Minimal slope conjecture of $F$-isocrystals

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Abstract The minimal slope conjecture, which was proposed by K. S. Kedlaya, asserts that two irreducible overconvergent $F$-isocrystals on a smooth variety are isomorphic to each other if both minimal slope constitutions of slope filtrations are isomorphic to each other. We affirmatively solve the minimal slope conjecture for overconvergent $F$-isocrystals on curves and for overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals on smooth varieties over finite fields.

Mathematics Subject Classification 14F30 (Primary) · 14G17 (Secondary)

1 Introduction

In this paper we study the minimal slope conjecture of overconvergent $F$-isocrystals which was proposed by Kedlaya [32]. At first let us explain the conjecture and our main results.

1.1 Slope filtrations

Let us fix the notation as follows:
- $p$: a prime number;
- $k$: a perfect field of characteristic $p$;
• \( R \): a complete discrete valuation field of mixed characteristic with residue field \( k = R/m \);
• \( K \): the field of fractions of \( R \);
• \( a \mapsto |a| \): a multiplicative valuation of \( K \);
• \( \sigma : K \to K \): a \( q \)-Frobenius on \( K \) for a positive power \( q \) of \( p \), that is, a \( p \)-adic continuous homomorphism of fields such that \( \sigma(a) \equiv a^q \pmod{m} \) for any \( a \in R \).

Let \( X \) be a scheme separated locally of finite type over \( \text{Spec} \ k \). Let \( M^\dagger \) be an overconvergent \( F \)-isocrystal on \( X/K \), and denote the convergent \( F \)-isocrystal on \( X/K \) associated to \( M^\dagger \) by \( M \). We say \( M \) admits a slope filtration if there exists an increasing filtration

\[
0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{r-1} \subsetneq M_r = M
\]

of \( M \) as convergent \( F \)-isocrystals on \( X/K \) such that

(i) \( M_i/M_{i-1} \) is nonzero and isoclinic of slope \( s_i \) and
(ii) \( s_1 < s_2 < \cdots < s_r \).

We call \( s_1 \) (resp. \( s_r \)) the minimal slope (resp. the maximal slope) of \( M \) when \( M \neq 0 \). If the Newton polygons of the Frobenius structure of a convergent \( F \)-isocrystal \( M \) are constant on a smooth scheme \( X \), \( M \) admits a slope filtration [32, Corollary 4.2], [44, Corollary 2.6]. It is known that, for any convergent \( F \)-isocrystal \( M \) on a smooth connected scheme \( X \), there exists an open dense subscheme \( U \) of \( X \) such that the Newton polygons of \( M \) are constant by Grothendieck’s specialization theorem (see [32, Theorem 3.1.2] and [44, Proposition 2.2]). Hence \( M \) always admits a unique slope filtration as convergent \( F \)-isocrystals after a certain shrinking of \( X \).

1.2 The minimal slope conjecture

Kedlaya proposed the following problem in [32, Remarks 5.14, 5.15], which we call the minimal slope conjecture.

**Conjecture 1.1** [32, Remark 5.14] Let \( X \) be a smooth connected scheme separated of finite type over \( \text{Spec} \ k \). Let \( M^\dagger \) and \( N^\dagger \) be irreducible overconvergent \( F \)-isocrystals on \( X/K \) such that both convergent \( F \)-isocrystals \( M \) and \( N \) associated to \( M^\dagger \) and \( N^\dagger \) admit the slope filtrations \( \{ M_i \} \) and \( \{ N_j \} \), respectively. Suppose that there is an isomorphism \( h : M_1 \to N_1 \) between the minimal slope constitution of slope filtrations of \( M \) and \( N \) as convergent \( F \)-isocrystals. Then there exists a unique isomorphism \( g^\dagger : M^\dagger \to N^\dagger \) of overconvergent
$F$-isocrystals such that the induced diagram

$$
\begin{align*}
\mathcal{M}_1 & \xrightarrow{h} \mathcal{N}_1 \\
\cap & \cap \\
\mathcal{M} & \xrightarrow{g} \mathcal{N}
\end{align*}
$$

is commutative in the category of convergent $F$-isocrystals on $X/K$.

If $X$ is proper, then $\mathcal{M}_1 = \mathcal{M} = \mathcal{M}^\dagger$ by the irreducibility. Hence the conjecture is trivially true. Ambrosi and D’Addezio proved the conjecture over a finite field $k$ only with the hypothesis of the nontriviality of the morphism $h : \mathcal{M}_1 \to \mathcal{N}_1$ when $\mathcal{N}^\dagger$ is of rank one [5, Theorem 1.1.1]. The version with the relaxed hypothesis on $h$ is called the stronger version of Kedlaya’s conjecture. They applied the result to a generalized Lang–Néron’s theorem on a finiteness of torsion points of Abelian varieties [5, Theorem 1.2.1].

In this paper we will study the following dual form of the minimal slope conjecture which implies Conjecture 1.1.

**Conjecture 1.2** (Dual form of the stronger version of Conjecture 1.1) With the notation in Conjecture 1.1 and renumbering the slope filtration such as

$$
\mathcal{M} = \mathcal{M}^0 \supseteq \mathcal{M}^1 \supseteq \mathcal{M}^2 \supseteq \cdots \supseteq \mathcal{M}^{r-1} \supseteq \mathcal{M}^r = 0
$$

with the sequence of slopes $s^0 > s^1 > \cdots > s^{r-1}$, suppose that there is a nontrivial morphism $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ between the maximal slope quotients as convergent $F$-isocrystals. Then there exists a unique isomorphism $g^\dagger : \mathcal{N}^\dagger \to \mathcal{M}^\dagger$ of overconvergent $F$-isocrystals such that the induced diagram

$$
\begin{align*}
\mathcal{N} & \xrightarrow{g} \mathcal{M} \\
\downarrow & \downarrow \\
\mathcal{N}/\mathcal{N}^1 & \xrightarrow{h} \mathcal{M}/\mathcal{M}^1
\end{align*}
$$

is commutative in the category of convergent $F$-isocrystals on $X/K$.

### 1.3 Results and strategies

In this paper we establish affirmative results for the dual form of the minimal slope conjecture.

**Theorem 1.3** (Corollary 6.5) If $C$ is a smooth connected curve over $\text{Spec} \, k$, then Conjecture 1.2 holds.
Our method is different from Ambrosi and D’Addezio’s monodromy group method in [5]. We may assume $C$ is affine and will compare the modules of global sections of $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ inside the modules of global sections of the maximal slope quotient $\mathcal{M}/\mathcal{M}^1$. The key ingredients are the notion of PBQ (pure of bounded quotient) for overconvergent $F$-isocrystals in both local and global theories, the notion of saturated overconvergent $F$-isocrystals in both local and global theories, and the opposite filtrations of $\varphi$-modules in the local theory.

The notion of PBQ was introduced by Chiarellotto and the author to give a necessary condition of Dwork’s conjecture on the comparison between Frobenius slopes and logarithmic growth for $\varphi$-$\nabla$-modules in [10] (see Remark 3.25). In this paper we globalize the notion of PBQ which requires the space of bounded solutions on the generic disc of the associated Frobenius-differential module at the generic point is pure of Frobenius slope. We introduce the notion of saturation for overconvergent $F$-isocrystals, which requires the module of global sections of $\mathcal{M}^\dagger$ is naturally included in that of the maximal slope quotient $\mathcal{M}/\mathcal{M}^1$. The notion of opposite filtrations of $\varphi$-modules were introduced by De Jong to study the homomorphisms of $p$-divisible groups on local rings of equal characteristic $p$ in [14]. Applying the local theory, we prove the rank of the maximal slope quotient of any overconvergent $F$-isocrystal which is included in $\mathcal{M}/\mathcal{M}^1$ is less than or equal to the rank of $\mathcal{M}/\mathcal{M}^1$. Then an overconvergent $F$-isocrystal on a curve is irreducible if and only if it is PBQ and saturated and the maximal slope quotient $\mathcal{M}/\mathcal{M}^1$ is irreducible (Proposition 6.2). So the given nontrivial morphism $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ is isomorphic and we can compare, in the level of global sections, the irreducible overconvergent $F$-isocrystals $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ in the maximal slope quotient $\mathcal{M}/\mathcal{M}^1$. Then we obtain that $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ coincide with each other by the PBQ property and the upper bound of ranks of maximal slope quotients.

Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. In the case of general dimensions we deal with the case where $k$ is a finite field and study the minimal slope problem for overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals which Abe introduced to establish the celebrated work on the $p$-adic Langlands correspondence and the companion theorem in [1, 2]. Our result for the general dimensions is as follows.

**Theorem 1.4** (Theorem 7.20) *Let $X$ be a smooth connected scheme separated of finite type over the spectrum $\text{Spec} \ k$ of a finite field $k$, and $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ irreducible overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals on $X$ which admit slope filtrations as convergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals. If $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ is a nontrivial morphism between the maximal slope quotients, then there exists an isomorphism $g^\dagger : \mathcal{N}^\dagger \to \mathcal{M}^\dagger$ of overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals.*
Abe and Esnault proved Lefschetz theorem for $\mathbb{Q}_p$-$F$-isocrystals which asserts an existence of a smooth curve $C_\alpha$ passing at any given closed point $\alpha$ in an open dense subscheme of $X$ such that the restriction of the given irreducible $\mathbb{Q}_p$-$F$-isocrystal on $X$ is again irreducible on $C_\alpha$ in [4]. They also applied Lefschetz theorem to the weight theory after Abe–Caro’s work in [3]. Then the coincidence of characteristic polynomials of Frobenius of $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ holds at each closed point $\alpha$ by our study of the minimal slope conjecture on curves. Then applying Čebatarev’s density theorem [1] we obtain an isomorphism $g^\dagger : \mathcal{N}^\dagger \rightarrow \mathcal{M}^\dagger$ of overconvergent $\mathbb{Q}_p$-$F$-isocrystals. However we can not show the compatibility between $g^\dagger$ and the given morphism $h$ at this moment (see Lemma 7.17).

Remark 1.5 D’Addezio solved Crew’s parabolicity conjecture [13, p. 460] and the minimal slope conjecture for arbitrary base field $k$, recently [15].

1.4 Further problems

The minimal slope conjecture seems to be not an $\ell$-adic but a $p$-adic own problem. However, the author also expects to find some consequences in $\ell$-adic theory from the $p$-adic theory with a view point of the companion theorem. In the theory of $p$-adic local systems on a variety of characteristic $p$ one has a naturally wider category that includes the category of local systems arising from geometries. The minimal slope constitution expands in its whole overconvergent $F$-isocrystal if it is irreducible. So the author asks a question that how one can directly recover the whole overconvergent $F$-isocrystal, e.g., its rank, from the minimal slope constitution of the slope filtrations.

Let $X$ be a smooth variety over $\text{Spec} \, k$, and $\mathcal{L}$ a unit-root convergent $F$-isocrystal on $X/K$. Is $\mathcal{L}$ a minimal slope constitution of slope filtration of an irreducible overconvergent $F$-isocrystal? What is the essential image of the natural functor

$$
\left( \begin{array}{c}
\text{overconvergent $F$-isocrystals on $X/K$} \\
\text{admitting slope filtration with the minimal slope 0}
\end{array} \right) \rightarrow \left( \begin{array}{c}
\text{unit-root convergent $F$-isocrystals on $X/K$}
\end{array} \right)
$$

taking the minimal slope constitution of the slope filtrations? A unit-root convergent $F$-isocrystal belonging to the essential image is called “geometric”. For example, the overconvergent $F$-isocrystal $\mathcal{K}\mathcal{L}_\psi$ on $\mathbb{G}_{m F_p}/\mathbb{Q}_p(\zeta_p)$ ($\zeta_p : p$-th root of unity) of rank 2 corresponding to the variation of Kloosterman sums.
\[
 a \mapsto - \sum_{x \in k^\times} \psi \circ tr_{k/\mathbb{F}_p}(x + a/x)
\]

includes a rank one unit-root convergent \( F \)-isocrystal \( L_{\psi} \) on \( \mathbb{G}_m \mathbb{F}_p / \mathbb{Q}_p(\xi_p) \) as a minimal slope constitution, where \( \psi \) is a nontrivial additive character of \( \mathbb{F}_p \) \([18, 23] [42, 6.2]\). One can regard \( L_{\psi} \) as a unit-root convergent \( F \)-isocrystal on \( \mathbb{A}^1 \mathbb{F}_p \), however the Frobenius acts by 1 at \( a = 0 \) and it is not the minimal slope constitution of irreducible objects by the weight reason. Is there an explicit list of rank one geometric unit-root convergent \( F \)-isocrystals on \( \mathbb{G}_m \mathbb{F}_p \)? In the list any positive tensor power \( L_{\psi} \otimes \bigotimes \) of \( L_{\psi} \) is included. We know such a geometric unit-root convergent \( F \)-isocrystal satisfies Dwork’s conjecture on the meromorphy of unit-root \( L \)-functions \([17]\) which was affirmatively solved by Wan \([46–48]\). In one sense our problem is a converse problem of Dwork’s conjecture. Another direction is to characterize geometric unit-root convergent \( F \)-isocrystals by asymptotic behaviors of infinite towers of ramifications along the points at which the Newton polygons jump. The work of Kramer-Miller seems to be in this direction \([34–36]\).

In conclusion the phenomena of slopes and their jumps are mysterious and interesting, which we should study.

### 1.5 Structure of this paper

In Sect. 2 we fix the notation of the local settings and recall the opposite filtration of \( \varphi \)-modules. In Sect. 3 we introduce and study the local and global properties of the PBQ filtrations of overconvergent \( F \)-isocrystals on a curve. In Sect. 4 we prove the local version of the minimal slope conjecture. In Sect. 5 we introduce the notion of saturated overconvergent \( F \)-isocrystals in general dimension. In Sect. 6 we estimate the rank of the maximal slope quotients applying results in Sect. 2 and prove the minimal slope conjecture in the case of curves. In Sect. 7 we study the minimal slope conjecture for overconvergent \( \mathcal{O}_p^\dagger \)-\( F \)-isocrystals on a smooth variety over a finite field. We put two appendices in order to understand overconvergent \( F \)-isocrystals well. We study Frobenius endomorphisms of (partially) \( \dagger \)-spaces, and the trace map of \( F \)-isocrystals with respect to finite base field changes.

### 2 Opposite slope filtrations

In this section we review the opposite filtration of \( \varphi \)-modules which was introduced by de Jong in \([14]\). The opposite filtration is one of the key ingredients to prove the minimal slope conjecture.
2.1 $\varphi$-$\nabla$-modules

Let $R$ be a complete discrete valuation ring with the fraction field $K$ of characteristic 0 and the residue field $k$ which is an arbitrary field of characteristic $p > 0$. We suppose that there exists a $q$-Frobenius on $K$ for a positive power $q$ of $p$ and denote the $\sigma$-fixed subfield by $K_\sigma$ (see Appendix A.1 for Frobenius endomorphisms). Let us fix notation as follows:

- $\mathcal{E}$: the Amice ring over $K$, that is,
  \[ \mathcal{E} = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \mid a_n \in K, \sup_n |a_n| < \infty, a_n \rightarrow 0 \text{ as } n \rightarrow -\infty \right\}; \]

- $\mathcal{R}$: the Robba ring over $K$, that is,
  \[ \mathcal{R} = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \mid a_n \in K, 0 < \exists \eta < 1 \text{ such that } |a_n| \eta^n \rightarrow 0 \text{ as } n \rightarrow -\infty, 0 < \forall \xi < 1 \text{ such that } |a_n| \xi^n \rightarrow 0 \text{ as } n \rightarrow \infty \right\}; \]

- $\mathcal{E}^\dagger$: the bounded Robba ring over $K$, that is,
  \[ \mathcal{E}^\dagger = \left\{ \sum_{n \in \mathbb{Z}} a_n t^n \in \mathcal{R} \mid \sup_n |a_n| < \infty \right\}; \]

- $K[[t]]_0$: the $K$-algebra of bounded functions on the unit disc $D(0, 1^-) = \{ |t| < 1 \}$, that is,
  \[ K[[t]]_0 \triangleq K \otimes_R R[[t]] = \left\{ \sum_{n=0}^{\infty} a_n t^n \in K[[t]] \mid \sup_n |a_n| < \infty \right\}. \]

Let $B$ be one of $\mathcal{E}$, $\mathcal{R}$, $\mathcal{E}^\dagger$ and $K[[t]]_0$. Then $B$ is furnished with

- $|\sum a_n t^n| = \sup_n |a_n|$, which is called the Gauss norm for $\sum_n a_n t^n \in B$;
- $d : B \rightarrow B dt$, which is a continuous $K$-derivation $d(\sum_n a_n t^n) = \sum_n n a_n t^{n-1} dt$;
- $\varphi : B \rightarrow B$, which is a $q$-Frobenius such that $\varphi|_K = \sigma$ and $\varphi(t) \equiv t^q \pmod{m}$. If we treat $\mathcal{E}$, $\mathcal{E}^\dagger$ and $K[[t]]_0$ (resp. $\mathcal{E}$ and $\mathcal{E}^\dagger$, resp. $\mathcal{R}$ and $\mathcal{E}^\dagger$) in the same time, then the Frobenius on $\mathcal{E}$ and $\mathcal{E}^\dagger$ (resp. $\mathcal{E}$, resp. $\mathcal{R}$) is an extension of that of $K[[t]]_0$ (resp. $\mathcal{E}^\dagger$, resp. $\mathcal{E}^\dagger$).

$\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) is a discrete valuation field with residue field $k((t))$ under the Gauss norm, and $K[[t]]_0$ is a principal ideal domain. We denote the integer ring of $\mathcal{E}$ (resp. $\mathcal{E}^\dagger$) by $\mathcal{O}_\mathcal{E}$ (resp. $\mathcal{O}_{\mathcal{E}^\dagger}$). Then $\mathcal{O}_\mathcal{E}$ (resp. $\mathcal{O}_{\mathcal{E}^\dagger}$) is a complete (resp. Henselian) discrete valuation ring and $\mathcal{E}^\dagger$ is algebraically closed in $\mathcal{E}$. 
Definition 2.1 Let $B$ be one of $E$, $E^\dagger$, and $K[[t]]_0$.

(1) A $\varphi$-module over $B$ is a free $B$-module $M$ of finite rank which is furnished with a $B$-linear bijection $\varphi_M : \varphi^* M \to M$.

(2) A $\nabla$-module over $B$ is a free $B$-module $M$ of finite rank which is furnished with a $K$-linear map $\nabla_M : M \to M dt$ such that $\nabla_M(am) = mad_a + a \nabla(m)$ for $a \in B$ and $m \in M$.

(3) A $\varphi$-$\nabla$-module over $B$ is a triplet $(M, \nabla_M, \varphi_M)$ such that $(M, \varphi_M)$ is a $\varphi$-module over $B$ and $(M, \nabla_M)$ is a $\nabla$-module over $B$ satisfying the diagram

\[
\begin{array}{ccc}
\varphi^* M & \xrightarrow{\varphi^* \nabla} & \varphi^*(M dt) \\
\varphi_M & \downarrow & \varphi_M \otimes \varphi \\
M & \xrightarrow{\nabla} & M dt
\end{array}
\]

is commutative.

$\varphi_M$ is called Frobenius and $\nabla_M$ is called a connection. We denote the category of $\varphi$-$\nabla$-modules over $B$ by $\Phi M_B^\nabla$.

Definition 2.2 Let $M$ be a $\varphi$-$\nabla$-module over $E$.

(1) An $E^\dagger$-lattice of $M$ is a $\varphi$-$\nabla$-module $M^\dagger$ over $E^\dagger$ such that $M \cong E \otimes_{E^\dagger} M^\dagger$ as $\varphi$-$\nabla$-modules $M$ over $E$.

(2) A $K[[t]]_0$-lattice of $M$ is a $\varphi$-$\nabla$-module $M_0$ over $K_\alpha[[x_\alpha]]_0$ such that $M \cong E \otimes_{K[[t]]_0} M_0$ as $\varphi$-$\nabla$-modules over $E$. One can also define the notion of $K[[t]]_0$-lattices of $\varphi$-$\nabla$-modules over $E^\dagger$.

The category $\Phi M_B^\nabla$ is a $K_\sigma$-linear Abelian category which is furnished with tensor products $\otimes_B$ and duals $M \mapsto M^\vee$ (see [40, Section 3]). The category does not depend on the choice of $q$-Frobenius $\varphi$ [42, Theorem 3.4.10]. The forgetful functor

\[
\Phi M_{E^\dagger} \to \Phi M_E^\vee, \quad M^\dagger \mapsto M = E \otimes_{E^\dagger} M^\dagger
\]

is fully faithful by [30, Theorem 5.1]. We use the notation $M$ (resp. $M^\dagger$) for a $\varphi$-$\nabla$-module over $E$ (resp. $E^\dagger$), and for a $\varphi$-$\nabla$-module $N^\dagger$ over $E^\dagger$ the image by the above forgetful functor is denoted by $N$. Note that the category of $\varphi$-modules over $E^\dagger$ (resp. $E$) is a $K_\sigma$-linear Abelian category, but, except unit-root objects below, it may depend on the choice of Frobenius and the natural functor from the category over $E^\dagger$ to that over $E$ is not fully faithful (see [40, Remark 2.2.7] for a counter example).

Definition 2.3 (1) A unit-root $\varphi$-module over $E$ is a $\varphi$-module $M$ over $E$ such that there exists a finite free $O_E$-submodule $L$ of $M$ satisfying $E \otimes_{O_E} L = M$.
and \( \varphi(L) \) generates \( L \) as an \( \mathcal{O}_E \)-module. A \( \varphi \)-module over \( E^\dagger \) (resp. a \( \varphi \)-\( \nabla \)-module over \( E \), resp. a \( \varphi \)-\( \nabla \)-module over \( E^\dagger \)) is unit-root if so is the associate \( \varphi \)-module over \( E \).

(2) A \( \varphi \)-module \( M \) over \( E \) is purely of slope \( \frac{m}{n} \) \((m, n \in \mathbb{Z}, n > 0)\) if \( M \otimes^\mathbb{Z} n(M) \) is a unit-root \( \varphi \)-module over \( E \). Here \( M \otimes^\mathbb{Z} n(M) \) means a tensor product of \( n \)-copies of \( M \) with a Frobenius \( \varphi \).

A \( \varphi \)-module over \( E^\dagger \) (resp. a \( \varphi \)-\( \nabla \)-module over \( E \), resp. a \( \varphi \)-\( \nabla \)-module over \( E^\dagger \)) is pure of slope \( \frac{m}{n} \) if so is the associate \( \varphi \)-module over \( E \).

**Proposition 2.4** (See [30, Remark 1.7.8], [9, Theorem 2.4])

(1) Let \( M \) be a \( \varphi \)-module over \( E \). Then \( M \) admits a slope filtration

\[
M = M^0 \supseteq M^1 \supseteq \cdots \supseteq M^{r-1} \supseteq M^r = 0
\]

as \( \varphi \)-modules over \( E \), namely, \( M^i / M^{i+1} \) has a unique Frobenius slope \( s^i \) with \( s^0 > s^1 > \cdots > s^{r-1} \). We call \( s_0 \) the maximal slope of \( M \), and \( M / M^1 \) the maximal slope quotient of \( M \).

(2) If \( M \) is a \( \varphi \)-\( \nabla \)-module over \( E \), then the slope filtration as \( \varphi \)-modules is a filtration as \( \varphi \)-\( \nabla \)-modules over \( E \).

We use the notation above for the slope filtrations of \( \varphi \)-\( \nabla \)-modules over \( E \).

## 2.2 Generalized series

In this subsection we assume that the residue field \( k \) of \( K \) is algebraically closed for simplicity, and \( \sigma \) is a \( q \)-Frobenius on \( K \) for a positive power \( q \) of \( p \) such that the natural map

\[
K \otimes W(F_q) W(k) \to K
\]

is an isomorphism of fields, where \( W(k) \) is the Witt-vectors ring with coefficients in \( k \). We also fix a \( q \)-Frobenius \( \varphi \) on \( E \) and \( E^\dagger \) defined by

\[
\varphi \left( \sum_{n \in \mathbb{Z}} a_n t^n \right) = \sum_{n \in \mathbb{Z}} \sigma(a_n) t^{q^n}.
\]

We introduce Hahn–Mal’cev–Neumann’s generalized power series rings \( \tilde{E} \), \( \tilde{E}^\dagger \) and give their fundamental properties. These rings were studied in [25, Section 4] [26, Section 4] [30, Section 2]. Let us regard \( \mathbb{Q} \) as a well-ordered Abelian group with respect to the usual Archimedean order \( \leq \), and define

\[
\tilde{E} = \left\{ \sum_{n \in \mathbb{Q}} a_n t^n \middle| \begin{array}{l}
a_n \in K, \sup_n |a_n| < \infty, |a_n| \to 0 \text{ (as } n \to -\infty), \\
\text{the support set } \{n \in \mathbb{Q} | |a_n| \geq \delta \} \text{ is well-ordered for any } \delta \in |K^\times|.
\end{array} \right\}
\]
\[ \tilde{E}^\dagger = \{ \sum_{n \in \mathbb{Q}} a_n t^n \in \tilde{E} \mid |a_n| \eta^n \to 0 \text{ (as } n \to -\infty) \text{ for some } 0 < \eta < 1. \}, \]

\[ k((t^\mathbb{Q})) = \left\{ \sum_{n \in \mathbb{Q}} a_n t^n \mid a_n \in k, \text{ the support set } \{ n \in \mathbb{Q} \mid a_n \neq 0 \} \text{ is well-ordered.} \right\} \]

\[ \tilde{\varphi} : \text{the } q\text{-Frobenius on } \tilde{E} \text{ (resp. } \tilde{E}^\dagger) \text{ defined by} \]

\[ \tilde{\varphi} \left( \sum_{n \in \mathbb{Q}} a_n t^n \right) = \sum_{n \in \mathbb{Q}} \sigma(a_n) t^{\eta n}. \]

Here the sum and the product in \( \tilde{E} \) (resp. \( \tilde{E}^\dagger \), resp. \( k((t^\mathbb{Q})) \)) are defined by

\[ \sum_n a_n t^n + \sum_n b_n t^n = \sum_n (a_n + b_n) t^n, \]

\[ \sum_n a_n t^n \times \sum_n b_n t^n = \sum_n \left( \sum_{l+m=n} a_l b_m \right) t^n, \]

which are well-defined by the following lemmas since \( K \) is a complete discrete valuation field. Indeed, the boundedness of elements of \( \tilde{E} \) and Lemma 2.5 (1) (resp. (2)) below implies the well-definedness of products (resp. the existence of inverses for nonzero elements). The case of \( k((t^\mathbb{Q})) \) is similar and it is algebraically closed by [25, Proposition 1]. \( \tilde{E}^\dagger \) is a \( K \)-subalgebra (resp. a subfield) of \( \tilde{E} \) by Lemma 2.6 (3), (4) (resp. (5) and Lemma 2.5 (3)).

**Lemma 2.5** Suppose \( I, J \) are well-ordered subsets of \( \mathbb{Q} \).

1. For any \( n \in \mathbb{Q} \), the set \( \{(i, j) \in I \times J \mid i + j = n\} \) is finite.
2. The sets \( I \cup J \) and \( I + J = \{i + j \mid i \in I, j \in J\} \) are well-ordered.
3. Suppose \( I \subset \mathbb{Q}_{>0} \). If \( mI \) denotes the sum \( I + \cdots + I \) of \( m \) copies of \( I \), then \( \cup_{m \geq 1} mI \) is well-ordered.

**Lemma 2.6** For an element \( a = \sum_n a_n t^n \in \tilde{E} \), let us define a map \( N_a : \mathbb{Z} \to \mathbb{Q} \cup \{\infty\} \) by

\[ N_a(l) = \min\{n \in \mathbb{Q} \mid \text{ord}_\pi(a_n) \leq l\} \cup \{\infty\}. \]

Here \( \pi \) is a uniformizer of \( K \), \( \text{ord}_\pi \) is an additive valuation of \( K \) normalized by \( \text{ord}_\pi(\pi) = 1 \), \( \mathbb{Q} \cup \{\infty\} \) is furnished with the standard order \( < \) such that \( n < \infty \) for all \( n \in \mathbb{Q} \), and \( n + \infty = \infty + n = \infty \) for any \( n \in \mathbb{Q} \cup \{\infty\} \).

1. If \( l \geq m \), then \( N_a(l) \leq N_a(m) \).
2. \( N_{\varphi(a)}(l) = q N_a(l) \) for any \( l \in \mathbb{Z} \).
3. For \( a, b \in \tilde{E} \), the inequality \( N_{ab}(l) \geq \inf_{i+j=l} (N_a(i) + N_b(j)) \) holds for any \( l \in \mathbb{Z} \). Note that it is sufficient to take only finite numbers of \( (i, j) \in \mathbb{Z} \times \mathbb{Z} \).
to compute the infimum above since the coefficients of $a$ and $b$ are bounded and the valuation of $K$ is discrete.

(4) For an element $a \in \mathcal{E}$, the following are equivalent.

(i) $a \in \mathcal{E}^\dagger$.

(ii) There exist $c, d \in \mathbb{Q}$ with $d > 0$ such that $N_a(l) \geq c - dl$ for any $l \in \mathbb{Z}$.

(iii) $N_a(l)/l$ is lower bounded on $l > 0$.

(5) Suppose $a = \sum_n a_n t^n \in \mathcal{E}^\dagger$ such that $|a_n| \leq 1$ for all $n \in \mathbb{Q}$ and $|a_n| < 1$ for all $n < 0$. Then there exists $d \in \mathbb{Q}_{>0}$ such that $N_a(l) \geq -dl$ for all $l \in \mathbb{Z}$. Moreover, $N_{a^m}(l) \geq -dl$ for any $l$.

Proof (4) Let $a = \sum_n a_n t^n \in \mathcal{E}$. Suppose $a \in \mathcal{E}^\dagger$. Then there exist $C, D \in \mathbb{Q}$ with $D > 0$ such that $\text{ord}_\pi(a_n) \geq C - Dn$ for any $n$. Then $N_a(l) \geq C/D - l/D$ for any $l$. The converse also holds. Hence we have (i) $\iff$ (ii). (ii) $\Rightarrow$ (iii) is trivial. Since $\sup_n |a_n| < \infty$, the converse (iii) $\Rightarrow$ (ii) holds.

Lemma 2.7 (1) $\mathcal{E}$ is a complete discrete valuation field under the valuation

$$\left| \sum_n a_n t^n \right| = \sup_n |a_n|$$

with the residue field $k((\mathbb{Q}))$.

(2) $\mathcal{E}^\dagger$ is a discrete valuation field such that the completion under the valuation in (1) is $\mathcal{E}$ and the integer ring $\mathcal{O}_{\mathcal{E}^\dagger}$ is Henselian.

(3) The following is a commutative diagram of extensions of discrete valuation fields with the same valuation group

$$\begin{array}{ccc}
\mathcal{E} & \to & \mathcal{E}^\dagger \\
K & \downarrow & \downarrow \\
\mathcal{E} & \to & \mathcal{E}^\dagger
\end{array}$$

such that $\mathcal{E}^\dagger \cap \mathcal{E} = \mathcal{E}^\dagger$ in $\mathcal{E}$ and the natural morphism

$$\mathcal{E}^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to \mathcal{E}$$

is injective.

(4) The $q$-Frobenius endomorphisms act compatibly in the diagram above. If $(-)_{\varphi}$ denotes the $\varphi$-fixed subset of $(-)$, then

$$\mathcal{E}_{\varphi} = (\mathcal{E}^\dagger)_{\varphi} = \mathcal{E} = (\mathcal{E}^\dagger)_{\varphi} = K_{\sigma}.$$  

Proof (1) $\mathcal{E}$ is complete by Lemma 2.5 (2).

