PERTURBATION OF IMAGE AND CONJUGATE DUALITY FOR VECTOR OPTIMIZATION

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Abstract. This paper aims at employing the image space approach to investigate the conjugate duality theory for general constrained vector optimization problems. We introduce the concepts of conjugate map and subdifferential by using two types of maximums. We also construct the conjugate duality problems via a perturbation method. Moreover, the separation condition is proposed by means of vector weak separation functions. Then, it is proved to be a new sufficient condition, which ensures the strong duality theorem. This separation condition is different from the classical regular conditions in the literature. Simultaneously, the application to a nonconvex multi-objective optimization problem is shown to verify our main results.

1. Introduction. Image space analysis (for short ISA) was initiated in [9] and carried on in [3, 6, 7, 12, 19, 22, 24]. Since then, it has been extensively used as a preliminary and auxiliary tool for investigating various mathematical topics like constrained extremum problems, variational inequalities. More generally, it can be used for any problem, which can be expressed under the form of the infeasibility of a parametric system. The infeasibility of a parametric system is characterized by the disjunction of two suitable subsets in the image space. Separation plays a key role in the image space. By virtue of separation, several theoretical aspects can be developed, such as alternative and saddle point optimality conditions [4, 9, 12–16, 26], scalarization [5], duality [8, 27, 28], regularity [6, 20, 21], Courant penalty methods [18], gap functions and error bounds [25].

Duality theories for constrained extremum problems and vector optimization problems based on image space approach have attracted much attention [8, 11, 12, 15, 17, 27, 28]. Gianessi et al. [12, 15] defined a family of vector Lagrangian
dual problems and established the strong duality theorem. Except for investigating
vector optimization problems by means of Lagrangian duality in the image space,
the conjugate duality is also an important aspect. The core technique of conjugate
duality is the perturbation method.

Bot [1] constructed conjugate dual problems for general scalar optimization prob-
lems. Under convexity assumptions, the stability of the primal problem (or equival-
ently, the perturbation function is subdifferentiable at zero point) was proven to
be a sufficient condition for strong duality theorem. Therefore, it becomes a main
issue to formulate sufficient conditions, called regularity conditions, which ensure
that the primal problem is stable. For unconstrained vector optimization problems,
Tanino [23] introduced new concepts of conjugate map and subdifferential for a
set-valued map. Under the convexity hypothesis, he proved that a regular condi-
tion, which is somewhat like the generalized interior point condition, ensures the
stability.

Motivated by [1, 12, 15, 23], we devote to investigate conjugate duality theory
for general vector optimization problems via image space approach. We introduce
conjugate maps, biconjugate maps and subdifferentials of a set-valued map. First,
we define a family of perturbation problems and introduce the optimal value maps
for the perturbation problems. Then, we construct the corresponding conjugate dual
problems by applying the conjugate maps of the optimal value maps. Also, we define
how a set-valued map is said to be weakly subdifferentiable. This concept arises from
weakening how a set-valued map is subdifferentiable in [23]. The existent regular
conditions in the literature [1,2,23] are generally related to convexity assumptions.
In order to establish the strong dual theorem for some nonconvex optimization
problems, we construct a new condition based on the separation argument in the
image space, which is called the separation condition. The separation condition is
proved to imply that the optimal value map of a perturbation problem is weakly
subdifferentiable. By virtue of this result, we further derive strong duality theorem.
In fact, the separation condition is proposed from a new point of view, which is
completely different from the regular conditions used in [1,2,23] and just reflects
the main features of image space approach.

The rest of this paper is organized as follows. In Section 2, we recall some sym-
bols, basic concepts and the main features of image space approach. In Section 3,
we introduce the concepts of conjugate maps, biconjugate maps and subdifferentials
according to two classes of maximums. In Section 4, we construct the conjugate
duality problems by perturbation approach. Moreover, we use the separation con-
dition to establish strong duality theorem and illustrate a nonconvex example.

2. Preliminaries. Let $X, Y$ and $Z$ be finite dimensional linear topological spaces,
where $Y$ is partially ordered by a pointed closed convex cone $C$. If there is no
confusion, we will always denote by $0$ the zero element in arbitrary finite dimensional
linear topological space. We set $C_0 := C \setminus \{0\}$ and denote by int $C$ the interior of
$C$. We always use the following symbols, which describe three order relations. For
any $a, b \in Y$,

$$a \prec b \iff b - a \in \text{int } C, \quad a \prec b \iff b - a \not\in \text{int } C;$$

$$a \leq b \iff b - a \in C_0, \quad a \nleq b \iff b - a \not\in C_0;$$

$$a \leq_C b \iff b - a \in C, \quad a \nleq_C b \iff b - a \not\in C.$$
In this paper, we pay attention to the following constrained vector optimization problem (P):

\[
\begin{align*}
\text{(P)} \quad & \min f(x) \quad \text{s.t.} \quad x \in S := \{x \in X : g(x) \in -D\}, \\
\end{align*}
\]

where \( f : X \to Y, \ g : X \to Z \) and \( D \) is a closed cone in \( Z \).

Motivated by the definition of scalar indicator function, we introduce the positive infinity \( +\infty \) and negative infinity \( -\infty \) in the linear topological space \( Y \). By convention, \( +\infty \) and \( -\infty \) satisfy

\[-\infty < y < +\infty, \ -\infty \preceq y \preceq +\infty, \ -\infty \leq y \leq +\infty, \ \forall y \in Y.\]

Then, for a nonempty set \( G \subset X \), we define the vector-valued indicator function \( \hat{d}_G : X \to Y \cup \{\pm \infty\} \) as

\[
\hat{d}_G(x) = \begin{cases}
0, & \text{if } x \in G, \\
+\infty, & \text{if } x \notin G.
\end{cases}
\]

For a set \( M \subset Y \cup \{\pm \infty\} \), we introduce several sets, let \( A_1(M) \) denote all points above \( M \) in terms of \( \prec \), \( A_2(M) \) denote all points above \( M \) in terms of \( \preceq \), respectively, by

\[
\begin{align*}
A_1(M) := \{y \in Y : y \succ y' \text{ for some } y' \in M\}; \\
A_2(M) := \{y \in Y : y \succeq y' \text{ for some } y' \in M\},
\end{align*}
\]

and let \( B_1(M) \) denote all points below \( M \) in terms of \( \prec \), let \( B_2(M) \) denote all points below \( M \) in terms of \( \preceq \), respectively, by

\[
\begin{align*}
B_1(M) := \{y \in Y : y \prec y' \text{ for some } y' \in M\}; \\
B_2(M) := \{y \in Y : y \preceq y' \text{ for some } y' \in M\}.
\end{align*}
\]

Now we recall the concepts of (weak) efficient points in the sense of minimums and maximums in the literature as follows.

