Non-uniform continuity on initial data for the two-component b-family system in Besov space

Xing Wu¹ · Cui Li² · Jie Cao¹

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Abstract
This paper studies a two-component b-family system, which includes the two-component Camassa-Holm system and the two-component Degasperis-Procesi system as special case. It is shown that the solution map of this system is not uniformly continuous on the initial data in Besov spaces $\mathcal{B}^{-1}_{p,r}(\mathbb{R}) \times \mathcal{B}^s_{p,r}(\mathbb{R})$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leq p, r < \infty$. Our result covers and extends the previous non-uniform continuity in Sobolev spaces $H^{s-1}(\mathbb{R}) \times H^s(\mathbb{R})$ for $s > \frac{5}{2}$ to Besov spaces (Nonlinear Anal., 2014, 111: 1-14). Compared with the generalized rotation b-family system considered by Holmes et al. (Z. Angew. Math. Mech., 2021), our non-uniform continuity is established in a broader range of Besov spaces.

Keywords Non-uniform dependence · Two component b-family system · Besov spaces

Mathematics Subject Classification 35B30 · 35G25 · 35Q53

1 Introduction
In this paper, we are concerned with the following two-component b-family system on $\mathbb{R}$:

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Xing Wu
ny2008wx@163.com

¹ College of Information and Management Science, Henan Agricultural University, Zhengzhou 450002, Henan, China

² School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450000, Henan, China
\[
\begin{align*}
\dot{m}_t &= u m_x + k_1 u_x m + k_2 \rho \rho_x, \\
\dot{\rho}_t &= k_3 (u \rho)_x, \\
m &= u - u_{xx}, \\
u(0, x) &= u_0, \rho(0, x) = \rho_0.
\end{align*}
\] (1.1)

which was introduced by Guha in [18]. As shown in [18], there are two cases about this system: (i) \(k_1 = b, k_2 = 2b\) and \(k_3 = 1\); (ii) \(k_1 = b + 1, k_2 = 2\) and \(k_3 = b\) with \(b \in \mathbb{R}\).

If \(k_1 = 2\) and \(k_3 = 1\), then system (1.1) becomes the following two-component Camassa-Holm system

\[
\begin{align*}
\dot{m}_t &= u m_x + 2 u_x m + \sigma \rho \rho_x, \\
\dot{\rho}_t &= (u \rho)_x, \\
m &= u - u_{xx},
\end{align*}
\] (1.2)

here \(\sigma = \pm 1\). System (1.2) was derived by Constantin and Ivanov [9] in the context of shallow water theory, and then has attracted much more attention. The local well-posedness for system (1.2) in Sobolev and Besov space were established in [9, 14, 16, 20]. The global existence of strong solutions and wave-breaking criteria were investigated in [14, 16, 19, 20], and the global weak solution has been obtained in [17]. Moreover, when \(\rho = 0\), (1.2) reduces to the classical Camassa-Holm equation modeling the unidirectional propagation of shallow water waves over a flat bottom. It was shown that the Camassa-Holm equation has a bi-Hamiltonian structure [2], is completely integrable [8] and can describe wave-breaking phenomena [5](namely, the wave remains bounded while its slope becomes infinite in finite time). The Cauchy problem of the Camassa-Holm equation was studied in the series of papers [3, 4, 6, 7, 11, 22, 32]. Danchin[11, 12] showed the local existence and uniqueness of strong solutions to Camassa-Holm equation with initial data in \(B^s_{p,r}\), for \(s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}\), \(1 \leq p \leq \infty, 1 \leq r < \infty\). The continuous dependence of the solution on the initial data has been supplemented by Li and Yin in [32].

If \(k_1 = 3\) and \(k_3 = 2\), then system (1.1) becomes the following two-component Degasperis-Procesi system

\[
\begin{align*}
\dot{m}_t &= u m_x + 3 u_x m + \sigma \rho \rho_x, \\
\dot{\rho}_t &= 2(u \rho)_x, \\
m &= u - u_{xx},
\end{align*}
\] (1.3)

which was first proposed in [39] as a natural generalization of the Degasperis-Procesi equation in the context of supersymmetry. The local well-posedness of system (1.3) was established in [15, 29, 42], the precise blow-up scenario and some blow-up rate of strong solutions were also presented in [29, 42]. Moreover, when \(\rho = 0\), (1.3) becomes the integrable Degasperis-Procesi equation with bi-Hamiltonian structure and admits traveling wave solutions [13]. The local well-posedness of Degasperis-Procesi equation in Sobolev and Besov spaces were established in [21, 43], and an inverse scattering method for smooth localized solutions developed for Degasperis-Procesi equation can be found in [10].
As an important part of the well-posedness theory, the continuity of solution suggests ways to solve the Cauchy problem of the equation under consideration. In fact, the non-uniform continuous dependence of the solution mapping on the initial data implies that the local well-posedness cannot be obtained by the contraction mappings principle since this would suggest that the solution is Lipschitz continuous. After the non-uniform dependence for some dispersive equations was studied by Kenig et al. [30], the issue of non-uniform continuity of solutions on initial data has attracted much more attention. Using constructed traveling wave solutions, Himonas et al. showed that the data-to-solution map of the Camassa-Holm equation can not be better than continuous in the Sobolev spaces $H^s$ for $s \geq 2$ on the circle [23] and for $s = 1$ on both the circle and the line [24]. We mention that these types of construction solutions are only suitable in Sobolev spaces with exponent less than $\frac{3}{2}$ on the line. Later, a sequences of the high-low frequency approximate solutions have been used to show that the solution map of the Camassa-Holm equation is not uniformly continuous in Sobolev spaces $H^s$ ($s > \frac{5}{2}$) on the line [25], which was applied earlier by Koch and Tzvetkov [31] to the Benjamin-Ono equation. Similar results can be obtained for the Degasperis-Procesi equation [26] and for the famous Novikov equation [27]. Recently, by developing a new approximation technique, the above results in Sobolev spaces were extended to the Besov space $B^s_{p,r}(\mathbb{R})$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leq p \leq \infty, 1 \leq r < \infty$ or $(s, p, r) = (\frac{3}{2}, 2, 1)[34, 35, 40, 41]$.

