Weyl group characters afforded by zero weight spaces

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In memory of Bert Kostant

Abstract

Let $G$ be a compact Lie group with Weyl group $W$. We give a formula for the character of $W$ on the zero weight space of any finite dimensional representation of $G$. The formula involves weighted partition functions, generalizing Kostant’s partition function. On the elliptic set of $W$ the partition functions are trivial. On the elliptic regular set, the character formula is a monomial product of certain co-roots, up to a constant equal to 0 or $\pm 1$. This generalizes Kostant’s formula for the trace of a Coxeter element on a zero weight space. If the long element $w_0 = -1$, our formula gives a method for determining all representations of $G$ for which the zero weight space is irreducible.

1 Introduction and statement of results

Let $G$ be a compact Lie group with maximal torus $T \subset G$. One of the oldest problems in representation theory is to decompose the representation of $G$ on the Hilbert space $L^2(G/T)$ of functions on $G/T$ which are square-integrable with respect to a $G$-invariant measure. From the Peter-Weyl theorem it follows that an irreducible representation $V$ of $G$ appears in $L^2(G/T)$ with multiplicity equal to the dimension of the space $V^T$ of $T$-invariant vectors in $V$. Kostant showed that $\dim V^T$ can be expressed in terms of his Partition Function, which counts the number of ways a weight can be expressed as a non-negative linear combination of positive roots. Therefore the $G$-decomposition of $L^2(G/T)$ is known, in principle (see section 1.4 below).

It is also natural to ask for the $G$-decomposition of $L^2(T)$, where $T$ is the homogenous space of all maximal tori in $G$. This problem has not been solved, even in principle, except for $\text{SU}_2$ (well-known, see below) and $\text{SU}_3$ (following from known results, see section 5.1.1).

More generally, on $G/T$ there is also a right action by the Weyl group $W$ commuting with the left $G$-action, so $L^2(G/T)$ is actually a representation of $G \times W$. If $V \in \text{Irr}(G)$ and $U \in \text{Irr}(W)$ then the multiplicity of $V \boxtimes U$ in $L^2(G/T)$ equals the multiplicity of $U$ in $V^T$. For example, the multiplicity of $V$ in $L^2(T)$ equals the multiplicity of the trivial character of $W$ in $V^T$. 
If $G = SU_2$ then $G/T = S^2$ and the nontrivial element of $W$ acts on $S^2$ via the antipodal map. The irreducible constituents of $L^2(S^2)$ are $V_\mu \otimes \varepsilon^m$ where $V_\mu$ has odd dimension $\mu = 2m + 1$, and $\varepsilon$ is the nontrivial character of $W$. Also, $T$ is the real projective plane and $V_\mu$ appears in $L^2(T)$ with multiplicity one if $\mu \in 1 + 4\mathbb{Z}$, zero otherwise.

For larger groups, the first results on the $W$-decomposition of $V^T$ were obtained in the 1970’s by Gutkin [14] and Kostant [21]. Much work has been done since (see section 1.5), but until now the character of $V^T$, for general $V \in \text{Irr}(G)$, was known on just one non-identity conjugacy class in $W$, namely the class $\text{cox}$ consisting of the Coxeter elements. This result was also obtained by Kostant. He showed that $\text{tr}(\text{cox}, V^T) \in \{-1, 0, 1\}$ and he gave a formula for the exact value, in terms of the $W$-action on a certain finite quotient of the character lattice of $T$.

In this paper we give, for any $G$, any $V \in \text{Irr}(G)$ and any $w \in W$ a formula for the character $\text{tr}(w, V^T)$, in terms of the highest weight of $V$. For those $w$ sharing certain properties with those of $\text{cox}$, we give a direct generalization of Kostant’s formula for $\text{tr}(\text{cox}, V^T)$ in terms of the $W$-action on other finite quotients of the character lattice of $T$.

To explain our formulas, it is useful to rank the elements of $W$ according to the dimension $d(w)$ of the fixed-point set of $w$ in $T$. At one extreme, the identity element $1_W$ has $d(1_W) = \dim T$ and Kostant’s partition function gives a formula for $\text{tr}(1_W, V^T)$, in principle. At the other extreme, $d(\text{cox}) = 0$ and we have seen there is a simple formula for $\text{tr}(\text{cox}, V^T)$. Thus we expect $d(w)$ to measure the complexity of $\text{tr}(w, V^T)$ as a function of the highest weight of $V$. Indeed, our formula for $\text{tr}(w, V^T)$ interpolates between Kostant’s formulas for $\text{tr}(1_W, V^T)$ and $\text{tr}(\text{cox}, V^T)$ and involves a weighted partition function $\mathcal{P}_w$ of rank equal to $d(w)$.

1.1 Some basic notation

We assume $G$ is connected and all normal subgroups of $G$ are finite. For technical reasons we also assume $G$ is simply connected, even though any $V$ with $V^T \neq 0$ factors through the quotient of $G$ by its center. Let $\mathfrak{g}$ and $\mathfrak{t}$ be the complexified Lie algebras of $G$ and $T$ and let $R$ be the set of roots of $T$ in $\mathfrak{g}$. Choose a set $R^+$ of positive roots in $R$, and let $\rho$ be half the sum of the roots in $R^+$. Let $P$ be the additive group indexing the characters of $T$; we write $e_\lambda : T \to S^1$ for the character indexed by $\lambda \in P$. Let $P_{++}$ be the set of dominant regular characters of $T$ with respect to $R^+$. For $\mu \in P_{++}$, let $V_\mu$ be the irreducible representation of $G$ with highest weight $\mu - \rho$. (This is consistent with the $SU_2$ example above, and greatly simplifies our formulas.) The sign character of $W$ is denoted by $\varepsilon$.

1.2 Elliptic traces on zero-weight spaces

When $d(w) = 0$ we say $w$ is elliptic. The character of $V^T$ on the elliptic set in $W$ has intrinsic meaning: It determines the virtual character of $V^T$ modulo linear combinations of induced representations from proper parabolic subgroups of $W$ [29]. Coxeter elements are elliptic; in $SU_n$ there are no others. For all $G \neq SU_n$ there are non-Coxeter elliptic elements. For example $W(E_8)$ has
30 elliptic conjugacy classes.

Assume \( w \in W \) is elliptic. We regard \( w \) as a coset of \( T \) in the normalizer \( N_G(T) \). Because \( w \) is elliptic, the elements of \( w \) are contained in a single conjugacy class \( w^G \) in \( G \) and we have

\[
\text{tr}(w, V^T_\mu) = \text{tr}(t, V^T_\mu),
\]

for any element \( t \in T \cap w^G \). Therefore \( \text{tr}(w, V^T_\mu) \) can be computed from the Weyl Character Formula, after cancelling poles arising from the set \( R^+_t = \{ \alpha \in R^+: e_\alpha(t) = 1 \} \). We obtain ([28] and section 2.1 below)

\[
\text{tr}(w, V^T_\mu) = \frac{1}{\Delta(t)} \sum_{v \in W^t} \varepsilon(v)e_{v\mu}(t)H_t(v\mu),
\]

where

\[
\Delta = e_\rho \prod_{\alpha \in R^+ \setminus R^+_t} (1 - e_{-\alpha}(t)), \quad W^t = \{ v \in W : v^{-1}R^+_t \subset R^+ \}, \quad H_t(v\mu) = \prod_{\alpha \in R^+_t} \frac{\langle v\mu, \check{\alpha} \rangle}{\langle \rho_t, \check{\alpha} \rangle}
\]

and \( \rho_t \) is the half-sum of the roots in \( R^+_t \). The monomials \( H_t \) are \( W \)-harmonic polynomials on \( t^* \). Their geometric meaning is discussed at the end of section 2.1.

Let \( \hat{G} \) be a compact Lie group dual to \( G \). Then \( P \) may be regarded as the lattice of one-parameter subgroups of a maximal torus \( \hat{T} \) of \( \hat{G} \). Let \( m \) be the order of \( w \) and let \( \hat{\mu} = \mu(e^{2\pi i/m}) \in \hat{T} \). Let \( C_G(t) \) and \( C_{\hat{G}}(\hat{\mu}) \) be the corresponding centralizers. From (2), one gets the following vanishing result.

**Theorem 1.1** If \( \dim C_G(t) < \dim C_{\hat{G}}(\hat{\mu}) \) then \( \text{tr}(w, V^T_\mu) = 0 \).

For example, if \( \mu \in mQ \), where \( m \) is the order of \( w \), then \( \hat{\mu} = 1 \), so Thm. 1.1 implies that \( \text{tr}(w, V^T_\mu) = 0 \). Hence if \( \mu \in nQ \), where \( n \) is the least common multiple of the orders of the elliptic elements in \( W \) then Thm. 1.1 implies that the character of \( V^T_\mu \) is a linear combination of induced characters from proper parabolic subgroups of \( W \).

Though useful, formula (2) does not tell the whole story. For example if \( w = \text{cox} \) it expresses \( \text{tr}(\text{cox}, V^T_\mu) \) as a sum of \( |W| \) terms, but we know from Kostant that cancellations put the actual trace in \( \{-1, 0, 1\} \). This is because Coxeter elements have an additional property shared by some but not all elliptic elements.

We say that \( w \in W \) is regular if the subgroup of \( W \) generated by \( w \) acts freely on \( R \) (cf. [35]). For elliptic regular elements we generalize Kostant’s formula as follows.

**Theorem 1.2** Assume \( w \) is elliptic and regular. Then \( \text{tr}(w, V^T_\mu) = 0 \) unless there exists \( v \in W \) such that \( v\mu \in \rho + mQ \), in which case

\[
\text{tr}(w, V^T_\mu) = \varepsilon(v) \prod_{\langle \mu, \check{\alpha} \rangle} \frac{\langle \mu, \check{\alpha} \rangle}{\langle \rho, \check{\alpha} \rangle},
\]

where the product is over the positive coroots \( \check{\alpha} \) of \( G \) for which \( \langle \mu, \check{\alpha} \rangle \in m\mathbb{Z} \).
Theorem 1.2 shows that the harmonic polynomial in $(2)$ is actually a harmonic monomial when viewed from the dual group $\hat{G}$. The key to Thm. 1.2 is that the assumed inequality in Thm. 1.1 holds automatically when (and only when) $w$ is both elliptic and regular. (See [33] and section 3.1 below.)

The dual group $\hat{G}$ was first used by D. Prasad in [26] to give a new interpretation of Kostant’s result for $\text{tr}(\text{cox}, V_\mu^T)$. In this case $m = h$ is the Coxeter number, $\hat{R}_\mu^+ = \emptyset$ and Thm. 1.2 becomes Kostant’s formula for $\text{tr}(\text{cox}, V_\mu^T)$.

At the opposite extreme, if the long element $w_0 = -1$ in $W$ then $w_0$ is an elliptic involution. Applying Thm. 1.2 to $w_0$ gives the following qualitative result (see section ??).

**Corollary 1.3** Assume $-1 \in W$ but $G$ is not $\text{SU}_2$. Then there are only finitely many $V \in \text{Irr}(G)$ for which $V^T$ is an irreducible representation of $W$.

For $\text{SU}_n$, $n \geq 2$, the symmetric powers $\text{Sym}^{kn}(\mathbb{C}^n)^T$, for $k = 0, 1, 2, \ldots$, afford the trivial and sign characters alternately. On the other hand, for $\text{SU}_4$ the two-dimensional representation of $W$ is all of $V_\mu^T$ for just one $\mu$ (see section 5.2). Cor. 1.3 can be sharpened to classify irreducible zero weight spaces of other groups. See section ?? which includes some history of this problem.

### 1.3 A general character formula for zero weight spaces

Now take any element $w \in W$. Let $S = (T_w)^\circ$ be the identity component of $T_w$ and let $L = C_G(S)$ be the centralizer of the torus $S$. There is an element $t \in T$ which is $L$-conjugate to an element of the coset $w$. We fix such a $t$ and consider the coset $tS \subset T$.

The positive roots are partitioned as $R^+ = R^+_t \sqcup R^+_1 \sqcup R^+_2$, where

\[
R^+_t = \{ \alpha \in R^+ : e_\alpha \equiv 1 \text{ on } tS \}, \\
R^+_1 = \{ \alpha \in R^+ : e_\alpha \equiv e_\alpha(t) \neq 1 \text{ on } tS \}, \\
R^+_2 = \{ \alpha \in R^+ : e_\alpha \text{ is nonconstant on } tS \}.
\]

As in the elliptic case, we set

\[
\rho_t = \frac{1}{2} \sum_{\alpha \in R^+_t} \alpha, \quad \Delta = e_\rho \prod_{\alpha \in R^+_1} (1 - e_{-\alpha}).
\]

\[
H_t(\lambda) = \prod_{\alpha \in R^+_t} \frac{\langle \lambda, \alpha \rangle}{\langle \rho_t, \alpha \rangle}, \quad W^{tS} = \{ v \in W : v^{-1}R^+_t \subset R^+ \}.
\]

The set $R^+_2 = \{ \beta_1, \ldots, \beta_r \}$ determines a weighted partition function $P_w$ as follows. Let $Y$ be the character lattice of $S$. For each $i = 1, \ldots, r$, the restriction of $e_{\beta_i}$ to $S$ is a non-trivial character $e_{\nu_i}$ for some $\nu_i \in Y$. Let $z_i = e_{-\beta_i}(t)$. Now for $\nu \in Y$, let

\[
P_w(\nu) = \sum z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}.
\]
where the sum runs over all $r$-tuples $(n_1, \ldots, n_r)$ of nonnegative integers such that $\sum_{i=1}^{r} n_i \nu_i = \nu$.

