On Semi-Rational Groups

Tzoor Plotnikov

Abstract
A finite group is called semi-rational if the distribution induced on it by any word map is a virtual character. In [AV11] Amit and Vishne give a sufficient condition for a group to be semi-rational, and ask whether it is also necessary. We answer this in the negative, by exhibiting two new criteria for semi-rationality, each giving rise to an infinite family of semi-rational groups which do not satisfy the Amit-Vishne condition. On the other hand, we use recent work of Lubotzky to show that for finite simple groups the Amit-Vishne condition is indeed necessary, and we use this to construct the first known example of an infinite family of non-semi-rational groups.

1 Introduction
For a finite group $G$ and a word $w = x_{i_1}^{n_1} x_{i_2}^{n_2} \cdots x_{i_\ell}^{n_\ell}$ in the free group on $r$ generators one can define the “word map” $w : G^r \to G$ by

$$w(g_1, \cdots, g_r) = g_{i_1}^{n_1} g_{i_2}^{n_2} \cdots g_{i_\ell}^{n_\ell},$$

namely, substituting each $x_i$ by $g_i$. One defines $N_{w,G} : G \to \mathbb{C}$ by

$$N_{w,G}(g) = \left| \{ (g_1, \cdots, g_r) \in G^r : w(g_1, \cdots, g_r) = g \} \right|,$$

the distribution which $w$ induces on $G$.

For every automorphism $\alpha \in \text{Aut}(G)$ and every $g \in G$ there is a bijection between the solution set of $w = g$ and the solution set of $w = \alpha(g)$ given by $(g_1, \cdots, g_r) \mapsto (\alpha(g_1), \cdots, \alpha(g_r))$. In particular, $N_{w,G}$ is a class function of $G$, and can be written as

$$N_{w,G} = \sum_{\chi \in \text{irr}(G)} N_{w,G}^\chi \cdot \chi$$

where by orthogonality of characters

$$N_{w,G}^\chi = \langle N_{w,G}, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} N_{w,G}(g) \overline{\chi(g)} = \frac{1}{|G|} \sum_{(g_1, \cdots, g_r) \in G^r} \overline{\chi(w(g_1, \cdots, g_r))},$$

with $\overline{\chi}$ being the complex conjugate of $\chi$. If for every irreducible character $N_{w,G}^\chi \in \mathbb{N}$ then $N_{w,G}$ is a character of the group $G$, and if $N_{w,G}^\chi \in \mathbb{Z}$ then $N_{w,G}$ is a difference of characters and we call it a generalized character, or a virtual character of $G$.

The author acknowledges the support of ISF 1031/17 grant of Ori Parzanchevski
**Definition 1.1.** A finite group $G$ is called semi-rational if $N_{w,G}$ is a generalized character for every $r \in \mathbb{N}$ and every $w \in F_r$.

In [AV11] Amit and Vishne provided several results on word maps from character theoretic point of view. They used an argument by Stanley ([Sta86], Exercise 7.69.j) to show

**Theorem 1.2 ([AV11], Proposition 3.2).** For every $w \in F_r$ and $\chi \in \text{Irr}(G)$ one has $N^2_{w,G} \in \mathbb{Z}[\{\chi(g) | g \in G\}]$. In particular, $N^k_{w,G} \in \mathbb{Z}[\omega]$ where $\omega$ is a primitive root of unity of order $|G|$.

**Remark 1.3.** The theorem Amit and Vishne prove in their work actually states that $N^2_{w,G} \in \mathbb{Z}[\{\psi(g) | g \in G, \psi \in \text{Irr}(G)\}]$, but examination of the proof shows that the stronger version given here is still viable, and we will use this stronger version in Proposition 2.1.

They use this theorem to prove a necessary and sufficient condition for a group to be semi-rational:

**Theorem 1.4 ([AV11], Corollary 3.3).** $N_{w,G}$ is a generalized character if and only if for every $g,h \in G$ generating the same cyclic subgroup $N_{w,G}(g) = N_{w,G}(h)$.

