Differential bundles in the category of smooth manifolds

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Abstract

A tangent category is a category equipped with an endofunctor that satisfies certain axioms which capture the abstract properties of the tangent bundle functor from classical differential geometry. Cockett and Cruttwell introduced differential bundles in 2017 as an algebraic alternative to vector bundles in an arbitrary tangent category. In this paper, we prove that differential bundles in the category of smooth manifolds are precisely vector bundles. In particular, this means that we can give a characterisation of vector bundles that exhibits them as models of a tangent categorical essentially algebraic theory.

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1 Introduction

A smooth vector bundle is a model of an algebraic theory in the category of smooth manifolds that satisfies an additional topological axiom. If q : E → M is the underlying projection of the vector bundle then these axioms include the data of a zero section ξ : M → E, an addition +q : E ×q E → E and a scalar multiplication •q : R × E → E that satisfy the appropriate axioms describing an R-module in the slice category over M. The additional topological axiom is that vector bundles are locally trivial. This means that q : E → M is locally isomorphic to a projection π0 : U × Rn → U for some open set U and natural number n. The advantage of using the local triviality condition as part of the definition of a smooth vector bundle is that it makes clear how to perform calculations using local coordinates. However, the local triviality condition axiomatises the existence of a trivialization, which is not an algebraic condition.

The main results of this paper are about differential bundles in a tangent category. A tangent category consists of a category X equipped with an endofunctor T on X that satisfies axioms which capture the abstract properties of the tangent bundle functor from classical differential geometry. The idea behind the definition of a differential bundle is to axiomatise the following fundamental property of vector bundles: if q : E → M is a vector bundle, and x ∈ M and v ∈ Ex is a vector in the fibre above x, then the tangent space Tq(v) is naturally

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identified with \( E_\ast \). In particular every differential bundle \( q : E \to M \) has a universal lift \( \mu : E_q \times_q E \to T(E) \) which is analogous to the map \( \mu(a, b) = \frac{dv}{dt}\big|_0 (a + t \cdot b) \) in the theory of vector bundles and a vertical lift \( \lambda : E \to T(E) \) given by \( \lambda(e) = \mu(0,e) \). (In Section 1 of [12] these maps are called the big and small vertical lifts respectively and the big lift also appears in 6.11 of [11].) Then the local triviality condition is replaced by the algebraic condition (in the sense of Freyd and Kelly [9]) that

\[
\begin{array}{ccc}
E_q \times_q E & \xrightarrow{\mu} & T(E) \\
\downarrow q \pi_0 & & \downarrow T(q) \\
M & \xrightarrow{0} & T(M)
\end{array}
\]

is a pullback preserved by iterated applications of the tangent bundle functor \( T \). The work in [9] and [2] has provided evidence that differential bundles are an appropriate generalisation of vector bundles and many of the results of classical differential geometry concerning vector bundles hold for differential bundles. However, there is no proof of the equivalence between the vector bundles and differential bundles in the category of smooth manifolds in the literature. In this paper, we give a proof of this result. Specifically in Section 4 we prove:

**Theorem 4.2.7.** The category of differential bundles (with linear or bundle morphisms) in the category of smooth manifolds is isomorphic to the category of smooth vector bundles (with linear or bundle morphisms, respectively).

Our proof makes use of a more general result, which proves that in a tangent category with negatives every differential bundle is the retract of a pullback of the tangent projection on its total space (Corollary 3.1.3). The category of vector bundles is closed to idempotent splittings and reindexing, so the result follows.

In addition to making rigorous the relationship between differential bundles and vector bundles, we describe some alternative characterisations of differential bundles. To do this, we make use of pre-differential bundles, which captures the equational fragment of the definition of a differential bundle. In Section 2 we show how a universal property on a pre-differential bundle induces an addition map. Later we give the following characterisation of differential bundles in the category of smooth manifolds:

**Corollary 3.3.9.** A pre-differential bundle \((q : E \to M, \xi, \lambda)\) in the category of smooth manifolds is a differential bundle if and only if

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & TE \\
\downarrow q & & \downarrow (\pi, Tq) \\
M \xrightarrow{(\xi,0)} E \times TM
\end{array}
\]

is a pullback.

We prove this using retractive display systems and strong differential bundles. A differential bundle is strong when the space of tangent vectors based at a zero vector decomposes as a biproduct of its horizontal and vertical parts. The precise definition of a strong differential bundle is in Definition 3.2.1. A proper retractive display system is a class of maps in a tangent category that satisfies certain axioms which capture the abstract properties of the class of submersions in the category of smooth manifolds. (A smooth function is a submersion whose derivative at every point is surjective.) The theory of retractive display systems and the main example (tangent categories where the tangent submersions form a display system) is developed in Section 1.2.

**Proposition 3.3.2.** If \( \mathcal{R} \) is a proper retractive display system on a tangent category \( \mathcal{X} \) and \((q : E \to M, \xi, \lambda)\) is a strong differential bundle in \( \mathcal{X} \), then \( q \) is in \( \mathcal{R} \).

### 1.1 Tangent categories

In this section, we recall the definition of a tangent category. A tangent category consists of a category \( \mathcal{X} \) equipped with a structure that axiomatises the properties of the tangent bundle functor from classical differential geometry. The idea of a tangent category originated in [13] and was further developed in [1]. We begin with the definition of an additive bundle which is a basic building block for the theory.

**Definition 1.1.1.** An additive bundle over \( M \) is a commutative monoid in the slice category \( \mathcal{X}/M \). Explicitly, an additive bundle consists of a projection \( q : E \to M \), an addition \( + : E_q \times_q E \to E \) and a zero \( \xi : M \to E \) satisfying the usual axioms for a commutative monoid.

We often use \( E_n \) to denote the \( n \)-fold pullback \( E_q \times_q E_q \times_q \cdots \times_q E_q \) and write the addition map as an infix operation \( \pi_0 +_q \pi_1 \).

**Definition 1.1.2.** If \( q \) and \( q' \) are additive bundles then an additive bundle morphism \( f : q \Rightarrow q' \) is a square

\[
\begin{array}{ccc}
E & \xrightarrow{f} & E' \\
\downarrow q & & \downarrow q' \\
M & \xrightarrow{f_0} & M'
\end{array}
\]
such that \((f_1 \pi_0 + q', f_1 \pi_1) = f_1 (\pi_0 + q \pi_1)\) and \(f_1 \xi = \xi' f_0\).

The following definition describes a specific type of limit that repeatedly occurs in the theory of tangent categories.

**Definition 1.1.3.** If \(T\) is an endofunctor on \(X\) then a limit diagram in \(X\) is a \(T\)-limit diagram if and only if it is preserved by \(T^n\) for all \(n \in \mathbb{N}\). We write \(T\)-pullback, \(T\)-equaliser etc. for the appropriate specialisations of this definition.

The following is definition 2.1 in [3].

**Definition 1.1.4.** A tangent category is a category \(X\) equipped with:

- an endofunctor \(T\) on \(X\)
- a natural transformation \(p : T \Rightarrow id_X\) such that each \(n\)-fold pullback \(T_n := T_p \times_p \ldots \times_p T\) exists and at each \(M\) the limit \(T_n(M)\) is a \(T\)-limit
- natural transformations \(0 : id \Rightarrow T, + : T_p \times_p T \Rightarrow T, \ell : T \Rightarrow TT\) and \(c : TT \Rightarrow TT\)

such that:

- for every object \(M\): \(p_M, 0_M\) and \(+M\) form an additive bundle \(B\)
- \((\ell, 0) : (T(M), p) \rightarrow (T^2(M), T(p))\) is an additive bundle morphism
- \((c, id) : (T^2(M), T(p)) \rightarrow (T^2(M), p)\) is an additive bundle morphism
- \(c^2 = id\), \(c \ell = \ell\), \(T(\ell)\ell = \ell\ell\), \(cT(c)c = cT(c)\ell + T(\ell)c\ell\)
- the lift \(\ell\) is universal: the diagram

\[
\begin{array}{ccc}
T_2(M) & \xrightarrow{p_0 + T(\ell)p_1} & T^2(M) \\
\downarrow_{Tp_0 = q_1} & & \downarrow_{T(p)} \\
M & \xrightarrow{0} & T(M)
\end{array}
\]

is a \(T\)-pullback.

**Definition 1.1.5.** A tangent category with negatives is a tangent category with a natural transformation \(- : T \Rightarrow T\) which makes all of the commutative monoids \((p_M, 0_M, +_M)\) abelian groups.

Examples of tangent categories include the category of smooth manifolds and the infinitesimally linear objects in a model of synthetic differential geometry (see [10]). For more details, see [1].

### 1.2 Retractive display systems

In the category of smooth manifolds, the projection for every vector bundle is a submersion. Submersions have useful \(T\)-stability properties - the \(T\)-pullback along any submersion exists and is itself a submersion, and they are stable under the tangent functor. Tangent display systems were introduced by Cockett and Cruttwell and axiomatise the class of submersions' \(T\)-stability properties in an arbitrary tangent category. In this paper, we consider an extension to tangent display systems that are closed to retracts, which we call a retractive display system.

**Definition 1.2.1.** A tangent display system is a class of maps \(D\) that is:

- stable under \(T\)-pullbacks: the \(T\)-pullback along any map \(d \in D\) exists and is contained in \(D\),
- stable under the tangent functor.

We call any tangent display system that is closed to retracts in the arrow category a retractive display system. If for all \(M, p_M \in D\), we call \(D\) a proper (retractive) display system.

In this section, we shall show that the submersions in the category of smooth manifolds give a retractive display system, and give a general construction of retractive display systems from display systems. We recall the definition of a submersion:

**Definition 1.2.2.** If \(A\) and \(B\) are smooth manifolds then a smooth function \(f : A \rightarrow B\) is a submersion if and only if the derivative \(Df|_a\) of \(f\) at every point \(a \in A\) is a surjective linear map.

