Remarks on Lubbock’s summation formulae

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The coefficients occurring in summation formulae of the Lubbock type are shown to be generalised Bernoulli polynomials which turn up in subdivision questions such as quantum field theory around a conical singularity and on spherical lunes. An image interpretation is made, generating functions are brought in and some trigonometric summations encountered. A formal Lubbock formula is introduced for a general delta operator.

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1. Introduction

Lubbock’s method is a reasonably well known means of summing a large number of terms by taking a weighted subset and adding a, hopefully small, correction. Textbook explanations can be found in Whittaker and Robinson, [1], Milne–Thomson, [2] and Gibb, [3]. Steffensen, [4], has a more analytical treatment and gives several variants based on the standard interpolations. In all cases the resulting formula involves a set of polynomials, which are actually generalised Bernoulli polynomials. This does not seem to have been noticed, and it is this fact I wish to enlarge on in this note.

2. The third formula of Lubbock type

The third formula of Lubbock type will be used to present the essentials of the calculation. It can be obtained from a subdivision of the Newton–Bessel interpolation but it is better to begin at an earlier stage and consider the problem of subtabulation which devolves upon the expression for the central difference operator, \( \delta_m \), for a general step, \( m \). (For subtabulation, \( m \) is usually set equal to \( 1/h \) with \( h \) an integer.) For a unit interval, \( \delta \equiv \delta_1 \).

Rather than work with function interpolation as such, it is more efficient to proceed purely operationally, e.g. de Morgan, [6], Steffensen [4] §18, Michel, [7], Hildebrand, [8].

For convenience, I repeat some textbook material. \( \delta_m \) is given in terms of the derivative \( D \) by,

\[
\delta_m = 2 \sinh \frac{mD}{2}
\]

and therefore,

\[
\delta_m = 2 \sinh \left( m \sinh^{-1} \frac{\delta}{2} \right),
\]

(1)

The expansion of the right-hand side in powers of \( \delta \) is accomplished by expanding \( \sinh mx \) in powers of \( \sinh x \). This can be assumed known from the corresponding trigonometric series classically derived by recursion. See e.g. Gregory, [9] p.74. The method and results are described in more detail by Poinsot, [10], based on similar

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2 Cohen and Hubbard, [5], in their development of an Adams–type multirevolution integration technique, encounter Lubbock’s equation.
work of Lagrange. The result is

\[
\delta_m = m\delta \left[ 1 + \frac{m^2 - 1}{3! \delta^2} \delta^2 - \frac{(m^2 - 1)(m^2 - 3^2)}{5! \delta^4} \delta^4 + \ldots \right].
\]

Lubbock’s formula is a subdivision of the central summation formula involving the sum operator, \( \mu \delta^{-1} \), where the mean operator, \( \mu \), is given by \( \mu = \cosh D/2 \) and

\[
\mu \delta^{-1} = \coth \frac{D}{2} = \frac{d}{dD} \log \delta = \mu \frac{d}{d\delta} \log \delta,
\]

with the same for \( \mu_m \delta^{-1} \), [11]. Direct computation yields,

\[
\log \delta_m = \log(m\delta) + \log \left[ 1 + \frac{m^2 - 1}{3! \delta^2} \delta^2 - \frac{(m^2 - 1)(m^2 - 3^2)}{5! \delta^4} \delta^4 + \ldots \right]
\]

\[
= \log(m\delta) + \frac{m^2 - 1}{24} \delta^2 - \frac{(m^2 - 1)(m^2 + 11)}{2880} \delta^4 + \ldots,
\]

and so,

\[
\mu_m \delta_m^{-1} = \frac{1}{m} \mu \delta^{-1} + \frac{m^2 - 1}{12m} \mu \delta - \frac{(m^2 - 1)(m^2 + 11)}{720m} \mu \delta^3 + \ldots
\]

\[
= \frac{1}{m} \mu \delta^{-1} + \sum_{\nu=1}^{\infty} Q_{2\nu}(m) \mu \delta^{2\nu - 1},
\]

introducing the coefficients, \( Q_{2\nu} \), of Steffensen, [4]. (He has \( m = 1/h \)).

Equation (3) applied to a function, \( F(x) \), is just a central summation version of Lubbock’s formula.

