BROODING OVER PION LASERS

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Abstract

Brooding over pions, wave-packets and Bose-Einstein correlations, we present a recently obtained analytical solution to a pion laser model, which may describe the final state of pions in high energy heavy ion collisions.

Introduction. At the planned RHIC accelerator pions, the lightest observable strongly interacting particles, will be produced in thousands in a unit interval of the longitudinal phase-space. At such high multiplicities, the bosonic nature of the pions and the resulting Bose-Einstein statistics may dominate the behaviour of the observable physical quantities, like the multiplicity distribution, the single-particle momentum spectra and the second order intensity correlation function (frequently referred to as the Bose-Einstein correlation function or BECF). Present experiments at CERN SPS involving central collisions of two almost completely stripped nuclei (heavy ions) already produce cca 600 similar pions in a unit value of rapidity, when \(Pb + Pb\) reactions are measured at 160 AGeV laboratory bombarding energy. At the RHIC collider, to be completed by 1999, thousands of pions could be produced in a unit rapidity interval in a 40 TeV collision of \(Au\) nuclei \[1, 2\]. This multiplicity is large enough for the higher order Bose-Einstein symmetrizations to reveal their effects on the observables. We report here about a simple model \[3, 4\] that is analytically solvable both in the dense and the rare Bose-gas limit. The model is based on stimulated emission of pion wave-packets that is induced by the overlap of the multi-pion wave-packet states. The presentation closely follows the lines of ref. \[3\].

Specification of the Pion-Laser Model. If the number of pions in a unit value of phase-space is large enough these bosons may condense into the same quantum state and a pion laser could be created \[3\]. Similarly to this process, when a large number of bosonic atoms are collected in a magnetic trap and cooled down to increase their density in phase-space, the bosonic nature of the atoms reveals itself in the formation of a Bose-Einstein condensate \[4\], a macroscopic quantum state. Such a condensation mechanism may provide the key to the formation of atomic lasers in condensed matter physics and to the formation of pion lasers in high energy particle and heavy ion physics, reviewed recently in refs \[4, 5\].

The density matrix of a generic quantum mechanical system is

\[
\hat{\rho} = \sum_{n=0}^{\infty} p_n \hat{n},
\]

(1)
Here the index \( n \) characterizes sub-systems with particle number fixed to \( n \), the multiplicity distribution is prescribed by the set of \( \{ p_n \}_{n=0}^{\infty} \), normalized as \( \sum_{n=0}^{\infty} p_n = 1 \). The density matrixes are normalized as \( \text{Tr} \hat{\rho} = 1 \) and \( \text{Tr} \hat{\rho}_n = 1 \), where

\[
\hat{\rho}_n = \int d\alpha_1...d\alpha_n \rho_n(\alpha_1, ..., \alpha_n) |\alpha_1, ..., \alpha_n\rangle \langle \alpha_1, ..., \alpha_n|
\]

and the states \( |\alpha_1, ..., \alpha_n\rangle \) denote properly normalized \( n \)-particle wave-packet boson states.

A wave packet creation operator is

\[
\alpha_i^\dagger = \int \frac{d^3p}{(\pi \sigma^2)^{3/2}} e^{-\frac{(p-p_i)^2}{2\sigma^2_i}} e^{-i\xi_i(p-p_i)+i\omega(p)(t-t_i)} \hat{a}^\dagger(p),
\]

where \( \alpha_i = (\xi_i, \pi_i, \sigma_i, t_i) \) refers to the parameters of the wave packet \( i \): the center in space, in momentum space, the width in momentum space and the production time, respectively. For simplicity we assume that all the wave packets are emitted at the same instant and with the same width, \( \alpha_i = (\pi_i, \xi_i, \sigma, t_0) \). The \( n \) boson states, normalized to unity, are given as

\[
|\alpha_1, ..., \alpha_n\rangle = \left( \sum_{\sigma^{(n)}} \prod_{i=1}^{n} \langle \alpha_i | \alpha_{\sigma_i} \rangle \right)^{-1/2} \alpha_n^\dagger ... \alpha_1^\dagger |0\rangle.
\]

Here \( \sigma^{(n)} \) denotes the set of all the permutations of the indexes \( \{1, 2, ..., n\} \) and the subscript sized \( \sigma_i \) denotes the index that replaces the index \( i \) in a given permutation from \( \sigma^{(n)} \).

