Contour Integral Representations for the Characters of Logarithmic CFTs

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Abstract:

We propose a contour integral representation for the one-point correlators at genus one of the primaries of a family of rational logarithmic conformal field theories.

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1 Introduction

Conformally invariant quantum field theories in two dimensions have been widely studied because of their relevance to diverse phenomena. Of considerable interest is the classification of conformal field theories (CFTs). There has been significant progress in classifying a subset of CFTs, namely the rational conformal field theories (RCFTs). These are theories in which the number of primary fields (with respect to the infinite dimensional Virasoro algebra or an extended chiral algebra) is finite.

Among the diverse approaches to the problem, the idea in Refs. [1]–[5] to consider the one-loop characters of the irreducible representations of the chiral algebra is a beautiful one. They propose to view these as the independent solutions of a differential equation of finite order. Since these characters transform into linear combinations of themselves under a modular transformation [6, 7], the differential equation itself must have definite covariance property under a modular transformation. CFTs with two or three characters have been classified in this approach. The differential equation has been known to be related to the existence of a null vector [1] (see also [8, 9] for a more recent and rigorous analysis).

CFTs have since been generalised to accommodate logarithmic conformal field theories (LCFTs). A characteristic feature of the theories in this class is the existence of a logarithmic branch cut in certain chiral correlation functions [10]. The theory studied in [10] is the first member of a series of perhaps the simplest LCFTs, the logarithmic $(p, 1)$ minimal models. These theories are rational with respect to a chiral $W$-algebra [11, 12]. However, not all the highest weight representations are completely decomposable [13]. LCFTs have been studied extensively since their discovery (a partial list is Refs. [14]–[20]). There are many examples where the nature of representations differ from the minimal family that we will focus on. For recent reviews, see, for example, Refs. [21, 22].

In this paper, we consider modular differential equation for the logarithmic $(p, 1)$ minimal models. These have been studied in Ref. [23], where it is shown that the vacuum torus am-
plitudes can be thought to arise as solutions to a modular differential equation. Our focus, however, will be on finding explicit integral representations for the vacuum torus amplitudes along the lines of Ref. [24]. The advantage of the integral representations is that they are explicit, one can in principle calculate the solution to any give order. Moreover, it is shown in the recent paper [9] that while the torus one-point functions in an RCFT satisfy a modular differential equation, the solutions are not always expressible in terms of standard transcendental functions. In view of this result the integral representations could be an effective way to find these amplitudes. They may also be useful in the calculation of the other correlation functions on the torus.

In the following, we review the logarithmic correlator of Ref. [10] in Sec.2 and write the integral representations of the hypergeometric differential equation. We also recall some relevant facts about the LCFTs. In Sec.3 we review the classification of RCFTs in terms of the modular differential equation. We also comment on the case for LCFTs. In Secs.4 and 5 the integral representations for the characters, more precisely, the vacuum torus amplitudes is proposed. We consider this in some detail for the simplest case of the (2,1) minimal model with $c = -2$ (Sec.4) and comment on the more general models (Sec.5). We end with some concluding remarks.

2 Logarithmic minimal models

An infinite family of logarithmic CFTs is the set of minimal models labelled by $(p,1)$ with $p = 2, 3, \cdots$ following the notation of BPZ [25]. The central charge of the $(p,1)$ model $M_p$ is

$$c_p = 1 - \frac{6(p-1)^2}{p} = 13 - \left( p + \frac{1}{p} \right)$$

(2.1)

and the degenerate fields $\phi_{r,s}$, $r, s \in \mathbb{Z}^+$ have the conformal dimensions

$$h_{r,s} = \frac{(sp - r)^2 - (p-1)^2}{4p}$$

(2.2)

and a null vectors at level $rs$. The set of primaries inside the Kac table for these models is empty. There is, however, a chiral $W$-algebra generated by fields of spin $(2p-1)$. Restricting the set of degenerate primaries to the range $s = 1$ and $1 \leq r \leq 3p-1$ yields a finite dimensional representation [21]. We may think of these fields as being on the boundary of an extended Kac table.

The first of these models corresponding to $p = 2$ has $c = -2$. The set of primaries $\phi_{r,1} \equiv \phi_r$ with $r = 1, \cdots, 5$ are in representations of the chiral $W_3$-algebra generated by a triplet of spin-3 fields [11, 12]: The fields $\phi_1$, $\phi_2$ and $\phi_3$ with weights 0, $-1/8$ and 0 respectively are singlet representations, and $\phi_4$ and $\phi_5$ with weights $3/8$ and 1 are doublets. Moreover, while the doublets as well as the singlet $\phi_2$ are irreducible representations of the Virasoro algebra,
the pair $\phi_1$ and $\phi_3$ (with $h_1 = h_3$) form an indecomposable Jordan block. For example, under the action of $L_0$:

$$
L_0 |\phi_1\rangle = h_1 |\phi_1\rangle,
L_0 |\phi_3\rangle = h_3 |\phi_3\rangle + |\phi_1\rangle.
$$

(2.3)

This is a characteristic feature of these logarithmic theories. In the model $\mathcal{M}_p$, there are $(p-1)$ pairs of equal weights that form $2 \times 2$ indecomposable Jordan blocks of the Virasoro algebra. These are the fields $(\phi_1, \phi_{2p-1}), (\phi_2, \phi_{2p-2}), \cdots, (\phi_{p-1}, \phi_{p+1})$, all (except the first) of which have negative weight $h \leq 0$ (equality for the first). The remaining fields $\phi_p$ and $\phi_{2p}, \cdots, \phi_{3p-1}$ are all in irreducible representations, with all except the first field having positive weights.

