We establish a weighted pointwise Jacobian determinant inequality on corank 1 Carnot groups related to optimal mass transportation akin to the work of Cordero-Erausquin, McCann and Schmuckenschl"ager. The weights appearing in our expression are distortion coefficients that reflect the delicate sub-Riemannian structure of our space including the presence of abnormal geodesics. Our inequality interpolates in some sense between Euclidean and sub-Riemannian structures, corresponding to the mass transportation along abnormal and strictly normal geodesics, respectively. As applications, entropy, Brunn-Minkowski and Borell-Brascamp-Lieb inequalities are established.

Keywords: Carnot group; Jacobian determinant inequality; optimal mass transportation; abnormal and normal geodesics; entropy inequality; Brunn-Minkowski inequality; Borell-Brascamp-Lieb inequality.

MSC: 53C17, 35R03, 49Q20.

1. Introduction

As a general framework of our results, let $(X, d, m)$ be a suitably regular geodesic metric measure space with topological dimension $N \in \mathbb{N}$ where the theory of optimal mass transportation can be successfully developed. Examples for such spaces include Riemannian and Finsler manifolds, see McCann [13] and Ohta [14], the Heisenberg group $\mathbb{H}^n$, see Ambrosio and Rigot [2], or even more general sub-Riemannian structures with ‘well-behaved’ cut locus, see Figalli and Rifford [8]. Let $\mu_0$ and $\mu_1$ be two probability measures on $X$ which are absolutely continuous w.r.t. the reference measure $m$, and let $\mu_s = (\psi_s)_# \mu_0$, $s \in [0, 1]$, be the unique displacement interpolation measure joining $\mu_0$ and $\mu_1$ throughout the so-called $s$-intermediate optimal transport map $\psi_s : X \to X$. Roughly speaking, for $s \in (0, 1)$ fixed, the Jacobian determinant inequality reads as

$$\left( \text{Jac}(\psi_s)(x) \right)^{\frac{1}{N}} \geq \tau_{x,s}^N(\theta_x) + \tau_1^N(\theta_x) \left( \text{Jac}(\psi)(x) \right)^{\frac{1}{N}} \text{ for } \mu_0\text{-a.e. } x \in X. \quad (1.1)$$

Here, $\text{Jac}(\psi_s)(x)$ and $\text{Jac}(\psi)(x)$ are interpreted as the Radon-Nikodym derivatives of $\mu_s$ and of $\mu_1$ w.r.t. the reference measure $m$, while $\tau_{x,s}^N$ is the distortion coefficient which encodes information on the geometric structure of the space $X$. Expressions of $\tau_{x,s}^N$ can be calculated in terms of the Jacobian of the exponential map or estimated in terms of a curvature condition. The expression $\theta_x$ can be given as a function of $d(x, \psi(x))$ or its derivatives.

The Jacobian determinant inequality (1.1) in the above general form has been considered first in the setting of complete Riemannian manifolds (endowed with the natural Riemannian distance and volume form) in the pioneering work of Cordero-Erausquin, McCann and Schmuckenschl"ager [6]. This result constituted the starting point of an extensive study of the geometry of metric measure spaces, while relation (1.1) became an equivalent formulation of the famous curvature-dimension condition $\text{CD}(K,N)$, due to Lott and Villani [12], and Sturm [16, 17], where $\tau_{x,s}^N$ is replaced by explicit expressions $\tau_{x,s}^{N,K}$, $K$ being the lower bound.
of the Ricci curvatures in the Riemannian setting. Namely, \( \tau^{N,K}_s \) is given by

\[
\tau^{N,K}_s(\theta) = \begin{cases} 
\frac{s^n}{\theta} \left( \sinh \left( \sqrt{-K_N} s \theta \right) / \sinh \left( \sqrt{-K_N} \theta \right) \right)^{1-\frac{s}{\theta}} & \text{if } K \theta^2 < 0; \\
\frac{s^n}{\theta} \left( \sin \left( \sqrt{K_N} s \theta \right) / \sin \left( \sqrt{K_N} \theta \right) \right)^{1-\frac{s}{\theta}} & \text{if } 0 < K \theta^2 < (N-1)\pi^2; \\
\frac{s^n}{\theta} \left( \sin \left( \sqrt{K_N} s \theta \right) / \sin \left( \sqrt{K_N} \theta \right) \right)^{1-\frac{s}{\theta}} & \text{if } K \theta^2 \geq (N-1)\pi^2,
\end{cases}
\]

and \( \theta = \theta_x \) is precisely the Riemannian distance \( d(x, \psi(x)) \).

Juillet [10] proved that the Lott-Sturm-Villani curvature-dimension condition does not hold for any pair of parameters \( (N, K) \) on the Heisenberg group \( \mathbb{H}^n \) (endowed with its usual Carnot-Carathéodory metric \( d_{CC} \) and \( L^{2n+1} \)-measure), which is the simplest sub-Riemannian structure. Accordingly, there were strong doubts on the validity of a sub-Riemannian version of the Jacobian determinant inequality in the sub-Riemannian context. However, by using a natural Riemannian approximation of the Heisenberg group as in Ambrosio and Rigot [2], the authors of the present paper proved (1.1) on \( \mathbb{H}^n \), see [3, 4], where the Heisenberg distortion coefficient \( \tau^{2n+1}_s : [0, 2\pi] \to [0, \infty] \) is defined by

\[
\tau^{2n+1}_s(\theta) = \begin{cases} 
\frac{s^n}{\theta} \left( \sin \left( \sqrt{K_N} s \theta \right) / \sin \left( \sqrt{K_N} \theta \right) \right)^{2n+1} & \text{if } \theta \in (0, 2\pi); \\
\frac{s^n}{\theta} \left( \sin \left( \sqrt{K_N} s \theta \right) / \sin \left( \sqrt{K_N} \theta \right) \right)^{2n+1} & \text{if } \theta = 0; \\
\frac{s^n}{\theta} \left( \sin \left( \sqrt{K_N} s \theta \right) / \sin \left( \sqrt{K_N} \theta \right) \right)^{2n+1} & \text{if } \theta = 2\pi,
\end{cases}
\]

and \( \theta = \theta_x \) is the 'vertical' derivative of \( d_{CC}(\psi(x), \cdot) \) at the point \( x \).

In the present paper we prove a Jacobian determinant inequality on corank 1 Carnot groups where the sub-Riemannian geometry is more complicated than the one of the model Heisenberg group \( \mathbb{H}^n \) due to the presence of abnormal geodesics and the 'anisotropic' structure of the cut locus. Our method is different from the one in [3, 4] as we obtain the Jacobian determinant inequality by an intrinsic approach, without using a Riemannian approximation. As in [3, 4], we apply our Jacobian determinant inequality to establish various functional and geometric inequalities in the present setting including entropy, Brunn-Minkowski and Borell-Brascamp-Lieb inequalities. These results should open up the way to considering the above inequalities in a broader context outside the realm of \( CD(K, N) \)-type conditions by replacing the coefficients \( \tau^{N,K}_s \) by expressions that are suitable for sub-Riemannian geometries.

In order to present our main result, let us fix some notation. We denote by \( G \) a \( k+1 \) dimensional corank 1 Carnot group with its Lie algebra \( \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \), where \( \dim \mathfrak{g}_1 = k \) and \( \dim \mathfrak{g}_2 = 1 \). The operation on \( \mathfrak{g} \) (in exponential coordinates on \( \mathbb{R}^k \times \mathbb{R} \)) can be given by

\[
x \circ y = \left( x_1 + y_1, ..., x_k + y_k, x_z + y_z - \frac{1}{2} \sum_{i,j=1}^{k} A_{ij} x_j y_i \right),
\]

where \( x = (x_1, ..., x_k, x_z) \), \( y = (y_1, ..., y_k, y_z) \), and \( A = [A_{ij}] \) is a \( k \times k \) real skew-symmetric matrix. Let \( e = (0^k, 0) \in \mathbb{R}^k \times \mathbb{R} \) be the neutral element in \( (G, \circ) \). The layers \( \mathfrak{g}_1 \) and \( \mathfrak{g}_2 \) are generated by the left-invariant vector fields

\[
X_i = \partial_{x_i} - \frac{1}{2} \sum_{j=1}^{k} A_{ij} x_j \partial_z, \quad i = 1, ..., k.
\]

Moreover, \([X_i, X_j] = A_{ij} \partial_z\). By the spectral theorem for skew-symmetric matrices one can consider the diagonalized representation of \( A \) given by

\[
A = \begin{bmatrix}
0_{k-2d} & \alpha_1 J \\
0 & \alpha_1 J \\
& \ddots \\
& & \alpha_d J
\end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\
-1 & 0
\end{bmatrix},
\]
where \(0 < \alpha_1 \leq \ldots \leq \alpha_d\), and \(0_{k-2d}\) is the \((k-2d) \times (k-2d)\) square null-matrix.

For further use, let us introduce the functions \(\phi_1, \phi_2 : [0, 2\pi] \times (0, 1) \to \mathbb{R}\) given by
\[
\phi_1(t, s) = \frac{\sin(ts/2)}{s} \quad \text{and} \quad \phi_2(t, s) = \frac{\sin(ts/2) - ts/2 \cos(ts/2)}{s}.
\]

To define the distortion coefficient, we introduce the set
\[
D = \left\{ p = (p_x, p_z) \in \mathbb{R}^{k+1} : |p_z| < \frac{2\pi}{\alpha_d} \text{ and } A p_x \neq 0_{g_k} \right\} \subset T^*_e G,
\]
where \(p_x = (p_{x}^1, p_{x}^2, \ldots, p_{x}^d) \in \mathbb{R}^{k-2d} \times \mathbb{R}^2 \times \cdots \times \mathbb{R}^2\), and let \(\overline{D}\) be the closure of \(D\). The distortion coefficient \(\tau^{k,\alpha}_s : \overline{D} \to \mathbb{R}\) on the Carnot group \((G, o)\) is defined by
\[
\tau^{k,\alpha}_s(p) = \begin{cases}
\left( \frac{\sum_{i=1}^{d} ||p_{x}^i||^2 \prod_{j \neq i} \sigma_1(\alpha_j p_z, s) \sigma_1(\alpha_j p_x, s) \sigma_2(\alpha_j p_z, s)}{\sum_{i=1}^{d} ||p_{x}^i||^2 \prod_{j \neq i} \sigma_1(\alpha_j p_x, 1) \sigma_1(\alpha_j p_z, 1) \sigma_2(\alpha_j p_z, 1)} \right)^{\frac{1}{s}} & \text{if } p \in D \& p_z \neq 0; \\
\infty & \text{if } p \in D \& p_z = 0; \\
& \text{if } A p_x \neq 0_{g_k} \& |p_z| = \frac{2\pi}{\alpha_d}; \\
& \text{if } A p_x = 0_{g_k},
\end{cases}
\]
where \(p = (p_x, p_z)\) and \(\alpha = (\alpha_1, \ldots, \alpha_d)\).

Let us consider two compactly supported probability measures \(\mu_0\) and \(\mu_1\) on \(G\) which are absolutely continuous w.r.t. \(\mathcal{L}^{k+1}\). Since the distribution \(\Delta = \{X_1, \ldots, X_k\}\) on the corank 1 Carnot group \(G\) is two-generating, there exists a unique map realizing the optimal transportation between the measures \(\mu_0\) and \(\mu_1\) w.r.t. the cost function \(d_{CC}^2/2\), see Figalli and Rifford [8, Proposition 4.2 and Theorem 3.2]; this map can be defined \(\mu_0\)-a.e. through a \(d_{CC}^2/2\)-concave function \(\varphi : G \to \mathbb{R}\) as
\[
\psi(x) := \begin{cases}
\exp_x(-d_\varphi(x)) & \text{if } x \in \mathcal{M}_\varphi \cap \text{supp}(\mu_0); \\
x & \text{if } x \in \mathcal{S}_\varphi \cap \text{supp}(\mu_0).
\end{cases}
\] (1.5)

Hereafter, \(d_{CC}\) is the Carnot-Carathéodory metric on \(G\) and the sets \(\mathcal{M}_\varphi\) and \(\mathcal{S}_\varphi\) denote the moving and static sets of the transportation, respectively; see Section 2 for details. For \(s \in (0, 1)\) fixed, we also introduce the \(s\)-interpolant optimal transport map as
\[
\psi_s(x) := \begin{cases}
\exp_x(-sd_\varphi(x)) & \text{if } x \in \mathcal{M}_\varphi \cap \text{supp}(\mu_0); \\
x & \text{if } x \in \mathcal{S}_\varphi \cap \text{supp}(\mu_0).
\end{cases}
\] (1.6)

Our main result reads as follows.

**Theorem 1.1. (Jacobian determinant inequality on Carnot groups)** Let \((G, o)\) be a \(k + 1\) dimensional corank 1 Carnot group, and assume that \(\mu_0\) and \(\mu_1\) are two compactly supported Borel probability measures on \(G\), both absolutely continuous w.r.t. \(\mathcal{L}^{k+1}\). Let \(s \in (0, 1)\) be fixed, \(\psi : G \to G\) be the unique optimal transport map transporting \(\mu_0\) to \(\mu_1\) associated to the cost function \(d_{CC}^2/2\) and \(\psi_s\) its \(s\)-interpolant map. Then the following Jacobian determinant inequality holds
\[
(Jac(\psi_s)(x))^{\frac{1}{s+1}} \geq \tau^{k,\alpha}_s(\theta_x) \tau^{k,\alpha}_s(\theta_x) (Jac(\psi)(x))^{\frac{1}{s+1}} \text{ for } \mu_0\text{-a.e. } x \in G,
\] (1.7)
where \(\theta_x = (p_x, p_z) \in T^*_e G\) is given by \(\exp_x(\theta_x) = x^{-1} \circ \psi(x)\).

Let us notice that if \(p = (p_x, p_z) \in D\), we have that
\[
\lim_{p_z \to 0} \tau^{k,\alpha}_s(p) = s^{\frac{k+3}{s+1}} \quad \text{and} \quad \lim_{p_z \to \pm 2\pi/\alpha_d} \tau^{k,\alpha}_s(p) = +\infty.
\]
Furthermore, monotonicity properties of the functions $d_1$ and $d_2$ show that

$$\tau_{s}^{k,\alpha}(p) \geq s^{\frac{k+3}{k+1}}$$

for all $s \in (0, 1)$, $p \in \overline{D}$.

(1.8)

Therefore, the measure contraction property $\text{MCP}(0, k + 3)$ proved by Rizzi [15] is formally a consequence of (1.7). Notice, however that we use Rizzi’s result to prove the absolute continuity of the interpolant measure $\mu_s = (\psi_s)_#\mu_0$ (see Proposition 2.5), needed in the proof of the Jacobian determinant inequality.

In our next remark we consider the situation when $G = \mathbb{H}^n$ is the $n$-dimensional Heisenberg group. In this case we have $k = 2n = 2d$ and $\alpha_i = 4$ for every $i \in \{1, ..., d\}$. In this case no abnormal geodesics appear and the Carnot distortion coefficient $\tau_s^{2n,\alpha}(p_x, p_z)$ reduces to the Heisenberg distortion coefficient $\tau_s^{2n+1}(4p_z)$, which is nothing but relation (1.2) (introduced in [4]). Consequently, most of the results of [4] will be covered in the present work.

Let us notice furthermore, that in general corank 1 Carnot groups, the coefficients $\tau_s^{k,\alpha}$ and $\tau_1^{k,\alpha}$ depend not only on the parameter $p_z$ (as in the Heisenberg group) but also on $\|p^i_x\|$, $i \in \{1, ..., d\}$, showing a more anisotropic character of the present geometric setting as compared to the Heisenberg group. As we shall see later, $\|p^i_x\|$ and $p_z$ can be obtained by differentiating $\frac{d_{cc}(\psi(x), \cdot)}{2}$ w.r.t. the horizontal vector fields from the distribution $\Delta$ and the vertical vector field $\partial_z$, respectively (see Lemma 2.2 below).

Our final remark is of technical nature, but the details will be clear by reading the proof of Theorem 1.1. In this proof, we shall distinguish the cases when the mass is transported along abnormal and strictly normal geodesics, respectively. On one hand, when the mass transport is realized along abnormal geodesics, it turns out that the Jacobian determinant inequality reduces to an Euclidean-type determinant inequality thus the distortion coefficient can be $\tau_s^{k,\alpha} = s$ as in the Euclidean framework. We notice that in this case the full Jacobian matrix of $\psi_s$ might not exist; however, since the matrix has a triangular structure, the Jacobian can be reduced to two parts of the diagonal which are well defined and inequality (1.7) makes sense.

