Finite group discretization of Yang-Mills and Einstein actions

Leonardo Castellani\textsuperscript{1,2} and Chiara Pagani\textsuperscript{1}

\textsuperscript{1} Dipartimento di Scienze e Tecnologie Avanzate, Università del Piemonte Orientale, Italy;
\textsuperscript{2} Dipartimento di Fisica Teorica and I.N.F.N
Via P. Giuria 1, 10125 Torino, Italy.

abstract@to.infn.it

Abstract

Discrete versions of the Yang-Mills and Einstein actions are proposed for any finite group. These actions are invariant respectively under local gauge transformations, and under the analogues of Lorentz and general coordinate transformations. The case $\mathbb{Z}_n \times \mathbb{Z}_n \times \ldots \times \mathbb{Z}_n$ is treated in some detail, recovering the Wilson action for Yang-Mills theories, and a new discretized action for gravity.

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1 Introduction

Discretization of field theories, and in particular of gravity, has a long history which we do not attempt to review here. Comprehensive accounts and reference lists can be found in [1]. The motivations to consider field theories on discrete structures are at least of two kinds. One is computational, as in quantum gauge theories, where the lattice approach yields nonperturbative information. The other is related to the mathematical consistency of the quantum theory, since “separating” the spacetime points removes the ultraviolet divergences: this is certainly useful in nonrenormalizable theories like ordinary Einstein quantum gravity.

According to a current paradigm, quantum gravity arises within string/brane theory [2], a consistent quantum theory that is expected to unify all known interactions. Then Einstein theory is to be considered a low energy effective field theory, part of the more fundamental “brane world” [3] and as such not needing to make sense at the quantum level by itself. But attempts to make it consistent are still worthwhile: after all Yang-Mills theory, which also emerges in the low energy regime of the string/brane theory, is a well defined quantum field theory without need of a “stringy” regularization.

On the other hand, string/brane theory also suggests that spacetime at short distances may not be smooth. For example gauge theories on noncommutative spaces are a low-energy limit of open strings in a background $B$-field, describing the fluctuations of the D-brane worldvolume [4, 5]. This has prompted many investigations in $*$-deformations of Yang-Mills theories [6] and has provided one of the bridges between string theory and noncommutative geometry (NCG). Reviews on NCG can be found in [7].

In this paper we construct a discretized Yang-Mills and Einstein action for any finite group $G$: this is a particular noncommutative setting, where noncommutativity does not concern coordinates between themselves, but only coordinates with differentials.

The spacetime points are replaced by isolated points labeling the group elements of $G$. The functions on these sets of points are endowed with differential calculi, due to the Hopf algebra induced by the group structure [8, 9, 10, 11]. Then one can construct the analogue of the vielbein, the connection, the curvature etc. on these finite group “manifolds”. In fact this program can be carried out for any Hopf algebra, quantum groups being a notable example (differential calculi on Hopf algebras were first constructed in [12]; for a review with applications to field theory see for ex. [13]). Here we use the differential calculi to define Yang-Mills and Einstein actions on any finite $G$, and we show that these actions are respectively invariant under

1 In fact in the brane world scenarios (see [3] for an early reference, and [4] for a more contemporary point of view) spacetime is considered as a dynamical brane within a higher dimensional space.

2 In contrast, there are very few studies for $*$-deformed gravity theories, the reason being that one lacks a “stringy” motivation. But it would be of interest to see how the intrinsic nonpolynomiality of gravity plays against the nonpolynomiality of the $*$ product.
the $G$-analogues of local gauge variations, and of Lorentz and general coordinate transformations. Previous investigations on finite $G$ field theories can be found in \[14, 15, 10, 11, 16\].

A possible application of this technique is to use finite group spaces as internal spaces in Kaluza-Klein (super)gravity or superstring theories. Harmonic analysis on finite group spaces being elementary, the reduction of the higher dimensional theory is easy to implement. In fact the Kaluza-Klein reduction of $M^4 \times Z_2$ gauge theories coupled to fermions yields a Higgs field in $d = 4$, with the correct spontaneous symmetry breaking potential and Yukawa couplings, see for ex. \[11\].

In Section 2 we give a review of the differential calculus on finite groups. Section 3 illustrates the general results for the case $G = (Z_n)^N$. In Section 4 we present the Yang-Mills action on finite groups. Section 5 contains two proposals for a gravity action on a finite group $G$, differing in the choice of tangent group, and the corresponding invariances are discussed; the $G = (Z_n)^N$ gravity action is treated in more detail. Some conclusions are included in Section 6.

2 Differential calculus on finite groups

Notations

Let $G$ be a finite group of order $n$ with generic element $g$ and unit $e$. Consider $Fun(G)$, the set of complex functions on $G$. An element $f$ of $Fun(G)$ is specified by its values $f_g \equiv f(g)$ on the group elements $g$, and can be written as

$$f = \sum_{g \in G} f_g x^g, \quad f_g \in \mathbb{C}$$

(2.1)

where the functions $x^g$ are defined by

$$x^g(g') = \delta^g_{g'}$$

(2.2)

Thus $Fun(G)$ is a $n$-dimensional vector space, and the $n$ functions $x^g$ provide a basis. $Fun(G)$ is also a commutative algebra, with the usual pointwise sum and product, and unit $I$ defined by $I(g) = 1, \forall g \in G$. In particular:

$$x^g x^{g'} = \delta_{g, g'} x^g, \quad \sum_{g \in G} x^g = I$$

(2.3)

The left action of the group on itself induces the (pullback) $\mathcal{L}_{g_1}$ on $Fun(G)$:

$$\mathcal{L}_{g_1} f(g_2) \equiv f(g_1 g_2)|_{g_2}, \quad \mathcal{L}_{g_1} : Fun(G) \to Fun(G)$$

(2.4)

where $f(g_1 g_2)|_{g_2}$ means $f(g_1 g_2)$ seen as a function of $g_2$. Similarly we can define the right action on $Fun(G)$ as:

$$(\mathcal{R}_{g_1} f)(g_2) = f(g_2 g_1)|_{g_2}$$

(2.5)
For the basis functions we find easily:

\[ \mathcal{L}_{g_1} x^g = x^{g^{-1}} g, \quad \mathcal{R}_{g_1} x^g = x^{gg_1^{-1}} \]  

Moreover:

\[ \mathcal{L}_{g_1} \mathcal{L}_{g_2} = \mathcal{L}_{g_2 g_1}, \quad \mathcal{R}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g g_2}, \quad \mathcal{L}_{g_1} \mathcal{R}_{g_2} = \mathcal{R}_{g_2} \mathcal{L}_{g_1} \]  

The \( G \) group structure induces a Hopf algebra structure on \( \text{Fun}(G) \), and this allows the construction of differential calculi on \( \text{Fun}(G) \), according to the techniques of ref. [12, 13]. We summarize here the main properties of these calculi. A detailed treatment can be found in [10].

