Optimal algorithms for linear-algebra by quantum inspiration

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Abstract

Recent results by Harrow et. al. [2], and by Ta-Shma [5], suggest that quantum computers may have an exponential advantage in solving a wealth of linear algebraic problems, over classical algorithms. Building on the quantum intuition of these results, we step back into the classical domain, and explore its usefulness in designing classical algorithms. We achieve an algorithm for solving the major linear-algebraic problems in time $O(n^{\omega+\nu})$ for any $\nu > 0$, where $\omega$ is the optimal matrix-product constant. Thus our algorithm is optimal w.r.t. matrix multiplication, and comparable to the state-of-the-art algorithm for these problems due to [1]. Being derived from quantum intuition, our proposed algorithm is completely disjoint from all previous classical algorithms, and builds on a combination of low-discrepancy sequences and perturbation analysis. As such, we hope it motivates further exploration of quantum techniques in this respect, hopefully leading to improvements in our understanding of space complexity and numerical stability of these problems.

1 Introduction

1.1 General

Recent results by Harrow et. al. [2], and by Ta-Shma [5], suggest that quantum computers may have an exponential advantage in solving a wealth of linear algebraic problems, over classical algorithms. Specifically, [2] present an algorithm that given a linear system of equations $Ax = b$, when $A$ is $O(1)$-sparse (i.e. constant number of non-zeros in each row), produces a quantum state $x$, which implicitly holds the solution to the set of equations. Using this state, one can compute certain predicates on the solution. The algorithm runs in time $O(\kappa \log^2(n)/\varepsilon)$, where $\kappa$ is the conditioning number of $A$, and $\varepsilon$ is the additive error to the solution, thus providing an exponential speed up over the best classical algorithm that one could hope for. A recent result by [5], shows that one can carry out such inversion in logarithmic quantum space, compared to the best known classical space complexity standing at $O(\log^2(n))$. In addition, [5] show how to to perform other linear-algebraic tasks such as SVD, and eigenvalue decomposition in quantum logarithmic space - tasks which are not known to be possible in classical space better than $O(\log^2(n))$.

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A turnkey for both algorithms above is the quantum procedure known as phase estimation [4], where given a unitary matrix, and one of its eigenvectors, it produces an estimate of the corresponding eigenvalue, to arbitrary precision. Here, we step back into the classical domain, and explore this routine in the context of solving the major linear algebraic-problems on a classical computer. We achieve an algorithm whose run-time bit-complexity is \(O(n^{\omega+\nu})\), for any \(\nu > 0\), where \(n^\omega\) is the optimal time for multiplying two integer matrices. Thus essentially, this is an optimal algorithm w.r.t. matrix multiplication, comparable with the state-of-the-art algorithm due to [1].

Our algorithm does not use any of the known classical methods but takes a randomized approach to spectral analysis, building on quantum intuition. On a technical level, we critically use low-discrepancy sequences, appended with linear perturbation analysis, to generate sufficient randomness, at a sufficiently low computational cost. As such, it offers a new route to linear-algebraic approach to spectral analysis, building on quantum intuition. On a technical level, we critically use low-discrepancy sequences, appended with linear perturbation analysis, to generate sufficient randomness, at a sufficiently low computational cost. As such, it offers a new route to linear-algebraic problems, both in respect to run-time analysis, numerical stability, and space complexity.

### 1.2 Main Results

We define eigenvector sampling as a distribution, in which every unique eigenvector is sampled with non-negligible probability, and returned with polynomially small error:

**Definition 1 Eigenvector sampling** Let \(A\) be a Hermitian \(n \times n\) matrix, with \(n\) unique eigenvalues \(\lambda_1, \ldots, \lambda_n\), and corresponding eigenvectors \(v_1, \ldots, v_n\). We say that distribution \(E = E(A, \delta) : \mathbb{C}^n \mapsto [0, 1]\) is an eigenvector sampler of \(A\), if for any \(k \in [n]\) w.p. \(\Omega(1/n)\), the sampler returns a unit vector \(v\), such that \(\|v - v_k\| \leq \delta\).

**Definition 2 Eigenvalue separation**

Let \(A\) be a Hermitian \(n \times n\) matrix, with eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_n\). \(A\) has eigenvalue spacing \(\Delta_\lambda\), if for any \(i \neq j\) we have \(|\{2\pi\lambda_i\} - \{2\pi\lambda_j\}| \geq \Delta_\lambda\) for all \(i \neq j\), where \(\{x\}\) is the fractional part of \(x\).

For example, any positive semidefinite matrix \(A\), with \(\|A\| \leq 1 - \Delta_\lambda\) which also has \(|\lambda_i - \lambda_j| \geq \Delta_\lambda\) for all \(i \neq j\), has eigenvalue separation \(\Delta_\lambda\).

**Theorem 1 Efficient Sampler**

For any \(\nu > 0\), there exists a randomized algorithm running in expected time \(\tilde{O}(n^\omega)\) that for any Hermitian positive-semidefinite matrix \(A, \|A\| = O(1)\), with eigenvalue separation at least \(2^{-\text{polylog}(n)}\), and given \(\delta = \Omega(1/\text{poly}(n))\) samples from eigenvector sampling \(E(A, \delta)\) w.p. \(\Omega(n^{-\nu})\).

Possessing an eigenvector sampler at one’s disposal, running in time \(\tilde{O}(n^\omega)\) does not guarantee, however, that one can generate all eigenvectors in reasonable time, i.e. achieve complete diagonalization at time \(\tilde{O}(n^\omega)\). In fact, the trivial arithmetic bound in this case is even \(O(n^{\omega+1})\). However, we show that in fact one can use a variant of our proposed eigenvector sampler, and run it in "parallel" in \(n\) separate processes, and w.h.p. collect all eigenvectors in \(\tilde{O}(n^\omega)\). Specifically we show:

**Theorem 2 Efficient Diagonalization**

For any \(\nu > 0\), there exists a randomized algorithm running in expected time \(O(n^{\omega+\nu})\) that for any Hermitian positive-semidefinite matrix \(A, \|A\| = O(1)\), with eigenvalue separation \(2^{-\text{polylog}(n)}\) and eigenvectors \(v_1, \ldots, v_n\) and given \(\delta = \Omega(1/\text{poly}(n))\) returns \(n\) vectors \(v'_1, \ldots, v'_n\), such that for all \(k \in [n]\)

\[\|v_k - v'_k\| \leq \delta.\]
Finally, we show that the translation of the spectral analysis problem into product of unitary matrices, is advantageous also in terms of bit-complexity. This is shown in section (3.2.5), thus providing an algorithm of bit complexity \(O(n^{\omega+\nu})\) for eigenvector sampling and complete diagonalization. We note that the algorithm above does not in fact require the matrix to be Hermitian: given a complex matrix \(A\), we can compute the absolute value of its singular values, by running the algorithm on \(A^\dagger A\) and returning the square root of the eigenvalues. Also, our algorithm can be easily generalized to all major linear-algebra operations, namely matrix-inversion, solving linear systems, etc.

1.3 Some Intuition

As stated before, our algorithm builds on the well-known quantum routine called ”phase-estimation” \[4\], which was devised in order to measure the eigenvalue corresponding to a quantum state, which is an eigenvector of some unitary matrix. Specifically, given a complex vector \(v\), the phase estimation routine returns the eigenvalue \(\lambda\) of an \(n \times n\) unitary matrix \(U\), corresponding to \(v\), up to arbitrary precision. Given a Hermitian matrix \(A\), we employ the phase estimation routine w.r.t. its exponentiated matrix \(e^{iA}\), as is usually the case, when one solves the Schrödinger equation. We employ the phase estimation routine as a "filter", essentially extracting the result up to 1 bit of precision, and then "post-select" on measuring a specific value "0" of this bit. In that case, one can check that the routine has the effect proportional to the following linear operator:

\[
\frac{I + U}{2}.
\]

This operator, when raised to some \(p\)-th power has the following effect: any eigenvector corresponding to an eigenvalue that is ”\(\varepsilon\)-close” to the origin of the unit \([0, 1]\) is attenuated by a scalar approximately \((1 - \varepsilon^2)^p\). So if there is just one eigenvector that is, say \(\varepsilon/2\)-close to the origin, whereas all other eigenvectors are at least \(\varepsilon\)-far, then applying the phase estimation operator \(O(1/\varepsilon^2)\) times will result in a constant relative gain of this eigenvector w.r.t. all others. Moreover, iterating the phase-estimation operator, say \(O(\log(n)/\varepsilon^2)\) will result in a polynomial gain of this eigenvector over all other vectors, and essentially the output will be "close" to being comprised of a single eigenvector.

From this simple observation, one could consider the following algorithm: draw a random vector \(v\), and assuming it has equal-magnitude components in all unique eigenvectors, randomly shift \(v\) by a random phase \(e^{2\pi i \theta}v\), and then apply the phase-estimation operator, in the hope of ”filtering-out” a single eigenvector. However, we immediately realize, that even ”reasonable” eigenvalue spacing can be as small as say, \(1/n^3\), one would require choosing the number of rotations to be some \(\text{poly}(n)\), which is prohibitively large, if we wish to attain \(O(n^\omega)\) complexity.

As such, we try to randomize the distribution of the eigenvalues so that it resembles a random sample of a set of i.i.d. uniform variables. We observe that if that were the case, then we could hope that in the above scheme we can sample w.p. \(\Omega(1/n)\) a random eigenvector of \(A\), i.e. achieve an \(E(A, \delta)\) sampler for some \(\delta = 1/\text{poly}(n)\), that runs in time \(O(n^\omega)\), regardless of eigenvalue spacing. Therefore, the randomization process is cardinal.

A first attempt - multiplying \(e^{2\pi i A}\) by some random unitary is undesirable, since the resulting unitary has no immediate connection to the original operator. One could also try to raise \(e^{iA}\) by a very large number, so that its eigenvalues become randomly distributed. Such a result is known for example, if the eigenvalues \(\lambda_i\) are irrational. In other words, multiplying the eigenvalues by a large random number should cause their residuals to converge to a random distribution. However, such
results stated in general as having zero deviation (discrepancy) from uniform i.i.d. distribution, in the limit of large $n$, whereas we are constrained to raising $e^{2\pi i A}$ to a power at most, say $2^{\text{polylog}(n)}$, and by that - to achieve error which is at most $1/\text{poly}(n)$.

