153                 | 18  |                |                      |

9. DISTRIBUTION OF GROWTH EXPONENTS AND FREQUENCY DIFFERENCES

Let give a summary of theoretical results for the evolution of moments. Assuming that $\langle \rho^n \rangle$ depends exponentially on $L$, we obtain a $(2n + 1)$th order algebraic equation for the growth exponent $x$. For $n = 1$ and $n = 2$, such equations are written out explicitly [8]:

$$x(x^2 + 4\delta) = 2W^2,$$

$$x(x^2 + 4\delta)(x^2 + 16\delta) = 42W^2x^2 + 96W^2\delta,$$

where $\delta$ is energy measured from the lower edge of the band and $W$ is the amplitude of the random potential.

The structure of equations for an arbitrary $n$th moment can be established based on the argumentation of Section 4 in [8]. Deep in the allowed and forbidden bands, we can retain only the diagonal elements in matrices (42) and (47) of that paper and their analogs for higher moments; this leads to the equation

$$\prod_{k=0}^{2n} [x - 2(n - k)\delta - B^k_n \delta^2] = O(\epsilon^{4 \delta^{2n-1}}),$$

where $\epsilon^2 = W^2/4\delta^2$, $\delta^2 = -\delta$, and $B^k_n = n(2n - 1) + 3k(k - 2n).$
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A similar equation near the edge of the band,

\[ x^{2n+1} = \sum_{k=0}^{k_{\text{max}}} C_k W^{2k} x^{2n+1-3k}, \]  

\[ k_{\text{max}} = \left[ \frac{2n+1}{3} \right], \]  

follows from the fact that, for \( x \sim \delta \sim \epsilon^2 \), all the terms of the equation are of the same order of magnitude and only combinations of \( \delta^{2n}\epsilon^{2m} \) with \( n \geq m \) are admissible, of which only \( \delta^{2n}\epsilon^{2n} \sim W^{2n} \) remain finite as \( \delta \to 0 \). The nontrivial roots of Eq. (39) are of order \( W^{2/3} \) and lead to incommensurate frequencies in the oscillations (2).

Deep in the allowed band, the parameters \( \delta \) and \( \epsilon \) are complex, and to reduce them to a real form, we have to make the replacements \( \delta \to i\delta \) and \( \epsilon \to -i\epsilon \). Then Eq. (38) gives a complete set of exponents for the extremely metallic regime (\( \epsilon^2 \ll \delta \)),

\[ x_n^k = 2i(n-k)\delta - B^k\epsilon^2 + O(\epsilon^4 / \delta), \]  

\[ k = 0, 1, \ldots, 2n. \]  

It is easily seen that all oscillation frequencies in (2) are integer multiples of \( \tilde{\Delta} = 2\delta \), that is, their incommensurability disappears. In the ideal case, all the differences of adjacent frequencies \( \Delta_1 = \omega_{n+1} - \omega_n \) should be equal to \( \tilde{\Delta} \); however, some harmonics could be not manifested due to their weak intensity, so that the differences \( \Delta_1 \) are “quantized”; i.e., they can be equal to \( \tilde{\Delta}, 2\tilde{\Delta}, 3\tilde{\Delta}, \) etc.

The distribution of the differences \( \Delta_n \) for 17 harmonics, which are obvious from Fig. 3, is shown in Fig. 11. It is easy to see that these harmonics split into two groups I and II localized near the values of \( \tilde{\Delta} \) and \( 2\tilde{\Delta} \), where \( \tilde{\Delta} \) is chosen from the best agreement. The absence of exact quantization should not cause any trouble, since it is implemented only in the extremal

---

**Fig. 11.** Distribution of differences between adjacent frequencies \( \Delta_n = \omega_{n+1} - \omega_n \); the values of \( s \) are indicated near the corresponding lines. The differences \( \Delta_n \) are divided into two groups, I and II, that are localized near \( \tilde{\Delta} \) and \( 2\tilde{\Delta} \).

**Fig. 12.** Distribution of growth exponents \( \alpha_s \) and their comparison with the sequence (41) for a proper choice of \( \epsilon^2 \). The values of \( s \) are indicated at the corresponding lines.

