Rational \( \mathcal{W} \) algebras from composite operators

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Abstract

Factoring out the spin 1 subalgebra of a \( \mathcal{W} \) algebra leads to a new \( \mathcal{W} \) structure which can be seen either as a rational finitely generated \( \mathcal{W} \) algebra or as a polynomial non-linear \( \mathcal{W}_\infty \) realization.
1 Introduction

Although a large set of $\mathcal{W}$ algebras and superalgebras is today available, a complete classification of such new symmetries is not yet obtained (for recent reviews see [1] and [2]). As far as the classical case is concerned, the known $\mathcal{W}$ algebras and superalgebras can be constructed as symmetries – conserved currents – of Toda and super-Toda theories [1, 3]. Each such $\mathcal{W}$ algebra is generated by a finite number of primary fields $W_{h_k}$ $(k = 1, ..., N)$ and is of polynomial type: this means that the Poisson brackets of two primary fields $W_{h_i}$ and $W_{h_j}$ have the following structure

$$\{W_{h_i}(z), W_{h_j}(w)\} = \sum_\alpha P_\alpha(W_{h_k}(w), \partial^\alpha W_{h_k}(w))\delta^{(\alpha)}(z - w) \quad (1.1)$$

the $P_\alpha$ being polynomials in the $W_{h_k}$ and their derivatives. Note that also the occurrence of $\mathcal{W}$ algebras in which the polynomials are replaced by rational functions in the $W_{h_k}$ and their derivatives, has been discussed in [4].

Besides the finitely generated $\mathcal{W}$ algebras, the $\mathcal{W}_\infty$ ones have also been considered [5]. In this second class of $\mathcal{W}$ algebras, the most popular one – and the oldest one – is the area-preserving diffeomorphism algebra on a two dimensional manifold, also denoted $w_\infty$. Different deformations of $w_\infty$ have already been obtained [6] and various realizations of such $\mathcal{W}_\infty$-algebras recently considered (see for example [7] for a review). Such structures show up rather naturally in conformal field theoretical approaches of the quantum Hall effect [8], of the black hole problem [9, 10] and also in the matrix models approach [11].

One of the purposes of this letter is to show that a large class of rational $\mathcal{W}$ algebras can be obtained from finitely generated polynomial $\mathcal{W}$ algebras. Let us, at this point, make precise or recall, some general definitions and properties on finitely generated $\mathcal{W}$ algebras. Indeed, there are two basic ingredients in the construction of a Toda theory [12]: a simple Lie algebra – or superalgebra – $\mathcal{G}$ and an $Sl(2)$ – or $OSp(1|2)$ – subalgebra. When $Sl(2)$ is chosen as the principal subalgebra of $\mathcal{G}$, then one gets the $\mathcal{G}$-abelian Toda model. The corresponding $\mathcal{W}$ algebra of conserved quantities is generated by primary fields of conformal spin $h = 2, ..., k$, these numbers being actually the degrees of the fundamental polynomial invariants of $\mathcal{G}$. As an example, for $\mathcal{G} = Sl(3)$, one gets the algebra usually denoted $\mathcal{W}_3$ or $\mathcal{W}(A_2)$ [1] and generated by the fields $W_2, W_3$. By extension, we will call such algebras abelian $\mathcal{W}$ algebras. When $Sl(2)$ is not principal in $\mathcal{G}$, the conserved quantities of the corresponding non abelian Toda model will generate what we will call a non abelian $\mathcal{W}$ algebra. In $Sl(3)$, there is only one $Sl(2)$ which is not principal in it, its Dynkin index is 1. The generators of the corresponding non abelian $\mathcal{W}$ algebra $\mathcal{W}(A_2, A_1)$ are, in addition to $W_2$, two conformal spin-$\frac{3}{2}$ fields $W_{3/2}^\pm$ and a spin-one field $W_1$; this algebra is also called the Bershadsky algebra [13].

One can remark [1] that most of the non abelian $\mathcal{W}$ algebras contain spin-one fields. It

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1 We denote by $\mathcal{W}(\mathcal{G}, \mathcal{K})$ the $\mathcal{W}$ algebra based on a Toda model characterized by the Lie algebra $\mathcal{G}$ and the $Sl(2)$ embedding principal in the subalgebra $\mathcal{K}$ of $\mathcal{G}$. In the case of the abelian Toda model where $\mathcal{K} = \mathcal{G}$, the corresponding $\mathcal{W}$ algebra is simply denoted by $\mathcal{W}(\mathcal{G})$.

