1. Introduction

The purpose of this paper is three-fold. On the one hand – and that was its original motivation – we establish an asymptotic estimate for the subgroup growth of Fuchsian groups, that is, groups $\Gamma$ of the form

$$
\Gamma = \left\langle x_1, \ldots, x_r, y_1, \ldots, y_s, u_1, v_1, \ldots, u_t, v_t \right| \quad (1)
$$

with integers $r, s, t \geq 0$ and $e_1, \ldots, e_s \geq 2$, and $a_1, \ldots, a_r \in \mathbb{N} \cup \{\infty\}$.

**Theorem A.** Let $\Gamma$ be a Fuchsian group such that

$$
\alpha(\Gamma) := \sum_i \left( 1 - \frac{1}{a_i} \right) + \sum_j \frac{2}{e_j} + 2(t - 1) > 0,
$$

and let

$$
\mu(\Gamma) = \sum_i \left( 1 - \frac{1}{a_i} \right) + s + 2(t - 1)
$$

be the hyperbolic measure of $\Gamma$. Then the number $s_n(\Gamma)$ of index $n$ subgroups in $\Gamma$ satisfies an asymptotic expansion

$$
s_n(\Gamma) \approx \delta L_\Gamma(n!)^{\mu(\Gamma)} \Phi_\Gamma(n) \left\{ 1 + \sum_{\nu=1}^{\infty} a_\nu(\Gamma)n^{-\nu/m_\Gamma} \right\}, \quad (n \to \infty). \quad (3)
$$

Here,

$$
\delta = \begin{cases} 
2, & \forall i : a_i \text{ finite and odd, } \forall j : e_j \text{ even} \\
1, & \text{otherwise},
\end{cases}
$$

$$
L_\Gamma = (2\pi)^{-1/2-\sum_i(1-1/a_i)}(a_1 \cdots a_r)^{-1/2} \exp \left( -\sum_{i \mid a_i} \frac{1}{2a_i} \right),
$$

$$
\Phi_\Gamma(n) = n^{3/2-\sum_i(1-1/a_i)} \exp \left( \sum_{i=1}^{r} \sum_{t \mid a_i, t \lt a_i} \frac{n^{t/a_i}}{t} \right),
$$

$$
m_\Gamma = [a_1, a_2, \ldots, a_r],
$$
and the $a_{\nu}(\Gamma)$ are explicitly computable constants depending only on $\Gamma$.

On the other hand, the proof of Theorem A requires the representation theoretic approach initiated in [26], and large parts of the present paper are concerned with various statistical aspects of symmetric groups, and rather subtle estimates for values and multiplicities of their characters, which are also of independent interest. In particular, we show the following.

**Theorem B.** Let $\varepsilon > 0$ and an integer $q$ be given, $n$ sufficiently large, and let $\chi$ be an irreducible character of $S_n$.

(i) We have $|\chi(c)| \leq (\chi(1))^{1-\delta}$ with

$$\delta = \left(\left(1 - 1/(\log n)\right)^{-1} \cdot \frac{12\log n}{\log(n/f)} + 18\right)^{-1},$$

where $c$ is any conjugacy class of $S_n$ with $f$ fixed points.

(ii) We have

$$\sum_{\pi \in S_n} |\chi(\pi)| \leq (\chi(1))^{\frac{1}{q} + \varepsilon} \sum_{\pi \in S_n} 1.$$

(iii) Let $m_{\chi}^{(q)}$ be the multiplicity of $\chi$ in the $q$-th root number function of $S_n$. Then

$$m_{\chi}^{(q)} \leq (\chi(1))^{1-2/q + \varepsilon}.$$

All these bounds are essentially best possible. In characteristic zero, estimates for character values and multiplicities are among the least understood topics in the representation theory of symmetric groups. In recent times, additional interest in this circle of problems was sparked by the theory of random walks on finite groups. In this context, Theorem B enables us to prove the following.

**Theorem C.** Let $c$ be a conjugacy class in $S_n$. Denote by $t_c(c)$ the least even integer such that $t_c(c)$ elements chosen at random from $c$ have, with probability $\geq 1 - \frac{1}{n}$, no common fixed point, and let $t_s(c)$ be the mixing time for the random walk generated by $c$. Then, for $n \geq 4000$, we have

$$t_c(c) \leq t_s(c) \leq 10t_c(c).$$

Theorem C establishes in full generality a conjecture of Roichman; cf. [28, Conj. 6.6]. For special choices of $c$, Roichman’s conjecture had already been known to hold: Diaconis and Shahshahani [8] established it for transpositions, Roichman [28] generalized their result to conjugacy classes having at least $cn$ fixed points, and Fomin and Lulov [11] established a character bound implying Theorem C for conjugacy classes having only cycles of the same length.

In the remainder of this introduction, we describe in more detail the organization of our paper, and the contents of individual sections. After a short introduction to random walks on finite groups generated by conjugacy classes and their connection with character theory, Section 2 establishes Theorem C and the first part of Theorem B. This includes the proof of a variety of preliminary character estimates for arbitrary conjugacy classes, which are used throughout the paper. The main tools here are the hook formula and the Murnaghan-Nakayama rule. Section 3 gives the proof of Theorem B (ii).
preparation for this argument, we derive a number of results concerning the statistics of symmetric groups, mostly dealing with the distribution of cycles in various subsets of $S_n$; cf. Subsection 3.1. Again, this group of results is also used in other sections. The theory developed up to this point would already enable us to estimate the subgroup growth of Fuchsian groups with $s = 0$; that is, none of the generators $y_1, \ldots, y_s$ in (1) are present. However, in order to deal with Fuchsian groups in full generality, we also need some insight into the growth behaviour of multiplicities of root number functions for symmetric groups, measured against the degree of the corresponding irreducible character; in particular, we have to establish Theorem B (iii). Section 4 is devoted to the proof of this and related results.

Proof and discussion of Theorem A are the principal themes of Section 5. Following the argument establishing Theorem A, we demonstrate that condition (2) is in fact necessary. More specifically, we show the following.

**Theorem D.** Let $\Gamma$ be as in (1) with $r = t = 0$ and $\alpha(\Gamma) < 0$ (that is, $\Gamma$ is the one-relator group associated with the defining relation $y_1^{e_1}y_2^{e_2} \cdots y_s^{e_s} = 1$). Then, as $n$ tends to infinity, we have

$$s_n(\Gamma) \sim K(n!)^{\mu(\Gamma) - \alpha(\Gamma)/2} \exp \left( \sum_{j=1}^{s} \sum_{\nu_j < e_j} \frac{n^{\nu_j/e_j}}{\nu_j} + \frac{\alpha(\Gamma) - 2\mu(\Gamma) + 2}{4} \log n \right).$$

This is the contents of Theorem 4, where also the constant $K$ is given explicitly. According to Theorem D, the subgroup growth of these one-relator groups is faster than might be expected in view of Theorem A. In Subsection 5.2 we discuss the explicit computation of the coefficients $a_{\nu}(\Gamma)$ in general, and, as an example, compute the first 22 of these coefficients for the triangle group $\Gamma(2, 3, 7)$, only 10 of which turn out to be non-vanishing. As a further application of Theorem B (iii), we also determine the subgroup growth of Demuškin groups (Poincaré groups of dimension 2). These are known to be one-relator groups essentially associated with a defining relation of the form

$$R = x_1^h [x_1, x_2][x_3, x_4] \cdots [x_{m-1}, x_m], \quad h \in \mathbb{N} \cup \{\infty\}, \quad p \geq 3 \text{ prime.}$$

In Subsection 5.4 we prove the following slightly more general result.

**Theorem E.** For integers $q \geq 1$ and $d \geq 2$, let

$$\Gamma_{q,d} = \left\langle x_1, y_1, \ldots, x_d, y_d \mid x_1^q, x_1y_1z_1^{-1}x_1 \cdots x_dy_d = 1 \right\rangle.$$

Then there exist explicitly computable constants $\gamma_{\nu}(\Gamma_{q,d})$, such that

$$s_n(\Gamma_{q,d}) \approx \delta n(n!)^{2d-2} \left\{ 1 + \sum_{\nu=1}^{\infty} \gamma_{\nu}(\Gamma_{q,d}) n^{-\nu} \right\}, \quad n \to \infty,$$

where

$$\delta = \begin{cases} 1, & q \text{ even} \\ 2, & q \text{ odd} \end{cases}.$$

Introduce an equivalence relation $\sim$ on the class of finitely generated groups via

$$\Gamma \sim \Delta : \Leftrightarrow s_n(\Gamma) = (1 + o(1))s_n(\Delta), \quad n \to \infty.$$
In [24] Theorem 3] a characterization in terms of structural invariants is given for the equivalence relation \( \sim \) on the class of groups \( \Gamma \) of the form

\[
\Gamma = G_1 \ast G_2 \ast \cdots \ast G_s \ast F_r
\]

with \( r, s \geq 0 \) and finite groups \( G_r \), and it is shown that each \( \sim \)-class of free products decomposes into finitely many isomorphism classes. Our final section is concerned with the analogous problems for Fuchsian groups.

**Theorem F.** The multiset \( \{a_1, a_2, \ldots, a_r\} \) together with the numbers \( \mu(\Gamma) \) and \( \delta \) form a complete system of invariants for the equivalence relation \( \sim \) on the class \( \mathcal{F} \) of all Fuchsian groups \( \Gamma \) satisfying \( \alpha(\Gamma) > 0 \).

Theorem F allows us to construct an infinite sequence of pairwise non-isomorphic Fuchsian groups, all of which are \( \sim\)-equivalent to the same Fuchsian group \( \Gamma \); in particular there cannot be a finiteness result for the relation \( \sim \) on \( \mathcal{F} \). The situation changes, if we take into account the full precision of \( \mathcal{B} \) in Theorem A. More specifically, consider three refinements of the equivalence relation \( \sim \) on \( \mathcal{F} \): (i) the relation \( \approx \) of strong equivalence defined via

\[
\Gamma \approx \Delta :\Leftrightarrow s_n(\Gamma) = s_n(\Delta)(1 + \mathcal{O}(n^{-A})) \text{ for every } A > 0,
\]

(ii) isomorphy, and (iii) equality of the system of parameters

\[
(r, s, t; a_1, a_2 \ldots, a_r, e_1, e_2, \ldots, e_s)
\]

in the Fuchsian presentation \( \mathcal{P} \), denoted \( \Gamma = \Delta \). Clearly,

\[
\Gamma = \Delta \Rightarrow \Gamma \approx \Delta \Rightarrow \Gamma \approx \Delta \Rightarrow \Gamma \sim \Delta.
\]

It can be shown that all these implications are in fact strict. For these relations, we have the following surprising result.

**Theorem G.** Each \( \approx \)-equivalence class of \( \mathcal{F} \) decomposes into finitely many classes with respect to \( \sim \); that is, each group \( \Gamma \in \mathcal{F} \) has only finitely many presentations of the form \( \mathcal{P} \), and is \( \approx \)-equivalent to at most finitely many non-isomorphic \( \mathcal{F} \)-groups.

**Some notation.** Permutations are denoted by \( \pi \), \( \sigma \), or \( \tau \). For \( \pi \in S_n \) and \( 1 \leq i \leq n \), let \( s_i(\pi) \) be the number of \( i \)-cycles of \( \pi \). The support \( \text{supp}(\pi) \) of \( \pi \) is the set of points moved by \( \pi \). For number partitions we mostly follow the conventions of [20] Chap. I, Sec. 1]. Specifically, a partition \( \lambda = (\lambda_1, \lambda_2, \ldots) \) is a weakly decreasing sequence of non-negative integers \( \lambda_j \), such that \( \lambda_j = 0 \) for \( j \) sufficiently large. The weight \( |\lambda| \) of \( \lambda \) is \( |\lambda| = \sum \lambda_j \), and the norm \( ||\lambda|| \) of \( \lambda \) is the largest \( j \) such that \( \lambda_j \neq 0 \). As usual, we write \( \lambda \vdash n \) for \( |\lambda| = n \), and say that \( \lambda \) is a partition of \( n \). For partitions \( \lambda, \mu \) we write \( \mu \subseteq \lambda \), if \( \mu_j \leq \lambda_j \) for all \( j \) (that is, the Ferrers diagram of \( \mu \) is contained in the Ferrers diagram of \( \lambda \)). For a partition \( \lambda \), we denote by \( \lambda' \) the conjugate partition: \( \lambda'_i = \max\{j : \lambda_j \geq i\} \) (that is, the Ferrers diagram of \( \lambda' \) is obtained from that of \( \lambda \) by reflection through the main diagonal). By \( \lambda \setminus \lambda_1 \) we mean the partition \( (\lambda_2, \lambda_3, \ldots) \) (that is, the Ferrers diagram of \( \lambda \setminus \lambda_1 \) is obtained from that of \( \lambda \) by deleting the first row). Whenever convenient, we shall denote the integer \( |\lambda| - \lambda_1 \) by \( \Delta \). For a partition \( \lambda \vdash n \), we denote by \( \chi_{\lambda} \) the irreducible character of \( S_n \) associated with \( \lambda \). For a finite group \( G \), let \( \text{Irr}(G) \) be the set of irreducible characters of \( G \). The usual scalar product on the space \( \mathbb{C}^G \) is denoted by \( \langle \cdot, \cdot \rangle_G \), or simply by \( \langle \cdot, \cdot \rangle \) if \( G \) is a symmetric group.
We use what we believe to be standard number theoretic notation. Specifically, the partition function is denoted \( p(n) \), \( \tau(n) \) and \( \sigma(n) \) are the number of divisors and the sum of divisors of \( n \), respectively, \( S(n, m) \) is the number of (set theoretic) partitions of an \( n \)-set into \( m \) non-empty parts (a Stirling number of the second kind). For integers \( m \) and \( n \), we denote their greatest common divisor and least common multiple by \( (m, n) \) respectively. For arithmetic functions \( f, g \), we denote their greatest common divisor and least common multiple by \( (f, g) \) and \( \text{lcm}(f, g) \). Asymptotic equivalence is denoted by \( \sim \): we write \( f(n) \sim g(n) \) to mean \( \lim_{n \to \infty} \frac{f(n)}{g(n)} = 1 \). We use \( \approx \) to denote asymptotic expansions in the sense of Poincaré; for instance we write

\[
f(n) \approx g(n) \left\{ 1 + \sum_{\nu=1}^{\infty} a_\nu n^{-\nu/q} \right\}, \quad (n \to \infty).
\]

Finally, we use some notation from probability theory: \( 1_X \) denotes the characteristic function for a subset \( X \subseteq \Omega \) of the sample space \( \Omega \), and \( \mathbf{E} \xi \) is the expected value of the random variable \( \xi \).

2. Character Estimates and Random Walks on Symmetric Groups

2.1. Roichman’s Conjectures. Let \( \mathbf{E} = E_1, E_2, E_3, \ldots \) be a Markov chain on a metric space \( X \). The random walk \( (x_k)_{k \geq 0} \) on \( X \) determined by \( \mathbf{E} \) is by definition the collection of all infinite paths on \( X \) with probability distribution induced by \( \mathbf{E} \). If \( X \) is finite, then \( \mathbf{E} \) can be determined by its transition matrix \( P \). In what follows, we will be interested in the case when \( X \) is a finite symmetric group given with the discrete metric.\(^1\) Let \( X = S_n \), and let \( 1 \neq c \subseteq S_n \) be a non-trivial conjugacy class. The random walk \( w_c \) generated by \( c \) has initial state \( x_0 = 1 \) and the transition matrix \( P_c = (p_{\pi, \sigma})_{\pi, \sigma \in S_n} \) where

\[
p_{\pi, \sigma}^c := \begin{cases} \frac{1}{|c|}, & \pi \sigma^{-1} \in c \\ 0, & \text{otherwise.} \end{cases}
\]

The distribution in the \( k \)-th step of \( w_c \) is given by the convolution formula

\[
P(x_k = \pi) = \frac{1}{|c|} \sum_{\sigma \in S_n} P(x_{k-1} = \pi \sigma^{-1}).
\]

More generally, for two functions \( f, g : S_n \to \mathbb{C} \), the convolution \( f \ast g : S_n \to \mathbb{C} \) is given by

\[
(f \ast g)(\pi) = \sum_{\sigma \in S_n} f(\sigma) g(\pi \sigma^{-1});
\]

in particular, \( P(x_k = \pi) \) is the \( k \)-fold convolution of the function \( \frac{1}{|c|} 1_c \). In the sequel we shall always take \( k \) even, to avoid parity problems. Given a norm \( \| \cdot \| \) on the complex algebra \( \mathbb{C}^{S_n} \) and \( \varepsilon > 0 \), we say that the random walk \( w_c \) has reached \( \varepsilon \)-equidistribution with respect to \( \| \cdot \| \) in step \( k \), if

\[
\left\| P(x_k = \pi) - \frac{2}{n!} 1_{A_n} \right\| < \varepsilon \cdot \left\| \frac{2}{n!} 1_{A_n} \right\|.
\]

\(^1\)Cf. [7] for more details on random walks on finite groups and their applications.
We define the statistical mixing time \( t_\text{s}(c) \) of \( w_c \) as the least even integer \( k \) for which \( w_c \) has reached \( \frac{1}{n} \)-equidistribution with respect to the \( \ell^2 \)-norm. A first lower bound for \( t_\text{s}(c) \) is given by the combinatorial mixing time \( t_\text{c}(c) \) of \( c \), that is, the least even integer \( k \), such that any \( k \) elements of \( c \) have no common fixed point with probability at least \( 1 - \frac{1}{n} \). In [28], Roichman conjectured that for every non-trivial conjugacy class \( c \subseteq A_n \),

\[
t_\text{s}(c) \ll t_\text{c}(c). \tag{4}
\]

His main result [28, Theorem 6.1] establishes this conjecture for classes \( c \) with \( cn \) fixed points. Roichman suggests an approach to the general conjecture (4) via a certain estimate for character values in symmetric groups. More precisely, he conjectures that, for every \( \epsilon > 0 \), \( n \) sufficiently large, each conjugacy class \( c \subseteq S_n \), and every partition \( \lambda \vdash n \)

\[
|\chi_\lambda(c)| \leq \chi_\lambda(1) \left( \max \left( \frac{\lambda_1}{n}, \frac{\|\lambda\|}{n}, \frac{1}{e} \right) \right)^{(1-\epsilon)n \log \frac{n}{n-\text{supp}(c)+1}}, \tag{5}
\]

which would imply (4). Unfortunately, as it stands, estimate (5) is false. This can be seen, for instance, as follows. For \( c \) fixed-point free, \( \lambda \) such that \( \lambda_1, \|\lambda\| \leq \frac{n}{e} \), and \( \epsilon = \frac{1}{2} \), (5) simplifies to

\[
|\chi_\lambda(c)| \leq \chi_\lambda(1)e^{-(n \log n)/2}. \tag{6}
\]

The right-hand side of (6) is bounded above by \( \sqrt{n!}n^{-n/2} < 1 \); that is, for \( c \) and \( \lambda \) as above, and \( n \) sufficiently large, it would follow that \( \chi_\lambda(c) = 0 \). Since the irreducible characters \( \{\chi_\lambda : \lambda \vdash n, \max(\lambda_1, \|\lambda\|) > n/e \} \) would generate the space of class functions on \( S_n \), this would imply that, for \( n \) sufficiently large, the characters

\[
\{\chi_\lambda : \lambda \vdash n, \max(\lambda_1, \|\lambda\|) > n/e \}
\]

would generate the space of class functions on the set of fixed-point free conjugacy classes of \( S_n \). Comparing the size of the former set with the dimension of the latter space, we find that, for large \( n \),

\[
2 \sum_{0 \leq \nu \leq n-n/e} p(\nu) \geq p(n) - p(n-1), \tag{7}
\]

where \( p(n) \) is the number of partitions of \( n \). The right-hand side can be estimated via the first term of Rademacher’s series expansion for \( p(n) \) (see for example [1, Theorem 5.1]) to give

\[
p(n) - p(n-1) \sim \frac{\pi e^\pi \sqrt{n}}{12 \sqrt{2n^{3/2}}}, \quad n \to \infty.
\]

Bounding the left-hand sum in (7) by means of the estimate\(^2\) \( p(n) < \frac{\pi}{\sqrt{6n}}e^\pi \sqrt{2n/3} \) we obtain

\[
2 \sum_{0 \leq \nu \leq n-n/e} p(\nu) \leq 2np(n - \lfloor n/e \rfloor) \leq 2\pi \sqrt{n/6}e^\pi \sqrt{2(n-1)/n}. \tag{8}
\]

From these two estimates it is clear that inequality (7) is violated for large \( n \).

However, the basic idea behind Roichman’s approach turns out to be correct. As a substitute for (5), we prove the following.

\(^2\)Cf., for instance, [17, Satz 7.6].
Theorem 1. For sufficiently large $n$, a non-trivial conjugacy class $c \subseteq S_n$, and a partition $\lambda \vdash n$, we have
\[ |\chi_\lambda(c)| \leq (\chi(1))^{1-1/6t_c(c)} \]  
and, for $1 \leq s_1(c) \leq n-2$,
\[ |t_c(c) - \frac{2 \log n}{\log(n/s_1(c))}| \leq 3, \]
whereas $t_c(c) = 2$ for $s_1(c) = 0$.

This result in turn allows us to establish Roichman’s conjecture \[4\] for the mixing time of random walks on symmetric groups.

Theorem 2. For $n \geq 4000$ and each non-trivial conjugacy class $c \subseteq S_n$, we have
\[ t_c(c) \leq t_s(c) \leq 10t_c(c). \]

The constants in Theorems 1 and 2 are most certainly not optimal, but we have not attempted to tighten our numerical estimates.

This section is organized as follows. In Subsection 2.2 we describe the general connection between random walks on finite groups and character estimates, and we explain, how Theorem 2 can be deduced from Theorem 1. The next subsection establishes certain elementary estimates for values and degrees of irreducible characters of symmetric groups, which will be used throughout the paper. Finally, Subsections 2.4 and 2.5 contain the proof of Theorem 1.

2.2. Character Theory and Random walks. Here, we describe the connection between character theory and probability measures on finite groups. For a more detailed presentation, see \[7\] and \[30\]. Let $G$ be a finite group. For a class function $f : G \to \mathbb{C}$ and an irreducible character $\chi$ of $G$, define the Fourier coefficient $\alpha_\chi(f)$ by means of the equation
\[ f(g) = \sum_\chi \alpha_\chi(f)\chi(g), \quad g \in G. \]
Since irreducible characters form a basis of the space of class functions, the Fourier coefficients exist and are uniquely defined by this equation. We first state without proof some facts about Fourier coefficients.

Lemma 1. 
(i) $\alpha_\chi(f) = \langle f, \chi \rangle$,
(ii) $\alpha_\chi(f \ast g) = \frac{\alpha_\chi(f)\alpha_\chi(g)}{\chi(1)}$,
(iii) $\sum_\chi |\alpha_\chi(f)|^2 = \frac{1}{|G|} \sum_{g \in G} |f(g)|^2$.

Using Lemma 1 we obtain a bound for the statistical mixing time of $w_c$.

Lemma 2. Let $c \subseteq G$ be a conjugacy class, and let $k$ be an integer. Then we have
\[ |G| \left\| \left( \frac{1}{|G|}1_c \right)^k - \frac{1}{|G|}1 \right\|_2^2 = \sum_{\chi \neq \chi_0} \frac{|\chi(c)|^{2k}}{(\chi(1))^{2k-2}}, \]
where \( \chi_0 \) is the trivial character.

**Proof.** Noting that \( \alpha_\chi(1) = \delta_{\chi,\chi_0} \), this follows from Lemma \[1\]. \qed

In contrast to \[7\], we will only consider the \( \ell^2 \)-norm. The passage from the \( \ell^2 \)-norm to \( \ell^p \)-norms with \( p \leq 2 \) is immediate by Hölder’s inequality; however, in the context of random walks, the passage from \( \ell^2 \) to \( \ell^\infty \) is possible as well.

**Lemma 3.** Let \( G \) be a finite group, and let \( f : G \to [0, \infty) \) be a probability measure. Then we have

\[
\left\| f * f - \frac{1}{|G|} \right\|_\infty \leq \left\| f - \frac{1}{|G|} \right\|_2^2.
\]

In particular, if \( w_c \) reaches \( \varepsilon \)-equidistribution with respect to the \( \ell^2 \)-norm after \( k \) steps, it reaches \( \varepsilon^2 \)-equidistribution with respect to the \( \ell^\infty \)-norm after \( 2k \) steps.

