Partial Polarization by Quantum Distinguishability

Mayukh Lahiri, Armin Hochrainer, Radek Lapkiewicz, Gabriela Barreto Lemos, and Anton Zeilinger

Vienna Center for Quantum Science and Technology (VCQ), Faculty of Physics, Boltzmannasse 5, University of Vienna, Vienna A-1090, Austria.

Institute for Quantum Optics and Quantum Information, Austrian Academy of Sciences, Boltzmannasse 3, Vienna A-1090, Austria.

Partial polarization is the manifestation of the correlation between two mutually orthogonal transverse field components associated with a light beam. We show both theoretically and experimentally that the origin of this correlation can be purely quantum mechanical. We perform a two-path first-order (single photon) interference experiment and demonstrate that the degree of polarization of the light emerging from the output of the interferometer depends on path distinguishability. We use two independent methods to control the distinguishability of the photon paths. While the distinguishability introduced in one of the methods can be erased by performing a suitable measurement on the superposed beam, the distinguishability introduced in the other method cannot be erased. We show that the beam is partially polarized only when both types of distinguishability exist. Our main result is the dependence of the degree of polarization on the inerasable distinguishability, which cannot be explained by the classical (non-quantum) theory of light.

The interference effect displayed by a quantum system or entity when sent through an interferometer is a key feature of quantum mechanics. If quantum entities of a particular kind are sent one at a time through a two-path interferometer and one can identify (even in principle) the path traversed by each of them, no interference occurs. The information that leads to the identification of the path is often called the path information. Because the common-sense understanding of a particle implies that the path traversed can always be identified, the particle behavior of the quantum entity is often interpreted as the complete availability of the path information, i.e., the complete distinguishability of the paths. On the other hand, when the path information is fully unavailable, i.e., when the paths are fully indistinguishable, perfect interference occurs—a characteristic of waves.

The wave-particle duality of a light quantum (photon) has been confirmed by numerous experiments. Furthermore, the relationship between the path distinguishability and the ability of light (or any quantum entity) to interfere has been studied in detail. The ability of light to interfere has also long been the subject of investigation in coherence theory, where it is quantified by the degree of coherence. It has been shown that under certain circumstances, it is possible to establish a relationship between the path distinguishability and the degree of coherence. The concepts of coherence theory are often used to analyze polarization properties of light beams. It is therefore natural to ask whether a connection between partial polarization and path distinguishability can be established.

We develop a two-path interferometer in which we use two independent methods for introducing path distinguishability in a controlled manner. We investigate both theoretically and experimentally how the polarization property of the light beam emerging from the output of the interferometer depends on the distinguishability of photon paths.

Let us begin by considering a thought experiment in which photon beams generated by two identical sources $Q_1$ and $Q_2$ are superposed by a lossless and balanced non-polarizing beam-splitter, BS (Fig. 1). Photons emerging from one of the outputs of BS are collected by a photo-detector, D. The device, M, attached to $Q_1$ determines with probability $1 - M^2$ whether $Q_1$ has emitted.

\[ \text{FIG. 1: Two identical sources, } Q_1 \text{ and } Q_2, \text{ emit indistinguishable linearly polarized photons. The photons generated by } Q_1 \text{ are sent through a polarization rotator, } \Gamma, \text{ and then superposed with the photons generated by } Q_2 \text{ by a beam splitter, BS. The superposed beam emerging from one of the outputs of BS is sent through a polarizer, } \Theta, \text{ and then detected by a photo-detector, D. The device, M, attached to } Q_1 \text{ determines with probability } 1 - M^2 \text{ whether } Q_1 \text{ has emitted.} \]

From one of the outputs of BS are collected by a photo-detector, D. Clearly, this system is equivalent to a two-path interferometer. The two paths are of equal length and we label them as 1 and 2. Photons emitted by $Q_1$ can only travel via path 1 and photons emitted by $Q_2$ can only travel via path 2. We assume that the photons emitted by $Q_1$ and $Q_2$ are linearly polarized in the same direction and are identical to each other. On path 1, we place a polarization rotator (for example, a half-waveplate), $\Gamma$, by which we can rotate the direction of the incident linear polarization by an arbitrary angle $\gamma$, where $\cos \gamma \geq 0$; the superposed beam emerging from BS is sent through a polarizer, $\Theta$, before arriving at D [Fig. 1]. The light emerging from $\Theta$ is linearly polarized along a direction that makes an angle $\theta$ with the polarization-direction of the light originally emitted by the sources. In this arrangement, $\Theta$ and D constitute...
the detection system that measures the degree of polarization of the superposed beam emerging from BS.

