PERTURBED INTERPOLATION FORMULAE AND APPLICATIONS

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Abstract. We employ functional analysis techniques in order to deduce that some classical and recent interpolation results in Fourier analysis can be suitably perturbed. As an application of our techniques, we obtain generalizations of Kadec’s 1/4–theorem for interpolation formulae in the Paley–Wiener space both in the real and complex case, as well as a perturbation result on the recent Radchenko–Viazovska interpolation result [24] and the Cohn–Kumar–Miller–Radchenko–Viazovska [37] result for Fourier interpolation with derivatives in dimensions 8 and 24. We also provide several applications of the main results and techniques, all relating to recent contributions in interpolation formulae and uniqueness sets for the Fourier transform.

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1. Introduction

A fundamental question in analysis is that of how to recover a function $f$ from some subset $\{f(x)\}_{x \in A}$ of its values, together with some information on its Fourier transform $\hat{f}: \mathbb{R} \to \mathbb{C}$, which we define to be

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x)e^{-2\pi ix\xi} \, dx.$$  

The perhaps most classical result in that regard is the Shannon–Whittaker interpolation formula: if $\hat{f}$ is supported on an interval $[-\delta/2, \delta/2]$, then

$$f(x) = \sum_{k=-\infty}^{\infty} f(k/\delta)\text{sinc}(\delta x - k),$$

where convergence holds both in $L^2(\mathbb{R})$ and uniformly, where we let $\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$.

In spite of this classical formula, a major recent breakthrough in regard to the problem of determining which conditions on the sets $A, B \subset \mathbb{R}$ imply that a function $f \in \mathcal{S}(\mathbb{R})$ is uniquely determined by its values at $A$ and the values of its Fourier transform at $B$
was made by Radchenko and Viazovska [24], where the authors prove that, whenever $f : \mathbb{R} \to \mathbb{R}$ is even and Schwartz, then

$$f(x) = \sum_{k=0}^{\infty} f(\sqrt{k})a_k(x) + \sum_{k=0}^{\infty} \hat{f}(\sqrt{k})\hat{a}_k(x).$$

Radchenko and Viazovska’s result and its techniques were somewhat inspired by Viazovska’s recent solution to the sphere packing problem in dimension 8 [31], and her subsequent work with Cohn, Kumar, Miller and Radchenko to solve the same problem in dimension 24 [9], as they include the usage of modular forms in order to construct some special functions with particular properties at the desired nodes of interpolation.

Subsequently to the Radchenko–Viazovska result, other recent works have successfully used a similar approach in order to construct interpolation and uniqueness formulae. Among those, we mention the following:

1. In [8], Cohn and Gonçalves use a modular form construction in order to obtain that there are $c_j > 0$, $j \in \mathbb{N}$, so that, for each $f \in S_{\text{rad}}(\mathbb{R}^{12})$ real,

$$f(0) - \sum_{j \geq 1} c_j f(\sqrt{2j}) = -\hat{f}(0) + \sum_{j \geq 1} c_j \hat{f}(\sqrt{2j}).$$

Such a formula enables the authors to prove a sharp version of a root uncertainty principle first raised by Bourgain, Clozel and Kahane [4] in dimension 12; see, e.g., [15, 13, 14] and the references therein for more information on this topic;

2. On the other hand, in [10], Cohn, Kumar, Miller, Radchenko and Viazovska develop upon the basic ideas of [24] to be able to prove universal optimality results about the $E_8$ and Leech lattices in dimensions 8 and 24, respectively. In order to do so, they prove interpolation formulae in such dimensions that involve the values of $f(\sqrt{2n})$, $f'(\sqrt{2n})$, $\hat{f}(\sqrt{2n})$, $\hat{f}'(\sqrt{2n})$, where $f$ is a radial, Schwartz function, and $n \geq n_0$, with $n_0 = 1$ if $d = 8$, and $n_0 = 2$ in case $d = 24$;

3. Finally, more recently, other developments in the theory of interpolation formulae given values on both Fourier and spatial side has been made by Stoller [29], who considered the problem of recovering any function in $\mathbb{R}^d$ from its restrictions and the restrictions of its Fourier transforms to spheres of radii $\sqrt{n}$, $n > 0$, for any $d > 0$. Moreover, we mention also the more recent work of Bondarenko, Radchenko and Seip [3], which generalizes Radchenko and Viazovska’s construction of the interpolating functions to prove interpolation formulae for some classes of functions $f$ that take into account the values of $\hat{f}$ at log $n/4\pi$, and the values of $f$ at a sequence $(\rho - 1/2)/i$, where $\rho$ ranges over non-trivial zeros of some $L-$function with positive imaginary part.

One fundamental point to stress is that, in a suitable way, all the previously mentioned results are related to some sort of summation formula, the most basic instance
of such being the classical Poisson summation formula
\[ \sum_{m \in \mathbb{Z}} f(m) = \sum_{n \in \mathbb{Z}} \hat{f}(n), \]
which is a particular case, for instance, of \((1.3)\) in case we set \(x = 0\). Clearly, the formula \((1.4)\) is also a manifestation of such a principle that implies rigidity between certain values of \(f\) and other values of \(\hat{f}\).

In that regard, these topics can be inserted into the framework of crystalline measures. Indeed, if we adopt the classical definition of a crystalline measure to be a distribution with locally finite support, such that its Fourier transform possesses the same support property, we will see that the Poisson summation formula implies, for instance, that the measure \(\delta_Z\) is not only a crystalline measure, but also \textit{self-dual}, in the sense that \(\delta_Z = \hat{\delta}_Z\) holds in \(\mathcal{S}'(\mathbb{R})\).

Outside the scope of interpolation formulae per se, we mention the works \([18, 19, 22]\), where the authors explore in a deeper level structural questions on crystalline measures. In particular, in \([22]\), Meyer exhibits examples of crystalline measures with self-duality properties, and uses modular forms to construct explicitly examples of non-zero self-dual crystalline measures \(\mu\) supported on \(\{\pm \sqrt{k + a}, k \in \mathbb{Z}\}\), for \(a \in \{9, 24, 72\}\).

We also mention the recent work of Kurasov and Sarnak \([17]\), where the authors, as a by-product of investigations of the additive structure of the spectrum of metric graphs, prove that there are exotic examples of \textit{positive} crystalline measures other than generalized Dirac combs.

Our investigation in this paper focuses on both classical and modern results in the theory of such interpolation formulae and crystalline measures. In generic terms, we are interested in determining when, given an interpolation formula such as \((1.2)\) or \((1.3)\), we can \textit{perturb} it suitably. That is, given a sequence of real numbers \(\{\varepsilon_k\}_{k \in \mathbb{Z}}\), under which conditions can we recover \(f\) from the values
\[ \{(f(s_n + \varepsilon_n), \hat{f}(\hat{s}_n + \varepsilon_n))\}_{n \in \mathbb{Z}}, \]
given that we can recover \(f\) from \(\{(f(s_n), \hat{f}(\hat{s}_n))\}_{n \in \mathbb{Z}}\)?

In this manuscript, the main idea is to study such perturbations of interpolation formulae for band-limited and Schwartz functions through functional analysis. Indeed, most of our considerations are based off the idea that, whenever an operator \(T : B \to B\), where \(B\) is a Banach space, satisfies that
\[ \|T - I\|_{B \to B} < 1, \]
then \(T\) is, in fact, a \textit{bijection} with continuous inverse \(T^{-1} : B \to B\). In fact, in all our considerations on interpolation formulae below, some form of this principle will be employed, and even the importance of other proofs and results in the paper, such as Theorem 1.5, arise naturally when trying to employ this principle to different contexts.

### 1.1. Perturbations and Interpolation formulae in the band-limited case.

The question of when we are able to recover the values of a function such that its Fourier transform is supported in \([-1/2, 1/2]\) from its values at \(n + \varepsilon_n\) is well-known, having
been asked by Paley and Wiener [23], where the authors prove that recovery – and also an associated interpolation formula – is possible as long as $\sup_n |\varepsilon_n| < \pi^{-2}$. Many results relate to the original problem of Paley and Wiener, but the most celebrated of them all is the so-called Kadec-$1/4$ theorem, which states that, as long as $\sup_n |\varepsilon_n| < \frac{1}{4}$, then one can recover any $f \in L^2(\mathbb{R})$ which has Fourier support on $[-1/2, 1/2]$ from its values at $n + \varepsilon_n$, $n \in \mathbb{Z}$; see [16] for Kadec’s original proof and [1] for a generalization.

Our first results provide one with a simpler proof of a particular range of Kadec’s result.

**Theorem 1.1.** Let $\{\varepsilon_k\}_{k \in \mathbb{Z}}$ be a sequence of real numbers and consider $L = \sup_k |\varepsilon_k|$. If $L < 1/2$ and

$$1 - \frac{\sin(\pi L)}{\pi L} + \frac{\pi L \sin \pi L}{3(1-L)} + \sin \pi L < 1,$$

then any function $f \in PW_\pi$ is completely determined by its values \{f(n+\varepsilon_n)\}_{n \in \mathbb{Z}} and there is $C = C(L) > 0$ such that

$$\frac{1}{C} \sum_{n \in \mathbb{Z}} |f(n+\varepsilon_n)|^2 \leq \|f\|_2^2 \leq C \sum_{n \in \mathbb{Z}} |f(n+\varepsilon_n)|^2,$$

for all $f \in PW_\pi$.

Moreover, there are functions $g_n \in PW_\pi(\mathbb{R})$ such that for every $f \in PW_\pi$, the following identity holds:

$$f(x) = \sum_{n \in \mathbb{Z}} f(n+\varepsilon_n)g_n(x),$$

where the right-hand side converges absolutely.

The condition in Theorem 1.1 is satisfied for $L < 0.239$, which possesses only a 0.011 gap to Kadec’s result. The main difference, however, that while Kadec’s proof relies on a clever expansion of the underlying functions in a different orthonormal basis, we have almost not used orthogonality in our considerations. We have, nonetheless, chosen not to pursue the path of exploring orthogonality in this question much deeper in order not to make the exposition longer.

We must also remark that, in the proof of such a result, one can use complex numbers for perturbations. The difference is that we have to take into account the sine of complex numbers, and the result would be $L < 0.2125$ instead of $L < 0.239$. This only falls very mildly short of the results in [1, Theorem 3], where $L < 0.218$ is achieved in the complex setting, and our methods of proof are relatively simpler in comparison to those of [1], where the authors must enter the realm of Lamb-Oseen functions and constants. Also, we do not make any use of the orthogonality, which could be exploited to improve on the current result.

As another application of the idea of inverting an operator, we mention a couple of results related to Vaaler’s interpolation formula. In [30], J. Vaaler proved, as means to study extremal problems in Fourier analysis, the following counterpart to the Shannon–Whittaker interpolation formula: let $f \in L^2(\mathbb{R})$, and suppose that $\hat{f}$ is supported on
Then
\[\frac{\sin^2(\pi x)}{\pi^2} \sum_{k \in \mathbb{Z}} \left\{ \frac{f(k)}{(x-k)^2} + \frac{f'(k)}{x-k} \right\}.\]

This can be seen as a natural tradeoff: (1.2) demands that we have information at \( \frac{1}{2} \mathbb{Z} \) in order to recover the functions \( f \) as stated above. On the other hand, Vaaler’s result only demands information at \( \mathbb{Z} \), but one must pay the price of also providing it for the derivative.

The first result concerning (1.6) is a direct deduction of its validity from the Shannon–Whittaker formula (1.2). We state it, for completeness, in the following form.

**Theorem 1.2.** Fix a sequence \( \{a_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z}) \). Consider the function \( f \in PW_{\pi}^2 \) given by
\[f(x) = \sum_{n \in \mathbb{Z}} a_n \text{sinc}(x - n),\]
for each \( x \in \mathbb{R} \). Then the interpolation formula
\[f(x) = 4 \frac{\sin^2(\frac{1}{2} \pi x)}{\pi^2} \sum_{j \in \mathbb{Z}} \left\{ \frac{a_{2k}}{(x-2k)^2} + \frac{b_{2k}}{x-2k} \right\}\]
holds, where the right-hand side converges uniformly on compact sets, and we let
\[b_k = \sum_{j \neq k} \frac{a_j}{k-j} (-1)^{k-j}.\]

As a main difference between our proof of Theorem 1.2 and the original proof in [30] is the absence of any significant use of the Fourier transform. Differently, however, from the de Branges spaces approach in [11], we do not delve deeply into any theory of function spaces, but rather we make use of classical operators in \( \ell^2(\mathbb{Z}) \) such as discrete Hilbert transforms and its properties.

Our final contribution in the realm of interpolation formulae for band-limited function is an appropriate perturbation of Vaaler’s formula (1.6). We mention that, to the best of our knowledge, this result in its present form is new. See, for instance, the remark following Corollary 2 in [11] together with [21, 27] for related discussion on sampling sequences with derivatives for \( PW_{\pi}^2 \).

**Theorem 1.3.** Let \( \{\varepsilon_k\}_{k \in \mathbb{Z}} \) be a sequence of real numbers and consider \( L = \sup_k |\varepsilon_k| \).
Suppose that \( L < 0.111 \). Then any function \( f \in PW_{2\pi} \) is completely determined by its values \( \{f(n + \varepsilon_n)\}_{n \in \mathbb{Z}} \) and those of its derivative \( \{f'(n + \varepsilon_n)\}_{n \in \mathbb{Z}} \), and there is \( C = C(L) > 0 \) such that
\[\frac{1}{C} \sum_{n \in \mathbb{Z}} (|f(n + \varepsilon_n)|^2 + |f'(n + \varepsilon_n)|^2) \leq \|f\|_2^2 \leq C \sum_{n \in \mathbb{Z}} (|f(n + \varepsilon_n)|^2 + |f'(n + \varepsilon_n)|^2),\]
for all \( f \in PW_{2\pi} \).

Moreover, there are functions \( g_n, h_n \in PW_{2\pi} \) so that, for all \( f \in PW_{2\pi} \), we have
\[f(x) = \sum_{n \in \mathbb{Z}} \left\{ f(n + \varepsilon_n)g_n(x) + f'(n + \varepsilon_n)h_n(x) \right\},\]
where convergence holds absolutely.

This result and its method of proof follow, essentially, the same basic ideas from Theorem 1.1 and its proof, with only an increase in technical difficulties, such as considering higher order analogues of the perturbed discrete Hilbert transforms we use for the proof of 1.1. We note also that these technical changes, together with the work of Littman [20], allow one to extend the perturbation results for arbitrarily many derivatives; see Theorem 6.1 for a discussion on that. In order to avoid the not so pleasant computations needed in order to prove such a result, and due to the fact that its proof follows the main ideas of the proofs of theorems 1.3 and 1.1, we omit it.

1.2. Perturbations of symmetric interpolation formulae. Moving on from band-limited functions to Schwartz functions instead, we face the fundamental question of determining whether formula (1.3) is rigid for its interpolation nodes or not. In other words, a fundamental question concerns conditions when we can replace a single interpolation node \( \sqrt{k} \) by a suitable perturbation of it, say \( \sqrt{k + \varepsilon_k} \), where \( \varepsilon_k \in (-1, 1) \).

Perhaps surprisingly, the idea of inverting an operator \( T \) when it is reasonably close to the identity still works in this context. The next result can thus be regarded as the main new feature of this paper, establishing criteria when we are allowed, not only to perturb one node in the interpolation formula, but all of them simultaneously.

**Theorem 1.4.** There is \( \delta > 0 \) so that, for each sequence of real numbers \( \{\varepsilon_k\}_{k \geq 0} \) such that \( \varepsilon_k \in (-1/2, 1/2), \varepsilon_0 = 0, \sup_{k \geq 0} |\varepsilon_k|(1 + k)^{5/4} < \delta \forall k \geq 0 \), there are sequences of functions \( \{\theta_j\}_{j \geq 0}, \{\eta_j\}_{j \geq 0} \) with

\[
|\theta_j(x)| + |\eta_j(x)| + |\hat{\theta}_j(x)| + |\hat{\eta}_j(x)| \lesssim (1 + j)^{O(1)}(1 + |x|)^{-10}
\]

and

\[
f(x) = \sum_{j \geq 0} \left( f(\sqrt{j + \varepsilon_j})\theta_j(x) + \hat{f}(\sqrt{j + \varepsilon_j})\eta_j(x) \right),
\]

for all \( f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \) real-valued functions.

In other words, we can perturb each interpolation node from \( \sqrt{k} \) to \( \sim \sqrt{k + k^{-5/4}} \) and still obtain a valid interpolation formula converging for all Schwartz functions. In fact, one does not strictly need that \( f \in \mathcal{S}(\mathbb{R}) \), but only that \( f, \hat{f} \) decay at least as fast as \( (1 + |x|)^{-M} \) for some sufficiently large \( M \gg 1 \).

As an immediate corollary of Theorem 1.4 we obtain that the continuous family of measures

\[
\mu_x = \frac{\delta_x + \delta_{-x}}{2} - \sum_{j \geq 0} \frac{\theta_j(x)}{2} \delta_{\pm \sqrt{j + \varepsilon_j}}
\]

possesses Fourier transform given by

\[
\hat{\mu}_x = \sum_{j \geq 0} \frac{\eta_j(x)}{2} \delta_{\pm \sqrt{j + \varepsilon_j}},
\]

whenever \( \{\varepsilon_i\}_{i \geq 0} \) satisfies the hypotheses of Theorem 1.4. This follows from the fact that \( \mu_x \) is even and real-valued, so that its distributional Fourier transform will also be
an even and real-valued distribution. Therefore, it suffices to test against even, real-valued functions $f$, and thus Theorem 1.4 gives us the asserted equality. This provides one with a new class of nontrivial examples of crystalline measures supported on both space and frequency on basically any set of the form $\pm \sqrt{k + \varepsilon_k}, |\varepsilon_k| \leq \delta k^{-5/4}$. This, in particular, aligns well with the recent examples from [3] and [17], which indicate that crystalline measures are, if not impossible, very hard to classify.

In order to prove Theorem 1.4, we need to find a suitable space to use the idea of inverting operators close to the identity. It turns out that, in analogy to Sobolev spaces, the weighted spaces $\ell^2_s(N)$ of sequences square summable against $n^s$ are natural candidates to work with, as it is well suited to accommodate the sequence

$$\{(f(\sqrt{k + \varepsilon_k}), \hat{f}(\sqrt{k + \varepsilon_k}))\}_{k \geq 0}$$

whenever $f, \hat{f}$ decay sufficiently fast. In order to prove some perturbation result – that is, a weaker version of Theorem 1.4 –, using the spaces $\ell^2_s(N)$ together with the polynomial growth bounds on $\{a_n\}_{n \geq 0}$ from (1.3) is already enough.

On the other hand, the fact that me may push the perturbations up until the $k^{-5/4}$ threshold needs a suitable refinement to the Radchenko–Viazovska [24] or even to the Bondarenko–Radchenko–Seip [3] bounds. The next result, thus, provides us with an additional exponential factor that mitigates growth of the interpolating functions.

**Theorem 1.5.** Let $b_n^\pm = a_n \pm \hat{a}_n$, where $\{a_n\}_{n \geq 0}$ are the basis functions in (1.3). Then there is an absolute constant $c > 0$ such that

$$|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n)e^{-c|x|},$$

$$|(b_n^\pm)'(x)| \lesssim n^{3/4} \log^{3/2}(1 + n)e^{-c|x|},$$

for all positive integers $n \in \mathbb{N}$.

The proof of such a result employs a mixture of the main ideas for the uniform bounds in [24] and [3], with the addition of an explicit computation of the best uniform constant bounding $|x|^k |b_n^\pm(x) + (b_n^\pm)'(x)|$ in terms of $k$ and $n$. In order to obtain such a constant, we employ ideas from characterizations of Gelfand–Shilov spaces, as in [7]. Finally, with a modification of the growth lemma for Fourier coefficients of 2–periodic functions, we are able to obtain a slight improvement over the growth stated in Theorem 1.5. As, however, this modification does not yield any improvement on the perturbation range stated in Theorem 1.4 we postpone a more detailed discussion about it to Corollary 4.7 below.

1.3. **Applications.** As a by-product of our method of proof for Theorem 1.4 we are able to deduce some interesting consequences in regard to some other interpolation formulae and uniqueness results. Indeed, it is a not so difficult task to adapt the ideas employed before to the contexts of interpolation formulae for odd functions. As remarked by Radchenko and Viazovska,
the following interpolation formula is available whenever $f : \mathbb{R} \to \mathbb{R}$ is odd and belongs to the Schwartz class:

$$f(x) = d_0^+(x)f'(0) + i\hat{f}'(0) + \sum_{n \geq 1} c_n(x)\frac{f(\sqrt{n})}{\sqrt{n}} - \hat{c}_n(x)\frac{\hat{f}(\sqrt{n})}{\sqrt{n}} ,$$

where the interpolating sequence $\{c_i\}_{i \geq 0}$ possesses analogous properties to those of $\{a_i\}_{i \geq 0}$, and the function $d_0^+(x) = \frac{\sin(\pi x^2)}{\sinh(\pi x)}$ is odd, real and so that it vanishes together with its Fourier transform at $\pm \sqrt{n}, n \geq 0$.

