On irreducible holonomy algebras of Riemannian supermanifolds

Anton S. Galaev*

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Abstract

Possible irreducible holonomy algebras \( g \subset \mathfrak{osp}(p, q|2m) \) of Riemannian supermanifolds under the assumption that \( g \) is a direct some of simple Lie superalgebras of classical type and possibly of a one-dimensional center are classified.

Keywords: Riemannian supermanifold, Levi-Civita superconnection, holonomy algebra, Berger superalgebra

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Contents

1 Introduction 2
2 The main results 3
3 Structure of the space \( \mathcal{R}(g) \) 5
4 Proof of the Main Theorem 9
A Prolongations of Lie algebras 16
B Holonomy algebras of pseudo-Riemannian manifolds 18
C Weak-Berger algebras 20

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Possible irreducible holonomy algebras of non-locally symmetric pseudo-Riemannian manifolds are classified by M. Berger in [4]. These algebras are the following: $\mathfrak{so}(p, q)$, $\mathfrak{so}(p, \mathbb{C})$, $\mathfrak{u}(r, s)$, $\mathfrak{su}(r, s)$, $\mathfrak{sp}(r, s)$, $\mathfrak{sp}(r, s) \oplus \mathfrak{sp}(1)$, $\mathfrak{sp}(r, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{sp}(r, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, $\mathfrak{spin}(4, 3) \subset \mathfrak{so}(4, 4)$, $\mathfrak{spin}(7, \mathbb{C}) \subset \mathfrak{so}(8, 8)$, $G_2 \subset \mathfrak{so}(7)$, $G_{2(2)} \subset \mathfrak{so}(4, 3)$ and $G_2^C \subset \mathfrak{so}(7, 7)$. This result, especially in the case of Riemannian manifolds, has a lot of consequences and applications both in geometry and physics, see Appendix B, the references [1, 3, 6, 19, 20] and the references therein.

Holonomy groups and holonomy algebras for superconnections on supermanifolds are introduced recently in [13]. In the present paper we generalize the result of M. Berger for the case of Riemannian supermanifolds.

For a supersubalgebra $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$ we define the vector superspace $\mathcal{R}(\mathfrak{g})$ of curvature tensors of type $\mathfrak{g}$ that consists of linear maps from $\Lambda^2 \mathbb{R}^{p,q|2m}$ to $\mathfrak{g}$ satisfying the Bianchi super identity. We call $\mathfrak{g}$ a Berger superalgebra if $\mathfrak{g}$ is spanned by the images of the elements of $\mathcal{R}(\mathfrak{g})$.

The holonomy algebra of a Riemannian supermanifold is a Berger superalgebra. Consequently, Berger superalgebras may be considered as the candidates to the holonomy algebras. A Berger superalgebra $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$ is called symmetric if the superspace $\nabla \mathcal{R}(\mathfrak{g})$, consisting of linear maps from $\mathbb{R}^{p,q|2m}$ to $\mathcal{R}(\mathfrak{g})$ satisfying the second Bianchi super identity, is trivial. If such $\mathfrak{g}$ is the holonomy algebra of a Riemannian manifold, then this manifold is locally symmetric. The classification of irreducible symmetric Berger superalgebras can be reduced from [40]. In the present paper we classify irreducible non-symmetric Berger supersubalgebras $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$ of the form

$$g = (\oplus i \mathfrak{g}_i) \oplus \mathfrak{j},$$

where $\mathfrak{g}_i$ are simple Lie superalgebras of classical type and $\mathfrak{j}$ is a trivial or one-dimensional center. The obtained list is the following:

$\mathfrak{osp}(p, q|2m)$, $\mathfrak{osp}(r|2k, \mathbb{C})$, $\mathfrak{u}(p_0, q_0|p_1, q_1)$, $\mathfrak{su}(p_0, q_0|p_1, q_1)$,
$\mathfrak{hosp}(r, s|k)$, $\mathfrak{hosp}(r, s|k) \oplus \mathfrak{sp}(1)$, $\mathfrak{osp}^{sk}(2k|r, s) \oplus \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{osp}^{sk}(2k|r) \oplus \mathfrak{sl}(2, \mathbb{C})$.

This list generalizes the list of irreducible holonomy algebras of non-locally symmetric pseudo-Riemannian manifolds, in the same time, we do not get the analogs of the important holonomy algebras $G_2$ and $\mathfrak{spin}(7)$.

Remark that in general a semi-simple Lie superalgebra $\mathfrak{g}$ has the form $\mathfrak{g} = \oplus_i (\mathfrak{g}_i \otimes \Lambda(n_i))$, where $\mathfrak{g}_i$ is a simple Lie superalgebra (either of classical or of Cartan type), and $\Lambda(n_i)$ is the
Grassmann superalgebra in $n_t$ variables \[22\]. Moreover, there exist representations of solvable Lie superalgebras in vector superspaces of dimensions greater than one \[22\]. Thus we consider only a part of irreducible subalgebras $g \subset \mathfrak{osp}(p, q|2m)$.

In Section 2 we formulate the main result of the paper. The remaining sections are dedicated to the proof. In Section 3 we describe the superspaces $\mathcal{R}(g)$ for subalgebras $g \subset \mathfrak{osp}(p, q|2m)$. Let $V = \mathbb{R}^{p, q|2m}$. We show that any element $R \in \mathcal{R}(g)$ satisfies $R|_{\Lambda^2 V_0} \in \mathcal{R}(\text{pr}_{\mathfrak{so}(p,q)} g_0)$ and $R|_{\Lambda^2 V_1} \in \tilde{R}(\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0)$, where $\mathcal{R}(\text{pr}_{\mathfrak{so}(p,q)} g_0)$ is the space of curvature tensors for the subalgebra $\text{pr}_{\mathfrak{so}(p,q)} g_0 \subset \mathfrak{so}(p,q)$ and $\tilde{R}(\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0)$ is a similar space for the subalgebra $\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0 \subset \mathfrak{sp}(2m, \mathbb{R})$. Next, any $R \in \mathcal{R}(g)_1$ satisfies $R(\cdot, \xi)|_{V_0} \in \mathcal{P}_\eta(\text{pr}_{\mathfrak{so}(p,q)} g_0)$ and $R(\cdot, x)|_{V_1} \in \mathcal{P}_\omega(\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0)$, where $x \in V_0$ and $\xi \in V_1$ are fixed, $\mathcal{P}_\eta(\text{pr}_{\mathfrak{so}(p,q)} g_0)$ is the space of weak curvature tensors for the subalgebra $\text{pr}_{\mathfrak{so}(p,q)} g_0 \subset \mathfrak{so}(p,q)$ and $\mathcal{P}_\omega(\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0)$ is a similar space for the subalgebra $\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0 \subset \mathfrak{sp}(2m, \mathbb{R})$. Elements of the space $\mathcal{P}_\eta(\mathfrak{so}(n))$ appear as a part of the curvature tensor of a Lorentzian manifold \[21\]. These properties of the space $\mathcal{R}(g)$ give strong conditions on the representation $g_0 \subset \mathfrak{so}(p,q) \oplus \mathfrak{sp}(2m, \mathbb{R})$ of the even part of $g$. We prove that under some assumption $\text{pr}_{\mathfrak{so}(p,q)} g_0 \subset \mathfrak{so}(p,q)$ is the holonomy algebra of a pseudo-Riemannian manifold and $\text{pr}_{\mathfrak{osp}(2m,\mathbb{R})} g_0 \subset \mathfrak{sp}(2m, \mathbb{R})$ satisfies a similar strong condition. These facts allow us to prove the classification theorem.

In Appendix A we classify irreducible subalgebras $g \subset \mathfrak{gl}(n, \mathbb{F})$ ($\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$) such that the skew-symmetric prolongation $g^{[1]} = \{ \varphi \in (\mathbb{F}^n)^* \otimes g | \varphi(x)y = -\varphi(y)x \text{ for all } x, y \in \mathbb{F}^n \}$ of $g$ is non-zero. Irreducible subalgebras $g \subset \mathfrak{so}(n, \mathbb{R})$ with non-zero skew-symmetric prolongation are classified in \[32\]. In \[32\] some applications of this classification are obtained.

In Appendix B we recall some facts about holonomy algebras of pseudo-Riemannian manifolds. In Appendix C we discuss reductive subalgebras $g \subset \mathfrak{so}(p,q)$ with $\mathcal{P}_\eta(g) \neq 0$. In Appendix D we consider reductive subalgebras $g \subset \mathfrak{osp}(2m, \mathbb{R})$ with $\tilde{R}(g) \neq 0$ and $\mathcal{P}_\omega(g) \neq 0$.

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## 2 The main results

Let $V = V_0 \oplus V_1$ be a real or complex vector superspace and $g \subset \mathfrak{gl}(V)$ a supersubalgebra. The space of algebraic curvature tensors of type $g$ is the vector superspace $\mathcal{R}(g) = \mathcal{R}(g)_0 \oplus \mathcal{R}(g)_1$, where

$$\mathcal{R}(g) = \left\{ R \in \Lambda^2 V^* \otimes g \left| R(X,Y)Z + (\pm 1)^{|X||Y|+|Z|}R(Y,Z)X + (\pm 1)^{|Y||X|+|Z|}R(Z,X)Y = 0 \right| \text{ for all homogeneous } X, Y, Z \in V \right\}.$$
Here \(|\cdot|\) ∈ \(\mathbb{Z}_2\) denotes the parity. The identity that satisfy the elements \(R \in \mathcal{R}(\mathfrak{g})\) is called the Bianchi super identity. Obviously, \(\mathcal{R}(\mathfrak{g})\) is a \(\mathfrak{g}\)-module with respect to the action

\[
A \cdot R = R_A, \quad R_A(X, Y) = [A, R(X, Y)] - (-1)^{|A||R|} R(AX, Y) - (-1)^{|A|(|R| + |X|)} R(X, AY),
\]

where \(A \in \mathfrak{g}, R \in \mathcal{R}(\mathfrak{g})\) and \(X, Y \in V\) are homogeneous.

Define the vector supersubspace

\[
L(\mathcal{R}(\mathfrak{g})) = \text{span}\{R(X, Y) | R \in \mathcal{R}(\mathfrak{g}), X, Y \in V\} \subset \mathfrak{g}.
\]

From (2) it follows that \(L(\mathcal{R}(\mathfrak{g}))\) is an ideal in \(\mathfrak{g}\). We call a supersubalgebra \(\mathfrak{g} \subset \mathfrak{gl}(V)\) a Berger superalgebra if \(L(\mathcal{R}(\mathfrak{g})) = \mathfrak{g}\).

If \(V\) is a vector space, which can be considered as a vector superspace with the trivial odd part, then \(\mathfrak{g} \subset \mathfrak{gl}(V)\) is a usual Lie algebra, which can be considered as a Lie superalgebra with the trivial odd part. Berger superalgebras in this case are the same as the usual Berger algebras.

**Proposition 2.1** [13] Let \(\mathcal{M}\) be a supermanifold of dimension \(n|m\) with a linear torsion-free connection \(\nabla\). Then its holonomy algebra \(\text{hol}(\nabla) \subset \mathfrak{gl}(n|m, \mathbb{R})\) is a Berger superalgebra.

Consider the vector superspace

\[
\mathcal{R}^\nabla(\mathfrak{g}) = \left\{ S \in V^* \otimes \mathcal{R}(\mathfrak{g}) \left| S_X(Y, Z) + (-1)^{|X||Y|+|Z|} S_Y(Z, X) + (-1)^{|Z||X|+|Y|} S_Z(X, Y) = 0 \right. \right\}.
\]

A Berger superalgebra \(\mathfrak{g}\) is called symmetric if \(\mathcal{R}^\nabla(\mathfrak{g}) = 0\). This is a generalization of the usual symmetric Berger algebras, see e.g. [36], and the following is a generalization of the well-known fact about smooth manifolds.

**Proposition 2.2** [13] Let \(\mathcal{M}\) be a supermanifold with a torsion-free connection \(\nabla\). If \(\text{hol}(\nabla)\) is a symmetric Berger superalgebra, then \((\mathcal{M}, \nabla)\) is locally symmetric (i.e. \(\nabla R = 0\)). If \((\mathcal{M}, \nabla)\) is a locally symmetric superspace, then its curvature tensor at any point is annihilated by the holonomy algebra at this point and its image coincides with the holonomy algebra.

The proof of the following proposition is as in [36].

**Proposition 2.3** Let \(\mathfrak{g} \subset \mathfrak{gl}(V)\) be an irreducible Berger superalgebra. If \(\mathfrak{g}\) annihilates the module \(\mathcal{R}(\mathfrak{g})\), then \(\mathfrak{g}\) is a symmetric Berger superalgebra.

Here is the Main Theorem of this paper.

**Theorem 2.1** Let \(\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)\) be an irreducible non-symmetric Berger supersubalgebra of the form (11). Then \(\mathfrak{g} \subset \mathfrak{osp}(p,q|2m)\) coincides with one of the Lie superalgebras from Table [4].
Table 1 Possible irreducible holonomy algebras $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$ of the from \([\mathfrak{g}]\) of Riemannian supermanifolds

| $\mathfrak{g}$ | $(p, q|2m)$ |
|----------------|------------|
| $\mathfrak{osp}(p, q|2m)$ | $(p, q|2m)$ |
| $\mathfrak{osp}(p|2k, \mathbb{C})$ | $(p, p|4k)$ |
| $\mathfrak{u}(p_0, q_0|p_1, q_1)$ | $(2p_0, 2q_0|2p_1 + 2q_1)$ |
| $\mathfrak{su}(p_0, q_0|p_1, q_1)$ | $(2p_0, 2q_0|2p_1 + 2q_1)$ |
| $\mathfrak{hosp}(r, s|k)$ | $(4r, 4s|4k)$ |
| $\mathfrak{hosp}(r, s|k) \oplus \mathfrak{sp}(1)$ | $(4r, 4s|4k)$ |
| $\mathfrak{osp}^s(2k|r, s) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $(2k, 2k|2r + 2s)$ |
| $\mathfrak{osp}^s(2k|r) \oplus \mathfrak{sl}(2, \mathbb{C})$ | $(4k, 4k|4r)$ |

3 Structure of the space $\mathcal{R}(\mathfrak{g})$

Consider a subalgebra $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m) = \mathfrak{osp}(V)$ and describe the space $\mathcal{R}(\mathfrak{g})$. Denote the supersymmetric metric on $V$ by $g$. It can be represented as the sum $g = \eta + \omega$, where $\eta$ is a pseudo-Euclidean metric of signature $(p, q)$ on $V^\mathbb{C} = \mathbb{R}^{p+q}$ and $\omega$ is a symplectic form on $V = \mathbb{R}^{2m}$. From the Bianchi super identity it follows that any $R \in \mathcal{R}(\mathfrak{g})$ satisfies

$$g(R(X, Y)Z, W) = (-1)^{|X||Y|(|Z||W|)}g(R(Z, W)X, Y)$$

for all homogeneous $X, Y, Z, W \in V$. This means that $R : \Lambda^2 V \to \mathfrak{g} \subset \Lambda^2 V$ is a symmetric map. In particular, $R$ is zero on the orthogonal complement $\mathfrak{g}^\perp \subset \Lambda^2 V$. Note that we identify $\mathfrak{osp}(V)$ with $\Lambda^2 V$, the element $X \wedge Y \in \mathfrak{osp}(V)$ is defined by $(X \wedge Y)Z = (-1)^{|Y||Z|}g(X, Z)Y - (-1)^{|Y||Z|+|X|}g(Y, Z)X$, where $X, Y, Z \in V$ are homogeneous.

