CONVERGENCE ACCELERATION ALGORITHM VIA AN EQUATION RELATED TO THE LATTICE BOUSSINESQ EQUATION

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Abstract. The molecule solution of an equation related to the lattice Boussinesq equation is derived with the help of determinantal identities. It is shown that this equation can for certain sequences be used as a numerical convergence acceleration algorithm. Numerical examples with applications of this algorithm are presented.

Key words. Lattice Boussinesq equation, convergence acceleration algorithm, molecule solution

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1. Introduction. In recent years, it has been found that integrable systems are closely connected to certain numerical algorithms. This observation allows a fresh look at the research in both fields, and a lot of interesting work has been done lately.

For example, one step of the QR algorithm is equivalent to the time evolution of the finite nonperiodic Toda lattice [30]. Moreover, Wynn’s celebrated ε-algorithm is nothing but the fully discrete potential KdV equation [21, 24]. The continuous-time Toda equation leads to a new algorithm for computing the Laplace transform of a given analytic function [20]. The discrete Lotka–Volterra system has applications in numerical algorithms for computing singular values [16, 15, 32]. As far as the links between integrable systems and convergence acceleration algorithms are concerned, more results have recently been achieved (see, e.g., [8, 12, 18, 19]).

On the other hand, rapid progress has been made recently in the study of discrete integrable systems. As a result, many new examples of discrete integrable systems have been found, among them the lattice Gel’fand–Dikii hierarchy [22, 23]. However, to the best of our knowledge, nothing has been done so far on designing convergence acceleration algorithms via new discrete integrable systems. This is what we want to do in this article. Our starting point will be the lattice Boussinesq equation [22, 23], and we want to use it for the design of a new convergence acceleration algorithm.

In view of the importance of both discrete integrable systems and sequence transformations in the context of convergence acceleration algorithms, it might be helpful – before discussing more specific details – to first mention some basic facts about discrete integrable systems and about sequence transformations, and to explain how discrete integrable systems can be a source of inspiration for sequence transformations.

Discrete integrable systems could be considered as a specific class of discrete systems which possesses an important property of what is usually called integrability. In particular, the existence of a Lax pair and a τ-function are two of the most important features of integrability shared by some famous numerical algorithms, such as Rutishauser’s qd-algorithm [20] and Wynn’s ε-algorithm [30]. Let us also remark that corresponding to different boundary conditions there are different versions of discrete
integrable systems available. An example is the famous Toda equation

\[
\frac{d^2 x_k}{dt^2} = e^{x_{k-1} - x_k} - e^{x_k - x_{k+1}}. \tag{1.1}
\]

In the infinite chain case with \( k = 0, \pm 1, \pm 2, \ldots \), we call (1.1) the infinite Toda lattice equation. Under periodic boundary condition \( x_{k+K_0} = x_k \) with \( k = 0, \pm 1, \pm 2, \ldots \) and fixed \( K_0 > 0 \). (1.1) is referred to as the periodic Toda lattice equation. If \( k = 0, 1, 2, \ldots \) with the boundary condition \( x_0(t) = -\infty \), we call (1.1) the semi-infinite Toda equation or infinite Toda molecule equation. If \( x_0(t) = -\infty \) and \( x_{N+1} = +\infty \), we call (1.1) the finite nonperiodic Toda equation or finite molecule Toda equation.

In connection with convergence acceleration algorithms, we are exclusively interested in the semi-infinite or infinite molecule case of (1.1), corresponding to the semi-infinite or infinite Toda molecule equation, and the solutions obtained in this way are called molecule solutions. As it turns out, our molecule solutions are closely related to sequence transformations.