Furthermore, the triangular structure of the Jacobian matrix will allow us to perform the necessary changes of variable in order to provide important applications (see e.g. the entropy and Borell-Brascamp-Lieb inequalities via a suitable Monge-Ampère equation). On the other hand, once the mass transport is along strictly normal geodesics, the distortion coefficient $\tau_s^{k,\alpha}$ encodes information on the genuine sub-Riemannian character of the Carnot group obtained by a careful analysis of the Jacobian for the exponential map. It could also happen that a positive part of the mass is transported along abnormal geodesics while the complementary mass is transported by strictly normal geodesics, so different formulas for $\tau_s^{k,\alpha}$ will be used in the same instance of the mass transportation; such a scenario will be presented in Example 3.1 (see also Figure 2).

The organization of the paper is as follows. The proof of Theorem 1.1 will be provided in Section 3 after a self-contained presentation of the needed technical details in Section 2, i.e., properties of the Carnot-Caratheodory metric $d_{cc}$, exponential map and its Jacobian, the cut locus, and the optimal mass transportation on corank 1 Carnot groups. We emphasize that the optimal mass transportation developed by Figalli and Rifford [8] for large classes of sub-Riemannian manifolds cannot be directly applied since the squared distance function $d_{cc}^2$ is not necessarily locally semiconcave outside of the diagonal of $G \times G$ which is crucial in [8] (e.g. the regularity of optimal mass transport maps $\psi$ and $\psi_s$, or the validity of the Monge-Ampère equation). Section 4 is devoted to applications, i.e., by the Jacobian determinant inequality we shall derive entropy inequalities, the Brunn-Minkovski inequality and the Borell-Brascamp-Lieb inequality on corank 1 Carnot groups.

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2. Preliminaries

2.1. Carnot-Carathéodory metric, geodesics and their Jacobians on Carnot groups. A horizontal curve on $G$ is an absolutely continuous curve $\gamma : [0, r] \to G$ for which there exist measurable functions $h_j : [0, r] \to \mathbb{R}$ ($j = 1, \ldots, k$) such that

$$\dot{\gamma}(s) = \sum_{j=1}^{k} h_j(s) X_j(\gamma(s)) \quad \text{a.e. } s \in [0, r].$$

The length of this curve is

$$l(\gamma) = \int_{0}^{r} \|\dot{\gamma}(s)\|\,ds = \int_{0}^{r} \sqrt{\sum_{j=1}^{k} h_j^2(s)}\,ds.$$ 

The classical Chow-Rashevsky theorem assures that any two points from the Carnot group can be joined by a horizontal curve. Thus we can equip the Carnot group $G$ with its natural Carnot-Carathéodory metric by

$$d_{CC}(x, y) = \inf\{l(\gamma) : \gamma \text{ is a horizontal curve joining } x \text{ and } y\},$$

where $x, y \in G$ are arbitrarily fixed.

Let $e = (0_{\mathbb{R}^k}, 0) \in \mathbb{R}^k \times \mathbb{R}$ be the neutral element in $(G, \circ)$. The left invariance of the vector fields in the distribution $\Delta = \{X_1, \ldots, X_k\}$ is inherited by the distance $d_{CC}$, thus

$$d_{CC}(x, y) = d_{CC}(e, x^{-1} \circ y) \quad \text{for every } x, y \in G.$$

Following Rizzi [15], the explicit form of the minimal geodesics on a general corank 1 Carnot group can be described as follows.

**Proposition 2.1.** (Rizzi [15]) On a corank 1 Carnot group $(G, \circ)$ the geodesics $s \mapsto \exp_e(sp) \in G$ starting from $e = (0_{\mathbb{R}^k}, 0)$, with direction

$$p = (p^0, p^1_x, \ldots, p^d_x, p_z) \in (\mathbb{R}^{k-2d} \times \mathbb{R}^2 \times \ldots \times \mathbb{R}^2) \times \mathbb{R} = T^*_e G$$

has the following equation

$$\exp_e(sp) : \begin{cases} 
\gamma^0(s) = p^0_x s, \\
\gamma^i(s) = \left(\frac{\sin (\alpha_i p_z s)}{\alpha_i p_z^2}\right) I + \frac{\cos(\alpha_i p_z s) - 1}{\alpha_i p_z^2} J \right) p^i_x, \quad s \in [0, 1], \\
\gamma_z(s) = \sum_{i=1}^{d} \|p^i_x\|^2 \frac{\alpha_i p_z s - \sin(\alpha_i p_z s)}{2\alpha_i p_z^2},
\end{cases} \quad (2.1)$$

when $p_z \neq 0$. When $p_z = 0$, the geodesic is

$$s \mapsto \exp_e(sp) = (p^0_x s, p^1_x s, \ldots, p^d_x s, 0), \quad s \in [0, 1]. \quad (2.2)$$

Hereafter, $I$ denotes the $2 \times 2$ unit matrix.

We notice that once $A$ has a non-trivial kernel, there are abnormal geodesics on $G$ which appear precisely when $Ap_x = 0$; their equations are given by

$$s \mapsto \exp_e(sp^0_x, 0, \ldots, 0, sp_z) = (sp^0_x, 0_{\mathbb{R}^{2d+1}}), \quad s \in [0, 1], \quad (2.3)$$

for every $p_z \in \mathbb{R}$. Note that each abnormal geodesic can be described uniquely by a normal geodesic as well, namely, letting $p_z = 0$ and $p^1_x = \ldots = p^d_x = 0_{\mathbb{R}^2}$; thus the map in (2.2) may define an abnormal geodesic.
We recall from Rizzi [15] that the Jacobian determinant of the exponential map is

\[
\text{Jac}(\exp_x)(p) = \begin{cases} 
\frac{2^{2d}}{\prod_{i=1}^{d} \alpha_i^2 p_i^{2d+2}} \sum_{i=1}^{d} \|p'_i\|^2 \prod_{j \neq i} \left( \sin \frac{\alpha_j p_j}{2} \right)^2 \sin \frac{\alpha_i p_i}{2} \times \\
\times \left( \sin \frac{\alpha_i p_i}{2} - \frac{\alpha_i p_i}{2} \cos \frac{\alpha_i p_i}{2} \right) & \text{if } p_z \neq 0; \\
\frac{1}{p_z^2} \sum_{i=1}^{d} \|p'_i\|^2 \alpha_i^2 & \text{if } p_z = 0.
\end{cases}
\]  

(2.4)

By left-invariance, the minimal geodesics on G starting from an arbitrary point \( x \in G \) are represented by \( s \mapsto \exp_x(sp) = x \circ \exp_e(sp) \), \( s \in [0,1] \), where the two covectors \( p \in T_x^*G \) and \( \tilde{p} \in T_e^*G \) can be identified. Moreover, since for every \( x \in G \) the left-translation \( L_x(y) = x \circ y \), \( y \in G \), is a volume-preserving map, it follows that

\[
\text{Jac}(\exp_x)(p) = \text{Jac}(\exp_p)(p) \quad \text{for every } p \in T_y^*G.
\]  

(2.5)

Given \( x, y \in G \) and assume that \( x = \exp_y(p) \) for some \( p = (p_x, p_z) = (p^0_x, p^1_x, \ldots, p^d_x, p_z) \in T_y^*G \). Then \( y = \exp_x(\tilde{p}) \), where \( \tilde{p} = (\tilde{p}^0_x, \tilde{p}^1_x, \ldots, \tilde{p}^d_x, \tilde{p}_z) \) is given by

\[
\begin{cases} 
\tilde{p}^i_x = -p^i_x, \\
\tilde{p}_z = (\cos(\alpha_i p_z) I + \sin(\alpha_i p_z)J) p^i_x, & i \in \{1, \ldots, d\}; \\
\tilde{p}_z = -p_z.
\end{cases}
\]

(2.6)

From now on, we assume the matrix \( A \) has the diagonal representation given in (1.4).

We notice that \( \Delta = \{X_1, \ldots, X_k\} \) is not a fat distribution whenever the kernel of \( A \) is non-trivial. Indeed, in this case we have \( T_x G \neq \Delta(x) + [X_j, \Delta](x) \) for every \( x \in G \) and \( j \in \{1, \ldots, k - 2d\} \). However, \( \Delta \) is two-generating, i.e.,

\[
T_x G = \Delta(x) + [\Delta, \Delta](x) \quad \text{for every } x \in G.
\]

For simplicity of notation, we reorganize the vector fields in \( T_x^*G \) as

\[
\begin{cases} 
X^0 = (X_1, \ldots, X_{k-2d}); \\
X^i = (X_{k-2d+2i-1}, X_{k-2d+2i}), & i \in \{1, \ldots, d\}; \\
Z = \partial_z.
\end{cases}
\]

(2.7)

We split the distribution \( \Delta \) on \( G \) into two types of vector fields; namely, \( \Delta_0 = \{X^0\} \) and \( \tilde{\Delta} = \{X^1, \ldots, X^d\} \). This splitting gives the following trivial representation of the distance function \( d_{\text{CC}}^* \):

**Lemma 2.1.** (Pythagorean rule) For every \((\xi, \eta, z), (\xi, \eta, z) \in \mathbb{R}^{k-2d} \times \mathbb{R}^{2d} \times \mathbb{R} \), we have

\[
d_{\text{CC}}^2((\xi, \eta, z), (\xi, \eta, z)) = d_{\text{CC}}^2((\xi, \eta, z)) + d_{\tilde{\text{CC}}}^2((\eta, z), (\eta, z)),
\]

where \( d_{\text{CC}} \) is the Euclidean metric in \( \mathbb{R}^{k-2d} \) while \( d_{\tilde{\text{CC}}} \) is the Carnot-Caratheodory distance on \( \mathbb{R}^{2d} \times \mathbb{R} \) w.r.t. to the distribution \( \tilde{\Delta} \) inherited from the original sub-Riemannian structure.

**Proof.** By the left-invariance of the metric \( d_{\text{CC}} \), we have

\[
d_{\text{CC}}^2((\xi, \eta, z), (\xi, \eta, z)) = d_{\text{CC}}^2(e, (\xi, -\eta, -z) \circ (\xi, \eta, z)).
\]

Let \( \gamma = (\gamma^0, \gamma, \ldots, \gamma^d, \gamma_z) : [0,1] \to G \) be the geodesic given by (2.1) or (2.2) joining \( e \) and the element \((\xi, -\eta, -z) \circ (\xi, \eta, z)\), having its initial vector \( p = (p^0_x, p^1_x, \ldots, p^d_x, p_z) \in \mathbb{R}^{k-2d} \times \mathbb{R} \times \cdots \times \mathbb{R}^{2d} \times \mathbb{R} \). We have that

\[
d_{\text{CC}}^2((\xi, \eta, z), (\xi, \eta, z)) = \sum_{i=0}^{d} \|p^i_x\|^2. 
\]

Note that \( \|p^0_x\|_{\mathbb{R}^{k-2d}} = d_{\text{CC}}(e, (0_{\mathbb{R}^{k-2d}}, -\eta, -z) \circ (0_{\mathbb{R}^{k-2d}}, \eta, z)) = d_{\text{CC}}((\eta, z), (\eta, z)) \)

which is realized precisely by the geodesic \( \tilde{\gamma} = (\gamma^1, \ldots, \gamma^d, \gamma_z) \), concluding the proof. \( \square \)
2.2. Cut locus. Let us consider the set

\[ D = \left\{ p = (p_x, p_z) \in \mathbb{R}^{k+1} : |p_z| < \frac{2\pi}{\alpha_d} \text{ and } A p_x \neq 0_{\mathbb{R}^k} \right\} \subset T_e^* G. \]

Rizzi [15, Lemma 16] proved that \( D \) is precisely the injectivity domain of parameters associated to geodesics joining the origin \( e \) to points of \( G \). We know that all points in the corank 1 Carnot group \( G \) can be reached by a minimal normal geodesic; namely, for every \( x \in G \) there exists a parametrization \( p \) in the closure of \( D \), i.e.,

\[ \overline{D} = \left\{ p = (p_x, p_z) \in \mathbb{R}^{k+1} : |p_z| \leq \frac{2\pi}{\alpha_d} \right\}, \]

which defines a minimal normal geodesic joining \( e \) and \( x \).

The cut locus of the origin \( e \) in \( G \) is

\[ \text{cut}_G(e) = \exp_e(\overline{D} \setminus D) = G \setminus \exp_e(D) \]

\[ = \left( \mathbb{R}^{k-2d} \times \{0_{\mathbb{R}^{d+1}}\} \right) \cup \left\{ \exp_e \left( p_x, \pm \frac{2\pi}{\alpha_d} \right) : p_x \neq 0_{\mathbb{R}^k} \right\}. \]

The set \( \mathbb{R}^{k-2d} \times \{0_{\mathbb{R}^{d+1}}\} \) in the above representation corresponds to the image of abnormal geodesics while the latter set contains the conjugate points to \( e \), see (2.4). By left-invariance, the cut locus of the point \( x \in G \) is

\[ \text{cut}_G(x) = L_x(\text{cut}_G(e)). \]

Note that \( \text{cut}_G(x) \) is closed and \( \mathcal{L}^{k+1}(\text{cut}_G(x)) = 0 \) for every \( x \in G \); moreover, by (2.6) it follows that \( y \in \text{cut}_G(x) \) if and only if \( x \in \text{cut}_G(y) \).

**Lemma 2.2.** Fix \( y \in G \) and let \( x = (x^0, x^1, \ldots, x^d, z) \notin \text{cut}_G(y) \). If \( x = \exp_y(p_0^0, p_1^1, \ldots, p_d^d, p_z) \) then we have

(i) \( X^0 \frac{d^2_{CC}(y, \cdot)}{2} |x = p_0^0 \) and \( Z \frac{d^2_{CC}(y, \cdot)}{2} |x = p_z \);

(ii) for every \( i \in \{1, \ldots, d\} \),

\[ X^i \frac{d^2_{CC}(y, \cdot)}{2} |x = [\cos(\alpha_i p_z) I - \sin(\alpha_i p_z) J] p_x^i. \]  

**Proof.** By exploring the left-invariance, it is enough to consider the case when \( e = 0 \). Let us introduce the auxiliary functions \( f, g : (-2\pi, 2\pi) \setminus \{0\} \to \mathbb{R} \) defined by

\[ f(t) = \frac{\sin^2 \left( \frac{t}{2} \right)}{\left( \frac{t}{2} \right)^2} \quad \text{and} \quad g(t) = \frac{t - \sin(t)}{\sin^2 \left( \frac{t}{2} \right)}, \quad t \in (-2\pi, 2\pi) \setminus \{0\}. \]

We consider the case when \( p_z \neq 0 \); the case \( p_z = 0 \) can be obtained by a limiting procedure. Since \( x \notin \text{cut}_G(e) \) and the cut locus is closed, there exists a small neighborhood \( V_x \) of \( x \) such that \( V_x \cap \text{cut}_G(e) = \emptyset \). Let \( w = (x^0_w, x^1_w, \ldots, x^d_w, z_w) = \exp_e \left( (p_w^0)_x, (p_w^1)_x, \ldots, (p_w^d)_x, (p_w)_z \right) \in V_x \) be arbitrarily fixed. By (2.1) (for \( s = 1 \)) we have that

\[ \|x^i_w\|^2 = \|(p_w)_x^i\|^2 f(\alpha_i (p_w)_z), \quad i \in \{1, \ldots, d\}. \]

Thus, one has

\[ d^2_{CC}(e, w) = \sum_{i=0}^{d} \|(p_w)_x^i\|^2 = \|x_0^0\|^2 + \sum_{i=1}^{d} \frac{\|x^i_w\|^2}{f(\alpha_i (p_w)_z)} \]

(i) By (2.10) we directly have that \( X^0 \frac{d^2_{CC}(e, \cdot)}{2} |x = 2x^0 \). Furthermore, the last component in (2.1) can be written as

\[ z_w = \sum_{i=1}^{d} \|(p_w)_x^i\|^2 \frac{\alpha_i (p_w)_z - \sin(\alpha_i (p_w)_z)}{2\alpha_i ((p_w)_z)^2} = \frac{1}{8} \sum_{i=1}^{d} \alpha_i \|x^i_w\|^2 g(\alpha_i (p_w)_z). \]
We may differentiate (2.10) and (2.11) w.r.t. the variable $z_w$ at the point $x$, obtaining
\[
Z(d_{CC}^2(e,\cdot))|_x = -\sum_{i=1}^d \alpha_i |x^i|^2 \frac{f'(\alpha_i p_z)}{f''(\alpha_i p_z)} (Z(p_w) z_x) \quad \text{and} \quad 1 = \frac{1}{8} \sum_{i=1}^d \alpha_i^2 |x^i|^2 g'(\alpha_i p_z) (Z(p_w) z_x).
\]
Note that $\frac{f'(\alpha)}{f''(\alpha)} = \frac{1}{2} g'(t)$; thus, the latter relations give at once that $Z(d_{CC}^2(e,\cdot))|_x = 2p_z$.

