Robust design and evaluation of phase codes for radar performance optimization with a finite alphabet constraint

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This letter considers the robust design of radar waveform with discrete phase codes under constant modulus constraint to enhance target detectability. To solve the resultant max-min non-convex optimization problem, a robust discrete phase coding algorithm via iterative technology (R-DPCA-IT) is developed which monotonically improves the worst-case SNR and has an upper bound ensuring to converge a finite value. At each iteration, the original multidimensional quadratic optimization problem is turned into multiple one-dimensional optimization problems solved globally via line search. Finally, the computational time and objective value of the proposed algorithm are assessed in comparison with the existing methods.

Introduction: Radar waveform design has been received wide attention in the literatures during the last decades. Additionally, high performance computer, cognitive radio technology, and powerful digital arbitrary waveform generators have paved the way for waveform design exploiting the priori information with respect to the operating environment and target to significantly enhance the target detection capability and tracking performance [1–8]. However, the cognitive systems are hard to obtain the accurate prior information. Recently, robust waveform design has received considerable attention in the literatures [2, 9–17]. For example, in [2], focusing on the worst-case signal-to-interference-plus-noise ratio (SINR) with colored interference, a robust design of a slow-time transmitter sequence under peak-to-average-power ratio (PAR) constraints is solved by semi-definite program (SDP) relaxation and randomization. Based on mutual information and minimum mean-squared error metric, a minimax robust waveform design has been introduced in [9]. For signal-dependent clutter suppression, [11] considered the joint design of the radar waveform and a Doppler filter bank using a related generalized Dinkelbach’s procedure to improve the worst-case SINR at the output of the filter array. Also aiming at improving the worst-case SINR, the authors focused on uncertainties both in the received useful signal component and interference covariance matrix [12]. For extended targets, [13–15] have introduced several methods for robust waveform design in the presence of signal-dependent interference. A robust transmit code and receive filter is jointly designed under PAR constraint and an energy constraint for an extended target to significantly enhance the target detection capability and tracking performance [9–17]. Most of the aforementioned design methods are based on SDP-related techniques which share a huge computation complexity probably limiting its usage from a practical point of view.

In this letter, we focus on the robust slow-time waveform with discrete phase modulation design under constant modulus constraint assuming that coloured interference with known covariance matrix. According to the prior information of the rough target Doppler frequency, a max-min quadratic optimization problem is found to improve the worst-case signal-to-noise ratio (SNR). To tackle the resultant non-convex problem, we propose a coordinate descent based (CD-based) algorithm to iteratively and sequentially update the slow-time code. Precisely, we split the multidimensional problem into a set of trigonometric function problems which can be solved by 1D search. Finally, the performance of the proposed algorithm is assessed via several numerical simulations. Results show the proposed algorithm is a reasonable trade-off between performance gains and complexity in comparison with the SDP-related and majorization-minimization (MM)-related [20] techniques.

System model and problem formulation: Consider a monostatic radar system which emits a coherent burst of N slow-time coded pulses denoted by $s = [s_1, s_2, \ldots, s_N]^T \in \mathbb{C}^N$, where $\mathbb{C}^N$ denotes the sets of N-dimensional vectors of complex numbers and $(\cdot)^T$ is the transpose operation. Each pulse at the receiver end is down-converted to baseband and then is sampled after matched filtering. Considering a far-field target located at the range-azimuth cell under test, the N-dimensional column vector $\mathbf{v} = [v_1, v_2, \ldots, v_N]^T$ of the observations can be written as [1]

$$\mathbf{v} = \mathbf{R}_s \circ \mathbf{p}_0 + \mathbf{n},$$

(1)

where $\mathbf{n}$ is a complex parameter accounting for the target radar cross section (RCS), channel propagation effects, and other terms involved into the radar range equation, $\mathbf{p}_0 = [1, e^{j2\pi f_0}, \ldots, e^{j2\pi (N-1)f_0}]^T$ is the Doppler frequency vector of the target where $f_0 = 2f_0$ is the normalized target Doppler frequency, with $T$ and $f_0$ denoting the pulse repetition interval (PRI) and the actual Doppler frequency of the target, respectively. $\mathbf{n} \in \mathbb{C}^N$ represents a zero-mean complex circular Gaussian vector with known positive definite covariance matrix, that is $\mathbb{E}[\mathbf{n}\mathbf{n}^H] = \mathbf{M}$. Where $(\cdot)^*$ and $\mathbb{E}[]$ denote the conjugate transpose and statistical expectation, respectively. If the target Doppler frequency is a priori known, the SNR of the signal is defined as [1]

