MAXIMAL FUNCTIONS AND MEASURES ON THE UPPER-HALF PLANE

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Abstract. We study weighted boundedness of Hardy-Littlewood-type maximal function involving Orlicz functions. We also obtain some sufficient conditions for the weighted boundedness of the Hardy-Littlewood maximal function of the upper-half plane.

1. Introduction

Our setting is the upper-half plane

\[ \mathcal{H} = \{ z = x + iy \in \mathbb{C} : x \in \mathbb{R}, \quad y > 0 \}. \]

For \( \omega \) a weight, that is a nonnegative locally integrable function on \( \mathcal{H} \), \( \alpha > -1 \), and \( 1 \leq p < \infty \), we denote by \( L^p_{\omega}(\mathcal{H}, dV_\alpha) \), the set of functions \( f \) defined on \( \mathcal{H} \) such that

\[ \| f \|_{L^p_{\omega,\alpha}} := \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z) < \infty \]

with \( dV_\alpha(x + iy) = y^\alpha dx dy \). We write \( L^p(\mathcal{H}, dV_\alpha) \) when \( \omega(z) = 1 \) for any \( z \in \mathcal{H} \), and put \( \| f \|_{p,\alpha} = \| f \|_{p,1,\alpha} \). We recall that the Bergman space \( A^p_\alpha(\mathcal{H}) \) is the closed subset of \( L^p(\mathcal{H}, dV_\alpha) \) consisting of holomorphic functions.

Let us recall that for any interval \( I \subset \mathbb{R} \), its associated Carleson square \( Q_I \) is the set \( Q_I := \{ z = x + iy \in \mathbb{C} : x \in I \text{ and } 0 < y < |I| \} \).

Let \( \Phi \) be a Young function, and \( \alpha > -1 \). For any interval \( I \subset \mathbb{R} \), define \( L^\Phi(Q_I, |Q_I|^{-1}\omega dV_\alpha) \) to be the space of all functions \( f \) such that

\[ \frac{1}{|Q_I|_{\omega,\alpha}} \int_{Q_I} \Phi(|f(z)|) \omega(z) dV_\alpha(z) < \infty \]

where \( |Q_I|_{\omega,\alpha} = \int_{Q_I} \omega(z) dV_\alpha(z) \). We define on \( L^\Phi(Q_I, |Q_I|^{-1}\omega dV_\alpha) \) the following Luxembourg norm

\[ \| f \|_{Q_I, \Phi,\omega,\alpha} := \inf \{ \lambda > 0 : \frac{1}{|Q_I|_{\omega,\alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z) \leq 1 \}. \]

When \( \Phi(t) = t^p \), \( 1 \leq p < \infty \), \( L^\Phi(Q_I, |Q_I|^{-1}\omega dV_\alpha) \) is just \( L^p(Q_I, |Q_I|^{-1}\omega dV_\alpha) \) in which case \( \| f \|_{Q_I, \Phi,\omega,\alpha} \) is just replaced by \( \left( \frac{1}{|Q_I|_{\omega,\alpha}} \int_{Q_I} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{1/p} \).

2000 Mathematics Subject Classification. Primary: 47B38 Secondary: 30H20, 42A61, 42C40.

Key words and phrases. Bekollé-Bonami weight, Carleson-type embedding, Dyadic grid, Maximal function, Upper-half plane.
Then the maximal function $M_{\Phi, \omega, \alpha}$ is defined as

$$M_{\Phi, \omega, \alpha}f(z) := \sup_{z \in Q_I} \|f\|_{Q_I, \Phi, \omega, \alpha}.$$  

More precisely, the supremum is taken over all intervals $I$ such that $z \in Q_I$. Our definition here is inspired from the one in the works [2, 6] and actually, it is a weighted version of the one studied in [6]. When $\omega = 1$, we simply write $\|f\|_{Q_I, \Phi, \alpha}$ for $\|f\|_{Q_I, \Phi, 1, \alpha}$ and $M_{\Phi, \alpha}$ for the corresponding maximal function. We observe that when $\Phi = 1$, $M_{\Phi, \omega, \alpha}$ coincides with the (weighted) Hardy-Littlewood maximal function $M_{\omega, \alpha}$ given by

$$M_{\omega, \alpha}f(z) = \sup_{I \subset \mathbb{R}} \frac{1}{|Q_I|^{1/\alpha}} \int_{Q_I} |f(z)| \omega(z) dV_\alpha(z).$$

The unweighted Hardy-Littlewood maximal function, that corresponds to $\omega = 1 = \Phi$ will be denoted $M_\alpha$.

We are interested in this paper in weighted boundedness of the maximal function $M_{\Phi, \omega, \alpha}$, that is the characterization of positive Borel measure $\mu$ and weight $\omega$ such that $M_{\Phi, \omega, \alpha}$ is bounded from $L^p(H, \omega dV_\alpha)$ to $L^q(H, d\mu)$. These estimates are quite useful in the estimate of other operators as the Bergman projection (see for example [4, 7]). We will also see that they can be used to obtain alternative characterizations of Bergman spaces. We observe that in [3], these characterizations were obtained for the weighted Hardy-Littlewood maximal function of the upper-half plane.

1.1. Statement of the results. Recall that a function from $[0, \infty)$ to itself is a Young function if it is continuous, convex and increasing, and satisfies $\Phi(0) = 0$ and $\Phi(t) \to \infty$ as $t \to \infty$. We assume all over the text that the Young function $\Phi$ is such that $\Phi(1) = 1$. We also recall that the function $t \mapsto \Phi'(t)$ is increasing, and $\Phi'(t) \simeq \frac{\Phi(t)}{t}$.

Given a Young function $\Phi$, we say it satisfies the $\Delta_2$ (or doubling) condition, if there exists a constant $K > 1$ such that, for any $t \geq 0$,

$$\Phi(2t) \leq K \Phi(t).$$

Let $1 < p < \infty$. We say a Young function $\Phi$ belongs to the class $B_p$, if it satisfies the $\Delta_2$ condition and there is a positive constant $c$ such that

$$\int_0^\infty \Phi(t) \frac{dt}{t^{1/p}} < \infty.$$  

For any set $E \subset H$, given a weight $\omega$, and $\alpha > -1$, we write

$$|E|_{\omega, \alpha} := \int_E \omega(z) dV_\alpha(z).$$

Recall that for $1 < p < \infty$ and $\alpha > -1$, a weight $\omega$ is said to belong to the Békollé-Bonami class $B_{p, \alpha}$, if $[\omega]_{B_{p, \alpha}} < \infty$, where

$$[\omega]_{B_{p, \alpha}} := \sup_{I \subset \mathbb{R}} \left( \frac{1}{|Q_I|^{1/\alpha}} \int_{Q_I} \omega(z) dV_\alpha(z) \right) \left( \frac{1}{|Q_I|^{1-\alpha'}} \int_{Q_I} \omega(z)^{1-\alpha'} dV_\alpha(z) \right)^{-1}. $$
For $p = 1$, a weight $\omega$ is said to belong to the Békollé-Bonami class $B_{1,\alpha}$, if
\[
[\omega]_{B_{1,\alpha}} := \sup_{z \in Q_I} \sup_{|Q_I| \alpha} \left( \frac{1}{|Q_I|} \int_{Q_I} \omega dV_\alpha \right) \omega(z)^{-1} < \infty.
\]
It is easy to see that $B_{1,\alpha}$ is a subset of $B_{2,\alpha}$ and
\[
[\omega]_{B_{2,\alpha}} \leq [\omega]_{B_{1,\alpha}}, \quad \text{for all } \omega \in B_{1,\alpha}.
\]
We define
\[
B_{\infty,\alpha} := \bigcup_{p > 1} B_{p,\alpha}.
\]
We have the following result for the weighted maximal function.

**Theorem 1.1.** Let $\alpha > -1$, $1 < p \leq q < \infty$, and $\omega$ a weight and $\mu$ a positive Borel measure on $\mathcal{H}$. Assume that $\Phi \in B_p$ and that $\omega \in B_{\infty,\alpha}$. Then the following assertions are equivalent.