(ii) In order to prove relation (2.8) we proceed in a similar way as in (i), by deriving (2.10) and (2.11) w.r.t. the corresponding variables. \hfill $\square$

A direct consequence of Lemma 2.2 is:

**Proposition 2.3.** Fix $x, y \in G$ such that $y \notin \text{cut}_G(x)$. If $\nabla = (X^0, X^1, \ldots, X^d, Z)$, then
\[
y = \exp_x \left( -\nabla \frac{d_{CC}^2(y,\cdot)}{2} \right). \tag{2.12}
\]

**Proof.** Let $x = \exp_y(p)$ for some $p = (p_x, p_z) = (p_x^0, p_x^1, \ldots, p_x^d, p_z) \in D$. According to Lemma 2.2, we have that
\[
-\nabla \frac{d_{CC}^2(y,\cdot)}{2} |_x \in (\bar{p}_x, \bar{p}_x^1, \ldots, \bar{p}_x^d, \bar{p}_z),
\]
where
\[
\begin{align*}
\bar{p}_x^i &= -p_x^i, \\
\bar{p}_x &= -\cos(\alpha_i p_z) I - \sin(\alpha_i p_z) J p_x^i, \quad i \in \{1, \ldots, d\}; \\
\bar{p}_z &= -p_z.
\end{align*}
\]
Thus, by relation (2.6) it follows that
\[
\exp_x \left( -\nabla \frac{d_{CC}^2(y,\cdot)}{2} \right) = \exp_x (\bar{p}_x, \bar{p}_x^1, \ldots, \bar{p}_x^d, \bar{p}_z) = y,
\]
which concludes the proof. \hfill $\square$

Our next proposition states that for fixed $y$ the function $x \mapsto d_{CC}^2(y, x)$ fails to be locally semiconvex at the points $x$ that are in the cut locus of $y$.

**Proposition 2.3.** Let $x \in \text{cut}_G(y)$ be such that $x = \exp_y(p_x, \pm \frac{2\pi}{\alpha_d})$ where $p_x = (p_x^0, p_x^1, \ldots, p_x^d)$ satisfies $A p_x \neq 0_{\mathbb{R}^d}$ and $p_x^{-d-q+1} = \ldots = p_x^d = 0_{\mathbb{R}}$, the number $q \in \{1, \ldots, d\}$ being the multiplicity of $\alpha_d$, i.e., $0 < \alpha_1 \leq \ldots \leq \alpha_{d-q} < \alpha_{d-q+1} = \ldots = \alpha_d$. Then
\[
\lim_{||v|| \to 0} \frac{d_{CC}^2(y, \exp_x(\nu)) + d_{CC}^2(y, \exp_x(-\nu)) - 2d_{CC}^2(y, x)}{||v||^2} = -\infty.
\]

**Proof.** We discuss only the case when $x = \exp_y(p_x, \pm \frac{2\pi}{\alpha_d})$. Due to the left-translation invariance of the Carnot-Carathéodory distance, it suffices to prove the claim for $y = e$ and $x \in \text{cut}_G(e)$. Let
\[
\nu = (0_{\mathbb{R}^{d-2}}, (v, 0), 0),
\]
with $|v| \neq 0$ small enough. Let $p^v = ((p^v)^0, (p^v)^1, \ldots, (p^v)^d, (p^v)_z) \in \overline{D}$ be the parametrization of the geodesic joining $e$ and $\exp_y(\nu)$, i.e., $x \circ \exp_e(\nu) = \exp_x(\nu) = \exp_e(p^v)$. Since $x = \exp_e(p_x, \pm \frac{2\pi}{\alpha_d})$ and $\exp_e(\nu) = \nu$ (see (2.2)), we have
\[
\exp_e(p^v) = \exp_x \left( p_x, \frac{2\pi}{\alpha_d} \right) \circ \nu.
\]
The latter relation and (2.1) imply in particular that
\[
||p^v|^i_x||^2 f(\alpha_i p^v)_z = ||p^v|^i_x||^2 f \left( \frac{2\pi}{\alpha_i \alpha_d} \right), \quad i \in \{1, \ldots, d-1\}; \tag{2.13}
\]
Due to the specific choice of the direction $\nu_X$ we obtain that the latter term is nothing but the first component of $f=\exp_x(v)f(\alpha_i)\alpha_i$, where the functions $f$ and $g$ are from (2.9). By the above relations we observe that

$$d^2_{CC}(e,\exp_x(\nu)) = \sum_{i=1}^d ||(p^{i})^2 = d^2_{CC}(e,\exp_x(-\nu)).$$

Therefore, in order to conclude the claim it suffices to prove

$$\lim_{v \to 0} \frac{d^2_{CC}(e,\exp_x(\nu)) - d^2_{CC}(e,x)}{v^2} = -\infty. \quad (2.16)$$

By construction, $(p^{i})^2 \leq \frac{2\pi}{\alpha_d}$. We notice that $(p^{i})^2 \neq \frac{2\pi}{\alpha_d}$; indeed, if $(p^{i})^2 = \frac{2\pi}{\alpha_d}$ then (2.14) contradicts the fact that $v \neq 0$. Moreover, we prove that

$$\lim_{v \to 0} \frac{(p^{i})^2}{v^2} = \frac{2\pi}{\alpha_d} \quad (2.17)$$

If we assume the contrary, i.e., $(p^{i})^2 \to c_0$ as $v \to 0$ (eventually taking a subsequence) for some $0 < c_0 < \frac{2\pi}{\alpha_d}$, then $g(\alpha_i) \to g(\alpha,c_0)$ as $v \to 0$, and the values of $g(\alpha,c_0)$ are all well defined for every $i \in \{1,\ldots,d\}$. If we apply relations (2.13) and $p^{d-q+1} = \ldots = p^{d} = 0_{\mathbb{R}^2}$ into (2.15), and take the limit in the latter relation as $v \to 0$, we obtain that

$$\sum_{i=1}^{d-q} \alpha_i ||p^{i1}||^2 \left(g(\alpha_i) \cdot g \left(\frac{2\pi}{\alpha_d}\right)\right) f \left(\alpha_i \frac{2\pi}{\alpha_d}\right) = 0.$$

Since $g$ is strictly monotone on $(0,2\pi)$, $A\mathcal{P}_x \neq 0_{\mathbb{R}^2}$ (thus $p^{0}_{\mathbb{R}^2} \mathcal{P}_x$ for some $i_0 \in \{1,\ldots,d-g\}$) and $f \left(\alpha \frac{2\pi}{\alpha_d}\right) > 0$ for every $i \in \{1,\ldots,d\}$, the latter relation leads us to a contradiction.

Let $R(\nu) = d^2_{CC}(e,\exp_x(\nu))$ be defined for every $v \in (-\varepsilon,\varepsilon)$ for $\varepsilon > 0$ small enough. We notice that $\exp_x(\nu) \notin \text{cut}_G(e)$ for $|v| \in (0,\varepsilon)$, thus $R$ is differentiable on $(-\varepsilon,\varepsilon) \setminus \{v\}$. Due to the specific choice of the direction $\nu$, we have that $R'(\nu) = X_{k-1}d^2_{CC}(e,\nu)\exp_x(\nu)$. Note that the latter term is nothing but the first component of $X^d d^2_{CC}(e,\nu)\exp_x(\nu)$. Since $\exp_x(\nu) = \exp_e(p^{i})^2$, by Lemma 2.2 and (2.14) we have that

$$X^d d^2_{CC}(e,\nu)\exp_x(\nu) = 2\cos(\alpha_d(p^{i})^2)I - \sin(\alpha_d(p^{i})^2)J(p^{i})^2d$$

$$= \left(\alpha_d(p^{i})^2 \cos \left(\frac{2\pi}{\alpha_d}\right)\right) v, \alpha_d(p^{i})^2 v$$.

Consequently, we have $R'(\nu) = \alpha_d(p^{i})^2 \cos \left(\frac{2\pi}{\alpha_d}\right) v$ and by the L’Hospital rule and (2.17) we obtain that

$$\lim_{v \to 0} \frac{d^2_{CC}(e,\exp_x(\nu)) - d^2_{CC}(e,x)}{v^2} = \lim_{v \to 0} \frac{R'(\nu)}{2v} = \lim_{v \to 0} \frac{\alpha_d^2(p^{i})^2}{2v} = -\infty,$$

which concludes the proof of (2.16).

\[\Box\]

2.3. The Jacobian of the exponential map along a reversed geodesic. Let $x, y \in G$ be such that $x \notin \text{cut}_G(y)$ and $\gamma : [0,1] \to G$ be the unique geodesic $\gamma(s) = \exp_x(sp)$ joining $x$ and $y$ for some $p \in D$. For every $s \in (0,1]$, let us introduce the Jacobian matrix

$$Y(s) = \frac{d(\exp_x)_{sp}}{s}.$$
According to (2.4), the matrix \( Y(s) \) is invertible for every \( s \in (0, 1] \). In the sequel, we are going to consider the reversed geodesic path \( s \mapsto \exp_y((1 - s)p_0) \), \( s \in [0, 1] \), where \( \exp_y p_0 = x \) and compute the ‘reverse’ of \( Y \), i.e.,

\[
Y(1 - s) = d(\exp_y)_{(1-s)p_0}, \quad s \in [0, 1).
\]  

(2.18)

Here, \( p_0 \in T^*_y G \) is given by means of \( p \in T_x^* G \) similarly as in (2.6). With these notations, we have

**Proposition 2.4.** Let \( x, y \in G \) be such that \( x \notin \text{cut}_G(y) \) and \( \gamma : [0, 1] \to G \) be the unique geodesic \( \gamma(s) = \exp_x(sp) \) joining \( x \) and \( y \) for some \( p \in T^*_y G \). For every \( s \in (0, 1) \), one has

\[
Y(1 - s) = \frac{1}{1 - s} Y(s) \text{Hess}_{x,y}(s) Y(1),
\]

where

\[
\text{Hess}_{x,y}(s) = \text{Hess} \frac{d^2_{\text{CC}}(\gamma(s), \cdot)}{2} \bigg|_x - s \text{Hess} \frac{d^2_{\text{CC}}(y, \cdot)}{2} \bigg|_x.
\]

In addition, \( \text{Hess}_{x,y}(s) \) is a positive semidefinite, symmetric matrix.

Let us note that in the above statement \( \text{Hess} = \nabla^2 \) denotes the (a priori not necessarily symmetric) Carnot Hessian, i.e.,

\[
\text{Hess} = \begin{bmatrix}
X_1X_1 & X_1X_2 & \cdots & X_1X_k & X_1Z \\
X_2X_1 & X_2X_2 & \cdots & X_2X_k & X_2Z \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
X_kX_1 & X_kX_2 & \cdots & X_kX_k & X_kZ \\
ZX_1 & ZX_2 & \cdots & ZX_k & ZZ
\end{bmatrix}.
\]

This notation will be used also later on.

A similar result to Proposition 2.4 has been proved by Cordero-Erausquin, McCann and Schmuckenschläger [6] on Riemannian manifolds by exploring properties of Jacobi fields. Since the theory of Jacobi fields in our setting is not (yet) available, we give a direct proof of Proposition 2.4. To do this, we need the following:

**Claim 2.1.** Let \( m \in \mathbb{N}, c, \eta_i : [0, 1] \to \mathbb{R}^m, i \in \{1, 2\}, \) be some differentiable maps with \( \eta_2(0) = 0 \) and a smooth function \( F : \mathbb{R}^{2m} \to \mathbb{R}^m \) in a neighborhood of \( (c(0), \eta_1(0)) \) such that \( t \mapsto F(c(t), \eta_1(t)) \) is constant near the origin. Then

\[
DF(c(t), \eta_1(t) + \eta_2(t))|_{t=0} = D_2 F(c(0), \eta_1(0)) \dot{\eta}_2(0).
\]

**Proof.** By assumption, we have near the origin that

\[
0 = DF(c(t), \eta_1(t)) = D_1 F(c(t), \eta_1(t)) \dot{c}(t) + D_2 F(c(t), \eta_1(t)) \dot{\eta}_1(t).
\]

By using the latter relation at \( t = 0 \) and \( \eta_2(0) = 0 \), we obtain

\[
DF(c(t), \eta_1(t) + \eta_2(t))|_{t=0} = D_1 F(c(0), \eta_1(0)) \dot{c}(0) + D_2 F(c(0), \eta_1(0)) (\dot{\eta}_1(0) + \dot{\eta}_2(0)) = D_2 F(c(0), \eta_1(0)) \dot{\eta}_2(0),
\]

which completes the proof. \( \square \)

**Proof of Proposition 2.4.** We first deal with the properties of the matrix \( \text{Hess}_{x,y}(s) \). By pure metric arguments, one can check that for every \( z \in G \) and \( s \in [0, 1] \) we have the inequality

\[
m_{x,y}^s(z) := d^2_{\text{CC}}(\gamma(s), z)/2 - sd^2_{\text{CC}}(y, z)/2 + s(1 - s)d^2_{\text{CC}}(x, y)/2 \geq 0.
\]

(2.20)

In the Riemannian setting this has been established first by Cordero-Erausquin, McCann and Schmuckenschläger [6, Claim 2.4]. Moreover, in (2.20) equality is realized precisely when \( z = x \); the same proof works in our setting as well.
Since $\gamma((0, 1]) \cap \text{cut}_G(x) = \emptyset$, it follows that $z \mapsto m_{x,y}^s(z)$ is twice differentiable at $x$ (see Proposition 2.2) and its gradient is

$$\nabla m_{x,y}^s(\cdot)|_x = \nabla \frac{d^2_{CC}(\gamma(s), \cdot)}{2}|_x - s\nabla \frac{d^2_{CC}(y, \cdot)}{2}|_x = 0_{g_{k+1}}, \quad (2.21)$$

while its Carnot Hessian $\nabla^2 m_{x,y}^s(\cdot)|_x = H_{x,y}(s)$ is positive semidefinite.

In order to prove the symmetry of $H_{x,y}(s)$, we verify that the Lie brackets $[W_1, W_2]m_{x,y}^s(\cdot)|_x$ vanish for every choice of $W_1, W_2 \in \Delta \cup \{Z\} = \{X_1, ..., X_k, Z\}$. Indeed, the Lie bracket is either trivial by definition or it is $Zm_{x,y}^s(\cdot)|_x$ up to a multiplicative constant (depending on the eigenvalues $\alpha_i$, $i \in \{1, ..., d\}$; but $Zm_{x,y}^s(\cdot)|_x = 0$ due to (2.21).

We now prove relation (2.19). Since $x \notin \text{cut}_G(y)$ and cut$_G(y)$ is closed, one may fix a curve $c : [0, 1] \to G$ with $c(0) = x$ and $\dot{c}(0) = w \in T_x G$ arbitrarily fixed such that $c([0, 1]) \cap \text{cut}_G(y) = \emptyset$. We notice that $s \mapsto \exp_{c(t)}\left(-s\nabla \frac{d^2_{CC}(y, \cdot)}{2}|_{c(t)}\right)$ is the unique geodesic joining $c(t)$ and $y$; indeed, for $s = 0$ we have $c(t)$, while for $s = 1$ one has precisely $y$ due to Proposition 2.2, see Figure 1.

