Abstract. The $n$-step mixed integer rounding (MIR) inequalities of Kianfar and Fathi [10] are valid inequalities for the mixed-integer knapsack set that are derived by using periodic $n$-step MIR functions and define facets for group problems. The mingling and 2-step mingling inequalities of Atamtürk and Günlük [4] are also derived based on MIR and they incorporate upper bounds on the integer variables. It has been shown that these inequalities are facet-defining for the mixed integer knapsack set under certain conditions and generalize several well-known valid inequalities. In this paper, we introduce new classes of valid inequalities for the mixed-integer knapsack set with bounded integer variables, which we call $n$-step mingling inequalities (for positive integer $n$). These inequalities incorporate upper bounds on integer variables into $n$-step MIR and, therefore, unify the concepts of $n$-step MIR and mingling in a single class of inequalities. Furthermore, we show that for each $n$, the $n$-step mingling inequality defines a facet for the mixed integer knapsack set under certain conditions. For $n = 2$, we extend the results of Atamtürk and Günlük on facet-defining properties of 2-step mingling inequalities. For $n \geq 3$, we present new facets for the mixed integer knapsack set. We also derive as a special case conditions under which the $n$-step MIR inequalities define facets for the mixed integer knapsack set. In addition, we show that $n$-step mingling can be used to generate new valid inequalities and facets based on covers and packs defined for mixed integer knapsack sets.
Mixed-integer rounding (MIR) is a simple yet powerful procedure for generating valid inequalities for mixed-integer programs (MIP) [13, 14, 15]. When used as cuts MIR inequalities are very effective for solving mixed-integer programs with unbounded integer variables. However, for problems with bounded variables lifting techniques tend to be more effective as they explicitly use the variable bound information, whereas the MIR procedure does not. In order to incorporate bound information on the integer variables into MIR Atamtürk and Günlük [4] introduce a simple procedure, called mingling. Mingling updates the coefficients of a base inequality after arranging terms suitably by using upper bounds of the variables. They also define 2-step mingling inequalities, which subsume MIR inequalities properly and the 2-step MIR of Dash and Günlük [5] under certain conditions. Mingling and 2-step mingling lead to strong valid inequalities for mixed-integer sets with bounded variables and facets of mixed-integer knapsack sets derived earlier by superadditive lifting techniques. In particular, mingling inequalities subsume continuous cover and reverse continuous cover inequalities of Marchand and Wolsey [11] for the mixed 0-1 knapsack problems; 2-step mingling inequalities subsume continuous integer knapsack cover and pack inequalities of Atamtürk [1, 3] for mixed integer knapsack problems.

In another direction, for a base mixed-integer constraint Kianfar and Fathi [10] introduce a different generalization of MIR called the \( n \)-step MIR inequalities. These inequalities are obtained by applying periodic \( n \)-step MIR functions [10] on the coefficients of a base inequality. They are facet-defining for the infinite group problem \([6, 7, 8, 9]\) and also for certain single-constraint mixed integer polyhedra. \( n \)-step MIR inequalities generalize the 2-step MIR inequalities of Dash and Günlük [5].

In this paper, we unify the concepts of mingling and \( n \)-step mixed-integer rounding to define a single class of valid inequalities, which we call the \( n \)-step mingling inequalities, for the mixed-integer knapsack set. These new inequalities incorporate upper bounds on integer variables into \((n-1)\)-step MIR with the mingling procedure. While the two-step mingling inequality does not subsume two-step MIR inequality properly, three-step inequality does. In general, \( n \)-step mingling inequalities subsume \((n-1)\)-step MIR inequalities.

It is important to note that although the \((n-1)\)-step MIR function is used to describe the coefficients of an \( n \)-step mingling inequality, the \( n \)-step mingling inequality is different from inequality one obtains by simply applying the \((n-1)\)-step MIR procedure on a mingling inequality. Indeed, as we show later in the paper, the \( n \)-step mingling inequality dominates the latter inequality.

We show that for each positive integer \( n \), the \( n \)-step mingling inequality defines a facet for the mixed-integer knapsack set under certain conditions. In the case of \( n = 2 \), our results extend the results of [4] on facet-defining
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properties of 2-step mingling. For \( n \geq 3 \), our results present new facets for the mixed integer knapsack set. We also derive conditions under which the \( n \)-step MIR inequalities of [10] define facets for the mixed-integer knapsack set as a special case. In addition, we show that \( n \)-step mingling can be used to generate new valid inequalities and facets based on covers and packs defined for a mixed-integer knapsack set.

We begin with a brief review of mingling in Sect. 2. In Sect. 3 we introduce the basic ideas of \( n \)-step mingling by presenting 3-step mingling inequality. We then present the general case of the \( n \)-step mingling inequality. In Sect. 4 we show that \( n \)-step mingling inequalities define new facets for the mixed-integer knapsack set under certain conditions, and also derive sufficient conditions under which the \( n \)-step MIR inequalities are facet-defining for this set. We present symmetric \( n \)-step mingling in Sect. 5 and \( n \)-step mingling cover and pack inequalities in Sect. 6. We conclude with a few final remarks in Sect. 7.

2. Mingling inequalities: a brief review

In this section we briefly review the mingling and 2-step mingling inequalities of Atamtürk and Günlük [4]. This review establishes the notation that will be used in the rest of the paper. First, recall that for the mixed-integer set defined as

\[
\sum_{i \in N} a_i x_i + s \geq b, \quad x \in \mathbb{Z}_+^N, \quad s \in \mathbb{R}
\]

the MIR inequality [13, 14, 15] with parameter \( \alpha > 0 \) (also known as the \( \alpha \)-MIR inequality) is

\[
\sum_{i \in N} \mu_{\alpha,b}(a_i) x_i + s \geq \mu_{\alpha,b}(b),
\]

where \( \mu_{\alpha,b} \) is the MIR function

\[
\mu_{\alpha,b}(t) := b^{(1)} \lfloor t/\alpha \rfloor + \min\{b^{(1)}, t^{(1)}\},
\]

where for any \( r \in \mathbb{R} \), \( r^{(1)} \) is defined as

\[
r^{(1)} := r - \alpha \lfloor r/\alpha \rfloor.
\]

We note that the nonnegativity of \( x_i, i \in N \), is necessary for the validity of the MIR inequality unless \( a_i/\alpha \in \mathbb{Z} \); see [5, 10, 12, 15].