2 A study of $Sl(2)$ embeddings in semi-simple Lie algebras $\mathcal{G}$ leading to $\mathcal{W}$ algebras without Kac-Moody component can be found in ref. [14].
is reasonable to expect the spin-one part in a $\mathcal{W}$ algebra to play a particular role. One can easily realize that these fields close linearly into a Kac-Moody algebra $\mathcal{W}_1$. Note that the horizontal (or finite part) of $\mathcal{W}_1$ is the commutant of the chosen $\mathfrak{sl}(2)$ in $\mathcal{G}$. Moreover, the $\mathcal{W}$ generators decompose into irreducible representations under the adjoint action of this Kac-Moody algebra. It is also known that in a super $\mathcal{W}$ algebra the spin 1/2 part, when it exists, can be factored out \cite{13}. More precisely, a meromorphic conformal field theory can be decomposed into the tensor product of a spin 1/2 part and a conformal field theory without such spin 1/2 fields. Similarly, one could wonder what will result from the factorization in $\mathcal{W}$ of its $\mathcal{W}_1$ subalgebra.

As is shown below, rational $\mathcal{W}$ structures appear when determining the commutant $\text{Com}_{\mathcal{W}_1} \mathcal{W}_1$ of $\mathcal{W}_1$ in $\mathcal{W}$ algebras. Each such a rational $\mathcal{W}$ algebra provides an explicit realization of a non-linear – but polynomial – $\mathcal{W}_\infty$ algebra. This new structure appears by assuming the primary fields, which are expressed as rational functions of the basic fields in the original rational $\mathcal{W}$ algebra, to be independent algebra generators. Furthermore, the explicit construction of the fields in the commutant naturally involves a derivative which is covariant with respect to the Kac-Moody transformations generated by $\mathcal{W}_1$.

After presenting the general properties of the commutant $\text{Com}_{\mathcal{W}} \mathcal{W}_1$ when $\mathcal{W}_1$ is restricted to be a $U(1)$ Kac-Moody, we work out explicitly the simplest case, the one associated to $\mathcal{W}(A_2, A_1)$ (the Bershadsky algebra). Then we illustrate the situation for non-abelian $\mathcal{W}_1$'s with the specific examples taken from $A_3$ non-abelian Toda models.

2 The $U(1)$ case

2.1 General features

We will consider in this section the case of a $\mathcal{W}$ algebra containing a spin-one field. Let $J(z)$ be the spin-one primary field under the $T_0(z)$ Virasoro field in $\mathcal{W}$. Their Poisson brackets are

$$\{T_0(z), T_0(w)\} = -2T_0(w)\delta'(z-w) + \partial T_0(w)\delta(z-w) + c\delta''(z-w)$$

$$\{T_0(z), J(w)\} = -J(w)\delta'(z-w) + \partial J(w)\delta(z-w)$$

$$\{J(z), J(w)\} = -\gamma\delta'(z-w) \quad (2.1)$$

The other primary fields $W^q_h(z)$ in $\mathcal{W}$ undergo the $T_0(z)$ action as follows

$$\{T_0(z), W^q_h(w)\} = -hW^q_h(w)\delta'(z-w) + \partial W^q_h(w)\delta(z-w) \quad (2.2)$$

where $h$ denotes the conformal dimension, while $q$ specifies the $U(1)$ charge carried by the primary field:

$$\{J(z), W^q_h(w)\} = qW^q_h(w)\delta(z-w), \quad (2.3)$$

This last relation reflects the action of the Kac-Moody part on the rest of the $\mathcal{W}$ algebra in accordance with refs. \cite{3,16}.

