Frustrated superfluids in a non-Abelian flux

Fadi Sun1,2,4, Junsen Wang1,3, Jinwu Ye1,2,4 and Youjin Deng 3
1 Department of Physics and Astronomy, Mississippi State University, MS, 39762, USA
2 Department of Physics, Capital Normal University,
Key Laboratory of Terahertz Optoelectronics, Ministry of Education,
and Beijing Advanced innovation Center for Imaging Technology, Beijing, 100048, China
3 CAS Center for Excellence and Synergetic Innovation Center in Quantum Information and Quantum Physics,
University of Science and Technology of China, Hefei, Anhui 230026, China
4 Kavli Institute of Theoretical Physics, University of California, Santa Barbara, Santa Barbara, CA 93106
(Dated: September 19, 2018)

We study possible superfluid states of the Rashba spin-orbit coupled (SOC) spinor bosons with the spin anisotropic interaction λ hopping in a square lattice. The frustrations from the non-abelian flux due to the SOC leads to novel spin-bond correlated superfluids. By using a recently developed systematic “order from quantum disorder” analysis, we not only determine the true quantum ground state, but also evaluate the mass gap in the spin sector at λ < 1, especially compute the the excitation spectrum of the Goldstone mode in the spin sector at λ = 1 which would be quadratic without the analysis. The analysis also leads to different critical exponents on the two sides of the 2nd order transition driven by a roton touchdown at λ = 1. The intimate analogy at λ = 1 with the charge neutral Goldstone mode in the pseudo-spin sector in the Bilayer quantum Hall systems at the total filling factor νσ = 1 are stressed. The experimental implications and detections of these novel phenomena in cold atoms loaded on a optical lattice are presented.

There were previous extensive research efforts to study the geometric frustrations on the boson superfluids in a triangular8 and a Kagome lattice8,12,13. There were also more recent investigations on the Abelian flux induced frustrations to the boson superfluids in a bipartite lattice14,15. However, the interplay of Rashba or Dresselhaus spin-orbit coupling (SOC) with interactions has not been studied in any bosonic superfluid systems yet. The SOC leads to a non-Abelian flux which can be characterized by various Wilson loops16.

The Rashba or Dresselhaus spin-orbit coupling (SOC) is ubiquitous in various 2d or layered non-centrosymmetric insulators, semi-conductor systems, metals and superconductors. The lattice regularization of any linear combination of the Rashba and Dresselhaus SOC ασ_x + βσ_y is the kinetic term in Eq. The anisotropy can be adjusted by the strains, the shape of the surface or gate electric fields. In the cold atom side, there were experimental advances to generate 2d Rashba SOC for the fermion 40K gas22,23 and the tunable quantum anomalous Hall (QAH) SOC for 87Rb atoms in a square lattice24. More recently, the optical lattice clock scheme has been implemented to generate very long lifetime 1d SOC for various atoms25–33. The 2d Rashba SOC with a long enough lifetime in a square lattice maybe generated in some near future cold atom experiments.

In this paper, we will investigate various spin-bond correlated Superfluids of Rashba spin-orbit coupled (SOC) spinor bosons in a square lattice weakly interacting with a spin-anisotropic interaction parameter λ. For the Rashba SOC, we focus on the anisotropic line (α = π/2, β). We find the SOC provides a completely new frustrating source than the well studied geometric frustrations and Abelian flux. It leads to various novel frustrated Superfluids even in a bipartite lattice. Our main results are summarized in the (β, λ) phase diagram in the Fig.1. There are various SF states such as PW-X, Z-x, PW-XY and ZY-x phases34 with not only off-diagonal long range order, but also different kinds of spin-bond correlated orders and various novel quantum phase transitions among these SF phases. We develop a novel, general and systematic "order from quantum disorder" analysis to not only determine the quantum ground state, to calculate the roton gap, but also the correction to the spectrum in the PW-X state, especially the dispersion of the SOC Goldstone mode in the PW-X state at λ = 1. When 0 < β < βc, by performing the analysis, we find the transition from the PW-X to the Z-x state is a novel second order one driven by the roton touchdown with different critical exponents on the two sides of the transition35. The roton mode at λ < 1 becomes the SOC Goldstone mode at the spin isotropic interaction point λ = 1 due to the U(1)_{soc} symmetry breaking. While the PW-X to the PW-XY is a second order Bosonic Lifshitz one with the anisotropic dynamic exponents (z_x = 1, z_y = 2) and an accompanying Z_2 transition breaking the P_y symmetry in the spin sector. It is driven by the softening of the superfluid Goldstone mode along the y direction. The two second order transitions meet at a Tetra-critical point. We also extend our analysis to β < β < π/2. The finite temperature phase transitions and the transitions from weak to strong coupling above all these phases in Fig.[1] are sketched. The intimate analogy at λ = 1 with the charge neutral Goldstone mode in the pseudo-spin sector in the Bilayer quantum Hall systems at the total filling factor νσ = 1 are stressed. The transitions driven by the roton touchdown are contrasted with those leading to superfluid to supersolid transitions in frustrated lattices. Dramatic differences than the frustrated superfluids in the QAH systems of spinor bosons in a square lattice are
listed. The experimental implications and detections of these novel phenomena in cold atoms are presented.

We study short-range interacting bosons hopping in a square lattice subject to non-abelian gauge potential

\[ H_U = -t \sum_{\langle i,j \rangle} b_i^\dagger (i \sigma_x) U_{ij}^{\sigma \sigma'} b_{j}^{\sigma'} + h.c. \]
\[ + \frac{U}{2} \sum_i (n_{i \uparrow}^2 + n_{i \downarrow}^2 + 2 \lambda n_{i \uparrow} n_{i \downarrow}) - \mu \sum_i n_i \]

(1)

where the \( U_1 = e^{i \alpha \sigma_x}, U_2 = e^{i \beta \sigma_y} \) are the non-abelian gauge fields put on the two links in the square lattice (Fig.1b). Setting \( \alpha = \beta = 0 \), Eqn.(1) reduces to the conventional spin 1/2 (two component) spinor boson Hubbard model. In this paper, we focus on various novel Superfluids along the anisotropic SOC line (a) \( \alpha = \pi/2, \beta \) and spin-anisotropic interaction \( \lambda \).

RESULTS

1. Mean field Analysis of the ground states in \( \beta, \lambda \) space. In the weak interaction limit \( U \ll t \), one can diagonalize the kinetic energy first, then treat the \( U \) as a small perturbation. Along the anisotropic line \( \alpha = \pi/2, 0 < \beta < \pi/2 \), the kinetic energy is:

\[ H_0 = -t \sum_i [b_i^\dagger (i \sigma_x) b_{i+x} + b_i^\dagger e^{i \lambda \sigma_y} b_{i+y} + h.c.] \]

(2)

which becomes \( H_0 = -2t \sum_k b_k^\dagger (\cos \beta \cos k_y - \sin k_x \sigma_x - \sin \beta \sin k_y \sigma_y) b_k \) in the \( k \) space.

The non-interacting Hamiltonian \( H_0 \) has the following \( Z_2 \) reflection symmetry: \( \mathcal{P}_x: k_y \rightarrow -k_y, \sigma_y \rightarrow -\sigma_y, \sigma_x \rightarrow \sigma_x, \sigma_z \rightarrow -\sigma_z \), \( \mathcal{P}_y: k_x \rightarrow -k_x, \sigma_y \rightarrow -\sigma_y, \sigma_z \rightarrow -\sigma_z \). \( \mathcal{P}_z: k_y \rightarrow -k_y, \sigma_x \rightarrow -\sigma_x, k_y \rightarrow -k_y, \sigma_y \rightarrow -\sigma_y, \sigma_z \rightarrow -\sigma_z \). This \( \mathcal{P}_z \) symmetry is also equivalent to a joint \( \pi \) rotation of orbital and spin around \( \hat{z} \) axis. All these discrete symmetries are respected by the interaction. In addition to the discrete symmetries, there is also a spin-orbital coupled \( U(1) \) symmetry at \( \lambda = 1 \).

When \( 0 < \beta < \beta_c \approx 0.29\pi \), there are two minima in the lower band at \( (\pm \pi/2, 0) \). When \( \beta_c < \beta < \pi/2 \), there are four minima in the lower band at \( (\pm \pi/2, \pm k_0) \) where \( \sin k_0 = \sqrt{\sin^2 \beta - \cos^2 \beta} \), \( 0 < k_0 < \pi/2 \). We will demonstrate our results on the left part \( 0 < \beta < \beta_c \) in Fig.1. Then at the end, we will outline our results to the right half \( \beta_c < \beta < \pi/2 \).

At the weak coupling, most of the bosons condense into:

\[ \Psi_{i,0} = \frac{1}{\sqrt{2N_s}} \left[ c_1 e^{i K r_i} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 e^{-i K r_i} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \]

(3)

where \( K = (\pi/2, 0) \) and \( c_1 \) and \( c_2 \) are two complex numbers subject to the normalization condition \( |c_1|^2 + |c_2|^2 = 1 \).

