A NECESSARY CONDITION FOR THE EXISTENCE OF GLOBAL-IN-TIME SOLUTIONS TO NAVIER-STOKES EQUATIONS

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Abstract. A necessary condition for the existence of global-in-time smooth solutions to 3D Navier-Stokes equations is studied in this paper.

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1. Introduction

This paper studies the existence and smoothness of solutions to the incompressible Navier-Stokes equations:

\[
\begin{aligned}
\partial_t u &= \Delta u - (u \cdot \nabla) u - \nabla p + f, \quad t \in [0, +\infty), \ x \in \Omega \subseteq \mathbb{R}^3, \\
\qquad u &\in L^2_\sigma(\Omega), \\
\qquad u(., 0) &= u_0.
\end{aligned}
\]  

(NS)

The prize problem \([3]\) of Clay Institute ask a proof of the existence of smooth functions \( u(x, t) \in (C^\infty(\mathbb{R}^3 \times [0, +\infty)))^3 \), \( p(x, t) \in C^\infty(\mathbb{R}^3 \times [0, +\infty)) \) satisfying the Navier-Stokes equations (NS). The solution are required to be not only global in time but also smooth at \( t = 0 \).

The regularity of weak/mild/strong solutions on the whole \( \mathbb{R}^3 \) was studied by Leray \([16]\) (1934), Foias \([6]\) (1989), Sohr \([23]\) (2001), Grujić \([15]\) (1998), Guberović \([9]\) (2010), Lemarié-Rieusset \([14]\) (2016) and Dong \([4]\) (2007), ...But, the smoothness in these results do not include at \( t = 0 \).

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The regularity of weak/strong solutions on bounded space domains was studied by Serrin [26] (1962), Kaniel (1967), Giga [27] (1986), Sohr [23] (2001). But, the smoothness in these results do not include at \( t = 0 \). In order to prove the smoothness of \( u \) at \( t = 0 \), Temam [25] (1995), Boyer [5] (2013) and Robinson [20] (2016) must use an additional initial condition, which is

\[
\forall i \in \mathbb{N}, u^{(i)}_0|_{\partial \Omega} = 0, \tag{1.0.2}
\]

where \( u^{(i)}_0 \) is defined at (2.0.11) in Definition 2.0.6.

The aim of this paper is to show that the necessary condition for solutions to the Navier-Stokes system (NS) satisfying \( u \in (C^\infty(\mathbb{R}^3 \times [0, +\infty))^3 \) is that

\[
\forall n \in \mathbb{N}, \forall v_n \in P_n L^2_\sigma(\Omega), \int_{\Omega} (v_n \cdot \nabla)v_n \cdot Av_n \, dx = 0, \tag{1.0.3}
\]

where \( \Omega \) is any bounded domain with \( C^2 \) boundary, \( A \) is the Stoke operator, \( P_n L^2_\sigma(\Omega) = \text{span} \{a_1, a_2, \cdots, a_n\} \) as in Lemma 2.0.4.

The paper is organized as follows:

- In the section 2, some preliminaries and notations are given.
- Section 3 studies the existence and smoothness of solutions to the problem (S) under the necessary condition (1.0.3):

\[
(S) \quad \begin{cases}
\partial_t \tilde{v} - \Delta \tilde{v} = -(\tilde{v} \cdot \nabla)\tilde{v} - (\tilde{\beta} \cdot \nabla)\tilde{\beta} - \nabla \tilde{p} + \tilde{\theta}, \\
\text{Domain} : (x, t) \in \Omega \times [0, +\infty), \text{where } \Omega \text{ is bounded}, \\
\text{Unknown} : (\tilde{v}, \tilde{\beta}), \text{where } \tilde{v}(., t) \in V(\Omega), \\
\text{Initial condition} : \tilde{v}(., 0) = 0.
\end{cases}
\]

The result in this section will be used in sections 4, 5.
- The section 4 studies the existence and smoothness of solutions to the Navier-Stokes equations (1.0.1) on bounded domains under the necessary condition (1.0.3).
- The section 5 studies the existence and smoothness of solutions to the Navier-Stokes equations (1.0.1) on the whole space under the necessary condition (1.0.3).

2. Notations and Preliminaries

- We will use \( \alpha \leq \gamma \) to mean that \( \alpha \leq c\gamma \) for some constant \( c \).
- \( u \cdot \nabla v := (u \cdot \nabla)v \).
- \( \partial_i^\ell v \) often is written by \( v^{(i)} \), where \( \partial_i^\ell v = \frac{\partial^\ell v}{\partial t^\ell} \).
- \( C^i_B(\Omega) := \{u \in C(\Omega) : D^\alpha u \text{ is bounded on } \Omega \text{ for } |\alpha| \leq i\} \).
- \( \mathbb{L}^k(\Omega) := (H^k(\Omega))^3; \mathbb{L}^p(\Omega) := (L^p(\Omega))^3. \)
- \( \|\cdot\|_{\mathbb{L}^p} := \|\cdot\|_{L^p(\Omega)} \).
- For \( \Omega \subseteq \mathbb{R}^3, V(\Omega) := \{u \in (H^1(\Omega))^3 : \text{div} u = 0\} \).
- For a bounded domain \( \Omega \subseteq \mathbb{R}^3 \),

\[
L^2_\sigma(\Omega) = \{v \in \mathbb{L}^2(\Omega) : \text{div} v = 0, v \cdot \tilde{n}|_{\partial \Omega} = 0\}. \tag{2.0.1}
\]

- \( L^2_\sigma(\mathbb{R}^3) = \{v \in L^2(\mathbb{R}^3) : \text{div} v = 0\} \).
- \( b(u, v, w) := \int_{\Omega} (u \cdot \nabla)v \cdot w \, dx \).
- \( P_n \) is a projection onto a finite-dimensional subspace as in Lemma 2.0.4.
Lemma 2.0.1. (Gagliardo-Nirenberg’s inequality)
Let $\Omega$ be a $C^3$-domain of $\mathbb{R}^3$ with compact boundary. Let $q \in [2, 6]$. Then, we have

1. There exists a constant $C_1(\Omega)$ such that
   $$\|u\|_{L^q(\Omega)} \leq C_1 \|u\|^{3/q-1/2}_{L^2(\Omega)} \|\nabla u\|^{3/2-3/q}_{L^2(\Omega)}, \forall r \geq 1, \forall u \in V(r\Omega).$$  \hfill (2.0.2)

2. There exists a constant $C_2(\Omega)$ such that
   $$\|\nabla u\|_{L^q(\Omega)} \leq C_2 \|\nabla u\|^{3/q-1/2}_{L^2(\Omega)} \|Au\|^{3/2-3/q}_{L^2(\Omega)}, \forall r \geq 1, \forall u \in V(r\Omega),$$
   where $A$ is the Stokes operator.

3. There exists a constant $C_3(\Omega)$ such that
   $$\|u\|_{L^q(\Omega)} \leq C_3 \|u\|^{3/q-1/2}_{L^2(\Omega)} \|\nabla u\|^{3/2-3/q}_{L^2(\Omega)} + C_3 \|u\|_{L^2(\Omega)}, \forall u \in H^2(\Omega).$$  \hfill (2.0.4)

Proof.

1. By Boyer & Fabrie [5] Proposition III.2.35, Remark III.2.17, there exists $C_1(\Omega)$ such that for every $\tilde{u} \in V(\Omega)$
   $$\|\tilde{u}\|_{L^q(\Omega)} \leq C_1 \|\tilde{u}\|^{3/q-1/2}_{L^2(\Omega)} \|\nabla \tilde{u}\|^{3/2-3/q}_{L^2(\Omega)}.$$  \hfill (2.0.5)
   Let $r > 1$ and $u \in V(r\Omega)$. Denote $\tilde{u}(x) := u(rx)$ for $x \in \Omega$. Then, $\tilde{u}$ satisfies \hfill (2.0.5). Besides, $\|\tilde{u}\|_{L^q(\Omega)} = r^{-3/p} \|u\|_{L^q(r\Omega)}$ and $\|\nabla \tilde{u}\|_{L^2(\Omega)} = r^{-3/(p+1)} \|\nabla u\|_{L^2(\Omega)}$. From this property and (2.0.5), it deduces
   $$r^{-3/q} \|u\|_{L^q(\Omega)} \leq C_1 r^{-3/(2p+1)} \|u\|^{3/q-1/2}_{L^2(\Omega)} \|\nabla u\|^{3/2-3/q}_{L^2(\Omega)}.$$  \hfill (2.0.6)

   It deduces the assertion (1).

2. By Boyer & Fabrie [5] Proposition III.2.35, there exists $C_2(\Omega)$ such that
   $$\forall \tilde{u} \in V(\Omega) \cap H^2(\Omega), \|\nabla \tilde{u}\|_{L^q(\Omega)} \leq C_2 \|\tilde{u}\|^{3/q-1/2}_{L^2(\Omega)} \|\tilde{u}\|^{3/2-3/q}_{H^2(\Omega)}.$$  \hfill (2.0.7)
   Besides, $\|\tilde{u}\|_{H^2(\Omega)} \leq C_3 \|A\tilde{u}\|_{L^2(\Omega)}$ by the property of Stoke operator (see [20] Theorem 2.23).

   So that there exists a constant $c_4 := C_2 C_3$ such that for every $\tilde{u} \in V(\Omega)$
   $$\|\nabla \tilde{u}\|_{L^q(\Omega)} \leq c_4 \|\tilde{u}\|^{3/q-1/2}_{L^2(\Omega)} \|A\tilde{u}\|^{3/2-3/q}_{L^2(\Omega)}.$$  \hfill (2.0.6)

   Let $r > 1$ and $u \in V(r\Omega)$. Denote $\tilde{u}(x) := u(rx)$ for $x \in \Omega$. Then, $\tilde{u}$ satisfies \hfill (2.0.6). Besides, $\|\nabla \tilde{u}\|_{L^q(\Omega)} = r^{-3/p+1} \|\nabla u\|_{L^q(\Omega)}$ and $\|A\tilde{u}\|_{L^2(\Omega)} = r^{-3/(p+2)} \|Au\|_{L^2(\Omega)}$. From this property and (2.0.6), it deduces (2.0.8) with $C_2 := c_4$.

3. By Nirenberg [18] (2.2) & Comment 5 at page 126, we get the inequality for the case $r = 1$. In the same manner as in the proof of assertions (1), we can prove that (2.0.4) holds for any $r \geq 1$, where the constant $C_3$ does not depend on $r$.

The proof is complete. \hfill □

Lemma 2.0.2. Let $\Omega$ be a bounded $C^2$-domain in $\mathbb{R}^3$. Let $v, u \in V(\Omega)$. Then
   $$\int_{\Omega} (u \cdot \nabla)v \cdot v \, dx = 0.$$  \hfill (2.0.7)

Proof. See Constantin & Foias [2] (6.18). \hfill □
Lemma 2.0.3.
Let \( \Omega \) be a connected bounded \( C^{m+2} \)-domain in \( \mathbb{R}^3 \) with \( m \in \mathbb{N} \). Let \( r \geq 1 \).
Consider the Stokes problem:
\[
\left\{ \begin{array}{l}
-\Delta v + \nabla p = f, \ x \in r\Omega, \\
v \in V(r\Omega).
\end{array} \right.
\] (2.0.8)

Then, there exists an orthonormal basis \( \mathbf{e}_i \) satisfying
\[
\text{Lemma 2.0.4.}
\]

Proof.
(1) See Boyer & Fabrie [5, Theorem IV.5.8].
(2) Consider the Stokes problem:
\[
\left\{ \begin{array}{l}
-\Delta \tilde{v} + \nabla \tilde{p} = \tilde{f}, \ x \in \Omega, \\
\tilde{v} \in V(\Omega).
\end{array} \right.
\] (2.0.10)

In view of Boyer & Fabrie [5, Theorem IV.5.8], for each \( i \leq m \), there exists a constant \( c_i \) satisfying
\[
\forall \tilde{f} \in H^1(\Omega), \ |D^{i+2}\tilde{v}|_{L^2(\Omega)}^2 \leq c_i |\tilde{f}|_{H^1(\Omega)}^2.
\]

(1) \( \mathcal{N} \) is an orthogonal basis in \( V(\Omega) \);
(2) \( a_j \in \mathbb{H}^k(\Omega) \cap V(\Omega) \) are eigenfunctions of Stokes operator.

We denote \( P_n \mathbb{H}^2_\sigma(\Omega) := \text{span}\{a_1, a_2, \ldots, a_n\} \), and \( P_n \) be the projection operator from \( \mathbb{H}^2(\Omega) \) onto \( P_n \mathbb{H}^2_\sigma(\Omega) \).

Proof See Robinson & Rodrigo & W. Sadowski [20, Chapter 2, Theorem 2.24].

The following theorem is a particular case of Aubin-Lions-Simon Theorem.
Theorem 2.0.5. Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. Assume that the embedding of $B_1$ in $B_2$ is continuous and that the embedding of $B_0$ in $B_1$ is compact. For $T > 0$, we define

$$E := \left\{ v \in L^\infty(0, T; B_0), \ \frac{dv}{dt} \in L^\infty(0, T; B_2) \right\}. $$

Then, the embedding of $E$ in $C([0, T]; B_1)$ is compact.
Proof. See the Aubin-Lions-Simon theorem in Boyer & Fabrie [5, Theorem II.5.16]. □

Definition 2.0.6. $u_0^{(i)}$ is defined by a recurrence

$$u_0^{(i+1)} = P\left( \Delta u_0^{(i)} - \sum_{r=0}^{i} \binom{i}{r} (u_0^{(r)} \cdot \nabla) u_0^{(i-r)} + \partial_t f(\cdot, 0) \right) \text{ for } i \geq 0, \quad (2.0.11)$$

where $u_0^{(0)} := u_0$ and $P$ is the Leray projector onto $L^2_\sigma(\Omega)$.

3. The Problem (S) with Initial Condition $v(\cdot, 0)|\Omega = 0$

3.1. The introduction of the problem (S).

Let $i^* \in \mathbb{N}$ and

- $\Omega \subset \mathbb{R}^3$ be a connected bounded open set with a smooth boundary.
- $\beta, \bar{\beta} \in (C^\infty(\Omega \times [0, +\infty)))^3$.
- $\forall i \in \mathbb{N}, \ \partial_t^2 \beta \in C([0, +\infty); L^2_\sigma(\Omega))$.
- $\forall i \leq i^*, \forall x \in \Omega, P(\partial_t^2 \bar{\beta}(x, 0)) = 0$,

where $P$ is the Leray projector onto $L^2_\sigma(\Omega)$.