First consider that space $\mathcal{R}(\mathfrak{g})_0$. Let $R \in (\Lambda^2 V^* \otimes \mathfrak{g})_0$. Define the following maps

$$A = \text{pr}_{\mathfrak{so}(p,q)} \mathfrak{g}_0 \circ R|_{\Lambda^2 V_0 \otimes \Lambda^2 V_1} : \Lambda^2 V_0 \otimes \Lambda^2 V_1 \to \mathfrak{pr}_{\mathfrak{so}(p,q)} \mathfrak{g}_0,$$

$$B = \text{pr}_{\mathbb{R}^{2m*} \otimes \mathbb{R}^{p+q}} \mathfrak{g}_1 \circ R|_{V_0 \otimes V_1} : V_0 \otimes V_1 \to \mathfrak{pr}_{\mathbb{R}^{2m*} \otimes \mathbb{R}^{p+q}} \mathfrak{g}_1,$$

$$C = \text{pr}_{\mathbb{R}^{2q*} \otimes \mathbb{R}^{2m}} \mathfrak{g}_1 \circ R|_{V_0 \otimes V_1} : V_0 \otimes V_1 \to \mathfrak{pr}_{\mathbb{R}^{2q*} \otimes \mathbb{R}^{2m}} \mathfrak{g}_1,$$

$$D = \text{pr}_{\mathfrak{sp}(2m, \mathbb{R})} \mathfrak{g}_0 \circ R|_{\Lambda^2 V_0 \otimes \Lambda^2 V_1} : \Lambda^2 V_0 \otimes \Lambda^2 V_1 \to \mathfrak{pr}_{\mathfrak{sp}(2m, \mathbb{R})} \mathfrak{g}_0.$$

In the definition of $B$ and $C$ we used the inclusion

$$\mathfrak{osp}(p, q|2m) \subset \mathfrak{gl}(p + q|2m, \mathbb{R}) = \mathfrak{gl}(p + q, \mathbb{R}) \oplus \mathfrak{gl}(m, \mathbb{R}) \oplus \mathbb{R}^{2m*} \otimes \mathbb{R}^{p+q} \oplus \mathbb{R}^{p+q*} \otimes \mathbb{R}^{2m}.$$

Since $R$ takes values in $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$, we obtain

$$\omega(C(x, \xi)y, \delta) = -\eta(y, B(x, \xi)\delta)$$

(4)
for all \( x, y \in V_0 \) and \( \xi, \delta \in V_1 \), i.e. the maps \( B \) and \( C \) define each other. Extend the maps \( A, B, C, D \) to \( \Lambda^2 V \) mapping the natural complements to zero. Then \( R = A + B + C + D \). In the matrix form we may write
\[
R(x, y) = -R(y, x) = \begin{pmatrix} A(x, y) & 0 \\ 0 & D(x, y) \end{pmatrix}, \quad R(\xi, \delta) = R(\delta, \xi) = \begin{pmatrix} A(\xi, \delta) & 0 \\ 0 & D(\xi, \delta) \end{pmatrix},
\]
where \( x, y \in V_0 \) and \( \xi, \delta \in V_1 \).

Writing down the Bianchi identity, we get that \( R \in \mathcal{R}(g)_0 \) if and only if the following conditions hold: \( A|_{\Lambda^2 V_0} \in \mathcal{R}(pr_{so(p,q)} g_0) \), \( D|_{\Lambda^2 V_1} \in \hat{R}(pr_{sp(2m, R)} g_0) \) (this space is defined in Appendix [D]),
\[
D(x, y)\xi + C(y, \xi)x + C(\xi, x)y = 0, \tag{5}
\]
\[
A(\xi, \delta)x - B(\delta, x)\xi + B(x, \xi)\delta = 0 \tag{6}
\]
for all \( x, y \in V_0 \) and \( \xi, \delta \in V_1 \).

Suppose that \( R \in \mathcal{R}(g)_0 \). Using \([3]\), we get
\[
\omega(D(x, y)\xi, \delta) = \eta(A(\xi, \delta)x, y), \tag{7}
\]
\[
\omega(C(x, \xi)y, \delta) = -\omega(C(y, \delta)x, \xi), \quad \eta(B(x, \xi)\delta, y) = \omega(B(y, \delta)\xi, x) \tag{8}
\]
for all \( x, y \in V_0 \) and \( \xi, \delta \in V_1 \). In particular, we see that \( A|_{\Lambda^2 V_1} \) and \( D|_{\Lambda^2 V_0} \) define each other. The meanings of the restrictions \([3]\) and \([3]\) on \( A|_{\Lambda^2 V_1} \) and \( D|_{\Lambda^2 V_0} \) are not so clear. On the other hand, if representation of \( g_0 \) is diagonal in \( V_0 \oplus V_1 \) (by this we mean that the both representations of \( g_0 \) on \( V_0 \) and \( V_1 \) are faithful), then \( A|_{\Lambda^2 V_1} \) and \( D|_{\Lambda^2 V_0} \) are given by \( D|_{\Lambda^2 V_1} \) and \( A|_{\Lambda^2 V_0} \), respectively. We will use this in many situations.

We turn now to the space \( \mathcal{R}(g)_1 \). Let \( R \in (\Lambda^2 V^* \otimes g)_1 \). Define the following maps
\[
A = pr_{so(p,q)} g_0 \circ R|_{V_0 \otimes V_1} : V_0 \otimes V_1 \to pr_{so(p,q)} g_0,
\]
\[
B = pr_{R^m \otimes l^p+q} g_1 \circ R|_{\Lambda^2 V_0 \oplus \Lambda^2 V_1} : \Lambda^2 V_0 \oplus \Lambda^2 V_1 \to pr_{R^m \otimes l^p+q} g_1,
\]
\[
C = pr_{R^p+q \otimes R^2m} g_1 \circ R|_{\Lambda^2 V_0 \oplus \Lambda^2 V_1} : \Lambda^2 V_0 \oplus \Lambda^2 V_1 \to pr_{R^p+q \otimes R^2m} g_1,
\]
\[
D = pr_{sp(2m, R)} g_0 \circ R|_{V_0 \otimes V_1} : V_0 \otimes V_1 \to pr_{sp(2m, R)} g_0.
\]
Since \( R \) takes values in \( g \subset \mathfrak{osp}(p, q|2m) \), we obtain
\[
\omega(C(x, y)z, \xi) = -\eta(z, B(x, y)\xi), \quad \omega(C(\xi, \delta)z, \theta) = -\eta(z, B(\xi, \delta)\theta) \tag{9}
\]
for all \( x, y, z \in V_0 \) and \( \xi, \delta, \theta \in V_1 \). Thus the maps \( B \) and \( C \) define each other. Extend the maps \( A, B, C, D \) to \( \Lambda^2 V \) mapping the natural complements to zero. Then \( R = A + B + C + D \). In
the matrix form we may write

\[
R(x, y) = -R(y, x) = \begin{pmatrix}
0 & B(x, y) \\
C(x, y) & 0
\end{pmatrix}, \quad R(\xi, \delta) = R(\delta, \xi) = \begin{pmatrix}
0 & B(\xi, \delta) \\
C(\xi, \delta) & 0
\end{pmatrix},
\]

where \( x, y \in V_0 \) and \( \xi, \delta \in V_1 \). Writing down the Bianchi identity, we get that \( R \in \mathcal{R}(g)_{\bar{1}} \) if and only if the following conditions hold:

\[
\begin{align*}
B(x, y)z + B(y, z)x + B(z, x)y &= 0, \\
C(\xi, \delta)\theta + C(\delta, \theta)\xi + C(\theta, \xi)\delta &= 0, \\
B(x, y)\xi + A(y, \xi)x + A(\xi, x)y &= 0, \\
C(\xi, \delta)x - D(\delta, x)\xi + D(x, \xi)\delta &= 0
\end{align*}
\]

for all \( x, y, z \in V_0 \) and \( \xi, \delta, \theta \in V_1 \). Let us fix \( \xi \in V_1 \). Using (10), we get

\[
\eta(B(x, y)\xi, z) + \eta(B(y, z)\xi, x) + \eta(B(z, x)\xi, y) = 0
\]

for all \( x, y, z \in V_0 \). Using this and (12), we conclude that for each fixed \( \xi \in V_1 \) it holds \( R(\cdot, \xi) \in \mathcal{P}_D(\text{pr}_{g_{\bar{1}}} g_0) \), this space is defined in Appendix [C]. Similarly, for each \( x \in V_0 \) it holds \( R(\cdot, x) \in \mathcal{P}_D(\text{pr}_{\text{sp}(2m, \mathbb{R})} g_0) \), this space is defined in Appendix [D]. This will be extremely useful especially in the case when the representation of \( g_0 \) or of some ideal of \( g_0 \) is diagonal in \( V_0 \oplus V_1 \).

Now we are able to prove the following proposition.

**Proposition 3.1** Let \( g \) be a simple (real or complex) Lie superalgebra admitting an even non-degenerate \( g \)-invariant bilinear supersymmetric form, i.e. such that the adjoint representation of \( g \) is orthosymplectic. Then \( \mathcal{R}(g) = \mathcal{R}(g)_0 \) is one-dimensional and it is spanned by the Lie superbrackets of \( g \).

**Proof.** First of all, from the Jacobi super identity it follows that \([\cdot, \cdot] \in \mathcal{R}(g)_0 \) for each simple Lie superalgebra. Note that the representation of \( g_0 \) is diagonal in \( g_0 \oplus g_1 \) (up to the center of \( g_0 \), which does not play a role).

First we prove that \( \mathcal{R}(g)_{\bar{1}} = 0 \). Suppose that \( g_0 \) contains at least two simple ideals \( h_1 \) and \( h_2 \). Let \( R \in \mathcal{R}(g)_{\bar{1}} \), then for each fixed \( \xi \in g_1 \) and any \( x \in h_1 \) we have \( R(x, \xi) \in h_1 \). On the other hand, for each fixed \( x \in h_1 \) we have \( \text{pr}_{g_0} \circ R(x, \cdot) \in \mathcal{P}_D(\text{pr}_{\text{sp}(g_1)} g_0) \), but the \( g_0 \)-module \( g_1 \) is a tensor product of irreducible representations of simple ideals in \( g_0 \) (unless \( g = \text{osp}(p, q|2) \) or \( g = \text{osp}(p|2, \mathbb{C}) \)), these cases can be considered in the same way as \( g = \text{osp}(1|2m, \mathbb{R}) \) below and
we get $R(g)\vert_1 = 0$). We have $pr_{g_0} \circ R(x, \cdot) = 0$ for all $x \in h_1$. Similarly, $pr_{g_0} \circ R(x, \cdot) = 0$ for all $x \in g_0$, i.e. $R = 0$.

We are left with the cases when $g = \text{osp}(1|2m, F)$ and $g = \text{osp}(2|2m, F)$, $F = \mathbb{R}$ or $\mathbb{C}$ (for other simple $g$ such that the semisimple part of $g_0$ is simple, the adjoint representation of $g$ is not orthosymplectic).

Consider the case $g = \text{osp}(1|2m, F)$, the case $g = \text{osp}(2|2m, F)$ is similar. Since the complexification of the adjoint representation of $g = \text{osp}(1|2m, \mathbb{R})$ is irreducible, it is enough to consider the adjoint representation of $g = \text{osp}(1|2m, \mathbb{C})$. Let $R \in R(g)\vert_1$. Then for each $x \in g_0 = \text{sp}(2m, \mathbb{C})$ it holds $pr_{g_0} \circ R(x, \cdot) \in \mathcal{P}_\omega(\text{sp}(2m, \mathbb{C}))$, and for each $\xi \in g_1 = \mathbb{C}^{2m}$ it holds $pr_{g_0} \circ R(\cdot, \xi) \in \mathcal{P}_\eta(\text{ad}_{\text{sp}(2m, \mathbb{C})})$. That is $R(g)\vert_1$ is contained in the diagonal form in the $\text{sp}(2m, \mathbb{C})$-module $(\mathbb{C}^{2m} \times \mathcal{P}_\eta(\text{ad}_{\text{sp}(2m, \mathbb{C})})) \oplus (\text{sp}(2m, \mathbb{C}) \otimes \mathcal{P}_\omega(\text{sp}(2m, \mathbb{C})))$. Suppose that $m \geq 2$. In [14] we prove that $\mathcal{P}_\eta(\text{ad}_{\text{sp}(2m, \mathbb{C})}) \simeq \text{sp}(2m, \mathbb{C})$ and any $P \in \mathcal{P}_\eta(\text{ad}_{\text{sp}(2m, \mathbb{C})})$ is of the form $P(\cdot) = [x, \cdot]$, where $x \in \text{sp}(2m, \mathbb{C})$. Note that $\mathcal{P}_\omega(\text{sp}(2m, \mathbb{C}))$ contains a submodule isomorphic to $\text{sp}(2m, \mathbb{C})$ and any element $P$ of this module is of the form $P(\cdot) = \xi \cdot$ for some $\xi \in \mathbb{C}^{2m}$ (for $\xi, \delta \in \mathbb{C}^{2m}$ the element $\xi \cdot \delta \in \text{sp}(2m, \mathbb{C})$ is defined by $(\xi \cdot \delta) \theta = \omega(\xi, \theta) \delta + \omega(\delta, \theta) \xi$). We conclude that $R(g)\vert_1$ is contained in the diagonal form in the $\text{sp}(2m, \mathbb{C})$-module $(\mathbb{C}^{2m} \times \mathcal{P}_\eta(\text{ad}_{\text{sp}(2m, \mathbb{C})})) \oplus (\text{sp}(2m, \mathbb{C}) \otimes \mathbb{C}^{2m})$.