For general \( \lambda \), we obtain the mean field interaction energy of the state:

\[ E_{int}^0 = \frac{U}{2N_s} \left( 1 + \frac{\lambda - 1}{2} - 1 \right) |(c_1 c_2^* + c_1^* c_2)|^2 \]

(4)

whose minimization leads to various mean field states. In the following section, we study \( \lambda < 1, \lambda = 1, \) and \( \lambda > 1 \) in detail.

2. Order from disorder analysis on the PW-X SF at \( 0 < \beta < \beta_c, \lambda < 1 \). The minimization of Eqn.(4) requires \( c_1 c_2^* + c_1^* c_2 = 0 \) which implies two sets of solutions (1) \( c_1 = 0 \) or \( c_2 = 0 \). (2) \( arg(c_1) = arg(c_2) = \pm \pi/2 \). So one must perform an order from disorder analysis to find the quantum ground state from this classically degenerate set of states. It turns out that due to the singularities at the plane wave solutions \( c_1 = 0 \) or \( c_2 = 0 \), it is difficult to perform such a analysis in the original basis Eqn.(4).

It is necessary to go to a different basis. We find it convenient if we introduce two \( Z \)-x states which satisfy \( Q \Psi_{Z,x,\pm}^0 = \pm \Psi_{Z,x,\pm}^0 \) with \( Q = \sum_i (-1)^i S_i^z \):

\[ \Psi_{Z,x,\pm}^0 = \frac{1}{\sqrt{2N_s}} \left[ e^{i K r_i} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \pm e^{-i K r_i} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right] \]

(5)

which can be used to re-parameterize Eqn.(1) as:

\[ \Psi_{i,0} = e^{i \phi/2} \cos(\theta/2) \Psi_{Z,x,+,0}^0 + e^{-i \phi/2} \sin(\theta/2) \Psi_{Z,x,-}^0 \]

(6)
which means that in Eq.8
\[
c_1 = |e^{i\phi/2} \cos(\theta/2) + e^{-i\phi/2} \sin(\theta/2)|/\sqrt{2},
\]
\[
c_2 = |e^{i\phi/2} \cos(\theta/2) - e^{-i\phi/2} \sin(\theta/2)|/\sqrt{2}
\] (7)

and the interaction energy becomes
\[
E_{\text{int}}^0 = \frac{U}{2N_s} \left( 1 + \frac{\gamma - 1}{2} \sin^2 \theta \right)
\] (8)

which determines \( \theta = \pi/2, 3\pi/2, \ldots \). However, we cannot determine \( \phi \) which forms a classically degenerate family. One must study quantum corrections to the ground state energy, namely, the order from disorder analysis.

Setting \( \theta = \pi/2 \), one can rewrite the spinor field into a condensation part plus a fluctuating part \( \Psi = \sqrt{N_0} \Psi_0 + \psi \) where \( N_0 \) is the number of condensate atoms. As shown in the Method section, the substitution leads to the expansion of the Hamiltonian \( H = H^{(1)} + H^{(2)} + \ldots \) where the superscript denotes the order in the fluctuations \( \psi \) and \( \cdots \) means high order. \( H^{(1)} = E^{(0)} \) is the classical ground state energy, setting \( H^{(1)} = 0 \) determines the values of the chemical potential \( \mu = -2t(1 + \cos \beta) + Un(1 + \lambda)/2 \). Diagonalizing \( H^{(2)} \) by a \( 8 \times 8 \) Bogoliubov transformation leads to
\[
H_2 = E_0^{(2)} + \sum_{\mathbf{k} \in \mathbb{RBZ}} \sum_{l=1}^{4} \omega_l(\mathbf{k}; \phi) \left( \beta^\dagger_{\mathbf{l}, \mathbf{k}} \beta_{\mathbf{l}, \mathbf{k}} + \frac{1}{2} \right)
\] (9)

and the quantum corrections to the ground state energy:
\[
E_{\text{GS}}(\phi) = E_0 + \frac{1}{2} \sum_{\mathbf{k} \in \mathbb{RBZ}} \sum_{l=1}^{4} \omega_l(\mathbf{k}; \phi)
\] (10)

where \( E_0 = E^{(0)} + E^{(2)}_0 \). We find that it picks up its minima at \( \phi = 0, \pi, \cdots \). So the “quantum order from disorder” mechanism selects the PW-X with \( c_1 = 1, c_2 = 0 \) or \( c_1 = 0, c_2 = 1 \) among the classical degenerate family tuned by \( \phi \).

(a) The Excitation spectrum of the PW-X state

After determining the quantum ground state to be the PW-X state, we now investigate its excitation spectrum and also its critical behaviours when approaching the three boundaries \( \beta \rightarrow \beta_c^-, \lambda \rightarrow 1^- \) and \( \beta \rightarrow 0^+ \) in Fig.1.

There are two linear gapless modes located at \( (0, 0) \) and \( (\pi, 0) \) respectively:
\[\omega_G(q_x) = \sqrt{n_0U(1 + \lambda)q_x^2 + A(\beta)q_y^2},\]
\[\omega_R(q_x) = \sqrt{n_0U(1 - \lambda)q_x^2 + A(\beta)q_y^2}\] (11)

where \( A(\beta) = \cos \beta - \sin^2 \beta \). The first is just the superficial Goldstone mode, the second is the roton mode where \( \mathbf{q} = (\pi, 0) \). As to be shown below, the roton mode will acquire a gap (Fig.3a) from the order from disorder mechanism

Because \( q_x \) direction is always non-critical, we can set \( q_x = 0 \) in the above equations. Because \( A(\beta_c) = 0 \), so the Goldstone mode becomes quadratic along the \( q_y \) direction at \( \beta_c \):
\[\omega_G(q_x = 0, q_y) = \sqrt{n_0U(1 + \lambda)/2q_y^2},\]
which is a bosonic Lifshitz transition with the anisotropic dynamic exponent \( z_x = 1, z_y = 2 \).

As \( \lambda \rightarrow 1^- \), the Goldstone mode remains un-critical, but the roton mode becomes a quadratic one:
\[\omega_R^0(q_x) = t \sqrt{q_x^2 + A(\beta)q_y^2} + A'(\beta)q_y^2\] (13)

where \( A'(\beta) = \cos \beta \). In the following, we will perform an order from disorder analysis on the roton mode.

(b) The Roton mass gap generated from the “order by disorder” mechanism in the PW-X state

Combining Eq.8 and Eq.10 we can extract the \( \theta \) and \( \phi \) dependence of the ground state energy around its minimum:
\[H_2 = E_0 + \frac{1}{2} A(\delta \theta)^2 + \frac{1}{2} B(\delta \phi)^2\]
where \( \theta = \pi/2 + \delta \theta \) and \( \phi = 0 + \delta \phi \). We obtain
\[A(\lambda) = (1 - \lambda)Un_0^2/2 \text{ analytically from Eq.8 and } B(\lambda) = b(n_0U)^2/t \approx \text{ can be numerically extracted from Fig.2a}.

![FIG. 2](image-url)

Because \( |n_0\delta \theta/2, \delta \phi| = \imath h \) forms a pair of conjugate variables. The “quantum order from disorder” mechanism leads to the roton gap:
\[\Delta_R = 2\sqrt{AB/n_0} = \sqrt{2(1 - \lambda)UB} \sim n_0U\sqrt{U/t}\] (15)

which is shown in Fig.4b to be a monotonically increasing function of \( \beta \). When \( \beta \rightarrow 0^+ \), the roton mode near \( (\pi, 0) \) becomes the Goldstone mode due to the breaking of \( U(1) \) at \( \beta = 0 \). Indeed, the roton gap vanishes \( \beta \rightarrow 0^+ \) as shown in Fig.4b.

As shown below, \( B \) is non-critical as \( \lambda \) approaches \( 1^- \), so critical behaviour of the roton gap \( \Delta_R \propto \sqrt{1 - \lambda} \) when \( \lambda < 1 \).

In fact, as shown in the method section, the order from disorder mechanism not only leads to the gap in Eq.15 but also modify the dispersion shown in Eq.18.

3. The Order from disorder analysis on the PW-X SF at \( \beta < \beta < \pi/2, \lambda = 1 \). When \( \lambda = 1 \) and \( \beta < \beta_c \), Eq.4 shows that any \( c_1 \) and \( c_2 \) give the
same interaction energy, thus they form a classically degenerate ground-state manifold. It was known that the Hamiltonian Eq(1) at $\alpha = \pi/2$ and $\lambda = 1$ has a spin-orbit coupled $U_{soc}(1)$ symmetry $Q = \sum_i (-1)^i e^{iS_i^\theta}$. In order to distinguish between an accidental degeneracy and the exact degeneracy, we introduce two Y-x states:

$$\Psi_{Y-x,\pm}^0 = \frac{1}{\sqrt{2N_s}} \left[ e^{iK r_i} (1 \pm ie^{-iK r_i}) \right] (16)$$

which satisfy $Q \Psi_{Y-x,\pm}^0 = \pm \Psi_{Y-x,\pm}^0$.

