ISOPERIMETRIC PLANAR CLUSTERS WITH INFINITELY MANY REGIONS

MATTEO NOVAGA, EMANUELE PAOLINI, EUGENE STEPANOV, AND VINCENZO MARIA TORTORELLI

Abstract. An infinite cluster $E$ in $\mathbb{R}^d$ is a sequence of disjoint measurable sets $E_k \subset \mathbb{R}^d$, $k \in \mathbb{N}$, called regions of the cluster. Given the volumes $a_k \geq 0$ of the regions $E_k$, a natural question is the existence of a cluster $E$ which has finite and minimal perimeter $P(E)$ among all clusters with regions having such volumes. We prove that such a cluster exists in the planar case $d = 2$, for any choice of the areas $a_k$ with $\sum \sqrt{a_k} < \infty$. We also show the existence of a bounded minimizer with the property $P(E) = \mathcal{H}^1(\partial E)$, where $\partial E$ denotes the measure theoretic boundary of the cluster. We also provide several examples of infinite isoperimetric clusters for anisotropic and fractional perimeters.

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1. INTRODUCTION

A finite cluster $E$ is a sequence $E = (E_1, \ldots, E_k, \ldots, E_N)$ of measurable sets, such that $|E_k \cap E_j| = 0$ for $k \neq j$, where $|\cdot|$ denotes the Lebesgue measure (usually called volume). The sets $E_j$ are called regions of the cluster $E$ and $E_0 := \mathbb{R}^d \setminus \bigcup_{k=1}^{\infty} E_k$ is called external region. We denote the sequence of volumes of the regions of the cluster $E$ as

$$\mathbf{m}(E) := (|E_1|, |E_2|, \ldots, |E_N|)$$

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and we call perimeter of the cluster the quantity

\[ P(E) := \frac{1}{2} \left[ P(E_0) + \sum_{k=1}^{N} P(E_k) \right], \]

where \( P \) is the Caccioppoli perimeter. A cluster \( E \) is called minimal, or isoperimetric, if

\[ P(E) = \min \{ P(F) : m(F) = m(E) \}. \]

In this paper we consider infinite clusters, i.e., infinite sequences \( E = (E_k)_{k \in \mathbb{N}} \) of essentially pairwise disjoint regions: \( |E_j \cap E_i| = 0 \) for \( i \neq j \) (this can be interpreted as a model for a soap foam). Note that a finite cluster with \( N \) regions, can also be considered a particular case of an infinite cluster for example by posing \( E_k = \emptyset \) for \( k > N \). Clusters with infinitely many regions of equal areas were considered in [12], where it has been shown that the honeycomb cluster is the unique minimizer with respect to compact perturbations. Infinite clusters have been considered also in [17,13,3], where the variational curvature is prescribed, and in [23], where existence of generalized minimizers for both finite and infinite isoperimetric clusters has been proven in the general setting of homogeneous metric measure spaces.

An interesting example of infinite cluster, detailed in Example 4.1 (see Figure 1) is the Apollonian packing of a circle (see [15]). In fact this cluster is composed by isoperimetric regions and hence should trivially have minimal perimeter among clusters with regions of the same areas. Actually, it turns out that this cluster has infinite perimeter and hence all clusters with same prescribed areas have infinite perimeter too. Note that very few explicit examples of minimal clusters are known [10,20,14,25,19]. Nevertheless, quite curiously, Apollonian packings give nontrivial examples of infinite isoperimetric clusters for fractional perimeters [4,7,6], as shown in Example 4.1. An even simpler example of an infinite isoperimetric planar cluster is given in Example 4.2 (see Figure 1 again) where the Caccioppoli perimeter is replaced by an anisotropic perimeter functional [16,21,22,5,8].

