Almost resolvable $k$-cycle systems with $k \equiv 2 \pmod{4}$

L. Wang$^{1,2}$ and H. Cao$^1$ *

$^1$ Institute of Mathematics, Nanjing Normal University, Nanjing 210023, China
$^2$ Department of Literature and Science, Suqian College, Suqian 223800, China

Abstract

In this paper, we show that if $k \geq 6$ and $k \equiv 2 \pmod{4}$, then there exists an almost resolvable $k$-cycle system of order $2kt + 1$ for all $t \geq 1$ except possibly for $t = 2$ and $k \geq 14$. Thus we give a partial solution to an open problem posed by Lindner, Meszka, and Rosa (J. Combin. Des., vol. 17, pp.404-410, 2009).

Key words: cycle system; almost resolvable cycle system

1 Introduction

In this paper, we use $V(H)$ and $E(H)$ to denote the vertex-set and the edge-set of a graph $H$, respectively. We denote the cycle of length $k$ by $C_k$ and the complete graph on $v$ vertices by $K_v$. A factor of a graph $H$ is a spanning subgraph whose vertex-set coincides with $V(H)$. If its connected components are isomorphic to $G$, we call it a $G$-factor. A $G$-factorization of $H$ is a set of edge-disjoint $G$-factors of $H$ whose edge-sets partition $E(H)$. An $r$-regular factor is called an $r$-factor. Also, a 2-factorization of a graph $H$ is a partition of $E(H)$ into 2-factors.

A $k$-cycle system of order $v$ is a collection of $k$-cycles whose edges partition $E(K_v)$. A $k$-cycle system of order $v$ exists if and only if $3 \leq k \leq v$, $v \equiv 1 \pmod{2}$ and $v(v-1) \equiv 0 \pmod{2k}$ [2, 6, 17, 22]. A $k$-cycle system of order $v$ is resolvable if it has a $C_k$-factorization. A resolvable $k$-cycle system of order $v$ exists if and only if $3 \leq k \leq v$, $v$ and $k$ are odd, and $v \equiv 0 \pmod{k}$, see [3, 4]. If $v \equiv 1 \pmod{2k}$, then a $k$-cycle system exists, but it is not resolvable. In this case, Vanstone et al. [19] started the research of the existence of almost resolvable $k$-cycle systems.

In a $k$-cycle system of order $v$, a collection of $(v-1)/k$ disjoint $k$-cycles is called an almost parallel class. In a $k$-cycle system of order $v \equiv 1 \pmod{2k}$, the maximum possible number

*Research supported by the National Natural Science Foundation of China under Grant 11571179, and the Priority Academic Program Development of Jiangsu Higher Education Institutions. E-mail: caohaitao@njnu.edu.cn
of almost parallel classes is \((v - 1)/2\), in which case a half-parallel class containing \((v - 1)/2k\) disjoint \(k\)-cycles is left over. A \(k\)-cycle system of order \(v\) whose cycle set can be partitioned into \((v - 1)/2\) almost parallel classes and a half-parallel class is called an almost resolvable \(k\)-cycle system, denoted by \(k\)-ARCS\((v)\). Lindner, Meszka, and Rosa [12] (also, Adams et al. [1]) presented the following open problem “The outstanding problem remains the construction of almost resolvable \(2k\)-cycle systems of order \(4k + 1\), since this will determine the spectrum for almost resolvable \(2k\)-cycle systems with the one possible exception of orders \(8k + 1\).” Since then, many authors have contributed to proving the following known conclusions.

**Theorem 1.1.** ([1, 5, 9–12, 14, 19–21]) Let \(n \equiv 1 \pmod{2^k}\). There exists a \(k\)-ARCS\((n)\) for any odd \(k \geq 3\) and any even \(k \in \{4, 6, 8, 10, 14\}\), except for \((k, n) \in \{(3, 7), (3, 13)\}\) and except possibly for \((k, n) \in \{(8, 33), (14, 57)\}\).

In this paper, we construct almost resolvable cycle systems of order \(2^{kt} + 1\) for any \(k \geq 18\) and \(k \equiv 2 \pmod{4}\) by using algebraic methods. Thus we have partially solved the above open problem given by Lindner et al. in [1, 12]. Combining the known results in Theorem 1.1, we will prove the following main result.