**Fig. 13.** For a thin wire, the transverse motion is quantized; this gives rise to \( N_0 \) discrete levels, which turn into one-dimensional subbands when the longitudinal motion is taken into account. The main contribution to the conductance oscillations comes from the upper filled subband.
metallic regime. In reality, the metallic regime is not extremal, and there are corrections to it, which are indicated in (40).

The growth exponents \( \alpha_n \) are determined by the real part of expression (40) and are integer multiples of \( \epsilon^2 \). Running over all possible values of \( n \) and \( k \), we obtain an infinite sequence of exponents, which looks as follows near the origin:

\[
\ldots, -3\epsilon^2, -\epsilon^2, 0, 2\epsilon^2, 3\epsilon^2, 6\epsilon^2, 8\epsilon^2, \ldots \tag{41}
\]

The exponents obtained as a result of processing the experimental data by formula (36) are shown in Fig. 12 with the opposite sign\(^9\) and well reproduce this sequence for an appropriate choice of \( \epsilon^2 \). The only exception is the absence of the quantity \( 3\epsilon^2 \), which is probably related with the low intensity of the corresponding lines. The fact that only the exponents in the neighborhood of the origin are manifested is naturally explained by the fact that harmonics with larger (in modulus) exponents are localized near the ends of the experimental range of magnetic fields and are not visible in its middle.

Note that the maximal exponent for the moment \( \langle \rho^n \rangle \) in the forbidden band is attained at \( k = 2n \),

\[
x_n^{\text{max}} = 2n\delta + n(2n - 1)e^2, \tag{42}
\]

and, in the allowed band for \( k = n \),

\[
x_n^{\text{max}} = n(n + 1)e^2. \tag{43}
\]

These results are consistent with the functional form for the log-normal regime

\[
x_n = an + bn^2/2 \tag{44}
\]

with the parameters \( a \) and \( b \) obtained in [8]: \( a = 2\delta - \epsilon^2 \) and \( b = 4\epsilon^2 \) in the forbidden band, and \( a = \epsilon^2 \) and \( b = 2\epsilon^2 \) in the allowed band.

10. MICROSCOPIC PICTURE

It is easy to see that quantization turns out to be more precise for the growth exponents \( \alpha_n \) (Fig. 12) than for the frequency differences \( \Delta_n \) (Fig. 11). This fact has a simple explanation.

A thin wire is a quasi-one-dimensional system in which the transverse motion is quantized; this quantization gives rise to \( N_0 \) discrete levels \( \epsilon_n^0 \), which, with

\[\epsilon_n^0 = \alpha^2 + k_n^2/2m, \tag{45}\]

whose states below the Fermi level \( E_F \) are occupied (Fig. 13). The magnetic field \( B \) most strongly affects the upper filled subband with the minimal Fermi energy \( \epsilon_F \) by restricting the motion in it to the length \( L \) determined by the condition

\[m\omega_B^2L^2 \sim \epsilon_F \sim E_0/N_0, \tag{46}\]

so that

\[L \sim a\frac{B_0}{B_N N_0}, \tag{47}\]

where \( E_0 = \hbar^2/ma^2 \) and \( B_0 = \phi_0/a^2 \) are atomic units of energy and magnetic field (\( \phi_0 = \pi\hbar c/e \) is the flux quantum). The conductance of the system is determined by the sum of the conductances of one-dimensional subbands whose oscillations decay exponentially with increasing \( L \) (see below). Therefore, the main contribution to the oscillations comes from the upper subband, for which the length \( L \) is minimal; however, the neighboring subbands also have some influence. For the frequency differences \( \Delta_n \), this effect significantly broadens the distribution (Fig. 11), since the oscillation frequency is determined by the Fermi momentum \( k_F \), the value of which is different in different subbands. As for the growth exponents \( \alpha_n \) (Fig. 12), the situation is completely different. The Fermi momenta for the upper subbands are small, and the slow particle approximation is valid for scattering by impurities.\(^\text{10}\), so the scattering amplitude is independent of momentum. The growth rates are directly related to the scattering amplitude and do not depend on the Fermi momenta of one-dimensional subbands. Therefore, the influence of adjacent subbands does not violate exact quantization, and Fig. 12 actually corresponds to a strictly one-dimensional system. Deviations from exact quantization are associated only with the nonextremality of the metallic regime and experimental errors.