For $\rho \neq 0$, and $b \in \mathbb{R}$, the Cauchy problem of system (1.1) in Sobolev space was first established by Liu and Yin [37] for $(u_0, \rho_0) \in H^s \times H^{s-1}$ with $s \geq 2$. Later, Lv and Wang [38] proved that the solution map is not uniformly continuous for $s > \frac{5}{2}$ by using the approximate solutions. The local well-posedness space was further enlarged, and established in Besov space $B^s_{p,r} \times B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leq p, r \leq \infty$ (however, for $r = \infty$, the continuity of the data-to-solution map is established in a weaker topology)[36, 44]. Some aspects concerning blow-up scenario, global solutions, persistence properties and propagation speed, see the discussions in [37, 45]. In the present paper, motivated by [34, 41], we aim at showing that the solution map of (1.1) is not uniformly continuous depending on the initial data in Besov spaces $B^s_{p,r} \times B^{s-1}_{p,r}$ with $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$, $1 \leq p, r < \infty$. However, one problematic issue is that we here deal with a coupled system with these two components of the solution in different Besov spaces. On the other hand, compared with Novikov equation with cubic nonlinearity [35], quadratic nonlinearity for system (1.1) weakens the attenuation we need. Therefore, the analysis of the two component $b$-family system would be somewhat more difficult.

For studying the non-uniform continuity of a two-component $b$-family system, it is more convenient to express (1.1) in the following equivalent nonlocal form

$$
\begin{align*}
\begin{cases}
    u_t - uu_x &= f(u) + g(\rho), \\
    \rho_t - k_3u\rho_x &= k_3 \rho u_x, \\
    u(0, x) = u_0, \rho(0, x) = \rho_0,
\end{cases}
\end{align*}
$$

(1.4)
where \( f(u) = f_1(u) + f_2(u) \) and
\[
\begin{align*}
  f_1(u) &= \partial_x (1 - \partial^2_x)^{-1} \left( \frac{k_1}{2} u^2 \right), \\
  f_2(u) &= \partial_x (1 - \partial^2_x)^{-1} \left( \frac{3 - k_1}{2} u^2 \right), \\
  g(\rho) &= \partial_x (1 - \partial^2_x)^{-1} \left( \frac{k_2}{2} \rho^2 \right).
\end{align*}
\]

Our main result is stated as follows.

**Theorem 1.1** Let
\[
  s > \max \left\{ 1 + \frac{1}{p}, \frac{3}{2} \right\}, \quad 1 \leq p \leq \infty, \ 1 \leq r < \infty.
\]

The solution map \((u_0, \rho_0) \rightarrow (u(t), \rho(t))\) of the initial value problem \((1.4)\) is not uniformly continuous from any bounded subset of \(B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R})\) into \(C([0, T]; B^s_{p,r}(\mathbb{R}) \times B^{s-1}_{p,r}(\mathbb{R}))\). More precisely, there exist two sequences of solutions \((u_n^1(t), \rho_n^1(t))\) and \((u_n^2(t), \rho_n^2(t))\) such that the corresponding initial data satisfy
\[
  \|u_n^1(t), u_n^2(t)\|_{B^s_{p,r}} + \|\rho_n^1(t), \rho_n^2(t)\|_{B^{s-1}_{p,r}} \leq 1,
\]
and
\[
  \lim_{n \to \infty} (\|u_n^1(0) - u_n^2(0)\|_{B^s_{p,r}} + \|\rho_n^1(0) - \rho_n^2(0)\|_{B^{s-1}_{p,r}}) = 0,
\]
but
\[
  \lim_{n \to \infty} \|u_n^1(t) - u_n^2(t)\|_{B^s_{p,r}} \gtrsim t, \quad \lim_{n \to \infty} \|\rho_n^1(t) - \rho_n^2(t)\|_{B^{s-1}_{p,r}} \gtrsim t, \quad t \in [0, T_0],
\]
with small positive time \(T_0\) for \(T_0 \leq T\).

**Remark 1.1** Since \(B^{\frac{s}{2}}_{2,2} = H^s\), our result covers and extends the previous non-uniform continuity of solutions on initial data in Sobolev spaces \(H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})\) for \(s > \frac{5}{2}\) [38] to Besov spaces.

**Remark 1.2** Recently, Holmes et al. [28] established the non-uniform continuity of the data-to-solution map to a generalized rotation b-family system under the product Besov space \(B^s_{p,r} \times B^{s-1}_{p,r}\), where \(s > \max\{1 + \frac{1}{p}, 2 - \frac{1}{p}\}, 1 \leq p \leq \infty, 1 \leq r < \infty\). Our result differs from that of [28] in the sense that our non-uniform continuity is established in a broader range of Besov spaces and we choose \(\rho_n^1(0)\) and \(\rho_n^2(0)\) different, which is important in the background of the two-component system.

**Notations** Given a Banach space \(X\), we denote the norm of a function on \(X\) by \(\|\cdot\|_X\), and
\[
  \| \cdot \|_{L^\infty_T(X)} = \sup_{0 \leq t \leq T} \| \cdot \|_X.
\]
For \( f = (f_1, f_2, ..., f_n) \in X \),
\[
\|f\|^2_X = \|f_1\|^2_X + \|f_2\|^2_X + ... + \|f_n\|^2_X.
\]
The symbol \( A \lesssim B \) means that there is a uniform positive constant \( C \) independent of \( A \) and \( B \) such that \( A \leq C B \).

## 2 Littlewood-Paley analysis

In this section, we will review the definition of Littlewood-Paley decomposition and nonhomogeneous Besov space, and then list some useful properties. For more details, the readers can refer to [1].

There exists a couple of smooth functions \((\chi, \varphi)\) valued in \([0, 1]\), such that \( \chi \) is supported in the ball \( B \triangleq \{ \xi \in \mathbb{R} : |\xi| \leq \frac{4}{3} \} \), \( \varphi \) is supported in the ring \( C \triangleq \{ \xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \} \). Moreover,
\[
\forall \xi \in \mathbb{R}, \, \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j} \xi) = 1, \\
\forall \xi \in \mathbb{R} \setminus \{0\}, \, \sum_{j \in \mathbb{Z}} \varphi(2^{-j} \xi) = 1, \\
|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j} \cdot) \cap \text{Supp } \varphi(2^{-j'} \cdot) = \emptyset, \\
j \geq 1 \Rightarrow \text{Supp } \chi(\cdot) \cap \text{Supp } \varphi(2^{-j} \cdot) = \emptyset.
\]

Then, we can define the nonhomogeneous dyadic blocks \( \Delta_j \) as follows:
\[
\Delta_j u = 0, \text{ if } j \leq -2, \quad \Delta_{-1} u = \chi(D) u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \\
\Delta_j u = \varphi(2^{-j} D) u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u), \text{ if } j \geq 0.
\]

**Definition 2.1** ([1]) Let \( s \in \mathbb{R} \) and \( 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}) \) consists of all tempered distribution \( u \) such that
\[
\|u\|_{B^s_{p,r}(\mathbb{R})} \triangleq \left\| \left( 2^{js} \|\Delta_j u\|_{L^p(\mathbb{R})} \right)_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})} < \infty.
\]

In the following, we list some basic lemmas and properties about Besov space which will be frequently used in proving our main result.