### Solution for a New Type of Density Matrix

There is one special density matrix, for which one can overcome the difficulty, related to the non-vanishing overlap of many hundreds of wave-packets, even in an explicit analytical manner. Namely, if one assumes, that we have a system, in which the emission probability of a boson is increased if there is another emission in the vicinity:

\[
\rho_n(\alpha_1, ..., \alpha_n) = \frac{1}{\mathcal{N}(n)} \prod_{i=1}^{n} \rho_1(\alpha_i) \left( \sum_{\sigma^{(n)}} \prod_{k=1}^{n} \langle \alpha_k | \alpha_{\sigma_k} \rangle \right)
\]

The coefficient of proportionality, \( \mathcal{N}(n) \), can be determined from the normalization condition. The density matrix of eq. (5) describes a quantum-mechanical wave-packet system with induced emission, and the amount of the induced emission is controlled by the overlap of the \( n \) wave-packets [4], yielding a weight in the range of \([1, n!]\). Although it is very difficult numerically to operate with such a wildly fluctuating weight, we were able to reduce the problem [4] to an already discovered “ring”-algebra of permanents for plane-wave outgoing states [4].

For the sake of simplicity we assume a non-relativistic, non-expanding static source at rest in the frame where the calculations are performed. This may correspond to the central, mid-rapidity piece of the highly relativistic fireball created at RHIC.

\[
\rho_1(\alpha) = \rho_x(\xi) \rho_p(\pi) \delta(t-t_0),
\]

\[
\rho_x(\xi) = \frac{1}{(2\pi R^2)^{3/2}} \exp(-\xi^2/(2R^2)),
\]

\[
\rho_p(\pi) = \frac{1}{(2\pi m T)^{3/2}} \exp(-\pi^2/(2m T)),
\]

(6)
and a Poisson multiplicity distribution \( p^{(0)}_n \) for the case when the Bose-Einstein effects are negligible:

\[
p^{(0)}_n = \frac{n^n_0}{n!} \exp(-n_0),
\]

This prescribes the multiplicity distribution of pion lasers in a very rare gas limit \([5]\), and completes the specification of the model. The plane-wave model, to which the multi-particle wave-packet model was reduced in ref. \([4]\), can be further simplified \([4]\) to a set of recurrence relations with the help of the so-called “ring-algebra” discovered first by S. Pratt in ref. \([5]\). The probability of finding events with multiplicity \( n \), as well as the single-particle and the two-particle momentum distribution in such events is given as

\[
p_n = \omega_n \left( \sum_{k=0}^{\infty} \omega_k \right)^{-1},
\]

\[
N^{(n)}_1(k_1) = \sum_{i=1}^{n} \frac{\omega_{n-i}}{\omega_n} G_i(1, 1),
\]

\[
N^{(n)}_2(k_1, k_2) = \sum_{i=1}^{n} \sum_{m=1}^{l-1} \frac{\omega_{n-l}}{\omega_n} [G_m(1, 1)G_{l-m}(2, 2) + G_m(1, 2)G_{l-m}(2, 1)],
\]

where \( \omega_n = p_n/p_0 \) and

\[
G_n(i, j) = n^n_0 h_n \exp(-a_n(k_i^2 + k_j^2) + g_n k_i k_j).
\]

Averaging over the multiplicity distribution \( p_n \) yields the inclusive spectra as

\[
G(1, 2) = \sum_{n=1}^{\infty} G_n(1, 2),
\]

\[
N_1(k_1) = \sum_{n=1}^{\infty} p_n N^{(n)}_1(k_1) = G(1, 1),
\]

\[
N_2(k_1, k_2) = G(1, 1)G(2, 2) + G(1, 2)G(2, 1).
\]

With the help of the notation

\[
\sigma^2_T = \sigma^2 + 2mT, \quad R^2_e = R^2 + \frac{mT}{\sigma^2_T},
\]

the recurrence relations that correspond to the solution of the ring-algebra \([3, 4]\) are obtained \([4]\) for the case of the multi-particle wave-packet model. These correspond to the pion laser model of S. Pratt when a replacement \( R \rightarrow R_e \) and \( T \rightarrow T_e = \sigma^2_T/(2m) \) is performed.

