Nonparametric Regression with Adaptive Truncation via a Convex Hierarchical Penalty

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Abstract

We consider the problem of non-parametric regression with a potentially large number of covariates. We propose a convex, penalized estimation framework that is particularly well-suited for high-dimensional sparse additive models. The proposed approach combines appealing features of finite basis representation and smoothing penalties for non-parametric estimation. In particular, in the case of additive models, a finite basis representation provides a parsimonious representation for fitted functions but is not adaptive when component functions possess different levels of complexity. On the other hand, a smoothing spline type penalty on the component functions is adaptive but does not offer a parsimonious representation of the estimated function. The proposed approach simultaneously achieves parsimony and adaptivity in a computationally efficient framework. We demonstrate these properties through empirical studies on both real and simulated datasets. We show that our estimator converges at the minimax rate for functions within a hierarchical class. We further establish minimax rates for a large class of sparse additive models. The proposed method is implemented using an efficient algorithm that scales similarly to the Lasso with the number of covariates and samples size.

1 Introduction and Motivation

Consider first univariate non-parametric function estimation from \( n \) pairs of observations \((x_1, y_1), \ldots, (x_n, y_n)\) with \( x_i, y_i \in \mathbb{R} \) for each \( i \). Assume that, for each \( i \), \( y_i = f(x_i) + \varepsilon_i \), where \( \varepsilon_i \) are i.i.d. with mean 0 and finite variance \( \sigma^2 \). There are many proposals for estimating \( f \); local polynomials (Stone, 1977), kernels (Nadaraya, 1964; Watson, 1964), splines (Wahba, 1990), and others. Here, we focus on basis expansions estimators (also known as projection estimators) (ˇCencov, 1962) which are arguably the simplest and among the most commonly used.

Let \( y = [y_1, \ldots, y_n]^T \in \mathbb{R}^n \) and \( x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n \) be the response and covariate vectors. For \( v \in \mathbb{R}^n \), let \( ||v||_n^2 = n^{-1} \sum_{i=1}^{n} v_i^2 \) be a modified \( \ell_2 \)-norm, referred to as the empirical norm. Projection estimators are solutions to linear regression problems based on a set of basis functions \( \{\psi_k\}_{k=1}^{\infty} \), along with a truncation level \( K \). More specifically, let \( \Psi_K \in \mathbb{R}^{n \times K} \) be the \( n \times K \) matrix with entries \( \Psi_K(i,k) = \psi_k(x_i) \) for \( k \leq K, i \leq n \). The basis expansion estimate of \( f \) is then given by \( \hat{f} = \sum_{k \leq K} \hat{\beta}_k^{proj} \psi_k \), where

\[
\hat{\beta}^{proj} = \arg\min_{\beta \in \mathbb{R}^K} \frac{1}{2} ||y - \Psi_K \beta||_n^2.
\] (1)

To asymptotically balance bias and variance, \( K \equiv K_n \) is allowed to vary with \( n \). Unfortunately, choosing the truncation level \( K \) can be difficult in practice; it depends on \( \sigma^2 \), properties of \( f \) (e.g., smoothness) and the choice of basis functions. Usually, \( K \) is chosen via split sample validation. For basis
expansions hierarchically ordered by some measure of complexity (i.e., $\psi_1$ less complex than $\psi_2$, etc, ...), projection estimators with small $K$ would also give a parsimonious representation of $f$.

The projection estimation approach extends easily to additive models (Hastie et al., 2009), where each $x_i = (x_{i1}, \ldots, x_{ip})^T$ is now a $p$-vector, and the true underlying model is believed to be of the form

$$ y_i = \sum_{j=1}^{p} f_j(x_{ij}) + \varepsilon_i. \tag{2} $$

The components of this model can be estimated by using a basis expansion in each component and solving the optimization problem

$$ \hat{\beta}_1^{A-proj}, \ldots, \hat{\beta}_p^{A-proj} = \arg\min_{\beta_j \in \mathbb{R}^{K_j}} \frac{1}{2} \left\| y - \sum_{j=1}^{p} \Psi_{K_j}^j \beta_j \right\|_n^2. \tag{3} $$

The estimate of $f_j$ is then $\hat{f}_j = \sum_{k=1}^{K_j} \hat{\beta}_j^{A-proj} \psi_k$.

For high-dimensional problems, when $p \gg n$, it is often assumed that for many components $f_j \equiv 0$. A popular choice in this scenario is to add a sparsity inducing penalty to the basis expansion framework (Ravikumar et al., 2009) and solve

$$ \hat{\beta}_1^{SPAM}, \ldots, \hat{\beta}_p^{SPAM} = \arg\min_{\beta_j \in \mathbb{R}^{K}} \frac{1}{2} \left\| y - \sum_{j=1}^{p} \Psi_{K_j}^j \beta_j \right\|_n^2 + \lambda \sum_{j=1}^{p} \left\| \Psi_{K_j}^j \beta_j \right\|_n. \tag{4} $$

This approach is known as Sparse Additive Modeling (SpAM). In practice, the same truncation level is used for each feature ($K_j \equiv K_i$) to keep computation tractable, even in low-dimensional additive models. When $f_j$ have widely different complexities, this strategy leads to poor estimates. In scenarios with only a moderate number of observations this issue often severely limits the effectiveness of predictive models built using SpAM. Addressing this limitation is one of our major motivations.

In this manuscript, we propose hierbasis, a penalized estimation method motivated by the projection estimator: In hierbasis, the truncation level is determined data-adaptively rather than being prespecified. The hierbasis framework can be applied to fit both univariate and multivariate models, as well as additive models with or without sparsity. We also discuss an extension of hierbasis for multivariate settings. When applied to univariate problems, hierbasis performs similarly to a standard basis expansion/projection estimator (with a little more regularization). However, for additive or sparse additive models, hierbasis automatically chooses a truncation level for each feature. These truncation levels will often differ between features based on the underlying complexity of the true $f_j$. This can vastly improve prediction accuracy of our model; it additionally allows us to maintain as much parsimony as possible in estimating each $\hat{f}_j$. We illustrate these advantages in our data example in section 3 — there, using a polynomial basis expansion, on average, we find 13 features with non-zero $\hat{f}_j$; of these 5 are linear, 7 are quadratic, and 1 is cubic. None were selected to have truncation level larger than 4. hierbasis is also computationally very efficient: It can be applied to problems with thousands of observations and features. In addition, hierbasis estimates attain minimax optimal rates under standard smoothness assumptions, for univariate, multivariate, and sparse additive models. In particular, the univariate hierbasis converges at the order of $O\left(n^{-\frac{2m}{2m+1}}\right)$ where $m$ is the degree of smoothness, and similarly the multivariate hierbasis attains the rate $O\left(n^{-\frac{2m}{2m+1}}\right)$. The sparse additive hierbasis, under a suitable compatibility condition, is shown to converge at $O\left\{\max\left(sn^{-\frac{2m}{2m+1}}, \frac{s\log p}{n}\right)\right\}$ where $s$ is the number of non-zero $f_j$. Even without the compatibility condition, additive hierbasis is consistent with convergence rate $O\left\{\max\left(sn^{-\frac{m}{2m+1}}, \frac{\log p}{n}\right)\right\}$. 

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The rest of this paper is organized as follows: Section 2 gives a formal description of hierbasis in the univariate case, as well as its extension to additive and sparse additive models. Section 3 contains an analysis of genomic data comparing hierbasis to SpAM. Section 4 gives an efficient algorithm for fitting hierbasis and its additive extension. In Section 5, we present a theoretical analysis of hierbasis, which also applies to a more general class of sparse additive models. Section 6 contains a simulation study exploring the operating characteristics of hierbasis, and comparing its performance to other non-parametric estimation methods. Concluding remarks are presenting in Section 7.

2 Methodology

Estimation via a basis expansion is a commonly used technique in nonparametric regression. The basis expansion is often truncated to achieve a parsimonious representations and control the bias-variance tradeoff. While separately tuning the truncation level over each parameter may be feasible for low-dimensional regressions, this approach becomes quickly infeasible for additive models where, the optimal truncation level requires searching over a subset of $\mathbb{R}^p$.

Our proposal is motivated by the need for an adaptive estimator that can select the truncation level in a data-driven manner. We achieve this goal through a penalized estimation formulation using a novel penalty. Our approach is particularly suitable for basis functions which possess a natural hierarchy, i.e., when basis functions $\{\psi_k\}_{k=1}^\infty$ become increasingly complex for higher values of $k$; examples of such basis functions include polynomial, trigonometric and wavelet basis functions and are depicted in Figure 1. To emphasize the hierarchical nature of our proposed penalized estimation framework and its motivation based on basis functions with natural hierarchy, we refer to it as the hierarchical basis expansion estimator, or, hierbasis.

2.1 The hierbasis Proposal

Consider first the univariate case and the projection estimator of Equation 1. As noted in Section 1, choosing the truncation level $K$ is key here: $K$ too small will result in a large bias, while $K$ too large will over-inflate variance. In particular, the balance necessitates that $K = O\left(n^{\frac{1}{2m+1}}\right) \ll n$, where $m$ relates to the smoothness of the underlying $f$. Our proposal, hierbasis, addresses this challenge by consider instead a complete basis with $K = n$ and using a penalized regression framework to data-adaptively choose the truncation level. More specifically, the hierbasis estimator is defined as

$$\hat{\beta}^{\text{hier}} = \argmin_{\beta \in \mathbb{R}^n} \frac{1}{2} \| y - \Psi_n \beta \|_n^2 + \lambda \Omega(\beta),$$

Figure 1: Examples of basis functions with natural hierarchical complexity; polynomial, trigonometric and wavelet basis functions are shown in the left, center and right panels, respectively.
where
\[
\Omega (\beta) = \sum_{k=1}^{n} w_k \| \Psi_{k:n} \beta_{k:n} \|_2^n ,
\]
with \( w_k = k^m - (k - 1)^m \). Here, \( \Psi_{k:n} \) denotes the submatrix of \( \Psi_n \) containing columns \( k, k+1, \ldots, n \), \( \beta_{k:n} \) is the subvector of \( \beta \) containing the \( k, k+1, \ldots, n \) entries, and \( m \) and \( \lambda \) are tuning parameters.

The hierarchical group lasso form (Zhao et al., 2009) of the \textbf{hierbasis} penalty, \( \Omega (\beta) \), will result in a solution \( \beta_{\text{hier}} \) with hierarchical sparsity: That is, if \( \beta_{hier}^k = 0 \) for some \( k \), then \( \beta_{hier}^{k'} = 0 \) for all \( k' > k \). For sufficiently large \( \lambda \), many entries of \( \beta_{hier} \) will be 0. For a given \( \lambda \), we define the \textit{induced truncation level} to be the minimal \( K \leq n \) such that \( \beta_{hier}^{k} = 0 \) for all \( k > K \). Unlike the simple basis expansion estimator, this truncation level is data-adaptive, not prespecified.

The \textbf{hierbasis} estimator is determined by two tuning parameters, \( m \) and \( \lambda \). \( m \) is analogous to the smoothness parameter in smoothing splines (Wahba, 1990), or the number of bounded derivatives estimator, this truncation level is data-adaptive, not prespecified.

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In the univariate setting, the simple basis expansion estimator (1) with truncation level chosen by split sample validation, is likely adequate. In contrast, \textbf{hierbasis} adds additional regularization to the function estimate (in addition to choosing a truncation level). The additional shrinkage may be helpful, as indicated in the empirical results of Section 6; however the benefit is generally relatively small. The true benefit of \textbf{hierbasis} comes in application to additive and multivariate problems described next.

### 2.2 Additive hierbasis

As noted in Section 1, the projection estimator is commonly used to fit additive models (3), often using the same set of basis functions \( \{ \psi_k \}_{k=1}^{\infty} \) for all features. Ideally, the additive projection estimator (3) is obtained by considering a different truncation level \( K_j \) for each feature. When \( p \) is small, this can be achieved by using split sample validation and searching over all combinations of \( (K_1, \ldots, K_p) \); however, the number of candidate models grows exponentially in \( p \) and becomes quickly unwieldy. Often, a single \( K = K_j \) for all \( j \) is used in practice. This difficulty in selecting the truncation level is the primary limitation of the projection estimator in additive and multivariate models. If the level of smoothness of each component is vastly different, then this single truncation level will result in some \( f_j \) estimates with too many degrees of freedom (giving overly variable function estimates), and others with too few (with insufficiently flexible estimates). A single choice of truncation level can thus lead to very poor regression estimates.

Our \textbf{hierbasis} proposal is designed to circumvent the above limitation of projection estimators in choosing the truncation level in models with multiple covariate. In particular, the \textit{additive hierbasis} is a straightforward extension of the univariate \textbf{hierbasis} (5), and is defined as the solution to

\[
\tilde{\beta}_1^{A-\text{hier}}, \ldots, \tilde{\beta}_p^{A-\text{hier}} = \arg\min_{\beta_j \in \mathbb{R}^n} \frac{1}{2} \| y - \sum_{j=1}^{p} \Psi_{n}^j \beta_j \|_n^2 + \lambda \sum_{j=1}^{p} \Omega_j (\beta_j) ,
\]

where
\[
\Omega_j (\beta_j) = \sum_{k=1}^{n} w_k \| \Psi_{k:n}^j \beta_{j,k:n} \|_n^2 ,
\]
with \( w_k = k^m - (k - 1)^m \). The function estimates are obtained as \( \hat{f}_j = \sum_{k=1}^{n} \tilde{\beta}_{kj}^{A-\text{hier}} \psi_k \).

The additive \textbf{hierbasis} solution (7) will result in \( \hat{f}_j \) estimates that are \textit{hierarchically sparse} for each \( j \). Specifically, for each \( j \), there is some minimal \( K_j \) such that for all \( k > K_j \), \( \tilde{\beta}_{kj}^{A-\text{hier}} = 0 \). In addition,
the major advantage of additive *hierbasis* is that the *induced truncation level* is feature-wise adaptive: \( K_j \) may be different for each feature \( j \). This important characteristic mitigates a major disadvantage of simple projection estimators. As a result the additive *hierbasis* allows us to balance goodness-of-fit and parsimony for each feature individually, without an exhaustive computational search.

The advantage of *hierbasis* over simple projection estimators becomes even more significant in high dimensions, when \( p \gg n \). For instance, the popular *SpAM* estimator (4) is generally obtained by using a single truncation level, which, as noted above, can result in poor estimators. Similar to *SpAM*, the *sparse additive* *hierbasis* for high-dimensional additive models employs an additional sparsity-inducing penalty (Yuan and Lin, 2006), and is defined as

\[
\hat{\beta}^{S_{-hier}}_1, \ldots, \hat{\beta}^{S_{-hier}}_p = \underset{\beta_j \in \mathbb{R}^n}{\text{argmin}} \frac{1}{2} \| y - \sum_{j=1}^p \psi_n^j \beta_j \|_2^2 + \lambda \sum_{j=1}^p \Omega_j (\beta_j) + \lambda^2 \sum_{j=1}^p \| \psi_n^j \beta_j \|_n, \tag{9}
\]

where \( \Omega_j (\beta_j) \) is defined as in (8).

An important feature of the the optimization problem for sparse additive *hierbasis* (9) is that the tuning parameters for the two penalty terms are linked (\( \lambda \) and \( \lambda^2 \)). This link is theoretically justified in Section 5. Briefly, for an oracle \( \lambda \), the choice of tuning parameters in (9) gives rate-optimal estimates. Our numerical experiments in Section 3 and 6 corroborate this finding and show that the above choice of tuning parameters results in strong predictive performance without requiring split-sample validation over a multi-dimensional space of tuning parameters.

As with *SpAM*, for sufficiently large \( \lambda \), the sparse additive *hierbasis* gives a sparse solution with most \( \hat{\beta}^{S_{-hier}}_j \equiv 0 \). The two estimators differ, however, in their nonzero estimates: non-zero \( \hat{\beta}^{S_{-hier}}_j \) are hierarchically sparse, with a data-driven feature-specific induced truncation level, whereas nonzero \( \hat{\beta}^{\text{SPAM}}_j \) in (4) all have the same complexity. This additional flexibility of sparse additive *hierbasis* proves critical in high-dimensional settings, and is achieved without paying a price in computational or sample complexity. Moreover, with the tuning parameters in (9), the additional flexibly of sparse additive *hierbasis* is achieved with the same number of tuning parameters as *SpAM*.