**Lemma 2.1** ([1])

1. Algebraic properties: \( \forall s > 0, \, B^s_{p,r}(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) is a Banach algebra. \( B^s_{p,r}(\mathbb{R}) \) is a Banach algebra \( \iff B^s_{p,r}(\mathbb{R}) \iff L^\infty(\mathbb{R}) \iff s > \frac{1}{p} \) or \( s = \frac{1}{p}, \, r = 1 \) with \( 1 \leq p < \infty \).
(2) For any $s > 0$ and $1 \leq p, r \leq \infty$, there exists a positive constant $C = C(s, p, r)$ such that
\[
\|uv\|_{B^s_{p,r}({\mathbb R})} \leq C \left( \|u\|_{L^\infty({\mathbb R})} \|v\|_{B^{s}_p({\mathbb R})} + \|u\|_{L^\infty({\mathbb R})} \|v\|_{B^s_{r}({\mathbb R})} \right).
\]

(3) Let $m \in \mathbb{R}$ and $f$ be an $\mathcal{S}^m$-multiplier (i.e., $f : \mathbb{R} \to \mathbb{R}$ is smooth and satisfies that $\forall \alpha \in \mathbb{N}$, there exists a constant $C_\alpha$ such that $|\partial^\alpha f(\xi)| \leq C_\alpha (1 + |\xi|)^{m-\alpha}$ for all $\xi \in \mathbb{R}$). Then the operator $f(D)$ is continuous from $B^s_{p,r}({\mathbb R})$ to $B^{s-m}_{p,r}({\mathbb R})$.

(4) Let $1 \leq p, r \leq \infty$ and $s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}$. Then we have
\[
\|uv\|_{B^{s-2}_{p,r}({\mathbb R})} \leq C \|u\|_{B^{s-2}_{p,r}({\mathbb R})} \|v\|_{B^{s-1}_{p,r}({\mathbb R})}.
\]

Lemma 2.2 ([1, 33]) Let $1 \leq p, r \leq \infty$. Assume that
\[
\sigma > -\min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\} \quad \text{or} \quad \sigma > -1 - \min\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\} \quad \text{if} \quad \text{div} \, v = 0.
\]

There exists a constant $C = C(p, r, \sigma)$ such that for any solution to the following linear transport equation:
\[
\partial_t f + v \partial_x f = g, \quad f|_{t=0} = f_0,
\]
the following statements hold:
\[
\|f(t)\|_{B^s_{p,r}} \leq \|f_0\|_{B^s_{p,r}} + \int_0^t \|g(\tau)\|_{B^s_{p,r}} d\tau + \int_0^t CV_p(v, \tau) \|f(\tau)\|_{B^s_{p,r}} d\tau
\]
or
\[
\sup_{s \in [0, t]} \|f(s)\|_{B^s_{p,r}} \leq Ce^{CV_p(v, t)} \left( \|f_0\|_{B^s_{p,r}} + \int_0^t \|g(\tau)\|_{B^s_{p,r}} d\tau \right),
\]
with
\[
V_p(v, t) = \begin{cases}
\int_0^t \|\partial_x v(s)\|_{B^s_{p,r}} ds, & \text{if } \sigma > 1 + \frac{1}{p} \text{ or } \{\sigma = 1 + \frac{1}{p} \text{ and } r = 1\}, \\
\int_0^t \|\partial_x v(s)\|_{B^s_{p,r}} ds, & \text{if } \sigma = 1 + \frac{1}{p} \text{ and } r > 1, \\
\int_0^t \|\partial_x v(s)\|_{B^s_{p,\infty} \cap L^\infty} ds, & \text{if } \sigma < 1 + \frac{1}{p}.
\end{cases}
\]

3 Non-uniform continuous dependence

In this section, we will give the proof of our main theorem. For brevity, we sometimes use $u_{0,n}^i, \rho_{0,n}^i$ to denote $u_n^i(0)$ and $\rho_n^i(0)$ respectively, $i = 1, 2$. 

\[ \mathcopyright \text{Springer} \]
Let \( \hat{\phi} \in C_0^\infty(\mathbb{R}) \) be an even, real-valued and non-negative function on \( \mathbb{R} \) and satisfy
\[
\hat{\phi}(x) = \begin{cases} 
1, & \text{if } |x| \leq \frac{1}{2}, \\
0, & \text{if } |x| \geq \frac{1}{2}.
\end{cases}
\]

Define the high frequency function \( f_n \) and the low frequency function \( g_n \) by
\[
f_n = 2^{-ns} \phi(x) \sin \left( \frac{17}{12} 2^n x \right), \quad g_n = 2^{-n} \phi(x), \quad n \gg 1.
\]

It has been showed in [34] that \( \| f_n \|_{B^\sigma_{p,r}} \lesssim 2^{n(\sigma-s)} \).

Let
\[(u^1_n(0), \rho^1_n(0)) = (f_n, 2^n f_n), \quad (u^2_n(0), \rho^2_n(0)) = (f_n + g_n, 2^n f_n + g_n),\]
then it is easy to verify that
\[
\| u^1_n(0), u^2_n(0) \|_{B^{s+\sigma}_{p,r}} \lesssim 2^{n\sigma} \text{ for } \sigma \geq -1 \quad \text{and}
\| \rho^1_n(0), \rho^2_n(0) \|_{B^{s+l}_{p,r}} \lesssim 2^{n(l+1)} \text{ for } l \geq -2, \quad (3.1)
\]

Consider the system (1.4) with initial data \((u^1_n(0), \rho^1_n(0))\) and \((u^2_n(0), \rho^2_n(0))\), respectively. According to the local well-posedness result in [36, 44], there exists corresponding solution \((u^1_n, \rho^1_n), (u^2_n, \rho^2_n)\) belonging to \( C([0, T]; B^s_{p,r} \times B^{s-1}_{p,r}) \) and has common lifespan \( T \approx 1 \). Moreover, by Lemma 2.1-2.2, there holds
\[
\| u^1_n \|_{L^\infty_T(B^s_{p,r})} + \| \rho^1_n \|_{L^\infty_T(B^{s+1}_{p,r}-1)} \lesssim \| u^1_n(0) \|_{B^s_{p,r}} + \| \rho^1_n(0) \|_{B^{s+1}_{p,r}-1} \lesssim 2^{nk}, \quad k \geq -1, \quad (3.2)
\]
\[
\| u^2_n \|_{L^\infty_T(B^{s+l}_{p,r})} + \| \rho^2_n \|_{L^\infty_T(B^{s+l-1}_{p,r})} \lesssim \| u^2_n(0) \|_{B^{s+l}_{p,r}} + \| \rho^2_n(0) \|_{B^{s+l-1}_{p,r}} \lesssim 2^{nl}, \quad l \geq -1. \quad (3.3)
\]