**Exterior differential**

A (first-order) differential calculus on \( \text{Fun}(G) \) is defined by a linear map \( d : \text{Fun}(G) \to \Gamma \), satisfying the Leibniz rule

\[ d(ab) = (da)b + a(db), \quad \forall a, b \in \text{Fun}(G); \]  

The “space of 1-forms” \( \Gamma \) is an appropriate bimodule on \( \text{Fun}(G) \), which essentially means that its elements can be multiplied on the left and on the right by elements of \( \text{Fun}(G) \). From the Leibniz rule \( da = d(Ia) = (dI)a + Ida \) we deduce \( dI = 0 \). Consider the differentials of the basis functions \( x^g \). From \( 0 = dI = d(\sum_{g \in G} x^g) = \sum_{g \in G} dx^g \) we see that only \( n-1 \) differentials are independent. Moreover every element of \( \Gamma \) can be expressed as a linear combination (with complex coefficients) of terms of the type \( x^g dx^{g'} \), since the commutations:

\[ dx^g x^{g'} = -x^g dx^{g'} + \delta_{g}^{g'} dx^g \]  

allow to reorder functions to the left of differentials.

**Partial derivatives**

Consider the differential of a function \( f \in \text{Fun}(g) \):

\[ df = \sum_{g \in G} f_g dx^g = \sum_{g \neq e} f_g dx^g + f_e dx^e = \sum_{g \neq e} (f_g - f_e) dx^g \equiv \sum_{g \neq e} \partial_g f dx^g \]  

We have used \( dx^e = - \sum_{g \neq e} dx^g \) (from \( \sum_{g \in G} dx^g = 0 \)). The partial derivatives of \( f \) have been defined in analogy with the usual differential calculus, and are given by

\[ \partial_g f = f_g - f_e = f(g) - f(e) \]  

Not unexpectedly, they take here the form of finite differences (discrete partial derivatives at the origin \( e \)).

3
Left and right covariance

A differential calculus is left or right covariant if the left or right action of $G$ ($\mathcal{L}_g$ or $\mathcal{R}_g$) commutes with the exterior derivative $d$. Requiring left and right covariance in fact defines the action of $\mathcal{L}_g$ and $\mathcal{R}_g$ on differentials: $\mathcal{L}_g db \equiv d(\mathcal{L}_g b), \forall b \in Fun(G)$ and similarly for $\mathcal{R}_g db$. More generally, on elements of $\Gamma$ (one-forms) we define $\mathcal{L}_g$ as:

$$\mathcal{L}_g (adb) \equiv d(\mathcal{L}_g b), \forall b \in \text{Fun}(G)$$

and similar for $\mathcal{R}_g$. A differential calculus is called bicovariant if it is both left and right covariant.

Left and right invariant one forms

As in usual Lie group manifolds, we can introduce a basis in $\Gamma$ of left-invariant one-forms $\theta^g$:

$$\theta^g \equiv \sum_{h \in G} x^h dx^h \quad (= \sum_{h \in G} x^h dx^{h^{-1}}),$$

It is immediate to check that indeed $\mathcal{L}_g \theta^g = \theta^g$. The relations (2.14) can be inverted:

$$dx^h = \sum_{g \in G} (x^h g - x^h) \theta^g$$

From $0 = dI = d \sum_{g \in G} x^g = \sum_{g \in G} dx^g = 0$ one finds:

$$\sum_{g \in G} \theta^g = \sum_{g, h \in G} x^h dx^{h^{-1}} = \sum_{h \in G} x^h \sum_{g \in G} dx^{h^{-1}} = 0$$

Therefore we can take as basis of the cotangent space $\Gamma$ the $n-1$ linearly independent left-invariant one-forms $\theta^g$ with $g \neq e$ (but smaller sets of $\theta^g$ can be consistently chosen as basis, see later).

Analogous results hold for right invariant one-forms $\zeta^g$:

$$\zeta^g = \sum_{h \in G} x^g dh^h$$

From the expressions of $\theta^g$ and $\zeta^g$ in terms of $xdx$, one finds the relations

$$\theta^g = \sum_{h \in G} x^h \zeta^{ad(h)g}, \quad \zeta^g = \sum_{h \in G} x^h \theta^{ad(h^{-1})g}$$

Conjugation

On $Fun(G)$ there are two natural involutions $\ast$ satisfying $(ab)^\ast = b^\ast a^\ast$ ($= a^\ast b^\ast$ since functions on $G$ commute):

$$(x^g)^\ast = x^g$$

$$(x^g)^\ast = x^{g^{-1}}$$
We use the slightly different symbol ⋆ for the second one. These conjugations are extended to the (first-order) differential calculus via the rule:

\[(x^h dx^k)^* = (dx^k)^*(x^h)^* \]  \hspace{1cm} (2.21)

and similar for ⋆. Then

\[(\theta^g)^* = -\theta^{g^{-1}}, \quad (\theta^g)^* = -\zeta^g \]  \hspace{1cm} (2.22)

**Commutations between \(x\) and \(\theta\)**

\[x^h dx^g = x^h \theta^{g^{-1}h} = \theta^{g^{-1}h}x^g \quad (h \neq g) \Rightarrow x^h \theta^g = \theta^g x^{hg^{-1}} \quad (g \neq e) \]  \hspace{1cm} (2.23)

implying the general commutation rule between functions and left-invariant one-forms:

\[f \theta^g = \theta^g \mathcal{R}_g f \]  \hspace{1cm} (2.24)

Thus functions do commute between themselves (i.e. \(\text{Fun}(G)\) is a commutative algebra) but do not commute with the basis of one-forms \(\theta^g\). In this sense the differential geometry of \(\text{Fun}(G)\) is noncommutative.

**Classification of bicovariant calculi**

The right action of \(G\) on the elements \(\theta^g\) is given by:

\[\mathcal{R}_h \theta^g = \theta^{ad(h)g}, \quad \forall h \in G \]  \hspace{1cm} (2.25)

where \(ad\) is the adjoint action of \(G\) on itself, i.e. \(ad(h)g \equiv hgh^{-1}\). Then bicovariant calculi are in 1-1 correspondence with unions of conjugacy classes (different from \(\{e\}\)) \[8\]: if \(\theta^g\) is set to zero, one must set to zero all the \(\theta^{ad(h)g}, \quad \forall h \in G\) corresponding to the whole conjugation class of \(g\).

We denote by \(G'\) the subset corresponding to the union of conjugacy classes that characterizes the bicovariant calculus on \(G\) \((G' = \{g \in G|\theta^g \neq 0\})\). Unless otherwise indicated, repeated indices are summed on \(G'\) in the following.

**Exterior product**

An exterior product, compatible with the left and right actions of \(G\), can be defined by

\[\theta^{g_1} \wedge \theta^{g_2} = \theta^{g_1} \otimes \theta^{g_2} - \theta^{g_1^{-1}g_2g_1} \otimes \theta^{g_1} \]  \hspace{1cm} (2.26)

where the tensor product between elements \(\rho, \rho' \in \Gamma\) is defined to have the properties \(\rho a \otimes \rho' = \rho \otimes a\rho', \quad a(\rho \otimes \rho') = (a\rho) \otimes \rho' \) and \((\rho \otimes \rho')a = \rho \otimes (\rho' a)\).