**Theorem 1.4** With notation as above, we have

$$\text{tr}(w, V^T_\mu) = \frac{1}{\Delta(t)} \sum_{v \in W tS} \varepsilon(v) e_{w \mu}(t) P_w(v \mu - \rho) H_{tS}(v \mu). \quad (3)$$

This is proved similarly to (2), but now we take the constant term along $S$ of the restriction of the character of $V_\mu$ to the coset $tS$. See section 4.

If $w = 1_W$, formula (3) becomes Kostant’s formula for $\dim V^T_\mu$. Some examples of intermediate partition functions $P_w$ are found in section ???.

### 1.4 Explicit results

For non-elliptic $w$, the formula (3) is difficult to compute explicitly as a function of $\mu$. The difficulty, measured by $d(w) = \dim S$, is in the weighted partition function $P_w$. The most difficult case is $\dim V^T_\mu$, which is known explicitly only for small groups. For general $G$, it was shown in [22] that the function $\mu \mapsto \dim V^T_\mu$ is a piecewise polynomial function, but there are no explicit formulas for these polynomials. For $w \neq 1$, the function $\mu \mapsto \text{tr}(w, V^T_\mu)$ also appears to be a piecewise polynomial function.

In section ??? we give explicit formulas for $\text{tr}(w, V^T_\mu)$ for all $w$ in the groups of rank two and also SU$_4$; these are the groups for which explicit polynomial formulas for $\dim(V^T_\mu)$ are known (to me).

For Spin$_8$ and $F_4$ we carry out the idea of 1.3 above find all $\mu$ for which $V^T_\mu$ is irreducible. These turn out to be just the known examples, coming from small representations [30].

For $E_7$ and $E_8$ the same idea, with more computation, should also find all irreducible $V^T_\mu$, but we do not address this here.

For $E_6$, $-1 \notin W$. Instead we explicitly compute $\text{tr}(w, V^T_\mu)$ for the elliptic triality $w$, using Thm. 1.2. When $\text{tr}(w, V^T_\mu) \neq 0$ the method of Cor. 1.3 applies. We find that the only possibilities for an irreducible representation of $W$ to be a zero weight space are the five known ones (if $\text{tr}(w, V^T_\mu) \neq 0$) and two possible additions (if $\text{tr}(w, V^T_\mu) = 0$). See section 6.3.

### 1.5 Earlier work

Zero-weight spaces have been much-studied in the past half-century. A recent survey of the problem is given in [15].

For classical groups, [1], [2], [3], [12], [14] use Schur-Weyl duality express the decomposition of $V^T_\mu$ in terms of induced representations of symmetric groups. For a fixed classical group, say $G =$
SU$_3$, this method involves multiplicities in induced representations of arbitrarily large symmetric groups.

For $G = G_2$, [23] gives a computer algorithm, but not a formula, for explicitly computing the character of $V^T_{\mu}$ for any given $\mu$.

For “small” $V_{\mu}$, the decomposition of $V^T_{\mu}$ is given explicitly in [14] (for SL$_n$), [30] (for types $D_n$ and $E_n$) and [31] (for types $B_n, C_n, G_2, F_4$).

### 1.6 Organization of the paper

Section 2 is purely about the Weyl character formula on torsion elements of $G$. These results are applied to elliptic traces in section 3. In section 4.1 we return to the Weyl character formula, now to study its values on cosets of subtori in $T$. These results are applied to the traces of general elements $w \in W$ in section 4.4. In section 5 we give explicit formulas for the character of $V^T_{\mu}$, for small groups. The final section 6 concerns irreducible zero weight spaces.

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2 The Weyl character formula and torsion elements

This section is purely about the Weyl Character formula. Notation is that of section 1.2.

2.1 The Weyl character formula and harmonic polynomials

It is known, from [28] or as a special case of Thm. 4.5 below, that for \( \mu \in P_{++} \) we have

\[
\text{tr}(t, V_\mu) = \frac{1}{\Delta(t)} \sum_{v \in W^t} \varepsilon(v)e_{v\mu}(t)H_t(v\mu).
\]  (4)
The function $\mu \mapsto H_t(\mu)$ and its $W$-translates $H_t^v(\mu) := H_t(v \mu)$ belong to the vector space $\mathcal{H}$ of $W$-harmonic polynomials on $t^*$. In [24] Macdonald showed that $\{ H_t^v : v \in W^t \}$ spans an irreducible representation of $W$, now called the truncated induction of the sign character of $W_t$ to $W$. Here $W_t$ is the Weyl group of the centralizer $C_G(t)$.

By a theorem of Borel (see [27, section 5] for a simple proof), the graded vector space $\mathcal{H}$ is canonically isomorphic to the homology of the flag manifold $B = G/T$. The translates $H_t^v$, for $v \in W^t$, correspond to the fundamental classes of the connected components of the fixed-point submanifold $B_t \subset B$. In this interpretation, equation (4), combined with the Borel-Weil theorem, is a special case of the Atiyah-Segal fixed-point theorem [4, Theorem (3.3)]. However the proof of (4) given in [28] or Thm. 4.5 below, is elementary; it boils down to the Weyl dimension formula for $C_G(t)$.

### 2.2 Traces of torsion elements in $G$

In this section we study the values of (4) on torsion elements of $G$.

**Proposition 2.1** Let $m$ be a positive integer. Suppose $t \in T$ has $\text{Ad}(t)$ of order $m$, and let $y$ be a coset of $mQ$ in $P$.

(a) If $|R_t^+| < |\tilde{R}_y^+|$, then $\text{tr}(t, V_{\mu}) = 0$ for all $\mu \in y \cap P_{++}$.

(b) If $|R_t^+| = |\tilde{R}_y^+|$ then there is a constant $C_{t,y} \in \mathbb{C}$ such that

$$\text{tr}(t, V_{\mu}) = C_{t,y} \prod_{\tilde{\alpha} \in R_t^+} \langle \mu, \tilde{\alpha} \rangle,$$  

for all $\mu \in y \cap P_{++}$

**Proof:** For any $v \in W$, the function $\mu \mapsto e_v(\mu)(t)$ is constant on each coset of $mQ$ in $P$. Hence, for each coset $y \in P/mQ$, we can define a function $\tau_y : W \rightarrow \mathbb{C}$ by $\tau_y(v) = e_v(t)$ for any $\mu \in y$. We set

$$K_y = \frac{1}{|W_t|} \sum_{v \in W} \varepsilon(v) \tau_y(v) H_t^v,$$

where $H_t^v(\mu) = H_t(v \mu)$. Thus $K_y$ is a harmonic polynomial on $t^*$, of degree $|R_t^+|$ and depending only on the coset $y \in P/mQ$. From equation (4) we have

$$\text{tr}(t, V_{\mu}) = \frac{1}{\Delta(t)} \cdot K_y(\mu), \quad \text{for all } \mu \in y. \quad (5)$$

One checks that

$$\tau_y(vu) = \tau_y(v) \quad \text{for all } u \in W_y. \quad (6)$$

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Equation (6) implies that under the action of $W$ on $H$ we have

$$K^u_y = \varepsilon(u)K_y, \quad \text{for all} \quad u \in W_y.$$  

It follows that $K_y$ is divisible by the polynomial

$$\prod_{\hat{\alpha} \in \hat{R}_y^+} \hat{\alpha}.$$  

(7)

Hence if $|R_t^+| < |\hat{R}_y^+|$ we have $K_y = 0$ and if $|R_t^+| = |\hat{R}_y^+|$ then $K_y$ is a constant multiple of the polynomial (7). This proves the Proposition.

2.3 Principal weights

We continue with $t \in T$ having $\text{Ad}(t)$ of order $m$. Now we consider certain families of weights.

An weight $\mu \in P$ is $m$-principal if the $W$-orbit of $\mu$ meets $\rho + mQ$. If $y \in P/mQ$ is the coset of an $m$-principal weight we can compute the constant $C_{t,y}$ appearing in Proposition 2.1, as follows.

Lemma 2.2 Assume $y = v\rho + mQ$ for some $v \in W$. Then we have

$$K_y(v\rho) = \varepsilon(v)\Delta(t).$$

Proof: One checks that for any $v \in W$ we have

$$K_{v\rho + mQ}(v\rho) = \varepsilon(v)K_{\rho + mQ}(\rho),$$

so we may assume $v = 1$.

From Weyl’s identity (cf. (21) below) applied to $W_t$ it follows that

$$\Delta(t) = \sum_{v \in W^t} \varepsilon(v)e_v(t)H(v\rho) = K_{\rho + mQ}(\rho),$$

as desired. □

Set

$$\hat{R}_m^+ = \{ \hat{\alpha} \in \hat{R}_t^+ : \langle \rho, \hat{\alpha} \rangle = m \}$$

Proposition 2.3 Let $t \in T$ have $\text{Ad}(t)$ of order $m$. Assume that

(a) $|R_t^+| = |\hat{R}_m^+|$, and

(b) $\mu \in P_{++}$ is $m$-principal.
Then
\[
\text{tr}(t, V_\mu) = \varepsilon(v) \prod_{\alpha \in \hat{R}_m^+} \frac{\langle v\mu, \alpha \rangle}{\langle \rho, \alpha \rangle},
\]
where \( v \) is any element of \( W \) for which \( v\mu \in \rho + mQ \).

**Proof:** From equation (5) we have
\[
\text{tr}(t, V_\mu) = \frac{K_y(\mu)}{\Delta(t)}.
\]
For each \( \alpha \in \hat{R}_y^+ \) let \( \epsilon_\alpha = \pm 1 \) be such that \( \epsilon_\alpha \cdot v\alpha \in \hat{R}_y^+ \). Sending \( \alpha \mapsto \epsilon_\alpha \cdot v\alpha \) is a bijection \( \hat{R}_y^+ \rightarrow \hat{R}_m^+ \), so we have \( |R_y^+| = |\hat{R}_y^+| \). Applying Lemma 2.2 and equation (7), we get
\[
\varepsilon(v)\Delta(t) = K_y(v\rho) = C_{t,y} \prod_{\alpha \in \hat{R}_y^+} \langle v^{-1}\rho, \alpha \rangle = C_{t,y} \prod_{\alpha \in \hat{R}_y^+} \langle \rho, v\alpha \rangle,
\]
so
\[
C_{t,y} = \frac{\varepsilon(v)\Delta(t)}{\prod_{\alpha \in \hat{R}_y} \langle \rho, v\alpha \rangle}.
\]
It follows that
\[
\text{tr}(t, V_\mu) = \varepsilon(v) \prod_{\alpha \in \hat{R}_y^+} \frac{\langle \mu, \alpha \rangle}{\langle \rho, v\alpha \rangle} = \varepsilon(v) \prod_{\alpha \in \hat{R}_y^+} \frac{\langle v\mu, \epsilon_\alpha \cdot v\alpha \rangle}{\langle \rho, \epsilon_\alpha \cdot v\alpha \rangle} = \varepsilon(v) \prod_{\alpha \in \hat{R}_m^+} \frac{\langle v\mu, \alpha \rangle}{\langle \rho, \alpha \rangle}.
\]

**Remark.** From Lemma 2.5 below, the stabilizer of \( \rho + mQ \) in \( P/mQ \) is the reflection subgroup \( W_m = \langle r_{\alpha} : \alpha \in \hat{R}_m^+ \rangle \). Since the polynomial
\[
\prod_{\alpha \in \hat{R}_m^+} \alpha
\]
transforms under the sign character of \( W_m \), it follows that the right side of (8) is independent of the choice of \( v \), as it must be.

The set
\[
T_m := \{ t \in T : \text{Ad}(t) \text{ has order } m \text{ and } |R_t| = |\hat{R}_m| \}
\]
may consist of several \( W \)-orbits. These must be separated by characters of \( G \). However, Prop. 2.3 implies that if \( \mu \in P_{++} \) is \( m \)-principal then the character of \( V_\mu \) is constant on \( T_m \).

Certain elements of \( T_m \) are obtained as follows. Let \( 2\hat{\rho} \) be the sum of the coroots \( \hat{\alpha} \in \hat{R}^+ \). If \( \eta \in S^1 \) has order \( 2m \), then the element \( t = 2\hat{\rho}(\eta) \) belongs to \( T_m \). For such elements, Prop. 2.3 gives the following character value.
Corollary 2.4 Let $m > 1$ be an integer. Assume that $\mu + mQ$ is $m$-principal and that $t = 2\rho(\eta)$, where $\eta \in S^1$ has order $2m$. Then we have

$$\text{tr}(t, V_\mu) = \varepsilon(v) \prod_{\tilde{\alpha} \in R^+_m} \frac{\langle v\mu, \tilde{\alpha} \rangle}{\langle \rho, \tilde{\alpha} \rangle},$$

where $v\mu \in \rho + mQ$.