(i) There exists a constant $C_1 > 0$ such that for any $f \in L^p(\mathcal{H}, \omega dV_\alpha)$,
\[
\left( \int_{\mathcal{H}} (\mathcal{M}_{\Phi,\omega,\alpha} f(z))^q d\mu(z) \right)^{1/q} \leq C_1 \left( \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{1/p}.
\]

(ii) There is a constant $C_2$ such that for any interval $I \subset \mathbb{R}$,
\[
\mu(Q_I) \leq C_2 |Q_I|^{\frac{q}{p-q}}.
\]

We have the following lower triangle case result.

**Theorem 1.2.** Let $\alpha > -1$, $1 < q < p < \infty$. Let $\omega$ be a weight and $\mu$ a positive Borel measure on $\mathcal{H}$. Assume that $\Phi \in B_p$ and that $\omega \in B_{\infty,\alpha}$. Then (3) holds if and only if the function
\[
K_\mu(z) := \sup_{I \subset \mathbb{R}, \text{ interval}, \, z \in Q_I} \frac{\mu(Q_I)}{|Q_I|^{\frac{q}{p-q}}}
\]
belongs to $L^s(\mathcal{H}, \omega dV_\alpha)$ where $s = \frac{p}{p-q}$.

We recall that the complementary function $\Psi$ of the Young function $\Phi$, is the function defined from $\mathbb{R}_+$ onto itself by
\[
\Psi(s) = \sup_{t \in \mathbb{R}_+} \{ ts - \Phi(t) \}.
\]
We remark that a Young function satisfies (2) if and only if its complementary function $\Psi$ satisfies
\[
\int_\epsilon^\infty \left( \frac{t^{p'}}{\Psi(t)} \right)^{p-1} \frac{dt}{t} < \infty.
\]
Here, and all over the text, $p'$ denotes the conjugate exponent of $p$. We recall that
\[
t \leq \Phi^{-1}(t) \Psi^{-1}(t) \leq 2t, \quad \text{for all } t > 0.
\]
The following result provides a sufficient condition for the off-diagonal boundedness of the Hardy-Littlewood maximal function $\mathcal{M}_\alpha$.
**Theorem 1.3.** Let $\alpha > -1$, $1 < p \leq q < \infty$. Let $\Phi \in B_p$ and denote by $\Psi$ its complementary function. Assume that $\omega$ is a weight and $\mu$ a positive Borel measure on $\mathcal{H}$ such that there is a positive constant $C$ for which for any interval $I \subset \mathbb{R}$,

\[
\|\omega^{-1}\|_{Q_I, \Psi, \mu(Q)} \leq C |Q_I|^{q/p}.
\]

Then there is a positive constant $K$ such that for any $f \in L^p(\mathcal{H}, \omega dV_\alpha)$,

\[
\left( \int_{\mathcal{H}} (M_{\alpha} f(z))^q d\mu(z) \right)^{1/q} \leq K \|f\|_{p, \alpha}.
\]

It is easy to see that for $1 < p < \infty$, and $r > 1$, $\Phi(t) = t^{(p' r)'}$ is in the class $B_p$. Thus we have the following corollary.

**Corollary 1.4.** Let $\alpha > -1$, $1 < p \leq q < \infty$. Assume that $\omega$ is a weight and $\mu$ a positive Borel measure on $\mathcal{H}$ such that for some $r > 1$, there is a positive constant $C$ for which for any interval $I \subset \mathbb{R}$,

\[
\left( \frac{1}{|Q_I|_{\alpha}} \int_{Q_I} \omega^{p'r} d\mu(z) \right)^{q/p'} \leq C |Q_I|^{q/p}.
\]

Then there is a positive constant $K$ such that for any $f \in L^p(\mathcal{H}, \omega dV_\alpha)$,

\[
\left( \int_{\mathcal{H}} (M_{\alpha} f(z))^q d\mu(z) \right)^{1/q} \leq K \|f\|_{p, \alpha}.
\]

We also obtain the following result.

**Theorem 1.5.** Let $\alpha > -1$, $1 < p \leq q < \infty$. Let $\mu$ be a positive measure and $\omega$ a weight such that there is a positive constant $C_1$ such that for any interval $I \subset \mathbb{R}$,

\[
\mu(Q_I) \leq C_1 |Q_I|^{q/p}.
\]

Then there is a constant $C_2 > 0$ such that for any function $f$,

\[
\left( \int_{\mathcal{H}} (M_{\alpha, \omega} f(z))^q d\mu(z) \right)^{1/q} \leq C \left( \int_{\mathcal{H}} |f(z)|^p M_{\alpha} \omega(z) d\mu(z) \right)^{1/p}.
\]

We also have the following.

**Theorem 1.6.** Let $\alpha > -1$, $1 < q < p < \infty$. Let $\omega$ be a weight and $\mu$ a positive Borel measure on $\mathcal{H}$. Assume that $\Phi \in B_p$ and that the function

\[
K_\mu(z) := \sup_{I \subset \mathbb{R}, I \text{ interval}, z \in Q_I} \frac{\mu(Q_I)}{|Q_I|_{\omega, \alpha}}
\]

belongs to $L^s(\mathcal{H}, (M_{\alpha} \omega) dV_\alpha)$ where $s = \frac{p}{q-p}$. Then (13) holds for any function $f$. 
1.2. Methods of proof. Our presentation is essentially based on discretization methods with some of our considerations being different from the ones in [6] where similar topics were considered in $\mathbb{R}^n$. For the proof of Theorem 1.1, we will prove the following inclusion,

$$\{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \omega, \alpha} f(z) > \lambda \} \subset \{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \omega, \alpha}^d f(z) > \frac{\lambda}{C_\alpha} \}$$

where $\mathcal{M}_{\Phi, \omega, \alpha}^d$ is the dyadic analogue of $\mathcal{M}_{\Phi, \omega, \alpha}$. We also use the usual covering of $\{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \omega, \alpha} f(z) > \frac{\lambda}{C_\alpha} \}$ and prove that under condition (4),

$$\mu \left( \{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \alpha} f(z) > \frac{\lambda}{C_\alpha} \} \right) \leq C \left( \{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \alpha}^d f(z) > \frac{\lambda}{C_\alpha} \} \right)^{q/p}.$$

In the proof of Theorem 1.2, we use a discretization of the integral and an extension of the Carleson Embedding Theorem. The Carleson Embedding Theorem needed here is a generalization of the classical one (see [5, 8]) as we replace the average

$$\frac{1}{|Q|_{\omega, \alpha}} \int_{Q} |f(z)| \omega(z) dV_{\alpha}(z)$$

by $\| f \|_{Q_I, \Phi, \omega, \alpha}$.

For the proof of Theorem 1.5 and Theorem 1.6, we also use discretization and the unweighted version of Theorem 1.1.

2. Proof of Theorem 1.1

For $\alpha > -1$, we say a weight $\omega$ is $\alpha$-doubling, if there are an increasing function $\varphi$ with $\varphi(1) = 1$ and a constant $K = K(\omega, \alpha) \geq 1$ such that for any interval $I \subset \mathbb{R}$ and any set $E \subset Q_I$,

$$\frac{|Q_I|_{\omega, \alpha}}{|E|_{\omega, \alpha}} \leq K \varphi \left( \frac{|Q_I|_{\alpha}}{|E|_{\alpha}} \right).$$

Let us start this section with the following lemma.

**Lemma 2.1.** Let $\alpha > -1$, let $\omega$ be a weight on $\mathcal{H}$, and assume that $\Phi$ is a Young function. Then for any compactly supported function $f$ and any $\lambda > 0$, there exists a family of maximal (with respect to inclusion) dyadic intervals $\{ I_j \}$ such that

$$\{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \omega, \alpha}^d f(z) > \lambda \} = \bigcup_j Q_{I_j}.$$

The above Carleson squares $Q_{I_j}$ are maximal such

$$\| f \|_{Q_{I_j}, \Phi, \omega, \alpha} > \lambda$$

for each $j$.