Note that:

\[\theta^g \wedge \theta^g = 0 \quad \text{(no sum on } g)\]  \hspace{1cm} (2.27)

Left and right actions on \(\Gamma \otimes \Gamma\) are simply defined by:

\[\mathcal{L}_h (\rho \otimes \rho') = \mathcal{L}_h \rho \otimes \mathcal{L}_h \rho', \quad \mathcal{R}_h (\rho \otimes \rho') = \mathcal{R}_h \rho \otimes \mathcal{R}_h \rho' \]  \hspace{1cm} (2.28)
Compatibility of the exterior product with \( \mathcal{L} \) and \( \mathcal{R} \) means that
\[
\mathcal{L}(\theta^i \wedge \theta^j) = \mathcal{L}\theta^i \wedge \mathcal{L}\theta^j, \quad \mathcal{R}(\theta^i \wedge \theta^j) = \mathcal{R}\theta^i \wedge \mathcal{R}\theta^j
\] (2.30)
Only the second relation is nontrivial and is verified upon use of the definition (2.24).
The generalization to exterior products of \( n \) one-forms is straightforward, see ref. [10]

**Exterior derivative**

Equipped with the exterior product we can define the *exterior derivative*
\[
d : \Gamma \to \Gamma \wedge \Gamma \quad (2.31)
\]
\[
d(a_k db_k) = da_k \wedge db_k, \quad (2.32)
\]
which can easily be extended to \( \Gamma \wedge \Gamma \) \((d : \Gamma \wedge \Gamma \to \Gamma \wedge (\Gamma \wedge (n+1)))\), and has the following properties:
\[
d(\rho \wedge \rho') = d\rho \wedge \rho' + (-1)^k \rho \wedge d\rho' \quad (2.33)
\]
\[
d(d\rho) = 0 \quad (2.34)
\]
\[
\mathcal{L}_g(d\rho) = d\mathcal{L}_g\rho \quad (2.35)
\]
\[
\mathcal{R}_g(d\rho) = d\mathcal{R}_g\rho \quad (2.36)
\]
where \( \rho \in \Gamma^\wedge k, \rho' \in \Gamma^\wedge n, \Gamma^\wedge 0 \equiv Fun(G) \). The last two properties express the fact that \( d \) commutes with the left and right action of \( G \).

**Tangent vectors**

Using (2.15) to expand \( df \) on the basis of the left-invariant one-forms \( \theta^g \) defines the (left-invariant) tangent vectors \( t_g \):
\[
df = \sum_{g \in G} f_g dx^g = \sum_{h \in G'} (\mathcal{R}_{h^{-1}} f - f) \theta^h \equiv \sum_{h \in G'} (t_h f) \theta^h \quad (2.37)
\]
so that the “flat” partial derivatives \( t_h f \) are given by
\[
t_h f = \mathcal{R}_{h^{-1}} f - f \quad (2.38)
\]
The Leibniz rule for the flat partial derivatives \( t_g \) reads:
\[
t_g(ff') = (t_g f)f' + \mathcal{R}_{g^{-1}}(f)t_g f' = (t_g f)\mathcal{R}_{g^{-1}} f' + ft_g f' \quad (2.39)
\]
In analogy with ordinary differential calculus, the operators \( t_g \) appearing in (2.37) are called (left-invariant) *tangent vectors*, and in our case are given by
\[
t_g = \mathcal{R}_{g^{-1}} - id \quad (2.40)
\]
### Fusion algebra

The tangent vectors satisfy the fusion algebra:

\[ t_g t_{g'} = \sum_h C^h_{g,g'} t_h \]  

where the structure constants are:

\[ C^h_{g,g'} = \delta^{h}_{g'} - \delta^h_g - \delta^h_{g'} \]  

and are \(ad(G)\) invariant:

\[ C^{ad(h)g_1}_{ad(h)g_2,ad(h)g_3} = C^{g_1}_{g_2,g_3} \]  

Defining:

\[ C^g_{g_1,g_2} \equiv C^g_{g_1,g_2} - C^g_{g_2,g_1} \delta^{-1}_{g_2}g - \delta^g_{g_1} \]  

the following fusion identities hold:

\[ C^k_{h_1,g} C^{h_2}_{k,g'} = C^h_{g,g'} C^{h_2}_{h_1,h} \]  

Thus the \( C \) structure constants are a representation (the adjoint representation) of the tangent vectors \( t \). Besides property (2.43) they also satisfy:

\[ C^g_{g_1,g_2} = C^{g_1}_{g_2^{-1}} \]  

### Cartan-Maurer equations, connection and curvature

From the definition (2.14) and eq. (2.24) we deduce the Cartan-Maurer equations:

\[ d\theta^g + \sum_{g_1,g_2} C^g_{g_1,g_2} \theta^{g_1} \wedge \theta^{g_2} = 0 \]  

where the structure constants \( C^g_{g_1,g_2} \) are those given in (2.42).

Parallel transport of the vielbein \( \theta^g \) can be defined as in ordinary Lie group manifolds:

\[ \nabla \theta^g = -\omega^g_{g'} \otimes \theta^{g'} \]  

where \( \omega^g_{g'} \) is the connection one-form:

\[ \omega^g_{g_2} = \Gamma^g_{g_3,g_2} \theta^{g_3} \]  

Thus parallel transport is a map from \( \Gamma \) to \( \Gamma \otimes \Gamma \); by definition it must satisfy:

\[ \nabla(a \rho) = (da) \otimes \rho + a \nabla \rho, \quad \forall a \in A, \quad \rho \in \Gamma \]
and it is a simple matter to verify that this relation is satisfied with the usual parallel transport of Riemannian manifolds. As for the exterior differential, $\nabla$ can be extended to a map $\nabla : \Gamma^n \otimes \Gamma \rightarrow \Gamma^{(n+1)} \otimes \Gamma$ by defining:

$$\nabla(\varphi \otimes \rho) = d\varphi \otimes \rho + (-1)^n \varphi \nabla \rho \quad (2.51)$$

Requiring parallel transport to commute with the left and right action of $G$ means:

$$L_h(\nabla \theta^g) = \nabla (L_h \theta^g) = \nabla \theta^g \quad (2.52)$$

$$R_h(\nabla \theta^g) = \nabla (R_h \theta^g) = \nabla \theta^{ad(h)g} \quad (2.53)$$

Recalling that $L_h(a \rho) = (L_h a)(L_h \rho)$ and $L_h(\rho \otimes \rho') = (L_h \rho) \otimes (L_h \rho')$, $\forall a \in A$, $\rho, \rho' \in \Gamma$ (and similar for $R_h$), and substituting (2.48) yields respectively:

$$\Gamma^{g_1}_{g_3,g_2} \in C \quad (2.54)$$

and

$$\Gamma^{ad(h)g_1}_{ad(h)g_3,ad(h)g_2} = \Gamma^{g_1}_{g_3,g_2} \quad (2.55)$$

Therefore the same situation arises as in the case of Lie groups, for which parallel transport on the group manifold commutes with left and right action iff the connection components are $ad(G)$ - conserved constant tensors. As for Lie groups, condition (2.55) is satisfied if one takes $\Gamma$ proportional to the structure constants. In our case, we can take any combination of the $C$ or $C$ structure constants, since both are $ad(G)$ conserved constant tensors. As we see below, the $C$ constants can be used to define a torsionless connection, while the $C$ constants define a parallelizing connection.