We now return to the lattice Boussinesq equation, which is the second equation of the lattice Gel’fand–Dikii hierarchy [22, 23]. The multisoliton solutions of the lattice Boussinesq equation defined on the elementary square were derived in [13, 31]. An ultradiscrete lattice Boussinesq equation and an alternate form of the discrete potential Boussinesq equation had been proposed and the multisoliton solutions of both equations had also been obtained [17]. The lattice Boussinesq equation is expressed as [22, 23]

\[
\frac{p^3 - q^3}{p - q + u_{l+1}^{m+1} - u_{l+2}^m} - \frac{p^3 - q^3}{p - q + u_{l+1}^{m+2} - u_{l+1}^{m+1}}
- u_{l+1}^{m+1} + u_{l+1}^m + u_{l+2}^m + u_{l+2}^{m+2} (p - q + u_{l+1}^{m+2} - u_{l+2}^{m+1})
+ u_{l+1}^m (p - q + u_{l+1}^m - u_{l+1}^m)
= (2p + q)(u_{l+1}^m + u_{l+2}^m) - (p + 2q)(u_{l+1}^m + u_{l+2}^m),
\]

which is equivalent to

\[
\frac{p^3 - q^3}{p - q + u_{l+1}^{m+1} - u_{l+2}^m} - \frac{p^3 - q^3}{p - q + u_{l+1}^{m+2} - u_{l+1}^{m+1}}
= (p + 2q + u_{l+1}^m - u_{l+1}^{m+2})(p - q + u_{l+1}^{m+2} - u_{l+1}^{m+1})
- (p + 2q + u_{l+1}^{m+1} - u_{l+1}^{m+2})(p - q + u_{l+1}^{m+1} - u_{l+1}^{m+1}). \tag{1.2}
\]

If we now set

\[
\bar{u}_l^m = u_l^m - pl - qm, \quad p^3 - q^3 = 1,
\]

we obtain from (1.2)

\[
\frac{1}{u_{l+1}^{m+1} - u_{l+2}^m} - \frac{1}{u_l^{m+2} - u_{l+1}^{m+1}}
= (\bar{u}_{l+1}^m - \bar{u}_{l+2}^{m+2})(\bar{u}_{l+1}^{m+2} - \bar{u}_{l+2}^m) - (\bar{u}_l^m - \bar{u}_{l+1}^{m+2})(\bar{u}_l^{m+1} - \bar{u}_{l+1}^{m+1}). \tag{1.3}
\]

With the help of the variable transformations

\[
n = -l, \quad k = m + l, \quad U_k^n = \bar{u}_l^m,
\]
we obtain from (1.3)

\[(U_{n+1}^{k+2} - U_{n+1}^{k+3}) - (U_{n+2}^{k+2} - U_{n+2}^{k+3})\]

\[= \frac{1}{U_{k+2}^{n+2} - U_{k+2}^{n+1}} - \frac{1}{U_{k+2}^{n+2} - U_{k+2}^{n+1}}.\]

This relationship can be simplified further yielding the following equation:

\[U_{k+3}^{n} = U_{k}^{n+1} - \frac{1}{(U_{k+2}^{n+2} - U_{k+2}^{n+1})(U_{k+1}^{n+2} - U_{k+1}^{n+1})}.\]  (1.4)

In this article, we will first derive the molecule solution of the two-dimensional difference equation (1.4). Then we will show that the resulting equation can be used for the acceleration of the convergence of computationally relevant sequences.

Our article is organized as follows: In section 2 we will derive the molecule solution of (1.4) with the help of determinantal identities. In section 3 we will provide a highly condensed review of the most basic features of sequence transformations. In section 4 we will use the results from section 2 to construct a new sequence transformation. We will also show that this transformation can be implemented by the lattice equation (1.4) with given initial values. In section 5 applications of this algorithm are presented. Section 6 is devoted to conclusions and discussions.

2. Molecule solution of the lattice equation (1.4). In this section, we study the molecule solution of the lattice equation (1.4) by Hirota’s bilinear method [14]. With the help of the dependent variable transformation

\[U_{k}^{n} = \frac{G_{k}^{n}}{F_{k}^{n}}.\]

we obtain the bilinear form of (1.4)

\[F_{k}^{n} G_{k}^{n+1} - F_{k+1}^{n+1} G_{k}^{n+1} = F_{k+1}^{n+1} F_{k}^{n+1},\]  (2.1)

\[F_{k}^{n+1} G_{k+1}^{n} - F_{k+1}^{n} G_{k+2}^{n} = F_{k+1}^{n} F_{k}^{n},\]  (2.2)

\[F_{k}^{n} F_{k+1}^{n+1} - F_{k+1}^{n+1} F_{k+2}^{n} = F_{k+3}^{n+1} F_{k-1}^{n+1}.\]  (2.3)

Equation (2.3) can be derived from (2.1)–(2.2) by eliminating the G’s.