In other words, \(f\) is a submersion if and only if for all \(a \in A\) and all \(v \in T(B)\) such that \(fa = pv\), there exists a \(w \in T(A)\) such that \(T(f)w = v\). This is a weakly universal cone over \(A \xrightarrow{f} B \xleftarrow{v} TB\): there exists at least one morphism into it for any other cone over the diagram.
**Definition 1.2.3.** We say that a commuting square is a weak pullback if for any \( x : X \to A \) and \( y : X \to B \) so that \( ax = yb \), there exists a map \( X \to W \) making the following diagram commute:

\[
\begin{array}{ccc}
X & \xrightarrow{\alpha} & W \\
\downarrow^{x} & & \downarrow^{f} \\
A & \xrightarrow{g} & B
\end{array}
\]

**Lemma 1.2.4.** Should the pullback of \( A \xleftarrow{f} C \xrightarrow{g} B \) exist, Definition 1.2.3 is equivalent to asking the induced map \((a, b) : W \to A \times_g B\) be a split epimorphism.

We take the submersion property for a map \( f \) using global elements (for all \( a \in A \) and all \( v \in T(B) \) such that \( fa = pv \), there exists a \( w \in T(A) \) such that \( T(f)w = v \)) and state it using generalized elements.

**Definition 1.2.5.** An arrow \( f : A \to B \) in a tangent category is a tangent submersion if and only if the naturality diagram

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TB \\
\downarrow^{p} & & \downarrow^{p} \\
A & \xrightarrow{f} & B
\end{array}
\]

is a weak \( T \)-pullback.

Following Lemma 1.2.4 in the case the pullback exists this is equivalent to asking for a section \( h : A_{f \times g}TB \to TA \) of the horizontal descent \((p, Tf) : TA \to A_{f \times g}TB\) (this is sometimes called a horizontal lift in differential geometry literature [4]). In smooth manifolds, the \( T \)-pullback along the projection \( p : T \Rightarrow id \) always exists, so to prove that every submersion is a tangent submersion it suffices to show the existence of a horizontal lift.

**Proposition 1.2.6.** In the category of smooth manifolds, the tangent submersions are precisely the classical smooth submersions.

**Proof.** There is an explicit construction of a horizontal lift for a classical smooth submersion in VII.1 of [5].

It is possible to show that the \( T \)-stability properties for submersions in the category of smooth manifolds follow from the general theory of weak pullbacks. We begin by showing that weak pullbacks satisfy a weakened version of the pullback lemma and then show that the retract of a weak pullback is a weak pullback (the first lemma may be found in [4]).

**Lemma 1.2.7** (Pullback lemma). Consider the diagram:

\[
\begin{array}{ccc}
\bullet & \xrightarrow{f} & \bullet \\
\downarrow^{(A)} & & \downarrow^{(B)} \\
\bullet & \xrightarrow{g} & \bullet
\end{array}
\]

If \( f, g \) are jointly monic, then \( (A) \) is a (weak) pullback if and only if the outer perimeter \((A) + (B)\) is a (weak) pullback. (Note that when \( (B) \) is a pullback, \( f, g \) are jointly monic.)

**Proof.** The proof for pullbacks holds for weak pullbacks.

**Lemma 1.2.8.** (Weak) pullbacks are closed to retracts.

**Proof.** Suppose that \( S' \) is a weak pullback, and \( S \) is a retract of it in the category of commuting squares. Consider the following diagram (suppressing the subscripts for \( s, r \)):

\[
\begin{array}{ccc}
Z & \xrightarrow{a} & A' \\
\downarrow^{x} & & \downarrow^{r} \\
A & \xrightarrow{r} & A \\
\downarrow^{s} & & \downarrow^{z} \\
B & \xrightarrow{s} & B \\
\downarrow^{y} & & \downarrow^{w} \\
C & \xrightarrow{w} & C \\
\downarrow^{x} & & \downarrow^{r} \\
D & \xrightarrow{r} & D
\end{array}
\]

Given a cone for \( S \), there is a corresponding cone for \( S' \) which induces a map \( Z \to A' \) and postcomposition with \( r_A \) gives the desired map into \( A \).
Using these lemmas, it is straightforward to prove the following $T$-stability properties hold for tangent submersions.

**Lemma 1.2.9.** In any tangent category $X$:

(a) Tangent submersions are closed to composition.
(b) Tangent submersions are closed to retracts.
(c) Any $T$-pullback of a tangent submersion is a tangent submersion.

**Proof.** (a) follows from Lemma 1.2.7 while (b) follows from Lemma 1.2.8. It remains to prove (c):

Suppose we have a $T$-pullback, where $u$ is a tangent submersion:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & M \\
\downarrow{v} & & \downarrow{u} \\
B & \xrightarrow{g} & N
\end{array}
\]

Then the following two diagrams are equal:

\[
\begin{array}{ccc}
TA & \xrightarrow{Tf} & TM & \xrightarrow{p} & M \\
\downarrow{Tv} & & \downarrow{Tu} & & \downarrow{u} \\
TB & \xrightarrow{Tg} & TN & \xrightarrow{p} & N
\end{array}
\quad
\begin{array}{ccc}
TA & \xrightarrow{p} & A & \xrightarrow{f} & M \\
\downarrow{Tv} & & \downarrow{u} & & \downarrow{u} \\
TB & \xrightarrow{p} & B & \xrightarrow{g} & N
\end{array}
\]

the left diagram is a weak pullback by composition. Therefore the outer perimeter of the right diagram is a weak pullback, and the right square is a pullback, so the left square is a weak pullback by the weak pullback lemma, as desired.

We see that the class of tangent submersions is closed to retracts in the arrow category, and is conditionally closed under $T$-reindexing (if the $T$-pullback of a tangent submersion exists, it is a tangent submersion). This leads to the following result:

**Proposition 1.2.10.** Let $X$ be a tangent category that allows for $T$-reindexing of the class of tangent submersions $\mathcal{R}$. Then the class of tangent submersions is a display system.

**Proof.** Any class of maps that is closed to $T$-reindexing is a tangent display system, and the class of submersions is closed to retracts in the arrow category.

In the category of smooth manifolds, where the class of smooth submersions is the canonical example of a proper tangent display system, this gives the following:

**Corollary 1.2.11.** The class of submersions in the category of smooth manifolds is a proper retractive display system.

We can also specify this result to a tangent category with $T$-pullbacks.

**Corollary 1.2.12.** The split tangent submersions in a tangent category in which all pullbacks exist and are $T$-limits form a retractive display system.

**Remark 1.2.13.** In synthetic differential geometry, the weak pullback used to define tangent submersions is equivalent to a weak lifting property. Recall that Day [6] characterised the orthogonal lifting property between two maps $f \perp g$ in $X$ as a pullback in Set:

\[
\begin{array}{ccc}
A & \xrightarrow{u} & X \\
\downarrow{f} & \equiv & \downarrow{g} \\
B & \xrightarrow{v} & Y
\end{array}
\quad
\begin{array}{ccc}
X(B, X) & \xrightarrow{X(id, g)} & X(B, Y) \\
\downarrow{X(f, id)} & & \downarrow{X(id, f)} \\
X(A, X) & \xrightarrow{X(id, g)} & X(A, Y)
\end{array}
\]

For a monoidal closed category, this can be strengthened a self-enriched orthogonal factorisation system by using the internal hom $[-, -]$. For a weak factorisation system, the weak lifting property (where the requirement that the lift is unique is dropped) is equivalently characterised by asking the commuting diagram on the right be a weak pullback.

In synthetic differential geometry (or any representable tangent category), the tangent functor is represented by pointed infinitesimal object $0 : 1 \to D$, so that $p_M = [0, M] : [D, M] \to [1, M] \simeq M$. Thus the condition that the naturality square for $p$ at $f$ be a weak pullback may be reinterpreted as a weak lifting property:

\[
\begin{array}{ccc}
1 & \xrightarrow{m} & M \\
\downarrow{\gamma} & \equiv & \downarrow{\gamma} \\
D & \xrightarrow{\gamma} & N
\end{array}
\quad
\begin{array}{ccc}
[D, M] & \xrightarrow{[id, f]} & [D, N] \\
\downarrow{[0, id]} & & \downarrow{[0, id]} \\
[1, M] & \xrightarrow{[id, f]} & [1, N]
\end{array}
\]
2 Differential bundles

In this section, we introduce the basic theory of differential bundles, along with some new characterisations which we shall use throughout the rest of this paper. In the first section, we pull out the purely equational fragment of the definition of differential bundles, which we call pre-differential bundles, to simplify the definition of differential bundles. In the second section, we introduce differential bundles and show they are precisely pre-differential bundles satisfying a universal property.

2.1 Pre-differential bundles

The original definition of a differential bundle consisted of an additive bundle and a lift satisfying various coherences and universal properties. In this section, we pull out the purely equational fragment of the definition of differential bundles, which we shall use throughout the rest of this paper. In the first section, we pull out the purely equational fragment of the definition of differential bundles, which we call pre-differential bundles, to simplify the definition.

Definition 2.1.1. A pre-differential bundle is a triple \((q : E \to M, \xi : M \to E, \lambda : E \to TE)\) so that \(q\xi = \text{id}\), \(\ell\lambda = T(\lambda)\lambda, p\lambda = \xi q, \) and \(\lambda \xi = q p\xi\).

