Spin Transport in a Quantum Wire

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( November 6, 2018)

We study the effect of electron-electron backscattering interactions on spin transport in a quantum wire. Even if these interactions have no significant effect on charge transport, they strongly influence the transport of spin. We use the quantum Boltzmann equation in the collision approximation to derive equations of motion for spin current and magnetization. In the limit of small perturbations from equilibrium, we explain the existence of ‘precessional’ and ‘diffusive’ behaviors. We also discuss the low-temperature non-linear decay of an uniform spin current outside the hydrodynamic regime.

PACS: 71.10.Pm, 72.10.-d, 75.70.Pa

I. INTRODUCTION

Recent developments in the study of one-dimensional (1D) physics have been aimed at observing the somewhat intriguing processes of spin-charge separation and charge fractionalization. The theoretical study of various interacting 1D systems predicts the decay of the electron into separate spin and charge excitations traveling at different velocities (see e.g. [1]). Also, the charge of the excitations is expected to renormalize in the presence of interactions. This phenomenon is called charge fractionalization [2–4]. This strongly non-Fermi liquid behavior is described by the Luttinger Liquid theory [5].

With the discovery of quasi-one-dimensional systems such as quantum Hall effect edge states, quantum wires, and more recently carbon nanotubes, there has been good experimental confirmation of this theory [6,7]. How-

ever, direct observation of charge fractionalization or spin-charge separation remains a challenge, and it is believed that spin transport may provide useful insight into the detection of the theoretically predicted separation between charge and spin [8–10]. For example, as seen here, backscattering interactions terms markedly affect spin transport, but do not affect charge transport.

Besides providing insight into the physics of Luttinger Liquids, spin transport has also useful technological applications [11]. Because of the long lifetime of spin excitations, the spin of the electron may be useful for information storage or as a transport element, and could have significant applications in fields such as quantum computation [12].

This paper investigates the effect of electron-electron interactions on spin transport in 1D. We consider the model of an ideal, infinitely long quantum wire with two transport channels: spin up and spin down. As is well-known, the forward scattering interactions are responsible for charge fractionalization and spin-charge separation, and significantly affect charge transport. By contrast, the backward scattering processes are marginally irrelevant under the renormalization group flow [13], and do not contribute to the transport of charge. However, the effect of backscattering processes on the spin dynamics can be quite dramatic. It yields two different regimes [8]: one characteristic for small temperatures where the magnetization vector spatially precesses about the direction of the spin current, and another one relevant for higher temperatures where spin diffusion dominates. In our analysis we will only take into account the effects of the backscattering processes; due to spin-charge separation, we expect these results to be unchanged by the forward scattering.

Equations of motion for the spin current and magnetization have been derived in [8], mostly phenomenologically. Here we provide a more general approach. The treatment we use follows closely the Kadanoff-Baym formalism [14], generalized to include the electron spin and chirality. We use the quantum Boltzmann equation in the collision approximation to write down a general form for the spin transport equations.

In the hydrodynamic limit of small perturbations from equilibrium, we retrieve the behavior predicted in [8]. In this limit we can identify the terms responsible for the precession of the magnetization and for the spin diffusion. We find the spin diffusion coefficient

\[ D_s = \frac{2\pi\hbar^3v_F^4}{u^2k_B T^2} \]

in qualitative agreement with the estimate of Ref. [8], but with a different numerical prefactor. This linearized hydrodynamics is manifestly spin-charge separated, giving us some confidence in the validity of the result.

In the presence of larger perturbations, non-linear terms appear as well. An analysis of the full equations of motion may be done numerically. For the particular case of an injected uniform spin current, we perform a numerical analysis of the equations at various temperatures. At high temperatures, we retrieve purely diffusive behavior. In the limit of zero temperature, however, the system ceases to show exponentially decaying behavior and we can find an analytic solution for the decay of the spin current. This solution is in very good agreement with the corresponding numerical results.
We check that the backscattering interactions cause no corrections to the charge and heat transport equations. A comment on the validity of these results is in order. It seems quite likely that the quantum Boltzmann approach is asymptotically exact in the hydrodynamic regime, due to the marginal irrelevance of backscattering in equilibrium (up to possible logarithmic corrections requiring a higher-order analysis). The low-temperature limit is, however, much less transparent. Indeed, the quantum Boltzmann description is almost certainly not exact in this regime. In particular, while the original formulation is manifestly spin-charge separated, the quantum Boltzmann equations are not, but become so in linear response. Thus the decay of the spin current found at low temperatures depends upon the form of the charge distribution functions defined in Section IV. Nevertheless, this dependence appears fairly innocuous, and leads to no obvious unphysical behavior for local quantities (e.g., magnetization, charge density, current, energy density). For this reason, and since the results are in agreement with expectations based on scaling, we find the quantum Boltzmann approximation reasonable an this limit as well.

In the following section we write down the spin transport equations. The derivation of these equations is presented in detail in the Appendix. In section III, we study the perturbative regime and the resulting precessional and diffusive behaviors. In section IV, using numerical and analytical methods, we study the time evolution of a uniform spin current injected into the system. We present the conclusions in section V.

