Implicit complexity via structure transformation

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Abstract

Implicit computational complexity, which aims at characterizing complexity classes by machine-independent means, has traditionally been based, on the one hand, on programs and deductive formalisms for free algebras, and on the other hand on descriptive tools for finite structures.

We consider here “uninterpreted” programs for the transformation of finite structures, which define functions over a free algebra $A$ once the elements of $A$ are themselves considered as finite structures. We thus bridge the gap between the two approaches above to implicit complexity, with the potential of streamlining and clarifying important tools and techniques, such as set-existence and ramification.

We illustrate this potential by delineating a broad class of programs, based on the notion of loop variant familiar from imperative program construction, that characterizes a generic notion of primitive-recursive complexity, without reference to any data-driven recurrence.

1 Introduction

Implicit computational complexity (ICC) strives to characterize complexity classes by resource-independent methods, thereby elucidating the nature of those classes and relating them to more abstract complexity measures, such as levels of descriptive or deductive abstractions. The various approaches to ICC fall, by and large, into two broad classes. One is descriptive complexity, which focuses on finite structures, and as such forms a branch of Finite Model Theory \cite{18}. Its historical roots go back at least to the characterization of Log-Space queries by recurrence \cite{16}, and of NP by existential set-quantification \cite{10}.

The other broad class in ICC focuses on computing over infinite structures, such as the natural numbers, strings, or lists, and uses programming and proof-theoretic methods to articulate resource-independent characterizations of complexity classes.

We argue here that computing over finite structures is, in fact, appropriate for implicit complexity over infinite structures as well. Our point of departure is the observation that inductive data-objects, such as natural numbers, strings and lists, are themselves finite structures, and that their computational behavior is determined by their internal makeup rather than by their membership in this or that infinite structure. For example, the natural number three is the structure (or more precisely partial-structure, see below)

\[
\mathcal{T}(3) \equiv 0 \circ \xrightarrow{s} \circ \xrightarrow{s} \circ \xrightarrow{s} \circ
\]

Lifting this representation, a function $f : \mathbb{N} \to \mathbb{N}$ is perceived as a mapping over finite second-order objects, namely the natural numbers construed as structures. This

\footnotesize{\textsuperscript{1}Indiana University and IRIF, Université Paris-Diderot}
\footnotesize{\textsuperscript{2}Research supported by LORIA Nancy and by Université de Lyon grant ANR-10-LABX-0070}
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view of inductive objects as finite structures is implicit already in long-standing repre-
resentations, such as the Church-Berarducci-Böhm lambda-coding of inductive data.

As a programming language of reference we propose a Turing-complete imperative
language ST for structure transformation, in the spirit of Gurevich’s ASMs.
We regard such programs as operating over classes of finite structures.

We illustrate the naturalness and effectiveness of our approach by delineating a
variant STV of ST, based on the notion of loop variants familiar from program de-
development and verification, and proving that it captures exactly primitive
recursion, in the strongest possible sense: all functions defined by recurrence over free
algebras are computable directly by STV programs, and all STV programs run in
time and space that are primitive-recursive in the size of the input.

We caution against confounding our approach with unrelated prior research ad-
ressing somewhat similar themes. Recurrence and recursion over finite structures
have been shown to characterize logarithmic space and polynomial time queries, re-
spectively, but the programs in question do not allow inception of new structure
elements, and so remain confined to linear space complexity, and are inadequate for
the kind of characterizations we seek. On the other hand, unbounded recurrence over
arbitrary structures has been considered by a number of authors, but always in the
traditional sense of computing within an infinite structure. Also, while the meta-finite
structures of merge finite and infinite components, both of those
are considered in the traditional framework, whereas we deal with purely finite struc-
tures, and the infinite appears via the consideration of collections of such structures.
Finally, the functions we consider are from structures to structures (as in ), and
are thus unrelated to the global functions of , which are (isomorphism-invariant) mappings that assigns to each structure a function over it.

2 General setting

2.1 Partial structures

We use the phrase vocabulary for a finite set \( V \) of function-identifiers and relation-
identifiers, with each identifier \( I \) assigned an arity \( \geq 0 \) denoted \( r(I) \). We refer to
nullary function-identifiers as tokens, and to ones of arity 1 as pointers.

By V-structure we’ll mean here a finite partial-structure over the vocabulary \( V \); that is, a V-structure \( S \) consists of a finite non-empty universe \( |S| \), for each function
identifier \( f \) of \( V \) a partial-function \( f_s : |S|^k \rightarrow |S| \), where \( k = r(f) \), and for each
relation-identifier \( Q \) of \( V \), a relation \( Q_s \subseteq |S|^k \), where \( k = r(Q) \). We refer to the
elements of \( |S| \) as \( S \)'s nodes.