### 2.3 Relationship to Existing Methods

The univariate *hierbasis* of Section 2.1 builds upon existing penalized methods for estimating regression functions. A popular penalized estimation method is the smoothing spline estimator (Wahba, 1990), which sets \( \psi_1, \ldots, \psi_n \) as a basis of \( n \) natural splines with knots at the observed \( \{ x_i \} \) and solves the following optimization problem

\[
\text{minimize} \quad \frac{1}{2} \| y - \Psi \beta \|_n^2 + \lambda \| C^{1/2} \beta \|_n^2, \tag{10}
\]

where \( C \in \mathbb{R}^{n \times n} \) and \( C_{j,k} = \int \psi_j^{(m+1)}(t) \psi_k^{(m+1)}(t) \, dt \), and where \( \psi^{(k)} \) is the derivative of \( \psi \) of order \( k \). The smoothing spline eliminates the dependence on the truncation level and has an efficient-to-compute closed form solution; however, its estimated functions are piecewise polynomial splines of degree \( m \) with \( n \) knots. As a result, smoothing spline estimates are not parsimonious, especially in multivariate settings. To achieve more parsimonious estimates, Mammen and van de Geer (1997) use a data-driven approach to select the knots in spline functions. Their locally adaptive regression splines use the same natural spline basis and solve

\[
\text{minimize} \quad \frac{1}{2} \| y - \Psi \beta \|_n^2 + \frac{\lambda}{m!} \| D \beta \|_1, \tag{11}
\]

where \( D \in \mathbb{R}^{(n-m-1) \times n} \) is defined as \( D_{i,j} = \psi_j^{(m)}(t_i) - \psi_j^{(m)}(t_{i-1}) \). This proposal of Mammen and van de Geer (1997) is closely related to the recent, more computationally tractable trend filtering proposal (Kim et al., 2009; Tibshirani, 2014).

Despite their appealing properties in the univariate setting, locally adaptive regression splines and trend filtering, are computationally difficult to extend to high-dimensional sparse additive models — even
for a single feature neither estimator has a closed-form solution. The SpAM estimator \( (4) \) overcomes this difficulty by using a fixed truncation level for all \( p \) components. As pointed out before, the main drawback of SpAM is that each of nonzero components in the additive model have the same level of complexity. The recently proposed sparse partially linear additive model (SPLAM) by Lou et al. (2014) partly mitigates this shortcomings by setting some of the nonzero components to linear functions. This is achieved by using a hierarchical penalty of the form \( \sum_{j=1}^{p} \lambda_1 \| \beta_j \|_2 + \lambda_2 \| \beta_j \|_2 \), where \( \beta_{j,1} \) is the coefficient of the linear term in the basis expansion and \( \beta_{j,1} = [\beta_{j,2}, \ldots, \beta_{j,K}] \in \mathbb{R}^{K-1} \). Depending on the value of tuning parameters \( \lambda_1 \) and \( \lambda_2 \), the first term in the above penalty sets the entire vector of coefficients for the \( j \)th feature to zero, whereas the second term only sets the \( K - 1 \) coefficients corresponding to higher-order terms to zero.

The additive and sparse additive hierbasis proposals of Section 2.2 can be seen as generalizations of SpAM and SPLAM, wherein the complexity of nonzero component are determined data-adaptively. More specifically, SpAM becomes a special case of sparse additive hierbasis if the weights in \( (8) \) are set to \( w_1 = 1 \) and \( w_k = 0 \) for \( k = 2, \ldots, K \). Similarly, with an orthogonal design matrix, \( \Psi_T \Psi/n = I_K \), SPLAM is a special case of sparse additive hierbasis that allows for another level of hierarchy with weights in \( (8) \) set to \( w_1 = w_2 = 1 \) and \( w_k = 0 \) for \( k = 3, \ldots, K \). Our theoretical analysis in Section 5.3 indicates that, in addition to the improved flexibility, the choice of weights in hierbasis result in optimal rates of convergence.

3 Analysis of Colitis Data

We apply hierbasis with logistic loss, in order to perform classification using gene expression measurements. Details on hierbasis with logistic loss are given in section 4.5. We consider the Colitis dataset (Burczynski et al., 2006) which has 22,283 gene expression measurements from peripheral blood mononuclear cells (PBMCs) sampled from 26 adults with ulcerative colitis and 59 with Crohn’s disease, available from GEO at accession number GDS 1615. The aim is to use gene expression measurements to distinguish between the two diseases.

Given the small sample size, we consider 1000 genes with the largest variance. We compare the performances of hierbasis to SpAM and the LASSO (Tibshirani, 1996), over 30 splits of the data into training and test sets, after standardizing each gene to have mean zero and variance one in the training set. We choose the tuning parameters using 5-fold CV in the training set and calculate the misclassification rate in the test set. We also calculate the sparsity for the model selected by CV defined as the proportion of fitted components which were identically zero. We use the parametrization for hierbasis given in \( (9) \) with \( m = 3 \). The maximum number of basis functions selected for each fitted component is 6 for hierbasis and for SpAM we fit multiple models with 2 to 6 basis functions. For computational reasons we did not use the full set of \( n = 75 \) basis vectors in hierbasis, and instead used only 6 basis functions. The use of smaller than \( n \) basis functions for hierbasis is further discussed in section 4.1.

The box-plots of misclassification error rates in the test set and sparsity are shown in Figure 2. The box-plots clearly show the superior performance of hierbasis over SpAM. Hierbasis appears to be comparable to the LASSO in terms of the MSE and gains a slight advantage in sparsity. In addition, each fitted function from hierbasis is monotonic and nearly as parsimonious as those linear fits from the lasso. This is demonstrated in Figure A.1 in Appendix A, where we plot some fitted functions for one split of the data. We also show SpAM estimates which are highly irregular and would indicate very complex non-linear relationships.

4 Computational Considerations and Extensions

4.1 Conservative Basis Truncation

The hierbasis proposal \( (5) \) uses a basis expansion with \( n \) basis functions. In practice, for any reasonable choice of \( \lambda \) the solution, \( \hat{\beta} \), will never have \( n \) nonzero entries. It will generally have very few non-zero
for $K < n$, then so long as $K \geq K_0$, the solution will be identical to that of the original proposal (5). Even when not identical, so long as $K$ is sufficiently large ($K \approx \frac{1}{2} n^{1/2} + 1$, where $a_n \approx b_n \iff a_n = Cb_n$ for some constant $C$) the theoretical properties of (5) will be maintained. This bound relies on the smoothness of the underlying $f$; choosing $K \approx \sqrt{n}$ gives a conservative upper bound which is independent of the underlying $f$. Additionally, as discussed in Section 4.2, by using $K$ rather than $n$ basis functions, the computational complexity decreases from $O(n^3)$ to $O(K^2n)$. A similar result holds for the sparse additive hierbasis with

$$\hat{\beta}_{S-hier}^{(K)} = \arg\min_{\beta \in \mathbb{R}^K} \frac{1}{2} \left\| y - \Psi_K \beta \right\|_2 + \lambda \sum_{k=1}^{K} \left\{ k^m - (k-1)^m \right\} \left\| \Psi_{k,K} \beta_{k,K} \right\|_n,$$

for $K < n$, then so long as $K \geq K_0$, the solution will be identical to that of the original proposal (5).

It is worth noting that it is easier to choose the pre-truncation level in (12) and (13) than the truncation level for the simple basis expansion estimator (Čencov, 1962) — the simple basis expansion requires an exact truncation level that is neither too large, nor too small. On the other hand, hierbasis only requires a basis that is not too small.

### 4.1.1 Algorithm for Solving hierbasis and sparse additive hierbasis

An appealing feature of hierbasis is that it can be efficiently computed. In fact, using the results of Jenatton et al. (2010), hierbasis can be computed via a one-step coordinate descent algorithm. We begin
by re-writing the optimization problem (12). Consider the decomposition $\Psi = UV$ such that $U \in \mathbb{R}^{n \times K}$ and $U^T U / n = I_K$. Then, by defining $\beta = V \beta$, the optimization problem (5) can be equivalently written as:

$$\min_{\beta \in \mathbb{R}^K} \frac{1}{2n} \left\| y - U \tilde{\beta} \right\|_2^2 + \lambda \sum_{k=1}^K w_k \left\| \beta_{k:K} \right\|_2,$$  \hspace{1cm} (14)

which is equivalent to solving

$$\min_{\beta \in \mathbb{R}^K} \frac{1}{2} \left\| U^T y / n - \beta \right\|_2^2 + \lambda \sum_{k=1}^K w_k \left\| \beta_{k:K} \right\|_2.$$  \hspace{1cm} (15)

With this formulation, we can directly apply the results of Jenatton et al. (2010), as detailed in Algorithm 1. The reformulation in (15) can also be used to efficiently solve the sparse additive extension (13).

Algorithm 1: One-Step coordinate descent for hierbasis

1: procedure hierbasis($y, U, \lambda, \{w_k\}_{k=1}^K$) \hspace{1cm} \text{\textcopyright Algorithm for Solving (15)}
2: \hspace{0.5cm} Initialize $\beta^1 = \ldots = \beta^K \leftarrow U^T y / n$
3: \hspace{0.5cm} for $k = K, \ldots, 1$ do
4: \hspace{1cm} Update $\beta_{k-1}^{k-1} \leftarrow \left(1 - \frac{w_k \lambda}{\| \beta_{k:K}^{k-1} \|^2}\right) + \beta_{k:K}^k$, where $(x)_+ = \max(x, 0)$
5: \hspace{0.5cm} end for
6: \hspace{0.5cm} return $\beta^1$
7: end procedure

via a block coordinate descent algorithm. Specifically, given a set of estimates $\{\beta^p\}_{p=1}^p$, we can fix all but one of the vectors $\beta_l$ and optimize over the non-fixed vector using Algorithm 1. Iterating until convergence yields the solution to problem (13), as described in Algorithm B.1 in Appendix B.

4.2 Convergence and Computational Complexity

As noted in Section 4.1.1, a closed form solution for the hierbasis optimization problem can be obtained by one pass of a coordinate descent algorithm as shown in Jenatton et al. (2010). The block coordinate descent algorithm for the sparse additive hierbasis has been extensively studied in the literature and is guaranteed to converge to the global optimum for convex problems.

Solving problem (5) requires a QR decomposition of the matrix $\Psi$ followed by the multiplication $U^T y$; these steps require $O(nK^2)$ and $O(nK)$ operations, respectively. However, these steps are only needed once for a sequence of $\lambda$ values. For additive hierbasis, $p$ such QR decompositions are needed once for the entire sequence of $\lambda$s.

By Proposition 2 of Jenatton et al. (2010), for a given $\lambda$, the optimization problem (15) can be solved in $O(K)$ operations. Each block update of the additive hierbasis requires a matrix multiplication $U_j^T r_{-j}$ followed by solving the proximal problem (B.1) (see Appendix B), which requires $O(nK)$ operations. Thus, the sparse additive hierbasis requires $O(npK)$ operations, which is equal to the computational complexity of the Lasso (Friedman et al., 2010) when $K = 1$.

The above computational complexity calculations indicate that hierbasis and sparse additive hierbasis can be solved very efficiently. Next, we report timing results for our R-language implementation of hierbasis on an Intel® CORE™ i5-3337U, 1.80 GHz processor. Solving the univariate hierbasis for an example with $K = n = 300$ takes a median time of 0.17 seconds. Solving the sparse additive hierbasis for the simulation setting of Section 6.2 on a grid of 50 $\lambda$ values takes a median time of 5.96 seconds.

4.3 Degrees of Freedom

For a regression with fixed design and $\epsilon \sim \mathcal{N}(0, \sigma^2 I_n)$, we consider the definition of degrees of freedom given by Stein (1981) $\text{df} = \frac{1}{\sigma^2} \sum_{i=1}^n \text{Cov}(y_i, \hat{y}_i)$, where $\hat{y}_i$ are the fitted response values. We apply
Claim 3.2 of Haris et al. (2015) to derive an unbiased estimate of df for the solution to the optimization problem (14), using the decomposition \( \Psi = UV \) from section 4.1.1. Let \( K_0 = \text{max}\{k : \beta_k \neq 0\} \), and let \( U_{K_0} \in \mathbb{R}^{n \times K_0} \) denote the first \( K_0 \) columns of \( U \). Furthermore, for a vector \( \nu \in \mathbb{R}^n \), define \( \nu_{k:K_0} \in \mathbb{R}^{K_0} \) as \( \nu_{k:K_0} = [0, 0, \ldots, 0, \nu_k, \nu_{k+1}, \ldots, \nu_{K_0}]^T \). We arrive at the following lemma.

**Lemma 4.1.** An unbiased estimator for the degrees of freedom of \( \hat{\beta} \), as defined in (5), is then given by

\[
\hat{d}_f = 1 + \text{trace} \left\{ U_{K_0} \left[ I_{K_0} + \sum_{k=1}^{K_0} \lambda w_k \left\{ \frac{\text{diag}(1_{k:K_0})}{\|\beta_{k:K_0}\|_2} - \frac{\beta_{k:K_0}\beta_{k:K_0}^T}{\|\beta_{k:K_0}\|_2^2} \right\} \right] U_{K_0}^T \frac{1}{n} \left( I_n - 1_n 1_n^T / n \right) \right\},
\]

where \( \text{diag}(\nu) \in \mathbb{R}^{n \times n} \) is a diagonal matrix with \( \nu \in \mathbb{R}^n \) on the main diagonal.

### 4.4 Non-additive Multivariate hierbasis

We begin by extending the hierbasis penalty to non-additive multivariate regression. To define the multivariate basis expansion, consider \( x \in \mathbb{R}^p \) and \( \nu \in \mathbb{Z}_+^p \) and let

\[
x^{\nu} = x_1^{\nu_1} x_2^{\nu_2} \ldots x_p^{\nu_p}.
\]

Then for functions \( f^0 : \mathbb{R}^p \rightarrow \mathbb{R} \), with the univariate basis functions \( \{\psi_j\}_{j=1}^\infty \) consider the following basis representation

\[
f^0(x) = \sum_{k=1}^{K} \psi_k(x^{\nu(k)}) \beta_k^0,
\]

where \( \|\nu_k\|_1 = 1 \) for \( k = 1, 2, \ldots, p \), \( \|\nu_m\|_1 = 2 \) for \( k = p+1, p+2, \ldots, \binom{p+2}{2} - 1 \) and so on. As in the univariate case, let \( \Psi \in \mathbb{R}^{n \times K} \) be the matrix with entries \( \psi_{i,k} = \psi_k(x_i^{\nu(k)}) \). Then, the multivariate hierbasis estimator is simply (5) with weights given by

\[
w_{q_k} = k^m - (k - 1)^m, \quad \text{where} \quad q_k = \binom{k+p-1}{p},
\]

and \( w_{q_k} = 0 \) for all other \( k \). Figure 3 demonstrates the multivariate hierbasis penalty for \( p = 2 \) and \( \psi_k \) the identity function, i.e. for \( z \in \mathbb{R} \), \( \psi_k(z) \equiv z \). It is clear from the figure that the multivariate hierbasis is a natural extension of the univariate penalty: when \( \psi_k(z) = z \), the fitted model can be a multivariate polynomial of any degree. With this choice of basis functions, multivariate hierbasis acts as a procedure for selecting the level of complexity of interaction models. It also follows that the multivariate hierbasis can be solved using Algorithm 1 with a single pass over the basis elements.

### 4.5 Extension to Classification

We can also extend hierbasis to the setting of binary classification via a logistic loss function. Let \( y \) with \( y_i \in \{-1, 1\} \) for \( i = 1, \ldots, n \) be the response. The logistic hierbasis is then obtained from the following modification of (5):

\[
(\beta_0, \hat{\beta}) = \arg \min_{\beta_0 \in \mathbb{R}, \beta \in \mathbb{R}^K} \frac{1}{2n} \sum_{i=1}^n \log (1 + \exp \{-y_i (\beta_0 + (\Psi \beta)_{y})\}) + \lambda \Omega(\beta),
\]

where \( \Omega(\beta) \) is an appropriate penalty.
Given a new \( x \in \mathbb{R} \), predicted values are given by \( \hat{\rho}(x) = 1/ \left[ 1 + \exp \left\{ -\hat{\beta}_0 - \sum_k \psi_k(x) \hat{\beta}_k \right\} \right] \). The extension of additive hierbasis to binary response can also be defined similarly, as

\[
(\beta_0, \{\hat{\beta}_j\}_{j=1}^p) = \arg \min_{\beta_0 \in \mathbb{R}, \{\beta_j\}_{j=1}^p \in \mathbb{R}^K} \frac{1}{2n} \sum_{i=1}^n \log \left( 1 + \exp \left[ -y_i \left\{ \beta_0 + \sum_{j=1}^p (\Psi^j \beta_j)_i \right\} \right] \right) + \lambda \sum_{j=1}^p \Omega_j(\beta_j) + \lambda^2 \sum_{j=1}^p \|\Psi^j \beta_j\|_n. \tag{20}
\]

The logistic hierbasis problem can be efficiently solved via a proximal gradient descent algorithm (Combettes and Pesquet, 2011); see Appendix B for details.

5 Theoretical Results

In this section we investigate asymptotic properties of hierbasis. In proving theoretical results about hierbasis, we combine previously developed ideas from empirical process theory and metric entropy with a number of novel results about general convergence rates of sparse additive models, and the metric entropy of our hierarchical class.

In particular, our new results in Section 6.2 allow one to establish convergence rates for a broad class of penalized sparse additive model estimators. Under a compatibility condition on the component features, these rates match the minimax lower bound for estimation of sparse additive models under independent component functions, established previously by Raskutti et al. (2009) — see Corollary 5.7.1. Thus, our additive and sparse additive hierbasis estimators are rate-optimal. On the other hand, with no assumptions on the component functions, we obtain rates that are the additive analog to assumption-free convergence rates for the Lasso (Chatterjee, 2013); this is established in Theorem 5.7. To our knowledge, assumption-free convergence rates have not been previously derived out for sparse additive models.