From the definition of a tangent category, we have that \(\ell\ell = T(\ell)\ell\), so that the pair \((T, \ell)\) is a weak comonad. (For the definition of weak comonad see for instance 1.1 of [16].) The condition that \(T(\lambda)\lambda = \ell T(\lambda)\) is precisely the same as the requirement that \(\lambda\) be a coalgebra of the weak comonad \((T, \ell)\). In fact if \(\lambda\) is a coalgebra of \((T, \ell)\) then \(p\lambda\) is an idempotent:

\[
p\lambda p\lambda = ppT(\lambda)\lambda = pp\ell\lambda = p\ell p\lambda = p\lambda
\]

and the condition that \(\xi q = p\lambda\) states that \(\xi, q\) is a splitting of the idempotent \(p\lambda\). This leads us to following proposition

Proposition 2.1.2. A pre-differential bundle is precisely a coalgebra of \((T, \ell)\) equipped with a chosen idempotent splitting \(\xi q = p\lambda\).

Proof. We have checked that \(p\lambda\) is always an idempotent, and \(\xi, q\) splits it by definition. All that remains is to check the condition that \(0\xi = \lambda\xi\).

\[
\lambda \xi = \lambda\xi q\xi = \lambda p\lambda \xi = pT(\lambda)\lambda \xi = p\ell \lambda \xi = 0p\lambda \xi = 0\xi q\xi = 0\xi
\]

There is a naturally defined category of pre-differential bundles, whose morphisms are pairs of maps \((f_1, f_0)\) which preserve the chosen idempotent splitting and preserve the coalgebra structure.

Definition 2.1.3. The category of pre-differential bundles in a tangent category \(X\) has:

- Objects: pre-differential bundles \((q : E \to M, \xi, \lambda)\)
- Morphisms: A map \(f : (q : E \to M, \xi, \lambda) \to (q' : E' \to M', \xi', \lambda')\) is given by a pair of maps \(f_1 : E \to E', f_0 : M \to M'\) so that the following diagrams commute.

\[
\begin{align*}
E & \xrightarrow{f_1} E' \\
M & \xrightarrow{f_0} M'
\end{align*}
\]

We can see that the base maps are redundant data, so we can treat the category of differential bundles as a category of coalgebras over a weak comonad with extra data (the chosen splitting of \(\lambda\)).

Proposition 2.1.4. The category of pre-differential bundles in \(X\) is isomorphic to the category of:

- Objects: pre-differential bundles \((q : E \to M, \xi, \lambda)\).
- Morphisms: a map \(f : (q : E \to M, \xi, \lambda) \to (q' : E' \to M', \xi', \lambda')\) is a coalgebra morphism \(f : \lambda \to \lambda'\).

Proof. It suffices to show that given a morphism of differential bundles \((f_1, f_0)\), the morphism between base spaces is determined by \(f_1\), which is immediate by the diagram:

\[
\begin{array}{ccc}
E & \xrightarrow{f_1} & E' \\
M & \xrightarrow{f_0} & M'
\end{array}
\]

thus \(f_0 = q' f_1 \xi\), and a linear bundle morphism is completely determined by a coalgebra morphism \(f_1 : \lambda \to \lambda'\).
2.2 Differential bundles

In this section, we deconstruct the definition of a differential bundle to show that it is precisely a pre-differential bundle satisfying a universal property, expressed as a single $T$-pullback diagram. We begin by inducing an additive bundle structure on a pre-differential bundle that satisfies Rosicky’s original diagram for the universality of the lift \[13\], and show the induced addition satisfies the same coherences that the vertical lift $\ell$ satisfies.

**Lemma 2.2.1.** Let $(q : E \to M, \xi, \lambda)$ be a pre-differential bundle in a tangent category such that:

- $n$-fold pullback powers of $q$ exist and are $T$-limits,
- (Rosicky’s universality diagram) the commuting square

$$
\begin{array}{ccc}
E & \xrightarrow{\lambda} & TE \\
\downarrow q & & \downarrow (T(q), p) \\
M & \xrightarrow{(\ell, 0)} & T(M) \times E
\end{array}
$$

is a $T$-pullback,

then there is a uniquely determined addition $+_q : E \times E \to E$ making $(q, +_q, \xi)$ an additive bundle and $(\lambda, \xi) : q \Rightarrow p$ and $(\lambda, 0) : q \Rightarrow T(q)$ additive bundle morphisms. Furthermore, when the tangent category has negatives, the additive bundle will have negatives.

**Proof.** First, note that $T^n(q)$ is a monomorphism because it is the pullback of a monomorphism. The addition $+_q$ is defined using the following commutative diagram:

$$
\begin{array}{ccc}
E_2 & \xrightarrow{\lambda \times \lambda} & T_2(E) \\
\downarrow +_q & & \downarrow +_p \\
E & \xrightarrow{\lambda} & T(E) \\
\downarrow q & & \downarrow (T(q), p) \\
M & \xrightarrow{(0, \ell)} & T(M) \times E \\
\downarrow id & & \downarrow +_p \times id \\
M & \xrightarrow{(0, 0), \xi} & T_2(M) \times E
\end{array}
$$

and so in particular $+_q$ is the unique addition on $q$ such that $\lambda +_q = +_p (\lambda \times \lambda)$. The left-hand square gives the identity $q(a +_q b) = qa = qb$. Post-composition by $\lambda$ gives the associativity, commutativity and unit laws.

Now that we have constructed an additive bundle structure, we must show that the bundle morphisms $(\lambda, \xi)$ and $(\lambda, 0)$ are additive. We observe that $\lambda(a +_q b) = \lambda a +_p \lambda b$ by construction and $p \lambda = \xi q$ because the middle square commutes. Thus $(\lambda, \xi) : (E, q) \to (TE, p)$ is a morphism of additive bundles. To show that $(\lambda, 0) : (E, q) \to (TE, T(q))$ is additive first observe that $T(q) \lambda = 0q$ because the middle square commutes. To show $\lambda$ preserves addition, compute:

$$
T(\lambda)(\lambda a +_q \lambda b) = T(\lambda)\lambda a +_p T(\lambda)\lambda b
= l\lambda a +_p l\lambda b
= l(\lambda a +_p \lambda b)
= l(\lambda(a +_q b))
= T(\lambda)(\lambda a +_q b)
$$

where $a, b \in E$ such that $qa = qb$.

In case the tangent category has negatives, we may induce the map $-_q$ from $-_p$ via the diagram:

$$
\begin{array}{ccc}
E & \xrightarrow{-q} & TE \\
\downarrow q & & \downarrow (T(q), p) \\
M & \xrightarrow{-(0, \xi)} & T(M) \times E \\
\downarrow -_p & & \downarrow -_p \times id \\
M & \xrightarrow{(0, 0), \xi} & T_2(M) \times E
\end{array}
$$

and postcomposition with $\lambda$ shows the necessary equations hold. □
Thus, we can see that Rosicky’s universality diagram uniquely determines the additive bundle structure in a differential bundle. We now give a proof that a morphism of pre-differential bundles preserves addition.

**Proposition 2.2.2.** Let \((q : E \rightarrow M, \xi, \lambda, (q', \xi', \lambda'))\) be a pair of differential bundles satisfying Rosicky’s universality diagram. Then any coalgebra morphism \(f : \lambda \rightarrow \lambda'\) gives rise to an additive bundle morphism \((f, f')\).

Proof. Note that \(q', \lambda'\) are jointly monic, then check post-composition for \(q'\)

\[ q'(f \pi_0 + q' f \pi_1) = q' f \pi_0 = f' q \pi_0 = f' q(\pi_0 + q \pi_1) = q' f(\pi_0 + q \pi_1) \]

Now post-composition by \(\lambda'\)

\[ \lambda'(f \pi_0 + q' f \pi_1) = \lambda' f \pi_0 + p\lambda' f \pi_1 = T(f)\lambda \pi_0 + T(f)\lambda \pi_1 \]

\[ = T(f)(\lambda \pi_0 + p\lambda \pi_1) = T(f)\lambda(\pi_0 + p\pi_1) = \lambda' f(\pi_0 + q \pi_1) \]

Therefore \((f, f')\) is an additive bundle morphism. \(\square\)

We now give the original definition of a differential bundle. A differential bundle is an additive bundle with a lift satisfying the same coherences with addition and universality conditions as the universal lift \(\ell\). Based on Lemma 2.2.1 we shall show that the universality conditions on \(\ell\) induce an addition map. Thus, a differential bundle is a pre-differential bundle satisfying some additional properties rather than having additional structure.

**Definition 2.2.3.** A differential bundle in a tangent category consists of arrows \(q : E \rightarrow M, +_q : E_2 \rightarrow E, \xi : M \rightarrow E\) and \(\lambda : E \rightarrow T(E)\) such that:

- \((q, \xi, \lambda)\) is a pre-differential bundle,
- \(n\)-fold pullbacks of \(q\) exist and are \(T\)-limits,
- \((\lambda, 0) : (E, q) \rightarrow (T(E), T(q))\) is an additive bundle morphism,
- \((\lambda, \xi) : (E, q) \rightarrow (T(E), p)\) is an additive bundle morphism,
- (Cockett-Cruttwell universality) the square

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\lambda \pi_0 + q \pi_1} & T(E) \\
q \pi_0 = q \pi_1 \downarrow & & \downarrow T(q) \\
M & \xrightarrow{0} & T(M)
\end{array}
\]

is a \(T\)-pullback.

The following proposition presents our new characterisations of differential bundles. The first is essentially the same as the original definition; the only difference is that it uses Rosicky’s universality diagram to induce an additive bundle structure. The second uses a rather opaque pullback diagram, that is then related to Rosicky’s and the original Cockett-Cruttwell universality condition.

**Proposition 2.2.4.** The following are equivalent

1. \((q, \xi, \lambda)\) is a differential bundle.