The minimal discrete harmonic in Fig. 3 corresponds to approximately four oscillations periods in the field range from 1 to 10 T. Its frequency is determined by the de Broglie wavelength \( \lambda \sim a\sqrt{N_0} \) in the

\(^9\) The exponents \( \alpha_n \) change sign when passing from the magnetic field \( B \) to the effective length of the system \( L \) (the right and left directions interchange). When passing from conductance to resistance, there is no change in the signs of \( \alpha \), since small fluctuations of these two quantities are proportional to each other.

\(^{10}\) Here the difference between the real system and the Anderson model considered in [8] is essential. In the Anderson model, the metallic regime corresponds to a high concentration of weak impurities, which can be considered by perturbation theory. In a real system, weak disorder is created by low concentration of strong impurities, for which the slow particle approximation is valid. This difference does not affect the results for the exponents, since, on the wavelength scale, the configuration of the random potential can be varied within wide limits without affecting the large-scale properties of the wave functions.
upper subband, and the number of oscillations when the field changes from \( B_{\text{min}} \) to 10\( B_{\text{min}} \) is given, in view of (47), by the estimate

\[ N_{\text{osc}} \sim \frac{B_0}{B_{\text{min}}} N_0 B_{\text{min}}, \]  

which, for \( B_0 \sim 10^4 \) T, \( B_{\text{min}} = 1 \) T, and \( N_{\text{osc}} = 4 \), gives

\[ N_0 \sim 2.5 \times 10^3. \]  

This is approximately half the number of atoms in the cross section of a wire 25 nm in diameter [5] and corresponds to a half-filled three-dimensional band. In this case, \( \lambda \sim 50a \) and the actual range of wire lengths

\[ L = 20 - 200a \]  

is within 310 nm [5].

Above, we assumed the condition \( \hbar \omega_0 \ll \varepsilon_F \), which is actually violated in the middle of the experimental range of fields. For strong fields, the estimate \( L \sim a(B_0/B)^{1/2} \) is more adequate, which follows from (46) after replacing \( \varepsilon_F \) by \( \hbar \omega_0 \). As a result, the lower bound in (50) shifts from 20a to 30a, which is not essential for our estimates.

As is clear from (23) and (25), the expression for \( \langle \rho \rangle \) contains a growing exponential \( e^{2\varepsilon_1} \) and oscillating terms decaying as \( e^{-\varepsilon_1} \). A similar picture is valid for higher moments: the maximal exponent (43) in the metallic regime is real and does not lead to oscillations, while the oscillating terms grow slower and are relatively small even for positive \( \alpha \). Therefore, for a typical value of \( \rho \), the situation is the same as for the moments, and, upon passing to the dimensionless conductance \( g = 1/\rho \), the oscillations turn out to be decreasing. If the typical values of \( \Delta_1 \sim \Delta_2 \sim 1 \) are accepted for the foreign leads, the oscillations of \( g \) in the region \( e^{2\varepsilon_1} \ll 1 \) turn out to be on the order of unity, which gives fluctuations on the order of \( e^{2}/\hbar \) for the dimensional conductance (Fig. 1). According to this picture, only the order of magnitude of fluctuations is universal, while their amplitude can be significantly affected by changing the Fermi level or the properties of the ideal leads.

11. CONCLUSIONS

In this paper, we have presented an accurate Fourier analysis of aperiodic oscillations of conductance (Fig. 1) that were discovered in the classical experiments by Webb and Washburn [5]. The results obtained in this work reconcile two alternative scenarios indicated at the beginning of the paper. On the one hand, the Fourier spectrum is practically discrete, which confirms the concept of [8], according to which aperiodic oscillations of conductance are determined by the superposition of incommensurate harmonics. On the other hand, the shape of the spectrum resembles discrete white noise, whose properties are close to those of continuous noise. A more detailed analysis has revealed the presence of a continuous component, the smallness of which is explained in Section 5.

