On the local boundedness of maximal H–monotone operators

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Abstract

In this paper we prove that maximal H-monotone operators \( T : \mathbb{H}^n \rightrightarrows V_1 \) whose domain is all the Heisenberg group \( \mathbb{H}^n \), are locally bounded. This implies that they are upper semicontinuous. As a consequence, maximal H-monotonicity of an operator on \( \mathbb{H}^n \) can be characterized by a suitable version of Minty’s type theorem.

Keywords: Heisenberg group; H-monotonicity; maximal H-monotonicity; Minty theorem

MSC: 47H05; 49J53

1 Introduction

Maximal monotone maps in Euclidean spaces \( \mathbb{R}^n \) and, more in general, in Hilbert spaces, play key roles in evolution equations and in other fields of functional analysis. The most notable example of a maximal monotone map in \( \mathbb{R}^n \) is provided by the subdifferential map \( \partial f \) associated to a convex function \( f : \mathbb{R}^n \to \mathbb{R} \).

The celebrated Minty theorem provides a characterization of maximal monotonicity (see [15]): given a monotone set-valued map \( T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \), then \( T \) is maximal monotone if and only if \( I + \lambda T \) is surjective onto \( \mathbb{R}^n \), for every \( \lambda > 0 \); in this case, the resolvent map \((I + \lambda T)^{-1}\) is single-valued and 1-Lipschitz continuous on \( \mathbb{R}^n \).

For operators defined on Carnot groups \( G \), a notion of H-monotonicity, and maximal H-monotonicity, has been introduced in [9]. This notion fits the monotonicity of maps in Euclidean spaces to the horizontal structure \( V_1 \) of \( G \). It arises naturally as the property fulfilled by the H-normal map \( \partial_H f \) associated to an H-convex function \( f : G \to \mathbb{R} \).

In the classical case, well-known regularity properties enjoyed by maximal monotone maps \( T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) are upper semicontinuity and local boundedness in the interior of the domain of \( T \); in particular, the proof of the latter relies essentially on the fact that any given ball of \( \mathbb{R}^n \) is contained in the convex hull of at most \( n + 1 \) points.

In this paper, we investigate maximal H-monotone operators \( T : \mathbb{H}^n \rightrightarrows V_1 \) defined on the Heisenberg group \( \mathbb{H}^n \), where \( V_1 \cong \mathbb{R}^{2n} \) denotes the first layer of the Lie algebra of \( \mathbb{H}^n \). An important example of a such operator is the horizontal normal map \( \partial_H f \) of an H-convex function \( f : \mathbb{H}^n \to \mathbb{R} \). When dealing with these operators, one has to face a much more intricate situation, due to the lack of the Euclidean geometry of the underlying setting. More...
precisely, we say that \( T : \mathbb{H}^n \Rightarrow V_1 \) is H-monotone if for every \( \eta \in \mathbb{H}^n \), \( \eta' \in H_\eta \), \( v \in T_\lambda(\eta) \) and \( v' \in T_\lambda(\eta') \) we have (see Definition 2.1)

\[
\langle v - v', \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0,
\]

where \( H_\eta \) is the horizontal plane through \( \eta \) and \( \xi \) is the canonical projection \( \xi_1(x, y, t) = (x, y) \in V_1 \), for every \( (x, y, t) \in \mathbb{H}^n \). The restriction \( \eta' \in H_\eta \) is an essential one, it implies that the notion of H-monotonicity provides information on the behaviour of the operator \( T \) at the point \( \eta \in \mathbb{H}^n \) only along the horizontal directions through \( \eta \). This restriction creates major difficulties in studying the properties of H-monotone maps. Despite the fact that several notions of convex hulls have been introduced in \( \mathbb{H}^n \) (\([8]\)), they seem not to be useful for our purpose.

The goal of this paper is to overcome the above indicated difficulties and study the local boundedness of maximal H-monotone maps. In Theorem 2.2 we show that, for an operator \( T \) with \( \text{dom}(T) = \mathbb{H}^n \), upper semicontinuity is equivalent to local boundedness. Our main result is the following:

**Theorem 1.1** Let \( T : \mathbb{H}^n \Rightarrow V_1 \) be a maximal H-monotone map, such that \( \text{dom}(T) = \mathbb{H}^n \). Then \( T \) is locally bounded.

This statement implies that \( T \) is upper semicontinuous. The proof of this theorem is considerably more involved when compared to the Euclidean framework. The statement recovers the same result as in the Euclidean case with considerably reduced assumptions as we can use information provided by the monotonicity only along horizontal directions. Our proofs require a deeper understanding of the horizontal geometry of \( \mathbb{H}^n \); in particular, the non-integrability of the horizontal bundle, or the so-called *twirling effect* (see \([3]\)) of horizontal planes, is used repeatedly in our considerations.

Theorem 1.1 sheds a new light on the regularity properties of a maximal H-monotone operator on \( \mathbb{H}^n \) and leads to the proof that any maximal H-monotone operator on \( \mathbb{H}^n \) can be characterized by a suitable version of Minty’s type theorem, thereby improving a previous result by two of the authors \([10]\).

**Theorem 1.2** Let \( T : \mathbb{H}^n \Rightarrow V_1 \) be an H-monotone map with \( \text{dom}(T) = \mathbb{H} \). Then the following two properties are equivalent:

i. \( T \) is maximal H-monotone;

ii. for every fixed \( \eta \in \mathbb{H}^n \) and \( \lambda > 0 \), the map \( (\xi_1 + \lambda T)|_{H_\eta} \) is surjective onto \( V_1 \).

As we will see in subsection 4.1, another application of our main theorem is the study of the regularity of the resolvent \( (\xi_1 + \lambda T)^{-1} : V_1 \Rightarrow \mathbb{H}^n \). Another forthcoming application (see \([4]\)), following the line of investigation in \([1]\), \([2]\), will target the study of the Hausdorff dimension of singular sets \( \Sigma_k(T) = \{ \eta \in \mathbb{H}^n : \dim(T(\eta)) \geq k \} \), for H-monotone maps \( T \) and integers \( k \).

2 Basic notions and preliminary results

2.1 The Heisenberg group \( \mathbb{H}^n \).

The Heisenberg group \( \mathbb{H}^n \) is the simplest Carnot group of step 2. In this section we will recall some of the necessary notation and background results used in the sequel. We will
focus only on those geometric aspects that are relevant to our paper. For a general overview of the subject we refer to [7].

The Lie algebra $\mathfrak{h}$ of $\mathbb{H}^n$ admits a stratification $\mathfrak{h} = V_1 \oplus V_2$ with $V_1 = \text{span}\{X_i, Y_i; \ 1 \leq i \leq n\}$ being the first layer of the so-called horizontal vector fields, and $V_2 = \text{span}\{T\}$ being the second layer which is one-dimensional. We assume $[X_i, Y_i] = -4T$ and the remaining commutators of basis vectors vanish. The exponential map $\exp : \mathfrak{h} \to \mathbb{H}^n$ is defined in the usual way. By these commutator rules we obtain, using the Baker-Campbell-Hausdorff formula, that $\mathbb{H}^n$ can be identified with $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ endowed with the non-commutative group law given by

$$\eta \circ \eta' = (x, y, t) \circ (x', y', t') = (x + x', y + y', t + t' + 2(\langle x', y \rangle - \langle x, y' \rangle)),$$

where $x, y, x'$ and $y'$ are in $\mathbb{R}^n$, $t \in \mathbb{R}$, and for $z, z' \in \mathbb{R}^n$, we have $\langle z, z' \rangle = \sum_{j=1}^n z_j z'_j$ the inner product in $\mathbb{R}^n$. Let us denote by $e$ the neutral element in $\mathbb{H}^n$. Transporting the basis vectors of $V_1$ from the origin to an arbitrary point of the group by a left-translation, we obtain a system of left-invariant vector fields written as first order differential operators as follows

$$X_j = \partial_{x_j} + 2y_j \partial_t, \quad j = 1, ..., n,$$
$$Y_j = \partial_{y_j} - 2x_j \partial_t, \quad j = 1, ..., n. \tag{1}$$

Via the exponential map $\exp : \mathfrak{h} \to \mathbb{H}$ we identify the vector $\sum_{i=1}^n (\alpha_i X_i + \beta_i Y_i) + \gamma T$ in $\mathfrak{h}$ with the point $(\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_n, \gamma)$ in $\mathbb{H}^n$; the inverse $\xi : \mathbb{H}^n \to \mathfrak{h}$ of the exponential map has the unique decomposition $\xi = (\xi_1, \xi_2)$, with $\xi_i : \mathbb{H}^n \to V_i$. Since we identify $V_1$ with $\mathbb{R}^2n$ when needed, $\xi_1 : \mathbb{H}^n \to V_1 \cong \mathbb{R}^{2n}$ is given by $\xi_1(x, y, t) = (x, y)$.