**Definition 2.1.** Given a set \( M \subset Y \cup \{\pm \infty\} \), \( \hat{y} \in Y \) is said to be a weak maximal point of \( M \) if \( \hat{y} \in M \) and \( \hat{y} \notin B_1(M) \) or \( \hat{y} \notin B_2(M) \) respectively. The set of all weak maximal points or maximal points of \( M \) is denoted by \( \text{IMax}_M \) or \( \text{IMax}_M \). Similarly, \( \hat{y} \in Y \) is said to be a weak minimal point or minimal point of \( M \) if \( \hat{y} \in M \) and \( \hat{y} \notin A_1(M) \) or \( \hat{y} \notin A_2(M) \) respectively. The set of all weak minimal points or minimal points of \( M \) is denoted by \( \text{IMin}_M \) or \( \text{IIMin}_M \).

**Remark 2.1.**

(i): \( \text{IMax} \emptyset = \emptyset, \ \text{IIMax} \emptyset = \emptyset \);

(ii): \( -\text{IMax}(-M) = \text{IMin}_M, \ -\text{IIMax}(-M) = \text{IIMin}_M \).

According to the concepts of efficient solutions and weak efficient solutions (see reference [10]), searching for a weak efficient solution of vector optimization problem (P) means finding a feasible point \( \bar{x} \in S \) such that \( f(\bar{x}) \in \text{IMin}_M \), where

\[
\text{IMin}_M := \text{IMin}\{f(x) : x \in S\},
\]

and searching for an efficient solution of vector optimization problem (P) means finding a feasible point \( \bar{x} \in S \) such that \( f(\bar{x}) \in \text{IIMin}_M \), where

\[
\text{IIMin}_M := \text{IIMin}\{f(x) : x \in S\}.
\]

Next, we recall the main features of image space analysis associated with (P). In general, we denote

\[
f_{\bar{x}}(x) := f(\bar{x}) - f(x), \ A_{\bar{x}}(x) := (f_{\bar{x}}(x), -g(x)).
\]
and introduce the set
\[ K_{\bar{x}} := \bigcup_{x \in X} A_{\bar{x}}(x) = A_{\bar{x}}(X). \]

\( K_{\bar{x}} \) is called the image of the problem (P), while \( Y \times Z \) is the image space. Denote
\[ H_1 = \text{int} \ C \times D, \quad H_2 = C_0 \times D. \]

Considering \( K_{\bar{x}} \) and \( H_1 \), we have
\[ \bar{x} \text{ is a weak efficient solution of (P)} \iff f(\bar{x}) - f(x) \in \text{int} \ C, \quad -g(x) \in D, \quad x \in X \text{ is impossible} \]
\[ \iff H_1 \cap K_{\bar{x}} = \emptyset. \]

(2)

Similarly, \( \bar{x} \) is an efficient solution of (P) if and only if \( H_2 \cap K_{\bar{x}} = \emptyset \). Obviously, the formula \( H_i \cap K_{\bar{x}} = \emptyset \ (i \in \{1, 2\}) \) gives a geometric way to characterize the optimality of a feasible point \( \bar{x} \).

3. Conjugate maps and subgradients. Conjugate maps, biconjugate maps, subgradients and subdifferentials have been proposed by scholars in the literature, which are important concepts in the analysis of set-valued maps.

Let \( \mathfrak{L}(X,Y) \) be the space of all linear operators which maps \( X \) into \( Y \). Because \( X \) and \( Y \) are finite dimensional spaces, then every linear operator in \( \mathfrak{L}(X,Y) \) is also continuous. \( \mathfrak{L}(X,Y) \) is regarded as a dual space of \( X \) with respect to \( Y \). For \( x \in X \) and \( T \in \mathfrak{L}(X,Y) \), \( Tx \) represents the value of \( T \) at \( x \). Also, we take \( F \) to be a set-valued map from \( X \) to \( Y \). Reasonably, a vector-valued function \( f \) from \( X \) to \( Y \) can be regarded as a set-valued map from \( X \) to \( Y \) such that each \( x \) is mapped into a single point set \( \{f(x)\} \).

**Definition 3.1.** For the set-valued map \( F : X \rightrightarrows Y \), a set-valued map \( F^* \) from \( \mathfrak{L}(X,Y) \) to \( Y \) defined by
\[ F^*(T) = \text{IMax} \bigcup_{x \in X} [Tx - F(x)] \quad \text{for} \quad T \in \mathfrak{L}(X,Y) \]
is called the conjugate map of \( F \) in the sense of weak maximal points. Furthermore, a set-valued map \( F^{**} \) from \( X \) to \( Y \) defined by
\[ F^{**}(x) = \text{IMax} \bigcup_{T \in \mathfrak{L}(X,Y)} [Tx - F^*(T)] \quad \text{for} \quad x \in X \]
is called the biconjugate map of \( F \) in the sense of weak maximal points. The conjugate map \( F^* \) and biconjugate map \( F^{**} \) of \( F \) in the sense of maximal points are defined analogously by changing \( \text{IMax} \) into \( \text{IIMax} \), namely,
\[ F^*(T) = \text{IIMax} \bigcup_{x \in X} [Tx - F(x)] \quad \text{for} \quad T \in \mathfrak{L}(X,Y) \]
and
\[ F^{**}(x) = \text{IIMax} \bigcup_{T \in \mathfrak{L}(X,Y)} [Tx - F^*(T)] \quad \text{for} \quad x \in X. \]