A new stress-energy tensor can be defined by subtracting from $T_0(z)$ a Sugawara term constructed with the $J(z)$ field, through the relation

$$T(z) = T_0(z) - \frac{1}{2\gamma} J(z)^2, \quad (2.4)$$
The first and third relations in (2.1) remain valid when shifting $T_0(z)$ into $T(z)$, while the second relation now gives
\[
\{T(z), J(w)\} = 0
\] (2.5)

Equation (2.2) then gives
\[
\{T(z), W_h^q(w)\} = -hW_h^q(w)\delta'(z - w) + \mathcal{D}W_h^q(w)\delta(z - w)
\] (2.6)

with
\[
\mathcal{D}W_h^q(w) = (\partial - \frac{q}{\gamma}J(w))W_h^q(w)
\] (2.7)

The presence of a covariant derivative associated to the Kac-Moody part of a $\mathcal{W}$ algebra has already been pointed out in ref. [16]. One can directly verify that for any positive integer $n$
\[
\{J(z), \mathcal{D}^nW_h^q(w)\} = q\mathcal{D}^nW_h^q(w)\delta(z - w)
\] (2.8)

which insures that under the ”inner automorphisms” generated by $J(z)$
\[
X(z) \to X(z) + \int dz'\alpha(z')\{J(z'), X(z)\} + \frac{1}{2}\int dz'dz''\alpha(z')\alpha(z'')\{J(z''), \{J(z'), X(z)\}\} + \ldots
\] (2.9)

the fields $\mathcal{D}^nW_h^q(z)$ transform as
\[
\mathcal{D}^nW_h^q(z) \to e^{\alpha(z)}\mathcal{D}^nW_h^q(z) \quad \text{with} \quad m = 0, 1, 2, \ldots
\] (2.10)
as must be the case for $\mathcal{D}$ a covariant derivative. One should stress that the use of the covariant derivative is particularly useful since it allows to write down in a compact form the Poisson brackets as we will do in the following.

It is easily shown that the commutant $\text{Com}_{\mathcal{W}}(J)$ of $J(z)$ – that is in other words the polynomial $\mathcal{W}$ subalgebra with elements having a zero Poisson bracket with $J(z)$ – is generated by $\partial^nT(z)$ and by the monomials
\[
(\mathcal{D}^{n_1}W_h^{q_1})(\mathcal{D}^{n_2}W_h^{q_2})\ldots(\mathcal{D}^{n_k}W_h^{q_k})
\] (2.11)

where $n_0$ and the $n_i$'s are non negative integers, and the charges $q_i$ satisfy the condition
\[
\sum_{i=1}^{k} q_i = 0
\] (2.12)

A remarkable feature of $\text{Com}_{\mathcal{W}}(J)$ is that it contains an infinite tower of primary fields of integral dimension. The primary fields can be specified by their order $n$, which counts the number of covariant derivatives in their leading term (subleading terms, having a lower number of covariant derivatives, should be added to get a primary field). A primary field of zeroth order is just given by the product $W_h^{q_0}$:
\[
W_h^{q_0}(z) = \prod_{j=1}^{k} W_h^{q_j}
\] (2.13)
with
\[ \sum_{j=1}^{k} q_j = 0 \quad \text{and} \quad \sum_{j=1}^{k} h_j = h \]  
(2.14)

with \( W^0_h \) primary in \( \mathcal{W} \) with respect to \( T_0(z) \) (cf. equation 2.2). Indeed, we can easily check that
\[ \{T(z), W^0_h(w)\} = -hW^0_h(w)\delta'(z - w) + \partial W^0_h(w)\delta(z - w) \]  
(2.15)

The primary fields of order \( n \) obtained from \( W^0_h \) have conformal dimension \( h + n \).

A primary field at first order is
\[ W^0_{h+1}(z) = \sum_{j=1}^{k} \alpha_j \mathcal{D}_j (\prod_{m=1}^{k} W^q_{h_m}) \]  
(2.16)

where \( \mathcal{D}_j \) denotes the covariant derivative applied to the \( j \)th term of the product, and the coefficients \( \alpha_j \) are such that
\[ \sum_{j=1}^{k} \alpha_j h_j = 0 \]  
(2.17)

At second order, the primary fields are of the form
\[ W^0_{h+2}(z) = \sum_{i,j=1}^{k} \alpha_{ij} \mathcal{D}_i \mathcal{D}_j (\prod_{m=1}^{k} W^q_{h_m}) + fT W^0_h \]  
(2.18)

with the two sets of condition:
\[ -fc + \sum_{j=1}^{k} \alpha_{jj} h_j = 0 \]  
(2.19)

and
\[ \alpha_{ii}(2h_i + 1) + \sum_{j \neq i} \alpha_{ij} h_j = 0 \quad \text{for } i = 1, \ldots, k \]  
(2.20)

in addition to the null charge restriction eq. (2.14a). The coefficients \( \alpha_{ij} \), as well as the \( \alpha_{ijl} \) introduced below, are assumed symmetric in the exchange of two indices.