We observe however, that actually, the randomization and filtering process can be redesigned so it requires a much weaker form of randomness. In fact, our analysis shows that we need the eigenvalues not to be $n$-wise independent, but even $s$-wise independent for $s = \text{polylog}(n)$. In this case, one can use techniques from the theory of random number generation, to derive the following: if one can make the eigenvalues be sampled independently and uniformly from some not-to-small interval, then w.h.p. it is sufficient to raise $e^{iA}$ to a power which is $\text{polylog}(n)$, and gain the property that any $s$-tuple of eigenvalues is distributed i.i.d. up to some small error, called discrepancy. To achieve approximate independent and uniform sampling of the eigenvalues (to which we have, of course no direct access), we perturbate the matrix by a small Gaussian perturbation $\varepsilon$ which is on one hand, small enough as not to deviate from the input by much, but on the other hand, large enough to achieve the requirement above.

Finally, once we have in place an algorithm that samples uniformly an eigenvector, out of $n$ possible eigenvectors, we arrive at the task of extracting all eigenvectors. Clearly, if each eigenvector is sampled w.p. $\theta(1/n)$, then repeating the algorithm $\theta(n \cdot \log(n))$ times, by the coupon-collector’s bound, would achieve the desirable result, but at too-large a cost. We circumvent this problem, by applying the eigenvector sampling routine ”in parallel” over $n$ copies of the same random vector, each rotated by a random complex phase. We then use fast matrix product to achieve asymptotic complexity, which is essentially the same as that of the eigenvector sampling routine.

The important observation in this case, however, is that using just a random phase between the different vectors, is sufficient to make the sampled eigenvector distributions independent, in some sense. In fact, we show a possibly weaker but sufficient property: whenever some eigenvector $v_i$ is sampled at some column $k$, the probability that it is sampled at another column $k' \neq k$ is at most $\theta(1/n)$, i.e. its marginal probability. This tells us that though correlations between columns may appear in general, in the event that a sampling event happens at some specific column (i.e. eigenvalue $v_i$ lands inside the filter at all trials, and all other eigenvalues miss the filter at least once) it would not happen very often at the other columns.

### 1.4 Previous work

In 2007, Demmel, Dimitriou, and Holtz [1] have shown how to compute a set of linear-algebraic problems, among which are $QR$-decomposition, matrix inversion, and solution of linear systems, in bit-complexity $O(n^{2+\nu})$ for any $\nu > 0$. Thus, essentially providing optimal run-time complexity, w.r.t. the matrix-product algorithm. However, the authors of [1] observe that their algorithm is only weakly stable in the following sense: if there exist two eigenvalues that are very close, then this could affect all eigenvectors, even those whose eigenvalues are well-separated.

In more precise term, given the matrix $A$, denote its upper-left block $A_{1,1}$, and bottom-right block by $A_{2,2}$. By [1] the conditioning of any eigenvector returned is upper-bounded by the minimal eigenvalue of:

$$\sigma_{\text{min}}(A_{1,1} \otimes I - I \otimes A_{2,2}).$$

Choosing some $A_{1,1}, A_{2,2}$ that commute, we have that the above is equal to the minimal difference $\min_{i,j} |\lambda_i^1 - \lambda_j^2|$, where $\lambda_i^1$ is the $i$-th eigenvalue of $A_{1,1}$, and $\lambda_j^2$ is the $j$-th eigenvalue of $A_{2,2}$. We can
further choose \(A_{1,1}, A_{2,2}\) so that exactly two eigenvalues, one of \(A_{1,1}\) and one of \(A_{2,2}\) are at distance \(\varepsilon\), whereas all others are at distance at least \(\sqrt{\varepsilon}\). In this case, the precision value for all eigenvectors will be at most \(\varepsilon\). The authors of \([1]\) then point out, that currently the only way to bypass this problem is to run another algorithm of complexity \(O(n^3)\).

Let us now analyze how our algorithm behaves on this example. If \(A_{1,2} = A_{2,1} = 0\) and the matrices \(A_{1,1}, A_{2,2}\) to have each a well-separated spectrum, then the spacing between any two eigenvalues of \(A\) is \(O(\sqrt{\varepsilon})\)-large except for a single pair of eigenvalues. Applying our algorithm to this example will generate each eigenvector with precision corresponding to the spacing of its eigenvalue: so any eigenvector except for the close pair will have conditioning \(O(\sqrt{\varepsilon})\). This is essentially the optimal situation one could expect: if two eigenvalues are exactly equal - one cannot expect an "informative" eigenvector to be returned. Hence our algorithm resolves the open question posed in \([1]\) in this respect.

## 2 Preliminaries

In this section we present the required technical tools for the analysis of the algorithm. These are centered around eigenvalue perturbation and low-discrepancy sequences, and most importantly the interaction of the two culminating in claim \([1]\). The proofs of these claims are usually deferred to the appendix.

### 2.1 Notation

1. The Hadamard transform: \(H[x, y] = [u, v]\), where \(u_i = \frac{x_i + y_i}{\sqrt{2}}, v_i = \frac{x_i - y_i}{\sqrt{2}}\).

2. \(\omega\) is the smallest constant such that multiplying two \(n \times n\) matrices can be performed in arithmetic complexity \(O(n^\omega)\).

3. For matrix \(A, \|A\|\) is the \(l_2\) operator-norm of the matrix.

4. Given a real number \(x\), we denote by \(\{x\}\) its fractional part in \([0, 1)\). Sometimes we shall use this notation to define sets - so it should be interpreted according to context.

### 2.2 Low-discrepancy sequences

Low discrepancy sequences are an immensely useful tool in random Monte-Carlo methods. Roughly speaking, these are sequences whose number of points that fall into an arbitrary set \(B\) is roughly proportional to the volume of \(B\).

**Definition 3 Multi-dimensional discrepancy**

For integer \(s\), put \(I^s = [0, 1)^s\). Given a sequence \(x = \{x_n\}_{n=1}^N\), with \(x_n \in I^s\) the discrepancy \(D_N(x)\) is defined as:

\[
D_N(x) = \sup_{B \in \mathcal{B}} \left\{ \frac{1}{N} \sum_{n=1}^N \chi_B(x_n) - vol(B) \right\},
\]

where \(\chi_B(x_n)\) is an indicator function which is 1 if \(x_n \in B\) and 0 otherwise, and \(\mathcal{B}\) is a non-empty family of Lebesgue-measurable subsets of \(I^s\).
Here, we shall define \( B \) as the set of all \( s \)-products of intervals \( \prod_{i=1}^{s} [u_{i}, v_{i}] \), with \( [u_{i}, v_{i}] \subseteq [0, 1) \).

Sometimes, given a sequence \( x \), we shall use the notation \( x \) as a distribution: it is the distribution derived by sampling uniformly an element of \( x \). Under this notation, we interpret the discrepancy \( D_{N}(x) \), as the absolute difference in probability that sampling uniformly from \( x \) generates an element in \( B \), and \( \text{vol}(B) \), optimized over all sets \( B \in B \). Our deployment of low-discrepancy sequences, stems however not from Monte Carlo motivation, but rather because they approximate well an independent distribution of uniform random variables.

**Fact 1** Let \( \{g_{n}\}_{n \in [N]} \) be a \( s \)-dimensional sequence \((g_{n} \in I^{s})\) with discrepancy \( D_{N} \). Let \( \{I_{i}\}_{i=1}^{s} \) be a set of \( s \) intervals of \([0, 1)\). For event \( A \) on the space of \( s \) binary variables \( \{x_{i}\}_{i=1}^{s} \), let \( P(A) \) be the probability function induced by setting, for all \( i \in [s] \), the variable \( x_{i} = 1 \) iff \( g_{n,i} \in I_{i} \), as \( n \) is chosen randomly in \([N] \). Then, for any such event \( A \), \( P(A) = P_{U}(A) \pm 2^{s}D_{N} \), where \( P_{U}(A) \) is the probability of \( A \) in case the variables \( x_{i} \) are i.i.d., and \( P(x_{i} = 1) = |I_{i}|. \)

**Proof:** Any such event \( A \) can be expressed as the union \( A = \bigcup A_{i} \) of at most \( 2^{s} \) disjoint events \( A_{i} \cap A_{j} = \emptyset \), where each \( A_{i} \) is of the form \( x_{1} = a_{1}, \ldots , x_{s} = a_{s} \), for \( a_{i} \in \{0, 1\} \). Let \( B_{i} = \prod_{k=1}^{s} [u_{k}, v_{k}] \), where for each \( k \in [s] \), \([u_{k}, v_{k}] = I_{i} \) if \( a_{k} = 1 \) and \([u_{k}, v_{k}] = [0, 1) \setminus I_{i} \), otherwise. Then by definition \( g_{n} \in B_{i} \) iff \( A_{i} \) occurs. Each such \( B_{i} \) is a legal \( s \)-dimensional box in \( I^{s} \), so by the discrepancy of \( \{g_{n}\}_{n \in [N]} \), we have \( P(g_{n} \in B_{i}) = P_{U}(A_{i}) \pm D_{N} \). Therefore, \( P(A) = \sum_{i} P(A_{i}) = \sum_{i} P(g_{n} \in B_{i}) = \sum_{i} P_{U}(A_{i}) \pm D_{N} = P_{U}(A) \pm 2^{s}D_{N}. \)

The above claim easily generalizes to sampling independently from several low-discrepancy sequences, as follows:

**Corollary 1** Let \( g_{1}, \ldots , g_{t} \) be \( t \) independent random samples from \( t \) \( s \)-dimensional sequences, each of discrepancy at most \( D_{N} \). Let \( \{I_{i}\}_{i=1}^{s} \) be a set of \( s \) intervals of \([0, 1)\). For event \( A \) on the space of \( s \cdot t \) binary variables \( \{x_{i,j}\}_{i,j} \) let \( P(A) \) be the probability of \( A \), induced by setting, for all \( i \in [s], j \in [t], x_{i,j} = 1 \) iff \( \{g_{j,i}\} \in I_{i} \). Then, for any such event \( A \), \( P(A) = P_{U}(A) \pm 2^{2st}D_{N} \), where \( P_{U}(A) \) is the probability of \( A \) in case the variables \( x_{i,j} \) are i.i.d. and \( P(x_{i,j} = 1) = |I_{i}|. \)