Set

\[\Psi_{k}(v_{n}) = \begin{vmatrix} v_{n} & v_{n+1} & \cdots & v_{n+k-1} \\ \Delta^{2} v_{n} & \Delta^{2} v_{n+1} & \cdots & \Delta^{2} v_{n+k-1} \\ \Delta^{4} v_{n} & \Delta^{4} v_{n+1} & \cdots & \Delta^{4} v_{n+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{2k-2} v_{n} & \Delta^{2k-2} v_{n+1} & \cdots & \Delta^{2k-2} v_{n+k-1} \end{vmatrix}, \quad k = 1, 2, \ldots,\]

\[\Psi_{-1}(v_{n}) = 0, \quad \Psi_{0}(v_{n}) = 1,\]

and

\[\Phi_{k}(v_{n}) = \begin{vmatrix} n & n+1 & \cdots & n+k-1 \\ v_{n} & v_{n+1} & \cdots & v_{n+k-1} \\ \Delta^{2} v_{n} & \Delta^{2} v_{n+1} & \cdots & \Delta^{2} v_{n+k-1} \\ \Delta^{4} v_{n} & \Delta^{4} v_{n+1} & \cdots & \Delta^{4} v_{n+k-1} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{2k-4} v_{n} & \Delta^{2k-4} v_{n+1} & \cdots & \Delta^{2k-4} v_{n+k-1} \end{vmatrix}, \quad k = 1, 2, \ldots,\]

\[\Phi_{-1}(v_{n}) = 0, \quad \Phi_{0}(v_{n}) = 1.\]
The solution of an initial value problem related to equations (2.1)–(2.3) is given below.

**Theorem 1** Given the initial values

\[ F_1^n = F_2^n = F_3^n = 1, \quad F_4^n = \Delta S_n, \quad G_1^n = 0, \quad G_2^n = n, \quad G_3^n = S_n. \]  

(2.4)

the solution of the bilinear equations (2.1)–(2.3) can be expressed as follows:

\[ F_{3k}^n = \Psi_{k-1}(\Delta^3 S_n), \quad G_{3k}^n = \Psi_k(S_n), \]  

(2.5)

\[ F_{3k+1}^n = \Psi_k(\Delta S_n), \quad G_{3k+1}^n = -\Psi_{k-1}(\Delta^4 S_n), \]  

(2.6)

\[ F_{3k+2}^n = \Psi_k(\Delta^2 S_n), \quad G_{3k+2}^n = \Phi_{k+1}(\Delta S_n). \]  

(2.7)

Proof: We consider the case \( k = 3m \) in (2.1)–(2.3). First, we prove the validity of the relationship

\[ F_{3m}^n G_{3m}^{n+1} - F_{3m}^{n+1} G_{3m}^n = F_{3m+1}^n F_{3m-1}^n. \]  

(2.8)

Define

\[ D_1 = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ S_n & S_{n+1} & \cdots & S_{n+m} \\ \Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+m} \\ \vdots & \vdots & \ddots & \vdots \\ \Delta^{2m-2} S_n & \Delta^{2m-2} S_{n+1} & \cdots & \Delta^{2m-2} S_{n+m} \end{vmatrix}. \]

Then we have the relations

\[ D_1 = \Psi_m(\Delta S_n) = F_{3m+1}^n, \]  

(2.9)

\[ D_1(1,1,m+1) = \Psi_{m-1}(\Delta^2 S_{n+1}) = F_{3m-1}^{n+1}, \]  

(2.10)

\[ D_1(1,1) = \Psi_m(S_{n+1}) = G_{3m+1}^n, \]  

