The Complete structure of the nonlinear $W_4$ and $W_5$ algebras from quantum Miura transformation

Chuan-Jie Zhu

International School for Advanced Studies (SISSA/ISAS)
and
INFN, Sezione di Trieste
via Beirut 2-4, I-34100 Trieste, Italy

ABSTRACT

Starting from the well-known quantum Miura transformation for the Lie algebra $A_n$, we compute explicitly the OPEs for $n = 3$ and 4. The primary fields with spin 3, 4 and 5 are found (for general $n$). By using these primary fields and the OPEs from quantum Miura transformation, we derive the complete structure of the nonlinear $W_4$ and $W_5$ algebras.
It is known that the quantum Miura transformation for the Lie algebra $A_n \simeq sl(n + 1)$ gives a quadratic nonlinear algebra \[1\]. This algebra is believed to be identical with the nonlinear extended conformal algebra $W_{n+1}$, generated by fields $W_k$’s with the integer $k$ ranging from 2 to $n + 1$. For $n = 1$ and $n = 2$, this gives the Virasoro algebra and the well-known Zamolodchikov’s nonlinear $W_3$ algebra \[1\]. For the general case such identification is not established explicitly. The problem with this identification comes from the fact that the basis fields in the quantum Miura transformation are not primary fields and the higher spin fields in $W_n$ are all primary fields (by definition). It is still an important open problem to find a primary basis in the quantum Miura transformation. Given the difficulty of this problem, in this paper we will establish such identification for $W_4$ and $W_5$ (commonly known as $W_4$ and $W_5$ algebras) by explicitly computing the operator product expansions (OPEs). In another word we will derive the complete structure of the nonlinear $W_4$ and $W_5$ algebras directly from the quantum Miura transformation. In fact the structure of the $W_4$ algebra is known in literature \[3,4\]. So our derivation serves as a non-trivial check to their results. The method we used was then applied to derive the more complicated $W_5$ algebra. As a first remark we note that most of our computations are done by computer symbolic calculation \textit{Mathematica} \[5\]. There exists also a \textit{Mathematica} package for computing and simplifying OPEs \[6\], but I didn’t make use of it in this paper.

1. The Quantum Miura Transformation

Let \{\vec{\varepsilon}_i, \ i = 1, 2, \cdots, n + 1\} be a set of vectors in an $n$-dimensional space. They are normalized as
\[
\vec{\varepsilon}_i \cdot \vec{\varepsilon}_j = \delta_{ij} - \frac{1}{n + 1},
\]
and satisfy the constraint
\[
\sum_{i=1}^{n+1} \vec{\varepsilon}_i = 0.
\]

1) I would like to thank A. Ganchev and De O. M. Werneck for giving me a copy of this reference and the file of the package.
Then the quantum Miura transformation is defined as

\[
R_{n+1}(z) = : (a \partial_z + \tilde{\varepsilon}_1 \cdot \partial_z \tilde{\phi}(z)) \cdots (a \partial_z + \tilde{\varepsilon}_{n+1} \cdot \partial_z \tilde{\phi}(z)) : = \sum_{k=0}^{n+1} U_k(z) (a \partial_z)^{n+1-k}.
\]

(3)

where \( : \) denotes normal ordering and \( a \) is a free parameter. We have for example

\[
U_0(z) = 1,
\]

\[
U_1(z) = \sum_{i=1}^{n+1} \tilde{\varepsilon}_i \cdot \partial_z \tilde{\phi}(z) = 0,
\]

(4)

\[
U_2(z) = \sum_{i<j} \tilde{\varepsilon}_i \cdot \partial_z \tilde{\phi}(z) \tilde{\varepsilon}_j \cdot \partial_z \tilde{\phi}(z) + a \sum_i (i-1) \tilde{\varepsilon}_i \cdot \partial_z^2 \tilde{\phi}(z).
\]

For general discussion about the fields \( U_k(z) \)’s and their algebras, please see refs. [1, 7]. The fact we will explicitly verified (for \( n = 3, 4 \)) is that the fields \( U_k(z) \)’s satisfy an algebra with quadratic defining relations

\[
U_k(z) U_l(w) = \sum_{m \geq 2} \frac{1}{(z-w)^m} \sum_{p+q=k+l-m} C_{kl}^{pq}(m) : U_p(z) U_q(w) : + : U_k(z) U_l(w),
\]

(5)

where the coefficient \( C \)'s are algebraic in \( a \).

2. The fundamental OPEs

To my knowledge the coefficient \( C \)'s are not known explicitly in general. I suspect if ref. [8] (I don’t have a copy of this reference) contains some explicit results for these coefficients. Nevertheless ref. [7] do have a general result for \( U_2(z) U_j(w) \) which is given as follows

\[
U_2(z) U_j(w) = \sum_{q=1}^j \frac{c_q}{(z-w)^{q+2}} U_{j-q}(w) + \frac{1}{(z-w)^2} U_j(z) + \frac{j-1}{(z-w)^2} U_j(w) + : U_2(z) U_j(w) :.
\]

(6)
where \( c_q \)'s are given by

\[
c_q = \frac{(n + 1 - j + q)!}{(n + 1 - j)!} \left( j - 1 + \frac{(q - 1)}{2} \left( \frac{1}{(n + 1)a^2} + n \right) \right) a^q. \tag{7}
\]