(2) We will prove $\mathcal{O}_{\mathcal{E}^\dagger}$ is Henselian. The rest is easy. Let $f(x) = x^s + a_1 x^{s-1} + \cdots + a_s \in \mathcal{O}_{\mathcal{E}^\dagger}[x]$ be a monic polynomial such that the image $\overline{f}(x)$
of \( f(x) \) in \( k((x^Q)) \) has a simple root \( \overline{\alpha} \in k((x^Q)) \). We may assume that \( \overline{\alpha} = 0 \). Since the residue field \( \mathcal{O}_{\mathcal{E}^+} \) is the algebraically closed field \( k((x^Q)) \), we may also assume that \( \mathcal{T}'(\overline{\alpha}) = 1 \). Let us take a unique element \( \alpha \) in the integer ring \( \mathcal{O}_{\mathcal{E}} \) of \( \mathcal{E} \) such that \( f(\alpha) = 0 \) and \( \alpha \) (mod \( m \mathcal{O}_{\mathcal{E}} \)) = \( \overline{\alpha} = 0 \). Indeed, such an \( \alpha \) exists because the integer ring \( \mathcal{O}_{\mathcal{E}} \) is complete. Let us fix constants \( c, d \in \mathbb{Q} \) with \( c \leq 0, d > 0 \) such that \( N_{\alpha_i}(l) \geq c - dl \) for any \( l \in \mathbb{Z} \) and all \( i \) by Lemma 2.6, where \( a_0 = 1 \) is the coefficient of \( x^n \). Our claim is the inequality

\[
N_{\alpha}(l) \geq -c + (2c - d)l
\]

for any \( l \in \mathbb{Z} \) by induction on \( l \). If \( l \leq 0 \), then \( N_{\alpha}(l) = \infty \). Suppose the equality holds for \( l \leq h \). Then there exist \( \alpha_h, \beta \in \mathcal{O}_{\mathcal{E}} \) with \( \alpha = \alpha_h + \pi^{h+1} \beta \) such that \( N_{\alpha_h}(l) \geq -c + (2c - d)l \) for any \( l \leq h \) and \( N_{\alpha_h}(h) = N_{\alpha_h}(h + 1) = N_{\alpha_k}(h + 2) \cdots \). Since

\[
N_{f(\alpha_h)}(h + 1) \\
\geq \min_i N_{\alpha_i \alpha^{s-i}}(h + 1) \\
\geq \min_i \min_{l_1 + l_2 = h + 1} \min_{l_1, l_2 \geq 0} \left( N_{\alpha_i}(l_1) + N_{\alpha^{s-i}}(l_2) \right) \\
\geq \min_i \min_{l_1 + l_2 = h + 1} \min_{l_1, l_2 \geq 0} \left( N_{\alpha_i}(l_1) + \sum_{j=1}^{s-i} N_{\alpha}(m_j) \right) \\
\geq \min \begin{cases} 
 c - dl_1 - (s - i)c + (2c - d)l_2 
 \quad \text{for} \quad 0 \leq i \leq s \\
 l_1 + l_2 = h + 1, l_2 \geq 0 \\
 \geq 0 \quad \text{if} \quad i \leq s - 2, \\
 l_1 > 0 \quad \text{if} \quad i = s - 1, \\
 = h + 1 \quad \text{if} \quad i = s 
\end{cases}
\]

by Lemma 2.6, the congruence \( f(\alpha_h) + \pi^{h+1} \beta f'(\alpha_h) \equiv f(\alpha_h + \pi^{h+1} \beta) \equiv 0 \) (mod \( \pi^{h+2} \)) implies an inequality

\[
N_{\pi^{h+1} \beta}(h + 1) \geq -c + (2c - d)(h + 1).
\]

Define \( \alpha_{h+1} \) by \( \alpha_h + \pi^{h+1} \beta \) removing the terms whose coefficients are 0 modulo \( \pi^{h+2} \). Then \( \alpha_{h+1} \) satisfies \( N_{\alpha_{h+1}}(l) \geq -c + (2c - d)l \) for any \( l \leq h + 1 \) and \( N_{\alpha_{h+1}}(h + 1) = N_{\alpha_{h+1}}(h + 2) = N_{\alpha_{h+1}}(h + 3) \cdots \). Therefore, \( \alpha \) belongs to \( \mathcal{E}^+ \) and \( \mathcal{O}_{\mathcal{E}^+} \) is Henselian.

(3) We prove the injectivity. For \( b_1, b_2, \ldots, b_r \in \mathcal{E} \), suppose \( a_1 b_1 + \cdots + a_r b_r = 0 \) in \( \mathcal{E} \) for \( a_1, \ldots, a_r \in \mathcal{E}^+ \). We may assume that the union of the
support sets of $a_1, \ldots, a_r$ has a nontrivial intersection with $\mathbb{Z}$. Take subseries $a'_i = \sum_{n \in \mathbb{Z}} a_{i,n} t^n$ of $a_i = \sum_{n \in \mathbb{Q}} a_{i,n} t^n$. Then $a'_i \in \mathcal{E}^{\dagger}$ for any $i$ and $a'_1 b_1 + \cdots + a'_r b_r = 0$. Hence $b_1, b_2, \ldots, b_r$ are linearly independent over $\mathcal{E}^{\dagger}$ if and only if they are linearly independent over $\mathcal{E}^{\dagger}$.

(4) By the universal property of Witt vectors rings, there exists a canonical isomorphism $\mathcal{E} = K_{\sigma} \otimes_{\mathbb{Z}[t]} W(k((t^\mathbb{Q})))$ with respect to the $q$-Frobenius $\varphi = id_{K_{\sigma}} \otimes \text{Frob}^s$ where Frob is the $p$-Frobenius of the Witt-vectors ring and $q = p^s$. The rest easily follow from it. $\square$

Remark 2.8 The field $\mathcal{E}$ (resp. $\mathcal{E}^{\dagger}$) plays a similar role with $\Gamma_2$ (resp. $\Gamma_{2,c}$) in [14, Section 4]. $\mathcal{E}$ (resp. $\mathcal{E}^{\dagger}$) is “larger” than $\Gamma_2$ (resp. $\Gamma_{2,c}$).

The following lemma is a generalization of [40, Proposition 2.2.2] which asserts the similar claim for Frobenius equations over $\mathcal{E}^{\dagger}$.

Proposition 2.9 Let $a_1, \ldots, a_s$ be elements in $O_{\mathcal{E}^{\dagger}}$. Suppose $y = \sum_n y_n t^n \in \mathcal{E}$ satisfies an Frobenius equation

$$\varphi^s(y) + a_1 \varphi^{s-1}(y) + \cdots + a_s y = 0.$$

Then $y$ belongs to $\mathcal{E}^{\dagger}$.

Proof We may assume that $|y| = 1$. Replacing $y$ by $t^m y$ for a sufficient large $m \geq 0$, we may assume that $|y_n| < 1$ for any $n < 0$ and that $|a_{i,n}| < 1$ for any $n < 0$ where $a_i = \sum_n a_{i,n} t^n$. Then there exists $d \in \mathbb{Q}$ with $d > 0$ such that $N_{a_i}(l) \geq -dl$ for any $l \in \mathbb{Z}$, where $d$ does not depend on $i$. Let us estimate $N_{a_i \varphi^{s-i}(y)}(l)/l$ on $l \in \mathbb{Z}$ when $a_i \neq 0$. If $l$ is a sufficiently large positive integer, then

$$N_{a_i \varphi^{s-i}(y)}(l)/l \geq \frac{1}{\min_{l_1, l_2 \geq 0}(N_{a_i}(l_1) + N_{\varphi^{s-i}(y)}(l_2))} \geq \min_{0 < l_2 \leq l} \left\{ -d + \frac{N_{\varphi^{s-i}(y)}(l_2)}{l_2} \right\} \cup \{-d\}$$

by Lemma 2.6. Suppose there exists a positive integer $m$ such that

(i) $N_y(l)/l > N_y(m)/m$ for any $l < m$ and
(ii) $N_y(m)/m < -d$.

If such an $m$ does not exist, then $N_y(l)/l$ is bounded on $l > 0$ and $y \in \mathcal{E}^{\dagger}$ by Lemma 2.5 (4). Since $y$ satisfies the given Frobenius equation, the following estimate

$$q^s N_y(m)/m = N_{\varphi^s(y)}/m = N_{a_1 \varphi^{s-1}(y) + \cdots + a_s y}(m)/m \geq -d + q^{s-1} N_y(m)/m$$
holds. Then it implies $N_y(m)/m \geq -d/(q^s - q^{s-1}) \geq -d$ and contradicts the hypothesis on $m$. Therefore $y \in \tilde{E}$. □

2.3 Opposite filtrations

Let us recall the opposite filtration in our context.

**Proposition 2.10** (cf. [14, Proposition 5.5 (ii)])

1. Let $M$ be a unit-root $\varphi$-module over $E$, put $\tilde{M} = \tilde{E} \otimes_E M$ to be a $\tilde{\varphi}$-module over $\tilde{E}$. Then $\tilde{M}$ is isomorphic to the trivial $\tilde{\varphi}$-module $\tilde{E} \otimes_E \dim M$.

2. Let $N^{\dagger}$ be a unit-root $\varphi$-module over $E^{\dagger}$, put $\tilde{N}^{\dagger} = \tilde{E}^{\dagger} \otimes_{E^{\dagger}} N^{\dagger}$ to be a $\tilde{\varphi}$-module over $\tilde{E}^{\dagger}$. Then $\tilde{N}^{\dagger}$ is isomorphic to the trivial $\tilde{\varphi}$-module $(\tilde{E}^{\dagger}) \otimes_{E^{\dagger}} \dim N^{\dagger}$.

**Proof** (1) Since the residue field $k((t^Q))$ of $\tilde{E}$ is algebraically closed by [25, Proposition 1], it contains an algebraic closure of the residue field $k((t))$ of $E$. Since $\tilde{E}$ is $p$-adically complete, the assertion holds by Dieudonné–Manin’s classification of $F$-spaces.

(2) Take a cyclic vector $e$ of the $\varphi$-module $N^{\dagger}$ [40, Lemma 3.1.4] and let $A$ be a representation matrix of the Frobenius $\varphi_{N^{\dagger}}$ with respect to the basis $e, \varphi_{N^{\dagger}}(e), \varphi_{N^{\dagger}}^2(e), \ldots, \varphi_{N^{\dagger}}^{s-1}(e)$, that is,

$$A = \begin{pmatrix}
1 & a_s \\
1 & a_{s-1} \\
& \ddots \\
& & 1 & a_1
\end{pmatrix} \in \text{GL}_s(\mathcal{O}_{E^{\dagger}}).$$

One can find a matrix $Y \in \text{GL}_s(\tilde{E})$ such that $A\varphi(Y) = Y$ by (1). If we put $Y^{-1} = (z_{i,j})$, then

$$\varphi(z_{i,j}) = z_{i,j+1} (1 \leq j \leq s - 1), \varphi(z_{i,s}) = a_1 z_{i,s} + a_2 z_{i,s-1} + \cdots + a_s z_{i,1}.$$

Hence $z_{i,1}$ satisfies the Frobenius equation

$$\varphi^s(z_{i,1}) = a_1 \varphi^{s-1}(z_{i,1}) + a_2 \varphi^{s-2}(z_{i,1}) + \cdots + a_s z_{i,1}$$

for any $i$. Since the unit-root condition implies $a_1, \ldots, a_s \in \mathcal{O}_{E^{\dagger}}$ and $|a_s| = 1$, we have $z_{i,1}$ is included in $\tilde{E}^{\dagger}$ by Proposition 2.9 and so that $Y \in \text{GL}_s(\tilde{E}^{\dagger})$. Therefore, $N^{\dagger}$ is a trivial $\tilde{\varphi}$-module over $\tilde{E}$. □
Proposition 2.11 (cf. [14, Proposition 5.5]) Let \( N^\dagger \) be a \( \varphi \)-module over \( \mathcal{E}^\dagger \), put \( \tilde{N}^\dagger = \tilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} N^\dagger \) to be a \( \tilde{\varphi} \)-module over \( \tilde{\mathcal{E}}^\dagger \). Then there is a filtration

\[
0 = \tilde{N}_0^\dagger \subset \tilde{N}_1^\dagger \subset \cdots \subset \tilde{N}_r^\dagger = \tilde{N}^\dagger,
\]

of \( \tilde{N}^\dagger \) satisfying

(i) \( \tilde{N}_i^\dagger / \tilde{N}_{i-1}^\dagger \) is pure of slope \( \lambda_i \);
(ii) \( \lambda_1 > \lambda_2 \cdots > \lambda_r \).

Moreover, if \( N = N^0 \supseteq N^1 \supseteq \cdots \supseteq N^r = 0 \) be a slope filtration of \( N = \mathcal{E} \otimes_{\mathcal{E}^\dagger} N^\dagger \), then \( s = r \) and

\[
\tilde{\mathcal{E}} \otimes_{\tilde{\mathcal{E}}^\dagger} \tilde{N}_i^\dagger / \tilde{N}_{i-1}^\dagger \cong \tilde{\mathcal{E}} \otimes_{\mathcal{E}} N^{i-1} / N^i
\]

for all \( i \) as \( \tilde{\varphi} \)-modules over \( \tilde{\mathcal{E}} \). Such an increasing filtration \( \{ \tilde{N}_i^\dagger \} \) is called the opposite slope filtration of \( \tilde{N}^\dagger \).

Proof Since the residue field \( k((t^{\mathcal{Q}})) \) of the complete discrete valuation field \( \tilde{\mathcal{E}} \) is algebraically closed, we have a decomposition \( \tilde{N} = \tilde{\mathcal{E}} \otimes_{\mathcal{E}^\dagger} N^\dagger \cong \oplus_i \tilde{\mathcal{E}} \otimes_{\mathcal{E}} N^i / N^i \) as \( \tilde{\varphi} \)-modules. Then the assertion follows from Lemma 2.12 below. \( \square \)

Lemma 2.12 (cf. [14, Corollary 5.7] (i)) Let \( \tilde{N}^\dagger \) be a nonzero \( \tilde{\varphi} \)-module over \( \tilde{\mathcal{E}}^\dagger \), \( \lambda_1 \) the maximal slope of \( \tilde{N} \), and \( \tilde{N} = \tilde{N}_1 \oplus \tilde{N} \) such that \( \tilde{N}_1 \) is the \( \tilde{\varphi} \)-submodule of \( \tilde{N} \) exactly of slope \( \lambda_1 \) and that \( \tilde{N} \) is the \( \tilde{\varphi} \)-submodule of \( \tilde{N} \) whose slopes are strictly less than \( \lambda_1 \). If \( \eta : \tilde{N}^\dagger \rightarrow \tilde{N} \) is the natural \( \tilde{\mathcal{E}}^\dagger \)-homomorphism defined by \( \tilde{N}^\dagger \subset \tilde{N} \rightarrow \tilde{N} / \tilde{N}_1 = \tilde{N} \), then \( \text{Ker}(\eta) \) is a \( \tilde{\varphi} \)-submodule of \( \tilde{N}^\dagger \) over \( \tilde{\mathcal{E}}^\dagger \) such that \( \tilde{\mathcal{E}} \otimes_{\tilde{\mathcal{E}}^\dagger} \text{Ker}(\eta) = \tilde{N}_1 \).

Proof It is sufficient to prove \( \text{Ker}(\eta) \neq 0 \). Since \( K' \otimes_K \text{Ker}(\eta) = \text{Ker}(\text{id}_{K'} \otimes \eta) \) for a finite extension \( K' \) of \( K \) with an extension of the \( q \)-Frobenius \( \sigma \), we may assume \( |q|^\lambda \in |K| \) by Lemma A.1 (3) in Appendix A. Hence we may assume that the maximal slope \( \lambda \) of \( \tilde{N} \) is 0. Suppose \( m \in \tilde{N} \setminus \{0\} \) satisfies \( \varphi_{\tilde{N}}(m) = m \). Such an \( m \) exists by Proposition 2.10 (1) since \( \tilde{N}^\dagger \neq 0 \) and \( \lambda = 0 \). Then Lemma 2.9 implies \( m \in \tilde{N}^\dagger \). Indeed, take a cyclic vector \( e \) of the dual \( (\tilde{N}^\dagger)^{\vee} \) of \( \tilde{N}^\dagger \) and \( B \in \text{GL}_s(\tilde{\mathcal{E}}^\dagger) \) is the representation matrix of Frobenius \( \varphi_{(\tilde{N}^\dagger)^{\vee}} \) with respect to the basis \( e, \varphi_{(\tilde{N}^\dagger)^{\vee}}(e), \ldots, \varphi_{(\tilde{N}^\dagger)^{\vee}}^{s-1}(e) \). Then all entries of \( B \) belongs to \( \mathcal{O}_{\tilde{\mathcal{E}}^\dagger} \) since all slopes of the dual \( \tilde{N}^\vee \) of \( \tilde{N} \) is \( \geq 0 \). If we use the dual basis of \( e, \varphi_{(\tilde{N}^\dagger)^{\vee}}(e), \ldots, \varphi_{(\tilde{N}^\dagger)^{\vee}}^{s-1}(e) \) as a basis of \( \tilde{N}^\dagger \), then the representation matrix of \( \varphi_{\tilde{N}^\dagger} \) is \( ^t B^{-1} \) where \( ^t B^{-1} \) means the transposition of the matrix \( B^{-1} \). Then, by the similar way of the proof of Proposition 2.10 (2), one can prove \( m \) belongs to \( \tilde{N}^\dagger \) by Lemma 2.9. Since all slopes of \( \tilde{N} \) is strictly less than 0, we have \( \eta(m) = 0 \). \( \square \)
Theorem 2.13 (cf. [14, Corollary 5.7]) Let $N^\dagger$ be a $\varphi$-module over $\mathcal{E}^\dagger$, put $\tilde{N}^\dagger = \tilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} N^\dagger$ to be a $\bar{\varphi}$-module over $\tilde{\mathcal{E}}^\dagger$ with the opposite slope filtration

$$0 = \tilde{N}_0^\dagger \subseteq \tilde{N}_1^\dagger \subseteq \cdots \subseteq \tilde{N}_n^\dagger = \tilde{N}^\dagger,$$

and $\tilde{\eta} : \tilde{N}^\dagger \to \tilde{\mathcal{E}}$ a nonzero injective $\tilde{\mathcal{E}}^\dagger$-homomorphism such that $\tilde{\eta} \circ \varphi \tilde{N}^\dagger = \bar{\varphi} \circ \bar{\varphi}^* (\tilde{\eta})$. Then the slope of $\tilde{N}_1^\dagger$ is 0 and $\dim_{\tilde{\mathcal{E}}^\dagger} \tilde{N}_1^\dagger = 1$.

Proof Since there exists a nonzero morphism $\tilde{\mathcal{E}} \otimes_{\tilde{\mathcal{E}}^\dagger} \tilde{N}_1^\dagger \to \tilde{\mathcal{E}}$ of $\varphi$-modules over $\tilde{\mathcal{E}}$, the slope of $\tilde{N}_1^\dagger$ should be 0. Then $\tilde{N}_1^\dagger$ has a basis $e_1, \ldots, e_{n_1}$ over $\tilde{\mathcal{E}}^\dagger$ such that $\varphi \tilde{N}_1^\dagger (e_i) = e_i$ for $1 \leq i \leq n_1$ by Proposition 2.10 (2). Then

$$\tilde{\eta} (1 \otimes e_i) \in \tilde{\mathcal{E}}^\dagger \varphi = K_{\sigma}.$$

for any $i$ by the commutativity of Frobenius and Lemma 2.7 (2). Since $\tilde{\eta}$ is injective, the equality $\dim_{\tilde{\mathcal{E}}^\dagger} \tilde{N}_1^\dagger = 1$ holds. \hfill \Box

Now we assume the residue field $k$ of $K$ is a perfect field of characteristic $p$ and $\sigma$ is a $q$-Frobenius on $K$ without any extra condition.

Theorem 2.14 Let $N^\dagger$ be a $\varphi$-$\nabla$-module over $\mathcal{E}^\dagger$, and $M$ a $\varphi$-$\nabla$-module over $\mathcal{E}$ such that $M$ has a unique slope. Let $\eta : N^\dagger \to M$ be an injective $\mathcal{E}^\dagger$-homomorphism which is compatible with Frobenius and connections. If $\eta(N^\dagger)$ generates $M$ as an $\mathcal{E}^\dagger$-space, then the maximal slope of $N$ coincides with the slope of $M$ and the equality $\dim_{\mathcal{E}} N / N^1 = \dim_{\mathcal{E}} M$ holds.

Proof Since the maximal slope and its dimension are stable under the scalar extension, we may assume that $k$ is algebraically closed and the natural map $K_{\sigma} \otimes_{W(F_q)} W(k) \to K$ is an isomorphism by Lemma A.1 (2) in Appendix A. Since the category $\Phi \mathbf{M}^\nabla_{\mathcal{E}^\dagger}$ is independent of the choice of $q$-Frobenius $\varphi$ with respect to $\sigma$ [42, Theorem 3.4.10], we may assume that $\varphi(t) = t^q$. We may also assume $M$ is unit-root. Indeed, if $s$ is the slope of $M$, then one can find a finite extension $K'$ of $K$ such that the $q$-Frobenius $\sigma$ extends on $L$ and there exists an element $b \in K'_\sigma$ with $|b| = |q|^s$ by Lemma A.1 (3). Put $\tilde{M} = \tilde{\mathcal{E}} \otimes_{\mathcal{E}} M$, $\tilde{N}^\dagger = \tilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} N^\dagger$. The injectivity of the natural map $\tilde{\mathcal{E}}^\dagger \otimes_{\mathcal{E}^\dagger} \mathcal{E} \to \tilde{\mathcal{E}}$ by Lemma 2.7 (3) implies the homomorphism $\tilde{\eta} : \tilde{N}^\dagger \to \tilde{M}$ induced by $\eta : N^\dagger \to M$ is injective. Therefore, the assertion follows from Theorem 2.13 since $\tilde{M}$ is isomorphic to $\tilde{\mathcal{E}}^\otimes \dim_{\mathcal{E}} M$ as $\bar{\varphi}$-modules by Proposition 2.10 (1) and $\dim_{\mathcal{E}} N / N^1 = \dim_{\tilde{\mathcal{E}}^\dagger} \tilde{N}_1^\dagger$ by Proposition 2.11. \hfill \Box

3 PBQ overconvergent $F$-isocrystals

We introduce the notion of the PBQ filtration for overconvergent $F$-isocrystals on a smooth curve. The notion has been already defined for local objects, $\varphi$-
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∇-modules, by Chiarellotto and the author in [10] in order to give a necessary condition of $\varphi$-$\nabla$-modules over $\mathcal{E}$ which satisfies Dwork’s conjecture on logarithmic growth. We will globalize the notion of PBQ modules in this section.

Let us keep the notation $k, R, K, \sigma$ in Sect. 2 such that the residue field $k$ of $K$ is supposed perfect and $\sigma$ is a $q$-Frobenius on $K$.

### 3.1 Settings

At first we fix our situation (see Appendix A for a brief introduction of $\dag$-algebras, their Frobenius endomorphisms and (over)convergent $F$-isocrystals).

Let us fix the notation as follows:

- $C$: a smooth connected affine curve over $\text{Spec } k$ such that $\overline{C}$ is a smooth completion of $C$ with an open immersion $j_C : C \to \overline{C}$;
- $\widehat{C}$: a smooth affine lift $\text{Spec } A_C$ of $C$ over $\text{Spec } R$;
- $\widehat{\mathcal{C}}$: the $p$-adic formal completion of $\mathcal{C}$.

Let us fix a projective smooth lift $\overline{C}$ of $C$ over $\text{Spec } R$ [22, III, Corollaire 7.4]. Then one can regard $\widehat{\mathcal{C}}$ as an open formal subscheme of the $p$-adic formal completion $\widehat{\mathcal{C}}$ over $\text{Spf } R$ by [45, Proposition 2.4.4 (i)]. Let us put

$$A^\dag_{C,K} = A^\dag_C \otimes_R K = \Gamma(\overline{C}\{\overline{\mathcal{C}}\}, j_{C\{\overline{\mathcal{C}}\}}^\dag \mathcal{O}_{\overline{C}\{\overline{\mathcal{C}}\}})$$

where $A^\dag_C$ is the $p$-adically weak completion of $A_C$

$$\widehat{A}_{C,K} = \widehat{A}_C \otimes_R K = \Gamma(\overline{C}\{\overline{\mathcal{C}}\}, j_{C\{\overline{\mathcal{C}}\}}^\dag \mathcal{O}_{\overline{C}\{\overline{\mathcal{C}}\}})$$

where $\widehat{A}_C$ is the $p$-adically formal completion of $A_C$.

Here $j_{C\{\overline{\mathcal{C}}\}}^\dag \mathcal{O}_{\overline{C}\{\overline{\mathcal{C}}\}}$ is the sheaf of functions on the rigid analytic space $\overline{C\{\overline{\mathcal{C}}\}} = \widehat{\mathcal{C}}$ over $\overline{C\{\overline{\mathcal{C}}\}}$. Note that $A^\dag_C$ and $\widehat{A}_C$ are independent of the choice of the lift $\mathcal{C}$ of $C$ up to $R$-isomorphisms, and are Noetherian integral domains by [21, Theorem]. Since $\mathcal{C}$ is smooth over $\text{Spec } R$, there exist a $q$-Frobenius

$$\varphi : A^\dag_{C,K} \to A^\dag_{C,K}$$

with $\varphi|_K = \sigma$ and a continuous derivation

$$d : A^\dag_{C,K} \to A^\dag_{C,K} \otimes_{A_C} \Omega^1_{A_C/R}$$
Any $q$-Frobenius $\varphi$ is faithfully flat since $C$ is smooth over $\text{Spec } k$. Let us define a $p$-adic lift $E_\eta$ of the function field $k(C)$ of $C$ by

$$E_\eta = \text{the } p\text{-adic completion of the localization } (A_C)_m$$

and then tensoring $\otimes K$,

where $\eta$ means the generic point of $C$. $E_\eta$ is an extension of complete valuation fields over $K$ having the same valuation group with $K$ such that the residue field of $E_\eta$ is the function field $k(C)$ of $C$. Then the Frobenius $\varphi$ and the derivation $d$ extend uniquely on $E_\eta$.

For a closed point $\alpha \in \overline{C}$ with the canonical $i_{\alpha, \overline{C}} : \alpha \to \overline{C}$, let us put

$$k_\alpha : \text{the function field of } \alpha;$$

$$K_\alpha : \text{the finite unramified extension of } K \text{ such that the residue field is } k_\alpha;$$

$$R_\alpha : \text{the integer ring of } K_\alpha;$$

$$x_\alpha \in \mathcal{O}_{\overline{C}}: \text{a lift of local coordinate of } \overline{C} \text{ at } \alpha.$$

When $\alpha \in C$, we have two natural commutative diagrams

$$\hat{A}_{C,K} \to K_\alpha[[x_\alpha]]_0 \quad A^\dagger_{C,K} \to A^\dagger_{C\setminus\{\alpha\},K} \to \mathcal{E}_{\alpha}^\dagger$$

$$E_\eta \to \mathcal{E}_{\alpha} \quad \hat{A}_{C,K} \to \hat{A}_{C\setminus\{\alpha\},K} \to \mathcal{E}_{\alpha}$$

where $\mathcal{E}_{\alpha}, \mathcal{E}_{\alpha}^\dagger, K_\alpha[[x_\alpha]]_0$ are the rings with coordinate $x_\alpha$ over $K_\alpha$ which are introduced in Sect. 2.1. Moreover, the $q$-Frobenius $\varphi$ and the derivation $d$ on $A^\dagger_{C,K}$ extend uniquely and compatibly on $K_\alpha$ and all rings in the diagrams above.

Let us introduce partially weakly complete finitely generated (w.c.f.g.) algebras in order to study the relation of various $\dagger$-algebras and $p$-adically complete algebras. Let $V$ be an open subscheme of $\overline{C}$ over $\text{Spec } k$ such that $C \subset V \subset \overline{C}$ with the open immersion $j_{C,V} : C \to V$ and consider the $K$-algebra

$$\hat{A}_{C,V,K} = \Gamma([V], j_{C,O_{\overline{C}}}) \sim = (\hat{A}_V[z]^\dagger / (yz - 1)) \otimes_R K.$$
Lemma 3.1 With the notation as above, let $\alpha$ be a closed point in $V \setminus C$. Consider the natural commutative diagram

$$
\xymatrix{
\widehat{A}^{\dagger}_{C \cup \{\alpha\}, V, K} \ar[r] \ar[d] & \widehat{A}^{\dagger}_{C, V, K} \ar[r] \ar[d] & \widehat{A}^{\dagger}_{C, V \setminus \{\alpha\}, K} \\
K_{\alpha}[\{x_{\alpha}\}]_{0} \ar[r] & \mathcal{E}^{\dagger}_{\alpha} \ar[r] & \mathcal{E}_{\alpha}.
}
$$

(1) The three vertical morphisms above are flat.
(2) $\widehat{A}^{\dagger}_{C \cup \{\alpha\}, V, K} \to \widehat{A}^{\dagger}_{C, V, K}$ is flat and $\widehat{A}^{\dagger}_{C, V, K} \to \widehat{A}^{\dagger}_{C, V \setminus \{\alpha\}, K}$ is faithfully flat.
(3) The equality $\widehat{A}^{\dagger}_{C \cup \{\alpha\}, V, K} = K_{\alpha}[\{x_{\alpha}\}]_{0} \cap \widehat{A}^{\dagger}_{C, V, K}$ (resp. $\widehat{A}^{\dagger}_{C, V, K} = \mathcal{E}^{\dagger}_{\alpha} \cap \widehat{A}^{\dagger}_{C, V \setminus \{\alpha\}, K}$) holds in $\mathcal{E}^{\dagger}_{\alpha}$ (resp. $\mathcal{E}_{\alpha}$).

Proof We may assume that $V$ is affine and $\alpha \in V, C = V \setminus \{\alpha\}$ by glueing after Tate’s acyclic theorem and the fact that $\{V_{\overline{\alpha}}\}$ is quasi-separated and quasi-compact. In this case $\widehat{A}^{\dagger}_{C \cup \{\alpha\}, V, K} = \widehat{A}_{V} = \widehat{A}^{\dagger}_{C \cup \{\alpha\}}$ and $\widehat{A}^{\dagger}_{C, V \setminus \{\alpha\}, K} = \widehat{A}_{C, K}$. Moreover, we may assume that our local coordinate $x_{\alpha}$ at $\alpha$ belongs to $A_{V}$ and the closed subscheme defined by $x_{\alpha} = 0$ in $V$ consists of only $\alpha$ after shrinking $V$.

(1) Let $I_{\alpha}$ be the maximal ideal of $A_{V}$ of the closed point $\alpha$. Since $R_{\alpha}[\{x_{\alpha}\}]$ is naturally isomorphic to the $I_{\alpha}$-adic completion of $\widehat{A}_{V}$, $R_{\alpha}[\{x_{\alpha}\}]$ is flat over $\widehat{A}_{V}$. Since $\widehat{A}^{\dagger}_{C, V, K}$ (resp. $\widehat{A}^{\dagger}_{C, V \setminus \{\alpha\}, K}$) is an integral domain and $\mathcal{E}^{\dagger}_{\alpha}$ (resp. $\mathcal{E}_{\alpha}$) is a field, the rest is trivial.