![Figure 1](image-url)  
**Figure 1.** The curve $c$ (starting from $x$), connected by geodesics with the point $y$

Let $\bar{p} : [0, 1] \to T_y G$ be a curve such that

$$\exp_{c(t)}\left(-s\nabla \frac{d^2_{CC}(y, \cdot)}{2}|_{c(t)}\right) = \exp_y((1 - s)\bar{p}(t)). \quad (2.22)$$

Let us observe that by (2.22) for $s = 0$ we have

$$c(t) = \exp_y(\bar{p}(t)) \quad \text{for all } t \in [0, 1].$$

In particular, for $t = 0$ we have that $x = c(0) = \exp_y(\bar{p}(0))$, i.e., $\bar{p}(0) = p_0$ and due to (2.18),

$$w = \dot{c}(0) = d(\exp_y)_{p_0}\dot{p}(0) = d(\exp_y)_{p_0}\bar{p}(0) = Y(1)\bar{p}(0). \quad (2.23)$$

Fix $s \in (0, 1)$. Now, we rewrite (2.22) into

$$\exp_{c(t)}(\eta_1(t) + \eta_2(t)) = \exp_y((1 - s)\bar{p}(t)), \quad (2.24)$$

where

$$\eta_1^t(t) = -\nabla \frac{d^2_{CC}(\gamma(s), \cdot)}{2}|_{c(t)} \text{ and } \eta_2^t(t) = \nabla \frac{d^2_{CC}(\gamma(s), \cdot)}{2}|_{c(t)} - s\nabla \frac{d^2_{CC}(y, \cdot)}{2}|_{c(t)}.$$ 

We are going to verify the assumptions of Claim 2.1 for the latter choices. First, due to Proposition 2.2, one has $t \mapsto \exp_{c(t)}(\eta_1^t(t)) = \gamma(s) = \text{constant}$, and due to (2.21), we also have $\eta_2^t(0) = 0$. Since we have $\eta_1^t(0) = -\nabla \frac{d^2_{CC}(\gamma(s), \cdot)}{2}|_x$, by Proposition 2.2 one has that
exp_x(\eta^k_t(0)) = \gamma(s) which is nothing but \gamma(s) = \exp_x(sp); thus \eta^k_t(0) = sp. Consequently, by differentiating relation (2.24) at t = 0 and using Claim 2.1 with \(F(q_1, q_2) = \exp_{q_1}(q_2)\) which is smooth around the point \((c(0), \eta^k_t(0)) = (x, sp)\), we obtain

\[
d(\exp_{c(0)}\eta^k_t(0))\dot{\eta}^k_t(0) = (1 - s)d(\exp_y((1 - s)\bar{\eta})(0)).
\]

Moreover,

\[
\dot{\eta}^k_t(0) = \left[\Hess_{CC}^2(\gamma(s), \cdot)\right]_x - s\Hess_{CC}^2(y, \cdot)\right]_x c(0).
\]

Finally, we recall by (2.23) that \(w = c(0) = \overline{\gamma}(1)\bar{\eta}(0)\) and due to (2.4), \(\overline{\gamma}(1)\) is invertible. Putting together the above computations, we have

\[
Y(s) \left[\Hess_{CC}^2(\gamma(s), \cdot)\right]_x - s\Hess_{CC}^2(\gamma(s), \cdot)\right]_x \right] w = (1 - s)\overline{\gamma}(1 - s)\overline{\gamma}(1)^{-1}w.
\]

Due to the arbitrariness of \(w\), the claim (2.19) follows.

2.4. Optimal mass transportation on corank 1 Carnot groups. We first recall some facts from Figalli and Rifford [8]. A function \(\varphi : G \rightarrow \mathbb{R}\) is \(c = d^2_{CC}/2\)-concave if there exist a nonempty set \(S \subset G\) and a function \(\varphi^c : S \rightarrow \mathbb{R} \cup \{-\infty\}\) with \(\varphi^c \neq -\infty\) such that

\[
\varphi(x) = \inf_{y \in S} \left\{\frac{1}{2}d^2_{CC}(x, y) - \varphi^c(y)\right\}.
\]

If \(\varphi\) is a \(d^2_{CC}/2\)-concave function, let

\[
\partial^c\varphi(x) = \left\{y \in S : \varphi(x) + \varphi^c(y) = \frac{1}{2}d^2_{CC}(x, y)\right\}
\]

be the \(c\)-superdifferential of \(\varphi\) at \(x\). For such a function \(\varphi\), let

\[
\mathcal{M}_\varphi = \{x \in G : x \notin \partial^c\varphi(x)\} \quad \text{and} \quad \mathcal{S}_\varphi = \{x \in G : x \in \partial^c\varphi(x)\}
\]

be the moving and static sets, respectively.

Let us fix \(\mu_0\) and \(\mu_1\) two compactly supported probability measures on \(G\) which are absolutely continuous w.r.t. \(\mathcal{L}^{k+1}\). According to [8, Theorem 2.3], there are two \(d^2_{CC}/2\)-concave, continuous functions \(\varphi, \varphi^c : G \rightarrow \mathbb{R}\) such that

\[
\varphi(x) = \min_{y \in \text{supp}(\mu_1)} \left\{\frac{1}{2}d^2_{CC}(x, y) - \varphi^c(y)\right\}
\]

and

\[
\varphi^c(y) = \min_{x \in \text{supp}(\mu_0)} \left\{\frac{1}{2}d^2_{CC}(x, y) - \varphi(x)\right\} \quad (2.25)
\]

and the optimal transport map is concentrated on the \(c\)-superdifferential of \(\varphi\). Since the distribution \(\Delta\) on the corank 1 Carnot group \(G\) is two-generating, it follows that \(d^2_{CC}\), locally Lipschitz on \(G \times G\), see Figalli and Rifford [8, Proposition 4.2, p. 136]. Therefore, applying the version of [8, Theorem 3.2, p. 130] with the weaker assumption for \(d^2_{CC}\) of being locally Lipschitz, there exists a \(d^2_{CC}/2\)-concave function \(\varphi : G \rightarrow \mathbb{R}\) such that \(\mathcal{M}_\varphi\) is open and \(\varphi\) is locally Lipschitz in a neighborhood of \(\mathcal{M}_\varphi \cap \text{supp}(\mu_0)\), thus \(\mu_0\)-a.e. differentiable in \(\mathcal{M}_\varphi\). Furthermore, for \(\mu_0\)-a.e. \(x\), there exists a unique optimal transport map defined \(\mu_0\)-a.e. by

\[
\psi(x) := \begin{cases} 
\exp_x(-d\varphi(x)) & \text{if } x \in \mathcal{M}_\varphi \cap \text{supp}(\mu_0); \\
X & \text{if } x \in \mathcal{S}_\varphi \cap \text{supp}(\mu_0),
\end{cases}
\]

and for \(\mu_0\)-a.e. \(x\) there exists a unique minimizing geodesic joining \(x\) and \(\psi(x)\) (or, equivalently, joining the element \(e\) with \(x^{-1} \circ \psi(x)\)). Hereafter, \(d\varphi(x)\) is identified by \(\nabla\varphi(x)\) whenever the latter exists, where \(\nabla = (X^0, X^1, ..., X^d, Z)\) is the Carnot gradient.

We notice that one cannot apply directly [8, Theorem 3.5, p. 132] of Figalli and Rifford to deduce the absolute continuity of the Wasserstein geodesic between \(\mu_0\) and \(\mu_1\) since in our case the semiconcavity assumption does not hold; however, we can recall the first part of their proof to conclude (based on [8, Theorem 3.2, p. 130] and [19, Corollary 7.22]) that
there is a unique Wasserstein geodesic \((\mu_s)_{s\in[0,1]}\) joining \(\mu_0\) and \(\mu_1\) given by the push-forward measure \(\mu_s = (\psi_s)_\#\mu_0\) for \(s \in [0,1]\), where

\[
\psi_s(x) := \begin{cases} 
\exp_s(-sd\varphi(x)) & \text{if } x \in \mathcal{M}_\varphi \cap \text{supp}(\mu_0); \\
 x & \text{if } x \in \mathcal{S}_\varphi \cap \text{supp}(\mu_0).
\end{cases}
\]  

(2.27)

In order to show the absolute continuity of the Wasserstein geodesic \(\mu_s\) we proceed in the following way.

**Proposition 2.5.** Let \(s \in (0,1)\). Consider the notations introduced above and the assumptions of Theorem 1.1. Under these conditions the interpolant measure \(\mu_s = (\psi_s)_\#\mu_0\) is absolutely continuous w.r.t. \(\mathcal{L}^{k+1}\).

**Proof.** We note that one cannot apply directly the result from Figalli and Rifford [8, Theorem 3.3] since the squared metric function \(d_G^2\) is not necessarily locally semiconcave on \(G \times G \setminus D_G\), where \(D_G = \{(x,x): x \in G\}\) is the diagonal of \(G \times G\). However, we can proceed exactly as in Figalli and Juillet [7, Theorem 1.2]; we just outline the proof. By the law of large numbers for empirical measures, we fix a sequence \(\{\mu^l_i\}_{i\in\mathbb{N}}\) of measures which weakly converge to \(\mu_1\), where \(\mu^l_1 = \frac{1}{l} \sum_{i=1}^l \delta_{y_i}\) for some \(y_i \in \text{supp}(\mu_1),\ i \in \{1,...,l\}\). Let \(\psi^l\) be the optimal transport map between \(\mu_0\) and \(\mu^l_1\), and \(\psi_s^l\) be the corresponding interpolant map. By applying the measure contraction property \(\text{MCP}(0,k+3)\) of Rizzi [15, Theorem 2], it turns out that for every \(S \subset \text{supp}(\mu_0)\) and \(l \in \mathbb{N}\) we have

\[
\mathcal{L}^{k+1}(\psi_s^l(S)) \geq (1-s)^{k+3}\mathcal{L}^{k+1}(S).
\]

By using suitable test-functions, the stability of the optimal transport, see Villani [19, Corollary 5.23, p. 79], implies that \(\psi_s^l\) converges to \(\psi_s\) \(\mu_0\)-a.e. It remains to pass to the limit as \(l \to \infty\) in order to prove the claim. \(\square\)

Before the proof of our main theorem in the next section let us indicate a technical difficulty that we need to address in the proof. This consists of the fact that in our setting the potential \(\varphi\) generating the optimal transportation map \(\psi\) via (2.26) is not locally semiconcave but only locally Lipschitz. Due to the lack of semiconcavity we do not have an Aleksandrov-type second order differentiability for \(\varphi\) and consequently, thus we do not know if \(\psi\) is differentiable almost everywhere. This regularity issue appears when we consider the transport of the mass along abnormal geodesics.

### 3. Proof of the Jacobian Determinant Inequality (Theorem 1.1)

Let \(s \in (0,1)\). We shall keep the previous notations. The proof is divided into two main parts: the static and moving cases, respectively. The latter case is also divided into two parts depending how the mass is transported, i.e., along abnormal or strictly normal geodesics.

#### 3.1. Static case.

We assume the static set \(\mathcal{S}_\varphi \cap \text{supp}(\mu_0) = \{x \in \text{supp}(\mu_0): x = \psi(x)\}\) has a positive \(\mu_0\)-measure. Note that \(\psi_s(x) = x\) for every \(x \in \mathcal{S}_\varphi\). On one hand, if we consider the density points of \(\mathcal{S}_\varphi\), we have that \(\text{Jac}(\psi)(x) = \text{Jac}(\psi_s)(x) = 1\) for \(\mu_0\)-a.e. \(x \in \mathcal{S}_\varphi\). On the other hand, if \(x \in \mathcal{S}_\varphi\), we have that \(\exp_{e}(\theta_x) = x^{-1} \circ \psi(x) = e, i.e., \theta_x = (p_x,p_x) = 0_{\mathbb{R}^{k+1}}\), thus \(A_{p_x} = 0_{\mathbb{R}^{k+1}}\). Therefore, by the definition of the distortion coefficient, we have \(\tau_{k+1}^{s,\alpha}(\theta_x) = s\) and \(\tau_{k+1}^{1,\alpha}(\theta_x) = 1 - s\), which concludes the proof of (1.7).

#### 3.2. Moving case.

We now assume that the moving set \(\mathcal{M}_\varphi \cap \text{supp}(\mu_0)\) has a positive \(\mu_0\)-measure. Due to (2.26), there exists a null \(\mathcal{L}^{k+1}\)-measure set \(C_0 \subset \mathcal{M}_\varphi \cap \text{supp}(\mu_0)\) such that for every \(x \in S := \mathcal{M}_\varphi \cap \text{supp}(\mu_0) \setminus C_0\) the function \(\varphi\) is differentiable at \(x\), the points \(x\) and \(\psi(x)\) can be joined by a unique minimizing geodesic and \(x^{-1} \circ \psi(x) = \exp_e(-\nabla \varphi(x))\), where

\[
\nabla \varphi(x) = (p_x,p_z),
\]
with
\[ p_x = (X^0 \varphi(x), X^1 \varphi(x), \ldots, X^d \varphi(x)) \]
and \( p_z = Z \varphi(x) \).

Let
\[ S_0 = \{ x \in S : \mathcal{A}p_x = 0_{\mathbb{R}^k}, \text{ where } p_x \text{ is from (3.1)} \}, \]
and
\[ S_1 = S \setminus S_0 = \{ x \in S : \mathcal{A}p_x \neq 0_{\mathbb{R}^k}, \text{ where } p_x \text{ is from (3.1)} \}. \]

We distinguish two cases.

3.2.1. Moving along abnormal geodesics. We assume that \( \mu_0(S_0) > 0 \). In terms of vector fields, the fact that \( \mathcal{A}p_x = 0_{\mathbb{R}^k} \) with \( p_x = (X^0 \varphi(x), X^1 \varphi(x), \ldots, X^d \varphi(x)) \) implies that \( X^1 \varphi(x) = \ldots = X^d \varphi(x) = 0_{\mathbb{R}^2} \) for a.e. \( x \in S_0 \). According to the explicit form of geodesics, see (2.2), we have
\[
\psi_s(x) = x \circ \exp_s(-s\nabla \varphi(x)) = x \circ \exp_s(-sX^0 \varphi(x), 0_{\mathbb{R}^{2d}}, Z \varphi(x))
\]
for a.e. \( x = (x_1, \ldots, x_k, z) \in S_0 \). In a similar way, one has
\[
\psi(x) = (x_1 - \partial_{x_1} \varphi(x), \ldots, x_{k-2d} - \partial_{x_{k-2d}} \varphi(x), x_{k-2d+1}, \ldots, x_k, z),
\]
for a.e. \( x = (x_1, \ldots, x_k, z) \in S_0 \).

We divide the proof into three steps.