To my knowledge the other general formula given in the same reference for \([U_3(0), U_j(k)]\) is not sufficient to give the OPE \(U_3(z)U_j(w)\). Because of the incompleteness of these results, we will compute all the OPEs explicitly for \(n = 3\) and \(4\). The computation is based on the following contraction rule

\[
\partial_z \phi_i(z) \partial_w \phi_j(w) = -\frac{\delta_{ij}}{(z - w)^2} + : \partial_z \phi_i(z) \partial_w \phi_j(w) :. \tag{8}
\]

The explicit realization of \(\varepsilon_i\)'s is not needed. All we needed are the relations (1) and (2). To compute the OPEs between the various \(U_k(z)\)'s we first compute

\[
(a\partial_z + \varepsilon_{n+1} \cdot \partial_z \phi(z))U_k(w) \equiv U_k^{n+1}(z, w), \tag{9}
\]

and then

\[
(a\partial_z + \varepsilon_n \cdot \partial_z \phi(z))U_k^{n+1}(z, w) \equiv U_k^n(z, w), \tag{10}
\]

and etc. In each step of the above computation, there involves only one contraction or differentiation. This can be easily done in Mathematica. The end result of this recursive computation gives

\[
U_k^1(z, w) = \sum_{l=0}^{n+1} U_l(z)U_k(w)(a\partial_z)^{n+1-l}. \tag{11}
\]

So the coefficient of the differential \((a\partial_z)^{n+1-l}\) gives the OPE \(U_l(z)U_k(w)\).
For $n = 3$ by doing the above computation explicitly we have

\[
U_2(z)U_3(w) \sim \frac{3(1 + 20a^2)}{2(z-w)^4} + \left( \frac{2}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) U_2(w) + \tilde{\Lambda}_1(w) + (z-w) P_1(w) + (z-w)^2 \tilde{\Lambda}_2(w),
\]

\[
U_2(z)U_4(w) \sim \frac{6a(1 + 20a^2)}{(z-w)^5} + \frac{4a}{(z-w)^3} U_2(w) + \left( \frac{3}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) U_3(w) + \tilde{\Lambda}_3(w) + (z-w) \tilde{\Lambda}_4(w),
\]

\[
U_2(z)U_5(w) \sim \frac{9a^2(1 + 20a^2)}{(z-w)^6} + \frac{(1 + 36a^2)}{4(z-w)^4} U_2(w) + \frac{3a}{(z-w)^3} U_3(w) + \left( \frac{4}{(z-w)^2} + \frac{1}{z-w} \partial_w \right) U_4(w) + \tilde{\Lambda}_5(w),
\]

\[
U_3(z)U_3(w) \sim -\frac{(1 + 20a^2)(1 + 36a^2)}{(z-w)^6} - \frac{2(1 + 12a^2)}{(z-w)^4} U_2(w) + \frac{1}{(z-w)^3} P_2(w) - \frac{1}{(z-w)^2} \left( 6a^2 \partial_w^2 U_2(w) - 2a \partial_w U_3(w) - 4U_4(w) + \tilde{\Lambda}_1(w) \right) + \frac{1}{z-w} P_3(w) + \tilde{\Lambda}_6(w),
\]
\[ U_3(z)U_4(w) \sim - \frac{3a(1 + 20a^2)(1 + 24a^2)}{(z - w)^7} - \frac{5a(1 + 12a^2)}{(z - w)^5}U_2(w) \]
\[ - \frac{1}{(z - w)^4} \left( a(1 + 12a^2)\partial_w U_2(w) + \frac{3}{2}(1 + 8a^2)U_3(w) \right) \]
\[ - \frac{1}{(z - w)^3} \left( 6a^3\partial_w^2 U_2(w) + \frac{1}{2}\partial_w U_3(w) + 4aU_4(w) + a\Lambda_1(w) \right) \]
\[ + \frac{1}{(z - w)^2} \left( \frac{a}{12}(1 - 24a^2)\partial_w^3 U_2(w) + 2a\partial_w U_4(w) - \frac{a}{2}\partial_w \Lambda_1(w) - \frac{1}{2}\Lambda_3(w) \right) \]
\[ - \frac{1}{z - w} \left( \frac{1}{2}a^3\partial_w^4 U_2(w) - a\partial_w^2 U_4(w) + a\Lambda_2(w) + \frac{1}{2}\Lambda_4(w) \right), \]

\[ U_4(z)U_4(w) \sim \frac{3(1 + 20a^2)(5 + 180a^2 + 2016a^4)}{32(z - w)^8} + \frac{5(1 + 12a^2)^2}{4(z - w)^6}U_2(w) + \frac{1}{(z - w)^5}P_4(w) \]
\[ + \frac{1}{(z - w)^4} \left( \frac{3}{4}a^2(7 + 60a^2)\partial_w^2 U_2(w) - 3a(1 + 6a^2)\partial_w U_3(w) \right) \]
\[ + 3(1 + 4a^2)U_4(w) + \frac{5 + 36a^2}{8}\Lambda_1(w) \right) + \frac{1}{(z - w)^3}P_5(w) + \frac{1}{z - w}P_6(w) \]
\[ + \frac{1}{(z - w)^2} \left( \frac{a^2}{16}(1 + 60a^2)\partial_w^4 U_2(w) - 3a^3\partial_w^3 U_3(w) + \frac{1}{2}(1 + 6a^2)\partial_w^2 U_4(w) \right) \]
\[ - \frac{3}{2}a\partial_w \Lambda_3(w) - \frac{1}{8}(1 - 36a^2)\Lambda_2(w) + 3a\Lambda_4(w) + 2\Lambda_5(w) - \frac{3}{4}\Lambda_6(w) \].

