Banks as Tanks:
A Continuous-Time Model of Financial Clearing*

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Abstract

We present a simple model of clearing in financial networks in continuous time. In the model, firms (banks) are represented as reservoirs (tanks) with liquid (money) flowing in and out. This approach provides a simple recursive solution to a classical static model of financial clearing introduced by Eisenberg and Noe (2001). The dynamic structure of our model helps to answer other related questions and, potentially, opens the way to handle more complicated dynamic financial networks. Also, our approach provides a useful tool for solving nonlinear equations involving linear system and max min operations similar to the Bellman equation for the optimal stopping of Markov chains and other optimization problems.

Keywords: Financial networks, clearing vector, continuous time, quasi-linear optimization.

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1 Introduction

Modern financial systems are richly interconnected networks of various institutions (banks, firms, etc.), which depend on each other. With most assets of one firm being liabilities of other financial institutions, such a system is naturally prone to “systemic risk” of contagions, in which a default by one firm can lead to defaults of many other, otherwise sound, firms. Optimal regulation of such networks is a subject of intensive debate (see, e.g., Yellen, 2013; Acemoglu, Ozdaglar, and Tahbaz-Salehi, 2015).

In a seminal contribution, Eisenberg and Noe (2001) introduced a framework for the study of financial systems in distress. They consider an environment in which all participants (henceforth, firms) default within a single clearing mechanism, and demonstrate that there always exists a (generically) unique “clearing payment vector” that satisfies natural requirements for the outcome of a clearing procedure. There is a finite algorithm to calculate the clearing vector. The Eisenberg–Noe approach has been successfully extended to incorporate liabilities of different seniority and other financial instruments and has become one of the cornerstones in the analysis of systemic financial risk. See El Bitar, K., Kabanov, Y., Mokbel, R. (2017) for a partial survey of recent results, Suzuki T. (2002), as an example of an earlier paper independent of Eisenberg and Noe (2001), and the current book of T. Hurd (2016).

In this paper, we develop a continuous-time model of financial clearing. We think of interconnected firms as reservoirs (tanks) filled with money, which flows in and out through pipes connecting the reservoirs. In the case of default or, more generally, in any circumstances that require “simultaneous” clearing, the pipes open and, given the assumptions about the intensity of the flows similar to assumptions of Eisenberg and Noe (2001), the resulting distribution of liquid in reservoirs at a finite time $T_*$ corresponds to the clearing payment vector.

The idea to use tanks describing the input/output behavior of a system is not quite original. There is, e.g., a large area of so-called stochastic fluid models using similar construction. Nevertheless, to the best of our knowledge, all these models were used for quite different purposes and we do not use any results from this area.

Formally, the clearing vector $p = (p_1, ..., p_n)$ in our model and that of Eisenberg and
Noe (2001) (EN model) satisfies the same system of equations

\[ p_i = \min(c_i + \sum_j p_j q_{ji}, b_i) \text{ for each } i = 1, 2, ..., n, \]  

or, in matrix form, \( p = \min(c + Q^T p, b) \), where \( p, c, \) and \( b \) are column \( n \)-dimensional vectors, the minimum is taken component-wise, \( n \) is the number of firms in the network, \( b_i \) represents the absolute liability of firm \( i \) to all other firms, \( b_{ij} \) is the liability (debt) of firm \( i \) to firm \( j \), \( b_i = \sum_j b_{ij} \), \( c_i \) represents the cash position (operating cash flow) of firm \( i \), \( Q = \{q_{ij}\} \) is a stochastic matrix representing relative liabilities (debts) of firm \( i \) to other firms, \( q_{ij} = b_{ij}/b_i \), and, finally, \( T \) is the transposition symbol. Thus \((Q^T p)_i\) equals the total amount received by firm \( i \) from other firms.

The main distinction of our model from EN model and all other related models, is that our focus is on a real mechanism of simultaneous payments which are implied by equation (1). Briefly, we introduce an artificial time interval \((0, T^*_s)\), which corresponds to an “instant” when these payments (transactions) should be implemented and this gives an opportunity to study different facets of these transactions. Note that even if all firms finally have enough cash to pay their debts, i.e., when \( p_i = b_i \) for all \( i \), the question of implementation of these payments remains to be nontrivial when some \( c_i < b_i \). On the other hand, even if \( c_i \ll b_i \) for all \( i \), at least some debts are always paid.

Importantly, our model demonstrates that the Eisenberg and Noe (2001) clearing mechanism does not require sophisticated planning and hands-on management on behalf of the regulators. In fact, allowing financially constrained firms to repay their debts at the maximum-available speed without any liquidity injections or guarantees will result in the (generically) unique clearing payment vector. While the existing clearing mechanisms address a number of important issues, our results show that, at least theoretically, there is no need in detailed regulation in a situation of financial distress if the mechanism of resolving simultaneous payments is set right.

Second, our results demonstrate the power of the continuous-time approach to static maximization problems. Our continuous-time model generates a discrete dynamical system with the following recursive structure: at each moment, there is a linear system of equations that determines the parameters of the next state of the system and the moment when the system switches to this new state. In a finite number of steps, the system reaches its ultimate outcome. All calculations are based on one easily programmable step.
Equations similar to (1) are ubiquitous in quasi-linear optimization and appear naturally in a number of economic contexts. First, all linear models, e.g., the Leontief closed and open models, with extra constraints will satisfy this equation. For another application, consider the classical problem of optimal stopping of a Markov chain with transition matrix \( Q \), where a decision maker observing the chain has at each time point two options, either to continue or to stop. In such a setting, the Bellman (optimality) equation takes the form \( v = \max(c + Qv, b) \), where vector \( v = (v_i) \) is the value function, i.e., \( v_i \) is the maximal possible expected reward over all possible stopping times if the Markov chain starts at state \( i \), \( c = (c_1, ..., c_n) \) is a vector that consists of current rewards \( c_i \) in state \( i \), and \( b = (b_1, ..., b_n) \) is a vector of terminal rewards, where \( b_i \) is the terminal reward if the Markov chain stops at state \( i \). Of course in equation (1) matrix \( Q \) and vector \( b \) are related and in the Bellman equation they are not, but in Section 6 we discuss a possible generalization when this relationship can be weakened or removed. Note also that in the Bellman equation maximum is taken instead of minimum and the straight matrix \( Q \) is used instead of the transposed one. 

To keep our paper short, we do not give a survey of a huge body of literature on banks interconnectedness and refer the reader to the mentioned papers and a comprehensive paper of Chen N., Liu X., Yao D. Yao (2016).

The rest of the paper is organized as follows. In Section 2 we introduce our continuous-time model. Section 3 contains our main results, Theorem 1 and 2. In subsection 3.3 we clarify the situation when and why the solution of (1) may be not unique and discuss briefly the related so-called Enron effect. In Section 4 we give a heuristic proof of both theorems, and Section 5 provides the main proofs and computational formulas. In subsection 5.2 we provide solution for a simple example with five states. In this subsection we also discuss the similarities and differences between our Flow algorithm and Fictitious Defaults algorithm presented in EN paper. In Section 6 we discuss the specific modeling choices that we make and potential directions for future research. The Appendix contains several technical proofs.

\[ \text{1A simple recursive algorithm, the } \textit{State Elimination Algorithm}, \text{ to solve this Bellman equation was developed in Sonin (1999, 2006). Later, it was modified to calculate the classical and generalized Gittins indices in Sonin (2008).} \]
2 Setup

In the Eisenberg and Noe (2001) model, there is a financial system composed of \( n \) firms (banks, agents, etc) with firm \( i \) owing amount \( b_{ij} \geq 0 \) to firm \( j \). The total liability of firm \( i \) is given by \( b_i = \sum_j b_{ij} \), and the vector \( b = (b_1, b_2, ..., b_n) \) represents the liabilities (debts) of all firms. Let \( Q = \{q_{ij}\} \) be a stochastic matrix with row \( i \) reflecting the proportions of the liabilities of firm \( i \), i.e., \( q_{ij} = b_{ij}/b_i \), called the relative liabilities matrix. Let \( c_i \) be the initial cash position of firm \( i \). The clearing vector \( p = (p_1, p_2, ..., p_n) \), where \( p_i \) is the total amount paid by firm \( i \) to all other firms, should satisfy the following two properties:

(A) **Priority and proportionality of debt claims.** With the total payment, \( p_i \), the firm \( i \) pays to \( j \) the fraction \( p_i q_{ij} \) in such a way that either its total debts are paid or all of its resources are exhausted.

(B) **Limited liability.** The total payment of any firm should never exceed the cash flow available to the firm, i.e., the initial cash plus money received from other firms.

Properties (A) and (B), taken together, imply that the clearing vector \( p \) satisfies equation (1). The existence of such a vector \( p \) follows from the Knaster–Tarski lattice version of the fixed-point theorem.

In our continuous-time model, for simplicity later called BaT (Banks as Tanks) model, the initial matrix of liabilities \( B = \{b_{ij}\} \) and the initial vector of cash positions \( c = (c_1, ..., c_n) \) have the same meaning; matrix \( B \) generates the stochastic matrix \( Q = \{q_{ij}\} \) of relative liabilities. We also will use the following terminology and notation. If for some bank \( b_i = 0 \) but there is \( k \) with \( q_{ki} > 0 \), then we define \( q_{ii} = 1 \) and \( q_{ij} = 0 \) for all \( j \neq i \), otherwise such bank can be removed from consideration. Notation \( Q_B, B \subseteq J \) means matrix obtained from stochastic matrix \( Q \) by deleting all rows and columns not in \( B \), \( I_B \) means an identity matrix of corresponding dimension. Vector \( v_B \) is defined similarly. Matrix \( Q_B \) is called **transient** if \( Q_B \) defines a transient Markov chain on a set \( B \) or, in other words, matrix \( I_B - Q_B \) is invertible, i.e. \( \det(I_B - Q_B) \neq 0 \), **ergodic** if \( Q_B \) defines an ergodic Markov chain on a set \( B \), i.e., all states in \( B \) are communicating and there are no transitions to states outside of \( B \).