Step 1: \( \varphi(\xi, \eta, z) \) is \( d_{\mathbb{R}^{k-2d}}^2/2 \)-concave on \( \mathbb{R}^{k-2d} \) for every \( (\eta, z) \in \mathbb{R}^{2d} \times \mathbb{R} \) fixed. Since \( \varphi \) is \( d_{CC}^2/2 \)-concave on \( G \), one has by (2.25) that for every \( (\xi, \eta, z) \in \mathbb{R}^{k-2d} \times \mathbb{R}^{2d} \times \mathbb{R} \),
\[
\varphi(\xi, \eta, z) = \min_{(\xi, \eta, z) \in \supp(\mu_1)} \left\{ \frac{1}{2} d_{CC}^2((\xi, \eta, z), (\xi, \eta, z)) - \varphi^c(\xi, \eta, z) \right\}
\]
Let \( \pi_1 : \mathbb{R}^{k-2d} \times \mathbb{R}^{2d} \times \mathbb{R} \to \mathbb{R}^{k-2d} \) be the projection \( \pi_1(\xi, \eta, z) = \xi \). For every \( \xi \in \pi_1(\supp(\mu_1)) \), let us introduce the compact set
\[
\Pi_\xi = \{(\eta, z) \in \mathbb{R}^{2d} \times \mathbb{R} : (\xi, \eta, z) \in \supp(\mu_1)\}.
\]
Let us fix \( (\eta, z) \in \mathbb{R}^{2d} \times \mathbb{R} \). We notice that the function \( \phi_{\eta, z} : \pi_1(\supp(\mu_1)) \to \mathbb{R} \cup \{-\infty\} \) defined by
\[
\phi_{\eta, z}(\xi) = \max_{(\eta, z) \in \Pi_\xi} \left\{ \varphi^c(\xi, \eta, z) - \frac{1}{2} d_{CC}^2((\eta, z), (\eta, z)) \right\}
\]
is well defined and \( \phi_{\eta, z} \neq -\infty \). Since
\[
\supp(\mu_1) = \bigcup_{\xi \in \pi_1(\supp(\mu_1))} (\xi, \Pi_\xi),
\]
by the Pythagorean rule (see Lemma 2.1) we have that for every \( \xi \in \mathbb{R}^{k-2d} \),
\[
\varphi(\xi, \eta, z) = \min_{\xi \in \pi_1(\supp(\mu_1))} \min_{(\eta, z) \in \Pi_\xi} \left\{ \frac{1}{2} d_{CC}^2((\xi, \eta, z), (\xi, \eta, z)) - \varphi^c(\xi, \eta, z) \right\}
\]
\[
= \min_{\xi \in \pi_1(\supp(\mu_1))} \min_{(\eta, z) \in \Pi_\xi} \left\{ \frac{1}{2} d_{\mathbb{R}^{k-2d}}^2(\xi, \xi) + \frac{1}{2} d_{CC}^2((\eta, z), (\eta, z)) - \varphi^c(\xi, \eta, z) \right\}
\]
\[
= \min_{\xi \in \pi_1(\supp(\mu_1))} \left\{ \frac{1}{2} d_{\mathbb{R}^{k-2d}}^2(\xi, \xi) + \min_{(\eta, z) \in \Pi_\xi} \left\{ \frac{1}{2} d_{CC}^2((\eta, z), (\eta, z)) - \varphi^c(\xi, \eta, z) \right\} \right\}
\]
which concludes the claim.
Step 2: For a.e. \( x = (\xi, \eta, z) \in S_0 \) one can identify the Jacobian determinants \( \text{Jac}(\psi_s)(x) \) and \( \text{Jac}(\psi)(x) \) with \( \det[I_{k-2d} - s\text{Hess}_s(\varphi)(x)] \) and \( \det[I_{k-2d} - \text{Hess}_s(\varphi)(x)] \), respectively, where \( I_{k-2d} \) is the \( (k-2d) \times (k-2d) \) unit matrix and \( \text{Hess}_s(\varphi)(\xi, \eta, z) \) is the usual Euclidean Hessian of the function \( \varphi(\cdot, \eta, z) \) at the point \( \xi \).

By Step 1, the \( d_{\text{R}^{k-2d}}^2/2 \)-concavity of \( \varphi(\cdot, \eta, z) \) is equivalent to the convexity of \( \xi \mapsto \frac{\|\xi\|^2_{\text{R}^{k-2d}}}{2} - \varphi(\xi, \eta, z) \) on \( \text{R}^{k-2d} \). In particular, by the Aleksandrov’s second differentiability theorem, the latter function is twice differentiable a.e., and its Hessian \( I_{k-2d} - \text{Hess}_s(\varphi)(\xi, \eta, z) \) is positive semidefinite and symmetric for a.e. \( \xi \in \text{R}^{k-2d} \); the same is true for \( I_{k-2d} - s\text{Hess}_s(\varphi)(\xi, \eta, z) \).

By (3.3) – if it exists– the formal Jacobian of \( \psi_s \) for a.e. \( x = (\xi, \eta, z) \in S_0 \) has the structure

\[
\left[ \begin{array}{cc}
A_s(x) & B_s(x) \\
0 & I_{2d+1}
\end{array} \right],
\]

where \( A_s(x) = I_{k-2d} - s\text{Hess}_s(\varphi)(\xi, \eta, z) \). Note however that \( B_s(x) \) might not exist since we have no information on the differentiability of \( \partial_i \varphi(\xi, \cdot, \cdot), i \in \{1, \ldots, k-2d\} \). We shall explain below that the existence of \( B_s(x) \) is not relevant as far as existence of the global Jacobi determinant is concerned.

Observe first that due to Proposition 2.5, the interpolant measure \( \mu_s = (\psi_s)\#\mu_0 \) is absolutely continuous w.r.t. \( L^{k+1} \); let \( \rho_s \) be its density function. Since the corank 1 Carnot group \( (G, d_{\text{CC}}, L^{k+1}) \) is a nonbranching metric measure space, both \( \psi \) and \( \psi_s \) are injective maps on a set of full measure, which requires a reinterpretation of the Monge-Ampère equations (3.5).

However, as we pointed out, the terms \( \text{Jac}(\psi_s)(x) \) and \( \text{Jac}(\psi)(x) \) may not exist on a set \( \tilde{S} \subset S_0 \) of positive measure, which requires a reinterpretation of the Monge-Ampère equations in (3.5); we shall consider only the first term since the other one works similarly.

First of all, \( \mu_s = (\psi_s)\#\mu_0 \) implies

\[
\int_G h(y)\,d\mu_s(y) = \int_G h(\psi_s(x))\,d\mu_0(x)
\]

for every Borel function \( h : G \to [0, \infty) \). In particular, for every measurable set \( \tilde{S} \subset S_0 \) with positive measure and Borel function \( h \) with \( \text{supp}(h) \subseteq \psi_s(\tilde{S}) \) we have

\[
\int_{\psi_s(\tilde{S})} h(y)\,d\rho_s(y) = \int_G h(y)\,\rho_s(y)\,dL^{k+1}(y) = \int_{\psi_s(\tilde{S})} h(y)\,\rho_s(y)\,dL^{k+1}(y).
\]

Let \( \pi_2 : \text{R}^{k-2d} \times \text{R}^{2d} \times \text{R} \to \text{R}^{2d+1} \) be the projection \( \pi_2(y) = \pi_2(y_1, y_2, y_3) = (y_2, y_3) \) and for every \( (y_2, y_3) \in \pi_2(\psi_s(\tilde{S})) \), let \( \Pi_{(y_2, y_3)} = \{y_1 \in \text{R}^{k-2d} : (y_1, y_2, y_3) \in \psi_s(\tilde{S})\} \). It is clear that \( \psi_s(\tilde{S}) = \cup_{(y_2, y_3)\in\pi_2(\psi_s(\tilde{S}))}(\Pi_{(y_2, y_3)}; y_2, y_3) \); by Fubini’s theorem it follows that

\[
\int_{\psi_s(\tilde{S})} h(y)\,\rho_s(y)\,dL^{k+1}(y) = \int_{\pi_2(\psi_s(\tilde{S}))} \left( \int_{\Pi_{(y_2, y_3)}} h(y)\,\rho_s(y)\,dL^{k-2d}(y_1) \right) \,dL^{2d+1}(y_2, y_3).
\]

We consider the change of variables \( y = (y_1, y_2, y_3) = \psi_s(x) \) with \( x = (\xi, \eta, z) \) which shows through (3.3) that \( y_1 = \pi_1(\psi_s(x)) \) and \( (y_2, y_3) = (\eta, z) \). Thus, \( dL^{k-2d}(y_1) = \det[A_s(x)]\,dL^{k-2d}(\xi) \) and \( \Pi_{(y_2, y_3)} = \pi_1(\tilde{S}_{\eta, z}; \eta, z) \) where \( \tilde{S}_{\eta, z} = \{\xi \in \text{R}^{k-2d} : (\xi, \eta, z) \in \tilde{S}\} \). Moreover, since \( \pi_2(\psi_s(\tilde{S})) = \pi_2(\tilde{S}) \), the latter term in the above relation becomes

\[
\int_{\pi_2(\tilde{S})} \left( \int_{\tilde{S}_{\eta, z}} h(\psi_s(x))\,\rho_s(\psi_s(x))\,\det[A_s(x)]\,dL^{k-2d}(\xi) \right) \,dL^{2d+1}(\eta, z)
\]

which is nothing but

\[
\int_{\tilde{S}} h(\psi_s(x))\,\rho_s(\psi_s(x))\,\det[A_s(x)]\,dL^{k+1}(x).
\]
The latter expression, relation
\[
\int_G h(\psi_s(x))d\mu_0(x) = \int_{\tilde{S}} h(\psi_s(x))\rho_0(x)dL^{k+1}(x)
\]
and (3.6) together with the arbitrariness of \( h \) and \( \tilde{S} \subset S_0 \) give that
\[
\rho_0(x) = \rho_s(\psi_s(x))\det[A_s(x)] \quad \text{for } \mu_0\text{-a.e. } x \in S_0.
\]
Consequently, (3.5) enables us to identify
\[
\text{Jac}(\psi_s)(x) := \det[A_s(x)] = \det[\bar{I}_{k-2d} - s\text{Hess}_\varphi(\varphi)(\xi, \eta, z)] \quad \text{for } \mu_0\text{-a.e. } x \in S_0,
\]
concluding the claim.

Step 3: proof of Theorem 1.1 concluded (abnormal mass transportation). Since \( I_{k-2d} - s\text{Hess}_\varphi(\varphi)(x) = (1-s)I_{k-2d} + s(I_{k-2d} - \text{Hess}_\varphi(\varphi)(x)) \), we may apply the concavity of \( \det(\cdot)\frac{1}{1+2s} \) on the cone of \((k-2d) \times (k-2d)\) positive semidefinite symmetric matrices, obtaining through Step 2 that
\[
\text{Jac}(\psi_s)(x)\frac{1}{1+2s} \geq 1 - s + s \text{Jac}(\psi_s)(x)\frac{1}{1+2s} \quad \text{a.e. } x \in S_0.
\]
Now, the concavity of the function \( t \mapsto t\frac{k-2d}{k+1} \), \( t > 0 \), gives that
\[
\text{Jac}(\psi_s)(x)\frac{k-2d}{k+1} = \left(\text{Jac}(\psi_s)(x)\frac{1}{1+2s}\right)^{\frac{k-2d}{k+1}}
\[
\geq \left(1 - s + s \text{Jac}(\psi_s)(x)\frac{1}{1+2s}\right)^{\frac{k-2d}{k+1}}
\[
\geq 1 - s + s \text{Jac}(\psi_s)(x)\frac{1}{1+2s} \quad \text{for a.e. } x \in S_0,
\]
which is exactly the required inequality (1.7).

3.2.2. Moving along strictly normal geodesics. We assume that \( \mu_0(S_1) > 0 \). The proof will be divided into four steps.

Step 1: \( \varphi \) admits a Hessian for a.e. \( x \in S_1 \).

It is well known that the Euclidean squared distance function \( d^2_{R^{k-2d}} \) is semiconcave on \( R^{k-2d} \times R^{k-2d} \), see Cannarsa and Sinestrari [5]. Moreover, since the distribution \( \bar{\Delta} = \{X_1, \ldots, X_k\} = \{X_{k-2d+1}, \ldots, X_k\} \) is fat on \( R^{2d+1} \), according to Figalli and Rifford [8, Proposition 4.1, pg. 136], the squared distance function \( d^2_{CC} \) is locally semiconcave on \( R^{2d+1} \times R^{2d+1} \setminus \bar{D} \), where \( \bar{D} \) denotes the diagonal of the set \( R^{2d+1} \times R^{2d+1} \), namely \( \bar{D} = \{(\eta, z), (\eta, z)\} : (\eta, z) \in R^{2d} \times R \) . Consequently, by the Pythagorean rule (see Lemma 2.1), the squared distance function \( d^2_{CC} \) is locally semiconcave on \( G \times G \setminus \bar{D} \), where
\[
\bar{D} = \{(\xi, \eta, z), (\xi', \eta, z)\} : \xi, \xi' \in R^{k-2d}, (\eta, z) \in R^{2d} \times R \}.
\]
In order to conclude the claim, we slightly modify the proof of [8, Theorem 3.2]. Namely, if \( x \in S_1 \) is arbitrarily fixed, we have that \( \mathcal{A}p_x \neq 0_k \) with \( p_x \) from (3.1), i.e., \( (x, \psi(x)) \notin \bar{D} \). In particular, if \( x = (\xi_x, \eta_x, z_x) \) and \( \psi(x) = (\xi_x', \eta_x', z_x') \) then \( \tilde{d}_{CC}((\eta_x, z_x), (\eta_x', z_x')) =: r_x > 0 \). By continuity reason, there exists an open neighborhood \( V_x \subset M_\varphi \cap \operatorname{supp}(\mu_0) \) of \( x \) such that \( \tilde{d}_{CC}((\eta_w, z_w), (\eta_w', z_w')) > \frac{r_x}{2} \) for every \( w = (\xi_w, \eta_w, z_w) \in V_x \) and \( \psi(w) = (\xi_w', \eta_w', z_w') \in \partial \varphi(w) \). Let \( \tilde{\varphi}_x : G \to R \) be defined by
\[
\tilde{\varphi}_x(w) = \inf \left\{ \frac{1}{2} \tilde{d}_{CC}(w, y) - \varphi^*(y) : y = (\xi_y, \eta_y, z_y) \in \operatorname{supp}(\mu_1), \tilde{d}_{CC}((\eta_w, z_w), (\eta_y, z_y)) > \frac{r_x}{2} \right\},
\]
where \( w = (\xi_w, \eta_w, z_w) \). Now, the locally semiconcavity of \( d^2_{CC} \) on \( G \times G \setminus \bar{D} \) is inherited by the \( d^2_{CC}/2 \)-concave function \( \tilde{\varphi}_x \) on \( V_x \). Moreover, one can observe that \( \tilde{\varphi}_x = \varphi \) on \( V_x \), thus \( \varphi \) is semiconcave on \( V_x \). By the Aleksandrov-Bangert theorem, see [6, Theorem 3.10, pg. 238], we conclude that \( \varphi \) admits a Hessian a.e. on \( V_x \), concluding the claim.
Step 2: $\psi(x) \notin \text{cut}_G(x)$ for a.e. $x \in S_1$.

Let us fix $x \in S_1$. We first observe that $|Z\varphi(x)| \leq \frac{2\pi}{\alpha_d}$. Indeed, if we assume that $|Z\varphi(x)| > \frac{2\pi}{\alpha_d}$ then $s \mapsto x^{-1} \circ \psi_s(x)$ turns to be a strictly normal geodesic (i.e., normal but abnormal, since $A_{p_x} \neq 0_{R^k}$) starting from $e$ and according to the Jacobian along the geodesic, see (2.4), the geodesic $s \mapsto x^{-1} \circ \psi_s(x)$ contains a conjugate point to $e$ for the value $s_x = \frac{2\pi}{\alpha_d|Z\varphi(x)|}$, i.e., the Jacobian vanishes at $s_x$. Since $0 < s_x < 1$, such geodesics cannot be minimizing after their first conjugate points, see Agrachev, Barilari and Boscaín [1, Corollary 8.45], which contradicts the minimality of $s \mapsto x^{-1} \circ \psi_s(x)$, $s \in [0,1]$. Therefore, $|Z\varphi(x)| \leq \frac{2\pi}{\alpha_d}$.

Fix $x \in S_1$ such that $\varphi$ has a Hessian at $x$ (which occurs a.e. on $S_1$, according to Step 1) and assume by contradiction that $\psi(x) \in \text{cut}_G(x)$. Since $A_{p_x} \neq 0_{R^k}$ (with $p_x$ from (3.1)), it follows that $x^{-1} \circ \psi(x) = \exp_{x}(-p_x, -p_x)$ where $|p_x| = |Z\varphi(x)| = \frac{2\pi}{\alpha_d}$. In particular, by the explicit representation (2.1) of the geodesic $\gamma = (\gamma^0, \gamma^1, ..., \gamma^d)$ given by $s \mapsto x^{-1} \circ \psi_s(x) = \exp_{x}(-s\nabla\varphi(x))$ and $|Z\varphi(x)| = \frac{2\pi}{\alpha_d}$ we have

$$
\gamma^{d-q+1}(0) = ... = \gamma^d(1) = 0_{R^2},
$$

where $q \in \{1, ..., d\}$ is the multiplicity of $\alpha_d$, see Proposition 2.3. We distinguish two cases:

**Case 1:** $X^{d-q+1}\varphi(x) = ... = X^d\varphi(x) = 0_{R^2}$. Since $\varphi$ admits a Hessian at $x \in S_1$, then $\varphi$ is differentiable in $x$ and the $c$-superdifferential $\partial c\varphi(x)$ contains the unique element $\psi(x)$; in particular,

$$
\varphi(z) - \varphi(x) \leq \frac{1}{2}(d_{CC}^2(\varphi(x), z) - d_{CC}^2(\varphi(x), x)) \quad \text{for every } z \in \text{supp}(\mu_0). \quad (3.9)
$$

Writing inequality (3.9) for $z = \exp_x(\nu)$ and $z = \exp_x(-\nu)$, respectively, it yields

$$
\lim_{||\nu|| \to 0} \frac{\varphi(\exp_x(\nu)) + \varphi(\exp_x(-\nu)) - 2\varphi(x)}{||\nu||^2} \leq \frac{1}{2} \lim_{||\nu|| \to 0} \frac{d_{CC}^2(\varphi(x), \exp_x(\nu)) + d_{CC}^2(\varphi(x), \exp_x(-\nu)) - 2d_{CC}^2(\varphi(x), x)}{||\nu||^2}.
$$

According to Proposition 2.3 the right hand side of the above inequality is $-\infty$ which contradicts the existence of the Hessian of $\varphi$ at $x$.