**Theorem 1.2.** If \(k \geq 6\) and \(k \equiv 2 \pmod{4}\), then there exists a \(k\)-ARCS\((2^{kt} + 1)\) for all \(t \geq 1\) except possibly for \(t = 2\) and \(k \geq 14\).

## 2 Preliminary

In this section, we present some preliminary notation and definitions, and provide lemma for the construction of a \(k\)-ARCS\((2^{kt} + 1)\) for \(k \equiv 2 \pmod{4}\). We point out that similar methods have been used for many years (see [8, 16, 18, 20, 21]).

Suppose \(\Gamma\) is an additive group and \(I = \{\infty_1, \infty_2, \ldots, \infty_f\}\) is a set which is disjoint with \(\Gamma\). We will consider an action of \(\Gamma\) on \(\Gamma \cup I\) which coincides with the right regular action on the elements of \(\Gamma\), and the action of \(\Gamma\) on \(I\) will coincide with the identity map. In other words, for any \(\gamma \in \Gamma\), we have that \(x + \gamma\) is the image under \(\gamma\) of any \(x \in \Gamma\), and \(x + \gamma = x\) holds for any \(x \in I\). Given a graph \(H\) with vertices in \(\Gamma \cup I\), the translate of \(H\) by an element \(\gamma\) of \(\Gamma\) is the graph \(H + \gamma\) obtained from \(H\) by replacing each vertex \(x \in V(H)\) with the vertex \(x + \gamma\). The development of \(H\) under a subgroup \(\Sigma\) of \(\Gamma\) is the collection \(\text{dev}_\Sigma(H) = \{H + x \mid x \in \Sigma\}\) of all translates of \(H\) by an element of \(\Sigma\).

For our constructions, we set \(\Gamma = \mathbb{Z}_l \times \mathbb{Z}_4\). Given a graph \(H\) with vertices in \(\Gamma\) and any pair \((r, s) \in \mathbb{Z}_4 \times \mathbb{Z}_4\), we set \(\Delta_{(r,s)}H = \{x - y \mid \{(x, r), (y, s)\} \in E(H)\}\). Finally, given a list \(\mathcal{H} = \{H_1, H_2, \ldots, H_t\}\) of graphs, we denote by \(\Delta_{(r,s)}H = \bigcup_{i=1}^{t} \Delta_{(r,s)}H_i\) the multiset union of the \(\Delta_{(r,s)}H_i\)’s.
Lemma 2.1. Let \( v = 2k + 1, k \equiv 2 \mod 4 \), and \( C = \{ F_1, F_2 \} \) where each \( F_i \) \((i = 1, 2)\) is a vertex-disjoint union of two cycles of length \( k \) satisfying the following conditions:

(i) \( V(F_i) = (\mathbb{Z}_k \times \mathbb{Z}_4) \setminus \{\infty\}\) \(\setminus\) \(\{(a_i, b_i)\}\) for some \((a_i, b_i)\) \(\in \mathbb{Z}_k \times \mathbb{Z}_4\), \(i = 1, 2\);

(ii) \( \infty \) has a neighbor in \( \mathbb{Z}_k \times \mathbb{Z}_4 \) for each \( j \) \(\in\) \(\mathbb{Z}_4\);

(iii) \( \Delta_{(p, p)} C = \mathbb{Z}_k \setminus \{0\} \) for each \( p \) \(\in\) \(\mathbb{Z}_4\);

(iv) \( \Delta_{(0,2)} C = \Delta_{(2,0)} C = \mathbb{Z}_k \setminus \{\pm d\} \), where \( d \) satisfies \( (d, \frac{k}{k}) = 1 \);

(v) \( \Delta_{(r,s)} C = \mathbb{Z}_k \) for each pair \((r, s)\) \(\in\) \(\mathbb{Z}_4 \times \mathbb{Z}_4\) satisfying \( r \neq s \) and \((r, s) \notin \{(0, 2), (2, 0)\}\).

Then, there exists a \( k \)-ARCS\((v)\).

