Towards spectral theory of the Nonstationary Schrödinger equation with a two-dimensionally perturbed one-dimensional potential

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Abstract

The Nonstationary Schrödinger equation with potential being a perturbation of a generic one-dimensional potential by means of a decaying two-dimensional function is considered here in the framework of the extended resolvent approach. The properties of the Jost solutions and spectral data are investigated.

1 Introduction

The Kadomtsev–Petviashvili equation in its version called KPI \cite{1-3}

\[(u_t - 6uu_{x_1} + u_{x_1x_1})_{x_1} = 3u_{x_2x_2},\]  

(1.1)

is a (2+1)-dimensional generalization of the celebrated Korteweg–de Vries (KdV) equation. As a consequence, the KPI equation admit solutions that behave at space infinity like the corresponding solutions of the KdV equation. For instance, if $u_1(t,x_1)$ obeys KdV, then $u(t,x_1,x_2) = u_1(t,x_1 + \mu x_2 + 3\mu^2 t)$ solves KPI for an arbitrary constant $\mu \in \mathbb{R}$. Thus, it is natural to consider solutions of (1.1) that are not decaying in all directions at space infinity but have 1-dimensional rays with behavior of the type of $u_1$. Without loss of generality, one can restrict to taking $\mu = 0$ (the generic case is reconstructed by means of the Galileo invariance of (1.1)).

Even though KPI has been known to be integrable for about three decades \cite{2,3}, the general theory is far from being complete. Indeed, the Cauchy problem for KPI with rapidly decaying initial data was first studied by using the Inverse Scattering Transform (IST) method in \cite{4-8}, the associated spectral operator being the nonstationary Schrödinger operator

\[\mathcal{L}(x,\partial_x) = i\partial_{x_2} + \partial_{x_1}^2 - u(x), \quad x = (x_1, x_2).\]  

(1.2)

However, it is known that the standard approach to the spectral theory of the operator (1.2), based on integral equations for the Jost solutions, fails for potentials with one-dimensional asymptotic behavior.

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In [9]-[16] the method of the extended resolvent was suggested as a way of pursuing a generalization of the IST that enables considering operators with nontrivial asymptotic behavior at space infinity.

The spectral theory for the simplest example of such potentials in [12] was developed in [17], where the Cauchy problem for the KPI equation was considered with initial data

\[ u(x) = u_1(x_1) + u_2(x), \tag{1.3} \]

\( u_1(x_1) \) being the value at time \( t = 0 \) of the one-soliton solution of the KdV equation and \( u_2(x) \) a smooth, real and rapidly decaying function on the \((x_1, x_2)\)-plane. In [17] the direct problem was studied by using a modified integral equation and it was shown that the modified Jost solution, in addition to the standard jump across the real \( k \)-axis, where \( k \) is the spectral parameter, also has a jump across a segment of the imaginary axis of the complex \( k \)-plane.

However, some essential properties of the Jost solutions and relations between spectral data were missing. In [18], in the framework of the extended resolvent approach, the problem was completely solved for the case where \( u_1 \) is a pure one-soliton potential. Here we extend the results to the case of generic (smooth and rapidly decaying) one-dimensional potential \( u_1 \). Some results are given here without proofs, which will be provided in a forthcoming publication.

2 Basic objects of the resolvent approach

Here we briefly review the basic elements of the extended resolvent approach. For further details, we refer the interested readers to [9]-[16].

Let \( A = A(x, \partial_x) \) denote a differential operator with kernel \( A(x, x') = A(x, \partial_x)\delta(x - x') \), \( \delta(x) = \delta(x_1)\delta(x_2) \) being the two-dimensional \( \delta \)-function. In what follows we consider differential operators whose kernels \( A(x, x') \) belong to the space \( S' \) of tempered distributions of the four real variables \( x \) and \( x' \). We introduce an extension of differential operators, i.e., to any differential operator \( A \) we associate the differential operator \( A(q) \) with kernel

\[ A(x, x'; q) = e^{-q(x-x')} A(x, x') = A(x, \partial_x + q)\delta(x - x'), \tag{2.1} \]

where \( x, x', q \in \mathbb{R}^2 \) and \( qx = q_1x_1 + q_2x_2 \). Such kernels form a subclass in the space \( S' \) of tempered distributions of six real variables. For generic elements \( A(x, x'; q) \) and \( B(x, x'; q) \) of this space \( S' \), we consider the standard composition rule

\[ (AB)(x, x'; q) = \int dx'' A(x, x''; q) B(x'', x'; q). \tag{2.2} \]

Since the kernels are distributions, this composition is neither necessarily defined for all pairs of operators nor necessarily associative.

An operator \( A \) can have an inverse in the sense of the composition law (2.2), say \( AA^{-1} = I \) or \( A^{-1}A = I \), where \( I \) is the unity operator in \( S' \), \( I(x, x'; q) = \delta(x - x') \).

The operation inverse to imbedding (2.1) consists in associating to any operator \( A(q) \), with kernel \( A(x, x'; q) \), its “hat-kernel”

\[ \hat{A}(x, x'; q) = e^{q(x-x')} A(x, x'; q). \tag{2.3} \]

In particular, for a differential operator \( A \) the following relations hold

\[ (\hat{A}B)(x, x'; q) = A(x, \partial_x)\hat{B}(x, x'; q), \quad (B\hat{A})(x, x'; q) = \hat{A}d(x', \partial_{x'})\hat{B}(x, x'; q), \tag{2.4} \]

where \( \hat{A}d \) is the dual operator to \( A \). In the following for equalities of the type (2.4) we shall use the notation

\[ \hat{AB} = \hat{A}\hat{B}, \quad B\hat{A} = \hat{B}\hat{A}. \tag{2.5} \]
2.1 Resolvent approach in the case of rapidly decaying potential

The operator extension $L(q)$ of $\mathcal{L}(x, \partial_x)$ in (1.2) is given by

$$L(q) = L_0(q) - U,$$  \hfill (2.6)

where $L_0$ has kernel

$$L_0(x, x'; q) = \left[ i(\partial_{x_2} + q_2) + (\partial_{x_1} + q_1)^2 \right] \delta(x - x'),$$  \hfill (2.7)

and $U$, which is called the potential operator, has kernel

$$U(x, x'; q) = u(x) \delta(x - x').$$  \hfill (2.8)

Note that we will always assume $u(x)$ to be real. The main object of our approach is the (extended) resolvent $M(q)$ of the operator $L(q)$, which is defined as the inverse of the operator $L$, i.e., $M$ satisfies

$$LM = ML = I.$$  \hfill (2.9)

The hat version (cf. (2.3)) of the extended resolvent for the “bare” operator $L_0$ is given by

$$\hat{M}_0(x, x'; q) = \frac{1}{2\pi i} \int_{k_3 = q_1} d k_\mathbb{R} \left[ \theta(x_2 - x'_2) - \theta(\ell_{23}(k) - q_2) \right] \Phi_0(x, k) \Psi_0(x', k),$$  \hfill (2.10)

(cf., for instance, [18] for details) where we introduced

$$\Phi_0(x, k) = e^{-i \ell(k) x}, \quad \Psi_0(x, k) = e^{i \ell(k) x},$$  \hfill (2.11)

and the two component vector

$$\ell(k) = (k, k^2), \quad k = k_\mathbb{R} + i k_3 \in \mathbb{C}.$$  \hfill (2.12)

Note that we use boldface font to indicate that the corresponding variables are complex valued. The functions $\Phi_0(x, k)$ and $\Psi_0(x, k)$ solve the nonstationary Schrödinger equation (1.2) and its dual in the case of zero potential, and they can be considered as the Jost solutions for this trivial case. Note also that they obey the conjugation property

$$\overline{\Phi_0(x, k)} = \Psi_0(x, \overline{k}).$$  \hfill (2.13)

Using notation (2.5), we can rewrite eqs. (2.10) for the case of zero potential in the form

$$\hat{L}_0 \hat{M}_0(q) = \hat{M}_0(q) \hat{L}_0 = I,$$  \hfill (2.14)

which shows that $\hat{M}_0(q)$ is a two-parametric set of Green’s functions of (1.2) and its dual.

From (2.10) one directly obtains that for $q_1 \neq 0$

$$\frac{\partial \hat{M}_0(q)}{\partial q_1} = \frac{i}{\pi} \int_{k_3 = q_1} d k_\mathbb{R} \kappa \delta(\ell_{23}(k) - q_2) \Phi_0(k) \otimes \Psi_0(k),$$  \hfill (2.14)

$$\frac{\partial \hat{M}_0(q)}{\partial q_2} = \frac{1}{2\pi i} \int_{k_3 = q_1} d k_\mathbb{R} \delta(\ell_{23}(k) - q_2) \Phi_0(k) \otimes \Psi_0(k),$$  \hfill (2.15)

where the direct product in (2.14), (2.15) is defined in the standard way as an operator with kernel

$$(\Phi_0(k) \otimes \Psi_0(k))(x, x') = \Phi_0(x, k) \Psi_0(x', k).$$  \hfill (2.16)
2.1.1 The resolvent and Hilbert identity

The resolvent of the operator $L$ in (2.6) with potential $u$ rapidly decaying can also be defined as the solution of the integral equations

$$
M = M_0 + M_0UM, \quad M = M_0 + MUM_0.
$$

(2.17)

Under a small norm assumption on the potential we expect that the solution $M$ exists and is unique (the same for both integral equations).