**Proof.** We have

\[
(f * f)(g) = \sum_{h \in G} f(h) f(gh^{-1})
\]

\[
= \sum_{h \in G} \left( f(h) - \frac{1}{|G|} \right) \left( f(gh^{-1}) - \frac{1}{|G|} \right) + \frac{2}{|G|} \sum_{h \in G} f(h) - \frac{1}{|G|}
\]

\[
= \frac{1}{|G|} + \sum_{h \in G} \left( f(h) - \frac{1}{|G|} \right) \left( f(gh^{-1}) - \frac{1}{|G|} \right),
\]

since \( \sum f(h) = 1 \). Applying the Cauchy-Schwarz-inequality to the last sum, our claim follows. \qed

We can now explain how to deduce Theorem \[2\] from Theorem \[1\]. Let \( c \subseteq S_n \) be a non-trivial conjugacy class. Arguing as in the proof of Lemma \[2\] we know that \( w_c \) reaches \( \frac{1}{n} \)-equidistribution with respect to the \( \ell^2 \)-norm after \( k \) steps, if and only if

\[
\sum_\chi \frac{\left| \chi(c) \right|^{2k}}{(\chi(1))^{2k-2}} \leq \frac{1}{n}.
\]

By Theorem \[1\] the left-hand side can be bounded above by

\[
\sum_\chi \left( \frac{\chi(1)}{\chi(1)} \right)^{2k - \frac{2k(1-1/(\log n))}{6t_c(c)}}.
\]

For \( k \geq 10t_c(c) \), this in turn is less than

\[
\sum_\chi \left( \frac{\chi(1)}{\chi(1)} \right)^{-5/4},
\]

and from \[26\] Theorem 1 we deduce that the latter quantity is \( O(n^{-5/4}) \). Hence, for \( n \) sufficiently large, we obtain the bound \( t_s(c) \leq 10t_c(c) \). We postpone the argument
leading to the lower bound $n \geq 4000$ to the end of the proof of Theorem 1 in Subsection 2.4.

2.3. Estimates for Character Values. Our main tools in this subsection are the hook formula for the dimension of $\chi_\lambda$ and the Murnaghan-Nakayama rule.

The hook formula. We have

$$\chi_\lambda(1) = \frac{n!}{\prod_{(i,j) \in \lambda} h_{i,j}},$$

where $h_{i,j}$ is the hook length of the box $(i,j)$.

The Murnaghan-Nakayama rule. Let $\pi = \sigma \gamma$ be the disjoint product of $\sigma \in S_{n-k}$ and a $k$-cycle $\gamma$. Then we have

$$\chi_\lambda(\pi) = \sum_{\tau} (-1)^{l(\tau)} \chi_{\lambda \setminus \tau}(\sigma).$$

Here, the sum extends over all rim hooks $\tau$ of size $k$ in $\lambda$, and $l(\tau)$ is the leg length of $\tau$.

Let $\lambda$ be a partition of $n$. By $sq(\lambda)$ we mean the side length of the largest square contained in the Ferrers diagram of $\lambda$; that is, the largest $j$ such that $\lambda_j \geq j$. Note that for $\lambda \vdash n$ we have

$$(sq(\lambda) - 1)sq(\lambda) \leq n - \lambda_1,$$

which we will apply mostly in the simpler version $sq(\lambda) \leq \sqrt{n - \lambda_1} + 1$. The quantity $sq(\lambda)$ leads to a useful inequality for $\chi_\lambda(1)$.

Lemma 4. Let $\lambda$ be a partition of $n$, and let $s = sq(\lambda)$. Then

$$\chi_\lambda(1) \geq \left( \frac{n}{s^2} \right) \left( \frac{s}{n} \right)^s (s^2)!. $$

Proof. Each of the $n - s^2$ points of $\lambda$ outside the maximal square lies in precisely $s$ hooks $h_{ij}$ with $i, j \leq s$, while the point $(i, j)$ with $i, j \leq s$ lies in exactly $i + j - 1$ such hooks. Hence,

$$\sum_{i,j \leq s} h_{ij} = s(n - s^2) + \sum_{i,j \leq s} (i + j - 1) = sn.$$

By the arithmetic-geometric inequality, this gives

$$\prod_{i,j \leq s} h_{ij} \leq \left( \frac{1}{s^2} \sum_{i,j \leq s} h_{ij} \right)^{s^2} = \left( \frac{n}{s} \right)^{s^2}.$$

Bounding the product of the hook lengths corresponding to points outside the maximal square by $(n - s^2)!$, our claim follows from the hook formula.

Our next result gives an upper bound for the modulus of character values $\chi_\lambda(c)$ in terms of $sq(\lambda)$ and the number of cycles of $c$. 

Lemma 5. Let \( c \subseteq S_n \) be a conjugacy class with \( c \) cycles, and let \( \lambda \vdash n \) be a partition. Then we have
\[
|\chi_{\lambda}(c)| \leq (2\text{sq}(\lambda))^c.
\]

Proof. If \( \mu \subseteq \lambda \) is a partition, then certainly \( \text{sq}(\mu) \leq \text{sq}(\lambda) \), hence, arguing by induction on \( c \) and applying the Murnaghan-Nakayama rule, it suffices to show that for any given \( k \), a partition \( \lambda \) has at most \( 2\text{sq}(\lambda) \) removable rim hooks of length \( k \). Let \( r \) be such a rim hook. The right-uppermost box of \( r \) is either to the right of the maximal square contained in the Ferrers diagram of \( \lambda \), or the left-lowest box of \( r \) is below the maximal square of \( \lambda \), or both. Since the right-uppermost box of a rim hook is always at the end of a row, while the left-lowest box is at the end of a column, our claim follows. \( \square \)

If \( \lambda_1 \) is of similar size as \( n \), Lemma 5 is of little use. In this case we will apply the following.

Lemma 6. Let \( \lambda \vdash n \) be a partition, and let \( c \subseteq S_n \) be a conjugacy class with \( c \) cycles and \( f \) fixed points. Then we have the bounds
\[
|\chi_{\lambda}(c)| \leq \chi_{\lambda \setminus \lambda_1}(1) \sum_{a, b \geq 0} \binom{f}{a} \binom{c - f}{b},
\]
(14)

and
\[
|\chi_{\lambda}(c)| \leq n \max_{\nu \leq n - \lambda_1} (2\sqrt{n - \lambda_1})^\nu \binom{c}{\nu},
\]
(15)

which improves on (14) if \( c \) is considerably smaller then \( n - \lambda_1 \).

Proof. Neglecting the sign in the Murnaghan-Nakayama rule, we see that the modulus of \( \chi_{\lambda}(c) \) is bounded above by the number of possible ways to completely deconstruct \( \lambda \) by removing rim hooks of sizes given by the cycle structure of \( c \). To prove the first estimate, we classify these deconstructions by means of the number \( a \) of fixed points of \( c \) contained within \( \lambda \setminus \lambda_1 \), and the corresponding number \( b \) of cycles of lengths \( \geq 2 \). Given \( a \) and \( b \), there are \( \binom{f}{a} \) ways to choose the fixed points of \( c \) to be removed from \( \lambda \setminus \lambda_1 \), and \( \binom{c - f}{b} \) ways to choose the corresponding set of cycles. Once these sets are given, their are at most \( \chi_{\lambda \setminus \lambda_1}(1) \) ways to remove these fixed points and cycles.

For the second bound we argue in a similar manner, this time bounding the number of deconstructions of \( \lambda \setminus \lambda_1 \) as in the proof of Lemma 5. \( \square \)

The next lemma will be useful in computing values of characters of small degrees.

Lemma 7. Let \( \lambda \vdash n \) be a partition, \( \mu = \lambda \setminus \lambda_1 \), and let \( \pi \in S_n \) be a permutation. Then
\[
\chi_{\lambda}(\pi) = \sum_{\tilde{\mu} \subseteq \mu, \tilde{\mu}_1 = 1} (-1)^{\tilde{\mu}} \sum_{c \subseteq S_{|\mu|-|\tilde{\mu}|}} \chi_{\mu, \tilde{\mu}}(c) \prod_{i \leq |\mu|} \binom{s_i(\pi)}{c_i},
\]
where \( c \) runs over all conjugacy classes of \( S_{|\mu|-|\tilde{\mu}|} \), \( \chi_{\mu, \tilde{\mu}}(c) \) denotes the number of ways to obtain \( \tilde{\mu} \) from \( \mu \) by removing rim hooks according to the cycle structure of \( c \), counted
with the sign prescribed by the Murnaghan-Nakayama rule, and \( c_i \) is the number of \( i \)-cycles of an element of \( \mathbf{c} \).

**Proof.** We may assume that \( \lambda_1 > 2|\mu| \), and that \( \pi \) contains a cycle of length \( > |\mu| \). For, if we replace \( \lambda_1 \) by \( \lambda_1 + 2n \), and add a cycle of length \( 2n \) to \( \pi \), both conditions are satisfied; on the other hand, the only way to remove a cycle of length \( 2n \) from the new partition is within the first row, hence \( \chi_{\lambda}(\pi) \) is not affected by these changes. Now we use the Murnaghan-Nakayama rule to remove all cycles of length \( \leq |\mu| \). If we are left with a partition not of the form \( \lambda = (\bar{\lambda}_1, 1, \ldots, 1) \), the remaining partition cannot be removed by deleting rim hooks of length \( > |\mu| \), so \( \bar{\mu} = \lambda \setminus \bar{\lambda}_1 \) can be assumed to be of the form \((1, \ldots, 1)\). Since \( \bar{\lambda} \) can be removed in precisely one way by deleting rim hooks of lengths \( > |\mu| \), and all but the last one are contained in the first row, the value of \( \chi_{\lambda}(\bar{\pi}) \) is \((-1)^{|\bar{\mu}| \lambda_1} \), where \( \bar{\pi} \) is the element obtained from \( \pi \) by removing all cycles of lengths \( \leq |\mu| \). The rim hooks which are removed and do not contain any box from the first row define a conjugacy class \( \mathbf{c} \) within \( S_{|\mu| - |\bar{\mu}|} \); counting all possible placements of the cycles of \( \mathbf{c} \) in \( \mu \setminus \bar{\mu} \) with the sign prescribed by the Murnaghan-Nakayama rule yields \( \chi_{\mu, \bar{\mu}}(\mathbf{c}) \). Finally, for fixed \( \mathbf{c} \), the set of \( i \)-cycles to be placed in \( \mu \setminus \bar{\mu} \) among all \( i \)-cycles of \( \pi \) can be chosen in \( \binom{n}{c_i} \) ways, and our claim follows. \( \square \)

Finally, we shall also need the following lower bounds for character degrees.

**Lemma 8.** Let \( \lambda \vdash n \) be a partition. Then
\[
\text{(i) } \chi_{\lambda}(1) \geq 2^{n/8}, \quad \|\lambda\| \leq \lambda_1 \leq 3n/4; \\
\text{(ii) } \chi_{\lambda}(1) \geq \left( \frac{\lambda_1}{n - \lambda_1} \right) \chi_{\lambda \setminus \lambda_1}(1), \quad \lambda_1 \geq n/2.
\]

**Proof.** (i) We distinguish the cases \( \lambda_1 \leq n/4, \lambda_1 \geq n/4 \) and \( \lambda_2 \leq n/8, \lambda_1 \geq n/4, \lambda_2 \geq n/8 \). In the first case it was shown in [26, Formula (23)] that \( \chi_{\lambda}(1) \geq (3/2)^{n/4} \geq 2^{n/8} \). In the second case, for any given \( \lfloor n/8 \rfloor \)-tuple \((t_1, \ldots, t_{\lfloor n/8 \rfloor})\) of 0’s and 1’s, we can start to deconstruct \( \lambda \) by choosing in the \( i \)-th step a box from the first row, if \( t_i = 1 \), and from \( \lambda \setminus \lambda_1 \), if \( t_i = 0 \). Hence, there are at least \( 2^{n/8} \) ways of deconstruction. In the final case, note that \( \chi_{\lambda}(1) \geq \chi_{\mu}(1) \) for any partition \( \mu \) contained in \( \lambda \); choosing \( \mu = (\lfloor n/4 \rfloor, \lfloor n/8 \rfloor) \), our claim follows by applying to \( \mu \) the argument used in the second case.

(ii) This follows as in the proof of [26, Formula (21)], observing that the hook product \( H[(\lambda_2, \ldots, \lambda_k)] \) equals \( \frac{(n - \lambda_1)!}{\chi_{\lambda \setminus \lambda_1}(1)!} \). \( \square \)

2.4. **Proof of Theorem 1.** The proof of Theorem 1 makes use of the following two auxiliary results, whose proofs will be given in the next subsection.

**Lemma 9.** Let \( \mathbf{c} \subseteq S_n \) be a non-trivial conjugacy class, and let \( \pi \) be the element visited by the random walk \( w_\mathbf{c} \) after \( 3t_\mathbf{c}(\mathbf{c}) \) steps. Then, for each \( k \geq 1 \), the probability that \( \pi \) has more than \( k \) fixed points is bounded above by
\[
\max \left( \frac{2^k}{(k-1)!}, \frac{2^{n/2}}{([n/2] - 1)!} \right).
\]
Lemma 10. Let $c_1, c_2 \subseteq S_n$ be non-trivial conjugacy classes with $f_1$ respectively $f_2$ fixed points. For $i = 1, 2$, let $x_i \in c_i$ be chosen at random. Then the probability that $x_1$ and $x_2$ have $l$ common fixed points, is at most

$$\binom{n}{l} \left( \frac{f_1 f_2}{n^2} \right)^l.$$

Moreover, the probability for $x_1 x_2$ to have $k$ cycles on $\text{supp}(x_1) \cup \text{supp}(x_2)$ is bounded above by

$$(\log n)^{k-1}/(k-1)!.$$ 

The proof of Theorem 1 now proceeds as follows. Define $g_1 : S_n \to [0, \infty)$ to be the density of the random walk $w_c$ after $3t_c(c)$ steps, and let $g_2$ be the corresponding density after $6t_c(c)$ steps. Using the fact that $g_1$ is a class function, we decompose $g_1$ as

$$g_1 = \sum_{e'} \alpha_{e'} 1_{e'},$$

and compute $g_2$ in the form

$$g_2(\pi) = \sum_{\sigma \in S_n} g_1(\sigma) g_1(\pi \sigma^{-1}) = \sum_{c_1, c_2} \alpha_{c_1} \alpha_{c_2} |c_1| |c_2| P_{k,l}(c_1, c_2)$$

From Lemmas 5, 9 and 10 we deduce that

$$|\langle g_2, \chi_\lambda \rangle| \leq \frac{1}{n!} \sum_{c_1, c_2} \alpha_{c_1} \alpha_{c_2} \sum_{c_1 \in c_1} (2\text{sq}(\lambda))^\# \text{cycles of } c_1 c_2$$

$$= \frac{1}{n!} \sum_{k,l} (2\text{sq}(\lambda))^{k+l} \sum_{c_1, c_2} \alpha_{c_1} \alpha_{c_2} |c_1| |c_2| P_{k,l}(c_1, c_2)$$

$$\leq \frac{1}{n!} \sum_{k,l} (2\text{sq}(\lambda))^{k+l} \sum_{f_1, f_2 \geq l} \sum_{c_1, c_2} \alpha_{c_1} \alpha_{c_2} |c_1| |c_2| \binom{n}{l} \left( \frac{f_1 f_2}{n^2} \right)^l \frac{(\log n)^{k-1}}{(k-1)!}$$

$$= \frac{1}{n!} \sum_{k,l} (2\text{sq}(\lambda))^{k+l} \binom{n}{l} \left( \frac{f_1 f_2}{n^2} \right)^l \frac{(\log n)^{k-1}}{(k-1)!}$$

$$\times \left( \sum_{c_1} \alpha_{c_1} |c_1| \right) \left( \sum_{c_2} \alpha_{c_2} |c_2| \right)$$

$$\leq \frac{1}{n!} \sum_{k,l} (2\text{sq}(\lambda))^{k+l} \binom{n}{l} \left( \frac{f_1 f_2}{n^2} \right)^l \frac{(\log n)^{k-1}}{(k-1)!}$$

$$\times \max \left( \frac{f_1 2f_1}{f_1!}, \frac{2n/2}{(\lceil n/2 \rceil - 1)!} \right) \max \left( \frac{f_2 f_2}{f_2!}, \frac{2n/2}{(\lceil n/2 \rceil - 1)!} \right).$$

Here, $P_{k,l}(c_1, c_2)$ denotes the probability that, for $x_1$ and $x_2$ chosen at random from $c_1$ respectively $c_2$, $x_1$ and $x_2$ have $l$ common fixed points, and the product $x_1 x_2$ has
precisely $k$ cycles on the remaining $n-l$ points. If $f_1$ is in the interval $[l, n/2]$, increasing $f_1$ by 1 changes the value of a summand by a factor
\[
2 \left( \frac{f_1 + 1}{f_1} \right)^{l+1} \frac{1}{f_1} \leq 2e,
\]
while each summand increases with $f_1$ in the range $n/2 \leq f_1 \leq n$. The same is true for $f_2$; hence, by symmetry, we obtain
\[
n! |\langle g_2, \chi_\lambda \rangle| \leq 80n^2 \sum_{k,l} (2^\text{sq}(\lambda))^{k+l} \left( \frac{n}{l} \right) \left( \frac{l^2}{n^2} \right) \frac{(\log n)^{k-1}}{(k-1)!} \left( \frac{2}{(l-1)!} \right)^2
\]
\[+ 80n^2 \sum_{k,l} (2^\text{sq}(\lambda))^{k+l} \left( \frac{n}{l} \right) \frac{(\log n)^{k-1}}{(k-1)!} \left( \frac{2^n/2}{(n/2 - 1)!} \right)^2 .
\]
For $n \to \infty$, the second sum tends to zero; thus, applying Stirling’s formula,
\[
n! |\langle g_2, \chi_\lambda \rangle| \leq 1 + 80n^2 \sum_{k,l} (2^\text{sq}(\lambda) \log n)^k \frac{1}{(k-1)!} \cdot \frac{(8^\text{sq}(\lambda)n^2)^l}{l!(l-1)!^2 n^{2l}}
\]
\[\leq 1 + 80n^5 \sum_{k,l} (2^\text{sq}(\lambda) \log n)^k \frac{1}{k!} \cdot \left( \frac{8e^3 \text{sq}(\lambda)}{ln} \right)^l .
\]
Since $\text{sq}(\lambda) \leq \sqrt{n}$, the second factor tends to zero, as $n$ tends to infinity, while the summation over $k$ yields an exponential series. Hence, we obtain the bound
\[
n! |\langle g_2, \chi_\lambda \rangle| \leq n^{2\text{sq}(\lambda)+7}.
\]
On the other hand, from Lemma 4 we deduce the bound $\chi_\lambda(1) \geq (\text{sq}(\lambda)/e)^{\text{sq}(\lambda)^2}$, and for $\chi_\lambda(1) > e^{3(\log n)^4}$, we deduce that
\[
n! |\langle g_2, \chi_\lambda \rangle| \leq \chi_\lambda(1)^{1/\log n}.
\]
By Lemma 1 (ii), we have
\[
n! |\langle g_2, \chi_\lambda \rangle| = n! \left| \left( \frac{1}{|c|} \mathbf{1}_c, \chi_\lambda \right) \right|^6 t_c(\chi) = \frac{|\chi_\lambda(c)|^{6t_c(\chi)}}{\chi_\lambda(1)^{6t_c(\chi)}},
\]
and together with our estimate for $|\langle g_2, \chi_\lambda \rangle|$ we obtain
\[
|\chi_\lambda(c)| \leq (\chi_\lambda(1))^{1-1/\log n}.
\]
Before establishing the upper bound for $|\chi_\lambda(c)|$ for characters associated to partitions $\lambda$ satisfying $n - \lambda_1 \leq 3 \log^4 n$, we prove the estimate for $t_c(\chi)$. Let $\xi_k$ be the random variable which for permutations $\pi_1, \ldots, \pi_k$ chosen independently at random from $c$ counts the number of points fixed by all of them. The probability that 1 is fixed by all of the permutations equals $\left( \frac{s_1(c)}{n} \right)^k$, whereas the probability that both 1 and 2 are fixed by all the permutations equals $\left( \frac{s_1(c)(s_1(c) - 1)}{n(n-1)} \right)^k$. 

Hence, \( E\xi_k = n \left( \frac{s_1(c)}{n} \right)^k \), and
\[
E(\xi_k^2) = n \left( \frac{s_1(c)}{n} \right)^k + n(n-1) \left( \frac{s_1(c)(s_1(c)-1)}{n(n-1)} \right)^k \leq E\xi_k + (E\xi_k)^2.
\]
Using the fact that \( \xi_k \) takes only integral values, we deduce the inequality
\[
1 - E\xi_k \leq P(\xi_k = 0) \leq 1 - E\xi_k + \frac{1}{2}(E\xi_k)^2;
\]
hence, we obtain for \( t_c(c) \) the bounds
\[
\min \left\{ k \in 2\mathbb{N} : \left( \frac{s_1(c)}{n} \right)^k \leq \frac{1}{n(n-1)} \right\} \leq t_c(c) \leq \min \left\{ k \in 2\mathbb{N} : \left( \frac{s_1(c)}{n} \right)^k \leq \frac{1}{n^2} \right\}.
\]
Since \( s_1(c) \leq n - 2 \) for every non-trivial class, the solutions of the equations
\[
\left( \frac{s_1(c)}{n} \right)^k = \frac{1}{n(n-1)} \quad \text{and} \quad \left( \frac{s_1(c)}{n} \right)^k = \frac{1}{n^2}
\]
differ by less than 1. Solving for \( k \) gives our claim.

Now we bound \( \chi_\lambda(c) \) using Lemma \( \S \). We have
\[
|\chi_\lambda(c)| \leq \sum_{a+2b+\leq n-\lambda_1} \left( \frac{s_1(c)}{a} \right) \left( \frac{n/2}{b} \right).
\]
Assume first that \( s_1(c) \leq n^{2/3} \). Then, using Lemma \( \S \) (ii),
\[
|\chi_\lambda(c)| \leq (n - \lambda_1)!^{1/2} \sum_{a+2b+\leq n-\lambda_1} n^{2a/3+b} \leq 2(n - \lambda_1)!^{3/2} \left( \frac{n}{2(n - \lambda_1)/3} \right) \leq \chi_\lambda(1)^{2/3+\varepsilon},
\]
which is sufficiently small, since \( t_c(c) \geq 2 \). If, on the other hand, \( s_1(c) > n^{2/3} \), replacing \( b \) by \( b - 1 \) and \( a \) by \( a + 2 \) changes the value of a summand by
\[
\frac{(s_1(c) - a)(s_1(c) - a - 1)b}{(a + 1)(a + 2)(n/2 - b + 1)} > n^{1/4},
\]
and, if \( a + 2b < n - \lambda_1 \), replacing \( a \) by \( a + 1 \) changes the value of a summand by \( \frac{s_1(c) - a}{a + 1} > n^{1/2} \), hence, for \( n \) sufficiently large, the whole sum over \( a \) and \( b \) is at most twice its greatest term. Again using Lemma \( \S \) we deduce from this that
\[
|\chi_\lambda(c)| \leq 2\chi_\lambda(1) \left( \frac{s_1(c)}{n - \lambda_1} \right)^{n/2} \leq 2\chi_\lambda(1) \left( \frac{s_1(c)}{n} \right)^{n-\lambda_1} \leq 2\chi_\lambda(1)^{1 - \frac{\log(n/s_1(c))}{\log n}},
\]
which is again sufficiently small by the lower bound for \( t_c(c) \).

We now sketch the computations needed to show that Theorem \( \S \) holds in fact for all \( n \geq 4000 \). In the form given above, the proof only applies to \( n \geq e^{40} \). First, in the deduction of Theorem \( \S \) from Theorem \( \S \), we can handle the characters \( \chi(n-1,1) \), \( \chi(2,1,1,...,1) \) separately, noting that for \( n \geq 4000 \) we have
\[
\sum_{\lambda} \chi_\lambda(1)^{-0.7} < \frac{1}{n}
\]
where the summation is extended over all partitions $\lambda \vdash n$ apart from $(n)$, $(n-1,1)$, $(2,1,1,\ldots,1)$ and $(1,1,\ldots,1)$. Following through the proof of Theorem 1 we find that the contribution of all characters $\chi_\lambda$ with $\lambda_1 \leq 3n/4$ of $n - \lambda_1 \leq n^{1/3}$ is sufficiently small. To close this gap, we estimate $|\langle g_2, \chi_\lambda \rangle|$ as above, but use Lemma 6 instead of Lemma 5 that is, the bound for $|\chi_\lambda(e)|$ now reads

$$
\min \left( \chi_\lambda(1), \max_{\nu \leq n - \lambda_1} \left( 2\sqrt{n - \lambda_1} \right)^\nu \binom{k+1}{\nu} \right) \quad \text{instead of} \quad (2s(q))^k; 
$$

as a result, we obtain a bound of sufficient quality for all characters $\chi_\lambda$ satisfying $\chi_\lambda(1) > 2e^{10}\log^2 n + 400\log n$; and we conclude the proof by noting that for $n \geq 4000$, the degree of a character $\chi_\lambda$ with $||\lambda|| \leq \lambda_1 < n - n^{1/3}$ is larger than this bound.