II. SPIN TRANSPORT EQUATIONS

The non-interacting low energy effective Hamiltonian for a spin-$\frac{1}{2}$ quantum wire is given by:

$$H_0 = \sum_\alpha \int dx \hbar v_F [\psi_{La}^\dagger \partial_x \psi_{La} - \psi_{Ra}^\dagger \partial_x \psi_{Ra}],$$

(1)

where $\alpha = \uparrow / \downarrow$ is the spin index, $R/L$ denotes the chirality of the right- and left- moving modes, and $v_F$ is the Fermi velocity. The quantum wire is taken to be infinitely long. We consider adding two types of local interactions that are allowed by SU(2) invariance. The forward scattering interaction is described by:

$$H_1 = -u_f \int dx J_R(x) J_L(x)$$

(2)

$$= -u_f \sum_{\alpha,\beta} \int dx \psi_{Ra}^\dagger(x) \psi_{Ra}(x) \psi_{L\beta}^\dagger(x) \psi_{L\beta}(x),$$

while the backward scattering interaction is:

$$H_2 = -u \int dx \bar{J}_R(x) \cdot \bar{J}_L(x)$$

(3)

$$= -\frac{u}{4} \sum_{\alpha,\beta,\gamma,\delta} \int dx \psi_{Ra}^\dagger(x) \bar{\sigma}_{\alpha\beta} \psi_{R\delta}(x) \cdot \psi_{L\gamma}^\dagger(x) \bar{\sigma}_{\gamma\delta} \psi_{L\delta}(x).$$

Of the quartic processes described by $H_1$ and $H_2$, only the backscattering processes will contribute to spin transport. The forward scattering terms are included as a check.

Given the above form for the interactions, we are interested in deriving equations for the generalized Wigner distribution functions $f^{\alpha_1\alpha_2}_r(p; x, t)$. For each chirality, these functions form a $2 \times 2$ matrix in the spin space. If we choose the $z$ axis as the reference axis, the diagonal components of this matrix give the distribution functions for the particles with the spin directed upward and downward along this axis. The off-diagonal components are related to the distributions of particles that have the spin directed along the $x$ and $y$ axes. In equilibrium the off diagonal components vanish, while the diagonal ones are equal and independent of space and time, and the Wigner matrix can be written as:

$$f^\alpha_r(p; x, t) = f^0_r(p) \delta_{\alpha\beta},$$

(4)

$$f^0_r(p) = \frac{1}{e^{\frac{\hbar v_F p_x}{k_B T}} + 1}.$$ 

(5)

We can relate the Wigner matrices to the actual spin current and magnetization, as well as with the total charge density and charge current. The local densities of the charge and of the spin components in the right and left moving channels are given by:

$$\mathbf{M}_r(\mathcal{X}) = \frac{1}{2} \sum_{\alpha_1,\alpha_2} \int_p f^{\alpha_1\alpha_2}_r(p; \mathcal{X}) \sigma^i_{\alpha_2\alpha_1},$$

(6)

$$n_r(\mathcal{X}) = \sum_\alpha \int_p f^{\alpha}_r(p; \mathcal{X}),$$

(7)

and the total spin current and magnetization are

$$\mathbf{J}_s = v_F (\mathbf{M}_R - \mathbf{M}_L),$$

$$\mathbf{M} = \mathbf{M}_R + \mathbf{M}_L.$$ 

(8)

For simplicity we chose to denote $(x, t)$ by $\mathcal{X}$ and $\int \frac{dp}{2\pi}$ by $\int_p$, and we will make use of this notation throughout the paper.

The total charge density and current can be written as

$$I = v_F (n_R - n_L),$$

$$\rho = n_R + n_L.$$ 

(9)

In equilibrium all the components of the spin current and magnetization vanish. If the system is perturbed, the effect of the interactions is to move the system towards its equilibrium position. The equations that describe this dynamics can be derived using a technique similar to [14].

The details of the derivation are presented in the Appendix.

The resulting generalized Boltzmann equations for the Wigner distribution functions $f^{\alpha_1\alpha_2}_r$ can be written as:
\[(\partial_t + v_F \partial_x) f^{\alpha \gamma}_{R}(p) = \frac{u}{2\hbar} i [f^{\alpha \gamma}_{R}(p) \int_q f^{\alpha 2}_L(q) - f^{\alpha \gamma}_L(q) f^{\gamma 2}_R(p)] + \frac{u^2}{8\hbar^2 v_F} \left\{ 2 f^{\alpha \alpha 2}_L(p) \left[ f^{\alpha \alpha 3}_L(p) \int_q f^{\alpha 3}_L(q) - f^{\alpha \alpha}_L(q) \right] - 2 \int_q f^{\alpha \alpha 2}_L(q) \left[ f^{\alpha \alpha 3}_L(q) f^{\alpha \alpha 4}_R(p) - f^{\gamma 2}_R(p) \right] \right. \\
+ \left. \int_q f^{\alpha \gamma}_L(q) f^{\alpha \alpha 3}_R(p) \left[ f^{\alpha \alpha 3}_L(q) - f^{\alpha \alpha}_R(p) \right] - \int_q f^{\alpha \gamma}_R(p) f^{\alpha \alpha 3}_L(q) \left[ f^{\alpha \alpha 3}_R(p) - f^{\alpha \alpha 3}_L(q) \right] \right\}, \]  

while the corresponding equation for \( f^{\alpha \alpha 2}_L \) can be obtained from Eq. (10) by interchanging \( L \) and \( R \) everywhere and by replacing \( (\partial_t + v_F \partial_x) \) by \( (\partial_t - v_F \partial_x) \). To simplify the notation, as all the distribution functions are evaluated at the same position \( x \) and at the same time \( t \), we dropped the spatial and temporal coordinates.