Let $N(x, y, t) = ((||x||^2 + ||y||^2)^2 + t^2)^{1/2}$ be the gauge norm in $\mathbb{H}^n$. It is an interesting exercise (see [11]) to check that the expression

$$d_H(\eta, \eta') = N((\eta')^{-1} \circ \eta)$$

satisfies the triangle inequality defining a metric on $\mathbb{H}^n$: this metric is the so-called Korányi-Cygan metric which is by left-translation and dilation invariance bi-Lipschitz equivalent to the Carnot-Carathéodory metric. Here, the non-isotropic Heisenberg dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$ for $\lambda > 0$ are defined by $\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t)$. The Korányi-Cygan ball of center $\eta_0 \in \mathbb{H}^n$ and radius $r > 0$ is given by $B_{\mathbb{H}^n}(\eta_0, r) = \{ \eta \in \mathbb{H}^n : d_H(\eta_0, \eta) \leq r\}$.

The horizontal structure relies on the notion of horizontal plane: given a point $\eta_0 \in \mathbb{H}^n$, the horizontal plane $H_{\eta_0}$ associated to $\eta_0 = (x_0, y_0, t_0)$ is the plane in $\mathbb{H}^n$ defined by

$$H_{\eta_0} = \{ \eta = (x, y, t) \in \mathbb{H}^n : t = t_0 + 2(\langle y_0, x \rangle - \langle x_0, y \rangle) \}.$$

This is the plane spanned by the horizontal vector fields $\{X_i, Y_i\}$ at the point $\eta_0$. We note that $\eta' \in H_{\eta}$ if and only if $\eta \in H_{\eta'}$.

### 2.2 Multivalued maps on $\mathbb{H}^n$

Let us consider a set-valued map $T : \mathbb{H}^n \rightrightarrows V_1$; we denote by $\text{dom}(T)$ the effective domain of $T$, i.e. the set $\{ \eta \in \mathbb{H}^n : T(\eta) \neq \emptyset \}$, and by $\text{gr}(T)$ the graph of $T$, i.e. $\{ (\eta, v) \in \mathbb{H}^n \times V_1 : \eta \in \text{dom}(T), v \in T(\eta) \}$.

Let $T : \mathbb{H}^n \rightrightarrows V_1$ be a set-valued map, with closed values, i.e. $T(\eta)$ is a closed set for every $\eta$. We recall (see [3] for this general setting) that $T$ is upper semicontinuous (briefly usc) at $\eta \in \mathbb{H}^n$ if, for every positive $\epsilon$, there exists $\delta > 0$ such that

$$T(\eta') \subseteq T(\eta) + B_{\mathbb{H}^n}(0, \epsilon), \quad \forall \eta' \in \mathbb{H}^n, \ d_H(\eta', \eta) < \delta,$$
where $T(\eta) + B_{\mathbb{R}^{2n}}(0, \epsilon)$ denotes the Minkowski sum of the two sets in $\mathbb{R}^{2n}$. If the operator $T$ is compact-valued, i.e. $T(\eta)$ is a compact for every $\eta$, then the usc of $T$ can be equivalently given as follows: if $\eta_k \to \eta$, and $v_k \in T(\eta_k)$, then there exists a subsequence $\{v_{k_n}\}$ such that $v_{k_n} \to v \in T(\eta)$. We say that $T$ is closed if $\text{gr}(T)$ is a closed subset of $\mathbb{H}^n \times V_1$.

Note that there is a gap between the dimension of the source and target spaces in this definition, unlike in the Euclidean case. Nevertheless, some basic properties follow in the same way as in the Euclidean setting. First, the properties of being upper semicontinuous, or closed, are related. Indeed,

**Remark 2.1** (see [3], Th. 16.12) Let $T : \mathbb{H}^n \Rightarrow V_1$. Then the following statements hold:

i. if $T$ is usc and closed-valued, then it is closed;

ii. if $T$ is closed, and $\text{rge}(T)$ is compact, then $T$ is upper semicontinuous.

Single-valued continuous functions map compact sets to compact sets. This property is also true for upper semicontinuous compact-set valued maps:

**Proposition 2.1** (see [3], Lemma 17.8) Let $T : \mathbb{H}^n \Rightarrow V_1$ be a compact-valued usc map. Then $T(K) \subset V_1$ is compact for every compact set $K \subset \mathbb{H}^n$.

### 2.2.1 $H$-monotone and $H$-cyclical monotone maps.

We say that $A \subset \mathbb{H}^n \times V_1$ is $H$-monotone (see [10]) if

$$
\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle \geq 0, \quad \forall (\eta, v) \in A, \ (\eta', v') \in A, \ \eta' \in H_\eta.
$$

We stress that in the previous definition, for every point $(\xi, v)$ in the set $A$, the $H$-monotonicity property gives us information about $A$ only in the horizontal directions $\{X_i(\xi), Y_i(\xi)\}_i$ through $\xi$; more precisely, (2) is equivalent to

$$
\langle \xi_1(\eta) - \xi_1(\eta \circ \exp(tw)), v - v' \rangle \geq 0, \quad \forall (\eta, v) \in A, \ (\eta \circ \exp(tw), v') \in A, \ t \in \mathbb{R}, \ w \in V_1,
$$

where, for every $w$ fixed, $t \mapsto \eta \circ \exp(tw)$ is the so called horizontal segment. This restriction gives rise to the most difficulties of our study.

We say that $A$ is maximal $H$-monotone if there are no $H$-monotone sets $B \subset \mathbb{H}^n \times V_1$ such that $A \subset B$ and there exists $(\eta, v) \in B$ such that $(\eta, v) \not\in A$. As usual, such notions of monotonicity and maxiamlity are inherited by the functions as follows:

**Definition 2.1** We say that a set-valued map $T : \mathbb{H}^n \Rightarrow V_1$ is an $H$-monotone map if $\text{gr}(T)$ is an $H$-monotone set, i.e. for every $\eta \in \mathbb{H}^n$, $\eta' \in H_\eta$, $v \in T(\eta)$ and $v' \in T(\eta')$ we have

$$
\langle v - v', \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0.
$$

We say that $T$ is strictly $H$-monotone, if for every $\eta \in \mathbb{H}^n$, $\eta' \in H_\eta$ with $\eta' \neq \eta$, $v \in T(\eta)$ and $v' \in T(\eta')$ in (3) we have a strict inequality. Moreover, we say that $T$ is maximal $H$-monotone if the set $\text{gr}(T)$ is maximal $H$-monotone.

A stronger version of the concept of monotonicity is the notion of cyclical monotonicity: in our context we say that $A \subset \mathbb{H}^n \times V_1$ is an $H$-cyclical monotone set (see Definition 6.1
in [9] if for every sequence \( \{ (\eta_i, v_i) \}_{i=0}^m \subset A \) such that \( \{ \eta_i \}_{i=0}^m \) is a closed H-sequence, i.e. \( \eta_i \in H_{\eta_{i+1}} \) and \( \eta_{m+1} = \eta_0 \), we have that
\[
\sum_{i=0}^m \langle \xi_1(\eta_{i+1}), v_i \rangle \leq \sum_{i=0}^m \langle \xi_1(\eta_i), v_i \rangle.
\]
Moreover, we say that \( A \) is maximal H–cyclically monotone if there are no H–cyclically monotone sets \( B \subset \mathbb{H}^n \times V_1 \) such that \( A \subset B \) and there exists \( (\eta, v) \in B \) such that \( (\eta, v) \notin A \).

A set-valued map \( T : \mathbb{H}^n \rightrightarrows V_1 \) is a (maximal) H-cyclically monotone map if \( \text{gr}(T) \) is a (maximal) H-cyclically monotone set.

Given a function \( u : \mathbb{H}^n \to \mathbb{R} \) we define the horizontal normal map of \( u \), \( \partial_H u : \mathbb{H}^n \Rightarrow V_1 \), by
\[
\partial_H u(\eta) = \{ p \in V_1 : u(\eta') \geq u(\eta) + \langle p, \xi_1(\eta') - \xi_1(\eta) \rangle, \forall \eta' \in H_\eta \}.
\]
It is well known that a function \( u : \mathbb{H}^n \to \mathbb{R} \) is H–convex (see [12]) if and only if \( \partial_H u(\eta) \) is non empty, for every \( \eta \). Moreover, for an H-convex function \( u \), we have that \( \partial_H u \) is H–cyclically monotone.

A cyclically monotone map has a better regularity since essentially it coincides with the horizontal normal map of an H-convex function. More precisely, in [9] the authors proved that if \( T : \mathbb{H}^n \rightrightarrows V_1 \) is an H-cyclically monotone map with \( \text{dom}(T) = \mathbb{H}^n \), then there exists an H-convex function \( u : \mathbb{H} \to \mathbb{R} \) such that \( \text{gr}(T) \subset \text{gr}(\partial_H u) \); if, in addition, \( T \) is maximal, then \( \text{gr}(T) = \text{gr}(\partial_H u) \).

