Universality far from equilibrium: From superfluid Bose gases to heavy-ion collisions

J. Berges,\textsuperscript{1,2} K. Boguslavski,\textsuperscript{1} S. Schlichting,\textsuperscript{3} and R. Venugopalan\textsuperscript{3}

\textsuperscript{1}Institut für Theoretische Physik, Universität Heidelberg, Philosophenweg 16, 69120 Heidelberg, Germany
\textsuperscript{2}ExtreMe Matter Institute EMMI, GSI Helmholtzzentrum, Planckstraße 1, 64291 Darmstadt, Germany
\textsuperscript{3}Brookhaven National Laboratory, Physics Department, Bldg. 510A, Upton, NY 11973, USA

Isolated quantum systems in extreme conditions can exhibit unusually large occupancies per mode. This over-population gives rise to new universality classes of many-body systems far from equilibrium. We present theoretical evidence that important aspects of non-Abelian plasmas in the ultra-relativistic limit admit a dual description in terms of a Bose condensed scalar field theory.

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\section{I. INTRODUCTION}

In recent years there have been important advances in understanding isolated quantum systems in extreme conditions far from equilibrium. Prominent examples include the (pre-)heating process in the early universe after inflation, the initial stages in collisions of ultra-relativistic nuclei at giant laboratory facilities, as well as table-top experiments with ultracold quantum gases. Even though the typical energy scales of these systems vastly differ, they can show very similar dynamical properties. Certain characteristic numbers can even be quantitatively the same. One may use this universality to learn from experiments with cold atoms aspects about the dynamics during the early stages of our universe \cite{1}. Ultraportant quantum gases are known to exhibit universal properties near unitarity in the presence of a very large scattering length $a$ \cite{2}. In this work we consider a different universal regime away from unitarity, which occurs far from equilibrium and has attracted much interest recently in the context of nonthermal fixed points \cite{3,4}. For an interacting Bose gas of density $n$ with an inverse coherence length described by the momentum $Q = \sqrt{16\pi a}$, this novel regime is characterized by an unusually large mode occupancy

\[ f(Q) \sim 1/\eta \] \hspace{1cm} (1)

in the dilute regime where $\eta = \sqrt{n a^3} \ll 1$. The average density $n = \int d^3 p / (2\pi)^3 f(p) \sim Q^3 / \eta$ becomes parametrically large, reflecting the underlying extreme nonequilibrium distribution $f(p)$ of modes. As a consequence of the large typical occupancies, the system is strongly correlated and its many-body properties become insensitive to the details of the underlying model or initial conditions. The far-from-equilibrium behavior can be described in terms of universal scaling exponents and scaling functions \cite{3,8}, similar to the description of critical phenomena in thermal equilibrium.

Such an over-occupation of modes can be found in a variety of systems in extreme conditions. In heavy-ion collisions at ultra-relativistic energies, a nonequilibrium plasma of highly occupied gluon fields with characteristic momentum $Q_s$ is expected to form shortly after the collision \cite{9,10}. While the running gauge coupling $\alpha(Q_s)$ is weak for sufficiently large $Q_s$, the system becomes strongly correlated because the typical gluon occupancy $g(Q_s) \sim 1/\alpha$ is large. Here the coupling plays the corresponding role to the diluteness parameter in (1). The characteristic gluon density is then parametrically $n_g \sim Q_s^3 / \alpha$ and the energy density for relativistic particles $\epsilon_g \sim Q_s^4 / \alpha$. Comparing the latter to a Stefan-Boltzmann law $\sim T^4$ with a typical thermal momentum $T$, one observes that the nonequilibrium distribution exhibits a smaller characteristic momentum $Q_s \sim \alpha^{1/4} T$ than the corresponding thermal system. Scaling behavior has been observed in this context in simulations of the space-time evolution of non-Abelian plasmas \cite{11,12}.