Proof: Let \( V(K_v) = (\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\} \). Note that \( 0, d, 2d, \ldots, (\frac{k}{k} - 1)d \) are \( \frac{k}{k} \) distinct elements since \( (d, \frac{k}{k}) = 1 \). Then we have the required half parallel class which is formed by the cycle \((0, 0), (d, 2), (2d, 0), (3d, 2), \ldots, ((\frac{k}{k} - 3)d, 0), ((\frac{k}{k} - 2)d, 2), ((\frac{k}{k} - 1)d, 0), (0, 2), (d, 0), (2d, 2), (3d, 0), \ldots, ((\frac{k}{k} - 3)d, 2), ((\frac{k}{k} - 2)d, 0), ((\frac{k}{k} - 1)d, 2)) \). By (i), we know that \( F_i \) is an almost parallel class. All the required \( k \) almost parallel classes are \( F_i + (l, 0), i = 1, 2, l \in \mathbb{Z}_k \).

Now we show that the half parallel class and the \( k \) almost parallel classes form a \( k \)-ARCS\((v)\). Let \( F' \) be a graph with the edge-set \( \{(a, 0), (a + d, 2)\}, \{(a, 0), (a - d, 2)\} | a \in \mathbb{Z}_k \} \) and \( \Sigma := \mathbb{Z}_k \times \{0\} \). Let \( F = dev_{\Sigma}(C) \cup F' \). The total number of edges – counted with their respective multiplicities – covered by the almost parallel classes and half parallel class of \( F \) is \( k(2k + 1) \), that is exactly the size of \( E(K_v) \). Therefore, we only need to prove that every pair of vertices lies in a suitable translate of \( C \) or in \( F' \). By (ii), an edge \( \{(z, j), \infty\} \) of \( K_v \) must appear in a cycle of \( dev_{\Sigma}(C) \).

Now consider an edge \( \{(z, j), (z', j')\} \) of \( K_v \) whose vertices both belong to \( \mathbb{Z}_k \times \mathbb{Z}_4 \). If \((j, j') \in \{(0, 2), (2, 0)\} \) and \( z - z' \in \{\pm d\} \), then this edge belongs to \( F' \). In all other cases there is, by (iii)-(v), an edge of some \( F_i \) of the form \( \{(w, j), (w', j')\} \) such that \( w - w' = z - z' \). It then follows that \( F_i + (-w' + z', 0) \) is an almost parallel class of \( dev_{\Sigma}(F_i) \) containing the edge \( \{(z, j), (z', j')\} \) and the proof is complete.

3 Main result

We first explain a notion which will be used in the proof of our construction. If a cycle \( C \) is the concatenation of the paths \( T_0, T_1, \ldots, T_m \) each of which can be obtained from a general formula, then we define \( C = (T_0, \ldots, T_i, \ldots, T_m) \), \( 0 \leq i \leq m \). For example, \( T = ((-2, 0), (2, 3), \ldots, ((-2 + i), 0), (2 + i, 3), \ldots, (-n, 0), (n, 3)) \), \( 0 \leq i \leq n - 2 \), means that \( T \) can be viewed as the concatenation of the paths \( T_0, T_1, \ldots, T_{n-2} \), where the general formula is \( T_i = ((-2 + i), 0), (2 + i, 3) \), \( 0 \leq i \leq n - 2 \).

Lemma 3.1. For any \( k \geq 18 \) and \( k \equiv 2 \mod 4 \), there exists a \( k \)-ARCS\((2k + 1)\).
Proof: Let \( v = 2k + 1 \), and \( k = 4n + 2 \), \( n \geq 4 \). We use Lemma 2.4 to construct a \( k\)-ARCS\((v)\) with \( V(K_v) = (\mathbb{Z}_k \times \mathbb{Z}_4) \cup \{\infty\} \). Three of the required parameters in (i) and (iv) of Lemma 2.4 are \((a_1, b_1) = (n, 0)\) and \(d = 2\). The other required parameters \(a_2, b_2\), and four cycles \(\{C_1, C_2\}\) in \(F_1\) and \(\{C_3, C_4\}\) in \(F_2\) for each \(n\) are listed as below.

