Hessian of Hausdorff dimension on purely imaginary directions
Martin Bridgeman, Beatrice Pozzetti, Andrés Sambarino, Anna Wienhard

To cite this version:
Martin Bridgeman, Beatrice Pozzetti, Andrés Sambarino, Anna Wienhard. Hessian of Hausdorff dimension on purely imaginary directions. Bulletin of the London Mathematical Society, 2022, 10.1112/blms.12612. hal-03027752

HAL Id: hal-03027752
https://hal.science/hal-03027752
Submitted on 27 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
HESSIAN OF HAUSDORFF DIMENSION ON PURELY IMAGINARY DIRECTIONS

MARTIN BRIDGEMAN, BEATRICE POZZETTI, ANDRÉS SAMBARINO, AND ANNA WIENHARD

Abstract. We extend classical results of Bridgeman-Taylor and McMullen on the Hessian of the Hausdorff dimension on quasi-Fuchsian space to the class of \( p_{1,1,2} \)-hyperconvex representations, a class introduced in [41] which includes small complex deformations of Hitchin representations and of \( \Theta \)-positive representations. We also prove that the Hessian of the Hausdorff dimension of the limit set at the inclusion \( \Gamma \rightarrow \text{PO}(n,1) \rightarrow \text{PU}(n,1) \) is positive definite when \( \Gamma \) is co-compact in \( \text{PO}(n,1) \) (unless \( n = 2 \) and the deformation is tangent to \( \mathcal{X}(\Gamma, \text{PO}(2,1)) \)).

Contents

1. Introduction .......................... 1
2. Anosov representations .......... 5
3. Thermodynamic formalism .. 10
4. Pressure forms .................. 13
5. Pluriharmonicity of length functions and its consequences 14
References .............................. 18

1. Introduction

One of the most interesting and well studied metrics on the Teichmüller space, the parameter space of hyperbolic structures on a closed surface \( S \) of genus \( g \geq 2 \), is the Weil-Petersson metric, a non-complete Riemannian metric. A celebrated result by B.-Taylor [14] and McMullen [36] gives a geometric interpretation of this metric in terms of dynamical invariants of quasi-Fuchsian representations.

Recall that the holonomy representation realizes the Teichmüller space \( \mathcal{T}(S) \) as a connected component of the character variety

\[
\mathcal{X}(\pi_1 S, \text{PSL}_2(\mathbb{R})) := \text{Hom}(\pi_1 S, \text{PSL}_2(\mathbb{R})) / \text{PSL}_2(\mathbb{R}),
\]

which, in turn, sits as a totally real submanifold of the complex character variety \( \mathcal{X}(\pi_1 S, \text{PSL}_2(\mathbb{C})) \), endowed with the complex structure \( J \) induced by the complex structure of the Lie group \( \text{PSL}_2(\mathbb{C}) \). A neighborhood of \( \mathcal{T}(S) \) in the complex character variety is given by quasi-Fuchsian space \( \Omega \mathcal{T}(S) \), the set of conjugacy classes

---

A. S. was partially financed by ANR DynGeo ANR-16-CE40-0025. B. P. and A. W. acknowledge funding by the Deutsche Forschungsgemeinschaft within the Priority Program SPP 2026 “Geometry at Infinity”. A. W. acknowledges funding by the European Research Council under ERC-Consolidator grant 614733, and by the Klaus-Tschira-Foundation. M. B. acknowledges funding by NSF Grants DMS-2005498.
Theorem 1.1 (B.-Taylor [14] -McMullen [36]). For each $\rho \in \mathcal{T}(S)$ and every differentiable curve $(\rho_t)_{t \in (-\epsilon, \epsilon)} \subset \mathcal{T}(S)$ with $\rho_0 = \rho$ it holds

$$\text{Hess} \ Hff(J \dot{\rho}) = \|\dot{\rho}\|_{WP}.$$
the Hausdorff dimension of the (image of the) limit curve $\xi^\rho: \partial \Gamma \to \mathcal{F}_a(G_C)$ for a(ny) Riemannian metric on $\mathcal{F}_a(G_C)$. It follows from Theorem 2.7 that $Hff_a$ is critical at $\mathcal{H}(S, G_R)$ and thus its Hausdorff is well defined.

**Corollary A.** For every $v \in T \mathcal{H}(S, G_R)$ and every $a \in \Pi$ one has

$$\text{Hess } Hff_a(Jv) = P^a(v).$$

Moreover, when $G_R = \text{PSL}_d(\mathbb{R})$ the Hessian of $Hff_a: \mathcal{X}_\Pi(\pi_1 S, \text{PSL}_d(\mathbb{C})) \to \mathbb{R}$, at a Hitchin point $\rho$, is strictly positive on every direction except $T_\rho \mathcal{H}_d(S)$, where it is degenerate. In particular the Hitchin locus is an isolated minimum for $Hff_a$.

The second statement follows directly from the first, together with the aforementioned non-degeneracy result by B.-Canary-Labourie-S. [12] for the spectral gap pressure form $P^a$.

Corollary A brings further evidence that the spectral gap pressure form is more geometric than the spectral radius pressure form, and shares more similarities to the classical Weil-Petersson metric. A key ingredient in its proof is the notion of the geometric than the spectral radius pressure form, and shares more similarities to the classical Weil-Petersson metric. A key ingredient in its proof is the notion of

Theorem B. Let $\Gamma$ be a word hyperbolic group with $\partial \Gamma$ homeomorphic to a circle and let $\rho \in \mathcal{X}_{\{a_1, a_2\}}(\Gamma, \text{PSL}_d(\mathbb{R}))$ be a regular point of the character variety $\mathcal{X}(\Gamma, \text{PSL}_d(\mathbb{C}))$. Then for every differentiable curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathcal{X}(\Gamma, \text{PSL}_d(\mathbb{R}))$ with $\rho_0 = \rho$ one has

$$\text{Hess } Hff_a(J\dot{\rho}) = P^a(\dot{\rho}).$$

Thanks to the work of [41, 40], Theorem B applies not only to Hitchin components but to $\Theta$-positive Anosov representations into indefinite orthogonal groups [24]. However, in all these cases, we do not know for which roots $a$ the associated pressure form is non-degenerate.

**Corollary B.** Let $\mathcal{P}(\Gamma, \text{SO}_0(p, q))$ denote the set of $\Theta$-positive Anosov representations. Then for every $v \in T \mathcal{P}(\Gamma, \text{SO}_0(p, q))$ and every $a \in \{a_1, \ldots, a_{p-2}\}$ one has

$$\text{Hess } Hff_a(Jv) = P^a(v).$$

The second main result of the paper is a generalization of B. [10] to Anosov representations of word hyperbolic groups that are not necessarily virtual surface groups.

**Theorem B.** Let $\Gamma$ be a word hyperbolic group and let $\rho \in \mathcal{X}_{\{a_1, a_2\}}(\Gamma, \text{PSL}_d(\mathbb{C}))$ be a smooth point. Assume moreover that

$$Hff_a: \mathcal{X}_{\{a_1, a_2\}}(\Gamma, \text{PSL}_d(\mathbb{C})) \to \mathbb{R}_+$$
is critical at \( \rho \). Then \( \text{Hess}_z \, \text{Hess}_x \) is positive semidefinite on a subspace of dimension at least half the real dimension of \( \mathcal{X}^b_{[\alpha, \omega]} (\Gamma, \text{PSL}_d(\mathbb{C})) \). If, furthermore, the pressure form \( P^n \) is non-degenerate, then there are no local maxima.

This theorem applies in particular to all convex cocompact Kleinian groups in \( \text{PSL}_2(\mathbb{C}) \), a result not covered by [10]. In general, admitting \((1, 1, 2)\)-hyperconvex representations is a relatively restrictive assumption on the group \( \Gamma \), as, for example, it implies that its boundary has dimension smaller than 2. However there are many classes of subgroups of \( \text{PSL}_2(\mathbb{C}) \) admitting \((1, 1, 2)\)-hyperconvex representations with Zariski dense image in \( \text{PSL}_d(\mathbb{C}) \). It is not yet known if the spectral gap pressure form is non-degenerate for such representations, and we hope that Theorem B will encourage research in this direction.

The proof of both Theorems A and B relies on pluriharmonicity of length functions. The mechanism behind both proofs is relatively versatile, and can be applied in many situations. As an example we use it in Corollary 5.4 to prove that, at a Fuchsian representation, the critical exponent \( h^{\omega_1}(\rho) \) relative to the first fundamental weight necessarily increases along purely imaginary deformations (see Section 5.5 for details).