They get as a corollary

**Corollary 1.5 ([AV11], Proposition 3.5).** If $g,h \in G$ lie in the same orbit of $\text{Aut}(G)$ whenever they generate the same cyclic subgroup then $G$ is semi-rational.

We call the condition of Corollary 1.5 the *Amit-Vishne condition*. In their article Amit and Vishne asked

**Question 1.6 ([AV11], Question 3.9).** Is the Amit-Vishne condition also necessary for a group to be semi-rational?

We provide in Section 2.1 two infinite families of semi-rational groups which do not satisfy the Amit-Vishne condition, answering Question 1.6 in the negative. These examples are based on Proposition 2.1 and Corollary 2.5 which gives new sufficient conditions for semi-rationality. We also use those new conditions to show that every group of order $pq$ is semi-rational.

On the other hand, we prove in Section 2.2 that the Amit-Vishne condition is indeed necessary for finite simple groups, and use this result to provide the first example of an infinite family of groups which are not semi-rational.

In Section 3 we provide several further questions and conjectures on the subject.

## 2 On The Amit-Vishne Condition

### 2.1 The Counter Examples

In [AV11], Amit and Vishne ask whether the condition in Corollary 1.5 is also necessary for a group to be semi-rational. We present in this section two infinite families of
semi-rational groups which do not satisfy the Amit-Vishne condition. In order to prove the semi-rationality of the groups, we prove new conditions for semi-rationality.

The first one is:

**Proposition 2.1.** For $G$ a finite group, if every irreducible character of degree $\geq 2$ takes values only in $\mathbb{Z}$ then $G$ is semi-rational.

**Proof.** Using Theorem 1.2, it is enough to show that for every word $w \in F_r$ and every one-dimensional character $\chi$ of $G$, one has $N_{w,G}^\chi \in \mathbb{Z}$.

For $\chi$ and $w$ as above, denote by $K$ the kernel of $\chi$, and write $\tilde{N}_{w,G}(gK) = \sum_{k \in K} N_{w,G}(gk)$. Then one has

$$N_{w,G}^\chi = \langle N_{w,G} \chi \rangle = \frac{1}{|G|} \sum_{g \in G} N_{w,G}(g) \chi(g) = \frac{1}{|G/K|} \sum_{gN \in G/K} \tilde{N}_{w,G}(gN) \chi(g) = \langle \tilde{N}_{w,G} \chi \rangle_{G/K},$$

where $\tilde{\chi}$ is the induced character on $G/K$. One easily sees that $\tilde{N}_{w,G} = |K|^{-1} N_{w,G/K}$, and therefore

$$N_{w,G}^\chi = |K|^{-1} N_{w,G/K}^\chi,$$

and since $G/K$ is abelian, $N_{w,G}^\chi \in \mathbb{Z}$. $\square$

**Proposition 2.2.** For every prime $p \geq 5$ the group $G_p = C_p \rtimes C_{p-1}$ is semi-rational but does not satisfy the Amit-Vishne condition.

**Proof.** We start by proving that $G_p$ is semi-rational, and for that we calculate the conjugacy classes:

One can write $G_p = \langle s, t | t^p = s^{p-1} = 1, s^{-1} t s = t^k \rangle$ for $k$ primitive in $(\mathbb{Z}/p\mathbb{Z})^\times$, and get that

$$(s^{\alpha t^\beta})^{-1} s^x (s^{\alpha t^\beta}) = s^{x - k^\beta \alpha},$$

and since the function $m : \mathbb{Z}/(p-1)\mathbb{Z} \to \mathbb{Z}/(p-1)\mathbb{Z}$ defined by $m(\beta) = \beta(1 - k^\beta)$ is surjective, the set $\{s^x, s^{x T}, \ldots, s^{x T^{p-2}}\}$ is a conjugacy class, where $T = (t)$. In addition, $(s^{\alpha t^\beta})^{-1} T (s^{\alpha t^\beta}) = k^x$, and since $\alpha \rightarrow k^\alpha x$ is also surjective, the conjugacy classes are exactly