Suppose \( Q_1 \) and \( Q_2 \) emit at the same rate and in such a way that only one photon exists in the system between an emission and a detection at D. Now, if path 2 is blocked, the probability amplitude of photo-detection is given by \( \alpha_2 = e^{i\phi_2} (\cos \theta - \gamma) / 2 \), where the phase \( \phi_2 \) depends on path length. Similarly, if path 1 is blocked, the probability amplitude of photo-detection will be \( \alpha_1 = e^{i\phi_1} (\cos \theta - \gamma) / 2 \). Suppose now that we attach (Fig. 1) to source \( Q_1 \) a device, M, that does not perform any measurement on the photons entering the interferometer but determines with a known probability whether \( Q_1 \) has emitted. Clearly, when both paths are open, there are three possible cases in which a photon can arrive at D: I) \( Q_1 \) emits and M reports the emission; II) \( Q_1 \) emits and M does not report the emission; III) \( Q_2 \) emits. Since the photons emitted by \( Q_1 \) and \( Q_2 \) are identical, possibilities II and III are indistinguishable. On the other hand, possibility I is fully distinguishable from possibilities II and III. For obtaining the total probability of detecting a photon at D, one therefore needs to add the probability associated with possibility I to the modulus square of the sum of probability amplitudes associated with II and III. Let us assume that when \( Q_1 \) has emitted a photon, the probability of M not reporting the emission is equal to \( M^2 \), where \( 0 \leq M \leq 1 \). The probability amplitudes associated with cases I, II, and III are then given by \( \alpha_1 \sqrt{1 - M^2}, \alpha_1 M, \) and \( \alpha_2 \), respectively. The probability of photo-detection at D is thus given by:

\[
\Phi = |\alpha_1 \sqrt{1 - M^2}|^2 + |\alpha_1 M + \alpha_2|^2
\]

\[
= \frac{1}{4} \left[ \cos^2 \theta + \cos^2 (\theta - \gamma) + 2 M \cos \theta \cos (\theta - \gamma) \cos (\phi_2 - \phi_1) \right],
\]

which is directly proportional to the photon counting rate at the detector.

It is to be noted that the path distinguishability has been introduced by two independent methods: a) by the polarization rotator \( \Gamma \), and b) by the device M. The path distinguishability introduced by \( \Gamma \) can be erased by the polarizer \( \Theta \), as is evident from the term \( \cos (\theta - \gamma) \) of Eq. (1). When \( \gamma = \pi / 2 \) and \( \theta = 0 \), i.e., when the maximum path distinguishability is introduced by \( \Gamma \) and the distinguishability is not erased by \( \Theta \), no interference occurs for any value of \( M \). It is clear that a measure of the path distinguishability (or, equivalently, indistinguishability) introduced by \( \Gamma \) can be represented by \( \cos \gamma \). On the other hand, the path distinguishability introduced by M cannot be erased by any other device. A measure of the path distinguishability introduced M is given by \( M \). When \( M = 0 \), M determines with complete certainty whether \( Q_1 \) has emitted. In this case, the paths are fully distinguishable and no interference occurs, irrespective of the orientation of \( \Gamma \).