If $\omega$ is $\alpha$-doubling, then

$$\| f \|_{Q_{I_j}, \Phi, \omega, \alpha} \leq \varphi(2^{2\alpha+\alpha}) K \lambda$$

for each $j$ where $\varphi$ and $K$ are the function and the constant in (17). Moreover, for any weight $\omega$,

$$\left| \{ z \in \mathcal{H} : \mathcal{M}_{\Phi, \omega, \alpha}^d f(z) > \lambda \} \right|_{\omega, \alpha} \leq C \int_{\{ z \in \mathcal{H} : |f(z)| > \frac{\lambda}{\Phi} \}} \frac{|f(z)|}{\lambda} \omega(z) dV_{\alpha}(z).$$
Proof. That \( \{ z \in \mathcal{H} : \mathcal{M}_{\Phi,\omega,\alpha}^d f(z) > \lambda \} \) is a union of maximal dyadic Carleson squares \( Q_{I_j} \) such that \( \| f \|_{Q_{I_j},\Phi,\omega,\alpha} > \lambda \) follows from the usual arguments.

Now assume that \( \omega \) is \( \alpha \)-doubling. To see that \( \| f \|_{Q_{I},\Phi,\omega,\alpha} \leq K \varphi(2^{2+\alpha}) \lambda \), observe that if \( I \) is such that \( Q_I \) is one of the maximal Carleson squares above, then \( \| f \|_{Q_I,\Phi,\omega,\alpha} \leq \lambda \), where \( I \) is a parent of \( I \). Using the convexity of \( \Phi \), we obtain that for any \( t > 0 \),

\[
L := \frac{1}{|Q_I|_{\omega,\alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\varphi(2^{2+\alpha}) K t} \right) \omega(z) dV_\alpha(z)
\leq \frac{\varphi(2^{2+\alpha}) K}{|Q_I|_{\omega,\alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\varphi(2^{2+\alpha}) K t} \right) \omega(z) dV_\alpha(z)
\leq \frac{1}{|Q_I|_{\omega,\alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{t} \right) \omega(z) dV_\alpha(z).
\]

Thus \( \| f \|_{Q_I,\Phi,\omega,\alpha} \leq \varphi(2^{2+\alpha}) K \| f \|_{Q_I,\Phi,\omega,\alpha} \).

Note that as for any Carleson square \( Q_{I_j} \) in the above family, we have \( \| f \|_{Q_{I_j},\Phi,\omega,\alpha} > \lambda \), it comes that

\[
\frac{1}{|Q_{I_j}|_{\omega,\alpha}} \int_{Q_{I_j}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z) > 1 \text{ for all } j.
\]

That is

\[
|Q_{I_j}|_{\omega,\alpha} \leq \int_{Q_{I_j}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z) \text{ for all } j.
\]

Hence

\[
\left| \left\{ z \in \mathcal{H} : \mathcal{M}_{\Phi,\omega,\alpha}^d f(z) > \lambda \right\} \right|_{\omega,\alpha} = \left| \bigcup_j Q_{I_j} \right|_{\omega,\alpha} = \sum_j |Q_{I_j}|_{\omega,\alpha}
\leq \sum_j \int_{Q_{I_j}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z)
\leq \int_{\mathcal{H}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z).
\]

Now following the usual arguments, we write \( f = f_1 + f_2 \) where

\[
f_1 := f \chi_{\{z \in \mathcal{H} : |f(z)| > \frac{\lambda}{2}\}}.
\]

Then

\[
\mathcal{M}_{\Phi,\omega,\alpha}^d f(z) \leq \mathcal{M}_{\Phi,\omega,\alpha}^d f_1(z) + \mathcal{M}_{\Phi,\omega,\alpha}^d f_2(z) \leq \mathcal{M}_{\Phi,\omega,\alpha}^d f_1(z) + \frac{\lambda}{2}.
\]

It follows that

\[
\left| \left\{ z \in \mathcal{H} : \mathcal{M}_{\Phi,\omega,\alpha}^d f(z) > \lambda \right\} \right|_{\omega,\alpha} \leq \left| \left\{ z \in \mathcal{H} : \mathcal{M}_{\Phi,\omega,\alpha}^d f_1(z) > \frac{\lambda}{2} \right\} \right|_{\omega,\alpha}
\leq C \int_{\{z \in \mathcal{H} : |f(z)| > \frac{\lambda}{2}\}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z).
\]
The proof is complete.

Let us prove the following level sets embedding.

**Lemma 2.2.** Let $\alpha > -1$ and let $\Phi$ be a Young function. Assume that $\omega$ is an $\alpha$-doubling weight. Let $f$ be a locally integrable function. Then for any $\lambda > 0$,

\[
\{ z \in H : M_{\Phi,\omega,\alpha} f(z) > \lambda \} \subset \{ z \in \mathcal{H} M_{\Phi,\omega,\alpha}^d f(z) > \frac{\lambda}{C} \}
\]

with \( C = 2K\varphi(2^{2+\alpha})(1 + (K\varphi(2^{2+\alpha}))^2) \), where $\varphi$ and $K$ are the function and the constant in (15).

**Proof.** The proof is essentially the same as the one of Lemma 3.4 in [3]. We give here the main modifications. Let us put

\[
E_\lambda := \{ z \in H : M_{\Phi,\omega,\alpha} f(z) > \lambda \}
\]

and

\[
E_\lambda^d := \{ z \in H : M_{\Phi,\omega,\alpha}^d f(z) > \frac{\lambda}{C} \}.
\]

Recall that there is a family $\{Q_{I_j}\}_{j \in \mathbb{N}_0}$ of maximal (with respect to the inclusion) dyadic Carleson squares such that

\[
\frac{\lambda}{C} < \| f \|_{Q_{I_j},\Phi,\omega,\alpha} \leq K\varphi(2^{2+\alpha})\frac{\lambda}{C} \quad \text{for each } j
\]

$K$ being the constant in (15). Moreover, $E_\lambda^d = \bigcup_{j \in \mathbb{N}} Q_{I_j}$.

Let $z \in E_\lambda$ and suppose that $z \notin E_\lambda^d$. Note that there is an interval $I$ (not necessarily dyadic) such that $z \in Q_I$ and

\[
\| f \|_{Q_I,\Phi,\omega,\alpha} > \lambda.
\]

Recall with [3, Lemma 2.3.] that $I$ can be covered by at most two adjacent dyadic intervals $J_1$ and $J_2$ ($J_1$ on the left of $J_2$) such that $|I| < |J_1| = |J_2| \leq 2|I|$. Hence $Q_I \subset Q_{J_1} \cup Q_{J_2}$ and we have that $z$ belongs only to one and only one of the Carleson boxes $Q_{J_1}$ and $Q_{J_2}$. Let us suppose that $z \in Q_{J_1}$, (in which case $z \notin Q_{J_2}$). Then $Q_{J_1} \cap E_\lambda^d = \emptyset$ or $Q_{J_1} \supset Q_{I_j}$ for some $j$ and in both cases, because of the maximality of the $I_j$s, we obtain that

\[
\| f \|_{Q_{J_1},\Phi,\omega,\alpha} \leq \frac{\lambda}{C}.
\]

For the other interval $J_2$, we have the following possibilities

- $J_2 = I_j$ for some $j$;
- $J_2 \subset I_j$ for some $j$;
- $J_2 \supset I_j$ for some $j$;
- $J_2 \cap B = \emptyset$.