As usual, the curvature arises from $\nabla^2$:

$$\nabla^2 \theta^g = -R^g \theta^g \otimes \theta^{g'} \quad (2.56)$$

$$R^{g_1}_{g_2} = d\omega^{g_1}_{g_2} + \omega^{g_1}_{g_3} \wedge \omega^{g_3}_{g_2} \quad (2.57)$$

The torsion $R^g$ is defined by:

$$R^{g_1} = d\theta^{g_1} + \omega^{g_1}_{g_2} \wedge \theta^{g_2} \quad (2.58)$$

Using the expression of $\omega$ in terms of $\Gamma$ and the Cartan-Maurer equations yields

$$R^{g_1}_{g_2} = (-\Gamma^{g_1}_{h,g_2} C^h_{g_1,g_4} + \Gamma^{g_1}_{g_3,h} \Gamma^h_{g_4,g_2} \theta^{g_3} \wedge \theta^{g_4} =$$

$$= (-\Gamma^{g_1}_{h,g_2} C^h_{g_3,g_4} + \Gamma^{g_1}_{g_3,h} \Gamma^h_{g_4,g_2} - \Gamma^{g_1}_{g_4,h} \Gamma^h_{g_4,g_3} \theta^{g_3} \otimes \theta^{g_4}$$

$$R^{g_1} = (-C^{g_1}_{g_2,g_3} + \Gamma^{g_1}_{g_2,g_3}) \theta^{g_2} \wedge \theta^{g_1} =$$

$$= (-C^{g_1}_{g_2,g_3} + \Gamma^{g_1}_{g_2,g_3} - \Gamma^{g_1}_{g_3,g_2} \theta^{g_2} \otimes \theta^{g_3} \quad (2.59)$$

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Thus a connection satisfying:
\[ \Gamma_{g_2,g_3}^{g_1} - \Gamma_{g_3,g_2}^{g_1} = C_{g_2,g_3}^{g_1} \] (2.60)
corresponds to a vanishing torsion \( R^q = 0 \) and could be referred to as a “Riemannian” connection.

On the other hand, the choice:
\[ \Gamma_{g_2,g_3}^{g_1} = C_{g_3,g_2}^{g_1} \] (2.61)
corresponds to a vanishing curvature \( R^q_{ij} = 0 \), as can be checked by using the fusion equations (2.45) and property (2.46). Then (2.61) can be called the parallelizing connection: finite groups are parallelizable.

**Tensor transformations**

Under the familiar transformation of the connection 1-form:
\[ (\omega^i_j)' = a^i_k \omega^k_l (a^{-1})^l_j + a^i_k d(a^{-1})^k_j \] (2.62)
the curvature 2-form transforms homogeneously:
\[ (R^i_j)' = a^i_k R^k_l (a^{-1})^l_j \] (2.63)

**Metric**

The metric tensor \( \gamma \) can be defined as an element of \( \Gamma \otimes \Gamma \):
\[ \gamma = \gamma_{i,j} \theta^i \otimes \theta^j \] (2.64)
Requiring it to be invariant under left and right action of \( G \) means:
\[ \mathcal{L}_h(\gamma) = \gamma = \mathcal{R}_h(\gamma) \] (2.65)
or equivalently, recalling \( \mathcal{L}_h(\theta^i \otimes \theta^j) = \theta^i \otimes \theta^j \), \( \mathcal{R}_h(\theta^i \otimes \theta^j) = \theta^{ad(h)i} \otimes \theta^{ad(h)j} \):
\[ \gamma_{i,j} \in C, \quad \gamma_{ad(h)i,ad(h)j} = \gamma_{i,j} \] (2.66)
These properties are analogous to the ones satisfied by the Killing metric of Lie groups, which is indeed constant and invariant under the adjoint action of the Lie group.

On finite \( G \) there are various choices of biinvariant metrics. One can simply take \( \gamma_{i,j} = \delta_{i,j} \), or \( \gamma_{i,j} = C^k_{i,j} C^{l}_{k,j} \).

For any biinvariant metric \( \gamma_{ij} \) there are tensor transformations (isometries) \( a^i_j \) under which \( \gamma_{ij} \) is invariant, i.e.:
\[ a^h_{h',k} \gamma_{h,k} a^k_{k'} = \gamma_{h',k'} \Leftrightarrow a^h_{h',k} \gamma_{h,k} = \gamma_{h',k'} (a^{-1})^k_k \] (2.67)
A class of isometries has been discussed in ref. [10]. In the case \( \gamma_{i,j} = \delta_{i,j} \) the isometries are clearly given by the usual orthogonal matrices.

**Lie derivative and diffeomorphisms**

The analogue of infinitesimal diffeomorphisms is found using general results valid for Hopf algebras [13, 19, 20, 17], of which finite groups are a simple example. As for differentiable manifolds, it can be expressed via the Lie derivative, which for finite groups takes the form:

\[
lt_g \rho = \left[ R_g^{-1} \rho - \rho \right]
\]  

(2.68)

where \( \rho \) is an arbitrary form field. Thus the Lie derivative along \( t_g \) coincides with the tangent vector \( t_g \).

As in the case of differentiable manifolds, the Cartan formula for the Lie derivative acting on \( p \)-forms holds:

\[
l_{t_g} = i_{t_g} d + d i_{t_g}
\]

(2.69)

see ref.s [13, 18, 19, 20, 17, 10].

Exploiting this formula, diffeomorphisms (Lie derivatives) along generic tangent vectors \( V \) can also be consistently defined via the operator:

\[
l_{V} = i_{V} d + d i_{V}
\]

(2.70)

This requires a suitable definition of the contraction operator \( i_{V} \) along generic tangent vectors \( V \), discussed in ref.s [18, 20, 17, 10].

We have then a way of defining “diffeomorphisms” along arbitrary (and \( x \)-dependent) tangent vectors for any tensor \( \rho \):

\[
\delta \rho = l_{V} \rho
\]

(2.71)

and of testing the invariance of candidate lagrangians under the generalized Lie derivative.

**Finite coordinate transformations**

The basis functions \( x^g \) defined in (2.2) are the “coordinates” of \( Fun(G) \). The most general coordinate transformation takes the form:

\[
x'^{g'} = \sum_{g \in G} x'^{g'}_{g} x^{g}
\]

(2.72)

where the \( n \times n \) matrix \( x'^{g'}_{g} \in GL(n, \mathbb{C}) \) is a constant invertible matrix. An example will be provided in the \( Z^N \times Z^N \times \ldots \times Z^N \) case. Let’s consider now the transformation of the differentials:

\[
dx'^{g'} = \sum_{g \in G} x'^{g'}_{g} dx^{g} = \sum_{g \neq e} x'^{g'}_{g} dx^{g} + x'^{g'}_{e} dx^{e} = \sum_{g \neq e} (x'^{g'}_{g} - x'^{g'}_{e}) dx^{g} = \sum_{g \neq e} \partial_{g} x'^{g'} dx^{g}
\]

(2.73)
a formula quite similar to the usual one, the only subtlety being that the index \( g \) does not include \( e \). The \((n - 1) \times (n - 1)\) constant matrix \( A^{g'}_g = \partial_g x^{g'} \) belongs then to \( GL(n - 1, \mathbb{C}) \). This holds for the universal calculus, with \( n - 1 \) independent differentials. Then the \( p \)-forms \( dx^{g_1} \wedge ... dx^{g_p} \) transform as covariant tensors. When the independent \( dx^g \) (or equivalently the independent \( \theta^g \)) are less than \( n - 1 \), the matrix \( A^{g'}_g \) is not constant any more, as we’ll see in the case of \( \mathbb{Z}^N \), and exterior products of coordinate differentials do not transform covariantly any more, due to noncommutativity of \( A^{g'}_g(x) \) with \( dx^g \).

