Well-posedness in Critical Spaces for the Density-dependent Incompressible Viscoelastic Fluid System

Huazhao Xie and Yunxia Fu

Abstract. We are concerned with the well-posedness of the density-dependent incompressible viscoelastic fluid system. By Schauder-Tychonoff fixed point argument, when \( \|1/\rho_0 - 1\|_{\dot{B}^{N/2}_p} \) is small, local well-posedness is showed to hold in Besov space. Furthermore, provided the initial data \((1/\rho_0 - 1, v_0, U_0 - I)\) is small under certain norm, we also get the existence of the global solution.

Keywords. Incompressible viscoelastic fluid; Well-posedness; Besov space; Oldroyd model; Fixed point theorem.

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1 Introduction

Viscoelastic fluids have a wide range of applications and hence have received a great deal of interest. Examples and applications of viscoelastic fluids include oil, liquid polymers, mucus, liquid soap, toothpaste, clay, ceramics, coatings, drug delivery systems for controlled drug release, viscoelastic blood flow past valves and so on, see [7] for more applications. The motion of a density-dependent incompressible viscoelastic fluid is described by the following inhomogeneous Oldroyd system:

\[
\begin{align*}
\rho_t + \text{div}(\rho v) &= 0, \quad (t, x) \in (0, +\infty) \times \mathbb{R}^N, \\
(vv)_t + \text{div}(\rho v \otimes v) + \nabla P &= \mu \Delta v + \text{div}(\rho U U^T), \\
U_t + v \cdot \nabla U &= \nabla v U, \\
\text{div} v &= 0
\end{align*}
\]  

(1.1)

supplemented with the initial data

\[
(\rho, v, U)|_{t=0} = (\rho_0, v_0, U_0),
\]  

(1.2)

where \( N \geq 2, \rho(t, x), v(t, x), P(t, x) \) and \( U(t, x) = (U^{ij}(t, x))_{N \times N} \) denote the density, velocity, hydrodynamic pressure and the deformation tensor respectively, the viscosity coefficient \( \mu > 0 \) is a constant.

In the context of hydrodynamics, the motion of the fluid flow is described by the particle trajectory \( x(t, X) \), where material points \( X \) are deformed to the spatial position \( x(t, X) \) at the time \( t \). The deformation tensor \( \tilde{U}(t, x) = \frac{\partial x}{\partial X}(t, X) \), when we work in Eulerian coordinate, we denote it by \( U(t, x) = \tilde{U}(t, X^{-1}(t, x)) \). Applying the chain rule, we see that \( U(t, x) \) satisfies the transport equation, which stands for

\[
U_t^{ij} + v^k \nabla_k U^{ij} = \nabla_k v^j U^{kj}, \quad \text{for} \quad i, j = 1, \cdots, N,
\]

where \( \nabla_i = \frac{\partial}{\partial x_i}, \quad U^{ij} = \frac{\partial x^i}{\partial X^j}, \quad \nabla_j v^i = (\nabla v)^{ij}. \)

We assume that the initial data satisfy the constrains

\[
\begin{align*}
\text{div} v_0 &= 0, \\
\text{div}(\rho_0 U_0^j) &= 0, \\
U_0^{ik} \nabla_i U_0^{ij} - U_0^{ij} \nabla_i U_0^{ik} &= 0.
\end{align*}
\]  

(1.3)
Using (1.3), it is easy to obtain that \( \text{div} (\rho U^T) = 0 \) and
\[
U^{ik} \nabla_i U^{ij} - U^{ij} \nabla_i U^{ik} = 0, \tag{1.4}
\]
hold for all time, see [8,12].

System (1.1) has been studied extensively. When the density is a constant, system (1.1) governs the homogeneous incompressible viscoelastic fluids, and there exist rich results in the literature for the global existence of classical solutions, see [2,9,10,11] and the references therein. Let \( H = U - I \) be the perturbation of deformation tensor \( U \), Lei et al. [9] find that
\[
\nabla_k H^{ij} - \nabla_j H^{ik} = H^{ij} \nabla_i H^{ik} - H^{ik} \nabla_i H^{ij}, \tag{1.5}
\]
which is useful to prove the global existence. When density \( \rho(t,x) \) is not a constant, the problem related to existence becomes more complicated and not much has been done. Qian and Zhang [12] got the well-posedness in critical spaces. Our assumptions and methods are different, we consider the problem in critical spaces. Hu and Wang in [13] considered the three-dimensional density-dependent incompressible viscoelastic fluids, they got the existence and uniqueness of the global strong solution with small initial data. Our assumptions and methods are different, we consider the problem in critical spaces.

We shall use scaling considerations to find which spaces are critical for (1.1). System (1.1) is invariant by the rescaling \( (\rho, v, P, U) \mapsto (\rho_l, v_l, P_l, U_l) \) with
\[
\rho_l(t,x) = \rho(l^2 t, l x), \quad v_l(t,x) = lv(l^2 t, l x), \quad P_l(t,x) = l^2 P(l^2 t, l x), \quad U_l(t,x) = l^2 U(l^2 t, l x).
\]
This motivates the following definition:

**Definition 1.1.** We say that a functional space is critical with respect to the scaling of the equations if the associated norm is invariant under the transformation: \( (\rho, v, P, U) \mapsto (\rho_l, v_l, P_l, U_l) \) (up to a constant independent of \( l \)).

In Sobolev spaces setting, the above definition would lead us to consider initial data in \( H^{N/p} \times \left( \dot{H}^{N/p-1} \right)^N \times \left( H^{N/p} \right)^N \). If \( \rho \) vanishes or becomes unbounded, system (1.1) degenerates, it is reasonable to assume that \( \rho_0 \in L^\infty \). For technical reasons, we suppose that the initial data belong to a somewhat smaller homogeneous Besov space \( \dot{B}^{N/p}_{p,1} \times \left( \dot{B}^{N/p-1}_{p,1} \right)^N \). Set
\[
\sigma = \frac{1}{\rho} - 1, \quad H = U - I,
\]
the system (1.1)-(1.2) can be reformulated as follows
\[
\begin{align*}
\sigma_t + v \cdot \nabla \sigma &= 0, \\
v^i_t + v \cdot \nabla v^i + (\sigma + 1) \nabla_i P &= \mu (\sigma + 1) \Delta v^i + \partial_k H^{jk} + H^{jk} \partial_j H^{ik}, \\
H^i_t + v \cdot \nabla H &= \nabla v (H + I), \\
\text{div } v &= 0, \\
(\sigma, v, H)|_{t=0} &= (\sigma_0, v_0, H_0).
\end{align*} \tag{1.6}
\]
The well-posedness of the system (1.1)-(1.2) is equivalent to the system (1.6).

The main results of this paper are as follows.