Finally, we also calculate the entropy of our hierarchical class (with matching upper and lower bounds Lemma 5.4 and Lemma 5.5). These new results allows us to establish that our univariate and sparse additive estimators, (5) and (9), are minimax rate-optimal within the hierarchical univariate and hierarchical sparse additive classes, respectively.

5.1 Entropy-based Rates

We begin by stating two results from the literature for establishing convergence rates. We then present our contributions in Sections 5.2 and 5.3. Firstly, Theorem 1 of Yang and Barron (1999) establishes an upper bound on convergence rates is given by Theorem 10.2 of van de Geer (2000). Here, we require a lower bound for the minimax rate subject to certain conditions. Secondly, a framework for establishing our contributions in Sections 5.2 and 5.3. Firstly, Theorem 1 of Yang and Barron (1999) establishes a

We first introduce some terminology and notation for the entropy of a set. For a set \( \mathcal{F} \) equipped with some metric \( d(\cdot, \cdot) \), the subset \( \{f_1, \ldots, f_N\} \subset \mathcal{F} \) is a \( \delta \)-cover if for any \( f \in \mathcal{F} \) \( \min_{1 \leq i \leq N} d(f, f_i) \leq \delta \). The log-cardinality of the smallest \( \delta \)-cover is the \( \delta \)-entropy of \( \mathcal{F} \) with respect to metric \( d(\cdot, \cdot) \). We denote by \( H(\delta, \mathcal{F}, Q) \), the \( \delta \)-entropy of a function class \( \mathcal{F} \) with respect to metric \( d(\cdot, \cdot) \). We denote by \( \left\| f \right\|_Q^2 = \int |f(x)|^2 \, dQ(x) \). For a fixed sample \( x_1, \ldots, x_n \) we denote by \( Q_n \) the empirical measure \( Q_n = \frac{1}{n} \sum_{i=1}^n \delta_{x_i} \) and use the short-hand notation \( \| \cdot \|_n = \| \cdot \|_{Q_n} \).

**Theorem 5.1** (Theorem 1, Yang and Barron (1999)). Consider the model

\[
y_i = f^0(x_i) + \varepsilon_i, \tag{21}
\]

with \( \varepsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2) \), \( x_i \overset{i.i.d.}{\sim} Q \). Assume the entropy condition

\[
H(\delta, \mathcal{F}, Q) = A_0 \delta^{-\alpha} \tag{22}
\]
holds for some function class \( F \) for \( \alpha \in (0, 2) \), and \( A_0 > 0 \). Then,

\[
\min_{\hat{f}} \max_{f \in F} \mathbb{E} \left\| \hat{f} - f^0 \right\|_Q^2 \geq A_1 n^{-\frac{2}{1+\alpha}},
\]

where the minimum is over the space of all measurable functions and \( A_1 \) is a constant that depends on \( A_0, \alpha \) and \( \sigma^2 \).

**Theorem 5.2** (Theorem 10.2, van de Geer (2000)). Consider the model (21), and define

\[
\hat{f} = \arg \min_{f \in \mathcal{F}_n} \frac{1}{2} \| y - f \|_n^2 + \lambda_n^2 \Omega(f|\mathcal{Q}_n),
\]

for some function class \( \mathcal{F}_n \) and semi-norm \( \Omega(\cdot|\mathcal{Q}_n) \) on \( \mathcal{F}_n \) which satisfy the entropy condition

\[
H(\delta, \{ f \in \mathcal{F}_n : \Omega(f|\mathcal{Q}_n) \leq 1 \}, \mathcal{Q}_n) \leq A_0 \delta^{-\alpha},
\]

for \( \alpha \in (0, 2) \). Then for

\[
\lambda_n^{-1} = n^{\frac{1}{1+\alpha}} \left\{ \Omega(f^*|\mathcal{Q}_n) \right\}^{\frac{2-\alpha}{2+\alpha}},
\]

and for any function \( f^*_n \in \mathcal{F}_n \), there is a constant \( c \) such that for all \( T \geq c \), with probability at least \( 1 - c \exp \left\{ -\frac{T^2}{c} \right\} \) we have

\[
\frac{1}{2} \| \hat{f} - f^0 \|_n^2 \leq \frac{5}{2} \max \left\{ 2 \| f^0 - f^*_n \|_n^2, C_0 \lambda_n^2 \Omega(f^*|\mathcal{Q}_n) \right\},
\]

where \( C_0 \) is a constant that depends on \( \alpha \) and \( T \).

We state Theorems 5.1 and 5.2 only for the sake of completeness. These results are well-known in the nonparametric literature and allow us to establish convergence rates of an estimator using only entropy bounds of the relevant function class. In the following section, we establish these entropy bounds for the hierbasis and multivariate hierbasis penalty.

### 5.2 Theoretical Results for hierbasis

In this section we prove minimax rates for univariate and multivariate hierbasis by specializing Theorems 5.1 and 5.2. We first introduce the nonparametric function classes for hierbasis. We then present the primary contribution of this section, that is establishing entropy bounds for the univariate and multivariate hierbasis function class. Using Theorem 5.1, these results immediately establish a lower bound on the minimax rate. For the upper bound, we use Theorem 5.2 and use an upper bound for the truncation error as a function of the truncation level \( K_n \). Proof of entropy results are presented in Appendix G; for completeness, we provide details for the upper bound in Appendix D.

We define the following function class for \( x \in \mathbb{R} \),

\[
\mathcal{F}_n = \left\{ f_{\beta}(x) = \sum_{k=1}^{K_n} \psi_k(x) \beta_k : \int \psi_k \psi_l dQ = 0 \text{ for } k \neq l, \int \psi_k^2 dQ = 1 \right\},
\]

and similarly define the multivariate function class \( \mathcal{F}_{p,n} \) for \( x \in \mathbb{R}^p \)

\[
\mathcal{F}_{p,n} = \left\{ f_{\beta}(x) = \sum_{k=1}^{K_n} \psi_k(x^{\nu_k}) \beta_k : \int \psi_k(x^{\nu_k}) \psi_l(x^{\nu_l}) dQ(x) = 0 \text{ for } k \neq l, \int \{ \psi_k(x^{\nu_k}) \}^2 dQ(x) = 1 \right\},
\]

where \( \nu_k \) is a \( p \)-vector of non-negative integers, \( x^{\nu_k} \) is as defined in (17) and \( Q \) is the probability measure associated with \( x \). In (28) and (29), we allow for the limiting case \( n = \infty \) where \( K_\infty = \infty \). With some abuse of notation for \( \beta \in \ell^2(\mathbb{R}) \), we define the notation \( \| \beta_{k:\infty} \|_2^2 = \sum_{l=k}^{\infty} \beta_l^2 \). The next subsection is dedicated to proving the main condition of Theorem 5.1 and 5.2, the entropy of the appropriate function classes for hierbasis.
5.2.1 Entropy Results for hierbasis

To specialize Theorems 5.1 and 5.2 for the analysis of hierbasis, we need to characterize \( H(\delta, F^M_n, Q) \) where \( F^M_n \) is the hierbasis function class (as defined in (30)), and establish an upper bound for \( H(\delta, \{ f\beta \in F_n : \Omega(\beta) \leq 1 \}, Q_n) \). In the next lemma, Lemma 5.3, we show that the calculation of \( H(\delta, F^M_n, Q) \) and \( H(\delta, \{ f\beta \in F_n : \Omega(\beta) \leq 1 \}, Q_n) \) is equivalent to an entropy calculation for a subset of \( \ell^2(\mathbb{R}) \) and \( \mathbb{R}^K \), respectively with respect to the usual \( \| \cdot \|_2 \) norm. This reduction allows us to use simple volume arguments and existing results for establishing the entropy conditions. The lemma considers the hierbasis penalty in full generality, i.e. the penalty (6) with any set of non-negative weights. This gives a similar reduction of entropy calculations for the multivariate case with little extra work.

**Lemma 5.3** (Reduction to \( \ell^2(\mathbb{R}) \) and \( \mathbb{R}^K \)). We denote by \( F^M_n \) [or \( F^M_p,n \)] the class of hierbasis (respectively multivariate hierbasis) functions where

\[
F^M_n = \{ f\beta \in F_n : \sum_{k=1}^{K_n} w_k \| \beta_{k;K_n} \|_2 \leq M \}, \quad F^M_p,n = \{ f\beta \in F_{p,n} : \sum_{k=1}^{K_n} w_k \| \beta_{k;K_n} \|_2 \leq M \}, \tag{30}
\]

and allow the limiting case \( n = \infty \). Then \( H(\delta, F^M_n \text{ or } F^M_{p,n}, Q) = H(\delta, H^w_{K_n}/M) \), the entropy of \( H^w_{K_n}/M \) with respect to the \( \| \cdot \|_2 \) norm, where

\[
H^w_{K_n}/M = \left\{ \beta \in \mathbb{R}^K_n : \sum_{k=1}^{K_n} \frac{w_k}{M} \| \beta_{k;K_n} \|_2 \leq 1 \right\}. \tag{31}
\]

Secondly, assume that the Gram matrix \( \Psi^T_{K_n} \Psi_{K_n}/n \) has a finite maximum eigenvalue of denoted by \( \Lambda_{\text{max}} \). Then, denoting \( H^w_{K_n} = H^w_{K_n}/M \), we have

\[
H \left( \delta, \{ f\beta \in F_n \text{ or } F_{p,n} : \sum_{k=1}^{K_n} w_k \| \Psi_{k;K_n} \beta_{k;K_n} \| \leq 1 \}, Q_n \right) \leq H(\delta/\sqrt{\Lambda_{\text{max}}}, H^w_{K_n}).
\]

Lemma 5.3 establishes the connections between entropy results for the function classes of interest and the set \( H^w_{K_n} \). It is easy to see that \( H(\delta, H^w_{K_n}/M) \) and \( H(\delta/\sqrt{\Lambda_{\text{max}}}, H^w_{K_n}) \) are proportional to \( H(\delta, H^w_{K_n}) \) where the proportionality constant depends on \( M \) and \( \sqrt{\Lambda_{\text{max}}} \), respectively. The next lemma establishes an upper bound for \( H(\delta, H^w_{K_n}) \) for univariate and multivariate hierbasis weights. This upper bound is all we need to specialize Theorem 5.2.

**Lemma 5.4** (An Upper Bound). For \( \delta \geq 0 \), for the region \( H^w_{K_n} \) with univariate hierbasis weights we have

\[
H \left( \delta, H^w_{K_n} \right) \leq U_{E,1} \delta^{-\frac{1}{m}}, \tag{32}
\]

and for the multivariate hierbasis weights (18) we have

\[
H \left( \delta, H^w_{K_n} \right) \leq U_{E,2} \delta^{-\frac{p}{m}}, \tag{33}
\]

for constants \( U_{E,1}, U_{E,2} > 0 \).

While Lemma 5.4 is sufficient for applying Theorem 5.2, to invoke Theorem 5.1 we need an exact value for the entropy up to a proportionality constant. A natural way to achieve this is to find a lower bound for the entropy which matches the upper bound, we do this in the following lemma.

**Lemma 5.5** (A Lower Bound). For \( \delta \in ([w_1 + \ldots + w_{K_n+1}]^{-m}, 1/2) \), for the region \( H^w_{K_n} \) with univariate hierbasis weights we have

\[
H(\delta, H^w_{K_n}) \geq L_{E,1} \delta^{-\frac{1}{m}}, \tag{34}
\]
and for the multivariate hierbasis weights (18) we have
\[ H(\delta, H^w_{K_n}) \geq L_{E,2} \delta^{-\frac{p}{m}}, \]
for constants \( L_{E,1}, L_{E,2} > 0 \) and where we assume, for simplicity, that \( K_n = qK' - 1 \equiv (K' + p - 1) - 1 \) for some \( K' \).

5.2.2 Specializing Theorems 5.1 and 5.2 for hierbasis

The following corollary establishes a lower bound for the minimax rate for estimating \( f_0 \), the true function which belongs to some function class \( \mathcal{F} \). We consider three different choices for \( \mathcal{F} \): 1) the hierbasis class; 2) the multivariate hierbasis class; and 3) the Sobolev class. To prove the result, we use the fact that if an upper bound for the convergence rates can be found that matches the lower bound, then we can conclude that our estimator is minimax.

**Corollary 5.5.1.** For the \( m \)th order hierbasis function class \( \mathcal{F}_\infty^M \equiv \{ f \in \mathcal{F}_\infty : \sum_{k=1}^{\infty} w_k \| \beta_k \|_2 \leq M \} \), where \( w_k = k^m - (k - 1)^m \), we have
\[ \min \max_{f \in \mathcal{F}_\infty^M} \mathbb{E} \left[ \| \hat{f} - f^0 \|_q^2 \right] \geq A_1 n^{-\frac{2m}{2m+1}}. \]  

(36)

For the \( m \)th order multivariate hierbasis class \( \mathcal{F}_{p,\infty}^M \equiv \{ f \in \mathcal{F}_{p,\infty} : \sum_{k=1}^{\infty} w_k \| \beta_k \|_2 \leq M \} \), where \( w_k \) are the weights defined in (18), we have
\[ \min \max_{f \in \mathcal{F}_{p,\infty}^M} \mathbb{E} \left[ \| \hat{f} - f^0 \|_q^2 \right] \geq A_2 n^{-\frac{2m}{2m+p}}. \]  

(37)

Finally, for the \( m \)th order Sobolev class \( \mathcal{F}_\text{Sob}^M \equiv \{ f \in \mathcal{F}_\infty : \sum_{k=1}^{\infty} (k^m \beta_k)^2 \leq M^2 \} \), we have
\[ \min \max_{f \in \mathcal{F}_\text{Sob}^M} \mathbb{E} \left[ \| \hat{f} - f^0 \|_q^2 \right] \geq A_3 n^{-\frac{2m}{2m+1}}. \]  

(38)

As the last step in our analysis, we next specialize Theorem 5.2 to establish an upper bound for the convergence rate of the univariate and multivariate hierbasis estimators. The following corollary demonstrates a number of interesting points. Firstly, we note that with respect to the empirical norm, \( \| \cdot \|_n \), our estimators achieve the minimax rate for the classes \( \mathcal{F}_\infty^M \) and \( \mathcal{F}_{p,\infty}^M \) (as defined in (30)). For the Sobolev class, \( \mathcal{F}_\text{Sob}^M \), if \( \sum_{k=1}^{\infty} w_k \| \beta_k \|_2 \leq C(M) \) for all \( f_\beta \in \mathcal{F}_\text{Sob}^M \), then hierbasis is minimax over the Sobolev class as well. This result also gives insight into the role of \( K_n \).

**Corollary 5.5.2.** Consider the model \( Y_i = f^0(x_i) + \epsilon_i \) for mean zero sub-gaussian noise \( \epsilon_i \). Define the univariate and multivariate hierbasis estimators as
\[ \hat{f}^\text{uni} = \arg \min_{f_\beta \in \mathcal{F}_n} \frac{1}{2} \| y - f_\beta \|_n^2 + \lambda^2 \Omega^\text{uni}(\beta), \quad \hat{f}^\text{multi} = \arg \min_{f_\beta \in \mathcal{F}_{p,n}} \frac{1}{2} \| y - f_\beta \|_n^2 + \lambda^2 \Omega^\text{multi}(\beta), \]
for \( p = 1 \) and \( p > 1 \), respectively, where \( \Omega^\text{uni} \) is the penalty (6) and \( \Omega^\text{multi} \) is the penalty (18). Assume that \( \max_k \| \psi_k \|_\infty = \psi_{\text{max}} < \infty \) and that the Gram matrix \( \Psi_{K_n}^T \Psi_{K_n} / n \) has a bounded maximum eigenvalue denoted by \( \Lambda_{\text{max}} \). Then,
1. For \( p = 1 \) and \( f^0 \in \mathcal{F}_\infty^M \) there is a constant \( c > 0 \) such that for all \( T \geq c \), we have with probability at least \( 1 - c \exp \left\{ -\frac{T^2}{2\sigma^2} \right\} \),
\[ \frac{1}{2} \| \hat{f}^\text{uni} - f^0 \|_n^2 \leq \frac{5}{2} \max \left\{ C_1 K_n^{-(2m-1)}, C_2 n^{-\frac{2m}{2m+1}} \right\}, \]
(39)

where \( C_1, C_2 > 0 \) are constants that depend on \( M, \psi_{\text{max}}, \Lambda_{\text{max}}, m \) and \( T \).
2. For $p = 1$, $f^0 \in \mathcal{F}_{Sob}^M$, there is a constant $c > 0$ such that for all $T \geq c$, we have with probability at least $1 - c \exp \left\{ -\frac{T^2}{c^2} \right\}$,

$$
\frac{1}{2} \| \hat{f}_{uni} - f^0 \|_n^2 \leq \frac{5}{2} \max \left\{ C_1 K_n^{-2(m-1)}, C_2 C_3 n^{- \frac{2m}{2m+1}} \right\},
$$

where $C_1, C_2 > 0$ are constants that depend on $M, \psi_{\max}, \Lambda_{\max}, m, T$ and, for $f^0 = \sum_{k=1}^{\infty} \psi_k \beta_k^0$, we have $C_3 \frac{2(m+1)}{n} = \sum_{k=1}^{\infty} w_k \| \beta_k^0 \|_2$.