They can be used to re-parameterize Eq(8) as:

$$\Psi_{i,0} = e^{i\phi/2} \cos(\theta/2) \Psi_{Y-x,\pm}^0 + e^{-i\phi/2} \sin(\theta/2) \Psi_{Y-x,-}^0$$

(17)

which means that in Eq(8):

$$c_1 = [(e^{i\phi/2} \cos(\theta/2) + e^{-i\phi/2} \sin(\theta/2))/\sqrt{2}],$$

$$c_2 = [i(e^{i\phi/2} \cos(\theta/2) - e^{-i\phi/2} \sin(\theta/2))/\sqrt{2}]$$

(18)

The $U(1)_{soc}$ symmetry at $\lambda = 1$ dictates that the energy should be independent of $\phi$. We expect that the classical degeneracy at the angle $\theta$ will be lifted by quantum effects, and a unique $\theta$ will be determined by the order-from-disorder mechanism.

We set $\phi = 0$ to spontaneously breaks the $U(1)_{soc}$ symmetry. Following the similar procedures as in the last section, we obtain Eqs(11) which pick up its minima at $\theta = \pi/2, \cdots$. So the "quantum order from disorder" mechanism still selects the PW-X state ($\theta = \pi/2, \phi = 0$). All the other exactly degenerate states can be generated by changing $\phi$. If we take $c_1 = 1, c_2 = 0$ PW-X state in Eq(8) then the $U(1)_{soc}$ related family is

$$\Psi^0(\phi) = \frac{1}{\sqrt{2N_s}} \left[ \cos \phi e^{iK r_i} (1 \pm \sin \phi e^{-iK r_i}) \right]$$

(19)

whose corresponding spin-orbital structure is $\langle \sigma_z \rangle = -\cos 2\phi, (\sigma_z) = 0, (\sigma_z) = -(-1)^s \cos 2\phi$. Setting $\phi = 0, \pi/2$ and $\phi = \pi/4, 3\pi/4$ recovers the two PW-X state and the Z-x,± state respectively.

By comparing $\Psi^{1,1}_i(\theta_1 = \pi/2, \phi_1)$ in Eq(8) with $\Psi^{1,1}_i(\theta_2, \phi_2 = 0)$ in Eq(17) (we used the subscript 1 for $\lambda < 1$ and 2 for $\lambda = 1$ case ), we can see the relations between the order from disorder variables in the two cases:

$$\theta_2 = \pi/2 - \phi_1$$

(20)

which just shows the $(\theta, \phi)$ exchanges in the Z-x basis in Eq(8) and the $Y - x$ basis in Eq(17).

(a) The Goldstone mode spectrum computed from the "order by disorder" mechanism in the PW-X SF state

The $U_{soc}(1)$ symmetry dictates no $\phi$ dependence, so the ground state energy an be written as:

$$E_{GS}(\theta) = E_1 + \frac{1}{2} B(\delta\theta)^2$$

(21)

where $E_1 = E_{GS}(\theta = \pi/2)$ and $\delta\theta = \theta - \pi/2$ and $B(\lambda = 1)$ can be extracted from Fig(2). The A term in Eq(13) vanishes due to the exact $U_{soc}(1)$ symmetry. Because $\lim_{\lambda \rightarrow 1} B(\lambda = 1) = B(\lambda = 1)$, so B(λ) is continuous at $\lambda = 1$.

As shown in the method section, the "quantum order from disorder" mechanism changes the quadratic dispersion in Eq(13) to a linear one: which is nothing but the SOC Goldstone mode due to the spontaneous breaking of the $U_{soc}(1)$ symmetry,

$$\omega_R = \sqrt{2Bt} \left[ q^2 + A'(\beta) q_n^2 \right]$$

(22)

where because $B \sim (n_0 U)^2 / t$, so the slope of the roton mode $\omega \sim \sqrt{n_0 U}$ is smaller than the superfluid velocity $c \sim \sqrt{n_0 U}$ in Eq(11) Fig.3b. Due to the wide momentum separation between the Goldstone and roton mode, the Goldstone mode is not affected by the order from disorder analysis. We observe that at the bosonic Lifshitz transition point $\beta_c$, in Fig.1b $A(\beta_c) = 0$, but $A'(\beta_c) > 0$, so the SF Goldstone mode in Eq(11) becomes quadratic signifying the transition, while the SOC Goldstone mode in Eq(22) remains linear along the $y$ direction. So the SOC Goldstone mode is non-critical across the bosonic Lifshitz transition. This fact is important to determine the finite temperature phase transitions above the $\lambda = 1$ line in Fig.7.

FIG. 3. Evolution of the low energy excitations as $\lambda$ changes from $\lambda < 1$ to $\lambda = 1$ at a fixed $0 < \beta < \beta_c$. (a) The roton gap $\Delta_R \sim \sqrt{1-\lambda}$ generated by the order from disorder mechanism at $\lambda < 1$. (b) The SOC Goldstone mode at $(-\pi/2, 0)$ due to the $U(1)_{soc}$ breaking computed by the order from disorder mechanism at $\lambda = 1$. (c) The roton gap $\Delta_R \sim \lambda - 1$ in the Z-x state at $\lambda > 1$. The superfluid Goldstone mode at $(\pi/2, 0)$ from the $U(1)_{soc}$ breaking remains non-critical through the transition.

4. The Z-x SF state at $\beta_c < \beta < \pi/2, \lambda > 1$.

When $\lambda > 1$ and $\beta < \beta_c$, the minimization of Eq(4) requires $c_1c_2^* + c_1^*c_2 = \pm 1$ which means $c_1 = \pm c_2 = e^{i\gamma}/\sqrt{2}$ where $\gamma$ is a global phase. So the 2 fold degenerate ground state is either $\Psi_{Z,x,+}$ with $c_1 = c_2 = 1/\sqrt{2}$ or $\Psi_{Z,x,-}$ with $c_1 = -c_2 = 1/\sqrt{2}$.

Let us pick up the $\Psi_{Z,x,+}$ state, we find its 4 excitation modes $\omega_{1,2,3,4}(k)$ in ascending order: $\omega_1(k)$ contains a Goldstone mode at $(0, 0)$ (Fig.3c), $\omega_2(k)$ contains a roton mode at $(0, 0)$ (Fig.3c) $\omega_{3,4}(k)$ are high energy bands (not shown in Fig.3c). Note that the Z-x state breaks the translation symmetry to two sites per unit cell, so
the Brillouin zone (BZ) in the PW-X state becomes the Reduced BZ (RBZ) with \(-\pi/2 < k_x < \pi/2, -\pi < k_y < \pi\).

We also extract the long wavelength limit of the SF Goldstone mode:
\[
\omega_1(k) = \sqrt{2n_0tU \left[ k_x^2 + A''(\beta, \lambda) k_y^2 \right]}
\] (23)
where \(k \in RBZ\) and \(A''(\beta, \lambda) = \cos \beta - \frac{\sin^2 \beta}{1 + n_0(\lambda - 1)/4t}\).

Because \(A''(\beta, \lambda = 1) = A(\beta)\), it recovers the SFGoldstone mode in Eq.11 at \(\lambda = 1\). However, when \(\lambda > 1\), by setting \(A''(\beta_0, \lambda) = 0\), one can see the Bosonic Lifshitz transition line shifts to \(\beta_0(\lambda) > \beta_c\) as shown in Fig.1b. We also obtain the critical behaviour of the roton mode \(\Delta_R = \omega_2(0, 0) \propto \lambda - 1\).

![FIG. 4. (a) The quantum critical behaviour of the roton gap \(\Delta_R \sim (1 - \lambda)^{\beta_c}\) as a function of \(\lambda\) at a fixed \(\beta = \pi/10\). \(\beta'_c = 1/2, \beta'_c = 1\) provide the first example of a QCP with different critical exponents on two sides of a QCP. However, as shown in Sec.4, the transition from the PW-XY to the ZY-x is also driven by the roton touchdown with equal exponents on both sides \(\beta'_c = \beta'_c = 1\). (b) The roton gap \(\Delta_R\) in red line calculated by order from disorder analysis Eq.(12) in the PW-X and PW-XY state when \(\lambda < 1\) is a monotonically increasing function of \(\beta\) at a fixed \(\lambda = 1/2\). The Green line is before order from disorder analysis. Both fixed \(t = 1, n_0U = 1\) and \(n_0 \approx 1\). Note that \(\beta_c \approx 0.288\pi\).](image)

5. The SFs and transitions on \(\beta_c < \beta < \pi/2\).

So far, we focused on the left part of the phase diagram Fig.1. In this section, we apply similar methods to study the SFs and transitions on the right half \(\beta_c < \beta < \pi/2\) of Fig.1.