We have the following result (see [10]) of Minty type in the case \( n = 1 \):

**Theorem 2.1** Let \( T : \mathbb{H} \rightrightarrows V_1 \) be an H-monotone map with \( \text{dom}(T) = \mathbb{H} \).

i. If \( T \) is maximal H-cyclically monotone, then the map \( (\xi_1 + \lambda T)|_{H_\eta} \) is surjective onto \( V_1 \) for every \( \eta \in \mathbb{H} \) and \( \lambda > 0 \).

ii. If the map \( (\xi_1 + \lambda T)|_{H_\eta} \) is surjective onto \( V_1 \) for every \( \eta \in \mathbb{H} \) for some \( \lambda > 0 \), then \( T \) is maximal H–monotone.

**Theorem 1.2** is a generalisation of Theorem 2.1, since we remove the H-cyclically monotone assumption in i., and show that the result holds in \( \mathbb{H}^n \). We note here, that every H–cyclically monotone set/map is an H-monotone set/map: the following example will convince the reader that the contrary is false, i.e. there exist maps that satisfies the assumption in Theorem 1.2 but not the assumption i. in Theorem 2.1.

**Example 2.1** Let us consider \( T : \mathbb{H}^1 \Rightarrow V_1 \) defined by
\[
T(x, y, t) = (3x, -2x + 4y).
\]
Then it follows (see Example 1 in [10] for the details) that \( T \) is maximal H-monotone, but not maximal H–cyclically monotone.

### 2.2.2 Usc and local boundedness for maximal H–monotone maps.

The purpose of this section is to establish the equivalence ofusc and the local boundedness of maximal H-monotone maps. Let us start with the following preliminary result.

**Proposition 2.2** Let \( T \) be a maximal H–monotone operator; then
Corollary 2.1

In particular, from the previous proposition, and from Proposition 2.1, we immediately get the following technical lemma:

**Lemma 2.1** Let us consider \( \eta, \eta' \in \mathbb{H}^n \) with \( \eta \neq \eta' \) and \( \eta' \in H_\eta \), and a sequence \( \{ \eta_k \}_k \subset \mathbb{H}^n \) with \( \eta_k \to \eta \) and \( \eta' \notin H_{\eta_k} \). Then there exists a sequence \( \{ \eta'_k \}_k \subset \mathbb{H}^n \) with the following properties:

a. \( \eta'_k \in H_{\eta'} \cap H_{\eta_k} \);

b. \( \eta'_k \to \eta' \);

c. \( \frac{\xi_1(\eta'_k) - \xi_1(\eta')}{\|\xi_1(\eta'_k) - \xi_1(\eta')\|} \to \frac{\xi_1(\eta) - \xi_1(\eta')}{\|\xi_1(\eta) - \xi_1(\eta')\|} \).

**Proof:** Let us suppose, without loss of generality, that

\[ \eta = e := (0,0,0), \quad \eta' = (x',y',0) \neq \eta; \]

moreover, \( \eta_k = (x_k,y_k,t_k) \). Since \( \eta' \in H_\eta \) and \( \eta_k \to e \), we have that \( \xi_1(\eta') \neq (0,0,0) \); we will suppose that \( x' \neq 0 \). Moreover, \( \xi_1(\eta_k) \neq \xi_1(\eta') \), for large \( k \); hence \( H_{\eta'} \cap H_{\eta_k} \neq \emptyset \). In addition, \( \eta_k \notin H_{\eta'} \), therefore,

\[ t_k + 2(\langle x',y_k \rangle - \langle y',x_k \rangle) \neq 0. \]
Our aim is to construct a sequence $\eta'_k$ satisfying conditions a., b. and c. Set $\eta'_k = (x'_k, y'_k, t'_k)$, where

$$x'_k = (1 + \varepsilon_k)x', \quad y'_k = (1 + \varepsilon_k)y' + A_k \varepsilon_k^2 x', \quad A_k = -\text{sgn}(t_k + 2((x', y_k) - (y', x_k))). \quad (5)$$

We will show that there exists a sequence $\{\varepsilon_k\}_k$, with $\varepsilon_k > 0$ and $\varepsilon_k \to 0$, such that $a - c$ hold. Indeed, for such sequence $\{\varepsilon_k\}_k$, condition c. is satisfied; indeed,

$$\frac{(\varepsilon_k x', \varepsilon_k y' + A_k \varepsilon_k^2 x')}{{\|\varepsilon_k x', \varepsilon_k y' + A_k \varepsilon_k^2 x'\|}} = \frac{(x', y' + A_k \varepsilon_k x')}{{\|x', y' + A_k \varepsilon_k x'\|}} \to \frac{(x', y')}{{\|x', y'\|}}.$$

Let us show that such a sequence does exist. The condition $\eta'_k \in H_{\eta'} \cap H_{\eta_k}$ is equivalent to the following:

$$t'_k = 2((y'_k, x'_k) - (x', y'_k)) = t_k + 2((y_k, x'_k) - (x_k, y'_k)). \quad (6)$$

Taking into account (5), the second equality in (6) becomes

$$a_k A_k \varepsilon_k^2 + b_k \varepsilon_k + c_k = 0,$$

where

$$a_k = (\|x'\|^2 - \langle x', x_k \rangle), \quad b_k = (\langle x', y_k \rangle - \langle y', x_k \rangle), \quad c_k = (t_k/2 + \langle x', y_k \rangle - \langle y', x_k \rangle).$$

For every $k$, sufficiently large, $a_k > 0$; moreover $c_k \neq 0$ since $\eta_k \not\in H_{\eta'}$. Hence we have two solutions

$$\varepsilon_k, \pm = \frac{-b_k \pm \sqrt{b_k^2 + 4a_k|c_k|}}{2A_k a_k}.$$

Since $c_k \to 0$, we have $\varepsilon_{k, \pm} \to 0$. For every $k$, we define

$$\varepsilon_k = \begin{cases} \varepsilon_{k,+} & \text{if } A_k > 0 \\ \varepsilon_{k,-} & \text{if } A_k < 0 \end{cases}$$

The sequence $\{\varepsilon_k\}$ satisfies the condition $\varepsilon_k > 0$ and $\varepsilon_k \to 0$, therefore the sequence $\{\eta'_k\}$ defined in (5) proves the assertion. \hfill \Box

**Theorem 2.2** Let $T : \mathbb{H}^n \rightrightarrows V_1$ be maximal $H$-monotone, with $\text{dom}(T) = \mathbb{H}^n$. Then $T$ is locally bounded if and only if $T$ is usc.

**Proof:** By Corollary 2.1 we need to prove only the “if ” part. We argue by contradiction. Suppose that $T$ is not usc. Then there exists $\{(\eta_k, v_k)\}_k \subset \mathbb{H}^n \times V_1$ with $(\eta_k, v_k) \to (\eta, v)$ with $v_k \in T(\eta_k)$, but with $v \not\in T(\eta)$. Since $T$ is maximal, there exists a point $\eta' \in H_\eta$ and exists $v' \in T(\eta')$ such that

$$\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle < 0. \quad (7)$$

Suppose now that there is a subsequence $\{\eta_{k_j}\}_j$ of $\{\eta_k\}_k$ such that $\eta_{k_j} \in H_{\eta'}$ then

$$\langle \xi_1(\eta_{k_j}) - \xi_1(\eta'), v_{k_j} - v' \rangle \geq 0, \quad \forall j;$$

taking the limit, we obtain $\langle \xi_1(\eta) - \xi_1(\eta'), v - v' \rangle \geq 0$ which contradicts (7). Hence, for large $k_0$, $\eta_k \not\in H_{\eta'}$ for $k \geq k_0$. In particular

$$\eta' \not\in H_{\eta_k}, \quad \forall k \geq k_0.$$
Now we define the sequence \( \{ \eta'_k \} \subset \mathbb{H}^n \) as in Lemma 2.1. By the local boundedness of \( T \), up to considering a subsequence, there exists \( \{ v'_k \} \), with \( v'_k \in T(\eta'_k) \) and \( v'_k \to v'' \in V_1 \). Since \( \eta'_k \in H_{\eta_k} \), by the H-monotonicity of \( T \) we have
\[
\langle \xi_1(\eta_k) - \xi_1(\eta'_k), v_k - v' \rangle \geq 0, \quad \forall k \geq k_0;
\]
passing to the limit we obtain
\[
\langle \xi_1(\eta) - \xi_1(\eta'), v - v'' \rangle \geq 0.
\]
(8)

This last inequality and (7) imply that \( v'' \neq v' \).