(2) Since $\widehat{A}^{\dagger}_{C \cup \{\alpha\}, V, K} = \Gamma(V_{\overline{\alpha}}, J_{C \cup \{\alpha\}}^{\dagger} \mathcal{O}_{\mathcal{C}_{\overline{\alpha}}}) \to \Gamma(V_{\overline{\alpha}}, J_{C \cup \{\alpha\}}^{\dagger} \mathcal{O}_{\mathcal{C}_{\overline{\alpha}}}) = \widehat{A}_{C, K}$ is flat, we have only to prove $\widehat{A}^{\dagger}_{C, V, K} \to \widehat{A}_{C, K}$ is faithfully flat. The faithful flatness holds because the $m$-adic topological ring $\widehat{A}^{\dagger}_{C, V}$ is a Zariski ring. Indeed, any element in $1 + m\widehat{A}^{\dagger}_{C, V}$ is a unit in $\widehat{A}^{\dagger}_{C, V}$ so that any maximal ideal of $\widehat{A}^{\dagger}_{C, V}$ includes $m\widehat{A}^{\dagger}_{C, V}$.

(3) We may assume that $\alpha$ is $k$-rational by gluing by sections of rigid analytic spaces since $\widehat{A}_{C, K} \cap \widehat{A}^{\dagger}_{C, V, L} = \widehat{A}^{\dagger}_{C, V, K}$ by the equality $\widehat{A}^{\dagger}_{C, V, L} = \widehat{A}^{\dagger}_{C, V, K} \otimes_{K} L$ for any base change by a finite unramified extension $L$ of $K$ with the residual extension $l$ of $k$ and $C_l = C \times_{\text{Spec}k} \text{Spec} l, V_l = V \times_{\text{Spec}k} \text{Spec} l$. Then we have only to prove, if $V = C \cup \{\alpha\}$, then the equality

$$
\widehat{A}_{V, K} = K_{\alpha}[\{x_{\alpha}\}]_{0} \cap \widehat{A}_{C, K}
$$

holds. Indeed, the equality above implies $\widehat{A}^{\dagger}_{C, V, K} = \mathcal{E}^{\dagger}_{\alpha} \cap \widehat{A}_{C, K}$ because, for $h = \sum n_{x_{\alpha}}^{c_{n}} \in \mathcal{E}_{\alpha}$ (resp. $\mathcal{E}^{\dagger}_{\alpha}$), the minus part $h^{(-)} = \sum_{n < 0} c_{n} x_{\alpha}^{n}$ belongs to $\widehat{A}_{C, K}$ (resp. $\widehat{A}^{\dagger}_{C, V, K}$) and $h - h^{(-)} \in \widehat{A}_{V, K}$. The equality follows from
the integral version of the identification
\[ \hat{\Lambda}_V = R[[x_\alpha]] \cap \hat{\Lambda}_C \]

which easily comes from the equality \( \Gamma(V, \mathcal{O}_V) = \{ f \in \Gamma(C, \mathcal{O}_C) \mid v_\alpha(f) \geq 0 \} \), where \( v_\alpha \) is a valuation of \( k(C) \) at \( \alpha \), since all rings in the integral version are \( p \)-adically complete. \( \square \)

**Lemma 3.2**

1. \( \hat{\Lambda}_{C,K} = \bigcap_{\alpha \in C} E_\eta \cap K_\alpha[[x_\alpha]]_0. \)
2. \( A_{C,K}^\dagger = \bigcap_{\alpha \in C^\dagger} E_\eta \cap K_\alpha[[x_\alpha]]_0 \bigcap_{\alpha \in C^\dagger} E_\eta \cap \hat{\mathcal{E}}_\alpha^\dagger. \)

**Proof** (1) Since all the rings are \( p \)-adically complete, it is sufficient to prove \( A_{C/mA_C} = \bigcap_{\alpha \in C} k(C) \cap k_\alpha[[x_\alpha]] \). Here \( k(C) \) is the field of functions of \( C \). It holds because \( A_{C/mA_C} \) is a normal integral domain.

(2) For any closed point \( \alpha \in \overline{C} \setminus C \), there exists an affine open subscheme \( V \) of \( \overline{C} \) such that \( \alpha \in V \) and \( \alpha \) is defined by a single element of \( \Gamma(V, \mathcal{O}_V) \) in \( V \).

By using the gluing argument, the first equality follows from Lemma 3.1 (3). The second equality follows from (1). \( \square \)

### 3.2 Frobenius slope filtration

Let \( \mathcal{M}^\dagger \) be an overconvergent \( F \)-isocrystal on \( C/K \), and put
\[
\begin{align*}
M^\dagger &= \Gamma(\overline{C}[\xi], \mathcal{M}^\dagger), \quad M = \hat{\Lambda}_{C,K} \otimes A^\dagger_{C,K} M^\dagger = \Gamma(\overline{C}, \mathcal{M}) \\
M_\eta &= E_\eta \otimes \hat{\Lambda}_{C,K} M, \\
M_\alpha^\dagger &= \mathcal{E}_\alpha \otimes A^\dagger_{C,K} M^\dagger, \quad M_\alpha = \mathcal{E} \otimes \hat{\Lambda}_{C,K} M \quad \text{for a closed point } \alpha \text{ of } \overline{C} \setminus C.
\end{align*}
\]

\( M^\dagger \) is a projective \( A^\dagger_{C,K} \)-module of finite rank with a \( K \)-connection \( \nabla_{M^\dagger} : M^\dagger \to M^\dagger \otimes_{A_C} \Omega_{A_C/K}^1 \) and an isomorphism \( \varphi_{M^\dagger} : \varphi^* M^\dagger \to M^\dagger \) which is called Frobenius such that \( (\varphi_{M^\dagger} \otimes \varphi) \circ \varphi^* \nabla_{M^\dagger} = \nabla_{M^\dagger} \circ \varphi_{M^\dagger} \). The similar hold for the \( \hat{\Lambda}_{C,K} \)-module \( M \) and the \( E_\eta \)-space \( M_\eta \). There are various natural isomorphisms
\[
\begin{align*}
\hat{\Lambda}_{C,K} \otimes A^\dagger_{C,K} M^\dagger &\cong M, \quad E_\eta \otimes \hat{\Lambda}_{C,K} M \cong M_\eta, \\
\mathcal{E}_\alpha \otimes A^\dagger_{C,K} M^\dagger &\cong M_\alpha^\dagger, \quad \mathcal{E}_\alpha \otimes \hat{\Lambda}_{C,K} M \cong \mathcal{E}_\alpha \otimes E_\eta M_\eta \cong M_\alpha,
\end{align*}
\]

of \( \varphi \)-\( \nabla \)-modules.
**Proposition 3.3** (See [30, Remark 1.7.8], [9, Theorem 2.4], and Proposition 2.4) There exists a slope filtration \( \{ M_{\eta}^i \} \) of \( M_{\eta} \) with respect to Frobenius \( \varphi_{M_{\eta}} \) as \( \varphi \)-\( \nabla \)-modules over \( E_{\eta} \) such that \( \{ E_\alpha \otimes_{E_{\eta}} M_{\eta}^i \} \) coincides with the slope filtration \( \{ M_{\alpha}^i \} \) of \( M_{\alpha} \) with respect to Frobenius \( \varphi_{M_{\alpha}} \).

### 3.3 Generic disc

Let \( B \) be one of \( E_{\eta} \) and \( E_\alpha \) for a closed point \( \alpha \) in \( \overline{C} \), and put \( * = \eta \) or \( \alpha \) respectively. We recall the notion of generic open unit disc (see [11, 2.5, 4.1, 4.6] and [9, 0.4]). Let \( B^\tau \) be the complete discrete valuation ring which is isomorphic to \( B \), but the coordinate \( x_* \) is replaced by \( t_* \) in \( B^\tau \) for \( * = \eta \) or \( \alpha \). In the case where \( * = \eta \) we fix a generically etale morphism \( C \to \mathbb{A}^1_{\mathbb{R}} \) over \( \text{Spec} \mathbb{R} \) and take the coordinate \( x_\eta = x \) which is the standard one of the affine line \( \mathbb{A}^1_{\mathbb{R}} \). Then \( B^\tau \) is also an extension of \( K \) as a complete discrete valuation field with the same valuation groups. We introduce two \( B^\tau \)-algebras \( B^\tau[[X - t_*]]_0 \) and \( \mathcal{A}_{B^\tau}(t_*, 1) \), called the ring of bounded functions on the unit open generic disc and a ring of analytic functions on the unit open generic disc respectively as follows:

\[
B^\tau[[X - t_*]]_0 = \left\{ \sum_n a_n(X - t_*)^n \in B^\tau[[X - t_*]] \mid \sup_n |a_n| < \infty \right\},
\]

\[
\mathcal{A}_{B^\tau}(t_*, 1) = \left\{ \sum_n a_n(X - t_*)^n \in B^\tau[[X - t_*]] \mid \sum_n a_n(X - t_*)^n \text{ is convergent,} \right. \\
\left. \left| X - t_* \right| < 1. \right\}
\]

For an element \( f \in B \), we put

\[
f^\tau = \sum_{n=0}^{\infty} \frac{d^n f}{dx^n}(t_*) \frac{(X - t_*)^n}{n!}
\]

where \( g(t_*) \) means the evaluation of \( g \) at \( x_* = t_* \) for \( g \in B \). We define a \( q \)-Frobenius \( \varphi^\tau \) on \( B^\tau[[X - t_*]]_0 \subset \mathcal{A}_{B^\tau} \) by

\[
\varphi^\tau|_{B^\tau} = \varphi \text{ under the isomorphism } B^\tau \to B \text{ (} t_* \mapsto x_* \text{),}
\]

\[
\varphi^\tau(X - t_*) = \varphi(x_*)^\tau - \varphi^\tau(t_*).
\]

Since \( \varphi(x_*)^\tau - \varphi^\tau(t_*) \in (X - t_*)B^\tau[[X - t_*]]_0, \varphi^\tau \) is well-defined.

**Lemma 3.4** The application \( f \mapsto f^\tau \) is a \( K \)-algebra homomorphism such that

\[
(i) \quad \left( \frac{d f}{d x_*} \right)^\tau = \frac{d f^\tau}{d X};
\]

\[\square\] Springer
(ii) $\varphi(f)^\tau = \varphi^\tau(f^\tau)$.

Let $M$ be a $\nabla$-module (resp. a $\varphi$-$\nabla$-module) $M$ over $B$. We define a $\nabla$-module (resp. a $\varphi$-$\nabla$-module) $M^\tau$ over $B[[X - t_*]][0]$ associated to $M$ by

$$M^\tau = B[[X - t_*]][0] \otimes_B M,$$

$$\nabla_{M^\tau}(a \otimes m) = a \otimes \nabla_M(m) + \frac{da}{dX} \otimes mdX,$$

(resp. $\varphi_{M^\tau}(a \otimes m) = \varphi^\tau(a) \otimes \varphi_M(m)$).

If the matrix representation of the connection $\nabla_M$ of $M$ of arbitrary order is given by

$$\nabla_M \left( \frac{d}{dx_*} \right)^n (e_1, \ldots, e_s) = (e_1, \ldots, e_s) C_n \quad C_n \in M_s(B)$$

for any nonnegative integer $n$, where $C_0$ is a unit matrix, then the connection of $M^\tau$ is given by

$$\nabla_{M^\tau} (1 \otimes e_1, \ldots, 1 \otimes e_s) = (1 \otimes e_1, \ldots, 1 \otimes e_s) C_1^\tau dX.$$

Hence the solution matrix of $M^\tau$, which is called the generic solution matrix of $M$ at the generic point $t_*$, is

$$Y = \sum_{n=0}^{\infty} C_n(t_*) \frac{(X - t_*)^n}{n!}.$$

**Definition 3.5** Let $M$ be a $\nabla$-module over $B$.

(1) $M$ is said to be solvable if, for any $0 < \eta < 1$,

$$\left| \frac{1}{n!} C_n \right| \eta^n \to 0 \quad \text{as} \quad n \to \infty.$$

Equivalently, $M$ is solvable if all entries of the solution matrix $Y$ above belong to $A_{B^\tau}(t_*, 1)$.

(2) $M$ is bounded if $M$ is solvable and satisfies

$$\sup_n \left| \frac{1}{n!} C_n \right| < \infty.$$

Equivalently, $M$ is bounded if all entries of the solution matrix $Y$ above belong to $B^\tau[[X - t_*]][0]$. 
**Proposition 3.6** The notion of solvability (resp. boundedness) of $\nabla$-modules over $B$ does not depend on the choice of basis $e_1, \ldots, e_s$ of $M$ and the choice of coordinate $x_\alpha$.

*Proof* The independence of the choice of basis follows from the fact that the map $f \mapsto f^\mathcal{T}$ preserves units. The independence of coordinates follows from the fact that the formal lift $\widehat{A_C}$ of $A_C$ is independent of the choices up to continuous isomorphisms. Hence, for another choice of coordinate $x'_\alpha$, one has a continuous isomorphism $B_{x'_\alpha} \cong B_{x_\alpha}$ so that $B_{e'_\alpha}[[X - t'_\alpha]]_0 \cong B_{x_\alpha}[[X - t_\alpha]]_0$ and $A_{B_{x_\alpha}}(t'_\alpha, 1) \cong A_{B_{x_\alpha}}(t_\alpha, 1)$. Here $B_{x'_\alpha}$ is the differential field with coordinate $x'_\alpha$. $\square$

**Proposition 3.7** The category of solvable (resp. bounded) $\nabla$-modules over $B$ is Abelian, and it is closed under tensor products and duals.

*Proof* One can easily see the existence of duals follows from the solvable $\nabla$-module over $B$ is bounded. The boundedness follows from [9, Lemma 1.7]. $\square$

Because one can choose the coordinate $x_\alpha$ at $\alpha$ as a coordinate of $E_\eta$, Proposition 3.6 implies the proposition below.

**Proposition 3.8** Let $M_\eta$ be a $\nabla$-module over $E_\eta$. $M_\eta$ is solvable (resp. bounded) if and only if so is $E_\alpha \otimes_{E_\eta} M_\eta$, for a closed point (all closed points) $\alpha$ of $\overline{C}$.

### 3.4 Bounded $\varphi$-$\nabla$ modules

In this subsection we study properties of bounded $\varphi$-$\nabla$-modules over $B = E_\eta$ or $E_\alpha$. At first we recall a well-known fact (see [9, Theorem 6.6] for example).

**Proposition 3.9** Let $M$ be a $\varphi$-$\nabla$-module over $B$. Then the following hold.

1. $M$ is solvable.
2. If $M$ is unit-root, then $M$ is bounded.

The following theorem is a characterization of bounded $\varphi$-$\nabla$-modules. Chiarellotto and the author proved it in the local case, i.e., $B = E_\alpha$ in [10, Theorem 4.1]. Here we prove the assertion in the case where $B = E_\eta$ by using the local result.

**Theorem 3.10** Let $M$ be a $\varphi$-$\nabla$-module over $B$ with the slope filtration $\{M^i\}$. Then $M$ is bounded if and only if $M \cong \bigoplus_i M^i / M^{i+1}$ as $\varphi$-$\nabla$-modules over $B$. $\square$
Let us define the \( p \)-adic completion \( \hat{E}^\text{perf}_\eta \) of the perfection of the residue field of \( E_\eta \) by

\[
\hat{E}^\text{perf}_\eta = \text{the } p \text{-adic completion of } \lim_{\rightarrow} \left( E_\eta \xrightarrow{\varphi} E_\eta \xrightarrow{\varphi} \cdots \right).
\]

One can regard \( \hat{E}^\text{perf}_\eta \) as a subfield of \( \tilde{E}_\alpha \) in Sect. 2.2 by the natural embedding of direct systems

\[
\left( E_\eta \xrightarrow{\varphi} E_\eta \xrightarrow{\varphi} \cdots \right) \rightarrow \left( \tilde{E}_\alpha \xrightarrow{\varphi} \tilde{E}_\alpha \xrightarrow{\varphi} \cdots \right).
\]

**Lemma 3.11** With the notation as above, \( E_\eta = \hat{E}^\text{perf}_\eta \cap \tilde{E}_\alpha \) in \( \tilde{E}_\alpha \) for any closed point \( \alpha \in \overline{C} \).

**Proof** Denote the perfection of \( (\ ) \) by \( (\ )^{\text{perf}} \), and the completion of \( k(C) \) along \( \alpha \) by \( k(C)_\alpha \). Since all the items are \( p \)-adically complete discrete valuation fields with the same valuation group, the assertion follows from the fact

\[
k(C) = k(C)^{\text{perf}} \cap k(C)_\alpha
\]

where the intersection is taken in \( (k(C)_\alpha)^{\text{perf}} \). Indeed, let \( D : k(C) \to k(C) \) be the derivation induced by the standard derivation of \( k(\mathbb{P}^1_k) \) and the finite separable morphism \( C \to \mathbb{P}^1_k \). Then the \( p \)-power subfield \( k(C)^p \) of \( k(C) \) is the kernel of \( D \) since \( [k(C) : k(C)^p] = p \). On the other hand, \( D \) extends on \( k(C)_\alpha \) and, if \( u \) belongs to \( (k(C)_\alpha)^p \), then \( D(u) = 0 \). Hence

\[
k(C)^p = k(C) \cap (k(C)_\alpha)^p.
\]

The desired equality easily follows. \( \square \)

**Proof of Theorem 3.10** Let

\[
A = \begin{pmatrix}
A_{11} & \cdots & A_{1r} \\
\vdots & \ddots & \vdots \\
0 & \cdots & A_{rr}
\end{pmatrix}
\]

be the representation matrix of Frobenius \( \varphi_{M_\eta} \) with respect to the slope filtration of \( M_\eta \). Since \( \hat{E}^\text{perf}_\eta \) is \( p \)-adically complete, one has a unique solution \( X \) which is a upper triangle matrix with entries in \( \hat{E}^\text{perf}_\eta \) satisfying

\[
A \varphi(X) = X \begin{pmatrix}
A_{11} & 0 \\
\vdots & \ddots \\
0 & A_{rr}
\end{pmatrix}, \quad X = \begin{pmatrix}
E & * \\
\vdots & \ddots \\
0 & E
\end{pmatrix}
\]
because all the slopes of $A_i$’s are different. Here $E$ denotes the unit matrix of certain size. On the other hand we know $X \in M_n(\hat{E}_\eta^\text{perf}) \subset M_n(\tilde{E}_\alpha)$ belongs to $M_n(E_\alpha)$ by [10, Theorem 4.1]. Hence $X$ belongs to $M_n(E_\eta)$ by Lemma 3.11. □

### 3.5 Maximally bounded quotients

The following theorem was studied using $p$-adic functional analysis in [11, Sect. 4.3]. In this paper we use the assertion only in the case with Frobenius structures.

**Theorem 3.12** [11, Théorème 4.3,5] Let $M$ be a solvable $\nabla$-module over $B$. Then there exists a unique minimal $\nabla$-submodule $M_b$ of $M$ over $B$ such that $M/M_b$ is nonzero and bounded. Here “minimal” means, if $N$ is a $\nabla$-submodule of $M$ over $B$ such that $M/N$ is bounded, then $M_b \subset N$. Moreover, if $M$ is a $\varphi$-$\nabla$-module over $B$, then $M_b$ is a $\varphi$-$\nabla$-submodule of $M$ over $B$.

**Proof** Let $I$ be a set of $\nabla$-submodules over $B$ of $M$ such that, for any $N \in I$, $M/N$ is bounded. Our claim in the first part is (i) $\cap N \in I_N \in I$ and (ii) $I \supseteq \{M\}$. Suppose $N_1, N_2 \in I$. Since the natural morphism $M/(N_1 \cap N_2) \to M/N_1 \oplus M/N_2$ is injective, $M/(N_1 \cap N_2)$ is bounded so that $N_1 \cap N_2 \in I$. Hence the finite dimensionality of $M$ implies the claim (i). In the case where $M$ is a $\varphi$-$\nabla$-module over $B$ the claim (ii) follows from Propositions 3.3 and 3.9. In general case it was proved in [11, Proposition 4.3.4].

If $M$ is $\varphi$-$\nabla$-module over $B$, then $M/\varphi^*M_b \cong \varphi^*(M/M_b)$ is also bounded, hence $\varphi^*M_b \in I$. Comparing with the dimensions, we have $\varphi^*M_b = M_b$. Hence, $M_b$ is a $\varphi$-$\nabla$-submodule of $M$. (See also [9, Proposition 4.8]). □

**Definition 3.13** For a solvable $\nabla$-module $M$ over $B$, the unique nontrivial quotient $M/M_b$ is called the maximally bounded quotient of $M$.

The following theorem is a bounded version of [9, Proposition 1.10] which implies the stability of logarithmic growth filtrations by scalar extensions.

**Proposition 3.14** Let $M_\eta$ be a solvable $\nabla$-module over $E_\eta$, $\alpha$ a closed point in $\overline{C}$ and choose the coordinates $x$ of $\hat{A}_C$ and $x_\alpha$ of $\mathcal{E}_\alpha$ such that $x = x_\alpha$. If one defines $E_\eta^\tau$-spaces $\text{Sol}(M_\eta^\tau)$ and $\text{Sol}(M_\eta^\tau)$ by

\[
\text{Sol}(M_\eta^\tau) = \begin{cases} 
\{ f : M_\eta^\tau \to A_{E_\eta^\tau}((t_\eta, 1), f \text{ is } E_\eta^\tau[[X - t_\eta]]_0\text{-linear such that } \\
\text{ } \text{ } f(\nabla(\frac{d}{dX})(m)) = \nabla(\frac{d}{dX})(f(m)) \} \\
\text{for any } m \in M_\eta^\tau.
\end{cases}
\]

\[
\text{Sol}_0(M_\eta^\tau) = \begin{cases} 
\{ f : M_\eta^\tau \to E_\eta^\tau[[X - t_\eta]]_0\text{-linear such that } \\
\text{ } \text{ } f(\nabla(\frac{d}{dX})(m)) = \nabla(\frac{d}{dX})(f(m)) \} \\
\text{for any } m \in M_\eta^\tau.
\end{cases}
\]
and defines $\mathcal{E}_\alpha^\tau$-spaces $\text{Sol}(M_\alpha^\tau)$ and $\text{Sol}_0(M_\alpha^\tau)$ for the solvable $\nabla$-module $M_\alpha = \mathcal{E}_\alpha \otimes_{E_\eta} M_\eta$ over $\mathcal{E}_\alpha$ similarly, then we have an isomorphism

$$\text{Sol}_0(M_\alpha^\tau) \cong \mathcal{E}_\alpha^\tau \otimes_{E_\eta^\tau} \text{Sol}_0(M_\eta^\tau).$$

under the natural isomorphism $\text{Sol}(M_\alpha^\tau) \cong \mathcal{E}_\alpha^\tau \otimes_{E_\eta^\tau} \text{Sol}(M_\eta^\tau)$.

Note that, if the matrix representation of $M_\eta$ is as in Definition 3.5, then there are natural $E_\alpha^\tau$-isomorphisms

$$\text{Sol}(M_\eta^\tau) \cong \left\{ y \in A_{E_\eta^\tau}(t_\eta, 1) \mid \frac{dy}{dX} = y C^\tau \right\},$$

and the same for $M_\alpha$.

**Proof of Proposition 3.14** Since the natural map $\mathcal{E}_\alpha^\tau \otimes_{E_\eta^\tau} \text{Sol}_0(M_\eta^\tau) \to \text{Sol}_0((\mathcal{E}_\alpha \otimes_{E_\eta} M_\eta)^\tau)$ is an injection of $\mathcal{E}_\alpha^\tau$-spaces, it is sufficient to prove that, if any linear combination of $f_1, \ldots, f_l \in \text{Sol}(M_\eta^\tau)$ over $E_\eta^\tau$ is not contained in $f_1, \ldots, f_l \in \text{Sol}_0(M_\eta^\tau)$, then any linear combination of $f_1, \ldots, f_l$ over $\mathcal{E}_\alpha^\tau$ is not contained in $\text{Sol}_0((\mathcal{E}_\alpha \otimes_{E_\eta} M_\eta)^\tau)$. Consider a linear sum $c_1 f_1 + \cdots + c_l f_l$ for some $c_1, \ldots, c_l \in \mathcal{E}_\alpha^\tau$. We will show that the linear sum does not belong to $\text{Sol}_0((\mathcal{E}_\alpha \otimes_{E_\eta} M_\eta)^\tau)$. Let $V$ be an $E_\eta^\tau$-subspace of $\mathcal{E}_\alpha^\tau$ generated by $c_1, \ldots, c_l$, and $d_1, \ldots, d_m$ a basis of $V$ over $E_\eta^\tau$. Then

$$c_1 f_1 + \cdots + c_l f_l = d_1 g_1 + \cdots + d_m g_m$$

for some $g_1, \ldots, g_m \in \text{Sol}(M_\eta^\tau)$. By our hypothesis either $g_i$ is not contained in $\text{Sol}_0(M_\eta^\tau)$ or $g_i = 0$ for all $i$. Since $V$ is a finite dimensional topological vector space over the complete topological field $E_\eta^\tau$, the topology of $V$ induced by the norm of $\mathcal{E}_\alpha^\tau$ coincides with the product topology induced by the isomorphism $V \cong E_\eta^\tau d_1 + \cdots + E_\eta^\tau d_m$. If $c_1 f_1 + \cdots + c_l f_l$ is bounded, then the sequence $g_{1,n} d_1 + \cdots + g_{m,n} d_m$ in $n$ is bounded in $V$ where $g_i = \sum_n g_{i,n} (X - t_\eta)^n$. This holds only in the case where $g_i = 0$ for all $i$. Hence we complete the proof.

**Corollary 3.15** With the notation as in Proposition 3.14, there is a natural isomorphism

$$(\mathcal{E}_\alpha \otimes_{E_\eta} M_\eta)^b \cong \mathcal{E}_\alpha \otimes_{E_\eta} M_\eta^b$$

as $\varphi$-$\nabla$-modules over $\mathcal{E}_\alpha$.  

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3.6 Generic PBQ filtrations

**Definition 3.16** A \( \varphi \)-\( \nabla \)-module over \( B \) is said to be PBQ (pure of bounded quotient) if the maximally bounded quotient \( M/M^b \) has a unique Frobenius slope. Note that the slope of \( M/M^b \) is the maximal slope of \( M \) by Proposition 3.9 (2).

**Lemma 3.17** (1) If \( M_1 \) and \( M_2 \) are PBQ \( \varphi \)-\( \nabla \)-modules over \( B \) of a same maximal slope, then so is the direct sum \( M_1 \oplus M_2 \).

(2) Let \( \theta : M \to N \) be a surjection of \( \varphi \)-\( \nabla \)-module over \( B \). If \( M \) is PBQ and \( N \neq 0 \), then \( N \) is also PBQ of the same maximal slope with \( M \).

**Proof** The assertions easily follow from the definition. \( \square \)

**Proposition 3.18** Let \( M \) be a \( \varphi \)-\( \nabla \)-module over \( B \), and \( N \) a unit-root \( \varphi \)-\( \nabla \)-module over \( B \). Then \( M \) is PBQ if and only if \( M \otimes_B N \) is PBQ.

**Proof** On the generic disc the unit-root object is trivial as differential modules, that is, there exists an \( F \)-space \( N^{t,0} \) over \( B^t \) such that \( N^t \cong N^{t,0} \otimes_B B^t[[X - t]]_0 \). Since \((M \otimes_B N)^t \) is a direct sum of \( \dim_B N \) copies of \( M^t \) as differential modules, the bounded solutions of \((M \otimes_B N)^t \) are \( \dim_B N \) copies of those of \( M^t \). Hence \((M \otimes_B N)/(M \otimes_B N)^b \cong (M/M^b) \otimes_B N \). Therefore, the assertion holds. \( \square \)

**Theorem 3.19** ([10, Corollary 5.5] in the case of \( E_\alpha \)) Let \( M \) be a \( \varphi \)-\( \nabla \)-module over \( B \).

(1) There exists a unique increasing filtration

\[
0 = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_r = M
\]

as \( \varphi \)-\( \nabla \)-modules over \( B \) such that

(i) \( P_i/P_{i-1} \) is PBQ:

(ii) if \( \lambda_i \) is the maximal slope of \( P_i \), then \( \lambda_1 > \lambda_2 > \cdots > \lambda_r \).

Moreover, if an increasing filtration of \( M \) satisfies the conditions (i), (ii), then it coincides with the filtration \( \{P_i\} \). The filtration \( \{P_i\} \) is called the PBQ filtration of \( M \).

(2) There is an isomorphism \( M/M^b \cong \bigoplus_{i=1}^r P_i/P_i^1 \) as \( \varphi \)-\( \nabla \)-modules over \( B \).

**Proof** (1) We prove the assertion by induction on the dimension of \( M \). If \( M/M^b \) has only one slope, then \( M \) is PBQ. If \( M/M^b \) has several Frobenius slopes, then one obtain the maximal \( \varphi \)-\( \nabla \)-submodule \( M' \) such that the image of \( M' \to M/M^b \) coincides with the direct summand of maximal slope of \( M/M^b \) by Proposition 3.10. Applying the induction hypothesis to \( M' \) we has a nontrivial submodule \( P_1 \) of \( M' \) which is PBQ and the image of \( P_1 \to M/M^b \)
coincides with the direct summand of maximal slope of $M/M^b$. Applying the induction hypothesis to the quotient $M/P_1$ again, we have a desired filtration of $M$.

Now we prove the uniqueness. Let $P_1$ and $P'_1$ be first steps of two filtrations satisfying the conditions (i) and (ii). Consider $Q = P_1 + P'_1$ in $M$ which is the image of $P_1 \oplus P'_1 \to M$. If $Q \neq P_1$, then the nontrivial surjection $P'_1 \to Q/P_1$ contradicts to the hypothesis that $P'_1$ is PBQ with the maximal slope $\lambda_1$ by Lemma 3.17 (2) since there is an isomorphism $M/M^1 \cong P/P^1$ of the maximal slope quotients. Hence $Q = P_1 = P'_1$.

(2) We prove the assertion by induction on the length $r$ of the PBQ filtration $\{P_i\}$ of $M$. If $r = 1$, then the assertion follows from the definition of PBQ. Suppose the assertion holds for $r - 1$ and put $M' = M/P_1$. Then there exists an exact sequence

$$P_1/P_1^b \to M/M^b \to M'/(M')^b \to 0$$

of $\varphi$-$\nabla$-modules over $B$. The left arrow is injective since the composite $\text{Ker}(M/M^b \to M'/(M')^b) \to M/M^1 \cong P_1/P_1^1$ is surjective by the maximality of the maximal slope $\lambda_1$ of $P_1$. Since $P_1$ is PBQ with the maximal slope $\lambda_1$, the assertion $M/M^b \cong \bigoplus_{i=1}^r P_i/P_i^1$ follows from the induction hypothesis and the maximality $\lambda_1$ of slopes of $M$.

**Definition 3.20** The PBQ $\varphi$-$\nabla$-module $P_1$ of the first step of the PBQ filtration of $M$ in Theorem 3.19 above is called the maximally PBQ submodule of $M$.

The same notion will be defined for $\varphi$-$\nabla$-modules over $E^\dagger_{\alpha}$, $K[[x_{\alpha}]]_0$ in Sect. 3.7 and for overconvergent $F$-isocrystals on $C/K$ in Sect. 3.8.

**Theorem 3.21** Let $M_\eta$ be a $\varphi$-$\nabla$-modules over $E_\eta$, and $\alpha$ a closed point in $\overline{C}$.

1. $M_\eta$ is PBQ if and only if $E_\alpha \otimes_{E_\eta} M_\eta$ is PBQ.
2. If $\{P_\eta,i\}$ is the PBQ filtration of $M_\eta$, then $\{E_\alpha \otimes_{E_\eta} P_\eta,i\}$ is the PBQ filtration of $E_\alpha \otimes_{E_\eta} M_\eta$.