(2.11)

\[ D_1(2,m+1) = \Psi_{m-1}(\Delta^3 S_n) = F_{3m}^n, \]  

(2.12)

\[ D_1(1,m+1) = \Psi_m(S_n) = G_{3m}^n, \]  

(2.13)

\[ D_1(2,1) = \Psi_{m-1}(\Delta^2 S_{n+1}) = F_{3m+1}^n, \]  

(2.14)

where \( D_1(j,k) \) and \( D_1(j,k|p,q) \) are \( m \)-th-order and \((m-1)\)-th-order determinants obtained by eliminating the \( j \)-th row and the \( k \)-th column from the \( D_1 \) and by eliminating the \( j \)-th and \( k \)-th rows and the \( p \)-th and \( q \)-th columns from the determinant \( D_1 \), respectively.

From the above results, we see that the bilinear equation (2.8) is equivalent to the Jacobi identity [10]

\[ D_1 D_1(1,1,m+1) = D_1(1,1) D_1(2,m+1) - D_1(1,m+1) D_1(2,1). \]

Next, we prove the validity of another relationship

\[ F_{3m+2}^n G_{3m-1}^{n+1} - F_{3m-1}^{n+1} G_{3m+2}^n = F_{3m+1}^{n+1} F_{3m}^n. \]  

(2.15)
According to the assumptions of theorem 1, we have

\[
lhs = F_{3m+2}^n G_{3m+1}^{n+1} - F_{3m+1}^n G_{3m+2}^n
\]

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_n & \Delta^3 S_{n+1} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-1} S_n & \Delta^{2m-1} S_{n+1} & \cdots & \Delta^{2m-1} S_{n+m}
\end{vmatrix}
\times
\begin{vmatrix}
n + 1 & n + 2 & \cdots & n + m \\
\Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_{n+1} & \Delta^3 S_{n+2} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-3} S_{n+1} & \Delta^{2m-3} S_{n+2} & \cdots & \Delta^{2m-3} S_{n+m}
\end{vmatrix}
\]

\[
- \begin{vmatrix}
n & n + 1 & \cdots & n + m \\
\Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_n & \Delta^3 S_{n+1} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-1} S_n & \Delta^{2m-1} S_{n+1} & \cdots & \Delta^{2m-1} S_{n+m}
\end{vmatrix}
\times
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_{n+1} & \Delta^3 S_{n+2} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-3} S_{n+1} & \Delta^{2m-3} S_{n+2} & \cdots & \Delta^{2m-3} S_{n+m}
\end{vmatrix}
\]

By Schwein’s determinantal identity \([1]\) we obtain

\[
lhs = \begin{vmatrix}
1 & 1 & \cdots & 1 \\
\Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_n & \Delta^3 S_{n+1} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-3} S_n & \Delta^{2m-3} S_{n+1} & \cdots & \Delta^{2m-3} S_{n+m}
\end{vmatrix}
\times
\begin{vmatrix}
\Delta S_{n+1} & \Delta S_{n+2} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_{n+1} & \Delta^3 S_{n+2} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-1} S_{n+1} & \Delta^{2m-1} S_{n+2} & \cdots & \Delta^{2m-1} S_{n+m}
\end{vmatrix}
\]

We obviously have

\[
\begin{vmatrix}
1 & 1 & \cdots & 1 \\
\Delta S_n & \Delta S_{n+1} & \cdots & \Delta S_{n+m} \\
\Delta^3 S_n & \Delta^3 S_{n+1} & \cdots & \Delta^3 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-3} S_n & \Delta^{2m-3} S_{n+1} & \cdots & \Delta^{2m-3} S_{n+m}
\end{vmatrix}
\]
Then we have the relations

\[
\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+m} \\
\Delta^4 S_n & \Delta^4 S_{n+1} & \cdots & \Delta^4 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m-2} S_n & \Delta^{2m-2} S_{n+1} & \cdots & \Delta^{2m-2} S_{n+m} \\
\end{array}
\]

\[
= \Psi_{m-1}(\Delta^3 S_n) = F_{3m}^n.
\]

Thus, \( \text{lhs} = F_{3m}^n F_{3m+1}^{n+1} \), and (2.15) is proved.

