OPTIMAL SOLVABILITY FOR THE FRACTIONAL $p$-LAPLACIAN WITH DIRICHLET CONDITIONS

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Abstract. We study a nonlinear, nonlocal Dirichlet problem driven by the fractional $p$-Laplacian, involving a $(p-1)$-sublinear reaction. By means of a weak comparison principle we prove uniqueness of the solution. Also, comparing the problem to 'asymptotic' weighted eigenvalue problems for the same operator, we prove a necessary and sufficient condition for the existence of a solution. Our work extends classical results due to Brezis-Oswald [7] and Diaz-Saa [10] to the nonlinear nonlocal framework.

1. Introduction

The present paper is devoted to the study of the following Dirichlet boundary value problem:

\begin{align}
(-\Delta)_p^s u &= f(x,u) \quad \text{in } \Omega \\
u &> 0 \quad \text{in } \Omega \\
u &= 0 \quad \text{in } \Omega^c.
\end{align}

Here $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) is a bounded domain with a $C^{1,1}$-boundary $\partial \Omega$, $p > 1$, $s \in (0,1)$ are real numbers, and the leading operator is the fractional $p$-Laplacian, defined for a sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$(-\Delta)_p^s u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x-y|^{N+ps}} \, dy,$$

where $B_\varepsilon(x)$ denotes the ball centered at $x \in \mathbb{R}^N$ with radius $\varepsilon > 0$. This is a nonlinear, nonlocal operator which in special cases reduces, up to a multiplicative constant, to the fractional Laplacian ($p = 2, s \in (0,1)$), to the $p$-Laplacian ($p > 1, s = 1$), and in particular to the Laplacian ($p = 2, s = 1$). The fractional $p$-Laplacian is degenerate if $p > 2$, singular if $p \in (1,2)$. An introduction to this operator and related problems can be found in [28].

Our hypotheses on the reaction $f$ are the following:

$\mathbf{H}$ $f : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ is a Carathéodory function s.t.

(i) $f(\cdot, t) \in L^\infty(\Omega)$ for all $t \in \mathbb{R}^+$;

(ii) there exists $c_0 > 0$ s.t. for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^+$

$$f(x,t) \leq c_0(1 + t^{p-1});$$

(iii) for a.e. $x \in \Omega$ the mapping

$$t \mapsto \frac{f(x,t)}{t^{p-1}}$$

is strictly decreasing in $\mathbb{R}_0^+$.

Remark 1.1. Some comments on hypotheses $\mathbf{H}$ are in order:

(a) The boundedness condition $\mathbf{H}$ (i) is obviously satisfied in the autonomous case, i.e., $f \in C(\mathbb{R}^+)$. 

(b) The growth condition $\mathbf{H}$ (ii) acts on the reaction from above only, as is the case in [7,10] but differently from previous results in the nonlinear setting such as [2,3,27].

(c) The strict monotonicity condition $\mathbf{H}$ (iii) classifies our reaction as a $(p-1)$-sublinear one.

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Example 1.2. We present here two examples of autonomous reactions satisfying $H$. First, for all $1 < q \leq p < r$, we recall the sub- and equidiffusive logistic reactions

$$f(t) = t^{r-1} - t^{r-1}.$$ 

Also, for any $\alpha \geq p - 1$, $r > p$ we have the exponentially growing reaction

$$f(t) = \begin{cases} 
  t^{p-1} - t^{r-1} & \text{if } t \in [0, 1] \\
  t^{p-1} - e^{\alpha - 1} & \text{if } t > 1,
\end{cases}$$ 

Note that both may have a super-critical growth from below (see [1] for other results related to supercritical fractional $p$-Laplacian equations).

The study of boundary value problems with sublinear reactions dates back to the classical work of Brezis and Oswald [7], dealing with the Laplacian as leading operator ($p = 2$, $s = 1$) and Dirichlet boundary conditions, with hypotheses analogous to $H$. In [7] the authors prove that the problem admits at most one solution, and provide a characteristic condition for the existence of such a solution (this is called in the current literature “optimal solvability”). In the following years, similar results have been proved for a variety of nonlinear local elliptic operators, such as the $p$-Laplacian with Dirichlet conditions [10] or Neumann conditions [15], or a general nonlinear operator with Robin conditions [12]. See also [26] for an alternative approach based on nonsmooth critical point theory.

In the nonlocal framework, we recall the related results for the fractional Laplacian in $\mathbb{R}^N$ [29], for the spectral fractional Laplacian in a bounded domain [24], for the fractional $p$-Laplacian with nonlocal Robin conditions [27], and a mixed local-nonlocal operator with Dirichlet conditions [2, 3]. We remark again that the last three contributions present partial results regarding the necessary condition for existence, and they both employ bilateral growth conditions on the reaction. A different type of optimal solvability result for the fractional $p$-Laplacian with Dirichlet conditions and a critical reaction was obtained in [6].

Our result is the first exact counterpart of [7, 10] for the Dirichlet fractional $p$-Laplacian, and to our knowledge it is new even for the linear case, i.e. for the Dirichlet fractional Laplacian ($p = 2$, $s \in (0, 1)$). We relate the solvability of problem (1.1) to the signs of two weighted 'eigenvalues', defined as follows. By $H$ (iii), for a.e. $x \in \Omega$ we may define

$$a_0(x) = \lim_{t \to 0^+} \frac{f(x, t)}{t^{p-1}}, \quad a_\infty(x) = \lim_{t \to \infty} \frac{f(x, t)}{t^{p-1}}.$$ 

We have for a.e. $x \in \Omega$

$$a_0(x) \geq f(x, 1) \geq a_\infty(x),$$

so by $H$ (i) we can find $C > 0$ s.t. $a_0 \geq -C$, $a_\infty \leq C$ in $\Omega$. On the contrary, $a_0 = +\infty$ and $a_\infty = -\infty$ may occur on non-null subsets of $\Omega$. For any measurable function $a$ defined in $\Omega$, possibly taking one of the values $\pm \infty$ (but not both), we set

$$\lambda_1(a) = \inf_{v \in W^{s,p}_0(\Omega) \setminus \{0\}} \frac{\|v\|^p - \int_{\{v \neq 0\}} a(x)|v|^p \, dx}{\|v\|^p} \in \mathbb{R} \cup \{\pm \infty\}$$

(see Section 2 for the notation). If $a \in L^q(\Omega)$ (for convenient $q > 1$), then $\lambda_1(a) \in \mathbb{R}$ is the principal eigenvalue of the following weighted eigenvalue problem:

$$\begin{aligned}
(-\Delta)_p^s v - a(x)|v|^{p-2}v &= \lambda |v|^{p-2}v \quad \text{in } \Omega \\
v &= 0 \quad \text{in } \Omega^c
\end{aligned}$$

(1.3)

For a discussion on weighted and nonweighted eigenvalues of $(-\Delta)_p^s$, see [5, 17, 23, 25]. Using eigenvalues as asymptotic thresholds for general nonlinear reactions is certainly not new in the study of elliptic problems driven by the fractional $p$-Laplacian, see for instance [14, 18–20] (the last one dealing with the logistic equation, with a result agreeing with those of the present paper in the sub- and equidiffusive case).

Our main result is the following:

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be a bounded domain with a $C^{1,1}$-boundary $\partial \Omega$, $p > 1$, $s \in (0, 1)$ be real numbers, $f$ satisfy $H$. Then, problem (1.1)

(i) has at most one solution;
(ii) has a solution iff \( \lambda_1(a_0) < 0 < \lambda_1(a_{\infty}) \).

We note that, for an autonomous reaction, the inequality in (ii) is equivalent to the following, where \( \lambda_1 > 0 \) denotes the principal eigenvalue of (1.3) with \( a = 0 \):

\[
\lim_{t \to -\infty} \frac{f(t)}{t^{p-1}} < \lambda_1 \quad \text{for all } a = 0.
\]

For a definition and a discussion on the notion of 'solution', as well as some basic properties of \((-\Delta)^s_p\), we refer to Section 2. In Section 3 we shall prove the uniqueness statement (i). Regarding the existence statement (ii), we will first prove in Section 4 that the condition is necessary by exploiting the properties of the eigenvalue problem (1.3) proved in [17,25]. Finally, in Section 5 we will tackle the more delicate issue of the sufficient condition for existence, by using a variational approach and introducing a sequence of auxiliary truncated problems.

