CDNNs: The coupled deep neural networks for coupling of the Stokes and Darcy-Forchheimer problems *

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Abstract

In this article, we present an efficient deep learning method called coupled deep neural networks (CDNNs) for coupled physical problems. Our method compiles the interface conditions of the coupled PDEs into the networks properly and can be served as an efficient alternative to the complex coupled problems. To impose energy conservation constraints, the CDNNs utilize simple fully connected layers and a custom loss function to perform the model training process as well as the physical property of the exact solution. The approach can be beneficial for the following reasons: Firstly, we sampled randomly and only input spatial coordinates without being restricted by the nature of samples. Secondly, our method is meshfree which makes it more efficient than the traditional methods. Finally, our method is parallel and can solve multiple variables independently at the same time. We give the theory to guarantee the convergence of the loss function and the convergence of the neural networks to the exact solution. Some numerical experiments are performed and discussed to demonstrate the performance of the proposed method.

Key words: Scientific computing, Machine learning, the Stokes equations, Darcy-Forchheimer problems, Beaver-Joseph-Saffman interface condition.

1 Introduction

The fluid flow between porous media and free-flow zones has extensive applications in hydrology, environmental science, and biofluid dynamics. A lot of researchers derive suitable mathematical and numerical models for fluid movement. The system can be viewed as a coupled problem with two physical systems interacting across an interface. The simplest mathematical formulation for the coupled problem is coupling of the Stokes and Darcy flow with proper interface conditions. The most suitable and popular interface conditions are called Beavers-Joseph-Saffman conditions.

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1 INTRODUCTION

However, Darcy’s law only provides a linear relationship between the gradient of pressure and velocity in the coupled model, which usually fails for complex physical problems. Forchheimer [2] conducted flow experiments in sand packs and recognized that for moderate Reynolds numbers ($Re > 0.1$ approximately), Darcy’s law is not adequate. He found that the pressure gradient and Darcy velocity should satisfy the Darcy-Forchheimer law. Since the great attention has received in the coupled model, a large number of traditional methods have been devoted to the coupled Stokes and Darcy flows problems [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. However, the difficulty of the complicated high dimensional coupled problems causes the limitation of traditional methods.

Owing to the enormous potential in approximating high-dimensional nonlinear maps [17, 18, 19, 20, 21, 22, 23, 24], deep learning has attracted growing attention in many applications, such as image, speech, text recognition and scientific computing [14, 15, 16]. Many works have arisen based on the function approximation capabilities of the feed-forward fully-connected neural network to solve initial/boundary value problems [31, 32, 33, 34] in the past decades. The solution to the system of equations can be obtained by minimizing the loss function, which typically consists of the residual error of the governing Partial Differential Equations (PDEs) along with initial/boundary values. Recently, Raissi etc. [35, 36, 37] developed Physics Informed Neural Networks (PINNs) [38, 39, 40, 41, 42, 43]. Moreover, Sirignano and Spiliopoulos proposed the Deep learning Galerkin Method [45] for solving high dimensional PDEs. Additionally, some recent works have successfully solved the second-order linear elliptic equations and the high dimensional Stokes problems [25, 26, 27, 28]. Although several excellent works have been performed in applying deep learning to solve PDEs, the topic for solving complicated coupled interface problems remains to be investigated.

Considering the performance of deep learning for solving PDEs, our contribution is to design the CDNNs as an efficient alternative model for complicated coupled physical problems. We can encode any underlying physical laws naturally as prior information to obey the law of physics. To satisfy the differential operators, boundary conditions and divergence conditions, we train the neural networks on batches of randomly sampled points. The method only inputs random sampling spatial coordinates without considering the nature of samples. Notably, we take the interface conditions as the constraint for the CDNNs. The approach is parallel and solves multiple variables independently at the same time. Specially, the optimal solution can be obtained by using the appropriate optimization method instead of a linear combination of basic functions. Furthermore, we validate the convergence of the loss function under certain conditions and the convergence of the CDNNs to the exact solution. Several numerical experiments are conducted to investigate the performance of the CDNNs.

The article is organized as follows: Section 2 introduces the coupled model and the relation methodology. Section 3 discusses the convergence of the loss function $J(\mathbf{U})$ and the convergence of the CDNNs to the exact solution. Section 4 reveals some numerical experiments to illustrate the efficiency of the CDNNs. The article ends with conclusion in section 5.
2 Methodology

Let $\Omega_S$ and $\Omega_D$ be two bounded and simply connected polygonal domains in $\mathbb{R}^2$ such that $\partial \Omega_S \cap \partial \Omega_D = \Gamma \neq \emptyset$ and $\Omega_S \cap \Omega_D = \emptyset$. Then, let $\Gamma_S := \partial \Omega_S \setminus \Gamma, \Gamma_D := \partial \Omega_D \setminus \Gamma$ and $\mathbf{n}_S$ as the unit normal vector pointing from $\Omega_S$ to $\Omega_D$, $\mathbf{n}_D$ as the unit normal vector pointing from $\Omega_D$ to $\Omega_S$, on the interface $\Gamma$ we have $\mathbf{n}_D = -\mathbf{n}_S$. In addition, $\mathbf{t}$ represents the unit tangential vector along the interface $\Gamma$. Figure 4 gives a schematic representation of the geometry.

When kinematic effects surpass viscous effects in a porous medium, the Darcy velocity $\mathbf{u}_D$ and the pressure gradient $\nabla p_D$ does not satisfy a linear relation. Instead, a nonlinear approximation, known as the Darcy-Forchheimer model, is considered. When it is imposed on the porous medium $\Omega_D$ with homogeneous Dirichlet boundary condition on $\Gamma_D$ the equations read:

$$\nabla \cdot \mathbf{u}_D = f_D, \quad \text{in } \Omega_D, \quad (1)$$

$$\frac{\mu}{\rho} \mathbf{K}^{-1} \mathbf{u}_D + \frac{\beta}{\rho} | \mathbf{u}_D | \mathbf{u}_D + \nabla p_D = \mathbf{g}_D, \quad \text{in } \Omega_D, \quad (2)$$

$$p_D = 0, \quad \text{on } \partial \Omega_D \setminus \Gamma, \quad (3)$$

where $\mathbf{K}$ is the permeability tensor, assumed to be uniformly positive definite and bounded, $\rho$ is the density of the fluid, $\mu$ is its viscosity and $\beta$ is a dynamic viscosity, all assumed to be positive constants. In addition, $\mathbf{g}_D$ and $f_D$ are source terms. We remark that in this context we exploit homogeneous Dirichlet boundary condition, in fact, we can also consider homogeneous Neumann boundary condition, i.e., $\mathbf{u}_D \cdot \mathbf{n}_D = 0$ on $\Gamma_D$ and the arguments used in this paper are still true.

The fluid motion in $\Omega_S$ is described by the Stokes equations:

$$-\nu \Delta \mathbf{u}_S + \nabla p_S = \mathbf{f}_S, \quad \text{in } \Omega_S, \quad (4)$$

$$\nabla \cdot \mathbf{u}_S = 0, \quad \text{on } \Omega_S, \quad (5)$$

$$\mathbf{u}_S = 0, \quad \text{on } \partial \Omega_S \setminus \Gamma, \quad (6)$$

where $\nu > 0$ denotes the viscosity of the fluid.

On the interface, we prescribe the following interface conditions

$$\mathbf{u}_S \cdot \mathbf{n}_S = \mathbf{u}_D \cdot \mathbf{n}_S, \quad \text{on } \Gamma, \quad (7)$$

$$p_S - \nu n_s \frac{\partial \mathbf{u}_S}{\partial n_S} = p_D, \quad \text{on } \Gamma, \quad (8)$$

$$-\nu \mathbf{t} \cdot \frac{\partial \mathbf{u}_S}{\partial n_S} = G \mathbf{u}_S \cdot \mathbf{t}, \quad \text{on } \Gamma. \quad (9)$$

Condition (7) represents continuity of the fluid velocity’s normal components, (8) represents the balance of forces acting across the interface, and (9) is the Beaver-Joseph-Saffman condition \[46\]. The constant $G > 0$ is given and is usually obtained from experimental data.

