THE NON-EXISTENCE OF COMPLEX SPHERE $S^n$ ($n > 2$)

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Abstract. We show the non-existence of complex structure on sphere with the standard round metric, of any dimension other than two, in particular, on $S^6$.

1. Introduction

Given $M^n$ a Riemannian manifold, an interesting question is that does there exist any complex structure on $M^n$ that makes $M^n$ a complex manifold? For example, are spheres complex manifolds? Of course $n$ needs being even and $M$ needs being orientable. Great work has been done in this field.

Ehresmann [6] showed that the 6-sphere admits an almost-complex structure.

H. Hopf [9] proved that $S^4$ and $S^8$ do not admit almost complex structures.

A. Kirchhoff [10] proved that there is an almost-complex structure on $S^6$.

Eckmann-Frohlicher [5] and Ehresmann-Liberman [7] proved that Kirchhoff’s almost complex structure on $S^6$ is not integrable to a complex structure.

Borel and Serre proved that $S^n$ admits an almost complex structure if and only if it is $S^2$ or $S^6$.

C. LeBrun [11] proved that there is no integrable orthogonal almost-complex structure on $S^6$.

The tough case is $S^6$.

Hirzebruch [8] in 1954 and Liberman[12] in 1955, Yau [13] in 1990 asked whether or not is there a complex structure on $S^6$? This last problem has been open for long time period before this writing.

Atiyah [1][2] gave very interesting geometric ”conceptual proof” and algebraic ”conceptual proof” to the non-existence of complex 6-sphere.

In this paper we prove that any almost-complex structure on $S^n$ ($n$ even, $n > 2$, including $S^6$ ) with standard round metric is not integrable therefore is not a complex manifold. Our approach is geometric analysis. The relevant explicit and detailed calculations depend on the underlying metric.

For a given complex manifold, the complex structure gives a canonical almost-complex structure. In the study of existence or non-existence of complex structure for a manifold, one naturally asks if there exits any almost-complex structure , and if yes, can it be ”integrated” to a complex structure.

An almost-complex structure $J$ on Riemannian manifold $M^n$ is an endomorphism of the tangent bundle $TM$ with $J^2 = -1$. It is easy to know that if $M$ has an almost-complex structure, then $M$ has even dimension $n$ and $M$ is orientable.

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Nijenhuis tensor $N_J$ for the almost complex structure $J$ is given by the following equation

\[(1.1) \quad N_J(X, Y) = [JX, JY] - J[X, JY] - J[JX, Y] - [X, Y],\]

for all smooth vector fields $X, Y$. The celebrated Newlander-Nirenberg theorem [14] implies that $N_J = 0$ if and only if $J$ is a canonical almost-complex structure of a complex manifold. An almost-complex structure $J$ is called integrable if $N_J$ vanishes. So in studying existence or nonexistence of complex structure on a manifold, one often studies existence or nonexistence of integrable almost-complex structures, or equivalently studies almost-complex structures and the related Nijenhuis tensors vanishing or non-vanishing property.

In this paper we show that any almost-complex structure on spheres of any dimension other than two, including open case $S^6$, with standard round metric, is not integrable by showing its Nijenhuis tensor does not vanish, therefore there is no complex structure on them with the standard round metric. We state and prove the results in the next section.

This is a cross list updated version for Ling [13] that was deposited into arXiv under category MA instead of category DG:

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2. Main Result and Its Proof

**Theorem 2.1.** Let $J$ be any almost-complex structure on (unit) sphere $S^n$, $n > 2$ ($n$ even), with the standard round metric. Then Nijenhuis tensor $N_J$ in (1.1) of the almost-complex structure $J$ does not vanish. Therefore $J$ is not integrable.

**Corollary 2.2.** $S^6$ with the standard round metric is not a complex manifold.

**Proof of Theorem 2.1.** We set the center of $S^n$ at the origin of $\mathbb{R}^{n+1}$, so

$S^n := \{ y \in \mathbb{R}^{n+1} \big| y_{\mathbb{R}^{n+1}} = 1 \}.$

That is, $S^n$ consists of $y = (y^1, \cdots, y^{n+1}) \in \mathbb{R}^{n+1}$ with the equation

$(y^1)^2 + \cdots + (y^{n+1})^2 = 1.$

Therefore

$y^1 dy^1 + \cdots + y^{n+1} dy^{n+1} = 0, \quad y \in S^n \subset \mathbb{R}^{n+1}.$

This equation is expressed by pairs between forms and vectors. Use the standard metric in $\mathbb{R}^{n+1}$ to lower down forms and (it happens the metric dual is the agebraic dual) and write equation in inner product of vectors, we have

$y^1 \frac{\partial}{\partial y^1} + \cdots + y^{n+1} \frac{\partial}{\partial y^{n+1}} = 0, \quad y \in S^n \subset \mathbb{R}^{n+1},$

that is

\[(2.1) \quad \sum_{i=1}^{n} y^i \frac{\partial}{\partial y^i} + y^{n+1} \frac{\partial}{\partial y^{n+1}} = 0, \quad y \in S^n \subset \mathbb{R}^{n+1}.
\]

Take the Stereographic projection local coordinates of $S^n$ at the north pole $(0, \cdots, 0, 1) \in \mathbb{R}^{n+1}$.