**Theorem 1.1.** For the system (1.6), let \( p \in [1, N] \), there exists a constant \( c = c(N) \), such that for any \( \nu_0 \in (\dot{B}^{N/p-1}_{p,1})^N \) with \( \text{div} \nu_0 = 0, H_0 \in (\dot{B}^{N/p}_{p,1})^N \) and \( \sigma_0 \in \dot{B}^{N/p}_{p,1} \) with \( \|\sigma_0\|_{\dot{B}^{N/p}_{p,1}} \leq \epsilon \), then there exists a positive time \( T \in (0, +\infty) \) such that system (1.6) has a unique solution with

\[
\sigma \in C([0, T]; \dot{B}^s_{p,1}), \quad v \in \left( C([0, T]; \dot{B}^{s-1}_{p,1}) \cap L^1([0, T]; \dot{B}^{s+1}_{p,1}) \right)^N,
\]

\[
H \in C([0, T]; \dot{B}^s_{p,1}) \quad \text{and} \quad \nabla P \in \left( L^1([0, T]; \dot{B}^{s-1}_{p,1}) \right)^N.
\]

In addition, for \( p = 2 \), we denote \( B^s := B^s_{2,1} \), there exists a constant \( \eta \) such that, if

\[
\|\sigma_0\|_{B^s} + \|H_0\|_{B^{s-1}} + \|\nu_0\|_{B^{s-1}} \leq \eta,
\]

then the system (1.6) has a unique global solution with \( (\sigma, v, H) \in \mathcal{H}^{N/2} \), where

\[
\mathcal{H}^s := \left( L^2(R^+; B^s) \cap C(R^+; B^s \cap B^{s-1}) \right) \times \left( L^1(R^+; B^{s+1}) \cap C(R^+; B^s \cap B^{s-1}) \right)^N \times \left( L^2(R^+; B^s) \cap C(R^+; B^s \cap B^{s-1}) \right)^{N^2},
\]

and \( \nabla P \in L^1(R^+; B^{s-1})^N \).

This paper is structured as follows. Section 2 is devoted to recall some basic results on Besov spaces. In Section 3, using Schauder-Tychonoff fixed point argument, we prove the existence and uniqueness of the local solution. At last, we concentrate on the proof of the global existence of solution.

Notation: Throughout the paper, \( C \) stands for a universal constant. \( Z'(\mathbb{R}^N) \) stands for the dual space \( Z(\mathbb{R}^N) = \{ f \in \varphi(\mathbb{R}^N) : D^\alpha f(0) = 0, \forall \alpha \in \mathbb{N}^N \text{ multi-index} \} \). The notation \( L_p^p(X) \) stands for the set of measurable functions on \((0, T)\) with values in \( X \), such that \( \| \cdot \|_X \in L^p(0, T) \), where \( X \) be a Banach space, \( p \in [1, +\infty] \).

## 2 Basic results on Besov spaces

In this section, we mainly review some results on Besov spaces. At first, we introduce the Littlewood-Paley theory. The homogeneous Littlewood-Paley decomposition relies upon a dyadic partition of unity: radial function \( \varphi \in \varphi(\mathbb{R}^N) \) supported in the shell \( C := \{ \xi \in \mathbb{R}^N \mid \frac{3}{4} \leq \xi \leq \frac{4}{3} \} \) such that

\[
\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1, \quad \text{for all } \xi \neq 0.
\]

The homogeneous dyadic blocks and low frequency cut-off are defined by

\[
\Delta_q f := \varphi(2^{-q}D)f, \quad \text{and} \quad S_q f := \sum_{k \leq q - 1} \Delta_k f \quad \text{for } q \in \mathbb{Z}.
\]

It is easy to verify the following properties hold:

\[
\Delta_q \Delta_k f \equiv 0 \quad \text{for } |q - k| \geq 2; \quad \Delta_q (S_{k-1} f \Delta_k f) \equiv 0 \quad \text{for } |q - k| \geq 5.
\]

The definition of the Besov space depend on the Littlewood-Paley decomposition.
Lemma 2.1. For \( s \in \mathbb{R}, \ (p, r) \in [1, +\infty]^2 \) and \( u \in \mathcal{Z}'(\mathbb{R}^N) \), we set
\[
\|u\|_{B^s_{p,r}} := \left\|2^s \|\Delta j u\|_p\right\|_{l^r},
\]
with the usual change if \( r = +\infty \). The homogeneous Besov space \( \dot{B}^s_{p,r} \) is defined by
\[
\dot{B}^s_{p,r} := \{u \in \mathcal{Z}'(\mathbb{R}^N) : \|u\|_{\dot{B}^s_{p,r}} < +\infty\}.
\]

Next, we introduce the Besov-Chemin-Lerner space \( \tilde{L}^k_T(\dot{B}^s_{p,r}) \) which is initiated in [1].

Definition 2.2. For \( s \in \mathbb{R}, \ (p, r, k) \in [1, +\infty]^3 \) and \( T \in (0, +\infty) \). The space \( \tilde{L}^k_T(\dot{B}^s_{p,r}) \) is defined by
\[
\tilde{L}^k_T(\dot{B}^s_{p,r}) := \{u \in L^k(0, T; \mathcal{Z}'(\mathbb{R}^N)) : \|u\|_{\tilde{L}^k_T(\dot{B}^s_{p,r})} < +\infty\},
\]
where
\[
\|u\|_{\tilde{L}^k_T(\dot{B}^s_{p,r})} := \left\{\int_0^T \|u(t)\|_{\dot{B}^s_{p,r}} dt\right\}^{1/k},
\]
with the usual change if \( r = +\infty \).

By virtue of Minkowski’s inequality, we have
\[
\|u\|_{\tilde{L}^k_T(\dot{B}^s_{p,r})} \leq \|u\|_{L^k_T(\dot{B}^s_{p,r})} \quad \text{if} \quad k \leq r,
\]
\[
\|u\|_{\tilde{L}^k_T(\dot{B}^s_{p,r})} \geq \|u\|_{L^k_T(\dot{B}^s_{p,r})} \quad \text{if} \quad k \geq r.
\]

In order to get the global existence result, we need to give the definition of hybrid Besov spaces.

Definition 2.3. For \( \mu > 0, r \in [1, +\infty] \) and \( s \in \mathbb{R} \), we define the hybrid Besov space \( B^{\mu r}_{p,s} \) as the set of functions \( u \) such that
\[
\|u\|_{B^{\mu r}_{p,s}} := \sum_{q \in \mathbb{Z}} 2^{qs} \max(\mu, 2^{-q})^{1 - \frac{s}{r}} \|\Delta q u\|_{L^2} < \infty.
\]

Let us list some important properties of the Besov spaces, see [3].