2.4 Dual groups and torsion elements

It will be helpful to study the $W$-action on $P/mQ$ in terms of the dual group of $G$.

The simply connected group $G$ has root datum $$(\mathcal{D}(G) = (P, R, \check{Q}, \check{R}).$$

Let $\hat{G}$ be a simple compact Lie group with dual root datum $$\mathcal{D}(\hat{G}) = (\check{Q}, \check{R}, P, R).$$

We fix a maximal torus $\hat{T}$ in $\hat{G}$ and identify $P$ with the lattice of one-parameter subgroups of $\hat{T}$.

Let $\pi : \hat{G}_{sc} \to \hat{G}$ be the simply connected covering map. The group $\hat{G}_{sc}$ has root datum $$\mathcal{D}(\hat{G}_{sc}) = (\check{P}, \check{R}, Q, R),$$

where $\check{P}$ is the lattice of one-parameter subgroups in a maximal torus of the adjoint group of $G$.

The kernel of $\pi$ is the center $\hat{Z}_{sc}$ of $\hat{G}_{sc}$ and we have a canonical isomorphism $\hat{Z}_{sc} \simeq P/Q$. The preimage torus $\hat{T}_{sc} = \pi^{-1}(\hat{T})$ may be identified with $Q \otimes S^1$ and the exact sequence

$$1 \to \hat{Z}_{sc} \to \hat{G}_{sc} \to \hat{G} \to 1.$$ (9)

restricts to a $W$-equivariant exact sequence on maximal tori:

$$1 \to \hat{Z}_{sc} \to \hat{T}_{sc} \to \hat{T} \to 1.$$ (10)

Now let $\zeta \in S^1$ have finite order $m$. Evaluation at $\zeta$ gives an isomorphism from $P/mP$ to the torsion subgroup $\hat{T}[m] := \{\tau \in \hat{T} : \tau^m = 1\}$. The preimage $\hat{T}_{sc}[m] := \pi^{-1}(\hat{T}[m])$ fits into the exact sequence

$$1 \to \hat{Z}_{sc} \to \pi^{-1}(\hat{T}[m]) \to \hat{T}[m] \to 1.$$ (11)

This sequence is $W$-equivariantly isomorphic to the canonical exact sequence

$$0 \to P/Q \to P/mQ \to P/mP \to 0.$$ (12)

For $\mu \in P$, let $W_\mu = W_y$ be the stabilizer of the coset $y = \mu + mQ$ under the $W$-action on $P/mQ$, and let $\check{R}^+_\mu = \{\tilde{\alpha} \in \check{R}^+ : \langle \mu, \tilde{\alpha} \rangle \in m\mathbb{Z}\}$. This set of roots depends only on the coset $y = \mu + mQ$.

We sometimes write $$\check{R}^+_y = \check{R}^+_\mu.$$

Let $\zeta \in \mathbb{C}^\times$ have order $m$ as above. Then $\check{R}^+_\mu$ is a set of positive roots for the connected centralizer $C_G^\circ(\mu(\zeta))$ and $W_\mu$ is the Weyl group of $C_G^\circ(\mu(\zeta))$. 

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Lemma 2.5 \textit{The group $W_{\mu}$ is generated by $\{r_{\alpha} : \check{\alpha} \in \check{R}_{\mu}^+\}$.}

\textbf{Proof:} This follows from the isomorphism $P/mQ \cong \pi^{-1} (\widehat{T}[m])$ and the fact that $\widehat{G}_{sc}$ is simply connected. \hfill \blacksquare

3 \hspace{10pt} \textbf{Elliptic traces on zero weight spaces}

We apply the results of the previous section to compute $\text{tr}(w, V_{\mu}^T)$ when $w \in W$ is elliptic. Ellipticity of $w$ is equivalent to any of the following: $T_w$ is finite; $w$ fixes no nonzero element of $P$; all elements of the coset $w \subset N$ are $T$-conjugate; $w$ itself is contained in a single $G$-conjugacy class. The unique $G$-conjugacy class containing $w$ is denoted by $w^G$.

One consequence of ellipticity is the following.

\textbf{Lemma 3.1} Assume $w \in W$ is elliptic. Let $t \in T \cap w^G$ and let $z$ belong to the center $Z$ of $G$. Then there exists $v \in W$ such that $tv = tz$.

\textbf{Proof:} Since $t \in w^G$, there exist $g \in G$ and $n \in w$ such that $t = gng^{-1}$. Since $z \in Z$ we also have $zt = gzn(g^{-1})$. But $zn \in w$ and $w$ is elliptic so there exists $s \in T$ such that $zn = sns^{-1}$. Hence $zt = gsn^{-1}g^{-1} = hth^{-1}$, where $h = gs$, so $zt$ and $t$ are $G$-conjugate elements of $T$. It follows that $zt$ and $t$ are $W$-conjugate, as claimed. \hfill \blacksquare

Another consequence is that, since an elliptic element $w \in W$ fixes no nonzero weight in $P$, we have $\text{tr}(w, V_{\mu}^T) = \text{tr}(w^G, V_{\mu})$.

From equation (4) it follows that

$$\text{tr}(w, V_{\mu}^T) = \frac{1}{\Delta(t)} \sum_{v \in W^t} \varepsilon(v)e_{v\mu}(t)H(v\mu). \quad (13)$$

for any $t \in T \cap w^G$.

By [35, Theorem 8.5], the character values $\text{tr}(w, V_{\mu}^T)$ are rational for all dominant regular weights $\mu$. From Proposition 2.1 we obtain

\textbf{Proposition 3.2} Assume $w \in W$ is elliptic. Let $t \in T \cap w^G$ have $\text{Ad}(t)$ of order $m$ and let $y \in P/mQ.$

(a) If $|R_t| < |\check{R}_y|$ then $\text{tr}(w, V_{\mu}^T) = 0$ for all $\mu \in y \cap P_{++}$.

(b) If $|R_t| = |\check{R}_y|$ then there is a constant $C_{t,y} \in \mathbb{Q}$ such that for all $\mu \in y \cap P_{++}$ we have

$$\text{tr}(w, V_{\mu}^T) = C_{t,y} \cdot \prod_{\check{\alpha} \in \check{R}_y} \langle \mu, \check{\alpha} \rangle.$$
3.1 Elliptic regular traces

We now assume \( w \in W \) is elliptic and regular, and will prove Thm. 1.2 of the introduction.

Let \( \widehat{G} \) be the dual group of \( G \), as in section 2.4. We need the following consequence of [33]: Let \( \hat{g} \in \widehat{G} \) have order \( m \). Then

\[
\dim C_\widehat{G}(\hat{g}) \geq \frac{|R|}{m},
\]

with equality if and only if \( \hat{g} \) is \( \widehat{G} \)-conjugate to \( \rho(e^{2\pi i/m}) \).

**Proposition 3.3** Assume \( w \in W \) is elliptic and regular of order \( m \) and let \( t \in T \cap w^G \). Then \( \text{Ad}(t) \) has order \( m \) and for all \( y \in P/mQ \) we have \( |R_t^+| \leq |\hat{R}_y^+| \), with equality if and only if \( y \) is \( m \)-principal.

**Proof:** It suffices to prove the result for one \( t \in T \cap w^G \). Let \( t = 2\hat{\rho}(\eta) \), where \( \eta \in \mathbb{C}^\times \) has order 2m. Then \( \text{Ad}(t) \) has order \( m \) and from [34, Prop. 8] we have \( t \in w^G \).

Since \( \text{Ad}(t) \) has order \( m \) it follows that \( \text{Ad}(g) \) has order \( m \) for every \( g \in w^G \). Let \( n \in w \). Since \( w \) is elliptic regular and \( \text{Ad}(n) \) has order \( m \), the group \( \langle n \rangle \) freely permutes the root spaces of \( T \) in \( g \) and have no nonzero invariants in \( t \). Therefore we have

\[
\dim C_G(t) = \dim C_G(n) = \frac{|R|}{m}.
\]

Let \( \zeta \in S^1 \) have order \( m \). Given any \( \nu \in P \), the element \( \nu(\zeta) \in \widehat{T}[m] \) depends only on the coset \( y = \nu + mP \in P/mP \). As in (12), we have a \( W \)-equivariant isomorphism

\[
P/mP \to \widehat{T}[m], \quad y \mapsto \hat{y} = \nu(\zeta), \quad \text{for any } \nu \in y.
\]

Now take \( y \in P/mQ \) and let \( k \) be the order of \( y + mP \) in the group \( P/mP \). Clearly \( k \) divides \( m \). Applying (14) gives the first inequality in the following.

\[
\dim C_\widehat{G}(\hat{y}) \geq \frac{|R|}{k} \geq \frac{|R|}{m} = \frac{|\hat{R}|}{m} = \dim C_G(t).
\]

Since \( G \) and \( \widehat{G} \) have the same rank, we obtain the inequality \( |R_t| \leq |\hat{R}_y| \).

If \( y = v\rho + mQ \) for some \( v \in W \), then applying [34, Prop. 8] to \( \widehat{G} \) shows that \( \hat{y} \in w^G \). Since \( w \) is elliptic regular of order \( m \), we have \( \dim C_\widehat{G}(\hat{y}) = |\hat{R}_y|/m \). Now (15) implies that \( |R_t| = |\hat{R}_y| \).

Conversely, if \( |R_t| = |\hat{R}_y| \) then both inequalities in (15) become equalities. Thus, \( k = m \) and \( \dim C_\widehat{G}(\hat{y}) = |\hat{R}_y|/m \). The condition for equality in (14) implies that \( \hat{y} \in w^G \). It follows that \( \hat{y} \) is \( W \)-conjugate to \( \rho(\zeta) \), which means the image \( y' \) of \( y \) in \( P/mP \) is \( W \)-conjugate to \( \rho + mP \). We may therefore assume \( y' = \rho + mP \) and it remains to prove that \( y \) is \( W \)-conjugate to \( \rho + mQ \).
In the exact sequence (11)

$1 \rightarrow \hat{Z}_{sc} \rightarrow \pi^{-1}(\hat{T}[m]) \xrightarrow{\pi} \hat{T}[m] \rightarrow 1,$

the stabilizer $W_{\hat{y}} = \{v \in W : \hat{y}^v = \hat{y}\}$ acts on the fiber $\pi^{-1}(\hat{y})$. Since $w$ is elliptic, this action of $W_{\hat{y}}$ on $\pi^{-1}(\hat{y})$ is transitive, by Lemma 3.1 applied to the group $\hat{G}_{sc}$. Passing to the exact sequence (12)

$0 \rightarrow P/Q \xrightarrow{m} P/mQ \rightarrow P/mP \rightarrow 0,$

this means that $W_{\hat{y}}$ acts transitively on the fiber in $P/mQ$ above $\rho + mP$. Since $\rho$ and $\rho + mQ$ belong to this fiber, the proof is complete. ■

Combining Cor. 2.4, Prop. 3.2 (a) and Prop. 3.3 we have proved Theorem 1.2 in the introduction, restated here.

**Theorem 3.4** Let $w \in W$ be elliptic and regular of order $m$ and let $y \in P/mQ$.

1. If $y$ is not in the $W$-orbit of $\rho + mQ$ then for all $\mu \in y \cap P_{++}$ we have

$$\text{tr}(w, V^T \mu) = 0.$$  

2. If $vy = \rho + mQ$ for some $v \in W$ then for all $\mu \in y \cap P_{++}$ we have

$$\text{tr}(w, V^T \mu) = \varepsilon(v) \prod_{\alpha \in \hat{R}_m^+} \langle v \mu, \alpha \rangle \langle \rho, \alpha \rangle.$$  

(17)

**Remark.** We have observed that the polynomial $\prod_{\alpha \in \hat{R}_m^+} \varepsilon(\alpha)$ transforms by the sign character under the reflection subgroup $W_m = \langle r_\alpha : \alpha \in \hat{R}_m^+ \rangle$, so that the product

$$\varepsilon(v) \prod_{\alpha \in \hat{R}_m^+} \langle v \mu, \alpha \rangle$$

is independent of the choice of $v \in W$ for which $v \mu \in \rho + mQ$. In fact there is a unique such $v$ for which $v^{-1} \hat{R}_m^+ \subset \hat{R}^+$. With this choice, (17) becomes

$$\text{tr}(w, V^T \mu) = \varepsilon(v) \prod_{\alpha \in \hat{R}_m^+} \langle \mu, \alpha \rangle \langle \rho, \alpha \rangle.$$  

(18)

where $\hat{R}_m^+ = \{\alpha \in \hat{R}^+ : \langle \mu, \alpha \rangle \in m\mathbb{Z}\}$. Version (18) has the computational advantage that each term in the product is positive.

### 4 A general character formula for zero weight spaces

The general formula for $\text{tr}(w, V^T \mu)$ interpolates between the cases where $\dim T_w = 0$ and $T_w = T$. 

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4.1 The Weyl Character Formula and subtori

We return to the Weyl Character Formula, now to analyze its restriction to cosets of subtori in $T$. At the moment there are no further specifications on these cosets.

