Steady viscous flows in an annulus between two cylinders produced by vibrations of the inner cylinder

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Abstract
We study the steady streaming between two infinitely long circular cylinders produced by small-amplitude transverse vibrations of the inner cylinder about the axis of the outer cylinder. The Vishik-Lyusternik method is employed to construct an asymptotic expansion of the solution of the Navier-Stokes equations in the limit of high-frequency vibrations for Reynolds numbers of order of unity. The effect of the Stokes drift of fluid particles is also studied. It is shown that it is nonzero not only within the boundary layers but also in higher order terms of the expansion of the averaged outer flow.

Keywords: steady streaming, oscillating boundary layers, asymptotic methods

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1. Introduction
In this paper, we study two-dimensional oscillating flows of a viscous incompressible fluid between two infinitely long circular cylinders. The outer cylinder is fixed, and the inner cylinder performs small-amplitude harmonic oscillations about the axis of the outer cylinder. It is well-known that high-frequency oscillations of the boundary of a domain occupied by a viscous fluid not only produce an oscillating flow, but can also lead to appearance of a steady flow, which is usually called the steady streaming (see, e.g., [1]). Here we are interested in this steady part of the flow. The basic parameters in our study are the inverse Strouhal number $\alpha$ and the dimensionless viscosity $\nu$ (the inverse Reynolds number), defined as

$$\alpha = \frac{V_0^*}{a\omega}, \quad \nu = \frac{1}{Re} = \frac{\nu^*}{V_0^* a} \quad (1.1)$$

where $V_0^*$ is the amplitude of the velocity of the oscillating cylinder, $a$ is its radius, $\omega$ is the angular frequency of the oscillations, and $\nu^*$ is the kinematic viscosity of the fluid. Parameter $\alpha$ measures the ratio of the amplitude of the displacement of the oscillating cylinder to its radius and is assumed to be small: $\alpha \ll 1$. One more dimensionless parameter which is widely used in literature is the ‘streaming Reynolds number’ $Re_s = V_0^{*2}/\omega \nu^* = \alpha/\nu$, which plays the role of the Reynolds number for the steady part of the flow. Note that for small $\alpha$, $Re_s = O(1)$ corresponds to $\nu = O(\alpha)$, i.e. to high Reynolds numbers ($Re = 1/\nu \gg 1$).

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The steady streaming produced by an oscillating cylinder both in an unbounded fluid and in a cylindrical container had been studied by many researchers. A review of early works can be found in [2]. Regular perturbation analysis had been used to study both an oscillating cylinder in an unbounded fluid and the case of the flow between two cylinders in [3, 4] for the flow regimes with \( \nu \gg 1 \) (\( \text{Re} \ll 1 \)). In [5], the method of matched asymptotic expansions had been used to construct an asymptotic solution for an oscillating cylinder in an unbounded fluid for \( \nu = O(1) \) (\( \text{Re} = O(1) \)). In [6], the theoretical results of [3, 4, 5] had been reconsidered, corrected with the Stokes drift, and compared with the experimental observations, which demonstrated a good agreement between the theoretical and experimental results. For high Reynolds numbers such that \( \text{Re}_s = O(1) \) or \( \text{Re}_s \gg 1 \), the steady streaming induced by an oscillating cylinder in an unbounded fluid had been studied in [7, 8, 9, 2, 10]. For \( \text{Re}_s = O(1) \), the steady flow outside the Stokes layer is governed by the steady Navier-Stokes equations, and it had been shown that for \( \text{Re}_s \gg 1 \) there is a double boundary layer near the oscillating cylinder and that the steady streaming takes the form of a jet-like flow along the axis of oscillations. In [11, 12], the steady flow between two coaxial cylinders produced by small-amplitude transverse oscillations of the inner cylinder had been studied in the case of \( \text{Re}_s = O(1) \) (and \( \text{Re}_s \gg 1 \)). The results of [12] are a good agreement with the experimental results of [13]. In [11, 12], the case of small \( \text{Re}_s \) corresponding to \( \nu = O(1) \) had also been briefly discussed.

The aim of this paper is to investigate steady streaming between two circular cylinders in the case of \( \nu = O(1) \). As was mentioned above, this problem had been treated before in [12], where an asymptotic solution of the Navier-Stokes equations had been constructed using the method of matched asymptotic expansions. Formulae given in [12] can be used to write down a (composite) uniformly valid expansion for the averaged stream function. For an oscillating cylinder in an unbounded fluid, this had been explicitly done in [5]. However, the asymptotic solutions for both a single cylinder (presented in [5]) and for two cylinders (presented in [12]) are incomplete because of the absence of the \( O(\alpha^{3/2}) \) term associated with the averaged outer flow in the expansion of the averaged stream function (this will be discussed in more detail in Section 5). In the present paper, we construct a uniformly valid expansion of the averaged stream function in powers of \( \alpha^{1/2} \) up to terms of order \( O(\alpha^{3/2}) \).

An interesting question which had not been discussed in [5, 12] is the relation of the expansions obtained there to the original problem of a steady streaming produced by an oscillating cylinder as it is seen by an observer fixed in space. In [5], the problem of a steady streaming produced by a cylinder which is placed in an oscillating flow had been considered. However, the steady component of the Eulerian velocity depends on the reference frame that is used to describe the flow. To clarify this point is one of the aims of this study.

A similar problem arises in relation to the results of [12], where a conformal mapping that maps the gap between two eccentric cylinders onto the annulus between two cylinders with a common axis was employed. The subsequent analysis had been done using the transformed coordinates that depend on time, and the latter leads to a question whether a transformation back to the physical coordinates will change the steady component of the flow. The results of the present study give an answer to this question.

Another question which is related to the previous two is the effect of the Stokes drift, which is understood here as a difference between the averaged Eulerian velocity field and the averaged
Lagrangian velocity (the velocity of fluid particles). The importance of Stokes drift is evident: (i) it is the Lagrangian velocity (rather than the Eulerian velocity) that is observed in experiments; (ii) it is the Lagrangian velocity that is invariant under the change of the frame of reference from the one fixed in the oscillating cylinder to the one fixed in space (and vice versa). The Stokes drift in various oscillating flows had been studied before by many authors (see, e.g., [14, 15, 4, 6]). However, there is still no clarity about when one can expect a nonzero effect of the Stokes drift in a flow produced by an oscillating cylinder. One of the aims of the present paper is to resolve this ambiguity.

To construct the asymptotic expansion of the solution of the Navier-Stokes equations for small $\alpha$, we employ the Vishik-Lyusternik method\(^1\) (see, e.g., [16, 17]) rather than the method of matched asymptotic expansions which is routinely used in fluid mechanics. In most cases that we know, at least first two terms in the uniformly valid asymptotic expansions produced by these two methods are the same. This does not mean that, given an expansion obtained by one of the two methods, it is easy to derive the same expansion using the other one. For example, although the asymptotic expansion constructed here using the Vishik-Lyusternik method can be obtained by the method of matched asymptotic expansions, this is not a straightforward procedure (it requires a certain transformation of the velocity field before the asymptotic procedure is started). In comparison with the method of matched asymptotic expansions, the Vishik-Lyusternik method involves more algebra in computing higher order terms, but it has two essential advantages: (i) it does not require the procedure of matching the inner and outer expansions and (ii) the boundary layer part of the expansion satisfies the condition of decay at infinity (in boundary layer variable) in all orders of the expansion, which is not the case in the method of matched asymptotic expansions where the boundary layer part usually does not decay and may even grow at infinity. The Vishik-Lyusternik method had been used to study viscous boundary layers at a fixed impermeable boundary by Chudov [18]. Recently, it has been applied to viscous boundary layers in high Reynolds number flows through a fixed domain with an inlet and an outlet [19] and to viscous flows in a half-plane produced by tangential vibrations on its boundary [20].

In the present paper we compute first two nonzero terms in the asymptotic expansion of the steady velocity produced by an oscillating inner cylinder and the corresponding expansion for the stream function. The steady Eulerian velocity field is then corrected with the Stokes drift. The results for an oscillating cylinder in an unbounded fluid are obtained by passing to the limit $R \to \infty$ (where $R$ is the ratio of the radius of the outer cylinder to the radius of the inner cylinder).

The outline of the paper is as follows. In Section 2, we formulate the mathematical problem. In Section 3, we describe the method of constructing the asymptotic expansion and derive the equations and boundary conditions that are to be solved. These equations are solved and then corrected with the Stokes drift in Section 4. Section 5 contains the discussion of the results.

2. Formulation of the problem

We consider a two-dimensional flow of a viscous incompressible fluid between two circular cylinders with radii $a$ and $b$ ($b > a$) produced by small translational vibrations of an inner cylinder about

\(^1\)Nayfeh refers to this method as the method of composite expansions [17].
the axis of the outer cylinder which is fixed in space. Let \( \mathbf{x}^* = (x^*, y^*) \) be Cartesian coordinates on the plane and let \( \mathbf{x}_0^*(t^*) = (x_0^*(t^*), 0) \) be the position of the centre of the inner cylinder at time \( t^* \). We assume that \( x_0^*(t^*) \) is oscillating in \( t^* \) with angular frequency \( \omega \) and period \( T = 2\pi/\omega \). Using \( 1/\omega \), \( a \), \( V_0^* \) and \( \rho a\omega V_0^* \) as the characteristic scales for time, length, velocity and pressure (\( \rho \) is the fluid density), we introduce the dimensionless variables \( \tau = \omega t^* \), \( \mathbf{x} = \frac{1}{a} \mathbf{x}^* \), \( \mathbf{x}_0 = \frac{1}{a} \mathbf{x}_0^* \), \( \mathbf{v} = \frac{1}{V_0^*} \mathbf{v}^* \) and \( p = \frac{\rho}{\rho a\omega V_0^*} \). The motion of the fluid is governed by the two-dimensional Navier-Stokes equations

\[
v_x + \epsilon^2 \left[ (v \cdot \nabla) v - \nu \nabla^2 v \right] = -\nabla p, \quad \nabla \cdot v = 0. \tag{2.1}
\]

Here \( \epsilon = \sqrt{a} \), \( \alpha \) and \( \nu \) are the inverse Strouhal number and the inverse Reynolds number (the dimensionless viscosity), defined by \( \frac{\alpha V_0^*}{\nu} \). The velocity of the fluid satisfies the standard no-slip condition on the surfaces of the cylinders

\[
\mathbf{v}(\mathbf{x}, \tau) \bigg|_{\text{outer cylinder}} = 0, \quad \mathbf{v}(\mathbf{x}, \tau) \bigg|_{\text{inner cylinder}} = \frac{1}{\epsilon^2} \mathbf{x}_0(\tau) = \frac{1}{\epsilon^2} \mathbf{x}_0(\tau) e_x. \tag{2.2}
\]

Here dots denote differentiation with respect to \( \tau \) and \( x_0(\tau) \) is a given function which prescribes the motion of the inner cylinder. In what follows we are interested in the asymptotic behaviour of periodic solution of Eqs. (2.1), (2.2) in the high-frequency limit \( \epsilon \to 0 \). We assume that the amplitude of the oscillations of the cylinder is \( O(\epsilon^2) \), i.e. \( x_0(\tau) = \epsilon^2 f(\tau) \) for some \( 2\pi \)-periodic function \( f \) and \( f = O(1) \). In what follows, we will consider \( f(\tau) \) given by

\[
f(\tau) = \text{Re} \left( C e^{i\tau} \right) \tag{2.3}
\]

where \( C \) is a complex constant having unit modulus (\( |C| = 1 \)).

