Abstract

We prove that if \( V \) is a \( C_2 \)-cofinite simple vertex operator algebra of CFT-type with a nonsingular invariant bilinear form and its an automorphism group \( G \) is finite, then an orbifold model \( V^G \) is also \( C_2 \)-cofinite.

1 Introduction

Orbifold models were first studied in [3] in 1985 in order to construct new conformal field theories. From a known rational conformal field theory \( V \) and its finite symmetry group \( G \), they took the set of \( G \)-fixed points \( V^G \), which is called an orbifold model, with expectation of nice algebraic properties including finiteness of isomorphic class of simple modules and modular invariant properties of their characters, see [4]. Separately from that, in order to solve the monstrous moonshine conjecture in the finite group theory, Borcherds [1] has introduced a concept of vertex (operator) algebra (shortly, we call VA or VOA), which is now understood as an algebraic aspect of conformal field theory. As a remarkable application of the theory of vertex operator algebras, a modular invariance property of conformal field theory is proved by Zhu [13] under the assumptions of \( C_2 \)-cofiniteness and the semisimplicity of modules (rationality in the terminology of vertex operator algebra). Therefore, it is now believed that a rational conformal field theory is corresponding to a rational \( C_2 \)-cofinite vertex operator algebra.

Among these two conditions on VOAs, the rationality implies the semisimplicity of the category of \( N \)-gradable modules and the \( C_2 \)-cofiniteness condition seems to be a finiteness condition on the number of isomorphism classes of simple modules. Therefore, the \( C_2 \)-cofiniteness condition plays a basic role to consider the representation theory with finite number of simple modules. For example, under this condition, we are always able to consider fusion products of modules (see [11]), and some kind of modular invariance properties of (pseudo) trace functions [10]. Moreover, roughly speaking, without this condition or a weaker condition, it is hard to argue the general problem.

\[ C_2 \text{-cofiniteness of orbifold models for finite groups} \]

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Abstract

We prove that if \( V \) is a \( C_2 \)-cofinite simple vertex operator algebra of CFT-type with a nonsingular invariant bilinear form and its an automorphism group \( G \) is finite, then an orbifold model \( V^G \) is also \( C_2 \)-cofinite.
In connection with orbifold models of vertex operator algebras, there is a natural conjecture (called orbifold conjecture) that if $V$ is a $C_2$-cofinite rational VOA and $G$ is a finite automorphism group, then the fixed point subVOA $V^G$ is still rational and $C_2$-cofinite. In order to consider this problem, by the above reason, it is natural to expect the $C_2$-cofiniteness of orbifold models first.

For this conjecture of $C_2$-cofiniteness, the author [12] gave an affirmative answer to the case with a solvable automorphism group $G$. Recently, using this result, the author and S. Carnahan gave an affirmative answer to the rationality and $C_2$-cofiniteness of orbifold models in [2]. So, the remaining is the case for a finite nonabelian simple automorphism group $G$. In this paper, we will give a complete answer to this conjecture.

**Main Theorem** Let $V$ be a $C_2$-cofinite simple vertex operator algebra of CFT-type with a nonsingular invariant bilinear form. If $G$ is a finite automorphism group of $V$, then $V^G$ is $C_2$-cofinite.

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## 2 Notation and Preliminary results

### 2.1 $G$-invariant $V$-internal fusion product

In this paper, we adopt the notation from [9] and [12]. Let $V$ be a VOA and $W$ its module. For $W$, we set

$$C_2(W) = \text{Span}_\mathbb{C}\{v_{-2}w \mid v \in V, w \in W\}$$

and if $\dim W/C_2(W) < \infty$, then we call $W$ “$C_2$-cofinite”. It is known that if $V$ is $C_2$-cofinite, then all finitely generated $V$-modules are $C_2$-cofinite. Let $\text{Irr}(G)$ be the set of irreducible characters of a group $G$. For $\phi \in \text{Irr}(G)$, $M_\phi$ denotes a $G$-module affording $\phi$. As Dong and Mason have shown in [5], if $G$ acts on a VOA $V$ faithfully, then $V$ decomposes into a direct sum:

$$V = \bigcoprod_{\phi \in \text{Irr}(G)} (V^\phi \otimes M_\phi) \quad (2.1)$$

of simple $V^G \times G$-modules $V^\phi \otimes M_\phi$, where $V^\phi$ is a nonzero simple $V^G$-module. We note that the dual space of $(V^\chi \otimes M_\chi)$ is $V^{\bar{\chi}} \otimes M_{\bar{\chi}}$, where $\bar{\chi}$ denotes the complex conjugate of $\chi$. From now on, to simplify the notation, $W^\chi$ denotes $V^\chi \otimes M_\chi$ for $\chi \in \text{Irr}(G)$.

A powerful tool in the representation theory is the rigidity property in the tensor product (or the fusion product $\boxtimes$ in VOA theory defined by intertwining operators), that is, we expect the composition

$$V^G \boxtimes W \xrightarrow{\iota_{W\boxtimes W^*}} (W \boxtimes W^*) \boxtimes W \xrightarrow{\phi} W \boxtimes (W^* \boxtimes W) \xrightarrow{1 \otimes \pi_W} W \boxtimes V^G$$

is an isomorphism.
for some simple object $W$ and its restricted dual $W^*$, where $\epsilon_W : V^G \to W \boxtimes W^*$ is an embedding and $\pi_W : W^* \boxtimes W \to V^G$ is a homomorphism such that $\pi_W \epsilon_W = 1_{V^G}$. In order to use this property, it is very important to find a natural embedding $\epsilon_W : V^G \to W \boxtimes W^*$.