Since one of our goals is to obtain solution \( p \) of (1), we assume that (A) and (B) hold as well. However, we allow each firm \( i \)'s parameters \( b_i, c_i, p_i \) to depend on time, \( 0 \leq t < \infty \). Thus, firm \( i \) at time \( t \) is described by the vector \( x_i(t) = (b_i(t), c_i(t), p_i(t)) \),
representing its remaining debt, current cash position, and amount repaid up to moment $t$, respectively. The initial position of firm $i$ is $x_i(0) = (b_i, c_i, 0)$. Since $p_i(t) = b_i - b_i(t)$, we can monitor only the coordinates $b_i(t)$ and $c_i(t)$. Thus, the state of the whole system can be described by a $3n$-dimensional, or $2n$-dimensional, vector $x(t) = (x_i(t), i = 1, \ldots, n)$. Vector $b(t) = (b_1(t), \ldots, b_n(t))$ represents the remaining debts of all firms and vector $c(t) = (c_1(t), \ldots, c_n(t))$ represents the cash positions of all firms. The existence of a generically unique solution in BaT model will follow automatically, and the possibility of multiple solutions and their meaning will be discussed in subsection 3.3.

One can visualize the continuous-time BaT model as follows. Each firm $i$ is a reservoir (tank) filled by a liquid with initial volume (level) $c_i, i = 1, \ldots, n$. Each tank is connected to all other tanks by incoming and outgoing pipes. The maximum possible rate of the flow through each pipe $(i, j)$ is its capacity. To fully specify the continuous-time model, we add assumptions (C1) and (C2) to assumptions (A) and (B).

(C1) The capacity of out-pipe $(i, j)$ is $q_{ij}$.

Assumption (C1) means that at any point in time, firm $i$ can pay to firm $j$ with rate proportional to its total (initial) debt. Thus, the total capacity of all out-pipes from firm $i$ is 1. (In Section 6 we discuss the possibility of replacing 1 by any $m_i, 0 < m_i < \infty$, and disengaging matrix $Q$ from obligations $b_{ij}$.)

For assumption (C2), we need to introduce another notion, which will play a key role in our further analysis, that of the partition of the set of all firms $J = \{1, 2, \ldots, n\}$ into three groups, $P(t) = (J_+(t), J_0(t), J_*(t))$ at each moment $t \geq 0$. The first group of firms $J_+(t)$ is called positive, and consists of those firms that, at time $t$, still have outstanding debts and have positive cash values. The next group $J_0(t)$, called zero, are those firms that still have debts and have zero cash value (they might be still paying their debts with money flowing to them from other firms). The last group, $J_*(t)$, called absorbing, includes those firms that have paid out all of their debts (and are still, possibly, receiving money from other firms). Thus

$$
\begin{align*}
J_+(t) &= \{i : b_i(t) > 0, c_i(t) > 0\}; \\
J_0(t) &= \{i : b_i(t) > 0, c_i(t) = 0\}; \\
J_*(t) &= \{i : b_i(t) = 0\}. 
\end{align*}
$$

We call the position of firm $i$ in this partition the status of firm $i$ at time $t$. 
Assumption (C2) specifies out-rates \( u_i(t) \) for all firms at all times \( t \geq 0 \):

\[(C2) \quad \begin{align*}
\text{If } i \in J_+(t), \text{ then } u_i(t) &= 1; \\
\text{If } i \in J_*(t), \text{ then } u_i(t) &= 0; \\
\text{If } i \in J_0(t), \text{ then the out-rate } u_i(t) \text{ equals to the in-rate } n_i(t), \text{i.e.,} \\
u_i(t) &= n_i(t), \quad n_i(t) = \sum_j u_j(t)q_{ji} = \sum_{j \in J_+(t)} q_{ji} + \sum_{j \in J_0(t)} u_j(t)q_{ji}.
\end{align*}\]

The out-rates completely define the in-rates, which allows us to concentrate on the former. At the same time, while assumption (C1) implies that the out-rates satisfy \( u_i \leq 1 \), the in-rates \( n_i \) are defined by the columns of the stochastic matrix \( Q \), so they can potentially be less or more than one.

One of the crucial steps in solving the continuous-time model is finding the out-rates for the empty reservoirs \( J_0(t) \), i.e., finding a solution to a linear system given by the first equalities in (3). For this group, the in-rates are necessarily equal to the out-rates, so we call them the equilibrium rates. We later show that for firms in \( J_0(t) \), these rates exist, are unique, and are always less than one. The latter ensures that (C1) and (C2) are compatible. Then, assumptions (A), (B) and (C1), (C2) jointly determine the following simple dynamics for each vector \( x_i(t) = (b_i(t), c_i(t), p_i(t)) \):

\[
\begin{align*}
p_i(t) &= \int_0^t u_i(s)ds, \quad b_i(t) = b_i - p_i(t), \\
c_i(t) &= c_i + \sum_j p_j(t)q_{ji} - p_i(t) = c_i + \int_0^t d_i(s)ds, \quad d_i(s) = n_i(s) - u_i(s).
\end{align*}
\]

The term \( d_i(s) = n_i(s) - u_i(s) \), with the in-rate at time \( s \), \( n_i(s) = \sum_j u_j(s)q_{ji} \), is called the balance rate because it defines whether \( c_i(s) \) is growing, declining, or constant. We denote by \( d(s) = (d_i(s)) \) the corresponding \( n \)-dimensional vector. We will see later that \( u_i(t) \) and hence \( n_i(t) \) and \( d_i(t) \) are constant on the intervals, where none status is changed.

**Example 1.** We will illustrate the general construction by considering the following simplified example with \( n = 5 \) and three parameters \( a, b \) and \( \varepsilon \). Note that for a more realistic example even with \( n = 5 \) manual calculations can be tedious. This example should clarify when there is no uniqueness of solution of equation (1) and why we need some assumptions in Theorem 1. In this example \( c = (1, \varepsilon, \varepsilon, \varepsilon, 0), \ b_{12} = a, b_{15} = 1 - a, b_{25} = 2b, b_{23} = 2(1 - b), b_{34} = 3, b_{42} = 4, \) all other \( b_{ij} = 0 \). We assume that \( 0 \leq a, b \leq 1, \)
and \( \varepsilon \) is a small number, \( 0 \leq \varepsilon \leq 1/18 \), though of course the full study can be done for all nonnegative \( \varepsilon \) and \( c_1 \). Then vector \( b = (1, 2, 3, 4, 0) \) and matrix \( Q^T \) is given by the first matrix below. Two other matrices \( Q_i^T \), with \( b = 1/2 \), will be used in subsection 5.2 where we solve this example for different values of parameters, and represent parts of matrix \( Q^T \) with rows and columns not from \( D \) being removed. Note that, despite the extreme simplicity of this example, with state 5 being absorbing, equal cash positions \( c_2 = c_3 = c_4 \), and primitive structure of transitions, it is not easy by a glance to write the unique payment vector \( p = (1, 2/3 + 6\varepsilon, 1/3 + 4\varepsilon, 1/3 + 5\varepsilon, 0) \), e.g., for \( a = 1/3, b = 1/2, \varepsilon > 0 \). This solution works for all \( 0 \leq \varepsilon \leq 1/18 \). On the other hand, a reader can immediately see all multiple solutions \( p = (1, s, s, s, 0) \) for the case \( a = b = \varepsilon = 0 \), with any \( 0 \leq s \leq 2 \).

\[
Q^T = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 1 - b & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
1 - a & b & 0 & 0 & 1
\end{bmatrix}, \quad Q^T_{\{3,4\}} = \begin{bmatrix}
0 & 0 \\
1 & 0
\end{bmatrix}, \quad Q^T_{\{2,3,4\}} = \begin{bmatrix}
0 & 0 & 1 \\
1/2 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\]

3 Main Results

3.1 Flow in BaT model

Before formulating our main results, we discuss informally what happens after all pipes are open at time 0. First, given that the whole system is “self-contained”, i.e., the total amount of cash in the system remains the same, the money will flow until there is at least one firm with a positive status, i.e., having an outstanding debt and a positive cash reserve. During this period, the total amount of debt will be decreasing with at least the unit rate, and thus, since the total amount of debt is finite, the process ends in a finite time \( T^* = \min \{ t : J_+(t) = \emptyset \} \). At this moment \( \sum_i c_i = \sum_i c_i(T^*) = \sum_{i \in J_+(T^*)} c_i(T^*) \). The last equality holds because \( J_+(T^*) = \emptyset \) and \( c_i(T^*) = 0 \) for all \( i \in J_0(T^*) \). This equalities also immediately imply that the set \( J_+(T^*) \) is always not empty even if all \( c_i \ll b_i \), and at least one \( c_i > 0 \). (Some other global type balances, e.g. \( \sum_i b_i = \sum_i p_i(T^*) + \sum_{i \in J_0(T^*)} b_i(T^*) \), can be written to check calculations.)