In the following, we shall firstly show that for the selected high frequency initial data \((u^1_n(0), \rho^1_n(0))\), the corresponding solution \((u^1_n, \rho^1_n)\) can be approximated by the initial data. More precisely, that is

**Proposition 3.1** Under the assumptions of Theorem 1.1, we have
\[
\| u^1_n - u^1_n(0) \|_{L^\infty_T(B^s_{p,r})} + \| \rho^1_n - \rho^1_n(0) \|_{L^\infty_T(B^{s-1}_{p,r})} \lesssim 2^{-\frac{s}{2}(s-\frac{3}{2})}. \quad (3.4)
\]

**Proof** Denote
\[
\epsilon = u^1_n - u^1_n(0), \quad \delta = \rho^1_n - \rho^1_n(0),
\]
then we can derive from (1.4) that \((\epsilon, \delta)\) satisfies

\[
\begin{align*}
\epsilon_t - u_n^1 \partial_x \epsilon &= (u_n^1 - u_{0,n}^1)\partial_x u_{0,n}^1 + [f(u_n^1) - f(u_{0,n}^1)] + [g(\rho_n^1) - g(\rho_{0,n}^1)] \\
&\quad + f(u_{0,n}^1) + g(\rho_{0,n}^1) + u_{0,n}^1 \partial_x u_{0,n}^1, \\
\delta_t - k_3u_n^1 \partial_x \delta &= k_3(u_n^1 - u_{0,n}^1)\partial_x \rho_{0,n}^1 + k_3u_{0,n}^1 \partial_x \rho_{0,n}^1 + k_3 \rho_{n}^1 \partial_x u_{n}^1, \\
\epsilon(0, x) &= 0, \quad \delta(0, x) = 0,
\end{align*}
\] (3.5)

Applying Lemma 2.2 yields

\[
\|\epsilon\|_{B^{s-1}_{p,r}} \lesssim \int_0^t \|\partial_x u_n^1\|_{B^{s-1}_{p,r}} \|\epsilon\|_{B^{s-1}_{p,r}} d\tau + \int_0^t \|f(u_n^1) - f(u_{0,n}^1), g(\rho_n^1) - g(\rho_{0,n}^1)\|_{B^{s-1}_{p,r}} d\tau \\
+ t \|f(u_{0,n}^1), g(\rho_{0,n}^1), u_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-1}_{p,r}},
\]

(3.6)

\[
\|\delta\|_{B^{s-2}_{p,r}} \lesssim \int_0^t \|\partial_x u_n^1\|_{B^{s-2}_{p,r}} \|\delta\|_{B^{s-2}_{p,r}} d\tau + \int_0^t \|u_n^1 - u_{0,n}^1\|_r \|\partial_x \rho_{0,n}^1, \rho_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-2}_{p,r}} d\tau \\
+ \int_0^t \|\rho_n^1 \partial_x u_n^1 - \rho_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-2}_{p,r}} d\tau + t \|u_{0,n}^1 \partial_x \rho_{0,n}^1, \rho_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-2}_{p,r}}.
\]

(3.7)

Using Lemma 2.1 and the fact that \(B^{s-1}_{p,r}(\mathbb{R})\) is a Banach algebra when \(s > \max\{1 + \frac{1}{p}, \frac{3}{2}\}\), we have

\[
\begin{align*}
\|(u_n^1 - u_{0,n}^1) \partial_x u_{0,n}^1\|_{B^{s-1}_{p,r}} &\lesssim \|u_n^1 - u_{0,n}^1\|_{B^{s-1}_{p,r}} \|\partial_x u_{0,n}^1\|_{B^{s-1}_{p,r}} \\
&\lesssim \|u_n^1 - u_{0,n}^1\|_{B^{s-1}_{p,r}} \|u_{0,n}^1\|_{B^{s-1}_{p,r}}, \\
\|f(u_n^1) - f(u_{0,n}^1)\|_{B^{s-1}_{p,r}} &\lesssim \|u_n^1 - u_{0,n}^1\|_{B^{s-1}_{p,r}} \|u_{0,n}^1\|_{B^{s-1}_{p,r}}, \\
\|g(\rho_n^1) - g(\rho_{0,n}^1)\|_{B^{s-2}_{p,r}} &\lesssim \|\rho_n^1 - \rho_{0,n}^1\|_{B^{s-2}_{p,r}} \|\rho_n^1\|_{B^{s-1}_{p,r}} \\
\|(u_n^1 - u_{0,n}^1) \partial_x \rho_{0,n}^1\|_{B^{s-2}_{p,r}} &\lesssim \|\partial_x \rho_{0,n}^1\|_{B^{s-2}_{p,r}} \|u_{0,n}^1 - u_{0,n}^1\|_{B^{s-1}_{p,r}} \\
&\lesssim \|u_n^1 - u_{0,n}^1\|_{B^{s-1}_{p,r}} \|\rho_{0,n}^1\|_{B^{s-1}_{p,r}}, \\
\|u_{0,n}^1 \partial_x \rho_{0,n}^1\|_{B^{s-2}_{p,r}} &\lesssim \|u_{0,n}^1 \partial_x \rho_{0,n}^1\|_{B^{s-2}_{p,r}} \|\rho_{0,n}^1\|_{B^{s-1}_{p,r}} \\
&\lesssim \|u_{0,n}^1\|_{B^{s-1}_{p,r}} \|\partial_x \rho_{0,n}^1\|_{L^\infty} \|\rho_{0,n}^1\|_{B^{s-1}_{p,r}} \\
&\lesssim 2^{-ns} 2^{n(\frac{1}{2} + \frac{3}{2})} + 2^{-3n} 2^n 2^{-ns} 2^n \lesssim 2^{-n(s - \frac{1}{2})}, \\
\|\rho_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-2}_{p,r}} &\lesssim \|\rho_{0,n}^1 \partial_x u_{0,n}^1\|_{B^{s-2}_{p,r}} \|\rho_{0,n}^1\|_{B^{s-1}_{p,r}} \\
&\lesssim 2^{n(s - \frac{3}{2})} \\
\|\rho_{0,n}^1, \partial_x u_{0,n}^1\|_{L^\infty} \|\rho_{0,n}^1, \partial_x u_{0,n}^1\|_{L^p} \lesssim 2^{-n(s - \frac{1}{2})}.
\end{align*}
\]
Again using Lemma 2.1 and the Banach algebra property of \( B^{s-1}_{p,r} \), one has