Since \( \eta'_k \in H_{\eta'_k} \), the monotonicity of \( T \) again gives
\[
\langle \xi_1(\eta'_k) - \xi_1(\eta'), v'_k - v' \rangle \geq 0, \quad \forall k \geq k_0;
\]
dividing by \( \| \xi_1(\eta'_k) - \xi_1(\eta') \| \) and passing to the limit, condition c. in Lemma 2.1 guarantees
\[
\langle \xi_1(\eta) - \xi_1(\eta'), v'' - v' \rangle \geq 0.
\]
(9)

Now summing the inequalities in (8) and in (9), we obtain an inequality in contradiction with (7). This concludes the proof. □

3 Local boundedness of maximal H-monotone operators.

It is well known that a maximal monotone operator \( T : \mathbb{R}^n \to \mathbb{R}^n \) is locally bounded. The proof relies essentially on the fact that, given any ball, there exist \( n+1 \) points whose convex hull contains the ball. In the case of operators \( T : \mathbb{H}^n \to V_1 \sim \mathbb{R}^{2n} \) the situation is much more involved. This section is essentially devoted to the proof Theorem 1.1. We first show that a maximal H-monotone operator defined on all \( \mathbb{H}^n \) is locally bounded on every vertical segment (see Proposition 3.1). We consider that this step is really the bulk of the paper. Secondly, we show that \( T \) inherits the local boundedness on every horizontal segment from the local boundedness of the vertical ones following an idea from [6].

Proposition 3.1 Let \( T : \mathbb{H}^n \to V_1 \) be a maximal H-monotone map such that \( \text{dom}(T) = \mathbb{H}^n \). Then the restriction of \( T \) to any vertical line is locally bounded, i.e. for every set of the type \( L := \{ \eta = (x, y, t) \in \mathbb{H}^n : t \in I \} \), with \( x \) and \( y \) fixed and \( I \subseteq \mathbb{R} \), \( I \) compact interval, there exists \( K = K(I) \) such that \( \text{diam}_{\mathbb{R}^{2n}} \{ T(L) \} \leq K \).

Proof: The proof is by contradiction. Assume that there exists one of these vertical segments on which \( T \) is not bounded. Without loss of generality we can assume that the segment is contained in the \( t \) axis. Moreover, we can assume that there exists a sequence of points on the \( t \) axis of the form \( \eta_k = (0, 0, h_k) \) such that \( h_k \to 0 \) and
\[
\lim_{k \to \infty} \text{diam}_{\mathbb{R}^{2n}} \{ T(\eta_k) \} = \infty.
\]
(10)

To obtain a contradiction we use a measure–theoretical argument as follows. Consider the sets:
\[
A_k = \{ \eta \in B_{H^n}(e, 5) : \exists \ u \in T(\eta) \ such \ that \ |u| > \frac{k}{2^n} \}.
\]
We will construct measurable subsets $S_k \subset A_k$ with the property that there exists a constant $c > 0$ such that for any $k$ we have
\[ \mathcal{L}^{2n+1}(S_k) > c, \] (11)
where $\mathcal{L}^{2n+1}$ is the Lebesgue measure in $\mathbb{H}^n$.

Assuming the existence of $S_k$ let us show how to get the desired contradiction. We consider the sets
\[ U_k = \bigcup_{m \geq k} S_{m}. \]
Then $U_k$ is measurable and it is decreasing: $U_{k+1} \subset U_k$ and $\mathcal{L}^{2n+1}(U_k) > c$. Let $\eta \in S$, then $\eta$ lies in infinitely many sets $S_k$. In particular there exists a sequence $h(k)$ of indexes $h(k) \to \infty$ such that $\eta \in S_{h(k)}$ for each $h(k)$. This implies that there exists $u_{h(k)} \in T(\eta)$ such that $\|u_{h(k)}\| \geq h(k)$. On the other hand $T(\eta)$ is compact by Proposition 2.2, which is a contradiction.

In the following we will construct the sets $S_k \subset A_k$ (first step) and will show the existence of a constant $c > 0$ (independent on $k$) for which (11) holds for any $k$ (second step).

**First step.** The construction of $S_k$ uses the measurable selection theorem (see e.g. [16]). Let us observe first that by (10) it follows that $A_k \neq \emptyset$ for any $k$. Moreover, for $k \geq 1$ there exists $h(k) \in \mathbb{N}$ such that
\[ \text{diam}_{\mathbb{R}^{2n}} \{ T(\eta_{h(k)}) \} \geq 10k. \]
To ease the notation we can assume that $h(k) = k$. We obtain a sequence $\{u_k\}_k$ with $u_k \in T(\eta_k)$ such that
\[ \|u_k\| \geq 10k. \] (12)
Let us consider the unit vector in $V_1$
\[ \omega_k = \frac{u_k}{\|u_k\|}, \] (13)
and the horizontal segment
\[ L_k = \{ \nu_k(t) := \eta_k \circ \exp(t\omega_k) \in \mathbb{H}^n : t \in [1, 2] \}. \]
We claim that $L_k \subset A_{10k}$. Indeed, let $\nu_k(t) \in L_k$ and $v_k \in T(\nu_k(t))$. By the $H$-monotonicity of $T$ we have
\[ \langle \xi_1(\nu_k(t)) - \xi_1(\eta_k), v_k - u_k \rangle \geq 0 \]
and hence, by (12) and (13)
\[ \langle v_k, t\omega_k \rangle \geq \langle u_k, t\omega_k \rangle \geq 10k. \]
Since $\omega_k$ is a unit vector by the Cauchy-Schwarz inequality we obtain
\[ \|v_k\| \geq \langle v_k, \omega_k \rangle \geq 10k. \] (14)

The idea of the proof is to enlarge the segment $L_k$ by glueing $2n$-dimensional sectors in the horizontal plane of each of its points. We will prove that by this construction we obtain an enlarged $(2n+1)$-dimensional set which is still a subset of $A_k$ and whose Lebesgue measure is bounded below by a uniform constant.
Let us consider $I \subset \mathbb{R}^{2n-1}$ given by
\[ I = [0, \pi] \times [0, \pi] \times \cdots \times [0, \pi] \times [0, 2\pi) \]
and the spherical coordinates $\omega : I \rightarrow S^{2n-1}$ given by $\omega(\Phi) = (\omega^1(\Phi), \ldots, \omega^{2n}(\Phi))$, for $\Phi = (\phi^1, \ldots, \phi^{2n-1})$.

Then these sectors are $2n$-dimensional and disjoint. We define the desired set $S_k$ by
\[ S_k = \{ \nu := \nu_k(t) \circ \exp(\rho \omega(\Phi)) \in \mathbb{H}^n : t \in T_k^{\omega(\Phi)} \}, \]
where $T_k^{\omega(\Phi)}$ is the subset of $\mathbb{H}^n$ defined by
\[ T_k^{\omega(\Phi)} = \{ \nu_k(t) \circ \exp(t \omega(\Phi)) \in \mathbb{H}^n : t \in I_k^{\omega(\Phi)} \}. \]
It is clear that for $k$ sufficiently large, by the construction, we have $S_k \subseteq B_{H^n}(e, 5)$. We claim first that $S_k \subseteq A_k$. To see this let $v = \nu_k(t) \circ \exp(\rho(\Phi))$ be an arbitrary point in $S_k$, and let $v \in T(\nu)$.

We intend to prove that $\|v\| \geq \frac{k}{\omega}$. This will be done, using the fact that $v \in H_{\nu_k(t)}$ and the monotonicity of $T$ by comparing $(\nu, v)$ to the point $(\nu_k(t), \tilde{v}_k(t))$, i.e.

$$\langle \xi_1(\nu) - \xi_1(\nu_k(t)), v - \tilde{v}_k(t) \rangle \geq 0$$

which implies

$$\rho(\omega(\Phi), v - \tilde{v}_k(t)) \geq 0$$

Let us note first that, if $\Phi = (\phi^1, \phi^2, \ldots, \phi^{2n-1})$ and $\Psi = (\psi^1, \psi^2, \ldots, \psi^{2n-1})$ belong to the same $(2n-1)$-cube $I^2$, then

$$\langle \omega(\Phi), \omega(\Psi) \rangle \geq 2^{-2n-1}/2.$$  \hspace{1cm} (21)

Indeed, from the expression of the left hand side of the previous inequality and taking into account the equation for the the spherical coordinates, we have

$$\sum_{i=1}^{2n} \omega^i(\Phi)\omega^i(\Psi) = \cos \phi^1 \cos \psi^1 +$$

$$+ \sin \phi^1 \cos \phi^2 \sin \psi^1 \cos \psi^2 +$$

$$+ \cdots +$$

$$+ \sin \phi^1 \sin \phi^2 \cdots \sin \phi^{2n-2} \cos \phi^{2n-1} \sin \psi^1 \sin \psi^2 \cdots \sin \psi^{2n-2} \cos \psi^{2n-1} +$$

$$+ \sin \phi^1 \sin \phi^2 \cdots \sin \phi^{2n-2} \sin \phi^{2n-1} \sin \psi^1 \sin \psi^2 \cdots \sin \psi^{2n-2} \sin \psi^{2n-1};$$

if we take the last two lines of the sum above we have

$$\omega^{2n-1}(\Phi)\omega^{2n-1}(\Psi) + \omega^{2n}(\Phi)\omega^{2n}(\Psi) =$$

$$= \sin \phi^1 \sin \phi^2 \cdots \sin \phi^{2n-2} \sin \psi^1 \sin \psi^2 \cdots \sin \psi^{2n-2} \cos(\phi^{2n-1} - \psi^{2n-1})$$

$$\geq \frac{1}{\sqrt{2}} \sin \phi^1 \sin \phi^2 \cdots \sin \phi^{2n-2} \sin \psi^1 \sin \psi^2 \cdots \sin \psi^{2n-2},$$

noticing that to obtain the previous inequality we use the fact that $\sin \psi^i$ and $\sin \phi^i$ are nonnegative. Iterating this argument, we finally get (21).