Moreover for each $R \in R(g)\vert_1$ there exist linear maps $\varphi : \mathbb{C}^{2m} \to \text{sp}(2m, \mathbb{C})$ and $\psi : \text{sp}(2m, \mathbb{C}) \to \mathbb{C}^{2m}$ such that $R(x, \xi) = [\varphi(\xi), x] = \psi(x) \cdot \xi$ for all $\xi \in \mathbb{C}^{2m}$ and $x \in \text{sp}(2m, \mathbb{C})$. Since $R(g)\vert_1$ is an $\text{sp}(2m, \mathbb{C})$-module, $\varphi$ and $\psi$ are proportional as the elements of $\mathbb{C}^{2m} \otimes \text{sp}(2m, \mathbb{C})$. To show that the equation $[\varphi(\xi), x] = \psi(x) \cdot \xi$ for all $\xi \in \mathbb{C}^{2m}$ and $x \in \text{sp}(2m, \mathbb{C})$ has only the trivial solution it is enough to decompose the $\text{sp}(2m, \mathbb{C})$-module $\mathbb{C}^{2m} \otimes \text{sp}(2m, \mathbb{C})$ into the direct sum of irreducible components and to check this equation for a non-zero representative of each component.

We have $\mathbb{C}^{2m} \otimes \text{sp}(2m, \mathbb{C}) = V_{3\pi_1} \oplus V_{\pi_1 + \pi_2} \oplus \mathbb{C}^{2m}$. Let $(\xi_\alpha)_{\alpha=-m,\ldots,-1,1,\ldots,m}$ be the standard basis of $\mathbb{C}^{2m}$, i.e. $\omega(\xi_\alpha, \xi_\beta) = \delta_{\alpha,-\beta}$. Then $\text{sp}(2m, \mathbb{C})$ is spanned be the elements of the form $\xi_\alpha \cdot \xi_\beta$. Let $\varphi = c\psi = \xi_1 \otimes \xi_1 \otimes \xi_1 \in V_{3\pi_1}$. Substituting to the equation $\xi = \xi_1$ and $x = \xi_1 \cdot \xi_1$, we get $0 = [\xi_1 \cdot \xi_1, \xi_1 \cdot \xi_1]$, on the other hand, $[\xi_1 \cdot \xi_1, \xi_1 \cdot \xi_1] = 2\xi_1 \cdot \xi_1 \neq 0$. Hence $\varphi = c\psi = \xi_1 \otimes \xi_1 \otimes \xi_1$ is not a solution of the equation. Similarly, taking $\varphi = c\psi = \sum_\alpha \xi_\alpha \otimes \xi_{-\alpha} \otimes \xi_1 \in \mathbb{C}^{2m}$, $\xi = \xi_1$ and $x = \xi_1 \cdot \xi_1$, we get that $\varphi = c\psi = \sum_\alpha \xi_\alpha \otimes \xi_{-\alpha} \otimes \xi_1$ is not a solution of the equation.

Finally, the $\text{sp}(2m, \mathbb{C})$-module $\mathbb{C}^{2m} \otimes \text{sp}(2m, \mathbb{C})$ contains the weight space of the weight $\pi_1 + \pi_2$ of dimension 2 and this space consists of the vectors $c_1 \xi_1 \otimes \xi_1 \otimes \xi_2 + c_2 \xi_2 \otimes \xi_1 \otimes \xi_1$, $c_1, c_2 \in \mathbb{C}$. Let $\varphi = c\psi$ equal to such vector. Taking $\xi = \xi_1$ and $x = \xi_2 \cdot \xi_1$, we get $c_1 = 0$; taking $\xi = \xi_2$ and $x = \xi_1 \cdot \xi_2$, we get $c_2 = 0$. Thus we may conclude that $R(g)\vert_1 = 0$. If $m = 1$, then $\mathbb{C}^2 \otimes \text{sp}(2, \mathbb{C}) = V_{3\pi_1} \oplus \mathbb{C}^2$. It is not hard to see that $\mathcal{P}_\omega(\text{sp}(2, \mathbb{C})) = \mathbb{C}^2$. And the further proof is the same.

Next we prove that $R(g)_0$ is one-dimensional.

Let $R \in R(g)_0$ be given as above by the linear maps $A, B, C, D$. We have seen that $A|_{\Lambda^2 V_0}$ (or $D|_{\Lambda^2 V_1}$) defines uniquely $A$ and $D$. We claim that it defines the whole $R$. Indeed, suppose
that $A = 0$ and $D = 0$. Let $\xi, \delta \in \mathfrak{g}_1$ and $x \in \mathfrak{g}_0$. We have $\xi \cdot R(x, \delta) = [\xi, R(x, \delta)]$. If $R(x, \delta) \neq 0$, then since $\mathfrak{g}$ is simple, there exists a $\xi$ such that $\xi \cdot R(x, \delta) \neq 0$. On the other hand, $\xi \cdot R \in \mathcal{R}(\mathfrak{g})_1 = 0$ and we get a contradiction, this proves the claim.

If the semisimple part of $\mathfrak{g}_0$ is simple, then $A|_{\Lambda^2 \mathfrak{v}_0}$ being an element in $\mathcal{R}(\text{ad}_{\mathfrak{g}_0})$ is proportional to the Lie brackets in $\mathfrak{g}_0$ \cite{36}. Since $A|_{\Lambda^2 \mathfrak{v}_0}$ defines uniquely $R \in \mathcal{R}(\mathfrak{g})_0$, we get that $\mathcal{R}(\mathfrak{g})_0$ is one-dimensional.

Suppose that $\mathfrak{g}$ is of type II and the semisimple part of $\mathfrak{g}_0$ is not simple, then $\bar{\mathcal{R}}(\mathfrak{g}_0 \subset \mathfrak{sp}(\mathfrak{g}_1))$ is one-dimensional (Appendix D). Hence $D|_{\Lambda^2 \mathfrak{v}_1}$ belongs to a one-dimensional space. Since $D|_{\Lambda^2 \mathfrak{v}_1}$ defines uniquely $R \in \mathcal{R}(\mathfrak{g})_0$, we get that $\mathcal{R}(\mathfrak{g})_0$ is one-dimensional.

Finally suppose that $\mathfrak{g}$ is of type I and the semisimple part of $\mathfrak{g}_0$ is not simple, i.e. $\mathfrak{g}_0 = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{z}$. Then obviously $A|_{\mathfrak{b}_1 \otimes \mathfrak{b}_2} = 0$, $A|_{\Lambda^2 \mathfrak{b}_1} = c_1[\cdot, \cdot]_{\mathfrak{b}_1}$, and $A|_{\Lambda^2 \mathfrak{b}_2} = c_2[\cdot, \cdot]_{\mathfrak{b}_2}$ are annihilated by $\mathfrak{g}_0$. Then $D|_{\Lambda^2 \mathfrak{v}_1}$ belongs to a two-dimensional space annihilated by $\mathfrak{g}_0$. On the other hand, $\mathfrak{g}_0$ may annihilate only a one-dimensional subspace in $\bar{\mathcal{R}}(\mathfrak{g}_0 \subset \mathfrak{sp}(\mathfrak{g}_1))$. Hence there exists a $c \in \mathbb{C}$ such that for each $R$ it holds $c_1 = c c_2$. Thus $\mathcal{R}(\mathfrak{g})_0$ is one-dimensional. The proposition is proved. □

4 Proof of the Main Theorem

**Lemma 4.1** Let $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m)$ be an irreducible subalgebra of the form (1). If $\mathcal{R}(\mathfrak{g})_1 = 0$, then $\mathcal{R}(\mathfrak{g})$ is a trivial $\mathfrak{g}$-module.

**Proof.** We have $(\mathfrak{g}_i)_1 \cdot \mathcal{R}(\mathfrak{g})_0 \subset \mathcal{R}(\mathfrak{g})_1 = 0$. Since each $\mathfrak{g}_i$ is simple of classical type, it holds $(\mathfrak{g}_i)_0 = [(\mathfrak{g}_i)_1, (\mathfrak{g}_i)_1]$. Consequently, $(\mathfrak{g}_i)_1 \cdot \mathcal{R}(\mathfrak{g})_0 = 0$. Suppose that $\mathfrak{z} \neq 0$. Since $\mathfrak{osp}(p, q|2m)_0 \cap \mathfrak{q}(2m, \mathbb{R}) = 0$, by the Schur Lemma $\mathfrak{z} = \mathbb{R} \mathfrak{j}$, where $\mathfrak{j}$ is an even complex structure on $V$. It is not hard to see that $J \cdot \mathcal{R}(\mathfrak{g}) = 0$. □

First we consider case by case simple real Lie superalgebras $\mathfrak{g}$ of classical type (they are classified in \cite{33}) and for each $\mathfrak{g}$ we find all irreducible representations $\mathfrak{g} \subset \mathfrak{osp}(p, q|2m) = \mathfrak{osp}(V)$ such that $\mathfrak{g}$ is a non-symmetric Berger supersubalgebra. We explain the way of the considerations and then give several examples demonstrating this proof.

We begin with the case when $\mathfrak{g}_0$ is of the form $\mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{z}$, where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ are simple and $\mathfrak{z}$ is trivial or one-dimensional. If $\mathfrak{g}$ is of type II, i.e. the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is irreducible, then $\mathfrak{g}_1$ is of the form $W_1 \otimes W_2$, $\mathfrak{h}_1 \subset \mathfrak{so}(W_1)$ and $\mathfrak{h}_2 \subset \mathfrak{sp}(W_2)$ are irreducible. If $\mathfrak{g}$ is of type I, i.e. the $\mathfrak{g}_0$-module $\mathfrak{g}_1$ is of the form $\mathfrak{g}_{-1} \oplus \mathfrak{g}_1$, where $\mathfrak{g}_{-1}$ and $\mathfrak{g}_1$ are irreducible $\mathfrak{g}_0$-modules, then there are two vector spaces $U_1$ and $U_2$ such that $\mathfrak{h}_1 \subset \mathfrak{gl}(U_1)$, $\mathfrak{h}_2 \subset \mathfrak{gl}(U_2)$ are irreducible, $\mathfrak{g}_{-1} = U_1^* \otimes U_2$, and $\mathfrak{g}_1 = U_2^* \otimes U_1$.

Consider several cases:
Case a. $\mathfrak{h}_1$ annihilates $V_1$. Suppose that $\mathfrak{g}$ is of type II. Since the inclusion $i : \mathfrak{g} \hookrightarrow \mathfrak{osp}(p, q|2m)$ is a Lie superalgebra homomorphism, the restriction $i|_{\mathfrak{g}_1} : \mathfrak{g}_1 = W_1 \otimes W_2 \rightarrow \mathfrak{osp}(V_1) = V_0 \otimes V_1$ is $\mathfrak{g}_0$-equivariant. In particular, it is $\mathfrak{h}_1$-equivariant. Since $\mathfrak{h}_1$ annihilates $W_2$ and $V_1$, we conclude that $V_0$ is a direct sum of $\mathfrak{h}_1$-submodules isomorphic to $W_1$ and of an $\mathfrak{h}_1$-trivial submodule. Similarly, if $\mathfrak{g}$ is of type I, then $V_0$ is a direct sum of $\mathfrak{h}_1$-submodules isomorphic to $U_1 \oplus U_1^\ast$ and of an $\mathfrak{h}_1$-trivial submodule.

Under the current assumption we have three cases:

Case a.1. $\mathfrak{h}_2$ annihilates $V_0$. Suppose that $\mathfrak{g}$ is of type II. By the above arguments, $V_1$ is a direct sum of $\mathfrak{h}_2$-submodules isomorphic to $W_2$ and of an $\mathfrak{h}_2$-trivial submodule. From Appendixes \textbf{C} and \textbf{D} we read that if $V_0$ contains more then one $\mathfrak{h}_1$-submodule isomorphic to $W_1$ and if $V_1$ contains more then one $\mathfrak{h}_2$-submodule isomorphic to $W_2$, then $\mathcal{P}_\eta(\mathfrak{h}_1 \subset \mathfrak{so}(V_0)) = 0$ and $\mathcal{P}_\eta(\mathfrak{h}_2 \subset \mathfrak{so}(V_1)) = 0$. Consequently, $\mathcal{R}(\mathfrak{g})_1 = 0$ and $\mathfrak{g}$ is symmetric. Thus either $V_0$ contains exactly one $\mathfrak{h}_1$-submodule isomorphic to $W_1$, or $V_1$ contains exactly one $\mathfrak{h}_2$-submodule isomorphic to $W_2$. Similarly, if $\mathfrak{g}$ is of type I, then $V_0$ contains exactly one $\mathfrak{h}_1$-submodule isomorphic to $U_1 \oplus U_1^\ast$ or $V_1$ contains exactly one $\mathfrak{h}_2$-submodule isomorphic to $U_2 \oplus U_2^\ast$. Next we check when such representation of $\mathfrak{g}$ exists. For this we may pass to the complexification of $\mathfrak{g}$ and of its representation. If the resulting representation is not irreducible, we take one of its irreducible components. Note that the type of $\mathfrak{g}$ may change.

Let $\mathfrak{g}$ be of type II. Suppose, for instance, that $V_1$ contains exactly one $\mathfrak{h}_2$-submodule isomorphic to $W_2$. Since $\mathfrak{g}_1 \otimes W_2$ contains only one submodule annihilated by $\mathfrak{h}_2$ and isomorphic to $W_2$ as the $\mathfrak{h}_1$-module and since the representation of $\mathfrak{g}_1$ on $V$ is $\mathfrak{g}_0$-equivariant, $\mathfrak{g}$ preserves the vector supersubspace $(\mathfrak{g}_1 \cdot V_1) \oplus V_1 \subset V$ and its even part contains only one $\mathfrak{h}_1$-submodule isomorphic to $W_1$. From the irreducibility of $V$ it follows that $V_0$ contains exactly one $\mathfrak{h}_1$-submodule isomorphic to $W_1$ and $V_1$ contains exactly one $\mathfrak{h}_2$-submodule isomorphic to $W_2$. Suppose that $V$ contains a non-trivial vector $v$ annihilated by $\mathfrak{g}_0$. Then the homogeneous components of $v$ are also annihilated by $\mathfrak{g}_0$ and we may assume that $v$ is homogeneous. Suppose that $v \in V_0$. Consider the map from $\mathfrak{g}_1$ to $V_1$ sending $\varphi \in \mathfrak{g}_1$ to $\varphi v$. This map is $\mathfrak{g}_0$-equivariant and since the $\mathfrak{g}_0$-modules $\mathfrak{g}_1$ and $V_1$ are not isomorphic, this map is zero. This shows that $Rv \subset V$ is an invariant subspace. Thus we get that $V_0 = W_1$ and $V_1 = W_2$. Such a representation of $\mathfrak{g}$ is either the identity one, or it does not exist. Let $\mathfrak{g}$ be of type I. By the similar arguments we get that $V_0 = U_1$ and $V_1 = U_2$. This happens e.g. for the complexification of the identity representation of $\mathfrak{su}(p_0, q_0|p_1, q_1)$.