For the sake of simplicity, we choose the path lengths such that \( \phi_2 - \phi_1 \) is equal to a multiple of \( 2\pi \), i.e., \( \cos (\phi_2 - \phi_1) = 1 \). In this case, it can be readily shown from Eq. (1) that when \( \theta = \gamma / 2 \), the probability \( \Phi \) attains its maximum value \( \Phi_{\text{max}} = (1 + M) \cos^2 (\gamma / 2) / 2 \); and when \( \theta = \gamma / 2 + \pi / 2 \), it attains the minimum value \( \Phi_{\text{min}} = (1 - M) \sin^2 (\gamma / 2) / 2 \). The degree of polarization of the beam generated by superposition is given by:

\[
P = \frac{\Phi_{\text{max}} - \Phi_{\text{min}}}{\Phi_{\text{max}} + \Phi_{\text{min}}} = \frac{M + \cos \gamma}{1 + M \cos \gamma}.
\]

It follows from Eq. (2) that both devices (\( \Gamma \) and M) must be used to introduce path distinguishability for generating a partially polarized (\( 0 < P < 1 \)) beam. If only one of the devices introduces path distinguishability, i.e., if either \( M = 1 \) or \( \cos \gamma = 1 \), the beam is always fully polarized (\( P = 1 \)). On the other hand, the beam is unpolarized (\( P = 0 \)) if and only if both devices introduce maximum path distinguishability, i.e., if and only if \( M = 0 \) and \( \cos \gamma = 0 \). The central feature of the thought experiment is the dependence of the degree of polarization on the inerasable distinguishability for a “fixed amount of the erasable distinguishability,” because the phenomenon is beyond the scope of the classical theory of optical fields.

We now discuss an experiment in which the above-mentioned phenomenon is observed. This experiment is based on the concept of so-called “induced coherence without induced emission” [20, 21]. Two identical nonlinear crystals, NL1 and NL2, are pumped by two mutually coherent pump beams, \( P_1 \) and \( P_2 \), respectively (Fig. 2). Each crystal converts a pump photon into a photon pair (signal and idler), each linearly polarized, by the process of parametric down-conversion. We denote the signal and the idler generated in NL1 by \( S_1 \) and \( I_1 \), re-
by $S_2$ and $I_2$, respectively. The idler beam, $I_1$, is sent through NL2 and is aligned with $I_2$ (Fig. 2). We ensure that the down-converted light is weak enough, so that it is highly improbable for photon pairs emitted by both crystals to be simultaneously present in the system. Under this condition the effect of stimulated emission at NL2 can be neglected. An attenuator (neutral density filter), $A$, is placed on the path of $I_1$ between NL1 and NL2; the transmission coefficient of $A$ can be varied. The signal beam $S_1$ is sent through a half-wave plate, $\Gamma$, such that its polarization direction can be rotated by a chosen angle $\gamma$. It is then superposed with $S_2$ by a non-polarizing beam-splitter, $BS$.

We first need to meet the condition under which the beams $S_1$ and $S_2$ interfere (i.e., the beams are mutually coherent) in absence of $A$ and $\Gamma$. According to the principles of quantum mechanics, if there exists any information leading to the identification of the path traversed by a signal photon emerging from BS, no interference occurs. Although the signal photons emitted by the two crystals are identical in all aspects (i.e., same polarization, frequency, etc.), it is possible to obtain the path information by the method of coincidence detection, simply because signal and idler of a down-converted photon pair are produced “simultaneously” at a particular crystal. However, it has been shown that under a certain condition this path information can be removed [21]. Suppose that $\tau_{S_1}$, $\tau_{S_2}$, and $\tau_{I_1}$ are the propagation times of $S_1$ from NL1 to D, of $S_2$ from NL2 to D, and of $I_1$ from NL1 to NL2. If $|\tau_{S_1} - \tau_{S_2} - \tau_{I_1}|$ is less than the coherence time of the down-converted light, it is not possible to distinguish between an $S_1$ photon and an $S_2$ photon at D in absence of $A$ and $\Gamma$. We set the optical path lengths in our experiment in such a way that this condition is met. We stress that measurement of coincidence counts is absolutely not required in order to observe the interference at D, because the unavailability of path information is alone enough for this purpose. In fact, no coincidence measurement is performed in our experiment. It must further be noted that $S_1$ and $S_2$ beams interfere not because of stimulated emission occurring at NL2 [20, 22].