The second, and slightly less obvious statement, is that the amounts paid by each firm,
i.e., \( p_i = p_i(T_*) \), \( i \in J \) at the end of the process, are solutions to the initial system (1). Indeed, the status of each firm \( i \) at this time is either zero or absorbing. If it is absorbing, then our process implies that the debt \( b_i \) is fully paid, \( p_i(T_*) = b_i \). If firm \( i \) at \( T_* \) has zero status, the debt \( b_i \) is not fully paid and \( c_i(T_*) = 0 \), but since payments at each time \( t \) are defined by out-rates proportional to capacities \( q_{ij} \) (see Assumptions (C1), and (C2)), the total payment \( p_i(T_*) \) is equal to the sum of the initial cash and the money obtained from other firms, i.e., \( c_i + \sum_j p_j(T_*)q_{ji} \). We call this solution of (1) the basic, or Flow, solution.

One of the important properties of the flow in the system is that any status change is irreversible, i.e., the following statement holds

**Proposition 1.** The only possible status change is: from the positive group \( J_+ \) to the absorbing group \( J_* \), or to the zero group \( J_0 \), and from the zero group \( J_0 \) to the absorbing group \( J_* \).

This property will follow easily from the Theorem 1 in this section.

Note that zero group \( J_0(t) \), generally variable in time, plays an important role in our model. Some of these firms may pay their debts in full staying in this group, which will result in their status changed to absorbing, and at the final stage, they may have positive cash values. “Real defaults” are those firms that end up in zero group, \( J_0(T_*) \), and thus will not pay their debts in full.

Note also that the evolution of the flow and partitions depend on initial vector \( c \), so it is possible, e.g., when not all \( c_i > 0 \), that some banks, among those with \( c_i = 0 \), are not involved in the flow at all, i.e., for them \( u_i(t) = 0 \) for all \( t \). Exactly this possibility is responsible for potential existence of multiple solutions and is discussed in Theorem 2. Briefly, every solution of (1) is a linear combination of a Flow solution with "Enron type" multiple solutions, see Example 1 above and subsection 3.3.

### 3.2 Theorem 1. Properties of the Flow Solution

Let us denote \( T_0 = 0 \) and \( T_k, k = 1, 2, ..., k_*, T_{k_*} = T_*, \) the times when at least one firm changes its status. Let the vector \( X_k = (T_k, P_k, b_k, c_k) \), where partition \( P_k = P(T_k) = (J_{+,k}, J_{0,k}, J_{*,k}) \), vectors \( b_k = (b_i(T_k)) \), \( c_k = (c_i(T_k)) \), \( i \in J \). Thus, \( X_k \) contains all the information about the system at moment \( T_k \).

To slightly simplify our presentation, we will assume that \( c_i > 0 \) for all \( i \), i.e., \( J_0 = \emptyset \).
Values of $u_i$ for $k = 0$ when $J_0(0) \neq \emptyset$, can be obtained using the modified partition $P_0 = P(0)$, (see subsection 5.2 and Big Bang effect in Section 6).

**Theorem 1.** (a) There is a finite time $T_* = \min \{ t : J_+(t) = \emptyset \}$, when all flows stop. Vector $p(T_*)$ solves equation (1), i.e., it is the clearing payment vector in both the EN and the BaT models. The time interval $[0, T_*)$ consists of a finite number $k_*$ of half-intervals $\Delta_k = [T_k, T_{k+1})$, $0 \leq k < k_*$, $T_0 = 0$, $T_{k_*} = T_*$. Times $T_k$ are the only times when at least one firm changes its status.

(b) Vector $X_k$ uniquely defines constant out-rates $u_{i,k}$ on interval $\Delta_k$, and vector $X_{k+1}$, $0 \leq k < k_*$. These out-rates, suppressing further indices $k$, $u_{i,k} = u_i$, are: $u_i = 1$ if $i \in J_{+,k}$, $u_i = 0$ if $i \in J_{+,k}$ and if $i \in J_{0,k}$ and $0 < k$, then $u_i = v_i$, where vector $v = (v_i)$ is obtained as a solution of a linear nonhomogeneous system

$$v = e_k + Q^T_k v, \text{ i.e. } v = (I_k - Q^T_k)^{-1} e_k,$$

vector $e_k = (e_{i,k})$, $e_{i,k} = \sum_{j \in J_{+,k}} q_{ji}, i \in J_{0,k}$, matrix $Q^T_k = Q^T_{J_{0,k}}$, i.e., is obtained from matrix $Q^T$ by deleting all rows $j$ and columns $j, j \notin J_{0,k}$.

(c) The following Monotonicity Property holds: solution $v = (v_i)$ of system (4) satisfies $0 < v_i < 1$; all the out-rates, and therefore all the in-rates, in all tanks are nonincreasing in $k$, i.e.,

$$u_{i,k} \geq u_{i,k+1}, \quad n_{i,k} \geq n_{i,k+1}, \quad i \in J, \ k = 0, 1, ...$$

Note that part (c) implies that if $u_i(t) < 1$ or $u_i(t) = 0$ for some $t$ and $i$, then $u_i(t') < 1, u_i(t') = 0$ for all $t' > t$ and hence any status change is irreversible, i.e., Proposition 1, formulated in the previous subsection, holds.

Equation (6) is referred to as the equilibrium equation and the corresponding $m$-dimensional vector $e_k$ with $m = |J_{0,k}|$ as the input vector on the interval $\Delta_k$.

### 3.3 Theorem 2. Uniqueness and Enron type solutions in Equation (1)

In our model we obtain a specific, basic solution of equation (1), whereas at the same time in Eisenberg and Noe (2001) and some other papers a possibility of multiple solutions is
mentioned. We will clarify this situation using simple facts from Markov chain theory. This makes the situation rather transparent.

We agreed before that if \( b_i = 0 \), i.e. \( i \in J_*(0) \), we assume that for this \( i \) there is at least one \( k \in J, b_{ki} > 0 \), and the \( i \)-th row of matrix \( Q \) consists of \( q_{ii} = 1 \) and \( q_{ij} = 0, i \neq j \). Note that matrix \( Q \) is defined only by liabilities matrix \( (b_{ij}) \) but the flow and solution(s) of equation \((\Pi)\) depend also on assets vector \( c = (c_1, ..., c_n) \). We are going to analyze the situation for all vectors \( c \).

Given vector \( c \), let us define a set of active banks \( A \equiv A(c) = A_1 \cup A_2 \cup ..., \) where \( A_1 = \{ i : c_i > 0 \}, A_2 = \{ i : \text{there is } k \in A_1, b_{ki} > 0 \}, \) etc. In other words, given \( c \), bank \( i \) is active if it will participate in the flow, i.e. if \( u_i(t) > 0 \) for some \( t \), or, if \( i \in J_*(0) \), there is \( k \) such that \( u_k(t) > 0 \) and \( q_{ki} > 0 \). Note also that if \( u_i(t) > 0 \) for some \( t \), then \( u_i(0) > 0 \). Thus, all banks in \( J_+(0) \) are active. Banks in \( J_0(0) \) and \( J_*(0) \) may be active or nonactive. Active banks form a closed set in terms of matrix \( Q \), i.e., submatrix \( Q_A \) is a stochastic matrix, and can be partitioned as \( A = A_* + R \), where \( A_* = A \cap J_*(0) \) is a set of absorbing states for matrix \( Q_A \) and \( R = A \setminus A_* \). The definition of set \( A \) implies that \( R \) is a union of transient states for matrix \( Q_A \) and possibly of some number of ergodic subclasses, i.e., closed subsets of communicating states. Since \( b_i > 0 \) for all \( i \in R \), there is no absorbing states in \( R \).

The set of nonactive banks \( A' = J \setminus A \) can be described as \( A' = \{i : c_i = 0, \text{ and } q_{ki} = 0 \text{ for all } k \in A\} \), and therefore is a subset of \( J_0(0) \) but since the inflow into these banks is zero, Flow solution has \( p_i = 0 \) for all \( i \in A' \). Set \( A' \) can be partitioned into three (possibly empty) sets: \( A' = A'_* + C + S \), defined as follows. The set \( A'_* = A' \cap J_*(0) \) is a set of absorbing states for matrix \( Q_{A'} \). Since \( b_i > 0 \) for all \( i \in A' \setminus A'_* \), there is no absorbing states in this set and therefore \( A' \setminus A'_* = C + S \), where \( C \) is the set of transient states, with transitions to set \( A \) also possible, and \( S \) is the union of ergodic subclasses. We call these subclasses in \( S \) swamps, (vortexes). Each swamp (vortex) \( S_* \), as an ergodic subclass with stochastic matrix \( Q^T_{S_*} \), has a unique invariant distribution \( \pi_* \) with \( \pi_{*i} > 0, i \in S_* \). Thus, vector \( \pi_* \) satisfies the equation \( \pi_* = Q^T_{S_*} \pi_* \) and if we consider \( \pi_* \) as \( n \)-dimensional vector with \( \pi_{*i} = 0 \) for all \( i \not\in S_* \), then also \( \pi_* = Q^T \pi_* \). If \( m \) is selected in such a way that \( m \cdot \pi_{*i} \leq b_i \), for all \( i \), i.e., if \( m = \min_{i \in S} b_i/\pi_{*i} \), then \( m > 0 \) and vector \( \pi_* = m \cdot \pi_* \) solves also equation \((\Pi)\) with values \( \pi_{*i} \leq b_i \) for all \( i \) and thus
satisfies the equality $p_{*i} = (Q^T p_*)_i$. If the number $k$ of the ergodic subclasses is more than one, they have an empty intersection and therefore any linear combination of such solutions with coefficients $0 \leq s_k \leq 1$ is also a solution of (1) with coordinates $p_{*i} \leq b_i$ for $i \in S$ and $p_{*i} = 0$ for $i \notin S$.