3. For $p > 1$, $f^0 \in \mathcal{F}_{p,\infty}^M$, assume that $p < 2m$ and define the integer $K'$ such that $K_n = qK' - 1 \equiv \binom{K'+p-1}{p} - 1$. Then there is a constant such that for all $T \geq c$, we have with probability at least $1 - c \exp \left\{ -\frac{T^2}{c^2} \right\}$,

$$
\frac{1}{2} \| \hat{f}_{multi} - f^0 \|_n^2 \leq \frac{5}{2} \max \left\{ C_1 K'^{-2(m-1)}, C_2 n^{- \frac{2m}{2m+p}} \right\},
$$

where $C_1, C_2 > 0$ are constants that depend on $M, \psi_{\max}, \Lambda_{\max}, m, p, T$.

The above result demonstrates that we can achieve the usual non-parametric rates as long as the truncation level $K_n$ satisfies $K_n^{-2(m-1)} \leq n^{- \frac{2m}{2m+1}}$. We note that since $K_n^{2m} \geq K_n^{-2(m-1)}$, an appropriate choice of truncation level would be any $K_n \geq n^{\frac{1}{2m+1}}$, which gives us $\sqrt{n}$ as a conservative truncation level.

### 5.3 Theoretical Results for Sparse Additive Models

In this section, we will establish the convergence rates of high-dimensional sparse additive models in terms of a general entropy condition. Raskutti et al. (2009) proved a lower bound for the minimax rates for estimation of sparse additive models assuming independent covariates; for completeness, we state this result as Theorem 5.6.

Our first contribution is an oracle inequality for an upper bound on the prediction error of additive models. This inequality establishes consistency for the estimators with slow convergence rates, specifically, these rates are $O(\nu_n^2)$ where $\nu_n^2$ is the minimax lower bound of Raskutti et al. (2009). We then proceed to state a compatibility condition which leads to two corollaries: firstly, it establishes convergence rates of the order of $O(\nu_n^2)$ and, secondly, it automatically establishes minimax rates for univariate regression as a special case of an additive model with $p = 1$. Our contributions in this section extend to a broad class of estimators and can be seen as the additive model analog of Theorem 5.2.

Let $f^0$ be the true function such that

$$Y_i = f^0(x_i) + \varepsilon_i, \text{ for } i = 1, 2, \ldots, n,$$  \hspace{1cm} (42)

where $\varepsilon_i$ is independent random mean-zero noise, $x_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,p}]^T \in \mathbb{R}^p$ for each $i = 1, 2, \ldots, n$. We denote by $f^*$ a sparse additive approximation to the function $f^0$,

$$f^*(x_i) = c^0 + \sum_{j=1}^{p} f_j^*(x_{i,j}) = c^0 + \sum_{j \in S} f_j^*(x_{i,j}),$$

where $S$, which we call the active set, is a subset of $\{1, \ldots, p\}$ and, $c^0 = E[\bar{Y}]$ where $\bar{Y}$ is the sample mean. To ensure identifiability we assume,

$$\sum_{i=1}^{n} f_j^*(x_{i,j}) = 0 \text{ for all } j = 1, 2, \ldots, p.$$  \hspace{1cm} (44)
Consider the estimator \( \hat{f} = \sum_{j=1}^{p} \hat{f}_j \), where,
\[
\hat{f}_1, \ldots, \hat{f}_p = \arg \min_{\{f_j\}_{j=1}^{p} \in \mathcal{F}}  \frac{1}{2n} \sum_{i=1}^{n} \left\{ Y_i - \hat{Y} - \sum_{j=1}^{p} f_j(x_{i,j}) \right\}^2 + \lambda_n \sum_{j=1}^{p} I(f_j),
\]
(45)
where \( I(\cdot) \) is a penalty of the form
\[
I(f_j) = \|f_j\|_n + \lambda_n \Omega(f_j),
\]
(46)
for a semi-norm \( \Omega(\cdot) \). We can think of \( \Omega(f_j) \) as a smoothness penalty for function \( f_j \).

**Theorem 5.6** (Theorem 1, Raskutti et al. (2009)). For \( n \) i.i.d. samples from the sparse additive model
\[
Y_i = \sum_{j \in S} f_j^0(x_{i,j}) + \varepsilon_i,
\]
(47)
where \( |S| = s \leq p/4 \), \( x_i \overset{iid}{\sim} Q \), \( \varepsilon_i \overset{iid}{\sim} \mathcal{N}(0, \sigma^2) \) and, \( f_j^0 \in \mathcal{F} \) where \( \mathcal{F} \) is a class satisfying the entropy condition
\[
H(\delta, \mathcal{F}, Q) = A_0 \delta^{-\alpha},
\]
(48)
with \( \alpha \in (0, 2) \). Further assume the covariates are independent, i.e. \( Q = \bigotimes_{j=1}^{p} Q_j \). Then for a constant \( C > 0 \),
\[
\min_{\{f_j\}_{j=1}^{p}} \max_{\{f_j^0\}_{j=1}^{p} \in \mathcal{F}} \mathbb{E} \left[ \left\| \sum_{j=1}^{p} \hat{f}_j - f_j^0 \right\|^2 \right] \geq \max \left\{ \frac{\sigma^2 s \log(p/s)}{32n}, C s \left( \frac{\sigma^2}{n} \right)^{\frac{2}{2+\alpha}} \right\},
\]
(49)
where the minimum is over the set of all measurable functions.

We next state the first key result of this section, which establishes an oracle inequality for additive models, as well as slow rates of convergence.

**Theorem 5.7.** Assume the model (42), with \( \max_i K^2 \left( \mathbb{E} \varepsilon_i^2 / K^2 - 1 \right) \leq \sigma_0^2 \), for some constants \( K \) and \( \sigma_0 \). Assume the entropy condition
\[
H(\delta, \{ f \in \mathcal{F} : \Omega(f) \leq 1 \}, Q_n) \leq A_0 \delta^{-\alpha},
\]
(50)
holds for \( \alpha \in (0, 2) \), for some function class \( \mathcal{F} \) and, some constant \( A_0 \). Then for the estimator (45), for
\[
\rho_n = \kappa \max \left( n^{-\frac{1}{2+\alpha}}, \sqrt{\frac{\log p}{n}} \right),
\]
(51)
and for \( \lambda_n \geq 4 \rho_n \) with probability at least \( 1 - 2 \exp(-c_1 n \rho_n^2) - c_2 \exp(-c_3 n \rho_n^2) \) we have
\[
\left\| \hat{f} - f^0 \right\|^2 + \lambda_n \sum_{j \in S^c} \| \hat{f}_j - f_j^* \|_n + \frac{3 \lambda_n^2}{2} \sum_{j \in S} \Omega(\hat{f}_j - f_j^*) \leq 3 \lambda_n \sum_{j \in S} \| \hat{f}_j - f_j^* \|_n + 4 \lambda_n^2 \sum_{j \in S} \Omega(f_j^*) + \| f^* - f^0 \|^2,
\]
(52)
where \( \kappa \geq c_2 \) and \( c_1 = c_1(A_0, \sigma_0), c_2 = c_2(A_0, \alpha, K, \sigma_0), c_3 \geq 1/c_2^2 \) are positive constants and \( S = \{ j : f_j^* \neq 0 \} \).

Furthermore, if the function class \( \mathcal{F} \) satisfies \( \sup_{f \in \mathcal{F}} \| f \|_n \leq R \), we have
\[
\frac{1}{2} \| \hat{f} - f^0 \|^2 \leq C_s \max \left( sn^{-\frac{1}{2+\alpha}}, s \sqrt{\frac{\log p}{n}} \right) + \frac{1}{2} \| f^* - f^0 \|^2,
\]
(53)
where \( C_s \geq 0 \) depends on \( \kappa, R \) and \( \max_j \Omega(f_j^*) \) and \( s = |S| \).
We are now ready to establish the fast rates of convergence for additive models, using the compatibility condition stated next. 

**Compatibility Condition:** We say that the compatibility condition is met for the set $S$, if for some constant $\phi(S) > 0$, and for all $f \in F = \{ f : f = \sum_{j=1}^{p} f_j \}$, satisfying

$$
\sum_{j \in S^c} \| f_j \|_n \leq 4 \sum_{j \in S} \| f_j \|_n,
$$

it holds that

$$
\sum_{j \in S} \| f_j \|_n \leq \sqrt{s} \| f \|_n / \phi(S).
$$

**Corollary 5.7.1.** Assuming the conditions of Theorem 5.7 and the compatibility condition is met for $S = \{ j : f_j^* \neq 0 \}$, then with probability at least $1 - 2 \exp \left( -c_1 \kappa n^{\frac{\alpha}{2+\alpha}} \right) - c_2 \exp \left( -c_3 \kappa n^{\frac{\alpha}{2+\alpha}} \right)$ we have

$$
\frac{1}{2} \| \hat{f} - f^0 \|_n^2 + \lambda_n \sum_{j \in S^c} \| \hat{f}_j - f_j^* \|_n + \lambda_n^2 \sum_{j \in S} \Omega(\hat{f}_j - f_j^*) \leq C_f \max \left( sn^{-\frac{\alpha}{2+\alpha}}, \frac{s \log p}{n} \right) + 2 \| f^* - f^0 \|_n^2,
$$

where $C_f \geq 0$ is a constant that depends on $\phi(S)$ and $\max_j \Omega(f_j^*)$ and $c_1, c_2$ and $c_3$ are the constants of Theorem 5.7.

**Corollary 5.7.2.** Assuming the conditions of Theorem 5.7 with $p = 1$, the compatibility condition holds trivially with $\phi(S) = 1$ and we have

$$
\frac{1}{2} \| \hat{f} - f^0 \|_n^2 \leq C_f n^{-\frac{\alpha}{2+\alpha}} + 2 \| f^* - f^0 \|_n^2,
$$

with probability at least $1 - 2 \exp \left( -c_1 \kappa n^{\frac{\alpha}{2+\alpha}} \right) - c_2 \exp \left( -c_3 \kappa n^{\frac{\alpha}{2+\alpha}} \right)$ for a constant $C_f \geq 0$ that depends on $\Omega(f^*)$.

6 Simulation Studies

6.1 Simulation for Univariate Regression

We begin with a simulation to compare the performance of **hierbasis** to smoothing splines (Wahba, 1990) and trend filtering (Kim et al., 2009; Tibshirani, 2014). Smoothing splines and trend filtering are implemented in the R packages **splines** (R Core Team, 2014) and **genlasso** (Arnold and Tibshirani, 2014), respectively.

We generate the data using (21) for different choices of the function $f^0$. The errors are generated as $\varepsilon \sim N_n(0, \sigma^2 I_n)$ where $\sigma^2$ satisfies $\text{SNR} = (n-1)^{-1} \sum_{i=1}^{n} (f^0(x_i))^2 / \sigma^2$, for a fixed Signal-to-Noise Ratio (SNR). For this simulation we consider the fixed design with $x_i = i/n$ for $i = 1, \ldots, n$. This was also done to facilitate comparison to trend filtering which can become substantially slow for random $x_i$, particularly when the covariates are not uniformly distributed over a closed interval. We consider four different choices of $f^0$ denoted by $g_t$ for $t = 1, 2, 3, 4$. for $n = 150$ and SNR of 2 or 3. The true functions $g_t$ are as follows:

$$
g_1(x) = -0.43 + 4.83x - 14.65x^2 + 11.76x^3,
g_2(x) = 0.23 - 8.44x + 45.20x^2 - 81.41x^3 + 46.59x^4,
g_3(x) = \exp(-5x + 1/2) - 2/5 \sinh(5/2), \quad g_4(x) = -\sin(7x - 0.4).
$$

We applied **hierbasis** to 100 $\lambda$ values linear on the log scale from $\lambda_{\max}$, for which $\hat{\beta} = 0$, down to $10^{-4}\lambda_{\max}$. We applied smoothing splines to a grid of 100 values for degrees of freedom from 10 to 1. Trend
filtering is applied to a sequence of lambda values, automatically selected by the its R implementation. For hierbasis and smoothing splines we fix \( m = 3 \). We fit trend filters of orders 1, 2 and 3.

For each simulation setting, we plot the Mean-Squared-Error (MSE) as a function of degrees of freedom (DoF) where we define \( \text{MSE} = \| f^0 - \hat{f} \|^2_n \) for a fitted model \( \hat{f} \). We also generate a test set of size \( n_{\text{test}} = 75 \). For each method, we find a \( \lambda^* \) which minimizes the prediction error on the test set. For this \( \lambda^* \), we evaluate the MSE and DoF for the fitted model and report them relative to the MSE and DoF of hierbasis, to be precise we report the ratios \( \frac{\text{MSE}}{\text{MSE}_{\text{hierbasis}}} \) and \( \frac{\text{DoF}}{\text{DoF}_{\text{hierbasis}}} \).

Figure 4 displays the MSE of hierbasis, smoothing splines, trend filtering of orders 1, 2 and 3 as a function of degrees of freedom. We also plot the results for fitting hierbasis with \( m = 1 \). Hierbasis appears to outperform the competitors in terms of MSE especially for polynomials. We observe comparable performance for the exponential and sine functions. This also provides empirical evidence for the theoretical results where we proved hierbasis to converge with rates comparable to smoothing splines.

Since the functions considered in this simulation are smooth, as expected, we see that hierbasis with \( m = 1 \) does not converge as fast as competing methods.

Figure A.2 shows examples of some fitted models for a fixed value of DoF. We see hierbasis seems to perform very well and is mostly robust to changes in the value of \( m \). The smoothing splines estimates are unable to do as well as hierbasis for the same number of effective degrees of freedom. In the bottom panel of Figure A.2, it is not surprising to observe the first order trend filter perform poorly due to model misspecification.

### 6.2 Simulation for Multivariate Additive Regression

We proceed with a simulation study to illustrate the performance of hierbasis in the additive setting. We perform a small simulation study to compare the performance of additive hierbasis to SpAM (Ravikumar et al., 2009). SpAM is implemented in the R package SAM (Zhao et al., 2014) which uses natural spline basis functions. To facilitate a fairer comparison, we also implement SpAM using a polynomial basis expansion, which we refer to as SpAM.pol. Due to a lack of R packages for sparsity-smoothness penalties (Meier et al., 2009) and SPLAM (Lou et al., 2014), we defer the comparison to these methods to future work.
We consider the simulation setting of Meier et al. (2009) with some modifications to have high dimensional data and smaller signal-to-noise ratio. We generate \( n = 200 \) samples for \( p = 500 \) features. The data is generated as follows:

\[
y_i = 5f_1(x_{i,1}) + 3f_2(x_{i,2}) + 4f_3(x_{i,3}) + 6f_4(x_{i,4}) + \varepsilon_i,
\]

where \( \varepsilon_i \) are i.i.d. normal such that SNR = 3 and

\[
f_1(x) = x, \quad f_2(x) = (2x - 1)^2, \quad f_3 = \frac{2\sin(2\pi x)}{2 - \sin(2\pi x)} \quad \text{and} \quad f_4(x) = 0.1\sin(2\pi x) + 0.2\cos(2\pi x) + 0.3\sin^2(2\pi x) + 0.4\cos^3(2\pi x) + 0.5\sin^3(2\pi x),
\]

and the covariates are i.i.d. Uniform(0,1). We implemented the parametrization (13), with \( m = 1 \) and a sequence of 50 \( \lambda \) values, decreasing linearly on the log-scale. We fix the maximum number of basis functions \( K_n = 20 \) for hierbasis and we implement SpAM with \( \{3, 5, 8, 10, 15, 20\} \) basis functions.

It is not surprising to observe superior performance of hierbasis over SpAM.poly in terms of MSE in Figure 5. However, we note in the same figure that hierbasis seems to even outperform SpAM. For small lambda values, i.e. more complex models, we observe lower MSE for SpAM with fewer basis functions. With low sparsity SpAM is able to control the variance of the estimator by the small number of basis functions used. Whereas hierbasis can control the variance by controlling smoothness via the \( \Omega_j(\cdot) \) penalty. For large lambda values, we obtain sparser models and hence control the variance. However, now the bias for SpAM is inflated when using fewer basis functions.

In Figure A.3, we show some of the fitted functions for both SpAM and HierBasis using the \( \lambda \) value which minimizes the test set error for SpAM with 3 and 15 basis functions.
Figure 5: Average MSE (over 100 simulated datasets) as a function of $\lambda$ for hierbasis vs SpAM (Left) and for hierbasis vs SpAM.poly (Right). The colored lines indicate results for hierbasis of order 1 (---) and, SpAM (SpAM.poly) with 3 (---), 5 (-----), 8 (---------) basis functions and SpAM (SpAM.poly) with 10 (-----), 15 (------) and, 20 (--------) basis functions.