**Proof** (1) follows from Corollary 3.15.
(2) follows from (1) and the uniqueness of PBQ filtrations by Theorem 3.19 (1). \qed

### 3.7 PBQ filtrations over $E^\dagger_{\alpha}$, and $K[[x]]_0$

What we deal are only local objects in this subsection, so we drop the subscript $\alpha$ from the notation $x_\alpha$, $E_\alpha$, $E^\dagger_{\alpha}$ as in Sect. 2. We recall the fact that the PBQ filtrations over $E$ descend to those over $K[[x]]_0$. 

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Definition 3.22 (1) [10, Definition 5.1] A $\varphi$-$\nabla$-module $M_0$ over $K[[x]]_0$ is said to be PBQ if $E \otimes_{K[[x]]_0} M_0$ is PBQ.

(2) [37, Section 12] A $\varphi$-$\nabla$-module $M^\dagger$ over $E^\dagger$ is said to be PBQ if $E \otimes_{E^\dagger} M^\dagger$ is PBQ.

Theorem 3.23 (1) [10, Theorem 5.6] Let $M_0$ be a $\varphi$-$\nabla$-module over $K[[x]]_0$. Then there exists a unique filtration $\{P_{0,i}\}$ of $M_0$ as $\varphi$-$\nabla$-modules over $K[[x]]_0$ such that $\{P_{0,i}\}$ is a $K[[x]]_0$-lattice of the PBQ filtration of $E \otimes_{K[[x]]_0} M_0$. Then there exists a unique filtration $\{P^\dagger_i\}$ of $\varphi$-$\nabla$-modules over $E^\dagger$ such that $\{P^\dagger_i\}$ is an $E^\dagger$-lattice of the PBQ filtration of $E \otimes_{E^\dagger} M^\dagger$. In particular, an irreducible $\varphi$-$\nabla$-module over $K[[x]]_0$ (resp. $E^\dagger$) is PBQ.

For a $\nabla$-module $M_0$ over $K[[x]]_0$ we define a $K$-space of solutions by

$$\text{Sol}(M_0) = \left\{ f : M_0 \to A_K(0, 1) \mid f \text{ is } K[[x]]_0\text{-linear such that } \frac{d}{dx} f(m) = f \left( \nabla \left( \frac{d}{dx} \right)(m) \right) \right\}$$

for any $m \in M_0$.

where $A_K(0, 1)$ is analytic on the open unit disk $|x| < 1$, and define a $K$-space of bounded solutions by

$$\text{Sol}_0(M_0) = \left\{ f : M_0 \to K[[x]]_0 \mid f \text{ is } K[[x]]_0\text{-linear such that } \frac{d}{dx} f(m) = f \left( \nabla \left( \frac{d}{dy} \right)(m) \right) \right\}$$

for any $m \in M_0$.

If $E \otimes_{K[[x]]_0} M_0$ is solvable, then $\dim_K \text{Sol}(M_0) = \text{rank}_{K[[x]]_0} M_0$ by Christol transfer theorem [12, Théorème 2]. Moreover, $\text{Sol}(M_0)$ is an $F$-space over $K$ whose Frobenius is defined by

$$F_{\text{Sol}(M_0)}(f) = F_{K[[x]]_0} \circ f \circ F^{-1}_{M_0} \text{ for } f \in \sigma^* \text{Sol}(M_0)$$

and $\text{Sol}_0(M_0)$ is an $F$-subspace over $K$ since the $q$-Frobenius $\varphi$ acts on $K[[x]]_0$. Note that $\text{Sol}(M_0)$ is the dual of the $F$-space $M_0 \otimes_{K[[x]]_0} K$ over $K$ ($K[[x]]_0 \to K$, $\sum_n a_n x^n \mapsto a_0$). Concerning to the bounded quotients, the proposition below holds which corresponds to a bounded Dwork’s conjecture on logarithmic growth versus Frobenius slopes (see Remark 3.25 below).

Proposition 3.24 Let $M_0$ be a PBQ $\varphi$-$\nabla$-module over $K[[x]]_0$ such that the maximal slope of $E \otimes_{K[[x]]_0} M_0$ is $\lambda^{\text{max}}$. Then, if $\text{Sol}(M_0)^{(-\lambda^{\text{max}})}$ denotes the
F-subspace of slope $-\lambda_{\text{max}}$ in the F-space $\text{Sol}(M_0)$ over $K$, then we have an equality
\[ \text{Sol}_0(M_0) = \text{Sol}(M_0)^{(-\lambda_{\text{max}})}. \]

In particular, there exists a unique $\varphi$-$\nabla$-module $L_0$ over $K[[x]]_0$ with a surjection $M_0 \to L_0$ of $\varphi$-$\nabla$-modules over $K[[x]]_0$ such that $L_0$ is isoclinic of slope $\lambda_{\text{max}}$ and that \( \text{rank}_{K[[x]]_0} L_0 = \text{Sol}(M_0)^{(-\lambda_{\text{max}})}. \)

Proof The equality follows from the proof of Dwork’s conjecture [37, Theorem A] (see Remark 3.25). Here we give a sketch of another proof. The natural morphism $\text{Sol}_0(M_0) \otimes_K M_0 \to K[[x]]_0 \otimes_K \text{Sol}_0(M_0)^{\vee}$ induces a surjection $M_0 \to K[[x]]_0 \otimes_K \text{Sol}_0(M_0)^{\vee}$ of $\varphi$-$\nabla$-modules over $K[[x]]_0$. If $\text{Sol}_0(M_0)$ has a slope different from $-\lambda_{\text{max}}$, then it is contradicts to the fact that $E \otimes_{K[[x]]_0} M_0$ is PBQ with the maximal slope $\lambda_{\text{max}}$. Hence $\text{Sol}_0(M_0) \subset \text{Sol}(M_0)^{(-\lambda_{\text{max}})}$. The opposite inclusion follows from [9, Theorem 6.17 (ii)]. In particular, $L_0 = K[[x]]_0 \otimes_K \text{Sol}_0(M_0)^{\vee}$. \(\square\)

Remark 3.25 In ‘70–’80 Dwork studied the logarithmic growth (log-growth for short) of solutions of $p$-adic linearly differential equations and proposed problems on the relation between log-growth and Frobenius slopes. After the work of Dwork, Christol and Robba, Chiarellotto and the author formulated the relation precisely, which is called Dwork’s conjecture [10, Conjectures 2.4, 2.5]. Let $\lambda$ be a nonnegative real number, and $K[[x]]_{\lambda}$ a $K[[x]]_0$-module of analytic functions of log-growth $\leq \lambda$ on the unit open disc defined by
\[ K[[x]]_{\lambda} = \left\{ \sum_n a_n x^n \in A_K(0, 1) \left| \limsup_{n \to \infty} \frac{|a_n|}{(n + 1)^{\lambda}} \leq \infty \right. \right\}. \]

For a $\varphi$-$\nabla$-module $M_0$ over $K[[x]]_0$, we define a $K$-space of solutions of $M_0$ of log-growth $\leq \lambda$ by
\[ \text{Sol}_{\lambda}(M_0) = \left\{ f : M_0 \to K[[x]]_\lambda \left| \frac{d}{dx} f(m) = f(\nabla (\frac{d}{dy})(m)) \right. \right. \text{for any } m \in M_0. \right\}, \]

and $\text{Sol}(M_0)^{\leq \lambda - \lambda_{\text{max}}}$ denotes the $F$-subspace of slope $\leq \lambda - \lambda_{\text{max}}$ in $\text{Sol}(M_0)$. Then Dwork’s conjecture asserts that, if a $\varphi$-$\nabla$-module $M_0$ over $K[[x]]_0$ is PBQ, then the equality
\[ \text{Sol}_{\lambda}(M_0) = \text{Sol}(M_0)^{\leq \lambda - \lambda_{\text{max}}} \]
holds for all $\lambda \geq 0$. Recently Ohkubo affirmatively solved the conjecture in [37, Theorems 0.1, 0.2]. More precisely, Ohkubo generalized Dwork’s conjecture.
for \( \varphi \cdot \nabla \)-modules over \( \mathcal{E}^\dagger \) and proved it. Since the bounded object is split by Frobenius slopes by Theorem 3.10 and the full faithfulness of the functor \( \Phi M^\dagger_{k[[x]]} \to \Phi M^\dagger_{\mathcal{E}} \) by [14, Theorem 9.1], the converse of Dwork’s conjecture also holds (see also [37, Section 14]).

### 3.8 Global PBQ filtration

Now we introduce a global PBQ filtration for overconvergent \( F \)-isocrystals on a curve. Let \( C \) be a smooth connected curve over \( \text{Spec} \ k \).

**Definition 3.26** An overconvergent \( F \)-isocrystal \( M^\dagger \) on \( C/\mathcal{K} \) is said to be PBQ if, for an affine open dense subscheme \( U \) of \( C \) with a smooth lift \( U = \text{Spec} \ A_U \) such that \( A_U^\dagger \) admits a \( q \)-Frobenius \( \varphi \) which is compatible with the \( q \)-Frobenius \( \sigma \) on \( K \), the associated \( \varphi \cdot \nabla \)-module \( M^\eta = E^\eta \otimes_{A_U^\dagger, K} \Gamma(\lbrack \widehat{C}, j_U^\dagger M^\dagger \rbrack) \) over \( E^\eta \) is PBQ.

In Definition 3.26 the definition of PBQ is independent of the choice of affine open subschemes \( U \). We will prove the following existence theorem which is a key role to study the minimal slope conjecture.

**Theorem 3.27** Let \( M^\dagger \) be an overconvergent \( F \)-isocrystal on \( C/\mathcal{K} \). Then there exists a unique filtration \( 0 = \mathcal{P}^\dagger_0 \subsetneq \mathcal{P}^\dagger_1 \subsetneq \cdots \subsetneq \mathcal{P}^\dagger_r = M^\dagger \) as overconvergent \( F \)-isocrystals on \( C/\mathcal{K} \) such that \( \{E^\eta \otimes_{A_U^\dagger, K} \Gamma(\lbrack \widehat{C}, j_U^\dagger \mathcal{P}^\dagger_i \rbrack)\} \) is the PBQ filtration of \( M^\eta = E^\eta \otimes_{A_U^\dagger, K} \Gamma(\lbrack \widehat{C}, j_U^\dagger M^\dagger \rbrack) \) for any affine open dense subscheme \( U \) of \( C \). The filtration \( \{\mathcal{P}^\dagger_i\} \) is called the PBQ filtration of \( M \).

**Corollary 3.28** An irreducible overconvergent \( F \)-isocrystal on \( C/\mathcal{K} \) is PBQ.

We will give an example of PBQ filtrations at the end of this section.

**Proposition 3.29** Let \( M^\dagger \) be an overconvergent \( F \)-isocrystal on \( C/\mathcal{K} \) admitting the slope filtration of \( M \), and \( 0 = \mathcal{P}^\dagger_0 \subsetneq \mathcal{P}^\dagger_1 \subsetneq \cdots \subsetneq \mathcal{P}^\dagger_r = M^\dagger \) the PBQ filtration of \( M^\dagger \). Then \( \bigoplus_{i=1}^r \mathcal{P}_i/\mathcal{P}^1_i \) is a quotient of \( M \) as convergent \( F \)-isocrystals on \( U/\mathcal{K} \). If we put \( M^b = \text{Ker}(M \to \bigoplus_{i=1}^r \mathcal{P}_i/\mathcal{P}^1_i) \), then the generic fiber \( E^\eta \otimes M/M^b \) of \( M/M^b \) is isomorphic to \( M^\eta/M^b_\eta \). The convergent \( F \)-isocrystal \( M/M^b \) is called the bounded quotient of \( M \).

**Proof** Since \( \mathcal{P}^\dagger_1 \) is the maximal PBQ submodule of \( M^\dagger \), one has \( \mathcal{P}_1/\mathcal{P}^1_1 = M/M^1 \). Hence, there exists a canonical projection \( M \to M/M^1 \oplus (M/\mathcal{P}_1)/(\mathcal{P}/\mathcal{P}^1) \cong \bigoplus_{i=1}^r \mathcal{P}_i/\mathcal{P}^1_i \) by induction on \( r \). The rest follows from Theorems 3.10 and 3.27.

Let us study several properties of PBQ overconvergent \( F \)-isocrystals.
Lemma 3.30 (1) If \( M_1 \) and \( M_2 \) are PBQ overconvergent F-isocrystals on \( C/K \) of a same maximal slope, then so is the direct sum \( M_1 \oplus M_2 \).

(2) Let \( \theta : M^\dagger \to N^\dagger \) be a surjection of overconvergent F-isocrystals on \( C/K \). If \( M^\dagger \) is PBQ and \( N^\dagger \neq 0 \), then \( N^\dagger \) is also PBQ. Moreover, the maximal slope of \( N_\dagger \) is equal to that of \( M_\dagger \).

Proof The assertions follow from Lemma 3.17.

Lemma 3.31 Let \( f : C' \to C \) be a finite etale morphism of smooth connected curves over \( \text{Spec} \, k \). Let \( M^\dagger \) (resp. \( N^\dagger \)) be an overconvergent F-isocrystal on \( C/K \) (resp. \( C'/K \)), and denotes the direct image of \( M^\dagger \) on \( C/K \) (resp. the inverse image of \( N^\dagger \) on \( C'/K \)) by \( f_* M^\dagger \) (resp. \( f^* N^\dagger \)).

(1) If \( N^\dagger \) is PBQ, then so is \( f^* N^\dagger \).

(2) If \( f_* M^\dagger \) is PBQ, then so is \( M^\dagger \).

Note that the inverse image of overconvergent isocrystals is defined in [6, Definitions 2.3.2 (iv)] and the direct image for finite etale morphisms is studied in [43, 5.1].

Proof (1) By the projection formula there is an isomorphism \( f_* f^* N^\dagger = f_* (j_{C'}^\dagger \mathcal{O}_{C'[\hat{L}]} \otimes j_{C'}^\dagger \mathcal{O}_{C'[\hat{L}]}) N^\dagger \). Since the direct image \( f_* (j_{C'}^\dagger \mathcal{O}_{C'[\hat{L}]}) \) of the constant overconvergent F-isocrystal \( j_{C'}^\dagger \mathcal{O}_{C'[\hat{L}]} \) is unit-root, the direct image \( f_* f^* N^\dagger \) is PBQ by Proposition 3.18. Since the constant overconvergent F-isocrystal \( j_{C}^\dagger \mathcal{O}_{C[\hat{L}]} \) is a direct summand of the direct image \( f_* (j_{C'}^\dagger \mathcal{O}_{C'[\hat{L}]}) \), \( N^\dagger \) is PBQ by Lemma 3.30.

(2) Since the adjoint morphism \( f^* f_* M^\dagger \to M^\dagger \) is surjective, the assertions follows from Lemmas 3.30 and 3.31.

Now we return to prove Theorem 3.27. The following are key lemmas to construct unit-root overconvergent F-isocrystals and overconvergent F-subisocrystals from several compatible generic and local data.

Lemma 3.32 Suppose that \( C \) is affine with a smooth lift \( \hat{C} = \text{Spec} \, A_C \) such that \( A_C^\wedge \) admits a q-Frobenius \( \varphi \) which is compatible with the q-Frobenius \( \sigma \) on \( K \). Let \( L_\eta \) be a unit-root \( \varphi \)-\( \nabla \)-module over \( E_\eta \) satisfying the conditions as follows:

(i) For any closed point \( \alpha \in C \), there exists a \( K_\alpha[\left[x_\alpha\right]]_0 \)-lattice \( L_{0,\alpha} \) of \( L_\alpha = E_\alpha \otimes_{E_\eta} L_\eta \).

(ii) For any closed point \( \alpha \in \overline{C} \setminus C \), there exists an \( \mathcal{E}_\alpha^\dagger \)-lattice \( L_\alpha^\dagger \) of \( L_\alpha = E_\alpha \otimes_{E_\eta} L_\eta \).

Then there exists a unit-root overconvergent F-isocrystal \( L^\dagger \) on \( C/K \) such that the \( \varphi \)-\( \nabla \)-module \( E_\eta \otimes_{A_C^\wedge} \Gamma(\overline{C}^\wedge, L^\dagger) \) over \( E_\eta \) is isomorphic to the given \( L_\eta \). \( L^\dagger \) is unique up to isomorphisms.
Proof Let $G_{k(C)}$ be the absolute Galois group of the function field $k(C)$ of $C$. By Katz correspondence between unit-root $F$-spaces and $p$-adic representations by [24, Proposition 4.1.1], one has a continuous representation

$$
\rho : G_{k(C)} \rightarrow \text{GL}_s(K_\sigma).
$$

By the condition (i) we know that $\rho$ is unramified at $\alpha \in C$ and $\rho$ has a finite local monodromy at $\alpha \in \overline{C} \setminus C$ by [41, Theorem 4.2.6]. Hence $\rho$ corresponds to an unit-root overconvergent $F$-isocrystals on $C/K$ which has a desired property by [41, Theorem 7.2.3]. The uniqueness follows from the equivalence of Katz correspondence. 

Lemma 3.33 (c.f. [32, Remark 5.10]) Suppose that $C$ is affine with a smooth lift $\mathcal{C} = \text{Spec} \, A_C$ such that $A^\dagger_C$ admits a $q$-Frobenius $\varphi$ which is compatible with the $q$-Frobenius $\sigma$ on $K$. Let $\mathcal{M}^\dagger$ be an overconvergent F-isocrystal on $C/K$, put $M^\dagger = \Gamma(\overline{\mathcal{C}}, \mathcal{M}^\dagger)$, and $L_\eta$ a $\varphi$-$\nabla$-submodule of $M_\eta = E_\eta \otimes_{A^\dagger_C,K} M^\dagger$ over $E_\eta$. Suppose that

(i) for any closed point $\alpha$ of $C$ there exists a $K_\alpha[[x_\alpha]]_0$-lattice $L_{0,\alpha}$ of $L_\alpha = \mathcal{E}_\alpha \otimes_{E_\eta} L_\eta$, and

(ii) for any closed point $\alpha$ of $\overline{C} \setminus C$ there exists an $\mathcal{E}_\alpha^\dagger$-lattice $L^\dagger_\alpha$ of $L_\alpha = \mathcal{E}_\alpha \otimes_{E_\eta} L_\eta$.

(Note that $L_{0,\alpha}$ (resp. $L^\dagger_\alpha$) is canonically a $\varphi$-$\nabla$-submodule of $M_{0,\alpha} = K_\alpha[[x_\alpha]]_0 \otimes_{A^\dagger_C,K} M^\dagger$ (resp. $M^\dagger_\alpha = \mathcal{E}_\alpha^\dagger \otimes_{A^\dagger_C,K} M^\dagger$) by full faithfulness).

Then there exists a unique overconvergent $F$-subisocrystal $\mathcal{L}^\dagger$ of $\mathcal{M}^\dagger$ on $C/K$ such that, if $L^\dagger = \Gamma(\overline{\mathcal{C}}, \mathcal{L}^\dagger)$, then $L_{\eta} = E_\eta \otimes_{A^\dagger_C,K} L^\dagger$, $L_{0,\alpha} = K_\alpha[[x_\alpha]]_0 \otimes_{A^\dagger_C,K} L^\dagger$ for any closed point $\alpha$ of $C$, and $L^\dagger_\alpha = \mathcal{E}_\alpha^\dagger \otimes_{A^\dagger_C,K} L^\dagger$ for any closed point $\alpha$ of $\overline{C} \setminus C$ as $\varphi$-$\nabla$-modules.

Proof First we prove the assertion in the case where $\dim_{E_\eta} L_{\eta} = 1$. When $\dim_{E_\eta} L_{\eta} = 1$, Lemma 3.32 implies that there exists an overconvergent $F$-isocrystal $\mathcal{L}^\dagger$ on $C/K$ which is determined by the data $L_{\eta}, L_\alpha$ ($\alpha \in C$) and $L^\dagger_\alpha$ ($\alpha \in \overline{C} \setminus C$). What we want is a nontrivial morphism $\mathcal{L}^\dagger \rightarrow \mathcal{M}^\dagger$ of overconvergent $F$-isocrystals which induces the conditions $L_{\eta} = E_\eta \otimes_{A^\dagger_C,K} L$ and so on. Replacing $\mathcal{M}^\dagger$ by $(\mathcal{L}^\dagger)^\vee \otimes_{\mathcal{L}^\dagger \otimes_{\overline{\mathcal{C}}} \mathcal{L}} \mathcal{M}^\dagger$, there exist

$(\eta)$ an element $e_\eta \in L_{\eta}$ such that $\varphi_{M_\eta}(e_\eta) = e_\eta$,

$(\alpha)_0$ an element $e_\alpha \in L_{0,\alpha}$ such that $\varphi_{M_\alpha}(e_\alpha) = e_\alpha$ for any closed point $\alpha$ of $C$,

$(\alpha)^\dagger$ an element $e_\alpha \in L^\dagger_\alpha$ such that $\varphi_{M_\alpha}(e_\alpha) = e_\alpha$ for any closed point $\alpha$ of $\overline{C} \setminus C$.
Since \( E_\alpha \otimes E_\eta L_\eta = E_\alpha \otimes K_\alpha[[x_\alpha]]_0 L_{0,\alpha} \) for any closed point \( \alpha \) of \( C \), there exists an element \( u_\alpha \in E_\alpha \) such that
\[
e_\eta = u_\alpha e_\alpha
\]

Since both Frobenius in the equality \( E_\alpha \otimes E_\eta M_\eta = E_\alpha \otimes K_\alpha[[x_\alpha]]_0 M_{0,\alpha} \) commute with each other, \( u_\alpha \) is contained in \( (K_\alpha)_\sigma \). We now change \( e_\alpha \) by \( u_\alpha e_\alpha \in L_{0,\alpha} \). The similar holds for \( e_\alpha \) for any closed point \( \alpha \) of \( \overline{C} \setminus C \). If \( e_\eta \) belongs to \( M^\dagger \), then it determines a morphism \( j_C^+ \mathcal{O}_{\overline{C}[\xi]} \rightarrow (L^\dagger)^\vee \otimes j_C^+ \mathcal{O}_{\overline{C}[\xi]} \mathcal{M}^\dagger \) of overconvergent \( F \)-isocrystals which satisfies the desired conditions. In order to prove \( e_\eta \in M^\dagger \) it is sufficient to prove
\[
M^\dagger = \bigcap_{\alpha \in C} M_\eta \cap M_{0,\alpha} \bigcap_{\alpha \in C \setminus C} M_\eta \cap M^\dagger_{\alpha}
\]
in \( M_\eta \). Since \( M^\dagger \) is a finite generated projective \( A^\dagger_{C,K} \)-module, the equality above follows from Lemma 3.2.

Now we treat the general case. Put \( s = \dim_{E_\eta} L_\eta \). Consider the \( s \)-th exterior product \( \wedge^s \mathcal{M}^\dagger \) of \( \mathcal{M}^\dagger \) and the data \( \wedge^s L_\eta, \wedge^s L_{0,\alpha}, \wedge^s L^\dagger_\alpha \). Then there exist an overconvergent \( F \)-isocrystal \( \mathcal{N}^\dagger \) of rank one and an injective morphism \( \mathcal{N}^\dagger \rightarrow \wedge^s \mathcal{M}^\dagger \) which is determined by the data \( \wedge^s L_\eta, \wedge^s L_{0,\alpha}, \wedge^s L^\dagger_\alpha \) by the former part of this proof. We define an overconvergent \( F \)-isocrystal \( \mathcal{L}^\dagger \) on \( C/K \) by
\[
\mathcal{L}^\dagger = (\mathcal{N}^\dagger)^\vee \otimes j_C^+ \mathcal{O}_{\overline{C}[\xi]} \ker(\mathcal{N}^\dagger \otimes j_C^+ \mathcal{O}_{\overline{C}[\xi]} \mathcal{M}^\dagger \rightarrow \wedge^{s+1} \mathcal{M}^\dagger).
\]
Here \( \mathcal{N}^\dagger \otimes \mathcal{M}^\dagger \rightarrow \wedge^{s+1} \mathcal{M}^\dagger \) is defined by \( n \otimes m \mapsto n \wedge m \). Then \( E_\eta \otimes A^\dagger_{C,K} \Gamma(\overline{C}[\xi], \mathcal{L}^\dagger) = L_\eta, K_\alpha[[x_\alpha]]_0 \otimes A^\dagger_{C,K} \Gamma(\overline{C}[\xi], \mathcal{L}^\dagger) = L_{0,\alpha} \) and \( E_\alpha \otimes A^\dagger_{C,K} \Gamma(\overline{C}[\xi], \mathcal{L}^\dagger) = L^\dagger_\alpha \) hold by our construction. Hence we obtain a desired overconvergent \( F \)-subisocrystal \( \mathcal{L}^\dagger \) of \( \mathcal{M}^\dagger \). The uniqueness follows from Lemma 3.32 and our construction. \( \square \)

**Proof of Theorem 3.27** We first prove the uniqueness. It is sufficient to prove the uniqueness on the first step \( \mathcal{P}^\dagger_1 \) of PBQ filtration. Suppose \( \mathcal{P}^\dagger_1 \) and \( (\mathcal{P}^\dagger_1') \) are first steps of two PBQ filtrations of \( \mathcal{M}^\dagger \). Then the associated \( \varphi-\nabla \)-modules \( P_{1,\eta} \) and \( P_{1,\eta} + P_{1,\eta}' \) over \( E_\eta \) coincides with each other in \( M_\eta \) by Theorem 3.19. Hence \( \mathcal{P}^\dagger_1 \) and \( (\mathcal{P}^\dagger_1') \) are equal.

Now we prove the existence. Since the problem is local on \( C \) by patching and the full faithfulness of restriction functors \([43, \text{Theorem 6.3.1}]\), we may assume that \( C \) is affine with a smooth lift \( C = \text{Spec} \ A_C \) such that \( A^\dagger_C \) admits a
q-Frobenius \( \varphi \) which is compatible with the \( q \)-Frobenius \( \sigma \) on \( K \). The unique existence of PBQ filtration of the \( \varphi \)-\( \nabla \)-module \( K_\alpha[[x_\alpha]]_0 \otimes_{A_{C,K}^\dagger} \Gamma(\overline{C}, \mathcal{M}^\dagger) \) for a closed point \( \alpha \in C \) (resp. \( E_\alpha^\dagger \otimes_{A_{C,K}^\dagger} \Gamma(\overline{C}, \mathcal{M}^\dagger) \) for a closed point \( \alpha \in \overline{C} \setminus C \)) gives the comparison data in Lemma 3.33 for each step of PBQ filtration by Theorem 3.23. Hence we obtain a desired PBQ filtration of \( \mathcal{M}^\dagger \) by Lemma 3.33.

3.9 Example

Let us give an example of PBQ filtration with two steps.

Suppose \( p \) is an odd prime number and fix an embedding \( \overline{\mathbb{Q}} \subset \mathbb{Q}_p \) from an algebraic closure \( \overline{\mathbb{Q}} \) of the field \( \mathbb{Q} \) of rational numbers to an algebraic closure \( \mathbb{Q}_p \) of the field \( \mathbb{Q}_p \) of \( p \)-adic numbers. Let \( \sigma \) be the \( p \)-Frobenius id on \( \mathbb{Q}_p \).

Let \( C = \mathbb{P}_F^1 \setminus \{ 0, 1, \infty \} \) be a smooth curve over \( \text{Spec} \mathbb{F}_p \) with the standard coordinate \( z \), \( \overline{C} = \mathbb{P}^1 \) the smooth completion of \( C \), \( X \) an affine scheme defined by the equation

\[
y^2 = x(x - 1)(x - z)
\]

over \( C \), and \( \overline{X} \) the Legendre family of elliptic curves which is a completion of \( X \) over \( C \). Let \( \mathcal{L}^\dagger \) be an overconvergent \( F \)-isocrystal on \( C/\mathbb{Q}_p \) of rank 2 which is defined by the first relative rigid cohomology of \( X/\mathbb{C} \), \( \chi \) a nontrivial quadratic character on \( C \) corresponding to the double cover defined by the equation \( w^2 = z(z - 1) \), and \( \mathcal{L}^\dagger(\chi) \) (resp. \( j_\mathcal{C}^\dagger \mathcal{O}_{\overline{C}}(\chi) \)) the twist of \( \mathcal{L}^\dagger \) (resp. \( j_\mathcal{C}^\dagger \mathcal{O}_{\overline{C}}(\chi) \)) by the rank 1 overconvergent \( F \)-isocrystal on \( C/\mathbb{Q}_p \) corresponding to \( \chi \). Note that \( \mathcal{L}^\dagger(\chi) \) is pure of weight 1 (recall the definition of weights in Definition 7.14). Since \( \mathcal{L}^\dagger \) (resp. \( \chi \)) ramifies tamely at 0, 1, \( \infty \) (resp. 0, 1) with exponents 1, 1, 1/2 (resp. 1/2, 1/2), respectively, over the projective line \( \overline{C} \), the support forgetting map \( H^1_{\text{rig}}(C/\mathbb{Q}_p, \mathcal{L}^\dagger(\chi)) \to H^1_{\text{rig}}(C/\mathbb{Q}_p, \mathcal{L}^\dagger(\chi)) \) is isomorphic and the first rigid cohomology \( H^1_{\text{rig},c}(C/\mathbb{Q}_p, \mathcal{L}^\dagger(\chi)) \) is of dimension 2 and pure of weight 2 by Euler–Poincaré formula. Moreover we have

(a) if \( p \equiv 1 \pmod{4} \), then the slopes of Frobenius action on \( H^1_{\text{rig}}(C/\mathbb{Q}_p, \mathcal{L}^\dagger(\chi)) \) are 0 and 2;
(b) if \( p \equiv 3 \pmod{4} \), then the slopes of Frobenius action on \( H^1_{\text{rig}}(C/\mathbb{Q}_p, \mathcal{L}^\dagger(\chi)) \) are 1 and 1.

Indeed, (a) and (b) follows from the fact that the arithmetic family \( \overline{Y} \) of singular \( K3 \) surfaces over \( \text{Spec} \mathbb{Z}[1/2] \) which is a minimal desingularization of
completion of the affine scheme $Y$ over $C$ defined by the equation

$$y^2 = z(z - 1)x(x - 1)(x - z)$$

has a complex multiplication of $\mathbb{Q}(\sqrt{-1})$. If the zeta function of the elliptic curve $E$ defined by $y^2 = x^3 - x$ over $\mathbb{F}_p$ is $Z(E/\mathbb{F}_p, t) = (1 - \pi_p t)(1 - \overline{\pi}_p t)/(1 - t)(1 - pt)$, then the zeta function of $Y/\mathbb{F}_p = Y \times_{\text{Spec} \mathbb{Z}[1/2]} \text{Spec} \mathbb{F}_p$ over $\mathbb{F}_p$ is given by

$$Z(Y/\mathbb{F}_p, t) = \frac{1}{(1 - t)(1 - p^2 t)(1 - pt)^2 (1 - \rho \pi_p^2 t)(1 - \overline{\rho} \pi_p^2 t)}$$

for a root $\rho$ of unity in $\mathbb{Q}(\sqrt{-1})$ by [39, the case A in pp. 290, 291] applying the work of [38]. Here $\overline{\pi}_p$ (resp. $\overline{\rho}$) is the complex conjugate of $\pi_p$ (resp. $\rho$). Hence the $L$-function of $L^+(\chi)$ is

$$L(C/\mathbb{F}_p, L^+(\chi), t) = (1 - \rho \pi_p^2 t)(1 - \overline{\rho} \pi_p^2 t)$$

since the Leray spectral sequence induces an isomorphism $H^1_{\text{rig,c}}(C/\mathbb{Q}_p, L^+(\chi)) \cong H^2_{\text{rig,c}}(Y/\mathbb{Q}_p)$ of $F$-spaces. More precisely, $\rho = 1$ if $p \equiv 1 \pmod{4}$ and $\rho = \sqrt{-1}$ if $p \equiv 3 \pmod{4}$ by calculating $\#Y(\mathbb{F}_p)$ modulo 4.