3. Bernoulli polynomials

The point I wish to make here is that, by inspection, the \( Q_{2\nu} \) can be recognised as generalised Bernoulli polynomials. More particularly,

\[
Q_{2\nu}(m) = \frac{1}{m(2\nu)!} B_{2\nu}^{(2\nu+1)}(\nu \mid m, 1),
\]

using the notation of Nörlund, [12]. \( 1 \) is the \( 2\nu \)-dimensional vector, \((1, 1, \ldots, 1)\).
Before proving this, I remark that this particular polynomial turns up in situations involving subdivisions, or coverings, such as quantum field theory around a cosmic string (subdivision of a circle), [13], or on a lune section of a divided sphere where it appears in the conformal anomaly, [14].

I approach the derivation of the relation (4) from the right–hand side by noting that the Bernoulli polynomial is simply related to a residue of a Barnes ζ–function at one of its poles, in fact the first. I elaborate on this by continuing the discussion in the Appendix to [13] which is concerned with the field theory around a conical singularity. I repeat a little of the material given there for smoothness of exposition.

The general definition of the Barnes ζ–function is,

\[
\zeta_r(s, a | \omega) = \frac{i\Gamma(1-s)}{2\pi} \int_L dz \frac{\exp(-az)(-z)^{s-1}}{\prod_{i=1}^r (1 - \exp(-\omega_i z))} = \sum_{m=0}^\infty \frac{1}{(a + m\omega)^s}, \quad \text{Re } s > d, \tag{5}
\]

where I refer to the components, \(\omega_i\), of the \(r\)-vector, \(\omega\), as the degrees or parameters. For simplicity, I assume that the \(\omega_i\) are real and positive. Often they are integers. If \(a\) is zero, the origin, \(m = 0\), is to be excluded. The contour, \(L\), is the standard Hankel one.

The pole structure of the ζ–function is,

\[
\zeta_r(s + l, a | \omega) \rightarrow \frac{N_l(r, a, \omega)}{s} + R_l(r, a, \omega) \quad \text{as } s \rightarrow 0, \tag{6}
\]

where \(1 \leq l \leq r\) and \(u = r/2\) if \(r\) is even and \(u = (r - 1)/2\) if \(r\) is odd. Barnes determined the residues in terms of generalised Bernoulli polynomials, although he did not refer to them as such.

I am not interested in the general case and simply set \(l = 1\) to write down the known expression, [15],

\[
N_1(r, a, m) = \frac{(-1)^{r+1}}{(r-1)!} \frac{1}{m} B_{r-1}^{(r)}(a | m, 1), \tag{7}
\]

which allows me to make contact with (4) on setting \(r = 2\nu + 1\) and \(a = \nu\) to give the conjecture,

\[
Q_{2\nu}(m) = N_1(2\nu + 1, \nu, m). \tag{8}
\]

Hence I turn to the pole structure of the contour integral, (5), near \(s = 1\) which is due entirely to the Γ–function.
Using the symmetry of the integrand, and collapsing the contour to a loop around the origin, the residue is easily picked out as,

\[ N_1(2\nu + 1, \nu, m) = \frac{1}{2\pi i} \int dD \frac{e^{\nu D}}{(e^D - 1)^{2\nu}(e^{mD} - 1)} \]

\[ = \frac{1}{2\pi i} \int dD \frac{e^{\nu D}e^{mD}}{(e^D - 1)^{2\nu}(e^{mD} - 1)} \]

\[ = \frac{1}{4\pi i} \int dD \frac{e^{\nu D}(e^{mD} + 1)}{(e^D - 1)^{2\nu}(e^{mD} - 1)} \]

\[ = \frac{1}{2^{2\nu+2}\pi i} \int dD \frac{\coth mD/2}{\sinh^{2\nu} D/2}. \]

Changing variables to \( \delta = 2 \sinh D/2 \) this becomes,

\[ N_1(2\nu + 1, \nu, m) = \frac{1}{2\pi i} \int d\delta \frac{1}{\delta^{2\nu}} \frac{\coth mD/2}{\cosh D/2 \sinh mD/2} \]

\[ = \frac{1}{2^{2\nu+2}\pi i} \int d\delta \frac{1}{\delta^{2\nu}} \mu^{-1}\mu_m\delta^{-1}_m, \quad \nu \geq 0, \tag{9} \]

which is the coefficient of \( \delta^{2\nu-1} \) in the expansion of \( \mu^{-1}\mu_m\delta^{-1}_m \) and this is just \( Q_{2\nu}(m) \) according to (3). Therefore (8), and hence (4), have been proved.

