Extending light-front holographic QCD using the ’t Hooft Equation

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Abstract

We show the ’t Hooft Equation and the light-front holographic Schrödinger Equation are complementary to each other in governing the transverse and longitudinal dynamics of colour confinement in quark-antiquark mesons. Together, they predict remarkably well the light, heavy-light and heavy-heavy meson spectroscopic data. The universal emerging hadronic scale of light-front holography, $κ \approx 0.5$ GeV, controls the transverse dynamics of confinement in all these mesons. In heavy-heavy mesons, it also coincides numerically with the ’t Hooft coupling which governs longitudinal confinement, thus reflecting the restoration of manifest 3-dimensional rotational symmetry.

Keywords: Light-front holography, ’t Hooft equation, Hadron spectroscopy, Longitudinal dynamics

1. Introduction

Although QCD is the accepted theory for the strong interactions, it is not yet possible to predict the experimentally observed hadron spectrum from first principles. This is due to our incomplete understanding of the non-perturbative aspects of QCD responsible for colour confinement. While much progress is being made with numerical simulations on the lattice [1], complementary insights into non-perturbative QCD can be obtained from the AdS/CFT duality [2, 3] which refers to a correspondence between gravitation (no $Λ$) QCD, and the scale $Λ_{QCD}$ in the running coupling, generated after perturbative renormalization beyond tree-level. However, if we neglect quark masses ($m \rightarrow 0$) and ignore quantum loops (no $Λ_{QCD}$), QCD possesses an underlying conformal symmetry.

Light-front holography, pioneered by Brodsky and de Téramond [11, 14–16], exploits this conformal limit in the Hamiltonian formulation of $(3+1)$-dim QCD on the light front, with $N_c = 3$. The valence meson light-front wavefunction then factorizes as:

$$\Psi(x, ζ, φ) = \frac{φ(ζ)}{\sqrt{2πζ}} e^{iLφ} X(x)$$

where $X(x) = \sqrt{x(1−x)}χ(x)$ and $ζ = \sqrt{x(1−x)} b_⊥$ with $b_⊥ = b_⊥ e^{iφ}$, being the transverse separation between the quark and the antiquark. $x = k^+/P^+$ is the fraction of the meson’s light-front momentum, $P^+$, carried by the quark and $L$ is the orbital angular momentum quantum number.

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The transverse mode, $\phi(\zeta)$, satisfies the holographic Schrödinger Equation:

$$
\left( \frac{d^2}{d\xi^2} - \frac{1 - 4L^2}{4\xi^2} + U_T(\zeta) \right) \phi(\zeta) = M_T^2 \phi(\zeta) \tag{3}
$$

where $J$ is the meson’s spin. Eq. (4) is the holographic potential at equal light-front time, $x^+ = 0$. While the derivation of Eq. (4) in QCD remains an open question, its form is uniquely fixed by the underlying conformal symmetry and a holographic mapping to AdS$_5$ [17]. In this mapping, the variable $\zeta$ is identified with the fifth dimension of AdS$_5$, and Eq. (3) becomes the wave equation for spin-J bosonic modes propagating in AdS$_5$ spacetime distorted by a quadratic dilaton field [18]. The emerging hadronic scale, $\kappa$, generates the meson masses in the absence of quark masses and $\Lambda_{QCD}$. The longitudinal mode, $X(x)$, is fixed by the holographic mapping of the electromagnetic (or gravitational) form factor in physical spacetime, resulting in $X(x) = \sqrt{x(1-x)}$ [19, 20].

The holographic Schrödinger Equation admits analytical solutions:

$$
\phi_{nT}(\zeta) \propto \zeta^{1/2+L} \exp \left( - \frac{\kappa^2 \zeta^2}{2} \right) L_{nT}^L(\kappa^2 \zeta^2) \tag{5}
$$

and

$$
M_T^2(n_T, J, L) = 4\kappa^2 \left( n_T + \frac{J + L}{2} \right) \tag{6}
$$

where $J = L + S$ and $S$ is the total quark-antiquark spin, i.e. $S = 0$ or 1.\(^1\) Importantly, Eq. (6) predicts that the pion is massless:

$$
M_\pi = M_T(0, 0, 0) = 0 \tag{7}
$$

just as expected in the chiral limit. Eq. (6) also correctly predicts the Regge-like linear dependence of the meson mass squared on the radial and orbital quantum numbers.