We essentially follow the approach of the original papers [7,10], but with some important differences typical of the nonlocal framework, which deserve to be laid out:

(a) In [10] a form of 'hidden convexity' (i.e., the energy functional is convex in the variable \( u^\frac{1}{s} \)) represents a useful tool for uniqueness. Though an equivalent form of convexity for the fractional energy was proved in [13] (see also [4]), we prefer to follow a different (and in our opinion simpler) approach, based on a discrete Picone’s inequality from [6] and a comparison argument from [19].

(b) Although enjoying good interior regularity, solutions of fractional order equations are generally singular at the boundary, which prevents a classical Hopf’s boundary lemma from holding, thus making it difficult to work with quotients between solutions (as extensively done in [7,10]). In fact, as proved in [21], global regularity of a solution \( u \) only amounts at \( u \in C^\alpha(\Omega) \) for some \( \alpha \in (0,s) \). Nevertheless, some useful boundary estimates on \( u \) in terms of the distance function \( d_\Omega(x) = \text{dist}(x,\Omega^c)^s \) and a related fractional Hopf’s lemma from [20] can be employed, along with the technical Lemma 2.4, to overcome such difficulty (in the linear and degenerate cases \( p \geq 2 \) a better regularity theory holds, see Remark 2.5).

(c) Due to the nonlocal nature of the operator, in the proof of the sufficient condition it is not immediate to compare the energies of solutions of (1.1) and the truncated problems, respectively. We deal with such issue by applying a special submodularity inequality from [16].

**Notations.** Throughout the paper we shall use the following notations:

- \( \mathbb{R}^+ = [0, \infty) \), \( \mathbb{R}^- = (-\infty, 0] \), and \( \mathbb{R}^+_0 = (0, \infty) \).
- \( A^c = \mathbb{R}^N \setminus A \) for all \( A \subset \mathbb{R}^N \).
- \( f \leq g \) in \( \Omega \) means that \( f(x) \leq g(x) \) for a.e. \( x \in \Omega \) (and similar expressions), for any two measurable functions \( f, g : \Omega \to \mathbb{R} \).
- \( f \vee g = \max\{f,g\} \), \( f \wedge g = \min\{f,g\} \), and \( f^\pm = (\pm f) \vee 0 \) are the positive and negative parts of \( f \), respectively.
- \( X_+ \) is the positive order cone of an ordered Banach space \( X \).
- \( \| \cdot \|_q \) denotes the standard norm of \( L^q(\Omega) \) (or \( L^q(\mathbb{R}^N) \), which will be clear from the context), for any \( q \in [1, \infty) \).
- \( \| \cdot \| \) is the reference norm defined in Section 2.
- Every function \( u \) defined in \( \Omega \) is identified with its 0-extension to \( \mathbb{R}^N \).
- \( J_p(t) = |t|^{p-2}t \) for all \( t \in \mathbb{R} \).
- \( C \) will denote positive universal constants whose value may change case by case.

2. Preliminaries

In this section we fix a functional-analytic framework for problem (1.1) and recall some technical results, referring the reader to [11] for details. From now on, \( \Omega, p, \) and \( s \) will be as in Section 1. First, for any measurable \( u : \mathbb{R}^N \to \mathbb{R} \) we define the Gagliardo seminorm

\[
[u]_{s,p} = \left( \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dx \, dy \right)^{\frac{1}{p}}.
\]
Then, we define the fractional Sobolev spaces
\[ W^{s,p}(\mathbb{R}^N) = \{ u \in L^p(\mathbb{R}^N) \colon |u|_{s,p} < \infty \}, \]
and
\[ W^{s,p}_0(\Omega) = \{ u \in W^{s,p}(\mathbb{R}^N) \colon u = 0 \text{ in } \Omega^c \}. \]
The latter is a uniformly convex, separable Banach space under the norm \( \| u \| = |u|_{s,p} \), with dual space \( W^{-s',p'}(\Omega) \). The embedding \( W^{s,p}_0(\Omega) \hookrightarrow L^q(\Omega) \) is compact for all \( q \in [1, p^*_s) \), where
\[
p^*_s = \begin{cases} \frac{Np}{N-mps} & \text{if } ps < N, \\ \infty & \text{if } ps \geq N.
\end{cases}
\]
The slight abuse of notation in the definition of \( W^{s,p}_0(\Omega) \) is justified by the fact that, if \( \partial \Omega \) is smooth enough, the above-defined space coincides with the completion of \( C^\infty_c(\Omega) \) with respect to the Gagliardo seminorm defined in \( \Omega \times \Omega \) (see for instance [5]). Also, \( W^{s,p}_0(\Omega) \) has a lattice structure and the following submodularity inequality holds for all \( u, v \in W^{s,p}_0(\Omega) \) (see [16, Remark 3.3]):
\begin{equation}
\| u \vee v \|_p + \| u \wedge v \|_p \leq \| u \|_p + \| v \|_p.
\end{equation}

We define the fractional \( p \)-Laplacian as an operator \((-\Delta)^p_s : W^{s,p}_0(\Omega) \to W^{-s',p'}(\Omega)\) s.t. for all \( u, \varphi \in W^{s,p}_0(\Omega) \)
\[
\langle (-\Delta)^p_s u, \varphi \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} j_p(u(x) - u(y))(\varphi(x) - \varphi(y)) \frac{dx dy}{|x-y|^{N+ps}}.
\]
Equivalently, \((-\Delta)^p_s\) is the gradient of the functional
\[
W^{s,p}_0(\Omega) \ni u \mapsto \| u \|_p^p.
\]
Such a definition agrees with the one given in Section 1 if \( u \) is smooth enough (see [21, Proposition 2.12]). It can be seen that \((-\Delta)^p_s\) is a continuous, \((S)_+\)-operator, i.e., whenever \( u_n \rightharpoonup u \) in \( W^{s,p}_0(\Omega) \) and
\[
\limsup_n \langle (-\Delta)^p_s u_n, u_n - u \rangle \leq 0,
\]
then \( u_n \rightharpoonup u \) in \( W^{s,p}_0(\Omega) \) (see for instance [14, Lemma 2.1]). Plus, for all \( u \in W^{s,p}_0(\Omega) \) we have
\[
\| u^\pm \|_p \leq \langle (-\Delta)^p_s u, \pm u \rangle.
\]
Now we focus on problem (1.1). First, we extend the reaction \( f \) by setting for all \((x, t) \in \Omega \times \mathbb{R}_0^-\)
\[
f(x, t) = f(x, 0).
\]
We say that \( u \in W^{s,p}_0(\Omega) \) is a (weak) solution of (1.1), if for all \( \varphi \in W^{s,p}_0(\Omega) \)
\begin{equation}
\langle (-\Delta)^p_s u, \varphi \rangle = \int_\Omega f(x, u) \varphi \, dx.
\end{equation}
Since \( H \) does not include a full growth condition on \( f(x, \cdot) \), the right-hand side of (2.2) may \emph{a priori} blow up for some \( u \in W^{s,p}_0(\Omega) \). We need some labor to show that (2.2) is well posed and can be reasonably assumed as a weak formulation of (1.1). We begin with a weak minimum principle:

\textbf{Lemma 2.1.} \textit{Let } \( H \) \textit{hold, } u \in W^{s,p}_0(\Omega) \textit{ satisfy (2.2). Then, } u \geq 0 \textit{ in } \Omega.

