Statistical mechanics of typical set decoding

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Abstract. – The performance of “typical set (pairs) decoding” for ensembles of Gallager’s linear code is investigated using statistical physics. In this decoding, error happens when the information transmission is corrupted by an untypical noise or two or more typical sequences satisfy the parity check equation provided by the received codeword for which a typical noise is added. We show that the average error rate for the latter case over a given code ensemble can be tightly evaluated using the replica method, including the sensitivity to the message length. Our approach generally improves the existing analysis known in information theory community, which was reintroduced by MacKay (1999) and believed as most accurate to date.

Triggered by active investigations on error correcting codes in both of information theory (IT) and statistical physics (SP) communities \[\textit{1} \textsuperscript{1} \textsuperscript{2} \textsuperscript{3} \textsuperscript{4} \textsuperscript{5} \textsuperscript{6} \textsuperscript{7} \textsuperscript{8} \textsuperscript{9} \textsuperscript{10} \], there is a growing interest in the relationship between IT and SP. Since it turned out that the two frameworks that have different backgrounds have investigated similar subjects, it is quite natural to expect that standard techniques known in one framework bring about remarkable developments in the other, and vice versa.

The purpose of this Letter is to present such an example. More specifically, we will show that a method to evaluate the performance of error correcting codes established in IT community \[\textit{1} \textsuperscript{2} \textsuperscript{3} \textsuperscript{4} \textsuperscript{5} \textsuperscript{6} \] can be generally improved by introducing the replica method. This serves as a direct answer to a question from IT researchers why the methods from physics always provide more optimistic evaluations than those known in IT literatures. In our formulation, the IT method is naturally linked to the existing SP analysis being parametrized by the number of replicas $\rho \geq 0$, which clearly explains how the IT and SP methods are related to each other.

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In a general scenario, the $N$ dimensional Boolean message $x \in \{0, 1\}^N$ is encoded to the $M(> N)$ dimensional Boolean vector $y^0$, and transmitted via a noisy channel, which is taken here to be a Binary Symmetric Channel (BSC) characterized by flip probability $p$ per bit; other transmission channels may also be examined within a similar framework. At the other end of the channel, the corrupted codeword is decoded utilizing the structured codeword redundancy.

The error correcting code that we focus on here is Gallager’s linear code $\mathcal{C}$. This code was originally introduced by Gallager about forty years ago but was almost forgotten soon after the proposal due to the technological limitations in those days. However, since the recent rediscovery by MacKay and Neal $\mathcal{C}$, this is now recognized as one of the best codes to date.

A code of this type is characterized by a randomly generated $(M-N) \times M$ Boolean sparse parity check matrix $H$, composed of $K$ and $C$ ($\geq 3$) non-zero (unit) elements per row and column, respectively. Encoding the message vector $x$, is carried out using the $M \times N$ generating matrix $G^T$, satisfying the condition $HG^T=0$, where $y^0=G^T x \pmod{2}$. The $M$ bit codeword $y^0$ is transmitted via a noisy channel, BSC in the current analysis; the corrupted vector $y = y^0 + n^0 \pmod{2}$ is received at the other end, where $n^0 \in \{0,1\}^M$ represents a noise vector with an independent probability $p$ per bit of having a value 1. Decoding is carried out by multiplying $y$ by the parity check matrix $H$, to obtain the syndrome vector $z = H y = H(G^T x + n^0) = H n^0 \pmod{2}$, and to find a solution to the parity check equation

$$H n = z \pmod{2},$$

for estimating the true noise vector $n^0$. One retrieves the original message using the equation

$$G^T x = y - n \pmod{2}; \ x \text{ becomes an estimate of the original message.}$$

Several schemes can be employed for solving Eq. (1). In recent years, the maximum a posterior (MAP) and the maximizer of posterior marginal (MPM) decodings which correspond to zero and the Nishimori’s temperatures, respectively, have been widely investigated $\mathcal{C}$, $\mathcal{C}$, $\mathcal{C}$, $\mathcal{C}$. However, we will here evaluate the performance of another scheme termed typical set (pairs) decoding, which was pioneered by Shannon $\mathcal{C}$ and reintroduced by MacKay $\mathcal{C}$ for analyzing the Gallager-type codes. Although this decoding method is slightly weaker to reduce the block or bit error rates, a rigorous analysis becomes easier than those for the above two decoding methods and investigation on it is now becoming popular in IT community $\mathcal{C}$, $\mathcal{C}$, $\mathcal{C}$.

