RIGIDITY IN ÉTALE MOTIVIC STABLE HOMOTOPY THEORY

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Abstract. For a scheme $X$, denote by $\mathcal{SH}(X^\wedge)$ the stabilization of the hypercompletion of its étale $\infty$-topos, and by $\mathcal{SH}_{\text{ét}}(X)$ the localization of the stable motivic homotopy category $\mathcal{SH}(X)$ at the (desuspensions of) étale hypercovers. For a stable $\infty$-category $C$, write $C^\wedge$ for the $p$-completion of $C$.

Finally we denote by $\mathcal{C}_p^\wedge$ for the presheaves satisfying descent with respect to all hypercovers. Equivalently, the canonical functor

$$c_p^\wedge : \mathcal{SH}(X^\wedge)_p^\wedge \to \mathcal{SH}_{\text{ét}}(X)_p^\wedge$$

is an equivalence of $\infty$-categories. This generalizes the rigidity theorems of Suslin-Voevodsky [SV96], Ayoub [Ayo14] and Cisinski-Déglise [CD16] to the setting of spectra. We deduce that under further regularity hypotheses on $X$, if $S$ is the set of primes not invertible on $X$, then the endomorphisms of the $S$-local sphere in $\mathcal{SH}_{\text{ét}}(X)$ are given by étale hypercohomology with coefficients in the $S$-local classical sphere spectrum:

$$[1][1/S], [1/1/S], [1/S], [1/S] : \mathcal{SH}_{\text{ét}}(X) \simeq \mathbb{H}^{1}_{\text{ét}}(X, 1[1/S]).$$

This confirms a conjecture of Morel.

The primary novelty of our argument is that we use the pro-étale topology [BS13] to construct directly an invertible object $1_p(1)[1] \in \mathcal{SH}(X^\wedge)_p^\wedge$ with the property that $c_p^\wedge(1_p(1)[1]) \simeq \Sigma^\infty \mathbb{G}_m \in \mathcal{SH}_{\text{ét}}(X)_p^\wedge$.

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1. Introduction

Étale motivic cohomology with finite coefficients invertible in the base coincides with étale cohomology. In more categorical terms, the canonical functor $D(X_\text{ét}, \mathbb{Z}/n) \to D\mathcal{M}_{\text{ét}}(X, \mathbb{Z}/n)$ is an equivalence, provided that $1/n \in \mathcal{O}(X)^\times$. This was proved for the case where $X$ is the spectrum of a field by Suslin-Voevodsky [SV96, Theorem 4.4] [Voe00, Proposition 3.3.3]. Versions of this result over more general bases were established by Ayoub for motives without transfers [Ayo14, Theorem 4.1] and by Cisinski-Déglise for motives with transfers [CD16, Theorem 4.5.2]. It is a natural question to ask if there is a “spectral” version of these results. The main aim of this article is to establish the following positive answer.

Theorem (see Theorem 6.6). Let $X$ be a locally $p$-étale finite scheme and $p$ a prime with $1/p \in X$.

Then the canonical functor $\mathcal{SH}(X^\wedge)_p^\wedge \to \mathcal{SH}_{\text{ét}}(X)_p^\wedge$ is an equivalence.

We recall the definitions of the terms in the above theorem in the following set of remarks.

Remark (spectral sheaves and hypercompletion). We denote by $X_\text{ét}$ the small étale site of $X$, i.e. the category of (finitely presented) étale $X$-schemes, with the Grothendieck topology given by the jointly surjective quasi-compact families. Associated with this we have the $\infty$-topos $\mathcal{Sh}(X_\text{ét})$ of sheaves of spaces on $X_\text{ét}$, i.e. presheaves of spaces satisfying étale descent. We denote by $\mathcal{Sh}(X_\text{ét})$ its hypercompletion; in other words these are the presheaves satisfying descent with respect to all hypercovers. Equivalently, equivalences are detected on homotopy sheaves [DHI04]. Finally we denote by $\mathcal{SH}(X^\wedge_\text{ét})$ the stabilization of the $\infty$-topos $\mathcal{Sh}(X^\wedge_\text{ét})$; equivalently this is the category of spectral hypersheaves: the category of

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1We will abuse notation and write $1/p \in X$ instead of $p \in \mathcal{O}(X)^\times$. 

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functors from $X^{op}_{et}$ to spectra, satisfying descent for all étale hypercovers (or, equivalently, being local for the family of weak equivalences detected by homotopy sheaves). See Section 2.2 for more about sheaves of spectra.

Remark (p-completion of stable categories). In spectral settings, there is no evident analog of “working with $\mathbb{Z}/n$ coefficients”. One standard resolution of this is to use p-completion, which is somewhat analogous to working with $\mathbb{Z}_p$-coefficients, where $\mathbb{Z}_p$ denotes the ring of p-adic integers. Let $C$ be a presentable stable ∞-category (or a triangulated category with small coproducts). For $X, Y \in C$ the set of homotopy classes of maps from $X$ to $Y$ is naturally an abelian group, and consequently for every $X \in C$ we have a canonical endomorphism $p = \text{id}_X \equiv \text{id}_X + \text{id}_X + \cdots + \text{id}_X$. We denote by $X/p$ the cofiber of (cone on) this endomorphism. We call a map $f : X \to Y \in C$ a p-equivalence if $\text{cone}(f)/p \simeq 0$, and we denote by $C^p$ the localization of $C$ at the p-equivalences; under our assumptions this exists and is in fact equivalent to the full subcategory of $C$ right orthogonal to all objects $X \in C$ such that $X/p \simeq 0$. See Section 2.1 for more about $p$-completion.

Remark (étale motivic stable homotopy theory). The motivic stable ∞-category $\mathcal{SH}(X)$ is obtained from the ∞-topos $\mathcal{P}(\text{Sm}_X)$ by (1) inverting the Čech nerves of Nisnevich coverings, (2) inverting all maps of the form $\mathbb{A}^1_\mathbb{Z} \to Y$ for $Y \in \text{Sm}_X$, and (3) passing to pointed objects, then stabilizing with respect to the endofunctor $\wedge^{\infty}$; see e.g. [Mor03, Section 5] [BH21, Sections 2.2 and 4.1]. The category $\mathcal{SH}_C(X)$ is constructed in exactly the same way, except that in step (2) we use the étale hypercoverings instead. Equivalently, $\mathcal{SH}_{et}(X)$ is obtained from $\mathcal{SH}(X)$ by inverting the maps of the form $\Sigma_{+}^{\infty} \mathcal{Y} \to \Sigma_{+}^{\infty-n} \mathcal{Y}$, for étale hypercoverings $\mathcal{Y} \to Y \in \text{Sm}_X$. The functor of taking the associated locally constant sheaf $e : \text{Sh}(\text{Sm}_{et}) \to \text{Sh}(\text{Sm}_{X, et})$ induces a functor $\mathcal{SH}(X^p) \to \mathcal{SH}(X)$, which after $p$-completion induces the equivalence in the theorem.

Remark ($p$-étale finiteness). We call a scheme $X$ $p$-étale finite if for every finite type scheme $Y/X$ there exists $n$ such that every finitely presented, qcqs étale $Y$-scheme $Z$ satisfies $c_{dp}(Z) \leq n$. We call $X$ locally $p$-étale finite if it admits a $p$-étale cover by $p$-étale finite schemes. This holds for example if $X$ is of finite type over a field of finite virtual $p$-étale cohomological dimension (this includes all finite fields, separably closed fields, numbers fields, and $\mathbb{R}$) or $\mathbb{Z}$. We call a scheme (locally) étale finite if it is (locally) $p$-étale finite for all primes $p$. See Definitions 2.11 and 5.8 and Examples 2.14 and 5.9.

With the above theorem at hand, we of course find that $[1, 1]_{\mathcal{SH}_{et}(X)} \simeq [1, 1]_{\mathcal{SH}(X)}$. This is significant, since the left hand side is a priori much more complicated than the right hand side, which is basically controlled by étale cohomology of $X$ and the classical stable homotopy groups. In general, one expects to learn essentially everything about a category $C$ by studying $C^p$ for all $p$, and also the rationalization $C_\mathbb{Q}$. Since $\mathcal{SH}_{et}(X) \simeq \mathcal{DM}(\text{et}, \mathbb{Q}) \simeq \mathcal{DM}(\text{X}, \mathbb{Q})$ is reasonably well understood, one might hope to patch together all of these computations to determine $[1, 1]_{\mathcal{SH}_{et}(X)}$; this was the original aim of the article. It has been fulfilled as follows.

Corollary (see Corollary 7.3). Let $X$ be locally étale finite, and $S$ be the set of primes not invertible on $X$. Assume that $X$ is regular, noetherian and finite dimensional. Then

$$[1, 1]_{\mathcal{SH}_{et}(X)_{1/S}} \simeq \mathbb{H}^0_{et}(X, 1_{1/S}),$$

where the right hand side denotes étale hypercohomology with coefficients in the (classical) sphere spectrum (in other words $\pi_0$ of the spectrum of global sections of the étale hypersheafification of the presheaf of spectra with value the classical sphere spectrum).

Proof strategy (for the main theorem). Suppose that the functor $e : \mathcal{SH}(X_\mathbb{Z}) \to \mathcal{SH}_{et}(X)$ is indeed an equivalence. As a basic sanity check, we should be able to write down an object $\mathbb{I}_p(1)[1] \in \mathcal{SH}(X_\mathbb{Z})$ such that $e(\mathbb{I}_p(1)[1]) \simeq \mathbb{G}_m$. In the abelian situation, say with $\mathbb{Z}/p^n$ coefficients (i.e. in $\mathcal{DM}(\text{et}, \mathbb{Z}/p^n)$), the corresponding sheaf is $\mu_{p^n}$, the sheaf of $p^n$-th roots of unity. In the spectral situation however, it is not so obvious (to the author) what the analogous object is. We know that $\mathbb{I}_p(1)[1]$ should be an invertible spectrum, that $\mathbb{I}_p(1)[1] \wedge \mathbb{Z}/p^n \simeq \mu_{p^n}[1]$, and by analogy with the abelian situation we might guess that $\mathbb{I}_p(1)[1] \simeq 1[1]$ if the base has all $p^n$-th roots of unity for all $n$.

It turns out that the construction of $\mathbb{I}_p(1)[1]$ is central to our proof of the main theorem, so let us pursue this further. The last condition gives a clue: even if we don’t know how to construct $\mathbb{I}_p(1)[1]$ directly, it seems to be a form of $1[1]$ in some sense, so we might try to construct it by descent. The problem is that since we are somehow working with $\mathbb{Z}/p^n$ coefficients for all $n$ at the same time, there will usually not be any étale cover after which the equivalence $\mathbb{I}_p(1)[1] \simeq 1[1]$ is achieved. Indeed we expect this to happen after all $p^n$-th roots of unity have been adjoined for all $n$, and this does not constitute
an étale cover. It is however a pro-étale cover. This suggests that we might wish to employ the pro-étale topology, as defined by Bhatt-Scholze [BS13]. We review this somewhat technical notion at the beginning of Section 3, but the upshot is that $\hat{Z}_p(1) := \lim_{\mu_p^\infty} \in \text{Sch}_X$ belongs to the pro-étale site $\text{X}_{\text{pro\-ét}}$, and we define

$$\hat{\mu}_p(1)[1] := \Sigma^\infty K(\hat{Z}_p(1), 1) \in \text{SH}(\text{X}_{\text{pro\-ét}})_p^\wedge$$

(note that $\hat{Z}_p(1)$ is a pro-étale form of $Z_p$, hence $\hat{\mu}_p(1)[1]$ is a pro-étale form of $\Sigma^\infty K(Z_p, 1) \simeq \mathbb{I}[1] \in \text{SH}(\text{X}_{\text{pro\-ét}})_p^\wedge$). This object has the expected properties, but it lives in the wrong category. However one may show that in good cases (e.g. $X = \text{Spec}(\mathbb{Z}[1/p])$), the functor $\text{SH}(X_{\text{pro\-ét}})_p^\wedge \to \text{SH}(\text{X}_{\text{pro\-ét}})_p^\wedge$ is fully faithful and $\hat{\mu}_p(1)[1]$ is in the essential image. This way we obtain $\hat{\mu}_p(1)[1] \in \text{SH}(\text{Spec}(\mathbb{Z}[1/p])_{\text{pro\-ét}})_p^\wedge$, and we define it over a general scheme by base change from $\text{Spec}(\mathbb{Z}[1/p])$.

