Bäcklund–Darboux transformations in Sato’s Grassmannian

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Abstract

We define Bäcklund–Darboux transformations in Sato’s Grassmannian. They can be regarded as Darboux transformations on maximal algebras of commuting ordinary differential operators. We describe the action of these transformations on related objects: wave functions, tau-functions and spectral algebras.

Classically, a Darboux transformation [6] of a differential operator $L$, presented as a product $L = QP$, is defined by exchanging the places of the factors, i.e. $L' = PQ$. Obviously all versions of Darboux transformations have the property that if $\Phi(x)$ is an eigenfunction of $L$, i.e. $L\Phi = \lambda\Phi$ then $P\Phi$ is an eigenfunction of $L'$, i.e. $L'P\Phi = \lambda P\Phi$. Motivated by this characteristic property we give a version of Darboux transformation directly on wave functions. The plane $W$ of Sato’s Grassmannian is said to be a Bäcklund–Darboux transformation of the plane $V$ iff the corresponding wave functions are connected by:

$$\Psi_W(x, z) = \frac{1}{g(z)} P(x, \partial_x) \Psi_V(x, z),$$

$$\Psi_V(x, z) = \frac{1}{f(z)} Q(x, \partial_x) \Psi_W(x, z)$$

for some polynomials $f$, $g$ and differential operators $P$, $Q$, or equivalently –

$$fV \subset W \subset \frac{1}{g}V.$$

To any plane $W \in Gr$ one can associate a maximal algebra $\mathcal{A}_W$ of commuting ordinary differential operators [3, 4, 5] (called a spectral algebra). Recall that a rank of $\mathcal{A}_W$ is the g.c.d. of the orders of the operators from $\mathcal{A}_W$. We prove that Bäcklund–Darboux transformations preserve the rank of the spectral algebra.

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Moreover if \( W \) is a Bäcklund–Darboux transformation of \( V \) such that \( \mathcal{A}_V = \mathbb{C}[L_V] \) for some operator \( L_V \) then every operator from \( \mathcal{A}_W \) is a Darboux transformation (in the sense of eq. (17) below) of an operator from \( \mathcal{A}_V \).

In our terminology the set of rational solutions of the KP hierarchy \([11, 13]\) coincides with the set of Bäcklund–Darboux transformations of the simplest plane \( H_+ = \text{span}\{z^n | n = 0, 1, \ldots \} \). The corresponding tau-function \( \tau_W \) is given by the so-called “superposition law for wobbly solitons” (see e.g. \([13, 15]\)). We generalize this formula for a Bäcklund–Darboux transformation \( W \) of an arbitrary plane \( V \) provided that the wave function \( \Psi_V(x, z) \) is well defined at all zeros of the polynomial \( f(z)g(z) \) (see Theorem \( 3 \) below). In a particular but important case which we use it is proven in \([1]\). The case when \( \Psi_V(x, z) \) is not well defined for some zero \( z = \lambda \) of \( f(z)g(z) \) is even more interesting (cf. \([3, 4]\)). We obtain a formula for \( \tau_W \) valid in the general situation (see Theorem \( 1 \) below).

A geometric interpretation of \( \text{Ker} P \) can be given using the so-called conditions \( C \) (introduced in \([15]\) for the rational solutions of KP). When the spectral curve \( \text{Spec} \mathcal{A}_V = \mathbb{C} \) (i.e. \( \mathcal{A}_V = \mathbb{C}[L_V] \)) the spectral curve \( \text{Spec} \mathcal{A}_W \) of \( W \) can be obtained from that of \( V \) by introducing singularities at points where the conditions \( C \) are supported – see \([13]\) and also \([3]\) (from \([3]\) it is known that \( \text{Spec} \mathcal{A}_W \) is an algebraic curve).

This paper may be considered as a part of our project \([3, 4]\) on the bispectral problem (see \([3]\)). Although here we do not touch the latter, the present paper arose in the process of working on \([3, 4]\). We noticed that many facts, needed in \([3, 4]\) can be naturally obtained in a more general situation. Apart from the applications to the bispectral problem we hope that some of the results can be useful elsewhere.

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The aim of this section is to recollect some facts and notation from Sato’s theory of KP-hierarchy \([12, 10, 13]\) needed in the paper. We use the approach of V. Kac and D. Peterson based on infinite wedge products (see e.g. \([10]\)) and the recent survey paper by P. van Moerbeke \([14]\).