One of our ideas is to consider $V^G \times G$-module as objects and define a restricted fusion product by using $G$-commutative intertwining operators. Furthermore, since our purpose is to prove $C_2$-cofiniteness of $W^\chi \subseteq V$, in order to ensure the existence of fusion products, we will restrict ourself to consider only intertwining operators which appear in $V$. More precisely, we introduce the following notation.

**Definition 1** Let $V$ be a VOA and $U$ a subVOA of $V$. For $U$-modules $U^1, U^2, W$, we call an intertwining operator $\mathcal{Y} \in \mathcal{I}(T_{U^1, U^2})$ “primitive $V$-internal” if there are embeddings $\tau_i : U^i \to V$ for $i = 1, 2$ and a homomorphism $\mu : V \to W$ such that

$$\mathcal{Y}(v, z)u = \mu(Y(\tau_1(v), z)\tau_2(u)).$$

(2.2)

We will call a linear combination of these primitive $V$-internal intertwining operators “$V$-internal operator”.

**Definition 2** For $V^G \times G$ modules $U$, $W$ and $T$, an intertwining operator $\mathcal{Y} \in \mathcal{I}(T_{U, W})$ is called to be $G$-commutative if $\mathcal{Y}(gu, z)gw = g(\mathcal{Y}(u, z)w)$ for $u \in U$, $w \in W$ and $g \in G$.

From the definition of $V$-internal operators, for a simple $V^G \times V$-modules $W^{\phi_i}$ ($i = 1, 2, 3$), every $G$-commutative $V$-internal operators $\mathcal{Y}$ of type $(W^{\phi_i})^*$ is a linear combination of $\{\mathcal{Y}_{g,h} \mid g, h \in G\}$, where $\mathcal{Y}_{g,h}$ is given by

$$\langle w, \mathcal{Y}_{g,h}(v, z)u \rangle = \langle w, Y(v^g, z)u^h \rangle$$

(2.3)

for $w \in W^{\phi_3}, v \in W^{\phi_1}, u \in W^{\phi_2}$. Furthermore, since $V$ is a direct sum of simple $V^G$-modules (2.1), the definition of $V$-internal operator implies that if there is a surjective $G$-commutative $V$-internal operator $\mathcal{Y}$ of type $(V_{W^{\phi_1}, W^{\phi_2}})^T$ then $T$ is a direct sum of simple $V \times G$-modules.

In this paper, we will only treat $V^G \times G$-modules and the set of $G$-commutative $V$-internal operators and we define a tensor product in this category. For example, for two simple $V^G \times G$-modules $W^{\phi_1}, W^{\phi_2}$ and a $V^G \times G$-module $T$, we consider the set of $G$-commutative $V$-internal operators $I_{V,G}(W^{\phi_1}, W^{\phi_2})$ of type $(W^{\phi_1}, W^{\phi_2})^T$ and define a fusion product $W^{\phi_1} \boxtimes_{V,G} W^{\phi_2}$ as an isomorphic class of maximal object $T$ with a surjective $G$-commutative $V$-internal operators. As we explained, since $T$ is always a direct sum of simple $V^G \times G$-modules and the multiplicity of $W^\chi$ in $T$ is less than or equal to $|G|^2$ by (2.3), $W^{\phi_1} \boxtimes_{V,G} W^{\phi_2}$ is well-defined as a finite direct sum of copies of $W^\phi$ with $\phi \in \text{Irr}(G)$.

**Remark 3** We don’t need the precise values for our arguments, but the multiplicity of $W^{\phi_3}$ in $W^{\phi_1} \boxtimes_{V,G} W^{\phi_2}$ is given by $\langle \phi_3, \phi_1 \phi_2 \rangle$. 

3
3 Rigidity

In order to prove $C_2$-finiteness of $V^G$, we will take a minimal counter example $G$ with the smallest order. Then $G$ has no proper normal subgroup. Furthermore, by the previous paper \[14\], $G$ is a nonabelian simple group. In this case, $G$ is of even order by Feit-Thompson’s theorem \[7\] and \text{Irr}(G)$ contains $\chi$ with $\bar{\chi} = \chi \neq 1$. We note $(W^\chi)^* = W^X = W^\chi$ since the invariant bilinear form on $V$ is $G$-invariant. As we explained in the previous section, $W^\chi \boxtimes_{V,G} W^x$ is a direct sum of $W^\phi$ with $\phi \in \text{Irr}(G)$. We choose $\mathcal{Y}^1 \in I_{V,G}(W^X \boxtimes_{V,G} W^X)$ to define a fusion product $W^\chi \boxtimes_{V,G} W^X$. Similarly, we also choose $\mathcal{Y}^2 \in I(W^X \boxtimes_{V,G} W^X \boxtimes_{V,G} W^X)$ and $\mathcal{Y}^3 \in I_{V,G}(W^X \boxtimes_{V,G} W^X \boxtimes_{V,G} W^x)$ to define fusion products $(W^X \boxtimes_{V,G} W^X) \boxtimes_{V,G} W^X$ and $W^X \boxtimes_{V,G} (W^X \boxtimes_{V,G} W^x)$, respectively. Since we are essentially treating only vertex operators in $V$ and $\langle w, \mathcal{Y}^1(v,z)u \rangle$ are linear combinations of $\langle w, Y(g(v),z)h(u) \rangle$ with $g, h \in G$ for $v, u \in V$ and we also have the similar results for $\mathcal{Y}^2$ and $\mathcal{Y}^3$ by considering the fusion products as a direct sum of simple $V \times G$-modules, $(\alpha, \mathcal{Y}^2(\mathcal{Y}^1(w^1, x-y)w^2, y)w^3), (\beta, \mathcal{Y}^3(w^1, x)\mathcal{Y}^1(w^2, y)w^3)$ and $(\beta, \mathcal{Y}^3(w^2, y)\mathcal{Y}^1(w^1, x)w^3)$ are all well-defined and expansions of the rational functions for $w^1 \in W^X$, $\alpha \in (W^X \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^X)^*$ and $\beta \in (W^X \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^X)^*$ in the regions $\{(x, y) \in \mathbb{C}^2 \mid 0 < |x-y| < |y|\}$, $\{(x, y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\}$ and $\{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < |y|\}$. To simplify the notation, we denote the rational function for $\langle \cdot, \cdot \rangle$ by $\langle \cdot, \cdot \rangle_f$ with the subscript $f$. We also have the maximality of fusion products $(W^X \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^X$ and $W^X \boxtimes_{V,G} (W^X \boxtimes_{V,G} W^x)$ and so there is an isomorphism