For notational brevity, we set $\mathbf{u} = (\mathbf{u}_S, \mathbf{u}_D, p_S, p_D)$ and recall the classical Sobolev spaces

$$X^0_S = \{ v_S \in [H^1(\Omega_S)]^d : v_S|_{\Gamma_S} = 0 \},$$
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Figure 1: Coupled domain with interface $\Gamma$.

$$\mathcal{Y}_D = \{q_D \in [W^{1,3/2}(\Omega_D)]^2 : q_D|_{\Gamma_D} = 0\},$$

$$\mathcal{X}_S = \{v_S \in X^0_S : \text{div } v_S = 0\},$$

where

$$H^k(\Omega) = \{v \in L^2(\Omega) : D^\alpha_v v \in L^2(\Omega), \forall \alpha : |\alpha| \leq k\},$$

and their norm

$$\|v\|_k = \sqrt{(v,v)_k} = \left\{ \sum_{|\alpha|=0}^k \int_{\Omega} (D^\alpha_v v)^2 dx \right\}^{1/2}, \quad \|v\|_{W_{k,p}} = \left\{ \sum_{|\alpha| \leq k} \|v\|_{L^p}^p \right\}^{1/p}.$$

Particularly,

$$\|v\|_k = \|v\|_{W^{k,2}}$$

where $k > 0$ is a positive integer and $\|v\|_0$ denotes the norm on $L^2(\Omega)$ or $(L^2(\Omega))^2$, $D^\alpha_v v$ is the generalized derivative of $v$. Moreover, $(\cdot, \cdot)_D$ represents the inner product in the domain $D$ and $<\cdot, \cdot>$ represents the inner product on the interface $\Gamma$.

To solve coupling of the Stokes and Darcy-Forchheimer problems, we propose the CDNNs in Figure 2. Further more, we give observations of the state variable $\mathbf{U}(x; \theta) = (U_S(x; \theta_1), U_D(x; \theta_2), P_S(x; \theta_3), P_D(x; \theta_4))$, which is the neural network solution to the coupled Stokes and Darcy-Forchheimer problem. $(\theta_1, \theta_3)$ and $(\theta_2, \theta_4)$ are the stacked parameters of $\theta$ for Stokes and Darcy respectively. The following constrained optimization procedure aims to reconstruct the parameters $\theta$ by minimizing the loss function

$$J[\mathbf{U}] = J_{\Omega_S \setminus \Gamma}[\mathbf{U}] + J_{\Omega_D \setminus \Gamma}[\mathbf{U}] + J_{\Gamma}[\mathbf{U}]. \quad (10)$$
The nodal values of the parameters in the input layer admitted by the deep learning model.

Furthermore, it should be noted that $J(\mathbf{U})$ can measure how well the approximate solution satisfies differential operators, divergence conditions, boundary conditions and interface conditions. Notice
that
\[ \|f(y)\|_{0,Y,\omega} = \int_{Y} |f(y)|^2 \omega(y)dy, \]
where \(\omega(y)\) is the probability density of \(y\) in \(Y\). Especially, if \(J(\mathbf{U}) = 0\) then \(\mathbf{U}\) is the solution to the coupled Stokes and Darcy-Forchheimer problems \([1, 9]\). Due to the infeasibility to estimate \(\theta\) by directly minimizing \(J(\mathbf{U})\) when integrated over a higher dimensional region, so we apply a sequence of randomly sampled points from domain instead forming mesh grid. The main steps of the CDNNs for the coupled Stokes and Darcy-Forchheimer equations are presented as Algorithm 1. Another noticeable point is that the term \(\nabla_{\theta}G(\theta^n, z^{(n)})\) is unbiased estimate of \(\nabla_{\theta}J(\mathbf{U}(\cdot; \theta^n))\) because the population parameters can be estimated by sample mathematical expectations.

3 Convergence

According to the definition of loss function \(J(\mathbf{U})\), it can measure how well \(\mathbf{U}\) satisfies the equations \([1, 9]\). Neural networks are a set of algorithms for classification and regression tasks inspired by the biological neural networks in brains. There have various types of neural networks with different neuron connection forms and architectures. According to the \([20]\), if there is only one hidden layer and output, the set of functions implemented by following networks with \(m_1, m_2, m_3\) and \(m_4\) hidden units for coupling of the Stokes and Darcy-Forchheimer problems

\[
[c_{U_D}^{m_1}(\varphi)]^d = \left\{ \Theta(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \right\}^{d = 1} = \sum_{i=1}^{m_1} \beta_i \varphi \left( \sum_{j=1}^{d} \sigma_{j,i} x_j + c_i \right),
\]

\[
[c_{U_D}^{m_2}(\zeta)]^d = \left\{ \Lambda(x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \right\}^{d = 1} = \sum_{i=1}^{m_2} \beta'_i \zeta \left( \sum_{j=1}^{d} \sigma'_{j,i} x_j + c'_i \right),
\]

\[
[c_{P_\beta}^{m_3}(\psi)]^d = \left\{ \Psi(t, x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \right\}^{d = 1} = \sum_{i=1}^{m_3} \beta''_i \psi \left( \sum_{j=1}^{d} \sigma''_{j,i} x_j + c''_i \right),
\]

\[
[c_{P_\gamma}^{m_4}(\gamma)]^d = \left\{ \Upsilon(t, x) : \mathbb{R}^d \rightarrow \mathbb{R}^d \right\}^{d = 1} = \sum_{i=1}^{m_4} \beta'''_i \gamma \left( \sum_{j=1}^{d} \sigma'''_{j,i} x_j + c'''_i \right),
\]

where \(\Theta(x) = (\Theta_1(x), \Theta_2(x), \ldots, \Theta_d(x))\), \(\Lambda(x) = (\Lambda_1(x), \Lambda_2(x), \ldots, \Lambda_d(x))\), \(\varphi, \zeta, \psi\) and \(\gamma\) are the shared activation functions of the hidden units in \(C^2(\Omega)\), bounded and non-constant. \(x_j\) is input, \(\beta_i, \beta'_i, \beta''_i, \beta'''_i, \sigma_{j,i}, \sigma'_{j,i}, \sigma''_{j,i}\) and \(\sigma'''_{j,i}\) are weights, \(c_i, c'_i, c''_i\) and \(c'''_i\) are thresholds of the neural networks.

More generally, we use the similar notation
\[
[c_{U_D}^{m_1}(\varphi)]^d \times [c_{U_D}^{m_2}(\zeta)]^d \times c_{P_\beta}(\psi) \times c_{P_\gamma}(\gamma)
\]
for the multi layer neural networks with arbitrarily large number of hidden units \(m_1, m_2, m_3\) and \(m_4\). In particular, the parameters and the activation function in each dimension of \([c_{U_D}^{m_1}(\varphi)]^d\) or
Algorithm 1: The CDNNs for the coupled problems

**Input:** \(\rho_{\omega_1}^{(n)} = \{x_s^n, x_D^n\}, \rho_{\omega_2}^{(n)} = \{r_s^n, r_D^n\}, \rho_{\omega_3}^{(n)} = \{r_T^n\}\), max iterations \(M\), learning rate \(\alpha\)

**Output:** \(\theta_{n+1}\)

1. Randomly generated sample points \(\rho^{(n)} = \{\rho_{\omega_1}^{(n)}, \rho_{\omega_2}^{(n)}, \rho_{\omega_3}^{(n)}\}\) from \((\Omega_S, \Omega_D)\), \((\partial \Omega_S \setminus \Gamma, \partial \Omega_D \setminus \Gamma)\) and \(\Gamma\) by the respective probability densities \(\omega_1, \omega_2\) and \(\omega_3\);
2. Initialize the parameters \(\theta\);
3. while iterations \(\leq M\) do
   
   read current;

   \[
   G(\rho^{(n)}, \theta^n) = G_S(\rho^{(n)}, \theta^n) + G_D(\rho^{(n)}, \theta^n) + G_T(\rho^{(n)}, \theta^n),
   \]

   where

   \[
   G_S(\rho^{(n)}, \theta^n) = \left( f_S + \nu \Delta U_S(x_s^n; \theta_1) - \nabla P_S(x_s^n; \theta_3) \right)^2
   \]

   \[
   + \left( \nabla \cdot U_S(x_s^n; \theta_1) \right)^2 + \left( U_S(r_s^n; \theta_1) \right)^2,
   \]