If $J_2 \supset I_j$ for some $j$ or $J_2 \cap E_\lambda^d = \emptyset$, then because of the maximality of the $I_j$s,

\[
\| f \|_{Q_{J_2},\Phi,\omega,\alpha} \leq \frac{\lambda}{C}.
\]
If \( J_2 = I_j \) for some \( j \), then
\[
\|f\|_{Q_{J_2}, \Phi, \omega, \alpha} \leq \frac{K \lambda \varphi(2^{2+\alpha})}{C}.
\]

If \( J_2 \subset I_j \) for some \( j \), then
\[
J_2 = I_j^-, \quad J_2 \subset I_j^- \text{ or } J_2 \subset I_j^+
\]
where \( I_j^- \) and \( I_j^+ \) denote the left and right halfs of \( I_j \) respectively. If \( J_2 \subset I_j^- \) or \( J_2 \subset I_j^+ \), then \( J_1 \cap I_j \neq \emptyset \), and this implies that \( J_1 \subset I_j \). Thus \( z \in Q_{J_1} \subset Q_{I_j} \subset E_{\lambda}^2 \) which contradicts the hypothesis \( z \notin E_{\lambda} \). Hence we can only have \( J_2 = I_j^- \) which leads to the estimate
\[
\|f\|_{Q_{J_2}, \Phi, \omega, \alpha} \leq \frac{(K \varphi(2^{2+\alpha}))^2}{C} \lambda.
\]

It follows from that above discussion and the convexity of \( \Phi \) that
\[
L := \frac{1}{|Q_{J_1}|_{\omega, \alpha}} \int_{Q_{J_1}} \Phi \left( \frac{|f|}{2K \varphi(2^{2+\alpha})} \left( \frac{1}{\lambda} + \frac{(K \varphi(2^{2+\alpha}))^2}{C} \right) \lambda \right) \omega dV_{\alpha}
\]
\[
\leq \frac{|Q_{J_1}|_{\omega, \alpha}}{|Q_{I_j}|_{\omega, \alpha}} \left( \frac{1}{|Q_{J_1}|_{\omega, \alpha}} \int_{Q_{J_1}} \Phi \left( \frac{|f|}{2K \varphi(2^{2+\alpha})} \right) \lambda \omega dV_{\alpha} \right) + \frac{|Q_{J_2}|_{\omega, \alpha}}{|Q_{J_1}|_{\omega, \alpha}} \left( \frac{1}{|Q_{J_2}|_{\omega, \alpha}} \int_{Q_{J_2}} \Phi \left( \frac{|f|}{2K \varphi(2^{2+\alpha})} \right) \lambda \omega dV_{\alpha} \right)
\]
\[
\leq \frac{K \varphi(2^{2+\alpha})}{|Q_{J_1}|_{\omega, \alpha}} \int_{Q_{J_1}} \Phi \left( \frac{|f|}{2K \varphi(2^{2+\alpha})} \right) \lambda \omega dV_{\alpha} + \frac{K \varphi(2^{2+\alpha})}{|Q_{J_2}|_{\omega, \alpha}} \int_{Q_{J_2}} \Phi \left( \frac{|f|}{2K \varphi(2^{2+\alpha})} \right) \lambda \omega dV_{\alpha}
\]
\[
\leq \frac{1}{2} \left( \frac{1}{|Q_{J_1}|_{\omega, \alpha}} \int_{Q_{J_1}} \Phi \left( \frac{|f|}{\lambda} \right) \lambda \omega dV_{\alpha} \right) + \frac{1}{2} \left( \frac{1}{|Q_{J_2}|_{\omega, \alpha}} \int_{Q_{J_2}} \Phi \left( \frac{|f|}{(K \varphi(2))^{3/2}} \right) \lambda \omega dV_{\alpha} \right)
\]
\[
\leq 1.
\]
Thus
\[
\|f\|_{Q_{J_1}, \Phi, \omega, \alpha} \leq 2\lambda \left( \frac{K \varphi(2^{2+\alpha})}{C} + \frac{(K \varphi(2^{2+\alpha}))^3}{C} \right) = \lambda.
\]

which clearly contradicts (17). The proof is complete. \( \square \)

We need the following estimate.

**Lemma 2.3.** Let \( \gamma \geq 1 \), and \( \alpha > -1 \). Let \( \sigma \) and \( \omega \) be weights, and \( \mu \) a positive measure on \( \mathcal{H} \). Assume that there is a constant \( C > 0 \) such that for any interval \( I \subset \mathbb{R} \),
\[
\mu(Q_I) \leq C|Q_I|_{\omega, \alpha}.
\]
Then for any function $f$ and any $t > 0$,

$$
\mu \left( \{ z \in \mathcal{H} : M_{\Phi, \sigma, \alpha}^d f(z) > t \} \right) \leq C \left( \{ z \in \mathcal{H} : M_{\Phi, \sigma, \alpha}^d f(z) > t \} \right)^{\gamma}.
$$

 Proof. Recall with Lemma 2.1 that there is family $\{I_j\}_j$ of dyadic maximal intervals such that

$$
\{ z \in \mathcal{H} : M_{\Phi, \sigma, \alpha}^d f(z) > t \} = \bigcup_j I_j.
$$

Thus

$$
\mu \left( \{ z \in \mathcal{H} : M_{\Phi, \sigma, \alpha}^d f(z) > t \} \right) = \sum_j \mu(I_j)
$$

$$
\leq C \sum_j |Q_{I_j}|_{\omega, \alpha}^{\gamma}
$$

$$
\leq C \left( \sum_j |Q_{I_j}|_{\omega, \alpha} \right)^{\gamma}
$$

$$
= C \left( \{ z \in \mathcal{H} : M_{\Phi, \sigma, \alpha}^d f(z) > t \} \right)^{\gamma}.
$$

2.1. A first proof of Theorem 1.1. The following result extends the classical estimate of the weighted Hardy-Littlewood maximal function.

**Lemma 2.4.** Let $1 < p < \infty$ and $\alpha > -1$. Assume that $\omega$ is a weight and $\Phi \in B_p$. Then there is a positive constant $C$ such that for any function $f$,

$$
\left( \int_\mathcal{H} (M_{\Phi, \omega, \alpha}^d f(z))^p \omega(z) dV_\alpha(z) \right)^{1/p} \leq C \left( \int_\mathcal{H} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{1/p}.
$$

Proof. Using the last part in Lemma 2.1 and that $\Phi \in B_p$, we obtain

$$
\| M_{\Phi, \omega, \alpha}^d f \|^p_{p, \omega, \alpha} = \int_0^\infty p\lambda^{p-1} \left| \{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > \lambda \} \right|_{\omega, \alpha} d\lambda
$$

$$
\leq \int_0^\infty p\lambda^{p-1} \left( \int_{\{ z \in \mathcal{H} : |f(z)| > \frac{1}{2} \}} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV_\alpha(z) \right) d\lambda
$$

$$
= p \int_{\mathcal{H}} \int_0^{2|f(z)|} \lambda^p \Phi \left( \frac{|f(z)|}{\lambda} \right) \frac{d\lambda}{\lambda^p} \omega(z) dV_\alpha(z)
$$

$$
= p \int_{\mathcal{H}} |f(z)|^p \left( \int_{1/2}^{\infty} \Phi(\lambda) \frac{d\lambda}{\lambda^p} \right) \omega(z) dV_\alpha(z)
$$

$$
= C \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z).
$$

Here $C = p \int_{1/2}^\infty \Phi(\lambda) \frac{d\lambda}{\lambda^p}$. \qed

Any weight in the Békollé-Bonami class, $B_{p, \alpha}$, $1 < p < \infty$ is an $\alpha$-doubling weight.

**Lemma 2.5.** Let $1 < p < \infty$ and $\alpha > -1$. Then any weight $\omega \in B_{p, \alpha}$ is $\alpha$-doubling, with doubling function $\varphi(t) = t^p$ and constant $K = |\omega| \mathcal{B}_{p, \alpha}$.\qed
Proof. Using Hölder’s inequality and the definition of the class $B_{p,\alpha}$, we obtain for any $E \subset Q_I$,

$$
\frac{|E|_\alpha}{|Q_I|_\alpha} \leq \frac{(\int_E \omega(z)d\nu_\alpha(z))^{1/p} \left( \int_{Q_I} \omega(z)^{-\frac{\alpha}{\alpha'+p}}d\nu_\alpha(z) \right)^{1/p'}}{|Q_I|_\alpha}
$$

$$
\leq \left( \frac{|E|_{\omega,\alpha}}{|Q_I|_{\omega,\alpha}} \right)^{1/p} \left( \frac{\int_{Q_I} \omega(z)d\nu_\alpha(z)}{|Q_I|_\alpha} \right)^{1/p'}
$$

$$
\leq \left( \frac{|E|_{\omega,\alpha}}{|Q_I|_{\omega,\alpha}} C_{p,\alpha} \right)^{1/p}.
$$

Hence

$$
\frac{|Q_I|_{\omega,\alpha}}{|E|_{\omega,\alpha}} \leq \left( \frac{|Q_I|_\alpha}{|E|_\alpha} \right)^p \omega_{B_{p,\alpha}}.
$$

□

Let us now prove Theorem 1.1.