Notice that the above formulae are not written as the form in (5). All the functions appeared in the right hand side are functions of \( w \) only. Nevertheless one can explicitly verify that the above OPEs fit the form in (5). Also we have included some regular terms in the OPEs. They are included in order to define the composite fields \( \Lambda_i(w) \)'s. For the \( W_4 \) algebra only composite fields with spin up to 6 are needed. The other terms denoted as \( P_i(w) \) are not given explicitly. They can easily be obtained from the symmetric property of the OPEs: \( U_i(z)U_i(w) = U_i(w)U_i(z) \).

For \( n = 4 \), similar OPEs between \( U_k(z) \)'s are also calculated but we will not give the explicit results here because the expressions are too long to write them. For illustration purpose we give only the OPEs of \( U_2(z) \) with other fields. These results are in agreement with the general formula (6). We have
Here we included only two regular terms just to fix the definition of $\tilde{\Lambda}_1(w)$ and $\tilde{\Lambda}_4(w)$. These two fields are needed to form primary fields from $U_i(w)$'s. In the next two sections we will use these OPEs (and the ones not explicitly written here) to derive the complete structure of the $W_4$ and $W_5$ algebras.

4. The algebra $W_4$

To begin with let us recall some generalities about conformal field theory. (See ref. [9] for a recent review.) A primary field $\phi_h(w)$ with dimension (spin) $h$ has the following OPE with the stress energy tensor $T(z)$:

$$T(z)\phi_h(w) \sim \left(\frac{h}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)\phi_h(w), \quad (14)$$

and the OPE of $T(z)$ with itself is

$$T(z)T(w) \sim \frac{c}{2(z-w)^4} + \left(\frac{2}{(z-w)^2} + \frac{1}{z-w}\partial_w\right)T(w). \quad (15)$$

Here $c$ is a free parameter called center charge. From (12) and (13) we can identify $U_2(z)$ with the stress energy tensor but the fields $U_3(w)$ and $U_4(w)$ are not primary.
fields with spin 3 and 4. Later we will redefine $U_i(w)$’s by adding some terms from the descendant fields of $U_2(z) \equiv T(z)$ and other fields such that the new fields are primary fields. To completely fix the freedom of redefining the descendant fields, we will ask all the fields appearing in the OPEs to be quasi-primary fields. This is so because the OPE of two quasi-primary fields has some nice properties [10]. The OPE of two quasi-primary fields $\phi^i$ and $\phi^j$ with integer conformal dimensions $h_i$ and $h_j$ takes the following general form:

$$
\phi^i(z)\phi^j(w) = \frac{\gamma^{ij}}{(z-w)^{h_i+h_j}} + \sum_k C^{ij}_k \sum_{n=0}^{\infty} \frac{a^{(ijk)}_n}{n!} \frac{\partial_w \phi^k(w)}{(z-w)^{h_i+h_j-h_k-n}}, \quad (16)
$$

where $k$ denotes all the possible quasi-primary fields occurring in the OPE (not necessarily containing singular parts), $\gamma^{ij}$ plays the role of a metric on the space of quasi-primary fields and $a^{(ijk)}_n$ are given by

$$
a^{(ijk)}_n = \frac{(h_i-h_j+h_k)_n}{(2h_k)_n}, \quad (17)
$$

with the notation $(x)_n = \Gamma(x+n)/\Gamma(x)$. Notice that for $h_i-h_j+h_k \leq 0$, the summation over $n$ truncates to a finite summation.

Because of this general formula we can symbolically write the OPE of $\phi^i(z)\phi^j(w)$ as

$$
\phi^i(z)\phi^j(w) \simeq \frac{\gamma^{ij}}{(z-w)^{h_i+h_j}} + \sum_k C^{ij}_k \frac{\phi^k(w)}{(z-w)^{h_i+h_j-h_k-n}}, \quad (18)
$$

What this formula actually means is (16). For example, the OPE $T(z)T(w)$ expanded up to $(z-w)^2$ is

$$
T(z)T(w) \sim 2 \left( \frac{1}{(z-w)^2} \frac{1}{z-w} \partial_w + \frac{3}{20} \partial_w^2 + \frac{1}{30} (z-w) \partial_w^3 + \frac{1}{168} (z-w)^2 \partial_w^4 \right) T(w)
+ \left( 1 + \frac{1}{2} (z-w) \partial_w + \frac{5}{36} (z-w)^2 \partial_w^2 \right) A_1(w) + (z-w)^2 A_2(w) + \frac{c/2}{(z-w)^4}, \quad (19)
$$

2) We use “$\simeq$” to signify this writing, whereas “$=$” or “$\sim$” is used for other purpose.
which can be symbolically written as

\[ T(z)T(w) \simeq \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \Lambda_1(w) + (z-w)^2 \Lambda_2(w). \quad (20) \]

Here the fields \( \Lambda_1(w) \) and \( \Lambda_2(w) \) are quasi-primary fields with spin 4 and 6 respectively. They are related to the fields \( \tilde{\Lambda}_1(w) \) and \( \tilde{\Lambda}_2(w) \) in (12). One should redefine these fields such that the OPEs takes the general form (16).

As a technical remark we mention that in (12) we give only the basic OPEs. In fact we will also need the OPEs between \( U_i(z) \)'s and \( \tilde{\Lambda}_1(w) \) and \( \tilde{\Lambda}_1(z) \) with itself. These can be computed easily by using the Wick theorem for the contraction involving composite fields. See [9] for details.