Boundary conditions (2.2) take the form

\[
\mathbf{v}(\mathbf{x}, \tau) \bigg|_{\text{outer cylinder}} = 0, \quad \mathbf{v}(\mathbf{x}, \tau) \bigg|_{\text{inner cylinder}} = f'(\tau) e_x. \tag{2.4}
\]

The time-dependent boundary of the inner cylinder can be described in the parametric form by the equations

\[
x = \cos \tilde{\theta} + \epsilon^2 f(\tau), \quad y = \sin \tilde{\theta} \tag{2.5}
\]

where \( \tilde{\theta} \in [0, 2\pi) \) is the parameter on the cylinder boundary. Now the boundary condition on the inner cylinder can be written as

\[
\mathbf{v} \bigg|_{x = \cos \tilde{\theta} + \epsilon^2 f(\tau)} = f'(\tau) e_x. \tag{2.6}
\]

Using the assumption that \( \epsilon \) is small, we expand \( u \) and \( v \) in Taylor’s series at point \( (x, y) = (\cos \tilde{\theta}, \sin \tilde{\theta}) \). This yields

\[
\mathbf{v} \bigg|_{x = \cos \tilde{\theta}, y = \sin \tilde{\theta}} + \epsilon^2 f(\tau) \partial_x \mathbf{v} \bigg|_{x = \cos \tilde{\theta}, y = \sin \tilde{\theta}} + \frac{\epsilon^4 f^2(\tau)}{2} \partial_x^2 \mathbf{v} \bigg|_{x = \cos \tilde{\theta}, y = \sin \tilde{\theta}} + \ldots = f'(\tau) e_x \tag{2.7}
\]

Note that each term on the left side of Eq. (2.7) is evaluated at the averaged position of the inner cylinder (where the axes of both cylinders coincide).
In polar coordinates \((r, \theta)\) with origin at the axis of the outer cylinder, Eqs. (2.1) take the form

\[
\begin{align*}
  u_r &= -p_r + \epsilon^2 \left( -uu_r - \frac{v}{r} u_\theta + \frac{v^2}{r} + \nu \left( \nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2} v \right) \right), \\
  v_r &= -\frac{1}{r} p\theta + \epsilon^2 \left[ -uv_r - \frac{v}{r} v_\theta - \frac{vv}{r} + \nu \left( \nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2} u_\theta \right) \right], \\
  u_r + \frac{u}{r} + \frac{1}{r} v_\theta &= 0,
\end{align*}
\]

(2.8)

(2.9)

(2.10)

where \(u\) and \(v\) are the radial and azimuthal components of the velocity, subscripts ‘\(r\)’, ‘\(\theta\)’ and ‘\(\tau\)’ denote partial derivatives, and where \(\nabla^2 = \partial_r^2 + (1/r) \partial_r + (1/r^2) \partial_\theta^2\). The boundary conditions at the outer cylinder are

\[
\begin{align*}
  u \bigg|_{r=R} &= 0, \\
  v \bigg|_{r=R} &= 0
\end{align*}
\]

(2.11)

where \(R = b/a\). The boundary condition \((2.7)\) at the inner cylinder takes the form

\[
\begin{align*}
  u \bigg|_{r=1} &= + \epsilon^2 f \left( Lu + \frac{\sin \theta}{r} v \right) \bigg|_{r=1} \\
  &+ \frac{\epsilon^4 f^2}{2} \left[ L^2 u - \frac{\sin^2 \theta}{r} u - 2 \sin \theta \frac{L v}{r^2} \frac{\sin \theta \cos \theta}{r^2} v \right] \bigg|_{r=1} + \ldots = f'(\tau) \cos \theta, \\
  v \bigg|_{r=1} &= + \epsilon^2 f \left( Lv - \frac{\sin \theta}{r} u \right) \bigg|_{r=1} \\
  &+ \frac{\epsilon^4 f^2}{2} \left[ L^2 v - \frac{\sin^2 \theta}{r} v - 2 \sin \theta \frac{L u}{r^2} \frac{2 \sin \theta \cos \theta}{r^2} u \right] \bigg|_{r=1} + \ldots = -f'(\tau) \sin \theta.
\end{align*}
\]

(3.2)

Here \(L = \cos \theta \partial_r - (\sin \theta / r) \partial_\theta\).

3. Asymptotic expansion

We seek a solution of \((2.8) - (2.13)\) in the form

\[
\begin{align*}
  u &= u^i(r, \theta, \tau, \epsilon) + \epsilon u^a(\xi, \theta, \tau, \epsilon) + \epsilon u^b(\eta, \theta, \tau, \epsilon), \\
  v &= v^i(r, \theta, \tau, \epsilon) + v^a(\xi, \theta, \tau, \epsilon) + v^b(\eta, \theta, \tau, \epsilon), \\
  p &= p^i(r, \theta, \tau, \epsilon) + p^a(\xi, \theta, \tau, \epsilon) + p^b(\eta, \theta, \tau, \epsilon).
\end{align*}
\]

(3.1)

(3.2)

(3.3)

Here \(\xi = (r-1)/\epsilon\) and \(\eta = (R-r)/\epsilon\) are the boundary layer variables. Functions \(u^i, v^i, p^i\) represent a regular expansion of the solution in power series in \(\epsilon\) (an outer solution), and \(u^a, v^a, p^a\) and \(u^b, v^b, p^b\) correspond to boundary layer corrections to this regular expansion. Superscripts ‘\(a\)’ and ‘\(b\)’ correspond to the boundary layers at the inner and outer cylinders respectively. We assume that the boundary layer part of the expansion rapidly decays outside thin boundary layers, i.e. \(u^a, v^a, p^a \rightarrow 0\) as \(\xi \rightarrow \infty\) and \(u^b, v^b, p^b \rightarrow 0\) as \(\eta \rightarrow \infty\).

3.1. Regular part of the expansion

To derive the equations governing the regular part of the expansion, it is convenient to work with the Cartesian form \((2.1)\) of the Euler equations. Later we can rewrite the results in polar coordinates. Let

\[
\begin{align*}
  v^i &= v^i_0 + \epsilon v^i_1 + \epsilon^2 v^i_2 + \ldots, \\
  p^i &= p^i_0 + \epsilon p^i_1 + \epsilon^2 p^i_2 + \ldots,
\end{align*}
\]

(3.4)
where \( v^i = v^i_x e_x + v^i_y e_y \) and \( v_k^i = v_k^i_x e_x + v_k^i_y e_y \) \((k = 0, 1, 2, \ldots)\). On substituting (3.4) in (2.11) and collecting terms of equal powers of \( \epsilon \), we find that the successive approximations \( v_k^i, p_k^i \) \((k = 0, 1, 2, \ldots)\) satisfy the equations:

\[
\partial_r v_k^i = -\nabla p_k^i, \quad \nabla \cdot v_k^i = 0 \tag{3.5}
\]

for \( k = 0, 1 \) and

\[
\partial_r v_k^i = -\nabla p_k^i + \left( -\sum_{l=0}^{k-2} (v_l^i \cdot \nabla) v_{k-l}^i + \nu \nabla^2 v_{k-2}^i \right), \quad \nabla \cdot v_k^i = 0. \tag{3.6}
\]

for \( k \geq 2 \). In what follows, we will use the following notation: for any \( 2\pi \)-periodic function \( F(\tau) \),

\[
F(\tau) = \bar{F} + \tilde{F}(\tau), \quad \bar{F} = \frac{1}{2\pi} \int_0^{2\pi} F(\tau) d\tau,
\]

i.e. \( \bar{F} \) is the mean value of \( F(\tau) \) and, by definition, \( \bar{F}(\tau) = F(\tau) - \bar{F} \) is the oscillating part of \( F(\tau) \).

**Leading-order terms.** Consider first Eqs. (3.5) for \( k = 0 \). We seek a solution \( v_0^i \) which is periodic in \( \tau \). Averaging the equation for \( v_0^i \) yields: \( \nabla \bar{p}_0^i = 0 \), which is the necessary condition for existence of periodic (in \( \tau \)) solutions for \( v_0^i \). Without loss of generality, we put \( \bar{p}_0^i = 0 \), i.e. in the leading order the pressure in the outer solution is purely oscillatory with zero mean value. The general solution of Eqs. (3.5) can be written as \( v_0^i = \bar{v}_0^i + \tilde{v}_0^i \) where \( \bar{v}_0^i = \nabla \phi_0 \) and \( \phi_0 \) has zero mean value and is the solution of the boundary value problem

\[
\nabla^2 \phi_0 = 0, \quad \phi_0|_{r=1} = f'(\tau) \cos \theta, \quad \phi_0|_{r=R} = 0 \tag{3.7}
\]

The boundary conditions for \( \phi_0 \) at \( r = 1 \) and \( r = R \) will be justified later.

On averaging the equations for \( v_2^i \) and the second of equations (3.5), we obtain

\[
(\bar{v}_0^i \cdot \nabla) \bar{v}_0^i + (\bar{v}_0^i \cdot \nabla) \tilde{v}_0^i = -\nabla \bar{p}_2^i + \nu \nabla^2 \tilde{v}_0^i, \quad \nabla \cdot \tilde{v}_0^i = 0. \tag{3.8}
\]

Since \( \bar{v}_0^i \) is irrotational, we can rewrite (3.8) as

\[
(\bar{v}_0^i \cdot \nabla) \bar{v}_0^i = -\nabla \Pi_0 + \nu \nabla^2 \bar{v}_0^i, \quad \nabla \cdot \bar{v}_0^i = 0, \tag{3.9}
\]

where \( \Pi_0 = \bar{p}_2^i + |\nabla \phi_0|^2/2 \). Equations (3.9) represent the time-independent Navier-Stokes equations that describe steady flows of a viscous incompressible fluid. Boundary conditions for \( \bar{v}_0^i \) will be specified later.

**First-order terms.** The solution of Eqs. (3.5) for \( k = 1 \) has the form \( v_1^i = \bar{v}_1^i + \tilde{v}_1 \) where \( \bar{v}_1^i = \nabla \phi_1 \) and \( \phi_1 \) has zero mean value and is the solution of the boundary value problem

\[
\nabla^2 \phi_1 = 0, \quad \phi_1|_{r=1} = a_1(\theta, \tau), \quad \phi_1|_{r=R} = b_1(\theta, \tau). \tag{3.10}
\]

Functions \( a_1(\theta, \tau) \) and \( b_1(\theta, \tau) \) will be defined later. Manipulations similar to those employed in derivation of Eqs. (3.9) lead to the following equations for \( \bar{v}_1^i \):
where $\Pi_1 = \tilde{p}_3 + (\nabla \phi_0 \cdot \nabla \phi_1)$. Boundary conditions for $\tilde{v}_i^1$ will be specified later.

*Second-order terms.* It will be proved later that

$$\tilde{v}_0^i \equiv 0 \quad \text{and} \quad \tilde{v}_1^i \equiv 0. \quad (3.12)$$

Using these and the fact that both $\tilde{v}_0^i$ and $\tilde{v}_1^i$ are irrotational, it can be shown that

$$\partial_\tau \tilde{v}_2^i = -\nabla Q_2, \quad \nabla \cdot \tilde{v}_2^i = 0,$$

where $Q_2 = \tilde{p}_3^i + \frac{(\nabla \phi_0)^2}{2} - \frac{(\nabla \phi_0)^2}{2}$. It follows that $\tilde{v}_2^i$ is irrotational, i.e. $\tilde{v}_2^i = \nabla \phi_2$, and $\phi_2$ is the solution of the boundary value problem

$$\nabla^2 \phi_2 = 0, \quad \phi_2|_{r=1} = a_2(\theta, \tau), \quad \phi_2|_{r=R} = b_2(\theta, \tau), \quad (3.13)$$

where functions $a_2(\theta, \tau)$ and $b_2(\theta, \tau)$ will be defined later.

The equations for $\tilde{v}_2^i$ can be written in the form

$$0 = -\nabla \Pi_2 + \nu \nabla^2 \tilde{v}_2^i, \quad \nabla \cdot \tilde{v}_2^i = 0, \quad (3.14)$$

where $\Pi_2 = \tilde{p}_4^i + \frac{\nabla \phi_0 \cdot \nabla \phi_2}{2} + \frac{(\nabla \phi_1)^2}{2}$ and where we have used the assumptions (3.12). Thus, the second order averaged velocity is described by the Stokes equations. Again, boundary conditions for $\tilde{v}_2^i$ will be specified later.

*Third-order terms.* Separating the oscillating part of the equation for $\tilde{v}_3^i$ and employing (3.12) and the fact that $\tilde{v}_0^i$ and $\tilde{v}_1^i$ are irrotational, we find that

$$\partial_\tau \tilde{v}_3^i = -\nabla Q_3,$$

where $Q_3 = \tilde{p}_3^i + \nabla \phi_0 \cdot \nabla \phi_3 - \nabla \phi_0 \cdot \nabla \phi_1$. This equation and the continuity equation $\nabla \cdot \tilde{v}_3^i = 0$ imply that $\tilde{v}_3^i = \nabla \phi_3$ where $\phi_3$ is the solution of the boundary value problem

$$\nabla^2 \phi_3 = 0, \quad \phi_3|_{r=1} = a_3(\theta, \tau), \quad \phi_3|_{r=R} = b_3(\theta, \tau). \quad (3.15)$$

Function $a_3(\theta, \tau)$ and $b_3(\theta, \tau)$ will be defined later.

The equations for $\tilde{v}_3^i$ can be derived in the same manner as the equations for $\tilde{v}_0^i$, $\tilde{v}_1^i$ and $\tilde{v}_2^i$. They are given by

$$0 = -\nabla \Pi_3 + \nu \nabla^2 \tilde{v}_3^i, \quad \nabla \cdot \tilde{v}_3^i = 0, \quad (3.16)$$

where $\Pi_3 = \tilde{p}_5^i + \frac{\nabla \phi_0 \cdot \nabla \phi_3}{2} + \frac{\nabla \phi_1 \cdot \nabla \phi_2}{2}$.

Thus, we have found that both the second and third order averaged velocities ($\tilde{v}_2^i$ and $\tilde{v}_3^i$) in the outer flow are solutions of the Stokes problem with boundary conditions which will be determined later.
3.2. Boundary layers

Boundary layer at the inner cylinder. We assume that

\begin{align*}
u^a &= u_0^a + \epsilon \nu_{1}^a + \ldots, \quad v^a = v_0^a + \epsilon v_{1}^a + \ldots, \quad p^a = p_0^a + \epsilon p_{1}^a + \ldots \quad (3.17)
\end{align*}

Now we use our assumption that \( u^b, v^b \) and \( p^b \) are nonzero only within a thin boundary layer near the outer cylinder and drop them from Eqs. (3.11)–(3.3). Then we substitute the resulting equations, as well as (3.4) and (3.17), into Eqs. (2.8)–(2.10) and take into account that \( u_k^0, v_k^0, p_k^0 \) (\( k = 0, 1, \ldots \)) satisfy the equations (3.5), (3.6). After that, we make the change of variables \( r = 1 + \epsilon \xi, \) expand every function of \( \epsilon \xi \) in Taylor’s series at \( \xi = 0 \) and collect terms of equal powers in \( \epsilon. \) This yields the following equations:

\begin{align*}
\nu_k^0 &= \nu_{k\xi}^0 + \nu_{k\xi\xi}^0 + \nu_{k\xi\theta}^0 + \nu_{k\theta\theta}^0, \\
p_k^0 &= \frac{1}{R} p_{k\theta}^0, \\
\nu_{k\xi}^0 + \nu_{k\xi\xi}^0 + \nu_{k\xi\theta}^0 + \nu_{k\theta\theta}^0 &= F_k^a, \\
\nu_{k\theta}^0 &= G_k^a, \\
\nu_{k\xi}^0 + \nu_{k\xi\xi}^0 + \nu_{k\xi\theta}^0 + \nu_{k\theta\theta}^0 &= H_k^a
\end{align*}

for \( k = 0, 1, \ldots \) Here \( F_0^a = 0, G_0^a = 0 \) and \( H_0^a = 0; \) for \( k > 0, \) functions \( F_k^a, G_k^a \) and \( H_k^a \) are defined in terms of \( \nu^0_k, \nu^1_k, \ldots, \nu^a_k, \ldots, \nu^a_k \) and \( u_0^0, \ldots, u_0^a. \) Explicit expressions for these functions are given in Appendix A (for \( k = 1, 2, 3 \)).