Hence, by (16), (17) and (20), and recalling that by definition of the set $S_k$ in (19) we have that $\Phi_k(t)$ and $\Phi$ lie in the same $(2n-1)$-cube $I^{\omega(k)},$

$$\|v\| \geq \langle v, \omega(\Phi) \rangle$$

$$\geq \langle \tilde{v}_k(t), \omega(\Phi) \rangle$$

$$= \|\tilde{v}_k(t)\|\langle \omega(\Phi_k(t)), \omega(\Phi) \rangle$$

$$\geq 10k2^{-2n-1}/2 > \frac{k}{2^n}.$$  \hspace{1cm} (21)

**Second step.** Our second claim is, that there exists a constant $c > 0$ with the property that

$$\mathcal{L}^{2n+1}(S_k) \geq c.$$  

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To prove this fact let us consider, for every $k$, the mapping

$$F_k = (F_k^1, \ldots, F_k^{2n+1}) : [1, 2] \times [1, 2] \times I^{2(n+1)} \to \mathbb{H}^n,$$

given by

$$F_k(t, \rho, \Theta) = \nu_k(t) \circ \exp(\rho \omega(\Theta)),$$

where $\nu_k(t)$ is as in (18) and $\Theta = (\theta^1, \ldots, \theta^{2n-1})$. Let $\Phi_k \in I$ be such that $\omega_k = \omega(\Phi_k)$.

Our aim is to show that if $\Theta$ is suitably chosen with respect to $\Phi_k$, then $|\det(JF_k(t, \rho, \Theta))|$ is bounded from below by a positive constant, where $JF$ is the Jacobian of the function $F_k$. Since $\omega_k$ is fixed, we can assume, without loss of generality and to simplify the computations, that $\omega_k = (1, 0, \ldots, 0)$, i.e. $\Phi_k = (0, \ldots, 0)$.

Recalling that $\eta_k = (0, 0, h_k)$, we obtain the formula

$$F_k(t, \rho, \Theta) = (F_k^1, \ldots, F_k^{2n+1})(t, \rho, \Theta) = \begin{pmatrix}
    t + \rho \cos \theta^1 \\
    \rho \sin \theta^1 \cos \theta^2 \\
    \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \cos \theta^4 \\
    \vdots \\
    \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \ldots \sin \theta^{2n-1} \\
    \rho \sin \theta^1 \sin \theta^2 \sin \theta^3 \ldots \cos \theta^{2n-1} \\
    h_k - 2t \rho \sin \theta^1 \sin \theta^2 \ldots \sin \theta^n \cos \theta^{n+1}
\end{pmatrix}.$$

Let us consider the Jacobian $JF_k$ of the function $F_k$. If $n = 1$, trivial computations show that $|\det(JF_k(t, \rho, \theta))| = 2\rho^2 |\sin \theta|$. In the general case, we note that the first three columns of $JF_k$ are

$$\begin{pmatrix}
    1 \\
    0 \\
    0 \\
    \vdots \\
    0 \\
    -2\rho \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1}
\end{pmatrix}, \begin{pmatrix}
    \cos \theta^1 \\
    \sin \theta^1 \cos \theta^2 \\
    \sin \theta^1 \sin \theta^2 \cos \theta^3 \\
    \vdots \\
    \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\
    \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\
    -2t \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1}
\end{pmatrix}, \begin{pmatrix}
    -\rho \sin \theta^1 \\
    \rho \cos \theta^1 \cos \theta^2 \\
    \rho \cos \theta^1 \sin \theta^2 \cos \theta^3 \\
    \vdots \\
    \rho \cos \theta^1 \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\
    \rho \cos \theta^1 \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\
    -2t \rho \cos \theta^1 \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1}
\end{pmatrix}.$$ 

In particular, the second and the third ones can be written as

$$\begin{pmatrix}
    \cos \theta^1 / \sin \theta^1 \\
    \cos \theta^2 \\
    \sin \theta^2 \cos \theta^3 \\
    \vdots \\
    \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\
    \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\
    -2t \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1}
\end{pmatrix}, \begin{pmatrix}
    -\sin \theta^1 / \cos \theta^1 \\
    \cos \theta^2 \\
    \sin \theta^2 \cos \theta^3 \\
    \vdots \\
    \prod_{i=2}^{2n-2} \sin \theta^i \cos \theta^{2n-1} \\
    \prod_{i=2}^{2n-2} \sin \theta^i \sin \theta^{2n-1} \\
    -2t \prod_{i=2}^n \sin \theta^i \cos \theta^{n+1}
\end{pmatrix}.$$ 

therefore, if we remove the first entry, we get two dependent columns. This means that, when computing the determinant of $JF_k$ starting from the first column, we have actually only one term to consider, namely

$$\det(JF_k(t, \rho, \Theta)) = -2\rho \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \cdot \det(Jw^\rho(\rho, \Theta)),$$
where \( w^{\rho}(\rho, \Theta) = \rho w(\Theta) \) denotes the 2n-dimensional spherical coordinates (see \((15)\)). By known computations,

\[
\det(J w^{\rho}(\rho, \Theta)) = \rho^{2n-1} \sin^{2n-2} \theta^1 \sin^{2n-3} \theta^2 \ldots \sin \theta^{2n-2};
\]

thus

\[
|\det(J F_k(t, \rho, \Theta))| = 2^\rho \rho^{2n} \prod_{i=1}^n \sin \theta^i \cos \theta^{n+1} \sin^{2n-2} \theta^1 \sin^{2n-3} \theta^2 \ldots \sin \theta^{2n-2}.
\]

We note that \( \det(J F_k(t, \rho, \Theta)) \neq 0 \) for a.e. \( \Theta \in I_0^{(k)} \). Let us consider the set

\[
C_k = T_{\rho}^{(k)} \times [1, 2] \times I_0^{(k)}.
\]

Since \( S_k = F([1, 2] \times [1, 2] \times I_0^{(k)}) \), we have that \( F(C_k) \subseteq S_k \). By the change of variable formula we have that

\[
\mathcal{L}^{2n+1}(S_k) \geq \int_{C_k} |\det(J F_k(t, \rho, \Theta))|dt d\rho d\theta^1 d\theta^2 \ldots d\theta^{2n-1} \geq c_0 \mathcal{L}^{2n+1}(C_k) := c.
\]

It is an exercise to show that \( c \) is a uniform constant which does not depend on \( k \), finishing the proof. \( \Box \)

We are now able to prove Theorem \( 1.1 \).

**Proof of Theorem 1.1** We show that \( T \) is bounded in a suitable neighbourhood of the origin. Let us consider the 4n segments in \( \mathbb{H}^n \):

\[
I_j^+ := \{(e_j, 0, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, \quad I_j^- := \{(-e_j, 0, s) \in \mathbb{H}^n : -1 \leq s \leq 1\},
\]

\[
I_{j+n}^+ := \{(0, e_j, s) \in \mathbb{H}^n : -1 \leq s \leq 1\}, \quad I_{j+n}^- := \{(0, -e_j, s) \in \mathbb{H}^n : -1 \leq s \leq 1\},
\]

where \( j = 1, 2, \ldots, n \). Here \( e_j \) denotes the \( n \)-tuple with 1 in the \( j \) position, and 0 otherwise.