Case a.2. $\mathfrak{h}_2$ annihilates $V_1$. In this case $\mathfrak{h}_1 \oplus \mathfrak{h}_2$ annihilates $V_1$. Let $U \subset V_0$ be an irreducible $\mathfrak{h}_1 \oplus \mathfrak{h}_2$-module. Since $U$ is not $\mathfrak{g}$-invariant, $\mathfrak{g}_1 \cdot U \neq 0$. On the other hand, $\mathfrak{g}_1 \cdot U \subset V_1$ is annihilated by $\mathfrak{h}_1 \oplus \mathfrak{h}_2$, i.e. $\mathfrak{g}_1 \otimes U$ contains a one-dimensional subspace annihilated by $\mathfrak{h}_1 \oplus \mathfrak{h}_2$. This may happens only if $U \simeq \mathfrak{g}_1$ (if $\mathfrak{g}$ is of type II), or if $U$ is isomorphic either to $\mathfrak{g}_{-1}$, or to $\mathfrak{g}_1$ (if $\mathfrak{g}$ is of type I). We show that such representations do not exist.
Case a.3. The representation of \( h_2 \) is diagonal in \( V_0 \oplus V_1 \). We may decompose \( V_0 \) as the direct sum \( V_0 = L_1 \oplus L_2 \oplus L_3 \oplus L_4 \) such that \( h_1 \oplus h_2 \) annihilates \( L_4 \), \( h_1 \) annihilates \( L_2 \), \( h_2 \) annihilates \( L_1 \), \( L_1 \) is a direct sum of \( h_1 \)-submodules isomorphic to \( W_1 \) (resp. \( U_1 \oplus U_1^* \), \( L_2 \) is an \( h_2 \)-submodule, and \( L_3 \) is an \( h_1 \oplus h_2 \)-submodule (such that each irreducible component of \( L_3 \) is faithful for both \( h_1 \) and \( h_2 \)). If \( L_3 \neq 0 \), then from Appendix C it follows that \( \mathcal{P}_g(\text{pr}_{\mathfrak{so}(L_1 \oplus L_2)} \mathfrak{g}) = 0 \) and this implies \( \mathcal{R}(\mathfrak{g})_1 = 0 \). Thus, \( L_3 = 0 \). Since \( \mathfrak{g} \) is a Berger algebra, one of the following holds:

1. There exists an \( R \in \mathcal{R}(\mathfrak{g})_0 \) such that for some \( x, y \in L_2 \) it holds \( 0 \neq R(x, y) \in h_2 \). Expressing \( R \) in terms of the maps \( A, B, C, D \) as above, we get \( A|_{\Lambda^2 L_2} \neq 0 \) and \( D|_{\Lambda^2 L_2} \neq 0 \). Hence \( A|_{\Lambda^2 V_1} \neq 0 \) and \( D|_{\Lambda^2 V_1} \neq 0 \). This implies that \( h_2 \subset \mathfrak{so}(L_2) \) is a Berger subalgebra and \( h_2 \subset \mathfrak{sp}(V_1) \) is a skew-Berger subalgebra.

2. There exists an \( R \in \mathcal{R}(\mathfrak{g})_0 \) such that for some \( \xi, \delta \in V_1 \) it holds \( 0 \neq R(\xi, \delta) \in h_2 \). This case is similar to Case 1 and we get the same conclusion.

3. There exists an \( R \in \mathcal{R}(\mathfrak{g})_1 \) such that for some \( x \in L_2 \) and \( \xi \in V_1 \) it holds \( 0 \neq R(x, \xi) \in h_2 \). This shows that \( h_2 \subset \mathfrak{so}(L_2) \) is a Berger subalgebra and \( h_2 \subset \mathfrak{sp}(V_1) \) is a skew-Berger subalgebra.

Thus \( L_2 \) is an irreducible \( h_2 \)-module or \( L_2 = L \oplus L^* \), where \( h_2 \subset \mathfrak{gl}(L) \) is irreducible. The same holds for \( V_1 \). Next, as in Case a.1, we show that from the irreducibility of \( \mathfrak{g} \subset \mathfrak{osp}(V) \) it follows that \( L_2 = 0 \) and we get a contradiction.

Case b. \( h_1 \) annihilates \( V_0 \). This case is analogous to Case a. Note that if \( \mathfrak{g} \) is of type II, then in this case \( h_1 \subset \mathfrak{sp}(V_1) \) and \( V_1 \) is a direct sum of \( h_1 \)-submodules isomorphic to \( W_1 \) and of a \( h_1 \)-trivial submodule. Since \( h_1 \subset \mathfrak{so}(W_1) \), we get that \( h_1 \subset \mathfrak{su}(W_1) \). As above we may consider Cases b.1, b.2, b.3.

We are left with the following case:

Case c. The representations of \( h_1 \) and \( h_2 \) are diagonal in \( V_0 \oplus V_1 \).

We may decompose \( V_0 \) as the direct sum \( V_0 = L_1 \oplus L_2 \oplus L_3 \oplus L_4 \) such that \( h_1 \oplus h_2 \) annihilates \( L_4 \), \( h_1 \) annihilates \( L_2 \), \( h_2 \) annihilates \( L_1 \), \( L_1 \) is an \( h_1 \)-submodule, \( L_2 \) is an \( h_2 \)-submodule, and \( L_3 \) is an \( h_1 \oplus h_2 \)-submodule. Let \( V_0 = L_1' \oplus L_2' \oplus L_3' \oplus L_4' \) be the similar decomposition. Since \( \mathcal{R}(\mathfrak{g})_1 \neq 0 \), we get that \( P_\eta(\text{pr}_{\mathfrak{so}(V_0)} \mathfrak{g}) \neq 0 \) and \( P_\omega(\text{pr}_{\mathfrak{sp}(V_1)} \mathfrak{g}) \neq 0 \). This shows that if \( L_1 \neq 0 \), then \( L_3 = 0 \). Moreover \( L_2 \neq 0 \), since the representation of \( h_1 \) is diagonal in \( V_0 \oplus V_1 \). Thus either \( L_3 = 0 \), or \( L_1 = 0 \) and \( L_2 = 0 \). Furthermore, if \( L_3 \neq 0 \), then \( h_1 \oplus h_2 \subset \mathfrak{so}(L_3) \) is a Berger subalgebra (which is irreducible or \( L_3 \) is of the form \( U \oplus U^* \)) and \( h_1 \oplus h_2 \subset \mathfrak{gl}(U) \) is irreducible. Similarly, if \( L_1 \neq 0 \) and \( L_2 \neq 0 \), then \( h_1 \subset \mathfrak{so}(L_1) \) and \( h_2 \subset \mathfrak{so}(L_2) \) are Berger subalgebras. The same statements we get for \( V_1 \) (instead of Berger subalgebras we get skew-Berger subalgebras).

If \( L_1 \neq 0 \), \( L_2 \neq 0 \), \( L_1' \neq 0 \), and \( L_2' \neq 0 \) then we show in the same way as in Case a.1 that the representation is not irreducible. If \( L_1 \neq 0 \), \( L_2 \neq 0 \), and \( L_3' \neq 0 \), then we show as for the adjoint representations that \( \mathcal{R}(\mathfrak{g})_1 = 0 \). If \( L_3 \neq 0 \), and \( L_3' \neq 0 \), then either \( h_1 \oplus h_2 \) does not appear as a Berger or skew-Berger algebra, or \( h_1 \oplus h_2 \) appears only as a reducible Berger and
a reducible skew-Berger algebra. In the last case the representation of \( g \) is not irreducible.

Note that in each case we get a decomposition of \( V \) into irreducible \( g_0 \)-modules. Then we ask if such representation of \( g \) exists, in other words, we should check if the obtained representation of \( g_0 \) can be extended to an irreducible representation of \( g \). This can be done by passing to the complex case. Then we may use the theory of representations of the complex simple Lie superalgebras [17, 18, 21, 22, 23, 31, 37, 38, 39]. Any irreducible representation \( g \subset \mathfrak{gl}(V) \) is the highest-weight representation \( V_\Lambda \) and the weight \( \Lambda \) is given by its labels on the Kac-Dynkin diagram of \( g \). There is a way to decompose the \( g_0 \)-module \( V_\Lambda \) into irreducible components. One \( g_0 \)-module \( V_\Lambda \) is obtained directly from \( \Lambda \). Then \( V_\Lambda \) must coincide with one of the irreducible \( g_0 \)-modules obtained by us. The weight \( \tilde{\Lambda} \) defines uniquely \( \Lambda \) and we need only to check that \( V_\Lambda \) consists exactly of the irreducible \( g_0 \)-modules obtained by us. We will demonstrate this technics in the examples below.

**Example 4.1** Let \( g \) be the real form of the complex simple Lie superalgebra \( F(4) \) with \( g_0 = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(7, \mathbb{R}) \) and \( g_1 = \mathbb{R}^2 \otimes \Delta \), where \( \Delta \cong \mathbb{R}^8 \) is the spinor representation of \( \mathfrak{so}(7, \mathbb{R}) \).

**Case a.1.** We have \( V_0 = \Delta \) and \( V_1 = \mathbb{R}^2 \). Note that \( V_0 \otimes V_1 = \mathfrak{osp}(8|2, \mathbb{R})_1 \), hence \([V_0, V_1] = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(8, \mathbb{R}) \). This shows that the representation of \( g \) on \( \mathbb{R}^{8|2} \) does not exist.

**Case a.2.** We consider it after Case a.3.

**Case a.3.** We have \( V_0 = \Delta \oplus L \), where \( \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{so}(L) \) is an irreducible Berger subalgebra, and \( V_1 = \mathbb{R}^2 \). As in Case a.1, such representation does not exist. We may prove it also in another way. First since \( \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{so}(L) \) is an irreducible Berger algebra, the only possible \( L \) are \( \mathbb{R}^3 \) and \( \mathbb{R}^5 \). Passing to the complexification we get that as \( g_0 = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \)-modules, \( V_0 = \mathbb{C}^8 \oplus L \), where \( \mathbb{C}^8 \) is the spinor representation of \( \mathfrak{so}(7, \mathbb{C}) \), \( L \) is either \( \mathbb{C}^3 \), or \( \mathbb{C}^5 \), and \( V_1 = \Pi \mathbb{C}^2 \). Note that neither \( g_1 \cap \mathbb{C}^8 \), nor \( g_1 \cap \mathbb{C}^2 \) contain any of the \( g_0 \)-modules \( \mathbb{C}^3 \) and \( \mathbb{C}^5 \).

This means that \( g_1 \cdot (\mathbb{C}^8 \oplus \Pi \mathbb{C}^2) \subset \mathbb{C}^8 \oplus \Pi \mathbb{C}^2 \), i.e. the vector supersubspace \( \mathbb{C}^8 \oplus \Pi \mathbb{C}^2 \subset V \) is \( g \)-invariant, hence \( L = 0 \). In fact, the method of the decomposition of a \( g_0 \)-module \( V \) into irreducible components discussed above is founded on the fact that if \( U \subset V \) is an irreducible \( g_0 \)-module, then \( g_1 \) takes it into some irreducible components of the tensor product \( g_1 \otimes U \).

Let us show how the above discussed method can be applied to our representations. The information about the irreducible representations of \( F(4) \) can be found in [37]. Any irreducible representation (with the highest weight \( \Lambda \)) of \( F(4) \) is given by the labels \((a_1, a_2, a_3, a_4)\) on the Kac-Dynkin diagram. Define the following number \( b = \frac{1}{3}(2a_1 - 3a_2 - 4a_3 - 2a_4) \). The labels must satisfy the conditions: \( b, a_2, a_3, a_4 \) are non-negative integers; if \( b = 0 \), then \( a_1 = \cdots = a_4 = 0 \); \( b \neq 1 \); if \( b = 2 \), then \( a_2 = a_4 = 0 \); if \( b = 3 \), then \( a_2 = 2a_4 + 1 \). The weight \( \tilde{\Lambda} \) is given by the labels \((b, a_2, a_3, a_4)\) on the Dynkin diagram of the Lie algebra \( \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{so}(7, \mathbb{C}) \). In our case \( \tilde{\Lambda} \) must be one of \((0, 0, 0, 1), (1, 0, 0, 0), (2, 0, 0, 0), (4, 0, 0, 0)\). The first two cases do not satisfy the conditions on the labels. The third case corresponds to the adjoint representation, which
is different from our ones. In the second case \( V \) contains a \( \mathfrak{g}_0 \)-module with the highest weight \((3, 1, 0, 0)\), while our representations do not contain such submodule.

Coming back to Case a.2 we get that in this case the representation is given by \( b = 1, a_2 = 0, a_3 = 0, \) and \( a_4 = 1 \). But the representations of \( F(4) \) with \( b = 1 \) do not exist.

Case b. Does not appear, since the representation of \( \mathfrak{so}(7, \mathbb{R}) \) in \( \Delta \) is not unitary.

Case c. Appendixes [2] and [4] show that the only representation of \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(7, \mathbb{R}) \) as a skew-Berger algebra is in the space \( \mathbb{R}^2 \otimes \Delta \), and Lie algebra \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(7, \mathbb{R}) \) does not appear as the Berger subalgebra of \( \mathfrak{so}(p, q) \). Thus, as we have seen, in this case \( \mathcal{R}(\mathfrak{g})_1 = 0 \).

Example 4.2 Let \( \mathfrak{g} \) be the real form of the complex simple Lie superalgebra \( \mathfrak{osp}(4|2, \alpha, \mathbb{C}) \) with \( \mathfrak{g}_0 = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{g}_1 = \mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2 \). Suppose that \( \mathfrak{g} \subset \mathfrak{osp}(p, q|2m) = \mathfrak{osp}(V) \) is an irreducible Berger supersubalgebra. We may consider several cases.