The resolvent satisfies the following analog of the Hilbert identity

$$
M' - M = -M'(L' - L)M,
$$

(2.18)

where $L'$ is another operator of the type (1.2) and $M'$ is its resolvent. Eq. (2.18) can be written in the form

$$
M(q') - M(q) = M(q')L_0(q')(M_0(q') - M_0(q))L_0(q)M(q),
$$

(2.19)

which can be used to obtain for the derivatives of the resolvent $\hat{M}(q)$ the following expressions

$$
\frac{\partial \hat{M}(q)}{\partial q_j} = \hat{M}(q) \frac{\partial \hat{M}_0(q)}{\partial q_j} \hat{L}_0(q), \quad j = 1, 2.
$$

(2.20)

Then by (2.13) and (2.15) for $q_1 \neq 0$

$$
\frac{\partial \hat{M}(q)}{\partial q_1} = i \frac{1}{\pi} \int_{k_3 = q_1} \, d k_R \, \hat{k} \delta(\ell_{23}(k) - q_2) \Phi(k) \otimes \Psi(k),
$$

(2.21)

$$
\frac{\partial \hat{M}(q)}{\partial q_2} = \frac{1}{2\pi i} \int_{k_3 = q_2} \, d k_R \, \delta(\ell_{23}(k) - q_2) \Phi(k) \otimes \Psi(k),
$$

(2.22)

where we introduced the functions

$$
\Phi(x, k) = \int dx' \left( \hat{L}_0^+(x', \partial_{x'}) \hat{G}(x, x', k) \right) \Phi_0(x', k),
$$

(2.23)

$$
\Psi(x', k) = \int dx \, \Psi_0(x, k) \hat{L}_0(x, \partial_x) \hat{G}(x, x', k),
$$

(2.24)

with $G(x, x', k)$ defined as a specific value of the resolvent itself

$$
G(x, x', k) = \hat{M}(x, x'; q) \big|_{q = \ell_3(k)} \equiv \hat{M}(x, x'; k_3, 2k_3 k_R).
$$

(2.25)

In what follows we consider the function $G(x, x', k)$ as the kernel of the operator $G(k)$ and the functions $\Phi(x, k)$ and $\Psi(x', k)$ as “vector” $\Phi(k)$ and “covector” $\Psi(k)$. For shortness we write equations of the type (2.23)–(2.24) omitting the $x, x'$-dependence, i.e. as

$$
\Phi(k) = G(k) \hat{L}_0 \Phi_0(k), \quad \Psi(k) = \Psi_0(k) \hat{L}_0 \hat{G}(k),
$$

(2.26)

$$
\hat{G}(k) = \hat{M}(q) \big|_{q = \ell_3(k)}.
$$

(2.27)

In order to study the discontinuity at $q = 0$ we introduce the following notation for the specific limits of the resolvent at this point of discontinuity

$$
G_{\pm}(x, x') = \lim_{q_2 \rightarrow \pm 0} \lim_{q_1 \rightarrow 0} \hat{M}(x, x'; q),
$$

(2.28)
where the limit \( q_1 \to 0 \) is independent of the sign. Using again the Hilbert identity \((2.19)\), we get

\[
G_+ - G_- = \frac{1}{2\pi i} \int dk \Phi_\pm (k) \otimes \Psi_\mp (k),
\]

where we introduced the functions \( \Phi_\pm (x, k) \) and \( \Psi_\mp (x, k) \), which are analogous to \((2.20)\), are defined by

\[
\Phi_\pm (k) = G_\pm \to L_0 \Phi_0(k), \quad \Psi_\mp (k) = \Psi_0(k) \to L_0 G_\pm.
\]

Next we consider in details the properties of all the objects introduced so far.

### 2.1.2 Properties of the Green’s function

Thanks to \((2.5)\) it is clear that \( G(k) \) defined in \((2.25)\) is a Green’s function of the operator \( L \) depending on the complex parameter \( k \), i.e., \( \hat{L} G(k) = G(k) \hat{L} = I \). Also, due to the reality of \( u(x) \), we have

\[
G(x, x', k) = G(x', x, k).
\]

Applying the reduction \((2.25)\) to equalities \((2.17)\) it follows that this function obeys the integral equations

\[
G(k) = G_0(k) + G_0(k) U G(k), \quad G(k) = G_0(k) + G(k) U G_0(k),
\]

where the Green’s function \( G_0(k) \) of the operator \( L_0 \) is defined by the general formula \((2.27)\) in terms of \( M_0 \). By \((2.32)\) one can check that

\[
g(x, x', k) = e^{i\ell_0(x-x')} g(x, x', k)\]

is a bounded function of its arguments and that \( \lim_{k \to \infty} g(x, x', k) = 0 \) if the potential \( u(x) \) decays rapidly enough.

The function \( G(x, x', k) \) is a continuously differentiable function of \( k \) in the whole complex plane \( \mathbb{C} \) with exception of the real axis \( k_\Re = 0 \). By \((2.21)\) and \((2.22)\) for \( k_\Re \neq 0 \) we have

\[
\frac{\partial G(k)}{\partial k_\Re} = \frac{\text{sgn} k_\Re}{2\pi i} \Phi(k) \otimes \Psi(k), \quad \frac{\partial G(k)}{\partial k_\Im} = \frac{\text{sgn} k_\Im}{2\pi} \Phi(k) \otimes \Psi(k),
\]

so that in the complex domain this Green’s function is analytic for \( k_\Re \neq 0 \) and discontinuous at the real axis.

Properties of the functions \( G_\pm (x, x') \) as well follow from the properties of the resolvent. Both of them are also Green’s functions of the operator \((1.2)\) and its dual, obey the conjugation property

\[
G_\pm (x, x') = G_{\mp}(x', x),
\]

and satisfy integral equations

\[
G_\pm = G_{0, \pm} + G_{0, \pm} U G_\pm, \quad G_\pm = G_{0, \pm} + G_\pm U G_{0, \pm},
\]

where \( G_{0, \pm} \) is given by

\[
G_{0, \pm} (x, x') = \frac{\pm \theta(\pm(x_2 - x'_2))}{2\pi i} \int dk \Phi_0(x, k) \Psi_0(x', k).
\]

Since the resolvent is discontinuous at \( q = 0 \), the limiting values of the Green’s function \( G(k) \) on the real axis, \( G^{\pm}(k) = G(k \pm i0) \), do not coincide, in general, with the Green’s functions...
\( G_\pm \). In order to find relations between them we again start from (2.19) choosing both \( q' \) and \( q \) to be real. Performing for \( q' \) and \( q \) proper limiting procedures one obtains

\[
G^\sigma(k) - G_\pm = \mp \frac{1}{2\pi i} \int dp \theta(\mp \sigma(k - p)) \Phi_\pm(p) \otimes (\Psi_0(p) \bar{\mathcal{L}}_0^* G^\sigma(k)),
\]

(2.37)

\[
G^\sigma(k) - G_\pm = \mp \frac{1}{2\pi i} \int dp \theta(\mp \sigma(k - p)) \left( G^\sigma(k) \bar{\mathcal{L}}_0^* \Phi_0(p) \right) \otimes \Psi_\pm(p), \quad \sigma = +, -,
\]

(2.38)

where in the rhs we used a shorthand notation analogous to that in 2.21.

2.1.3 Jost and advanced/retarded solutions and bilinear representation for the resolvent

It is straightforward to check that the functions \( \Phi(x, k) \) and \( \Psi(x, k) \) defined by (2.26) obey the nonstationary Schrödinger equation with potential \( u(x) \) and its dual, i.e., \( \bar{\mathcal{L}} \Phi(k) = \Psi(k) \bar{\mathcal{L}} = 0 \), and, thanks to (2.31), they satisfy the conjugation property

\[
\Phi(x, k) = \Psi(x, k).
\]

(2.39)

The integral equations for these functions

\[
\Phi(k) = \Phi_0(k) + G_0(k) U \Phi(k), \quad \Psi(k) = \Psi_0(k) + \Psi(k) U G_0(k),
\]

(2.40)

follow from (2.32). Finally, they obey the orthogonality relation

\[
\frac{1}{2\pi} \int dx \, \Psi(x, k + p) \Phi(x, k) = \delta(p), \quad k \in \mathbb{C}, \quad p \in \mathbb{R}.
\]

(2.41)

The properties of the functions \( \Phi_\pm(x, k) \) and \( \Psi_\pm(x, k) \) are derived analogously from their definition (2.30). They also satisfy \( \bar{\mathcal{L}} \Phi_\pm(k) = \Psi_\pm(k) \bar{\mathcal{L}} = 0 \), the integral equations

\[
\Phi_\pm(k) = \Phi_0(k) + G_{0,\pm} U \Phi_\pm(k), \quad \Psi_\pm(k) = \Psi_0(k) + \Psi_\pm(k) U G_{0,\pm},
\]

(2.42)

and the conjugation property

\[
\Phi_\pm(x, k) = \Psi_\mp(x, k).
\]

(2.43)