The paper discovers many qualitative moments that confirm the correctness of the interpretation presented. The frequencies of discrete lines weakly depend on the treatment procedure, which proves their objective origin. The “natural” origin of \( f(\lambda) \) found when eliminating shift oscillations confirms the correctness of the analogy with one-dimensional systems. The same is evidenced by the manifestations of exponential growth of the amplitude of the harmonics. The phase distribution of the coefficients \( A_\nu \) is consistent with their supposed randomness. The distribution of growth exponents and frequency differences is consistent with the theoretical results for the metallic regime. Microscopic estimates confirm the picture presented.

Universal conductance fluctuations have been discussed in many papers (see, for example, [20–40] and references therein), and it would be interesting to process the results of other experiments in the spirit of the present study.

APPENDIX A

Solution of Equation (19)

We restrict the analysis to the metallic region when typical values of \( \rho \) are small. Considering the eigenvalue problem for the operator on the right-hand side of Eq. (19) and restricting ourselves to the lowest order in \( \rho \), we have the equation

\[ -\lambda P = P_\rho + \rho P_{\rho\rho}. \]  

(A.1)

Assuming that \( \rho \) varies in the range from zero to \( R \), we impose the finiteness condition at \( \rho = 0 \) and the zero boundary condition at \( \rho = R \). Then the eigenvalues and eigenfunctions have the form

\[ \lambda_\nu = \mu_\nu^2/4R, \quad e_\nu(\rho) = J_0(\mu_\nu\sqrt{\rho/R}). \]  

(A.2)

where \( \mu_\nu \) are the roots of the Bessel function \( J_0(\cdot) \). Solving Eq. (19) with the initial condition (22) by expanding \( P(\rho) \) in terms of eigenfunctions (A.2), we have

\[ P(\rho,t) = \int_0^\infty \mu d\mu e^{-\mu^2} J_0(2\mu\sqrt{\rho/R}) J_0(2\mu\sqrt{\rho_0/R}), \]  

(A.3)

where \( t = \alpha(L - L_0) \) and it is taken into account that, for large \( R \), the spectrum of the eigenvalues \( \mu_\nu \) becomes quasicontinuous and the summation over \( s \) can be replaced by integration with respect to \( \mu \) using the asymptotics \( \mu_\nu = \pi s + \text{const} \) for large \( s \). Calculating the integral in (A.3), we obtain

\[ P(\rho,t) = \frac{1}{t} \exp \left\{ -\rho + \rho_0 \right\} I_0 \left( 2\sqrt{\rho_0} \right), \]  

(A.4)
where \( I_0(x) = J_0(ix) \). For \( \rho \leq t \) and \( t \gg \rho_0 \), distribution (A.4) transforms into (21), and, for \( \rho \approx \rho_0 \) and \( t \ll \rho_0 \), it becomes Gaussian,

\[
P(\rho, t) = \left( \frac{1}{4\pi \rho_0 t} \right)^{1/2} \exp \left\{- \frac{(\rho - \rho_0)^2}{4\rho_0 t} \right\}. \tag{A.5}
\]

The closeness of (A.4) to the Gaussian distribution allows us to characterize it by the first two moments. Multiplying (19) by \( \rho^n \) and integrating with respect to \( \rho \), we obtain evolution equations for the moments of the distribution \( P(\rho) \); their solution for the initial condition (22) has the form

\[
\langle \rho \rangle = -\frac{1}{2} + \frac{1 + 2\rho_0}{2} e^{2t}, \tag{A.6}
\]

\[
\langle \rho^2 \rangle = \frac{1}{3} - \frac{1 + 2\rho_0}{2} e^{2t} + \left[ \rho_0^2 + \frac{1 + 2\rho_0}{2} - \frac{1}{3} \right] e^{4t}.
\]