Note that in the intermediate range, we established a bound somewhat weaker than Theorem 1 in particular, we do not claim that Theorem 1 holds for all $n \geq 4000$. However, it may well be true that both theorems are in fact true for all integers $n$ without any exception.

### 2.5. Proof of Lemmas 9 and 10

To complete the proof of Theorem 1 it remains to establish Lemmas 9 and 10.

**Proof of Lemma 9.** Let $a$, $b$ be integers, and let $\pi_1, \ldots, \pi_{3t(c)} \in c$ be elements chosen at random. Denote by $P(a,b)$ the probability that the points $1, \ldots, a$ are fixed by all the $\pi_i$, and that, for each $\beta$ with $a+1 \leq \beta \leq a+b$, there is some $i$ such that $\pi_i$ moves $\beta$, while the product $\pi_1 \cdots \pi_{3t(c)}$ fixes $\beta$. We have

$$
P(a,0) = P(\pi_1 \text{ fixes } 1, \ldots, a)^{3t(c)} \leq P(\pi_1 \text{ fixes } 1)^{3at(c)} = \left( \left( \frac{s_1(c)}{n} \right)^{t(c)} \right)^{3^a}.
$$

By the definition of $t(c)$, we have $\left( \frac{s_1(c)}{n} \right)^{t(c)} < \frac{1}{n}$; hence, $P(a,0)$ is bounded by $n^{-3^a}$. 

Next, consider $P(0,b)$. The product $\pi_1 \cdots \pi_{3t(c)}$ fixes $\beta$ if and only if

$$
(\pi_1 \cdots \pi_{3t(c)-1})(\beta) = \pi_{3t(c)}^{-1}(\beta).
$$

Let $h \in \operatorname{Sym}([n] \setminus \{(\pi_1 \cdots \pi_{3t(c)-1})(\gamma) : \gamma \leq b, \gamma \neq \beta\})$ be chosen at random. Then replacing $\pi_{3t(c)}$ by $\pi_{3t(c)}^h$ does not alter $(\pi_1 \cdots \pi_{3t(c)-1})(\gamma)$ for $\gamma \leq b, \gamma \neq \beta$, while $(\pi_1 \cdots \pi_{3t(c)-1})(\beta) = (\pi_{3t(c)}^h)^{-1}(\beta)$ holds with probability $\frac{1}{n-b-1}$ or 0. Since the $\pi_i$ are chosen from a conjugacy class, conjugating with a random element from some subgroup does not affect the equidistribution of the $\pi_i$, hence, we obtain $P(0,b) \leq (n-b+1)^b$. Finally, a permutation $\pi \in S_n$ that fixes the points $1, \ldots, a$ can be viewed as a permutation on $n-a$ elements, thus $P(a,b) \leq n^{-3a}(n-a-b+1)^b$. If $a+b \leq n/2$, we deduce $P(a,b) \leq \frac{n^b}{n^{b+1}}$, and, summing over all pairs $a, b$ with $a + b \geq k$ we obtain our claim. Note that $P$ is decreasing in both $a$ and $b$; hence, if $a + b > n/2$, we may replace the pair $(a, b)$ by some pair $(a', b')$ satisfying $a' + b' = \lfloor n/2 \rfloor$ and use our estimate for the latter pair. Since the probability that there exist $k$ points which are fixed by the product $\pi_1 \cdots \pi_{3t(c)}$ is at most $\binom{n}{k}$ times the probability that $\pi_1 \cdots \pi_{3t(c)}$ fixes the points $1, \ldots, k$, our claim follows. \qed
Proof of Lemma \[10\]. The probability that both \(x_1\) and \(x_2\) fix 1 equals \(\frac{f_1 f_2}{n^2}\), and the conditional probability

\[
P(x_1 \text{ and } x_2 \text{ fix 1} \mid \exists a_1, \ldots, a_k : a_i \neq 1, \, x_1 \text{ and } x_2 \text{ fix } a_i, \forall i \leq k)
\]
is smaller, thus, the first claim of the lemma follows. Now let \(x_1, x_2\) be chosen at random from \(c_1\) and \(c_2\), respectively. Assume that not both of them fix 1, without loss we assume that \(x_1(1) \neq 1\). Then 1 lies in a cycle of length \(i\) of \(x_1 x_2\) if and only if

\[
(x_2(x_1 x_2)^{i-1})(1) = x_1^{-1}(1)
\]
with \(i\) chosen minimal among all positive integers with this property. Choose an element \(h \in \text{Sym}(\{n\} \setminus \{x_2(x_1 x_2)^{j}(1) : 0 \leq j \leq i - 2\})\) at random, and replace \(x_1\) by \(x_1^h\). Then \(16\) becomes true with probability \(\frac{1}{n - i + 1}\). Increasing \(i\) until \(16\) happens to hold, we obtain one cycle of \(x_1 x_2\). Next, choose some point outside this cycle, and repeat the procedure, where \(h\) is to be chosen in such a way that \(h\) fixes all points in all cycles already determined as well as the points already constructed in the current cycle. In this way, we obtain the number \(c\) of cycles of \(x_1 x_2\) as the value returned by the following Stochastic Algorithm

(i) Set \(m := n\), \(i := 0\) and \(c := 0\).
(ii) Increase \(i\) by 1.
(iii) With probability \(1 - \frac{1}{m - i + 1}\), go to (ii); otherwise continue with (iv).
(iv) Set \(m := m - i\), \(c := c + 1\) and \(i := 0\).
(v) If \(m = 0\), stop and return \(c\); otherwise go to (ii).

Let \(P\) be the probability that \(x_1 x_2\) has \(k\) cycles, and that their lengths are \(c_1, \ldots, c_k\). Then, in step (iii) of the algorithm, the second possibility was chosen \(k\) times, and the probabilities were \(\frac{1}{n - c_1 + 1}, \frac{1}{n - c_1 - c_2 + 1}, \ldots, \frac{1}{n - c_1 - \cdots - c_k + 1}\), respectively. Hence, \(P\) is bounded above by the product of these probabilities; and, writing \(i_j := c_{k-j+1} + \cdots + c_k + 1\), we obtain

\[
P(x_1 x_2 \text{ has } k \text{ cycles}) \leq \sum_{1=i_1<i_2<\cdots<i_k\leq n} \prod_{j=1}^{k} \frac{1}{i_j}
\]

\[
\leq \frac{1}{(k - 1)!} \sum_{2 \leq i_2, \ldots, i_k \leq n} \prod_{j=1}^{k} \frac{1}{i_j}
\]

\[
= \frac{1}{(k - 1)!} \left( \sum_{2 \leq i \leq n} \frac{1}{i} \right)^{k-1}
\]

\[
\leq \frac{(\log n)^{k-1}}{(k - 1)!},
\]

which proves our claim. \(\square\)
3. Character estimates for elements of prescribed order

As it stands, Theorem 1 is not strong enough to obtain an asymptotic estimate for the subgroup growth of Fuchsian groups. When combined with the methods of Subsection 2.3, it could be used to determine the asymptotics of $s_n(\Gamma)$ for certain Fuchsian groups $\Gamma$, namely those given by a presentation of the form

$$\Gamma = \left\langle x_1, \ldots, x_r \mid x_1^{a_1} = x_2^{a_2} = \cdots = x_r^{a_r} = x_1x_2\cdots x_r = 1 \right\rangle$$

where $a_i \geq 2$ and $r \geq 73$. Unfortunately, this would exclude all better known examples in this class. From the point of view of an application to the subgroup growth of Fuchsian groups, the weakness of Theorem 1 is caused by its generality. Combining instead an estimate by Fomin and Lulov \(11\) for character values $\chi_\lambda(\pi)$ where all cycle lengths of $\pi$ are equal, with combinatorial arguments plus the estimates of Subsection 2.3, we shall derive the following sharper estimate.

**Proposition 1.** Let $q \geq 2$ be an integer, $\varepsilon > 0$, and let $n$ be sufficiently large. Then, for every partition $\lambda \vdash n$, we have

$$\sum_{\pi \in S_n, \pi^q = 1} |\chi_\lambda(\pi)| \leq (\chi_\lambda(1))^{1/2} \varepsilon |\text{Hom}(C_q, S_n)|.$$  \hspace{1cm} (17)

We begin with some results concerning the statistical distribution of permutations in $S_n$ and in wreath products, which will be used in the proof of Proposition 1 as well as in the next section.

### 3.1. Statistics of the symmetric group.

For integers $n \geq 1$ and $q \geq 2$, define $N(n, q)$ to be the number of elements $\pi \in S_n$ with $\pi^q = 1$. Furthermore, for integers $s_t$ with $t|q$ and $t < q$, define $N(n, q, s_1, \ldots, s_T)$, to be the number of elements $\pi \in S_n$ with $\pi^q = 1$ and $s_t(\pi) = s_t$ for all $t|q, t < q$.

**Lemma 11.** Let $q \geq 2$ be an integer, $n$ sufficiently large, and let $s_1, \ldots, s_T$ be given in such a way that $n \equiv 1 \cdot s_1 + \cdots + T \cdot s_T \pmod q$. Then we have the estimate

$$\frac{N(n, q, s_1, \ldots, s_T)}{N(n, q)} \leq q^{\sigma(q)} \prod_{t|q, t < q} \left( \frac{en^{t/q}}{ts_t} \right)^{s_t},$$  \hspace{1cm} (18)

where $\sigma(q)$ denotes the sum of divisors of $q$.

**Proof.** Put $S := \sum_{t|q} t s_t$. We have

$$N(n, q, s_1, \ldots, s_T) = \frac{n!}{((n-S)/q)!s_1!\cdots s_T!s_1^{s_1}\cdots s_T^{s_T} T^{s_T} \cdot q^{(n-S)/q}}.$$

Let

$$\tilde{s}_t := \begin{cases} s_t, & s_t \leq 2en^{t/q}, \\ s_t \mod q/t, & s_t > 2en^{t/q} \quad (t|q, t < q), \end{cases}$$
where \( s_t \mod q/t \) takes values in \( \{0, 1, \ldots, q/t - 1\} \), and put \( \tilde{S} := \sum_{t|q} t \tilde{s}_t \). Then we get

\[
\frac{N(n, q, s_1, \ldots, s_T)}{N(n, q)} \leq \frac{N(n, q, s_1, \ldots, s_T)}{N(n, q, \tilde{s}_1, \ldots, \tilde{s}_T)}
= \frac{((n - \tilde{S})/q)! \tilde{s}_1! \cdots \tilde{s}_T!}{((n - S)/q)! s_1! \cdots s_T!} T^{n - \tilde{S}}/q \cdot q^{q - S}/q
\leq \prod_{t > 2en/q} q^{q/t} \left( \frac{en/t}{ts_t} \right)^{s_t}
\leq q^{\sigma(q)} \prod_{t > 2en/q} \left( \frac{en/t}{ts_t} \right)^{s_t},
\]
as claimed.

For a conjugacy class \( c \) of \( S_n \), denote by \( C_{S_n}(c) \) the centralizer of \( c \) in \( S_n \). Then \( C_{S_n}(c) \) is isomorphic to a direct product of the form

\[
C_{S_n}(c) \cong \prod_{d \mid q} C_d \wr S_{s_d(\sigma)},
\]
where \( \sigma \) is some element of \( c \).

**Lemma 12.** Let \( q \) and \( l \) be integers, and let \( c \in S_n \) be a conjugacy class with \( c^q = 1 \).

(i) The number of \( \pi \in S_n \) with at least \( k \) cycles is bounded above by

\[
n!(\log n)^{k-1}/((k - 1)!).
\]

(ii) The number of \( \pi \in C_{S_n}(c) \) with at least \( k \) cycles is bounded above by

\[
|C_{S_n}(c)| \left( \frac{3q \log n}{k} \right)^{k/q},
\]

provided that \( k \geq (\log n)^3 \) and \( n \geq n_0(q) \).

**Proof.** (i) Let \( \pi \in S_n \) be chosen at random. Then 1 is a fixed point of \( \pi \) with probability \( \frac{1}{n} \). If it is not a fixed point, then it lies in a 2-cycle with probability \( \frac{1}{n-1} \), and, more generally, the conditional probability for 1 to lie in a cycle of length \( c \), provided that it lies in a cycle of length at least \( c \) is \( \frac{1}{n - c + 1} \). Arguing now as in the proof of Lemma 10 establishes our claim.

(ii) Consider a single direct factor \( G := C_d \wr S_{s_d(\sigma)} \) of \( C_{S_n}(c) \), and let \( \phi : G \to S_{s_d(\sigma)} \) be the canonical projection. Let \( c \) be a cycle in \( G \). Then, the projection \( \pi \) of \( c \) in \( S_{s_d(\sigma)} \) is a cycle, too, and there are at most \( d \) cycles \( c_1, \ldots, c_d \in G \) which have the same image in
$S_{s_d}(\sigma)$. We deduce that the probability that a permutation $\pi$, chosen at random in $G$, has $k$ cycles is at most the probability that a permutation chosen at random in $S_{s_d}(\sigma)$ has $\lceil k/d \rceil$ cycles. Together with part (i) of this lemma, we obtain

$$\frac{1}{|C_{S_n}(c)|} \left| \left\{ \pi \in C_{S_n}(c) : |\text{Orbits}(\pi)| \geq k \right\} \right| \leq \sum_{\sum d_i q = k} \prod_{d_i q} \min \left( 1, \frac{(\log n)^{\kappa_d - 1}}{(\kappa_d - 1)!} \right)$$

$$\leq k^{q(q)} \max_{\sum d_i q = k} \prod_{d_i q} \min \left( 1, \frac{(\log n)^{\kappa_d - 1}}{(\kappa_d - 1)!} \right)$$

$$\leq k^{q(q)} \max_{\kappa_d \geq 3 \log n} \frac{(\log n)^{\kappa_d - 1}}{(\kappa_d - 1)!},$$

If we replace $\kappa_d$ by $\kappa_d + \frac{q}{d}$, and $\kappa_q$ by $\kappa_q - 1$, a single summand is changed by a factor

$$\frac{(\log n)^{\frac{q}{d}}(\kappa_q - 1)}{\kappa_d(\kappa_d + 1) \cdots (\kappa_d + q/d - 1)},$$

which is less than 1, provided that $\kappa_d > (\log n)\sqrt{\kappa_q}$. Hence, the maximum is attained for some tuple $(\kappa_1, \ldots, \kappa_q)$ satisfying $\kappa_q \geq k/q - q\sqrt{k} \log n$. From this we deduce

$$\frac{1}{|C_{S_n}(c)|} \left| \left\{ \pi \in C_{S_n}(c) : |\text{Orbits}(\pi)| \geq k \right\} \right| \leq k^{q(q)} \frac{(\log n)^{k/q - q\sqrt{k}(\log n) - 1}}{[k/q - q\sqrt{k}(\log n) - 1)!}$$

$$\leq \left( \frac{3q \log n}{k} \right)^{k/q},$$

provided that $k \geq (\log n)^3$ and $n \geq n_0(q)$. \square

**Lemma 13.** Let $\pi \in S_n$ be chosen at random, and let $d, d_1, d_2$ be positive integers.

(i) As $n \to \infty$, the distribution of $s_d(\pi)$ converges to a Poisson distribution with mean $\frac{1}{d}$, and we have

$$\frac{1}{n!} \sum_{\pi \in S_n} (s_d(\pi))^q \to \sum_{\nu = 1}^q d^{-\nu} S(q, \nu), \quad n \to \infty,$$

where the $S(q, \nu)$ are Stirling numbers of the second kind.

(ii) As $n \to \infty$, the random variables $s_{d_1}(\pi)$ and $s_{d_2}(\pi)$ are asymptotically independent.

**Proof.** (i) Let $P(d, k)$ be the probability that for $\pi$ chosen at random from $S_n$, $\pi$ contains the $d$-cycles (12, $d$), $(d + 1, \ldots, 2d), \ldots, ((k - 1)d + 1, \ldots, kd)$. Then, as $P(d, k) = \frac{(n - kd)!}{n!}$, we have for $k \leq n/d$,

$$\frac{1}{n!} \sum_{\pi \in S_n} \binom{s_d(\pi) k}{k} = \frac{n!}{d^k k!(n - kd)!} P(d, k) = \frac{1}{d^k k!}.$$
On the other hand, for a random variable $\xi$ which has Poisson distribution with mean $1/d$, we have

$$E\left(\xi \choose k\right) = \sum_{\nu=0}^{\infty} \left(\nu \choose k\right) \frac{e^{-1/d}}{d^\nu \nu!} = \frac{1}{d^k k!} \sum_{\nu=k}^{\infty} \frac{1}{(\nu-k)! (d^\nu - k)} = \frac{1}{d^k k!}.$$

We conclude that the first $[n/d]$ moments of $\xi$ and $s_d(\pi)$ coincide, hence, by the method of moments,\(^3\) the distributions are identical, proving the first assertion. Let $\xi$ be a random variable with mean $1/d$ and Poisson distribution. Then $E\xi^q$ is the expected number of $q$-multisets in $[\xi]$. A set $S$ of $\nu$ elements gives rise to $S(q, \nu)$ different multisets $M$, such that $M = S$ as sets, and the computation of $E(\xi^\nu)$ shows, that the expected number of $\nu$-sets is $d^{-\nu}$, whence the second assertion.

(ii) For $i \leq t, i \neq d$, fix integers $e_i$. We compute the conditional expectation

$$E\left(\left(\begin{array}{c} s_d(\pi) \\ k \end{array} \right) \bigg| s_i(\pi) = e_i, i \leq t, i \neq d \right) = \frac{(n-\mathcal{E})!}{d^k k! (n-\mathcal{E}-kd)!} P(d, k),$$

where $\mathcal{E} = \sum_{q \neq i \leq t} i e_i$. As $n$ tends to infinity, the right hand side converges to $\frac{1}{d^k n!}$, proving our claim.

\(\square\)

Our next group of results describes the distribution of cycles in permutations of prescribed order. The proof makes use of the following purely analytic result.

**Lemma 14.** Let $P(z) = \sum_{k=1}^{q} a_k z^k$ be a real polynomial, and let $Q(z) = \sum_{k=1}^{q} |a_k| z^k$. Assume that $P(z) \neq \pm Q(\pm z)$, and that $\{k : a_k \neq 0\}$ has greatest common divisor 1. Define the sequences $(b_n), (b^+_n)$ by means of the equations

$$e^{P(z)} = \sum_{n=0}^{\infty} b_n z^n/(n!), \quad e^{Q(z)} = \sum_{n=0}^{\infty} b^+_n z^n/(n!).$$

Then we have $|b_n| < b^+_n e^{-cn^{1/q}}$ for some $c > 0$ and sufficiently large $n$.

**Proof.** We first claim that there is some constant $c$, such that for all real numbers $r$ sufficiently large, and all complex numbers $z$ with $|z| = r$, we have $R_P(z) \leq Q(r) - cr$. For otherwise we would have $R a_k z^k \geq |a_k| r^k - cr$, that is, $|\frac{z}{|z|} - \zeta| < \varepsilon$ for some $(2k)$-th root of unity $\zeta$, and $\varepsilon$ arbitrarily small. Since by assumption the set $\{k : a_k \neq 0\}$ has greatest common divisor 1, we deduce that $|\arg z| < \varepsilon$ or $|\arg z - \pi| < \varepsilon$. However, the assumptions $P(z) \neq \pm Q(\pm z)$ and $a_k \neq 0$ for at least one odd $k$ imply, that in these cases $R a_k z^k$ was negative for at least one value of $k$. From this we obtain that $|e^{P(z)}| \leq e^{Q(r) - cr}$ for some $c > 0$ and all $z$ with $|z| = r$. Let $r_n$ be the solution of the equation $r Q'(r) = n$. Then we deduce from Cauchy’s bound that $\beta_n \leq \frac{e^{Q(r_n) - cr_n}}{r_n^{Q(r_n)}(r_n)^{rn}}$, while from \cite{II}, Corollary II we obtain the lower bound $\beta^+_n \geq \frac{e^{Q(r_n)}}{r_n^{rn} c}$ with some absolute constant $c$. From these bounds and the asymptotics $r_n \sim q n/(qa q)$ the lemma follows. \(\square\)

**Lemma 15.** Let $q \geq 2$ and $e_t \geq 0 (t|q, t < q)$ be integers.

\(^3\)Cf., for instance, \cite{I}.
(i) There exist constants $\alpha_{e_1,\ldots,e_T}^{(d)}$, such that, for all $n \geq 1$,

$$\sum_{\pi \in S_n} \prod_{t \mid q} \left( s_t(\pi) \right)^{e_t} = \sum_{\nu=0}^{D} \alpha_{e_1,\ldots,e_T}^{(\nu)} \frac{n!}{(n-\nu)!} |\text{Hom}(C_q, S_{n-\nu})|,$$  

(19)

where $D = \sum_t t e_t$. The coefficients $\alpha_{e_1,\ldots,e_T}^{(d)}$ are recursively determined by means of the equations

$$\alpha_{e_1,\ldots,e_q}^{(d)} = \begin{cases} \frac{1}{d} \sum_{t \mid q} \sum_{t < q} \left( e_t \right) \alpha_{e_1,\ldots,e_{t-\nu},\ldots,e_T}^{(d-t)}, & d \geq 1 \\ 1, & d = 0. \end{cases}$$

(ii) We have

$$\sum_{\pi \in S_n} \prod_{t \mid q} \left( s_t(\pi) \right)^{e_t} = \left( 1 + O(n^{-1/q}) \right) \alpha_{e_1,\ldots,e_T}^{D} n^{D/q}.$$  

(iii) If $q$ is even, then there exists a constant $c > 0$ such that, for $n$ sufficiently large,

$$\sum_{\pi \in S_n} \prod_{t \mid q} (-1)^{(t-1)s_t(\pi)} \left( s_t(\pi) \right)^{e_t} < e^{-cn^{1/q}} |\text{Hom}(C_q, S_{n-\nu})|.$$  

(20)

Proof. (i) It will be convenient to allow $t$ to run over all divisors of $q$, setting $e_q := 0$. Abbreviate the left-hand side of (19) as $S_{e_1,\ldots,e_q}(n)$, and let $\pi$ be a permutation in $S_n$ with $\pi^q = 1$. Then $n$ occurs in some cycle of $\pi$ of length $t$ for some $t \mid q$. Let $\pi' \in S_{n-t}$ be the permutation resulting from $\pi$ by deleting the cycle containing $n$. We have $s_d(\pi') = s_d(\pi)$ for $d \neq t$, and $s_t(\pi') = s_t(\pi) - 1$, that is,

$$\prod_{d \mid q} \left( s_d(\pi') \right)^{e_d} = \left( s_t(\pi') + 1 \right)^{e_t} \prod_{d \mid q} \left( s_d(\pi') \right)^{e_d}.$$  

The $t$-cycle containing $n$ can be chosen in $\frac{(n-1)!}{(n-t)!}$ ways; hence, we obtain for $n \geq 1$ the recursion formula

$$S_{e_1,\ldots,e_q}(n) = \sum_{t \mid q} \frac{(n-1)!}{(n-t)!} \sum_{\nu=0}^{e_t} \left( e_t \right) S_{e_1,\ldots,e_{t-\nu},\ldots,e_q}(n-t),$$  

(21)

where

$$S_{e_1,\ldots,e_q}(0) = \begin{cases} 1, & (e_1, \ldots, e_q) = (0, \ldots, 0) \\ 0, & \text{otherwise}, \end{cases}$$

and $S_{e_1,\ldots,e_q}(n) = 0$ if one of the $e_t$ or $n$ is negative. Introducing the exponential generating functions

$$G_{e_1,\ldots,e_q}(z) = \sum_{n=0}^{\infty} S_{e_1,\ldots,e_q}(n) \frac{z^n}{(n!)},$$
multiplying (21) by \( \frac{n^{-1}}{(n-1)!} \), and summing over \( n \geq 1 \), this recurrence relation translates into the differential equation

\[
S'_{e_1,\ldots,e_q}(z) - \left( \sum_{t|q} z^{t-1} \right) S_{e_1,\ldots,e_q}(z) = \sum_{t|q} z^{t-1} \sum_{\nu=1}^{e_t} \binom{e_t}{\nu} S_{e_1,\ldots,e_t-\nu,\ldots,e_q}(z).
\]

Integrating the latter equation, we find that

\[
S_{e_1,\ldots,e_q}(z) = \exp \left( \sum_{t|q} \frac{z^t}{t} \right) \sum_{t|q} \sum_{\nu=1}^{e_t} \binom{e_t}{\nu} \int_0^z \left( \zeta^{t-1} \exp \left( - \sum_{t|q} \frac{\zeta^t}{t} \right) S_{e_1,\ldots,e_t-\nu,\ldots,e_q}(\zeta) \right) d\zeta
\]

+ \exp \left( \sum_{t|q} \frac{z^t}{t} \right), \quad (22)

where the value of the integration constant has been determined by comparing the coefficients of \( z \). We claim that there exist polynomials \( P_{e_1,\ldots,e_q}(z) \), such that

\[
S_{e_1,\ldots,e_q}(z) = P_{e_1,\ldots,e_q}(z) \exp \left( \sum_{t|q} \frac{z^t}{t} \right). \quad (23)
\]

The proof is by induction on \( e = \sum_{t|q} e_t \). If \( e_t = 0 \) for all \( t \), then \( S_{0,\ldots,0}(n) = |\text{Hom}(C_q, S_n)| \), that is,

\[
S_{0,\ldots,0}(z) = \exp \left( \sum_{t|q} \frac{z^t}{t} \right),
\]

and (23) holds with \( P_{0,\ldots,0}(z) = 1 \). Suppose now that our claim holds for all tuples \((e'_1,\ldots,e'_q)\) with \( \sum_{t|q} e'_t < e \), and some \( e \geq 1 \), and let \((e_1,\ldots,e_q)\) be a tuple with \( \sum_{t|q} e_t = e \). Inserting (23) into the right-hand side of (22), we find that

\[
S_{e_1,\ldots,e_q}(z) = \exp \left( \sum_{t|q} \frac{z^t}{t} \right) \sum_{t|q} \sum_{\nu=1}^{e_t} \binom{e_t}{\nu} \int_0^z \zeta^{t-1} P_{e_1,\ldots,e_t-\nu,\ldots,e_q}(\zeta) d\zeta + \exp \left( \sum_{t|q} \frac{z^t}{t} \right),
\]

that is, (23) holds for \((e_1,\ldots,e_q)\) with

\[
P_{e_1,\ldots,e_q}(z) = 1 + \sum_{t|q} \sum_{\nu=1}^{e_t} \binom{e_t}{\nu} \int_0^z \zeta^{t-1} P_{e_1,\ldots,e_t-\nu,\ldots,e_q}(\zeta) d\zeta.
\]

Comparing coefficients in (23) and in the recurrence relation determining the polynomials \( P_{e_1,\ldots,e_q}(z) \), the assertions of the lemma follow.