Consider the space of formal Laurent series in \( z^{-1} \)

\[
\mathcal{V} = \left\{ \sum_{k \in \mathbb{Z}} a_k z^k \mid a_k = 0 \text{ for } k \gg 0 \right\}.
\]

We define the fermionic Fock space \( F \) consisting of formal infinite sums of semi-infinite monomials

\[
z^{i_0} \wedge z^{i_1} \wedge \ldots
\]

such that \( i_0 < i_1 < \ldots \) and \( i_k = k \) for \( k \gg 0 \). Let \( gl_\infty \) be the Lie algebra of all \( \mathbb{Z} \times \mathbb{Z} \) matrices having only a finite number of non-zero entries. One can define a
representation $r$ of $\mathfrak{gl}_\infty$ in the fermionic Fock space $F$ as follows

$$ r(A)(z^{i_0} \wedge z^{i_1} \wedge \ldots) = A z^{i_0} \wedge z^{i_1} \wedge \ldots + z^{i_0} \wedge A z^{i_1} \wedge \ldots + \ldots. \quad (1) $$

The above defined representation $r$ obviously cannot be continued on the Lie algebra $\tilde{\mathfrak{gl}}_\infty$ of all $\mathbb{Z} \times \mathbb{Z}$ matrices with finite number of non-zero diagonals. If we regularize it by

$$ \hat{r}(D)(z^{i_0} \wedge z^{i_1} \wedge \ldots) = [(d_{i_0} + d_{i_1} + \ldots) - (d_0 + d_1 + \ldots)](z^{i_0} \wedge z^{i_1} \wedge \ldots) \quad (2) $$

for $D = \text{diag}(\ldots, d_{-1}, d_0, d_1, \ldots)$ and by (1) for an off-diagonal matrix this will give a representation of a central extension $\hat{\mathfrak{gl}}_\infty = \tilde{\mathfrak{gl}}_\infty \oplus \mathbb{C} c$ of $\mathfrak{gl}_\infty$. Here the central charge $c$ acts as multiplication by 1. Introduce also the shift matrices $\Lambda_k$ for $k \in \mathbb{Z}$ representing the multiplication by $z^k$ in the basis $\{z^i\}_{i \in \mathbb{Z}}$ of $V$. Then $\hat{r}(\Lambda_k)$ generate a representation of the Heisenberg algebra:

$$ [\hat{r}(\Lambda_n), \hat{r}(\Lambda_m)] = n \delta_{n,-m}. $$

There exists a unique isomorphism (see [10] for details):

$$ \sigma: F \to B = \mathbb{C}[[t_1, t_2, t_3, \ldots]] \quad (3) $$

$$ \sigma(\hat{r}(\Lambda_n)) = \frac{\partial}{\partial t_n}, \quad \sigma(\hat{r}(\Lambda_{-n})) = n t_n, \quad n > 0, \quad (4) $$

known as the boson-fermion correspondence ($B$ is called a bosonic Fock space).

Sato’s Grassmannian $\text{Gr}$ [12, 7, 13] consists of all subspaces $W \subset V$ which have an admissible basis

$$ w_k = z^k + \sum_{i<k} w_{ik} z^i, \quad k = 0, 1, 2, \ldots $$

To a plane $W \in \text{Gr}$ we associate a state $|W\rangle \in F$ as follows

$$ |W\rangle = w_0 \wedge w_1 \wedge w_2 \wedge \ldots $$

A change of the admissible basis results in a multiplication of $|W\rangle$ by a non-zero constant. Thus we define an embedding of $\text{Gr}$ into the projectivization of $F$ which is called a Plücker embedding. One of the main objects of Sato’s theory is the tau-function of $W$ defined as the image of $|W\rangle$ under the boson-fermion correspondence [3]:

$$ \tau_W(t) = \sigma(|W\rangle) = \langle 0 | e^{H(t)} | W\rangle, \quad (5) $$

where $H(t) = -\sum_{k=0}^{\infty} t_k \hat{r}(\Lambda_k)$. Another important function connected to $W$ is the Baker or wave function

$$ \Psi_W(t, z) = e^{\sum_{k=1}^{\infty} t_k z^k \tau(t - [z^{-1}]) / \tau(t)}, \quad (6) $$

where $[z^{-1}]$ is the vector $(z^{-1}, z^{-2}/2, \ldots)$. We often use the notation $\Psi_W(x, z) = \Psi_W(t, z)|_{t_1=x, t_2=t_3=\ldots=0}$. 