For small \( \rho \), the expressions are simplified,

\[
\langle \rho \rangle = \rho_0 + t, \quad \langle \rho^2 \rangle = \rho_0^2 + 4\rho_0 t + 2t^2, \tag{A.7}
\]

\[
\sigma^2 = 2\rho_0 t + t^2,
\]

and correspond to distribution (A.4). These results are valid for \( L > L_0 \). The description of evolution in the range \( 0 < L < L_0 \) is complicated by the need to satisfy two conditions (20) and (22), which is possible only under certain restrictions on the realization of the random potential. These restrictions are imposed on the interval \( (0, L_0) \) as a whole and are not significant for small \( L \) and \( L \) close to \( L_0 \). In the first case, we have \( \langle \rho \rangle = \sigma \) in accordance with distribution (21). In the second case, the situation is determined by the fact that the broadening of the distribution occurs symmetrically when \( L \) deviates to the right or left of \( L_0 \). For the quantity \( \langle \rho \rangle \), conditions (20) and (22) are automatically satisfied if \( \rho_0 \) is chosen equal to the mean value of distribution (21) for \( L = L_0 \). As a result of the aforesaid, we arrive at the picture shown in Fig. 8.

**APPENDIX B**

**Evolution of \( \langle \rho \rangle \) for the Initial Condition (22)**

The calculation of \( \langle \rho \rangle \) technically reduces to studying the evolution of the second moments for the transfer matrix with complex elements \( T_\gamma \) [8]:

\[
\begin{align*}
\langle z_1^{(0)} \rangle & = \langle T_{11}^{(0)} \rangle, \\
\langle z_2^{(0)} \rangle & = \langle T_{11}^{(0)} T_{12}^{(0)*} \rangle, \\
\langle z_3^{(0)} \rangle & = \langle T_{11}^{(0)*} T_{12}^{(0)} \rangle, \\
\langle z_4^{(0)} \rangle & = \langle T_{12}^{(0)*} \rangle.
\end{align*}
\tag{B.1}
\]

They satisfy the system of difference equations, whose general solution has the form [8]

\[
\begin{align*}
\langle z_1^{(0)} \rangle & = C_0 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} + \sum_{i=1}^{3} C_i \begin{pmatrix} e_2(x_i) \\ e_3(x_i) \end{pmatrix} \exp(x_i t),
\end{align*}
\tag{B.2}
\]

where \( x_1, x_2, x_3 \) are the roots of the first equation in (37) and

\[
e_2(x) = \frac{\mathcal{A} x + \mathcal{B}}{p(x)}, \quad e_3(x) = \frac{\mathcal{A} x + \mathcal{B}^*}{p(x)},
\]

\[
\mathcal{A} = 2\varepsilon^2 - 2i\Delta, \quad \mathcal{B} = 4\alpha\Delta + 4i\varepsilon^2(\alpha - \Delta), \tag{B.3}
\]

\[
p(x) = x^2 + 2\varepsilon^2 x + 4\Delta^2.
\]

Here \( \alpha = -\Delta_2\delta, \Delta = \Delta_1\delta, \delta \) and \( \varepsilon \) were defined after (23),

\[
\Delta_1 = \frac{1}{2} \left( \frac{k}{k} - \frac{k'}{k} \right), \quad \Delta_2 = \frac{1}{2} \left( \frac{k + k'}{k} \right),
\tag{B.4}
\]

where \( \frac{k}{k'} = k \) and \( \frac{k}{k'} = k \) are the Fermi momenta in the system under study and the ideal leads connected to it. In contrast to [8], as the initial condition, we take the general transfer matrix

\[
T = \begin{pmatrix}
\sqrt{\rho + 1} & e^{i\varphi} & \sqrt{\rho} e^{i\varphi} \\
\sqrt{\rho} e^{-i\varphi} & \sqrt{\rho + 1} & e^{-i\varphi} \\
e^{i\varphi} & e^{-i\varphi} & \sqrt{\rho + 1} + 1 & e^{-i\varphi}
\end{pmatrix}.
\tag{B.5}
\]

with \( \rho = \rho_0 \) and \( \psi = \theta - \varphi \), rather than the identity matrix. The quantity \( z_4^{(n)} \) immediately determines \( \langle \rho \rangle \), giving a general expression for the latter:

\[
\langle \rho \rangle = C_0 + C_1 e^{\varphi} + C_2 e^{-\varphi} + C_3 e^{\varphi},
\tag{B.6}
\]

where

\[
\begin{align*}
C_0 & = \frac{1}{2}, \quad C_1 = (-1)^{i+1} \left[ 1 + 2\rho_0 \right] \frac{Q}{2Q} \\
& + K_i \frac{R_i}{Q} + K_2 \frac{S_i}{Q}, \quad i = 1, 2, 3
\end{align*}
\tag{B.7}
\]