(ii) By [24 Eq. (22)], we have

\[
\frac{|\text{Hom}(C_q, S_n)|}{|\text{Hom}(C_q, S_{n-k})|} = (1 + \mathcal{O}(n^{-1/q})) n^{k(1-1/q)}.
\]

Together with part (i), our claim follows.

(iii) Denote the left-hand side of (20) by \( S_{e_1,\ldots,e_q}^*(n) \). In this case, we obtain the recurrence relation

\[
S_{e_1,\ldots,e_q}^*(n) = \sum_{t|q} (-1)^{t-1} \frac{(n-1)!}{(n-t)!} \sum_{\nu=0}^{e_t} \binom{e_t}{\nu} S_{e_1,\ldots,e_t-\nu,\ldots,e_q}^*(n-t).
\]

\[\text{Cf., for instance, [9] Prop. 1}.\]
Arguing as in part (i), the corresponding exponential generating function $S_{e_1,\ldots,e_q}(z)$ is found to satisfy
\[
S_{e_1,\ldots,e_q}(z) = P_{e_1,\ldots,e_q}(z) \exp \left( \sum_{t\in\mathbb{N}} \frac{(-1)^{t-1}z^t}{t} \right)
\]
with certain polynomials $P_{e_1,\ldots,e_q}$. Our claim follows from this and Lemma 14. □

Lemma 16. Define the polynomials $P_{e_1,\ldots,e_q}$ as in the proof of Lemma 15. Then we have
\[
P_{e_t}(z) = 1 + \frac{z^t}{t},
\]
\[
P_{e_1+e_2}(z) = 1 + \frac{z^{t_1}}{t_1} + \frac{z^{t_2}}{t_2} + \frac{t_1 + t_2}{(t_1t_2)^2}z^{t_1+t_2},
\]
\[
P_{k\cdot e_t}(z) = \sum_{\nu=0}^k S(k+1,\nu+1) \frac{z^{t\nu}}{\nu},
\]
where $\bar{e}_t$ denotes the tuple with $e_t = 1$ and $e_d = 0$ for $d \neq t$.

Proof. The first two equations follow directly from the definition. The third equation is established by induction on $k$. For $k = 1$ the claim is already proven. Assuming the result for $P_{\kappa\cdot e_t}(z)$ with $\kappa < k$ and some $k \geq 2$, we find that
\[
P_{k\cdot e_t}(z) = 1 + \sum_{\nu=1}^k \binom{k}{\nu} \int_0^z \zeta^{t-1}P_{(k-\nu)\cdot e_t}(\zeta) d\zeta
\]
\[
= 1 + \sum_{\nu=1}^k \binom{k}{\nu} \sum_{\mu=0}^{k-\nu} S(k-\nu+1,\mu+1) \frac{z^{(\mu+1)t}}{(\mu+1)t^{\mu+1}}
\]
\[
= 1 + \sum_{\mu=0}^{k-1} \frac{z^{(\mu+1)t}}{t^{\mu+1}} \cdot \frac{1}{\mu+1} \sum_{\nu=1}^{k-\mu} \binom{k}{\nu} S(k-\nu+1,\mu+1).
\]

Hence, our claim would follow from the identity
\[
(\mu + 1)S(k + 1,\mu + 2) = \sum_{\nu=1}^{k-\mu} \binom{k}{\nu} S(k - \nu + 1,\mu + 1),
\]
which can be seen to hold as follows: the left-hand side counts the number of partitions of a set with $k + 1$ points, with one special point distinguished, into $\mu + 2$ non-empty parts with one part distinguished, which does not contain the special point. On the right-hand side, we first determine the size $\nu$ of the distinguished part, then select the points for this part avoiding the distinguished point, and finally partition the remaining $k + 1 - \nu$-set into $\mu + 1$ non-empty parts. The number of possibilities in the latter case clearly matches the combinatorial description of the left-hand side, and the result follows. □
3.2. Proof of Proposition 11

Lemma 17. Let \( \pi \in S_n \) be a permutation consisting only of cycles of length \( q \). Then, for every irreducible character \( \chi \), we have \( |\chi(\pi)| \leq \sqrt{n} \left( \chi(1) \right)^{1/q} \).

This follows from [11, Theorem 1.1] together with the Murnaghan-Nakayama rule.

Lemma 18. Let \( q \geq 2 \) be an integer, let \( \pi \in S_n \) be a permutation with \( s \) cycles of lengths different from \( q \), and let \( \lambda \) be a partition of \( n \). Then

\[
|\chi_{\lambda}(\pi)| \leq \sqrt{n} \left( 2 \text{sq}(\lambda) \right)^{(q-1)/q} \left( \chi_{\lambda}(1) \right)^{1/q}.
\]

Proof. For every partition \( \mu \) denote by \( N(\mu, \lambda) \) the number of ways to obtain \( \mu \) from \( \lambda \) by stripping off \( s \) rim hooks of lengths according to the cycle structure of \( \pi \), ending in an element \( \pi_0 \) containing only cycles of length \( q \). By the Murnaghan-Nakayama rule, we have

\[
\chi_{\lambda}(1) \geq \sum_{\mu} N(\mu, \lambda) \chi_{\mu}(1).
\]

Neglecting the sign in the Murnaghan-Nakayama rule, and applying Lemma 17, Hölder’s inequality, and the estimate (25), we get

\[
|\chi_{\lambda}(\pi)| \leq \sum_{\mu} N(\mu, \lambda) |\chi_{\mu}(\pi_0)|
\leq \sqrt{n} \sum_{\mu} N(\mu, \lambda) \left( \chi_{\mu}(1) \right)^{1/q}
\leq \sqrt{n} \left( \sum_{\mu} N(\mu, \lambda) \right)^{(q-1)/q} \left( \sum_{\mu} \chi_{\mu}(1) N(\mu, \lambda) \right)^{1/q}
\leq \sqrt{n} \left( \sum_{\mu} N(\mu, \lambda) \right)^{(q-1)/q} \left( \chi_{\lambda}(1) \right)^{1/q}.
\]

Arguing as in the proof of Lemma 17, we see that the \( s \) cycles of length different from \( q \) can be chosen in at most \( (2 \text{sq}(\lambda))^s \) ways, that is,

\[
\sum_{\mu} N(\mu, \lambda) \leq (2 \text{sq}(\lambda))^s,
\]

and our claim follows. \( \square \)

Lemma 19. Let \( q \geq 2 \) be an integer, let \( \pi \in S_n \) be a permutation with cycle lengths \( \leq q \), and let \( \lambda \vdash n \) be a partition of \( n \) satisfying \( \lambda_1 \geq \|\lambda\| \). Set \( \Delta = n - \lambda_1 \), and let \( \varepsilon > 0 \) be given. Then there exists a constant \( C = C(q, \varepsilon) \), such that, for \( \Delta \in [C, n/C] \) and \( n \) sufficiently large,

\[
|\chi_{\lambda}(\pi)| \leq \chi_{\lambda}(1)^{\frac{1}{q} + \varepsilon} \prod_{t < q} E(n, \Delta, t, s_t(\pi)).
\]
where

\[ E(n, \Delta, t, s) = \begin{cases} \left( \frac{(\Delta+2t-1)/2q}{n^{t/q}} \right)^s, & \text{if } ts \leq \Delta \\ \left( \frac{s}{\Delta \frac{2t-2q+1}{2q^n}} \right)^\Delta, & \text{if } ts > \Delta. \end{cases} \]

**Proof.** The proof is by induction on the minimum \( m(\pi) \) of the cycle lengths in \( \pi \) from \( q \) down to 1. For \( m(\pi) = q \), our claim follows immediately from Lemma 17. Suppose that our assumption holds for all \( \pi \) with \( m(\pi) \geq t + 1 \) and some \( t \in [q-1] \). Write \( \mu = \lambda \setminus \lambda_1 \). For a partition \( \nu \subseteq \lambda \) denote by \( N(\nu, \lambda) \) the number of ways to obtain \( \nu \) from \( \lambda \) by removing \( s_t(\pi) \) rim hooks of length \( t \), let \( \pi \in S_n \) be a permutation with \( m(\pi) = t \), and let \( \pi_0 \in S_{n-ts_t(\pi)} \) be a permutation whose cycle structure is the same as that of \( \pi \) with all \( t \)-cycles removed. Then we have

\[ |\chi_{\lambda}(\pi)| \leq \sum_{\nu} N(\nu, \lambda) |\chi_{\lambda}(\pi_0)|. \]

We claim that for a partition \( \nu \) such that \( N(\nu, \lambda) \neq 0 \),

\[ |\chi_{\nu}(\pi_0)| \leq \left( \chi_{\lambda}(1) \right)^{2\varepsilon} \left( x_{\nu}(1) \right)^{\frac{1}{2} + \varepsilon} \prod_{t < \tau < q} E(n, \Delta, \tau, s_\tau(\pi)). \]

Indeed, if \( \Delta' = n - ts_t(\pi) - \nu_1 \geq \varepsilon \Delta \), then this assumption holds (even without the factor \( \left( \chi_{\lambda}(1) \right)^{\varepsilon} \) by the inductive hypothesis; otherwise, using Lemma 8 and the fact that \( \Delta \leq n/C \), we get

\[ |\chi_{\nu}(\pi_0)| \leq \chi_{\nu}(1) \leq n^{\varepsilon \Delta} \leq n^{\Delta} \leq \left( \chi_{\lambda}(1) \right)^{2\varepsilon}. \]

Setting

\[ E := \prod_{t < \tau < q} E(n, \Delta, \tau, s_\tau(\pi)), \]

this gives

\[ |\chi_{\lambda}(\pi)| \leq E\left( \chi_{\lambda}(1) \right)^{\varepsilon} \sum_{\nu} N(\nu, \lambda) \left( x_{\nu}(1) \right)^{1/q} = E\left( \chi_{\lambda}(1) \right)^{\varepsilon} \sum_{\nu} N(\nu, \lambda) \left( x_{\nu}(1) \right)^{1/q}. \]

(26)

Put \( \kappa := \nu \setminus \nu_1 \). Given \( a \), we bound \( x_{\nu}(1) \) as follows: we choose a set \( I \) of \( |\kappa| = \Delta - at \) integers in \([n - ts_t(\pi)]\), and then count the number of ways of removing all boxes of \( \nu \) in such a way that, in the \( i \)-th step, a box outside the first row is removed if and only if \( i \in I \). We claim that the latter number is bounded by \( \chi_{\kappa}(1) \leq \chi_{\mu}(1) N(\kappa, \mu)^{-1} \). Indeed, refining the removal of a rim hook into a sequence of removals of single boxes, we see that there are at least \( N(\kappa, \mu) \) ways to obtain \( \kappa \) from \( \mu \) by removing single boxes, while there are \( \chi_{\kappa}(1) \) ways to remove \( \kappa \) completely by deleting boxes. Hence, we have \( \chi_{\kappa}(1) \geq N(\kappa, \mu) \chi_{\kappa}(1) \), from which the last claim follows. Since \( I \) can be chosen in \( \binom{n - ts_t(\pi)}{\Delta - at} \) different ways, we obtain that

\[ \chi_{\nu}(1) \leq \left( \frac{n - ts_t(\pi)}{\Delta - at} \right) \chi_{\mu}(1) N(\kappa, \mu)^{-1}. \]

(27)
Next, if \( \nu_1 = \lambda_1 - (s_t(\pi) - a)t \), \( N(\nu, \lambda) \) is the number of ways to remove \( s_t(\pi) - a \) rim hooks from the first row of \( \lambda \), and \( a \) rim hooks from \( \mu \). The position in the sequence of steps where a rim hook is removed from the first row can be chosen in \( \binom{s_t(\pi)}{a} \) ways, thus

\[
N(\nu, \lambda) = \left( \binom{s_t(\pi)}{a} \right) N(\kappa, \mu).
\] (28)

Inserting (27) and (28) into (26), we get

\[
|\chi(\pi)| \leq E(\chi(1)) \varepsilon \sum_{a \leq s_t(\pi)} \sum_{\kappa \subseteq \mu \atop \kappa \ni \Delta - at} \left( \binom{s_t(\pi)}{a} \right) \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} N(\kappa, \mu)^{1 - 1/q} \left( \chi(1) \right)^{1/q}.
\]

Finally,

\[
\sum_{\kappa \subseteq \mu \atop \kappa \ni \Delta - at} N(\kappa, \mu)
\]

is the number of ways to remove \( a \) rim hooks from \( \mu \), which can be done in at most \( (2s_q(\mu))^a \leq (2\sqrt{\Delta})^a \) ways, as we saw in the proof of Lemma 2. Applying Hölder’s inequality, and observing the fact that the number of partitions of \( \Delta \) is \( e^{c\sqrt{\Delta}} \leq (\chi(1))^\varepsilon \), we obtain

\[
|\chi(\pi)| \leq E(\chi(1)) \varepsilon \sum_{a \leq s_t(\pi)} \left( \binom{s_t(\pi)}{a} \right) \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} (2\sqrt{\Delta})^{(1 - 1/q)a} \left( \chi(1) \right)^{1/q}.
\]

Since by Lemma 8 (ii) we have for \( C > 3/\varepsilon \)

\[
\left( \chi(1) \right)^{1/q + \varepsilon} \geq \left( \frac{n^{\Delta}}{\Delta!} \right)^{1/q} \left( \chi(1) \right)^{1/q},
\]

we obtain

\[
\frac{|\chi(\pi)|}{(\chi(1))^{1/q + \varepsilon}} \leq E \left( \frac{\Delta!}{n^{\Delta}} \right)^{1/q} \sum_{a \leq s_t(\pi)} \left( \binom{s_t(\pi)}{a} \right) \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} (2\sqrt{\Delta})^{(1 - 1/q)a}. \] (29)

Since by assumption \( \Delta \geq C \), the number \( s_t(\pi) \leq n \) of summands is of order at most \( (\chi(1))^\varepsilon \); in particular, we can estimate the sum over \( a \) by its largest term. We now distinguish two cases, according to whether \( \Delta \geq ts_t(\pi) \) or \( \Delta < ts_t(\pi) \).

Case 1: \( \Delta \geq ts_t(\pi) \). We first note that terms of the order of magnitude \( e^{c\Delta} \) can be neglected on the right-hand side of (29), since they are bounded above by \( (\chi(1))^\varepsilon \); in particular, \( \binom{s_t(\pi)}{a} \leq 2^{\Delta/t} \) and \( 2^a \leq 2^{\Delta/t} \) are absorbed into the term \( (\chi(1))^\varepsilon \). We now split the summation over \( a \) into the ranges \( a \leq \varepsilon s_t(\pi) \), \( \varepsilon s_t(\pi) < a < s_t(\pi) - \varepsilon \Delta \) and
\( a \geq s_t(\pi) - \varepsilon \Delta \). In the last case, we have

\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( n - ts_t(\pi) \right)^{1/q} \Delta^{\frac{a-1}{2q}} \leq \left( \frac{\Delta! n^{\Delta - ts_t(\pi) + \varepsilon \Delta t}}{n^\Delta (\Delta - t s_t(\pi) + \varepsilon \Delta t)!} \right)^{1/q} \Delta^{\frac{a-1}{2q}} s_t(\pi)
\]

\[
\leq n^{\varepsilon \Delta t/q} \left( \frac{\Delta^{1 + \frac{a}{q} - \frac{1}{2q}}}{n^{t/q}} \right)^{s_t(\pi)}
\]

\[
\leq \left( \chi(1) \right)^{\varepsilon} \left( \frac{\Delta^{1 + \frac{a}{q} - \frac{1}{2q}}}{n^{t/q}} \right)^{s_t(\pi)}
\]

Hence, every term in this range is of the desired magnitude, and therefore this part of the sum is sufficiently small.

Next, we turn our attention to terms with \( a \leq \varepsilon s_t(\pi) \). In this case, we obtain

\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( n - ts_t(\pi) \right)^{1/q} \Delta^{\frac{a-1}{2q}} \leq \left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{n}{\Delta} \right)^{1/q} \Delta^{\frac{a-1}{2q}} s_t(\pi)
\]

\[
\leq \left( \chi(1) \right)^{\varepsilon}.
\]

Finally, consider the range \( \varepsilon s_t(\pi) < a < s_t(\pi) - \varepsilon \Delta \). It suffices to consider the case where the terms in this range are not dominated by terms in the other ranges, that is, we may assume that the maximal term lies within this range. If we increase \( a \) by 1, a single summand in (24) is changed by a factor

\[
F(a) := \left( \frac{(n - ts_t(\pi) - \Delta + at) \cdots (n - ts_t(\pi) - \Delta + at - t + 1)}{(\Delta - at + 1) \cdots (\Delta - at + t)} \right)^{1/q} \Delta^{1/2 - 1/2q}
\]

\( F(a) \) is monotonically increasing, hence there is a unique positive solution \( x_0 \) of the equation \( F(x) = 1 \), and the value \( a_{\text{max}} \) for which the corresponding summand is maximal, differs from \( x_0 \) by at most 1. Using the bounds for \( a \), we find that

\[
F(a) \asymp \frac{n^{t/q}}{\Delta^{\frac{a-1}{2q}}} \quad a \in (\varepsilon s_t(\pi), s_t(\pi) - \varepsilon \Delta);
\]

hence, we can neglect this range, unless the expression on the right-hand side, which does not depend on \( a \), is \( \asymp 1 \); that is, \( n \asymp \Delta^{(q+2t-1)/2t} \). In the latter case, we have

\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( n - ts_t(\pi) \right)^{1/q} \Delta^{\frac{a-1}{2q}} \leq \left( \frac{\Delta^{at/q + \frac{a-1}{2q}}}{n^{at/q}} \right)^{\frac{\Delta q + 2t - 1}{q} \Delta}
\]

\[
\leq \left( \frac{\Delta q + 2t - 1}{n^{2t}} \right)^{\frac{\Delta}{2q}}
\]

\[
\leq e^{\varepsilon \Delta}
\]

\[
\leq \left( \chi(1) \right)^{\varepsilon}.
\]
Case 2: \( ts_t(\pi) > \Delta \). Note that we have \( a \leq \Delta/t \). We begin with \( a \) in the range \( a \geq (1 - \varepsilon)\Delta/t \). Then we have
\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{a} \right)^{1/q} \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} \Delta^{\frac{a-1}{2q}a} \leq \left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{\Delta} \right)^{1/q} \left( \frac{n - ts_t(\pi)}{\varepsilon \Delta} \right)^{1/q} \Delta^{\frac{a-1}{2q}a}
\]
\[
\leq (\chi_\lambda(1))^{e \varepsilon \Delta} \left( \frac{s_t(\pi)}{\Delta} \right)^{e \varepsilon \Delta},
\]
which is the desired result.

If \( a \leq \varepsilon \Delta/t \), then we have
\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{a} \right)^{1/q} \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} \Delta^{\frac{a-1}{2q}a} \leq \left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{\Delta} \right)^{1/q} \left( \frac{n}{\Delta} \right)^{1/q} \Delta^{\frac{a-1}{2q}a}
\]
\[
\leq (\chi_\lambda(1))^{e \varepsilon \Delta} \left( \frac{s_t(\pi)}{\Delta} \right)^{e \varepsilon \Delta},
\]
which is less than \((\chi_\lambda(1))^{e \varepsilon \Delta} E(n, \Delta, t, s_t(\pi))\).

Finally, if \( a \in (\varepsilon \Delta/t, (1 - \varepsilon)\Delta/t) \), increasing \( a \) by 1 changes a single summand by a factor
\[
F(a) := \frac{(s_t(\pi) - a)((n - \Delta - ts_t(\pi) + at) \cdots (n - \Delta - ts_t(\pi) + at - t + 1))^{1/q}}{(a + 1)\Delta^{1/2 - 1/2q}((\Delta - at + 1) \cdots (\Delta - at + t))^{1/q}}
\]
\[
\leq \frac{n^{t/q} \Delta^{1/2 - 1/q + 1/2q}}{s_t(\pi)};
\]
and, as in the first case, the summands corresponding to these values of \( a \) can be neglected, unless the last expression is \( \asymp 1 \). If this is the case, we compute a single summand to be
\[
\left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{a} \right)^{1/q} \left( \frac{n - ts_t(\pi)}{\Delta - at} \right)^{1/q} \Delta^{\frac{a-1}{2q}a} \leq (\chi_\lambda(1))^{e \varepsilon \Delta} \left( \frac{\Delta!}{n^\Delta} \right)^{1/q} \left( \frac{s_t(\pi)}{\Delta^{1/2 + 1/2q}} \right)^{a} \left( \frac{n^{1/q}}{\Delta^{1/2}} \right)^{\Delta - at}
\]
\[
\leq (\chi_\lambda(1))^{e \varepsilon \Delta} \left( \frac{s_t(\pi)}{\Delta^{1/2 - t/q + 1/2q + t/q}} \right)^{a}
\]
\[
\leq (\chi_\lambda(1))^{e \varepsilon \Delta}.
\]

\[\Box\]

We are now in a position to prove Proposition 111. Let \( \lambda \) be a partition of \( n, q \geq 2 \) an integer, and \( \pi \in S_n \) a permutation such that \( \pi^q = 1 \). Then \( s_t(\pi) = 0 \), unless \( t|q \). We prove 111, using estimates in different ranges for cycle numbers of \( \pi \) and \( \Delta = n - \lambda_1 \).

