Stable and Unstable Circular Strings in Inflationary Universes

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Abstract

It was shown by Garriga and Vilenkin that the circular shape of nucleated cosmic strings, of zero loop-energy in de Sitter space, is stable in the sense that the ratio of the mean fluctuation amplitude to the loop radius is constant. This result can be generalized to all expanding strings (of non-zero loop-energy) in de Sitter space. In other curved spacetimes the situation, however, may be different.

In this paper we develop a general formalism treating fluctuations around circular strings embedded in arbitrary spatially flat FRW spacetimes. As examples we consider Minkowski space, de Sitter space and power law expanding universes. In the special case of power law inflation we find that in certain cases the fluctuations grow much slower than the radius of the underlying unperturbed circular string. The inflation of the universe thus tends to wash out the fluctuations and to stabilize these strings.
1 Introduction

It is generally believed that cosmic strings can be formed in the early history of the universe. This may take place via the Kibble mechanism (see for instance Ref. [1]) before, in or after the inflationary era and/or via spontaneous quantum nucleation [2] during inflation. After formation the evolution (and fate) of the cosmic strings depend on the further evolution of the universe and on the specific qualitative and quantitative features of the strings themselves. Since cosmic strings may be responsible for galaxy formation it is important to understand the evolution of strings through the inflationary era and into the more "stable" radiation dominated universe. This problem is conveniently discussed in the framework of "test-strings" in fixed spatially flat Friedman-Robertson-Walker (FRW) spacetimes, neglecting backreactions from the strings on the universe.

The main obstacle is the highly non-linear nature of the equations of motion for a string in a curved background. As a consequence one often has to rely on numerical calculations to obtain qualitative results. There are exceptions, however. In some highly symmetric curved spacetimes it is possible to solve the equations of motion analytically if a suitable ansatz for the string (shape) is made. As examples we mention stationary strings in stationary spacetimes and oscillating circular strings in axially symmetric spacetimes.

The purpose of the present paper is to consider small fluctuations around circular strings in spatially flat FRW spacetimes. The evolution of the fluctuations as compared to the evolution of the unperturbed underlying circular string, we believe, is relevant for the evolution of closed strings in general. In some cases the strings will turn out to be stable, while in others they will eventually collapse into black holes or disintegrate into smaller closed strings.

Our motivation is, besides the possible implications in a cosmological scenario as discussed above, the recent interest in the stability properties of cosmic and fundamental strings in curved spacetimes. Among a variety of papers we mention the followings: Loustó and Sánchez [3] discussed the string-instabilities by considering first order fluctuations around the center of mass of a small (fundamental) string in the background of a black hole. A similar analysis was carried out by the author [4] for a large (cosmic) string winding around a black hole in the equatorial plane. String-instabilities in de Sitter space were discussed by Garriga and Vilenkin [5, 6, 7], concentrating
on the circular zero loop-energy nucleated strings \cite{2}. The analogue problem
for circular strings of non-zero loop-energy was considered in Ref.\cite{4}, con-
centrating on the question of boundedness/unboundedness of the comoving
fluctuations in the two regions $r \to 0$ and $r \to \infty$ (r is the physical radius
of the unperturbed circular string).

The paper is organized as follows: in Section 2 we give the equation of
motion determining the physical radius of the circular string in the FRW
background, and we derive the linearized equations for the fluctuations in
the two directions perpendicular to the string world-sheet. In Section 3 we
first recall the results obtained in flat Minkowski space. We then discuss
the results for the various types of expanding strings in de Sitter space and
finally we consider power law inflationary universes. Section 4 contains our
conclusions.

Throughout the paper we use sign-conventions of Misner-Thorne-Wheeler
\cite{8} and units where $G = 1$, $c = 1$ and the string tension $(2\pi\alpha')^{-1} = 1$.

2 The Circular String Fluctuations

In the zero thickness limit a cosmic string is described by the Nambu-Goto
action:

$$S = \int d\tau d\sigma \sqrt{-\det G_{AB}},$$

(2.1)

where $G_{AB}$ is the induced metric on the world-sheet:

$$G_{AB} = g_{\mu\nu} x^\mu_A x^\nu_B.$$

(2.2)

In this paper we are interested in configurations describing circular strings
with small fluctuations in a spatially flat FRW spacetime. The FRW line
element is usually written in the form:

$$ds^2 = -dt^2 + a^2(t)[dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2],$$

(2.3)

where $t$ is the cosmic time and $a(t)$ is the scale factor.