Lemma 2.1. For \( s \in \mathbb{R}, \ (p, r) \in [1, +\infty] \). The following inequalities hold true:

(i) there exists a universal constant \( C \) such that
\[
C^{-1} \|u\|_{B^s_{p,r}} \leq \|\nabla u\|_{B^{s-1}_{p,r}} \leq C\|u\|_{B^s_{p,r}}; \tag{2.1}
\]

(ii) if \( s_1, s_2 \leq \frac{N}{p} \) and \( s_1 + s_2 > N \max(0, \frac{2}{p} - 1) \),
\[
\|uv\|_{B^{s_1+s_2-N}{p,1}} \leq C\|u\|_{B^{s_1}_{p,1}}\|v\|_{B^{s_2}_{p,1}}; \tag{2.2}
\]

(iii) if \( s_1 \leq \frac{N}{p}, s_2 < \frac{N}{p} \) and \( s_1 + s_2 \geq N \max(0, \frac{2}{p} - 1) \),
\[
\|uv\|_{B^{s_1+s_2-N}{p,\infty}} \leq C\|u\|_{B^{s_1}_{p,\infty}}\|v\|_{B^{s_2}_{p,\infty}}. \tag{2.3}
\]
Lemma 2.2. Let $1 \leq k, p, q, q_1, q_2 \leq \infty$ with $\frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q}$, then

(i) if $f \in \dot{L}_{T}^{q_1}(\dot{B}_{p,1}^{s_1})$, $g \in \dot{L}_{T}^{q_2}(\dot{B}_{p,1}^{s_2})$ and $s_1, s_2 \leq \frac{N}{p}$, $s_1 + s_2 > N \max (0, \frac{2}{p} - 1)$,

$$
\|fg\|_{L_{T}^{q}(\dot{B}_{p,1}^{s_1+s_2-\frac{N}{p}})} \leq C \|f\|_{L_{T}^{q_1}(\dot{B}_{p,1}^{s_1})} \|g\|_{L_{T}^{q_2}(\dot{B}_{p,1}^{s_2})}; 
$$

(2.4)

(ii) if $f \in \dot{L}_{T}^{q_1}(\dot{B}_{p,1}^{s_1})$, $g \in \dot{L}_{T}^{q_2}(\dot{B}_{p,1}^{s_2})$ and $s_1 \leq \frac{N}{p}$, $s_2 < \frac{N}{p}$, $s_1 + s_2 \geq N \max (0, \frac{2}{p} - 1)$,

$$
\|fg\|_{L_{T}^{q}(\dot{B}_{p,1}^{s_1+s_2-\frac{N}{p}})} \leq C \|f\|_{L_{T}^{q_1}(\dot{B}_{p,1}^{s_1})} \|g\|_{L_{T}^{q_2}(\dot{B}_{p,1}^{s_2})}; 
$$

(2.5)

(iii) for $f \in \dot{L}_{T}^{k}(\dot{B}_{p,1}^{s}), s \in \mathbb{R}$ and $\epsilon \in (0, 1)$, we have

$$
\|f\|_{\dot{L}_{T}^{k}(\dot{B}_{p,1}^{s})} \leq C\frac{\|f\|_{\dot{L}_{T}^{k}(\dot{B}_{p,1}^{s})}}{\epsilon} \log \left(\frac{\|f\|_{\dot{L}_{T}^{k}(\dot{B}_{p,1}^{s})} + \|f\|_{\dot{L}_{T}^{k}(\dot{B}_{p,1}^{s})}}{\|f\|_{\dot{L}_{T}^{k}(\dot{B}_{p,1}^{s})}}\right). 
$$

(2.6)

Lemma 2.3. (see [4]) Let $0 < R_1 < R_2$ and $\psi \in C_{0}^{\infty}(\mathbb{R}^{N})$ be supported in the annulus $C(0, R_1, R_2)$. For all indices $s, t, p, \alpha, \alpha_2$ and $\alpha_2$ such that $1 \leq p, \alpha, \alpha_2, \alpha_2' \leq +\infty$, $1/\alpha + 1/\alpha_2 = 1/\alpha_2$, $t \leq N/p + 1$, $1 \leq s \leq N/p + 1$ and $s + t > 1$, there exists $c_{q} \in \mathbb{Z}$ such that $\sum_{q} c_{q} \leq 1$ and

$$
\|\text{div}[a, \psi(2^{-q}D)]\nabla B\|_{L_{T}^{q_1}(L^{p})} \leq C c_{q} 2^{-q(s+t-2N/p)} \|\nabla A\|_{L_{T}^{q_2}(\dot{B}_{p,1}^{s})} \|\nabla B\|_{L_{T}^{q_2}(\dot{B}_{p,1}^{s})}. 
$$

(2.7)

Now we prove a useful estimate.

Lemma 2.4. Let $1 < p \leq N, 1 \leq \alpha \leq +\infty$, and $u$ be a solution of

$$
d - \text{div}(a \nabla u) = f, 
$$

(2.8)

where the diffusion coefficient $a(t, x) \in \dot{L}_{T}^{\infty}(\dot{B}_{p,1}^{N})$ and $0 < a < a(t, x) < \overline{a}$. Then the following estimate holds

$$
\|\nabla u\|_{L_{T}^{q}(\dot{B}_{p,1}^{s})} \leq C_{p}\|f\|_{L_{T}^{q}(\dot{B}_{p,1}^{-2})} + C_{p}\|\nabla a\|_{L_{T}^{q}(\dot{B}_{p,1}^{-2})} \|\nabla u\|_{L_{T}^{q}(\dot{B}_{p,1}^{s})}. 
$$

(2.9)

Proof. The proof of this lemma lies on standard energy estimates for dyadic blocks. Applying $\Delta_{j}$ to (2.8) and denoting $R_j := -\text{div}([a, \Delta_{j}]\nabla u)$, we obtain

$$
-d\text{div}(a\nabla \Delta_{j} u) = \Delta_{j} f + R_j. 
$$

Multiplying both sides of the above equation by $|\Delta_{j} u|^{p-2} \Delta_{j} u$ and integrating over $\mathbb{R}^{N}$, we have

$$
- \int_{\mathbb{R}^{N}} \text{div}(a\nabla \Delta_{j} u)|\Delta_{j} u|^{p-2} \Delta_{j} u \, dx = \int_{\mathbb{R}^{N}} (\Delta_{j} f + R_j)|\Delta_{j} u|^{p-2} \Delta_{j} u \, dx. 
$$

From [4], we have the following inequality

$$
C_{2}\left(\frac{p-1}{p^2}\right)2^{2j}\|\Delta_{j} u\|_{L^{p}}^{p} \leq - \int_{\mathbb{R}^{N}} \text{div}(a\nabla \Delta_{j} u)|\Delta_{j} u|^{p-2} \Delta_{j} u \, dx, 
$$

using the above inequality and Hölder inequality, we get

$$
C_{2}2^{2j}\|\Delta_{j} u\|_{L^{p}}^{p} \leq \|\Delta_{j} u\|_{L^{p}}^{p-1}(\|R_j\|_{L^{p}} + \|\Delta_{j} f\|_{L^{p}}). 
$$

By integration with respect to $t$ over $(0, T)$, we have

$$
C_{p}\|\nabla u\|_{L_{T}^{q}(\dot{B}_{p,1}^{s})} \leq \|f\|_{L_{T}^{q}(\dot{B}_{p,1}^{-2})} + \|\nabla a\|_{L_{T}^{q}(\dot{B}_{p,1}^{-2})} \|\nabla u\|_{L_{T}^{q}(\dot{B}_{p,1}^{s})}, 
$$

here we have used (2.1) and (2.7).
Remark 2.1. We also have an estimate for solutions of (2.8) in \( \dot{L}^p_T(\dot{B}^{\frac{N}{p}}_{p,\infty}) \), with the same assumptions as in Lemma 2.4, the following inequality holds:

\[
\| \nabla u \|_{\dot{L}^p_T(\dot{B}^{\frac{N}{p}}_{p,\infty})} \leq C_p \| f \|_{\dot{L}^p_T(\dot{B}^{\frac{N}{p}}_{p,\infty})} + C_p \| \nabla v \|_{\dot{L}^p_T(\dot{B}^{\frac{N}{p}}_{p,\infty})} \| \nabla u \|_{\dot{L}^p_T(\dot{B}^{\frac{N}{p}}_{p,\infty})}.
\] (2.10)

The proof goes along the lines of the proof of Lemma 2.4, and use the following inequality which can be found in [5],

\[
\sup_j 2^{-2j}\| \text{div} (\{a, \Delta_j \} \cdot \nabla u) \|_{L^p} \leq C \sup_j 2^{-2j}\| \nabla a \|_{\dot{B}^{\frac{N}{p}}_{p,\infty}} \| \nabla u \|_{\dot{B}^{\frac{N}{p}}_{p,\infty}}.
\]

We state the classical estimates in Besov space for the transport and heat equations, see [3].