(a) When \(\lambda > 1\), the \(Q\) takes one of the four minima \((\pm \pi/2, \pm k_0)\) shown in Fig.1a2 and the \(\phi\) is determined by \(k_0\) and given in Eq.(25). Then \(\langle \sigma_x \rangle = -\cos \phi(\beta), \langle \sigma_y \rangle = \sin \phi(\beta), \langle \sigma_z \rangle = 0\). This is the PW-XY state in Fig.1b where the spin is in the \(XY\) plane. It breaks the \(P_x\) symmetry. There is a quantum phase transition from the plane wave superfluid at \(\beta < \beta_c\) (PW-X) to the plane wave superfluid at \(\beta_c < \beta < \pi/2\) (PW-XY) which is in the Ising university class in the spin sector. As shown in Sec.2 and confirmed also below, this Ising transition in the spin sector is just affiliated to the bosonic Lifshitz transition at \(\beta = \beta_c\) with the dynamic exponents \((z_x = 1, z_y = 2)\).

Writing \(\Psi = \Phi_{0, PW-XY} + \psi\), following similar procedures as in the PW-X phase, we are able to find the excitation spectrum above the PW-XY state. We find a gapless SF mode at \((0, 0)\) which is due to the \(U(1)_c\) symmetry breaking. We also find three gapped modes at \((\pi, 0)\), \((\pi, 2k_0)\) and \((0, 2k_0)\) (Fig.1a2). In the following, we will study the critical behaviors of these modes (1) across the phase transition from the PW-XY to the ZY-x transition through the \(\lambda = 1\) line where there is the enhanced \(U(1)_{soc}\) symmetry. (2) across the phase transition from the PW-X to the PW-XY through the bosonic Lifshitz transition at \(\beta = \beta_c\). (3) approaching to the right axis \(\beta \rightarrow \pi/2\).

At a fixed \(\beta_c < \beta < \pi/2\), as \(\lambda \rightarrow 1\) along the red dashed line in Fig.1b, the gapless linear SF mode remain un-critical, the roton gap at \((\pi, 0)\) decreases to zero as \(\Delta_R \sim 1 - \lambda\), the mode becomes linear, it is just the SOC Goldstone mode due to the \(U(1)_{soc}\) symmetry breaking at \(\lambda = 1\). We find the roton gap at \((\pi, 2k_0)\) also decreases to zero, but the mode becomes quadratic. As to be shown below, an order from disorder analysis at \(\lambda = 1\) will open a gap to this quadratic mode. However, the roton gap at \((0, 2k_0)\) remains as \(\lambda \rightarrow 1^-\).

At a fixed \(\lambda < 1\), as \(\beta \rightarrow \beta_c^+\) in Fig.1b, the gapless linear SF mode at \((0, 0)\) also becomes quadratic along the \(y\) direction as in Eq.(12) so confirm the transition at \(\beta_c^+\) is a second order bosonic Lifshitz one with the dynamic exponent \((z_x = 1, z_y = 2)\). The roton gap at \((\pi, 0)\) decreases to zero, the mode becomes linear, matches the behaviours of the roton mode from the left (before applying the order from disorder analysis) shown in Fig.1b. Of course, as \(\beta \rightarrow \beta_c^+\), the \(k_0 \rightarrow 0\), the modes at \((\pi, 2k_0)\) and \((0, 2k_0)\) become the same as those at \((\pi, 0)\) and \((0, 0)\) respectively. As \(\beta \rightarrow \pi/2\) in Fig.1b, \(k_0 \rightarrow \pi/2\), only the mode at \((\pi, \pi)\) becomes a gapless linear one (as to be shown in the next section, this mode becomes the Goldstone mode due to the \(\tilde{U}_z(1)\) symmetry breaking at \(\lambda < 1, \beta = \pi/2\)), while the modes at \((0, \pi)\) and \((\pi, 0)\) remain gapped.

(b) Order from quantum disorder at \(\lambda = 1\). We find that in addition to the exact \(U(1)_{soc}\) symmetry, there is also a spurion \(U(1)\) symmetry leading to a classically degenerate family of states. By a similar order from disorder analysis as in the \(\beta < \beta_c, \lambda = 1\) case, we show that the quantum ground state is the PW-XY state up to all the other states related by the exact \(U(1)_{soc}\) symmetry (Fig.1b). Then there exists two gapless modes which are due to the \(U(1)_c\) and \(U(1)_{soc}\) symmetry breaking respectively.

Indeed, as approaching from below \(\lambda \rightarrow 1^-\), as said above, the roton gap \(\Delta_R\) at \((\pi, 0)\) decreases to zero as \(\Delta_R \sim 1 - \lambda\), the mode becomes linear at \(\lambda \rightarrow 1\) which is precisely the SOC Goldstone mode due to the \(U(1)_{soc}\) symmetry breaking. The roton gap at \((\pi, 2k_0)\) also decreases to zero, but the mode becomes quadratic which is due to the spurion \(U(1)\) symmetry at \(\lambda = 1\). The order from disorder analysis at \(\lambda = 1\) opens a gap to the quadratic mode at \((\pi, 2k_0)\). So \((\pi, 2k_0)\) remains un-critical across the PW-XY to the ZY-x transition.

As \(\lambda \rightarrow 1^-\), \(\beta \rightarrow \pi/2\), namely approaching the Abelian point with a \(\pi\) flux, all the three modes touch zero: the
modes at $(0, \pi), (\pi, 0)$ become linear, while the $(\pi, \pi)$ mode becomes quadratic (which, as to be shown below, will become a linear mode after an order from disorder analysis). These facts recover those found at the $\pi$ flux Abelian point in $^{15,35}$ and will be discussed again in Sec.6.

(c) When $\lambda > 1$. $\mathbf{Q}_1$ can take one of the two values $\mathbf{Q}_1 = (\pi/2, \pm k_0)$ in Fig.1, then $\mathbf{Q}_2 = (-\pi/2, \pm k_0)$. It leads to the 4-fold degenerate ZY-x phase with the momentum $(\pi, 0)$ in Fig.1, where the spin is in the ZY plane as shown in Eq.(27) and Fig.8. Following the similar procedures as in the Z-x state on the left in Fig.1b, in the RBZ, we find a SF Goldstone mode $\omega_1(k)$ and also a roton mode $\omega_2(k)$ with a roton gap $\Delta_R^+ = \omega_2(0, 0) \propto \lambda - 1$.

6. The physical picture from the left and right Abelian points

(a) Left Abelian point: The analogy at $\lambda = 1$ to the neutral gapless mode in the Bilayer Quantum Hall system (BLQH)

At the left Abelian point ($\beta = 0, \lambda = 1$), the spin $SU(2)$ symmetry becomes evident in the rotated basis $\tilde{S}_i = (S_i^x, (1)^{k} S_i^y, (1)^k S_i^z)$. The breaking of this $SU(2)$ leads to the quadratic FM spin wave mode $\omega \sim k^2$. However, at any $\beta > 0$, the spin $SU_{s}(2)$ symmetry is reduced to the $U(1)_{soc}$. So the FM spin wave becomes a linear Goldstone mode at any $\beta > 0$. Very similar phenomena happen in the bilayer quantum Hall systems (BLQH) at the total filling factor $\nu_T = \frac{15}{12}$. At $d = 0$, there exists an exact $SU_{s}(2)$ symmetry dictating a $\omega \sim k^2$ neutral gapless mode. At any finite distance $d > 0$, a capacitive term reduces the spin $SU(2)$ symmetry to a easy plane $U(1)$ symmetry, therefore transfers the quadratic FM mode to a linear Goldstone mode due to the breaking of the $U(1)$ symmetry. In BLQH, the capacitive term is due to the difference between inter- and intra-layer Coulomb interaction which explicitly breaks the $SU(2)$ to $U(1)$ at any $d > 0$. Here the capacitive term Eq.(21) is generated by the order from disorder mechanism. As shown in Sec.3 and detailed in the SM, it is this capacitive term which transforms the otherwise quadratic dispersion to the linear Goldstone mode as dictated by the above exact symmetry breaking analysis.

Of course, in the $\mathbf{S}$ basis, moving up along the left axis ($\lambda > 1$) is the Ising limit, down ($\lambda < 1$) is the easy plane limit where there is another spin Goldstone mode due to the $U(1)_{z}$ symmetry breaking. Using the relation $\tilde{S}_i = (1)^{k} S_i^z$, this Goldstone mode becomes the one at $(\pi, 0)$ in the original basis, which is precisely observed in the PW-X phase in the limit $\lambda < 1, \beta \rightarrow 0^+$ as discussed in Sec.2. Indeed, the roton gap vanishes $\beta \rightarrow 0^+$ as shown in Fig.3.

(a) Right Abelian point: Four linear gapless modes reduce to two at $\lambda = 1$.