We insist on referring to partial-structures since we consider partiality to be a core
component of our approach. For example, we shall identify each string in \( \{0, 1\}^\ast \) with a
structure over the vocabulary with a token \( e \) and pointers \( 0 \) and \( 1 \). So \( 011 \) is identified
with the four element structure

\[
e \circ \rightarrow \circ \rightarrow \circ \rightarrow \circ
\]

Here 0 is interpreted as the partial-function defined only for the leftmost element, and
1 as the partial-function defined only for the second and third elements.

We might, in fact, limit attention to vocabularies without relation identifiers, since
a k-ary relation \( Q \) \( (k > 0) \) can be represented by its support, that is the k-ary partial-
\begin{equation*}
\sigma(x_1, \ldots, x_k) = \begin{cases} 
\text{df} & \text{if } \vec{x} \in Q \\
\text{undefined} & \text{else}
\end{cases}
\end{equation*}

Thus, for instance, \( Q \) is empty iff \( \sigma \) is empty (which is not the case if relations are represented by their characteristic functions). Note that by using the support rather than the characteristic function we bypass the traditional representation of truth values by elements, and obtain a uniform treatment of functional and relational structure revisions (defined below), as well as initiality conditions.

A tuple of structures is easily presentable as a single structure. Given structures \( S_1, \ldots, S_k \), where \( S_i \) is a \( V_i \)-structure, let \( V \) be the disjoint union of \( V_1, \ldots, V_k \), and let \( S_1 \oplus S_2 \oplus \cdots \oplus S_k \) be the \( V \)-structure whose universe is the disjoint union of \( |S_i| \) \((i = 1..k)\), and where the interpretation of an identifier of \( V_i \) is the same as it is in \( S_i \), i.e. is empty/undefined on \( |S_j| \) for every \( j \neq i \).

2.2 Accessible structures and free structures

The \textit{terms over} \( V \), or \( V \)-terms, are generated by the closure condition: if \( r(f) = k \) and \( \alpha_1, \ldots, \alpha_k \) are terms, then so is \( f \alpha_1 \cdots \alpha_k \). (We use parentheses and commas for function application only at the discourse level.) Note that we do not use variables, so our “terms” are all closed. The \textit{height} \( h(t) \) of a term \( t \) is the height of its syntax-tree: \( h(f \alpha_1 \cdots \alpha_k) = 1 + \max \{ h(\alpha_i) | i = 1..k \} \).

Given a \( V \)-structure \( S \) the value of a \( V \)-term \( \alpha \) in \( S \), \( [\alpha]_S \), is defined as usual by recurrence on \( \alpha \): If \( r(f) = k \), then \( [f \alpha_1 \cdots \alpha_k]_S = f_s([\alpha_1]_S, \ldots, [\alpha_k]_S) \). We say that a term \( \alpha \) \textit{denotes} its value \( v \), and also that it is an \textit{address} for \( v \).

A node of a \( V \)-structure \( S \) is \textit{accessible} if it is the value in \( S \) of a \( V \)-term. The \textit{height} of an accessible node \( a \) is the minimum of the heights of addresses of \( a \). A structure \( S \) is \textit{accessible} when all its nodes are accessible. If, moreover, every node has a unique address we say that \( S \) is \textit{free}.

A \( V \)-structure \( T \) is a \textit{term-structure} if

1. its universe consists of \( V \)-terms; and

2. if \( f \alpha_1, \ldots, \alpha_k \in |T| \) then \( \alpha_1, \ldots, \alpha_k \in |T| \) and \( f_s(\alpha_1, \ldots, \alpha_k) = f \alpha_1 \cdots \alpha_k \).

From the definitions we have

\textbf{Proposition 1} A \( V \)-structure \( S \) is free iff it is isomorphic to a term \( V \)-structure.

Note that if \( V \) is functional (no relation identifiers), then for each \( V \)-term \( q \) we have a free term-structure \( T(q) \) consisting of the sub-terms of \( q \) (\( q \) included). Each \( T(q) \) can be represented as a dag of terms, whose terminal nodes are tokens. It will be convenient to fix a reserved token, say \( \bullet \), that will denote in each structure \( T(q) \) the term \( q \) as a whole.

3 Structure-transformation programs

Programs operating on structures and transforming them are well known, for example from Gurevich’s Abstract State Machines [14, 15, 6]. We define a version of such programs, giving special attention to basic execution steps (structure revisions).
3.1 Structure revisions

We consider the following basic operations on \( V \)-structures, transforming a \( V \)-structure \( S \) to a \( V \)-structure \( Q \) which, aside from the changes indicated below, is identical to \( S \).

- **Function-revisions**
  
  1. A *function-extension* is an expression \( f_{\alpha_1} \cdots \alpha_k \downarrow \beta \). The intent is that if \( [\alpha_1]_s, \ldots, [\alpha_k]_s, [\beta]_s \) are all defined, but \( [f(\alpha_1, \ldots, \alpha_k)]_s \) is undefined, then \( f(\{\alpha_1\}_s, \ldots, [\alpha_k]_s) = [\beta]_s \). \( f \) is the eigen-function of the extension.
  
