ON THE AXIOM OF PLANES AND THE AXIOM OF SPHERES IN THE ALMOST HERMITIAN GEOMETRY

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We prove analogues of Cartan’s criterion for constancy of sectional curvature to an arbitrary almost Hermitian manifold. As a consequence we establish for such a manifold analogues of a Cartan’s theorem. Our results generalize some theorems in [2; 5; 7, 8].

1. Introduction. Let \( N \) be an \( n \)-dimensional submanifold of an \( m \)-dimensional Riemannian manifold \( M \) with Riemannian metric \( g \) and let \( \nabla \) and \( \tilde{\nabla} \) be the Levi-Civita connections on \( M \) and \( N \), respectively. It is well known, that the equation

\[
\alpha(x, y) = \tilde{\nabla}_x y - \nabla_x y,
\]

where \( x, y \in \mathfrak{X}N \), defines a normal-bundle-valued symmetric tensor field, called the second fundamental form of the immersion. The submanifold \( N \) is said to be totally umbilical, if

\[
\alpha(x, y) = g(x, y)H
\]

for all \( x, y \in \mathfrak{X}N \), where \( H = (1/n)\text{trace } \alpha \) is the mean curvature vector of \( N \) in \( M \). In particular, if \( \alpha \) vanishes identically, \( N \) is called a totally geodesic submanifold of \( M \).

For \( x \in \mathfrak{X}N, \xi \in \mathfrak{X}N^\perp \) we write \( \tilde{\nabla}_x \xi = -A_x \xi + D_x \xi \), where \( -A_x \xi \) (respectively, \( D_x \xi \)) denotes the tangential (respectively, the normal) component of \( \tilde{\nabla}_x \xi \). A normal vector field \( \xi \) is said to be parallel, if \( D_x \xi = 0 \) for each \( x \in \mathfrak{X}N \).

The manifold \( M \) is said to satisfy the axiom of \( n \)-planes (respectively, \( n \)-spheres), where \( n \) is a fixed integer \( 2 \leq n < m \) if for each point \( p \in M \) and for any \( n \)-dimensional subspace \( \alpha \) of \( T_pM \) there exists an \( n \)-dimensional totally geodesic submanifold \( N \) (respectively an \( n \)-dimensional totally umbilical submanifold \( N \) with non-zero parallel mean curvature vector) containing \( p \), such that \( T_pN = \alpha \).

In his book on Riemannian geometry [1] E. Cartan proved the following theorem.

Theorem. Let \( M \) be an \( m \)-dimensional Riemannian manifold, \( m > 2 \), which satisfies the axiom of planes for some \( n, 2 \leq n < m \). Then \( M \) has constant sectional curvature.

In [4] Leung and Nomizu have substituted the axiom of \( n \)-planes with the axiom of \( n \)-spheres and have proved a generalization of the above mentioned Cartan’s theorem.

Analogous results for Kaehler manifolds have been proved in [2; 5; 8] and in [7] it has been studied a similar problem for some almost Hermitian manifolds.

2. Preliminaries. Let \( M \) be an \( m \)-dimensional Riemannian manifold with Riemannian metric \( g \) and let \( \nabla \) be its Levi-Civita connection. The curvature tensor \( R \) associated with \( \nabla \) has the following properties:

1) \( R(X, Y) = -R(Y, X) \) for \( X, Y \in T_pM \),
2) \( R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \) for \( X, Y, Z \in T_pM \),

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3) $R(X, Y, Z, U) = -R(X, Y, U, Z)$ for $X, Y, Z, U \in T_pM$, where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$.

The curvature of a two dimensional plane in $T_pM$ with an orthonormal basis $X, Y$ is defined by $K(X, Y) = R(X, Y, Y, X)$.

It is easy to compute, that if $N$ is a totally geodesic submanifold of $M$ or a totally umbilical submanifold of $M$ with parallel mean curvature vector, then $R(x, y, z, \xi) = 0$ for all vectors $x, y, z \in T_pM, \xi \perp T_pM$ and for each point $p \in N$.