Also \( \Phi_\pm(x, k) \) and \( \Psi_\pm(x, k) \) satisfy an orthogonality relation, i.e.,

\[
\frac{1}{2\pi} \int dx \, \Psi_\pm(x, k + p) \Phi_\mp(x, k) = \delta(p), \quad k, p \in \mathbb{R}.
\]

(2.44)

The properties of the Jost solutions enable us to reconstruct \( M(q) \) from (2.21) in the form

\[
\tilde{M}(x, x'; q) = \frac{1}{2\pi i} \int \frac{dk}{k_{\pm} = q_1} \left[ \theta(x_2 - x'_2) - \theta(\ell_{23}(k) - q_2) \right] \Phi(x, k) \Psi(x', k),
\]

(2.45)

that generalizes (2.11) to the case of nonzero potentials. Continuity of the resolvent at \( q_1 = 0 \) when \( q_2 \neq 0 \) implies that

\[
\int dk \, \Phi^+(k) \otimes \Psi^+(k) = \int dk \, \Phi^-(k) \otimes \Psi^-(k)
\]

(2.46)
where we introduced for the limiting values of the Jost solutions at the real axis the notation
\[ \Phi^\pm(x, k) = \Phi(x, k \pm i0), \quad \Psi^\pm(x, k) = \Psi(x, k \pm i0). \] (2.47)

Representation (2.45) plays a crucial role in the resolvent approach since it enables us to express all objects of the spectral theory in terms of the Jost solutions. In particular, thanks to (2.25), for the Green’s function of the Jost solutions we get
\[ G(x, x', k) = \frac{1}{2\pi i} \int dp \left[ \theta(x_2 - x'_2) - \theta(k_3 p) \right] \Phi(x, p + k) \Psi(x', p + k). \] (2.48)

Moreover, from (2.45) one can get the following expression of the advanced/retarded Green’s functions in terms of the limiting values of the Jost solutions
\[ G^\pm(x, x', k) = \pm \theta(\pm(x_2 - x'_2)) \frac{1}{2\pi i} \int dp r^\sigma(p, k) \Psi^\sigma(x', k), \] (2.49)

where \( \sigma = +, - \) and we used notation (2.47). Note that condition (2.46) guarantees that the \( G^\pm \)'s are independent of the sign \( \sigma \) in the rhs.

From the bilinear representation (2.45) we also obtain the following of completeness relation the Jost solutions
\[ \frac{1}{2\pi} \int_{x'_2 = x_2} dk \Re \Psi(x', k) \Phi(x, k) = \delta(x_1 - x'_1). \] (2.50)

2.1.4 Relations among Jost and advanced/retarded solutions. Spectral data
We obtained the bilinear representations (2.48) and (2.49) of the Green’s functions for the Jost solutions of the Green’s function for the advanced/retarded solutions. We derived also equations (2.37) and (2.38) relating these Green’s functions. These relations can be exploited for deriving relations among the advanced/retarded and Jost solutions on the real axis and to use them to introduce the scattering data. In fact, applying \( \overleftarrow{L}_0 \Phi_0(k) \) to (2.37) from the right and \( \overrightarrow{L}_0 \Psi_0(k) \) to (2.38) from the left, recalling definitions (2.26) and (2.30), we get (notice that \( k \in \mathbb{R}, \sigma = +, - \))
\[ \Phi^\sigma(k) = \int dp \Phi^\pm(p) r^\sigma_\pm(p, k), \quad \Psi^\sigma(k) = \int dp \overrightarrow{r}^\pm_\sigma(p, k) \Psi^\sigma_\pm(p), \] (2.51)

where
\[ r^\sigma_\pm(p, k) = \delta(p - k) \pm \theta(\pm(p - k)) \overrightarrow{r}^\sigma(p, k), \quad p, k \in \mathbb{R}, \] (2.52)
\[ \overrightarrow{r}^\sigma(p, k) = \frac{\Psi_0(p) \overrightarrow{L}_0 G^\sigma(k) \overrightarrow{L}_0 \Psi_0(k)}{2\pi i} = \frac{\Psi_0(p) \overrightarrow{L}_0 \Phi^\sigma(k)}{2\pi i}, \] (2.53)

are the scattering data. Recalling notations (2.23), (2.24) and (2.26), the “expectation values” at the numerator have the following explicit expressions
\[ \Psi_0(p) \overrightarrow{L}_0 G^\sigma(k) \overrightarrow{L}_0 \Phi_0(k) = \int dx \int dx' \Psi_0(x, p) \left( \mathcal{L}_0(x', \partial_x) \mathcal{L}^*_0(x, \partial_x') G^\sigma(x, x', k) \right) \Phi_0(x', k), \] (2.54)
\[ \Psi_0(p) \overrightarrow{L}_0 \Phi^\sigma(k) = \int dx \Psi_0(x, p) \mathcal{L}_0(x, \partial_x) \Phi^\sigma(x, k), \] (2.55)
showing that they are functions of \( p \) and \( k \) only.

In order to get the advanced/retarded solutions in terms of the boundary values of the Jost ones we use (2.48) and the limiting values of \((2.58)\) on the real axis to obtain

\[
\mathcal{G}^\sigma(k) - \mathcal{G}^\pm = \mp \frac{1}{2\pi i} \int dp \phi(\pm \sigma(p-k)) \Phi^\sigma(p) \otimes \Psi^\sigma(p), \quad \sigma = +, -, \quad (2.56)
\]

and in the same way as above we readily derive

\[
\Phi^\pm(k) = \int dp \Phi^\sigma(p) r^\sigma_\pm(p,k), \quad \Psi^\pm(k) = \int dp \Psi^\sigma(p) r^\sigma_\pm(p,k). \quad (2.57)
\]

Now inserting these expressions into \((2.51)\) and taking into account the orthogonality properties \((2.41)\) and \((2.44)\) of the Jost and advanced/retarded solutions we derive that the spectral data obey the following characterization equations \((2.58)\)

\[
\int dp r^\sigma_\pm(p,k) r^\sigma_\pm(p,k') = \delta(k-k'), \quad (2.58)
\]

\[
\int dp r^\sigma_\pm(k',p) r^\sigma_\pm(k,p) = \delta(k-k'), \quad \sigma = +, -, \quad (2.59)
\]

\[
\int dp r^\sigma_\pm(p,k) r^\sigma_\pm(p,k') = \int dp r^\sigma_\pm(p,k) r^\sigma_\pm(p,k'). \quad (2.60)
\]

If we introduce the alternative scattering data

\[
F^\sigma(k,k') = \int dp r^\sigma_\pm(p,k) r^\sigma_\pm(p,k'), \quad (2.61)
\]

we can express the discontinuity of the Jost solutions across the real axis as follows

\[
\Phi^\sigma(k) = \int dp \Phi^{-\sigma}(p) F^{-\sigma}(p,k), \quad \Psi^\sigma(k) = \int dp F^\sigma(k,p) \Psi^{-\sigma}(p). \quad (2.62)
\]

Thanks to \((2.60)\) these scattering data are independent of the choice of the + and − sign in the r.h.s. of \((2.61)\). These scattering data satisfy the characteristic equations

\[
(F^\sigma)^\dagger = F^{\sigma}, \quad F^{-\sigma} = (F^{\sigma})^{-1} \quad (2.63)
\]

to which we have to add the requirement that \( F^{\sigma} \) can be decomposed in the products \((2.52)\) of two sets of triangular operators \((2.52)\). Here again the first equation in \((2.63)\) is related to the reality requirement for \( u(x) \) while the second equation is a regularity requirement.

The inverse problem can be formulated in the standard way by using the analyticity properties of the Jost solutions. In \([13]\) we also demonstrated that the inverse problem can be formulated in terms of the resolvent itself. Here we skip this for shortness, as well as many other results that follows from the extended resolvent approach.

## 3 Case of one-dimensional potential

### 3.1 Resolvent approach

Now we consider the imbedding of the standard one-dimensional scattering transform in terms of two-dimensional differential operator with one-dimensional potential. Specifically, we consider the extended differential operator

\[
L_1(q) = L_0(q) - U_1, \quad U_1(x,x';q) = u_1(x_1)\delta(x - x'), \quad (3.1)
\]
where $L_q$ is defined in [2.7]. For the Jost solution of the nonstationary Schrödinger equation and its dual we use notation

\begin{align}
\varphi(x, k) &= e^{-i k^2 x} \Phi_1(x_1, k) = e^{-i k x_1 - i k^2 x} \chi_1(x_1, k), \\
\psi(x, k) &= e^{i k^2 x} \Psi_1(x_1, k) = e^{i k x_1 + i k^2 x} \xi_1(x_1, k),
\end{align}

where $\Phi_1(x_1, k)$ and $\Psi_1(x_1, k)$ are the Jost solutions of the one-dimensional Sturm-Liouville operator, namely $\chi_1$ is defined via the integral equation

\begin{equation}
\chi_1(x_1, k) = 1 + \int_{-\infty}^{x_1} dy \frac{e^{2i k (y - y_1)} - 1}{2i k} u_1(y_1) \chi_1(y_1, k)
\end{equation}

and a similar equation defines the dual Jost solution. Functions $\varphi(x, k)$ and $\psi(x, k)$, as well as their boundary values at the real axis, obey a conjugation property like (2.39), i.e.,