  2. A *function-contraction* is an expression \( f_{\alpha_1} \cdots \alpha_k \uparrow \). The intent is that \( f_{\alpha_1} \cdots \alpha_k \uparrow \) is undefined.

- **Relation-revisions**

  Relation revisions may be viewed as a special case of function-revisions, given the functional representation of relations described above. We mention them explicitly since they are used routinely.

  1. A *relation-extension* is an expression \( R \downarrow (\alpha_1, \ldots, \alpha_k) \) where \( R \) is a \( k \)-ary relation identifier. The intent is that if each \( [\alpha_i]_s \) is defined, then \( [R]_s \) is \( [R]_S \) augmented with the tuple \( (\{\alpha_1\}_s, \ldots, [\alpha_k]_s) \) (if not already there). \( R \) is the eigen-relation of the extension.
  
  2. A *relation-contraction* is an expression \( R \uparrow (\alpha_1, \ldots, \alpha_k) \). The intent is that if each \( [\alpha_i]_s \) is defined, then \( [R]_c \) is \( [R]_s \) with the tuple \( ([\alpha_1]_s, \ldots, [\alpha_k]_s) \) removed (if there).

- **Node-revisions**

  1. A *node-inception* is an expression of the form \( c \downarrow \), where \( c \) is a token. The intent is that, if \( [c]_s \) is undefined, then \( |Q| \) is \( |S| \) augmented with a new node \( \nu \) denoted by \( c \) (i.e. \( [c]_Q = \nu \)). A traditional alternative notation is \( c := \text{new} \). Assigning \( \nu \) to a compound address \( f_{\alpha_1} \cdots \alpha_k \) can be viewed as an abbreviation for \( c \downarrow; f_{\alpha_1} \cdots \alpha_k \downarrow c; c \uparrow \), where \( c \) is a fresh token.
  
  2. A *node-deletion* is an expression of the form \( c \uparrow \), where \( c \) is a token. The intent is that \( Q \) is obtained from \( S \) by removing the node \( \nu = \{c\} \) (if defined), and removing all tuples containing \( \nu \) from each \( R_s \) (\( R \) a relation-identifier) and from the graph of each \( f_s \) (\( f \) a function identifier). Again, a more general form of node-deletion, \( f\nu \uparrow \), can be implemented as the composition of a function-extension \( c \downarrow f\nu \) and \( c \uparrow \), \( c \) a fresh token.

  Deletions are needed, for example, when the desired output structure has fewer nodes than the input structure ("garbage collection").

We refer to the operations above collectively as *revisions*. Revisions cannot be split into smaller actions. On the other hand, a function-extension and a function-contraction can be combined into an *assignment*, i.e. a phrase of the form \( f\nu := \beta \). This can be viewed as an abbreviation, with \( b \) a fresh token, for the composition of four revisions:

\[
b \downarrow \beta; \ f\nu \uparrow; \ f\nu \downarrow b; \ b \uparrow
\]
3.2 ST Programs

Our programming language ST consists of guarded iterative programs built from structure revisions. Uninterpreted programs over a vocabulary V normally refer to an expansion W of V, as needed to implement algorithms and to generate output. We refer from now to such an expansion W.

- A test is one of following types of phrases.
  1. A convergence-expression !α, where α is an address. This is intended to state that the address α is defined for the current values of the function-identifiers. Thus ¬!α states that α is undefined in the current structure.
  2. An equation α = β where α and β are addresses. This is intended to state that both addresses are defined and evaluate to the same node.
  3. A relational-expression Rα₁· · ·αk, where R is a k-ary relation-identifier and each αi is an address. By the convention above, this may be construed as a special case of the equation σRα₁· · ·αk = α₁.

- A guard is a boolean combination of tests.

Given a vocabulary V the V-programs of ST are generated inductively as follows (we omit the reference to V when un-needed).

1. A structure-revision is a program.
2. If P and Q are programs then so is P; Q.
3. If G is a guard and P, Q are programs, then if [G] {P} {Q} and do [G] {P} are programs.

3.3 Program semantics

Given a vocabulary V a V-configuration (cfg) is a V-structure. Given a V-structure Q and W ⊇ V, we write QW for the expansion of Q to W with all identifiers in W−V interpreted as empty (everywhere undefined functions and empty relations). For a program P over V we define the binary yield relation ⇒P between V-configurations by recurrence on P. For P a structure-revision the definition follows the intended semantics described informally above. The cases for composition, branching, and iteration, are straightforward as usual.

Let Φ : C ↦ C′ be a partial-mapping from a class C of V-structures to a class C′ of V′-structures. A W-program P computes Φ if for every S ∈ C, S ⇒P Q for some W-expansion Q of Φ(S).

The vocabulary V′ of the output structure T need not be related to the input vocabulary V.

We shall focus mostly on programs as transducers. Note that all structure revisions refer only to accessible structure nodes. It follows that non-accessible nodes play no role in the computational behavior of ST programs. We shall therefore focus from now on accessible structure only.