Boundary layer at the outer cylinder. Let \( u^b = u_0^b + \epsilon u_{1}^b + \ldots, \quad v^b = v_0^b + \epsilon v_{1}^b + \ldots \) and \( p^b = p_0^b + \epsilon p_{1}^b + \ldots \) The same procedure as before produces the following sequence of equations:

\begin{align*}
\nu_k^b &= \nu_{k\nu}^b + \frac{1}{R} p_{k\nu}^b = F_k^b, \\
p_{k\nu}^b &= G_k^b, \\
-\nu_{k\nu}^b + \frac{1}{R} p_{k\nu}^b &= H_k^b
\end{align*}

for \( k = 0, 1, \ldots \) Here \( F_0^b = 0, G_0^b = 0 \) and \( H_0^b = 0; \) explicit expressions for \( F_k^b, G_k^b \) and \( H_k^b \) are given in Appendix A (for \( k = 1, 2, 3 \)).

We require that in all orders the boundary layer corrections to the outer solution rapidly decay outside boundary layers, i.e. (for each \( k = 0, 1, \ldots \))

\begin{align*}
u_k^a, \nu_k^b, p_k^a, p_k^b \to 0 \quad \text{as} \quad \xi \to \infty \quad \text{and} \quad u_k^a, v_k^a, p_k^b \to 0 \quad \text{as} \quad \eta \to \infty.
\end{align*}

3.3. Boundary conditions

Now we take our expansions of the velocity in the outer flow and in the boundary layers, substitute them into (2.11)–(2.13) and collect terms of equal powers in \( \epsilon. \) This produces the following boundary conditions:

\begin{align*}
u_k^a|_{r=1} + u_k^a|_{\xi=0} &= Q_k, \\
u_k^a|_{r=1} + v_k^a|_{\xi=0} &= S_k, \\
u_k^a|_{r=R} + u_k^b|_{\nu=0} &= 0, \\
u_k^a|_{r=R} + v_k^b|_{\eta=0} &= 0.
\end{align*}
for $k = 0, 1, \ldots$. Here

$$Q_0 = f'(\tau) \cos \theta, \quad S_0 = -f'(\tau) \sin \theta,$$

functions $Q_k$ and $S_k$ for $k > 0$ depend on $\phi_0', \ldots, \phi_{k-1}', v_0', \ldots, v_{k-1}'$ and $u_0', \ldots, u_{k-1}'$ and are given in Appendix A (for $k = 1, 2, 3$). Note that boundary conditions for $\phi_0$ at $r = 1$ and $r = R$ in the boundary value problem (3.7) follow directly from (3.25) and (3.27) with $k = 0$.

4. Analysis of the asymptotic equations

4.1. Leading order equations

**Outer flow.** The solution of (3.7) that describes the (leading order) oscillating outer flow is

$$\phi_0 = -\frac{f'(\tau)}{R^2 - 1} \left( r + \frac{R^2}{r} \right) \cos \theta. \quad (4.1)$$

**Inner cylinder.** Consider now Eqs. (3.18)–(3.20) for $k = 0$. The condition of decay at infinity (in variable $\xi$) for $p_0^a$ and Eq. (3.19) have a consequence that $p_0^a \equiv 0$. Equation (3.18) simplifies to the standard heat equation

$$v_0^a \tau = \nu v_0^a \xi \xi. \quad (4.2)$$

Boundary condition for $v_0^a$ at $\xi = 0$ follows from (3.21) (with $k = 0$):

$$v_0^a|_{\xi=0} = -v_0^i|_{r=1} - f'(\tau) \sin \theta = -\frac{2R^2}{R^2 - 1} \Re (iCe^{i\tau}) \sin \theta. \quad (4.3)$$

The solution of (4.2) subject to the boundary conditions (4.3) and (3.24) is given by

$$v_0^a = -\frac{2R^2}{R^2 - 1} \Re (iCe^{-\gamma \xi + i\tau}) \sin \theta, \quad \gamma = \frac{1 + i}{\sqrt{2\nu}}. \quad (4.4)$$

It follows from (4.4) that $\bar{v}_0^a = 0$. Thus, in the leading order the boundary layer at the inner cylinder is a purely oscillatory Stokes layer. This fact implies that the boundary condition for $\bar{v}_0^a$ at $r = 1$ (that is obtained by averaging the condition (3.26)) is $\bar{v}_0^i|_{r=1} = 0$. Similarly, averaging the condition (3.25) yields $\bar{u}_0^i|_{r=1} = 0$. Thus, we have

$$\bar{v}_0^i|_{r=1} = 0. \quad (4.5)$$

The normal velocity $u_0^a$ is determined from Eq. (3.20):

$$u_0^a(\xi, \theta, \tau) = \int_{\xi}^{\infty} v_0^a(\xi', \theta, \tau) d\xi' = -\frac{2R^2}{R^2 - 1} \Re \left( \frac{i}{\gamma} Ce^{-\gamma \xi + i\tau} \right) \cos \theta. \quad (4.6)$$

Here the constant of integration is chosen so as to guarantee that $u_0^a(\xi, \theta, \tau)$ decays as $\xi \to \infty$. $u_0^a|_{\xi=0}$ gives us the boundary condition for the next approximation of the outer solution. Indeed, according to (3.25) for $k = 1$, we must have

$$u_0^i(r, \theta, \tau)|_{r=0} = -u_0^a(\xi, \theta, \tau)|_{\xi=0} = \frac{2R^2}{R^2 - 1} \Re \left( \frac{i}{\gamma} Ce^{i\tau} \right) \cos \theta. \quad (4.7)$$
This equation defines function $a_1(\theta, \tau)$ in (3.10).

**Outer cylinder.** Consider now Eqs. (3.21)–(3.23) for $k = 0$. An analysis similar to what we did for the boundary layer at the inner cylinder results in the formula

$$v_0^b = -\frac{2}{R^2 - 1} \operatorname{Re} \left( i C e^{-\gamma \theta + \nu \xi} \right) \sin \theta. \quad (4.8)$$

As before, the radial velocity $u_0^b$ is determined from the incompressibility condition (3.23): 

$$u_0^b = -\frac{2}{R(R^2 - 1)} \operatorname{Re} \left( \frac{i}{\gamma} C e^{-\gamma \xi + \nu \xi} \right) \cos \theta. \quad (4.9)$$

Again, the constant of integration is chosen so as to guarantee the decay of $u_0^b$ as $\eta \to \infty$. $u_0^b|_{\eta=0}$ gives us the boundary condition for the next approximation of the outer solution:

$$u_1^i|_{r=1} = -u_0^b|_{\eta=0} = -\frac{2}{R(R^2 - 1)} \operatorname{Re} \left( \frac{i}{\gamma} C e^{\nu \xi} \right) \cos \theta. \quad (4.10)$$

This equation defines function $b_1(\theta, \tau)$ in (3.10).

It follows from (4.8) that $\vec{v}_0^b = 0$, i.e. in the leading order the boundary layer at the outer cylinder is purely oscillatory. This, in turn, implies that the boundary condition for $\vec{v}_0^b$ at $r = R$ (obtained by averaging the condition (3.28)) is $\vec{v}_0^b|_{r=R} = 0$. Similarly, Eq. (3.27) yields $\vec{u}_0^i|_{r=R} = 0$. Hence,

$$\vec{v}_0^i|_{r=R} = 0. \quad (4.11)$$

**Averaged outer flow.** Equations (3.9) and boundary conditions (4.5) and (4.11) imply that $\vec{v}_0^i \equiv 0$, i.e there is no steady streaming in the leading order of the expansion. This justifies our earlier assumption about $\vec{v}_0^i$.

### 4.2. First order equations

**Oscillatory outer flow.** Since now we know functions $a_1(\theta, \tau)$ and $b_1(\theta, \tau)$ (defined by Eqs. (4.7) and (4.10)), we can solve problem (3.10). The solution is given by

$$\phi_1 = -\frac{2R}{(R^2 - 1)^2} \left( (R + 1)r + \frac{R^3 + 1}{r} \right) \operatorname{Re} \left( \frac{i}{\gamma} C e^{\nu \xi} \right) \cos \theta. \quad (4.12)$$

**Inner cylinder.** Consider Eqs. (3.18)–(3.20) for $k = 1$. The fact that $G_1^a = 0$ (Eq. (A.4) in Appendix A) and the same arguments as before lead us to conclusion that $p_1^i \equiv 0$. Hence, Eq. (3.18) reduces to

$$v_1^a_{\tau} - \nu v_1^a_{\xi \xi} = -\cos \theta f'(\tau)v_0^a + \nu v_0^a. \quad (4.13)$$

Averaging in $\tau$ and integrating in variable $\xi$ twice, we find that

$$\bar{v}_1^a = -\frac{1}{\nu} \cos \theta \int_0^\infty f'(\tau)v_0^a d\xi'. \quad (4.14)$$
Here the constants of integration are chosen so as to ensure that $\tilde{v}_1^a \to 0$ as $\xi \to \infty$. It follows from the definition of averaging that $\tilde{f}_1'(\tau)f_2(\tau) = -\tilde{f}_1(\tau)f_2'(\tau)$ for any $2\pi$-periodic functions $f_1$ and $f_2$. Employing this property in (4.14), we obtain

$$\tilde{u}_1^a = \frac{1}{\nu} \cos \theta \int_\xi^\infty \tilde{f}(\tau)\tilde{v}_0^a d\xi' = \cos \theta \int_\xi^\infty \tilde{f}(\tau)\tilde{v}_0^a d\xi' = -\cos \theta \tilde{f}(\tau)\tilde{v}_0^a, \quad (4.15)$$

Here we used the fact that $\tilde{v}_0^a$ satisfies Eq. (4.2). Substitution of (2.3) and (4.4) in (4.15) yields

$$\tilde{v}_1^a = \frac{1}{2\sqrt{2}\nu} \frac{R^2}{R^2 - 1} e^{-\xi/\sqrt{2}nu} \left( \cos(\xi/\sqrt{2}nu) - \sin(\xi/\sqrt{2}nu) \right) \sin 2\theta. \quad (4.16)$$

The oscillatory part $\tilde{v}_1^a$ of $\tilde{v}_1^a$ satisfies the equation

$$\tilde{v}_1^a - \nu \tilde{v}_1^a \xi = -\cos \theta \tilde{f} \tilde{v}_0^a + \nu \tilde{v}_0^a, \quad (4.17)$$

the condition of decay at infinity and the boundary condition

$$\tilde{v}_1^a \xi = -\frac{2R(R^3 + R + 2)}{(R^2 - 1)^2} Re \left( iC \gamma e^{i\tau} \right) \sin \theta - \cos \theta \tilde{f} \tilde{v}_0^a \xi = 0, \quad (4.18)$$

which follows from the oscillatory part of (3.20) and from Eq. (4.12). Standard but tedious calculations result in

$$\tilde{v}_1^a = -\cos \theta \tilde{f} \tilde{v}_0^a + w, \quad w = -\frac{2R^2}{R^2 - 1} Re \left[ iC \gamma \left( \frac{R^3 + R + 2}{R(R^2 - 1)} \frac{1}{2} - \frac{\xi}{2} \right) e^{-\gamma x + i\tau} \right] \sin \theta. \quad (4.19)$$

We do not give an explicit formula for $\tilde{v}_0^a$ as we do not use it in what follows. Both $\tilde{u}_1^a$ and $\tilde{u}_1^a$ are computed using Eq. (3.20). We have

$$\tilde{u}_1^a = \int_\xi^\infty \tilde{v}_1^a d\xi = -\frac{R^2}{R^2 - 1} e^{-\xi/\sqrt{2}nu} \sin(\xi/\sqrt{2}nu) \cos 2\theta, \quad (4.20)$$

$$\tilde{u}_1^a = \partial_\theta \left( \cos \theta \tilde{f} \tilde{v}_0^a \right) + h, \quad h = -\frac{2R^2}{R^2 - 1} Re \left[ iC \gamma \left( \frac{R^3 + R + 2}{R(R^2 - 1)} \frac{2\gamma}{2} - \frac{3\xi}{2} \right) e^{-\gamma x + i\tau} \right] \cos \theta. \quad (4.21)$$

**Outer cylinder.** Similar analysis, applied to Eqs. (3.21)–(3.23), yields

$$\tilde{v}_1^b = -\frac{2\sin \theta}{R(R^2 - 1)} Re \left[ iC \gamma \left( \frac{2R^3 + 2R + 1}{R^2 - 1} + \frac{\eta}{2} \right) e^{-\gamma x + i\tau} \right], \quad (4.22)$$

$$\tilde{u}_1^b = \frac{2\cos \theta}{R(R^2 - 1)} Re \left[ iC \gamma \left( \frac{4R^3 + 4R + 1}{R^2 - 1} + \frac{3\eta}{2} \right) e^{-\gamma x + i\tau} \right]. \quad (4.23)$$

Equations (4.22) and (4.23) imply that $\tilde{v}_1^b \equiv 0$ and $\tilde{u}_1^b \equiv 0$, i.e. in contrast with the inner cylinder, **there is no first-order steady boundary layer at the outer cylinder.**

\(^2\tilde{f} \tilde{v}_0^a\) as a function of $\tau$ is purely oscillatory with double frequency of oscillation and therefore, it produces zero contribution to all quantities which will be of interest to us.
Averaged outer flow. On averaging boundary conditions (3.27) and (3.28) (for \( k = 1 \)) and using the fact that \( \bar{u}_0^0 = 0 \) and \( \bar{v}_1^0 = 0 \), we find that
\[
\bar{v}_1^1|_{r=R} = 0.
\] (4.24)
Further, averaging boundary conditions (3.25) and (3.26) and using (4.6) and (4.15), we obtain
\[
\bar{u}_1^1|_{r=1} = -\bar{u}_0^1|_{\xi=0} = 0, \quad \bar{v}_1^1|_{r=1} = -\bar{v}_0^1|_{\xi=0} - \cos \theta f(f(\tau)v_0^2|_{\xi=0}) = 0.
\]
These, together with (4.24), imply that Eqs. (3.11) should be solved with zero boundary conditions, which, in turn, leads to a conclusion that \( \bar{v}_1^1 \equiv 0 \). This justifies our earlier assumption and means that there is no steady outer flow in the first order of the expansion.