Next, we prove the third relationship:

\[
F_{3m}^n F_{3m+2}^{n+1} - F_{3m+1}^n F_{3m+2}^n = F_{3m+3}^n F_{3m+1}^{n+1}.
\]

(2.16)

Define

\[
D_2 \equiv \begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+m} \\
\Delta^4 S_n & \Delta^4 S_{n+1} & \cdots & \Delta^4 S_{n+m} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2m} S_n & \Delta^{2m} S_{n+1} & \cdots & \Delta^{2m} S_{n+m} \\
\end{array}
\]

Then we have the relations

\[
D_2 = \Psi_m(\Delta^3 S_n) = F_{3m+3}^n,
\]

(2.17)

\[
D_2(1, m + 1|1, m + 1) = \Psi_{m-1}(\Delta^2 S_{n+1}) = F_{3m+1}^{n+1},
\]

(2.18)

\[
D_2(1|1) = \Psi_m(\Delta^2 S_{n+1}) = F_{3m+2}^{n+1},
\]

(2.19)

\[
D_2(m + 1|m + 1) = \Psi_{m-1}(\Delta^3 S_n) = F_{3m}^n,
\]

(2.20)

\[
D_2(1|m + 1) = \Psi_m(\Delta^3 S_n) = F_{3m+2}^n,
\]

(2.21)

\[
D_2(m + 1|1) = \Psi_{m-1}(\Delta^3 S_{n+1}) = F_{3m+1}^{n+1}.
\]

(2.22)

From the above results, we see that the bilinear equation (2.16) is nothing but the Jacobi identity [10].

\[
D_2D_2(1, m + 1|1, m + 1) = D_2(1|1)D_2(m + 1|m + 1) - D_2(1|m + 1)D_2(m + 1|1).
\]

The proof of other cases of (2.17)-(2.23) can be obtained in a similar way. □

3. General properties of sequence transformations. Many calculations produce results that are actually sequences whose rate of convergence is governed by one or several parameters. Unfortunately, it often happens that the rate of convergence of such a sequence \( \{S_n\} \) is so slow that the determination of a sufficiently accurate approximation to its limit \( S = S_\infty \) by increasing the index \( n \) does not work in practice. Another frequently occurring problem is that such a sequence \( \{S_n\} \) does not necessarily produce a convergent result as \( n \rightarrow \infty \) even if it actually corresponds to a meaningful mathematical object with a well-defined numerical value.
In such a situation, it can be extremely helpful to apply a so-called sequence transformation, which transforms the original sequence \{S_n\} to a new sequence \{S'_n\} with hopefully better numerical properties according to

\[ T: \{S_n\} \rightarrow \{S'_n\}. \]

In rudimentary form, sequence transformations have been known for centuries. Their older history is reviewed in an article [7] and a monograph [3] by Brezinski. More recent developments are discussed in two articles by Brezinski [5, 6]. There is also an extensive bibliography by Brezinski [4] containing more than 6000 references up to 1991.

The active research on sequence transformations is documented by the fact that in recent years quite a few specialized monographs or longer reviews have been published, for example the ones by Brezinski and Redivo Zaglia [9], Sidi [28], and Weiner [33]. Numerous other references can be found in [35, Appendix B].

Sequence transformations try to achieve an acceleration of convergence or a summation in the case of divergence by purely numerical means. Since, however, a computational algorithm can involve only a finite number of arithmetic operations, a sequence transformation \( T \) can associate only a finite subset of the input sequence \( \{S_n\} \) with an element \( S'_m \) of the transformed sequence.