It is easy to prove the formula for the transformation of the partial derivatives:

\[
\partial_{g'} = \sum_{g \neq e} (A^{-1})_g^{g'} \partial_g
\]  

(2.74)

**Haar measure and integration**

Since we want to define actions (integrals on \( p \)-forms), we must now define integration of \( p \)-forms on finite groups.

Let us start with integration of functions \( f \). We define the integral map \( h \) as a linear functional \( h : Fun(G) \mapsto \mathbb{C} \) satisfying the left and right invariance conditions:

\[
h(\mathcal{L}_g f) = h(f) = h(\mathcal{R}_g f)
\]

(2.75)

Then this map is uniquely determined (up to a normalization constant), and is simply given by the “sum over \( G \)” rule:

\[
h(f) = \sum_{g \in G} f(g)
\]

(2.76)

Next we turn to define the integral of a \( p \)-form. Within the differential calculus we have a basis of left-invariant 1-forms, which allows the definition of a biinvariant volume element. In general for a differential calculus with \( m \) independent tangent vectors, there is an integer \( p \geq m \) such that the linear space of \( p \)-forms is 1-dimensional, and \((p + 1)\)-forms vanish identically\(^3\). This means that every product of \( p \) basis one-forms \( \theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} \) is proportional to one of these products, that can be chosen to define the volume form \( \text{vol} \):

\[
\theta^{g_1} \wedge \theta^{g_2} \wedge ... \wedge \theta^{g_p} = \epsilon^{g_1,g_2,...,g_p} \text{vol}
\]

(2.77)

where \( \epsilon^{g_1,g_2,...,g_p} \) is the proportionality constant. The volume \( p \)-form is obviously left invariant. As shown in ref. \((11)\) it is also right invariant, and the proof is based on the \( ad(G) \) invariance of the \( \epsilon \) tensor:

\[\epsilon^{ad(g)h_1,...,ad(g)h_p} = \epsilon^{h_1,...,h_p} . \]

Having identified the volume \( p \)-form it is natural to set

\[
\int f \text{vol} \equiv h(f) = \sum_{g \in G} f(g)
\]

(2.78)

\(^3\)with the exception of \( \mathbb{Z}_2 \), see ref. \((11)\)
and define the integral on a \( p \)-form \( \rho \) as:

\[
\int \rho = \int \rho_{g_1 \ldots g_p} g_1 \wedge \ldots \wedge g_p =
\int \rho_{g_1 \ldots g_p} \epsilon^{g_1 \ldots g_p} \text{vol} \equiv 
\sum_{g \in G} \rho_{g_1 \ldots g_p}(g) \epsilon^{g_1 \ldots g_p}
\quad (2.79)
\]

Due to the biinvariance of the volume form, the integral map \( \int : \Gamma^p \mapsto C \) satisfies the biinvariance conditions:

\[
\int \mathcal{L}_g f = \int f = \int \mathcal{R}_g f
\quad (2.80)
\]

Moreover, under the assumption that \( d(\theta^{g_2} \wedge \ldots \wedge \theta^{g_p}) = 0 \), i.e. that any exterior product of \( p - 1 \) left-invariant one-forms \( \theta \) is closed, the important property holds:

\[
\int df = 0
\quad (2.81)
\]

with \( f \) any \((p - 1)\)-form: \( f = f_{g_2 \ldots g_p} \theta^{g_2} \wedge \ldots \wedge \theta^{g_p} \). This property, which allows integration by parts, has a simple proof (see ref. \([10]\)).

3 Calculus on \( Z_n \times \ldots \times Z_n \)

We apply here the general theory to products of cyclic groups. For simplicity we assume the order of these cyclic groups to be the same.

We start with \( Z_n \) and then generalize to products.

**Calculus on \( Z_n \)**

*Elements of \( Z_n \):* \( u^i = \{ e, u, u^2, \ldots u^{n-1} \} \), with \( u^0 = u^n = e \).

*Basis of dual functions on \( Z_n \):* \( x^{u^i} = \{ x^e, x^u, x^{u^2}, \ldots, x^{u^{n-1}} \} \). Left and right actions coincide since the group is abelian, i.e. \( \mathcal{L}_u x^{u^i} = x^{u_i-j} = \mathcal{R}_u x^{u^i} \).

*Alternative basis.* It is convenient to use a basis of functions that reproduce the algebra of the \( Z_n \) elements \( u^i \). This basis is given by \( y^i = \sum_{j=0}^{n-1} q^{ij} x^{u^j} \), where \( q \equiv (-1)^{n} \) is the \( n \)-th root of unity. Thus \( y^i y^j = y^{i+j}, y^0 = I \). For example \( y^1 = y \) is given by

\[
y = x^e + qx^u + q^2 x^{u^2} + \ldots + q^{n-1} x^{u^{n-1}}
\quad (3.82)
\]

Using \( \sum_{j=0}^{n-1} q^{ij} = n \delta_{i0} \) one finds the inverse transformation: \( x^{u^i} = \frac{1}{n} \sum_{j=0}^{n-1} q^{-ij} y^j \).

Finally the \( G \) action is: \( \mathcal{L}_u y^i = \mathcal{R}_u y^i = q^{ij} y^j \).
Conjugation classes: \{e\}, \{u\}, \{u^2\}, \ldots, \{u^{n-1}\}. Unions of different (nontrivial) conjugation classes give rise to different calculi. We’ll use the differential calculus corresponding to the single conjugation class \{u\}.

Left-invariant one-forms: \( \theta^u = \sum_{j=0}^{n-1} x^{u^{j+1}} dx^{u^j} \). In the one-dimensional bicovariant calculus we are interested in, all the \( \theta^u \) are set to zero, except

\[
\theta^u = \sum_{j=0}^{n-1} x^{u^{j+1}} dx^{u^j} = \sum_{j=1}^{n-1} (x^{u^{j+1}} - x^u) dx^{u^j} = -\theta^e
\]  

Thus the only independent left (and right)-invariant one-form is \( \theta^u \).

Inversion formula: \( dx^{u^i} = (x^{u^{i+1}} - x^{u^i}) \theta^u \), or in the y basis: \( dy^i = (R_{u^{-1}} y^i - y^i) \theta^u = (q^{-1} - 1) y^i \theta^u \). Whereas \( \theta^u \) cannot be expressed by means of a single differential \( dx^{u^i} \), it can be expressed in terms of a single \( dy^i \): \( \theta^u = \frac{1}{q^{-1} - 1} y^{n-i} dy^i \). Therefore any \( dy^i \) can be used as basis for one-forms. We’ll choose for simplicity \( dy \).