The cycle \(C_1\) is the concatenation of the paths \(T_1, T_2, T_3,\) and \(T_4\), where
\[
T_1 = (0, 3), (0, 0), (1, 3));
T_2 = ((-2, 0), (2, 3), \ldots, ((2 + i) 0), (2 + i, 3), \ldots, (n, 0), (n, 3)), 0 \leq i \leq n - 2;
T_3 = ((-n, 1), (n, 2), \ldots, (n - i, 1), (n - i, 2), \ldots, (2, 1), (2, 2)), 0 \leq i \leq n - 2;
T_4 = ((-1, 1), (0, 2), (0, 1)).
\]
The cycle \(C_2\) is the concatenation of the paths \(\infty, T_1, T_2, T_3,\) and \(T_4\), where
\[
T_1 = ((1, 1), (-1, 3), \ldots, (1 + i, 3), \ldots, (n, 1), (-n, 3)), 0 \leq i \leq n - 1;
T_2 = ((n - 1, 0), (-1, 0), (-2, 2));
T_3 = ((1, 0), (-3, 2), \ldots, (1 + i, 0), (3 + i, 2), \ldots, (n - 2, 0), (n, 2)), 0 \leq i \leq n - 3;
T_4 = ((1, 2), (-1, 2)).
\]
To construct \(C_3\) and \(C_4\), we start with \(k = 18\). Here, \((a_2, b_2) = (3, 2)\).
\[
C_3 = ((0, 0), (0, 1), (1, 0), (1, 2), (2, 0), (2, 1), (2, 2), (3, 0), (3, 1), (3, 2), (3, 3)), 0 \leq i \leq n - 2;
\]
\[
C_4 = ((\infty, (-2, 0), (-3, 0), (-1, 0), (2, 0), (-4, 1), (4, 1), (-1, 1), (3, 1), (-3, 1), (4, 2), (1, 3), (3, 3)), 0 \leq i \leq n - 2;
\]
For \(k > 18\), the cycle \(C_4\) is the concatenation of the paths \(T_1, T_2, T_3,\) and \(T_4\), where
\[
T_1 = ((0, 0), (0, 1), (1, 0));
T_2 = ((-2, 1), (2, 0), \ldots, ((2 + i, 1), (2 + i, 0), \ldots, (-n, 1), (n, 0)), 0 \leq i \leq n - 2;
T_3 = ((-n, 2), (n, 3), \ldots, ((n - i, 2), (n - i, 3), \ldots, (2, 2), (3, 3)), 0 \leq i \leq n - 2;
T_4 = ((-1, 2), (0, 3), (0, 2)).
\]
Next, we consider the last cycle \(C_4\). We distinguish the following two cases.

**Case 1:** \(k \equiv 2 \pmod{8}\) and \(k \geq 26\). Here, \((a_2, b_2) = (\frac{n + 2}{2}, 2)\).

The cycle \(C_4\) is the concatenation of the paths \(\infty, T_1, T_2, \ldots, T_8\), where
\[
T_1 = ((-3, 0), (-\frac{n + 2}{2}, 0), \ldots, ((1 - i, 0), (\frac{n + 2}{2} + i, 0), \ldots, (1, 0), (-n, 0)), 0 \leq i \leq \frac{n + 2}{2}.
\]

For the paths \(T_2, T_3, \ldots, T_8\), we distinguish the following three subcases.

**Case 1.1:** \(k \equiv 2 \pmod{24}\) and \(k \geq 26\).
\[
T_2 = ((n - 1, 1), (2, 1), (n, 1), (-1, 1));
T_3 = ((n - 2, 1), (3, 1), (n - 4, 1), (5, 1), (n - 3, 1), (1, 1), \ldots, ((n - 2 - 3i, 1), (3 + 3i, 1), (n - 4 - 3i, 1), (5 + 3i, 1), (n - 3 - 3i, 1), (1 + 3i, 1), \ldots, ((\frac{n + 2}{2} - 1), 1), (\frac{n + 2}{2} + 1), ((\frac{n + 2}{2} + i, 1), (\frac{n + 2}{2} + i, 1), (\frac{n + 2}{2} + 1), (\frac{n + 10}{6} + 1)), 0 \leq i \leq \frac{n + 2}{6};
T_3 = ((n + 3, 1), (\frac{n + 2}{6}, 1), (\frac{n + 2}{6} - 1, 1), (\frac{n + 2}{6} + 2));
T_4 = ((\frac{n + 2}{6} + i, 2), (\frac{n + 2}{6} + i, 2), \ldots, ((\frac{n + 2}{6} - 1, 2), (\frac{n + 2}{6} + i, 2), \ldots, (1, 2), (n, 2)), 0 \leq i \leq \frac{n + 2}{6};
T_6 = ((-n - 1, 3), (-2, 3), (-n - 3, 1), (1, 3));
T_7 = ((-n - 2, 3), (-3, 3), (-n - 4, 3), (-5, 3), (-n - 3, 3), (-1, 3), \ldots, ((n - 2 - 3i, 3), (3 + 3i, 3), (3 + 3i, 3)), 0 \leq i \leq \frac{n + 2}{6};
\]
\[\begin{align*}
(−n−4−3t), 3, (−5+3t), 3, (−n−3−3t), 3, (−1+3t), 3, 
(−n−8−3t), 3, (−n−6−3t), 3, (−n+4−3t), 3, 
(−n+2−3t), 3, (−n+8−3t), 3, (−n+3−3t), 3), \quad 0 ≤ i ≤ \frac{n+1}{2}.
\end{align*}\]