$y = y = (y^1, \cdots, y^n, y^{n+1}) \in S^n : \longrightarrow x = (x^1, \cdots, x^n) \in \mathbb{R}^n.$
It is easy to see that

\[ y^i = \frac{2}{|x|^2 + 1} x^i, \quad 1 \leq i \leq n, \quad \text{and} \quad y^{n+1} = \frac{|x|^2 - 1}{|x|^2 + 1}, \]

where for \( x = (x^1, \ldots, x^n) \in \mathbb{R}^n, |x|^2 = (x^1)^2 + \cdots + (x^n)^2 \). For convenience we make notations

\[ \mu := \frac{2}{|x|^2 + 1}, \quad \nu = \frac{1}{\mu}. \]

Then above (2.2) can be rewrite as

\[ y^i = \mu x^i, \quad 1 \leq i \leq n, \quad \text{and} \quad y^{n+1} = 1 - \mu. \]

\{\frac{\partial}{\partial x^i}|_y\}_{i=1}^n \text{ is the natural basis for the tangent space } T_y S^n \text{ of } S^n \text{ at } y \in S^n \backslash \{\text{the north pole and the south pole}\}.

In the following when we say \( y \in S^n \), \( y \) is neither the north pole nor south pole.

There is no problem for this because our calculations are local ones.

For each \( 1 \leq i \leq n \), we have the following equations

\[
\frac{\partial}{\partial y^i} = \nu \frac{\partial}{\partial x^i} = \frac{|x|^2 + 1}{2} \frac{\partial}{\partial x^i},
\]

and

\[
\frac{\partial}{\partial y^{n+1}} = \frac{\partial}{\partial y^{n+1}} (\frac{y^p}{x^p}) \frac{\partial}{\partial x^p} = \frac{\partial}{\partial y^{n+1}} (\frac{1}{\mu} y^p) \frac{\partial}{\partial x^p} = \frac{1}{\mu} \frac{\partial}{\partial x^i} = \nu \frac{\partial}{\partial x^i}.
\]

that is

\[\frac{\partial}{\partial y^{n+1}} = \sum_{p=1}^n \nu x^p \frac{\partial}{\partial x^p} = \sum_{p=1}^n \frac{|x|^2 + 1}{2} x^p \frac{\partial}{\partial x^p}.\]

By (2.4) and (2.5) we have

\[\frac{\partial}{\partial y^i}|_y = \frac{|x|^2 + 1}{2} \frac{\partial}{\partial x^i}|_y \in T_y S^n, \quad \frac{\partial}{\partial y^{n+1}}|_y = \sum_{p=1}^n \frac{|x|^2 + 1}{2} x^p \frac{\partial}{\partial x^p}|_y \in T_y S^n.\]

Now for almost-complex structure tensor \( J \) on \( S^n \), let \( (J_i^j)|_y \) be the matrix of the \( J \) under the natural basis \( \{\frac{\partial}{\partial x^i}|_y\}_{i=1}^n \) of the tangent space \( T_y S^n \) of \( S^n \) at \( y \in S^n \), namely

\[ J|y : T_y S^n \to T_y S^n, \quad J|_y \frac{\partial}{\partial x^i} = J_i^j|_y \frac{\partial}{\partial x^j}, \quad y \in S^n.\]
here and in this paper we will use the convention of taking sums over duplicated
indexes from 1 to \(n\). unless otherwise stated, or emphasized. By (2.6), we have
\[
J \frac{\partial}{\partial y^i} = \nu J \frac{\partial}{\partial x^i}, \quad 1 \leq i \leq n, \quad J \frac{\partial}{\partial y^{n+1}} = \nu x^j J \frac{\partial}{\partial x^j}.
\]

Take above (2.3), (2.4) and (2.5) into (2.1) We have the following equa
tions.
\[
\begin{align*}
\left\{ \mu x^i \right\} \left\{ \nu \frac{\partial}{\partial x^i} \right\} + \left\{ (1 - \mu) \right\} \left\{ \nu x^j \frac{\partial}{\partial x^j} \right\} &= 0.
\end{align*}
\]
By (2.7), (2.8) can also be written as
\[
\begin{align*}
\sum_{i=1}^{n+1} y^i J \frac{\partial}{\partial y^i} &= 0.
\end{align*}
\]
From (2.8), we have
\[
\nu x^i J \frac{\partial}{\partial x^i} = 0.
\]
Since \(\nu \neq 0\), we have
\[
x^i J \frac{\partial}{\partial x^i} = 0,
\]
\[
J^k_i x^i \frac{\partial}{\partial x^k} = 0.
\]
We denote \(\frac{\partial}{\partial x^i}\) by \(\partial_i\) and \(\nabla_{\frac{\partial}{\partial x^i}}\) by \(\nabla_i\) for convenience.