$$\rho : (W^X \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^X \rightarrow W^X \boxtimes_{V,G} (W^X \boxtimes_{V,G} W^x)$$

such that

$$\langle \beta, \rho(\mathcal{Y}^2(\mathcal{Y}^1(w^1, x-y)w^2, y)w^3) \rangle_f = \langle \beta, \mathcal{Y}^3(w^1, x)\mathcal{Y}^1(w^2, y)w^3 \rangle_f$$

(3.2)

on $x, y$ for $w^1, w^2, w^3 \in W^X$ and $\beta \in (W^X \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^x)^*$. Since $\pi(Y(v,z)w)$ defines a $G$-commutative $V$-internal operator $\pi Y$ for the projection $\pi : V \rightarrow V^G$, we have a projection

$$\pi_x : W^x \boxtimes_{V,G} W^x \rightarrow V^G,$$

which coincides with the one defined by inner product.

Let $\{e^1, ..., e^k\}$ be an orthonormal basis of $M_x$. We use the notation $v^i$ to denote $v \otimes e^i \in V^x \otimes M_x$ for $v \in V^x$. Using these notation and the ideas in the previous section, the elements in $\epsilon(x)(V^G)$ for the embedding $\epsilon_x : V^G \rightarrow W^X \boxtimes_{V,G} W^x$ are easily given as linear combinations of the following elements:

Lemma 4 For any $\mathcal{Y} \in I_{V,G}(W^X \boxtimes_{V,G} W^x)$ and any $v, u \in V^x$, we have

$$\sum_{i=1}^k \mathcal{Y}(v^i, z)u^i \in (W^X \boxtimes_{V,G} W^x)^G[[z, z^{-1}]]$$

and $\pi_x(\mathcal{Y}(v^s, z)u^t) = 0$ for $s \neq t$. In particular, $W^X \boxtimes_{V,G} W^x$ contains $V^G$ as a $V^G$-submodule.
Since $g \in G$ is an automorphism preserving inner products, the action of $g$ on $\text{Space}_G \{v \otimes e^1, \ldots, v \otimes e^k\}$ is given by an orthogonal maxtrix $A_g = (a_{ij}(g))$ satisfying $^tA_gA_g = I_k$. We note that $A_g$ does not depend on the choice of $v \in V^\chi$. Set $\mathcal{Y}(v^i, z) = \sum v^i_m z^{-m-1}$. Then we have $\sum g(v^i_m u^i) = \sum_i g(v^i_m)g(u^i) = \sum a_{ij}(g)v^i_m a_{ih}(g)u^h = \sum_j v^i_m u^j$. Hence we have $\sum v^i_m u^i \in (W^x \boxtimes_{V,G} W^x)^G$ for any $m$. Similarly, we have $|G|\pi(v^i_m u^i) = \sum_{g \in G} g(v^i_m u^i) = \sum_i (\sum_{g \in G} a_{si}(g)a_{ij}(g))v^i_m u^i = 0$ for $s \neq t$, see [6].

We also note that $(W^x \boxtimes_{V,G} W^x)^G$ is isomorphic to $V^G$, since $W^x$ is a simple $V^G \otimes G$-module. Therefore, we may assume that $\mathcal{Y}^1$ and $\mathcal{Y}^2$ satisfy

$$\sum_{i=1}^{k} \mathcal{Y}^1(v^i, z)u^i = \sum_{i=1}^{k} Y(v^i, z)u^i, \quad \pi(\mathcal{Y}^1(v^i, z)u^i) = \pi(Y(v^i, z)u^i)$$

and

$$\mathcal{Y}^2(\sum_{i=1}^{k} \mathcal{Y}^1(v^i, x - y)u^i, y)h^r = Y(\sum_{i=1}^{k} Y(v^i, x - y)u^i, y)h^r$$

for $v, u, h \in V^\chi$. Let $\epsilon_\chi : V^G \to W^x \boxtimes_{V,G} W^x$ be an injection with an identity map $\pi_\chi : V^G \to W^x \boxtimes_{V,G} W^x \to V^G$. We choose $\mathcal{Y}^1$ satisfying $\pi_\chi(\mathcal{Y}^1(w, z)u) = \pi(Y(w, z)u)$. By these maps, we have the following diagram.