Our last result is another application of the previous techniques in a rank one situation. Recall that a representation \( \rho : \Gamma \to \text{PU}(n, 1) \) is convex-cocompact if and only if it is projective Anosov when \( \text{PU}(n, 1) \) is considered as a subgroup of \( \text{PGL}_{n+1}(\mathbb{C}) \) through the standard inclusion. Moreover, for \( \gamma \in \Gamma \) the real length of the associated closed geodesic is the spectral radius \( \omega_1 \left( \lambda(\rho(\gamma)) \right) \).

If \( \rho : \Gamma \to \text{PU}(n, 1) \) is convex co-compact then we denote by \( \text{Hff}_{\chi \text{CH}^n}(\rho) \) the Hausdorff dimension the limit set of \( \Gamma \) in the visual boundary of the complex-hyperbolic space, with respect to a visual metric. A celebrated result by Sullivan [45] in real-hyperbolic space, and further extended to arbitrary negatively curved manifolds by Yue [48], asserts that if \( \rho : \Gamma \to \text{PU}(n, 1) \) is convex-co-compact then
\[
\text{Hff}_{\chi \text{CH}^n}(\Gamma) = h^{\omega_1}(\rho).
\]

Assume now that \( \Gamma \) is a co-compact lattice of \( \text{PSO}(n, 1) \) and denote by \( \iota \) the inclusion
\[
\iota : \Gamma \to \text{PSU}(n, 1),
\]
where the inclusion of \( \text{PSO}(n, 1) \to \text{PSU}(n, 1) \) is given by extending the coefficients. Bourdon [6] proved that \( \iota \) is a global rigid minima for \( \text{Hff}_{\chi \text{CH}^n} \) among convex co-compact representations of \( \Gamma \) in \( \text{PU}(n, 1) \). In section 5.4 we prove the following strengthening.

**Theorem C.** Assume \( \iota \) is a regular point of the character variety \( \mathcal{X}(\Gamma, \text{PSU}(n, 1)) \), then \( \text{Hess}_\iota \, \text{Hff}_{\chi \text{CH}^n} \) is positive definite in any direction not tangent to \( \mathcal{X}(\Gamma, \text{PSO}(n, 1)) \).

If \( n > 2 \) then Mostow’s classical rigidity result states that \( \iota \) is an isolated point of \( \mathcal{X}(\Gamma, \text{PSO}(n, 1)) \) so Theorem C implies that the Hessian of the Hausdorff dimension at \( \iota \) is positive definite.

**Remark 1.2.** If \( \rho \) is a small deformation of \( \iota \) in \( \text{PSU}(n, 1) \) then it is a convex co-compact subgroup of \( \text{PSU}(n, 1) \) and moreover, by P.-S.-W. [41, Corollary 8.5], the Hausdorff dimension of the limit set for a Riemannian metric on \( \mathcal{CH}^n \) also coincides with \( h^{\omega_1}(\rho) \), so in Theorem C we can consider \( \text{Hff}_{\chi \text{CH}^n} \) either as the Hausdorff dimension for a visual metric or for a Riemannian metric.
Outline of the paper. In Section 2 we discuss the background on Anosov representations needed in the paper: after reviewing the basic definitions we discuss, in Section 2.2, the results of [41] which singled out $(1,1,2)$-hyperconvex representations. In Section 2.3 we recall the basic facts about higher rank Teichmüller theories needed to deduce Corollaries A and B from Theorem A; finally in Section 2.4 we discuss an important dynamical viewpoint on Anosov representations: these can be thought of as reparametrizations of the geodesic flow of $\Gamma$, and are thus amenable to the thermodynamic formalism. In Section 3 we discuss the thermodynamic formalism needed to define, in Section 4, the pressure forms. We conclude the paper in Section 5 introducing the main technical tool of the paper, pluriharmonicity of dynamical intersection, which is directly used to prove Theorems A, B and C.

2. Anosov representations

In this section we introduce the necessary background on Anosov representations, and recall how they give rise to reparametrizations of the geodesic flow.

2.1. Basic notions. We recall the Cartan and the Jordan-Lyapunov projections and the characterization of Anosov representations we are going to use.

Let $G$ be a semisimple real-algebraic Lie group of non-compact type with finite center, for $K = \mathbb{R}$ or $\mathbb{C}$ denote by $G_K$ the group of the $K$-points of $G$. Fix a maximal compact subgroup $K < G_K$ with Lie algebra $\mathfrak{k}$. We denote by $E < \mathfrak{k}$ a Cartan subalgebra, and by $\Delta \subset \mathfrak{e}^+$ a choice of simple roots. This corresponds to the choice of a Weyl chamber in $E$, which we will denote by $E^+$. In the case $G = \text{PSL}_d$ we identify $E^+$ with

$$\{ (x_1, \ldots, x_d) \in \mathbb{R}^d | x_1 \geq \ldots \geq x_d, \sum x_i = 0 \}.$$

Every element $g \in G_K$ can be written as a product

$$g = k_1 \exp (\sigma(g)) k_2$$

for $k_1, k_2 \in K$ and a unique element $\sigma(g) \in E^+$, the Cartan projection of $g$. If $G_\mathbb{R} = \text{PSL}_d(\mathbb{R})$, the numbers $\sigma_i(g)$ are the logarithms of the square roots of the eigenvalues of the symmetric matrix $g^t g$. If $a \in \Delta$ then we denote by $\omega_a$ its associated fundamental weight.

Let $\Theta \subset \Delta$ be a subset of simple roots. We denote by $P_\Theta < G$ the associated parabolic subgroup, by $\tilde{P}_\Theta$ the opposite associated parabolic group and by

$$E_\Theta := \bigcap_{a \in \Delta \setminus \Theta} \ker(a)$$

the Lie algebra of the center of the Levi group $P_\Theta \cap \tilde{P}_\Theta$. It comes equipped with the natural projection

$$p_\Theta : E \to E_\Theta$$

parallel to $\bigcap_{a \in \Theta} \ker a$. Finally let $E^*_\Theta < E^*$ be the subspace generated by the fundamental weights associated to elements in $\Theta$

$$E^*_\Theta := \langle \omega_a | a \in \Theta \rangle = \{ \varphi \in \mathbb{E}^* | \varphi \circ p_\Theta = \varphi \}.$$

One has that $E^*_\Delta = E$ and $P_\Delta$ is a minimal parabolic subgroup.

Let $\Gamma$ be a finitely generated discrete group and denote by $||$ the word length for a fixed finite symmetric generating set.
Definition 2.1. Let $\Theta \subset \Delta$. A representation $\rho : \Gamma \to G_K$ is $\Theta$-Anosov if there exist positive constants $c, \mu$ such that for all $\gamma \in \Gamma$ and $a \in \Theta$ one has

$$a\left(\sigma(\rho(\gamma))\right) \geq \mu|\gamma| - c.$$  \hspace{1cm} (2)

A $\{a_1\}$-Anosov representation $\rho : \Gamma \to \PGL_d(K)$ will be called projective Anosov.

Note that this is not the original definition given in Labourie and Guichard-W. [32, 23], but a characterization due to Kapovich-Leeb-Porti and Bochi-Potrie-S. [27, 4]. Note also that there is a recent characterization by Kassel-Potrie [29] only in terms of the Jordan-Lyapunov projection (see below for the definition) rather than the Cartan projection.

Anosov representations are quasi-isometric embeddings, thus in particular they are injective and have discrete image. It was proven in [27] (see also [4]) that only word hyperbolic groups admit Anosov representations; we will denote by $B_\Gamma$ the Gromov boundary of the group $\Gamma$.

A key property of Anosov representations is the existence of equivariant boundary maps with good dynamical properties [32, 23, 22, 27, 4]. With our definition, the existence of boundary maps for such representations is a Theorem of [27] and [4]. From now on we will restrict ourselves, without loss of generality, to self-opposite subsets $\Theta \subset \Delta$.

Theorem 2.2 (Kapovich-Leeb-Porti [27]). Let $\rho : \Gamma \to G_K$ be $\Theta$-Anosov. Then there exist a unique dynamics preserving, continuous, transverse equivariant boundary map

$$\xi^\Theta : \partial \Gamma \to G_K/P_\Theta.$$

If $G = \PGL_d$ and $\Theta = \{a_r\}$, then $G_{a_r}/P_\Theta = S_r(K^d)$ and we write $\xi^r = \xi^\{a_r\}$.

It was proven in [23] that it is possible to reduce the study of general $\{a\}$-Anosov representations to projective Anosov representations. Indeed one can use the following result by Tits, since for the representations $\Lambda_a$ below one has

$$a_1\left(\sigma(\Lambda_a(\rho(\gamma)))\right) = a\left(\sigma(\rho(\gamma))\right).$$

Proposition 2.3 (Tits [46]). For every $a \in \Delta$ there exists an irreducible proximal representation $\Lambda_a : G_R \to \PGL_d(R)$ whose highest restricted weight is $l_\omega_a$ for some $l \in \mathbb{N}$.