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The Baker function $\Psi_W(x, z)$ contains the whole information about $W$ and hence about $\tau_W$, as the vectors $w_k = \partial_k^k \Psi_W(x, z)|_{x=0}$ form an admissible basis of $W$. We can expand $\Psi_W(t, z)$ in a formal series as

$$\Psi_W(t, z) = e^{\sum_{k=1}^{\infty} t_k z^k \left( 1 + \sum_{k>0} a_k(t) z^{-k} \right)}.$$  \hspace{1cm} (7)

Introduce also the pseudo-differential operators $K_W(t, \partial_x) = 1 + \sum_{j>0} a_j(t) \partial_x^{-j}$ such that $\Psi_W(t, z) = K_W(t, \partial_x) e^{\sum_{k=1}^{\infty} t_k z^k}$ (the wave operator) and $P = K_W \partial_x K_W^{-1}$. Then $P$ satisfies the following infinite system of non-linear differential equations

$$\frac{\partial}{\partial t_k} P = \left[ P^k_+, P \right],$$  \hspace{1cm} (8)

called the KP hierarchy and $\Psi_W(x, z)$ is an eigenfunction of $P(x, \partial_x)$:

$$P \Psi_W(x, z) = z \Psi_W(x, z).$$  \hspace{1cm} (9)

A very important object connected to the plane $W$ is the algebra $A_W$ of polynomials $f(z)$ that leave $W$ invariant:

$$A_W = \{ f(z) | f(z) W \subset W \}.  \hspace{1cm} (10)$$

For each $f(z) \in A_W$ one can show that there exists a unique differential operator $L_f(x, \partial_x)$, the order of $L_f$ being equal to the degree of $f$, such that

$$L_f \Psi_W(x, z) = f(z) \Psi_W(x, z).$$  \hspace{1cm} (11)

Explicitly we have

$$L_f = K_W f(\partial_x) K_W^{-1}.  \hspace{1cm} (12)$$

We denote the commutative algebra of these operators by $A_W$, i.e.

$$A_W = \{ L_f | L_f \Psi_W = f \Psi_W, f \in A_W \}.  \hspace{1cm} (13)$$

Obviously $A_W$ and $A_W$ are isomorphic. We call $A_W$ spectral algebra corresponding to the plane $W$. Following I. Krichever \[11\] we introduce the rank of $A_W$ to be the dimension of the space of joint eigenfunctions of the operators from $A_W$. It coincides with the greatest common divisor of the orders of the operators $L_f$.

We also use the notation $Gr^{(N)} := \{ W \in Gr | z^N \in A_W \}$. It coincides with the subgrassmannian of solutions of the so-called $N$-th reduction of the KP hierarchy.

\section{In this section we introduce our basic definition of Bäcklund–Darboux transformation in the Sato’s Grassmannian.}

The classical Darboux transformation is defined on ordinary differential operators in the variable $x$, presented in a factorized form $L = Q P$; it exchanges the places of the factors, i.e. the image of $L$ is the operator $\overline{T} = P Q$. The next classical lemma answers the question when the factorization $L = Q P$ is possible (see e.g. \[1\]).
Lemma 1 \( L \) can be factorized as
\[
L = QP \text{ iff } \text{Ker}P \subset \text{Ker}L.
\]
(14)

In this case
\[
\text{Ker}Q = P(\text{Ker}L).
\]
(15)

A slightly more general construction is the following one. For operators \( L \) and \( P \) such that the kernel of \( P \) is invariant under \( L \), i.e.
\[
L(\text{Ker}P) \subset \text{Ker}P
\]
(16)
we consider the transformation
\[
L \mapsto \overline{L} = PLP^{-1}.
\]
(17)

The fact that \( \overline{L} \) is a differential operator follows from Lemma 1. Indeed, \( L(\text{Ker}P) \subset \text{Ker}P \) is equivalent to \( \text{Ker}P \subset \text{Ker}(PL) \). If \( h \) is the characteristic polynomial of the linear operator \( L|_{\text{Ker}P} \) then \( h(L|_{\text{Ker}P}) = 0 \), i.e.
\[
\text{Ker}P \subset \text{Ker}h(L).
\]

This shows that
\[
h(L) = QP, \quad h(\overline{L}) = PQ
\]
for some operator \( Q \).

Now we come to our basic definition.

Definition 1 We say that a plane \( W \) (or the corresponding wave function \( \Psi_W(x, z) \)) is a Bäcklund–Darboux transformation of the plane \( V \) (respectively wave function \( \Psi_V(x, z) \)) iff there exist (monic) polynomials \( f(z), g(z) \) and differential operators \( P(x, \partial_x), Q(x, \partial_x) \) such that
\[
\Psi_W(x, z) = \frac{1}{g(z)}P(x, \partial_x)\Psi_V(x, z),
\]
(18)
\[
\Psi_V(x, z) = \frac{1}{f(z)}Q(x, \partial_x)\Psi_W(x, z).
\]
(19)

Here necessarily \( \text{ord}P = \text{deg}g \) and \( \text{ord}Q = \text{deg}f \). The polynomial \( g(z) \) can be chosen arbitrary but of the same degree because the wave function \( \Psi_W(x, z) \) is determined up to a multiplication by a formal series of the form \( 1 + \sum_{k=1}^{\infty} a_k z^{-k} \).