If \( \Delta = 0 \), the assertion is trivial, and if \( 1 \leq \Delta \leq n^\varepsilon \), and \( s_t(\pi) \geq 2\varepsilon n^{t/q} \) for some \( t \), then we use the trivial estimate \( \chi_\lambda(\pi) \leq \chi_\lambda(1) \leq n^\Delta \), together with Lemma 111 to see that the contribution of such terms to the sum in question is \( \leq e^{-cn^{1/q}} \). If on the other hand
If \( s_t(\pi) < 2en^{t/q} \) for all \( t \), we estimate \( |\chi_{\lambda}(\pi)| \) as follows. As in the proof of Lemma 6, we choose \( a_t \) cycles of length \( t \) without boxes from the first row, and obtain

\[
|\chi_{\lambda}(\pi)| \leq \Delta! \sum_{\sum t a_t \leq \Delta} \prod_{t} \left(2en^{t/q}/a_t \right) \leq \Delta! \Delta^{\tau(q)} (2en)^{\Delta/q} \leq (\Delta!)^2 (\chi_{\lambda}(1))^{1/q},
\]

and this is of the desired order of magnitude, since

\[
(\chi_{\lambda}(1))^{3\varepsilon} \geq \left( \frac{n}{\Delta} \right)^{3\varepsilon \Delta} \geq n^{2\varepsilon \Delta} \geq \Delta^{2\Delta}.
\]

Next, we consider the case where \( \Delta \) is in the range \([n^\varepsilon, n/C]\) for some sufficiently large constant \( C \). Assume first, that there is some \( t \) such that \( s_t(\pi) < 2en^{t/q} \). Then we have \( E(n, \Delta, t, s_t(\pi)) \leq (\chi_{\lambda}(1))^\varepsilon \). For, either \( \Delta < t s_t(\pi) \), which implies

\[
s_t(\pi) < 2en^{t/q} \leq \Delta^{1/2} n^{t/q} \leq \Delta \frac{2t+1}{2q} n^{t/q};
\]

or \( ts_t(\pi) \leq \Delta \leq t s_t(\pi)^{1+1/2q} \), in which case

\[
\Delta \frac{2t+1}{2q} \leq \Delta^{1-1/(2q)} \leq (s_t(\pi))^{1-1/(4q^2)} \leq n^{t/q};
\]

or, finally, \( \Delta > ts_t(\pi)^{1+1/2q} \), which implies

\[
E(n, \Delta, t, s_t(\pi)) \leq n^{s_t(\pi)} \leq 2^\Delta \leq (\chi_{\lambda}(1))^\varepsilon.
\]

Hence, disregarding the factor corresponding to such a value of \( t \) does not change the estimate in Lemma 19 significantly. From this observation and Lemmas 11 and 19, we obtain for \( \pi \) of order dividing \( q \)

\[
\frac{N(q, n, s_1(\pi), \ldots s_T(\pi))}{N(q, n)} |\chi(\pi)| \leq (\chi_{\lambda}(1))^{\frac{1}{4} + \varepsilon} \prod_{t/q \leq t < q \atop s_t(\pi) \geq 2en^{t/q}} \left( \frac{en^{t/q}}{ts_t(\pi)} \right)^{s_t(\pi)} E(n, \Delta, t, s_t(\pi)).
\]

If \( t \) is such that \( s_t(\pi) > \Delta \), then

\[
\left( \frac{en^{t/q}}{s_t(\pi)} \right)^{s_t(\pi)} E(n, \Delta, t, s_t(\pi)) = \left( \frac{en^{t/q}}{s_t(\pi)} \right)^{s_t(\pi) - \Delta} \Delta^{\frac{2t-2t^{1+1}}{2q^{t}}} \Delta < 1;
\]

if \( \Delta/t \leq s_t(\pi) \leq \Delta \), we find that

\[
\left( \frac{en^{t/q}}{s_t(\pi)} \right)^{s_t(\pi)} E(n, \Delta, t, s_t(\pi)) = \left( \frac{en^{t/q}}{s_t(\pi)} \right)^{s_t(\pi) - \Delta} \Delta^{\frac{2t-2t^{1+1}}{2q^{t}}} \Delta \leq e^{c\Delta} \left( \frac{\Delta}{n^{t/q}} \right)^{(1-\frac{1}{2})\Delta} \Delta^{\frac{2t-2t^{1+1}}{2q^{t}}} \Delta \leq e^{c\Delta} \left( \frac{\Delta}{n^{t/\pi}} \right) \Delta \leq (\chi_{\lambda}(1))^\varepsilon,
\]
since $t$ is bounded by $q/2$. Finally, if $ts_t(\pi) \leq \Delta$, we have

$$\left( \frac{en^{t/q}}{s_t(\pi)} \right)^{s_t(\pi)} \bar{E}(n, \Delta, t, s_t(\pi)) = \left( e^{\Delta \frac{a^{2t-1}}{2q}} \right)^{s_t(\pi)} \leq \left( e^{\Delta \frac{1-1/4q}{s_t(\pi)}} \right)^{s_t(\pi)},$$

and either $s_t(\pi) < \Delta^{1-1/4q}$, in which case this factor is $\leq (\chi_{\lambda}(1))^\varepsilon$, or $s_t(\pi) \geq \Delta^{1-1/4q}$, in which case it is less than 1. Hence, in any case the right-hand side of (30) is bounded by $(\chi_{\lambda}(1))^\varepsilon$. Summing over all possible values for the $s_t(\pi)$ gives an additional factor $\leq n^{r(q)}$, which is absorbed into $(\chi_{\lambda}(1))^\varepsilon$ as well. Hence, for these characters (17) holds.

Finally, we have to consider partitions with $\Delta > n/C$. By Lemma 8 this implies $\chi_{\lambda}(1) \geq e^m$. By Lemma 13 we have

$$|\chi_{\lambda}(\pi)| \leq (2\sqrt{n})^{\frac{1}{4} - \varepsilon} (\chi_{\lambda}(1))^{\frac{1}{4} + \varepsilon},$$

where $s = \sum_{t<q} s_t(\pi)$. Together with Lemma 11 we deduce

$$\frac{|\chi_{\lambda}(\pi)|N(n, q, s_1(\pi), \ldots, s_T(\pi))}{(\chi_{\lambda}(1))^{1/4+\varepsilon} N(n, q)} \leq (4n)^{\frac{a+2t-1}{2q}} \prod_{t|q \ t<q} \left( e^{n^{t/q}} \right)^{s_t(\pi)}$$

$$= \prod_{t|q \ t<q} \left( e^{4en^{\frac{a+2t-1}{2q}}} \right)^{s_t(\pi)}.$$

In the last product, all factors with $s_t(\pi) > 4en^{\frac{a+2t-1}{2q}}$ are bounded by 1, while for $t$ such that $s_t(\pi) \leq 4en^{\frac{a+2t-1}{2q}}$ the corresponding factor is at most $n^{1-\frac{1}{4q}} < (\chi_{\lambda}(1))^\varepsilon$. Hence, also in this case, the left-hand side of (17) has the desired order of magnitude, and the proof of Proposition 1 is complete.

4. The multiplicities of root number functions

In order to be able to estimate the subgroup growth of Fuchsian groups, we also need to establish certain properties of the multiplicities of root number functions for symmetric groups. These are summarized in our next two results. For a positive integer $q$, define the $q$-th root number function $r_q : S_n \to \mathbb{N}_0$ via

$$r_q(\pi) := |\{\sigma \in S_n : \sigma^q = \pi\}|,$$

and, for each irreducible character $\chi$ of $S_n$, let

$$m_{\chi}^{(q)} := \langle r_q, \chi \rangle$$

be the multiplicity of $\chi$ in $r_q$. It is known that the functions $r_q$ are proper characters, that is, the $m_{\chi}^{(q)}$ are non-negative integers; cf. 29.

**Proposition 2.** Let $q \geq 2$ be an integer, $\varepsilon > 0$, $n \geq n_0(\varepsilon)$, and let $\lambda \vdash n$ be a partition with corresponding character $\chi_{\lambda}$.

(i) We have $m_{\chi_{\lambda}}^{(q)} < (\chi_{\lambda}(1))^{1-2/q+\varepsilon}$. 

(ii) Given a partition \( \mu \vdash \Delta \), there exists some constant \( C^q_{\mu} \), depending only on \( \mu \) and \( q \), such that, for \( \lambda \setminus \lambda_1 = \mu \) and \( n \) sufficiently large, we have \( m^{(q)}_{\lambda \Delta} = C^q_{\mu} \). In particular, we have

\[
C^q_{(1)} = \tau(q) - 1,
\]

\[
C^q_{(2)} = \frac{1}{2}(\sigma(q) + \tau(q)^2 - 3\tau(q) + \tau_{\text{odd}}(q)),
\]

\[
C^q_{(1,1)} = \frac{1}{2}(\sigma(q) + \tau(q)^2 - 3\tau(q) - \tau_{\text{odd}}(q)) + 1,
\]

where \( \tau_{\text{odd}}(q) \) is the number of odd divisors of \( q \).

(iii) For a partition \( \mu \vdash \Delta \), \( q \) odd, and sufficiently large \( n \), we have \( m^{(q)}_{\lambda \Delta} = 0 \).

**Proposition 3.** Let \( \varepsilon > 0 \) be given, let \( q \geq 2 \) be an integer, \( \Delta \geq \Delta_0(q, \varepsilon) \) and \( n \geq n_0(q, \Delta, \varepsilon) \). Then, for partitions \( \mu \vdash \Delta \) and \( \lambda \vdash n \) with \( \lambda \setminus \lambda_1 = \mu \), we have that

\[
\left| \frac{m^{q}_{\lambda \Delta!}}{|\chi_{\mu(1)} Q_\Delta(q)|} - 1 \right| < \varepsilon,
\]

where \( Q_\Delta(q) = \sum_{i=1}^{\Delta} S(\Delta, \Delta - i) q^i \).

This section is devoted to the proofs of these results.

4.1. **Proof of Proposition 2.** We begin by translating the problem of bounding \( m^{(q)}_{\lambda \Delta} \) from an algebraic into a combinatorial question.

**Lemma 20.** For \( q \geq 2 \) and \( \chi \in \text{Irr}(S_n) \), we have

\[
m^{(q)}_{\chi} \leq \sum_{c^\pi = 1} \langle 1, |\chi_{\downarrow C_{S_n}(c)}| \rangle_{C_{S_n}(c)},
\]

where the summation extends over all conjugacy classes \( c \) in \( S_n \), whose orders divide \( q \), and \( C_{S_n}(c) \) denotes the centralizer of some element \( \pi \in c \).

**Proof.** It is shown in [29] that, for every class \( c \), there exists a linear character \( \phi_c \) of \( C_{S_n}(c) \) such that \( r_q = \sum_{c^\pi = 1} \phi_c^{\uparrow S_n} \). By Frobenius reciprocity this implies

\[
m^{(q)}_{\chi} = \sum_{c^\pi = 1} \langle \phi_c^{\uparrow S_n}, \chi \rangle_{S_n}
\]

\[
= \sum_{c^\pi = 1} \langle \phi_c, \chi_{\downarrow C_{S_n}(c)} \rangle_{C_{S_n}(c)}
\]

\[
\leq \sum_{c^\pi = 1} \langle 1, |\chi_{\downarrow C_{S_n}(c)}| \rangle_{C_{S_n}(c)}.
\]

\[\square\]
Proof of Proposition 2 (i) Let $\varepsilon > 0$ be given. We first consider characters $\chi_\lambda$ with $\chi_\lambda(1) > n^{1-\varepsilon^2/4}$, starting from the formula

$$m^{(q)}_{\chi_\lambda} \leq \sum_{c^q=1} \frac{1}{|C_{S_n}(c)|} \sum_{\pi \in C_{S_n}(c)} |\chi_\lambda(\pi)|.$$  

The number of summands of the outer sum is $\leq n^{\pi(q)}$ and therefore negligible. In the inner sum, we bound $\chi_\lambda(\pi)$ by either using Lemma 5 or via the trivial bound $\chi_\lambda(1)$, depending on the number of cycles of $\pi$. Estimating the number of elements $\pi \in C_{S_n}(c)$ with $k$ cycles using Lemma 12 (ii) for $k \geq (\log n)^3$, and trivially otherwise, we obtain

$$m^{(q)}_{\chi_\lambda} \leq \left(\chi_\lambda(1)\right)^{\varepsilon} (2\sqrt{n})^{(\log n)^3} + \left(\chi_\lambda(1)\right)^{\varepsilon} \sum_{k \geq (\log n)^3} \min\left((2\sqrt{n})^k, \chi_\lambda(1)\right) \frac{(3q \log n)^{k/q}}{[k/q]!}.$$  

The first summand is negligible. The greatest term of the sum is coming from one of $k = \left\lfloor \frac{2 \log \chi_\lambda(1)}{\log n} \right\rfloor$ and $k = \left\lceil \frac{2 \log \chi_\lambda(1)}{\log n} \right\rceil$, and these terms differ by a factor $n$ at most; hence, using Stirling’s formula, we obtain

$$m^{(q)}_{\chi_\lambda} \leq \left(\chi_\lambda(1)\right)^{1+\varepsilon} e^{-\frac{2}{\varepsilon^2} \log \chi_\lambda(1) \frac{\log \chi_\lambda(1)}{\log n}} \leq \left(\chi_\lambda(1)\right)^{1-\frac{2}{\varepsilon^2} + 2\varepsilon},$$  

and our claim is proven in this case. In particular, setting $\Delta = n - \lambda_1$, our first claim holds true for all characters $\chi_\lambda$ belonging to a partition $\lambda$ with $\Delta > n^{1-\varepsilon^2/2}$.

Next, we consider the range $(\log n)^4 \leq \Delta \leq n^{1-\varepsilon^2/2}$. In this range we estimate $\chi_\lambda(\pi)$ by means of Lemma 6 and Equation 15, and we bound the number of centralizer elements with $k$ cycles again via Lemma 12 (ii), or trivially. In this way, we obtain

$$m^{(q)}_{\chi_\lambda} \leq \left(\chi_\lambda(1)\right)^{\varepsilon} + \left(\chi_\lambda(1)\right)^{\varepsilon} \sum_{k \geq (\log n)^3} \min\left((4\sqrt{\Delta})^k, \chi_\lambda(1)\right) \frac{(3q \log n)^{k/q}}{[k/q]!} \leq \left(\chi_\lambda(1)\right)^{1+3\varepsilon} e^{-\frac{2}{\varepsilon^2} \log \chi_\lambda(1) \frac{\log \chi_\lambda(1)}{\log \Delta}}.$$  

On the other hand, from Lemma 8 (ii) we see that $\frac{\log \log \chi_\lambda(1)}{\log \Delta} \geq 1$ for all $\Delta < n/5$, hence, our claim holds in this case as well.

Finally, for $\Delta < (\log n)^4$, we see from Lemma 12 (ii) that we may neglect the contribution of permutations with at least $(\log n)^3$ cycles. For the remaining ones we have $\chi_\lambda(\pi) \leq (\log n)^{3\Delta}$ by Lemma 6 Equation 14, which is less then $(\chi_\lambda(1))^{\varepsilon}$. This completes the proof of Proposition 2 (i).

(ii) We describe the computation of $m^{(q)}_{\chi_\lambda}$ for bounded $\Delta$. We have

$$m^{(q)}_{\chi_\lambda} = \langle \chi_\lambda, r_q \rangle = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi)q.$$  

Let $\mu \vdash \Delta$ be a partition. By Lemma 7 there exists a polynomial $P_\mu(x_1, \ldots, x_\Delta)$, such that, for every $n$ and $\lambda \vdash n$ with $\lambda \setminus \lambda_1 = \mu$, we have $\chi_\lambda = P_\mu(s_1, \ldots, s_\Delta)$. Moreover, if
$x_d$ has weight $d$, then $P_\mu$ has weight at most $|\mu|$. We have

$$s_d(\pi^q) = \sum_{\kappa \text{ cycles of length } \kappa} (\kappa, q)s_\kappa(\pi), \quad (31)$$

since the $q$-th power of a cycle of length $\kappa$ consists of $(\kappa, q)$ cycles, each of length $\frac{\kappa}{(\kappa, q)}$. Thus, we have to compute

$$\frac{1}{n!} \sum_{\pi \in S_n} Q_{\mu, q}(s_1(\pi), s_2(\pi), \ldots, s_{\Delta q}(\pi))$$

for a certain polynomial $Q_{\mu, q}$ of weight at most $\Delta q$.

By Lemma 13, the last expression converges to some constant. Hence, $m_{\chi_{\lambda}}^{(q)}$ converges to some real number, but as $m_{\chi_{\lambda}}^{(q)}$ is integral, it has to become constant for $n$ sufficiently large.

We now consider the cases $\mu = (1), (2)$ and $(1, 1)$. If $\lambda = (n - 1, 1)$, we have

$$m_{\chi_{\lambda}}^{(q)} = \frac{1}{n!} \sum_{\pi \in S_n} \chi_\lambda(\pi^q) = \frac{1}{n!} \sum_{\pi \in S_n} s_1(\pi^q) - 1 = \frac{1}{n!} \sum_{\pi \in S_n} \sum_{d|q} ds_d(\pi) - 1.$$

By Lemma 13, the expected number of $d$-cycles is $\frac{1}{d^2}$; hence, $m_{\chi_{\lambda}}^{(q)} = \tau(q) - 1$. Next, for $\lambda = (n - 2, 2)$, Lemma 7 yields

$$\chi_\lambda = \frac{s_1^2}{2} - \frac{3s_1}{2} + s_2.$$

Inserting (31), we deduce

$$m_{\chi_{\lambda}}^{(q)} = \frac{1}{n!} \sum_{\pi \in S_n} \left( \frac{1}{2} \left( \sum_{d|q} ds_d(\pi) \right)^2 - \frac{3}{2} \sum_{d|q} ds_d(\pi) \right) + \sum_{d|q, (2d, q) = d} ds_d(\pi)$$

$$= \frac{1}{2} \sum_{d_1, d_2|q} \frac{1}{n!} \sum_{\pi \in S_n} (d_1d_2s_{d_1}(\pi)s_{d_2}(\pi)) - \frac{3}{2} \sum_{d|q} \frac{1}{n!} \sum_{\pi \in S_n} ds_s(\pi)$$

$$+ \sum_{d|q, (2d, q) = d} \frac{1}{n!} \sum_{\pi \in S_n} ds_d(\pi).$$

If $d_1 \neq d_2$ then, by Lemma 13, $s_{d_1}(\pi)$ and $s_{d_2}(\pi)$ are asymptotically independent for $\pi \in S_n$ chosen at random, and have mean values $\frac{1}{d_1}$ and $\frac{1}{d_2}$, respectively, while $(s_d(\pi))^2$ has mean value $\frac{1}{d} + \frac{1}{d^2}$. Hence, the first sum is asymptotically equal to $\sigma(q) + \tau(q)^2$. The second sum equals $\tau(q)$, whereas the last sum yields $|\{d|q : (2d, q) = d\}|$, which equals $\tau_{\text{odd}}(q)$. We deduce that, as $n \to \infty$,

$$m_{\chi_{\lambda}}^{(q)} \to \frac{1}{2} \left( \sigma(q) + \tau(q)^2 - 3\tau(q) + \tau_{\text{odd}}(q) \right),$$

which implies our claim, since $m_{\chi_{\lambda}}^{(q)}$ is always integral. A similar computation leads to the value of $m_{\chi_{\lambda}}^{(q)}$ for $\lambda = (n - 2, 1, 1)$. 


(iii) For \( q \) odd, we have
\[
m_{\chi, q} = \sum_{\pi \in S_n} \chi_{\lambda'}(\pi^q) = \sum_{\pi \in S_n} \chi_{\lambda}(\pi^q) \varepsilon(\pi) = \sum_{\pi \in S_n} \chi_{\lambda}(\pi^q) \varepsilon(\pi),
\]
where \( \varepsilon \) is the sign character. As in the proof of part (ii), we can write \( m_{\chi, q} \) as a linear combination of sums of the form
\[
\frac{1}{n!} \sum_{\pi \in S_n} \varepsilon(\pi) \prod_{j=1}^{lq} (s_j(\pi)) e_j.
\]
Observe first that the contribution coming from permutations \( \pi \) with \( \sum_{j=1}^{lq} j s_j(\pi) > n/2 \) is \( o(1) \) by Lemma 13. The sum over permutations \( \pi \) with \( \sum_{j=1}^{lq} j s_j(\pi) \leq n/2 \) vanishes, since it can be expressed as a linear combination of sums of the form
\[
\sum_{\pi \in S_n} \varepsilon(\pi) = \pm \sum_{\pi \in S_{n-S}} \varepsilon(\pi) = 0,
\]
where \( S = \sum_{j=1}^{lq} s_j \). Hence, for \( q \) odd and \( n \) large, \( m_{\chi, q} < 1 \) and therefore vanishes. □

4.2. Proof of Proposition \ref{proposition:random-walks}. We compute the scalar product \( \langle r_q, \chi_{\lambda} \rangle \), and evaluate \( \chi_{\lambda} \) using Lemma 7 to obtain
\[
m_{\chi, q} = \frac{\chi_{\mu}(1)}{n!} \sum_{\pi \in S_n} \left( s_1(\pi^q) \right)^{\Delta} + O \left( \frac{\Delta \chi_{\mu}(1)}{n!} \sum_{i \leq \Delta-1} \sum_{\pi \in S_n} \left( s_1(\pi^q) + \ldots + s_{\Delta}(\pi^q) \right) \right). \tag{32}
\]
By Lemma 13 we have as \( n \to \infty \)
\[
\frac{1}{n!} \left| \left\{ \pi \in S_n : s_1(\pi) = a \right\} \right| \sim \frac{e^{-1/t}}{ta!},
\]
and the events “\( s_1(\pi) = a \)” and “\( s_\nu(\pi) = b \)” are asymptotically independent. From this together with Equation (31) we deduce
\[
\frac{1}{n!} \sum_{\pi \in S_n} \left( s_1(\pi^q) \right)^{\Delta} = \frac{1}{n!} \sum_{\pi \in S_n} \left( \sum_{t \mid q} ts_t(\pi) \right)^{\Delta} \sim \sum_{d_t \mid q} \prod_{t \mid q} \left( \frac{1}{n!} \sum_{\pi \in S_n} ts_t(\pi) \right)^{d_t} \sum_{\nu=1}^{\infty} \frac{e^{-1/t} (ta)^{d_t}}{t^\nu \nu!} \sum_{\nu=1}^{\infty} \prod_{t \mid q} t^{d_t} \sum_{\nu=1}^{\infty} S(d_t, \nu) t^{-\nu}. \]
The last quantity can be written as
\[ \sum_{d_1 + \cdots + d_q} \prod_{t \mid q} Q_{d_t}(t), \tag{33} \]
where
\[ Q_n(t) = \sum_{\nu=0}^{n-1} S(n, n - \nu) t^\nu. \]

For fixed \( n \), the sequence \( S(n, m) \) is unimodal in \( m \); define \( m_0 \) to be the least \( m \) with \( S(n, m_0) = \max_n S(n, m) \). Kanold \[15\] has shown that
\[ m_0 \sim \frac{n}{\log n}, \]
morover, it can be deduced from his proof that
\[ \sum_{m=(1+\varepsilon)m_0}^{(1-\varepsilon)m_0} S(n, m) \sim \sum_{m=1}^{n} S(n, m), \quad n \to \infty. \tag{34} \]

This estimate implies \( Q_{d_t}(t) \leq (t/q)^{d_t(1-\varepsilon)} Q_{d_t}(q) \) for all \( t \mid q \) and \( d_t \geq d_0(\varepsilon) \).

Next, we establish the inequality \( Q_n(x) Q_{n'}(x) \leq Q_{n+n'}(x) \) for all real positive \( x \) and \( n, m \geq 1 \). More precisely, we show that each single coefficient of \( Q_n(x) Q_{n'}(x) \) is less than or equal to the corresponding coefficient of \( Q_{n+n'}(x) \), which implies our claim.

Computing the \( m \)-th coefficient explicitly, we have to show that
\[ \sum_{i+j=m} S(n, n'-i) S(n', n'-j) \leq S(n+n', n+n'-m), \]
which is true since the right-hand side is the number of partitions of a set with \( n+n' \) elements into \( n+n'-m \) parts, while the left-hand side is the number of these partitions that respect some fixed partition of the large set into two sets with \( n \) and \( n' \) elements.

Together with \( \text{(33)} \), we deduce that, for \( d \) sufficiently large, and a fixed tuple \( d_1, \ldots, d_q \) with \( d_1 + \cdots + d_q = \Delta \),
\[ \prod_{t \mid q} Q_{d_t}(t) \leq \prod_{t \mid q}(t/q)^{2d_t/3} Q_{d_t}(q) \leq (2/3)^{\Delta-d_q} \prod_{t \mid q} Q_{d_t}(q) \leq (2/3)^{\Delta-d_q} Q_\Delta(q). \]

We now split the sum \( \text{(33)} \) into three ranges, according to whether \( d_q = \Delta, \Delta - C\tau(q) \leq d_q \leq \Delta - 1 \), or \( d_q < \Delta - C\tau(q) \), and we want to show that the sum over the latter two ranges is negligible compared to the first term. From the last estimate we find that, for \( \varepsilon > 0 \) and \( \Delta > \Delta_0(q, \varepsilon) \), there is some \( C = C(\varepsilon) \), such that\(^5\)
\[ \sum_{d_1 + \cdots + d_q} \prod_{t \mid q} Q_{d_t}(t) \leq Q_\Delta(q) \sum_{\nu \geq C\tau(q)} (2/3)^\nu \left( \frac{\nu + \tau(q) - 1}{\tau(q) - 1} \right) \leq \varepsilon Q_\Delta(q). \]

\(^5\)Complex integration shows that we may take \( C(\varepsilon) = 10 + 7 \log \varepsilon^{-1} \).
In the range $\Delta - C\tau(q) \leq d_q \leq \Delta - 1$, the number of summands is bounded, and their sum can be estimated by $CQ_{\Delta-1}(q)$, and we obtain
\[
\sum_{\sum_{t|q} \pi_t = \Delta} \prod_{t^q \neq \Delta} Q_{d_t}(t) \leq \varepsilon Q_{\Delta}(q) + CQ_{\Delta-1}(q).
\]
Now we use the inequalities (cf. [15, Satz 1])
\[
\frac{(m + 1)^n}{m!} \left( 1 - \frac{m}{(1 - 1/m)^n} \right) \leq S(n, m) \leq \frac{(m + 1)^n}{m!},
\]
to see that $S(n, n - \mu) q^\mu$ in monotonically decreasing with $\mu$ for $n$ sufficiently large and $\mu > n - \frac{n}{2\log n}$; in particular, we have
\[
\sum_{\mu > n - n/(2\log n)} S(n, n - \mu) q^\mu < \frac{1}{n} Q_n(q).
\]
Since $S(\Delta + 1, \Delta - \mu) \geq (\Delta - \mu) S(\Delta, \Delta - \mu)$, the latter inequality implies
\[
Q_{\Delta}(q) \geq \sum_{\mu \leq \Delta - \Delta/(2\log \Delta)} S(\Delta, \Delta - \mu) q^\mu
\]
\[
\geq \frac{\Delta}{2\log \Delta} \sum_{\mu \leq \Delta - \Delta/(2\log \Delta)} S(\Delta - 1, \Delta - \mu) q^\mu
\]
\[
\geq \frac{\Delta}{2\log \Delta} \left( 1 - \frac{1}{\Delta - 1} \right) Q_{\Delta-1}(q),
\]
and we deduce that
\[
Q_{\Delta}(q) \leq \frac{1}{n!} \sum_{\pi \in S_n} (s_1(\pi^q))^\Delta \leq (1 + \varepsilon) Q_{\Delta}(q),
\]
provided that $\Delta \geq \Delta_0(q, \varepsilon)$ and $n \geq n_0(\Delta, q, \varepsilon)$. Estimating the error term in (32) in a way similar to our treatment of the main term, and using the fact that
\[
\left( \frac{s_1(\pi^q)}{\Delta} \right) = \frac{1}{\Delta!} (s_1(\pi^q))^\Delta + O(s_1(\pi^q)^{\Delta-1}),
\]
Proposition 5 follows.