Recall that the Jordan decomposition states that every $g \in G_K$ can be written uniquely as a commuting product $g = g_e g_h g_n$, where $g_e$ is elliptic, $g_h$ is $\mathbb{R}$-split and $g_n$ is unipotent. The Jordan-Lyapunov projection $\lambda : G_K \to \mathbb{E}^+$ is defined by the logarithm of the eigenvalues of $g_h$ with multiplicities and in decreasing order. If $G = \PGL_d$, this corresponds to the logarithm of the modulus of the roots of the characteristic polynomial of $g$ with multiplicities and in decreasing order, and we denote by

$$\lambda(g) = (\lambda_1(g), \ldots, \lambda_d(g)) \in \{(x_1, \ldots, x_d) \in \mathbb{R}^d | x_1 \geq \ldots \geq x_d, \sum x_i = 0\}$$

its coordinates.

We will denote by $\Lambda_\rho \subset \mathbb{E}^+$ the limit cone of the subgroup $\rho(\Gamma) < G_K$. This is the cone given by

$$\Lambda_\rho := \overline{\{x \in \mathbb{R}^+ | \lambda(\rho(\gamma)) \gamma \in \Gamma\}}.$$
It was proven by Benoist [3] that, provided \( \rho(\Gamma) \) is Zariski dense, \( \Lambda_\rho \) is convex and has non-empty interior.

For every functional \( \varphi \in E^* \) that is positive on the limit cone \( \Lambda_\rho \) we denote by \( h^\varphi(\rho) \) the critical exponent of the Dirichlet series

\[
s \mapsto \sum_{\gamma \in \Gamma} e^{-s \varphi(\sigma(\rho(\gamma)))},
\]

it can be computed as

\[
h^\varphi(\rho) = \inf \left\{ s : \sum_{\gamma \in \Gamma} e^{-s \varphi(\sigma(\rho(\gamma)))} < \infty \right\} = \sup \left\{ s : \sum_{\gamma \in \Gamma} e^{-s \varphi(\sigma(\rho(\gamma)))} = \infty \right\}.
\]

2.2. Hyperconvex representations. We begin with the following definition from [41]:

**Definition.** A \( \{a_1, a_2\}\)-Anosov representation \( \rho : \Gamma \to \operatorname{PGL}_d(\mathbb{K}) \) is called \( (1,1,2) \)-hyperconvex if for every triple of pairwise distinct points \( x, y, z \in \partial \Gamma \) one has

\[
(\xi^1_\rho(x) \oplus \xi^1_\rho(y)) \cap \xi^{d-2}_\rho(z) = \{0\}.
\]

The following is a direct consequence of the uniqueness of boundary maps:

**Lemma 2.4.** The complexification of a real hyperconvex representation is hyperconvex (over \( \mathbb{C} \)).

An important property of \( (1,1,2) \)-hyperconvex representations, established in [41] is that, for these representations, the Hausdorff dimension of the limit curve for a Riemannian metric on \( \mathbb{P}(\mathbb{K}^d) \) computes the critical exponent for the first simple root. If \( \rho = \{a_1\}\)-Anosov, the root \( a_1 \) is positive on the limit cone (recall Equation (2)) and thus its critical exponent is well defined. We then have the following.

**Theorem 2.5** (P.-S.-W. [41]). Let \( \rho : \Gamma \to \operatorname{PGL}_d(\mathbb{K}) \) be \( (1,1,2) \)-hyperconvex, then

\[
\dim_{\text{Haus}} (\xi^1(\partial \Gamma)) = h^2_{\rho^1}.
\]

A second important property of \( (1,1,2) \)-hyperconvex representations into \( \operatorname{PSL}_d(\mathbb{R}) \) was established in P.-S.-W. [41] (and independently in Zimmer-Zhang [49]) is the following: if \( \Gamma \) is such that \( \partial \Gamma \) is homeomorphic to a circle, then the image of the boundary map \( \xi^1_\rho \) is a \( C^1 \)-curve. As a result we get

**Theorem 2.6** (P.-S.-W. [41]). Let \( \rho : \Gamma \to \operatorname{PSL}_d(\mathbb{R}) \) be \( (1,1,2) \)-hyperconvex. If \( \partial \Gamma \) is homeomorphic to a circle then \( \dim_{\text{Haus}} (\xi^1(\partial \Gamma)) = 1 \).

Thus, for fundamental groups of surfaces, the Hausdorff dimension is constant and minimal on the real \( (1,1,2) \)-hyperconvex locus.

2.3. Higher rank Teichmüller Theory. A higher Teichmüller space is a union of connected components of a character variety \( X(\pi_1 \Sigma, G_\mathbb{R}) \) only consisting of Anosov representations.

Historically, the first family of higher Teichmüller spaces are Hitchin components. They arise whenever \( G_\mathbb{R} \) is a center free real-split simple Lie group. In this case there is a unique principal subalgebra \( \mathfrak{sl}_2(\mathbb{R}) \subset \mathfrak{g}_\mathbb{R} \) characterized by the property that the centralizer of \( \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \) has minimal dimension (Kostant [31]). The Hitchin component \( \mathcal{H}(\Sigma, G_\mathbb{R}) \subset X(\pi_1 \Sigma, G_\mathbb{R}) \) is the connected component containing Fuchsian representations: the composition of the holonomy of a hyperbolization.
π₁S → PSL₂(ℝ) and the morphism PSL₂(ℝ) → Gₘ induced by the inclusion of the principal subalgebra. Representations in the Hitchin component are Anosov with respect to the minimal parabolic subgroup [32]. Furthermore, representations in the Hitchin component are hyperconvex:

**Theorem 2.7** ([32, 41, 44]). Let Gₘ be a simple split center-free real group. For every ρ ∈ Η(S, Gₘ) and a ∈ Π the representation Λₐρ : π₁S → PSL(V, ℝ) is (1, 1, 2)-hyperconvex.

Proof. This was established, for the groups Gₘ = PSLₐ(ℝ), PSp(2n, ℝ), PSO(n, n + 1) or the split form of the exceptional complex Lie group G₂, by Labourie [32], for G = SO(n, n) by P.-S.-W. [41, Theorem 9.9]. The general case follows from S. [44, Remark 5.14]. □

The second family of higher Teichmüller spaces are spaces of maximal representations in Hermitian Lie groups Gₘ [16]. Our results here do not apply in this setting. Maximal representations are, in general, only Anosov with respect to one root a, which therefore doesn’t belong to the Levi-Anosov subspace. Even though we know that for maximal representations the critical exponent hₐ is constant and equal to one ([40, Theorem 1.2]), it is not clear if a spectral gap pressure metric Pₐ can be constructed in this case. Moreover, since maximal representations are, in general, not (1,1,2)-hyperconvex, it is not known if, for complex deformations in ρ : Γ → Gₘ, the critical exponent hₐ equals the Hausdorff dimension of the limit set.

Conjecturally there are two further families of higher Teichmüller spaces, given by Θ-positive representations as introduced in [24, 47]. Θ-positive representations exist when Gₘ is locally isomorphic to SO(p, q), p < q, or when Gₘ belongs to a special family of exceptional Lie groups. In a forthcoming article, Guichard, Labourie and W. [25] prove that Θ-positive representations are Θ-Anosov; in particular, in the case of SO(p, q), p < q, Θ-positive representations are Anosov with respect to the first p − 1 roots. Since this article is not yet available, we will here consider Θ-positive Anosov representations, and use the following result from [40].

**Theorem 2.8** ([40, Theorem 10.1]). Let ρ : Γ → SO(p, q) be a Θ-positive Θ-Anosov representation. For every a ∈ {a₁, . . . , aₚ₋₂} the representation Λₐρ : π₁S → PSL(V, ℝ) is (1, 1, 2)-hyperconvex.

Note that when Gₘ admits a Θ-positive structure, Guichard and W. conjectured several years ago, see also [24, 47], that then there exist additional connected components (namely the conjectured components of Θ-positive representations) in the representation variety, which are not distinguished by characteristic numbers. This conjecture has been proven by Collier [17] in the case of Gₘ = SO(n, n + 1) and by Aparicio-Arroyo, Bradlow, Collier, García-Prada, Gothen, and Oliveira [1] in the case of Gₘ = SO(p, q) using methods from the theory of Higgs bundles.

---

²In fact, Labourie proved this for Gₘ = PSLₐ(ℝ), which implies the result also for symplectic groups and odd-dimensional orthogonal groups. Fock and Goncharov gave a characterization of representations in the Hitchin component as positive representations, from which the Anosov property can be deduced with a little work.
2.4. Reparametrizations of geodesic flows. In this section we describe a very useful dynamical viewpoint on Anosov representations from S. [43] and B.-Canary-Labourie-S. [11], which makes them amenable to the thermodynamic formalism: Any Anosov representation gives rise to a reparametrization of the geodesic flow.