With this preliminary out of the way, our proof is actually a fairly straightforward adaptation of the arguments from [CD16]. We can summarize it as follows.

1. Construct $\hat{\mu}_p(1) \in \text{SH}(X_{\text{pro\-ét}})_p^\wedge$.
2. Construct a natural map $\sigma : \mathbb{G}_m \to e(\hat{\mu}_p(1)[1]) \in \text{SH}(\text{Sm}_{X_{\text{pro\-ét}}})_p^\wedge$ and prove that if $E \in \text{SH}(X_{\text{pro\-ét}})_p^\wedge$ then $e(E)$ is local with respect to the family of maps $\sigma \wedge \text{id}_Y, Y \in \text{Sm}_X$.
3. Prove homotopy invariance and proper base change for $E \in \text{SH}(X_{\text{pro\-ét}})_{\text{pro\-ét}}$ is an equivalence.
4. Prove that $\Sigma^\infty \sigma \in \text{SH}(X_{\text{pro\-ét}})$ is an equivalence.

Once these steps are achieved, we conclude from (4) that $\text{SH}_{\text{pro\-ét}}(X) \simeq L_{K,S}\text{SH}(\text{Sm}_{X_{\text{pro\-ét}}})_p^\wedge$, where the right hand side denotes the localization at the family of maps from (2) and also at $Y \times \mathbb{A}^1 \to Y$. Steps (2,3) then imply that $e : \text{SH}(X_{\text{pro\-ét}})_p^\wedge \to \text{SH}_{\text{pro\-ét}}(X)_p^\wedge$ is fully faithful. Essential surjectivity follows from the fact that both sides satisfy proper base change (as established for the left hand side in (3) and for the right hand side by Ayoub), via an argument of Cisinski-Déglise [CD16, Proof of Theorem 4.5.2].

Of steps (2–4), the most interesting one is probably (4). Since $\sigma$ is stable under base change, using a localization argument we may reduce to the case where $X$ is the spectrum of a separably closed field of characteristic $\neq p$. In this situation we construct a map $\tau : \hat{\mu}_p(1)[1] \simeq \mathbb{I}[1] \to \mathbb{G}_m \in L_{K,S}\text{SH}(\text{Sm}_{X_{\text{pro\-ét}}})_p^\wedge$, which induces an inverse to $\sigma$ in $D\text{MH}_{\text{pro\-ét}}(k, \mathbb{Z}/p)$. Since $[1, \mathbb{I}]_{L_{K,S}\text{SH}(\text{Sm}_{X_{\text{pro\-ét}}})q_p^\wedge \simeq Z_p$ by (3), we deduce from this that $\tau : \mathbb{I}[1] \to \mathbb{I}[1] \in L_{K,S}\text{SH}(\text{Sm}_{X_{\text{pro\-ét}}})_p^\wedge$ is an equivalence. This implies that $\Sigma^\infty \sigma$ is an equivalence by a general result about symmetric monoidal categories [Bac18b, Lemma 22].

**Organization.** In Section 2 we collect some preliminaries about $p$-completion, spectral sheaves, and étale cohomological dimension. In Section 3 we use the pro-étale topology to construct the twisting spectrum $\hat{\mu}_p(1)$ and establish its properties, achieving step (1). In Section 4 we prove some “standard facts” about étale cohomology with spectral coefficients. This achieves steps (2) and (3). In Section 5 we prove/recall some essentially well-known facts about the functor $X \to \text{SH}_{\text{pro\-ét}}(X)$. Then we carry out step (4) and hence conclude the proof of the main theorem in Section 6. We collect some applications in Section 7.

**Necessity of the étale finiteness hypothesis.** We prove our main result (and hence all applications) under the assumption of (local) “$p$-étale finiteness”. While this is satisfied quite often in practice, it is an unsatisfying hypothesis, since the rigidity theorems of Cisinski-Déglise and Ayoub do not need it. Essentially the only part of the proof where we need the hypothesis is in step (3). That is to say, the author has been unable to prove (for example) that $\text{SH}(X_{\text{pro\-ét}})_p^\wedge \to \text{SH}(\mathbb{A}^1 \times X_{\text{pro\-ét}})^\wedge$ is fully faithful (for $1/p \in X$) without the assumption that $X$ is étale finite.

We also often use the notion of “uniformly bounded étale cohomological dimension $\leq n$”, which is slightly stronger than the usual notion of “locally of étale cohomological dimension $\leq n$”. Some of our intermediate results can probably be strengthened to hold under the weaker assumption; we chose not to do this because all our examples satisfy the stronger conclusion anyway.

**Use of $\infty$-categories.** This article is written in the language of $\infty$-categories, as set out in [Lur09, Lur16]. This is mostly inconsequential: apart from Sections 3 and 5, everything can be formulated at the level of triangulated categories, and for Sections 3 and 5 a translation into the language model categories is straightforward.

**Notation.** We denote by $\text{Map}(A, B)$ the mapping space between objects in an $\infty$-category, by $\text{map}(A, B)$ the mapping spectrum in a stable $\infty$-category, and by $\overline{\text{map}}(A, B)$ the internal mapping spectrum in a closed symmetric monoidal stable $\infty$-category. We put $[A, B] = \pi_0\text{Map}(A, B)$. We write $D(X)$ for the strong dual of an object $X$ in a symmetric monoidal category, if it exists.
We use homological notation for $t$-structures, see e.g. [Lur16, Section 1.2.1]. For any $\infty$-category $\mathcal{C}$ (other than stable $\infty$-categories with a $t$-structure), we denote by $\mathcal{C}_{\leq 0}$ the subcategory of 0-truncated objects [Lur09, Definition 5.5.6.1]. In particular for a site $\mathcal{C}$, $\text{Shv}(\mathcal{C})_{\leq 0}$ denotes the 1-category of sheaves of sets on $\mathcal{C}$.

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2. Preliminaries

We collect some essentially well-known results.

2.1. $p$-completion. Throughout we fix a stable, presentably symmetric monoidal $\infty$-category $\mathcal{C}$ and a (strongly) dualizable object $A$. For closely related results, see [MNN17, Sections 2 and 3]. We let $\mathcal{C}[A^{-1}] = \{X \mid X \otimes A \simeq 0\} \subset \mathcal{C}$. Denote by $\mathcal{C}_{A-\text{tors}}$, the left orthogonal of $\mathcal{C}[A^{-1}]$, and by $\mathcal{C}_A^\perp$ the right orthogonal.

Lemma 2.1. The inclusion $\mathcal{C}[A^{-1}] \subset \mathcal{C}$ is both reflective and co-reflective.

Proof. The functor $\otimes A : \mathcal{C} \to \mathcal{C}$ has both a right and a left adjoint (namely tensoring with $DA$), hence preserves limits and colimits. It follows that $\mathcal{C}[A^{-1}]$ is presentable (e.g. use [Lur09, Proposition 5.5.3.12]) and the inclusion preserves limits and colimits. The result follows now by the adjoint functor theorem [Lur09, Corollary 5.5.2.9].

By [BG16, Remark 6], we thus have a recollement situation, and in particular there is a canonical equivalence $\mathcal{C}_{A-\text{tors}} \simeq \mathcal{C}_A^\perp$. We can identify $\mathcal{C}_{A-\text{tors}}, \mathcal{C}_A^\perp$ more explicitly.

Lemma 2.2. 

(1) The category $\mathcal{C}_{A-\text{tors}} \subset \mathcal{C}$ is the localising subcategory generated by objects of the form $DA \otimes X$ for $X \in \mathcal{C}$.

(2) The category $\mathcal{C}_A^\perp$ is the localization of $\mathcal{C}$ at the maps $f : X \to Y \in \mathcal{C}$ such that $f \otimes A$ is an equivalence.

Proof. (1) Let $\mathcal{C}'$ be the localizing subcategory generated by objects of the form $DA \otimes X$. Clearly $\mathcal{C}' \subset \mathcal{C}_{A-\text{tors}}$. The inclusion $\mathcal{C}' \to \mathcal{C}$ has a right adjoint by presentability. In order to conclude that $\mathcal{C}' = \mathcal{C}_{A-\text{tors}}$ it suffices to prove that if $X \in \mathcal{C}_{A-\text{tors}}$ and $[Y \otimes DA, X] = 0$ for all $Y \in \mathcal{C}$, then $X \simeq 0$. The assumption implies that $X \otimes A \simeq 0$, and hence $X \in \mathcal{C}[A^{-1}]$. The result follows.

(2) The functor $\text{Fun}(\Delta^1, \mathcal{C}) \to \text{Fun}(\Delta^1, \mathcal{C}), f \mapsto f \otimes \text{id}_A$ is accessible. Hence the localization exists; denote it by $LC \subset \mathcal{C}$. Note that $f \otimes A$ is an equivalence if and only if cone$(f) \otimes A \simeq 0$. Consequently $LC$ is the right orthogonal of $\mathcal{C}[A^{-1}]$. This concludes the proof.

Example 2.3. Any stable symmetric monoidal $\infty$-category receives a symmetric monoidal functor from finite spectra. In particular the case $A = 1/p$ always applies. In this case $\mathcal{C}[A^{-1}]$ consists of the uniquely $p$-divisible objects, $\mathcal{C}_{p-\text{tors}} := \mathcal{C}_{A-\text{tors}}$ consists of the $p$-torsion objects, and $\mathcal{C}_p^\perp := \mathcal{C}_A^\perp$ consists of the $p$-complete objects. In particular we see that if $C$ is compactly generated then so is $C_{p-\text{tors}}$, and hence so is the equivalent category $\mathcal{C}_p^\perp$. We call the maps $f$ such that $f \otimes A$ is an equivalence, i.e. such that $f/p$ is an equivalence, $p$-equivalences.

We have the following obvious but comforting results.

Lemma 2.4. Let $F : \mathcal{C} \to \mathcal{D}$ be any stable functor of stable $\infty$-categories. Then $F$ preserves $p$-equivalences.

Proof. A map $\alpha : X \to Y$ is a $p$-equivalence if and only if $\alpha/p : X/p \to Y/p$ is an equivalence. Here $p : X \to X$ denotes the sum of $p$ times the identity map, and $X/p$ the cofiber. Similarly for $Y$. Being stable, $F$ preserves the identities, the $Ab$-enrichment of the homotopy categories, and cofibers. The result follows.

Consequently, $F$ canonically induces a functor $\mathcal{C}_p^\perp = F : \mathcal{C}_p^\perp \to \mathcal{D}_p^\perp$.