With our techniques, we are able to prove an analogous result to Theorems 1.5 and 1.4 for the odd interpolation formula. Also, with our techniques, we are able to perturb the Cohn–Kumar–Miller–Radchenko–Viazovska interpolation results with derivatives in dimensions 8 and 24 in a suitable range, as polynomial growth bounds for such interpolating functions are available in [10]; see theorems 5.10 and 5.12 for more details.

Another interesting application of our techniques delves a little deeper into functional analysis techniques. Indeed, in order to prove that the operator that takes the set of values $\{f(\sqrt{k})\}_{k \geq 0}$, $\{\hat{f}(\sqrt{k})\}_{k \geq 0}$ to the sequences

$$\{f(\sqrt{k} + \varepsilon_k)\}_{k \geq 0}, \{\hat{f}(\sqrt{k} + \varepsilon_k)\}_{k \geq 0}$$

is bounded and close to the identity on a suitable $\ell^2(N) \times \ell^2(N)$ space, we explore two main options, which are Schur’s test and the Hilbert–Schmidt test. Although there is no direct relation between them, Schur’s test seems to hold, in generic terms, for more operators than the Hilbert–Schmidt test, and for that reason we employ the former in our proof of Theorem 1.4. On the other hand, the Hilbert–Schmidt test has the advantage that, whenever an operator is bounded in the Hilbert–Schmidt norm, it is automatically a compact operator. This allows us to use many more tools derived from the theory of Fredholm operators, and, in particular, deduce a sort of interpolation/uniqueness result in case $\varepsilon_0 \neq 0$, which is excluded by Theorem 1.4 above; see Theorem 5.3 below for such an application.

The perhaps most interesting and nontrivial application of Theorem 1.4 and its techniques is to the problem of Fourier uniqueness for powers of integers. In [25], we have proven a preliminary result on conditions on $(\alpha, \beta), 0 < \alpha, \beta, \alpha + \beta < 1$, so that the only $f \in S(\mathbb{R})$ such that

$$f(\pm n^\alpha) = \hat{f}(\pm n^\beta) = 0$$

is $f \equiv 0$. In particular, we prove that, if $\alpha = \beta$, then we can take $\alpha < 1 - \frac{\sqrt{2}}{2}$.

By an approximation argument, a careful analysis involving Laplace transforms and the perturbation techniques and results above, we are able to reprove such a result for $\alpha = \beta$ in the $\alpha < \frac{\varepsilon}{2}$ range in case $f$ is real and even by a completely different method than that in [25]. Although the current method does not yield any improvement over [25, Theorem 1], we believe it is a promising path towards proving that the wished uniqueness result holds in the $0 < \alpha, \beta < \frac{1}{2}$ range. We refer the reader to Corollary 5.8 below and the discussion that succeeds it for more precise statements.
1.4. Organization. We comment briefly on the overall display of our results throughout the text. In Section 2 below, we discuss generalities on background results needed for the proofs of the main Theorems, going over results in the theory of band-limited functions, modular forms and functional analysis. Next, in Section 3 we prove, in this order, theorems 1.1, 1.2 and 1.3 about band-limited perturbed interpolation formulae. We then prove, in Section 4 Theorem 1.4, by first discussing the proof Theorem 1.5 in §4.1. We then discuss the applications of our main results and techniques in Section 5 and finish the manuscript with Section 6, talking about some possible refinements and open problems that arise from our discussion throughout the paper.

2. Preliminaries

2.1. Band-limited functions. We start by recalling some basic facts about band-limited functions. Given a function \( f \in L^2(\mathbb{R}) \), we say that it is band-limited if its Fourier transform satisfies that \( \text{supp}(\hat{f}) \subset [-M,M] \) for some \( M > 0 \). In this case, we say that \( f \) is band-limited to \([-M,M]\).

It is a classical result due to Paley and Wiener that a function \( f \in L^2(\mathbb{R}) \) is band-limited if and only if it is the restriction of an entire function \( F : \mathbb{C} \to \mathbb{C} \) to the real axis, and the function \( F \) is of exponential type; that is, there exists \( \sigma > 0 \) so that, for each \( \varepsilon > 0 \),

\[
|F(z)| \leq C_{\varepsilon} e^{(\sigma+\varepsilon)|z|},
\]

for all \( z \in \mathbb{C} \). From now on we will abuse notation and let \( F = f \) whenever there is no danger of confusion, and we may also write \( f \in PW_\sigma \) (Paley–Wiener space) to denote the space of functions with such properties.

Besides this fact, we will make use of some interpolation formulae for those functions. Namely,

1. **Shannon–Whittaker interpolation formula.** For each \( f \in L^2(\mathbb{R}) \) band-limited to \([-\frac{1}{2},\frac{1}{2}]\), the following formula holds:

\[
f(x) = \sum_{n \in \mathbb{Z}} f(n) \text{sinc}(x - n),
\]

where \( \text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \) and the sum above converges both in \( L^2(\mathbb{R}) \) and uniformly on compact sets of \( \mathbb{C} \).

2. **Vaaler interpolation formula.** For each \( f \in L^2(\mathbb{R}) \) band-limited to \([-1,1]\), the following formula holds:

\[
f(x) = \left( \frac{\sin(\pi x)}{\pi} \right)^2 \sum_{n \in \mathbb{Z}} \left[ \frac{f(n)}{(x-n)^2} - \frac{f'(n)}{x-n} \right],
\]

where the right-hand side converges both in \( L^2(\mathbb{R}) \) and uniformly on compact sets of \( \mathbb{C} \).

For more details on these classical results, see, for instance, [30], [20], [23], [28] and [32].

2.2. Modular forms. In order to prove the improved estimates on the interpolation basis for the Radchenko–Viazovska interpolation result, we will need to make careful
computations involving certain modular forms defining the interpolating functions. For that purpose, we gather some of the facts we will need in this subsection.

We denote by $\mathbb{H} = \{z \in \mathbb{C} : \text{Im}(z) > 0\}$ the upper half plane in $\mathbb{C}$. The special feature of this space is that the group $SL_2(\mathbb{R})$ of matrices with real coefficients and determinant 1 acts naturally on it through Möbius transformations: for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, $z \in \mathbb{H} \Rightarrow \gamma z = \frac{az + b}{cz + d} \in \mathbb{H}$.

For our purposes, it will suffice to look at the subgroup $PSL_2(\mathbb{Z}) = SL_2(\mathbb{Z})/\{\pm I\}$.

Some elements of this group will be of special interest to us. Namely, we let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

This already allows us to define the most valuable subgroup of $SL_2(\mathbb{Z})$ for us: the group $\Gamma_0$ is defined then as the subgroup of $SL_2(\mathbb{Z})$ generated by $S$ and $T^2$. This group has 1 and $\infty$ as cusps, and its standard fundamental domain is given by $D = \{z \in \mathbb{H} : |z| > 1, \text{Re}(z) \in (-1, 1)\}$.

With these at hand, we define modular forms for $\Gamma_0$. For that purpose, we will use the following notation for the Jacobi theta series:

$$\vartheta(z, \tau) = \sum_{n \in \mathbb{Z}} \exp(\pi i n^2 \tau + 2\pi i n z).$$

We are interested in some of its Nullwerte, the so-called Jacobi theta series. These are defined in $\mathbb{H}$ by

$$\Theta_2(\tau) = \exp\left(\frac{\pi i \tau}{4}\right) \vartheta\left(\frac{1}{2}, \tau\right),$$

$$\Theta_3(\tau) = \vartheta(0, \tau)(=: \theta(\tau)), $$

$$\Theta_4(\tau) = \vartheta\left(\frac{1}{2}, \tau\right).$$

These functions satisfy the identity $\Theta_4^4 = \Theta_2^4 + \Theta_4^4$. Moreover, under the action of the elements $S$ and $T$ of $SL_2(\mathbb{Z})$, they transform as

$$(-iz)^{-1/2} \Theta_2(-1/z) = \Theta_4(z), \quad \Theta_2(z + 1) = \exp(i\pi/4)\Theta_2(z),$$

$$(-iz)^{-1/2} \Theta_3(-1/z) = \Theta_3(z), \quad \Theta_3(z + 1) = \Theta_4(z),$$

$$(-iz)^{-1/2} \Theta_4(-1/z) = \Theta_2(z), \quad \Theta_4(z + 1) = \Theta_3(z).$$

(2.1)

These functions allow us to construct the classical lambda modular invariant given by

$$\lambda(z) = \frac{\Theta_2(z)^4}{\Theta_3(z)^4}.$$
Using the nome \( q = q(z) = e^{\pi i z} \), the lambda invariant can be alternatively rewritten as

\[
\lambda(z) = 16q \times \prod_{k=1}^{\infty} \left( \frac{1 + q^{2k}}{1 + q^{2k-1}} \right)^8 = 16q - 128q^2 + 704q^3 + \cdots.
\]

Besides this alternative formula, this is also invariant under the action of elements of the subgroup \( \Gamma(2) \subset SL_2(\mathbb{Z}) \) of all matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) so that \( a \equiv b \equiv 1 \mod 2 \), \( c \equiv d \equiv 0 \mod 2 \). Besides this invariance, (2.1) gives us immediately that

\[
\lambda(z + 1) = \frac{\lambda(z)}{1 - \lambda(z)}, \quad \lambda\left(-\frac{1}{z}\right) = 1 - \lambda(z).
\]

We then define the modular invariant function for \( \Gamma_\theta \) to be

\[
J(z) = \frac{1}{16} \lambda(z)(1 - \lambda(z)).
\]

From (2.3), we obtain immediately that \( J \) is invariant under the action of elements of \( \Gamma_\theta \); i.e.,

\[
J(z + 2) = J(z), \quad J\left(-\frac{1}{z}\right) = J(z).
\]

Other properties of the functions \( \lambda \) and \( J \) that we may eventually need will be proved throughout the text.

Finally, we mention that, for the proof in \([4]\) we will need to use the so-called \( \theta \)-automorphy factor defined, for \( z \in \mathbb{H} \) and \( \gamma \in \Gamma_\theta \), as

\[
j_\theta(z, \gamma) = \frac{\theta(z)}{\theta(\gamma z)}.
\]

With this in hands, we defined a slash operator of weight \( k/2 \) to be

\[
(f|_{k/2}\gamma)(z) = j_\theta(z, \gamma)^k f\left(\frac{az + b}{cz + d}\right),
\]

where \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \). These slash operators induce other sign slash operators given by

\[
(f|_{k/2}\gamma) = \chi_\varepsilon(\gamma)(f|_{k/2}\gamma),
\]

where we let \( \chi_\varepsilon \) be the homomorphism of \( \Gamma_\theta \) so that \( \chi_\varepsilon(S) = \varepsilon, \chi_\varepsilon(T^2) = 1 \).

For more information on the functions \( \lambda, J \) and the automorphy factors we just defined, we refer the reader to \([3]\) and \([24, \text{Section 2}]\); see also \([2], [34]\).

2.3. Functional analysis. We also recall some classical facts in functional analysis that will be useful throughout our proof.

As our main goal and strategy throughout this manuscript is to prove that a small perturbation of the identity is invertible, we must find ways to prove that the operators arising in our computations are bounded. To that extent, we use two major criteria to prove boundedness – and therefore to prove smallness of the bounding constant. These are:
(1) *Hilbert-Schmidt test.* Let $H$ be a Hilbert space, and let there be given a linear operator $T : H \to H$. If $T$ satisfies additionally that
\[
\sum_{i,j} |\langle Te_j, e_i \rangle|^2 < +\infty
\]
for some orthonormal basis $\{e_i\}_{i \in \mathbb{Z}}$ of $H$, then the operator $T$ is bounded. Moreover,
\[
\|T\|_{H \to H}^2 \leq \sum_{i,j} |\langle Te_j, e_i \rangle|^2 =: \|T\|_{HS}^2.
\]

(2) *Schur test.* Let $(a_{ij})_{i,j \geq 0}$ denote an infinite matrix. Suppose that there are two sequences $\{p_i\}_{i \geq 0}$ and $\{q_i\}_{i \geq 0}$ of positive real numbers so that
\[
\sum_{i \geq 0} |a_{ij}|q_i \leq \lambda p_j,
\]
\[
\sum_{j \geq 0} |a_{ij}|p_j \leq \mu q_i,
\]
for some positive constants $\mu, \lambda > 0$. Then the operator $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ given by $a_{ij} = \langle Te_i, e_j \rangle$ (where $\{e_i\}_{i \geq 0}$ denotes the standard orthonormal basis of $\ell^2(\mathbb{N})$) extends to a *bounded* linear operator. Moreover,
\[
\|T\|_{\ell^2 \to \ell^2} \leq \sqrt{\mu \lambda}.
\]

Both tests will play a major role in the deduction of the validity of perturbed interpolation versions of the Radchenko–Viazovska result. The main difference is that, while Schur’s test generally gives one boundedness for more operator, the Hilbert-Schmidt test imposes stronger conditions on the operator. In fact, let us denote by $T \in \mathcal{H}S(H)$ the fact that $\|T\|_{HS} < +\infty$. A classical consequence of this fact is that $T \in \mathcal{K}(H)$; that is, $T$ is compact.

This fact will be used when proving that a suitable version of our interpolation results holds for small perturbations of the origin. See, for instance, [5, Chapter 6]

### 2.4. Notation.
We will use Vinogradov’s modified notation throughout the text; that is, we write $A \lesssim B$ in case there is an absolute constant $C > 0$ so that $A \leq C \cdot B$. If the constant $C$ before depends on some set of parameters $\lambda$, we shall write $A \lesssim_{\lambda} B$.

On the other hand, we shall also use the big-$O$ notation $f = O(g)$ if there is an absolute constant $C$ such that $|f| \leq C \cdot g$, although the usage of this will be restricted mostly to sequences. We may occasionally use as well the standard Vinogradov notation $a \ll b$ to denote that there is a (relatively) large constant $C > 1$ such that $a \leq C \cdot b$.

We shall also denote the spaces of sequences decaying polynomially as
\[
\ell^2_{s}(\mathbb{N}) = \left\{ \{a_n\}_{n \in \mathbb{N}} : \sum_{n \in \mathbb{N}} |a_n|^2 n^{2s} < +\infty \right\}.
\]
Finally, we always normalize our Fourier transform as
\[ \hat{f}(\xi) = \mathcal{F}f(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx. \]

3. Perturbed Interpolation Formulae for Band-Limited Functions

3.1. Perturbed forms of the Shannon–Whittaker formula and Kadec’s result. Fix a sequence \( \varepsilon = \{\varepsilon_k\}_{k \in \mathbb{Z}} \) of real numbers such that \( \sup_k |\varepsilon_k| < 1 \). We wish to obtain a criterion based solely on the value of \( L = \sup_n |\varepsilon_n| \) such that the sequence \( \{n + \varepsilon_n\}_{n \in \mathbb{Z}} \) is completely interpolating in \( \text{PW}_\pi \), i.e., for every sequence \( a = \{a_n\} \in \ell^2(\mathbb{Z}) \) there is a unique \( f \in L^2(\mathbb{R}) \) of exponential type \( \tau(f) \leq \pi \) that satisfies
\[ f(n + \varepsilon_n) = a_n. \]

Our goal here is to obtain a simple proof of such a criterion going through new and simple ideas. We will fall short of the \( 1/4 \) proven by Kadec by approximately 0.11, but it illustrates the power of our perturbation scheme and does not go through the theory of exponential bases.

In this particular case, we need to invert in \( \ell^2(\mathbb{Z}) \) the operator given by
\[ A_\varepsilon(a)(n) = \sum_{k \in \mathbb{Z}} a_k \text{sinc}(n + \varepsilon_n - k), \]
where
\[ \text{sinc}(x) = \frac{\sin \pi x}{\pi x}. \]

The fact \( A_\varepsilon \) is invertible will follow from proving that it is a close perturbation of the identity whenever \( L \) is sufficiently small.

3.1.1. Auxiliary perturbations of the Hilbert transforms. Given a sequence \( a = \{a_k\}_{k \in \mathbb{Z}} \), we define the following operators, which are kin to the discrete Hilbert transform:
\[ \mathcal{H}_\varepsilon(a)(n) = \sum_{k \neq n} \frac{(-1)^{n-k}a_k}{n + \varepsilon_n - k}, \]
\[ \mathcal{H}_0(a)(n) = \sum_{k \neq n} \frac{(-1)^{n-k}a_k}{n - k}. \]

We start by comparing these two objects:
\[ \mathcal{H}_0(a)(n) - \mathcal{H}_\varepsilon(a)(n) = \sum_{k \neq n} (-1)^{n-k}a_k \left( \frac{1}{n - k} - \frac{1}{n + \varepsilon_n - k} \right) \]
\[ = \varepsilon_n \sum_{k \neq n} (-1)^{n-k}a_k \frac{1}{(n - k)(n + \varepsilon_n - k)}. \]
This identity then gives us
\[
|H_0(a)(n) - H_\epsilon(a)(n)| \leq |\epsilon_n| \sum_{k \neq n} |a_k| \frac{1}{|n - k|^2} \frac{|n - k|}{|n + \epsilon_n - k|} \\
\leq \frac{|\epsilon_n|}{1 - |\epsilon_n|} \sum_{k \neq n} |a_k| \frac{1}{|n - k|^2}.
\]

This means that, in norm, one can compare these two operators. Indeed, it is a classical result that the operator norm of $H_0$ is $\pi$, and by Plancherel the operator norm of the transformation

\[
S(a) = \sum_{k \neq n} a_k \frac{1}{|n - k|^2}
\]
is $\pi^2/3$. This in turn implies

\[
\|H_\epsilon\| \leq \pi + \frac{\pi^2}{3} \sup_n |\epsilon_n| - \frac{1}{1 - \sup_n |\epsilon_n|}.
\]

3.1.2. Norm estimates of the perturbation. It is worth noticing the the estimate (3.1) is very crude, as it is meant to depend only on $L = \sup_n |\epsilon_n|$. For instance, if $\{\epsilon_n\}_{n \in \mathbb{Z}}$ is a constant sequence, then the norm $\|H_\epsilon\|$ is equal to $\pi$. We also note that the fact that we obtain invertibility by means of perturbations of small norm of a invertible operator does not take into account other factors, such as cancellation.

In order to apply our perturbation scheme to the operator $A_\epsilon$, we need to bound the following family of operators:

\[
P_\epsilon(a)(n) = \sum_{k \in \mathbb{Z}} a_k (\text{sinc}(n + \epsilon_n - k) - \delta_{n,k}).
\]

We may rewrite them as

\[
P_\epsilon(a)(n) = (\text{sinc}(\epsilon_n) - 1)a_n + \sum_{k \neq n} a_k (\text{sinc}_n(n + \epsilon_n - k))
\]

\[
= (\text{sinc}(\epsilon_n) - 1)a_n + \sum_{k \neq n} a_k \frac{(-1)^{n-k} \sin \pi \epsilon_n}{\pi(n + \epsilon_n - k)}
\]

This implies, on the other hand,

\[
P_\epsilon(a)(n) = (\text{sinc}(\epsilon_n) - 1)a_n + \left(\frac{\sin \pi \epsilon_n}{\pi}\right) H_\epsilon(a)(n),
\]

which in turn implies that

\[
\|P_\epsilon\| \leq \sup_n |\text{sinc}(\epsilon_n) - 1| + \sup_n \left|\frac{\sin \pi \epsilon_n}{\pi}\right| \|H_\epsilon\|
\]

\[
\leq \sup_n |\text{sinc}(\epsilon_n) - 1| + \sup_n \left|\sin \pi \epsilon_n\right| + \frac{\pi \sup_n |\sin \pi \epsilon_n| \sup_n |\epsilon_n|}{3 (1 - \sup_n |\epsilon_n|)}.
\]

Since $A_\epsilon = P_\epsilon + Id$, whenever

\[
1 - \text{sinc}(L) + |\sin \pi L| + \frac{\pi L \sin \pi L}{3 (1 - L)} < 1,
\]
we will have that $A_\varepsilon$ is invertible. In particular, a routine numerical evaluation implies that $L < 0.239$ satisfies the inequality above. Let then $A_\varepsilon^{-1} : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ be the inverse of $A_\varepsilon$, which is continuous by the considerations above. We know, by the Shannon–Whittaker interpolation formula \[1.2\] that $A_\varepsilon$ takes $\{f(k)\}_{k \in \mathbb{Z}}$, for $f \in \text{PW}_\pi$, to $\{f(k + \varepsilon_k)\}_{k \in \mathbb{Z}}$. This is enough to prove the assertion about recovery, and as such implies that

$$\sum_{n \in \mathbb{Z}} |f(n + \varepsilon_n)|^2$$

is an equivalent norm to the usual $L^2 -$norm on $\text{PW}_\pi$, by [33] Theorem 1.13.