Given a hyperbolic group $\Gamma$ we denote by $\mathbb{U}\Gamma$ the Gromov geodesic flow; this is a metric space endowed with a topologically transitive flow $\phi$ whose periodic orbits correspond to conjugacy classes in $\Gamma$. If $\Gamma$ admits an Anosov representation then $\phi$ is moreover metric Anosov [11]. Note that, if $\Gamma$ is the fundamental group of a compact negatively curved manifold $M$, we can choose $\mathbb{U}\Gamma = UM$; more generally, whenever $\Gamma$ admits an Anosov representation, its geodesic flow can be explicitly constructed with the aid of the associated boundary maps [11, Theorem 1.10].

If $\alpha > 0$, we denote by $\text{Hol}_\alpha(\mathbb{U}\Gamma, \mathbb{R})$ the space of $\alpha$-H"older continuous functions on $\mathbb{U}\Gamma$ and by $\text{Hol}(\mathbb{U}\Gamma, \mathbb{R})$ is the space of all H"older continuous functions. If $f \in \text{Hol}(\mathbb{U}\Gamma, \mathbb{R})$ and $a \in [\Gamma]$ is a conjugacy class, then we define the $f$-period of $a$ by

$$\ell_f(a) = \int_0^{\ell(a)} f(\phi_t(x))dt$$

where $x \in a$. If $f \in \text{Hol}_\alpha(\mathbb{U}\Gamma, \mathbb{R})$, we obtain a new flow $\phi^f$ on $\mathbb{U}\Gamma$ called the reparametrization of $\phi$ by $f$. The flow $\phi^f$ is given by the formula

$$\phi^f_t(x) = \phi^f_{k_f(x,t)}(x)$$ (3)

where $k_f(x,t) = \int_0^t f(\phi_s(x))ds$ for all $x \in X$ and $t \in \mathbb{R}$. The flow $\phi^f$ is H"older equivalent to $\phi$ and if $a \in [\Gamma]$, then $\ell_f(a)$ is the period of $a$ in the flow $\phi^f$.

In [11, Section 4] B.-Canary-Labourie-S. associate to any projective Anosov representation a reparametrization of the geodesic flow $\mathbb{U}\Gamma$. They prove the following statement, the second part is proved in [11, Section 6].

**Proposition 2.9 ([11]).** Let $\rho : \Gamma \to \text{PGL}_d(\mathbb{R})$ be a projective Anosov representation. Then there exists a positive H"older-continuous function $f_{\rho}^{\lambda_1} : \mathbb{U}\Gamma \to \mathbb{R}_{>0}$ such that for every conjugacy class $[\gamma] \in [\Gamma]$ one has

$$\ell_{\gamma}(f_{\rho}^{\lambda_1}) = \lambda_1(\rho(\gamma)).$$

Moreover, if $\{\rho_a\}_{a \in D}$ is an analytic family of such representations, then one can choose $f_{\rho_a}^{\lambda_1}$ so that the function $u \mapsto f_{\rho_a}^{\lambda_1}$ is analytic.

Proposition 2.9 together with Tits Proposition 2.3 directly give the following from Potrie-S. [39], where $K = \mathbb{R}$ case is treated. When $K = \mathbb{C}$ the result follows at once by considering $G_C$ as a real group. Recall equation (1) for the definition of $p_\Theta : E \to E_\Theta$.

**Corollary 2.10 ([39, Cor. 4.5]).** Let $\rho : \Gamma \to G_K$ be $\Theta$-Anosov, then there exists a positive H"older-continuous function $f_{\rho}^{\lambda} : \mathbb{U}\Gamma \to E_\Theta$ such that for every conjugacy class $[\gamma] \in [\Gamma]$ one has

$$\ell_{\gamma}(f_{\rho}^{\lambda}) = p_\Theta \left( \lambda(\rho(\gamma)) \right).$$

Moreover, if $\{\rho_a\}_{a \in D}$ is an analytic family of such representations, then one can choose $f_{\rho_a}^{\lambda}$ so that the function $u \mapsto f_{\rho_a}^{\lambda}$ is analytic.

Thus, Corollary 2.10 readily implies that if $\rho$ is $\Theta$-Anosov then for every $\varphi \in (E_\Theta)^*$ that is strictly positive on $\Lambda_\rho - \{0\}$ there exists a reparametrization of the
geodesic flow of $\Gamma$ whose periods are given by
\[ \varphi(\lambda(\rho(\gamma))) \].

namely, if we denote by $f^x_\rho = \varphi(f^\theta_\rho)$ then one considers the flow $\phi^{f^x_\rho}$. We will need in the following that, in this situation, the critical exponent $h^x(\rho)$ is also the entropy of the flow $\phi^{f^x_\rho}$. This can be found for example in Ledrappier [33], S. [43] and on Glorieux-Monclair-Tholozan [21] for the general version.

**Proposition 2.11.** Let $\rho : \Gamma \rightarrow G_K$ be $\Theta$-Anosov. For each $\varphi \in E^{\ast}_\Theta$ strictly positive on $\Lambda_\rho - \{0\}$ it holds that
\[
h^x(\rho) = \lim_{T \rightarrow \infty} \frac{\log \# \{ \gamma \in [\Gamma] \mid \varphi(\lambda(\rho(\gamma))) < T \}}{T}.
\]

This applies, in particular, to the root $a_1$ if a representation $\rho$ is $(1,1,2)$-hyper-convex.

### 3. Thermodynamic Formalism

We now briefly describe the thermodynamic formalism introduced by Bowen, Ruelle, Parry, Pollicott (among others), and in particular the pressure function on the space of Hölder observables on a metric space with a Hölder flow (see [42]). This will then be used, in Section 4, to define various pressure forms $P^x$ on subsets of the representation variety $X(\Gamma, PSL_d(\mathbb{R}))$ by assigning to each representation $\rho$ the Hölder function $f^x_\rho$ on the geodesic flow space $U\Gamma$ of the group.

For a moment we forget about representations and let $X$ be a compact metric space with a Hölder continuous flow $\phi = \{\phi_t : X \rightarrow X\}_{t \in \mathbb{R}}$ without fixed points. We denote by $O$ the collection of periodic orbits of the flow $\phi$. For $a \in O$, we let $\ell(a)$ be the length of the periodic orbit $a$.

As in Section 2.4 we denote by $\text{Hol}_\alpha(X, \mathbb{R})$ the space of $\alpha$-Hölder continuous functions on $X$ for some $\alpha > 0$, and we set the $f$-period of $a \in O$ to be
\[
\ell_f(a) = \int_0^{\ell(a)} f(\phi_t(x)) dt.
\]

Two maps $f, g \in \text{Hol}_\alpha(X, \mathbb{R})$ are called Livšic cohomologous if there exists $U : X \rightarrow \mathbb{R}$ such that, for all $x \in X$, then
\[
f(x) - g(x) = \left. \frac{\partial}{\partial t} \right|_{t=0} U(\phi_t x).
\]

It follows that if $f$ and $g$ are Livšic cohomologous then $\ell_f(a) = \ell_g(a)$ for all $a \in O$. If $f \in \text{Hol}_\alpha(X, \mathbb{R}_+)$, we denote by $\phi^f$ the reparametrization of $\phi$ by $f$, which is the flow on $X$ defined by (3).

We let $\mathcal{M}_\phi$ be the set of $\phi$-invariant probability measures on $X$. In particular if $\delta_a$ is the Lebesgue measure on the periodic orbit $a$, then $\delta_a = \delta_0/\ell(a) \in \mathcal{M}_\phi$. For $\mu \in \mathcal{M}_\phi$ we denote by $h(\phi, \mu)$ its metric entropy. Then, for $f \in \text{Hol}_\alpha(X, \mathbb{R})$, the topological pressure is
\[
P(f) = \sup_{m \in \mathcal{M}_\phi} \left\{ h(\phi, m) + \int_X f dm \right\}.
\]

\(^3\text{Recall that for every } \varphi \in (E^{\ast}_\Theta)^* \text{ one has } \varphi \circ p_{\Theta} = \varphi.\)
Note that the topological pressure $P$ depends on the flow $\phi$, but we will omit this in the notation. The topological pressure $P$ depends on the flow $\phi$, but we will omit this in the notation. The topological entropy of a flow is given by $h_{\text{top}}(\phi) = P_0(0)$. A measure $m_\phi$ that attains this supremum is called an equilibrium state for $\phi$ and an equilibrium state for the zero function is called a measure of maximal entropy.

We note that $P(\phi)$ only depends on the Livšic cohomology class of $\phi$.