All the commonly occurring sequence transformations \( T \) can be represented by an infinite set of doubly indexed quantities \( T_k^{(n)} \) with \( k, n \in \mathbb{N}_0 \). In the literature, the superscript \( n \) typically indicates the minimal index occurring in the finite subset \( \{S_n, S_{n+1}, \ldots, S_{n+\ell}\} \) with \( \ell = \ell(k) \) of sequence elements which are used for the computation of the transform \( T_k^{(n)} \), and the subscript \( k \) – usually called the order of the transformation – is a measure for the complexity of \( T_k^{(n)} \). Moreover, the \( T_k^{(n)} \) are gauged in such a way that \( T_0^{(n)} \) always corresponds to an untransformed sequence element according to \( T_0^{(n)} = S_n \).

The basic assumption of all sequence transformations is that the elements of a slowly convergent or divergent sequence \( \{S_n\} \), which could be the partial sums \( S_n = \sum_{k=0}^{n} a_k \) of an infinite series, can for all indices \( n \) be partitioned into a (generalized) limit \( S \) and a remainder or truncation error \( R_n \) according to \( S_n = S + R_n \). If the sequence \( \{S_n\} \) converges to its limit \( S \), the remainders \( R_n \) can be made negligible by increasing \( n \) as much as necessary. But many sequences converge so slowly that this does not work in practice. Increasing the index \( n \) also does not help in the case of a divergent sequence.

Alternatively, one can try to improve convergence or accomplish a summation by computing approximations to the remainders \( R_n \) which are then eliminated from the sequence elements \( S_n \), yielding a new sequence with elements \( S'_n = S + R'_n \). At least conceptually, this is what a sequence transformation tries to do.

Some transformations – for example the so-called Levin-type transformations discussed in [34] – possess closed form expressions. However, the vast majority of all known sequence transformations are defined by a recursive scheme.

The probably best known example of such a transformation is Wynn’s \( \varepsilon \)-algorithm [36], which corresponds to the following nonlinear two-dimensional recursive scheme:
\[ \varepsilon^{(n)}_{k} = \varepsilon^{(n+1)}_{k-1} + \frac{1}{\varepsilon^{(n+1)}_{k} - \varepsilon^{(n)}_{k}^{-1}}, \quad k, n \in \mathbb{N}_0. \] (3.1b)

The elements \( \varepsilon^{(n)}_{2k} \) with even subscripts provide approximations to the (generalized) limit \( S \) of the sequence \( \{S_n\} \) to be transformed, whereas the elements \( \varepsilon^{(n)}_{2k+1} \) with odd subscripts are only auxiliary quantities which diverge if the whole process converges.

A compact FORTRAN 77 program for the \( \varepsilon \)-algorithm as well as the underlying computational algorithm is described in [33, section 4.3], and in [25, p. 213], a translation of this FORTRAN 77 program to C can be found.

The two-dimensional recursive scheme (3.1) for the \( \varepsilon \)-algorithm was derived by making some assumptions about the mathematical nature of the truncation errors or remainders \( R_n = S_n - S \) of the elements of the sequence \( \{S_n\} \) which is to be transformed. Other sequence transformations defined by recursive schemes were also constructed by trying to eliminate the remainders of appropriate model sequences.

But a converse approach is also possible. We can take an equation in two discrete variables – for example, one based on integrable systems – and analyze if and under which conditions this equation can be used as a starting point for the construction of a sequence transformation.

Let us conclude this section with some useful terminology that is typical of the literature on sequence transformations. Assume that a sequence \( \{S_n\} \), which converges to some limit \( S \), satisfies

\[ \lim_{n \to \infty} \frac{S_{n+1} - S}{S_n - S} = \lim_{n \to \infty} \frac{R_{n+1}}{R_n} = \rho. \] (3.2)

If \( 0 < |\rho| < 1 \) holds, we say that the sequence \( \{S_n\} \) converges linearly; if \( \rho = 1 \) holds, we say that this sequence converges logarithmically; and if \( \rho = 0 \) holds, we say that it converges hyperlinearly. Of course, \( |\rho| > 1 \) implies that the sequence \( \{S_n\} \) diverges. Simple examples of linearly, logarithmically, and hyperlinearly convergent sequences are the partial sums of the geometric series \( 1/(1 - z) = \sum_{\nu=0}^{\infty} z^\nu \), of the Dirichlet series \( \zeta(s) = \sum_{\nu=0}^{\infty} (\nu + 1)^{-s} \) for the Riemann zeta function, and of the power series \( \exp(z) = \sum_{\nu=0}^{\infty} z^\nu/\nu! \) for the exponential function, respectively.