Lemma 2.5. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction of stable $\infty$-categories, with $F$ fully faithful. Then there is an induced adjunction $F_p ^\perp : \mathcal{C}_p^\perp \rightleftarrows \mathcal{D}_p^\perp : G_p^\perp$, with $F_p ^\perp$ still fully faithful. The essential image of $F_p^\perp$ consists of those $X \in \mathcal{D}_p^\perp$ such that $X/p \in \mathcal{D}$ lies in the essential image of $F$.
Proof. Only the last statement requires proof. If \( X = F_p^n(Y) \) then \( X/p = F(Y)/p \) is of the claimed form. Conversely, let \( X \in D^+_p \) with \( X/p \) in the essential image of \( F \). We wish to show that \( F_p^nG_pX \to X \) is an equivalence, for which it suffices to show that it is a \( p \)-equivalence. Since \( F, G \) commute with taking the cofiber of multiplication by \( p \), it thus suffices to show that \( FGX/p \to X/p \) is an equivalence. This is true by assumption. \( \square \)

2.2. Sheaves of spectra. Given an \( \infty \)-topos \( X \) we denote by \( \mathcal{SH}(X) \) the category of spectral sheaves on \( X \), i.e. limit-preserving functors \( X^{op} \to \mathcal{SH} \). This is a presentable \( \infty \)-category [Lur18, Remark 1.3.6.1]. The category \( \mathcal{SH}(X) \) is equivalent to the stabilization of \( X \) [Lur18, Remark 1.3.3.2], and so is in particular stable. One puts

\[
\mathcal{SH}(X)_{\leq 0} = \{ E \in \mathcal{SH}(X) \mid \Omega^{\infty}E \simeq \ast \}.
\]

This defines the non-positive part of a right complete \( t \)-structure on \( \mathcal{SH}(X) \), with heart the category of abelian group objects in \( X_{\leq 0} \) [Lur18, Proposition 1.2.1.1]. We write \( \underline{\mathcal{E}}(E) \) for the homotopy sheaves of \( E \in \mathcal{SH}(X) \). The \( t \)-structure is nondegenerate provided that \( X \) is hypercomplete (see [Lur09, Section 6.5] for this notion); if \( X \) is furthermore locally of cohomological dimension \( \leq n \) for some \( n \) then the \( t \)-structure is left complete [Lur18, Corollary 1.3.3.11]. If \( X \) is an \( \infty \)-topos we denote by \( \hat{X} \) its hypercompletion; somewhat abusively if \( C \) is a site then we write \( \text{Shv}(C^\wedge) \) instead of \( \text{Shv}(C) \wedge \). Similarly \( \text{Shv}(C^\wedge) \) means \( \text{SH}(\text{Shv}(C^\wedge)) = \text{SH}(\text{Shv}(C^\wedge)) \). If \( g^* : X \to Y \) is (the left adjoint of) a geometric morphism, then there is an induced adjunction \( g^* : \mathcal{SH}(X) \rightleftarrows \mathcal{SH}(Y) : g_* \), with \( g^* \) \( t \)-exact [Lur18, Remark 1.3.2.8].

Recall the notion of coherent and locally coherent \( \infty \)-topoi from [Lur18, Definition A.2.1.6]. Coherence is stable under hypercompletion [Lur18, Proposition A.2.2.2]. If \( C \) is a “finitary” site (see [Lur18, Section A.3.1]), then \( \text{Shv}(C) \) is locally coherent and coherent [Lur18, Proposition A.3.1.3]. In particular any “reasonable” topology on schemes yields a locally coherent topos, and any object represented by a qcqs scheme is coherent.

It is well-known that if \( X \) is a locally coherent \( \infty \)-topos and \( X \in X \) is coherent, then sheaf cohomology on \( X \) commutes with filtered colimits. The next result is an equally well-known generalization of this.

Lemma 2.6. Let \( X \) be a locally coherent topos and \( X \in X \) be coheren. Then \( \text{map}(\Sigma^{\infty}X_+, \bullet) : \mathcal{SH}(X)_{\leq 0} \to \mathcal{SH} \) commutes with filtered colimits.

Proof. Immediate consequence of the same statement for \( n \)-truncated spaces [Lur18, Corollary A.2.3.2(1)], using that \( \Omega^{\infty} : \mathcal{SH}(X) \to X \) preserves filtered colimits (since filtered colimits commute with finite limits in any \( \infty \)-topos [Lur09, Example 7.3.4.7], filtered colimits of \( \Omega \)-spectra are \( \Omega \)-spectra). \( \square \)

Recall that an object \( X \) in an \( \infty \)-topos \( X \) is said to have cohomological dimension \( \leq n \) for a family of sheaves \( \{ F_n \in \text{SH}(X)^{\wedge} \} \) if \( H^i(X, F_n) = 0 \) for all \( i > n \) and all \( n \) [CM19, Definition 2.8]. Recall also that \( E \in \mathcal{SH}(X) \) is said to be postnikov-complete if the natural map \( E \to \lim E_i \) is an equivalence.

Lemma 2.7. Let \( X \) be an \( \infty \)-topos, \( Y \in \mathcal{SH}(X) \) postnikov complete, \( X \in X \) of cohomological dimension \( \leq n \) for \( \{ \underline{\mathcal{E}}(Y_i) \} \) of \( Y \). Then

(1) We have \( [\Sigma^{\infty}X_+, Y] \simeq [\Sigma^{\infty}X_+, Y_{\leq n}] \).
(2) If \( Y \in \mathcal{SH}(X)_{\geq m} \) then \( \text{map}(\Sigma^{\infty}X_+, Y) \in \mathcal{SH}_{\geq m, \leq n} \).

Suppose that postnikov towers converge in \( X \) and \( X \) is of cohomological dimension \( \leq n \). Then also

(3) \( \Sigma^{\infty}X_+ \in \mathcal{SH}(X) \) is compact.

Alternatively, suppose that objects in \( \mathcal{SH}(X) \) of the form \( Y/p \) are postnikov complete, and \( X \) is of \( p \)-cohomological dimension \( \leq n \). Then

(4) \( \Sigma^{\infty}X_+/p \in \mathcal{SH}(X) \) is compact.

Proof. We will conflate \( X \) and \( \Sigma^{\infty}X_+ \) for notational simplicity.

(1) By assumption \( Y \simeq \lim_{i \in \mathbb{Z}} Y_{\leq i} \). Consequently we have the Milnor exact sequence \( 0 \to \lim_{i \in \mathbb{Z}} \pi_1 \text{Map}(X, Y_{\leq i}) \to \pi_0 \text{Map}(X, Y) \to \lim_{i \in \mathbb{Z}} \pi_0 \text{Map}(X, Y_{\leq i}) \to 0 \) [GJ09, Proposition VI.2.15]. It is thus enough to show that \( \{ \pi_1 \text{Map}(X, Y_{\leq i}) \} \) stabilizes, and \( \{ \pi_0 \text{Map}(X, Y_{\leq i}) \} \) stabilizes at \( i = n \). The first statement follows from the second applied to \( \Omega Y \). To prove the second statement, it suffices to show that \( [X, Y_{\leq i}] \to [X, Y_{\leq i}] \) is an isomorphism for \( i \geq n \). We have the fiber sequence \( \underline{\mathcal{E}}_{i+1}(Y)[i + 1] \to Y_{\leq i+1} \to Y_{\leq i} \) which induces an exact sequence

\[
[X, \underline{\mathcal{E}}_{i+1}(Y)[i + 1]] \to [X, Y_{\leq i+1}] \to [X, Y_{\leq i}] \to [X, \underline{\mathcal{E}}_{i+1}(Y)[i + 2]].
\]

The two outer terms vanish for \( i \geq n \) by assumption, whence the result.

(2) We need to prove that \( [X[i], Y] = 0 \) for \( i < m - n \), or equivalently \( [X, Y]^n \) is of cohomological dimension \( \leq n \). But \( [X, Y]^n = [X, Y^\wedge_n] = [X, 0] = 0 \), by (1). The result follows.
(3) Let \( \{Y_i\} \) be a filtered system in \( SH(\mathcal{X}) \). We have
\[
[X, \colim_i Y_i] \cong [X, \tau_\leq n \colim_i Y_i] \cong [X, \colim_i \tau_\leq n Y_i] \cong \colim_i [X, \tau_\leq n Y_i] \cong \colim_i [X, Y_i],
\]
which is the desired result. Here we have used (1) for the first and last equivalence, and for the second equivalence we use that \( \tau_\leq n : SH(\mathcal{X}) \to SH(\mathcal{X}) \) preserves filtered colimits by [Lur18, Proposition 1.3.2.7(2)], and the third equivalence is Lemma 2.6.

(4) Essentially the same argument applies, using that \( [X/p, Y] \cong [X, Y/p[-1]] \) and \( Y/p \) has \( p^2 \)-torsion homotopy sheaves.

\[ \square \]

If the assumption of condition (4) holds (i.e. every spectrum of the form \( E/p \) is postnikov complete), then we shall say that \( SH(\mathcal{X}) \) is \( (p) \)-postnikov complete. Let us note the following fact.

Remark 2.8. Suppose that \( \mathcal{X} \) is hypercomplete and there exists \( d \) such that \( \mathcal{X} \) is locally of \((p-)\)cohomological dimension \( \leq d \). Then \( SH(\mathcal{X}) \) is \((p-)\)postnikov complete. This follows from [CM19, Proposition 2.10].

If \( f : \mathcal{C} \to \mathcal{D} \) is a functor of \( t \)-categories, we call \( f \) of \textit{finite cohomological dimension} \((\leq n)\) if \( f(\mathcal{C}_{\geq 0}) \subset \mathcal{D}_{2-n} \). We will similarly say that \( f \) is of \textit{finite \((p-)\)cohomological dimension} \((\leq n)\) if \( f(\mathcal{C}_{\geq 0}/p) \subset \mathcal{D}_{2-n} \). For example in the situation of Lemma 2.7, the functor \( (1)_{\Sigma^\infty X, \bullet} : SH(\mathcal{X}) \to SH \) is of \((p-)\)cohomological dimension \( \leq n \).

\[ \text{Lemma 2.9. Let } f : \mathcal{C} \to \mathcal{D} \text{ be a stable functor between } t \text{-categories. Assume that } f \text{ is of finite } (p-)\text{cohomological dimension and vanishes on bounded above objects (modulo } p), \text{ and that } \mathcal{D} \text{ is non-degenerate (on } p\text{-torsion objects). Then } f \cong 0. \]

Proof. Let \( E \in \mathcal{C} \) and put \( E' = E \) (respectively \( E' = E/p \)). For any \( i \) we have the fiber sequence \( E'_{\geq i} \to E' \to E'_{<i} \) and hence we get a fiber sequence \( f(E'_{\geq i}) \to f(E') \to 0 \). We conclude that \( f(E') \) is \( \infty \)-connective (\( f \) being of finite cohomological dimension), and hence zero.

\[ \square \]

2.3. \textit{Étale cohomological dimension}. We start with the following variant of [CD16, Theorem 1.1.5]. Here and everywhere in this article, cohomological dimension of schemes refers to \( étale \) cohomological dimension.

\[ \text{Theorem 2.10 (Gabber, Cisinski-Dégallise). Let } X \text{ be a quasi-separated, noetherian scheme of dimension } d. \text{ Let } d' = \text{pcd}_p(X) := \sup_{x \in X} \text{cd}_p(k(x)) \text{ and } \text{pcd}(X) = \sup_p \text{pcd}_p(X). \text{ Then } \text{cd}_p(X) \leq 3d + d' + 3 \text{ and } \text{cd}_1(X) \leq 3d + \text{pcd}(X) + 3. \]