7 Conclusion

In this paper we introduced hierbasis, a novel approach to non-parametric regression and high dimensional models. Recall the original motivation: for non-parametric regression, especially additive models, we require an estimator that can adapt to function complexity in a data-adaptive way. We showed that state-of-the-art methods like SpAM and SPLAM are unable to do that effectively. More data adaptive proposals, such as the sparsity smoothness penalty of Meier et al. (2009), come at a cost of highly complex fitted models even for simple underlying surfaces. The use of hierarchical penalty allows us to adaptively fit simple models for simple functions as shown in Sections 6 and 6.2.

Our theoretical analyses in Section 5 not only show that hierbasis rates are faster than any of the existing methods but also establish fast convergence rates for a broad class sparse additive estimators, where the sparsity smoothness penalty is one special case. A similar result was proved by Raskutti et al. (2012); however, they considered independent component functions in a RKHS. Thus smoothness penalties that are not a norm of some Hilbert space are not covered by their formulation.

The R package HierBasis, available on https://github.com/asadharis/HierBasis, implements the methods described in this paper.

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Figure A.1: Plots of some fitted functions for a single split of the colitis data into training and test sets for hierbasis (---), LASSO (----), SpAM (-----) with 3 basis functions.

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A Additional Figures for Simulation Studies and Data Analysis

In this Appendix we present some additional figures referenced in Section 3 and 6.

Figure A.1 shows examples of some fitted functions for one split of the dataset into training and test sets.

Figure A.2 shows examples of some fitted models for a fixed value of DoF.

In Figure A.3, we show some of the fitted functions for both SpAM and HierBasis using the \( \lambda \) value which minimizes the test set error for SpAM with 3 and 15 basis functions.

B Algorithms for Additive and Logistic hierbasis

Here we give an algorithm for additive and sparse-additive hierbasis as well as an algorithm for logistic hierbasis. We use a block-wise coordinate descent algorithm for solving additive and sparse additive hierbasis. This algorithm cyclically iterates through features, and for each feature applies the univariate solution detailed in Algorithm 1. The exact details are given in Algorithm B.1.

We also give an algorithm for solving logistic hierbasis based on proximal gradient descent. To begin let \( L(\beta_0, \beta) = \frac{1}{m} \sum_{i=1}^{n} \log (1 + \exp [-y_i \{ \beta_0 + (\Psi \beta)_i \}]) \). We denote by \( \nabla L(\beta_0, \beta) \), the derivative of \( L \) at the point \( (\beta_0, \beta) \in \mathbb{R}^{K+1} \). Algorithm B.2 presents the steps for solving (19). The algorithm for additive logistic hierbasis can be similarly derived and is omitted in the interest of brevity.
Figure A.2: Scatterplots of simulated data along with true and estimated functions. The top row includes plots of the simulated data along with the true function used for generating the data. The other three rows show the fitted functions for each method and the degrees of freedom corresponding to the fitted model — hierBasis, smoothing splines and trend filtering are shown in rows 2-4, respectively. For trend filtering and hierbasis $m = 1$, 2 and 3 are shown in green (---), blue (---) and red (---). Only the available R implementation with order $m = 3$ is shown for smoothing splines.
Figure A.3: The first 4 component functions of the simulation study from Section 6.2. The estimates of \textit{hierbasis} are shown in green (—), whereas \textit{SpAM} fitted with 20 and \textit{SpAM} with 3 basis functions are shown in In blue (——) and red (-----), respectively. In each case, the tuning parameter leading to the smallest MSE was used.

\textbf{Algorithm B.1} Block coordinate descent for additive \textit{hierbasis}

1: \textbf{procedure} \textsc{Additive.hierbasis}(\textit{y}, \{\textit{Ψ}\textsubscript{\textit{j}}\}\textsubscript{\textit{j}=1}^\textit{p}, \textit{λ}, \{\textit{w}\textsubscript{\textit{k}}\}\textsubscript{\textit{k}=1}^\textit{K}, \textit{max.iter})

2: \hspace{1em} Initialize \textit{β}\textsubscript{\textit{j}} ← 0 for \textit{j} = 1, ..., \textit{p}

3: \hspace{1em} \textbf{while} \textit{l} ≤ \textit{max.iter} \textbf{and} not converged \textbf{do}

4: \hspace{2em} for \textit{j} = 1, ..., \textit{p} \textbf{do}

5: \hspace{3em} \textbf{Set} \textit{r}\textsubscript{\textit{j}} ← \textit{y} − \sum_{\textit{j}' \neq \textit{j}} \textit{Ψ}\textsubscript{\textit{j}'}\textit{β}\textsubscript{\textit{j}'}

6: \hspace{2em} Update \textit{β}\textsubscript{\textit{j}} ← \arg \min_{\textit{β} \in \mathbb{R}^\textit{K}+1} \frac{1}{2} \| \textit{r}\textsubscript{\textit{j}} - \textit{Ψ}\textsubscript{\textit{j}}\textit{β} \|^2 \textit{+} \lambda \sum_{\textit{k}=1}^{\textit{K}} \| \tilde{\textit{w}}\textsubscript{\textit{k}} \| \textit{Ψ}\textsubscript{\textit{j}''}\textsubscript{\textit{k}}\textit{β}\textsubscript{\textit{j''}\textsubscript{\textit{k}}} \|^2 \textit{n}

7: \hspace{2em} \textbf{end for}

8: \hspace{1em} \textbf{end while}

9: \hspace{1em} \textbf{return} \textit{β}\textsubscript{1}, ..., \textit{β}\textsubscript{\textit{p}}

10: \textbf{end procedure}

\textbf{Algorithm B.2} Proximal gradient descent for logistic \textit{hierbasis}

1: \textbf{procedure} \textsc{Logistic.hierbasis}(\textit{y}, \textit{Ψ}, \textit{λ}, \{\textit{w}\textsubscript{\textit{k}}\}\textsubscript{\textit{k}=1}^\textit{K}) \triangleright Algorithm for Solving (19)

2: \hspace{1em} Initialize (\textit{β}\textsubscript{0}\textsubscript{0}, \textit{β}\textsubscript{0}\textsuperscript{l})

3: \hspace{1em} \textbf{for} \textit{l} = 1, 2, ... \textbf{until} convergence \textbf{do}

4: \hspace{2em} Select a step size \textit{t}\textsubscript{\textit{l}} via line search

5: \hspace{2em} Update

6: \hspace{2em} \hspace{1em} (\textit{β}\textsubscript{0}\textsubscript{0}, \textit{β}\textsubscript{\textit{l}}) ← \arg \min_{(\textit{β}\textsubscript{0}, \textit{β}) \in \mathbb{R}^{\textit{K}+1}} \frac{1}{2} \| (\textit{β}\textsubscript{0}, \textit{β}) - ((\textit{β}\textsubscript{\textit{l}-1}, \textit{β}\textsubscript{\textit{l}-1}) - \textit{t}\textsubscript{\textit{l}} \nabla \textit{L}(\textit{β}\textsubscript{\textit{l}-1}, \textit{β}\textsubscript{\textit{l}-1})) \|_2^2 \textit{+} \lambda \Omega(\textit{β})

7: \hspace{2em} \textbf{end for}

8: \hspace{1em} \textbf{return} (\textit{β}\textsubscript{\textit{l}}, \textit{β}\textsubscript{\textit{l}})

9: \textbf{end procedure}
C Proofs for Section 5.2.1

Proof of Lemma 5.3. Firstly, we have for \( f_1, f_2 \in F_n \)
\[
\|f_1 - f_2\|_Q^2 = \int (f_1 - f_2) dQ = \int \left\{ \sum_{k=1}^{K_n} \psi_k(x) (\beta_k^{[1]} - \beta_k^{[2]}) \right\}^2 dQ
\]
\[
= \int \left\{ \sum_{k=1}^{K_n} \psi_k^2(x) (\beta_k^{[1]} - \beta_k^{[2]})^2 + \sum_{k \neq l} \psi_k(x) \psi_l(x) (\beta_k^{[1]} - \beta_k^{[2]}) (\beta_l^{[1]} - \beta_l^{[2]}) \right\} dQ
\]
\[
= \|\beta^{[1]} - \beta^{[2]}\|_2^2,
\]
where the final equality follows due to the orthonormality of \( \psi_k \). Similarly for \( f_1, f_2 \in F_{p,n} \) we can show that \( \|f_1 - f_2\|_Q^2 = \|\beta^{[1]} - \beta^{[2]}\|_2^2 \). Thus if \( \{\beta^1, \ldots, \beta^N\} \) is the smallest \( \delta \)-cover of \( H_{K_n}^{w/M} \) then the functions \( f_\beta \) associated with \( \{\beta^1, \ldots, \beta^N\} \) form the smallest \( \delta \)-cover with respect to the \( L_Q \) norm. This can be extended to the case \( n = \infty \). This proves the first part.

Secondly, note that for \( f_1, f_2 \in F_n \) (or \( F_{p,n} \)) we have
\[
\|f_1 - f_2\|_n = (\beta^{[1]} - \beta^{[2]})^T \Psi_n^T \Psi_n (\beta^{[1]} - \beta^{[2]}) \leq \Lambda_{\text{max}} \|\beta^{[1]} - \beta^{[2]}\|_2^2,
\]
(C.1)
thus if \( \{\beta^1, \ldots, \beta^N\} \) is the smallest \( \delta \)-cover for \( H_{K_n}^{w} \), then the associated functions \( \{f^1, \ldots, f^N\} \) is a \( \sqrt{\Lambda_{\text{max}}} \delta \) cover of \( \{f_\beta \in F_n (or \ F_{p,n}) : \sum_{k=1}^{K_n} w_k \|\Psi_k:K_{n}\beta_k:K_n\| \leq 1\} \) with respect to the \( Q_n \) metric. Since this is a cover and not the smallest cover, we have
\[
H(\sqrt{\Lambda_{\text{max}}} \delta, \{f_\beta \in F_n (or \ F_{p,n}) : \sum_{k=1}^{K_n} w_k \|\Psi_k:K_{n}\beta_k:K_n\| \leq 1\}, Q_n) \leq H(\delta, H_{K_n}^{w}),
\]
and since the inequality holds for all \( \delta > 0 \), we can select \( \delta = \delta'/\sqrt{\Lambda_{\text{max}}} \) giving us the result. \( \square \)

Proof of Lemma 5.4. For the Ellipsoid \( E_{K_n}^{w} \) where
\[
E_{K_n}^{w} = \left\{ \beta \in \mathbb{R}^{K_n} : \sum_{k=1}^{K_n} \beta_k^2 (w_1 + \ldots + w_k)^2 \leq 1 \right\},
\]
(C.2)
we show that \( H_{K_n}^{w} \subset E_{K_n}^{w} \) in Lemma F.1. Dumer (2006) proved an upper bound for ellipsoids which we state in Appendix G.1. For the special case of \( w_k = k^m - (k-1)^m \), this theorem yields the desired upper bound as shown in Corollary G.1.1. Therefore we have \( H(\delta, H_{K_n}^{w}) \leq H(\delta, E_{K_n}^{w}) \leq U_{E,1} \delta^2 \).

Similarly, we can consider the special case of multivariate hierbasis weights in Corollary G.1.2, which gives us the result \( H(\delta, H_{K_n}^{w}) \leq H(\delta, E_{K_n}^{w}) \leq U_{E,2} \delta^2 \). \( \square \)

Proof of Lemma 5.5. Let \( d \) be the integer such that \( (w_1 + \ldots + w_{d+1})^{-1} \leq \delta \leq (w_1 + \ldots + w_d)^{-1} \) for \( \delta \in ((w_1 + \ldots + w_{K_n+1})^{-1}, 1) \). Note that since \( \delta \geq (w_1 + \ldots + w_{K_n+1})^{-1}, d \leq K_n \). We define the truncated hierbasis region as
\[
\vec{H}_d^{w} = \left\{ \beta \in H_{K_n}^{w} : \beta_j = 0 \ \forall \ j \geq d + 1 \right\}.
\]
(C.3)
Then we have that \( H_{d}^{w} \subset \vec{H}_d^{w} \subset H_{K_n}^{w} \) where \( \vec{H}_d^{w} \) is simply viewing \( \vec{H}_d^{w} \) as a subset of \( \mathbb{R}^d \). Let \( B_n(r) \) be the \( n \)-ball of radius \( r \). By Lemma F.2, we have \( B_d([w_1 + \ldots + w_d]) \subset H_d^{w} \). The lower bound of the entropy of a ball can be obtained by a simple volume argument. Since \( (w_1 + \ldots + w_d)^{-1} \geq \delta \) then \( B_d(\delta) \subset B_d([w_1 + \ldots + w_d]) \) and hence
\[
H(\delta/2, H_{d}^{w}) \geq H(\delta/2, B_d(\delta)) \geq \log \frac{Vol(B_d(\delta))}{Vol(B_d(\delta/2))} = d \log(2).
\]
Since the above inequality holds for $\delta \leq 1$, for $\delta \in ([w_1 + \ldots + w_{K_n}]^{-1}, 1/2)$ we have $H(\delta, H_d^w) \geq d \log(2)$.

Now for the univariate case we have $(w_1 + \ldots + w_{d+1})^{-1} = (d + 1)^{-m} \leq \delta \Rightarrow (d + 1) \geq \delta^{-\frac{1}{m}}$ and hence we have

$$H(\delta, H_d^w) \geq d \log(2) \geq (\delta^{-\frac{1}{m}} - 1) \log(2) = \delta^{-\frac{1}{m}} \left(1 - \delta^{1/m}\right) \log(2) \geq \delta^{-\frac{1}{m}} \left(1 - \frac{1}{2^{1/m}}\right) \log(2).$$

Now for the multivariate case, the argument is slightly different due to presence of zero weights. As before, there is some $d'$ such that $(w_1 + \ldots + w_{q_{d'}})^{-1} \leq \delta \leq (w_1 + \ldots + w_{q_{d'}})^{-1}$ and hence $d = q_{d'} - 1$. Note that by assumption we have $K_n = q_{K'} - 1$ and hence $\delta \geq (w_1 + \ldots + w_{q_{K'}})^{-1}$ which implies that $d' \leq K'$ and hence $d \leq K_n$. Finally we have that since $w_1 + \ldots + w_{q_{d'}-1} = w_1 + \ldots + w_{q_{d'}-1} = (d' - 1)^m$, therefore $d' - 1 \geq \delta^{-\frac{1}{m}}$. Now we have that

$$H(\delta, H_d^w) \geq d \log(2) = (q_{d'} - 1) \log(2) = \left\{\left(\frac{d' + p - 1}{p}\right) - 1\right\} \log(2) \geq \left\{\left(\delta^{-\frac{1}{m}} + p\right) - 1\right\} \log(2) = \delta^{-\frac{1}{m}} \left(1 + p \delta^{1/m}\right) - \delta^{-\frac{1}{m}} \log(2) \geq \delta^{-\frac{1}{m}} \left(\frac{1}{p} - \frac{\delta^{1/m}}{p}\right) \log(2) \geq \delta^{-\frac{1}{m}} A \log(2),$$

where the last inequality follows from the fact that $g(\delta) > 0$ for all $\delta \in (0, 1)$.

\[\Box\]

### D Details for Corollary 5.5.2

#### D.1 Univariate Case

Firstly, if $f^0(x) = \sum_{k=1}^{\infty} \psi_k(x) \beta_k^0$ then we select $f_n^*(x) = \sum_{k=1}^{K_n} \psi_k(x) \beta_k^0 \in F_n$. Secondly, we note that for the \texttt{hierbasis} estimator we have $\Omega(f_n^*|Q_n) = \Omega^{uni}(\beta_{1,K_n}^0)$. For brevity we will drop the dependence on $\beta^0$ and denote $\Omega^{uni}(\beta_{1,K_n}^0)$ by $\Omega$. Thus we have

$$\lambda_n^2 \Omega(f_n^*|Q_n) = n^{-\frac{2}{2+\alpha}} \Omega^{-\frac{2-\alpha}{2+\sigma}} \Omega = n^{-\frac{2}{2+\sigma}} \Omega^{\frac{2\alpha}{2+\alpha}} = n^{-\frac{2m}{2m+1}} \Omega^{\frac{2}{2m+1}}.$$  

For the term $\Omega(\beta_{1,K_n}^0)$ we have

$$\Omega(\beta_{1,K_n}^0) = \sum_{k=1}^{K_n} w_k \sqrt{\left(\beta_{1,K_n}^0\right)^T \Psi_{k,K_n}^T \Psi_k \Psi_{k,K_n} \beta_{1,K_n}^0} \leq \psi_{\max} \sum_{k=1}^{K_n} w_k \|\beta_{1,K_n}^0\|_2 \leq \psi_{\max} M,$$

for $f^0 \in F^M_{\infty}$. For $F^M_{\infty}$, we do have the above bound and hence we keep the $\Omega$ term in the inequality.
For the truncation error we note that

\[ \|f^0 - f_n^*\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{k=K_n+1}^{\infty} \psi_k(x_i) \beta_k^0 \right\}^2 \leq \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=K_n+1}^{\infty} \beta_k^0 \right)^2 \]

\[ \leq \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=K_n+1}^{\infty} \frac{k^m}{k^m} |\beta_k^0| \right)^2 \]

\[ \leq \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( M \sqrt{\sum_{k=K_n+1}^{\infty} \frac{1}{k^{2m}}} \right)^2 = \psi_{\text{max}}^2 M^2 \sum_{k=K_n+1}^{\infty} \frac{1}{k^{2m}}, \]

where the last inequality follows from the proof of Lemma F.1. The result now follows since

\[ \sum_{k=K_n+1}^{\infty} k^{-2m} \leq \left\{ (2m - 1)(K_n + 1)^{2m-1} \right\}^{-1} \leq \frac{1}{2m - 1} \frac{1}{K_n^{2m-1}}. \]