**Proof:** The proof follows from the above claim, by defining \( A \) as a disjoint union of at most \( 2^{st} \) events \( A_{j} \), where each \( A_{j} = \bigwedge_{k=1}^{t} A_{j,k} \), and each \( A_{j,k} \) itself is a boolean CNF on \( s \) variables of the form \( x_{1,k} = a_{1,k} \land \ldots \land x_{s,k} = a_{s,k} \). By independence of \( g_{i}(n) \), all \( t \) \( s \)-tuples of variables are independent, so

\[
P(A_{j}) = \prod_{k=1}^{t} P(A_{j,k}),
\]

where \( A_{j,k} \) is the boolean CNF corresponding to the \( k \)-th CNF of the \( j \)-th event. By the above claim,

\[
P(A_{j,k}) = P_{U}(A_{j,k}) \pm 2^{s}D_{N},
\]

thus

\[
P(A_{j}) = \prod_{k=1}^{t} (P_{U}(A_{j,k}) \pm 2^{s}D_{N}) = \prod_{k=1}^{t} P_{U}(A_{j,k}) \pm t \cdot 2^{s}D_{N} = P_{U}(A_{j}) \pm t \cdot 2^{s}D_{N}.
\]

Therefore \( P(A) = \sum_{j} P(A_{j}) = \sum_{j} P_{U}(A_{j}) \pm t \cdot 2^{s}D_{N} = P_{U}(A) \pm 2^{st} \cdot t \cdot 2^{s} \cdot D_{N} = P_{U}(A) \pm 2^{st}D_{N}. \)

An important technical tool is a method for generating low-discrepancy sequences on the unit cube, while using only very small variance. While it is well known that if one samples \( g \) uniformly
from the grid \( n/N \) for \( n \in [N] \), the resulting sequence \( \{gn/N\}_{n \in [N]} \) is low discrepancy, we would like to be able to sample \( g \) from a much tighter range. This, in order to prove that small-scale eigenvalue perturbation, which does not change the input by much, causes the sequence generated by multiples of these perturbed eigenvalues to have low-discrepancy. This is the focus of the following lemma.

**Lemma 1** We are given integer \( N \), with prime divisor \( M = \theta(N^a) \) for some constant \( a > 0 \), and an integer \( s = O(\log(N)) \). There exists a constant \( b > 0 \) such that the following holds: Let \( g = (g_1, \ldots, g_s) \in \mathbb{N}^s \), such that each coordinate \( g_i \) is independently chosen uniformly on some interval \( I_i \subseteq [N] \) of size \( M \). Let \( S(g) \) be the sequence

\[
S(g) = \left\{ \frac{gn}{N} \right\}_{n=0}^{N-1}.
\]

Then

\[
\Pr \left( \left| \frac{1}{N} - \frac{1}{N^b} \right| \geq 1 - N^{-b} \right) \geq 1 - N^{-b}.
\]

### 2.3 Eigenvalue perturbation

Eigenvalue perturbation is a well-developed theory, examining the behavior of eigen-systems under additive perturbation, usually much smaller compared to the original system. Here we apply the standard analysis on symmetrized Gaussian perturbation to show that one can derive a simple form for the distribution of the perturbed eigenvalues.

**Fact 2** *Low-complexity perturbation of well-separated spectrum*

Let \( A \) be a Hermitian matrix with eigenvalues \( O(1) = \lambda_1 > \lambda_2 \ldots > \lambda_m \), and eigenvalue separation \( \Delta \lambda \leq \varepsilon \). Let \( G \) be the complex Gaussian matrix, with entries that are i.i.d. complex 0-mean, 1-std normally distributed. For any constant \( a > 1 \) the following holds: the eigenvalues \( \lambda_i' \) of \( A + \varepsilon^a \cdot (G + G^\dagger) \) are each distributed as \( X_i + Y_i \), where \( X_i = \mathcal{N}(\lambda_i, \varepsilon^{2a}) \), \( Y_i \) is some r.v. with \( \mathbb{E} (|Y_i|^2)^{1/2} = O(\varepsilon^{2a-1}) \), and the variables \( \{X_i\}_{i=1}^n \) are independent.

**Fact 3** Let \( A \) be a Hermitian matrix with eigenvalues \( \lambda_i \), \( \Delta \lambda \geq \varepsilon \), and let \( v_1, \ldots, v_n \) be the unique eigenvectors of \( A \). Then for any \( \mathcal{E} \), \( \|\mathcal{E}\| \leq \varepsilon^2 \), the eigenvectors \( v_i' \), and their corresponding eigenvalues \( \lambda_i' \) of \( A + \mathcal{E} \), have \( \|v_i - v_i'| \| \leq \varepsilon \), and \( \|\lambda_i - \lambda_i'| \| \leq \varepsilon^2 \), for all \( i \in [n] \).

This fact can be readily derived similarly to fact (2): for perturbation of magnitude at most \( \varepsilon^2 \), the first-order deviation in eigenvalues is \( O(\varepsilon^2) \) and eigenvectors is \( O(\varepsilon) \).

### 2.4 Interaction of low-discrepancy sequences and eigenvalue perturbation

Here we describe, arguably the main technical tool used in the analysis of the algorithm, combining eigenvalue perturbation and low-discrepancy sequences. It shows that if one has a set of eigenvalues \( \lambda_1, \ldots, \lambda_s \), whose distribution, following perturbation, \( \lambda_1', \ldots, \lambda_s' \), is approximately independent Gaussian, with sufficiently high variance, then w.h.p. the corresponding sequence \( \{m\lambda_1'\}, \ldots, \{m\lambda_s'\} \) is low-discrepancy. In the main proof of the algorithm, we shall then employ this claim and balance the requirement for sufficient variance (to induce low-discrepancy) with not-too-large variance to prevent the perturbed eigenvalues from becoming too close.
Claim 1 Given integer $n$, and $\varepsilon = 2^{-\text{polylog}(n)}$ there exists an integer constant $l$ and integer $M = 2^{\text{polylog}(n)}$ such that the following holds: If $\lambda_1, \ldots, \lambda_n$ be $n$ random variables, where each $\lambda_i$ is distributed according to $X_i + Y_i$, where for all $i \in [n]$ the variables $X_i$ are i.i.d. and $X_i \sim N(c_i, \varepsilon^2)$, for $c_i \in \mathbb{R}$, and $\mathbb{E} |Y_i|^2 = O(\varepsilon^2)$, then for any subset of $s = \text{polylog}(n)$ variables $\lambda_1, \ldots, \lambda_s$, and randomly chosen $m \in [M]$, we have w.p. $1 - O(\varepsilon)$ that the $s$-dimensional sequence $S(\lambda') \equiv \{\{m\lambda'_1\}, \ldots, \{m\lambda'_s\}\}_{m \in [M]}$ has discrepancy $O(\varepsilon)$.

2.5 Technical Estimates

Fact 4 Approximating a Gaussian by a convex sum of uniform distributions

Let $g = (g_1, \ldots, g_n)$ be standard Gaussian measure in $\mathbb{R}^n$. Then $g$ is equal to a convex sum of two distributions: $(1 - p)D_U + p \cdot D_V$, where $D_U$ is the $n$-fold distribution of independent variables $z_1, \ldots, z_n$, where each $z_i, |z_i| \leq \sqrt{n}$, and $z_i$ is the convex sum $z_i = \sum_{j=1}^m t_j w_j$ of $m > n^2$ i.i.d. uniformly distributed variables, with $w_j \sim U[I_j]$, with $|I_j| = 1/m$, and $p = O(n^2/m)$.

Fact 5 Efficient approximation of exponentiated matrix

Given a Hermitian $n \times n$ matrix $A$, $\|A\| = O(1)$, and error parameter $\varepsilon = 2^{-\text{polylog}(n)}$, one can generate a matrix $U_A$ such that $\|e^{iA} - U_A\| \leq \varepsilon$, in time $\tilde{O}(n^\omega)$.

Proof: Put $s = \log(\varepsilon) = \text{polylog}(n)$, and let $U_A$ be the Taylor approximation of $e^{iA}$ up to the first $s$ terms. The approximation error can be bounded by:

$$\left\|e^{iA} - \sum_{m=0}^{s-1} \frac{(iA)^m}{m!} \right\| \leq \sum_{m=s}^{\infty} \frac{\|A^m\|}{m!} = \sum_{m=s}^{\infty} \frac{\|A\|^m}{m!} = O\left(\sum_{m=s}^{\infty} \frac{1}{m!}\right) = O(2^{-s}) = \varepsilon,$$

where we have used the fact that $\|A\| = O(1)$. The complexity of the approximation is comprised of $\text{polylog}(n)$ matrix multiplications for a total of $\tilde{O}(n^\omega)$.

Fact 6 Random unit vectors have well-balanced entries

Let $A$ be an $n \times n$ Hermitian matrix with $n$ unique eigenvectors $v_i$, and $\varepsilon = \Omega(1/\text{poly}(n))$. Let $v \in \mathbb{C}^n$ be a vector chosen uniform w.r.t. the Haar measure on the unit sphere in $\mathbb{C}^n$. For any $i \in [n]$ let $|v| = \langle v, v_i \rangle$. Then w.p. at least $1 - \varepsilon$, for all pairs $i, j \in [n]$, $|v^i| = O(\log(n))$.

3 Efficient eigenvector sampling

Here we prove theorem (1), and provide an algorithm for sampling approximately uniformly the eigenvectors of a matrix with well-separated eigenvalues.