   \[
   G_D(\rho^{(n)}, \theta^n) = \left( \frac{\mu}{\rho} K^{-1} U_D(x_D^n; \theta_2) + \frac{\beta}{\rho} \left| U_D(x_D^n; \theta_2) \right| U_D(x_D^n; \theta_2) + \nabla P_D(x_D^n; \theta_4) - g_D \right)^2
   \]

   \[
   + \left( f_D - \nabla \cdot U_D(x_D^n; \theta_2) \right)^2 + \left( P_D(x_D^n; \theta_4) \right)^2,
   \]

   and

   \[
   G_T(\rho^{(n)}, \theta^n) = \left( U_S(x; \theta_1) \cdot \mathbf{n}_S - U_D(x; \theta_2) \cdot \mathbf{n}_S \right)^2
   \]

   \[
   + \left( P_S(x; \theta_3) - \nu \mathbf{n}_S \frac{\partial U_S(x; \theta_1)}{\mathbf{n}_S} - P_D(x; \theta_4) \right)^2
   \]

   \[
   + \left( - \nu t \frac{\partial U_S(x; \theta_1)}{\mathbf{n}_S} - G U_S(x; \theta_1) \cdot \mathbf{t} \right)^2,
   \]

   and

   \[
   \theta^{n+1} = \theta^n - \alpha \nabla G(\rho^{(n)}, \theta^n).
   \]

   if \(\lim_{n \to \infty} \| \nabla \theta G(\rho^{(n)}, \theta_n) \| = 0\) then
      return the parameters \(\theta_{n+1}\);
   else
      go back to the beginning of current section.
The parameters of the CDNNs can be formalized as follows
\[
\begin{align*}
\theta_1^k &= (\beta_1, \ldots, \beta_n, \sigma_{11}, \ldots, \sigma_{dn}, c_1^1, \ldots, c_n^1), \\
\theta_2^k &= (\beta_1', \ldots, \beta_n', \sigma_{11}', \ldots, \sigma_{dn}', c_1', \ldots, c_n'), \\
\theta_3^k &= (\beta_1'', \ldots, \beta_n'', \sigma_{11}'', \ldots, \sigma_{dn}'', c_1'', \ldots, c_n''), \\
\theta_4^k &= (\beta_1''', \ldots, \beta_n''', \sigma_{11''}, \ldots, \sigma_{dn''}, c_1''', \ldots, c_n'''),
\end{align*}
\]
where \( k = 1, 2, \ldots, d, \theta_1 \in \mathbb{R}^{(2+d)nd}, \theta_2 \in \mathbb{R}^{(2+d)nd}, \theta_3 \in \mathbb{R}^{(2+d)n} \) and \( \theta_4 \in \mathbb{R}^{(2+d)n} \).

In the next two subsections, we prove that the neural network \( \bar{U}^n \) with \( n \) hidden units for \( \bar{U}^S, \bar{U}^D, \bar{P}^S \) and \( \bar{P}^D \) satisfy the differential operators, boundary conditions, divergence conditions and interface conditions arbitrarily well for sufficiently large \( n \). More importantly, we confirm that there exists \( \bar{U}^n \in [\mathcal{C}_U^S(\varphi)]^d \times [\mathcal{C}_U^D(\zeta)]^d \times \mathcal{C}_P^S(\psi) \times \mathcal{C}_P^D(\gamma) \) such that \( J(\bar{U}^n) \to 0 \) as \( n \to \infty \). Another significant consideration, we give the convergence of \( \bar{U}^n \to \bar{u} \) as \( n \to \infty \) where \( \bar{u} \) is the exact solution to the coupled equations (1-9).

### 3.1 Convergence of the loss function \( J(\bar{U}) \)

In this subsection, we prove the CDNNs \( \bar{U} \) can make the loss function \( J(\bar{U}) \) arbitrarily small.

**Assumption 3.1.** \( \nabla \mathbf{v}(x), \nabla \mathbf{x}(x) \) and \( \nabla q(x) \) are locally Lipschitz with Lipschitz coefficient that they have at most polynomial growth on \( \mathbf{v}(x) \) and \( q(x) \). Then, for some constants \( 0 \leq q_i \leq \infty (i = 1, 2, 3, 4) \) we have
\[
\begin{align*}
|\Delta \mathbf{v}(x; \theta) - \Delta \mathbf{v}(x)| &\leq (|\nabla \mathbf{v}(x; \theta)|^{q_2} + |\nabla \mathbf{v}(x)|^{q_2/2})|\nabla \mathbf{v}(x; \theta) - \nabla \mathbf{v}(x)|, \\
|\nabla q(x; \theta) - \nabla q(x)| &\leq (|Q(x; \theta)|^{q_3} + |q(x)|^{q_3/2})|Q(x; \theta) - q(x)|, \\
|\nabla \mathbf{v}(x; \theta) - \nabla \mathbf{v}(x)| &\leq (|V(x; \theta)|^{q_4} + |\mathbf{v}(x)|^{q_4/2})|V(x; \theta) - \mathbf{v}(x)|.
\end{align*}
\]

**Theorem 3.1.** Under the Assumption 3.1, there exists a neural network \( \bar{U} \in [\mathcal{C}_U^S(\varphi)]^d \times [\mathcal{C}_U^D(\zeta)]^d \times \mathcal{C}_P^S(\psi) \times \mathcal{C}_P^D(\gamma) \), satisfying
\[
J(\bar{U}) \leq C \epsilon^2, \ \forall \epsilon > 0,
\]
where \( C \) depends on the data \( \{\Omega_S, \Omega_D, \Gamma, \mu, \rho, \beta, K^{-1}, \omega_1, \omega_2, \omega_3, f_D, g_D, f_S\} \).

**Proof.** From Theorem 3 of [20], we obtain that there exists \( \bar{U} \in [\mathcal{C}_U^S(\varphi)]^d \times [\mathcal{C}_U^D(\zeta)]^d \times \mathcal{C}_P^S(\psi) \times \mathcal{C}_P^D(\gamma) \) which are uniformly 2-dense on compacts of \( C^2(\Omega_S) \times C^2(\Omega_D) \times C^1(\Omega_S) \times C^1(\Omega_D) \). It means that for \( \bar{u}(x) \in C^2(\Omega_S) \times C^2(\Omega_D) \times C^1(\Omega_S) \times C^1(\Omega_D) \), \( \forall \epsilon > 0 \), we confirm that
\[
\begin{align*}
\max \sup_{a \leq 2} |\partial_x^2 \mathbf{u}_S(x; \theta_1) - \partial_x^2 \mathbf{u}_S(x)| < \epsilon, \\
\max \sup_{a \leq 2} |\partial_x^2 \mathbf{u}_D(x; \theta_2) - \partial_x^2 \mathbf{u}_D(x)| < \epsilon.
\end{align*}
\]
Firstly, we recall the form and discuss the convergence of conjugate numbers $r$. According to the Assumption 3.1, by using the Hölder inequality and Young inequality, setting conjugate numbers $r_1$ and $r_2$ such that $\frac{1}{r_1} + \frac{1}{r_2} = 1$, it follows that

$$\int_{\Omega} |\Delta U(x;\theta_1) - \Delta u(x)|^2 d\omega_1(x)$$

$$\leq \int_{\Omega} \left( |\nabla U(x;\theta_1)|^{l_1} + |\nabla u(x)|^{l_2} \right) \left( |\nabla U(x;\theta_1) - \nabla u(x)|^2 \right) d\omega_1(x)$$

$$\leq \left[ \int_{\Omega} \left( |\nabla U(x;\theta_1)|^{l_1} + |\nabla u(x)|^{l_2} \right) d\omega_1(x) \right]^{1/r_1} \times \left[ \int_{\Omega} \left( |\nabla U(x;\theta_1) - \nabla u(x)|^{2r_2} d\omega_1(x) \right) \right]^{1/r_2}$$

$$\leq \left[ \int_{\Omega} \left( |\nabla U(x;\theta_1) - \nabla u(x)|^{l_1} + |\nabla u(x)|^{l_1l_2} \right) d\omega_1(x) \right]^{1/r_1} \times \left[ \int_{\Omega} \left( |\nabla U(x;\theta_1) - \nabla u(x)|^{2r_2} d\omega_1(x) \right) \right]^{1/r_2}$$

$$\leq C \left( \epsilon^{l_1} + \sup_{\Omega} |\nabla u(x)|^{l_1l_2} \right) \epsilon^2,$$