Suppose $p \equiv 1 \pmod{4}$. If we fix the unit root element $\pi_p$ under the fixed embedding $\mathbb{Q}(\sqrt{-1}) \subset \mathbb{Q}_p$ then there exists a nontrivial extension

$$0 \to L^+(\chi) \to M^+ \to j^+_C \mathcal{O}_{\widehat{C}(\pi_p^2)} \to 0$$

of overconvergent $F$-isocrystals on $C/\mathbb{Q}_p$ by (a) where $j^+_C \mathcal{O}_{\widehat{C}(\pi_p^2)}$ denotes the twist of $j_C \mathcal{O}_{\widehat{C}(\pi_p^2)}$ by $\pi_p^2$ time Frobenius. Since the maximal generic slope of $L^+(\chi)$ is 1 and the slope of $j^+_C \mathcal{O}_{\widehat{C}(\pi_p^2)}$ is 0, $M^+$ has two steps of the PBQ filtration as above.

### 4 A local version of the minimal slope conjecture

In this section we will study a local version of the minimal slope conjecture, Theorem 4.2. Let us keep the notation in Sect. 2.

#### 4.1 A local version

**Definition 4.1** A $\varphi$-$\nabla$-module $M^+$ over $\mathcal{E}^+$ is saturated if the canonical morphism $M^+ \to M/M^1$ is injective.
In this section we will prove the theorem below, which is easily deduced from Theorem 4.11.

**Theorem 4.2** Let $M^\dagger$ and $N^\dagger$ be $\varphi\nabla$-modules over $E^\dagger$, and $h : N/N^1 \to M/M^1$ a morphism of $\varphi\nabla$-modules over $E$. Suppose either

(i) both $M^\dagger$ and $N^\dagger$ are irreducible and $h$ is nontrivial, or

(ii) both $M^\dagger$ and $N^\dagger$ are saturated and PBQ and $h$ is an isomorphism.

Then there exists a unique isomorphism $g^\dagger : N^\dagger \to M^\dagger$ of $\varphi\nabla$-modules over $E$ such that the induced morphism $N/N^1 \to M/M^1$ between the maximal slope quotients by $g^\dagger$ coincides with the given $h$.

### 4.2 Properties of saturated $\varphi\nabla$-modules

**Definition 4.3** Let $M$ be a $\varphi\nabla$-module over $E$ which has a unique slope of Frobenius $\varphi_M$, $N^\dagger$ a $\varphi\nabla$-module over $E^\dagger$, and $N^\dagger \to M$ an injection of $E^\dagger$-spaces which commutes with connections and Frobenius. A $\varphi\nabla$-module $N^\dagger$ over $E$ is quasi-saturated in $M$ if $N^\dagger$ generates $M$ as an $E$-space, in other words, the associated morphism $N \to M$ of $\varphi\nabla$-modules over $E$ is surjective.

We give several properties on the notion of saturated and quasi-saturated. Let $M$ be a $\varphi\nabla$-module over $E$ which has a unique slope, and $N^\dagger$ a nontrivial $\varphi\nabla$-module over $E^\dagger$. Let $\theta^\dagger : N^\dagger \to M$ be an $E^\dagger$-morphism which commutes with connections and Frobenius, and $L^\dagger$ the image of $\theta^\dagger$. Since $L^\dagger$ is of finite dimension over $E$, the following proposition holds.

**Proposition 4.4** With the notation as above, suppose $\theta^\dagger$ is nontrivial. Then $L^\dagger$ is a nontrivial $\varphi\nabla$-module over $E$ by the induced connection and Frobenius. In particular, any irreducible $\varphi\nabla$-module over $E^\dagger$ is saturated.

**Proposition 4.5** With the notation as above, suppose $\theta^\dagger$ is injective. Then the maximal slope of $N^\dagger$ coincides with the slope of $M$.

*Proof* Let $P^\dagger$ be the maximally PBQ submodule of $N^\dagger$. If the maximal slope of $P$, which is equal to that of $N$, is not equal to the slope of $M$, then $\theta^\dagger|_{P^\dagger}$ is a zero map by Lemma 3.17 (2). This contradicts the injectivity of $\theta^\dagger$.  

**Proposition 4.6** With the notation as above, suppose the induced morphism $\theta : N \to M$ from $\theta^\dagger$ is surjective.

1. $L^\dagger$ is quasi-saturated in $M$ and the inequality $\dim_E L/L^1 \geq \dim_E M$ holds.
2. If $\dim_E L/L^1 = \dim_E M$, then $L^\dagger$ is saturated.

*Proof* (1) We may suppose that $\theta^\dagger$ is injective by Proposition 4.4, and then the maximal slope of $L$ coincides with that of $M$. Since $L^\dagger$ is the image of $\theta^\dagger$, $L^\dagger$ generates $M$ as an $E$-space. Moreover, the induced morphism $L \to M$ is surjective as a morphism of $\varphi\nabla$-modules over $E$.
(2) Since \( L \to M \) is surjective by definition of \( L^\dagger \), the hypothesis implies the natural morphism \( L/L^1 \to M \) is an isomorphism. Hence \( L^\dagger \) is saturated. \( \square \)

**Proposition 4.7** Let \( M^\dagger \) be a \( \varphi\nabla \)-module over \( \mathcal{E}^\dagger \), and \( L^\dagger \) a \( \varphi\nabla \)-module over \( \mathcal{E}^\dagger \) which is defined by the image of the canonical morphism \( \theta^\dagger : M^\dagger \to M/M^1 \).

1. The induced morphism \( M/M^1 \to L/L^1 \) from \( \theta \) is an isomorphism.
2. \( L^\dagger \) is saturated.
3. If \( M^\dagger \) is saturated, then \( \theta^\dagger : M^\dagger \to L^\dagger \) is an isomorphism.

**Proof** Since \( \theta^\dagger : M^\dagger \to L^\dagger \) is surjective, the induced morphism \( M/M^1 \to L/L^1 \) by \( \theta \) is surjective. (1) and (2) follows from Proposition 4.6. (3) is trivial. \( \square \)

**Definition 4.8** For a \( \varphi\nabla \)-module \( M^\dagger \) over \( \mathcal{E}^\dagger \), we define the saturation \( M^{\text{sat},\dagger} \) of \( M^\dagger \) by the \( \varphi\nabla \)-module \( L^\dagger \) in Proposition 4.7.

### 4.3 Saturated versus quasi-saturated

**Proposition 4.9** Let \( M \) be a \( \varphi\nabla \)-module over \( \mathcal{E} \), and \( N^\dagger \) a \( \varphi\nabla \)-module over \( \mathcal{E}^\dagger \). If \( N^\dagger \) is quasi-saturated in \( M \), then the equality below holds:

\[
\dim_{\mathcal{E}} N/N^1 = \dim_{\mathcal{E}} M.
\]

In particular, \( N^\dagger \) is saturated.

**Proof** The assertion follows from Theorem 2.14 in Sect. 2. \( \square \)

**Corollary 4.10** Let \( N^\dagger \) be a \( \varphi\nabla \)-module over \( \mathcal{E}^\dagger \). Then the following conditions are equivalent.

1. \( N^\dagger \) is irreducible.
2. \( N^\dagger \) is PBQ and saturated and \( N/N^1 \) is irreducible.

**Proof** (i) \( \Rightarrow \) (ii): Suppose \( M \) is a nontrivial quotient of \( N/N^1 \) as a \( \varphi\nabla \)-module over \( \mathcal{E} \). Since \( N^\dagger \) is irreducible, the natural homomorphism \( N^\dagger \to M \) is injective and its image is a \( \varphi\nabla \)-module over \( \mathcal{E}^\dagger \) by Proposition 4.4. Since \( N^\dagger \) is quasi-saturated in \( M \), we have \( M = N/N^1 \) by Proposition 4.9 so that \( N/N^1 \) is irreducible. The rest hold by Theorem 3.23.

(ii) \( \Rightarrow \) (i): Suppose \( L^\dagger \) is a nontrivial subobject of \( N^\dagger \). Since \( L^\dagger \subset N^\dagger \subset N/N^1 \), the maximal slope of \( L \) is equal to that of \( N \) by Proposition 4.5. Since \( N/N^1 \) is irreducible, the maximal slope of \( N/L \) is less than the maximal slope of \( N \) if \( N/L \) is not a zero object. Applying Lemma 3.17 (2) to the surjection \( N^\dagger \to N^\dagger/L^\dagger \), we have \( L^\dagger = N^\dagger \). Hence \( N^\dagger \) is irreducible. \( \square \)
Now we give a version of Theorem 4.2 for general $\varphi$-$\nabla$-modules over $E^\dagger$.

**Theorem 4.11** Let $M^\dagger$ (resp. $N^\dagger$) be a $\varphi$-$\nabla$-module over $E^\dagger$ with the PBQ filtration $0 = P^\dagger_0 \subsetneq P^\dagger_1 \subsetneq \cdots \subsetneq P^\dagger_r = M^\dagger$ (resp. $0 = Q^\dagger_0 \subsetneq Q^\dagger_1 \subsetneq \cdots \subsetneq Q^\dagger_s = N^\dagger$), and $M/M^b$ (resp. $N/N^b$) the maximally bounded quotient of $M$ (resp. $N$). If there exists an isomorphism $h : N/N^b \to M/M^b$ of $\varphi$-$\nabla$-modules over $E$, then $r = s$ and there is a unique isomorphism $g^\dagger : \bigoplus_{i=1}^r Q^\sat_i \to \bigoplus_{i=1}^r P^\sat_i$ such that the induced diagram

$$
\begin{array}{ccc}
\bigoplus_{i=1}^r Q^\sat_i & \xrightarrow{g} & \bigoplus_{i=1}^r P^\sat_i \\
\downarrow & & \downarrow \\
N/N^b & \xrightarrow{h} & M/M^b
\end{array}
$$

is commutative.

**Proof** Since $M/M^b \cong \bigoplus_{i=1}^r P_i/P_i^1$ and $N/N^b \cong \bigoplus_{i=1}^s Q_i/Q_i^1$ by Theorems 3.19 and 3.23 and since $h$ is an isomorphism, we have $r = s$ and $h|_{Q_i/Q_i^1} : Q_i/Q_i^1 \to P_i/P_i^1$ is an isomorphism of $\varphi$-$\nabla$-modules over $E$ for any $i$. Hence it is sufficient to prove the assertion when $r = s = 1$.

Suppose $r = s = 1$, that is, both $M^\dagger$ and $N^\dagger$ are PBQ. We may assume both $M^\dagger$ and $N^\dagger$ are saturated by Proposition 4.7. Let $L^\dagger$ be the image of $M^\dagger \oplus N^\dagger \to M/M^1$ ($(m, n) \mapsto m + h(n)$). Then $L^\dagger$ is the PBQ with the same maximal slope of $M$ and includes both $M^\dagger$ and $N^\dagger$. Since $L^\dagger$ is quasi-saturated in $M/M^1$, we have

$$\dim_E L/L^1 = \dim_E M/M^1$$

by Proposition 4.9 so that the maximal slope of the quotient $E \otimes_{E^\dagger} L^\dagger/M^\dagger$ is less than the maximal slope of $L$ if $L^\dagger \neq M^\dagger$. Since $L^\dagger$ is PBQ, Lemma 3.17 (2) implies $L^\dagger/M^\dagger = 0$, hence $L^\dagger = M^\dagger$. The same holds for $N^\dagger$ and we have an identification $N^\dagger = M^\dagger = L^\dagger$ in $M/M^1$. Therefore, there is an isomorphism $g^\dagger : N^\dagger \to M^\dagger$ which makes the given diagram commutative. The uniqueness of $g^\dagger$ follows from the PBQ property of $N^\dagger$ by Lemma 3.17 (2).

**Proof of Theorem 4.2** In the case (i) the nontrivial morphism $h$ is an isomorphism by Corollary 4.10. Then the assertion easily follows from Theorem 4.11.

**5 Saturated overconvergent $F$-isocrystals**

In this section we introduce saturated overconvergent $F$-isocrystals.
5.1 Saturated overconvergent $F$-isocrystals

Let $X$ be a smooth connected scheme separated of finite type over $\text{Spec} \ k$, and $\overline{X}$ a completion of $X$ over $\text{Spec} \ k$ with the canonical open immersion $j_{X,\overline{X}} : X \to \overline{X}$. Let $\mathcal{M}^\dagger$ be an overconvergent $F$-isocrystal on $X/K$ with respect to Frobenius $\sigma$, $\mathcal{M}^\dagger_U = j^\dagger_{U,\overline{X}} \mathcal{M}^\dagger$ the restriction of $\mathcal{M}^\dagger$ on $U/K$ as an overconvergent $F$-isocrystal, and $\mathcal{M}_U$ the convergent $F$-isocrystal on $U/K$ associated to $\mathcal{M}^\dagger$. For an affine open dense subscheme $U$ of $X$, let us take an affine smooth lift $\mathcal{U} = \text{Spec} \ A_U$ over $\text{Spec} \ k$, $\overline{\mathcal{U}}$ a completion of $\mathcal{U}$ over $\text{Spec} \ R$, $\overline{U} = \overline{\mathcal{U}} \times_{\text{Spec} \ R} \text{Spec} \ k$ the reduction of $\overline{\mathcal{U}}$, and $\hat{U}$ the $p$-adic completion of $\overline{U}$. Such a lift $\mathcal{U}$ exists by [19, Théorème 6]. Let us put $\mathcal{M}^\dagger_{\mathcal{U}} = \Gamma(\overline{\mathcal{U}}[\hat{\mathcal{V}}], \mathcal{M}^\dagger_{\mathcal{U}})$, $\mathcal{M}_U = \Gamma(\overline{U}[\hat{\mathcal{V}}], \mathcal{M}_U)$ with the slope filtration $\{M^i_U\}$ if $\mathcal{M}_U$ admits the slope filtration.

**Definition 5.1** An overconvergent $F$-isocrystal $\mathcal{M}^\dagger$ is saturated if the natural morphism $M^\dagger_{\mathcal{U}} \to M_U/M^1_U$ is injective for any affine open dense subscheme $U \subset X$ such that $\mathcal{M}$ admits the slope filtration.

**Proposition 5.2** Let $\mathcal{M}^\dagger$ be an overconvergent $F$-isocrystal on $X/K$. Let $V$ be an affine open dense subscheme of $X$ such that the associated convergent $F$-isocrystal $\mathcal{M}^\dagger_V$ to $\mathcal{M}^\dagger_V$ admits a slope filtration. Then $\mathcal{M}^\dagger$ is saturated if and only if $M^\dagger_V \to M_V/M^1_V$ is injective.

**Proof** Suppose that $M^\dagger_V \to M_V/M^1_V$ is injective. It is sufficient to prove, for any affine open dense subscheme $U$ of $V$, the natural morphism $M^\dagger_U = A^\dagger_{U,K} \otimes_{A^\dagger_{V,K}} M^\dagger_V \to M_U/M^1_U$ is injective since the top horizontal arrow of the commutative diagram

$$
\begin{array}{ccc}
M^\dagger_W & \to & M^\dagger_{V \cap W} \\
\downarrow & & \downarrow \\
M_W/M^1_W & \to & M_{V \cap W}/M^1_{V \cap W}
\end{array}
$$

is injective for any affine open dense subscheme $W$ of $X$.

**Lemma 5.3** Suppose that $X$ is affine with an affine smooth lift $\mathcal{X} = \text{Spec} \ A_X$ over $\text{Spec} \ R$ such that $A^\dagger_X$ admits a $q$-Frobenius $\varphi$ which is compatible with the $q$-Frobenius $\sigma$ on $K$. Let $\mathcal{M}^\dagger$ be an overconvergent $F$-isocrystal on $X/K$ such that the associated convergent $F$-isocrystal $\mathcal{M}$ to $\mathcal{M}^\dagger$ admits the slope filtration. Then there exists an overconvergent $F$-isocrystal $\mathcal{L}^\dagger$ on $X/K$ with an isomorphism

$$
\Gamma(\overline{\mathcal{X}}[\hat{\mathcal{X}}], \mathcal{L}^\dagger) \cong \text{Im}(M^\dagger \to M/M^1)
$$
of $A_{X,K}^\dagger$-modules such that the isomorphism commutes with connections and Frobenius.

**Proof** Since $A_{X,K}^\dagger$ is Noetherian [21, Theorem], the image $L^\dagger = \text{Im}(M^\dagger \to M/M^1)$ is a finitely generated $A_{X,K}^\dagger$-module. Since the $q$-Frobenius $\varphi$ on $A_{X,K}^\dagger$ is flat (see Appendix A.2) and $\text{id} \otimes \varphi : A_{X,K}^\dagger \otimes A_{X,K}^\dagger, \varphi \to A_{X,K}^\dagger$ is an isomorphism, the natural morphism $A_{X,K}^\dagger \otimes A_{X,K}^\dagger, \varphi \to \hat{A}_{X,K}$ is injective where $A_{X,K}^\dagger \otimes A_{X,K}^\dagger, \varphi$ means the extension by $\varphi : A_{X,K}^\dagger \to A_{X,K}^\dagger$ (resp. $\hat{A}_{X,K} \to \hat{A}_{X,K}$). By the compatibility of the $A_{X,K}^\dagger$-linear homomorphism $M^\dagger \to M/M^1$ with the connections and the Frobenius the integrable $K$-connection $\nabla_{M^\dagger}$ and Frobenius $\varphi_{M^\dagger}$ induce an integrable $K$-connection $\nabla_{L^\dagger} : L^\dagger \to L^\dagger \otimes_{A_X} \Omega_{A_X/R}^1$ and an isomorphism $\varphi_{L^\dagger} : \varphi^*L^\dagger \to L^\dagger$ such that $\varphi_{L^\dagger}$ is horizontal. Hence the connection $\nabla_{L^\dagger}$ is overconvergent and the sheafification $L^\dagger$ of $L^\dagger$ is our desired overconvergent $F$-isocrystal on $X/K$ by [6, Théorème 2.5.7] (see Theorem A.3). □

Let us continue the proof of Proposition 5.2. Suppose $U$ is an open dense subscheme of $V$. Our claim is that, if $N_{U}^\dagger$ is an overconvergent $F$-isocrystal on $U/K$ such that

$$\Gamma(J\hat{X}, N_{U}^\dagger) \cong \text{Ker}(M_{U}^\dagger \to M_{U}/M_{U}^1),$$

then $N_{U}^\dagger = 0$. Here the right hand side is a kernel of the natural homomorphism and it determines an overconvergent $F$-isocrystal on $U/K$ by Lemma 5.3 since the category of overconvergent $F$-isocrystals is Abelian. Since $M_{U}^\dagger$ is a restriction of $M^\dagger$ on $U$ and $N_{U}^\dagger$ is a subobject of $M_{U}^\dagger$, there exists an overconvergent $F$-isocrystal $N^\dagger$ on $X/K$ whose restriction on $U$ is $N_{U}^\dagger$ by [29, Proposition 5.3.1]. By our construction $N_V \to M_V/M_V^1$ is a zero map by the full faithfulness of the restriction functor from the category of convergent $F$-isocrystals on $V$ to that on $U$ [29, Theorem 5.2.1] [31, Theorem 4.2.1]. On the other hand the composition

$$N_{V}^\dagger = \Gamma(J\hat{X}, N_{V}^\dagger) \subset M_{V}^\dagger \to M_{V}/M_{V}^1$$

is injective. Hence $N_{U}^\dagger = 0$.

**Corollary 5.4** Let $U$ be an open dense subscheme of $X$. An overconvergent $F$-isocrystal $M^\dagger$ on $X/K$ is saturated if and only if so is the restriction $M_{U}^\dagger$ on $U/K$. □
**Corollary 5.5** An irreducible overconvergent $F$-isocrystal on $X/K$ is saturated.

*Proof* If $\mathcal{M}^\dagger$ is irreducible, then so is $\mathcal{M}^\dagger_U$ for arbitrary affine open dense subscheme $U$ of $X$ by [29, Proposition 5.3.1]. $\mathcal{M}^\dagger_U$ is saturated by Proposition 5.2 and Lemma 5.3 because one can find an affine open dense subscheme $U$ of $X$ on which the hypothesis in Lemma 5.3 holds.  

**Proposition 5.6** Let $(\mathcal{M}^\dagger, F\mathcal{M}^\dagger)$ be an overconvergent $F$-isocrystal on $X/K$.

(1) Let $f : Y \to X$ be a finite etale morphism of connected schemes over $\text{Spec } k$. Then $\mathcal{M}^\dagger$ is saturated if and only if so is the inverse image $f^*\mathcal{M}^\dagger$ as an overconvergent $F$-isocrystal on $Y/K$.

(2) Suppose $L$ (resp. $R_L$) is a finite extension of $K$ with a residue field $l$ (resp. the integer ring of $L$), $\sigma_L : L \to L$ an extension of the $n$-th power $\sigma^n$ of the $q$-Frobenius $\sigma$ on $K$. Let us put $X_l = X \times_{\text{Spec } k} \text{Spec } l$ to be the base extension of $X$, and $\mathcal{M}^\dagger_{X_l/L}$ the induced overconvergent $F$-isocrystal on $X_l/L$ with respect to $\sigma_L$ by the inverse image of $(\mathcal{M}^\dagger, F^n\mathcal{M}^\dagger)$. Then $\mathcal{M}^\dagger$ is saturated if and only if so is $\mathcal{M}^\dagger_{X_l/L}$ on each connected component of $X_l$.

*Proof* We may assume that $X$ is affine and $\mathcal{M}$ admits the slope filtration by Corollary 5.4. Take an affine smooth lift $\text{Spec } A_X$ of $X$ over $\text{Spec } R$.

(1) Since $Y$ is finite etale over $X$, there exists a finite $A_X$-algebra $A_Y$ such that the weak completion $A_Y ^\dagger$ is finite etale over $A_X ^\dagger$ by Jacobian criterion of etaleness. The assertion follows from the fact that $A_Y ^\dagger$ is faithfully flat over $A_X ^\dagger$ and the natural morphism $A_Y ^\dagger \otimes A_X ^\dagger \hat{A}_X \to \hat{A}_Y$ is an isomorphism.

(2) The assertion follows from Proposition 5.2 and the fact that the morphism $A_X ^\dagger \to A_{X_l} \cong L \otimes A_{X,l} ^\dagger \hat{A}_X \to \hat{A}_{X_l}$ is an isomorphism.  

**Remark 5.7** In the previous version of this paper we define an saturated overconvergent $F$-isocrystal on a curve $C$ if the natural morphism $M^\dagger \to M_\eta/M_\eta^\dagger$ is injective. Here $M_\eta$ is the generic $\varphi$-$\nabla$-module over $E_\eta$ (see Sect. 3.1 for the notation). It is equivalent to that of Definition 5.1. Indeed, the natural morphism $M_U/M_U^\dagger \to M_\eta/M_\eta^\dagger$ is injective for any affine dense subscheme $U$ of $C$.

### 5.2 Saturation

**Proposition 5.8** Let $\mathcal{M}^\dagger$ be an overconvergent $F$-isocrystal on $X/K$. Then there exists a saturated overconvergent $F$-isocrystal $\mathcal{L}^\dagger$ on $X/K$ with a surjective morphism $\theta^\dagger : \mathcal{M}^\dagger \to \mathcal{L}^\dagger$ as overconvergent $F$-isocrystals such that
the induced morphism $\mathcal{M}/\mathcal{M}^1 \to \mathcal{L}/\mathcal{L}^1$ between the maximal slope quotient is an isomorphism as convergent $F$-isocrystals. If a saturated overconvergent $F$-isocrystal $(\mathcal{L}')^\dagger$ on $X/K$ satisfies the similar properties, then there exists a unique isomorphism $\xi: (\mathcal{L}')^\dagger \to \mathcal{L}^\dagger$ of overconvergent $F$-isocrystals.

**Proof** For an affine open dense subscheme $U$ of $X$ in Lemma 5.3, let $\mathcal{L}_U^\dagger$ be the overconvergent $F$-isocrystal on $U/K$ in Lemma 5.3. Since $\mathcal{L}_U^\dagger$ is a quotient of $\mathcal{M}^\dagger$ and $L_U$ generates $M_U/M_U^1$ as an $\hat{A}_{U,K}$-module, $\mathcal{L}_U^\dagger$ is saturated and the induced morphism $M_U/M_U^1 \to L_U/L_U^1$ is an isomorphism.

If one has another $(\mathcal{L}')^\dagger$ on $U/K$ with the same properties, then

$$(L')^\dagger_U \to L'_U/(L')^1_U \cong M/M^1 \cong L_U/L_U^1 \leftarrow L^\dagger_U$$

induces a bijection of $A^\dagger_{U,K}$-modules between $L^\dagger_U$ and $(L')^\dagger_U$ since there exist compatible surjections from $M^\dagger_U$ to both $(L')^\dagger_U$ and $L^\dagger_U$. Hence we have a canonical isomorphism $\xi_U: (\mathcal{L}')^\dagger_U \to \mathcal{L}_U^\dagger$ of overconvergent $F$-isocrystals on $U/K$. The glueing method works by the canonical uniqueness of $\mathcal{L}_U^\dagger$ and Corollary 5.4, and we obtain the desired overconvergent $F$-isocrystal $\mathcal{L}^\dagger$ on $X/K$.

**Definition 5.9** For an overconvergent $F$-isocrystal $\mathcal{M}^\dagger$ on $X/K$, we define the saturation $\mathcal{M}^{\text{Sat.}^\dagger}$ of $\mathcal{M}^\dagger$ by the overconvergent $F$-isocrystal $\mathcal{L}^\dagger$ on $X/K$ in Proposition 5.8.

### 6 In the case of curves

In this section we prove the minimal slope conjecture in the case of curves. Let $C$ be a smooth connected curve over $\text{Spec} \ k$, and $\overline{C}$ a smooth completion of $C$.

#### 6.1 Ranks of the maximal slope quotients

**Proposition 6.1** Let $\mathcal{M}$ be an isoclinic convergent $F$-isocrystal on $C/K$, and $\mathcal{N}^\dagger$ be an overconvergent $F$-isocrystals on $C/K$ such that the associated convergent $F$-isocrystal $\mathcal{N}$ on $C/K$ admits the slope filtration $\{\mathcal{N}^i\}$. Suppose there exists a surjective morphism $\mathcal{N} \to \mathcal{M}$ of convergent $F$-isocrystal such that the induced morphism $\Gamma(\overline{C}[\mathcal{N}, j_U^\dagger N^\dagger]) \to \Gamma(\overline{U}[\mathcal{N}, M])$ of global sections is injective for an affine open dense subscheme $U$ of $C$. Then the maximal slope of $\mathcal{N}$ coincides with the slope of $\mathcal{M}$ and the equality below holds:

$$\text{rank } \mathcal{N}/\mathcal{N}^1 = \text{rank } \mathcal{M}.$$
Proof. Fix a notation as in Sect. 3.1. If $C = \overline{C}$, the assertion is trivial. Hence we may assume that $C$ is affine with a smooth lift $\bar{C} = \text{Spec} \ A_C$. First we prove the coincidence between the slope of $\mathcal{M}$ and the maximal slope of $\mathcal{N}$. Let $\mathcal{P}^\dagger$ be the maximally PBQ overconvergent $F$-subisocrystal of $\mathcal{N}^\dagger$. Then $\mathcal{P}$ admits the slope filtration by [44, Proposition 2.7]. Since the induced morphism $\mathcal{P} \to \mathcal{M}$ is nontrivial, the maximal slope of $\mathcal{P}$ must be equal to the slope of $\mathcal{M}$ by Lemma 3.30. This yields the maximal slope of $\mathcal{N}$ is equal to the slope of $\mathcal{M}$.

By the hypothesis of the surjectivity of $\mathcal{N} \to \mathcal{M}$ the inequality $\text{rank} \frac{\mathcal{N}}{\mathcal{N}^1} \geq \text{rank} \ \mathcal{M}$ always holds. Suppose $\text{rank} \frac{\mathcal{N}}{\mathcal{N}^1} > \text{rank} \ \mathcal{M}$ holds.

Let $P^\dagger$ be the maximally PBQ overconvergent $F$-subisocrystal of $\mathcal{N}^\dagger$. Then $P^\dagger$ admits the slope filtration by \cite[Proposition 2.7]{44}. Since the induced morphism $P^\dagger \to M$ is nontrivial, the maximal slope of $P^\dagger$ must be equal to the slope of $M$ by Lemma 3.30. This yields the maximal slope of $P$ is equal to the slope of $M$.

Let us define a nontrivial convergent $F$-isocrystal $H$ on $C/K$ by

$$H = \text{Ker}(\mathcal{N} \to \mathcal{M})$$

and $L$ the maximally PBQ convergent $F$-subisocrystal of $H$. The convergent version of PBQ filtrations also exists similarly to Theorems 3.19 and 3.27. By our hypothesis the maximal slope of $H$ and hence $L$ is that of $M$. We will construct the maximally PBQ overconvergent $F$-subisocrystal of $L^\dagger$ such that the convergent $F$-isocrystal on $C/K$ associated to $L$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_s$ be all distinct closed points of $\overline{C} \setminus C$. Let us put $N^\dagger_C = \Gamma(\overline{C}[\mathcal{\overline{\mathcal{\overline{C}}}}], \mathcal{M}_C) = \Gamma(\overline{C}[\mathcal{\overline{\mathcal{\overline{C}}}}], \mathcal{M})$ and $L_C = \Gamma(\overline{C}[\mathcal{\overline{\mathcal{\overline{C}}}}], \mathcal{L})$. For any closed point $\alpha_i$ we put

$$H^\dagger_{\alpha_i} = \text{Ker}(E^\dagger_{\alpha_i} \otimes A^\dagger_{C,K} N^\dagger_C \to E_{\alpha_i} \otimes \Delta_{C,K} M_C).$$

Then $H^\dagger_{\alpha_i}$ is a $\phi$-$\nabla$-module over $E^\dagger_{\alpha_i}$. Let us take the maximally PBQ $\phi$-$\nabla$-module $L^\dagger_{\alpha_i}$ of $H^\dagger_{\alpha_i}$ over $E^\dagger_{\alpha_i}$. Then our claim is as follows.

Claim. Let us define $H_{\alpha_i} = E_{\alpha_i} \otimes E^\dagger_{\alpha_i} H^\dagger_{\alpha_i}$ and $L_{\alpha_i} = E_{\alpha_i} \otimes E^\dagger_{\alpha_i} L^\dagger_{\alpha_i}$ for $1 \leq i \leq s$.

1. $\dim E_{\alpha_i} H_{\alpha_i}/H_{\alpha_i}^1 = \text{rank} \mathcal{H}/\mathcal{H}^1$.

2. The natural commutative diagram

$$\begin{array}{ccc}
0 & \to & H^\dagger_{\alpha_i} \\
\downarrow & & \downarrow \\
0 & \to & E_{\alpha_i} \otimes \Delta_{C,K} H_C \\
& & \downarrow \\
& & E_{\alpha_i} \otimes \Delta_{C,K} N^\dagger_C \\
& & \to \\
& & E_{\alpha_i} \otimes \Delta_{C,K} M_C \\
& & \to 0
\end{array}$$

with exact rows induces an injection

$$L_{\alpha_i} \to E_{\alpha_i} \otimes \Delta_{C,K} L_C$$

of $\phi$-$\nabla$-modules over $E_{\alpha_i}$.
(3) The injection $L_{\alpha_i} \rightarrow \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C$ is an isomorphism.