4.3. Second order equations

Oscillatory outer flow. To find the oscillatory part of the second-order outer flow we need to solve problem (3.13) for the velocity potential \( \phi_2 \). Functions \( a_2(\theta, \tau) \) and \( b_2(\theta, \tau) \) which appear in (3.13) are determined by boundary conditions (3.25) and (3.27) for \( k = 2 \). With the help of Eq. (3.25) with \( k = 0 \) and the continuity equations for \( v_0^0 \) and \( v_0^1 \), boundary condition (3.25) for \( k = 2 \) can be reduced to
\[
u u_0^0|_{\xi=0} + \partial_\theta (\sin \theta f u_0^1|_{r=1}) = -u_0^1|_{\xi=0} + \cos 2\theta f f'.
\] (4.25)
The oscillatory part of (4.25) gives us \( a_2(\theta, \tau) \):
\[
a_2(\theta, \tau) = -\bar{u}_1^0|_{\xi=0} + \cos 2\theta f f'.
\] (4.26)
The second term on the right side of (4.26) will be ignored because it makes zero contribution to all quantities which we are interested in. The oscillatory part of (3.27) gives us \( b_2(\theta, \tau) \):
\[
b_2(\theta, \tau) = -u_1^0|_{\eta=0}.
\] (4.27)
Substituting (4.21) in (4.26) and (4.23) in (4.27) (and ignoring the second term on the right side of (4.26) as well as the first term in (4.21)), we solve problem (3.13). The solution is given by
\[
\phi_2 = -\frac{f \cos \theta}{(R^2-1)(R-1)^2} \left( R^4 - 2R^3 + 6R^2 - 2R + 1 \right).
\] (4.28)
Inner cylinder. Consider now Eqs. (3.18)–(3.20) for \( k = 2 \). First we note that it follows from Eqs. (4.2) and (4.6) that \( u_0^0|_{\tau=0} = \nu u_0^2|_{\xi=0} \), which, in turn, implies that \( G_2^a \equiv 0 \) (see Eq. (A.4) in Appendix A). Hence, Eq. (3.19) reduces to \( p_2^a = 0 \). This equation and the condition of decay at infinity imply that \( p_2^a \equiv 0 \). Equation (3.18) takes the form
\[
u v_0^a - \nu u_0^a = F_2^a.
\] (4.29)
Averaging yields the equation \( \bar{v}_2^a = -\frac{1}{\nu} F_2^a \) where \( F_2^0(\xi, \theta) \) is obtained by averaging Eq. (A.2). The solution of this equation that satisfies the condition of decay at infinity can be written as
\[
\bar{v}_2^a = -\frac{1}{\nu} \int_\xi^\infty F_2^a(\xi', \theta) d\xi' d\xi'.
\] (4.30)
Similarly, it can be shown that \( u(4.40) \) is in agreement with earlier results of Duck and Smith (Eq. (3.33) in [12]) and Haddon and C.

Lengthy, but standard calculations result in the formula
\[
\bar{v}_2 = \frac{R e^{-\xi} \sin(2\theta)}{4(R^2 - 1)^2} \left[ 2R^3 e^{-\xi} - (5R^3 - R) \xi \left( \cos \xi - \sin \xi \right) + 4R^3 \cos \xi + (11R^3 + R - 4) \sin \xi \right]
\]

where \( \xi = \xi/\sqrt{2\nu} \). The oscillatory part of \( v_2^i \) can be obtained by separating the oscillatory part of Eq. (4.29) and solving it. We will not do it as we are only interested in the steady part of the solution. Equation (3.20) for \( k = 2 \) is used to obtain \( \bar{u}_2^i \):
\[
\bar{u}_2^i = \frac{\sqrt{2\nu} R e^{-\xi} \cos 2\theta}{4(R^2 - 1)^2} \left[ 2R^4 e^{-\xi} + (14R^3 - 6R) \xi \sin \xi + (12R^3 - 4) \sin \xi + (20R^3 - 4) \cos \xi \right] \cos 2\theta.
\]

**Outer cylinder.** Similar calculations result in the following formulae for \( \bar{v}_2^i \) and \( \bar{u}_2^i \):
\[
\bar{v}_2^i = \frac{e^{-\eta} \sin 2\theta}{2R(R^2 - 1)^2} \left[ e^{-\eta} - 2\eta (\cos \eta - \sin \eta) + 8 \sin \eta + 2 \cos \eta \right],
\]
\[
\bar{u}_2^i = \frac{\sqrt{2\nu} e^{-\eta} \cos 2\theta}{2R^2(R^2 - 1)^2} \left[ e^{-\eta} + 4\eta \sin \eta + 8 \sin \eta + 12 \cos \eta \right],
\]

where \( \eta = \eta/\sqrt{2\nu} \).

**Averaged outer flow.** The steady part of the second-order outer flow is determined from the Stokes equations (3.14). Averaging (4.25) and using yields (4.20), we obtain
\[
\bar{v}_2^i|_{r=1} = 0.
\]

Boundary condition for \( v_2^i \) is obtained by averaging (3.26). It can be shown that it reduces to
\[
\bar{v}_2^i|_{r=1} = -\frac{3}{2} \frac{R^4}{(R^2 - 1)^2} \sin 2\theta.
\]

Similarly, it can be shown that
\[
\bar{v}_2^i|_{r=R} = -\bar{u}_2^i|_{r=0} = 0,
\]
\[
\bar{v}_2^i|_{r=R} = -\bar{u}_2^i|_{r=0} = -\frac{3}{2} \frac{1}{R(R^2 - 1)^2} \sin 2\theta.
\]

It is convenient to introduce stream function \( \bar{\psi}_2^i \) such that \( \bar{v}_2^i = \frac{1}{\bar{\psi}_2^i} \psi_2^i \) and \( \bar{u}_2^i = -\bar{\psi}_2^i \theta \). Then the Stokes equations (3.14) and the boundary conditions (4.35)–(4.38) take the form
\[
\nabla^2 \bar{\psi}_2^i = 0, \quad \bar{\psi}_2^i|_{r=1} = \bar{\psi}_2^i|_{r=R} = 0,
\]
\[
\bar{\psi}_2^i|_{r=1} = \frac{3}{2} \frac{R^4}{(R^2 - 1)^2} \sin 2\theta, \quad \bar{\psi}_2^i|_{r=R} = \frac{3}{2} \frac{1}{R(R^2 - 1)^2} \sin 2\theta.
\]

The solution of (4.39) is given by
\[
\bar{v}_2^i = \frac{3}{4(R^2 - 1)^4} \left( C_1 + \frac{C_2}{r^2} + C_3 r^2 + C_4 r^4 \right) \sin 2\theta
\]

where \( C_k \) \( (k = 1, \ldots, 4) \) depend only on \( R \) and are given in Appendix A (Eq. (A.25)). Equation (4.40) is in agreement with earlier results of Duck and Smith (Eq. (3.33) in [12]) and Haddon and Riley (Eq. (3.1) in [11]). Formulae for \( \bar{u}_2^i \) and \( \bar{v}_2^i \) can be easily obtained from Eq. (4.40).
4.4. Third order equations

Here we are interested only in the averaged outer flow. However, to find it, we need boundary conditions which come from the averaged boundary layers.

**Inner cylinder.** Averaging Eqs. (3.18) and (3.19) for \(k = 3\), we get

\[
\bar{v}^{3}_{3\xi\xi} = \frac{1}{\nu} \bar{p}^{a}_{3\theta} - \frac{1}{\nu} \bar{F}^{a}_{3} \quad \text{and} \quad \bar{p}^{a}_{3\xi} = \bar{G}^{a}_{3}, \tag{4.41}
\]

where \(\bar{F}^{a}_{3}\) and \(\bar{G}^{a}_{3}\) are obtained by averaging Eqs. (A.3) and (A.6). We do not need explicit solutions of Eqs. (4.41) as all we need is \(\bar{v}^{3}_{3,\xi=0}\). First, we integrate the second equation (4.41). Then we insert \(\bar{p}^{a}_{3}\) into the first equation (4.41) and integrate it twice. This yields

\[
\bar{v}^{3}_{3} = -\frac{1}{\nu} \int_{\xi'}^{\xi} \int_{\xi''}^{\xi'} \bar{G}^{a}_{3\theta}(\xi''''', \theta) \, d\xi'''' \, d\xi'' - \frac{1}{\nu} \int_{\xi'}^{\xi} \int_{\xi''}^{\xi'} \bar{F}^{a}_{3}(\xi'', \theta) \, d\xi'' \, d\xi'. \tag{4.42}
\]

**Outer cylinder.** Similarly, it can be shown that

\[
\bar{v}^{3}_{3} = -\frac{1}{\nu} \int_{\eta'}^{\eta} \int_{\eta''}^{\eta'} \bar{G}^{b}_{3\theta}(\eta''''', \theta) \, d\eta'''' \, d\eta'' - \frac{1}{\nu} \int_{\eta'}^{\eta} \int_{\eta''}^{\eta'} \bar{F}^{b}_{3}(\eta'', \theta) \, d\eta'' \, d\eta'. \tag{4.43}
\]

Here \(\bar{F}^{b}_{3}\) and \(\bar{G}^{b}_{3}\) are obtained by averaging Eqs. (A.12) and (A.15).

**Averaged outer flow.** The steady part of the third-order outer flow is a solution of (3.16) subject to appropriate boundary conditions. These boundary conditions are obtained by averaging (3.25)–(3.28) for \(k = 3\). Averaging Eqs. (3.25) and (3.27) yields the equations \(\bar{u}^{i}_{3}\big|_{r=1} = -\bar{u}^{a}_{2}\big|_{\xi=0}\) and \(\bar{u}^{i}_{3}\big|_{r=R} = -\bar{u}^{a}_{2}\big|_{\eta=0}\). Substituting (4.42) and (4.43) into these equations, we obtain

\[
\bar{u}^{i}_{3}\big|_{r=1} = -\frac{\nu R(11 R^3 - 2)}{2(R^2 - 1)^2} \cos 2\theta, \quad \bar{u}^{i}_{3}\big|_{r=R} = \frac{13\nu R^2}{2R^2 - 12 R^2 + 4 R - 4} \cos 2\theta. \tag{4.44}
\]

On averaging Eq. (3.26) for \(k = 3\) and using zeroth- and first-order boundary conditions, the boundary condition for \(\bar{v}^{i}_{3}\) at \(r = 1\) can be simplified to

\[
\bar{v}^{i}_{3}\big|_{r=1} = -\frac{\nu R(11 R^3 - 2)}{2(R^2 - 1)^2} \cos 2\theta \left( v^{i}_{1r}\big|_{r=1} + u^{a}_{2}\big|_{\xi=0} \right) - \cos^2 \theta \frac{\nu R^2}{2} v^{a}_{1}\big|_{\eta=0} - \cos^2 \theta \frac{\nu R^3}{6} v^{a}_{0}\big|_{\xi=0}. \tag{4.45}
\]

Substituting here Eq. (4.42) and the explicit formulae for \(v^{i}_{0}, v^{i}_{1}, u^{a}_{0}, u^{a}_{1}, v^{a}_{1}, v^{a}_{2}, \) etc., we find that

\[
\bar{v}^{i}_{3}\big|_{r=1} = \sqrt{2\nu} K_{1}(R) \sin 2\theta, \quad K_{1}(R) = \frac{R(53 R^4 - 53 R^3 - 12 R^2 + 4 R - 4)}{4(R - 1)(R^2 - 1)^2}. \tag{4.45}
\]

Similarly, by averaging (3.28) and using (4.43), we obtain

\[
\bar{v}^{i}_{3}\big|_{r=R} = -\sqrt{2\nu} K_{2}(R) \sin 2\theta, \quad K_{2}(R) = \frac{4 R^5 + R^4 - R^3 + 14 R - 14}{4 R^2(R - 1)(R^2 - 1)^2}. \tag{4.46}
\]
On introducing stream function $\tilde{\psi}_3^i$ such that $\bar{u}_3^i = \frac{1}{\nu} \tilde{\psi}_3^i\theta$ and $\bar{v}_3^i = -\tilde{\psi}_3^i\tau$, Eqs. (3.16) and boundary conditions (4.44)–(4.46) can be written as

$$\nabla^4 \tilde{\psi}_3^i - 1 = 0,$$

$$\tilde{\psi}_3^i_{|r=1} = -\frac{2\nu R(11R^3 - 2)}{4(R^2 - 1)^2} \sin 2\theta,$$

$$\tilde{\psi}_3^i_{|r=R} = \frac{13}{4R^2(R^2 - 1)^2} \sin 2\theta,$$

$$\tilde{\psi}_3^i_{|r=0} = -\frac{2\nu K_1(R)}{2} \sin 2\theta,$$

$$\tilde{\psi}_3^i_{|r=\infty} = \frac{2\nu K_2(R)}{2} \sin 2\theta.$$ (4.47)

The solution of this boundary value problem is given by

$$\tilde{\psi}_3^i = \frac{2\nu}{8R(R - 1)(R^2 - 1)^2} \left( D_1 + \frac{D_2}{r^2} + D_3r^2 + D_4r^4 \right) \sin 2\theta.$$ (4.48)

where $D_k$ ($k = 1, \ldots, 4$) depend on $R$ only and are given by Eq. (A.26) in Appendix A. Explicit expressions for $\tilde{u}_3^i$ and $\tilde{v}_3^i$ can now be easily obtained from (4.48). 