Equipped with these general knowledges we now start our final mission. Setting \( U_2(z) \equiv T(z) \), we define new fields \( W(w) \) and \( U(w) \) as follows

\[
W(w) = U_3(w) + c_1 \partial_w T(w), \\
U(w) = U_4(w) + c_2 \partial_w U_3(w) + c_3 \partial^2_w T(w) + c_4 \tilde{\Lambda}_1(w). \quad (21)
\]

The primary field conditions are\(^3\)

\[
T(z)W(w) \simeq \frac{3}{(z-w)^2} W(w) + \Lambda_3(w) + (z-w) \Lambda_4(w), \\
T(z)U(w) \simeq \frac{4}{(z-w)^2} U(w) + \Lambda_5(w), \quad (22)
\]

where the fields \( \Lambda_i(w) \)'s are quasi-primary. By using the OPEs in (12) we get the

\[ T(z)W(w) \sim 3 \left( \frac{1}{(z-w)^2} + \frac{1}{z-w} \partial_w + \frac{1}{14} \partial^2_w + \frac{1}{84} (z-w) \partial^3_w \right) W(w) \]

\[ + \left( 1 + \frac{2}{5} (z-w) \partial_w \right) \Lambda_3(w) + (z-w) \Lambda_4(w), \]

\[ T(z)U(w) \sim 4 \left( \frac{1}{(z-w)^2} + \frac{1}{z-w} \partial_w + \frac{1}{24} \partial^2_w \right) U(w) + \Lambda_5(w). \]

3) Notice that these formulas actually mean the following
following unique solution for \( c_i \)’s

\[
\begin{align*}
    c_1 &= -a, \\
    c_2 &= -\frac{a}{2}, \\
    c_3 &= \frac{-30 + 19c + 2c^2}{120(22 + 5c)}, \\
    c_4 &= \frac{(2 - 9c)}{20(22 + 5c)},
\end{align*}
\]

(23)

where \( c \) is the central charge: \( c = 3(1 + 20a^2) \). This result is in agreement with the general result for the spin-3 and 4 primary fields given in [11]. Later in the next section we will give the general formula for spin-3, 4 and 5 primary fields. The fields \( \Lambda_i(w) \) are related to \( \tilde{\Lambda}_i(w) \). The explicit relations are not quite illuminating to merit displaying.

Having found the primary fields we are now ready to compute the other three OPEs. The computation is just a complicated algebraic calculation which can be done by computer. The final results are

\[
\begin{align*}
\frac{W(z)W(w)}{-(7 + c)/10} &\simeq \frac{c/3}{(z - w)^6} + \frac{2}{(z - w)^4} T(w) + \frac{32}{(22 + 5c)(z - w)^2} \frac{\Lambda_1(w)}{U(w)} - \frac{10}{(7 + c)(z - w)^2} - \frac{10}{(7 + c)} \Lambda_6(w), \\
\frac{W(z)U(w)}{10(22 + 5c)} &\simeq -\frac{(c + 2)(7c + 114)}{10(22 + 5c)} \frac{W(w)}{(z - w)^4} - \frac{26(c + 2)}{5(22 + 5c)(z - w)^2} \frac{\Lambda_3(w)}{U(w)} - \frac{(7c + 114)}{10(22 + 5c)} \frac{\Lambda_4(w)}{(z - w)^2}, \\
\frac{U(z)U(w)}{300(22 + 5c)} &\simeq \frac{c/4}{(z - w)^8} + \frac{2}{(z - w)^6} T(w) + \frac{42}{(22 + 5c)(z - w)^4} \frac{\Lambda_1(w)}{U(w)} + \frac{90(c^2 + c + 218)}{(2 + c)(7c + 114)(z - w)^4} \frac{U(w)}{\Lambda_5(w)} + \frac{3(19c - 582)}{10(2 + c)(7c + 114)} \frac{\Lambda_2(w)}{(z - w)^2} + \frac{120}{(2 + c)(7c + 114)} \frac{\Lambda_5(w)}{(z - w)^2} + \frac{225(22 + 5c)}{96(9c - 2)} \frac{\Lambda_6(w)}{(z - w)^2} + \frac{96(9c - 2)}{(2 + c)(22 + 5c)(7c + 114)} \frac{\Lambda_7(w)}{(z - w)^2},
\end{align*}
\]

(24)

where the field \( \Lambda_7(w) \) (which is quasi-primary) is defined as the regular part in the
following OPE
\[
T(z)\Lambda_1(w) \simeq \frac{(22 + 5c)}{5} \frac{T(w)}{(z-w)^1} + \frac{4}{(z-w)^2} \Lambda_1(w) + \Lambda_7(w).
\] (25)

Up to some normalization factors for the fields \(W(w)\) and \(U(w)\), these are the OPEs given in [3,4] for the nonlinear \(W_4\) algebra.