Moreover, we recall the definitions of subgradients and subdifferentials.
Definition 3.2. Let \( \hat{x} \in X \) and \( \hat{y} \in F(\hat{x}) \). An element \( T \in \mathcal{L}(X, Y) \) is said to be a subgradient of \( F \) at \( (\hat{x}; \hat{y}) \) in the sense of weak maximal points or maximal points if
\[
T \hat{x} - \hat{y} \in \text{IMax} \bigcup_{x \in X} [Tx - F(x)]
\]
or
\[
T \hat{x} - \hat{y} \in \text{IIMax} \bigcup_{x \in X} [Tx - F(x)].
\]
The set of all subgradients of \( F \) at \( (\hat{x}; \hat{y}) \) in the sense of weak maximal points or maximal points is called the corresponding subdifferential of \( F \) at \( (\hat{x}; \hat{y}) \) and is denoted by \( \partial F(\hat{x}; \hat{y}) \) or \( \partial_{II} F(\hat{x}; \hat{y}) \). Moreover, we set \( \partial F(\hat{x}) = \bigcup_{\hat{y} \in F(\hat{x})} \partial F(\hat{x}; \hat{y}) \) and \( \partial_{II} F(\hat{x}) = \bigcup_{\hat{y} \in F(\hat{x})} \partial_{II} F(\hat{x}; \hat{y}) \). When \( \partial F(\hat{x}) \neq \emptyset \) or \( \partial_{II} F(\hat{x}) \neq \emptyset \), \( F \) is said to be weakly subdifferentiable at \( \hat{x} \) in the sense of weak maximal points or maximal points.

4. Conjugate duality. In this section, we define the perturbation functions and the corresponding optimal value maps and optimal solution maps. We analyze some properties of the optimal value maps based on the perturbation of image. Moreover, we establish the perturbation problems of the primal problem (P). By virtue of the optimal value maps of perturbation problems, we construct the dual problem of the primal problem (P). Without convexity assumptions, strong duality theorem is derived under the separation condition, which ensures the disjoint of two suitable subsets \( \mathcal{K}_x \) and \( \mathcal{H}_i \) \( (i \in \{1, 2\}) \) in the image space.

Problem (P) has the following equivalent form
\[
\min_{x \in S} f(x) = \min_{x \in X} (f(x) + \hat{\delta}_S(x))
\]
We denote \( F(x) := f(x) + \hat{\delta}_S(x) \) for convenience. As we know, searching for a weak efficient solution implies finding a weak minimal point of the set \( f(S) \), and searching for an efficient solution implies finding a minimal point of the set \( f(S) \), then we have
\[
\text{IMin} \ (P) = \text{IMin}\{f(x) : x \in S\} = \text{IMin}\{F(x) : x \in X\}
\]
or
\[
\text{IIMin} \ (P) = \text{IIMin}\{f(x) : x \in S\} = \text{IIMin}\{F(x) : x \in X\}.
\]

We introduce a perturbation parameter \( u \in Z \) in the feasible set \( S \). The corresponding perturbation feasible sets \( S(u) \) are defined as
\[
S(u) := \{x \in X : g(x) \in u - D\}, \forall u \in Z.
\]
Moreover, we define the perturbation function \( F_0(x, u) : X \times Z \rightarrow Y \cup \{\pm \infty\} \) as
\[
F_0(x, u) := f(x) + \hat{\delta}_{S(u)}(x).
\]
It can be easily observed that \( S(0) = S \) and \( F_0(x, 0) = F(x) \). Based on the perturbation function, we obtain a series of perturbation problems \( (P_u) \) associated with the perturbation parameter \( u \in Z \) formulated by
\[
(P_u) \quad \min_{x \in X} F_0(x, u),
\]
where the meaning of “\( \min \)” contains two cases as we have taken into account for the primal problem (P), that is \( (P_u) \) means to solve
\[
\text{IMin} \ (P_u) := \text{IMin}\{F_0(x, u) : x \in X\}
\]
or
\[
\text{IIMin} \ (P_u) := \text{IIMin}\{F_0(x, u) : x \in X\}.
\]
Now we define the following optimal value map for \((P_u)\) in the sense of weak minimal points or minimal points, which is a set-valued map from \(Z\) to \(Y\) given by
\[
W(u) := \text{IMin}(P_u) = \text{IMin}\{F_0(x, u) : x \in X\} = \text{IMin}\{f(x) : x \in S(u)\}
\]
or
\[
V(u) := \text{IIMin}(P_u) = \text{IIMin}\{F_0(x, u) : x \in X\} = \text{IIMin}\{f(x) : x \in S(u)\}.
\]

The optimal solution map in the sense of weak minimal points or minimal points, which is a set-valued map from \(Z\) to \(X\), is also defined by
\[
\Phi(u) := \{x \in S(u) : f(x) \in W(u)\}
\]
or
\[
\Psi(u) := \{x \in S(u) : f(x) \in V(u)\}.
\]

Then, the set of optimal values for the primal problem \((P)\) in the sense of weak minimal points or minimal points is equal to \(W(0)\) or \(V(0)\). In fact, we calculate
\[
\text{IMin}(P) = \text{IMin}\{F(x) : x \in X\} = \text{IMin}\{F_0(x, 0) : x \in X\} = W(0).
\]

Similarly, we obtain \(\text{IIMin}(P) = V(0)\). There are some properties of the optimal value maps. To this end, we define the perturbation image and extended perturbation image for \((P)\) associated with efficient solutions and weak efficient solutions given by

\[
\mathcal{K}_{(\bar{x}, u)} := \{(a, b) : a = f(\bar{x}) - f(x), b = -g(x) + u, x \in X\},
\]

\[
\mathcal{E}^{W}_{(\bar{x}, u)} := \{(a, b) : a \preceq f(\bar{x}) - f(x), b \leq_D -g(x) + u, x \in X\},
\]

\[
\mathcal{E}^{E}_{(\bar{x}, u)} := \{(a, b) : a \preceq f(\bar{x}) - f(x), b \leq_D -g(x) + u, x \in X\}.
\]

The relation between the optimal value map and the extended perturbation image is stated in the following proposition.