4. The other formulae

I turn to Lubbock’s original formula of 1829. As shown first by de Morgan, [6] §184, see also Sprague, [16], and Steffensen, [4] §18, the problem becomes that of developing the forwards summation operator, \( \Delta^{-1}_m \), in powers of \( \Delta \). Formally,

\[ \Delta^{-1}_m = \frac{1}{(1+\Delta)^m - 1} = \frac{1}{m} \Delta^{-1} + \Lambda_1 + \Lambda_2 \Delta + \Lambda_3 \Delta^2 \ldots, \tag{10} \]

where the \( \Lambda_\nu \) are rational functions of \( m \).

With the earlier calculation in mind, the expression for the residue, (7), manipulates to, on setting \( \Delta = e^D - 1 \),

\[ N_1(r + 1, a, m) = \frac{1}{2\pi i} \int dD \frac{e^{aD}}{(e^D - 1)^r(e^{mD} - 1)} \]

\[ = \frac{1}{2\pi i} \int d\Delta \frac{(1+\Delta)^{a-1}}{\Delta^r} \frac{1}{(1+\Delta)^m - 1}, \tag{11} \]

4 I have changed notation to a more suggestive one by the substitution \( z = -D \).
which, if $a = 1$, gives the coefficient of $\Delta^{r-1}$ in (10), i.e., our answer,

$$\Lambda_r (m) = N_1 (r + 1, 1, m) = \frac{1}{r!} \frac{1}{m} B^{(r+1)} (1 \mid m, 1).$$  \hspace{1cm} (12)$$

The second Lubbock type formula requires the central operator expansion (Steffensen, [4] §18.211),

$$\delta_m^{-1} = \frac{1}{m} \delta^{-1} + P_2 \delta + P_4 \delta^3 + P_6 \delta^5 + \ldots ,$$  \hspace{1cm} (13)

and again the contour integral is rearranged,

$$N_1 (2\nu + 1, a, m) = \frac{1}{2\pi i} \oint dD \frac{e^{aD}}{(e^D - 1)^{2\nu} (e^{mD} - 1)}$$

$$= \frac{1}{2^{2\nu+2} \pi i} \oint dD \frac{e^{aD} e^{-\nu D} e^{-mD/2}}{\sinh^{2\nu} D/2 \sinh mD/2}$$

$$= \frac{1}{2^{2\nu+2} \pi i} \oint dD \frac{\cosh D/2}{\sinh^{2\nu} D/2 \sinh mD/2}.$$

On the last line, I have set $a = \nu + (m - 1)/2$ and used the antisymmetry of the integrand to introduce the factor of $\cosh D/2 \equiv \mu$ so that, again changing variables to $\delta$, gives,

$$N_1 (2\nu + 1, (2\nu + m - 1)/2, m) = \frac{1}{2\pi i} \oint d\delta \frac{1}{\delta^{2\nu}} \delta_m^{-1} ,$$  \hspace{1cm} (14)

which entails the coefficients,

$$P_{2\nu} (m) = \frac{1}{m (2\nu)!} B^{(2\nu+1)}_{2\nu} (\nu + (m - 1)/2 \mid m, 1).$$  \hspace{1cm} (15)

5. Other expressions

By applying Lubbock’s subdivision method to the three standard interpolation formulae, Steffensen, [4] §15.159, obtains representations of the $\Lambda_{\nu}$, $P_{2\nu}$ and $Q_{2\nu}$ in terms of factorials, which I quote,

$$\Lambda_{\nu} (1/h) = \frac{1}{\nu!} \sum_{0}^{h-1} \left( \frac{x}{h} \right)^{\nu}$$

$$P_{2\nu} (1/h) = \frac{1}{(2\nu)!} \sum_{-(h-1)/2}^{(h-1)/2} \left( \frac{x}{h} \right)^{[2\nu]}$$  \hspace{1cm} (16)
$Q_{2\nu}(1/h) = \frac{1}{(2\nu)!} \sum_{-\frac{(h-2)}{2}}^{\frac{(h-2)}{2}} \left(\frac{x}{h}\right)^{[2\nu+1]-1}.$ 

(18)

Here, $h$ is assumed integral and positive. The summations (over $x$) proceed in unit steps.