Now consider the mixed-integer knapsack set with a single continuous variable and upper bounds on the integer variables

\[
K_\geq := \left\{ (x, s) \in \mathbb{Z}_+^{|N|} \times \mathbb{R}_+ : \sum_{i \in I} a_i x_i + \sum_{j \in J} a_j x_j + s \geq b, \quad x \leq u \right\},
\]

where \( (I, J) \) is a partitioning of \( N \) with \( a_i > 0 \) for \( i \in I \) and \( a_j < 0 \) for \( j \in J \). The upper bound on each integer variable can be either a positive integer or infinity. We assume in this section that \( b \geq 0 \). In Section 5 we will consider the case \( b \leq 0 \). It is clear that the MIR inequality (1) for \( K_\geq \) does not utilize the upper bounds \( u \).
The mingling inequality for $K_\geq$ does use the upper bounds $u_i$ and is derived as follows. Let $I^+ := \{1, \ldots, n\}$ be a subset of $\{i \in I : a_i > b\}$ indexed in non-increasing order of $a_i$’s, and

$$\bar{J} := \left\{ j \in J : a_j + \sum_{i \in I^+} a_i u_i < 0 \right\}.$$

For $j \in J \setminus \bar{J}$, we define a set $I_j$, an integer $k_j$, and the numbers $\bar{u}_{ij}$ such that $0 \leq \bar{u}_{ij} \leq u_i$ for $i \in I_j$ as follows:

$$I_j := \{1, \ldots, p(j)\}, \quad \text{where } p(j) := \min \left\{ p \in I^+ : a_j + \sum_{i=1}^{p} a_i u_i \geq 0 \right\};$$

$$k_j := \min \left\{ k \in \mathbb{Z}_+ : a_j + \sum_{i=1}^{p(j)-1} a_i u_i + a_{p(j)} k \geq 0 \right\};$$

$$\bar{u}_{ij} := \begin{cases} u_i, & \text{if } i < p(j), \\ k_j, & \text{if } i = p(j). \end{cases}$$

On the other hand for $j \in \bar{J}$, we simply let $I_j := I^+$, $p(j) := n$, $k_j := u_n$, and $\bar{u}_{ij} := u_i$ for $i \in I_j$. Also we define

$$\delta_j := \min \left\{ b, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij} \right\} \quad \text{for } j \in J,$$

and, therefore, we have $0 \leq \delta_j \leq b$ for $j \in J \setminus \bar{J}$, and $\delta_j < 0$ for $j \in \bar{J}$. Also for $i \in I$, let $J_i := \{j \in J : i \in I_j\}$; therefore, $J_i = \emptyset$ for $i \in I \setminus I^+$. Atamtürk and Günlük [4] prove that the mingling inequality

$$\sum_{i \in I^+} b \left[ x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j \right] + \sum_{i \in I \setminus I^+} a_i x_i + \sum_{j \in J} \delta_j x_j + s \geq b$$

(2)

is valid for $K_\geq$ and facet-defining for $\text{conv}(K_\geq)$ provided that $b - \min \{ \delta_j : j \in \bar{J}\} \geq \max \{a_i : a_i > b, i \in I \setminus I^+\}$. For $K_\geq$ they also introduce the 2-step mingling inequality

$$\sum_{i \in I^+} \mu_{\alpha,b}(b) \left[ x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j \right] + \sum_{i \in I \setminus I^+} \mu_{\alpha,b}(a_i) x_i + \sum_{j \in J} \mu_{\alpha,b}(\delta_j) x_j + s \geq \mu_{\alpha,b}(b),$$

(3)

which is valid for any $\alpha > 0$ such that $\alpha [b/\alpha] \leq \alpha_0 := \min \{a_i : i \in I^+\}$. They show that if $b > 0$, $J = \emptyset$, $I^+ = \{i \in I : a_i \geq \alpha [b/\alpha]\}$, and $\alpha = a_k$ for some $k \in I$, then inequality (3) becomes the continuous integer cover inequality [1] obtained by superadditive lifting [2], which has been shown to be facet-defining for $\text{conv}(K_\leq)$ in Atamtürk [1].

Observe that for $I^+ = \emptyset$, the two-step mingling inequality (3) reduces to the $\alpha$-MIR inequality (1); and if $J = \emptyset$ and $\alpha = \alpha_0$, then the two-step mingling inequality (3) reduces to the mingling inequality (2). Note that the 2-step mingling inequality (3) cannot be obtained by simply applying the MIR procedure with $\mu_{\alpha,b}$ on the mingling inequality (2) by considering
\[ x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j \] as a single integer variable because this expression is not necessarily nonnegative.

3. n-STEP MINGLING INEQUALITIES

In this section we introduce the n-step mingling inequalities. In order to present the basic ideas on an easier case, we first prove the validity of the 3-step mingling based on 2-step mingling. After we establish this base case, we use induction on \( n \) to prove the validity of \( n \)-step mingling for \( n \geq 3 \).

Let us define some new notation that will be used throughout the paper. Let \( \alpha = \{\alpha_1, \alpha_2, \ldots\} \) be a fixed sequence in \( \mathbb{R}_{>0} \). Then for \( r \in \mathbb{R} \) we define the following recursive remainders with respect to \( \alpha \)

\[
    r^{(q)} := r^{(q-1)} - \alpha_q \left\lfloor r^{(q-1)} / \alpha_q \right\rfloor,
\]

where \( r^{(0)} = r \). Based on the definition above, for any integer \( q \geq 1 \) and \( b \in \mathbb{R}_+ \) we can define a partitioning of \( \mathbb{R} \) as follows: for \( m = 0, \ldots, q - 1 \)

\[
    T^q_m := \{ t \in \mathbb{R} : t^{(k)} < b^{(k)}, \ k = 1, \ldots, m, \ t^{(m+1)} \geq b^{(m+1)} \};
\]

\[
    T^q_q := \{ t \in \mathbb{R} : t^{(k)} < b^{(k)}, \ k = 1, \ldots, q \}.
\]

For instance, for \( q = 1 \)

\[
    t \in \begin{cases} 
    T^1_0 & \text{if } t^{(1)} \geq b^{(1)}, \\
    T^1_1 & \text{if } t^{(1)} < b^{(1)};
\end{cases}
\]

and for \( q = 2 \)

\[
    t \in \begin{cases} 
    T^2_0 & \text{if } t^{(1)} \geq b^{(1)}, \\
    T^2_1 & \text{if } t^{(1)} < b^{(1)} \text{ and } t^{(2)} \geq b^{(2)}, \\
    T^2_2 & \text{if } t^{(1)} < b^{(1)} \text{ and } t^{(2)} < b^{(2)}.
\end{cases}
\]

3.1. 3-step mingling. Following the notation in Atamtürk and Günlük [4], for \( j \in J \) define \( \tilde{u}_{ij} := \bar{u}_{ij} - 1 \) if \( \delta_j = b \) and \( i = p(j) \); and \( \tilde{u}_{ij} := \bar{u}_{ij} \) otherwise. We also define

\[
    \tilde{\delta}_j := \begin{cases} 
    0, & \text{if } \delta_j = b, \\
    \delta_j, & \text{otherwise}.
\end{cases}
\]

In the process of proving validity of the 2-step mingling inequality in [4], it is proved that the inequality

\[
    \sum_{i \in I^+} \alpha_1 \left[ \frac{b}{\alpha_1} \right] \left[ x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j \right] + \sum_{i \in I^+} \sum_{j \in J} a_i x_i + \sum_{j \in J} \tilde{\delta}_j x_j + s \geq b \quad (4)
\]

is valid for the set \( K \geq \text{if } \alpha_1 \left[ b/\alpha_1 \right] \leq \alpha_0 := \min \{ a_i : i \in I^+ \} \). Inequality (4) can be relaxed to
is satisfied. Therefore, we will have

\[-\alpha_1 \left( \frac{b}{\alpha_1} \right) + \sum_{i \in I^+} \alpha_1 \left[ \frac{b}{\alpha_1} \right] x_i - \sum_{j \in J} \bar{u}_{ij} x_j + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i \]

\[\geq b^{(1)}, \]

which can also be written as follows because when \(\delta_j = b\) we have \(\tilde{\delta}_j = 0\) and \(\bar{u}_{ij} = \bar{u}_{ij} - 1.\)

\[-\alpha_1 \left( \frac{b}{\alpha_1} \right) + \sum_{i \in I^+} \alpha_1 \left[ \frac{b}{\alpha_1} \right] x_i - \sum_{j \in J} \bar{u}_{ij} x_j + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i \]

\[+ \sum_{i \in I \setminus I^+} \alpha_1 \left[ \frac{\tilde{\delta}_j}{\alpha_1} \right] x_j + \sum_{j \in J} \alpha_1 \left[ \frac{\delta_j}{\alpha_1} \right] x_j + \sum_{j \in J} \delta_j^{(1)} x_j + s \geq b^{(1)}, \]