At third order, the primary fields are given by
\[ W^0_{h+3}(z) = \sum_{i \leq j \leq l} \alpha_{ijl} \mathcal{D}_i \mathcal{D}_j \mathcal{D}_l (\prod_{m=1}^{k} W^q_{h_m}) + \sum_{j=1}^{k} g_j T \mathcal{D}_j (\prod_{m=1}^{k} W^q_{h_m}) + f(\partial T) W^0_h \]  
(2.21)

The vanishing of the coefficients of the \( \delta'' \) term in the Poisson brackets implies the relations
\[ \alpha_{iii}(3h_i + 3) + \sum_{j \neq i} \alpha_{iij} h_j = 0 \]  
\[ \alpha_{ijj}(2h_j + 1) + \alpha_{jji}(2h_j + 1) + \sum_{k \neq i,j} \alpha_{ijk} h_k = 0 \quad \text{for } i \neq j \]  
(2.22)
and the vanishing of the coefficients of $\delta'''$, $\delta''''$ leads respectively to
\[
\begin{align*}
 cg_j - (3h_j + 1)\alpha_{jjj} - \sum_{i \neq j} h_i \alpha_{iji} & = 0 \\
 -fc + \sum_i \alpha_{iii} h_i & = 0
\end{align*}
\]
(2.23)

A similar set of equations can be constructed at any order and it can be immediately checked that it always admits solutions. Moreover, at least for the invariants generated by bilinear products of two fields with opposite charges, the solution is unique.

2.2 An example: the U(1) Commutant of the Bershadsky algebra $\mathcal{W}(A_2, A_1)$

Let us treat for illustration the case of the Bershadsky algebra. This algebra contains one spin-two field (the stress-energy tensor $T(z)$), one spin-one field (the current $J(z)$) and a couple of spin-$\frac{3}{2}$ fields $W_\pm(z)$ of opposite charge under the $U(1)$ current. The commutation relations are:
\[
\begin{align*}
\{J(z), J(w)\} & = \frac{3}{2} c \delta'(z - w) \\
\{T(z), T(w)\} & = -2T(w)\delta'(z - w) + \partial T(w)\delta(z - w) + \frac{c}{2} \delta'''(z - w) \\
\{T(z), J(w)\} & = 0 \\
\{T(z), W_\pm(w)\} & = -\frac{3}{2} W_\pm(w)\delta'(z - w) + (\mathcal{D}W_\pm)(w)\delta(z - w) \\
\{J(z), W_\pm(w)\} & = \pm \frac{3}{2} W_\pm(w)\delta(z - w) \\
\{W_+(z), W_-(w)\} & = (T - c\mathcal{D}^2)(w)\delta(z - w) \\
\{W_+(z), W_+(w)\} & = \{W_-(z), W_-(w)\} = 0
\end{align*}
\]
(2.24)

Notice that the Poisson brackets can be compactly written using the covariant derivative introduced in the previous section:
\[
\mathcal{D}W_\pm = (\partial \mp \frac{1}{c} J)W_\pm
\]

Clearly the commutant of the $U(1)$ current contains the stress-energy tensor $T$, its derivatives, and the bilinear products $W^{(p,q)} = (\mathcal{D}^p W_+)(\mathcal{D}^q W_-)$, with $p, q$ non-negative integers. As explained in the previous paragraph, the fields $W^{(p,q)}$ and $T$ are the building blocks from which one constructs an infinite tower of primary fields $W_{3+n}$ ($n \geq 0$) of integral dimension $3 + n$, one for each value of $n$; at the first orders we get, in a specific normalization,
\[
\begin{align*}
W_3 & = W_+ \cdot W_- \\
W_4 & = W_+ \cdot \mathcal{D}W_- - W_- \cdot \mathcal{D}W_+
\end{align*}
\]
\[
W_5 = \frac{3}{2} \mathcal{D}^2 W_+ \cdot W_+ - 4 \mathcal{D} W_+ \cdot \mathcal{D} W_+ + \frac{3}{2} \mathcal{D}^2 W_+ \cdot W_+ + \frac{9}{c} T \cdot W_3
\]

\[
W_6 = (\mathcal{D} W_+) \cdot (\mathcal{D}^2 W_-) - (\mathcal{D} W_-) \cdot (\mathcal{D}^2 W_+) - \frac{1}{6} \partial^2 W_4 - \frac{13}{2c} T \cdot W_4
\quad (2.25)
\]

Analogous formulae hold for \( n > 3 \).