The problem (S), which is investigated in this section, is

$$\begin{align*}
\left\{ \begin{array}{ll}
\partial_t \tilde{v} - \Delta \tilde{v} &= -(\tilde{v} \cdot \nabla) \tilde{v} - (\bar{\beta} \cdot \nabla) \tilde{v} - (\bar{v} \cdot \nabla) \beta - \nabla \bar{p} + \bar{\theta}, \\
\tilde{v}(\cdot, 0) &= \tilde{v}_0, \\
\tilde{v} &\in L^2_\sigma(\Omega),
\end{array} \right. \quad (3.1.2)
\end{align*}$$

The Galerkin method will be used to deal with the problem (S). The basis is used for Galerkin equations is as in Lemma 2.0.1. The $n$-th order Galerkin problem corresponding to the problem (S) is given by

$$\begin{align*}
\left\{ \begin{array}{ll}
\partial_t \tilde{v}_n + A \tilde{v}_n &= P_n(-\nabla \cdot \tilde{v}_n - \beta^2 \cdot \nabla \tilde{v}_n - \nabla \cdot \tilde{v} \beta + \bar{\theta}), \\
\tilde{v}_n(\cdot, 0) &= 0, \\
\tilde{v}_n &\in V(\Omega),
\end{array} \right. \quad (3.1.3)
\end{align*}$$

where $A$ is the Stokes operator, and $P_n$ is the projection onto the $P_n L^2_\sigma(\Omega)$ (see Lemma 2.0.4).

Pre-estimate of $\|\tilde{v}_n\|_2^2$ is as at (3.2.0):

$$\forall T > 0, \sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T} \|\tilde{v}_n(\cdot, t)\|_2^2 < +\infty. \quad (3.1.4)$$

Using this pre-estimate, in the same manner as in Boyer & Fabrie [5], Robinson & Rodrigo & Sadowski [20], we can prove the following statement:

For each $n \in \mathbb{N}$, the system (3.1.3) possesses a unique solution $\tilde{v}_n$ satisfying

$$\tilde{v}_n \in (C^\infty(\Omega \times [0, +\infty)))^3 \text{ and } \partial_t^i \tilde{v}_n \in C([0, +\infty); V(\Omega)) \forall i \in \mathbb{N}. \quad (3.1.5)$$
3.2. The boundedness of Galerkin approximations.

Proposition 3.2.1.

(1) Let \( n \in \mathbb{N} \) arbitrarily. Let \( \tilde{v}^{(0)}_{no} \) and \( \tilde{v}^{(i)}_{no} \) be defined by a recurrence
\[
\tilde{v}^{(i+1)}_{no} = P_n \left( \Delta \tilde{v}^{(i)}_{no} + \partial_i^* \tilde{\theta}(., 0) \right) - \sum_{r=0}^{i} \left( \tilde{v}^{(r)}_{no} \cdot \nabla \tilde{v}^{(i-r)}_{no} + \tilde{\beta}^{(r)}_{o} \cdot \nabla \tilde{v}^{(i-r)}_{no} + \tilde{v}^{(r)}_{no} \cdot \nabla \tilde{\beta}^{(i-r)}_{o} \right), \forall i \in \mathbb{N},
\]

where \( \tilde{\beta}^{(r)}_{o} := \partial_i^* \tilde{\beta}(., 0) \), and \( P_n \) is given as in Lemma 2.0.4.

Then for every \( i \leq i^* + 1 \),
\[ \forall x \in \Omega, \tilde{v}^{(i)}_{no}(x) = 0. \] (3.2.2)

(2) For every \( n \in \mathbb{N} \), \( 0 \leq i \leq i^* + 1 \), \( x \in \Omega \),
\[ \partial_i^* \tilde{v}^{(i)}_{no}(x, 0) = 0. \] (3.2.3)

(3) Let \( \tilde{v}^{(0)}_{o} \) and \( \tilde{v}^{(i)}_{o} \) be defined by a recurrence
\[
\tilde{v}^{(i+1)}_{o} = \mathcal{P} \left( \Delta \tilde{v}^{(i)}_{o} + \partial_i^* \tilde{\theta}(., 0) \right) - \sum_{r=0}^{i} \left( \tilde{v}^{(r)}_{o} \cdot \nabla \tilde{v}^{(i-r)}_{o} + \tilde{\beta}^{(r)}_{o} \cdot \nabla \tilde{v}^{(i-r)}_{o} + \tilde{v}^{(r)}_{o} \cdot \nabla \tilde{\beta}^{(i-r)}_{o} \right), \forall i \in \mathbb{N},
\]

where \( \tilde{\beta}^{(r)}_{o} := \partial_i^* \tilde{\beta}(., 0) \), and \( \mathcal{P} \) is the Leray projection onto \( L^2_{\Omega}(\Omega) \).

Then for every \( i \leq i^* + 1 \),
\[ \forall x \in \Omega, \tilde{v}^{(i)}_{o}(x) = 0. \] (3.2.5)

Proof.

(1) Let us prove \( \tilde{v}^{(i)}_{no} \) by induction.

Firstly, one sees that \( \tilde{v}^{(0)}_{no} := 0 \). Assume by induction that \( \tilde{v}^{(i)}_{no} \) holds for all \( i \leq \hat{i} \).

Then,
\[
\tilde{v}^{(\hat{i}+1)}_{no} = P_n \left( \Delta \tilde{v}^{(\hat{i})}_{no} + \partial_i^* \tilde{\theta}(., 0) \right) - \sum_{r=0}^{\hat{i}} \left( \tilde{v}^{(r)}_{no} \cdot \nabla \tilde{v}^{(\hat{i}-r)}_{no} + \tilde{\beta}^{(r)}_{o} \cdot \nabla \tilde{v}^{(\hat{i}-r)}_{no} + \tilde{v}^{(r)}_{no} \cdot \nabla \tilde{\beta}^{(\hat{i}-r)}_{o} \right) = \quad \text{(3.2.2)}
\]
\[ P_n(0) = P_n(\partial_i^* \tilde{\theta}(., 0)) = P_n(\mathcal{P}(\partial_i^* \tilde{\theta}(., 0))) = 0 \] by (3.1.1).

Therefore, \( \tilde{v}^{(i)}_{no} \) holds for \( i = \hat{i} + 1 \).

Hence, \( \tilde{v}^{(i)}_{no} \) holds for all \( i \leq i^* + 1 \).

(2) One sees that \( \tilde{v}^{(i)}_{o} \) equals to \( \tilde{v}^{(i)}_{no} \) where \( \tilde{v}^{(i)}_{no} \) is given at (3.2.1). Combining this fact and (3.2.2), it yields (3.2.3).

(3) By the same arguments as in the proof of the assertion (1), it deduces the assertion (3).

The proof is complete. \( \square \)

The uniform boundedness of \( \|\tilde{v}_n\|_2 \) is as in the following statement.
Proposition 3.2.2. Let (3.1.7) hold and \( i^* > 6 \). Assume that \( \tilde{v}_n \) is the solution to the problem (3.1.3) for each \( n \in \mathbb{N} \). Then,

\[
(1) \quad \forall T > 0, \sup_n \sup_{0 \leq t \leq T} \| \tilde{v}_n(\cdot, t) \|^2 < +\infty, \tag{3.2.6}
\]

\[
(2) \quad \forall T > 0, \sup_n \int_0^T \| \nabla \tilde{v}_n(\cdot, t) \|^2 dt < +\infty. \tag{3.2.7}
\]

Proof.

(1) Let \( T > 0 \).

Take the \( L^2 \)-inner product of the first equation of Galerkin system (3.1.3) with \( \tilde{v}_n \). Using (2.0.7), Hölder inequality, (3.1.1) and Gagliardo-Nirenberg’s inequality (2.0.4), one gets

\[
\frac{d}{dt} \| \tilde{v}_n \|_2^2 + 2 \| \nabla \tilde{v}_n \|_2^2 \leq 2 \| b(\tilde{v}_n, \tilde{v}_n) \| + 2 \| \tilde{\beta} \|_2 \| \tilde{v}_n \|_2^2 + 2 \| \tilde{\beta} \|_2 \| \tilde{v}_n \|_2^2 + \| \tilde{\beta} \|_2 \| \tilde{v}_n \|_2^2 \\
\leq 0 + \| \tilde{\beta} \|_6 \| \tilde{v}_n \|_2 \| \nabla \tilde{v}_n \|_2^2 + \| \tilde{\beta} \|_2 \| \tilde{v}_n \|_2^2 + \| \tilde{\beta} \|_2 \| \tilde{v}_n \|_2^2 \\
\leq \| \tilde{v}_n \|_2^2 \| \nabla \tilde{v}_n \|_2 + \| \tilde{v}_n \|_2^2.
\]

Using Young’s inequality, one obtains

\[
\frac{d}{dt} \| \tilde{v}_n \|_2^2 + 2 \| \nabla \tilde{v}_n \|_2^2 \leq (c_1 \| \tilde{v}_n \|_2^2 + \| \nabla \tilde{v}_n \|_2^2) + (c_1 + \| \tilde{v}_n \|_2^2).
\]

Therefore, there exists a constant \( c_2 \) such that

\[
\forall n \in \mathbb{N}, \forall t \in [0, T], \frac{d}{dt} \| \tilde{v}_n(\cdot, t) \|_2^2 + \| \nabla \tilde{v}_n(\cdot, t) \|_2^2 \leq c_2 \| \tilde{v}_n(\cdot, t) \|_2^2 + c_2, \tag{3.2.8}
\]

where the initial value \( \| \tilde{v}_n(\cdot, 0) \|_2^2 \) is zero by the initial condition in (3.1.3).

Omitting the second term, one has a Gronwall’s inequality. So that

\[
\forall t \in [0, T], \forall n \in \mathbb{N}, \| \tilde{v}_n(\cdot, t) \|_2^2 \leq e^{c_2 T} (\| \tilde{v}_n(\cdot, 0) \|_2^2 + c_2 T) = e^{c_2 T} c_2 T =: c_3.
\]

It deduces (3.2.6).

(2) Integrating (3.2.6) in \( t \) between 0 and \( T \), it follows that

\[
\forall n \in \mathbb{N}, \| \tilde{v}_n(\cdot, T) \|_2^2 - 0 + \int_0^T \| \nabla \tilde{v}_n \|_2^2 dt \leq c_2 c_3 T + c_2 T =: c_4.
\]

It implies (3.2.7).

The proof is complete. \( \square \)

The uniform boundedness of the sequence \( (\partial_t^i \tilde{v}_n) \) is as in the following lemma.

Lemma 3.2.3. Let (3.1.7) hold and \( i^* > 6 \). Assume that \( \tilde{v}_n \) is the solution to the problem (3.1.3) for each \( n \in \mathbb{N} \).

Then,

\[
(1) \quad \forall T > 0, \sup_n \sup_{0 \leq t \leq T} \sup_{i \leq t \leq t+1} \| \partial_t^i \tilde{v}_n(\cdot, t) \|_{H^1(\Omega)} < +\infty; \tag{3.2.9}
\]

\[
(2) \quad \forall T > 0, \sup_n \sup_{0 \leq t \leq T} \sup_{i \leq t \leq t+1} \| A \partial_t^i \tilde{v}_n(\cdot, t) \|_{L^2(\Omega)} < +\infty; \tag{3.2.10}
\]

\[
(2) \quad \forall T > 0, \sup_n \sup_{0 \leq t \leq T} \sup_{i \leq t \leq t+1} \| \partial_t^i \tilde{v}_n(\cdot, t) \|_{H^2(\Omega)} < +\infty. \tag{3.2.11}
\]
Proof. Let $T > 0$.
(1) Proof of (3.2.9).
First, let us prove the following claim.

- **Claim 1.**
  1. For every $0 \leq i \leq i^* + 1$, there exists a constant $c_{0,i}$ such that
     \[ \forall n \in \mathbb{N}, \forall t \in [0, T], \quad \|\partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \leq c_{0,i}. \tag{3.2.12} \]
  2. For every $0 \leq i \leq i^* + 1$, there exists a constant $c_{1,i}$ such that
     \[ \forall n \in \mathbb{N}, \forall t \in [0, T], \quad \|\nabla \partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \leq c_{1,i}. \tag{3.2.13} \]
  3. For every $0 \leq i \leq i^* + 1$, there exists a constant $b_{2,i}$ such that
     \[ \forall n \in \mathbb{N}, \int_0^T \|A\partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \, dt \leq b_{2,i}. \tag{3.2.14} \]

- **Proving Claim 1 by induction**
  1. **Proving (3.2.12) holds for $i = 0$.**
     Observing that (3.2.12) is valid for $i = 0$ by (3.2.10).
  2. **Proving (3.2.13) holds for $i = 0$.**
     Take the $L^2$-inner product of the first equation in (3.1.3) with $\tilde{v}_n$. Then, using (1.0.3), Hölder’s inequality, (3.1.1) and Gagliardo-Nirenberg’s inequality, one gets for every $t \in [0, T]$ and $n \in \mathbb{N}$,
     \begin{align*}
     \frac{d}{dt} \|\nabla \tilde{v}_n\|_2^2 + 2\|A\tilde{v}_n\|_2^2 &\leq \int_\Omega (\tilde{v}_n \cdot \nabla) \tilde{v}_n \cdot A\tilde{v}_n \, dx + \|\tilde{\beta}\|_6 \|\nabla \tilde{v}_n\|_3 \|A\tilde{v}_n\|_2 + \\
     &\quad + \|\tilde{v}_n\|_6 \|\nabla \tilde{\beta}\|_3 \|A\tilde{v}_n\|_2 + \|\tilde{\theta}\|_2 \|A\tilde{v}_n\|_2 \\
     &\leq 0 + \|\nabla \tilde{v}_n\|_2 \|A\tilde{v}_n\|_2^2 + \|\nabla \tilde{v}_n\|_2 \|A\tilde{v}_n\|_2 + \|A\tilde{v}_n\|_2.
     \end{align*}

     Using Young’s inequality, simplifying terms of $\|A\tilde{v}_n\|_2^2$, one obtains
     \[ \forall n \in \mathbb{N}, t \in [0, T], \quad \frac{d}{dt} \|\nabla \tilde{v}_n\|_2^2 + \|A\tilde{v}_n\|_2^2 \leq c_1 \|\nabla \tilde{v}_n\|_2^2 + c_1. \tag{3.2.15} \]

     Omitting the second term, one has a Gronwall’s inequality. So that
     \[ \forall n \in \mathbb{N}, t \in [0, T], \quad \|\nabla \tilde{v}_n\|_2^2 \leq e^{c_1T}(\|\nabla \tilde{v}_n(\cdot, 0)\|_2^2 + c_1 T) \leq e^{c_1T}c_1 T =: c_2. \]

     Hence, (3.2.13) is valid for $i = 0$.