5. The \(W_5\) algebra

One can repeat all the above computations for \(W_5\). Before putting \(n = 4\) let us study the problem of finding primary fields with spin 3, 4 and 5. To solve the primary field condition we need also the OPEs \(U_2(z)\tilde{\Lambda}_1(w)\) and \(U_2(z)\tilde{\Lambda}_4(w)\). These OPEs are found to be \((T(z) = U_2(z))\)

\[
T(z)\tilde{\Lambda}_1(w) \sim \frac{3c}{(z-w)^6} + \frac{(8 + c)}{(z-w)^4} T(w) + \frac{3\partial_w T(w)}{(z-w)^2} + \left(\frac{4}{(z-w)^2} + \frac{1}{z-w} \partial_w\right) \tilde{\Lambda}_1(w),
\]

\[
T(z)\tilde{\Lambda}_4(w) \sim \frac{7(n - 1)ac}{(z-w)^7} + \frac{(n + 1)(10 + c)a}{(z-w)^5} T(w) + \frac{(24 + c)}{2(z-w)^4} U_3(w)
\]

\[+ \frac{3}{(z-w)^3} \partial_w U_3(w) + \frac{2(n - 1)a}{(z-w)^3} \tilde{\Lambda}_1(w) + \left(\frac{5}{(z-w)^2} + \frac{1}{z-w} \partial_w\right) \tilde{\Lambda}_4(w),\] (26)

where \(c\) is the central charge: \(c = n(1 + (n + 1)(n + 2)a^2)\). From this equation and eq. (6) one can prove that the following fields are primary fields:

\[
W(w) = U_3(w) - \frac{1}{2} (n - 1)a \partial_w T(w),
\]

\[
U(w) = U_4(w) - \frac{1}{2} (n - 2) a \partial_w U_3(w) + \frac{(n - 1)(n - 2)}{4n(n + 1)(n + 2)} \frac{(2c^2 + (16 + n)c - 10n)}{(22 + 5c)} \partial_w^2 T(w)
\]

\[+ \frac{(n - 1)(n - 2)}{2n(n + 1)(n + 2)} ((12 + 5n)c - 2n) \tilde{\Lambda}_1(w),\]

\[
V(w) = U_5(w) - \frac{1}{2} (n - 3) a \partial_w U_4(w) + \frac{3(n - 2)(n - 3)}{4n(n + 1)(n + 2)} \frac{(c^2 + (22 + n)c - 18n)}{(114 + 7c)} \partial_w^2 U_3(w)
\]

\[+ \frac{3(n - 1)(n - 2)(n - 3)}{72n(n + 1)(n + 2)} \frac{(2c^2 + (64 + 9n)c - 42n)a}{(114 + 7c)} \partial_w^3 T(w)
\]

\[+ \frac{(n - 2)(n - 3)}{n(n + 1)(n + 2)} \frac{(20 + 7n)c - 6n}{(114 + 7c)} \left(\frac{1}{4} (n - 1)a \partial_w \tilde{\Lambda}_1(w) - \tilde{\Lambda}_4(w)\right).\] (27)
Surely for \( n = 3 \) we get back eq. (11). For \( n = 4 \) the above formulae gave all the primary fields. Explicitly these primary fields are

\[
T(w) = U_2(w), \\
W(w) = U_3(w) - \frac{3a}{2} \partial_w U_2(w), \\
U(w) = U_4(w) - a \partial_w U_3(w) + \frac{(c^2 + 10c - 20)}{40(22 + 5c)} \partial_w^2 U_2(w) \frac{(4c - 1)}{5(22 + 5c)} \tilde{\Lambda}_1(w), \\
V(w) = U_5(w) - \frac{a}{2} \partial_w U_4(w) + \frac{(c^2 + 26c - 72)}{80(114 + 7c)} \partial_w^2 U_3(w) \\
- \frac{a(c^2 + 50c - 84)}{240(114 + 7c)} \partial_w^3 T(w) + \frac{3a(2c - 1)}{10(114 + 7c)} \partial_w \tilde{\Lambda}_1(w) - \frac{2(2c - 1)}{5(114 + 7c)} \tilde{\Lambda}_4(w).
\]

From these primary fields we can compute the \( W_5 \) algebra. The only difficulty is that one should compute a lot of OPEs involving composite fields and then introduce more composite fields until one exhausts all possible composite fields with highest spin 8. In total there are 25 independent composite fields which are needed in the OPEs of \( W_5 \). All of them are defined to be quasi-primary. In the course of presenting the various OPEs we will also include some regular terms in the OPEs to define some of these composite fields. The definition of the rest composite fields will be given after we give all the fundamental OPEs. Firstly the OPEs with the stress energy tensor \( T(z) \) state that all the fields \( W(w), U(w) \) and

---

4) These fields are constructed from the following normal ordered products:

- spin 4: \( T^2 \);
- spin 5: \( TW \);
- spin 6: \( TT'', TW', TU, W^2, T^3 \);
- spin 7: \( TW'', TU', TV, WU, T^2W \);
- spin 8: \( TT^{(4)}, TW^{(3)}, TU'', TV', WW'', WU', WV, U^2, T^2T'', T^2W', T^2U, TW^2, T^4 \).
\(V(w)\) are primary fields:

\[
T(z)T(w) \simeq \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2} T(w) + \sum_{i=1}^{3} (z-w)^i \Lambda_i(w),
\]

\[
T(z)W(w) \simeq \frac{3}{(z-w)^2} W(w) + \sum_{i=0}^{3} (z-w)^i \Lambda_{i+4}(w),
\]

\[
T(z)U(w) \simeq \frac{4}{(z-w)^2} U(w) + \sum_{i=0}^{2} (z-w)^i \Lambda_{i+8}(w),
\]

\[
T(z)V(w) \simeq \frac{5}{(z-w)^2} V(w) + \Lambda_{11}(w) + (z-w) \Lambda_{12}(w).
\]