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1Of course, if C is a proper class (in the sense of Godel-Bernays set theory), then the mapping defined by P is a proper-class.
3.4 Examples

1. Concatenation by splicing. The following program computes concatenation over \(\{0, 1\}^*\). It takes as input a pair \(T(u) \oplus T(v)\) of structures, where the nil and two successor identifiers are \(e, 0, 1\) for \(T(u)\) and \(\hat{e}, \hat{0}, \hat{1}\) for \(T(v)\). The output is \(T(u \cdot v)\), with vocabulary \(e, 0, 1\).

   a ↓ e;                        % moving token a to end of input 1
   do ![0a ∨ !1a]
      { a := 0a; a := 1a};       % note: only one of 0a, 1a is defined
   0a ↓ 0e; 1a ↓ 1e; a := 0a; a := 1a; % hooking 0/1 to input 2
   do ![0a ∨ !1a]               % copying input 2 to 0/1
      { 0a ↓ 0a; 1a ↓ 1a; a ↓ 0a; a ↓ 1a }

2. Concatenation by copying. The previous program uses no inception, as it splices the second argument over the first. The following program copies the second argument over the first, thereby enabling a repeated and modular use of concatenation, as in the multiplication example below.

   a ↓ e;                        % moving a to end of input 1
   do ![0a ∨ !1a]
      { a := 0a; a := 1a }
   b ↓ e;                       % copy of input 2 incepted after input 1
   do ![0b ∨ !1b]
      { c ↓;                  
        if ![0b]
           { 0a ↓ c; a ↓ c; b := 0b }
           { 1a ↓ c; a ↓ c; b := 1b }
        c↑
      }

3. String multiplication is the function that for inputs \(w \in \{0, 1\}^*\) and \(n \in \mathbb{N} = \{s\}^*\) returns \(nw = \text{the result of concatenating } n \text{ copies of } w\). This is computed by the following program, which takes as input a pair \(T(n) \oplus T(w)\) of structures, with vocabularies \(\{z, s\}\) and \(\{e, 0, 1\}\) respectively, and output vocabulary \(\{e, 0, 1\}\).
i ↓ z; a ↓ e;
do [!si]
\{ i := si;
b := e;
\} % iterate over numerical input

do [|!0b ∨ !1b]
\{c ↓;
if ![0b]
\{0a ↓ c; b := 0b\}
\{1a ↓ c; b := 1b\};
a := c; c ↑
\}

3.5 Computability

Since guarded iterative programs are well known to be sound and complete for Turing computability, the issue of interest here is articulating Turing computability in the ST setting. Consider a Turing transducer over an I/O alphabet Σ, with full alphabet Γ ⊃ Σ, set of states Q, start state s, print state p, and transition function δ. The input \( w = σ_1⋯σ_k ∈ Σ^* \) is taken to be the structure \( e ◦ σ_1 − → ◦ ⋯ ◦ σ_k − → ◦ \).

Define \( V_M \) to be the vocabulary with \( e, c \) and each state in \( Q \) as tokens; and with \( r \) and each symbol in \( Γ \) as pointers. Thus the program vocabulary is broader than the input vocabulary, both in representing \( M \)'s machinery, and with auxiliary components. The intent is that a configuration \( (q, σ_1⋯σ_i⋯σ_k) \) (i.e. with \( σ_i \) cursor ed) be represented by the \( V_M \)-structure
\[
\begin{array}{cccc}
o & σ_1 & ⋯ & σ_k \\
e & q & ⋯ & c \\
\end{array}
\]

All remaining tokens are undefined.

The program simulating \( M \) implements the following phases:

1. Convert the input structure into the structure for the initial configuration, and initialize \( c \) to the initial input element, and \( r \) to be the destructor function for the input string.
2. Main loop: configurations are revised as called for by \( δ \). The pointer \( r \) is used to represent backwards cursor movements. The loop’s guard is \( p \) (the “print” state) being undefined.
3. Convert the final configuration into the output.

4 STV: programs with variants

4.1 Loop variants

A variant is a finite set \( T \) of function- and relation-identifiers of positive arity, to which we refer as \( T \)'s components.
Given a vocabulary $V$ the $V$-programs of STV are generated inductively as follows, in tandem with the notion of a variant $T$ being terminating in an STV-program $P$. Again, we omit the reference to $V$ when it is clear or irrelevant.

1. A structure-revision over $V$ is a program. A variant $T$ is terminating in any revision except for a function- or relation-extension whose eigen function/relation is a component of $T$.

2. If $P$ and $Q$ are STV-programs with $T$ terminating, then so is $P; Q$.

3. If $G$ is a guard and $P, Q$ are STV-programs with $T$ terminating, then so is $\text{if}[G]\{P\}\{Q\}$.

4. If $G$ is a guard, and $P$ is a STV-program with $S$ and $T$ terminating variants, then $\text{do}\,[G]\,[S]\{P\}$ is a STV-program, with $T$ terminating.

We write $\text{STV}(W)$ for the programming language consisting of STV-programs over vocabulary $W$, and omitting $W$ when in no loss of clarity.