Moreover, by writing

$$A_\varepsilon^{-1}(b)(k) = \sum_{n \in \mathbb{Z}} b_n \cdot \rho_{k,n},$$

we have immediately

\begin{equation}
\sum_{n \in \mathbb{Z}} f(n + \varepsilon_n) \rho_{k,n} = f(k),
\end{equation}

and $\sup_n (\sum_{k \in \mathbb{Z}} |\rho_{k,n}|^2) \lesssim 1$. If $(A_\varepsilon^{-1})^* : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ denotes the adjoint of the inverse of $A_\varepsilon$, then we see that

$$\|(A_\varepsilon^{-1})^*(\text{sinc}_x(k))\|_{\ell^2(\mathbb{Z})} \lesssim \|A_\varepsilon^{-1}\|_{\ell^2 \to \ell^2},$$

where the implicit constant does not depend on $x$, and we let $\text{sinc}_x(k) := \text{sinc}(x - k)$. Therefore, by letting $g_n(x) = \sum_{k \in \mathbb{Z}} \rho_{k,n}\text{sinc}(x - k)$, we have

$$\sup_{x \in \mathbb{R}} \left( \sum_{n \in \mathbb{Z}} |g_n(x)|^2 \right)^{1/2} \lesssim 1,$$

and thus, by the previous considerations, the sum $\sum_{n \in \mathbb{Z}} f(n + \varepsilon_n)g_n(x)$ converges absolutely. As $\langle (A_\varepsilon^{-1})^*(\text{sinc}_x(k)), f(n + \varepsilon_n) \rangle = \langle \text{sinc}_x(k), A_\varepsilon^{-1}(f(n + \varepsilon_n)) \rangle = f(x)$ by Shannon–Whittaker, this implies

$$f(x) = \sum_{n \in \mathbb{Z}} f(n + \varepsilon_n)g_n(x),$$

as desired. This finishes the proof of Theorem 1.1.

3.2. From Shannon to Vaaler: the proof of Theorem 1.2. We now concentrate in proving that the usual Shannon–Whittaker interpolation formula implies Vaaler’s celebrated interpolation result with derivatives [30].

Indeed, as proving that the interpolation formula of Theorem 1.2 converges uniformly on compact sets of $\mathbb{C}$ is a routine computation, given that $\{a_k\}_{k \in \mathbb{Z}}, \{b_k\}_{k \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$, we shall omit this part and focus on proving that the asserted equality holds.

Given a sequence $a = \{a_k\}_{k \in \mathbb{Z}}$, we define the following operators:

$$\mathcal{H}(a)(k) = \frac{1}{\pi} \sum_{0 \neq j \in \mathbb{Z}} \frac{a_{k-j}}{j} = \frac{1}{\pi} \sum_{k \neq j \in \mathbb{Z}} \frac{a_j}{k - j},$$

$$\mathcal{H}_1(a)(k) = \frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{a_{k-j}}{j + \frac{1}{2}} = \frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{a_j}{k - j + \frac{1}{2}}.$$
It is known that both $\mathcal{H}$ and $\mathcal{H}_1$ are bounded operators in $\ell^2(\mathbb{Z})$, with $\mathcal{H}_1$ being also unitary with $\mathcal{H}_2$ its inverse being given by

$$\mathcal{H}_2(a)(k) = -\frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{a_{j-k}}{j-k - \frac{1}{2}} = \frac{1}{\pi} \sum_{j \in \mathbb{Z}} \frac{a_j}{j - k + \frac{1}{2}}.$$ 

Given a function $f \in PW_{\pi}$, as a consequence of the Shannon–Whittaker interpolation formula we obtain, for every $k \in \mathbb{Z}$, that

$$f'(k) = \sum_{j \neq k} \frac{f(j)}{k-j}(-1)^{k-j}.$$ 

We consider three sequences, as follows:

$$a(k) = f(2k-1), \quad b(k) = f(2k), \quad c(k) = f'(2k).$$

We have, thus,

$$c(k) = f'(2k) = \sum_{j \neq 2k} \frac{f(j)}{2k-j}(-1)^{2k-j} = \frac{1}{2} \sum_{j \neq k} \frac{f(2j)}{k-j} - \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{f(2j-1)}{k-j + \frac{1}{2}}$$

$$= \frac{1}{2} \sum_{j \neq k} \frac{b(j)}{k-j} - \frac{1}{2} \sum_{j \in \mathbb{Z}} \frac{a(j)}{k-j + \frac{1}{2}} = \frac{\pi}{2} \mathcal{H}(b)(k) - \frac{\pi}{2} \mathcal{H}_1(a)(k).$$

This means that, for every $k \in \mathbb{Z},$

$$\mathcal{H}_1(a)(k) = \mathcal{H}(b)(k) - \frac{2}{\pi} c(k).$$

Since $\mathcal{H}_2$ is the inverse of $\mathcal{H}_1$, this can be rewritten as

$$a(k) = (\mathcal{H}_2 \circ \mathcal{H})(b)(k) - \frac{2}{\pi} \mathcal{H}_2(c)(k).$$

We know, by the Shannon–Whittaker interpolation formula, that

$$f(x) = \sum_{k \in \mathbb{Z}} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)}.$$ 

This implies, on the other hand,

$$f(x) = \sum_{k \in \mathbb{Z}} f(2k) \frac{\sin \pi(x-2k)}{\pi(x-2k)} + \sum_{k \in \mathbb{Z}} (\mathcal{H}_2 \circ \mathcal{H})(b)(k) - \frac{2}{\pi} \mathcal{H}_2(c)(k) \frac{\sin \pi(x-2k+1)}{\pi(x-2k+1)}$$

$$= \sum_{k \in \mathbb{Z}} b(k) \frac{\sin \pi x}{\pi(x-2k)} + \sum_{k \in \mathbb{Z}} (\mathcal{H}_2 \circ \mathcal{H})(b)(k) \frac{\sin \pi(x-2k+1)}{\pi(x-2k+1)}$$

$$- \frac{2}{\pi} \sum_{k \in \mathbb{Z}} \mathcal{H}_2(c)(k) \frac{\sin \pi(x-2k+1)}{\pi(x-2k+1)} = A(x) + B(x) + C(x).$$

We shall investigate each term $A, B$ and $C$ thoroughly in order to obtain our final result.
3.2.1. Determining C. By considering the family of functions $h_j \in PW_\pi$ – which satisfy the important property $h_j(k) = 0$, if $k \in 2\mathbb{Z}$ – given by

$$h_j(z) = \frac{\sin^2(\frac{1}{2}\pi z)}{\pi^2(z - 2j)},$$

we obtain

$$C(x) = -2 \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} f'(2j) \frac{\sin \pi(x - 2k + 1)}{\pi^2(j - k + \frac{1}{2})} \frac{1}{\pi(x - 2k + 1)}$$

$$= 4 \sum_{j \in \mathbb{Z}} f'(2j) \sum_{k \in \mathbb{Z}} \frac{1}{\pi^2((2k - 1) - 2j)} \frac{1}{\pi(x - 2k - 1)}$$

$$= 4 \sum_{j \in \mathbb{Z}} f'(2j) \sum_{k \in \mathbb{Z}} h_j(2k - 1) \frac{\sin \pi(x - 2k - 1)}{\pi(x - 2k - 1)}$$

$$= 4 \sum_{j \in \mathbb{Z}} f'(2j) \sum_{k \in \mathbb{Z}} h_j(k) \frac{\sin \pi(x - k)}{\pi(x - k)}.$$  

Notice that one can use Fubini’s theorem to justify all the changes of order of summation, by the fact that $h_j \in PW_\pi$. By applying the Shannon-Whittaker interpolation to $h_j$, we have

$$C(x) = 4 \sum_{j \in \mathbb{Z}} f'(2j) \frac{\sin^2(\frac{1}{2}\pi x)}{\pi^2(x - 2j)}$$

3.2.2. Determining B. For the second term, we expand

$$B(x) = \sum_{k \in \mathbb{Z}} H_2 \circ \mathcal{H}(b)(k) \frac{\sin \pi(x - 2k + 1)}{\pi(x - 2k + 1)}$$

$$= \frac{1}{\pi} \sum_{k \in \mathbb{Z}} \frac{\sin \pi(x - 2k + 1)}{\pi(x - 2k + 1)} \sum_j \frac{\mathcal{H}(b)(j)}{j - k + \frac{1}{2}}$$

$$= \frac{1}{\pi^2} \sum_{k \in \mathbb{Z}} \frac{\sin \pi(x - 2k + 1)}{\pi(x - 2k + 1)} \sum_j \sum_{l \neq j} \frac{b(l)}{(j - k + \frac{1}{2})(j - l)}.$$  

By Fubini’s theorem, this implies

$$B(x) = \frac{1}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \sum_{j \neq l} \frac{1}{j - l} \sum_{k \in \mathbb{Z}} \frac{1}{j - k + \frac{1}{2}} \frac{\sin \pi(x - 2k + 1)}{\pi(x - 2k + 1)}$$

$$= \frac{1}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \sum_{j \neq l} \frac{2}{j - l} \sum_{k \in \mathbb{Z}} \frac{1}{2j - 2k + 1} \frac{\sin \pi(x - 2k + 1)}{\pi(x - 2k + 1)}$$

$$= \frac{1}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \sum_{j \neq l} \frac{2}{j - l} \frac{\sin^2(\frac{1}{2}\pi x)}{\pi^2} = \frac{\sin^2(\frac{1}{2}\pi x)}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \sum_{j \neq 0} \frac{1}{j(j + l - \frac{1}{2})}.$$  

But it is a well-known fact that the summation formula

$$g(z) = \sum_{j \neq 0} \frac{1}{j(j + z)} = \frac{\psi(1 + z) - \psi(1 - z)}{z},$$
holds, where \( \psi(z) = \frac{d}{dz} \log \Gamma(z) \) is the digamma function. This implies, on the other hand,

\[
B(x) = \frac{2 \sin^2\left(\frac{1}{2} \pi x\right)}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \frac{\psi(1 + l - \frac{x}{2}) - \psi(1 - l + \frac{x}{2})}{2l - x}.
\]

3.2.3. Determining \( A + B \). Using that \( \sin(2x) = 2 \sin x \cos x \), we obtain

\[
A(x) = -\frac{2 \sin^2\left(\frac{1}{2} \pi x\right)}{\pi^2} \sum_{l \in \mathbb{Z}} b(l) \pi \cot(\pi \frac{x}{2}) \frac{1}{(x - 2l)^2}.
\]

The digamma function satisfies the following functional equations, which we shall make use of:

\[
\psi(1 - z) = \psi(z) + \pi \cot \pi z,
\]

\[
\psi(1 + z) = \psi(z) + \frac{1}{z}.
\]

Using these relations with \( z = \frac{x}{2} - l \) in the equations above, we obtain readily

\[
A(x) + B(x) = \frac{4 \sin^2\left(\frac{4}{\pi} \pi x\right)}{\pi^2} \sum_{j \in \mathbb{Z}} \left\{ \frac{f(2j)}{(x - 2j)^2} + \frac{f'(2j)}{x - 2j} \right\}.
\]

3.2.4. \( A + B + C \). Summing the analysis undertaken for the terms above, we have

\[
f(x) = A(x) + B(x) + C(x) = \frac{4 \sin^2\left(\frac{4}{\pi} \pi x\right)}{\pi^2} \sum_{j \in \mathbb{Z}} \left\{ \frac{f(2j)}{(x - 2j)^2} + \frac{f'(2j)}{x - 2j} \right\}.
\]

This finishes the proof of Theorem 1.2.

3.3. Perturbations of Interpolation Formulae with derivatives. By the arguments in the previous section, the formula we just derived for \( PW_{2\pi} \), i.e.,

\[
f(x) = \frac{\sin^2(\pi x)}{\pi^2} \sum_{k \in \mathbb{Z}} \left\{ \frac{f(k)}{(x - k)^2} + \frac{f'(k)}{x - k} \right\},
\]

converges in compact sets of \( \mathbb{C} \). We fix, for shortness, the notation

\[
g(x) = \frac{\sin^2(\pi x)}{\pi^2 x^2}, h(x) = \frac{\sin^2(\pi x)}{\pi^2 x},
\]

which means we can read Vaaler’s interpolation as

\[
f(x) = \sum_{k \in \mathbb{Z}} \{ f(k)g(x - k) + f'(k)h(x - k) \}.
\]

Because of uniform convergence, we can differentiate term by term in the above formula. This implies, thus,

\[
f'(x) = \sum_{k \in \mathbb{Z}} \{ f(k)g'(x - k) + f'(k)h'(x - k) \}.
\]

We record, for completeness, the formulae for the derivatives of \( g', h' \):

\[
g'(x) = \frac{2 \sin(\pi x)(\pi x \cos(\pi x) - \sin(\pi x))}{\pi^2 x^3},
\]

\[
h'(x) = \frac{\sin(\pi x)(2 \pi x \cos(\pi x) - \sin(\pi x))}{\pi^2 x^2},
\]
and for $n \in \mathbb{Z}$,

$$g_n = h'_n = 0, \quad g'_n = h_n = \delta_0.$$  

Our goal now is to invert the operator $A = A_\varepsilon$ defined in $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ by

$$A_1(a, b)_n = \sum_{k \in \mathbb{Z}} a_k \cdot g(n + \varepsilon_n - k) + \sum_{k \in \mathbb{Z}} b_k \cdot h(n + \varepsilon_n - k),$$

$$A_2(a, b)_n = \sum_{k \in \mathbb{Z}} a_k \cdot g'(n + \varepsilon_n - k) + \sum_{k \in \mathbb{Z}} b_k \cdot h'(n + \varepsilon_n - k),$$

where $A(a, b) = (A_1(a, b), A_2(a, b))$ for $(a, b) \in \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$. Furthermore, we wish to establish a criterion that depends only on $L = \sup |\varepsilon_n|$. For that purpose, we estimate when the operator norm of $A_\varepsilon - Id$ from $\ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z})$ to itself is small, in terms of $L$.

### 3.3.1. Auxiliary perturbations for the derivative case

Given a sequence $a = \{a_k\}_{k \in \mathbb{Z}}$, we define the following operators:

$$H^p_0(a)_n = \sum_{k \neq n} \frac{a_k}{(n + \varepsilon_n - k)^p},$$

and denote by $H^p_\varepsilon$ the operator associated to the sequence $\varepsilon_n = 0, \forall n \in \mathbb{Z}$. In an analogous manner to the proof of Theorem $1.1$, we compare:

$$H^p_0(a)_n - H^p_\varepsilon(a)_n = \sum_{k \neq n} a_k \left( \frac{1}{(n-k)^p} - \frac{1}{(n+\varepsilon_n-k)^p} \right)$$

$$= \sum_{j=0}^{p-1} \binom{p}{j} \varepsilon_n^{p-j} \sum_{k \neq n} a_k \frac{1}{(n-k)^p(n-k)^{p-j}}.$$  

Therefore,

$$|H^p_0(a)_n - H^p_\varepsilon(a)_n| \leq \sum_{j=0}^{p-1} \binom{p}{j} \varepsilon_n^{p-j} \sum_{k \neq n} a_k \frac{|n-k|^p}{|n-k|^{2p-j}(|n-k|-|\varepsilon_n|)^p},$$

$$\leq \frac{1}{(1-|\varepsilon_n|)^p} \sum_{j=0}^{p-1} \binom{p}{j} \varepsilon_n^{p-j} \epsilon S^{2p-j}(a^*_n),$$

where

$$S^q(a)_n = \sum_{k \neq n} \frac{a_k}{|n-k|^q},$$

and $a^* = \{|a_n|\}$. Let us consider $S(p) = \max\{|S^q|, q = 1, \ldots, 2p\}$. Since $S^{q+1}(a^*_n) \leq S^q(a^*_n)$, we have

$$|H^p_0(a)_n - H^p_\varepsilon(a)_n| \leq \frac{S^{p+1}(a^*_n)}{(1-|\varepsilon_n|)^p} \sum_{j=0}^{p-1} \binom{p}{j} \varepsilon_n^{p-j}$$

$$= \frac{(1+|\varepsilon_n|)^p - 1}{(1-|\varepsilon_n|)^p} S^{p+1}(a^*_n).$$
This means that we have the following estimate on the norm of the perturbed operator:
\[
\|H_p^\varepsilon\| \leq \gamma_p(L),
\]
where we let
\[
\gamma_p(L) = \|H_0^\varepsilon\| + \frac{(1 + L)^p - 1}{(1 - L)^p} \|S^{p+1}\|.
\]
Now, in order to estimate the value of \(\gamma_p(L)\), we resort to [20, Corollary 2], which gives us that
\[
\|H_0^\varepsilon\| = \frac{(2\pi)^m b_m}{m!}.
\]
where \(b_m\) is the maximum of \(|B_m(x)|\) when \(x \in [0, 1]\), and \(B_m\) denotes the \(m\)-th Bernoulli polynomial. Therefore,
\[
\|H_0^\varepsilon\| = \pi, \|H_2^\varepsilon\| = \frac{\pi^2}{3}, \|H_3^\varepsilon\| = \frac{\pi^3}{9\sqrt{3}}.
\]
On the other hand, by Plancherel’s theorem it is easy to see that
\[
\|S^p\| = 2\zeta(p).
\]
Thus, by a straightforward calculation,
\[
\|P_\varepsilon\| \leq \sqrt{2} \max\{|g(L) - 1|, |h'(L) - 1|, |g'(L)|, |h(L)|\} + \frac{\sin(\pi L)^2}{\pi^2} \|G_\varepsilon\|,
\]
we obtain
\[
\|H_1^\varepsilon\| \leq \pi + \frac{L}{1 - L} \pi^2 \frac{2}{3},
\]
\[
\|H_2^\varepsilon\| \leq \frac{\pi^2}{3} + 2 \left(\frac{L^2 + 2L}{(1 - L)^2}\right) \zeta(3),
\]
\[
\|H_3^\varepsilon\| \leq \frac{\pi^3}{9\sqrt{3}} + \left(\frac{L^3 + 3L^2 + 3L}{(1 - L)^3}\right) \frac{\pi^4}{45}.
\]
3.3.2. Norm estimates of the perturbations in the derivative case. In order to invert the operator \(A_\varepsilon\), we estimate the norm of \(P_\varepsilon = A_\varepsilon - Id = (P_1, P_1)\), where
\[
P_1(a, b)_n = \sum_{k \in \mathbb{Z}} a_k \cdot (g(n + \varepsilon_n - k) - \delta_{n,k}) + \sum_{k \in \mathbb{Z}} b_k \cdot h(n + \varepsilon_n - k),
\]
\[
P_2(a, b)_n = \sum_{k \in \mathbb{Z}} a_k \cdot g'(n + \varepsilon_n - k) + \sum_{k \in \mathbb{Z}} b_k \cdot (h'(n + \varepsilon_n - k) - \delta_{n,k}).
\]
By a straightforward calculation,
\[
P_1(a, b)_n = (g(\varepsilon_n) - 1)a_n + \frac{\sin(\pi \varepsilon_n)^2}{\pi^2} H_1^2(a)_n + h(\varepsilon_n)b_n + \frac{\sin(\pi \varepsilon_n)^2}{\pi^2} H_1^1(b)_n,
\]
\[
P_2(a, b)_n = g'(\varepsilon_n)a_n + \frac{2 \sin(\pi \varepsilon_n)(\pi \varepsilon_n \cos(\pi \varepsilon_n) - \sin(\pi \varepsilon_n))}{\pi^2} H_1^3(a)
\]
\[
+ (h'(\varepsilon_n) - 1)b_n + \frac{\sin(\pi \varepsilon_n)(2\pi \varepsilon_n \cos(\pi \varepsilon_n) - \sin(\pi \varepsilon_n))}{\pi^2} H_2^2(b).
\]
Thus,
\[
\|P_\varepsilon\| \leq \sqrt{2} \max\{|g(L) - 1|, |h'(L) - 1|, |g'(L)|, |h(L)|\} + \frac{\sin(\pi L)^2}{\pi^2} \|G_\varepsilon\|,
\]
where \( G_\varepsilon = G = (G_1, G_2) \) and
\[
G_1(a, b)_n = \mathcal{H}_\varepsilon^1(a) + \mathcal{H}_\varepsilon^1(b),
\]
\[
G_2(a, b)_n = \frac{2(\pi \varepsilon_n \cos(\pi \varepsilon_n) - \sin(\pi \varepsilon_n))}{\sin(\pi \varepsilon)} \mathcal{H}_\varepsilon^2(a) + \frac{(2\pi \varepsilon_n \cos(\pi \varepsilon_n) - \sin(\pi \varepsilon_n))}{\sin(\pi \varepsilon)} \mathcal{H}_\varepsilon^2(b).
\]
By taking \( L < 1/4 \) and using the Cauchy-Schwarz inequality, we have
\[
\|G_\varepsilon\|^2 / 2 \leq \max\{\|\mathcal{H}_\varepsilon^1\|, \|\mathcal{H}_\varepsilon^2\|\}^2
\]
\[
+ \max \left\{ \left( \frac{2(\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \|\mathcal{H}_\varepsilon^1\|^2, \left( \frac{(2\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \|\mathcal{H}_\varepsilon^2\|^2 \right\}
\]
\[
\leq \max\{\gamma_1(L)^2, \gamma_2(L)^2\}
\]
\[
+ \max \left\{ \left( \frac{2(\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \gamma_3(L)^2, \left( \frac{(2\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \gamma_2(L)^2 \right\}.
\]
We note that we have abused the notation \( \|G_\varepsilon\| \) to denote the operator norm of \( G_\varepsilon \) when defined on \( \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \). One can further check that, for \( 0 \leq L < 1/4 \),
\[
|g(L) - 1| < |h'(L) - 1|, |h(L)| < |g'(L)|, \gamma_1(L)^2 < \gamma_2(L)^2 \quad \text{and}
\]
\[
\left( \frac{2(\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \gamma_3(L)^2 < \left( \frac{(2\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \gamma_2(L)^2,
\]
which means, in turn,
\[
\|G_\varepsilon\| \leq \gamma_2(L) \sqrt{2 \left( 1 + \left( \frac{2\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \right)},
\]
and directly implies the estimate
\[
\|P_\varepsilon\| \leq 1 - \frac{\sin(\pi L)(2\pi L \cos(\pi L) - \sin(\pi L))}{\pi^2 L^2} + \frac{2\sin(\pi L)(\sin(\pi L) - \pi L \cos(\pi L))}{\pi^2 L^3}
\]
\[
+ \frac{\sin(\pi L)^2}{\pi^2} \left( \frac{\pi^2}{3} + 2 \left( \frac{L^2 + 2LL - 1}{L - L^2} \right) \zeta(3) \right) \sqrt{2 \left( 1 + \left( \frac{(2\pi L \cos(\pi L) - \sin(\pi L))}{\sin(\pi L)} \right)^2 \right)}.
\]
By evaluating the last expression on the right-hand side above numerically, we obtain that we can go up to \( L < 0.111 \) and maintain \( \|P_\varepsilon\| < 1 \). By invoking again [33, Theorem 1.13], we see immediately that
\[
\sum_{n \in \mathbb{Z}} \left( |f(n + \varepsilon_n)|^2 + |f'(n + \varepsilon_n)|^2 \right)
\]
yields an equivalent norm for \( PW_{2\pi} \), as long as \( \sup_n |\varepsilon_n| < 0.111 \).
Moreover, as \( \mathcal{A}_x^{-1} : \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \) is bounded, the same argument as in the proof of Theorem 1.1 shows that there are \( \varrho_{k,n}, \vartheta_{k,n}, \varrho'_{k,n}, \vartheta'_{k,n} \) such that
\[
f(k) = \sum_{n \in \mathbb{Z}} f(n + \varepsilon_n) \varrho_{k,n} + f'(n + \varepsilon_n) \vartheta_{k,n},
\]
\[
f'(k) = \sum_{n \in \mathbb{Z}} f(n + \varepsilon_n) \varrho'_{k,n} + f'(n + \varepsilon_n) \vartheta'_{k,n},
\]
and \( \sup_n \left( \sum_{k \in \mathbb{Z}} \left| \varrho_{k,n} \right|^2 + \left| \vartheta_{k,n} \right|^2 + \left| \varrho'_{k,n} \right|^2 + \left| \vartheta'_{k,n} \right|^2 \right) \leq 1 \). By using the adjoint \( (\mathcal{A}_x^{-1})^* : \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \times \ell^2(\mathbb{Z}) \) in an analogous manner to that of the proof of Theorem 1.1 together with (3.9) and (1.6), we obtain the asserted existence of the functions \( g_n, h_n \in PW_2 \pi \) so that
\[
f(x) = \sum_{n \in \mathbb{Z}} f(n + \varepsilon_n) g_n(x) + f'(n + \varepsilon_n) h_n(x),
\]
where the right-hand side converges absolutely, as desired. This proves the desired perturbation of Vaaler’s interpolation formula, given in Theorem 1.3.