(5)

On the other hand, the 2-step mingling inequality (3) in its open form can be written as

\[-b^{(1)} \left( \frac{b}{\alpha_1} \right) + \sum_{i \in I^+} b^{(1)} \left[ \frac{b}{\alpha_1} \right] x_i - \sum_{j \in J} \bar{u}_{ij} x_j + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i \]

\[+ \sum_{i \in I \setminus I^+} b^{(1)} \left[ \frac{\delta_j}{\alpha_1} \right] x_j + \sum_{j \in J} b^{(1)} \left[ \frac{\tilde{\delta}_j}{\alpha_1} \right] x_j + \sum_{j \in J} \delta_j^{(1)} x_j + s \geq b^{(1)}, \]

(6)

The last two inequalities imply that the following inequality is valid for any \(b^{(1)} \leq \gamma \leq \alpha_1.\)

\[-\gamma \left( \frac{b}{\alpha_1} \right) + \sum_{i \in I^+} \gamma \left[ \frac{b}{\alpha_1} \right] x_i - \sum_{j \in J} \bar{u}_{ij} x_j + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i + \sum_{i \in I \setminus I^+} \frac{a_i}{\alpha_1} x_i \]

\[+ \sum_{j \in J} \gamma \left[ \frac{\delta_j}{\alpha_1} \right] x_j + \sum_{j \in J} \gamma \left[ \frac{\tilde{\delta}_j}{\alpha_1} \right] x_j + \sum_{j \in J} \gamma \left[ \frac{\delta_j}{\alpha_1} \right] x_j + \sum_{j \in J} \delta_j^{(1)} x_j + s \geq b^{(1)}, \]

\[\gamma = \alpha_2 \left[ \frac{\delta_j^{(1)}}{\alpha_2} \right] \text{ as long as the condition } \alpha_2 \left[ \frac{b^{(1)}}{\alpha_2} \right] \leq \alpha_1 \]

is satisfied. Therefore, we will have
For the 3-step mingling inequality (8) can be written in a compact form as

\[
-\alpha_2 \left[ \frac{b^{(1)}}{\alpha_2} \right] x_i + \sum_{i \in I^+} \alpha_2 \left[ \frac{b^{(1)}}{\alpha_2} \right] x_i - \sum_{j \in J} \bar{a}_{ij} x_j + \sum_{\delta \in I_1^+} \alpha_2 \left[ \frac{\delta_j}{\alpha_1} \right] x_j \]

Now applying the MIR function \( \mu_{\alpha_2, b^{(1)}}^1 \) to (7) we arrive at the valid the 3-step mingling inequality:

\[
\sum_{i \in I^+} b^{(2)} \left[ \frac{b^{(1)}}{\alpha_2} \right] \left[ \frac{b}{\alpha_1} \right] x_i - \sum_{j \in J} \bar{a}_{ij} x_j + \sum_{\delta \in I_1^+} \alpha_2 \left[ \frac{\delta_j}{\alpha_1} \right] x_j \geq b^{(1)} \cdot (7)
\]

For \( \alpha = (\alpha_1, \alpha_2) \) and \( b \), defining

\[
\mu_{\alpha, b}^2 (t) = \begin{cases} 
\left[ \frac{b^{(1)}}{\alpha_2} \right] \left[ \frac{t}{\alpha_1} \right] b^{(2)} & \text{if } t \in I_0^2 \\
\left[ \frac{b^{(1)}}{\alpha_2} \right] \left[ \frac{t}{\alpha_1} \right] b^{(2)} + \left[ \frac{t^{(1)}}{\alpha_2} \right] b^{(2)} & \text{if } t \in I_1^2 \\
\left[ \frac{b^{(1)}}{\alpha_2} \right] \left[ \frac{t}{\alpha_1} \right] b^{(2)} + \left[ \frac{t^{(1)}}{\alpha_2} \right] b^{(2)} + t^{(2)} & \text{if } t \in I_2^2.
\end{cases}
\]

the 3-step mingling inequality (8) can be written in a compact form as

\[
\sum_{i \in I^+} \mu_{\alpha, b}^2 (b) x_i - \sum_{j \in J} \bar{a}_{ij} x_j + \sum_{i \in I^+} \mu_{\alpha, (a_i)}^2 x_i + \sum_{j \in J} \mu_{\alpha, (\delta_j)}^2 x_j \geq \mu_{\alpha, b}^2 (b)
\]
Hence, we have derived the 3-step mingling inequality, which is valid for $K \geq$ for $\alpha = (\alpha_1, \alpha_2)$ satisfying $\alpha_i [b^{i-1}/\alpha_i] \leq \alpha_{i-1}$ for $i = 1, 2$.

3.2. $n$-step mingling. The 2-step MIR function $\mu_{\alpha,b}^2$ used above is the special case of the $n$-step MIR function for $n = 2$. The $n$-step MIR function was introduced in [10] as a tool for producing $n$-step MIR inequalities for a set that is slightly different from $K \geq$, i.e., $Y = \{ x \in \mathbb{Z}^+ : \sum_{i=1}^N a_i x_i + a_1 z = b, z \in \mathbb{Z} \}$. With a small modification, for $K \geq$ we define the $n$-step MIR function for $\alpha = (\alpha_1, \ldots, \alpha_n)$ as follows:

$$\mu_{\alpha,b}^n(t) = \begin{cases} \sum_{k=1}^m \prod_{l=k+1}^m \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{t^{(l-1)}}{\alpha_k} \right] b^{(n)} \prod_{l=m+2}^n \left[ \frac{t^{(m)}}{\alpha_{m+1}} \right] b^{(n)} \quad \text{if } t \in \mathcal{T}_m^n; \\
\sum_{k=1}^n \prod_{l=k+1}^n \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{t^{(l-1)}}{\alpha_k} \right] b^{(n)} + t^{(n)} \quad \text{if } t \in \mathcal{T}_n^n 
\end{cases}$$

Accordingly, the $n$-step MIR inequality for $K \geq$ is

$$\sum_{i \in \mathbb{N}} \mu_{\alpha,b}^n(a_i) x_i + s \geq \mu_{\alpha,b}^n(b). \quad (10)$$

Clearly, inequality (10) does not use the information on upper bounds $u$.