### 4.2 Semantics of STV-programs

The semantics of STV-programs is defined as for programs of ST, with the exception of the looping construct do. A loop $\text{do}\,[G]\,[T]\{P\}$ is entered if $G$ is true in the current state, and is re-entered if $G$ is true in the current state, and the previous pass executes at least one contraction for some component of the variant $T$. Thus, as $\text{do}\,[G]\,[T]\{P\}$ is executed, no component of $T$ is extended within $P$ (by the syntactic condition that $T$ is terminating in $P$), and is contracted at least once for each iteration, save the last (by the semantic condition on loop execution).

### 4.3 String duplication

The following program duplicates a string given as a structure: the output structure has the same nodes as the input, but with functions appearing in duplicate. The algorithm has two phases: a first loop, with the variant consisting collectively of the functions, creates two new copies of the string (while depleting the input function in the process). A second loop restores one of the two copies to the original identifiers, thereby allowing the duplication to be useful within a larger program that refers to the original identifiers. Function duplication in arbitrary structures is more complicated, and will be discussed below.
\[ a := e; \]
\[
\text{do}[!0a \lor !1a][0,1] \quad \% 0/1 copied to \hat{0}/\hat{1} and \hat{0}/\hat{1} \]
\[
\{ \text{b} \downarrow a; \quad \text{while being consumed as variant} \}
\]
\[
\text{if}[!0a] \quad \{
\quad \{
\hat{0}(a) \downarrow 0a; \quad \hat{0}(a) \downarrow 0a; \quad a := 0a; \quad 0b \uparrow \}
\quad \{
\hat{1}(a) \downarrow 1a; \quad \hat{1}(a) \downarrow 1a; \quad a := 1a; \quad 1b \uparrow \}
\}
\]
\[
a := e; \quad \hat{0}/\hat{1} \text{ restored to } 0/1 \]
\[
\text{do}[!0a \lor !1a][\hat{0},\hat{1}] \quad \{
\text{if}[!0a] \quad \{
0a \downarrow \hat{0}a; \quad \hat{0}a \uparrow; \quad a := 0a; \}
\quad \{
1a \downarrow \hat{1}a; \quad \hat{1}a \uparrow; \quad a := 1a; \}
\}
\]

The ability of STV programs to duplicate structures (for now only string structures) is at the core their ability to implement recurrence, so be discussed below.

### 4.4 Further examples

1. **Concatenation.** Using string duplication, we can easily convert the concatenation examples of §3.4 to STV. The changes are similar for the splicing and for the copying programs. The programs are preceded by the duplication of each of the two inputs. The copy of 0, 1 is then used as guard for the first loop, and is depleted by an entry in each cycle. The copy of \( \hat{0}, \hat{1} \) is used as guard for the second loop, and is similarly depleted.

2. **Multiplication.** The program of §3.4 is preceded by a duplication of the string input. The outer loop has \( s \) as a variant, which is depleted by a contraction in each cycle of the current \( si \). The inner loop has the copy of 0, 1 as variant.

3. **Exponentiation.** A program transforming the structure for \( 1^n e \) to the structure for \( 1^{2^n} \) is obtained by combining the programs for duplication and concatenation. Using for the input vocabulary a token \( z \) and a pointer \( s \), and for output a token \( y \) and a pointer \( t \), The program first initializes the output to the structure for \( 1 \). The main loop has \( !se \) as guard and \( s \) as variant. The body triplicates its initial \( t \), and uses one copy as variant for an inner loop that concatenates the other two copies.

### 5 Programs for structure expansions

In this section we describe programs that expand arbitrary (finite) structures in important ways.
5.1 Enumerators

Given a $W$-structure $S$ we say that a pair $(a, e)$, with $a \in |S|$ and $e : |S| \to |S|$, is an enumerator for $S$ if for some $n$ the sequence

$$a, e(a), \ldots, e^n(a)$$

consists of all accessible nodes of $S$, without repetitions, and $e^{n+1}(a)$ is undefined. An enumerator is monotone if the value of a term never precedes the value of its subterms. This is guaranteed if the value of a term of height $h$ never precedes the value of terms of height $< h$.

**Theorem 2** For each vocabulary $W$ there is a program that for $W$-structures $S$ as input yields an expansion of $S$ with a monotone enumerator $E$.

**Proof.** The program maintains, in addition to the identifiers in $W$, four auxiliary identifiers, as follows.

- A token $a$, intended to set the head $a$ of the enumerator.
- A pointer $e$, intended to denote a (repeatedly growing) initial segment of the intended enumerator $e$;
- A set identifier $E$, intended to denote the set of nodes enumerates by $e$ so far.
- A pointer $d$ intended to list, starting from a token $b$, some accessible nodes not yet listed in $e$; these are to be appended to $e$ at the end of each loop-cycle.
- A token $f$, intended to serve as a flag to indicate that the last completed cycle has added some elements to $E$.