The primary fields \( W_{3+n} \) with \( n \geq 2 \) are not independent: they can be expressed as rational functions of \( T, W_3, W_4 \) and their derivatives, as the following reasoning shows. At first one can notice that \( W^{(p,q)} \) for \( q \geq 1 \) are linear combinations of \( V^{(p')} \equiv W_{(p',0)} \) and their derivatives; besides that, due to the properties of the covariant derivatives, the following quadratic relations hold:

\[
V^{(p+1)} \cdot V^{(0)} = V^{(0)} \cdot \partial V^{(p)} - V^{(p)} \cdot \partial V^{(0)} + V^{(p)} \cdot V^{(1)}
\quad (2.26)
\]

They show that the fields \( V^{(p')} (p' \geq 2) \) are obtained as rational functions of

\[
\begin{align*}
V^{(0)} & = W_3 \\
V^{(1)} & = \frac{1}{2} (\partial W_3 - W_4)
\end{align*}
\quad (2.27)
\]

The \( U(1) \)-Commutant algebra \( \text{Com}_W(J) \) of the Bershadsky algebra has therefore the structure of a finitely generated rational \( W \)-algebra of type \( (2,3,4) \). It is explicitly given by \( T, W_3, W_4 \) satisfying:

\[
\begin{align*}
\{T(z), T(w)\} & = -2T(w)\delta'(z - w) + \partial T(w)\delta(z - w) + \frac{1}{2} \delta''(z - w) \\
\{T(z), W_{3+n}(w)\} & = -(3 + n)W_{3+n}(w)\delta'(z - w) + \partial W_{3+n}(w)\delta(z - w) \\
\{W_3(z), W_3(w)\} & = 2W_4(w)\delta'(z - w) - \partial W_4(w)\delta(z - w) \\
\{W_3(z), W_4(w)\} & = -2W_3(w)\delta''(z - w) - 4\partial W_3(w)\delta''(z - w) \\
& + \frac{2}{7} [W_5 - 16 T \cdot W_3 + 9 \partial^2 W_3](w)\delta'(z - w) \\
& - \frac{2}{7} \partial [W_5 - 16 T \cdot W_3 + 20 \partial^2 W_3](w)\delta(z - w) \\
\{W_4(z), W_4(w)\} & = 6W_4(w)\delta''(z - w) - 9 \partial W_4(w)\delta''(z - w) \\
& + [ -8 W_6 + \frac{17}{3} \partial^2 W_4 + 6 W_3 \cdot W_3 - \frac{116}{3} T \cdot W_4 ](w)\delta'(z - w) \\
& + \partial [4 W_6 - \frac{4}{3} \partial^2 W_4 - 3 W_3 \cdot W_3 + \frac{58}{3} T \cdot W_4](w)\delta(z - w)
\end{align*}
\quad (2.28)
\]

where \( n \) is a non-negative integer; the explicit expression of the primary fields \( W_{5,6} \) in terms of \( T, W_{3,4} \) can be computed from \( (2.25,2.27) \). We get

\[
\begin{align*}
W_5 & = \frac{1}{4 W_3} (7 \Psi + 14 W_3 \partial W_4 - 8 W_3 \partial^2 W_3 + 36 T W_3^2) \\
W_6 & = -\frac{1}{12 W_3^2} (3 W_4 \Psi + 3 W_3 \partial \Psi + 8 W_3^2 \partial^2 W_4 - 6 W_3^2 \partial^3 W_3 + 52 T W_3^2 W_4)
\end{align*}
\quad (2.29)
\]
Here $\Psi$ is a 8-dimensional polynomial in $W_3, W_4$ and their derivatives:

$$
\Psi = W_4^2 - 2W_3 \partial W_4 - (\partial W_3)^2 + 2W_3 \partial ^2 W_3
$$

(2.30)

The above formulae are given for the central charge $c = 1$. In the classical case the algebra for any other value of the central charge can be obtained by simply rescaling the fields and the Poisson brackets as well.