### D.2 Multivariate Case

Now we assume that \( f^0(x) = \sum_{k=1}^{\infty} \psi_k(x^{\nu_k}) \beta_k^0 \) for \( x \in \mathbb{R}^p \). Then we take \( f_n^*(x) = \sum_{k=1}^{K_n} \psi_k(x^{\nu_k}) \beta_k^0 \). Now by the same calculations as in the univariate case, we have

\[ \lambda_n^2 \Omega(f_n^*Q_n) = n^{-\frac{2m}{2m+p}} \Omega_{2m+p} \leq n^{-\frac{2m}{2m+p}} [\psi_{\text{max}} M]^{\frac{2p}{2m+p}}. \]

For the truncation error we note that \( K_n = q_{K'} - 1 \) and hence

\[ \|f^0 - f_n^*\|^2 = \frac{1}{n} \sum_{i=1}^{n} \left\{ \sum_{k=K_n+1}^{\infty} \psi_k(x_i^{\nu_k}) \beta_k^0 \right\}^2 \leq \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=K_n+1}^{\infty} \beta_k^0 \right)^2 \]

\[ = \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{k=K_n+1}^{\infty} \frac{K^m}{K^m} |\beta_k^0| + \sum_{k=K_n+1}^{\infty} \frac{(K' + 1)^m}{(K' + 1)^m} |\beta_k^0| + \ldots \right)^2 \]

\[ = \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{R=K'}^{\infty} \frac{R^m}{R^m} \sum_{k=K'}^{\infty} |\beta_k^0| \right)^2 \]

\[ = \psi_{\text{max}}^2 \frac{1}{n} \sum_{i=1}^{n} \left( \sum_{R=K'}^{\infty} \frac{1}{R^{2m}} \sqrt{\sum_{R=K'}^{\infty} \frac{R^m}{R^m} \sum_{k=K'}^{\infty} |\beta_k^0|} \right)^2 \]

\[ \leq \psi_{\text{max}}^2 M^2 \sum_{R=K'}^{\infty} \frac{1}{R^{2m}} \leq \frac{M^2 \psi_{\text{max}}^2}{2m - 1} \frac{1}{(K')^{2m-1}}. \]

### E Proof of Theorem 5.7

Recall that \( \{\hat{f}_j\}_{j=1}^p \in \mathcal{F} \) where \( \mathcal{F} \) is some arbitrary univariate function class. We denote the functions \( \hat{f}(x) = \sum_{j=1}^{p} \hat{f}_j(x_j) \) and \( f^0(x) = \sum_{j=1}^{p} f^0_j(x_j) \) for \( x = [x_1, \ldots, x_p]^T \in \mathbb{R}^p \). For the proof of Theorem 5.7, \( \lambda_n \) and \( \rho_n \) are functions of \( n \) but for convenience we will simply write \( \lambda, \rho \). We begin the proof of Theorem 5.7 with a basic inequality.
Lemma E.1 (Basic Inequality). For any function $f^* = \sum_{j=1}^{p} f_j^*$, where $f_j^* \in \mathcal{F}$ and, the solution $\hat{f}$ of (45), we have the following basic inequality

$$\frac{1}{2} \| \hat{f} - f^0 \|^2_n + \lambda I_p(\hat{f}) \leq | \langle \varepsilon, \hat{f} - f^* \rangle_n | + \lambda I_p(f^*) + | \varepsilon | \sum_{j=1}^{p} \| \hat{f}_j - f_j^* \|_n + \frac{1}{2} \| f^* - f^0 \|^2_n, \quad \text{(E.1)}$$

where $\langle \varepsilon, f \rangle_n = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f(x_i)$, $\varepsilon = \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i$ and $I_p(f) = \sum_{j=1}^{p} I(f_j) = \sum_{j=1}^{p} \| f_j \|_n + \lambda \Omega(f_j)$ for an additive function $f$.

Proof. We have

$$\frac{1}{2n} \sum_{i=1}^{n} \left[ Y_i - \bar{Y} - (\hat{f} - f^0)(x_i) \right]^2 + \lambda I_p(\hat{f}) \leq \frac{1}{2n} \sum_{i=1}^{n} \left[ Y_i - \bar{Y} - f^*(x_i) \right]^2 + \lambda I_p(f^*).$$

$$\iff \frac{1}{2n} \sum_{i=1}^{n} \left[ \varepsilon_i + c^0 - \bar{Y} - (\hat{f} - f^0)(x_i) \right]^2 + \lambda I_p(\hat{f}) \leq \frac{1}{2n} \sum_{i=1}^{n} \left[ \varepsilon_i + c^0 - \bar{Y} - (f^* - f^0)(x_i) \right]^2 + \lambda I_p(f^*).$$

$$\Rightarrow \frac{1}{2n} \sum_{i=1}^{n} \left[ \varepsilon_i + c^0 - \bar{Y} \right]^2 + (\hat{f} - f^0)(x_i) - 2[\varepsilon_i + c^0 - \bar{Y}](\hat{f} - f^0)(x_i) + \lambda I_p(\hat{f})$$

$$\leq \frac{1}{2n} \sum_{i=1}^{n} \left[ \varepsilon_i + c^0 - \bar{Y} \right]^2 + (f^* - f^0)(x_i) - 2(\varepsilon_i + c^0 - \bar{Y})(f^* - f^0)(x_i) + \lambda I_p(f^*)$$

$$\Rightarrow \frac{1}{2} \| \hat{f} - f^0 \|^2_n - \langle \varepsilon + c^0 - \bar{Y}, \hat{f} - f^0 \rangle_n + \lambda I_p(\hat{f})$$

$$\leq \frac{1}{2} \| f^* - f^0 \|^2_n - \langle \varepsilon + c^0 - \bar{Y}, f^* - \hat{f} - f^0 \rangle_n + \lambda I_p(f^*)$$

$$\Rightarrow \frac{1}{2} \| \hat{f} - f^0 \|^2_n - \langle \varepsilon + c^0 - \bar{Y}, f^* - \hat{f} \rangle_n + \lambda I_p(\hat{f})$$

$$\leq \frac{1}{2} \| f^* - f^0 \|^2_n - \langle \varepsilon + c^0 - \bar{Y}, f^* - \hat{f} \rangle_n - \langle \varepsilon + c^0 - \bar{Y}, f^* - f^0 \rangle_n + \lambda I_p(f^*),$$

which implies

$$\frac{1}{2} \| \hat{f} - f^0 \|^2_n + \lambda I_p(\hat{f}) \leq \frac{1}{2} \| f^* - f^0 \|^2_n - \langle \varepsilon + c^0 - \bar{Y}, f^* - \hat{f} \rangle_n + \lambda I_p(f^*).$$

$$\Rightarrow \frac{1}{2} \| \hat{f} - f^0 \|^2_n + \lambda I_p(\hat{f}) \leq | \langle \varepsilon, \hat{f} - f^* \rangle_n | + \sum_{j=1}^{p} \langle c^0 - \bar{Y}, \hat{f}_j - f_j^* \rangle_n + \lambda I_p(f^*) + \frac{1}{2} \| f^* - f^0 \|^2_n$$

$$\Rightarrow \frac{1}{2} \| \hat{f} - f^0 \|^2_n + \lambda I_p(\hat{f}) \leq | \langle \varepsilon, \hat{f} - f^* \rangle_n | + | c^0 - \bar{Y} | \sum_{j=1}^{p} \| \hat{f}_j - f_j^* \|_n + \lambda I_p(f^*) + \frac{1}{2} \| f^* - f^0 \|^2_n.$$

Now for the second term note that:

$$| c^0 - \bar{Y} | = \left| \frac{1}{n} \sum_{i=1}^{n} (c^0 - Y_i) \right| = \left| \frac{1}{n} \sum_{i=1}^{n} \left\{ c^0 - c^0 - \sum_{j=1}^{p} f_j^0(x_{i,j}) - \varepsilon_i \right\} \right| = | \varepsilon |.$$

Which leads us to

$$\frac{1}{2} \| \hat{f} - f^0 \|^2_n + \lambda I_p(\hat{f}) \leq | \langle \varepsilon, \hat{f} - f^* \rangle_n | + \lambda I_p(f^*) + | \varepsilon | \sum_{j=1}^{p} \| \hat{f}_j - f_j^* \|_n + \frac{1}{2} \| f^* - f^0 \|^2_n. \quad \text{(E.2)}$$

□
Lemma E.2 (Bounding the term $|\hat{\varepsilon}|$). For $\varepsilon = [\varepsilon_1, \ldots, \varepsilon_n]$ such that $E(\varepsilon_i) = 0$ and
\[
K^2 \left( \mathbb{E} \varepsilon_i^2 / K^2 - 1 \right) \leq \sigma_0^2, \tag{E.3}
\]
for all $\kappa > 0$ and
\[
\rho = \kappa \max \left( n^{-\frac{1}{2+\alpha}}, \sqrt{\frac{\log p}{n}} \right), \tag{E.4}
\]
we have that with probability at least $1 - 2 \exp \left( - \frac{nt^2}{8(K^2 + \sigma_0^2)} \right)$,
\[
|\hat{\varepsilon}| \leq \rho, \tag{E.5}
\]
for a constant $c_1$ that depends on $K$ and $\sigma_0$.

Proof. By Lemma 8.2 of van de Geer (2000) (with $\gamma_n = 1/n$) we have for all $t > 0$
\[
P \left( \left| \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i \right| \geq t \right) \leq 2 \exp \left\{ - \frac{nt^2}{8(K^2 + \sigma_0^2)} \right\}.
\]
The result follows by setting $t = \rho$.

Lemma E.3 (Bounding the term $|\langle \varepsilon, \hat{f} - f^* \rangle|_n$). For $\lambda \geq 4\rho$ where $\rho = \kappa \max \left( n^{-\frac{1}{2+\alpha}}, \sqrt{\frac{\log p}{n}} \right)$ and for some constant $\kappa$, if
\[
H(\delta, \{ f \in \mathcal{F} : \Omega(f) \leq 1 \}, Q_n) \leq A_0 \delta^{-\alpha}, \tag{E.6}
\]
we then have with probability at least $1 - c_2 \exp \left( -c_3 n \rho^2 \right)$
\[
|\langle \varepsilon, \hat{f}_j - f^*_j \rangle|_n \leq \rho \| \hat{f}_j - f^*_j \|_n + \rho \lambda \Omega(\hat{f}_j - f^*_j), \tag{E.7}
\]
for all $j = 1, \ldots, p$ and positive constants $c_2$ and $c_3$.

Proof. Firstly, for $\mathcal{F}_0 = \{ f \in \mathcal{F} : \Omega(f) \leq 1 \}$ we have by assumption a $\delta$ cover $f_1, \ldots, f_N$ such that for all $f \in \mathcal{F}_0$ we have $\min_{j \in \{1, \ldots, N\}} \| f_j - f \|_n \leq \delta$. Now we are interested in the set $\mathcal{F}_{0,\lambda} = \{ f \in \mathcal{F} : \lambda \Omega(f) \leq 1 \}$. Firstly, for a function $f \in \mathcal{F}_{0,\lambda}$,
\[
\min_{j \in \{1, \ldots, N\}} \| f - f_j / \lambda \|_n = \min_{j \in \{1, \ldots, N\}} \frac{1}{\lambda} \| \lambda f - f_j \|_n \leq \frac{\delta}{\lambda},
\]
because $\Omega(\lambda f) = \lambda \Omega(f) \leq 1 \implies \lambda f \in \mathcal{F}_0$. This means that the set $\{ f_1 / \lambda, \ldots, f_N / \lambda \}$ is a $\delta / \lambda$ cover of the set $\mathcal{F}_{0,\lambda}$.

This implies that $H(\delta, \mathcal{F}_0, Q_n) \leq A_0 \delta^{-\alpha} \implies H(\delta / \lambda, \mathcal{F}_{0,\lambda}, Q_n) \leq A_0 \delta^{-\alpha}$ or equivalently $H(\delta, \mathcal{F}_{0,\lambda}, Q_n) \leq A_0 (\delta \lambda)^{-\alpha}$. Finally, since $\{ f \in \mathcal{F} : I(f) \leq 1 \} \subset \{ f \in \mathcal{F} : \Omega(f) \leq \lambda^{-1} \}$ we have
\[
H(\delta, \{ f \in \mathcal{F} : I(f) \leq 1 \}, Q_n) \leq A_0 (\delta \lambda)^{-\alpha} \leq A_1 (\delta \rho)^{-\alpha},
\]
since $\lambda^{-1} \leq \rho^{-1} / 4$.

We now apply Lemma I.1 to the class $\left\{ \frac{\hat{f}_j}{I(f)} : f \in \mathcal{F} \right\}$ with $T = \sqrt{n} \rho^{1+\alpha}/2$. We have for $\kappa > c_2$ sufficiently large
\[
P \left( \sup_{f_j \in \mathcal{F}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \varepsilon_i f_j(x_i) \right\} \geq \rho \right) \leq c_2 \exp \left( - \frac{n \rho^2}{c_2^2} \right),
\]
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and by the union bound we have

\[
P \left( \max_{j \in \{1, \ldots, p\}} \sup_{f_j \in \mathcal{F}} \frac{|\langle \varepsilon, f_j \rangle_n|}{\|f_j\|_n^{1-\alpha/2} I^{\alpha/2}(f_j)} \geq \rho \right) \leq c_2 \exp\left( -\frac{n\rho^2}{c_2^2} \right) \\
\leq c_2 \exp\left( -\frac{n\rho^2}{c_2^2} + \log p \right) \\
= c_2 \exp\left\{ -n\rho^2 \left( \frac{1}{c_2^2} - \frac{\log p}{n\rho^2} \right) \right\} \\
\leq c_2 \exp\left( -c_3 n\rho^2 \right),
\]

for some constant \( c_3 \geq \frac{1}{c_2^2} - \frac{\log p}{n\rho^2} \). Finally, we show that \( c_3 > 0 \). This follows from the fact that \( \frac{1}{c_2^2} - \frac{\log p}{n\rho^2} > 0 \Leftrightarrow n\rho^2 > c_2^2 \log p \). This holds since \( n\rho^2 \geq \kappa^2 \log p \) and \( \kappa > c_2 \). Thus, we have with probability at least \( 1 - c_2 \exp\left( c_3 n\rho^2 \right) \) for all \( j \in \{1, 2, \ldots, p\} \)

\[
|\langle \varepsilon, \tilde{f}_j - f_j^* \rangle_n| \equiv |\langle \varepsilon, \tilde{\Delta}_j \rangle_n| \leq \rho \|\tilde{\Delta}\|_n^{1-\alpha/2} I^{\alpha/2}(\tilde{\Delta}_j) \\
= \rho \|\tilde{\Delta}_j\|_n \left\{ 1 + \frac{\lambda \Omega(\tilde{\Delta}_j)}{\|\tilde{\Delta}_j\|_n} \right\}^{\alpha/2} \leq \rho \|\tilde{\Delta}_j\|_n + \rho \lambda \Omega(\tilde{\Delta}_j),
\]

where the last inequality follows by Bernoulli’s inequality.

\[ \square \]

E.1 Using the Active Set

So far we have shown that, for \( \lambda \geq 4\rho \), with probability at least \( 1 - 2 \exp\left( -\frac{n\rho}{c_1} \right) - c_2 \exp\left( -c_3 n\rho^2 \right) \), the following inequality holds

\[
\|\tilde{f} - f^0\|_n^2 + 2\lambda \sum_{j=1}^{p} I(\tilde{f}_j) \leq 2|\langle \varepsilon, \tilde{f} - f^* \rangle_n| + 2|\varepsilon| \sum_{j=1}^{p} \|\tilde{\Delta}_j\|_n + 2\lambda \sum_{j=1}^{p} I(f_j^*) + \|f^* - f^0\|_n^2 \\
\leq \left\{ \sum_{j=1}^{p} 2\rho \|\tilde{\Delta}_j\|_n + 2\rho \lambda \Omega(\tilde{\Delta}_j) \right\} + \left\{ 2\rho \sum_{j=1}^{p} \|\tilde{\Delta}_j\|_n \right\} \\
+ \left\{ 2\lambda \sum_{j=1}^{p} I(f_j^*) \right\} + \|f^* - f^0\|_n^2 \\
\Rightarrow \|\tilde{f} - f^0\|_n^2 + 2\lambda \sum_{j=1}^{p} I(\tilde{f}_j) \leq \sum_{j=1}^{p} \left\{ \lambda \|\tilde{\Delta}_j\|_n + \frac{\lambda^2}{2} \Omega(\tilde{\Delta}_j) + 2\lambda \|f_j^*\|_n + 2\lambda^2 \Omega(f_j^*) \right\} + \|f^* - f^0\|_n^2.
\]

For notational convenience we will exclude the \( \|f^* - f^0\|_n^2 \) term in the following manipulations. If \( S \) is the active set then we have on the right hand side,

\[
\text{RHS} = \lambda \sum_{j \in S} \left\{ \|\tilde{\Delta}_j\|_n + \frac{\lambda^2}{2} \Omega(\tilde{\Delta}_j) + 2\|f_j^*\|_n + 2\lambda \Omega(f_j^*) \right\} + \lambda \sum_{j \in S^c} \left\{ \|\tilde{f}_j\|_n + \frac{\lambda}{2} \Omega(\tilde{f}_j) \right\} \\
\leq \lambda \sum_{j \in S} \left\{ \|\tilde{\Delta}_j\|_n + \frac{\lambda^2}{2} \Omega(\tilde{\Delta}_j) + 2\|\tilde{\Delta}_j\|_n + 2\|\tilde{f}_j\|_n + 2\lambda \Omega(f_j^*) \right\} + \lambda \sum_{j \in S^c} \left\{ \|\tilde{f}_j\|_n + \frac{\lambda^2}{2} \Omega(\tilde{f}_j) \right\} \\
= 3 \sum_{j \in S} \lambda\|\tilde{\Delta}_j\|_n + 2 \sum_{j \in S} \lambda^2 \Omega(f_j^*) + \sum_{j \in S^c} \lambda\|\tilde{f}_j\|_n + \frac{1}{2} \sum_{j \in S^c} \lambda^2 \Omega(\tilde{f}_j) + 2 \sum_{j \in S} \lambda\|\tilde{f}_j\|_n + \frac{1}{2} \sum_{j \in S} \lambda^2 \Omega(\tilde{\Delta}_j),
\]
where the inequality holds by the decomposition \( |f_j|^n = |f_j^* - \hat{f}_j + \hat{f}_j|^n \leq |\hat{\Delta}_j|^n + |\hat{f}_j|^n \).