Our main result, Theorem 3.1, states that if \( d = 2 \) (planar case), given any sequence of positive numbers \( a = (a_1, a_2, \ldots, a_k, \ldots) \) such that \( \sum_{k=0}^{\infty} \sqrt{a_k} < +\infty \), there exists a minimal cluster \( E \) in \( \mathbb{R}^2 \) with \( m(E) = a \). The assumption on \( a \) is
necessary to have at least a competitor cluster with finite perimeter. The proof
relies on two facts which are only available in the planar case: the isodiametric
inequality for connected sets and the semicontinuity of the length of connected sets
(Golab theorem).

2. Notation and preliminaries

2.1. Perimeters and boundaries. For a set \( E \subset \mathbb{R}^d \) with finite perimeter one
 can define the reduced boundary \( \partial^* E \) which is the set of boundary points \( x \) where
the outer normal vector \( \nu_E(x) \) can be defined. One has \( D1_E = \nu_E \cdot \mathcal{H}^{d-1}, \partial^* E \)
where \( 1_E \) is the characteristic function of \( E \) and \( D1_E \) is its distributional derivative
(the latter is a vector valued measure and its total variation is customarily denoted
by \( |D1_E| \)). The measure theoretic boundary of a measurable set \( E \) is defined by
\[
\partial E := \{ x \in \mathbb{R}^d : 0 < |E \cap B_\rho(x)| < |B_\rho(x)| \quad \text{for all } \rho > 0 \}.
\]
The corresponding notions for clusters can be defined as follows:
\[
\partial^* E := \bigcup_{k=1}^{+\infty} \bigcup_{j=0}^{k-1} \partial^* E_k \cap \partial^* E_j,
\]
\[
\partial E := \{ x \in \mathbb{R}^d : 0 < |E_k \cap B_\rho(x)| < |B_\rho(x)| \\
\quad \text{for all } \rho > 0 \text{ and some } k = k(\rho, x) \in \mathbb{N} \}.
\]
Clearly \( \partial^* E \subseteq \partial E \) because given \( x \in \partial^* E \) there exists \( k \) such that \( x \in \partial^* E_k \), while
\( \partial E_k \subseteq \partial E \) for all \( k \). Also it is easy to check that \( \partial E \) is closed (and it is the closure
of the union of all the measure theoretic boundaries \( \partial E_k \)). Moreover the following
result holds true.

**Proposition 2.1.** If \( E \) is a cluster with finite perimeter, then \( P(E) = \mathcal{H}^{d-1}(\partial^* E) \).

**Proof.** Consider the sets \( X_n \), for \( 1 \leq n \leq \infty \), defined by
\[
X_n := \{ x \in \mathbb{R}^d : \# \{ k \in \mathbb{N} : x \in \partial^* E_k \} = n \}
\]
(notice that \( k = 0 \in \mathbb{N} \), the external region, is included in the count). It is clear
that \( X_n = \emptyset \) for all \( n \geq 3 \) because in every point of \( \partial^* E_k \) there is an approximate
tangent hyper-plane which can only be shared by two regions.

We claim that \( \mathcal{H}^{d-1}(X_1) = 0 \). To this aim suppose by contradiction that
\( \mathcal{H}^{d-1}(X_1) > 0 \). Then there exists a \( j \in \mathbb{N} \) such that
\[
|D1_{E_j}|(X_1) = \mathcal{H}^{d-1}(X_1 \cap \partial^* E_j) > 0,
\]
because \( X_1 \) is contained in the countable union \( \bigcup_{j=0}^\infty X_1 \cap \partial^* E_j \). Hence there is a
subset \( A \subset X_1 \cap \partial^* E_j \) such that \( D1_{E_j}(A) \neq 0 \). Notice that \( \sum_{k=0}^\infty 1_{E_k} = 1 \), hence
also \( \sum_k D1_{E_k} = 0 \). Since \( D1_{E_j}(A) \neq 0 \) there must exist at least another index
\( k \neq j \) such that \( D1_{E_k}(A) \neq 0 \), hence \( \mathcal{H}^{d-1}(A \cap \partial^* E_k) > 0 \). But then
\[
\emptyset \neq A \cap \partial^* E_k \subset X_1 \cap \partial^* E_j \cap \partial^* E_k, \quad j \neq k,
\]
contrary to the definition of \( X_1 \), which proves the claim.
In conclusion, the union of all reduced boundaries $\partial^* E_k$ is contained in $X_2$ up to a $\mathcal{H}^{d-1}$-negligible set. Hence