In order to argue the typical set decoding, we first introduce the definition of typical. Due to the law of large number, a noise vector $n$ generated by the BSC satisfies a condition

$$\left| \frac{1}{M} \sum_{i=1}^{M} n_i - p \right| \leq \epsilon_M,$$

with a high probability for large $M$ and any sequence of positive number $\epsilon_M \sim \mathcal{O}(M^{-\gamma})$ ($0 < \gamma < 1/2$). We define that a vector $n$ is classified as typical when this condition is satisfied. We also call the set of all typical vectors the typical set.

Then, one can define the typical set decoding as a scheme to select a vector $n$ that belongs to the typical set and satisfies Eq. (1), as an estimate of the true noise $n^0$. In the case that there are two or more typical vectors satisfy Eq. (1), it is decided that an error is automatically declared $\mathcal{C}$. For this scheme, there can happen two types of decoding error; the first possibility, referred to the type I error here, takes place when the true noise $n^0$ is not typical, while the other one, termed the type II error, is declared when there are two or more typical vectors that satisfy Eq. (1) in spite that the true noise $n^0$ is typical. It can be shown that the probability for the type I error, $P_I$, vanishes in the limit $M \to \infty$. Therefore, we will here focus on the evaluation of the probability for the type II error, $P_{II}$.

To proceed further, it is convenient to employ the binary expression for bit sequences rather than Boolean one utilizing a mapping $\{0, 1, +\} \to \{+1, -1, \times\}$. This makes it possible
to introduce the error indicator function that becomes one when error happens and zero, otherwise, as

$$
\Delta \left( n^0, H \right) = \lim_{\rho \to +0} \mathcal{V}_{NF}^\rho \left( n^0, H \right),
$$

(3)

where

$$
\mathcal{V}_{NF}^\rho \left( n^0, H \right) \equiv \text{Tr}_{n \not= n^0} \prod_{\mu=1}^{M-N} \delta \left( \prod_{l \in \mathcal{L}^{(\mu)}} n_{l}^0, \prod_{l \in \mathcal{L}^{(\mu)}} n_l \right) \delta \left( \sum_{l=1}^{M} n_{l} - M \tanh F \right),
$$

(4)

where we have introduced the gauge transform $n_l \to n_{l}^0 n_l$ in the last form of Eq. (3) for further convenience, and where 1 denotes the $M$ dimensional vector all the elements of which are 1. Eq. (4) denotes the set of all vectors that differs from $n^0$ in the intersection of the typical set and the solution space of Eq. (1). The field $F = (1/2) \ln [(1 - p)/p]$ and $\mathcal{L}^{(\mu)}$ represents the level of the channel noise and the set of indices that have non-zero elements in $\mu$ th row in the parity check matrix $H$, respectively.

From the definition, the probability of the type II error for a given matrix $H$ is given as

$$
P_{II}(H) = \langle \Delta \left( n^0, H \right) \delta \left( \sum_{l=1}^{M} n_{l}^0 - M \tanh F \right) \rangle_{n^0},
$$

where $\langle \cdot \cdot \cdot \rangle_{n^0} = \text{Tr}_{n \not= n^0} (\cdot \cdot \cdot) \exp[\sum_{l=1}^{M} n_{l}^0]/(2 \cosh F)^M$. Since the parity check matrix $H$ is generated such that it is natural to evaluate the average of $P_{II}(H)$ over an ensemble of codes for given parameters $K$ and $C$ as a performance measure for the code ensemble. Employing Eq. (5), the average is given as

$$
P_{II} = \lim_{\rho \to +0} \exp \left[ -M \mathcal{E}(\rho) \right],
$$

where

$$
\mathcal{E}(\rho) = \frac{1}{M} \ln \left\{ \left\langle \mathcal{V}_{NF}^\rho \left( n^0, H \right) \delta \left( \sum_{l=1}^{M} n_{l}^0 - M \tanh F \right) \right\rangle_{n^0} \right\},
$$

(5)

for large $M$. Here, $\langle \cdot \cdot \cdot \rangle_{H}$ represents an average over the uniform distribution of the parity check matrix for a given choice of parameters $K$ and $C$.