On the left hand side we have

\[
\text{LHS} = \| \hat{f} - f^0 \|^2_n + 2\lambda \sum_{j \in S} \left\{ \| \hat{f}_j \|^n + \lambda \Omega(\hat{f}_j) \right\} + 2\lambda \sum_{j \in S^c} \left\{ \| \hat{f}_j \|^n + \lambda \Omega(\hat{f}_j) \right\} \\
\geq \| \hat{f} - f^0 \|^2_n + 2\lambda \sum_{j \in S} \left\{ \| \hat{f}_j \|^n + \lambda \Omega(\hat{\Delta}_j) - \lambda \Omega(f_j^* \right\} + 2\lambda \sum_{j \in S^c} \left\{ \| \hat{f}_j \|^n + \lambda \Omega(\hat{f}_j) \right\},
\]

where the inequality follows from the triangle inequality \( \Omega(\hat{f}_j) + \Omega(f_j^*) \geq \Omega(\hat{\Delta}_j) \) since \( \Omega(\cdot) \) is a semi-norm.

By re-arranging the terms we obtain the inequality

\[
\| \hat{f} - f^0 \|^2_n + 2\lambda \sum_{j \in S} \left\{ \| \hat{f}_j \|^n + \frac{3\lambda^2}{2} \Omega(\hat{f}_j) \right\} + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\hat{\Delta}_j) \leq 3\lambda \sum_{j \in S} \| \hat{\Delta}_j \|^n + 4\lambda^2 \sum_{j \in S} \Omega(f_j^*) + \| f^* - f^0 \|^2_n
\]

which implies that

\[
\| \hat{f} - f^0 \|^2_n + \lambda \sum_{j \in S^c} \| \hat{\Delta}_j \|^n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\hat{\Delta}_j) \leq 3\lambda \sum_{j \in S} \| \hat{\Delta}_j \|^n + 4\lambda^2 \sum_{j \in S} \Omega(f_j^*) + \| f^* - f^0 \|^2_n.
\]

(E.8)

This implies the slow rates for convergence for \( \lambda \geq 4\rho \) and \( s = |S| \)

\[
\frac{1}{2} \| \hat{f} - f^0 \|^2_n \leq s \lambda \left\{ 3R + 2\lambda \max_j \Omega(f_j^*) \right\} + \frac{1}{2} \| f^* - f^0 \|^2_n.
\]

(E.9)

This completes the proof of Theorem 5.7. In the next section we prove the oracle inequality with fast rates via the compatibility condition.

### E.2 Using the Compatibility Condition

Recall the compatibility condition for \( f = \sum_{j=1}^P f_j \), whenever

\[
\sum_{j \in S^c} \| f_j \|^n \leq 4 \sum_{j \in S} \| f_j \|^n,
\]

(E.10)

then we have

\[
\sum_{j \in S} \| f_j \|^n \leq \sqrt{s} \| f \|_n / \phi(S).
\]

(E.11)

Once we assume the compatibility condition we can prove Corollary 5.7.1 by considering the following two cases.

**Case 1:** \( \lambda \sum_{j \in S} \| \hat{\Delta}_j \|^n \geq 4\lambda^2 \sum_{j \in S} \Omega(f_j^*) \) in which case we have

\[
\| \hat{f} - f^0 \|^2_n + \lambda \sum_{j \in S^c} \| \hat{\Delta}_j \|^n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\hat{\Delta}_j) \leq 4\lambda \sum_{j \in S} \| \hat{\Delta}_j \|^n + \| f^* - f^0 \|^2_n,
\]

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hence for the function $\hat{f} - f^* = \sum_{j=1}^p \tilde{\Delta}_j$ (E.10) holds and hence by the compatibility condition we have

$$
\|\hat{f} - f^0\|^2_n + \lambda \sum_{j \in S^c} \|\tilde{\Delta}_j\|_n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\tilde{\Delta}_j) \leq \frac{4\lambda\sqrt{s}}{\phi(S)} \|\hat{f} - f^0\|_n + \|f^* - f^0\|^2_n
\leq \frac{4\lambda\sqrt{s}}{\phi(S)} \|\hat{f} - f^0\|_n + \frac{4\lambda\sqrt{s}}{\phi(S)} \|f^* - f^0\|_n + \|f^* - f^0\|^2_n
\leq 2 \left\{ \frac{2\lambda\sqrt{2s}}{\phi(S)} \right\} \left( \frac{\|\hat{f} - f^0\|_n}{\sqrt{2}} \right) + 2 \left\{ \frac{2\lambda\sqrt{s}}{\phi(S)} \right\} (\|f^* - f^0\|_n) + \|f^* - f^0\|^2_n
\leq \frac{4\lambda^2(2s)}{\phi^2(S)} + \frac{\|\hat{f} - f^0\|_n^2}{2} + \frac{4\lambda^2s}{\phi^2(S)} + \|f^* - f^0\|^2_n + \|f^* - f^0\|^2_n
\leq \frac{12\lambda^2s}{\phi^2(S)} + \|\hat{f} - f^0\|_n^2 + 2\|f^* - f^0\|^2_n,
$$

where we use the inequality $2ab \leq a^2 + b^2$ and this implies that

$$
\frac{1}{2}\|\hat{f} - f^0\|_n^2 + \lambda \sum_{j \in S^c} \|\tilde{\Delta}_j\|_n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\tilde{\Delta}_j) \leq \frac{12s\lambda^2}{\phi^2(S)} + 2\|f^* - f^0\|^2_n. \quad (E.12)
$$

**Case 2:** $\lambda \sum_{j \in S} \|\tilde{\Delta}_j\|_n \leq 4\lambda^2 \sum_{j \in S} \Omega(f_j^*)$ in which case we have

$$
\|\hat{f} - f^0\|_n^2 + \lambda \sum_{j \in S^c} \|\tilde{\Delta}_j\|_n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\tilde{\Delta}_j) \leq 16\lambda^2 \sum_{j \in S} \Omega(f_j^*) + \|f^* - f^0\|^2_n
\leq 16s\lambda^2 \max_j \Omega(f_j^*) + \|f^* - f^0\|^2_n,
$$

which implies

$$
\frac{1}{2}\|\hat{f} - f^0\|_n^2 + \lambda \sum_{j \in S^c} \|\tilde{\Delta}_j\|_n + \frac{3\lambda^2}{2} \sum_{j \in S} \Omega(\tilde{\Delta}_j) \leq 16s\lambda^2 \max_j \Omega(f_j^*) + 2\|f^* - f^0\|^2_n. \quad (E.13)
$$

### F Constraining the hierbasis Penalty Region

Recall the following definitions

$$
H_{K_n} = \left\{ \beta \in \mathbb{R}^{K_n} : \sum_{k=1}^{K_n} w_k \|\beta_k : K_n\|_2 \leq 1 \right\}, \quad \text{and}
$$

$$
E_{K_n} = \left\{ \beta \in \mathbb{R}^{K_n} : \sum_{k=1}^{K_n} \beta_k^2 (w_1 + \ldots + w_k)^2 \leq 1 \right\}. \quad (F.1)
$$

**Lemma F.1.** For the regions $H_{K_n}^u$ and $E_{K_n}^u$ as defined in (F.1) and (F.2), respectively, we have $H_{K_n}^u \subseteq E_{K_n}^u$ for all $n \geq 1$ and non-negative weights.
Proof. It is sufficient to show
\[
\left( \sum_{k=1}^{K_n} w_k \beta_{k:K_n} \right)^2 \leq \left( \sum_{k=1}^{K_n} \beta_k \right)^2 (w_1 + \ldots + w_k)^2 \leq \left( \sum_{k=1}^{K_n} w_k \beta_{k:K_n} \right)^2 .
\]
We now have
\[
\left( \sum_{k=1}^{K_n} w_k \beta_{k:K_n} \right)^2 = \sum_{m=1}^{K_n} w_m^2 \beta_{m:K_n}^2 + 2 \sum_{m<k} w_m w_m \beta_{m:K_n} \beta_{k:K_n} \left\| \beta_{m:K_n} \right\| \left\| \beta_{k:K_n} \right\| \geq \sum_{l=1}^{K_n} \sum_{m=1}^{K_n} w_m^2 \beta_l^2 \right\| \beta_{m:K_n} \right\| \left\| \beta_{k:K_n} \right\| \geq \sum_{l=1}^{K_n} \sum_{m=1}^{K_n} \beta_l^2 \sum_{k=1}^{K_n} \sum_{m<k} \sum_{l=1}^{K_n} w_m \beta_{m:K_n} \beta_{k:K_n} \left\| \beta_{m:K_n} \right\| \left\| \beta_{k:K_n} \right\| \geq \sum_{l=1}^{K_n} \beta_l^2 \left( \sum_{m=1}^{K_n} w_m^2 + 2 \sum_{m=1}^{K_n} \sum_{k=1}^{K_n} w_m \beta_{m:K_n} \beta_{k:K_n} \right) = \sum_{l=1}^{K_n} \beta_l^2 \left( \sum_{m=1}^{K_n} w_m \right)^2 .
\]

Lemma F.2. For the region \( H_{K_n}^w \) as defined in (F.1), we have the inclusion \( B_{K_n}^w \subset H_{K_n}^w \) where
\[
B_{K_n}^w = \left\{ \beta \in \mathbb{R}^{K_n} : \sum_{k=1}^{K_n} \beta_k^2 \leq (w_1 + \ldots + w_{K_n})^{-2} \right\} . \tag{F.3}
\]
Proof. Let \( \beta \in B_{K_n}^w \) and we for brevity we denote \( \| \cdot \| = \| \cdot \|_2 \). Then for \( \beta \in B_{K_n}^w \)
\[
1 \geq \| \beta \| (w_1 + w_2 + \ldots + w_{K_n}) \geq \| \beta \| \left( \sum_{m=1}^{K_n} \beta_{m:K_n} \right) \geq \| \beta \| \left( \sum_{m=1}^{K_n} \beta_{m:K_n} \right) \geq w_1 \| \beta_{1:K_n} \| + w_2 \| \beta_{2:K_n} \| + \ldots + w_{K_n} \| \beta_{K_n:K_n} \|,
\]
which implies that \( \beta \in H_{K_n}^w \). \( \square \)

In Figure F.1, we demonstrate the above two lemma’s for \( K_n = 2 \) for the special case of \( w_k = k^m - (k-1)^m \).

G Some Entropy Results for Ellipsoids

In this section we establish some entropy results for the ellipsoid (C.2) and the circle (F.3) which will allow us to establish entropy rates for the hierbasis, penalty region \( H_{K_n}^w \).

Since \( K_n \) can potentially be \( \infty \) (or arbitrarily large), we need a way to handle this dimension. It turns out that this can be done using a simple argument which we demonstrate in the following theorem.

G.1 An Upper Bound

Theorem G.1. (Dumer, 2006) For any \( \theta \in (0, 1/2) \), the \( \delta \)-entropy of the ellipsoid \( E_{K_n}^w \) satisfies the following inequality
\[
H(\delta, E_{K_n}^w) \leq \sum_{k=1}^{d-1} \log \left( \frac{1}{\delta \sum_{l=1}^{K_n} w_l} \right) + \mu \theta \log(3/\theta) , \tag{G.1}
\]
where \( \mu_\theta \leq K_n \) is the largest integer such that \( w_1 + \ldots + w_{\mu_\theta} < (\sqrt{1 - \theta})^{-1} \) and \( d \leq K_n + 1 \) is the largest integer such that \( w_1 + \ldots + w_d \leq \delta^{-1} \). If \( \delta^{-1} \leq w_1 \) then \( H(\delta, E_{K_n}^w) = 0 \) holds trivially.

**Corollary G.1.1 (Sobolev Ellipsoids).** For Theorem G.1, let \( w_k = k^m - (k - 1)^m \). Then we have the following upper bound:

\[
H(\delta, E_{K_n}^w) \leq U_E \delta^{-1/m},
\]

for some constant \( U_E \) which only depends on \( m \) and \( \theta \).

**Proof.** Firstly, we note that with this definition of \( w_k \), we can let \( K_n = \infty \). Thus if we can show that \( H(\delta, E_{\infty}^w) \leq U \delta^{-1/m} \) then the result follows since \( E_{K_n}^w \subset E_{\infty}^w \) for all \( K_n < \infty \).

Now we have \( w_1 + \ldots + w_{\mu_\theta} = \mu_\theta^m \), hence

\[
\mu_\theta^m < \frac{\delta^{-1}}{\sqrt{1 - \theta}} \Rightarrow \mu_\theta \log(3/\theta) < \log(3/\theta) \left( \frac{\delta^{-1}}{\sqrt{1 - \theta}} \right)^{1/m} = U_1 \delta^{-1/m}.
\]

Now for the second part we use the fact that \( w_1 + \ldots + w_{d-1} = (d - 1)^m \leq \delta^{-1} < d^m \) and we obtain

\[
\sum_{j=1}^{d-1} \log \left( \frac{1}{\delta^j} \right) = (d - 1) \log(\delta^{-1}) + \log \left[ \frac{1}{\{(d - 1)\}^m} \right] \leq \delta^{-1/m} \log(\delta^{-1}) - m \log \{(d - 1)!\}
\]

\[
\leq \delta^{-1/m} \left[ \log(\delta^{-1}) - m\delta^{1/m} \log \{(d - 1)!\} \right] \leq \delta^{-1/m} \left[ \log(d^m) - m\delta^{1/m} \log \{(d - 1)!\} \right]
\]

\[
\leq \delta^{-1/m} m \log(d) - d \log(d - 1) \log \{(d - 1)!\}.
\]

Now by sterling’s inequality we have for all \( d \in \{1, 2, \ldots \} \)

\[
\log(d + 1) - \frac{\log(d)}{d + 1} \leq \log(d + 1) - \frac{\log(\sqrt{2\pi}d^{d+1/2}e^{-d})}{d + 1}
\]

\[
= \log(d + 1) + \frac{d}{d + 1} \left( \frac{\log(\sqrt{2\pi})}{d + 1} - \frac{d + 1/2}{d + 1} \log(d) \right)
\]

\[
\leq \log(d + 1) + 1 - \frac{d + 1 - 1 + 1/2}{d + 1} \log(d)
\]

\[
= 1 + \log \left( \frac{d + 1}{d} \right) + (1/2) \frac{\log(d)}{d + 1} \leq 1 + \log(2) + 1.
\]
This implies that
\[
    \sum_{j=1}^{d-1} \log \left( \frac{1}{\delta j^m} \right) \leq \delta^{-1/m} m \left[ \log(d) - d^{-1} \log((d-1)!) \right] \leq U_2 \delta^{-1/m}. 
\]

**Corollary G.1.2** (Multivariate hierbasis). For Theorem G.1, let \( w_{q_k} = k^m - (k-1)^m \) where for a fixed dimension \( p \) we define \( q_k = \sum_{l=1}^{k} \left( \frac{l+p-2}{l-1} \right) = \left( \frac{k+p-1}{p} \right) \) and all other \( w_k = 0 \). Then we have the following upper bound:
\[
    H(\delta, E_{K_n}^w) \leq U_E \delta^{-p/m}, \tag{G.3}
\]
for some constant \( U_E \) which only depends on \( m \) and \( \theta \).