In this letter we present first theoretical evidence that important universal properties of these very different systems far from equilibrium can agree. Since symmetries and underlying scattering processes for scalar and gauge theories show profound differences, the observation of universal dynamics in this case is highly non-trivial. We employ the largest real-time lattice gauge theory simulations to date to analyze the space-time evolution of non-Abelian plasmas in the limit of ultra-relativistic energies \cite{13}. These results are compared to the nonequilibrium dynamics of scalar Bose fields. To put our studies in the context of heavy-ion collision experiments, in both cases we consider longitudinally expanding systems in three space dimensions. We consider a relativistic bosonic field theory with weak coupling parameter $\lambda$, for which suitable atomic model Hamiltonians may be constructed \cite{14}. We will see below that characteristic low-momentum properties turn out to be in the same universality class as the corresponding non-relativistic theory. Moreover, we generalize our bosonic system to include $N$ real-valued field components to study the possible dependence of the universality class on $N$. For the non-Abelian gluon fields we consider two different colors, since no indications for a significant dependence on a larger number of colors have been reported so far.

\footnote{Here three spatial dimensions are considered. To ease the comparison with high-energy particle physics, natural units will be employed where Planck’s constant over $2\pi \hbar$, the speed of light $(c)$ and Boltzmann’s constant $(k_B)$ are set to one.}

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II. DYNAMIC UNIVERSALITY CLASS

Universal scaling behavior of many-body systems in vacuum or thermal equilibrium are known to be efficiently classified in terms of universality classes, which are characterized by scaling exponents and scaling functions. Depending on the range of momenta, in equilibrium one distinguishes between infrared and ultraviolet fixed points and associated scaling properties. In general, scaling behavior of systems far from equilibrium can exhibit several distinct momentum regimes. Explicit examples for isolated systems starting from extreme conditions have been given in the context of wave turbulence [3-5, 17, 18].

For the following presentation, we classify dynamic universality classes in terms of scaling properties of a time-dependent distribution function and refer for further discussions in terms of nonthermal renormalization group fixed points to the literature (see e.g., [4]). For longitudinally expanding systems the distribution function depends on proper time $t$, on transverse momentum $p_T$, and on longitudinal momentum $p_z = \nu/t$ where $\nu$ denotes the rapidity wave number. In the universal regime the distribution is then determined by a time-independent scaling function $f_S$, an overall scaling with time with scaling exponent $\alpha$, and two exponents $\beta$ and $\gamma$ for the scaling behavior of transverse and longitudinal momentum:

$$f(p_T, p_z, t) = (Qt)^\alpha f_S \left( (Qt)^\beta p_T, (Qt)^\gamma p_z \right).$$

Systems belonging to the same universality class have the same values for $\alpha$, $\beta$ and $\gamma$ as well as the same form of $f_S(p_T, p_z)$ in a given inertial range of momenta.

The distribution function reflects properties of equal-time correlation functions of the underlying quantum field theory. For the massless scalar theory with quartic interaction $\lambda (\sum_a \Phi_a(x) \Phi_a(x))^2$/4!$N$ for the $a = 1, \ldots, N$ field components, the anti-commutator expectation value $F(x, x', t, t') = \sum_a \langle [\Phi_a(x, t), \Phi_a(x', t')] \rangle/2N$ determines the distribution. Using in spatial Fourier space the notation $\tilde{F}(p, t) = \partial_t \partial_t' \tilde{F}(p; t, t')|_{t=t'}$ and $\bar{F}(p; t) \equiv (\partial_t \tilde{F}(p; t, t') + \partial_{t'} \tilde{F}(p; t, t'))|_{t=t'}/2$ it reads

$$f(p, t) + 1/2 = \left( F(p; t) - \bar{F}(p; t) \right)^{1/2}. \quad (3)$$

A similar definition can also be given for the $SU(2)$ symmetric gauge field theory if an additional (Coulomb) gauge fixing is applied [15]. We emphasize that the interpretation of $f(p_T, p_z, t)$ as a single-particle distribution may be misleading in the over-populated regime, in particular for the gauge theory. However, our results indicate that the scaling properties can be efficiently classified in terms of the underlying correlation functions. We will also assume that the quantum field theory and the corresponding classical-statistical field theory are in the same universality class for sufficiently high occupations, and employ standard lattice simulation techniques to compute the nonequilibrium time evolution [15]. The agreement is well established for critical phenomena in thermal equilibrium and has been verified explicitly for the nonequilibrium scalar field theory in the highly occupied regime, where $f(p_T, p_z, t) \gg 1/2$ such that the ‘quantum-half’ in (3) becomes insignificant [3].