In the right $\pi$ flux Abelian point ($\beta = \pi/2, \lambda = 1$), the spin $SU(2)$ symmetry becomes evident in the rotated basis $\tilde{S}_i = ((1)^{k} S_i^x, (1)^{k} S_i^y, (1)^{k} S_i^z)$ which was used in $^{15,35}$. In the non-Abelian gauge at ($\alpha = \pi/2, \beta = \pi/2$), we find a spurious $U(1)$ symmetry which leads to a degenerate family of states connecting $\mathbf{K}_1 = (\pi/2, \pi/2)$ and $\mathbf{K}_3 = -\mathbf{K}_1$. A similar order from disorder analysis as in Sec.2 shows that quantum corrections select the PW-XY state to be the ground state up to the spin $SU(2)$ symmetry. Taking the the PW-XY state as the simplest quantum ground state, we find there are three gapless linear modes at $\mathbf{K}_1 = (\pi/2, \pi/2)$, $\mathbf{K}_2$ and $\mathbf{K}_1$, one quadratic mode at $\mathbf{K}_3$. By performing a similar order from quantum disorder analysis as that in Sec.3, we find the quantum corrections change the quadratic mode at $\mathbf{K}_3$ to a linear one. This is consistent with the results achieved in $^{15}$ using an Abelian gauge which explicitly keeps the spin $SU(2)$ symmetry. In $^{53}$, we showed that there are strong quantum fluctuations induced by the coupling between the off-diagonal long range order in the SF sector and the magnetic orders in the spin order (This is in sharp contrast to the case at the left Abelian point discussed above where there is no such coupling.) The complete symmetry breaking $U(1)_z \times SU(2)_z \rightarrow 1$ leads to 4 linear gapless modes. When transforming the $\theta = \pi$ Abelian gauge to the present non-Abelian gauge with ($\alpha = \pi/2, \beta = \pi/2$), the 90$^\circ$ co-planar state in $^{15}$ becomes the PW-XY state, the 4 linear modes are shifted to the 4 minima at ($\pm \pi/2, \pm \pi/2$).

When slightly away from the right $\pi$ flux Abelian point, the 4 minima shift to ($\pm \pi/2, \pm k_0$), only 2 of the 4 gapless modes survive, one due to the $U(1)_z$ symmetry breaking at $(0, 0)$, another due to the $U(1)_{soc}$ symmetry breaking at $(\pi, 0)$. While the modes at $(0, 2k_0)$ and $(\pi, 2k_0)$ acquire gaps.

Of course, in the $z$ basis, moving up along the right axis ($\lambda > 1$) is the Ising limit, down ($\lambda < 1$) is the easy plane limit where there is another spin Goldstone mode due to the $U(1)_{z}$ symmetry breaking. Using the relation $\tilde{S}_i = ((1)^{k} S_i^x + $, this spin Goldstone mode becomes the one at $(\pi, \pi)$ in the original basis, which is precisely observed in the PW-X Y phase in the limit $\lambda < 1, \beta \rightarrow \pi/2$ as discussed in Sec.5. Indeed, in this limit, only $(0, 0)$ and $(\pi, \pi)$ modes become linear gapless ones, but $(\pi, 0)$ and $(0, \pi)$ remain gapped.

7. Finite temperature phase transitions above all the SF phases and transitions from the weak to strong coupling

Now we briefly discuss the finite temperature phases and phase transitions above all the SFs in Fig.1b. Of course, due to the correlated spin-bond orders of the SFs, there are also other interesting phase transitions associated with the restorations of these spin-bond correlated orders at a finite temperature shown in Fig.5 and Fig.8. The nature of these finite temperature transitions can be qualitatively determined by the degeneracy of the ground states: PW-X($d = 2$), Z-x SF($d = 2$), PW-XY($d = 4$), ZY-x SF($d = 4$). At $\lambda = 1$, due to the breaking of $U(1)_{soc}$, there is also a KT transition $T_{KT}^{soc}$ in Fig.7.
coupling, the two SF states break both the $T^\text{SF}$ and $Z\text{-}x$ SF respectively. The transition temperature $T_2 \sim \Delta_R \sim |1-\lambda|^{\beta_0}$ scales as the roton gap at $(\pi,0)$ in Fig.6. There is a KT transition at $T^\text{KT}$ to restore the $U(1)_{\text{soc}}$ symmetry at $\lambda = 1$. Then there is another higher KT transition at $T^\text{KT}_s$ from the SF to a normal fluid (NF). (b) $\beta_c < \beta < \pi/2$, There is a $Z_4$ clock melting transition in the PW-XY SF and ZY-x SF respectively. The transition temperature $T_4 \sim \Delta_R \sim |1-\lambda|$ scales as the roton gap at $(\pi,0)$.

![Diagram](image)

FIG. 5. The quantum phase transitions at $T = 0$ are driven by roton touchdowns. The finite temperature phase transitions in (a) $0 < \beta < \beta_c$. There is a Ising melting transition in the PW-XY SF and Z-x SF respectively. The temperature transitions $T_2 \sim \Delta_R \sim |1-\lambda|^{\beta_0}$ scales as the roton gap at $(\pi,0)$ in Fig.6. There is a KT transition at $T^\text{KT}$ to restore the $U(1)_{\text{soc}}$ symmetry at $\lambda = 1$. Then there is another higher KT transition at $T^\text{KT}_s$ from the SF to a normal fluid (NF). (b) $\beta_c < \beta < \pi/2$, There is a $Z_4$ clock melting transition in the PW-XY SF and ZY-x SF respectively. The transition temperature $T_4 \sim \Delta_R \sim |1-\lambda|$ scales as the roton gap at $(\pi,0)$.

![Diagram](image)

FIG. 6. The quantum phase transitions at $T = 0$ are driven by the softening of SF along the $y$ direction (namely the bosonic Lifshitz transition with $(z_x = 1, z_y = 2)$) in Eq.12. The finite temperature phase transitions in (a) $\lambda < 1$. There are KT transitions at $T^\text{KT}_s$ from the SF to a normal fluid (NF). There is also a Ising and $Z_4$ clock melting transition with $T_3 \sim \Delta_3(\pi,0)$ and $T_4 \sim \Delta(\pi,\text{2k}_x)$ above the PW-XY SF and PW-XY SF respectively. Note that as $\beta \to \beta^*_c$, $T_4 \sim \Delta_3(\pi,0)$ and $\beta \to \pi/2$. On the left and right axis, there is also a KT transition due to the $U(1)$ symmetry breaking in the $SU(2)$ basis and $SU(2)$ basis respectively. (b) $\lambda > 1$. Note that as shown in Fig.1b, $\beta_0 > \beta_c$.

Now we briefly mention the transitions from the various SFs in Fig.1b at weak coupling to the magnetic phases at the strong coupling. At $\lambda = 1$, at the weak coupling, the two SF states break both the $U(1)_c$ order and also the $U(1)_{\text{soc}}$ symmetry, therefore supports two kinds of linear Goldstone modes shown in Fig.3b. However, at the strong coupling, to the leading order of $\lambda^2/U$, it was shown that the the Y-x state is the exact ground state not only free of frustrations, but also quantum fluctuations. It is a gapped Mott state which keep both $U(1)$ symmetry. So the two SFs to the Y-x state transition may go through the Y-x SF phase which still breaks the $U(1)_c$ symmetry, and restores the $U(1)_{\text{soc}}$ symmetry.
DISCUSSIONS

In fact, all the quantum phases and phase transitions in Fig.1 can be summarized as in the following global picture: The SOC parameter $\beta$ splits the phase diagram into a 1D SOC regime where there are only two minima at $(0, \pm \pi/2)$ which are independent of the SOC parameter $\beta$ and a 2D SOC regime where there are 4 minima at $(\pm \pi/2, \pm k_0)$ which depend on the SOC parameter $\beta$). The two regimes are connected by the bosonic Lifshitz transition driven by the softening of SF Goldstone mode with the dynamic exponent $(z = 1, z_y = 2)$ and the accompanying $Z_2$ transition breaking the $P_x$ symmetry in the spin sector. The spin anisotropy parameter $\lambda$ splits each regime into two: below $\lambda = 1$ the states are plane wave (PW) states, above $\lambda = 1$, the states are stripe ones breaking translational symmetries of the lattice. They are connected by second order transitions driven by the roton touchdowns. The Tetra-critical point in the $\lambda = 1$ line can be considered as the bosonic version of the fermionic topological transitions also connecting the two Abelian points with flux 0 and $\pi$ driven by the collisions of 2d Dirac fermions or 3d Weyl fermions. The extension to generic Rashba SOC parameter $(\alpha, \beta)$ which tunes the strengths of the frustrations will be presented in a separate publication. Of course, any $\alpha \neq \pi/2$ breaks the $U(1)_{\text{soc}}$ symmetry explicitly.