A preliminary program-segment sets $a$ and $f$ to be the node denoted by one of the $V$-tokens (there must be one, or else there would be no accessible nodes), and defines $e$ to list any additional nodes denoted by tokens. (The value of $f$ is immaterial, only $f$ being defined matters.) Note that $e$, $d$ and $E$ are initially empty by default.

The main loop starts by re-initializing $d$ to empty, using string duplication described above, resetting $f$ to undefined (i.e. false), and duplicating $e$ as needed for the following construction. Each pass then adds to $d$ all nodes that are obtained from the current values in $e$ by applications of $W$’s functions, and that are not already in $E$. That is, for each unary function-id $g$ of $W$ a secondary loop travels through $e$, using an auxiliary token $t_1$. When $g$ applied to an entry is not in $E$, the value of that output is appended to both $a$ and $E$. The guard of that loop is $!et_1$, and the variant is $e$.

For function identifiers $g$ of arity $> 1$ the process is similar, except that nested loops are required, with additional duplications of $e$ ahead of each loop. Whenever a new node is appended to $a$, the token $f$ is set to be defined (say as the current value of $a$).

When every non-nullary function-identifier of $W$ is treated, the list $a$ is appended to $e$, leaving $a$ empty. $\square$

In §3.4 we gave a program for duplicating a string. Using an enumerator, a program using the same method would duplicate, for the accessible nodes, each structure function. Namely, to duplicate a $k$-ary function denoted by $f$ to one denoted by $f'$, the
program’s traverses \( k \) copies of the enumerator with \( k \) tokens \( c_1 \ldots c_k \), and whenever \( f c_1 \ldots c_k \) is defined, the program defines \( f' c_1 \ldots c_k \Downarrow f c_1 \ldots c_k \).

Observe that an enumerator for a structure usually ceases to be one with the execution of a structure revision; for example, a function contract may turn an accessible node into an inaccessible one. This can be repaired by accompanying each revision by an auxiliary program tailored to it, or simply by redefining an enumerator whenever one is needed.

5.2 Quasi-inverses

We shall need to refer below to decomposition of inductive data, i.e. inverses of constructors. While in general structure functions need not be injective, we can still have programs for quasi-inverses, which we define as follows\(^2\).

For a relation \( R \subseteq A \times B \) and \( a \in A \), define \( R' a \triangleq \{ b \in B \mid a R b \} \). We call a partial-function \( f : A \to B \) a choice-function for \( R \) if \( f \subseteq R \) and \( f(a) \) is defined whenever \( R' a \neq \emptyset \). A partial-function \( g : A \to B \) is a quasi-inverse of \( f \) if it is a choice function for the relation \( f^{-1} \). When \( f \) is \( r \)-ary, i.e. \( A = \times_{i=1}^{r} A_i \), \( g \) can be construed as an \( r \)-tuple of functions \( \langle g_1 \ldots g_r \rangle \). We write \( f^{-i} \) for \( g_i \). If \( f \) is injective then its unique quasi-inverse is its inverse \( f^{-1} \).

**Theorem 3** For each vocabulary \( W \) there is a program that for each \( W \)-structure \( S \) as input yields an expansion of \( S \) with quasi-inverses for each non-nullary \( W \)-function.

**Proof.** The proof of Theorem\(^2\) can be easily modified to generate quasi-inverses for each structure function, either in tandem with the construction of an enumerator, or independently. Namely, whenever the program in the proof of Theorem\(^2\) adds a node \( x = g(x_1 \ldots x_k) \) to \( a \) and \( E \) (where \( k = r(g) \)), our enhanced program defines \( g^{-i}(x) = x_i \) \((i = 1 \ldots k)\).

Note that, contrary to enumerators, quasi-inverses are easy to maintain through structure revisions. An extension of a function \( f \) can be augmented with appropriate extensions of \( f \)’s quasi-inverses, and a contraction of \( f \) with appropriate contractions of those quasi-inverses.

6 A generic delineation of primitive recursion

6.1 Recurrence over inductive data

Recall that the schema of recurrence over \( \mathbb{N} \) consists of the two equations

\[
\begin{align*}
f(0, \vec{x}) &= g_0(\vec{x}) \\
f(sn, \vec{x}) &= g_s(n, \vec{x}, f(n, \vec{x}))
\end{align*}
\]

(1)

More generally, given a free algebra \( \mathbb{A} = \mathbb{A}(C) \) generated from a finite set \( C \) of constructors, recurrence over \( \mathbb{A} \) has one equation per constructor:

\[
\begin{align*}
f(c(z_1, \ldots, z_k), \vec{x}) &= g_c(\vec{x}, z_1, \ldots, z_k) \\
where \quad y_j &= f(z_j, \vec{x}) \quad (j = 1 \ldots k, \ k = r(c))
\end{align*}
\]

(2)

\(^2\)A common equivalent definition is that \( f \circ g \circ f = f \).