     Integrating both sides (3.2.15) in $t$ between 0 and $T$ yields
     \[ \|\nabla \tilde{v}_n(\cdot, T)\|_2^2 - \|\nabla \tilde{v}_n(\cdot, 0)\|_2^2 + \int_0^T \|A\tilde{v}_n(\cdot, t)\|_2^2 \, dt \leq c_3, \forall n \in \mathbb{N}. \]

     Hence, (3.2.14) holds for $i = 0$.

- **Assumption by induction.**
  Take $i$ such that $1 \leq i \leq i^* + 1$. Assume by induction that (3.2.10)-(3.2.13) hold for every $0 \leq i \leq i-1$. Then, there exist constants $c_{0,0}, \cdots, b_{2,i-1}$ such that for each $i \leq i-1$,
  \[ \forall n \in \mathbb{N}, \forall t \in [0, T], \quad \|\partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \leq c_{0,i}, \]
  \[ \forall n \in \mathbb{N}, \forall t \in [0, T], \quad \|\nabla \partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \leq c_{1,i}, \]
  \[ \forall n \in \mathbb{N}, \int_0^T \|A\partial_t^i \tilde{v}_n(\cdot, t)\|_2^2 \, dt \leq b_{2,i}. \tag{3.2.16} \]
— Proving (3.2.12) holds for \( i = \hat{i}. \)

One differentiates the Galerkin equation (3.1.8) \( \hat{i} \) times with respect to \( t \) to get \( \tilde{v}_{n}^{(i+1)} + A\tilde{v}_{n}^{(i)} = (P_n(\phi_n))^{(i)}. \) So that

\[
\tilde{v}_{n}^{(i+1)} + A\tilde{v}_{n}^{(i)} = P_n(\phi_n^{(i)}),
\]

where

\[
\phi_n^{(i)} := - \sum_{r=0}^{\hat{i}} \left( \tilde{v}_{n}^{(i)}, \nabla \tilde{v}_{n}^{(i-r)} + \tilde{\beta}_{n}^{(r)} \cdot \nabla \tilde{v}_{n}^{(i-r)} + \tilde{v}_{n}^{(r)} \cdot \nabla \tilde{\beta}_{n}^{(i-r)} \right) + \tilde{\theta}_{n}^{(i)}.
\]  (3.2.17)

Taking the \( L^2 \)-inner product of (3.2.17) with \( \tilde{v}_{n}^{(i)} \), one deduces

\[
\frac{d}{dt} \| \tilde{v}_{n}^{(i)} \|_2^2 + 2 \| \nabla \tilde{v}_{n}^{(i)} \|_2^2 \leq 2 \sum_{r=0}^{\hat{i}} \left( \sum_{i=0}^{\hat{i}} |b(\tilde{v}_{n}^{(r)}, \tilde{v}_{n}^{(i-r)}, \tilde{v}_{n}^{(i)})| + 2 \| \tilde{\theta}_{n}^{(i)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2 \right)
\]

\[
= 2 \sum_{r=0}^{\hat{i}} \left( \sum_{i=0}^{\hat{i}} |b(\tilde{v}_{n}^{(r)}, \tilde{v}_{n}^{(i-r)}, \tilde{v}_{n}^{(i)})| \right) + 2 \sum_{r=0}^{\hat{i}} \left( \sum_{i=0}^{\hat{i}} |b(\tilde{v}_{n}^{(r)}, \tilde{\beta}_{n}^{(i-r)}, \tilde{v}_{n}^{(i)})| \right) + 2 \| \tilde{\theta}_{n}^{(i)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2.
\]

Here, \( \gamma_1 \)-\( \gamma_3 \) are estimated by using (2.0.7), Gagliardo-Nirenberg’s inequality (2.0.2), (3.1.1) and (3.2.10) as follows

\[
\gamma_1 \leq |b(\tilde{v}_{n}^{(i)}, \tilde{v}_{n}^{(i)}| + \sum_{r=0}^{\hat{i}-1} |b(\tilde{v}_{n}^{(i)}, \tilde{v}_{n}^{(i-r)}, \tilde{v}_{n}^{(i)})| + |b(\tilde{v}_{n}^{(i)}, \tilde{\beta}_{n}^{(i)}|,
\]

\[
\leq 0 + \sum_{r=1}^{\hat{i}-1} \| \tilde{v}_{n}^{(i)} \|_6 \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_6 + \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{v}_{n}^{(i)} \|_2 + \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2.
\]

\[
\gamma_2 \leq \sum_{r=0}^{\hat{i}} \| \tilde{\beta}_{n}^{(r)} \|_6 \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_3 \leq \sum_{r=0}^{\hat{i}} \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{\beta}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2 + \sum_{r=1}^{\hat{i}} \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{\beta}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2 + \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\gamma_3 \leq \sum_{r=0}^{\hat{i}} \| \tilde{\beta}_{n}^{(r)} \|_6 \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_3 \leq \sum_{r=0}^{\hat{i}} \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{\beta}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2 + \sum_{r=1}^{\hat{i}} \| \nabla \tilde{v}_{n}^{(i-r)} \|_2 \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{\beta}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2 + \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2
\]

\[
\leq \| \tilde{v}_{n}^{(i)} \|_2 + \| \tilde{v}_{n}^{(i)} \|_2 \| \nabla \tilde{v}_{n}^{(i)} \|_2.
\]
Therefore,
\[ \frac{d}{dt} \| \hat{v}^{(\tilde{i})} \|_2^2 + 2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 \leq \| \hat{v}^{(\tilde{i})} \|_2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 + \| \hat{v}^{(\tilde{i})} \|_2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 + \| \hat{v}^{(\tilde{i})} \|_2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 \]
\[ + \| \hat{v}^{(\tilde{i})} \|_2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 + \| \nabla \hat{v}^{(\tilde{i})} \|_2 + \| \hat{v}^{(\tilde{i})} \|_2. \]

Applying Young inequality for terms on the right-hand sides, one gets
\[ \frac{d}{dt} \| \hat{v}^{(\tilde{i})} \|_2^2 + 2 \| \nabla \hat{v}^{(\tilde{i})} \|_2 \leq \left[ c_3 \| \hat{v}^{(\tilde{i})} \|_2^2 + \frac{\| \nabla \hat{v}^{(\tilde{i})} \|_2^2}{4} \right] + \left[ c_3 + \frac{\| \nabla \hat{v}^{(\tilde{i})} \|_2^2}{4} \right] + c_3 + \| \hat{v}^{(\tilde{i})} \|_2^2. \]

where \( \| \hat{v}^{(\tilde{i})} \|_2^2 \leq \| \hat{v}^{(\tilde{i})} \|_2^2 + 1 \). Therefore, there exists a constant \( c_4 \) such that
\[ \forall n \in \mathbb{N}, \forall t \in [0, T], \frac{d}{dt} \| \hat{v}^{(\tilde{i})} (.., t) \|^2 + \| \nabla \hat{v}^{(\tilde{i})} (.., t) \|_2 \leq c_4 \| \hat{v}^{(\tilde{i})} (.., t) \|_2^2 + c_4, \quad (3.2.19) \]
where \( \| \hat{v}^{(\tilde{i})} (.., 0) \|^2 = 0 \) by (3.2.18).
Absorbing the second term, one gets a Gronwall’s inequality. So that
\[ \forall n \in \mathbb{N}, \forall t \in [0, T], \| \hat{v}^{(\tilde{i})} (.., t) \|^2 \leq e^{c_4 T} \left( \| \hat{v}^{(\tilde{i})} (.., 0) \|^2 + c_4 T \right) = e^{c_4 T} c_4 T. \]

(3.2.20)

So that (3.2.12) holds for \( i = \tilde{i} \).

- Proving (3.2.19) holds for \( i = \tilde{i} \).

Taking the \( L^2 \)-inner product of the equation (3.2.17) with \( A \hat{v}^{(\tilde{i})} \) yields
\[ \frac{d}{dt} \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 + 2 \| \nabla \hat{v}^{(\tilde{i})} \|_2^2 \leq 2 \sum_{r=0}^{\tilde{i}} | b(\hat{v}^{(r)}(\tilde{r}), \hat{v}^{(\tilde{i}-\tilde{r})}, A \hat{v}^{(\tilde{i})}) | + \gamma_4 \]
\[ + 2 \sum_{r=0}^{\tilde{i}} | b(\hat{\tilde{v}}^{(r)}(\tilde{r}), \hat{v}^{(\tilde{i}-\tilde{r})}, A \hat{v}^{(\tilde{i})}) | + 2 \sum_{r=0}^{\tilde{i}} | b(\hat{v}^{(r)}(\tilde{r}), \hat{\tilde{v}}^{(\tilde{i}-\tilde{r})}, A \hat{v}^{(\tilde{i})}) | + 2 \| \hat{\tilde{v}}^{(i)} \|_2 \| A \hat{v}^{(\tilde{i})} \|_2. \]
\[ \gamma_4, \gamma_5, \gamma_6 \]

Here, \( \gamma_4 - \gamma_6 \) are estimated by using Gagliardo-Nirenberg’s inequality (Lemma 3.2.4, 3.2.10, 3.2.13) and (3.2.20) as follows
\[ \gamma_4 \leq \sum_{r=0}^{\tilde{i}} \| \hat{v}^{(r)} \|_6 \| \nabla \hat{v}^{(\tilde{i}-\tilde{r})} \|_3 \| A \hat{v}^{(\tilde{i})} \|_2 \]
\[ \leq \sum_{r=0}^{\tilde{i}} \| \nabla \hat{v}^{(r)} \|_2 \| \nabla \hat{v}^{(\tilde{i}-\tilde{r})} \|_2 \| A \hat{v}^{(\tilde{i}-\tilde{r})} \|_2 \| A \hat{v}^{(\tilde{i})} \|_2 \]
\[ \leq \| \nabla \hat{v}^{(\tilde{i})} \|_2 \| A \hat{v}^{(\tilde{i})} \|_2 \leq \sum_{r=0}^{\tilde{i}-1} \| A \hat{v}^{(\tilde{i}-\tilde{r})} \|_2 \| A \hat{v}^{(\tilde{i})} \|_2 + \| \nabla \hat{v}^{(\tilde{i})} \|_2 \| A \hat{v}^{(\tilde{i})} \|_2. \]
\[ \gamma_5 \leq \sum_{r=0}^{\tilde{i}} \| \hat{\tilde{v}}^{(r)} \|_6 \| \nabla \hat{v}^{(\tilde{i}-\tilde{r})} \|_3 \| A \hat{v}^{(\tilde{i})} \|_2 \]
\[
\begin{align*}
&\leq \sum_{r=0}^{i} \| \nabla \tilde{v}_n^{(i-r)} \|_2^2 \| A \tilde{v}_n^{(i-r)} \|_2 + \sum_{r=1}^{i} \| A \tilde{v}_n^{(i-r)} \|_2 \| A \tilde{v}_n^{(i)} \|_2, \\
&\leq \| \nabla \tilde{v}_n^{(i)} \|_2^2 \| A \tilde{v}_n^{(i)} \|_2 + \sum_{r=1}^{i} \| A \tilde{v}_n^{(i-r)} \|_2 \| A \tilde{v}_n^{(i)} \|_2, \\
&\gamma_6 \leq \sum_{r=0}^{i} \| \tilde{v}_n^{(i)} \|_2 \| \nabla \tilde{v}_n^{(i-r)} \|_6 \| A \tilde{v}_n^{(i)} \|_2 \leq \sum_{r=0}^{i} \| \tilde{v}_n^{(i)} \|_2 \| \nabla \tilde{v}_n^{(i-r)} \|_2 \| A \tilde{v}_n^{(i)} \|_2, \\
&\leq \sum_{r=0}^{i-1} \| A \tilde{v}_n^{(i-r)} \|_2 + \| \nabla \tilde{v}_n^{(i)} \|_2 \| A \tilde{v}_n^{(i)} \|_2 \leq \| A \tilde{v}_n^{(i)} \|_2 + \| \nabla \tilde{v}_n^{(i)} \|_2 \| A \tilde{v}_n^{(i)} \|_2 + 1.
\end{align*}
\]

Therefore,

\[
\frac{d}{dt} \| \nabla \tilde{v}_n^{(i)} \|_2^2 + 2 \| A \tilde{v}_n^{(i)} \|_2^2 \leq 2 \left[ c_5 \| \nabla \tilde{v}_n^{(i)} \|_2^2 + \| A \tilde{v}_n^{(i)} \|_2^2 \right] + \\
+ \left[ c_5 \| \nabla \tilde{v}_n^{(i)} \|_2^2 \| A \tilde{v}_n^{(i-r)} \|_2 \right] + \left[ c_6 \sum_{r=1}^{i} \| A \tilde{v}_n^{(i-r)} \|_2 \right] + \left[ c_5 + \| \nabla \tilde{v}_n^{(i)} \|_2 \right] + \left[ c_5 + \| A \tilde{v}_n^{(i)} \|_2 \right],
\]

where \( \| A \tilde{v}_n^{(i-r)} \|_2 \leq \| A \tilde{v}_n^{(i-r)} \|_2^2 + 1 \).

Therefore, there exists a constant \( c_6 \), such that

\[
\frac{d}{dt} \| \nabla \tilde{v}_n^{(i)} \|_2^2 + \| A \tilde{v}_n^{(i)} \|_2^2 \leq c_6 \left( 1 + \| A \tilde{v}_n^{(i-r)} \|_2 \| \nabla \tilde{v}_n^{(i)} \|_2^2 + \| A \tilde{v}_n^{(i-r)} \|_2^2 \right) + c_6 \sum_{r=1}^{i} \| A \tilde{v}_n^{(i-r)} \|_2^2 + c_6, \quad (3.2.21)
\]

where the initial value of \( \| \nabla \tilde{v}_n^{(i)} \|_2^2 \) is zero by (3.2.24).

Absorbing the second term, one gets a Gronwall’s inequality. So that for every \( n \in \mathbb{N} \), \( t \in [0, T] \),

\[
\| \nabla \tilde{v}_n^{(i)} (\cdot, t) \|_2^2 \leq e^{c_6 T + c_6 \int_0^T \| A \tilde{v}_n \|_2^2 dt} \left( 0 + c_6 \sum_{r=1}^{i} \int_0^T \| A \tilde{v}_n^{(i-r)} \|_2^2 dt + c_6 T \right).
\]

By this fact and (3.2.16), there exists a constant \( c_7 \) such that

\[
\forall t \in [0, T], \forall n \in \mathbb{N}, \quad \| \nabla \tilde{v}_n^{(i)} (\cdot, t) \|_2^2 \leq c_7. \quad (3.2.22)
\]

Therefore, (3.2.13) holds for \( i = i \).