Coming back to the $c = -2$ model $\mathcal{M}_2$, the field $\phi_2$ has a null vector at level two. This leads to a differential equation for the four-point correlation function \cite{10}

$$
\left\langle \phi_2(z_1)\phi_2(z_2)\phi_2(z_3)\phi_2(z_4) \right\rangle \sim [(z_1 - z_2)(z_3 - z_4)]^{1/4} [\xi(1 - \xi)]^{1/4} F(\xi)
$$

(2.4)

with

$$
\xi(1 - \xi) \frac{d^2 F}{d\xi^2} + (1 - 2\xi) \frac{dF}{d\xi} - \frac{1}{4} F = 0,
$$

(2.5)

where, $\xi$ is the cross-ratio and we have only displayed the chiral part of the correlator. This is a hypergeometric differential equation with $a = b = 1/2$ and $c = 1$. The two solutions

$$
I_1 = F(a, b, c; \xi) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_1^\infty dt t^{a-c}(1 - t)^{c-b-1}(t - \xi)^{-a},
$$

$$
I_2 = \xi^{1-c}F(b - c + 1, a - c + 1, 2 - c; \xi)
$$

$$
= \frac{\Gamma(2 - c)}{\Gamma(1 - a)\Gamma(1 + a - c)} \int_0^\xi dt t^{a-c}(1 - t)^{c-b-1}(\xi - t)^{-a},
$$

(2.6)

for this choice of parameters are not independent, and indeed can be obtained from one another by a change of integration variable. The second solution, therefore, is to be replaced by one with a logarithmic dependence. The Jordan block structure of the two $h = 0$ fields of the theory was deduced from it in Ref. \cite{10}.

In order to obtain the logarithmic piece, we may replace the parameters $a$, $b$ and $c$ by adding an infinitesimal piece $\varepsilon$ to them (one can do this for $c$ only, but this is more general). Now consider the linearly independent combinations $\tilde{I}_1 \equiv (I_1 + I_2) / 2 = I_1$ and $\tilde{I}_2 \equiv \lim_{\varepsilon \to 0} (I_1 - I_2) / \varepsilon \sim \ln \xi I_1(\xi) + P(\xi)$, where $P(\xi)$ is an infinite series in $\xi$ that can be determined. A similar regularisation to obtain the logarithmic solution was done in Ref. \cite{27}.

\footnote{Indeed as argued in Ref. \cite{26} the logarithmic dependence can arise only in non-unitary theories.}
3 The modular differential equation

In a rational conformal field theory, the number of primaries with respect to the conformal Virasoro algebra or a larger chiral algebra is finite. As a result, the partition function as well as the correlation functions on an arbitrary genus Riemann surface, may be expressed as a sum of products of a finite number of holomorphic and anti-holomorphic building blocks. These are functions of the various moduli. For the one loop partition function, these holomorphic functions are also the (appropriately defined) characters of the representations of the symmetry algebra [6]. Under a modular transformation of the torus, the characters transform as linear combinations of themselves and thus provide a representation of the modular group of the genus-one surface.

The authors of Refs. [1]–[4] proposed to classify RCFTs in terms of their characters. The finite number \( n \) of the characters are to be regarded as the independent solutions of a differential equation of order \( n \). Given the transformation properties of the characters, this differential equation must be modular invariant. In order to write this equation, one must take note of the fact that the derivative \( \partial_\tau \equiv \frac{\partial}{\partial \tau} \) does not transform covariantly under a modular transformation. Instead, one applies the covariant derivative on a modular form of degree 2\( k \)

\[
  D_k = \partial_\tau - \frac{i\pi k}{6} E_2(\tau),
\]

(3.1)

(where \( E_2(\tau) \) is the second Eisenstein series), appropriately to write the most general modular invariant differential equation (MDE) of order \( n \):

\[
  D_n^2 \chi + \sum_{k=0}^{n-1} f_k(\tau) D_{\tau}^k \chi = 0. 
\]

(3.2)

In the above \( f_k(\tau) \) are modular forms of weight \( 2(n-k) \). Let \( \chi_1(\tau), \cdots, \chi_n(\tau) \) be \( n \) linearly independent solutions of the MDE. The coefficients are then \( f_k(\tau) = (-)^{n-k} W_k(\tau)/W(\tau) \), where

\[
  W_k(\tau) = \det \begin{pmatrix}
    \chi_1 & \chi_2 & \cdots & \chi_n \\
    D_{\tau} \chi_1 & D_{\tau} \chi_2 & \cdots & D_{\tau} \chi_n \\
    \vdots & \vdots & \ddots & \vdots \\
    D_{\tau}^{k-1} \chi_1 & D_{\tau}^{k-1} \chi_2 & \cdots & D_{\tau}^{k-1} \chi_n \\
    D_{\tau}^{k+1} \chi_1 & D_{\tau}^{k+1} \chi_2 & \cdots & D_{\tau}^{k+1} \chi_n \\
    \vdots & \vdots & \ddots & \vdots \\
    D_{\tau}^n \chi_1 & D_{\tau}^n \chi_2 & \cdots & D_{\tau}^n \chi_n 
  \end{pmatrix}
\]

(3.3)

and \( W(\tau) \equiv W_n(\tau) \) is the Wronskian.

As explained in [2,4], the classification of RCFTs is then characterised by two numbers: \( n \), the number of characters or the order of the MDE and \( \ell \), the number of zeros\(^2\) of the Wronskian.

\(^2\)Due to the presence of the orbifold points in the moduli space, \( 6\ell \), and not \( \ell \) is required to be an integer.
The number $\ell$ can in turn be expressed in terms of $n$, the central charge $c$ and the weights $h_\alpha$ of the primary fields as $\ell = \frac{1}{2}n(n-1) + \frac{1}{4}nc - 6 \sum h_\alpha$.