### III. Perturbative Expansion

If the deviations from equilibrium are small, we can write the matrices of Wigner distribution functions as a sum of equilibrium diagonal matrices and small perturbations, \( f^{\alpha \beta}_L(p) = f^{\alpha \beta}_{0L}(p) \delta_{\alpha \beta} + f^{\alpha \beta}_L(p) \). We use an integral form of Eq. (10) that is given in the Appendix in the equation (A20). We expand this form in \( f^{\alpha \beta}_{0L} \), so that in the terms proportional to \( u^2 \) we keep only the linear terms in \( F \). We make use of the fact that \( f^{0}_R(p) + f^{0}_L(p) = 1 \) and we obtain

\[
(\partial_t \pm v_F \partial_x) \int_p f^{\alpha \alpha 2}_L(p; \mathcal{X}) = \pm \frac{u}{2\hbar} \sum_{\gamma} \int_q f^{\alpha \gamma}_L(q; \mathcal{X}) \left[ \int_p f^{\gamma \alpha 2}_L(p; \mathcal{X}) + \int_p f^{\gamma \alpha 2}_R(p; \mathcal{X}) \right] + \frac{u^2}{2\hbar^2 v_F} \int_q \left[ f^{\alpha \alpha 2}_L(q; \mathcal{X}) - f^{\alpha \alpha 2}_R(q; \mathcal{X}) \right] \left[ \int_p f^{0}_R(p) f^{0}_L(p) \right].
\]

We use Eq. (8) and

\[
\int_p f^{0}_R(p) f^{0}_L(p) = \int \frac{dp}{2\pi} \frac{1}{e^{\frac{\epsilon_{pF}}{k_B T}} + 1} \frac{1}{e^{\frac{-\epsilon_{pF}}{k_B T}} + 1} = \frac{1}{2\pi k_B T} \frac{k_B T}{v_F}
\]

(12)

to write

\[
(\partial_t \pm v_F \partial_x) \mathbf{M}^{\alpha \beta}_L(p; \mathcal{X}) = \pm \frac{u}{2\hbar} \sum_{j,k} \mathbf{M}^{\alpha \beta}_{0L}(p; \mathcal{X}) \mathbf{M}^{jk}_L(p; \mathcal{X}) e^{ki} + \frac{u^2}{4\pi \hbar^3 v_F} [\mathbf{M}^{\alpha \beta}_L(p; \mathcal{X}) - \mathbf{M}^{\alpha \beta}_{0L}(p; \mathcal{X})].
\]

The temperature dependence of the second term only is a result of an explicit dependence of this term on the form of the equilibrium distribution functions, while the first term linear in \( u \) is independent of this form.

In terms of the total spin current and magnetization, Eq. (13) becomes:

\[
(\partial_t \pm v_F \partial_x) \mathbf{J}_s(\mathcal{X}) + v_F^2 \partial_x \mathbf{M}(\mathcal{X}) = \frac{u}{\hbar} \mathbf{M}(\mathcal{X}) \times \mathbf{J}_s(\mathcal{X}) - u^2 \frac{k_B T}{2\pi \hbar v_F} \mathbf{J}_s(\mathcal{X}).
\]

(14)

We also note that the spin continuity equation still holds:

\[
(\partial_t + v_F \partial_x) \mathbf{J}_s(\mathcal{X}) + \partial_x \mathbf{M}(\mathcal{X}) = 0.
\]

(15)

Equation (14) is qualitatively similar to the corresponding result in (8), up to a numerical factor in front of the \( \mathbf{J}_s \) term.

The charge dynamics is unaffected. We obtain from Eqs. (5, 9, 11):

\[
(\partial_t \pm v_F \partial_x) \mathbf{J}_c(\mathcal{X}) = 0,
\]

(16)

\[
(\partial_t \pm v_F \partial_x) \mathbf{J}_s(\mathcal{X}) + v_F^2 \partial_x \mathbf{M}(\mathcal{X}) = 0.
\]

(17)

As expected, this result holds as well in the non-perturbative regime, and it can be exactly derived using the more general equation (A20).

We analyze Eq. (14) to extract the physics of the system. If the first term of the RHS of the equation dominates, as it is the case for zero or small temperatures, the spin transport is ballistic. In the low frequency linear response limit, the time derivative of \( \mathbf{J}_s \) can be neglected. The spin current is thus constant in time, and the space dependence of the magnetization is a precession about the direction of the spin current (8).

In the opposite limit of large temperature, the second term will dominate and the spin transport will be mainly diffusive. We retrieve the diffusion equation:
From Eq. (18) we derive:

$$v_F^2 \partial_x^2 \mathbf{M}(x) + u^2 \frac{k_B T}{2 \pi \hbar^2 v_F^2} \partial_t \mathbf{M}(x) = 0,$$

(18)

and we thus extract the spin diffusion coefficient:

$$D_s = \frac{2 \pi \hbar^3 v_F^2}{u^2 k_B T}.$$

(19)

This differs only by a factor of 8 from the value derived in [8].

The crossover temperature between diffusive and pre-

cessional spin transport can be estimated as a function of

the equilibrium distribution function are small, and we

This is because at each momentum, the variations from

Boltzmann equations for the system are easily solvable.

This indicates that the system evolves towards an equilibrium

state described by the usual Fermi distributions.