Proof (1) Since $N_{\alpha_i} = \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} N_{\alpha_i}^\dagger \rightarrow \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} M_C$ is surjective, we have

$$\text{dim} \mathcal{E}_{\alpha_i} H_{\alpha_i} / H_{\alpha_i}^1 = \text{dim} \mathcal{E}_{\alpha_i} N_{\alpha_i} / N_{\alpha_i}^1 - \text{rank} M = \text{rank} \mathcal{H} / \mathcal{H}^1$$

applying Proposition 4.9 to $N_{\alpha_i}^\dagger / H_{\alpha_i}^\dagger \subset \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C$.

(2) Since $\mathcal{E}_{\alpha_i}^\dagger \rightarrow \mathcal{E}_{\alpha_i}$ is a field extension, the natural morphism $H_{\alpha_i} \rightarrow \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C$ is injective. The image of $L_{\alpha_i}$ is included in $\mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C$ by Lemma 3.17 (2) and Theorem 3.21 (2). Indeed, the maps from $L_{\alpha_i}$ to the PBQ subquotients induced by the PBQ filtration of $\mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C$ is a zero map if the maximal slopes are different.

(3) Note that the slope filtration is strict for morphisms of $F$-isocrystals and it is compatible of localization. Hence there is a sequence of isomorphisms

$$L_{\alpha_i} / L_{\alpha_i}^1 \cong H_{\alpha_i} / H_{\alpha_i}^1 \cong \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C / \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} H_C^1 \cong \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C / \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C^1$$

by (1) and the global-local compatibility of PBQ filtrations by Theorems 3.21 (2) and 3.27. Since both $L_{\alpha_i} / L_{\alpha_i}^1$ and $\mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C / \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C^1$ are PBQ with isomorphic maximally slope quotients, the injection $L_{\alpha_i} \rightarrow \mathcal{E}_{\alpha_i} \otimes \hat{A}_{C,K} L_C$ is surjective by Lemma 3.17 (2). Note that one can not directly apply Theorem 4.2 by lack of saturation.

Now applying Lemma 3.33 or [32, Remark 5.10] to the compatible data $L_C, L_{\alpha_i}^\dagger (i = 1, \ldots, s)$ in the overconvergent $F$-isocrystal $N^\dagger$, we have an overconvergent $F$-isocrystal $L^\dagger$ on $C / K$ such that $L^\dagger$ is a subobject of $N^\dagger$ and that the convergent $F$-isocrystal $L$ on $C / K$ associated to $L^\dagger$ is $L_C$. Then we have an injection

$$L_C^\dagger = \Gamma(\overline{\mathcal{C}}, L^\dagger) \rightarrow N_C^\dagger \rightarrow M_C.$$ 

But the associated morphism $L_C \rightarrow M_C$ is a zero map, and it is a contradiction. Therefore, the equality $\text{rank} N / N^1 = \text{rank} M$ holds. This completes a proof of Proposition 6.1.

\begin{prop}
Let $M^\dagger$ be an overconvergent $F$-isocrystal on $C / K$. Then the following conditions are equivalent.

(i) $M^\dagger$ is irreducible.

(ii) $M^\dagger$ is PBQ and saturated and $M_U / M_U^1$ is irreducible for any open dense subscheme $U$ of $C$ such that $M_U$ admits the slope filtration.

\end{prop}
Proof The similar proof works with Corollary 4.10 by Corollary 3.28, Lemma 3.30, Corollary 5.4 and Proposition 6.1.

Example 6.3 Let 0 → \( L^\dagger \) → \( M^\dagger \) → \( N^\dagger \) → 0 be an exact sequence of overconvergent \( F \)-isocrystals on \( C/K \). Suppose \( L^\dagger \) and \( N^\dagger \) are irreducible with the maximal slopes \( s_L \) and \( s_N \), respectively. Then the following hold.

1. If \( s_L < s_N \), then \( M^\dagger \) is not saturated. \( M^\dagger \) is a nontrivial extension if and only if \( M^\dagger \) is PBQ.
2. If \( s_L = s_N \), then \( M^\dagger \) is PBQ and saturated, but \( M/M^1 \) is not irreducible. \( M^\dagger \) is a nontrivial extension if and only if so is \( M/M^1 \).
3. If \( s_L > s_N \), then the exact sequence gives the PBQ filtration of \( M^\dagger \). \( M^\dagger \) is a nontrivial extension if and only if \( M^\dagger \) is saturated. Moreover, if furthermore \( M^\dagger \) is of rank 2, then the sequence is split by [31, Theorem 4.2.1] since the induced sequence as convergent \( F \)-isocrystals on \( C/K \) is split.

6.2 The minimal slope conjecture on curves

Theorem 6.4 Let \( M^\dagger \) (resp. \( N^\dagger \)) be an overconvergent \( F \)-isocrystal on \( C/K \) such that \( M \) (resp. \( N \)) admits the Frobenius filtration, and 0 = \( P_0^\dagger \subsetneq P_1^\dagger \subsetneq \cdots \subsetneq P_r^\dagger = M^\dagger \) (resp. 0 = \( Q_0^\dagger \subsetneq Q_1^\dagger \subsetneq \cdots \subsetneq Q_s^\dagger = N^\dagger \)) the PBQ filtration of \( M^\dagger \) (resp. \( N^\dagger \)), and \( M/M^b = \bigoplus_i P_i/P_i^1 \) (resp. \( N/N^b = \bigoplus_j Q_j/Q_j^1 \)) the maximally bounded quotient of \( M \) (resp. \( N \)) (Proposition 3.29). If there exists an isomorphism \( h : N/N^b \rightarrow M/M^b \) of overconvergent \( F \)-isocrystals, then \( r = s \) and there is a unique isomorphism \( g : \bigoplus_{i=1}^r Q_i^{sat,\dagger} \rightarrow \bigoplus_{i=1}^r P_i^{sat,\dagger} \) such that the induced diagram

\[
\begin{array}{ccc}
\bigoplus_{i=1}^r Q_i^{sat} & \xrightarrow{g} & \bigoplus_{i=1}^r P_i^{sat} \\
\downarrow & & \downarrow \\
N/N^b & \xrightarrow{h} & M/M^b
\end{array}
\]

is commutative.

Proof The proof is similar to that of the local case, Theorem 4.11, by applying Proposition 6.1 and Lemma 3.30 (2) instead of Proposition 4.9 and Lemma 3.17 (2), respectively.

Corollary 6.5 Let \( M^\dagger \) and \( N^\dagger \) be overconvergent \( F \)-isocrystals on \( C/K \) such that the associated convergent \( F \)-isocrystal \( M \) and \( N \) admit the slope filtrations, and \( h : N/N^\dagger \rightarrow M/M^1 \) a morphism between the maximal slope quotients as convergent \( F \)-isocrystals. Suppose either

(i) \( M^\dagger \) and \( N^\dagger \) are irreducible and \( h \) is nontrivial, or
(ii) $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ are PBQ and saturated and $h$ is an isomorphism.

Then there is a unique isomorphism

$$g^\dagger : \mathcal{N}^\dagger \rightarrow \mathcal{M}^\dagger$$

of overconvergent $F$-isocrystals on $C/K$ such that the induced morphism $\mathcal{N}/\mathcal{N}^1 \rightarrow \mathcal{M}/\mathcal{M}^1$ by $g^\dagger$ coincides with the given $h$.

**Proof** In the case (i) the nontrivial morphism $h$ is an isomorphism by Proposition 6.2. The assertion follows from Theorem 6.4. $\Box$

**Remark 6.6**

(1) Suppose $C$ is projective. Then $\mathcal{M}^\dagger = \mathcal{M}$. If furthermore $\mathcal{M}$ is irreducible and admits a slope filtration, then $\mathcal{M}^1 = 0$. Therefore the minimal slope conjecture is trivially true in this case.

(2) The hypothesis that $\mathcal{M}$ and $\mathcal{N}$ admit slope filtrations in Theorem 6.4 and Corollary 6.5 is redundant. It is enough to assume that there exists a convergent $F$-subisocrystal $\mathcal{M}^1$ such that $\mathcal{M}/\mathcal{M}^1$ is isoclinic and all slopes of $\mathcal{M}^1$ at any point of $C$ is less than the slope of $\mathcal{M}/\mathcal{M}^1$, and the same for $\mathcal{N}$. However, if one takes a sufficiently small open dense subscheme $U$ of $C$, then both $\mathcal{M}$ and $\mathcal{N}$ admit slope filtrations [32, Theorem 3.1.2, Corollary 4.2], [44, Proposition 2.2, Corollary 2.6]. The existence of $g^\dagger$ follows from the full faithfulness of the restriction functor of overconvergent $F$-isocrystals [43, Theorem 6.3.1].

### 6.3 Lefschetz condition

Let $X$ be a scheme separated of finite type over $\text{Spec } k$, $\alpha = \text{Spec } k_\alpha$ a closed point of $X$ with the closed immersion $i_{\alpha,X} : \alpha \rightarrow X$, and $\mathcal{M}^\dagger$ an irreducible overconvergent $F$-isocrystal on $X/K$. Let us consider the following condition (LC), called Lefschetz condition, for $(X, \alpha, \mathcal{M}^\dagger)$:

There exist a smooth curve $C_\alpha$ over $\text{Spec } k$ and a morphism $i_{C_\alpha,X} : C_\alpha \rightarrow X$ over $\text{Spec } k$ such that there exists a $k_\alpha$-rational point of $C_\alpha$ which maps to $\alpha$ by $i_{C_\alpha,X}$ (say $C_\alpha$ is passing at $\alpha$ and denote the $k_\alpha$-rational point in $C_\alpha$ also by $\alpha$) and that the restriction $i_{C_\alpha,X}^*\mathcal{M}^\dagger$ on $C_\alpha$ is irreducible.

The question on an existence of such curves for $\mathcal{M}^\dagger$ has an affirmative answer more generally in the case where $k$ is finite and $\mathcal{M}^\dagger$ is an overconvergent $\overline{Q}_p$-$F$-isocrystal in [4, Theorem 0.3] (see Theorem 7.19). It will be used in order to prove our main theorem (Theorem 7.20). See [32, Conjecture 5.19] for the detail of the problem. The next proposition follows from Theorem 6.4.
Proposition 6.7 With the notation as above, let $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ be overconvergent $F$-isocrystals on $X/K$ such that the associated convergent $F$-isocrystals $\mathcal{M}$ and $\mathcal{N}$ admit the slope filtrations, and $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ a nontrivial morphism as convergent $F$-isocrystals. Suppose that $\mathcal{M}^\dagger$ (resp. $\mathcal{N}^\dagger$) is irreducible and the triplet $(X, \alpha, \mathcal{M}^\dagger)$ (resp. $(X, \alpha, \mathcal{N}^\dagger)$) satisfies the condition (LC) with respect to a morphism $i_{C_{a,M},X} : C_{a,M} \to X$ (resp. $i_{C_{a,N},X} : C_{a,N} \to X$). Let $Q_{C_{a,M}}^\dagger$ (resp. $P_{C_{a,N}}^\dagger$) be the maximally PBQ overconvergent $F$-subisocrystal of $i_{C_{a,M},X}^{*}\mathcal{N}^\dagger$ (resp. $i_{C_{a,N},X}^{*}\mathcal{M}^\dagger$). Then there exists a unique surjective (resp. injective) morphism

$$g_{C_{a,M}}^\dagger : Q_{C_{a,M}}^\dagger \to i_{C_{a,M},X}^{*}\mathcal{M}^\dagger$$

(resp. $g_{C_{a,N}}^\dagger : i_{C_{a,N},X}^{*}\mathcal{N}^\dagger \to P_{C_{a,N}}^\dagger$) of overconvergent $F$-isocrystals on $C_{a,M}/K$ (resp. $C_{a,N}/K$) such that the induced morphism between the maximal slope quotients from $g_{C_{a,M}}^\dagger$ (resp. $g_{C_{a,N}}^\dagger$) is the given $i_{C_{a,M},X}^{*}h$ (resp. $i_{C_{a,N},X}^{*}h$).

Proof Suppose $\mathcal{M}^\dagger$ (resp. $\mathcal{N}^\dagger$) is irreducible. Then $\mathcal{M}/\mathcal{M}^1$ (resp. $\mathcal{N}/\mathcal{N}^1$) is irreducible by Proposition 6.2 so that $h$ is surjective (resp. injective). Hence there exists a unique surjective (resp. injective) morphism $g_{C_{a,M}}^\dagger$ (resp. $g_{C_{a,N}}^\dagger$) which is compatible with the given $i_{C_{a,M},X}^{*}h$ (resp. $i_{C_{a,N},X}^{*}h$) by Theorem 6.4. □

Corollary 6.8 Suppose furthermore that both $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ are irreducible and that both $(X, \alpha, \mathcal{M}^\dagger)$ and $(X, \alpha, \mathcal{N}^\dagger)$ satisfy the condition (LC). If $K_{\alpha}$ is an unramified extension of $K$ with residue field $k_{\alpha}$, then there exists an isomorphism

$$g_{\alpha}^\dagger : i_{\alpha,X}^{*}\mathcal{N}^\dagger \to i_{\alpha,X}^{*}\mathcal{M}^\dagger$$

of $F$-spaces over $K_{\alpha}$.

Proof Note that, if one can take the same curve $C_{\alpha} \to X$ for both $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$, then the assertion follows just from Corollary 6.5 (1). Keep the notation as in Proposition 6.7. Both inequalities $\text{rank } \mathcal{M}^\dagger \leq \text{rank } Q_{C_{a,M}}^\dagger \leq \text{rank } \mathcal{N}^\dagger$ and $\text{rank } \mathcal{N}^\dagger \leq \text{rank } P_{C_{a,N}}^\dagger \leq \text{rank } \mathcal{M}^\dagger$ hold by Proposition 6.7. Hence $\text{rank } \mathcal{M}^\dagger = \text{rank } \mathcal{N}^\dagger$. We have an isomorphism $g_{C_{a,M}}^\dagger : i_{C_{a,M},X}^{*}\mathcal{N}^\dagger \to i_{C_{a,M},X}^{*}\mathcal{M}^\dagger$ by

$$i_{C_{a,M},X}^{*}\mathcal{N}^\dagger \xleftarrow{\sim} Q_{C_{a,M}}^\dagger \xrightarrow{\sim} i_{C_{a,M},X}^{*}\mathcal{M}^\dagger$$
such that it induces $i^*_{C_*M_*X} (h)$. It induces an isomorphism $g^\dagger_\alpha : i^*_{\alpha, X} N^\dagger_\alpha \to i^*_{\alpha, X} M^\dagger_\alpha$ as $F$-spaces over $K_\alpha$. □

**Corollary 6.9** With the hypothesis in Proposition 6.7, suppose furthermore that $M^\dagger_\alpha$ is irreducible and there is a closed point $\alpha$ of $X$ such that $(X, \alpha, M^\dagger_\alpha)$ satisfies the condition (LC). If $N^\dagger_\alpha$ is unit-root, then $M^\dagger_\alpha$ is unit-root and there exists a unique surjection

$$g^\dagger : N^\dagger_\alpha \to M^\dagger_\alpha$$

of overconvergent $F$-isocrystals on $X/K$ such that the induced morphism of convergent $F$-isocrystals from $g^\dagger$ coincides with the given $h$.

**Proof** By the hypothesis $i^*_{\alpha, X} M^\dagger_\alpha$ is unit-root by Proposition 6.7. Hence $M^\dagger_\alpha$ is unit-root (i.e., $M^1_\alpha = 0$) and the assertion follows from the full faithfulness of the functor from the category of overconvergent $F$-isocrystals to that of convergent $F$-isocrystals [31, Theorem 4.2.1]. □

7 The case of finite fields

In this section we study the minimal slope conjecture on varieties of arbitrary dimension over finite fields. Let $\overline{\mathbb{Q}}_p$ be an algebraic closure of the field $\mathbb{Q}_p$ of $p$-adic numbers. Let $q$ be a positive power of $p$, $k$ the field of $q$ elements, $R = W(k)$ the ring of Witt vectors with coefficients in $k$, and $K$ the field of fractions of $R$ with the $q$-Frobenius $\sigma = \text{id}_K$. We regard $K$ as a subfield of $\overline{\mathbb{Q}}_p$ and identify an algebraic closure $\overline{K}$ of $K$ with $\overline{\mathbb{Q}}_p$.

7.1 $F$-isocrystals with $\overline{\mathbb{Q}}_p$-structures

We recall the definition of $\overline{\mathbb{Q}}_p$-$F$-isocrystals which are introduced by Abe in [1, 1.4, 4.2.1] (see also [4, 1.1] and [32, 9.2]). Let $X$ be a scheme locally of finite type over $\text{Spec} \ k$.

**Definition 7.1** Let $L$ be a finite extension of $K$ in $\overline{\mathbb{Q}}_p$. An overconvergent $L$-$F_q$-isocrystal $M^\dagger_\alpha$ on $X$ is an overconvergent $F$-isocrystal $M^\dagger_\alpha$ on $X/K$ with respect to the $q$-Frobenius $\sigma$ such that $M^\dagger_\alpha$ is furnished with a $K$-algebra homomorphism

$$L \to \text{End}_{F\text{-Isoc}}(M^\dagger_\alpha),$$

where $\text{End}_{F\text{-Isoc}}(M^\dagger_\alpha)$ is the $K$-space of endomorphisms of $M^\dagger_\alpha$ as overconvergent $F$-isocrystals on $X/K$. Note that the $q$ of $F_q$ indicates $q$-Frobenius. The
$K$-algebra homomorphism $L \to \text{End}_{F\text{-Isoc}}(\mathcal{M}^\dagger)$ is called an $L$-structure. A morphism of overconvergent $L$-$F_q$-isocrystals on $X$ is a morphism of overconvergent $F$-isocrystals on $X/K$ which commutes with $L$-structures. The category of overconvergent $L$-$F_q$-isocrystals on $X$ is denoted by $F_q\text{-Isoc}^\dagger(X) \otimes L$.

We also define the notion of convergent $L$-$F_q$-isocrystals on $X$ similarly and denote their category by $F_q\text{-Isoc}(X) \otimes L$.

**Example 7.2** (1) An overconvergent $F$-isocrystal on $X/K$ is furnished with the natural $K$-structure since the category of overconvergent $F$-isocrystals on $X/K$ is $K$-linear by $\sigma = \text{id}_K$.

(2) Let $L$ be an extension of $K$ in $\mathbb{Q}_p$. We regard $L$ as an $F$-space over $K$ with respect to $\sigma$ by $F_L = \text{id}_L$. Then, for an overconvergent $F$-isocrystal $\mathcal{M}^\dagger$ on $X/K$, the $j^\dagger_X \mathcal{O}_{|X|} \otimes_K L$-module $\mathcal{M}^\dagger \otimes_K L$ is furnished with an $L$-structure which is defined by

$$a = \text{id}_{\mathcal{M}^\dagger} \otimes a\text{id}_L : \mathcal{M}^\dagger \otimes_K L \to \mathcal{M}^\dagger \otimes_K L$$

for $a \in L$. The category $F_q\text{-Isoc}^\dagger(X) \otimes L$ is an $L$-linear Abelian category. We will show that it is furnished with tensor products, duals and the unit object $j^\dagger_X \mathcal{O}_{|X|} \otimes_K L$ in the propositions below.

Let us study tensor products, duals, and extensions of $F$-isocrystals with $\mathbb{Q}_p$-structures, and introduce an invariant $\mathbb{Q}_p$-rank below. For an overconvergent $F$-isocrystal $\mathcal{M}^\dagger$, $\text{rank}(\mathcal{M}^\dagger)$ means the rank of locally free $j^\dagger_X \mathcal{O}_{|X|}$-module $\mathcal{M}^\dagger$ on each connected component of $X$.

**Proposition 7.3** Let $\mathcal{M}^\dagger$ be an overconvergent $L_i$-$F_q$-isocrystal on $X$, and $L = L_1 L_2$ the field of composite between $L_1$ and $L_2$ in $\mathbb{Q}_p$. Let us put $\mathcal{M}^\dagger = (\mathcal{M}_1^\dagger \otimes \mathcal{M}_2^\dagger) \otimes L_1 \otimes_K L_2$ where $\mathcal{M}_1^\dagger \otimes j^\dagger_X \mathcal{O}_{|X|} \mathcal{M}_2^\dagger$ is the tensor product as an overconvergent $F$-isocrystal on $X/K$ with $L_1 \otimes_K L_2$-actions and $L_1 \otimes_K L_2 \to L$ ($a \otimes b \mapsto ab$).

(1) If we define $\nabla_{\mathcal{M}^\dagger} = (\nabla_{\mathcal{M}_1^\dagger} \otimes \text{id}_{\mathcal{M}_2^\dagger} + \text{id}_{\mathcal{M}_1^\dagger} \otimes \nabla_{\mathcal{M}_2^\dagger}) \otimes \text{id}_L$, then the pair $(\mathcal{M}^\dagger, \nabla_{\mathcal{M}^\dagger})$ is an overconvergent isocrystal on $X/K$ such that

$$\frac{\text{rank}(\mathcal{M}^\dagger)}{\deg(L/K)} = \frac{\text{rank}(\mathcal{M}_1^\dagger)}{\deg(L_1/K)} \times \frac{\text{rank}(\mathcal{M}_2^\dagger)}{\deg(L_2/K)}$$

on each connected component of $X$.

(2) If we define $F_{\mathcal{M}^\dagger} : F^* \mathcal{M}^\dagger \to \mathcal{M}^\dagger$ by $F_{\mathcal{M}^\dagger} = F_{\mathcal{M}_1^\dagger \otimes \mathcal{M}_2^\dagger} \otimes \text{id}_L$ where

$$F^* \mathcal{M}^\dagger = F^* (\mathcal{M}_1^\dagger \otimes \mathcal{M}_2^\dagger) \otimes L_1 \otimes_K L_2 L,$$

then $F_{\mathcal{M}^\dagger}$ is horizontal with respect to the connections. In particular, the triplet $(\mathcal{M}^\dagger, \nabla_{\mathcal{M}^\dagger}, F_{\mathcal{M}^\dagger})$ is an overconvergent $F$-isocrystal on $X/K$. 

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For an element $a \in L$, we define $a : M^\dagger \to M^\dagger$ by $a = \text{id}_{M_1} \otimes M_2 \otimes \text{aid}_L$, then it defines an $L$-structure on $M^\dagger$.

In particular, $M^\dagger$ is an overconvergent $L$-$F_q$-isocrystal on $X$.

**Proposition 7.4** Let $M^\dagger$ be an overconvergent $L$-$F_q$-isocrystal on $X$. We define an action of $L$ on $(M^\dagger)^\vee = \text{Hom}_{j_X^\dagger \mathcal{O}_X}[M^\dagger, j_X^\dagger \mathcal{O}_X]$ by

$$(a, \eta) \mapsto a\eta(\eta)(a m) = \eta(am) \text{ for } m \in M^\dagger,$$

for $a \in L$ and $\eta \in (M^\dagger)^\vee$, then it is an $L$-structure on $(M^\dagger)^\vee$. In particular, $(M^\dagger)^\vee$ is an overconvergent $L$-$F_q$-isocrystal on $X$ such that

$$\frac{\text{rank}((M^\dagger)^\vee)}{\deg(L/K)} = \frac{\text{rank}(M^\dagger)}{\deg(L/K)}$$

on each connected component of $X$. Moreover there is a perfect pairing

$$(M^\dagger \otimes_K L) \otimes ((M^\dagger)^\vee \otimes_K L) \to j_X^\dagger \mathcal{O}_X \otimes_K L$$

of overconvergent $L$-$F_q$-isocrystals on $X$. In other words, $(M^\dagger)^\vee$ is the dual of $M^\dagger$ in $F_q$-$\text{Isoc}^\dagger(X) \otimes L$.

**Proposition 7.5** Let $L_1 \subset L_2$ be extensions of $K$ in $\bar{\mathbb{Q}}_p$. We define a functor $F_q$-$\text{Isoc}^\dagger(X) \otimes L_1 \to F_q$-$\text{Isoc}^\dagger(X) \otimes L_2$ of scalar extensions by

$$\theta_{L_1, L_2} : M^\dagger \mapsto (M^\dagger \otimes_K L_2) \otimes_{L_1 \otimes_K L_2} L_2 = M^\dagger \otimes_{L_1} L_2.$$

Then the equality

$$\frac{\text{rank}(\theta_{L_1, L_2}(M^\dagger))}{\deg(L_2/K)} = \frac{\text{rank}(M^\dagger)}{\deg(L_1/K)}$$

holds on each connected component of $X$. Moreover, $\theta_{L_1, L_2} = \theta_{L_2, L_3} \circ \theta_{L_1, L_2}$ for $L_1 \subset L_2 \subset L_3$ in $\bar{\mathbb{Q}}_p$.

**Definition 7.6** We define the category $F_q$-$\text{Isoc}^\dagger(X) \otimes \bar{\mathbb{Q}}_p$ by the 2-colimit of $F_q$-$\text{Isoc}^\dagger(X) \otimes L$ over all finite extensions $L$ in $\bar{\mathbb{Q}}_p$ by $\theta_{L_1, L_2}$ in the previous proposition. An object of $F_q$-$\text{Isoc}^\dagger(X) \otimes \bar{\mathbb{Q}}_p$ is said to be an overconvergent $\bar{\mathbb{Q}}_p$-$F_q$-isocrystals on $X$. We define the $\bar{\mathbb{Q}}_p$-rank of an overconvergent $\bar{\mathbb{Q}}_p$-$F_q$-isocrystal $M^\dagger$ on $X$ by

$$\bar{\mathbb{Q}}_p \text{-rank}(M^\dagger) = \frac{\text{rank}(M^\dagger)}{\deg(L/K)}.$$
on each connected component of $X$. We also define the category $F_q$-Isoc$(X) \otimes \overline{Q}_p$ of convergent $\overline{Q}_p$-$F_q$-isocrystals on $X$ similarly.

Lemma 7.7 Let $M^\dagger$ be an irreducible overconvergent $\overline{Q}_p$-$F_q$-isocrystal on $X$ which is represented by an overconvergent $L$-$F_q$-isocrystal on $X$ for an extension $L$ of $K$ in $\overline{Q}_p$. Then there exists an irreducible overconvergent $F$-isocrystal $N^\dagger$ on $X/K$ such that $M^\dagger$ is a direct sum of a finite number of copies of $N^\dagger$ as an overconvergent $F$-isocrystal on $X/K$.

Proof Let $N^\dagger$ be a nontrivial irreducible subobject of $M^\dagger$ as an overconvergent $F$-isocrystal on $X/K$ with respect to the $q$-Frobenius $\sigma$. The irreducibility $M^\dagger$ implies that the natural morphism $N^\dagger \otimes_K L \to M^\dagger$ as overconvergent $F$-isocrystals on $X/K$ is surjective where $L$ is regarded as an $F_q$-space over $K$ such that $F_L = \text{id}_L$. Hence $M^\dagger$ is a direct sum of a finite number of copies of $N^\dagger$. \hfill \Box

Let $k_0$ be a subfield of $k$ with $\text{deg}(k/k_0) = n$, and $K_0$ the unramified extension of $\overline{Q}_p$ with the residue field $k_0$ and the $q_0$-Frobenius $\sigma_0 = \text{id}_{K_0}$ where $q_0^n = q$. Since $K$ is finite and unramified over $K_0$, the smooth formal scheme over $R$ can be regarded as a smooth formal scheme over the integer ring $R_0$ of $K_0$. Hence, for an overconvergent $L$-$F_q$-isocrystal $M^\dagger$ on $X$, 

$$(M_0^\dagger, \nabla_{M_0^\dagger}) = (\oplus_{i=0}^{n-1}(F_q^i)\ast M^\dagger, \oplus_{i=0}^{n-1}(F_q^i)\ast \nabla_{M^\dagger})$$

$F_{M_0^\dagger}(a_0 \otimes_{\sigma_0} m_0, \ldots, a_{n-1} \otimes_{\sigma_0} m_{n-1})$

$$= (F_{M^\dagger}(a_{n-1} \otimes_{\sigma_0} m_{n-1}), a_0 \otimes_{\sigma_0} m_0, \ldots, a_{n-2} \otimes_{\sigma_0} m_{n-2})$$

has a structure of overconvergent $L$-$F_{q_0}$-isocrystals on $X$.

Lemma 7.8 Suppose that $L$ admits an extension $\sigma_0$ of $q_0$-Frobenius $\sigma_0$ on $K_0$. The $L$-structure on $(M_0^\dagger, \nabla_{M_0^\dagger}, F_{M_0^\dagger})$ which is induced from that on $M^\dagger$ is given by 

$$(a_0 \otimes_{\sigma_0} m_0, \ldots, a_{l-1} \otimes_{\sigma_0} m_{l-1})$$

$$\mapsto (ba_0 \otimes_{\sigma_0} m_0, \ldots, a_0^{n-1}(b)a_{n-1} \otimes_{\sigma_0} m_{n-1}).$$

for $b \in L$. In particular, the $K_0$-structure induced by the $q_0$-Frobenius coincides with $K_0 \subset L \to \text{End}_{F\text{-Isoc}}(M_0^\dagger)$.

Proposition 7.9 [1, Corollary 1.4.11] With the notation as above, the functor

$$\Pi_{q/q_0} : F_q\text{-Isoc}^\dagger(X) \otimes \overline{Q}_p \to F_{q_0}\text{-Isoc}^\dagger(X) \otimes \overline{Q}_p : \mathcal{M}^\dagger \mapsto \mathcal{M}_0^\dagger$$
is an equivalence of categories. Moreover, the equalities hold

\[ \overline{\mathbb{Q}}_p\text{-rank}(\mathcal{M}^\dagger) = \frac{\text{rank}_{j^\dagger}\mathcal{O}_{Y|X}(\mathcal{M}^\dagger)}{\text{deg}(L/K)} = \frac{\text{rank}_{j^\dagger}\mathcal{O}_{Y|X}(\mathcal{M}_0^\dagger)}{\text{deg}(L/K_0)} = \overline{\mathbb{Q}}_p\text{-rank}(\mathcal{M}_0^\dagger), \]

on each connected component of \( X \).

**Proof** For any finite extension \( L' \) of \( K \), there exists a finite extension \( L \) of \( L' \) which admits a \( q_0 \)-Frobenius \( \sigma_0 \) which is an extension of \( \sigma_0 \) on \( K_0 \) by Lemma A.1 (3) in Appendix A. For an object \( \mathcal{N}^\dagger \) of \( F_{q_0}\text{-Isoc}^\dagger(X) \otimes L \), if \( \pi_{(K,\sigma)/(K_0,\sigma_0)}^* \) is the pull back functor defined in Appendix B, then \( \pi_{(K,\sigma)/(K_0,\sigma_0)}^*\mathcal{N}^\dagger \) has a natural \( K \)-structure induced by the \( q \)-Frobenius so that it admits a \( K \otimes K_0 L \)-structure. Let us define an overconvergent \( L \)-\( F_q \)-isocrystal \( \mathcal{M}^\dagger \) on \( X \) by

\[ \mathcal{M}^\dagger = (\pi^*_{(K,\sigma)/(K_0,\sigma_0)}\mathcal{N}^\dagger) \otimes K \otimes K_0 L \]

with Frobenius \( F_{\mathcal{M}^\dagger} = (\pi^*_{(K,\sigma)/(K_0,\sigma_0)}F_{\mathcal{N}^\dagger})^n \otimes \text{id}_L \), where \( K \otimes K_0 L \to L (b \otimes a \mapsto ba) \). One can easily verify this correspondence \( \mathcal{N}^\dagger \mapsto \mathcal{M}^\dagger \) is a quasi-inverse of the given functor.