From the general theory of differential equations, one would expect the above formalism to extend to the case where there are logarithmic solutions. However, unlike ordinary RCFTs, the characters of the irreducible representations of LCFTs by themselves do not provide a representation of the modular group [28]. The character of a primary is also the trace (upto a phase) of $q^{L_0}$ over the module above it. Now, the action of $q^{L_0}$ on the indecomposable representation in Eq. (2.3) gives us [28]

$$q^{L_0} \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix} \sim \begin{pmatrix} q^{L_0(1)} & 0 \\ \ln q & q^{L_0(3)} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_3 \end{pmatrix}, \quad (3.4)$$

where, the superscripts on $L_0$ refer to its action on the two field in the Jordan block. In taking a trace to obtain the characters, one gets two identical series in $q$, equal to the character over the module over $\phi_1$. Fortunately, there is another set of objects, namely the one-point correlation functions on the torus, or the vacuum torus amplitudes (VTA), which still close under modular transformations. MDEs were originally written for the VTAs in [1].

Following [23, 28], the analysis of Mathur et al can be generalised and applied to LCFTs. The solutions of the MDE are now the vacuum torus amplitudes $T(\tau)$. In [2], it was assumed that the highest weights are all distinct $h_\alpha \neq h_\beta$ for $\alpha \neq \beta$. Now it is relaxed to allow for degenerate weights, but the solutions are assumed to the distinct $T_\alpha \neq T_\beta$ for $\alpha \neq \beta$. The rest of the analysis is the same as in the case of RCFTs. In particular, LRCFTs can also be classified by $n$ and $\ell$.

It turns out to be convenient to change variable from $\tau$ to

$$\lambda = \left( \frac{\vartheta_2(\tau)}{\vartheta_3(\tau)} \right)^4 = 16q^{1/2} \left( 1 - 8q^{1/2} + 44q + \cdots \right), \quad (3.5)$$

where $q = \exp(2\pi i \tau)$ and $\vartheta$'s are the standard Jacobi theta-functions [1, 3]. The above maps (six copies of) the moduli space to the complex plane which has no orbifold singularity. In fact, $\lambda$ is the parameter that appears in the elliptic equation defining the torus. The variable $q$ can in turn be written as $q = \left( \frac{\lambda}{16} \right)^2 \left( 1 + \lambda + 232 \left( \frac{\lambda}{16} \right)^2 + O(\lambda^3) \right)$. Using the properties of the theta-functions, one finds that under the generators $T$ and $S$ of the modular group:

$$\lambda(\tau + 1) = \frac{\lambda(\tau)}{\lambda(\tau) - 1}, \quad \lambda \left( -\frac{1}{\tau} \right) = 1 - \lambda(\tau). \quad (3.6)$$

Notice that $T^2 : \lambda \rightarrow \lambda$, therefore, $\lambda$ really parametrises the subgroup $\Gamma(2)$ of the modular group. When written in terms of $\lambda$, the coefficients of the MDE are rational functions and therefore, the MDE is a differential equation of the Fuchsian type. This is a real advantage of going over to the variable $\lambda$. For details and additional comments, see Ref. [29].

In the case of the RCFTs, this fact has been used in [2], where theories with two and three characters were analysed in detail. In the former case, the MDE is a hypergeometric equation...
The authors of [24] exploit the Fuchsian character of the MDE written in terms of $\lambda$ to write an explicit set of solutions as contour integrals — one does not really need to know the MDE. The building blocks are the Feigin-Fuchs-Dotsenko-Fateev (DFFF) type contour integral [30]:

$$J \equiv (\lambda(1-\lambda))^\alpha J(\lambda)$$

$$\sim (\lambda(1-\lambda))^\alpha \int \prod_{i=1}^{n_1} dt_i \prod_{k=1}^{n_2} ds_k \prod_{i=1}^{n_1} [t_i(t_i-1)(t_i-\lambda)]^a \prod_{k=1}^{n_2} [s_k(s_k-1)(s_k-\lambda)]^b$$

$$\prod_{i<j} (t_i-t_j)^{-2a/b} \prod_{k<l} (s_k-s_l)^{-2b/a} \prod_{i<k} (t_i-s_k)^{-2b},$$

where,

$$\alpha = \frac{1}{3} \left( -n_1(1+3a) - n_2(1+3b) + \frac{a}{b} n_1(n_1-1) + \frac{b}{a} n_2(n_2-1) + 2n_1n_2 \right)$$

These integrals are invariant under modular $S$ and $T$ transformations up to changes in integration limits and phase factors. The limits of the integration are chosen such that the $T$ transformation leaves the integral invariant up to a phase. These lead to the following set of $n = (n_1+1)(n_2+1)$ independent integrals $J_{AB}$:

$$J_{AB} \equiv (\lambda(1-\lambda))^\alpha J_{AB}(\lambda)$$

$$= (\lambda(1-\lambda))^\alpha \prod_{i=1}^{A} \int_{0}^{\lambda} dt_i \prod_{j=A+1}^{n_1} \int_{1}^{\lambda} dt_i \prod_{k=1}^{B} \int_{0}^{\lambda} ds_k \prod_{l=B+1}^{n_2} \int_{1}^{\lambda} ds_l$$

$$\prod_{i=1}^{n_1} [t_i(t_i-1)(t_i-\lambda)]^a \prod_{k=1}^{n_2} [s_k(s_k-1)(s_k-\lambda)]^b$$

$$\prod_{i<j} (t_i-t_j)^{-2a/b} \prod_{k<l} (s_k-s_l)^{-2b/a} \prod_{i<k} (t_i-s_k)^{-2b},$$

which go into each other under the modular transformations.

