A PDF PSA, or Never gonna set _xscale again - guilty feats with logarithms.

John C. Forbes

1 Flatiron Institute, Simons Foundation, 162 Fifth Avenue, New York, NY 10010
*E-mail: jforbes@flatironinstitute.org

1 April 2020

ABSTRACT
In the course of doing astronomy, one often encounters plots of densities, for example probability densities, flux densities, and mass functions. Quite frequently the ordinate of these diagrams is plotted logarithmically to accommodate a large dynamic range. In this situation, I argue that it is critical to adjust the density appropriately, rather than simply setting the x-scale to ‘log’ in your favorite plotting code. I will demonstrate the basic issue with a pedagogical example, then mention a few common plots where this may arise, and finally some possible exceptions to the rule.

To put it explicitly, in general
\[
\int_a^b p(x)dx \neq \int_{\log_{10} a}^{\log_{10} b} p(10^u)du,
\]
where \( u = \log_{10} x \), because the change of variables requires substituting \( dx = \ln(10)xdu \), not just \( du \). However, when plotting the PDF of \( x \) with respect to a logarithmically-scaled \( x \)-axis, the area under the curve is in fact the incorrect right-hand side of Equation (2).

2 GENERIC PEDAGOGICAL EXAMPLE
To illustrate the argument, I’m going to look in some detail at a density function which is constant in log-space, but of course behaves quite differently in linear space. I will then plot the PDF of this variable in a variety of ways. I argue that some of them, which are common in astronomy, are misleading if not outright wrong.

Consider a random variable \( x \) whose PDF \( f(x) \propto 1/x \) between \( a \) and \( b \) with \( 0 < a < b \). The normalized PDF is therefore
\[
f(x) = \begin{cases} 
0 & \text{for } x < a \text{ or } x > b \\
\frac{1}{\ln(b/a)} & \text{for } a \leq x \leq b
\end{cases}
\]
From this, we can compute the distribution of the random variable \( u = \log_{10} x \), which will become useful momentarily. Defining\(^1\) \( g(u) \) to be the PDF of \( u \), under a change of

\(^1\) Instead of defining a new function, e.g. \( f \) or \( g \), for every random variable, one could simply denote them all with e.g. \( p \), so we \( \) would have \( p(x) \) and \( p(u) \). Slightly more formally these would be denoted \( p_X(x) \) and \( p_U(u) \), but the subscript is often suppressed. For clarity, I’ll just use \( f \) and \( g \) for now.
Figure 1. The pedagogical example from section 2. The three panels show three different ways one could imagine plotting a log-uniform probability density. Also shown are normalized histograms analogous to each curve. Panel c shows the distribution in log-space, i.e. where this distribution is uniform. Panel a shows the same distribution in linear space. Panel b shows what happens if you take the code that produces panel a, and simply set the x-axis to be logarithmic without changing anything else. This produces a plot that visually suggests one is more likely to, for instance, draw a value within a factor of 2 of $x = 1$, than within a factor of 2 of $x = 10$. This is wrong - these two probabilities are equal, as suggested in panel c. From the overplotted histogram, we can see that this can be understood as implicitly using changing bin sizes.

Figure 2 shows several more examples of plots of densities one may encounter in astronomy. The plots are shown according to how clear they are, and how correct they are. These are slightly different things – plots may technically have the correct labels, but be misleading for the reasons we discuss here (as in Fig. 1b).

3 ANTICIPATED FAQ

In my subfield, everyone plots $X$ vs $Y$ in a way I guess you’d object to. Should I really fight with my coauthors over this?

Probably not! There are certainly situations where avoiding the problem illustrated in Figure 1 might cause more confusion than it would avoid.

Hang on, don’t you plot $\Sigma$ vs. $r$ all the time? What gives?