Now, let $M$ be a $2m$-dimensional almost Hermitian manifold with Riemannian metric $g$ and almost complex structure $J$.

A subspace $\alpha$ in $T_pM$ is said to be holomorphic (respectively, antiholomorphic or totally real) if $J\alpha = \alpha$ (respectively $J\alpha \perp \alpha$). For the dimension $k$ of a holomorphic (respectively, antiholomorphic) subspace $\alpha$ of $T_pM$ we have $k = 2n, 1 \leq n \leq m$ (respectively, $1 \leq k \leq m$). If the holomorphic (respectively, antiholomorphic) sectional curvature in each point $p \in M$, i.e. the curvature of a holomorphic (respectively, antiholomorphic) 2-dimensional subspace $\alpha$ of $T_pM$ does not depend on $\alpha$, then $M$ is said to be of pointwise constant holomorphic (respectively, antiholomorphic) sectional curvature.

A connected Riemannian (respectively, Kaehler) manifold of global constant sectional curvature (respectively, of constant holomorphic sectional curvature) is called a real-space-form (respectively, a complex-space-form).

An almost Hermitian manifold is said to be an $RK$-manifold, if $R(X, Y, Z, U) = R(JX, JY, JZ, JU)$ for all $X, Y, Z, U \in T_pM, p \in M$.

For a two dimensional subspace $\alpha$ of $T_pM$ with an orthonormal basis $X, Y$ the angle $\theta \in [0, \pi/2]$ between $\alpha$ and $J\alpha$ is defined by $\cos \theta = |g(X, JY)|$.

We shall need the following theorems:

**Theorem A** [3]. Let $M$ be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$, and let $T : (T_pM)^4 \rightarrow \mathbb{R}$ be a four-linear mapping, which satisfies the conditions:

1) for all $X, Y, Z, U \in T_pM$

\[ T(X, Y, Z, U) = -T(Y, X, Z, U) , \]
\[ T(X, Y, Z, U) + T(Y, Z, X, U) + T(Z, X, Y, U) = 0 , \]
\[ T(X, Y, Z, U) = -T(X, Y, U, Z) ; \]

2) $T(X, Y, Y, X) = 0$, where $X, Y$ is a basis of an arbitrary two dimensional subspace $\alpha$ in $T_pM$, for which the angle between $\alpha$ and $J\alpha$ is one of the numbers $0, \pi/4, \pi/2$. Then $T = 0$.

Let for all $X, Y, Z, U \in T_pM$

\[ R_1(X, Y, Z, U) = g(X, U)g(Y, Z) - g(X, Z)g(Y, U) , \]
\[ R_2(X, Y, Z, U) = g(X, JU)g(Y, JZ) - g(X, JZ)g(Y, JU) - 2g(X, JY)g(Z, JU) . \]

**Theorem B** [3]. If $M$ is a $2m$-dimensional $RK$-manifold, $m \geq 2$, with pointwise constant holomorphic sectional curvature $c$ and with pointwise constant antiholomorphic sectional curvature $K$, then the curvature tensor $R$ has the form

\[(2.1) \quad R = KR_1 + (c - K)R_2/3 . \]
As is proved in [6], if the curvature tensor of a 2m-dimensional connected almost Hermitian manifold has the form (2.1) and if \( m \geq 3 \), then \( c \) and \( K \) are global constants. On the other hand, it is proved in [3], that if the curvature tensor of an almost Hermitian manifold \( M \) of dimension \( 2m \geq 4 \) has the form (2.1) with global constants \( c \) and \( K \), then either \( M \) is of constant holomorphic sectional curvature \( c = K \) or \( M \) is a Kaehler manifold of constant holomorphic sectional curvature. Hence we have:

**Theorem C.** Let \( M \) be a connected RK-manifold of dimension \( 2m \geq 6 \). If \( M \) has pointwise constant holomorphic sectional curvature and pointwise constant antiholomorphic sectional curvature, then \( M \) is one of the following:

1) a real-space-form;
2) a complex-space-form.