Independent differential in the y basis:

\[
d y = (q^{-1} - 1) y \theta^u = \sum_{j=0}^{n-1} (q^{j-1} - q^j) x^{u^j} \theta^u;
\theta^u = \frac{1}{q^{-1} - 1} y^{n-1} dy = \sum_{j=0}^{n-1} \frac{1}{q^{j-1} - q^j} x^{u^j} dy
\]  

Thus any one-form can be expanded on the \( \theta^u \) (vierbein) basis or on the “coordinate” basis \( dy \). The transition from one basis to the other is given by the 1 \( \times \) 1 vielbein components:

\[
\theta^u_y = \frac{1}{q^{-1} - 1} y^{n-1} \equiv J
\]  

Commutations:

\[
f\theta^u = \theta^u R_u f, \quad f dy = dy R_u f \quad \Rightarrow \quad x^{u^i} \theta^u = \theta^u x^{u^{i-1}}, \quad x^{u^i} dy = dy x^{u^{i-1}}
\]  

Partial derivatives:

\[
d f = (t_u f) \theta^u = (\partial_y f) dy
\]  

where \( t_u \) is the tangent vector \( t_u \equiv R_{u^{-1}} - id \) and \( \partial_y \) is the curved partial derivative:

\[
\partial_y f = (R_{u^{-1}} f - f) J
\]  

Exterior products

\[
\theta^u \wedge \ldots \wedge \theta^u = 0
\]  

and similar for products of \( dy \).

Cartan-Maurer equation:

\[
d \theta^u = 0
\]  

Torsion and curvature vanish for any connection \( \omega^u_u = c \theta^u \). 

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Integration: the volume form is $\theta^u$, and the integral on any 1-form $\rho$ is:

$$\int \rho = \int \rho_u \theta^u = \int \rho_u \, vol = \sum_{g \in \mathbb{Z}_n} \rho_u(g)$$  \hspace{1cm} (3.91)

Integration by parts holds since:

$$\int df = \int (t_u f) \theta^u = \sum_{g \in \mathbb{Z}_n} (\mathcal{R}_{u^{-1}} f - f)(g) = 0$$  \hspace{1cm} (3.92)

Conjugation

Using \((2.19)\) or \((2.20)\):

$$(x^u)^* = x^{u'}, \quad (x^{u'})^* = x^{u'-1}, \quad y^* = y^{-1}, \quad y^* = y$$  \hspace{1cm} (3.93)

Calculus on \((\mathbb{Z}_n)^N\)

The basis functions are just tensor products of the basis functions of the single $\mathbb{Z}_n$ factors: $x^{u_1} \otimes \ldots \otimes x^{u_N}$ etc. We use the bicovariant calculus corresponding to the union of the $N$ conjugation classes \(\{u \otimes e \otimes \ldots \otimes e\}, \ldots \{e \otimes e \otimes \ldots \otimes u\}\). Then the $N$ left-invariant one-forms are: $\theta^{u \otimes e \otimes \ldots \otimes e}, \ldots \theta^{e \otimes e \otimes \ldots \otimes u}$, and $\theta^{e \otimes e \otimes \ldots \otimes e}$ is minus their sum. For short we label the $N$ independent vielbeins $\theta$ as: $\theta^1, \theta^2, \ldots \theta^N$, the corresponding tangent vectors as $t_1, t_2, \ldots t_N$, etc. Moreover the $N$ special group elements $(u \otimes e \otimes \ldots \otimes u), \ldots (e \otimes e \otimes \ldots \otimes u)$ are likewise denoted $u_1, \ldots u_N$, and the $N$ special $y$ coordinate functions follow the same notation: $y^1 = (y \otimes id \ldots \otimes id)$ etc. Thus, for example:

$$df = (\mathcal{R}_{u_1^{-1}} f - f) \theta^1 + \ldots + (\mathcal{R}_{u_N^{-1}} f - f) \theta^N$$  \hspace{1cm} (3.94)

and

$$dy^i = (q^{-1} - 1)y^i \theta^i, \quad \theta^i = \frac{1}{q^{-1} - 1}(y^i)^{n-1} dy^i$$  \hspace{1cm} (3.95)

the vielbein components being therefore diagonal $\theta^i_j = \frac{1}{q^{-1} - 1}(y^i)^{n-1} \delta^i_j$.

Commutations between one-forms and functions are simply given by

$$f \theta^i = \theta^i \mathcal{R}_{u_i} f, \quad f dy^i = dy^i \mathcal{R}_{u_i} f$$  \hspace{1cm} (3.96)

Curved partial derivatives:

$$\partial_y^i f = (\mathcal{R}_{u_i^{-1}} f - f) J_i, \quad J_i = \frac{1}{q^{-1} - 1}(y^i)^{n-1}$$  \hspace{1cm} (3.97)

The exterior products are antisymmetric (as for any abelian group, see the defining formula \((2.26)\)). The Cartan-Maurer equations still read $d \theta^i = 0$. The volume form can be taken to be

$$vol = \theta^1 \wedge \ldots \wedge \theta^N$$  \hspace{1cm} (3.98)

and the $\epsilon$ tensor in this case coincides with the usual Levi-Civita alternating tensor. Integration by parts holds because of $d \theta^i = 0$. 

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4 Yang-Mills theory on finite $G$

Notations: $G$ indicates the gauge group, which we take to be a unitary Lie group.

Gauge potential

The dynamical field of finite $G$ gauge theory is a matrix-valued one-form $A(x) = A_i(x)\theta^i$. As in the usual case, $G$-gauge transformations are defined as

$$A' = -(dG)G^{-1} + GAG^{-1}$$

where $G(x)$ is a $G$ unitary group element ($G^\dagger = G^{-1}$) in some irrep, depending on the coordinates $x$ of the finite $G$ group manifold. The $\dagger$ conjugation acts as $*$ on the $x$ coordinates. In components:

$$A'_h = - (\partial_h G)R_{h^{-1}}G^{-1} + GA_h R_{h^{-1}}G^{-1}$$

Notice that $\partial_h G$ is a finite difference of group elements, and therefore $A_h$ must belong to the $G$ group algebra, rather than to the $G$ Lie algebra.

Hermitian conjugation

We define hermitian conjugation on matrix valued one forms $A$ as follows:

$$A^\dagger = (A_h\theta^h)^\dagger \equiv (\theta^h)^* A_h^\dagger = -\theta^{h^{-1}}A_h^\dagger = -\theta^h A_{h^{-1}}$$

where $\dagger$ acts as hermitian conjugation on the matrix structure of $A_h$ and as $*$ conjugation on the $Fun(G)$ entries of the matrix.