Note that a composition of two Bäcklund–Darboux transformations is again a Bäcklund–Darboux transformation. For example the bispectral potentials of \( \mathbb{R} \) can be obtained by one Bäcklund–Darboux transformation in contrast to the finite number of “rational” Darboux transformations of Duistermaat and Grünbaum.

Simple consequences of Definition 1 are the identities
\[
PQ\Psi_W(x, z) = f(z)g(z)\Psi_W(x, z),
\]
(20)
\[
QP\Psi_V(x, z) = f(z)g(z)\Psi_V(x, z),
\]
(21)
i.e. the operator $\mathcal{L} = PQ \in \mathcal{A}_W$ is a Darboux transformation of $L = QP \in \mathcal{A}_V$. Obviously (18) implies the inclusion

$$gW \subset V. \quad (22)$$

Conversely, if (22) holds there exists $P$ satisfying (18). Therefore a definition equivalent to Definition 1 is the following one.

**Definition 2** A plane $W$ is a Bäcklund–Darboux transformation of a plane $V$ iff

$$fV \subset W \subset \frac{1}{g}V \quad (23)$$

for some polynomials $f(z), g(z)$.

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In this section we study the behavior of the spectral algebra under Bäcklund–Darboux transformations. The following simple lemma will be useful.

**Lemma 2** In the notation of (23) and (10)

$$fgA_V \subset A_W \subset \frac{1}{fg}A_V. \quad (24)$$

The proof is obvious from (23). $\square$

**Proposition 1** The Bäcklund–Darboux transformations preserve the rank of the spectral algebras, i.e. if $W$ is a Bäcklund–Darboux transformation of $V$ then $\text{rank} \mathcal{A}_W = \text{rank} \mathcal{A}_V$.

**Proof.** Let $\text{rank} \mathcal{A}_V = N$. Then Lemma 1 implies that $fg \in \mathcal{A}_V$ and therefore $\deg fg = Nj, j \in \mathbb{N}$. The right inclusion in (24) gives $N \mid \text{rank} \mathcal{A}_W$. Because $\text{rank} \mathcal{A}_V = N$, $\mathcal{A}_V$ contains polynomials of degrees $ka + lb$, for $k, l \in \mathbb{Z}_{\geq 0}$ with $(a, b) = N$. The left inclusion in (24) implies that $\mathcal{A}_W$ contains polynomials of degrees $ka + lb + Nj$, for $k, l \in \mathbb{Z}_{\geq 0}$, i.e. $\text{rank} \mathcal{A}_W | N$ and therefore $\text{rank} \mathcal{A}_W = N$. $\square$

The most important case of algebras $\mathcal{A}_V$ of rank $N$ is

$$\mathcal{A}_V = \mathbb{C}[z^N], \quad \mathcal{A}_V = \mathbb{C}[L_V] \quad (25)$$

for some natural number $N$ and a differential operator $L_V$ of order $N$ (see [3]). This corresponds to the case when the spectral curve $\text{Spec} \mathcal{A}_V$ of $V$ is $\mathbb{C}$. We shall describe $\mathcal{A}_W$ for a Bäcklund–Darboux transformation $W$ of $V$ for which (25) holds. First observe that due to (21) we have

$$f(z)g(z) = h(z^N), \quad (26)$$

$$QP = h(L_V) \quad (27)$$

for some polynomial $h$. 6
Proposition 2 If $A_V = \mathbb{C}[L_V]$, ord$L_V = N$ then

$$A_W = \{ u \in \mathbb{C}[z^N] \mid u(L_V)\text{Ker}P \subset \text{Ker}P \},$$

$$A_W = \{ Pu(L_V)^{P-1} \mid u \in A_W \}. \tag{28}$$

Proof. Since $A_W \subset \mathbb{C}[z]$ the right inclusion of (24) with $A_V = \mathbb{C}[z^N]$ and $f(z)g(z) = h(z^N)$ implies $A_W \subset \mathbb{C}[z^N]$. Let $u(z^N) \in A_W$ and $L$ be the corresponding operator from $A_W$ (see (12)), such that

$$L\Psi_W(x, z) = (z^N)\Psi_W(x, z).$$

Using (15) we compute

$$L\Psi_W(x, z) = L \frac{1}{g(z)} P\Psi_V(x, z) = \frac{1}{g(z)} LP\Psi_V(x, z)$$

and

$$u(z^N)\Psi_W(x, z) = u(z^N) \frac{1}{g(z)} P\Psi_V(x, z)$$

$$= \frac{1}{g(z)} Pu(z^N)\Psi_V(x, z) = \frac{1}{g(z)} Pu(L_V)\Psi_V(x, z).$$

Therefore

$$LP = Pu(L_V).$$

The operator $L = Pu(L_V)^{P-1}$ is differential iff $u(L_V)\text{Ker}P \subset \text{Ker}P$ and obviously it belongs to $A_W$. \qed

Thus the determination of $A_W$ is reduced to the following finite-dimensional problem:

For the linear operator $L_V |_{\text{Ker}(L_V)}$ find all polynomials $u(z^N)$ such that the subspace $\text{Ker}P$ of $\text{Ker}h(L_V)$ is invariant under the operator $u(L_V)$.