### 3. Criteria for constancy of the holomorphic and the antiholomorphic curvature at one point.

**Lemma 1.** Let \( M \) be an almost Hermitian manifold with dimension \( 2m, m \geq 2 \) and for a point \( p \in M \)

\[(3.1) \quad R(X, JX, JX, Y) = 0 \]

holds for all \( X, Y \in T_p M \), with \( g(X, Y) = g(X, JY) = 0 \). Then \( M \) has constant holomorphic sectional curvature at \( p \) and

\[(3.2) \quad R(X, Y, Y, X) = R(JX, JY, JY, JX) , \]

where \( X, Y \) are as above.

**Proof.** Taking two arbitrary unit vectors \( X, Y \) in \( T_p M \) with \( g(X, Y) = g(X, JY) = 0 \) and applying (3.1) for the vectors \( X + \alpha Y, \alpha X - Y \), we obtain

\[(3.3) \quad H(X) - \alpha^2 H(Y) + (\alpha^2 - 1)R(X, JX, JY, Y) + (\alpha^2 - 1)R(X, JY, JX, Y) + \alpha^2 K(X, JY) - K(JX, Y) = 0 , \]

where \( H(X) = R(X, JX, JX, X) \) denotes the holomorphic sectional curvature, determined by \( X \).

Let \( \alpha = 1 \):

\[(3.4) \quad H(X) - H(Y) + K(X, JY) - K(JX, Y) = 0 . \]

From (3.3) and (3.4) it follows

\[(3.5) \quad H(Y) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(X, JY) . \]

Analogously \( H(X) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(JX, Y) \).

Substituting \( X \) by \( JX \) and \( Y \) by \( JY \) we get

\[(3.6) \quad H(X) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(X, JY) . \]

From (3.5) and (3.6) we see that

\[(3.7) \quad H(X) = H(Y) \]

and combining this with (3.4) we find (3.2).
Let \( m > 2 \) and \( U, V \) be arbitrary unit vectors in \( T_pM \). We choose \( X \) in span\{\( U, JU \)\} \( \cap \) \( \text{span}\{\( V, JV \)\} \). According to (3.7) we have \( H(U) = H(X) = H(V) \) and the Lemma is proved in the case \( m > 2 \). In the case \( m = 2 \) we put \( c = H(X) = H(Y) \) and using (3.6) we see that \( H(\alpha X + \beta Y) = c \), where \( \alpha^2 + \beta^2 = 1 \). Hence it is not difficult to find that the holomorphic sectional curvature in \( p \) is a constant.

The following lemma is trivial.

**Lemma 2.** Let \( \alpha \) be a two dimensional subspace in \( T_pM \) such that the angle between \( \alpha \) and \( J\alpha \) is \( \pi/4 \). Then \( \alpha \) has an orthonormal basis \( X, (JX + U)/\sqrt{2} \), where \( X, U \) are unit vectors in \( T_pM \) with \( g(X, U) = g(X, JU) = 0 \).

**Lemma 3.** Let \( M \) be a \( 2m \)-dimensional almost Hermitian manifold, \( m \geq 2 \) and for each point \( p \in M \) (3.1) holds for all \( X, Y, Z \in T_pM \) with \( g(X, Y) = g(X, JY) = 0 \). Then \( M \) is an RK-manifold.