**Proposition 4.1.** (i): For \(\bar{x} \in X\) and any \(u \in Z\), we have
\[
W(u) = f(\bar{x}) - \text{IMax}\{a : (a, b) \in \mathcal{E}^{W}_{(\bar{x}, u)}, b = 0_Z\}.
\]

If in addition, \(\bar{x}\) is a weak efficient solution of \((P)\), then
\[
\text{IMax}\{a : (a, b) \in \mathcal{E}^{W}_{(\bar{x}, u)}, b = 0_Z\} \subseteq W(0_Z) - W(u).
\]

(ii): For \(\bar{x} \in X\) and any \(u \in Z\), we have
\[
V(u) = f(\bar{x}) - \text{IIMax}\{a : (a, b) \in \mathcal{E}^{E}_{(\bar{x}, u)}, b = 0_Z\}.
\]

If in addition, \(\bar{x}\) is an efficient solution of \((P)\), then
\[
\text{IIMax}\{a : (a, b) \in \mathcal{E}^{E}_{(\bar{x}, u)}, b = 0_Z\} \subseteq V(0_Z) - V(u).
\]

**Proof.** We prove the first statement and the second is similar. Since
\[
\text{IMax}\{a : (a, b) \in \mathcal{E}^{W}_{(\bar{x}, u)}, b = 0_Z\}
\]
\[
= \text{IMax}\{f(\bar{x}) - f(x) : 0_Z \leq_D -g(x) + u, x \in X\}
\]
\[
= \text{IMax}\{f(\bar{x}) - f(x) : g(x) \in u - D, x \in X\}
\]
\[
= f(\bar{x}) - \text{IIMin}\{f(x) : g(x) \in u - D, x \in X\}
\]
\[
= f(\bar{x}) - W(u),
\]
we obtain $W(u) = f(\bar{x}) - \text{IMax}\{a : (a, b) \in \mathcal{E}_{(\bar{x}, u)}^{W}, \ b = 0_Z\}$. Additionally, if $\bar{x}$ is a weak efficient solution of (P), then $f(\bar{x}) \in W(0_Z)$ and

$$\text{IMax}\{a : (a, b) \in \mathcal{E}_{(\bar{x}, u)}^{W}, \ b = 0_Z\} = f(\bar{x}) - W(u) \subseteq W(0_Z) - W(u).$$

The proof is completed. □

Both the efficient solutions and weak efficient solutions of the perturbation problem $(P_u)$ can be characterized by the perturbation extended image.

**Proposition 4.2.** (i): For $\bar{x} \in X$, we have $f(\bar{x}) \in W(u)$ if and only if

$$\mathcal{E}_{(\bar{x}, u)}^{W} \cap (C \times \{0_Z\}) = \emptyset.$$  \hfill (3)

(ii): For $\bar{x} \in X$, we have $f(\bar{x}) \in V(u)$ if and only if

$$\mathcal{E}_{(\bar{x}, u)}^{E} \cap (C \times \{0_Z\}) = \emptyset.$$  \hfill (4)

**Proof.** The proof of the two statements are similar, we only prove the first one. In order to prove the necessity, we use reduction to absurdity, i.e., suppose that (3) does not hold, then there exists $x \in X$ such that $a \prec f(\bar{x}) - f(x)$, $b \leq D - g(x) + u$ and $(a, b) \in C \times \{0_Z\}$. Hence we obtain

$$f(\bar{x}) - f(x) \in a + \text{int} \ C \subseteq C + \text{int} \ C \subseteq \text{int} \ C$$

and $0_Z = b \in -g(x) + u - D$. It follows that $x \in S(u)$ and $f(x) \prec f(\bar{x})$, which contradicts $f(\bar{x}) \in W(u)$.

Now we prove the sufficiency. To this end, we also prove by contradiction, i.e., suppose $f(\bar{x}) \notin W(u)$, then there exists $x \in S(u)$ such that $f(x) \prec f(\bar{x})$. Because $x \in S(u)$ indicates $0_Z \leq D - g(x) + u$, and $f(x) \prec f(\bar{x})$ implies $0_Y \prec f(\bar{x}) - f(x)$, then we get $(0_Y, 0_Z) \in \mathcal{E}_{(\bar{x}, u)}^{W}$ and $(0_Y, 0_Z) \in C \times \{0_Z\}$, which is a contradiction with (3). □

Now we define the dual problem to (P) as

$$\text{IMax}(D) := \text{IMax} \bigcup_{T \in \mathcal{L}(Z, Y)} [-W^*(T)]$$

or

$$\text{IIMax}(D) := \text{IIMax} \bigcup_{T \in \mathcal{L}(Z, Y)} [-V^*(T)].$$

Furthermore, according to the definition of biconjugate map, we claim

$$\text{IMax}(D) = \text{IMax} \bigcup_{T \in \mathcal{L}(Z, Y)} [-W^*(T)]$$

$$= \text{IMax} \bigcup_{T \in \mathcal{L}(Z, Y)} [T \cdot 0 - W^*(T)]$$

$$= W^{**}(0)$$

or analogously,

$$\text{IIMax}(D) = V^{**}(0).$$

The weak duality property holds naturally according to the structures of primal problems and duality problems.

**Theorem 4.1.** (i): For any $y \in \text{IMin} (P)$ and any $\hat{y} \in \text{IMax}(D)$, we have $y \nless \hat{y}$. 
(ii): For any \( y \in \text{IIMin}(P) \) and any \( \hat{y} \in \text{IMax}(D) \), we have
\[ y \not\in \hat{y}. \]

Proof. We just prove (i), and the proof of (ii) is similar. Since \( \text{IIMin}(P) = W(0) \) and \( \text{IMax}(D) = \text{Max}(\bigcup_{T \in \mathcal{L}(Z,Y)}[-W^*(T)]) \), then it follows from \( W^*(T) = \text{IMax}(\bigcup_{u \in Z} [Tu - W(u)]) \) that
\[ -y \not\in \hat{y}, \forall y \in W(0), \forall T \in \mathcal{L}(Z,Y), \forall \hat{y} \in -W^*(T) \]
holds, or equivalently, \( y \not\in \hat{y} \) holds. So the proof is completed. \( \square \)

Before we derive strong duality theorem, we need to introduce the class of vector weak separation functions in the image space, which is a crucial tool to realize the separation between the two subsets \( \mathcal{K}_z \) and \( \mathcal{H}_z (i \in \{1,2\}) \).