Expansion of the factorials in powers of $x$ allows the summations to be done in terms of ordinary Bernoulli polynomials with the coefficients being differentials of zero. The results are given by Steffensen, [4] §15 and I do not repeat them here. Cohen and Hubbard, [5], give an equivalent treatment and related formulae are derived by Cvijović and Srivastava, [17]. Of course, actual evaluation gives the same answer as the other methods, i.e. rational functions of the integer $h$, which can then be allowed to take any value, e.g. $h = 1/m$.

6. An image interpretation

Not surprisingly, one can view the identity of the expressions (16), (17) and (18) of the previous section with the Bernoulli forms in terms of images. The particular transformation responsible is, [12] p.169, (see also [15] for the $\zeta$-function equivalent),

$$\sum_{s=0}^{h-1} B_{n+1}^{(n+1)}(a + \frac{s}{h} \mid \frac{1}{h}, 1) = h B_{n+1}^{(n+1)}(a \mid \frac{1}{h}, 1),$$

(19)

the left–hand side of which can be regarded as an image sum over lattice points on a divided interval.$^5$

The factorial structure,

$$B_{n+1}^{(n+1)}(a) = (a-1)(a-2)\ldots(a-n) \equiv (a-1)^{(n)} = a^{(n+1)-1},$$

is standard, [12], hence, for $\nu = n$, (19) reads,

$$h B_{n+1}^{(n+1)}(a \mid \frac{1}{h}, 1) = \sum_{s=0}^{h-1} \left(\frac{1}{h} \right)^{(n)}.$$

(20)

Setting $a$ to the various values ($a = 1$ is simplest) reveals the required identities.

As usual in image calculations, the function of $h$ resulting from an explicit summation can be continued away from integral $h$, which is the same statement as in the previous section.

$^5$ (19) is an example of a more general relation and results from an ancient, textbook trigonometric identity or, [12], by finite difference properties of the Bernoulli polynomials.
7. Generating functions

The image interpretation has actually been available for a number of years. In [18] and [19] the modulated cosec sums,

\[ S_\nu(m, w) \equiv \sum_{l=1}^{m-1} \cos \left( \frac{2\pi w l}{m} \right) \csc^{2\nu} \left( \frac{\pi l}{m} \right), \]  

(21)

with twisting \( w \in \mathbb{Z} \), were evaluated in terms of Bernoulli polynomials by a contour method equivalent to the one just described. This provides an alternative viewpoint because the relation with the object used here, (7), is,

\[ S_\nu(m, w) = (-1)^{\nu+1} 2^{2\nu} m N_1(2\nu + 1, w + \nu, m) . \]  

(22)

The left–hand sides of the expressions, (3), (13) and (10), of the Lubbock formulae are essentially generating functions for the \( Q \), \( P \) and \( \Lambda \) quantities and, therefore, the generating function derived in [18], in a different context, can be checked, at least for the functions, \( Q \), to which we are restricted by the sums, (21), and the relation, (22). (See the Appendix.)

A generating function is, (I set \( \zeta = imD/2 \) in [18] for a closer comparison),

\[ S(D; m, w) \equiv \sum_{\nu=1}^{\infty} (-1)^\nu \sinh^{2\nu}(D/2) S_\nu(m, w) \]
\[ = m \left( \frac{1}{m} - \frac{\cosh(mDf)}{\sinh mD/2} \tanh(D/2) \right), \]  

(23)

where \( f = w/m - 1/2 \). Comparison of the top line with the right–hand sides of the various Lubbock formulae shows that one should set, algebraically,

\[ 2 \sinh D/2 = \delta , \]

so that a rearrangement of (23) gives the Lubbock equation.\(^6\)

For example, from (3), the generating function for the third type function \( Q_{2\nu} \) is,

\[ (\mu^{-1} \delta)(\mu_m \delta_m^{-1}) = \frac{\tanh D/2}{\tanh mD/2} , \]

\(^6\) These generating functions are not the same as the standard ones for the generalised Bernoulli polynomials to be found in Nörlund, [12], say, which are concerned only with the lower index on \( B_n^\mu \), for fixed \( n \).
which is the same result obtained from (23) on putting \( w = 0 \), the untwisted value.