4.5. Stokes drift

So far we have discussed the Eulerian velocity. However, the velocity observed in experiments is the velocity of fluid particles, i.e. the Lagrangian velocity. It is well-known that in oscillatory flows the observed averaged Lagrangian velocity differs from the averaged Eulerian velocity, and the difference between these two is known as the Stokes drift. Below we discuss the effect of the Stokes drift on the averaged Lagrangian velocity.

The motion of fluid particles is governed by the ordinary differential equation

$$\frac{dx}{d\tau} = \epsilon^2 v(x, \tau, \epsilon),$$ (4.49)

The velocity field $v(x, \tau, \epsilon)$ is the solution of the Navier-Stokes equations, which is $2\pi$-periodic in $\tau$ and has a nonzero average. We have already computed first three terms in the uniformly valid asymptotic expansion of $v(x, \tau, \epsilon)$. Now we are interested in constructing an asymptotic expansion of the solution of (4.49) for small $\epsilon$. We introduce the slow time $t = \epsilon^2 \tau$, assume that $x = x(\tau, t, \epsilon)$ and substitute this in (4.49). This gives us the equation

$$x_t = \epsilon^2 (v(x, \tau, \epsilon) - x_t).$$

It is shown in Appendix B that the solution of this equation can be presented in the form $x(x_0, \tau) = \tilde{x}(x_0, \tau, t, \epsilon) + \bar{x}(x_0, t, \epsilon)$ where $\tilde{x}(x_0, \tau, t, \epsilon)$ represents purely oscillatory part of the motion of the fluid particle whose averaged position at $t = 0$ was $x_0$ and $\bar{x}(x_0, t, \epsilon)$, is the solution of the equation

$$x_t = \bar{v}(x, \epsilon) + \tilde{v}(x, \epsilon).$$

This equation describes slow motion of this particle due to the steady part of the Eulerian velocity field, $v(x, \epsilon)$, and the Stokes drift velocity, $\tilde{v}(x, \epsilon)$. If we denote the Lagrangian velocity of fluid particles by superscript $L$ and the Eulerian velocity by superscript $E$, then our results can be summarized as follows. Our asymptotic expansion for the averaged Eulerian velocity has the form

$$\bar{v}^E = \epsilon^2 \left[ \bar{v}^2 + \bar{v}_1^2 \right] + O(\epsilon^3), \quad \tilde{v}^E = \epsilon \tilde{v}_1^0 + \epsilon^2 \left[ \tilde{v}_2^0 + \tilde{v}_1^1 + \tilde{v}_1^2 \right] + O(\epsilon^3).$$ (4.50)
It is shown in Appendix B that the Lagrangian velocity of fluid particles is given by

\[ \bar{u}^L = \epsilon^2 \bar{u}_2 + O(\epsilon^3), \quad \bar{v}^L = \epsilon^2 \left[ \bar{v}_2^i + \bar{v}_2^a + \bar{v}_2^b + \bar{v}_2^c \right] + O(\epsilon^3). \tag{4.51} \]

where the Stokes drift velocity of the fluid particles \( \bar{v}_2^a \) is given by Eq. (B.33) in Appendix B. Comparing (4.50) with (4.51), we observe that the Stokes drift eliminates (i) \( \bar{u}_2^i \) from the first equation (4.50) and (ii) the \( O(\epsilon) \) term from the second equation (4.50). It also results in the additional \( O(\epsilon^2) \) term \( \bar{v}_2^3 \) in the expansion of the azimuthal velocity. Thus, the \( O(\epsilon) \) steady boundary layer at the inner cylinder disappears when we take account of the Stokes drift. This is a consequence of the fact that the steady Lagrangian velocity rather than the steady Eulerian velocity are invariant to the change of reference frame. More precisely, the steady Eulerian velocity is not invariant in the following sense: it can be shown that in the reference frame fixed in the inner cylinder, there would be no \( O(\epsilon) \) steady boundary layer at the inner cylinder and an \( O(\epsilon) \) steady boundary layer would appear near the outer cylinder (which is oscillating in this reference frame). It can be also shown that the steady Lagrangian velocity is the same both in the oscillating reference frame and in the fixed one. We note here that if one uses the method of matched asymptotic expansions, then although it is possible, it is not easy to detect the existence of an \( O(\epsilon) \) steady boundary layer near the inner or outer cylinders. Fortunately, the presence or absence of the \( O(\epsilon) \) steady boundary layer do not affect the outer flow: the \( O(\epsilon) \) steady outer flow is zero in both cases.

Further calculations with the help of the known formulae for \( \bar{v}_2^a, \bar{v}_2^b \) and \( \bar{v}_2^c \) show that \( \bar{v}^L \) can be written as

\[ \bar{v}^L = \epsilon^2 \left[ \bar{v}_2^i + \bar{v}_2^a + \bar{v}_2^b \right] + O(\epsilon^3), \tag{4.52} \]

where

\[ (\bar{v}_2^a)^L = -\frac{R^4 \sin 2\theta}{4(R^2 - 1)^2} F'(\tilde{\xi}), \quad (\bar{v}_2^b)^L = -\frac{\sin 2\theta}{4R(R^2 - 1)^2} F'(\tilde{\eta}), \tag{4.53} \]

where \( F'(z) = e^{-z} \left( 3e^{-z} + 8 \cos z + 8 \sin z \right), \tilde{\xi} = \xi/\sqrt{2\nu} \) and \( \tilde{\eta} = \eta/\sqrt{2\nu} \).

4.6. Asymptotic expansion for stream function

To rewrite our asymptotic expansion of terms of the averaged stream function, we first observe that

\[ \psi = \psi_0 + \epsilon \left[ \psi_1 + \psi_2^a + \psi_2^b \right] + \epsilon^2 \left[ \psi_2^i + \psi_1^a + \psi_1^b \right] + \epsilon^3 \left[ \psi_1^a + \psi_2^a + \psi_2^b \right] + O(\epsilon^4), \]

where \( \psi_k^j \) is such that \( u_k^j = \psi_k^j - \psi_k^0 \) and \( v_k^j = -\psi_k^j \) for \( k = 0, 1, \ldots \) and where \( \psi_k^a, \psi_k^b \) are defined as

\[ \psi_k^a = \int_{\xi}^{\infty} v_k^a(\xi, \theta, \tau) d\xi, \quad \psi_k^b = -\int_{\eta}^{\infty} v_k^b(\eta, \theta, \tau) d\eta \]

for \( k = 0, 1, \ldots \) Similarly, we have \( \tilde{\psi} = \epsilon^2 \left[ \tilde{\psi}_2^a + \tilde{\psi}_2^b \right] + \epsilon^3 \left[ \tilde{\psi}_3^a + \tilde{\psi}_2^a + \tilde{\psi}_2^b \right] + O(\epsilon^4) \) and

\[ \tilde{v}^L = \epsilon^2 \tilde{v}_2^a + \epsilon^3 \left[ (\tilde{v}_2^a)^L + (\tilde{v}_2^a)^L + (\tilde{v}_2^b)^L \right] + O(\epsilon^4). \tag{4.54} \]

In the last formula, \( (\tilde{v}_2^a)^L \) and \( (\tilde{v}_2^b)^L \) are obtained from (4.53) and given by

\[ (\tilde{v}_2^a)^L = \frac{\sqrt{2\nu} R^4 e^{-\frac{\xi}{\sqrt{2\nu}}}}{4(R^2 - 1)^2} F(\tilde{\xi}) \sin 2\theta, \quad (\tilde{v}_2^b)^L = -\frac{\sqrt{2\nu}}{4R(R^2 - 1)^2} F(\tilde{\eta}) \sin 2\theta. \tag{4.55} \]
$(\bar{\psi}_3)^L$ is the stream function for the third-order Lagrangian velocity (see Appendix B):

$$
(\bar{\psi}_3)^L = \frac{\sqrt{2\nu}}{8R(R - 1)(R^2 - 1)^4} \left( D_1 + \frac{\hat{D}_2}{r^2} + D_3r^2 + D_4r^4 \right) \sin 2\theta,
$$

(4.56)

where $D_1$, $\hat{D}_2$, $D_3$ and $D_4$ are constants (given by (A.26) and (B.41)).

Thus, the Stokes drift produces a nonzero contribution to the third order outer flow given by the second terms on the right sides of Eqs. (B.37) and (B.38). Note that the Stokes drift corrections to the outer flow in the lower order approximations are all zero. Also, it follows from Eqs. (B.37) and (B.38) that the Stokes drift contribution to $(\bar{\psi}_3)^L$ vanishes in the limit $R \to \infty$, which means that if there were no outer cylinder, the third order Stokes drift correction to the outer flow would be zero too. This can also be shown independently by treating the problem on a steady flow produced by an oscillating cylinder in an unbounded fluid. Thus, the appearance of a nonzero Stokes drift correction to the outer flow is caused by the presence of the outer cylinder.

5. Discussion

Equation (4.54) together with Eqs. (4.40), (4.55)–(4.56) represent the first two non-zero terms in the asymptotic expansion of the stream function for the averaged Lagrangian velocity. Let us first discuss the domain of applicability of formula (4.54).

**Domain of applicability.** Our asymptotic expansion is formally valid for $\epsilon \ll 1$ and for $\nu = O(1)$. In practice, we may expect that it will be valid for all $\epsilon$ and $\nu$ such that the contribution of the $O(\epsilon^3)$ term to the right side of Eq. (4.54) is smaller than the contribution of the $O(\epsilon^2)$ term. For each value of $R > 1$, this requirement corresponds to a domain in the space of parameters $\epsilon$ and $\nu$.

It is convenient to rewrite Eq. (4.54) in the form

$$
\bar{\psi}^L = \epsilon^2 \left[ \Phi_0(R, r) + \epsilon \sqrt{\nu} \Phi_1(R, r, \nu, \epsilon) \right] \sin 2\theta + O(\epsilon^4)
$$

where $\Phi_0 = \bar{\psi}_2^L / \sin 2\theta$ and $\Phi_1 = \frac{1}{\sqrt{\nu}} \left[ (\bar{\psi}_3^L)^L + (\bar{\psi}_2^L)^L_{\xi=(r-1)/\epsilon} + (\bar{\psi}_2^L)^L_{\eta=(R-r)/\epsilon} \right] / \sin 2\theta$.

Consider now the following quantity

$$
\chi(R, \nu, \epsilon) = \epsilon \sqrt{\nu} \max_{r \in [1, R]} |\Phi_1(R, r, \nu, \epsilon)| / \max_{r \in [1, R]} |\Phi_0(R, r)|,
$$

which measures the magnitude of the second nonzero term relative to the first term. We expect that our theory will work for all $R$, $\nu$, $\epsilon$ such that $\chi(R, \nu, \epsilon) < 1$, and the smaller $\chi$ is, the better the theory should work. The level curves $\chi(R, \nu, \epsilon) = 1$ in the $(\epsilon, \nu)$ plane for several values of $R$ are shown in Fig. 1. For each $R$ the domain, where we expect that the theory is valid, lies below the corresponding curve. Of course, the ‘domains of applicability’ shown in Fig. 1 are very approximate. Nevertheless Fig. 1 gives us an idea of where our theory might work. In particular, it shows that for $\nu \geq 1$ the interval in $\epsilon$ within which the theory is applicable is very narrow, but it becomes much wider for smaller values of $\nu$. Figure 1 also shows that the ‘domain of applicability’ shrinks when we reduce the radius $R$ of the outer cylinder. Our theory is also unapplicable to the case of high Reynolds numbers when the dimensionless viscosity $\nu$ is comparable with $\epsilon$. The
Figure 1: The curves \( \kappa(R, \nu, \epsilon) = 1 \) for \( R = 2, 3, 5 \) and 10. The dashed line corresponds to \( \nu = \epsilon \). For each \( R \), the ‘domain of applicability’ of the theory lies below the corresponding curve, but above the dashed line.

dashed-line curve in Fig. 1 corresponds to \( \nu/\epsilon = 1 \), and the area below this curve is not covered by the present theory.