Before proceeding any further, it is worthy of mentioning general properties of the exponent $\mathcal{E}(\rho)$. First, $P_{II}$ is expected to vanish in the limit $M \to \infty$ for a sufficiently small noise $p$. The highest noise level $p_c$ for this is termed the error threshold $\epsilon$. This happens when $\mathcal{E}(0) = \lim_{\rho \to +0} \mathcal{E}(\rho) > 0$. The value of $\mathcal{E}(0) > 0$ represents the sensitivity of $P_{II}$ to the message length and serves as a performance measure of the code ensemble when $M$ is finite. Next, since $\mathcal{V}_{NF}(n^0, H) = 0, 1, 2, \ldots$, $\mathcal{V}_{NF}^\rho (n^0, H)$ increases with respect to $\rho$, which implies the exponent $\mathcal{E}(\rho)$ becomes a decreasing function of $\rho \geq 0$. This is linked to an inequality

$$
\frac{\partial \mathcal{E}(\rho)}{\partial \rho} = - \frac{1}{M} \left\langle \left( S_{NF}(n^0, H) \mathcal{V}_{NF}^\rho (n^0, H) \delta \left( \sum_{l=1}^{M} n_{l}^0 - M \tanh F \right) \right)_{n^0} \right\rangle_{H} < 0,
$$

(6)

where $S_{NF}(n^0, H) = \ln \mathcal{V}_{NF}(n^0, H)$ is the entropy representing the number of wrong solutions for Eq. (2) belonging to the typical set. One can also show that $\partial^2 \mathcal{E}(\rho)/\partial \rho^2 < 0$, which implies $\mathcal{E}(\rho)$ is a convex function of $\rho$.

We are now ready to connect the current argument to the existing analysis of the typical set decoding. Since $\mathcal{E}(0) \geq \mathcal{E}(1)$, one can obtain a lower bound of $p_e$ from the condition $\mathcal{E}(1) = 0$. For $\rho = 1$ in Eq. (3), it is convenient to insert an identity $1 = \int d\omega \delta \left( \sum_{l=1}^{M} n_{l} - M \omega \right)$
in the final form of Eq. (3). Then, for a sequence \( \mathbf{n} \) that satisfies \((1/M) \sum_{l=1}^{M} n_l = \omega \), one obtains

\[ \langle \text{Tr}_\mathbf{n} \delta \left( \sum_{l=1}^{M} n_l^0 - M \tanh F \right) \rangle \frac{n_0}{M} \sim \exp \left[-MK(\omega, F)\right], \]

where \( K(\omega, F) = \left(\frac{1+\omega}{2}\right) H \left( \frac{2 \tanh F}{1+\omega} \right) + \left(\frac{1-\omega}{2}\right) \ln 2 - H(\tanh F) \) and \( H(x) = \frac{(1+x)}{2} \ln \frac{1+x}{2} - \frac{(1-x)}{2} \ln \frac{1-x}{2} \). Further, the remaining average required in Eq. (3) is evaluated as \( \langle \text{Tr}_\mathbf{n} \delta \left( \sum_{l=1}^{M} n_l - M\omega \right) \prod_{\mu=1}^{N} \delta \left( 1; \sum_{l} n_l \right) \rangle \sim \exp \left[ MR(\omega) \right] \). The exponent \( R(\omega) \) is termed the weight enumerator \( \chi \). This provides an averaged distribution of the distances between the true noise \( \omega \) and other vectors that satisfy Eq. (1) in the current context \( [1, 9] \). This corresponds to Eq. (4.7) in \( [1] \).