\textit{Proof.} First, for all \( \delta > 0 \) we can find \( C_\delta > 0 \) s.t. for a.e. \( x \in \Omega \) and all \( t \in [0, \delta] \)
\begin{equation}
f(x, t) \geq -C_\delta t^{p-1}.
\end{equation}
Indeed, by \( H \) (i) (iii) we have
\[
f(x, t) \geq \frac{f(x, \delta)}{\delta^{p-1}} t^{p-1} \geq -\frac{\| f(\cdot, \delta) \|_{\infty} \delta^{p-1}}{\delta^{p-1}},
\]
as claimed. Moreover, by letting \( t \to 0^+ \) in (2.3) and recalling the extended definition of \( f \), we have for a.e. \( x \in \Omega \) and all \( t \in \mathbb{R}^- \)
\begin{equation}
f(x, t) \geq 0.
\end{equation}
Now test (2.2) with \(-u^- \in W^{s,p}_0(\Omega) \) and use (2.4):
\[
\| u^- \|_p \leq \langle (-\Delta)^p_s u, -u^- \rangle = \int_{\{u < 0\}} f(x, u) u \, dx \leq 0.
\]
So, \( u^- = 0 \), which concludes the proof.

\[\square\]

As pointed out in Section 1, an important role in nonlocal problem is played by the distance function defined for all \( x \in \mathbb{R}^N \) by

\[ d_\Omega(x) = \text{dist}(x, \Omega^c). \]

Indeed, we have the following regularity result and boundary estimates:

**Lemma 2.2.** Let \( \mathbf{H} \) hold, \( u \in W^{s,p}_0(\Omega) \) satisfy (2.2). Then,

(i) \( u \in C^\alpha(\overline{\Omega}) \) for some \( \alpha \in (0, s) \);

(ii) there exists \( C > 0 \) s.t. in \( \Omega \)

\[ 0 \leq u \leq C d_\Omega^s. \]

**Proof.** We assume \( ps < N \), the remaining cases being easily solved by fractional Sobolev embeddings (see [11, Theorems 6.9, 6.10]). By Lemma 2.1 we have \( u \geq 0 \) in \( \Omega \). So, arguing as in [8, Theorem 3.3] and using only the growth condition from above \( \mathbf{H} \) (ii), we see that \( u \in L^q(\Omega) \) for all \( q > 1 \). Then fix

\[ r > \max \left\{ \frac{N}{ps}, \frac{1}{p - 1} \right\}, \]

so that \( u^{p-1} \in L^r(\Omega) \). By \( \mathbf{H} \) (ii) again and arguing as in [5, Theorem 3.1], we get that \( u \in L^\infty(\Omega) \). Now \( \mathbf{H} \) (ii) and (2.3) (with \( \delta = \|u\|_\infty \)) imply

\[ f(\cdot, u) \in L^\infty(\Omega). \]

Thus, by [21, Theorem 1.1] we have (i) and by [21, Theorem 4.4] we have (ii).

Next, we improve Lemma 2.1 to a strong minimum principle, incorporating a fractional Hopf-type property (see [9] for a similar result in the pure power case):

**Lemma 2.3.** Let \( \mathbf{H} \) hold, \( u \in W^{s,p}_0(\Omega) \setminus \{0\} \) satisfy (2.2). Then,

\[ \inf_{\Omega} \frac{u}{d_\Omega^s} > 0. \]

**Proof.** By Lemma 2.2 we have \( u \in C^\alpha(\overline{\Omega}) \setminus \{0\} \). By \( \mathbf{H} \) (i) we may set for all \( t \in \mathbb{R} \)

\[ g(t) = \frac{\|f(\cdot, u\|_\infty)\|_\infty}{\|u\|_{\infty}^{p-1}} j_p(t). \]

Clearly \( g \in C(\mathbb{R}) \cap BV_{\text{loc}}(\mathbb{R}) \) (being monotone). By (2.2) and arguing as in Lemma 2.1, we have weakly in \( \Omega \)

\[ (-\Delta)_p^s u + g(u) \geq 0 = g(0). \]

By [20, Theorem 2.6] we conclude.

In particular, by Lemma 2.3 we have \( u > 0 \) in \( \Omega \). In conclusion, we see that (2.2) is well posed and that any function \( u \neq 0 \) satisfying (2.2) actually solves (1.1).

The following technical result, which in a sense simplifies some arguments in [6,19,27], will be very useful in the next sections (recall that we tacitly identify functions defined in \( \Omega \) with their 0-extensions to \( \mathbb{R}^N \)):

**Lemma 2.4.** Let \( u, v \in W^{s,p}_0(\Omega) \cap C(\overline{\Omega}), C > 1 \) be s.t. in \( \Omega \)

\[ \frac{d_\Omega^s}{C} \leq u, v \leq C d_\Omega^s. \]

Then,

\[ \frac{u^p}{v^{p-1}} \in W^{s,p}_0(\Omega). \]

**Proof.** First we note that in \( \Omega \)

\[ 0 < \frac{u}{v} = \frac{u}{d_\Omega^s} \frac{d_\Omega^s}{v} \leq C^2, \]

so \( u/v \in L^\infty(\Omega) \). Since \( u \in L^\infty(\Omega) \), we immediately see that also

\[ \frac{u^p}{v^{p-1}} \in L^\infty(\Omega), \]
while the same function vanishes in $\Omega^c$. There remains to show that the Gagliardo seminorm of $u^p/v^{p-1}$ is finite. To do that, we first show that there exists $C > 0$ s.t. for all $x, y \in \mathbb{R}^N$

(2.6) \[ \left| \frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}} \right| \leq C|u(x) - u(y)| + C|v(x) - v(y)|. \]

First, by symmetry and by recalling that all involved functions vanish in $\Omega^c$, we may assume $x, y \in \Omega$ and $u(x) \geq u(y)$. Indeed, if $x \in \Omega$ and $y \notin \Omega$, by (2.5) we have

\[ \left| \frac{u(x)^p}{v(x)^{p-1}} \right| \leq C|u(x)|. \]

So, if $x, y \in \Omega$, by monotonicity of the maps $t \mapsto t^{p-1}$, $t^{p-2}$ in $\mathbb{R}_0^+$ and Lagrange’s rule we have

(2.7) \[ \left| \frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}} \right| \leq \frac{|u(x)^p - u(y)^p|}{v(x)^{p-1}v(y)^{-p}} + \frac{u(y)^p}{v(x)^{p-1}}|v(x)^{p-1} - v(y)^{p-1}| \leq \frac{u(x)^p}{v(x)^{p-1}}|u(x) - u(y)| + (p-1)u(y)^p \frac{(v(x)^{p-2}) \vee (v(y)^{p-2})}{v(x)^{p-1}v(y)^{p-1}}|v(x) - v(y)|. \]

The first term is easily estimated by recalling that $u/v \in L^\infty(\Omega)$. For the second term, we distinguish two cases:

(a) if $v(x)^{p-2} \geq v(y)^{p-2}$, then we have

\[ \frac{u(y)^p}{v(x)^{p-1}} \frac{(v(x)^{p-2}) \vee (v(y)^{p-2})}{v(x)^{p-1}v(y)^{-p}} \leq \frac{u(y)^p}{v(x)v(y)^{p-1}} \leq \frac{u(x)^p}{v(x)^{p-1}} \frac{u(y)^{p-1}}{v(y)^{p-1}} \leq \frac{\|u\|}{v} : \]

(b) if $v(x)^{p-2} < v(y)^{p-2}$, then we have

\[ \frac{u(y)^p}{v(x)^{p-1}} \frac{(v(x)^{p-2}) \vee (v(y)^{p-2})}{v(x)^{p-1}v(y)^{-p}} \leq \frac{u(y)^p}{v(x)^{p-1}v(y)^{-p}} \leq \frac{u(x)^p}{v(x)^{p-1}} \frac{u(y)^{p-1}}{v(y)} \leq \frac{\|u\|}{v^\infty}. \]

Plugging the above estimates into (2.7), we have

\[ \left| \frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}} \right| \leq \frac{\|u\|}{v} \left| u(x) - u(y) \right| + (p-1)\frac{\|u\|}{v^\infty} \left| v(x) - v(y) \right|, \]

so (2.6) is proved. Now, integrating (2.6) and using the elementary inequality

\[ (a + b)^p \leq 2^{p-1}(a^p + b^p) \quad \text{for all } a, b \in \mathbb{R}^+, \]

we get

\[ \int_{\mathbb{R}^N \times \mathbb{R}^N} \left| \frac{u(x)^p}{v(x)^{p-1}} - \frac{u(y)^p}{v(y)^{p-1}} \right|^p \frac{dx dy}{|x - y|^{N+ps}} \leq C\|u\|^p + C\|v\|^p < \infty. \]

So we conclude that $u^p/v^{p-1} \in W^s_p(\Omega)$.

\[ \Box \]

**Remark 2.5.** In the degenerate and linear cases $p \geq 2$, much more can be said about the regularity of weak solutions. First, regarding pure Hölder regularity, we have $u \in C^s(\Omega)$. More important, let us introduce the weighted Hölder spaces

\[ C^s_\alpha(\Omega) = \left\{ u \in C^0(\Omega) : \frac{u}{d_\Omega^\alpha} \text{ has a } \alpha\text{-Hölder continuous extension to } \overline{\Omega} \right\}, \]