$$\{1\}, T - \{1\}, sT, \ldots, s^{p-2} T.$$

Now, since $T \triangleleft G_p$ and $G_p/T \cong C_{p-1}$, there are $p - 1$ one-dimensional characters of $G_p$ pulled from $C_{p-1}$, and by column orthogonality one gets the full character table of $G_p$:

| $\chi$ | $T - \{1\}$ | $sT$ | $s^2 T$ | $\ldots$ | $s^{p-2} T$ |
|-------|---------------|------|---------|----------|----------------|
| $\chi_1$ | 1 | 1 | 1 | 1 | \ldots | 1 |
| $\chi_j$ | 1 | 1 | $e^{\frac{2\pi i}{p-1}}$ | $e^{\frac{4\pi i}{p-1}}$ | $\ldots$ | $e^{\frac{2(p-1)\pi i}{p-1}}$ |
| $\chi_{p-1}$ | 1 | 1 | $e^{\frac{2(p-2)\pi i}{p-1}}$ | $e^{\frac{4(p-2)\pi i}{p-1}}$ | $\ldots$ | $e^{\frac{2(p-2)^2 \pi i}{p-1}}$ |
| $\chi$ | $p - 1$ | $-1$ | 0 | $\ldots$ | 0 |
By Proposition 2.1, $G_p$ is semi-rational.

Next, we show that $G_p$ does not satisfy the Amit-Vishne condition: Suppose that $\alpha$ is an automorphism of $G_p$ sending $s$ to $s^{-1}$. Since $T$ is a normal $p$-Sylow subgroup, the image of $t$ must be in $T$. Denote $\alpha(t) = t^n$, so

$$\alpha(ts) = \alpha(t)\alpha(s) = t^n s^{p^n - 2} = s^{p^n - 2} k^{2p - 2} n$$

and on the other hand

$$\alpha(ts) = \alpha(st^k) = s^{p^n - 2} t^{nk},$$

which together gives $nk \equiv nk^{p^n - 2} \pmod{p}$. Since $n$ and $k$ are prime to $p$, we get $k^{p^n - 3} \equiv 1 \pmod{p}$, which is a contradiction, since $k$ is primitive, and $p \geq 5$. So there is no automorphism sending $s$ to $s^{-1}$, and $G_p$ does not satisfy the Amit-Vishne condition.

For the next counter-example to Question 1.6 we first prove the following condition:

**Proposition 2.3.** If for $g, h \in G$ which generate the same cyclic subgroup there exists a normal subgroup $N \trianglelefteq G$ such that $G/N$ is semi-rational and such that $gN \subseteq O(g)$ and $hN \subseteq O(h)$ then $N_{w,G}(g) = N_{w,G}(h)$.

**Remark 2.4.** Here $O(g)$ denotes the orbit of $g$ under $\text{Aut}(G)$.

**Proof.** Denote $H = G/N$ and $\pi : G \rightarrow H$ the quotient map. Then the following diagram commutes:

$$\begin{array}{ccc}
G & \xrightarrow{\pi} & H \\
\downarrow w & & \downarrow w \\
G & \xrightarrow{\pi} & H
\end{array}$$

By going through the upper branch, every element $x \in H$ has $|N|^|N_{w,H}(x)|$ preimages in $G'$, and by going through the lower branch $x$ has $\sum_{y \in \pi^{-1}(x)} N_{w,G}(y)$ preimages in $G$. Putting this together for $x = gN$ and $x = hN$ we get

$$|N|^|N_{w,H}(gN)| = \sum_{y \in gN} N_{w,G}(y)$$

and

$$|N|^|N_{w,H}(hN)| = \sum_{y \in hN} N_{w,G}(y)$$

respectively. But since $g, h$ generate the same subgroup in $G$, so do $gN$ and $hN$ in $H$, giving $N_{w,H}(gN) = N_{w,H}(hN)$ since $H$ is semi-rational. Therefore

$$\sum_{y \in gN} N_{w,G}(y) = \sum_{y \in hN} N_{w,G}(y) \quad (*)$$

and since $gN \subseteq O(g)$, for every $y \in gN$ one has $N_{w,G}(g) = N_{w,G}(y)$, so $(*)$ becomes

$$|N| \cdot N_{w,G}(g) = |N| \cdot N_{w,G}(h),$$

yielding $N_{w,G}(g) = N_{w,G}(h)$. 

\[\square\]
This gives our second semi-rationality condition:

Corollary 2.5. Suppose there exists $N \triangleleft G$ such that $gN \subseteq O(g)$ for every $g \notin N$, and that for every $g, h \in N$ generating the same subgroup one has $N_{w,G}(g) = N_{w,G}(h)$. Then $G$ is semi-rational.