– Proving (3.2.14) holds for \( i = i \).
From (3.2.21) and (3.2.22), it deduces that
\[
\frac{d}{dt} \| \nabla \tilde{v}_n^{(i)} \|_2^2 + \| A \tilde{v}_n^{(i)} \|_2^2 \leq c_6 (1 + \| A \tilde{v}_n \|_2^2) c_7 + c_6 \sum_{r=1}^i \| A \tilde{v}_n^{(i-r)} \|_2^2 + c_6. \tag{3.2.23}
\]

Integrating both sides in time between 0 and \( T \), and using (3.2.16), it deduces
\[
\| \nabla \tilde{v}_n^{(i)} (., T) \|_2^2 - 0 + \int_0^T \| A \tilde{v}_n^{(i)} \|_2^2 dt \leq c_8.
\]

Hence, (3.2.14) holds for \( i = \hat{i} \).

- **Results of induction.**

So that (3.2.12) - (3.2.14) hold for \( 0 \leq i \leq i^* + 1 \). It deduces (3.2.14).

(2) **Proofs of (3.2.10), (3.2.11).**

Let us prove the following claim.

- **Claim 2.**

  For \( 0 \leq i \leq i^* \), there exists a constant \( c_{2,i} \) such that
  \[
  \forall n \in \mathbb{N}, \forall t \in [0, T], \| A \tilde{v}_n^{(i)} (., t) \|_2^2 \leq c_{2,i}. \tag{3.2.24}
  \]

- **Proving Claim 2 by induction.**

  - Let us prove that (3.2.24) holds for \( i = 0 \).

    By (3.2.10) and (3.2.13), it deduces that
    \[
    \forall n \in \mathbb{N}, \forall t \in [0, T], \frac{d}{dt} \| \nabla \tilde{v}_n (., t) \|_2^2 + \| A \tilde{v}_n (., t) \|_2^2 \leq c_6
    \]
    \[
    \| A \tilde{v}_n \|_2^2 \leq - \frac{d}{dt} \| \nabla \tilde{v}_n \|_2^2 + c_9
    \]
    \[
    \| A \tilde{v}_n \|_2^2 \leq 2 \int_\Omega | \partial_t \nabla \tilde{v}_n | | \nabla \tilde{v}_n | dx + c_9.
    \]

    Using Cauchy’s inequality and (3.2.9), one deduces
    \[
    \forall n \in \mathbb{N}, \forall t \in [0, T], \| A \tilde{v}_n \|_2^2 \leq \| \partial_t \nabla \tilde{v}_n \|_2^2 + \| \nabla \tilde{v}_n \|_2^2 + c_9 \leq C_1 + C_1 + c_9.
    \]

    Therefore, (3.2.24) holds for \( i = 0 \).

    - Take \( \hat{i} \in \mathbb{N} \) such that \( 1 \leq \hat{i} \leq i^* \). Assume by induction that (3.2.24) holds for \( 0 \leq i \leq \hat{i} - 1 \). Then, there are \( c_{2,0}, \cdots, c_{2,\hat{i}-1} \) such that
      \[
      \forall i \leq \hat{i} - 1, \forall n \in \mathbb{N}, \forall t \in [0, T], \| A \tilde{v}_n^{(i)} (., t) \|_2^2 \leq c_{2,i}. \tag{3.2.25}
      \]

    - Let us prove that (3.2.24) holds for \( i = \hat{i} \).

      By the same arguments as in the proof of the assertion (1), one gets (3.2.24). By this inequality and (3.2.24), one obtains
      \[
      \frac{d}{dt} \| \nabla \tilde{v}_n^{(\hat{i})} \|_2^2 + \| A \tilde{v}_n^{(\hat{i})} \|_2^2 \leq c_{10} \text{ for every } n \in \mathbb{N} \text{ and } t \in [0, T].
      \]

      Using the Cauchy’s inequality and (3.2.9), one gets
      \[
      \forall n \in \mathbb{N}, \forall t \in [0, T], \| A \tilde{v}_n^{(\hat{i})} \|_2^2 \leq \frac{d}{dt} \| \nabla \tilde{v}_n^{(\hat{i})} \|_2^2 + c_{10}
      \]
      \[
      \leq 2 \int_\Omega | \partial_t \nabla \tilde{v}_n^{(\hat{i})} | | \nabla \tilde{v}_n^{(\hat{i})} | dx + c_{10}
      \]
      \[
      \leq \| \nabla \tilde{v}_n^{(\hat{i}+1)} \|_2^2 + \| \nabla \tilde{v}_n^{(\hat{i})} \|_2^2 + c_{10} \leq C_1^2 + C_1^2 + c_{10}.
      \]
3.3. The existence and the regularity of solutions to the problem (S).

Consider the problem (S) at (3.1.2) and the problem at (3.1.3). For each

\[ (\tilde{v}, \tilde{p}) \]

the problem (S) has a classical solution \( \tilde{v} \) for every \( \tilde{t} \).

The proof of Lemma is complete. \( \square \)

3.3.1. Let (3.1.1) hold and \( i^* > 6 \). Let \( \tilde{v}_n \) be the solution to the problem (3.1.3) for each \( n \in \mathbb{N} \).

Then, the problem (S) has a classical solution \((\tilde{v}, \tilde{p})\) satisfying

\[ \tilde{v} \in C^{i^* - 4}(\Omega \times [0, +\infty)) \cap C([0, +\infty); V(\Omega)), \] (3.3.1)

\[ \tilde{p} \in C^{i^* - 3}(\Omega \times [0, +\infty)); \] (3.3.2)

\[ \forall \tilde{t} \leq i^* - 4, \partial_{i^*} \tilde{v}(\cdot, 0) = 0; \] (3.3.3)

and there exists a subsequence of \((\tilde{v}_n)\), still denoted by \((\tilde{v}_n)_n\), such that

\[ \partial_{i^*} \tilde{v}_n \] converges to \( \partial_{i^*} \tilde{v} \) in \( L^\infty(0, T; \mathbb{H}^2(\Omega)) \), \( \forall \tilde{t} \leq i^* - 4. \) (3.3.4)

Here, \( \partial_{i^*} \tilde{v} \) is the classical partial derivative of \( \tilde{v} \) for every \( i \leq i^* - 4 \).

Proof. Let \( T > 0 \) arbitrarily.

The proof is split into some steps.

\* Step 1: Proof of the existence of \( \partial_{i^*} \tilde{v} \) in \( \mathbb{H}^1(\Omega \times (0, T)) \) for \( 0 \leq j \leq i^* \).

Let \( Q_T := \Omega \times (0, T) \subset \mathbb{R}^4 \). By the Rellich- Kondrachov theorem (see Adams [1], VI-Theorem 6.2),

\[ H^2(Q_T) \hookrightarrow H^1(Q_T) \] is compact. (3.3.5)

By (3.2.11), there exists a constant \( \bar{c}_2(\Omega) \) such that

\[ \forall n \in \mathbb{N}, \forall \tilde{t} \leq i^*, \forall \tilde{t} \in [0, T], \| \partial_{i^*} \tilde{v}_n(\cdot, \tilde{t}) \|_{\mathbb{H}^2(\Omega)} \leq \bar{c}_2. \] (3.3.6)

So that \((\tilde{v}_n)_n\) is a bounded sequence in \( \mathbb{H}^2(Q_T) \). By the compact embedding (3.3.5), there exist a subsequence of \((\tilde{v}_n)_n\), also denoted by \((\tilde{v}_n)_n\), and a function \( \tilde{v} \) in \( \mathbb{H}^1(Q_T) \) such that

\[ (\tilde{v}_n)_n \] converges to \( \tilde{v} \) in \( \mathbb{H}^1(Q_T) \). (3.3.7)

Consider the sequence \((\partial_{i^*} \tilde{v}_n)_n\) corresponding to the subsequence \((\tilde{v}_n)_n\) at (3.3.7). By (3.3.6), the sequence \((\partial_{i^*} \tilde{v}_n)_n\) is bounded in \( \mathbb{H}^2(Q_T) \). By this fact, (3.3.5)-(3.3.7), there exists a subsequence of the subsequence \((\tilde{v}_n)_n\), also denoted by \((\tilde{v}_n)_n\), such that

\[ (\partial_{i^*} \tilde{v}_n)_n \] converges to \( \partial_{i^*} \tilde{v} \) in \( \mathbb{H}^1(Q_T) \),

where \( \partial_{i^*} \tilde{v}_n := D^{(0,0,0,1)} \tilde{v} \).
Step 2: Proving that \( \star \)

Consider the sequence \((\tilde{v}_n)\), still denoted by \((\tilde{v}_n)\), such that

\[
(\tilde{v}^{(i)}_n) \text{ converges to } \tilde{v}^{(i)} \text{ in } H^1(Q_T) \text{ for } 0 \leq i \leq i^*,
\]

where \(\tilde{v}^{(i)}_n := D^{(0,0,0,j)}\tilde{v}\).

\[ \star \quad \text{Step 2: Proving that } \partial_t v \in L^\infty(0, T; H^2(\Omega)). \]

Consider the sequence \((\tilde{v}_n)\) at \((3.3.8)\). By \((3.3.6)\), \((\tilde{v}_n)\) is bounded in \(C([0, T]; H^2(\Omega)) \subset L^\infty(0, T; H^2(\Omega))\) and \(\partial_t \tilde{v}_n\) is bounded in the space \(L^\infty(0, T; H^1(\Omega))\). From this fact, Theorem 2.0.5 and \((3.3.8)\), there exist a subsequence of \((\tilde{v}_n)\), still denoted by \((\tilde{v}_n)\), such that

\[
(\tilde{v}_n) \text{ converges to } \tilde{v} \text{ in the space } C([0, T]; H^1(\Omega)).
\]

Besides, \(\tilde{v}^{(i)}_n \in C([0, T]; V(\Omega))\) by \((3.1.5)\). It follows that

\[
(\tilde{v}^{(i)}_n) \text{ converges to } \tilde{v}^{(i)} \text{ in } C([0, T]; V(\Omega)) \text{ for every } i \leq i^*.
\]

Consider the sequence \((\tilde{v}_n)\) at \((3.3.9)\). By \((3.3.6)\) and \((3.1.5)\), \((\tilde{v}^{(i)}_n)\) is bounded in \(L^\infty(0, T; H^2(\Omega))\) for every \(0 \leq i \leq i^*\). Besides, by the Sobolev embedding, \(\|\tilde{v}^{(i)}_n(., t)\|_{L^\infty(\Omega)} \leq c(\Omega)\|\tilde{v}^{(i)}_n(., t)\|_{H^2(\Omega)}\) for every \(t \in [0, T]\). So that the sequence \((\tilde{v}^{(i)}_n \cdot \nabla \tilde{v}^{(i-\tau)}_n)\) is bounded in \(L^\infty(0, T; H^1(\Omega))\) and the sequence \((\partial_t \tilde{v}^{(i)}_n \cdot \nabla \tilde{v}^{(i-\tau)}_n)\) is bounded in \(L^\infty(0, T; L^2(\Omega))\) for \(0 \leq i \leq i^* - 1\). By Theorem 2.0.5 and \((3.3.8)\), there exists a subsequence of \((\tilde{v}_n)\), still denoted by \((\tilde{v}_n)\), such that

\[
(\tilde{v}^{(i)}_n \cdot \nabla \tilde{v}^{(i-\tau)}_n) \text{ converges to } (\tilde{v}^{(i)} \cdot \nabla \tilde{v}^{(i-\tau)}) \text{ in } C([0, T]; L^2(\Omega)).
\]

So that for every \(i \leq i^* - 1\)

\[
\phi^{(i)}_n \text{ converges to } \phi^{(i)} \text{ in } C([0, T]; L^2(\Omega)),
\]

where \(\phi^{(i)}_n\) is given at \((3.2.18)\) and

\[
\phi^{(i)} = -\sum_{r=0}^{i} \left(\begin{array}{c} i \\ r \end{array}\right) \tilde{p}^{(r)} \cdot \nabla \tilde{p}^{(i-r)} + \tilde{\beta}^{(r)} \cdot \nabla \tilde{p}^{(i-r)} + \tilde{v}^{(r)} \cdot \nabla \tilde{\beta}^{(i-r)} + \tilde{\beta}^{(i)},
\]

Besides, one has

\[
\|P_n \phi^{(i)} - P \phi^{(i)}\|_2 \leq \|P_n(\phi^{(i)} - \phi^{(i)})\|_2 + \|P_n - P\| \phi^{(i)}\|_2 \\
\leq \|\phi^{(i)}_n - \phi^{(i)}\|_2 + \|P - P_n\| \phi^{(i)}\|_2.
\]

It follows that

\[
(P_n \phi^{(i)}_n) \text{ converges to } P \phi^{(i)} \text{ in } L^\infty(0, T; L^2(\Omega)).
\]
Differentiate the first equation in (3.1.3) with respect to time \( i \) times, one deduces

\[
A\tilde{v}^{(i)} = -\tilde{v}^{(i+1)} + P_n\phi^{(i)}
\tag{3.3.14}
\]

From this fact, (3.3.10) and (3.3.13), it deduces that for every \( i \leq i^* - 1 \)

\[
A\tilde{v}^{(i)} \text{ converges to } -\tilde{v}^{(i+1)} + \mathcal{P}\phi^{(i)} \text{ in } L^\infty(0, T; L^2(\Omega)),
\]

and so that \( A\tilde{v}^{(i)} \) is a Cauchy sequence in \( L^\infty(0, T; L^2(\Omega)) \). Combining this fact and (3.2.20), it follows that \( (\tilde{v}^{(i)}_n)_n \) is a Cauchy sequence in the space \( L^\infty(0, T; H^2(\Omega)) \), and hence \( (\tilde{v}^{(i)}_n)_n \) converges in this space to a function, which can be proved to be \( \tilde{v}^{(i)} \). Therefore, for every \( i \leq i^* - 1 \)

\[
\tilde{v}^{(i)} \text{ converges to } \tilde{v}^{(i)} \text{ in the space } L^\infty(0, T; H^2(\Omega)).
\tag{3.3.15}
\]

**Step 3: Proving \( \tilde{v} \in (C^{i^*-4}(\Omega \times [0, T]))^3 \).** — Similarly as in the proof of Lemma 3.2.1, one gets

\[
\forall i \leq i^* - 1, \quad \tilde{v}^{(i+1)} + A\tilde{v}^{(i)} = P_n(\phi^{(i)})
\]

\[
\mathcal{P}(-\tilde{v}^{(i+1)} + \Delta\tilde{v}^{(i)} + P_n\phi^{(i)}) = 0.
\]

Passing to the limit, by (3.3.15) and (3.3.13), one obtains

\[
\mathcal{P}(-\tilde{v}^{(i+1)} + \Delta\tilde{v}^{(i)} + \phi^{(i)}) = 0 \text{ in } L^\infty(0, T; L^2(\Omega)).
\]

By Helmholtz-Leray decomposition (see Temam [24] Theorem 1.5), there exists \( p_i \in L^\infty(0, T; H^1(\Omega)) \) such that \( -\tilde{v}^{(i+1)} + \Delta\tilde{v}^{(i)} + \phi^{(i)} = \nabla p_i \). Hence,

\[
\forall i \leq i^* - 1, -\Delta\tilde{v}^{(i)} + \nabla p_i = -\tilde{v}^{(i+1)} + \phi^{(i)} \text{ in } L^\infty(0, T; L^2(\Omega)),
\tag{3.3.16}
\]

where \( \tilde{v}^{(i)} \in C([0, T]; V(\Omega)) \) by (3.3.10), and \( \phi^{(i)} \) is given at (3.3.12).