\[
\| f_1(u_{0,n}) \|_{B^{s-1}_{p,r}} \lesssim \| (u_{0,n})^2 \|_{B^{s-1}_{p,r}} \lesssim \| u_{0,n} \|_{L^\infty} \| u_{0,n} \|_{B^{s-1}_{p,r}} \leq 2^{-n(s+\frac{1}{2})},
\]

\[
\| f_2(u_{0,n}) \|_{B^{s-1}_{p,r}} \lesssim \| (\partial_x u_{0,n})^2 \|_{B^{s-1}_{p,r}} \lesssim \| \partial_x u_{0,n} \|_{L^\infty} \| \partial_x u_{0,n} \|_{B^{s-1}_{p,r}} \leq 2^{-n(s-\frac{1}{2})},
\]

\[
\| g(\rho_{0,n}) \|_{B^{s-1}_{p,r}} \lesssim \| (\rho_{0,n})^2 \|_{B^{s-1}_{p,r}} \lesssim \| \rho_{0,n} \|_{L^\infty} \| \rho_{0,n} \|_{B^{s-1}_{p,r}} \leq 2^{-n(s-\frac{1}{2})},
\]

\[
\| u_{0,n} \partial_x u_{0,n} \|_{B^{s-1}_{p,r}} \lesssim \| u_{0,n} \|_{L^\infty} \| \partial_x u_{0,n} \|_{B^{s-1}_{p,r}} + \| u_{0,n} \|_{B^{s-1}_{p,r}} \| \partial_x u_{0,n} \|_{L^\infty} \leq 2^{-ns}.
\]

For the term

\[
\rho_n \partial_x u_n - \rho_{0,n} \partial_x u_{0,n} = (\rho_n - \rho_{0,n}) \partial_x u_n + \rho_{0,n} \partial_x (u_n - u_{0,n}).
\]

Following the same procedure of estimates as above, we find that

\[
\| (\rho_n - \rho_{0,n}) \partial_x u_n \|_{B^{s-1}_{p,r}} \lesssim \| \rho_n - \rho_{0,n} \|_{B^{s-1}_{p,r}} \| \partial_x u_n \|_{B^{s-1}_{p,r}} \lesssim \| \rho_n - \rho_{0,n} \|_{B^{s-1}_{p,r}} \| u_n \|_{B^{s-1}_{p,r}},
\]

\[
\| \rho_{0,n} \partial_x (u_n - u_{0,n}) \|_{B^{s-1}_{p,r}} \lesssim \| \partial_x (u_n - u_{0,n}) \|_{B^{s-1}_{p,r}} \| \rho_{0,n} \|_{B^{s-1}_{p,r}} \lesssim \| u_n - u_{0,n} \|_{B^{s-1}_{p,r}} \| \rho_{0,n} \|_{B^{s-1}_{p,r}}.
\]

Denote

\[
X_s = \| \epsilon \|_{B^{s}_{p,r}} + \| \delta \|_{B^{s-1}_{p,r}}.
\]

taking the above estimates into (3.6)–(3.7), we get

\[
X_s - 1 \lesssim \int_0^1 X_{s-1}(\| u_{0,n} \|_{B^{s}_{p,r}} + \| \rho_{0,n} \|_{B^{s-1}_{p,r}}) + 2^{n(s-\frac{1}{2})},
\]

since \( \{u_{1,n}, \rho_{1,n}\} \) is bounded in \( B^{s}_{p,r} \times B^{s-1}_{p,r} \), which together with the Gronwall Lemma imply

\[
X_{s-1} \lesssim 2^{-n(s-\frac{1}{2})}.
\]

Combining with (3.2) for \( k = 1 \) and the interpolation inequality, we obtain that

\[
X_s \lesssim X_{s-1} \frac{1}{2} X_{s+1} \frac{1}{2} \lesssim 2^{-\frac{n}{2}(s-\frac{1}{2})} 2^{n} \lesssim 2^{-\frac{n}{2}(s-\frac{1}{2})}.
\]

Thus we have complete the proof of Proposition 3.1. \( \square \)

In order to obtain the non-uniformly continuous dependence property for the system (1.4), we will show that for the constructed initial data \( (u_{1,n}(0), \rho_{1,n}(0)) \) with small perturbation, it can not approximate to the solution \( (u_{n}, \rho_{n}(0)) \).
Proposition 3.2  Under the assumptions of Theorem 1.1, we have

\[ \|u_n^2 - u_0^2 - n - t\psi^2_0\|_{B^{p,r}_{n+1}} + \|\rho_n^2 - \rho_0^2_{n} - tw_0^n\|_{B^{p,r}_{n+1}} \lesssim t^2 + 2^{-n}\min(s-\frac{3}{2}, \frac{1}{2}) \tag{3.8} \]

here, \( \psi_0^n = u_{0,n}^2 \partial_{x} u_{0,n} \), \( w_0^n = k_3 u_{0,n}^2 \partial_{x} \rho_{0,n}^2 \).

**Proof** Firstly, due to (3.1) and making full use of the product estimates in Lemma 2.1, for \( \sigma \geq -1 \), we have

\[ \|\psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|u_{0,n}^2\|_{L^\infty} \|\partial_{x} u_{0,n}^2\|_{B^{p,r}_{n+1}} + \|u_{0,n}^2\|_{B^{p,r}_{n+1}} \|\partial_{x} u_{0,n}^2\|_{L^\infty} \lesssim (2^{-ns} + 2^{-n}) 2^n(\sigma+1) + 2^{\alpha} (2^{-ns} 2^n + 2^{-n}) \lesssim 2^{\alpha} \tag{3.9} \]
\[ \|\psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|u_{0,n}^2\|_{L^\infty} \|\partial_{x} \rho_{0,n}^2\|_{B^{p,r}_{n+1}} + \|u_{0,n}^2\|_{B^{p,r}_{n+1}} \|\partial_{x} \rho_{0,n}^2\|_{L^\infty} \lesssim (2^{-ns} + 2^{-n}) 2^n(\sigma+2) + 2^{\alpha} (2^{n-s} 2^n + 2^{-n}) \lesssim 2^{n(\sigma+1)} \tag{3.10} \]
\[ \|\psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|\partial_{x} \rho_{0,n}^2\|_{B^{p,r}_{n+1}} \|u_{0,n}^2\|_{B^{p,r}_{n+1}} \lesssim 2^{-n} \tag{3.11} \]