$$\begin{array}{ccc}
V^G \boxtimes_{V,G} W^x & \Downarrow \epsilon_\chi \boxtimes_{V,G} 1_W \\
\downarrow & & \downarrow \\
(W^x \boxtimes_{V,G} W^x) \boxtimes_{V,G} W^x & \xrightarrow{\rho} & W^x \boxtimes_{V,G} (W^x \boxtimes_{V,G} W^x) \\
\downarrow \pi_\chi \boxtimes_{V,G} 1_W & & \downarrow 1_W \boxtimes_{V,G} \pi_\chi \\
V^G \boxtimes_{V,G} W^x & \xrightarrow{\pi_\chi} & W^x \boxtimes_{V,G} V^G
\end{array}$$

**Theorem 5 (Rigidity)** $(1_W \boxtimes_{V,G} \pi_\chi)\rho(\epsilon_\chi \boxtimes_{V,G} 1_W)$ is an isomorphism.

**[Proof]** Clearly, the above homomorphisms are commutative with actions of $V^G \times G$. Since $W^x$ is a simple $V^G \times G$-module, it is enough to show $(1_W \boxtimes_{V,G} \pi_\chi)\rho(\epsilon_\chi \boxtimes_{V,G} 1_W) \neq 0$. As we mentioned, $\sum_{i=1}^{k} \mathcal{Y}^1(v^i, z)u^i \in V^G[[z, z^{-1}]]$ for $v, u \in V^\chi$. We can choose homogenous elements $v, u, h, w \in V^\chi$ so that $u_{2wt(u) - 1}h = 1$ and $\langle w, v \rangle \neq 0$. Then since $\pi(Y(u^i, z)w^j) = 0$ for $i \neq j$, we have

$$\langle w^j, (1_W \boxtimes_{V,G} \pi_\chi)\rho(\mathcal{Y}^2(\sum_{i=1}^{k} \mathcal{Y}^1(v^i, x - y)u^i, y)h^j) \rangle_f = \langle w^j, (1_W \boxtimes_{V,G} \pi_\chi) \sum_{i=1}^{k} \mathcal{Y}^2(v^i, x)\mathcal{Y}^1(u^i, y)h^j \rangle_f$$

$$= \langle w^j, \sum_{i=1}^{k} \mathcal{Y}^2(v^i, x)\pi_\chi(\mathcal{Y}^1(u^i, y)h^j) \rangle_f$$

$$= \langle w^j, \mathcal{Y}^3(v^j, x)\pi(\mathcal{Y}^1(w^j, y)h^j) \rangle_f.$$
Since we can recognize its expansion region from the products of intertwining operators, we will omit $\rho$ from now on. We summarize the above result as a corollary.

**Corollary 6**  There is $0 \neq s \in \mathbb{C}$ such that for $\theta \in W^\times$, we have

$$
\langle \theta, \sum_{i=1}^{k} Y(Y(v^i, x-y)u^i, y)w^r \rangle_f = s \langle \theta, \sum_{i=1}^{k} Y(Y(u^i, y)w^r) \rangle_f
$$

for $v, u, w \in V^\times$ and $r \in \{1, \ldots, k\}$.

**Remark 7**  The above does not mean the rigidity of Borcherds identity

$$
\langle \theta, \sum_{i=1}^{k} (v^i u^i) w^r \rangle \sim \langle \theta, \sum_{j=0}^{\infty} \left(\frac{n}{j}\right) (-1)^j \{v^r_{n-j} \pi(u_{m+j} w^r) - (-1)^n u^r_{n+m-j} \pi(v^r w^r)}\rangle_f
$$

However, we will get this identity under some stronger conditions.

## 4 Proof of the main theorem

### 4.1 Borcherds identity

Let $\theta \in W^\times$, $v, u, w \in V^\times$ be homogeneous elements. We choose $r \in \{1, \ldots, k\}$. As it is well known, there is a rational function $t(x, y)$ such that

$$
t(x, y) = \langle \theta, Y(\sum_{i=1}^{k} Y(v^i, x-y)u^i, y)w^r \rangle_f
$$

(4.1) in the region $0 < |x-y| < |y|$. Since $t(x, y)$ is defined by elements in $V$ and vertex operator $Y$ of $V$, there are $a, b, c \in \mathbb{Z}$ and a homogeneous polynomial $f(x, y)$ such that

$$
t(x, y) = \frac{f(x, y)}{x^a y^b (x-y)^c}.
$$

(4.2)

From the expansions of $t(x, y)$ in the regions $\{(x, y) \in \mathbb{C}^2 \mid 0 < |x-y| < |y|\}$, $\{(x, y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\}$ and $\{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < |y|\}$, we get $a \leq \text{wt}(u) + \text{wt}(w)$, $b \leq \text{wt}(v) + \text{wt}(u)$ and $c \leq \text{wt}(v) + \text{wt}(w)$. Allowing negative values for $a, b, c, w$, we put $f(x, y) = \sum_{i=0}^{d} r_i x^{l-i} y^i$ with $r_0 \neq 0 \neq r_t$ and $f(x, y)$ has no zero (pole) at $x-y$. The total degree of $t(x, y)$ is $t = a - b - c$, which coincides with $\text{wt}(\theta) - \text{wt}(v) - \text{wt}(u) - \text{wt}(w)$. By the property of the rigidity (Theorem 5), we have

$$
t(x, y) = s \langle \theta, Y(v^r, x)\pi(Y(u^r, y)w^r) \rangle_f
$$

(4.3) in the region $\{(x, y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\}$ for some $0 \neq s \in \mathbb{C}$. We also have a similar expansion in the region $\{(x, y) \in \mathbb{C}^2 \mid 0 < |x| < |y|\}$.