**Case 1.2:** \(k \equiv 10 \pmod{24}\) and \(k ≥ 34\).

\(T_2 = ((n−1, 1), (1, 1), (n−2, 1), (−1, 1));\)

\(T_3 = ((n, 1), (4, 1), (n−5, 1), (3, 1), (n−4, 1), (2, 1), \ldots, (n−3, 1), (4+3t, 1), (n−5−3t, 1), (3+3t, 1), \)
\(n−4−3t, 1), (2+3t, 1), \ldots, (n−14−3t, 1), (n−6−3t, 1), (n+2−3t, 1), (n+8−3t, 1), (n+16−3t, 1), \)
\(0 ≤ i ≤ \frac{n+14−3t}{6};\)

\(T_4 = ((\frac{n+10−3t}{2}, 1), (\frac{n+6−3t}{2}, 1), (\frac{n+4−3t}{2}, 1), (\frac{n+2−3t}{2}, 1), (\frac{n−4−3t}{2}, 1), (\frac{n−6−3t}{2}, 1));\)

\(T_5 = ((\frac{n+4−3t}{2}, 2), (\frac{n+8−3t}{2}, 2), \ldots, (\frac{n+2−3t}{2}−i, 2), (\frac{n+8−3t}{2}+i, 2), \ldots, (1, 2), (n, 2), 0 ≤ i ≤ \frac{n−10−3t}{2};\)

\(T_6 = ((−n−1, 3), (−1, 3), (−n−2, 3), (1, 3));\)

\(T_7 = ((−n−3, 3), (−4, 3), (−n−5, 3), (−3, 3), (−n−4, 3), (−2, 3), \ldots, (−n−3, 3), (−4+3t, 3), \)
\(−(n−5−3t), 3), (−(3+3t), 3), (−(n−4−3t), 3), (−(2+3t), 3), \ldots, (−\frac{n+14−3t}{3}, (−\frac{n+8−3t}{3}, (−\frac{n+4−3t}{3}, \)
\(−\frac{n+2−3t}{3}, (−\frac{n+8−3t}{3}, 3), 0 ≤ i ≤ \frac{n−10−3t}{6};\)

\(T_8 = ((−\frac{n+8−3t}{2}, 3), (−\frac{n+2−3t}{2}, 3), (−\frac{n+8−3t}{2}, 3), (−\frac{n+4−3t}{2}, 3), (−\frac{n+2−3t}{2}, 3)).\)

**Case 3:** \(k \equiv 18 \pmod{24}\) and \(k ≥ 42\).

\(T_2 = ((n−1, 1), (1, 1), (n−2, 1), (−1, 1));\)

\(T_3 = ((n, 1), (4, 1), (n−5, 1), (3, 1), (n−4, 1), (2, 1), \ldots, (n−3, 1), (4+3t, 1), (n−5−3t, 1), (3+3t, 1), \)
\(n−4−3t, 1), (2+3t, 1), \ldots, (n−14−3t, 1), (n−6−3t, 1), (n+2−3t, 1), (n+8−3t, 1), (n+16−3t, 1), \)
\(0 ≤ i ≤ \frac{n+14−3t}{6};\)

\(T_4 = ((\frac{n+10−3t}{2}, 1), (\frac{n+6−3t}{2}, 1), (\frac{n+4−3t}{2}, 1), (\frac{n+2−3t}{2}, 1), (\frac{n−4−3t}{2}, 1), (\frac{n−6−3t}{2}, 1));\)