For a circular string the physical radius is given by:

$$f(t) = ra(t)$$

(2.4)
and it is therefore convenient, for our purposes, to write eq. (2.3) in terms of \( f \):

\[
ds^2 = -(1 - H^2 f^2)dt^2 - 2H f \, dt df + df^2 + f^2 d\theta^2 + f^2 \sin^2 \theta d\phi^2,
\]

(2.5)

where we also introduced the Hubble function:

\[
H(t) = \frac{1}{a(t)} \frac{da}{dt}.
\]

(2.6)

The unperturbed circular string with time dependent physical radius is obtained by the ansatz:

\[
t = \tau, \quad f = f(\tau), \quad \theta = \pi/2, \quad \phi = \sigma.
\]

(2.7)

The physical radius \( f(\tau) \) is to be determined by the equations of motion corresponding to the action (2.1). Using eq. (2.2) with the spacetime metric (2.5) and the ansatz (2.7) we find:

\[
\ddot{f} - 2H f \dot{f}^3 + (6H^2 f^2 - 1)\dot{f}^2 + 3H f(1 - 2H^2 f^2)\dot{f}
- f^2 \dot{H} + 2H^4 f^4 - 3H^2 f^2 + 1 = 0,
\]

(2.8)

where a dot denotes differentiation with respect to \( t = \tau \). This equation gives the physical string radius as a function of the cosmic time in a spatially flat FRW spacetime \cite{9,10,11}. Since we are only interested in timelike string evolution, we furthermore have the constraint:

\[
G_{\tau \tau} = (\dot{f} - H f)^2 - 1 < 0.
\]

(2.9)

The general solution of eq. (2.8), for arbitrary \( H(\tau) \), does not seem to be available. The complete solution is in fact only known in a few very special cases like Minkowski space \((H = 0)\) and de Sitter space \((H = \text{const.} \equiv H_o \neq 0)\), where the integrability is guaranteed by the existence of the extra Killing vector \( \partial/\partial t \) (besides \( \partial/\partial \phi \)) in the metric (2.5). In many other cases it is however possible to find special solutions and/or asymptotic solutions. We will return to that question in Section 3, but for the moment we just assume that we have a solution \( f(\tau) \) for a spacetime \( H(\tau) \) and we now address the question of small fluctuations around it.

A covariant formalism of ”linearized” fluctuations around arbitrary string configurations in arbitrary curved spacetime backgrounds was developed by
From the two world-sheet tangent vectors:
\[ \dot{x}^\mu = (1, \dot{f}, 0, 0), \quad x'^\mu = (0, 0, 0, 1), \] (2.10)
we introduce two vectors \( n_\mu^\|, n_\mu^\perp \), perpendicular to the string world-sheet:
\[ n_\perp^\mu = (0, 0, 1, 0), \]
\[ n_\| = \frac{1}{\sqrt{1 - (\dot{f} - Hf)^2}} (\dot{f} - Hf, 1 + Hf(\dot{f} - Hf), 0, 0), \] (2.12)
fulfilling \( (R, S) \) takes the values " \( \perp \) " and " \( \| \) "):
\[ g_\mu\nu n_\perp^\mu n_\perp^\nu = \delta_{RS}, \quad g_\mu\nu n_\| x^A, A = 0, \] (2.13)
as well as the completeness relation:
\[ g^{\mu\nu} = G^{AB} x^A x^B, \] (2.14)
The general physical fluctuation is then expressed as:
\[ \delta x^\mu = n_\perp^\mu \delta x^\perp + n_\| ^\mu \delta x^\|, \] (2.15)
where \( \delta x^\perp \) and \( \delta x^\| \) are the comoving fluctuations perpendicular to the string plane and in the string plane, respectively. To first order in \( \delta x^\mu \) it can now be shown that the comoving fluctuations fulfill the following matrix equation [12]:
\[ \square \delta x_R + 2\mu_{RS} A(\delta x^S), A + (\nabla_A \mu_{RS} A) \delta x^S - \mu_{RT} A A^T \delta x^S + \frac{2}{G_C} \Omega_{R} A B \Omega_{SAB} \delta x^S - h_{AB} x^A x^B \Omega_{RSA} n_\perp^\mu n_\perp^\nu \delta x^\perp = 0. \] (2.16)
Here \( h_{AB} \) and \( G_{AB} \) are the intrinsic (Polyakov) and induced metric, respectively, which in our case are identical since we are working with the Nambu-Goto action. \( \Omega_{RAB} \) and \( \mu_{RSA} \) are the second fundamental form and normal fundamental form, respectively [15]:
\[ \Omega_{RAB} = g_\mu\nu n_\| x^A, A \nabla_\rho x^B, \] (2.17)
$$\mu_{RSA} = g_{\mu\nu} n_{RS}^{\mu} x_{\nu}^B \nabla_\rho n_{S}^\rho,$$