First suppose that the representation of none of the Lie algebras \( \mathfrak{sl}(2, \mathbb{R}) \) is diagonal in \( V_0 \oplus V_1 \). Using the fact that the representation of \( \mathfrak{sl}(2, \mathbb{R}) \) in \( \mathbb{R}^2 \) is symplectic and the arguments of Case a.1, we get that \( V_1 = \mathbb{R}^2 \) as the \( \mathfrak{sl}(2, \mathbb{R}) \)-module, and \( V_0 = \mathbb{R}^2 \otimes \mathbb{R}^2 \) as the \( \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \)-module. To analyse such representations we turn to the complex case. Then we may use the theory of representations of the complex simple Lie superalgebras \( \mathfrak{osp}(4|2, \alpha, \mathbb{C}) \) [21]. Any representation of \( \mathfrak{osp}(4|2, \alpha, \mathbb{C}) \) is given by the numbers \((a_2, a_1, a_3)\) (the labels on the Kac-Dynkin diagram of \( \mathfrak{osp}(4|2, \alpha, \mathbb{C}) \)) such that \( a_2, a_3 \) and the number \( b = \frac{1}{1+\alpha}(2a_1 - a_2 - a_3) \) are non-negative integers. Furthermore, if \( b = 0 \), then \( a_1 = a_2 = a_3 = 0 \); if \( b = 1 \), then \( \alpha(a_3 + 1) = \pm(a_2 + 1) \). In [21] it is shown that \( V \) contains the following \( \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \)-submodules: \((b, a_2, a_3)\), \((b-1, a_2 \pm 1, a_3 \pm 1)\), \((b-1, a_2 \pm 1, a_3 \mp 1)\), \((b-2, a_2 \pm 2, a_3)\), \((b-2, a_2, a_3 \pm 2)\), \((b-2, a_2, a_3)\), \((b-3, a_2 \pm 1, a_3 \pm 1)\), \((b-3, a_2 \pm 1, a_3 \mp 1)\), \((b-4, a_2, a_3)\) (the representations are given by the labels on the Dynkin diagram of \( \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \)). In our case \((b, a_2, a_3)\) is one of \((1, 0, 0)\), \((0, 1, 0)\), \((0, 0, 1)\), \((0, 1, 1)\), \((1, 1, 0)\), \((1, 0, 1)\). In the first case \( \alpha = 1 \) and we get the identity representation of \( \mathfrak{osp}(4|2, \mathbb{R}) \); the second, the third and the forth cases are not possible; in the last two cases \( V \) contains the representations \((0, 2, 1)\) and \((0, 1, 2)\) that give the contradiction. Thus the only possible representation is the identity representation of \( \mathfrak{osp}(4|2, \mathbb{R}) \).

Next we suppose that a number of the representation of the Lie algebras \( \mathfrak{sl}(2, \mathbb{R}) \) are diagonal in \( V_0 \oplus V_1 \). By the same arguments as in Cases a.3 and b we show that if \( \mathfrak{g} \) is a Berger superalgebra, then it is symmetric.

The representations of the simple Lie superalgebras \( \mathfrak{g} \) such that the semisimple part of \( \mathfrak{g}_0 \) is simple can be considered in the same way. The situation becomes simpler, since the representation of \( \mathfrak{g}_0 \) is diagonal in \( V_0 \oplus V_1 \) (except for the identity representations of \( \mathfrak{osp}(1|2m, \mathbb{R}) \) and \( \mathfrak{osp}(2|2m, \mathbb{R}) \)). We immediately conclude that \( \text{pr}_{\mathfrak{so}(V_0)} \mathfrak{g}_0 \subset \mathfrak{so}(V_0) \) is a Berger subalgebra and \( \text{pr}_{\mathfrak{sp}(V_1)} \mathfrak{g}_0 \subset \mathfrak{sp}(V_1) \) is a skew-Berger subalgebra.
Example 4.3 Consider the Lie superalgebra \( g = \mathfrak{pe}(n, \mathbb{R}) \). Recall that \( \mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R}) \) and \( \mathfrak{g}_1 = \mathbb{O}^2 \mathbb{R}^n \oplus \Lambda^2 \mathbb{R}^n^* \). Suppose that \( g \subset \mathfrak{osp}(p, q|2m) = \mathfrak{osp}(V) \) is an irreducible non-symmetric Berger algebra. Then \( \text{pr}_{\mathfrak{so}(V_0)} \mathfrak{g}_0 \subset \mathfrak{so}(V_0) \) is a Berger algebra and \( \text{pr}_{\mathfrak{sp}(V_1)} \mathfrak{g}_0 \subset \mathfrak{sp}(V_1) \) is a skew-Berger algebra. Appendixes [2] and [3] show that \( V_0 \) and \( V_1 \) should be contained in the following list: \( \mathfrak{sl}(n, \mathbb{R}), \mathbb{R}^n \oplus \mathbb{R}^n^*, \mathbb{O}^2 \mathbb{R}^n \oplus \mathbb{O}^2 \mathbb{R}^n^*, \Lambda^2 \mathbb{R}^n \oplus \Lambda^2 \mathbb{R}^n^*, \Lambda^3 \mathbb{R}^6 (n = 6) \), and \( \Lambda^4 \mathbb{R}^8 \) (n = 8). To study these representations we turn to the complexification. Suppose, for example, that \( V_0 = \mathbb{C}^n \oplus \mathbb{C}^n^* \). The \( \mathfrak{g}_0 \)-submodule \( \mathfrak{g}_1 \cdot \mathbb{C}^n \) must coincide with an irreducible component from the list obtaining by the complexification of the above one. We have \( \mathfrak{g}_1 \otimes \mathbb{C}^n = V_{3\pi_1} \oplus V_{\pi_1+\pi_2} \oplus V_{\pi_1+\pi_n-2} \oplus \mathbb{C}^n^* \). Hence, \( \mathfrak{g}_1 \cdot \mathbb{C}^n = \mathbb{C}^n^* \subset V_1 \). The tensor product \( \mathfrak{g}_1 \otimes \mathbb{C}^n^* \) does not contain \( \mathbb{C}^n^* \). This means that the vector supersubspace \( \mathbb{C}^n \oplus \mathbb{C}^n^* \) (where \( \mathbb{C}^n^* \subset V \)) of \( V \) is \( g \)-invariant and we get a contradiction. All the other representatives from the above list can be considered in a similar way.

Thus we conclude that if \( g \) is a simple real Lie superalgebra and there exists an irreducible representation \( g \subset \mathfrak{osp}(p, q|2m) \) such that \( g \) is a non-symmetric Berger subalgebra, then this representation is the identity one of \( g \) and \( g \) is one of the following Lie algebras with their identity representation: \( \mathfrak{osp}(p, q|2m), \mathfrak{osp}(p|2m, \mathbb{C}), \mathfrak{su}(p_0, q_0|p_1, q_1) \) and \( \mathfrak{ho}sp(p, q|m) \). These Lie superalgebras with their identity representations are non-symmetric Berger superalgebras, since they contain, respectively, the subalgebras \( \mathfrak{so}(p, q), \mathfrak{so}(p, \mathbb{C}), \mathfrak{su}(p_0, q_0) \) and \( \mathfrak{ho}sp(p, q) \), which are non-symmetric Berger algebras.

Suppose that \( g \) is a simple real Lie superalgebra and there exists an irreducible representation \( g \subset \mathfrak{osp}(p, q|2m) \) such that \( g \oplus \mathfrak{z} \) is a non-symmetric Berger supersubalgebra, where \( \mathfrak{z} \) is a supersubalgebra of \( \mathfrak{osp}(p, q|2m) \) commuting with \( g \). Since \( g \) is not contained in \( q(n, \mathbb{R}) \), i.e. it does not commute with an odd complex structure, by the Schur Lemma for representations of Lie superalgebras, \( \mathfrak{z} \) is either \( \mathbb{R}J \), where \( J \) is an even complex structure, or \( \mathfrak{z} = \mathfrak{sp}(1) \), i.e. \( \mathfrak{z} \) is spanned by an even quaternionic structure \( J_1, J_2, J_3 \). From Appendix [2] it follows that if \( \mathfrak{h} \subset \mathfrak{so}(p, q) \) is an irreducible subalgebra different from several exceptions, then if \( \mathfrak{h} \oplus \mathfrak{z} \) is a Berger algebra, then \( \mathfrak{h} \) is a Berger algebra. Similarly, from Appendix [3] it follows that if \( \mathfrak{h} \subset \mathfrak{sp}(2m, \mathbb{R}) \) is an irreducible subalgebra then if \( \mathfrak{h} \oplus \mathfrak{z} \) is a skew-Berger algebra, then \( \mathfrak{h} \) is a skew-Berger algebra. The same note holds for representations in \( U \oplus U^* \), where \( \mathfrak{h} \subset \mathfrak{sl}(U) \) is irreducible. This shows that the above method can be applied also to irreducible subalgebras \( \mathfrak{g} \oplus \mathfrak{z} \subset \mathfrak{osp}(p, q|2m) \), where \( \mathfrak{g} \) is simple. We obtain the identity representations of \( \mathfrak{u}(p_0, q_0|p_1, q_1), \mathfrak{ho}sp(p, q|m) \oplus \mathbb{R}J \) and \( \mathfrak{ho}sp(p, q|m) \oplus \mathfrak{sp}(1) \). Since \( \mathfrak{u}(p_0, q_0|p_1, q_1) \) contains \( \mathfrak{u}(p_0, q_0) \), it is a non-symmetric Berger superalgebra. Since \( \mathcal{R}(\mathfrak{sp}(p, q) \oplus \mathbb{R}J) = \mathcal{R}(\mathfrak{sp}(p, q)), \mathcal{R}(\mathfrak{so}(m, \mathbb{H}) \oplus \mathbb{R}J) = \mathcal{R}(\mathfrak{so}(m, \mathbb{H})) \), and \( J \) acts diagonally in \( V_0 \oplus V_1 \), we get that \( \mathcal{R}(\mathfrak{ho}sp(p, q|m) \oplus \mathbb{R}J) = \mathcal{R}(\mathfrak{ho}sp(p, q|m)) \), i.e. \( \mathfrak{ho}sp(p, q|m) \oplus \mathbb{R}J \) is not a Berger superalgebra. Generalizing the curvature tensor of the quaternionic projective space (with indefinite metric) [1], we define the curvature
tensor $R \in \mathcal{R}(\mathfrak{hosp}(p,q|m) \oplus \mathfrak{sp}(1))$ by

$$R(X,Y) = -\frac{1}{2} \sum_{a=1}^{3} g(J_{a}X,Y)J_{a} + \frac{1}{4}(X \wedge Y + \sum_{a=1}^{3} J_{a}X \wedge J_{a}Y),$$

where $X,Y \in V$. The restriction of $R$ to $\Lambda^{2} \mathbb{R}^{4p,4q}$ coincides with the curvature tensor of the quaternionic projective space, its image is not contained in $\mathfrak{sp}(p,q)$. This shows that $R \notin \mathcal{R}(\mathfrak{hosp}(p,q|m))$. Thus, $\mathcal{R}(\mathfrak{hosp}(p,q|m)) \neq \mathcal{R}(\mathfrak{hosp}(p,q|m) \oplus \mathfrak{sp}(1))$ and $\mathfrak{hosp}(p,q|m) \oplus \mathfrak{sp}(1)$ is a non-symmetric Berger superalgebra.

Let now $\mathfrak{g}^{1} \subset \mathfrak{gl}(V^{1})$ and $\mathfrak{g}^{2} \subset \mathfrak{gl}(V^{2})$ be two irreducible real supersubalgebras. Consider the tensor product of these representations $\mathfrak{g} = \mathfrak{g}^{1} \oplus \mathfrak{g}^{2} \subset \mathfrak{gl}(V^{1} \otimes V^{2}) = \mathfrak{gl}(V)$. Suppose that $\mathfrak{g} \subset \mathfrak{osp}(V)$. Note that if the real vector superspaces $V^{1}$ and $V^{2}$ admit complex structures commuting, respectively, with the elements of $\mathfrak{g}^{1}$ and $\mathfrak{g}^{2}$, then the representation of $\mathfrak{g} = \mathfrak{g}^{1} \oplus \mathfrak{g}^{2}$ in $V^{1} \otimes V^{2}$ is reducible and we consider its representation in $V^{1} \otimes_{\mathbb{C}} V^{2}$. For the even and odd parts of $V^{1} \otimes V^{2}$ we have

$$(V^{1} \otimes V^{2})_{0} = V_{0}^{1} \otimes V_{0}^{2} \oplus V_{1}^{1} \otimes V_{1}^{2}, \quad (V^{1} \otimes V^{2})_{1} = V_{0}^{1} \otimes V_{1}^{2} \oplus V_{1}^{1} \otimes V_{0}^{2}.$$  

This shows that if the even and odd parts of both $V^{1}$ and $V^{2}$ are non-trivial, then the representation of $\mathfrak{g}_{0}$ is diagonal in $(V^{1} \otimes V^{2})_{0} \oplus (V^{1} \otimes V^{2})_{1}$. Consequently, $\text{pr}_{\mathfrak{so}(V^{1} \otimes V^{2})_{0}} \mathfrak{g}_{0}$ is a Berger algebra and $\text{pr}_{\mathfrak{so}(V^{1} \otimes V^{2})_{1}} \mathfrak{g}_{0}$ is a skew-Berger algebra. Using the arguments as above, it is easy to see that if $\mathfrak{g}$ is a Berger superalgebra, then it is symmetric. The same works for the tensor product of several representations (we may assume that $\mathfrak{g}^{2}$ is a direct sum of simple Lie superalgebras and $V^{2}$ is a tensor products of irreducible representations of these Lie superalgebras).

Next, we assume that the even and odd parts of $V^{1}$ are non-trivial and $V^{2}$ is either purely even or purely odd. If $V^{2}$ is purely odd, then $V^{1} \otimes V^{2} = \Pi V^{1} \otimes \Pi V^{2}$, where $\Pi V^{2}$ is purely even. Thus we may assume that $V^{2}$ is purely even, i.e. $V^{2}$ is a usual vector space. Since $\mathfrak{g} = \mathfrak{g}^{1} \oplus \mathfrak{g}^{2} \subset \mathfrak{osp}(V^{1} \otimes V^{2})$, we get that either $\mathfrak{g}^{1} \subset \mathfrak{osp}(V^{1})$ and $\mathfrak{g}^{2} \subset \mathfrak{so}(V^{2})$, or $\mathfrak{g}^{1} \subset \mathfrak{osp}^{sk}(V^{1})$ and $\mathfrak{g}^{2} \subset \mathfrak{sp}(V^{2})$.