Thus, given an initial assets vector $c$, a swamp is a set, where all its members are: nonactive, have no cash, and have positive debts but only to themselves. As a result, they have a possibility to “run money in a circle”, paying partially their debts, at the same time keeping zero cash positions. These transactions can be considered also as a partial debts restructuring, i.e. a transformation of an initial debt vector $b_1$ into a new debt vector $b_2 \leq b_1$. We have shown above that multiple solutions do exist if there is at least one swamp among nonactive banks. Later we prove also the “only if” part.

Generally, the simplest examples of swamps are situations where there are indices $i$ and $j$ such that $b_{ij} = b_i > 0, b_{ji} = b_j > 0$, and $b_{si} = b_{sj} = b_{is} = b_{js} = 0$ for all $s \neq i, j$, or when there are three indices $i, j$ and $k$ with similar properties, as in Example 1, where we obtain a swamp $\{2, 3, 4\}$ with two active banks $\{1, 5\}$, when $a = b = \varepsilon = 0$.

Though swamp seems a mathematical abstraction, it nevertheless brings to mind real situations of Enron type, where a financial or commercial enterprise is seeking to inflate its value by creating a network of artificial debtors and reporting its high value based on these data, neglecting to mention its own obligations. “Special Purpose Entities were created to mask significant liabilities from Enron’s financial statements. These entities made Enron seem more profitable than it actually was, ...” Dan Ackman, “Enron the Incredible”. Forbes.com, Jan. 17, 2002.

Note, that by Theorem 1, given $c$, with at least one $c_i > 0$, there is always a nontrivial Flow solution of (1) with values $p_i = 0$ for $i \notin A(c)$. The full formulation of results about uniqueness is given by

**Theorem 2**  Given vector $c$ and corresponding active set $A = A(c)$, there is a unique basic (flow) solution $p$ with $p_i = 0$ for $i \notin A(c)$. If set $S$ in the decomposition of nonactive set $A' = A'_* + C + S$ is an empty set, then the basic solution is the only solution of (1). If set $S \neq \emptyset$, i.e., if there is at least one swamp, then there are multiple solutions $p'$ of the form $p' = p + \sum k s_k p_{sk}$, where $p$ is basic solution, $0 \leq s_k \leq 1$, and each $p_{sk}$ is obtained through an invariant distribution $\pi_{*k}$ for a corresponding ergodic subclass.
Note, that our basic solution corresponds to the least, and solution with all \( s_k = 1 \) in Theorem 2 to the greatest clearing payment vector in EN 2001 and El Bitar at all 2016 papers. They coincide if there is no swamps.

4 Proofs of Theorems 1 and 2

We give the formal proof of parts (a) and (b) of Theorem 1, which are relatively straightforward, in the next Section. The monotonicity property, part (c), is more technical and is relegated to the Appendix. Here, we provide a heuristic proof of Theorem 1 which clarifies the dynamic evolution of the status groups \( J_+(t), J_0(t) \) and \( J_*(t) \), and a full proof of Theorem 2.

We start with some technicalities. Given a transient matrix \( Q \), i.e. a matrix with invertible \( (I - Q) \), matrix \( N = (I - Q)^{-1} \) is called the fundamental matrix for the transient Markov chain defined by matrix \( Q \). The entries of this matrix \( n(x, y) \) have a well-known probabilistic interpretation as the expected number of visits to \( y \), starting in \( x \) before the exit from the corresponding set. Matrix \( N \) satisfies also the equalities

\[
N = \sum_{n=0}^{\infty} Q^n = (I - Q)^{-1}; \quad N = I + QN = I + NQ. \tag{8}
\]

Because of equation (1) we are working mainly with matrices \( Q^T \) and \( Q_B^T \) for some \( B \subseteq J \). If matrix \( Q_B \) is transient, i.e., \( (I - Q_B) \) is invertible, then matrix \( (I - Q_B^T) \) is also invertible. Lemma 1 collects some standard results about matrices \( Q_B^T \) except point (d) that seems obvious and can be relatively easily proved.

**Lemma 1** If matrix \( Q_B \) is ergodic, then

a) the equation \( v = Q_B^Tv \) has a nontrivial solution, an invariant distribution \( \pi_* \).

If matrix \( Q_B \) is transient, then

b) the equation \( v = Q_B^Tv \) has only trivial solution. The equation \( v = e + Q_B^Tv, e \neq 0 \), has a unique solution, \( v = (I - Q_B^T)^{-1}e = N_B^T e \).

(c) Two solutions \( v \) and \( v' \) of equations \( v = e + Q_B^Tv, \) and \( v' = e' + Q_B^Tv' \), satisfy \( v \leq v' \) if \( e \leq e' \).

(d) Suppose the solution of an equation \( v = e + Q_B^Tv \), satisfies the inequalities \( v_i \leq b_i \) for all \( i \). Then the solution of an equation \( u = \min(e + Q_B^Tu, b) \) coincides with \( v \).
Now, let us look at the global relations between the three status groups evolving in
time. First, since every out-rate is someone’s in-rate, the total balance of outflows (the
sum of out-rates) and inflows (the sum of in-rates) for the whole system is always zero,
i.e., at any time $t$ we have
\[ \sum_{j \in J} u_i = \sum_{j \in J} n_i, \tag{9} \]
or, equivalently, $\sum_{j \in J} d_i = 0$, where $d_i = n_i - u_i$. Note that $d_i = 0$ for all $i \in J_0(t)$ at
any moment $t$, so the total inflow in this group is always equal to the total outflow, i.e.,
the total balance is zero. Since all $u_i = 0$ in $J_*(t)$ group, outflow in this group is always
zero, and inflow is always nonnegative. When it is positive, then (9) implies that the
total balance for the positive group is negative, so the total volume (sum of cash) here is
decreasing, going partly to the absorbing group and partly to the zero group. Potentially,
there are four possibilities of status change for a firm: 1) from the positive group $J_+$ to
the absorbing group $J_*$, 2) from $J_+$ to the zero group $J_0$, 3) from $J_0$ to $J_*$, and, 4)from
$J_0$ to $J_+$. Proposition 1, based on part (c) of Theorem 1, implies that this last transition,
from $J_0$ to $J_+$, is impossible.

The heuristic idea of the proof of part (c) of Theorem 1 is as follows: let $r_k = r$ be
the index of the bank that changed its status at time $T$. In cases 1) and 3) the input
from $r$ to every firm drops to zero and by point c) of Lemma 1 the equilibrium rates will
drop. Then all in-rates (and out-rates) in all firms will drop. If case 2), the move of firm
$r$ means that $d_r < 0$, i.e., $n_r < 1$ and, as a result of increasing the size of the zero group,
input from $r$ to every firm drops to the lower, though positive, value. Thus, equilibrium
out-rates will drop for all firms in the zero group, and hence in-rates (and out-rates) in
all firms will drop. These considerations are formalized in the Appendix.

Proof of Theorem 2. In the previous section we explained and proved the nonunique-
ness part of Theorem 2. It remains to show that, given vector $c$: 1) the absence of swamps
in the set of nonactive states $A'$ implies that any solution of (11) has $p_i = 0$ for all $i \in A'$,
and 2) a solution of (11), such that $p_i = 0$ for all $i \in A'$, is unique, and therefore is given
by Flow solution.
1) First, let us show that for any solution of (11), $p_i = 0$ for all $i \in C$, where $C$ is the set of
transient states in the decomposition $A' = A_* + C + S$ of the set of nonactive banks. Since
c_i = 0 for $i \in C$, for such $i$ and a solution of (11), we have $p_i = \min((Q^Tp)_i, b_i)$. Since
$q_{ki} = 0$ for all $k \in A$, the equality for these $p_i$ and $i$ can be written as $p_i = \min((Q_C^T p)_i, b_i)$. But since $Q_C$ is a transient matrix, an equation $p = Q_C^T p$ has only a trivial solution and then, using point (d) of Lemma 1 we obtain that $p_i = 0$ for all $i \in C$. We also always have $p_i = 0$ for $i \in A'$. Therefore, if $S = \emptyset$, then $p_i = 0$ for all $i \in A'$.

The proof of 2) follows from the next lemma, that seems rather obvious. We omit the formal proof of this lemma.

**Lemma 2** If given vectors $c_1$ and $c_2$, vectors $p_1$ and $p_2$ are the solutions of (1) and $c_2 \leq c_1$, then $p_2 \leq p_1$ for all $i \in A(c_1)$.

Note also that if $c_2 \leq c_1$, then $A(c_2) \subseteq A(c_1)$, and if there are swamps in $A'(c_1)$ then in this set it is possible that $p_{1i} = 0$ and $p_{2i} > 0$. Lemma 2 immediately implies that if there are two solutions of equation (1) with the same active set, then they coincide on this set. This proves the remaining part of Theorem 2.

In the next section we prove Theorem 1 and obtain simple computational formulas describing the evolution of the system in the continuous-time model.

## 5 Discrete Dynamical System

### 5.1 Proof of part (b) of Theorem 1

One of the main advantages of the continuous-time BaT model is that it immediately generates a homogeneous discrete dynamical system $(X_k)$, where time moments $k = 0, 1, ..., k_s - 1$ correspond to moments of status change $T_k$ in the continuous-time model, and vector $X_k$ contains all information about the system at this moment, i.e., $X_k = (T_k, P_k = P(T_k), b_{i,k} = b_i(T_k), c_{i,k} = c_i(T_k), i \in J)$. Part (b) of Theorem 1 can be restated as follows: state $X_k$ uniquely defines the next state $X_{k+1} = G(X_k), k = 0, 1, ..., k_s - 1$.