(for $\alpha = 0$, the extension is simply continuous). It can be seen that any weak solution $u$ of (1.1) satisfies $u \in C^s_\alpha(\Omega)$ for some $\alpha \in (0, s)$, and if $u \neq 0$ then

\[ u \in \text{int}(C^s_\alpha(\Omega)_+) \]

(see [22, Theorem 1.1] and [20, Theorem 2.7]). Obviously, also the proofs of Lemma 2.4 above and of some of the following results are simpler in such a case. On the other hand, in the singular case $p \in (1, 2)$ this regularity theory is not available so far. Nevertheless, Lemmas 2.3 and 2.4 above hold the same and permit to face both the degenerate and the singular cases.
3. Uniqueness

In this section we prove that the solution of problem (1.1), if any, is unique. The argument is similar to that of [19, Theorem 2.8] and makes a crucial use of Lemma 2.4 and the following discrete Picone’s inequality from [6, Proposition 2.2]:

\[(3.1) \quad j_p (a - b) \left( \frac{c^p}{ap-1} - \frac{d^p}{bp-1} \right) \leq |c - d|^p \text{ for all } a, b \in \mathbb{R}^+, c, d \in \mathbb{R}^+.\]

Our uniqueness result is the following:

**Proposition 3.1.** Let \( H \) hold. Then, (1.1) has at most one solution.

**Proof.** Let \( u, v \in W_0^{s,p}(\Omega) \setminus \{0\} \) satisfy (2.2). By Lemmas 2.2, 2.3 we have \( u, v \in C^\alpha(\Omega) \), and we can find \( C > 1 \) s.t. in \( \Omega \)

\[\frac{d_\Omega^*}{C} \leq u, v \leq Cd_\Omega^*.\]

So, by Lemma 2.4 we have

\[\frac{u^p}{vp-1}, \frac{v^p}{vp-1} \in W_0^{s,p}(\Omega).\]

Set \( w = (u^p - v^p)^+ \), then by the relations above

\[\frac{w}{vp-1} = \left( \frac{u^p}{vp-1} - v \right)^+ \in W_0^{s,p}(\Omega),\]

and similarly \( w/vp-1 \in W_0^{s,p}(\Omega) \). Testing (2.2) with such functions and recalling \( H \) (iii), we have

\[(3.2) \quad \left\langle (-\Delta)^s_p u, \frac{w}{vp-1} \right\rangle - \left\langle (-\Delta)^s_p v, \frac{w}{vp-1} \right\rangle = \int_{\{u>v\}} \left( \frac{f(x, u)}{u^{p-1}} - \frac{f(x, v)}{v^{p-1}} \right) (u^p - v^p) \, dx \leq 0.\]

To proceed, we prove that for all \( x, y \in \mathbb{R}^N \)

\[(3.3) \quad j_p (u(x) - u(y)) \left( \frac{w(x)}{u(x)^{p-1}} - \frac{w(y)}{u(y)^{p-1}} \right) \geq j_p (v(x) - v(y)) \left( \frac{w(x)}{v(x)^{p-1}} - \frac{w(y)}{v(y)^{p-1}} \right),\]

(with the usual convention that any function defined in \( \Omega \) is identified with its 0-extension to \( \mathbb{R}^N \)). Indeed, four cases may occur:

(a) if \( u(x) > v(x), u(y) > v(y) \), then by applying (3.1) twice we have

\[j_p (u(x) - u(y)) \left( \frac{w(x)}{u(x)^{p-1}} - \frac{w(y)}{u(y)^{p-1}} \right) = |u(x) - u(y)|^p - j_p (u(x) - u(y)) \left( \frac{v(x)^p}{u(x)^{p-1}} - \frac{v(y)^p}{u(y)^{p-1}} \right) \]

\[\geq |u(x) - u(y)|^p - |v(x) - v(y)|^p \geq j_p (v(x) - v(y)) \left( \frac{w(x)}{v(x)^{p-1}} - \frac{w(y)}{v(y)^{p-1}} \right) - |v(x) - v(y)|^p \]

\[= j_p (v(x) - v(y)) \left( \frac{w(x)}{v(x)^{p-1}} - \frac{w(y)}{v(y)^{p-1}} \right);\]

(b) if \( u(x) > v(x), u(y) \leq v(y) \), then

\[\frac{u(y)}{u(x)} \leq \frac{v(y)}{v(x)}.\]
and since $j_p$ is increasing and $(p - 1)$-homogeneous in $\mathbb{R}$, we have
\[
j_p(u(x) - u(y)) \left( \frac{w(x)}{u(x)^{p-1}} - \frac{w(y)}{u(y)^{p-1}} \right) = j_p(u(x) - u(y)) \frac{u(x)^p - v(x)^p}{u(x)^{p-1}}
\]
\[
= j_p \left( 1 - \frac{u(y)}{u(x)} \right) (u(x)^p - v(x)^p)
\]
\[
\geq j_p \left( 1 - \frac{v(y)}{v(x)} \right) (u(x)^p - v(x)^p)
\]
\[
= j_p(v(x) - v(y)) \frac{u(x)^p - v(x)^p}{v(x)^{p-1}}
\]
\[
= j_p(v(x) - v(y)) \left( \frac{w(x)}{v(x)^{p-1}} - \frac{w(y)}{v(y)^{p-1}} \right);
\]

(c) if $u(x) \leq v(x)$, $u(y) > v(y)$, then
\[
\frac{u(x)}{u(y)} \leq \frac{v(x)}{v(y)}
\]

and, similarly to the previous case, we have
\[
j_p(u(x) - u(y)) \left( \frac{w(x)}{u(x)^{p-1}} - \frac{w(y)}{u(y)^{p-1}} \right) = -j_p(u(x) - u(y)) \frac{u(y)^p - v(y)^p}{u(y)^{p-1}}
\]
\[
\geq -j_p(v(x) - v(y)) \frac{u(y)^p - v(y)^p}{v(y)^{p-1}}
\]
\[
= j_p(v(x) - v(y)) \left( \frac{w(x)}{v(x)^{p-1}} - \frac{w(y)}{v(y)^{p-1}} \right);
\]

(d) if $u(x) \leq v(x)$, $u(y) \leq v(y)$, then we have $w(x) = w(y) = 0$ and (3.3) holds trivially. Integrating (3.3) in $\mathbb{R}^N \times \mathbb{R}^N$ we have
\[
\langle (-\Delta)_p^a u, \frac{w}{u^{p-1}} \rangle \geq \langle (-\Delta)_p v, \frac{w}{v^{p-1}} \rangle,
\]
which along with (3.2) forces
\[
\int_{\{u>v\}} \left( \frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}} \right) (u^p - v^p) \, dx = 0.
\]

By $\mathbf{H}$ (iii) (strict monotonicity), the integrand above is negative, so we deduce that $\{u>v\}$ has 0-measure, i.e., $u \leq v$ in $\Omega$. Similarly we see that $u \geq v$ in $\Omega$, and thus $u = v$. \qed

4. Existence/II: necessary condition

In this section we assume the existence of a solution of (1.1), and we prove that
\[
\lambda_1(a_0) < 0 < \lambda_1(a_\infty),
\]
with $\lambda_1(a_0), \lambda_1(a_\infty) \in \mathbb{R} \cup \{\pm \infty\}$ defined by (1.2). We begin with the inequality on the right:

**Lemma 4.1.** Let $\mathbf{H}$ hold and $u$ be a solution of (1.1). Then, $\lambda_1(a_\infty) > 0$.

**Proof.** From Lemmas 2.2, 2.3 we know that $u \in C^\infty(\Omega)$ satisfies (2.2) and there exists $C > 1$ s.t. in $\Omega$
\[
\frac{d_{\Omega}^a}{C} \leq u \leq C d_{\Omega}^a.
\]
In particular $u \in L^\infty(\Omega)$. Set for all $x \in \Omega$
\[
a(x) = \frac{f(x,\|u\|_\infty)}{|\|u\|_\infty|^{p-1}}.
\]
By $\mathbf{H}$ (i) (iii) we have $a \in L^\infty(\Omega)$ and for a.e. $x \in \Omega$
\[
f(x,u) \frac{u^p}{u^{p-1}} \geq a(x) > a_\infty(x),
\]