Let $V^{1}$ and $V^{2}$ be a complex vector superspace and a complex vector space, respectively, and let $V = V^{1} \otimes V^{2}$. Let $g_{1}$ be a supersymmetric bilinear forms on $V^{1}$ and $g_{2}$ be a symmetric bilinear form on $V^{2}$ From the results of [35] it follows that $\mathcal{R}(\mathfrak{sl}(V^{1}) \oplus \mathfrak{sl}(V^{2}) \oplus \mathbb{C}) \simeq V^{*} \otimes V^{*}$. Any $\tau \in V^{*} \otimes V^{*}$ defines the curvature tensor $R_{\tau}$ by

$$R_{\tau}(x_{1} \otimes x_{2}, u_{1} \otimes u_{2}) = A(x_{1} \otimes x_{2}, u_{1} \otimes u_{2}) + B(x_{1} \otimes x_{2}, u_{1} \otimes u_{2}),$$

where $A(x_{1} \otimes x_{2}, u_{1} \otimes u_{2}) \in \mathfrak{sl}(V^{1}) \oplus \mathbb{C}$, $B(x_{1} \otimes x_{2}, u_{1} \otimes u_{2}) \in \mathfrak{sl}(V^{2}) \oplus \mathbb{C}$, and for $v_{1} \in V^{1}$ and $v_{2} \in V^{2}$ we have

$$A(x_{1} \otimes x_{2}, u_{1} \otimes u_{2})v_{1} = (-1)^{|v_{1}||u_{1}|} \tau(x_{1}, x_{2}, v_{1}, u_{2})u_{1} - (-1)^{(|v_{1}|+|u_{1}|)|x_{1}|} \tau(u_{1}, u_{2}, v_{1}, x_{2})x_{1},$$

$$B(x_{1} \otimes x_{2}, u_{1} \otimes u_{2})v_{2} = (-1)^{|v_{1}||u_{2}|} \tau(x_{1}, x_{2}, u_{1}, v_{2})u_{1} - (-1)^{(|v_{1}|+|u_{2}|)|x_{1}|} \tau(u_{1}, u_{2}, v_{1}, x_{2})x_{1},$$
\[ B(x_1 \otimes x_2, u_1 \otimes u_2)v_2 = \tau(x_1, x_2, u_1, v_2)u_2 - (-1)^{|u_1||x_1|}\tau(u_1, u_2, x_1, v_2)x_2. \]

In particular, \( \text{tr}(B(x_1 \otimes x_2, u_1 \otimes u_2)) = (\tau(x_1, x_2, u_1, u_2) - (-1)^{|u_1||x_1|}\tau(u_1, u_2, x_1, u_2)) \). If \( q > 1 \), then by the same arguments as in [30, 4] it can be shown that if \( R_\tau \in \mathcal{R}(\mathfrak{osp}(V^1) \oplus \mathfrak{so}(V^2)) \subset \mathcal{R}(\mathfrak{sl}(V^1) \oplus \mathfrak{sl}(V^2) \oplus \mathbb{C}) \) then it is given by \( \tau \in \circ 2 V^* \) such that \( \tau(x_1, x_2, u_1, u_2) = c g_1(x_1, u_1) g_2(x_2, u_2) \), where \( c \in \mathbb{C} \). Hence, \( \mathcal{R}(\mathfrak{osp}(V^1) \oplus \mathfrak{so}(V^2)) = \mathcal{R}(\mathfrak{osp}(V^1) \oplus \mathfrak{so}(V^2))_0 \) is one-dimensional. Thus \( \mathfrak{osp}(V^1) \oplus \mathfrak{so}(V^2) \subset \mathfrak{osp}(V^1 \otimes V^2) \) is a symmetric Berger superalgebra and \( \mathcal{R}(\mathfrak{g}) = 0 \) for any proper supersubalgebra \( \mathfrak{g} \subset \mathfrak{osp}(V^1) \oplus \mathfrak{so}(V^2) \subset \mathfrak{osp}(V^1 \otimes V^2) \).

Similarly, if \( g_1 \) is a super-skew-symmetric bilinear forms on \( V^1 \) and \( g_2 \) is a symplectic form on \( V^2 \), then \( \mathcal{R}(\mathfrak{osp}^{sk}(V^1) \oplus \mathfrak{sp}(V^2)) = \mathcal{R}(\mathfrak{osp}^{sk}(V^1) \oplus \mathfrak{sp}(V^2))_0 = \mathbb{C} R_\tau \), where \( \tau(x_1, x_2, u_1, u_2) = g_1(x_1, u_1) g_2(x_2, u_2) \). The same hold if \( V^1 \) and \( V^2 \) are real.

Thus we are left with the cases \( \mathfrak{g}^2 = \mathfrak{sl}(2, \mathbb{R}) \) and \( \mathfrak{g}^2 = \mathfrak{sl}(2, \mathbb{C}) \) (in the last case \( \mathfrak{g}^1 \subset \mathfrak{osp}^{sk}(V^1) \) admits a complex structure and we consider the representation of \( \mathfrak{g}^1 \oplus \mathfrak{g}^2 \) in \( V^1 \otimes \mathbb{C} V^2 \)). Suppose that \( \mathfrak{g}^2 = \mathfrak{sl}(2, \mathbb{R}) \). We have \( \mathfrak{g}^1 \subset \mathfrak{osp}^{sk}(V^1) \). Since there are no reductive Lie algebras \( \mathfrak{h} \) such that \( \mathfrak{h} \oplus \mathfrak{sl}(2, \mathbb{R}) \) appears both as a Berger subalgebra of \( \mathfrak{so}(p, q) \) and as a skew-Berger subalgebra of \( \mathfrak{sp}(2m, \mathbb{C}) \), there are no ideal in \( \mathfrak{g}^1 \) that acts diagonally in \( V_0^1 \oplus V_1^1 \). Hence \( \mathfrak{g}^1 \subset \mathfrak{osp}^{sk}(V^1) \) is the identity representation of the Lie superalgebra \( \mathfrak{g}^1 \) (this representation must exist). Appendixes [3] and [4] and the condition \( \mathcal{R}(\mathfrak{g})_1 \neq 0 \) imply \( \mathfrak{g}^1 = \mathfrak{osp}^{sk}(V^1) = \mathfrak{osp}^{sk}(2m|r, s) \). Similarly, if \( \mathfrak{g}^2 = \mathfrak{sl}(2, \mathbb{C}) \), then \( \mathfrak{g}^1 = \mathfrak{osp}^{sk}(2m|r, \mathbb{C}) \). We get the following two algebras \( \mathfrak{osp}^{sk}(2m|r, s) \oplus \mathfrak{sl}(2, \mathbb{R}) \subset \mathfrak{osp}(\mathbb{R}^{2m|r,s} \otimes \mathbb{R}^2) \) and \( \mathfrak{osp}^{sk}(2m|r) \oplus \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{osp}(\mathbb{C}^{2m|r} \otimes \mathbb{C}^2) \). The second representation is the complexification of the identity representation of \( \mathfrak{hosp}(r, r|m) \oplus \mathfrak{sp}(1) \), hence the second representation gives us a non-symmetric Berger superalgebra. The complexification of the first representation is the second one, hence the first representation gives us a non-symmetric Berger superalgebra.

The theorem is proved. \( \square \)

## A Prolongations of Lie algebras

Here we classify irreducible subalgebras \( \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{F}) \) (\( \mathbb{F} = \mathbb{R} \) or \( \mathbb{C} \)) such that the skew-symmetric prolongation

\[
\mathfrak{g}^{[1]} = \{ \varphi \in (\mathbb{F}^n)^* \otimes \mathfrak{g} | \varphi(x)y = -\varphi(y)x \text{ for all } x, y \in \mathbb{F}^n \}
\]

of \( \mathfrak{g} \) is non-zero.

Irreducible subalgebras \( \mathfrak{g} \subset \mathfrak{so}(n, \mathbb{R}) \) with non-zero skew-symmetric prolongations are classified in [32]. These subalgebras are exhausted by the whole orthogonal Lie algebra \( \mathfrak{so}(n, \mathbb{R}) \) and by the adjoint representations of compact simple Lie algebras.

Irreducible subalgebras \( \mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{C}) \) with \( \mathfrak{g}^{[1]} \neq 0 \) are classified in [14]. We give this list in Table 2. To get this result it was used that \( \mathfrak{g}^{[1]} \neq 0 \) coincides with \( \Pi(\mathfrak{g} \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(1)} \) and the
fact that the whole Cartan prolong $g_* = \Pi V \oplus g \oplus (g \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(1)} \oplus (g \subset \mathfrak{gl}(0|n, \mathbb{C}))^{(2)} \oplus \cdots$ is an irreducible transitive Lie superalgebra with the consistent $\mathbb{Z}$-grading and $g_1 \neq 0$. All such $\mathbb{Z}$-graded Lie superalgebras are classified in \[22\].

Table 2 Irreducible subalgebras $g \subset \mathfrak{gl}(n, \mathbb{C})$ with $g^{[1]} \neq 0$.

| $g$                      | $V$                          | $g$                      | $V$                          |
|-------------------------|------------------------------|-------------------------|------------------------------|
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathbb{C}^n$, $n \geq 3$  | $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^n$, $n \geq 3$  |
| $\mathfrak{gl}(n, \mathbb{C})$ | $\mathbb{C}^n$, $n \geq 2$  | $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{C}$ | $\mathbb{C}^n \otimes \mathbb{C}^n$, $n \geq 3$  |
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathbb{C}^2 \mathbb{C}^n$, $n \geq 3$  | $\mathfrak{so}(n, \mathbb{C})$ | $\mathbb{C}^n$, $n \geq 4$  |
| $\mathfrak{gl}(n, \mathbb{C})$ | $\mathbb{C}^2 \mathbb{C}^n$, $n \geq 3$  | $\mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{C}$ | $\mathbb{C}^n$, $n \geq 4$  |
| $\mathfrak{sl}(n, \mathbb{C})$ | $\mathbb{A}^2 \mathbb{C}^n$, $n \geq 5$  | $\mathfrak{sp}(2n, \mathbb{C}) \oplus \mathfrak{C}$ | $\mathbb{C}^{2n}$, $n \geq 2$  |
| $\mathfrak{gl}(n, \mathbb{C})$ | $\mathbb{A}^2 \mathbb{C}^n$, $n \geq 5$  | $\mathfrak{g}$ is simple | $\mathfrak{g}$  |
| $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathfrak{sl}(m, \mathbb{C}) \oplus \mathfrak{C}$ | $\mathbb{C}^n \otimes \mathbb{C}^m$, $n, m \geq 2$  | $\mathfrak{g} \oplus \mathfrak{C}$, $\mathfrak{g}$ is simple | $\mathfrak{g}$  |

Now we classify irreducible subalgebras $g \subset \mathfrak{gl}(n, \mathbb{R})$ with $g^{[1]} \neq 0$. Let $g \subset \mathfrak{gl}(n, \mathbb{R})$ be such subalgebra. If this representation is absolutely irreducible, i.e. $\mathbb{R}^n$ does not admit a complex structure commuting with the elements of $g$, then $g \otimes \mathbb{C} \subset \mathfrak{gl}(n, \mathbb{C})$ is an irreducible subalgebra, and $(g \otimes \mathbb{C})^{[1]} = g^{[1]} \otimes \mathbb{C} \neq 0$. Note that if the representation $g \subset \mathfrak{gl}(n, \mathbb{R})$ is different from the adjoint one, then $(g \otimes \mathbb{C})^{(1)} \neq 0$, and $g^{(1)} \neq 0$. We may use Table B from \[7\].

Suppose that the representation $g \subset \mathfrak{gl}(n, \mathbb{R})$ is non-absolutely irreducible, i.e. $E = \mathbb{R}^n$ admits a complex structure $J$ commuting with the elements of $g$. In this case the complexified space $E \otimes \mathbb{C}$ admits the decomposition $E \otimes \mathbb{C} = V \oplus \bar{V}$, where $V$ and $\bar{V}$ are the eigenspaces of the extension of $J$ to $E \otimes \mathbb{C}$ corresponding to the eigenvalues $i$ and $-i$, respectively. The Lie algebra $g \otimes \mathbb{C}$ preserves this decomposition. Consider the ideal $g_1 = g \cap J g \subset g$. Since $g$ is reductive, there is an ideal $g_2 \subset g$ such that $g = g_1 \oplus g_2$. The Lie algebra $g_1 \otimes \mathbb{C}$ admits the decomposition $g_1 \otimes \mathbb{C} = g_1' \oplus g_1''$ into the eigenspaces of the extension of $J$ to $g_1 \otimes \mathbb{C}$ corresponding to the eigenvalues $i$ and $-i$, respectively. It is easy to see that $g_1'$ annihilates $V$, $g_1''$ annihilates $\bar{V}$, and $g_2 \otimes \mathbb{C}$ acts diagonally in $V \oplus \bar{V}$. We immediately conclude that $(g \otimes \mathbb{C})^{[1]} = (g_1' \subset \mathfrak{gl}(\bar{V}))^{[1]} \oplus (g_1'' \subset \mathfrak{gl}(V))^{[1]}$. It is clear that the representation of $g_1' \oplus (g_2 \otimes \mathbb{C})$ in $\bar{V}$ is irreducible. If $\dim g_2 \geq 1$, then this representation is of the form of the tensor product of irreducible representations of $g_1'$ and $g_2 \otimes \mathbb{C}$. Obviously, in this case $(g_1' \subset \mathfrak{gl}(\bar{V}))^{[1]} = 0$, similarly $(g_1'' \subset \mathfrak{gl}(V))^{[1]} = 0$. We conclude that $\dim g_2 \leq 1$, and $g \subset \mathfrak{gl}(n, \mathbb{R}) = \mathfrak{gl}(\mathbb{R}^n, \mathbb{C})$ is a complex subalgebra considered as the real one. Thus we obtain Table \[3\] ($H_n(\mathbb{C}) = \{A \in \text{Mat}_n(\mathbb{C})|A^* = A\}$, $S_n(\mathbb{H}) = \{A \in \text{Mat}_n(\mathbb{H})|A^* = A\}$ and $A_n(\mathbb{H}) = \{A \in \text{Mat}_n(\mathbb{H})|A^* = -A\}$).
Table 3  Irreducible subalgebras $\mathfrak{g} \subset \mathfrak{gl}(n, \mathbb{R})$ with $\mathfrak{g}^{[1]} \neq 0$ different from algebras of Table 2