here we set $l_1 \vee l_2 = \max\{l_1, l_2\}$. In the same way,

$$\int_{\Omega} \left( |\nabla P_S(x;\theta_3) - \nabla p_S(x)|^2 d\omega_1(x) \right)$$

$$\leq \int_{\Omega} \left( |P_S(x;\theta_3)|^{l_3} + |p_S(x)|^{l_4} \right) \left( P_S(x;\theta_3) - p_S(x) \right)^2 d\omega_1(x)$$

$$\leq \left[ \int_{\Omega} \left( |P_S(x;\theta_3)|^{l_3} + |p_S(x)|^{l_4} \right) d\omega_1(x) \right]^{1/r_3} \times \left[ \int_{\Omega} \left( |P_S(x;\theta_3) - p_S(x)|^{2r_4} d\omega_1(x) \right) \right]^{1/r_4}$$

$$\leq \left[ \int_{\Omega} \left( |P_S(x;\theta_3) - p_S(x)|^{l_3} + |p_S(x)|^{l_3l_4} \right) d\omega_1(x) \right]^{1/r_3} \times \left[ \int_{\Omega} \left( |P_S(x;\theta_3) - p_S(x)|^{2r_4} d\omega_1(x) \right) \right]^{1/r_4}$$

$$\leq C \left( \epsilon^{l_3} + \sup_{\Omega} |p_S(x)|^{l_3l_4} \right) \epsilon^2,$$

where $\frac{1}{r_3} + \frac{1}{r_4} = 1$ and $l_3 \vee l_4 = \max\{l_3, l_4\}$. 

\[ (18) \]

\[ (19) \]
For the boundary condition, we have
\[
\int_{\partial \Omega_h \setminus \Gamma} |U_S(x; \theta_1) - u_S(x)|^2 \, d\omega_2(x) \leq C \varepsilon^2. \tag{23}
\]

Owing to (21) - (23), we can conclude that
\[
J_{\Omega_h \setminus \Gamma}(\overline{U}) = \left\| f_S + \nu \Delta U_S(x; \theta_1) - \nabla P_S(x; \theta_3) \right\|_{0, \Omega_h, \Omega_1}^2 + \left\| \nabla \cdot U_S(x; \theta_1) \right\|_{0, \partial \Omega_h \setminus \Gamma, \omega_2}^2 \leq \nu \left\| \Delta U_S(x; \theta_1) - \Delta u_S(x) \right\|_{0, \Omega_h, \Omega_1}^2 + \left\| \nabla P_S(x; \theta_3) - \nabla p_S(x) \right\|_{0, \Omega_h, \Omega_1}^2 + \left\| \nabla \cdot U_S(x; \theta_1) \right\|_{0, \partial \Omega_h \setminus \Gamma, \omega_2}^2 \leq \nu \int_{\Omega_h} |\Delta U_S(x; \theta_1) - \Delta u_S(x)|^2 \, d\omega_1(x) + \int_{\Omega_h} |\nabla P_S(x; \theta_3) - \nabla p_S(x)|^2 \, d\omega_1(x) \tag{24}
\]
\[
+ \int_{\Omega_h} |\nabla \cdot \left( U_S(x; \theta_1) - u_S(x) \right)|^2 \, d\omega_1(x)
\]
\[
+ \int_{\partial \Omega_h \setminus \Gamma} |U_S(x; \theta_1) - u_S(x)|^2 \, d\omega_2(x)
\]
\[
\leq C \varepsilon^2.
\]

Next, we remain to prove the convergence of $J_{\Omega_D \setminus \Gamma}(\overline{U})$ and $J_\Gamma(U)$. We know that
\[
J_{\Omega_D \setminus \Gamma}(\overline{U}) = \left\| f_D - \nabla \cdot U_D(x; \theta_2) \right\|_{0, \Omega_D, \Omega_1}^2 + \left\| P_D(x; \theta_4) \right\|_{0, \partial \Omega_D \setminus \Gamma, \omega_2}^2 + \left\| \frac{\mu}{\rho} K^{-1} U_D(x; \theta_2) + \frac{\beta}{\rho} U_D(x; \theta_2) \right\|_{0, \partial \Omega_D \setminus \Gamma, \omega_2}^2 \leq \left\| \frac{\mu}{\rho} K^{-1} U_D(x; \theta_2) + \frac{\beta}{\rho} U_D(x; \theta_2) \right\|_{0, \Omega_D, \Omega_1}^2 \tag{25}
\]
From (19), we have
\[
\int_{\Omega_D} \left| \nabla P_D(x; \theta_4) - \nabla p_D(x) \right|^2 \, d\omega_1(x)
\]
\[
\leq \int_{\Omega_D} \left( |P_D(x; \theta_4)|^s + |p_D(x)|^s \right) \left( |P_D(x; \theta_4) - p_D(x)|^s \right) \, d\omega_1(x), \tag{26}
\]
which can be updated by using the Hölder inequality and Young inequality, thus we have
\[
\left[ \int_{\Omega_D} \left( |P_D(x; \theta_4)|^s + |p_D(x)|^s \right) \, d\omega_1(x) \right]^{1/s} \times \left[ \int_{\Omega_D} \left( |P_D(x; \theta_4) - p_D(x)|^s \right) \, d\omega_1(x) \right]^{1/s} \leq \left[ \int_{\Omega_D} \left( |P_D(x; \theta_4) - p_D(x)|^s + |p_D(x)|^s \right) \, d\omega_1(x) \right]^{1/s} \times \left[ \int_{\Omega_D} \left( |P_D(x; \theta_4) - p_D(x)|^s \right) \, d\omega_1(x) \right]^{1/s} \leq C \left( \varepsilon^s + \sup_{\Omega_D} |p_D(x)|^s \right) \varepsilon^2, \tag{27}
\]
where $\frac{1}{r_s} + \frac{1}{r_a} = 1$ and $l_5 \lor l_6 = \max\{l_5, l_6\}$.

Next we prove the boundedness of term $|U_D(x; \theta_2)|U_D(x; \theta_2)$,

$$
\int_{\Omega_D} \left( |U_D(x; \theta_2)|U_D(x; \theta_2) - |u_D(x)|u_D(x) \right)^2 d\omega_1(x)
$$

$$
= \int_{\Omega_D} \left( U_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) + |u_D(x)| \left( U_D(x; \theta_2) - u_D(x) \right) \right)^2 d\omega_1(x)
$$

$$
= \int_{\Omega_D} \left( U_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) \right)^2 d\omega_1(x) + \int_{\Omega_D} \left( |u_D(x)| \left( U_D(x; \theta_2) - u_D(x) \right) \right)^2 d\omega_1(x)
$$

$$
+ 2 \int_{\Omega_D} \left( u_D(x) \left( U_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) \right) \left( U_D(x; \theta_2) - u_D(x) \right) d\omega_1(x),
$$

where

$$
\int_{\Omega_D} \left( U_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) \right)^2 d\omega_1(x)
$$

$$
\leq \left[ \int_{\Omega_D} \left( U_D(x; \theta_2) \right)^{2r} d\omega_1(x) \right]^{1/r} \times \left[ \int_{\Omega_D} \left( |U_D(x; \theta_2)| + |u_D(x)| \right)^{2r} d\omega_1(x) \right]^{r}
$$

$$
\leq \left[ \int_{\Omega_D} \left( \left( U_D(x; \theta_2) - u_D(x) \right) + u_D(x) \right)^{2r} d\omega_1(x) \right]^{1/r} \times \left[ \int_{\Omega_D} \left( |U_D(x; \theta_2)| + |u_D(x)| \right)^{2r} d\omega_1(x) \right]^{r},
$$

by using the Hölder inequality and Young inequality, setting conjugate numbers $r_7$ and $r_8$ such that $\frac{1}{r_7} + \frac{1}{r_8} = 1$.