**Case 2:** there exists $i_0 \in \{d-q+1, ..., d\}$ such that $X^{i_0}\varphi(x) \neq 0_{R^2}$. By rotating the vector $X^{i_0}\varphi(x)$ by any angle in $\mathbb{R}^3$, one can see from (2.1) that we can construct infinitely many minimal geodesics joining $e$ and $\exp_{x}(-\nabla\varphi(x)) = x^{-1} \circ \psi(x)$ which differ by each other in the $i_0^{th}$ component. But the latter statement contradicts the fact that $x \in S_1 \subset S$, (in particular, the uniqueness of the geodesic joining $e$ to $x^{-1} \circ \psi(x)$, $x \in S$), see Figalli and Rifford [8, Theorem 3.2].

**Step 3:** $\psi_s$ and $\psi$ are differentiable a.e. on $S_1$; moreover, for a.e. $x \in S_1$,

$$
d\psi_s(x) = Y_x(s) \left[ \text{Hess} \frac{d_{CC}^2(\psi_s(x), \cdot)}{2} \right]_x - s \text{Hess} \varphi(x), \quad (3.10)
$$

$$
d\psi(x) = Y_x(1) \left[ \text{Hess} \frac{d_{CC}^2(\psi(x), \cdot)}{2} \right]_x - \text{Hess} \varphi(x), \quad (3.11)
$$

where

$$
Y_x(s) = d(\exp_x)_{-s\nabla\varphi(x)}, \quad s \in (0,1].
$$

For the first part, we recall that for every $x \in S_1$,

$$
\psi(x) = x \circ \exp_{e}(-\nabla\varphi(x)) \quad \text{and} \quad \psi_s(x) = x \circ \exp_{e}(-s\nabla\varphi(x)),
$$

see (2.26) and (2.27), respectively. Since $\psi(x) \notin \text{cut}_G(x)$ for a.e. $x \in S_1$ (Step 2), thus $-\nabla\varphi(x)$ belongs to the injectivity domain $D$ of $\exp_{e}$, and $\varphi$ has a Hessian a.e. on $S_1$ (Step 1), it follows that $\psi$ and $\psi_s$ are differentiable at a.e. $x \in S_1$.
In order to prove (3.10) and (3.11), we need a discrete version of Claim 2.1:

**Claim 3.1.** Let \( m \in \mathbb{N} \), \( F : \mathbb{R}^{2m} \to \mathbb{R}^m \) be a smooth function in a neighborhood of \((x, y) \in \mathbb{R}^{2m}\) and \( \{x_n\}, \{y_n\}, \{z_n\} \subseteq \mathbb{R}^m \) be three sequences satisfying the following properties:

(a) \( \lim_{n \to \infty} x_n = x \) and \( x_n \neq x \) for every \( n \in \mathbb{N} \);

(b) \( \lim_{n \to \infty} y_n = y \) and \( F(x_n, y_n) = F(x, y) \) for every \( n \in \mathbb{N} \);

(c) \( \lim_{n \to \infty} z_n = 0 \) and \( \lim_{n \to \infty} \|x_n - x\|_{\mathbb{R}^m} = v \in \mathbb{R}^m \).

Then

\[
\lim_{n \to \infty} \frac{F(x_n, y_n + z_n) - F(x, y)}{\|x_n - x\|_{\mathbb{R}^m}} = D_2 F(x, y) v. \tag{3.12}
\]

The proof of the claim is left as an exercise to the interested reader.

We shall apply the above claim to prove only (3.10) since the proof of (3.11) works in a similar way. To do this, without loss of generality, we can fix a Lebesgue density point \( x \in S_1 \) in the differentiability set of \( \varphi \), i.e., where \( \varphi \) is twice differentiable (thus both \( \nabla \varphi(x) \) and \( \text{Hess} \varphi(x) \) exist).

Since \( x \) is a Lebesgue density point of \( S_1 \), we can find a linearly independent frame \( \{v_i : i = 1, \ldots, k+1\} \) at \( x \), such that there exist sequences \( \{x_{n,i}\} \subseteq \mathbb{R}^{k+1} \setminus \text{cut}_G(\psi_s(x)) \) in the differentiability set of \( \varphi \) such that for every \( i \in \{1, \ldots, k+1\} \):

\[
\lim_{n \to \infty} x_{n,i} = x, \quad \text{and} \quad \lim_{n \to \infty} \frac{x_{n,i} - x}{\|x_{n,i} - x\|_{\mathbb{R}^{k+1}}} = v_i. \tag{3.13}
\]

Fix \( i \in \{1, \ldots, k+1\} \). We shall apply Claim 3.1 with the smooth function \( F(w, q) = \exp_u(q) \) in a neighborhood of the point \((x, y) := (x, -s \nabla \varphi(x)) \) and three sequences \( x_{n, i}, y_{n,i} := -\nabla d_{CC}^2(\psi_s(x), \cdot)_{x_{n,i}} \) and \( z_{n,i} := -y_{n,i} - s \nabla \varphi(x_{n,i}) \). We clearly have that \( \psi_s(x_{n,i}) = F(x_{n,i}, y_{n,i} + z_{n,i}) \). According to Proposition 2.2, we have that \( F(x_{n,i}, y_{n,i}) = \psi_s(x) = F(x, y) \) for every \( n \in \mathbb{N} \) and \( \lim_{n \to \infty} y_{n,i} = -\nabla d_{CC}^2(\psi_s(x), \cdot)_{x} \) for every \( n \in \mathbb{N} \). The latter relation and (3.14) give that \( \lim_{n \to \infty} z_{n,i} = -y + s \nabla \varphi(x) = 0 \) and (3.15) and (3.13) yield that

\[
\lim_{n \to \infty} \frac{\psi_s(x_{n,i}) - \psi_s(x)}{\|x_{n,i} - x\|_{\mathbb{R}^{k+1}}} = d_{CC}^2(\psi_s(x), \cdot)_{x} - s \text{Hess} \varphi(x) v_i. \tag{3.16}
\]

Thus, (3.12) together with (3.16) reads as

\[
d_{CC}^2(\psi_s(x), \cdot)_{x} - s \text{Hess} \varphi(x) v_i = D_2 F(x, y) v.
\]

Since \( \text{span}\{v_1, \ldots, v_{k+1}\} = \mathbb{R}^{k+1} \), the latter relation yields (3.10).

**Step 4:** proof of Theorem 1.1 concluded (strictly normal mass transportation). We recall by Proposition 2.4 that the Hessian

\[
H_{x, \psi}(s) := \text{Hess} d_{CC}^2(\psi_s(x), \cdot)_{x} - s \text{Hess} d_{CC}^2(\psi(x), \cdot)_{x}
\]
is a \((k + 1) \times (k + 1)\) type positive semidefinite, symmetric matrix. Since
\[
\nabla \frac{d^2_{\text{CC}}(\psi(x), \cdot)}{2} |_x - \nabla \varphi(x) = 0 \quad \text{for a.e. } x \in S_1,
\]
a similar argument as in the first part of the proof of Proposition 2.4 and the \(d^2_{\text{CC}}/2\)-concavity of \(\varphi\) gives that \(\text{Hess} \frac{d^2_{\text{CC}}(\psi(x), \cdot)}{2} |_x - \text{Hess} \varphi(x)\) is also a positive semidefinite, symmetric matrix for a.e. \(x \in S_1\). Thus, by the concavity of \(\det(\cdot)^{1/r}\) on the set of \((k + 1) \times (k + 1)\) type positive semidefinite, symmetric matrices one has
\[
(\text{Jac}(\psi)(x))^{\frac{1}{r+1}} = \det \left( Y_x(s) \frac{\text{Hess} \frac{d^2_{\text{CC}}(\psi(x), \cdot)}{2} |_x - s \text{Hess} \varphi(x)}{1 - s} \right)^{\frac{1}{r+1}}
\]
\[
= \det(Y_x(s))^{\frac{1}{r+1}} \det \left[ (1 - s) \frac{H_x(\psi(x))(s) - H_x(\varphi(x))}{1 - s} + s \left( \frac{\text{Hess} \frac{d^2_{\text{CC}}(\psi(x), \cdot)}{2} |_x - \text{Hess} \varphi(x)}{1 - s} \right) \right]^{\frac{1}{r+1}}
\]
\[
\geq \det(Y_x(s))^{\frac{1}{r+1}} \left( (1 - s) \det \left[ \frac{H_x(\psi(x))(s) - H_x(\varphi(x))}{1 - s} \right]^{\frac{1}{r+1}} + s \det \left( \frac{\text{Hess} \frac{d^2_{\text{CC}}(\psi(x), \cdot)}{2} |_x - \text{Hess} \varphi(x)}{1 - s} \right)^{\frac{1}{r+1}} \right)
\]
where \(Y_x\) corresponds to \(Y_x\) via (2.19).

On one hand, by (2.5) we have that
\[
\det(Y_x(s)Y_x(1)^{-1}) = \frac{\text{Jac}(\exp_e)(-s \nabla \varphi(x))}{\text{Jac}(\exp_e)(-\nabla \varphi(x))},
\]
thus by (2.4),
\[
s \det(Y_x(s)Y_x(1)^{-1})^{\frac{1}{r+1}} = \tau_{s;k}^{k,\alpha}(\theta_x),
\]
where \(\theta_x = -\nabla \varphi(x) \in D\). On the other hand, by the definition of \(Y_x\) (see (2.19)) and relation (2.6) we also have
\[
(1 - s) \det(Y_x(s)Y_x(1)^{-1})^{\frac{1}{r+1}} = \tau_{1-s}^{k,\alpha}(\theta_x).
\]
Combining the above facts we obtain the required Jacobian inequality (1.7). \(\Box\)

**Remark 3.1.** (a) Step 2 is the most fastidious part of the proof whenever the mass transportation is realized along abnormal geodesics, see §3.2.1. Note that reversing the roles of the metrics \(d^2_{\text{CC}}/2\) and \(d^2_{\text{CC}},\) a similar argument as in Step 1 shows that \(\varphi(\xi, \cdot, \cdot)\) is a \(d^2_{\text{CC}}/2\)-concave function on \(\mathbb{R}^{2d} \times \mathbb{R}\) \((\xi \in \mathbb{R}^{k-2d} \text{ is fixed})\). However, since \((x, \psi(x)) \in D\) for every \(x \in S_0\) (see (3.2) and (3.8)) and we only know that \(d^2_{\text{CC}}\) is locally semiconcave on \(\mathbb{R}^{2d+1} \times \mathbb{R}^{2d+1} \setminus D,\) where \(\bar{D} = \pi_2(D)\), no conclusion can be drawn in general for the locally semiconcavity of \(\varphi(\xi, \cdot, \cdot)\) on \(\pi_2(S_0)\). Thus, no higher regularity is known for \(\varphi(\xi, \cdot, \cdot)\) which justifies the block-decomposition of the Jacobian matrix of \(\psi\) in order to interpret and compute its determinant.

(b) If \(S_0 \subset G\) is open and \(\varphi\) is smooth enough on \(S_0\) (say \(C^1\)), one can see that \(X^1 \varphi(x) = \ldots = X^d \varphi(x) = 0\) for every \(x \in S_0\) (see §3.2.1) implies the fact that \(\varphi\) does not depend on the components \(x_{k-2d+1}, \ldots, x_k, z\), i.e., the Jacobian of \(\psi\) can be calculated in the usual way on \(S_0\); in particular, the following example falls into this framework.

We conclude this section by constructing two measures and the optimal transportation map between them such that a positive mass is transported along abnormal geodesics while the complementary mass is transported along strictly normal geodesics, respectively.
Example 3.1. Let $G = \mathbb{R}^m \times \mathbb{H}^d$ be the $m + 2d + 1$ dimensional corank 1 Carnot group endowed with its natural group operation inherited by the Euclidean space $\mathbb{R}^m$ and Heisenberg group $\mathbb{H}^d$; in our setting, $k = m + 2d$ and $\alpha_i = 4$ for every $i \in \{1, ..., d\}$ in (1.4). Let $a \in \mathbb{R}^m \setminus \{0\}$ and $b \in \mathbb{R}^{2d} \setminus \{0\}$ and consider the potentials $\varphi_0, \varphi_1 : G \to \mathbb{R}$ defined by

$$
\varphi_0(x_1, x_2) = \langle a, x_1 \rangle_{\mathbb{R}^m} \quad \text{and} \quad \varphi_1(x_1, x_2) = -(b, z_{x_2})_{\mathbb{R}^{2d}}
$$

for every $(x_1, x_2) = (x_1, (z_{x_1}, t_{x_1})) \in \mathbb{R}^m \times \mathbb{H}^d$, where $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$ denotes the usual inner product in $\mathbb{R}^m$. Moreover, let $\varphi_j^0, \varphi_j^1 : G \to \mathbb{R}$ be defined by

$$
\varphi_j^0(y_1, y_2) = -\frac{1}{2}||a||_{\mathbb{R}^m}^2 - \langle a, y_1 \rangle_{\mathbb{R}^m} \quad \text{and} \quad \varphi_j^1(y_1, y_2) = -\frac{1}{2}||b||_{\mathbb{R}^{2d}}^2 + \langle b, z_{y_2} \rangle_{\mathbb{R}^{2d}}
$$

for every $(y_1, y_2) = (y_1, (z_{y_1}, t_{y_2})) \in \mathbb{R}^m \times \mathbb{H}^d$. If $d_{CC}$ is the Carnot-Carathéodory metric on $G$, one has for every $(x_1, x_2) \in G$ that

$$
\varphi_j(x_1, x_2) = \inf_{(y_j, y_2) \in \mathbb{R}^m \times \mathbb{H}^d} \left\{ \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \varphi_j^0(y_1, y_2) \right\}, \quad j \in \{0, 1\},
$$

where we explore Lemma 2.1, and Ambrosio and Rigot [2, Example 5.7, p.287] in the case $j = 1$.

Accordingly, $\varphi_j$ are $d_{CC}^2/2$-concave functions on $G$, for $j \in \{0, 1\}$. If $\varphi = \min\left\{ \varphi_0, \varphi_1 \right\}$ and $\varphi^c = \max\left\{ \varphi_0^c, \varphi_1^c \right\}$, we claim that for every $(x_1, x_2) \in G$,

$$
\varphi(x_1, x_2) = \inf_{(y_1, y_2) \in \mathbb{R}^m \times \mathbb{H}^d} \left\{ \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \varphi^c(y_1, y_2) \right\}, \quad (3.17)
$$

To see this, let $(x_1, x_2) \mapsto \eta(x_1, x_2)$ be the function at the right hand side of (3.17). First, we have by definition that $\varphi_j^c(y_1, y_2) \leq \varphi_j^c(y_1, y_2)$ for every $(y_1, y_2) \in G$ and $j \in \{0, 1\}$. Accordingly, $\varphi_j(x_1, x_2) \geq \eta(x_1, x_2)$ for every $(x_1, x_2) \in G$ and $j \in \{0, 1\}$, i.e., $\varphi \geq \eta$.