As $\lambda \to 0$

$$J_{AB}(\lambda) \sim \lambda^{\alpha+\Delta_{AB}},$$

where,

$$\Delta_{AB} = A(1+2a) + B(1+2b) - \frac{a}{b} A(A-1) - \frac{b}{a} B(B-1) - 2AB.$$  

Let us observe that $\Delta_{00} = 0$ and

$$\sum_{A=0}^{n_1} \sum_{B=0}^{n_2} (\alpha + \Delta_{AB}) = \frac{1}{6} n(n-1).$$
From the behaviour of the character $\chi_\alpha$ associated with a primary field of weight $h_\alpha$:

$$\chi_\alpha \sim \lambda^{2h_\alpha - \frac{c}{12}}, \quad (3.13)$$

in the $\lambda \to 0$ limit, one identifies the sets $\{2h_\alpha - c/12\}$ and $\{\alpha + \Delta_{AB}\}$ and finds that

$$\ell \equiv \frac{nc}{4} - 6 \sum h_\alpha + \frac{n(n-1)}{2} = 0. \quad (3.14)$$

Therefore, only the characters of those RCFTs with $\ell = 0$ may be expressed as DFFF integrals.

As shown in [24], only for RCFTs with at most five characters, the leading asymptotic behaviour of the DFFF integrals suffice to prove that they are the characters. When the number of characters $n \geq 6$, one needs the next to leading order terms in the power series expansion in $q$ in order to match them with the expected power series of the characters.

4 Contour integrals for the ‘characters’ of the $c = -2$ theory

We have mentioned earlier that in the case of the logarithmic minimal CFTs, the solutions of the modular differential equation are the vacuum torus amplitudes, rather than the characters. We would expect that the integral representations, which provide a set of solutions to the MDE, to represent the vacuum torus amplitudes. In the following, we propose the set of integrals for the $(p,1)$ series of logarithmic minimal models $\mathcal{M}_p$. Mukhi at al [24] gave a prescription to identify the parameters $(A, B, a, b)$ in Eq. (3.9) to those of the $(p, q)$ minimal models [25]. While this does not directly apply to the logarithmic family $\mathcal{M}_p$, one may still determine these parameters consistently, and indeed, the integrals are simpler for the LCFTs.

Let us first consider the $c = -2$ model $\mathcal{M}_2$ that we have discussed in Sec. 2. The fifth order modular differential equation for the vacuum torus amplitudes $T$ was derived [23] from the existence of a null vector in the vacuum module of the chiral $W$-algebra:

$$D_5^5 T(\tau) + \sum_{k=0}^{4} f_k(\tau) D_4^k T(\tau) = 0. \quad (4.1)$$

From the expression (3.14), one easily finds that in this theory $\ell = 0$. Thus the Wronskian does not have a zero for any finite value of $\tau$, and this model ought to be among those for which the solutions of the MDE admit an integral representation.

We have $n = 5 = (n_1 + 1)(n_2 + 1)$. Therefore, either $n_1$ or $n_2$ must be zero. This in turn implies that either $A$ or $B$ is zero. Since there is an invariance under the exchange of the pairs $(A, a) \leftrightarrow (B, b)$, we may choose $B = 0$ without any loss of generality. Following [24], our proposal is thus:

$$(A, a) = (r - 1, -5/8), \quad (B, b) = (0, 5/2), \quad (4.2)$$
using which we get the expected values for \( \alpha = \frac{1}{6} \equiv -\frac{\nu}{12} \) and \( \Delta_{AB} = 2h_{r,1} \).

Explicitly, we find the following basis for the integrals:\[3\]:

\[
J_{00} = \left[ \lambda(1 - \lambda) \right]^{1/6} \int_1^\infty dt_1 \cdots dt_4 \prod_{i=1}^4 \left[ t_i(t_i - 1)(t_i - \lambda) \right]^{-5/8} \prod_{i<j} (t_i - t_j)^{1/2}
\]

\[
J_{10} = \left[ \lambda(1 - \lambda) \right]^{1/6} \int_0^\lambda dt_1 \int_1^\infty dt_2 dt_3 dt_4 \prod_{i=1}^4 \left[ t_i(t_i - 1)(t_i - \lambda) \right]^{-5/8} \prod_{i<j} (t_i - t_j)^{1/2}
\]

\[
J_{20} = \left[ \lambda(1 - \lambda) \right]^{1/6} \int_0^\lambda dt_1 dt_2 \int_1^\infty dt_3 dt_4 \prod_{i=1}^4 \left[ t_i(t_i - 1)(t_i - \lambda) \right]^{-5/8} \prod_{i<j} (t_i - t_j)^{1/2}
\] \quad (4.3)

\[
J_{30} = \left[ \lambda(1 - \lambda) \right]^{1/6} \int_0^\lambda dt_1 dt_2 dt_3 \int_1^\infty dt_4 \prod_{i=1}^4 \left[ t_i(t_i - 1)(t_i - \lambda) \right]^{-5/8} \prod_{i<j} (t_i - t_j)^{1/2}
\]

\[
J_{40} = \left[ \lambda(1 - \lambda) \right]^{1/6} \int_0^\lambda dt_1 \cdots dt_4 \prod_{i=1}^4 \left[ t_i(t_i - 1)(t_i - \lambda) \right]^{-5/8} \prod_{i<j} (t_i - t_j)^{1/2}
\]

Since, this is a theory with five characters, it will suffice to match the leading asymptotic behaviour in order to identify the DFFF integrals with the vacuum torus amplitude. The leading \( \lambda \) dependence suggests that \( J_{40} \) corresponds to the vacuum torus amplitude of the field \( \phi_{A+1} \). However, both \( J_{00} \) and \( J_{20} \) correspond to fields with \( h = 0 \) and the weight of the field for \( J_{40} \), \( h = 1 \), differs from these by an integer. So a change of basis may be necessary. Moreover, the set above cannot be linearly independent and we will need to put in the logarithmic solution corresponding to the degenerate roots \( h = 0 \) of Eq. (1).