The argument for the validity of the 3-step mingling inequality can be generalized to prove the validity of what we will refer to as the $n$-step mingling inequality for $K \geq$ (Theorem 1 below). For $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ satisfying $\alpha_k \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \leq \alpha_{k-1}$ for $k = 1, \ldots, n - 1$, the $n$-step mingling inequality is

$$\sum_{i \in \mathcal{I}^+} \mu_{\alpha,b}^{n-1}(b) \left[ x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j \right] + \sum_{i \in \mathcal{I} \setminus \mathcal{I}^+} \mu_{\alpha,b}^{n-1}(a_i) x_i + \sum_{j \in J} \mu_{\alpha,b}^{n-1}(\delta_j) x_j + s \geq \mu_{\alpha,b}^{n-1}(b). \quad (11)$$

We see that the $n$-step mingling inequality makes explicit use of the upper bound information. Observe that for $\mathcal{I}^+ = \emptyset$, $n$-step mingling inequality (11) reduces to the $(n-1)$-step MIR inequality (10). As in 2-step mingling, the $n$-step mingling inequality (for any $n$) cannot be obtained by simply applying $(n-1)$-step MIR function $\mu_{\alpha,b}^{n-1}$ on (2) by considering $[x_i - \sum_{j \in J_i} \bar{u}_{ij} x_j]$ as a single variable because this expression is not necessarily nonnegative.

**Remark 1.** Note that $(n-1)$-step MIR function is used to express the coefficients of the $n$-step mingling inequality. However, we should emphasize, that the $n$-step mingling inequality is different from inequality one obtains by simply applying the $(n-1)$-step MIR procedure on a mingling inequality. Indeed, the $n$-step mingling inequality dominates the latter as shown below.

By collecting terms, let us rewrite the mingling inequality (2) in its unmingled form

$$\sum_{i \in \mathcal{I}^+} b x_i + \sum_{i \in \mathcal{I} \setminus \mathcal{I}^+} a_i x_i + \sum_{j \in J} \left( \delta_j - \sum_{i \in J_i} \bar{u}_{ij} \right) x_j + s \geq b$$
so that each term has a nonnegative variable. Then, the \((n - 1)\)-step MIR inequality for the mingling inequality is

\[
\sum_{i \in I^+} \mu_{\alpha, b}^{n-1}(b) x_i + \sum_{i \in I^+} \mu_{\alpha, b}^{n-1}(a_i) x_i + \sum_{j \in J} \mu_{\alpha, b}^{n-1} \left( \delta_j - \sum_{i \in I_j} \bar{u}_{ij} \right) x_j + s \geq \mu_{\alpha, b}(b).
\]

Inequality (12) differs from the \(n\)-step mingling inequality (11) only in the coefficients of \(x_j, j \in J\). Comparing these coefficients, by subadditivity of the \((n - 1)\)-step MIR function [10], we see that

\[
\mu_{\alpha, b}^{n-1}(\delta_j) \leq \mu_{\alpha, b}^{n-1} \left( \delta_j - \sum_{i \in I_j} b \bar{u}_{ij} \right) + \mu_{\alpha, b}^{n-1} \left( \sum_{i \in I_j} b \bar{u}_{ij} \right)
\]

\[
\leq \mu_{\alpha, b}^{n-1} \left( \delta_j - \sum_{i \in I_j} b \bar{u}_{ij} \right) + \mu_{\alpha, b}^{n-1}(b) \sum_{i \in I_j} \bar{u}_{ij}.
\]

Therefore, the \((n - 1)\)-step MIR inequality for the mingling inequality is dominated by the \(n\)-step mingling inequality (11). Example 1 shows that the domination is strict.

**Theorem 1.** For \(n \geq 2\), the \(n\)-step mingling inequality (11) is valid for \(K \geq 1\) for a sequence \(\alpha\) satisfying \(\alpha_k \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \leq \alpha_{k-1} \) for \(k = 1, \ldots, n - 1\).

**Proof.** The case for \(n = 2\) was proved in [4]. For \(n \geq 3\) consider the inequality

\[
\begin{align*}
-\alpha_{n-1} \prod_{l=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] + \alpha_{n-1} \left[ \frac{b^{(n-2)}}{\alpha_{n-1}} \right] + \sum_{i \in I^+} \alpha_{n-1} \prod_{l=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \sum_{j \in J} \left[ \frac{a_i^{(n-1)}}{\alpha_{n-1}} \right] x_i & \\
+ \sum_{m=0}^{n-3} \sum_{i \in I^+} \sum_{k=1}^{m} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{a_i^{(k-1)}}{\alpha_k} \right] + \sum_{l=m+2}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{a_i^{(n-1)}}{\alpha_{m+1}} \right] x_i & \\
+ \sum_{m=0}^{n-3} \sum_{j \in J} \sum_{k=1}^{m} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(k-1)}}{\alpha_k} \right] + \sum_{l=m+2}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(m-1)}}{\alpha_{m+1}} \right] x_j & \\
+ \sum_{j \in J} \sum_{k=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(k-1)}}{\alpha_k} \right] \left[ \frac{\delta_j^{(n-2)}}{\alpha_{n-1}} \right] x_j + s & \geq b^{(n-2)}.
\end{align*}
\]
We have already shown in Sect. 3.1 that (13) and the \( n \)-step mingling inequality (11) are valid for \( n = 3 \) (inequality (13) for \( n = 3 \) reduces to inequality (7)). Now we use induction on \( n \). As the induction hypothesis we assume inequality (13) and the \( n \)-step mingling inequality are valid. We prove that inequality (13) is valid if \( n \) is replaced with \( n + 1 \) and then by applying a 1-step MIR function we prove that the \((n + 1)\)-step mingling inequality is valid. Inequality (13) can be relaxed to the following inequality much like the way (4) is relaxed in Sect. 3.1:

\[
- \alpha_{n-1} \prod_{l=1}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] + \alpha_{n-1} + \sum_{i \in I^+} \alpha_{n-1} \prod_{l=1}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ x_i - \sum_{j \in J_i} \tilde{u}_{ij} x_j \right] \\
+ \sum_{m=0}^{n-2} \sum_{i \in I^+} \left( \alpha_{n-1} \sum_{k=1}^{m} \prod_{l=k+1}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ \frac{a_i(k-1)}{\alpha_k} \right] + \prod_{l=m+2}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ \frac{a_i(m)}{\alpha_{m+1}} \right] \right) x_i \\
+ \sum_{m=0}^{n-2} \sum_{j \in J} \left( \alpha_{n-1} \sum_{k=1}^{m} \prod_{l=k+1}^{n-1} \left[ \frac{\delta_j(k-1)}{\alpha_l} \right] \left[ \frac{a_j(m)}{\alpha_{m+1}} \right] \right) x_j \\
+ \sum_{j \in J} \left( \alpha_{n-1} \sum_{k=1}^{m} \prod_{l=k+1}^{n-1} \left[ \frac{\delta_j(n-1)}{\alpha_l} \right] \left[ \frac{a_j(m)}{\alpha_{m+1}} \right] \right) x_j + s \geq b^{(n-1)}.
\]

(14)