$$P(E) = \frac{1}{2} \sum_{k=0}^{+\infty} P(E_k) = \frac{1}{2} \sum_{k=0}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap X_2) =$$

$$= \frac{1}{2} \sum_{k=0}^{+\infty} \sum_{j \neq k} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) = \sum_{k=0}^{+\infty} \sum_{j=k+1}^{+\infty} \mathcal{H}^{d-1}(\partial^* E_k \cap \partial^* E_j) =$$

$$= \mathcal{H}^{d-1} \left( \bigcup_{k=0}^{+\infty} \bigcup_{j=k+1}^{+\infty} \partial^* E_k \cap \partial^* E_j \right) = \mathcal{H}^{d-1}(\partial^* E)$$

as claimed.

2.2. **Auxiliary results.** In the following theorem we collect known existence and regularity results for finite minimal clusters from [20] [18].

**Theorem 2.2** (existence and regularity of planar $N$-clusters). Let $a_1, a_2, \ldots, a_N$ be given positive real numbers. Then there exists a minimal $N$-cluster $E = (E_1, \ldots, E_N)$ in $\mathbb{R}^2$, with $|E_k| = a_k$ for $k = 1, \ldots, N$. If $E$ is a minimal $N$-cluster and $d = 2$, then $\partial E$ is a pathwise connected set composed by circular arcs or line segments joining in triples at a finite number of vertices. Moreover in this case $P(E) = \mathcal{H}^1(\partial E)$.

The statement below gives isodiametric inequality for planar finite clusters,

**Proposition 2.3** (diameter estimate). If $E$ is an $N$-cluster in $\mathbb{R}^2$ and $\partial E$ is pathwise connected, then

$$\text{diam} \, \partial E \leq P(E).$$

*Proof.* Since $\partial E$ is pathwise connected, given any two points $x, y \in \partial E$ we find that $|x - y| \leq \mathcal{H}^1(\partial E) = P(E)$. $\square$

Another ingredient will be the following statement on cluster truncation,

**Proposition 2.4** (cluster truncation). Let $E = (E_1, \ldots, E_k, \ldots)$ be a (finite or infinite) cluster and let $T_N E$ be the $N$-cluster $(E_1, \ldots, E_N)$. Then

$$P(T_N E) \leq P(E).$$

*Proof.* For measurable sets $E, F$ the inequality

$$P(E \cup F) + P(E \cap F) \leq P(E) + P(F)$$

holds, hence if $|E \cap F| = 0$, one has

$$P(E) = P((E \cup F) \cap (\mathbb{R}^d \setminus F)) \leq P(E \cup F) + P(F).$$
It follows that
\[ 2P(T_N E) = \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{n} E_k\right) \]
\[ \leq \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + P\left(\bigcup_{k=n+1}^{\infty} E_k\right) \]
\[ \leq \sum_{k=1}^{n} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) + \sum_{k=n+1}^{\infty} P(E_k) \]
\[ = \sum_{k=1}^{\infty} P(E_k) + P\left(\bigcup_{k=1}^{\infty} E_k\right) = 2P(E) \]
as claimed. \[\square\]

**Lemma 2.5.** Let \( E \) be a measurable set and \( \Omega \) an open connected set. If \( \partial E \cap \Omega = \emptyset \), then either \( |\Omega \cap E| = 0 \) or \( |\Omega \setminus E| = 0 \).