As elucidated in Ref., there is just one sign difference on the down spin hopping between the the Rashba SOC model Eq.1 and the quantum anomalous Hall (QAH) model. But this sign difference makes the two models show dramatically different phenomena. In the QAH model, the system has the $[C_4 \times C_4]D$ symmetry, at $h = 0$, it has an extra anti-unitary Reflection $Z_2$ symmetry. The transition driven by roton dropping is a first order one, the quantum Lifshitz transition driven by the softening of the SF Goldstone mode is a second order one. The two transitions meet at a topological Tri-critical (TT) point which separates the $N = 2$ XY-AFM from the $N = 4$ XY-AFM at $h = 0$ line. Both phases are gapped in the spin sector and break the $[C_4 \times C_4]D$ symmetry, but keeps the Reflection symmetry, so can only be distinguished by the topology of the BEC condensation momenta on the two sides of the transition instead of any symmetry principles. At the TT, the condensation momenta are along the two crossing lines $k_x = \pm k_y$ and the dispersion also becomes flat along the two lines. In the Rashba SOC model, the system has the three unitary $P_x, P_y, P_z$ symmetries at any $\lambda$. It has an extra $U(1)_{\text{soc}}$ symmetry at $\lambda = 1$. The transition driven by roton touchdown is a second order one, the quantum Lifshitz transition driven by the softening of the SF Goldstone mode is also a second order one. The two transitions meet at a Tetra-critical point at $\beta_c$ which separates the PW-X phase from the PW-XY phase at the $\lambda = 1$ line. Both phases support the gapless linear spin Goldstone mode due to the $U(1)_{\text{soc}}$ breaking, the PW-X phase keeps the $P_x$ symmetry, but the PW-XY phase breaks the $P_x$ symmetry. Both break the $P_y, P_z$ symmetry, so there is an accompanying $P_x$ symmetry breaking across the T critical point. Across the T point at $\beta_c$ (Fig.1a), there is just $N = 2$ BEC condensation momenta split into $N = 4$ BEC condensation momenta instead having any continuous distributions of BEC condensation momenta or flat directions in the dispersion relation, so the transition is just a conventional symmetry breaking transition instead of a topological one. The Rashba model in a Zeeman field at $\lambda = 1$ will be presented in a separate publication. Of course, any Zeeman field breaks explicitly the $U(1)_{\text{soc}}$ symmetry. The Rashba SOC (QAH) model is topological trivial (non-trivial). So this work is complementary to that in the QAH model studied in Ref. The results to be achieved may shed lights on each other. It remains interesting to investigate possible connections between the many body phenomena of weakly interacting spinor bosons and the topology of the underlying non-interacting energy bands.

The PW-X to Z-x ( or PW-XY to ZY-x ) phase transition as shown in Fig.3 can also be contrasted with a superfluid (SF) to a supersolid (SS) transition of a single component hard core boson in a triangular lattice slightly away from 1/3 ( or 2/3 ) filling, also driven by a roton touchdown. The main difference here is: In the former, the roton touchdown is tuned by the spin anisotropic parameter $\lambda$ inside a superfluid phase, at any fillings within a weak coupling limit. The translational symmetry breaking in Z-x ( or ZY-x ) comes from the spin-orbit coupling. While in the latter, the roton touchdown is tuned by $t/V_1$ where $V_1$ is the nearest neighbor (NN) interaction, near the SF to Mott transition at 1/3 ( or 2/3 ) filling. The solid component comes from the underlying Mott phase at 1/3 ( or 2/3 ) filling. The SF component comes from the interstitials ( or vacancies ) relative to the Mott phase.

Methods

1. The spin-orbital structure and the degeneracy of the Z-x and ZY-x state.

When $\beta > \beta_c$, as shown in Fig.M1a, the spectrum has 4 minima at $K_1 = (\pi, 0)$, $K_2 = (-\pi, 0)$, $K_3 = (2\pi, 0)$, $K_4 = (-\pi, -2\pi)$, where $k_0 = \arcsin \sqrt{\sin^2 \beta - \cot^2 \beta}$. The corresponding spinor is

$$\chi_i = \frac{1}{\sqrt{2}} \left(-e^{i\phi_i}\right), \quad i = 1, 2, 3, 4 \quad (25)$$

where $\phi_1 = \arcsin(\sin \beta \sqrt{1 + \sin^2 \beta}) = \phi_0$, $\phi_2 = \pi - \phi_0$, $\phi_3 = \pi + \phi_0$, $\phi_4 = -\phi_0$.

The 4-fold degenerate ZY-x state can be written as

$$\Psi_{1\pm 2}^{\text{ZY-x}} = \frac{1}{\sqrt{2}} e^{-iK_1 r_i} \chi_1 \pm \frac{1}{\sqrt{2}} e^{i\phi_0} e^{-iK_2 r_i} \chi_2,$$

$$\Psi_{3\pm 4}^{\text{ZY-x}} = \frac{1}{\sqrt{2}} e^{-iK_1 r_i} \chi_3 \pm \frac{1}{\sqrt{2}} e^{i\phi_0} e^{-iK_2 r_i} \chi_4 \quad (26)$$

whose spin-orbital structure can be easily evaluated as

$$\langle \Psi_{1\pm 2}^{\text{ZY-x}} | \tilde{d} | \Psi_{1\pm 2}^{\text{ZY-x}} \rangle = (0, -\sin \phi_0, \pm(1)^i \cos \phi_0)$$

$$\langle \Psi_{3\pm 4}^{\text{ZY-x}} | \tilde{d} | \Psi_{3\pm 4}^{\text{ZY-x}} \rangle = (0, \pm \sin \phi_0, \pm(1)^i \cos \phi_0) \quad (27)$$
which are shown in the following figure.

It is easy to see that the state differs by translation by one lattice site along the row, but reverses the Y component of the spin along the column. If setting \( k_0 = 0, \phi_0 = 0 \), the \( Y \) component vanishing, the 4-fold degenerate \( ZY-x \) reduces to the 2-fold degenerate \( Z \)-x state in the Fig.1b.

2. The kinetic energy and currents in all the 4 phases in Fig.1.

Using the method introduced, we will evaluate the conserved density current in all the 4 phases in Fig.1. In the context of 2d charge-vortex duality, the vortex currents in the dual lattice gives the boson densities in the direct lattice.

We can write the kinetic energy in Eq.2 as:

\[
\mathcal{H}_\text{hop} = \sum_{i,\mu} (\psi_i^\dagger H_\mu \psi_{i+\mu} + h.c.),
\]

where \( H_x = -ie^{i\pi/2} \sigma_z = -it \sigma_x, H_y = -ie^{i\beta} \sigma_y = -t(\cos \beta + i \sin \beta) \sigma_y \).

Along a given bond \((i,i+\mu)\):

\[
\psi_i^\dagger H_\mu \psi_{i+\mu} = K - iI
\]

where \( K \) is the kinetic energy and \( I \) is the current flowing along the bond.

As shown in Fig.1, for \( \lambda \leq 1 \), the ground-state is a PW state

\[
\psi_i = e^{iK_i - r_i} \chi_1,
\]

where, as listed in Eq.25, the \( k_0 \) and \( \phi_0 \) take the piecewise form

\[
k_0 = \begin{cases} 0, & \beta < \beta_c \\ \arcsin \sqrt{\sin^2 \beta - \cot^2 \beta}, & \beta > \beta_c \end{cases}
\]

and

\[
\phi_0 = \begin{cases} 0, & \beta < \beta_c \\ \arctan(\sin \beta \sqrt{1 + \sin^2 \beta}), & \beta > \beta_c \end{cases}
\]

By using the relations between \( k_0 \) and \( \phi_0 \)

\[
\tan k_0 = \tan \beta \sin \phi_0, \cos \phi_0 = \frac{1}{\sqrt{1 + \sin^2 \beta \sin^2 k_0}}
\]

We find

\[
\psi_i^\dagger H_x \psi_{i+\mu} = \begin{cases} -t, & \beta < \beta_c \\ \frac{-t}{\sin \beta \sqrt{1 + \sin^2 \beta}} \leq -t, & \beta > \beta_c \end{cases}
\]

and

\[
\psi_i^\dagger H_y \psi_{i+\mu} = \begin{cases} -t \cos \beta, & \beta < \beta_c \\ \frac{-t \sin \beta}{\sqrt{1 + \sin^2 \beta}} \leq -t \cos \beta, & \beta > \beta_c \end{cases}
\]

So there are no currents in both the PW-X and PW-XY states. It is easy to show that the two equalities hold at the quantum Lifshitz transition point \( \beta_c \).

For \( \lambda > 1 \), the 4 fold degenerate ground-state were written in Eq.27. We find the results are the same as the \( \lambda \leq 1 \) case. So the kinetic energies stay the same across the second order transition at \( \lambda = 1 \).

3. The gap and Goldstone mode generated by the order from disorder mechanism

In this method section, we provide a unified scheme to compute the mass generation of the roton mode at \( \lambda < 1 \) and the computation of spectrum of the Goldstone mode at \( \lambda = 1 \) by the order-from-disorder mechanism. The relations between the parametrization in the PW-X at \( \lambda < 1 \) and \( \lambda = 1 \) has been established in Eq.20. Here, we may just use the order from disorder variables in the \( \lambda < 1 \) case. Since \( B(\lambda) \) in Eq.14 and Eq.11 is continuous at \( \lambda = 1 \), so we only need focus on \( \lambda < 1 \) case and then take \( \lambda \to 1^{-} \) limit. We need to establish relation between the \( \phi \) variable in Eq.14 in the order from disorder analysis and the original Bose fields in Eq.1.