\begin{equation}
\varphi(x, k) = \psi(x, \bar{k}), \quad \varphi^\pm(x, k) = \psi^\mp(x, k), \quad k \in \mathbb{R}
\end{equation}

and satisfy orthogonality and completeness relations of the form

\begin{align}
\frac{1}{2\pi} \int dx_1 \psi(x, k + p) \varphi(x, k) &= \frac{\delta(p)}{t_1(k)}, \quad p \in \mathbb{R}, \quad k \in \mathbb{C}, \\
\int dx_1 \psi(x, i\kappa_j) \varphi(x, k) &= \int dx_1 \psi(x, k) \varphi(x, i\kappa_j) = 0, \quad |k_\| < \kappa_j, \\
\int dx_1 \psi(x, i\kappa_j) \varphi(x, i\kappa_j') &= \frac{i\delta_{j,j'}}{t_j}, \\
\frac{1}{2\pi} \int_{x_2}^{x_2} d\kappa R t_1(k) \varphi(x, k) \psi(x', k) - \\
&- i \sum_{j=1}^{N} t_j \theta(\kappa_j - |k_\|) \varphi(x, i\kappa_j) \psi(x', i\kappa_j) \bigg|_{x_2} = \delta(x_1 - x'_1)
\end{align}

where $t_1(k)$ is the transmission coefficient of the one-dimensional problem, with poles at points $\pm i\kappa_j$ (for definiteness, we assume $\kappa_j > 0$ for all $j = 1, \ldots, N$), and residues at these points given by

\begin{equation}
t_{\pm j} = \text{res}_{k = \pm i\kappa_j} t_1(k).
\end{equation}

Another set of discrete spectral data for the one-dimensional problem is given by the coefficients $b_j$, defined by one of the following equalities

\begin{equation}
\Phi_1(x_1, i\kappa_j) = b_j \Phi_1(x_1, -i\kappa_j), \quad \Psi_1(x_1, -i\kappa_j) = b_j \Psi_1(x_1, i\kappa_j).
\end{equation}

These coefficients are real and nonzero and such that

\begin{equation}
\text{sgn}(it_j b_j) = -1.
\end{equation}

By [3,11] we have

\begin{equation}
\varphi(x, i\kappa_j) = b_j \varphi(x, -i\kappa_j), \quad \varphi(x, i\kappa_j) = b_j \overline{\varphi(x, i\kappa_j)}.
\end{equation}
3.2 Resolvent

The resolvent of the one-dimensional $L$-operator that depends on $q_2$ as a parameter is given by

\[
\tilde{M}_1(x, x'; q) = \frac{1}{2\pi i} \int_{\mathbb{R}} d\mathbf{k}_\alpha \left[ \theta(x_2 - x_2') - \theta(2\mathbf{k}_\alpha - q_2) \right] t_1(\mathbf{k}) \varphi(x, \mathbf{k}) \psi(x', \mathbf{k}) - \sum_j \theta(x_j^2 - q_j^2) t_j(x_2 - x_j^2) \varphi(x, i\alpha_j) \psi(x', i\alpha_j).
\] (3.14)

We decompose this kernel into the sum of a regular and a singular part, so that

\[
\tilde{M}_1(q) = \tilde{M}_{1,\text{reg}}(q) + \sum_j \Gamma_j(q) \varphi(i\alpha_j) \otimes \psi(i\alpha_j),
\] (3.15)

with $x$-independent functions

\[
\Gamma_j(q) = \frac{t_j \text{sgn} q_1}{2\pi i} \log \frac{q_2 + 2iq_1(q_1 - \alpha_j)}{q_2 + 2iq_1(q_1 + \alpha_j)}, \quad j = 1, \ldots, N,
\] (3.16)

where we use for the logarithm the following definition

\[
\log z = \log |z| + i \arctan \frac{\Im z}{\Re z} + i\pi \theta(-z_\Re) \text{sgn} z_\Im.
\] (3.17)

With such definition, it follows that the extended resolvent in the case of one-dimensional potential has logarithmic singularities at the points $q = (\pm\alpha_j, 0)$ with a cut along $q_2 = 0$, since $\Gamma_j(q_1, +0) - \Gamma_j(q_1, -0) = -t_j \theta(\alpha_j - |q_1|)$.

3.3 Properties of the resolvent and of the Green’s function.

For the discontinuity of the resolvent at $q_2 = 0$ we get from (3.14)

\[
\tilde{M}_1(q_1, +0) - \tilde{M}_1(q_1, -0) = -\sum_j t_j \theta(\alpha_j - |q_1|) \varphi(i\alpha_j) \otimes \psi(i\alpha_j).
\] (3.18)

For all other values of $q$ the resolvent (3.14) has derivatives with respect to $q$ of the form of (2.14), (2.15). For instance

\[
\frac{\partial \tilde{M}_1(q)}{\partial q_2} = \frac{1}{2\pi i} \int_{\mathbb{R}} d\mathbf{k}_\alpha \delta(\ell_{x_2}(\mathbf{k}) - q_2) t_1(\mathbf{k}) \varphi(\mathbf{k}) \otimes \psi(\mathbf{k}),
\] (3.19)

which at the vicinity of the point $q_1 = 0$ has to be considered in the distributional sense. Thus, we can introduce the Green’s function $G_1(x, x', \mathbf{k})$ of the Jost solutions by using a reduction analogous to (2.24). Then, from (3.14) we get the bilinear representation

\[
G_1(x, x', \mathbf{k}) = \frac{1}{2\pi i} \int d\alpha \left[ \theta(x_2 - x_2') - \theta(\mathbf{k}_\alpha - \alpha - \mathbf{k}_\Re) \right] t_1(\alpha + i\mathbf{k}_\Im) \times
\]

\[
\varphi(x, \alpha + i\mathbf{k}_\Im) \psi(x', \alpha + i\mathbf{k}_\Im) - \sum_j t_j \theta(\alpha_j - |\mathbf{k}_\Im|) \left[ \theta(x_2 - x_2') - \theta(-\mathbf{k}_\Re \mathbf{k}_\Im) \right] \varphi(x, i\alpha_j) \psi(x', i\alpha_j),
\] (3.20)

\[
- \sum_j t_j \theta(\alpha_j - |\mathbf{k}_\Im|) \left[ \theta(x_2 - x_2') - \theta(-\mathbf{k}_\Re \mathbf{k}_\Im) \right] \varphi(x, i\alpha_j) \psi(x', i\alpha_j),
\] (3.21)
that generalizes to the case of generic potential \( u_1 \) the Green’s function used in \( \cite{17} \) and \( \cite{18} \) for pure one-soliton potential. This Green’s function obeys the conjugation property \( \cite{2.41} \) and the function \( g_1(x, x', k) = e^{i\ell(k)(x-x')} G_1(x, x', k) \) is bounded and decaying as \( k \to \infty \).

Taking into account that the resolvent obeys \( \cite{2.32} \) with \( U = U_1 \) we get that the Green’s function obeys integral equations of the type \( \cite{2.32} \),

\[
G_1(k) = G_0(k) + G_0(k) U_1 G_1(k), \quad G_j(k) = G_0(k) + G_j(k) U_1 G_0(k),
\]

it is analytic when \( k_\theta k_\Im \neq 0 \) and in this region, in analogy with \( \cite{2.33} \), satisfies

\[
\frac{\partial G_j(k)}{\partial k_\Im} = \frac{\text{sgn} k_\Im}{2\pi} t_j(k)\varphi(k) \otimes \psi(k).
\]

The discontinuity across the imaginary axis is equal to

\[
G_1(+0 + i k_\Im) - G_1(-0 + i k_\Im) = -\text{sgn} k_\Im \sum_j t_j(\theta \omega_j - |k_\Im|)\varphi(i\omega_j) \otimes \psi(i\omega_j).
\]

Decomposition \( \cite{3.15} \) also gives

\[
G_1(k) = G_{1, reg}(k) + \sum_j \gamma_j(k)\varphi(i\omega_j) \otimes \psi(i\omega_j),
\]

where

\[
\gamma_j(k) = \Gamma_j(\ell_\Im(k)),
\]

so that by \( \cite{3.30} \)

\[
\gamma_j(k) = \frac{t_j\text{sgn} k_\Im}{2\pi i} \log \frac{k - i\omega_j}{k + i\omega_j}.
\]

This proves that the Green’s function, in addition to the standard discontinuity at the real axis, has also a discontinuity at the imaginary axis when \( |k_\Im| < \max_j \omega_j \). Inside the quadrants \( k_\Re k_\Im \neq 0 \) the Green’s function is continuous up to the borders. Note that since \( \overrightarrow{\nabla} \varphi(i\omega_j) = 0 \), the regular part \( G_{1, reg}(k) \) is still a Green’s function.