Let \( X, Y \) be schemes locally of finite type over \( \text{Spec} \ k \) and \( f : X \to Y \) be a morphism of schemes. Then the usual pull back functor \( f^* \) of overconvergent \( F \)-isocrystals induces the pull back functor

\[ f^* : F_q\text{-Isoc}^\dagger(Y) \otimes \overline{\mathbb{Q}}_p \to F_q\text{-Isoc}^\dagger(X) \otimes \overline{\mathbb{Q}}_p. \]

Then, for an overconvergent \( \overline{\mathbb{Q}}_p \)-\( F_q \)-isocrystal \( \mathcal{N}^\dagger \) on \( Y \), we have

\[ \overline{\mathbb{Q}}_p\text{-rank}(f^*\mathcal{N}^\dagger) = \overline{\mathbb{Q}}_p\text{-rank}(\mathcal{N}^\dagger) \]

on each connected component.

### 7.2 Changes of base fields

Let \( k_n \) be a finite extension of \( k \) such that \( \text{deg}(k_n/k) = n \), \( K_n \) the unramified extension of \( K \) in \( \overline{K} \) with the residue field \( k_n \) and denote the \( q \)-Frobenius of \( K_n \) by the same symbol \( \sigma \). For a scheme \( X \) locally of finite type over \( \text{Spec} \ k \), let us put \( X_n = X \times_{\text{Spec} \ k} \text{Spec} \ k_n \) with the projection \( \pi_n : X_n \to X \). We
define the push forward functor by

$$\pi^*_{n, p} : F_{q^n} \text{-Isoc}^\dagger(X_n) \otimes \mathbb{Q}_p \to F_q \text{-Isoc}^\dagger(X) \otimes \mathbb{Q}_p$$

$$\mathcal{M}^\dagger \mapsto \pi((K_n, \sigma^n)/(K, \sigma))\ast \mathcal{M}^\dagger = \pi((K_n, \sigma)/(K, \sigma))\ast M_0^\dagger,$$

where $\pi((K_n, \sigma^n)/(K, \sigma))\ast$ and $\pi((K_n, \sigma)/(K, \sigma))\ast$ are the push forward functor defined in Appendix B and $M_0^\dagger$ is the $\mathbb{Q}_p$-isocrystal on $X_n$ introduced just before Lemma 7.8. If $\pi^*_{(K_n, \sigma^n)/(K, \sigma)}$ is the pull back functor defined in Appendix B, then $\pi^*_{(K_n, \sigma^n)/(K, \sigma)}\ast \mathcal{M}^\dagger$ is furnished with an extra $K_n$-structure arising from the $q^n$-Frobenius structures. We define the pull back functor by

$$\pi^*_{n, p} : F_q \text{-Isoc}^\dagger(X) \otimes \mathbb{Q}_p \to F_{q^n} \text{-Isoc}^\dagger(X_n) \otimes \mathbb{Q}_p$$

$$\mathcal{N}^\dagger \mapsto (\pi^*_{(K_n, \sigma^n)/(K, \sigma)}\ast \mathcal{N}^\dagger) \otimes_{K_n \otimes_K L} L$$

in which two $K_n$-structures from the $q^n$-Frobenius and from the $\mathbb{Q}_p$-structure are identified by tensoring $\otimes_{K_n \otimes_K L} L$. One can easily see the definition of the pull back functor $\pi^*_{n, p}$ does not depends on the choice of $L$. Then $\pi^*_{n, p}$ is a left adjoint of $\pi^*_{n, p}\ast$. Moreover the equalities

$$\mathbb{Q}_p\text{-rank}(\pi^*_{n, p} \ast \mathcal{M}^\dagger) = \frac{\text{rank}(\pi^*_{n, p} \ast \mathcal{M}^\dagger)}{\deg(L/K)} = \frac{\text{rank}(\mathcal{M}^\dagger) \deg(K_n/K)^2}{\deg(L/K)}$$

$$= \mathbb{Q}_p\text{-rank}(\mathcal{M}^\dagger) \times n,$$

$$\mathbb{Q}_p\text{-rank}(\pi^*_{n, p} \ast \mathcal{N}^\dagger) = \frac{\text{rank}(\pi^*_{n, p} \ast \mathcal{N}^\dagger)}{\deg(L/K_n)} = \frac{\text{rank}(\mathcal{N}^\dagger) \deg(L/K)}{\deg(L/K)} = \mathbb{Q}_p\text{-rank}(\mathcal{N}^\dagger)$$

hold on each connected component.

**Proposition 7.10** Suppose $X$ is separated of finite type over $\text{Spec} k$. Let $\mathcal{M}^\dagger$ be an overconvergent $L$-$F_q$-isocrystal $\mathcal{M}^\dagger$ on $X$ for a finite extension $L$ over $K_n$ in $\mathbb{Q}_p$. Then the rigid cohomology $H^i_{\text{rig}}(X/K, \mathcal{M}^\dagger)$ is an $F_q$-space over $K$ with an $L$-structure (an $L$-$F_q$-space over $K$ for short)

$$L \to \text{End}_{F_{\text{sp}}(H^i_{\text{rig}}(X/K, \mathcal{M}^\dagger))}$$

induced from the $L$-structure on $\mathcal{M}^\dagger$. The base change homomorphism of rigid cohomology induces an isomorphism

$$H^i_{\text{rig}}(X_n/K_n, \pi^*_{n, p} \ast \mathcal{M}^\dagger) \to (K_n \otimes_K H^i_{\text{rig}}(X/K, \mathcal{M}^\dagger)) \otimes_{K_n \otimes_K L} L \cong H^i_{\text{rig}}(X/K, \mathcal{M}^\dagger)$$
of $L$-$F^n_q$-spaces over $K_n$. Here the right hand side has a $K_n$-space structure by $K_n \subset L$, and the Frobenius on the right hand side is $F^n_{H_{n\text{rig}}^i(X/K, M^\dagger)}$. The results do not depend on the choice of $L$ in the category of $\mathbb{Q}_p$-$F_q^n$-spaces over $K$. The same hold for the rigid cohomology $H^i_{\text{rig,c}}(X/K, M^\dagger)$ with compact supports.

### 7.3 Lefschetz trace formula

Let $X$ be a connected scheme separated of finite type over $\text{Spec } k$ which is pure of dimension $d$, and $M^\dagger$ an object in $F$-$\text{Isoc}^\dagger(X) \otimes \mathbb{Q}_p$ such that $M^\dagger$ is represented as an overconvergent $L$-$F_q^n$-isocrystal on $X$ for an extension $L$ of $K$ in $\mathbb{Q}_p$. Then, for any closed point $\alpha$ in $X$ with $\text{deg}(k_\alpha/k) = n$ such that $K_\alpha \subset L$, the inverse image $i^*_{\alpha, X} M^\dagger$ is an $L$-$F_q^n$-space over $K_\alpha$, that is, an $F$-isocrystal on $\alpha/K$ with respect to $q$-Frobenius $\sigma$ and with an $L$-structure. For the linearization of Frobenius, we define a $\mathbb{Q}_p$-$F_{q^n}$-space $(M^\dagger_{\alpha, \mathbb{Q}_p}, F_{M^\dagger_{\alpha, \mathbb{Q}_p}})$ over $K_\alpha$ by the colimit of

$$\left(\mathcal{M}^\dagger_{\alpha, L}, F_{\mathcal{M}^\dagger_{\alpha, L}}\right) = \left(\pi^*_{(K_\alpha, \sigma^n)/(K, \sigma)}(i^*_{\alpha, X} \mathcal{M}^\dagger) \otimes K_n \otimes L, (\pi^*_{(K_\alpha, \sigma^n)/(K, \sigma)}(i^*_{\alpha, X} (F_{M^\dagger}))) \otimes \text{id}_L\right)^n,$$

over all finite extensions $L$ of $K$ in $\mathbb{Q}_p$, that is, the object of the quasi-inverse functor of $(i^*_{\alpha, X} \mathcal{M}^\dagger, i^*_{\alpha, X} (F_{M^\dagger}))$ in Proposition 7.9. By definition we have

$$\mathbb{Q}_p\text{-rank}(M^\dagger_{\alpha, \mathbb{Q}_p}) = \mathbb{Q}_p\text{-rank}(i^*_{\alpha, X} \mathcal{M}^\dagger) = \mathbb{Q}_p\text{-rank}(\mathcal{M}^\dagger).$$

The pair $(M^\dagger_{\alpha, L}, F_{M^\dagger_{\alpha, L}})$ is regarded both as an $L$-space of dimension $\mathbb{Q}_p\text{-rank}(M^\dagger_{\alpha, \mathbb{Q}_p})$ with an $L$-linear endomorphism $F_{M^\dagger_{\alpha, L}}$ and as a $K$-space of dimension $\mathbb{Q}_p\text{-rank}(M^\dagger_{\alpha, \mathbb{Q}_p})\text{deg}(L/K)$ with a $K$-linear endomorphism $F_{M^\dagger_{\alpha, L}}$. On the other hand the pair $(\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}, F_{\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}})$ is just a $\mathbb{Q}_p$-space with a $\mathbb{Q}_p$-linear endomorphism $F_{\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}}$.

**Lemma 7.11** With the notation as above, the following hold.

1. **The natural homomorphism**

   $$(i^*_{\alpha, X} \mathcal{M}^\dagger, i^*_{\alpha, X} (F_{M^\dagger}))^n \to (\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}, F_{\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}}) \quad m \mapsto m \otimes 1$$
induces an isomorphism of $L$-spaces with an $L$-linear endomorphism.

(2) For any $k_n$-rational point $\beta$ mapping to the closed point $\alpha$ with the composite $i_{\beta, X} = i_{\alpha, X} \circ i_{\beta, \alpha} : \beta \rightarrow \alpha \rightarrow X$ of morphisms, we have

$$
\text{Tr}_L \left( F_{i_{\beta, X}^{\dagger}; i_{\beta, X}^* \mathcal{M}^\dagger}^n \right) = \text{Tr}_L \left( F_{\mathcal{M}_{\alpha, L}^{\dagger}; \mathcal{M}_{\alpha, L}^\dagger} \right),
$$

$$
\text{Tr}_K \left( F_{i_{\beta, X}^{\dagger}; i_{\beta, X}^* \mathcal{M}^\dagger}^n \right) = \text{Tr}_K \left( F_{\mathcal{M}_{\alpha, L}^{\dagger}; \mathcal{M}_{\alpha, L}^\dagger} \right),
$$

where $\text{Tr}_E$ means the trace of $E$-linear homomorphisms.

(3) If $I_K (L, \overline{\mathbb{Q}}_p)$ is a set of $K$-algebra homomorphisms from $L$ to $\overline{\mathbb{Q}}_p$, then the equality

$$
\text{Tr}_K \left( F_{\mathcal{M}_{\alpha, \overline{\mathbb{Q}}_p}^{\dagger}; \mathcal{M}_{\alpha, \overline{\mathbb{Q}}_p}^\dagger} \right) = \sum_{\tau \in I_K (L, \overline{\mathbb{Q}}_p)} \tau \left( \text{Tr}_L \left( F_{\mathcal{M}_{\alpha, L}^{\dagger}; \mathcal{M}_{\alpha, L}^\dagger} \right) \right)
$$

holds.

Proof (2) Let $\tau : K_{\alpha} \rightarrow K_{\beta}$ be the $K$-algebra isomorphism which is the lift of the $k$-algebra isomorphism $K_{\alpha} \rightarrow K_{\beta}$. Then $i_{\beta, X}^{\dagger} = i_{\alpha, (i_{\alpha, X} \mathcal{M}^\dagger)}$ and

$$
F_{i_{\beta, X}^* \mathcal{M}^\dagger} = (\text{id}_{i_{\alpha, X} \mathcal{M}^\dagger} \otimes \tau) \circ F_{i_{\beta, X}^* \mathcal{M}^\dagger} \circ (\text{id}_{i_{\alpha, X} \mathcal{M}^\dagger} \otimes \tau^{-1}).
$$

Hence it implies the assertion.

(3) follows from (2) and the isomorphism $L \otimes_K L \cong \prod_{\tau \in I_K (L, \overline{\mathbb{Q}}_p)} \tau (L)$. \qed

The Lefschetz trace formula for overconvergent $\overline{\mathbb{Q}}_p$-$F$-isocrystals follows from the Lefschetz trace formula of usual overconvergent $F$-isocrystals [20, Théorème 6.2].

**Proposition 7.12** [1, Theorem A.3.2] With the notation as above, the trace formula

$$
\sum_{\beta \in X (k_n)} \text{Tr}_L \left( F_{i_{\beta, X}^* \mathcal{M}^\dagger}^n \right) = \sum_{i = 0}^{2d} (-1)^i \text{Tr}_L \left( F_{H_{\text{rig}, c}^i}^n ; H_{\text{rig}, c}^i (X/K, \mathcal{M}^\dagger) \right)
$$

holds where $X (k_n)$ is the set of $k_n$-rational points of $X$.

Proof The idea of this proof is same with that of [1, Theorem A.3.2]. When $\dim (X) = 0$, the assertion is trivial. In general dimensional cases, it is sufficient.
to prove that the right hand side is 0 when \( X(k_n) = \emptyset \) by applying the excision sequence

\[
\cdots \to H^i_{\text{rig},c}((X \setminus Z)/K, \mathcal{M}^\dagger) \to H^i_{\text{rig},c}(X/K, \mathcal{M}^\dagger) \to H^{i+1}_{\text{rig},c}((X \setminus Z)/K, \mathcal{M}^\dagger) \to \cdots
\]

of rigid cohomology with compact supports, where \( Z \) is a closed subscheme of \( X \) containing the finite set \( X(k_n) \). The exact sequence above is a sequence of \( L\)-\( F_q\)-spaces over \( K \).

Suppose \( X(k_n) = \emptyset \). The right hand side of the equality

\[
\sum_{\tau \in I_K(L, \overline{\mathbb{Q}}_p)} \tau \left( \sum_{i=0}^{2d} (-1)^i \text{Tr}_L \left( F^n_{H^i_{\text{rig},c}} ; H^i_{\text{rig},c}(X/K, \mathcal{M}^\dagger) \right) \right) = 0
\]

vanishes by the Lefschetz trace formula for usual overconvergent \( F\)-isocrystals [20, Théorème 6.2], and hence

\[
\sum_{\tau \in I_K(L, \overline{\mathbb{Q}}_p)} \tau \left( \sum_{i=0}^{2d} (-1)^i \text{Tr}_K \left( F^n_{H^i_{\text{rig},c}} ; H^i_{\text{rig},c}(X/K, \mathcal{M}^\dagger) \right) \right) = 0.
\]

For a nonzero element \( \lambda \) in \( L \), we define an \( F_q\)-space \( L(\lambda) \) over \( K \) with an \( L\)-structure by

\[
\begin{cases}
L(\lambda) = L : \text{as an } K\text{-space} \\
F_{L(\lambda)}(m) = \lambda m \text{ for } m \in L \\
L \times L(\lambda) \to L(\lambda) \quad (a, m) \mapsto am.
\end{cases}
\]

Then \( \mathcal{M}^\dagger \otimes_L L(\lambda) \) is an overconvergent \( L\)-\( F_q\)-isocrystal on \( X \), and in particular the action of Frobenius is given by

\[
F_{\mathcal{M}^\dagger \otimes_L L(\lambda)} = \lambda F_{\mathcal{M}^\dagger}.
\]

From the assumption \( X(k_n) = \emptyset \) we also have

\[
\sum_{\tau \in I_K(L, \overline{\mathbb{Q}}_p)} \tau \left( \sum_{i=0}^{2d} (-1)^i \lambda \text{Tr}_L \left( F^n_{H^i_{\text{rig},c}} ; H^i_{\text{rig},c}(X/K, \mathcal{M}^\dagger) \right) \right) = 0.
\]
for any \( \lambda \in L^\times \). Since \( \sum_{\tau \in I_K(L, \mathbb{Q}_p)} \tau \) is a trace map of the extension \( L \) over \( K \) of characteristic 0 and \( \lambda \) is arbitrary, we have the desired vanishing
\[
\sum_{i=0}^{2d} (-1)^i \text{Tr}_L \left( F^n_{H_{i}^{\text{rig}, c}} ; H_{i}^{\text{rig}, c}(X/K, \mathcal{M}^\dagger) \right) = 0.
\]
\[
\sum_{i=0}^{2d} (-1)^i \text{Tr}_L \left( F^n_{H_{i}^{\text{rig}, c}} ; H_{i}^{\text{rig}, c}(X/K, \mathcal{M}^\dagger) \right) = 0.
\]

\[\square\]

Let us define the \( L \)-function \( L(X, \mathcal{M}^\dagger, t) \) of \( \mathcal{M}^\dagger \) by
\[
L(X, \mathcal{M}^\dagger, t) = \prod_{n=1}^{\infty} \prod_{\alpha : \text{closed points of } X, \deg(k_\alpha/k) = n} \det_{\mathbb{Q}_p} \left( 1 - t^n F_{\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}} ; \mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p} \right)^{-1}
\]
\[
= \exp \left( \sum_{n=1}^{\infty} \sum_{\beta \in X(k_n)} \text{Tr}_{\mathbb{Q}_p} \left( F^n_{i_{\beta, X}^i \mathcal{M}^\dagger} ; i_{\beta, X}^i \mathcal{M}^\dagger \right) t^n \right) \in \mathbb{Q}_p[[t]],
\]
where \( \det_E(1 - tf) \in E[t] \) means a characteristic polynomial of an \( E \)-linear endomorphism \( f \). Note that the second equality holds by Lemma 7.11.

**Corollary 7.13** [1, Corollary A.3.3] With the notation as above, we have
\[
L(X, \mathcal{M}^\dagger, t) = \prod_{i=0}^{2d} \det_{\mathbb{Q}_p} \left( 1 - t F_{H_{i}^{\text{rig}, c}} ; H_{i}^{\text{rig}, c}(X/K, \mathcal{M}^\dagger) \right)^{(-1)^{i+1}}.
\]

### 7.4 Čebyatarev density theorem

Now we recall weights of overconvergent \( \mathbb{Q}_p \)-\( F \)-isocrystals (see [32, Section 10] for a brief introduction of weights). Let \( \iota : \mathbb{Q}_p \to \mathbb{C} \) be an isomorphism of fields.

**Definition 7.14** Let \( X \) be a connected scheme separated of finite type over \( \text{Spec } k \).

1. An overconvergent \( \mathbb{Q}_p \)-\( F \)-isocrystal \( \mathcal{M}^\dagger \) on \( X \) is \( \iota \)-pure of weight \( w \in \mathbb{Z} \) if, for any closed point \( \alpha \) in \( X \) with \( \deg(k_\alpha/k) = n \), any reciprocal root of the polynomial
\[
\det_{\mathbb{Q}_p} \left( 1 - t F_{\mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p}} ; \mathcal{M}^\dagger_{\alpha, \mathbb{Q}_p} \right) \in \mathbb{Q}_p[t]
\]
has a complex absolute value $q^{nw/2}$ under the isomorphism $\iota$. An overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystal on $X$ is $t$-pure if it is $t$-pure of weight $w$ for some $w \in \mathbb{Z}$.

(2) An overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystal $\mathcal{M}^\dagger$ on $X$ is $t$-mixed if it is a successive extension of overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals of $t$-pure.

The following proposition is the Čebatěrev density theorem for overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals [1, Proposition A.4.1].

**Proposition 7.15** Let $X$ be a smooth connected scheme separated of finite type over Spec $k$, and $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ irreducible overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals on $X$ such that $(\mathcal{N}^\dagger)^{\vee} \otimes \mathcal{M}^\dagger$ is $t$-pure. Suppose that there is an open dense subscheme $U$ of $X$ such that, for any closed point $\alpha$ of $U$ with $\deg(k(\alpha)/k) = n$, there is an isomorphism

$$
(\mathcal{M}^\dagger\alpha, \overline{\mathbb{Q}}_p, F_{\mathcal{M}^\dagger\alpha, \overline{\mathbb{Q}}_p}) \cong (\mathcal{N}^\dagger\alpha, \overline{\mathbb{Q}}_p, F_{\mathcal{N}^\dagger\alpha, \overline{\mathbb{Q}}_p})
$$

as $\overline{\mathbb{Q}}_p\cdot F_q^n$-spaces over $K_\alpha$. Then the following hold.

1. There exists an isomorphism $g^\dagger : \mathcal{N}^\dagger \to \mathcal{M}^\dagger$ of overconvergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals on $X$.

2. Suppose furthermore that both $\mathcal{M}^\dagger$ and $\mathcal{N}^\dagger$ admit slope filtrations as convergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals and that there exists an isomorphism $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ between the maximal slope quotients as convergent $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystals, and that either the canonical homomorphism

$$
\text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* M^\dagger, \overline{\mathbb{Q}}_p) \to \text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* (M/\mathcal{M}^1))
$$

or

$$
\text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* N^\dagger, \overline{\mathbb{Q}}_p) \to \text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* (N/\mathcal{N}^1))
$$

is surjective for some positive integer $m$. Then there exists a unique isomorphism $g^\dagger : \mathcal{N}^\dagger \to \mathcal{M}^\dagger$ such that the induced morphism of the maximal slope quotients by $g^\dagger$ coincides with the given $h$. Here $\text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* M^\dagger, \overline{\mathbb{Q}}_p)$ (resp. $\text{End}_{\overline{\mathbb{Q}}_p\cdot F_q\text{-Isoc}}(\pi_m^* (M^\dagger/\mathcal{M}^1))$) is the $\overline{\mathbb{Q}}_p$-space of endomorphisms of $\pi_m^* M^\dagger$ (resp. $\pi_m^* (M^\dagger/\mathcal{M}^1)$) as an overconvergent (resp. a convergent) $\overline{\mathbb{Q}}_p\cdot F_q$-isocrystal on $X_m$, and the same for $N^\dagger$.

**Proof** (1) This is a proof in [1, Proposition A.4.1]. Denote the dimension of $X$ by $d$. Note that $(\mathcal{N}^\dagger)^{\vee} \otimes \mathcal{M}^\dagger$ is $t$-pure of weight 0 by the hypothesis. Since
there is an isomorphism

\[
\text{Hom}_{\mathbb{Q}_p - F\text{-Isoc}}(N^\dagger, M^\dagger) \cong H^0_{\text{rig}}(X/K, (N^\dagger)^\vee \otimes M^\dagger)^{F_{H^0_{\text{rig}}}} = 1
\]

\[
= H^0_{\text{rig}}(U/K, (N^\dagger)^\vee \otimes M^\dagger)^{F_{H^0_{\text{rig}}}} = 1
\]

and both \( M^\dagger \) and \( N^\dagger \) is irreducible on \( X \), we have only to prove the nonvanishing of the right hand side. Here \( \text{Hom}_{\mathbb{Q}_p - F\text{-Isoc}}^\dagger \) is a group of morphisms as overconvergent \( \mathbb{Q}_p - F\text{-isocrystals} \) on \( X \), the \( F_{H^0_{\text{rig}}} = 1 \) means the Frobenius invariant subspace of the 0-th rigid cohomology, and the second equality follows from the full faithfulness of the restriction functor of overconvergent \( F \)-isocrystals \([43, \text{Theorem 6.3.1}]\). By Poincaré duality \([27, \text{Theorem 1.2.3}]\) it is sufficient to prove an existence of a nontrivial cocycle in \( H^2_{\text{rig}, c}(U/K, (\mathcal{N}^\dagger)^\vee \otimes \mathcal{N}^\dagger) \) on which Frobenius \( F_{H^2_{\text{rig}, c}} \) acts by \( q^d \).

Because \((\mathcal{N}^\dagger)^\vee \otimes \mathcal{N}^\dagger\) is \( t \)-pure of weight 0, the weights of Frobenius \( F_{H^j_{\text{rig}, c}} \) on \( H^j_{\text{rig}, c}(U/K, (\mathcal{M}^\dagger)^\vee \otimes \mathcal{N}^\dagger) \) are \( \leq j \) \([28, \text{Theorem 5.3.2}]\). Hence what we want is that \( L(U, (\mathcal{M}^\dagger)^\vee \otimes \mathcal{N}^\dagger, u) \) has a factor \( 1 - q^d u \) in the denominator. This holds since

\[
L(U, (\mathcal{M}^\dagger)^\vee \otimes \mathcal{N}^\dagger, u) = L(U, (\mathcal{M}^\dagger)^\vee \otimes \mathcal{M}^\dagger, u)
\]

by the hypothesis of coincidence of \( \mathbb{Q}_p - F\text{-spaces} \) at each closed point \( \alpha \in U \).

(2) Since there exists an isomorphism \((g')^\dagger : N^\dagger \to \mathcal{M}^\dagger \) in (1), the condition of endomorphisms for \( \pi^*_{\mathcal{M}^\dagger} \mathcal{M}^\dagger \) is equivalent to that for \( \pi^*_{\mathcal{N}^\dagger} \mathcal{N}^\dagger \).

Suppose that

\[
\text{End}_{\mathbb{Q}_p - F\text{-Isoc}}(\pi^*_{\mathcal{M}^\dagger} \mathcal{M}^\dagger) \to \text{End}_{\mathbb{Q}_p - F\text{-Isoc}}(\pi^*_{\mathcal{N}^\dagger} (\mathcal{N}^\dagger/\mathcal{N}^{1}))
\]

is surjective. By the hypothesis we can find a morphism \( \tau_m : \pi^*_m \mathcal{N}^\dagger \to \pi^*_m \mathcal{M}^\dagger \) which induces \( \pi^*_m (g')^{-1} \circ h \) on \( \pi^*_m \mathcal{N}^\dagger/\mathcal{N}^{1} \). By our construction \( \pi^*_m (g')^{-1} \circ \tau_m : \pi^*_m \mathcal{N}^\dagger \to \pi^*_m \mathcal{M}^\dagger \) is a morphism of overconvergent \( \mathbb{Q}_p - F_{q^m}\text{-isocrystals} \) on \( X_m \) such that the induced morphism of the maximal slope quotients by \( \pi^*_m (g')^{-1} \circ \tau_m \) coincides with the given
\( \pi^*_{m, \mathbb{Q}_p} (h) \). Let us consider the composite

\[
g^\dagger : N^\dagger \to \pi_*(Km, \sigma^m)/(K, \sigma) \to \pi_* \pi^* (\pi^* (Km, \sigma^m)/(K, \sigma)) N^\dagger \to \pi_* \pi^* (Km, \sigma^m)/(K, \sigma) N^\dagger
\]

\[
\pi_*(\pi^* (\pi_*(Km, \sigma^m)/(K, \sigma))) N^\dagger \to \pi_*(\pi^* (\pi_*(Km, \sigma^m)/(K, \sigma))) M^\dagger
\]

\[
\frac{1}{m} \text{Tr}_{(Km, \sigma^m)/(K, \sigma)} M^\dagger
\]

of the morphisms of overconvergent \( L-F_q \)-isocrystals on \( X \) for a sufficiently large finite extension \( L \) of \( K \) in \( \mathbb{Q}_p \), where \( \text{ad}_{(Km, \sigma^m)/(K, \sigma)} \) and \( \text{Tr}_{(Km, \sigma^m)/(K, \sigma)} \) are defined in Appendix B. Since the induced morphism of the maximal slope quotients by \( g^\dagger \) coincides with the given \( h \) by Lemma B.1, \( g^\dagger \) is the desired isomorphism by the irreducibility of \( M^\dagger \) and \( N^\dagger \).

The corollary below follows from Corollary 6.8.

**Corollary 7.16** Let \( M^\dagger \) and \( N^\dagger \) be irreducible overconvergent \( \mathbb{Q}_p-F_q \)-isocrystals on \( X \) such that \( (N^\dagger)^\vee \otimes \mathbb{M}^\dagger \) is \( \iota \)-pure and admit slope filtrations, and \( h : N^\dagger / N^1 \to M^\dagger / M^1 \) a nontrivial morphism of convergent \( \mathbb{Q}_p-F_q \)-isocrystals. Suppose that there is an open dense subscheme \( U \) of \( X \) such that, for any closed point \( \alpha \) of \( U \), both \((X, \alpha, M^\dagger)\) and \((X, \alpha, N^\dagger)\) satisfy the condition \((\text{LC}_{\mathbb{Q}_p}(X, \alpha, M^\dagger))\):

1. There exist a smooth curve \( C_\alpha \) over \( \text{Spec } k \) and a morphism \( i_{C_\alpha, X} : C_\alpha \to X \) over \( \text{Spec } k \) such that \( C_\alpha \) is passing at \( \alpha \) and that the restriction \( i^*_{C_\alpha, X} \mathbb{M}^\dagger \) on \( C_\alpha \) is an irreducible overconvergent \( \mathbb{Q}_p-F_q \)-isocrystal on \( C_\alpha \).

Then the following hold.

1. There exists an isomorphism \( g^\dagger : N^\dagger \to M^\dagger \) of overconvergent \( \mathbb{Q}_p-F_q \)-isocrystals on \( X \).
2. Suppose furthermore that either the canonical homomorphism

\[
\text{End}_{\mathbb{Q}_p-F-\text{Isoc}} (\pi^*_{m, \mathbb{Q}_p} \mathbb{M}^\dagger) \to \text{End}_{\mathbb{Q}_p-F-\text{Isoc}} (\pi^*_{m, \mathbb{Q}_p} (\mathbb{M} / \mathbb{M}^1)) \text{ or }
\]

\[
\text{End}_{\mathbb{Q}_p-F-\text{Isoc}} (\pi^*_{m, \mathbb{Q}_p} N^\dagger) \to \text{End}_{\mathbb{Q}_p-F-\text{Isoc}} (\pi^*_{m, \mathbb{Q}_p} (N / N^1))
\]

is surjective for some positive integer \( m \). Then there exists a unique isomorphism \( g^\dagger : N^\dagger \to M^\dagger \) such that the induced morphism of the maximal slope quotients by \( g^\dagger \) coincides with the given \( h \).
Applying Corollary 6.8, there exists an isomorphism

\[(\mathcal{M}_{\alpha, \overline{\Omega}_p}, F_{\mathcal{M}_{\alpha, \overline{\Omega}_p}}) \cong (\mathcal{N}_{\alpha, \overline{\Omega}_p}, F_{\mathcal{N}_{\alpha, \overline{\Omega}_p}})\]

of \(\overline{\Omega}_p\)-\(F_q\)-spaces over \(K_\alpha\) for any closed point \(\alpha \in U\) such that \(\deg(k_\alpha/k) = n\). Hence the assertions follow from Proposition 7.15. □

We give a sufficient condition of the bijectivity of endomorphisms in Corollary 7.16 (2).

**Lemma 7.17** Let \(X, Y\) be schemes separated of finite type over \(\text{Spec} \ k\) such that \(X\) is connected and \(Y\) is nonempty, and \(f : Y \rightarrow X\) a morphism. For an overconvergent \(\overline{\Omega}_p\)-\(F_q\)-isocrystal \(\mathcal{M}^\dagger\) on \(X\) which admits a slope filtration, both canonical homomorphisms

\[
\text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(\mathcal{M}^\dagger) \rightarrow \text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(f^*\mathcal{M}^\dagger)
\]

and

\[
\text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(\mathcal{M}/\mathcal{M}^1) \rightarrow \text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(f^*(\mathcal{M}/\mathcal{M}^1))
\]

are injective. In particular, the canonical homomorphism \(\text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(\mathcal{M}^\dagger) \rightarrow \text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(\mathcal{M}/\mathcal{M}^1)\) is bijective if the bijectivity \(\text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(\mathcal{M}^\dagger) \cong \text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(i_{\alpha, C}^*\mathcal{M}^\dagger)\) holds for a closed point \(\alpha \in X\) and a smooth curve \(i_{\alpha, C} : C_\alpha \rightarrow X\) passing at \(\alpha\) such that \(i_{\alpha, C}^*\mathcal{M}^\dagger\) is irreducible.