To prove part (b), it is sufficient to describe this mapping (transformation) $G$. Since the system is homogenous, we need to program only one step of this evolution.

If, on some interval $\Delta_k = [T_k, T_{k+1})$, the out-rates vector $u = (u_i, i \in J)$ is known, then Assumptions (A), (B) and (C1), (C2) determine the following simple dynamics for each vector $x_i(t) = (b_i(t), c_i(t), p_i(t))$ on this interval. To simplify notation, we suppress further indices $k$, denoting $u_{i,k} = u_i$, $\Delta_k = \Delta$, $T_k = T$, $T_{k+1} = T'$, partitions $P_k = P$, $P_{k+1} = P'$,
current state as $X_k = X$ and $X_{k+1} = G(X) = X'$. The formulas below are obtained from (4) with constant values $u_j$, and as we mentioned before, we can only use $b_i(t)$ and $c_i(t)$, since $p_i(t) = b_i - b_i(t)$:

$$b_i(t) = b_i(T) - u_i(t - T),$$
$$c_i(t) = c_i + \sum_j p_j(t)q_{ji} - p_i(t) = c_i(T) + d_i(t - T), d_i = n_i - u_i,$$

$$n_i = \sum_j u_jq_{ji} = \sum_{j \in J_+} q_{ji} + \sum_{j \in J_0} u_jq_{ji}.$$  

(10)

We recall that the $n$-dimensional vector $d = (d_i)$ is called the balance rate vector.

The time of the next status change $T'$ is the first time when one of those quantities $b_i(t), c_i(t)$, which were positive at moment $T$, will reach zero for the first time. Assuming current out-rates $u_i$, we denote the potential moment $T + s_i$ when firm $i$ will repay its debt, and the potential moment $T + t_i$ when firm $i$ reaches the zero cash position. Equations (10) immediately imply that

$$s_i = b_i(T)/u_i, \text{ if } b_i > 0,$$
$$t_i = -c_i(T)/d_i \text{ if } c_i > 0, d_i < 0, \ldots = +\infty \text{ if } c_i = 0 \text{ or } d_i \geq 0,$$

$$T' = T + t', \ t' = \min_i (s_i, t_i).$$  

(11)

Note that, although in (11) the $s_i$ are calculated only for $i \in J_+ \cup J_0$ and $t_i$ only for $i \in J_+$, for computational purposes it is more convenient to consider $s = (s_i)$ and $t = (t_i)$ as $n$-dimensional vectors, taking all missing coordinates equal to $+\infty$. Note also that for $i \in J_0$ we have $c_i(t) = 0$ for all $t \in \Delta$, so for them $t_i = +\infty$.

Without loss of generality, we can assume that at each moment only one firm can change its status. Since there is no moment in time when two or more firms change their status, we can focus on this generic case. Accommodating a general case would require minor modifications.

Let $r_k = r$ be the index of the firm that changed its status at $T'$. As we discussed above, there are four possible evolutions of a firm’s status: 1) from the positive group $J_+$ to the absorbing group $J_*$, 2) from $J_+$ to the zero group $J_0$, 3) from $J_0$ to $J_*$, and, finally, 4) from $J_0$ to $J_+$. Part (c) of Theorem 1 implies that this transition, from $J_0$ to $J_+$, is impossible. This is what happens in each of cases 1)–3):
1) In this case, moment $t' = s_r$, positive firm $r$ becomes absorbing, and we obtain the state $X' = (T', P', b'_i = b_i(T'), c'_i = c_i(T'))$, where $P' = P(T') = (J_+ \setminus r, J_0, J_+ \cup r)$.

2) In this case, moment $t' = t_r$, positive firm $r$ becomes a zero firm, and we obtain the state $X' = (T', P', b'_i, c'_i)$, where $P' = (J_+ \setminus r, J_0 \cup r, J_+ \cup r)$.

3) In this case, moment $t' = s_r$, zero firm $r$ becomes absorbing, and we obtain the state $X' = (T', P', b'_i, c'_i)$, where $P' = (J_+ \setminus r, J_0 \cup r, J_+ \cup r)$.

In all three cases, the values $b'_i = b_i(T')$ and $c'_i = c_i(T')$ for all $i$ are defined by the formulas (10) with $t = T'$.

Now we explain how to obtain the out-rates vector $u = (u_i, i = 1, ..., n)$ for each interval $\Delta_k = [T_k, T_{k+1})$, $k = 0, 1, 2, ...$. For the positive and absorbing groups, they are equal to 1 and 0 respectively, so we have to discuss only the zero group. When $k = 0$, the initial state is $X(0) = (T_0 = 0, P_0 = P(T_0), x_i(0) = (b_i, c_i, 0))$. In this section we consider only the case when $J_0(0) = \emptyset$, leaving the more complicated case when $J_0(0) \neq \emptyset$ to the Appendix. Note that we can always obtain the case $J_0 = \emptyset$ assuming that initially a small amount of money $\varepsilon > 0$ is given to each firm.

When, at moment $t = 0$, all firms have positive cash, i.e., our initial partition is $J_+ = J, J_0 = \emptyset$, then, by Assumption (C2), all out-rates $u_i = 1$, and for each $i$ on some, perhaps small, interval $(0, s)$ we have simple relations obtained from formula (10) with all $u_i = 1$. Then, we can use (11) with $u_i = 1$ to obtain $T_1$, and after that, as was described above in mapping $G$, we can obtain state $X_1 = X(T_1)$. Now it is possible that $J_0(T_1) \neq \emptyset$, but now we can turn to the case when $k > 0$. Note that formula (6) from Theorem 1 part (b) is nothing more than a rewritten formula (3) of Assumption (C2) and by point (b) of Lemma 1 the solution of (6) is given by the equality $u = N^T_k e_{+,k}$. The only remaining question is properties in part (c) of Theorem 1. A heuristic explanation of its validity was given in the previous section and a rigorous proof is given in the Appendix.

### 5.2 Solution of Example 1. Comparison of Fictitious Defaults and Flow algorithms

We will consider three versions of Example 1 for different values of parameters. More detailed calculations for this and for more general cases can be found in Presman 2017. Note that for all $a > 0$ or $\varepsilon > 0$ all banks are active. If $a = \varepsilon = 0$, then only $\{1, 5\}$ are
active, and there are no swamps. Only if \( a = b = 0, \varepsilon > 0 \), then there is a swamp \( \{2, 3, 4\} \).

First, we apply the Flow algorithm implied by BaT model and after that the Fictitious Defaults (FD) algorithm developed in EN 2001. We describe FD algorithm in a simple and concise form that will reveal the similarities and differences with Flow algorithm.

Applying Flow algorithm, we prefer to use coordinate presentation instead of matrix equations, though we will indicate the appropriate matrices. The reason for that is that the program implementation and update can be done for each bank in parallel, i.e., given its status and \( n_i \), one can obtain \( u_i \) and critical numbers \( s_i, t_i \) without extra information about other banks. After that one can obtain the time of the next status change \( T' \), see the last equality in (11), and the next state of the system.

**Example 1A**, \( a = 1/3, b = 1/2, \varepsilon > 0 \). Then \( P_0 = (\{1, 2, 3, 4\}; 0; 5) \), i.e. \( J_0 = \emptyset \), and therefore this is a regular case covered by Theorem 1. Since 5 is an absorbing bank with \( u_5 = 0 \), we follow further only the first four coordinates of all vectors. Then vector \( u = (1, 1, 1, 1) \), vector \( n = (0, a + 1, 1/2, 1) \) and vector \( d = n - u \) is: \((-1, a, -1/2, 0)\). Thus on the first interval \([0, T_1]\) we have \( b_i(t) = b_i - t \) for all \( i \), \( c_1(t) = 1 - t, c_2(t) = \varepsilon + at, c_3(t) = \varepsilon - t/2, c_4(t) = c_4 \). Then for small \( \varepsilon \), \( T_1 \) is the first moment when \( c_3(t) \) hits zero and hence \( T_1 = 2\varepsilon \).

Now \( P_1 = P(T_1) = (\{1, 2, 4\}; 3; 5) \), vector \( u = (1, 1, u_3 = n_3, 1) \), vector \( n = (0, a + 1, 1/2, n_4 = u_3) \), and therefore \( u_3 = n_3 = 1/2 \) and thus vector \( d = (-1, a, 0, -1/2) \). Hence on the second interval \([T_1, T_2]\) we have: \( b_i(t) = b_i(T_1) - u_i(t - T_1), c_1(t) = c_1 - t, c_2(t) = c_2(T_1) + a(t - T_1), c_3(t) = 0, c_4(t) = \varepsilon - (t - T_1)/2 \). Then \( T_2 \) is the first moment when one of the terms \( b_i(t), c_1(t), c_2(t), c_4(t) \) hits zero. Since \( \varepsilon \) is small, \( T_2 = \min(t : c_4(t) = 0) \), i.e. \( T_2 = 4\varepsilon \).

Now \( P_2 = P(T_2) = (\{1, 2\}; \{3, 4\}; 5) \). Continuing in a similar fashion, we obtain vector \( u = (1, 1, 1/2, 1/2) \). (Formally we have to solve equation (6) with matrix \( Q_{3,4}^T \), second matrix in (5), and vector \( e = (1/2, 0) \).) Then vector \( d = (-1, -1/6, 0, 0) \), and \( T_3 = \min(t : c_2(t) = 0) = 18\varepsilon \).