Proof. We use the ideas of [CD16, Theorem 1.1.5]. Let \( D = 3d + d' + 3 \) (respectively \( D = 3d + \text{pcd}(X) + 3 \)). We need to show that \( H^i_{\text{ét}}(X, F) = 0 \) for every \( p \)-torsion (respectively every \( étale \) sheaf (of abelian groups) \( F \) on \( X \) and every \( i > D \). Since \( X \) is qcqs, \( étale \) cohomology commutes with filtered colimits of sheaves. Hence we may assume that \( F \) is constructible. As in the reference, we may reduce to \( F \) being a sheaf of \( Z/p \)-modules (respectively, \( Z/p \)-modules for some prime \( p \) or \( \mathbb{Q} \)-modules). For the case of a \( \mathbb{Q} \)-module the claim follows from [CD16, Lemma 1.1.4]. Hence suppose that \( F \) is a \( Z/p \)-module. Let \( Z = X \otimes_Z \mathbb{Z}/p \) and \( U = X \setminus Z \). Let \( i \) be the closed and \( j \) the open immersion. We have the distinguished triangle \((*) \) \( i'_*R^i j'_*F \to F \to j_*R^i j_*F \). From this we get in particular that \( R^j j_*j^*F \otimes \mathbb{Z}/p = 0 \) for \( b > 2d + d' \) (see the reference; this uses [ILO14, Lemma XVIII-A.2.2]) and hence \( R^i j_*F \) is concentrated in cohomological degrees \( \leq 2d + d' + 2 \). Considering \((*)\), it is enough to show that \( H^0_{\text{ét}}(Z, R^i j_*F) = 0 \) and \( H^1_{\text{ét}}(U, j_*F) = 0 \) for \( i > D \). For \( U \), this is [ILO14, Lemma XVIII-A.2.2] again. For \( Z \) we use that \( \text{cd}_p(Z) \leq \dim Z + 1 \leq d + 1 \) [GAV72, Theorem X.5.1] (and [Sta18, Tag 02UZ]).

\[ \square \]

Abstracting from this, we will make use of the following notion.

\[ \text{Definition 2.11. We say that a scheme } S \text{ is of uniformly bounded } p\text{-}étale \text{ cohomological dimension } \leq n \text{ if for every finitely-presented, } \text{qcqs } étale \text{ scheme } Y/X \text{ we have } \text{cd}_p(Y) \leq n; \text{ uniformly bounded } étale \text{ cohomological dimension } \leq n \text{ is defined similarly.} \]

We recall the following well-known facts.

\[ \text{Lemma 2.12. Let } f : X \to S \text{ be a morphism of schemes.} \]

\begin{enumerate}
\item \( \text{If } f \text{ is quasi-finite, then } \text{pcd}_p(X) \leq \text{pcd}_p(S) \text{ and } \text{pcd}(X) \leq \text{pcd}(S). \)
\item \( \text{If } f \text{ is finite type and } S \text{ is quasi-compact, then } \text{pcd}_p(X) < \infty \text{ (respectively } \text{pcd}(X) < \infty \text{) as soon as } \text{pcd}_p(S) < \infty \text{ (respectively } \text{pcd}(S) < \infty \text{).} \)
\end{enumerate}

Proof. This follows from the fact that if \( L/K \) is an extension of fields, then \( \text{cd}_p(L) \leq \text{cd}_p(K) + \text{trdeg}(L/K) \) [Sha72, Theorem 28 of Chapter 4].

\[ \square \]
Corollary 2.13. If $X$ is noetherian, $\dim X < \infty$ and $\pcd(X) < \infty$ (respectively $\pcd_p(X) < \infty$), then $X$ is of uniformly bounded ($p$-)étale cohomological dimension. If $Y/X$ is finite type, then also $Y$ is uniformly of bounded ($p$-)étale cohomological dimension.

Proof. By Lemma 2.12(2), $Y$ satisfies the same assumptions as $X$. Hence we may assume that $Y = X$.

Let $U/X$ be étale. We have $\pcd_p(U) \leq \pcd_p(X)$ by Lemma 2.12(1). Since also $\dim U \leq \dim X$, it follows from Theorem 2.10 that if $U$ is moreover quasi-separated and finite type (whence noetherian), we have $\cd_p(U) \leq \pcd_p(X) + 3 \dim X + 3$. The argument for $\cd$ in place of $\pcd_p$ is the same. This concludes the proof.

Example 2.14. The assumptions of Corollary 2.13 hold for $X = \Spec(R)$, where $R$ is a field of finite étale cohomological dimension (e.g., separably closed fields, finite fields, unorderable number fields), or a strictly henselian noetherian local ring [ILO14, Lemma XVIII-A.1.1]. They hold étale-locally on $X = \Spec(R)$ for $R$ a field of finite virtual étale cohomological dimension (e.g., number fields), or $R = \mathbb{Z}$ (cover $\Spec(\mathbb{Z})$ by $\Spec(\mathbb{Z}[1/2,x]/x^2 + 1)$).

Let us note the following permanence property.

Lemma 2.15. Let $\{S_i\}_i$ be a pro- (qcqs scheme) with finitely presented, affine étale transition morphisms. If $S_0$ is of uniformly bounded ($p$-)étale cohomological dimension, then so is $S := \lim_i S_i$.

Proof. Suppose $S_0$ is of uniformly bounded $p$-étale cohomological dimension $\leq n$. Then the same holds for $S_i$ for all $i$. If $X/S$ is étale and qcqs, then $X = \lim_i X_i$ for some system of qcqs étale schemes $X_i/S_i$. Since étale cohomology of qcqs schemes commutes with cofiltered limits [GAV72, Theorem VII.5.7], we conclude that $\cd_p(X) \leq n$. The argument for étale cohomological dimension is the same. The result follows.

Let us summarize the following convenient properties.

Lemma 2.16. Let $X$ be (étale locally) of uniformly bounded étale cohomological dimension. Then $\SH(X^\wedge_{\et})$ is postnikov-complete and compactly generated. In fact if $Y \subseteq X_{\et}$ is qcqs (of finite étale cohomological dimension), then $\Sigma^\infty_+ Y$ is compact.

If $X$ is only (étale locally) of uniformly bounded p-étale cohomological dimension, then $\SH(X^\wedge_{\et})$ is $p$-postnikov complete and $\SH(X^\wedge_{\et})^\wedge_p$ is compactly generated. In fact if $Y \subseteq X_{\et}$ is qcqs (of finite $p$-étale cohomological dimension), then $\Sigma^\infty_+ Y/p \subseteq \SH(X^\wedge_{\et})^\wedge_p$ is compact.

Proof. The ($p$-)postnikov completeness follows from Remark 2.8. Lemma 2.7(3,4) now show that $\Sigma^\infty_+ Y$ (respectively $\Sigma^\infty_+ Y/p$) is compact in $\SH(X^\wedge_{\et})$. We deduce that $\Sigma^\infty_+ Y/p \subseteq \SH(X^\wedge_{\et})^\wedge_p$ is compact, as follows:

\[ [Y/p, \colim_i T_i]_{\SH(X^\wedge_{\et})}^\wedge_p \cong [Y[1], (\colim_i T_i)/p]_{\SH(X^\wedge_{\et})}^\wedge_p \]

\[ \cong [Y[1], (\colim_i T_i)/p]_{\SH(X^\wedge_{\et})} \]

\[ \cong [Y/p, \colim_i T_i]_{\SH(X^\wedge_{\et})} \]

\[ \cong \colim_i [Y/p, T_i]_{\SH(X^\wedge_{\et})} \]

\[ \cong \colim_i [Y[1], T_i/p]_{\SH(X^\wedge_{\et})} \]

\[ \cong \colim_i [Y[1], T_i/p]_{\SH(X^\wedge_{\et})}^\wedge_p \]

\[ \cong \colim_i [Y/p, T_i]_{\SH(X^\wedge_{\et})}^\wedge_p \]

In other words we use that the forgetful functor $\SH(X^\wedge_{\et})^\wedge_p \to \SH(X^\wedge_{\et})$ commutes with the formation of $(\colim_i -)/p$.

3. Construction of the twisting spectrum

We will make use of some of the convenient properties of the pro-étale topology [BS13]. Recall that a morphism of schemes $X \to Y$ is called weakly étale if $X \to Y$ is flat and $\Delta : X \to X \times_Y X$ is also flat. The pro-étale site of $X$ consists of those weakly étale $X$-schemes of cardinality smaller than some (large enough) bound; the coverings are the fqc coverings. We denote the pro-étale topos of $X$ (with respect to the implicit cardinality bound) by $\mathcal{X}_{pro\et}$. We spell out some salient properties.
Lemma 3.1. Let $X$ be a topos. If $X \in \mathcal{X}$ is a weakly contractible [BS13, Definition 3.2.1] object, then $X$ has cohomological dimension 0.

Proof. It suffices to prove that $H^0(X, \bullet)$ is exact. Let $\nu : SH(\mathcal{X})^\land \to SH(X^\land_{proet})$ be a sheaf of sheaves. We claim that $\nu(\mathcal{X}) \in SH(X^\land_{proet})$ is left-complete and compactly generated. Consequently the composite $\nu$ is in the essential image of $SH(\mathcal{X})^\land \to SH(X^\land_{proet})$.

Remark 3.2 (Hoyois). The notion of a weakly contractible object in a 1-topos [BS13, Definition 3.2.1] is essentially the same as that of an object of homotopy dimension $\leq 0$ in an infty-topos [Lur09, Definition 7.2.1.1]. Lemma 3.1 above is essentially a special case of [Lur09, Corollary 7.2.2.30].

Corollary 3.3. For any scheme $X$, postnikov towers converge in $Shv(X^\land_{proet})$ and $SH(X^\land_{proet})$ is left-complete and compactly generated.

Proof. $X^\land_{proet}$ is generated by coherent weakly contractible objects [BS13, Proposition 4.2.8]. It suffices to prove that $\nu U : Shv(X^\land_{proet}) \to Shv$ is fully faithful [BS13, Lemma 5.1.2]. A sheaf $F \in Shv(X^\land_{proet})$ is called classical if it is in the essential image of $\nu$. Note that a sheaf is classical if and only if it is continuous for pro-étale systems of affine schemes [BS13, Lemma 5.1.2]. For a t-category $C$, we denote by $C_-$ the subcategory of (homologically) bounded above objects.

Lemma 3.4. The functor $\nu : SH(X^\land_{proet})_- \to SH(\mathcal{X})^\land_-$ is fully faithful. If postnikov towers converge in $X^\land_{proet}$, then the functor $\nu : SH(X^\land_{proet}) \to SH(X^\land_{proet})$ is also fully faithful.

Proof. Let $U \in X^\land_{proet}$ be qcqs (e.g. affine). By Lemma 2.6, the functors $map(\Sigma_0^\land U, \bullet) : SH(X^\land_{proet})_{\leq 0} \to SH$ preserve filtered colimits. Consequently the composite $\alpha = \nu_* \nu^* : SH(X^\land_{proet})_{\leq 0} \to SH(X^\land_{proet})_{\leq 0}$ preserves filtered colimits. For $E \in SH(X^\land_{proet})^\circ$ we have $\alpha(E) \simeq E$ by [BS13, Corollary 5.1.6].

Proposition 3.5. The essential image of $\nu : SH(X^\land_{proet})_\leq 0 \to SH(\mathcal{X})^\land_\leq 0$ consists of those spectra $E \in SH(X^\land_{proet})_\leq 0$ with classical homotopy sheaves, and the functor $\nu_*$ is t-exact when restricted to such objects. If $X$ has étale locally uniformly bounded étale cohomological dimension, then the functor $\nu : SH(X^\land_{proet}) \to SH(X^\land_{proet})$ has essential image those spectra with classical homotopy sheaves, and $\nu_*$ is t-exact on the entire essential image of $\nu$.

Proof. Clearly spectra in the essential image of $\nu$ have classical homotopy sheaves; we need to prove the converse.

The functor $\nu_* : SH(X^\land_{proet})^\circ \to SH(X^\land_{proet})^\circ$ is an equivalence onto the subcategory of classical sheaves, with inverse given by $\nu_*$ (by Lemma 3.4). The result for spectra with bounded homotopy sheaves follows immediately. As in the proof of Lemma 3.4, the functor $\nu_*$ preserves filtered colimits in $SH(X^\land_{proet})_{\leq 0}$, and hence so does $\nu_* \nu_*$. Since the subcategory of $SH(X^\land_{proet})_{\leq 0}$ consisting of spectra with classical homotopy sheaves is generated under filtered colimits by bounded spectra with classical homotopy sheaves, we find that $\nu_* \nu_* \simeq id$ is an equivalence on this subcategory. This first statement follows.