Similarly the generating function for the \( P_{2\nu} \) is,

\[
\delta \delta_m^{-1} = \frac{\sinh D/2}{\sinh mD/2},
\]

which follows from (23) on setting \( w = (m - 1)/2 \), \textit{i.e.} \( f = -1/2m \).

The generating function nature of the Lubbock formulae is actually quite clear in their original formulation. Using the factorial expressions, (16), (17) and (18) to obtain the generating functions would simply be reversing a step in their derivation. Some enlightenment can be had by applying this reconstruction to the more general relation, (20), as follows,

\[
\sum_{n=0}^{\infty} \frac{1}{n!} h B_n^{(n+1)}(a \mid \frac{1}{h}, 1) \Delta^n = \sum_{s=0}^{h-1} \sum_{n=0}^{\infty} \frac{1}{n!} (a - 1 + \frac{s}{h})^{(n)} \Delta^n
\]

\[
= \left( a - 1 + \frac{s}{h} \right) \Delta^n = \sum_{s=0}^{h-1} (1 + \Delta)^{a - 1 + s/h} (24)
\]

which confirms equation (11) if \( h = 1/m \).

The sums, (21), can be manipulated and evaluated in various ways. They have an interesting history, some of which is outlined in the above references. Cvijović and Srivastava, [17], give further information, also on related summations. In [20] they apply the contour technique to obtain many other closed forms of a similar type.

In the Appendix I revisit a variant of the contour method and apply it to the integral (11) which does not fall totally into the class (21).

8. Computation

There are choices when computing the \( \Lambda, P \) and \( Q \). One direct method has been given in the section 5. However, one needs the values of the relevant differentials of zero which themselves have to be evaluated, say by recursion or direct expansion, and the formulae are not particularly illuminating. Such a calculation of the \( Qs \) was effectively performed, case by case, by Woolhouse, [21] eqn.(a), and this is the earliest reference I know for these ‘conical’ polynomials.
If (4), (15) and (12) are used, the equations for the Bernoulli polynomials involved are given in Nörlund [12]. Especially one finds the polynomial,

$$ B^{(n+1)}_\nu(x \mid m, 1) = \sum_{s=0}^{\nu} m^s \binom{\nu}{s} B_s B^{(n)}_{\nu-s}(x), \quad (25) $$

where \( B^{(n)}_\nu(x) \equiv B^{(n)}_\nu(x \mid 1) \) is the, more frequently studied, Bernoulli polynomial whose degrees are all unity and is given by the polynomial in \( x \),

$$ B^{(n)}_\nu(x) = \sum_{s=0}^{\nu} \binom{\nu}{s} x^s B^{(n)}_{\nu-s}. \quad (26) $$

Here \( B^{(n)}_\nu \equiv B^{(n)}_\nu(0) \) are generalised Bernoulli numbers which can be calculated in various ways. A recursion formula is, [12] p.195 equn. (14),

$$ B^{(n)}_\nu = \sum_{s=1}^{\nu} (-1)^s \binom{\nu}{s} B_s B^{(n)}_{\nu-s}, $$

with \( B^{(n)}_0 = 1 \) and the ordinary Bernoulli numbers, \( B_s \), are needed. Special values of \( x \) can make life easier. For example,

$$ B^{(n+1)}_\nu(1) = \frac{n - \nu}{n} B^{(n)}_\nu. \quad (27) $$

Such an evaluation is easily programmed and quite rapid but could be viewed as overkill since it involves special cases of a more general construction. Individual methods have been proposed for these particular cases. For example, de Morgan, [6], derives the \( \Lambda_\nu \) by successive recursion based on the relation,

$$ \binom{m}{1} \Lambda_\nu + \binom{m}{2} \Lambda_{\nu-1} + \binom{m}{3} \Lambda_{\nu-2} + \ldots \binom{m}{\nu+1} \Lambda_0 = 0, $$

(cf Steffensen, [4] p.140).

The recursions for \( P \) and \( Q \) are similar, but the coefficients are more complicated, involving the central factorials of \( m/2 \), [4].