Similarly, we can obtain

$$
\int_{\Omega_D} \left( u_D(x) \left( U_D(x; \theta_2) - u_D(x) \right) \right)^2 d\omega_1(x)
$$

$$
\leq \left[ \int_{\Omega_D} \left( u_D(x) \right)^{r_9} \right]^{1/r_9} \times \left[ \int_{\Omega_D} \left( U_D(x; \theta_2) - u_D(x) \right)^{2r_{10}} d\omega_1(x) \right]^{1/r_{10}},
$$

$$
(30)
$$
here $\frac{1}{r_9} + \frac{1}{r_{10}} = 1$. Furthermore, we set $\frac{1}{r_{11}} + \frac{1}{r_{12}} = 1$ and $\frac{1}{r_{13}} + \frac{1}{r_{14}} = 1$,
\[
\int_{\Omega_D} |u_D(x)| u_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) \left( U_D(x; \theta_2) - u_D(x) \right) d\omega_1(x)
\]
\[
\leq \left[ \int_{\Omega_D} \left( U_D(x; \theta_2) \left( |U_D(x; \theta_2)| - |u_D(x)| \right) \right)^{r_{11}} d\omega_1(x) \right]^{1/r_{11}}
\times \left[ \int_{\Omega_D} \left( u_D(x) \left( U_D(x; \theta_2) - u_D(x) \right) \right)^{r_{12}} d\omega_1(x) \right]^{1/r_{12}}
\]
\[
\leq \left[ \int_{\Omega_D} \left( \left( U_D(x; \theta_2) - u_D(x) \right) + u_D(x) \right)^{r_{11}r_{13}} d\omega_1(x) \right]^{1/r_{11}r_{13}}
\times \left[ \int_{\Omega_D} \left( u_D(x) \left( U_D(x; \theta_2) - u_D(x) \right) \right)^{r_{12}r_{14}} d\omega_1(x) \right]^{1/r_{12}r_{14}}.
\]

According to the inequalities (32) - (31), we conclude
\[
\int_{\Omega_D} \left( |U_D(x; \theta_2)| U_D(x; \theta_2) - |u_D(x)| u_D(x) \right)^2 d\omega_1(x)
\]
\[
\leq (x^2 + \sup_{\Omega_D} |u_D(x)|)^2 + \sup_{\Omega_D} |u_D(x)|^2 + 2\epsilon \sup_{\Omega_D} |u_D(x)| (\epsilon + \sup_{\Omega_D} |u_D(x)|)^2.
\]

For the boundary condition, we know
\[
\int_{\partial\Omega_D \setminus \Gamma} |U_D(x; \theta_2) - u_D(x)|^2 d\omega_2(x) \leq C \epsilon^2.
\]

Combining the equations (26) - (33), we obtain
\[
J_{\Omega_D \setminus \Gamma}(U) = \left\| u_D - \nabla \cdot U_D(x; \theta_2) \right\|^2_{0, \partial\Omega_D, \omega_1} + \left\| P_D(x; \theta_4) \right\|^2_{0, \partial\Omega_D \setminus \Gamma, \omega_2}
\]
\[
+ \left\| u_D(x; \theta_2) - u_D(x; \theta_2) \right\|^2_{0, \partial\Omega_D \setminus \Gamma, \omega_2} + \left\| \frac{\mu K^{-1}}{\rho} U_D(x; \theta_2) - \frac{\beta}{\rho} U_D(x; \theta_2) + \nabla P_D(x; \theta_4) - g_D \right\|^2_{0, \partial\Omega_D, \omega_1}
\]
\[
\leq \int_{\Omega_D} \left( \nabla \cdot \left( U_D(x; \theta_2) - u_D(x) \right) \right)^2 d\omega_1(x) + \int_{\partial\Omega_D \setminus \Gamma} \left( P_D(x; \theta_4) - p_D(x) \right)^2 d\omega_2(x)
\]
\[
+ \beta \int_{\partial\Omega_D \setminus \Gamma} \left( |U_D(x; \theta_2)| U_D(x; \theta_2) - |u_D(x)| u_D(x) \right) d\omega_1(x)
\]
\[
+ \frac{\mu}{\rho} \int_{\partial\Omega_D \setminus \Gamma} \left( u_D(x; \theta_2) - u_D(x) \right)^2 d\omega_1(x)
\]
\[
+ \int_{\partial\Omega_D \setminus \Gamma} \left( \nabla P_D(x; \theta_4) - \nabla p_D(x) \right)^2 d\omega_1(x)
\]
\[
\leq C \epsilon^2.
\]

The loss in interface is referred in (11). According to the Assumption 3.1, we know that
\[
\int_{\Gamma} \left( |\nabla U_D(x; \theta_2) - \nabla u_D(x)|^2 d\omega_3(x)
\]
\[
\leq \int_{\Gamma} \left( |U_D(x; \theta_2)|^2 + |u_D(x)|^2 \right) \left( U_D(x; \theta_2) - u_D(x) \right)^2 d\omega_3(x),
\]
\[
(35)
\]
which can be updated by using the Hölder inequality and Young inequality, thus we have

\[
\left[ \int_{\Gamma} \left( |U_D(x; \theta_3)|^{l_7} + |u_D|^{l_8} \right)^{r_{15}} \, d\omega_3(x) \right]^{1/r_{15}} \times \left[ \int_{\Gamma} (U_D(x; \theta_3) - u_D(x))^{2r_{16}} \, d\omega_3(x) \right]^{1/r_{16}} \leq \left[ \int_{\Gamma} \left( |U_D(x; \theta_3) - u_D(x)|^{l_7} + |u_D(x)|^{l_8} \right)^{r_{15}} \, d\omega_3(x) \right]^{1/r_{15}} \\
\times \left[ \int_{\Gamma} (U_D(x; \theta_3) - u_D(x))^{2r_{16}} \, d\omega_3(x) \right]^{1/r_{16}} \leq C \left( l_7 + \sup_{\Gamma} |u_D(x)|^{l_8} \right) \epsilon^2,
\]

where \( \frac{1}{r_{15}} + \frac{1}{r_{16}} = 1 \) and \( l_7 \lor l_8 = \max\{l_7, l_8\} \).

Above all, we can obtain

\[
J_T(\mathbf{U}) = \left\| U_S(x; \theta_1) \cdot \mathbf{n}_S - U_D(x; \theta_2) \cdot \mathbf{n}_S \right\|_{0, \Gamma, \omega_3}^2 + \left\| P_S(x; \theta_3) - \nu \mathbf{n}_S \frac{\partial U_S(x; \theta_1)}{\partial \mathbf{n}_S} - P_D(x; \theta_4) \right\|_{0, \Gamma, \omega_3}^2 \\
+ \left\| - \nu t \frac{\partial U_S(x; \theta_1)}{\partial \mathbf{n}_S} - G U_S(x; \theta_1) \cdot \mathbf{t} \right\|_{0, \Gamma, \omega_3}^2 \\
\leq \left\| U_S(x; \theta_2) - u_S(x) \right\|_{0, \Gamma, \omega_3}^2 \left\| \mathbf{n}_S \right\|_{0, \Gamma, \omega_3}^2 + \left\| U_D(x; \theta_2) - u_D(x) \right\|_{0, \Gamma, \omega_3}^2 \left\| \mathbf{n}_S \right\|_{0, \Gamma, \omega_3}^2 \\
+ \left\| P_S(x; \theta_3) - p_S(x) \right\|_{0, \Gamma, \omega_3}^2 + \left\| P_D(x; \theta_4) - p_D(x) \right\|_{0, \Gamma, \omega_3}^2 \\
+ \nu \left\| \mathbf{n}_S \right\|_{0, \Gamma, \omega_3}^4 \left\| \nabla U_S(x; \theta_1) - \nabla u_S(x; \theta_1) \right\|_{0, \Gamma, \omega_3}^2 + G \left\| \mathbf{t} \right\|_{0, \Gamma, \omega_3}^2 \left\| U_S(x; \theta_1) - u_S(x) \right\|_{0, \Gamma, \omega_3}^2 \\
+ \nu \left\| \mathbf{t} \right\|_{0, \Gamma, \omega_3}^2 \left\| \nabla U_S(x; \theta_1) - \nabla u_S(x; \theta_1) \right\|_{0, \Gamma, \omega_3}^2 \left\| \mathbf{n}_S \right\|_{0, \Gamma, \omega_3}^2 \\
\leq C \epsilon^2,
\]

which completes the proof. \( \square \)

### 3.2 Convergence of the CDNNs to the exact solution

In the last subsection, we have proved the convergence of the loss function. In this subsection, we remain to discuss the convergence of the CDNNs to the exact solution. According to the Galerkin
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method, the neural networks satisfy

\[ \nabla \cdot U^n_D - f_D = 0, \text{ in } \Omega_D, \quad (38) \]
\[ \frac{\mu}{\rho} K^{-1} U^n_D + \frac{\beta}{\rho} \mid U^n_D \mid U^n_D + \nabla P^n_D - g_D = 0, \text{ in } \Omega_D, \quad (39) \]
\[ P^n_D = 0, \text{ on } \partial \Omega_D \setminus \Gamma, \quad (40) \]
\[ -\nu \Delta U^n_S + \nabla P^n_S - f_s = 0, \text{ in } \Omega_S, \quad (41) \]
\[ \nabla \cdot U^n_S = 0, \text{ on } \Omega_S, \quad (42) \]
\[ U^n_S = 0, \text{ on } \partial \Omega_S \setminus \Gamma, \quad (43) \]
\[ U^n_S \cdot n_S - U^n_D \cdot n_S = 0, \text{ on } \Gamma, \quad (44) \]
\[ P^n_S - \nu n_s \frac{\partial U^n_S}{\partial n_S} - P^n_D = 0, \text{ on } \Gamma, \quad (45) \]
\[ -\nu \eta \frac{\partial U^n_S}{\partial n_S} - G U^n_S \cdot t = 0, \text{ on } \Gamma. \quad (46) \]

Based on the above system of equations, we give the following assumption and theorem to guarantee the convergence of the CDNNs to the exact solution.