2. \((q, \xi, \lambda)\) is a pre-differential bundle, all \(T\)-pullback powers of \(q\) exist and the diagrams:

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & TE \\
\downarrow q & & \downarrow (Tq, p) \\
M & \xrightarrow{(0, \ell)} & TM \times E
\end{array}
\]

(2)

are \(T\)-pullbacks (where the additive bundle structure on \(T(q)\) is induced by Lemma 2.2.1).

3. \((q, \xi, \lambda)\) is a pre-differential bundle, all \(T\)-pullback powers of \(q\) exist and the diagram:

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\lambda \pi_0 + q \pi_1} & TE \\
\downarrow q \pi_0 = q \pi_1 & & \downarrow Tq \\
M & \xrightarrow{0} & TM
\end{array}
\]

(3)

are \(T\)-pullbacks (where the additive bundle structure on \(T(q)\) is induced by Lemma 2.2.1).

4. \((q, \xi, \lambda)\) is a pre-differential bundle, all \(T\)-pullback powers of \(q\) exist and the diagram:

\[
\begin{array}{ccc}
E_2 & \xrightarrow{T(\lambda) \pi_0 + T(p) \pi_1} & T^2(E) \\
\downarrow q \pi_0 = q \pi_1 & & \downarrow (T^2(q), T(p)) \\
M & \xrightarrow{(0, 0, \ell)} & T^2(M) \times T(E)
\end{array}
\]

(4)

is a \(T\)-pullback.
**Proof.** We prove the chain of equivalences holds.

(1 ⇒ 2) We need only show that Eq. (2) is a \( T \)-pullback. If \( a : A \rightarrow T^{n+1}(E) \) satisfies \( T^{n+1}(q)a = T^n(0qp)a \) and \( T^n(p)a = T^n(\xi qp)a \) then by the universality of the lift there exists \((u, v) : A \rightarrow T^n(E_2)\) such that

\[
T^n(\lambda)u + T^{n+1}(q)T^n(0)v = a
\]

and \( T^n(q)u = T^n(q)v = T^n(\xi q)p.a \). However by postcomposing the displayed equation with \( T^n(p) \) we obtain that actually \( v = T^n(p)a = T^n(\xi q)p.a \) and so the displayed equation is equivalently \( T^n(\lambda)u = a \). But this means that \( u : A \rightarrow T^n(E) \) is the factorisation we require. Moreover \( T^n(\lambda) \) is a monomorphism because \( T^n(\mu) \) is a monomorphism and so the factorisation is unique.

(2 ⇒ 1) By Lemma 2.2.1, we have a uniquely determined additive bundle so that \((+, q, \xi), (\lambda, \xi) : q \rightarrow p, (\lambda, 0) : q \rightarrow Tq\) are additive bundle morphisms. The universality of the vertical lift holds for the induce addition by our assumption, thus \((\lambda, q, \xi)\) is a differential bundle.

(3 ⇒ 2) We first exhibit Rosicky’s universality diagram Eq. (2) as a retract of the diagram Eq. (3):

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\ell_{\lambda q \lambda + T(\mu)0}} & T^2(E) \\
\downarrow{\pi_1} & & \downarrow{p} \\
E & \xrightarrow{\lambda} & T(E) \\
\downarrow{q} & \downarrow{T(q, p)} & \downarrow{(T^2(q), T(p))} \\
M & \xrightarrow{(0, \xi)} & T(M) \times E \\
\downarrow{id} & \downarrow{p} & \downarrow{0} \\
M & \xrightarrow{(0, 0, \xi)} & T^2(M) \times T(E)
\end{array}
\]

Pullbacks are closed to retracts, thus Rosicky’s universality condition holds (Eq. (2) is a pullback) and we may induce an additive bundle structure as in Lemma 2.2.1.

Now observe that in the diagram

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\ell_{\lambda q \lambda + T(\mu)0}} & T^2(E) \\
\downarrow{q \pi_1} & & \downarrow{Tq} \\
T(\ell q) & \xrightarrow{T(\lambda)} & T^2(E) \\
\downarrow{(T^2(q), T(p))} & & \downarrow{(T^2(q), T(p))} \\
TM & \xrightarrow{T(\xi)} & T^2M \times TE
\end{array}
\]

the outer perimeter is Eq. (3) and the right square is a \( T \)-pullback, so by the pullback lemma the left square is a \( T \)-limit.

(2 ⇒ 3) This also follows from the pullback lemma - as Eq. (3) and Eq. (2) are \( T \)-pullbacks the composite Eq. (4) is a \( T \)-pullback.

\[\square\]

When a tangent category has negatives, a pre-differential bundle that satisfies Rosicky’s universality diagram is a differential bundle.

**Corollary 2.2.5.** In a tangent category with negatives, a pre-differential bundle \((q : E \rightarrow M, \xi, \lambda)\) is a differential bundle if and only if \( n \)-fold \( T \)-pullback powers of \( q \) exist and the diagram

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & T(E) \\
\downarrow{q} & \downarrow{T(q, p)} & \downarrow{(T(q, p))} \\
M & \xrightarrow{(\xi, 0)} & T(M) \times E
\end{array}
\]

is a \( T \)-pullback.

**Proof.** The reverse implication follows by Proposition 2.2.4 to prove the forwards implication it suffices to prove

\[
\begin{array}{ccc}
E_2 & \xrightarrow{\ell_{\lambda q \lambda + T(\mu)0}} & TE \\
\downarrow{q \pi_1} & & \downarrow{Tq} \\
M & \xrightarrow{0} & TM
\end{array}
\]

is universal. Suppose that \( a : A \rightarrow T^{n+1}(E) \) satisfies \( T^{n+1}(q)a = T^n(0qp)a \). We need to show that there exists a unique factorisation of \( a \) through \( T^n(\mu) : T^n(E_2) \rightarrow T^{n+1}(E) \). The difference \( a - T^{n+1}(q)T^n(0)p.a : A \rightarrow T^{n+1}(E) \) satisfies:-
Furthermore we assume that the following pullback
\[ T^{n+1}(q)(a - T^n(\xi(q)) \cdot T^n(0)p)a = T^n(0)p)a \]
and
\[ T^n(p)(a - T^{n+1}(\xi(q)) \cdot T^n(0)p)a = T^n(p)a - T^n(q) \cdot T^n(0)p)a = T^n(\xi(q))p)a \]
so there exists an \( e : A \to T^n(E) \) such that \( T^n(\lambda)e = a - T^n(0)p)a \) and \( T^n(q)e = T^n(q)p)a \). Now \( (e, T^n(p)a) : A \to T^n(E_2) \) is the factorisation we require:
\[
T^n(\mu)(e, T^n(p)a) = T^n(\lambda p_0 + T^n(0)p_1)(e, T^n(p)a) = T^n(\lambda)e + T^{n+1}(q) \cdot T^n(0)p)a = (a - T^{n+1}(q) \cdot T^n(0)p)a + T^{n+1}(q) \cdot T^n(0)p)a = a
\]
and this factorisation is unique because \( T^n(\lambda) \) and \( T^n(0) \) are monomorphisms.

Thus, differential bundles are pre-differential bundles satisfying a universality condition, so they are a full-subcategory of pre-differential bundles.

**Definition 2.2.6.** The category of differential bundles is the full subcategory of the category of pre-differential bundles satisfying the Rosicky and Cockett-Cruttwell universality conditions.

**Lemma 2.2.7.** Every morphism of differential bundles preserves addition.

*Proof.* Every differential bundle is a pre-differential bundle that satisfies Rosicky’s universality diagram, so this follows immediately from Proposition 2.2.2. \( \square \)

## 3 Relating differential bundles and tangent projections

In this section, we prove the general results from which we deduce that every differential bundle in the category of smooth manifolds is a vector bundle. Our general strategy is to demonstrate that a differential bundle \( (q : E \to M, \xi, \lambda) \) is a linear retract of a pullback of the tangent bundle on \( E \). In fact we prove this twice using two different sets of assumptions. On the one hand in Section 3.1 we work in a tangent category with negatives in which the pullback of the differential bundle \( p : T(E) \to E \) along \( \xi \) exists. On the other hand in Section 3.2 we work in a general tangent category and assume that \( (q, \xi, \lambda) \) is a strong differential bundle (defined in Definition 3.2.1) of which all differential bundles in a tangent category with negatives are examples. Then in Section 3.3 we use retractive display systems (see Section I.2) to characterise differential bundles as pre-differential bundles satisfying a single pullback diagram.

### 3.1 Differential bundles as retracts

In this section, we consider a fixed differential bundle \( (q : E \to M, \xi, \lambda) \) in a tangent category \( X \) with negatives. Furthermore we assume that the following pullback
\[
\begin{array}{ccc}
T_M(E) & \xrightarrow{\iota_M} & T(E) \\
\pi_M \downarrow & & \downarrow p \\
M & \xrightarrow{\xi} & E
\end{array}
\]
exists in \( X \), \( \pi_M \) is a differential bundle and \( (\iota_M, \xi) \) is a linear morphism of differential bundles. Under the above assumptions we prove that \( q \) is a retract of the pullback \( \pi_M \):
\[
\begin{array}{ccc}
T_M(E) & \xrightarrow{\iota_M} & T_M(E) \\
\pi_M \downarrow & & \downarrow q \\
M & \xrightarrow{\xi} & M
\end{array}
\]
where the map between the base spaces is the identity. First, we describe the idempotent whose splitting defines this retract.