Fix an element $t \in T$ and a subtorus $S \subset T$. We recall more notation from the introduction: The set of positive roots $R^+$ is partitioned as

$$R^+ = R_{tS}^+ \sqcup R_1 \sqcup R_2,$$

where

$$R_{tS}^+ = \{ \alpha \in R^+ : e_{\alpha} \equiv 1 \text{ on } tS \},$$
$$R_1 = \{ \alpha \in R^+ : e_{\alpha} \equiv e_{\alpha}(t) \neq 1 \text{ on } tS \},$$
$$R_2 = \{ \alpha \in R^+ : e_{\alpha} \text{ is nonconstant on } tS \}.$$

Also, we set

$$\rho_{tS} = \frac{1}{2} \sum_{\alpha \in R_{tS}^+} \alpha, \quad H_{tS}(\lambda) = \prod_{\alpha \in R_{tS}^+} \frac{\langle \lambda, \check{\alpha} \rangle}{\langle \rho_{tS}, \check{\alpha} \rangle}, \quad W^{tS} = \{ v \in W : v^{-1}R_{tS}^+ \subset R^+ \},$$

$$\Delta = e_{\rho} \prod_{\alpha \in R_1} (1 - e_{-\alpha}), \quad S_0 = \{ s \in S : e_{\alpha}(ts) \neq 1 \ \forall \alpha \in R_2 \}. $$

Let $W_{tS}$ be the subgroup of $W$ generated by $\{ r_\alpha : \alpha \in R_{tS}^+ \}$. The product mapping $W_{tS} \times W^{tS} \rightarrow W$ is a bijection.

Lemma 4.1 Let $tS$ be a coset of the subtorus $S \subset T$. For all $\mu \in P_{++}$ and $s \in S_0$ we have

$$\text{tr}(ts, V_\mu) = \sum_{v \in W_{tS}} \epsilon(v) \cdot H_{tS}(v\mu) \cdot e_{v\mu-\rho}(ts) \prod_{\alpha \in R_1} (1 - e_{-\alpha}(t)) \prod_{\beta \in R_2} (1 - e_{-\beta}(ts)).$$

Proof: Let $T'$ be the covering torus of $T$ with character group $\frac{1}{2}P$ (cf. [6, VI.3.3]).

We again write $e_\lambda : T' \rightarrow S^1$ for the character of $T'$ corresponding to $\lambda \in \frac{1}{2}P$.

The Weyl group $W$ acts on $T'$ via its action on $\frac{1}{2}P$, and $W$ acts on the character ring $\mathbb{C}[T']$ as $(w \cdot f)(t) = f(t^w)$. Let $A$ and $A_{tS}$ be the operators on $\mathbb{C}[T']$ given by

$$A(f) = \sum_{w \in W} \epsilon(w) w \cdot f, \quad A_{tS}(f) = \sum_{w \in W_{tS}} \epsilon(w) w \cdot f.$$

Then

$$A(f) = \sum_{v \in W_{tS}} \sum_{u \in W_{tS}} \epsilon(uv)(uv \cdot f) = \sum_{v \in W_{tS}} \epsilon(v) A_{tS}(v \cdot f).$$
In particular for $\mu \in P$ we have

$$A(e_\mu) = \sum_{v \in W \cap S} \varepsilon(v) A(tS(e_\mu)).$$  \hspace{1cm} (19)

When restricted to $T$, the character $\chi_\mu$ of $V_\mu$ may be regarded as a function on $T'$, via the covering map $T' \to T$. It is given by the traditional expression of Weyl’s Character Formula:

$$\chi_\mu = \frac{A(e_\mu)}{A(e_\rho)}.$$  \hspace{1cm} (20)

On the right side of (20) the numerator and denominator are functions on $T'$ whose zeros must be cancelled in order to evaluate $\chi_\mu$.

From Weyl’s identity (applied to both $W$ and $W \cap S$), we have

$$A(e_\rho) = D \cdot D = D \cdot A(tS(e_\rho)),$$  \hspace{1cm} (21)

where

$$D = \prod_{\alpha \in R^+_t} (e_{\alpha/2} - e_{-\alpha/2}), \quad D = \prod_{\alpha \in R^+_t \cup R^+_s} (e_{\alpha/2} - e_{-\alpha/2}), \quad e_\rho = \prod_{\alpha \in R^+_t} e_{\alpha/2}$$

are all elements of $\mathbb{C}[T']$.

Recall $t \in T$ has been fixed. Now we also fix $s \in S_0$ and let $x \in T'$ be a lift of $ts$. Also let $z \in T'$ be arbitrary. For all $u \in W \cap S$, $v \in W \cap S$ and $\mu \in \frac{1}{2}P$ we have

$$e_{uv\mu}(xz) = e_{uv}(x) \cdot e_{uv\mu}(z).$$

It follows that

$$A(tS(vu))(xz) = e_{vu}(x) \cdot A(tS(vu))(z).$$

From (21) we have

$$A(e_\rho)(xz) = D(tS)(xz) \cdot D(tS)(xz).$$

For all $\alpha \in R^+_t$ we have $e_\alpha(st) = 1$, so $e_{\alpha/2}(x) = \pm 1$. It follows that $e_\rho(z) = \pm 1$ and that

$$D(tS)(xz) = e_\rho(z) \cdot D(tS)(z) = e_\rho(z) A(tS(e_\rho))(z),$$

so

$$A(e_\rho)(xz) = e_\rho(z) \cdot D(tS)(xz) \cdot A(tS(e_\rho))(z).$$  \hspace{1cm} (22)

So far $x$ is fixed but we have made no restriction on $z$. Since the set of elements in $T'$ on which no root $= 1$ is dense in $T'$, we can choose a sequence $z_n \to 1$ in $T'$ such that $e_\alpha(xz_n) \neq 1$ for all $n$ and all $\alpha \in R$.

From (20) and (22) we have that

$$\chi_\mu(xz_n) = \frac{1}{e_\rho(z_n) \cdot D(tS)(xz_n)} \sum_{v \in W \cap S} \varepsilon(v) e_{vu}(x) \frac{A(tS(vu))(z_n)}{A(tS(e_\rho)(z_n)).}$$
Letting $n \to \infty$ we get, from the Weyl dimension formula applied to the (connected) centralizer $C_L(t)$, where $L = C_G(S)$, that

$$\chi_\mu(x) = \frac{1}{e_{\rho S}(x) \cdot D^{tS}(x)} \sum_{v \in W^{tS}} \varepsilon(v) H_{tS}(v\mu) e_{v\mu}(x).$$

Since

$$e_{\rho S} \cdot D^{tS} = e_\rho \cdot \prod_{\alpha \in R_{tS}^1 \sqcup R_{tS}^2} (1 - e_{-\alpha}),$$

it follows that

$$\chi_\mu(x) = \sum_{v \in W^{tS}} \frac{\varepsilon(v) H_{tS}(v\mu)}{\prod_{\alpha \in R_{tS}^1 \sqcup R_{tS}^2} (1 - e_{-\alpha}(x))} e_{v\mu - \rho}(x).$$

Since $\mu$ and all $v\mu - \rho$ lie in $P$ we may replace $x$ by its projection $st \in T$. Decomposing the product according to $R_{tS}^1 \sqcup R_{tS}^2$ we obtain the formula in Lemma 4.1.

\section*{4.2 Characters and constant terms}

Given $w \in W$, let $S = (T_w)^o$ be the identity component of the fixed-point group $T_w := \{ t \in T : t^w = t \}$, and let $Y$ be the character lattice of $S$.

The rank of $Y$ equals the rank of the character lattice $P_w = \{ \lambda \in P : w\lambda = \lambda \}$ of the quotient torus $T/[w,T]$, where $[T,w] = \{ t^w \cdot t^{-1} : t \in T \}$.

For $s \in S \cap [T,w]$ one checks that $s^m = 1$, where $m$ is the order of $w$. This implies that the composite mapping

$$S \hookrightarrow T \longrightarrow T/[T,w]$$

has finite kernel, hence is surjective. It follows that the restriction map $P \to Y$ is injective on $P_w$.

Recall that we regard $w$ as a subset of $N$. Fix $n \in w$. From the surjectivity of (23) it follows that every element of $w$ is $T$-conjugate to an element of $nS$. Indeed, given any $t \in T$, there exist $s \in S$ and $z \in T$ such that $s = (z^w \cdot z^{-1}) \cdot t$, so we have

$$z(nt)z^{-1} = n(z^w \cdot z^{-1})t = ns.$$ 

Let $\mathbb{C}[S]$ be the character ring of $S$. Every element $\varphi \in \mathbb{C}[S]$ uniquely expressed as a finite linear combination

$$\varphi = \sum_{\nu \in Y} \hat{\varphi}(\nu)e_\nu$$

with coefficients $\hat{\varphi}(\nu) \in \mathbb{C}$. The constant-term functional $\int_S : \mathbb{C}[S] \to \mathbb{C}$ is given by

$$\int_S \varphi(s) \, ds = \hat{\varphi}(0).$$
Lemma 4.2 Let $V$ be a finite-dimensional representation of $G$ and let $w \in W$. Then the trace of $w$ on the zero weight space $V^T$ is given by

$$tr(w, V^T) = \int_S tr(ns, V) \, ds,$$

for any choice of $n \in w$.

**Proof:** For $\lambda \in P$, let $V^\lambda = \{ v \in V : tv = e^\lambda(t)v \ \forall t \in T \}$. We have $nV^\lambda = V^{w_\lambda}$. It follows that for all $s \in S$ we have

$$tr(ns, V) = \sum_{\lambda \in P_w} tr(ns, V^\lambda) = \sum_{\lambda \in P_w} tr(n, V^\lambda) \cdot e^\lambda(s).$$

Since the restriction map $P \to Y$ is injective on $P_w$ it follows that

$$\int_S tr(ns, V) \, ds = tr(n, V^T) = tr(w, V^T).$$

Since $n \in C_G(S)$, there exists $g \in C_G(S)$ such that $n^g \in T$. We set

$$t := n^g \in T.$$

Since $g$ centralizes $S$, we have $(ns)^g = ts$ for all $s \in S$. It follows that

$$tr(ns, V) = tr(ts, V),$$

as functions on $S$. From Lemma 4.2 we have

**Proposition 4.3** Let $w \in W$ and let $S = (T_w)^\circ$ be the connected fixed-point subtorus of $w$ in $T$. Then there exists $t \in T$ which is $C_G(S)$-conjugate to an element of $w$ and for any such $t$ the trace of $w$ on the zero weight space $V^T$ is given by

$$tr(w, V^T) = \int_S tr(ts, V) \, ds.$$

### 4.3 Formal Characters

In this section we extend the functional $\int_S$ to certain rational functions on $S$, using the theory of formal characters [15, 22.5].

Let $S$ be a compact (or algebraic) torus with character lattice $Y$ and coordinate algebra $\mathbb{C}[S]$. The Fourier expansion

$$\varphi = \sum_{\nu \in Y} \hat{\varphi}(\nu)e_{\nu}, \quad \text{for } \varphi \in \mathbb{C}[S]$$
defines a $C$-algebra isomorphism (Fourier transform) $\varphi \mapsto \hat{\varphi}$ from $C[S]$ to the group algebra $C[Y]$, where $C[S]$ has the pointwise product and $C[Y]$ has the convolution product:

$$\phi \ast \psi(\nu) = \sum_{\lambda + \mu = \nu} \phi(\lambda)\psi(\mu).$$

(24)

Via this isomorphism we identify the constant term functional $\int_{S} : C[S] \to \mathbb{C}$ with the evaluation functional $\phi \mapsto \phi(0) : C[Y] \to \mathbb{C}$.

Let us be given nonzero elements $\nu_{1}, \ldots, \nu_{r} \in Y$, not necessarily distinct. Let $\Gamma \subset Y$ be the semigroup generated by $\{\nu_{1}, \ldots, \nu_{r}\}$.

We let $C((Y))$ be the set of functions $\phi : Y \to \mathbb{C}$ supported on finitely many sets of the form $\lambda - \Gamma$ for some $\lambda \in Y$. The convolution product (24) extends to $C((Y))$ and $C[Y]$ is a subalgebra of $C((Y))$. We extend the functional $\int_{S}$ to $C((Y))$ by setting

$$\int_{S} \phi = \phi(0).$$

For any complex number $z$ and $i = 1, \ldots, r$ the Fourier transform of $1 - ze^{-\nu_{i}}$ becomes invertible in $C((Y))$; its inverse is the function whose support is contained in $-\mathbb{N}\nu_{i}$ and whose value at $-n\nu_{i}$, for $n \in \mathbb{N}$, is $z^{n}$. More generally, for

$$h = \prod_{i=1}^{r} (1 - z_{i}e_{-\nu_{i}}) \in C[S]$$

(25)

the convolution-inverse of $\hat{h}$ given by the weighted partition function

$$\hat{\nu}^{-1}(\nu) = P(\nu) = \sum z_{1}^{n_{1}}z_{2}^{n_{2}} \cdots z_{r}^{n_{r}}$$

(26)

where the sum runs over all $(n_{1}, \ldots, n_{r}) \in \mathbb{N}^{r}$ such that $\sum_{i=1}^{r} n_{i}\nu_{i} = \nu$. Note that $\hat{h}^{-1}$ is supported on $-\Gamma$ and that $(\hat{e}_{\nu} \ast \hat{h}^{-1})(0) = P(\nu)$.

