Transformation Properties of Classical and Quantum Laws under Some Nonholonomic Spacetime Transformations

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Abstract

Nonrelativistic Newton and Schrödinger equations remain correct not only under holonomic but also under nonholonomic transformations of the spacetime coordinates. Here we study the properties of transformations which are holonomic in the space coordinates while additionally transforming the time in a path-dependent way. This makes them nonholonomic in spacetime. The resulting transformation formulas of physical quantities establish relations between different physical systems. Furthermore we point out certain differential-geometric features of these relations.

1 Introduction

Based on the pioneering work of Poincaré\textsuperscript{1} and Sundman\textsuperscript{2} in celestial mechanics, Kustaanheimo and Stiefel\textsuperscript{3} solved the Kepler problem in 1965 by using a local transformation of space and time which possesses no global counterpart. This nonintegrable transformation regularizes the singular three-dimensional Kepler problem by mapping it to the four-dimensional harmonic oscillator. Motivated by the discovery of the dynamical symmetry group O(4,2)\textsuperscript{4}, Duru and Kleinert\textsuperscript{5,6} showed in 1979 that these transformations can also be implemented in path integrals. Generalizations of this procedure have led to relations between Green functions of many different quantum systems, thereby producing previously unknown solutions\textsuperscript{7}.

Here we discuss differential-geometric properties of a special class of these transformations, in which the spatial part of the transformations is holonomic. This excludes the treatment of the Kepler problem in three dimensions, but admits all one-dimensional DK transformations (see Chaper 14 of Ref. 7).

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Consider the movement of a point mass $m$ on a $D$-dimensional Riemann manifold under the influence of external potentials. Both classical and quantum mechanical equations are form invariant under ordinary holonomic space transformations. Locally, some initial space coordinates $q^i$ are changed into $Q^\lambda$ ($i, \lambda = 1, \ldots, D$) according to

$$dq^i = e^i_\lambda(Q) dQ^\lambda,$$

where the coefficients $e^i_\lambda(Q)$ obey the integrability conditions of Schwarz

$$\partial_\mu e^i_\lambda(Q) - \partial_\lambda e^i_\mu(Q) = 0.$$  

This is why (1) is integrable and possesses the global form

$$q^i = q^i(Q).$$

In the following we investigate what happens to this form invariance if the time $t$ is transformed under a local transformation to a new time $s$ via

$$\frac{dt}{ds} = f(Q),$$

where the positive but otherwise arbitrary function $f(Q)$ depends on the final space coordinates. The combined local space and time transformation

$$\begin{pmatrix} dt \\ dq^i \end{pmatrix} = \begin{pmatrix} f(Q) & 0 \\ 0 & e^i_\lambda(Q) \end{pmatrix} \begin{pmatrix} ds \\ dQ^\lambda \end{pmatrix}$$

is characterized by transformation coefficients $E^I_\Lambda$ ($I, \Lambda = 0, \ldots, D$) which no longer fulfill the time part of the spacetime extension of the integrability condition (2), making (4) nonholonomic in spacetime.

## 2 Classical Mechanics

Consider a trajectory $q^i(t)$ of a point mass $m$ in a $D$-dimensional Riemann manifold with metric $g^{ij}_{(i)}(q)$ under the influence of a vector potential $A^i_{(i)}(q)$ and a scalar potential $V^{(i)}(q)$. This problem may be solved by integrating the equations of motion, but it may often be simplified by subjecting the system to nonholonomic spacetime transformations of the type (3) and (4). They relate the trajectory $q^i(t)$ to some trajectory $Q^\lambda(s)$ in a space with metric $g^{ij}_{(f)}(Q)$, vector potential $A^i_{(f)}(Q)$ and scalar potential $V^{(f)}(Q)$ as is depicted in Fig. 1.

The holonomic space transformation (3) changes the metric in accordance with its tensor character. The nonholonomic time transformation (4) turns out to produce an additional conformal factor$^8$. Together we end up with

$$g^{ij}_{\lambda\mu}(Q) = f(Q)e^i_\lambda(Q)e^j_\mu(Q)g^{ij}_{ij}(q(Q)).$$
The vector potential is transformed by the holonomic space transformation (3) as usual. In contrast to the metric, it is unchanged by the nonholonomic time transformation (4) as it is coupled to the velocity. Together we obtain

$$ A^{(f)}_{\chi}(Q) = e^{i\chi(Q)} A^{(i)}_{i}(q(Q)). $$

If the energy of the orbit $q^i(t)$ is denoted by $E^{(i)}$, the transformation of the scalar potential is

$$ V^{(f)}(Q) = f(Q) \left\{ V^{(i)}(q(Q)) - E^{(i)} \right\}. $$

Note that the initial energy $E^{(i)}$ plays the role of a strength parameter in an additional potential proportional to $-f(Q)$. The transformation of the trajectory is given by