Comparison with experiments. Schematic picture of the averaged Lagrangian flow is shown in Fig. 2. The flow is symmetric relative to the axis of oscillation (the \( x \)-axis). There are four stagnation points in the flow (two on the \( x \) axis and two on the \( y \) axis) and three of them are shown in Fig. 2. The stagnation points are symmetric relative to the \( x \) and \( y \) axes and their distance from the origin is denoted by \( r_s \) in Fig. 2 (note that the thickness of the inner vortex structure is \( r_s - 1 \)). Examples of the streamlines produced by formula (4.54) are shown in Fig. 3. The streamlines shown in Figures 3a, 3b and 3c (which correspond to \( \epsilon = 0.05, 0.06 \) and 0.07 for \( \nu = 0.5 \) and \( R = 4 \)) are similar and qualitatively consistent with the experimental observations (see, e.g., [6], [21]). Figures 3a, 3b and 3c show that the thickness and the intensity of the inner vortex structure increases with \( \epsilon \). At some value of \( \epsilon \) the inner vortex structure becomes dominating (Fig. 3d). This is inconsistent with the experimental observations and corresponds to the situation where the \( O(\epsilon^3) \) term in (4.54) is larger than (or comparable to) the \( O(\epsilon^2) \) term. Our theory is not valid in this situation.

Typical profiles of the azimuthal velocity are shown in Fig. 4. Qualitatively the behaviour of the azimuthal velocity is similar to the velocity profiles measured in the experiments by Bertelsen et. al. [6], but there is no quantitative agreement, because in the experiments \( \epsilon \approx 0.3 \), which is outside the area of applicability of our theory.

The averaged flow structure does not change for a wide range of values of \( R \). Figure 5 shows the dependence of the distance of stagnation points from the origin \( r_s \) on the radius of the outer cylinder \( R \) and on \( \epsilon \). Qualitatively the behaviour of \( r_s \) (as a function of \( R \)) agrees with experimental observation of Bertelsen et. al. [6]. The structure of the averaged flow changes slightly for smaller values of \( R \) (when \( R \) is not very different from 1). In this case, the boundary layer at the outer cylinder results in appearance of an additional narrow vortex system near the outer cylinder. Figure 6 shows the streamlines for the Lagrangian velocity for \( \nu = 0.4, \epsilon = 0.048 \) and \( R = 1.7 \). One can see a narrow boundary layer at the outer cylinder which is similar to the boundary layer at the inner cylinder but considerably weaker.
Figure 2: Typical streamlines of the averaged Lagrangian flow.

Figure 3: The streamlines of the averaged Lagrangian flow for $\nu = 0.5$, $R = 4$
One cylinder in the fluid that extends to infinity. In the limit $R \to \infty$, the stream function for the Lagrangian velocity reduces to

$$\bar{\psi}^L = \bar{\psi}^i_2 + \epsilon^2 \left[ \bar{\psi}^i_3 + (\bar{\psi}^a_2)^L \right] + O(\epsilon^4),$$  \hspace{1cm} (5.1)

where

$$\bar{\psi}^i_2 = \frac{3}{4} \left( 1 - \frac{1}{r^2} \right) \sin 2\theta, \quad \bar{\psi}^i_3 = \frac{\sqrt{2\nu}}{8} \left( -75 + \frac{53}{r^2} \right) \sin 2\theta, \quad (\bar{\psi}^a_2)^L = \frac{\sqrt{2\nu}}{4} F(\tilde{\xi}) \sin 2\theta.$$

Wang [5] studied the steady flow produced by a fixed circular cylinder placed in an oscillating flow. He applied the method of matched asymptotic expansions and obtained a uniformly valid expansion of the stream function under the same assumptions as in the present paper. His theory however was not directly applicable to the steady streaming flow produced by a cylinder oscillating in the fluid which is at rest at infinity, because his expansion of the Eulerian velocity field is not invariant under the appropriate change of the reference frame. In order to obtain the invariant velocity field, one should consider the Lagrangian velocity which is different from the Eulerian velocity by the Stokes drift velocity of the fluid particles. This had been understood first by Skavlem and Tjøtta [4], and Bertelsen et. al. [6] had corrected Wang’s theory by taking account of the Stokes drift.

However, the averaged stream function for the Eulerian velocity in Wang’s theory (given by Eq. (3.36) in [5] and Eq. (25) in [6]) is incomplete because of the absence of the $O(\epsilon^3)$ term associated with the averaged outer flow. In order to compute this term, it is necessary to obtain first the second nonzero term in the inner expansion of the averaged stream function and then, following Van Dyke’s recipe [22], match the first two terms of the outer expansion with the first two terms of the inner expansion. This will result in a correct $O(\epsilon^3)$ outer flow term, which then can be incorporated in the composite formula valid for the whole flow domain. It can be shown that if this is done, then the uniformly valid expansion for the averaged stream function becomes

$$\bar{\psi} = \epsilon^2 \left\{ \frac{3}{4} \left( 1 - \frac{1}{r^2} \right) + \epsilon \frac{\sqrt{2\nu}}{4} \left[ \frac{1}{2} \left( -75 + \frac{49}{r^2} \right) + G(\xi) \right] \right\} \sin 2\theta + O(\epsilon^4)$$

where $G(\xi) = e^{-\tilde{\xi}}(e^{-\tilde{\xi}} + 12 \cos \tilde{\xi} + 8 \sin \tilde{\xi} + 4 \tilde{\xi} \sin \tilde{\xi})$. Exactly the same formula can be obtained by the Vishik-Lyusternik method. If we now take into account the Stokes drift, we obtain Eq. (5.1).
This proves that (at least) up to $O(\epsilon^3)$ terms, the averaged stream function for the Lagrangian velocity is the same both in the oscillating (with the cylinder) reference frame and in the fixed (in space) reference frame.

A remark on the results of Duck and Smith [12]. As was mentioned earlier, the expression for $\bar{\psi}_2$ obtained here agrees with formula (3.33) in [12] (as well as with equation (3.1) in [11]). However, the boundary layer parts of the expansions are different. The most essential difference is that in our expansion of the Eulerian velocity there is a nonzero steady boundary layer at the inner cylinder in the first order in $\epsilon$, while it is present only in the second order in [12]. This is a result of the transformation of coordinates employed in [12]. The transformation maps the gap between two eccentric cylinders onto the annulus between two cylinders whose axes coincide and it is a time-dependent transformation. In order to obtain our expansion, one needs to do the following: (i) to write down a composite expansion using the formulae for inner and outer expansions obtained in [12], (ii) to perform the inverse transformation of the coordinates, and (iii) to expand the resulting velocity field in Taylor’s series in $\epsilon$. The result will almost certainly coincide with our expansion of the averaged stream function for the Eulerian velocity up to $O(\epsilon^2)$ terms.

Remarks of the steady streaming theories for high Reynolds numbers. There are quite a few papers dealing with the steady streaming in an unbounded fluid produced by an oscillating circular cylinder (see [7, 8, 9, 2, 10]) at high Reynolds numbers such that $Re_s = O(1)$ or $Re_s \gg 1$ (in our notation this means that $\nu/\epsilon^2 = O(1)$ or $\nu/\epsilon^2 \ll 1$, respectively). In all these papers, coordinate systems oscillating with the cylinder are used, i.e. what is being really solved is the problem about a steady streaming flow produced by the fixed cylinder placed in an unbounded oscillating flow. As the present study shows, the problem with a fixed cylinder is not equivalent to the problem with an oscillating cylinder not only within the boundary layer but also in the outer flow. In fact, the situation is even more difficult, because for $\nu/\epsilon^2 = O(1)$ (and even more so for $\nu/\epsilon^2 \ll 1$) it is unclear how to solve the problem in the frame of reference fixed in space. In the present study, we employed the standard trick: using the fact that the amplitude of oscillations was small we expanded the solution in Taylor’s series about the averaged position of the oscillating cylinder and thus transferred the no-slip boundary conditions at the moving surface of the oscillating cylinder to the fixed surface of the cylinder at its averaged position. This allowed us to formulate our asymptotic expansion in terms of a sequence of boundary value problems with fixed boundaries. The thickness of the boundary layer on the inner cylinder that appeared in our expansion is $O(\epsilon)$, which is much larger than the $O(\epsilon^2)$ displacement of the inner cylinder from its averaged position, and this is what justifies the transfer of boundary conditions from the moving boundary to the fixed one. If however we considered the case of $\nu = \epsilon^2$ ($Re_s = 1$), the thickness of the Stokes layer would be $O(\epsilon^2)$, i.e. of the same order as the displacement of the cylinder, and therefore it would be impossible to justify the transfer of boundary conditions. Even if we worked in the frame of reference fixed with the oscillating cylinder, we would encounter the problem of calculating the Stokes drift velocity in the boundary layer because of non-analytic dependence of the boundary layer velocity on $\epsilon$.

In [11] and [12], the same problem as in the present paper, i.e. the steady flow between two cylinders produced by small-amplitude oscillations of the inner cylinder, had been studied in the case of $Re_s = O(1)$ (and $Re_s \gg 1$). Haddon and Riley [11] had realised the impossibility of the transfer of the boundary conditions from the moving surface to a fixed one in this flow regime and addressed the problem by employing two different coordinate systems for boundary layers on the
inner and outer cylinders. In the end, however, they had applied the transfer of the boundary conditions at the inner cylinder for the steady outer flow, and it is unclear whether this can be justified. In \cite{12}, a conformal mapping that maps the gap between two eccentric cylinders onto the annulus between two cylinders with a common axis was employed. The subsequent analysis had been done using the transformed coordinates. Neither the inverse transformation to physical coordinates, nor the Stokes drift had been computed, and as the above discussion indicates, these are the questions where potential problems may arise. Thus, in spite of a considerable progress in this area and a good agreement with experimental results achieved by the theory (see, e.g., \cite{12}), there are still certain unanswered questions concerning the steady streaming at high Reynolds numbers.

References

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Functions \( F_k^i, G_k^i \) and \( H_k^a \) for \( k = 1, 2, 3 \) in Eqs. \( (3.18)-(3.20) \) are given by

\[
F_1^a = -u_0^a |_{r_1} v_0^a + \nu v_0^2 + \xi_0^a, \\
F_2^a = -u_0^a |_{r_1} v_0^a - (u_1^a |_{r_1} + \xi u_0^a |_{r_1}) v_0^a - \partial_\theta \left( \frac{(v_0^a)^2}{2} + v_0^a |_{r_1} v_0^a \right) \\
- u_0^a v_0^a - u_0^a v_0^a + \nu (v_0^a - \xi v_0^2 + v_0^2 + v_0^0 - v_0^0) + \xi p_0^2 - \xi^2 p_0^2, \\
F_3^a = -u_0^a |_{r_1} v_0^a - (u_1^a |_{r_1} + \xi u_0^a |_{r_1}) v_0^a - \left( u_2^a |_{r_1} + \xi u_1^a |_{r_1} + \xi_1^2 u_0^a |_{r_1} \right) v_0^a \\
- \partial_\theta \left( v_0^a |_{r_1} v_0^a + (v_1^a |_{r_1} + \xi v_0^a |_{r_1}) v_0^a + v_0^0 v_0^a - \frac{(v_0^a)^2}{2} - \xi v_0^a |_{r_1} v_0^a \right) \\
- u_0^a v_0^a - u_0^a v_0^a - v_0^0 v_0^a + u_0^a - u_0^a |_{r_1} v_0^a - (u_1^a |_{r_1} + \xi u_0^a |_{r_1}) v_0^a - v_0^a |_{r_1} u_0^a \\
- u_0^a v_0^a - \xi u_0^a |_{r_1} v_0^a + \xi p_0^2 - \xi^2 p_0^2 + \xi^3 p_0^2
\]

6. Appendix A. Explicit expressions for the right hand sides of Eqs. (3.18)-(3.20), (3.21)-(3.23), (3.24) and (3.26)
\[ G^a_1 = 0, \quad G^a_2 = -\left( u^a_0 - \nu u^a_{00} \right), \]
\[ G^a_3 = -\left( u^a_{1r} - \nu u^a_{00} \right) - u^a_{1r=1} u^a_{00} - \xi v^a_{00,0} - \xi v^a_0 + (v^a_0)^2 + 2v^a_1 + \nu \left( u^a_{00} - 2u^a_{00} \right), \]
\[ H^a_1 = \xi v^a_{00,0} - u^a_0, \quad H^a_2 = \xi v^a_{00,0} - u^a_1 + \xi u^a_0, \quad H^a_3 = \xi^2 v^a_{00,0} - \xi^2 u^a_1 - \xi^2 u^a_0. \]