**Lemma 4.4** Let $h \in C[S]$ have the form (25) and suppose $f, g \in C[S]$ are such that $h \cdot f = g$. Then

$$\int_{S} f(s) \, ds = \sum_{\nu \in Y} \hat{g}(\nu)P(\nu),$$

where $P(\nu)$ is given by (26).

**Proof:** We have $\hat{g} = \hat{f} \cdot \hat{h} = \hat{f} \ast \hat{h}$ in $C[Y]$ hence also in $C((Y))$. Since $\hat{h}$ is invertible in $C((Y))$ it follows that

$$\hat{f} = \hat{g} \ast \hat{h}^{-1} = \sum_{\nu \in Y} \hat{g}(\nu)\hat{e}_{\nu} \ast \hat{h}^{-1},$$

so we have

$$\int_{S} f(s) \, ds = \hat{f}(0) = \sum_{\nu} \hat{g}(\nu)(\hat{e}_{\nu} \ast \hat{h}^{-1})(0) = \sum_{\nu} \hat{g}(\nu)P(\nu).$$

$\blacksquare$
4.4 The character formula

We apply the results (and notation) of Section 4.1 to the situation of 4.2.

Enumerate $R_2^S = \{ \beta_1, \ldots, \beta_r \}$, let $\nu_i \in Y$ be the restriction of $\beta_i$ to $S$, and let $z_i = e_{-\beta_i}(t)$. As in (26), for $v \in Y$ we set

$$\mathcal{P}_w(v) = \sum z_1^{n_1} z_2^{n_2} \cdots z_r^{n_r}$$

where the sum runs over all $(n_1, \ldots, n_r) \in \mathbb{N}^r$ such that $\sum_{i=1}^r n_i \nu_i = v$. If there are no such $r$-tuples $(n_i)$ then $\mathcal{P}_w(v) = 0$.

**Theorem 4.5** The trace of $w \in W$ on the zero weight space $V^T_\mu$ is given by

$$\text{tr}(w, V^T_\mu) = \frac{1}{\prod_{a \in R_1^T}(1 - e^{-a}(t))} \sum_{v \in W^T_S} \varepsilon(v) H_{ts}(v\mu)e_{v\mu - \rho}(t)\mathcal{P}_w(v\mu - \rho),$$

(27)

where $S = (T_w)^o$ and $t \in T$ is $C_G(S)$-conjugate to an element of $w$ (see Proposition 4.3).

**Proof:** From Proposition 4.3 we have

$$\text{tr}(w, V^T_\mu) = \int_S \text{tr}(ts, V_\mu) \, ds.$$ 

From Lemma 4.1 we have on $S_0$ the relation

$$\text{tr}(ts, V_\mu) = \sum_{v \in W^T_S} \frac{\varepsilon(v) \cdot H_{ts}(v\mu)}{\prod_{a \in R_1^T}(1 - e^{-a}(t))} \cdot \frac{e_{v\mu - \rho}(ts)}{\prod_{\beta \in R_2^T}(1 - e^{-\beta}(ts))}.$$ 

For $v \in W^T_S$ we set

$$c_v = \varepsilon(v) \cdot \frac{H_{ts}(v\mu)e_{v\mu - \rho}(t)}{\prod_{a \in R_1^T}(1 - e^{-a}(t))}.$$ 

In Prop. 4.4 we take $f, g, h \in \mathbb{C}[S]$ defined by

$$f(s) = \text{tr}(ts, V_\mu), \quad g(s) = \sum_{v \in W^T_S} c_v e_{v\mu - \rho}(s), \quad h(s) = \prod_{\beta \in R_2^T}(1 - e^{-\beta}(ts)).$$

Since $f \cdot h = g$, we have from Lemma 4.4 that

$$\text{tr}(w, V^T_\mu) = \int_S \text{tr}(ts, V_\mu) \, ds = \hat{f}(0) \quad \text{(Prop. 4.3)}$$

$$= \hat{g} \ast \hat{h}^{-1}(0) = \sum_{v \in W^T_S} c_v \hat{e}_{v\mu - \rho} \ast \hat{h}^{-1}(0) = \sum_{v \in W^T_S} c_v \hat{\mathcal{P}}_w(v\mu - \rho).$$

Theorem 1.4 is proved. ■

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4.4.1 A special case

Let \( J \) be a subset of the simple roots in \( R^+ \), let \( R^+_J \) be the positive roots spanned by \( J \) and \( W_J = \langle r_\alpha : \alpha \in R^+_J \rangle \). Suppose \( w \) is a Coxeter element in \( W_J \). Then \( S = (\cap_{\alpha \in J} \ker e_\alpha)^0 \). Let \( G_J \) be the derived group of \( C_G(S) \). We may choose \( n \in w \cap G_J \), and let \( t \) be a \( G_J \)-conjugate of \( n \). Then \( t \) is a regular element of \( G_J \), so we have

\[
R^+_tS = \emptyset, \quad R^+_1S = R^+_j, \quad R^+_2S = R^+ - R^+_j, \quad H^+_tS \equiv 1. \tag{28}
\]

In this situation (27) becomes

\[
\text{tr}(w, V^*_\mu) = \frac{1}{\prod_{\alpha \in R^+_j} (1 - e^{-\alpha}(t))} \sum_{\nu \in W} \varepsilon(\nu)e_{\nu\mu - \rho}(t)P_w(\nu\mu - \rho). \tag{29}
\]

5 Small Groups

In this section we work out the complete character formula for \( \text{tr}(w, V^*_\mu) \) in the cases where Kostant’s partition function has a closed quasi-polynomial expression (that I am aware of). This means \( G \) is one of \( SU_3, Sp_4, G_2, SU_4 \).

We use the following notation. For \( a \in \mathbb{Z} \), let \((a)_2 = 0\) if \( a \) is even, \((a)_2 = 1\) if \( a \) is odd. For \( n \in \{3, 4, 6\} \) let

\[
(a)_n = \begin{cases} 
1 & \text{if } a \equiv 1 \mod n \\
-1 & \text{if } a \equiv -1 \mod n \\
0 & \text{if } \gcd(a, n) > 1.
\end{cases}
\]

5.1 Rank-Two Groups

For the rank-two groups, let \( \alpha \) and \( \beta \) be simple roots, where \( \alpha \) is a short root for \( Sp_4 \) and is a long root for \( G_2 \). \( \alpha_0 \) is the highest root and \( \beta_0 \) is the highest short root. Let \( \omega_\alpha \) and \( \omega_\beta \) be the corresponding fundamental weights, so that \( \rho = \omega_\alpha + \omega_\beta \). For \( \mu \in P_++ \), we have

\[
\mu = a\omega_\alpha + b\omega_\beta \tag{30}
\]

where \( a, b \) are positive integers. The character of \( W \) afforded by \( V^*_\mu \) will be expressed in terms of quasi-polynomial functions of \((a, b)\).

Recall that \( \mathbb{N} = \{0, 1, 2, 3, \ldots\} \).

5.1.1 \( SU_3 \)

Here \( V^*_\mu \neq 0 \) if and only if \( a \equiv b \mod 3 \). Since \( V \) and its dual have isomorphic zero weight spaces, we may assume that \( a \leq b \). The conjugacy classes in \( W = S_3 \) are represented by \( 1 = 1_W \),
a reflection $r$, and a Coxeter element $\text{cox}$. We find

$$\dim V^T_\mu = a$$
$$\text{tr}(r, V^T_\mu) = (a)_2 \cdot (-1)^{b+1}$$
$$\text{tr}(\text{cox}, V^T_\mu) = (a)_3$$

The computations are summarized as follows. Kostant’s partition function $P$ is given by $P(m\alpha + n\beta) = 1 + \min(n, m)$, for $n, m \in \mathbb{N}$. From this one easily computes the above value for $\dim V^T_\mu$.

The value for $\text{tr}(\text{cox}, V^T_\mu)$ is obtained from Kostant’s formula. Setting $\zeta = \eta^2$, we get

$$\text{tr}(\text{cox}, V^T_\mu) = \text{tr}(\tilde{\rho}(\zeta), V_\mu) = \frac{\zeta^a - \zeta^{-a}}{\zeta - \zeta^{-1}} = (a)_3.$$

For $r = r_\alpha$, we have $S = \tilde{\lambda}(S^1)$, where $\tilde{\lambda}(z) = \text{diag}(z, z, z^{-2})$, and we may take $t = \tilde{\rho}(-1) = \text{diag}(-1, 1, -1)$. We have (cf. 28)

$$R^t_{iS} = \emptyset, \quad R^t_1 = \{\alpha\}, \quad R^t_2 = \{\beta, \rho\}.$$ 

We have $Y = \mathbb{Z}_\nu$, where $e_\nu(\tilde{\lambda}(z)) = z$, $z_1 = \beta(t) = -1$, $z_2 = \rho(t) = 1$, $\nu_1 = \nu_2 = 3\nu$ and

$$P_\nu(n\nu) = \begin{cases} 1 - (n/3)_2 & \text{if } n \in 3\mathbb{N} \\ 0 & \text{if } n \notin 3\mathbb{N} \end{cases}$$

From 29 we get

$$\text{tr}(r, V^T_\mu) = \frac{1}{2} \sum_{v \in W} \varepsilon(v) e_\nu(t) P_\nu(v\mu - \rho)$$

$$= \frac{1}{2} \left\{ (-1)^{a+b} \left[ 1 - \left( \frac{a + 2b - 3}{3} \right)_2 \right] - (-1)^b \left[ 1 - \left( \frac{a + 2b - 3}{3} \right)_2 \right] \right\}$$

$$= (-1)^{b+1} (a)_2 \left( \frac{a + 2b}{3} \right)_2 = (-1)^{b+1} (a)_2.$$ 

Using the representation theory of $SU_2$, one can also obtain $\text{tr}(r, V^T_\mu)$ by calculating multiplicities of weights $k\alpha$ for $k \in \mathbb{N}$. For $SU_3$ this is feasible because the partition function $P$ is so simple.

**Remarks:**

1) $V^T_\mu$ is a multiple of the regular representation $\text{Reg}$ of $W$ if and only if $a \in 6\mathbb{Z}$, in which case $V^T_\mu$ is $a/6$ copies of $\text{Reg}$.

2) $V^T_\mu$ is irreducible if and only if $a \in 1, 2$. If $a = 1$, then $V^T_\mu = \text{Sym}^{3d}(\mathbb{C}^3)^*$ where $b = 1 + 3d$, and we have $V^T_\mu \simeq \epsilon^d$. If $a = 2$ then $V^T_\mu \simeq \varrho$. Thus, each irrep of $W$ is of the form $V^T_\mu$ for infinitely many $\mu \in P_{++}$. 

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In this section we explicate our character formula for $G = \text{Sp}_4$. The simple roots are $\alpha, \beta$ where $\alpha$ is short. Writing $\mu = a\omega_\alpha + b\omega_\beta$, we have $V_\mu^T \neq 0$ iff $a$ is odd. For $\eta \in S^1$ of order eight, let

$$ (a)_8 = \frac{\eta^a - \eta^{-a}}{\eta - \eta^{-1}} = \begin{cases} 
1 & \text{if } a \equiv 1, 3 \mod 8 \\
-1 & \text{if } a \equiv 5, 7 \mod 8.
\end{cases} $$

The conjugacy classes in $W$ are represented by $1_W, r_\alpha, r_\beta, \text{cox}^2, \text{cox}$.

For $\mu = a\omega_\alpha + b\omega_\beta$ the values of $\text{tr}(w, V_\mu^T)$ are shown in the following table.

| $w$ | $\text{tr}(w, V_\mu^T)$ |
|-----|---------------------|
| $1_W$ | $\frac{1}{2} (ab + (b)_2)$ |
| $r_\alpha$ | $\frac{(-1)^{b+1}}{2} \cdot [b + (a)_4(b)_2]$ |
| $r_\beta$ | $\begin{cases} (a)_4 & \text{if } b \in 2 + 4\mathbb{N} \\
(b)_4 & \text{if } a + b \in 2 + 4\mathbb{N} \\
0 & \text{otherwise} \end{cases}$ |
| $\text{cox}^2$ | $\frac{(a)_4}{2} [b + a(b)_2]$ |
| $\text{cox}$ | $\begin{cases} -(a)_8(a)_4 & \text{if } b \in 2 + 4\mathbb{N} \\
(a)_8(b)_4 & \text{if } a + b \in 2 + 4\mathbb{N} \\
0 & \text{otherwise} \end{cases}$ |

The calculations are summarized as follows.

The formula for $\dim V_\mu^T$ is an exercise in [7]. The traces of $\text{cox}$ and $\text{cox}^2$ come from Theorem 3.4.