Let

\[ \begin{aligned}
z_n &= u_n^2 - u_0^2 - n - t\psi_0^n, \\
\omega_n &= \rho_n^2 - \rho_0^2_{n} - tw_0^n,
\end{aligned} \tag{3.12} \]

then we can derive from (1.4) that \( (z_n, \omega_n) \) satisfies

\[ \begin{aligned}
\partial_t z_n - u_n^2 \partial_{x} z_n &= (z_n + t\psi_0^n)\partial_{x} u_0^2_n + tu_n^2 \partial_{x} \psi_0^n + f(u_n^2) + g(\rho_n^2) \\
\partial_t \omega_n - k_3 u_n^2 \partial_{x} \omega_n &= k_3 (z_n + t\psi_0^n) \partial_{x} \rho_0^2_n + k_3 tu_n^2 \partial_{x} \psi_0^n + k_3 \rho_n^2 \partial_{x} u_n^2,
\end{aligned} \tag{3.12} \]

Applying Lemma 2.1, using (3.3), (3.9)–(3.10), we arrive at

\[ \|z_n\|_{B^{p,r}_{n+1}} \lesssim \|z_n\|_{B^{p,r}_{n+1}} \|\partial_{x} \rho_0^2_n\|_{B^{p,r}_{n+1}} \lesssim 2^n \|z_n\|_{B^{p,r}_{n+1}}, \tag{3.13} \]
\[ \|z_n\|_{B^{p,r}_{n+1}} \lesssim \|z_n\|_{B^{p,r}_{n+1}} \|\partial_{x} \rho_0^2_n\|_{B^{p,r}_{n+1}} \lesssim \|z_n\|_{B^{p,r}_{n+1}}, \tag{3.14} \]
\[ \|\psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|\psi_0^n\|_{B^{p,r}_{n+1}} \|\partial_{x} \rho_0^2_n\|_{B^{p,r}_{n+1}} \lesssim 2^{-n} 2^n \lesssim C, \tag{3.15} \]
\[ \|\psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|\psi_0^n\|_{B^{p,r}_{n+1}} \|\partial_{x} \rho_0^2_n\|_{B^{p,r}_{n+1}} \lesssim 2^{-n} \tag{3.16} \]
\[ \|u_n^2 \partial_{x} \psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|u_n^2 \|_{B^{p,r}_{n+1}} \|\partial_{x} \psi_0^n\|_{B^{p,r}_{n+1}} \lesssim 2^{-n} 2^n \lesssim C, \tag{3.17} \]
\[ \|u_n^2 \partial_{x} \psi_0^n\|_{B^{p,r}_{n+1}} \lesssim \|u_n^2 \|_{B^{p,r}_{n+1}} \|\partial_{x} \psi_0^n\|_{B^{p,r}_{n+1}} \lesssim 2^{-n} \tag{3.18} \]

It needs to pay more attention to deal with the term \( k_3 \rho_n^1 \partial_{x} u_n^1 \) and it can be can be decomposed as

\[ \rho_n^2 \partial_{x} u_n^2 = \rho_n^2 \partial_{x} z_n + (\omega_n + \rho_0^2_n) \partial_{x} u_0^2_n + t(\rho_n^2 \partial_{x} \psi_0^n + \psi_0^n \partial_{x} u_0^2_n). \]
With (3.3), (3.9)–(3.10) at hand, by Lemma 2.1, we find that

\[
\begin{align*}
\|\rho_n^2 \partial_x z_n\|_{B^{s-1}_{p,r}} &\lesssim \|\rho_n^2\|_{B^{s-1}_{p,r}} \|\partial_x z_n\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \\
\|\rho_n^2 \partial_x z_n\|_{B^{s-2}_{p,r}} &\lesssim \|\partial_x z_n\|_{B^{s-2}_{p,r}} \|\rho_n^2\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \\
\|\omega_n \partial_x u_0^2\|_{B^{s-1}_{p,r}} &\lesssim \|\omega_n\|_{B^{s-1}_{p,r}} \|\partial_x u_0\|_{B^{s-1}_{p,r}} \lesssim \|\omega_n\|_{B^{s}_{p,r}}, \\
\|\omega_n \partial_x u_0^2\|_{B^{s-2}_{p,r}} &\lesssim \|\omega_n\|_{B^{s-2}_{p,r}} \|\partial_x u_0\|_{B^{s-1}_{p,r}} \lesssim \|\omega_n\|_{B^{s}_{p,r}}, \\
\|\rho_{0,n}^2 \partial_x u_0^2\|_{B^{s-1}_{p,r}} &\lesssim \|\rho_{0,n}^2\|_{B^{s}_{p,r}} \|\partial_x u_0\|_{B^{s-1}_{p,r}} \lesssim \|\rho_{0,n}^2\|_{B^{s}_{p,r}} \|\partial_x u_0\|_{B^{s}_{p,r}} L_{\infty} \\
&\lesssim (2^{-ns} 2^n + 2^{-n}) \cdot 1 + 1 \cdot (2^{-ns} 2^n + 2^{-n}) \lesssim 2^{-n \min\{s-1, 1\}}.
\end{align*}
\]  

(3.19) \hspace{1cm} (3.20) \hspace{1cm} (3.21) \hspace{1cm} (3.22) \hspace{1cm} (3.23)

Applying Lemma 2.2 to the second equation of (3.12), using the fact that \{u_n^2\} is bounded in \(L_{\infty}^{\infty} (B^s_{p,r})\), firstly with (3.14), (3.16), (3.18), (3.20), (3.22), (3.24), (3.26), (3.28), we infer that

\[
\|\omega_n\|_{B^{s-2}_{p,r}} \leq C \int_0^t (\|z_n\|_{B^{s}_{p,r}} + \|\omega_n\|_{B^{s-2}_{p,r}}) d\tau + Ct^2 2^{-n} + C 2^{-n \min\{s-1, 1\}},
\]

(3.29)

and again combining with (3.13), (3.15), (3.17), (3.19), (3.21), (3.23), (3.25), (3.27), we obtain that

\[
\|\omega_n\|_{B^{s-1}_{p,r}} \leq C \int_0^t (\|z_n\|_{B^s_{p,r}} + \|\omega_n\|_{B^{s-1}_{p,r}}) d\tau + C \int_0^t 2^n \|z_n\|_{B^{s-1}_{p,r}} d\tau + Ct^2 + C 2^{-n \min\{s-1, 1\}}.
\]

(3.30)