**Lemma 3.1.1.** The arrow \( \chi : T_M(E) \to T_M(E) \) uniquely determined by the equation
\[
\iota_M \chi = \iota_M - \xi(\lambda(q)) \iota_M
\]
is idempotent and linear.
Proof. The sum on the right hand side is well-typed because
\[
pT(\xi q)\iota_M = \xi q p_M = \xi q \pi_M = \xi \pi_M = p_M
\]
and the arrow \(\chi\) is factors through \(T_M(E)\) because \(p_M \chi = p_M = \xi \pi_M\). To see that \(\chi\) is idempotent:
\[
\iota_M \chi \chi = (\iota_M - p T(\xi q)\iota_M) \chi = \iota_M \chi - p T(\xi q)\iota_M \chi = (\iota_M - p T(\xi q)) (\iota_M - p T(\xi q)) = (\iota_M - p T(\xi q)) \iota_M = \iota_M \chi
\]
and so \(\chi \chi = \chi\) because \(\iota_M\) is a monomorphism. To see that \(\chi\) is linear:
\[
T(\iota_M) \lambda_M \chi = \iota_M \chi = \ell(\iota_M - p T(\xi q))\ell_M = \ell(\iota_M - T(\xi q))\ell_M = T(\iota_M) \lambda_M - T(\xi q) T(\iota_M) \lambda_M = T(\iota_M) T(\chi) \lambda_M
\]
and so \(\lambda_M \chi = T(\chi) \lambda_M\) because \(T(\iota_M)\) is a monomorphism. Note that \(T(\iota_M) \lambda_M = \ell_M\) because \((\iota_M, \xi)\) is a linear morphism of differential bundles.

Next, we show that the idempotent \(\chi\) splits.

**Proposition 3.1.2.** If \(q : E \to M\) is a differential bundle in a tangent category with negatives and the pullback \(T_M(E)\) exists then
\[
E \xrightarrow{(\lambda, q)} T_M(E) \xrightarrow{\chi} T_M(E)
\]
is an equaliser that is preserved by any functor.

Proof. Let \(a : A \to T_M(E)\) such that \(\chi a = a\). First we show that there exists \(e : A \to E\) such that \((\lambda, q)e = a\). To this end observe that
\[
T(q) \iota_M a = T(q) \iota_M \chi a = T(q) (\iota_M - p T(\xi q)) a = 0 q p_M a
\]
and also \(p_M a = \xi q p_M a\) by the definition of \(T_M(E)\). Therefore by the universality of the vertical lift as described in Corollary 3.1.3. there exists a unique \(e : A \to E\) such that \(\iota_M a = \lambda e\). This is the arrow \(e\) that we require:
\[
\iota_M (\lambda, q)e = \lambda e = \iota_M a
\]
and so \((\lambda, q)e = a\) because \(\iota_M\) is a monomorphism. To check this is unique let \(f : A \to E\) be an arrow such that \((\lambda, q)f = a\). Then
\[
\iota_M (\lambda, q)f = \iota_M a \implies \lambda f = \lambda e \implies f = e
\]
and so \((\lambda, q)\) is the equaliser of \(\chi\) and \(id\). Since \(\chi\) is idempotent the equaliser \((\lambda, q)\) is preserved by any functor. \(\square\)

**Corollary 3.1.3.** If \(K\) is the factorisation of \(\chi\) through \((\lambda, p)\) then
\[
T_M(E) \xrightarrow{K} E \xrightarrow{(\lambda, q)} T_M(E)
\]
is a retract over the fixed base space \(M\).

**Corollary 3.1.4.** If \((q : E \to M, \xi, \lambda)\) is a differential bundle and the pullback differential bundle \(\pi_M\) exists then \((q, \xi, \lambda)\) is a retract of a pullback of a tangent bundle.
3.2 Strong differential bundles as retracts

In this section, we re-interpret the results of Section 3.1 in an arbitrary tangent category. This process reveals a third universality condition satisfied by differential bundles in tangent categories with negatives, which states the space of tangent vectors based at a zero vector decomposes as a biproduct of its horizontal and vertical parts. We call differential bundles satisfying this third universality condition strong, and show that in a tangent category with negatives any differential bundle \((q : E \to M, \xi, \lambda)\) is strong if the \(T\)-pullback \(E_q \times_p TM \to M\) exists, from which Corollary 3.1.3 follows.

Definition 3.2.1. A differential bundle \((q : E \to M, \xi, \lambda)\) is strong if and only if

\[
\begin{array}{ccc}
E_q \times_p TM & \xrightarrow{\lambda \pi_0 + \pi T(\xi) \pi_1} & T(E) \\
q\pi_0 = p\pi_1 & \downarrow & \downarrow\
M & \xrightarrow{\xi} & E
\end{array}
\]

is a \(T\)-pullback.

We now make it rigorous that tangent vectors on \(T M \to E\) exists, we could characterise the bundle \(T_M(E)\) as an idempotent splitting. The bundle \(T_M(E)\) is the pullback of \(p_E\) along \(\xi : M \to E\), and a strong differential bundle characterises this pullback as a biproduct \(T M_p \times q E\), so there is a canonical linear idempotent that splits as \((q : E \to M, \xi, \lambda)\).

Proof.

(a) To check that \(q\pi_0 : E_q \times_p TM \to M\) is a differential bundle use the same construction found in corollary 5.9 of [3]. In particular by Corollary 3.2.3 it suffices to prove that the following diagram is a \(T\)-limit

\[
\begin{array}{ccc}
(E_q \times_p TM) & \xrightarrow{(\lambda \times 1)_{\pi_0} + T(\xi)_{\pi_1} \pi_0} & T(E_q \times_p TM) \\
q\pi_0 & \downarrow & \downarrow\
M & \xrightarrow{(\xi, \lambda)} & TM
\end{array}
\]

but this follows immediately by the commutation of limits.

(b) Now we show that \(E_q \times_p TM\) is a coproduct. First note that \(id_M : M \to M\) is the zero object in the category of differential bundles over \(M\) and that each differential bundle \(q : E \to M\) has a zero morphism \((\xi, id_M) : 1_M \Rightarrow q\) which is preserved by any linear morphism. The coproduct diagram is:

\[
E \xrightarrow{id = (id, 0q)} E_q \times_p TM \xrightarrow{\xi_1 = (\xi, id)} TM
\]

because for any pair of linear bundle morphisms \(f : E \to Z, g : TM \to Z\) the map \((f\pi_0 + g\pi_1)\) satisfies:

\[(f\pi_0 + g\pi_1)(\xi, id) = (f\xi + g id) = (\xi z + q + g) = g\]

and

\[(f\pi_0 + g\pi_1)(id, 0q) = (f\pi_1 + g0q) = (f\pi_1 + g \xi q) = f\]

which are the equations expressing the universal property of a coproduct. Next, we check the biproduct identities:

\[
\begin{align*}
\pi_{00} &= \pi_0(id, 0q) = id \\
\pi_{10} &= \pi_1(id, 0q) = 0q \\
\pi_{01} &= \pi_0(\xi, id) = \xi p \\
\pi_{11} &= \pi_1(\xi, id) = id
\end{align*}
\]

and so \(E_q \times_p T(M)\) is the biproduct of \(q : E \to M\) and \(p : T(M) \to M\) in the category of differential bundles over \(M\).

\(\square\)

In the previous section, we proved that in a tangent category with negatives and the pullback along \(p\) always exists, we could characterise the bundle \(T_M(E)\) as an idempotent splitting. The bundle \(T_M(E)\) is the pullback of \(p_E\) along \(\xi : M \to E\), and a strong differential bundle characterises this pullback as a biproduct \(T M_p \times q E\), so there is a canonical linear idempotent that splits as \((q : E \to M, \xi, \lambda)\).
Corollary 3.2.3. Let \((q : E \to M, \xi, \lambda)\) be a strong differential bundle in a tangent category, and assume \(T\)-pullback powers of \(E_q \times_p TM\) exist. Then there is an idempotent splitting in the category of differential bundles:

\[
\begin{array}{ccc}
E_q \times_p TM & \xrightarrow{\pi_0} & E \\
q \circ \pi_0 & \downarrow & \downarrow q \\
M & \xrightarrow{q \circ \pi_1} & M
\end{array}
\]

Furthermore, if \(T\)-pullback powers of \(E_q \times_p TM \to M\) exist, this is a linear splitting in the category of differential bundles above \(M\).

In a tangent category with negatives, any differential bundle will be strong, provided the \(T\)-pullback of the projection along the tangent projection exists. This means the strong universality condition may be seen as “the other side” of the Cockett-Cruttwell universality condition, and both follow from Rosicky’s universality condition.

Lemma 3.2.4. If \((q : E \to M, \xi, \lambda)\) is a pre-differential bundle in a tangent category with negatives and the \(T\)-pullback \(E_q \times_p T(M)\) exists then

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & T(E) \\
q \downarrow & \downarrow \pi_1 & \downarrow q \\
M & \xrightarrow{(\xi, 0)} & E \times TM
\end{array}
\]

is a \(T\)-pullback if and only if

\[
\begin{array}{ccc}
E_q \times_p T(M) & \xrightarrow{\lambda \pi_0 + \pi_1 (\xi) \pi_1} & T(E) \\
q \downarrow & \downarrow \pi_1 & \downarrow q \\
M & \xrightarrow{\xi} & E
\end{array}
\]

is a \(T\)-pullback.