Now, let us prove by induction on \( s \) that

\[
\tilde{v}^{(i)} \in L^\infty(0, T; H^s(\Omega)) \text{ for } 0 \leq s \leq i^*, \text{ for } 0 \leq i + s \leq i^*.
\tag{3.3.17}
\]

- First, one sees that (3.3.17) holds for \( s = 2 \) by (3.3.10) and (3.3.13).
- Take \( \bar{s} \in \mathbb{N} \) such that \( 2 \leq \bar{s} < i^* \). Assume by induction on \( s \) that (3.3.17) holds for \( 0 \leq s \leq \bar{s} \). Then,

\[
\tilde{v}^{(i)} \in L^\infty(0, T; H^\bar{s}(\Omega)) \text{ for } i + \bar{s} \leq i^*.
\tag{3.3.18}
\]

- Now take \( i \in \mathbb{N} \) such that \( i + (\bar{s} + 1) \leq i^* \). Let us show that

\[
\tilde{v}^{(i)} \in L^\infty(0, T; H^{\bar{s}+1}(\Omega)).
\]

From \( i + (\bar{s} + 1) \leq i^* \) and \( r \leq i \), it deduces \( i+1 + \bar{s} \leq i^* \), \( i+r + \bar{s} \leq i^* \) and \( r + \bar{s} \leq i^* \), and so that \( \tilde{v}^{(i+1)}, \tilde{v}^{(i+r)}, \tilde{v}^{(r)} \in L^\infty(0, T; H^{\bar{s}}(\Omega)) \) by (3.3.18). Besides, \( \|\tilde{v}^{(r)}(., t)\|_{L^\infty(\Omega)} \leq c(\Omega)\|\tilde{v}^{(r)}(., t)\|_{H^2(\Omega)} \) for every \( t \in [0, T] \), by the Sobolev embedding. Therefore,

\[
\tilde{v}^{(r)} \cdot \nabla \tilde{v}^{(i+r)}, \tilde{v}^{(r)} \cdot \nabla \tilde{v}^{(i+r)} \in L^\infty(0, T; H^{\bar{s}+1}(\Omega))
\]

Hence \( \phi^{(i)} \in L^\infty(0, T; H^{\bar{s}+1}(\Omega)) \). Applying (2.0.3) (the regularity of the Stoke problem) for the equation (3.3.10), it follows that \( \tilde{v}^{(i)} \in L^\infty(0, T; H^{\bar{s}+1}(\Omega)) \). Therefore, (3.3.17) holds for \( s = \bar{s} + 1 \).

- The proof by induction for (3.3.12) is complete.
By (3.3.17), one gets
\[ \tilde{v}(i^*-s) \in L^\infty(0, T; H^s(\Omega)) \] for every \( s \leq i^*. \)

Hence, \( v \in H^{i^*(\Omega \times [0, T])}. \)

By the Sobolev embedding theorem (see Adams [1, Theorem 5.4-Part I-Case C]), after possibly being redefined on a set of measure zero, \( \tilde{v} \) satisfies
\[ \tilde{v} \in (C^{i^*-3}(\tilde{\Omega} \times [0, T]))^3. \]
Therefore,
\[ \tilde{v} \in (C^{i^*-4}(\bar{\Omega} \times [0, T]))^3, \]
and \( v^{(i)} \) is a classical derivative of \( \tilde{v} \) with respect to \( t \) for \( i \leq i^* - 3. \)

\[ \textit{Step 4: Proving (3.3.1)-(3.3.4).} \]

Consider (3.3.16) with \( i = 0 \). Denote \( \tilde{p} \) for \( \tilde{p}_0 \), one gets
\[ \partial_t \tilde{v} - \Delta \tilde{v} = -\tilde{v} \cdot \nabla \tilde{v} - \tilde{\beta} \cdot \nabla \tilde{\beta} - \tilde{\theta} - \nabla \tilde{p}. \]

From this fact and (3.3.19), it implies that
\[ \tilde{p} \in C^{i^*-5}(\bar{\Omega} \times [0, T]). \]

By (3.3.10), \( \tilde{v} \in C([0, T]; V(\Omega)). \) So that \( \tilde{v}(., t) \in V(\Omega) \) for every \( t \in [0, T]. \)

Also by (3.3.10), the sequence \( (\tilde{v}_n(., 0)) \) converges to \( \tilde{v}(., 0) \) in \( V(\Omega). \)
Therefore, \( \|\tilde{v}(., 0)\|_2 = 0 \) since \( \|\tilde{v}_n(., 0)\|_2 = 0 \) for every \( n \in \mathbb{N} \) by the initial condition in (3.1.3). Besides, \( \tilde{v}(., 0) \in C(\bar{\Omega}) \) by (3.3.19). It follows that \( \tilde{v}(., 0) = 0 \) on \( \Omega. \) From this fact, (3.3.20), (3.3.22), (3.3.19), and (3.3.21), it deduces (3.3.1)-(3.3.2).

From (3.3.10) and (3.2.3), it deduces (3.3.3).
From (3.3.15), it follows (3.3.4).

The proof is complete.

\[ \square \]

4. The Navier-Stokes equations on bounded domains

Consider the Navier-Stokes equations (NS) at (1.0.1) in the case that
\[ \Omega \subset \mathbb{R}^3 \]
be a connected bounded open set with a smooth boundary, \( u_0 \in (C^\infty(\Omega))^3 \cap L^2(\Omega), \)
\[ f \in (C^\infty(\Omega) \times [0, +\infty))^3. \]

The Navier-Stokes system will be replaced by the system \((S^\#)\) as in the following definition

\[ \text{Definition 4.0.1.} \]

(1) Take \( i^* \in \mathbb{N}. \) The function \( \beta : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^3 \) is defined by
\[ \beta(x, t) := \sum_{k=0}^{i^*+1} \frac{u_o^{(k)}(x)}{k!} t^k, \quad \forall (x, t) \in \Omega \times [0, +\infty), \]
where \( u_o^{(k)} \) is given at (2.0.11).

(2) The function \( \theta : \Omega \times [0, +\infty) \rightarrow \mathbb{R}^3 \) is defined by
\[ \theta := -\partial_t \beta + \Delta \beta - (\beta \nabla)\beta + f. \]
Proposition 4.0.2. Let \( u_o, f \) satisfy (4.0.1) and \( i^* > 6 \).

1. Let \( u_o^{(i)} \) be given at (2.0.11). Then,
\[
u_o^{(i)} \in (C^\infty(\bar{\Omega}))^3 \cap L^2_\sigma(\Omega)
\]
for every \( i \in \mathbb{N} \).

2. Let \( \beta, \theta \) be given at (4.0.2) and (4.0.3). Then,
\[
\beta, \theta \in (C^\infty(\bar{\Omega} \times [0, +\infty)))^3, \quad \partial_t \beta \in C([0, +\infty); L^2_\sigma(\Omega)) \quad \forall i \in \mathbb{N}.
\]
\[
\partial_t \beta(., 0) = u_o^{(i)} \quad \text{for every } 0 \leq i \leq i^* + 1.
\]

For \( i \leq i^* \), the Leray projection of \( \partial_t \theta(., 0) \) onto \( L^2_\sigma(\Omega) \) is zero:
\[
\forall i \leq i^*, \mathcal{P}(\partial_t \theta(., 0)) = 0.
\]

Proof.

1. By Temam [24, Remark 1.6], for every \( m \in \mathbb{N} \), there exists a constant \( c(m, \Omega) \) such that
\[
||\mathcal{P}u||_{H^m(\Omega)} \leq c||u||_{H^m(\Omega)}, \quad \forall u \in H^m(\Omega).
\]
Combining this property with (4.0.11) and the assumption (4.0.1), it follows that
\[
\forall i \in \mathbb{N}, \forall m \in \mathbb{N}, u_o^{(i)} \in H^m(\Omega) \cap L^2_\sigma(\Omega).
\]
By the Sobolev embedding (see Adam [1, Theorem 5.4, Part I-Case C]), one gets (4.0.5).

2. From (4.0.2), (4.0.3), and (4.0.4), it implies the assertion (2).

3. Let \( i \leq i^* \). Differentiating (4.0.3) \( i \) time with respect to \( t \), and taking the Leray projection, one has
\[
\mathcal{P} \left( \partial_t \theta(., 0) \right) = \mathcal{P} \left( - \beta_o^{(i+1)} + \Delta \beta_o^{(i)} - \sum_{r=0}^{i} \binom{i}{r} \beta_o^{(r)} \cdot \nabla u_o^{(i-r)} + \partial_t f(., 0) \right),
\]
where \( \beta_o^{(k)} := \partial_t^k \beta(., 0) = u_o^{(k)} \) for every \( 0 \leq k \leq i^* + 1 \) by (4.0.2). From this fact and (2.0.11), it deduces
\[
\mathcal{P} \left( \partial_t \theta(., 0) \right) = \mathcal{P} \left( - u_o^{(i+1)} + \Delta u_o^{(i)} - \sum_{r=0}^{i} \binom{i}{r} u_o^{(r)} \cdot \nabla u_o^{(i-r)} + \partial_t f(., 0) \right)
\]
\[
= - u_o^{(i+1)} + \mathcal{P} \left( \Delta u_o^{(i)} - \sum_{r=0}^{i} \binom{i}{r} u_o^{(r)} \cdot \nabla u_o^{(i-r)} + \partial_t f(., 0) \right)
\]
\[
= - u_o^{(i+1)} + u_o^{(i+1)} = 0.
\]

The existence of global-in-time solutions to the system (S\#) is as in the following statement.
Lemma 4.0.3. Let \((4.0.1)\) be valid.
Then, the system \((S')\) has a solution \((v, p)\) satisfying
\[
v \in \left(C^{r-4}(\bar{\Omega} \times [0, +\infty))\right)^3, \quad p \in C^{r-5}(\bar{\Omega} \times [0, +\infty)).
\] (4.0.8)

Proof. By \((4.0.6)-(4.0.7)\), \(\beta\) satisfies \((3.1.1)\) with \(\beta\) instead of \(\tilde{\beta}\), and \(\theta\) satisfies \((3.1.1)\) with \(\theta\) instead of \(\tilde{\theta}\). Besides, the system \((S')\) has the same form as the system \((S)\). So that, the results for \(\tilde{v}\) in Section 3 also hold for \(v\) in this section.

From Lemma \((3.3.1)\) it deduces the assertion. \(\square\)

The existence of a smooth global-in-time solution to the Navier-Stokes (NS) (at \((1.0.1)\)) is as in the below statement.

Theorem 4.0.4. Let \((4.0.1)\) be valid.

Then, the Navier-Stokes system \((NS)\) has a unique solution \((u, p)\) satisfying
\[
u \in \left(C^\infty(\Omega \times [0, +\infty))\right)^3, \quad p \in C^\infty(\Omega \times [0, +\infty)).
\] (4.0.9)

Here, the uniqueness of \(p\) is up to a constant.

Proof.

– By Lemma \((4.0.3)\) the system \((S')\) possesses a solution \((v, p)\) satisfying \((4.0.8)\).

Put \(u := v + \beta\). Substituting \(v = u - \beta\) and the expression \((4.0.3)\) of \(\theta\) into the first equation of the problem \((S')\) yields
\[
\partial_t (u - \beta) - \Delta (u - \beta) = -(u - \beta) \cdot \nabla (u - \beta) - \beta \cdot \nabla (u - \beta) - (u - \beta) \cdot \nabla \beta - \nabla p - \partial_t \beta + \Delta \beta - \beta \cdot \nabla \beta + f.
\]

Simplifying this equation, it deduces
\[
\partial_t u = -u \cdot \nabla u + \Delta u - \nabla p + f. \tag{4.0.10}
\]

Besides, \(u(., 0) = v(., 0) + \beta(., 0) = 0 + u_o = u_o\). Furthermore, \(u \in L^2(\Omega)\) since \(v \in V(\Omega)\) and \(\beta \in L^2(\Omega)\). So that \((u, p)\) is a solution to the Navier-Stokes equations \((NS)\).

From \((4.0.6)-(4.0.8)\) and the fact \(u = v + \beta\), it deduces that \((u, p)\) satisfies
\[
(u, p) \in \left(C^{r-4}(\Omega \times [0, +\infty))\right)^3 \times C^{r-5}(\Omega \times [0, +\infty)).
\]

Since \(r^*\) is arbitrary, it follows that \((u, p)\) satisfies \((4.0.9)\).

– Now, let us prove the uniqueness of solution.

Assume that the Navier-Stokes system \((NS)\) has two solutions \((u_1, p_1)\), \((u_2, p_2)\) belonging to \((C^\infty(\Omega \times [0, +\infty))\)^3 \times \(C^\infty(\Omega \times [0, +\infty))\).