Light-front holography needs to be extended to accommodate non-zero quark masses that generate the physical pion mass. This was originally done using a prescription by Brodsky and de Téramond (BdT) [21], resulting in a first-order shift to the mass spectrum given by

$$
\Delta M_{BdT} = \int \frac{dx}{x(1-x)} \times X_{BdT}^2(x) \left( \frac{m_q^2}{x} + \frac{m_{\pi q}^2}{1-x} \right) \tag{8}
$$

where

$$
X_{BdT}(x) = \sqrt{x(1-x)} \times \exp \left( -\frac{(1-x)m_q^2 + x m_{\pi q}^2}{2\kappa^2 x(1-x)} \right) \tag{9}
$$

so that $M_\pi = \Delta M_{BdT}$. Similarly, $M_K = \Delta M_{BdT}$ when the strange quark is taken into account. Using the BdT prescription, a global fit to the spectroscopic data of light hadrons,\(^2\) using $m_{u/d} = 0.046$ GeV and $m_s = 0.357$ GeV, yields $\kappa = 0.523 \pm 0.024$ GeV [22]. The BdT prescription, together with a universal $\kappa \approx 0.5$ GeV, have been widely used in a successful phenomenology of light mesons [11, 25–32]. The same prescription has also been used to accommodate heavy quarks, leading to the conclusion that $\kappa \propto \sqrt{m_Q}$, where $m_Q$ is the heavy quark mass, in order to be consistent with Heavy Quark Effective Theory (HQET) [33] and spectroscopic data [34–36]. In other words, when the BdT prescription is used for heavy quarks, the universality of $\kappa$ seems to be lost. Refs. [31, 32] attempts to prevent this by using a new scale $\lambda \neq \kappa$ in Eq. (9), thus hinting at the possibility that the longitudinal mode is the solution of Schrödinger-like Equation different from Eq. (3). The idea to use the ’t Hooft Equation to go beyond the BdT prescription was first proposed in Ref. [37], with the goal of predicting the meson decay constants. Very recently, Refs. [38, 39] also go beyond the BdT prescription using a phenomenological longitudinal confinement potential, first proposed in Ref. [40] in the context of basis light-front quantization. While both Refs. [38, 39] focus on the chiral limit and the phenomenon of chiral symmetry breaking, Ref. [38] extends their analysis to heavy mesons in their ground state, and discusses the relation of their approach to the ’t Hooft Equation.

In this letter, we show that the ’t Hooft Equation is complementary to, and consistent with, the holographic Schrödinger Equation. Together, they capture the main features of 3-dimensional confinement dynamics in (non-exotic) mesons and successfully predict their full spectrum.

2. The ’t Hooft Equation

In an earlier approach [41], ’t Hooft derived a Schrödinger-like equation for the longitudinal mode, starting from the QCD Lagrangian in (1 +

\(^1\)In light-front holography, the quark spin wavefunction is assumed to decouple from the confinement dynamics.

\(^2\)Supersymmetric light-front holography [22–24] provides a unified framework for baryons and mesons/tetraquarks.
1)-dim in the $N_c \gg 1$ approximation. This Lagrangian now contains two mass scales: the quark mass and the gauge coupling. The resulting ‘t Hooft Equation is:

$$\left(\frac{m^2}{x} + \frac{m_q^2}{1-x}\right) \chi(x) + U_L(x)\chi(x) = M_L^2\chi(x), \quad (10)$$

with

$$U_L(x)\chi(x) = g^2P\int dy \frac{\chi(x) - \chi(y)}{(x-y)^2} \quad (11)$$

where $P$ denotes the principal value prescription and $g = g_s\sqrt{N_c}$ is the (finite) ‘t Hooft coupling with mass dimensions which plays the role of $\Lambda_{\text{QCD}}$. Together with the quark masses, it generates the meson masses. The ‘t Hooft potential, Eq. (11), is derived by summing an infinite number of planar ladder and rainbow diagrams at $x^+ = 0$, and in the light-front gauge, $A^+ = 0$.

Using the fact that, $k^+ = xP^+$, is conjugate to the light-front distance, $x^-$, the Fourier transformation of Eq. (11) yields

$$U_L(x^-) = \frac{g^2}{2}P^+|x^-|, \quad (12)$$

and, since $x^+ = 0$, we can rewrite Eq. (12) as

$$U_L(b_q) = g^2P^+|b_q|, \quad (13)$$

where we have chosen the notation $x^3 \equiv b_q$. Therefore, in the meson’s rest frame, where $P^+ = M$, the ‘t Hooft potential corresponds to the Coulomb potential which is linear in one space dimension.

The end-point analysis of the ‘t Hooft Equation with $m_q = m_q = m$, using the ansatz $\chi(x) = x^\beta(1-x)^\beta$ yields the transcendental equation:

$$\frac{m^2\pi}{g^2} - 1 + \pi\beta \cot\pi\beta = 0. \quad (14)$$

We note that, in the chiral limit, when $m \to 0$ with $g$ fixed, then Eq. (14) implies that $\beta = 0$ so that $X(x) = \sqrt{x(1-x)}$, i.e. the longitudinal mode of light-front holography is reproduced. In the same limit, it is known [38, 42] that the ‘t Hooft Equation predicts that $M^2 = m_{u/d}$, which is consistent with the Gell-Mann-Oakes-Renner relation [43]. It is also known [44] that, in the heavy quark limit, the ‘t Hooft Equation predicts that $f_M \propto m_Q^{1/2}$, as expected from Heavy-Quark-Effective-Theory (HQET) [45]. It is also worth noting that in a carefully constrained conformal limit, $m \to 0$ and $g \to 0$, the ‘t Hooft Equation possesses a gravity dual in AdS$_3$ [46]. In our approach, these results carry over to (3+1)-dim since the holographic Schrödinger Equation gives no contribution to the pion mass (see Eq. (7)) and the meson decay constant is only sensitive to the meson wavefunction, Eq. (2), evaluated at $\zeta = 0$ [19].