However, it should be emphasized here that one can evaluate \( E(1) \) without introducing the weight variable \( \omega \). Moreover, it is evident that the tightest estimate (exact value) of \( \rho_\omega \) can be obtained by evaluating \( E(0) = \lim_{\omega \rightarrow +0} E(\rho) \). This can be carried out by the replica method, which gives rise to a set of order parameters \( q_{\alpha, \beta, \ldots, \gamma} = (1/M) \sum_{l=1}^{M} Z_l n_l^\beta \ldots n_l^\gamma \), where \( \alpha, \beta \ldots \) represent replica indices and the variable \( Z_l = 1 \ldots M \) comes from enforcing the restriction \( C \) connections per index as in \( [1] \).

Further calculation requires a certain ansatz about the symmetry of the order parameters. As a first approximation we assume replica symmetry (RS) in the following order parameters and their conjugate variables

\[ q_{\alpha, \beta, \ldots, \gamma} = q \int dx \, \pi(x) \, x^\alpha, \quad \hat{q}_{\alpha, \beta, \ldots, \gamma} = \hat{q} \int d\hat{x} \, \hat{\pi}(\hat{x}) \hat{x}^\alpha, \quad \text{(7)} \]

where \( l \) denotes the number of replica indices, \( q \) and \( \hat{q} \) are normalization variables for defining \( \pi(\cdot) \) and \( \hat{\pi}(\cdot) \) as distributions. Unspecified integrals are carried out over the interval \([-1, 1]\).

Originally, the summation \( \text{Tr}_n \times 1 \) excludes the case \( n = 1 \); but one can show that for large \( M \) limit, this becomes identical to the full summation in the non-ferromagnetic phase, where \( \pi(x) \neq \delta(x-1) \) and \( \hat{\pi}(\hat{x}) \neq \delta(\hat{x}-1) \). In addition, we employ Morita’s scheme \( [12] \) which in this case converts the restricted annealed average with respect to \( \mathbf{n}^0 \) to a quenched one:

\[ \frac{1}{M} \ln \left( \langle \cdots \times \delta \left( \sum_{l=1}^{M} n_l^0 - M \tanh F \right) \rangle \right) \frac{n_0}{M} = \frac{1}{M} \langle \ln(\cdots) \rangle_{\mathbf{n}^0}, \quad \text{(8)} \]

to simplify the calculation of the average over \( \mathbf{n}^0 \) in Eq. (3) considerably, and obtain

\[ E(\rho) = \text{Ext}^*_\{q, q_\pi(\pi), \pi(\cdot), G\} \left\{ -\frac{C}{K} q^K \int \prod_{i=1}^{K} dx_i \pi(x_i) \left( \frac{1 + \sum_{i=1}^{K} x_i}{2} \right)^\rho \right. \]

\[ - \left. \ln \left( \int \prod_{\mu=1}^{C} d\hat{x}_\mu \, \hat{\pi}(\hat{x}_\mu) \left( \text{Tr}_{n=1} e^{G n^0 n} \prod_{\mu=1}^{C} \left( \frac{1 + \hat{x}_\mu}{2} \right)^\rho \right) \right) n^0 \right\} + Cq\hat{q} \int dx \, \hat{\pi}(\hat{x}) \left( \frac{1 + \hat{x}}{2} \right)^\rho + \left( \frac{C}{K} - C \right) + \rho G \tanh F, \quad \text{(9)} \]

where \( \langle \cdots \rangle_{n^0} = \text{Tr}_{n=1} \times 1 \langle \cdots \rangle \) and \( \text{Ext}^*_\{\cdots\} \) denotes the functional extremization excluding the possibility of \( \pi(x) = \delta(x-1) \) and \( \hat{\pi}(\hat{x}) = \delta(\hat{x}-1) \) as is introduced in \( [1] \).