(4.1)
the first inequality being strict on a non-null subset of $\Omega$. Define $\lambda_1(a) \in \mathbb{R}$ according to (1.2), then the infimum is attained at some $v \in W_0^{s,p}(\Omega)$ with $\|v\|_p = 1$. Since $|v| \in W_0^{s,p}(\Omega)$ with $\|v\|_p \leq \|v\|$, we may assume $v \geq 0$ in $\Omega$. In particular,

$$0 < \|v\|^p = \int_{\Omega} (\lambda_1(a) + a(x)) v^p \, dx.$$ 

So $m = \lambda_1(a) + a \in L^\infty(\Omega)$ is a weight function s.t. $m^+ \neq 0$. By [17, Proposition 3.3] and arguing as in Lemmas 2.2, 2.3, we see that $v \in C^\alpha(\overline{\Omega})_+$ is unique and, for a possibly bigger $C > 1$, we have in $\Omega$

$$\frac{d_m^V}{C} \leq v \leq Cd_m^V.$$ 

Equivalently, $v$ is the unique positive, $L^p(\Omega)$-normalized principal eigenfunction of (1.3). Thanks to the estimates above, we can find $\tau > 0$ s.t. $u < \tau v$ in $\Omega$, hence in particular $\|u\|_p < \tau$. Then, $\tau v \in W_0^{s,p}(\Omega) \cap C^\alpha(\overline{\Omega}) \setminus \{0\}$ satisfies weakly in $\Omega$

$$(-\Delta)_p^s(\tau v) = (\lambda_1(a) + a(x))(\tau v)^{p-1}.$$ 

By Lemma 2.4 we have

$$\frac{u^p}{(\tau v)^{p-1}} \in W_0^{s,p}(\Omega).$$

Testing (2.2) and (4.2) with convenient functions, and using (4.1) and the normalization of $v$, we have

$$\left\langle (-\Delta)_p^s u, u - \frac{(\tau v)^p}{(\tau v)^{p-1}} \right\rangle + \left\langle (-\Delta)_p^s (\tau v), \tau v - \frac{u^p}{(\tau v)^{p-1}} \right\rangle$$

$$= \int_{\Omega} \frac{f(x, u)}{u^{p-1}} \, dx + \int_{\Omega} (\lambda_1(a) + a(x))(\tau v)^{p-1} (\tau v)^p - u^p \, dx$$

$$= \int_{\Omega} \frac{f(x, u)}{u^{p-1}} \, dx + \lambda_1(a) \int_{\Omega} (\tau v)^{p-1} \, dx$$

$$< \lambda_1(a) (\tau^p - \|u\|_p^p).$$

Besides, by the aforementioned Picone’s inequality (3.1) we have

$$\left\langle (-\Delta)_p^s u, u - \frac{(\tau v)^p}{(\tau v)^{p-1}} \right\rangle + \left\langle (-\Delta)_p^s (\tau v), \tau v - \frac{u^p}{(\tau v)^{p-1}} \right\rangle$$

$$= \|u\|^p + \|\tau v\|^p$$

$$- \int_{\mathbb{R}^N \times \mathbb{R}^N} j_p(u(x) - u(y))(\frac{(\tau v(x))^p}{u(x)^{p-1}} - \frac{(\tau v(y))^p}{u(y)^{p-1}}) \frac{dx \, dy}{|x - y|^{N+ps}}$$

$$- \int_{\mathbb{R}^N \times \mathbb{R}^N} j_p(\tau v(x) - \tau v(y))(\frac{u(x)^p}{(\tau v(x))^{p-1}} - \frac{u(y)^p}{(\tau v(y))^{p-1}}) \frac{dx \, dy}{|x - y|^{N+ps}} \geq 0.$$ 

Concatenating the inequalities above, we get

$$\lambda_1(a)(\tau^p - \|u\|_p^p) > 0,$$

which along with $\tau > \|u\|_p$ implies $\lambda_1(a) > 0$. Finally, we note that by (4.1) we have for all $w \in W_0^{s,p}(\Omega)$ with $\|w\|_p = 1$

$$\|w\|^p - \int_{\{w \neq 0\}} a_\infty(x)|w|^p \, dx \geq \|w\|^p - \int_{\{w \neq 0\}} a(x)|w|^p \, dx \geq \lambda_1(a).$$

Taking the infimum over $w$, we find

$$\lambda_1(a_\infty) \geq \lambda_1(a) > 0$$

and conclude. \hfill \Box

**Remark 4.2.** In the proof of Lemma 4.1 we have seen, en passant, that the mapping $a \mapsto \lambda_1(a)$ defined in (1.2) is monotone nonincreasing with respect to the pointwise ordering in $\Omega$. If $a_\infty \in L^\infty(\Omega)$ (for instance when $f$ satisfies a global growth condition), we would get $\lambda_1(a_\infty) > \lambda_1(a)$ reasoning as in [17, Proposition 4.2].

The argument for the inequality on the left is simpler:
Lemma 4.3. Let $\mathbf{H}$ hold and $u$ be a solution of (1.1). Then, $\lambda_1(a_0) < 0$.

Proof. If $a_0 = \infty$ on a non-null subset of $\Omega$, then by (1.2) we have $\lambda_1(a_0) = -\infty$ and there is nothing to prove. So we may assume $a_0 < \infty$ in $\Omega$. By $\mathbf{H}$ (iii) and (2.3) (with $\delta = 1$) we have for a.e. $x \in \Omega$

$$a_0(x) > f(x,1) \geq -C_1.$$  

As in Lemma 4.1 we have $u \in C^\alpha(\Omega)$ and $u > 0$ in $\Omega$. Testing (2.2) with $u \in W^{s,p}_0(\Omega)$ and using $\mathbf{H}$ (iii) we have

$$\|u\|^p = \langle (-\Delta)_p^s u, u \rangle = \int_\Omega f(x,u)u \, dx < \int_\Omega a_0(x)u^p \, dx.$$  

So by (1.2) we have

$$\lambda_1(a_0) \leq \frac{\|u\|^p - \int_\Omega a_0(x)u^p \, dx}{\|u\|^p} < 0$$

and we conclude. \qed

Summarizing Lemmas 4.1, 4.3, we have the following necessary condition for existence:

Proposition 4.4. Let $\mathbf{H}$ hold and (1.1) have a solution. Then, $\lambda_1(a_0) < 0 < \lambda_1(a_{\infty})$.

5. Existence/II: sufficient condition

The most delicate part of this study consists in proving that $\lambda_1(a_0) < 0 < \lambda_1(a_{\infty})$ implies existence of a solution to (1.1). Following [7, 10], we use a variational approach. Recalling the extended definition of $f$, set for all $(x, t) \in \Omega \times \mathbb{R}$

$$F(x, t) = \int_0^t f(x, \tau) \, d\tau,$$

and for all $u \in W^{s,p}_0(\Omega)$

$$\Phi(u) = \frac{\|u\|^p}{p} - \int_\Omega F(x, u) \, dx.$$  

Under hypotheses $\mathbf{H}$, and especially due to the lack of a growth condition on $f(x, \cdot)$ from below, we cannot expect $\Phi$ to be Gâteaux differentiable, in fact $\Phi$ is not even continuous in $W^{s,p}_0(\Omega)$. So, in the following lemmas we will explore the properties of $\Phi$:

Lemma 5.1. Let $\mathbf{H}$ hold. Then, $\Phi : W^{s,p}_0(\Omega) \to \mathbb{R} \cup \{\infty\}$ is sequentially weakly l.s.c.

Proof. By $\mathbf{H}$ (ii) we have for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^+$

$$F(x, t) \leq \int_0^t c_0(1 + \tau^{p-1}) \, d\tau = c_0 \left( t + \frac{t^p}{p} \right),$$

while for all $t \in \mathbb{R}^-$ we have by $\mathbf{H}$ (i) and (2.4)

$$F(x, t) = \int_0^t f(x, 0) \, d\tau \leq 0.$$  

The estimates above imply for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

(5.1)  

$$F(x, t) \leq C(1 + |t|^p).$$

By (5.1) and the continuous embedding $W^{s,p}_0(\Omega) \hookrightarrow L^p(\Omega)$, we have $\Phi(u) > -\infty$ for all $u \in W^{s,p}_0(\Omega)$. Now let $(u_n)$ be a sequence in $W^{s,p}_0(\Omega)$ s.t. $u_n \rightharpoonup u$ in $W^{s,p}_0(\Omega)$. Then

$$\liminf_n \|u_n\|^p \geq \|u\|^p.$$  

Passing if necessary to a subsequence, we may assume that $u_n \to u$ in $L^p(\Omega)$ (by the compact embedding $W^{s,p}_0(\Omega) \hookrightarrow L^p(\Omega)$), that $u_n(x) \to u(x)$ for a.e. $x \in \Omega$ and there is $g \in L^p(\Omega)$ s.t. $|u_n| \leq g$ in $\Omega$ for all $n \in \mathbb{N}$. By continuity of the Nemytskii operator, we have for a.e. $x \in \Omega$

$$\lim_n F(x, u_n(x)) = F(x, u(x)).$$  

Besides, by (5.1) we have for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$

$$F(x, u_n(x)) \leq C(1 + g(x)^p).$$
We consider the three integrals separately:

\[ \lim \sup_n \int_{\Omega} F(x, u_n) \, dx \leq \int_{\Omega} F(x, u) \, dx. \]

Thus,

\[ \lim \inf_n \Phi(u_n) \geq \Phi(u), \]

which proves that \( \Phi \) is sequentially weakly l.s.c.