Proof. First, we prove the forward implication. So let \(a : A \to T^n + 1(E)\) so that \(T^n(p)a = T^n(q \pi p)a\). Then \(a \circ T^n(\xi \pi) a\) satisfies:-

- \(T^n + 1(q)(a \circ T^n(\xi \pi) a) = T^n + 1(q) a \circ T^n + 1(q \pi) a = T^n(0 \pi) a\)
- \(T^n(p)(a \circ T^n(\xi \pi) a) = T^n(p) a = T^n(q \pi p) a\)

and so using the first bullet point in the statement of this lemma we induce an \(a' : A \to T^n E\) such that \(T^n(\lambda) a' = a \circ T^n(\xi \pi) a\). Now \((a', T^n(q) a) : A \to E_q \times_p TM\) is the factorisation we need because:

\[
(T^n(\lambda) \pi_0 + T^n(\xi) \pi_1)(a', T^n + 1(\xi) \pi_1) = (a \circ T^n(\xi \pi) a) + T^n(\xi \pi p) T^n + 1(\xi) a
\]

which is unique because \(T^n(\lambda)\) and \(T^n + 1(\xi)\) are monomorphisms. Next, we prove the converse implication. So let \(a : A \to T^n + 1 E\) be such that \(T^n + 1(q) a = T^n(0 \pi) a\) and \(T^n(p) a = T^n(q \pi p) a\). By combining the latter equation with the assumption contained second bullet point of the statement of this lemma we induce a pair of maps \((\tilde{a}_0, \tilde{a}_1) : A \to T^n(E_q \times_p TM)\) so that

\[
T^n(\lambda) \tilde{a}_0 + T^n(p) T^n + 1(\xi) \tilde{a}_1 = a
\]

holds. Post-composing both sides of this equation with \(T^n + 1(q)\) gives \(\tilde{a}_1 = T^n + 1(q) a = T^n(0 \pi p) a\) and so the displayed equation is in fact \(T^n(\lambda) \tilde{a}_0 = a\) which shows that \(\tilde{a}_0 : A \to T^n(E)\) is the factorisation we require. This factorisation is unique because \(T^n(\lambda) \pi_0 + T^n(\xi) T^n + 1(\xi) \pi_1\) and \(T^n + 1(\xi)\) are monomorphisms.

Corollary 3.2.5. Let \((q : E \to M, \xi, \lambda)\) be a pre-differential bundle in a tangent category with negatives such that:-

- \(n\)-fold pullback powers of \(q\) exist and are \(T\)-limits and
- the pullback \(E_q \times_p TM\) exists and is a \(T\)-pullback.

Then \((\lambda, q, \xi)\) is a strong differential bundle if and only if

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & T(E) \\
q \downarrow & \downarrow \pi_1 & \downarrow q \\
M & \xrightarrow{(\xi, 0)} & E \times TM
\end{array}
\]

is a \(T\)-pullback.
Recall that in the category of smooth manifolds, $p_M$ is a submersion so $T$-pullbacks along $p_M$ exist for all $M$, thus we have the following corollary.

**Corollary 3.2.6.** In the category of smooth manifolds, every differential bundle is strong.

### 3.3 Differential bundles and retractive display systems

The condition that a differential bundle $(q : E \rightarrow M, \xi, \lambda)$ has $T$-pullback powers of its projection $q$ is necessary to ensure it has a coherent additive bundle structure. In a display differential bundles were considered - these are differential bundles $(q : E \rightarrow M, \xi, \lambda)$ in a display tangent category $(X, \mathcal{D})$ satisfying $q \in \mathcal{D}$, which guarantees the existence of $T$-pullback powers of $q$ (along with some convenient re-indexing properties). In this section we explore how the strong universality condition of Section 3.2 interacts with the retractive display systems of Section 1.2, and find that every strong differential bundle is displayed.

**Definition 3.3.1.** Let $X$ be a tangent category with a display system $\mathcal{D}$. We say that $(q, \xi, \lambda)$ is a $\mathcal{D}$-displayed differential bundle if $q \in \mathcal{D}$.

Now suppose our tangent category has a proper retractive display system $R$. Because a strong differential bundle naturally splits a linear idempotent of a pullback of a tangent projection, every strong differential bundle is $R$-displayed.

**Proposition 3.3.2.** If $R$ is a proper retractive display system on a tangent category $X$ and $(q : E \rightarrow M, \xi, \lambda)$ is a strong differential bundle in $X$, then $q$ is in $R$.

**Proof.** This follows immediately from Lemma 3.2.2. Let $(q : E \rightarrow M, \xi, \lambda)$ be a strong differential bundle and consider the following diagrams:

- The left diagram exhibits $(q\pi_0, (\xi, 0), (\lambda \times \ell))$ as the pullback of the tangent projection on the total space $p_E$, and the right diagram exhibits $(q, \lambda, \xi)$ is the splitting of the linear idempotent $\iota_0\pi_0$ on $(q\pi_0, (\xi, 0), (\lambda \times \ell))$, so we have $q$ is $R$-displayed.

We can use Proposition 3.3.2 to show that every differential bundles in a tangent category with negatives and a retractive display system is $R$-displayed. We first prove a lemma that holds in a tangent category with negatives and display system.

**Lemma 3.3.3.** In a tangent category with negatives and a proper display system $D$, every differential bundle is strong.

**Proof.** Consider a differential bundle $(q : E \rightarrow M, \xi, \lambda)$, by Corollary 3.2.5 this is a strong differential bundle if and only if the $T$-pullback $E_q\times_pTM$ exists, but this holds as $p \in \mathcal{D}$.

The following corollary is a straightforward application of the two previous results (Proposition 3.3.2 and Lemma 3.3.3).

**Corollary 3.3.4.** In a tangent category with negatives and a proper retractive display system $\mathcal{R}$, every differential bundle $(q : E \rightarrow M, \xi, \lambda)$ has its projection $q \in \mathcal{R}$.

When we apply this corollary to the category of smooth manifolds, where $\mathcal{R}$ is the class of submersions, we have the following:

**Corollary 3.3.5.** In the category of smooth manifolds where the class of submersions is $\mathcal{R}$, every differential bundle is $\mathcal{R}$-display.

Now that we have shown that the projection of a differential bundle in the category of differential bundles is a submersion, we may rewrite the lift in local coordinates.

**Proposition 3.3.6.** Let $(q : E \rightarrow M, \xi, \lambda)$ be a differential bundle in the category of smooth manifolds. Then the lift $\lambda$ may be rewritten in local coordinates as $\lambda(u, a) = (u, 0, 0, a)$.
Proof. We can use the implicit function theorem to write \( \lambda \) in terms of local co-ordinates (see [9] for details). The equations \( T(q) \lambda = 0 \) and \( p \lambda = \xi q \) imply that \( \lambda(u, a) = (u, 0, 0, \Lambda(u, a)) \) for some \( \Lambda(u, a) \in \mathbb{R}^n \). Next we write the universality of the lift in terms of local coordinates:

\[
\begin{array}{cccc}
B^k \times \mathbb{R}^n & \xrightarrow{(\pi_0, 0, 0, \pi_1)} & B^k \times \mathbb{R}^n \times \mathbb{R}^k \times \mathbb{R}^n \\
E & \xrightarrow{\lambda} & T(E) & \\
\pi_0 & \downarrow & (0, \xi) & \downarrow (\xi, 0) \\
M & \xrightarrow{(\xi, 0)} & T(M) \times E & \xrightarrow{\xi} E \\
B^k & \xrightarrow{(id, id, 0)} & B^k \times \mathbb{R}^k \times B^k \times \mathbb{R}^n
\end{array}
\]

where \( B^k \) is the unit ball in \( \mathbb{R}^k \) and so \( \lambda(u, a) = (u, 0, 0, a) \).

In case a tangent category with negatives has a proper retractive display system \( \mathcal{D} \), every differential bundle is \( \mathcal{D} \)-display. Because we have shown the equivalence of various universality conditions, we can use the retract-closed property to force various pullbacks to be \( T \)-pullbacks. We use this to give a simplified definition of a differential bundle that is simpler in practice to verify.

**Lemma 3.3.7.** Let \( (q : E \to M, \xi, \lambda) \) be a pre-differential bundle in a tangent category with negatives and a proper display system \( \mathcal{D} \) (as defined in Definition 1.2.1). If either of the diagrams in the statement of Lemma 3.2.4 is a pullback then both diagrams are \( T \)-pullbacks.

**Proof.** By Lemma 3.2.4, if one is a pullback the other is. If the strong universality diagram is a pullback it is a \( T \)-pullback because \( \pi \in \mathcal{D} \). Then by Lemma 3.3.7 both diagrams are \( T \)-pullbacks.

The category of smooth manifolds is a tangent category with negatives. Furthermore, the surjective submersions form a proper retractive display system \( \mathcal{D} \) as defined in Definition 1.2.1. The following result describes a situation where the characterisation of differential bundles further simplifies. First, we remove the requirement that pullback powers of \( q \) exist. Second, we remove the requirement that the diagram expressing the universality of the lift is a \( T \)-limit (although we still require it be a limit).

**Proposition 3.3.8.** Let \( \mathcal{K} \) be a tangent category with negatives equipped with a proper retractive system \( \mathcal{D} \). A pre-differential bundle \( (q : E \to M, \xi, \lambda) \) is a differential bundle if and only if

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & T(E) \\
\downarrow q & & \downarrow T(p) \\
M & \xrightarrow{(\xi, 0)} & E \times TM \\
\end{array}
\]

is a pullback.

**Proof.** The forward implication holds by Corollary 2.2.5 so it remains to prove the reverse implication. First Lemma 3.3.7 implies that both

\[
\begin{array}{ccc}
E \times_p TM & \xrightarrow{\nu} & TE \\
\downarrow q\pi_0 = p\pi_1 & & \downarrow \rho \\
M & \xrightarrow{\xi} & E \\
\end{array}
\]

and

\[
\begin{array}{ccc}
E & \xrightarrow{\lambda} & TE \\
\downarrow q \pi_0 & & \downarrow \rho \\
M & \xrightarrow{(\xi, 0)} & E \times TM \\
\end{array}
\]

are \( T \)-limits. Then \( q \in \mathcal{D} \) because \( \mathcal{D} \) is closed to retracts, so \( T \)-pullback powers of \( q \). By Corollary 2.2.5 \( (q, \xi, \lambda) \) is a differential bundle.