Let us now compute the subleading terms. We consider \( J_{00} \) first. Making a change of variable to bring the limits of the integrals in standard form, we rewrite it as (we have not put the specific values of \( \alpha, a \) and \( b \) at this stage):

\[
J_{00} = \lambda^a \int_0^1 \left[ \lambda(1 - \lambda) \right]^{1/6} \prod_{i=1}^4 \left[ t_i^{-2-3a+\frac{2a}{2}(n_1-1)}(1-t_i)^a \prod_{i<j} (t_i - t_j)^{-\frac{2a}{2}} F(\lambda),
\]

where,

\[
F(\lambda) = (1-\lambda)^a \prod_{i=1}^{n_1} (1-t_i \lambda)^a = 1 - \left( \alpha + a \sum_{i=1}^{n_1} t_i \right) \lambda
\] \quad (4.5)

\[
+ \left( \frac{\alpha(\alpha - 1)}{2} + a\alpha \sum_{i=1}^{n_1} t_i + \frac{a(a - 1)}{2} \sum_{i=1}^{n_1} t_i^2 + a^2 \sum_{i<j} t_i t_j \right) \lambda^2 + \mathcal{O}(\lambda^3)
\]

\[3\]In the special case of Eq. (4.1) for \( B = 0 \) this type of integrals were first considered by Selberg and generalised by Aomoto [31].
The integrals that one needs to evaluate at each order in $\lambda$, when the expansion (4.3) is substituted in Eq. (4.4), have fortunately been evaluated in [30]. Following the notation there, let us define:

$$\{f(t_i)\} = \prod_{i=1}^{n} \int_{0}^{1} dt_i t_i^\gamma (1 - t_i)^\delta \prod_{i<j} (t_i - t_j)^{2\rho} f(t_i), \quad (4.6)$$

where $\gamma, \delta, \rho$ are arbitrary constants and $f(t_i)$ is a function of $t_i$.

The expansion of $J_{00}$ in $\lambda$ is therefore:

$$J_{00}(\lambda) = \lambda^n \left[ \{1\} - (\alpha\{1\} + n_1 a\{t_1\}) \lambda + \left( \frac{\alpha(\alpha - 1)}{2} \{1\} + a a n_1 \{t_1\} \right. \right. \right.$$  
$$\left. \left. + \frac{a(a - 1)}{2} n_1 \{t_1^2\} + a^2 n_1(n_1 - 1) \{t_1 t_2\} \right) \lambda^2 + O(\lambda^3) \right], \quad (4.7)$$

where $\{1\}$ is the overall normalisation constant which we may choose to factor out and define the normalised integrals $\tilde{f}(t_i) = \{f(t_i)\}/\{1\}$.

The integrals (4.6) are special cases of the more general DFFF integrals. The case $f(\{t_i\}) = 1$, i.e., the normalisation constant, was evaluated by Selberg:

$$\{1\} = \prod_{m=0}^{n-1} \frac{\Gamma((m + 1)\rho) \Gamma(1 + \gamma + m\rho) \Gamma(1 + \delta + m\rho)}{\Gamma(\rho) \Gamma(2 + \gamma + \delta + (n - 1 + m)\rho)}. \quad (4.8)$$

This result was generalised by Aomoto [31] to the elementary symmetric polynomials

$$P_k^{(n)}(\{t_i\}) = \sum_{1 \leq j_1 < \cdots < j_k \leq n} \prod_{i=1}^{k} t_{j_i} = \frac{1}{\Gamma(k + 1) \Gamma(n - k + 1)} \sum_{\sigma_n} \prod_{i=1}^{k} t_{\sigma(n,i)} \quad (4.9)$$

of degree $k$ and found to be:

$$\overline{P_k^{(n)}}(\{t_i\}) = \frac{\Gamma(n + 1)}{\Gamma(k + 1) \Gamma(n - k + 1)} \frac{\Gamma \left( \frac{n + 1}{\rho} + n \right) \Gamma \left( \frac{n + \delta + 2}{\rho} + 2n - k - 1 \right)}{\Gamma \left( \frac{n + 1}{\rho} + n - k \right) \Gamma \left( \frac{n + \delta + 2}{\rho} + 2n - 1 \right)}. \quad (4.10)$$

Let us list the ones we need for calculation up to order $\lambda^2$ for completeness: From the above we find:

$$\overline{t_1} = \frac{1}{n} \overline{P_1^{(n)}} = \frac{\gamma + 1 + (n - 1)\rho}{\gamma + \delta + 2 + 2(n - 1)\rho},$$
$$\overline{t_1 t_2} = \frac{2}{n(n - 1)} \overline{P_2^{(n)}} = \frac{(\gamma + 1 + (n - 1)\rho) (\gamma + 1 + (n - 2)\rho)}{(\gamma + \delta + 2 + 2(n - 1)\rho) (\gamma + \delta + 2 + (2n - 3)\rho)}, \quad (4.11)$$

and in addition, the following integral has been evaluated in [30]:

$$\overline{t_1^2} = \overline{t_1 t_2} + \frac{(1 + n\rho) (\gamma + 1 + (n - 1)\rho) (\delta + 1 + (n - 1)\rho)}{(\gamma + \delta + 2 + 2(n - 1)\rho) (\gamma + \delta + 2 + (2n - 3)\rho) (\gamma + \delta + 3 + 2(n - 1)\rho)}. \quad (4.12)$$
The relevant constants in this case are $\gamma = -13/8$, $\delta = -5/8$ and $\rho = 1/4$: hence $\tau_1 = 1/10, \tau_1\tau_2 = -1/80$ and $\tau_1 = 7/80$. Let us return to the normalisation constant, the exact value of which is not important at this point, later. Suffice to say that it is a finite constant. This brings us to the expansion (determined up to an overall normalisation constant):

$$J_{00} \sim \lambda^{1/6} \left(1 + \frac{1}{12} \lambda + \frac{43}{1152} \lambda^2 + \mathcal{O}(\lambda^3)\right),$$

$$\sim q^{1/12} \left(1 + \mathcal{O}(q^3)\right).$$ (4.13)

where we have used Eq.(3.5) in writing the last expression in terms of $q$.