**Assumption 3.2.** We assume \((u_S, u_D) \in C^\xi(\bar{\Omega}_S) \times C^\xi(\bar{\Omega}_D) \) where \( \xi > 2 \) with itself and its first derivative bounded in \( \bar{\Omega}_S \times \bar{\Omega}_D \). Moreover, for every \( n \in N \), \( U^n_S \times U^n_D \in C^{1,2}(\bar{\Omega}_S) \cap X_S \times C^{1,2}(\bar{\Omega}_D) \cap L^3(\bar{\Omega}_D)^2 \). We assume that the subspace \( X^n_S \times L^3(\Omega_S)^2 \times L^2(\Omega_S) \times Y^n_D \subset [\mathcal{E}_{U_S}(\varphi)]^d \times [\mathcal{E}_{U_D}(\zeta)]^d \times \mathcal{E}_{P_S}(\psi) \times \mathcal{E}_{P_D}(\gamma) \) satisfies the discrete inf-sup condition.

**Theorem 3.2.** Under the Assumption 3.2 and Theorem 3.1, the neural network \( U^n_S \) can converge strongly to \( u_S \) in \( L^2 \), \( P^n_S \), \( U^n_D \) and \( P^n_D \) can converge strongly to \( p_S, u_D \) and \( p_D \) in \( H^{-1} \). In addition, if the sequences \( \{U^n_S\}_{n \in N}, \{P^n_S\}_{n \in N}, \{U^n_D\}_{n \in N} \) and \( \{P^n_D\}_{n \in N} \) are uniformly bounded and equicontinuous in \( \Omega_S \) and \( \Omega_D \), they can converge to \( u_S, p_S, u_D \) and \( p_D \) respectively.

**Proof.** Firstly, we give the weak formulation for \((38)-(46)\). Multiplying \((41)\) by \( V^n_S \in X_S \cap [\mathcal{E}_{U_S}(\varphi)]^d \) and \((42)\) by \( Q^n_S \in \mathcal{E}_{P_S}(\psi) \) and integration by parts yields

\[ \nu(\nabla U^n_S, \nabla V^n_S)_{\Omega_S} - \nu < \nabla U^n_S \cdot n_S, V^n_S > \gamma -(P^n_S, \nabla \cdot V^n_S)_{\Omega_S} + < P^n_S, V^n_S \cdot n_S > \gamma = (f_s, V^n_S)_{\Omega_S}, \quad (47) \]
\[ (\nabla \cdot U^n_S, Q^n_S)_{\Omega_S} = 0. \quad (48) \]

Multiplying \((38)\) by \( Q^n_D \in Y_D \cap \mathcal{E}_{P_D}(\gamma) \) and \((39)\) by \( V^n_D \in [\mathcal{E}_{U_D}(\zeta)]^d \), it then follows from integration by parts that

\[ -(U^n_D, \nabla Q^n_D)_{\Omega_D} + < U^n_D \cdot n_D, Q^n_D > \gamma = (f_D, Q^n_D)_{\Omega_D}, \quad (49) \]
\[ \frac{\mu}{\rho} (K^{-1} U^n_D, V^n_D)_{\Omega_D} + \frac{\beta}{\rho} (U^n_D, U^n_D, V^n_D)_{\Omega_D} - (P^n_D, \nabla \cdot V^n_D)_{\Omega_D} + < P^n_D, V^n_D \cdot n_D > \gamma = (g_D, V^n_D)_{\Omega_D}. \quad (50) \]
Considering the interface conditions, simple algebraic calculation yields
\[
< \nabla U^n_S \nabla S, V^n_S > = < n_S \nabla U^n_S \cdot n_S, V^n_S \cdot n_S > + < n_S \nabla U^n_S \cdot t, V^n_S \cdot t >
\]
which gives by employing interface conditions (45) and (46)
\[
< P^n_S I - \nu \nabla U^n_S \nabla S, V^n_S > = < P^n_S I - \nu \nabla U^n_S \cdot n_S > + G < U^n_S \cdot t, V^n_S \cdot t >. \quad (51)
\]

For convenience of presentation, we introduce the nonlinear operator \( A : L^3(\Omega_D)^2 \to L^3(\Omega_D)^2 \) defined by
\[
A(V) = \frac{\mu K^{-1}}{\rho} V + \frac{\beta}{\rho} |V| V.
\]
The definition of (52) gives
\[
(A(U^n_D), U^n_D)_{\Omega_D} \geq C(\|U^n_D\|_0^2 + \|U^n_D\|_3^2).
\]
According to the Assumption 3.2 and (50),
\[
\|P^n_D\|_0 \leq C \sup_{V^D \in L^3(\Omega_D)^2} \frac{(A(U^n_D), V^n_D)_{\Omega_D} - (g_D, V^n_D)_{\Omega_D}}{\|V^n_D\|_{L^3(\Omega_D)}} \leq C(\|U^n_D\|_0 + \|U^n_D\|_3 + \|g_D\|_0).
\]
Taking \( V^n_S = U^n_D, Q^n_S = P^n_S, V^n_D = U^n_D \) and \( Q^n_D = P^n_D \) in (47)-(50) and adding the resulting equations (51) yields
\[
\nu(\nabla U^n_S, \nabla U^n_S)_{\Omega_S} + G < U^n_S \cdot t, U^n_S \cdot t > + \frac{1}{2} (A(U^n_D), U^n_D)_{\Omega_D} = (f_S, U^n_S)_{\Omega_S} + (f_D, P^n_D)_{\Omega_D} + \frac{1}{2} (g_D, U^n_D)_{\Omega_D}.
\]
According to the definition of the \( H^1 \) norm, the *Young* inequality and the *Poincaré* inequality, we can obtain
\[
\|U^n_S\|_1^2 + \|U^n_D\|_0^2 + \|U^n_D\|_3^3 \leq C(\|f_S\|_0\|U^n_S\|_0 + \|f_D\|_0\|P^n_D\|_0 + \|g_D\|_0\|U^n_D\|_0)
\]
\[
\leq C(\|f_S\|_0\|U^n_S\|_0 + \|f_D\|_0\|U^n_D\|_0^2 + \|U^n_D\|_3^3 + \|g_D\|_0\|U^n_D\|_0) + \|g_D\|_0\|U^n_D\|_0
\]
\[
\leq C(\|f_D\|_0^2 + \|f_D\|_3^3 + \|f_S\|_0^2 + \|g_D\|_0^2).
\]
Thus we have
\[
\|P^n_D\|_0 \leq C(\|U^n_D\|_0 + \|U^n_D\|_3^3 + \|g_D\|_0)
\]
\[
\leq C(\|f_D\|_0 + \|f_D\|_3^3 + \|f_S\|_0 + \|f_S\|_0^2 + \|g_D\|_0 + \|g_D\|_0^2).
\]
According to the Assumption \[3.2\] - \[47\] - \[48\] and \[51\] 
\[
\|P_S^n\|_0 \leq C \sup_{\mathbf{v}_S \in \mathbf{X}_S} \|\langle f_S, \mathbf{v}_S^n \rangle - \langle \nabla U_S^n, \nabla V_S^n \rangle - \langle P_D^n, \mathbf{V}_S^n \cdot \mathbf{n}_S \rangle \rangle - G \mathbf{U}_S^n \cdot \mathbf{t}, \mathbf{V}_S^n \cdot \mathbf{t} \rangle
\]
\[
\leq C(\|f_S\|_0 + \|\nabla U_S^n\|_0 + \|P_D^n\|_0 + \|U_S^n\|_0)
\]
\[
\leq C(\|f_D\|_0 + \|f_D^n\|_0 + \|f_S\|_0 + \|f_S^n\|_0 + \|g_D\|_0 + \|g_D^n\|_0).
\]