Definition 4.1. Let \( \Omega \) be a set of parameters with \((\Theta, \Lambda) \in \Omega\). A class of vector-valued functions \( \omega: Y \times Z \times \Omega \to Y \), which depends on the parameter \((\Theta, \Lambda) \in \Omega\), is called a class of vector weak separation functions corresponding to weak efficient solutions or efficient solutions if and only if
\[ w(u, v; \Theta, \Lambda) \geq_C 0, \forall (u, v) \in \mathcal{H}_1, \forall (\Theta, \Lambda) \in \Omega \quad (4) \]
or\[ w(u, v; \Theta, \Lambda) \geq_C 0, \forall (u, v) \in \mathcal{H}_2, \forall (\Theta, \Lambda) \in \Omega. \quad (5) \]

In order to introduce a suitable class of vector weak separation functions in Definition 4.1, we need to extend the classical concept of positive polar cone (also called duality cone) to vector case.

Definition 4.2. As in Section 3, Let \( \mathcal{L}(Z,Y) \) denote the family of all linear continuous operators from \( Z \) to \( Y \) and \( \mathcal{L}(Y,Y) \) denote the family of all linear continuous operators from \( Y \) to \( Y \). The vector positive polar cone (also called vector duality cone) of \( D \) with respect to \( C \) is defined by
\[ D^*_C := \{ \phi \in \mathcal{L}(Z,Y) : \phi d \geq_C 0, \forall d \in D \}, \quad (6) \]
similarly, the vector positive polar cone of \( C \) with respect to \( C \) is defined as
\[ C^*_C := \{ \varphi \in \mathcal{L}(Y,Y) : \varphi c \geq_C 0, \forall c \in C \}. \quad (7) \]

Based on Definition 4.2, we easily observe that linear vector separation functions are both the class of vector weak separation functions corresponding to weak efficient solutions and efficient solutions. We give a simple proof here. Let \( \Omega := C^*_C \times D^*_C \), the linear vector function \( \omega: Y \times Z \times \Omega \to Y \) is given by
\[ w(u, v; \Theta, \Lambda) := \Theta u + \Lambda v, \quad \Theta \in C^*_C, \quad \Lambda \in D^*_C. \]

Since we get \( \Theta u \geq_C 0 \) and \( \Lambda v \geq_C 0 \) for any \((u, v) \in \mathcal{H}_3 = \text{int} C \times D \) and \((u, v) \in \mathcal{H}_2 = C_0 \times D \) according to (6) and (7). Then, (4) and (5) hold.

In particular, if we take \( \Theta = I \in C^*_C \), where \( I \) is the unit operator from \( Y \) to \( Y \), then we obtain the following sufficient optimality conditions.

Theorem 4.2. (i): If there exists \( \bar{\Lambda} \in D^*_C \) such that
\[ w(u, v; I, \bar{\Lambda}) \neq 0, \forall (u, v) \in \mathcal{K}_x, \quad (8) \]
then \( \bar{x} \) is a weak efficient solution of \((P)\).

(ii): If there exists \( \bar{\Lambda} \in D^*_C \) such that
\[ w(u, v; I, \bar{\Lambda}) \neq 0, \forall (u, v) \in \mathcal{K}_x, \quad (9) \]
then \( \bar{x} \) is an efficient solution of \((P)\).
Proof. We only prove (i), and (ii) can be proved analogously. Suppose that (8) holds. Since \( \bar{\Lambda} \in D^*_C \), \((u,v) \in \mathcal{H}_1 = \text{int} \, C \times D \), then, for any \((u,v) \in \mathcal{H}_1 \), we get
\[
 u + \bar{\Lambda} v \in \text{int} \, C + C \subseteq \text{int} \, C.
\]
It results from (10) that
\[
w(u,v;I,\bar{\Lambda}) = u + \bar{\Lambda} v \succ 0, \ \forall (u,v) \in \mathcal{H}_1.
\]
Thus, we obtain \( \mathcal{K}_\bar{x} \cap \mathcal{H}_1 = \emptyset \) due to (8) and (11). Then it follows from the equivalence between (1) and (2) that \( \bar{x} \) is a weak efficient solution of (P).

In fact, from the proof above, condition (8) indicates \( \mathcal{K}_\bar{x} \cap \mathcal{H}_1 = \emptyset \) and condition (9) indicates \( \mathcal{K}_\bar{x} \cap \mathcal{H}_2 = \emptyset \). So conditions (8) and (9) ensure the existence of a disjoint between the two subsets \( \mathcal{K}_\bar{x} \) and \( \mathcal{H}_i \) \( (i \in \{1,2\}) \) in the image space. For this reason, we call them separation conditions. In fact, the separation conditions are not strict.

We give a nonconvex constrained multi-objective optimization problem satisfying such separation conditions to illustrate Theorem 4.2.