**Proof.** Notice that \( \Omega \setminus \partial E = A_0 \cup A_1 \), where
\[ A_0 := \{ x \in \Omega : |B_{\rho}(x) \cap E| = 0 \text{ for some } \rho > 0 \}, \]
\[ A_1 := \{ x \in \Omega : |B_{\rho}(x) \setminus E| = 0 \text{ for some } \rho > 0 \}. \]
It is clear that \( A_0 \) and \( A_1 \) are open disjoint sets, and if \( \partial E \cap \Omega = \emptyset \) their union is the whole set \( \Omega \). If \( \Omega \) is connected, then either \( A_0 \) or \( A_1 \) is equal to \( \Omega \) which means that either \( |\Omega \cap E| = 0 \) or \( |\Omega \setminus E| = 0 \). \( \square \)

### 3. Main result

The statement below provides existence of infinite planar isoperimetric clusters.

**Theorem 3.1** (existence). Let \( a = (a_1, \ldots, a_k, \ldots) \) be a sequence of nonnegative numbers such that \( \sum_{k=1}^{\infty} \sqrt{a_k} < \infty \). Then there exists a minimal cluster \( E \) in \( \mathbb{R}^2 \), with \( m(E) = a \) satisfying additionally

(3) \[ \bigcup_{k=1}^{\infty} E_k \text{ is bounded}, \]
(4) \[ \partial E \text{ is pathwise connected}, \]
(5) \[ \mathcal{H}^1(\partial E \setminus \partial^* E) = 0. \]

**Remark 3.2.** In view of (3) and Proposition 2.1 for the minimal cluster provided by the above Theorem 3.1 one has

(6) \[ P(E) = \mathcal{H}^1(\partial E) = \mathcal{H}^1(\partial^* E). \]

Of course there exists a set with finite perimeter \( E \) such that \( P(E) < \mathcal{H}^1(\partial E) \) hence (3) is false for general, non minimal, clusters.

It is interesting to note that, as shown in example 4.3, there exists a finite cluster \( E \) satisfying (3), for which one does not have \( P(E_k) = \mathcal{H}^1(\partial E_k) \) for all \( k \). It would be interesting to see whether these equalities hold for minimal clusters.
Proof. Let \( \bar{p} := 2\sqrt{\pi} \sum_{k=1}^{\infty} \sqrt{a_k} < +\infty \), and

\[
p := \inf\{ P(E) : E \text{ cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k, \ k = 1, 2, \ldots, n, \ldots \},
p_n := \inf\{ P(E) : E \text{ n-cluster in } \mathbb{R}^2 \text{ with } |E_k| = a_k, \ k = 1, \ldots, n \}
\]

so that a cluster \( E \) with measures \( m(E) = a \) is minimal if and only if \( P(E) = p \), while an \( n \)-cluster \( E \) with measures \( |E_k| = a_k \) for \( k = 1, \ldots, n \) is minimal if and only if \( P(E) = p_n \).

If \( E \) is a competitor for \( p \), then \( T_n E \) is a competitor for \( p_n \) and, by Proposition 2.4, one has \( P(T_n E) \leq P(E) \). Hence \( p_n \leq p \). Moreover one can build a competitor for \( p \) which is composed by circular disjoint regions \((B_1, \ldots, B_j, \ldots)\), where \( B_j \) are disjoint balls of radii \( \frac{a_j}{\pi} \), to find that \( p \leq \bar{p} < +\infty \).