Assume henceforth, $\theta$ satisfies $\langle \theta, V^G_2 W^\times \rangle = 0$. Here and further $A_{-2}B$ denotes the subspace spanned by $\{a_{-2}b \mid a \in A, b \in B\}$ for subsets $A, B$ of $V$. Then there is no
positive powers of $y$ in the expansion in (4.3) in the region $0 < |x - y| < |y|$. This implies $t - a - b \leq 0$. As a result, we have

$$
\text{wt}(\theta) = t + \text{wt}(v) + \text{wt}(u) + \text{wt}(w) - a - b - c \leq \text{wt}(v) + \text{wt}(u) + \text{wt}(w) - c. \tag{4.4}
$$

In particular, if we take $0$ with $\text{wt}(\theta) \geq \text{wt}(v) + \text{wt}(u) + \text{wt}(w)$, then $c \leq 0$ and $t(x,y)$ has no pole at $x - y$. In other words, $\langle \theta, \sum_{i=1}^{k}(v^i_n u^m r) \rangle = 0$ for $n \geq 0$. Under this assumption, $t(x,y)$ has the same expansions with only finitely many terms of $x^p y^q$ in the region $\{(x,y) \in \mathbb{C}^2 \mid 0 < |x| < |y|\}$ and $\{(x,y) \in \mathbb{C}^2 \mid 0 < |y| < |x|\}$. In particular, we have

$$
\langle \theta, v^r_n \pi(u^m w^r) \rangle = \langle \theta, u^r_n \pi(v^m w^r) \rangle. \tag{4.5}
$$

Since the expansion in $0 < |y| < |x|$ is a finite sum, we can replace $x^p y^q$ by $y^{-(x-y)/p} x^p y^q = \sum_{j=0}^{\infty} \binom{n}{p} (x-y)^j y^{p+q-j}$. For example, the expansion of $t(x,y)$ in the region $0 < |y| < |x|$ is

$$
\sum_{n,m \in \mathbb{Z}, \text{finite}} \langle \theta, v^r_n \pi(u^m w^r) \rangle x^{n-1} y^{m-1} = \sum_{n,m \in \mathbb{Z}, \text{finite}} \langle \theta, v^r_n \pi(u^m w^r) \rangle \sum_{j=0}^{\infty} \binom{n-1}{j} (x-y)^j y^{m-2-n-j} = \sum_{j=0}^{\infty} \sum_{p \in \mathbb{Z}, \text{finite}} \langle \theta, \sum_{n \in \mathbb{Z}, \text{finite}} \binom{n-1}{j} v^r_n \pi(u^r_{p-1-n-j} w^r) \rangle (x-y)^j y^{-p-1},
$$

where $p = m + n + 1 + j$. This should be equal to the expansion

$$
\sum_{j,p \in \mathbb{Z}} \langle \theta, \sum_{i=1}^{k} (v^i_{n,j-1} u^i_p) w^r \rangle (x-y)^j y^{-p-1}
$$

of $t(x,y)$ in the region $0 < |x - y| < |y|$ up to nonzero scalar multiples (which we denote by $\sim$). We exchange the indexes $n$ and $j$ to come back to the ordinary indexes. We note $n \geq 0$ because $j$ is non-negative. Then by (4.5), we have

$$
\langle \theta, \sum_{i=1}^{k} (v^i_{n,j-1} u^i_p) w^r \rangle \sim \sum_{n \in \mathbb{Z}, \text{finite}} \langle \theta, v^r_n \pi(u^r_{p-1-n-j} w) \rangle
$$

$$
= \langle \theta, \sum_{j \leq n-1} v^r_n \pi(u^r_{p-1-n-j} w) \rangle + \langle \theta, \sum_{n-1 < j} v^r_n \pi(u^r_{p-1-n-j} w) \rangle
$$

$$
= \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle + \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p-1-n-j} w) \rangle
$$

$$
= \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle + \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p-1-n-j} w) \rangle
$$

$$
= \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle + \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle
$$

$$
= \langle \theta, \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle + \sum_{j=0}^{\infty} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle
$$

where we have omitted the superscript $r$ of $v^r$ and others. Therefore, we have:

**Theorem 8** [Borcherd’s identity]

If $\langle \theta, V^2 GW \rangle = 0$ and $\text{wt}(\theta) > \text{wt}(v) + \text{wt}(u) + \text{wt}(w)$, then there is $0 \neq s \in \mathbb{C}$ such that for $r = 1, \ldots, k$, we have

$$
\langle \theta, \sum_{j=0}^{\infty} \binom{n}{j} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle = \rho \langle \theta, \sum_{j=0}^{\infty} \binom{n}{j} (\frac{n+j}{n}) v^r_{n,j} \pi(u^r_{p+n} w) \rangle - \rho u^r_{n+p-j} \pi(v^r_{p-j} w). \tag{4.6}
$$
4.2 Finiteness of $|V^\chi : V^\chi_2 V^G|$  

In this section, we will prove the following proposition.