Let \(v_1 := u_1 - \beta, v_2 := u_2 - \beta\). Then \(v_1, v_2\) satisfy the system \((4.0.4)\):
\[
\begin{align*}
\partial_t v_1 - \Delta v_1 &= -(v_1 \cdot \nabla)v_1 - (\beta \cdot \nabla)v_1 - (v_1 \cdot \nabla)\beta - \nabla p_1 + \theta, \\
\partial_t v_2 - \Delta v_2 &= -(v_2 \cdot \nabla)v_2 - (\beta \cdot \nabla)v_2 - (v_2 \cdot \nabla)\beta - \nabla p_2 + \theta.
\end{align*}
\] (4.0.11)

Let \(w := v_2 - v_1\). Subtracting two equations at \((4.0.11)\) yields
\[
\partial_t w - \nu \Delta w = -w \cdot \nabla v_1 - v_2 \cdot \nabla w - (\beta \cdot \nabla)w - (w \cdot \nabla)\beta - \nabla (p_2 - p_1).
\]

Taking the \(L^2\)-inner product of this equation with \(w\), using \((2.0.21)\) and Hölder’s inequality, one gets
\[
\frac{1}{2} \frac{d}{dt} \|w\|^2_2 + \|\nabla w\|^2_2 \leq \|w\|^2_2 \|\nabla v_1\|_2 + \|\beta\|_6 \|\nabla w\|_2 \|w\|_3 + \|w\|^3_2 \|\nabla \beta\|^2_2.
\]
Using Gagliardo-Nirenberg’s inequality \([2.0.4]\) and Young’s inequality, then simplifying terms of \(\|\nabla w\|_2^2\), one obtains

\[
\frac{d}{dt}\|w\|_2^2 + 2\|\nabla w\|_2^2 \leq \|w\|_2^4 \|\nabla w\|_2 \|\nabla v_1\|_2 + \|\nabla w\|_2 \|w\|_2^2 \|\nabla w\|_2 + \|w\|_2^4 \|\nabla w\|_2^2.
\]

\[
\frac{d}{dt}\|w\|_2^2 \leq c_1 \|w\|_2^2 \|\nabla v_1\|_2^2 + c_1 \|w\|_2^4 + c_1 \|w\|_2^2.
\]

\[
\frac{d}{dt}\|w\|_2^2 \leq c_1 (\|\nabla v_1\|_2^2 + 2)\|w\|_2^2,
\]

where \(w(0,0) = v_2(0,0) - v_1(0,0) = (u_2(0,0) - \beta) - (u_1(0,0) - \beta) = u_o - u_o = 0\).

This is a Gronwall’s inequality, so that

\[
\forall t \in [0, +\infty), \|w(\cdot,t)\|_2^2 \leq e^{c_1} \int_0^t (\|\nabla (v_2 - \beta)\|_2^2 + 2) \|w(\cdot,0)\|_2^2 dt = 0.
\]

Besides, \(w\) is continuous on \(\Omega \times [0, +\infty)\). Therefore, \(w = 0\) on \(\Omega \times [0, +\infty)\), and so that

\[
v_2 = v_1 \text{ on } \Omega \times [0, +\infty).
\]

Hence, \(u_2 = u_1\) on \(\Omega \times [0, +\infty)\).

From \([4.0.12]\) and \([1.0.11]\), one gets \(\nabla (p_2 - p_1) = 0\) on \(\Omega \times [0, +\infty)\). Since \(\Omega\) is connected, there exists a constant \(c_1\) such that \(p_2 - p_1 = c_4\) on \(\Omega \times [0, +\infty)\). Therefore, the uniqueness of \(p\) is up to a constant.

The proof is complete. \(\square\)

5. The Navier-Stokes equations on the whole space

5.1. New unknown \(v\) with initial condition \(v(\cdot, 0) \equiv 0\).

Consider the incompressible Navier-Stokes equations (NS) (at \([1.0.1]\)) in the case that

\[
\begin{align*}
&u_o \in (C(\mathbb{R}^3))^3 \cap \mathbb{H}^i(\mathbb{R}^3) \cap L^2(\mathbb{R}^3), \forall i \in \mathbb{N}, \\
f \in (C^\infty(\mathbb{R}^3 \times [0, +\infty)))^3, \\
&\partial^j_t f \in C([0, +\infty), \mathbb{H}^i(\mathbb{R}^3)), \forall i \in \mathbb{N}, \forall j \in \mathbb{N}.
\end{align*}
\]

Definition 5.1.1.

1. In this section, let \((u_o^{(k)})_k\) be defined as at \([2.0.11]\) with \(\Omega = \mathbb{R}^3\).
2. Take \(i^* \in \mathbb{N}\). The function \(\beta : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3\) is defined by

\[
\beta(x,t) := \sum_{k=0}^{i^*+1} u_o^{(k)}(x) \frac{t^k}{k!}, \quad \forall (x,t) \in \mathbb{R}^3 \times [0, +\infty).
\]

3. The function \(\theta : \mathbb{R}^3 \times [0, +\infty) \to \mathbb{R}^3\) is defined by

\[
\theta := -\partial_t \beta + \Delta \beta - (\beta \nabla) \beta + f.
\]

4. The problem \((N)\) for unknown \((v, p)\) is

\[
\begin{align*}
\partial_t v - \Delta v &= -(v \cdot \nabla) v - (\beta \cdot \nabla) v - (v \cdot \nabla) \beta - \nabla p + \theta, \\
v(\cdot, 0) &= v_o = 0, \\
\text{div } v &= 0,
\end{align*}
\]

where \((x,t) \in \mathbb{R}^3 \times [0, +\infty)\).

5. For each \(k \in \mathbb{N}\), the problem \((N_k)\) for unknown \((v_k, p_k)\) is

\[
\begin{align*}
\partial_t v_k - \Delta v_k &= -(v_k \cdot \nabla) v_k - (\beta \cdot \nabla) v_k - (v_k \cdot \nabla) \beta - \nabla p_k + \theta, \\
v_k(\cdot, 0) &= v_{o,k} = 0, \\
v_k(\cdot, t) &\in V(\Omega) \cup V(\Omega_k),
\end{align*}
\]
Proposition 5.1.2. Let \((x, t) \in \Omega_k \times [0, +\infty). \) Here, \(\Omega_k := \{x \in \mathbb{R}^3: |x| < k\}.

(6) The \(n\)-th order Galerkin problem corresponding to the problem \((N_k)\) is

\[
\begin{aligned}
\partial_t v_{kn} + \mathcal{A} v_{kn} &= P_{kn}(- (v_{kn} \cdot \nabla) v_{kn} - (\beta \cdot \nabla) v_{kn} - (v_{kn} \cdot \nabla) \beta + \theta), \\
v_{kn}(. , 0) &= 0, \\
v_{kn} &\in V(\Omega_k),
\end{aligned}
\]

\[(5.1.6)\]

where \(P_{kn}\) is the projection from \(L^2(\Omega_k)\) onto the space spanned by the first \(n\) functions in the basis \(\mathcal{N}\) in \(L^2(\Omega_k)\) as in Lemma 2.0.3.

Proof.

(1) Let \(u^{(i)}_\sigma\) be given at (2.0.11) with \(\Omega = \mathbb{R}^3\). Then,

\(\mathcal{A}^{(i)} = (C^\infty(\mathbb{R}^3))^3 \cap H^2(\mathbb{R}^3) \cap L^2_0(\mathbb{R}^3)\) for every \(i, j \in \mathbb{N}\).

(2) Let \(\beta, \theta\) be given at (5.1.2) and (5.1.3). Then,

\[
\begin{align*}
\beta &\in (C^\infty(\mathbb{R}^3 \times [0, +\infty)))^3, \partial_t^j \beta \in C([0, +\infty); H^2(\mathbb{R}^3) \cap L^2_0(\mathbb{R}^3)) \forall i, j \in \mathbb{N}, \\
\theta &\in (C^\infty(\mathbb{R}^3 \times [0, +\infty)))^3, \partial_t^i \theta \in C([0, +\infty); H^2(\mathbb{R}^3)) \forall i, j \in \mathbb{N}, \\
\partial_t^j \beta(., 0) &= u^{(i)}_\sigma \quad \text{for every} \, 0 \leq i \leq i^* + 1.
\end{align*}
\]

(3) For every \(i \leq i^*\), the Leray projection of \(\mathcal{A}^{(i)} \theta(., 0) \in L^2(\mathbb{R}^3)\) onto \(L^2_0(\mathbb{R}^3)\) is zero:

\[
\forall i \leq i^*, \quad P(\mathcal{A}^{(i)} \theta(., 0)) = 0. 
\]

(4) For every \(i \leq i^*\), the Leray projection of \(\mathcal{A}^{(i)} \theta(., 0) \in L^2(\Omega_k)\) onto \(L^2_0(\Omega_k)\) is zero:

\[
\forall i \leq i^*, \quad \forall k \in \mathbb{N}, \quad P_k(\mathcal{A}^{(i)} \theta(., 0)) = 0. 
\]

Proof.

(1) By Robinson & Rodrigo & Sadowski [20 Lemma 2.9], the Leray projector \(P\) commutes with partial derivative on the whole space. Therefore, for any multi-index \(\alpha\) and \(w \in H^{|\alpha|}(\mathbb{R}^3)\),

\[
\|D^\alpha P(w)\|_{L^2(\mathbb{R}^3)} = \|P(D^\alpha w)\|_{L^2(\mathbb{R}^3)} \leq \|D^\alpha w\|_{L^2(\mathbb{R}^3)}
\]

It follows that

\[
\forall k \in \mathbb{N}, \forall w \in H^k(\mathbb{R}^3), \quad \|P(w)\|_{H^k(\mathbb{R}^3)} \leq \|w\|_{H^k(\mathbb{R}^3)}
\]

\[(5.1.12)\]

Let \(i \in \mathbb{N}\). By (5.1.1), \(u^{(i)}_{\sigma} \in H^{i+2}(\mathbb{R}^3)\) for every \(s \in \mathbb{N}\). Using (5.1.12) and (2.0.11), one gets \(u^{(i)}_{\sigma} \in H^s(\mathbb{R}^3)\) for every \(s \in \mathbb{N}\). Using the Sobolev embedding (see Adams [1 Theorem 5.4, Part I-Case C]), one obtains \(u^{(i)}_{\sigma} \in (C^\infty(\mathbb{R}^3))^3\).

It deduces the assertion (1).

(2) From (5.1.2), (5.1.3) and the assertion (1), it deduces the assertion (2).

(3) Let \(i \leq i^*\). One differentiates \((5.1.3)\) \(i\) times with respect to \(t\), and then takes the Leray projector to get

\[
\mathcal{P}(\mathcal{A} \theta(., 0)) = \mathcal{P}\left(- \partial_t \beta^{(i+1)} + \Delta \beta^{(i)} - \sum_{r=0}^{i} \left(\begin{array}{c}
\beta^{(r)} \\
\mathcal{P}(\mathcal{A} \theta)(., 0)
\end{array}\right) + \partial_t \beta^{(i+1)} f(., 0)\right),
\]

where \(\beta^{(k)} := \partial_t^k \beta(., 0) = u^{(k)}_{\sigma} \) for every \(0 \leq k \leq i + 1\) by (5.1.9).
From this fact and (7.1.1), one gets

\[
P\left(\partial_t^i \theta(.,0)\right) = P\left(-u_0^{(i+1)} + \Delta u_0^{(i)} - \sum_{r=0}^{i} \binom{i}{r} u_0^{(r)} \cdot \nabla u_0^{(i-r)} + \partial_t^i f(.,0)\right)
\]

\[
= -u_0^{(i+1)} + P\left(\Delta u_0^{(i)} - \sum_{r=0}^{i} \binom{i}{r} u_0^{(r)} \cdot \nabla u_0^{(i-r)} + \partial_t^i f(.,0)\right)
\]

\[
= -u_0^{(i+1)} + u_0^{(i+1)} = 0
\]

(4) Let \(i \leq i^*\) and \(k \in \mathbb{N}\).

By Lemma 2.4.3, there is an orthonormal basis \(N_k\) in \(L^2_2(\Omega_k)\) such that \(b \in V(\Omega_k) \cap (C^\infty(\Omega_k))^3\) for every \(b \in N_k\).

Let \(b \in N_k\) arbitrarily. The function \(b : \Omega_k \rightarrow \mathbb{R}^3\) is extended by zero outside \(\Omega_k\) and such extended function is denoted by \(\tilde{b} : \mathbb{R}^3 \rightarrow \mathbb{R}^3\). Then, \(\tilde{b} \in L^2_3(\mathbb{R}^3)\) and \(\int_{\mathbb{R}^3} \partial_t^i \theta(.,0) \cdot \tilde{b} \, dx = 0\) by (5.1.10). Therefore, \(\int_{\Omega_k} \partial_t^i \theta(.,0) \cdot b \, dx = 0\).

Hence, the Leray projection of \(\partial_t^i \theta(.,0) \in L^2(\Omega_k)\) onto \(L^2_2(\Omega_k)\) is zero.

The proof is complete. \(\square\)

5.2. The boundedness of the sequence of approximations \((v_k)_k\).

The sequence \((v_k)_k\) satisfies some properties as in the following statement

**Proposition 5.2.1.** Let \(u_o, f\) satisfy (5.1.1), \(i^* > 6\). Then, we have

(1) for each \(k \in \mathbb{N}\), the system \((N_k)\) has a solution \((v_k,p_k)\) satisfying

- \(v_k \in (C^{i^*-4}(\Omega_k \times [0, +\infty)))^3 \cap C([0, +\infty); V(\Omega_k))\),
- \(\tilde{p} \in C^{i^*-5}(\Omega_k \times [0, +\infty))\);
- \(\forall i \leq i^* - 4, \partial_t^i v_k(.,0) = 0\); \hspace{1cm} (5.2.1)

and there exists a subsequence of \((v_{kn})_n\), still denoted by \((v_{kn})_n\), such that

\[
\partial_t^i v_{kn} \text{ converges to } \partial_t^i v_k \text{ in } L^\infty(0,T; H^2(\Omega_k)), \forall i \leq i^* - 1, \hspace{1cm} (5.2.2)
\]

Here, \(\partial_t^i v_k\) is the classical partial derivative of \(v_k\) for every \(i \leq i^* - 4\), and \(v_{kn}\) is the solution of the system \((5.1.1)\).

(2) the solutions \((v_k)_k\) satisfies

\[
\forall T > 0, \exists C_1 \in (0, +\infty), \sup_{k \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{i \leq i^* - 4} \|\partial_t^i v_k(.,t)\|_{H^1(\Omega_k)} \leq C_1. \hspace{1cm} (5.2.3)
\]

**Proof.**

For each \(k \in \mathbb{N}\), by Proposition (5.1.2), the function \(\beta\) on \(\Omega_k \times [0, +\infty)\) satisfies the condition (5.1.1) with \(\beta\) instead of \(\tilde{\beta}\), the function \(\theta\) on \(\Omega_k \times [0, +\infty)\) also satisfies the condition (5.1.1) with \(\tilde{\theta}\) instead of \(\theta\). Besides, the system \((N_k)\) has the form of the system \((S)\), which is introduced at (5.1.2) in Section 3. So that in this section we can apply results in Section 3 about solutions \(\tilde{v}\) for \(v_k\) instead of \(\tilde{\tilde{v}}\).

By Lemma 5.3.1 for each \(k \in \mathbb{N}\), the system \((N_k)\) has a solution \((v_k,p_k)\) satisfying (5.2.1)-(5.2.2).