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In this section we shall describe the tau-function of the Bäcklund–Darboux transformation $W$ of $V$ in terms of the tau-function of $V$ and $\text{Ker}P$.

Let $V \in G_{\nu(N)}$, i.e. $z^N V \subset V$ and $L_V\Psi_V(x, z) = z^N\Psi_V(x, z)$ for some operator $L_V$ of order $N$. (We do not suppose that $A_V = \mathbb{C}[z^N]$ but only that $\mathbb{C}[z^N] \subset A_V$.) Let $W$ be a Bäcklund–Darboux transformation of $V$ such that (15, 19, 26, 27) hold. Let us fix a basis $\{ \Phi_i(x) \}_{0 \leq i \leq d-1}$ of $\text{Ker}(L_V)$ (where $d = \deg h$). The kernel of $P$ is a subspace of $\text{Ker}(L_V)$. We fix a basis of $\text{Ker}P$

$$f_k(x) = \sum_{i=0}^{dN-1} a_{ki} \Phi_i(x), \quad 0 \leq k \leq n - 1. \tag{30}$$

We can suppose that $P$ and $g$ are monic. Eq. (15) implies (see e.g. [9])

$$\Psi_W(x, z) = \frac{Wr(f_0(x), \ldots, f_{n-1}(x), \Psi_V(x, z))}{g(z)Wr(f_0(x), \ldots, f_{n-1}(x))}. \tag{31}$$
where $W_r$ denotes the Wronski determinant. When we express $f_k(x)$ by (30) we obtain
\[
\Psi_W(x, z) = \frac{\sum \det A^I W_r(\Phi_I(x)) \Psi_I(x, z)}{\sum \det A^I W_r(\Phi_I(x))}.
\] (32)
The sum is taken over all $n$-element subsets
\[
I = \{i_0 < i_1 < \ldots < i_{n-1}\} \subset \{0, 1, \ldots, dN - 1\}
\]
and here and further we use the following notation:
\[
A = (a_{ki})_{0 \leq k \leq n-1, 0 \leq i \leq dN-1}
\]
is the matrix from (30) and
\[
A^I = (a_{ki})_{0 \leq k \leq n-1, 0 \leq i \leq n-1}
\]
is the corresponding minor of $A$,
\[
\Phi_I(x) = \{\Phi_{i_0}(x), \ldots, \Phi_{i_{n-1}}(x)\}
\]
is the corresponding subset of the basis $\{\Phi_i(x)\}$ of $\text{Ker} h(L_V)$ and
\[
\Psi_I(x, z) = \frac{W_r(\Phi_I(x), \Psi_V(x, z))}{g(z) W_r(\Phi_I(x))}
\] (33)
is the Bäcklund–Darboux transformation of $V$ with a basis of $\text{Ker} P f_k(x) = \Phi_{i_k}(x)$.

So if we know how $V$ transforms when the basis $\{f_k(x)\}$ of $\text{Ker} P$ is a subset of the basis $\{\Phi_i(x)\}$ of $\text{Ker} h(L_V)$, the formula (32) gives us $\Psi_W(x, z)$ for an arbitrary Bäcklund–Darboux transformation $W$.

We shall obtain a similar formula for the tau-functions as well.

Let $\tau_V(0)$ and $\tau_W(0)$ be nonzero and let us normalize them to be equal to 1 (recall that the tau-function is defined up to a multiplication by a constant). We set
\[
\Delta_I = W_r(\Phi_I(x))|_{x=0}.
\]
Denote by $\tau_I$ the tau-function corresponding to the wave function (33), also normalized by $\tau_I(0) = 1$. Then we have the following theorem.

**Theorem 1** In the above notation
\[
\tau_W(t) = \frac{\sum \det A^I \Delta_I \tau_I(t)}{\sum \det A^I \Delta_I}.
\] (34)
For the proof we have to introduce some more terminology. These are the so-called conditions $C$ (cf. [15]). They are conditions (or equations) that should be imposed on a vector $v \in V$ in order to belong to $gW$ (recall (22)).