$$\text{SNR}(s) = |\alpha_0|^2 s^H \mathbf{R}_s,$$

(2)

where $\mathbf{R}_s = \mathbf{M}^{-1} \odot (\mathbf{p}_0 \mathbf{p}_0^H)^*$ while $(\cdot)^*$ denotes complex conjugate. Note that $\mathbf{R}_s$ is positive definite due to $s^H \mathbf{R}_s > 0$ for any $s \neq 0$. Besides, $\mathbf{R}_s$ is a Hermite matrix because of $\mathbf{M}$ and $(\mathbf{p}_0 \mathbf{p}_0^H)^*$ are both Hermite matrices.

While lack of a priori information, we consider a robust system (i.e. worst-case SNR) accounting for $v_k, k = 1, \ldots, K$ with the number of all possible Doppler shifts $K$. Hence, Equation (2) can be transformed into

$$\text{SNR}_r(s) = \min_{k=1}^K \text{SNR}_k(s),$$

(3)

with $\text{SNR}_k(s) = |\alpha_0|^2 s^H \mathbf{R}_k s$ and $\mathbf{R}_k = M^{-1} \odot (\mathbf{p}_k \mathbf{p}_k^H)^*$. Therein, from practical considerations, a finite alphabet phase code is adopted due to the limited number of bits are available in digital waveform generators. Finally, we can formulate the waveform design problem with the constant modulus constraint as follows:

$$\mathcal{P}_1 = \left\{ \begin{array}{l}
\max_{s \in \mathbb{C}^N} \min_{k=1}^K |s_k| \mathbf{R}_k s \\
\text{s.t.} \quad |s_k| = 1, \\
\text{arg } s_k \in \left[0, \frac{\pi}{2}, \ldots, \frac{\pi}{2}(M - 1) \right], \quad k = 1, \ldots, N,
\end{array} \right.$$  

(4)

where $M$ denotes the number of discrete alphabet. Note that $\mathcal{P}_1$ is a general NP-hard problem which can not be solved in polynomial time. Herein, we proposed a CD based algorithm to monotonically increase the objective value of the original NP-hard problem.

Robust discrete phase coding algorithm via iterative technology (R-DPCA-IT): In this section, we focus on solving the problem $\mathcal{P}_1$ in a polynomial time. An inspection of the $\mathcal{P}_1$ reveals that the constraints function concerning $(s_1, s_2, \ldots, s_N)$ are separate. Consequently, a CD-based [18, 19] R-DPCA-IT is proposed to sequentially optimize $(s_1, s_2, \ldots, s_N)$ to maximize the worst-case output SNR. Specifically, we sequentially optimize $s$, while other codes in $s$ are fixed until all codes in $s$ have been updated. Let $s^{(m)}$ and $\text{SNR}^{(m)} = \text{SNR}_r(s^{(m)})$, respectively, denote the solution of problem $\mathcal{P}_1$ and the worst-case SNR at the $m$th iteration. Hence, $s^{(m)}$ can be obtained from

$$\mathcal{P}_2^{(m)} = \left\{ \begin{array}{l}
\max_{s_k \in \mathbb{C}^N} \min_{k=1}^K |s_k| \mathbf{R}_k(s_k + s_k^{(m)}) \\
\text{s.t.} \quad |s_k| = 1, \\
\text{arg } s_k \in \left[0, \frac{\pi}{2}, \ldots, \frac{\pi}{2}(M - 1) \right],
\end{array} \right.$$  

(5)
**Algorithm 1 R-DPCA-IT**

Input: \([R_i]_{i=1}^{M}, N, \epsilon\) 

Output: 

1. Initialization: \(n = 0, x^{(0)}\) and \(\text{SNR}^{(0)}_{T}\) by Equation (3); 
2. \(n := n + 1, i = 0\); 
3. \(i := i + 1\); 
4. Compute \([a_{ik}], [a_{ik}],\) and \([a_{ik}]\), and find the optimal solution \(\psi_{\star}^{n}\) of Equation (8); 
5. Update \(s_{i} = e^{2\pi i \psi_{\star}^{n}}\); 
6. If \(\epsilon < N\), return to Step 3; Otherwise go to Step 7; 
7. Update \(\epsilon^{(0)}\) and compute \(\text{SNR}^{(0)}_{T}\) by Equation (3); 
8. If \(|\text{SNR}^{(0)}_{T} - \text{SNR}^{(0)}_{T}^{\star}| \leq \epsilon\), output \(s' = \epsilon^{(0)}\); Otherwise return to Step 2.