\(^{(1)}\) The weight enumerator is usually introduced for the distance between codewords \([1, 1, 1]\). However, since \( y^0 \cdot y^1 = n^0 \cdot n^1 \mod 2 \) holds for two sets of Boolean vectors \( (y^0, n^0) \) and \( (y^1, n^1) \) that satisfy \( y = y^0 + n^0 = y^1 + n^1 \mod 2 \), the distance between the noise vectors \( n^0 \) and \( n^1 \) is identical to that for the codewords \( y^0 \) and \( y^1 \).
Two analytical solutions of $\pi(x)$ and $\tilde{\pi}(\tilde{x})$ can be obtained in the limit $K, C \to \infty$, keeping the code rate $R = N/M = 1 - C/K$ finite:

1. $\pi(x) = \frac{1}{2}[(1 + \tanh F)\delta(x - \tanh F) + (1 - \tanh F)\delta(x + \tanh F)]$, $\tilde{\pi}(\tilde{x}) = \delta(\tilde{x})$

2. $\pi(x) = \frac{1}{2}[\delta(x - 1) + \delta(x + 1)]$, $\tilde{\pi}(\tilde{x}) = \frac{1}{2}[\delta(\tilde{x} - 1) + \delta(\tilde{x} + 1)]$.

One can show that both of these are locally stable against perturbations to the RS solutions providing $\mathcal{E}(\rho) = \rho \cdot [H(\tanh F) - (1 - R) \ln 2]$ and $\mathcal{E}(\rho) = H(\tanh F) - (1 - R) \ln 2$, respectively. Selecting the relevant branch that has the lower exponent for $\rho \geq 1$ and taking the limit $\rho \to 0$ [5], one obtains the exponent as

$$\mathcal{E}(0) = \lim_{\rho \to +0} \mathcal{E}(\rho) = \left\{ \begin{array}{ll} (R_c - R) \ln 2, & R < R_c, \\ 0, & R > R_c, \end{array} \right. \quad (10)$$

where $R_c = 1 + p \log_2 p + (1 - p) \log_2 (1 - p)$ corresponds to Shannon’s limit [5].

Note that in the vicinity of $R \sim R_c$, this exponent exceeds the upper bound of possible reliability function that represents the vanishing rate of the decoding error probability for the best code [1, 5]. However, this does not imply any contradiction because the current analysis is just for $P_{I_1}$ while the convergence rate of $P_I$ is slower than that of the reliability function.

For finite $K$ and $C$, one can obtain $\mathcal{E}(\rho)$ via numerical methods. Similar to the case of $K, C \to \infty$, there generally appear two branches of solutions:

1. continuous distributions for $\pi(x)$ and $\tilde{\pi}(\tilde{x})$, for which $\lim_{\rho \to +0} \mathcal{E}(\rho) = 0$.

2. $\rho$ independent frozen distributions $\pi(x) = \frac{1}{2}[(1 + b)\delta(x - 1) + (1 - b)\delta(x + 1)]$, $\tilde{\pi}(\tilde{x}) = \frac{1}{2}[(1 + \tilde{b})\delta(\tilde{x} - 1) + (1 - \tilde{b})\delta(\tilde{x} + 1)]$.

The parameters $b$ and $\tilde{b}$ are determined from the extremization problem [3] by setting $\rho = 1$, the functional extremization with respect to $\pi(\cdot)$ and $\tilde{\pi}(\cdot)$ is then reduced to that for the first moments $b = \int dx \pi(x)$ and $\tilde{b} = \int d\tilde{x} \tilde{\pi}(\tilde{x})$. The exponent of this branch is completely frozen to that for $\rho = 1$ as $\mathcal{E}(\rho) = \mathcal{E}(1)$ for $\forall \rho > 0$. Although the distributions of the two branches look quite different, their exponents coincide at $\rho = 1$ in any situation.

Note that the frozen branch corresponds to the conventional IT analysis [1, 2], and would provide the exact estimate in absence of other solutions. However, in order to take an appropriate limit $\lim_{\rho \to +0} \mathcal{E}(\rho)$, one has to select the dominant branch for $\rho \geq 1$ [3] among the existing solutions, and the frozen branch does not necessarily provide the correct exponent for $\rho \to +0$. Actually, the scenario suggested by our analysis supports this statement (Fig. 1).

When the channel noise $p$ is sufficiently high (Fig. 1 (a)), the exponent for the continuous branch is monotonically decreasing with respect to $\rho$ which implies this is the dominant branch for $\rho \geq 1$. This provides $\lim_{\rho \to +0} \mathcal{E}(\rho) = 0$. However, for lower $p$, $\mathcal{E}(\rho)$ of the continuous branch is maximized to a positive value for a certain parameter $\rho_{\text{opt}}$ (Fig. 1 (b)).