(2.18)

where $\nabla_\rho$ is the spacetime covariant derivative. $\Box$ and $\nabla_A$ are the world-sheet d’Alambertian and covariant derivative, respectively, determined by the world-sheet metric (2.2). Finally $R_{\mu\rho\sigma\nu}$ is the Riemann tensor of the metric (2.5), and since we are using a somewhat unconventional parametrization for the FRW spacetime we have listed the non-vanishing components in the Appendix.

Generally eqs. (2.16) constitute an extremely complicated set of coupled partial differential equations. Fortunately some simplifications arise for the unperturbed circular strings under consideration here. Using the explicit expressions for the tangent vectors and normal vectors (2.10)-(2.12), it is easily seen that all components of the normal fundamental form vanish, while:

$$\Omega_{\perp AB} = 0, \quad \Omega_{\parallel AB} \Omega_{\parallel AB} = \frac{2 (1 + H f (\dot{f} - H f))}{f^2} \frac{1}{1 - (\dot{f} - H f)^2}.$$

(2.19)

The relevant projections of the Riemann tensor, appearing in eqs. (2.16) are evaluated using the explicit expressions of $R_{\mu\rho\sigma\nu}$ given in the Appendix. After a little algebra one finds:

$$G^{AB} x^\mu_A x^\nu_B R_{\mu\rho\sigma\nu} n_{\perp}^\rho n_{\perp}^\sigma = \frac{\dot{H}}{(f - H f)^2 - 1} - 2 H^2,$$

(2.20)

$$G^{AB} x^\mu_A x^\nu_B R_{\mu\rho\sigma\nu} n_{\parallel}^\rho n_{\parallel}^\sigma = \frac{2 (\dot{f} - H f)^2 - 1}{1 - (\dot{f} - H f)^2} \dot{H} - 2 H^2,$$

(2.21)

while the cross terms ($\propto n_{\perp}^\rho n_{\parallel}^\sigma$) vanish. We eventually end up with two separated partial differential equations in the form:

$$\Box \delta x_{\perp} + V_{\perp}(\tau) \delta x_{\perp} = 0,$$

(2.22)

$$\Box \delta x_{\parallel} + V_{\parallel}(\tau) \delta x_{\parallel} = 0,$$

(2.23)

where:

$$V_{\perp} = \frac{\dot{H}}{1 - (\dot{f} - H f)^2} + 2 H^2,$$

(2.24)

$$V_{\parallel} = \frac{1 - 2 (\dot{f} - H f)^2}{1 - (\dot{f} - H f)^2} \dot{H} + \frac{2 (1 + H f (\dot{f} - H f))^2}{f^2} \frac{1}{1 - (\dot{f} - H f)^2} + 2 H^2,$$

(2.25)
and the d’Alambertian is given by:

$$
\Box = \frac{1}{\sqrt{-G}} \partial_A (\sqrt{-G} \ G^{AB} \partial_B )
\begin{align*}
&= \frac{\partial_\tau^2}{1 - (\dot{f} - H f)^2} + \frac{\partial_\sigma^2}{f^2} - H \frac{1 - 2(\dot{f} - H f)^2}{1 - (\dot{f} - H f)^2} \partial_\tau,
\end{align*}
(2.26)
$$

using eqs. (2.2), (2.7) as well as eq. (2.8) to eliminate the $\ddot{f}$-terms.