As far as the Jost solutions \( \varphi \) and \( \psi \) are concerned, the integrals in definitions \( \cite{3.15} \) are diverging, as expected. They can be conveniently regularized for \( k_\Im \neq 0 \) by means of the following limiting procedure

\[
\varphi(x, k) = \lim_{\varepsilon \to +0} \int dx' \overrightarrow{G_1(k)}(x, x') e^{-i\ell(k) x + i\varepsilon k_\Im x', k_\Im x'},
\]

\[
\psi(x, k) = \lim_{\varepsilon \to +0} \int dx' e^{i\ell(k) x' - i\varepsilon k_\Im x'} \overrightarrow{G_0(k)}(x', x).
\]

From \( \cite{3.27} \) we get the discontinuity of the resolvent with respect to \( q_2 \) when \( q_1 \neq 0 \). Denoting the corresponding boundary values of the resolvent as

\[
\lim_{q_2 \to \pm 0} \left[ M_1(q) \right]_{q_1 = k_\Im} = G_1(\pm 0 + i k_\Im),
\]

we get from \( \cite{3.21} \)

\[
G_1(x, x', \pm 0 + i k_\Im) = \frac{1}{2\pi i} \int d\alpha \left[ \theta(x_2 - x'_2) - \theta(k_\Im \alpha) \right] t_1(\alpha + i k_\Im) x \times \varphi(x, \alpha + i k_\Im) \psi(x', \alpha + i k_\Im) - \sum_j t_j \theta(|k_\Im|) \left[ \theta(x_2 - x'_2) - \theta(\pm k_\Im) \right] \varphi(x, i\omega_j) \psi(x', i\omega_j),
\]

\[\text{for } q_1 \neq 0.\]
Correspondingly, decomposition (3.25) gives
\[ G_1^j(\pm 0 + i k\alpha) = G_{1,\text{reg}}^j(i k\alpha) + \sum_j \gamma_j(\pm 0 + i k\alpha) \varphi(i \kappa_j) \otimes \psi(i \kappa_j), \] (3.33)
where \( \gamma_j(\pm 0 + i k\alpha) \) is defined by (3.27).

If we introduce the advanced/retarded Green’s functions in analogy to (2.28), by (3.14) we get the following bilinear representation for these Green’s functions in terms of the Jost solutions on the real axis
\[ G_{1,\pm}(x, x') = \pm \theta(\pm(x_2 - x'_2)) \left( \frac{1}{2\pi i} \int d\alpha \, t_1^\sigma(\alpha) \varphi^\sigma(x, \alpha) \psi^\sigma(x', \alpha) - \sum_j t_j \varphi(x, i \kappa_j) \psi(x', i \kappa_j) \right), \] (3.34)
where \( \sigma = +, - \), and, as before, we used the upper index \( \sigma \) for the boundary values of the Jost solutions and \( t_1(k) \) at the real \( k \)-axis. It is clear that the advanced/retarded Green’s functions are independent of the choice of \( \sigma \). These Green’s functions obey the conjugation property (2.34).

The advanced/retarded solutions also must be defined by the analog of relations (2.30) of the regular case, i.e.,
\[ \varphi_{\pm}(x, k) = \int dx' \, (G_{1,\pm} \leftarrow \mathcal{L}_0)(x, x') e^{-it(k)x'}, \] (3.35)
\[ \psi_{\pm}(x, k) = \int dx' \, e^{it(k)x'} (\mathcal{L}_0 G_{1,\pm})(x', x). \] (3.36)

For the boundary values of the Green’s function of the Jost solutions at the real axis, from (3.21) we obtain
\[ G_1^j(x, x', k) = \frac{1}{2\pi i} \int d\alpha \, [\theta(x_2 - x'_2) - \theta(\alpha - k)] \, t_1^\sigma(\alpha) \varphi^\sigma(x, \alpha) \psi^\sigma(x', \alpha) - \sum_j t_j \theta(x_2 - x'_2) - \theta(-\sigma k) \varphi(x, i \kappa_j) \psi(x', i \kappa_j), \quad k \in \mathbb{R}. \] (3.37)
These functions are finite for all \( k \) but discontinuous at \( k = 0 \). By (3.34) and (3.37)
\[ G_1^j(k) - G_{1,\pm} = \frac{\mp 1}{2\pi i} \int d\alpha \, \theta(\pm \sigma(\alpha - k)) t_1^\sigma(\alpha) \varphi^\sigma(\alpha) \otimes \psi^\sigma(\alpha) \pm \theta(\mp \sigma k) \sum_j t_j \varphi(i \kappa_j) \otimes \psi(i \kappa_j), \quad \sigma = +, -. \] (3.38)

We observe that
\[ G_{1,\pm}(x, x') = \pm \theta(\pm(x_2 - x'_2)) \left( \frac{1}{2\pi i} \int d\alpha \, \varphi_{\pm}(x, \alpha) \psi_{\pm}(x', \alpha) - \sum_j t_j \varphi(x, i \kappa_j) \psi(x', i \kappa_j) \right), \] (3.39)
that proves independence of (3.34) on the sign \( \sigma \) in the r.h.s. It also proves that the set of advanced/retarded solutions is not complete.
The advanced/retarded solutions and the limiting values of the Jost solutions at the real axis are related to each other by means of the following relations

$$\varphi_\pm(k) = \int dp \varphi^\sigma(p) r^\sigma_\pm(p,k), \quad t^\sigma_1(k) \varphi^\sigma(k) = \int dp \varphi_\pm(p) r^\sigma_\pm(p,k), \quad (3.40)$$

where

$$r^\sigma_\pm(p,k) = \delta(k-p)[\theta(\pm\sigma k) + \theta(\mp\sigma k) t_1(\sigma k)] + \theta(\mp\sigma k) \delta(k+p) r^\sigma_\mp(k). \quad (3.41)$$

Note that

$$r^\sigma_\pm(p,k) = \delta(k-p)[\theta(\mp\sigma k) + \theta(\pm\sigma k) t_1(\sigma k)] + \delta(k+p) \theta(\pm\sigma k) r^\sigma_\mp(k), \quad (3.42)$$

and

$$\varphi_\pm(k) = \psi_\mp(k), \quad (3.43)$$

which allows us to write down the corresponding relations for dual solutions.

Introducing alternative spectral data like in (2.61) we have (cf. (2.62))

$$f^\sigma = r^\sigma_\mp r^\sigma_\pm, \quad (3.44)$$

we have (cf. (2.63))

$$\varphi^\sigma T^\sigma_1 = \varphi^{-\sigma} f^{-\sigma}, \quad T^\sigma_1 \psi^\sigma = f^\sigma \psi^{-\sigma}, \quad (3.45)$$

where $T^\sigma_1(p,k) = \delta(p) t^\sigma_1(k)$ and $f^\sigma$ obeys properties (cf. (2.64)).

$$f^\sigma = f^\sigma, \quad f^{-\sigma} = T^{-1\sigma}(f^\sigma)^{-1} T^\sigma_1. \quad (3.46)$$

4. Inverse scattering transform on nontrivial background: two-dimensional perturbation of the one-dimensional potential

4.1 Resolvent

Now we start studying the operator $\mathcal{L}$ with potential given in (1.3). In order to investigate the properties of this operator we introduce its extension $L(q)$ like in (2.6) and the inverse to this extension, i.e., the resolvent, by (2.10). Correspondingly, the resolvent $M(q)$ obeying (2.11) will be defined through one of the integral equations

$$M(q) = M_1(q) + M_1(q) U_2 M(q), \quad M(q) = M_1(q) + M(q) U_2 M_1(q), \quad (4.1)$$

where $U_2(x, x'; q) = u_2(x) \delta(x-x')$. We assume $u_2(x)$ to be real, smooth, and rapidly decaying with respect to both variables $(x_1, x_2)$. Moreover we assume it to be “small” in the sense that solutions $M(x, x'; q)$ of the both equations in (4.1) exist in $S'(\mathbb{R}^6)$. Their properties with respect to variables $q$ are inherited from the corresponding properties of the resolvent $M_1(q)$. For instance, $M(q)$ is a continuous function of $q$ when $q \neq 0$ and $q_2 \neq 0$. The effective tool for investigating the properties of the resolvent is given by the Hilbert identity (2.13), which we can also write in the form

$$M'(q') - M(q) = M'(q') L'_1(q')(M'_1(q') - M_1(q)) L_1(q) M(q), \quad (4.2)$$
so that in analogy with (2.20) we get for the derivatives of the hat-kernel (2.8) of the resolvent
\[ \frac{\partial \hat{M}(q)}{\partial q_j} = \hat{M}(q) \mathcal{L}_j \frac{\partial \hat{M}(q)}{\partial q_j} \hat{M}(q), \quad j = 1, 2, \quad q_2 \neq 0. \]

Then using (3.19) we get at \( q_2 \neq 0 \) equalities of the kind (2.24), (2.22) for derivatives of the \( \hat{M}(q) \):
\[ \frac{\partial \hat{M}(q)}{\partial q_1} = \frac{i}{\pi} \int d_k \delta(\ell_{23}(k) - q_2) t_1(k) \Phi(k) \otimes \Psi(k), \quad (4.3) \]
\[ \frac{\partial \hat{M}(q)}{\partial q_2} = \frac{1}{2\pi i} \int d_k \delta(\ell_{23}(k) - q_2) t_1(k) \Phi(k) \otimes \Psi(k), \quad (4.4) \]

while now the Jost solutions are defined as (cf. (2.26))
\[ \Phi(k) = \mathcal{G}(k) \mathcal{L}_1 \varphi(k), \quad \Psi(k) = \psi(k) \mathcal{L}_1 \mathcal{G}(k), \quad (4.5) \]

where the Green’s function \( \mathcal{G}(x, x', k) \) of the Jost solutions is defined as value of the resolvent by (2.21) in the same way as in the decaying case.