**Proof.** We put \( T(X, Y, Z, U) = R(X, Y, Z, U) - R(JX, JY, JZ, JU) \) for all \( X, Y, Z, U \in T_pM \). Obviously \( T \) has the property (1) of Theorem A. Let \( \alpha \) be a subspace in \( T_pM \) such that the angle between \( \alpha \) and \( J\alpha \) is \( \theta \). If \( \theta = 0 \), \( \alpha \) is a holomorphic plane and if \( \alpha = \text{span}\{X, JX\} \) we have \( T(X, JX, JX, X) = 0 \). If \( \alpha = \pi/2 \), \( \alpha \) is an antiholomorphic plane and we can choose two vectors \( X, Y \in T_pM \) such that \( \alpha = \text{span}\{X, JY\} \), \( g(X, Y) = g(X, JY) = 0 \). According to Lemma 1 we have \( T(X, Y, X, X) = 0 \). Let \( \theta = \pi/4 \) and let \( X, (JX + U)/\sqrt{2} \) be an orthonormal basis of \( \alpha \) as in Lemma 2. Then \( T(X, JX + U, JX + U, X) = 0 \). According to Theorem A we have \( T = 0 \), which proves our assertion.

**Lemma 4.** Let \( M \) be a \( 2m \)-dimensional almost Hermitian manifold, \( m \geq 2 \) and for a point \( p \in M \)

\[
(3.8) \quad R(X, Y, Y, Z) = 0
\]

holds for all \( X, Y, Z \in T_pM \) with \( g(X, Y) = g(X, JY) = g(X, Z) = g(Y, Z) = 0 \). Then \( M \) has constant antiholomorphic sectional curvature at \( p \).

**Proof.** According to Lemma 1 \( M \) has constant holomorphic sectional curvature \( c \) at \( p \). We apply (3.8) for the vectors \( X + JX, Y, JX - X \) where \( X, Y \in T_pM \) are arbitrary unit vectors with \( g(X, Y) = g(X, JY) = 0 \) and we get \( K(JX, Y) = K(X, Y) \).

As in the proof of Lemma 1 we have

\[
(3.9) \quad H(X) = R(X, JX, JY, Y) + R(X, JY, JX, Y) + K(X, JY) \nonumber \n
\]

Hence applying the first Bianchi’s identity we obtain

\[
(3.10) \quad H(X) = 2R(X, JX, JY, Y) + R(JX, JY, X, Y) + K(X, JY) \nonumber \n
\]

The substitution of \( Y \) with \( JY \) in (3.9) gives

\[
H(X) = R(X, JX, JY, Y) - R(X, JY, JX, Y) + K(X, Y) \nonumber \n
\]

Combining this with (3.10) we derive

\[
(3.11) \quad 2H(X) = 3R(X, JX, JY, Y) + K(X, Y) + K(X, JY) \nonumber \n
\]

Let \( m = 2 \). We put \( K = K(X, Y) \) and from (3.9), (3.10), (3.11) we have \( R(X, JX, JY, Y) = \frac{2}{3}(c - K) \); \( R(X, JY, JX, Y) = \frac{1}{3}(c - K) \); \( R(JX, JY, X, Y) = \frac{1}{3}(K - c) \).
We put $R' = KR_1 + \frac{c}{3} R_2$. A simple calculation shows that $R(X_1, X_2, X_3, X_4) = R'(X_1, X_2, X_3, X_4)$, whenever $X_1, X_2, X_3, X_4$ are chosen among the vectors $X, JX, JY, Y$. Consequently $R = R'$ and the Lemma is proved in the case $m = 2$.

Let $m > 2$. We choose a unit vector $Z$, normal to $X, JX, JY, Y$. Because of (3.8), from $R(X + Z, Y, Y, X - Z) = 0$ we get

\begin{equation}
K(X, Y) = K(Y, Z).
\end{equation}

Let $m = 3$. We shall show that

\begin{equation}
R(X, JX, Y, Z) = R(X, Y, Z, JX) = 0
\end{equation}

and the case $m = 3$ will follow as the case $m = 2$. From $R(\alpha X + JZ, \alpha JX - Z, \alpha JX - Z, Y) = 0$, where $\alpha$ takes the values 1 and $-1$, we find

\begin{equation}
R(X, JX, Z, Y) + R(X, Z, JX, Y) = 0
\end{equation}

and from $R(Y, X + JX, X + JX, Z) = 0$ it follows

\begin{equation}
R(X, Y, Z, JX) + R(X, Z, Y, JX) = 0.
\end{equation}

Using (3.14), (3.15) and the properties of the curvature tensor we get $R(X, Y, Z, JX) = 0$ and together with (3.14) this gives (3.13).