In the following, we shall estimate \(z_n\) in \(B^{s-1}_{p,r}\) and \(B^s_{p,r}\), respectively. With the aid of Lemma 2.1 and (3.3), (3.9), one has

\[
\|z_n \partial_x u_0^2\|_{B^{s}_{p,r}} \lesssim \|z_n\|_{L_{\infty}} \|\partial_x u_0^2\|_{B^{s}_{p,r}} + \|z_n\|_{B^{s}_{p,r}} \|\partial_x u_0^2\|_{B^{s}_{p,r}} L_{\infty} \\
\lesssim \|z_n\|_{B^{s-1}_{p,r}} \|\partial_x u_0^2\|_{B^{s}_{p,r}} + \|z_n\|_{B^{s}_{p,r}} \|\partial_x u_0^2\|_{B^{s}_{p,r}}
\]

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\[ \lesssim 2^n \|z_n\|_{B^{s-1}_{p,r}} + \|z_n\|_{B^{s}_{p,r}}, \]  
(3.31) 
\[ \|z_n \partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s-1}_{p,r}} \|\partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s-1}_{p,r}}, \]  
(3.32) 
\[ \|v_0^n \partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|v_0^n\|_{B^{s-1}_{p,r}} \|\partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \leq C, \]  
(3.33) 
\[ \|v_0^n \partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|v_0^n\|_{B^{s-1}_{p,r}} \|\partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \leq 2^{-n}, \]  
(3.34) 
\[ \|u_{0,n}^2 \partial_x v_0^n\|_{B^{s-1}_{p,r}} \lesssim \|u_{0,n}^2\|_{B^{s-1}_{p,r}} \|\partial_x v_0^n\|_{B^{s-1}_{p,r}} + \|u_{0,n}^2\|_{B^{s-1}_{p,r}} \|\partial_x v_0^n\|_{B^{s-1}_{p,r}} \leq C, \]  
(3.35) 
\[ \|u_{0,n}^2 \partial_x v_0^n\|_{B^{s-1}_{p,r}} \lesssim \|u_{0,n}^2\|_{B^{s-1}_{p,r}} \|\partial_x v_0^n\|_{B^{s-1}_{p,r}} \leq 2^{-n}. \]  
(3.36) 
For the term \( f(u_n^2) = f_1(u_n^2) + f_2(u_n^2) \), we have from Lemma 2.1 and (3.3) that
\[ \|f_1(u_n^2)\|_{B^{s}_{p,r}} \lesssim \|(u_n^2)^2\|_{B^{s-1}_{p,r}} \lesssim \|u_n^2\|_{B^{s-1}_{p,r}}^2 \lesssim 2^{-2n}, \]  
(3.37) 
while it needs to be more careful to deal with \( f_2(u_n^2) \). By making full use of the structure of \( u_n^2 \), we find that
\[ f_2(u_n^2) = \frac{3 - k_1}{2} \partial_x (1 - \partial_x^2)^{-1} (\partial_x (u_n^2 + u_{0,n}^2) \partial_x z_n) \]
\[ + \frac{3 - k_1}{2} \partial_x (1 - \partial_x^2)^{-1} (t \partial_x (u_n^2 + u_{0,n}^2) \partial_x v_0^n) \]
\[ + \frac{3 - k_1}{2} \partial_x (1 - \partial_x^2)^{-1} ((\partial_x u_{0,n}^2)^2), \]
and
\[ \|f_{2,1}\|_{B^{s}_{p,r}} \lesssim \|\partial_x (u_n^2 + u_{0,n}^2)\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \]  
(3.38) 
\[ \|f_{2,1}\|_{B^{s-1}_{p,r}} \lesssim \|\partial_x (u_n^2 + u_{0,n}^2)\|_{B^{s-2}_{p,r}} \lesssim \|z_n\|_{B^{s-1}_{p,r}}, \]  
(3.39) 
\[ \|f_{2,2}\|_{B^{s}_{p,r}} \lesssim \|\partial_x (u_n^2 + u_{0,n}^2)\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \]  
(3.40) 
\[ \|f_{2,2}\|_{B^{s}_{p,r}} \lesssim \|\partial_x (u_n^2 + u_{0,n}^2)\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \]  
(3.41) 
\[ \|f_{2,3}\|_{B^{s}_{p,r}} \lesssim \|\partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s}_{p,r}}, \]  
(3.42) 
\[ \|f_{2,3}\|_{B^{s-1}_{p,r}} \lesssim \|\partial_x u_{0,n}^2\|_{B^{s-1}_{p,r}} \lesssim \|z_n\|_{B^{s-1}_{p,r}}, \]  
(3.43)
\[ g(\rho_n^2) = \frac{k_2}{2} \partial_x (1 - \partial_x^2)^{-1}(\omega_n(\rho_n^2 + \rho_0^2, n)) + \frac{k_2}{2} \partial_x (1 - \partial_x^2)^{-1}(t w_0^n(\rho_n^2 + \rho_0^2, n)) \]

\[ + \frac{k_2}{2} \partial_x (1 - \partial_x^2)^{-1}( (\rho_0^2, n)^2) , \]

and

\[ \|g_1\|_{B^r_{p, r}} \lesssim \|\omega_n(\rho_n^2 + \rho_0^2)\|_{B^{r-1}_{p, r}} \lesssim \|\omega_n\|_{B^{r-1}_{p, r}} , \] (3.44)

\[ \|g_1\|_{B^{r-1}_{p, r}} \lesssim \|\omega_n(\rho_n^2 + \rho_0^2)\|_{B^{r-2}_{p, r}} \lesssim \|\omega_n\|_{B^{r-2}_{p, r}} , \] (3.45)

\[ \|g_2\|_{B^r_{p, r}} \lesssim \|w_0^n(\rho_n^2 + \rho_0^2)\|_{B^{r-1}_{p, r}} \leq C , \] (3.46)

\[ \|g_2\|_{B^{r-1}_{p, r}} \lesssim \|w_0^n(\rho_n^2 + \rho_0^2)\|_{B^{r-2}_{p, r}} \lesssim 2^{-n} , \] (3.47)

\[ \|g_3\|_{B^r_{p, r}} \lesssim \|(\rho_0^2, n)^2\|_{B^{r-1}_{p, r}} \lesssim \|\rho_0^2, n\|_{L^\infty} \|\rho_0^2, n\|_{B^{r-1}_{p, r}} \]

\[ \lesssim (2^{-ns} 2^n + 2^{-n}) \cdot 1 \lesssim 2^{-n \min\{s, 1\}} , \] (3.48)

\[ \|g_3\|_{B^{r-1}_{p, r}} \lesssim \|(\rho_0^2, n)^2\|_{B^{r-1}_{p, r}} \lesssim \|\rho_0^2, n\|_{L^\infty} \|\rho_0^2, n\|_{B^{r-1}_{p, r}} \]

\[ \lesssim (2^{-ns} 2^n + 2^{-n}) \cdot 2^{-\frac{3}{2}} \lesssim 2^{-n \min\{s - \frac{1}{2}, \frac{3}{2}\}} . \] (3.49)