Let us fix an admissible basis $\{v_k\}_{k \geq 0}$ of $V$ and set
\[
V(n) = \bigoplus_{k=0}^{n-1} \mathbb{C}v_k
\]
We define a linear map
\[ C: V \rightarrow V(n) \]
by defining it on the wave function of \( V \)
\[ C\Psi_V(x, z) = \sum_{k=0}^{n-1} f_k(x)v_k, \quad (35) \]
where \( \{ f_k(x) \} \) is the basis of \( \text{Ker} P \). The point is that \( C \) acts on the variable \( z \). If we choose \( v_k = \partial^k_x \Psi_V(x, z)|_{x=0} \) then
\[ Cv = C\partial^p_x \Psi_V(x, z)|_{x=0} = \partial^p_x C\Psi_V(x, z)|_{x=0} = \sum_{k=0}^{n-1} f^{(p)}_k(0)v_k. \]

Let \( V_C \) be the kernel of \( C \), i.e.
\[ V_C = \{ v \in V \mid Cv = 0 \}. \]
Then the description of \( gW \) is straightforward (cf. [15]).

**Lemma 3** \( W = \frac{1}{g} V_C. \)

**Proof.** First we show that \( gW \subset V_C \). Indeed,
\[ C(g(z)\Psi_W(x, z)) = C(P(x, \partial_x)\Psi_V(x, z)) = P(x, \partial_x)C\Psi_V(x, z) \]
\[ = P(x, \partial_x) \sum_{k=0}^{n-1} f_k(x)v_k = 0, \quad \text{because } f_k \in \text{Ker} P. \]

On the other hand the vectors \( g(z)\partial^j_x \Psi_W(x, z)|_{x=0} \) can be expressed in the form
\[ v_{j+n} + \sum_{k<j+n} d_{jk}v_k, \quad j \geq 0, \]
i.e. the plane \( gW \) maps one to one on the plane \( \bigoplus_{j \geq n} C v_j \). But the same is true for the plane \( V_C \) as \( \text{Im} C = V(n) \) (because \( \text{det}(C|_{V(n)}) = Wr(f_k(x))|_{x=0} \neq 0 \)). \( \square \)

**Corollary 1** \( W \) has an admissible basis
\[ w_j = \frac{1}{g(z)} \left( 1 - C_{(n)}^{-1} C \right) v_{j+n}, \quad j \geq 0, \quad (36) \]
where \( C_{(n)} = C|_{V(n)}. \)

**Proof.** \( C \left( 1 - C_{(n)}^{-1} C \right) = C - C_{(n)}C_{(n)}^{-1} C = 0 \) since \( \text{Im} C = V(n). \) \( \square \)

\( V_C \) can also be interpreted as the intersection of the kernels of certain linear functionals on \( V \) (\( \text{pr}_k \circ C: V \rightarrow C v_k \equiv C \)) which form an \( n \)-dimensional linear space. We denote it by abuse of notation again by \( C \).
Lemma 4 Any condition \( c \in C \) gives rise to a function
\[
f(x) = \langle c, \Psi_V(x, z) \rangle
\] (37)
from \( \text{Ker} P \), and vice versa.

The proof follows immediately from the definition (33). \( \square \)

We define linear functionals \( \chi_i \) and \( c_k \) on \( V \) by
\[
\begin{align*}
\langle \chi_i, \Psi_V(x, z) \rangle &= \Phi_i(x), \quad 0 \leq i \leq dN - 1, \\
\langle c_k, \Psi_V(x, z) \rangle &= f_k(x), \quad 0 \leq k \leq n - 1,
\end{align*}
\]
(38) (39)
i.e. \( c_k = \sum_{i=0}^{dN-1} a_{ki} \chi_i \).

We can now give the proof of Theorem 4. The basis (36) of \( W \) can be written as
\[
w_j = \frac{1}{g(z)} (v_j + n - \sum_{0 \leq k \leq n-1, 0 \leq i \leq dN-1} (C^{-1}_n A)_{ki} \langle \chi_i, v_j + n \rangle v_k), \quad j \geq 0.
\]