The category of smooth manifolds has a retractive display system - the class of smooth submersions - so, in particular, we have the following corollary.
Corollary 3.3.9. A pre-differential bundle \( (q : E \to M, \xi, \lambda) \) in the category of smooth manifolds is a differential bundle if and only if

\[
\begin{array}{ccc}
E & \xrightarrow{} & TE \\
\downarrow^q & & \downarrow^{(q, Tq)} \\
M & \xrightarrow{(\xi, 0)} & E \times TM
\end{array}
\]

is a pullback.

The goal of this paper is to show that every differential bundle in the category of smooth manifolds is a vector bundle. Using Lemma 3.3.7, that a pre-differential bundle satisfying with an additive bundles structure is insufficient, thus demonstrating the necessity of the universality conditions on differential bundles.

## 4 Differential bundles in smooth manifolds

In this section we prove that the category \( VBun \) of (smooth) vector bundles is isomorphic to the category \( DBun(SMan) \) of differential bundles in the category of smooth manifolds. In Section 4.1 we define a functor \( \Psi : VBun \to DBun(SMan) \). Then in Section 4.2 we define a functor \( \Phi : DBun(SMan) \to VBun \) and show that \( \Psi \) and \( \Phi \) are inverses.

In this paper we follow Definition 5.9 in [15] and work with manifolds that may have different dimensions in different connected components. One advantage of this definition is that the category of smooth manifolds is idempotent complete. The traditional definition of smooth manifold insists that all of the local co-ordinate systems have the same dimension even across different connected components (see for instance Definition 2.1 of [7] and the pure manifolds of 1.1 of [11]). The main result of this paper (differential bundles in the category of smooth manifolds are vector bundles) also holds for the traditional definition of a manifold and all of our proofs remain unchanged in this case.

### 4.1 Vector bundles are differential bundles

In this section we define a functor \( \Psi : VBun \to DBun(SMan) \) from the category of smooth vector bundles to the category of differential bundles in the category of smooth manifolds. The main result of this paper is that \( \Psi \) is invertible which we prove in Section 4.2. Since we allow our manifolds to have different dimensions in different connected components, it is natural to allow the dimension of the fibres of our vector bundles to have different dimensions in different connected components also. Therefore the definition of a vector bundle that we use is a slight generalisation of the definition in Section 12.3 of [15].

**Definition 4.1.1.** A (smooth) vector bundle consists of a map \( q : E \to M \) in the category of smooth manifolds such that:

- for all \( m \in M \) each fibre \( q^{-1}(m) \) is a vector space
- for all \( m \in M \) there exists an open neighbourhood \( U \) of \( m \), a natural number \( r \) and a fibre-preserving diffeomorphism \( \phi : q^{-1}(U) \to U \times \mathbb{R}^r \) such that for all \( x \in U \) the map \( \phi|_{q^{-1}(x)} \) is a vector space isomorphism.

In the particular case where \( E \) and \( M \) have constant global dimensions \( l \) and \( k \) respectively the third condition implies that every fibre of a vector bundle \( q : E \to M \) has dimension \( l - k \). (I.e. the dimension of the fibres is globally constant and we recover the definition of rank-(\( l - k \)) vector bundle in Section 12.3 of [15].) Note that in the general case the dimension of the fibres of \( q \) is still constant within each connected component of \( M \) but is not necessarily globally constant.

**Proposition 4.1.2.** Every vector bundle in the category of smooth manifolds is a differential bundle.

**Proof.** If \( q : E \to M \) is a vector bundle and \( m \in M \) then we can write an element of \( E \) in local coordinates as \( (m, a) \) and an element of \( T(E) \) in local coordinates as \( (m, a, v, b) \) where \( m \in M \), \( a, b \in \mathbb{R}^r \) and \( v \in \mathbb{R}^k \) where \( r \) is the dimension of \( q^{-1}(m) \) and \( k \) is the dimension of \( M \) in the component containing \( m \). Then \( +_q : E_2 \to E \) is given by \( +_q(m, a, b) = (m, a + b) \) where \( \xi : M \to E \) by \( \xi(m) = (m, 0) \) and \( \lambda : E \to T(E) \) by \( \lambda(m, a) = (m, 0, 0, a) \).

First, we check the axioms of a pre-differential bundle. The equality \( q \xi = id \) is immediate. Next

\[
\lambda \xi(m) = \lambda(m, 0) = (m, 0, 0, 0) = 0(m, 0) = 0 \xi(m)
\]

and

\[
T(\lambda) \lambda(m, a) = T(\lambda)(m, 0, 0, a) = (m, 0, 0, 0, 0, 0, a) = \ell(m, 0, 0, a) = \ell \lambda(m, a)
\]

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because \( T(\lambda)(m, a, v, b) = (m, 0, 0, v, 0, 0, b) \). Now we check the universality of the lift. So suppose that \( \bar{a} : A \rightarrow T(E) \) satisfies \( T(q)\bar{a} = 0qp\bar{a} \) and \( p\bar{a} = \xi qp\bar{a} \) as in the diagram

\[
\begin{array}{cccc}
A & \xrightarrow{f} & E & \xrightarrow{\lambda} T(E) \\
\downarrow \bar{a} & & \downarrow q & \downarrow (T(q), p) \\
M & \xrightarrow{(0, \xi)} & T(M) \times E
\end{array}
\]

and suppose that \( \bar{a} = (m, a, v, b) \) in local coordinates. Then the condition \( T(q)\bar{a} = 0qp\bar{a} \) implies that \( v = 0 \) and the condition \( p\bar{a} = \xi qp\bar{a} \) implies that \( a = 0 \). Therefore there exists a factorisation \( f : A \rightarrow E \) given by in local co-ordinates by \((m, a)\) which is unique because \( \lambda \) is a monomorphism.

Proposition \[\text{4.1.2}\] formulates the lift \( \lambda \) in terms of local coordinates. The following remark reformulates the lift in terms of the scalar multiplication.

Remark 4.1.3. The function \( T(\bullet, q) \) is given in local co-ordinates by the following formula:

\[
(s, t) \bullet_q^T (x, a, v, b) = (x, s \bullet_q a, v, t \bullet_q a + s \bullet_q b)
\]

therefore

\[
\partial_0^q! \bullet_q^T 0a = (0, 1) \bullet_q^T (x, a, 0, 0) = (x, 0, 0, a) = \lambda a
\]

where \( 0^q \) is the additive unit of \( \mathbb{R} \) and \( \partial : \mathbb{R} \rightarrow T(\mathbb{R}) \) is defined by \( x \mapsto (x, 1) \). Therefore \( \lambda = \partial_0^q! \bullet_q^T 0 \).

Lemma 4.1.4. The function defined in Proposition \[\text{4.1.2}\] extends to a functor \( \Psi : VBun \rightarrow DBun(SMan) \) from the category of vector bundles to the category of differential bundles in the category of smooth manifolds.

Proof. The action of \( \Psi \) on objects is given in Proposition \[\text{4.1.2}\]. The action of \( \Psi \) on arrows is the identity function. For this to make sense we need to check that if

\[
\begin{array}{ccc}
E_0 & \xrightarrow{f} & E_1 \\
\downarrow \Phi_0 & & \downarrow \Phi_1 \\
M_0 & \xrightarrow{\psi} & M_1
\end{array}
\]

is a morphism of vector bundles then \( \lambda_1 f = T(f)\lambda_0 \) where \( \lambda_i \). We use the formulation of the lifts \( \lambda_i \) in terms of the scalar multiplication given in Remark \[\text{4.1.3}\].

\[
\begin{align*}
\lambda_1 f &= T(\bullet)(\partial_0^q!, 0)f \\
&= T(\bullet)(\partial_0^q!, 0f) \\
&= T(\bullet)(\partial_0^q!, T(f)0) \\
&= T(\bullet)(id \times T(f))(\partial_0^q!, 0) \\
&= T(\bullet)(id \times f)(\partial_0^q!, 0) \\
&= T(f\bullet)(\partial_0^q!, 0) = T(f)\lambda_0
\end{align*}
\]

where the penultimate equality follows from the fact that \( f \) preserves scalar multiplication. □

4.2 Differential bundles in smooth manifolds are vector bundles

In Section \[\text{4.1}\] we constructed a functor \( \Psi : VBun \rightarrow DBun(SMan) \) from the category of smooth vector bundles to the category of differential bundles in the category of smooth manifolds. In this section we construct a functor \( \Phi : DBun(SMan) \rightarrow VBun \) and show that \( \Psi \) and \( \Phi \) are inverses. Our general strategy is to recall that the category of vector bundles is closed under pullback and idempotent splittings. Then we can apply Corollary \[\text{3.1.3}\] to obtain our result. So first we recall that the pullback of a vector bundle is a vector bundle.