**Proposition 2.1.** Let \( s \in (-1 - N \min(\frac{k}{p}, \frac{1}{p}), 1 + \frac{N}{p}) \), and \( 1 \leq p, r \leq +\infty \), and \( s = 1 + \frac{N}{p} \), if \( r = 1 \). Let \( u \) be a solenoidal vector such that \( \nabla v \in L^1_T(\dot{B}^{\frac{N}{p}}_{p,r} \cap L^\infty) \). Assume that \( u_0 \in \dot{B}^s_{p,r} \), \( g \in L^1_T(\dot{B}^s_{p,r}) \), and \( f \) solves

\[
\begin{cases}
\partial_t u + v \cdot \nabla u = g, \\
u|_{t=0} = u_0.
\end{cases}
\] (2.11)

Then for any \( t \in [0, T] \), we have

\[
\| u \|_{\dot{L}^p_T(\dot{B}^{s}_{p,r})} \leq C V(t) \left( \| u_0 \|_{\dot{B}^s_{p,r}} + \int_0^t e^{-CV(\tau)} \| g(\tau) \|_{\dot{B}^s_{p,r}} d\tau \right),
\]

where \( V(t) := \int_0^t \| \nabla v(\tau) \|_{\dot{B}^{s/p}_{p,r} \cap L^\infty} d\tau \). If \( r < +\infty \), then \( u \in C([0, T]; \dot{B}^s_{p,r}) \).

**Proposition 2.2.** Let \( s \in \mathbb{R} \) and \( 1 \leq k, p, r \leq +\infty \). Assume that \( u_0 \in \dot{B}^s_{p,r} \), \( f \in \dot{L}^k_T(\dot{B}^{s-k/2+\frac{N}{k}}_{p,r}) \), and \( u \) solves

\[
\begin{cases}
\partial_t u - \mu \Delta u = f, \\
u|_{t=0} = u_0.
\end{cases}
\] (2.12)

Denote \( 1/k_2 = 1 + 1/k_1 - 1/k \). Then there exist positive constants \( c \) and \( C = C(N) \) such that for all \( k_1 \in [k, +\infty] \), we have

\[
\| u \|_{\dot{L}^p_T(\dot{B}^{s-k/2+k_1}_{p,\infty})} \leq C \left\{ \sum_{q \in \mathbb{Z}} 2^{qs} \| \Delta q u_0 \|_{L^r} \left( \frac{1 - e^{-c\mu T 2^{s-k_2} k_1}}{c\mu k_1} \right)^{1/2} \\
\quad + \sum_{q \in \mathbb{Z}} 2^{q(s-2+2/k)} \| \Delta q f \|_{\dot{L}^p_T(\dot{L}^r)} \left( \frac{1 - e^{-c\mu T 2^{s-k_2} k_2}}{c\mu k_2} \right)^{1/2} \right\}.
\]

Moreover, there holds

\[
\mu \frac{1}{k_1} \| u \|_{\dot{L}^p_T(\dot{B}^{s-k/2+k_1}_{p,\infty})} \leq C \left( \| u_0 \|_{\dot{B}^{s}_{p,r}} + \mu \frac{1}{k_2} \| f \|_{\dot{L}^p_T(\dot{B}^{s-k/2+k_1}_{p,\infty})} \right).
\]

If \( r < +\infty \), then \( u \) belongs to \( C([0, T]; \dot{B}^s_{p,r}) \).

We also give an estimate for the linear hyperbolic and parabolic coupled system

\[
\begin{cases}
c_t + v \cdot \nabla c + \Lambda d = f, \\
d_t + v \cdot \nabla d - \mu \Delta d - \Lambda c = g,
\end{cases}
\] (2.13)

where \( \Lambda = (-\Delta)^{1/2} \).
Proposition 2.3. (see [6]) Let \((c, d)\) be a solution of (2.13) on \([0, T]\) with initial data \((c_0, d_0)\), \(1 - N/2 < s \leq 1 + N/2\) and \(V(t) = \int_0^t \|v\|_{\dot{B}_{\infty,p}^{s+1}}^p \, dt\). The following estimate holds on \([0, T]\)

\[
\|c(t)\|_{\dot{B}_{\infty,1}^s} + \|d(t)\|_{\dot{B}_{1,p}^{-s-1}} + \mu \int_0^t \left( \|c(\tau)\|_{\dot{B}_{\infty,1}^s} + \|d(\tau)\|_{\dot{B}_{1,p}^{-s-1}} \right) \, d\tau 
\leq C e^{CV(t)} \left( \|c_0\|_{\dot{B}_{\infty,1}^s} + \|d_0\|_{\dot{B}_{1,p}^{-s-1}} + \int_0^t e^{-CV(\tau)} \left( \|f(\tau)\|_{\dot{B}_{\infty,1}^s} + \|g(\tau)\|_{\dot{B}_{1,p}^{-s-1}} \right) \, d\tau \right),
\]

where \(C = C(N, s)\).

3 Well-posedness in critical spaces

For the system (1.6), when the initial density is small in \(\dot{B}_N^{\frac{N}{p}}\), the local well-posedness can be obtained by means of the following form of Schauder-Tychonoff fixed point argument. Furthermore, when initial data \((\sigma_0, v_0, H_0)\) is small under certain norm, we can obtain the global existence result.

Theorem 3.1. (Hukuhara) Let \(K\) be a convex subset of a locally convex topological linear space \(E\), and \(\Phi\) be a continuous self-mapping of \(K\). If \(\Phi(K)\) is contained in a compact subset of \(K\), then \(\Phi\) has a fixed point in \(K\).

Let us briefly enumerate the main steps of the proof: In the first step, we show the local existence problem amounts to find a fixed point for some map \(\Phi\). In the next two steps, we state various \(a \text{ priori}\) estimates for \(\Phi\). In the fourth step, we show that Hukuhara’s theorem indeed applies. Step five is devoted to the uniqueness. At last, when the initial data is small under certain norm, we give the proof of global existence.