We start our evolution at time $t_0$ from over-populated initial conditions,

$$f(p, t_0) = \frac{n_0}{\lambda} \Theta\left( Q - \sqrt{p_T^2 + (\xi_0 p_z)^2} \right). \quad (4)$$
where $n_0$ parametrizes the initial amplitude and $\xi_0$ the initial anisotropy of the distribution function. Following the procedure of Ref. [18] typical values employed for scalars are $n_0 = 35$, $\xi_0 = 1$ and $Q t_0 = 10^4$. Universal results are independent of the choice of the initial conditions and the coupling, and we vary $t_0$, $n_0$, $\xi_0$ and $\lambda$ to verify this. The typical lattices employed for the scalar theory range from $96^2 \times 512$ to $192^2 \times 786$, while for the gauge theory we use $256^2 \times 2048$ and refer to Ref. [15] for different sets of parameters.

Fig. 1 shows scalar field theory results for the reduced distribution

$$\tilde{f}(p_T, t) = \frac{t}{t_0} \int \frac{dp_z}{2\pi Q} f(p_T, p_z, t)$$

integrated over rapidity wave number $\nu = tp_z$. It is plotted as a function of the transverse momentum for $N = 4$ at different times. One observes that it quickly becomes approximately time-independent. With (2) follows

$$\tilde{f}(p_T, t) \sim (Q t)^{\alpha - \gamma + 1} \int \frac{dp_z}{(2\pi)} f_S(Q t) \tilde{p}_T$$

such that time independence implies the scaling relations

$$\alpha - \gamma + 1 = 0 , \quad \beta = 0 .$$

Remarkably, the very same relations have been observed in Ref. [15] for the non-Abelian gauge theory. We have explicitly verified that the same relations hold in the scalar field theory case also for $N = 2$.

In order to clarify the physics that determines the universality class, we note that Fig. 1 also exhibits a rather accurate power law $\tilde{f}(p_T, t) \sim 1/p_T$ for $p_T \lesssim Q$. Here $p_T \tilde{f}(p_T, t)$ is constant in transverse momentum as well as in time. This means that the modes are uniformly distributed in the transverse plane and the corresponding number density $\sim \int d^2 p_T \tilde{f}(p_T, t)$ is conserved at each transverse momentum separately. Since the longitudinal momenta are red-shifted because of expansion, one expects for typical momenta $p_T \gg p_z$. Accordingly, the energy density should scale as $\sim \int d^2 p_T p_T \tilde{f}(p_T, t) = \text{const}$ in the universal regime. This implies that in addition to the number density the energy density is conserved.

The existence of a scaling solution where both energy and particle number are conserved is remarkable when contrasted to the discussion in isotropic systems without longitudinal expansion [17]. There is no single scaling solution conserving both energy and particle number in that case. Instead, a dual cascade emerges such that in a given momentum range only one conservation law constrains the scaling solution [3, 6, 19]. The fact that for longitudinally expanding systems both conservation laws are effective in the same momentum regime works in favor of a large universality class encompassing rather different systems. For instance, the exponent $\beta = 0$ is entirely fixed by enforcing both conservation laws without further knowledge about the underlying dynamics. Consequently, one may expect the same exponent to appear in a larger class of isolated systems, which are dominated by number conserving processes and undergo a longitudinal expansion.