Let us assume that two sequences \( \{S_n\} \) and \( \{S'_n\} \) converge to the same limit \( S \). We say that the sequence \( \{S'_n\} \) converges more rapidly than \( \{S_n\} \) if

\[ \lim_{n \to \infty} \frac{S'_n - S}{S_n - S} = 0. \] (3.3)

4. A convergence acceleration algorithm. In this section, we propose a new sequence transformation and show that this transformation can be derived via the lattice equation (1.4).

We now consider a new sequence transformation defined as the following ratio of
determinants:

\[
T_k^{(n)} = \begin{vmatrix}
S_n & S_{n+1} & \cdots & S_{n+k} \\
\Delta^2 S_n & \Delta^2 S_{n+1} & \cdots & \Delta^2 S_{n+k} \\
\Delta^4 S_n & \Delta^4 S_{n+1} & \cdots & \Delta^4 S_{n+k} \\
\vdots & \vdots & \ddots & \vdots \\
\Delta^{2k} S_n & \Delta^{2k} S_{n+1} & \cdots & \Delta^{2k} S_{n+k}
\end{vmatrix}
\]  

(4.1)

Obviously this transformation is a particular case of the E-transformation which was first derived by Schneider [27] and later rederived independently by Hávıe [11] and Brezinski [2]. From the kernel of the E-transformation [9], the kernel of the transformation (4.1) is obtained in the theorem below.

**Theorem 2** A necessary and sufficient condition that \( T_k^{(n)} = S \) for all \( n \) is that

\[
S_n = S + a_1 \Delta^2 S_n + a_2 \Delta^4 S_n + \cdots + a_k \Delta^{2k} S_n,
\]

where \( a_i \) with \( i = 1, \ldots, k \) are some constants.

Theorem 1 implies that the transformation (4.1) can be implemented via the lattice equation

\[
U_{n+3}^k = U_{n+1}^k - \frac{1}{(U_{n+2}^k - U_{n+1}^k)(U_{n+1}^k - U_{n+1}^k)} \quad k = 1, 2, \ldots, n = 0, 1, \ldots
\]

(4.2)

with initial conditions

\[
U_1^n = 0, \quad U_2^n = n, \quad U_3^n = S_n, \quad n = 0, 1, \ldots
\]

(4.3)

We have

\[
T_k^{(n)} = U_{3k+3}^n.
\]

5. Numerical experiments. In this section, we will show how the convergence of some of some slowly convergent example sequences can be accelerated by the new algorithm (4.2)–(4.3).

**Example 1.** We consider the linearly convergent sequence

\[
S_n = 2^n \sin \left( \frac{\pi}{2^n} \right),
\]

which converges to \( S = \pi = 3.14159 26535 89793 \ldots \) The corresponding transformation results are presented in Table 5.1.

**Example 2.** We consider the sequence

\[
S_n = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k}
\]
Table 5.1
Numerical results of example 1.

| n  | $T_0^{(n)}$ | $T_1^{(n)}$ | $T_2^{(n)}$ | $T_3^{(n)}$ | $T_4^{(n)}$ |
|----|-------------|-------------|-------------|-------------|-------------|
| 1  | 2.00000     | 3.16790     | 3.14158     | 3.14159     | 3.14159     |
| 2  | 2.82843     | 3.14304     | 3.14159     | 3.14159     | 3.14159     |
| 3  | 3.06146     | 3.14168     | 3.14159     | 3.14159     | 3.14159     |
| 4  | 3.12145     | 3.14159     | 3.14159     | 3.14159     | 3.14159     |
| 5  | 3.13655     | 3.14159     | 3.14159     | 3.14159     | 3.14159     |