The behavior of \( \Phi \) at infinity is governed by \( a_\infty \):

**Lemma 5.2.** Let \( H \) hold and \( \lambda_1(a_\infty) > 0 \). Then, \( \Phi \) is coercive in \( W^{s,p}_0(\Omega) \).

**Proof.** Using \( H \) (iii) and de l'Hôpital's rule, we see that for a.e. \( x \in \Omega \)

\[ \lim \frac{F(x, t)}{t^p} = \frac{a_\infty(x)}{p}. \tag{5.2} \]

We aim at proving that

\[ \lim_{\|u\| \to \infty} \Phi(u) = +\infty. \]

Arguing by contradiction, let \( (u_n) \) be a sequence in \( W^{s,p}_0(\Omega) \) s.t. \( \|u_n\| \to \infty \) and \( \Phi(u_n) \leq C \) for some \( C \in \mathbb{R} \) and for all \( n \in \mathbb{N} \). Then, by \( (5.1) \) we have for all \( n \in \mathbb{N} \)

\[ \frac{\|u_n\|^p}{p} \leq C + \int_{\Omega} F(x, u_n) \, dx \leq C(1 + \|u_n\|^p). \]

So, we have that \( \|u_n\|^p \to \infty \), as well. Now, for all \( n \in \mathbb{N} \) set

\[ \rho_n = \|u_n\|, \quad v_n = \frac{u_n}{\rho_n} \in W^{s,p}_0(\Omega). \]

Then clearly \( \rho_n \to \infty \) and \( \|v_n\|^p = 1 \) for all \( n \in \mathbb{N} \). As above we have for all \( n \in \mathbb{N} \)

\[ \frac{\|v_n\|^p}{p} \leq C \frac{1 + \rho_n^p}{\rho_n^p} \leq C, \]

so \( (v_n) \) is bounded in \( W^{s,p}_0(\Omega) \). By the reflexivity of \( W^{s,p}_0(\Omega) \) and the compact embedding \( W^{s,p}_0(\Omega) \hookrightarrow L^p(\Omega) \), possibly passing to a subsequence, we have that \( v_n \rightharpoonup v \) in \( W^{s,p}_0(\Omega) \) and \( v_n \to v \) in \( L^p(\Omega) \). Hence, we have in particular

\[ \|v\|^p = 1. \tag{5.3} \]

Passing if necessary to a further subsequence, we have \( v_n(x) \to v(x) \) for a.e. \( x \in \Omega \), with dominated convergence in \( L^p(\Omega) \). We claim that

\[ \lim \sup_n \int_{\Omega} \frac{F(x, \rho_n v_n)}{\rho_n^p} \, dx \leq \int_{\{v > 0\}} \frac{a_\infty(x)v^p}{p} \, dx. \tag{5.4} \]

Indeed, for all \( n \in \mathbb{N} \) we have

\[ \int_{\Omega} \frac{F(x, \rho_n v_n)}{\rho_n^p} \, dx = \int_{\{v > 0\}} \frac{F(x, \rho_n v_n^+)}{\rho_n^p} \, dx + \int_{\{v \leq 0\}} \frac{F(x, \rho_n v_n^+)}{\rho_n^p} \, dx + \int_{\{v_n \leq 0\}} \frac{F(x, \rho_n v_n)}{\rho_n^p} \, dx \]

\[ = I_n^1 + I_n^2 + I_n^3. \]

We consider the three integrals separately:

1. In the set \( \{v > 0\} \) we have a.e. \( v_n > 0 \) for all \( n \in \mathbb{N} \) big enough, so by \( (5.2) \) and \( \rho_n \to \infty \) we get

\[ \lim_n \frac{F(x, \rho_n v_n)}{\rho_n^p} = \lim_n \frac{F(x, \rho_n v_n)}{(\rho_n v_n)^p} v_n^p = \frac{a_\infty(x)v^p}{p}. \]

Thus, by applying Fatou’s lemma, as in Lemma 5.1, we have

\[ \lim \sup_n I_n^1 \leq \int_{\{v > 0\}} \frac{a_\infty(x)v^p}{p} \, dx. \]
In \( \{ v \leq 0 \} \) we have \( v_n^+ \to 0 \), with dominated convergence in \( L^p(\Omega) \), so
\[
\lim_n \int_{\{v \leq 0\}} (v_n^+)^p \, dx = 0,
\]
which in turn implies, along with (5.1),
\[
\limsup_n I_n^2 \leq C \limsup_n \int_{\{v \leq 0\}} 1 + \left( \frac{\rho_n v_n^+}{\rho_n} \right)^p \, dx \\
\leq C \lim_n \left( \frac{1}{\rho_n} + \int_{\{v < 0\}} (v_n^+)^p \, dx \right) = 0.
\]
(c) Finally, in \( \{ v_n \leq 0 \} \) for all \( n \in \mathbb{N} \) we have
\[
F(x, \rho_n v_n) = f(x, 0) \rho_n v_n,
\]
so by (2.4)
\[
\limsup_n I_n^2 \leq \limsup_n \int_{\{v < 0\}} f(x, 0) \frac{v_n}{\rho_n} \, dx \leq 0.
\]
Adding up the relations above, we find (5.4). Now, by \( v_n \to v \) in \( W_0^{a,p}(\Omega) \) and (5.4) we have
\[
\frac{\|v\|^p}{p} \leq \liminf_n \frac{\|v_n\|^p}{p} \\
\leq \limsup_n \frac{1}{\rho_n} \left( \Phi(u_n) + \int_{\Omega} F(x, u_n) \, dx \right) \\
\leq \limsup_n \left( \frac{C}{\rho_n} + \int_{\Omega} \frac{F(x, \rho_n v_n)}{\rho_n} \, dx \right) \\
\leq \int_{\{v > 0\}} a_\infty(x) v^p \, dx.
\]
Recalling (1.2) (with \( a = a_\infty \)), by the inequality above and by the fact that \( \|v^+\| \leq \|v\| \), we have
\[
\lambda_1(a_\infty) \|v^+\|^p_p \leq \|v^+\|^p - \int_{\{v > 0\}} a_\infty(x) v^p \, dx \leq 0.
\]
Since by assumption \( \lambda_1(a_\infty) > 0 \), we have \( v \leq 0 \) in \( \Omega \). But then again, by (5.5), we have
\[
\|v\|^p \leq \int_{\{v > 0\}} a_\infty(x) v^p \, dx = 0.
\]
So \( v = 0 \), against (5.3). This contradiction proves that \( \Phi \) is coercive in \( W_0^{a,p}(\Omega) \). \( \square \)

On the other hand, \( a_0 \) determines the behavior of \( \Phi \) near the origin:

**Lemma 5.3.** Let \( \textbf{H} \) hold and \( \lambda_1(a_0) < 0 \). Then, there exists \( \bar{u} \in W_0^{a,p}(\Omega) \) s.t. \( \Phi(\bar{u}) < 0 \).