Proof of Theorem 1.1. We start by observing that for any interval $I \subset \mathbb{R}$, $\||\chi_{Q_I}\|_{Q_I,\Phi,\omega,\alpha} = 1$. Thus taking $f = \chi_{Q_I}$, we obtain

$$
\mu(Q_I)^{1/q} \leq \left( \int_{Q_I} (M_{\Phi,\omega,\alpha}(\chi_{Q_I})(z))^q d\mu(z) \right)^{1/q} \leq C \|\chi_{Q_I}\|_{p,\omega,\alpha} = C|Q_I|_{\omega,\alpha}^{1/p}
$$

which leads to (4).

Conversely, assume that (4) holds. Using Lemma 2.5 Lemma 2.2, we first obtain

$$
L := \int_{\mathcal{H}} (M_{\Phi,\omega,\alpha}f(z))^q d\mu(z)
$$

$$
= \int_0^\infty q \lambda^{q-1} \mu \{ \{ z \in \mathcal{H} : M_{\Phi,\omega,\alpha}f(z) > \lambda \} \} d\lambda
$$

$$
\leq \int_0^\infty q \lambda^{q-1} \mu \{ \{ z \in \mathcal{H} : M_{\Phi,\omega,\alpha}^d f(z) > \lambda \} \}^{q/p} d\lambda
$$

$$
\leq \int_0^\infty q \lambda^{q-1} \left[ \left\{ z \in \mathcal{H} : M_{\Phi,\omega,\alpha}^d f(z) > \lambda \right\} \right]^{q/p} \omega_{\omega,\alpha} d\lambda
$$

$$
\leq C^{q-p} \|M_{\Phi,\omega,\alpha}^d f\|\omega,\alpha^{q-p} \int_0^\infty q \lambda^{p-1} \left[ \left\{ z \in \mathcal{H} : M_{\Phi,\omega,\alpha}^d f(z) > \lambda \right\} \right]^{q/p} \omega_{\omega,\alpha} d\lambda
$$

$$
= C^{q-p} \|M_{\Phi,\omega,\alpha}^d f\|\omega,\alpha^{q-p}.
$$

Where we have also used the inequality

$$
t^p \left[ \left\{ z \in \mathcal{H} : M_{\Phi,\omega,\alpha}^d f(z) > t \right\} \right]_{\omega,\alpha} \leq \|M_{\Phi,\omega,\alpha}^d f\|\omega,\alpha^{p}.
We easily conclude with the help of Lemma 2.4 that
\[
\left( \int_{\mathcal{H}} (\mathcal{M}_{\Phi,\omega,\alpha} f(z))^q \, d\mu(z) \right)^{1/q} \leq C \left\| \mathcal{M}_{\Phi,\omega,\alpha} f \right\|_{p,\omega,\alpha}^{1/p} \leq C \left( \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{1/p}.
\]

The proof is complete. □

In the case 1 < p = q < ∞, ω = 1 and µ = V_α, we obtain as a consequence, the following characterization of weighted Bergman spaces of the upper-half plane.

**Corollary 2.6.** Let 1 < p < ∞, and α > −1. Assume Φ ∈ B_p. Then for any analytic function f on H, the following are equivalent.

(i) f ∈ L^p(\mathcal{H}, dV_\alpha).

(ii) \(\mathcal{M}_{\Phi,\alpha} f \in L^p(\mathcal{H}, dV_\alpha)\).

**Proof.** That (i)⇒ (ii) is a special case of Theorem 1.1. To see that (ii)⇒ (i), observe that the Mean Value Theorem applied to the disc inscribed in the Carleson box Q_I with centre z ∈ H allows one to see that there is a constant C > 0 such that
\[
|f(z)| \leq C |Q_I|_\alpha \int_{Q_I} |f(w)| dV_\alpha(w).
\]

It is then enough to prove that there is a constant K > 0 such that for any interval I ⊂ \mathbb{R},
\[
\frac{1}{|Q_I|_\alpha} \int_{Q_I} |f(w)| dV_\alpha(w) \leq K \left\| f \right\|_{Q_I,\Phi,\alpha}.
\]

Using the convexity of Φ, Jensen’s inequality, and putting λ = \left\| f \right\|_{Q_I,\Phi,\alpha}, we easily obtain
\[
\Phi \left( \int_{Q_I} \frac{|f(w)|}{\lambda} dV_\alpha(w) \right) |Q_I|_\alpha \leq \int_{Q_I} \Phi \left( \frac{|f(w)|}{\lambda} \right) \, dV_\alpha(w) \leq 1.
\]

Hence
\[
\frac{1}{|Q_I|_\alpha} \int_{Q_I} |f(w)| dV_\alpha(w) \leq \lambda \Phi(1) = \lambda.
\]

The proof is complete. □

### 2.2. A second proof of Theorem 1.1

We consider the following system of dyadic grids,

\[\mathcal{D}^3 := \{2^j ([0, 1) + m + (-1)^j \beta) : m \in \mathbb{Z}, \, j \in \mathbb{Z}\}, \text{ for } \beta \in \{0, 1/3\}.\]

For \(\beta = 0\), \(\mathcal{D}^0\) is the standard dyadic grid of \(\mathbb{R}\), simply denoted \(\mathcal{D}\).

We recall with [7] that given an interval I ⊂ \(\mathbb{R}\), there is a dyadic interval J ∈ \(\mathcal{D}^3\) for some \(\beta \in \{0, 1/3\}\) such that I ⊆ J and |J| ≤ 6|I|. It follows that
if \( \omega \) is \( \alpha \)-doubling then in particular, we have that \( \|Q_I\|_{\omega, \alpha} \leq \varphi(6)K|Q_I|_{\omega, \alpha} \) where \( \varphi \) and \( K \) are given in (13). Hence putting \( \lambda := \|f\|_{Q_I, \Phi, \omega, \alpha} \), we obtain

\[
\frac{1}{|Q_I|_{\omega, \alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\varphi(6)K\lambda} \right) \omega(z) dV(z) \leq \frac{\varphi(6)K}{|Q_I|_{\omega, \alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\varphi(6)K\lambda} \right) \omega(z) dV(z) \\
\leq \frac{1}{|Q_I|_{\omega, \alpha}} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV(z) \leq 1.
\]

Thus for any locally integrable function \( f \),

\[
\mathcal{M}_{\Phi, \omega, \alpha} f(z) \leq C \sum_{\beta \in \{0,1/3\}} \mathcal{M}_{\Phi, \omega, \alpha}^{d, \beta} f(z), \quad z \in \mathcal{H}
\]

where \( \mathcal{M}_{\Phi, \omega, \alpha}^{d, \beta} \) is defined as \( \mathcal{M}_{\Phi, \omega, \alpha} \) but with the supremum taken only over dyadic intervals of the dyadic grid \( D^\beta \). When \( \omega \equiv 1 \), we use the notation \( \mathcal{M}_{\Phi, \alpha}^{d, \beta} \), and if moreover, \( \beta = 0 \), we just write \( \mathcal{M}_{\Phi, \alpha}^{d} \).

It follows from the above observation that to prove that (3) holds, it is enough to prove that this inequality holds for \( \mathcal{M}_{\Phi, \omega, \alpha}^{d, \beta} \), \( \beta = 0, 1/3 \).