From Proposition \( 3.1 \) there is \( K > 0 \) such that \( T(I_j^+), T(I_j^-) \subseteq B_{\mathbb{R}^{2n}}(0, K) \), for every \( j = 1, \ldots, 2n \). Let \( r \in (0, 1) \) small enough such that, for every \( \xi \in B_{\mathbb{H}^n}(0, r) \) and for every \( j = 1, \ldots, 2n \), we have \( H_\xi \cap I_j^+, H_\xi \cap I_j^- \neq \emptyset \) : we note that by the continuity of the map \( \xi \mapsto H_\xi \) such \( r > 0 \) exists since the claim holds for \( \xi = 0 \). Now, for any \( \xi = (x, y, t) \in B_{\mathbb{H}^n}(0, r) \) we define \( \xi^+, \xi^-, v_j^+ \) and \( v_j^- \) by

\[
\xi_j^+ = \xi \circ \exp(v_j^+(\xi)) = H_\xi \cap I_j^+, \quad \xi_j^- = \xi \circ \exp(v_j^-(\xi)) = H_\xi \cap I_j^-, \quad j = 1, 2, \ldots, 2n.
\]

Straightforward computations show that \( v_j^+ \) coincide with one of the vectors from the following list

\[
(e_j - x, -y), \quad (-e_j - x, -y), \quad (-x, e_j - y), \quad (-x, -e_j - y),
\]

thus \( \|v_j^\pm\| \leq 2 \), for every \( j \), and for every \( \xi \in B_{\mathbb{H}^n}(0, r) \).

From the \( H \)-monotonicity of \( T \), we have that

\[
\langle u, v_j^+(\xi) \rangle \leq \langle u, v_j^+(\xi) \rangle \leq K \|v_j^+(\xi)\| \leq 2K, \quad (22)
\]

for every \( u \in T(\xi), u_j \in T(\xi_j), \) and for every \( j = 1, \ldots, 2n \). The inequalities \( (22) \) imply that \( T(\xi) \) is contained in the polyhedron \( P(\xi) \) defined by:

\[
P(\xi) := \{ u \in V_1 : \langle u, v_j^+(\xi) \rangle \leq 2K, \langle u, v_j^-(\xi) \rangle \leq 2K \quad j = 1, \ldots, 2n \}.
\]
Note that there is no \( v \in \mathbb{R}^{2n} \setminus \{0\} \) such that the half-space \( \{ u \in \mathbb{R}^{2n} : \langle v, u \rangle \leq 0 \} \) contains all the vectors \( \{ v_j^\pm \}_j \), as a consequence, the set \( P(\xi) \) turns out to be a polytope, i.e. it is bounded. Indeed, on the contrary, if \( v \in \mathbb{R}^{2n} \setminus \{0\} \) is such that \( tv \in P(\xi) \), for every \( t \geq 0 \), then \( \langle v, v_j^\pm(\xi) \rangle \leq 0 \), i.e. the set \( \{ v_j^\pm(\xi) \}_j \) belongs to the half-space \( \{ u : \langle v, u \rangle \leq 0 \} \), a contradiction. The continuity of the maps \( \xi \mapsto v_j^\pm(\xi) \), for every \( j \), entails, in particular, that the set-valued map \( \xi \mapsto P(\xi) \) is upper semicontinuous; thus, if \( r \) is small enough, there exists \( K' \geq 2K \) such that

\[
P(\xi) \subseteq B_{\mathbb{R}^{2n}}(0, K'), \quad \forall \xi \in B_{\mathbb{H}^n}(0, r).
\]

This implies that \( T(\xi) \subseteq B_{\mathbb{R}^{2n}}(0, K') \), for all \( \xi \in B_{\mathbb{H}^n}(0, r) \), therefore \( T \) is locally bounded at the origin. \( \square \)

Clearly, Theorem 2.2 and Theorem 1.1 give

**Corollary 3.1** Let \( T : \mathbb{H}^n \rightrightarrows V_1 \) be a maximal \( H \)-monotone map, such that \( \text{dom}(T) = \mathbb{H}^n \). Then \( T \) is locally bounded and upper semicontinuous.

### 4 On Minty’s theorem.

This section we apply our main result in Theorem 1.1 in order to prove a horizontal version of Minty’s theorem. In the following, for a given operator \( T : \mathbb{H}^n \rightrightarrows V_1 \) and \( \lambda > 0 \), we denote by \( T_\lambda : \mathbb{H}^n \rightrightarrows V_1 \) the operator

\[
T_\lambda = \xi_1 + \lambda T.
\]

It is clear that if \( T \) is \( H \)-monotone, then \( T_\lambda \) is strictly \( H \)-monotone. We recall that in [10] the authors prove Theorem 2.1 a result of Minty type in the case \( n = 1 \). Now, our aim is to prove Theorem 1.2. In comparison to Theorem 2.1, we will remove the \( H \)-cyclically monotone assumption in i., and we also show that the result holds for \( \mathbb{H}^n \). We note that in the Example 2.1 we have a map that satisfies the assumption in Theorem 1.2, but not the assumption i. in Theorem 2.1.

In order to prove the following result, we will follow the idea in [5] by using degree-theoretical arguments for set valued maps [14]; the results needed in the proof are collected in the Appendix of [5].

**Proof of Theorem 1.2**

Let us first prove that i. implies ii., which is the more difficult part. Let \( T \) be maximal \( H \)-monotone with \( \text{dom}(T) = \mathbb{H}^n \). Let us fix \( \eta \in \mathbb{H}^n \) and \( \lambda > 0 \). We consider the linear projection map \( \pi : H_\eta \rightarrow V_1 = \mathbb{R}^{2n} \) defined by \( \pi(x, y, t) = (x, y) \). Note that since we restricted the projection to a hyperplane we have that \( \pi \) is bijective and we denote by \( \pi^{-1} : \mathbb{R}^{2n} \rightarrow H_\eta \) its inverse. We introduce the following notations: \( \tilde{T}_\lambda \) is the operator \( \tilde{T}_\lambda = T_\lambda \circ \pi^{-1} : \mathbb{R}^{2n} \rightrightarrows V_1 \) and \( \pi(\zeta) = \tilde{\zeta}, \forall \zeta \in H_\eta \). We have to prove that \( \tilde{T}_\lambda \) is surjective.

Let us fix \( p_0 \in V_1 \cong \mathbb{R}^{2n} : \) we show that it is possible to find \( R_0 > 0 \) large enough such that

\[
p_0 \in \tilde{T}_\lambda (B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)) : \quad (23)
\]

in particular, we show this for

\[
R_0 > \|p_0\| + \|\xi_1(\eta)\| + \lambda \sup\{\|v_\eta\| : v_\eta \in T(\eta)\}.
\]

(24)
Note that the fact that the expression on the right in the above inequality is finite follows from local boundedness of \( T \).

**Step 1.** In order to prove (23), we show first that

\[
\deg_{SV} \left( \tilde{T}_\lambda - p_0, B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), 0 \right) = 1,
\]

(25)

where \( \deg_{SV} \) denotes the degree function for set-valued maps. We consider the parametric set-valued map \( \mathcal{F} : [0, 1] \times B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \to V_1 \) defined by

\[
\mathcal{F}(\alpha, \tilde{\zeta}) = \tilde{\zeta} - p_0 + \lambda \alpha T(\pi^{-1}(\tilde{\zeta})),
\]

for all \( \alpha \in [0, 1] \), \( \tilde{\zeta} \in B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \).

First we note that, by Proposition 2.2, the map \( \mathcal{F} \) is convex-valued and compact-valued, i.e. for every fixed \( (\alpha, \tilde{\zeta}) \in [0, 1] \times B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \), the set \( \mathcal{F}(\alpha, \tilde{\zeta}) \) is compact and convex in \( \mathbb{R}^{2n} \). Moreover, Corollary 2.1 and Corollary 3.1 imply that

\[
\{ \cup \mathcal{F}(\alpha, \tilde{\zeta}) : (\alpha, \tilde{\zeta}) \in [0, 1] \times B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \}
\]

is compact in \( \mathbb{R}^{2n} \). Finally, Corollary 2.1 implies that the map \( (\alpha, \tilde{\zeta}) \mapsto \mathcal{F}(\alpha, \tilde{\zeta}) \) is usc from \( [0, 1] \times B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \) into \( 2^{\mathbb{R}^{2n}} \setminus \{ \emptyset \} \).