For $w = r_\alpha$ we have $S = \tilde{\beta}_0(S^1)$, where $\tilde{\beta}_0$ is the highest coroot. Then $Y = \mathbb{Z}_\nu$ with $\nu(\tilde{\beta}_0(z)) = z$, and we can take $t = \tilde{\alpha}(i)$, where $i^2 = -1$. Then

$$ R_{1S}^t = \emptyset, \quad R_{1S}^t = \{\alpha\}, \quad R_{2S}^t = \{\beta, \alpha + \beta, 2\alpha + \beta\}. $$

Each root in $R_{2S}^t$ restricts to $2\nu$ on $S$. The partition function is given by

$$ P_{r_\alpha}(n\nu) = \sum_{(n_1, n_2, n_3)} (-1)^{n_1+n_3}, $$
summed over \( \{(n_1, n_2, n_3) \in \mathbb{N}^3 : 2n_1 + 2n_2 + 2n_3 = n\} \), so

\[
\mathcal{P}_{r_\alpha}(n\nu) = \begin{cases} 
(-1)^{n/2}(1 + \lfloor n/4 \rfloor) & \text{if } n \in 2\mathbb{N} \\
0 & \text{if } n \notin 2\mathbb{N}.
\end{cases}
\]

In the sum

\[
\text{tr}(r_\alpha, V^T_\mu) = \frac{1}{\Delta(t)} \sum_{v \in W} \varepsilon(v)e_{v\mu}(t)\mathcal{P}_{r_\alpha}(v\mu - \rho),
\]

only the terms for \( v = 1, r_\alpha, r_\beta, r_\alpha r_\beta \) are nonzero and we get

\[
\text{tr}(r_\alpha, V^T_\mu) = (a)_4 \left[ \mathcal{P}_{r_\alpha}(a + 2b - 3) - (-1)^b \mathcal{P}(a - 3) \right],
\]

which simplifies to the formula above. The trace of \( r_\beta \) is obtained similarly.

**Remarks.**

1. The irreps of \( \text{Sp}_4 \) with nonzero weight spaces are those factoring through \( \text{SO}_5 \). We see that the long element \( w_0 = \text{cox}^2 \) has nonzero trace on \( V^T \) for every irrep \( V \) of \( \text{SO}_5 \). This means the virtual character \( V^T \) is never a linear combination of induced representations from proper parabolic subgroups of \( W \) (and in particular is never the regular representation).

In fact, this holds for any irrep \( V \) of \( \text{SO}_{2n+1}, n \geq 1 \). Indeed, \( P/2Q = A \sqcup (\rho + A) \), where \( A = Q/2Q \). The \( W \)-stabilizer of \( \rho \) in \( \rho + A \) is the symmetric group \( S_n \), so \( W \) is transitive on \( \rho + A \). Hence \( \text{tr}(w_0, V^T) \neq 0 \) by Thm. 3.4.

2. Back to \( \text{Sp}_4 \). Comparing \( \dim V^T_\mu \) and \( \text{tr}(w_0, V^T_\mu) \) shows that \( w_0 \) is a scalar on \( V^T_\mu \) if and only if \( a = 1 \) or \( b = 1 \). In the former case \( V^T_\mu \) is the degree \( b - 1 \) harmonic polynomials on the five-dimensional orthogonal representation and \( w_0 \) is trivial on \( V^T_\mu \). In the latter case \( V^T_\mu \) is the degree \( a - 1 \) polynomials on the four-dimensional symplectic representation and \( w_0 \) acts on \( V^T_\mu \) by the scalar \( (a)_4 \).

3. The trivial representation of \( W \) appears in \( V^T_\mu \) with multiplicity

\[
\langle \text{triv}, V^T_\mu \rangle = \begin{cases} 
\frac{b}{4} \left[ \frac{a}{4} \right] & \text{if } b \in 4\mathbb{N} \\
\frac{1}{4} \left( b \left[ \frac{a}{4} \right] + (a)_4(1 - (a)_8) \right) & \text{if } b \in 2 + 4\mathbb{N} \\
\frac{1}{4}(b + (a)_4) \left[ \frac{a}{4} \right] & \text{if } a + b \in 4\mathbb{N} \\
\frac{1}{4} \left( (b + (a)_4) \left[ \frac{a}{4} \right] + (b)_4(1 + (a)_8) \right) & \text{if } a + b \in 2 + 4\mathbb{N}
\end{cases}
\]

The reflection representation appears with multiplicity

\[
\langle \text{refl}, V^T_\mu \rangle = \frac{1}{8}[a - (a)_4][b - (b)_2(a)_4].
\]
5.1.3 \( G_2 \)

In this section we explicate our character formula for \( G \) of type \( G_2 \).

For relatively prime positive integers \( m, n, \) and \( q \in \mathbb{Q} \), let \( P_{mn}(q) \) be the number of solutions \((x, y) \in \mathbb{N} \times \mathbb{N}\) of the equation \( mx + ny = q \). For any integer \( k \), we have the constant-term formula

\[
\int_{S^1} \frac{z^k}{(z^m - z^{-m})(z^n - z^{-n})} \, dz = P_{mn} \left( \frac{k - m - n}{2} \right).
\]

Now \( P_{mn}(q) = 0 \) unless \( q \in \mathbb{N} \), in which case \( P_{mn}(q) \) is given by Popoviciu’s formula [25] (see also [5, 1.3])

\[
P_{mn}(q) = \frac{q}{mn} - \left\{ \frac{qm'}{n} \right\} - \left\{ \frac{qn'}{m} \right\} + 1,
\]

where \( m', n' \) are any integers such that \( mm' + nn' = 1 \), and \( \{ r \} = r - \lfloor r \rfloor \) denotes the fractional part of a rational number \( r \).

For \( G_2 \) we will need just two of these partition functions, \( P_{12} \) and \( P_{23} \), which have the more explicit formulas

\[
P_{12}(q) = 1 + \left\lfloor \frac{q}{2} \right\rfloor, \quad P_{23} = 1 + \left\lfloor \frac{q}{2} \right\rfloor - \left\lceil \frac{q}{3} \right\rceil.
\]

The conjugacy classes in \( W \) are represented by

\[
1_W, \ r_\alpha, \ r_\beta, \ \text{cox}^3, \ \text{cox}^2, \ \text{cox}.
\]

For \( \mu = a\omega_\alpha + b\omega_\beta \) the values of \( \text{tr}(w, V_\mu^T) \) are shown in the following table.
As far as I know, the dimension of $V^T_\mu$ for $G_2$ was first given explicitly in [2]. The formulas therein evidently take positive integer values without any sign cancellations. On the other hand, they have nine different cases, according to congruences of $(a,b) \mod (2,6)$. A more compact expression for $\dim V^T_\mu$ was obtained by Vergne, using coordinates $(m,n)$ where $\mu - \rho = m\alpha + n\beta$ (see [22]). If we change coordinates to $(a,b)$ as in (30) then Vergne’s formula for $\dim V^T_\mu$ simplifies to the one given above.

The character values on the reflections come from Theorem 4.5.

The elliptic character values are expressed as harmonic quasi-polynomials as in (2) and also as piecewise monomials determined by the action of $W$ on $P/mP$, as in Thm. 1.2.

The latter is analyzed by recalling that $W$ is the isometry group of the quadratic form $q : P \to \mathbb{Z}$
given by \( q(\mu) = 3a^2 + 3ab + b^2 \), for which \( q(\rho) = 7 \). For each \( m \in \{2, 3, 6\} \), reduction modulo \( m \) gives a quadratic form \( q_m : P/mP \rightarrow \mathbb{Z}/m\mathbb{Z} \) and a homomorphism from \( W \) to the isometry group \( O(q_m) \) of \( q_m \) on \( P/mP \). For all \( m \), we have \( \text{tr}(\text{cox}^{6/m}, V^T_\mu) \neq 0 \) only if \( q_m(\mu) = q_m(\rho) = 1 \).

The form \( q_2 \) is anisotropic; \( W \) is transitive on nonzero vectors in \( P/2P \) and \( \text{tr}(\text{cox}^3, V^T_\mu) \neq 0 \) if and only if \( q_2(\mu) = 1 \).

The form \( q_3 \) is degenerate, with radical spanned by \( \omega_\alpha + 3P \). The map \( W \rightarrow O(q_3) \) realizes \( W \) as the Borel subgroup of \( GL(P/3P) \) stabilizing the radical line. The orbit \( W \rho \) consists of the six vectors in \( P/3P \) with \( q_3(\mu) = 1 \); these are the cosets of the short roots.

For \( m = 6 \) the vectors with \( q_6 = 1 \) form two \( W \)-orbits, represented by \( \rho + 6P \) and \( \beta + 6P \). It follows that \( \text{tr}(\text{cox}, V^T_\mu) \neq 0 \) if and only if \( q_6(\mu) = 1 \) and \( \mu \) is not in the coset of a root.

**Remarks.**

1) \( V^T_\mu \) is a multiple of the regular representation of \( W \) if and only if \( (a, b) \equiv (0, 0) \mod (2, 6) \).

2) Let \( W_2 \simeq W_{A_1 \times A_1} \) be a subgroup of \( W \) generated by reflections about a pair of orthogonal roots. Then \( V^T_\mu \) is induced from a character of \( W_2 \) if and only if \( b \in 3\mathbb{Z} \).

3) \( V^T_\mu \) is irreducible for \( W \) only for the trivial, seven-dimensional and adjoint representations. Indeed, from the monomial formula for \( \text{tr}(\text{cox}^3, V^T_\mu) \) we find that the list of \( \mu \in P_{++} \) for which \( \text{tr}(w_0, V^T_\mu) \) belongs to the set \( \{ \chi(\text{cox}^3) : \chi \in \text{Irr}(W) \} \) = \{±1, ±2\}, is just

\[
(a, b) = (1, 1), \quad (1, 2), \quad (2, 1), \quad (3, 2).
\]

The first three cases are those with \( V^T_\mu \) irreducible. In the last case \( \mu = 3\omega_\alpha + 2\omega_\beta \), we have \( \text{tr}(\text{cox}^3, V^T_\mu) = -2 \) and \( \text{tr}(\text{cox}^2, V^T_\mu) = +1 \). As these are not the values of any \( \chi \in \text{Irr}(W) \), it follows that \( V^T_\mu \) cannot be irreducible for \( W \).

### 5.2 SU₄

For \( G = SU_4 \), let \( \alpha, \beta, \gamma \) be simple roots with \( \alpha \) and \( \gamma \) orthogonal to each other. Let \( \omega_\alpha, \omega_\beta, \omega_\gamma \) be the corresponding fundamental weights, so that \( \rho = \omega_\alpha + \omega_\beta + \omega_\gamma \).

For a dominant regular weight \( \mu \), the character of \( V^T_\mu \) is expressed as a function of coordinates \( (a, b, c) \) where \( a, b, c \) are positive integers such that

\[
\mu = a\omega_\alpha + b\omega_\beta + c\omega_\gamma.
\]

Replacing \( V_\mu \) by its dual does not change the \( W \)-module structure of \( V^T_\mu \), so we may assume that

\[
a \leq c.
\]

Set

\[
d := \frac{c - a}{2}, \quad f := b - d, \quad s := \frac{c + a}{2}.
\]
Then $V^T_\mu \neq 0$ if and only if $f \in 1 + 2\mathbb{Z}$.

Index the conjugacy classes in $W = S_4$ by partitions $[\lambda_1 \lambda_2 \cdots]$ of 4. For example $[1111]$ is the identity element and $[4] = \text{cox}$.

The character values are shown in the following table.

| $w$ | $\text{tr}(w, V^T_\mu)$ |
|-----|--------------------------|
| $[1111]$ | $\frac{1}{2} \begin{cases} ab(a+b) & f \leq 1 \\ a(b+c+1-d^2) & f > 1 \end{cases}$ |
| $[211]$ | $\frac{1}{2} \begin{cases} a \cdot [d]_{2N} - b \cdot [s]_{2N} & f \leq -1 \\ (1 + s(f)_4) \cdot [d]_{2N} - (1 + d(f)_4) \cdot [s]_{2N} & f > -1 \end{cases}$ |
| $[22]$ | $\frac{1}{2} \begin{cases} (-1)^{q} b \cdot [c]_{1+2N} - (-1)^{c} a \cdot [d]_{2N} & \\ (a)_3 \cdot [(2c+f)]_{3N} - [f]_{3N} \end{cases}$ |
| $[31]$ | $\frac{1}{2} \begin{cases} (-1)^{a} \delta \left(\frac{s}{2}\right) + (-1)^{b} \delta \left(\frac{b+s}{2}\right) - (-1)^{c} \delta \left(\frac{d}{2}\right) & \\ (a)_3 \cdot [(2c+f)]_{3N} - [f]_{3N} \end{cases}$ |

where for any $A \subset \mathbb{Z}$ and $x \in \mathbb{Q}$ we use the notation

$[x]_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \not\in A \end{cases}$

$\delta(x) = \begin{cases} (-1)^x & \text{if } x \in \mathbb{N} \\ 0 & \text{if } x \not\in \mathbb{N} \end{cases}$

The dimension of $V^T_\mu$ was obtained in [22, Thm. 6.1], using coordinates different from ours. Our standing condition $a \leq c$ puts us in case (2) (if $f \leq 1$) or (4) (if $f > 1$) of [loc. cit.].

We give here just the computation for the trace of a reflection on $V^T_\mu$, leaving the other (easier) cases to the reader.

We take $w = r_\beta$, $S = \{\text{diag}(x, y, y, z) : xy^2z = 1\}$, $t = \text{diag}(-1, -1, 1, 1) = \bar{\alpha}(-1)$. Then $R^+_t S = \emptyset$, $R^+_1 S = \{\beta\}$, $R^+_2 S = \{\alpha, \gamma, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma\}$. For $s \in S$ we set $\chi(s) = x/y$, $\eta(s) = y/z$, $(\chi \eta)(s) = x/z$.