Since the potentials in eqs. (2.22)-(2.23) depend on $\tau$ only, it is convenient to Fourier expand the comoving fluctuations:

$$
\delta x_R(\tau, \sigma) = \sum_{n \in \mathbb{Z}} C_n R(\tau) e^{i n \sigma},
(2.27)
$$

where $C_{n R} = C^*_{-n R}$ and the tilde denotes summation for $|n| \neq 0, 1$ only. The zero modes and the $|n| = 1$ modes are excluded from the summation since they do not correspond to ”real” physical fluctuations on a circular string [5-7]. They describe spacetime translations and rotations, that do not change the circular shape of the string.

The Fourier expansions reduce eqs. (2.22)-(2.23) to ordinary second order differential equations determining $C_{n R}(\tau)$. In the next section we consider these equations in various cases of cosmological interest.

3 Special Cases

In this section we use the general formalism of Section 2 in some special cases. As our first example we consider Minkowski space. This case has already been discussed in the literature (Appendix A of Ref.[6]), but we include it here to get an independent check of equations (2.22)-(2.23). We then consider expanding strings in de Sitter space and as an example of other families of inflationary backgrounds, we consider power law expanding universes.

3.1 Minkowski Space

For Minkowski space the Hubble function equals zero and equation (2.8), determining the radius of the unperturbed circular string reduces to [10]:

$$
\ddot{f} f - \dot{f}^2 + 1 = 0.
(3.1)
$$
The solution is just a trigonometric function:

\[ f(\tau) = r_m \cos\left(\frac{\tau - \tau_0}{r_m}\right), \tag{3.2} \]

where \( r_m \) is the maximal radius obtained at \( \tau = \tau_0 \). The string then contracts and collapses at \( \tau = \tau_0 + \pi r_m / 2 \). We can now easily calculate the fluctuations around the configuration (3.2). The Fourier transformed versions of eqs. (2.22)-(2.23), using also eqs. (2.24)-(2.27), reduce to:

\[ \ddot{C}_{n\perp} + n^2 C_{n\perp} = 0, \tag{3.3} \]
\[ \ddot{C}_{n\parallel} + \left(n^2 - \frac{2}{\cos^2(\tau - \tau_0)}\right)C_{n\parallel} = 0, \tag{3.4} \]

where we have rescaled the time parameter \( \tau \to \tau r_m \). These two equations are solved by:

\[ C_{n\perp}(\tau) = A_{n\perp} \cos n\tau + B_{n\perp} \sin n\tau, \tag{3.5} \]
\[ C_{n\parallel}(\tau) = A_{n\parallel}[n \cos n(\tau - \tau_0) + \tan(\tau - \tau_0) \sin n(\tau - \tau_0)] + B_{n\parallel}[n \sin n(\tau - \tau_0) - \tan(\tau - \tau_0) \cos n(\tau - \tau_0)]. \tag{3.6} \]

Not surprisingly the fluctuations in the direction perpendicular to the plane of the string are completely finite and regular. For the fluctuations in the plane of the string we see that they blow up for \( \tau \to \tau_0 + \pi / 2 \), that is when the unperturbed string collapses. Note that the fluctuations (3.5)-(3.6) are the comoving fluctuations, i.e. the fluctuations as seen by an observer travelling with the unperturbed circular string. For an external observer the fluctuations (3.6) should be multiplied by the Lorentz contraction factor \( \sqrt{1 - \dot{f}^2} \) (generally \( \sqrt{1 - (\dot{f} - Hf)^2} \)). In the present case this factor exactly cancels the divergent terms in eq. (3.6), so that all fluctuations in eqs. (3.5)-(3.6) are finite and regular for an external observer. This is however still not the final picture: to discuss the stability of the circular shape the relevant quantity is the ratio of the fluctuation amplitude to the radius of the unperturbed circular string. Since the radius of the string goes to zero during the collapse, it follows that for suitable initial conditions this ratio will blow up, and the string will develop cusps and probably disintegrate into smaller loops. For other initial conditions the ratio will stay approximately constant until the string falls into its own Schwarzschild radius, and the string will collapse into a black hole in a "regular" way. These possibilities were discussed in more detail in Ref. [6].
3.2 de Sitter Space