5. Subgroup growth of Fuchsian groups

5.1. The generic case. Let $r, s, t \geq 0$ be integers, $a_1, \ldots, a_r \geq 2$ in $\mathbb{N} \cup \{\infty\}$, and let $e_1, \ldots, e_s \geq 2$ be integral. Define the group $\Gamma = \Gamma(t; a_1, \ldots, a_r; e_1, \ldots, e_s)$ associated with these data by\footnote{$\Gamma$ is a Fuchsian group in the most general sense met in the literature; cf., for instance, [19] Prop. 5.3 or [21] Sec. II.7.}
\[
\Gamma = \left\langle x_1, \ldots, x_r, y_1, \ldots, y_s, u_1, v_1, \ldots, u_t, v_t \right| x_1^{a_1} = x_2^{a_2} = \cdots = x_r^{a_r} = x_1 x_2 \cdots x_r y_1^{e_1} y_2^{e_2} \cdots y_s^{e_s} [u_1, v_1][u_2, v_2] \cdots [u_t, v_t] = 1 \rightangle.
\]
Define
\[
\mu(\Gamma) = \sum_{i=1}^{r} \left( 1 - \frac{1}{a_i} \right) + s + 2(t - 1),
\]
\[
\alpha(\Gamma) = \mu(\Gamma) - \sum_{j=1}^{s} \left( 1 - \frac{2}{e_j} \right),
\]
\[
m_{\Gamma} = [a_1, \ldots, a_r].
\]

The main result of this section provides an asymptotic expansion for the number of index \( n \) subgroups of these groups.

**Theorem 3.** Let \( \Gamma \) be given as in (35), and suppose that \( \alpha(\Gamma) > 0 \). Then the number \( s_n(\Gamma) \) of index \( n \) subgroups in \( \Gamma \) satisfies an asymptotic expansion
\[
s_n(\Gamma) \approx \delta L_{\Gamma}(n!)^{\mu(\Gamma)} \Phi_{\Gamma}(n) \left\{ 1 + \sum_{\nu \geq 1} a_\nu(\Gamma) n^{-\nu/m_{\Gamma}} \right\}, \quad (n \to \infty).
\]

Here,
\[
\delta = \begin{cases} 
2, & \forall i : a_i \text{ finite and odd}, \forall j : e_j \text{ even} \\
1, & \text{otherwise},
\end{cases}
\]
\[
L_{\Gamma} = (2\pi)^{-1/2-\sum_i (1-1/a_i)} (a_1 a_2 \cdots a_r)^{-1/2} \exp \left( -\sum_{i=2i} \frac{1}{2a_i} \right),
\]
\[
\Phi_{\Gamma}(n) = n^{3/2-\sum_i (1-1/a_i)} \exp \left( \sum_{i=1}^{r} \sum_{t|a_i, t < a_i} \frac{n^{t/a_i}}{t} \right),
\]
and the \( a_\nu(\Gamma) \) are explicitly computable constants depending only on \( \Gamma \).

**Corollary 1.** Let \( \Gamma \) be as in (35), and suppose that \( \alpha(\Gamma) > 0 \). Then we have
\[
\frac{s_{n+1}(\Gamma)}{s_n(\Gamma)} \sim (n+1)^{\mu(\Gamma)}, \quad (n \to \infty);
\]
in particular, \( s_n(\Gamma) \) is strictly increasing for sufficiently large \( n \).

**Proof of Theorem 3** The proof proceeds in three steps: first we express \( h_n(\Gamma) := |\text{Hom}(\Gamma, S_n)|/n! \) in character theoretic terms; next, we use results from Sections 3 and 4 to obtain an asymptotic estimate for \( h_n(\Gamma) \). The assertions of the theorem are then deduced by means of an asymptotic method for divergent power series due to Bender [2].

Let \( w \) be a segment of
\[
R = x_1 x_2 \cdots x_r y_1^{e_1} y_2^{e_2} \cdots y_s^{e_s} [u_1, v_1] [u_2, v_2] \cdots [u_t, v_t],
\]
where powers \( y_j^{e_j} \) and commutators are treated as single letters. For such a segment \( w \) and \( \pi \in S_n \), define \( N_w(\pi) \) to be the number of solutions of the equation \( w = \pi \), subject
to the conditions $x_i^{a_i} = 1$ for those $i$ for which $x_i$ occurs in $w$. Observing that $N_w(\pi)$ is a class function, define coefficients $\beta^{(w)}_\chi$ via the expansion

$$N_w(\pi) = \prod_{i=1}^{\nu(w)} \frac{|\text{Hom}(C_{a_i}, S_n)|}{n!} \sum_{\chi \in \text{Irr}(S_n)} \beta^{(w)}_\chi \chi(\pi), \quad \text{(36)}$$

where the product is extended over those $i \in [r]$, such that $x_i$ occurs in $w$. Moreover, for a positive integer $q$ and an irreducible character $\chi$, we set

$$\alpha^{(q)}_\chi := \frac{1}{|\text{Hom}(C_q, S_n)|} \sum_{\chi(1) = 1} \chi(\pi).$$

We claim that

$$\beta^{(w)}_\chi = (n!)^{\{j : y_j \in w\} + 2\{k : u_k \in w\}} \prod_{x_i \in w} \alpha^{(a_i)}(x_i) \prod_{y_j \in w} m^{(e_j)}(y_j) \frac{\prod_{x_i \in w} \alpha^{(a_i)}(x_i) \prod_{y_j \in w} m^{(e_j)}(y_j)}{(\chi(1))^{l(w) - 1}}, \quad \text{(37)}$$

where $m^{(q)}_i = \langle r_q, \chi \rangle$, and $l(w)$ is the number of different letters occurring in $w$. We prove (37) by induction on $l(w)$. If $w$ has length 1, (37) holds by the definition of $\alpha^{(a_i)}(1)$, the definition of $m^{(e_j)}$, or [26] Lemma 1, according to which letter occurs in $w$. Similarly, the induction step breaks into three cases with regard to the new letter; we only have to treat the case where the new letter is $x_i$ for some $2 \leq i \leq r$, the other cases follow directly from Lemma 1(ii). We have

$$N_{wx_i}(\pi) = \sum_{\sigma \in S_n} N_w(\sigma) \cdot |\text{Hom}(C_{a_i}, S_n)| \sum_{\chi} \alpha^{(a_i)}(\chi) \chi(\sigma^{-1} \pi)$$

$$= \prod_{i=1}^{\nu(w)} \frac{|\text{Hom}(C_{a_i}, S_n)|}{n!} \sum_{\sigma} \sum_{\chi} \alpha^{(a_i)}(\chi) \beta^{(w)}_\chi \chi(\sigma) \chi(\sigma^{-1} \pi)$$

$$= \prod_{i=1}^{\nu(w)} \frac{|\text{Hom}(C_{a_i}, S_n)|}{n!} \sum_{\chi} \alpha^{(a_i)}(\chi) \beta^{(w)}_\chi \chi(\sigma) \sum_{c_1, c_2} |c_1| \cdot |c_2| \frac{\chi(c_1) \chi(c_2) \chi(\pi)}{n!}$$

$$= \prod_{i=1}^{\nu(w)} \frac{|\text{Hom}(C_{a_i}, S_n)|}{n!} \sum_{\chi} \alpha^{(a_i)}(\chi) \beta^{(w)}_\chi \chi(\sigma) \chi(\pi).$$

Here, we have used orthogonality, and the fact\(^7\) that the number of solutions in a finite group $G$ of the equation $x_1 \cdot x_2 = g$ with $x_i \in c_i$ is given by

$$\frac{|c_1| \cdot |c_2|}{n!} \sum_{\chi} \chi(c_1) \chi(c_2) \chi(g^{-1}).$$

From this, (37) follows.

Now take $w$ as the full relation $R$, and set $\pi = 1$ in the definition of $\beta^{(w)}_\chi$ to obtain

$$h_n(\Gamma) = (n!)^{s+2t-2} \prod_{i=1}^{r} |\text{Hom}(C_{a_i}, S_n)| \sum_{\chi \in \text{Irr}(S_n)} \prod_{i=1}^{s} \alpha^{(a_i)}_\chi \prod_{j=1}^{s} m^{(e_j)}_\chi \frac{(\chi(1))^{r+s+2t-2}}{(\chi(1))^{r+s+2t-2}}. \quad \text{(38)}$$

\(^7\) Cf., for instance, [3] Prop. 9.33 or [16] Theorem 6.3.1.
We concentrate on the sum over characters. Let $A > 0$ be given, and split the sum into three parts $\sum_1, \sum_2, \sum_3$, according to whether $\lambda_1 \geq n - A$, $\|\lambda\| \geq n - A$, or $\lambda_1, \|\lambda\| < n - A$. Note that the first two cases are mutually exclusive for $n > 2A + 1$. For $\varepsilon > 0$ and $n$ sufficiently large, we have by Propositions 11 and 2 (i)

\[
\sum_3 < \sum_{\|\lambda\|, \lambda_1 < n - A} (\chi_\lambda(1))^{\sum_i 1/a_i + \sum_j (1 - 2/e_j) - 2s - 2t + 2 + \varepsilon} \\
\leq 2 \sum_{\|\lambda\| \leq \lambda_1 < n - A} (\chi_\lambda(1))^{-\alpha(\Gamma) + \varepsilon}.
\]

If $\lambda_1 > 3n/4$, Lemma 8 (ii) gives $\chi_\lambda(1) > (n - \lambda_1)^{-\alpha(\Gamma)}$, thus

\[
\sum_{\text{3n/4} < \nu < n - A} (\chi_\lambda(1))^{-\alpha(\Gamma) + \varepsilon} < \sum_{3n/4 < \nu < n - A} \left( \frac{\nu}{n - \nu} \right)^{-\alpha(\Gamma) + \varepsilon} p(n - \nu) \ll \left( \frac{n - A}{A} \right)^{-\alpha(\Gamma) + \varepsilon};
\]

whereas for $\lambda_1 \leq 3n/4$, Lemma 8 (i) implies $\chi_\lambda(1) > 2^{n/8}$, hence

\[
\sum_{\|\lambda\| \leq \lambda_1 \leq 3n/4} (\chi_\lambda(1))^{-\alpha(\Gamma) + \varepsilon} < 2^{-n/8} p(n) < 2^{-n/9}.
\]

We conclude that $\sum_3 \ll n^{-\alpha(\Gamma)A + \varepsilon}$.

Next, we consider $\sum_2$. Suppose that $\delta = 2$. Then every permutation of order $a_i$, every $e_j$-th power of a permutation and every commutator is even, hence $\chi_\lambda(\pi) = \chi_{\lambda'}(\pi)$ for $\pi$ with $\pi^{a_i} = 1$, and $m_{\lambda,i}^{(e_j)} = m_{\lambda',i}^{(e_j)}$, and we obtain $\sum_2 = \sum_1$ in this case. If, on the other hand, $\delta = 1$, then either there is some $i$ such that $a_i$ is even, or there is some $j$ such that $e_j$ is odd. In the first case, write

\[
\alpha_{\lambda,i}^{(a_i)} = \frac{1}{|\text{Hom}(C_{a_i}, S_n)|} \sum_{\pi^{a_i} = 1} \chi_\lambda(\pi) = \frac{1}{|\text{Hom}(C_{a_i}, S_n)|} \sum_{\pi^{a_i} = 1} \sum_{\prod_{l|a_i} (-1)^{(l-1)s_l(\pi)} (s_l(\pi))^{e_l}} \gamma_{e_1, \ldots, e_\Delta} \prod_{l|a_i} (-1)^{(l-1)s_l(\pi)} (s_l(\pi))^{e_l},
\]

where $\Delta = n - \|\lambda\|$, and the coefficients $\gamma_{e_1, \ldots, e_\Delta}$ are determined via Lemma 7. Interchanging summations and applying Lemma 13 (ii), we see that $\alpha_{\lambda,i}^{(a_i)} < e^{-c n^{1/a_i}}$, and $\sum_2$ is negligible. In the second case, we have $m_{\lambda,i}^{(e_j)} = 0$ for some $i$ and $n$ sufficiently large by Proposition 2 (iii), hence $\sum_3$ vanishes. By what we have shown so far,

\[
\sum_1 + \sum_2 + \sum_3 = \delta \sum_1 + O(n^{-A\alpha(\Gamma) + \varepsilon}).
\]

To deal with $\sum_1$, we fix a partition $\lambda \vdash n$ with $\lambda_1 \geq n - A$. Then $\prod_{j=1}^s m_{\lambda,\lambda}^{(e_j)}$ is ultimately constant, and $\chi_\lambda(1)$ is a polynomial in $n$ of degree $n - \lambda_1$. Using Lemmas 7
and \( \Gamma_0(i) \), we compute

\[
\alpha^{(a_i)}_{\lambda, \gamma} = \frac{1}{|\text{Hom}(C_{a_i}, S_n)|} \sum_{\pi a_i = 1} \chi_\lambda(\pi)
\]

\[
= \frac{1}{|\text{Hom}(C_{a_i}, S_n)|} \sum_{b a_i = 1} \sum_{t a_i \leq \Delta} \gamma_{e_1, \ldots, e_{a_i}} \prod_{t | a_i} (s_t(\pi))^{e_t}
\]

\[
= \sum_{\sum_{t | a_i \leq \Delta}} \gamma_{e_1, \ldots, e_{a_i}} \sum_{k \leq \sum_{t | a_i} t e_t} \alpha^{(k)}_{e_1, \ldots, e_{a_i}} \frac{n!}{(n-k)!} \frac{|\text{Hom}(C_{a_i}, S_{n-k})|}{|\text{Hom}(C_{a_i}, S_n)|}.
\]

By \( \Gamma \) Eq. (22) \] we have, for every finite group \( G \) and each fixed \( k \), an asymptotic expansion

\[
\frac{|\text{Hom}(G, S_n)|}{|\text{Hom}(G, S_{n-k})|} \approx n^{k(1-1/m)} \exp \left( \sum_{\nu=1}^{\infty} Q_\nu^{(k)} n^{-\nu/m} \right), \quad (n \to \infty),
\]

(39)

where the coefficients \( Q_\nu^{(k)} \) are given in \( \Gamma \) after Equation (22). Putting \( G = C_{a_i} \), we find that

\[
\alpha^{(a_i)}_{\lambda, \gamma} \approx n^{\Delta/a_i} \left( \sum_{\nu=1}^{\infty} A_{\nu}^{a_i} n^{-\nu/a_i} \right), \quad n \to \infty.
\]

Inserting these results into \( \Gamma \), we obtain an asymptotic formula

\[
\beta^{(R)}_{\lambda, \gamma} = (n!)^{s+2\nu} n^{(1-\nu+s+2\iota+\sum_{1/a_i})} \Delta \left( \sum_{\nu=0}^{\infty} B_{\nu}^{R} n^{-\nu/m_f} \right), \quad n \to \infty.
\]

For fixed \( A \), there are only finitely many partitions \( \lambda \vdash n \) with \( \Delta \leq A \), hence, we obtain an asymptotic expansion for \( \sum_1 \) with leading term given by the partition \( \lambda = (n) \). In this case, \( \alpha^{(a_i)}_{\lambda(n)} = m_{\lambda(n)}^{(e_i)} = 1 \), thus, \( \beta^{(R)}_{\lambda(n)} = (n!)^{s+2\nu} \), and therefore, as \( n \to \infty \),

\[
|\text{Hom}(\Gamma, S_n)| = N_R(1) \approx \delta(n!)^{s+2\nu-1} \prod_{i=1}^{r} |\text{Hom}(C_{a_i}, S_n)| \left( 1 + \sum_{\nu=1}^{\infty} C_{\nu}(\Gamma) n^{-\nu/m_f} \right).
\]

Using the asymptotic expansion \( \Gamma \) Theorem 5 \] for \( |\text{Hom}(G, S_n)| \), the main term of the last expression is found to be

\[
\delta L_{\Gamma}(n!)^{\mu(\Gamma)+1} n^{-1} \Phi_{\Gamma}(n),
\]

with \( \mu(\Gamma), L_{\Gamma}, \Phi_{\Gamma} \) and \( \delta \) as defined above. Mimicking the proof of \( \Gamma \) Prop. 1 \] and using our assumption that \( \mu(\Gamma) > 0 \), we find that, for each fixed \( K \geq 1 \),

\[
\sum_{k=K}^{n-K} \frac{h_k(\Gamma) h_{n-k}(\Gamma)}{h_n(\Gamma)} \ll n^{-K}$\mu(\Gamma)\$.
\]

Combining the latter estimate with the Lemma from \( \Gamma \) Sec. 3 \] (cf. also \( \Gamma \)) and the transformation formula\(^8\)

\[
s_n(\Gamma) = n h_n(\Gamma) - \sum_{k=1}^{n-1} h_{n-k}(\Gamma) s_k(\Gamma),
\]

(40)

\(^8\) Cf. \( \Gamma \) Prop. 1 \] or \( \Gamma \) Prop. 1 \]. A far reaching generalization of this counting principle is found in \( \Gamma \).
now gives
\[
\frac{s_n(\Gamma)}{n h_n(\Gamma)} \approx 1 + \sum_{k=1}^{\infty} d_k(\Gamma) \frac{h_{n-k}(\Gamma)}{h_n(\Gamma)}, \quad (n \to \infty),
\]
where \(d_k(\Gamma)\) is the coefficient of \(z^k\) in the formal power series \((\sum_{n \geq 0} h_n(\Gamma) z^n)^{-1}\). Expanding \(h_{n-k}(\Gamma) h_n(\Gamma)\) by means of the asymptotic formula for \(h_n(\Gamma)\) and the Taylor-series of \(\Phi_{\Gamma}(n)\), the theorem follows.

The condition \(\alpha(\Gamma) > 0\) in Theorem 3 is essentially necessary. It can be violated in one of two ways: either \(\mu(\Gamma) \leq 0\), or \(\mu(\Gamma) > 0\), but there are sufficiently many large values among the \(e_j\) to keep \(\alpha(\Gamma)\) small. Here, we deal with the first possibility; cf. Subsection 5.3 for the second case.

The groups \(\Gamma\) with a presentation of the form (35) and \(\mu(\Gamma) \leq 0\) naturally fall into three classes, according to whether \(\mu(\Gamma) = 0\), and either \(s = 0\), or \(e_j = 2\) for all \(j \leq s\); or \(\mu(\Gamma) < 0\). We will show that in each of these cases the assertion of Theorem 3 fails to hold.

(i) \(\mu(\Gamma) = 0\), and either \(s = 0\) or \(e_j = 2\) for all \(j \leq s\). Then \(\Gamma\) is virtually abelian of rank 2, hence, \(s_n(\Gamma) \ll n^2\); cf. [19, Chapter III, Prop. 7.10]. This would certainly contradict the assertion of Theorem 3 provided that \(r \neq 0\). If \(r = 0\), we have \(\Gamma = \langle x, y \mid [x, y] = 1 \rangle\) or \(\Gamma = \langle x, y \mid x^2 y^2 = 1 \rangle\), and \(s_n(\Gamma) = \sigma(n)\) in both cases, whereas Theorem 3 would predict \(s_n(\Gamma) \sim n^{3/2}\).

(ii) \(\mu(\Gamma) = 0\), and \(e_1 > 2\), say. Then either \(r = 2\), \(a_1 = a_2 = 2\), or \(r = 0\), \(s = 2\), \(t = 0\). In the first case, \(\Gamma\) maps homomorphically onto \(C_2 \ast C_{e_1}\), whereas in the second case, \(\Gamma\) maps homomorphically onto \(C_{e_1} \ast C_{e_2}\); that is, in both cases \(\Gamma\) maps onto a free product with negative Euler characteristic, and therefore \(s_n(\Gamma) \gg s_n(C_2 \ast C_3) \gg (n!)^{1/6}\), while Theorem 3 would predict that \(\Gamma\) is of subexponential growth.

(iii) \(\mu(\Gamma) < 0\). In this case, \(\Gamma\) is finite, and \(s_n(\Gamma)\) is ultimately 0, which contradicts the assertion of Theorem 3 as well.

5.2. Computation of the coefficients \(a_\nu(\Gamma)\). In this section we describe how the coefficients \(a_\nu(\Gamma)\) can be computed explicitly. We begin by giving explicit values to some of the quantities shown to exist in the previous sections.

Combining Lemmas 7, 15, and 16, we evaluate the constants \(\alpha_{\lambda\lambda}^{(q)}\) for partitions \(\lambda \vdash n\) with \(\Delta \leq 2\).