By Lemma 5.2.2, one gets

\[
\forall T > 0, \exists C_1 \in (0, +\infty), \sup_{k \in \mathbb{N}} \sup_{0 \leq t \leq T} \sup_{i \leq i^* - 4} \|\partial_t^i v_{kn}(.,t)\|_{H^1(\Omega)} \leq C_1. \hspace{1cm} (5.2.4)
\]
Observing that the inequalities, which are used in the proof of Lemma 5.2.3, are Hölder’s inequality, Young’s inequality, and Gagliardo-Nirenberg’s inequalities (2.0.2)–(2.0.4). So that the constants $C_1$ are uniform for all $\Omega_k$. From this fact and 

\[ (5.2.4) \]

it follows that

\[ (5.2.5) \]

Consider the subsequence at (5.2.2). Passing to the limit as $n \to \infty$ for (5.2.5), one gets (5.2.4). The boundedness of the sequence $(\partial_t^k v_k)_k$ is as in the following statement

**Proposition 5.2.2.** Let $u_\alpha, f$ satisfy (5.1.1), $i^* > 6$. Then, we have

1. For each $s \leq i^*-4$ and $T > 0$, the sequence $\left( \|\partial_t^{i^*-4-s} v_k\|_{C([0,T];H^s(\Omega_k))} \right)_k$ is bounded.
2. For each $T > 0$, the sequence $\left( \|v_k\|_{H^{i^*-4}(\Omega_k \times [0,T])} \right)_k$ is bounded.

**Proof.** Let $T > 0$.

Denote $I := i^*-4$. Let us prove by induction on $s$ that there exist constants $\tilde{c}_1, \ldots, \tilde{c}_I$ such that

\[ (5.2.6) \]

where

\[ (5.2.7) \]

- First, one sees that (5.2.8) holds for $s = 0, 1$ by (5.2.4).
- Take $\tilde{s}$ such that $2 \leq \tilde{s} \leq I$. Assume by induction that (5.2.8) holds for every $s \leq \tilde{s} - 1$. Then, there exist constants $\tilde{c}_0, \ldots, \tilde{c}_{\tilde{s}-1}$ such that

\[ (5.2.8) \]

Then, for every $k \in \mathbb{N}$, $t \in [0,T]$, $s \leq \tilde{s} - 1$ and $i + s \leq I$,

\[ (5.2.9) \]

where $\tilde{c}_s = \tilde{c}_0 + \cdots + \tilde{c}_{\tilde{s}-1}$.

- Let us prove that (5.2.8) holds for $s = \tilde{s}$.

Take $i \in \mathbb{N}$ such that

\[ (5.2.10) \]

Differentiating (5.1.3) $i$ times with respect to $t$, it implies that for each $t \in [0,T]$

\[ (5.2.11) \]
Applying Lemma 2.4.3 (the regularity of the Stoke problem), there exists a constant $c_2$, which is uniform for all $\Omega_k$, such that for every $t \in [0, T]$, \[
\|D^\delta v_k^{(i)}(\cdot, t)\|_{L^2(\Omega_k)} \leq c_2\|v_k^{(i+1)}\|_{H^{\delta-2}(\Omega_k)} + c_2\|\theta^{(i)}\|_{H^{\delta-2}(\Omega_k)}
\] \[+ c_2\sum_{r=0}^{j} \delta^{(j)} v_k^{(r)}(v_k^{(i-r)} + \beta(r) \nabla v_k^{(i-r)} + \nabla \theta^{(i-r)})\|_{H^{\delta-2}(\Omega_k)}
\] (5.2.12)

The terms on the right-hand side are estimated as follows:

(a) By (5.2.10), \((i+1) + (s-2) < i + s \leq I\). Therefore, by (5.2.13), \[
\|v_k^{(i+1)}\|_{H^{\delta-2}(\Omega_k)} \leq \bar{c}_{s-2}.
\] (5.2.13)

(b) By Hölder’s inequality, Gagliardo-Nirenberg’s inequality (Lemma 2.0.1), \[
\sum_{\mid \alpha \mid = \bar{s}-2} \|v_k^{(r)}\|_{D^\alpha \nabla v_k^{(i-r)}\|L^4(\Omega_k)} \leq \sum_{\mid \alpha \mid = \bar{s}-2} \|v_k^{(r)}\|_{L^4(\Omega_k)}\|D^\alpha \nabla v_k^{(i-r)}\|_{L^4(\Omega_k)}
\] \[\leq \sum_{\mid \alpha \mid = \bar{s}-2} \|v_k^{(r)}\|_{L^2}^\frac{1}{2} \|D^\alpha \nabla v_k^{(i-r)}\|_{L^4(\Omega_k)}^\frac{1}{2} (\|D^\alpha \nabla v_k^{(i-r)}\|_{L^4(\Omega_k)}^\frac{1}{2} \|D^\alpha \nabla v_k^{(i-r)}\|_{L^4(\Omega_k)}^\frac{1}{2} + \|D^\alpha \nabla v_k^{(i-r)}\|_{L^4(\Omega_k)}^\frac{1}{2}).
\]

Besides, by (5.2.10) one has \((i-r) + (\bar{s}-2+1) \leq i + (\bar{s}-1) \leq I\), \((i-r) + (\bar{s}-2+2) \leq i + \bar{s} \leq I\). Applying (5.2.12) and (5.2.13) yields \[
\sum_{\mid \alpha \mid = \bar{s}-2} \|v_k^{(r)}\|_{D^\alpha \nabla v_k^{(i-r)}\|L^4(\Omega_k)} \leq c_0 \bar{c}_1 (c_{\bar{s}-1} \mu_k(\bar{s})^\frac{1}{2} + \bar{c}_{\bar{s}-1}) \leq \mu_k(\bar{s})^\frac{1}{2} + 1.
\]

Similarly, one can prove that \[
\sum_{\mid \alpha \mid \leq \bar{s}-2} \|D^\alpha (v_k^{(r)} \cdot \nabla v_k^{(i-r)})\|_{L^2(\Omega_k)} \leq \mu_k(\bar{s})^\frac{1}{2} + 1.
\]

So that \[
c_2 \sum_{r=0}^{j} \mathcal{C}_r \|v_k^{(r)} \cdot \nabla v_k^{(i-r)}\|_{H^{\delta-2}(\Omega_k)} \leq \mu_k(\bar{s})^\frac{1}{2} + 1.
\] Using Young’s inequality, it deduces that \[
c_2 \sum_{r=0}^{j} \mathcal{C}_r \|v_k^{(r)} \cdot \nabla v_k^{(i-r)}\|_{H^{\delta-2}(\Omega_k)} \leq \frac{1}{4\mu_k(\bar{s})} + c_3.
\] (5.2.14)

(c) In the same manner to establish (5.2.14), there exists $c_4$ such that \[
c_2 \sum_{r=0}^{j} \mathcal{C}_r \|\beta(r) \cdot \nabla v_k^{(i-r)}\|_{H^{\delta-2}(\Omega_k)} \leq \frac{1}{4\mu_k(\bar{s})} + c_4.
\] (5.2.15)

(d) In the same manner to establish (5.2.14), there exists $c_5$ such that \[
c_2 \sum_{r=0}^{j} \mathcal{C}_r \|v_k^{(r)} \cdot \nabla \beta^{(i-r)}\|_{H^{\delta-2}(\Omega_k)} \leq c_5.
\] (5.2.16)

Hence, from (5.2.12), (5.2.14), there exists constants $c_6$ such that \[
\forall i \leq I - \bar{s}, \forall k \in \mathbb{N}, \forall t \in [0, T], \|D^\delta v_k^{(i)}\|_{L^2(\Omega_k)} \leq \frac{1}{4\mu_k(\bar{s})} + c_6
\] \[
\forall i \leq I - \bar{s}, \forall k \in \mathbb{N}, \sup_{t \in [0, T]} \|D^\delta v_k^{(i)}\|_{L^2(\Omega_k)} \leq \frac{1}{4\mu_k(\bar{s})} + c_6
\]
\[
\forall k \in \mathbb{N}, \sum_{i=0}^{l-s} \sup_{t \in [0,T]} \| D^s v_k^{(i)} \|_{L^2(\Omega_h)} \leq \frac{I}{2I} \mu_k(s) + I_{C_6} \ni \exists \forall k \in \mathbb{N}, \mu_k(s) \leq \frac{1}{2} \mu_k(s) + c_7
\]

\[
\forall k \in \mathbb{N}, \mu_k(s) - \frac{1}{2} \mu_k(s) \leq c_7
\]

\[
\forall k \in \mathbb{N}, \mu_k(s) \leq 2c_7.
\]

So that, (5.2.6) holds for \( \bar{s} \). The proof of (5.2.6) is complete.

Therefore, for each \( s \leq i^* - 4 \), the sequence \( \sup_{t \in [0,T]} \| \partial_t^{i^*-4-s} v_k(\cdot, t) \|_{H^s(\Omega_h)} \) is bounded. Besides, \( \partial_t^{i^*-4-s} v_k \in C([0, T]; H^s(\Omega_h)) \) by (5.2.4). It deduces the assertion (1).

From the assertion (1), it deduces the assertion (2). \( \square \)

5.3. The existence and the regularity of solutions.

The existence of solutions to the problem (N) (at (5.1.4)) is as in the following statement

**Lemma 5.3.1.** Let (5.1.4) be valid, \( i^* \geq 12 \).

Then, the problem (N) has a solution \((v, p)\) satisfying

\[
\cdot v \in (C^{i^*-8}(\mathbb{R}^3 \times [0, +\infty))^3), \quad p \in C^{i^*-9}(\mathbb{R}^3 \times [0, +\infty));
\]

\[
\forall s \leq i^* - 6, \partial_t^{i^*-6-s} v \in C([0, +\infty); H^s(\mathbb{R}^3)).
\]

**Proof.** Let \( T > 0 \) arbitrarily.

(1) - Let \( h_o \) be a function \( \mathbb{R} \rightarrow [0, 1] \) satisfying

\[
h_o \in C^\infty(\mathbb{R}) \text{ and } h_o(s) = \begin{cases} 1 & \text{for } s \leq \frac{1}{4} \\ 0 & \text{for } s \geq \frac{3}{4} \end{cases}
\]

Take \( k \in \mathbb{N} \). Let \( \eta_k : \mathbb{R}^3 \rightarrow [0, 1] \) be the function defined by

\[
\eta_k(x) = h_o(|x| - (k - 1))
\]

Obviously,

\[
\forall i \in \mathbb{N}, \exists c > 0, \forall k \in \mathbb{N}, \sup_{|\alpha| \leq i, x \in \mathbb{R}^3} |D^\alpha \eta_k(x)| < c.
\]

One sees that \( v_k \) is defined on the bounded domain \( \Omega_k \) and \( v_k \in (C^{i^*-4}(\bar{\Omega}_k \times [0, T]))^3 \) by (5.2.4). From this function, a function \( \dot{v}_k \) on \( \mathbb{R}^3 \) is defined

\[
\forall k \in \mathbb{N}, \dot{v}_k(x, \cdot) := \begin{cases} \eta_k(x)v_k(x, \cdot) & \text{for } x \in \Omega_k, \\ 0 & \text{for } x \in \mathbb{R}^3 \setminus \Omega_k. \end{cases}
\]

Obviously, \( \dot{v}_k \in (C^{i^*-4}(\mathbb{R}^3 \times [0, T]))^3 \).

By Proposition 5.2.2 and (5.3.4), the sequence \( (\dot{v}_k)_k \) is bounded in the space \( H^t(\mathbb{R}^3 \times (0, T)) \). By the Rellich-Kondrachov theorem (see Adams [1] Theorem 6.2-Part II), there exist a subsequence of \( (\dot{v}_k)_k \), still denoted by \( (\ddot{v}_k)_k \), and \( v \) such that

\[
(\ddot{v}_k)_k \text{ converges to } v \text{ in the space } (C^{i^*-7}(\mathbb{R}^3 \times (0, T)))^3.
\]

Here, \( v \in (C_B^{i^*-7}(\mathbb{R}^3 \times (0, T)))^3 \), so that

\[
v \text{ belongs to the space } (C_B^{i^*-8}(\mathbb{R}^3 \times [0, T]))^3.
\]
NECESSARY CONDITION FOR GLOBAL SOLUTION TO NAVIER-STOKES EQUATIONS}

Besides, \( \tilde{v}_k = v_k \) on \( B_x \times (0, T) \), and so that \( D^\alpha \partial^\beta_t \tilde{v}_k = D^\alpha \partial_t^\beta v_k \) on \( B_x \times (0, T) \) for \( |\alpha| \leq 4, i \leq 1 \). By the pointwise convergence \( (\ref{equation:3.3}) \) and the fact that \( (v_k, p_k) \) solves the problem \( (N_k) \)\( (\ref{equation:1.1}) \), one gets

- \( D^\alpha v(x, t) = \lim_{k \rightarrow \infty} D^\alpha \tilde{v}_k(x, t) = \lim_{k \rightarrow \infty} D^\alpha v_k(x, t) \).
- \( \text{div} \ v(x, t) = \lim_{k \rightarrow \infty} \text{div} \ \tilde{v}_k(x, t) = \lim_{k \rightarrow \infty} \text{div} v_k(x, t) = 0 \).
- \( \nabla \times \varphi(x, t) = \lim_{n \rightarrow \infty} \nabla \times \tilde{\varphi}_n(x, t) = \lim_{k \rightarrow \infty} \nabla \times \varphi_k(x, t) = 0 \).

where

\[ \varphi := \partial_t v - \Delta v + (v \cdot \nabla) v + (\beta \cdot \nabla)v + (v \cdot \nabla) \beta - \theta, \]

\[ \tilde{\varphi}_n := \partial_t \tilde{v}_n - \Delta \tilde{v}_n + (\tilde{v}_n \cdot \nabla) \tilde{v}_n + (\beta \cdot \nabla) \tilde{v}_n + (\tilde{v}_n \cdot \nabla) \beta - \theta, \]

\[ \varphi_k := \partial_t v_k - \Delta v_k + (v_k \cdot \nabla) v_k + (\beta \cdot \nabla) v_k + (v_k \cdot \nabla) \beta - \theta. \]

So that for all \( (x, t) \in \mathbb{R}^3 \times (0, T) \)

- \( D^\alpha v(x, t) = \lim_{k \rightarrow \infty} D^\alpha v_k(x, t) \).
- \( \text{div} v(x, t) = 0 \).
- \( \nabla \times \varphi(x, t) = 0 \).