Algorithm 1 Given are $\delta, \nu, n \times n$ Hermitian positive-semidefinite matrix $A$ with eigenvalue separation $\Delta$ and $\|A\| = O(1)$. Let $a = 1/\nu$, $p = 8 \cdot \log^{2a}(n) \cdot \log(n^3/\delta)$, $t = \frac{\nu \cdot \log(n)}{\log(\log(n))}$. Put $D_N = \varepsilon = \min \left\{ 2^{-\text{polylog}(n)}, \Delta \right\}$. Apply claim (1), to $n, \varepsilon$ and compute integers $1 < l = O(1)$, $M = 2^{\text{polylog}(n)}.$

1. Set $A_0 = A$, and let $U_0$ be the Taylor approximation of $e^{2\pi i A_0}$ using fact (5).

2. Randomly choose a vector $v_0$ on the unit sphere in $\mathbb{C}^n$ w.r.t. the Haar measure. Set $v = v_0$.

3. Execute $t$ times:
(a) Perturbate $A$: set $A_k = A_{k-1} + \varepsilon^i (G + G^\dagger)$, where $G$ is standard complex Gaussian.

(b) Compute a Taylor approximation of $e^{2\pi i A_k}$, with error parameter $\varepsilon/M$, using fact (5), and denote by $U_k$.

(c) Choose uniformly at random $m \in M$.

(d) Compute $U_k^m$ by repeated squaring.

(e) Repeat $p$ times: Generate the $2n$ vector $[w_0, w_1] = H \left( \frac{1}{\sqrt{2}} [v, U_k^m \cdot v] \right)$.

(f) Set $v = w_0/\|w_0\|$.

4. If $\| (v^U_0 v) \cdot v - U_0 \cdot v \| \leq \delta$, return $v$, else set $v = v_0$, and go to (3).

3.1 Eigenvalue filtering

Step (3) filters eigenvectors according to the location of their corresponding eigenvalue. This is done via the quantum phase estimation routine. We shall require two technical estimates in this respect, whose proof is deferred to the appendix.

Fact 7 Let $v = \sum_{i=1}^n \alpha_i v_i$, be the eigen-decomposition of any vector $v$, i.e. $v_i$ is the $i$-th eigenvector of $A$, corresponding to eigenvalue $\lambda_i$. For any integer $m$, and given $v$ as above, after a single application of step (3e), the projection of $v$ on the $i$-th eigenspace is $|\alpha'_i|^2 \cdot \left( \frac{1 + \cos(2\pi m \lambda_i)}{2} \right)^p$.

Let $a_{i,j}$ denote the scaling of the $i$-th eigenvector at the $j$-th iteration of step (3e):

$$a_{i,j} = \left( 1 + \cos(2\pi m_j \lambda_i) \right)^p.$$ (2)

From this point on denote $x_{i,j} = \{m_j \lambda_i\}$.

For any $n \in \mathbb{N}$, let $B' \subseteq B \subseteq [0,1)$ be the following sets:

$$B = \frac{1}{2} \left[ -\frac{1}{\log^4(n)}, \frac{1}{\log^4(n)} \right], \quad B' = \frac{1}{\sqrt{2\log(n^3/\delta)}} \cdot B$$

Fact 8 Let $p = 8 \cdot \log^2(n) \cdot \log(n^3/\delta)$ be the global constant defined above. For any $i \in [n], j \in [t]$, if $x_{i,j} \in B'$, then $a_{i,j} \geq 1/e$, and if $x_{i,j} \notin B$, then $a_{i,j} = O(\delta/n^3)$.

3.2 Putting it all together: proof of theorem [1]

Proof: First, we note that the convergence criterion ensures, that the algorithm returns only vectors that are $\delta$-close to eigenvectors of $A$. We now have to show that this criterion is met in reasonable time, and that the eigenvector returned is close to being uniformly random. At a high level, the proof is as follows: we define events

$$E_0 \supseteq E_1 \supseteq E_2, E_3$$

$E_0$ is the event that the values $x_{i,j}$ behave "randomly" enough, $E_1$ is the event that a single eigenvalue falls into $B$ at all trials, whereas all other eigenvalues fall outside $B$ for at least one trial, $E_2$ is the event that the "chosen" eigenvalue of $E_1$, actually falls into $B'$ for all trials. $E_3$ is the event that in the initial vector $v$, the relative scaling of each eigenvector, is approximately uniform. We then show that conditioned on $E_2 \cap E_3$, the algorithm samples from $E(A, \delta)$. This event will be shown to happen with probability $\Omega(n^{-\nu})$ which is sufficient for theorem [1].


3.2.1 The event $E_0$

Assume first that the unitary approximation of step (3b) is without error. We shall then account for this error in the next paragraph on the stability of the eigen-decomposition. Let $P_j$ denote some partitioning of the set of eigenvalues, at the $j$-th iteration of step (3) into sets, each of size polylog($n$).

Let $E_0$ denote the event in which for every part $p_j \in P_j$, and for all $j \in [t]$, the uniform distribution on $\{x_{i,j}\}_{i \in p_j}$ has discrepancy at most $D_M$, and for any step $j$, the $i$-th eigenvalue of $A_j$, $\lambda_i'$ compares to the $i$-th eigenvalue of $A_0$ as $|\lambda_i' - \lambda_i| < \varepsilon/2$. We claim:

**Fact 9** $P(E_0) \geq 1 - 2^{-\text{polylog}(n)}$.

**Proof:** The number of iterations is $t = O(\log(n))$. At each iteration we condition on all previous iterations holding the properties promised by claim (1), i.e. that any subset of polylog($n$) eigenvalues is drawn w.h.p. from a sequence of discrepancy $O(D_M)$. In addition we condition on the magnitude of deviation of each eigenvalue being most $\varepsilon^2$. The latter occurs w.p. $1 - 2^{-\text{polylog}(n)}$ by fact (3), since $l > 1$.

Hence, at any given step $k$, conditioned on the success of all previous $k-1$ draws, the eigenvalues deviate from the original eigenvalues by at most $(k - 1) \cdot \varepsilon^2 < \varepsilon/2$. Hence, the spacing between the eigenvalues of the matrix $A_{k-1}$ is at least $\Delta_\lambda - 2\varepsilon/2 \geq \varepsilon$. Therefore, by fact (2), at the $k$-th application of step (3a), the eigenvalues $\lambda_i'$ of $A_k$ are distributed as $X_i + Y_i$, where $X_i \sim N(\lambda_i, \varepsilon^2)$, and $Y_i$ has $\mathbb{E} [Y_i^2]^{1/2} = O(\varepsilon^2)$. Therefore, by claim (1) each part $p_j \in P_j$ has discrepancy $O(D_M)$ w.p. $1 - O(D_M)$. By the union bound over all parts $p_j \in P_j$, and all trials $j \in [t]$ we have $P(E_0) \geq 1 - t \cdot n \cdot O(D_M) = 1 - 2^{-\text{polylog}(n)}$.

**Stability of eigen-decomposition** In the event $E_0$, we note that by fact (3) at each application of step (3a), the eigenvectors of the perturbed matrix $A_k$ deviate from the eigenvectors of $A_0$, $v_1, \ldots, v_n$ by $O(\varepsilon)$. In addition, by fact (5) the approximation $U_k$ of $e^{iA_k}$ at step (3b) is with error $O(\varepsilon/M)$. Therefore, the approximation error of $e^{2\pi i A_k \cdot m}$ is at most $O(\varepsilon)$ for all $m \in [M]$. So the eigenvectors of $U_k^m$ relate to $v_1, \ldots, v_n$ with error $O(\varepsilon)$.

Hence, given an eigen-decomposition of the vector $v$ according to the eigenvectors of $A_0$, $v = \sum_i \alpha_i v_i$, we can rewrite a decomposition of $v$, according to the eigen-vectors of $U_k^m$, $v_i'$, using the same coefficients, while absorbing the error into a new term of norm at most $\varepsilon$, i.e.

$$v = \sum_i \alpha_i v_i' + \mathcal{E}, \quad \|\mathcal{E}\| = O(\varepsilon).$$

Therefore, up to error at most $O(t\varepsilon)$, we can consider all iterations of steps (2)-(3) as acting on the same set of eigenvectors $v_i$. So from now on, we shall assume that the eigenvectors remain the same throughout the algorithm, and account for the error at the final step. Furthermore, sorting the eigenvalues of $A_0$ in descending order $\lambda_1 > \lambda_2 > \ldots > \lambda_n$, then since by the same fact (3) these eigenvalues do not cross each other, we can uniquely refer to the eigenvalue of $A_k$ corresponding to $\lambda_i$, as the eigenvalue of $A_k$ closest to the eigenvalue $\lambda_i$ of $A_0$. Hence, we can say that the random variables $x_{i,j}$, for all $j \in [t]$, correspond to the $i$-th eigenvector of $A_0$.

3.2.2 The event $E_1$

**Fact 10** $P(E_1|E_0) = \Omega(1)$
Proof: Put \( s = \log^{6a}(n) \), and condition on \( E_0 \). This implies that whenever we consider a part \( p_j \in P_j \) at any iteration \( j \in [t] \) we can assume it is sampled with discrepancy \( O(D_M) \). Fixing such a part \( p_j \in P_j \), at a specific iteration \( j \), by fact (1) the probability that for a random \( m \in [M] \), the number of indices \( k \in [s] \), for which \( x_{i_k} \in B \), deviates from expectation \( |B| = 1/\log^a(n) \), by more than \( 1/\log^{2a}(n) \) is at most what it should be for a random draw, up to an error at most \( 2^s \cdot O(D_M) = 2^{-\Omega(\log^{2a}(n))} \).

Using the Hoeffding bound, for each \( p_j \in P_j \) we have:

\[
\Pr \left( \left| \frac{1}{|p_j|} \sum_{x_{i_k} \in B} \lambda_{i_k} \in p_j - \frac{1}{\log^a(n)} \right| \geq \frac{1}{\log^{2a}(n)} \right) \leq 2^{-0.5(1/\log^{2a}(n))s^2} + 2^s \cdot D_N = 2^{-\Omega(\log^{2a}(n))}
\]

(4)

Using this bound, we would now like to study the distribution of the number of eigenvalues that "survive" in the interval \( B \) for various times \( j \in [t] \). This, because the "surviving" eigenvalues experiences a relative gain, w.r.t. all the eigenvalues mapped outside of \( B \), and therefore, the "last surviving" eigenvalue in \( B \) will also be the eigenvalue whose eigenvector is returned by the algorithm.