Differentiating the above equation (2.10) yields
\[
(\nabla_i x^p) J^q_p \partial_q + x^p (\nabla_i J^q_p) \partial_q + x^p J^q_p \nabla_i \partial_q = 0,
\]
where covariant derivatives are using the Levi-Civita connection determined by the
standard round metric on \(S^n\) that is induced from the metric of \(\mathbb{R}^{n+1}\). The above
equation yields
\[
J^q_i x^p \partial_i J^q_p - \mu x^p J^q_i x^p = 0,
\]
and
\[
J^q_i x^p \partial_i J^q_p - \mu |x|^2 J^q_i = 0.
\]
Therefore we have the following equation for \(1 \leq i, j, k, p \leq n\).
\[
J^j_i = \xi x^p \partial_i J^j_p,
\]
for \(|x| \neq 1\), where
\[
\xi := \frac{|x|^2 + 1}{|x|^2 - 1}, \quad \text{for } |x| \neq 1.
\]
Take \(N_J\) is the Nijenhuis tensor in (1.1) and under the basis \(\{\partial_i\}_{i=1}^n\) let
\[
N_J(\partial_i, \partial_j) = N^k_{ij} \partial_k.
\]
It is known that
\[
N^k_{ij} = J^p_i (\partial_p J^k_j - \partial_j J^k_p) - J^p_j (\partial_p J^k_i - \partial_i J^k_p).
\]
We now show the Nijenhuis tensor \(N_J\) does not vanish.
Differentiating the equation (2.11), we have
\[ \partial_p J_j^k = -(\xi - 1)^2 x^p x^q \partial_j J^k_q + \xi \partial_j J^k_p + \xi x^q \partial_p \partial_j J^k_q. \]
and
\[ (2.12) \quad \partial_p J_j^k - \partial_j J^k_p = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ -x^p J_j^k + x^j J^k_p \right\} \]
for all \( x \in \mathbb{R}^n, |x| \neq 1, x \neq 0. \)

Now for \( x \in \mathbb{R}^n \) with \( |x| = 1 \), we take sequence let \( x^\epsilon_i = x^i + \text{sign}(x^i)\epsilon \) for \( 1 \leq i \leq n \) for any \( \epsilon > 0 \), where \( \text{sign}(a) = 1 \) if \( a \geq 0 \) and \( \text{sign}(a) = -1 \) if \( a < 0 \).

Then \( x^\epsilon = (x_1^\epsilon, \cdots, x_n^\epsilon) \in \mathbb{R}^n \) and \( |x^\epsilon| > |x|^2 = 1, \) \( x^\epsilon \to x \) as \( \epsilon \to 0 \). So (2.12) holds for \( x^\epsilon \) with \( \epsilon > 0 \). Note all functions in (2.12) are continuous. After passing to the limit we have (2.12) holding for \( |x| = 1 \).

Therefore (2.12) holds for all \( x \in \mathbb{R}^n, x \neq 0 \).

Therefore for all \( x \in \mathbb{R}^n, x \neq 0 \), we have the following equation.
\[ (2.13) \quad N_{ij}^i = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ x^p J^p_i J^j_k - x^p J^p_i J^k_j + \delta^k_i x^i - \delta^i_k x^j \right\}. \]

Therefore for each \( 1 \leq i \leq n, x \neq 0 \).
\[ N^i_{ij} = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ x^p J^p_j J^i_k - x^p J^p_i J^j_k + \delta^j_i x^i - \delta^i_j x^j \right\} \]
\[ = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ x^p J^p_j J^i_k - x^p J^p_i J^j_k + \delta^j_i x^i - x^j \right\}, \]
and
\[ \sum_j N^i_{ij} = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ J^i_j \sum_{j,p} x^p J^p_j - \sum_{j,p} x^p J^p_i J^j_k + x^i - \sum_j x^j \right\} \]
\[ = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ J^i_j \sum_{j,p} x^p J^p_j - \sum_{j,p} x^p J^p_i J^j_k - \sum_{j \neq i} x^j \right\}. \]

Therefore
\[ \sum_i \sum_j N^i_{ij} \]
\[ = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ \sum_i J^i_j \sum_{j,p} x^p J^p_j - \sum_{j=p} \sum_i x^p J^p_i J^j_k - \sum_i \sum_j x^j \right\}, \]
\[ = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ 0 \sum_{j,p} x^p J^p_j + \sum_{j=p} x^p \delta^p_j - \sum_i \sum_{j \neq i} x^j \right\}, \]
\[ = \frac{2}{|x|^2(|x|^2 + 1)} \left\{ \sum_j x^j - \sum_i \sum_{j \neq i} x^j \right\}. \]

Therefore
\[ \sum_i \sum_j N^i_{ij} \mid_{x=(1, \cdots, 1)} = \frac{2(2 - n)}{n(n^2 + 1)} \neq 0, \quad \text{if} \ n \neq 2. \]

Therefore Nijenhuis tensor \( N_J \) does not vanish, otherwise the above sum would be zero since each term is zero.
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