The conservation laws indicate that modes are redistributed along the longitudinal direction. This has been analyzed in detail for the corresponding gauge theory in Ref. [15]. Fig. 2 shows our results for the scalar theory for intermediate $p_T \sim Q/2$, where the rescaled distribution as a function of the rescaled longitudinal momentum is given for different times. According to (2) this fixed point distribution should be independent of time, and indeed we find that all data collapses onto a single curve using the scaling exponents

$$\alpha = -2/3 , \quad \gamma = 1/3$$
to few percent accuracy in accordance with the scaling relation (6). The very same exponents (7) were found to characterize the gauge theory [15, 20]. For the latter we also give our results for the normalized fixed point distribution in Fig. 2. The Gaussian shaped curve with width squared $\sigma^2 = \int dp/p^2 f(t_1)/\int dp f(t_1)$ for the scalar theory is seen to accurately agree with the corresponding gauge theory case. These results are a striking manifestation of universality in this characteristic intermediate transverse momentum regime.

III. BOSE CONDENSATION

In the following we analyze the deep infrared as well as the high momentum regime of both theories in more detail to search for possible different infrared and ultraviolet fixed points. For non-expanding systems a nonthermal infrared fixed point has been observed in the context of wave turbulence in scalar field theories [9–11]. In that case, one of the striking consequences is the presence of a zero mode showing Bose condensation far from equilibrium [7, 8, 22]. Similar speculations have also been put forward for over-populated gauge theories [21] and condensation in that case is debated.

Fig. 3 shows the scalar distribution function at vanishing momentum for different times. The data is for $N = 4$ but we find corresponding results also for $N = 2$. Apart from the characteristic $\sim 1/p_T$ behavior at intermediate momenta discussed above, one observes a distinct infrared regime below the momentum scale where the occupation number becomes larger than $\sim 1/\lambda$. The infrared spectrum can be characterized by a steep power law $f \sim 1/p^5$, and we observe that the distribution function is isotropic in this regime. We conclude that the extreme occupancies at low-momenta enhance scattering rates to overcome the longitudinal expansion. The scaling behavior agrees with the inverse particle cascade found in non-relativistic Bose gases without expansion, where the observed scaling exponent characterizes superfluid turbulence in three spatial dimensions [4–8].

In Fig. 4 we show our results for the gauge theory. For the shorter times that were accessible because of limited computational resources there is no obvious strong enhancement of infrared modes as seen for the scalar field theory. Apart from the characteristic $\sim 1/p_T$ behavior discussed above, one notes the emergence of a slight infrared enhancement. However, the concept of a gauge dependent number distribution may become problematic at low momenta and further studies in this regime should concentrate on gauge invariant correlation functions. For the scalar theory we also note a shift of the $\sim 1/p_T$ regime to lower momenta at later times, giving rise to a flat distribution range at higher momenta which will be studied elsewhere. For the accessible shorter times we cannot attribute a similar regime to the gauge theory.

Since for the scalar theory the nonthermal infrared fixed point has been shown to catalyze the formation of a Bose-Einstein condensate in non-expanding systems [7, 8], we repeat the corresponding analysis for our case and search for a zero mode which scales with volume, i.e. $F(p = 0; t) \sim V$. This function oscillates and we take the period average to illustrate the evolution of this observable. The results are shown in Fig. 5 where we present the time evolution of the condensate observable for different system sizes. For the initial conditions [1] there is no condensate at time $t_0$. Accordingly, Fig. 5 shows at early times the decrease of the ratio $F(p = 0; t)/(VQ^2)$ as the volume is increased. However, after a transient regime the ratio becomes volume independent signaling the presence of a time-dependent condensate evolving as $\sim (t/t_0)^{-1/3}$. The formation of a coherent zero mode over the entire volume is truly remarkable in view of the longitudinal expansion of the system.

IV. CONCLUSIONS

The observed universality challenges our understanding of the thermalization process of the Quark Gluon Plasma in the limit of very high energies, where the gauge coupling is weak and the dynamics starts from an over-populated regime. Since the underlying perturbative scattering processes are very different for gauge field and scalar degrees of freedom, our findings point to a much more general principle. Here the classification of strongly correlated quantum many-body systems in terms of nonequilibrium universality classes and associated scaling properties represents a crucial step. Such a development also opens intriguing new perspectives to experimentally access universal properties of systems in extreme conditions with the help of quantum degenerate gases.
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