From Theorem 4.5 we have

$$\text{tr}(r_\beta, V^T_\mu) = -\frac{1}{2} \sum_{v \in W} \varepsilon(v) \cdot (-1)^{(v_\mu, \bar{\alpha})} \mathcal{P}_w(v \mu - \rho).$$

Here the partition function $\mathcal{P}_w$ is given by

$$\mathcal{P}_w(v) = \sum_{(p,q,r)} (-1)^r,$$
where the sum is over those \((p, q, r) \in \mathbb{N}^3\) such that
\[
2p\chi + 2q\eta + r(\chi + \eta) = \nu. \tag{32}
\]
If \(\nu\) is the restriction to \(S\) of \(v\mu - \rho\) then (32) means that
\[
2p + r = A, \quad 2q + r = C, \tag{33}
\]
where \(v\mu - \rho = A\alpha + B\beta + C\gamma\). In particular, \(r \equiv A \equiv C \mod 2\). Setting \(n = \min\{A, C\}\) we have \((-1)^r = (-1)^n\) and we find that
\[
\mathcal{P}_w(v\mu - \rho) = P(n) \cdot [A - C]_{2\mathbb{Z}},
\]
where
\[
P(n) = (-1)^n \left(1 + \left\lfloor \frac{n}{2} \right\rfloor \right).
\]
and \([x]_{2\mathbb{Z}} = 1\) if \(x \in 2\mathbb{Z}\), zero otherwise.

We next find those \(v \in W\) for which (33) has a solution. Write \(v^{-1}\) as a permutation \(v_1v_2v_3v_4\), sending \(i \mapsto v_i\). The values of \(v^{-1}\omega_\alpha\) and \(v^{-1}\omega_\gamma\) are determined by \(v_1\) and \(v_4\), respectively. When \(v_1 = 4\) we have \(A < 0\) so there is no solution to (33). Likewise, when \(v_4 = 1, 2\) we have \(C < 0\) (since \(a \leq c\), in the case \(v_4 = 2\)), so there is again no solution to (33).

The remaining possibilities are as follows: For each pair \((v_1, v_4)\) there are two \(v\)'s, with opposite values of \(\varepsilon(v)\). In each row the top \(v\) has sign \(\varepsilon(v) = +1\) and the bottom \(v\) has sign \(\varepsilon(v) = -1\), and \(m = (f - 3)/2\).

| \(v_1\) | \(v_4\) | \(n = \min\{A, C\}\) | \(|A - C|\) | \(v^{-1}\) | \(\langle \mu, v^{-1}\omega_\alpha \rangle\) |
|------|------|----------------|--------|--------|----------------|
| 1    | 4    | \(m + s\)     | \(d\)  | 1234   | \(a\)      |
|      |      |                |        | 1324   | \(a + b\)  |
| 1    | 3    | \(m\)         | \(s\)  | 1423   | \(a + b + c\)|
|      |      |                |        | 1243   | \(a\)      |
| 2    | 4    | \(m + d\)     | \(s\)  | 2314   | \(b\)      |
|      |      |                |        | 2134   | \(-a\)     |
| 2    | 3    | \(m\)         | \(d\)  | 2143   | \(-a\)     |
|      |      |                |        | 2413   | \(b + c\)  |
| 3    | 4    | \(m - f\)     | \(b + s\) | 3124 | \(-a - b\) |
|      |      |                |        | 3214   | \(-b\)     |

Label the rows of this table by the pairs \((i, j) = (v_1, v_4)\). For each \((i, j)\) let \(m_{ij}, A_{ij}, C_{ij}\) be the corresponding values of \(m, A, C\) and set
\[
M_{ij} = P(m_{ij}) \cdot [A_{ij} - C_{ij}]_{2\mathbb{Z}}.
\]
After further simplification, (31) becomes
\[
\text{tr}(r\beta, V^T_w) = (-1)^a \left\{ M_{13} + M_{24} - M_{14} - M_{23} - (-1)^b \cdot M_{34} \right\}
= (-1)^a \left\{ [P(m) + P(m + d)] \cdot [s]_{2\mathbb{Z}} - [P(m + s) + P(m)] \cdot [d]_{2\mathbb{Z}} - (-1)^b P(m - f) \cdot [s - d]_{1+2\mathbb{Z}} \right\}
\]
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Breaking into cases for $s, d$ even/odd, we get the values for $\text{tr}(r_\beta, V^{T}_\mu)$ in the table above.

Remarks:

1. $V^{T}_\mu$ is a multiple of the regular representation when $a, b, c \in 2\mathbb{Z}$ and either $a$ or $c$ is in $3\mathbb{Z}$. For example, if $(a, b, c) = (2, 4, 12)$ then $\dim V^{T}_\mu = 24$ so $V^{T}_\mu$ is exactly the regular representation of $W$.

2. The table of $\mu \in P_{++}$ for which $V^{T}_\mu$ is irreducible is as follows.

| $(a, b, c)$          | $V^{T}_\mu$ |
|---------------------|------------|
| $(1, 3, 1)$         | $\varnothing_2$ |
| $(1, 1, 4k + 1)$    | $\varepsilon^k$ |
| $(2, 1, 4k + 2)$    | $\varepsilon^k \otimes \varnothing$ |
| $(1, 2, 4k + 3)$    | $\varepsilon^{k+1} \otimes \varnothing$ |

Here, $k \in \mathbb{N}$. $\varnothing$ and refl$_2$ are the reflection and two-dimensional irreducible representations of $S_4$, respectively. As Kostant and Gutkin discovered for $S_n$, each irrep of $S_4$ appears as the zero weight space of exactly one constituent of $\otimes^4 \mathbb{C}^4$. In fact, all but refl$_2$ appear in infinitely many higher tensor powers as well.

3. We have $d = 0$ exactly when $V_\mu$ is self-dual. In this case $\mu = (a, b, a)$ with $b$ odd, so $s = a$ and $f = b$. The formulas simplify to

$$
\text{tr}(1^4, V^{T}_\mu) = \frac{1}{2}a(ab + 1)
$$
$$
\text{tr}(211, V^{T}_\mu) = \frac{1}{2}((a)_2 + a \cdot (b)_4)
$$
$$
\text{tr}(22, V^{T}_\mu) = \frac{1}{2}(-1)^{a+1}[a + b \cdot (a)_2]
$$
$$
\text{tr}(31, V^{T}_\mu) = (a)_3([2a + b]_{3N} - [b]_{3N})
$$

$$
\text{tr}(4, V^{T}_\mu) = \begin{cases} 
1 & \text{if } a + b \in 2 + 4\mathbb{N} \\
-1 & \text{if } a \in 2 + 4\mathbb{N} \\
0 & \text{otherwise}
\end{cases}
$$

6 Irreducible zero weight spaces

It was shown already in [14] and [21] that every irrep of the symmetric group $S_n$ appears as $V^{T}$ for some irrep of $\text{SL}_n$. Since, from [21] we have $\text{tr}(\text{cox}^G, V^T_\mu) \in \{-1, 0, +1\}$, it follows that the trace of an $n$-cycle on any irrep of $S_n$ also lies in $\{-1, 0, +1\}$.

When they first met, Kostant asked Lusztig if he could prove this last fact for $S_n$ directly. Lusztig observed that since an $n$-cycle generates its own centralizer in $S_n$, one need only find $n$ irreps of $S_n$ for which $\text{tr}(\text{cox}) \neq 0$. The exterior powers of the reflection representation fit the bill.

In any irreducible Weyl group $W$ with Coxeter number $h$, it is still true that a Coxeter element generates its own centralizer and one can again find $h$ irreps (no longer exterior powers of the
reflection representation) of $W$ on which $\text{cox}$ has nonzero trace. Thus for any simple $G$ we have:

1. The trace of $\text{cox}^G$ on any irrep of $G$ lies in $\{-1, 0, +1\}$;
2. The trace of $\text{cox}$ on any irrep of $W$ lies in $\{-1, 0, +1\}$.

One may ask if (1) $\Rightarrow$ (2) as it did for the symmetric group. That is, does every irrep of $W$ appear as $V^T$ for some irrep of $G$? We have seen the answer is negative, for $G = \text{Sp}_4$ and $G_2$. In this section we will see it is also negative for $D_4, F_4$ and $E_6$.

So the question, raised in [16], becomes: which irreps of $W$ appear as $V^T$ for some irrep $V$ of $G$?

Theorem 3.4 leads to a method for answering this question in the case that $-1 \in W$. This means $w_0 = -1$ acts by a scalar $\pm 1$ on any irreducible representation of $W$. Since $w_0$ is an elliptic involution, Thm. 3.4 implies that $V^T_\mu$ can only be irreducible if $\mu + 2Q$ belongs the the $W$-orbit of $\rho + 2Q$ in $P/2Q$, in which case there is $v \in W$ such that $v^{-1} \tilde{R}_2^+ = \tilde{R}_\mu^+ \subset \tilde{R}^+$ and

$$\dim V^T_\mu = \varepsilon(v) \prod_{\alpha \in \tilde{R}_\mu^+} \frac{\langle \mu, \alpha \rangle}{\langle \rho, \alpha \rangle}. \quad (34)$$

Since $\mu \in P_{++}$, each factor $\langle \mu, \alpha \rangle$ is strictly positive and increases when we add a fundamental weight to $\mu$. Hence each coset in $P/2Q$ contains only finitely many $\mu$’s for which the product in (34) is the dimension of an irreducible representation of $W$. Thus one arrives at a finite list $M$ of possible $\mu$’s for which $V^T_\mu$ can be irreducible. Computation of other character values for $\mu \in M$ can show that $V^T_\mu$ is also reducible for certain $\mu \in M$.

For Spin$_8$ and $F_4$ this eliminates all but the known irreducible $V^T_\mu$ as we will see. For $E_6$ we use the elliptic triality to do the same, up to two possible exceptions.

### 6.1 Spin$_8$

We label the Dynkin diagram of type $D_4$ as

```
 1 2 3
4
```

and a weight $\mu = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4 \in P$ will be written as

$$\mu = \begin{pmatrix} a & b & c \end{pmatrix}. $$

We have $\rho \in Q$. It follows that the $W$-orbit of $\rho$ in $P/2Q$ lies in $Q/2Q$. Now $\mu \in Q$ if and only if $a \equiv c \equiv d \mod 2$. We will show that the only $\mu \in Q \cap P_{++}$ with $V^T_\mu$ irreducible are the known ones (cf. [30]):

$$\mu = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 & 3 & 1 & 1 & 3 & 1 & 3 & 1 & 2 & 1 \end{pmatrix}. \quad (35)$$

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These $V_\mu$ are the trivial, spherical harmonics of degree two on the three eight dimensional orthogonal representations of $G$, and the adjoint representation, respectively.

The set
\[ \widehat{R}_\mu^+ = \{ \tilde{\alpha} \in \widehat{R}^+ : \langle \mu, \tilde{\alpha} \rangle \in 2\mathbb{Z} \} \]
depends only on the coset
\[ \mu + 2P =: \begin{pmatrix} a & b & c \\ d \end{pmatrix}_{2P} \in P/2P. \]

The $W$-orbit of $\rho + 2P$ in $P/2P$ consists of the nonzero cosets
\[ \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 \end{pmatrix}_{2P}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 \end{pmatrix}_{2P}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 \end{pmatrix}_{2P}. \quad (36) \]

The sign character $\varepsilon$ is trivial on the stabilizer in $W$ of each the cosets (36). It follows that the sign of $\text{tr}(w_0, V_\mu^T)$ is constant on the fiber in $P/2Q$ above each of these cosets.

The values of $\text{tr}(w_0, V_\mu^T)$ are shown in the table below

| $\mu + 2P$ | $2^5 \text{tr}(w_0, V_\mu^T)$ |
|------------|-------------------------------|
| $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 \end{pmatrix}_{2P}$ | $(a + b)(b + c)(b + d)(a + b + c + d)$ |
| $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 \end{pmatrix}_{2P}$ | $-b(a + b + c)(a + b + d)(b + c + d)$ |
| $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 \end{pmatrix}_{2P}$ | $-acd(a + 2b + c + d)$ |

In each of these cosets, we find that besides the $\mu$ listed in (35), there is only one other $\mu \in P_{++}$ (up to diagram symmetry) for which
\[ \text{tr}(w_0, V_\mu^T) \in \{ \chi(w_0) : \chi \in \text{Irr}(W) \} = \{ 1, 2, 3, \pm 4, 6, \pm 8 \} \]
and also $| \text{tr}(w_0, V_\mu^T) | = \dim V_\mu^T$, namely
\[ \mu = \begin{pmatrix} 5 & 1 & 1 \\ 1 \end{pmatrix}. \]

For this $\mu$ we have $\text{tr}(w_0, V_\mu^T) = \dim(V_\mu^T) = 6$. However, $V_\mu$ is the degree-four spherical harmonics, whose zero weight space can be written down explicitly and seen to be reducible for $W$.

6.2 $F_4$

For the group $G$ of type $F_4$, we use our formula for $\text{tr}(w_0, V_\mu^T)$ to determine all irreps $V_\mu$ for which $V_\mu^T$ is irreducible for $W$. We show this occurs only for the known cases $\dim V_\mu = 1, 26, 52$. 