**Proof** The former assertion follows from the fact that the induced morphisms

\[
\begin{align*}
H^0_{\text{rig}}(X/K, (\mathcal{M}^\dagger)^\vee \otimes \mathcal{M}^\dagger) &\rightarrow H^0_{\text{rig}}(Y/K, (f^*\mathcal{M}^\dagger)^\vee \otimes f^*\mathcal{M}^\dagger), \\
H^0_{\text{conv}}(X/K, (\mathcal{M}/\mathcal{M}^1)^\vee \otimes \mathcal{M}/\mathcal{M}^1) &\rightarrow H^0_{\text{conv}}(Y/K, (f^*(\mathcal{M}/\mathcal{M}^1))^\vee \otimes f^*(\mathcal{M}/\mathcal{M}^1))
\end{align*}
\]

of \(L\)-\(F_q\)-spaces over \(K\) are injective by the hypothesis of connectedness. Here \(H^0_{\text{conv}}(X/K, (\mathcal{M}/\mathcal{M}^1)^\vee \otimes \mathcal{M}/\mathcal{M}^1)\) is the 0-th convergent cohomology, that is, the space of horizontal sections of the convergent isocrystal \((\mathcal{M}/\mathcal{M}^1)^\vee \otimes \mathcal{M}/\mathcal{M}^1\). The second assertion follows from the former assertion and the bijectivity of \(\text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(i_{\alpha, C}^*\mathcal{M}^\dagger) \rightarrow \text{End}_{\overline{\Omega}_p\text{-}F\text{-Isoc}}(i_{\alpha, C}^*\mathcal{M}/\mathcal{M}^1))\) by Corollary 6.5. □

### 7.5 Consequences from the companion theorem

After Abe’s celebrated work on \(p\)-adic Langlands correspondence and the companion theorem in the case of curves [1,2], Deligne’s companion conjecture between \(\ell\)-adic coefficients and \(p\)-adic coefficients in general dimension
[16, Conjecture 1.2.10] is one of the hottest problems in arithmetic geometry. (See [4,13,33] for details.)

Using Lefschetz theorem for isocrystals with tame ramifications along boundaries, Abe and Esnault obtained a weight thereom.

**Theorem 7.18** [4, Theorem 2.7] Let X be a smooth connected scheme separated of finite type over Spec k. Then any $\mathbb{Q}_p - F_q$-isocrystal on X is $i$-mixed. In particular, any irreducible $\mathbb{Q}_p - F_q$-isocrystal on X is $i$-pure.

Then Abe and Esnault established the Lefschetz theorem below after the companion theorem. Note that one can relax the condition on finite determinant of $\mathbb{Q}_p - F_q$-isocrystals by twisting a character [2, Theorem 6.1] and the Lefschetz theorem in [4] is more general which asserts an existence of a curve passing at given finite closed points and on which the pull back is irreducible.

**Theorem 7.19** [4, Theorem 3.10] Let X be a smooth connected scheme separated of finite type over Spec k, and $\mathcal{M}^\dagger$ an irreducible $\mathbb{Q}_p - F_q$-isocrystal on X. There exists a dense open subscheme U of X such that, for any closed point $\alpha \in U$, the triplet $(X, \alpha, \mathcal{M}^\dagger)$ satisfies the condition $(LC_{\mathbb{Q}_p})$ in Corollary 7.16.

As a consequence we have an affirmative answer to Kedlaya’s question “Minimal slope conjecture” [32, Remark 5.14] (Conjectures 1.1, 1.2) for $\mathbb{Q}_p - F$-isocrystals on smooth varieties over finite fields by Theorems 7.19, 7.18, Corollary 7.16 and Lemma 7.17.

**Theorem 7.20** Let X be a smooth connected scheme separated of finite type over Spec k, and $\mathcal{M}^\dagger, \mathcal{N}^\dagger$ irreducible overconvergent $\mathbb{Q}_p - F_q$-isocrystals on X admitting slope filtrations of $\mathcal{M}$ and $\mathcal{N}$. Suppose that there exists a nontrivial morphism $h : \mathcal{N}/\mathcal{N}^1 \to \mathcal{M}/\mathcal{M}^1$ between the maximal slope quotients as convergent $\mathbb{Q}_p - F$-isocrystals. Then the following hold.

1. There exists an isomorphism $g^\dagger : \mathcal{N}^\dagger \to \mathcal{M}^\dagger$ of overconvergent $\mathbb{Q}_p - F_q$-isocrystals on X.
2. Suppose furthermore that, after a finite extension of k, there exists a smooth curve $i_{\alpha,C} : C_{\alpha} \to X$ passing at a closed point $\alpha \in X$ such that $i_{\alpha,C,X}^* \mathcal{M}^\dagger$ is irreducible and that the restriction map

$$\text{End}_{\mathbb{Q}_p - F-\text{Isoc}}(\mathcal{M}^\dagger) \to \text{End}_{\mathbb{Q}_p - F-\text{Isoc}}(i_{\alpha,C,X}^* \mathcal{M}^\dagger)$$

is bijective. Then the isomorphism $g^\dagger$ in (1) induces a natural commutative diagram

$$\begin{align*}
\mathcal{N} & \xrightarrow{g} \mathcal{M} \\
\downarrow & \downarrow \\
\mathcal{N}/\mathcal{N}^1 & \to \mathcal{M}/\mathcal{M}^1
\end{align*}$$

of convergent $\mathbb{Q}_p - F$-isocrystals.
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Appendix A: Frobenius

A.1  Frobenius on a complete discrete valuation field

Let $R$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with the residue field $k = R/m$ which is not necessarily perfect, and $K$ the field of fractions of $R$. For a positive power $q$ of $p$, a $q$-Frobenius $\sigma$ on $K$ is a continuous endomorphism

$$\sigma : K \to K$$

such that $\sigma(a) \equiv a^q \pmod{m}$ for $a \in R$. We define the $\sigma$-invariant subfield $K_\sigma$ by

$$K_\sigma = \{a \in K \mid \sigma(a) = a\}.$$

Remark that the same letter $\sigma$ is used for the $q$-Frobenius which acts on an extension $K'$ of $K$.

Lemma A.1 (c.f. [44, Remark 2.1]) Suppose $K$ admits a $q$-Frobenius $\sigma$.

1. $K_\sigma$ is a complete discrete valuation field with a finite residue field of cardinal $\leq q$. In particular, $K_\sigma$ is a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers.

2. There exists a finite unramified extension $K'$ of $K$ such that $K'$ admits a $q$-Frobenius $\sigma$ which is an extension of $\sigma$ on $K$, the residue field $(K')_\sigma$ is a field of $q$-elements and the valuation group of $(K')_\sigma$ is same with that of $K$. If furthermore $k$ is perfect, then the natural map $(K')_\sigma \otimes_{W(F_q)} W(k') \to K'$ is bijective where $k'$ is the residue field of $K'$ and $\sigma = \text{id}_{(K')_\sigma} \otimes \text{Frob}_p^q = \text{id}_{(K')_\sigma} \otimes \text{Frob}_q$ for $q = p^s$. Here $W(k')$ means the Witt vector ring with coefficients in $k'$ and $\text{Frob}_p$ (resp. $\text{Frob}_q$) is the canonical $p$-Frobenius (resp. $q$-Frobenius) on $W(k')$.

3. For a finite extension $L$ of $\mathbb{Q}_p$, there exist a finite extension $K'$ of $K$ and a positive integer $n$ which satisfy the following:

(i) $K'$ admits a $q$-Frobenius $\sigma$ which is an extension of $\sigma$ on $K$, 

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(ii) the residue field of \((K')_{\sigma^n}\) is a field of \(q^n\)-elements,

(iii) \(L \subset (K')_{\sigma^n}\).

In order to prove Lemma A.1 we introduce extensions \(\hat{K}^{\text{perf}} \subset \hat{K}^{\text{perf,ur}}\) of \(K\) as complete discrete valuation fields with the same valuation group to \(K\) such that the residue field of \(\hat{K}^{\text{perf}}\) (resp. \(\hat{K}^{\text{perf,ur}}\)) is a perfection of \(k\) (resp. an algebraic closure of \(k\)). Moreover, \(\hat{K}^{\text{perf}}\) and \(\hat{K}^{\text{perf,ur}}\) admit a \(q\)-Frobenius \(\sigma\) which is compatible with extensions of \(K\). The field \(\hat{K}^{\text{perf}}\) is defined by the \(p\)-adic completion of the inductive limit of the inductive system

\[
K \rightarrow K \rightarrow K \rightarrow \cdots
\]

where \(K = \lim_{\rightarrow}(K \rightarrow \sigma(K) \rightarrow \cdots)\) and the \(q\)-Frobenius on \(\hat{K}^{\text{perf}}\) which is compatible to the Frobenius \(\sigma\) on \(K\) by the system \(\sigma = (\sigma \rightarrow \sigma \rightarrow \sigma \rightarrow \cdots)\) of Frobenius. \(\hat{K}^{\text{perf,ur}}\) is the \(p\)-adic completion of the maximally unramified extension of \(\hat{K}^{\text{perf}}\). Then the \(q\)-Frobenius extends uniquely on \(\hat{K}^{\text{perf,ur}}\) and it is denoted by the same symbol \(\sigma\).

**Proof of Lemma A.1**

(1) One can easily see that \(K_\sigma\) is a complete discrete valuation field. If \(a \in K_\sigma \cap R\), then \(a\) satisfies the congruence \(a^q \equiv a \mod m\). Hence the residue field of \(K_\sigma\) is finite of cardinal \(\leq q\).

(2) Since \((\hat{K}^{\text{perf,ur}})_\sigma\) is a finite extension of \(\bigotimes_p \mathbb{Q}_p\) with the residue field of \(q\) elements and the valuation ring of \((\hat{K}^{\text{perf,ur}})_\sigma\) coincides that of \(\hat{K}^{\text{perf,ur}}\), the composite field \(K' = (\hat{K}^{\text{perf,ur}})_\sigma K\) is a desired field. When the residue field \(k\) is perfect, then \(W(k)\) is the canonically subring of \(K\). Comparing the ramification, one has a natural isomorphism \((K')_\sigma \otimes W(\mathbb{F}_q) W(k) \rightarrow K'\).

(3) Replacing \(K\) and \(L\) by finite extensions respectively, we may assume that \(K\) admits a \(q\)-Frobenius such that \((\hat{K}^{\text{perf,ur}})_{\sigma^n} = K_{\sigma^n}\) by the proof of (2), the cardinal of the residue field of \(L\) is \(q^n\), and \(L\) is a totally ramified extension of \(K_{\sigma^n}\). Since the valuation group of \((\hat{K}^{\text{perf,ur}})_{\sigma^n}\) coincides with that of \(\hat{K}^{\text{perf,ur}}\) and hence of \(K\), the natural map \(L \otimes_{K_{\sigma^n}} K \rightarrow L K\) is an isomorphism of fields. Then \(K' = L K\) and \(\sigma = \text{id}_L \otimes \sigma\) are our desired field and \(q\)-Frobenius.

\(\square\)

**A.2 Frobenius on \(\dagger\)-spaces associated to affine smooth schemes**

Let \(X\) be an affine smooth integral scheme separated of finite type over \(\text{Spec } k\). Suppose there exist an affine smooth integral scheme \(X = \text{Spec } A_X\) of finite type over \(\text{Spec } R\) such that \(X' = \text{Spec } A_X \times_{\text{Spec } R} \text{Spec } k = X\). Note that an affine smooth scheme separated of finite type over \(\text{Spec } k\) always admits a smooth lift over \(\text{Spec } R\) [19, Théorème 6]. Suppose \(A_X = R[x_1, \ldots, x_N]/I\). If \(\hat{A}_X\) and \(A^\dagger_X\) are

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the $p$-adically formal completion of $A_X$ and the $p$-adically weak completion of $A_X$ (i.e., a weakly complete finitely generated (w.c.f.g.) algebra over $(R, \mathfrak{m})$), then they are defined by
\[
\widehat{A}_X = R[x_1, \ldots, x_N] / I R[x_1, \ldots, x_N],
\]
\[
\overset{\dagger}{A}_X = R[x_1, \ldots, x_N] / I R[x_1, \ldots, x_N],
\]
respectively, where $R[x_1, \ldots, x_N]$ is the $m$-adic completion of $R[x_1, \ldots, x_N]$, and
\[
R[x_1, \ldots, x_N] = \lim_{\lambda \to 1^+} R[x_1, \ldots, x_N]_\lambda,
\]
\[
R[x_1, \ldots, x_N]_\lambda = \left\{ f \in R[x_1, \ldots, x_N] \mid f \text{ is convergent on the closed ball } \max_i |x_i| \leq \lambda. \right\}.
\]

Then $A_X \subset \widehat{A}_X$ are Noetherian [21, Theorem] and integral domains. Indeed the localization $(A_X)_m$ of $A_X$ at $mA_X$ is analytically irreducible and analytically unramified, and the natural morphism $\widehat{A}_X \to (A_X)_m$ is injective. In addition, $\widehat{A}_X$ (resp. $A_X^\dagger$) is furnished a continuous integrable derivation $d : \widehat{A}_X \to \widehat{A}_X \otimes_{A_X} \Omega^1_{A_X/R}$ (resp. $d : A_X^\dagger \to A_X^\dagger \otimes_{A_X} \Omega^1_{A_X/R}$) which is an extension of the $R$-derivation on $A_X$. It is known that $A_X^\dagger \subset \widehat{A}_X$ are independent of the choice of the lift $A_X$ up to continuous $R$-isomorphisms by the approximation theorem for w.c.f.g. algebras in [7, Theorem 2]. We do not discuss the functorialities here (see [45, Section 2] details).

A continuous endomorphism $\varphi$ on $\widehat{A}_X$ (resp. $A_X^\dagger$) is said to be a $q$-Frobenius with respect to $\sigma$ if it satisfies $\varphi(a) \equiv a^q \pmod{m}$ and $\varphi|R = \sigma$. The $q$-Frobenius always exists on $\widehat{A}_X$ (resp. $A_X^\dagger$) by formal smoothness (resp. and [7, Theorem 2]) [45, Section 2.4] and it is faithfully flat since $X$ is regular.

Let $\overline{X}$ be the Zariski closure of the immersion $X \to A^N_R \subset \mathbb{P}^N_R$ determined by $x_1, \ldots, x_N$ where $\mathbb{P}^N_R$ is the $N$-dimensional projective space over Spec $R$, $\overline{X}$ the closed fiber of $\overline{X}$ with the canonical open immersion $j_X : X \to \overline{X}$, $\widehat{\overline{X}}$ the $p$-adic formal completion of $\overline{X}$, and $\widehat{\overline{X}}_K$ the Raynaud’s generic fiber of $\overline{X}$ with the specialization morphism $sp : \widehat{\overline{X}}_K \to \widehat{\overline{X}}$ of locally ringed $G$-spaces. For a locally closed subset $Z \subset \widehat{\overline{X}}$, we put $]Z[\widehat{\overline{X}}] = sp(Z)$ to be the tube of $Z$ in $\widehat{\overline{X}}_K$.

Since $\widehat{\overline{X}}$ is smooth around $X$, there exists a strict neighborhood $W_0$ of $]X[\widehat{\overline{X}}$ in $]\overline{X}[\overline{X}$ such that $W_0$ is a smooth affinoid space over Spm $K$. The following lemma is an essential part of [6, Proposition 2.5.5].
Lemma A.2 Let \( \varphi \) be a \( q \)-Frobenius \( \varphi \) on \( A_X^\dagger \). By replacing \( W_0 \) by a sufficiently small strict neighborhood of \( ]X[\overline{\mathcal{X}} \) in \( ]X[\overline{\mathcal{X}} \), for any strict neighborhood \( W \) of \( ]X[\overline{\mathcal{X}} \) in \( W_0 \), there exists a strict neighborhood \( W' \) of \( ]X[\overline{\mathcal{X}} \) in \( W \) such that the \( q \)-Frobenius \( \varphi \) induces a morphism \( \varphi : W' \to W \) of rigid analytic spaces over \( \text{Spm} \, K \).

Instead of recalling the definition of overconvergent \( F \)-isocrystals [6, Chapter 2] (see [8, 10.2, 12.1] in the general case without assuming the global embedding of \( \mathcal{X} \) into a smooth formal scheme), we give an equivalent condition of the definition of overconvergent \( F \)-isocrystals in affine smooth cases.

Theorem A.3 [6, Corollaire 2.5.8] Let \( \varphi \) be a \( q \)-Frobenius \( \varphi \) on \( A_X^\dagger \). The global section functor \( \Gamma(]X[\overline{\mathcal{X}}, -) \) induces an equivalence between the category of overconvergent \( F \)-isocrystals on \( X/K \) with respect to \( \sigma \) and the category of \( A_X^\dagger \)-modules \( M^\dagger \) of finite type which is furnished with an integrable connection \( \nabla : M^\dagger \to M^\dagger \otimes_{A_X} \Omega^1_{A_X/K} \) and a horizontal isomorphism \( \varphi_{M^\dagger} : (\varphi^*M^\dagger, \varphi^*\nabla) \to (M^\dagger, \nabla) \) called Frobenius. Note that \( M^\dagger \) is a projective \( A_X^\dagger \)-module.

Appendix B: Change of base fields and Frobenius

In this appendix we recall the push forward functor and the pull back functor between the categories of \( F \)-isocrystals under the base extensions and changing Frobenius by its powers. Suppose \( k \) is an arbitrary perfect field of characteristic \( p \). Let \( L \) be a finite extension of \( K \) with the residue field \( l \) such that \( L \) admits an extension \( \sigma \) of the \( q \)-Frobenius \( \sigma \) on \( K \) (note that we use the same notation \( \sigma \)). Let \( X \) be a smooth scheme separated of finite type over \( \text{Spec} \, k \) with a completion \( \overline{\mathcal{X}} \) of \( X \) over \( \text{Spec} \, k \), and put the scalar extension \( X_l = X \times_{\text{Spec} \, k} \text{Spec} \, l \) and the projection \( \pi_{l/k} : X_l \to X \). For a positive integer \( n \), we define the pull back functor

\[
\pi^*_\sigma^{(L,\sigma^n)/(K,\sigma)} : \begin{cases} \\
(\text{overconvergent } F \text{-isocrystals on } X/K \text{ with respect to } \sigma) \\
(\text{overconvergent } F \text{-isocrystals on } X_l/L \text{ with respect to } \sigma^n) 
\end{cases}
\]

and the push forward functor

\[
\pi_{(L,\sigma^n)/(K,\sigma)}^* : \begin{cases} \\
(\text{overconvergent } F \text{-isocrystals on } X_l/l \text{ with respect to } \sigma^n) \\
(\text{overconvergent } F \text{-isocrystals on } X/K \text{ with respect to } \sigma) 
\end{cases}
\]
as follows. Locally on $X$, there exists a smooth lift $X = \text{Spec} \: A_X$ over $\text{Spec} \: R$ and a $q$-Frobenius $\varphi$ on the $p$-adically weak completion $A_X^\dagger$ of $A_X$ which is compatible to $\sigma$ (Appendix A.2). Then the functors are defined by

$$\pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal M^\dagger, \nabla_{\mathcal M^\dagger}, F_{\mathcal M^\dagger}) = (L \otimes_K \mathcal M^\dagger, \text{id}_L \otimes \nabla_{\mathcal M^\dagger}, \text{id}_L \otimes_{\sigma^n} (\sigma^n \otimes F_{\mathcal M^\dagger}^{n}))$$

where $\otimes_{\sigma^n}$ means the scalar extension by $\sigma^n : L \to L$, and

$$\pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal N^\dagger, \nabla_{\mathcal N^\dagger}, F_{\mathcal N^\dagger}) = (\oplus_{i=0}^{n-1} (\varphi^i)^* (\pi_* \mathcal N^\dagger, \pi_* \nabla), F_{\pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal N^\dagger)}(a_0 \otimes_{\sigma} m_0, \ldots, a_{n-1} \otimes_{\sigma} m_{n-1})) = (F_{\mathcal N^\dagger}(a_{n-1} \otimes_{\sigma} m_{n-1}), a_0 \otimes_{\sigma} m_0, \ldots, a_{n-2} \otimes_{\sigma} m_{n-2}).$$

Then $\pi_{(L,\sigma^n)/(K,\sigma)}^*$ is a left adjoint of $\pi_{(L,\sigma^n)/(K,\sigma)}$. For an overconvergent $F$-isocrystal $\mathcal M^\dagger$ on $X/K$, the adjoint $\text{ad}_{(L,\sigma^n)/(K,\sigma)} : \mathcal M^\dagger \to \pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal M^\dagger)$ of $\pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal M^\dagger) \to \pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal M^\dagger)$ is given by

$$\text{ad}_{(L,\sigma^n)/(K,\sigma)}(m) = (m, F_{\mathcal M^\dagger}^{-1}(m), \ldots, F_{\mathcal M^\dagger}^{-n+1}(m))$$

where $F_{\mathcal M^\dagger}^{-i} : \mathcal M^\dagger \to (\varphi^i)^* \mathcal M^\dagger$ is the inverse of $F_{\mathcal M^\dagger}^i = F_{\mathcal M^\dagger} \circ \cdots \circ (\varphi^{i-1})^* (F_{\mathcal M^\dagger}) : (\varphi^i)^* \mathcal M^\dagger \to \mathcal M^\dagger$. We also define a trace map $\text{Tr}_{(L,\sigma^n)/(K,\sigma)} : \pi_{(L,\sigma^n)/(K,\sigma)}^*(\mathcal M^\dagger) \to \mathcal M^\dagger$ by

$$\text{Tr}_{(L,\sigma^n)/(K,\sigma)}(a_0 \otimes_{\sigma} m_0, a_1 \otimes m_1, \ldots, a_{n-1} \otimes m_{n-1}) = \sum_{i=0}^{n-1} \text{tr}_{L/K} (\sigma^i(a_i)) F_{\mathcal M^\dagger}^i(1 \otimes m_i),$$

where $\text{tr}_{L/K} : L \to K$ is the trace of finite extension of fields. Since $\text{tr}_{L/K}$ commutes with the $q$-Frobenius $\sigma$, the trace map $\text{Tr}_{(L,\sigma^n)/(K,\sigma)}$ is well-defined and commutes with connections and Frobenius. Hence it is a morphism of overconvergent $F$-isocrystals on $X/K$ with respect to $\sigma$.

**Lemma B.1** (1) As morphisms of (over)convergent $F$-isocrystals on $X/K$ with respect to $\sigma$, the following identity holds:

$$\text{Tr}_{(L,\sigma^n)/(K,\sigma)} \circ \text{ad}_{(L,\sigma^n)/(K,\sigma)} = n \text{deg}(L/K) \text{id}.$$
(2) Let $K \subset L \subset M$ be a sequence of finite extensions with the compatible $q$-Frobenius $\sigma$. Then the associativities below hold:

\[
\pi^*(M,\sigma^{mn})/(K,\sigma) = \pi^*(M,\sigma^{mn})/(L,\sigma^m) \circ \pi^*(L,\sigma^m)/(K,\sigma),
\]
\[
\pi(M,\sigma^{mn})/(K,\sigma) = \pi(L,\sigma^m)/(K,\sigma) \circ \pi(M,\sigma^{mn})/(L,\sigma^m),
\]
\[
ad(M,\sigma^{mn})/(K,\sigma) = \pi(L,\sigma^m)/(K,\sigma) \circ \pi(M,\sigma^{mn})/(L,\sigma^m) \circ \pi(L,\sigma^m)/(K,\sigma)
\]
\[
\text{Tr}(M,\sigma^{mn})/(K,\sigma) = \text{Tr}(L,\sigma^m)/(K,\sigma) \circ \pi(L,\sigma^m)/(K,\sigma) \circ \pi(L,\sigma^m)/(K,\sigma).
\]

Remark B.2 In this appendix we define the functors $\pi^*(L,\sigma^n)/(K,\sigma)$ and $\pi(L,\sigma^n)/(K,\sigma)$ only for smooth schemes. Using the method of hypercoverings \cite[2.3.2 (iii), 2.3.7]{Berthelot} \cite[10.2, 12.1]{Chiarellotto}, one can define the same for arbitrary schemes locally of finite type over $\text{Spec} \, k$.

References

1. Abe, T.: Langlands correspondence for isocrystals and the existence of crystalline companions for curves. J. Am. Math. Soc. 31(4), 921–1057 (2018)
2. Abe, T.: Langlands program for $p$-adic coefficients and the petits camarades conjecture. J. Reine Angew. Math. 734, 59–69 (2018)
3. Abe, T., Caro, D.: Theory of weights in $p$-adic cohomology. Am. J. Math. 140(4), 879–975 (2018)
4. Abe, T., Esnault, H.: A Lefschetz theorem for overconvergent isocrystals with Frobenius structure. Ann. Ec. Nor. Sup. 4e série 52, 1243–1264 (2019)
5. Ambrosi, E., D’Addezio, M.: Maximal tori of monodromy groups of $F$-isocrystals and an application to abelian varieties. Algebraic Geom. arXiv:1811.08423
6. Berthelot, P.: Cohomologie rigide et cohomologie rigide à support propre, part 1. Prépublication IRMAR 96-03. https://perso.univ-rennes1.fr/pierre.berthelot/
7. Bosch, S.: A rigid analytic version of M. Artin’s theorem on analytic equations. Math. Ann. 255, 395–404 (1981)
8. Chiarellotto, B., Tsuzuki, N.: Cohomological descent of rigid cohomology for etale coverings. Rendiconti di Padova 109, 63–215 (2003)
9. Chiarellotto, B., Tsuzuki, N.: Logarithmic growth and Frobenius filtrations for solutions of $p$-adic differential equations. J. Inst. Math. Jussieu 8(3), 465–505 (2009)
10. Chiarellotto, B., Tsuzuki, N.: Log-growth filtration and Frobenius slope filtration of $F$-isocrystals at the generic and special points. Doc. Math. 16, 33–69 (2011)
11. Christol, G.: Modules différentiels et équations différentielles $p$-adiques. Queen’s Papers In Pure and Applied Mathematics, vol. 66. Queen’s University, Kingston (1983)
12. Christol, G.: Un théorème de transfert pour les disques singuliers réguliers, Cohomologie $p$-adique. Astérisque, 119-120, SMF, pp. 151–168 (1984)
13. Crew, R.: $F$-isocrystals and their monodromy groups. Ann. Scient. Éc. Norm. Sup. 25, 429–464 (1992)
14. Jong, J.A.: Homomorphisms of Barsotti–Tate groups and crystals in positive characteristic. Invent. Math. 134(2), 301–333 (1998)
15. D’Addezio, M.: Parabolicity conjecture of $F$-isocrystals. arXiv:2012.12879
16. Deligne, P.: La conjecture de Weil. II. Publ. Math. IHÉS 52, 313–428 (1981)
17. Dwork, B.: Normalized period matrices. II. Ann. Math. 2(98), 1–57 (1973)
18. Dwork, B.: Bessel functions as $p$-adic functions of the argument. Duke Math. J. 41(4), 711–738 (1974)
19. Elkik, R.: Solutions d’équations à coefficients dans un anneau hensélien. Ann. Sci. École Norm. Sup. 6, 553–603 (1973)
20. Étesse, J.-Y., Le Stum, B.: Fonctions $L$ associées aux $F$-isocristaux surconvergents, I: Interprétation cohomologique. Math. Ann. 296, 557–576 (1993)
21. Fulton, W.: A note on weakly complete algebras. Bull. Am. Math. Soc. 75, 591–593 (1969)
22. Grothendieck, A.: Revêtements étals et groupe fondamental (SGA1). In: Lecture Notes in Math., vol. 224. Springer (1971)
23. Haessig, C.D.: $L$-functions of symmetric powers of Kloosterman sums (unit root $L$-functions and $p$-adic estimates). Math. Ann. 369(1–2), 17–47 (2017)
24. Katz, N.M.: $p$-adic properties of modular schemes and modular forms. In: Modular Functions of One Variable, III (Proceedings of the International Summer School, University of Antwerp, Antwerp, 1972). Lecture Notes in Mathematics, vol. 350, pp. 69–190. Springer, Berlin (1973)
25. Kedlaya, K.S.: Power series and $p$-adic algebraic closures. J. Number Theory 89, 324–339 (2001)
26. Kedlaya, K.S.: Slope filtrations revisited. Doc. Math. 10, 447–525 (2005). (errata, ibid. 12, 361–362 (2007))
27. Kedlaya, K.S.: Finiteness of rigid cohomology with coefficients. Duke Math. J. 134, 15–97 (2006)
28. Kedlaya, K.S.: Fourier transforms and $p$-adic “Weil II”. Compos. Math. 142, 1426–1450 (2006)
29. Kedlaya, K.S.: Semistable reduction for overconvergent $F$-isocrystals, I: unipotence and logarithmic extensions. Compos. Math. 143, 1164–1212 (2007)
30. Kedlaya, K.S.: Slope filtration for relative Frobenius. Astérisque 319, 259–301 (2008)
31. Kedlaya, K.S.: Semistable reduction for overconvergent $F$-isocrystals, II: a valuation-theoretic approach. Compos. Math. 144, 657–672 (2008)
32. Kedlaya, K.S.: Notes on isocrystals. J. Number Theory 237, 353–394 (2022)
33. Kramer-Miller, J.: The monodromy of $F$-isocrystals with log-decay. arXiv:1612.01164
34. Kramer-Miller, J.: Slope filtrations of $F$-isocrystals and logarithmic decay. Math. Res. Lett. 28(1), 107–125 (2021)
35. Kramer-Miller, J.: The monodromy of unit-root $F$-isocrystals with geometric origin. to appear in Compos. Math.
36. Ohkubo, S.: Logarithmic growth filtrations for $(\varphi, \nabla)$-modules over the bounded Robba ring. Compos. Math. 157(6), 1265–1301 (2021)
37. Shioda, T., Inose, H.: On singular K3 surfaces. In: Bailey, W.L., Shioda, T. (eds.) Complex analysis and algebraic geometry, pp. 119–136. Cambridge University Press, Cambridge (1977)
38. Stienstra, J., Beukers, F.: On the Picard–Fuchs equation and the formal Brauer group of certain elliptic K3-surfaces. Math. Ann. 271(2), 269–304 (1985)
39. Tsuzuki, N.: The overconvergence of morphisms of étale $\varphi$-$\nabla$-spaces on a local field. Compos. Math. 103(2), 227–239 (1996)
40. Tsuzuki, N.: Finite local monodromy of overconvergent unit-root $F$-isocrystals on a curve. Am. J. Math. 120(6), 1165–1190 (1998)
41. Tsuzuki, N.: Slope filtration of quasi-unipotent overconvergent $F$-isocrystals. Ann. Inst. Fourier (Grenoble) 48(2), 379–412 (1998)
42. Tsuzuki, N.: Morphisms of $F$-isocrystals and the finite monodromy theorem for unit-root $F$-isocrystals. Duke Math. J. 111(3), 385–418 (2002)
43. Tsuzuki, N.: Constancy of Newton polygons of $F$-isocrystals on Abelian varieties and isotriviality of families of curves. J. Inst. Math. Jussieu 20(2), 587–625 (2021)
44. Put, M.: The cohomology of Monsky and Washnitzer. Mémoires de la S.M.F. 2e série 23, 33–59 (1986)
46. Wan, D.: Dwork’s conjecture of unit root zeta functions. Ann Math. 150, 867–927 (1999)
47. Wan, D.: Higher rank case of Dwork’s conjecture. J. Am. Math. Soc. 13, 807–852 (2000)
48. Wan, D.: Rank one case of Dwork’s conjecture. J. Am. Math. Soc. 13, 853–908 (2000)

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