A few comments are in order: the algebra $Com_W(J)$ has a new structure with respect to the standard $\mathcal{W}$ algebras: its closure is not on polynomials, but on rational functions of the fields and their derivatives. Such algebra provides a specific realization of an underlying non-linear, but of polynomial type, $\mathcal{W}_\infty$ algebra, given by the Poisson brackets of the primary fields $W_{3+n}$ (and of $T$) among themselves, considered now as independent algebra generators. Such a $\mathcal{W}_\infty$ algebra is non-linear as it can be immediately seen from the relations (2.28). Moreover it is a genuine $\mathcal{W}_\infty$ algebra, i.e. it is not possible to find out a finite subset of primary fields which, together with the stress-energy tensor, close the algebra in a polynomial way. The latter statement can be immediately verified by inspecting the Poisson brackets of the above introduced monomials $W^{(p,q)}$ among themselves. The situation here should be compared with that of standard finitely generated $\mathcal{W}$-algebras: in that case [17] to get rid of the non-linear character of the $\mathcal{W}$ algebras one is lead to introduce a linear $\mathcal{W}_\infty$ algebra, promoting non-linear terms to be new primary fields. Here, to get rid of the non-polynomial character of the finitely generated $\mathcal{W}$ algebra, we promote the rationally expressed primary fields to be new fields; the resulting $\mathcal{W}_\infty$ algebra is in that case non-linear.

Finally one should notice that the central charge is degenerated, namely it appears only in the Poisson brackets of the stress-energy tensor with itself.

It is evident that, even if we have worked out explicitly the construction for the simplest example, the same structure holds in more general cases.

3 The non abelian Kac-Moody case

Let us now discuss the non abelian case. For this purpose, we will treat two examples, namely the algebra $\mathcal{W}(A_3, A_1 \oplus A_1)$ associated to the WZNW reduction $Sl(2)_2 \subset Sl(4)$ (the $Sl(2)$ subscript is the Dynkin index of the embedding) – admitting a Kac-Moody subalgebra $\mathcal{W}_1 = Sl(2)$ – and the algebra $\mathcal{W}(A_3, A_1)$ associated to the WZNW reduction $Sl(2)_1 \subset Sl(4)$ – admitting a Kac-Moody subalgebra $\mathcal{W}_1 = Sl(2) \oplus U(1)$. Finally, we will give some indications on the general structure for the non abelian case.

3.1 The example of $\mathcal{W}(A_3, A_1 \oplus A_1)$

Consider the $\mathcal{W}$ algebra $\mathcal{W}(A_3, A_1 \oplus A_1)$ generated by three spin-one fields $J^i(z)$ and four spin-two fields $W^i(z)$ and $T(z)$ with $i = 1, 2, 3$. The Poisson brackets of this $\mathcal{W}$ algebra are given by

$$
\{T(z), T(w)\} = -2T(w)\delta'(z-w) + \partial T(w)\delta(z-w) + c\delta'''(z-w)
$$

$$
\{T(z), J^i(w)\} = 0
$$
\[
\{J^i(z), J^j(w)\} = c(D^{ij})_w \delta(z-w)
\]
\[
\{J^i(z), W^j(w)\} = -\epsilon^{ijk} W^k(w) \delta(z-w)
\]
\[
\{T(z), W^i(w)\} = -2W^i(w) \delta'(z-w) + (DW)^i(w) \delta(z-w)
\]
\[
\{W^i(z), W^j(w)\} = -\frac{1}{2} ((D^3_w)^{ji} - 2T \cdot D_w^{ji} - \partial T \cdot \eta^{ji})_w \delta(z-w)
\]

(3.1)

where \(\epsilon^{ijk}\) is the completely antisymmetric tensor of rank 3 such that \(\epsilon^{123} = 1\). The tensor \(\eta_{ij} (\equiv \eta^{ij})\) is the diagonal matrix \(\text{Diag}(-1, 1, 1)\) and is used for lowering (raising) the indices.