where \(s^{(n)} = [s^{(n)}_{1}, \ldots, s^{(n)}_{1}, 0, s^{(n-1)}_{1}, \ldots, s^{(n-1)}_{1}]^{T}\), and \(s_{i} = [0, \ldots, s_{i}, \ldots, 0]^{T}\). Further, the objective function of problem \(P_{a_{ik}}^{(n)}\) can be expanded as 

\[
\min_{k=1}^{M} |a_{ik}|^{2} \left[ R_{i,k}(s_{i})^{2} + s^{(n)}_{ik} R_{i,k}(s^{(n)}) \right] + 2\Re \left[ a_{ik} s_{i} \right],
\]

with \(a_{ik} = |a_{ik}| R_{i,k}(s_{i})^{|s_{i}|} \) and \(R_{i,k,i}\) the \((i, j)\)th entry of \(R_{i}\), where \(|\theta(x)|\) represents the real part of \(x\). Imposing the constraint \(|s_{i}| = 1\), Equation (5) can be transformed into 

\[
P_{s_{i}} \left\{ \begin{array}{ll}
\max \min_{a_{ik}, k=1, \ldots, M} & a_{ik} + 2\Re \left[ a_{ik} s_{i} \right] \\
\text{s.t.} & |s_{i}| = 1, \\
& \arg s_{i} \in \left[ \frac{2\pi}{M} \right], \ldots, \left[ \frac{2\pi}{M} (M - 1) \right].
\end{array} \right.
\]

Further, the following inequality holds true 

\[
\text{SNR}^{(n-1)}_{T} \leq v(\Omega_{a_{ik}}^{(n)}) \leq \ldots \leq v(\Omega_{a_{ik}}^{(0)}) = \text{SNR}^{(n)}_{T}.
\]

The formal description of the alternating optimization procedure is reported in Algorithm 1. It is worth observing that the total computational complexity is associated with the size of \(s\) and \(K\). More in detail, each iteration involves the solution of \(P_{s_{i}}\) which requires to compute the \(a_{ik}\) with the computational complexity of \(O(N^{2}K)\).

**Simulation:** In this section, we assess the performance of the proposed algorithm also compared with the SDP-related and MM-related technique [20]. Specifically, the interference covariance matrix \(M\) is modelled as \(M_{i,j} = \rho^{i+j}\) [1], where one-lag correlation coefficient \(\rho = 0.8\). We set \(|\alpha|^{2}=0\ \text{dB}\) and \(\nu_{i} = 0.1 + k\Delta v\) with \(\Delta v=0.025\) for \(k = 1, \ldots, K\), and pulses number \(N = 64\). Besides, we consider the exit condition \(\epsilon = 10^{-6}\) for R-DPCA-IT, resort to the CVX toolbox [21] to solve the SDP and set the number of random trial to 500 involved in randomization approach. The running computation time is analyzed using Mat-
can be obtained by $P_d = Q(\sqrt{\frac{2\text{SNR}_T}{\gamma^2}}, \sqrt{-2\ln P_{fa}})[1]$, where $Q(\cdot, \cdot)$ denotes the Marcum Q function of order 1 and $P_{fa}$ stands for a desired value of the false alarm probability. In Figure 3, we fix $K = 4$, and plot the detection probability $P_d$ assuming $P_{fa} = 10^{-6}$ against $|\alpha_0|^2$ (in dB) for $M = 2, 64$. As expected, the larger the $M$ and $|\alpha_0|^2$ the bigger $P_d$ for all considered algorithms. In particular, R-DPCA-IT gain superior detection performance (increase detection probability by 10%) than SDP and MM under the same values of $M = 2$.

**Conclusion:** In this letter, we have addressed the design problem of radar waveform with discrete phase under constant modulus and proposed the R-DPCA-IT to sequentially improve the worst-case SNR. Results have shown that the proposed algorithm achieves a reasonable trade-off between performance gains and complexity.

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