Functions \( F^b_k, G^a_k \) and \( H^a_k \) for \( k = 1, 2, 3 \) in Eqs. (3.21)–(3.23) are given by

\[ F^b_1 = -\frac{\nu}{R} v^0_0 - \frac{\eta}{R^2} v^0_0, \quad F^b_2 = u^b_{0r=1} v^b_{n+1} + (u^b_{1r=1} - \eta u^b_{0r=1}) v^b_{0n} + \frac{1}{R} \partial_\theta \left( \frac{\eta}{R^2} v^b_{00} - v^b_0 \right) \]
\[ + \frac{1}{R} u^b_{0r=1} v^b_0 + \frac{1}{R} \partial_\theta \left( \frac{\eta}{R^2} v^b_{00} - v^b_0 \right) - \frac{\eta}{R^2} p^b_{10} - \frac{\eta^2}{R^3} v^b_0, \quad F^b_3 = u^b_{0r=1} v^b_n + \left( u^b_{1r=1} - \eta u^b_{0r=1} \right) v^b_{1n+1} + \left( u^b_{1r=1} - \eta u^b_{0r=1} \right) v^b_{0n} + \frac{1}{R} \partial_\theta \left( \frac{\eta}{R^2} v^b_{00} - v^b_0 \right) \]
\[ + \frac{1}{R} u^b_{0r=1} v^b_0 + \frac{1}{R} \partial_\theta \left( \frac{\eta}{R^2} v^b_{00} - v^b_0 \right) - \frac{\eta}{R^2} p^b_{10} - \frac{\eta^2}{R^3} v^b_0, \quad G^b_1 = 0, \quad G^b_2 = u^b_0 - \nu v^b_{00}, \quad G^b_3 = u^b_{1r} - \nu v^b_{00} - u^b_{1r=1} v^b_0 + u^b_{0r=1} v^b_0 \]
\[ - \frac{1}{R} \left( (v^b_0)^2 + 2v^b_{1r=1} v^b_0 \right) + \nu \left( \frac{1}{R} u^b_{0} + \frac{2}{R^2} v^b_{00} \right), \quad H^b_1 = -\frac{\eta}{R^2} v^b_{00} - \frac{1}{R} v^b_0, \quad H^b_2 = -\frac{\eta}{R^2} v^b_{00} - \frac{\eta^2}{R^3} v^b_{00} - \frac{1}{R} v^b_{11} - \frac{\eta}{R^2} v^b_{0}, \quad H^b_3 = -\frac{\eta}{R^2} v^b_{00} - \frac{\eta^2}{R^3} v^b_{00} - \frac{\eta^3}{R^4} v^b_{00} - \frac{1}{R} u^b_{0} - \frac{\eta}{R^2} v^b_{11} - \frac{\eta^2}{R^3} u^b_{0}. \]

Functions \( Q^b_k \) and \( S^b_k \) in Eqs. (3.25) and (3.26) are given by

\[ Q_1 = 0, \quad (A.19) \]
We seek the solution of (B.1) and (B.2) in the form

\[
Q_2 = -\cos \theta f \left( u^a_0 |_{r=1} + u^a_0 |_{\xi=0} \right) + \sin \theta f u^a_0 |_{r=1} \tag{A.20}
\]

\[
Q_3 = -\cos \theta f \left( u_i^1 |_{r=1} + u^a_1 |_{\xi=0} \right) + \sin \theta f \left( u^a_1 |_{r=1} + u^a_1 |_{\xi=0} \right)
- \sin \theta f \left( v^i_1 |_{r=1} + v^a_1 |_{\xi=0} \right) - \cos^2 \theta \frac{f^2}{2} u^a_0 |_{\xi=0} - \sin \theta \cos \theta f^2 u^a_0 |_{\xi=0}, \tag{A.21}
\]

\[
S_1 = -\cos \theta f (\tau) u^a_0 |_{\xi=0}, \tag{A.22}
\]

\[
S_2 = -\cos \theta f \left( v^i_0 |_{r=1} + v^a_0 |_{\xi=0} \right) + \sin \theta f \left( v^a_0 |_{r=1} + v^a_0 |_{\xi=0} \right)
+ \sin \theta f u^a_i |_{r=1} - \cos^2 \theta \frac{f^2}{2} v^a_0 |_{\xi=0}. \tag{A.23}
\]

\[
S_3 = -\cos \theta f \left( v^i_1 |_{r=1} + v^a_1 |_{\xi=0} \right) + \sin \theta f \left( v^a_1 |_{r=1} + v^a_1 |_{\xi=0} \right)
+ \sin \theta f \left( u^a_i |_{r=1} + u^a_1 |_{\xi=0} \right) - \cos^2 \theta \frac{f^2}{2} v^a_0 |_{\xi=0}
+ \sin \theta \cos \theta f^2 v^a_0 |_{\xi=0} - \sin^2 \theta \frac{f^2}{2} v^a_0 |_{\xi=0} - \cos^3 \theta \frac{f^3}{6} v^a_0 |_{\xi=0}. \tag{A.24}
\]

Explicit formulae for constants $C_1$, $C_2$, $C_3$ and $C_4$ in Eq. (4.40) are

\[
C_1 = R^6 (R^2 + 2) + 2R^2 + 1, \quad C_2 = -R^8 - R^2, \\
C_3 = -R^8 (2R^2 + 1) - R^2 - 2, \quad C_4 = R^4 + 1. \tag{A.25}
\]

Constants $D_1$, $D_2$, $D_3$ and $D_4$ in Eq. (4.48) are given by

\[
D_1 = -(75R^{10} - 75R^9 + 138R^8 - 158R^7 \\
+ 106R^6 - 134R^5 + 131R^4 - 79R^3 + 80R^2 - 40R + 40), \\
D_2 = R^2 (53R^8 - 53R^7 + 32R^6 - 44R^5 - 39R^4 - 51R^3 + 52R^2 - 14R + 14), \\
D_3 = 2(75R^8 - 77R^7 + 58R^6 - 74R^5 + 59R^4 - 52R^3 + 53R^2 - 40R + 40), \\
D_4 = -(75R^6 - 79R^5 + 31R^4 - 43R^3 + 44R^2 - 40R + 40). \tag{A.26}
\]

7. Appendix B. Calculation of the averaged Lagrangian velocity

In polar coordinates, Eq. (4.49) is equivalent to

\[
r_{\tau} = \epsilon^2 \left[ u(r, \theta, \tau) - r_{\tau} \right]. \tag{B.1}
\]

\[
r_{\theta} = \epsilon^2 \left[ v(r, \theta, \tau) - r_{\theta} \right]. \tag{B.2}
\]

We already know that

\[
u = v^i_0 + \epsilon \left( v^i_0 + u^a_0 + u^b_0 \right) + \epsilon^2 \left( v^i_2 + u^a_2 + u^b_2 \right) + \ldots, \tag{B.3}
\]

\[
v = v^i_0 + v^a_0 + v^b_0 + \epsilon \left( v^i_1 + v^a_1 + v^b_1 \right) + \epsilon^2 \left( v^i_2 + v^a_2 + v^b_2 \right) + \ldots \tag{B.4}
\]

We seek the solution of (B.1) and (B.2) in the form

\[
r = r_0 + \epsilon r_1 + \epsilon^2 r_2 + \ldots, \quad \theta = \theta_0 + \epsilon \theta_1 + \epsilon^2 \theta_2 + \ldots \tag{B.5}
\]
On substituting (B.3)–(B.5) in (B.1) and (B.2) and collecting terms of equal powers in $\epsilon$, we obtain the following sequence of equations:

$$r_{0r} = 0, \quad r_0\theta_{0r} = 0, \quad (B.6)$$
$$r_{1r} = 0, \quad r_0\theta_{1r} + r_1\theta_{0r} = 0, \quad (B.7)$$

and

$$\partial_rr_n + \partial_t r_{n-2} = U_{n-2}, \quad (B.8)$$
$$\sum_{m=0}^{n-2} r_m\partial_r\theta_{n-m} + \sum_{m=0}^{n-2} r_m\partial_t\theta_{n-m} = V_{n-2}, \quad (B.9)$$

for $n = 2, 3, \ldots$. Functions $U_n$ and $V_n$ are obtained by substitution of (B.3)–(B.5) in the right sides of Eqs. (B.1) and (B.2) and subsequent Taylor’s expansion of all terms. There is a subtle technical trick here. Functions which depend on boundary layer variables $\xi = (r - 1)/\epsilon$ and $\eta = (R - r)/\epsilon$ are not analytic at $\epsilon = 0$. Therefore, for these functions we use Taylor’s expansions at $r = r_0 + \epsilon r_1$ rather than at $r = r_0 + \epsilon r_1$. For example,

$$u_0^a(\xi, \theta, \tau) = u_0^a\left(\frac{r_0 - 1}{\epsilon} + r_1 + \epsilon r_2 + \ldots, \theta_0 + \epsilon \theta_1 + \ldots, \tau\right)$$
$$= u_0^a(\xi^*, \theta_0, \tau) + \epsilon \left[r_2 u_0^a(\xi^*, \theta_0, \tau) + \theta_1 u_0^a(\xi^*, \theta_0, \tau) + \ldots\right]$$

where $\xi^* = \frac{r_0 - 1}{\epsilon} + r_1$. Below we use the following notation: $\cdot |_{r = r_0, \theta = \theta_0}$ and $\cdot |_{s} = (\cdot) |_{\xi = \frac{r_0 - 1}{\epsilon} + r_1, \theta = \theta_0}$ or $\cdot |_{s} = (\cdot) |_{\eta = \frac{R - r_0}{\epsilon} - r_1, \theta = \theta_0}$ depending on whether the quantity being evaluated is a function of $\xi$ or $\eta$. The explicit formulæ for $U_n$ and $V_n$ are

$$u_0^a(\xi, \theta, \tau) = u_0^a|_{r = r_0, \theta = \theta_0} = u_0^a|_{s}, \quad (B.10)$$
$$u_1 = \left[u_1^a + r_1 u_{0r}^a + \theta_1 u_{0\theta}^a\right]|_{r_0} + u_0^a|_{s} + u_0^a|_{s}, \quad (B.11)$$
$$V_1 = \left[u_1^a + r_1 u_{0r}^a + \theta_1 u_{0\theta}^a\right]|_{r_0} + \left[v_1^b + r_2 u_{0\xi_1}^a + \theta_1^b v_{1\theta_0}^a\right]|_{s} + \left[v_1^b - r_2 v_{0\eta_1}^a + \theta_1 v_{1\theta_0}^a\right]|_{s}, \quad (B.12)$$
$$U_2 = \left[u_2^a + r_1 u_{1r}^a + \theta_1 u_{1\theta}^a + \frac{1}{2} \left(r_2^2 u_{0rr}^a + 2 r_1 \theta_1 u_{0r\theta}^a + \theta_1^2 u_{0\theta\theta}^a\right) + r_2 u_{0r}^a + \theta_2 u_{0\theta}^a\right]|_{r_0}$$
$$+ \left[u_1^a + r_2 u_{0\xi_1}^a + \theta_1 u_{0\theta}^a\right]|_{s} + \left[u_1^a - r_2 u_{0\eta_1}^a + \theta_1 u_{0\theta}^a\right]|_{s}, \quad (B.13)$$
$$V_2 = \left[v_2^b + r_1 v_{1r}^b + \theta_1 v_{1\theta}^a + \frac{1}{2} \left(r_2^2 v_{0rr}^b + 2 r_1 \theta_1 v_{0r\theta}^b + \theta_1^2 v_{0\theta\theta}^b\right) + r_2 v_{0r}^b + \theta_2 v_{0\theta}^b\right]|_{r_0}$$
$$+ \left[v_1^a + r_2 v_{0\xi_1}^a + \theta_1 v_{0\theta}^a + \frac{1}{2} \left(r_2^2 v_{0rr}^a + 2 r_1 \theta_1 v_{0r\theta}^a + \theta_1^2 v_{0\theta\theta}^a\right) + r_3 v_{0\xi_1}^a + \theta_2 v_{0\theta}^a\right]|_{s} + \left[v_1^a - r_2 v_{0\eta_1}^a + \theta_1 v_{0\theta}^a + \frac{1}{2} \left(r_2^2 v_{0rr}^a + 2 r_1 \theta_1 v_{0r\theta}^b + \theta_1^2 v_{0\theta\theta}^a\right) + r_3 v_{0\eta_1}^a + \theta_2 v_{0\theta}^b\right]|_{s}. \quad (B.14)$$

It follows from Eqs. (B.6) and (B.7) that $r_0$, $\theta_0$, $r_1$ and $\theta_1$ do not depend on fast time $\tau$, i.e. $r_0 = r_0(t)$, $\theta_0 = \theta_0(t)$, $r_1 = r_1(t)$ and $\theta_1 = \theta_1(t)$. Equations (B.10) and the fact that $\bar{u}_{i0}^a = 0,$
\( \bar{v}^i_0 = 0, \bar{v}^v_0 = 0, \bar{v}^b_0 = 0 \) imply that \( \bar{U}_0 = 0 \) and \( \bar{V}_0 = 0 \). Then, averaging Eqs. (B.8) and (B.9) (for \( n = 2 \)), we find that \( v_{0t} = 0 \) and \( r_0/\theta_0 = 0 \). Therefore, \( r_0 = \text{const} \) and \( \theta_0 = \text{const} \), and we use \( r_0 \) and \( \theta_0 \) to identify fluid particles: \( (r(r_0, \theta_0, t, \tau), \theta(r_0, \theta_0, t, \tau)) \) represents the trajectory of a fluid particle whose averaged (in \( \tau \)) position at \( t = 0 \) was \( (r_0, \theta_0) \). From Eq. (B.11), we deduce that \( \bar{U}_1 = 0 \). Averaging Eq. (B.3) for \( n = 3 \), we find that \( r_1 = 0 \), so that \( r_1 = \text{const} \). We choose \( r_1 = 0 \). Note that this implies that \( (\cdot)|_a = (\cdot)|_b \) in Eqs. (B.12)-(B.14). Averaging Eq. (B.9) for \( n = 3 \) yields

\[
 r_0\theta_{1t} = \left[ \bar{v}^{a}_1 + \bar{r}_2\bar{v}^{a}_0\xi \right]_0 - \left[ \bar{r}_2\bar{v}^{b}_0\eta \right]_0. \tag{B.15}
\]