By \( (\ref{equation:3.7}) \), \( \nabla \times \varphi \in (C_B^\infty(\mathbb{R}^3 \times [0, T]))^3 \). From this fact and \( (\ref{equation:3.10}) \), it deduces \( \nabla \times \varphi(x, t) = 0 \) for every \( (x, t) \in \mathbb{R}^3 \times [0, T] \). So that for each \( t \in [0, T] \), \( \varphi(., t) \) is a \( C^1 \)- conservative vector field in \( \mathbb{R}^3 \) by Marsden & Tromba \[17\], Chapter 8-Theorem 7, and so that there exists a function \( p(., t) \) such that \( \nabla(-p(., t)) = \varphi(., t) \). Hence, for every \( (x, t) \in \mathbb{R}^3 \times [0, T] \)

\[ \partial_t v - \Delta v + (v \cdot \nabla) v + (\beta \cdot \nabla) v + (v \cdot \nabla) \beta - \theta = -\nabla p. \]

Consider the sequence \( (\tilde{v}_k)_k \) at \( (\ref{equation:3.3}) \). By \( (\ref{equation:5.3.3}) \), \( (\ref{equation:5.3.4}) \) and Proposition \( (\ref{proposition:2.2}) \), the sequence \( (\tilde{v}_k)_k \) is bounded in the space \( C([0, T]; \mathbb{H}^3(\mathbb{R}^3)) \subset L^\infty(0, T; \mathbb{H}^2(\mathbb{R}^3)) \), and the sequence \( (\partial_t \tilde{v}_k)_k \) is bounded in the space \( L^\infty(0, T; \mathbb{H}^1(\mathbb{R}^3)). \) By Theorem \( (\ref{theorem:2.0.3}) \), there exists a subsequence of \( (\tilde{v}_k)_k \), still denoted by \( (\tilde{v}_k)_k \), converges in \( C([0, T]; \mathbb{H}^1(\mathbb{R}^3)) \). Combining this fact and the convergence \( (\ref{equation:5.3.6}) \), it follows that \( (\tilde{v}_k)_k \) converges to \( v \) in \( C([0, T]; \mathbb{H}^1(\mathbb{R}^3)) \). Hence,

- \( v \in C([0, T]; \mathbb{H}^1(\mathbb{R}^3)) \)
- \( (\tilde{v}_k(., 0))_k \) converges to \( v(., 0) \) in the space \( \mathbb{H}^1(\mathbb{R}^3) \).

Besides, \( \tilde{v}_k(., 0) \) is bounded in \( \mathbb{R}^3 \) since \( v_k(., 0) = 0 \) on \( \Omega_k \) (see \( (\ref{equation:5.1.5}) \)). Combining this fact, \( (\ref{equation:5.3.13}) \) and \( (\ref{equation:5.3.7}) \), it deduces that

\[ \forall x \in \mathbb{R}^3, \ v(x, 0) = 0. \]

Because \( (\ref{equation:5.3.11}) \), \( (\ref{equation:5.3.14}) \), \( (\ref{equation:5.3.9}) \), \( (\ref{equation:5.3.7}) \) hold for all \( T > 0 \), it deduces \( (\ref{equation:5.3.4}) \).

By Proposition \( (\ref{proposition:5.2.2}) \) and \( (\ref{equation:5.3.3}) \), one has the sequence

\[ (\|\partial_t^{i_0 - s - 4} \tilde{v}_k\|_{C([0, T]; \mathbb{H}^s(\mathbb{R}^3))})_k \]

is bounded for every \( s \leq i_0 - 4, \ T > 0 \).

Apply Theorem \( (\ref{theorem:2.0.3}) \) ( Aubin-Lions-Simon Theorem), one obtains

\[ \partial_t^{i_0 - s - 6} v \in C([0, T]; \mathbb{H}^s(\mathbb{R}^3)) \]

for every \( s \leq i_0 - 6, \ T > 0 \). Since \( T > 0 \) is
arbitrary, it follows that $\partial_t^{i^*-6} v \in C([0, +\infty); H^s(\mathbb{R}^3))$ for every $s \leq i^*-6$.
It deduces \([5.3.2]\).

The existence of solutions to the Navier-Stokes equations \((1.0.1)\) is as in the following statement

**Theorem 5.3.2.** Let $\Omega = \mathbb{R}^3$ and \([5.1.1]\) be valid.
Then, the Navier-Stokes system \((NS)\) has a unique solution \((u, p)\) satisfying

\[
\begin{align*}
\text{\bullet } u &\in (C^\infty(\mathbb{R}^3 \times [0, +\infty))^3, \quad p \in C^\infty(\mathbb{R}^3 \times [0, +\infty)); \\
\text{\bullet } \forall i \in \mathbb{N}, j \in \mathbb{N}, \partial_t^i u \in C([0, +\infty); H^j(\mathbb{R}^3)).
\end{align*}
\]

Here, the uniqueness of $p$ is up to a constant.

**Proof.**

1. Let $i^* \geq 12$. The system \((N)\) has a unique solution $(v, p)$ satisfying \([5.3.1]-5.3.2\).

   Put

   \[
u = u + \beta \quad (**)
\]

   Substituting $v = u - \beta$ and the expression \([5.1.3]\) of $\theta$ into the first equation of the problem \((N)\) yields

   \[
\partial_t (u - \beta) - \Delta (u - \beta) = -(u - \beta) \cdot \nabla (u - \beta) - (u - \beta) \cdot \nabla (u - \beta) - \nabla p - \partial_t \beta + \Delta \beta - \beta \cdot \nabla \beta + f.
\]

   Simplifying this equation, one obtains $\partial_t u = -u \cdot \nabla u + \Delta u - \nabla p + f$. Besides, $u(.,0) = v(.,0) + \beta(.,0) = 0 + u_0 = u_o$ and $\text{div } u = \text{div } (v + \beta) = 0$. So that $(u, p)$ is a solution to the Navier-Stokes equations \((NS)\).

   From \([5.3.17], \ 5.1.7\), and Lemma \([5.3.1]\) it follows that

   \[
\begin{align*}
\text{\bullet } u &\in (C^{i^*-8}(\mathbb{R}^3 \times [0, +\infty))^3, \quad p \in C^{i^*-9}(\mathbb{R}^3 \times [0, +\infty)); \\
\text{\bullet } \forall s \leq i^*-6, \partial_t^{i^*-6-s} u \in C([0, +\infty); H^s(\mathbb{R}^3)).
\end{align*}
\]

2. Since this fact holds for all $i^* > 10$, it follow \([5.3.13], 5.3.16\).

   Now, let us prove the uniqueness of solutions

   Let $(u_2, p_2)$ be a second solution to the Navier-Stokes system \((NS)\) \((1.0.1)\) on $\mathbb{R}^3 \times [0, +\infty)$ satisfying \([5.3.15], \ 5.3.16\).

   Denote $w = u_2 - u$.

   Two solutions satisfy the first equation at \((1.0.1)\):

   \[
\begin{align*}
\partial_t u - \nu \Delta u = -u \cdot \nabla u - \nabla p + f; \\
\partial_t u_2 - \nu \Delta u_2 = -u_2 \cdot \nabla u_2 - \nabla p_2 + f.
\end{align*}
\]

   Subtracting the first equation from the second equation, then taking the inner product of the result with $w$, one gets

   \[
\begin{align*}
\int_{\mathbb{R}^3} \partial_t w \cdot w \ dx - \nu \int_{\mathbb{R}^3} \Delta w \cdot w \ dx &= -\int_{\mathbb{R}^3} (w \cdot \nabla u) \cdot w \ dx - \int_{\mathbb{R}^3} (u_2 \cdot \nabla w) \cdot w \ dx, \\
\frac{d}{dt} \int_{\mathbb{R}^3} |w|^2 \ dx + 2\nu \int_{\mathbb{R}^3} |\nabla w|^2 \ dx &\leq 2 \int_{\mathbb{R}^3} |\nabla u| |w|^2 \ dx.
\end{align*}
\]

   Using Hölder’s inequality, Gagliardo-Nirenberg’s inequality and Young’s inequality, one obtains

   \[
\begin{align*}
\frac{d}{dt} \|w\|_*^2 + 2\nu \|\nabla w\|_*^2 &\leq \|\nabla u\|_2 \|w\|_2^2 \\
\frac{d}{dt} \|w\|_*^2 + 2\nu \|\nabla w\|_*^2 &\leq \|\nabla u\|_2 \|w\|_2^2 \|\nabla w\|_2^2
\end{align*}
\]
\[ \frac{d}{dt} \|w\|_2^2 + 2\nu \| \nabla w\|_2^2 \leq c_2 \| \nabla u\|_2 \|w\|_2^2 + 2\nu \| \nabla w\|_2^2 \]
\[ \frac{d}{dt} \|w\|_2^2 + 2\nu \| \nabla w\|_2^2 \leq c_2 \| \nabla u\|_2 \|w\|_2^2 + 2\nu \| \nabla w\|_2^2 \]

Let \( T > 0 \) arbitrarily. For every \( t \in [0, T] \), one gets
\[ \frac{d}{dt} \|w\|_2^2 \leq c_3 \|w\|_2^2, \quad (5.3.19) \]
where \( c_3 := c_2 \sup_{0 \leq t \leq T} \| \nabla u\|_2^2 \). This is a Gronwall’s inequality, so that
\[ \forall t \in [0, T], \|w(\cdot, t)\|_2^2 \leq e^{c_3 t} \|w(\cdot, 0)\|_2^2 = 0, \]
where \( w(\cdot, 0) = u_2(\cdot, 0) - u(\cdot, 0) = u_o - u_o = 0 \) on \( \mathbb{R}^3 \).

Hence, \( \|w(\cdot, t)\|_2^2 = 0 \) for every \( t \in [0, +\infty) \).

From this property and the fact that \( w \in (C(\mathbb{R}^3 \times [0, +\infty]))^3 \), it deduces \( w \) vanishes on \( \mathbb{R}^3 \times [0, +\infty) \). So that \( u_2 = u \) on \( \mathbb{R}^3 \times [0, +\infty) \). Hence, the existence of \( u \) is unique.

From (5.3.18) and the uniqueness of \( u \), one gets \( \nabla (p_2 - p) = 0 \). Combining this property with the fact \( (p_2 - p) \in C^1(\mathbb{R}^3 \times [0, +\infty)) \) yields that there exists a constant \( c_4 \) such that \( p_2 = p + c_4 \) on \( \mathbb{R}^3 \times [0, +\infty) \). Hence, the existence of \( p \) is unique up to a constant.

\[ \square \]

References

[1] R. A. Adams, *Sobolev Spaces*. Academic Press, 1975.
[2] P. Constantin and C. Foias, *Navier-Stokes equations*. University of Chicago Press, Chicago, IL, 1988.
[3] C. L. Fefferman, *Existence and Smoothness of the Navier-Stokes Equation*. Millennium problems. Clay Mathematics Institute.
[4] H. Dong and D. Du, *On the local smoothness of solutions of the Navier-Stokes equations*. J. math. fluid mech. 9, 139–152 (2007).
[5] F. Boyer and P. Fabrie, *Mathematical Tools for the Study of the Incompressible Navier-Stokes Equations and Related Models-Theorem V.2.10*. Springer Science- Business Media New York, 2013.
[6] C. Foias & R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*. J. Funct. Anal. 87, 359-369, 1989.
[7] Y. Giga, *Time and spatial analyticity of solutions of the Navier-Stokes equations*. Comm. Partial Differential Equation 8, p.929-948, 1983.
[8] Y. Giga, *Solutions for semilinear parabolic equations in \( L^p \) and regularity of weak solutions of the Navier-Stokes system*. J. Differential equation 62 (1986), no. 2, 186-212.
[9] R. Guberovič, *Smoothness of Koch-Tataru solutions to the Navier-Stokes equation revisited*. Discrete Contin. Dyn. Syst., 27 (2010) 231-236.
[10] M. Hieber & J. C. Robonson & Y. Shibata, *Mathematical Analysis of the Navier-Stokes Equations*. Springer, Italy, 2017.
[11] T. Kato, *Strong \( L^p \) solutions of the Navier-Stokes equations in \( \mathbb{R}^n \) with applications to weak solutions*. Math. Zeit. 187, 471-480, 1951.
[12] O.A. Ladyzhenskaya, *The Mathematical Theory of Viscous Incompressible flow*. Gordon and Breach, New York, 1969.
[13] O.A Ladyzhenskaya & G.A. Seregin, *On partial regularity of suitable weak solutions to the three-dimensional Navier-Stokes equations*. J. Math. Fluid Mech. 1, 356-387, 1999.
[14] Pierre Gilles Lemarié- Rieusset, *The Navier-Stokes Problem in the 21st Century*. CRC Press, 2016.
[15] Z. Grujić and I. Kukavica, *Space analyticity for the Navier-Stokes and related equations with initial data in \( L^p \)*. J. Funct. Anal., 152 (1998) 447-466.
[16] J. Leray *Essai sur le mouvement d’un liquide visqueux emplissant l’espace*. Acta Math. J. 63, 193–248. 1934.

[17] Marsden & Tromba *Vector Calculus*. W. H. Freeman and Company 2003

[18] L. Nirenberg *On elliptic partial differential equations* Annali della Scuola Normale Superiore di Pisa (3) 13: 115–162. 1959.

[19] D. Le, Vu T. Nguyen *Global and blow up solutions to cross diffusion systems on 3D domains* Proc. Amer. Math. Soc. 144 (2016), no. 11, 4845- 4859.

[20] J. C. Robinson, J. L. Rodrigo, W. Sadowski *The Three-Dimensional Navier-Stokes Equations*. Cambridge University Press, 2016.

[21] Seregin, Gregory *Lecture Notes On Regularity Theory For The Navier-Stokes Equations*. World Scientific, 2015.

[22] J. Serrin *On the interior regularity of weak solutions of the Navier-Stokes equations*. Arch. Rational Mech. Anal. 9 (1962), 187-195.

[23] H. Sohr *The Navier-Stokes Equations, An Elementary Functional Analytic Approach. Chapter V-Theorem 1.8.2*. Springer Basel AG, 2001.

[24] R. Temam *Navier-Stokes Equations - Theory and Numerical Analysis*. AMS Chelsea Publishing, 2001.

[25] R. Temam *Navier-Stokes Equations and Nonlinear Functional Analysis. Chapter 6-Condition (6.21)*. Siam, 1995.

[26] James Serrin *On the interior regularity of weak solutions of Navier-Stokes equations*. Arch. Rational Mech., Anal. 9 (1962), 187-195.

[27] Y. Giga *Solutions for semilinear parabolic equations in $L^p$ and regularity of weak solutions of the Navier-Stokes system*. J. Differential Equation 62 (1986), no 2, 186-212.

[28] T. Tsai *Lectures on Navier-Stokes Equations*. American mathematical Society, 2018.

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