4. Perturbations of Fourier interpolation on the real line

4.1. Improved estimates on the interpolation basis. As our goal is to obtain the perturbations of the formula
\[
f(x) = \sum_{n \geq 0} \left[ f(\sqrt{n}) a_n(x) + \hat{f}(\sqrt{n}) \hat{a}_n(x) \right]
\]
to as large as possible, we must improve the decay estimates for the interpolating functions \( a_n \). In [24, Section 5], the authors prove that \( a_n/n^2 \) is uniformly bounded in \( n \geq 0, x \in \mathbb{R} \). In order to be able to make the perturbations larger, we need to improve that result substantially, as even the refined bound \( |a_n| = O(n^{1/4} \log^{3/2}(1 + n)) \) from [3] does not seem to be enough for our purposes. This first subsection is, therefore, devoted to the proof of Theorem 1.5.

In order to prove this result we will employ the moral idea behind the characterization of Gelfand-Shilov spaces. These are spaces where, in a nutshell, both function and Fourier transform decay as fast as the negative exponential of a certain monomial. The following result connects these spaces with specific decay on function and Fourier side for certain Schwartz norms. See, e.g., [7, Theorem 2.3] for a proof.

**Lemma 4.1.** Let \( A, B, r, s > 0 \) be positive constants. The following assertions are equivalent:

1. There is \( C > 0 \) such that
\[
\sup_{x \in \mathbb{R}} |x^\alpha \varphi(x)| \leq CA^\alpha (\alpha!)^r, \quad \sup_{\xi \in \mathbb{R}} |\xi^\beta \hat{\varphi}(\xi)| \leq CB^\beta (\beta!)^s,
\]
for all \( \alpha, \beta \in \mathbb{Z} \);

2. There is \( C' > 0 \) such that
\[
|\varphi(x)| \leq C' e^{-\theta|x/A|^{1-r}}, \quad |\hat{\varphi}(\xi)| \leq C' e^{-\Omega|\xi/B|^{1-s}},
\]

where \( \theta, \Omega > 0 \).
for all \( x, \xi \in \mathbb{R} \).

We will use this result together with explicit estimates on \( \{b_n^\pm\}_{n \geq 0} \), in the same spirit as in [21]. Indeed, let \( \varepsilon \in \{\pm\} \) be a sign. In [24], the authors consider the generating functions

\[
\sum_{n=0}^{\infty} g_n^\varepsilon(z) e^{i\pi n\tau} =: K_\varepsilon(\tau, z),
\]

where \( g_n^\varepsilon \) are weakly holomorphic modular forms of weight \( 3/2 \) with growth and coefficient properties so that the functions

\[
b_n^\varepsilon(x) = \frac{1}{2} \int_{-1}^{1} g_n^\varepsilon(z) e^{i\pi x z} dz
\]

are eigenvectors of the Fourier transform associated to the eigenvalues \( \varepsilon \) satisfying

\[
b_n^\varepsilon = a_n \pm \hat{a}_n, \quad \{a_n\}_{n \geq 0} \text{ defined as in [13]}
\]

We mention, for completeness, the following result:

**Theorem 4.2** (Theorem 3 in [21]). The following assertions hold:

\[
K_+ (\tau, z) = \frac{\theta(\tau)(1-2\lambda(\tau))\theta(z)^3 J(z)}{J(z) - J(\theta)},
\]

\[
K_- (\tau, z) = \frac{\theta(\tau) J(\tau) \theta(z)^3 (1-2\lambda(z))}{J(z) - J(\theta)},
\]

(4.1)

where \( \theta, J \) and \( \lambda \) are as previously defined. Moreover, \( K_\varepsilon(\tau, z) \) are meromorphic functions with poles at \( \tau \in \Gamma_\theta z \), and the right-hand side of (4.1) converges for all \( \tau \) with large enough imaginary part.

The authors then define the natural candidate for the generating function for the \( \{b_n^\varepsilon\}_{n \geq 0} \) to be

\[
F_\varepsilon(\tau, x) = \frac{1}{2} \int_{-1}^{1} K_\varepsilon(\tau, z) e^{i\pi x z} dz,
\]

which is defined, a priori, for each fixed \( x \in \mathbb{R} \) and \( \{\tau \in \mathbb{H} : \forall k \in \mathbb{Z}, |\tau - 2k| > 1 \} \supset \mathcal{D} + 2\mathbb{Z} \), where \( \mathcal{D} \) is the standard fundamental domain for \( \Gamma_\theta \). By Theorem 4.2 there holds that, whenever \( \text{Im}(\tau) > 1 \),

\[
F_\varepsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x) e^{i\pi n\tau}.
\]

(4.3)

As \( F_\varepsilon(\tau, x) \) admits an analytic continuation to \( \mathbb{H} \) (see [24, Proposition 2]), they are able to extend (4.3) to the entire upper half space \( \mathbb{H} \). Moreover, the following functional equations hold:

\[
F_\varepsilon(\tau, x) - F_\varepsilon(\tau + 2, x) = 0,
\]

\[
F_\varepsilon(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon\left( \frac{-1}{\tau}, x \right) = e^{i\pi x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}.
\]

(4.4)
The proof of Theorem 1.5 follows the same essential philosophy as the proof of [24, Theorem 4]: in order to bound each of the terms $b_n^\pm$, we bound, uniformly on $x \in \mathbb{R}$, the analytic function $F_\pm(\tau, x)$. Relating the two bounds is achieved by the following Lemma, originally attributed to Hecke (see [24, Lemma 1] and [2, Lemma 2.2(ii)]):

**Lemma 4.3.** Let $f : \mathbb{H} \to \mathbb{C}$ be a $2$–periodic analytic function admitting an absolutely convergent Fourier expansion

$$f(\tau) = \sum_{n \geq 0} c_n e^{i\pi n \tau}.$$  

Suppose, additionally, that for some $\alpha > 0$ it satisfies that $|f(\tau)| \leq C \text{Im}(\tau)^{-\alpha}$, for $\text{Im}(\tau) < \epsilon_0$. Then, for all $n > \frac{1}{\epsilon_0}$,

$$|c_n| \leq \tilde{C} n^\alpha.$$  

Moreover, if $n > \frac{\alpha}{\pi \epsilon_0}$, the improved estimate

$$|c_n| \leq C' \left( \frac{e\pi}{\alpha} \right) n^\alpha$$

holds.

**Proof of Lemma 4.3.** As $f$ is analytic on $\mathbb{H}$ and its Fourier series expansion converges absolutely, an application of Fubini’s theorem gives us that

$$2c_n = \int_{-1+i/n}^{1+i/n} f(\tau)e^{-i\pi n \tau} d\tau.$$  

The right hand side is, nonetheless, bounded in absolute value by

$$\int_{-1}^{1} Cn^\alpha e^{-\pi} dt = 2Ce^{-\pi}n^\alpha,$$

which follows from the growth restriction on $f$ near the boundary of $\mathbb{H}$. The first assertion follows then with $\tilde{C} = 2Ce^{-\pi}$. For the second one, we compute instead

$$2c_n = \int_{-1+i\frac{\alpha}{\pi n}}^{1+i\frac{\alpha}{\pi n}} f(\tau)e^{-i\pi n \tau} d\tau.$$  

Estimating the absolute value of this integral with the given condition yields that $|c_n| \leq C' \left( \frac{e\pi}{\alpha} \right) n^\alpha$, as wished.

We are now ready to prove Theorem 1.5:

**Proof of Theorem 1.5.** We consider the functions

$$F_\pm^k(\tau, x) := x^k F_\pm(\tau, x).$$

By Lemma 4.3 if we prove that, for some $\Delta > 0$,

$$|F_\pm^k(\tau, x)| \leq C^k(k!)\text{Im}(\tau)^{-k/2-\Delta},$$

for all $k \geq 1$, then we will have that

$$\sup_{x \in \mathbb{R}} |x^k b_n^\pm(x)| \leq \tilde{C} n^{\Delta n^{k/2}(k!)}. $$
As \( b_n^\varepsilon = \varepsilon \hat{b}_n \), Lemma 4.3 then implies that each of the functions \( b_n^\varepsilon \) decays like
\[
|b_n^\varepsilon(x)| \lesssim n^\Delta \varepsilon^{-|x|/\sqrt{\pi}},
\]
which is the content of Theorem 1.3. Therefore, we focus on proving a suitable version of (4.5). By the functional equation for \( F_\varepsilon \), we see that \( F_\varepsilon^k \) is a 2-periodic function on \( \mathbb{H} \) that satisfies the functional equation
\[
F_\varepsilon^k(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon^k(-1/\tau, x) = x^k(e^{i\pi x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}).
\]
The strategy, in analogy to that in \[24\], is of splitting in cases: if \( \tau \in \mathcal{D}, \) then estimates for \( F_\varepsilon^k \) are available directly by analytic methods. Otherwise, we need to use (4.0) to obtain the bound (4.5) for all \( \tau \in \mathbb{H}. \)

More explicitly, we have the following:

**Proposition 4.4.** There is a positive constant \( C > 0 \) such that, for each \( k \geq 0 \) odd, the inequality
\[
|F_\varepsilon^k(\tau, x)| \leq C^k(k!)(1 + \text{Im}(\tau))^{-k/2}
\]
holds, whenever \( \tau \in \mathcal{D}. \)

This Proposition can be directly compared to \[24\] Lemma 4. In fact, it is nothing but a carefully quantified version of it.

**Proof of Proposition 4.4.** As the proof follows thoroughly the main ideas in Lemma 4 in \[24\], we will mainly focus on the points where we have to sharpen bounds.

We see directly from the definition of \( F_\varepsilon^k \) that we are allowed to consider only values of \( \tau \in \mathcal{D}_1 = \mathcal{D} \cap \{ \tau \in \mathbb{H} : \text{Re}(\tau) \in (-1, 0) \} \). By subsequent considerations from that reduction, we see that the bound
\[
|x^k F_\varepsilon(\tau, x)| \leq 10 \int_\ell |K_\varepsilon(\tau, z)| |x^k(e^{-\pi x^2 \text{Im}(\tau)} + |z|^{-1/2} e^{-\pi x^2 \text{Im}(-1/z)})| \, |dz|
\]
holds, where \( \ell \) is the path joining \( i \) to 1 on the upper half space, defined to be
\[
\ell = \left\{ w \in \mathcal{D} : \text{Re}(J(w)) = \frac{1}{64}, \text{Im}(J(w)) > 0 \right\}.
\]
An explicit computation gives us that the maximal value of
\[
x^k e^{-\pi x^2 \text{Im}(z)}
\]
is attained at \( x = \left( \frac{k}{2\pi \text{Im}(z)} \right)^{1/2} \). Therefore, as any \( z \in \ell \) has norm bounded from above and below by absolute constants, we find that there is \( C > 0 \) so that
\[
|F_\varepsilon^k(\tau, x)| \leq C^{k/2} \cdot \left( \frac{k}{2\pi e} \right)^{k/2} \int_\ell |K_\varepsilon(\tau, z)| |\text{Im}(z)|^{-k/2} \, |dz|.
\]
We have then three regimes to consider:

**Case 1:** \(|\tau - i| < 1/10\). Notice that if we prove that the proposition holds for any \( \tau \in \mathbb{H} \) so that \(|\tau - i| = \frac{1}{10}\), we can use the maximum modulus principle on \( F_\varepsilon^k \) on that circle to conclude that the proposition holds inside as well. Moreover, by the functional equation (4.0), we see that the proposition holds for \( \mathcal{A} = \{ \tau \in \mathbb{H} : |\tau - i| = 1/10, |\tau| \leq 1 \} \),
in case it holds for the image of the circle arc \( \mathcal{A} \) under the action of \( S \). But a simple computation shows that \( S \mathcal{A} \) is just another circle arc contained (up to endpoints) in \( \{ \tau \in \mathcal{D}_1 : \frac{1}{4} > |\tau - i| > \frac{1}{10} \} \). This shows that in order to prove the proposition for this case, it suffices to show it for the other cases.

**Case 2:** \( |\tau - i| > \frac{1}{10} \), \( \text{Im}(\tau) > \frac{1}{2} \). For this case, we use the fact that \( |K_\varepsilon(\tau, z)| \lesssim |\theta(z)|^3 \lesssim \text{Im}(z)^{-2e^{-\pi/\text{Im}(z)}} \) for \( z \in \ell \), \( \text{Im}(\tau) > \frac{1}{2} \), with constants independent of \( \tau \). Using this bound in (4.10) yields

\[
|F^k_x(\tau, x)| \leq (1 + |x|^{k+2})e^{-c|x|} \lesssim C^k \left( \frac{k + 2}{e} \right)^{k+2},
\]

for some \( C > 0 \). Applications of Stirling’s formula imply that this bound is controlled by \( C^k(k!) \), with \( C_1 > 0 \) an absolute constant. This shows the result in this case.

**Case 3:** \( |\tau - i| > \frac{1}{10} \), \( \text{Im}(\tau) \leq \frac{1}{2} \). Again, we resort to the estimates in the proof of Lemma 4 in [24]: there, the authors prove that

\[
|K_+(\tau, z)| \lesssim \text{Im}(\tau)^{-1/2} \left| J(\tau) \right|^{3/8} |J(z)|^{5/8} \text{Im}(z)^{-3/2} / |J(z) - J(\tau)|,
\]

\[
|K_-(\tau, z)| \lesssim \text{Im}(\tau)^{-1/2} \left| J(\tau) \right|^{7/8} |J(z)|^{1/8} \text{Im}(z)^{-3/2} / |J(z) - J(\tau)|.
\]

Due to the not-so-symmetric nature of these bounds, we focus on the one for \( K_+ \), and the analysis for \( K_- \), as well as the bounds, will be almost identical for the other, and thus the details will be omitted.

Taking advantage of the explicit structure of the curve we are integrating over (4.8), and the fact that there is an absolute constant \( C > 0 \) so that \( \text{Im}(z)^{-1} \leq C \log(1+|J(z)|) \) plus that \( z \in \ell \iff J(z) = 1/64 + it, \ t \in \mathbb{R} \),

\[
\int_\ell |K_+(\tau, x)| \text{Im}(z)^{-k/2} |dz| \leq C^{k/2} \text{Im}(\tau)^{-1/2} \int_0^\infty \frac{|J(\tau)|^{3/8} t^{-3/8} \log^{(k-1)/2}(1 + t)}{\sqrt{1 + t^2} / |J(\tau)|^2} \ dt.
\]

\[
= C^{k/2} \text{Im}(\tau)^{-1/2} \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1 + t^2}} \ dt.
\]

Now, the last integral in (4.10) can be estimated as follows: as \( k - 1 \) is even, by using that \( \log(1 + ab) \leq \log(1 + a) + \log(1 + b) \) whenever \( a, b > 0 \), the integral

\[
\int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2}(1 + t|J(\tau)|)}{\sqrt{1 + t^2}} \ dt
\]

is bounded by

\[
\sum_{i=0}^{k-1} \left( \frac{k-1}{2} \right) \log^i(1 + |J(\tau)|) \int_0^\infty \frac{t^{-3/8} \log^{(k-1)/2-i}(1 + t)}{\sqrt{1 + t^2}} \ dt.
\]
Each summand above can be easily estimated. Indeed, \((k-1/2) \leq 2k/2\) trivially, \(\log(1 + |J(\tau)|) \leq C'\text{Im}(\tau)^{-1}\), and the integrals can be explicitly bounded in terms of Gamma functions. In fact, we first split the integrals in question as

\[
\left(\int_0^1 + \int_1^\infty\right) \frac{t^{-3/8} \log^{(k-1)/2-i}(1+t)}{\sqrt{1+t^2}} dt.
\]

For the first part, we simply bound the integrand by \(t^{-3/8} \log(2) (k-1/2) - i (1 + t) \sqrt{1 + t^2}\), and this yields us a bound uniform in \(k\). For the second, we change variables \(\log(1 + t) \mapsto \rightarrow s\) in (4.11) above. A simple computation shows that it is bounded by \(10 \int_0^\infty e^{-3s/8} s^{(k-1)/2 - i} ds \lesssim C'' \int_0^\infty e^{-r} r^{(k-1)/2 - i} dr = C'' \Gamma\left(\frac{k-1}{2} - i + 1\right)\).

Thus, (4.11) is bounded by \(C'\text{Im}(\tau)^{(1-k)/2} \Gamma\left(\frac{k-1}{2}\right)\).

Putting together the estimates in (4.10) and (4.9) and using Stirling’s formula for the approximation of \(\Gamma\), we conclude that

\[|F^k(\tau, x)| \leq C''(k!)\text{Im}(\tau)^{-k/2},\]

which was the content of the proposition. \(\square\)

In order to finish the proof of Theorem 1.5, we first notice that \(F^k\) is \(2\)-periodic, so we lose no generality in assuming that \(\tau \in \{z \in \mathbb{H} : \text{Re}(z) \in [-1, 1]\} = S_1\). If \(\text{Re}(\tau) \in [-1, 1]\), then we have two cases:

1. If \(\tau \in \mathcal{D}\), we can use Proposition 4.4 directly, and the decay obtained by the assertion of the Proposition remains unchanged;
2. If \(\tau \in S_1 \setminus \mathcal{D}\), the strategy is to use (4.6) to reduce it to the previous case. In fact, we define the \(\Gamma\) cocycle \(\left\{\phi_A^k\right\}_{A \in \Gamma_\theta}\) by

\[
\phi_{T_2}^k(\tau, x) = 0,
\]

\[
\phi_S^k(\tau, x) = x^k(e^{i\pi x^2\tau} + \varepsilon(-i\tau)^{-1/2}e^{i\pi x^2(-1/\tau)}),
\]

thogether with the cocycle relation

\[
(4.12) \quad \phi^k_{AB} = \phi^k_A + \phi^k_A|B.
\]

For a fixed \(\tau \in S_1 \setminus \mathcal{D}\), we associate \(\tau' \in \mathcal{D}\) through the following process: let

\[
(4.13) \quad \begin{cases} 
\gamma_0 = \tau, \\
\gamma_I = -\frac{1}{\gamma_I} - 2n_I,
\end{cases}
\]

where \(n_I = \left\lfloor \frac{(-1/\gamma_{I-1})+1}{2}\right\rfloor\). We define \(m = m(\tau)\) to be the smallest positive integer so that \(\gamma_m \in \mathcal{D}\). In this case, we let \(\gamma_m(\tau) =: \tau'\). In other words, we
have that the sequence

\[
\begin{align*}
\tau_0 &= \tau', \\
\tau_{i+1} &= -\frac{1}{\tau_i} + 2n_i
\end{align*}
\]

(4.14)

satisfies the hypotheses of Lemma 3 in [24]. We therefore have that \(|\tau_j| > 1\), \(\text{Im}(\tau_j)\) is nonincreasing and \(\text{Im}(\tau_j) \leq \frac{1}{|\tau_j - 1|}\). An inductive procedure shows us that

\[
\gamma_{m-i} = -\frac{1}{\tau_i}
\]

In particular, the sequence \(\{\tau_i\}_{i \geq 0}\) is in fact finite, with at most \(m(\tau)\) terms.