Wow, I’m flattered you’re familiar with my work! I do often plot the surface density of mass, $\Sigma$, in disks as a function of cylindrical radius $r$, whereas the differential amount of mass per unit radius is actually $2\pi r \Sigma$. This partly falls into the case covered in the previous question, i.e. everyone who studies the density distribution of disks plots $\Sigma$ vs. $r$. In addition though, $\Sigma$ is a physically meaningful quantity, related directly, for example, to the self-gravity of the disk or the expected star formation rate. On top of that, $\Sigma$ itself often has an exceptionally simple form for as-yet poorly-understood reasons, namely $\Sigma \propto \exp(-r/r_s)$, where $r_s$ is some scale-length. This leads us to conclude that even though some things may be interpreted as densities, the differential distribution of mass (literal or probability) is not always the most important point to convey in a plot.
Figure 2. A set of annotated plots showing some of the pitfalls discussed in this work. Plots further to the right are more correct, and plots further up are clearer in their meaning. These axes don’t necessarily align, since a plot can be labelled correctly but still be misleading, and a plot can be labelled incorrectly even if the author’s point is quite clear.

Fine, I’ll give you $\Sigma$ vs $r$, but shortly after you wrote this “paper,” weren’t you a co-author on some work plotting power spectral density (PSD) per unit frequency vs. log-frequency, exactly analogous to the problematic panel in Figure 1?

Yes, and I did raise this exact issue with my co-authors. We decided that this also falls under the first case raised in this FAQ. In particular, one of people’s few intuitions for power spectra is that a white noise power spectrum should be flat, i.e. have a constant PSD. If we were to make the adjustment I advocate for in this work, a plot of a white noise power spectrum would not be flat, but rather...
would be inversely proportional to the frequency. To avoid conflicting with people’s intuition, i.e. to keep white noise spectra flat, we kept the plots of power per frequency vs. log frequency. So I tried, but not that hard!

I’m looking at Figure 2, and I don’t understand what you have against $\nu F_{\nu}$ plots. I thought they were intended to address exactly the problem you’re pointing out.

Yes, $\nu F_{\nu} = \nu dF/d\nu = dF/d\ln \nu$, so one can easily see by eye in a $\nu F_{\nu}$ plot against log $\nu$ where most of the flux in the spectrum is being emitted. My only objection (which is why it’s closer to the origin in Figure 2, but still in the upper-right quadrant) is that $dF/d\ln \nu$ is not quite the same as $dF/d\log \nu$. The difference is just a constant factor of ln10, and one usually doesn’t care too much about the normalization of these plots. Nonetheless I would personally prefer if the units were ergs per second per square centimeter per dex, as opposed to ergs per second per square centimeter per e-folding of $\nu$ (the de facto units of $\nu F_{\nu}$), or ergs per second per square centimeter (the not-quite-right label people often use on plots of $\nu F_{\nu}$).

So what’s your opinion of plots of Janskys vs. log $\lambda$?

Let’s just say they would be off the chart in Figure 2.

Is this what you’ve been working on instead of responding to my email?

I actually wrote most of this in 2017, so if you’ve been waiting on a reply for that long, sorry! You should probably ping me again.

How did you make the plots look cartoonish?

import matplotlib.pyplot as plt
with plt.xkcd():
  # usual plotting code here.

4 TAKEAWAY POINTS
This is a quick summary of the points I tried to raise.

– Probability and Probability Density are different quantities with different units.

– When you’re plotting a PDF, remember that the units on the y-axis do have dimensions in general. In particular, they should be something like “probability per unit-whatever-is-on-the-x-axis.” If the thing on the x-axis is logarithmic, your y-axis should probably be something per dex.

– There are many exceptions to the latter point. Clarity and not confusing your readers is more important than whether I am personally annoyed by your plot.

– When plotting something that is very similar to a histogram against a continuous variable, it’s rare that the y-axis should be “Probability.”

– If you’re plotting a PDF, you can usually be more explicit with your label than just “PDF.” In particular, make sure there is no ambiguity about exactly which variable’s PDF is being plotted.

– Make sure that your PDFs integrate to one.

ACKNOWLEDGEMENTS
To be submitted to the Astro-Pedantic Journal on April 1, 2020 for a bit of levity in these unsettling times.