Unlike the holographic Schrödinger Equation, the ‘t Hooft Equation must be solved numerically. Following Ref. [37], we expand the longitudinal mode onto a Jacobi polynomial basis:

$$\chi(x) = \sum_n c_n f_n(x) \quad (15)$$

with

$$f_n(x) = N_n x^{\tilde{\beta}_1}(1-x)^{\tilde{\beta}_2} P_n^{(2\tilde{\beta}_2,2\tilde{\beta}_1)}(2x-1), \quad (16)$$

where $P_n^{(2\tilde{\beta}_2,2\tilde{\beta}_1)}$ are the Jacobi polynomials and [47]

$$N_n = \sqrt{\frac{(2n+\tilde{\beta}_1+\tilde{\beta}_2)!}{\Gamma(n+\tilde{\beta}_1+\tilde{\beta}_2)}} \times \sqrt{\frac{\Gamma(n+\tilde{\beta}_1+\tilde{\beta}_2)}{\Gamma(n+1)}} \quad (17)$$

with $\tilde{\beta}_1 \equiv 2\beta_1$ and $\tilde{\beta}_2 \equiv 2\beta_2 + 1$. The resulting matrix representation of Eq. (10) can then be diagonalized numerically. Note that we require our predictions to be independent of the choice of basis, i.e. to remain stable with respect to variations in $\beta_{1,2}$.

3. Predicting the meson spectrum

We compute the meson mass spectrum using [37, 40]

$$M^2(n_L, n_T, J, L) = M_{T}^2(n_T, J, L) + M_{L}^2(n_L) \quad (18)$$

where $M_T^2(n_T, J, L)$ and $M_L^2(n_L)$ are the eigenvalues of Eq. (3) and Eq. (10) respectively. Using the light-front parity and charge conjugation operators given in Ref. [48], we predict the parity and charge conjugation quantum numbers to the meson to be $P = (-1)^{J+1}$ and $C = (-1)^{L+S+n_L}$ respectively. We note that $n_L \geq n_T + L$, i.e. in any hadron, an orbital and radial excitations in the transverse dynamics is always accompanied by an excitation in the longitudinal dynamics. Before showing our numerical predictions, we make two comments that are important to interpret our results.

First, we use the universal holographic mass scale: $\kappa = 0.523$ GeV for all mesons. Besides
Using Eq. (18), we predict that
\[ g \text{ heavy mesons and } \]

\[ U \text{ Eq. (24): } \]

the linear term dominates the right-hand-side of potentials respectively. For heavy-heavy mesons, corresponding to the holographic and ‘t Hooft potentials.

\[ \text{for heavy-heavy mesons, } \]

\[ \text{for heavy-light mesons. This can be understood starting from the general relation between a light-front and an instant-form potential in the confinement region (where kinetic energy is minimal): } \]

\[ \text{using the BdT prescription for light mesons [11]. The BdT prescription can then be thought as resulting from the ‘t Hooft Equation with a weak longitudinal coupling, } g \ll \kappa; \text{ see Table 1. Our predictions for the heavy-heavy mesons are shown in Fig. 1. The agreement with data is very good. The quality of agreement with data is similar to that achieved using the BdT prescription for light mesons [11].} \]

\[ \text{Fig. 2. The agreement with data is also very good, with } g = \kappa, \text{ as we anticipated from the restoration of rotational symmetry. Finally, our predictions for heavy-light mesons are shown in Fig. 3. As can be seen, the agreement with data is good, although less impressive (even though the maximum discrepancy never exceeds 10%) than for the light and heavy-heavy mesons. For heavy-light mesons, the data prefer } g > \kappa; \text{ see Table 1. Interestingly, } g \text{ deviates from } \kappa \text{ in opposite directions for light and heavy-light mesons. The underlying reason for this remains to be explored.} \]

\[ \text{We emphasize that, except for pseudoscalar mesons in their ground states (for which } M_T = 0), \text{ the precise locations and slopes of the Regge trajectories of all other mesons are sensitive to both } g \text{ and } \kappa. \text{ Therefore the universality of } \kappa \text{ across the full spectrum is non-trivial.} \]

\[ \text{4. Conclusions} \]

\[ \text{We have shown that the meson spectrum can be very well described by using the holographic Schrödinger Equation in conjunction with the ‘t Hooft Equation. We find that the emerging hadronic scale of light-front holography remains universal across the full spectrum. For heavy-heavy mesons, it coincides with the ‘t Hooft coupling, as expected from the restoration of manifest 3-dimensional rotational symmetry in the non-relativistic limit.} \]

\[ \text{Table 1: The quark masses and ‘t Hooft couplings in GeV. Note that we use } \kappa = 0.523 \text{ GeV for all mesons.} \]
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