In the limit of temperatures that are much larger than

the applied chemical potential, the numerical analysis in-

dicates that the system evolves towards an equilibrium
distribution function matrix. From Eq. (10) we derive

$$\partial_t f_R^{\uparrow \downarrow}(p) = \pm \frac{u^2}{4 \hbar^2 v_F^2} \left\{ f_R^{\uparrow}(p) f_R^{\downarrow}(p) \int_q [f_L^{\downarrow}(q) - f_R^{\downarrow}(q)] + \int_q f_L^{\downarrow}(q) f_R^{\uparrow}(p) [f_R^{\uparrow}(p) - f_R^{\downarrow}(p)] + \int_q f_R^{\uparrow}(p) f_R^{\downarrow}(p) - f_R^{\downarrow}(p) \int_q f_L^{\uparrow}(q) \right\},$$

(26)

$$\partial_t \mathbf{J}^z_s = -u^2 \frac{k_B T}{2 \pi \hbar^3 v_F^2} \mathbf{J}^z_s.$$

(27)

with the solution

$$\mathbf{J}^z_s(t) = \mathbf{J}^z_s(0)e^{-tD},$$

(28)

where $D = \frac{u^2 k_B T}{2 \pi \hbar^3 v_F^2}$.

This is confirmed by the numerical analysis. When we

plot the logarithm of the current as a function of time

we obtain straight lines, consistent with the assumption

that the current is decaying exponentially. If we set all

the constants to 1 the slope of the decay is just $-u^2 T$.

We compare the numerical results with the theoretical

estimates for an exponential decay (represented in the

figure by the dashed lines) and the results are in very

good agreement.
B. Zero temperature regime

In the opposite limit of zero temperature (or $\mu \gg k_B T$) the numerical analysis indicates strong deviations from the purely exponential decay. This can be understood as follows: at low temperatures the perturbations cannot be treated as small with respect to the equilibrium Fermi function. As an example, in Fig.2 we plot the distribution functions for the case we described in Eqs.(21-25), at $t = 0$ and our numerical predictions for the form of these distribution functions at later times. In the range $|p| < \mu/hv_F$ the difference from the equilibrium distributions can be as high as 1, therefore making the perturbative approach of section III inconsistent.

We note that instead of moving toward the unperturbed Fermi distributions, in the region $|p| < \mu/hv_F$ the value of the distribution functions is a constant. At very large times this constant goes to 1/2. This type of behavior is entirely consistent with the physics of the system. In one dimension with linear dispersion, for the specific form of the interactions we consider, two interacting electrons can only transfer spin while the momenta of the electrons are preserved. Thus, the sum of the numbers of spin up and spin down right moving electrons at each momentum has to remain constant in time.

FIG. 1. The logarithm of the spin current plotted as a function of time. The interaction strength has been taken equal to 0.3, the chemical potential $\mu = 1$ and the curves denoted by 1, 2, 3 and 4 correspond to the temperatures of $T = 4$, 7, 10 and 20 respectively.

FIG. 2. The distribution functions for a) spin up right movers, b) spin down right movers c) spin up left movers d) spin down left movers depicted at $t = 0$ and at subsequent times. After sufficiently long times, the values for all the individual distribution functions will be constant in the region $-\mu/hv_F < p < \mu/hv_F$ and equal to 1/2. Here we set all the constants to 1 and we chose $\mu = 2$, $T = 0.1$ and $u = 0.3$.

Taking this observation into account, we can also solve analytically the equations of motion for the spin current in the limit of zero temperature. We define $f_R^{\uparrow} = \frac{1}{2}(f_R^{\uparrow\uparrow} - f_R^{\downarrow\downarrow})$ and $f_R = f_R^{\uparrow\uparrow} + f_R^{\downarrow\downarrow}$. In these new variables Eq.(26) becomes:

$$\partial_t f_{R/L}^s(p) = \frac{u^2}{4hv_F^2} \left\{ \int_q f_{R/L}^s(q) \left[ f_{R/L}(p) - \frac{1}{2}[f_{R/L}(p)]^2 \right] - f_{R/L}(p) \left[ \int_q f_{L}/R(q) - \frac{1}{2} \int_q [f_{L}/R(p)]^2 \right] \right.\
\left. + 2[f_{R/L}(p)]^2 \int_q f_{L}/R(q) - 2 \int_q [f_{L}/R(q)]^2 f_{R/L}(p) \right\}.$$ (29)

For the case defined by Eqs.(21-25), the form of the analytical solution can be inferred from the results of the numerical analysis. In Fig.3 we plot $f_{R/L}^c/s$ at $t = 0$ and subsequent times as obtained from the numerical analysis.

Using Eq.(29), we note that $\partial_t f_{R/L}^c = 0$. Therefore these distribution functions are constant in time and at any time we can use for them the values we fixed at $t = 0$. For this particular case,

$$f_R^c(p) = \Theta(-p + \frac{\mu}{hv_F}) + \Theta(-p - \frac{\mu}{hv_F})$$ (30) and

$$f_L^c(p) = \Theta(p - \frac{\mu}{hv_F}) + \Theta(p + \frac{\mu}{hv_F}).$$ (31)
Then Eqs. (24) simplify, so that for $|p| < \mu/\hbar v_F$:

$$
\partial_t f_{R/L}^s(p) = \frac{u^2}{4\hbar^2 v_F} \left( \frac{1}{2} \int_q f_{R/L}^s(q) - \frac{\mu}{2\pi \hbar v_F} f_{R/L}^s(p) \right) + 2\int_q f_{R/L}^s(q) - 2\int_q [f_{R/L}^s(q)]^2 f_{R/L}^s(p),
$$

while

$$
\partial_t f_{R/L}^c(p) = 0
$$

for other values of $p$.