Now let $X$ be étale-locally of uniformly bounded étale cohomological dimension. Postnikov towers converge in $X^\land_{proet}$ (see Lemma 2.16). Hence $\nu_* : SH(X^\land_{proet}) \to SH(X^\land_{proet})$ is fully faithful by Lemma 3.4. Let $U \in X^\land_{proet}$ have cohomological dimension $< n$ and let $E \in SH(X^\land_{proet})_{> n}$ have classical homotopy sheaves. I claim that $[U, \nu_*, E] = 0$. To see this, we note that $Map(U, \nu_* E) \simeq \lim_\nu Map(U, \nu_*(E_{\leq 1}))$ (since postnikov towers converge in $SH(X^\land_{proet})$). Consequently the composite $\nu_* \nu_*$ preserves filtered colimits in $SH(X^\land_{proet})_{\leq 0}$, and hence so does $\nu_* \nu_*$. Since the subcategory of $SH(X^\land_{proet})_{\leq 0}$ consisting of spectra with classical homotopy sheaves is generated under filtered colimits by bounded spectra with classical homotopy sheaves, we find that $\nu_* \nu_* \simeq id$ is an equivalence on this subcategory. This first statement follows.

Let $\bar{x}$ be a geometric point of $X$. Then $X_{\bar{x}} = \lim_\alpha X_\alpha$, where $\{X_\alpha\}$ is a cofiltered system of affine étale $X$-schemes of bounded cohomological dimension (here we use that $X$ is étale-locally uniformly of
bounded étale cohomological dimension), say bounded by \( n \geq 0 \). We deduce that if \( E \in \mathcal{SH}(X^\wedge_{\text{proet}}) \) has classical homotopy sheaves, then

\[
\pi_0(\nu_*E)(X) = \colim_a [X_a, \nu_*E] \simeq \colim_a [X_a, \nu_*(E_{\leq n})] = \pi_0(\nu_*(E_{\leq n}))(X)
\]

\[
\simeq \nu_*(\pi_0(E_{\leq n}))(X) \simeq \nu_*(\pi_0(E))(X),
\]

using the t-exactness statement for bounded above spectra (which we already proved) again. We have thus shown that \( \nu_* \) is t-exact on arbitrary spectra with classical homotopy sheaves. Since \( \nu^* \) is t-exact (as always) and \( X^\wedge_{\text{proet}} \) is hypercomplete (by definition), this shows that \( \nu^* \nu_* \Rightarrow \text{id} \) is an equivalence on spectra with classical homotopy sheaves, which concludes the proof.

After these preparatory remarks, we come to our application of the pro-étale topology: the construction of the twisting spectrum. Given a sheaf of groups \( F \) on \( X_{\text{proet}} \), we denote by \( K(F, 1) \in \text{Shv}(X^\wedge_{\text{proet}}) \) the sheaf obtained from the presheaf \( U \mapsto K(F(U), 1) \), where \( K(F(U), 1) \) is the Eilenberg-MacLane space of the group \( F(U) \).

**Theorem 3.6.** Let \( X \) be a scheme and \( 1/p \in X \). Let \( \hat{Z}_p(1) := \lim_n \mu_{p^n} \in \text{Shv}(X^\wedge_{\text{proet}})^{\leq 0} \) and \( \hat{\Pi}_p(1)[1] = (\Sigma^\infty K(\hat{Z}_p(1), 1))_p^c \in \mathcal{SH}(X^\wedge_{\text{proet}})^{\leq 0}_p \).

1. \( \hat{Z}_p(1) \) is a representable by a weakly étale \( X \)-scheme, and so stable under base change.
2. \( \hat{\Pi}_p(1) \) is stable under base change, and an invertible object of \( \mathcal{SH}(X^\wedge_{\text{proet}})^{\leq 0}_p \).
3. If \( X \) has all \( p^n \)-th roots of unity for all \( n \), then \( \hat{\Pi}_p(1) \simeq \mathbb{1} \in \mathcal{SH}(X^\wedge_{\text{proet}})^{\leq 0}_p \).
4. \( \hat{\Pi}_p(1) \) lies in the essential image of \( \nu^* : \mathcal{SH}(X_{et})^{\leq 0}_p \rightarrow \mathcal{SH}(X^\wedge_{\text{proet}})^{\leq 0}_p \). The same holds for its \( \otimes \)-inverse.

**Proof.** (1) Attaching \( p \)-th roots of unity is an étale extension away from \( p \) [Neu13, Corollary 10.4]. Consequently \( \mu_p \) is an étale \( X \)-scheme, and \( \hat{Z}_p(1) \) is represented by the same limit, taken in the category of schemes. Finally pullback of schemes is a limit, so commutes with limits, so the scheme \( \hat{Z}_p(1) \) is stable under pullback.

(2) Stability is clear. For invertibility, note first that if \( f : Y \rightarrow X \) is weakly étale, then \( f^* \) has a left adjoint \( f_! \) which satisfies a projection formula. It follows that \( f^* \text{map}(A, B) \simeq \text{map}(f^*A, f^*B) \). This implies that being invertible is pro-étale local on \( X \). Let \( X' \) be obtained by attaching all \( p^n \)-th roots of unity to \( X \), for all \( n \). By construction \( X' \rightarrow X \) is pro-(finite étale), and in fact a covering map [Sta18, Tag 090N]. We may thus replace \( X' \) by \( X \) and so assume that \( X \) has all \( p^n \)-th roots of unity for all \( n \). It thus suffices to show (3).

(3) In this situation \( \mu_{p^n} \simeq \mathbb{Z}/p^n \) and \( \hat{Z}_p(1) \simeq \hat{Z}_p \) (defined to be \( \lim \mathbb{Z}/p^n \) taken in \( \text{Shv}(X^\wedge_{\text{proet}}) \)). Note that \( 1 \in \hat{Z}_p \) defines a map \( S^1 \rightarrow K(\hat{Z}_p, 1) \) which we shall show is a stable \( p \)-equivalence. As a preparatory remark, let \( F \in \text{Ab}(X_{\text{proet}}) \) be any sheaf of abelian groups. Then \( K(F, 1) \in \text{Shv}(X^\wedge_{\text{proet}}) \) is a sheaf of spaces with \( \pi_0(K(F, 1))(Y) = H^0_{\text{proet}}(Y, F) \), \( \pi_1(K(F, 1))(Y) = F(Y) \) and \( \pi_1(K(F, 1))(Y) = 0 \) else, for all \( Y \in X_{\text{proet}} \). In particular, if \( Y \) is \( w \)-contractible, then \( K(F, 1)(Y) \simeq K(F, 1)(Y) \) by Lemma 3.1. Let \( f : F \rightarrow G \in \text{Ab}(X_{\text{proet}}) \) and assume that for each \( w \)-contractible \( Y \), the map \( F(Y) \rightarrow G(Y) \) is a derived \( p \)-completion (i.e., \( \text{Hom}(\mathbb{Z}/p^\infty, F(Y)) = 0 = \text{Hom}(\mathbb{Z}/p^\infty, G(Y)) \) and \( G(Y) \simeq \text{Ext}(\mathbb{Z}/p^\infty, F(Y)) \)). Then \( K(F, 1)(Y) \rightarrow K(G, 1)(Y) \) is a \( p \)-equivalence in the classical sense [BK87, VI.2.2], and consequently \( \Sigma^\infty(K(F, 1)(Y)) \rightarrow \Sigma^\infty(K(G, 1)(Y)) \) is a \( p \)-equivalence of spectra as follows from [BK87, Proposition 5.3(i)]. This implies that \( \Sigma^\infty(f) \in \mathcal{SH}(X^\wedge_{\text{proet}}) \) is a \( p \)-equivalence: it suffices to show that \( \text{cof}(\Sigma^\infty(f)/p) \) has vanishing homotopy sheaves, which follows from our assumption about \( w \)-contractible \( Y \) and the fact that \( \Sigma^\infty = L_{\text{proet}}^{\Sigma_+} : \mathcal{SH}(X^\wedge_{\text{proet}}) \rightarrow \mathcal{SH}(X^\wedge_{\text{proet}}) \), using Lemma 3.7 below. Here \( \Sigma_+ : \mathcal{P}(X^\wedge_{\text{proet}}) \rightarrow \mathcal{SH}(\mathcal{P}(X^\wedge_{\text{proet}})) \) is the presheaf level suspension spectrum functor and \( L_{\text{proet}} : \mathcal{SH}(\mathcal{P}(X^\wedge_{\text{proet}})) \rightarrow \mathcal{SH}(X^\wedge_{\text{proet}}) \) is the pro-étale hypersheafification functor.

To apply this remark to our case, note that \( S^1 \simeq K(\mathbb{Z}_p, 1) \in \mathcal{SH}(X^\wedge_{\text{proet}}) \), where \( \mathbb{Z}_p \in \mathcal{SH}(X^\wedge_{\text{proet}})^{\leq 0} \) denotes the constant sheaf. We have \( H^0_{\text{proet}}(Y, \mathbb{Z}) = M(|Y|, \mathbb{Z}) \), where \( M \) denotes the set of continuous maps between two topological spaces (we view \( \mathbb{Z} \) as discrete); see Lemma 3.8 below. Also \( H^0_{\text{proet}}(Y, \hat{Z}_p) = \lim_n H^0_{\text{proet}}(Y, \mathbb{Z}/p^n) = \lim_n M(|Y|, \mathbb{Z}/p^n) \), by the same Lemma (or [BS13, Lemma 4.2.12]). We thus need to prove that \( M(|Y|, \mathbb{Z}) \rightarrow \lim_n M(|Y|, \mathbb{Z}/p^n) \) is a derived \( p \)-completion of abelian groups. Since the source has no \( p \)-torsion, for this it is enough to show that \( M(|Y|, \mathbb{Z}) \rightarrow M(|Y|, \mathbb{Z}/p^n) \) is surjective for every \( n \) (clearly the kernel is \( p^n M(|Y|, \mathbb{Z}) \)) [BK87, Section VI.2.1, bottom of page 166]. But this is clear: if \( s : \mathbb{Z}/p^n \rightarrow \mathbb{Z} \) is any set-theoretic section of \( \mathbb{Z} \rightarrow \mathbb{Z}/p^n \) then composition with \( s \) induces a section of \( M(|Y|, \mathbb{Z}) \rightarrow M(|Y|, \mathbb{Z}/p^n) \).
(4) It suffices to treat the case $X = \text{Spec}(\mathbb{Z}/p\mathbb{Z})$. By Example 2.14, $X$ has étale-locally uniformly bounded étale cohomological dimension. Hence by Lemma 2.16, Lemma 3.4 and Proposition 3.5, $\nu^* : \mathcal{SH}(X^\alpha_{proet}) \to \mathcal{SH}(X^\alpha_{proet})$ is fully faithful with essential image the spectra with classical homotopy sheaves. Lemma 2.5 now implies that it is sufficient (and necessary) to show that $\hat{i}_p(1)/p$ and its dual $D(\hat{i}_p(1))/p$ have classical homotopy sheaves. By [BS13, Lemma 5.1.4] this is pro-étale local on $X$ (note that taking strong duals commutes with base change), so we may assume that $\hat{i}_p(1)$ is $p$-equivalent to $1$, in which case the claim is clear.

Lemma 3.7. Let $C$ be a site and $F$ a presheaf of pointed sets on $C$. Suppose that for every $X \in C$ there exists a covering $\{Y_i \to X\}_i$ with $F(Y_i) = *$ for all $i$. Then $aF = *$.