We define \( Q^\beta \) by

\[
Q^\beta := \{Q = Q_I : I \in D^\beta\}.
\]

**Definition 2.7.** Let \( \alpha > -1 \) and \( \omega \) be a weight. For any \( \gamma \geq 1 \), a sequence of positive numbers \( \{\lambda_Q\}_{Q \in Q^\gamma} \) is called a \((\omega, \alpha, \gamma)\)-Carleson sequence, if there is a constant \( C > 0 \) such that for any \( R \in Q^\gamma \),

\[
\sum_{Q \subseteq R} \lambda_Q \leq C|R|^{\gamma}_{\omega, \alpha}.
\]

For \( Q_I \), we denote by \( T_I \) its upper half. That is

\[
T_I := \{z = x + iy \in \mathbb{C} : x \in I \text{ and } \frac{|I|}{2} < y < |I|\}.
\]

Recall that for \( 1 < p < \infty \) and \( \alpha > -1 \), a weight \( \omega \) is said to belong to the Békollé-Bonami class \( B_{p, \alpha} \), if \( [\omega]_{B_{p, \alpha}} < \infty \), where

\[
[w]_{B_{p, \alpha}} := \sup_{I \subset \mathbb{R}, I \text{ interval}} \left( \frac{1}{|Q_I|_{\alpha}} \int_{Q_I} \omega(z) dV_{\alpha}(z) \right) \left( \frac{1}{|Q_I|_{\alpha}} \int_{Q_I} \omega(z)^{-1/p'} dV_{\alpha}(z) \right)^{p-1}.
\]

For \( p = \infty \), we say \( \omega \in B_{\infty, \alpha} \), if

\[
[w]_{B_{\infty, \alpha}} := \sup_{I \subset \mathbb{R}, I \text{ interval}} \frac{1}{|Q_I|_{\omega, \alpha}} \int_{Q_I} \mathcal{M}_{\alpha}(\omega\chi_Q_I) dV_{\alpha}(z) < \infty.
\]

The following is proved in [1].

**Lemma 2.8.** Let \( \alpha > -1 \), and \( 1 < p < \infty \). Then the class \( B_{\infty, \alpha} \) contains \( B_{p, \alpha} \). Moreover, for any weight \( \omega \in B_{p, \alpha} \),

\[
[w]_{B_{\infty, \alpha}} \leq [w]_{B_{p, \alpha}}.
\]

One can also prove that for any weight \( \omega \in B_{\infty, \alpha} \), the sequence \( \{|Q_I|_{\omega, \alpha}\}_{I \in D} \) is a \((\omega, \alpha, 1)\)-Carleson sequence.
**Lemma 2.9.** Let $\alpha > -1$, and $\omega \in B_{\infty, \alpha}$. Then for any $J \in D^\beta$, 
\[
\sum_{I \subseteq J, I \in D^\beta} |Q_I|_{\omega, \alpha} \leq C_\alpha [\omega]_{B_{\infty, \alpha}} |Q_J|_{\omega, \alpha}.
\]

**Proof.** We have 
\[
\sum_{I \subseteq J, I \in D^\beta} |Q_I|_{\omega, \alpha} = \sum_{I \subseteq J, I \in D^\beta} \frac{|Q_I|_{\omega, \alpha}}{|Q_I|_{\alpha}} |Q_I|_{\alpha} = C_\alpha \sum_{I \subseteq J, I \in D^\beta} \frac{|Q_I|_{\omega, \alpha}}{|Q_I|_{\alpha}} |T_I|_{\alpha} \\
\leq C_\alpha \sum_{I \subseteq J, I \in D^\beta} \int_{T_I} M_\alpha (\omega \chi_{Q_I}) dV_\alpha \\
\leq C_\alpha [\omega]_{B_{\infty, \alpha}} |Q_J|_{\omega, \alpha}.
\]

□

**Theorem 2.10.** Let $\omega$ be a weight on $H$ and $\gamma \geq 1$. Assume \{$\lambda_Q$\}_{Q \in Q^b} is a sequence of positive numbers. Assume that there exists some constant $A > 0$ such that for any Carleson square $Q^0 \in Q^b$, 
\[
\sum_{Q \subseteq Q^0, Q \in Q^b} \lambda_Q \leq A |Q^0|_{\omega, \alpha}^\gamma.
\]

Then there exists a constant $B > 0$ such that for all $p \in (1, \infty)$, and any $\Phi \in B_p$, 
\[
\sum_{Q, Q \in Q^b} \lambda_Q \|f\|_{Q, \Phi, \omega, \alpha}^p \leq B \|M_{\Phi, \omega, \alpha} f\|_{p, \omega, \alpha}^p.
\]

**Proof.** For simplicity of presentation, we assume that $\beta = 0$. We will also need the following inequality.
\[(20)\]
\[
\lambda^p \{z \in H : M_{\Phi, \omega, \alpha} f(z) > \lambda\} \leq |M_{\Phi, \omega, \alpha} f\|_{p, \omega, \alpha}^p.
\]

We can suppose that $f > 0$. As in the case of $\Phi = 1$ in \[8\], we read \[
\sum_{Q \in \mathcal{Q}} \lambda_Q \|f\|_{Q, \Phi, \omega, \alpha}^p \leq \|M_{\Phi, \omega, \alpha} f\|_{p, \omega, \alpha}^p.
\]

Let $Q^*_t$ be the set of maximal dyadic Carleson squares $R$ with respect to the inclusion so that $\|f\|_{R, \Phi, \omega, \alpha} > t$. Then
\[
\bigcup_{R \in Q^*_t} R = \{z \in H : M_{\Phi, \omega, \alpha} f(z) > t\}.
\]
It follows from the hypothesis on the sequence \( \{ \lambda_Q \}_{Q \in \mathcal{Q}} \) that
\[
\mu(Q_t) = \sum_{Q \in \mathcal{Q}_t} \lambda_Q \leq \sum_{R \in \mathcal{Q}_t^*} \sum_{Q \subseteq R} \lambda_Q \\
\leq A \sum_{R \in \mathcal{Q}_t^*} |R|_{\omega, \alpha}^\gamma \\
\leq A \left\{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > t \right\}_{\omega, \alpha}^\gamma.
\]
Hence using (20), we obtain
\[
S := \sum_{Q \in \mathcal{Q}} \lambda_Q \| f \|_{Q, \Phi, \omega, \alpha}^p \\
\leq A \int_0^\infty p \gamma t^{p\gamma - 1} \left\{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > t \right\}_{\omega, \alpha}^\gamma dt \\
= A \int_0^\infty p \gamma t^{p - 1} \left\{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > t \right\}_{\omega, \alpha} \\
\left( t^p \left\{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > t \right\}_{\omega, \alpha} \right)^{\gamma - 1} dt \\
\leq A \gamma \| M_{\Phi, \omega, \alpha}^d \|_{p, \omega, \alpha}^{p(\gamma - 1)} \int_0^\infty p \{ z \in \mathcal{H} : M_{\Phi, \omega, \alpha}^d f(z) > t \}_{\omega, \alpha} t^{p - 1} dt \\
\leq A \gamma \| M_{\Phi, \omega, \alpha}^d \|_{p, \omega, \alpha}^{p \gamma}.
\]
The proof is complete. \( \square \)

We can now present our second proof of Theorem 1.1.

**Proof of Theorem 1.1.** We only check that (3) holds under (4). As observed above, it is enough to prove that there exists a positive constant \( C \) such that for any \( \beta \in \{0, 1/3\} \), and any function \( f \),
\[
\int_{\mathcal{H}} (M_{\Phi, \omega, \alpha}^d f(z))^q d\mu(z) \leq C \left( \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{q/p}.
\]
This will follow from the following result.

**Lemma 2.11.** Let \( \alpha > -1, 1 < p \leq q < \infty, \omega \in B_{\infty, \alpha}, \) and \( \mu \) a positive Borel measure on \( \mathcal{H} \). Assume that \( \Phi \in B_p \) and that (4) holds. Then there exists a positive constant \( C \) such that for any \( \beta \in \{0, 1/3\} \) and any function \( f \),
\[
\int_{\mathcal{H}} (M_{\Phi, \omega, \alpha}^d f(z))^q d\mu(z) \leq C \left( \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_\alpha(z) \right)^{q/p}.
\]