Proposition 2.6. For every prime $p \geq 3$ the group $C_{p^2} \times C_p = \langle s, t \mid t^{p^2} = s^p = 1, s^{-1}ts = t^{p+1} \rangle$ is semi-rational but does not satisfy the Amit-Vishne condition.

Proof. The same kind of calculation done in Proposition 2.2 shows that $gT = [g] \subseteq O(g)$ for every $g \notin T$, where $T = \langle t^p \rangle$.

For $g \in T$ we show that for every $y$ prime to the order of $g$ one can construct an automorphism of $G = C_{p^2} \times C_p$ sending $g$ to $g^y$, therefore giving $N_{w,G}(g) = N_{w,G}(g^y)$. It is enough to show this for $g = t^p$. Consider $\eta$ defined by $t \mapsto t^r$ and $s \mapsto s$. Since $t^rs = st^{(p+1)y} = s(t^r)^{p+1}$, $\eta$ extends uniquely to a homomorphism of $G$, and since $t^r, s \in im(\eta)$ with $y$ prime to the order of $t$, $\eta$ is surjective and hence an automorphism. Additionally, $\eta(t^p) = (t^r)^p = (t^p)^y$, as we wanted.

In conclusion, the conditions in Corollary 2.5 are satisfied, and therefore $G$ is semi-rational.

We show now that $G$ does not satisfy the Amit-Vishne condition, as there is no automorphism sending $s$ to $s^{-1}$: Suppose $\alpha$ is such an automorphism. Since $\alpha(t)$ needs to be of order $p^2$, one gets $\alpha(t) = t^v$ for some $1 \leq y < p^2$ prime to $p$. So

$$\alpha(ts) = t^v s^{p-1} = s^{p-1} t^{(p+1)y}$$

and

$$\alpha(ts) = \alpha(st^{p+1}) = s^{p-1} t^{(p+1)y}$$

gives together $(p+1)^y = (p+1)(p+1)^{p-2} = (p+1)(p+2) \equiv 1 - 2p \pmod{p^2}$

which is a contradiction, since for $p \geq 3, 1 - 2p \not\equiv 1 \pmod{p^2}$, and therefore $G$ does not satisfy the Amit-Vishne condition.

We use Corollary 2.5 to provide another semi-rationality result:

Proposition 2.7. The group $G = C_p \times C_q$ is semi-rational for $p, q$ primes and $p \equiv 1 \pmod{q}$.

Proof. We can write $G = \langle s, t \mid t^p = s^q = 1, s^{-1}ts = t^k \rangle$ for some $k$ of order $q$ modulo $p$. By a straightforward computation, the conjugacy class of $g$ is $gT = g(t)$ for every $g \notin T$.

In addition, we show that for every $g, h \in G$ generating the same subgroup there is an automorphism sending $g$ to $h$. It is enough to show that $t$ can be mapped to $t^r$ by an automorphism for every $y \neq 0$. This is true since the conditions $t \mapsto t^r$ and $s \mapsto s$ extend uniquely to an automorphism. This automorphism surely sends $t$ to $t^r$.

Therefore $N_{w,G}(g) = N_{w,G}(h)$ for every $g, h \in T$ generating the same subgroup.

The condition for Corollary 2.5 hold, and therefore $G$ is semi-rational.
2.2 The Finite Simple Group Case

Even though the Amit-Vishne condition is not a necessary condition for semi-rationality in the general case, for finite simple groups Question 1.6 has a positive answer.