For all \( j > 1 \), let \( S_j \subseteq [n] \) denote the set of indices \( i \in [n] \) for which \( x_{i_k} \in B \) for all \( 1 \leq k < j \). We will show that the size \( |S_j| \) decays exponentially fast in \( j \), up to the point where very few eigenvalues remain in \( B \).

Our proof strategy is to regroup the "surviving" eigenvalues from the previous round, and partition them as a group into small low-discrepancy sets, so that the number of "survivors" decays exponentially fast with the number of trials. This will be essentially a "tournament" between the eigenvalues. For technical purposes, we will not explore the decay beyond the point where there are \( O(\text{polylog}(n)) \) "survivors" \( B \) because then the Hoeffding bound (4) above will not work. Instead, for the very high regime, we shall apply discrepancy bounds directly.

By definition, the number of trials \( t \) is such that \( (\log^a(n))^t = n \). Hence \( t = \log(n)/(a \cdot \log(\log(n))) \).

Consider now the following process: For \( j = 1 \) choose some arbitrary partition \( P_1 \). For each \( 1 < j < t \) if \( |S_{j-1}| \geq 2s \) choose a partition \( P_j \), such that all indices \( i \in S_{j-1} \) are partitioned together as a group, separately from all other indices. We assume here for simplicity, that the number of survivors is divisible by \( s \), but if it doesn’t we can have each group be as large as \( 2s \), with only a constant change to the discrepancy \( D_M \). If \( |S_{j-1}| < 2s \) we choose an arbitrary partition \( P_j \) such that \( S_{j-1} \) is contained in one part \( p_j \in P_j \). In both cases, all indices \( [n] - S_{j-1} \) are partitioned arbitrarily. For any \( 1 \leq j \leq t \) if \( |S_j| \geq s \) then bound (4) holds so

\[
|S_j| = |S_{j-1}| \cdot \left( \frac{1}{\log^a(n)} \pm \frac{1}{\log^{2a}(n)} \right),
\]

(5)

Let now \( t' \) be the random variable representing the number of trials so that \( |S_{t'}| < 2s \). By equation (4) and the union bound over trials, we have that w.p. at least \( 1 - 2^{-\text{polylog}(n)} \) we have \( \log^a(n)t' = n/s \). In other words, it is exactly the number of trials for which the expected number of "survivors" is exactly \( s \).

We now denote \( |S_{t'}| = s' \), and estimate the probability that precisely a single element \( i \in [n] \), has \( x_{i,j} \in B \), for all remaining trials \( j \in [t] \). By \( E_0 \) in each trial \( t \), we draw independently an element from a low-discrepancy sequence. Therefore, by corollary (1) the number of elements that are located in \( B \) for all iterations \( t' < j < t \) is distributed as \( B(s', 1/(\log^a(n))^{t-t'}) = B(s, 1/s) \), up to error at most \( 2^{2a(t-t')} \cdot D_M \). Since by assumption \( s' = \theta(s) \) w.h.p., then w.p. \( \Omega(1) \) a single element of \( S_{t'} \) is mapped to \( B \) for all last \( t - t' \) iterations. Let us denote this event by \( E_1 \), i.e. after precisely \( t \) iterations, a single element \( i \) has \( x_{i,j} \in B \) for all \( j \in [t] \). We say that index \( i \) is chosen by event \( E_1 \), and denote by \( E_{1,i} \) this event. Hence \( E_1 = \bigcup_{i=1}^n E_{1,i} \). We observe that \( P(E_1 | E_0) = \Omega(1) \).
3.2.3 Conditioned on $E_1$ the selected eigenvalue is uniformly distributed

For any $i \in [n]$, let $E_{2,i} \subseteq E_{1,i}$, denote the subset of event $E_{1,i}$ in which $x_{i,j} \in B'$ for all $j \in [t]$. We estimate the probability:

$$P(E_{2,i}|E_1) = \text{Prob}(x_{i,j} \in B' \forall j \in [t] \mid E_1),$$

and claim

Fact 11

$$P(E_{2,i}|E_1) = \Omega(n^{-v-1}).$$

Proof: Recall that $E_1 = E_0 \cap B$, where $B$ is the event that a single value is mapped to $B$ for all steps $1 \leq j \leq t$. So,

$$P(E_{2,i}|E_1) = \text{Prob}(x_{i,j} \in B' \forall j \in [t] \mid E_0 \cap B) = \frac{\text{Prob}(x_{i,j} \in B' \forall j \in [t] \mid E_0) \cdot P(E_0)}{\text{Prob}(E_0 \cap B)}$$

In words, the first factor in the numerator above is the probability that $x_{i,j}$ falls into $B'$ for all trials, and that it is a “sole survivor”, i.e. all other $i$’s fall outside $B$ in at least one iteration. We observe that this probability, up to a small error, the probability that $x_{i,j}$ is in $B'$ for all iterations, whereas any other $i' \neq i$, $i' \in S_{\nu}$, falls outside $B$ for at least one of the last $t' - t'$ iterations, conditioned on $|S_{\nu}| = \theta(s)$. This because, by the analysis of $E_0$, the probability that the first $t'$ iterations reduce the number of survivors to $\theta(s)$ is almost 1, i.e. $1 - O(n^{-k})$, for any constant $k > 0$. Hence the numerator is at least:

$$P(E_0) \cdot \left( \text{Prob}(x_{i,j} \in B' \forall j \in [t] \land \forall i' \neq i, i' \in S_{\nu} \exists j, t' < j \leq t \text{ s.t. } x_{i',j} \notin B \mid E_0 \land |S_{\nu}| = \theta(s)) - O(n^{-k}) \right),$$

Let us apply corollary (1) to the set $S_{\nu}$. Since we condition on $|S_{\nu}| = \theta(s)$ then:

$$P(E_{2,i}|E_1) \geq \frac{\text{Prob}(x_{i,j} \in B' \forall j \in [t] \mid E_0) \cdot P(E_1) - O(n^{-k})}{\text{Prob}(E_0 \cap B)} \geq \text{Prob}(x_{i,j} \in B' \forall j \in [t] \mid E_0) - O(n^{-k}) = |B'|^t - O(n^{-k})$$

where in the last equality we have used the low discrepancy by corollary (1) for $s = 1$. By our choice of parameters we have:

$$P(E_{2,i}|E_1) \geq |B'|^t - 2^t D_M - O(n^{-k}) = \theta(n^{-1-v}). \quad (6)$$

3.2.4 The event $E_2 \cap E_3$

Fix $\varepsilon = n^{-v}$. By the fact (6) w.p. at least $1 - \varepsilon$, for any randomly chosen initial vector $v$, we have

$$|\alpha_i|^2 \leq O(\log(n)) \cdot |\alpha_j|^2,$$

for all pairs $i, j \in [n]$. Define this event as $E_3$. On the other hand, conditioned on $E_{2,i}$, we have by fact (8) that the cumulative scaling of the $i$-th eigenvector is:

$$\prod_{j=1}^{t} a_{i,j} \geq e^{-t} = \Omega(1/n),$$
whereas for any \( k \neq i \), we have that \( x_{k,j} \notin B \) for some \( j \in [t] \), and hence
\[
\prod_{j=1}^{t} a_{k,j} \leq 1^{t-1} \cdot \delta n^{-3} = \delta n^{-3}.
\]

We note that the relative scaling above does not change following normalization at step (3f). Hence, conditioned on \( E_{2,i} \cap E_{3} \), we have for the eigenvector \( v_i \) corresponding to eigenvalue \( \lambda_i \):
\[
\left\| (v_j U_0 v) \cdot v - U_0 v \right\| \leq t \cdot \varepsilon + O(\log(n)) \cdot \frac{1}{\prod_{j=1}^{t} a_{i,j}} \cdot \sum_{k \neq i} |\alpha_k|^2 \prod_{j=1}^{t} a_{k,j} \leq o(\delta) + n^3 \cdot \delta n^{-3} = O(\delta),
\]

Therefore, the convergence criterion holds, and the algorithm returns \( v \), that is \( O(\delta) \) close to the \( i \)-th eigenvector. Accumulating the probabilities of \( E_0, E_1, E_2 \) we get that for any \( i \in [n] \):
\[
P(E_{2,i}) = P(E_{2,i}|E_1)P(E_1) = \theta(n^{-\nu - 1}).
\]

By independence of \( E_2 \) and \( E_3 \), we have \( P(E_{2,i} \cap E_3) = \theta(n^{-\nu - 1}) \). Hence, conditioned on \( E_2 \cap E_3 \), the algorithm samples from \( E(A, \delta) \), for the parameter \( \delta \) of the input. Moreover, since \( P(E_1) = \Omega(1) \), then \( P(E_2 \cap E_3) = \sum_{i=1}^{n} P(E_{2,i} \cap E_3) = \Omega(n^{-\nu}) \). Therefore, w.p. \( \Omega(n^{-\nu}) \) the algorithm samples from \( E(A, \delta) \).

Finally, the running time of the algorithm is comprised, at each iteration of at most \( O(\log(M)) = \text{polylog}(n) \) matrix products, each of which is \( O(n^\omega) \) arithmetic operations. Hence, the overall expected arithmetic running time is \( O(n^\omega) \).

### 3.2.5 Bit-complexity

We would now like to claim, that the algorithm can be executed in bit-complexity that is \( \tilde{O}(n^\omega) \). We note that multiplying two \( b \)-bit integers together can be done in bit-complexity \( O(b \cdot \log^2(b)) \), hence if we use registers of size at most \( \text{polylog}(n) \), we can absorb the overhead into the bit-complexity function \( \tilde{O}(n^\omega) \). We realize that we must handle a quantized version of the following operations: Perturbation and discrepancy analysis, product of unitaries, and division:

1. **Matrix product:** we start with input of bit size \( b = \text{polylog}(n) \), such that \( 2^{-b} < 1/M^3 \ll \delta \). Whenever we encounter a product of unitary matrices, we extract the top \( b \) bits of each entry of the output, introducing, by unitarity, an error of magnitude at most \( \sqrt{n} 2^{-b} \), thus keeping the number of bits \( b \) throughout the computation.