The resolvent is discontinuous at \( q_2 = 0 \), as inherited by the same properties of \( M_1(q) \). Hence, we will consider separately the cases \( q_1 = 0 \) and \( q_1 \neq 0 \). The boundary values of the resolvent in the first case, \( \lim_{q_2 \to \pm 0} \hat{M}(q)|_{q_1 = 0} \), define the advanced/retarded Green’s functions like in (2.25). In the case \( q_1 \neq 0 \) for the boundary values of the resolvent thanks to (2.24) we have in analogy with (2.30) that
\[ \lim_{q_2 \to \pm 0} \left( \hat{M}(q) \right|_{q_1 = k_0} = \mathcal{G}(\pm 0 + i k_3). \quad (4.6) \]

### 4.2 The Green’s function.

By (1.1) and definition (2.24) this Green’s function satisfies integral equations
\[ \mathcal{G}(k) = \mathcal{G}_1(k) + \mathcal{G}_1(k) U_2 \mathcal{G}(k), \quad \mathcal{G}(k) = \mathcal{G}_1(k) + \mathcal{G}(k) U_2 \mathcal{G}_1(k). \quad (4.7) \]

Taking into account that \( \mathcal{L} = \mathcal{L}_1 - U_2 \), one check that \( \mathcal{G}(k) \) satisfies the differential equations \( \hat{L} \mathcal{G}(k) = \mathcal{G}(k) \hat{L} = I \) and the conjugation property (2.31). From (1.1), (1.6), it is clear that the analyticity properties of the Green’s function \( \mathcal{G}(k) \) are inherited from \( \mathcal{G}_1(k) \). This means that it is analytic in the region \( k_0 \neq 0 \), i.e., it obeys (2.26) in this region. The Green’s function satisfies the conjugation property (2.31) and possesses the standard cut at \( k_0 = 0 \) and additional cut at \( k_0 = 0 \) when \( |k_3| < \max_j \rho_j \). Inside the quadrants \( k_0 \neq 0 \) \( \mathcal{G}(k) \) is continuous up to the borders, as follows from the properties of \( \mathcal{G}_1(k) \). In particular, note that
\[ \lim_{k_3 \to +0} \mathcal{G}(\pm 0 + i k_3) = \lim_{k \to +0} \mathcal{G}^\sigma(k) \equiv \mathcal{G}^\sigma(\pm 0), \quad \sigma = +, - \quad (4.8) \]

As always in order to study properties of the resolvent at points of discontinuity we use the Hilbert identity. By (4.6) this can be written as discontinuity of the Green’s function across the imaginary axis:
\[ \mathcal{G}(+0 + i k_3) - \mathcal{G}(0 + i k_3) = -\text{sgn} k_3 \sum_j t_j \theta(\rho_j - |k_3|) \times \]
\[ \times \left( \mathcal{G}(\pm 0 + i k_3) \mathcal{L}_1 \varphi(i \rho_j) \right) \otimes \left( \psi(i \rho_j) \mathcal{L}_1 \mathcal{G}(\mp 0 + i k_3) \right) \quad (4.9) \]
Eq. (4.9) suggests introducing the functions $\Phi_j(x, k_3)$ and $\Psi_j(x, k_3)$ by means of
\[ \Phi_j(k_3) = G(+0 + i k_3) \overrightarrow{L} \varphi(i x_j), \quad \Psi_j(k_3) = \psi(i x_j) \overleftarrow{L} \overline{G}(+0 + i k_3), \] (4.10)
that thanks to (3.13) obey
\[ \overline{\Phi}_j(x, k_3) = b_j \Phi_j(x, -k_3), \quad \overline{\Psi}_j(x, k_3) = \frac{\Phi_j(x, -k_3)}{b_j}. \] (4.11)

Now by (4.9) (bottom sign) we derive for the discontinuity the following expression
\[ G(+0 + i k_3) - G(-0 + i k_3) = -\sum_{l,m} \theta(x_l - |k_3|) \Phi_l(k_3)(A(k_3)^{-1})_{lm} \otimes \Psi_m(k_3) \] (4.12)
where we introduced the diagonal matrix $A(k_3)$ with elements
\[ A_{lm}(k_3) = \frac{\delta_{lm}}{t_l \text{sgn} k_3} - \theta(\min\{x_l, x_m\} - |k_3|) \left( \psi(i x_l) \overrightarrow{L} \varphi(i x_m) \right). \] (4.13)

Therefore the discontinuity of the Green’s function at the imaginary axis is given not in terms of Jost solutions but in terms of functions $\Phi_j(x, k_3)$ and $\Psi_j(x, k_3)$, which are solutions of the nonstationary Schrödinger equation and its dual different from the Jost ones.

One can prove that, under the assumption of unique solvability of the integral equation for the Green’s function, the matrix $A(k_3)\dagger$ is non-singular. Also, one can easily show that
\[ A(k_3)\dagger = BA(-k_3)B^{-1}, \] (4.14)
where we introduced the diagonal matrix $B = \text{diag}\{b_1, \ldots, b_N\}$. Note that $B$ is Hermitian, since the $b_j$’s in (4.11) are real. Preserving for the limiting values at $k_3 = \pm 0$ of the matrix the usual notation, $A^\pm = \lim_{k_3 \to \pm 0} A(k_3)$, we get by (4.13)
\[ A^\sigma_m = \frac{\sigma \delta_{lm}}{t_l} - \left( \psi(i x_l) \overrightarrow{L} \varphi(i x_m) \right). \] (4.15)

These constant matrices are invertible and, due to (4.14), satisfies the conjugation property
\[ (A^\sigma)\dagger = BA^{-\sigma}B^{-1}, \quad \sigma = +, -. \] (4.16)

### 4.3 The Jost solutions.

The functions $\Phi(x, k)$ and $\Psi(x, k)$ introduced in (4.9) are the Jost solutions of the nonstationary Schrödinger equations and its dual with potential (4.3), satisfying the integral equations
\[ \Phi(k) = \varphi(k) + G_1(k)U_2\Phi(k), \quad \Psi(k) = \psi(k) + \Psi(k)U_2G_1(k). \] (4.17)

Thanks to the properties of the Green’s function $G(k)$, the Jost solutions are analytic functions of $k \in \mathbb{C}$ when $k \not\in k_3 \neq 0$ and in the generic situation they have the standard discontinuity at the real axis, $k_3 = 0$, and an additional discontinuity at the segment of the imaginary axis: $k_3 = 0$, $|k_3| \leq \max_j x_j$. Let us consider $|k_3| \neq x_j$ for any $j$. The functions $\varphi(k)$ and $\psi(k)$ are continuous at $k_3 = 0$, while the Jost solution $\Phi(k)$ is not and the discontinuity is given by
\[ \Phi(x, +0 + i k_3) - \Phi(x, -0 + i k_3) = \sum_l \theta(x_l - |k_3|) \Phi_l(x, k_3) w_l(k_3), \] (4.18)
where we introduced functions $w_l(k_3)$ as

$$w_l(k_3) \equiv t_l \text{sgn} k_3 \left( \psi(i \varepsilon_l) \overrightarrow{L}_1 G(-0 + i k_3) \overrightarrow{L}_1 \varphi(i k_3) \right).$$  (4.19)

Analogously, the discontinuity of the Jost solution $\Psi(k)$ of the dual equation is given by

$$\Psi(x, +0 + i k_3) - \Psi(x, -0 + i k_3) = \sum_l \theta(\varepsilon_l - |k_3|) \Psi_l(x, k_3) w_l(-k_3).$$  (4.20)

Note that

$$w_l(-k_3) = \frac{t_l}{b_l} \text{sgn} k_3 \theta(\varepsilon_l - |k_3|) \left( \psi(i k_3) \overrightarrow{L}_1 G(-0 + i k_3) \overrightarrow{L}_1 \varphi(i \varepsilon_l) \right).$$  (4.21)

Thus we see by (4.18) that the discontinuity of the Jost solution across the imaginary axis is given in terms of the functions $\Phi_l(x, k_3), \Psi_l(x, k_3)$ introduced in (4.10). In the same way as for the Jost solutions we prove that these functions also satisfy the differential equations $\overrightarrow{L} \Phi_l(k_3) = \Psi_l(k_3) \overrightarrow{L} = 0$ that follows also from (4.18), (4.20). They generalize functions $\varphi(i \varepsilon_l)$ and $\psi(i \varepsilon_l)$ for the case $u_2 \neq 0$. This results in a nontrivial dependence of $\Phi_l$ and $\Psi_l$ on $k_3$. By (1.9) we need these functions on the interval $|k_3| < \varepsilon_l$ only. In what follows we call them the auxiliary Jost solutions. As a consequence of the properties of the total Green’s function, these solutions are discontinuous at $k_3 = 0$. Their conjugation property is given in (4.11).