Now let $m > 3$. We take arbitrary antiholomorphic spaces $\alpha, \beta$ in $T_pM$ with orthonormal basis $X, Y$ and $Z, U$ respectively such that $X \perp Y, JY$ and $Z \perp U, JU$. Let $V, W$ be unit vectors in $\text{span}\{X, JX\} \perp \text{span}\{Z, JZ\} \perp \text{span}\{U, JU\}$ and $V \perp W, JW$. According to (3.12)

\begin{equation}
K(X, V) = K(V, W) = K(V, Z).
\end{equation}

Let $A \in \text{span}\{V, JV\} \cap \text{span}\{Z, JZ\} \cap \text{span}\{U, JU\}$ be a unit vector. From (3.12)

\begin{equation}
K(V, Z) = K(Z, A) = K(Z, U).
\end{equation}

Analogously

\begin{equation}
K(X, V) = K(X, Y).
\end{equation}

From (3.16), (3.17), (3.18) it follows $K(X, Y) = K(Z, U)$ and the Lemma is proved.

4. The main results. Let $M$ be a $2m$-dimensional almost Hermitian manifold, $m \geq 0$.

Axiom of holomorphic $2n$-planes (respectively, $2n$-spheres). For each point $p \in M$ and for any $2n$-dimensional holomorphic subspace $\alpha$ of $T_pM$ there exists a totally geodesic submanifold $N$ (respectively, a totally umbilical submanifold $N$ with nonzero parallel mean curvature vector) containing $p$ such that $T_pN = \alpha$.

Axiom of antiholomorphic $n$-planes (respectively, $n$-spheres). For each point $p \in M$ and for any $n$-dimensional antiholomorphic subspace $\alpha$ of $T_pM$ there exists a totally geodesic submanifold $N$ (respectively, a totally umbilical submanifold $N$ with nonzero parallel mean curvature vector) containing $p$ such that $T_pN = \alpha$.

Theorem 1. Let $M$ be a $2m$-dimensional almost Hermitian manifold, $m \geq 2$. If $M$ satisfies the axiom of holomorphic $2n$-planes or the axiom of holomorphic $2n$-spheres for
some \( n, 1 \leq n < m \), then \( M \) is an RK-manifold with pointwise constant holomorphic sectional curvature.

Proof. The condition gives \( R(X, JX, JX, Y) = 0 \) for all vectors \( X, Y \in T_pM \) with \( g(X, Y) = g(X, JY) = 0 \) and for each point \( p \in M \) and the Theorem follows from Lemma 1 and Lemma 3.

Theorem 2. Let \( M \) be a \( 2m \)-dimensional almost Hermitian manifold, \( m \geq 2 \). If \( M \) satisfies the axiom of antiholomorphic \( n \)-planes or the axiom of antiholomorphic \( n \)-spheres for some \( n, 2 \leq n \leq m \), then \( M \) is an RK-manifold with pointwise constant holomorphic sectional curvature and with pointwise constant antiholomorphic sectional curvature and consequently the curvature tensor has the form (2.1).

Proof. By the condition it follows \( R(X, Y, Y, Z) = 0 \) for each point \( p \in M \) and for all \( X, Y, Z \in T_pM \) with \( g(X, Z) = g(Y, Z) = (X, Y) = g(X, JY) = 0 \). Now the Theorem follows from Lemmas 1, 3 and 4.

By Theorem C and Theorem 2 we derive

Theorem 3. Let \( M \) be a \( 2m \)-dimensional connected almost Hermitian manifold, \( m \geq 3 \). If \( M \) satisfies the axiom of antiholomorphic \( n \)-planes or the axiom of antiholomorphic \( n \)-spheres for some \( n, 2 \leq n \leq m \), then \( M \) is one of the following:

1) a real-space-form;
2) a complex-space-form.

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