Let us introduce the following auxiliary quantities:

\[
\gamma_\pm = \frac{1}{2} \left( 1 + x \pm \sqrt{1 + 2x} \right), \quad x = R^2_e \sigma^2_T
\]

The general analytical solution for the multiplicity distribution of the model is given through the generating function of the multiplicity distribution \( p_n \)

\[
G(z) = \sum_{n=0}^{\infty} p_n z^n = \exp \left( \sum_{n=1}^{\infty} C_n (z^n - 1) \right),
\]
where $C_n$ is

$$C_n = \frac{n_0^n}{n} \left[ \frac{\Delta}{x} - \frac{\Delta^2}{x^2} \right]^{-3},$$

(18)

together with the general analytic solution for the functions $G_n(1, 2)$:

$$G_n(1, 2) = j_n e^{-\frac{n}{2\sigma_T^2}} \left[ (\Delta_k^2)^2 + (\Delta_{k_1}^2)^2 \right],$$

(19)

$$j_n = n^n \left[ \frac{b_n}{\sigma_T^2} \right]^\frac{3}{2},$$

(20)

The detailed proof that the analytic solution to the multi-particle wave-packet model is indeed given by the above equations is described in ref. [4]. The representation of eq. (17) indicates that the quantities $C_n$-s are the so-called combinants [10, 11, 12] of the probability distribution of $p_n$ and in our case their explicit form is known for any set of model parameters, as given by eqs. (18, 19, 20). One can prove [4], that the mean multiplicity $\langle n \rangle = \sum_{i=1}^{\infty} n p_n = \sum_{i=1}^{\infty} i C_i$. The large $n$ behavior of $nC_n$ depends on the ratio of $n_0/\gamma_+^2$, since for large values of $n$, we always have $(\gamma_-/\gamma_+)^2 << 1$. The critical value of $n_0$ is

$$n_c = \gamma_+^2 = \left[ \frac{1 + x + \sqrt{1 + 2x}}{2} \right]^{\frac{3}{2}}.$$

(21)

If $n_0 < n_c$, one finds $\lim_{n \to \infty} nC_n = 0$ and $\langle n \rangle < \infty$, if $n_0 > n_c$ one obtains $\lim_{n \to \infty} nC_n = \infty$ and $\langle n \rangle = \infty$, finally, if $n_0 = n_c$ one finds $\lim_{n \to \infty} nC_n = 1$ and $\langle n \rangle = \infty$. The divergence of the mean multiplicity $\langle n \rangle$ is related to condensation of the wave-packets to the wave-packet state with $\pi = 0$, i.e. to the wave-packet state with zero mean momentum [4], if $n_0 \geq n_c$.

The multiplicity distribution of eq. (17) is studied at greater length in ref. [4].

**Dense Bose-gas Limit:** This wave-packet model exhibits a lasing behavior in the very dense Bose-gas limit, which corresponds to an optically coherent behavior, characterized by a vanishing enhancement of the two-particle intensity correlations at low momentum, $C(k_1, k_2) = 1$, a case which is described in greater details in ref. [4].

**Rare gas limiting case.** Large source sizes or large effective temperatures correspond to the $x >> 1$ limiting case, where the general analytical solution of the model, presented above, becomes particularly simple and the exclusive and inclusive spectra and correlation functions can be obtained analytically to leading order in $1/x << 1$. From eq. (11) one obtains that

$$G_n(1, 2) = j_n \exp \left[ -\frac{n}{2\sigma_T^2} (k_1^2 + k_2^2) - \frac{\Delta^2}{2n} \Delta k^2 \right],$$

(22)

$$j_n = \frac{n^{5/2} C_n}{(\pi \sigma_T^2)^2}, \quad C_n = \frac{n_0^n}{n^4} \left( \frac{2}{x} \right)^{\frac{1}{2}(n-1)},$$

(23)

where $\Delta k = k_1 - k_2$. We can see from eq. (23), that the higher order corrections will contribute to the observables with reduced effective temperatures and reduced effective radii.