Now we are in the position to apply the mentioned degree-theoretical arguments for set valued maps. According to the above discussion, it follows that our map \( \mathcal{F}(\alpha, \cdot) \) is a homotopy of class \((P)\) (see [14] and also Appendix in [5]). The argument is based on the application of Theorem 6.2 in [5]. In order to apply this statement we need to show that the constant curve \( \gamma : [0, 1] \to \mathbb{R}^{2n} \), defined by \( \gamma(\alpha) = 0 \), is such that

\[
\gamma(\alpha) \notin \mathcal{F}(\alpha, \partial B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0)), \quad \forall \alpha \in [0, 1].
\]

(26)

We show (26) through arguing by contradiction: suppose that for some \( \alpha \) there exists \( \tilde{\zeta} \in \partial B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0) \) such that

\[
0 \notin \mathcal{F}(\alpha, \tilde{\zeta}) = \tilde{\zeta} - p_0 + \lambda \alpha T(\pi^{-1}(\tilde{\zeta})),
\]

i.e. \( p_0 = \xi_1(\zeta) + \lambda \alpha w_\zeta \) for some \( w_\zeta \in T(\zeta), \zeta \in H_\eta \) and \( \zeta \in \partial B_{\mathbb{H}^n}(\eta, R_0) \). This implies that, for every \( v_\eta \in T(\eta) \), we have

\[
p_0 - \xi_1(\eta) - \lambda \alpha v_\eta = \xi_1(\zeta) - \xi_1(\eta) + \lambda \alpha (w_\zeta - v_\eta).
\]

Multiplying the previous vector equality by \( (\xi_1(\zeta) - \xi_1(\eta)) \) we obtain

\[
\langle \xi_1(\zeta) - \xi_1(\eta), p_0 - \xi_1(\eta) - \lambda \alpha v_\eta \rangle = \| \xi_1(\zeta) - \xi_1(\eta) \|^2 + \lambda \alpha \langle \xi_1(\zeta) - \xi_1(\eta), w_\zeta - v_\eta \rangle.
\]

The H–monotonicity of \( T \) implies

\[
\| \xi_1(\zeta) - \xi_1(\eta) \|^2 \leq \| \xi_1(\zeta) - \xi_1(\eta), p_0 - \xi_1(\eta) - \lambda \alpha v_\eta \| \leq \| \xi_1(\zeta) - \xi_1(\eta) \| \cdot \| p_0 - \xi_1(\eta) - \lambda \alpha v_\eta \|
\]

hence

\[
R_0 \leq \| p_0 - \xi_1(\eta) - \lambda \alpha v_\eta \|
\]

This contradicts (23) and hence (26) holds. The homotopy invariance property for \( \mathcal{F} \) (see Theorem 6.2 in [5]) gives that

\[
\deg_{SV} (\mathcal{F}(\alpha, \cdot), B_{\mathbb{R}^{2n}}(\tilde{\eta}, R_0), \gamma(\alpha))
\]
Step 2. By Step 1 and the definition of $\deg_{SV}$, for small $\varepsilon > 0$, one has that

$$\deg_B(f_\varepsilon - p_0, B_{R^{2n}}(\tilde{\eta}, R_0), 0) = 1,$$

where $f_\varepsilon : B_{R^{2n}}(\tilde{\eta}, R_0) \to R^{2n}$ is a continuous approximate selector of the upper semicontinuous set-valued map $\tilde{T}_\lambda$ such that

$$f_\varepsilon(\tilde{\zeta}) \in \tilde{T}_\lambda \left( B_{R^{2n}}(\tilde{\zeta}, \varepsilon) \cap B_{R^{2n}}(\tilde{\eta}, R_0) \right) + B_{R^{2n}}(0, \varepsilon), \quad \forall \tilde{\zeta} \in B_{R^{2n}}(\tilde{\eta}, R_0),$$

see Proposition 6.1 in [5]. Let $\varepsilon = \frac{1}{k}$ and let $\phi_k := f_{1/k}$, $k \in \mathbb{N}$. First of all, from (28) and the properties of the Brouwer degree function $\deg_B$ (see Theorem 6.1 in [5]), we have that for every $k \in \mathbb{N}$ there exists $\zeta_k \in B_{R^{2n}}(\tilde{\eta}, R_0)$ such that $p_0 = \phi_k(\zeta_k)$. Up to a subsequence, we may assume that $\zeta_k \to \tilde{v} \in B_{R^{2n}}(\tilde{\eta}, R_0)$. On the other hand, by relation (29), we have that

$$p_0 = \phi_k(\zeta_k) \in \tilde{T}_\lambda \left( B_{R^{2n}}(\tilde{\zeta}_k, 1/k) \cap B_{R^{2n}}(\tilde{\eta}, R_0) \right) + B_{R^{2n}}(0, 1/k),$$

i.e., there exists $\tilde{v}_k \in B_{R^{2n}}(\tilde{\zeta}_k, 1/k) \cap B_{R^{2n}}(\tilde{\eta}, R_0)$ and $p_k \in B_{R^{2n}}(0, 1/k)$ such that $p_0 \in \tilde{T}_\lambda(\tilde{v}_k) + p_k$. Clearly, $\tilde{v}_k \to \tilde{v}$ and $p_k \to 0$ as $k \to \infty$. Let us consider the sequence $\{ (p_0 - p_k, \tilde{v}_k) \} \in \text{graph}(\tilde{T}_\lambda)$; by the use of $\tilde{T}_\lambda$ we get that $p_0 \in \tilde{T}_\lambda(\tilde{v})$.

Finally, we claim that $\tilde{v} \in B_{R^{2n}}(\tilde{\eta}, R_0)$. To see this, let us assume, by contradiction, that $\tilde{v} \in \partial B_{R^{2n}}(\tilde{\eta}, R_0)$. Then, $p_0 \in \tilde{T}_\lambda(\tilde{v})$ is equivalent to $0 \in \tilde{T}_\lambda(\tilde{v}) - p_0 = \mathcal{F}(1, \tilde{v})$, which contradicts relation (26). Consequently, $\tilde{v} \in B_{R^{2n}}(\tilde{\eta}, R_0)$; therefore we obtain (23), which concludes the proof of the first implication $i. \Rightarrow \; ii$ of Theorem 1.2.

Let us prove that $ii. \Rightarrow \; i$. The proof is essentially in [10], where the case $n = 1$ is considered; however, for the sake of completeness, we include it. Let $T : \mathbb{H}^n \Rightarrow V_1$ be a set-valued H-monotone map, with domain $\mathbb{H}^n$, such that, for every $\eta_0 \in \mathbb{H}^n$,

$${\text{rge}}(T_\lambda)|_{H_{\eta_0}} = V_1.$$ 

We argue by contradiction and suppose that $T$ is not maximal H-monotone. Then there exist $\eta_0 \in \mathbb{H}^n$, and $w \notin T(\eta_0)$ such that, for every $\eta \in H_{\eta_0}$, and $v \in T(\eta)$,

$$\langle w - v, \xi_1(\eta_0) - \xi_1(\eta) \rangle \geq 0.$$ (30)

Without loss of generality, we assume that $\eta_0 = e$: in fact, via a left translation, the map $\eta \mapsto T(\eta_0 \circ \eta)$ has the same properties of $T$. From the assumptions, for $\lambda = 1$ we have $\text{rge}(T + \xi_1)|_{H_e} = V_1$; therefore

$$w = \tilde{v} + \xi_1(\tilde{\eta}),$$ (31)
for some \( \tilde{\eta} \in H_e \) and \( \tilde{v} \in T(\tilde{\eta}) \). From (31), choosing \( \eta = \tilde{\eta} \) in (30), we obtain
\[-\langle \xi_1(\tilde{\eta}), \xi_1(\tilde{\eta}) \rangle \geq 0,\]
i.e., \( \xi_1(\tilde{\eta}) = 0 \). Since \( \tilde{\eta} \in H_e \), we deduce that \( \tilde{\eta} = 0 \), and \( w = \tilde{v} \in T(e) \), contradicting our assumption on \( w \). This concludes the proof of Theorem 1.2. \( \square \)

4.1 Lipschitz continuity of the resolvent operator in the Hausdorff metric.

In this subsection we are interested in studying the regularity of the resolvent \( Q_\lambda \) of a maximal H–monotone operator \( T \) defined by
\[ Q_\lambda = (\xi_1 + \lambda T)^{-1} : V_1 \rightrightarrows \mathbb{H}^n. \]

First, we have to recall that if \( T \) is maximal H–monotone and \( \eta \in \mathbb{H}^n \), then the map \( T_\lambda \mid_{\mathbb{H}_\eta} \) is not injective, in general, and hence \( (T_\lambda \mid_{\mathbb{H}_\eta})^{-1} : V_1 \rightrightarrows \mathbb{H}_\eta \) is not single-valued (see Example 4.1 below). Using the strictly H–monotonicity of the operator \( T_\lambda \), the only information we have is that, for every \( \eta' \in \mathbb{H}_\eta \),
\[ T_\lambda(\eta) \cap T_\lambda(\eta') = \emptyset. \]

We note that, for every fixed \( v \), \( Q_\lambda(v) \) is a closed subset of \( \mathbb{H}^n \), since it is the inverse image via the usc map \( T_\lambda \) of a point. Moreover, Theorem 1.2 implies that for every fixed \( v \in V_1 \) and \( \eta \in \mathbb{H}^n \), there exists at least one point \( \eta' \in \mathbb{H}_\eta \) such that \( v \in T_\lambda(\eta') \), i.e. \( \eta' \in Q_\lambda(v) \). Therefore \( H(0,0,h) \cap Q_\lambda(v) \neq \emptyset \), for every \( h \in \mathbb{R} \). Hence \( Q_\lambda(v) \) is unbounded for every fixed \( v \in V_1 \). We summarize this discussion in the following:

**Remark 4.1** Let \( T : \mathbb{H}^n \rightrightarrows V_1 \) be a maximal H–monotone map with \( \text{dom}(T) = \mathbb{H}^n \). Then, for every \( \lambda > 0 \), the resolvent \( Q_\lambda : V_1 \rightrightarrows \mathbb{H}^n \) is closed–valued, and \( Q_\lambda(v) \) is unbounded for every \( v \in V_1 \).