Now \( P_3 = P(T_3) = (1; \{2, 3, 4\}; 5) \) and \( u_1 = 1 \). To obtain equilibrium rates \( u_2, u_3, u_4 \) we have the equalities: \( u_2 = n_2 = 1/3 + u_4, u_3 = n_3 = u_2/2, u_4 = n_4 = u_3 \). (Formally we have to solve equation (5) with matrix \( Q_{2,3,4}^T \), third matrix in (5), and vector \( e = (1/3, 0, 0) \).) Then vector \( u = (1, 2/3, 1/3, 1/3) \) and \( T_4 = \min(t : c_1(t) = b_1(t) = 0) = T_* = 1. \)
Therefore, the final partition is \( P(T_*) = (\emptyset; \{2,3,4\}; \{1,5\}) \) and tracking the values of \( p_i(T_k) \) for \( k = 1,2,3,4 \), it is easy to obtain the unique solution \( p = (1, 2/3 + 6\varepsilon, 1/3 + 4\varepsilon, 1/3 + 5\varepsilon, 0) \). This solution works for all \( 0 < \varepsilon \leq 1/18 \). We can check this solution using equation (1) with \( p_1 = 1, p_5 = 0 \) and \( p_i = c_i + (Q^T)\varepsilon < b_i \) for \( i = 2, 3, 4 \). The final positions are \( b(T_*) = (0, 4/3 - 6\varepsilon, 8/3 - 4\varepsilon, 11/3 - 5\varepsilon, 0) \) and \( c(T_*) = (0, 0, 0, 0, 1 + 3\varepsilon) \).

For \( \varepsilon = 1/18 \) vector \( p = (1, 1, 10/18, 11/18, 0) \).

**Example 1B.** Let \( \varepsilon = 0, a = 2/3, b = 1/2 \). This case is not covered by Theorem 1 since \( J_0 = \{2,3,4\} \) is not an empty set. Formally at time 0 we have \( P_0 = (1; \{2,3,4\}; 5) \).

If we were to try to solve equation (5) for the equilibrium rates \( u_2,u_3,u_4 \), we would have a system \( u_2 = n_2 = a + u_4, u_3 = n_3 = u_2/2, u_4 = n_4 = u_3 \). The unique solution is \( u_2 = 2/3, u_3 = u_4 = 1/3 \), but then \( n_2 = 4/3, d_2 = 4/3 - 2/3 = 2/3 \) and therefore the value \( c_2(t) > 0 \) for all \( t > 0 \), i.e., bank 2 is instantly no longer in \( J_0 \). It means that in this case at moment 0 bank 2 should be classified as positive and we have to use the modified partition \( P(0) = (\{1,2\}, \{3,4\}; 5) \). Then \( u_4 = u_2 = 1 \) and for the states 3, 4 we have a system \( u_3 = n_3 = 1/2, u_4 = n_4 = u_3 \). Then \( d_2 = n_2 - u_2 = 2/3 + 1/2 - 1 = 1/6 \) and therefore the value \( c_3(t) = t/6 > 0 \) is growing from 0. The moment \( T_1 \) is the moment when \( c_1(t) = b_1(t) \) hits zero, and hence \( T_1 = 1 \). At this moment \( c_2(t) = 1/6 \) and on the next and the last interval \( [T_1, T_2] \) we have: \( u_1 = 0, u_2 = 1, u_3 = u_4 = 1/2, d_2 = 1/2 - 1 = -1/2 \) and \( T_2 = \min(t: c_2(t) = 1/6 - (T_2 - T_1)/2 = 0) \), i.e., \( T_* = T_2 = T_1 + 1/3 = 4/3 \). The final partition is \( P(T_*) = (\emptyset; \{2,3,4\}; \{1,5\}) \), and the unique solution \( p = (1, 4/3, 2/3, 2/3, 0) \).

We explain how to modify initial partition \( P_0 \) when \( J_0 \) is not an empty set for the general case in the Appendix.

**Example 1C.** In cases 1A and 1B above we had a unique solution. Now we modify our example to obtain a non unique payment vector, though rather trivial. Let \( b_{15} = 1, b_{23} = 2, b_{34} = 3, b_{42} = 4 \), all other \( b_{ij} = 0 \), and \( c = (1, 0, 0, 0, 0) \). Thus vector \( b = (1, 2, 3, 4, 0) \) and we have matrix \( Q^T \) with parameters \( a = b = 0 \). Now set \( \{2,3,4\} \) is a swamp. The banks in this set have no cash, have debts only to themselves, and nobody outside has debts to them. It is easy to see that then any vector \( (1,s,s,s,0) \) with \( 0 \leq s \leq 2 = \min(b_i, i = 2, 3, 4) \) is a payment vector. This is true, of course, because the equation \( (I - Q^T)p = p \) has a nontrivial solution \( (0,1,1,1,0) \). If we change the debts for bank 4 to, e.g., \( b_{42} = b_{43} = 2 \), then we obtain payment vectors of type \( (1, s, 2s, 2s, 0) \) corresponding
to the invariant distribution for the corresponding Markov chain with $0 \leq s \leq 3/2$. Note
that now we have $s \leq 3/2$, because of constraints $s \leq 2, 2s \leq 3, 2s \leq 4$ given by debts.

Before to explain how the Example 1A would be solved by the Fictitious Defaults (FD) algorithm presented in EN paper, we prefer to describe this algorithm in a brief, compact form for the case when all $c_i > 0$. First, given vector $p = (p_i), p_i \geq 0$, we define a mapping $\Phi : p \rightarrow p'$ by formula $p' = \min(c + (Q^Tp, b)$. Any solution of (1) is a fixed point of this mapping. To proceed, we need one simple proposition. As before we will keep track of only first four coordinates. The $n$-dimensional vector $p$ is a solution of (1) and coincides with $p(0)$, where $p(0) = b, b$ is “input” vector $e(1) = (e_i(1))$. Note, that according to Proposition 2, we have $r_i(1) \leq b_i, i \in D_1$. Now extend vector $r(1)$ to n-dimensional vector $s(1), s_i(1) = r_i(1), i \in D(1), s_i(1) = b_i, i \notin D(1)$, and obtain vector $p(2) = \Phi(s(1))$. Proposition 2 implies that $p_\ast \leq p(2) \leq p(1)$. Let $D(2) = \{i : p_i(2) < b_i\}$. Since $p(2) \leq p(1)$, we have $D(2) \supseteq D(1)$. If $D(2) \setminus D(1) = \emptyset$, stop, vector $p(2)$ is a solution of (1) and coincides with $p_\ast$. Otherwise, make the following transformation of vector $p(k)$ to $p(k + 1)$. First, solve linear system on a set $D_1$

$$(I_1 - Q_1^T)r(1) = e_1, \quad (e_i(1)) = c_i + \sum_{j \notin D_1} q_{ij}b_j, i \in D_1$$

(12) i.e., obtain vector $r(1) = (I_1 - Q_1^T)^{-1}e_1$, where matrix $Q_1 = Q_{D_1}$, and “input” vector $e(1) = (e_i(1))$. Note, that according to Proposition 2, we have $r_i(1) \leq b_i, i \in D_1$. Now extend vector $r(1)$ to n-dimensional vector $s(1), s_i(1) = r_i(1), i \in D(1), s_i(1) = b_i, i \notin D(1)$, and obtain vector $p(2) = \Phi(s(1))$. Proposition 2 implies that $p_\ast \leq p(2) \leq p(1)$. Let $D(2) = \{i : p_i(2) < b_i\}$. Since $p(2) \leq p(1)$, we have $D(2) \supseteq D(1)$. If $D(2) \setminus D(1) = \emptyset$, stop, vector $p(2)$ is a solution of (1) and coincides with $p_\ast$. Otherwise, repeat the previous step, i.e., solve linear system on a set $D_2$, etc. Since sequence of sets $D(k)$ is strictly increasing, the basic payment vector will be obtained in at most $n$ steps. We omit an easy proof of optimality.

**Now we present solution of Example 1A using FD algorithm.** We have $p(0) = b = (1, 2, 3, 4, 0)$. As before we will keep track of only first four coordinates. Then $p(1) = \Phi(p(0)) = \min(c + Q^Tb, b) = \min((1, \varepsilon + 1/3 + 4, \varepsilon + 1, \varepsilon + 3), (1, 2, 3, 4)), i.e.
Now we have to solve linear system (12) with matrix $Q^T_{\{3,4\}}$ (the second matrix in (5)) and “input” vector $e(1) = (e_3, e_4) = (\varepsilon + 1, \varepsilon)$. Then vector $r(1) = (\varepsilon + 1, 2\varepsilon + 1)$, vector $s(1) = (1, 2, \varepsilon + 1, 2\varepsilon + 1)$, and then vector $p(2) = \Phi(s(1)) = (1, 3\varepsilon + 4/3, 2\varepsilon + 1, 2\varepsilon + 1)$.

Now set $D(2) = \{2, 3, 4\}$. Now we have to solve linear system (12) with matrix $Q^T_{\{2,3,4\}}$ (the third matrix in (5)) and “input” vector $e(2) = (\varepsilon + 1/3, \varepsilon, \varepsilon)$. Then vector $r(2) = (6\varepsilon + 2/3, 4\varepsilon + 1/3, 5\varepsilon + 1/3)$, vector $s(2) = (1, 6\varepsilon + 2/3, 4\varepsilon + 1/3, 5\varepsilon + 1/3)$, and then vector $p(3) = \Phi(s(2)) = (1, 6\varepsilon + 2/3, 4\varepsilon + 1/3, 5\varepsilon + 1/3)$. Now $D(3) = D(2)$, i.e., $p(3)$ is a solution of equation (1).