**Proof.** Let \( a \geq 2 \). To each integer \( k \), we associate the level set
\[
\Omega_k := \{ z \in \mathcal{H} : a^k < M_{\Phi, \omega, \alpha}^d f(z) \leq a^{k+1} \}.
\]
We observe with Lemma 2.11 that \( \Omega_k \subset \cup_{j=1}^\infty Q_{I_{k,j}} \), where \( Q_{I_{k,j}} (I_{k,j} \in D^\beta) \) is a dyadic Carleson square maximal (with respect to the inclusion) such that \( \| f \|_{Q_{I_{k,j}}, \Phi, \omega, \alpha} > a^k \).
Using the above observations and condition (4), we obtain
\[
\int_{\mathcal{H}} (\mathcal{M}_{\Phi,\omega,\alpha}^{d,\beta} f(z))^q d\mu(z) = \sum_k \int_{\Omega_k} (\mathcal{M}_{\Phi,\omega,\alpha}^{d,\beta} f(z))^q d\mu(z) \\
\leq a^q \sum_k a^{kq} \mu(\Omega_k) \\
\leq a^q \sum_{k,j} a^{kq} \mu(Q_{k,j}) \\
\leq a^q \sum_{k,j} \|f\|^q_{q_{k,j},\Phi,\omega,\alpha} \mu(Q_{k,j}) \\
\leq C \sum_{k,j} \|f\|^q_{q_{k,j},\Phi,\omega,\alpha} |Q_{k,j}|^{q/p} \\
\leq C \left( \sum_{k,j} \|f\|^p_{p_{k,j},\Phi,\omega,\alpha} |Q_{k,j}|^{w_{\alpha}} \right)^{q/p}.
\]

As \( \omega \in \mathbb{B}_{\infty,\alpha} \), we have that the sequence
\[
(21) \quad \lambda_Q = \begin{cases} |Q|^{w_{\alpha}} & \text{if } Q = Q_{k,j} \text{ for some } k,j \\ 0 & \text{otherwise} \end{cases}
\]
is a \((\omega, 1, \alpha)\)-Carleson sequence. Hence using Theorem 2.10 and Lemma 2.4, we obtain
\[
\int_{\mathcal{H}} (\mathcal{M}_{\Phi,\omega,\alpha}^{d,\beta} f(z))^q d\mu(z) \leq C \left( \sum_{k,j} \|f\|^p_{p_{k,j},\Phi,\omega,\alpha} |Q_{k,j}|^{w_{\alpha}} \right)^{q/p}.
\]
\[
\int_{\mathcal{H}} |f(z)|^p \omega(z) dV_{\alpha}(z) \right)^{q/p}.
\]

The proof is complete.

3. Proof of Theorem 1.2

Let us start by proving the following result.

**Lemma 3.1.** Let \( 1 < q < p < \infty \). Let \( \omega \in \mathbb{B}_{\infty,\alpha} \), and \( \mu \) be a positive measure on \( \mathcal{H} \). Assume that \( \Phi \in \mathcal{B}_p \) and that the function

\[
(22) \quad K_p(z) := \sup_{I \subseteq \mathbb{R}} \frac{\mu(Q_I)}{|Q_I|^{w_{\alpha}}}
\]

belongs to \( L^s(\mathcal{H}, \omega dV_{\alpha}) \) where \( s = \frac{p}{p-q} \). Then there is a positive constant \( C > 0 \) such that for any \( f \in L^p(\mathcal{H}, \omega dV_{\alpha}) \), and any \( \beta \in \{0, \frac{1}{3}\} \),

\[
(23) \quad \left( \int_{\mathcal{H}} (\mathcal{M}_{\Phi,\omega,\alpha}^{d,\beta} f(z))^q d\mu(z) \right)^{1/q} \leq C \left( \int_{\mathcal{H}} |f(z)|^p \omega(z) dV_{\alpha}(z) \right)^{1/p}.
\]
Proof. Let $a \geq 2$. To each integer $k$, we associate the set
\[
\Omega_k := \{ z \in \mathcal{H} : a^k < \mathcal{M}^{d,\beta}_{\Phi,\omega,\alpha} f(z) \leq a^{k+1} \}.
\]
We recall with Lemma 2.1 that $\Omega_k \subset \bigcup_{j=1}^{\infty} Q_{I_{k,j}}$, where $Q_{I_{k,j}} (I_{k,j} \in \mathcal{D}^\beta)$ is a dyadic cube maximal (with respect to the inclusion) such that
\[
\|f\|_{Q_{I_{k,j}},\Phi,\omega,\alpha} > a^k.
\]
For $\beta \in \{0, \frac{1}{3}\}$, we define
\[
K_{d,\mu}^{\beta}(z) := \sup_{I \in \mathcal{D}^\beta, z \in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)}.
\]
Then $K_{d,\mu}^{\beta}(z) \leq K_{\mu}(z)$ for any $z \in \mathcal{H}$. Hence, that $K_{\mu}(z)$ belongs to $L^{p/(p-q)}(\mathcal{H}, \omega dV_{\alpha})$ implies that $K_{d,\mu}^{\beta} \in L^{p/(p-q)}(\mathcal{H}, \omega dV_{\alpha})$.

Proceeding as in the second proof of Theorem 1.1, we obtain at first that
\[
\int_{\mathcal{H}} (\mathcal{M}^{d,\beta}_{\Phi,\omega,\alpha} f(z))^q d\mu(z) = \sum_k \int_{\Omega_k} (\mathcal{M}^{d,\beta}_{\Phi,\omega,\alpha} f(z))^q d\mu(z) \leq a^q \sum_{k,j} \|f\|^q_{Q_{I_{k,j}},\Phi,\omega,\alpha} \mu(Q_{I_{k,j}}).
\]
Now using Hölder’s inequality, we obtain
\[
\int_{\mathcal{H}} (\mathcal{M}^{d,\beta}_{\Phi,\omega,\alpha} f(z))^q d\mu(z) \leq a^q \sum_{k,j} \|f\|^q_{Q_{I_{k,j}},\Phi,\omega,\alpha} \mu(Q_{I_{k,j}}) \leq a^q A^{q/p} B^{1/s}
\]
where
\[
A := \sum_{k,j} \|f\|^p_{Q_{I_{k,j}},\Phi,\omega,\alpha} |Q_{I_{k,j}}|_{\omega,\alpha}
\]
and
\[
B := \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^s |Q_{I_{k,j}}|_{\omega,\alpha}.
\]
The estimate of $A$ is already obtained in the second proof of Theorem 1.1. Let us estimate $B$. As $\omega$ is $\alpha$-doubling, we obtain

$$B := \left( \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|} \right)^s |Q_{I_{k,j}}| \omega,\alpha \right)^s \lesssim \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|} \right)^s |T_{I_{k,j}}| \omega,\alpha \lesssim \sum_{k,j} \int_{T_{I_{k,j}}} \left( K_{\beta,\mu}^\beta(z) \right)^s \omega(z) dV_\alpha(z) \leq \int_{\mathcal{H}} \left( K_{\beta,\mu}^\beta(z) \right)^s \omega(z) dV_\alpha(z) = C \|K_{\beta,\mu}^\beta\| s,\omega,\alpha < \infty.$$ 

The proof of the lemma is complete. \(\square\)

We can now prove the Theorem 1.2.

**Proof of Theorem 1.2.** The proof of the sufficiency follows from Lemma 3.1 and the observations made at the beginning of this section. Let us prove the necessity. For this, we first check that the Hardy-Littlewood maximal function is pointwise dominated by the generalized maximal function. Indeed, let $I \subset \mathbb{R}$ be an interval and put $\lambda := \|f\|_{Q_I,\Phi,\omega,\alpha}$. Using the convexity of $\Phi$, we obtain

$$\Phi \left( \frac{1}{|Q_I|} \int_{Q_I} \frac{|f(z)|}{\lambda} \omega(z) dV(z) \right) \leq \frac{1}{|Q_I|} \int_{Q_I} \Phi \left( \frac{|f(z)|}{\lambda} \right) \omega(z) dV(z) \leq 1.$$ 

Thus for any $z \in \mathcal{H}$, and for any interval $I \subset \mathbb{R}$ such that $z \in Q_I$,

$$\frac{1}{|Q_I|} \int_{Q_I} |f(w)| \omega(w) dV(w) \leq \|f\|_{Q_I,\Phi,\omega,\alpha}.$$ 

Hence

$$\mathcal{M}_{\omega,\alpha} f(z) \leq \mathcal{M}_{\Phi,\omega,\alpha} f(z)$$

for any locally integrable function $f$. It follows that if (3) holds for the operator $\mathcal{M}_{\Phi,\omega,\alpha}$, it also holds for $\mathcal{M}_{\Phi,\alpha}$. That (3) holds for $\mathcal{M}_{\Phi,\alpha}$ implies that the function given by (5) belongs to $L^{p/(p-q)}(\mathcal{H},\omega dV_\alpha)$ is proved in 3. The proof is complete. \(\square\)

### 4. Proof of Theorem 1.3

We will be using discretization once more.