For de Sitter space the Hubble function is constant \( H = \text{const.} \equiv H_o \neq 0 \), so that eq. (2.8) has no explicit time dependence and is therefore integrable. The complete solution is actually known, and may be found in Ref.\[17\] (in Ref.\[17\] the solution is written in terms of a time parameter related to the cosmic time by elliptic theta-functions). The fluctuations around circular strings in de Sitter space were discussed by the author in Ref.\[4\], using the static parametrization and the conformal gauge. Here we shall give some more results.

A circular string configuration in de Sitter space of special interest is provided by the nucleated string of Basu, Guth and Vilenkin \[2\]. The nucleated string has zero loop-energy and is explicitly given by:

\[
f(\tau) = \frac{1}{H_o} \sqrt{e^{2H_o(\tau-\tau_o)} + 1}, \tag{3.7}
\]

where \( \tau_o \) is a constant, i.e. it expands with the same rate as the universe (for \( \tau - \tau_o >> 1/H_o \)). The explicit form (3.7) was obtained using the instanton method \[2\], but it is easily shown that eq. (2.8) is fulfilled, as it should be. It is convenient to write the solution in terms of the time parameter \( \tau_c \):

\[
\tau - \tau_o = \frac{1}{H_o} \log \tan(\tau_c - \tau_o), \tag{3.8}
\]

so that \( \tau \in [\tau_o, \tau_o + \pi/2] \). The solution (3.7) is now:

\[
f(\tau_c) = \frac{1}{H_o \cos(\tau_c - \tau_o)}. \tag{3.9}
\]

The string expands from horizon size at \( \tau_c = \tau_o \) towards infinity for \( \tau_c \to \tau_o + \pi/2 \). In the \((\tau_c, \sigma)\) coordinates the Fourier transformed versions of eqs. (2.22)-(2.23) reduce to:

\[
\frac{d^2C_{nR}}{d\tau_c^2} + \left(n^2 - \frac{2}{\cos^2(\tau_c - \tau_o)}\right)C_{nR} = 0. \tag{3.10}
\]

Two remarks are worth doing here: the two fluctuation equations take exactly the same form. This is somewhat surprising since, although the de Sitter spacetime is isotropic, the isotropy is broken by the plane circular string.
Secondly, the two equations take exactly the same form as the $C_{n||}$-equation in Minkowski space (3.4), so they are both solved by eq. (3.6). The physical interpretation is however completely different here. Eq. (3.4) describes the fluctuations around a collapsing string in Minkowski space. After Lorentz contraction we found that the ratio of the fluctuation amplitude to the string radius blew up, leading to unstable strings. Eq. (3.10), on the other hand, describes the fluctuations around an expanding string in de Sitter space. The Lorentz contraction plays no role here (for $\tau - \tau_0 >>> 1/H_0$), and by comparing eqs. (3.6) and (3.9) we see that the ratio of the mean fluctuation amplitude to the string radius is constant (c.f. Ref.[7]), indicating that the string is stable.

The solution (3.7) is a very special one (zero loop-energy), so it is not obvious whether the results for the fluctuations can be generalized to other expanding circular strings in de Sitter space. The evolution of arbitrary circular strings in de Sitter space was discussed in Ref.[17]. An expanding circular string, that has passed the horizon from the inside, expands with the same rate as the universe ($H_0 f >>> 1$):

$$f(\tau) \propto e^{H_0 \tau}$$ (3.11)

In this case the fluctuation equations become approximately:

$$\ddot{C}_{nR} + H_0 \dot{C}_{nR} - 2H_0^2 C_{nR} \approx 0,$$ (3.12)

with solutions:

$$C_{nR}(\tau) \approx A_{nR} e^{H_0 \tau} + B_{nR} e^{-2H_0 \tau}.$$ (3.13)

So, due to the $A_{nR}$-terms, the fluctuations grow with the same rate as the universe, and the ratio of the fluctuation amplitude to the radius of the circular string (3.11) is constant for these strings also.