In this situation, the solution for $0 < \rho < \rho_{\text{opt}}$ is physically wrong because inequality [6] does not hold. The frozen replica symmetry breaking (RSB) solution [4] (a one step RSB ansatz under the constraint $(1/M)\mathbf{n}^a \cdot \mathbf{n}^b = 1$ for replica indices $a$ and $b$ in the same subgroup) is a suitable scheme for obtaining a consistent solution. Employing this 1RSB solution, one finds $\mathcal{E}(\rho) = \mathcal{E}(\rho_{\text{opt}})$ for $0 < \rho < \rho_{\text{opt}}$, which implies $\lim_{\rho \to +0} \mathcal{E}(\rho) = \mathcal{E}(\rho_{\text{opt}}) > 0$ indicating a vanishing behaviour $P_{I_1} \sim \exp[-M\mathcal{E}(\rho_{\text{opt}})]$. These imply that the critical condition determining the error threshold $p_c$ is given by $\partial \mathcal{E}(\rho)/\partial \rho |_{\rho \to +0} = 0$, being computed for the continuous solution. Employing the gauge transform [6], one can show that the variational parameter $G$ in Eq. (9)
Fig. 1. – Appropriate limits for $\lim_{\rho \to +0} \mathcal{E}(\rho)$ in the case of finite $K$ and $C$. The solution that has the lower exponent for $\rho \geq 1$ should be selected as the relevant branch, which is drawn as a thick curve or line in each case. For $p \geq p_c$, (a), the continuous solution is relevant while the 1(frozen)RSB solution which emerges from this solution at $\rho = \rho_g$ provides an appropriate exponent $\mathcal{E}(\rho_g)$ for $p_b \leq p < p_c$ (b). For $0 \leq p < p_b$ (c), the frozen (RS) solution is relevant. In the limit $K, C \to \infty$, the situation (b) does not appear.

Fig. 2. – (a): Numerically computed $\mathcal{E}(\rho)$ of the continuous branch for $p = 0.0915, 0.0990$ for $K = 6$ and $C = 3$ ($R = 1/2$). Symbols and error bars are obtained from 50 numerical solutions. Curves are computed via a quadratic fit. For $p = 0.0915$, $\mathcal{E}(\rho)$ is maximized to a positive value $\mathcal{E}(\rho_g) \simeq 2.5 \times 10^{-3}$ for $\rho_g \simeq 0.5$ while it vanishes at $\rho \simeq 1$ as is suggested in the IT literature. On the other hand, for $p = 0.0990$, our predicted threshold, it is maximized to zero at $\rho \simeq 0$, which implies that this is the correct threshold. (b): Comparison of the estimates of $p_c$ between the IT and the current methods is summarized in a table. The estimates for the IT method are taken from [1]. The numerical precision is up to the last digit for the current method. Shannon’s limit denotes the highest possible $p_c$ for a given code rate.

enforcing $\sum_{l=1}^{M} n_l^0 n_l = M \tanh F$ coincides with $F$ in this limit. Then, the critical condition is summarized as

$$F \tanh F - \frac{1}{M} \left\langle \ln \left[ \frac{\text{Tr}}{n \neq 1} \prod_{\mu=1}^{M-N} \delta \left( 1; \prod_{l \in E(\mu)} n_l \right) e^{F \sum_{l=1}^{M} n_l^0 n_l} \right] \right\rangle_{H^0} = 0, \quad (11)$$

which is identical to what has been obtained for the phase boundary of the ferro-paramagnetic transition along the Nishimori’s temperature predicted by the existing replica analysis [2].