$$ q^i(t) = q^i(Q(s)), $$

where the relation between both time coordinates $t$ and $s$ is obtained by integrating (4) along the trajectory:

$$ t = \int_0^s ds f(Q(s)). $$

### 3 Quantum Mechanics

Let the quantum mechanical system of the point mass $m$ in a space with metric $g^{(i)}_{ij}(q)$, vector potential $A^{(i)}_{i}(q)$, and scalar potential $V^{(i)}(q)$ be described by the time evolution amplitude $G_c^{(i)}(q; t; q_0, t_0)$. This can be calculated by solving a Schrödinger equation or evaluating a path integral. The nonholonomic spacetime transformations (3) and (4) make it possible to obtain $G_c^{(i)}(q; t; q_0, t_0)$ from an analogous amplitude.
in the transformed system. The respective transformation properties of the physical quantities are illustrated in Fig. 2.

The quantum mechanical transformations of the metric and the vector potential turn out to coincide with the classical ones \((6)\) and \((7)\). However, in contrast to the classical problem, the scalar potential of the transformed system contains an additional quantum correction term proportional to \(\hbar^2\)

\[
V^{(f)}(Q) = f(Q) \left\{ V^{(i)}(q(Q)) - E^{(i)} \right\} + \frac{\hbar^2}{m} \left\{ \frac{2 - D}{8} \Gamma^{(f)\lambda\mu}(Q) \frac{\partial \mu f(Q)}{f(Q)} \right. \\
+ g^{(f)\lambda\mu}(Q) \left[ \frac{(D - 2)(D - 6)}{32} \frac{\partial \lambda f(Q)\partial \mu f(Q)}{f(Q)^2} + \frac{D - 2}{8} \frac{\partial \lambda\partial \mu f(Q)}{f(Q)} \right] \right\},
\]

where \(\Gamma^{(f)\lambda\nu}(Q)\) is the Christoffel symbol associated with the metric \(g^{(f)\lambda\mu}(Q)\). Note that the quantum corrections happen to be absent for \(D = 2\) and reduce for \(D = 1\) to known results.

An analysis\(^8\) of the differential geometric properties of the nonholonomic space-time transformations \((8)\) and \((9)\) in the framework of Riemann-Cartan differential geometry\(^9,10\) reveals that the quantum corrections can be expressed in terms of curvature and torsion of the transformed spacetime. If \(R^{(f)}(Q), \bar{R}^{(f)}(Q)\) and \(S^{(f)}(Q) = S^{(f)\lambda\mu}(Q)\) denote Cartan’s curvature scalar, Riemann’s curvature scalar, and contracted Cartan’s torsion tensor, respectively, one has instead of \((8)\)

\[
V^{(f)}(Q) = f(Q) \left\{ V^{(i)}(q(Q)) - E^{(i)} \right\} + V^{(qu)}(Q)
\]

with the quantum correction

\[
V^{(qu)}(Q) = f(Q) \left\{ \frac{\hbar^2}{m} \left[ \frac{4 - D^2}{8} \left( S^{(f)}(Q) \right)^2 + \frac{D - 2}{16} \left( R^{(f)}(Q) - \bar{R}^{(f)}(Q) \right) \right] \right\}.
\]

It is not astonishing that the paths in quantum mechanics whose quantum fluctuations probe the neighborhood of the classical orbits are sensitive to curvature and

Figure 2: *Nonholonomic space and time transformation relating the metrics, external potentials and time evolution amplitudes of two quantum systems with each other.*
torsion of spacetime.

The relation between the initial time evolution amplitude \( G^{(i)}_c(q, t; q_0, t_0) \) and the corresponding amplitude \( G^{(f)}_{c, E (0)} (Q(q), s; Q(q_0), 0) \) in the transformed system is given by the DK transformation\(^8\) which essentially contains two Fourier transformations:

\[
G^{(i)}_c(q, t; q_0, t_0) = \left[ f (Q(q)) f (Q(q_0)) \right]^{\frac{e^{-i\pi}}{2\pi\hbar}} \int_{-\infty}^{+\infty} \frac{dE^{(i)}}{2\pi\hbar} \int_{0}^{+\infty} ds \times \exp \left\{ -\frac{i}{\hbar} E^{(i)} (t - t_0) \right\} G^{(f)}_{c, E (0)} (Q(q), s; Q(q_0), 0). \quad (14)
\]

4 Outlook

It will be interesting to extend this discussion to allow for nonholonomic space transformations which include the Kustaanheimo-Stiefel case (which by themselves are discussed in Kleinert’s lecture and in Ref. 11), and to completely general nonholonomic spacetime transformations.

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