\(^3\)We use infix notation for binary relations.
The set \( \text{PR}(A) \) of \textit{primitive recursive functions} over \( A \) is generated from the constructors of \( A \) (for example zero and successor for \( \mathbb{N} \)), by recurrence over \( A \) and explicit definitions. Using standard codings, it is easy to see that any non-trivial (i.e. infinite) algebra can be embedded in any other. Consequently, the classes \( \text{PR}(A) \) are essentially the same for all non-trivial \( A \), and we refer to them jointly as \( \text{PR} \). A natural question is whether there is a generic approach, unrelated to free algebras, that delineates the class \( \text{PR} \).

The recurrence schema (for \( \mathbb{N} \)) was seemingly initiated by the interest of Dedekind in formalizing arithmetic, and articulated by Skolem [24]. It was studied extensively (e.g. [21]), and generalized to all admissible structures [3]. Our aim here is to characterize the underlying notion of primitive recursion generically, via uninterpreted programs. We delineate a natural variant of \( ST, STV \) which is sound and complete for \( \text{PR} \). That is, on the one hand every \( STV \) program terminates in time primitive-recursive in the size of the input structure. On the other hand, \( STV \) captures \( \text{PR} \) in two ways: any instance of recurrence over a free algebra can be implemented directly by an \( STV \) program; and every \( ST \) program that runs in \( \text{PR} \) resources in the size of the input structure can be transformed into an extensionally equivalent \( STV \) program.

Recurrence is guaranteed to terminate because it consumes its recurrence argument. The very same consumption phenomenon is used, in a broad and generic sense, in the Dijkstra-Hoare program verification style, in the notion of a \textit{variant} [12, 8, 26]. Our core idea is to use a generic notion of program variants in lieu of recurrence arguments taken from free algebras.

### 6.2 Resource measures

We first identify appropriate notions of size measures for structures. We focus on accessible structures, since non-accessible nodes remain non-accessible through revisions and are inert through the execution of any program. Consequently they do not affect the time or space consumption of computations.

We take the size \( \#S \) of an accessible \( V \)-structure \( S \) to be the count of tuples of nodes that occur in the structure’s relations and (graphs of) functions. Note that this is in tune with our use of variants, which are consumed not by the elimination of nodes, but by the contraction of functions and relations. Moreover, we believe that the size of functions and relations is an appropriate measure in general, since they convey more accurately than the number of nodes the information contents of a structure.

Note that for word-structures, i.e. \( T(w) \) for \( w \in \Sigma^* \) (\( \Sigma \) an alphabet) the total size of the structure’s functions is precisely the length or \( w \), so in this important case our measure is identical to the count of nodes.

Suppose \( V \) is a vocabulary with all identifiers of arity \( \leq r \). If \( S \) is a \( V \)-structure of size \( k \), then the number of accessible nodes is \( O(k^r) \). Conversely, if the number of accessible nodes is \( a \), then the size is \( O(a^{r+1}) \). It follows that the distinction between our measure and node-count does not matter for super-polynomial complexity.

We say that a program \( P \) runs \textit{within time} \( t : \mathbb{N} \to \mathbb{N} \) if for all structures \( S \), the number of configurations in a complete trace of \( P \) on input \( S \) is \( \leq t(\#S) \); it runs
within space $s : \mathbb{N} \rightarrow \mathbb{N}$ if for all $S$, all configurations in an execution trace of $P$ on input $S$ are of size $\leq s(\#S)$.

We say that $P$ runs in PR if it runs within time $t$, for some PR function $t$, or — equivalently — within space $s$, for some PR function $s$.

### 6.3 PR-soundness of STV-programs

We assign to each STV-program $P$ a primitive-recursive function $b_P : \mathbb{N} \rightarrow \mathbb{N}$ as follows. The aim is to satisfy Theorem 4 below.

- If $P$ is an extension or an inception revision, then $b_P(n) = 1$; if $P$ is any other revision then $b_P(n) = 0$.
- If $P$ is $S;Q$ then $b_P(n) = b_Q(b_S(n))$.
- If $P$ is $\text{if}[G][S]\{Q\}$ then $b_P(n) = \max[b_S(n), b_Q(n)]$.
- If $P$ is $\text{do}[G][T]\{Q\}$ then $b_P(n) = b^{[n]}_Q(n)$.

**Theorem 4** If $P$ is an STV-program computing a mapping $\Phi_P$ between structures, and $S$ is a structure, then

$$\#\Phi_P(S) \leq b_P(\#S)$$

**Proof.** Structural induction on $P$.

- If $P$ is a revision, then the claim is immediate by the definition of $b_P$.
- If $P$ is $S;Q$ then

$$\#\Phi_P(S) = \#\Phi_Q(\Phi_S(S)) \leq b_Q(\#\Phi_S(S)) \quad (\text{IH for } Q) \leq b_Q(b_S(\#S)) \quad (\text{IH for } S, b_Q \text{ is non-decreasing}) = b_P(\#S)$$