Matter fields and covariant derivative

Matter fields $\psi$ transform in an irrep of $G$:

$$\psi' = G\psi, \quad (\psi^\dagger)' = \psi^\dagger G^\dagger = \psi^\dagger G^{-1}$$

and their covariant derivative, defined by

$$D\psi = d\psi + A\psi, \quad D\psi^\dagger = d\psi^\dagger - \psi^\dagger A$$

transforms as it should: $(D\psi)' = G(D\psi)$, $(D\psi^\dagger)' = (D\psi^\dagger)G^{-1}$. Requiring compatibility of hermitian conjugation with the covariant derivative, i.e. $(D\psi)^\dagger = D\psi^\dagger$, implies:

$$A^\dagger = -A$$

that is, $A$ must be an antihermitian connection. In components this means:

$$A_h^\dagger = R_{h^{-1}}A_{h^{-1}}$$
Field strength

The field strength $F$ is formally defined as usual:

$$F = dA + A \wedge A = d(A_{k}\theta^k) + A_{h}\theta^h \wedge A_{k}\theta^k =$$

$$= (\partial_{h}A_{k})\theta^h \wedge \theta^k + A_{k}d\theta^k + A_{h}(\mathcal{R}_{h^{-1}}A_{k})\theta^h \wedge \theta^k =$$

$$= [\partial_{h}A_{k} - A_{j}C^j_{hk} + A_{h}(\mathcal{R}_{h^{-1}}A_{k})]\theta^h \wedge \theta^k$$

(4.106)

and varies under gauge transformations (4.99) as

$$F' = GFG^{-1}$$

(4.107)

or in components:

$$F'_{hk} = GF_{hk}\mathcal{R}_{h^{-1}}G^{-1}, \quad (F'^\dagger_{hk})' = (\mathcal{R}_{h^{-1}}G)F'^\dagger_{hk}G^{-1}$$

(4.108)

Action

Due to the above gauge variations of $F$, the following action is invariant under gauge transformations:

$$A_{YM} = \sum_{G} Tr(F_{hk}F'^\dagger_{hk})$$

(4.109)

Link variables

Introducing the link field $U_{h}(x)$:

$$U_{h}(x) \equiv I + A_{h}(x)$$

(4.110)

transforming as

$$U'_{h} = GU_{h}\mathcal{R}_{h^{-1}}G^{-1}$$

(4.111)

the $F$ components take the form

$$F_{hk} = U_{h}\mathcal{R}_{h^{-1}}U_{k} - U_{k}\mathcal{R}_{k^{-1}}U_{khk^{-1}}$$

(4.112)

Requiring $U_{h}$ to be unitary ($U'^\dagger_{h} = U_{h}^{-1}$), and substituting (4.112) in the action (4.109) leads to a suggestive result

$$A_{YM} = -\sum_{G} Tr[U_{h}(\mathcal{R}_{h^{-1}}U_{k})(\mathcal{R}_{k^{-1}}U_{khk^{-1}}U_{k}^{-1} + \text{herm. conjugate})]$$

(4.113)

(a constant term $2 \sum_{G} Tr I$ has been dropped). When the finite group $G$ is abelian, the action (4.113) reduces to the Wilson action. In particular this happens for $G = Z_{N} \times Z_{N} \times Z_{N} \times Z_{N}$, a result already obtained in ref. [14].

Fermion coupling

We can add a Dirac term for fermions $\psi$:

$$A_{Dirac} = \sum_{G} \psi'^\dagger \gamma^0 \gamma^h D_{h}\psi$$

(4.114)

invariant under global Lorentz transformations $SO(dimG')$ and local $G$ gauge transformations.
5 Invariant $G$-gravity actions

We have now all the ingredients necessary for the construction of gravity actions on finite group spaces, invariant under the analogues of diffeomorphisms and local Lorentz rotations.

What we aim for is a dynamical theory of vielbein fields whose “vacuum” solution describes the finite $G$ manifold. Then the dynamical vielbeins $V^a$ are not left-invariant any more, being a deformation of the $\theta$ one-forms:

$$V^a = \sum_{g \in G} V^a_g(x) \theta^g$$

(5.115)

The vielbein components $V^a_g$ along the rigid basis $\theta$ are assumed to be invertible $x$-dependent matrices. The inverse we denote as usual by $V^g_a$.

In addition we also consider the spin connection 1-form $\omega^{g_1}_{g_2}$ as an independent field (first order formulation). The $\omega$ field equations will then determine the expression of $\omega$ in terms of the vielbein field.

We propose two different actions for the $G$-analogue of gravity:

$$A_G = \int R \det(V^a_i) vol(G) = \sum_{g \in G} R \det(V^a_i)$$

(5.116)

and

$$A_G = \int R \epsilon_{a_1...a_p} V^{a_1}_{i_1} ... V^{a_p}_{i_p} \theta^{i_1} \wedge ... \wedge \theta^{i_p} = \sum_{g \in G} R \det(\epsilon(V^a_i))$$

(5.117)

In both actions the scalar $R$ is the finite group analogue of the Gaussian curvature:

$$R \equiv V^a_h (R_{h^{-1}k^{-1}} V^k_b) R^{ab}_{hk}, \quad R^{ab}_{hk} \equiv \gamma^{bc} R^a_c h k$$

(5.118)

where $\gamma^{bc} = \delta^{bc}$, and the curvature components on the rigid basis $\theta$ are defined by $R^a_b = R^a_{b \ h k} \theta^h \otimes \theta^k$:

$$R^a_{b \ h k} = \partial_h \omega^a_{b \ k} - \partial_k \omega^a_{b \ h k} - \omega^a_{c \ h} C^c_{h \ k} + \omega^a_{c \ k} (R_{h^{-1}c \ h k} - \omega^c_{d \ k} (R_{k^{-1}c \ b \ h k})$$

(5.119)

where the constants $C$ are given by the $G$-antisymmetrization of the $C$ constants (cf. eq. (2.44)): $C^a_{h, k} \equiv C^a_{h, k} - C^c_{h, k}.$

The determinant in (5.116) is the usual determinant of the $m \times m$ matrix $V^a_i$, while the “determinant” in (5.117) is computed via the $\epsilon$ tensor of eq. (2.77), i.e. $\det(\epsilon(V^a_i)) = \epsilon^{a_1...a_p} \epsilon_{a_1...a_p} V^{a_1}_{i_1} ... V^{a_p}_{i_p}$, $p$ being the order of the volume form.

Invariances of $A_G$

Both actions are invariant under the local field transformations:

$$V^{a'}_h' = a^{b'}_b V^b_h$$

(5.120)

$$\omega^{a'}_{c'}' = a^{b'}_b \omega^b_{c'} (a^{-1})^c_e + a^{b'}_c d(a^{-1})^c_e$$

(5.121)
where $a$ is an $x$-dependent matrix that rotates the metric and the $\epsilon$ tensor into themselves. For the first action $a$ simply belongs to $SO(m)$, while for the second it belongs to a subgroup of $O(m)$ that preserves the $\epsilon$ tensor of eq. (2.77). These are then the local tangent invariances of the two actions. Note that the two actions coincide in the case of $G = (Z_n)^N$, the $\epsilon$ tensor in (2.77) becoming just the usual alternating tensor.

Proof: under the above transformation the curvature components vary according to eq. (2.63):

\[(R^b_{c'\ h k})' = a^b_{\ b'}(R_{h^{-1}k^{-1}a^{c'}c})R^{bc}_{\ h k}\] (5.122)

(use also (2.67)), so that the Gaussian curvature in (5.118) is indeed invariant. So is $\det (V^a_i)$ if $a \in SO(m)$, and the first action $A_G$ is therefore invariant under a local $SO(m)$ tangent group. Similarly $\det_\epsilon (V^a_i)$ is a scalar under the $O(m)$ transformations conserving $\epsilon$, a subgroup of $O(m)$.