Proposition 9  \[ \dim(W^\chi/(L(-1)W^\chi + W^\chi_2 V^G)) < \infty. \]

[Proof] We first note that $V^\chi_2 W^\chi \subseteq L(-1)W^\chi + W^\chi_2 V^G$. Suppose that the proposition is false, then for any integer $n$, there is a $\theta \in W^\chi$ with $\wt(\theta) > n$ such that $\langle \theta, L(-1)W^\chi + W^\chi_2 V^G \rangle = 0$. Since $V$ is $C_2$-cofinite, all $V$-modules are $\mathbb{N}$-gradable. Therefore, for a subset $B$ which generates $V$ as a subVA, we have $\dim V/B_{-2} V < \infty$ as we mentioned in [12]. Since $W^\chi$ generates a $G$-invariant subVA of $V$ contains $V^G$, $W^\chi$ generates $V$. Since $V/C_2(V)$ is finite, $V$ has a finite set of generators. Therefore, there is $p^1 \in \mathbb{N}$ such that $B = \oplus_{i=0}^{p^1} W^\chi_i$ generates $V$ as a subVA. Since the $V^G$-parts in $B_{-2} V$ come from only $\pi(B_{-2} W^\chi)$, we have $\dim V^G/\pi(B_{-2} W^\chi) < \infty$. So, there is $p^2 \in \mathbb{N}$ such that $\pi(B_{-2} W^\chi)$ contains $\prod_{i=p^2}^{\infty} V^G$. Set $p = \max\{p^1, p^2\}$. Under these setting, we will prove the following lemma, which contradicts the choice of $\theta$.

Lemma 10 There are $\alpha \in V^G$, $w \in B$ with $\wt(\alpha) \leq p$ and $m \in \mathbb{Z}$ such that $\langle \theta^r, \alpha_m w^r \rangle \neq 0$

In particular, $\wt(\theta) \leq 2p$.

[Proof] Since $\theta \neq 0$ and $W^\chi$ is generated by $B = \oplus_{i=0}^{p^1} W^\chi_i$ as a $V^G$-module, there are $\alpha \in V^G$, $w \in B$ and $m \in \mathbb{Z}$ such that $\langle \theta, \alpha_m w \rangle \neq 0$. We choose $\alpha$ and $w$ with the minimal total weight $\wt(\alpha) + \wt(w)$. If $\wt(\alpha) > p$, then there are $\beta^{(j)} \in B$, $u^{(j)} \in W^\chi$ such that $\alpha = \sum_j \pi(\beta^{(j)} u^{(j)})$. Then since $\langle \theta, \pi(\beta^{(j)} u^{(j)}) \rangle = 0$ for some $j$, we may assume $\alpha = \pi(\beta^{(j)} u^{(j)})$. We denote $\beta^{(j)}$, $u^{(j)}$ by $\beta$, $u$, respectively. Furthermore, using skew symmetry, we have $\alpha = \sum_{j=0}^{\infty} \frac{(-1)^j L(-1)^j}{j!} \pi(u_{-2-j} \beta)$. For $j \geq 1$, we have $\langle \theta^r, \pi(L(-1)^j u_{-2-j} \beta) \rangle = (-m_2)(\theta^r, \pi(L(-1)^j u_{-2-j} \beta)) = 0$ from the minimality of $\wt(\alpha) + \wt(w)$. Therefore, we may replace $\alpha$ by $\pi(u_{-2} \beta)$. We again use the notation $\beta^j$ and $u^j$ to denote $\beta \otimes e^j$ and $u \otimes e^j$, respectively. Since we have taken $\theta$ with large enough weight, we can apply Borcherds identity (4.6). From the choice of $u$, $\beta$ and the minimality of $\wt(\alpha) + \wt(w)$, we have

\[
0 \neq \langle \theta^r, \sum_{k=1}^{\infty} (u_{-2}^j \beta^j) m \rangle \wedge^\chi \sum_{j=0}^{\infty} (-1)^j (u_{-2-j}^r \pi(\beta_{m+j} \beta) - (-1)^{-2} \beta_{m-2-j} \pi(u_j^r w^r))
\]

\[
\sim \sum_{j=0}^{\infty} (-1)^j \pi((u_j^r w^r))
\]

Since $\langle \theta^r, L(-1)^j W^\chi \rangle = 0$, we have $\langle \theta^r, \pi(u_j^r w^r) m_{-2-j} \beta^r \rangle \neq 0$ for some $j$. Then $\beta^r \in B$ and $\wt(\pi(u_j^r w^r)) + \wt(\beta^r) < \wt(\pi(u_{-2} \beta)) + \wt(w) = \wt(\alpha) + \wt(w)$, which contradicts the choice of $\alpha$ and $w$. \]
4.3 Coefficient functions

We quote the most part of the proof from [12] by replacing simple $V^G$-modules by simple $V^G \times G$-modules. Since $L(-1)C_2(W^\chi) \subseteq C_2(W^\chi)$, $W^\chi/C_2(W^\chi)$ is a finitely generated $\mathbb{C}[L(-1)] \times G$-module by Proposition 3. Let $D$ be the inverse image in $W^\chi$ of the $L(-1)$-torsion submodule of $W^\chi/C_2(W^\chi)$. Then there is a set of free generators $\{\alpha^i : i = 1, \ldots, t\}$ such that

$$W^\chi = (\oplus_{i=1}^t \mathbb{C}[L(-1)]\alpha^i \otimes M_\chi) \oplus D.$$ 

If $W^\chi$ is $C_2$-cofinite, then so is $V^G$ by the main theorem in [11] since $(W^\chi)^* \cong W^\chi$. So we may assume that $W^\chi$ is not $C_2$-cofinite. Furthermore, since $\dim(W^\chi/(L(-1)W^\chi + V^G_2W^\chi)) < \infty$, we have $t \geq 1$. Choose $\alpha^1$ so that $\text{wt}(\alpha^1)$ is the minimal weight of elements in $W - D$ and we denote $\alpha^1$ by $\alpha$ and set $K = (\oplus_{i=2}^t \mathbb{C}[L(-1)]\alpha^i) \otimes M_\chi \oplus D$.