To show that we use a result of Lubotzky [Lub14]:

**Theorem 2.8** ([Lub14], Theorem 1). If $A$ is a subset of a finite simple group $G$, then there exists a word $w \in F_k$ for some $k$ such that $im(w) = A$ if and only if $1 \in A$ and $\alpha(A) = A$ for every $\alpha \in Aut(G)$.

**Remark 2.9.** One can even get $w \in F_2$, but unlike Theorem 2.8 this requires the classification theorem of finite simple groups.

**Proposition 2.10.** For finite simple groups the Amit-Vishne condition is sufficient and necessary. Namely, $G$ is semi-rational if and only if for every $g, h \in G$ generating the same subgroup, $\alpha(g) = h$ for some $\alpha \in Aut(G)$.

**Proof.** Suppose that $G$ does not satisfy the Amit-Vishne condition. Namely, there are $g, g' \in G$ generating the same subgroup such that $\alpha(g) \neq g'$ for every $\alpha \in Aut(G)$. Denote by $O(g')$ the orbit of $g'$ under the action of $Aut(G)$, and $A = G - O(g')$. Then $1 \in A$ since $1 \notin O(g')$, and $\alpha(A) = A$ for every $\alpha \in Aut(G)$.

So by Theorem 2.8, there exists $w \in F_k$ such that $im(w) = A$. But $g \in A$ and $g' \notin A$, and therefore

$$N_{w,G}(g) > 0 = N_{w,G}(g').$$

By Theorem 1.4, $N_{w,G}$ is not a generalized character, so $G$ is not semi-rational. \qed

We use this result to prove that $PSL_2(p)$ is not semi-rational for $p \geq 11$:

**Proposition 2.11.** For every prime $p \geq 11$ the group $PSL_2(p)$ is not semi-rational.

**Proof.** A direct computation shows that $PSL_2(11)$ and $PSL_2(13)$ are not semi-rational, since for $w = x^2y^3$ the function $N_{w,PSL_2(11)}$ and $N_{2,PSL_2(13)}$ are not generalized characters of $PSL_2(11)$ and $PSL_2(13)$ respectively.

For $p \geq 17$ we prove that $PSL_2(p)$ does not satisfy the Amit-Vishne condition. Let $g$ be of the form $g = \left( \begin{array}{cc} x & \frac{p-1}{2} \\ \frac{p-1}{2} & x^{-1} \end{array} \right) \in PSL_2(p)$ for $x \in \mathbb{F}_p$ such that the order of $g$ is $\frac{p-1}{x}$. This occurs for $x$ of order $\frac{p-1}{2}$ for $p \equiv 3(\text{mod} 4)$ and for $x$ of order $p-1$ for $p \equiv 1(\text{mod} 4)$. Choose $s$ prime to $\frac{p-1}{2}$ such that $g^s = \left( \begin{array}{cc} x^s & x^{-s} \\ x^{-s} & x^s \end{array} \right) \notin \{g, g^{-1}\}$.

It is known that every automorphism of $PSL_2(p)$ is induced as a conjugation by a matrix from $PGL_2(p)$ ([Wil09], Section 3.3.4). Suppose that there exists an automorphism sending $g$ to $g'$. So we can write $g^s = A^{-1}gA$ for some $A \in PGL_2(p)$. But looking at the action of $PGL_2(p)$ on $\mathbb{F}_p \cup \{\infty\}$, the matrices $g, g'$ fix only 0 and $\infty$, and therefore $A$ either fixes 0 and $\infty$ or switches between the two.
If \( A \) fixes 0 and \( \infty \), then \( A \) is of the form \( A = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \), and if \( A \) switches between 0 and \( \infty \), then \( A \) is of the form \( A = \begin{pmatrix} b & a \\ a & b \end{pmatrix} \). But

\[
\left( \begin{array}{cc} a & b \\ b & a \end{array} \right)^{-1} \left( \begin{array}{cc} x & x^{-1} \\ x^{-1} & x \end{array} \right) \left( \begin{array}{cc} a & b \\ b & a \end{array} \right) = \left( \begin{array}{cc} x & x^{-1} \\ x^{-1} & x \end{array} \right) \neq g^s,
\]

and

\[
\left( \begin{array}{cc} b & a \\ a & b \end{array} \right)^{-1} \left( \begin{array}{cc} x & x^{-1} \\ x^{-1} & x \end{array} \right) \left( \begin{array}{cc} b & a \\ a & b \end{array} \right) = \left( \begin{array}{cc} x^{-1} & x \\ x & x^{-1} \end{array} \right) \neq g^s.
\]

Whence \( PSL_2(p) \) does not satisfy the Amit-Vishnue condition, and being simple they are not semi-rational by Proposition 2.10.