As $p$ is reduced further, the position of the maximum $\rho_g$ moves to the right and exceeds $\rho = 1$ at another critical noise rate $p_b$. This implies that below $p_b$, the limit $\rho \to +0$ is governed by the frozen (RS) solution that is identical to what is given by the conventional IT analysis (Fig. 1(c)). However, this situation is realized only sufficiently below from the threshold and the solution is of no use for direct evaluation of $p_c$ although it provides a lower bound.
Finally, we examined the case of $K = 6$ and $C = 3$ to demonstrate the accuracy of the estimated threshold. We numerically evaluated $\mathcal{E}(\rho)$ of the continuous branch for $p = 0.0915$, a recent highly accurate estimate of the error threshold for this parameter choice \cite{1} and for $p = 0.0990$, which is the threshold predicted by the replica method \cite{14, 8}. The numerical results are obtained by approximating $\pi(\cdot)$ and $\hat{\pi}(\cdot)$ using $10^6$ dimensional vectors and iterating the saddle point equations until convergence. The obtained results are shown in Fig. 2 (a); it indicates $\max_{\rho} \mathcal{E}(\rho) \simeq 2.5 \times 10^{-3}$ for $p = 0.0915$ while $\mathcal{E}(\rho)$ is maximized (to zero) at $\rho \simeq 0$ for $p = 0.0990$, suggesting a tighter estimate for the error threshold than those reported so far. Comparison in other parameter choices is also summarized in Fig. 2 (b).

In summary, we have investigated the performance of the typical set decoding for ensembles of Gallager’s codes. We have shown that the direct evaluation of the average type II error probability over the ensemble becomes possible employing the replica method. The link to the existing IT analysis which is based on the weight enumerator is also clarified. Although the weight enumerator does not play a crucial role for determination of the error threshold in the current analysis, it still remains a key factor for the error rate in low $R$ regions. Analysis of it from a view point of statistical physics is under way \cite{13}.

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REFERENCES

[1] S. Aji, H. Jin, A. Khandekar, D.J.C. MacKay and R.J. McEliece, BSC Thresholds for Code Ensembles Based on “Typical Pairs” Decoding, preprint, (1999).
[2] T.M. Cover and J.A. Thomas, Elements of Information Theory, Wiley (New York), (1991).
[3] R.G. Gallager, IRE Trans. Info. Theory, IT-8, 21 (1962).
[4] D.J. Gross and M. M´ezard, Nucl. Phys. , B240, 431 (1984).
[5] J.L. van Hemmen and R.G. Palmer, J. Phys. A: Math. and Gen., 12, 563 (1979).
[6] Y. Kabashima and D. Saad, Europhys. Lett., 44, 668 (1998); 45, 97 (1999).
[7] Y. Kabashima, T. Murayama and D. Saad, Phys. Rev. Lett., 84, 1355 (2000); T. Murayama, Y. Kabashima, D. Saad and R. Vicente, Phys. Rev. E, 62, 1577 (2000).
[8] Y. Kabashima, N. Sazuka, K. Nakamura and D. Saad, cond-mat/0010173 (2000).
[9] D.J.C. MacKay, IEEE Trans. on Info. Theor, 45, 399 (1999); D.J.C. MacKay and R.M. Neal, Electronic Lett., 33, 457 (1997).
[10] R.J. McEliece, The Theory of Information and Coding, Addison-Wesley (Reading, MA), (1977).
[11] R.J. McEliece and J. Omura, IEEE Trans. on Infor. Theor, 23, 611 (1977).
[12] T. Morita, Math. Phys. 5, 1401, (1964); R. Kühn, Z. Phys. B 100, 231 (1996).
[13] J. van Mourik, D. Saad and Y. Kabashima, preprint (2001).
[14] K. Nakamura, Y. Kabashima and D. Saad, cond-mat/0010073 (2000).
[15] H. Nishimori, J. Phys. Soc. of Japan, 62, 2973 (1993).
[16] H. Nishimori and K.Y.M. Wong, Phys. Rev. E, 60, 132 (1999).
[17] T. Richardson, A. Shokrollahi and R. Urbanke, Design of provably good low-density parity check codes, preprint (1999)
[18] P. Rujáin, Phys. Rev. Lett., 70, 2968 (1993).
[19] C.E. Shannon, Bell Sys. Tech. J., 27, 379 (1948); 27, 623 (1948).
[20] C.E. Shannon, The Mathematical Theory of Information, University of Illinois Press (Urbana, IL), (1949); reprinted (1998).
[21] N. Sourlas, Nature, 339, 693 (1989).
[22] N. Sourlas, Euro.Phys.Lett., 25, 159 (1994).