To check the converse inequality, fix $(x_1, x_2) \in G$ arbitrarily and assume that $\varphi_0(x_1, x_2) \leq \varphi_1(x_1, x_2)$. Then for every $(y_1, y_2) \in G$, we have

$$
\varphi_0(x_1, x_2) \leq \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \varphi_0^c(y_1, y_2);
$$

$$
\varphi_0(x_1, x_2) \leq \varphi_1(x_1, x_2) \leq \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \varphi_1^c(y_1, y_2).
$$

 Consequently, for every $(y_1, y_2) \in G$, one has

$$
\varphi_0(x_1, x_2) \leq \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) + \min\{ -\varphi_0^c(y_1, y_2), -\varphi_1^c(y_1, y_2) \}
$$

$$
= \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \max\{ \varphi_0^c(y_1, y_2), \varphi_1^c(y_1, y_2) \}
$$

$$
= \frac{1}{2} d_{CC}^2((x_1, x_2), (y_1, y_2)) - \varphi^c(y_1, y_2).
$$

Taking the infimum on the right w.r.t. $(y_1, y_2) \in G$, we obtain $\varphi_0(x_1, x_2) \leq \eta(x_1, x_2)$, i.e., $\varphi(x_1, x_2) \leq \eta(x_1, x_2)$, which concludes the proof of (3.17). In particular, (3.17) implies that $\varphi$ is a $d_{CC}^2/2$-concave function on $G$.

Let $G^0 = \{(x_1, x_2) = (x_1, (z_1, t)) \in \mathbb{R}^m \times \mathbb{H}^d : \langle (a, b), (x_1, z_1) \rangle_{\mathbb{R}^m \times \mathbb{H}^d} = 0 \}$ be the hyperplane separating $\mathbb{R}^m \times \mathbb{R}^{2d+1}$ into two halfspaces $G^- = \{(x_1, (z_1, t)) \in \mathbb{R}^m \times \mathbb{H}^d : \langle (a, b), (x_1, z_1) \rangle_{\mathbb{R}^m \times \mathbb{H}^d} \leq 0 \}$ and $G^+ = G \setminus G^-$. It follows that

$$
\varphi(x_1, x_2) = \begin{cases} 
\varphi_0(x_1, x_2) & \text{if } (x_1, x_2) \in G^-; \\
\varphi_1(x_1, x_2) & \text{if } (x_1, x_2) \in G^+.
\end{cases}
$$
and $\varphi$ is differentiable on $G \setminus G^0$. Let $\psi : G \to G$ be the optimal transport map generated by the $d_{CC}^2/2$-concave function $\varphi$; by Proposition 2.1 we have for every $(x_1, x_2) \in G \setminus G^0$ that

$$
\psi(x_1, x_2) = \exp_{(x_1, x_2)}(-\nabla \varphi(x_1, x_2)) = \begin{cases} 
\psi_0(x_1, x_2) & \text{if } (x_1, x_2) \in G^- \setminus G^0; \\
\psi_1(x_1, x_2) & \text{if } (x_1, x_2) \in G^+,
\end{cases}
$$

where

$$
\psi_0(x_1, x_2) = \exp_{(x_1, x_2)}(-a, 0_{\mathbb{R}^{d+1}}) = (x_1 - a, x_2)
$$

and

$$
\psi_1(x_1, x_2) = \exp_{(x_1, x_2)}((0_{\mathbb{R}^m}, b, 0) = (x_1, x_2 * (b, 0));
$$

here, 's' denotes the group operation on $\mathbb{H}^d$.

Let $\mu_0 = 1_B \mathcal{L}^{m+2d+1}$, where $B \subset G$ is a closed ball centered at $0_{\mathbb{R}^{m+2d+1}}$ with $\mathcal{L}^{m+2d+1}(B) = 1$ and $\mu_1 = \psi_\# \mu_0$. Note that every element of $B$ belongs to the moving set $M_\varphi$. Moreover, one can see that $\text{supp}(\mu_1) = \psi_0(B \cap G^- \setminus G^0) \cup \psi_1(B \cap G^+)$, and the sets $S_0$ and $S_1$ appearing in the proof of Theorem 1.1 correspond to the two half balls $B \cap G^-$ and $B \cap G^+$ (up to null measure sets), respectively. Consequently, the optimal mass transportation map $\psi$ translates the mass from $S_0$ along abnormal (Euclidean) geodesics into $\tilde{S}_0 = (-a, 0_{\mathbb{R}^{m+2d+1}}) + S_0$, while the mass from $S_1$ is transported along strictly normal (Heisenberg) geodesics into a distorted half ball $\tilde{S}_1 = \{(x_1, x_2 * (b, 0)) : (x_1, x_2) \in B \cap G^+\}$, see Figure 2.

![Figure 2](image)

**Figure 2.** The half balls $S_0$ and $S_1$ are transported along abnormal and strictly normal geodesics into the sets $\tilde{S}_0$ and $\tilde{S}_1$, respectively.

4. Applications

Having the Jacobian determinant inequality (1.7), we can prove several functional and geometric inequalities on corank 1 Carnot groups.

Let us denote by $\rho_0$, $\rho_1$ and $\rho_s$ the density functions (w.r.t. $\mathcal{L}^{k+1}$) of the absolutely continuous, compactly supported measures $\mu_0$, $\mu_1 = \psi_\# \mu_0$ and $\mu_s = (\psi_\# \mu_0)$, respectively. In fact, we have the Monge-Ampère equations

$$
\rho_0(x) = \rho_s(\psi_s(x)) \text{Jac}(\psi_s)(x) \quad \text{and} \quad \rho_0(x) = \rho_1(\psi(x)) \text{Jac}(\psi)(x) \quad \text{for } \mu_0 \text{-a.e. } x \in G. \quad (4.1)
$$

These equations can be deduced in a standard way both in the static case (see §3.1) and moving case with optimal mass transport along strictly normal geodesics (see §3.2.2), while in the case of abnormal transportation we provided a proper interpretation of them (see §3.2.1, Step 2).

Due to (4.1) we may reformulate the Jacobian determinant inequality (1.7) as

$$
(\rho_s(\psi_s(x)))^{\frac{1}{1-s}} \geq \frac{k^\alpha}{1-s}(\theta_x)(\rho_0(x))^{\frac{1}{1+s}} + \frac{k^\alpha}{s}(\theta_x)(\rho_1(\psi(x)))^{\frac{1}{1+s}}, \quad (4.2)
$$
which holds \( \mu_0 \) a.e. on the restricted set \( G_0 = \{ x \in G : \rho_0(x) > 0 \} \). Observe that by definition \( G_0 \) is of full measure in \( \text{supp}(\mu_0) \). For a fixed \( s \in (0,1) \) we restrict \( G_0 \) to the injectivity domain of \( \psi \) and \( \psi_s \) which will be still of full measure in \( \text{supp}(\mu_0) \). Moreover, we may exclude those points \( x \in S_1 \) from \( G_0 \) for which \( x^{-1} \circ \psi(x) \in \text{cut}_G(e) \), see Step 2 in §3.2.2, still obtaining a full measure set in \( \text{supp}(\mu_0) \) which prevents the blow-up of coefficients \( \tau_{k,\alpha}^{-1}(\theta_z) \) and \( \tau_{k,\alpha}^{-1}(\theta_x) \), respectively.

4.1. **Entropy inequalities.** Let \( (G, \circ) \) be a \( k + 1 \) dimensional corank 1 Carnot group and \( U : [0, \infty) \rightarrow \mathbb{R} \) be a function. The \( U \)-entropy of an absolutely continuous measure \( \mu \) w.r.t. \( L^{k+1} \) on \( G \) is defined by \( \text{Ent}_U(\mu|L^{k+1}) = \int_G U(\rho(x)) \, dL^{k+1}(x) \), where \( \rho = \frac{d\mu}{dx} \) is the density function of \( \mu \).

**Theorem 4.1. (Entropy inequality)** Under the same assumptions as in Theorem 1.1, if \( U : [0, \infty) \rightarrow \mathbb{R} \) is a function such that \( U(0) = 0 \) and \( t \mapsto t^{k+1}U\left(\frac{1}{t^{k+1}}\right) \) is non-increasing and convex, the following entropy inequality holds:

\[
\text{Ent}_U(\mu_s|L^{k+1}) \leq (1-s) \int_G \left( \frac{\tilde{\tau}_{k,\alpha}^{-1}(\theta_x)}{\tilde{\tau}_{k,\alpha}^{-1}(\theta_z)} \right)^{k+1} U \left( \frac{\rho_0(x)}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right) \, dL^{k+1}(x) + s \int_G \left( \frac{\tilde{\tau}_{k,\alpha}^{-1}(\theta_x)}{\tilde{\tau}_{k,\alpha}^{-1}(\theta_z)} \right)^{k+1} U \left( \frac{\rho_1(y)}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right) \, dL^{k+1}(y),
\]

where \( \tilde{\tau}_{k,\alpha} = s^{-1}\tau_{k,\alpha} \).

**Proof.** Since the function \( t \mapsto t^{k+1}U\left(\frac{1}{t^{k+1}}\right) \) is non-increasing, relation (4.2) implies that for a.e. \( x \in G_0 \) we have

\[
\frac{U(\rho_s(\psi_s(x)))}{\rho_s(\psi_s(x))} \leq \left( \frac{\tau_{k,\alpha}(\theta_x)(\rho_0(x))^{-1/(k+1)} + \tau_{k,\alpha}(\theta_x)(\rho_1(\psi(x)))^{-1/(k+1)} \right)^{k+1} \times U\left( \left( \frac{\tau_{k,\alpha}(\theta_x)(\rho_0(x))^{-1/(k+1)} + \tau_{k,\alpha}(\theta_x)(\rho_1(\psi(x)))^{-1/(k+1)} \right)^{-(k+1)} \right).
\]

Recalling relation \( s\tilde{\tau}_{k,\alpha} = \tau_{k,\alpha} \) and using the convexity of \( t \mapsto t^{k+1}U\left(\frac{1}{t^{k+1}}\right) \), the latter term can be estimated from above, obtaining

\[
\frac{U(\rho_s(\psi_s(x)))}{\rho_s(\psi_s(x))} \leq (1-s)\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_x)}{\rho_0(x)} \right)^{k+1} U \left( \frac{\rho_0(x)}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right) + s\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_x)}{\rho_1(\psi(x))} \right)^{k+1} U \left( \frac{\rho_1(\psi(x))}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right)
\]

for a.e. \( x \in G_0 \). An integration of the above inequality on \( G_0 \) w.r.t. the measure \( \mu_0 = \rho_0L^{k+1} \) gives

\[
\int_{G_0} U(\rho_s(\psi_s(x))) \, dL^{k+1}(x) \leq (1-s) \int_{G_0} \left( \frac{\tilde{\tau}_{k,\alpha}^{-1}(\theta_x)}{\tilde{\tau}_{k,\alpha}^{-1}(\theta_z)} \right)^{k+1} U \left( \frac{\rho_0(x)}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right) \, dL^{k+1}(x) + s \int_{G_0} \left( \frac{\tilde{\tau}_{k,\alpha}(\theta_x)}{\tilde{\tau}_{k,\alpha}(\theta_z)} \right)^{k+1} U \left( \frac{\rho_1(\psi(x))}{\left( \frac{\tilde{\tau}_{k,\alpha}(\theta_z)}{\tilde{\tau}_{k,\alpha}(\theta_x)} \right)} \right) \, dL^{k+1}(x).
\]

Recall that \( \psi_s \) and \( \psi \) are injective on \( G_0 \); thus we can perform the changes of variables \( z = \psi_s(x) \) and \( y = \psi(x) \), respectively, and by the Monge-Ampère equations (4.1) we obtain the required entropy inequality.

□
Corollary 4.1. Under the same assumptions as in Theorem 4.1, we have the following uniform entropy inequality:

$$\text{Ent}_U(\mu_i | \mathcal{L}^{k+1}) \leq (1 - s)^3 \int_G U \left( \frac{\rho_0(x)}{(1-s)^2} \right) d\mathcal{L}^{k+1}(x) + s^3 \int_G U \left( \frac{\rho_1(y)}{s^2} \right) d\mathcal{L}^{k+1}(y).$$

Proof. Since $t \mapsto \frac{d_1(t,s)}{d_1(t,1)}$ is increasing on $(0,2\pi)$ for every $s \in (0,1)$, $i \in \{1,2\}$, and $\lim_{t \to 0} \frac{d_1(t,s)}{d_1(t,1)} = 1$, $\lim_{t \to 0} \frac{d_2(t,s)}{d_2(t,1)} = s^2$, see [4, Lemma 2.1], we obtain

$$\tau^{k,\alpha}_s(\theta_x) \geq s^{k+1} \text{ for all } s \in (0,1) \text{ and } x \in G_0. \quad (4.3)$$

Thus, for the weights $\tilde{\tau}^{k,\alpha}_s$ we obtain

$$\left( \tilde{\tau}^{k,\alpha}_s(\theta_x) \right)^{k+1} \geq s^2 \text{ for all } s \in (0,1) \text{ and } x \in G_0. \quad (4.4)$$

Since the map $t \mapsto t^{k+1}U \left( \frac{1}{t^{k+1}} \right)$ is non-increasing, the desired inequality directly follows from Theorem 4.1. \qed

Remark 4.1. As a particular case of Theorem 4.1 and Corollary 4.1, we may choose various particular entropies for $U$, as the Rényi-type entropy, Shannon entropy, kinetic-type entropy or Tsallis entropy.

4.2. Brunn-Minkowski inequalities. Let $(G, \circ)$ be a connected, simply connected nilpotent Lie group of (topological) dimension $N$, and $\mu$ be a Haar measure on $G$. By extending a result of Leonardi and Masnou [11] from Heisenberg groups, Tao [18] proved that for every nonempty and bounded open sets $A, B \subset G$ the multiplicative Brunn-Minkowski inequality holds on $(G, \circ)$:

$$\mu(A \circ B) \frac{1}{k+1} \geq \mu(A) \frac{1}{k+1} + \mu(B) \frac{1}{k+1}. \quad (4.5)$$

In particular, this inequality is also valid on any $k+1$ dimensional corank 1 Carnot group $G$ with $N = k+1$ and $\mu = \mathcal{L}^{k+1}$.

In the sequel, we prove geodesic Brunn-Minkowski inequalities on corank 1 Carnot groups. To do this, let $A, B \subset G$ be two nonempty sets. In the sequel we want to quantify the Carnot distortion coefficients which characterize the sets $A$ and $B$. For this reason we introduce the notations

$$\tau^{k,\alpha}_s(A, B) = \sup_{A_0, B_0 (x,y) \in A_0 \times B_0} \inf \{ \tau^{k,\alpha}_s(p) : \exp_e(p) = x^{-1} \circ y \} \quad (4.6)$$

and

$$\tilde{\tau}^{k,\alpha}_s(A, B) = \sup_{A_0, B_0 (x,y) \in A_0 \times B_0} \inf \{ \tilde{\tau}^{k,\alpha}_s(p) : \exp_e(p) = x^{-1} \circ y \} = s \tau^{k,\alpha}_s(A, B), \quad (4.7)$$

where $A_0$ and $B_0$ are nonempty, full measure subsets of $A$ and $B$, respectively. Recalling relation (2.6) between the parameters of the exponential map joining $e$ to $x \in G$ and $x^{-1} \in G$, respectively, the following symmetry properties hold:

$$\tau^{k,\alpha}_s(x,y) = \tau^{k,\alpha}_s(y,x) \text{ and } \tilde{\tau}^{k,\alpha}_s(x,y) = \tilde{\tau}^{k,\alpha}_s(y,x) \text{ for all } x, y \in G. \quad (4.8)$$

For every $s \in [0,1]$ and $x, y \in G$, the set of $s$-intermediate points between $x$ and $y$ is

$$Z_s(x,y) = \{ z \in G : d_{CC}(x,z) = sd_{CC}(x,y), \ d_{CC}(z,y) = (1-s)d_{CC}(x,y) \}. \quad (4.9)$$

We clearly have the antisymmetry property

$$Z_s(x,y) = Z_{1-s}(y,x) \text{ for all } x, y \in G \text{ and } s \in [0,1].$$

The notion of $s$-intermediate points can be extended to the nonempty sets $A, B \subset G$ as

$$Z_s(A,B) = \bigcup_{(x,y) \in A \times B} Z_s(x,y).$$
Theorem 4.2. (Weighted and non-weighted Brunn-Minkowski inequalities) Let $(G,\circ)$ be a $k+1$ dimensional corank 1 Carnot group, $s \in (0,1)$, and $A$ and $B$ be two nonempty measurable sets of $G$. Then the following inequalities hold:

(i) $L^{k+1}(Z_s(A,B)) \geq \tau^{k,\alpha}_s(A,B)L^{k+1}(A) + \tau^{k,\alpha}_s(A,B)L^{k+1}(B)$;
(ii) $L^{k+1}(Z_s(A,B)) \geq \tau^{k,\alpha}_s(A,B)L^{k+1}(A) + \tau^{k,\alpha}_s(A,B)L^{k+1}(B)$;
(iii) $L^{k+1}(Z_s(A,B)) \geq \left(\frac{1}{4}\right)^{\frac{1}{k+1}} \left((1-s)\tau^{k+1}(A) + s\tau^{k+1}(B)\right)^{\frac{1}{k+1}}$.