In de Sitter space the physical radius of the horizon is constant ($= 1/H_0$) so in principle there could also be expanding strings that never cross the horizon. Such strings would expand much slower than the universe and would therefore be completely different from the strings discussed above. This type of circular strings has actually been found [18]:

$$f(\tilde{\tau}) = \frac{1}{H_0 \sqrt{2}} \tanh \frac{\tilde{\tau}}{\sqrt{2}},$$ (3.14)
where the time parameter $\tilde{\tau}$ is introduced by:

$$t = \tau = \frac{1}{H_0} [\tilde{\tau} + \log(1 - \frac{1}{\sqrt{2}} \tanh \frac{\tilde{\tau}}{\sqrt{2}})].$$  \hspace{1cm} (3.15)

Notice that $f(0) = 0$, $f(\infty) = 1/(H_0 \sqrt{2})$ and $df/d\tilde{\tau} > 0$. For this solution the fluctuation equations become:

$$\frac{d^2 C_{n\perp}}{d\tilde{\tau}^2} + (n^2 - \tanh^2 \frac{\tilde{\tau}}{\sqrt{2}})C_{n\perp} = 0; \hspace{1cm} (3.16)$$

$$\frac{d^2 C_{n\parallel}}{d\tilde{\tau}^2} + (n^2 - \tanh^2 \frac{\tilde{\tau}}{\sqrt{2}} - \coth^2 \frac{\tilde{\tau}}{\sqrt{2}})C_{n\parallel} = 0. \hspace{1cm} (3.17)$$

These two equations can be solved explicitly in terms of hypergeometric functions in the variable $z \equiv -\sinh^2 \frac{\tilde{\tau}}{\sqrt{2}}$. For our purposes it is however sufficient to consider the somewhat simpler approximate solutions. Starting from $f = 0$ at $\tilde{\tau} = 0$ the string size grows steeply towards the asymptotic size $f = 1/(H_0 \sqrt{2})$. Then eqs. (3.16)-(3.17) reduce to:

$$\frac{d^2 C_{n\perp}}{d\tilde{\tau}^2} + (n^2 - 1)C_{n\perp} \approx 0, \hspace{1cm} (3.18)$$

$$\frac{d^2 C_{n\parallel}}{d\tilde{\tau}^2} + (n^2 - 2)C_{n\parallel} \approx 0. \hspace{1cm} (3.19)$$

For the "shape-changing" modes $|n| \geq 2$ (c.f. the discussion after eq. (2.27)) the solutions are simple trigonometric functions. It follows that also in this case the ratio of the mean fluctuation amplitude to the radius of the circular string is constant.

### 3.3 Power Law Inflation

In this subsection we consider power law inflationary universes. The scale factor is given by:

$$a(\tau) = a_p \tau^p, \hspace{1cm} (3.20)$$

where $a_p$ is a dimensionfull constant, which is introduced to ensure that $a(\tau)$ is dimensionless. The dynamics of the unperturbed circular strings in the backgrounds (3.20) was investigated in Ref.[11], using both numerical and
analytical methods. For $0 < p < 1$, which includes as special cases matter and radiation dominated universes, the circular string dynamics is very similar to Minkowski space: expanding strings reach a maximal size after which they collapse. For $p > 1$, corresponding to the inflationary universes, the dynamics is much more interesting, so we restrict ourselves to that case in the following.

The Hubble function of the spacetime (3.20) is $H(\tau) = p/\tau$ and for $p > 1$ there is an event horizon with physical radius:

$$f_{EH}(\tau) = \frac{\tau}{p-1}. \quad \text{(3.21)}$$

Circular strings outside this horizon tend to expand with the same rate as the universe ($f \gg \tau$):

$$f(\tau) \propto \tau^p; \quad p > 1. \quad \text{(3.22)}$$

This is similar to de Sitter space. For such strings the fluctuation equations become:

$$\ddot{C}_{nR} + \frac{p}{\tau} \dot{C}_{nR} - \frac{p(2p - 1)}{\tau^2} C_{nR} \approx 0, \quad \text{(3.23)}$$

with solutions:

$$C_{nR}(\tau) \approx A_{nR} \tau^p + B_{nR} \tau^{1-2p}, \quad \text{(3.24)}$$

and the fluctuations grow with the same rate as the unperturbed strings, which again grow with the same rate as the universe. These strings are therefore stable in the same sense as the expanding strings in de Sitter space, c.f. Subsection 3.2.