The Bose condensation Eq.8 inspires us to parameterize the most general Bose field in the polar-like coordinate system as

\[
\Psi_i = e^{i\chi} \sqrt{\eta_1 e^{iK_i - r_i} + c_2 \eta_2 e^{-iK_i - r_i}}
\]

where one can parameterize the two coefficients \( c_{1,2} \) in the most general forms as in Eq.7

\[
c_1 = [e^{i\phi/2} \cos(\theta/2) + e^{-i\phi/2} \sin(\theta/2)]/\sqrt{2}, \quad c_2 = [e^{i\phi/2} \cos(\theta/2) - e^{-i\phi/2} \sin(\theta/2)]/\sqrt{2}
\]

and the two components spinors \( \eta_{1,2} \) as:

\[
\eta_1 = \left( e^{i\phi/2} \cos(\theta/2), \quad e^{-i\phi/2} \sin(\theta/2) \right), \quad \eta_2 = \left( e^{i\phi/2} \cos(\theta/2), \quad e^{-i\phi/2} \sin(\theta/2) \right)
\]

Thus we can write \( \Psi_i = \Psi_i(\rho, \chi, \theta, \phi, \phi_1, \phi_2) \).

Since we already found the quantum ground state is the PW-X state which corresponds to the saddle point values:

\[
\Psi_i = (\rho_0, 0, \pi/2, 0, -\pi/2, 0, \pi/2, 0).
\]

One can write down the quantum fluctuations around the saddle point as:

\[
\Psi_i = \Psi_{i,0} + (\delta \rho, \delta \chi, \delta \theta, \delta \phi, \delta \phi_1, \delta \phi_2)
\]
Now, we can separate the Bose field into the condensation part plus the quantum fluctuation part in the polar coordinate system.

\[
\Psi_i = \sqrt{\frac{\rho_0}{2}} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{i\mathbf{K} \cdot \mathbf{r}_i} \\
+ \sqrt{\frac{\rho_0}{8}} \left( \begin{array}{c} \delta \rho + i\chi \\ -\delta \theta + i\phi \end{array} \right) \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{i\mathbf{K} \cdot \mathbf{r}_i} \\
+ \sqrt{\frac{\rho_0}{8}} \left( -\delta \theta + i\phi \right) \left( \begin{array}{c} 1 \\ 1 \end{array} \right) e^{-i\mathbf{K} \cdot \mathbf{r}_i},
\]

where the third line explicitly contains the order from disorder variable \( \phi \).

As written down in the previous sections, in the original Cartesian coordinate, we have used:

\[
\Psi_i = \sqrt{\frac{N_0}{2}} \left( \begin{array}{c} 1 \\ -1 \end{array} \right) e^{i\mathbf{K} \cdot \mathbf{r}_i} + \left( \frac{\delta \psi_{\uparrow}}{\delta \psi_{\downarrow}} \right).
\]

After comparing both the \( e^{i\mathbf{K} \cdot \mathbf{r}_i} \) and the \( e^{-i\mathbf{K} \cdot \mathbf{r}_i} \) components in the two equations Eq.41 and Eq.42, one can express the order from disorder variable \( \phi \) in terms of the original Bose field:

\[
\phi(q) = -\frac{i}{\sqrt{2\rho_0}} \left[ \delta \psi_{\mathbf{K} + q, \uparrow} + \delta \psi_{\mathbf{K} + q, \downarrow} - \delta \psi_{\mathbf{K} - q, \uparrow} - \delta \psi_{\mathbf{K} - q, \downarrow} \right]
\]

(43)

where as expected, only the quantum fluctuations near \( -\mathbf{K} \) appear in the relation.

Thus we can express the quantum correction coming from order-from-disorder mechanism in terms of the original Bose fields as

\[
\delta H = \sum_i \frac{B}{2} \phi_i^2 + \sum_q \frac{B}{2} \phi^q \delta \phi^q - \delta \phi^q
\]

(44)

which, we must stress, only holds near \( -\mathbf{K} \), so it will not affect the Goldstone mode near \( \mathbf{K} \).

Finally, after combining with \( H_2 \) in Eq.11 we arrive at the Order from disorder (OFD) corrected Hamiltonian:

\[
H_{OFD} = H^{(2)} + \delta H = \sum_q \Psi_q^\dagger (\mathbf{M} + \delta \mathbf{M}) \Psi^q
\]

(45)

where the \( 4 \times 4 \) matrix \( \mathbf{M} \) was already obtained from the PW-X calculation leading to Eq.31 and the \( \delta \mathbf{M} \) in Eq.34 can be written as in a \( 4 \times 4 \) matrix form:

\[
\delta \mathbf{M} = \frac{B}{2n_0} \left( \begin{array}{cccc} 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & 1 & 1 \end{array} \right)
\]

(46)

We diagonalize the order from disorder corrected Hamiltonian Eq.35 by a \( 4 \times 4 \) Bogoliubov transformation:

\[
\begin{align*}
H_{OFD} &= \sum_{\mathbf{q} \in BZ} \omega_{l,q} \left( \beta^\dagger_{l,q} \beta_{l,q} + \frac{1}{2} \right)
\end{align*}
\]

(47)

where \( \omega_{l,q} < \omega_{2,q} \). So only one need to focus on \( \omega_1 \).

From the complete form of the dispersion \( \omega_1 (\mathbf{K} + \mathbf{q}) \), we can extract the long wavelength limit of the roton mode as

\[
\omega_R(q) = \sqrt{\Delta_R + BR [\frac{q^2}{4} + (\cos \beta - C \sin \beta^2)q^2_\perp]}
\]

(48)

where \( \Delta_R = \omega_R(q = 0) = \sqrt{2BU(1 - \lambda)} \) is the roton gap in Eq.15 generated by the order-from-disorder mechanism. The two coefficients \( B_R = 2B/\rho_0 + \rho_0 U(1 - \lambda) \) and \( C = 1 + \frac{BU[4t(1 - \lambda)]}{2BU[4tU(1 - \lambda)]} \) represent the corrections to the roton dispersion. Obviously, setting \( B = 0 \), Eq.48 recovers the second equation in Eq.11.

Of course, due to the momentum separation as stressed below Eq.11, the superfluid Goldstone mode \( \omega_1 (\mathbf{q}) \) dictated by the \( U_c (1) \) symmetry breaking remains the same as listed in Eq.11 as

Going beyond Eq.15, Eq.18 not only contains the roton gap \( \Delta_R \) in Eq.15 but also the correction to the dispersion relation due to the order from disorder formalism. To the best of our knowledge, this is the first " order from disorder " calculation of the correction to the dispersion relation beyond a gap calculation.

By taking \( \lambda \to 1^- \) limit in Eq.18 the roton gap vanishes and recover the linear dispersion in Eq.22.

Acknowledgements

F. Sun and J. Y acknowledge AFOSR FA9550-16-1-0412 for supports. The work at KITP was supported by NSF PHY11-25915. J.W and Y.D thank the support by NSFC under Grant No. 11625522 and the Ministry of Science and Technology of China (under grants 2016YFA0301604).

1 Murthy, G., Arovas, D. & Auerbach, A. Superfluids and supersolids on frustrated two-dimensional lattices. Phys. Rev. B 55, 3104 (1997).