| $\mathfrak{g}$ | $V$ | $\mathfrak{g}$ | $V$ |
|----------------|-----|----------------|-----|
| $\mathfrak{sl}(n, \mathbb{R})$ | $\mathbb{R}^n$, $n \geq 3$ | $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R}) \oplus \mathbb{R}$ | $\mathbb{R}^n \otimes \mathbb{R}^n$, $n \geq 3$ |
| $\mathfrak{gl}(n, \mathbb{R})$ | $\mathbb{R}^n$, $n \geq 2$ | $\mathfrak{sl}(n, \mathcal{H}) \oplus \mathfrak{sl}(n, \mathcal{H}) \oplus \mathbb{R}$ | $\mathbb{H}^n \otimes \mathbb{H}^m$, $n, m \geq 1$ |
| $\mathfrak{sl}(n, \mathbb{R})$ | $\odot^2 \mathbb{R}^n$, $n \geq 3$ | $\mathfrak{sl}(n, \mathbb{H}) \oplus \mathfrak{sl}(n, \mathbb{H})$ | $\mathbb{H}^n \otimes \mathbb{H}^n$, $n \geq 2$ |
| $\mathfrak{gl}(n, \mathbb{R})$ | $\odot^2 \mathbb{R}^n$, $n \geq 3$ | $\mathfrak{sl}(n, \mathcal{H}) \oplus \mathfrak{sl}(n, \mathcal{H}) \oplus \mathfrak{sl}(n, \mathbb{R})$ | $\mathbb{H}^n \otimes \mathbb{H}^n$, $n \geq 2$ |
| $\mathfrak{sl}(n, \mathcal{H})$ | $S_n(\mathbb{H})$, $n \geq 2$ | $\mathfrak{sl}(n, \mathbb{C}) \oplus \mathbb{R}$ | $\mathbb{H}_n(\mathbb{C})$, $n \geq 3$ |
| $\mathfrak{gl}(n, \mathcal{H})$ | $S_n(\mathbb{H})$, $n \geq 2$ | $\mathfrak{so}(p, q) \oplus \mathbb{R}$ | $\mathbb{R}^{p+q}$, $p + q \geq 4$ |
| $\mathfrak{sl}(n, \mathbb{R})$ | $\Lambda^2 \mathbb{R}^n$, $n \geq 5$ | $\mathfrak{so}(p, q) \oplus \mathbb{R}$ | $\mathbb{R}^{p+q}$, $p + q \geq 4$ |
| $\mathfrak{gl}(n, \mathbb{R})$ | $\Lambda^2 \mathbb{R}^n$, $n \geq 5$ | $\mathfrak{sp}(2n, \mathbb{R}) \oplus \mathbb{R}$ | $\mathbb{R}^{2n}$, $n \geq 2$ |
| $\mathfrak{sl}(n, \mathcal{H})$ | $\mathfrak{A}_n(\mathbb{H})$, $n \geq 3$ | $\mathfrak{g}$ is a real form of a simple complex Lie algebra | $\mathfrak{g}$ |
| $\mathfrak{gl}(n, \mathcal{H})$ | $\mathfrak{A}_n(\mathbb{H})$, $n \geq 3$ | $\mathfrak{g} \oplus \mathbb{R}$, $\mathfrak{g}$ is a real form of a simple complex Lie algebra | $\mathfrak{g}$ |
| $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(m, \mathbb{R}) \oplus \mathbb{R}$ | $\mathbb{R}^n \otimes \mathbb{R}^m$, $n, m \geq 2$ | $\mathfrak{g}$ | $\mathfrak{g}$ |
| $\mathfrak{sl}(n, \mathbb{R}) \oplus \mathfrak{sl}(n, \mathbb{R})$ | $\mathbb{R}^n \otimes \mathbb{R}^n$, $n \geq 3$ | $\mathfrak{g}$ | $\mathfrak{g}$ |

B Holonomy algebras of pseudo-Riemannian manifolds

Let $\mathfrak{g} \subset \mathfrak{so}(p, q) = \mathfrak{so}(V)$ be a subalgebra, where $V = \mathbb{R}^{p,q}$ is the pseudo-Euclidean space ($p$ is the number of minuses of the metric). The space of curvature tensors of type $\mathfrak{g}$ is defined as follows

$$ \mathcal{R}(\mathfrak{g}) = \left\{ R \in \Lambda^2 V^* \otimes \mathfrak{g} \left| \begin{array}{c} R(X, Y)Z + R(Y, Z)X + R(Z, X)Y = 0 \\ \text{for all } X, Y, Z \in V \end{array} \right. \right\}. $$

The above identity is called the Bianchi identity. The subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ is called a Berger subalgebra if it is spanned by the images of the elements $R \in \mathcal{R}(\mathfrak{g})$. From the Ambrose-Singer Theorem it follows that the holonomy algebra of a pseudo-Riemannian manifold is a Berger algebra.

Consider the vector space

$$ \mathcal{R}^V(\mathfrak{g}) = \left\{ S \in V^* \otimes \mathcal{R}(\mathfrak{g}) \left| \begin{array}{c} S_X(Y, Z) + S_Y(Z, X) + S_Z(X, Y) = 0 \\ \text{for all } X, Y, Z \in V \end{array} \right. \right\}. $$

If $\mathfrak{g} \subset \mathfrak{so}(V)$ is a Berger subalgebra and $\mathcal{R}^V(\mathfrak{g}) = 0$, then $\mathfrak{g}$ is called a symmetric Berger algebra, otherwise $\mathfrak{g}$ is called a non-symmetric Berger algebra. It is known that if $\mathfrak{g}$ is a reductive Lie algebra and $\mathfrak{g}$-module $\mathcal{R}(\mathfrak{g})$ is trivial, then $\mathcal{R}^V(\mathfrak{g}) = 0$. It is obvious that if $\mathfrak{g}$ is the holonomy algebra of a pseudo-Riemannian manifold $(M, g)$ and $\mathcal{R}^V(\mathfrak{g}) = 0$, then $(M, g)$ is locally symmetric, i.e., its curvature tensor is parallel.

A subalgebra $\mathfrak{g} \subset \mathfrak{so}(V)$ is called weakly-irreducible if it does not preserve any proper non-degenerate subspace of $V$. The next theorem is a simple version of the DeRham-Wu decompo-
Theorem B.1 Let \( g \subset so(V) \) be an irreducible Berger subalgebra, then there is a decomposition
\[
V = V_0 \oplus V_1 \oplus \cdots \oplus V_r
\]
into a direct sum of pseudo-Euclidean subspaces and a decomposition
\[
g = g_1 \oplus \cdots \oplus g_r
\]
into a direct sum of ideals such that \( g_i \) annihilates \( V_j \) if \( i \neq j \) and \( g_i \subset so(V_i) \) is a weakly-irreducible Berger subalgebra.

Irreducible non-symmetric Berger subalgebras \( g \subset so(V) \) classified Marcel Berger in [4]. Later it was proved that all these algebras can be realised as holonomy algebras of pseudo-Riemannian manifolds [7]. We list these subalgebras in Table 4.

Table 4 Irreducible Berger subalgebras \( g \subset so(p,q) \)

| \( g \)          | \( (p,q) \)          | \( g \)          | \( (p,q) \)          |
|------------------|----------------------|------------------|----------------------|
| so\((p,q)\)      | \((p,q)\)            | sp\((r,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})\) | \((4r,4r)\)          |
| so\((p,\mathbb{C})\) | \((p,p)\)            | spin\((7,\mathbb{R})\) | \((0,8)\)            |
| u\((r,s)\)       | \((2r,2s)\)          | spin\((4,3)\)    | \((4,4)\)            |
| su\((r,s)\)      | \((2r,2s)\)          | spin\((7,\mathbb{C})\) | \((8,8)\)            |
| sp\((r,s)\)      | \((4r,4s)\)          | \(G_2\)          | \((0,7)\)            |
| sp\((r,s) \oplus sp\((1)\) | \((4r,4s)\)          | \(G^*_{2(2)}\)  | \((4,3)\)            |
| sp\((r,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})\) | \((2r,2r)\)          | \(G^C_2\)       | \((7,7)\)            |

If \( p = 0 \), then weakly-irreducible subalgebras are the same as the irreducible ones. Weakly-irreducible not irreducible Berger subalgebras \( g \subset so(p,q) \) are classified only if \( p = 1 \), if \( g \subset u(1,q) \subset so(2,2q) \) and there are some partial results in the neural signature \((p,p)\), see the review [11].

Let \( g \subset so(V) \) be a reductive weakly-irreducible Berger subalgebra. If \( g \) is not irreducible, then it preserves a degenerate subspace \( W \subset V \). Consequently, \( g \) preserves the isotropic subspace \( L = W \cap W^\perp \). Since \( g \) is totally reducible, there exists a complementary invariant subspace \( L' \subset V \). Since \( g \) is weakly-irreducible, the subspace \( L' \) is degenerate. If \( L' \) is not isotropic, then \( g \) preserves the kernel of the restriction of the metric to \( L' \) and \( g \) preserves a complementary subspace in \( L' \) to this kernel, which is non-degenerate. Hence \( L' \) is isotropic and \( V = L \oplus L' \) is the direct sum of isotropic subspaces. This can happen only if \( p = q \). The metric on \( V \) allows to identify \( L' \) with the dual space \( L^* \) and the representations of \( g \) on \( L \) and \( L' \) are dual. This shows
that the representation $g \subset gl(L)$ is irreducible. Let $R \in \mathcal{R}(g)$. From the Bianchi identity it follows that $R(x, y) = 0$ and $R(\varphi, \psi) = 0$ for all $x, y \in L$ and $\varphi, \psi \in L^*$. Moreover, for each fixed $\varphi \in L^*$ it holds $R(\cdot, \varphi) \in (g \subset gl(L))^{(1)}$, where $(g \subset gl(L))^{(1)}$ is the first prolongation for the representation $g \subset gl(L)$ (similarly, for each fixed $x \in L$ it holds $R(\cdot, x) \in (g \subset gl(L^*))^{(1)}$). Consequently, $(g \subset gl(L))^{(1)} \neq 0$ and $g \subset gl(L)$ is contained in Table B from [7].

All weakly-irreducible (and irreducible) reductive symmetric Berger subalgebras $g \subset so(p, q)$ are listed in [5]. We do not give this list here as it is too long.

### C Weak-Berger algebras

Let $g \subset so(p, q) = so(V)$ be a subalgebra. Denote by $\eta$ the pseudo-Euclidian metric on $V$. The vector space

$$\mathcal{P}_\eta(g) = \left\{ P \in V^* \otimes g \mid \eta(P(X)Y, Z) + \eta(P(Y)Z, X) + \eta(P(Z)X, Y) = 0 \right\}$$

is called the space of weak-curvature tensors of type $g$. A subalgebra $g \subset so(V)$ is called a weak-Berger algebra if $h$ is spanned by the images of the elements $P \in \mathcal{P}(h)$. It is not hard to see that if $R \in \mathcal{R}(g)$ and $x \in V$ is fixed, then $R(\cdot, x) \in \mathcal{P}_\eta(g)$. In particular, any Berger algebra is a weak-Berger algebra. The converse statement is not obvious, it is proved recently in [24].

**Theorem C.1** Let $g \subset so(p, q)$ be an irreducible weak-Berger subalgebra, then it is a Berger subalgebra.

Remark that in the origin theorem $g$ is a subalgebra of $so(n)$. The above result immediately follows from the complexification process described in [24].

The spaces $\mathcal{P}_\eta(g)$ for irreducible Berger subalgebras $g \subset so(n, \mathbb{R})$ are found in [15]. This result can be easily extended to the case of subalgebras $g \subset so(p, q)$. In particular, it is proved that if $g = g^1 \oplus g^2 \subset so(V^1 \otimes V^2) = so(V)$, where $g^1 \subset gl(V^1)$ and $g^2 \subset gl(V^2)$ are irreducible, then $\mathcal{P}_\eta(g^1 \subset so(V)) = \mathcal{P}_\eta(g^2 \subset so(V)) = 0$, unless the complexification of $g \subset so(V)$ coincides with $sp(2m, \mathbb{C}) \oplus sl(2, \mathbb{C}) \subset so(4m, \mathbb{C})$. In the last case $\mathcal{P}_\eta(sl(2, \mathbb{R}) \subset so(V)) = 0$ and $\mathcal{P}_\eta(sp(2m, \mathbb{R}) \subset so(V)) = (sp(2m, \mathbb{C}) \subset sl(2m, \mathbb{C}))^{(1)}$.

In [15] it is shown that if $g \subset so(n, \mathbb{R})$ is an irreducible subalgebra and $\mathcal{P}_\eta(g) \neq 0$ or $\mathcal{R}(g) \neq 0$, then either $g$ is a Berger subalgebra, or $g = sp(\frac{n}{2}) \oplus \mathbb{R}J$. A similar result holds for irreducible subalgebras $g \subset so(p, q)$.

Let $g \subset gl(L)$ be an irreducible subalgebra. Then $g$ is a weakly-irreducible subalgebra of $so(L \oplus L^*) = so(p, p)$, where $p = \dim V$. Let $\eta$ be the natural metric on $L \oplus L^*$. It is easy to see that $P \in \mathcal{P}_\eta(g)$ if and only if $pr_{gl(L)} \circ P|_L \in (g \subset gl(L))^{(1)}$ and $pr_{gl(L^*)} \circ P|_L \in (g \subset gl(L^*))^{(1)}$.
Thus if \( g \subset \mathfrak{so}(L \oplus L^*) \) is a weak-Berger subalgebra, then \((g \subset \mathfrak{gl}(L))^{(1)} \neq \{0\}\) and \( g \subset \mathfrak{gl}(L) \) is given in Table B from [7].

D Skew-Berger subalgebras of \( \mathfrak{sp}(2m, \mathbb{R}) \)

Let \( V \) be a real or complex vector space and \( g \subset \mathfrak{gl}(V) \) a subalgebra. The space of skew-symmetric curvature tensors of type \( g \) is defined as follows

\[
\mathcal{R}(g) = \left\{ R \in \bigotimes^2 V^* \otimes g \left| \begin{array}{c}
R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0 \\
\text{for all } X,Y,Z \in V
\end{array} \right. \right\}.
\]

The subalgebra \( g \subset \mathfrak{gl}(V) \) is called a \textit{skew-Berger subalgebra} if it is spanned by the images of the elements \( R \in \mathcal{R}(g) \). Obviously \( \mathcal{R}(g) = \mathcal{R}(g \subset \mathfrak{gl}(\Pi V)) \) and \( g \subset \mathfrak{gl}(V) \) is a skew-Berger algebra if and only if \( g \subset \mathfrak{gl}(\Pi V) \) is a Berger superalgebra.