\(T_5 = ((\frac{n+4−3t}{2}, 2), (\frac{n+8−3t}{2}, 2), \ldots, (\frac{n+2−3t}{2}−i, 2), (\frac{n+8−3t}{2}+i, 2), \ldots, (1, 2), (n, 2), 0 ≤ i ≤ \frac{n−10−3t}{2};\)

\(T_6 = ((−n−1, 3), (−1, 3), (−n−2, 3), (1, 3));\)

\(T_7 = ((−n−3, 3), (−4, 3), (−n−5, 3), (−3, 3), (−n−4, 3), (−2, 3), \ldots, (−n−3, 3), (−4+3t, 3), \)
\(−(n−5−3t), 3), (−(3+3t), 3), (−(n−4−3t), 3), (−(2+3t), 3), \ldots, (−\frac{n+14−3t}{3}, (−\frac{n+8−3t}{3}, (−\frac{n+4−3t}{3}, \)
\(−\frac{n+2−3t}{3}, (−\frac{n+8−3t}{3}, 3), 0 ≤ i ≤ \frac{n−10−3t}{6};\)

\(T_8 = ((−\frac{n+10−3t}{2}, 3), (−\frac{n+6−3t}{2}, 3), (−\frac{n+4−3t}{2}, 3), (−\frac{n+2−3t}{2}, 3), (−\frac{n+8−3t}{2}, 3)).\)

**Case 2:** \(k \equiv 6 \pmod{8}\) and \(k ≥ 22\).

The cycle \(C_4\) is the concatenation of the paths \(∞, (−\frac{n+1}{2}, 0), T_1, T_2, \ldots, T_8, \) where

\(T_1 = ((−\frac{n+1}{2}, 0), (−\frac{n+1}{2}, 0), \ldots, (−\frac{n+1}{2}−i, 0), (−\frac{n+1}{2}+i, 0), \ldots, (−1, 0), (−n, 0), 0 ≤ i ≤ \frac{n+1}{2};\)

\(T_2 = ((n−1, 1), (−1, 1));\)

For the paths \(T_3, T_4, \ldots, T_8, \) we distinguish the following three subcases.

**Case 2.1:** \(k \equiv 6 \pmod{24}\) and \(k ≥ 30\).

\(T_3 = ((n−2, 1), (2, 1), (n, 1), (3, 1), (n−4, 1), (1, 1), \ldots, (n−2−3t, 1), (2+3t, 1), (n−3, 1), (3+3t, 1), \)
\(n−4−3t, 1), (1+3t, 1), \ldots, (n−2−3t, 1), (2+3t, 1), (n−3, 1), (3+3t, 1), \)
\(n−4−3t, 1), (1+3t, 1), \ldots, (n−2−3t, 1), (2+3t, 1), (n−3, 1), (3+3t, 1), \)
\(0 ≤ i ≤ \frac{n+15}{2};\)

\(T_4 = ((\frac{n+4−3t}{2}, 1), (\frac{n+8−3t}{2}, 1), (\frac{n+12−3t}{2}, 1), (\frac{n+16−3t}{2}, 1), (\frac{n+20−3t}{2}, 1), (\frac{n+24−3t}{2}, 1), \)
\(0 ≤ i ≤ \frac{n+15}{2};\)

\(T_5 = ((\frac{n+4−3t}{2}, 2), (\frac{n+8−3t}{2}, 2), \ldots, (\frac{n+2−3t}{2}−i, 2), (\frac{n+8−3t}{2}+i, 2), \ldots, (1, 2), (n, 2), 0 ≤ i ≤ \frac{n+5}{2};\)

\(T_6 = ((−n−1, 3), (1, 3));\)
$T_7 = ((-n-2),3,(-2,3),(-n,3),(-3,3),(-n-4,3),(-1,3),\ldots,(-n-2-3i),3,(-2+3i),3,$
$(-n-3i),3,(-3+3i),3,(-n-4-3i),3,(-1+3i),3,\ldots,(-n-2-3i),3,(-2+3i),3,$
$(-n-3i),3,(-3+3i),3,(-n-4-3i),3,(-1+3i),3,\ldots,(-n-2-3i),3,(-2+3i),3,(-n-3i),3),
(-n-3i),3,(-3+3i),3,(-n-4-3i),3,(-1+3i),3,\ldots,(-n-2-3i),3,(-2+3i),3,(-n-3i),3),
0 \leq i \leq n-11/6$;
$T_8 = ((-n-2),3,(-n+5),3,(-n-5),3,(-n-5),3,(-n-7),3,(-n-3),3))$.