Once the appropriate optical path lengths are chosen, the attenuator ($A$) and the half-wave plate ($\Gamma$) are placed at their positions (Fig. 2). The degree of polarization of the superposed signal beam emerging from one of the outputs of BS is determined. Below we provide both the theoretical analysis and the experimental results.

The photons $S_1$ and $S_2$ are generated as linearly polarized in the same direction $x$, say; $I_1$ and $I_2$ are linearly polarized along the direction $x'$, say [23]. The quantum state of light (interaction picture [12]) generated by a crystal is given by the well-known formula

$$\hat{a}_{I_{x'}} = [T\hat{a}_{I_{x'}} + R'\hat{a}_{Qx'}]e^{i\phi_I},$$

where $T$ is the complex amplitude transmission coefficient of $A$, $|T|^2 + |R'|^2 = 1$, $\hat{a}_{Qx'}$ represents the vacuum field at the unused port of the beam-splitter (the attenuator $A$), and $\phi_I$ is a phase factor due to propagation of $I_1$ from NL1 to NL2. It follows from Eqs. (3) and (4) that the quantum state of light generated in this system is given by

$$|\Psi\rangle = \hat{U}_2 \hat{U}_1 |\text{vac}\rangle$$

$$= \langle \text{vac} | + (g_1 |x\rangle_{S_1} + g_2 e^{-i\phi_I} T^* |x\rangle_{S_2}) |x'\rangle_{I_1}$$

$$+ g_2 e^{-i\phi_I} R^* |x\rangle_{S_1} |x'\rangle_0,$$

where $|\text{vac}\rangle$ is the vacuum state, $|x\rangle_{S_j} = \hat{a}_{S_jx}^\dagger |\text{vac}\rangle$ represents an $x$-polarized signal photon, $|x'\rangle_{I_1} = \hat{a}_{I_{x'}}^\dagger |\text{vac}\rangle$ represents an $x'$-polarized idler photon, $|x'\rangle_0 = \hat{a}_{Qx'}^\dagger |\text{vac}\rangle$, $a_0(x'x')_0 = 1$, and we have neglected the higher order terms. The positive-frequency part of the quantized field components, associated with the superposed beam emerging from one of the outputs of the beam-splitter, can be represented by

$$\hat{E}^{(+)}_{S_x} = e^{i\phi_{S_1}} (\cos \gamma \hat{a}_{S_1x} - \sin \gamma \hat{a}_{S_1y}) + ie^{i\phi_{S_2}} \hat{a}_{S_{2x}},$$

$$\hat{E}^{(+)}_{S_y} = e^{i\phi_{S_1}} (\sin \gamma \hat{a}_{S_1x} + \cos \gamma \hat{a}_{S_1y}) + ie^{i\phi_{S_2}} \hat{a}_{S_{2y}},$$

where $y$ is the Cartesian direction orthogonal to $x$, $\phi_{S_1}$ and $\phi_{S_2}$ are the phase changes associated with the propagation from NL1 to BS and from NL2 to BS, respectively, and we have included the action of $\Gamma$ on $S_1$. The degree of polarization can be determined by using the formula

$$P = \{1 - 4 \det G^{(1)} / [\text{tr} G^{(1)}]^{2}\}^{1/2},$$

where $\text{det}$ and $\text{tr}$ represent the determinant and the trace of a matrix, respectively, and $G^{(1)}$ is a $2 \times 2$ correlation matrix whose elements are given by [10, 15, 25]

$$G^{(1)}_{pq} = (\langle \Psi | \hat{E}^{(-)}_p \hat{E}^{(+)}_q |\Psi\rangle.$$
degree of polarization of the superposed signal beam is given by
\[
P = \left\{ \cos^2 \gamma + |T|^2 (\sin^2 \gamma + \cos^2 \gamma \cos^2 \beta) \right. \\
+ 2|T| \cos \gamma \cos \beta \right\} \frac{1}{1 + |T| \cos \gamma \cos \beta},
\]
where \( \beta = \phi_{S2} - \phi_{S1} - \arg(T) + \arg(g_2) - \arg(g_1) \).
If we set \( \cos \beta = 1 \), Eq. (9) reduces to the form
\[
P = \frac{|T| + \cos \gamma}{1 + |T| \cos \gamma}.
\]