and

\[
Q_1 = (x_2 - x_3) p(x_1), \quad Q_2 = (x_1 - x_3) p(x_2), \quad Q_3 = (x_1 - x_2) p(x_3),
\]

\[
Q = Q_1 - Q_2 + Q_3 = x_1^2 (x_2 - x_3) - x_2^2 (x_3 - x_1) + x_3^2 (x_1 - x_2).
\]

\[
R_i = [x_2 p(x_3) - x_3 p(x_2)] p(x_1), \quad R_2 = [x_1 p(x_3) - x_3 p(x_1)] p(x_2), \quad R_3 = [x_1 p(x_2) - x_2 p(x_1)] p(x_3),
\]

\[
S_1 = [p(x_3) - p(x_2)] p(x_1), \quad S_2 = [p(x_1) - p(x_3)] p(x_2), \quad S_3 = [p(x_2) - p(x_1)] p(x_3),
\]

\[1^1\] Equation (19) has the same form if we set \( L = L_0 + l \) or \( L = L_0 - l \) and consider the evolution with respect to \( l \). This is clear from the scheme for deriving similar equations (see Appendix A in [41]).
\[ K_1 = \frac{e^2 \sin \psi - \Delta \cos \psi}{4[\alpha \Delta^2 + \epsilon^2(\alpha - \Delta)]} \sqrt{\rho_0(1 + \rho_0)}. \]
\[ K_2 = \frac{e^2(\alpha - \Delta)\cos \psi + \alpha \Delta \sin \psi}{2[\alpha \Delta^2 + \epsilon^2(\alpha - \Delta)]} \sqrt{\rho_0(1 + \rho_0)}. \]  

Using the asymptotics for the roots \(x_1, x_2, \) and \(x_3\) in the metallic regime [8], we arrive at the results (23) and (25). The first of them is valid for “natural” ideal leads,12 which differ from the studied system only by the absence of the random potential in them; in this case, \(k = k', \Delta_1 = 0\), and the oscillations appear in the first order in the small parameter \(\epsilon^2/\delta\). The result (25) is valid for foreign leads when \(\Delta_1 \neq 0\) and oscillations occur in the zero order in \(\epsilon^2/\delta\). Using the asymptotics of \(x_1, x_2, \) and \(x_3\) in the “critical” region [8] leads to the result (24), which is valid near the edge of the initial band; it is given for “natural” leads, since the situation for foreign leads is sufficiently characterized by the formulas given in [8].

**APPENDIX C**

**On the Choice of the Natural Origin**

According to the Onsager relations, conductance is an even function of the magnetic field \(B\), and, when \(x = B\), the function \(f(x)\) in (3) is even. Let us choose a smoothing function in the form \(G(x - a) + G(x + a)\) with even \(G(x)\), i.e., in a form symmetrized with respect to \(x\). Then the Fourier transform is real and coincides, up to sign, with its modulus; therefore, it does not contain shift oscillations. Now we eliminate the function \(G(x + a)\). The resulting Fourier transform \(F(a)\) of the function \(f(x)G(x - a)\) is complex; its real part is half the previous one and does not contain shift oscillations.13 The latter are also absent in \(\text{Im}F(a)\), since they equally affect the real and imaginary parts. After the shift \(x \rightarrow x + a\), the Fourier transform takes the form

\[ F(a) = e^{iax} \int f(x + a)G(x)e^{-iax} \, dx \]

and the resulting integral for \(a = \mu_0\) corresponds to that considered in Section 2, while the factor \(e^{iax}\) corresponds to the shift oscillations. However, the sign of \(a\) obtained in this case does not correspond to that found empirically.

The reason for the contradiction is that the Onsager symmetry distinguishes not only the value \(B = 0\), but also \(B = \infty\); it is the latter value that corresponds to the empirical situation. When choosing

\[ x = 1/B, \]

we can repeat all the previous arguments; however, the increments of \(B\) and \(x\) have opposite signs, and, for a qualitative agreement with Section 2, we should change the sign of \(a\). This will give rise to a factor \(e^{iax}\) with the correct sign of \(a\).

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