The covariant derivative is here

\[
D^{ij} = \eta^{ij} \partial + \frac{1}{c} \epsilon^{ijk} J^k
\]

(3.2)

The transformations of the fields \(J^i\) and \(W^i\) with respect to the infinitesimal parameter \(\lambda^i(z)\) are given by

\[
\delta W^i = \epsilon_{jik} \lambda^j W^k
\]
\[
\delta J^i = c \partial \lambda^i + \epsilon_{jik} \lambda^j J^k
\]

The application of the covariant derivative on \(W^i\) leads to a covariant transformation property; in fact one gets \((n\ being\ an\ integer)\):

\[
\{J^i(z), (D^n W)^j(w)\} = -\epsilon^{ijk} (D^n W)^k(w) \delta(z-w)
\]

(3.3)

and, in terms of infinitesimal transformations generated by \(\lambda^i\):

\[
\delta (D^n W)^i = \epsilon_{jki} (D^n W)^k
\]

Notice that since the Virasoro generator \(T\) commutes with the Kac-Moody fields \(J^i\), the covariant derivative \(DT\) in equation (3.1) has to be understood as \(\partial T\).

The equations in (3.1) are similar as those in (2.3) and (2.4): they look formally just like the abelian case once the fields are accomodated into multiplets. In particular the analysis of the conformal dimensions of the derivatives fields \(D^n W\) is completely analogous to that done in sect. 2 for the abelian case. Moreover, (3.1b, d) show that the fields \(W^i(z)\) are primary fields with conformal dimension 2 with respect to the Virasoro generator \(T_0(z) = T(z) + \frac{1}{2} J^2(z)\).

Now, the covariant form of the \(W\) algebra with respect to its Kac-Moody subalgebra allows us to determine easily the non-linear \(W_\infty\) commutant of the Kac-Moody subalgebra \(SL(2)\) in the algebra \(W(A_3, A_1 \oplus A_1)\). Indeed, one can immediately check that bilinear invariants can be obtained from the scalar products \(\delta^i (D^p \bar{W}) \cdot (D^q \bar{W})(w)\) (where the upperscript \(t\) denotes transposition); indeed we have

\[
\{\bar{J}(z), \delta^i (D^p \bar{W}) \cdot (D^q \bar{W})(w)\} = 0
\]

(3.4)

Therefore, the elements in the commutant are generated as before by \(\partial^* T\) and by \(\delta^i (D^p \bar{W}) \cdot (D^q \bar{W})(z)\), with \(p, q, r\) non-negative integers.

The primary fields in the commutant are given by the same formulas as in the abelian case, with just the replacements \(W_+ \rightarrow \bar{W}\) and \(W_- \rightarrow \delta^i \bar{W}\). Notice that \(\delta^i \bar{W} \cdot \bar{W}\) has the same conformal dimension \((h = 3)\) as the commutant in the Bershadsky algebra.
3.2 The example of $\mathcal{W}(A_3, A_1)$

Now, we consider the example of the $\mathcal{W}$ algebra $\mathcal{W}(A_3, A_1)$ generated by four spin-one fields $J^i(z)$ ($i = 1, 2, 3$) and $J^0(z)$, four spin-$\frac{3}{2}$ fields $\hat{W}_\pm(z) = (W^u_\pm(z), W^d_\pm(z))$ (indices $u$ and $d$ stands for up and down) and one spin-two field $T(z)$. The Poisson brackets of this $\mathcal{W}$ algebra are given by

$$
\{T(z), T(w)\} = -2T(w)\delta'(z - w) + \partial T(w)\delta(z - w) + c\delta'''(z - w)
$$

$$
\{T(z), \hat{J}(w)\} = \{T(z), J^0(w)\} = 0
$$

$$
\{J^i(z), J^j(w)\} = \epsilon^{ik}_j J^k(w)\delta(z - w) - \frac{1}{2}\delta'(z - w)
$$

$$
\{J^0(z), J^0(w)\} = -\delta'(z - w)
$$

$$
\{\hat{J}(z), J^0(w)\} = 0
$$

$$
\{T(z), \hat{W}_\pm(w)\} = -\frac{3}{2}\hat{W}_\pm(w)\delta'(z - w) + (D\hat{W}_\pm)(w)\delta(z - w)
$$