Table 5.2
Numerical results of example 2.

| n  | $T_0^{(n)}$ | $T_1^{(n)}$ | $T_2^{(n)}$ | $T_3^{(n)}$ | $T_4^{(n)}$ | $T_5^{(n)}$ |
|----|-------------|-------------|-------------|-------------|-------------|-------------|
| 1  | 1.00000     | 0.70588     | 0.69381     | 0.69318     | 0.69314     | 0.69314     |
| 2  | 0.50000     | 0.68817     | 0.69294     | 0.69313     | 0.69314     | 0.69314     |
| 3  | 0.83333     | 0.69557     | 0.69323     | 0.69315     | 0.69314     | 0.69314     |
| 4  | 0.58333     | 0.69178     | 0.69311     | 0.69314     | 0.69314     | 0.69314     |
| 5  | 0.78333     | 0.69399     | 0.69316     | 0.69314     | 0.69314     | 0.69314     |

of the partial sums of the alternating series, which converges to $S = \ln 2 = 0.69314 71805 \ldots$.

The corresponding transformation results are presented in Table 5.2.

**Example 3.** We consider the logarithmically convergent sequence

$$S_n = \sum_{k=1}^{n} \frac{1}{k^2}$$

of the partial sums of the Dirichlet series for $\zeta(2)$, which converges to $S = \frac{\pi^2}{6} = 1.64493 406684 \ldots$ The corresponding transformation results are presented in Table 5.3.

**Note**
The results given above show that our new algorithm accelerates convergence in the first two cases but not in the third case. As is well known [9], Wynn’s $\varepsilon$-algorithm, which corresponds to the discrete potential KdV equation, can accelerate linear convergence and the convergence of alternating series but fails to accelerate logarithmic convergence. Apparently, the discrete Boussinesq equation and the discrete potential KdV equation have similar properties from the integrable systems’ point of view. We therefore guess that the algorithm based on (1.4) related to the lattice Boussinesq equation has the similar acceleration properties as the $\varepsilon$-algorithm. The numerical results given above are in agreement with our conjecture.

6. Conclusions and discussions. In this article, we construct the molecule solution of (1.4) related to the lattice Boussinesq equation by Hirota’s bilinear method. We show that this equation can be used as a numerical convergence acceleration algorithm. Our numerical experiments show that this algorithm is effective for linearly convergent sequences and alternating series but fails in the case of logarithmic convergence. Also we have studied the confluent form of (1.4) in [29].

It is known that the fully discrete potential KdV equation and the lattice Boussinesq equation are the first two equations of the Gel’fand–Dikii hierarchy. Now we know that both of the equations have connections with convergence acceleration algorithms. Therefore, it is a natural idea to investigate further whether higher order
Table 5.3
Numerical results of example 3.

| n  | $T_0^{(n)}$ | $T_1^{(n)}$ | $T_2^{(n)}$ | $T_3^{(n)}$ | $T_4^{(n)}$ | $T_5^{(n)}$ | $T_6^{(n)}$ | $T_7^{(n)}$ |
|----|--------------|--------------|--------------|--------------|--------------|--------------|--------------|--------------|
| 1  | 1.00000      | 1.38462      | 1.49536      | 1.54487      | 1.57198      | 1.58872      | 1.59990      | 1.60782      |
| 2  | 1.25000      | 1.45686      | 1.52776      | 1.56266      | 1.59608      | 1.60511      | 1.61164      | 1.61721      |
| 3  | 1.36111      | 1.49794      | 1.54871      | 1.57512      | 1.59112      | 1.60175      | 1.60925      | 1.61486      |
| 4  | 1.42361      | 1.52436      | 1.56334      | 1.58432      | 1.59738      | 1.60625      | 1.61261      | 1.61721      |
| 5  | 1.46361      | 1.54276      | 1.57412      | 1.59138      | 1.60234      | 1.60990      | 1.61542      | 1.62011      |

members of the Gel’fand–Dikii hierarchy have relationships to other convergence acceleration algorithms. We will consider these problems in the future.

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