2 R. G. Melko, et al, Phys. Rev. Lett. 95, 127207 (2005).

3 D. Heidarian and K. Damle, Phys. Rev. Lett. 95, 127206 (2005);

4 S. Wessel and M. Troyer, Phys. Rev. Lett. 95, 127205 (2005).

5 A.A. Burkov and L. Balents, Phys. Rev. B 72, 134502 (2005).

6 R. G. Melko, A. Del Maestro, A. A. Burkov, Phys. Rev. B 74, 214517 (2006).

7 Longhua Jiang and Jinwu Ye, J. Phys. Cond. Matt. 18, 6907 (2006).
K. Sengupta, S. V. Isakov, and Y. B. Kim, Phys. Rev. B 73, 245103 (2006).
6. S. V. Isakov, S. Wessel, R. G. Melko, K. Sengupta, and Y. B. Kim, Phys. Rev. Lett. 97, 147202 (2006).
7. K. Damle and T. Senthil, Phys. Rev. Lett. 97, 067202 (2006).
8. Lih-King Lim, C. Morais Smith, and Andreas Hemmerich, Phys. Rev. Lett. 100, 130402 (2008).
9. G. M"uller and N. R. Cooper, Phys. Rev. A 82, 063625 (2010).
10. Fadi Sun, Junsen Wang, Jinwu Ye and Youjin Deng, Abelian flux induced magnetic frustrations of spinor bo-
     son superfluids on a square lattice, [arXiv:1711.06304].
11. Fadi Sun, Jinwu Ye, Wu-Ming Liu, Phys. Rev. A 92, 043609 (2015).
12. Y. A. Bychkov and E.I. Rashba, J. Phys. C 17, 6039 (1984)
13. Jinwu Ye, Yong Baek Kim, A. J. Millis, B. I. Shraiman, Y. A. Bychkov and E.I. Rashba, J. Phys. C 17, 6039 (1984).
14. Fadi Sun, Junsen Wang, Jinwu Ye and Youjin Deng, Phys. Rev. A 92, 147202 (2006).
15. Lianghui Huang, Zengming Meng, Pengjun Wang, Jing Zhang, Experimental realization of a two-
     dimensional synthetic spin-orbit coupling in ultracold Fermi gases, Nature Physics 12, 540-544 (2016). However, the heating is still serious which prevents the observation of many body phenomena due to the Rashba SOC for the alkali atoms. So the physics observed in is still at single particle physics.
16. Zengming Meng, Lianghui Huang, Peng Peng, Donghao Li, Liangchao Chen, Yong Xu, Chuanwei Zhang, Pengjun Wang, Jing Zhang, Experimental observation of topological band gap opening in ultracold Fermi gases with two-dimensional spin-orbit coupling, Phys. Rev. Lett. 117, 235304 (2016).
17. Zhan Wu, Long Zhang, Wei Sun, Xiao-Tian Xu, Bao-Zong Wang, Si-Cong Ji, Youjin Deng, Shuai Chen, Xiong-Jun Liu, Jian-Wei Pan, Realization of Two-Dimensional Spin-orbit Coupling for Bose-Einstein Condensates, Science 354, 83-88 (2016). Indeed, the lifetime of SOC $^{87}$Rb BEC was already made as long as 300ms in this work and improved to $\sim 1s$ recently, so the current experiment set-up the stage to observe any possible many body phenomena at a weak interaction where the heating rate is well under control.
18. O. Boada, A. Celi, J. I. Latore, and M. Lewenstein, Quantum Simulation of an Extra Dimension, Phys. Rev. Lett. 108, 133001 C Published 29 March 2012; Featured in Physics: Cultivating Extra Dimensions, Published 29 March 2012; A. Celi, P. Massignan, J. Ruseckas, N. Goldman, I. B. Spielman, G. Juzelius, and M. Lewenstein, Synthetic Gauge Fields in Synthetic Dimensions, Phys. Rev. Lett. 112, 043001 C Published 28 January 2014.
19. Michael L. Wall, Andrew P. Koller, Shuming Li, Xibo Zhang, Nigel R. Cooper, Jun Ye, Ana Maria Rey, Syn-
     thetic Spin-Orbit Coupling in an Optical Lattice Clock, Phys. Rev. Lett. 116, 035301 (2016).
20. L. F. Livi, G. Cappellini, M. Diem, L. Franchi, C. Clivati, M. Frittelli, F. Levi, D. Calonico, J. Catani, M. Inguscio, L. Fallani, Synthetic dimensions and spin-orbit coupling with an optical clock transition, Phys. Rev. Lett. 117, 220401 C Published 23 November 2016. Editors’ Suggestion.
21. S. Koikowitz, S.L. Bromley, T. Bothwell, M.L. Wall, G.E. Martl, A.P. Koller, X. Zhang, A.M. Rey, J. Ye, Spin-orbit coupled fermions in an optical lattice clock, Nature 542, 667C70 (02 February 2017) doi:10.1038/nature20811.
22. Fengzhao Alex An, Eric J. Meier, Bryce Gadway, Direct observation of chiral currents and magnetic reflection in atomic flux lattices, [arXiv:1609.00467].
23. Nathaniel Q. Burdick, Yijun Tang, and Benjamin L. Lev, Long-Lived Spin-Orbit-Coupled Degenerate Dipolar Fermi Gas, Phys. Rev. X 6, 031022 C Published 17 August 2016. Here we still use the same notation used in. The PW-X means the plane wave state with the spin polarization along the X direction. The PW-Y means the plane wave state with the spin polarization along the XY direction. In the Z-x ( or ZY-x ) phase, the first Capital letter indicates the spin polarization, the second small letter indicates the orbital order.
24. It maybe common that the exponents are different on two sides of topological phase transition, but it seems this is a first example which happens across a QPT. It is a new phenomenon due to the frustrations from the SOC.
25. Because the bosons can condense at one of the two minima in Fig.1a1. In the following, the momentum is measured relative to this condensation point.
26. Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger, Many-body physics with ultracold gases, Rev. Mod. Phys. 80, 885 (2008).
27. Both Abelian points contain symmetry connecting the two Abelian points along $\lambda = 1$ line in Fig.1.
28. M. Kozuma, et.al, Coherent Splitting of Bose-Einstein Condensed Atoms with Optically Induced Bragg Diffraction, Phys. Rev. Lett. 82, 871 (1999); J. Stenger, et al, Bragg Spectroscopy of a Bose-Einstein Condensate, Phys. Rev. Lett. 82, 4569 (1999); D. M. Stamper-Kurn et al, Excitation of Phonons in a Bose-Einstein Condensate by Light Scattering, Phys. Rev. Lett. 83, 2876 - 2879 (1999); J. Heinze, et.al, Phys. Rev. Lett. 88, Excitation Spectrum of a Bose-Einstein Condensate, 120407, (2002); S. B. Papp, et.al, Bragg Spectroscopy of a Strongly Interacting Rb 85 Bose-Einstein Condensate, Phys. Rev. Lett. 101, 135301 (2008).
29. P. T. Ernst, et al, Probing superfluids in optical lattices by momentum-resolved Bragg spectroscopy, Nature Physics 6, 56 (2010).
T. Stoferle et al., Transition from a Strongly Interacting 1D Superfluid to a Mott Insulator, Phys. Rev. Lett. 92, 130403 (2004).

G. Birkl, et al., Bragg Scattering from Atoms in Optical Lattices, Phys. Rev. Lett. 75, 2823 (1995); M. Weidemüller, et al., Bragg Diffraction in an Atomic Lattice Bound by Light, Phys. Rev. Lett. 75, 4583 (1995), Local and global properties of light-bound atomic lattices investigated by Bragg diffraction, Phys. Rev. A 58, 4647 (1998). J. Ruostekoski, C. J. Foot, and A. B. Deb, Light Scattering for Thermometry of Fermionic Atoms in an Optical Lattice, Phys. Rev. Lett. 103, 170404 (2009).

Si-Cong Ji, Long Zhang, Xiao-Tian Xu, Zhan Wu, Youjin Deng, Shuai Chen, and Jian-Wei Pan, Softening of Roton and Phonon Modes in Bose-Einstein Condensate with Spin-Orbit Coupling, Phys. Rev. Lett. 114, 105301 C Published 9 March 2015,

Jinwu Ye, J.M. Zhang, W.M. Liu, K.Y. Zhang, Yan Li, W.P. Zhang, Light scattering detection of various quantum phases of ultracold atoms in optical lattices, Phys. Rev. A 83, 051604 (R) (2011). See comments below.

Jinwu Ye, K.Y. Zhang, Yan Li, Yan Chen and W.P. Zhang, Optical Bragg, atom Bragg and cavity QED detections of quantum phases and excitation spectra of ultracold atoms in bipartite and frustrated optical lattices, Ann. Phys. 328 (2013), 103-138.

Kinast, J. et al. Heat Capacity of a Strongly Interacting Fermi Gas. Science 307, 1296 (2005).

Ku, M. J. H. et al. Revealing the Superfluid Lambda Transition in the Universal Thermodynamics of a Unitary Fermi Gas. Science 335, 563 (2012).

Gemelke, N., Zhang X., Huang C. L., and Chin, C. In situ observation of incompressible Mott-insulating domains in ultracold atomic gases, Nature (London) 460, 995 (2009).

S. Sachdev, Quantum Phase transitions, 2nd edition, Cambridge University Press, 2011

Fa-Di Sun, Xiao-Lu Yu, Jinwu Ye, Heng Fan, W. M. Liu, Topological Quantum Phase Transition in Synthetic Non-Abelian Gauge Potential: Gauge Invariance and Experimental Detections, Scientific Reports 3, 2119 (2013).

Fadi Sun and Jinwu Ye, Type I and Type II Weyl fermions, Topological depletions and sub-leading scalings across topological phase transitions, Phys. Rev. B 96, 035113 (2017).

Fadi Sun, Junsen Wang, Jinwu Ye, Shuai Chen and Youjin Deng, Symmetry protected bosonic topological phase transitions: Quantum Anomalous Hall system of spinor bosons in a square lattice, arXiv:1711.11580.

Junsen Wang, Fadi Sun, Jinwu Ye and Youjin Deng, preprint in preparation.

For reviews of bilayer quantum Hall systems, see S. M. Girvin and A. H. MacDonald, in Perspectives in Quantum Hall Effects, edited by S. Das Sarma and Aron Pinczuk (Wiley, New York, 1997).

Jinwu Ye, Excitonic superfluid to pseudo-spin density wave transition in bilayer quantum Hall systems, Phys. Rev. Lett. 98, 236802 (2007). Fractional charges and quantum phase transitions in imbalanced bilayer quantum Hall systems, Phys. Rev. Lett. 97, 236803 (2006); Mutual Composite Fermion and Composite Boson approaches to balanced and imbalanced bi-layer quantum Hall system: An electronic analogy of the Helium 4 system, Annals of Physics, 323, 580 (2008).