We use the formula (3)
\[
\tau_W(t) = \sigma(w_0 \wedge w_1 \wedge w_2 \wedge \ldots)
\]
and expand all \( w_j \):
\[
\begin{align*}
\tau_W(t) &= \sum_{r=-1}^{n-1} \sum_{0 \leq k_i \leq n-1, 0 \leq i \leq dN-1} \left( C^{-1}_n A \right)_{k_{i0}} \cdots \left( C^{-1}_n A \right)_{k_{i1}} (-1)^{r+1}
\times \sum_{n \leq j_0 < \ldots < j_r} \sigma \left( \frac{1}{g} (\chi_{i_0}, v_{j_0}) v_{k_0} \wedge \ldots \wedge \frac{1}{g} (\chi_{i_r}, v_{j_r}) v_{k_r} \wedge \ldots \right)
\end{align*}
\]
(the term \( \frac{1}{g} (\chi_i, v_j) v_{k_s} \) is on the \( (j_s - n + 1) \)-st place in the wedge product). Let \( (k_0, \ldots, k_{n-1}) \) be a permutation of \( \{0, 1, \ldots, n-1\} \). Noting that
\[
\sum_{0 \leq i \leq dN-1} (C^{-1}_n A)_{ki} \langle \chi_i, v_j \rangle = \delta_{kj} \quad \text{for} \quad 0 \leq k, j \leq n - 1
\]
we can insert
\[
\begin{align*}
\sum_{0 \leq k_i \leq n-1, 0 \leq i \leq dN-1} &\left( C^{-1}_n A \right)_{k_{i+1} \ldots} \cdots \left( C^{-1}_n A \right)_{k_{i_1} \ldots} \\
n \leq j_{r+1} < \ldots < j_{n-1} \leq n-1
\end{align*}
\]
in the above expression for \( \tau_W(t) \). Then
\[
\tau_W(t) = \sum_{(k_0, \ldots, k_{n-1})} \sum_{0 \leq i_0, \ldots, i_{n-1} \leq dN-1} \left( C^{-1}_n A \right)_{k_{i0}} \cdots \left( C^{-1}_n A \right)_{k_{i_{n-1}}}
\]
\[
\times \sigma \left( R \left( \frac{1}{g} \right) r(\chi_i_0) \cdots r(\chi_{i_{n-1}})(v_{k_0} \wedge \ldots \wedge v_{k_{n-1}} \wedge v_n \wedge v_{n+1} \wedge \ldots) \right),
\]
where the operator \( R(\frac{1}{g}) \) acts as a group element
\[
R \left( \frac{1}{g} \right) (u_0 \wedge u_1 \wedge \ldots) = \frac{1}{g} u_0 \wedge \frac{1}{g} u_1 \wedge \ldots
\]
and \( r(\chi_i) \) is a contracting operator:
\[
r(\chi_i)(u_0 \wedge u_1 \wedge \ldots) = \sum_{j \geq 0} (-1)^j \langle \chi_i, u_j \rangle u_0 \wedge u_1 \wedge \ldots \hat{u}_j \wedge \ldots
\]
(the hat on \( u_j \) means as usually that it is omitted).

By the antisymmetry we obtain
\[
\tau_W(t) = \sum \det \left( C_{(n)}^{-1} A \right)^I R \left( \frac{1}{g} \right) r(\chi_I) \tau_V(t),
\]
where the sum is over the subsets
\[
I = \{ i_0 < \ldots < i_{n-1} \} \subset \{ 0, 1, \ldots, dN - 1 \}
\]
and
\[
r(\chi_I) = r(\chi_i_0) \cdots r(\chi_{i_{n-1}}).
\]
For the special Bäcklund–Darboux transformation \( \tau_I(t) \) with \( f_k(x) = \Phi_{i_k}(x) \) we have
\[
c_k = \chi_{i_k}, \quad A = (\delta_{i_k})_{0 \leq k \leq n-1}, \quad \det C_{(n)} = \text{Wr}(\Phi_I(x))|_{x=0} = \Delta_I.
\]
Now (40) implies
\[
\tau_I(t) = \frac{1}{\Delta_I} R \left( \frac{1}{g} \right) r(\chi_I) \tau_V(t).
\]
Noting that
\[
\left( C_{(n)}^{-1} A \right)^I = C_{(n)}^{-1} A^I
\]
and
\[
\det C_{(n)} = \text{Wr}(f_k(x))|_{x=0} = \sum \det A^I \Delta_I
\]
completes the proof of Theorem 1. \( \square \)

Suppose that linear functionals \( \chi_i \) can be defined on all \( z^k \), e.g. \( \chi_i \) are of the form:
\[
\chi_i = \sum_{j \geq 0} \alpha_{ij} \partial_z^j|_{z = \lambda_i}
\]
(41)
for \( \lambda_i \neq 0 \) (cf. [13]). We recall that (see e.g. [9])
\[
\text{Ker} \prod_{i=1}^r (L_V - \lambda_i^N)^{d_i} = \text{span} \left\{ \partial_z^k \psi_V(x, z) \bigg|_{z = \varepsilon \lambda_i} \right\}_{0 \leq j \leq N-1, 1 \leq i \leq r, 0 \leq k_i \leq d_i - 1},
\]
with \( \varepsilon = e^{2\pi i/N} \), when \( \Psi_V(x, z) \) is well defined for \( z = \varepsilon^j \lambda \) (cf. eq. (6)). Then we can put
\[
f_k(t) = \langle c_k, \Psi_V(t, z) \rangle, \quad 0 \leq k \leq n - 1
\]
and \( f_k(t) \) can be thought as obtained from \( f_k(x) \) by applying the KP flows. In this case \( \tau_W \) is given by the following theorem.