We can however find other types of expanding strings in the spacetimes (3.20). In the power law inflationary universes the event horizon size increases proportionally to $\tau$, eq. (3.21). In principle this allows the existence of monotonically expanding circular strings, that never pass the horizon from the inside. A string solution of this type is in fact provided by:

$$f(\tau) = \frac{\tau}{\sqrt{2p(p-1)}}, \quad p > 1. \quad \text{(3.25)}$$

It is easily checked that this solution fulfills eqs. (2.8) and (2.9) as well as:

$$\dot{f} > 0, \quad f(\tau) < f_{EH}(\tau). \quad \text{(3.26)}$$
We note in passing that the solution (3.25) is actually well-defined for \( p < 0 \) also. In that case it is convenient to consider instead:

\[
\hat{f}(\tau) = \frac{\tau_o - \tau}{\sqrt{2p(p - 1)}}; \quad p < 0
\]  

(3.27)

which solves eq. (2.8) in the spacetime with scale factor:

\[
\hat{a}(\tau) = \frac{a_p}{(\tau_o - \tau)^{|p|}}; \quad p < 0
\]  

(3.28)

In this case eq. (3.27) describes a collapsing string (for \( \tau \to \tau_o \)) in a super-inflationary universe (\( \dot{H} > 0 \)) which blows up for \( \tau \to \tau_o \).

But let us return to the expanding string (3.25) in the power law inflationary universe. The potentials (2.24)-(2.25) become:

\[
V_{\perp}(\tau) = \frac{2p^3}{(p + 1)\tau^2}, \quad V_{\parallel}(\tau) = \frac{2p(2p^2 - 1)}{(p + 1)\tau^2}
\]  

(3.29)

and the Fourier transformed fluctuation equations take the form:

\[
\ddot{C}_{n\perp} + \frac{1}{\tau}\dot{C}_{n\perp} + \frac{1}{\tau^2}[n^2(p^2 - 1) - p^2]C_{n\perp} = 0,
\]  

(3.30)

\[
\ddot{C}_{n\parallel} + \frac{1}{\tau}\dot{C}_{n\parallel} + \frac{1}{\tau^2}[n^2(p^2 - 1) - 2p^2 + 1]C_{n\parallel} = 0.
\]  

(3.31)

These two equations are solved by:

\[
C_{n\perp}(\tau) = A_{n\perp}\tau^{\alpha_+} + B_{n\perp}\tau^{\alpha_-},
\]  

(3.32)

\[
C_{n\parallel}(\tau) = A_{n\parallel}\tau^{\beta_+} + B_{n\parallel}\tau^{\beta_-},
\]  

(3.33)

where:

\[
\alpha_\pm = \pm\sqrt{n^2(1 - p^2) + p^2},
\]  

(3.34)

\[
\beta_\pm = \pm\sqrt{n^2(1 - p^2) + 2p^2 - 1}.
\]  

(3.35)

Notice that for \( p > 1 \) and \( n \geq 2 \) (which is the case we are interested in, c.f. the discussion after eq. (2.27)):

\[
\alpha_+^2 - 1 = (1 - n^2)(p^2 - 1) < 0, \quad \beta_+^2 - 1 = (2 - n^2)(p^2 - 1) < 0.
\]  

(3.36)
so that:

\[ \alpha_\pm^2 < 1, \quad \beta_\pm^2 < 1. \]  

(3.37)