Let \( \omega \) be the standard symplectic form on \( \mathbb{R}^{2m} \). A subalgebra \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) is called \textit{weakly-irreducible} if it does not preserve any proper non-degenerate subspace of \( \mathbb{R}^{2m} \). The next theorem is an analog of Theorem [B.1]. In fact the both theorems are partial cases of the Wu Theorem for Riemannian supermanifolds proved in [13].

\textbf{Theorem D.1} Let \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) be an irreducible skew-Berger subalgebra, then there is a decomposition

\[ V = V_0 \oplus V_1 \oplus \cdots \oplus V_r \]

into a direct sum of symplectic subspaces and a decomposition

\[ g = g_1 \oplus \cdots \oplus g_r \]

into a direct sum of ideals such that \( g_i \) annihilates \( V_j \) if \( i \neq j \) and \( g_i \subset \mathfrak{sp}(V_i) \) is a weakly-irreducible Berger subalgebra.

Irreducible skew-Berger subalgebras \( g \subset \mathfrak{gl}(n, \mathbb{C}) \) are classified in [14]. In Table 5 we list irreducible skew-Berger subalgebras \( g \subset \mathfrak{sp}(2m, \mathbb{C}) \).
Table 5  Irreducible skew-Berger subalgebras $g \subset \mathfrak{sp}(2m,\mathbb{C}) = \mathfrak{sp}(V)$

| $g$          | $V$                      | restriction |
|-------------|--------------------------|-------------|
| $\mathfrak{sp}(2m,\mathbb{C})$ | $\mathbb{C}^{2m}$        | $n \geq 1$  |
| $\mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{so}(m,\mathbb{C})$ | $\mathbb{C}^2 \otimes \mathbb{C}^m$ | $m \geq 3$  |
| $\mathfrak{spin}(12,\mathbb{C})$ | $\Delta_{12}^+ = \mathbb{C}^{32}$ |             |
| $\mathfrak{sl}(6,\mathbb{C})$ | $\Lambda^3 \mathbb{C}^6 = \mathbb{C}^{20}$ |             |
| $\mathfrak{sp}(6,\mathbb{C})$ | $V_{\pi_3} = \mathbb{C}^{14}$ |             |
| $\mathfrak{so}(n,\mathbb{C}) \oplus \mathfrak{sp}(2q,\mathbb{C})$ | $\mathbb{C}^n \otimes \mathbb{C}^{2q}$ | $n \geq 3$, $q \geq 2$ |
| $G_2^C \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathbb{C}^7 \otimes \mathbb{C}^2$ |             |
| $\mathfrak{so}(7,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$ | $\mathbb{C}^8 \otimes \mathbb{C}^2$ |             |

We get now the list of irreducible skew-Berger subalgebras $g \subset \mathfrak{sp}(2m,\mathbb{R})$. Let $V$ be a real vector space and $g \subset \mathfrak{gl}(V)$ an irreducible subalgebra. Consider the complexifications $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ and $g_\mathbb{C} = g \otimes_\mathbb{R} \mathbb{C} \subset \mathfrak{gl}(V_\mathbb{C})$. It is easy to see that $\mathcal{R}(g_\mathbb{C}) = \mathcal{R}(g) \otimes_\mathbb{R} \mathbb{C}$. Recall that the subalgebra $g \subset \mathfrak{gl}(V)$ is called absolutely irreducible if $g \otimes \mathbb{C} \subset \mathfrak{gl}(V_\mathbb{C})$ is irreducible and it is called not absolutely irreducible otherwise. The last situation appears if and only if there exists a complex structure $J$ on $V$ commuting with the elements of $g$. Then $V$ can be considered as a complex vector space and $g \subset \mathfrak{gl}(V)$ can be considered as a complex irreducible subalgebra. Consider also the natural representation $i : g_\mathbb{C} \to \mathfrak{gl}(V)$ in the complex vector space $V$. The following proposition is the analog of Proposition 3.1 from [36].

**Proposition D.1** Let $V$ be a real vector space and $g \subset \mathfrak{gl}(V)$ an irreducible subalgebra.

1. If the subalgebra $g \subset \mathfrak{gl}(V)$ is absolutely irreducible, then $g \subset \mathfrak{gl}(V)$ is a skew-Berger algebra if and only if $g_\mathbb{C} \subset \mathfrak{gl}(V_\mathbb{C})$ is a skew-Berger algebra.

2. If the subalgebra $g \subset \mathfrak{gl}(V)$ is not absolutely irreducible and if $(i(g_\mathbb{C}))^{[1]} = 0$, then $g \subset \mathfrak{gl}(V)$ is a skew-Berger algebra if and only if $Jg = g$ and $g \subset \mathfrak{gl}(V)$ is a complex irreducible skew-Berger algebra.

From this and Proposition 3.1 from [36] it follows that if the subalgebra $g \subset \mathfrak{sp}(V)$ is absolutely irreducible and $g_\mathbb{C} \subset \mathfrak{sp}(V_\mathbb{C})$ is both a skew-Berger and a Berger algebra, then the subalgebra $g \subset \mathfrak{sp}(V)$ is a skew-Berger algebra if and only if it is a Berger algebra. Similarly, if the subalgebra $g \subset \mathfrak{sp}(V)$ is not absolutely irreducible, $(i(g_\mathbb{C}))^{[1]} = (i(g_\mathbb{C}))^{[1]} = 0$, and $g \subset \mathfrak{gl}(V)$ is both a complex irreducible skew-Berger and Berger algebra, then $g \subset \mathfrak{sp}(V)$ is a real skew-Berger algebra if and only if it is a real Berger algebra. Remark that for any irreducible proper subalgebra $g \subset \mathfrak{sp}(2m,\mathbb{C})$ it holds $g^{[1]} = g^{(1)} = 0$. Note also that if $h = h_0 \oplus h_1$ is a simple Lie superalgebra, then the Lie superbrackets restricted to $h_1 \times h_1$ define an element in
$\mathfrak{h}_0 \subset \mathfrak{gl}(\mathfrak{h}_1)$, the image of this element spans $\mathfrak{h}_0$, thus, $\mathfrak{h}_0 \subset \mathfrak{gl}(\mathfrak{h}_1)$ is a skew-Berger algebra. All these facts allow us to classify irreducible skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R})$ different from the complex irreducible skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{C})$ considered as real ones. The result is given in Table 6.

Table 6  Irreducible skew-Berger subalgebras $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V)$ different from algebras from Table 5

| $\mathfrak{g}$ | $V$ | restriction |
|----------------|-----|-------------|
| $\mathfrak{sp}(2m, \mathbb{R})$ | $\mathbb{R}^{2m}$ | $n \geq 1$ |
| $\mathfrak{sp}(p, q)$ | $\mathbb{H}^{p,q}$ | $p + q \geq 2$ |
| $\mathfrak{u}(p, q)$ | $\mathbb{C}^{p,q}$ | $p + q \geq 2$ |
| $\mathfrak{su}(p, q)$ | $\mathbb{C}^{p,q}$ | $p + q \geq 2$ |
| $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{so}(p, q)$ | $\mathbb{R}^2 \otimes \mathbb{R}^{p,q}$ | $p + q \geq 2$ |
| $\mathfrak{sp}(1) \oplus \mathfrak{so}(n, \mathbb{H})$ | $\mathbb{H}^n$ | $n \geq 2$ |
| $\mathfrak{so}(2, 10)$ | $\Delta_+^{2,10} = \mathbb{R}^{32}$ | |
| $\mathfrak{so}(6, 6)$ | $\Delta_+^{6,6} = \mathbb{R}^{32}$ | |
| $\mathfrak{so}(6, \mathbb{H})$ | $\Delta_0^6 = \mathbb{H}^8$ | |
| $\mathfrak{sl}(6, \mathbb{R})$ | $\Lambda^3\mathbb{R}^6 = \mathbb{R}^{20}$ | |
| $\mathfrak{su}(1, 5)$ | $\{\omega \in \Lambda^3\mathbb{C}^6 | \ast w = w\}$ | |
| $\mathfrak{su}(3, 3)$ | $\{\omega \in \Lambda^3\mathbb{C}^6 | \ast w = w\}$ | |
| $\mathfrak{sp}(6, 3)$ | $\mathbb{R}^{14} \subset \Lambda^3\mathbb{R}^6$ | |
| $\mathfrak{so}(p, q) \oplus \mathfrak{sp}(2q, \mathbb{R})$ | $\mathbb{R}^{p,q} \otimes \mathbb{R}^{2q}$ | $p + q \geq 3, q \geq 2$ |
| $G_2 \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\mathbb{R}^7 \otimes \mathbb{R}^2$ | |
| $G_2^* \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\mathbb{R}^7 \otimes \mathbb{R}^2$ | |
| $\mathfrak{so}(7, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\Delta_7 \otimes \mathbb{R}^2$ | |
| $\mathfrak{so}(3, 4) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\Delta_{3,4} \otimes \mathbb{R}^2$ | |
| $\mathfrak{so}(2, 5) \oplus \mathfrak{su}(2)$ | $\mathbb{C}^4 \otimes \mathbb{C}^2$ | |
| $\mathfrak{so}(1, 6) \oplus \mathfrak{su}(2)$ | $\mathbb{C}^4 \otimes \mathbb{C}^2$ | |
| $\mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ | |
| $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{R})$ | $\mathbb{C}^2 \otimes \mathbb{R}^2$ | |

Let $\mathfrak{g} \subset \mathfrak{sp}(2m, \mathbb{R}) = \mathfrak{sp}(V)$ be a reductive weakly-irreducible subalgebra. Suppose that $\mathfrak{g}$ is not irreducible. As in Appendix B we may show that $V$ is of the form $V = L \oplus L^*$, where $\mathfrak{g} \subset \mathfrak{gl}(L)$ is irreducible. If $\mathfrak{g} \subset \mathfrak{sp}(V)$ is a skew-Berger subalgebra, then $(\mathfrak{g} \subset L)[1] \neq \{0\}$, and $\mathfrak{g}$ is contained in Table 3 or 2 from Appendix A.

Let $V$ be a complex or real vector space with a symplectic form $\omega$. Let $\mathfrak{g} \subset \mathfrak{sp}(V)$ be a
subalgebra. The vector space
\[ \mathcal{P}_\omega(g) = \left\{ P \in V^* \otimes g \left| \begin{array}{c}
\eta(P(X)Y, Z) + \eta(P(Y)Z, X) + \eta(P(Z)X, Y) = 0 \\
\text{for all } X, Y, Z \in V
\end{array} \right. \right\} \]
is called the space of skew-symmetric weak-curvature tensors of type \( g \). A subalgebra \( g \subset \mathfrak{sp}(V) \) is called a skew-symmetric weak-Berger algebra if \( g \) is spanned by the images of the elements \( P \in \mathcal{P}_\omega(g) \). It is not hard to see that if \( R \in \mathcal{R}(g) \) and \( X \in V \) is fixed, then \( R(\cdot, X) \in \mathcal{P}_\omega(g) \). In particular, any skew-Berger algebra is a skew-symmetric weak-Berger algebra. The converse statement gives the following theorem.

**Theorem D.2** Let \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) be an irreducible skew-symmetric weak-Berger subalgebra, then it is a skew-Berger subalgebra.

The proof of this theorem is a modified copy of the proof from [24] of Theorem C.1. If the representation \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) is not absolutely irreducible, then \( \mathcal{P}_\omega(g) \) is isomorphic to \( (g_C \subset \mathfrak{gl}(m, \mathbb{C})) \) and the proof follows from Appendix A and [14]. If the representation \( g \subset \mathfrak{sp}(2m, \mathbb{R}) \) is absolutely irreducible, then we need a classification of irreducible skew-symmetric weak-Berger subalgebras \( g \subset \mathfrak{sp}(2m, \mathbb{C}) \). It can be achieve in the same way as the classification of irreducible weak-Berger subalgebras \( g \subset \mathfrak{so}(n, \mathbb{C}) \). In fact, in [24] it is obtained a necessary condition for an irreducible subalgebra \( g \subset \mathfrak{so}(n, \mathbb{C}) \) to be a weak-Berger subalgebra, then there were classified all subalgebras \( g \subset \mathfrak{gl}(n, \mathbb{C}) \) satisfying this condition and the subalgebras \( g \subset \mathfrak{sp}(2m, \mathbb{C}) \) were noted. It is easy to see that a skew-symmetric weak-Berger subalgebras \( g \subset \mathfrak{sp}(2m, \mathbb{C}) \) satisfies the same necessary condition. Thus the proof follows immediately. □

The spaces \( \mathcal{P}_\omega(g) \) for irreducible weak-Berger subalgebras \( g \subset \mathfrak{sp}(2n, \mathbb{R}) \) can be found by methods of [15]. In particular, it is can be proved that if \( g = g^1 \oplus g^2 \subset \mathfrak{sp}(V^1 \otimes V^2) = \mathfrak{sp}(V) \), where \( g^1 \subset \mathfrak{gl}(V^1) \) and \( g^2 \subset \mathfrak{gl}(V^2) \) are irreducible, then \( \mathcal{P}_\omega(g^1 \subset \mathfrak{sp}(V)) = \mathcal{P}_\omega(g^2 \subset \mathfrak{sp}(V)) = 0 \), unless the complexification of \( g \subset \mathfrak{sp}(V) \) coincides with \( \mathfrak{so}(n, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sp}(2n, \mathbb{C}) \). In the last case \( \mathcal{P}_\omega(\mathfrak{sl}(2, \mathbb{C}) \subset \mathfrak{sp}(V)) = 0 \) and \( \mathcal{P}_\omega(\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{sp}(V)) = (\mathfrak{so}(n, \mathbb{C}) \subset \mathfrak{sl}(n, \mathbb{C})) \).

Let \( g \subset \mathfrak{gl}(L) \) be an irreducible subalgebra. Then \( g \) is a weakly-irreducible subalgebra of \( \mathfrak{sp}(L \oplus L^*) = \mathfrak{sp}(2m, \mathbb{R}) \), where \( 2m = \dim V \). Let \( \omega \) be the natural symplectic form \( L \oplus L^* \). It is easy to see that \( P \in \mathcal{P}_\omega(g) \) if and only if \( \text{pr}_{\mathfrak{gl}(L)} \circ P|_L \in (g \subset \mathfrak{gl}(L)) \) and \( \text{pr}_{\mathfrak{gl}(L^*)} \circ P|_L \in (g \subset \mathfrak{gl}(L^*)) \). Thus if \( g \subset \mathfrak{sp}(L \oplus L^*) \) is a skew-symmetric weak-Berger subalgebra, then \( (g \subset \mathfrak{gl}(L)) \neq \{0\} \) and \( g \subset \mathfrak{gl}(L) \) is given in Table 3 or 2 from Appendix A.
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