**FIG. 3.** The distribution functions a) $f_{R}^s(p)$, b) $f_{L}^s(p)$, c) $f_{R}^c(p)$, d) $f_{L}^c(p)$ at $t = 0$ and at subsequent times. The value for $\mu$ has been set to 2 while the value for $u$ is 0.3. We note that the “charge” distribution functions are time independent, while the “spin” distribution functions are constant on the interval $(-\mu, \mu)$, and decrease to zero after long enough times.

From Eqs. (24), as well as from the numerical analysis plotted in Fig. 4, we note that at $t = 0$, $f_{R}^s(p,t) + f_{L}^s(p,t) = 0$, which implies that $f_{R}^s(p,t) + f_{L}^s(p,t) = 0$ at all times. Moreover we can pick a solution of the form

$$
f_{R/L}^s(p,t) = \begin{cases} 
\pm \lambda(t), & |p| < \mu/\hbar v_F, \\
0, & \text{otherwise.} 
\end{cases}
$$

with $\lambda(0) = 1/2$. Therefore the equation for $\lambda$ becomes:

$$
\frac{d\lambda}{dt} = -\frac{u^2 \mu}{4\pi \hbar^3 v_F^2} (\lambda + 4\lambda^3).
$$

Noting that $J_z = \int p [f_{R}^s(p,t) - f_{L}^s(p,t)] = 4\lambda(t)\mu/2\pi \hbar v_F$, we obtain a time dependence for the spin current of the form:

$$
J_z(t) = \frac{J_z(0)}{\sqrt{2e^t/t_0}},
$$

where $J_z(0) = 2\mu/2\pi \hbar v_F$ and $t_0 = 2\pi \hbar^3 v_F/\mu u^2$.

This result agrees very well with the numerical analysis for all values of the parameters in the system. Below we plot the numerical versus the analytical result, for a particular set of values for $T$, $\mu$ and $u$.

**FIG. 4.** The spin current is plotted as a function of time, for $T = 0.1$, $\mu = 2$ and $u = 0.3$. Both the numerical and analytical result are given; the dashed lines corresponds to the estimate of Eq. (35), while the full line is the numerical result.

We note that for short times ($t \ll t_0$) the form of the decay is a power law $J_z(t) = (J_z(0))/\sqrt{1 + 2e^t/t_0}$, while for large enough times $t \gg t_0$ we retrieve a purely exponential decay $J_z(t) = (J_z(0))/(e^{-2t/2t_0})$.

**C. Intermediate temperature regime**

We can perform the numerical analysis for the intermediate regime of temperature when the temperature is comparable to the applied chemical potential. We study the crossover between the high temperature and low temperature regime by plotting the rate of change $\mathcal{R}$ of the log of the current at large enough time scales as a function of $\mu$ or $T$. As seen in the previous sections, in the zero temperature limit we expect this to be a constant $\mathcal{R} = -1/2t_0 \sim -\mu u^2/2$, while in the high temperature limit $\mathcal{R} = -D \sim -u^2 T$.

In Fig. 4 a) we plot $\mathcal{R}$ as a function of $T$, while keeping the chemical potential fixed to $\mu = 1$, and in b) we plot $\mathcal{R}$ as a function of of $\mu$ while keeping the temperature constant $T = 1$. 
FIG. 5. The rate of change of the log of the spin current with respect to
a) the applied chemical potential $\mu$ when the temperature is held constant $T = 1$.
b) the temperature $T$, when the chemical potential is held constant to $\mu = 1$.

As we showed before, when $T$ is held fixed and $\mu$ is small $R$ goes to a constant value $D \sim -u^2 T = -0.09$. Similarly, when $\mu$ is fixed and $T$ is small $R$ becomes $1/2u_0^2 \sim -u^2 \mu/2 = -0.045$. The figures above also portray the crossover between the two regimes.

V. CONCLUSION

We analyzed the effects of backscattering interactions on spin transport in a one dimensional quantum wire. Using the quantum Boltzmann equation we wrote down the equations of motion for the spin current and magnetization. We found that in the hydrodynamic limit of small perturbations, the backscattering processes generate terms in the equation of motion responsible for bulk precession of magnetization as well as for spin diffusion. We computed the diffusion coefficient.

We also analyzed both numerically and analytically the time evolution of a uniform spin current injected into the system. In the limit of high temperatures, this shows an exponential decay controlled by the values of the temperature and of the interaction strength. In the zero temperature limit, the analytic results indicate a more complex behavior characterized by a transition between a power law behavior at small times to an exponential decay behavior controlled by the strength of the applied chemical potential and by the strength of the interactions, at large enough times.

VI. ACKNOWLEDGEMENTS

This research has been supported by NSF-DMR-9985255, the Sloan and Packard foundations, and by Ferrando-Fithian Fellowship and Parsons Foundation Fellowship.