The first subleading term in the $q$ expansion is thus zero, which implies the existence of a null vector at the first level. This is expected of the identity field. Since the identity field is unique in a CFT, we identify $J_{00}$ to the one-point function of the identity field $\phi_1$ at one loop.

It is straightforward to repeat the steps above to find the subleading terms in the other integrals. We find (see Eq.(5.5) to order $\lambda^2$:

$$J_{10} \sim q^{-1/24} \left(1 + q + \mathcal{O}(q^2)\right),$$

$$J_{30} \sim q^{11/24} \left(1 + q + \mathcal{O}(q^2)\right),$$

$$J_{40} \sim q^{13/12} \left(1 + q + \mathcal{O}(q^2)\right),$$ (4.14)

up to the overall normalisation constants which we can determine (see Eq.(5.6)). The fact that the coefficient of the term $q^{1/2}$ vanishes in all cases provides a non-trivial check.

So far we have not discussed the integral $J_{20}$. This is supposed to describe the VTA of the partner of the identity field in the Jordan block and exhibit logarithmic behaviour. The normalisation plays an important part here. The two sets of variables for the two integrals require different redefinitions to bring them to the standard form in Eqs.(4.6) (see Eq.(5.5)). The normalisation for the second set has a factor of $\Gamma(2 + \gamma + \delta + \rho)$ corresponding to $m = 0$ in Eq.(4.8). The argument for the values of the parameters in the second integral vanishes, leading to a divergent factor of $\Gamma(0)$ in the denominator. This would make $J_{20}$ in (4.3) vanish. Interestingly, this is not quite the case. Even though the (normalised) coefficients of the terms of order one and $\lambda$ are finite, the same at order $\lambda^2$ diverges. Moreover, the divergence is due to a vanishing factor in the denominator that is exactly the argument of the gamma function mentioned above. This renders it finite. Put in a different way, the divergence is an artefact of separating out the normalisation factor. The expansion of the integral $J_{20}$ as a power series in $\lambda$ actually starts at order $\lambda^2$. The same conspiracy of factors give finite coefficients for all the higher order terms (since the same factors are part of the integrals (4.10) that appear at subsequent orders). Thus we find an infinite series in $\lambda$, the leading power of $\lambda$ (and hence $q$) of which cannot correspond to the field $\phi_3$ with $h_3 = 0$. Indeed, since there is only one root of the indicial equation with this power, $J_{20}$ cannot be an independent solution of the MDE, rather it must be the same as $J_{40}$ up to an overall constant. Moreover, this being a model with five
fields, the leading order behaviour suffices to identify the corresponding character uniquely. In principle, one should also be able to check this explicitly, at least term by term in the power series. However, in practice, this requires the values of the integrals \((4.6)\) with polynomials of increasing order. Only a subset of these integrals corresponding to the elementary symmetric polynomials \((4.9)\) have been evaluated [31]. One can check from Eq.\((4.10)\) that the series from \(J_{20}\) is of the expected form.

We, therefore, need to substitute \(J_{20}\) by a linearly independent solution. To this end, we need a regularisation scheme to replace the vanishing factor by \(\varepsilon\) and in addition we propose \(\alpha \rightarrow \alpha + \eta\). One may think of the latter as an analytic continuation in the central charge. As a result, we get \(J_{A0}(\lambda, \varepsilon, \eta)\) as a series in \(\lambda\), \(\varepsilon\) and \(\eta\). We find that if we set \(\varepsilon = \eta\) and consider the linear combination

\[
\tilde{J}_{20} \equiv \lim_{\varepsilon \to 0} \left[ \frac{1}{\varepsilon^2} J_{20} + \left( a_0 + \frac{a_1}{\varepsilon} \right) J_{00} + \left( b_0 + \frac{b_1}{\varepsilon} + \frac{b_2}{\varepsilon^2} \right) J_{40} \right],
\]

(4.15)

where, \(a_0, \ldots, b_1\) are appropriately defined constants, the unwanted divergent terms cancel out. The resultant finite terms behave like \(\ln \lambda\) times an infinite series in \(\lambda\) with the leading power of order one. This is the linearly independent logarithmic solution of the MDE.

We are now in a position to compare with the results of Ref.\([23, 28]\), where the authors constructs the characters of the \(c = -2\) triplet algebra. These are expressed in terms of the Dedekind eta- and generalized theta-functions:

\[
\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),
\]

\[
\theta_{\nu,k}(q) = \sum_{n=-\infty}^{\infty} q^{(2kn+\nu)^2/4k},
\]

\[
\partial \theta_{\nu,k}(q) = \sum_{n=-\infty}^{\infty} (2kn + \nu)q^{(2kn+\nu)^2/4k}.
\]

The characters of the four irreducible highest weight representations of the chiral algebra, namely, the identity \((h = 0)\) and the three fields with conformal weights \(h = -1/8, 3/8\) and 1, are:

\[
\chi_0(q) = (\theta_{1,2} + \partial \theta_{1,2})/\eta = 2q^{1/12} \left( 1 + q^2 + 4q^3 \cdots \right),
\]

\[
\chi_{-1/8}(q) = \theta_{0,2}/\eta = q^{-1/24} \left( 1 + q + 4q^2 + 5q^3 \cdots \right),
\]

\[
\chi_{3/8}(q) = \theta_{2,2}/\eta = 2q^{11/24} \left( 1 + q + 2q^2 + 3q^3 + \cdots \right),
\]

\[
\chi_1(q) = (\theta_{1,2} - \partial \theta_{1,2})/\eta = 4q^{13/12} \left( 1 + q + q^2 + 2q^3 + \cdots \right).
\]

(4.17)

Using these results, the corresponding vacuum torus amplitudes have been proposed in [23] by analysing the modular differential equation order by order in \(q\). One may take the VTAs of
the identity and the fields \( \phi_2, \phi_4 \) and \( \phi_5 \) to be the same as the respective characters above. In addition, the linearly independent solution

\[
T(q) = \ln q \partial q_1 / \eta
\]

may be taken to be the VTA of the field \( \phi_3 \). Of course, as has been stressed in [23, 28], there is no canonical choice of basis in this case and the proposal above is one possible consistent choice.