**Proof.** Using \( \textbf{H} \) (iii) and de l’Hôpital’s rule, we see that for a.e. \( x \in \Omega \)
\[
\lim_{t \to 0^+} \frac{F(x, t)}{t^p} = \frac{a_0(x)}{p}.
\]
By (1.2), we can find \( v \in W_0^{a,p}(\Omega) \) s.t.
\[
\|v\|^p - \int_{\{v \neq 0\}} a_0(x) |v|^p \, dx < 0.
\]
By a density argument, and replacing if necessary \( v \) with \( |v| \), we may assume \( v \in L^\infty(\Omega)_+ \). By (5.6), in \( \{ v > 0 \} \) we have for a.e. \( x \in \Omega \)
\[
\lim_{\varepsilon \to 0^+} \frac{F(x, \varepsilon v)}{\varepsilon^p} = \frac{a_0(x) v^p}{p}.
\]
Besides, given \( \delta > \|v\|_\infty \), by (2.3), we have in \( \{ v > 0 \} \) and for all \( \varepsilon \in (0, 1) \)
\[
\frac{F(x, \varepsilon v)}{\varepsilon^p} = \int_{0}^{\varepsilon v} \frac{f(x, t)}{\varepsilon^p} \, dt \geq - C_\delta \|v\|_\infty^p.
\]
So we can apply Fatou’s lemma and (5.7) and find
\[
\liminf_{\varepsilon \to 0^+} \int_{\{v > 0\}} \frac{F(x, \varepsilon v)}{\varepsilon^p} dx \geq \int_{\{v > 0\}} \frac{a_0(x)v^p}{p} dx > \frac{\|v\|^p}{p}.
\]
Then, for all \(\varepsilon > 0\) small enough we have
\[
\int_{\{v > 0\}} \frac{F(x, \varepsilon v)}{\varepsilon^p} dx > \frac{\|v\|^p}{p}.
\]
Now we set \(\bar{u} = \varepsilon v \in W^{s, p}_0(\Omega)\) and compute
\[
\Phi(\bar{u}) = \frac{\varepsilon^p\|v\|^p}{p} - \int_{\{v > 0\}} F(x, \varepsilon v) dx
\]
\[
= \varepsilon^p \left(\frac{\|v\|^p}{p} - \int_{\{v > 0\}} \frac{F(x, \varepsilon v)}{\varepsilon^p} dx\right) < 0,
\]
thus concluding.
\(\Box\)

**Remark 5.4.** In Lemmas 5.1, 5.2, and 5.3 above we did not use the *strict* monotonicity of \(H\) (iii). In fact, all the results in this Section can be proved, with minor adjustments, under the weaker condition (2.3) in place of \(H\) (iii).

The final step consists in proving the sufficient condition for existence:

**Proposition 5.5.** Let \(H\) hold and \(\lambda_1(a_0) < 0 < \lambda_1(a_\infty)\). Then, (1.1) has a solution.

**Proof.** From Lemmas 5.1, 5.2 we know that \(\Phi : W^{s, p}_0(\Omega) \to \mathbb{R} \cup \{\infty\}\) is sequentially weakly l.s.c. and coercive. By reflexivity of \(W^{s, p}_0(\Omega)\), there exists \(u \in W^{s, p}_0(\Omega)\) s.t.
\[
(5.8) \quad \Phi(u) = \inf_{v \in W^{s, p}_0(\Omega)} \Phi(v).
\]
We may assume \(u \geq 0\) in \(\Omega\). Otherwise, we replace \(u\) with \(u^+ \in W^{s, p}_0(\Omega)\). Indeed, by (2.4) we have
\[
\Phi(u^+) = \frac{\|u^+\|^p}{p} - \int_{\{u > 0\}} F(x, u) dx
\]
\[
\leq \frac{\|u\|^p}{p} - \int_{\{u > 0\}} F(x, u) dx + \int_{\{u \leq 0\}} f(x, 0) u dx \leq \Phi(u).
\]
Also, by Lemma 5.3 we have
\[
\Phi(u) \leq \Phi(\bar{u}) < 0,
\]
hence \(u \in W^{s, p}_0(\Omega) \setminus \{0\}\). Due to the lack of differentiability of \(\Phi\), we cannot infer immediately that \(u\) satisfies (2.2) and afterwards apply the regularity theory developed in Section 2. Instead, we prove independently that in (5.8) we may assume
\[
(5.9) \quad u \in L^\infty(\Omega).
\]
To this end, following [7], we introduce a sequence of truncated reactions by setting for all \(k \in \mathbb{N}\) and all \((x, t) \in \Omega \times \mathbb{R}\)
\[
f_k(x, t) = f(x, t^+) \vee (-k(t^+)^{p-1}).
\]
By \(H\), it is immediately seen that \(f_k : \Omega \times \mathbb{R} \to \mathbb{R}\) is a Carathéodory function for every \(k \in \mathbb{N}\). By \(H\) (i) we have for all \(t \in \mathbb{R}\)
\[
f_k(\cdot, t) \in L^\infty(\Omega).
\]
Plus, by \(H\) (ii) we have for a.e. \(x \in \Omega\) and all \(t \in \mathbb{R}^+\)
\[
-k^{p-1} \leq f_k(x, t) \leq (c_0 \vee k)(1 + t^{p-1}),
\]
while by (2.4) we have for a.e. \(x \in \Omega\) and all \(t \in \mathbb{R}^-\)
\[
|f_k(x, t)| \leq \|f(\cdot, 0)\|_\infty.
\]
In conclusion, for any \(k \in \mathbb{N}\) we can find \(c_k > 0\) s.t. for a.e. \(x \in \Omega\) and all \(t \in \mathbb{R}\)
\[
(5.10) \quad |f_k(x, t)| \leq c_k(1 + |t|^{p-1}).
\]
By $H$ (iii) we have for a.e. $x \in \Omega$ and all $0 < t < t'$
\[
\frac{f_k(x, t)}{t^p - 1} = \frac{f(x, t)}{t^p - 1} = \frac{f(x, t')}{{t'}^p - 1} \vee (-k) \geq \frac{f(x, t')}{{t'}^p - 1} \vee (-k) = \frac{f_k(x, t')}{(t')^p - 1},
\]
i.e., the map
\[
t \mapsto \frac{f_k(x, t)}{t^p - 1}
\]
is nonincreasing in $\mathbb{R}^+_0$, as well. Thus, $f_k$ satisfies $H$ (but the strict monotonicity in $H$ (iii)), and in addition the bilateral growth condition (5.10). In addition, we have for all $k \in \mathbb{N}$, a.e. $x \in \Omega$, and all $t \in \mathbb{R}$ the following useful inequality:
\[
(5.11) \quad f_k(x, t) \geq f_{k+1}(x, t) \geq f(x, t).
\]
By monotonicity, for all $k \in \mathbb{N}$ we may define two measurable functions by setting for a.e. $x \in \Omega$
\[
a_0^k(x) = \lim_{t \to 0^+} \frac{f_k(x, t)}{t^p - 1}, \quad a_\infty^k(x) = \lim_{t \to \infty} \frac{f_k(x, t)}{t^p - 1}.
\]
Some remarks on the sequences $(a_0^k)$, $(a_\infty^k)$ are now in order. First we focus on $(a_0^k)$. From (5.11) we have for a.e. $x \in \Omega$ and all $t \in \mathbb{R}^+_0$
\[
\frac{f_k(x, t)}{t^p - 1} \geq \frac{f(x, t)}{t^p - 1}.
\]
Passing to the limit as $t \to 0^+$ gives
\[
a_0^k(x) \geq a_0(x),
\]
hence in particular $a_0^k$ is bounded from below in $\Omega$. Now define $\lambda_1(a_0^k)$ as in (1.2). We have already seen (in Remark 4.2) that the map $a \mapsto \lambda_1(a)$ is nonincreasing, so by the main assumption we have for all $k \in \mathbb{N}$
\[
(5.12) \quad \lambda_1(a_0^k) \leq \lambda_1(a_0) < 0.
\]
The case for $(a_\infty^k)$ is subtler. First note that, by (5.10), we have $a_\infty^k \in L^\infty(\Omega)$ for all $k \in \mathbb{N}$. Also, dividing (5.11) by $t^p - 1$ and then letting $t \to \infty$, we get for all $k \in \mathbb{N}$ and a.e. $x \in \Omega$
\[
(5.13) \quad a_\infty^k(x) \geq a_\infty^{k+1}(x) \geq a_\infty(x).
\]
In fact we have for a.e. $x \in \Omega$
\[
(5.14) \quad \lim_{k} a_\infty^k(x) = a_\infty(x).
\]
Indeed, by (5.13) the sequence $(a_\infty^k(x))$ is nonincreasing, hence the limit above exists, and in addition
\[
\lim_{k} a_\infty^k(x) \geq a_\infty(x).
\]
Now fix $M > a_\infty(x)$. We can find $T > 0$ (depending on $x$) s.t. for all $t \geq T$
\[
\frac{f(x, t)}{t^p - 1} < M,
\]
For any such $t$, choose $k \in \mathbb{N}$ s.t.
\[
k \geq \frac{\|f(\cdot, t)\|_\infty}{t^p - 1},
\]
so we have
\[
\frac{f_k(x, t)}{t^p - 1} = \frac{f(x, t)}{t^p - 1} < M.
\]
Finally, let $t \to \infty$ to get
\[
a_\infty^k(x) \leq M,
\]
which completes the proof of (5.14). The next step consists in proving that
\[
(5.15) \quad \lim_{k} \lambda_1(a_\infty^k) = \lambda_1(a_\infty) > 0.
\]
By (5.13) and the monotonicity of $a \mapsto \lambda_1(a)$ (Remark 4.2 again), we see that $(\lambda_1(a_\infty^k))$ is a nondecreasing sequence, hence the limit above exists. Still by (5.13) we have
\[
\lim_{k} \lambda_1(a_\infty^k) \leq \lambda_1(a_\infty).
\]
Now, if \( \lambda_1(a^k_{\infty}) \to \infty \), then \( \lambda_1(a_{\infty}) = \infty \) and there is nothing to prove. So, let us assume that \( (\lambda_1(a^k_{\infty})) \) is bounded from above. For all \( k \in \mathbb{N} \) there exists \( v_k \in W^{s,p}_0(\Omega) \) s.t. \( \|v_k\|_p = 1 \) and
\[
\|v_k\|^p - \int_{\{v_k \neq 0\}} a^k_{\infty}(x)|v_k|^p \, dx < \lambda_1(a^k_{\infty}) + \frac{1}{k}.
\]
By (5.10) and (5.13), we can find \( M > 0 \) s.t. \( a^k_{\infty} \leq M \) in \( \Omega \), for all \( k \in \mathbb{N} \). So the last inequality gives for all \( k \in \mathbb{N} \)
\[
\|v_k\|^p < \int_{\{v_k \neq 0\}} a^k_{\infty}(x)|v_k|^p \, dx + \lambda_1(a^k_{\infty}) + \frac{1}{k} 
\leq M\|v_k\|^p + C < C.
\]
Therefore, \((v_k)\) is bounded in \( W^{s,p}_0(\Omega) \). By reflexivity and the compact embedding \( W^{s,p}_0(\Omega) \hookrightarrow L^p(\Omega) \), passing to a subsequence if necessary, we have \( v_k \rightharpoonup v \) in \( W^{s,p}_0(\Omega) \) and \( v_k \to v \) in \( L^p(\Omega) \). Hence in particular \( \|v\|_p = 1 \).
By weak convergence we have
\[
\liminf_k \|v_k\|_p \geq \|v\|_p.
\]
Also, by (5.14), \( a^k_{\infty}(x)|v_k(x)|^p \to a_{\infty}(x)|v(x)|^p \) for a.e. \( x \in \Omega \) with dominated convergence from above, so by Fatou’s lemma
\[
\limsup_k \int_{\{v_k \neq 0\}} a^k_{\infty}(x)|v_k|^p \, dx \leq \int_{\{v \neq 0\}} a_{\infty}(x)|v|^p \, dx.
\]
Therefore, using also (1.2), we have
\[
\lambda_1(a_{\infty}) \leq \|v\|^p - \int_{\{v \neq 0\}} a_{\infty}(x)|v|^p \, dx
\leq \liminf_k \left( \|v_k\|^p - \int_{\{v_k \neq 0\}} a^k_{\infty}(x)|v_k|^p \, dx \right)
\leq \lim_k \left( \lambda_1(a^k_{\infty}) + \frac{1}{k} \right) = \lim_k \lambda_1(a^k_{\infty}),
\]
hence we have (5.15).
Now, by (5.12) and (5.15) we can fix \( k \in \mathbb{N} \) s.t.
\[
\lambda_1(a^k_{\infty}) < 0 < \lambda_1(a^k_{\infty}).
\]
Set for all \((x, t) \in \Omega \times \mathbb{R}\)
\[
F_k(x, t) = \int_0^t f_k(x, \tau) \, d\tau,
\]
and for all \( v \in W^{s,p}_0(\Omega) \)
\[
\Phi_k(v) = \frac{\|v\|^p}{p} - \int_\Omega F_k(x, v) \, dx.
\]
Arguing as above and applying Lemmas 5.1, 5.2, and 5.3 (recalling Remark 5.4), we find \( u_k \in W^{s,p}_0(\Omega)_+ \setminus \{0\} \) s.t.
\[
(5.16) \quad \Phi_k(u_k) = \inf_{v \in W^{s,p}_0(\Omega)} \Phi_k(v).
\]
But now, by (5.10) we have \( \Phi_k \in C^1(W^{s,p}_0(\Omega)) \) with derivative given for all \( v, \varphi \in W^{s,p}_0(\Omega) \) by
\[
\langle \Phi_k'(v), \varphi \rangle = \langle (-\Delta)_s^p v, \varphi \rangle - \int_\Omega f_k(x, v)\varphi \, dx.
\]
So we can differentiate in (5.16) and see that \( u_k \) is a weak solution (in the sense of Section 2) of the following problem:
\[
\left\{ \begin{array}{ll}
(-\Delta)_s^p u_k = f_k(x, u_k) & \text{in } \Omega, \\
u_k = 0 & \text{in } \Omega^c.
\end{array} \right.
\]
Reasoning as in Lemmas 2.2 and 2.3, we see that \( u_k \in C^\alpha(\overline{\Omega}) \) and there exists \( C > 1 \) s.t. in \( \Omega \)
\[
\frac{d\Omega}{C} \leq u_k \leq Cd\Omega.
\]
Define
\[ \underline{u}_k = u \land u_k, \quad \overline{u}_k = u \lor u_k. \]