Lemma 4.2.1. Let \( q_0 : E_0 \rightarrow M_0 \) be a vector bundle with addition \(+_0\), zero \( \xi_0 \) and scalar multiplication \( \bullet_0 \). If

\[
\begin{array}{ccc}
E_1 & \xrightarrow{\iota} & E_0 \\
\downarrow q_1 & & \downarrow q_0 \\
M_1 & \xrightarrow{\iota} & M_0
\end{array}
\]

is a pullback then \( q_1 : E_1 \rightarrow M_1 \) is a vector bundle with addition \(+_1\), zero \( \xi_1 \) and multiplication \( \bullet_1 \) such that:-
function. For this to make sense we need to check that if \( \phi \) is a
pre-differential bundle axiom \( \lambda \xi \) and multiplication \( \bullet \). Proof. The action of \( \Phi \) on objects is defined in Proposition 4.2.5. Note that the scalar multiplication is the unique one satisfying \( \lambda(r \bullet_a a) = r \bullet_a \lambda a \) because \( \lambda \) is a monomorphism. The action of \( \Phi \) on arrows is the identity function. For this to make sense we need to check that if \( f \) and \( f_0 \) satisfy

\[
\begin{array}{ccc}
E_0 & \xrightarrow{f} & E_1 \\
\downarrow{g_0} & & \downarrow{g_1} \\
M_0 & \xrightarrow{f_0} & M_1
\end{array}
\]

Lemma 4.2.2. Let \( q_1 : E_1 \to M_1 \) be a vector bundle with addition \( +_1 \), zero \( \xi \) and multiplication \( \bullet_1 \). If

\[
\begin{array}{cc}
E_1 & \xrightarrow{\phi} E_1 \\
\downarrow{q_1} & \downarrow{q_1} \\
M_1 & \xrightarrow{M_1}
\end{array}
\]

is an idempotent vector bundle endomorphism over the fixed base space \( M \) then its image \( q_2 : E_2 \to M_1 \) is a vector bundle with addition \( +_2 \), zero \( \xi \) and multiplication \( \bullet _2 \) such that:-

- \( S(a +_2 b) = Sa +_1 Sb \)
- \( S \xi_2 = \xi_1 \)
- \( S(r \bullet_2 a) = r \bullet_1 Sa \)

where \( S \) is the equaliser of \( \phi \) and \( id \) and where \( a, b \in E_2 \) such that \( q_2(a) = q_2(b) \).

Proof. Since the morphism between base spaces is the identity, we only need to consider the image of \( \phi \). Following Proposition 1 in [14] it suffices to check that \( \phi \) has locally constant rank. However since \( \phi \) is a projection its rank is equal to its trace. Since the trace is continuous and the rank takes integer values, we conclude that the rank of \( \phi \) is locally constant.

Remark 4.2.3. The proof of Lemma 4.2.2 remains unchanged for the traditional definition of a smooth manifold because the map between the base spaces is the identity.

The following example shows that the above result does not necessarily hold if \( (\phi, id) \) is not idempotent.

Example 4.2.4. The kernel of the vector bundle morphism

\[
(\phi, id) : (\mathbb{R} \times \mathbb{R}, \pi_0) \to (\mathbb{R} \times \mathbb{R}, \pi_0)
\]

where \( \phi : (x, r) = (x, r \bullet x) \) is the union of \( \{0\} \times \mathbb{R} \) and \( (\mathbb{R} \setminus \{0\}) \times \{0\} \) in the category of topological bundles. Therefore the kernel does not necessarily exist in the category of smooth vector bundles if \( \phi_1 \) is not idempotent.

Now we combine our previous results to prove the main result of this paper.

Proposition 4.2.5. If \( (q : E \to M, +, \xi, \lambda) \) is a differential bundle in the category of smooth manifolds then \( q \) is the projection of a vector bundle with addition \( +_2 \), zero \( \xi \) and scalar multiplication \( \bullet \) satisfying \( \lambda(r \bullet_a a) = r \bullet_a \lambda a \).

Proof. Corollary 3.1.3 shows that \( q \) is a retract of a pullback of \( p : T(E) \to E \). First apply Lemma 4.2.1 with \( \iota_0 = \iota_M \) and second apply Lemma 4.2.2 with \( S = (\lambda, q) \). Since \( \lambda = \iota_M(\lambda, q) \) therefore \( q : E \to M \) is a vector bundle with addition \( +_2 \), zero \( \xi_2 \) and scalar multiplication \( \bullet_2 \) such that:-

- \( \lambda(a +_2 b) = \lambda a +_p \lambda b \),
- \( \lambda \xi_2 = 0 \xi \),
- \( \lambda(r \bullet_2 a) = r \bullet_a \lambda a \).

Therefore \( +_2 = + \) because \( \lambda : (E, q) \to (T(E), p) \) is an additive bundle morphism. Also \( \xi_2 = \xi \) because of the pre-differential bundle axiom \( \lambda \xi = 0 \xi \).

Lemma 4.2.6. The function defined in Proposition 4.2.5 extends to a functor \( \Phi : DBun(SMan) \to VBun \) from the category of differential bundles in the category of smooth manifolds to the category of vector bundles.

Proof. The action of \( \Phi \) on objects is defined in Proposition 4.2.5. Note that the scalar multiplication is the unique one satisfying \( \lambda(r \bullet_a a) = r \bullet_a \lambda a \) because \( \lambda \) is a monomorphism. The action of \( \Phi \) on arrows is the identity function. For this to make sense we need to check that if \( f \) and \( f_0 \) satisfy

\[
\begin{array}{ccc}
E_0 & \xrightarrow{f} & E_1 \\
\downarrow{g_0} & & \downarrow{g_1} \\
M_0 & \xrightarrow{f_0} & M_1
\end{array}
\]
and \( \lambda f = T(f)\lambda \) then \( f \) preserves the addition and scalar multiplication of \( \Phi(q_0) \). To see that the addition is preserved we refer to Proposition 2.16 of \([2]\). To see that the scalar multiplication is preserved we calculate:

\[
\lambda f(r \cdot_0 a) = T(f)(r \cdot_0 a) = T(f)(r \cdot_1 \lambda a) = r \cdot_1 T(f)\lambda a = r \cdot_1 \lambda fa = \lambda(r \cdot_1 fa)
\]

and so \( f(r \cdot_0 a) = r \cdot_1 fa \) because \( \lambda \) is a monomorphism.

**Theorem 4.2.7.** The category of differential bundles (with linear or bundle morphisms) in the category of smooth manifolds is isomorphic to the category of smooth vector bundles (with linear or bundle morphisms, respectively).

**Proof.** We show that the functors \( \Psi \) and \( \Phi \) (defined in Proposition 4.2.5 and Lemma 4.2.6 respectively) are inverses. Since the action of both \( \Phi \) and \( \Psi \) on arrows is the identity function we only need to consider the action on objects. Since \( \Phi \) and \( \Psi \) leave the projection, addition and zero section unchanged it in fact only remains to consider the lift and scalar multiplication.

In one direction let \((q : E \to M, \xi, +, \lambda)\) be a differential bundle in the category of smooth manifolds. Proposition 4.2.5 implies that \( \Phi(q) \) has scalar multiplication satisfying \( \lambda \cdot = \cdot_\rho(id \times \lambda) \). Remark 4.1.3 implies that \( \Psi \Phi(q) \) has lift \( \lambda_2 = T(\bullet)(\partial \xi^!0, 0) \). We need to show that \( \lambda = \lambda_2 \):

\[
T(\lambda)\lambda_2 = T(\lambda)T(\bullet)(\partial \xi^!0, 0) = T(\cdot_\rho)T(id \times \lambda)(\partial \xi^!0, 0) = T(\cdot_\rho)(T(\lambda)(\partial \xi^!0, 0)\lambda = \ell\lambda = T(\lambda)\lambda
\]

and so \( \lambda_2 = \lambda \) because \( T(\lambda) \) is a monomorphism.

In the other direction let \((q : E \to M, \xi, +, \bullet)\) be a vector bundle. Remark 4.1.3 implies that \( \Psi(q) \) is a differential bundle with lift given by \( \lambda(m, a) = (m, 0, 0, a) \) in local coordinates. Proposition 4.2.5 implies that \( \Phi \Psi(q) \) is a vector bundle with scalar multiplication \( \bullet \) satisfying \( \lambda \cdot_2 = \cdot_\rho(id \times \lambda) \). We need to show that \( \bullet = \cdot_2 \):

\[
\lambda(r \cdot_2 (m, a)) = r \cdot_\rho \lambda(m, a) = r \cdot_\rho (m, 0, 0, a) = (m, 0, 0, r \cdot a) = \lambda(r, r \cdot a) = \lambda(r \cdot (m, a))
\]

and so \( \bullet = \cdot_2 \) because \( \lambda \) is a monomorphism.

We conclude by showing that it was in fact necessary to use the universality of the lift in the proof of Proposition 4.2.5.

**Example 4.2.8.** We give an example of a pre-differential bundle (as defined in Definition 2.1.1) in the category of smooth manifolds that is not a vector bundle. Let \( q : \mathbb{R}^2 \to \mathbb{R} \) be defined by \( q(x, y) = (1 - \delta(y))x + \delta(y)x^3 \) where \( \delta \) is a smooth and monotonic bump function that is 0 for \( y \leq 0 \) and 1 for \( y \geq 1 \). To see that \( q \) cannot be the projection of a vector bundle recall that every vector bundle is, in particular, a submersion. However, the derivative of \( q \) the point \((0, 1)\) vanishes so \( q \) is not a submersion.

Now we show that \( q \) is a pre-differential bundle with lift \( \lambda(x, y) = (q(x, y), 0, 0, y) \) and zero section \( \xi(z) = (z, 0) \). First we check:

- \( q\xi(z) = q(z, 0) = (1 - \delta(0))z + \delta(0)z^3 = z \)
- \( \lambda\xi(z) = \lambda(z, 0) = (z, 0, 0, 0) = 0\xi(z) \)

and to check that \( T(\lambda)\lambda = \ell\lambda \) we first note that in general:

\[
T(\lambda)(x, y, v, w) = (q(x, y), 0, 0, y, (1 - \delta(y) + \delta(y)3x^2)v + (x^2 - 1)\delta'(y)wxw, 0, 0, w)
\]

and so

\[
T(\lambda)\lambda(x, y) = T(\lambda)(q(x, y), 0, 0, y) = (q(x, y), 0, 0, 0, 0, 0, 0) = \ell(q(x, y), 0, 0, y) = \ell\lambda(x, y)
\]

as required.

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