It is to be noted that Eq. (10) is strikingly similar to Eq. (2). This is due to the following reasons: When \( I_1 \) beam passes through \( A \), its intensity drops by a factor of \( |T|^2 \). Since a photon cannot be broken into further fractions, an idler photon can either be fully transmitted or fully blocked by \( A \). The probability of an idler photon being transmitted through \( A \) is therefore equal to \( |T|^2 \). If the idler photon is transmitted, one loses the path information for the signal photon completely (in absence of \( \Gamma \)); and if it is blocked, the path information for the signal photon is fully available. Because of this, the attenuator, \( A \), of Fig. 2 plays the role of the device, \( M \), shown in Fig. 1. The half-wave plate, \( \Gamma \), plays the same role in both cases. Therefore, if we set the interferometric phase to a multiple of \( 2\pi \), the degree of polarization of the superposed signal beam is obtained from Eq. (2) just by replacing \( M \) with \( |T| \).

In the experiment (Fig. 2), the value of \( |T| \) is changed by inserting neutral density filters (A) of different values of transmittance in \( I_1 \) beam. For each choice of \( T \) and \( \gamma \), the interferometric phase is set equal to a multiple of \( 2\pi \) by varying the path-length of \( S_1 \) with the mirror \( M_5 \) placed on a piezo-driven stage. Polarization state tomography is performed on the superposed signal beam by using a quarter waveplate, \( q \), a polarizer, \( \Theta \), and a single-photon counting module, \( D \). The photon counting rate at \( D \) is recorded for 15 sec in each measurement. The data are corrected for background (dark counts), and the matrix \( \mathcal{G} \) is determined by the maximum likelihood technique for a single-qubit system [20–22]. The degree of polarization is then calculated by using Eq. (7).

The experimental results are shown in Fig. 3(a). When \( \gamma = 90^\circ \) and \( |T| = 0 \), the experimentally measured degree of polarization is slightly more than zero; this is due to the fact that the non-polarizing beam splitter (BS) is in practice slightly polarizing. Also for \( \gamma \neq 0 \) and \( |T| = 0 \), we do not obtain completely polarized light; this is because under the experimental conditions, the \( I_1 \) beam suffers losses at the various optical components through which it passes. The solid curves of Fig. 3(a) are obtained by computation considering all these experimental imperfections. A comparison of Figs. 3(a) and 3(b) shows that the predictions of the thought experiment (Fig. 1) have been practically realized in the actual experiment (Fig. 2).

It is important to understand that the attenuator, \( A \), introduces path distinguishability for the signal photons without interacting with them. This path distinguishability (quantified by \( |T|^2 \)) cannot be erased by introducing any device that interacts with the signal photons (in this context, see [29]). This is what renders this experiment beyond the scope of the classical theory of optical fields.

In the special case of \( \gamma = \pi/2 \), the degree of polarization is equal to \( |T| \) (see the curves labeled by \( \gamma = 90^\circ \) in Fig. 3). This value of the degree of polarization cannot be increased by, for example, frequency filtering the superposed beam, i.e., by enhancing the coherence time of the light.

In the classical theory, light is considered as an electromagnetic wave and its properties are described by the fluctuating electromagnetic field associated with it. Partial polarization of a light beam is considered to be a manifestation of correlation between the transverse field components ([11], Sec. 10.9). The classical theory of partial polarization is based on the assumption that stochastic fluctuations are always associated with optical fields. We have demonstrated that partial polarization of a light beam can be solely due to the wave-particle duality of photons, i.e., solely due to the quantum nature of light. However, many simplest-order correlation properties of the partially polarized beam generated in our experiment can be successfully explained by the use of the classical theory of statistical optics. It is thus clear that even if several properties of the beam can be explained by assuming the existence of correlations between stochastic
classical fields, the origin of such correlations may not always be explained by the classical theory of radiation, but only by quantum mechanics.

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