This implies that

\[
m + 1 \leq 4m - 2 \leq 2\text{Im}(\tau)^{-1}.
\]

(4.15)

We will use (4.15) in the following computation with the cocycle condition. We write \(\tau' = A\tau\), where \(A \in \Gamma_\theta\) is of the form \(A = ST^{2n_m}ST^{2n_m-1}S \cdots T^{2n_1}S\).

As \(\{\phi_A^k\}_{A \in \Gamma_\theta}\) satisfies the cocycle condition (4.12), the proof of Lemma 3 in [24] gives us that

\[
\text{Im}(\tau')^{1/4}|\phi_A^k(\tau')| \leq \sum_{j=1}^m |\text{Im}(\tau_j)^{1/4}|\phi_A^k(\tau_j)|.
\]

By the definition of \(\phi_S^k\), we see that

\[
|\phi_S^k(\tau_j, x)| \leq CT \left(\frac{k + 1}{2}\right) (\text{Im}(\tau_j)^{-k/2} + |\tau_j|^{-1/2}\text{Im}(-1/\tau_j)^{-k/2}).
\]

(4.16)

As \(\gamma_{m-i} = -\frac{1}{\tau_i} = \tau_{i+1} - 2n_i\), \(|\tau_j| > 1\), and the sequence \(\text{Im}(\tau_j)\) is nonincreasing, the right-hand side of (4.16) is bounded from above by \(C\cdot\Gamma((k+1)/2)\text{Im}(\tau)^{-k/2}\).

From (4.15), it follows that

\[
|\phi_A^k(\tau')\text{Im}(\tau')^{1/4} \leq CT \left(\frac{k + 1}{2}\right) \text{Im}(\tau)^{-k/2} \left(\sum_{j=1}^m \text{Im}(\tau_j)^{1/4}\right).
\]

If we use the aforementioned facts about \(\text{Im}(\tau_j)\), we will see that, in fact,

\[
|\phi_A^k(\tau')\text{Im}(\tau')^{1/4} \leq CT \left(\frac{k + 1}{2}\right) \text{Im}(\tau)^{-k/2}m(\tau)^{3/4}.
\]

(4.17)

Now, using the functional equation for \(F^k_\epsilon\) implies

\[
F^k_\epsilon - (F^k_\epsilon)\cdot A = \phi_A^k,
\]

which then gives us

\[
|F^k_\epsilon(\tau, x)||\text{Im}(\tau)|^{1/4} \leq |\text{Im}(\tau')|^{1/4}|F^k_\epsilon(\tau', x)| + |\phi_A^k(\tau', x)||\text{Im}(\tau')|^{1/4}.
\]
Denoting $\text{Im}(\tau') =: I(\tau)$ and using Proposition 4.4 and (4.17) to estimate this expression, it follows that

$$|F_k^\varepsilon(\tau, x)| \leq \text{Im}(\tau')^{-k/2-\frac{1}{4}} \left( C^k (k!) \cdot I(\tau)^{1/4} + \Gamma((k + 1)/2) m(\tau)^{3/4} \right).$$

In order to estimate (4.18), we must resort not only to Lemma 4.3 and its proof, but also to the following estimate of the average values of $m(\tau)$ and $I(\tau)$, recently available by the work of Bondarenko, Radchenko and Seip. We refer the reader to Propositions 6.6 and 6.7 in [3] for a proof.

**Lemma 4.5.** Whenever $y \in (0, 1/2)$, we have

$$\int_{-1}^{1} I(x + iy)^{1/4} \lesssim 1$$

and

$$\int_{-1}^{1} m(x + iy)^{3/4} \lesssim \log^{3/2}(1 + y^{-1}).$$

An application of Lemma 4.5 together with the bound (4.18) to the proof of the first bound in Lemma 4.3 implies

$$\sup_{x \in \mathbb{R}} |x^k b_n^\pm(x)| \lesssim C^k n^{1/4} n^{k/2} \log^{3/2}(1 + n)(k!)$$

for $n > \frac{1}{c_0}, k \geq 1$. Also, in case $n \geq \frac{k}{\pi c_0}$, the sharper bound

$$\sup_{x \in \mathbb{R}} |x^k b_n^\pm(x)| \lesssim (C')^k n^{1/4} n^{k/2} \log^{3/2}(1 + n)(k!)^{1/2}$$

holds instead. We now employ then the main idea of proof of Lemma 4.1: we seek to optimize in $k > 0$. Indeed, let us start by optimizing (4.19). We postpone the discussion on the improved bound (4.20) to a later remark.

Notice that we may assume $|x| \geq C' \sqrt{n}$, as for if $|x| < C' \sqrt{n}$, the bound (4.19) with $k = 0$ gives us already the result, as $1 \lesssim e^{-|x|/\sqrt{n}}$. If we then set $k = \frac{|x|}{C' \sqrt{n}}$, where $C' > 0$ will be a fixed positive constant, whose exact value shall be determined later, we have that

$$|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \cdot \exp(k \log(C n^{1/2}) + k \log(k) - k \log|x|)$$

The exponential term above is

$$\exp \left( \frac{|x|}{C'} \log(C n^{1/2}) + \frac{|x|}{C' \sqrt{n}} (\log(|x|) - \log(C' \sqrt{n})) - \frac{|x|}{C'} \sqrt{n} \log |x| \right)
= \exp \left( \frac{|x|}{C' \sqrt{n}} \log \left( \frac{C}{C'} \right) \right).$$

We only need to set $C' \geq 2C$ above, and this quantity will grow like $\exp(-c|x|/\sqrt{n})$. This finishes the first assertion in Theorem 1.5.

For the second one, we notice that the proof above adapts in many instances. Indeed, if we shift our attention to the function $\partial_x F_k^\varepsilon(\tau, x)$ instead, we will see that, in an almost identical fashion to that of the proof of Proposition 4.4 we are able to prove that, for
all $\tau \in D$, 
\[
|\partial_x F^k_{\varepsilon}(\tau, x)| \lesssim C^k(k!) \Im(\tau)^{-\frac{k+1}{2}}.
\]
On the other hand, the partial derivative $\partial_x$ of the cocycle $\{\phi^k_A\}_{A \in \Gamma_\mu}$ is itself a cocycle with respect to the same slash operator. Moreover, for $A = S$, the following formula holds:
\[
\partial_x \phi^k_S(\tau, x) = (2\pi i)x^{k+1} \left( \tau e^{\pi ix^2\tau} + i\varepsilon(-i\tau)^{-3/2}e^{\pi ix^2(-1/\tau)} \right).
\]
In that case, using the notation from above for the elements $\tau', \tau_j \in \mathbb{H}$ associated to $\tau \in \mathbb{H} \cap \{|z| \leq 1\}$, we see that
\[
\Im(\tau')^{1/4}\Im(\tau')^{1/4} \Im(\tau')^{1/4}\Im(\tau')^{1/4} + \sum_{j=1}^m \Im(\tau_j)^{1/4}\Im(\tau_j)^{1/4}\Im(\tau_j)^{1/4}.
\]
For $j \in \{0, 1, 2, \ldots, m\}$, the definition of our new cocycle implies
\[
|\partial_x \phi^k_S(\tau_j, x)| \lesssim \Gamma \left( \frac{k+3}{2} \right) \left( \Im(\tau_j)^{-\frac{k+1}{2}} + \Im(\tau_j)^{-3/2} \Im(\tau_j+1)^{-\frac{k+1}{2}} \right)
\]
\[
\leq \Gamma \left( \frac{k+3}{2} \right) \Im(\tau)^{-\frac{k+1}{2}}.
\]
This follows as before from the fact that $\Im(\tau_j+1) = \frac{\Im(\tau_j)}{|\tau_j|^2} \geq \Im(\tau)$ and that $|\tau_j| > 1$.
Analyzing the functional equations for $\partial_x F^k_{\varepsilon}(\tau, x)$ in the same way as before readily gives that
\[
|\partial_x F^k_{\varepsilon}(\tau, x)| \leq C^k \Im(\tau)^{-\frac{k+1}{2}}(k!) \left( \Im(\tau)^{1/4} + m(\tau)^{3/4} \right).
\]
Lemma 1.5 and the considerations employed for $F^k_{\varepsilon}$ apply almost verbatim here, and thus we conclude that
\[
|(b_n^\pm)'(x)| \lesssim n^{3/4} \log^{3/2}(1+n)e^{-c|x|/\sqrt{n}},
\]
as wished.

As a consequence of Theorem 1.5, we are able to establish the following bound for the interpolation basis taking account both decay and zeros.

**Corollary 4.6.** Let $\{a_n\}$ be the interpolation sequence of functions from 1.5. Then there is $c > 0$ so that
\[
|a_n(x)| \lesssim n^{3/4} \log^{3/2}(1+n) \text{dist}(|x|, \sqrt{n})e^{-\frac{|x|}{\sqrt{n}}},
\]
for all positive integers $n \in \mathbb{N}$. 

\[\square\]
Proof. We simply use the fundamental theorem of calculus to the $a_n$: without loss of generality, we suppose $x > 0$. We then have:

$$|a_n(x)| = |a_n(x) - a_n(\sqrt{n}) + \delta_{n,m}| \leq \int_{\sqrt{n}}^{x} |a_n'(x)| \, dx + \delta_{n,m}$$

$$\leq n^{3/4} \log^{3/2}(1 + n) \text{dist}(x, \sqrt{n}) e^{-c|x|/\sqrt{n}} + \delta_{n,m}$$

$$\lesssim n^{3/4} \log^{3/2}(1 + n) \text{dist}(x, \sqrt{n}) e^{-c|x|/\sqrt{n}},$$

as the $\delta_{m,n}$ factor is only one if $|x| \in [\sqrt{n}, \sqrt{n + 1})$, where $1 \lesssim e^{-c|x|/\sqrt{n}}$. \[\square\]

Remark. Although the exponential bound $n^{1/4} \log^{3/2}(1 + n)e^{-c|x|/\sqrt{n}}$ suffices for our purposes, below we sketch how to deduce a slightly improved decay for the interpolation basis $\{a_n\}_{n \geq 0}$.

We again wish to optimize (4.20). If we set $k = \frac{|x|^2}{C'n}$, where $C' > 0$ will be chosen soon, we have that

$$|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \cdot \exp(k \log(Cn^{1/2}) + k \log(k^{1/2}) - k \log|x|).$$

This bound holds as long as $\pi n \gtrsim k \geq 1$. If instead $k < 1$, that means, $|x| \leq \sqrt{C'} \sqrt{n}$, we use the bound in either (4.19) or (4.20) for $k = 0$, which yields $|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \lesssim n^{1/4} \log^{3/2}(1 + n)e^{-c|x|^2/n}$, for $c > 0$.

On the other hand, in case $k > 1$, the first exponential term above becomes

$$\exp\left(\frac{|x|^2}{C'n} \log(Cn^{1/2}) + \frac{|x|^2}{C'n} (\log(|x|) - \log(\sqrt{C'n})) - \frac{|x|^2}{C'n} \log|x|\right)$$

$$= \exp\left(\frac{|x|^2}{C'n} \log\left(\frac{C}{\sqrt{C'}}\right)\right).$$

We only need to set $C' \geq (2C)^2$ above, and this quantity will grow like $\exp(-c|x|^2/n)$.

For the remaining $|x| > \sqrt{C'n}$ case, we need to refine the analysis of the proof of Lemma 4.3 and Theorem 1.5. Indeed, it is easy to see that if $n \in (2^{-j}\alpha, 2^{1-j}\alpha)$, $j \geq 1$, then evaluating the Fourier coefficients of a 2-periodic function $f : \mathbb{H} \rightarrow \mathbb{C}$ such that $|f(\tau)| \lesssim \text{Im}(\tau)^{-\alpha} (I(\tau)^{1/4} + m(\tau)^{1/4})$ for $\text{Im}(\tau) \leq 1$ as $2c_n = \int_{-1+\sqrt{2}\pi n}^{1+\sqrt{2}\pi n} f(\tau)e^{-\tau i\pi \tau} \, d\tau$ implies

$$|c_n| \lesssim \left(\frac{2\pi e^{1/2\alpha}}{\alpha}\right)^\alpha n^{\alpha} \log^{3/2}(1 + n).$$

Using this new bound in (4.18), we obtain that, when $n \in (2^{-j-1}k, 2^{-j}k)$,

$$|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \cdot \exp\left(k \left(\frac{j}{2} + \log(C \sqrt{n}) + \log(k^{1/2}) - \log|x|\right)\right).$$

This suggests that we take $k = \frac{|x|^2}{C'^2 n}$, which is admissible to the condition $n \in (2^{-j-1}k, 2^{-j}k)$ if $|x| \sim \sqrt{C'n}2^{3j}$. A similar computation to the ones above implies that

$$|b_n^\pm(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \exp\left(-c\frac{|x|^2}{2n}\right) \lesssim n^{1/4} \log^{3/2}(1 + n) \exp(-c'|x|).$$
whenever $C' \gg C$. The next corollary then follows as a natural consequence.

**Corollary 4.7.** Let $a_n : \mathbb{R} \to \mathbb{R}$ be the interpolating functions in the Radchenko-Viazovska interpolation formula. Then there are $c, C > 0$ so that

$$|a_n(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \left( e^{-c|x|^2/n} 1_{|x| < Cn} + e^{-c|x|^2/1_{|x| > Cn}} \right),$$

for each $n \geq 1$.

Indeed, the application of Lemma 4.3 requires that we take $n \geq C$, for some $C > 0$ some absolute constant. In order to prove such a result for $n \lesssim 1$, we may simply use the definition of $b_n^\pm$ as a Laplace transform of a the weakly holomorphic modular form $g_n^\pm$. Indeed, in order to extend Corollary 4.7 to $n = 0$, we write

$$a_0(x) = \tilde{a}_0(x) = \frac{1}{4} \int_{1}^{1} \theta(z)^3 e^{\pi x^2 z} \, dz.$$ 

In order to prove that $a_0$ decays exponentially, we employ a similar technique to that of [24 Proposition 1]. Indeed, we have

$$|\theta(z)|^3 \lesssim \text{Im}(z)^{-2} e^{-\pi/\text{Im}(z)} \quad \text{for } z \to \pm 1$$

and moreover that $|\theta(z)| \lesssim 1$ whenever $z \in \mathbb{H}, |z| = 1$. We also suppose without loss of generality that $x > 0$. This implies that, for $\delta > 0$,

$$|a_0(x)| \lesssim \int_{0}^{\delta} \frac{e^{-1/(2t)}}{t^2} \, dt + e^{-\pi x^2 \delta} \lesssim e^{-\pi x^2} + e^{-\pi x^2 \delta}.$$ 

We then choose, for $x \gg 1$, $\delta = \frac{1}{\sqrt{2\pi x}}$. This implies that $|a_0(x)| \lesssim e^{-\sqrt{2\pi x}}$, which is the desired bound. For other bounded values of $n$ such a proof can be easily adapted.

### 4.2. Proof of the main result.

Let

$$\ell^2_s(\mathbb{N}) = \{ (a_n)_{n} \in \ell^2(\mathbb{N}) : (n^s a_n)_{n} \in \ell^2(\mathbb{N}) \}.$$ 

Let $I : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \to \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ denote the identity operator. Recall the Radchenko-Viazovska interpolation result: for $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ a real function,

$$f(x) = \sum_{n \geq 0} (f(\sqrt{n}) a_n(x) + \hat{f}(\sqrt{n}) \tilde{a}_n(x)),$$ 

where $a_n : \mathbb{R} \to \mathbb{R}$ is a sequence of interpolating functions independent of the Schwartz function $f$. In particular,

$$f(\sqrt{k}) = \sum_{n \geq 0} (f(\sqrt{n}) a_n(\sqrt{k}) + \hat{f}(\sqrt{n}) \tilde{a}_n(\sqrt{k})).$$

In fact, for any pair of sequences $(\{x_i\}, \{y_i\})$ decaying sufficiently fast and satisfying the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} x_n^2 = \sum_{n \in \mathbb{Z}} y_n^2,$$

the function

$$\Phi(t) = \sum_{n \geq 0} (x_n a_n(t) + y_n \tilde{a}_n(t))$$

...
is well-defined and satisfies that \( \Theta(\sqrt{k}) = x_k, \hat{\Theta}(\sqrt{k}) = y_k \). In fact, let \((\{x_i\}, \{y_i\}) \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})\) for \( s > 0 \) sufficiently large. The operator

\[
T : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})
\]
given by \( T = (T^1, T^2) \), where

\[
T^1(\{x_i\}, \{y_i\})_k = \sum_{n \geq 0} (x_n a_n(\sqrt{k}) + y_n \bar{a}_n(\sqrt{k})),
\]

\[
T^2(\{x_i\}, \{y_i\})_k = T^1(\{y_i\}, \{x_i\})_k,
\]

has an explicit form: indeed, for \( k \geq 1 \), we have

\[
T^1(\{x_i\}, \{y_i\})_k = x_k, \; T^2(\{x_i\}, \{y_i\}) = y_k.
\]

For \( k = 0 \), we have

\[
T^1(\{x_i\}, \{y_i\})_0 = \frac{x_0 + y_0}{2} - \sum_{n \geq 1} x_{n^2} + \sum_{n \geq 1} y_{n^2},
\]

\[
T^2(\{x_i\}, \{y_i\})_0 = \frac{x_0 + y_0}{2} - \sum_{n \geq 1} y_{n^2} + \sum_{n \geq 1} x_{n^2}.
\]

In particular, it is then easy to see that \( T = I \) whenever \((\{x_i\}, \{y_i\})\) satisfy the Poisson relation (4.22). Inspired by this fact, we define the perturbed operator associated to a sequence \( \varepsilon_k > 0, k \in \mathbb{N} \), to be

\[
\tilde{T} \text{ defined on } \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}),
\]

where \( \tilde{T} = (\tilde{T}^1, \tilde{T}^2) \), with

\[
\tilde{T}^1(\{x_i\}, \{y_i\})_k = \sum_{n \geq 0} (x_n a_n(\sqrt{k + \varepsilon_k}) + y_n \bar{a}_n(\sqrt{k + \varepsilon_k})),
\]

\[
\tilde{T}^2(\{x_i\}, \{y_i\})_k = \tilde{T}^1((\{y_i\}, \{x_i\})_k,
\]

for \( k \geq 1 \), and \( \tilde{T}^1(\{x_i\}, \{y_i\})_0 = x_0, \tilde{T}^2(\{x_i\}, \{y_i\})_0 = y_0 \). A fundamental fact we will need for our proof is that this operator is bounded from \( \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \). One way to see this will be provided in the proof of our main theorem, by showing that the operator norm \( \|I - \tilde{T}\|_{\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})} < +\infty \). This is, incidentally, our main device to prove our result: if

\[
\|I - \tilde{T}\|_{\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})} < 1,
\]

then \( \tilde{T} \) is an invertible operator defined on \( \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \). Therefore, its inverse

\[
\tilde{T}^{-1} : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})
\]

is well-defined and bounded. In particular, for \( f \in S_{\text{even}}(\mathbb{R}) \) real, given the lists of values

\[
f(0), f(\sqrt{1 + \varepsilon_1}), f(\sqrt{2 + \varepsilon_2}), \ldots,
\]
and proves the existence of two sequences of functions
\( f \) for two sequences \( \{ x_i \}_i, \{ y_i \}_i \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \) so that
\[
\hat{T}(\{ x_i \}_i, \{ y_i \}_i) = (\{ f(\sqrt{k + \varepsilon_k}) \}_k, \{ \hat{f}(\sqrt{k + \varepsilon_k}) \}_k).
\]

But we also know that
\[
\hat{T}(\{ f(\sqrt{i}) \}_i, \{ \hat{f}(\sqrt{i}) \}_i) = T(\{ f(\sqrt{i}) \}_i, \{ \hat{f}(\sqrt{i}) \}_i) = (\{ f(\sqrt{k + \varepsilon_k}) \}_k, \{ \hat{f}(\sqrt{k + \varepsilon_k}) \}_k).
\]

This implies \( x_j = f(\sqrt{j}), y_j = \hat{f}(\sqrt{j}) \). By writing the \( k \)-th entry of the inverse of \( \hat{T} \) as
\[
\hat{T}^{-1}(\{ w_i \}_i, \{ z_i \}_i)_k = \sum_{j \geq 0} (\gamma_{j,k} w_j + \hat{\gamma}_{j,k} z_j),
\]
for two sequences \( \{ \gamma_{j,k} \}_{j,k \geq 0}, \{ \hat{\gamma}_{j,k} \}_{j,k \geq 0} \) so that \( |\gamma_{j,k}| + |\hat{\gamma}_{j,k}| \lesssim (j/k)^s \), we must have
\[
(4.24) \quad f(\sqrt{k}) = \sum_{j \geq 0} (\gamma_{j,k} f(\sqrt{j + \varepsilon_j}) + \hat{\gamma}_{j,k} \hat{f}(\sqrt{j + \varepsilon_j})).
\]

This implies, by (1.3), that we can recover \( f \) from its values and those of its Fourier transform at \( \sqrt{k + \varepsilon_k} \). Moreover, as the adjoint of \( \hat{T}^{-1} \) is also bounded from \( \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \) to itself, we conclude that, for \( s \gg 1 \) sufficiently large and \( f, \hat{f} \) both being \( \mathcal{O}(1 + |x|^{-10s}) \), we can use Fubini’s theorem in (1.3) together with (4.24). This proves the existence of two sequences of functions \( \{ \theta_j \}_{j \geq 0}, \{ \eta_j \}_{j \geq 0} \) so that
\[
|\theta_j(x)| + |\eta_j(x)| + |\hat{\theta}_j(x)| + |\hat{\eta}_j(x)| \lesssim (1 + j)^s (1 + |x|)^{-10}
\]
and
\[
f(x) = \sum_{j \geq 0} \left( f(\sqrt{j + \varepsilon_j}) \theta_j(x) + \hat{f}(\sqrt{j + \varepsilon_j}) \eta_j(x) \right).
\]

Thus, we focus on the proof of the invertibility of \( \hat{T} \), for \( s > 0 \) suitably chosen.