Proof. First of all, we notice that if $Z_s(A,B)$ is not measurable, $L^{k+1}(Z_s(A,B))$ will denote the outer Lebesgue measure of $Z_s(A,B)$.

(i) We first assume that both $A$ and $B$ have finite $L^{k+1}$-measures. If both sets have null measure, we have nothing to prove; thus, we may assume that $\max\{L^{k+1}(A),L^{k+1}(B)\} > 0$. The proof is divided into three steps.

Step 1: one has $\tau^{k,\alpha}_s(A,B) < \infty$ and $\tau^{k,\alpha}_s(A,B) < \infty$. By (4.6), if $\tau^{k,\alpha}_s(A,B) = +\infty$, we have in particular that $x^{-1} \circ y \in \text{cut}_G(e)$ for a.e. $(x,y) \in A \times B$. Therefore, $0 = L^{k+1}(\text{cut}_G(e)) \geq L^{k+1}(A^{-1} \circ B)$. Thus, by the multiplicative Brunn-Minkowski inequality (4.5) it follows that $L^{k+1}(A) = L^{k+1}(B) = 0$, which contradicts our initial assumption.

Step 2: the case $L^{k+1}(A) \neq 0 \neq L^{k+1}(B)$. Let $\mu_0 = \frac{1}{2\pi \tau^{k+1}(A) L^{k+1}}, \mu_1 = \frac{1}{2\pi \tau^{k+1}(B) L^{k+1}}$ and the Rényi entropy $U(r) = -r^{\frac{1}{\tau^{k+1}}}$ ($r \geq 0$) in Theorem 4.1; thus the entropy inequality and relations (4.6) and (4.7) imply that

$$\int_{\psi_s(A)} \rho_s(z)^{1-\frac{1}{\tau^{k+1}}}d\mathcal{L}^{k+1}(z) \geq \tau^{k,\alpha}_s(A,B) \int_A \rho_0^{1-\frac{1}{\tau^{k+1}}}d\mathcal{L}^{k+1} + \tau^{k,\alpha}_s(A,B) \int_B \rho_1^{1-\frac{1}{\tau^{k+1}}}d\mathcal{L}^{k+1} = \tau^{k,\alpha}_s(A,B)\mathcal{L}^{k+1}(A)^{\frac{1}{\tau^{k+1}}} + \tau^{k,\alpha}_s(A,B)\mathcal{L}^{k+1}(B)^{\frac{1}{\tau^{k+1}}}.$$  

By Hölder’s inequality one has that

$$\int_{\psi_s(A)} \rho_s(z)^{1-\frac{1}{\tau^{k+1}}}d\mathcal{L}^{k+1}(z) \leq \left(\int_{\psi_s(A)} \rho_s(z)d\mathcal{L}^{k+1}(z)\right)^{1-\frac{1}{\tau^{k+1}}} \left(\int_{\psi_s(A)} d\mathcal{L}^{k+1}(z)\right)^{\frac{1}{\tau^{k+1}}} = \mathcal{L}^{k+1}(\psi_s(A))^{\frac{1}{\tau^{k+1}}}.$$  

Since $\psi_s(A) \subset Z_s(A,B)$, the claim follows.

Step 3: the case $L^{k+1}(A) = 0 \neq L^{k+1}(B)$ or $L^{k+1}(A) \neq 0 = L^{k+1}(B)$. In fact, our claim reduces to proving that for every $x \in G$, we have

$$L^{k+1}(Z_s(\{x\},B)) \geq \left(\tau^{k,\alpha}_s(\{x\},B)\right)^{k+1}\mathcal{L}^{k+1}(B).$$  

The latter inequality follows by an approximation argument. In fact, if $\{\varepsilon_n\}_{n \in \mathbb{N}}$ is a decreasing sequence converging to 0, by Step 2 we have for every $n \in \mathbb{N}$ that

$$L^{k+1}(Z_s(B(x,\varepsilon_n),B))^{\frac{1}{\tau^{k+1}}} \geq \tau^{k,\alpha}_s(B(x,\varepsilon_n),B)^{\frac{k+2}{\tau^{k+1}}} + \tau^{k,\alpha}_s(B(x,\varepsilon_n),B)\mathcal{L}^{k+1}(B)^{\frac{1}{\tau^{k+1}}}.$$  

where $B(x,r) = \{y \in G : d_{CC}(x,y) \leq r\}$. By using the monotone convergence theorem one can prove that

$$\lim_{n \to \infty} L^{k+1}(Z_s(B(x,\varepsilon_n),B)) = L^{k+1}(Z_s(\{x\},B)) \text{ and } \lim_{n \to \infty} \tau^{k,\alpha}_s(B(x,\varepsilon_n),B) = \tau^{k,\alpha}_s(\{x\},B),$$  

which proves (4.10). If $A$ or $B$ has infinite $L^{k+1}$-measure, we apply again an approximation argument.

(ii) & (iii) These properties follow by (i) combined with the universal lower bound (4.3) for $\tau^{k,\alpha}_s$ and the $p$-mean inequality (4.11), respectively. \hfill \Box

The main result of Rizzi [15] concerning the measure contraction property on corank 1 Carnot groups is a direct consequence of the Brunn-Minkowski inequality (Theorem 4.2):
Corollary 4.2. (Measure contraction property) Let $(G,α)$ be a $k+1$ dimensional corank $1$ Carnot group. Then the measure contraction property $\text{MCP}(0,k+3)$ holds on $G$, i.e., for every $s \in [0,1]$, $x \in G$ and nonempty measurable set $E \subset G$,

$$\mathcal{L}^{k+1}(Z_s(\{x\},E)) \geq (\tau_s^{k,\alpha}(\{x\},E))^{k+1} \mathcal{L}^{k+1}(E) \geq s^{k+3} \mathcal{L}^{k+1}(E).$$

Remark 4.2. Due to Corollary 4.2, the measure contraction property $\text{MCP}(0,k+3)$ and our Jacobian determinant inequality on any $k+1$ dimensional corank $1$ Carnot group can be deduced from each other (via Proposition 2.5 and Theorem 4.2, respectively).

4.3. Borell-Brascamp-Lieb inequalities. In order to formulate our Borell-Brascamp-Lieb inequalities we introduce the notion of the $p$-mean, which for two non-negative numbers $a, b$ and weight $s \in (0,1)$ is defined as

$$M_s^p(a, b) = \begin{cases} 
(1 - s)a^p + s b^p)^{1/p} & \text{if } ab \neq 0, \\
0 & \text{if } ab = 0,
\end{cases}$$

with the conventions $M_s^{-\infty}(a, b) = \min\{a, b\}$; $M_0^0(a, b) = a^{1-s} b^s$; and $M_s^{+\infty}(a, b) = \max\{a, b\}$ if $ab \neq 0$ and $M_s^{-\infty}(a, b) = 0$ if $ab = 0$. According to Gardner [9, Lemma 10.1], one has

$$M_s^p(a, b)M_s^q(c, d) \geq M_s^p(ac, bd),$$

(4.11)

for every $a, b, c, d \geq 0$, $s \in (0,1)$ and $p, q \in \mathbb{R}$ such that $p + q \geq 0$ with $\eta = \frac{pq}{p+q}$ when $p$ and $q$ are not both zero, and $\eta = 0$ if $p = q = 0$.

Theorem 4.3. (Weighted Borell-Brascamp-Lieb inequality) Fix $s \in (0,1)$ and $p \geq -\frac{1}{k+1}$. Let $f, g, h : G \to [0,\infty)$ be Lebesgue integrable functions with the property that for all $(x, y) \in G \times G, z \in Z_s(x, y),$

$$h(z) \geq M_s^p \left( \frac{f(x)}{(\tau_s^{k,\alpha}(y, x))^{k+1}}, \frac{g(y)}{(\tau_s^{k,\alpha}(x, y))^{k+1}} \right),$$

(4.12)

Then the following inequality holds:

$$\int_G h \geq M_s^{\frac{p}{1+(k+1)p}} \left( \int_G f, \int_G g \right).$$

Proof. We give just a sketch of the proof; for details one may consult [4, Proof of Theorem 1.3]. If $\int_G f = 0$ or $\int_G g = 0$, the conclusion trivially follows due to our convention for the $p$-mean. When both integrals are strictly positive, we divide the proof into three steps.

Step 1: compactly supported, normalized functions. Let $f, g : G \to [0,\infty)$ be compactly supported and normalized functions, i.e., $\int_G f = \int_G g = 1$. By identifying $f$ and $g$ with the density functions of the probability measures $\mu_0$ and $\mu_1$, the Jacobian inequality (4.2) holds in the following form

$$(\rho_s(\psi_s(x))^{-\frac{1}{k+1}} \geq \tau_s^{k,\alpha}(\theta_2) (f(x))^{-\frac{1}{k+1}} + \tau_s^{k,\alpha}(\theta_2) (g(\psi(x))^{-\frac{1}{k+1}} \quad \text{for a.e. } x \in G_0.$$  

(4.13)

Combining inequality (4.13) with the assumption (4.12) one can conclude that

$$h(\psi_s(x)) \geq \rho_s(\psi_s(x)) \quad \text{for a.e. } x \in G_0.$$  

By multiplying the latter inequality by $\text{Jac}(\psi_s)(x) > 0$ a.e. $x \in G_0$ and integrating over $G_0$ w.r.t. $\mathcal{L}^{k+1}$, a change of variables $y = \psi_s(x)$ gives that

$$\int_G h \geq \int_{G_0} \rho_s(\psi_s(x))\text{Jac}(\psi_s)(x)d\mathcal{L}^{k+1}(x) = \int_{G_0} f = 1.$$  


Step 2: compactly supported, non-normalized functions. Let \( f, g : G \to [0, \infty) \) be compactly supported functions such that \( 0 < \int_G f < \infty \) and \( 0 < \int_G g < \infty \). We consider the normalized functions \( \tilde{f} = f \left( \int_G f \right)^{-1}, \tilde{g} = g \left( \int_G g \right)^{-1} \) and \( \tilde{h} = \left( M_s^{p/(k+1)p} \left( \int_G f, \int_G g \right) \right)^{-1} h \). One can apply inequality (4.11) for the \( p \)-means and Step 1 to the rescaled functions \( \tilde{f}, \tilde{g} \) and \( \tilde{h} \), obtaining that \( \int_G \tilde{h} \geq 1 \), which is equivalent to the desired inequality.

Step 3: integrable functions. The conclusion from Step 2 can be extended to any integrable functions \( f, g \) (with the property that \( 0 < \int f < \infty \) and \( 0 < \int g < \infty \)) by approximating \( f \) and \( g \) in \( L^1(G) \) from below by upper semicontinuous compactly supported functions.

Remark 4.3. (a) When \( \int_G f = \infty \) or \( \int_G g = \infty \) by a similar approximation argument as in Step 3 of the previous proof we can obtain that \( \int_G h = \infty \).

(b) Observe that Theorem 4.3 holds as well under weaker conditions, namely, if inequality (4.12) holds only for those \( x, y \in G \) for which \( f(x) > 0 \) and \( g(y) > 0 \).

As a direct consequence of Theorem 4.3, inequality (4.4) and the monotonicity of the \( p \)-mean we can formulate the following weaker Borell-Brascamp-Lieb-type inequality:

Corollary 4.3. (Uniformly weighted Borell-Brascamp-Lieb inequality) Fix \( s \in (0, 1) \) and \( p \geq -\frac{1}{k+1} \). Let \( f, g, h : G \to [0, \infty) \) be Lebesgue integrable functions satisfying

\[
h(z) \geq M_p^s \left( \frac{f(x)}{(1-s)^2}, \frac{g(y)}{s^2} \right) \quad \text{for all } (x, y) \in G \times G, z \in Z_s(x, y).
\]

Then the following inequality holds:

\[
\int_G h \geq M_s^{p/(k+1)p} \left( \int_G f, \int_G g \right).
\]

Corollary 4.4. (Non-weighted Borell-Brascamp-Lieb inequality) Fix \( s \in (0, 1) \) and \( p \geq -\frac{1}{k+3} \). Let \( f, g, h : G \to [0, \infty) \) be Lebesgue integrable functions satisfying

\[
h(z) \geq M_p(f(x), g(y)) \quad \text{for all } (x, y) \in G \times G, z \in Z_s(x, y).
\]

Then the following inequality holds:

\[
\int_G h \geq \frac{1}{4} M_s^{p/(k+1)p} \left( \int_G f, \int_G g \right). \tag{4.16}
\]

Proof. By the \( p \)-mean inequality (4.11) and assumption (4.15), we have

\[
4h(z) = M_p^p(f(x), g(y)) M_s^{\frac{1}{2}} \left( \frac{1}{(1-s)^2}, \frac{1}{s^2} \right) \geq M_s^{p/(k+1)} \left( \frac{f(x)}{(1-s)^2}, \frac{g(y)}{s^2} \right),
\]

for every \( x, y \in G \) and \( z \in Z_s(x, y) \). By the assumption \( p \geq -\frac{1}{k+3} \), we have \( \frac{p}{2p+1} \geq -\frac{1}{k+1} \), so we can apply Corollary 4.3 for the setting \( \tilde{h} = 4h \), \( \tilde{f} = f \), \( \tilde{g} = g \) and \( \tilde{p} = \frac{p}{2p+1} \), obtaining the desired inequality.

Remark 4.4. (a) All three versions of the Borell-Brascamp-Lieb inequality imply a corresponding Prékopa-Leindler-type inequality by setting \( p = 0 \) and using the convention \( M_s^0(a, b) = a^{1-s} b^s \) for all \( a, b \geq 0 \) and \( s \in (0, 1) \).

(b) The Brunn-Minkowski inequalities in Theorem 4.2 can be obtained alternatively from Theorem 4.3 whenever \( \mathcal{L}^{k+1}(A) \neq 0 \neq \mathcal{L}^{k+1}(B) \). Indeed, let \( p = +\infty \), and choose the functions \( f(x) = \left( \frac{1}{1-s} \right)^{k+1} 1_A(x) \), \( g(y) = \left( \frac{1}{s} \right)^{k+1} 1_B(y) \) and \( h(z) = 1_{Z_s(A,B)}(z) \).
With these choices assumption (4.12) holds at the points $x, y \in G$ where $f(x) > 0$ and $g(y) > 0$ and due to Remark 4.3(b) we may apply Theorem 4.3, obtaining
\[
\mathcal{L}_k^{k+1}(Z_s(A, B)) \geq M_s^{k+1} \left( \left( \frac{\tau_{1-s}^{k,a}(A, B)}{\tau_{1-s}^{k,a}(A, B)} \right)^{k+1} \mathcal{L}_k^{k+1}(A), \left( \frac{\tau_{s}^{k,a}(A, B)}{\tau_{s}^{k,a}(A, B)} \right)^{k+1} \mathcal{L}_k^{k+1}(B) \right)
\]
\[= \left( \frac{\tau_{1-s}^{k,a}(A, B)}{\tau_{1-s}^{k,a}(A, B)} \right)^{k+1} \mathcal{L}_k^{k+1}(A) \mathcal{L}_k^{k+1}(B) + \left( \frac{\tau_{s}^{k,a}(A, B)}{\tau_{s}^{k,a}(A, B)} \right)^{k+1} \mathcal{L}_k^{k+1}(A) \mathcal{L}_k^{k+1}(B),
\]
which concludes the proof.

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Mathematisches Institut, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland.

Email: zoltan.balogh@math.unibe.ch

Institute of Applied Mathematics, Óbuda University, Bécsi út 96, 1034 Budapest, Hungary; Department of Economics, Babeș-Bolyai University, Str. Teodor Mihali 58-60, 400591 Cluj-Napoca, Romania.

Email: alex.kristaly@econ.ubbcluj.ro

Mathematisches Institut, Universität Bern, Sidlerstrasse 5, 3012 Bern, Switzerland.

Email: kinga.sipos@math.unibe.ch