Cohen and Hubbard, [5], point out that the values of the polynomials \( (-1)^\nu m \Lambda_\nu(m) \) at \( m = 0 \) are the Adams coefficients of the Adams–Bashforth process of integrating differential equations. Equations (25) and (27) (or [12] §27) give for these coefficients the various forms,

$$ \frac{(-1)^\nu}{\nu!} B^{(\nu)}_\nu(1) = -\frac{(-1)^\nu}{(\nu-1)\nu!} B^{(\nu-1)}_\nu = \frac{(-1)^\nu}{\nu!} \int_0^1 dt (t-1) \ldots (t-\nu+1), $$

which can be compared with equation (6.2.9) in Hildebrand, [8].
9. Extension to a generalised Lubbock formula

It is clear that many of the preceding notions can be extended from the standard difference operators to a general delta operator, denoted by \( \theta \), considered as a formal power series in \( D \),

\[
\theta = \phi(D) = \frac{a}{1!}D + \frac{b}{2!}D^2 + \frac{c}{3!}D^3 + \frac{d}{4!}D^4 + \ldots ,
\]

(28)

i.e. \( \phi(0) = 0 \).

Proceeding by analogy, I define \( \theta_m \equiv \phi(mD) \equiv \phi_m \) and, using the contour integral essentially as bookkeeping, the coefficient of \( \theta^\nu \) in the expansion of the generating function, \( m \theta \theta_m^{-1} \), is the polynomial,

\[
Y_\nu(m) = \frac{m}{2\pi i} \oint dD \frac{\phi'}{\phi^\nu \phi_m} ,
\]

so that,

\[
m \theta \theta_m^{-1} = 1 + Y_1(m) \theta + Y_2(m) \theta^2 + \ldots .
\]

(29)

I do this in a purely formal way, not enquiring yet as to the numerical significance of the ‘summation’ \( \theta^{-1} \), so I can refer to (29) as a generalised Lubbock formula, with no specific application.

I list a few early polynomials in terms of the parameters defining the delta operator \( \theta \), (28),

\[
Y_1 = \frac{b}{2a^2} (m - 1) , \quad Y_2 = \frac{2ac - 3b^2}{12a^4} (m^2 - 1) ,
\]

\[
Y_3 = \frac{1}{24a^6} (m - 1)(a^2d - 4abc + 3b^3) m^2 + (a^2d - 6abc + 6b^3)(m + 1) .
\]

The choice \((a, b, c, \ldots) = (1, 1, 1, \ldots)\) gives \( \theta = \Delta \) and the \( Y_\nu \) reproduce the \( \Lambda_\nu \) functions, [4] p.140. The choice \((a, b, c, d, \ldots) = (1, 0, 1, 0, \ldots)\) corresponds to \( \theta = \delta \) and results in the \( P_{2\nu} \) functions, [4] p.144, as an algebraic check.

Since \( Y_\nu(1) = 0 \), if \( \phi(D) \) is an odd or even function \( Y_\nu(-1) \) also vanishes showing that, in this case, \( Y_\nu(m) \) has a factor of \((m^2 - 1)\).

As another example consider Steffensen’s particular delta operator,

\[
\theta = \frac{E^\alpha (E^\beta - 1)}{\beta} ,
\]

(30)

which is the most general divided difference of the first order. Giving \( \alpha \) and \( \beta \) appropriate values produces the standard difference operators.
It is only algebra to compute from the contour the explicit form,

\[ Y_\nu(m, \alpha, \beta) = \frac{\beta^{\nu+1}}{\nu!} \left( \nu \lambda B_{\nu-1}(a \mid m, 1) + B_{\nu}(1 + a \mid m, 1) \right) \]

\[ = \frac{\beta^{\nu+1}}{\nu!} \left( (1 + \lambda) B_{\nu}(1 + a \mid m, 1) - \lambda B_{\nu+1}(a \mid m, 1) \right), \]

(31)

where \( a = \lambda (1 - \nu - m) \) and I have set \( \lambda = \alpha/\beta \). A recursion relation has been used to get the second line. The \( Y_\nu \) can be explicitly evaluated for a given \( \nu \), but the expressions are not especially transparent.