A comparison with (4.17) justifies our proposal for the vacuum torus amplitudes to the DFFF integrals: \( J_{00} \) to the identity, \( J_{10} \) to \( \phi_2 \), \( J_{30} \) to \( \phi_4 \) and \( J_{40} \) to \( \phi_5 \). In addition, in both cases, we have a solution with logarithmic dependence as the vacuum torus amplitude of the field \( \phi_3 \). In [23], Eq.(4.18) is obtained by subtracting \( \chi_0 \) and \( \chi_1 \) and multiplying by a \( \ln q \) piece. If we substitute the integral representation that we have already identified, we find:

\[
\ln q (\chi_0 - \chi_1) = \ln q (\lambda) \left[ \lambda (1 - \lambda) \right]^{1/2} \int_1^{\infty} \prod_{i=1}^{4} dt_i (t_i(t_i - 1)(t_i - \lambda))^{- \frac{5}{2}} \left( 1 - \frac{\lambda^2}{t_i} \right) \prod_{i<j} (t_i - t_j)^{1/2}
\]

This is evidently of the form that one obtains from the combination (4.15). The new set of integrals solves the modular differential equation and is closed under modular transformations.

5 Contour integrals for the ‘characters’ of the \( c_{p,1} \) models

The contour integral representation for the vacuum torus amplitudes of the \( c = -2 \) logarithmic conformal field theory generalises to the other logarithmic minimal models \( \mathcal{M}_{p \geq 3} \) of the \( (p,1) \) family. In this section, we will briefly indicate how to extend our proposal. First, we compute that the value of \( \ell \) for any model of the series and find that:

\[
\ell = \frac{(3p - 1)(3p - 2)}{2} + \frac{3p - 1}{4} \left( 1 - \frac{6(p - 1)^2}{p} \right) - 6 \sum_{r=1}^{3p - 1} \frac{(p - r)^2 - (p - 1)^2}{4p} = 0.
\]

So, there is no obstruction for an integral representations for the vacuum torus amplitudes for these theories. Next we need to identify the set \( (A, B, a, b) \) with the parameters of \( \mathcal{M}_p \). The order of the MDE for the vacuum torus amplitudes is \( 3p - 1 \), and hence, \((n_1 + 1)(n_2 + 1) = 3p - 1\). Unlike the case of the \( c = -2 \) model, a priori there no reason now for \( B \) to be zero. Nevertheless, unless \( B = 0 \), we are led to a contradiction.

To see this, let us recall that the set \( \Delta_{AB} \) is to be identified with \( 2h_{r,1} \). Since only \( A \) and \( B \) can depend on \( r \), this leads to the relation

\[
A + 2pB = r - 1,
\]

after we have identified

\[
a = \frac{3 - 4p}{4p}, \quad b = -\frac{3 - 4p}{2},
\]

13
following [24]. The fact that $A = 0$ for the maximum value of $B$ and $r \leq 3p - 1$, gives the bound $2pB \leq 3p - 2$. The only allowed integer values are therefore $B = 0, 1$ for any $p \geq 2$. If $B = 1$, we get $n_2 = 1$. Substituting this into the relation $α = -\frac{c}{12}$ and using (3.8) we get a quadratic equation for $n_1$, the solutions of which are $n_1 = 5p - 2$ and $n_1 = 3p - \frac{3}{2}$. On the other hand, from the condition $(n_1 + 1)(n_2 + 1) = 3p - 1$, we find $n_1 = \frac{3}{2}(p - 1)$. Consistency of these require $p = 1/7$ or $p = 0$, neither of which is acceptable. Hence $B = n_2 = 0$ for all the $(p, 1)$ models.

The following relation between the two sets of parameters:

$$(A, a, n_1) = \left( r - 1, \frac{3 - 4p}{4p}, 3p - 2 \right), \quad (B, b, n_2) = \left( 0, \frac{4p - 3}{2}, 0 \right), \quad (5.4)$$

gives rise to correct values for $Δ_{AB} = \frac{(p-r)^2-(p-1)^2}{2p}$ and $α = -\frac{c}{12}$. A basis for the DFF integrals is (3.9) with the values of the parameters as above. However, this set cannot be linearly independent.

We see from the conformal weights (2.2), that the pairs of integrals $J_{A0}$ and $J_{A'0}$ with $A + A' = 2p - 2$, $A = 0, 2, \ldots, (p - 2)$ have the same leading power of $λ$. This is consistent with the Jordan block structure of these models discussed in Sec.2. Moreover, the leading powers of $λ$ of $J_{A0}$ for $A = 2p, \ldots, 3p - 2$ differ from the former sets respectively by $2r = 2, 4, \ldots, 2p - 2$, all even integers. This suggests linear relations involving the triads of integrals $(J_{r-1,0, J_{2r-r-1,0, J_{2p+r-1,0}}})$. The field $ϕ_r$ is an eigenfunction of $L_0$ and we expect $J_{r-1,0}$ to be its VTA.