**Lemma 3.1** (S. [43, Lemma 2.4]). Let $\phi$ be a Hölder continuous flow on a compact metric space $X$ and $f \in \text{Hol}_\alpha(X, \mathbb{R}_+)$. Then

$$P(-hf) = 0$$

if and only if $h = h_{\text{top}}(\phi f)$. Moreover, if $m$ is an equilibrium state of $-h_{\text{top}}(\phi)f$, then $fm$ is a positive multiple of a measure of maximal entropy for the flow $\phi f$.

We now restrict to transitive metric Anosov flows. In the manifold setting a metric Anosov flow $\phi$ corresponds to a standard Anosov flow where the unit tangent bundle of $X$ has a $\phi$-invariant decomposition $T_1(X) = E_- \oplus E_0 \oplus E_+$ where $E_-$ is contracting under the flow, $E_0$ is the direction of the flow and $E_+$ is contracting under the flow reverse flow of $\phi$ (see [38] for details). We have the following theorem of Livšic.

**Theorem 3.2** (Livšic’s Theorem, [35]). Let $\phi$ be a transitive metric Anosov flow. If $f \in \text{Hol}_\alpha(X, \mathbb{R})$ then $\ell_f(a) = 0$ for all $a \in O$ if and only if $f$ is Livšic cohomologous to 0.

It follows that for metric Anosov flows, the Livšic cohomology class of $f$ is determined by its periods.

Given $f \in \text{Hol}_\alpha(X, \mathbb{R})$ we let $R_T(f) = \{a \in O \mid \ell_f(a) \leq T\}$.

Then we have the following:

**Theorem 3.3** (Bowen [7], Bowen-Ruelle [9], Pollicott [38]). Let $\phi$ be a transitive metric Anosov flow and $f \in \text{Hol}_\alpha(X, \mathbb{R}_+)$ nowhere vanishing. Then

$$h(f) = \lim_{T \to \infty} \frac{\log \#R_T(f)}{T} = h_{\text{top}}(\phi f)$$

is finite and positive. Moreover for all $g \in \text{Hol}_\alpha(X, \mathbb{R})$ there exists a unique equilibrium state $m_g$ for $g$. The measure of maximal entropy $\mu_\phi$ for the flow $\phi$ is

$$\mu_\phi = \lim_{T \to \infty} \frac{1}{\#R_T(1)} \sum_{a \in R_T(1)} \frac{\delta_a}{\ell(a)}.$$

Furthermore for Anosov flows the derivatives of the Pressure function satisfy the following.

**Proposition 3.4** (Parry-Pollicott [37]). Let $\phi$ be a transitive metric Anosov flow and $f, g \in \text{Hol}_\alpha(X, \mathbb{R})$. Then

(i) The function $t \to P(f + tg)$ is analytic

(ii) The first derivative satisfies

$$\frac{\partial P(f + tg)}{\partial t} \bigg|_{t=0} = g dm_f,$$

where $m_f$ is the equilibrium state for $f$. 
(iii) If \( \int g \, dm_f = 0 \) (mean-zero) then
\[
\frac{\partial^2 P(f + t g)}{\partial t^2} \bigg|_{t=0} = \lim_{T \to \infty} \int_0^T \left( \int_0^T g(\phi_s(x)) \, ds \right)^2 \, dm_f(x) = \text{Var}(g, m_f).
\]

(iv) If \( \text{Var}(g, m_f) = 0 \) then \( g \) is Livščič cohomologous to zero.

Using the above, in [36] McMullen defined the Pressure semi-norm as follows. We let \( \mathcal{P}(X) \) be the space of pressure zero functions, i.e.
\[ \mathcal{P}(X) = \{ F \in \text{Hol}(X, \mathbb{R}) \mid P(F) = 0 \}. \]
Then by Proposition 3.4(ii), the tangent space to \( \mathcal{P}(X) \) at \( F \) can be identified with
\[ T_F(\mathcal{P}(X)) = \left\{ g \in \text{Hol}(X, \mathbb{R}) \mid \int g \, dm_F = 0 \right\}, \]
where \( m_F \) is the equilibrium state for \( F \). Then the pressure semi-norm of \( g \in T_F(\mathcal{P}(X)) \) is
\[ \mathbf{P}(g) = -\frac{\text{Var}(g, m_F)}{\int F \, dm_F}. \]
By Proposition 3.4 it follows that \( \mathbf{P}(g) \) only depends on the Livščič-cohomology class \([g]\) and is positive definite in the sense that it is zero if and only if \([g] = 0\). Therefore it can be considered as a (positive-definite) metric on the space of Livščič cohomology classes.

The dynamical intersection is defined in [11] as follows; if \( f, g \in \text{Hol}_c(X, \mathbb{R}) \) are positive, then their dynamical intersection is
\[
\mathbf{I}(f, g) = \lim_{T \to \infty} \frac{1}{\# R_T(f)} \sum_{a \in R_T(f)} \ell_g(a) \ell_f(a) = \frac{\int \mathcal{L}_{m_{-h_f}} g \, dm_{-h_f}}{\int \mathcal{L}_{m_{-h_f}} f \, dm_{-h_f}}. \tag{4}
\]
The last equality follows from [11, Sec. 3.4]. Similar definitions have been studied in different situations, for example by Bonahon [5], Burger [15] and Knieper [30].

The renormalized dynamical intersection is
\[ \mathbf{J}(f, g) := \frac{h(g)}{h(f)} \mathbf{I}(f, g). \]

**Proposition 3.5** (B.-Canary-Labourie-S. [11, Proposition 3.8]). For every pair of positive Hölder-continuous functions \( f \) and \( g \) one has \( \mathbf{J}(f, g) \geq 1 \). In particular \( \mathbf{J}(f, \cdot) \) is critical at \( f \) which gives
\[
\frac{\partial}{\partial t} \bigg|_{t=0} \log h(f_t) = \frac{\partial}{\partial t} \bigg|_{t=0} \mathbf{I}(f, f_t), \tag{5}
\]
where \((f_t)_{t \in (-\epsilon, \epsilon)}\) is a \( C^1 \) curve of positive Hölder-continuous functions with \( f_0 = f \).

Then we have:

**Theorem 3.6** (B.-Canary-Labourie-S. [11]). Let \( \phi \) be a transitive metric Anosov flow on a compact metric space \( X \). If \( f_t \in \text{Hol}(X, \mathbb{R}), t \in (-1, 1) \) is a 1-parameter family and \( F_t = -h_{f_t} f_t \) then
\[
\frac{\partial^2}{\partial t^2} \bigg|_{t=0} \mathbf{J}(f_0, f_t) = \mathbf{P}(\dot{F}_0)
\]
The following proposition characterizes degenerate vectors for the second derivative of $J$.

**Proposition 3.7** (B.-Canary-Labourie-S. [11, Lemma 9.3]). Let $(f_t)_{t \in (-\varepsilon,\varepsilon)}$ be a $C^1$ curve of positive Hölder-continuous functions. Then $(\partial^2/\partial t^2)|_{t=0}J(f_0,f_t) = 0$ if and only if for every periodic orbit $\tau$ one has

$$\frac{\partial}{\partial t} \bigg|_{t=0} \Phi(f_t) = 0.$$

### 4. Pressure forms

Now we will apply the thermodynamic formalism to representations. For this we make use of the interpretation of a $\Theta$-Anosov representation as a reparametrization of the geodesic flow as explained in Section 2.4.

Given any functional $\varphi \in E_0^*$ that is positive on the limit cone, one can associate a reparametrization $f^\varphi_\rho$ of the geodesic flow on $\Gamma$. Here we describe in detail two special cases of this construction which play an important role in the paper.

#### 4.1. Spectral radius pressure form.

Let $\rho, \eta$ be two projective Anosov representations (with possibly different target groups). They both give rise to reparametrizations $f^\omega_\rho$ and $f^\omega_\eta$, where $\omega_1$ is the first fundamental weight.

We define the spectral radius dynamical intersection of the two projective-Anosov representations $\rho, \eta$ to be the dynamical intersection between $f^\omega_\rho$ and $f^\omega_\eta$:

$$\Gamma^\omega(\rho, \eta) = I(f^\omega_\rho, f^\omega_\eta).$$

Analogously we define $J^\omega(\rho, \eta)$. Moreover, given a $C^1$ curve $(\rho_t)_{t \in (-\varepsilon,\varepsilon)}$ of projective Anosov representations the spectral radius pressure norm of $\rho_0$ is defined by

$$P^\omega(\rho_0) = \left. \frac{\partial^2}{\partial t^2} \right|_{t=0} J^\omega(\rho_0, \rho_t) \geq 0.$$

The spectral radius pressure norm induces a positive semidefinite symmetric bilinear two form at the smooth points of $\{a_1, a_2\}$-Anosov representations. However positive semi-definiteness is as far as thermodynamics goes, and one needs geometric arguments to establish non-degeneracy. In [11] B.-Canary-Labourie-S. prove non-degeneracy under some mild assumptions, giving

**Theorem 4.1** (B.-Canary-Labourie-S. [11, Theorem 1.4]). Let $\Gamma$ be word hyperbolic, and $\mathcal{G}_\mathbb{R} < \text{PGL}_d(\mathbb{R})$ be reductive. The spectral radius pressure form is an analytic Riemannian metric on the space $\mathcal{C}_0(\Gamma, \mathcal{G}_\mathbb{R})$ of conjugacy classes of $\mathcal{G}_\mathbb{R}$-generic, regular, irreducible, projective Anosov representations.