Generally both algorithms have some similarities and differences. Fixed point for a mapping is an equilibrium for a subsystem. Though the equations (12) seem similar to the equations in (6) but they are different since input vectors are defined differently. FD algorithm defines payments on some iterative steps, Flow algorithm defines rates for payments on intervals of constant statuses. FD requires possibly $n$ steps versus possibly $2n$ steps for Flow algorithm but computational complexity has the same order and substantial parts of calculates in the latter can be done in parallel.

6 Discussion

6.1 Possible Extensions

While our setup uses the initial setting of Eisenberg and Noe (2001) as the starting point, our model can be naturally generalized to work with many extensions. For example, a modification of our algorithm can work with liabilities or shares of different seniorities. First, we can assume that matrix $Q$ is not obtained by the equalities $q_{ij} = b_{ij}/b_i$, and just represents the priority of payments. Then of course we need the following modifications. Now all out-pipes from tank $i$ are not closed simultaneously but one by one, when the corresponding debt is paid. I.e. $(i, j)$ pipe is closed when debt $b_{ij}$ is paid. Corresponding equations and the times of status change can be easily modified. If we assume that some debts should be paid not just faster but before other payments, then at the initial moment not all $(i, j)$ pipes are open but only for $j$ in the senior (for $i$) class. When these debts are paid, then the other group of $(i, j)$ pipes is open, etc.
Similarly, while we assumed, that all “positive” firms have the same total out-capacities equal to one, the analysis is easily extended to heterogeneous rates (if the regulator considers it important to prioritize some payments). Note that, although the dynamics of the continuous-time model will be changed, the clearing vector will be the same if proportionality of all payments will be the same. Of course, if the requirement of proportionality of payments is changed, then not only will the dynamics of the continuous-time model be changed, but the clearing vector as well.

Another observation is that the continuous-time model may be naturally provided with microfoundations, a noncooperative game of \( n \) strategic players. In equilibrium, agents will be simultaneously transferring the prescribed amount, unless someone stops paying according to the schedule. If someone misses a payment, then it stops receiving payments from other agents.

The next observation is that because of the irreversibility property not all firms can leave group \( J_{+}(0) \) before \( T_{+} \), we can have a raw estimate \( T_{+} \leq \max_{i} b_{i} \). If the total capacity is changed from 1, as in our basic setup, to \( m \) then \( T_{+} \) will be changed to \( T_{+}/m \).

Let us discuss the following potential question: how much extra money \( x_{i} \geq 0 \) should be given to each (or some) firm to avoid all defaults? Note that the sum of unpaid debts, i.e., \( \sum_{i \in J} k_{i} \), \( k_{i} = b_{i} - p_{i}(T_{+}) \) can be substantially more than \( X = \sum_{i \in J} x_{i} \). Let us consider the following mechanism. Change the initial cash positions of all firms in \( J_{0}(T_{+}) \) from \( c_{i} \) to \( c'_{i} = c_{i} + k_{i} \), and run the continuous-time model again. Intuitively it is clear and can be proved that with these new initial positions, all debts will be paid and it is possible that some of these firm will have at the end positive cash positions \( c_{i}(T'_{+}) \). Intuitively it is clear and can be proved that if now the initial cash positions of all firms in \( J_{0}(T_{+}) \) are changed from \( c_{i} \) to \( c'_{i} = c_{i} + x_{i} \), where \( x_{i} = k_{i} - c_{i}(T'_{+}) \), e.g., by a central bank or a private consortium, then all these firms will finish with zero cash positions but, at the same time, fully paying their debts. Obviously \( 0 \leq x_{i} \leq k_{i} \).

6.2 Big Bang Effect at \( t=0 \)

The initial moment of time, \( t = 0 \), is exceptional if at this moment there is at least one zero firm. The reason for that is as follows. As we explained in the previous section, if at time \( t = 0 \) all firms have a positive amount of cash, then there is no need, initially, to
solved the equilibrium equations system, and moment $T_1$ is the first moment when one of
the firms either pays its debt or its cash position hits zero level. After that, the reasoning
of the previous section can be applied. However, if at time zero there are some zero firms
then, informally, there is a Catch-22 situation. Some zero firms at moment $t = 0$ should
be classified as positive if the input rates for them exceed one, because this means that
at the “next moment after zero” they instantly become positive. Equivalently, for these
firms the balance rate $d_i(0) > 0$. But to find these balance rates we need to know which
firms are positive and which are in the zero group. Note that in part (b) of Theorem 1,
formula (6) is valid for $k > 0$. In fact, this point is true for $k = 0$ also if “real” $J_+$ and
$J_0$ are known and used in the equilibrium equations, not formally defined by the initial
levels.

There are two ways to resolve this situation. One is by using a special procedure to
reveal these firms (see the Appendix), or, alternatively, by assuming that all zero firms
received a small positive amount $\varepsilon$ prior to $t = 0$ and then, during the “probe” time
interval $\Delta_0 = [0, t_m)$ with length of order $\varepsilon$, we can proceed as described in Section 5.

Then during this interval each of the initially zero firms will “reveal” its real status: the
cash values of “real-zero” firms, having balance rate negative, very soon hit zero level
during a series of close moments $t_1, ..., t_m$ ($m$ can be zero). “Real-positive” firms will
remain positive on interval $(0, t_m)$, and therefore will remain positive on a “long” time
interval at least until a more distant $T_1$. This means that on a very short interval, we
may have moments $t_1, t_2, ..., t_m$ of fast status changes, and at $t_m$ all the real statuses are
revealed. In Sections 3 and 5 we assumed, for the sake of brevity, that such moment $t_m$
exists, and plays the role of moment $T_1$. We prove the general case in the Appendix.

6.3 Debts Restructuring. Open Problems

A reader can notice that if in examples 1A,B,C we change, e.g., the total debts for banks
$b_2 = 2, b_3 = 3, b_4 = 4$ to values $b_2 = 1, b_3 = 2, b_4 = 3$ or to any $b_2 > 2, b_3 > 3, b_4 > 4$ in such
way that the corresponding rows of matrix $Q$ remain the same, then though the payment
vector will be changed, the final positions of all banks, vectors $c(T'_*)$, will be the same.

For a similar reason, Big Bang cosmology assumes that life of the Universe starts after the “Plank
epoch”, a minimal period of time, has passed.
In other words, part of debts can be “cancelled”. This raises a potential question “What is the minimal debt vector leading to the same final asset positions?” and how to obtain this vector and, more generally, how to change the debt structure, and thus matrix $Q$, to obtain simpler solution leading to the same result. The possible answer can be obtained using the structure of multiple solutions described in Theorem 2.

The banks in the model considered in this paper do not have a choice of payment or nonpayment of their debts. It seems an intriguing possibility to consider game type situations where banks have some choices about payment/nonpayment or the timing of their payments while facing potential rewards or penalties. Such a game may be especially appropriate to analyze situations during periods of financial crises.

Finally, as we mentioned in Introduction, there is a structural similarity between equation (1) and the standard Bellman equation in dynamic optimization. The “tank” interpretation for Bellman equation is again possible, though in this case the capacities for out-pipes and in-pipes should switch their roles as the direct stochastic matrix is used in this equation, rather than the transposed one. Furthermore, we will need a central tank that will, from time to time, add money to some tanks in a recursive way defined by the Bellman equation. The final position of all tanks should give both the value function and optimal stopping set.

7 Conclusion

In this paper, we develop a continuous-time model of clearing in financial networks. This approach provides an intuitive and simple recursive solution to a classical static model of financial clearing introduced by Eisenberg and Noe (2001). The same approach provides a useful tool to solve nonlinear equations involving a linear system and max min operations similar to the Bellman equation for the optimal stopping of Markov chains and other optimization problems. Finally, this approach allows to resolve simultaneous or nearly simultaneous payments in a time of financial distress by allowing to “stretch” an instant moment into a finite time interval. Practical implementation of such a mechanism is probably possible.
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Appendix

A1 Computational Formulas for the Dynamic System

Using matrix multiplication with full size $n \times n$ matrices, mapping $G$ defining the transformation $X(k+1) = G(X(k))$, can described as follows. As before, we skip the index $k$. Let $A \subseteq J$ and $D_A$ be a diagonal matrix with diagonal coefficients $\lambda_i = 1$ if $i \in A$ and 0 otherwise, $D_+ = D_{J+}, D_0 = D_{J_0}$. Let $1_A$ be a $n$-dimensional vector with $i$-th coordinate equal 1 if $i \in A$ and 0 otherwise, $1_+ = 1_{J+}, 1_0 = 1_{J_0}$, thus, e.g., $1_+$ is an $n$-dimensional vector with $i$ coordinate equal to 1 if $i \in J_+$ and 0 otherwise. Then, given partition $P = (J_+, J_0, J_*)$ matrix $Q^T_k$ from (6), obtained by elimination in matrix $Q^T$ of all rows and columns not in $J_0$, can be represented as a full size matrix $Q^T_0 = D_0 Q^T D_0$. Then input vector from (4) $e$ as $n$-dimensional vector can be represented as $e = D_0 Q^T 1_+$ with coordinate $e_i$ given by formula (5) if $i \in J_0$ and 0 otherwise. Finally, the $n$-dimensional vector $u$ can be represented as

$$u = (I - Q^T_0)^{-1}(1_+ + e) = (I - Q^T_0)^{-1}(I + D_0 Q^T)1_+ = AB 1_+. \quad (A1)$$

Thus, summing up the steps, given state $X = (T, P = (J_+, J_0, J_*), b_i(T), c_i(T), i \in J)$ to obtain next state $X'$, one has to:

1) Using partition $P$, generate matrix $D_0$, vector $1_+$. Obtain out-rate vector $u$ using (A1).