2. **Gaussian perturbation:** Instead of the continuous Gaussian perturbation, we use a quantized version: We start by using some uniform random bits to approximate Gaussian random variables. Let \( b' = \log(\varepsilon^{-2a}) = \text{polylog}(n) \), where \( \varepsilon^{-2a} \) is the parameter that appears in fact (2). We take registers of \( b' \) i.i.d. uniform bits, regard them as a real value \( z \in [0, 1] \) and compute for them an approximation \( \Phi^{-1}(z) \), where \( \Phi(z) \) is the standard Gaussian CDF in time \( O(\text{poly}(b')) \). By fact (12) this results in a variable \( \hat{z} \) that is \( \varepsilon^{-2a} \) close \( \Phi^{-1}(z) \). Therefore
\[
\text{Prob}(\hat{z} < z_0) = \text{Prob}(\Phi^{-1}(z) \pm 2^{-b'} < z_0) = \text{Prob}(z < \Phi(z_0 \pm 2^{-b'})) =
\]
\[
\text{Prob}(z < \Phi(z_0 \pm 2^{-b'})) = \text{Prob}(z < \Phi(z_0) + O(2^{-b'})) = \Phi(z_0) + O(2^{-b'}). 
\]
In this case, the approximation-error of the Gaussian perturbation added to each $A_k$ can be absorbed into the error term of fact (2), and hence the analysis continues from there as in the precise case. Thus, we can assume that the quantized perturbation has the same effect as the continuous one.

3. **Discrepancy:** Since we raise any unitary to a power $t$ most $M$, the added error to the eigenvalues of $e^{2\pi i A_t}$, as a result of the truncation to $b$ bits, is at most $M \cdot \sqrt{n 2^{-b}} = O(M^{-1.9})$. Thus, the quantized and continuous sequences $x_{i,j}$ are at pointwise distance at most $O(M^{-1.9})$. Hence, by lemma 2.5 of (3), the discrepancy of the resulting “digitized” sequence is at most $O(D_M) + O(2^b M^{-1.9}) = O(D_M)$.

4. **Division:** Finally, regarding the division operation, while it is not numerically stable in general, in case we have $E_2 \cap E_3$ it is numerically stable, because then $\|w_0\| = \Omega(1)$, by fact (5). In that case, division can be performed with the same $b$ bits of precision, adding error at most $O(\delta)$.

**4 Complete diagonalization**

Here we prove theorem (2). We propose the following algorithm:

**Algorithm 2** Given $\delta, \nu$, and $n \times n$ Hermitian positive-semidefinite matrix $A$, $\|A\| = O(1)$, with eigenvalue separation $\Delta\lambda_i$, choose the parameters as in algorithm (1). Let $A_0 = A$, and $U_0$ be the Taylor approximation of $e^{2\pi i A_0}$ using fact (5). Iterate until all $n$ unique eigenvectors of $A_0$ are sampled:

1. Randomly choose a vector $v$ on the unit sphere in $\mathbb{C}^n$ w.r.t. the Haar measure.

2. Randomly choose $n$ complex constants on $S^1$: $\{e^{2\pi i \theta_j}\}_{j=1}^n$, where each $\theta_j$ is uniform on the set $\{\frac{m\pi}{M}\}_{m=0}^{M-1}$. Set $D = \text{diag}[e^{2\pi i \theta_1}v, e^{2\pi i \theta_2}v, \ldots, e^{2\pi i \theta_n}v]$.

3. Execute $t$ times:
   
   (a) Perturbate $A$: set $A_k = A_{k-1} + \varepsilon(G + G^\dagger)$, where $G$ is standard complex Gaussian.
   
   (b) Compute a Taylor approximation $U_k$ of $e^{2\pi i A_k}$, with error parameter $\varepsilon/M$ using fact (5).
   
   (c) Choose uniformly at random $m \in M$.
   
   (d) Compute $U_k^m$ by repeated squaring.
   
   (e) Repeat $p$ times: Generate the $n \times 2n$ matrix $[W_0, W_1] = H\left(\frac{1}{\sqrt{2}}[V, U_k^m \cdot V \cdot D^m]\right)$. Set $V$ as the matrix in which we normalize each column of $W_0$, separately.

4. Return all columns $v$ of $V$ for which: $\|(v^\dagger U_0 v) \cdot v - U_0 \cdot v\| \leq \delta$.

**Proof:** By theorem (1) we have that for each column $k \in [n]$, w.p. $\Omega(1)$, at some point during the first $n^\nu$ iterations, event $E_2 \cap E_3$ occurs, and the algorithm samples from $E(A, \delta)$. Suppose one could show that the index $i$ for which the above holds, is independent between the columns of $V$. If that were the case, then $O(n^\nu)$ iterations of steps (1)-(4) would sample independently $\Omega(n)$ indices out of $[n]$, so by the coupon-collector’s theorem, after $O(n^\nu \log(n))$ repetitions of these steps, all $n$ eigenvectors will be returned, w.h.p. That would imply that the overall complexity is at most $O(\log(n))$ times that of

14
algorithm (1), or $\tilde{O}(n^{\omega+\nu})$. Choosing appropriate constant $\nu' < \nu$ in the analysis of the first algorithm we can force the overall complexity of the algorithm to $O(n^{\omega+\nu})$.

Since the eigenvalues sampled between different iterations of (1)-(4) are independent, it is sufficient to show independence for a given iteration. Let $c_k$ denote the index sampled from $E(A, \delta)$, following an event $E_2 \cap E_3$, at the $i$-th column, at some fixed iteration. The variable $c_k$ gets values in $[n]$, and an additional value 0, in case no eigenvalue is sampled, i.e. event $E_2 \cap E_3$ did not occur at the $k$-th column during this iteration. We would now like to show that for any $i, k \in [n]$, and values for $c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n$,

$$Pr(c_k = i | c_1, \ldots, c_{k-1}, c_{k+1}, \ldots, c_n) = Pr(c_k = i) = P(E_{2,i}) = \theta(n^{-\nu-1}),$$

(7)

where the probability is over all random coins tossed during the algorithm, and the last equality is known by theorem (1). However, a simpler, still sufficient condition may be devised:

$$Pr(c_k = i | \exists k' \neq k, \text{ s.t. } c_{k'} = i) = O(n^{-\nu-1}).$$

(8)

This condition is sufficient because by the union bound, the probability that index $i$ is chosen at some column $k' \neq k$ by $E(A, \delta)$ is at most $(n-1) \cdot O(n^{-\nu-1}) = O(n^{-\nu})$, and so the probability of index $i$ being chosen at index $k$ by $E(A, \delta)$, given that it was not chosen by $E(A, \delta)$ for all $k' \neq k$ is $\Omega(n^{-\nu-1})$. In this case, the coupon-collector’s bound holds as if the indices sampled by the various columns are independent.

Let $\rho_{i,j,k} = \{m_j(\lambda_i + \theta_k)\}$. At step (3c) at each iteration $j$ of algorithm (2), the $i$-th eigenvector at the $k$-th column is multiplied by a factor $\frac{1 + \cos(2\pi \rho_{i,j,k})}{2}$. Therefore, $c_k = i$ if $\rho_{i,j,k} \in B'$ for all $j \in [t]$, and for any $i' \neq i, \rho_{i,j,k} \notin B$, for some $j \in [t]$. For any $k \in [n]$ consider the probability measure $f$ of $\bigcup_{i,j} \rho_{i,j,k}$ conditioned on $\bigcup_{i,j,k', \neq k} \rho_{i,j,k'}$.

$$f \left( \bigcup_{i,j} \rho_{i,j,k} | \bigcup_{i,j,k', \neq k} \rho_{i,j,k'} \right) = g \left( \bigcup_{i,j} (\{m_j - \rho_{i,j,k'}\} | \bigcup_{i,j,k', \neq k} \rho_{i,j,k'}) = \bigcup_{i,j} \{m_j \cdot (\theta_k - \theta_k')\} | \bigcup_{i,j,k', \neq k} \rho_{i,j,k'} \right)$$

where $g$ is the distribution on $\{m_j(\theta_k - \theta_k')\}_j$, conditioned on the same set of variables as $f$. We note here, that the difference $\rho_{i,j,k} - \rho_{i,j,k'}$ depends only on $j$ and not on $i$, hence it can be written as a probability measure on $t$ variables, instead of $n \cdot t$. But since $\theta_k'$ is uniform on $\frac{2\pi m}{M}$ and independent of the variables $\{\lambda_i\}_{i=1}^n, \{\theta_k\}_{k=1}^n, \text{ and } \{m_j\}_{j=1}^t$, then

$$f \left( \bigcup_{i,j} \rho_{i,j,k} | \bigcup_{i,j,k', \neq k} \rho_{i,j,k'} \right) = g \left( \{m_j \cdot (\theta_k - \theta_k')\}_j \right).$$

(9)

Let us now return to inequality (8) and condition it further on the values of $\rho_{i,j,k'}$. To prove inequality (8) it suffices to show that for every $n(n-1)t$ values of $\bigcup_{i,j,k', \neq k} \rho_{i,j,k'}$, for which $\exists k' \neq k$, s.t. $c_{k'} = i$, we have:

$$Pr \left( c_k = i \exists k' \neq k, \text{ s.t. } c_{k'} = i, \bigcup_{i,j,k', \neq k} \rho_{i,j,k'} \right) = O(n^{-\nu-1}).$$

But by definition, the condition $c_k = i$ implies in particular, that $\forall j \in [t] \rho_{i,j,k} \in B'$. Hence the above can be upper-bounded by:

$$Pr \left( \forall j \in [t] \rho_{i,j,k} \in B' \exists k' \neq k, \text{ s.t. } c_{k'} = i, \bigcup_{i,j,k', \neq k} \rho_{i,j,k'} \right)$$
Choose some $k' \neq k$, such that $c_{k'} = i$. By assumption, we know that such $k'$ exists. The above is equal to:

$$Pr \left( \forall j \in [t] \{ \rho_{i,j,k} - \rho_{i,j,k'} \} \in Z_j \| \{ \rho_{i,j,k'} \}_{i,j,k' \neq k} \right) ,$$

where $Z_j$ is some interval of size $|\mathcal{B}'|$, determined by the value of $\rho_{i,j,k'}$. Using equation (9), the above is equal to:

$$Pr \left( \forall j \in [t] \{ m_j(\theta_k - \theta_{k'}) \} \in Z_j \right) .$$