On the other hand the \( n \)-step mingling inequality in its open form can be written as

\[
- b^{(n-1)} \prod_{l=1}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] + b^{(n-1)} + \sum_{i \in I^+} b^{(n-1)} \prod_{l=1}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ x_i - \sum_{j \in J_i} \tilde{u}_{ij} x_j \right] \\
+ \sum_{m=0}^{n-2} \sum_{i \in I^+} \left( b^{(n-1)} \sum_{k=1}^{m} \prod_{l=k+1}^{n-1} \left[ \frac{a_i(k-1)}{\alpha_k} \right] + \prod_{l=m+2}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ \frac{a_i(m)}{\alpha_{m+1}} \right] \right) x_i \\
+ \sum_{i \in I^+} \left( b^{(n-1)} \sum_{k=1}^{m} \prod_{l=k+1}^{n-1} \left[ \frac{a_i(k-1)}{\alpha_k} \right] + \prod_{l=m+2}^{n-1} \left[ \frac{b(l-1)}{\alpha_l} \right] \left[ \frac{a_i(m)}{\alpha_{m+1}} \right] \right) x_i.
\]
We see that inequality (16) is the same as inequality (13) where
\[ + \sum_{m=0}^{n-2} \sum_{j \in J} \left( b^{(n-1)} \sum_{k=1}^{m} \prod_{l=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(k-1)}}{\alpha_k} \right] + \prod_{l=m+2}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(m)}}{\alpha_{m+1}} \right] \right) x_j \]
\[ + \sum_{j \in J} \left( b^{(n-1)} \sum_{k=1}^{n-1} \prod_{l=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(k-1)}}{\alpha_k} \right] + \delta_j^{(n-1)} \right) x_j + s \geq b^{(n-1)}. \]

The only difference between (14) and (15) is that the multiplier \( \alpha_{n-1} \) in front of the terms in (14) is replaced with \( b^{(n-1)} \) in (15). Therefore the same valid inequality is valid if the multiplier \( \alpha_{n-1} \) in (14) is replaced with any \( \gamma \) satisfying \( b^{(n-1)} \leq \gamma \leq \alpha_{n-1} \). In particular, since \( b^{(n-1)} \leq \alpha_n \left[ \frac{b^{(n-1)}}{\alpha_n} \right] \leq \alpha_{n-1} \) we can replace the multiplier \( \alpha_{n-1} \) in (14) with \( \alpha_n \left[ \frac{b^{(n-1)}}{\alpha_n} \right] \) to arrive at the valid inequality
\[ - \alpha_n \prod_{i=1}^{n} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] + \alpha_n \left[ \frac{b^{(n-1)}}{\alpha_n} \right] + \sum_{i \in I^{+}} \alpha_n \prod_{l=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{a_i^{(k-1)}}{\alpha_k} \right] + \prod_{l=m+2}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{a_i^{(m)}}{\alpha_{m+1}} \right] \right) x_i \]
\[ + \sum_{i \in I^{+}} \left( \alpha_n \prod_{k=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{a_i^{(k-1)}}{\alpha_k} \right] + a_i^{(n-1)} \right) x_i \]
\[ + \sum_{j \in J} \left( \alpha_n \prod_{k=1}^{n-1} \left[ \frac{b^{(l-1)}}{\alpha_l} \right] \left[ \frac{\delta_j^{(k-1)}}{\alpha_k} \right] + \delta_j^{(n-1)} \right) x_j + s \geq b^{(n-1)}. \]

We see that inequality (16) is the same as inequality (13) where \( n \) is replaced with \( n+1 \). Now applying \( \mu_{\alpha_n, b^{(n-1)}} \) on inequality (16) gives the \( (n+1) \)-step mingling inequality
\[ \sum_{i \in I^{+}} \mu_{\alpha_i, b} \left[ x_i - \sum_{j \in I} \bar{u}_{ij} x_j \right] + \sum_{i \in I^{+}} \mu_{\alpha_i, b} (a_i) x_i + \sum_{j \in J} \mu_{\alpha, b} (\delta_j) x_j + s \geq \mu_{\alpha, b} (b), \]
which concludes the proof.
Remark 2 (A recursive formula for the $n$-step MIR function). If we generalize the notation introduced at the beginning of Sect. 3, we can write the $n$-step MIR function in a recursive form. Having the sequence of parameters \( \{\alpha_1, \alpha_2, \ldots\} \), we can generalize the notation \( r^{(q)} \) to \( r^{(\alpha_p, \ldots, \alpha_q)} \) for \( 1 \leq p \leq q \) as follows

\[
r^{(\alpha_p, \ldots, \alpha_q)} := \begin{cases} r^{(\alpha_p, \ldots, \alpha_{q-1})} - \alpha_q \frac{r^{(\alpha_p, \ldots, \alpha_{q-1})}}{\alpha_q}, & \text{if } p \leq q, \\ r - \alpha_q \frac{r}{\alpha_q}, & \text{if } p = q. \end{cases}
\]

Therefore \( r^{(q)} = r^{(\alpha_1, \ldots, \alpha_q)} \). Using this notation it is easy to show that the $n$-step MIR function defined explicitly in Sect. 3.2 can be written recursively as follows

\[
\mu^n_{(\alpha_1, \ldots, \alpha_n)}(t) = \mu^{n-1}_{(\alpha_2, \ldots, \alpha_n), b(\alpha_1)} \left( \frac{t}{\alpha_1} \right) + \min \left\{ \mu^{n-1}_{(\alpha_2, \ldots, \alpha_n), b(\alpha_1)} \left( \frac{t}{\alpha_1} \right), \mu^{n-1}_{(\alpha_2, \ldots, \alpha_n), b(\alpha_1)} \left( 1 \right) \right\},
\]

where \( \mu^1_{\alpha_1, b}(t) = b(\alpha_1) \left( \frac{t}{\alpha_1} \right) + \min \{ b(\alpha_1), t(\alpha_1) \} \) as defined in (1).

4. $n$-STEP MINGLING FACETS FOR MIXED-INTEGER KNAPSACK SETS

As our next main result, in this section we prove that, for any $n$, the $n$-step mingling inequalities are facet-defining for the mixed integer knapsack set under certain conditions. This makes $n$-step mingling a new way to generate facets for this set. Facets generated by $n$-step mingling for $n \geq 3$ (and also $n = 2$ where $J = \emptyset$) were not introduced in the literature before.

**Theorem 2.** For $n \geq 2$, the $n$-step mingling inequality (11) is facet-defining for $\text{conv}(K_\geq)$ if the following conditions are satisfied:

(i) \( b^{(n-1)} > 0 \) and \( \alpha_k = a_i \), where \( i_k \in I \setminus I^+ \) for \( k = 1, \ldots, n-1 \);

(ii) \( I^+ = \{ i \in I : a_i \geq \alpha_1 \left[ \frac{b}{\alpha_1} \right] \text{ and } \alpha_{k-1} \geq \alpha_k \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \text{ for } k = 2, \ldots, n-1 \}; \)

(iii) \( u_i \geq \left[ \frac{b}{\alpha_i} \right] - \left[ \frac{\min \{ 0, j \in J \} \alpha_i}{\alpha_i} \right] \text{ and } u_{ik} \geq \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \text{ for } k = 2, \ldots, n-1. \)

**Proof.** The validity of $n$-step mingling was proved in Theorem 1. Regarding condition (i), define \( I_\alpha \) as the subset of \( I \) that its corresponding coefficients are the parameters \( \alpha_1, \ldots, \alpha_{n-1} \), i.e. \( I_\alpha := \{ i_1, \ldots, i_{n-1} \} \). Below we list \( |I| + |J| + 1 \) affinely independent points in \( K_\geq \) on the face defined by (11). For each point we only describe the nonzero \( x \) components.