Step 1. Construction of the functional \(\Phi\)

Define \(\Phi\) by \((\sigma, v, H) = \Phi(a, u, \xi)\), where \((\sigma, v, H)\) is the solution of the linear problem

\[
\begin{align*}
\sigma_t + u \cdot \nabla \sigma &= 0, \\
v_t - \mu \Delta v &= G - (a + 1)\nabla P, \\
H_t + u \cdot \nabla H &= \nabla u(\xi + I), \\
\text{div} v &= 0, \\
(\sigma, v, H)|_{t=0} &= (\sigma_0, v_0, H_0),
\end{align*}
\]

with \(\sigma = \sigma(a)\) and

\[
G := -u \cdot \nabla u + \mu a \Delta u + \text{div} \xi + \xi^T \cdot \nabla \xi.
\]

Step 2. \(a \text{ priori}\) estimates

We shall prove that for suitably small \(T\) and \(\|\sigma_0\|_{\dot{B}_p^{\frac{N}{p}}}\), the functional \(\Phi\) has a fixed point in the Banach space

\[
\hat{B}_T^p := \tilde{L}_T^\infty(\dot{B}_p^{\frac{N}{p}}) \times (\tilde{L}_T^\infty(\dot{B}_p^{\frac{N}{p}-1}) \cap \tilde{L}_T^1(\dot{B}_p^{\frac{N}{p}+1}))^N \times \tilde{L}_T^\infty(\dot{B}_p^{\frac{N}{p}})^N.
\]
Let $E_0 := \|v_0\|_{B^{s,1}_{p,1}} + \|H_0\|_{B^{s,1}_{p,1}}$ and $\|\sigma_0\|_{B^{s,1}_{p,1}} = R_0$. $C_0 > 0$ and $(R, \eta) \in (0,1)^2$ to be fixed hereafter, we denote

$$A = \{ (\sigma, v, H) \in \tilde{E}^p_T : \|\sigma\|_{L^p_T(B^{s,1}_{p,1})} \leq R, \|v\|_{L^p_T(B^{s+1}_{p,1})} + \|v\|_{L^p_T(B^{s,1}_{p,1})} \leq \eta,$$

$$\|v\|_{L^p_T(B^{s-1,1}_{p,1})} + \|H\|_{L^p_T(B^{s,1}_{p,1})} \leq C_0 E_0 \}.$$ 

We claim that if $T, R, R_0$ and $\eta$ are small enough, $\Phi$ maps $A$ to $A$.

In what follows, we assume $(a, u, \xi) \in A$ and denote $s := \frac{N}{p}$, $U(t) := \int_0^t \|\nabla u(\tau)\|_{B^{s,1}_{p,1}} \, dt$ for convenience. Using Proposition 2.1 we have

$$\|\sigma\|_{L^p_T(B^{s,1}_{p,1})} \leq C_{U(T)} \|\sigma_0\|_{B^{s,1}_{p,1}} \leq e^{C_R} R_0,$$  \hspace{1cm} (3.2)

and

$$\|H\|_{L^p_T(B^{s,1}_{p,1})} \leq e^{C_{U(T)}} \left( \|H_0\|_{B^{s,1}_{p,1}} + \|\nabla u(I + \xi)\|_{L^p_T(B^{s,1}_{p,1})} \right)$$

$$\leq e^{C_R} \left( \|H_0\|_{B^{s,1}_{p,1}} + \|\nabla u\|_{L^p_T(B^{s,1}_{p,1})} + C\|\xi\|_{L^p_T(B^{s,1}_{p,1})} \right)$$

$$\leq e^{C_R} \left( \|H_0\|_{B^{s,1}_{p,1}} + \eta + C\eta C_0 E_0 \right).$$ \hspace{1cm} (3.3)

Taking advantage of Proposition 2.1 and Lemma 2.2 we obtain

$$\|v\|_{L^\infty_T(B^{s-1}_{p,1})} \leq C \left( \|v_0\|_{B^{s-1,1}_{p,1}} + \|G\|_{L^1_T(B^{s-1}_{p,1})} \right.$$  

$$+ \|\nabla P\|_{L^1_T(B^{s-1}_{p,1})} + \|a\|_{L^1_T(B^{s-1}_{p,1})} \right)$$

$$\leq C \left( \|v_0\|_{B^{s-1,1}_{p,1}} + \|G\|_{L^1_T(B^{s-1}_{p,1})} \right.$$  

$$+ (1 + \|a\|_{L^\infty_T(B^{s,1}_{p,1})}) \|\nabla P\|_{L^1_T(B^{s-1}_{p,1})} \right).$$ \hspace{1cm} (3.4)

Applying div on both sides of (3.1) and by the incompressible condition $\text{div} \, v = 0$, we have

$$\text{div} ((a + 1)\nabla P) = \text{div} G.$$

Since $B^{N/p}_{p,1} \to B^{N/p}_{p,\infty} \cap L^\infty$, therefore $1 - R \leq 1 - \|a\|_{L^\infty_T(B^{s-1}_{p,1})} \leq 1 + a \leq 1 + R$.

Choosing $R < \frac{1}{2}$, then using Lemma 2.2 to the above equation, we get

$$\|\nabla P\|_{L^1_T(B^{s-1}_{p,1})} \leq C \|G\|_{L^1_T(B^{s-1}_{p,1})} + C R \|\nabla P\|_{L^1_T(B^{s-1}_{p,1})},$$ \hspace{1cm} (3.5)

here we have used Lemma 2.2. Choosing $R < 1/2$ so small that $CR < 1$, (3.5) becomes

$$\|\nabla P\|_{L^1_T(B^{s-1}_{p,1})} \leq C \|G\|_{L^1_T(B^{s-1}_{p,1})},$$ \hspace{1cm} (3.6)

Thus, combining (3.3) and (3.6) yield that

$$\|v\|_{L^\infty_T(B^{s-1}_{p,1})} \leq C \left[ \|v_0\|_{B^{s-1,1}_{p,1}} + \|G\|_{L^1_T(B^{s-1}_{p,1})} \right].$$ \hspace{1cm} (3.7)

By the definition of $G$ and Lemma 2.2 we infer that

$$\|G\|_{L^1_T(B^{s-1}_{p,1})} \leq C \left( \|u\|_{L^2_T(B^{s-1}_{p,1})} + \|a\|_{L^\infty_T(B^{s-1}_{p,1})} \right)$$

$$+ CT \|\xi\|_{L^1_T(B^{s,1}_{p,1})} (1 + \|\xi\|_{L^1_T(B^{s,1}_{p,1})}).$$ \hspace{1cm} (3.8)
Then (3.4) becomes
\[ \| \nabla P \|_{L_p^1(\tilde{B}_{p,1}^{-1})} \leq C[R\eta + \eta^2 + TC_0E_0(1 + C_0E_0)]. \]

Combining (3.3), (3.7) and (3.8), we have
\[ \| u \|_{L_p^2(\tilde{B}_{p,1}^{-1})} + \| H \|_{L_p^2(\tilde{B}_{p,1}^{-1})} \leq Ce^{C_0}\left[ E_0 + \eta(1 + C_0E_0) \right] + C(1 + R)[R\eta + \eta^2 + TC_0E_0(1 + C_0E_0)]. \]

By Proposition 2.2, we can obtain
\[ \| u \|_{L_p^2(\tilde{B}_{p,1}^{-1})} + \| H \|_{L_p^2(\tilde{B}_{p,1}^{-1})} \leq C \sum_{q \in \mathbb{Z}} 2^{q(s-1)}\| \Delta_q u_0 \|_{L_p^r(1 - e^{-c2^{qs}T})} + C\| G - (a + 1)\nabla P \|_{L_p^1(\tilde{B}_{p,1}^{-1})} \]
\[ \leq C \sum_{q \in \mathbb{Z}} 2^{q(s-1)}\| \Delta_q u_0 \|_{L_p^r(1 - e^{-c2^{qs}T})} + C(1 + R)[R\eta + \eta^2 + TC_0E_0(1 + C_0E_0)] \]

here we have used (3.6) and (3.8).