Up to now, we obtain

\[
\{U_S^n\}_{n \in \mathbb{N}} \text{ is uniformly bounded in } H^1(\Omega_S). \tag{53}
\]

By using the uniformly boundedness of \(U_S^n\), we can extract a subsequence \(\{U_S^n\}_{n \in \mathbb{N}}\) of \(U_S^n\) which converge weakly in \(H^1(\Omega_S)\). Due to the compact embedding \(H^1(\Omega_S) \hookrightarrow L^2(\Omega_S)\), we have

\[
\lim_{n \to \infty} \|U_S^n - u_S\|_{0,\Omega_S} = 0.
\]

Similarly, we know \(L^3(\Omega_D) \subset L^2(\Omega_D)\), which implies that

\[
\{U_D^n\}_{n \in \mathbb{N}} \text{ is uniformly bounded in } L^2(\Omega_D). \tag{54}
\]

Due to the compact embedding \(L^2(\Omega_D) \hookrightarrow H^{-1}(\Omega_D)\), we have

\[
\lim_{n \to \infty} \|U_D^n - u_D\|_{-1,\Omega_D} = 0.
\]

The convergence of the \(P_S^n\) and \(P_D^n\) as follows

\[
\lim_{n \to \infty} \|P_S^n - P_S\|_{-1,\Omega_S} = 0, \quad \lim_{n \to \infty} \|P_D^n - P_D\|_{-1,\Omega_D} = 0.
\]

For all these reasons, \(\{U_S^n\}_{n \in \mathbb{N}}\) converge strongly to \(u_S\) in \(L^2\), \(\{P_S^n\}_{n \in \mathbb{N}}, \{U_D^n\}_{n \in \mathbb{N}}\) and \(\{P_D^n\}_{n \in \mathbb{N}}\) converge strongly to \(p_S, u_D\) and \(p_D\) in \(H^{-1}\). More generally, by the well known Arzelà-Ascoli theorem we can conclude that \(\{U_S^n\}_{n \in \mathbb{N}}, \{P_S^n\}_{n \in \mathbb{N}}, \{U_D^n\}_{n \in \mathbb{N}}\) and \(\{P_D^n\}_{n \in \mathbb{N}}\) converge uniformly to \(u_S, p_S, u_D\) and \(p_D\) respectively.

\[
4 \text{ Numerical Experiments}
\]

The section presents several numerical tests to confirm the proposed theoretical results. We start with three examples with known exact solution to test the efficiency of the proposed method, where the permeability for the third example is highly oscillatory. Then, the fourth example with no exact solution shows the application of the proposed method to high contrast permeability problem. This section concludes with a physical flow. The numerical examples presented below could violate the interface conditions \[8\] and \[9\] \[17\], that is, \(8\) and \(9\) are replaced by

\[
p_S - \nu n_S \frac{\partial u_S}{\partial n_S} = p_D + g_1, \quad \text{on } \Gamma, \tag{55}
\]
\[
-\nu t \frac{\partial u_S}{\partial n_S} = G u_S \cdot t + g_2, \quad \text{on } \Gamma, \tag{56}
\]

to deal with this case, the variational formulation has only a small change: The equation \[51\] now includes the two terms \(- < g_1, \mathbf{V}_S \cdot \mathbf{n}_S > \Gamma - < g_2, \mathbf{V}_S \cdot \mathbf{t} > \Gamma\) on the right side. In addition,
we utilize 16 neurons in each hidden layer and apply the relative $L^1$ error ($errL^1 : \frac{\|r-R\|_{L^1}}{\|r\|_{L^1}}$) and relative $L^2$ error ($errL^2 : \frac{\|r-R\|_0}{\|r\|_0}$) to reflect the accuracy between the results of the CDNNs and the exact solution ($r$: the exact solution; $R$: the neural network).

4.1 Test 1

In this subsection we study the performance of the CDNNs for the benchmark problem presented in [47]. This problem is defined for $\Omega_S = (0, 1)^2$, $\Omega_D = (0, 1) \times (1, 2)$ and the interface $\Gamma = \{0 < x < 1, y = 1\}$ as

$$\begin{align*}
u_S &= \begin{pmatrix} x^2 \pi \sin(2\pi y) (x-1)^2 \\ -2x \sin(y \pi)^2 (2x-1)(x-1) \end{pmatrix}, \\
\nu_D &= \begin{pmatrix} \sin(\pi x) \sin(\pi y) \\ -2x \sin(y \pi)^2 (2x-1) \end{pmatrix},
\end{align*}$$

and

$$\begin{align*}
u_S &= (\cos(1)-1) \sin(1) + \cos(y) \sin(x), \\
\nu_D &= \sin(\pi x) \cos(\pi y).
\end{align*}$$

Similar to [47], we fix $K$ to be the identity tensor in $\mathbb{R}^{2 \times 2}$, $\mu = \rho = \beta = \nu = 1$. Due to the interface conditions (8) and (9) are violated, we exploit the interface conditions (55) and (56), where $g_1$ and $g_2$ can be computed by the exact solution. Specifically, the errors converge as the hidden layer increases in Figure 3(a). Figure 3(b) reveals that the change of data has no significant influence on errors once the size is larger than $10^2$. In particular, Figures 4-5 and Table 1 show the details of the results, this is consistent with our theory.
Figure 3: The influence of different hidden layers and different training data on $\text{err}L^2$ (Test 1).
Figure 4: The contrast of the exact solution and the CDNNs (Test 1).
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Figure 5: The point-wise errors (Test 1).

4.2 Test 2

In this example, we consider $\Omega_S = (0, 1)^2$, $\Omega_D = (0, 1) \times (1, 2)$ and the interface $\Gamma = \{0 < x < 1, y = 1\}$ with an analytical solution presented in [47]. We set $K$ to be the identity tensor in $\mathbb{R}^{2 \times 2}$, $\mu = \rho = \beta = \nu = 1$, and the exact solution is given by

$$u_S = \left( -\cos^2\left(\frac{\pi x}{2}\right) \sin\left(\frac{\pi y}{2}\right) \right), \quad p_S = \frac{\pi}{4} \cos\left(\frac{\pi x}{2}\right)\left(\sin\left(\frac{\pi y}{2}\right) + \frac{\pi y}{2}\right)$$

and

$$u_D = \left( -\frac{1}{8} \sin\left(\frac{\pi x}{2}\right) \right), \quad p_D = -\frac{\pi}{4} \cos\left(\frac{\pi x}{2}\right) y.$$ 

Naturally, the corresponding $f_S$, $f_D$, and $g_D$ can be calculated by the exact solution. Note that this example satisfies the interface conditions (7) - (9). According to Test 1, we choose appropriate data and hidden layer to solve the second example. Figures 6 - 7 and Table 2 show the accuracy of the CDNNs for solving the coupled problems in detail.
Figure 6: The contrast of the exact solution and the results of the CDNNs (Test 2).
Table 2: The relative errors of Test 2.

|          | 1 layer | 2 layers | 3 layers |
|----------|---------|----------|----------|
|          | \(U_S\) | \(P_S\)  | \(U_D\)  | \(P_D\)  |
| \(\text{err}_{L^1}\) | \(1.82 \times 10^{-2}\) | \(1.19 \times 10^{-1}\) | \(1.57 \times 10^{-2}\) | \(1.08 \times 10^{-2}\) |
| \(\text{err}_{L^2}\) | \(3.66 \times 10^{-2}\) | \(1.23 \times 10^{-1}\) | \(4.62 \times 10^{-2}\) | \(1.11 \times 10^{-2}\) |
|          | \(U_S\) | \(P_S\)  | \(U_D\)  | \(P_D\)  |
|          | \(2.27 \times 10^{-4}\) | \(1.74 \times 10^{-3}\) | \(1.04 \times 10^{-4}\) | \(4.67 \times 10^{-5}\) |
| \(\text{err}_{L^2}\) | \(4.21 \times 10^{-4}\) | \(2.00 \times 10^{-3}\) | \(3.18 \times 10^{-4}\) | \(5.55 \times 10^{-5}\) |
|          | \(U_S\) | \(P_S\)  | \(U_D\)  | \(P_D\)  |
|          | \(1.65 \times 10^{-4}\) | \(1.01 \times 10^{-3}\) | \(1.13 \times 10^{-4}\) | \(7.69 \times 10^{-5}\) |
| \(\text{err}_{L^2}\) | \(3.37 \times 10^{-4}\) | \(1.50 \times 10^{-3}\) | \(3.44 \times 10^{-4}\) | \(8.32 \times 10^{-5}\) |

Figure 7: The point-wise errors (Test 2).