To proceed further, we need to compute \( \bar{r}_2\bar{v}^{a}_0\xi \) and \( \bar{r}_2\bar{v}^{b}_0\eta \). First we observe that the averaged part of \( r_2 \) does not make any contribution to these two terms, i.e.

\[
 \left. \bar{r}_2\bar{v}^{a}_0\xi \right|_0 = \left. \bar{r}_2\bar{v}^{b}_0\eta \right|_0 = \left. \bar{r}_2\bar{v}^{b}_0\eta \right|_0. \]

The oscillatory part of \( r_2 \) satisfies the equation

\[
 \var{r}_2 = U_0 = u_0^i|_0 = u_0^v(r_0, \theta_0, \tau)
\]

(whic is simply the oscillatory part of Eq. (B.8) for \( n = 2 \)). Substitution of (4.1) yields

\[
 \var{r}_2 = \frac{f(\tau)}{R^2 - 1} \left( \frac{R^2}{R^2 - 1} - 1 \right) \cos \theta_0. \tag{B.16}
\]

This implies that

\[
 \left. \bar{r}_2\bar{v}^{a}_0\xi \right|_0 = \frac{\cos \theta_0}{R^2 - 1} \left( \frac{R^2}{R^2 - 1} - 1 \right) f\bar{v}^{a}_0\xi, \quad \left. \bar{r}_2\bar{v}^{b}_0\eta \right|_0 = \frac{\cos \theta_0}{R^2 - 1} \left( \frac{R^2}{R^2 - 1} - 1 \right) f\bar{v}^{b}_0\eta.
\]

On the right sides of these equations we have functions \( v^{a}_0\xi \) and \( v^{b}_0\eta \) which are nonzero only within boundary layers with thickness \( O(\epsilon) \). It is therefore natural (and consistent with our asymptotic expansion for the velocity) to replace \( r_0 \) by \( 1 + \epsilon\xi \) in the first equation and by \( R - \epsilon\eta \) in the second equation (where \( \xi_0 = (r_0 - 1)/\epsilon \) and \( \eta_0 = (R - r_0)/\epsilon \)) and expand everything in Taylor’s series. This leads to the expressions

\[
 \left. \bar{r}_2\bar{v}^{a}_0\xi \right|_0 = \cos \theta_0 f\bar{v}^{a}_0\xi \left( 1 - \frac{2\xi_0 R^2}{R^2 - 1} + O(\epsilon^2) \right), \tag{B.17}
\]

\[
 \left. \bar{r}_2\bar{v}^{b}_0\eta \right|_0 = \epsilon \frac{2\epsilon_0}{R(R^2 - 1)} \cos \theta_0 f\bar{v}^{b}_0\eta + O(\epsilon^2). \tag{B.18}
\]

Now we substitute \( O(1) \) terms from (B.17) and (B.18) in Eq. (B.15) and move higher order terms to appropriate higher order equations. Equation (B.15) becomes

\[
 r_0\theta_{1t} = \left[ \bar{v}^{i}_1 + \cos \theta_0 f\bar{v}^{a}_0\xi \right]_0. \tag{B.19}
\]

The first term on the right side of this equation represents the averaged Eulerian velocity, while the second term is associated with the Stokes drift. Finally, substitution of (4.15) reduces this equation to \( r_0\theta_{1t} = 0 \), which, in turn, implies that \( \theta_1 = \text{const} \). Again, we choose \( \theta_1 = 0 \). Thus, the averaged Lagrangian velocity has no \( O(\epsilon) \) boundary layer term. Averaging the equation for \( r_4 \) yields

\[
 \var{r}_4 = \left[ \bar{v}^{i}_4 + \bar{r}_2\bar{u}^{i}_{0r} + \theta_2\bar{u}^{i}_{0\eta} + \bar{r}_2\bar{u}^{a}_0\xi + \bar{r}_2\bar{u}^{a}_0\xi + \bar{r}_2\bar{u}^{b}_0\eta \right]_0 + \left[ \bar{v}^{i}_1 + \bar{r}_2\bar{u}^{i}_{0r} \right]_0 - \left[ \bar{r}_2\bar{v}^{b}_0\eta \right]_0. \tag{B.19}
\]
Now we need an explicit formula for $\tilde{\theta}_2$. The equation for $\tilde{\theta}_2$ is

$$\tilde{\theta}_2 = \frac{1}{r_0} \left( v_0^i |_0 + i v_0^a |_0 + v_0^b |_0 \right).$$

On substituting Eqs. (4.1), (4.4), (4.8) and integrating over $\tau$, we obtain

$$\tilde{\theta}_2 = \tilde{\theta}_1 + \tilde{\theta}_2 + \tilde{\theta}_3$$

(B.20)

$$\tilde{\theta}_2 = \frac{\sin \theta_0 \left( R^2 - 1 \right)}{R^2 - 1 - r_0} \left( \frac{R^2}{r_0^2} + 1 \right) f(\tau), \quad \text{(B.21)}$$

$$\tilde{\theta}_2^a = -\frac{2R^2 \sin \theta_0}{R^2 - 1} \text{Re} \left( C e^{-\gamma_0 + i \tau} \right) + O(\epsilon), \quad \text{(B.22)}$$

$$\tilde{\theta}_2^b = -\frac{2 \sin \theta_0}{R(R^2 - 1)} \text{Re} \left( C e^{-\gamma \eta_0 + i \tau} \right) + O(\epsilon). \quad \text{(B.23)}$$

Now we return to Eq. (B.19). It follows from (4.1), (4.6), (4.9), (B.16), (B.21)–(B.23) that

$$\tilde{r}_2 v_{0r} |_0 = 0, \quad \tilde{r}_2 v_{0i} |_0 = 0, \quad \tilde{r}_2^a v_{0a} |_0 = 0, \quad \tilde{r}_2^b v_{0b} |_0 = 0.$$  

(B.24)

$$\tilde{r}_2^a v_{0a} |_0 = \sin \theta_0 v_0^a + O(\epsilon), \quad \tilde{r}_2^b v_{0b} |_0 = -\cos \theta_0 v_0^b + O(\epsilon). \quad \text{(B.25)}$$

After substitution of (B.24) and (B.25) in Eq. (B.19), it reduces to

$$\tilde{r}_2 = \tilde{u}_2(r_0, \theta_0). \quad \text{(B.26)}$$

Note that the right side of this equation is different from what we would have if we used the averaged Eulerian velocity: the second order term in the expansion of $\tilde{u}$ is $\tilde{u}_2^i + \tilde{u}_1^b$ rather than $\tilde{u}_2^b$. This means that the Stokes drift kills the boundary layer term.

Now let us derive equation for $\tilde{\theta}_2$. On averaging the equation for $\theta_4$ and using (B.14), we obtain

$$r_0 \tilde{\theta}_2 \left[ v_2^i + r_2 v_{0r}^i + \tilde{\theta}_2 v_{0i}^o \right] + \left[ v_2^b + r_2 v_{0r}^b + \frac{1}{2} r_2^b v_{0b}^b + r_3 v_{0c}^b + \tilde{\theta}_2 v_{0b}^o \right]$$

$$+ \left[ v_2^b - r_2 v_{0r}^b + \frac{1}{2} r_2^b v_{0b}^b - r_3 v_{0c}^b + \tilde{\theta}_2 v_{0b}^o \right] - r_2 \tilde{\theta}_2 + T_2,$$  

where all quantities on the right side of this equation are evaluated at $(r_0, \theta_0)$ and where

$$T_2 = -\cos \theta_0 \frac{2 \xi_0 R^2}{R^2 - 1} f(v_0^a) + \cos \theta_0 \frac{2 \eta_0}{R(R^2 - 1)} f(v_0^b). \quad \text{(B.28)}$$

represent the contribution that comes from $O(\epsilon)$ terms in (B.17) and (B.18). To proceed further, we need an explicit formula for $\tilde{r}_3$. We have the equation $\tilde{r}_3 = u_4^i |_0 + u_3^a |_0 + u_0^b |_0$, from which, on substituting (4.6), (4.9), (4.12) and integrating over $\tau$, we obtain

$$\tilde{r}_3 = \tilde{r}_3^i + \tilde{r}_3^a + \tilde{r}_3^b,$$  

(B.29)

$$\tilde{r}_3^i = -\frac{2 R}{(R^2 - 1)^2} \left[ R + 1 - \frac{R^3 + 1}{r^2} \right] \text{Re} \left( \frac{C}{\gamma} e^{i \tau} \right) \cos \theta_0,$$  

(B.30)

$$\tilde{r}_3^a = -\frac{2 R^2}{R^2 - 1} \text{Re} \left( \frac{C}{\gamma} e^{-\gamma_0 + i \tau} \right) \cos \theta_0,$$  

(B.31)

$$\tilde{r}_3^b = \frac{2}{R(R^2 - 1)} \text{Re} \left( \frac{C}{\gamma} e^{-\gamma \eta_0 + i \tau} \right) \cos \theta_0.$$  

(B.32)
After tedious but elementary calculations with the help of (B.16), (B.20), (B.23), (B.29)–(B.32), Eq. (B.27) can be reduced to

$$r_0 \hat{\theta}_{2\tau} = \tilde{v}_2^s + \tilde{v}_2^f$$

where $\tilde{v}_2^s$ represents the Stokes drift velocity of the fluid particles and is given by

$$\tilde{v}_2^s = \cos \theta_0 \left[ f v_{0\xi}^a - f v_0^a \right] + \frac{2 \cos \theta_0}{R^2 - 1} \left[ R^2 \left[ g v_{0\xi}^a - \xi_0 f v_0^a - Re \left( \frac{C}{\gamma e^{-\gamma \xi_0 + i\tau}} \right) \right] v_{0\xi}^a \right]$$

$$+ \frac{1}{R} \left[ g v_{0\eta}^b - \eta_0 f v_0^b - Re \left( \frac{C}{\gamma e^{-\gamma \eta_0 + i\tau}} \right) \right] v_{0\eta}^b \right]. \quad (B.33)$$

Here $g(\tau) = Re(C e^{i\tau}/\gamma)$. In order to obtain the $O(\epsilon^3)$ term in the expansion for the stream function, we need to compute the components of the averaged third-order Lagrangian velocity in the outer flow. Averaging the equations for $r_3$ and $\theta_3$ and ignoring all boundary layer terms, we obtain

$$\bar{r}_{3\tau} = \bar{u}_3 + r_3^0 u_{1r}^0 + r_3^0 u_{0r}^0 + \theta_3^0 u_{1\theta}^0 + \theta_3^0 u_{0\theta}^0, \quad (B.34)$$

$$r_0 \bar{\theta}_{3\tau} = -r_0^3 \theta_{3\tau} + \bar{v}_3 + r_0^3 v_{1r}^0 + r_3^0 v_{0r}^0 + \bar{\theta}_3 + r_0^3 \bar{v}_{1\theta}^0 + \bar{\theta}_3 + r_0^3 \bar{v}_{0\theta}^0. \quad (B.35)$$

All terms on the right sides of these equations are evaluated at $(r_0, \theta_0)$. Now we need to find $\bar{v}_3^i$. The equation for $\bar{\theta}_3^i$ is obtained from (B.9) for $n = 3$ by taking its oscillatory part and ignoring all boundary layer terms:

$$r_0 \bar{\theta}_{3\tau} = \bar{v}_3^i (r_0, \theta_0, \tau).$$

Inserting (4.17) into this equation and integrating over $\tau$, we find that

$$\bar{\theta}_3^i = \frac{2R}{(R^2 - 1)^2} \left( (R + 1) + \frac{R^3 + 1}{r^2} \right) g(\tau) \sin \theta. \quad (B.36)$$

Substitution of (B.11), (B.12), (B.14), (B.15), (B.16), (B.21), (B.30) and (B.36) in Eqs. (B.34) and (B.35) yields

$$\bar{r}_{3\tau} = \bar{u}_3 + \frac{4R}{(R^2 - 1)^2} \int \frac{g \cos 2\theta}{r^3}, \quad (B.37)$$

$$r_0 \bar{\theta}_{3\tau} = \bar{v}_3 + \frac{4R}{(R^2 - 1)^2} \int \frac{g \sin 2\theta}{r^3}. \quad (B.38)$$

Further calculations yield

$$\left( \bar{u}_3 \right)^L = \frac{\sqrt{2\nu}}{4R(R - 1)(R^2 - 1)^2} \left( \frac{D_1}{r} + \frac{D_2}{r^3} + D_3 r + D_4 r^3 \right) \cos 2\theta, \quad (B.39)$$

$$\left( \bar{v}_3 \right)^L = \frac{\sqrt{2\nu}}{4R(R - 1)(R^2 - 1)^2} \left( \frac{\hat{D}_2}{r^3} - D_3 r^2 - 2D_4 r^3 \right) \sin 2\theta, \quad (B.40)$$

where

$$\hat{D}_2 = D_2 - 4R^2(R - 1)(R^2 - 1). \quad (B.41)$$

Hence,

$$\left( \bar{v}_3^s \right)^L = \frac{\sqrt{2\nu}}{8R(R - 1)(R^2 - 1)^4} \left( D_1 + \frac{\hat{D}_2}{r^2} + D_3 r^2 + D_4 r^4 \right) \sin 2\theta \quad (B.42)$$