Since $\theta_k, \theta_{k'}$ are independently uniform on $\{ \frac{2\pi m}{M} \}_{m \in [M]}$, then so is $\{ \theta_k - \theta_{k'} \}$. Hence, by lemma (1), the 1-dimensional sequence $\{ m(\theta_k - \theta_{k'}) \}_{m \in [M]}$ has discrepancy $D_M$ w.p. $1 - 1/poly(M)$ over choices of $\theta_k, \theta_{k'}$, so the above is at most

$$\prod_{j=1}^{t} |Z_j| + 2^t D_M \leq |\mathcal{B}'|^t + 2^t D_M = O(n^{-\nu-1}) \forall i \in [n].$$

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6 Appendix

6.1 Eigenvalue perturbation

6.1.1 Proof of fact (2)

Proof: We perform first-order variational analysis on the following equation:

\[ Av_k = \lambda_k v_k, \]

where \( A(t) = A + t \cdot (G + G^\top) \), and approximate \( \lambda'_k, v'_k \) using standard Taylor series expansion, truncated to first-order:

\[
\begin{align*}
\lambda'_k &= \lambda_k + \varepsilon \dot{\lambda}_k + \frac{1}{2} \ddot{\lambda}_k \varepsilon^2 + \ldots + \frac{1}{t!} \lambda_i^{(t)} \varepsilon^t + \ldots = \lambda_k + \varepsilon \dot{\lambda}_k + O(\varepsilon^{2a-1}) \\
v'_k &= v_k + \varepsilon \dot{v}_k + \frac{1}{2} \ddot{v}_k \varepsilon^2 + \ldots + \frac{1}{t!} v_i^{(t)} \varepsilon^t + \ldots = v_k + \varepsilon \dot{v}_k + O(\varepsilon^{2a-2})
\end{align*}
\]

(10) (11)

It is well-known that the following equations hold:

\[
\begin{align*}
\dot{\lambda}_k &= v_k^\top A(t) v_k \\
\dot{v}_k &= \sum_{j \neq k} \frac{v_j}{\lambda_k - \lambda_j}.
\end{align*}
\]

(12) (13)

We observe that since \( G \) is invariant under the diagonalizing basis of \( A \), then so is the measure \( G + G^\top \). In this case we can assume that \( v_k = e_k \) for all \( k \in [n] \), where \( e_k \) is the \( k \)-th standard basis vector. Hence, the first-order derivatives behave as:

\[
\begin{align*}
\dot{\lambda}_k &= G_{k,k} \\
\dot{v}_k &= \sum_{j \neq k} \frac{v_j}{\lambda_k - \lambda_j}.
\end{align*}
\]

(14) (15)

Setting \( X_i = \lambda_i + \varepsilon \alpha G_{i,i} \) we have by equation (12) that the \( X_i \)'s are independent and distributed as stated by the claim. In addition we have \( \lambda'_i = X_i + Y_i \) with \( Y_i \) having bounded second moment as claimed.

\[ \blacksquare \]

6.2 Multi-dimensional discrepancy

6.2.1 Proof of claim (1)

Proof: Using \( l \) first as parameter, let \( p \) be the minimal prime that is at least \( \varepsilon^{-0.5l} \), and let \( M = p^3 \). By the prime number theorem, we can assume that \( \varepsilon^{-1.5l} \leq M \leq \varepsilon^{-1.5l} \). For any \( z \in [0, 1) \) let \( z^M \) be the number closest to \( z \) in the grid \( m/M, m \in [M] \). We will first see how to simplify the analysis by disregarding the \( Y_i \) components in the r.v.'s. Then, we will reduce the Gaussian distribution of the \( X_i \) components to uniform distribution over relatively large intervals for lemma (1) to work. This will allow us to claim that the r.v.’s \( \{X_i^m\}_{m \in M} \) behave well, i.e. with low discrepancy. Then, we will see how the matrix perturbation helps us disregard the fact that the \( X_i \)'s do not fall exactly on the \( m/M \) grid.
Removal of non-independent component. Consider the r.v.’s $X_i + Y_i$. For sufficiently large (constant) $l$, and using Chebyshev’s inequality and the union bound over all $i \in [n]$, w.p. $1 - \varepsilon'^3$ we have $|Y_i| \leq M^{-1.3}$ for all $i$, for some constant $a' > 0$, and so

$$\left| \{m \lambda'_i \} - \{m X_i \} \right| \leq m \cdot M^{-1.3} \leq M^{-0.3}$$

for all $m \in M$. Suppose we show that the discrepancy of the sequence

$$S(X) = \{ \{m X_1 \}, \ldots, \{m X_s \} \}_{m=0}^M,$$

is at most some value $D_M(S(X))$, w.p. at least $1 - \varepsilon a_1$, for constant $a_1 > 0$. Then by the union bound over these last two events, and by lemma 2.5 of [3], we have:

$$\Pr(D_M(S(\lambda')) \leq D_M(S(X)) + O(2^a \cdot M^{-0.3})) \geq 1 - \varepsilon a_2,$$

for some constant $a_2 > 0$, and where

$$S(\lambda') = \{ \{m \lambda'_1 \}, \ldots, \{m \lambda'_s \} \}_{m=0}^{M-1}.$$

Reducing the Gaussian distribution to uniform sampling Let us now upper-bound the discrepancy of the sequence $S(X)$. Since each $X_i \sim \mathcal{N}(\lambda_i, \varepsilon^2)$ then by fact (4), choosing there $m = 1/\varepsilon$ we have that w.p. at least $1 - \varepsilon^2 \cdot \varepsilon$ each $X_i$ is uniformly and independently drawn from an interval of size $\Omega(\varepsilon^{l+1})$.

$$\varepsilon^{l+1} = M^{-(l+1)/1.5l} \geq M^{-2/3}, \forall a > 0$$

then each $\lambda'_i$ is sampled uniformly and independently from an interval of size at least $M^{-2/3}$. In fact, w.l.o.g. we can assume that each interval size is precisely $M^{-2/3}$. (this, by increasing the number of intervals as necessary, and reducing the resulting approximation error of fact (4). Because of this, and since $M = p^3$ for prime $p$, we can apply lemma (1), using $N = M, a = 1/3$. We have:

$$\Pr(D_M(S(XP^M)) = O(M^{-a_3})) \geq 1 - \varepsilon a_3,$$

for constant $a_3 > 0$.

Treating the residual w.r.t. the $M$-grid Consider now, for each $i \in [s]$ the number $r_i = X_i - X_i^M$. Rewriting the sequence $X_i$ via the variables $r_i$ we have:

$$\{X_i \cdot m\} = \{\{X_i^M \cdot m\} + \{r_i \cdot m\}\} = \{\{X_i^M \cdot m\} + z_i\},$$

Since the variables $X_i$ are i.i.d. uniform on an intervals of size $M^{-2/3}$, then w.p. at least $1 - O(M^{-d})$ for some $d > 0$, each variable $X_i$ is at distance at least $1/M$ from the edge of its corresponding interval $I_i$, and so the variables $r_i$ are independent of the variables $X_i^M$. Hence, in this case, we can write

$$S(X) = \{S(X^M) + Z\},$$

where $Z$ is some distribution on $[0, 1]^s$ independent of $S(X^M)$. In words, sampling uniformly $m \in [M]$ and returning the $m$-th element of $S(X)$, is the same distribution as sampling uniformly $m \in [M]$, computing the $m$-th element $v$ of $S(X^M)$ and returning $\{v + z\}$, where $z$ is sampled independently from $Z$. Since the discrepancy $D_M()$ is invariant under shifts modulo 1, and $S(X)$ is a convex mixture of shifted sequences $S(X^M)$, we get that

$$\Pr(D_M(S(X)) = D_M(S(X^M)) = O(M^{-a_3})) = 1 - O(M^{-a_3}) - O(M^{-d}) = 1 - O(M^{-a_4}),$$

(18)
for some constant $a_4 > 0$.

**Concluding the proof:** By equations (16) and (18) we get for some $a_5 > 0$ that w.p. $1 - O(\varepsilon^{a_5})$:

$$D_M(S(\lambda')) \leq D_M(S(X)) + O(M^{-0.49}) = O(\varepsilon^{a_6}),$$

for some constant $a_6 > 0$. Taking $c = \min \{a_5, a_6\}$, we have that

$$\text{Prob}(D_M(S(\lambda')) = O(\varepsilon^{c})) = 1 - O(\varepsilon^c).$$

One can check that $c = \alpha l$ for some constant $\alpha < 1$. Hence there exists a constant integer $l$, such that $c = 1$, and the proof follows.