The regular terms in the above OPEs define the composite fields \(\Lambda_i(w)\) \((i = 1, 2, \cdots, 12)\) which are all quasi-primary. Secondly the OPEs of \(W(z)\) with other fields get a little bit complicated but still comprehensible. They are

\[
\frac{W(z)W(w)}{c_3} \simeq \frac{c}{3(z-w)^6} + \frac{2}{(z-w)^4} T(w) - \frac{320}{(68+7c)} \frac{U(w)}{(z-w)^2}
\]

\[
+ \frac{32}{(22+5c)} \Lambda_1(w) + \Lambda_{13}(w) + (z-w)^2 \Lambda_{14}(w),
\]

\[
W(z)U(w) \simeq - \frac{4(2+c)(23+c)}{5(22+5c)} \frac{W(w)}{(z-w)^4} - \frac{208(2+c)(23+c)}{5(22+5c)(114+7c)} \frac{\Lambda_4(w)}{(z-w)^2}
\]

\[
+ \frac{5}{(z-w)^2} V(w) - \frac{4(23+c)}{5(22+5c)} \frac{\Lambda_5(w)}{z-w} + \Lambda_{15}(w) + (z-w) \Lambda_{16}(w),
\]

\[
W(z)V(w) \simeq - \frac{(116+3c)(22+5c)}{20(114+7c)} \frac{U(w)}{(z-w)^4} + \frac{3(844-43c)}{1000(114+7c)} \frac{\Lambda_2(w)}{(z-w)^2}
\]

\[
- \frac{2(22+3c)}{(114+7c)} \frac{\Lambda_8(w)}{(z-w)^2} + \frac{3(2c-1)(68+7c)}{200(114+7c)} \frac{\Lambda_{13}(w)}{(z-w)^2}
\]

\[
- \frac{2(22+191c)}{25(22+5c)(114+7c)} \frac{\Lambda_{19}(w)}{(z-w)^2} - \frac{2(116+3c)}{5(114+7c)} \frac{\Lambda_9(w)}{z-w} + \Lambda_{17}(w),
\]

where \(c_3 = -\frac{68+7c}{80}\). As before some regular terms are given explicitly in order to set the definition of the fields \(\Lambda_i(w)\) \((i = 13, \cdots, 17)\) which are all quasi-primary. The definition of the spin-6 quasi-primary fields \(\Lambda_{19}(w)\) will be given later.
For the last three OPEs we will write them one by one. Firstly \(U(z)U(w)\) is given by

\[
\frac{U(z)U(w)}{c_4} \simeq \frac{c}{4(z-w)^8} + \frac{2}{(z-w)^6} T(w) + \frac{90(128 - 70c - c^2)}{(2 + c)(23 + c)(68 + 7c)(z-w)^3} U(w) + \frac{42}{(22 + 5c)(z-w)^4} \Lambda_1(w) + \frac{1}{(z-w)^2} \left( \frac{3(19c^2 - 362c - 7496)}{10(2 + c)(23 + c)(68 + 7c)} \Lambda_2(w) \right) + \frac{120(118 - 7c)}{(2 + c)(23 + c)(68 + 7c)} \Lambda_8(w) + \frac{9(22 + 5c)}{2(2 + c)(23 + c)} \Lambda_{13}(w) + \frac{72(38 + 3c)(4c - 1)}{(2 + c)(23 + c)(22 + 5c)(68 + 7c)} \Lambda_{19}(w) + \Lambda_{18}(w),
\]

(31)

where \(c_4 = \frac{(2 + c)(23 + c)(68 + 7c)}{300(22 + 5c)}\) and the regular term defines the quasi-primary field \(\Lambda_{18}(w)\). Secondly \(U(z)V(w)\) is given by

\[
U(z)V(w) \simeq \frac{(2 + c)(23 + c)(116 + 3c)}{100(114 + 7c)} \frac{W(w)}{(z-w)^6} + \frac{(23 + c)(116 + 3c)}{75(114 + 7c)} \frac{\Lambda_5(w)}{(z-w)^3} + \frac{1}{(z-w)^4} \left( \frac{33(2 + c)(23 + c)(116 + 3c)}{50(114 + 7c)^2} \Lambda_4(w) + \frac{(70272 + 9340c + 204c^2 + 11c^3)}{4(22 + 5c)(114 + 7c)} V(w) \right) + \frac{1}{(z-w)^2} \left( \frac{(7796 + 1196c + 29c^2)}{(22 + 5c)(114 + 7c)} \Lambda_{11}(w) - \frac{(334 + 37c)}{5(114 + 7c)} \Lambda_{15}(w) \right) + \frac{(2 + c)(1224c^2 + 23921c - 28834)}{25(22 + 5c)(114 + 7c)^2} \Lambda_{21}(w) + \frac{(2 + c)(297c^2 - 4934c - 231256)}{2100(22 + 5c)(114 + 7c)} \Lambda_{16}(w) \right) + \frac{1}{z-w} \left( \frac{(13320 + 262c + 11c^2)}{5(22 + 5c)(114 + 7c)} \Lambda_{12}(w) - \frac{3(116 + 3c)}{5(114 + 7c)} \Lambda_{16}(w) \right) + \frac{6(2 - 9c)(23 + c)(116 + 3c)}{25(22 + 5c)(114 + 7c)^2} \Lambda_{22}(w) + \frac{3(c - 28)(23 + c)(116 + 3c)}{125(114 + 7c)^2} \Lambda_{17}(w) \right).
\]

(32)