---

**Proof of invertibility of \( \hat{T} \).** We use, for this part, the Schur test. That is, define the infinite matrices
\[
A = \{ A_{ij} \}_{i,j \geq 0} \quad \text{and} \quad \hat{A} = \{ \hat{A}_{ij} \}_{i,j \geq 0}
\]
by
\[
A_{ij} = (a_j(\sqrt{i + \varepsilon_i} - \delta_{ij}) \times (i/j)^s, \quad \hat{A}_{ij} = \hat{a}_j(\sqrt{i + \varepsilon_i})(i/j)^s.
\]

For a given vector \( (x, y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \), we write then
\[
B(x, y) = (A \cdot x + \hat{A} \cdot y, A \cdot y + \hat{A} \cdot x),
\]
or, in matrix notation,
\[
B = \left( \begin{array}{cc} A & \hat{A} \\ \hat{A} & A \end{array} \right).
\]

Notice that the operator norm of \( \hat{T} - I \) acting on \( \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \) is, by virtue of our definitions, the same as the operator norm of \( B \) acting on \( \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \). Therefore, it will suffice to impose bounds on this latter quantity.
By Schur’s test, it suffices to find $\alpha, \beta > 0$ and positive sequences $\{p_i\}_{i \geq 0}, \{q_i\}_{i \geq 0}$ so that the following inequalities hold:

$$
\sum_{j > 0} (i/j)^s \times [(a_j(\sqrt{i + \varepsilon_i}) - \delta_{ij})|p_j| + |\tilde{a}_j(\sqrt{i + \varepsilon_i})|q_j] \leq \alpha p_i,
$$

$$
\sum_{j > 0} (i/j)^s \times [(a_j(\sqrt{i + \varepsilon_i}) - \delta_{ij})|q_j| + |\tilde{a}_j(\sqrt{i + \varepsilon_i})|p_j] \leq \alpha q_i,
$$

$$
\sum_{i > 0} (i/j)^s \times [(a_j(\sqrt{i + \varepsilon_i}) - \delta_{ij})|p_i| + |\tilde{a}_j(\sqrt{i + \varepsilon_i})|q_i] \leq \beta p_j,
$$

$$
\sum_{i > 0} (i/j)^s \times [(a_j(\sqrt{i + \varepsilon_i}) - \delta_{ij})|q_i| + |\tilde{a}_j(\sqrt{i + \varepsilon_i})|p_i] \leq \beta q_j.
$$

(4.25)

Now, we make the Ansatz that, for all $i > 0, p_i = q_i = i^\theta$, for some real number $\theta \in \mathbb{R}$. By making use of Theorem 1.5 we know that

$$
|a_j(\sqrt{i + \varepsilon_i}) - \delta_{ij}| + |\tilde{a}_j(\sqrt{i + \varepsilon_i})| \lesssim \frac{\varepsilon_i}{\sqrt{i}} j^{3/4} e^{-c \sqrt{i/j}}.
$$

Therefore, (4.25) reduces to verifying

$$
\sum_{j > 0} (i/j)^s \times j^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4} e^{-c \sqrt{i/j}} \leq \alpha i^\theta,
$$

$$
\sum_{i > 0} (i/j)^s \times i^\theta \times \frac{\varepsilon_i}{\sqrt{i}} j^{3/4} e^{-c \sqrt{i/j}} \leq \beta j^\theta.
$$

(4.26)

Estimate of the first term in (4.26). For this term, we rewrite it as

$$
i^{s-1/2} \times \varepsilon_i \left( \sum_{j > 0} j^{3/4-s} e^{-c \sqrt{i/j} j^\theta} \right).
$$

In order to estimate this last sum, we break it into $j < i^{1/3}$ and $j > i^{1/3}$ contributions. Therefore,

$$
\sum_{j > 0} j^{3/4-s} e^{-c \sqrt{i/j} j^\theta} \lesssim i^{1/3} \max(3/4-s+\theta,0) e^{-c i^{1/3}} + \sum_{j > i^{1/3}} j^{3/4-s} e^{-c \sqrt{i/j} j^\theta}.
$$

(4.27)

Because of the presence of the exponential, the first term is always bounded by an absolute constant times $i^\theta$, so we treat it as negligible. For the second term, notice that the summand is bounded by a constant times $\int_{j^{3/4-s+\theta}}^{j^{3/4-s+\theta}+\varepsilon} e^{\sqrt{j/x} - \sqrt{i/j}} dx$. Indeed, the ratio between both is bounded by

$$
\int_j^{j+1} (x/j)^{3/4-s+\theta} e^{\sqrt{i/j} - \sqrt{i/x}} dx \leq 2^{3/4-s+\theta} \sup_{x \in [j,j+1)} e^{\frac{\sqrt{i/x} - \sqrt{i/j}}{\sqrt{x}}} \lesssim_{s,\theta} 1,
$$

$$
\int_j^{j+1} \varepsilon^{3/4-s+\theta} e^{\sqrt{i/j} - \sqrt{i/x}} dx \leq 2^{3/4-s+\theta} \varepsilon^{\frac{\sqrt{i}}{\sqrt{x}}} \lesssim_{s,\theta} 1.
$$
as \( j > i^{1/3} \). Thus, we obtain that the second term on the right-hand side of (4.27) is bounded by

\[
\int_{i^{1/3}}^\infty x^{3/4-s+\theta} e^{-c\sqrt{i/x}} \, dx = \int_0^{i^{1/3}} (1 + 1/y)^{3/4-s+\theta} y^{-2} e^{-c\sqrt{y}} \, dy
\]

\[
\lesssim_{s,\theta} \int_0^{i^{1/3}} y^{-11/4+s-\theta} e^{-c\sqrt{y}} \, dy = i^{7/4-s+\theta} \int_0^{i^{2/3}} y^{-11/4+s-\theta} e^{-c\sqrt{y}} \, dy
\]

\[
\lesssim_{s,\theta} i^{7/4-s+\theta},
\]

as long as \(-11/4 + s - \theta > -1\), that is, \( \theta < s - 7/4 \). Thus, the first term in (4.26) is bounded under such a condition by

\[
C_{s,\theta} i^{s-\frac{7}{4} - s + \theta} = i^{s+\theta} \epsilon_i.
\]

In order for this last quantity to be less than \( \alpha i^{\theta} \), we must have \( \epsilon_i \lesssim_{s,\theta} \alpha i^{-\frac{5}{4}} \). We will assume that we have this bound while estimating the second term.

Estimate for the second term in (4.26). For the second term, the strategy is similar, only now the estimates become somewhat simpler by the arithmetic of the bounds given by Theorem 1.5. Indeed, the second term in (4.26) is bounded by

\[
c_{s,\theta} \sum_{i>0} i^{s-\frac{7}{4}-s+\theta} e^{-c\sqrt{i/j}}.
\]

Similarly as before, each summand above is bounded by \( \int_1^{i+1} x^{s+\theta-\frac{7}{4}} e^{-c\sqrt{x}} \, dx \). Thus, the expression within the parenthesis above is bounded by

\[
\int_1^\infty x^{s+\theta-\frac{7}{4}} e^{-c\sqrt{x}} \, dx \lesssim_{s,\theta} j^{s+\theta-\frac{7}{4}} \int_0^\infty x^{s+\theta-\frac{7}{4}} e^{-c\sqrt{x}} \, dx.
\]

This last integral converges given that \( s + \theta - \frac{7}{4} > -1 \iff s + \theta > \frac{3}{4} \). In the end, we obtain that the second term in (4.26) is bounded by \( c_{s,\theta} j^\theta \) if these conditions on \( s,\theta \) hold.

Finally, we gather these two estimates to get that, if \( s - \theta > \frac{7}{4}, s + \theta > \frac{3}{4} \) and if \( \epsilon_i < \gamma i^{-\frac{7}{4}} \) for \( \gamma > 0 \) sufficiently small, then both terms of (4.26) are bounded by small constants times \( i^\theta \) and \( j^\theta \). Notice that picking \( s = 10 \) and \( \theta > 0 \) sufficiently small yields that both conditions above hold true, and thus the result follows from Schur’s test, as previously indicated.

As mentioned in the beginning of this manuscript, the usage of Schur’s test here was instrumental in order to expand the range of our perturbations. In fact, in §5.1 we employ the Hilbert–Schmidt test successfully to our operator \( \tilde{T} \) and obtain that, as long as there is \( \delta > 0 \) such that \( \epsilon_i \lesssim i^{-\frac{7}{4} - \delta} \), then \( \tilde{T} \) is bounded on \( \ell^2_s(\mathbb{N}) \times \ell^2_j(\mathbb{N}) \), for \( s \) sufficiently large, but we seem to be unable to include \( 5/4 \) in our considerations with the Hilbert–Schmidt method.

On the other hand, we will see in that subsection that the Hilbert–Schmidt method provides us with a way to suitably perturb the origin, a feature we could not obtain with Schur’s test.
5. Applications of the main results and techniques

5.1. Interpolation formulae perturbing the origin. In the main results of this manuscript, the only interpolation node that remains unchanged in every scenario is 0. One of the reasons for that is aesthetic: we are concerned mainly with even functions here, so the origin keeps a sense of symmetry. The other main reason is technical: we recall that the operator

$$T : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \to \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$$

given by $T = (T^1, T^2)$, where

$$T^1(\{x_i\}, \{y_i\})_k = \sum_{n \geq 0} (x_n a_n(\sqrt{k}) + y_n \overline{a_n}(\sqrt{k})),
\quad T^2(\{x_i\}, \{y_i\})_k = T^1(\{y_i\}, \{x_i\})_k,$$  

for $k \geq 0$, is the identity when restricted to the set of pairs of sequences satisfying the Poisson summation formula

$$\sum_{n \in \mathbb{Z}} x_n^2 = \sum_{n \in \mathbb{Z}} y_n^2.$$  

For general sequences, the first entries of this operators possess a correction factor due to the lack of Poisson summation. Indeed, it is not difficult to verify that $\dim(\ker(T)) = \dim(\coker(T)) = 1$ from the explicit definitions. Therefore, we can no longer prove invertibility.

Nonetheless, we also remark that a direct computation shows that the range of $T$ is closed. Therefore, $T$ satisfies all conditions to be a Fredholm operator.

Let us then define a new perturbed operator $S$ defined on $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$, such that

$$S^1(\{x_i\}, \{y_i\})_k = \sum_{n \geq 0} (x_n a_n(\sqrt{k + \varepsilon_k}) + y_n \overline{a_n}(\sqrt{k + \varepsilon_k})),
\quad S^2(\{x_i\}, \{y_i\})_k = S^1(\{y_i\}, \{x_i\})_k,$$  

for all $k \geq 0$, where $\varepsilon_k > 0$, $\forall k \geq 0$. We denote by $e_n \in \ell^2_s(\mathbb{N})$ the vector consisting of $n^{-s}$ on the $n$–th entry, and zero otherwise. With this definition, the set

$$\{(e_n, 0) : n \in \mathbb{N} \} \cup \{(0, e_n) : n \in \mathbb{N} \}$$

forms an orthonormal basis of $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$. Thus,

$$\|A\|_{HS(\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}))} = \sum_{n \in \mathbb{N}} (\|A(e_n, 0)\|_{(s,s)}^2 + \|A(0, e_n)\|_{(s,s)}^2),$$

where we denote by $\| \cdot \|_{(s,s)}$ the norm of $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$. Let then $A = I - \tilde{T}$.

**Claim 5.1.** $\|A\|_{HS(\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}))} < +\infty$ holds whenever there is $\delta > 0$ so that $|\varepsilon_k| \lesssim k^{-\frac{s}{2} - \delta}$, $\forall k \geq 1$.  

Proof of Claim 5.1. As mentioned before, we can write the identity on $\ell_2^s(\mathbb{N}) \times \ell_2^s(\mathbb{N})$ as

$$I(\{x_i\}, \{y_i\}) = ((x_0, \mathfrak{G}(1), \mathfrak{G}(\sqrt{2}), \ldots), (y_0, \hat{\mathfrak{G}}(1), \hat{\mathfrak{G}}(\sqrt{2}), \ldots)),$$

where we define the function $\mathfrak{G}$ as in (4.23). With this notation, the operator $\tilde{T}$ becomes

$$\tilde{T}(\{x_i\}, \{y_i\}) = ((x_0, G(\sqrt{1} + \varepsilon_1), G(\sqrt{2} + \varepsilon_2), \ldots), (y_0, \hat{G}(\sqrt{1} + \varepsilon_1), \hat{G}(\sqrt{2} + \varepsilon_2), \ldots)).$$

Therefore, evaluating at the basis vectors gives us

$$(I - \tilde{T})(e_n, 0) = ((0, n^{-s}(a_n(\sqrt{1}) - a_n(\sqrt{1} + \varepsilon_1)), n^{-s}(a_n(\sqrt{2}) - a_n(\sqrt{2} + \varepsilon_2)), \ldots), (0, 0, \ldots)).$$

We readily see then that

$$\|I - \tilde{T}\|_{HS(\ell_2^s(\mathbb{N}) \times \ell_2^s(\mathbb{N}))}^2 = \sum_{n>0} \left( \sum_{k \geq 0} (1 + k)^{2s}(1 + n)^{-2s}|a_n(\sqrt{k}) - a_n(\sqrt{k + \varepsilon_k})|^2 \right)$$

$$+ \sum_{n>0} \left( \sum_{k \geq 0} (1 + k)^{2s}(1 + n)^{-2s}|\hat{a}_n(\sqrt{k}) - \hat{a}_n(\sqrt{k + \varepsilon_k})|^2 \right).$$

From Theorem 1.5, we know that

$$|a_n(\sqrt{k}) - a_n(\sqrt{k + \varepsilon_k})| \leq \int_{\sqrt{k}}^{\sqrt{k + \varepsilon_k}} |a'_n(t)| \, dt \leq \frac{C \varepsilon_k}{\sqrt{k}} n^{3/4} e^{-c \sqrt{k/n}},$$

(5.2)

for some $c > 0$ and $k \geq 1$. Analogously,

$$|\hat{a}_n(\sqrt{k}) - \hat{a}_n(\sqrt{k + \varepsilon_k})| \leq \frac{C \varepsilon_k}{\sqrt{k}} n^{3/4} e^{-c \sqrt{k/n}}.$$

These estimates plus the condition on the $\varepsilon_k$ imply that (5.1) may be bounded from above by an absolute constant times

$$\sum_{n \geq 0} \left( \sum_{k \geq 1} k^{2s} k^{-\frac{3}{2} - 2\delta} \cdot k^{-1} e^{-c \sqrt{k/n}} \right) n^{\frac{3}{2} - 2s}.$$

In order to prove convergence, we first investigate the inner sum. A Riemann sum approach together with a change of variables shows that this is bounded by a constant times

$$(1 + n)^{2s - \frac{3}{2} - 2\delta} \left( \int_0^\infty t^{2s - \frac{3}{2} - 2\delta} \cdot t^{-1} e^{-c \sqrt{t}} \, dt \right) =: (1 + n)^{2s - \frac{3}{2} - 2\delta} I_{s, \delta}.$$
Clearly, the inner integral converges given that \( s > \frac{5}{4} + \delta \). Putting these estimates together with (5.1) and using Fubini, we obtain that

\[
\|I - \tilde{T}\|_{HS(\ell_2^2(\mathbb{N}) \times \ell_2^2(\mathbb{N}))}^2 \leq I_{s, \delta} \left( \sum_{n \geq 0} (1 + n)^{-1 - 2\delta} \right) < +\infty,
\]

as desired. \( \square \)

As a direct corollary, we see that, for each \( \delta > 0 \), there is \( a > 0 \) so that, if \( |\varepsilon_i| \leq ai^{-\frac{5}{4} - \delta} \forall i > 0 \), then

\[
\|A\|_{HS(\ell_2^2(\mathbb{N}) \times \ell_2^2(\mathbb{N}))} < 1.
\]

In particular, we shall make use of the fact that \( T \) is a Fredholm operator by means of such an inequality, with aid of the following result:

**Theorem 5.2** (Theorems 2.8 and 2.10 in [26]). Let \( \Phi(X, Y) \) denote the set of bounded Fredholm operators between Banach spaces \( X \) and \( Y \). If \( A \in \Phi(X, Y) \) and \( K \in \mathcal{K}(X, Y) \) is a compact operator, then \( A + K \in \Phi(X, Y) \) and \( \iota(A) = i(A + K) \), where we define the index \( i : \Phi(X, Y) \to \mathbb{N} \) by \( \iota(A) = \dim(\ker(A)) - \dim(\coker(A)) =: \alpha(A) - \beta(A) \).

Furthermore, if \( \|K\|_{op} \) is small enough, then it also holds that \( \alpha(A + K) \leq \alpha(A) \).

Notice that we may write \( S - T = \tilde{T} - I + K_0 \), where \( K_0 \) has finite rank and bounded, and thus also compact. Therefore, \( S = T + (S - T) = T + (\tilde{T} - I) + K_0 \) can be written as sum of a Fredholm operator \( T \) and a compact operator \( \tilde{T} - I + K_0 \). This already implies that, modulo a finite-dimensional subspace, the sequences \( \{(f(\sqrt{k} + \varepsilon_k)), \{\hat{f}(\sqrt{k} + \varepsilon_k)\}\} \) determine the sequences \( \{(f(\sqrt{k})), \{\hat{f}(\sqrt{k})\}\} \). That is, we can determine the function \( f \in \mathcal{S}_{even}(\mathbb{R}) \) from its (Fourier-)values as \( \sqrt{k + \varepsilon_k} \), modulo subtracting functions belonging to a finite-dimensional space.

If, however, we make \( |\varepsilon_k| < \varepsilon k^{-\frac{4}{5} - \delta} \), with \( \varepsilon \) small enough, and \( |\varepsilon_0| \ll 1 \), we get that the operator norms of both \( I - \tilde{T} = A \) and \( K_0 \) can be made arbitrarily small. Thus, \( i(S) = i(T + (S - T)) = i(T) = 0 \iff \alpha(S) = \beta(S) \), and, moreover, \( \alpha(S) \leq \alpha(T) \), as the Hilbert–Schmidt norm of the difference is small. Thus, either \( \alpha(S) = \beta(S) = 0 \), in which case we can perfectly invert the operator \( S \), or \( \alpha(S) = \beta(S) = 1 \), which implies that there is essentially at most one function \( f_0 \in \mathcal{S}_{even}(\mathbb{R}) \) that vanishes at \( \sqrt{k + \varepsilon_k} \). As \( \{(f(\sqrt{k} + \varepsilon_k)), \{\hat{f}(\sqrt{k} + \varepsilon_k)\}\} \in \text{im}(S) \) for every real \( f \in \mathcal{S}_{even}(\mathbb{R}) \), we have proved the followin result.

**Theorem 5.3.** Let \( T, S, \{x_i\}_{i \geq 0} \) be as above. Then one of the following holds:

1. Either \( S \) is an isomorphism from \( \ell_2^2(\mathbb{N}) \times \ell_2^2(\mathbb{N}) \) onto itself, and thus the values \( \{(f(\sqrt{j} + \varepsilon_j)), \{\hat{f}(\sqrt{j} + \varepsilon_j)\}\} \)
determine any real function \( f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \):

(2) Or \( \ker(S) \) has dimension one, and therefore \( S \) is an isomorphism from \( \ker(S) \) onto \( \im(S) \).