Recall that a representation $\rho : \Gamma \to \mathcal{G}_\mathbb{R}$ is $\mathcal{G}_\mathbb{R}$-generic if its Zariski closure contains elements whose centralizer is a maximal torus in $\mathcal{G}_\mathbb{R}$, and it is regular if it is a smooth point of the algebraic variety $\text{Hom}(\Gamma, \mathcal{G}_\mathbb{R})$.

#### 4.2. Spectral gap pressure form.

We now consider two $\{a_1, a_2\}$-Anosov representations $\rho, \eta$ (with possibly different target groups). As explained in Section 2.4 they define reparametrizations $f^{a_1}_\rho$ and $f^{a_2}_\eta$ of the geodesic flow.

We define the spectral gap dynamical intersection of $\rho$ and $\eta$ to be the dynamical intersection between $f^{a_1}_\rho$ and $f^{a_2}_\eta$:

$$\Gamma^a(\rho, \eta) = I(f^{a_1}_\rho, f^{a_2}_\eta),$$
and analogously for $J^a_1(\rho, \eta)$. Given a $C^1$ curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)}$ of such $\{a_1, a_2\}$-representations the spectral gap pressure norm of $\rho_0$ is defined by

$$P^a_\rho(\rho_0) = \frac{\partial^2}{\partial t^2} \biggl|_{t=0} J^a_1(\rho_0, \rho_t) \geq 0.$$  

The spectral gap pressure norm induces a semidefinite symmetric bilinear two form on smooth points of $\{a_1, a_2\}$-Anosov representations. This looks very similar to the spectral radius pressure norm. It is, however, in general harder to check when the spectral gap pressure form is non-degenerate. As far as the authors know this has, so far, only been established for the Hitchin component in $PSL_d(\mathbb{R})$:

**Theorem 4.2** (B.-Canary-Labourie-S. [12, Theorem 1.6]). Let $G_\mathbb{R}$ denote either $PSL_d(\mathbb{R})$, $PSp(2n, \mathbb{R})$, $PSO(n, n+1)$ or the split form of the exceptional complex Lie group $G_2$. Then the spectral gap pressure form is positive definite on the Hitchin component $\mathcal{H}(S, G_\mathbb{R})$.

### 4.3. Vanishing directions.
Complex conjugation of matrices is an external automorphism of $PSL_d(\mathbb{C})$ and thus induces an involution

$$\tau : \mathfrak{X}(\Gamma, PSL_d(\mathbb{C})) \rightarrow \mathfrak{X}(\Gamma, PSL_d(\mathbb{C}))$$

whose fixed point set contains $\mathfrak{X}(\Gamma, PSL_d(\mathbb{R}))$ if $\rho \in \mathfrak{X}(\Gamma, PSL_d(\mathbb{R}))$ is a regular point, then the differential $d_{\rho} \tau$ splits the tangent space as a sum of purely imaginary vectors and the tangent space to the real characters:

$$T_\rho \mathfrak{X}(\Gamma, PSL_d(\mathbb{C})) = \ker(d_{\rho} \tau + \text{id}) \oplus T_\rho \mathfrak{X}(\Gamma, PSL_d(\mathbb{R}));$$

the almost complex structure $J$ of $\mathfrak{X}(\Gamma, PSL_d(\mathbb{C}))$ interchanges this splitting.

With a standard symmetry argument (see for example B.-Canary-S. [13, Section 5.8]), we get:

**Lemma 4.3.** Let $\rho : \Gamma \rightarrow PSL_d(\mathbb{R})$ be $\{a_1\}$-Anosov and let $v$ be a purely imaginary direction at $\rho$. Then $P^a_\rho(v) = 0$. If $\rho$ is moreover $\{a_2\}$-Anosov, then $P^a_\rho(v) = 0$.

**Proof.** Let us prove on the second statement, the first one being analogous. Consider a differentiable curve $(\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathfrak{X}(\{a_1, a_2\})(\Gamma, PSL_d(\mathbb{C}))$ such that $\rho_0 = \rho$, $\dot{\rho}_0 = v$ and $\tau \rho_t = -\rho_t$. For every conjugacy class $[\gamma] \in [\Gamma]$, the functions

$$t \mapsto \ell_\gamma(f^a_{\rho_t}) = (\lambda_1 - \lambda_2)(\rho_t(\gamma)) \quad \text{and} \quad t \mapsto h(f^a_{\rho_t})$$

are invariant under $t \mapsto -t$ and are thus critical at 0. Consequently, for every conjugacy class, the function $t \mapsto h(f^a_{\rho_t})\ell_\gamma(f^a_{\rho_t})$ is critical at 0 and hence Proposition 3.7 implies that $P^a_\rho(v) = 0$. \hfill $\square$

### 5. Pluriharmonicity of length functions and its consequences

In this section we prove the main results stated in the Introduction.

#### 5.1. Pluriharmonic length functions.
If $\rho, \eta \in \mathfrak{X}_\Theta(\Gamma, G_\mathbb{C})$ and $\varphi \in (E_\Theta)^\ast$ is strictly positive on $(\Lambda_\rho \cup \Lambda_\eta) - \{0\}$, then one can define their $\varphi$-dynamical intersection by

$$I^\varphi(\rho, \eta) = I(f_{\rho}^\varphi, f_{\eta}^\varphi) = \lim_{T \to \infty} \frac{1}{\#R_T(f_{\rho}^\varphi)} \sum_{[\gamma] \in R_T(f_{\rho}^\varphi)} \varphi \left( \frac{\lambda(\eta(\gamma))}{\lambda(\rho(\gamma))} \right),$$

where $f_{\rho}^\varphi = \varphi(f_{\rho}^\Theta)$ is given by Corollary 2.10.
Recall that a function is \textit{pluriharmonic} if it is locally the real part of a holomorphic function. The argument from B.-Taylor \cite[Section 5]{Taylor} applies directly and one has the following result.

**Proposition 5.1.** Consider $\rho \in \mathfrak{X}_{\Theta}(\Gamma, G_{\mathbb{C}})$ and $\varphi \in (E_{\Theta})^{\ast}$ that is strictly positive in $\Lambda_{\rho} - \{0\}$. Then the function

$$I_{\rho}^{\varphi} = I^{\varphi}(\rho, \cdot) : \mathfrak{X}_{\Theta}(\Gamma, G_{\mathbb{C}}) \to \mathbb{R}$$

is pluriharmonic (when defined).

Recall from Potrie-S. \cite[Corollary 4.9]{Potrie} that the map $\eta \mapsto \mathbb{P}(\rho_{\Theta}(\Lambda_{\eta}))$ is continuous on $\mathfrak{X}_{\Theta}(\Gamma, G_{\mathbb{C}})$, when considering the Hausdorff topology on compact subsets of $\mathbb{P}((E_{\Theta})^{\ast})$. Thus the domain of definition of $I_{\rho}^{\varphi}$ is an open subset of $\mathfrak{X}_{\Theta}(\Gamma, G_{\mathbb{C}})$ that contains, in particular, $\rho$. The proposition implies then that $I_{\rho}^{\varphi}$ is (defined and) pluriharmonic on a neighborhood of $\rho$.