Proof. Any pointed map $F \to G$ with $G$ a pointed sheaf must be zero. The result follows from the Yoneda lemma since $a$ is left adjoint to the inclusion of sheaves into presheaves.

Lemma 3.8. Let $S$ be a set. Define presheaves $F_1, F_2$ on the category of schemes, via $F_1(X) = S$ and $F_2(X) = M(|X|, S)$, the set of continuous maps from the underlying topological space of $X$ to $S$ (viewed as a discrete topological space). Then

1. The canonical map $F_1 \to F_2$ is a Zariski equivalence.
2. $F_2$ is a sheaf in the fppf topology.

Proof. The map $F_1 \to F_2$ is clearly injective. We show that it induces a surjection on the sheafification, whence (1). To do so, given $f \in F_2(X)$ we have to find a Zariski cover $\{U_a\}_a$ of $X$ and elements $f_a \in F_1(U_a)$ with $f_a$ mapping to $f|_{U_a}$. The image of $F_1 \to F_2$ consists of the constant functions; hence the cover $X = \bigsqcup_{a \in S} F^{-1}([s])$ works.

Note that $M(|X|, S)$ is the set of locally constant functions from $|X|$ to $S$. This condition is clearly Zariski local, so $F_2$ is a Zariski sheaf. To prove that $F_2$ is an fppf sheaf, it thus suffices to prove that $F_2$ has descent for faithfully flat morphisms $\alpha : X \to Y$ of affine schemes [Sta18, Tag 0301]. The canonical map $[X \times_Y X] \to |X| \times_Y |X|$ is surjective, as is $X \to Y$; this implies that an arbitrary function $f : |X| \to S$ descends to $Y$ if and only if the two pullbacks to $|X \times_Y X|$ agree, and uniquely so; in other words the presheaf of arbitrary (not necessarily continuous) functions into $S$ is a sheaf. Finally for $U \subseteq |Y|$, we have that $U$ is open if and only if $\alpha^{-1}(U)$ is open [Sta18, Tag 0256]; this implies that $f$ is continuous if and only if $f \circ \alpha$ is. This concludes the proof.

Theorem 3.6 applies in particular if $X = \text{Spec}(\mathbb{Z}/p\mathbb{Z})$ in which case $\nu^*$ is fully faithful (by Example 2.14, Lemma 2.16 and Lemma 3.4).

Definition 3.9. We put $\hat{i}_p(1) = \nu_* \hat{i}_p(1) \in \mathcal{SH}(\text{Spec}(\mathbb{Z}/p\mathbb{Z})^\alpha_{et})$. For general schemes $X$ with $1/p \in X$ there is a unique morphism $f : X \to \text{Spec}(\mathbb{Z}/p\mathbb{Z})$ and we define $\hat{i}_p(1) = f^* \hat{i}_p(1) \in \mathcal{SH}(X^\alpha_{et})$.

It follows that $\hat{i}_p(1) \in \mathcal{SH}(X^\alpha_{et})$ is stable under base change, invertible, and $\nu^* \hat{i}_p(1) = \hat{i}_p(1)$. We offer the following further plausibility check.

Lemma 3.10. We have $\hat{i}_p(1) \land \mathbb{H}(\mathbb{Z}/p^n) \simeq \mathbb{H}_p^n \in \mathcal{SH}(X^\alpha_{et})$.

Proof. For $E \in \mathcal{SH}(X^\alpha_{et})$, let $\mathbb{H}(E, \mathbb{Z}/p^n) = \pi_* (E \land \mathbb{H}(\mathbb{Z}/p^n))$. By hypercompleteness, what we have to show is the following: $\mathbb{H}(\hat{i}_p(1), \mathbb{Z}/p^n) = 0$ for $i \neq 0$, and $\mathbb{H}(\hat{i}_p(1), \mathbb{Z}/p^n) \simeq \mathbb{H}_p^n$. The first condition we can check on the stalks, so assume that $X$ has all $p^m$-th roots of unity for all $m$. Then $\hat{i}_p(1) \simeq \hat{i}_p^n$ and so the claim is clear.

To determine $\mathbb{H}(\hat{i}_p(1), \mathbb{Z}/p^n)$, we may work in $\mathcal{SH}(X^\alpha_{proet})$ instead (since $\nu^*$ is t-exact and $\nu^* \mathcal{SH}$ is fully faithful). We can model $K(\hat{Z}_p(1), 1)$ by the bar construction on $\hat{Z}_p(1)$. This implies that the homotopy sheaves of $\hat{i}_p(1)[1] \land \mathbb{H}(\mathbb{Z}/p^n)$ are given by the sheafifications of $U \mapsto H_1(\hat{Z}_p(1)(U), \mathbb{Z}/p^n)$, where on the right hand side we mean ordinary (reduced) group homology. Since $H_1(A, \mathbb{Z}/p^n) = A/p^nA$ for $A$ any abelian group, we find that $\mathbb{H}(K(\hat{Z}_p(1), 1), \mathbb{Z}/p^n) = \mathbb{Z}_p(1)/p^n \simeq \mathbb{Z}/p^n$ (where the last isomorphism can be checked pro-étale locally, whence assuming that there are all roots of unity). This concludes the proof.

Suppose $S$ is a scheme and $1/p \in S$. Let $C_n : (\mathbb{A}^1 \setminus 0)_S \to (\mathbb{A}^1 \setminus 0)_S$ be given by raising the coordinate to the $p^n$-th power. Since $1/p \in S$ this is étale. For each $n$ we have a commutative diagram

\[
\begin{array}{ccc}
(\mathbb{A}^n \setminus 0)_S & \xrightarrow{C_1} & (\mathbb{A}^n \setminus 0)_S \\
C_{n+1} \downarrow & & \downarrow C_n \\
(\mathbb{A}^n \setminus 0)_S & \xrightarrow{C_n} & (\mathbb{A}^n \setminus 0)_S.
\end{array}
\]
Hence we obtain an inverse system \( \{ C_n \}_n \) over \((A^1 \setminus 0)_S\) with limit \( C = C_S := \lim_n C_n \in (A^1 \setminus 0)_{S, prod}.\)

**Proposition 3.11.** The object \( C \) is canonically a \( \mathbb{Z}_p(1) \)-torsor, and hence is classified by an element \( \sigma = \sigma_0 \cdot [*, K(\mathbb{Z}_p(1), 1)]_{SH(\mathbb{A}^1 \setminus 0)_{S, prod}} \). It is stable under base change. Moreover if \( i_1 : S \to (A^1 \setminus 0)_S \) is the inclusion at the point 1, then \( i_1^* (\sigma) = * \).

**Proof.** Since \( C \) is representable, it is stable under base change. Note that \( C_n = S[t, t^{-1}, u/(w^n - t)] \) it follows immediately that \( i_1^* C_n = \mu_{p^n} \) and so \( i_1^* C \) is the trivial torsor. It remains to explain the \( \mathbb{Z}_p(1) \)-torsor structure. We have the multiplication map \( (A^1 \setminus 0) \times (A^1 \setminus 0) \to A^1 \setminus 0 \). Restricting the first factor to \( \mu_{p^n} \) and \( C_n \to C_n \). The structure map \( C_n \to A^1 \setminus 0 \) is equivariant for the trivial action by \( \mu_{p^n} \) on the target. Taking the inverse limit we obtain an action \( \mathbb{Z}_p(1) \times C \to C \), and the structure map \( C \to A^1 \setminus 0 \) is equivariant. To prove that this is a torsor, it remains to show that the shearing map \( \mathbb{Z}_p(1) \times C \to C \) is an isomorphism. Since limits commute, for this it is enough to show that each \( C_n \) is a \( \mu_{p^n} \)-torsor, which is clear. \( \square \)

Upon stabilization and \( p \)-completion, we obtain a map \( \sigma' = (\Sigma^\infty \sigma)_p \) \( \Rightarrow \mathbb{I} \to \mathbb{Z}_p(1)[1] \in SH((A^1 \setminus 0)_{S, prod})^p \), stable under base change. If \( A^1_{S, prod} \) is étale-locally of uniformly bounded étale cohomological dimension, e.g. \( S = \text{Spec} (\mathbb{Z}[1/p]) \), by Lemma 3.4 there is a unique map \( \sigma \) \( \Rightarrow \mathbb{I} \to \mathbb{Z}_p(1)[1] \in SH((A^1 \setminus 0)_{S, prod})^p \) with \( \nu^* (\sigma) = \sigma' \). This map \( \sigma \) is also stable under base change whenever defined.

**Definition 3.12.** Let \( S/\mathbb{Z}[1/p] \) be a base scheme. We denote by 
\[
\sigma : \mathbb{I} \to \mathbb{Z}_p(1)[1] \in SH((A^1 \setminus 0)_{S, ét})^p
\]
the map obtained from the one constructed above by base change to \( S \).

4. **Complements on étale cohomology**

**Lemma 4.1.** Consider a cartesian square
\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
f' \downarrow & & f \downarrow \\
X' & \xrightarrow{g} & X
\end{array}
\]
with \( g \) étale. Then 
\[
g^* f_* \simeq f'_* g'^* : SH(Y'_p) \to SH(X'_p).
\]

**Proof.** Clear by existence of left adjoints to étale pullback, stable under base change. \( \square \)

**Lemma 4.2.** Let \( f : X \to Y \) be a qcqs morphism with \( X, Y \) of uniformly bounded \( p \)-étale cohomological dimension. Then \( f_* : SH(X_{ét}^p) \to SH(Y_{ét}^p) \) preserves colimits and has finite \( p \)-cohomological dimension: there exists \( N \) such that \( f_* (SH(X_{ét}^p)) \subset SH(Y_{ét}^p) \) for all \( i \).

**Proof.** Since \( f \) is qcqs, \( f^* = X \times_Y \bullet \) preserves qcqs schemes. It follows now from Lemma 2.16 that \( f^* : SH(Y_{ét}^p) \to SH(X_{ét}^p) \) preserves compact generators. Consequently \( f_* \) preserves colimits.

Let \( X \) be of uniformly bounded \( p \)-étale cohomological dimension \( \leq N \). Let \( A \in Y_{ét} \) be qcqs and \( E \in SH(X_{ét}^p) \). Then \( map(A, f_* E/p) \simeq map(f^* A, E/p) \in SH_{\geq N} \), by Lemma 2.7(2) (and Lemma 2.16). Since \( A \) was arbitrary, this implies that \( f_* E/p \in SH(Y_{ét}^p)_{\geq N} \). This concludes the proof. \( \square \)

**Corollary 4.3** (homotopy invariance). Let \( X \) be a scheme, \( 1/p \in X \) and suppose that there is an étale cover \( \{ X_\alpha \to X \}_\alpha \) such that for each \( \alpha \), both \( X_\alpha \) and \( A^1 \times X_\alpha \) are of uniformly bounded \( p \)-étale cohomological dimension. Then \( q^* : SH(X_{ét}^p) \to SH(A^1 \times X_{ét}^p) \) is fully faithful.