### 4.3.1 Behavior of the Green’s function, Jost solutions, and spectral data at the points $\pm i \varepsilon_j$

As it was shown in (4.16) and (5.26), both $M_1(q)$ and $G_1(k)$ have logarithmic singularities at all points $q = (\pm i \varepsilon_j, 0)$, or correspondingly $k = \pm i \varepsilon_j, j = 1, \ldots, N$. It is clear that these singularities affect the Green’s function $G(k)$ as well as Jost solutions, auxiliary Jost solutions and spectral data.

Let us choose some $\varepsilon_j$ and $k$ belongs to some neighborhood of a point $i \varepsilon_j$ or of a point $-i \varepsilon_j$ that does not include other points $\pm i \varepsilon_l, l \neq j$, or points on the real axis. Then we introduce

$$G_{1,j}(k) = G_1(k) - \gamma_j(k) \varphi(k) \otimes \psi(k),$$  (4.22)

which, thanks to (5.26) is finite in the vicinities of $\pm i \varepsilon_j$, while can be discontinuous there in correspondence to (4.24). This regularization is such that $e^{it(k)(x-x')}G_{1,j}(x, x', k)$ is bounded on the $x$-plane. Now we define a new Green’s function of the nonstationary Schrödinger equation with potential (4.5) by means of one of the integral equations,

$$G_j(k) = G_{1,j}(k) + G_{1,j}(k)U_2 G_j(k),$$  (4.23)

$$\tilde{G}_j(k) = \tilde{G}_{1,j}(k) + \tilde{G}_{1,j}(k) \tilde{U}_2 \tilde{G}_j(k).$$  (4.24)

Thus $G_j(k)$ is a regularization of $G(k)$ in the neighborhood of $\pm i \varepsilon_j$ and is such that

$$G(k) = G_j(k) \tilde{G}_j(k) + \gamma_j(k) \tilde{G}_j(k) \otimes \psi(k),$$  (4.25)

where we use notations (4.10) for the Jost solutions and introduce in analogy

$$\tilde{G}_j(k) = G_j(k) \overrightarrow{L}_1 \varphi(k), \quad \tilde{G}_j(k) = \psi(k) \overrightarrow{L}_1 G_j(k).$$  (4.26)
The functions $\tilde{\Phi}_j(k)$ and $\tilde{\Psi}_j(k)$ are also solutions of the nonstationary Schrödinger equation and its dual with potential \( (1.3) \). These functions are bounded in vicinities of the points $\pm i \kappa_j$, though their limits at these points can be dependent on the sign of $k \Re$. The same properties hold for function $g_j(k) = \left( \psi(k) \overrightarrow{L}_1 \mathcal{G}_j(k) \overrightarrow{L}_1 \varphi(k) \right)$. (4.27)

Thus we see that behavior of the Green’s function and of the Jost solutions in vicinities of points $\pm i \kappa_j$ is determined by behavior of the function $g_j(k)$ at these points. If $g_j(k) = o(1), k \sim i \kappa_j$, (4.28) the Green’s function has logarithmic singularities at points $\pm i \kappa_j$ and the Jost solutions are bounded at these points and their limits depend on the sign of $k \Re$. On the other hand, if $g_j(k) = O(1), k \sim i \kappa_j$, (4.29) and the limit (also depending on the way of the limiting procedure) is different from zero, then

$$G(k) = \mathcal{G}_j(k) - \frac{1}{g_j(k)} \tilde{\Phi}_j(k) \otimes \tilde{\Psi}_j(k), \quad k \sim \pm i \kappa_j,$$

(4.30) and

$$\Phi(k) = o(1), \quad \Psi(k) = o(1), \quad k \sim \pm i \kappa_j,$$

(4.31) and in the r.h.s.’s we have terms of order $1/\log$.

In what follows we assume that condition (4.29) is fulfilled for all $j = 1, \ldots, N$. In particular, under such an assumption, one can show that

$$\Phi_j(\pm \kappa_j) = \Psi_j(\pm \kappa_j) = 0, \quad j = 1, \ldots, N$$

(4.32) while the values $\Phi_m(\pm \kappa_j)$ and $\Psi_m(\pm \kappa_j)$ of the auxiliary solutions for $m$ such that $\kappa_m > \kappa_l$ are finite and different from zero, and

$$\psi(i \kappa_j) \overrightarrow{L}_1 \Phi_m(\pm \kappa_j) = 0, \quad \Psi_m(\pm \kappa_j) \overrightarrow{L}_1 \varphi(i \kappa_j) = 0.$$  

(4.33) Taking into account (4.20) and (4.33) one gets

$$A(\pm \kappa_j)_{jm} = A(\pm \kappa_j)_{mj} = \frac{\pm \delta_{jm}}{t_j},$$

(4.34) that coincide with values of $A_{jm}(k_3)$ and $A_{jm}(k_3)$ when $k_3 > \min \{ \kappa_j, \kappa_m \}$. In the same way we derive

$$A(\pm \kappa_j)^{-1} = A(\pm \kappa_j)^{-1} = \pm \delta_{jm} t_j.$$  

(4.35) All other matrix elements of matrices $A(k_3)$ and $A^{-1}(k_3)$ have finite limits at $k_3 = \pm \kappa_j$.

Finally, one can prove that under condition (4.29)

$$w_l(\pm \kappa_j)_{jm} = 0 \quad \text{for all } j \text{ and } l: \kappa_j \leq \kappa_l.$$  

(4.36) The properties of the Jost solutions and spectral data derived so far enable us to reconstruct the auxiliary Jost solutions in terms of the boundary values of the Jost ones. Indeed,
the knowledge of derivatives of Green's function and auxiliary Jost solutions with respect to $k_3$ allows to obtain

$$G(\pm 0 + i k_3) = G^*(\pm 0) + \frac{\sigma}{2\pi} \int_0^{k_3} da t_1(i\alpha) \Phi(\pm 0 + i\alpha) \otimes \Psi(\pm 0 + i\alpha), \quad \sigma = \text{sgn} k_3, \quad (4.37)$$

$$\theta(x_j - |k_3|) \Phi_j(k_3) = \frac{\text{sgn} k_3}{2\pi} \int_{x_j, \text{sgn} k_3}^{k_3} da t_1(i\alpha) \Phi(+0 + i\alpha) \sum_l b_l w_l(-\alpha) A_{lj}(\alpha). \quad (4.38)$$

Also, from the expression for the derivative of $A_{mj}(k_3)$ one gets

$$(A^{-1}(k_3))_{jm} = t_m \text{sgn} k_3 \delta_{jm} + \theta(\min\{x_m, x_j\} - |k_3|) \frac{\text{sgn} k_3}{2\pi} b_m \times \int_{\min\{x_m, x_j\}, \text{sgn} k_3}^{k_3} da t_1(i\alpha) w_j(\alpha) w_m(-\alpha). \quad (4.39)$$

It is straightforward to verify that the properties of the Jost solutions and spectral data described above are compatible with representations (4.38) and (4.39).

Finally, one can derive bilinear representation of the resolvent in terms of the Jost solutions

$$\hat{M}(x, x'; q) = \frac{1}{2\pi i} \int_{k_3 = q_1} d k_R \left[ \theta(x_2 - x'_2) - \theta(2 k_R k_3 - q_2) \right] t_1(k) \Phi(x, k) \Psi(x', k) - \sum_l \left[ \theta(\min\{x_l, x_j\} - |q_1|) (A(q_1)^{-1})_{lm} \Psi_l(x, q_1) \Psi_m(x', q_1) \right]. \quad (4.40)$$

Note that the corresponding kernel $M(x, x'; q)$ defined by (2.34) belongs to $S'$. Continuity of the resolvent at $q_1 = 0$ in the case where $q_2 \neq 0$ implies that

$$\frac{1}{2\pi i} \int dk t_1^+(k) \Phi^+(k) \otimes \Psi^+(k) - \sum_{lm} (A^+)^{-1}_{lm} \Phi^+_l \otimes \Psi^+_m =$$

$$= \frac{1}{2\pi i} \int dk t_1^-(k) \Phi^-(k) \otimes \Psi^-(k) + \sum_{lm} (A^-)^{-1}_{lm} \Phi^-_l \otimes \Psi^-_m \quad (4.41)$$

where we introduced

$$\Phi^\pm_l(x) = \lim_{k_3 \to 0} \Phi_l(x, k_3), \quad \Psi^\pm_l(x) = \lim_{k_3 \to 0} \Psi_l(x, k_3). \quad (4.42)$$

Thanks to (2.39) and (1.11) these limiting values obey the following conjugation properties:

$$\overline{\Phi^\pm_l(x, k)} = \overline{\Psi^\mp_l(x, k)}, \quad \overline{\Phi^+_l(x)} = b_l \Psi^+_l(x), \quad k \in \mathbb{R}, \quad l = 1, \ldots, N. \quad (4.43)$$

The bilinear representation (4.40) for the resolvent leads, recalling (2.25), to the following bilinear representation for the Green's function of the Jost solutions:

$$G(x, x', k) = \frac{1}{2\pi i} \int dk' \left[ \theta(x_2 - x'_2) - \theta(\text{sgn} k_3 k') \right] t_1(k') + k) \Phi(x, k' + k) \Psi(x', k' + k) - \sum_l \left[ \theta(\min\{x_l, x_j\} - |k_3|) (A(k_3)^{-1})_{lm} \times \right.$$

$$\times \Phi_l(x, k_3) \Psi_m(x', k_3), \quad (4.44)$$
that generalizes (3.21). Below we use the bilinear representation to obtain relations between the Jost and advanced/retarded solutions.