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APPENDIX A:

1. Equations of motion for the imaginary time Green’s function

Given the non-interacting low energy effective Hamiltonian for a spin-$\frac{1}{2}$ quantum wire described in Eq. (1):

$$H_0 = \sum_\alpha \int dx \; \hbar v_F [\psi_{L\alpha}^\dagger i \partial_x \psi_{L\alpha} - \psi_{R\alpha}^\dagger i \partial_x \psi_{R\alpha}],$$  \hspace{1cm} (A1)

and the local interactions defined in Eqs. (2,3):

$$H_1 = -u_f \int dx J_R(x) J_L(x) = -u_f \sum_{\alpha, \beta} \int dx \psi_{R\alpha}^\dagger(x) \psi_{R\alpha}(x) \psi_{L\beta}^\dagger(x) \psi_{L\beta}(x),$$  \hspace{1cm} (A2)

and

$$H_2 = -u \int dx \vec{J}_R(x) \cdot \vec{J}_L(x) = -u \sum_{\alpha, \beta, \gamma, \delta} \int dx \psi_{R\alpha}^\dagger(x) \bar{\sigma}_{\alpha\beta} \psi_{R\beta}(x) \cdot \bar{\sigma}_{\gamma\delta} \psi_{L\delta}(x),$$  \hspace{1cm} (A3)

we derive the transport equations for the spin.

We start with the equations for $\psi_{r\alpha}$:

$$i\hbar \partial_t \psi_{r\alpha}(x, t) = [H(t), \psi_{r\alpha}(x, t)],$$  \hspace{1cm} (A4)

which implies

$$i\hbar \partial_t - \hbar v_F (1)^r \partial_x \psi_{r\alpha}(x, t) = \left[ -\frac{u}{2} \delta_{rR} \sum_\gamma \psi_{L\gamma}^\dagger(x, t) \psi_{L\alpha}(x, t) \psi_{R\gamma}(x, t), \right. \hspace{1cm} (A5)

\left. + \left( \frac{u}{4} - u_f \right) \delta_{rR} \sum_\gamma \psi_{L\gamma}^\dagger(x, t) \psi_{L\alpha}(x, t) \psi_{R\gamma}(x, t) \psi_{R\alpha}(x, t) \right] + (R \leftrightarrow L),$$

where $r = R/L$, $\epsilon_R = 1$ and $\epsilon_L = 2$.

We define the imaginary time Green’s functions to be:

$$G_{r_1 r_2}^{\alpha_1 \alpha_2} (\mathcal{X}_1; \mathcal{X}_2) = -i \frac{<T[S] \psi_{r_1 \alpha_1}(\mathcal{X}_1) \psi_{r_2 \alpha_2}(\mathcal{X}_2)>}{<T[S]>},$$  \hspace{1cm} (A6)

$$G_2 (\mathcal{X}_1^{\alpha_1}, \mathcal{X}_2^{\alpha_2}, \mathcal{X}_3^{\alpha_3}, \mathcal{X}_4^{\alpha_4}) = -\frac{<T[S] \psi_{r_1 \alpha_1}(\mathcal{X}_1) \psi_{r_2 \alpha_2}(\mathcal{X}_2) \psi_{r_3 \alpha_3}(\mathcal{X}_3) \psi_{r_4 \alpha_4}(\mathcal{X}_4)>}{<T[S]>},$$

where $\alpha_1, \alpha_2 = \uparrow / \downarrow$, $r_i = R/L$, and $\mathcal{X}_i$ stands for $x_i, t_i$. Also, $T$ denotes imaginary time ordering, and $S$ is given by

$$S = \exp \left[ -\frac{i}{\hbar} \sum_{\sigma, \gamma = \uparrow / \downarrow} \int dx \int_0^\infty dt \bar{U}^{\sigma\gamma}_{r r'} (\mathcal{X}) \psi_{\sigma r}(\mathcal{X}) \psi_{\gamma r'} (\mathcal{X}) \right].$$

Using Eq. (A6) we can derive the equations of motions for the imaginary time Green’s functions:

$$\hbar [\partial_t - (1)^{r_1} v_F \partial_x ] G_{r_1 r_2}^{\alpha_1 \alpha_2} (\mathcal{X}_1, \mathcal{X}_2) + i \sum_{r_1, \alpha} U_{r_1 r}^{\alpha_1 \alpha} (\mathcal{X}_1) G_{r_2 r}^{\alpha_2 \alpha} (\mathcal{X}_2, \mathcal{X}_1) = -i\hbar \delta(\mathcal{X}_1 - \mathcal{X}_2) \delta_{\alpha_1 \alpha_2} \delta_{r_1 r_2} +$$

$$\left[ -\frac{u}{2} \delta_{r_1 R} \sum_{\alpha} G_2 (\mathcal{X}_1^{\alpha}, \mathcal{X}_1^{\alpha} , \mathcal{X}_2^{\alpha}, \mathcal{X}_1^{\alpha}) + \left( \frac{u}{4} - u_f \right) \delta_{r_1 R} \sum_{\alpha} G_2 (\mathcal{X}_1^{\alpha}, \mathcal{X}_1^{\alpha} , \mathcal{X}_2^{\alpha}, \mathcal{X}_1^{\alpha}) \right] (R \leftrightarrow L).$$  \hspace{1cm} (A7)
Because momentum conservation implies $G_{RL}^{\alpha_1\alpha_2} = 0$, for simplicity we will denote $G_{RL}^{\alpha_1\alpha_2} = G_r^{\alpha_1\alpha_2}$.