Applying Lemma 2.2 firstly together with (3.32), (3.34), (3.36), (3.37), (3.39), (3.41), (3.43), (3.45), (3.47), (3.49) to the first equation of (3.12), using the fact that \(\{u_n^2\}\) is bounded in \(L_T^\infty(B^s_{p, r})\), we infer that

\[ \|z_n\|_{B^{r-1}_{p, r}} \leq C \int_0^t (\|z_n\|_{B^{r-1}_{p, r}} + \|\omega_n\|_{B^{r-2}_{p, r}}) d\tau + C t 2^{-n} + C 2^{-n \min\{s - \frac{1}{2}, \frac{3}{2}\}} , \] (3.50)

and again combining with (3.31), (3.33), (3.35), (3.37), (3.38), (3.40), (3.42), (3.44), (3.46), (3.48), we obtain

\[ \|z_n\|_{B^r_{p, r}} \leq C \int_0^t (\|z_n\|_{B^r_{p, r}} + \|\omega_n\|_{B^{r-1}_{p, r}}) d\tau + C \int_0^t 2^n \|z_n\|_{B^{r-1}_{p, r}} d\tau + C t^2 + C 2^{-n \min\{s, 1\}} . \] (3.51)

Using Gronwall Lemma to (3.23) and (3.43) imply

\[ \|z_n\|_{B^{r-1}_{p, r}} + \|\omega_n\|_{B^{r-2}_{p, r}} \leq C t^2 2^{-n} + C 2^{-n \min\{s - \frac{1}{2}, \frac{3}{2}\}} , \]
which together with (3.30) and (3.51) yield that
\[ \|z_n\|_{B_{p,r}^s} + \|\omega_n\|_{B_{p,r}^{s-1}} \leq Ct^2 + C2^{-n \min\left\{ \frac{3}{2}, \frac{1}{2} \right\}}. \]

Thus, we have finished the proof of Proposition 3.2. \( \square \)

**Proof of Theorem 1.1** It is obvious that
\[
\|u_{0,n}^2 - u_{0,n}^1\|_{B_{p,r}^s} = \|g_n\|_{B_{p,r}^s} \leq C2^{-n},
\]
\[
\|\rho_{0,n}^2 - \rho_{0,n}^1\|_{B_{p,r}^{s-1}} = \|g_n\|_{B_{p,r}^{s-1}} \leq C2^{-n},
\]
which means that
\[
\lim_{n \to \infty} (\|u_{0,n}^2 - u_{0,n}^1\|_{B_{p,r}^s} + \|\rho_{0,n}^2 - \rho_{0,n}^1\|_{B_{p,r}^{s-1}}) = 0.
\]

However, according to Proposition 3.1 and Proposition 3.2, we get
\[
\|\rho_n^2 - \rho_n^1\|_{B_{p,r}^{s-1}} = \|\omega_n + tw_0^n + gn + \rho_{0,n}^1 - \rho_{0,n}^1\|_{B_{p,r}^{s-1}} \\
\geq t\|w_0^n\|_{B_{p,r}^{s-1}} - 2^{-n} - t^2 - 2^{-n \min\left\{ \frac{3}{2}, \frac{1}{2} \right\}} - 2^{-n(3/2)}.
\]
\[ (3.52) \]

Notice that
\[
w_0^n = k_3u_{0,n}^2\partial_x \rho_{0,n}^2 = k_3(f_n + gn)\partial_x(2^n f_n + gn)
= k_3f_n\partial_x(2^n f_n) + k_3 gn\partial_x(2^n f_n) + k_3 f_n\partial_x gn + k_3 g_n\partial_x gn.
\]

With the aid of Lemma 2.1 and the Banach algebra property of $B_{p,r}^{s-1}$, we find that
\[
\|f_n\partial_x(2^n f_n)\|_{B_{p,r}^{s-1}} \lesssim \|f_n\|_{L^\infty}\|\partial_x(2^n f_n)\|_{B_{p,r}^{s-1}} + \|f_n\|_{B_{p,r}^{s-1}}\|\partial_x(2^n f_n)\|_{L^\infty}
\lesssim 2^{-ns}2^n + 2^{-ns}2^{-n}2^{-ns}2^n \lesssim 2^{-n(s-1)},
\]
\[
\|f_n\partial_x gn\|_{B_{p,r}^{s-1}} \lesssim \|f_n\|_{B_{p,r}^{s-1}}\|\partial_x gn\|_{B_{p,r}^{s-1}} \lesssim 2^{-2n},
\]
\[
\|g_n\partial_x gn\|_{B_{p,r}^{s-1}} \lesssim \|g_n\|_{B_{p,r}^{s-1}}\|\partial_x gn\|_{B_{p,r}^{s-1}} \lesssim 2^{-2n}.
\]

However, using the fact that $\Delta_j(g_n\partial_x(2^n f_n)) = 0$, $j \neq n$ and $\Delta_n(g_n\partial_x(2^n f_n)) = g_n\partial_x(2^n f_n)$ for $n \geq 5$, direct calculation shows that for $n \gg 1$,
\[
\|g_n\partial_x(2^n f_n)\|_{B_{p,r}^{s-1}} = 2^{n(s-1)}\|g_n\partial_x(2^n f_n)\|_{L^p}
= 2^{-n}\phi(x)\partial_x\phi(x) \sin\left(2 \pi \frac{17}{12} 2^n x\right) + 2^{-n}\phi^2(x) \cos\left(2 \pi \frac{17}{12} 2^n x\right) \|L^p
\gtrsim \|2^{-n}\phi^2(x) \cos\left(2 \pi \frac{17}{12} 2^n x\right) \|_{L^p} - 2^{-n} \to 17 \frac{2 \pi}{12} \frac{\int_0^{2\pi} |\cos x|^p dx}{2\pi} \|\phi^2(x)\|_{L^p},
\]
\[ \square \]
by the Riemann Theorem.
Taking the above estimates into (3.52) yields

\[ \liminf_{n \to \infty} \| \rho_n^2 - \rho_n^1 \|_{B_{t}^{p,r-1}} \gtrsim t \quad \text{for } t \text{ small enough}. \]

Similarly, we have

\[ \liminf_{n \to \infty} \| u_n^2 - u_n^1 \|_{B_{t}^{r,v}} \gtrsim t \quad \text{for } t \text{ small enough}. \]

This completes the proof of Theorem 1.1. \qed

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**Declarations**

**Conflicts of interest** Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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