4.4 Discontinuity of the resolvent at \( q = 0 \)

First, we introduce advanced/retarded Green’s functions as specific limits of the resolvent in analogy to (2.28). It is straightforward to prove that they satisfy the differential equations \( \mathcal{L} \mathcal{G} = \mathcal{E} = I \) and obey (2.34) and integral equations (2.35). In order to find difference among these Green’s function we use the Hilbert identity, (cf. (2.30)).

The bilinear representation (4.40) for the resolvent, thanks to (2.28), gives a representation for the advanced/retarded Green’s functions in terms of the Jost solutions on the real axis:

\[
\mathcal{G}_\sigma(x, x') = \pm \theta(\pm (x_2 - x'_2)) \left( \frac{1}{2\pi i} \int dk \ t_1^\sigma(k) \Phi^\sigma(x, k) \Psi^\sigma(x', k) - \sigma \sum_{l, m} (A^\sigma)^{-1}_{lm} \Phi^\sigma_l(x) \Psi^\sigma_m(x') \right),
\]

where the standard and auxiliary advanced/retarded solutions are defined as

\[
\Phi^\pm(k) = \mathcal{L}_1 \varphi^\pm(k), \quad \Phi^\pm,j = \mathcal{L}_1 \varphi(i \varepsilon_j),
\]

\[
\Psi^\pm(k) = \psi^\pm(k) \mathcal{L}_1 \mathcal{G}^\pm, \quad \Psi^\pm,j = \psi(i \varepsilon_j) \mathcal{L}_1 \mathcal{G}_\pm.
\]

These functions are solutions of the nonstationary Schrödinger equation with potential \( u \), obey conjugation properties (2.43) and also are such that

\[
\Phi^\pm,j = b_j \Psi^\mp,j.
\]

We mention that the l.h.s. of (4.45) is independent of the sign \( \sigma = +, - \) due to condition (4.41).

4.4.1 Relations among the Green’s functions.

Another limiting procedure at point \( q = 0 \) for the resolvent \( M(q) \) is given by the limiting values of the Green’s function \( \mathcal{G}(k) \) at the real axis. For these boundary values we use notation \( \mathcal{G}^\sigma(k) \). The difference between these limiting values and the advanced/retarded Green’s functions, like in the case of the decaying potential, can be presented in two forms. The first one follows from the limits \( k \rightarrow k \pm i0 \) in (4.44) and (4.45):

\[
\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \frac{1}{2\pi i} \int dk' \theta(\pm \sigma(k' - k)) \ t_1^\sigma(k') \Phi^\sigma(k') \otimes \Psi^\sigma(k') \pm 
\]

\[
\pm \theta(\mp \sigma k) \sum_{l, m} (A^\sigma)^{-1}_{lm} \Phi^\sigma_l(x) \otimes \Psi^\sigma_m(x').
\]

A second set of relations, obtained from the Hilbert identity, is given by

\[
\mathcal{G}^\sigma(k) - \mathcal{G}_\pm = \frac{1}{2\pi i} \int d\alpha \theta(\pm \sigma(\alpha - k)) \int dp \ \Phi^\sigma(p) r^\sigma(p, \alpha) \otimes \psi^\sigma(\alpha) \mathcal{L}_1 \mathcal{G}^\sigma(k) \pm 
\]

\[
\pm \theta(\mp \sigma k) \sum_j t_j \Phi^\pm,j \otimes \psi(i \varepsilon_j) \mathcal{L}_1 \mathcal{G}^\sigma(k),
\]

and the analogous equality that can be derived from this last one by conjugation.
4.5 Spectral data.

From (4.49) and (4.50) we obtain the following relations between the Jost and advanced/retarded solutions

$$t_1^+(k) \Phi^\sigma(k) = \int dp \Phi_\pm(p) \mathcal{R}_\pm^r(p, k) + \sum_j \Phi_{\pm,j} \mathcal{R}_{\pm,j}^r(k), \quad \sigma = +, -,$$

$$\Phi_m^\sigma = \int dp \Phi_\pm(p) \mathcal{R}_\pm^\sigma(p, k) + \theta(\pm \sigma) \Phi_{\pm,m} + \theta(\mp \sigma) \sum_j t_j \Phi_{\pm,j} A_{jm}^\sigma,$$

where we used (4.13) in the last line and where we introduced the spectral data

$$\mathcal{R}_\pm^r(p, k) = (r_\pm^r R_\pm^r)(p, k) \equiv \int d\alpha r_\pm^r(p, \alpha) R_\pm^r(\alpha, k),$$

where $R_\pm^\sigma$ is a triangular operator

$$R_\pm^\sigma(p, k) = \delta(p - k) \mp \theta(\pm \sigma)(p - k) R_\pm^\sigma(p, k),$$

with

$$R_\sigma^r(p, k) = \mp t^r_1(k) {\psi^\sigma(p) \dot{L}_1} \Phi^\sigma(k) = \pm t^r_1(k) {\psi^\sigma(p) U_2 \Phi^\sigma(k)}.$$ 

These relations combine (2.51) and (3.40). Eqs. (4.51), (4.52) give the boundary values of the Jost solutions in terms of the advanced/retarded ones. Inverse relations are given by

$$\Phi_\pm(k) = \int dk' \Phi^\sigma(k') \mathcal{R}_\pm^\sigma(k, k') - 2\pi i \sigma \sum_{l,m} \Phi^\sigma_l(A^\sigma)_{lm}^{-1} \frac{1}{b^\sigma_m} \mathcal{R}_{\pm,m}^\sigma(k),$$

$$\Phi_{\pm,j}(k) = \frac{-b_j}{2\pi it_j} \int dk' \Phi^\sigma(k') \mathcal{R}_{\pm,j}^\sigma(k') + \theta(\pm \sigma) \Phi_{\sigma,j} + \theta(\mp \sigma) \sum_l \Phi^\sigma_l(A^\sigma)^{-1}_{lj}.$$ 

If we introduce alternative spectral data in analogy with (2.61), (2.62) and (3.44), (3.45), we can write

$$t_1^\sigma(k) \Phi^\sigma(k) = \int dp \Phi^{-\sigma}(p) F^{-\sigma}(p, k) + 2\pi i \sigma \sum_{l,m} \Phi^{-\sigma}_l(A^{-\sigma})_{lm}^{-1} \frac{1}{b_m} F_{m}^{-\sigma}(k),$$

$$\Phi_j^\sigma = \int dp \Phi^{-\sigma}(p) \mathcal{F}_j^{-\sigma}(p, k) + 2\pi i \sigma \sum_{l,m} \Phi^{-\sigma}_l(A^{-\sigma})_{lm}^{-1} \frac{1}{b_m} F_{mj}^{-\sigma},$$
where

\[
\mathcal{F}^{-\sigma}(k, k') = \int dk'' R_\pm^\sigma(k'', k) R_\pm^\sigma(k', k') - \sum_j \frac{b_j}{2\pi it_j} R_\pm^\sigma(j, k) R_\pm^\sigma(k')
\]

(4.65)

\[
\mathcal{F}_l^{-\sigma}(k') = -\frac{\theta(\mp\sigma)b_l}{2\pi it_l} R_\pm^\sigma(k, k') \pm \frac{\theta(\mp\sigma)}{2\pi i} \sum_{m=1}^N b_lA^-_{lm} R_\pm^\sigma(k, m) + \\
+ \int dk'' \overline{R}_\pm^\sigma(k'', k') R_\pm^\sigma(k', k')
\]

(4.66)

\[
\tilde{\mathcal{F}}^{-\sigma}_j(k) = -\frac{\theta(\pm\sigma)b_j}{2\pi it_j} R_\pm^\sigma(k, k) \pm \sum_l b_j\theta(\mp\sigma) \frac{A^-_{lj}}{2\pi i} R_\pm^\sigma(k, l) + \\
+ \int dk' \overline{R}_\pm^\sigma(k', k) R_\pm^\sigma(k', k')
\]

(4.67)

\[
\mathcal{F}_{lj}^{-\sigma} = -\frac{\theta(\pm\sigma)b_l}{2\pi it_l} \delta_{lj} + \frac{\theta(\mp\sigma)}{2\pi i} \sum_{m=0}^N b_lA^-_{lm} A^-_{mj} + \int dk' \overline{R}_\pm^\sigma(k') \overline{R}_\pm^\sigma(k').
\]

(4.68)

Characterization equations for spectral data will be given in a forthcoming publication.

5 Conclusion

Existence of the bilinear representation (4.45) is one of the main advantages of the resolvent approach. This relation gives the extended resolvent in terms of the Jost solutions. These solutions themselves or any other relevant solutions of the linear problem under consideration (in our case (1.2)) and its dual are given as specific reductions of the corresponding Green’s functions. At the same time, the Green’s functions, in their turn, are values of the resolvent in some specific points. This property supplies us with the simple and regular way of deriving relations between different kinds of solutions of the linear problem and construction of the scattering data.

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