**Lemma 21.** Let \(q \geq 2\) be an integer. Then we have
\[
\alpha_{\chi_{(n-1,1)}}^{(q)} = |\text{Hom}(C_q, S_n)|^{-1} \sum_{\pi^q=1} \chi_{(n-1,1)}(\pi) = n \frac{|\text{Hom}(C_q, S_{n-1})|}{|\text{Hom}(C_q, S_n)|},
\]
and for $q$ even
\[
\alpha_{\chi(n-2,2)}^{(q)} = \frac{n^2}{2} \frac{\Hom(C_q, S_n)}{|\Hom(C_q, S_n)|}
\]
\[
\alpha_{\chi(n-2,1,1)}^{(q)} = 1 + \frac{n^2}{2} \frac{|\Hom(C_q, S_n-2)|}{|\Hom(C_q, S_n)|}
\]
whereas for $q$ odd
\[
\alpha_{\chi(n-2,2)}^{(q)} = -1 + \frac{n(n-1)}{2} \frac{|\Hom(C_q, S_n-2)|}{|\Hom(C_q, S_n)|},
\]
\[
\alpha_{\chi(n-2,1,1)}^{(q)} = \frac{n(n-1)}{2} \frac{|\Hom(C_q, S_n-2)|}{|\Hom(C_q, S_n)|}.
\]

We now use \([39\) and \([23\) Theorem 6\] to compute the first terms of the asymptotic series for $\alpha_{\chi(n-2,2)}^{(q)}$. We obtain
\[
\alpha_{\chi(n-2,2)}^{(q)} = n^{2/q}(1 + R_q(n^{-1/q}))(1 - 2 \sum_{\nu=1}^{q+3} \tilde{Q}_\nu^{(q)} n^{-\nu/q} + S_q(n^{-1/q})) + \mathcal{O}(n^{-4/5}),
\]
where the polynomials $R_q, S_q$ are given as follows.

| $q$  | $R_q(z)$                                                                 | $S_q(z)$                                                                 |
|------|--------------------------------------------------------------------------|--------------------------------------------------------------------------|
| 2    | $\frac{1}{3}z^2 - \frac{3}{4}z^3 + \frac{1}{8}z^4 + \frac{1}{32}z^5$   | $\frac{3}{4}z^2 - \frac{7}{8}z^3 + \frac{23}{64}z^4 + \frac{5}{128}z^5$ |
| 3    | $\frac{1}{5}z^3 - \frac{2}{5}z^5 + \frac{3}{16}z^6$                     | $\frac{1}{5}z^4 - \frac{4}{5}z^5 + \frac{17}{100}z^6$                    |
| 4    | $\frac{1}{5}z^4 - \frac{1}{5}z^6$                                       | $\frac{3}{16}z^4 + \frac{3}{8}z^5 + \frac{25}{64}z^6 + \frac{33}{64}z^7$ |
| 5    | $\frac{1}{5}z^5$                                                        | $\frac{3}{25}z^8$                                                       |
| 6    | $\frac{1}{5}z^6 - \frac{1}{12}z^9$                                      | $\frac{1}{12}z^6 + \frac{1}{6}z^7 + \frac{1}{4}z^8 - \frac{55}{216}z^9$ |
| 7    | $\frac{1}{2}z^7$                                                       | 0                                                                       |
| 8, 12| $\frac{1}{q}z^q$                                                        | $\frac{3}{q^2} (z^q + z^{q+2} + z^{q+3})$                               |
| 9    | $\frac{1}{9}z^9$                                                        | $\frac{1}{27}z^{12}$                                                   |
| 10, 18| $\frac{1}{q}z^q$                                                      | $\frac{3}{q^2} (z^q + z^{q+3})$                                        |
| 14, 16| $\frac{1}{q}z^q$                                                      | $\frac{3}{q^2}z^q$                                                   |
| $q \geq 20$, even | $\frac{1}{q}z^q$ | $\frac{3}{q^2}z^q$ |
| $q \geq 11$, odd  | $\frac{1}{q}z^q$ | 0 |

Note that by Lemma \([21\] we have $\alpha_{\chi(n-2,1,1)}^{(q)} = 1 + \alpha_{\chi(n-2,2)}^{(q)}$; that is, the asymptotic series above contains all necessary information for both characters.
As an example, consider the triangle group

$$\Gamma = \left\langle x, y, z \mid x^2 = y^3 = z^7 = xyz = 1 \right\rangle.$$ 

Here, $$\mu(\Gamma) = \alpha(\Gamma) = \frac{1}{42}$$ and $$m_\Gamma = 42$$. We first show how to determine the contribution of characters $$\chi_\lambda$$ with $$\Delta > 2$$. By Lemmas 7 and 15 (ii), we see that, for fixed $$\Delta$$ and $$q$$ prime,

$$\alpha_q^{(q)}(\Gamma) = \frac{1}{\Delta^q}.$$ 

Hence, for the triangle group $$\Gamma$$ we obtain

$$\beta(R)_{\chi_\lambda} = \left(1 + O(n^{-1/\Delta})\right) \frac{(\chi_\lambda, \lambda_1(1))^{3n^{41\Delta/42}/(\Delta!)^3}}{\chi(1)}.$$ 

Setting $$\pi = 1$$ in (36), we obtain

$$|\text{Hom}(\Gamma, S_n)| = \frac{|\text{Hom}(C_2, S_n)||\text{Hom}(C_3, S_n)||\text{Hom}(C_7, S_n)|}{n!} \times \left(\sum_{\Delta \leq 2} \chi_\lambda(1) \beta(R)_{\chi_\lambda} + \sum_{3 \leq \Delta \leq 22} \frac{(\chi_\lambda, \lambda_1(1))^{3n^{41\Delta/42}}/(\Delta!)^3}{\chi(1)} + O(n^{-23/42})\right).$$ 

Given our previous work in this section, the sum over $$\Delta$$ can be computed to whatever length is required. We have cut the sum after the term $$\Delta = 22$$, since this is the smallest precision bringing to bear all phenomena occurring in such computations at arbitrary scale. For $$\Delta = 0$$, we have $$\beta(R)_{\chi_\lambda} = 1$$, whereas for $$1 \leq \Delta \leq 2$$, we use the asymptotic for $$\alpha_q^{(q)}$$ computed above, to obtain

$$\sum_{1 \leq \Delta \leq 2} \chi_\lambda(1) \beta(R)_{\chi_\lambda} = 1 + n^{-1/42} + \frac{1}{2} n^{-1/21} + \frac{1}{2} n^{-11/21} + O(n^{-23/42}).$$ 

From Lemma 7, we obtain, for $$\Delta$$ fixed and $$n \to \infty$$, the asymptotic estimate

$$\chi_\lambda(1) = \left(1 + O(n^{-1})\right) \frac{\chi_\lambda, \lambda_1(1)n^\Delta}{\Delta!},$$ 

which implies

$$\sum_{3 \leq \Delta \leq 22} \frac{(\chi_\lambda, \lambda_1(1))^{3n^{41\Delta/42}}}/(\Delta!)^3 \chi(1) = \left(1 + O(n^{-1})\right) \sum_{3 \leq \Delta \leq 22} \frac{(\chi_\lambda, \lambda_1(1))^{2n^{-\Delta/42}}}/(\Delta!)^2$$

$$= \left(1 + O(n^{-1})\right) \sum_{3 \leq \Delta \leq 22} \frac{n^{-\Delta/42}}{\Delta!};$$

and combining these estimates we obtain

$$|\text{Hom}(\Gamma, S_n)| = \frac{1}{n!} |\text{Hom}(C_2, S_n)||\text{Hom}(C_3, S_n)||\text{Hom}(C_7, S_n)| \times \left(1 + \sum_{\Delta = 1}^{22} \frac{n^{-\Delta/42}}{\Delta!} + \frac{1}{2} n^{-11/21} + O(n^{-23/42})\right).$$
For $1 \leq k \leq 22$, we compute $s_k(\Gamma)$ using the software package GAP [12], and obtain

$$s_k(\Gamma) = \begin{cases} 
1, & k = 1, 8, 9 \\
2, & k = 7 \\
3, & k = 15 \\
9, & k = 14, 21 \\
13, & k = 22 \\
0, & \text{otherwise}, 
\end{cases} \quad 1 \leq k \leq 22.$$ 

From these values, $h_k(\Gamma)$ and hence $d_k(\Gamma)$ are easily computed for $1 \leq k \leq 22$. The first three coefficients of the asymptotic series for $|\text{Hom}(C_q, S_n)|$ are given in [23], after Corollary 2. Putting together the various expansions, our final result is that

$$s_n(\Gamma) = \frac{(2\pi)^{-\frac{33}{4}} e^{-\frac{1}{4}}}{\sqrt{42}} (n!)^{\frac{1}{4}} n^{-\frac{1}{4}} \exp\left(n^{1/2} + n^{1/3} + n^{1/7}\right) \times \left\{ 1 - \frac{2}{7} n^{-1/6} - \frac{1}{8} n^{-4/21} - \frac{1}{9} n^{-3/14} - \frac{113}{147} n^{-1/3} - \frac{23}{140} n^{-5/14} + \frac{319}{8064} n^{8/21} + \frac{1}{72} n^{-17/42} + \frac{1}{162} n^{-3/7} + \frac{745}{8232} n^{-1/2} - \frac{28309}{64680} n^{11/21} + O(n^{-23/42}) \right\}.$$ 

5.3. One-relator groups. The result of this subsection, apart from its inherent interest, also demonstrates that certain Fuchsian groups $\Gamma$ with $\mu(\Gamma) > 0$ and $\alpha(\Gamma) < 0$ have a much faster growth than would be predicted by Theorem 3. Consider a one-relator group

$$\Gamma = \left\langle y_1, y_2, \ldots, y_s \mid y_1^{e_1} y_2^{e_2} \cdots y_s^{e_s} = 1 \right\rangle, \quad (42)$$

and let $\bar{\Gamma} := C_{e_1} * C_{e_2} * \cdots * C_{e_s}$.

**Theorem 4.** Let $\Gamma$ be as in (42), and suppose that

$$\alpha(\Gamma) = -2 + \sum_{1 \leq j \leq s} \frac{2}{e_j} < 0.$$ 

Then, as $n$ tends to infinity, we have

$$s_n(\Gamma) \sim s_n(\bar{\Gamma}) \sim K(n!)^{\mu(\Gamma) - \alpha(\Gamma)/2} \exp\left(\sum_{j=1}^{s} \sum_{\nu < e_j}^{\frac{n^{\nu / e_j}}{\nu}} + \frac{\alpha(\Gamma) - 2\mu(\Gamma) + 2\log n}{4} \right),$$

where

$$K = \frac{\exp\left(-\sum_{j}^{\frac{(2e_j)^{-1}}{2e_j}}(2e_j)^{-1}\right)}{(2\pi)^{\frac{33}{4} + 2\mu(\Gamma) - \alpha(\Gamma)} \sqrt{e_1 e_2 \cdots e_s}}.$$
Proof. We have
\[
|\text{Hom}(\Gamma, S_n)| = \left| \left\{ (\pi_1, \ldots, \pi_s) \in S_n^s : \pi_1 \pi_2 \cdots \pi_s = 1 \right\} \right|
\]
\[
= \sum_{c_1, \ldots, c_s} r_{e_1}(c_1) r_{e_2}(c_2) \cdots r_{e_s}(c_s) N(c_1, \ldots, c_s), \tag{43}
\]
where
\[
N(c_1, \ldots, c_s) := \left| \left\{ (\pi_1, \ldots, \pi_s) \in S_n^s : \pi_1 \pi_2 \cdots \pi_s = 1, \pi_i \in c_i (1 \leq i \leq s) \right\} \right|.
\]

The contribution in (43) of the term corresponding to \(c_1 = c_2 = \cdots = c_s = 1\) is
\[
|\text{Hom}(C_{e_1}, S_n)| \cdot |\text{Hom}(C_{e_2}, S_n)| \cdots |\text{Hom}(C_{e_s}, S_n)| = |\text{Hom}(\bar{\Gamma}, S_n)|.
\]
We shall show that the sum over the remaining terms is of lesser order of magnitude. Since \(N(c_1, \ldots, c_s)\) is invariant under permutation of its arguments, we may assume that \(|c_1| \geq |c_j|\) for all \(j\). Hence,
\[
N(c_1, \ldots, c_s) = \left| \left\{ (\pi_2, \ldots, \pi_s) \in S_n^{s-1} : \pi_2 \pi_3 \cdots \pi_s \in c_1, \pi_j \in c_j (2 \leq j \leq s) \right\} \right|
\]
\[
\leq |c_2| \cdot |c_3| \cdots |c_s|
\]
\[
\leq \prod_{1 \leq j \leq s} |c_j|^{1-\delta_j}
\]
for any choice of non-negative real numbers \(\delta_1, \delta_2, \ldots, \delta_s\) such that \(\sum_j \delta_j = 1\). For a non-empty set \(J \subseteq [s]\), define \(S_J := \sum_{j \in J} 1/e_j\). By our assumption, \(S_J < 1\), and, by definition, \(\sum_{j \in J} 1/(S_J e_j) = 1\). Using the above estimate for \(N(c_1, \ldots, c_s)\) with \(\delta_j = 1/(S_J e_j)\), dividing Equation (43) by \(|\text{Hom}(\bar{\Gamma}, S_n)|\), and interchanging product and sum, we find that
\[
0 \leq \frac{|\text{Hom}(\Gamma, S_n)|}{|\text{Hom}(\bar{\Gamma}, S_n)|} - 1 \leq \sum_J \sum_{c_1, \ldots, c_s} \prod_{j \in J} r_{e_j}(c_j)|c_j|^{1-1/(S_J e_j)}
\]
\[
= \sum_J \prod_{j \in J} r_{e_j}(c)|c|^{1-1/(S_J e_j)}
\]
\[
\sum_{2 \leq \ell \leq n} \sum_{c \text{ moves } \ell \text{ points}} \frac{r_{e_j}(c)|c|^{1-1/(S_J e_j)}}{|\text{Hom}(C_{e_j}, S_n)|} = \left( \frac{n}{\ell} \right)^{1-1/(S_J e_j)} \frac{|\text{Hom}(C_{e_j}, S_{n-\ell})|}{|\text{Hom}(C_{e_j}, S_n)|} \sum_c r_{e_j}(c)|c|^{1-1/(S_J e_j)}, \tag{44}
\]
where the innermost sum extends over all fixed-point free conjugacy classes of \( S_{\ell} \). From [23 Corollary 2] we deduce that

\[
\frac{|\text{Hom}(C_{e_j}, S_{n-\ell})|}{|\text{Hom}(C_{e_j}, S_n)|} \ll \left( \left( \frac{n}{\ell} \right)! \right)^{-1/(1/e_j)}.
\]

Applying the Cauchy-Schwarz inequality, we find for the innermost sum in (44) that

\[
\sum_c r_{e_j}(c)|c|^{1/(S_{\ell}e_j)} \leq \left( \sum_c (r_{e_j}(c))^2 |c| \right)^{1/2} \left( \sum_c |c|^{2/(S_{\ell}e_j)} \right)^{1/2}
\]

\[
\leq \left( \sum_{\chi \in \text{Irr}(S_{\ell})} \left( \chi(1) \right)^{2-\frac{4}{S_{\ell}e_j}+\varepsilon} \right)^{1/2} \left( \sum_c |c|^{2/(S_{\ell}e_j)} \right)^{1/2},
\]

where we have used Proposition [24] (i) to estimate the first factor. If \( S_{\ell}e_j > 2 \), then we bound the second factor by \((\ell!)^{2/(S_{\ell}e_j)+\varepsilon}\); otherwise, each summand is \leq 1, and

\[
\sum_c |c|^{2/(S_{\ell}e_j)} \ll (\ell!)^\varepsilon.
\]

In both cases,

\[
\sum_c r_{e_j}(c)|c|^{1-1/(S_{\ell}e_j)} \leq \max\left( (\ell!)^{1-\frac{1}{e_j}-\frac{1}{S_{\ell}e_j}+\varepsilon}, (\ell!)^{\frac{1}{S_{\ell}e_j}+\varepsilon} \right).
\]

(45)

Putting (45) back into (44), we find that the left-hand side of (44) is bounded above by

\[
\sum_{2 \leq \ell \leq n} \left( \frac{n}{\ell} \right)^{1-1/(S_{\ell}e_j)} \left[ \left( \frac{n}{\ell} \right)! \right]^{-1/(1/e_j)} \max\left( (\ell!)^{1-\frac{1}{e_j}-\frac{1}{S_{\ell}e_j}+\varepsilon}, (\ell!)^{\frac{1}{S_{\ell}e_j}+\varepsilon} \right)
\]

\[
= \sum_{2 \leq \ell \leq n} \left( \frac{n}{\ell} \right)^{\frac{1}{e_j}-\frac{1}{S_{\ell}e_j}} \max\left( (\ell!)^{-\frac{1}{S_{\ell}e_j}+\varepsilon}, (\ell!)^{-\frac{1}{\ell}+\varepsilon} \right).
\]

For sufficiently large \( n \), increasing \( \ell \) by 1 decreases a summand by at least a factor \( \frac{1}{2} \); hence, as \( n \to \infty \),

\[
\sum_{2 \leq \ell \leq n} \sum_c \frac{r_{e_j}(c)|c|^{-1/(S_{\ell}e_j)}}{|\text{Hom}(C_{e_j}, S_n)|} \ll n^{-2(\frac{1}{S_{\ell}e_j}-\frac{1}{e_j})},
\]

and, therefore,

\[
|\text{Hom}(\Gamma, S_n)| \sim |\text{Hom}(\bar{\Gamma}, S_n)|, \quad n \to \infty.
\]

Since \( \chi(\bar{\Gamma}) = \sum_j 1/e_j - s + 1 < 0 \), and since the proof of [24 Proposition 1] only depends on an asymptotic estimate, we infer in particular that

\[
\sum_{0 < k < n} \left( \frac{n}{k} \right) \frac{|\text{Hom}(\Gamma, S_k)||\text{Hom}(\Gamma, S_{n-k})|}{|\text{Hom}(\Gamma, S_n)|} \to 0 \text{ as } n \to \infty.
\]

Combining this fact with the transformation formula (40) and (31 Theorem 3), we find that

\[
s_n(\Gamma) \sim |\text{Hom}(\Gamma, S_n)|/(n-1)! \sim |\text{Hom}(\bar{\Gamma}, S_n)|/(n-1)! \sim s_n(\bar{\Gamma}).
\]

The explicit asymptotic formula given for \( s_n(\bar{\Gamma}) \) results from [24 Theorem 1].
Remark. Theorem 4 was proved in [27, Example 1(ii)] for \( \alpha(\Gamma) < -1 \) (note that the invariant \( \alpha(\Gamma) \) defined in [27, Sec. 2] is \( \frac{2(\Gamma)+2}{2} \) in our present notation).

5.4. Demuškin groups. Let \( p \) be a prime. A pro-\( p \)-group \( \Gamma \) is termed a Poincaré group of dimension \( n \), if \( \Gamma \) has cohomological dimension \( n \), and the algebra \( H^*(\Gamma) \) is finite dimensional and satisfies Poincaré duality. A Poincaré group of dimension 2 is called a Demuškin group. These are one-relator groups

\[
\Gamma = \langle x_1, \ldots, x_m | R(x_1, \ldots, x_m) = 1 \rangle, \quad m = \dim H^1(\Gamma),
\]

and, for \( p \neq 2 \), the defining relation may be taken to be

\[
R = x_1^h[x_1, x_2][x_3, x_4] \cdots [x_{m-1}, x_m], \quad h \in \mathbb{N} \cup \{\infty\},
\]

with the understanding that \( x_1^h = 1 \) if \( h = \infty \); cf [4]–[6] and [18]. For \( m = 2 \), these groups are metacyclic, and their subgroup growth can be computed using the methods of [13]. The similarity of \( R \) to a surface group relation would allow us to estimate \( s_n(\Gamma) \) asymptotically for \( m \geq 6 \), using only tools from [26]. The case \( m = 4 \) however needs a more careful analysis. As another application of our estimates for multiplicities of root number functions, we prove the following result.

**Theorem 5.** For integers \( q \geq 1 \) and \( d \geq 2 \), let

\[
\Gamma_{q,d} = \langle x_1, y_1, \ldots, x_d, y_d | x_1^{q-1}[x_1, y_1][x_2, y_2] \cdots [x_d, y_d] = 1 \rangle.
\]

Then there exist explicitly computable constants \( \gamma_\nu(\Gamma_{q,d}) \), such that

\[
s_n(\Gamma_{q,d}) \approx \delta n(n!)^{2d-2} \left\{ 1 + \sum_{\nu=1}^{\infty} \gamma_\nu(\Gamma_{q,d}) n^{-\nu} \right\}, \quad (n \to \infty),
\]

where

\[
\delta = \begin{cases} 1, & q \text{ even} \\ 2, & q \text{ odd}. \end{cases}
\]

The proof runs parallel to the proof of Theorem 3 once we have established the following.

**Lemma 22.** Let \( q \geq 2 \) an integer. For a partition \( \lambda \vdash n \), define the coefficient \( l^{(q)}_{\chi\lambda} \) by means of the equation

\[
\left| \{ (\sigma, \tau) \in S_n^2 : \sigma^{q-1}[\sigma, \tau] = \pi \} \right| = n! \sum_{\lambda \vdash n} l^{(q)}_{\chi\lambda}(\pi).
\]

Then we have \( |l^{(q)}_{\chi\lambda}| \leq \sqrt{m^{(q)}_{\chi\lambda}(1)} \). Moreover, for a partition \( \mu \vdash l \), and a partition \( \lambda \vdash n \) with \( \lambda \setminus \lambda_1 = \mu \), the quantity \( \chi\lambda(1)l^{(q)}_{\chi\lambda} \) is a constant depending only on \( \mu \), provided \( n \) is sufficiently large.
Proof. Writing the equation $x^{k-1}[x, y] = \pi$ as $x^k(x^{-1})^y = \pi$, we see that the number of solutions can be computed as

$$\sum_c |C_{S_n}(c)| \cdot \{|\sigma, \tau \in c : \sigma^n \tau = \pi\}| = \sum_c |c| \sum_{\lambda \vdash n} \frac{\chi(c)\chi(c^k)\chi(\pi)}{\chi(1)},$$

that is,

$$l_{\chi^q}^{(q)} = \frac{1}{n!} \sum_c |c| \frac{\chi(c)\chi(c^q)}{\chi(1)} \leq \frac{1}{n!\chi(1)} \left( \sum_c |c| (\chi(c))^2 \right)^{1/2} \left( \sum_c |c| (\chi(c^q))^2 \right)^{1/2}$$

$$\leq \frac{1}{\sqrt{n!\chi(1)}} \left( \frac{1}{n!} \sum_c |c| (\chi(c^q))^2 \right)^{1/2} = \sqrt{\frac{m_{\chi^q}}{\chi(1)}}.$$

For the second claim we have to compute

$$\frac{1}{n!} \sum_{\pi \in S_n} \frac{\chi(\pi)\chi(\pi^k)}{\chi(1)}.$$

We express $\chi(\pi)$ as a polynomial in the functions $s_i(\pi)$. Then $\chi(\pi)\chi(\pi^k)$ is also a polynomial in these functions, and our claim follows from Lemma 13. □

As an example, consider $\lambda = (n-1, 1)$. Then

$$\chi(\pi)\chi(\pi^k) = (s_1(\pi) - 1)\left( \sum_{t|q} ts_1(\pi) - 1 \right).$$

The expected value of the first factor is 0, and it is stochastically independent of $s_2, \ldots, s_q$. Hence, we have

$$l_{\chi^q}^{(q)} = \frac{1}{\chi(1)n!} \sum_{\pi \in S_n} \left( (s_1(\pi))^2 - s_1(\pi) \right),$$

and the computations leading to the second assertion in Proposition 2(ii) give $l_{\chi^q}^{(q)} = \frac{1}{\chi(1)} = \frac{1}{n-1}$. 

For the second claim we have to compute

$$\frac{1}{n!} \sum_{\pi \in S_n} \frac{\chi(\pi)\chi(\pi^k)}{\chi(1)}.$$
For $\lambda = (n - 2, 2)$, we find in the case $q$ even
\[
\chi_\lambda(\pi)\chi_\lambda(\pi^q) = \frac{1}{4}(s_1(\pi))^4 - \frac{3}{2}(s_1(\pi))^3 + \frac{13}{4}(s_1(\pi))^2 - 3s_1(\pi) + 1
\]
\[+ s_2(\pi)\left(-2(s_1(\pi))^2 + 6s_1(\pi) - 4\right) + (s_2(\pi))^2\left((s_1(\pi))^2 - 3s_1(\pi) + 5\right) - 2(s_2(\pi))^3
\]
\[+ \left(\frac{1}{2}(s_1(\pi))^2 - \frac{3}{2}s_1(\pi) - s_2(\pi) + 1\right)
\left(\frac{1}{2}\sum_{t \geq 3} t^2(s_\ell(\pi))^2 - \frac{3}{2}\sum_{t \geq 3} ts_\ell(\pi) + \sum_{t \geq 2} ts_\ell(\pi)\right).
\]

Note that the expected value of $A$ is 0, hence, since the second factor contains only terms $s_t(\pi)$ with $t \geq 3$, it is stochastically independent of the first factor, and the expectation of the last summand vanishes; using Lemma 13, we find that the remaining terms vanish as well. Dealing in a similar way with the other cases, we obtain

| $\ell^{(q)}_\lambda(n-2,2)$ | $q$ even | $q$ odd |
|--------------------------|----------|---------|
| $l^{(q)}_{\lambda(n-2,2)}$ | 0        | $\frac{1}{n^2 - 3n}$ |
| $l^{(q)}_{\lambda(n-2,1,1)}$ | 13       | $\frac{1}{2(n^2 - 3n + 2)}$ |

As an example for the computation of the coefficients $\gamma_\nu(\Gamma_q, d)$, we consider the case $d = 2$ and $q \geq 2$. From the values given above we deduce the estimate
\[
\frac{h_n(\Gamma)}{(n!)^2} = 1 + \frac{1}{(n - 1)^3} + \left\{\frac{26}{(n^2 - 3n + 2)^3}, \frac{18}{(n^2 - 3n + 2)^3}, \frac{q}{(n^2 - 3n + 2)^3} \right\} + \mathcal{O}(n^{-9}).
\]

For small values of $k$, we compute $h_k(\Gamma_q, 2)$ as follows:

\[
\begin{align*}
\chi(\Gamma_{q,2}) = 1, & \quad h_2(\Gamma_{q,2}) = \begin{cases} 8, & (q, 2) = 1 \\
4, & (q, 2) = 2 \end{cases}, & \quad h_3(\Gamma_{q,2}) = \begin{cases} 72, & (q, 6) = 1 \\
45, & (q, 6) = 2 \\
63, & (q, 6) = 3 \\
36, & (q, 6) = 6 \end{cases}, \\
h_4(\Gamma_{q,2}) = \begin{cases} 1424, & (q, 6) = 1 \\
720, & (q, 6) = 2 \\
1280, & (q, 6) = 3 \\
576, & (q, 6) = 6 \end{cases}, & \quad h_5(\Gamma_{q,2}) = \begin{cases} 37192, & (q, 30) = 1 \\
21092, & (q, 30) = 2 \\
36040, & (q, 30) = 3 \\
35792, & (q, 30) = 5 \\
20840, & (q, 30) = 6 \\
19692, & (q, 30) = 10 \\
34640, & (q, 30) = 15 \\
19440, & (q, 30) = 30. \end{cases}
\end{align*}
\]

Proceeding as in Subsection 5.1, we obtain
\[
s_n(\Gamma_{q,2}) = \delta n(n!)^2 R(n),
\]
where $\delta$ is as in Theorem 5 and

\[
R(n) = \begin{cases}
    1 - n^{-1} - 7n^{-2} - 56n^{-3} - 1237n^{-4} - 33573n^{-5} + O(n^{-6}), & (q, 30) = 1 \\
    1 - n^{-1} - 3n^{-2} - 37n^{-3} - 623n^{-4} - 19460n^{-5} + O(n^{-6}), & (q, 30) = 2 \\
    1 - n^{-1} - 7n^{-2} - 47n^{-3} - 1111n^{-4} - 32826n^{-5} + O(n^{-6}), & (q, 30) = 3 \\
    1 - n^{-1} - 7n^{-2} - 56n^{-3} - 1237n^{-4} - 32173n^{-5} + O(n^{-6}), & (q, 30) = 5 \\
    1 - n^{-1} - 3n^{-2} - 28n^{-3} - 497n^{-4} - 19541n^{-5} + O(n^{-6}), & (q, 30) = 6 \\
    1 - n^{-1} - 3n^{-2} - 37n^{-3} - 623n^{-4} - 18060n^{-5} + O(n^{-6}), & (q, 30) = 10 \\
    1 - n^{-1} - 7n^{-2} - 47n^{-3} - 1111n^{-4} - 31426n^{-5} + O(n^{-6}), & (q, 30) = 15 \\
    1 - n^{-1} - 3n^{-2} - 28n^{-3} - 497n^{-4} - 18141n^{-5} + O(n^{-6}), & (q, 30) = 30.
\end{cases}
\]

Note that the series for $s_n(\Gamma_{q, 2})$ is far more dependent on $q$ and therefore on $\Gamma_{q, 2}$ itself than the series for $h_n(\Gamma_{q, 2})$.