**Proof.** Firstly, since \( w_1 = 1 \) the entropy is 0 for \( \delta \geq 1 \) and hence we will restrict ourselves to \( \delta \in (0, 1) \). We note that we must have \( \mu_\theta = q_{k_1} - 1 \) for some integer \( k_1 \). This is because all weights after \( q_{k_1}-1 \) are zero until \( w_{q_k} \). Now we have by definition
\[
    w_1 + \ldots + w_{q_{k_1}-1} = (k_1 - 1)^m \leq \left( \delta \sqrt{1 - \theta} \right)^{-1},
\]
and we have
\[
    \mu_\theta = q_{k_1} - 1 = \left( \frac{k_1 + p - 1}{p} \right) - 1 < \left( \frac{k_1 + p - 1}{p} \right) - 1 \leq \left( \frac{\delta \sqrt{1 - \theta}}{\delta} \right)^{-1/m} + p \delta^{-1/m} \leq \delta^{-1/m} \left( \delta \sqrt{1 - \theta} \right)^{-1/m} + p \delta^{-1/m}.
\]
where the second line follows from the inequality \( \binom{n}{k} \leq n^k / k! \). This implies that for \( \delta \in (0, 1) \)
\[
    \mu_\theta \log(3/\theta) \leq U_1 \delta^{-\frac{p}{m}}.
\]

Similarly, there is an integer \( k_2 \) such that \( d - 1 = q_{k_2} - 1 \). Which means that \( (k_2 - 1)^m \leq \delta^{-1} \leq k_2^m \). For the other term we have
\[
    \sum_{k=1}^{d-1} \log \left( \frac{1}{\delta \sum_{l=1}^{k} a_l} \right) = (d - 1) \left\{ \log(\delta^{-1}) - \frac{\sum_{k=1}^{d-1} \log(\sum_{l=1}^{k} a_l)}{d - 1} \right\}
\]
\[
    = (d - 1) \left\{ \log(\delta^{-1}) - \frac{(q_2 - 1) \log(1^m) + (q_3 - q_2) \log(2^m) + \ldots + (q_{k_2} - 1 - q_{k_2-1}) \log(q_{k_2-1}^m)}{d - 1} \right\}
\]
\[
    = (d - 1) \left[ \log(\delta^{-1}) - \frac{m(f(k_2) - \log(q_{k_2-1})}{d - 1} \right],
\]
where \( f(k_2) = (q_2 - 1) \log(1) + (q_3 - q_2) \log(2) + \ldots + (q_{k_2} - q_{k_2-1}) \log(q_{k_2-1}) = \sum_{l=1}^{k_2} (q_l - q_{l-1}) \log(q_{l-1}) \). Hence we have
\[
    \sum_{j=1}^{d-1} \log \left( \frac{1}{\delta \sum_{l=1}^{j} a_l} \right) \leq (d - 1) \left[ \log(k_2^m) - \frac{m(f(k_2) - \log(q_{k_2-1})}{d - 1} \right]
\]
\[
    = m(d - 1) \left[ \log(k_2) - \frac{f(k_2) - \log(q_{k_2-1})}{q_{k_2-1}} \right].
\]

Now by induction we can show that \( \frac{f(k_2) - \log(q_{k_2})}{q_{k_2} - 1} \geq \frac{\log(k_2-1)!}{k_2} \) which implies that

\[
m(d-1) \left\{ \log(k_2) - \frac{f(k_2) - \log(q_{k_2})}{q_{k_2} - 1} \right\} \leq m(d-1) \left[ \log(k_2) - \frac{\log((k_2-1)!)}{k_2} \right] \\
\leq (d-1)m \{ 2 + \log(2) \} .
\]

Finally, we note that

\[
d - 1 = q_{k_2} - 1 = \left( \frac{k_2 + p - 1}{p} \right) - 1 < \left( \frac{k_2 + p - 1}{p} \right)
\leq \frac{(k_2 + p - 1)^p}{p!} \leq \frac{(\delta^{-1/m} + p)^p}{p!} = \delta^{\frac{1}{m}} \frac{(1 + p\delta^{1/m})^p}{p!} \leq \delta^{\frac{1}{m}} \frac{(1 + p)^p}{p!}.
\]

\[\square\]

**H Proof of Theorem 5.2**

**Proof.** By definition

\[
\frac{1}{2} \|f - y\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq \frac{1}{2} \|f_n^* - y\|_n^2 + \lambda_n^2 \Omega(f_n^*|Q_n),
\]

which leads to the following inequality

\[
\frac{1}{2} \|\hat{f} - f_0\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq |\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + \frac{1}{2} \|f_n^* - f_0\|_n^2 + \lambda_n^2 \Omega(f_n^*|Q_n) , \quad (H.1)
\]

where \( \langle \varepsilon, f \rangle_n = \frac{1}{n} \sum_{i=1}^n \varepsilon_i f(x_i) \). Via the simple decomposition \( \|\hat{f} - f_n^*\|_n^2 \leq 2\|\hat{f} - f_0\|_n^2 + 2\|f_0 - f_n^*\|_n^2 \) we obtain

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq \|\hat{f} - f_0\|_n^2 + \|f_0 - f_n^*\|_n^2 + 2\lambda_n^2 \Omega(\hat{f}|Q_n)
\leq \|\hat{f} - f_0\|_n^2 + \|f_0 - f_n^*\|_n^2 + 2 \left\{ |\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + \frac{1}{2} \|f_n^* - f_0\|_n^2 + \lambda_n^2 \Omega(f_n^*|Q_n) \right\}
\leq 2|\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + 2\lambda_n^2 \Omega(f_n^*|Q_n) + \|f_0 - f_n^*\|_n^2 + 2 \left\{ 4|\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + 4\lambda_n^2 \Omega(f_n^*|Q_n), 2\|f_0 - f_n^*\|_n^2 \right\} .
\]

Thus our basic inequality is given by

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 2 \max \left\{ 2|\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + 2\lambda_n^2 \Omega(f_n^*|Q_n), \|f_0 - f_n^*\|_n^2 \right\} . \quad (H.2)
\]

Hence from the basic inequality either \( \frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 2\|f_0 - f_n^*\|_n^2 \) which implies the result or

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 4|\langle \varepsilon, \hat{f} - f_n^* \rangle_n| + 4\lambda_n^2 \Omega(f_n^*|Q_n) . \quad (H.3)
\]

Now note that \( H(\delta, \{ f \in F_n : \Omega(f|Q_n) \leq 1 \}, Q_n) \leq A_1 \delta^{-\alpha} \) implies

\[
H \left( \delta, \left\{ \frac{f - f_n^*}{\Omega(f|Q_n) + \Omega(f_n^*|Q_n)} : f \in F_n \right\} , Q_n \right) \leq \tilde{A}_1 \delta^{-\alpha} . \quad (H.4)
\]

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Thus we invoke Lemma 8.4 of van de Geer (2000) and conclude that with probability at least 1 – \( c \exp\{-\frac{T^2}{c}\} \) for constants \( c \) and \( T \geq c \), we have

\[
|\langle \varepsilon, \hat{f} - f_n^* \rangle_n| \leq \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ \Omega(\hat{f}|Q_n) + \Omega(f_n^*|Q_n) \right\}^{\frac{\alpha}{2}}. \tag{H.5}
\]

Define the set \( T \) as

\[
T = \left\{ \sup_{f \in \mathcal{F}_n} \langle \varepsilon, f - f_n^* \rangle_n \leq \frac{T}{\sqrt{n}} \|f - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ \Omega(\hat{f}|Q_n) + \Omega(f_n^*|Q_n) \right\}^{\frac{\alpha}{2}} \right\}, \tag{H.6}
\]

then on the set \( T \) we have

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 4 \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ \Omega(\hat{f}|Q_n) + \Omega(f_n^*|Q_n) \right\}^{\frac{\alpha}{2}} + 4\lambda_n^2 \Omega(f_n^*|Q_n).
\]

Which means we have either

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 8\lambda_n^2 \Omega(f_n^*|Q_n), \tag{H.7}
\]

which is of the desired form or

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 8 \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ \Omega(\hat{f}|Q_n) + \Omega(f_n^*|Q_n) \right\}^{\frac{\alpha}{2}}. \tag{H.8}
\]

We now consider (H.8) only.

**H.1 Case 1: \( \Omega(\hat{f}|Q_n) \geq \Omega(f_n^*|Q_n) \)**

In this case we have

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 8 \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ 2\Omega(\hat{f}|Q_n) \right\}^{\frac{\alpha}{2}}. \tag{H.9}
\]

which gives us

\[
\lambda_n^2 \Omega(\hat{f}|Q_n) \leq 8 \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \left\{ 2\Omega(\hat{f}|Q_n) \right\}^{\frac{\alpha}{2}} \quad \Leftrightarrow \quad \left\{ \Omega(\hat{f}|Q_n) \right\}^{1 - \frac{\alpha}{2}} \leq 2^{3 + \frac{\alpha}{2}}Tn^{-\frac{1}{2}}\lambda_n^{-2} \|\hat{f} - f_n^*\|_n^{1 - \frac{\alpha}{2}} \quad \Leftrightarrow \quad \Omega(\hat{f}|Q_n) \leq \left(2^{3 + \frac{\alpha}{2}}Tn^{-\frac{1}{2}}\lambda_n^{-2}\right)^{\frac{2\alpha}{2\alpha - \alpha}} \|\hat{f} - f_n^*\|_n.
\]

Plugging this into the right hand side of (H.9) and solving for \( \|\hat{f} - f_n^*\|_n \) we obtain

\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 \leq T \frac{2}{2\alpha - 2} \left(2^{3 + \frac{\alpha}{2}}Tn^{-\frac{1}{2}}\lambda_n^{-2}\right)^{\frac{2\alpha}{2\alpha - \alpha}} \|\hat{f} - f_n^*\|_n^{\alpha/2} \Rightarrow \frac{1}{2} \|\hat{f} - f_n^*\|_n \leq T \frac{2}{2\alpha - 2} \left(2^{3 + \frac{\alpha}{2}}Tn^{-\frac{1}{2}}\lambda_n^{-2}\right)^{\frac{2\alpha}{2\alpha - \alpha}} \Rightarrow \frac{1}{2} \|\hat{f} - f_n^*\|_n \leq C_1n^{-\frac{2}{2\alpha - \alpha}}\lambda_n^{-\frac{4\alpha}{2\alpha - \alpha}} = C_1\lambda_n^2 \Omega(f_n^*|Q_n),
\]

where \( C_1 = T \frac{4}{2\alpha - 2} \frac{1^{1+\alpha}}{2\alpha - \alpha} \) and recall the definition \( \lambda_n^{-1} = n^{\frac{1}{2\alpha}} \left\{ \Omega(f_n^*|Q_n) \right\}^{\frac{2\alpha}{2\alpha - \alpha}} \).
In this case we have
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 + \lambda_n^2 \Omega(\hat{f}|Q_n) \leq 8 \frac{T}{\sqrt{n}} \|\hat{f} - f_n^*\|_n^{1-\frac{\alpha}{2}} \{2 \Omega(f_n^*|Q_n)\}^{\frac{\alpha}{2}}.
\] (H.10)
From which we directly get
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 \leq 8 T n^{-\frac{1}{2}} \|\hat{f} - f_n^*\|_n^{1-\frac{\alpha}{2}} \{2 \Omega(f_n^*|Q_n)\}^{\frac{\alpha}{2}} \Rightarrow
\]
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 \leq 2^{3+\frac{\alpha}{2}} T n^{-\frac{1}{2}} \{\Omega(f_n^*|Q_n)\}^{\frac{\alpha}{2}} \Rightarrow
\]
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n \leq 2^{\frac{3+\alpha}{2}} T^{\frac{n}{1-\alpha}} n^{-\frac{1}{2-\alpha}} \{\Omega(f_n^*|Q_n)\}^{\frac{\alpha}{2-\alpha}} \Rightarrow
\]
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n \leq C_2 n^{-\frac{2}{1-\alpha}} \{\Omega(f_n^*|Q_n)\}^{\frac{2\alpha}{2(1-\alpha)}} = C_2 \lambda_n^2 \Omega(f_n^*|Q_n),
\]
where \(C_2 = T^{\frac{1}{1-\alpha}} 2^{\frac{14+\alpha}{2(1-\alpha)}}\). Thus we have shown that on the set \(\mathcal{T}\) we have
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n \leq \max\{8, C_1, C_2\} \lambda_n^2 \Omega(f_n^*|Q_n) = C_0 \lambda_n^2 \Omega(f_n^*|Q_n).
\] (H.11)
We have shown that with probability at least \(1 - c \exp(-\frac{T^2}{c})\) we have the inequality
\[
\frac{1}{2} \|\hat{f} - f_n^*\|_n^2 \leq \max\{2\|f^0 - f_n^*\|_n^2, C_0 \lambda_n^2 \Omega(f_n^*|Q_n)\}
\] (H.12)
To complete the proof we note that
\[
\frac{1}{2} \|\hat{f} - f^0\|_n^2 \leq \|\hat{f} - f_n^*\|_n^2 + \|f_n^* - f^0\|_n^2
\]
\[
\leq 2 \max\{2\|f^0 - f_n^*\|_n^2, C_0 \lambda_n^2 \Omega(f_n^*|Q_n)\} + \|f_n^* - f^0\|_n^2
\]
\[
\leq \frac{5}{2} \max\{2\|f^0 - f_n^*\|_n^2, C_0 \lambda_n^2 \Omega(f_n^*|Q_n)\}.
\]
\[
\square
\]

I Variation of Lemma 8.4 of van de Geer (2000)

Lemma I.1. Assume that \(\sup_{f \in \mathcal{F}} \|f\|_{Q_n} \leq R\) for some univariate function class \(\mathcal{F}\). Given the entropy bound
\[
H(\delta, \mathcal{F}, Q_n) \leq A_0(\delta \rho)^{-\alpha},
\] (I.1)
for some \(\alpha \in (0, 2)\) and constant \(A_0\) and that
\[
\max_{i=1,\ldots,n} K^2 \left(\mathbb{E} \varepsilon_i^2 | K^2 - 1\right) \leq \sigma_0^2.
\] (I.2)
Then for some constant \(c\) depending on \(A_0, \alpha, K\) and \(\sigma_0\), we have for all \(T \geq c\),
\[
P \left(\sup_{f \in \mathcal{F}} \left|\frac{1}{n} \sum_{i=1}^n \varepsilon_i f(x_i)\right| \geq T^{\rho^{-\alpha}/2} \sqrt{n} \right) \leq c \exp\left(-\frac{T^{2\rho^{-\alpha}}}{c^2}\right).
\] (I.3)
Proof. We have

\[
\int_0^\delta H^{1/2}(u, \mathcal{G}, Q_n) \, du \leq A_1 \gamma^{-\alpha/2} \delta^{1-\alpha/2}.
\]

For \( C_1 \geq 1 \) by Corollary 8.3 of van de Geer (2000) we have,

\[
P \left( \sup_{g \in \mathcal{G}, \|g\|_{Q_n} \leq \delta} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i g(x_i) \right| \geq 2 CC_1 A_1 \frac{\gamma^{-\alpha/2}}{\sqrt{n}} \delta^{1-\alpha/2} \right) \leq C \exp \left( -C_1^2 A_1^2 \gamma^{-\alpha} \right).
\]

Now we apply the peeling device (see van de Geer (2000)) to the class \( \mathcal{G} \). Let \( T = 2C_1 CA_1^{2-\alpha/2} \),

\[
P \left( \sup_{g \in \mathcal{G}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i g(x_i) \right| \geq T^{\gamma^{-\alpha/2}} \sqrt{n} \right) = P \left( \bigcup_{s=1}^\infty \sup_{g \in \mathcal{G}, \|g\|_{Q_n} \leq \frac{R}{2^n}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i g(x_i) \right| \geq T^{\gamma^{-\alpha/2}} \sqrt{n} \right)
\]

\[
\leq \sum_{s=1}^\infty P \left( \sup_{g \in \mathcal{G}, \|g\|_{Q_n} \leq \frac{R}{2^n}} \left| \frac{1}{n} \sum_{i=1}^n \varepsilon_i g(x_i) \right| \geq T^{\gamma^{-\alpha/2}} \sqrt{n} \right) \leq 2 C_1 A_1 \frac{\gamma^{-\alpha/2}}{\sqrt{n}} \left( \frac{R}{2^n} \right)^{1-\alpha/2}
\]

\[
\leq \sum_{s=1}^\infty C \exp \left\{ -C_1^2 A_1^2 \gamma^{-\alpha} \left( \frac{R}{2^n} \right)^{-\alpha} \right\} = \sum_{s=1}^\infty C \exp \left\{ -C_1^2 A_1^2 (2^{-s} - 1)^{-\alpha} \right\} \leq c \exp \left( -\frac{T^2 \gamma^{-\alpha}}{c^2} \right).
\]