Finally the last and the most complicated OPE \(V(z)V(w)\) is given by
\[
\frac{V(z)V(w)}{c_5} \approx \frac{c}{5(z-w)^{10}} + \frac{2}{(z-w)^8}T(w) + \frac{52}{(22 + 5c) (z-w)^6} \Lambda_1(w) \\
+ \frac{60(70272 + 9340c + 204c^2 + 11c^3)}{(2 + c)(23 + c)(68 + 7c)(114 + 7c)(z-w)^6} U(w) \\
+ \frac{1}{(z-w)^4} \left( \frac{3(1507824 + 248948c + 14880c^2 + 181c^3)}{2(2 + c)(23 + c)(116 + 3c)(114 + 7c)} \Lambda_{13}(w) \\
+ \frac{24(1148c^4 + 86853c^3 + 1942364c^2 + 14490156c - 3744688)}{(2 + c)(23 + c)(116 + 3c)(22 + 5c)(68 + 7c)(114 + 7c)} \Lambda_{19}(w) \\
+ \frac{1491c^4 + 55276c^3 - 1884932c^2 - 79552928c - 747091776}{10(2 + c)(23 + c)(116 + 3c)(68 + 7c)(114 + 7c)} \Lambda_{2}(w) \\
+ \frac{120(3767568 + 452876c + 11520c^2 + 187c^3)}{(2 + c)(23 + c)(116 + 3c)(68 + 7c)(114 + 7c)} \Lambda_{8}(w) \right) + \frac{1}{(z-w)^2} \left( \frac{4(11c^2 - 306c - 13656)}{(2 + c)(116 + 3c)(114 + 7c)} \Lambda_{14}(w) - \frac{48000}{(2 + c)(23 + c)(68 + 7c)} \Lambda_{17}(w) \\
+ \frac{3(2 + c)(23 + c)(116 + 3c)(22 + 5c)(68 + 7c)(114 + 7c)}{40(609c^4 - 29492c^3 - 1718284c^2 - 49796224c - 465449792)} \Lambda_{10}(w) \\
+ \frac{64(114 + 7c)}{(116 + 3c)(22 + 5c)} \Lambda_{18}(w) + \frac{48(1 - 2c)(578 + 19c)}{(2 + c)(23 + c)(116 + 3c)(114 + 7c)} \Lambda_{24}(w) \\
+ \frac{768(10972 - 84704c + 171793c^2 + 17652c^3 + 504c^4)}{(2 + c)(23 + c)(116 + 3c)(22 + 5c)^2(68 + 7c)(114 + 7c)} \Lambda_{25}(w) + C_3 \Lambda_3(w) \right) ,
\]

where \(c_5\) and the other two big coefficients are given by

\[
c_5 = -\frac{(2 + c)(23 + c)(116 + 3c)(68 + 7c)}{24000(114 + 7c)},
\]

\[
C_3 = \frac{8(5687552448 - 4443765376c - 535589308c^2 - 13386012c^3 + 236551c^4 + 4165c^5)}{175(2 + c)(23 + c)(116 + 3c)(22 + 5c)(68 + 7c)(114 + 7c)},
\]

\[
C_{20} = \frac{8(1555590208 - 7472235776c - 1362435108c^2 - 56078572c^3 + 273491c^4 + 37380c^5)}{15(2 + c)(23 + c)(116 + 3c)(22 + 5c)^2(68 + 7c)(114 + 7c)}.
\]

To finish the presentation we also need to give the definition of the other quasi-primary fields \(\Lambda_i(w)\) \((i = 19, \ldots, 25)\). These quasi-primary fields appear in
the regular terms of the OPEs involving quasi-primary fields. We have

\[
T(z)\Lambda_1(w) \simeq 22 + 5 c \frac{T(w)}{(z-w)} + \frac{4}{(z-w)^2}\Lambda_1(w) + \Lambda_{19}(w) + (z-w)^2\Lambda_{20}(w),
\]

\[
W(z)\Lambda_1(w) \simeq 48 \frac{W(w)}{(z-w)^4} + \frac{6}{(z-w)^2}\Lambda_4(w) - \frac{4}{z-w}\Lambda_5(w)
\]

\[
+ \left( \Lambda_{21}(w) + \frac{64}{35}\Lambda_6(w) \right) + (z-w)\Lambda_{22}(w),
\]

\[
T(z)\Lambda_8(w) \simeq 24 + c \frac{T(w)}{(z-w)^4} U(w) + \frac{6}{(z-w)^2}\Lambda_8(w) + \Lambda_{23}(w),
\]

\[
T(z)\Lambda_{13}(w) \simeq \frac{46}{63} \frac{T(w)}{(z-w)^6} - \frac{3520}{3(68+7c)} \frac{U(w)}{(z-w)^4} + \frac{6}{(z-w)^2}\Lambda_{13}(w) + \Lambda_{24}(w),
\]

\[
T(z)\Lambda_{19}(w) \simeq \frac{8(22+5c)}{15} \frac{T(w)}{(z-w)^6} + \frac{6}{(z-w)^2}\Lambda_{19}(w) + \Lambda_{25}(w).
\]

(35)

Of course, these OPEs involving quasi-primary fields aren’t all the OPEs needed in the derivation of the \(W_5\) algebras.