For each \( n \geq 1 \) consider a minimal \( n \)-cluster \( F^n \) with \( |F^n_k| = a_k \) for \( k \leq n \), \( F^n_k := \emptyset \) for \( k > n \) so that \( P(F^n) = p_n \). Hence, by Proposition 2.3 up to translations we might and shall suppose that all the regions \( F^n_k \) of all the clusters \( F^n \) are contained in a ball of radius \( \bar{p} \). In fact:

\[
\bar{p} \geq p \geq \sup_n p_n = \sup_n P(F^n) \geq \sup_n \text{diam } \partial F^n.
\]

Up to a subsequence we can hence assume that the first regions \( F^n_k \) converge to a set \( E_1 \) in the sense that their characteristic functions \( 1_{F^n_k} \) converge to the characteristic function \( 1_{E_1} \) in the Lebesgue space \( L^1(\mathbb{R}^2) \) (we call this convergence \( L^1 \) convergence of sets). Analogously, up to a subsequence also the second regions \( F^n_2 \) converge in \( L^1 \) sense to a set \( E_2 \), and in this way we define inductively the sets \( E_k \) for all \( k \geq 1 \). Then there exists a diagonal subsequence with indices \( j_n \) such that for all \( k \) one has \( F^{n_j}_k \to E_k \) in \( L^1 \) for all \( k \geq 1 \) as \( j \to +\infty \).

Consider the cluster \( E \) with the components \( E_k \) defined above. By continuity we have \( \mathbf{m}(E) = a \) because \( F^{n_j}_k \to E_k \) in \( L^1 \) as \( j \to +\infty \) and \( |F^{n_j}_k| = a_k \) for all \( j \).

We claim that the union of all the regions of \( F^{n_j} \) also converges to the union of all the regions of \( E \). For all \( \varepsilon > 0 \) take \( N \) such that \( \sum_{k=N+1}^{\infty} a_k \leq \varepsilon \) and notice that

\[
\bigcup_{k=1}^{\infty} E_k \triangle \bigcup_{k=1}^{\infty} F^{n_j}_k \subseteq \bigcup_{k=1}^{N} (E_k \triangle F^{n_j}_k) \cup \bigcup_{k=N+1}^{\infty} E_k \cup \bigcup_{k=N+1}^{\infty} F^{n_j}_k.
\]

Hence

\[
\limsup_j \left[ \sum_{k=1}^{\infty} \left| E_k \triangle \bigcup_{k=1}^{\infty} F^{n_j}_k \right| \right] \leq \lim_j \sum_{k=1}^{N} |E_k \triangle F^{n_j}_k| + 2\varepsilon = 2\varepsilon.
\]

Letting \( \varepsilon \to 0 \) we obtain the claim.

By lower semicontinuity of perimeter:

\[
P(E_k) \leq \liminf_{j \to +\infty} P(F^{n_j}_k) \quad \text{and} \quad P\left( \bigcup_{k=1}^{+\infty} E_k \right) \leq \liminf_{j \to +\infty} P\left( \bigcup_{k=1}^{+\infty} F^{n_j}_k \right)
\]

and hence \( P(E) \leq \liminf_j P(F^{n_j}) \leq p \) proving that \( E \) is actually a minimal cluster. Since all the regions \( F^{n_j}_k \) are equi-bounded we obtain (3).

We are going to prove (5). By Theorem 2.2 the minimal \( n \)-cluster \( F^n \) has a measure theoretic boundary \( \partial F^n \) which is a compact and connected set such that \( P(F^n) = \mathcal{H}^1(\partial F^n) \). Up to a subsequence, the compact sets \( \partial F^{n_j} \), being uniformly
bounded, converge with respect to the Hausdorff distance, to a compact set \( K \). Without loss of generality suppose \( n_j \) is labeling this new subsequence.

We claim that \( \partial E \subseteq K \). In fact for any given \( x \in \partial E \) and any \( \rho > 0 \) there exists \( k = k(\rho) \) such that \( B_\rho(x) \cap E_k \) and \( B_\rho(x) \setminus E_k \) both have positive measure.