2) Given vector $u$, obtain, using the last formula in (10), in-rate vector $n = Q^T u$ and the balance vector $d = n - u = (Q^T - I)u$.

3) Calculate vectors $s = (s_i), t = (t_i)$ using the formulas in (11). Calculate $t' = \min_{i \in J}(s_i, t_i)$, and index $r = \arg\min$. Then $T' = T + t'$.

4) Using $r$, obtain the new partition $P'$ using the following rules obtained in Section 5:

   If $t' = s_r$, and $r \in J_+$, i.e., positive firm $r$ becomes absorbing, the new partition is $P' = P(T') = (J_+ \setminus r, J_0, J_* \cup r)$. If $t' = s_r$, and $r \in J_0$, i.e., zero firm $r$ becomes absorbing, the new partition is $P' = (J_+, J_0 \setminus r, J_* \cup r)$. If $t' = t_r$, and $r \in J_+$, i.e., positive firm $r$ becomes a zero firm, the new partition is $P' = (J_+ \setminus r, J_0 \cup r, J_*).

5) Obtain new state $X' = (T', P', b'_i = b_i(T'), c'_i(T'), i \in J)$, where $b'_i, c'_i$ are calculated using formulas (10) with $t = T + t'$.
A2 Proof of Theorem \[ (c) \]

We prove part (c) of Theorem 1 by induction on \( k \), where \( k \) is the number of the interval \( \Delta_k = [T_k, T_{k+1}), k = 0, 1, \ldots \). When \( k = 0 \), the initial state is \( X(0) = (T_0 = 0, P_0 = P(T_0), x_i(0) = (b_i, c_i, 0)) \). If, at moment \( t = 0 \), in the initial partition \( J_0 = \emptyset \), then the transition from state \( X_0 \) to \( X_1 \) was described in section 5. More complicated case when \( J_0 \neq \emptyset \) will be considered in part 3 of this Appendix. With \( k > 0 \), as before we skip further indication to \( k \), denoting \( u_i, k = u_i, \Delta_k = \Delta, T_k \) as \( T, T_{k+1} \) as \( T' \), and partitions \( P_k = P, P_{k+1} = P' \). We denote other vectors, coordinates, partition and matrices before status change, i.e., on the interval \( \Delta_{k-1} \), as \( b, c, P, J_0 = J_{0, k-1}, Q_0^T = Q_{0, k-1}^T \) and the values after change as \( b', c', P', J'_0 = J_{0, k}, Q'_0 = Q_{0, k}^T \), etc. We denote the solution of the equilibrium equations on the interval \( \Delta_{k-1} \) as \( (w_i) \), and on the interval \( \Delta_k \) as \( (v_i) \). Our induction statement includes that \( 0 < w_i < 1 \). If we prove that \( w_i > v_i \) for all \( i \in J'_0 \), this will imply point (c) for the next interval.

Let the previous partition \( P = (J_+, J_0, J_\star) \), the previous input vector \( (e_i) \), new input vector \( (e'_i) \), and let \( r \) be the index of the firm that changed its status at moment \( T \). We start with the more difficult case 2), where the status of \( r \) is changed from \( + \) to 0, and the size of the matrix in the equilibrium equations is increased by one. We have \( J'_+ = J_+ \cap r, J'_0 = J_0 \cup r \). If \( Q_0 \) has dimension \( m \), then the new matrix \( Q'_0 \) has dimension \( m + 1 \) and \( u_r = 1 \) is changed to an extra unknown variable \( v_r \). Thus the equilibrium equations system (6) for \( v_i \) is

\[
v_i = e'_i + \sum_{j \in J_0} v_j q_{ji} + v_r q_{ri}, \quad i \in J_0; \quad v_r = e'_r + \sum_{j \in J_0} v_j q_{jr},
\]

where \( e'_i = \sum_{j \in J_+} q_{ji} = e_i - q_{ri} \), \( e'_r = \sum_{j \in J_+} q_{jr} = e_r \). The status of \( r \) was changed from \( + \) to 0 not without a reason. It was changed because the cash position of firm \( r \) has reached zero, \( c_r(T) = 0 \). Thus \( c_r(t) > 0 \) for \( t < T \) and hence, by formula (10), \( d_r < 0 \). Thus,

\[
-d_r = 1 - n_r(T) = 1 - \left( \sum_{j \in J_+} q_{jr} + \sum_{j \in J_0} w_j q_{jr} \right) = \varepsilon_r > 0.
\]

The equilibrium equations system (6) for \( w_i \) can be represented as follows, using artificial variable \( w_r = 1 \), equality (A3) and equality \( e_r = \sum_{j \in J_+} q_{jr} \):

\[
w_i = e_i + \sum_{j \in J_0} w_j q_{ji} + w_r q_{ri} - q_{ri}, i \in J_0; \quad w_r = e_r + \sum_{j \in J_0} w_j q_{jr} + \varepsilon_r.
\]
Using equalities $e_i = e'_i + q_{ri}, e'_r = e_r$ and subtracting equations for $v_i$, given in (A2), from equations in (A4), we obtain the following equation for the vector $y = w - v = ((w - v)_i), i \in (J_0 \cup r),$

$$y_i = \sum_{j \in J_0} y_j q_{ji} + y_r q_{ri}, \quad i \in J_0; \quad y_r = \sum_{j \in J_0} y_j q_{jr} + \varepsilon_r.$$  \hfill (A5)

Since $\varepsilon_r > 0$, using Lemma 1, we obtain $y_i \geq 0$ for all $i \in (J_0 \cup r)$, i.e., part (c) of Theorem 1 holds in case 2).

In case 1), + to ∗, we have $J'_+ = J_+ \setminus r, J'_0 = J_0, J_* = J_* \cup r$. Thus $Q'_0 = Q_0$ but vector $e$ is diminished because firm $r$ closed its out-pipes at moment $T$, i.e., $e'_i = e_i - q_{ri} \leq e_i$. Then, similarly to case 2), subtracting the equation for $w_i$ from the equation for $u_i$, and using Lemma 1 again, we obtain that $y_i \geq 0$ for all $i \in J'_0$, i.e., part (c) of Theorem 1 holds in case 1) as well.

In case 3), 0 to ∗, we have $J'_+ = J_+ \setminus r, J'_0 = J_0 \setminus r, J_* = J_* \cup r$, and all inputs $e_i, i \in J_0$ remain the same. The new matrix $Q'_0$ has dimension $m - 1$ and, similarly to cases 1) and 2), the system for the vector $y = w - v = ((w - v)_i), J_0 \setminus r$ has a similar form and $y_i \geq 0$, i.e., point (c) of Theorem 1 holds also in case 3).

The monotonicity property for in-rates $n_i$ follows from the fact, proven above, that all out-rates from the previous interval to the next are decreasing.

### A3 Modification of the Initial Partition for $t = 0$

Let $P(0) = (J_+, J_0, J_*)$ be the initial partition, and $L_0$ denote the unknown set of firms from $J_0$ which at the “next” moment will be positive, and then respectively should have out-rates $u_i(0) = 1$, thus not those defined by equilibrium equations for the initial partition.

**Lemma A3.** There is a subset $L_0 \subset J_0$ such, that for a modified partition $P^M = (J'_+, J'_0, J_*)$ with $J'_+ = J_+ \cup L_0, J'_0 = J_0 \setminus L_0$, the values of the solutions of the equilibrium equations for this partition $P^M$ on the initial interval $\Delta_0$, satisfy (i) $n_i = u_i < 1, i \in J'_0$; and (ii) $n_i > 1$ for $i \in L_0$.

Thus, the $u_i$ for the first interval $\Delta_0$ should be defined by the modified partition and for all other intervals according to Theorem 1. We skip the formal proof of Lemma A3 but outline the following algorithm to obtain the set $L_0 \subset J_0$:
Step 1. Solve the equilibrium equations (6) for the initial partition \( P(0) \) to obtain the initial vector \( w(0) = (w_i, i \in J_0) \). If \( w_i < 1 \) for all \( i \in J_0 \), then we are done and Lemma \[A3\] holds with \( L_0 = \emptyset \). Otherwise, set \( L_1 = \{i : w_i \geq 1\} \) is not empty.

Step 2. Define a new partition \( P(1) = (J_+(1) = J_+ \cup L_1, J_0(1) = J_0 \setminus L_1) \). Solve the equilibrium equations for this partition. Let \( v_i \) be a new solution. By point (c) of Theorem 1 and Lemma 1, the initial solution and new solution satisfy \( n_i(w) \geq n_i(v) \) for all \( i \), and in particular \( w_i \geq v_i, i \in J_0(1) \). Calculate \( n_i(v) \) for all \( i \in L_1 \). If they all exceed 1, we are done, \( L_1 = L_0 \). If not, then set \( L_2 = \{i : n_i(v) > 1\} \subset L_1 \) is still not empty. Then repeat Step 2, i.e., define the new partition \( P(2) = (J_+(2) = J_+ \cup L_2, J_0(2) = J_0 \setminus L_2) \), solve the equilibrium equations for this new partition, and so on. Eventually, we obtain a sequence of sets \( J_0 \supseteq L_1 \supseteq L_2 \supseteq \ldots \supseteq L_0 \). Note that if \( L_1 \neq \emptyset \), then by definition of \( L_1 \) it is impossible that \( L_0 = \emptyset \). Vector \( u_i, i \in J \) obtained for the modified partition satisfies both Assumptions (C1) and (C2) to specify all out-rates for the first interval.