**Example 4.1.** Take \( X = \mathbb{R} \), \( Y = \mathbb{R}^2 \) and \( Z = \mathbb{R}^2 \). Let \( C = \mathbb{R}^2_+ \) and \( D = \mathbb{R}^2_+ \). For a vector \( x \) in finite dimensional Euclidean space, we use \( x^\top \) to denote the transpose of \( x \). We set \( f(x) = (f_1(x), f_2(x)) \) with
\[
f_1(x) = x^2, \quad f_2(x) = \left( \frac{1}{16} \right)^x.
\]
And we set \( g(x) = (g_1(x), g_2(x)) \) with
\[
g_1(x) = -x^2, \quad g_2(x) = \left\{ \begin{array}{ll}
 \frac{-1}{x^2} & \text{if } x \neq 1 \\
 0 & \text{if } x = 1.
\end{array} \right.
\]
By direct calculation, we obtain the feasible set \( S = ]-\infty, 1[ \). Take a point \( \bar{x} = 0 \) for example, then \( f(\bar{x}) = f(0) = (0,1)^\top \), obviously the matrix
\[
\bar{\Lambda} = \left( \begin{array}{cc}
 0 & 0 \\
 1 & 0
\end{array} \right)
\]
(12)
is a vector positive polar cone of \( \mathbb{R}^2_+ \) with respect to \( \mathbb{R}^2_+ \). We will verify that \( \bar{\Lambda} \) satisfies (8) and (9), namely,
\[
 f(\bar{x}) - f(x) + \bar{\Lambda}(-g(x)) \notin \text{int} \, \mathbb{R}^2_+, \ \forall x \in \mathbb{R}
\]
and
\[
 f(\bar{x}) - f(x) + \bar{\Lambda}(-g(x)) \notin \mathbb{R}^2_+ \setminus \{(0,0)^\top\}, \ \forall x \in \mathbb{R}.
\]
Now we pay attention to the following equations
\[
f(\bar{x}) - f(x) + \bar{\Lambda}(-g(x))
\]
\[
= \left( \begin{array}{cc}
 0 & 1 \\
 1 & 0
\end{array} \right) - \left( \begin{array}{c}
 x^2 \\
 \left( \frac{1}{16} \right)^x
\end{array} \right) + \left( \begin{array}{cc}
 0 & 0 \\
 1 & 0
\end{array} \right) \left( -\frac{1}{x^2} \right) \text{ or } 0
\]
\[
= \left( \begin{array}{cc}
 -x^2 & 1 - \left( \frac{1}{16} \right)^x + x^2
\end{array} \right).
\]
Since \(-x^2 \leq 0\) for all \( x \in \mathbb{R} \), it results
\[
(-x^2, 1 - \left( \frac{1}{16} \right)^x + x^2)^\top \notin \text{int} \, \mathbb{R}^2_+, \ \forall x \in \mathbb{R}.
\]
When \(-x^2 = 0\), then we have \( x = 0 \) and \( 1 - \left( \frac{1}{16} \right)^x + x^2 = 0 \). As a consequence,
\[
(-x^2, 1 - \left( \frac{1}{16} \right)^x + x^2)^\top \notin \mathbb{R}^2_+ \setminus \{(0,0)^\top\}, \ \forall x \in \mathbb{R}.
\]
(13) and (14) show that the separation conditions (8) and (9) hold. By Theorem 4.2, \( \bar{x} = 0 \) is both an efficient solution and a weak efficient solution.

Take another point \( \hat{x} = 1 \) for example, then \( f(\hat{x}) = f(1) = (1, \frac{1}{16})^T \), obviously the matrix
\[
\bar{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
is a vector positive polar cone of \( \mathbb{R}_+^2 \) with respect to \( \mathbb{R}_+^2 \). If \( x \neq 1 \), we get
\[
\begin{align*}
f(\hat{x}) - f(x) + \bar{M}(-g(x)) &= \left( \frac{1}{16}, -\frac{1}{16}x \right) - \left( \frac{x^2}{16} \right) + \left( 0 \ 1 \right) \left( \frac{x^2}{x-1} \right) \\
&= \left( 1 - x^2 - \frac{1}{16}x \right) - \left( \frac{x^2}{16} \right),
\end{align*}
\]
and if \( x = 1 \), we have
\[
f(\hat{x}) - f(x) + \bar{M}(-g(x)) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]

For \( x \neq 1 \), we calculate \( \frac{1}{16} - \frac{1}{16}x \geq 0 \Rightarrow x > 1 \). But when \( x > 1 \), we get \( 1 - x^2 - \frac{1}{x-1} < 0 \). So these results deduce that
\[
f(\hat{x}) - f(x) + \bar{M}(-g(x)) \notin \mathbb{R}_+^2 \setminus \{ (0, 0)^T \}, \ \forall x \in \mathbb{R},
\]
and thus
\[
f(\hat{x}) - f(x) + \bar{M}(-g(x)) \notin \text{int} \mathbb{R}_+^2, \ \forall x \in \mathbb{R}
\]
because of \( \text{int} \mathbb{R}_+^2 \subseteq \mathbb{R}_+^2 \setminus \{ (0, 0)^T \} \). Therefore, we verified that \( \bar{M} \) satisfies (8) and (9). According to Theorem 4.2, \( \hat{x} = 1 \) is also both an efficient solution and a weak efficient solution. In fact, we can easily observe from Figure 1 that the points 0 and 1 are both efficient solutions and weak efficient solutions.

![Figure 1](image)

**Figure 1.** The red curve shows the set of objective function values.

Now, we prove a lemma, which states that the optimal value maps for \((P_u)\) are weakly subdifferentiable under the separation conditions. This lemma is a basis to establish strong duality theorem.
Lemma 4.1. (i): If there exists $\bar{\Lambda} \in D_C^*$ such that
\[ w(u, v; I, \bar{\Lambda}) \neq 0, \ \forall (u, v) \in K_x, \] \tag{15} \]
then $W$ is weakly subdifferentiable at $u = 0$ in the sense of weak maximal points.

(ii): If there exists $\bar{\Lambda} \in D_C^*$ such that
\[ w(u, v; I, \bar{\Lambda}) \neq 0, \ \forall (u, v) \in K_x, \] \tag{18} \]
then $V$ is weakly subdifferentiable at $u = 0$ in the sense of maximal points.

Proof. We just prove (i), and the proof of (ii) is similar. If there exists $\bar{\Lambda} \in D_C^*$ such that (15) holds, then
\[ f(\bar{x}) - f(x) \neq \bar{\Lambda}g(x), \ \forall x \in X. \] \tag{16} \]
For any $u \in Z$ and any $x \in S(u)$, we have $g(x) \in u - D$, then $u - g(x) \in D$. Since $\bar{\Lambda} \in D_C^*$, by the definition of vector positive polar cone, we get
\[ \bar{\Lambda}(u - g(x)) \in C. \] \tag{17} \]
It follows from (16) and (17) that, for any $u \in Z$ and any $x \in S(u),$
\[ \bar{\Lambda}u \nsucceq f(\bar{x}) - f(x). \] \tag{18} \]
Using proof by contradiction, we suppose that there exists a $u \in Z$ and a $x \in S(u)$ such that (18) holds, namely $\bar{\Lambda}u \prec f(\bar{x}) - f(x)$. Then, we have
\[ f(\bar{x}) - f(x) - \bar{\Lambda}u \in \text{int} C. \] \tag{19} \]
Immediately, (17) and (19) deduce
\[ f(\bar{x}) - f(x) - \bar{\Lambda}g(x) \]
\[ = (f(\bar{x}) - f(x) - \bar{\Lambda}u) + (\bar{\Lambda}u - \bar{\Lambda}g(x)) \]
\[ \in \text{int} C + C \subseteq \text{int} C. \]
Thus, we obtain $f(\bar{x}) - f(x) \succeq \bar{\Lambda}g(x)$, which is a contradiction with (16).