Taking \( C_0 = 6C, R_0 \leq \frac{\eta}{2}, \eta \) small such that \( e^{n\eta} \leq \frac{\eta}{2}, \eta(1 + C_0E_0) \leq E_0 \)
and choosing \( R < 1/2 \) small enough such that \( (1 + R)(R\eta + \eta^2) \leq \min\{E_0, \frac{\eta}{2}, \frac{\eta}{2}\} \).

Next, we choose \( T \) small such that \( CT(1 + R)(1 + C_0E_0) \leq \min\{\frac{1}{4}, \frac{\eta}{2}, \frac{\eta}{2}\} \) and
\[ C \sum_{q \in \mathbb{Z}} 2^{q(s-1)}\| \Delta_q u_0 \|_{L_p^r(1 - e^{-c2^{qs}T})} \leq \frac{\eta}{2}. \]

Then
\[ \| \sigma \|_{L_p^2(\tilde{B}_{p,1}^{s+1})} \leq R, \quad \| u \|_{L_p^2(\tilde{B}_{p,1}^{s+1})} + \| H \|_{L_p^2(\tilde{B}_{p,1}^{s+1})} \leq \eta, \]
\[ \| u \|_{L_p^2(\tilde{B}_{p,1}^{s+1})} + \| H \|_{L_p^2(\tilde{B}_{p,1}^{s+1})} \leq C_0E_0, \quad \| \nabla P \|_{L_p^1(\tilde{B}_{p,1}^{s+1})} < \infty. \]

Therefore, the functional \( \Phi \) maps \( \mathcal{A} \) to \( \mathcal{A} \).

Step 3. Time derivatives
The compactness of \( \Phi \) will be supplied by the following lemma.

**Lemma 3.1.** Denote \( (\sigma, u, H) := \Phi(a, u, \xi) \), let \( (a, u, \xi) \) be in \( \mathcal{A} \) with \( T, \eta, C_0, R_0 \)
and \( R \) chosen according to Step 2. Then \( \sigma_t, H_t \in L_p^2(\tilde{B}_{p,1}^{-1}) \) and \( u_t \in L_p^2(\tilde{B}_{p,1}^{-1} + \tilde{B}_{p,1}^{s+2+\alpha}) \) for any \( \alpha \in [-1, 1] \) such that \( \alpha > \max(2 - 2N/p, 2 - N) \). Moreover, there exists a constant \( \overline{C} \) depending on \( T, \eta, C_0, R_0, R \) and \( E_0 \) such that
\[ \| \sigma_t \|_{L_p^2(\tilde{B}_{p,1}^{-1})} + \| H_t \|_{L_p^2(\tilde{B}_{p,1}^{-1})} + \| u_t \|_{L_p^2(\tilde{B}_{p,1}^{s+1} + \tilde{B}_{p,1}^{s+2+\alpha})} \leq \overline{C}. \]

The similar proof can be found in [3,12], we omit here.

Step 4. The fixed point argument
We first introduce a functional space
\[ Y_T^\rho := \tilde{L}_p^\rho(\tilde{B}_{p,1}^{s+1}) \times (\tilde{L}_p^\rho(\tilde{B}_{p,1}^{-1}) \cap \tilde{L}_p^2(\tilde{B}_{p,1}^{s+1}) \cap \mathcal{M}_T(\tilde{B}_{p,1}^{s+1}))^N \times \tilde{L}_p^\rho(\tilde{B}_{p,1}^{s+1})^{N^2}, \]
where \( \mathcal{M}_T(\tilde{B}_{p,1}^{s+1}) \) stands for the space of bounded measures on \([0, T]\) with values in \( \tilde{B}_{p,1}^{s+1} \). The space \( Y_T^\rho \) endowed with the norm
\[ \| (a, u, \xi) \|_{Y_T^\rho} := \| (a, \xi) \|_{L_p^\rho(\tilde{B}_{p,1}^{s+1})} + \| u \|_{L_p^\rho(\tilde{B}_{p,1}^{s+1}) \cap L_p^2(\tilde{B}_{p,1}^{s+1})} + \int_0^T d\| u \|_{L_p^\rho(\tilde{B}_{p,1}^{s+1})} \]

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is a Banach space. Furthermore, $Y_p^\varphi$ is the dual space of
\[ X_p^\varphi := \tilde{L}_1^1(B_{p,1}^\infty) \times (\tilde{L}_p^1(B_{p,1}^\infty) + \tilde{L}_2^2(B_{p,1}^{2,q}) + C([0,T];B_{p,1}^{2,q-1}))^N \times \tilde{L}_1^1(B_{p,1}^\infty)^N, \]
where $\tilde{L}_1^1(B_{p,1}^\infty)$ stands for the completion of $\mathcal{A}(0,T;\mathbb{R}^N)$ under the norm of $\tilde{L}_1^1(B_{q,\infty})$, and that $X_p^\varphi$ is a separable Banach space.

Let $\mathcal{C}$ be as in Lemma 3.1. We denote
\[ D := \{(a,u,\xi) \in \mathcal{A}; \|(a_t,\xi_t)\|_{L_1^1(B_{p,1}^{3,1})} + \|u_t\|_{L_2^2(B_{p,1}^{3,1}+B_{p,1}^{3,2})} \leq \mathcal{C}\}. \]

Since $Y_p^\varphi$ is the dual space of a Banach space, we gather that $Y_p^\varphi$ endowed with the weak star topology is a convex topological linear space. Obviously, $\mathcal{A}$ is a convex subset of $Y_p^\varphi$. From Lemma 3.1 we know that $\Phi(\mathcal{A}) \subset \mathcal{D}$. Since $\mathcal{D} \subset \mathcal{A}$, it is clear that $\Phi$ is a self-mapping of $\mathcal{D}$. Just like the proof in \[3,12\], we can obtain the continuity and compactness of $\Phi$. Then Hukuhara’s theorem ensures that the map $\Phi$ has a fixed point $(\sigma,v,H) \in \mathcal{A}$, which is a solution of $\text{(1.6)}$.

Furthermore, we can check that the right hand sides of $\text{(1.6)}$ and $\text{(1.6)}$ belong to $L_1^1(B_{p,1}^\infty)$, then Proposition 2.1 and Proposition 2.2 insure $\sigma,H,v \in C([0,T];B_{p,1}^\infty)$.