4.3 Test 3

In this subsection, we solve coupling of the Stokes and Darcy-Forchheimer problems with highly oscillatory permeability over domains \(\Omega_S = (0, 1) \times (0, 1/2)\), \(\Omega_D = (0, 1) \times (1/2, 1)\) and the interface \(\Gamma = \{0 < x < 1, \ y = 1/2\}\) presented in [47]. Here we set \(\mu = \rho = \beta = \nu = 1\), \(K^{-1} = gI\) and \(g\) is
defined by
\[
\varrho = \frac{2 + 1.8\sin(2\pi x/\varepsilon)}{2 + 1.8\sin(2\pi y/\varepsilon)} + \frac{2 + 1.8\sin(2\pi y/\varepsilon)}{2 + 1.8\sin(2\pi x/\varepsilon)},
\]
where \(\varepsilon = 1/16\). The profile of \(\varrho\) is shown in Figure 8. The exact solution is given by
\[
\begin{align*}
\mathbf{u}_S &= \left(\begin{array}{c}
16y\cos(\pi x)^2(y^2 - 0.25) \\
8\pi\cos(\pi x)\sin(\pi x)(y^2 - 0.25)^2
\end{array}\right), & p_S &= x^2
\end{align*}
\]
and
\[
\begin{align*}
\mathbf{u}_D &= \left(\begin{array}{c}
\sin(2\pi x)\cos(2\pi y) \\
-\cos(2\pi x)\sin(2\pi y)
\end{array}\right), & p_D &= \cos(2\pi x)\cos(2\pi y).
\end{align*}
\]
We calculate the relative errors in Table 3 to reflect ability of the CDNNs for solving the coupled problems with highly oscillatory permeability. Figures 9 - 10 reveal that the CDNNs handle the highly oscillatory permeability coupled problems without losing accuracy.

![Figure 8: The value of \(\varrho\) (Test 3).](image)

| 400 sampled points | 1 layer | \(U_S\) | \(P_S\) | \(U_D\) | \(P_D\) |
|---------------------|---------|--------|--------|--------|--------|
| \(\text{err}L_1\)  | 3.59 \times 10^{-1} | 5.03 \times 10^0  | 5.52 \times 10^{-2} | 7.59 \times 10^{-2} |
| \(\text{err}L_2\)  | 6.79 \times 10^{-1} | 5.20 \times 10^0  | 1.73 \times 10^{-1} | 7.07 \times 10^{-2} |
| 2 layers | \(U_S\) | \(P_S\) | \(U_D\) | \(P_D\) |
| \(\text{err}L_1\)  | 8.42 \times 10^{-4} | 9.41 \times 10^{-3} | 1.15 \times 10^{-3} | 1.32 \times 10^{-3} |
| \(\text{err}L_2\)  | 1.65 \times 10^{-3} | 1.05 \times 10^{-2} | 3.43 \times 10^{-3} | 1.56 \times 10^{-3} |
| 3 layers | \(U_S\) | \(P_S\) | \(U_D\) | \(P_D\) |
| \(\text{err}L_1\)  | 1.89 \times 10^{-4} | 3.04 \times 10^{-3} | 2.97 \times 10^{-4} | 6.70 \times 10^{-5} |
| \(\text{err}L_2\)  | 3.65 \times 10^{-4} | 3.37 \times 10^{-3} | 8.98 \times 10^{-4} | 8.40 \times 10^{-5} |
Figure 9: The contrast of the exact solution and the results of the CDNNs (Test 3).
4.4 Test 4

The problems that we have studied so far have the exact solution. In this example, we consider coupling of the Stokes and Darcy-Forchheimer problems with no exact solution over $\Omega_S = (-1/2, 3/2) \times (0, 2)$, $\Omega_D = (-1/2, 3/2) \times (-2, 0)$ and the interface $\Gamma = \{-1/2 < x < 3/2, \ y = 0\}$. Specifically, in the Stokes region, the Dirichlet boundary condition is given by Kovasznay flow \[48\],

$$u_S = \begin{pmatrix} 1 - e^{\lambda x} \cos(2\pi y) \\ \frac{\lambda}{2\pi} e^{\lambda x} \sin(2\pi y) \end{pmatrix},$$

where $\lambda = \frac{-8\pi^2}{1 + \sqrt{1 + 64\pi^2}}$. Moreover, we set $\mu = \rho = \beta = \nu = 1$ and $g_D = 0$, $f_D = 0$, $f_S = 0$. In addition, $p_D$ satisfies the homogeneous Dirichlet boundary condition along $y = -2$, otherwise it has an homogeneous Neumann boundary condition. The permeability is taken to be $K = \varepsilon I$ and the $\varepsilon = 10^4$. Since the exact solution for this example is not available, we provide $L^2$ error of interface to demonstrate the accuracy of the CDNNs in Table 4. Obviously, the error decreases gradually with the increasing of the hidden layers. Furthermore, Figures 11 - 12 display the exact solution and the results of CDNNs in detail.
Table 4: The error in interface of Test 4 (K=10000).

|       | Interface1 | Interface2 | Interface3 |
|-------|------------|------------|------------|
| 1 layer | $6.49 \times 10^{-2}$ | $9.14 \times 10^{-2}$ | $3.03 \times 10^{-2}$ |
| 2 layers | $2.31 \times 10^{-2}$ | $4.74 \times 10^{-2}$ | $2.38 \times 10^{-3}$ |
| 3 layers | $2.12 \times 10^{-2}$ | $1.28 \times 10^{-2}$ | $4.94 \times 10^{-4}$ |

Figure 11: The results of CDNNs (Test 4).

Figure 12: The velocity of Stokes and Darcy (Test 4).
4.5 Test 5

We conclude this section with a physical flow, where \( \Omega_S = (0,1) \times (1,2) \), \( \Omega_D = (0,1)^2 \) and the interface \( \Gamma = \{0 < x < 1, \, y = 1\} \). In \( \Omega_S \), the boundaries of the cavity are walls with no-slip condition, except for the upper boundary where a uniform tangential velocity \( \mathbf{u}_S(x, 2) = (1, 0)^T \) is imposed, which is driven cavity flow. And more precisely, we enforce homogeneous Neumann and Dirichlet boundary conditions, respectively, on \( \Gamma_{D,N} = \{x = 0 \text{ or } y = 0\} \) and \( \Gamma_{D,D} = \{x = 1\} \).

In addition, we set \( K \) to be the identity tensor in \( \mathbb{R}^{2 \times 2} \), \( \mu = \rho = \beta = \nu = 1 \), and \( f_D = 0, \, f_S = 0, \, g_D = 0 \). The results of the CDNNs are depicted in Figure 13. More vividly, we display the velocity flows of free-flow and porous media zones in Figure 13.

Figure 13: The results of the CDNNs (Test 5).

(a) Stokes velocity. (b) Darcy velocity.

Figure 14: The velocity flows of Stokes and Darcy (Test 5).
5 Conclusions

In this article, we proposed the CDNNs to study the coupled Stokes and Darcy-Forchheimer problems. This method can avoid many limitations of the traditional methods, such as decoupling, grid construction and the complicated interface conditions. Specially, we provide the convergence of the loss function and the convergence of the CDNNs to the exact solution. The numerical results are consistent with our theory sufficiently. Moreover, we leave the following issues subject to our future works, 1) Combining data-driven with model-driven to solve the high dimensional coupled problems; 2) Considering the specific size of the networks through theoretical analysis; 3) Combining traditional numerical methods with deep learning to solve more complicated high dimensional coupled problems.

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