In particular, any real function \( f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \) is uniquely determined by

\[
\{(f(\sqrt{j} + \varepsilon_j)), \{\hat{f}(\sqrt{j} + \varepsilon_j)\}\},
\]

together with the value of

\[
\frac{\langle\{(f(\sqrt{j} + \varepsilon_j)), \{\hat{f}(\sqrt{j} + \varepsilon_j)\}\}, \{\{\alpha_i\}, \{\beta_i\}\}\rangle_{(s,s)}}{\|\{\alpha_i\}, \{\beta_i\}\|_{(s,s)}^2},
\]

where \((\{\alpha_i\}, \{\beta_i\}) \in \ker(S)\) is a generator for the kernel of \( S \).

Notice that the first option in Theorem 5.3 yields immediately an interpolation formula, in the spirit of (4.23). For the second one, the operator is now only invertible if restricted to \( \ker(S) \), and now the process of recovering \( f \in \mathcal{S}_{\text{even}}(\mathbb{R} : \mathbb{R}) \) has to take into account the inner product with the kernel vector and the structure of the range.

5.2. Uniqueness for small powers of integers. Let \( \alpha \in (0, 1/2) \). We are interested in determining when the only function \( f \in \mathcal{S}_{\text{even}}(\mathbb{R}) \) that vanishes together with its Fourier transform at \( \pm n^\alpha \) is the identically zero function.

Indeed, we would like to study the natural operator that sends the sequence of values at the roots of integers \( \{(f(\sqrt{R})_k, \{\hat{f}(\sqrt{R})_k\}\} \) to the sequence \( \{(f(n^\alpha)_n, \{\hat{f}(n^\alpha)_n\}\). Our goal is to show that this operator is injective. In order to do that, we will first study simpler operators.

Indeed, let \( K_0 \in \mathbb{N} \) be a fixed positive integer. Fix a set of \( 2K_0 \) positive real numbers \( t_1 < t_2 < \cdots < t_{2K_0} \) such that \( t_1 > \sqrt{K_0} \) and none of the \( t_j \) can be written as a square root of a positive integer. We fix \( s > 0 \) sufficiently large and define the operator \( T_{K_0} : \ell^2_0(\mathbb{N}) \times \ell^2_0(\mathbb{N}) \rightarrow \ell^2_0(\mathbb{N}) \times \ell^2_0(\mathbb{N}) \)

\[
(x_i, y_i) \mapsto ((x_0, \Theta(t_1), \Theta(t_2), \ldots, \Theta(t_{2K_0}), x_{K_0+1}, x_{K_0+2}, \ldots),
\]

\[
(y_0, \hat{\Theta}(t_1), \hat{\Theta}(t_2), \ldots, \hat{\Theta}(t_{2K_0}), y_{K_0+1}, y_{K_0+2}, \ldots)).
\]

Here, we denoted by \( \Theta \) the function defined as in (4.23).

Lemma 5.4. For any \( K_0 \geq 1 \), the operator \( T_{K_0} \) is bounded and injective.

Proof. We begin with the boundedness assertion. As \( T_{K_0} \) differs only in the first \( K_0 \) coordinates from an iteration of the shift operator

\[
s(((x_i)_i, (y_i)_i) = ((0, x_0, x_1, \ldots), (0, y_0, y_1, \ldots)),
\]
boundedness follows from boundedness of the operator that maps a pair of sequences \((\{x_i\}, \{y_i\}) \in \ell_2^\infty(\mathbb{N}) \times \ell_2^\infty(\mathbb{N})\) into
\[
(\langle x_0, \mathcal{G}(t_1), \mathcal{G}(t_2), \ldots, \mathcal{G}(t_{2K_0}), 0, \ldots \rangle, \langle y_0, \mathcal{G}(t_1), \mathcal{G}(t_2), \ldots, \mathcal{G}(t_{2K_0}), 0, \ldots \rangle).
\]
As \(\mathcal{G}, \mathcal{G} \in L^\infty(\mathbb{R})\) for any pair of sequences \(\{x_i\}, \{y_i\}\), with bounds depending only on the \(\ell_2^\infty(\mathbb{N})\)-norms of the sequences, it follows that this new finite-rank operator is bounded.

The injectivity part is subtler. Indeed, fix a pair of sequences \((\{x_i\}, \{y_i\}) \in \ell_2^\infty(\mathbb{N}) \times \ell_2^\infty(\mathbb{N})\), and suppose that \(T_{K_0}(\{x_i\}, \{y_i\}) = 0\). It follows that the special function \(\mathcal{G}(t)\) is a linear combination of \(a_1, \ldots, a_{K_0}, \hat{a}_1, \ldots, \hat{a}_{K_0}\). In order to analyze such functions, we will need to investigate further the intrinsic form of the interpolating functions \(a_n\), and thus those of \(b_n^\pm\).

Indeed, it follows from the Fourier expansion of \(g_n^\pm\) near infinity and the formula
\[
b_n^\pm(x) = \frac{1}{2} \int_{-1}^{1} g_n^\pm(z)e^{\pi i x z} \, dz
\]
that, whenever \(|x| > \sqrt{\pi}\), it can also be represented as
\[
b_n^\pm(x) = \sin(\pi x^2) \int_{0}^{\infty} g_n^\pm(1 + it)e^{-\pi x^2 t} \, dt.
\]
As \(a_n = (b_n^+ + b_n^-)/2\) and \(\hat{a}_n = (b_n^+ - b_n^-)/2\), we see that the Fourier invariant part of our function \(g\) may be written as
\[
(\mathcal{G} + \mathcal{G})(x) = \sin(\pi x^2) \int_{0}^{\infty} \left( \sum_{j=1}^{K_0} \alpha_j g_j^\pm(1 + it) \right) e^{-\pi x^2 t} \, dt,
\]
for some sequence \(\alpha_j\) of real numbers, and an analogous identity holds for the \(-1\)-eigenvalue part \(\mathcal{G} - \mathcal{G}\), with \(g_n^-\) instead. We recall that the weakly holomorphic modular forms \(g_n^\pm\) satisfy that
\[
g_n^+(z) = \theta(z)^3 P_n^+(1/J(z)),
g_n^-(z) = \theta(z)^3 (1 - 2\lambda(z)) P_n^-(1/J(z)),
\]
where the monic polynomials \(P_n^-, P_n^+\) are of degree \(n\). Therefore, there are polynomials \(Q, R\) of degree \(\leq K_0\) such that
\[
\mathcal{G} + \mathcal{G} = \sin(\pi x^2) \int_{0}^{\infty} \theta(1 + it)^3 Q(1 + it) e^{-\pi x^2 t} \, dt
\]
\[
\mathcal{G} - \mathcal{G} = \sin(\pi x^2) \int_{0}^{\infty} \theta(1 + it)^3 (1 - 2\lambda(1 + it)) R(1 + it) e^{-\pi x^2 t} \, dt.
\]
(5.3)
Before moving forward, we need the following result:
Lemma 5.5. The factors $\theta(1+it)^3$ and $(1-2\lambda(1+it))$ do not change sign for $t \in (0, \infty)$, and the function $1/J(1+it)$ is real-valued and monotonic for $t \in (0, \infty)$.

Proof. By using (2.1), we get that
$$\theta(1+it) = \sum_{n \in \mathbb{Z}} (-1)^n e^{-\pi n^2 t} = \sum_{n \in \mathbb{Z}} e^{-4\pi n^2 t} - \sum_{n \in \mathbb{Z}} (-1)^n e^{-2(2n+1)^2 t}.$$  
We now consider the function $f_t(x) = e^{-\pi (2x)^2 t}$. Then the sum above equals
$$\sum_{n \in \mathbb{Z}} f_t(n) - \sum_{n \in \mathbb{Z}} f_t(n + 1/2).$$  
By the Poisson summation formula, the difference above equals
$$\frac{1}{2\sqrt{t}} \left( \sum_{n \in \mathbb{Z}} e^{-\pi (\frac{n}{\sqrt{t}})^2} - \sum_{n \in \mathbb{Z}} e^{\pi in} e^{-\pi (\frac{n}{\sqrt{t}})^2} \right) = \frac{1}{\sqrt{t}} \sum_{n \text{ odd}} e^{-\pi (\frac{n}{\sqrt{t}})^2} \geq 0.$$  
This proves the first assertion.

For the second, we simply see from (2.2) that $\lambda(1+z)$ has only nonpositive coefficients in its $q$–series expansion. This implies that $\lambda(1+it)$ is nonpositive for $t \in (0, \infty)$, which implies that $1-2\lambda(1+it)$ is always nonnegative.

Finally, for the third assertion, we notice that, as $J(1+z) = \frac{1}{16} \lambda(1+z)(1-\lambda(1+z))$, and thus, from the analysis above, the $q$–series expansion of $J(1+z)$ contains only nonpositive coefficients. Therefore, the function $\frac{1}{J(1+it)}$ is nonpositive for $t \in (0, \infty)$, and it is monotonically decreasing there. This finishes the proof. \hfill \Box

By Lemma 5.5, we get that the part of the integrand in the expressions above multiplying the $e^{-\pi x^2 t}$ factor changes sign at most $K_0 + 1$ times. Notice that we can embed both integrals in (5.3) into the framework of Laplace transforms: denoting
$$Q(t) = \theta(1+it)^3 Q(1+it), \ R(t) = \theta(1+it)^3 (1-2\lambda(1+it)) R(1+it),$$
we are interested in studying the positive zeros of $\mathcal{L}[Q](\pi x^2), \mathcal{L}[R](\pi x^2)$, where
$$\mathcal{L}[\phi](s) = \int_0^\infty \phi(t) e^{-st} dt$$
denotes the Laplace transform of $\phi$ evaluated at the point $s$. We may reduce even further our task to studying the positive zeros of $\mathcal{L}[Q], \mathcal{L}[R]$. The following result, a version of the Descartes rule for the Laplace transform, is the tool we need to bound the number of positive zeros of such expressions as a function of their number of changes of signs.

Proposition 5.6 (Descartes rule for the Laplace transform). Let $\phi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that its Laplace transform $\mathcal{L}[\phi]$ converges on some open half-plane $\text{Re}(s) > s_0$. Then the number of zeros of $\mathcal{L}[\phi]$ on the interval $(s_0, +\infty)$ is at most number of sign changes of $\phi$.

Proof. The proof follows by induction on the number of sign changes of the function $\phi$. Indeed, if $\phi \equiv 0$, it follows easily that the Laplace transform $\mathcal{L}[\phi] \geq 0$, with equality if and only if $\phi \equiv 0$. 


Suppose now that $\phi$ changes sign $n + 1$ times on $(0, \infty)$. Number its zeros on the positive half-line as $s_0 < s_1 < \cdots < s_n$. Then $L[\phi]$ has as many zeros as $e^{s_0 t}L[\phi](t) = F(t)$. The derivative of $F$ is then given by

$$F'(t) = -\int_0^\infty (s - s_0)\phi(s)e^{-(s-s_0)t}\,ds = e^{s_0 t}L[(s-s_0)\phi(s)](t).$$

Notice that the new smooth function $(s-s_0)\phi(s)$ still satisfies the same properties as $\phi$, but now has exactly $n$ sign changes. By inductive hypothesis, $F'$ has at most $n$ zeros, which, by the mean value theorem, implies that $F$ has at most $n + 1$ zeros. This finishes the proof. \qed

Using this claim for $Q, R$, we see that their respective Laplace transform possess at most $K_0$ zeros on the interval $(\sqrt{K_0}, +\infty)$. With this information, we can already finish: from (5.3), the functions $\Theta \pm \tilde{\Theta}$ can only vanish at at most $K_0$ points on the interval $(\sqrt{K_0}, \infty)$ which are not roots of positive integers, in case $\Theta \not\equiv 0$. But, according to our assumption that $(\{x_i\}, \{y_i\}) \in \ker(T_{K_0})$, we have $\Theta(t_j) = \tilde{\Theta}(t_j) = 0, j = 1, \ldots, 2K_0$. By the properties we chose for the sequence $t_j$, $\Theta \equiv 0$, and thus the map $T_{K_0}$ is injective. \qed

We need one more result in order to use our methods to infer results about uniqueness for small powers of integers. In contrast to the full perturbation case of our main theorem, we must prove that the injective operators $T_{K_0}$ are also somewhat stable with respect to injectivity under perturbations. In order to do this, the following result is essential.

**Lemma 5.7.** The range of $T_{K_0}$ is closed.

**Proof.** Suppose the sequence in $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ given by $\{T_{K_0}(\{x_i\}, \{y_i\})\}_{j \geq 0}$ is a Cauchy sequence. This implies that the sequence $\{\{x_i\}_{i=0,K_0+1}^{2K_0}, \{y_i\}_{i=0,K_0+1}^{2K_0}\}_{j \geq 0}$ is a Cauchy sequence, and therefore it converges to a certain limiting sequence

$$\{(x_i)_{i=0,K_0+1}^{2K_0}, (y_i)_{i=0,K_0+1}^{2K_0}\} \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}).$$

Define, thus, the $4K_0 \times 2K_0$ matrix $A_{K_0}$ given by taking

$$(a_1(t_j), a_2(t_j), \ldots, a_{2K_0}(t_j), a_{2K_0+1}(t_j), \tilde{a}_1(t_j), \tilde{a}_2(t_j), \ldots, \tilde{a}_{2K_0}(t_j))$$

and

$$(\tilde{a}_1(t_j), \tilde{a}_2(t_j), \ldots, \tilde{a}_{2K_0}(t_j), a_1(t_j), a_2(t_j), \ldots, a_{2K_0}(t_j))$$

to be its lines, for $j = 1, \ldots, 2K_0$. We first claim that this matrix is injective. Indeed,

$$\tilde{\Theta}(t) = \sum_{i=1}^{K_0} (x_i a_i(t) + y_i \tilde{a}_i(t))$$

vanishes, together with its Fourier transform, at $t_j, j = 1, \ldots, 2K_0$, where $(\{x_i\}_{i=1}^{K_0}, \{y_i\}_{i=1}^{K_0})$ belongs to $\ker(A_{K_0})$. By the proof of Lemma 5.4 this implies $x_i = y_i = 0, i = 1, \ldots, K_0$.

As $A_{K_0}$ is injective, there is a constant $c_{K_0} > 0$ so that

$$\|A_{K_0}v\|_{4K_0} \geq c_{K_0} \|v\|_{2K_0},$$

(5.4)
where we denote by $d$ the usual euclidean norm on a $d$–dimensional space. Translating to our original problem, as $\{T_{K_0}\{x_i^j\}, \{y_i^j\}\}_{j \geq 0}$ is a Cauchy sequence in $\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$,

$$\{\{x_i^j\}\}_{i=0,K_0+1,...}, \{y_i^j\}_{i=0,K_0+1,...}$$

is a convergent sequence, and thus we get that the sequences

$$\sum_{i=1}^{K_0} (x_i^j a_i(t_j) + y_i^j \hat{g}_i(t_j)), j = 1, \ldots, 2K_0$$

are also Cauchy in $k \geq 0$. By (5.4), $\{(x_i^j)_{i=1}^{K_0}, \{y_i^j\}_{i=1}^{K_0}\}_{k \geq 0}$ is Cauchy. This implies that there is a limiting sequence $(\{x_i\}, \{y_i\}) \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$ so that

$$T_{K_0}(\{x_i^j\}, \{y_i^j\}) \rightarrow T_{K_0}(\{x_i\}, \{y_i\}), \text{ as } j \rightarrow \infty.$$ 

This finishes the proof. 

We are finally able to prove the following uniqueness result:

**Corollary 5.8.** Let $\alpha \in (0, \frac{\pi}{4})$. There exists $c_{\alpha} > 0$ so that $\forall c < c_{\alpha}$, if $f \in \mathcal{S}_{\text{even}}(\mathbb{R})$ is a real function that vanishes together with its Fourier transform at $\pm c_{\alpha} n^\alpha$, then $f \equiv 0$.

**Proof.** We start by noticing that, whenever $n \in \mathbb{N}$ is sufficiently large, then there is $m \in \mathbb{N}$ so that $|\sqrt{n} - cm^\alpha| \lesssim c_{\alpha} n^{-\frac{\alpha}{2}}$. Indeed, we simply let $m = \lfloor (n/c^2 \alpha)^{\frac{1}{\alpha-1}} \rfloor$. We get that

$$|\sqrt{n} - cm^\alpha| = c\alpha \int_{\lfloor (n/c^2 \alpha)^{1/(2\alpha)} \rfloor}^{(n/c^2 \alpha)^{1/(2\alpha)}} t^{\alpha-1} dt \lesssim c_{\alpha} n^{-\frac{\alpha-1}{2\alpha}}.$$ 

In particular, if $\frac{2\alpha-1}{2\alpha} < -\frac{\alpha}{4} - \frac{1}{2} \iff \alpha < \frac{2}{9}$, then for all $n \geq n_0(\alpha)$, there exists $m \in \mathbb{N}$ so that we can write $m^\alpha = \sqrt{n} + \varepsilon_n$, where $\varepsilon_n$ satisfies the conditions of Theorem 5.4. Let us single out the sequence of numbers selected above, which we index as $\{m(n)^\alpha\}_{n \geq n_0(\alpha)}$. We then consider the operator $T_{n_0(\alpha)}$ associated to some sequence of $2n_0(\alpha)$ positive real numbers $t_j, j = 1, \ldots, 2n_0(\alpha)$, satisfying the hypotheses of Lemma 5.4.

We claim that the perturbed operator

$$\tilde{T}_{n_0(\alpha)} : \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \rightarrow \ell^2_s(\mathbb{N})$$

is injective. Indeed, from Lemma 5.7 there must exist a constant $C_{n_0}$ so that

$$\|T_{n_0} v\|_{(s,s)} \geq C_{n_0} \|v\|_{(s,s)}$$

holds for all $v \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N})$. But, by the same calculation as in the previous subsection, we have that

$$\|\tilde{T}_{n_0(\alpha)} - T_{n_0(\alpha)}\|_{HS(\ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}))} < C_{n_0}/2$$
are an arbitrary pair of exponents, we notice that we can still pick closed range and is injective. This readily implies that the sequence \((n^n)\) determines uniquely the sequence \(\{\hat{c}_i^n\}\), so that

\[
\rho \left( \sum_{i=0}^{\infty} C_i^n + 2 \right) \leq n^n \implies \exists k \geq 0, \forall n \geq 0, \, c_k \geq 0.
\]

One can inquire about the importance of such a result, as in [25] we have shown that the uniqueness result stated in Corollary 5.8 hold for \(\alpha \in (0, 1 - \sqrt{2}/2)\), which is significantly larger than the range stated here. Nonetheless, Corollary 5.8 gives us automatic results. Indeed, if one manages to prove that for all \(\delta > 0\) there is \(\epsilon > 0\) so that, if \(|e_k| \leq \epsilon, \forall k \in \mathbb{N}\), then

\[
\|I - T\|_{op} < \delta,
\]

it implies automatically that we can extend the results in Corollary 5.8 to the full diagonal range \(\alpha \in (0, 1/2)\).

We also note that Corollary 5.8 is not all we can say about the problem of determining the best exponents \((\alpha, \beta)\) so that

\[
f(\pm n^\alpha) = \hat{f}(\pm n^\beta) = 0, \, f \in \mathcal{S}_{even}(\mathbb{R}) \Rightarrow f \equiv 0.
\]

Indeed, we can easily go further than the diagonal case exposed above: if \(\alpha, \beta \in (0, 2/9)\) are an arbitrary pair of exponents, we notice that we can still pick \(n_0 \in \mathbb{N}\) so that for each \(n > n_0 = n_0(\alpha, \beta)\), there exists a pair \((m_1(n), m_2(n)) \in \mathbb{N}^2\) so that

\[
|cm_1(n)^\alpha - \sqrt{n}| + |cm_2(n)^\beta - \sqrt{n}| \lesssim c^{1/\alpha} \alpha^{\frac{\alpha - 1}{2\alpha}} + c^{1/\beta} \beta^{\frac{\beta - 1}{2\beta}}.
\]
This induces us to consider the operator
\[ T_{n_0(\alpha, \beta)} : \ell^2_s(N) \times \ell^2_s(N) \to \ell^2_s(N) \times \ell^2_s(N) \]
\[
\{(x_i), (y_i)\} \mapsto ((x_0, \mathcal{G}(ck^0), \mathcal{G}(ck^2), \ldots, \mathcal{G}(ck_{2n_0})), \mathcal{G}(m_1(n_0 + 1)\alpha), \mathcal{G}(m_1(n_0 + 2)\alpha), \ldots),
\]
\[
(y_0, \mathcal{G}(cl^0_1), \mathcal{G}(cl^2_1), \ldots, \mathcal{G}(cl_{2n_0}^2), \mathcal{G}(m_2(n_0 + 1)\beta), \mathcal{G}(m_2(n_0 + 2)\beta), \ldots))
\]
(5.7)
for two sequences of integers \((k_j, l_j), j = 1, \ldots, 2n_0\), so that \(|t_j - ck^0_j| + |t_j - cl^0_j|\) is sufficiently small for all \(j \in [0, 2n_0]\), where we select \(t_j, j = 1, \ldots, 2n_0\) satisfying the hypotheses of Lemma 5.3.