Step 5. The proof of uniqueness

We assume that $(\sigma^i,v^i,H^i,\nabla P^i) \in \tilde{E}_p^\varphi \times \tilde{L}_1^1(B_{p,1}^{N-1})^N$ ($i = 1,2; p \in [1,N]$) are two solutions of $\text{(1.6)}$ with the same initial data. Set
\[ (\delta\sigma,\delta v,\delta H,\nabla \delta P) = (\sigma^1 - \sigma^2, v^1 - v^2, H^1 - H^2, \nabla P^1 - \nabla P^2). \]

Then $(\delta\sigma,\delta v,\delta H,\nabla \delta P)$ satisfies
\[ \begin{cases} 
\delta\sigma_t + v^2 \cdot \nabla \delta\sigma = \delta G_1, \\
\delta v_t - \mu \Delta \delta v = \delta G_2, \\
\delta H_t + v^2 \cdot \nabla \delta H = \delta G_3,
\end{cases} \quad (3.9) \]
where
\[ \begin{align*}
\delta G_1 & := -\delta v \nabla \sigma^1, \\
\delta G_2 & := \mu \sigma^1 \Delta u^1 - v^1 \cdot \nabla u^1 - (\sigma^1 + 1) \nabla P^1 + \text{div} H^1 + \text{div} (H^1 H^{1T}) - \mu \sigma^2 \Delta u^2 + v^2 \cdot \nabla u^2 - \text{div} H^2 - \text{div} (H^2 H^{2T}) + (\sigma^2 + 1) \nabla P^2, \\
\delta G_3 & := \nabla \delta u + \nabla v^1 \cdot H^1 - \nabla v^2 \cdot H^2 - \delta u \cdot \nabla H^1.
\end{align*} \]

In what follows, we set $V_i(t) := \int_0^t ||v^i(\tau)||_{B_{p,1}^{3,1}} d\tau$ for $i = 1,2$, and denote by $A_T$ a constant depending on $\|(\sigma^1,\sigma^2)\|_{L_1(T;B_{p,1}^{3,1})}$ and $\|H^1, H^2\|_{L_1(T;B_{p,1}^{3,1})}$. Since for any $p \in [1,N]$, $\tilde{E}_p^\varphi \subseteq \tilde{E}_p^N$. So we take $p = N$ in the sequel. Applying the Proposition 2.1 and Lemma 2.1 we get that for any $t \in [0,T]$,
\[ \|\delta\sigma(t)\|_{\tilde{B}_{N,\infty}}^N \leq e^{CV^2(T)} \int_0^t \|\delta v\|_{\tilde{B}_{N,\infty}}^N \|\sigma^1\|_{\tilde{B}_{N,\infty}}^N d\tau, \]
\[ \|\delta H(t)\|_{\tilde{B}_{N,\infty}}^N \leq e^{CV^2(T)} \int_0^t \left(\|v\|_{\tilde{B}_{N,\infty}}^N (1 + \|H^1\|_{\tilde{B}_{N,\infty}}^N) + \|v^2\|_{\tilde{B}_{N,\infty}}^N \|\delta H\|_{\tilde{B}_{N,\infty}}^N \right) d\tau. \]
By the above estimates and Gronwall’s inequality, we obtain
\[
\|\delta\sigma\|_{B^1_{N,\infty}} + \|\delta H\|_{B^0_{N,\infty}} \leq e^{CV^2(T)} \int_0^T \|\delta v\|_{L^1_t(B^1_{N,1})} (1 + \|\sigma^1\|_{B^1_{N,1}} + \|H^1\|_{B^1_{N,1}}) d\tau. \tag{3.10}
\]
Using Proposition 2.2, we know
\[
\|\delta v\|_{L^1_t(B^1_{N,1})} + \|\delta v\|_{L^1_t(B^0_{N,\infty})} \leq \|\delta G_2\|_{L^1_t(B^1_{N,1})}. \tag{3.11}
\]
Since \(\text{div} (\delta G_2) = \text{div} (\delta v_t - \mu \Delta \delta v) = 0\), thus
\[
\text{div} ((\sigma^1 + 1)\nabla \delta P) = \text{div} F,
\]
where
\[
0 < c_1 \leq \sigma^1 + 1 \leq c_2,
\]
\[
F = \mu \delta \sigma \Delta v^1 + \mu \sigma^2 \Delta \delta v - v^1 \nabla \delta v - \delta v \nabla v^2 - \delta \sigma \nabla P^2 + \text{div} \delta H + \text{div} (H^1 H^{1T} - H^2 H^{2T}).
\]
By using the estimate (2.10) to the above equation, we have
\[
\|\nabla \delta P\|_{B^{-1}_{N,\infty}} \leq C\|F\|_{B^1_{N,\infty}} + C\|\sigma^1\|_{B^1_{N,1}} \|\nabla \delta P\|_{B^{-1}_{N,\infty}}. \tag{3.12}
\]
Choosing \(C\|\sigma^1\|_{B^1_{N,1}} \leq \frac{1}{2}\), by the definition of \(F\), we obtain
\[
\|\delta G_2\|_{B^{-1}_{N,\infty}} \leq C\|\delta v\|_{B^0_{N,\infty}} (\|v^1\|_{B^1_{N,1}} + \|v^2\|_{B^1_{N,1}}) + C\|\sigma^2\|_{B^1_{N,1}} \|\delta v\|_{B^1_{N,\infty}} + \|\delta H\|_{B^0_{N,\infty}} + C\|\delta \sigma\|_{B^1_{N,1}} (\|v^1\|_{B^1_{N,1}} + \|\nabla P^2\|_{B^{-1}_{N,1}})
+ \|\text{div} (H^1 H^{1T} - H^2 H^{2T})\|_{B^{-1}_{N,\infty}}.
\]
We can take \(T \leq T\) small enough such that for any \(t \in [0, T]\),
\[
\|\nabla P\|_{L^1_t(B^1_{N,1})} + \|(v^1, v^2)\|_{L^1_t(B^3_{N,1}) \cap L^2_t(B^1_{N,1})} + \|(\sigma^1, \sigma^2)\|_{L^\infty_t(B^3_{N,1})} \ll 1.
\]
Therefore, (3.11) becomes
\[
\|\delta v\|_{L^1_t(B^1_{N,\infty})} \leq A_T \int_0^t \left( \|\delta \sigma\|_{B^0_{N,\infty}} (1 + \|v^1\|_{B^1_{N,1}} + \|\nabla P^2\|_{B^0_{N,\infty}}) + \|\delta H\|_{B^0_{N,\infty}} \right) d\tau. \tag{3.13}
\]
Using (2.6), we obtain
\[
\|\delta v\|_{L^1_t(B^1_{N,\infty})} \leq C \|\delta v\|_{L^1_t(B^1_{N,\infty})} \log \left( e + C_T \|\delta v\|_{L^1_t(B^1_{N,\infty})} \right),
\]
where \(C_T = \|\delta v\|_{L^1_t(B^0_{N,\infty})} + \|\delta v\|_{L^1_t(B^1_{N,\infty})}\). It yields that for any \(t \in [0, T]\),
\[
\|\delta v\|_{L^1_t(B^1_{N,\infty})} \leq A_T \int_0^t \left( \|\delta \sigma\|_{L^1_t(B^1_{N,\infty})} (1 + \|v^1\|_{B^1_{N,1}} + \|\nabla P^2\|_{B^1_{N,1}}) \log(e + C_T \|\delta v\|_{L^1_t(B^1_{N,\infty})}^{-1}) \right) d\tau. \tag{3.14}
\]
Since \( \int_0^1 \| \nabla P^2 \|_{B_{2,1}^1} \ll 1 \) for any \( t \in [0, T] \), \( 1 + \| v^1 \|_{\dot{B}_{3,2}^1} \) is integrable on \([0, T]\), and
\[
\frac{1}{r} \log(e + C_T r^{-1}) \, dr = +\infty,
\]
from Osgood lemma we know \((d \sigma, d \delta v, d H, \nabla \delta P) = (0, 0, 0, 0)\) on \([0, T]\). Using Continuity argument, we can prove \((\sigma^1, v^1, H^1, \nabla P^1) = (\sigma^2, v^2, H^2, \nabla P^2)\) on \([0, T]\).