**Proof.** Consider $\eta \in \mathfrak{X}_{\Theta}(\Gamma, G_{\mathbb{C}})$ such that $\varphi|_{\Lambda_{\eta} - \{0\}}$ is strictly positive. It follows then from Bochi-Potrie-S. \cite[Proposition 5.11]{Bochi} that there exists a neighborhood $\mathcal{U}$ of $\eta$ such that the constants in Definition 2.1 hold for every $\psi \in \mathcal{U}$. This implies that the convergence in the definition of $I^{\varphi}(\rho, \cdot)$ is uniform on compact subsets of its domain of definition. For each $T > 0$, the truncated sum in equation (6) is the real part of a holomorphic function and thus Theorem 1.23 from Axler-Bourdon-Ramey \cite{Axler} yields the result. \hfill $\square$

5.2. **Proof of Theorem A.** Let $\rho \in \mathfrak{X}(\pi_{1} S, \text{PSL}_{d}(\mathbb{R}))$ be $(1, 1, 2)$-hyperconvex and assume that it is a regular point of the character variety $\mathfrak{X}(\pi_{1} S, \text{PSL}_{d}(\mathbb{R}))$. Consider a tangent vector $v \in T_{\rho}\mathfrak{X}(\pi_{1} S, \text{PSL}_{d}(\mathbb{R}))$. Note that then $Jv$ is a purely imaginary tangent direction in $T_{\rho}\mathfrak{X}(\pi_{1} S, \text{PSL}_{d}(\mathbb{C}))$. Thus, Lemma 4.3 implies that for any $C^{1}$ curve $(\rho_{t})_{t \in (-\varepsilon, \varepsilon)}$ with $\rho_{0} = \rho$, $\rho_{0} = Jv$ and $T\rho_{t} = \rho_{-t}$ we have

$$0 = P_{\rho_{t}^{\ast}}(Jv) = \frac{\partial^{2}}{\partial t^{2}} \bigg|_{t=0} J^{\rho_{t}^{\ast}}(\rho_{0}, \rho_{t}). \quad (7)$$

Recall that $I(f, f) = 1$ and that if $\rho$ is $(1, 1, 2)$-hyperconvex then Theorem 2.6 states that $h_{\rho}^{\ast} = 1$. Moreover, as observed in the proof of Lemma 4.3, $h_{\rho_{t}^{\ast}}(\rho_{0}) = 0$, so developing the last term of equation (7) one obtains

$$0 = \text{Hess}_{\rho}(h_{\rho}^{\ast})(Jv) + \text{Hess}_{\rho} I_{\rho}^{\rho_{t}^{\ast}}(Jv).$$

Proposition 5.1 states that $I_{\rho}^{\rho_{t}^{\ast}}$ is pluriharmonic, so $\text{Hess}_{\rho} I_{\rho}^{\rho_{t}^{\ast}}(Jv) = -\text{Hess}_{\rho} I_{\rho}^{\rho_{t}^{\ast}}(v)$ and thus

$$\text{Hess}_{\rho}(h_{\rho}^{\ast})(Jv) = \text{Hess}_{\rho} I_{\rho}^{\rho_{t}^{\ast}}(v).$$

Lemma 2.4 implies that, at least for small $t$, $\rho_{t}$ is $(1, 1, 2)$-hyperconvex (over $\mathbb{C}$) and thus Theorem 2.5 yields $h_{\rho_{t}^{\ast}}(\rho_{t}) = \text{Hff}_{\rho_{t}}(\rho_{t})$. Finally, since $h_{\rho}^{\ast} = 1$ in a neighborhood of $\rho$ in $\mathfrak{X}(\pi_{1} S, \text{PSL}_{d}(\mathbb{R}))$ one has

$$\text{Hess}_{\rho} I_{\rho}^{\rho_{t}^{\ast}}(v) = P_{\rho_{t}^{\ast}}(v).$$

The result follows.
5.3. Proof of Theorem B. By Theorem 2.5 $H_{\hat{f}1} = h^{a1}$ in a neighborhood of $\rho$, and thus by assumption, the latter is critical at $\rho$. Since $J^{a1}(\rho, \cdot)$ is also critical at $\rho$ (Proposition 3.5) one concludes that $I_{\rho}^{a1}$ is critical at $\rho$ and thus its Hessian is well defined.

By Proposition 5.1 $I_{\rho}^{a1}$ is pluriharmonic and thus one has (as before) that for every $v \in T_{\rho} \chi(\Gamma, \text{PSL}_d(C))$

$$\text{Hess}_{\rho} I_{\rho}^{a1}(Jv) = - \text{Hess}_{\rho} I_{\rho}^{a1}(v).$$

One concludes that the $(+, 0, -)$ signature of $\text{Hess}_{\rho} I_{\rho}^{a1}$ is of the form $(p, 2k, p)$ for some $p \leq \dim \chi(\Gamma, \text{PSL}_d(C))$. Moreover, by Theorem 3.6 one has

$$0 \leq P^{a1}(Jv) = \text{Hess}_{\rho} h^{a1}(Jv) - h_{\rho}^{a1} \text{Hess}_{\rho} I_{\rho}^{a1}(v),$$

so that $\text{Hess}_{\rho} I_{\rho}^{a1}(v) \geq 0$ implies $\text{Hess}_{\rho} h^{a1}(Jv) \geq 0$. In particular $\text{Hess}_{\rho} h^{a1}$ is positive semidefinite on a subspace of dimension at least

$$\dim \chi(\Gamma, \text{PSL}_d(C)) - p \geq \frac{1}{2} \dim \chi(\Gamma, \text{PSL}_d(C))$$

and the theorem is proven.

5.4. Proof of Theorem C. Let $\Gamma$ be a co-compact lattice in $\text{PSO}(n, 1)$ such that the inclusion $\iota: \Gamma \rightarrow \text{PSO}(n, 1)$ defines, after extending coefficients, a regular point of the character variety $\chi(\Gamma, \text{PSU}(n, 1))$. Theorem 2.2 (and the Remark following it) in Cooper-Long-Thistlethwaite [18] assert that $\iota$ is then a regular point of the $\text{PSL}_{n+1}(R)$ character variety $\chi(\Gamma, \text{PSL}_{n+1}(R))$.

Moreover, since $\text{so}(n, 1)$ is the fixed point set of an involution in $\text{sl}_{n+1}(R)$, one has the decomposition $\text{sl}_{n+1}(R) = \text{so}(n, 1) \oplus s$ with $[s, s] \subset \text{so}(n, 1)$. One readily sees that

$$\text{su}(n, 1) = \text{so}(n, 1) \oplus is \subset \text{sl}_{n+1}(C).$$  \hspace{1cm} (8)

The twisted cohomology $H^1(\Gamma, \text{sl}_{n+1}(R))$ splits as

$$H^1(\Gamma, \text{sl}_{n+1}(R)) = H^1(\Gamma, \text{so}(n, 1)) \oplus H^1(\Gamma, s).$$

Consequently, by equation (8) the subspace $H^1(\Gamma, s) \subset H^1(\Gamma, \text{sl}_{n+1}(C))$ is sent bijectively to $H^1(\Gamma, is)$ when multiplied by the complex structure $J$, i.e.

$$J: H^1(\Gamma, s) = H^1(\Gamma, is).$$  \hspace{1cm} (9)

We will need the following generalization of Crampon [19].

**Theorem 5.2** (Potrie-S. [39, Theorem 7.2]). Assume $\rho \in \chi(\Gamma, \text{PSL}_{n+1}(R))$ has finite kernel and divides a proper open convex set of $P(R^{n+1})$. Then the entropy

$$h^{a1}(\rho) \leq n - 1$$

and equality holds only if $\rho$ has values in $\text{PSO}(n, 1)$.

This has the following useful consequence.

**Corollary 5.3.** The spectral radius pressure form $P^{a1}$ on $\chi(\Gamma, \text{PGL}_{n+1}(R))$ is non-degenerate at $\iota$.

**Proof.** When $n = 2$ this follows directly from Theorem 4.1, but if $n > 2$, the embedding $\text{so}(n, 1) \subset \text{sl}_{n+1}(R)$ is not $\text{PGL}_{n+1}(R)$-generic so, even though $\iota(\Gamma)$ is
irreducible, we need additional arguments. Nevertheless, by Theorem 5.2, the entropy function \( \rho \mapsto h^{\omega_i}(\rho) \) is critical at \( \iota \), so by Proposition 3.7 one only needs to verify that the set
\[
\{ d_\iota \omega^\gamma : [\gamma] \in [\Gamma] \}
\]
spans the cotangent space \( \mathfrak{T}^*_\iota \mathfrak{X}(\Gamma, \text{PSL}_{n+1}(\mathbb{R})) \), where \( \omega^\gamma : \mathfrak{X}(\Gamma, \text{PGL}_{n+1}(\mathbb{R})) \to \mathbb{R} \) is the function
\[
\rho \mapsto \omega_1 \left( \chi(\rho(\gamma)) \right).
\]
As \( \iota \) is irreducible and projective Anosov, this is the content of B.-Canary-Labourie-S. [11, Proposition 10.1].

Consider then \( v \in H^1(\Gamma, \iota) \subset \mathfrak{T}_\iota \mathfrak{X}(\Gamma, \text{PSL}_{n+1}(\mathbb{R})) \), by equation (9) the purely imaginary vector \( J \cdot v \in T_{\iota} \mathfrak{X}(\Gamma, \text{PSL}_{n+1}(\mathbb{C})) \) belongs to \( H^1(\Gamma, \text{PSU}(n, 1)) \) and represents thus a non-trivial infinitesimal deformation of \( \iota \) inside \( \text{PSU}(n, 1) \). As in Lemma 4.3 we choose a differentiable curve \( (\rho_t)_{t \in (-\varepsilon, \varepsilon)} \subset \mathfrak{X}(\Gamma, \text{PSU}(n, 1)) \) with \( \rho_0 = \iota \) and \( \dot{\rho}_0 = Jv \) and \( \tau \rho_t = \rho_{-t} \).