Since \( |B_\rho(x) \cap F^n_k| \rightarrow |B_\rho(x) \cap E_k| > 0 \) and \( |B_\rho(x) \setminus F^n_k| \rightarrow |B_\rho(x) \setminus E_k| > 0 \) for \( j = j(\rho) \) sufficiently large by Lemma \([2.5]\) there is a point \( x'_k \in B_\rho(x) \cap \partial F^n_k \). As \( \rho \to 0 \) the sequence \( x'_k \) converges to \( x \) and since \( \partial F^n_k \subseteq \partial F^n_j \) we conclude that \( x \in K \).

The sets \( \partial F^n \) are connected, hence, by the classical Gōlb theorem on semi-continuity of one-dimensional Hausdorff measure over sequences of connected sets (see \([2\)\) theorem 4.4.17\) or \([24\)\) theorem 3.3\) for its most general statement and a complete proof), one has

\[
\mathcal{H}^1(K) \leq \liminf_n \mathcal{H}^1(\partial F^n)
\]

and \( K \) is itself connected. Summing up and using Proposition \([2.1]\)

\[
P(E) = \mathcal{H}^1(\partial E) \leq \mathcal{H}^1(\partial E) \leq \mathcal{H}^1(K) \\
\leq \liminf_n P(F^n) \leq \limsup_n p_n \leq \rho \leq P(E)
\]

hence \( \mathcal{H}^1(\partial E) = \mathcal{H}^1(K) = \mathcal{H}^1(\partial E) \), \( p_n \to p \) and \((5)\) follows.

Finally, to prove that \( \partial E \) is connected, it is enough to show \( \partial E = K \). We already know that \( \partial E \subseteq K \) so we suppose by contradiction that there exists \( x \in K \setminus \partial E \). Take any \( y \in E \). The set \( K \) is arcwise connected by rectifiable arcs, since it is a compact connected set of finite one-dimensional Hausdorff measure (see e.g. \([9\)\) lemma 3.11\) or \([2\)\) theorem 4.4.7\)), in other words, there exists an injective continuous curve \( \gamma: [0, 1] \to K \) with \( \gamma(0) = x \) and \( \gamma(1) = y \). Since \( \partial E \) is closed in \( K \) there is a small \( \varepsilon > 0 \) such that \( \gamma([0, \varepsilon]) \subseteq K \setminus \partial E \) and hence \( \mathcal{H}^1(K \setminus \partial E) > 0 \) contrary to \( \mathcal{H}^1(K) = \mathcal{H}^1(\partial E) \); this contradiction shows the last claim and hence concludes the proof. \( \square \)

4. Some examples

We collect here some interesting examples of infinite planar clusters.

**Example 4.1** (Apollonian packing). A cluster \( E \), as depicted in Figure \([1]\) can be constructed so that each region \( E_k = B_{r_k}(x_k) \), \( k \neq 0 \), is a ball contained in the ball \( B_1 = \mathbb{R}^2 \setminus E_0 \). The balls can be chosen to be pairwise disjoint and such that the measure of \( B_1 \setminus \bigcup_{k=1}^{\infty} E_k = 0 \) (see \([13]\)).

Clearly such a cluster must be minimal because each region \( E_k \) has the minimal possible perimeter among sets with the given area and the same is true for the complement of the exterior region \( E_0 \) which is their union. The boundary \( \partial E \) of such a cluster is the residual set, i.e. the set of zero measure which remains when the balls \( E_k \) are removed from the large ball \( B_1 \):

\[
\partial E = \bigcup_{k=0}^{+\infty} \partial B_{r_k}(x_k) = B_1 \setminus \bigcup_{k=1}^{+\infty} B_{r_k}(x_k).
\]

Unfortunately the residual set of such a cluster has Hausdorff dimension \( d > 1 \) (see \([13]\)) and hence the cluster \( E \) cannot have finite perimeter.
Figure 2. An example of a cluster $E$ with finite perimeter such that $P(E) = \mathcal{H}^1(\partial E)$ but $P(E_3) < \mathcal{H}^1(\partial E_3)$.