Since we have proven that (18) holds for any $u \in Z$ and any $x \in S(u)$, then
\[ -f(\bar{x}) \preceq -\bar{\Lambda}u - f(x), \ \forall u \in Z, \forall x \in S(u). \] \tag{20} \]
It follows from $W(u) = \text{IMin}\{f(x) : x \in S(u)\}$ and (20) that
\[ -f(\bar{x}) \preceq -\bar{\Lambda}u - y, \ \forall u \in Z, \forall y \in W(u). \] \tag{21} \]
Moreover, by Theorem 4.2, if there exists $\bar{\Lambda} \in D^*_C$, such that (15) holds, then $\bar{x}$ is a weak efficient solution of (P), that is $f(\bar{x}) \in \text{IMin}(P) = W(0)$. So (21) indicates
\[ -\bar{\Lambda}0 - f(\bar{x}) = -f(\bar{x}) \in \text{IMax} \bigcup_{u \in Z} [-\bar{\Lambda}u - W(u)]. \] \tag{22} \]
According to the definition of subgradient, (22) implies that $-\bar{\Lambda}$ is a subgradient of $W$ at $(0; f(\bar{x}))$ in the sense of weak maximal points, namely $-\bar{\Lambda} \in \partial f W(0; f(\bar{x})).$

Therefore, $\partial f W(0) \neq \emptyset$ and we proved that $W$ is weakly subdifferentiable at $u = 0$ in the sense of weak maximal points. \qed
From (23) and (25), we obtain

\[
W(u) = \{f(x) : x \in \Phi(u)\} = \{(x^2, (\frac{1}{10})^T : x \in \Phi(u)\},
\]

\[
V(u) = \{f(x) : x \in \Psi(u)\} = \{(x^2, (\frac{1}{10})^T : x \in \Psi(u)\}.
\]

Since \(\bar{x} = 0\) is both a weak efficient solution and an efficient solution, which has been shown in Example 4.1, then \(f(0) \in \text{IMin}(P) = W(0)\) and \(f(0) \in \text{IMin}(P) = V(0)\), where \(0 = (0, 0)^T \in \mathbb{R}^2\). Taking the matrix \(\bar{\Lambda}\) as in (12), we calculate

\[
-\bar{\Lambda} - f(0) - f(0) = \begin{pmatrix} 0 \\ -1 \end{pmatrix}
\]

and

\[
-\bar{\Lambda}u - W(u) = - \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - W(u)
\]

\[
= \left\{ \left( -x^2 - u_1 \right) : x \in \Phi(u) \right\}, \quad \forall u \in \mathbb{R}^2.
\]

(23) and (24) indicate

\[-\bar{\Lambda} - f(0) \neq y, \quad \forall u \in \mathbb{R}^2, \forall y \in -\bar{\Lambda}u - W(u).\]

So \(-\bar{\Lambda} - f(0) \in \text{IMax} \cup_{u \in \mathbb{R}^2} \{ -\bar{\Lambda}u - W(u) \}\), that is \(-\bar{\Lambda} \in \partial_t W(0; f(0))\). Then, we get \(\partial_t W(0) \neq \emptyset\). Similarly, we calculate

\[-\bar{\Lambda}u - V(u) = \left\{ \left( -x^2 - u_1 \right) : x \in \Psi(u) \right\}, \quad \forall u \in \mathbb{R}^2.
\]

(25) For convenience, we introduce the following condition \((C_1)\):

\[(C_1) \quad \forall u \in \mathbb{R}^2, \quad 0 \in \Psi(u) \Rightarrow -u_1 \leq 0.
\]

From (23) and (25), we obtain

\[-\bar{\Lambda}u - V(u) - (-\bar{\Lambda} - f(0)) = \left\{ \left( -x^2 - u_1 \right) : x \in \Psi(u) \right\}, \quad \forall u \in \mathbb{R}^2.
\]

Since \(-x^2 < 0\) for \(x \in \mathbb{R} \setminus \{0\}\), and \(-x^2 = 0, \quad -u_1 - (\frac{1}{10})^T + 1 = -u_1\) for \(x = 0\), we conclude that if the condition \((C_1)\) holds, then \(-\bar{\Lambda} - f(0) \in \text{IMax} \cup_{u \in \mathbb{R}^2} \{ -\bar{\Lambda}u - V(u) \}\) and \(-\bar{\Lambda} \in \partial_t V(0; f(0))\). Thus, we get \(\partial_t V(0) \neq \emptyset\). Moreover, the condition \((C_1)\) is equivalent to the following condition \((C_2)\)

\[(C_2) \quad \forall u \in \mathbb{R}^2, \quad -u_1 > 0 \Rightarrow 0 \notin \Psi(u).
\]

So, the rest work is to prove that the condition \((C_2)\) holds. For this purpose, we first calculate the optimal solution map \(\Psi(u)\) for all the cases such that \(u_1 < 0\).

Case 1: \(u_1 < 0, \quad u_2 < 0\), for \(S(u) \neq \emptyset\), there are four subcases.

(i): If \(\sqrt{-u_1} \leq \frac{1}{u_2} + 1\), then \(S(u) = [\frac{1}{u_2} + 1, 1]\) and \(\Psi(u) = [\frac{1}{u_2} + 1, 1]\);

(ii): If \(-\sqrt{-u_1} < \frac{1}{u_2} + 1 \leq \sqrt{-u_1}, \quad \sqrt{-u_1} < 1\), then \(S(u) = [\sqrt{-u_1}, 1]\) and \(\Psi(u) = [\sqrt{-u_1}, 1]\);

(iii): If \(\frac{1}{u_2} + 1 \leq -\sqrt{-u_1}, \quad -\sqrt{-u_1} < 1\), then \(S(u) = [\frac{1}{u_2} + 1, -\sqrt{-u_1}] \cup [\sqrt{-u_1}, 1]\) and \(\Psi(u) = [\sqrt{-u_1}, 1]\);

(iv): If \(\frac{1}{u_2} + 1 \leq -\sqrt{-u_1}, \quad 1 \leq \sqrt{-u_1}\), then \(S(u) = [\frac{1}{u_2} + 1, -\sqrt{-u_1}]\) and \(\Psi(u) = \emptyset\).