By Lemma 4.3 we have that
\[
0 = P_{\iota}^{\omega_i}(Jv) = \frac{\partial^2}{\partial t^2} \bigg|_{t=0} J^{\omega_i}(\iota, \rho_t).
\]
Expanding the second term, and using that both \( h^{\omega_i}(\rho_t) \) and \( I^{\omega_i}(\rho_t) \) are critical at \( t = 0 \) (as in the proof of Lemma 4.3) and that \( I^{\omega_i} \) is pluriharmonic, we get
\[
0 = \text{Hess}_{\iota}(h^{\omega_i})(Jv) - (n-1) \text{Hess}_{\iota}(I^{\omega_i})(v).
\]
On the other hand
\[
P_{\iota}^{\omega_i}(v) = \text{Hess}_{\iota}(h^{\omega_i})(v) + (n-1) \text{Hess}_{\iota}(I^{\omega_i})(v).
\]
Which in turn gives
\[
\text{Hess}_{\iota}(h^{\omega_i})(Jv) = P_{\iota}^{\omega_i}(v) - \text{Hess}_{\iota}(h^{\omega_i})(v) > 0,
\]
since \( P_{\iota}^{\omega_i}(v) > 0 \) by Corollary 5.3, and \(- \text{Hess}_{\iota}(h^{\omega_i})(v) \geq 0 \) since by Theorem 5.2 \( \iota \) is a global maxima of \( h^{\omega_i} \) among deformations in \( \text{PSL}_{n+1}(\mathbb{R}) \). The result then follows.

5.5. **The Hessian of the entropy at the Fuchsian locus of the Hitchin component.** Applying the same techniques as in the last section we can also show the following result on the Hitchin component.

**Corollary 5.4.** Let \( \iota \in \mathscr{H}_d(S) \) be a representation \( \pi_1 S \to \text{PSL}_2(\mathbb{R}) \to \text{PSL}_d(\mathbb{R}) \) in the embedded Teichmüller space. Then \( \text{Hess}(h^{\omega_i}_{\iota}) \) is positive definite on purely imaginary directions of \( T_{\iota} \mathfrak{X}(\pi_1 S, \text{PSL}_d(\mathbb{C})) \).

**Proof.** We mimic the last paragraph. In this case the pressure form \( P^{\omega_i} \) is positive definite on \( T_{\iota} \mathfrak{X}(S, \text{PSL}_d(\mathbb{R})) \) directly by Theorem 4.1. One gets, through the same arguments, that
\[
\text{Hess}_{\rho}(h^{\omega_i})(Jv) = P^{\omega_i}(v) - \text{Hess}(h^{\omega_i})(v).
\]
As we already observed, the first term on the right hand side is positive by Theorem 4.1, while \( \text{Hess}(h^{\omega_i})(v) \leq 0 \) since, by Potrie-S. [39, Theorem A], Fuchsian representations are maxima for the entropy within the Hitchin locus. The corollary follows.
We refer the reader to Dey-Kapovich [28] (see also Ledrappier [33] and Link [34]) for an interpretation of the critical exponent $h^\omega_1(p)$ as the Hausdorff dimension of the limit set with respect to a visual metric, i.e. a metric with respect to which the group action is conformal.

Finally, it would be interesting to relate Corollary 5.4, or an analog of it, to the recent work by Dai-Li [20] studying the translation lengths on the symmetric space of $\text{PSL}_d(\mathbb{C})$, when one deforms a Fuchsian representation along its Hitchin fiber.

References

[1] Marta Aparicio-Arroyo, Steven Bradlow, Brian Collier, Oscar García-Prada, Peter B. Gothen, and André Oliveira. $\text{SO}(p,q)$-Higgs bundles and higher Teichmüller components. *Invent. Math.*, 218(1):197–299, 2019. (Cited on page 8.)

[2] S. Axler, P. Bourdon, and W. Ramey. *Harmonic function theory*. Number 137 in Graduate Texts in Mathematics. Springer Verlag, 1992. (Cited on page 15.)

[3] Y. Benoist. Propriétés asymptotiques des groupes linéaires. *Geom. Funct. Anal.*, 7(1):1–47, 1997. (Cited on page 7.)

[4] J. Bochi, R. Potrie, and A. Sambarino. Anosov representations and dominated splittings. *Jour. Europ. Math. Soc.*, 21(11):3343–3414, 2019. (Cited on pages 2, 6, and 15.)

[5] F. Bonahon. The geometry of Teichmüller space via geodesic currents. *Invent. Math.*, 92:139–162, 1988. (Cited on page 12.)

[6] M. Bourdon. Sur le biraport au bord des CAT(−1)-espaces. *Publ. Math. I.H.E.S.*, 83:95–104, 1996. (Cited on page 4.)

[7] R. Bowen. Periodic orbits of hyperbolic flows. *Amer. J. Math.*, 94:1–30, 1972. (Cited on page 11.)

[8] R. Bowen. Hausdorff dimension of quasi-circles. *Publ. Math. I.H.E.S.*, 50(1):11–25, 1979. (Cited on page 2.)

[9] R. Bowen and D. Ruelle. The ergodic theory of Axiom A flows. *Invent. Math.*, 29:181–202, 1975. (Cited on page 11.)

[10] M. Bridgeman. Hausdorff dimension and the Weil-Petersson extension to quasifuchsian space. *Geom. & Top.*, 14(2):799–831, 2010. (Cited on pages 3 and 4.)

[11] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. The pressure metric for Anosov representations. *Geom. Funct. Anal.*, 25(4):1089–1179, 2015. (Cited on pages 2, 9, 12, 13, and 17.)

[12] M. Bridgeman, R. Canary, F. Labourie, and A. Sambarino. Simple roots flows for Hitchin representations. *Geom. Dedic.*, 192:57–86, 2018. (Cited on pages 2, 3, and 14.)

[13] M. Bridgeman, R. Canary, and A. Sambarino. An introduction to Pressure metrics on higher Teichmüller spaces. *Ergod. Th. & Dynam. Sys.*, 38(6):2001–2035, 2018. (Cited on page 14.)

[14] M. Bridgeman and E. Taylor. An extension of the Weil-Petersson metric to quasi-Fuchsian space. *Math. Annalen*, 341:927–943, 2008. (Cited on pages 1, 2, and 15.)

[15] M. Burger. Intersection, the Manhattan curve, and Patterson-Sullivan Theory in rank 2. *Inter. Math. Research. Not.*, 7, 1993. (Cited on page 12.)

[16] M. Burger, A. Iozzi, and A. Wienhard. Surface group representations with maximal Toledo invariant. *Ann. of Math.*, 172(1):517–566, 2010. (Cited on page 8.)

[17] Brian Collier. $\text{SO}(n,n+1)$-surface group representations and their Higgs bundles. *ASENS.*, 53(6):1561–1616, 2020. (Cited on page 8.)

[18] D. Cooper, D. D. Long, and M. Thistlethwaite. Flexing closed hyperbolic manifolds. *Geom. & Top.*, 11:2413–2440, 2007. (Cited on page 16.)

[19] M. Crampon. Entropies of compact strictly convex projective manifolds. *J. of Mod. Dyn.*, 3:511–547, 2009. (Cited on page 16.)

[20] S. Dai and Q. Li. Dominations results in $n$-Fuchsian fibers in the moduli space of Higgs bundles. *https://arxiv.org/pdf/2005.13960.pdf*, 2020. (Cited on page 18.)

[21] O. Glorieux, D. Monclair, and N. Tholozan. Hausdorff dimension of limit set for projective Anosov groups. In preparation, 2018. (Cited on page 10.)

[22] F. Guéritaud, O. Guichard, F. Kassel, and A. Wienhard. Anosov representations and proper actions. *Geom. & Top.*, 21:485–584, 2017. (Cited on page 6.)
Martin Bridgeman  
Boston College  
Department of Mathematics  
Chestnut Hill, MA 02467 United States of America  
bridgen@bc.edu

Beatrice Pozzetti  
Ruprecht-Karls Universität Heidelberg  
Mathematisches Institut, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany  
pozzetti@mathi.uni-heidelberg.de

Andrés Sambarino  
Sorbonne Université  
IMJ-PRG (CNRS UMR 7586)  
4 place Jussieu 75005 Paris France  
andres.sambarino@imj-prg.fr

Anna Wienhard  
Ruprecht-Karls Universität Heidelberg  
Mathematisches Institut, Im Neuenheimer Feld 205, 69120 Heidelberg, Germany  
HITS gGmbH, Heidelberg Institute for Theoretical Studies  
Schloss-Wolfsbrunnenweg 35, 69118 Heidelberg, Germany  
wienhard@uni-heidelberg.de