**Proof.** Let \( E \in SH(X_{ét}^p) \). We wish to prove that \( E \to q_* q^* E \) is a \( p \)-equivalence, or equivalently an equivalence mod \( p \). We may thus assume that \( E \) has \( p^2 \)-torsion homotopy sheaves. By Lemma 4.1 we may assume that \( X \) and \( A^1 \) are of uniformly bounded \( p \)-étale cohomological dimension. If \( E \in SH(X_{ét}^p) \), the result is [GAV72, Corollaire XV.2.2]. The result for all bounded above spectra follows by taking colimits via Lemma 4.2. The general case follows from Lemma 2.9. \( \square \)

**Corollary 4.4** (proper base change). Consider a cartesian square
\[
\begin{array}{ccc}
Y' & \xrightarrow{g'} & Y \\
f' \downarrow & & f \downarrow \\
X' & \xrightarrow{g} & X
\end{array}
\]
with \( f \) proper. Assume that there is an étale cover \( \{ X_\alpha \to X \}_\alpha \) such that each \( X_\alpha, X'_\alpha, Y_\alpha, Y'_\alpha \) is of uniformly bounded \( p \)-étale cohomological dimension. (Here \( Y_\alpha := Y \times_X X_\alpha \), and so on.) Then \( g^*f_* \simeq f'_*(g^*E) \) is a \( p \)-equivalence, i.e., an equivalence mod \( p \). We may thus replace \( E \) by \( E/p \) and assume that \( E \) has \( p^2 \)-torsion homotopy sheaves. By Lemma 4.1 we may assume that \( X, X', Y, Y' \) are of uniformly bounded \( p \)-étale cohomological dimension. If \( E \in \mathcal{SH}(\hat{Z}^n_{\et}) \), the result follows from [GAV72, Theorem XII.5.1]. The functors \( f_*, f'_* \) preserve colimits by Lemma 4.2, so we get the result for all bounded above spectra. Moreover by the same lemma, all our functors are of finitely cohomological dimension. Thus we are done by Lemma 2.9.

We now come to the analog of Corollary 4.3 for \( \sigma \). Thus let \( S \) be a scheme and denote by \( q : (A^1 \setminus \{0\})_S \to S \) the canonical map. Denote by \( i : S \to (A^1 \setminus \{0\})_S \) the inclusion at the point 1. For \( E \in \mathcal{SH}(S^n_{\et}) \) we consider the map \( q_*q^*E \to q_*(i_*i^*q^*E) \simeq E \), where the first map is the unit of adjunction and the equivalence just comes from \( qi = \text{id} \). We denote by \( E^{\text{sm}} \) the fiber of \( q_*q^*E \to E \). Now let \( E \in \mathcal{SH}(S^n_{\et}) \). Consider the morphism \( \sigma_E : \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \to q_*q^*E \) constructed as follows. Since \( \hat{\mathbb{F}}_p(1) \) is invertible, we have \( \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \simeq D(\hat{\mathbb{F}}_p(1)[1]) \wedge E \). Now by adjunction we need to construct a map \( q^*(D(\hat{\mathbb{F}}_p(1)[1]) \wedge E) \simeq D(\hat{\mathbb{F}}_p(1)[1]) \wedge q^*(E) \), or equivalently a map \( q^*E \to \hat{\mathbb{F}}_p(1)[1] \wedge q^*(E) \). We take \( i^*E \sigma_E \), where \( \sigma \in \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \) is the map of Definition 3.12. Note further that the composite \( \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \to q_*i^*q^*E \to q^*E \simeq q_*i_*i^*q^*E \) is trivialized: this follows from the fact that the \( \hat{\mathbb{Z}}(1) \)-torsor \( i^*C \) is trivial (see Proposition 3.11). Consequently \( \hat{\sigma}_E \) factors through \( E^{\text{sm}} \), yielding finally \( \sigma_E : \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \to \mathcal{E}^{\text{sm}} \).

**Proposition 4.5** (\( \sigma \)-locality). Let \( X \) be a scheme, \( 1/p \in X \) and suppose that there is an étale cover \( \{ X_\alpha \to X \}_\alpha \) such that each \( X_\alpha \) and \( (A^1 \setminus \{0\})_X \times_X X_\alpha \) are of uniformly bounded \( p \)-étale cohomological dimension. Then for every \( E \in \mathcal{SH}(X^n_{\et}) \), the map \( \sigma_E : \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \to \mathcal{E}^{\text{sm}} \) is an equivalence.

**Proof.** We are trying to prove that a certain map is a \( p \)-equivalence. Since all functors involved are stable, we may replace \( E \) by \( E/p \); hence we assume that \( E \) is \( p^2 \)-torsion and need to prove that the appropriate map is a plain equivalence. Using Lemma 4.1, we may assume that \( X, A^1_X \) are of uniformly bounded \( p \)-étale cohomological dimension. Recall that tensoring with an invertible object is an equivalence, so preserves colimits; hence \( \text{map}(\hat{\mathbb{F}}_p(1)[1], \bullet) \) preserves colimits. Note also that \( D(\hat{\mathbb{F}}_p(1)) \in \mathcal{SH}(X^n_{\et}) \), since this object is \( p \)-étale locally equivalent to \( \hat{\mathbb{F}}_p(1) \). Using Lemmas 4.2 and 2.16 we may apply Lemma 2.9. Consequently we may assume that \( E \) is bounded above, which using cocontinuity we immediately reduce to \( E \in \mathcal{SH}(X^n_{\et}) \). We may further assume that \( E \) corresponds to a sheaf of \( \mathbb{Z}/p \)-vector spaces.

We have defined \( E^{\text{sm}} \) as a summand of \( q_*q^*E \), where \( q = (A^1 \setminus \{0\})_S \to S \) is the projection. Since \( \mathbb{A}^1 \) is covered by two copies of \( A^1 \) with intersection \( A^1 \setminus 0 \), using homotopy invariance for étale cohomology [GAV72, Corollaire XV.2.2], we find that \( E^{\text{sm}} \) is also a summand of \( r_*r^*E^{[-1]} \), where \( r : \mathbb{A}^1 \to S \) is the projection. From proper base change [GAV72, Theorem XII.5.1] we deduce that \( E^{\text{sm}} \) is stable under arbitrary base change (for \( E \in \mathcal{SH}(X^n_{\et}) \)). Of course \( \text{map}(\hat{\mathbb{F}}_p(1)[1], E) \simeq D(\hat{\mathbb{F}}_p(1)[1]) \wedge E \) is also stable under base change. Using that geometric points serve as stalks for the étale topology, we reduce to the case where \( X = \text{Spec}(k), k \) a separably closed field. In this case \( \mathcal{SH}(X^n_{\et}) \) is just the topos of spaces, and in particular \( E \) corresponds to just a \( \mathbb{Z}/p \)-vector space. Using that étale cohomology commutes with filtered colimits, we reduce to the case \( E = HZ/p \). In this case \( E^{\text{sm}} \simeq H^1(A^1 \setminus 0, \mathbb{Z}/p)[-1] \simeq HZ/p[-1] \) [Gro77, Proposition VII.1.1(ii)]. Since also \( \hat{\mathbb{F}}_p(1) \simeq \hat{\mathbb{Z}} \) we find that similarly \( \text{map}(\hat{\mathbb{F}}_p(1)[1], HZ/p) \simeq HZ/p[-1] \). Thus the map \( \sigma_E \) we are trying to show is an equivalence corresponds simply to a map \( \sigma : \hat{\mathbb{Z}} \to \mathbb{Z}/p \), which is an isomorphism if and only if it is non-zero. In fact \( \hat{\mathbb{Z}}(\mathbb{A}^1 \setminus 0, \mathbb{Z}/p) \) classifies \( \mathbb{Z}/p \)-torsors, \( \sigma \) corresponds to such a torsor, and we need to show this torsor is non-zero. I claim that \( \sigma \) corresponds to \( C_1 \) (from the end of Section 3), which is clearly non-trivial.

It thus remains to prove the claim. It suffices to show that the map \( \sigma_1 \wedge HZ/p : HZ/p \to \hat{\mathbb{F}}_p(1)[1] \wedge HZ/p \simeq H\mu_p \in \mathcal{SH}(A^1 \setminus 0, X^n_{\et}) \) classifies \( C_1 \); here \( \sigma_1 : \hat{\mathbb{F}}_p(1) \to \hat{\mathbb{F}}_p(1) \) is the map from the end of Section 3. This claim is stable under base change, so it suffices to prove this for \( X = \text{Spec}(\mathbb{Z}[1/p]) \), and hence by fully faithfulness of \( \nu^* \) we may prove it in \( X_{proet} \) instead. Recall that in this context the map \( \sigma_1 \) is given by \( \Sigma^\infty \sigma_0 \), where \( \sigma_0 : \Sigma^1 \to K(\hat{\mathbb{Z}}_p(1)) \simeq \Omega^\infty H\hat{\mathbb{Z}}_p(1)[1] \) classifies the torsor \( C \). We obtain the diagram

\[
\begin{array}{cccc}
\hat{\mathbb{F}}_p(1)[1] & \xrightarrow{\sigma_1 \wedge HZ} & \hat{\mathbb{F}}_p(1)[1] & \xrightarrow{\nu^*} & \hat{\mathbb{Z}}_p(1)[1] \\
\end{array}
\]

\[\xrightarrow{\sigma_1 \wedge HZ} \xrightarrow{\nu^*} H\hat{\mathbb{Z}}_p(1)[1],\]

\[\text{Recall that we denote by } D(\cdot) \text{ the passage to strong duals.} \]
where \( \eta \) is the unit map and \( \epsilon \) the co-unit of adjunction. By construction, the composite map is adjoint to \( \sigma_0 \). The map \( \epsilon \) is a \( p \)-equivalence, by Lemma 3.10. The claim follows. \( \square \)

Remark 4.6 (Clausen). Taking \( E = \mathbb{I} \), we learn in particular that \( q_* (\mathbb{I}) \simeq \mathbb{I} \vee C \), where \( C \simeq \mathbb{I}_{p}(\mathbb{I})[1] \). A sufficiently well-developed form of the six functors formalism for étale cohomology with spectral coefficients should allow one to prove ab initio that \( C \) is invertible, circumventing our somewhat awkward construction using the pro-étale topology.

5. The motivic category \( \mathcal{SH}_{\text{et}}(\bullet) \)

Recall that a pre-motivic category is a functor \( \mathcal{C} : \text{Sch}^{op} \to \text{Cat}_{\infty} \), satisfying certain properties. Chiefly among them: each \( \mathcal{C}(X) \) is presentable, for each \( f : X \to Y \) the functor \( f^* : \mathcal{C}(Y) \to \mathcal{C}(X) \) has a right adjoint \( f_* \). If \( f \) is smooth, there is a left adjoint \( f_! \). The smooth base change formula holds. Typically one requires all \( \mathcal{C}(X) \) to be presentably symmetric monoidal and all \( f^* \) to be symmetric monoidal functors. Then the smooth projection formula is required to hold. One then often asks for \( \mathcal{A}1 \)-invariance \((\mathcal{A}1 \simeq * )\) and \( \mathbb{P}1 \)-stability \((\mathbb{P}1 \) is an invertible object in the symmetric monoidal structure). Usually each \( \mathcal{C}(X) \) is also required to be stable; in this situation one may ask that \( \mathcal{C} \) should satisfy localization: any decomposition \( Z, U \subset X \) into an open subset and closed complement should induce a recollement. See [CD09] for a careful statement. If this holds, many further properties follow, and one says that \( \mathcal{C} \) satisfies the full six functors formalism.

The assignment \( S \mapsto \mathcal{SH}(\text{Sm}_{S,\text{et}}^\wedge) \) defines a premotivic, stable presentably symmetric monoidal category not satisfying any of the further assumptions. We let \( \mathcal{SH}_{\text{et}}^S(\bullet) \) be the \( \mathcal{A}1 \)-localization of \( \mathcal{SH}(\text{Sm}_{S,\text{et}}^\wedge) \) and \( \mathcal{SH}_{\text{et}}(S) \) the \( \mathbb{P}1 \)-stabilization. We recall the following fundamental result.

Theorem 5.1 (Ayoub). The premotivic categories \( \mathcal{SH}_{\text{et}}^S(\bullet), \mathcal{SH}_{\text{et}}(\bullet) \) satisfy localization. Hence \( \mathcal{SH}_{\text{et}}(\bullet) \) satisfies the full six functors formalism.