Note. The two actions above correspond to two different definitions of the volume of the deformed $G$ manifold:

\[\text{vol} (\tilde{G}) \equiv \int \det (V^a_i) \text{vol} (G)\] (5.123)

and

\[\text{vol} (\tilde{G}) \equiv \int \epsilon_{a_1...a_p} V^a_{i_1}...V^a_{i_p} \theta^{i_1} \wedge ... \wedge \theta^{i_p}\] (5.124)

with the same local symmetries as for the corresponding actions. In the second case, we are not using the seemingly more natural expression $\int \epsilon_{a_1...a_m} V^a_{i_1}...V^a_{i_m} \theta^{i_1} \wedge ... \wedge \theta^{i_p}$ since it is not invariant under (5.120).

Both $A_G$ are also invariant under infinitesimal diffeomorphisms generated by the Lie derivative $\ell_V$ along an arbitrary tangent vector $V$, if integration by parts holds. Indeed the variation of any action $A$ reads:

\[\delta A = \int \ell_V (p\text{-form}) = \int [\text{div}_V (p\text{-form}) + i_V d(p\text{-form})] = 0\] (5.125)

since $d(p\text{-form}) = 0$ and $\int d(p - 1\text{-form}) = 0$.

The invariance under the finite coordinate transformations (2.72) is somewhat trivial, since there are no “world” indices in the definition of $A_G$ that transform under it. It would be possible, in principle, to introduce world indices by expressing the deformed vielbein $V^a_i$ as

\[V^a_i (x) \equiv V^a_{\alpha} (x) \theta^\alpha_i (x)\] (5.126)

where $\theta^\alpha_i$ is a given matrix (function of $x$) whose inverse is defined by $\theta^i \equiv \theta^\alpha_i dx^\alpha$. Then the dynamical fields are the vielbein components $V^a_{\alpha}$ transforming under both the local tangent group and the finite diffeomorphisms. However this would clearly
be artificial, since anyhow in (5.116) and (5.117) we have to refer explicitly to a preferred frame of reference, spanned by the $\theta^i$. The reason is that only in terms of this frame we are able to express the commutation rules as in eq. (2.24). A general mixture of the $\theta$ has a complicated commutation rule with a generic function $f$, hardly suited for constructing an invariant action. If one could use a basis of differentials $dx$ for the various forms one could give meaning to finite coordinate invariance: in general this is not fruitful since again the differentials have no simple commutations with the functions. A notable exception is provided by $G = (\mathbb{Z}_n)^N$ (and in general by any abelian group) as we discuss below. In this case the differentials $dy$ have the same commutation rules with functions as the $\theta$, and we find an action entirely written in terms of differentials $dy$ and components $V^a_\alpha$ explicitly invariant under linear general coordinate transformations.

**Field equations**

Varying the actions $A_G$ with respect to the vielbein $V^f_j$ yields the analogue of the Einstein equations, respectively

$$R^i_j + R_{kh}(V^h_a R^a_{bh} V^j_k) (\det V)^{-1} V^k_c V^j_b - R V^i_c = 0$$

(5.127)

and

$$R^i_j \det V + R_{kh}(V^h_a R^a_{bh} \det V) V^k_c V^j_b - p R e^j i_2 \ldots i_p \epsilon_c a_2 \ldots a_p V^a_{ij_2} \ldots V^a_{ip} = 0$$

(5.128)

where $R^i_j$, the analogue of the Ricci tensor, is defined by

$$R^i_j = V^h_a V^j_a (R_{a^{-1} h^{-1}} V^k_h) R^a_{bh}$$

(5.129)

and $R = V^c_j R^j_c$.

Similarly varying the actions (5.116), (5.117) with respect to $\omega^a_i$ yields a system of linear equations for all the components of the spin connection. We will write it explicitly in the case of $(\mathbb{Z}_n)^N$.

**Gravity action on $(\mathbb{Z}_n)^N$**

In this case we can write the action

$$A = \int R \det V \, d^N y$$

(5.130)

where now the curvature scalar is given by

$$R \equiv V^\alpha_a (R_{\alpha^{-1} h^{-1}} V^\beta_h) R^{ab}_{\alpha \beta}, \quad R^{ab}_{\alpha \beta} \equiv \gamma^{bc} R^a_{c \alpha \beta}$$

(5.131)

and the curvature components are defined in the usual way as $R^{ab} \equiv R^{ab}_{\alpha \beta} dy^\alpha \wedge dy^\beta$:

$$R^{ab}_{\alpha \beta} = \partial_a \omega^b_\alpha + \omega^c_\alpha (R_{c a} \omega_b^\beta) - (\alpha \leftrightarrow \beta)$$

(5.132)
The determinant in (5.130) is the usual determinant of the vielbein field $V^a_\alpha$, and the volume element is the usual $d^N y \equiv \epsilon_{\alpha_1 \ldots \alpha_N} dy^{\alpha_1} \wedge \ldots \wedge dy^{\alpha_N}$. Since differentials and $\theta$ have the same commutations with functions (see eq. (3.96)), the action is again invariant under the local $SO(N)$ transformations $(V^b_\beta)' = a^b_b V^b_\beta$ and (5.121). The action $A$ is also invariant under the finite coordinate transformations:

$$y'^{\alpha} = y_0^{\alpha} I + y_{\beta 1}^{\alpha} y^\beta$$

with $y_0^{\alpha}, y_{\beta 1}^{\alpha}$ real constants, so that the transition function $y'^\alpha(y)$ is linear in the old coordinates $y^\beta$. The curved derivatives $\partial_\beta y^\alpha$ are then constant, and commute therefore with all the differentials $dy$. The volume $d^N y$ transforms under (5.133) with the determinant of the curved derivative matrix (jacobian) while $det V$ transforms with the inverse jacobian, so that $det V d^N y$ is invariant under (5.133). The field equations are as in eq. (5.127), after replacing all $j, h, k$ indices with curved indices $\alpha, \beta, \gamma$. The “vacuum” solution $R^{ab}_{\alpha\beta} = 0$ corresponds to the vielbein $V^a_\alpha = \delta^a_\alpha$ which describes the rigid $(Z_N)^N$ manifold, the discrete analogue of flat euclidean space.

Calculating $\omega$ in terms of $V$

Varying the action (5.130) with respect to $\omega^{ab}_\beta$ yields the analogue of the torsion equation:

$$J_\alpha (id - R_\alpha) W^{\alpha \beta}_a b + (R_{\beta - 1} \omega^c_\alpha) W^{\beta \alpha}_a c + q R_\alpha [\omega^c_\alpha W^{\alpha \beta}_c b] = 0$$

(no sum on $\beta$) with

$$W^{\alpha \beta}_a b \equiv [V^a_\alpha (R_{\alpha - 1} \beta - 1 V^\beta_\beta) - (\alpha \leftrightarrow \beta)] \ det V$$

That a solution for $\omega$ exists can be verified in simple cases, as for example $G = Z_2 \times Z_2$. A simplifying assumption consists in taking $\omega^{ab}_\beta$ to be antisymmetric in $a, b$.

6 Conclusions

Differential calculi on discrete spaces are a powerful tool in the formulation of field theories living in such spaces. These calculi are in general noncommutative, and are constructed algebraically. We can expect them to be of relevance also in understanding the noncommutative field theories arising from string/brane theory.

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