Remark 2.12. After the completion of this paper, it was pointed to us by R. Guralnick that our Proposition 2.11 overlaps with results in [GS15].

3 Further Questions

Definition 3.1. For a group \( G \), denote \( Z^\ast(G) = \{ g \in G : \alpha(g) = g \forall \alpha \in Aut(G) \} \) and call it the absolute center of the group.

Since the conditions for semi-rationality require powers of \( g \) prime to the order of \( g \), only elements of order 3 or more can provide counter-example to semi-rationality. Therefore groups with absolute center of exponent larger than 2 are of natural interest.

Consider the groups \( C_p \rtimes C_{q^m} \) for \( p, q > 2 \) primes with \( p \equiv 1 (\text{mod} \ q) \) and \( m \geq 2 \). One can write

\[
C_p \rtimes C_{q^m} = \langle s, t | t^p = s^{q^m} = 1, s^{-1}ts = t^k \rangle
\]

where \( k \) is of order \( q \) modulo \( p \). It can be shown by a straightforward argument that \( Z^\ast(C_p \rtimes C_{q^m}) = \langle s^{q^{m-1}} \rangle \). So the exponent of the absolute center of \( C_p \rtimes C_{q^m} \) has exponent \( q \).

Indeed several of the groups of the form \( C_p \rtimes C_{q^m} \) are not semi-rational. Below is a list of such groups with a word for which \( \text{N}_{w, G} \) is not a generalized character, provided by direct computations:

\[
\begin{align*}
C_7 \rtimes C_9 & : w = x_1x_2x_1^2x_2^5 \\
C_{11} \rtimes C_{25} & : w = x_1x_2^4x_1^3x_2^3 \\
C_{13} \rtimes C_9 & : w = x_1x_2^4x_1x_2^7 \\
C_9 \rtimes C_9 & : w = x_1x_2^4x_1x_2^8 \\
C_{29} \rtimes C_{49} & : w = x_1x_2x_1^3x_2^6 \\
C_{31} \rtimes C_9 & : w = x_1x_2^7x_1^{-8}x_2^4 \\
C_{37} \rtimes C_9 & : w = x_1x_2^{-10}x_1^{-10}x_2^4.
\end{align*}
\]

This raises the questions:

Question 3.2. Is every group of the form \( C_p \rtimes C_{q^m} \) not semi-rational for \( m \geq 2 \)?
Question 3.3. Is every group with absolute center of exponent larger than 2 not semi-rational?

We note that $C_{19} \rtimes C_9$ is isomorphic to a normal subgroup of $C_{19} \rtimes C_{18}$, and hence a normal subgroup of a semi-rational group need not be semi-rational itself. A related question is:

Question 3.4. Is every quotient of a semi-rational group semi-rational itself?

References

[AV11] Alon Amit and Uzi Vishne. Characters and solutions to equations in finite groups. Journal of Algebra and its Applications, 10(04):675–686, 2011.

[GS15] Robert Guralnick and Pavel Shumyatsky. On rational and concise words. Journal of Algebra, 429:213–217, 2015.

[Lub14] Alexander Lubotzky. Images of word maps in finite simple groups. Glasgow Mathematical Journal, 56(2):465–469, 2014.

[Sta86] Richard P Stanley. Enumerative combinatorics, wadsworth publ. Co., Belmont, CA, 1986.

[Wil09] Robert Wilson. The finite simple groups, volume 251. Springer Science & Business Media, 2009.

Einstein Institute of Mathematics
The Hebrew University of Jerusalem
Jerusalem 91904
Israel

tzoor.plotnikov@mail.huji.ac.il