**Theorem 2** If \( f_k(t) \) are as above and \( g(z) = z^n \), then
\[
\tau_W(t) = \frac{W_r(f_k(t))}{W_r(f_k(0))} \tau_V(t).
\]

**Proof** uses the differential Fay identity \( [1] \) (see also \([14]\)):
\[
W_r(\Psi_V(t, z_0), \ldots, \Psi_V(t, z_n)) \tau_V(t)
= \prod_{0 \leq j < i \leq n} (z_i - z_j) \cdot \exp \left( \sum_{k=0}^{\infty} \sum_{i=0}^{n} t_k z_i^k \right) \tau_V \left( t - \sum_{i=0}^{n} [z_i^{-1}] \right),
\]
where \([z^{-1}] = (z^{-1}, z^{-2}/2, z^{-3}/3, \ldots)\). After introducing the vertex operator
\[
X(t, z) = \exp \left( \sum_{k=1}^{\infty} t_k z^k \right)
\exp \left( - \sum_{k=1}^{\infty} \frac{1}{kz^k} \partial_t \right)
\]
the RHS can be written in the form
\[
z_0^n \cdots z_n^n X(t, z_n) X(t, z_{n-1}) \cdots X(t, z_0) \tau_V(t)
= z_n^n X(t, z_n) \left( W_r(\Psi_V(t, z_0), \ldots, \Psi_V(t, z_{n-1})) \tau_V(t) \right).
\]
We apply the condition \( c_0 \) to the variable \( z_0, c_1 \) to \( z_1, \ldots, c_{n-1} \) to \( z_{n-1} \), and set \( z_n = z \). We obtain
\[
W_r(f_0(t), \ldots, f_{n-1}(t), \Psi_V(t, z)) \tau_V(t)
= z^n X(t, z) \left( W_r(f_0(t), \ldots, f_{n-1}(t)) \tau_V(t) \right).
\]
But \( [31] \) with \( g(z) = z^n \) imply
\[
\Psi_W(t, z) = \frac{W_r(f_0(t), \ldots, f_{n-1}(t), \Psi_V(t, z))}{z^n W_r(f_0(t), \ldots, f_{n-1}(t))}
\]
(KP flows applied to \( [31] \)). Because the tau-function is determined from \( [6] \) up to a multiplication by a constant, \( [13] \) follows (when \( \tau_V(0) = \tau_W(0) = 1 \)). \( \square \)

**Example 1** Let us consider the simplest plane in the Sato’s Grassmannian \( V = H^+ = \text{span}\{ z^i | i = 0, 1, \ldots \} \). Then
\[
\psi_V(t, z) = \exp \sum t_k z^k, \quad L_V = \partial_z, \quad \tau_V(t) = 1.
\]
Every linear functional on \( H^+ \) is a linear combination of conditions of the type
\[
e(k, \lambda) = \partial^k_{z|z=\lambda}
\]
and \( h \left( L(0) \right) = h(\partial_x) \) is an operator with constant coefficients. The set of rational solutions of the KP hierarchy \([11, 13]\) coincides with the set of Bäcklund–Darboux transformations of \( H_+ \). The formula (43) with \( \tau_V = 1 \) is called a “superposition law for wobbly solitons” (cf. \([15]\), eq. (5.7)). The so called polynomial solutions of KP \([7, 13]\) correspond to the case \( h(z) = z^d \), i.e. all conditions are supported at 0. □

Without any constraints on conditions \( C \), there is a weaker version of Theorem 2.

**Proposition 3** For \( g(z) = z^n \)

\[
\tau_W(x) = \frac{W_r(f_k(x))}{W_r(f_k(0))} \tau_V(x),
\]

where \( \tau_W(x) = \tau_W(x,0,0,\ldots) \).

**Proof.** Formula (6) implies

\[
\Psi_W(x, z) = e^{xz} \left( 1 - \partial_x \log \tau_W(x) z^{-1} + \cdots \right). \tag{44}
\]

On the other hand

\[
z^{-n}(\partial_x^n + p_1(x) \partial_x^n - 1 + \cdots + p_0(x)) \Psi_V(x, z) = e^{xz} \left( 1 - \partial_x \log \tau_V(x) z^{-1} + p_1(x) z^{-1} + \cdots \right).
\]

Comparing the coefficients at \( z^{-1} \) and noting that

\[
p_1(x) = -\partial_x \log W_r(f_k(x))
\]

completes the proof. □

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