The very rare gas limiting case corresponds to keeping only the leading $n = 1$ order terms in the above equations. The multiplicity distribution is a Poisson distribution with $\langle n \rangle = n_0$ and
no influence from stimulated emission. The momentum distribution is a Boltzmann distribution, and the exclusive and inclusive momentum distributions coincide. The leading order two-particle Bose-Einstein correlation function is a Gaussian correlation function with a constant intercept parameter of $\lambda = 1$ and with a momentum-independent radius parameter of $R_e = R_e$.

The two-particle exclusive correlation functions can also be evaluated by applying a Gaussian approximation to the leading order corrections in the $x >> 1$ limiting case:

$$C_2^{(n)}(k_1, k_2) = \frac{n^2}{n(n-1)} \frac{N_2^{(n)}(k_1, k_2)}{N_1^{(n)}(k_1) N_1^{(n)}(k_2)} = 1 + \lambda_K \exp \left(-R_{K,s}^2 \Delta k_s^2 - R_{K,o}^2 \Delta k_o^2 \right),$$

where $K = 0.5(k_1 + k_2)$, the side and outwards directions are introduced utilizing the spherical symmetry of the source as $\Delta k_s = \Delta k - K(\Delta k \cdot K)/(K \cdot K)$ and $\Delta k_o = K(\Delta k \cdot K)/(K \cdot K)$, similarly to refs. [13]. The mean-momentum dependent intercept and radius parameters are

$$\lambda_K = 1 + \frac{2}{(2\pi)^2} \left[ 1 - 2^{(5/2)} \exp \left( -\frac{K^2}{\sigma_T^2} \right) \right],$$

$$R_{K,s}^2 = R_e^2 + \frac{1}{(2\pi)^2} \left[ R_e^2 - \sqrt{2} \exp \left( -\frac{K^2}{\sigma_T^2} \right) \left( (n + 2)R_e^2 + \frac{2}{\sigma_T^2} \right) \right],$$

$$R_{K,o}^2 = R_{K,s}^2 + \frac{n}{x^2} \frac{K^2}{\sigma_T^2} \exp \left( -\frac{K^2}{\sigma_T^2} \right).$$

Thus the symmetrization results in a momentum-dependent intercept parameter $\lambda_K$ that starts from a $\lambda_{K=0} < 1$ value at low momentum and increases with increasing momentum. Already in the first paper about the pion laser model, ref. [4], a reduction of the exact intercept parameter was observed and interpreted as the onset of a coherent behavior in the low momentum modes. First, a partially coherent system is created, characterized by $\lambda_K < 1$, and if the density of pions is further increased, one finds a fully developed pion-laser with $\lambda_K = 0$, see ref. [4] for analytic considerations.

Observe that the radius parameter at low mean momentum decreases while the radius parameter at high mean momentum increases, as compared to $R_e$. The radius parameter of the exclusive correlation function thus becomes momentum-dependent even for static sources! This effect is more pronounced for higher values of the fixed multiplicity $n$, in contrast to the momentum dependence of $\lambda_K$ that is independent of $n$.

Last, but not least, a specific term appears in the two-particle exclusive correlation function, that contributes only to the out direction, which, in case of spherically symmetric sources, may be identified with the direction of the mean momentum [13]. This directional dependence is related only to the direction of the relative momentum as compared to the direction of the mean momentum, and does not violate the assumed spherical symmetry of the boson source. The effect vanishes both at very low or at very high values of the mean momentum $K$, according to eq. (27).

**Highlights:** Our new analytic result is that multi-boson correlations generate momentum-dependent radius and intercept parameters even for static sources, as well as induce a special directional dependence of the correlation function. The effective radius parameter of the two-particle correlation function is reduced for low values and enlarged for large values of the mean momentum in the rare gas limiting case, as compared to the case when multi-particle symmetrization effects are neglected. From the numerical studies, described in ref. [3], we find that for an
extended, hot and rare gas of a few hundred pions, the reduction of the radius parameters at low momentum is found to be the most apparent effect. The directional dependence of the radius parameters and the enhancement of the radii at high momentum is characteristic for a small, cold pion gas with only a handful of particles in it. These results can be understood qualitatively by an enhancement of the wave-packets in the low momentum modes, due to multi-particle Bose-Einstein symmetrization effects, as the system starts to approach the formation of a laser, characterized by the appearance of partial optical coherence in the low momentum modes.

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