- The point \( s = b^{(n-1)} \); \( x_{ik} = \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \) for \( k = 1, \ldots, n-1 \);

- For each \( i_m \in I_\alpha \) where \( m \in \{ 1, \ldots, n-1 \} \), the point \( s = 0 \); \( x_{ik} = \left[ \frac{b^{(k-1)}}{\alpha_k} \right] \) for \( k = 1, \ldots, m-1 \); \( x_{im} = \left[ \frac{b^{(m-1)}}{\alpha_m} \right] \);
For each $i \in I^+ \setminus I_\alpha$, the point
\[ s = 0; \quad x_i = 1; \]

For each $i \in I \setminus (I^+ \cup I_\alpha)$ where $a_i \in \mathcal{I}_m^{n-1}$ and $m \in \{0, \ldots, n-2\}$, the point
\[ s = 0; \quad x_i = 1; \quad x_{i_k} = \left\lceil \frac{b_{(k-1)}}{\alpha_k} \right\rceil - \left\lfloor \frac{a_{(k-1)}}{\alpha_k} \right\rfloor \text{ for } k = 1, \ldots, m + 1; \]

For each $i \in I \setminus (I^+ \cup I_\alpha)$ where $a_i \in \mathcal{I}_m^{n-1}$, the point
\[ s = b_i^{(n-1)} - a_i^{(n-1)}; \quad x_i = 1; \quad x_{i_k} = \left\lceil \frac{b_{(k-1)}}{\alpha_k} \right\rceil - \left\lfloor \frac{a_{(k-1)}}{\alpha_k} \right\rfloor \text{ for } k = 1, \ldots, n - 1; \]

For each $j \in J$ where $\delta_j < b$ and $\delta_j \in \mathcal{I}_m^{n-1}$ and $m \in \{0, \ldots, n-2\}$, the point
\[ s = 0; \quad x_j = 1; \quad x_i = \bar{u}_{ij} \text{ for } i \in I_j; \quad x_{i_k} = \left\lceil \frac{b_{(k-1)}}{\alpha_k} \right\rceil - \left\lfloor \frac{\delta_{(k-1)}}{\alpha_k} \right\rfloor \text{ for } k = 1, \ldots, m + 1; \]

For each $j \in J$ where $\delta_j < b$ and $\delta_j \in \mathcal{I}_m^{n-1}$, the point
\[ s = b_i^{(n-1)} - \delta_j^{(n-1)}; \quad x_j = 1; \quad x_i = \bar{u}_{ij} \text{ for } i \in I_j; \quad x_{i_k} = \left\lceil \frac{b_{(k-1)}}{\alpha_k} \right\rceil - \left\lfloor \frac{\delta_{(k-1)}}{\alpha_k} \right\rfloor \text{ for } k = 1, \ldots, n - 1; \]

For each $j \in J$ where $\delta_j = b$, the point
\[ s = 0; \quad x_j = 1; \quad x_i = \bar{u}_{ij} \text{ for } i \in I_j; \]

It is not difficult to verify that given the conditions (i) to (iii) each point belongs to $\text{conv}(K_\geq)$ and satisfies (11) at equality. Furthermore it is easy to arrange these points as rows of a nonsingular lower triangular matrix (by suitable arrangement of columns). Therefore, these points are affinely independent, which concludes the proof. □ □

Note that condition (i) ensures that $\alpha_k < \alpha_0$ for $k = 1, \ldots, n - 1$. This is a natural choice, because as by condition (ii) we have $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-1}$ and $I^+ = \{i \in I : a_i \geq \alpha_1 [b/\alpha_1]\}$, if $i_p \in I^+$, we must have $i_1, \ldots, i_p \in I^+$ and $b \leq a_{i_1} = a_{i_2} = \ldots = a_{i_p}$ (or $b \leq \alpha_0 = \alpha_1 = \alpha_2 = \ldots = \alpha_p$). It is easy to verify that if this is true and $\bar{J} = \emptyset$, the $n$-step mingling inequality with $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ reduces to $(n - p)$-step mingling inequality with $\alpha = (\alpha_{p+1}, \ldots, \alpha_{n-1})$. However, if $\bar{J} \neq \emptyset$, inequalities are not comparable.

We also note that 2-step mingling inequalities are observed to be facet-defining in [4] when $\bar{J} = \emptyset$, in which case they are equivalent to the continuous cover inequalities for $K_\geq$, which have been shown to be facet-defining by superadditive lifting of a simple MIR inequality in [1]. Therefore, for the particular case of 2-step mingling, Theorem 2 extends the facet-defining property to the case where $\bar{J} \neq \emptyset$ with an alternative direct proof.

Below we present a numerical example illustrating the facets obtained by mingling, 2-step, and 3-step mingling and compare them with a direct application of 1-step and 2-step MIR on the mingling inequality. Higher step mingling inequalities can also be generated similarly.
Example 1. Consider the set $K_>$ defined by the inequality

$$37x_1 + 33x_2 + 31x_3 + 15x_4 + 13x_5 + 6x_6 - 63x_7 - 82x_8 - 107x_9 + s \geq 25$$

and let $u$ be the vector of upper bounds and $u_1 = u_2 = u_3 = 1$. We have $I = \{1, 2, 3, 4, 5, 6\}$ and $J = \{7, 8, 9\}$. For $I^+ = \{i \in I : a_i > b\} = \{1, 2, 3\}$, we have $J = \{9\}$, $I_7 = \{1, 2\}$, $I_8 = I_9 = \{1, 2, 3\}$, and so $J_1 = J_2 = \{7, 8, 9\}$ and $J_3 = \{8, 9\}$. Also $\delta_7 = 7$, $\delta_8 = 19$, $\delta_9 = -6$. Then the corresponding mingling inequality (2) is

$$25x_1 + 25x_2 + 25x_3 + 15x_4 + 13x_5 + 6x_6 - 43x_7 - 56x_8 - 81x_9 + s \geq 25, \quad (17)$$

which is facet-defining based on the choice of $I^+$ and Proposition 2 of [4]. In order to write a 2-step mingling inequality, we may choose $\alpha_1 = a_4 = 15$. So that we have a facet-defining inequality, based on Theorem 2, we choose $I^+ = \{i \in I : a_i \geq \alpha_1 [b/\alpha_1]\} = \{1, 2, 3\}$. Then the sets $\bar{I}$, $I_j$, and $J_i$ and the values of $\delta_j$’s are the same as above and conditions (i) and (ii) of Theorem 2 are satisfied. For condition (iii) to hold, we need $u_4 \geq \lceil \frac{b}{\alpha_1} \rceil = \lceil \min\{\delta_j : j \in J\} \rceil = 2$. In this case, $b, a_5 \in I_0^T$ and $a_4, a_6, \delta_7, \delta_8, \delta_9 \in I_1^T$. So the corresponding 2-step mingling inequality (3) is

$$20x_1 + 20x_2 + 20x_3 + 10x_4 + 10x_5 + 6x_6 - 33x_7 - 46x_8 - 61x_9 + s \geq 20, \quad (18)$$

which defines a facet if $u_4 \geq 2$. Based on Theorem 2 the list of 10 affinely independent feasible points that lie on this facet is as follows. They are arranged in a form that shows the lower triangular structure, and hence affine independence.

$$\begin{bmatrix}
x_1 & x_2 & x_3 & x_4 & s & x_5 & x_6 & x_7 & x_8 & x_9 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 10 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 4 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 3 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 6 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 2 & 1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

On the other hand, the MIR inequality for the mingling inequality (17) is

$$20x_1 + 20x_2 + 20x_3 + 10x_4 + 10x_5 + 6x_6 - 28x_7 - 36x_8 - 51x_9 + s \geq 20,$$

which is strictly dominated by the 2-step mingling inequality (18) as shown in Remark 1.