- The case for $P$ of the form $\text{if}[G][S]\{Q\}$ is immediate.
- If $P$ is $\text{do}[G][T]\{Q\}$ then $\Phi_P(S)$ is $\Phi_Q^{[m]}(S)$ for some $m$. By the definition of variants, and the semantics of looping, $m$ is bounded by the size of $T$, which is bounded by the size of $S$. So

$$\#\Phi_P(S) = \#\Phi_Q^{[m]}(S) \quad \text{for some } m \leq \#S \leq b^{[m]}_Q(\#S) \quad \text{IH, } b_Q \text{ is non-decreasing} \leq b^{[n]}_Q(\#S) \quad \text{where } n = \#S \quad \text{since } b_Q \text{ is terminating} = b_P(\#S)$$

From Theorem 4 we obtain the soundness of STV-programs for PR:

**Theorem 5** Every STV-program runs in PR space, and therefore in PR time.
6.4 Completeness of STV-programs for PR

We finally turn to the completeness of STV for PR. The easiest approach would be to prove that STV is complete for PR(N), and then invoke the coding of primitive recurrence over any free algebra in PR(N). This, however, would fail to establish a direct representation of generic recurrence by STV-programs, which is one of the raisons d'être of STV. We therefore follow a more general approach.

Lemma 6 For each free algebra \( A(C) \), each instance of recurrence over \( A \) as in (2) above (with \( \vec{x} = x_1, \ldots, x_m \)), the following holds. Given STV-programs for the functions \( g_c \), there is an STV-program \( P \) that maps the structure \( T(w) \oplus T(x_1) \oplus \cdots \oplus T(x_m) \) to \( T(f(w, x_1, \ldots, x_m)) \).

Proof. The program \( P \) gradually constructs a pointer \( r \) that maps each node \( \nu \) of \( T(w) \) to the root of the structure \( T(f(u, \vec{x})) \), where \( u \) is the sub-term of \( w \) denoting \( \nu \) (it is uniquely defined since \( A \) is a free algebra). \( P \) starts by constructing a monotone enumerator for the structure \( T(w) \), as well as inverses for all constructors, by Theorems 2 and 3. (Since \( w \) is a term of a free algebra, a quasi-inverse of a constructor is an inverse). The main loop of \( P \) then scans that enumerator, using a token; reaching the end of the enumerator is the guard, and the enumerator itself is the variant.

For each node \( \nu \) encountered on the enumerator, \( P \) first identifies the constructor \( c \) defining \( \nu \), which is unique since \( w \in A(C) \). This identification is possible by testing for equality with the tokens, and — that failing — testing, for non-nullary constructor \( f \), the definability of the first inverse \( f^{-1} \). Since the enumerator is monotone, \( r \) is already defined for the values \( z_1 = f^{-1}(\nu), \ldots, z_k = f^{-k}(\nu) \) (\( k = r(c) \)). \( P \) can thus invoke the program \( P_c \) for the function \( g_c \), adapted to the disjoint union of

1. The structures \( T(x_j) \);
2. The structures spanned by the \( z_j \)'s, i.e. for each \( j \) the substructure of the input consisting of the sub-terms of \( z_j \);
3. The structures \( r(z_j) \) already obtained.

\( r(\nu) \) is then set to be the root of the result.

The program’s final output is then \( r \bullet \); that is the structure yielded for the program’s given recurrence argument. □

Theorem 7 For each free algebra \( A \), the collection of STV-programs is complete for PR(\( A \)).

Proof. The proof proceeds by induction on the PR definition of \( f \). The cases where \( f \) is a constructor are trivial. For explicit definitions, and more particularly composition, we need to address the need of duplicating substructures, for which we have programs, as explained in §5.1.

Finally, the case of recurrence is treated in Lemma 6. □

Theorem 7 establishes a simple and direct mapping of PR function definitions, over any free algebra, to STV programs. Another angle on the completeness of STV for PR refers directly to ST-programs (i.e. to programs without variants):
Corollary 8 For every ST-program $P$ running in PR resources, and defining a structure transformation $\Phi$, there is an STV-program $Q$ that computes $\Phi$.

Proof. Recall from §6.2 that the size of a structure, measured in size of functions and relations, is polynomial in the number of nodes. It follows that $P$ runs in time PR in the input’s number of nodes.

Suppose now that $P$’s input is a $V$-structure, and that $P$ operates within time $f(n)$, where $f$ is a PR function over $\mathbb{N}$.

Let $Q$ be the composition of the following STV-programs:

1. A program that expands each $V$-structure $S$ with an enumerator $(a, e)$, as in Theorem 2. The constructed enumerator $e$ is a list without repetition of the nodes of $S$. I.e., $e$ is essentially $T(n)$, where $n$ is the number of nodes in $S$.

2. A program that takes as input the structure $T(n)$ constructed in (1), and outputs $T(f(n))$ with, say, $t$ as the output’s successor function. Such a program exists by Theorem 7 applied to the free algebra $\mathbb{N}$.

3. The given ST-program $P$, with each loop assigned as variant a copy of $t$, and each loop-body preceded by a function-contraction of $t$.

Then $Q$ computes the same structure-transformation as $P$. $\square$

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