**6. Discussion**

From our explicit computation we see that there always exists a unique spin-\(j\) primary field for \(j = 3, 4\) and \(5\). The general formula is given by eq. (27). Here we explain why this is so. For \(U_3(w)\) there are two anomalous terms (the central term \(1/(z-w)^5\) and \(T(w)/(z-w)^3\)) to be cancelled but we only have one freedom by adding \(\partial_w U_2(w)\) to \(U_3(w)\). Nevertheless these two anomalous terms are related as shown by the following general observation: the OPE \((j > 2)\)

\[
T(z)W_j(w) \sim \frac{c_j}{(z-w)^{j+2}} + \left( \frac{j}{(z-w)^2} + \frac{1}{z-w}\partial_w \right) W_j(w),
\]

(36)

satisfies the Jacobi identity:

\[
[[L_m, L_n], W_j(p)] + ([W_j(p), L_m], L_n] - (m \leftrightarrow n) = 0.
\]

(37)

only for \(c_j = 0\). So the vanishing of one anomalous term insures the vanishing of the other term because the original algebra surely satisfies the Jacobi identities.
and redefinition of fields doesn’t spoil this property. For $U_4(w)$ there are four anomalous terms but only three of them are independent. Here we have three terms: $\partial_w U_3(w)$, $\partial_w^2 U_2(w)$ and $\tilde{\Lambda}_1(w)$ to be added to $U_4(w)$. So a unique spin-4 primary field always exists. For higher spin fields a simple counting of freedoms can’t prove the existence and uniqueness of higher spin primary fields.

What we can learn about the general structure of nonlinear $W$-algebras from our explicit results? By looking at the explicit OPEs we can guess about the first few terms in $W_j(z)W_j(w)$ as the following

$$W_j(z)W_j(w) \simeq \frac{c/j}{(z-w)^{2j}} + \frac{2}{(z-w)^{2(j-1)}} T(w) + \frac{2(5j+1)}{(22+5c)(z-w)^{2(j-2)}} \tilde{\Lambda}_1(w) + \ldots.$$  

(38)

where $W_j(w)$ is a spin-$j$ ($j > 2$) primary field. There is nothing special about the first term because it just set the normalization for $W_j(w)$. The second term can be proved by considering the central term in the following Jacobi identity

$$[[W_j(m), W_j(n)], L_p] + ([L_p, W_j(m)], W_j(n)) - (m \leftrightarrow n) = 0.$$  

(39)

The other term in (38) is an extrapolation from our explicit results. It is known to be true in $W(2,\delta)$ algebras [3]. A proof could be found by extending (or just following) their computations.

As a further remark we notice that these three terms in $W_j(z)W_j(w)$ are content independent, meaning that the structure constants only depend on the spin of the primary field $W_j(w)$ and don’t depend on the content of the algebra, i.e. how many basic fields constitute the algebra. We conjecture that these three structure constants are the only content independent ones. This conjecture is supported by our explicit results for the nonlinear $W_4$ and $W_5$ algebras. It is also true in $WB_2$, a nonlinear extended conformal algebra with a spin-4 primary field which is associated with the simple group $B_2$ or $C_2$.

One other aspect of the OPEs of the $W_n$ algebras is that there is a selection rule. This is closely related to the automorphisms of $W$-algebras which was discussed
extensively in ref. [12]. The primary fields fall into two sets: the even set consisting of even spin primary fields and the odd set consisting of odd spin primary fields. The OPEs of $\text{(even)} \times \text{(even)}$ and $\text{(odd)} \times \text{(odd)}$ fields give only even fields and the OPEs of $\text{(even)} \times \text{(odd)}$ fields give only odd fields. Presumably this selection rule is also presented in $W_n$-algebras [12].

The author would like to thank Prof. R. Iengo and Dr. D. P. Li for interesting discussions. This work at SISSA/ISAS was supported by an INFN post-doctoral fellowship.

Note added: After I finish this paper, I became aware of two papers by K. Hornfeck [13, 14] which also studied the $W_5$ algebra by using Jacobi identities. In paper [14], some structure constants are also computed from quantum Miura transformation.
REFERENCES

1. S. L. Lukyanov, Funct. Anal. Appl. 22 (1990) 1
2. A. B. Zamolodchikov, Theor. Math. Phys. 65 (1985) 1205
3. R. Blumenhagen, M. Flohr, A. Kliem, W. Nahm, A. Recknagel and R. Varnhagen, Nucl. Phys. B361 (1991) 255
4. H. G. Kausch and G. M. T. Watts, Nucl. Phys. B354 (1991) 740
5. S. Wolfram, Mathematica, A System for Doing Mathematics by Computer, 2nd edition (Addison-Wesley, 1991)
6. K. Thielemans, Intern. J. Mod. Phys. C2 (1991) 787
7. V. A. Fateev and S. L. Lukyanov, Intern. J. Mod. Phys. A3 (1988) 507
8. V. A. Fateev and S. L. Lukyanov, Additional symmetries and exactly solvable models in two-dimensional conformal field theory,” parts I, II and III, Sov. Sci. Rev. A. Phys. 15 (1990) 1
9. P. Bouwknegt and K. Schoutens, Phys. Reps. 223 (1993) 183
10. W. Nahm, in Recent developments in conformal field theories, eds. S. Randjbar-Daemi, E. Sezgin and J. B. Zuber (World Scientific, 1990); Conformal quantum field theories in two dimensions (World Scientific, to be published)
11. G. M. T. Watts, Nucl. Phys. B320 (1989) 648
12. A. Konecker, Automorphisms of W-algebras and extended rational conformal field theories, preprint BONN-HE-92-37 (November 1992) to be published in Nucl. Phys. B
13. K. Hornfeck, Phys. Lett. B275 (1992) 355
14. K. Hornfeck, W-algebras with set of primary fields of dimensions (3, 4, 5) and (3, 4, 5, 6), preprint KCL-TH-9209 or DFTT-70/92 (December 1992)