The 3-step mingling inequality can be constructed by choosing $\alpha_1 = a_4 = 15$ and $\alpha_2 = a_6 = 6$. For it to be facet-defining, the choice of $I^+$ will be the same as above. All other sets will remain the same as well. In order to satisfy condition (iii) of Theorem 2, we need $u_4 \geq 2$ and $u_6 \geq \lceil \frac{b(1)}{\alpha_2} \rceil = 2$. 


In this case, $b \in I^2_0$ and $a_4, a_5, a_6, \delta_7, \delta_8, \delta_9 \in I^2_2$. Then the corresponding 3-step mingling inequality (9) is

$$16x_1 + 16x_2 + 16x_3 + 8x_4 + 8x_5 + 4x_6 - 27x_7 - 36x_8 - 49x_9 + s \geq 16,$$

and it defines a facet if $u_4, u_6 \geq 2$. The list of 10 affinely independent feasible points that lie on this facet according to Theorem 2 is as follows:

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 2 & 0 & 1 & 0 & 0 & 0 & 1
\end{bmatrix}$$

On the other hand, the 2-step MIR inequality for the mingling inequality (17) is

$$16x_1 + 16x_2 + 16x_3 + 8x_4 + 8x_5 + 4x_6 - 22x_7 - 28x_8 - 41x_9 + s \geq 16,$$

which is strictly dominated by the 3-step mingling inequality (19) as shown in Remark 1.

Theorem 2 also makes it possible to derive sufficient facet-defining conditions for the $n$-step MIR inequalities of [10]. In the $(n+1)$-step mingling inequality, if we choose $I^+ = \emptyset$, we obtain the $n$-step MIR inequality. Using this fact and based on Theorem 2, we can state the following result about facet-defining property of $n$-step MIR inequalities for $K_{\geq}$:

**Corollary 1.** The $n$-step MIR inequality (10) defines a facet for $\text{conv}(K_{\geq})$ if the following conditions are satisfied:

(i) $b^{(n)} > 0$, $\alpha_k = a_{ik}$ where $i_k \in I$ for $k = 1, \ldots, n$;
(ii) $a_i \leq \alpha_1 \left[ b/\alpha_1 \right]$ for all $i \in I$, and $\alpha_{k-1} \geq \alpha_k \left[ b^{(k-1)}/\alpha_k \right]$ for $k = 2, \ldots, n$;
(iii) $u_i \geq \left[ b/\alpha_1 \right] - \left[ \min\{a_{ij} : j \in J\} / \alpha_1 \right]$ and $u_{ik} \geq \left[ b^{(k-1)}/\alpha_k \right]$ for $k = 2, \ldots, n$.

5. Symmetric $n$-step mingling inequalities

In this section we give a symmetric form of the $n$-step mingling inequalities for $K_{\geq}$ for the case $b \leq 0$. The approach is similar to the one used in [4] and is based on the correspondence between valid inequalities and facets for $K_{\geq}$ and

$$K_{\leq} = \{(x, t) \in \mathbb{Z}_+^N \times \mathbb{R}_+ : ax \leq b + t, x \leq u \}.$$
Theorem 3. For $K_n$ then we have 0 if and only if inequality $(a - \pi)x \leq b - \pi_0 + t$ is valid for $K_\leq$. Moreover, $\pi x + s \geq \pi_0$ is facet-defining for $\text{conv}(K_\geq)$ if and only if $(a - \pi)x \leq b - \pi_0 + t$ is facet-defining for $\text{conv}(K_\leq)$.

To write the symmetric $n$-step mingling inequality we update the coefficients of $x_i$, $i \in I$, in the base inequality of $K_\geq$ using the upper bounds of $x_j$, $j \in J$. Let $J^- := \{1, \ldots, m\}$ be a non-empty subset of $\{j \in J : a_j < b\}$ indexed in nondecreasing order of $a_j$’s, and $\bar{I} := \{i \in I : a_i + \sum_{i \in J^-} a_j u_j > 0\}$. For any $i \in I \setminus \bar{I}$, we define a set $J_i$, an integer $k_i$, and the numbers $0 \leq \bar{u}_{ji} \leq u_j$ for $j \in J_i$ as follows:

$$J_i := \{1, \ldots, p(i)\}, \quad p(i) := \min \left\{ p \in J^- : a_i + \sum_{j=1}^{p} a_j u_j \leq 0 \right\};$$

$$k_i := \min \left\{ k \in \mathbb{Z}_+ : a_i + \sum_{j=1}^{p(i)-1} a_j u_j + a_p k \leq 0 \right\};$$

$$\bar{u}_{ji} := \begin{cases} u_i, & \text{if } j < p(i), \\ k_i, & \text{if } j = p(i). \end{cases}$$

For $i \in I$, we let $J_i := J^-$, $p(i) := m$, $k_i := u_m$, and $\bar{u}_{ji} := u_j$ for $j \in J_i$. As a result, if we define

$$\lambda_i := \min \left\{ -b, -a_i - \sum_{j \in J_i} a_j \bar{u}_{ji} \right\} \quad \text{for } i \in I,$$

then we have $0 \leq \lambda_i \leq -b$ for $i \in I \setminus \bar{I}$, and $\lambda_i < 0$ for $i \in \bar{I}$. Also for $j \in J$, let $I_j := \{i \in I : j \in J_i\}$; therefore, $I_j = \emptyset$ for $j \in J \setminus J^-$. The symmetric $n$-step mingling inequality is defined as

$$\sum_{j \in J^-} \left( a_j + \mu_{\alpha_n}^{-1} (-b) \right) \left[ x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i \right] + \sum_{j \in J \setminus J^-} \left( a_j + \mu_{\alpha_n}^{-1} (-a_j) \right) x_j$$

$$+ \sum_{i \in I} \left( a_i + \sum_{j \in I_i} a_j \bar{u}_{ji} + \mu_{\alpha_n}^{-1} (\lambda_j) \right) x_i + s \geq b + \mu_{\alpha_n}^{-1} (-b). \quad (20)$$

Theorem 3. For $n \geq 2$, the symmetric $n$-step mingling inequality (20) is valid for $K_\geq$ (with $b \leq 0$) for a sequence $\alpha$ satisfying $\alpha_k \left[ \frac{(-b)^{(k-1)}}{\alpha_k} \right] \leq \alpha_{k-1}$ for $k = 1, \ldots, n - 1$, where $\alpha_0 := \max\{a_j : j \in J^-\}$. It is facet-defining for $\text{conv}(K_\geq)$ if the following conditions are satisfied:

(i) $(-b)^{(n-1)} > 0$ and $\alpha_k = -a_{j_k}$ where $j_k \in J \setminus J^-$ for $k = 1, \ldots, n - 1$;

(ii) $J^- = \{j \in J : a_j \leq \alpha_1 \ | b/\alpha_1 | \}$ and $\alpha_{k-1} \geq \alpha_k \left[ \frac{(-b)^{(k-1)}}{\alpha_k} \right]$ for $k = 2, \ldots, n - 1$;

(iii) $u_{ji} \geq \left[ \frac{-b}{\alpha_1} \right] - \left[ \frac{\min \{\lambda_i : i \in I\}}{\alpha_1} \right]$ and $u_{jk} \geq \left[ \frac{(-b)^{(k-1)}}{\alpha_k} \right]$ for $k = 2, \ldots, n - 1$. 