We introduce the self energy for the right- and left- moving particles by the relation

$$G_{RL}^{-1\alpha_1\alpha_2}(X_1, X_2) = G_{0RL}^{-1\alpha_1\alpha_2}(X_1, X_2) - \Sigma_{RL}^{\alpha_1\alpha_2}(X_1, X_2),$$  \tag{A8}$$

where $G_0$ is the Green’s function in the absence of the backscattering interactions between particles.

The self energy can be computed perturbatively. We generalize the method of [14] to include chirality and spin indices. In the collision approximation (second order perturbation theory in the interaction), the self energy is

$$\Sigma_{RL}^{\alpha_1\alpha_2}(X_1, X_2) = \frac{i}{2\hbar} G_{L/R}^{\alpha_1\alpha_2}(X_1, X_2) \delta(X_1 - X_2) - \frac{i}{\hbar} \left( \frac{1}{4} - u_f \right) \sum_\alpha G_{L/R}^{\alpha\alpha}(X_1, X_2) \delta(X_1 - X_2) \delta_{\alpha_1\alpha_2}$$  \tag{A9}$$

+ \frac{u^2}{4\hbar^2} \sum_{\alpha, \beta} G_{L/R}^{\alpha_1\alpha_2}(X_1, X_2) G_{L/R}^{\alpha_2\beta}(X_2, X_1) G_{R/L}^{\beta\alpha}(X_1, X_2)$$

$$- \frac{u(u - 4u_f)}{8\hbar^2} \sum_{\alpha, \beta} G_{L/R}^{\alpha_1\alpha_2}(X_1, X_2) G_{L/R}^{\alpha_2\beta}(X_2, X_1) G_{R/L}^{\beta\alpha}(X_1, X_2)$$

$$- \frac{u(u - 4u_f)}{8\hbar^2} \sum_{\alpha, \beta} G_{R/L}^{\alpha_1\alpha_2}(X_1, X_2) G_{L/R}^{\alpha_2\beta}(X_2, X_1) G_{R/L}^{\beta\alpha}(X_1, X_2)$$

$$+ \frac{(u - 4u_f)^2}{16\hbar^2} \sum_{\alpha, \beta} G_{R/L}^{\alpha_1\alpha_2}(X_1, X_2) G_{L/R}^{\alpha_2\beta}(X_2, X_1) G_{R/L}^{\beta\alpha}(X_1, X_2).$$

2. Equations of motion for real time Green’s functions

Following [14] we perform the analytic continuation from the imaginary time to real time. We define real time Green’s functions

$$g_r^{\alpha_1\alpha_2}(X_1, X_2) = -i < T[\psi_{r\alpha_1}(X_1)\psi_{r\alpha_2}^\dagger(X_2)] >,$$  \tag{A10}$$

$$g_r^{\alpha_1\alpha_2 >}(X_1, X_2) = -i < \psi_{r\alpha_1}(X_1)\psi_{r\alpha_2}^\dagger(X_2) >,$$  \tag{A11}$$

$$g_r^{\alpha_1\alpha_2 <}(X_1, X_2) = i < \psi_{r\alpha_2}^\dagger(X_2)\psi_{r\alpha_1}(X_1) >.$$  \tag{A12}$$

We also define

$$g_r^{\alpha_\beta}(x, t) \equiv g_r^{\alpha_\beta}(x; t; x, t),$$  \tag{A13}$$

and we introduce the Fourier transforms:

$$g_r^{\alpha_\beta < (>)}(p, \omega; x, t) \equiv \int dx' \int dt' e^{-ipx'} e^{i\omega t'} \left[ g_r^{\alpha_\beta}(x + x', t + t'/2; x - x', t - t'/2) \right].$$  \tag{A14}$$

We work in the collision approximation. We also assume that the space and time dependence of the Green functions of the system $g(X_1, X_2)$ vary slowly with respect to the center of mass coordinates $x = (x_1 + x_2)/2$ and $t = (t_1 + t_2)/2$ [14]. Thus the equations of motion for the real time Green’s functions become:

$$\hbar \left[ \partial_t \pm v_F \partial_x \right] g_{RL}^{\alpha_1\alpha_2 <}(X) = \pm \sum_\gamma \left\{ \frac{u}{2} \left[ g_r^{\alpha_1\gamma <}(X) g_{RL}^{\gamma\alpha_2 <}(X) - g_r^{\alpha_1\gamma <}(X) g_{RL}^{\gamma\alpha_2 <}(X) \right] \right\}$$

$$\pm \frac{u^2}{8\hbar} \sum_{\gamma, \alpha_3, \alpha_4} \int dp_1 \int \omega_1 \left\{ g_{L}^{\alpha_1\gamma <}(p_1, \omega_1; X) g_{RL}^{\gamma\alpha_2 <}(p_2) g_{RL}^{\alpha_3\alpha_4 >}(p_3, \omega_3; X) g_{RL}^{\alpha_3\alpha_4 >}(p_4, \omega_4; X) - \text{<} R \leftrightarrow L \text{> <} p_1 + p_2 - p_3 - p_4 \text{> 2} \pi \delta(\omega_1 + \omega_3 - \omega_2 - \omega_4) \right\}.$$
\[ \int_{-\infty}^{\infty} \frac{dx}{x} \left( \left[ g_L^{\alpha_1 \alpha_2}(p_1, w_1; \mathcal{X})g_L^{\alpha_3 \alpha_4}(p_2, w_2; \mathcal{X})g_R^{\alpha_4 \gamma}(p_3, w_3; \mathcal{X})g_R^{\gamma \alpha_2}(p_4, w_4; \mathcal{X}) \right] - (R \leftrightarrow L) \right) \int_{\omega} 2\pi \delta(p_1 + p_3 - p_2 - p_4)2\pi \delta(\omega_1 + \omega_3 - \omega_2 - \omega_4). \] (A15)

Here we denoted \( \int_{-\infty}^{\infty} \frac{dx}{x} \) by \( \int_\omega \), and similarly for \( \omega \). As before we chose \( \mathcal{X} \) to denote \( (x, t) \). For simplicity we set the high energy cutoff in the momentum and frequency integrals at infinity, as the energy scale beyond which the Luttinger Liquid physics ceases to be valid is much larger than the other energy scales involved in the problem.