By the lattice structure of \( W^{s,p}_0(\Omega) \) we have \( u_k, \overline{u}_k \in W^{s,p}_0(\Omega) \). Also, since \( 0 \leq u_k \leq u \) in \( \Omega \), we clearly have \( \underline{u}_k \in L^\infty(\Omega) \). Now we claim that
\begin{equation}
(5.17) \quad \Phi(\underline{u}_k) \leq \Phi(u).
\end{equation}

Indeed, by (5.16) we have
\[ \Phi_k(\underline{u}_k) \leq \Phi_k(u_k) \]
By the inequality above and (5.11) we have
\[ \|u_k\|_p - \|\overline{u}_k\|_p \leq \int_{\{u > u_k\}} (F_k(x, u_k) - F_k(x, u)) \, dx \]
\[ = \int_{\{u > u_k\}} \int_u^{u_k} f_k(x, t) \, dt \, dx \]
\[ \leq \int_{\{u > u_k\}} \int_u^{u_k} f(x, t) \, dt \, dx \]
\[ = \int_{\{u > u_k\}} (F(x, u_k) - F(x, u)) \, dx \]
\[ = \int_{\Omega} F(x, u_k) \, dx - \int_{\Omega} F(x, u) \, dx. \]
Besides, by the submodularity inequality (2.1) we have
\[ \frac{\|u_k\|_p^p}{p} - \frac{\|\overline{u}_k\|_p^p}{p} \leq \frac{\|u\|_p^p}{p} + \frac{\|u_k\|_p^p}{p}. \]
Concatenating the last relations we have
\[ \frac{\|u_k\|_p^p}{p} - \frac{\|u\|_p^p}{p} \leq \int_{\Omega} F(x, u_k) \, dx - \int_{\Omega} F(x, u) \, dx, \]
which is equivalent to (5.17). Thus, replacing if necessary \( u \) with \( u_k \) in (5.8), we finally get (5.9). By (2.3) with \( \delta > \|u\|_\infty \) and \( H \) (ii), we have for a.e. \( x \in \Omega \) and all \( t \in [0, \delta] \)
\[ |f(x, t)| \leq C(1 + t^{p-1}). \]
Such local bilateral growth condition allows us to differentiate in (5.8), thus finding that \( u \in W^{s,p}_0(\Omega) \setminus \{0\} \) satisfies (2.2). Then we can apply Lemmas 2.2, 2.3 and conclude that \( u \in C^\alpha(\overline{\Omega}) \) and \( u > 0 \) in \( \Omega \), hence \( u \) solves (1.1).

**Conclusion.** Simply lining up Propositions 3.1, 4.4, and 5.5 we have the complete proof of Theorem 1.3.

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