6.2.2 Proof of lemma (1)

**Proof:** For integers $P, s$ put $C^*_s(P)$ as the set of all vectors in $\mathbb{Z}^s$ with entries in $[-P/2, P/2) \cap \mathbb{Z}$, excluding the all-zero vector. Following Niederreiter [3] for $h \in \mathbb{Z}^s$

$$r(h) = \prod_{i=1}^{s} r(h_i),$$

where $r(h_i) = \max(1, |h_i|)$. For $g = (g_1, \ldots, g_s) \in \mathbb{Z}^s$, we denote:

$$R(g, P) = \sum_{h \cdot g = 0(\mod P), h \in C^*_s(P)} r(h)^{-1}.$$

By theorem [5.10] of [3] when each $g_i$ is randomly chosen on the entire interval $[P]$, for prime $P$, then

$$\mathbb{E}_g[R(g, P)] = O\left(\frac{\log^s(P)}{P}\right).$$

Since $R(g, P) \geq 0$ for all vectors $g$, then

$$\text{Prob}_g\left(R(g, P) \geq \frac{\log^s(P)}{\sqrt{P}}\right) \leq \frac{1}{\sqrt{P}}.$$

or

$$\text{Prob}_g\left(R(g, P) \leq \frac{\log^s(P)}{\sqrt{P}}\right) \geq 1 - \frac{1}{\sqrt{P}}. \quad (19)$$

Let us use the above equation to upper-bound the discrepancy of $S(g)$. Recall that $M$ is a prime divisor of $N$, with $M = \theta(N^a)$. We first observe that:

$$R(g, N) = \sum_{h \cdot g = 0(\mod N), h \in C^*_s(N)} r(h)^{-1} \leq \sum_{h \cdot g = 0(\mod M), h \in C^*_s(M)} r(h)^{-1} + \sum_{h \cdot g = 0(\mod M), h \in C^*_s(N), \exists i \text{ s.t. } |h_i| \geq M} r(h)^{-1}.$$

We note that the second term is at most

$$\frac{s}{M} \sum_{h \in \mathbb{Z}^{s-1}} r(h)^{-1} = \frac{s \cdot \log^{s-1}(N)}{M}.$$
Regarding the first term, \( h \cdot g = 0 \pmod{M} \) if and only if \( h \cdot (g \pmod{M}) = 0 \pmod{M} \), and since each \( g_i \) is uniform on some interval of size \( M \), then \( g_i \pmod{M} \) is uniform on \([M]\). Since \( M \) is prime, we can apply equation (19) to the first term with \( P = M \). Using this equation and the upper-bound on the second term we have:

\[
Prob \left( R(g, N) \leq \frac{2\log^s(M)}{\sqrt{M}} \right) \geq 1 - 1/\sqrt{M}.
\] (20)

According to theorem 5.6 of [3], the discrepancy of the sequence \( \{gn/N\}_{n=0}^{N-1} \) is upper-bounded by:

\[
D_N(S(g)) \leq \frac{s}{N} + 2^{-s} \cdot R(g, N).
\] (21)

Plugging equation (20) into equation (21) implies:

\[
Prob \left( D_N(S(g)) \leq \frac{2\log^s(M)}{\sqrt{M}} \right) \geq 1 - 1/\sqrt{M}.
\]

Choosing \( b = a \cdot 0.49 \), finishes the proof.

6.3 Technical Estimates

6.3.1 Proof of fact (4)

Proof: Let us partition the interval \([-\sqrt{n}/2, \sqrt{n}/2]\) into \( m \cdot \sqrt{n} \) equal intervals, each of size \( 1/m \), and let \( p_k \) denote the point of maximal absolute value in the \( k \)-th interval. Consider the real Gaussian PDF \( \Phi(x) : R \mapsto [0, 1] \): for any pair of neighboring intervals \( p_k, p_{k+1} \), we have

\[
|\Phi(p_k) - \Phi(p_{k+1})| \leq \Phi(p_k) \cdot \left( 1 - \frac{e^{-(p_{k+1}-p_k)^2}}{e^{-(p_k)^2}} \right) \leq 1 - e^{-2p_k/m-1/m^2}
\]

\[
1 - e^{-\sqrt{n}/m-1/m^2} = O(\sqrt{n}/m).
\]

Set \( t_j = \Phi(p_j) \). Then by the above, for any \( j \in [m] \), we have \( \max \{|t_j - t_{j-1}|, |t_j - t_{j+1}|\} = O(\sqrt{n}/m) \).

Let \( z_i : R \mapsto [0, 1] \) denote the following function, which is a sum of uniform distributions on large intervals, of size \( 1/m \) each:

\[
z_i = \sum_{j=1}^{m\sqrt{n}} t_j U[p_j - sgn(p_j)/(m), p_j].
\]

Then for each \( g_i \), we have that \( g_i(x) - z_i(x) > 0 \) for all \( x \in R \), and in addition:

\[
|g_i - z_i| = \left| g_i - \sum_{j=1}^{m\sqrt{n}} \Phi(p_j)U[p_j - sgn(p_j)/m, p_j] \right| \leq \frac{1}{m} \sum_{j=1}^{m\sqrt{n}} \max \{|t_j - t_{j+1}|, |t_j - t_{j-1}|\} + 2 \cdot e^{-n} = O(n/m).
\]

Hence each \( g_i \) may be written as a convex combination \((1-p) \cdot \hat{z}_i + p \cdot y_i\), where \( \hat{z}_i \) is the normalized function \( z_i \), and \( p = O(n/m) \). Since the variables \( g_i \) are independent, and the variables \( \hat{z}_i \) are independent, then the \( n \)-fold distribution of \( g_1, \ldots, g_n \), can be written as a convex sum of the of distribution \( \hat{z}_1, \ldots, \hat{z}_n \), and another distribution, occuring w.p. \( O(n^2/m) \).
6.3.2 Proof of fact 7

**Proof:** By definition, and using Dirac notation:

\[ U^m |v\rangle \sim e^{2\pi i A \cdot m} |v\rangle = \sum_{k=1}^{n} \alpha_k e^{2\pi i m \cdot \lambda_k} |v_k\rangle. \]

Hence:

\[
\frac{1}{\sqrt{2}} [v, \hat{U}_A |v\rangle] = \frac{1}{\sqrt{2}} |0\rangle \otimes |v\rangle + \frac{1}{\sqrt{2}} |1\rangle \otimes \hat{U}_A |v\rangle = \frac{1}{\sqrt{2}} |0\rangle \sum_{k=1}^{n} \alpha_i |v_i\rangle + \frac{1}{\sqrt{2}} |1\rangle \sum_{k=1}^{n} \alpha_i e^{2\pi i m \cdot \lambda_k} |v_k\rangle.
\]

Applying the Hadamard transformation \( H \) we get:

\[
\frac{1}{2} \sum_{k=1}^{n} \alpha_k \left( \left( 1 + e^{2\pi i m \cdot \lambda_k} \right) |0\rangle + \left( 1 - e^{2\pi i m \cdot \lambda_k} \right) |1\rangle \right) \otimes |v_k\rangle =
\]

\[
\frac{1}{2} |0\rangle \otimes \sum_{k=1}^{n} \alpha_k \left( 1 + e^{2\pi i m \cdot \lambda_k} \right) |v_k\rangle + \frac{1}{2} |1\rangle \otimes \sum_{k=1}^{n} \alpha_k \left( 1 - e^{2\pi i m \cdot \lambda_k} \right) |v_k\rangle.
\]

Setting \( v = w_0 \) amounts to post-selecting on the first qubit being \( |0\rangle \). Hence, at step each step, the projection onto the \( i \)-th eigenspace is multiplied by the factor

\[
\left| 1 + e^{2\pi i (m \cdot \lambda_k)} \right|^2 = \frac{1 + \cos(2\pi m \lambda_k)}{2},
\]

so after \( p \) iterations the overall factor is \( \left( \frac{1 + \cos(2\pi m \lambda_k)}{2} \right)^p \).

6.3.3 Proof of fact 8

**Proof:** By fact (7) and Taylor series approximation:

\[ a_{i,j}(x_{i,j}) = \left[ 1 + \frac{1}{2} \cos(2\pi x_{i,j}) \right]^p = \left[ 1 + \frac{1}{2} \left( 1 - \frac{(2\pi x_{i,j})^2}{2!} + \frac{(2\pi x_{i,j})^4}{4!} + \ldots \right) \right]^p. \]

Hence,

\[
\left[ 1 - \frac{(2\pi x_{i,j})^2}{4} \right]^p \leq a_{i,j} \leq \left[ 1 - (2\pi x_{i,j})^2 (1/4 - 1/48) \right]^p.
\]

For each \( i \in [n], j \in [t] \), for which \( x_{i,j} \in B' \) we have:

\[
a_{i,j} \geq \left( 1 - \frac{1}{4} \left( \frac{1}{\log^a(n) \cdot \sqrt{2\log(n^3/\delta)}} \right) 2^\frac{8\log^{2a}(n) \log(n^3/\delta)}{} \right) = \frac{1}{e},
\]

On the other hand, for any \( i, j \), for which \( x_{i,j} \notin B' \), we have

\[
a_{i,j} \leq \left( 1 - \left( \frac{1}{4} - \frac{1}{48} \right) \left( \frac{1}{\log^a(n)} \right) 2^\frac{8\log^{2a}(n) \log(n^3/\delta)}{} \right) \leq \left( \frac{1}{e} \right) \log(n^3/\delta) \leq \delta/n^3.
\]
Fact 12 Let $\Phi(x) : \mathbb{R} \to [0, 1]$ be the standard Gaussian CDF. For any integer $b > 0$ there exists an algorithm running in $\text{poly}(b)$, that for each $z \in [0, 1]$ represented to $b = \text{polylog}(n)$ bits of precision, returns $\hat{z}$, such that $|\hat{z} - \Phi^{-1}(z)| \leq 2^{-b}$.

For the above it is sufficient to use any root-finding algorithm (like Householder, or Newton-Raphson) for the function $\Phi(x) = z$. Since $\Phi$ is $d$-continuously differential for any integer $d$, some $O(\log(n))$ iterations of these algorithms suffice to achieve exponentially small error, in the number of bits.

6.3.4 Proof of fact 6

Proof: The Haar measure on the unit sphere in $\mathbb{C}^n$ can be realized by drawing a standard Gaussian vector and normalizing to unity. Let $U$ be the diagonalizing matrix of $A$. Let $x = (x_1, \ldots, x_n)$ be the Gaussian vector drawn, before normalization. Then since $U$ is unitary then $U \cdot x$ is also Gaussian-distributed, i.i.d., each by $\mathcal{N}(0, 1)$. Hence for each $i \in [n]$, $z_i \equiv \langle x | v_i \rangle$ is also distributed as a complex Gaussian.

Now, using the Gaussian function, for any $i \in [n]$, w.p. $1 - n^{-k}$, we have $|z_i| \leq \sqrt{k \cdot \log(n)}$, and therefore, by the union bound, w.p. $1 - n^{-k+2}$ for all $i, j \in [n]$, $|z_i|/|z_j| \leq \sqrt{k \cdot \log(n)}$. Since normalization does not change these ratios, then w.p. $1 - n^{-k+2}$, for all $i, j$ we have

$$\frac{|\alpha_i|}{|\alpha_j|} = \frac{|x_i|}{|x_j|} \leq \sqrt{k \cdot \log(n)} = \sqrt{2 \log(n) - \log(\varepsilon)}.$$