As we mentioned in the introduction, if we consider the resolvent in our context, we are very far from the Euclidean situation where the resolvent map \( (I + \lambda T)^{-1} \) of a maximal monotone set-valued map \( T : \mathbb{R}^n \rightrightarrows \mathbb{R}^n \) is single-valued on \( \mathbb{R}^n \) and 1-Lipschitz continuous. However, in this line of investigation, it is useful to think about the notion of multivalued Lipschitz map.

Let \( Q : V_1 \rightrightarrows \mathbb{H}^n \) be a closed–valued multivalued map. We recall (see Definition 9.26 in [15]) that \( Q \) is **Lipschitz continuous in the Hausdorff metric**, if \( \text{dom}(Q) = V_1 \) and there exists a positive \( k \) such that
\[ Q(v') \subseteq Q(v) + B_\mathbb{H}^n(0,k\|v' - v\|), \quad \forall v, v' \in V_1. \]

We have the following regularity result for our resolvent:

**Proposition 4.1** Let \( T : \mathbb{H}^n \rightrightarrows V_1 \) be a maximal H–monotone map with \( \text{dom}(T) = \mathbb{H}^n \). Then, for every \( \lambda > 0 \), the resolvent \( Q_\lambda \) is 1-Lipschitz continuous in the Hausdorff metric.

**Proof:** Let us consider \( v \) and \( v' \) in \( V_1 \), with \( v \neq v' \). For every \( \eta \in Q_\lambda(v) \), i.e.
\[ v \in \xi_1(\eta) + \lambda T(\eta), \quad (32) \]
In fact, for positive $k$ and $T$, let us consider \( \lambda T \mapsto H \)-convex (see \cite{12}). The associated horizontal subgradient is \( \lambda T \mapsto H \)-monotone. First, it is possible to prove that there exist $\lambda > 0$, such that for every $\eta, \eta' \in \mathbb{H}$:

\[
\langle v - \xi_1(\eta) - \xi_1(\eta'), \xi_1(\eta) - \xi_1(\eta') \rangle \geq 0
\]

and hence

\[
\|v - v'\| \geq \langle v - v', \xi_1(\eta) - \xi_1(\eta') \rangle \geq \|\xi_1(\eta) - \xi_1(\eta')\|.
\]

The previous inequality implies that for every $\eta \in Q_\lambda(v)$ there exists $\eta' \in Q_\lambda(v')$ such that $d_H(\eta, \eta') \leq \|v - v'\|$. This implies 1-Lipschitz continuity of $Q_\lambda$ in the Hausdorff metric. The claim is proved.

Let us conclude with the following example presented in \cite{10}:

**Example 4.1** Let us consider the gauge function $N : \mathbb{H} \to \mathbb{R}$ defined as

\[
N(x, y, t) := ((x^2 + y^2)^2 + t^2)^{1/4}.
\]

It is known that this function is $\mathbb{H}$-convex (see \cite{12}). The associated horizontal subgradient map $\partial_H N$ is given by

\[
\partial_H N(x, y, t) = \begin{cases} \{B_{\mathbb{R}^2}(0, 1) \} \quad & (x, y, t) = (0, 0, 0) \\ \left\{ \frac{1}{N^3(x, y, t)} (x(x^2 + y^2) + yt, y(x^2 + y^2) - xt) \right\} \quad & (x, y, t) \neq (0, 0, 0). \end{cases}
\]

For every fixed $\lambda > 0$, let the map $T_\lambda := \xi_1 + \lambda \partial_H N : \mathbb{H} \mapsto V_1$ that is maximal strictly $\mathbb{H}$-monotone. First, it is possible to prove that there exist $\eta, \eta' \in \mathbb{H}$, and $\eta, \eta' \in H_{\eta''}, \eta \neq \eta'$, such that

\[
T_\lambda(\eta) \cap T_\lambda(\eta') \neq \emptyset.
\]

Secondly, it is clear that $T_\lambda$ is not a Lipschitz continuous map, i.e. it does not exist a positive $k$ such that

\[
T_\lambda(\eta') \subseteq T_\lambda(\eta) + B_{\mathbb{R}^2} \left( 0, k d_H(\eta, \eta') \right), \quad \forall \eta, \eta' \in \mathbb{H}.
\]

In fact, for $\eta' = (0, 0, 0)$ and $\eta = (x, y, 0)$ the previous inclusion is false.

If we are interested in $Q_\lambda = (\xi_1 + \lambda \partial_H N)^{-1} : V_1 \mapsto \mathbb{H}$, an easy calculation gives

\[
Q_\lambda(0, 0) = \{(0, 0, t) \in \mathbb{H} : t \in \mathbb{R}\}.
\]

Now, let us consider $(x, y, t) \neq (0, 0, 0)$ and $v \neq (0, 0)$ with $(x, y, t) \in Q_\lambda(v)$, i.e. $v = T_\lambda(x, y, t)$; straightforward computations lead to the following

\[
\epsilon^2 \geq \|v\|^2 = \|T_\lambda(x, y, t)\|^2 = (x^2 + y^2) \left( 1 + \frac{\lambda^2}{N^2(x, y, t)} \right) + \frac{2\lambda}{N^3(x, y, t)} (x^2 + y^2);
\]

hence $(x, y, t) \in Q_\lambda(v)$ implies $\|(x, y)\| \leq \epsilon$. Moreover, since $T_\lambda$ is surjective on every horizontal plane $H_\eta$ and in particular on $H_{(0,0,t)}$, we obtain that for every $t \neq 0$ there exists $(x, y)$ such that $v \in T_\lambda(x, y, t)$. These prove that $0 \neq \|v\| \leq \epsilon$ gives

\[
Q_\lambda(v) \text{ is unbounded}, \quad Q_\lambda \subset \{(x, y, t) \in \mathbb{H} : \|(x, y)\| \leq \epsilon\}.
\]
References

[1] G. Alberti, L. Ambrosio, A geometrical approach to monotone function in \( \mathbb{R}^n \). *Math. Z.*, 230, 259–316, 1999.

[2] G. Alberti, L. Ambrosio, P. Cannarsa, On the singularities of convex functions *Manuscripta Math.*, 76, 421–435, 1992.

[3] C. D. Aliprantis, K. C. Border, Infinite Dimensional Analysis, *Springer*, 1999.

[4] Z.M. Balogh, A. Calogero, V. Penso, R. Pini, Singular sets of maximal \( H \)-monotone operators. *In preparation*, 2016.

[5] Z.M. Balogh, A. Calogero, A. Kristály, Sharp comparison and maximum principles via horizontal normal mapping in Heisenberg groups. *Journal of Functional Analysis*, 269, 2669–2708, 2015.

[6] Z.M. Balogh, M. Rickly, Regularity of convex functions on Heisenberg groups. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 2, no. 4, 847–868, 2003.

[7] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie Groups and Potential Theory for their Sub–Laplacians. *Springer*, 2007.

[8] A. Calogero, G. Carcano, R. Pini, Twisted convex hulls in the Heisenberg group. *J. Convex Anal.*, 14:607–619, 2007.

[9] A. Calogero, R. Pini, \( c \)-horizontal convexity on Carnot groups. *J. Convex Anal.*, 19:541–567, 2012.

[10] A. Calogero, R. Pini, On Minty’s theorem in the Heisenberg group. *Nonlinear Analysis*, 104:12–20, 2014.

[11] J. Cygan, Subadditivity of homogeneous norms on certain nilpotent Lie groups. *Proc. Amer. Math. Soc.*, 83:69–79, 1981.

[12] D. Danielli, N. Garofalo, D.M. Nhieu, Notions of convexity in Carnot groups. *Comm. Anal. Geom. 11*, no. 2., 263–341, 2003.

[13] L.C. Evans and R.F. Gariepy, Measure Theory and Fine Properties of Functions. *CRC Press*, 1991.

[14] S. Hu, N.S. Papageorgiou, Generalizations of Browder’s degree theory. *Trans. Amer. Math. Soc.* 347, no. 1, 233–259, 1995.

[15] R.T. Rockafellar, R.J.-B. Wets, Variational Analysis, *Springer*, 2004.

[16] D.H. Wagner, Survey of measurable selection theorems. *SIAM J. Control and Optim.* 15, 859–893, 1977.