$$
\{J^i(z), \hat{W}_\pm(w)\} = \sigma^i \hat{W}_\pm(w)\delta(z - w)
$$

$$
\{J^0(z), \hat{W}_\pm(w)\} = \pm \hat{W}_\pm(w)\delta(z - w)
$$

$$
\{\hat{W}_+(z) \hat{\otimes} \hat{W}_+(w)\} = \{\hat{W}_-(z) \hat{\otimes} \hat{W}_-(w)\} = 0
$$

$$
\{\hat{W}_+(z) \hat{\otimes} \hat{W}_-(w)\} = -\frac{1}{2}[D^2 - T](w)\delta(z - w)
$$

(3.5)

where the $\sigma_i$ are the Pauli matrices satisfying the algebra

$$
[\sigma^i, \sigma^j] = 2\epsilon^{ij}_k \sigma^k
$$

and $D = \partial - \hat{J} \cdot \hat{\sigma} - qJ^0$ is the covariant derivative. The commutant is in this case generated by $\partial^r T$ and by $\hat{W}_+(z) \hat{\otimes} (D^p \hat{W}_-)(z)$, with $p, q, r$ non-negative integers. The structure is again similar to the one given in the previous examples, and in particular even in this case the first composite invariant operator $\hat{W}_+(z) \hat{\otimes} (\hat{W}_-)(z)$ has conformal dimension equal to 3.

Conclusions

We have shown that, as soon as a classical $\mathcal{W}$ algebra admits a Kac-Moody part, the set of fields commuting with the spin 1 part generates a rational $\mathcal{W}$ subalgebra. This one can also be seen as a realization of a non-linear, but polynomial, $\mathcal{W}_\infty$-algebra.

The exhibited structures deserve more detailed studies. In particular the quantum version has to be considered; in this case it is reasonable to expect the central charge to play a crucial role. Another interesting question concerns the possible exploitation of the covariant derivatives extensively used in this approach for the construction of 2-dimensional integrable models.
References

[1] L. Feher, L. O’Raifeartaigh, P. Ruelle, I. Tsutsui and A. Wipf, Physics Reports 222 (1992) 1, and references therein.

[2] P. Bouwknegt and K. Schoutens, Physics Reports 223 (1993) 183, and references therein.

[3] L. Frappat, E. Ragoucy and P. Sorba, preprint ENSLAPP-AL-391/92, to appear in Comm. Math. Phys. (1993).

[4] L. Feher, L. O’Raifeartaigh, P. Ruelle and I. Tsutsui, Phys. Lett B283 (1992) 243 and preprint Bonn-HE-93-14.

[5] I. Bakas, Phys. Lett. B228 (1989) 57; Comm. Math. Phys. 134 (1990) 487.

[6] H. Aratyn, L.A. Ferreira, J.F. Gomes and H. Zimmermann, Phys. Lett. B293 (1992) 67.

[7] C.N. Pope, Lectures given at Trieste Summer School in High En. Physics 1991, preprint CPT-TAMU-103/91.

[8] A. Cappelli, C.A. Trugenberger and G.R. Zemba, Nucl. Phys. B396 (1993) 465.

[9] J. Ellis, N. Mavromatos and D. Nanopoulos, Phys. Lett. B267 (1991) 465; Phys. Lett. B272 (1991) 261.
S. de Alwis and J. Lykken, Phys. Lett. B269 (1991) 264.

[10] F. Yu and Y.S. Wu, Phys. Rev. Lett. 68 (1992) 2996.

[11] L. Bonora and C.S. Xiong, preprint SISSA 57/93/EP.

[12] A.N. Leznov and M.V. Saveliev, Acta Appl. Math., 116 (1989) 1.

[13] M. Bershadsky, Comm. Math. Phys. 139 (1991) 71.

[14] P. Bowcock and G.M.T. Watts, Nucl. Phys. B379 (1992), 63.

[15] P. Goddard and A. Schwimmer, Phys. Lett. B214 (1988), 209.

[16] F.A. Bais, T. Tjin and P. van Driel, Nucl. Phys. B357 (1991), 632.

[17] C.N. Pope, L.J. Romans and X. Shen, Phys. Lett. B236 (1990), 173.