Then $W^\chi/K \cong \mathbb{C}[L(-1)]\alpha \otimes M_\chi$.

The key idea is to denote elements $a_{-n+\text{wt}(a)+\text{wt}(b)-\text{wt}(\alpha)-1}b$ in $W^\chi$ as

$$f(n)\alpha_{-n-1}1 \quad \text{modulo } K$$

for $n \in \mathbb{Z}$ and consider $f(n)$ as functions of $n \in \mathbb{Z}$. From now on, for $a, b \in V$, we always use $M$ to denote $\text{wt}(a) + \text{wt}(b) - \text{wt}(\alpha)$. We note that since $\text{wt}(a_{-x-M-1}b) < \text{wt}(\alpha)$ for $x \in \mathbb{Z}_{<0}$, $a_{-x+M-1}b \in K$ by the choice of $\alpha$. Namely, $f(x) = 0$ for $x \in \mathbb{Z}_{<0}$. In order to obtain $f(x)$ by inner products, we take $\theta = \prod \theta^n \in \prod W^n_\chi$ so that $\langle \theta, V^G_2W^\chi + K \rangle = 0$ and

$$\langle \theta, \alpha_{-x-1}1 \rangle = 1$$

for all $x \in \mathbb{N}$. We note that $\langle \theta, L(-1)\alpha_{-1}1 \rangle = x\langle \theta, \alpha_{-x-1}1 \rangle = x$. Then we have

$$\langle \theta, a_{-x-\text{wt}(a)+\text{wt}(b)-1}b \rangle = f(x).$$

So we will consider the set $\text{Map}(\mathbb{N}, \mathbb{C})$ of all maps from $\mathbb{Z}$ to $\mathbb{C}$ satisfying $f(n) = 0$ for $n \in \mathbb{Z}_{<0}$. Let $\mathcal{F}_0$ and $\mathcal{F}_1$ be the spaces of coefficients $f(x)$ of $a_{-x+M-1}b$ at $\alpha_{-x-1}1$ modulo $K$ for $a \in V^G, b \in W^\chi$ and $a \in W^\chi, b \in V^G$, that is,

$$\mathcal{F}_0 = \text{Span}_\mathbb{C} \left\{ f \in \text{Map}(\mathbb{N}, \mathbb{C}) \mid \exists a \in V^G, \exists b \in W^\chi \text{ s.t. } \langle \theta, a_{-x+M-1}b \rangle = f(x) \text{ for } x \in \mathbb{N} \right\},$$

$$\mathcal{F}_1 = \text{Span}_\mathbb{C} \left\{ f \in \text{Map}(\mathbb{N}, \mathbb{C}) \mid \exists a \in W^\chi, \exists b \in V^G \text{ s.t. } \langle \theta, a_{-x+M-1}b \rangle = f(x) \text{ for } x \in \mathbb{N} \right\}.$$

For a map $f : \mathbb{N} \rightarrow \mathbb{C}$, we introduce two linear operators $S$ and $T$ as follows:

$$Sf(n) = (-1)^n \sum_{k=0}^n \binom{n}{k} (-1)^k f(n - k) \quad \text{for } n \in \mathbb{N},$$

$$Tf(n) = (-1)^n f(n) \quad \text{for } n \in \mathbb{N}.$$

Clearly, $S^2 = T^2 = \text{id}$. We also have the following by induction.

Lemma 11 [12]

$$(ST)^k f(n) = \sum_{j=0}^n \binom{n}{j} k^j f(n - j) \text{ for } k = 1, \ldots$$
We consider the set of functions with finite supports and the set of polynomials.

\[ \mathcal{F}_{\text{finite}} = \{ f : \mathbb{N} \to \mathbb{C} \mid |\{ n \in \mathbb{N} : f(n) \neq 0 \}| < \infty \} \]

\[ \mathcal{F}_{\text{poly}} = \{ f \in \mathbb{C}[x] \}. \]

An important element of \( \mathcal{F}_{\text{finite}} \) is a Kronecker delta function \( \delta^i(x) \) which is defined by \( \delta^i(0) = 1 \) and 0 otherwise. Clearly, \( \{ \delta^i(x) \mid i \in \mathbb{N} \} \) is a basis of \( \mathcal{F}_{\text{finite}} \).

**Lemma 12** We have:
1. \( \mathcal{F}_{\text{finite}} \) and \( \mathcal{F}_{\text{poly}} \) are all \( \mathbb{C}[x] \)-invariant.
2. \( T(\mathcal{F}_{\text{finite}}) = \mathcal{F}_{\text{finite}} \)
3. \( S(\mathcal{F}_{\text{finite}}) = \mathcal{F}_{\text{poly}} \)
4. \( T(\mathcal{F}_0) = \mathcal{F}_0 \)

**Proof** Since \((xf)(n) = nf(n)\), we have (1). Since \( T(\delta^i) = (-1)^i \delta^i \), we have (2). The direct calculation shows \((S\delta^i)(x) = (-1)^n \sum_{k=0}^x \binom{x}{k} (-1)^k \delta(x-k) = (-1)^x \binom{n}{x} (-1)^{-x} = (-1)^i \binom{\hat{n}}{i} \cdot (x-1) \cdots (x-i+1) \in \mathcal{F}_{\text{poly}} \). Thus we have (3). For (4), as we have shown in [12], \( \mathcal{F}_0 \) is a \( \mathbb{C}[x] \)-invariant. Therefore, if \( f \in \mathcal{F}_0 \) has a support \( \{n_1, \ldots, n_r\} \), then since \( (x^j f)(n_i) = n^i_j f(n_i) \) and \( x^j f \in \mathcal{F}_0 \) for \( j = 0, 1, \ldots, r \), we have \( \delta^{n_i} \in \mathcal{F}_0 \) for \( i = 1, \ldots, r \). We hence have \( T(f) = \sum_{i=1}^{r} (-1)^{n_i} f(n_i) \delta^{n_i} \in \mathcal{F}_0 \).