Lemma 1 ([4]). The inequality $\pi x + s \geq \pi_0$ is valid for $K_\geq$ if and only if inequality $(a - \pi)x \leq b - \pi_0 + t$ is valid for $K_\leq$. Moreover, $\pi x + s \geq \pi_0$ is facet-defining for $\text{conv}(K_\geq)$ if and only if $(a - \pi)x \leq b - \pi_0 + t$ is facet-defining for $\text{conv}(K_\leq)$.
Proof. The base inequality of $K_{\geq}$ when $b \leq 0$ can be written as $\sum_{j \in J} -a_j x_j + \sum_{i \in I} -a_i x_i \leq -b + s$ in the $K_{\leq}$ form. The corresponding $K_{\geq}$ form for this according to the Lemma 1 is $\sum_{j \in J} -a_j x_j + \sum_{i \in I} -a_i x_i + s \geq -b$. The n-step mingling inequality for this inequality is

$$\sum_{j \in J^-} \mu_{\alpha,-b}^{n-1} (-b) \left[ x_j - \sum_{i \in I_j} \bar{u}_{ji} x_i \right] + \sum_{j \in J \setminus J^-} \mu_{\alpha,-b}^{n-1} (-a_j) x_j$$

$$+ \sum_{i \in I} \mu_{\alpha,-b}^{n-1} (\lambda_j) x_i + s \geq \mu_{\alpha,-b}^{n-1} (-b). \quad (21)$$

Translating this inequality to the original $K_{\leq}$ form using Lemma 1 gives the symmetric n-step mingling inequality. The facet-defining conditions are the direct result of Theorem 2 and Lemma 1. \hfill \Box \Box

As a special case, for $J^- = \emptyset$ the symmetric n-step mingling inequality (20) reduces

$$\sum_{i \in N} \left( a_i + \mu_{\alpha,-b}^{n-1} (-a_i) \right) x_i + s \geq b + \mu_{\alpha,-b}^{n-1} (-b), \quad (22)$$

which is equivalent to the negative $(n-1)$-step MIR inequality in [10].

6. n-STEP MINGLING COVER AND PACK INEQUALITIES

Inequalities described in the previous sections can be used in connection with complementing bounded variables to derive n-step mingling generalizations of cover and pack inequalities [3] for mixed-integer knapsack sets. Consider the mixed-integer knapsack set with bounded integer variables

$$K_u^{\leq} := \left\{ (x, s) \in \mathbb{Z}_+^N \times \mathbb{R} : \sum_{i \in N} a_i x_i \leq b + s, \ x \leq u \right\},$$

where $a_i > 0$ for all $i \in N$. A subset $C$ of $N$ is a cover if $\beta := \sum_{i \in C} a_i u_i - b > 0$. After substitution $\bar{x}_i = u_i - x_i$, $i \in C$, the defining inequality of $K_u^{\leq}$ can be written as

$$\sum_{i \in C} a_i \bar{x}_i + \sum_{j \in N \setminus C} \bar{a}_j x_j + s \geq \beta. \quad (23)$$

Now for $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$ where $\alpha_{k-1} \geq \alpha_k \left[ \frac{b^{(k-1)}}{\alpha_k} \right]$ for $k = 2, \ldots, n-1$, by letting $I^+ \subseteq \{ i \in C : a_i \geq \alpha_1 \left[ \beta/\alpha_1 \right] \}$, and $J = N \setminus C$, the n-step mingling inequality for the base inequality (23) can be written as

$$\sum_{i \in I^+} \mu_{\alpha,\beta}^{n-1} (\beta) \left( \bar{x}_i - \sum_{j \in J_i} \bar{u}_{ji} x_j \right) + \sum_{i \in C \setminus I^+} \mu_{\alpha,\beta}^{n-1} (a_i) \bar{x}_i$$

$$+ \sum_{j \in J} \mu_{\alpha,\beta}^{n-1} \left( \min \{ \beta, a_j + \sum_{i \in I_j} a_i \bar{u}_{ij} \} \right) x_j + s \geq \mu_{\alpha,\beta}^{n-1} (\beta). \quad (24)$$
We call inequality (24) the n-step mingling cover inequality. For \( n = 2 \) inequality (24) reduces to the continuous integer cover inequality \([1, 3]\).

Alternatively, a subset \( P \) of \( N \) is called a pack if \( \theta := b - \sum_{i \in P} a_i u_i > 0 \). After substitution \( \bar{x}_i = u_i - x_i, i \in P \), the defining inequality of \( K^u_\leq \) can be written as

\[
\sum_{i \in P} a_i \bar{x}_i + \sum_{j \in N \setminus P} -a_j x_j + s \geq -\theta.
\]

For \( \alpha = (\alpha_1, \ldots, \alpha_{n-1}) \) where \( \alpha_{k-1} \geq \alpha_k \left\lfloor \frac{b(k-1)}{\alpha_k} \right\rfloor \) for \( k = 2, \ldots, n-1 \), by letting \( J^- \subseteq \{ j \in N \setminus P : a_j \geq \alpha_1 \left\lfloor \theta/\alpha_1 \right\rfloor \} \), and \( I = P \), the symmetric n-step mingling inequality for the base inequality above can be written as

\[
\sum_{j \in J^-} \left( -a_j + \mu_{\alpha, \theta}^{n-1}(\theta) \right) \left[ x_j - \sum_{i \in I_j} \bar{u}_{ji} \bar{x}_i \right] + \sum_{j \in J^- \setminus I^-} \left( -a_j + \mu_{\alpha, \theta}^{n-1}(a_j) \right) x_j
\]

\[
+ \sum_{i \in P} \left( a_i - \sum_{j \in J_i} a_j \bar{u}_{ji} + \mu_{\alpha, \theta}^{n-1} \left( \min \left\{ \theta, -a_i + \sum_{j \in J_i} a_j \bar{u}_{ji} \right\} \right) \right) x_i + s \geq -\theta + \mu_{\alpha, \theta}^{n-1}(\theta).
\]

We call inequality (25) the n-step mingling pack inequality. For \( n = 2 \) inequality (25) reduces to the continuous integer pack inequality \([1, 3]\). Facet-defining conditions for inequalities (24) and (25) can be easily derived using Theorem 2.

7. Concluding remarks

n-step mingling not only unifies two recent directions of research based on mixed-integer rounding (n-step MIR and mingling) but also generates new valid inequalities and facets for the mixed-integer knapsack set utilizing the bounds on the variables and period MIR functions. The facet-defining property of n-step mingling inequalities for mixed integer knapsack sets suggests that these inequalities can be effective as cutting planes for solving MIPs. A particularly appealing feature of the n-step mingling inequalities is that, while their derivation is involved, they can be described in a simple compact form and be implemented using the combination of existing mingling and n-step MIR routines and constraint aggregation \([12]\) routines to generate base inequalities.

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