If \( \alpha_\pm^2 < 0 \) (\( \alpha_\pm \) purely imaginary) the fluctuations (3.32) are represented by regularly oscillating trigonometric functions in the form \( \sim \cos(\alpha_\pm |\log \tau|) \). If \( 0 < \alpha_\pm^2 < 1 \) (\( \alpha_\pm \) real) the fluctuations grow with a power of \( \tau \), but always strictly slower than \( \tau \). Finally if \( \alpha_\pm^2 = 0 \) (this is only possible for very special values of \( p \)) eq. (3.32) is not the general solution of eq. (3.30). Instead we find that the fluctuations grow as the logarithm of \( \tau \). In any case we find that the ratio of the \( C_{n\perp} \)-fluctuation amplitude to the radius of the unperturbed circular string (which grows proportionally to \( \tau \), eq. (3.25)) is a decreasing function of \( \tau \). The same conclusion is drawn for the fluctuations \( C_{n\parallel} \). This behaviour is very different from the behaviour found in de Sitter space. For the expanding string solutions (3.25) the \( |n| \geq 2 \) fluctuation modes are washed out by the inflationary expansion of the universe. The circular shape thus becomes more and more stable in this case.

4 Conclusion

In conclusion we have studied small fluctuations around circular strings in spatially flat FRW spacetimes. Our main results are the two separated second order differential equations (2.22)-(2.23) determining the fluctuations around a circular configuration, fulfilling eq. (2.8), in an arbitrary spacetime of the form (2.3) (or (2.5)). As special cases we considered Minkowski, de Sitter and arbitrary power law expanding universes. In de Sitter space we found that for all expanding strings the ratio of the mean fluctuation amplitude to the radius of the unperturbed circular string is constant. In power law inflationary universes, on the other hand, we found special expanding solutions becoming more and more "circular".

In principle it is straightforward to consider any other FRW spacetime: first one has to solve eq. (2.8), or at least find some special solutions, and then the linear equations (2.22)-(2.23) have to be analyzed. Our formalism can of course easily be generalized to describe fluctuations around spherical membranes also. In fact, eq. (2.16) holds for an arbitrary defect of arbitrary dimension embedded in an arbitrary curved spacetime of higher dimension.
5 Appendix

In this appendix we give the explicit expressions for the non-vanishing components of the Christoffel symbol and Riemann tensor, corresponding to the line element (2.5).

The metric:

\[ g_{tt} = -(1 - H^2 f^2), \quad g_{tf} = -H f, \quad g_{ff} = 1, \quad g_{\theta\theta} = f^2, \quad g_{\phi\phi} = f^2 \sin^2 \theta. \]

(5.1)

The Christoffel symbol:

\[ \Gamma^t_{tt} = H^3 f^2, \quad \Gamma^t_{tf} = -H^2 f, \quad \Gamma^t_{ff} = H, \quad \Gamma^f_{fj} = H^2 f, \quad \Gamma^f_{tf} = -H^3 f^2; \]

\[ \Gamma^f_{\phi\phi} = \sin^2 \theta, \quad \Gamma^f_{\theta\theta} = f^2 \sin^2 \theta, \quad \Gamma^f_{\phi\theta} = \frac{\Gamma^f_{\theta\phi}}{f}, \quad \Gamma^f_{\phi\phi} = \sin^2 \theta \cot \theta. \]

(5.2)

The Riemann tensor:

\[ R_{tt} = -H, t - H^2, \quad R_{tt\phi\phi} = \sin^2 \theta R_{\theta\theta\phi}, \quad R_{tt\phi\phi} = f^2 (H^4 f^2 - H^2 - H^2), \sin^2 \theta, \]

\[ R_{\theta\phi\phi} = H^2 f^4 \sin^2 \theta, \quad R_{\phi\theta\phi} = \sin^2 \theta R_{\phi\theta\phi} = -H^3 f^2 \sin^2 \theta, \]

\[ R_{\phi\phi\phi} = \sin^2 \theta R_{\phi\phi\phi} = H^2 f^2 \sin^2 \theta. \]

(5.3)

The Ricci tensor:

\[ R_{tt} = (H^2 f^2 - 3)H, t - 3H^2 (1 - H^2 f^2), \quad R_{tt} = -H f (H, t + 3H^2), \]

\[ R_{ff} = H, t + 3H^2, \quad R_{\phi\phi} = \sin^2 \theta R_{\phi\phi} = f^2 (H, t + 3H^2) \sin^2 \theta. \]

(5.4)

The scalar curvature:

\[ R = 6 (H, t + 2H^2). \]

(5.5)
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