**Lemma 13** \( S(\mathcal{F}_0) = \mathcal{F}_1 \). In particular, \( \mathcal{F}_1 \subseteq \mathcal{F}_{\text{poly}} \).

**Proof** As we have explained in [12], the operator \( S \) comes from the skew-symmetry, that is, for \( f(x) = \langle \theta, a_{-x+M-1}b \rangle \), we have
\[
\langle \theta, b_{-x+M-1}a \rangle = \langle \theta, (-1)^{x+M} \sum_{k=0}^{\infty} \frac{L(-1)^k}{k!} (-1)^k a_{-(x-k)+M-1}b \rangle = \langle \theta, (-1)^{x+M} \sum_{k=0}^{x} \frac{(\hat{\chi})k}{k!} (-1)^k f(x-k) \alpha_{x-k+1} \rangle
\]
since \( L(-1)^k a_{x-k+M-1}b \in K \) for \( k > x \),
\[
= \langle \theta, (-1)^{x+M} \sum_{k=0}^{x} \frac{(\hat{\chi})k}{k!} (-1)^k f(x-k) \alpha_{x-1} \rangle = \langle \theta, (-1)^{x+M} Sf(x) \alpha_{x-1} \rangle.
\]
Thus we have \( \mathcal{F}_1 \subseteq S(\mathcal{F}_0) \) and \( \mathcal{F}_0 \subseteq S(\mathcal{F}_1) \). Since \( S^2 = 1 \), we have the equality.

**Lemma 14** \( T(\mathcal{F}_1) \subseteq \mathcal{F}_{\text{finite}} + \mathcal{F}_1 \).

**Proof** For \( v, u, w \in V^x \), we set \( N = \text{wt}(w) + \text{wt}(v) + \text{wt}(u) \). Since
\[
\text{wt}((v_{-x+h+N}u)_{-h}w) = x
\]
for any \( h \in \mathbb{Z} \), for any \( x > N \) and \( h \geq 0 \), we have
\[
0 = \langle \theta, \sum_{i=1}^{\delta} (v_{-x+h+N}u')_{-h}w \rangle
\]
\[
= \langle \theta, \sum_{j=0}^{\infty} ((-1)^{x+h+N}j) \alpha_{x+h+N-j} \pi(u^r_{-h+j}w^r) \rangle
\]
\[
- \langle \theta, \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \text{wt}(v) \cdot \text{wt}(u)-1 \rangle \alpha_{x+h+N-j} \pi(u^r_{-h+j}w^r) \rangle
\]
\[
- \langle \theta, \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \text{wt}(v) \cdot \text{wt}(u)-1 \rangle \alpha_{x+h+N-j} \pi(u^r_{-h+j}w^r) \rangle
\]
by the Borcherds identity (Theorem 8).

We note \( v_jw = 0 \) for \( j \geq Q = \text{wt}(v) + \text{wt}(w) \). Let us consider a \( Q \times Q \) matrix

\[
A := \left( (-1)^{h-j+N} \binom{-x+h+N}{j} \right)_{h,j=0,\ldots,Q-1}
\]

consisting of coefficients of \((-1)^x u^r_{-x-2+N-j}(v^r w^r)\). It is easy to see \( \det A = \pm 1 \) since \((-^x_j) - \binom{x}{j-1} = \binom{x}{j-1}\). Therefore, for each \( 0 \leq m < Q \), there are polynomials \( \lambda^m_r(x) \in \mathbb{C}[x] \) such that

\[
\langle \theta, \sum_{r=0}^{Q-1} \lambda^m_r(x) \left( \sum_{j=0}^{N+h+2} (-x+h+N) \binom{-x+h+N}{j} (-1)^j v^r_{-x+h-N-j}(u^r_{-2-h+j}w^r) \right) \rangle = (-1)^x \langle \theta, u^r_{-x-2+N-m} \pi(v^r_m w^r) \rangle
\]

for \( x > N \). Since all terms of the left side are in \( \mathcal{F}_1 \), the equation (4.7) for \( x > N \) implies that

\[
T(\langle \theta, u^r_{-x-2+N-m} \pi(v^r_m w^r) \rangle) \in \mathcal{F}_1 + \mathcal{F}_{\text{finite}}.
\]

Furthermore, since \( \{ \langle \theta, u^r_{-x-2+N-m} \pi(v^r_m w^r) \rangle \mid v, u, w \in V^x, m \in \mathbb{Z} \} \) spans \( \mathcal{F}_1 \), we have the desired result.

Then we have \( STST(F_0) = STS(F_0) = ST(F_1) \subseteq S(\mathcal{F}_{\text{finite}} + \mathcal{F}_1) \subseteq \mathcal{F}_{\text{poly}} + \mathcal{F}_{\text{finite}} \).

However, by the direct calculation, we have

\[
(ST)^2 \delta^k(x) = \sum_{j=0}^{x} \binom{x}{j} 2^j \delta^k(x-j) = \binom{x}{k} 2^{x-k} \quad \text{for} \quad x \geq 0,
\]

which is not a sum of a polynomial and a function with finite support. So, we have a contradiction.

This completes the proof of the main theorem.

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