**Proof.** As seen before, we only need to establish the inequality (10) for the dyadic maximal function $\mathcal{M}_{d,\alpha}^{\beta,\beta}$, $\beta \in \{0, 1/3\}$.

Let $a \geq 2$. To each integer $k$, we associate the set

$$\Omega_k := \{ z \in \mathcal{H} : a^k < \mathcal{M}_{d,\alpha}^{\beta} f(z) \leq a^{k+1} \}.$$
As a special case of Lemma 2.1, we have that $\Omega_k \subset \bigcup_{j=1}^{\infty} Q_{I_{k,j}}$, where $Q_{I_{k,j}}$ ($I_{k,j} \in D_\beta$) is a dyadic cube maximal (with respect to the inclusion) such that
\[
\frac{1}{|Q_{I_{k,j}}|} \int_{Q_{I_{k,j}}} |f(z)|dV_\alpha(z) > a^k.
\]

Proceeding as in the second proof of Theorem 1.1, we obtain
\[
\int_{\mathcal{H}} (\mathcal{M}_{\alpha}^{d,\beta} f(z))^q d\mu(z) = \sum_k \int_{\Omega_k} (\mathcal{M}_{\alpha}^{d,\beta} f(z))^q d\mu(z)
\]
\[
\leq a^q \sum_k a^{kq} \mu(\Omega_k)
\]
\[
\leq a^q \sum_{k,j} a^{kq} \mu(Q_{I_{k,j}})
\]
\[
\leq a^q \sum_{k,j} \left( \frac{1}{|Q_{I_{k,j}}|} \int_{Q_{I_{k,j}}} |f(z)|dV_\alpha(z) \right)^q \mu(Q_{I_{k,j}}).
\]

Now let $\Psi$ be the complementary function of the Young function $\Phi$. Recall the following Hölder’s inequality:
\[
(24) \quad \frac{1}{|Q_1|} \int_{Q_1} |(fg)(z)|dV_\alpha(z) \leq \|f\|_{Q_1,\Phi,\alpha} \|g\|_{Q_1,\Psi,\alpha}
\]

Using the above generalized Hölder’s inequality and (9), we obtain
\[
\int_{\mathcal{H}} (\mathcal{M}_{\alpha}^{d,\beta} f(z))^q d\mu(z) \leq a^q \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha} \|f\|_{Q_{I_{k,j}},\Psi,\alpha}^{q/p} \mu(Q_{I_{k,j}}).
\]
It follows using Lemma 2.4 that

$$\int_{\mathcal{H}} (M_{\alpha}^{d,\beta} f(z))^q d\mu(z) \leq C \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^q |Q_{I_{k,j}}|^{q/p}$$

$$\leq C \left( \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^p |Q_{I_{k,j}}|^{q/p} \right)^{q/p}$$

$$\leq C \left( \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^p T_{I_{k,j}} |^{q/p} \right)^{q/p}$$

$$= C \left( \int_{\mathcal{H}} (M_{\Phi,\alpha}^{d,\beta} (\omega f)(z))^p dV_{\alpha} \right)^{q/p}$$

$$\leq C \left( \int_{\mathcal{H}} |(\omega f)(z)|^p dV_{\alpha}(z) \right)^{q/p}.$$

The proof is complete. □

5. Proof of Theorem 1.5 and Theorem 1.6

To prove the inequality (13), again, we only need to prove that the same inequality holds when $M_{\beta}$ is replaced by its dyadic counterparts $M_{\Phi,\alpha,\beta}$. Recall that if $a \geq 2$, then we associate to each integer $k$, the set

$$\Omega_k := \{ z \in \mathcal{H} : a^k < M_{\Phi,\alpha} f(z) \leq a^{k+1} \}$$

and we have that $\Omega_k \subset \bigcup_{j=1}^\infty Q_{I_{k,j}}$, where $Q_{I_{k,j}} (I_{k,j} \in \mathcal{D}_{\beta})$ is a dyadic cube maximal (with respect to the inclusion) such that

$$\| f \|_{Q_{I_{k,j}},\Phi,\alpha} > a^k.$$

**Proof of Theorem 1.5** Following the same decomposition as in the proof of Theorem 1.1 and using the assumption on the measure $\mu$, we obtain

$$\int_{\mathcal{H}} (M_{\Phi,\alpha} f(z))^q d\mu(z) = \sum_k \int_{\Omega_k} (M_{\Phi,\alpha} f(z))^q d\mu(z)$$

$$\leq C \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^q \mu(Q_{I_{k,j}})$$

$$\leq C \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^q |Q_{I_{k,j}}|^{q/p}$$

$$= C \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^q \left( \frac{|Q_{I_{k,j}}|_{\omega,\alpha}}{|Q_{I_{k,j}}|_{\alpha}} \right)^{q/p}$$

$$= C \sum_{k,j} \| f \|_{Q_{I_{k,j}},\Phi,\alpha}^{1/p} \left( \frac{|Q_{I_{k,j}}|_{\omega,\alpha}}{|Q_{I_{k,j}}|_{\alpha}} \right)^{q/p}.$$
This leads us to
\[
\int_{H} (\mathcal{M}_{\Phi,\alpha} f(z))^q d\mu(z) \leq C \left( \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^q \right)^{q/p} \\
\leq C \left( \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^p \right)^{q/p} \\
\leq C \left( \int_{H} (\mathcal{M}_{\Phi,\alpha} f(z))^{1/p} (z)^p dV_{\alpha}(z) \right)^{q/p} \\
\leq C \left( \int_{H} |f(z)|^p \mathcal{M}_{\alpha}(z) dV_{\alpha}(z) \right)^{q/p}.
\]

Proof of Theorem 1.6. Using the same notations as above, we first obtain
\[
\int_{H} (\mathcal{M}_{\Phi,\alpha} f(z))^q d\mu(z) = \sum_k \int_{\Omega_k} (\mathcal{M}_{\Phi,\alpha} f(z))^q d\mu(z) \\
\leq C \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^q \mu(Q_{I_{k,j}}) \\
= C \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^q \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} |Q_{I_{k,j}}|_{\omega,\alpha}.
\]

An easy application of Hölder’s inequality to the last term in right of the above inequalities gives us
\[
\int_{H} (\mathcal{M}_{\Phi,\alpha} f(z))^q d\mu(z) \leq C \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^q \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} |Q_{I_{k,j}}|_{\omega,\alpha} \\
\leq CA^{q/p} B^{p-q}.
\]

where
\[
A := \sum_{k,j} \|f\|_{Q_{I_{k,j}},\Phi,\alpha}^p |Q_{I_{k,j}}|_{\omega,\alpha}
\]
and
\[
B := \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^{\frac{q}{p-q}} |Q_{I_{k,j}}|_{\omega,\alpha}.
\]

Following the lines of the proof of Theorem 1.5, we see that
\[
A \leq C \int_{H} |f(z)|^p \mathcal{M}_{\alpha}(z) dV_{\alpha}(z).
\]
The estimate of the term $B$ is quite harmless, we easily obtain

$$B := \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^{\frac{p}{p-q}} |Q_{I_{k,j}}|_{\omega,\alpha}$$

if we set

$$B := \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^{\frac{p}{p-q}} |Q_{I_{k,j}}|_{\omega,\alpha} |Q_{I_{k,j}}|_{\alpha}$$

$$\leq \sum_{k,j} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^{\frac{p}{p-q}} |Q_{I_{k,j}}|_{\omega,\alpha} |T_{I_{k,j}}|_{\alpha}$$

$$\leq \sum_{k,j} \int_{T_{I_{k,j}}} \left( \frac{\mu(Q_{I_{k,j}})}{|Q_{I_{k,j}}|_{\omega,\alpha}} \right)^{\frac{p}{p-q}} \frac{|Q_{I_{k,j}}|_{\omega,\alpha}}{|Q_{I_{k,j}}|_{\alpha}} dV_{\alpha}(z)$$

$$\leq \int_{H^p} (K_{\mu}(z))^{\frac{p}{p-q}} M_{\omega}(z) dV_{\alpha}(z).$$

The proof is complete. □

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