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We label the Dynkin diagram for $F_4$ as

$$1 \ 2 \ \Rightarrow \ 3 \ 4.$$  

We have $P = Q$ and the quotient $X := P/2Q = P/2P$ is an $\mathbb{F}_2$-vector space with basis $\bar{\omega}_i := \omega_i + 2P$, $i = 1, \ldots, 4$. We write $\mu = (a, b, c, d)$ for $\mu = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4 \in P$ and $(a, b, c, d)_2$ for the coset $\mu + 2P \in X$.

We have

$$2P \subset Q_\ell \subset P,$$

where $Q_\ell$ is the lattice spanned by the long roots (note $Q_\ell$ was called $Q$ in section 6.1). The $W$-action on $X$ preserves the subspace $Y := Q_\ell/2P$ and gives an isomorphism

$$W/\{\pm 1\} \xrightarrow{\sim} \text{GL}(X, Y) = \{g \in \text{GL}(X) : gY = Y\}.$$  

The latter group has three orbits in $X$, of sizes 1, 3, 12, the latter being $X \setminus Y = W(\rho + 2P)$. It follows that if $\mu = (a, b, c, d)$ then $\text{tr}(w_0, V^T_\mu) \neq 0$ if and only if $c$ and $d$ are not both even. In this case there is $v \in W$ such that $\mu \in v\rho + 2P$ and we have

$$\text{tr}(w_0, V^T_\mu) = \varepsilon(v) \prod_{\alpha \in \hat{R}_+} \frac{\langle \mu, \alpha \rangle}{\langle \rho, \alpha \rangle} = \frac{\varepsilon(v)}{2^{15} \cdot 3^2 \cdot 5} \prod_{\alpha \in v\hat{R}_+} \langle \mu, \alpha \rangle.$$

where we choose $v$ so that $v\hat{R}_2^+ = \hat{R}_\mu^+$. We list all positive coroots $\hat{\alpha}_1, \ldots, \hat{\alpha}_{24}$ as linear forms $A_i(\mu) = \langle \mu, \hat{\alpha}_i \rangle$, as follows.

| $A_1 = a$ | $A_2 = b$ | $A_3 = c$ | $A_4 = d$ |
| $A_5 = a + b$ | $A_6 = b + c$ | $A_7 = c + d$ |  |
| $A_8 = a + b + c$ | $A_9 = 2b + c$ | $A_{10} = b + c + d$ |  |
| $A_{11} = 2b + c + d$ | $A_{12} = a + b + c + d$ | $A_{13} = a + 2b + c$ |  |
| $A_{14} = 2b + 2c + d$ | $A_{15} = a + 2b + c + d$ | $A_{16} = 2a + 2b + c$ |  |
| $A_{17} = a + 2b + 2c + d$ | $A_{18} = 2a + 2b + c + d$ |  |  |
| $A_{19} = a + 3b + 2c + d$ | $A_{20} = 2a + 2b + 2c + d$ |  |  |
| $A_{21} = 2a + 3b + 2c + d$ |  |  |  |
| $A_{22} = 2a + 4b + 2c + d$ |  |  |  |
| $A_{23} = 2a + 4b + 3c + d$ |  |  |  |
| $A_{24} = 2a + 4b + 3c + 2d$ |  |  |  |

We have

$$\text{tr}(w_0, V^T_\mu) = \frac{\varepsilon(v)}{2^{15} \cdot 3^2 \cdot 5} A_{i_1} A_{i_2} \cdots A_{i_{10}},$$

where $v\hat{R}_2^+ = \{A_{i_1}, A_{i_2}, \ldots, A_{i_{10}}\}$.

These are shown in the table below, where the cosets in $X \setminus Y$ are labelled by $(a, b, c, d)_2$, with
\[(c, d)_2 \neq (0, 0)_2.\]

In each of the above cosets we next find all \(\mu\) for which
\[\text{tr}(w_0, V^T_\mu) \in \{\chi(w_0) : \chi \in \text{Irr}(W)\} = \{1, 2, \pm 4, 6, -8, 9, 12, -16\}.\]

In fact from all of the the twelve cosets there are only five such \(\mu\). In these cases \(V_\mu\) is small enough to compute \(\dim V^T_\mu\), as shown in the next table.

| \(\mu\) | \(\text{tr}(w_0, V^T_\mu)\) | \(\dim V^T_\mu\) |
|---|---|---|
| \((1, 1, 1, 1)\) | 1 | 1 |
| \((2, 1, 1, 1)\) | -4 | 4 |
| \((1, 1, 1, 2)\) | 2 | 2 |
| \((2, 2, 1, 1)\) | 4 | 228 |
| \((1, 1, 1, 2)\) | 12 | 12 |

The first three cases, where \(\dim V_\mu = 1\), 52, 26 are known to have irreducible zero weight spaces. In the case \(\mu = (2, 2, 1, 1)\), \(V^T_\mu\) is clearly reducible. In the last case \(\mu = (1, 1, 1, 2)\) we note that \(W\) has a unique 12-dimensional irreducible character \(\chi_{12}\), and \(\chi_{12}(\text{co}x) = 1\). On the other hand, the co-root \(\check{\beta} = 2\check{\alpha}_1 + 4\check{\alpha}_2 + 3\check{\alpha}_3 + \check{\alpha}_4\) has \(\langle \mu, \check{\beta} \rangle = 12\), which means that \(\text{tr}(\text{co}x, V^T_\mu) = 0\), by Kostant [21] (or Prop. 3.2 above). It follows that \(V^T_{(2,2,1,1)}\) is reducible. This completes the determination of all \(\mu\) for which \(V^T_\mu\) is irreducible, for the case \(G = F_4\).

### 6.3 \(E_6\)

In this section we compute \(\text{tr}(w, V^T_\mu)\) where \(w = \text{co}x^4\) is the elliptic regular element of order three. We will see that this almost determines the irreducible zero weight spaces for \(E_6\).
Since $\rho \in Q$, all $\mu \in P_{++}$ for which $V^T_\mu \neq 0$ belong to $Q$. Hence we consider the action of $W$ on $Q/3Q$. Let $q$ be the $W$-invariant quadratic form $q : Q \to \mathbb{Z}$ such that $q(\alpha) = 2$ for every $\alpha \in R$.

Passing to $Q/3Q$ gives a quadratic form $q_3 : Q/3Q \to \mathbb{F}_3$ with radical $3P/3Q$ and nondegenerate quotient $Q/3P$. This gives an isomorphism $\varphi : W \times \{I\} \xrightarrow{\sim} O(Q/3P)$.

One computes $q(\rho) = 78$, so $\rho + 3P$ is isotropic. Since $w_0 \rho = -\rho$, it follows that $W$ is transitive on the set of nonzero isotropic vectors in $Q/3P$. Arguing as in the last step of the proof of Prop. 3.3, it follows that $W$ is also transitive on the nonzero isotropic vectors in $Q/3Q$.

From this discussion and Theorem 4.5 we obtain

**Proposition 6.1** For $w = \text{cox}^4$ we have $\text{tr}(w, V^T_\mu) \neq 0$ if and only if $q_3(\mu + 3Q) = 0$. In this case there exists $v \in W$ for which $v\mu \in \rho + 3Q$ and we have

$$\text{tr}(w, V^T_\mu) = \frac{\varepsilon(v)}{3^5 \cdot 63} \cdot \prod_{\alpha \in \hat{R}_\mu^+} \langle \mu, \hat{\alpha} \rangle. \quad (37)$$

To compute $\text{tr}(w, V^T_\mu)$ explicitly, we label the Dynkin diagram as $1 \ 2 \ 3 \ 4 \ 5$ and write a weight $\mu = a\omega_1 + b\omega_2 + c\omega_3 + d\omega_4 + e\omega_5 + f\omega_6 \in P$ as

$$\mu = a_b_c_d_e_f.$$

As linear forms on $P$, we list the coroots as

\[
\begin{array}{ccc}
A_1 &=& a \\
A_2 &=& b \\
A_3 &=& a + b \\
A_4 &=& b + c \\
A_5 &=& a + b + c \\
A_6 &=& b + c + f \\
A_7 &=& a + b + c + d \\
A_8 &=& a + b + c + f \\
A_9 &=& a + b + c + d + f \\
A_{10} &=& a + b + 2c + d + f \\
A_{11} &=& a + 2b + 2c + d + f \\
A_{12} &=& a + 2b + 2c + d + e + f \\
B_1 &=& c \\
B_2 &=& f \\
B_3 &=& c + f \\
B_4 &=& b + c + d \\
B_5 &=& b + c + d + f \\
B_6 &=& a + b + c + d + e \\
B_7 &=& b + 2c + d + f \\
B_8 &=& a + b + c + d + e + f \\
B_9 &=& a + b + 2c + d + e + f \\
B_{10} &=& a + 2b + 2c + 2d + e + f \\
B_{11} &=& a + 2b + 3c + 2d + e + f \\
B_{12} &=& a + 2b + 3c + 2d + e + 2f \\
C_1 &=& e \\
C_2 &=& d \\
C_3 &=& d + e \\
C_4 &=& c + d \\
C_5 &=& c + d + e \\
C_6 &=& c + d + f \\
C_7 &=& b + c + d + e \\
C_8 &=& c + d + e + f \\
C_9 &=& b + c + d + e + f \\
C_{10} &=& b + 2c + d + e + f \\
C_{11} &=& b + 2c + 2d + e + f \\
C_{12} &=& a + b + 2c + 2d + e + f \\
\end{array}
\]

Note that $\hat{R}_\mu^+$ depends only on the $\mathbb{F}_3$-line in $Q/3P$ containing $\mu + 3P$. Let $X$ be the set of $q_3$-isotropic lines in $Q/3P$. We have $|X| = 40$, but we can reduce the calculation further.

Let $\vartheta$ be the involution of $P/3Q$ arising from the nontrivial symmetry of the Dynkin diagram. Then $X = Y \sqcup Z$ where $Y = \{y_1, \ldots, y_{16}\}$ consists of the $\vartheta$-fixed lines and $Z = \{z_1, \vartheta z_1, \ldots, z_{12}, \vartheta z_{12}\}$ are the remaining lines.

The table of $\text{tr}(w, V^T_\mu)$ is structured as follows. In each isotropic line in $Q/3P$, we give a vector $x = \sum c_i\omega_i \mod 3P$ where $+, 0, -$ correspond to $c_i = +1, 0, -1 \in \mathbb{F}_3$. Next we give an element
\( v \in W \) such that \( v \hat{R}_3^+ \subset \hat{R}^+ \) and \( x = v \rho + 3P \). Finally we give the numerator of \( \text{tr}(w, V^T) \) for any \( \mu \in x \cap P_{++} \). If \( \mu \in -x \cap P_{++} \) then \( \text{tr}(w, V^T_{\mu}) \) is the negative of the rightmost column in row \( x \) evaluated at \( \mu \). If \( \mu \in z_i \) then \( \text{tr}(w, V^T_{\mu}) \) is obtained from \( \text{tr}(w, V^T_{\mu}) \) by interchanging \( A_i \leftrightarrow C_i \).
We turn now to irreducible zero weight spaces for $E_6$.

As an abstract group, we have $W \simeq SO_5(3)$, via the twisted mapping $\varepsilon \varphi$. The Steinberg representation $81_p$ of $SO_5(3)$ and its twist $81_p' = \varepsilon \otimes 81_p$, are the only irreducible representations of $W$ whose character vanishes on $w$. We conclude a dichotomy:
**Proposition 6.2** If \( V^T_\mu \) is irreducible for \( W \) then exactly one of the following holds.

(i) \( q_3(\mu) = 0 \) and \( \text{tr}(w, V^T_\mu) \neq 0 \).

(ii) \( q_3(\mu) \neq 0 \) and \( V^T_\mu \simeq 81_p \) or \( 81'_p \).

I do not know if case (ii) ever occurs, but it seems unlikely.

Up to duality, there are five known representations \( V_\mu \) for which \( V^T_\mu \) is irreducible. These are small representations (cf. [30]).

| \( \mu \) | \( V^T_\mu \) | \( \text{tr}(w, V^T_\mu) \) |
|---|---|---|
| 1 1 1 1 | 1_p | 1 |
| 1 1 1 1 | 6_p | −3 |
| 2 1 1 1 | 20_p | 2 |
| 4 1 1 1 | 24_p | 6 |
| 2 2 1 1 | 64_p | −8 |

The representation \( 64_p \) is another Steinberg representation, via the isomorphism \( W \simeq O^-_6(2) \). However, because \( −1 \notin W \), there is no analogue of Prop. 6.2 for \( m = 2 \).

Using the table of \( \text{tr}(w, V^T_\mu) \) and the method used for \( D_4 \) and \( F_4 \), we find that (38) is the complete list of irreducible zero weight spaces with \( \text{tr}(w, V^T_\mu) \neq 0 \). We conclude

**Proposition 6.3** The only irreducible representations of \( W(E_6) \) afforded by a zero weight space of the compact Lie group \( E_6 \) are \( 1_p, 6_p, 20_p, 24_p, 64_p \) and possibly \( 81_p, 81'_p \).

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