By the same strategy outlined in the proof of Corollary 5.8, the Hilbert-Schmidt norm as operators acting on \(\ell^2_s(N) \times \ell^2_s(N)\) of the difference \(T_{n_0(\alpha, \beta)} - T_{n_0(\alpha, \beta)}\) is arbitrarily small, as long as we make the value of \(c = c(\alpha, \beta)\) smaller. As a consequence, \(T_{n_0}\) is also injective and its range is closed. These considerations prove, therefore, the following:

**Corollary 5.9.** Let \(\alpha, \beta \in (0, 2/9)\). Then there is \(c_{\alpha, \beta} > 0\) so that for all \(c < c_{\alpha, \beta}\), if \(f \in S_{\text{even}}(\mathbb{R})\) is a real function that vanishes at \(\pm cn^\alpha\) and its Fourier transform vanishes at \(\pm cn^\beta\), then \(f \equiv 0\).

**Remark.** In the end, we do not quite attain the primary goal of this section of proving Fourier uniqueness results for the sequences \((\{\pm n^\alpha\}, \{\pm n^\beta\})\), but only a slightly weaker version of it, with a small constant \(c(\alpha, \beta)\) in front. The main reason for that in the proofs above is the location of the positive reals \(t_i\) : although their exact values do not matter in the end, it is crucial, in order to use Proposition 5.6 that they lie after the node \(n_0\). We must therefore either force \(n_0\) not to be too large in order not to make the norm of the matrix \(A_{K_0}\) too small, or fix them from the beginning and make the perturbations of \(T_{K_0}\) fall closer to it. In any case, this implies nontrivial use of the constant \(c\) multiplying the sequences \((\{\pm n^\alpha\}, \{\pm n^\beta\})\).

We believe that further studying operators resembling \(T_{K_0}\) above and their injectivity properties could yield better results in this regard. In order not to make this exposition even longer, we will not pursue this matter any further.

5.3. The Cohn-Kumar-Miller-Radchenko-Viazovska result and perturbed interpolation formulae with derivatives. As another illustration of our main technique, we prove that the interpolation formulae with derivatives in dimension 8 and 24 from [10] can be suitably perturbed.

Indeed, we first recall one of the main results of [10]: let \((d, n_0)\) be either \((8, 1)\) or \((24, 2)\). Then every \(f \in S_{\text{rad}}(\mathbb{R}^d)\) can be uniquely recovered by the sets of values
\[
\{f(\sqrt{2}n), f'(\sqrt{2}n), \hat{f}(\sqrt{2}n), \hat{f}'(\sqrt{2}n)\}, n \geq n_0,
\]
satisfy that
\[ f(x) = \sum_{n \geq n_0} f(\sqrt{2n}) a_n(x) + \sum_{n \geq n_0} f'(\sqrt{2n}) b_n(x) + \sum_{n \geq n_0} \hat{f}(\sqrt{2n}) \hat{a}_n(x) + \sum_{n \geq n_0} \hat{f}'(\sqrt{2n}) \hat{b}_n(x). \]

(5.8)

We also have uniform estimates on the functions \( a_n, \hat{a}_n, b_n, \hat{b}_n \) : indeed, there is \( \tau > 0 \) so that
\[ \sup_{l \in \{0,1,2\}} \sup_{x \in \mathbb{R}^d} (1 + |x|)^{100} \left( |a_n^{(l)}(x)| + |\hat{a}_n^{(l)}(x)| + |b_n^{(l)}(x)| + |\hat{b}_n^{(l)}(x)| \right) \lesssim n^\tau, \]
for all \( n \in \mathbb{N} \). Here and throughout this section, we shall denote by \( g'(x) \) the derivative of the (radial) function \( g \) regarded as a one-dimensional function.

By [10] Theorem 1.9, we know that the matrices
\[ M_n(x) = \begin{pmatrix} a_n(x) & a_n'(x) & \hat{a}_n(x) & \hat{a}_n'(x) \\ b_n(x) & b_n'(x) & \hat{b}_n(x) & \hat{b}_n'(x) \\ \hat{a}_n(x) & \hat{a}_n'(x) & a_n(x) & a_n'(x) \\ \hat{b}_n(x) & \hat{b}_n'(x) & b_n(x) & b_n'(x) \end{pmatrix} \]

satisfy that \( M_n(\sqrt{2m}) = \delta_{mn} I_{4 \times 4} \). As we know that the map that takes a vector of sufficiently rapidly decaying sequences
\[ (\{a_n\}, \{\beta_n\}, \{\hat{a}_n\}, \{\hat{\beta}_n\}) \]
on to the function
\[ f(x) = \sum_{n \geq n_0} \left( a_n a_n(x) + \beta_n b_n(x) + \hat{a}_n \hat{a}_n(x) + \hat{\beta}_n \hat{b}_n(x) \right) \]
is, in fact, injective (and moreover an isomorphism if we consider the set of all arbitrarily rapidly decaying sequences), we shall make use of this function in our estimates. Indeed, we have that the map that takes the quadruple of sequences
\[ (\{a_n\}, \{\beta_n\}, \{\hat{a}_n\}, \{\hat{\beta}_n\}) \]
on to
\[ (f(\sqrt{2n}), f'(\sqrt{2n}), \hat{f}(\sqrt{2n}), \hat{f}'(\sqrt{2n}))_{n \geq n_0} \]
is, in fact, the identity. Another way to represent this map is as the series
\[ \sum_{n \geq n_0} (a_n, \beta_n, \hat{a}_n, \hat{\beta}_n) \cdot M_n(\sqrt{2n}). \]

We define, therefore, the operator that takes the same quadruple onto
\[ (f(\sqrt{2n + \varepsilon}), f'(\sqrt{2n + \varepsilon}), \hat{f}(\sqrt{2n + \varepsilon}), \hat{f}'(\sqrt{2n + \varepsilon}))_{n \geq n_0}. \]
In the alternative notation, this operator, which we shall denote by \( \Xi \), is given by
\[ \sum_{n \geq n_0} (a_n, \beta_n, \hat{a}_n, \hat{\beta}_n) \cdot M_n(\sqrt{2n + \varepsilon}). \]
As before, we seek to prove that $\mathfrak{T}$ is invertible when defined over some space
\[ \ell^2_s(\mathbb{N}) \times \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N}) =: (\ell^2(\mathbb{N}))^4, \]
where we may take $s \gg 1$ sufficiently large. As our aim here is not to establish the sharpest possible results, but only to prove that we may perturb the aforementioned interpolation formulae, we shall make use of the Hilbert–Schmidt test, as in (5.10) above. Indeed, we wish to prove that
\[ \|I - \mathfrak{T}\|_{HS((\ell^2(\mathbb{N}))^4)} < 1. \]

A simple computation with the Hilbert–Schmidt norm using (5.10) shows that this quantity is bounded by
\[
\sum_{m,n>0} m^{2s} n^{-2s} (|a_n(\sqrt{2m}) - a_n(\sqrt{2m + \varepsilon_m})|^2 + |\hat{a}_n(\sqrt{2m}) - \hat{a}_n(\sqrt{2m + \varepsilon_m})|^2 + |b_n(\sqrt{2m}) - b_n(\sqrt{2m + \varepsilon_m})|^2 + |\hat{b}_n(\sqrt{2m}) - \hat{b}_n(\sqrt{2m + \varepsilon_m})|^2 + |b'_n(\sqrt{2m}) - b'_n(\sqrt{2m + \varepsilon_m})|^2 + |\hat{b}'_n(\sqrt{2m}) - \hat{b}'_n(\sqrt{2m + \varepsilon_m})|^2).
\]

By (5.9) and the mean value theorem, the sum above is bounded by (an absolute constant times)
\[
\sum_{m,n>0} m^{2s} n^{-2s} m^{-100} n^{2\varepsilon^2 m}. \]
The sum above is representable as a product of a sum in $m$ and one in $n$. The one in $n$ is convergent if $s > \tau + 1$. We then fix such a value of $s$. For such values, the second sum is
\[
\sum_{m>0} m^{2s-100} \varepsilon^2 m,
\]
which converges in case $\varepsilon_m \lesssim m^{49-s}$. For all such sequences, the difference $I - \mathfrak{T}$ is a Hilbert–Schmidt operator. Moreover, if $\varepsilon_m \leq \delta m^{49-s}$ for $\delta > 0$ sufficiently small, we will have $\|I - \mathfrak{T}\|_{HS((\ell^2(\mathbb{N}))^4)} < 1$. Summarizing, we have shown the following result:

**Theorem 5.10.** There is $C_0 > 0$ so that the following holds: there is $\delta > 0$ so that, for each sequence $\varepsilon_k$ so that $|\varepsilon_k| < \delta k^{-C_0}$, then any function $f \in S_{rad}(\mathbb{R}^d)$ is uniquely determined by the values
\begin{equation}
(5.11) \quad \left( f(\sqrt{2n} + \varepsilon_k), f'(\sqrt{2n} + \varepsilon_k), \hat{f}(\sqrt{2n} + \varepsilon_k), \hat{f}'(\sqrt{2n} + \varepsilon_k) \right)_{n \geq n_0},
\end{equation}
where we let $(d, n_0) = (8, 1)$ or $(24, 2)$.

In the same spirit of (4.2) one can obtain an interpolation formula with the values (5.11) from Theorem 5.10.

We remark that, in the same way that we undertook our analysis for the Radchenko-Viazovska interpolating functions, we expect the functions $a_n, b_n$ in [10], Theorem 1.9 should also satisfy some exponential-like decay. This fact, although possible, should be sensibly more technically involved than Theorem 1.5 due to the more complicated
nature of the construction of the interpolating functions with derivatives in dimensions 8 and 24.

5.4. Perturbed interpolation formulae for odd functions. Finally, in the same spirit of the results in Section 4, we briefly comment on interpolation formulae for odd functions. Recall the following results from [24, Section 7]:

**Theorem 5.11** (Theorem 7 in [24]). There exist sequences of odd functions \( d^\pm_m : \mathbb{R} \to \mathbb{R} \), \( m \geq 0 \), belonging to the Schwartz class so that

\[
\hat{d}^\pm_m = \pm i d^\pm_m, \quad \hat{d}^\pm_m(\sqrt{n}) = \delta_{n,m} \sqrt{n}, \quad n \geq 1.
\]

Moreover, \( \lim_{x \to 0} \frac{d^+_m(x)}{x} = \delta_{0,m} \). These functions satisfy the uniform bound

\[
|d^+_n(x)| \lesssim n^{5/2}, \quad \forall x \in \mathbb{R}, \quad n \geq 0,
\]

and, finally, for each odd and real Schwartz function \( f : \mathbb{R} \to \mathbb{R} \),

\[
f(x) = d^+_0(x) \frac{f'(0)}{2} + \sum_{n \geq 1} \left( c_n(x) \frac{\hat{f}(\sqrt{n})}{\sqrt{n}} - \bar{c}_n(x) \frac{\hat{f}(\sqrt{n})}{\sqrt{n}} \right),
\]

where \( c_n = (d^+_n + d^-_n)/2 \), and the right-hand side of the sum above converges absolutely.

As a direct consequence, we see that any real, odd, Schwartz function on the real line is determined uniquely by the union of its values at \( \sqrt{n} \) and the values of its Fourier transform at \( \sqrt{n} \) with \( f'(0) \) and \( \hat{f}'(0) \). By employing the results in Section 4, we will show that we can actually recover any such function from \( \{ f(\sqrt{n} + \varepsilon_n) \}_{n \geq 1} \cup \{ \hat{f}(\sqrt{n} + \varepsilon_n) \}_{n \geq 1} \cup \{ f'(0) \} \cup \{ \hat{f}'(0) \} \) instead.

Indeed, first of all, we start by noticing that the same techniques employed to refine the uniform estimates from Radchenko–Viazovska [24] can be applied to the functions \( d^+_n \), as they are defined in a completely analogous way to the \( b^+_n \) from Section 4. By carrying out the same kind of estimates, we are able to obtain

\[
|d^+_n(x)| \lesssim n^{3/4} \log^{3/2}(1 + n) e^{-\varepsilon' |x|}/\sqrt{n}, \quad \forall x \in \mathbb{R}, \quad n \geq 1,
\]

for some absolute constant \( \varepsilon' > 0 \). By the same analysis of the \( \partial_x \)–partial derivative of the generating function used in (4.11) this readily implies that the derivatives of the \( d^+_n \) satisfy morally the same decay; in fact, \( |(d^+_n)'(x)| \lesssim n^{5/4} \log^{3/2}(1 + n) e^{-\varepsilon'' |x|}/\sqrt{n}, \quad \forall x \in \mathbb{R}, \quad n \geq 1 \), with \( \varepsilon'' > 0 \) another absolute constant.

We consider now the operator that takes a pair of sequences \( \{ \alpha_n \}, \{ \beta_n \} \in \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}), s > 0 \) to be chosen, into

\[
\left\{ \sum_{n \geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m + \varepsilon_m}) \right\}_{m \geq 0},
\]

where we abbreviate \( C_n(x) = \left( \frac{c_n(x)}{\sqrt{n}}, \frac{\bar{c}_n(x)}{\sqrt{n}} \right) \). Let us denote this operator by \( \mathcal{V} \).

From (5.12) and the fact that the function \( d^+_0(x) = \frac{\sin(\pi x^2)}{\sinh(\pi x)} \) vanishes together with its Fourier transform at \( \pm \sqrt{n} \), \( n \in \mathbb{N} \), we know that the identity operator on \( \ell^2_s(\mathbb{N}) \times \ell^2_s(\mathbb{N}) \)
may be written as
\[
\left\{ \sum_{n \geq 0} (\alpha_n, \beta_n) C_n(\sqrt{m}) \right\}_{m \geq 0}.
\]

Therefore, the techniques from \(\S 4.2\) \(\S 5.3\) and \(5.1\) together with our previous considerations in this subsection, allow us to deduce the following result:

**Theorem 5.12.** There is \(\delta > 0\) so that, in case \(|\varepsilon_n| \leq \delta n^{-\frac{7}{4}}\), then for each \(f \in \mathcal{S}_{\text{odd}}(\mathbb{R})\) real, the values
\[
(f(\sqrt{1 + \varepsilon_n}), f(\sqrt{2 + \varepsilon_2}), \ldots)
\]
and
\[
(\hat{f}(\sqrt{1 + \varepsilon_n}), \hat{f}(\sqrt{2 + \varepsilon_2}), \ldots)
\]
allow us to recover uniquely the values \((f(1), f(\sqrt{2}), f(\sqrt{3}), \ldots)\) and \((\hat{f}(1), \hat{f}(\sqrt{2}), \hat{f}(\sqrt{3}), \ldots)\). In particular, given the values
\[
\{f(\sqrt{n} + \varepsilon_n)\}_{n \geq 1} \cup \{\hat{f}(\sqrt{n} + \varepsilon_n)\}_{n \geq 1} \cup \{f'(0)\} \cup \{\hat{f}'(0)\},
\]
we can uniquely recover any real function \(f \in \mathcal{S}_{\text{odd}}(\mathbb{R})\).

As previously mentioned, we do not carry out the details here, for their similarities with the proof of theorems \(1.5\) and \(1.4\).

### 6. Comments and Remarks

In this section, we gather some remarks about the problems and techniques discussed, as well as state some results we expect to be true.

#### 6.1. Maximal perturbed Interpolation Formulae for Band-limited functions.

In Section 3, we have seen how our basic functional analysis techniques can be employed in order to deduce new interpolation formulae for band-limited functions. Although Kadec’s proof also uses the basic fact that, whenever a perturbation of the identity is sufficiently small, then we can basically ‘invert’ an operator, he then proceeds to find that the set of exponentials \(\{\exp(2\pi i(n + \varepsilon_n)x)\}_{n \geq 0}\) is a Riesz basis for \(L^2(-1/2, 1/2)\) if \(\sup_n |\varepsilon_n| < 1/4\) by means of orthogonality considerations. Indeed, one key strategy in his estimates is to expand in the different complete orthogonal system
\[
\{1, \cos(2\pi nt), \sin((2n - 1)\pi t)\}_{n \geq 1}
\]
and use the properties of this expansion. Our results, as much as they do not come so close to Kadec’s threshold, follow a slightly different path: instead of using the orthogonality of a different system, we choose to work directly with discrete analogues of the Hilbert transform and estimate over those. Although we do not reach – by a 0.011 margin – the sharp \(1/4\)–perturbation result, one advantage of our approach is that it yields bounds for perturbing any kind of interpolation formulae with derivatives. Indeed, following the line of thought of Vaaler, many authors have investigated the property of recovering the values of a function \(f \in L^2(\mathbb{R})\) band-limited to \([-k/2, k/2]\) from the values of its \((k - 1)\)–first derivatives (see, e.g., [20] and [12]). Our approach
in [3] in order to prove Theorem 1.3 generalizes easily to the case of several derivatives by an easy modification. It can be summarized as follows:

**Theorem 6.1.** There is $L(k) > 0$ so that if $\sup_{n \in \mathbb{Z}} |\varepsilon_n| < L(k)$, then any function $f \in L^2(\mathbb{R})$ band-limited to $[-k/2, k/2]$ is uniquely determined by the values of

$$f^{(l)}(n + \varepsilon_n), \; n \in \mathbb{Z}, l = 0, 1, \ldots, k - 1.$$ 

A natural question that connects our results to Kadec’s results is about the best value of $L(k)$ so that Theorem 6.1 holds. We do not have evidence to back any concrete conjecture, but we find possible that the threshold $L(k) = \frac{1}{4}$ is kept for higher values of $k \in \mathbb{N}$. We speculate that, in order to prove such a result, one would need to find an appropriate hybrid of our techniques and Kadec’s techniques (see for instance Section 10 in [33, Chapter 1]), taking into account properties of the discrete Hilbert transforms as well as orthogonality results.

### 6.2. Theorem 1.5, optimal decay rates for interpolating functions and maximal perturbations.

In Theorem 1.5, we have improved the uniform bound obtained by Radchenko and Viazovska [24] and, more recently, the sharper uniform bound by Bondarenko, Radchenko and Seip [3] on the interpolating functions $a_n$ to one that decays with $x$; namely, we have that

$$|a_n(x)| \lesssim n^{1/4} \log^{3/2}(1 + n) \left( e^{-\varepsilon|x|^2/n} 1_{|x| < Cn} + e^{-c|x|} 1_{|x| > Cn} \right),$$

holds for all $n \in \mathbb{N}$, where $C, c > 0$ are two fixed positive constants. Although this improves the decay rates from before, the power $n^{1/4}$ found here and in [3] in the growth seems likely not to be optimal; to that regard, we pose the following:

**Question 1.** What is the best decay rate for $a_n$ as in Theorem 1.5? Can one prove that $\sup_{x \in \mathbb{R}} |a_n(x)| = O(1)$ in $n$?

This conjectured growth seems to be the best possible, due to the recent findings of Bondarenko–Radchenko–Seip [3], which show that, for each $N \gg 1$, the average

$$\frac{1}{N + 1} \sum_{k \leq N} |a_k(x)|^2$$

grows slower than some power of $\log N$.

Notice that, by a simple modification of the computations made in [1,2], an affirmative answer to Question 1 yields an immediate improvement in the range of $\varepsilon_i$ that we allow for the theorems in [1,2]. Indeed, we get automatically that $|\varepsilon_i| \lesssim i^{-1}$ is allowed in such results. On the other hand, this seems to be the best possible result one can achieve with our current methods, as the mean value theorem implies that $\sup_{x \in \mathbb{R}} |a_n'(x)| \gtrsim \sqrt{n}$.

In particular, all indicates that one needs a new idea in order to prove the following conjecture:
Conjecture 6.2 (Maximal perturbations). Let $f \in S_{\text{even}}(\mathbb{R})$ be a real function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| < \theta$, $\forall i \in \mathbb{N}$, $f$ can be uniquely recovered from its values

$$f(0), f(\sqrt{1 + \varepsilon_1}), f(\sqrt{2 + \varepsilon_2}), \ldots,$$

together with the values of its Fourier transform

$$\hat{f}(0), \hat{f}(\sqrt{1 + \varepsilon_1}), \hat{f}(\sqrt{2 + \varepsilon_2}), \ldots.$$

It might not be an easy task to prove Conjecture 6.2 even with a new idea starting from our techniques, but we believe that the following version stands a chance of being more tractable with the current methods:

Conjecture 6.3 (Maximal perturbations, weak form). Let $f \in S_{\text{even}}(\mathbb{R})$ be a real function. Then, for each $a > 0$, there is $\delta > 0$ so that, if $|\varepsilon_i| \leq \delta k^{-a}$, then $f$ can be uniquely recovered from its values

$$f(0), f(\sqrt{1 + \varepsilon_1}), f(\sqrt{2 + \varepsilon_2}), \ldots,$$

together with the values of its Fourier transform

$$\hat{f}(0), \hat{f}(\sqrt{1 + \varepsilon_1}), \hat{f}(\sqrt{2 + \varepsilon_2}), \ldots.$$

In this framework, the results in §4.2 may be regarded as partial progress towards this conjecture. Notice that, by the remarks of §5.2, both versions of the conjecture imply that for each $\alpha \in (0, 1/2)$, there is $c_\alpha > 0$ so that if an even, real Schwartz function $f$ satisfies that $f(c_1 n^\alpha) = \hat{f}(c_2 n^\beta) = 0$ and $c_1 < c_\alpha$, $c_2 < c_\beta$, then $f \equiv 0$. These results can be compared, for instance, with our previous results in [24].

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