One could now implement the image form, (20), to give, for the term in brackets in (31),

\[ h(1 + \lambda) \sum_{s=0}^{h-1} \left( a + \frac{s}{h} \right)^{(\nu)} - h\lambda \sum_{s=0}^{h-1} \left( a - 1 + \frac{s}{h} \right)^{(\nu)}, \quad h = 1/m \]

or

\[ h(1 + \lambda) \sum_{s=0}^{h-1} \left( a + 1 + \frac{s}{h} \right)^{\nu+1-1} - h\lambda \sum_{s=0}^{h-1} \left( a + \frac{s}{h} \right)^{\nu+1-1}. \]

I note that the ‘factorials’ (poweroids) for this operator, essentially the Gould polynomials, are,

\[ x^{\nu} \equiv x(x - \nu\alpha - \beta)(x - n\alpha - 2\beta) \ldots (x - \nu\alpha - (\nu - 1)\beta) \]

\[ = x^{\nu} \beta^{\nu-1} e^{-\nu\lambda \beta} \left( \frac{x}{\beta} \right)^{(\nu)-1}, \]

(32)

but I take the analysis no further.

9. Comments

There is nothing particularly deep about these results. They are presented simply as an interesting link between interpolation theory and the Barnes \( \zeta \)-function. Indeed, finite difference considerations play a role in the theory of the latter, [15,22].

The main technicality used here is the very elementary one of extracting a term in a series as a residue. A more logical treatment would begin from the contour integrals, (14), (11), (9), and rewrite them in a fashion resembling (5) in order to obtain the Bernoulli form.
Continuing in the spirit of making connections, instead of computing the integral (11) as a residue at the origin, I deform the contour to bring in the other singularities which are all poles if $a$ is an integer. To this end the integral is first rewritten as,

$$N_1(r + 1, a, m) = \frac{1}{2\pi i} \oint dE \frac{E^{a-1}}{(E - 1)^r \ E^m - 1}, \quad (33)$$

where the contour initially circles $E = 1$ but is then blown up so as to encounter the other poles, whose contribution will be the entire integral if I assume, as I do, that $m + r > a$ for convergence at infinity, where the contour finally ends up. These poles lie on the unit circle at $E = \rho^j, (j = 1, 2, \ldots, m - 1)$ where $\rho$ is a fundamental $m$th root of unity.

The evaluation of the integral is straightforward and yields, after noting (7),

for even and odd $r$ separately, the finite image sums, $(a \in \mathbb{Z})$,

$$\sum_{j=1}^{m-1} \cos \left(\frac{2\pi j}{m} (a - \nu)\right) \csc^2 \left(\frac{\pi j}{m}\right) = \frac{2^{2\nu}}{(2\nu)!} \frac{1}{m} B_{2\nu+1}(a \ | \ m, 1)$$

$$\sum_{j=1}^{m-1} \sin \left(\frac{2\pi j}{m} (a - \nu + \frac{1}{2})\right) \csc^{2\nu-1} \left(\frac{\pi j}{m}\right) = \frac{2^{2\nu-1}}{(2\nu - 1)!} \frac{1}{m} B_{2\nu-1}(a \ | \ m, 1). \quad (34)$$

The even case is just (21) with the twisting $w = a - \nu$. The odd case is possibly novel. The equations can be rearranged in various ways.

A check follows on setting $a = \nu$ when the second line leads to the Bernoulli identity

$$2B_{2\nu-1}^{(2\nu)}(\nu \ | \ m, 1) = (2\nu - 1) B_{2\nu-2}^{(2\nu-1)}(\nu - 1 \ | \ m, 1),$$

which is a consequence of a finite difference relation and the connection,

$$B_{2\nu-1}^{(2\nu)}(\nu \ | \ m, 1) = -B_{2\nu-1}^{(2\nu)}(\nu - 1 \ | \ m, 1),$$

itself a result of the general parity relation, [12],

$$B_{2\nu-1}^{(2\nu)}(\nu - 1 \ | \ m, 1) = -B_{2\nu-1}^{(2\nu)}(m + \nu \ | \ m, 1),$$

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7 This condition is the equivalent of the periodicity of the integrand in the contour method, used in earlier works, which allows the cancellation of oppositely directed infinite lines.
combined with another finite difference property,

\[ B_{2\nu-1}^{(2\nu)}(m + \nu \mid m, 1) - B_{2\nu-1}^{(2\nu)}(\nu \mid m, 1) = m(2\nu - 1)B_{2\nu-2}^{(2\nu-1)}(\nu \mid 1) = 0. \]

There does not seem to be any relation similar to (34) when \( a \) is not integral, e.g. a half-integer, corresponding to the \( P_{2\nu} \) functions.

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