Case 2: \(u_1 < 0, \quad u_2 > 0\), for \(S(u) \neq \emptyset\), there are three subcases.
sufficiency. If We just prove (i), and the proof of (ii) is analogous. We first prove the Proof.

For the primal problem and its duality problem, we have Proposition 4.3.

\( 0 \not\in \text{equivalent to the weak subdifferentiability of optimal values maps at } \frac{1}{u_2} \).

(26) implies \( \hat{y} \in W^*(\hat{T}) \). According to \( \hat{y} \in W(0) \) and the definition of \( W^*(T) \), we get

\( y \neq -\hat{y}, \forall T \in \mathcal{L}(Z, Y), \forall y \in W^*(T). \) (27)

(27) is equivalent to \( \hat{y} \neq -y, \forall T \in \mathcal{L}(Z, Y), \forall y \in W^*(T) \), this combined with \( \hat{y} \in -W^*(\hat{T}) \) concludes \( \hat{y} \in \text{Imax} \cup_{T \in \mathcal{L}(Z, Y)} [-W^*(T)] = \text{Imax}(D). \) So we proved

\( \hat{y} \in \text{Imin}(P) \cap \text{Imax}(D) \neq \emptyset. \)

Now we prove the necessity. Suppose \( \text{Imin}(P) \cap \text{Imax}(D) \neq \emptyset \), then there exists \( \hat{y} \in \text{Imin}(P) = W(0) \) and \( \hat{y} \in \text{Imax}(D) = \text{Imax} \cup_{T \in \mathcal{L}(Z, Y)} [-W^*(T)] \), it follows that there exists \( \hat{T} \in \mathcal{L}(Z, Y) \) such that \( \hat{y} \in -W^*(\hat{T}) \). Therefore, we have

\( \hat{T}0 - \hat{y} \in W^*(\hat{T}) = \text{Imax} \cup_{u \in Z} [\hat{T}u - W(u)]. \) (28)

\( \hat{y} \in W(0) \) and (28) indicate \( \hat{T} \in \partial W(0; \hat{y}) \), and so \( W \) is weakly subdifferentiable at \( u = 0 \) in the sense of weak maximal points.

Finally, Lemma 4.1 and Proposition 4.3 immediately derive the strong duality theorem, which shows that separation conditions are sufficient to ensure zero duality gaps.

Theorem 4.3. (i): If there exists \( \hat{\lambda} \in D^*_C \) such that

\( w(u, v; I, \hat{\lambda}) \neq 0, \forall (u, v) \in K_{\hat{x}}, \)

then \( f(\bar{x}) \in \text{Imin}(P) \cap \text{Imax}(D) \neq \emptyset \) and \(-\hat{\lambda}\) is an optimal solution of \( \text{Imax}(D) \).
(ii): If there exists \( \bar{\Lambda} \in D_C^* \) such that
\[
w(u, v; I, \bar{\Lambda}) \not\geq 0, \quad \forall (u, v) \in K_{\bar{x}},
\]
then \( f(\bar{x}) \in \text{IIMin}(P) \cap \text{IIMax}(D) \neq \emptyset \) and \(-\bar{\Lambda}\) is an optimal solution of \( \text{IIMax}(D) \).

Proof. We just prove (i), and (ii) can be proved analogously. In the proof of Lemma 4.1, we conclude \( f(\bar{x}) \in W(0) \) and \(-\bar{\Lambda} \in \partial_{\bar{\Lambda}} W(0; f(\bar{x}))\). Then, as the same as the technique of the proof for sufficiency in Proposition 4.3, we get \( f(\bar{x}) \in -W^*(-\bar{\Lambda}) \) and
\[
f(\bar{x}) \in \text{IIMin}(P) \cap \text{IIMax}(D) \neq \emptyset.
\]
Furthermore, \( f(\bar{x}) \in \text{IIMax}(D) = \text{IIMax}(\bigcup_{T \in \mathbb{Z}_Y} [-W^*(T)]) \) and \( f(\bar{x}) \in -W^*(-\bar{\Lambda}) \) indicate that \(-\bar{\Lambda}\) is an optimal solution of \( \text{IIMax}(D) \). The proof is completed. \( \square \)

It is worth mentioning that the strong duality theorem above enable us to find optimal solutions and optimal values of duality problems. We still consider the problem in Example 4.1. Since there exists
\[
\bar{\Lambda} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\]
such that the separation conditions (8) and (9) are satisfied for \( \bar{x} = 0 \), then according to Theorem 4.3, we conclude that \( f(0) \in \text{IIMax}(D) \) and \(-\bar{\Lambda}\) is an optimal solution of \( \text{IIMax}(D) \), \( f(0) \in \text{IIMax}(D) \) and \(-\bar{\Lambda}\) is an optimal solution of \( \text{IIMax}(D) \).

Similarly, since there exists
\[
\bar{M} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}
\]
such that the separation conditions (8) and (9) are satisfied for \( \hat{x} = 1 \), then applying Theorem 4.3, we conclude that \( f(1) \in \text{IIMax}(D) \) and \(-\bar{M}\) is an optimal solution of \( \text{IIMax}(D) \), \( f(1) \in \text{IIMax}(D) \) and \(-\bar{M}\) is an optimal solution of \( \text{IIMax}(D) \).

5. Conclusions. Image space approach is a powerful tool to deal with vector optimization problems, and conjugate duality is a classical duality method. At present, there has not been related works on investigating conjugate duality theory via this approach. Based on the features of the image space, we constructed a new condition (the separation condition). Then we proved that the separation condition is a sufficient condition for establishing the strong duality theorem without convexity assumptions. Furthermore, we illustrated a nonconvex multi-objective optimization problem to verify our results.

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