However we can consider the fractional (non local) perimeter $P_s$ defined by

$$P_s(E) = \int_E \int_{\mathbb{R}^2 \setminus E} \frac{1}{|x-y|^{2+s}} \, dx \, dy$$

to define the corresponding non local perimeter $P_s(E)$ of the cluster $E$ by means of definition (2) with $P_s$ in place of $P$. If $r_k$ is the radius of the $k$-th disk of the cluster it turns out (see [3]) that the infimum of all $\alpha$, such that the series $\sum_k r_k^\alpha$ converges, is equal to $d$, the Hausdorff dimension of $\partial E$. Since $d \in (1, 2)$ for all $s < 2 - d$ we have

$$\sum_k r_k^{2-s} < +\infty$$

and since $P_s(B_r) = C \cdot r^{2-s}$ (with $0 < C < +\infty$) we obtain $P_s(E) < +\infty$ for such $s$.

It is well known (see [11]) that the solution to the fractional isoperimetric problem is given by balls, hence $E$ provides an example of an infinite minimal cluster with respect to the fractional perimeter $P_s$.

Example 4.2 (Anisotropic isoperimetric packing). We can find a similar example if we consider an anisotropic perimeter such that the isoperimetric problem has the square (instead of the circle) as a solution. If $\phi$ is any norm on $\mathbb{R}^2$ one can define the perimeter $P_\phi$ which is the relaxation of the following functional defined on regular sets $E \subset \mathbb{R}^2$:

$$P_\phi(E) = \int_{\partial E} \phi(\nu_E(x)) \, d\mathcal{H}^1(x)$$

where $\nu_E(x)$ is the exterior unit normal vector to $\partial E$ in $x$. If $\phi(x,y) = |x| + |y|$ (the Manhattan norm) it is well known that the $P_\phi$-minimal set with prescribed area (i.e. the Wulff shape) is a square with sides parallel to the coordinated axes (which is the ball for the dual norm). It is then easy to construct an infinite cluster $E = (E_1, \ldots, E_k, \ldots)$ where each $E_k$ is a square and also the union of all such squares is a square, see figure [11]. By iterating such a construction it is not difficult to realize that given any sequence $a_k$, $k = 1, \ldots, n, \ldots$ of numbers such that their sum is equal to 1 and each number is a power of $\frac{1}{4}$ it is possible to find a cluster $E$ with $m(E) = a$ such that each $E_k$ is a square and the union $\bigcup_k E_k$ is the unit square.

Example 4.3 (Cantor circles). See Figure and [11] example 2 pag. 59. Take a rectangle $R$ divided in two by segmente $S$ on its axis. Let $C$ be a Cantor set with positive measure constructed on $S$. Consider the set $E_3$ which is the union of the balls with diameter on the intervals composing the complementary set $S \setminus C$. Let $E_1$
and $E_2$ be the two connected components of $R \setminus E_3$. It turns out that the 3-cluster $E = (E_1, E_2, E_3)$ has finite perimeter and the perimeter of $E$ is represented by the Hausdorff measure of the boundary:

$$P(E) = H^1(\partial E).$$

However the same is not true for each region. In fact the boundary $\partial E_3$ of the region $E_3$ includes $C$ and hence

$$P(E_3) < H^1(\partial E_3).$$

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(Matteo Novaga, Emanuele Paolini, Vincenzo Tortorelli) Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127, Pisa

Email address, Matteo Novaga: matteo.novaga@unipi.it

Email address, Emanuele Paolini: emanuele.paolini@unipi.it

(Eugene Stepanov) Scuola Normale Superiore, Piazza dei Cavalieri 6, Pisa, Italy and St.Petersburg Branch of the Steklov Mathematical Institute of the Russian Academy of Sciences, St.Petersburg, Russia and Faculty of Mathematics, Higher School of Economics, Moscow

Email address, Eugene Stepanov: stepanov.eugene@gmail.com

Email address, Vincenzo Maria Tortorelli: tortorelli@dm.unipi.it