### 6. Finiteness Results

Call two finitely generated groups $\Gamma$ and $\Delta$ equivalent, denoted $\Gamma \sim \Delta$, if

\[s_n(\Gamma) = (1 + o(1))s_n(\Delta), \quad (n \to \infty).\]

In [24, Theorem 3] a characterization in terms of structural invariants is given for the equivalence relation $\sim$ on the class of groups $\Gamma$ of the form

\[\Gamma = G_1 \ast G_2 \ast \cdots \ast G_s \ast F_r\]

with $r, s \geq 0$ and finite groups $G_\sigma$, and it is shown that each $\sim$-class decomposes into finitely many isomorphism classes. Here we are concerned with the analogous problems for Fuchsian groups.

**Theorem 6.**

(i) Let $\Gamma = C_{a_1} \ast \cdots \ast C_{a_k} \ast F_r$ and $\Delta = C_{b_1} \ast \cdots \ast C_{b_l} \ast F_{r'}$ be free products of cyclic groups such that $s_n(\Gamma) \asymp s_n(\Delta)$. Then $r = r'$ and \(\{a_1, \ldots, a_k\} = \{b_1, \ldots, b_l\}\) as multisets.

(ii) Let

\[
\Gamma = \left\langle x_1, \ldots, x_r, y_1, \ldots, y_s, u_1, v_1, \ldots, u_t, v_t \right| \right.
\]

\[x_1^{a_1} = \cdots = x_r^{a_r} = x_1 \cdots x_r y_1^{c_1} \cdots y_s^{c_s} [u_1, v_1] \cdots [u_t, v_t] = 1\]

and

\[
\Delta = \left\langle x_1, \ldots, x_{r'}, y_1, \ldots, y_{s'}, u_1, v_1, \ldots, u_{t'}, v_{t'} \right| \right.
\]

\[x_1^{a'_{i}} = \cdots = x_{r'}^{a'_{r'}} = x_1 \cdots x_{r'} y_1^{c'_1} \cdots y_{s'}^{c'_{s'}} [u_1, v_1] \cdots [u_{t'}, v_{t'}] = 1\]

be Fuchsian groups, such that $\alpha(\Gamma), \alpha(\Delta) > 0$. Then $\Gamma \sim \Delta$ if and only if

(a) The multisets $\{a_i : 1 \leq i \leq r\}$ and $\{a'_i : 1 \leq i \leq r'\}$ coincide,

(b) $\mu(\Gamma) = \mu(\Delta)$,

(c) $\delta = \delta'$. 

Let
\[ \Gamma = \langle y_1, \ldots, y_s \mid y_1^{e_1} y_2^{e_2} \cdots y_s^{e_s} = 1 \rangle \]
and
\[ \Delta = \langle y_1, \ldots, y_{s'} \mid y_1^{e'_1} y_2^{e'_2} \cdots y_{s'}^{e'_{s'}} = 1 \rangle \]
be two one-relator groups with \( \alpha(\Gamma), \alpha(\Delta) < 0 \). Then the following assertions are equivalent:
(a) \( \Gamma \cong \Delta \).
(b) \( s = s' \) and \( \{e_1, \ldots, e_s\} = \{e'_1, \ldots, e'_{s'}\} \) as multisets.
(c) \( \hat{\Gamma} \cong \hat{\Delta} \), where the hat denotes profinite completion.

The proof of Theorem 6 requires the following two auxiliary results.

**Lemma 23.** Let \( A = \{a_1, \ldots, a_k\} \) and \( B = \{b_1, \ldots, b_l\} \) be multisets of integers, such that
\[
\sum_{d|a_i} \frac{1}{a_i} = \sum_{d|b_i} \frac{1}{b_i}
\]
for all \( d \geq 2 \). Then \( A = B \).

**Proof.** We argue by induction on \( n = k + l \). For \( n \leq 1 \), there is nothing to show. Assume that our claim holds for all multisets \( A', B' \) with \( |A'| + |B'| \leq n - 1 \), and let \( A, B \) be multisets as above. Let \( d \) be the greatest integer, such that \( \sum_{d|a_i} 1/a_i > 0 \). Then \( d = \max A \), and the value of the sum is \( |\{i : a_i = \max A\}|/\max A \). The same holds for \( B \), hence the greatest element of both multisets as well as the multiplicity of this maximum coincide. Deleting these elements in both multisets yields a pair of multisets \( A', B' \) of smaller cardinality, which are equal by the induction hypothesis. Hence we deduce \( A = B \). \( \square \)

**Lemma 24.** Given positive integers \( k \) and \( l \), disjoint tuples of variables \( \tilde{x}_i = (x_{i1}, \ldots, x_{ia_i}) \) for \( 1 \leq i \leq k \), words \( w_1(\tilde{x}_1), \ldots, w_k(\tilde{x}_k) \), and (possibly empty) words \( v_{ij}(\tilde{x}_i) \) for \( 1 \leq i \leq k \) and \( 1 \leq j \leq l \), and a permutation \( \sigma \in S_k \), define
\[
\Gamma = \langle \tilde{x}_1, \ldots, \tilde{x}_k \mid w_1(\tilde{x}_1) \cdots w_k(\tilde{x}_k) = v_{ij}(\tilde{x}_i) = 1 \mid 1 \leq i \leq k, 1 \leq j \leq l \rangle
\]
and
\[
\Gamma_\sigma = \langle \tilde{x}_1, \ldots, \tilde{x}_k \mid w_{\sigma(1)}(\tilde{x}_{\sigma(1)}) \cdots w_{\sigma(k)}(\tilde{x}_{\sigma(k)}) = v_{ij}(\tilde{x}_i) = 1 \mid 1 \leq i \leq k, 1 \leq j \leq l \rangle.
\]
Then \( \Gamma \) and \( \Gamma_\sigma \) have isomorphic profinite completions.

**Proof.** For \( 1 \leq i \leq k \) and a finite group \( G \), let
\[
N_i(G)(g) := \left| \left\{ \tilde{x}_i \in G^{a_i} : w_i(\tilde{x}_i) = g, v_{i1}(\tilde{x}_i) = \cdots = v_{il}(\tilde{x}_i) = 1 \right\} \right|, \quad g \in G.
\]
Since \( N_i(G) \) is a class function, we can introduce Fourier coefficients \( \alpha_{\chi,i} \) via
\[
N_i(G)(g) = \sum_{\chi \in \text{Irr}(G)} \alpha_{\chi,i}(g), \quad g \in G.
\]
Then, using orthogonality, we have
\[
|\text{Hom}(\Gamma, G)| = \sum_{g_1, \ldots, g_k \in G, g_1 g_2 \cdots g_k = 1} \prod_{1 \leq i \leq k} N_i(G)(g_i)
\]
\[
= \sum_{c_1, \ldots, c_k \subseteq G} \frac{|c_1| \cdots |c_k|}{|G|} \sum_{\chi} \sum_{\chi_1, \ldots, \chi_k} \frac{\chi(c_1) \cdots \chi(c_k)}{(\chi(1))^k-2} \chi_1(c_1) \alpha_{\chi_1,1} \cdots \chi_k(c_k) \alpha_{\chi_k,k}
\]
\[
= |G|^{k-1} \sum_{\chi} \frac{\alpha_{\chi,1} \cdots \alpha_{\chi,k}}{(\chi(1))^{k-2}}.
\]
Since this computation leads to the same character formula when replacing \( \Gamma \) by \( \Gamma_\sigma \), we deduce that, for each finite group \( G \),
\[
|\text{Hom}(\Gamma, G)| = |\text{Hom}(\Gamma_\sigma, G)|;
\]
in particular, \( s_n(\Gamma) = s_n(\Gamma_\sigma) \) for all \( n \). Writing
\[
|\text{Hom}(\Gamma, G)| = \sum_{U \leq G} |\text{Epi}(\Gamma, U)|
\]
and using M"obius inversion in the subgroup lattice of \( G \), this gives
\[
|\text{Epi}(\Gamma, G)| = \sum_{U \leq G} \mu(U, G)|\text{Hom}(\Gamma, U)|,
\]
thus also
\[
|\text{Epi}(\Gamma, G)| = |\text{Epi}(\Gamma_\sigma, G)|.
\]
(46)

For \( n \in \mathbb{N} \), define finite groups \( G_n \) and \( G_n^\sigma \) via
\[
G_n = \Gamma / \bigcap_{(\Gamma : \Delta) \leq n} \Delta, \quad G_n^\sigma = \Gamma_\sigma / \bigcap_{(\Gamma_\sigma : \Delta) \leq n} \Delta.
\]
From (46) we know in particular, that \( G_n \) is a homomorphic image of \( \Gamma_\sigma \). Let \( N \) be the kernel of such a projection map \( \phi : \Gamma_\sigma \to G_n \). Since \( s_\nu(\Gamma_\sigma) = s_\nu(G_n) \) for \( \nu \leq n \), we have
\[
N \leq \bigcap_{(\Gamma_\sigma : \Delta) \leq n} \Delta,
\]
hence, \( G_n^\sigma \) is a homomorphic image of \( G_n \). By symmetry, \( G_n \) and \( G_n^\sigma \) are isomorphic. By the universal properties of \( G_n \) and \( G_n^\sigma \),
\[
\hat{\Gamma} \cong \lim_{\leftarrow} G_n \cong \lim_{\leftarrow} G_n^\sigma \cong \hat{\Gamma}_\sigma,
\]
as claimed. □
Proof of Theorem 6. (i) By [24, Theorem 1], the assumption $s_n(\Gamma) \approx s_n(\Delta)$ is equivalent to the assertion that
\[
(n!)^{-\chi(\Gamma)} \exp \left( \sum_{i=1}^{k} \sum_{d|a_i} \frac{n^{d/a_i}}{d} + \frac{r + \chi(\Gamma) + 1}{2} \log n \right) \asymp (n!)^{-\chi(\Delta)} \exp \left( \sum_{i=1}^{l} \sum_{d|b_i} \frac{n^{d/b_i}}{d} + \frac{r' + \chi(\Delta) + 1}{2} \log n \right).
\]
Comparing orders of magnitude as in the proof of [24, Theorem 3], we find first that $\chi(\Gamma) = \chi(\Delta)$, then, successively, that
\[
\sum_{t|a_i} t = \sum_{t|b_i} t, \quad t \geq 2,
\]
and, finally, that $r = r'$. Our claim follows now from Lemma 23.

(ii) By Theorem 3 the assertion that $\Gamma \sim \Delta$ is equivalent to
\[
\delta s_n(C_{a_1} \ast \cdots \ast C_{a_r} \ast F_{s+2t}) \sim \delta' s_n(C_{a'_1} \ast \cdots \ast C_{a'_r} \ast F_{s'+2t'}), \quad (n \to \infty).
\]
By part (i), the latter assertion is equivalent to the conjunction of
\[
C_{a_1} \ast \cdots \ast C_{a_r} \ast F_{s+2t} \cong C_{a'_1} \ast \cdots \ast C_{a'_r} \ast F_{s'+2t'},
\]
and $\delta = \delta'$, whence our claim.

(iii) The equivalence of (a) and (b) follows from Theorem 4 and part (i). Since (c) obviously implies (a), it suffices to show that (b) implies (c); but this follows immediately from Lemma 24 upon setting $l = 0$ and $w_i(x_i) = x_i^{e_i}$.

Denote by $\mathcal{F}$ the class of all groups $\Gamma$ having a presentation of the form (35) with $\alpha(\Gamma) > 0$. We distinguish three refinements of the equivalence relation $\sim$ on $\mathcal{F}$: (i) the relation $\approx$ of strong equivalence defined via
\[
\Gamma \approx \Delta :\Leftrightarrow s_n(\Gamma) = s_n(\Delta)(1 + \mathcal{O}(n^{-A})) \text{ for every } A > 0,
\]
(ii) isomorphy, and (iii) equality of the system of parameters
\[
(r, s, t; a_1, a_2, \ldots, a_r, e_1, e_2, \ldots, e_s)
\]
in the Fuchsian presentation (35), denoted $\Gamma = \Delta$. Clearly,
\[
\Gamma = \Delta \Rightarrow \Gamma \cong \Delta \Rightarrow \Gamma \approx \Delta \Rightarrow \Gamma \sim \Delta.
\]
(47)
All these implications are in fact strict. To see this, define
\[
\Gamma_j = \left( x, y, z, u \mid R_j(x, y, z, u) = 1 \right), \quad 1 \leq j \leq 3,
\]
where
\[
R_j := \begin{cases} 
[x, y][z, u], & j = 1 \\
[x, y]z^2u^2, & j = 2 \\
x^2y^2z^2u^2, & j = 3.
\end{cases}
\]
Then $\Gamma_1$ and $\Gamma_2$ are isospectral, that is, $s_n(\Gamma_1) = s_n(\Gamma_2)$ for all $n$ (in particular, $\Gamma_1 \approx \Gamma_2$),
but $\Gamma_1 \not\approx \Gamma_2$; and $\Gamma_2 \cong \Gamma_3$ but $\Gamma_2 \not\approx \Gamma_3$. Our next result implies that $\approx$ is a much finer equivalence relation than $\sim$. It appears that the asymptotic series carries most of the structural information on Fuchsian groups which can be detected via subgroup growth.

**Theorem 7.** Each $\approx$-equivalence class of $F$ decomposes into finitely many classes with respect to $\cong$; that is, each group $\Gamma \in F$ has only finitely many presentations of the form \((\ref{35})\), and is $\approx$-equivalent to at most finitely many non-isomorphic $F$-groups.

**Corollary 2.** Let $\Gamma \in F$ be given by a representation as in \((\ref{35})\).

(i) The set \(\{\Delta \in F : \Delta \sim \Gamma\}/\cong\) is finite if and only if one of the following holds:

(a) $s = t = 0$,
(b) $s = 1$, $t = 0$, $\sum_{i=1}^{r} (1 - \frac{1}{a_i}) < 2$,
(c) $s + 2t = 2$, $r = 1$.

(ii) The set \(\{\Delta \in F : \Delta \sim \Gamma\}/\cong\) is infinite, but \(\{\Delta \in F : s_n(\Delta) = (1 + O(n^{-2\mu(\Gamma)})) s_n(\Gamma)\}/\cong\) is finite, if and only if the following three conditions hold:

(a) $s + 2t + \sum_{i=1}^{r} (1 - \frac{1}{a_i}) \geq 3$,
(b) $a_r$ is odd for $1 \leq i \leq r$,
(c) $e_j = 2$ for $1 \leq j \leq s$ with at most one exception, and for the exceptional index $j_0$ (if it occurs) we have $e_{j_0} = 2p^{\prime-1}$ for some prime $p$.

*Proof of Theorem 7.* Let \((r, s, t; a_1, a_2, \ldots, a_r, e_1, e_2, \ldots, e_s)\) be a given set of parameters, let $\Gamma$ be the corresponding group and assume that $\alpha(\Gamma) > 0$. We have to show that there are only finitely many tuples \((r', s', t'; a'_1, a'_2, \ldots, a'_r, e'_1, e'_2, \ldots, e'_s)\), such that for the corresponding group $\Delta$ we have $\Gamma \approx \Delta$. Before computing the coefficients of the asymptotic series for $s_n(\Gamma)$ and $s_n(\Delta)$, we show that we may assume without loss that $h_{\nu}(\Gamma) = h_{\nu}(\Delta)$ for $\nu = 2, 3$. In fact, from Theorem 4 (ii) we infer that $r + s + 2t = r' + s' + 2t'$, hence, $|\text{Hom}(\Delta, S_\nu)| \leq (\nu!)^{r+s+2t}$, that is, there are only finitely many choices for $h_{\nu}(\Delta)$ for each fixed $\nu$. Hence, in the sequel we may assume that $d_{\nu}(\Gamma) = d_{\nu}(\Delta)$ for $\nu = 2, 3$, where the $d_{\nu}$ are given as in \((\ref{44})\), and that $h_n(\Gamma) = (1 + O(n^{-3\mu(\Gamma)}) h_n(\Delta)$.

Theorem 4 (ii) already implies that the multisets \(\{a_i : 1 \leq i \leq r\}\) and \(\{a'_i : 1 \leq i \leq r'\}\) coincide. From Propositions 4 and 2 (i), and Equation \((38)\), we see that

\[
h_n(\Gamma) = (n!)^{s+2t-2} \prod_{i=1}^{r} |\text{Hom}(C_{a_i}, S_n)| \left\{ \sum_{\chi \in \pi \lambda} \prod_{i=1}^{r} \frac{\alpha^{(a_i)}(\chi) \prod_{j=1}^{s} m_{\lambda}^{(e_j)}}{\chi(1)^{r+s+2t-2}} + O(n^{-3\mu(\Gamma)}) \right\}.
\]

\[ (48) \]

In view of Proposition 2 (ii), the contribution of partitions $\lambda$ with $3 \leq \Delta \leq 3\mu(\Gamma)/\alpha(\Gamma)$ is of lesser order than the error term, hence, we can compute $h_n(\Gamma)$ up to a relative error of order $n^{-3\mu(\Gamma)}$ using only coefficients already computed in the previous sections. Inserting the values for $\alpha^{(q)}_{\lambda\lambda}$ computed in Subsection 5.2, and the multiplicities as given
Moreover, the contribution of the character $\chi$ expanding into the right-hand side of (48), we obtain

\[
\begin{align*}
h_\Gamma(n) &= \delta(n)!^{s+2t-2} \prod_{i=1}^{r} |\text{Hom}(C_{a_i}, S_n)| \left(1 + (n-1)^{-(r+s+2t-2)} \prod_{i=1}^{r} H_{1,a_i}(n) \prod_{j=1}^{s} (\tau(e_j) - 1) \\
&\quad + \left(\frac{n^2 - 3n + 2}{2}\right)^{-(r+s+2t)} \prod_{1 \leq i \leq r, 2 \nmid a_i} \frac{n}{n-1} H_{2,a_i}(n) \prod_{1 \leq i \leq r, 2 \nmid a_i} (H_{2,a_i}(n) - 1) \\
&\quad \times \prod_{j=1}^{s} \frac{1}{2}(\sigma(e_j) + (\tau(e_j))^2 - 3\tau(e_j) + \tau_{\text{odd}}(e_j)) + O(n^{-\mu(\Gamma)}) \right)
\end{align*}
\]

where

\[
H_{i,q}(n) := \binom{n}{i} \frac{|\text{Hom}(C_q, S_{n-i})|}{|\text{Hom}(C_q, S_n)|}.
\]

From [23] Theorem 6, we see that

\[
(n-1)^{-(r+s+2t-2)} \prod_{i=1}^{r} H_{1,a_i}(n) \asymp n^{-\mu(\Gamma)},
\]

and all other contributions to the asymptotic series in (49) are of lesser order, hence, expanding $h_\Delta(n)$ in the same way, we find that $\Gamma \approx \Delta$ implies

\[
\prod_{j=1}^{s} (\tau(e_j) - 1) = \prod_{j=1}^{s'} (\tau(e'_j) - 1).
\]

Moreover, the contribution of the character $\chi(n-1,1)$ to $h_\Gamma(n)$ and $h_{\Delta}(n)$ are identical. Next we consider terms of order $n^{-2\mu(\Gamma)}$. Arguing as for terms of order $n^{-\mu(\Gamma)}$, we find that $h_\Gamma(n) = (1 + o(n^{-2\mu(\Gamma)}))h_\Delta(n)$ is equivalent to (50) and

\[
\prod_{j=1}^{s} \left(\sigma(e_j) + (\tau(e_j))^2 - 3\tau(e_j) + \tau_{\text{odd}}(e_j)\right)
\]

\[
\quad + \prod_{j=1}^{s} \left(\sigma(e_j) + (\tau(e_j))^2 - 3\tau(e_j) - \tau_{\text{odd}}(e_j) + 2\right)
\]

\[
= \prod_{j=1}^{s'} \left(\sigma(e'_j) + (\tau(e'_j))^2 - 3\tau(e'_j) + \tau_{\text{odd}}(e'_j)\right)
\]

\[
\quad + \prod_{j=1}^{s'} \left(\sigma(e'_j) + (\tau(e'_j))^2 - 3\tau(e'_j) - \tau_{\text{odd}}(e'_j) + 2\right).
\]
For a fixed tuple \((e_1, \ldots, e_s)\), there are only finitely many tuples \((e'_1, \ldots, e'_{s'})\) with \(s' \leq s + 2t\), solving (51). Indeed, the left-hand side is bounded by some constant, whereas the right-hand side is bounded below by its greatest factor, as all factors occurring in the last equation are \(\geq 1\); thus \(e'_j\) is bounded for all \(j\).

\[\square\]

**Proof of Corollary 2** \(\dagger\) (i) Using Theorem 3 one checks in each of these cases that \(\{\Delta : \Delta \sim \Gamma\}\) is indeed finite. On the other hand, if none of the conditions (a)–(c) is satisfied, one easily computes that the groups

\[
\Delta_e := \left\{ x_1, \ldots, x_r, y_1, \ldots, y_{s+2t} \mid x_1^{a_1} = \cdots = x_r^{a_r} = x_1 \cdots x_r y_1^2 \cdots y_s^2 = 1 \right\}
\]

satisfy \(\alpha(\Delta_e) \geq \frac{2}{e}\), and, if necessary, adjusting the parity of \(e\), we have \(\Delta_e \sim \Gamma\) for infinitely many \(e\). By Theorem 4 there is an infinite sequence \((e_\nu)_{\nu \geq 1}\), such that \(\Delta_{e_\nu} \sim \Gamma\), while \(\Delta_{e_\nu} \not\sim \Delta_{e_\mu}\) for \(\nu \neq \mu\), which implies our claim.

(ii) In the proof of Theorem 7, we have seen that \(s_n(\Delta) = (1 + \mathcal{O}(n^{-2\mu(\Gamma)}))s_n(\Gamma)\) is equivalent to the conjunction of \(\Delta \sim \Gamma\) and

\[
\prod_{j=1}^{s} (\tau(e_j) - 1) = \prod_{j=1}^{s'} (\tau(e'_j) - 1).
\]

From this equation and part (i) it is easy to see that for a group \(\Gamma\) satisfying (a)–(c) the described sets are of the claimed cardinality. Now assume that \(\Gamma\) is a group such that \(\{\Delta \in \mathcal{F} : \Delta \sim \Gamma\}\) is infinite, while \(\{\Delta \in \mathcal{F} : s_n(\Delta) = (1 + \mathcal{O}(n^{-2\mu(\Gamma)}))s_n(\Gamma)\}\) is finite. If \(\delta(\Gamma) = 1\), and \(\{\Delta : \Delta \sim \Gamma\}\) is infinite, there are infinitely many integers \(e'\), such that \((e', 2, 2, \ldots, 2)\) solves this equation, and \(\delta(\Delta_{e'}) = \delta(\Gamma)\). Define the integer \(q = q(\Gamma)\) via

\[
q := \left( \prod_{j=1}^{s} (\tau(e_j) - 1) \right) + 1.
\]

If \(q\) is not 1 or prime, say \(q = a \cdot b\) with \(a, b \geq 2\), then \(s_n(\Delta_{2a\cdot b}) = (1 + \mathcal{O}(n^{-2\mu(\Gamma)}))s_n(\Gamma)\) for all odd primes \(p\), and, by Theorem 7, there are infinitely many non-isomorphic groups among the groups \(\Delta\) defined in this way. Hence, we may assume that all \(e_j\) are even, \(\tau(e_j) = 2\) for all \(j\) with at most one exception, and for the exceptional index \(j_0\), we have that \(\tau(e_{j_0})\) is either 2 or \(p + 1\) for some prime \(p\). This implies our claim. \(\square\)

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