Let us examine the behaviour of $J_{A0}$. There are two sets of integrals with different limits. After changing variables to bring them to the standard form we find:

$$J_{A0} = \lambda^{α + A(1 + 2a - a(A-1)/b)} \prod_{i=1}^{A} \int_0^{1} ds_i \left(s_i(s_i - 1)^{a} \prod_{i<j} (s_i - s_j)^{-2a/b} \right)$$

$$\times \prod_{k=A+1}^{n_1} \int_0^{1} du_k u_k^{-3+2a(n_1-1)/b} (1 - u_k)^a \prod_{k<l} (u_k - u_l)^{-2a/b}$$

$$\times (1 - λ)^{\frac{n_1}{2}} \prod_{i=1}^{A} (1 - λs_i)^{a} \prod_{k=A+1}^{n_1} (1 - λu_k)^{a} \prod_{i,k} (1 - λs_i u_k)^{-2a/b},$$

which may be expanded to the desired order in λ. Consider the overall normalisation factor in Eq. (5.5):
\[ A = p, \cdots, 2p - 2, \] all these, and only these, integrals have a divergent factor of \( \Gamma(0) \) in the denominator from \( n = 0, 1, \cdots, p - 2 \) in Eq.\((5.6)\) respectively! Recall that these were to correspond to the VTA for the Jordan block partner of the first set (in reverse order).

As in the \( c = -2 \) case discussed earlier, this would have made the integrals vanish. However, just like in that case, a zero appears in the denominator of the coefficient of a higher order term in \( \lambda \), so as to give a finite expression! Indeed, the coefficient of term at order \( \lambda^2 \) in the expansion of the integral in \( J_{2p-2,0} \) does diverge and this makes it finite. Similarly in \( J_{2p-2-A,0} \) (\( A = 0, \cdots, p \)), one can check that precisely the coefficient of the term \( \lambda^{2A+2} \) (we are counting modulo the power due to the weight of the field) in \( \text{(4.10)} \) has a divergent factor from \( k = 2A + 2 \) that makes it finite. The same factor is part of the integrals that appear in subsequent orders. Thus the behaviour of these integrals is not right to describe the VTAs of the fields \( \phi_{p+1}, \cdots, \phi_{2p-1} \), but rather they ought to be equal to \( J_{2A+2,0} \) (\( A = 0, \cdots, p \)). Therefore, as in the \( c = -2 \) model, we propose to regularise the triad \( (J_{00}, J_{2p-2-A,0}, J_{2A+2,0}) \) and replace the redundant second integral by a linear combination (see Eq.\((4.15)\)) which exhibits logarithmic dependence in \( \lambda \) and hence in \( q \). This prescription provides the required ‘logarithmic characters’.

In the general case, there is another issue to resolve. Just the leading order behaviour is not enough to establish that these are indeed the vacuum torus amplitudes. Since the number of ‘characters’ \( 3p - 1 > 5 \) for \( p > 2 \), one also needs to analyse the nonleading terms in this case. This could be done in a fashion similar to the \( c = -2 \) case. Let us consider the integral \( J_{00} \) which ought to correspond to the VTA of the identity field. Using Eqs.\((4.1)\)–\((4.12)\), we find

\[
J_{00} \sim \lambda^\alpha \left( 1 + \frac{1}{2} \alpha \lambda + \frac{1}{8} \alpha \left( \alpha + \frac{13}{8} \right) \lambda^2 + \mathcal{O}(\lambda^3) \right), \quad (5.7)
\]

where \( \alpha = -\frac{c}{12} = \frac{1}{2} \left( p + \frac{1}{p} - \frac{13}{6} \right) \). Hence, using Eq.\((3.5)\), we find that the coefficients of the \( q^{1/2} \) and \( q \) terms vanish, so that \( J_{00} \sim q^{-c/24} (1 + \mathcal{O}(q^2)) \) as expected of the identity field. This can be repeated for the other integrals.

6 Conclusions

We have proposed a contour integral representation for the vacuum torus amplitudes (one point function on the torus) of the logarithmic \((p, 1)\) minimal models, generalising earlier results for the characters of ordinary conformal field theories. We have studied the simplest \((2, 1)\) model with central charge \( c = -2 \) in some detail and identified the candidate integrals for the other models of the family. In fact, in the case of these logarithmic family of \((p, 1)\) minimal models, the integral representation (being of the Selberg-Aomoto family, rather than the more general type considered by Feigin-Fuchs and Dotsenko-Fateev) is simpler than the well known RCFT case.

For the irreducible representations, as well as for the fields in the indecomposable Jordan block that are the eigenfunctions of \( L_0 \), we find the integral representations by extending the
known results of the RCFTs. The same procedure applied to the partner fields in the Jordan block may seem to naively yield a vanishing expression. These are, however, the VTAs of primaries, the weights of which differ from that of the Jordan cell by an integer. We propose to replace these by the corresponding logarithmic characters obtained by a regularisation prescription. The regularisation scheme, however, is chosen for simplicity and could perhaps be improved upon for better understanding.

The modular differential equation for the $c = -2$ model was analysed as a power series in $q$ in [23]. It may be interesting to study the MDE in $\lambda$, as this is a Fuchsian equation. In principle, one can solve for the recursion relations and obtain the series solutions. This approach, as well as the integral representation to generate the series could be useful in view of the fact that the solutions of the MDEs for the VTAs are not always expressible in terms of known transcendental functions [9].

Let us close with a few brief remarks. First, there is a well defined relation between the minimal models RCFTs and the affine SU(2) CFTs. This helps to relate the characters of the two theories. The logarithmic extension of SU(2)$_k$ theories have been studied by Nichols [32], who shows that the Hamiltonian reduction of the SU(2)$_{k=0}$ theory yields the logarithmic $c = -2$ model. The contour integral representation may be useful in extending this connection to the entire logarithmic family. It is interesting to note that the contour integrals for the characters of the ordinary SU(2)$_k$ models are also given by Selberg type integrals ($B = 0$). Secondly, the modular differential approach to the characters have been extended to the genus two case in Ref. [33], where the MDE is written for one of the three complex moduli (keeping the others fixed). Since genus two surfaces are hyperelliptic, in terms of their representation as a sphere with six branch points, the MDE is Fuchsian. A contour integral representation for the genus two characters should therefore be possible. Finally, it may be an interesting problem to extend this approach to the various other known logarithmic conformal field theories.

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