3. The Quantum Boltzmann Equation

We introduce the generalized Wigner distribution functions by:

\[ f^{\alpha_1 \alpha_2}_R(p; x, t) = \int dx' e^{-ipx'} \left[ -ig^{\alpha_1 \alpha_2}_{r}(x + x', t; t - x', t) \right] = \int_\omega \left[ g^{\alpha_1 \alpha_2}_{r}(p, \omega; x, t) \right]. \] (A16)

The equations for these functions can be derived from Eq. (A15) realizing that \( \mathcal{X} \):

\[ g^{\alpha_1 \alpha_2}_r(p, \omega; \mathcal{X}) = f^{\alpha_1 \alpha_2}_r(p; \mathcal{X})a^{\alpha_1 \alpha_2}_L(p, \omega), \] (A17)

\[ g^{\alpha_1 \alpha_2}_r(p, \omega; \mathcal{X}) = [\delta_{\alpha_3 \alpha_4} - f^{\alpha_1 \alpha_2}_r(p; \mathcal{X})]a^{\alpha_1 \alpha_2}_L(p, \omega), \] (A18)

\[ a^{\alpha_1 \alpha_2}_L/R(p, \omega) = 2\pi \delta(\omega \pm v_F p). \] (A19)

We thus obtain:

\[ \left( \partial_t + v_F \partial_x \right) \int f^{\alpha_1 \alpha_2}_R(p) = \frac{u}{2\hbar} \sum_\gamma \int_{p_1, p_2} [f^{\alpha_1 \gamma}_R(p_1)f^{\alpha_2 \gamma}_L(p_2) - f^{\alpha_1 \gamma}_L(p_1)f^{\alpha_2 \gamma}_R(p_2)] + \frac{u^2}{8\hbar^2 v_F} \sum_\gamma, \alpha_3, \alpha_4 \int_{p_1, p_2} \left\{ \left[ f^{\alpha_1 \gamma}_L(p_1)[\delta_{\alpha_3 \alpha_4} - f^{\alpha_2 \alpha_4}_R(p_2)]f^{\alpha_3 \alpha_4}_R(p_2)[\delta_{\alpha_3 \alpha_4} - f^{\alpha_3 \alpha_4}_L(p_1)] \right] \right. \\
\left. - [\delta_{\alpha_1 \gamma} - f^{\alpha_1 \gamma}_L(p_1)]f^{\alpha_2 \alpha_4}_R(p_2)[\delta_{\alpha_3 \alpha_4} - f^{\alpha_3 \alpha_4}_R(p_2)]f^{\alpha_3 \alpha_4}_L(p_1)] - (R \leftrightarrow L) \right\}. \] (A20)

Using the same approximation of slowly varying distributions functions we can write these equations so they describe the change in the distribution functions at each momentum \( p \):

\[ \left( \partial_t + v_F \partial_x \right) f^{\alpha_1 \alpha_2}_R(p) = \frac{u}{2\hbar} \int f^{\alpha_1 \gamma}_R(p) \int f^{\alpha_2 \gamma}_L(q) - \int f^{\alpha_1 \gamma}_L(q)f^{\alpha_2 \gamma}_R(p) \] \[ + \frac{u^2}{8\hbar^2 v_F} \left\{ 2f^{\alpha_1 \alpha_2}_R(p) \int f^{\alpha_3 \alpha_4}_L(q) - f^{\alpha_3 \alpha_4}_R(q) - 2 \int f^{\alpha_1 \alpha_2}_R(q) \left[ f^{\alpha_3 \alpha_4}_R(p)f^{\alpha_3 \alpha_4}_L(q) - f^{\alpha_1 \gamma}_R(p) \right] \right\}. \] (A21)

\[ \left( \partial_t - v_F \partial_x \right) f^{\alpha_1 \alpha_2}_R(p) = \frac{u}{2\hbar} \int f^{\alpha_1 \gamma}_R(p) \int f^{\alpha_2 \gamma}_L(q) - \int f^{\alpha_1 \gamma}_L(q)f^{\alpha_2 \gamma}_R(p) \] \[ + \frac{u^2}{8\hbar^2 v_F} \left\{ 2f^{\alpha_1 \alpha_2}_R(p) \int f^{\alpha_3 \alpha_4}_L(q) - f^{\alpha_3 \alpha_4}_R(q) - 2 \int f^{\alpha_1 \alpha_2}_R(q) \left[ f^{\alpha_3 \alpha_4}_R(p)f^{\alpha_3 \alpha_4}_L(q) - f^{\alpha_1 \gamma}_R(p) \right] \right\}. \] (A22)

These are the generalized Boltzmann equations for a one dimensional spinfull electron gas.