**Case 2.2:** $k \equiv 14 \pmod{24}$ and $k \geq 38$.

$T_3 = ((n-2),1,2,1,1,1,\ldots,(-n-2-3i),1,2+3i,1,(-n-3i),1,3+3i,1,$
$(-n-4-3i),1,3+3i,1,\ldots,(-n-2-3i),1,2+3i,1,(-n-3i),1,3+3i,1,(-n-4-3i),1,3+3i,1),
0 \leq i \leq n-15$;
$T_4 = ((n+3),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1),
(-n+3),1,(-n-7),1,(-n-7),1,\ldots,(-n-2),3,(-n-2),3,\ldots,(-n-2),3,\ldots,(-n-2),3),
0 \leq i \leq n-15$;
$T_5 = ((-n-3),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3),
(-n-7),3,(-n-7),3,(-n-7),3,\ldots,(-n-2),3,(-n-2),3,\ldots,(-n-2),3,\ldots,(-n-2),3)\enspace.$

**Case 2.3:** $k \equiv 22 \pmod{24}$ and $k \geq 22$.

$T_3 = ((n-2),1,2,1,1,1,\ldots,(-n-2-3i),1,2+3i,1,(-n-3i),1,3+3i,1,$
$(-n-4-3i),1,3+3i,1,\ldots,(-n-2-3i),1,2+3i,1,(-n-3i),1,3+3i,1,(-n-4-3i),1,3+3i,1),
0 \leq i \leq n-15$;
$T_4 = ((n+3),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1,(-n-7),1),
(-n+3),1,(-n-7),1,(-n-7),1,\ldots,(-n-2),3,(-n-2),3,\ldots,(-n-2),3,\ldots,(-n-2),3),
0 \leq i \leq n-15$;
$T_5 = ((-n-3),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3,(-n-7),3),
(-n-7),3,(-n-7),3,(-n-7),3,\ldots,(-n-2),3,(-n-2),3,\ldots,(-n-2),3,\ldots,(-n-2),3)\enspace.$

Now we have constructed a $k$-ARCS($2k+1$) for any $k \geq 18$ and $k \equiv 2 \pmod{4}$. That is enough to prove our main result by using any of the two known recursive constructions which can be found in [12] and [9], respectively.

In the first recursive construction from [12], the authors start with a commutative quasigroup of order $2t$ with holes of size $2$ (see [13]), then give each vertex weight $k$ and use a $k$-ARCS($2k+1$) (exists by assumption) and a $C_k$-factorization of the complete bipartite graph $K_{k,k}$ (see [15]) as input designs to get a $k$-ARCS($2kt+1$) for all $t \geq 3$ and even integer $k \geq 8$. In the second recursive construction from [9], the authors use a $k$-ARCS($2k+1$) and a $k$-cycle frame of type $(2k)^t$ (all exist by assumption) to get a $k$-ARCS($2kt+1$). Note that Buratti et al. [7] have proved that there exists a $k$-cycle frame of type $(2k)^t$ for all $t \geq 3$ when $k$ is even, and $t \geq 4$ when $k$ is odd. So for any even integer $k \geq 4$, we may use the second recursive construction to obtain a $k$-ARCS($2kt+1$) for all $t \geq 3$ if there is a $k$-ARCS($2k+1$).
Actually, a commutative quasigroup of order $2t$ with holes of size 2 and a $C_k$-factorization of $K_{k,k}$ can lead to a $k$-cycle frame of type $(2k)^t$ when $k \geq 8$ is even and $t \geq 3$. Thus the two recursive constructions are the same when $k$ is even and $k \geq 8$ although they use different notations. We state it in the following theorem.

**Theorem 3.2.** ([9, 12]) Let $k \equiv 0 \pmod{2}$ and $k \geq 8$. If there exists a $k$-ARCS$(2k + 1)$, then there exists a $k$-ARCS$(2kt + 1)$ for all $t \geq 1$ except possibly for $t = 2$.

At last, we prove our main result.