Step 6. Global existence with small initial data.

Note that \( \| \cdot \|_{\dot{B}_{s,2}^0} \approx \| \cdot \|_{\dot{B}_{s-1}^{-1} \cap \dot{B}_s^1} \) and \( \| \cdot \|_{\dot{B}_{s+2}^0} = \| \cdot \|_{\dot{B}_s^1} \). The above steps ensure that there exists a positive time \( T_1 \) and a unique solution \((\sigma, v, H) \in \mathcal{H}^T_{T_1}, T \leq T_1\) such that
\[
\| (\sigma, H, v) \|_{\mathcal{H}^T_{T_1}} \leq C_1 (\| \sigma_0 \|_{\dot{B}_{s+2}^1} + \| H_0 \|_{\dot{B}_{s+2}^1} + \| v_0 \|_{\dot{B}_{s+2}^1}),
\]
(3.15)
where \( C_1 > 1 \), and
\[
| (\sigma, v, H) |_{\mathcal{H}^T_{T_1}} := \| (\sigma, H)(t) \|_{L_x^\infty(B_{s+2}^1)} + \| v(t) \|_{L_x^\infty(B_{s+2}^1)} + \mu \left( \| (\sigma, H)(t) \|_{L_x^1(B_{s+2}^1)} + \| v(t) \|_{L_x^1(B_{s+2}^1)} \right).
\]

We define \( d_{ij} = -\Lambda^{-1} \nabla_j v^i \), then \( v^i = \Lambda^{-1} \nabla_j d_{ij} \). By applying \(-\Lambda^{-1} \nabla_j \) to \((1.6)\), we get
\[
\partial_t d_{ij} - \mu \Delta d_{ij} + \Lambda^{-1}(\nabla_j \nabla_k H^{ij}) = \Lambda^{-1}(v \cdot \nabla v^i - H^{ik} \nabla_i H^{ik}).
\]
(3.16)
Equation \((1.5)\) implies
\[
\Lambda^{-1}(\nabla_j \nabla_k H^{ik}) = -\Lambda H^{ij} - \Lambda^{-1}(H^{ij} \nabla_i H^{ik} - H^{ik} \nabla_i H^{ij}).
\]
(3.17)
By plugging \((3.17)\) into \((3.16)\), from \((1.6)\) we get
\[
\begin{cases}
\sigma_t + v \cdot \nabla \sigma = 0, \\
\partial_t d_{ij} + v \cdot \nabla d_{ij} - \mu \Delta d_{ij} - \Lambda H^{ij} = G, \\
\partial_t H^{ij} + v \cdot \nabla H^{ij} + \Lambda d_{ij} = F,
\end{cases}
\]
(3.18)
where
\[
F = \nabla_k v^j H^{kj},
\]
\[
G = v \cdot \nabla(-\Lambda^{-1} \nabla_j v^i) + \Lambda^{-1} \nabla_j [v \cdot \nabla v^i + (\sigma + 1) P] - \mu \sigma \Delta v^i - H^{ik} \nabla_j H^{ik} + \Lambda^{-1} \nabla_k (H^{ij} \nabla_i H^{ik} - H^{ik} \nabla_i H^{ij}).
\]
We shall prove that there exists a constant \( M \geq C_1 \) such that, if the Besov norm of initial data \( \alpha := \| \sigma_0 \|_{\dot{B}_{s+2}^1} + \| v_0 \|_{\dot{B}_{s+2}^1} + \| H_0 \|_{\dot{B}_{s+2}^1} \) is small enough, then the solution of \((1.6)\) satisfies
\[
| (\sigma, v, H) |_{\mathcal{H}^{S/2}} \leq M \alpha
\]
for any \( T \in [0, +\infty) \).

Assume \((3.19)\) holds for \( t \in (0, \hat{T}] \). \((3.18)\) ensures that \( \hat{T} \) is bounded from blow by \( T_1 \) and there exists a constant \( T_2 > 0 \) such that \( | (\sigma, v, H) |_{\mathcal{H}^{S/2}} \leq M \alpha \) for any \( T \in [0, \hat{T} + T_2] \). Since
\[
\partial_t ((\sigma + 1) \partial_t P) = \partial_t ( -v \cdot \nabla v^i + \mu \sigma \Delta v^i + H^{jk} \partial_j H^{ik}),
\]
with \( c_1 \leq \sigma + 1 \leq c_2 \),
using Lemma 2.21, we obtain
\[ \| \nabla P \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C \| v \|^2_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} + C \mu \| \sigma \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \| v \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})}
+ C \| H \|^2_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} + C \| v \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \| \nabla P \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})}
\leq C(C_1 M \alpha)^2 + C C_1 M \| \nabla P \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})}, \]

choose \( \alpha \) so small that \( C C_1 M \alpha \leq 1/2 \), we have
\[ \| \nabla P \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C(C_1 M \alpha)^2. \]

Now, we estimate \( \| F \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \), since
\[ \| F \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C \| H \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \| v \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C(C_1 M \alpha)^2. \]

For the term of \( G \), using (3.20) we have
\[ \| G \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C \| \sigma \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} (\| \nabla P \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})})
+ C \| v \|^2_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} + C \| H \|^2_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} \leq C(C_1 M \alpha)^2. \]

For the equation (1.6), according to Proposition 2.3, we have
\[ \| (\sigma, v, H) \|_{H^{\frac{N}{2}}_t} \leq C e^{C\| v \|_{L^2_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})}^2} (\| \sigma_0 \|_{\dot{B}^{\frac{N}{2}}_{\infty, \infty}} + \| v_0 \|_{\dot{B}^{\frac{N}{2}}_{\infty, \infty}}
+ \| H_0 \|_{\dot{B}^{\frac{N}{2}}_{\infty, \infty}} + \| F \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})} + \| G \|_{L^1_t(\dot{B}^{\frac{N}{2}}_{\infty, \infty})}), \]
for \( T \in [0, \tilde{T} + T_2] \). Thus, we obtain
\[ \| (\sigma, v, H, \nabla P) \|_{H^{\frac{N}{2}}_t} \leq C e^{C C_1 M \alpha (\alpha + C_1^2 M^2 \alpha^2)}, \]
so choosing \( M = \max(4C, C_1) \) and making the following assumptions:
\[ C_1^2 M^2 \alpha \leq 1, \quad e^{C C_1 M \alpha} \leq 2, \quad C C_1 M \alpha < \frac{1}{2}, \]
then (3.19) holds for \( T \in [0, \tilde{T} + T_2] \), hence for any \( T \in [0, +\infty) \), it is followed from a bootstrap argument. The proof is complete.

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Corresponding author: Huazhao Xie,
Department of Mathematics, Henan University of Economics and Law, Zhengzhou, 450002, China
E-mail: hzh_xie@yahoo.com.cn

Yunxia Fu,
Department of Basic Courses, PLA Commanding Communications Academy, Wuhan 430010, China
E-mail: fuyunxia2005@yahoo.com.cn