**Proof of Theorem 1.2** Combining Theorems 1.1, 3.2, and Lemma 3.1 we can obtain the conclusion.

**Acknowledgments** We would like to thank the anonymous referees for their careful reading of the manuscript and many constructive comments and suggestions that greatly improved the readability of this paper.

**References**

[1] P. Adams, E. J. Billington, D. G. Hoffman, and C. C. Lindner, The generalized almost resolvable cycle system problem, *Combinatorica* 30 (2010), 617-625.

[2] B. Alspach and H. Gavlas, Cycle decompositions of $K_n$ and $K_n - I$, *J. Combin. Theory Ser. B* 81 (2001), 77-99.

[3] B. Alspach and R. Häggkvist, Some observations on the Oberwolfach problem, *Journal of Graph Theory* 9 (1985), 177-187.

[4] B. Alspach, P. J. Schellenberg, D. R. Stinson, and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, *J. Combin. Theory Ser. A* 52 (1989), 20-43.

[5] E. J. Billington, D. G. Hoffman, C. C. Lindner, and M. Meszka, Almost resolvable minimum coverings of complete graphs with 4-cycles, *Australas. J. Combin.* 50 (2011), 73-85.

[6] M. Buratti, Rotational $k$-cycle systems of order $v < 3k$; another proof of the existence of odd cycle systems, *J. Combin. Des.* 11 (2003), 433-441.

[7] M. Buratti, H. Cao, D. Dai, and T. Traetta, A complete solution to the existence of $(k, \lambda)$-cycle frames of type $g^2$, *J. Combin. Des.* 25 (2017), 197-230.

[8] M. Buratti and G. Rinaldi, On sharply vertex transitive 2-factorizations of the complete graph, *J. Combin. Theory Ser. A* 111 (2005), 245-256.

[9] H. Cao, M. Niu, and C. Tang, On the existence of cycle frames and almost resolvable cycle systems, *Discrete Math.* 311 (2011), 2220-2232.

[10] I. J. Dejter, C. C. Lindner, M. Meszka, and C. A. Rodger, Corrigendum/addendum to: almost resolvable 4-cycle systems, *J. Combin. Math. Combin. Comput.* 66 (2008), 297-298.

[11] I. J. Dejter, C. C. Lindner, C. A. Rodger, and M. Meszka, Almost resolvable 4-cycle systems, *J. Combin. Math. Combin. Comput.* 63 (2007), 173-181.
[12] C. C. Lindner, M. Meszka, and A. Rosa, Almost resolvable cycle systems—an analogue of Hanani triple systems, *J. Combin. Des.* **17** (2009), 404-410.

[13] C. C. Lindner and C. A. Rodger, Design Theory, CRC Press, Boca Raton, FL, 1997, 208pp.

[14] M. Niu and H. Cao, More results on cycle frames and almost resolvable cycle systems, *Discrete Math.* **312** (2012), 3392-3405.

[15] W. L. Piotrowski, The solution of the bipartite analogue of the Oberwolfach problem, *Discrete Math.* **97** (1991), 339-356.

[16] A. Rosa, On certain valuations of the vertices of a graph, in: Theory of Graphs, Internat. Sympos., Rome, 1966, Gordon and Breach/Dunod, New York/Paris, 1967, pp. 349-355.

[17] M. Šajna, Cycle decompositions: complete graphs and fixed length cycles, *J. Combin. Des.* **10** (2002), 27-78.

[18] T. Traetta, A complete solution to the two-table Oberwolfach problems, *J. Combin. Theory Ser. A* **120** (2013), 984-997.

[19] S. A. Vanstone, D. R. Stinson, P. J. Schellenberg, A. Rosa, R. Rees, C. J. Colbourn, M. W. Carter, and J. E. Carter, Hanani triple systems, *Israel J. Math.* **83** (1993), 305-319.

[20] L. Wang and H. Cao, Completing the spectrum of almost resolvable cycle systems with odd cycle length, *Discrete Math.* **341** (2018), 1479-1491.

[21] L. Wang, S. Lu, and H. Cao, Further results on almost resolvable cycle systems and the Hamilton-Waterloo problem, *J. Combin. Des.* **26** (2018), 27-47.

[22] S. Wu and M. Buratti, A complete solution to the existence problem for 1-rotational $k$-cycle systems of $K_n$, *J. Combin. Des.* **17** (2009), 283-293.