Proof. The localization axiom is verified in [Ayo07, Corollaire 4.5.47]. What is implicit here is that a topology \( \tau \) has been fixed, which is allowed to be the étale topology: see the beginning of Section 4.5 in the reference. Localization together with the remaining standard properties implies the full six functors formalism; see Chapter 1 of the reference. \( \square \)

We also wish to treat the continuity axiom: usually this asks that for certain pro-schemes \( X = \lim_i X_i \) and any \( E \in \mathcal{C}(X_0) \) we have \([\mathbb{I}, E_X]_{\mathcal{C}(X)} \simeq \text{colim}_i [\mathbb{I}, E_{X_i}]_{\mathcal{C}(X_i)} \). We shall say that \( p \)-continuity holds if \([\mathbb{I}, E_X/p]_{\mathcal{C}(X)} \simeq \text{colim}_i [\mathbb{I}, E_{X_i}/p]_{\mathcal{C}(X_i)} \).

We begin with the following abstract result. It is a spectral analog of a considerable weakening of [CD16, Lemma 1.1.12].

Lemma 5.2. Let \( I \) be an essentially small filtering category and \((\mathcal{C}_i)_{i \in I}\) a system of sites with colimit \( \mathcal{C} \). Let \( \mathcal{X}_i = \text{Shv}(\mathcal{C}_i)\wedge, \mathcal{X} = \text{Shv}(\mathcal{C})\wedge \). Suppose given for each \( i \) a generating family \( \mathcal{G}_i \subset \mathcal{X}_i \). Write \( f^*_i : \mathcal{X}_i \to \mathcal{X} \) for the pullback, and \( \mathcal{G} \) for the canonical generating family of \( \mathcal{X} \). Assume the following:

1. For each \( i \in I \), each \( X \in \mathcal{G}_i \) is coherent. Each \( X \in \mathcal{G} \) is coherent.
2. For each \( \alpha : i \to j \in I \), the functor \( \alpha^* \) has a left adjoint \( f_!^\alpha \) preserving coverings.
3. For each \( \alpha : i \to j \in I \), \((i \to j)^* \mathcal{G}_j \subset \mathcal{G}_i \). For each \( i \), \( f^*_i(\mathcal{G}_i) \subset \mathcal{G} \).

Let \( X \in \mathcal{G}, X = \lim_i X_i \) for some family of objects \( \{X_i \in \mathcal{G}_i\} \). Let \( F \in \mathcal{SH}(X_0) \). Assume that one of the following conditions holds:

a. \( F \in \mathcal{SH}(X_0) \leq N \) for some \( N \).

b. \( \mathcal{SH}(X), \mathcal{SH}(\mathcal{X}_i) \) are \( (p-) \)-postnikov complete for all \( i \) and there is \( N \) such that \( X \) and each of the \( X_i \) has \( (p-) \)-cohomological dimension \( \leq N \).

Then

\[ (f_!^\alpha F')(X) \simeq \text{colim}_i F'(i \to 0)\mathcal{G}_i X_i) \).

Here \( F' = F \) in case (a) and \( F' = F/p \) in case (b).

Proof. We will put \( f = f_0 \), etc.

a. Consider \( f^*_{\text{pre}} : \mathcal{SH}(\mathcal{P}(\mathcal{C})) \to \mathcal{SH}(\mathcal{P}(\mathcal{C})) \). It suffices to show that \( f^*_{\text{pre}} \) preserves \( N \)-truncated (whence automatically hypercomplete) sheaves. Since \( F \) is \( N \)-truncated so is \( f^*_{\text{pre}} F \) and we need to show it is a sheaf (automatically hypercomplete). By the coherence assumption, this happens if (and only if) \( f^*_{\text{pre}} F \) (1) takes finite coproducts to products, and (2) satisfies descent for morphisms of the form \( f^*_j(Y_j \to X_j) \), where \( Y_j \to X_j \in \mathcal{C}_j \) is a covering. Since finite limits commute with filtered colimits
The functor $\mathcal{P}$ version also does. The claim about truncations follows by the same argument.

(b) Note that

$$[X, f^*F'] \simeq [X, (f^*F')_{\leq N}] \simeq \colim_i [X_i, F'_{\leq N}] \simeq \colim_i [X_i, F'],$$

using both case (a) and Lemma 2.7(1). This was to be proved. □

In order to apply the above lemma, we need some preparations.

Lemma 5.3. Let $\{f_\alpha : S_\alpha \to S\}_\alpha$ be an étale cover. The functors $\{f_\alpha\}_\alpha$ form a conservative collection for $\mathcal{SH}(\text{Sm}^\wedge_{S, \text{et}}), \mathcal{SH}^{\Sigma^1}_{S, \text{et}}(\bullet)$ and $\mathcal{SH}_{et}(\bullet)$.

Proof. Since Čech nerves of étale covers have been inverted, the functors $\{f_\alpha\}_\alpha$ have jointy dense image. The result follows. □

Let $X \in \text{Sm}_S$. We have a continuous map of sites $e_X : \text{X}_{et}^\wedge \to \text{Sm}_{S, \text{et}}$ inducing $e_X : \mathcal{P}(X_{et}) \subseteq \mathcal{P}(\text{Sm}^\wedge_{S, \text{et}}) : e_X$.

Lemma 5.4. The functor $e_X$, preserves étale-hyperlocal equivalences (that is, those maps of presheaves of spaces or spectra which become equivalences when taking the associated hypercomplete sheaf).

Proof. Since étale-hyperlocal equivalences can be tested on stalks, and any stalk of $X_{et}$ is also a stalk of $\text{Sm}_{S, \text{et}}$, the result follows. □

Corollary 5.5. The functor $e_X, : \text{Shv}(\text{Sm}^\wedge_{S, \text{et}}) \to \text{Shv}(X_{et}^\wedge)$ preserves colimits and truncations. Similarly for $e_X^\wedge : \text{Shv}(\text{Sm}^\wedge_{S, \text{et}}) \to \mathcal{SH}(X_{et}^\wedge)$.

Proof. Colimits of sheaves are computed as colimit of presheaves (i.e. sectionwise) followed by hyper-sheafification. Since the presheaf version of $e_X$ preserves colimits, Lemma 5.4 implies that the sheaf version also does. The claim about truncations follows by the same argument. □

Lemma 5.6. Suppose that every smooth $S$-scheme is étale locally of uniformly bounded $(p)$-étale cohomological dimension. Then $(p)$-postnikov towers converge in $\mathcal{SH}(\text{Sm}^\wedge_{S, \text{et}})$, and all the objects $\Sigma^\infty X_+$ for $X \in \text{Sm}_S$ with $\text{cd}(X) < \infty$ are compact. If $\text{cd}_p X < \infty$ then $\Sigma^\infty X_+/p$ is compact both in $\mathcal{SH}(\text{Sm}^\wedge_{S, \text{et}})$ and in $\mathcal{SH}(\text{Sm}^\wedge_{S, \text{et}})_p$.

Proof. The collection of functors $e_X$ for various $X \in \text{Sm}_S$ is clearly conservative and commutes with limits, and also with truncations by Corollary 5.5. It is hence enough to show that $\mathcal{SH}(X_{et}^\wedge)$ $(p)$-postnikov complete, objects of finite cohomological dimension are compact in $\mathcal{SH}(X_{et}^\wedge)$, and that $e_X$ preserves filtered colimits. The first two statements are proved in Lemma 2.16, and the last one is Corollary 5.5. □

Corollary 5.7. Under the assumptions of Lemma 5.6, $\mathcal{SH}_{et}(S)$ (respectively $\mathcal{SH}_{et}(S)_p$) is compactly generated by $\Sigma^\infty X_+ \wedge G^{\wedge n}_m$ (respectively $\Sigma^\infty X_+ \wedge G^{\wedge n}_m/p$) for $n \in \mathbb{Z}$ and $X \in \text{Sm}_S$ with $\text{cd}(X) < \infty$ (respectively $\text{cd}_p(X) < \infty$). A similar statement holds for $\mathcal{SH}^{\Sigma^1}$.

Proof. Compact generators are preserved under stabilization with respect to a symmetric object [BH21, Proof of Lemma 4.1] and $\mathbb{A}^1$-localization (or more generally any localization at a family of maps between compact objects). The same proof works for $\mathcal{SH}^{\Sigma^1}$. □

Definition 5.8. We say that a scheme $S$ is $(p)$-étale finite if every finite type $S$-scheme is of uniformly bounded $(p)$-étale cohomological dimension. We call $S$ locally $(p)$-étale finite if there exists an étale cover $\{S_\alpha \to S\}_\alpha$ with each $S_\alpha$ $(p)$-étale finite.

Example 5.9. $S$ is $(p)$-étale finite whenever Corollary 2.13 applies. In particular all the schemes from Example 2.14 are (locally) étale finite.

Proposition 5.10. Let $S_0$ be qcqs and $(p)$-étale finite, and $\{S_i\}_i$ a pro-scheme with each $S_i$ étale and affine over $S_0$. Then $\mathcal{SH}(\text{Sm}^\wedge_{S, \text{et}}), \mathcal{SH}^{\Sigma^1}_{S, \text{et}}(\bullet)$ and $\mathcal{SH}_{et}(\bullet)$ satisfy $(p)$-continuity for the pro-system $\{S_i\}_i$. 

Proof. By Lemma 2.15, $S$ is $(p)$-étale finite. We wish to apply Lemma 5.2(b) with $C_i = \text{Sm}_{S_i,\text{ét}}$. $X_0$ some smooth, quasi-separated $S_0$-scheme. We can do this by Lemma 5.6 (which says that the required postnikov towers converge) and the definition of $(p)$-étale finiteness (which ensures that the $X_i$ have bounded $(p)$-étale cohomological dimension). We conclude that if $f : S \to S_0$ is the projection, then for $E \in \mathcal{SH}(\text{Sm}_{S,\text{ét}})$ and $X$ smooth and quasi-separated, we have $(f^*E')(f^*X) \simeq \text{colim}(f^*X)$. (Here $E' = E$ or $E' = E/p$, as appropriate.) In particular, $\mathcal{SH}(\text{Sm}_{S,\text{ét}})$ satisfies $(p)$-continuity for this system (take $X_0 = S_0$).

Since smooth quasi-separated schemes generate our categories, we conclude also that $f^* : \mathcal{SH}(\text{Sm}_{S_0,\text{ét}}) \to \mathcal{SH}(\text{Sm}_{S,\text{ét}})$ preserves $A^1$-local objects. This implies continuity for $\mathcal{SH}_{S_0,\text{ét}}$. Since (filtered) colimits of spectra commute with finite limits, the functor $f^* : \mathcal{SH}_{S,\text{ét}}(S_0) \to \mathcal{SH}_{S,\text{ét}}(S)$ preserves $G_m$-$\Omega$-spectra. This implies continuity for $\mathcal{SH}_{S,\text{ét}}(\bullet)$. □

Corollary 5.11. Let $S$ be locally $(p)$-étale finite. Then the family of functors $i_{S_0}^* : F(S) \to F(S_0)$ is conservative, where $F$ is one of $\mathcal{SH}(\text{Sm}_{S_0,\text{ét}}), \mathcal{SH}_{S_0,\text{ét}}(\bullet)$ or $\mathcal{SH}_{S_0,\text{ét}}(\bullet)$ (respectively their $p$-completions), $\bar{x}$ runs through geometric points of $S$ and $S_0$ denotes the strict henselization.

Proof. Let $\{S_0\}$ be an étale covering by schemes which are qcqs and $(p)$-étale finite. Any geometric point of some $S_0$ is also a geometric point of $S$, hence by Lemma 5.3 we may assume $S$ qcqs and $(p)$-étale finite. Let $\{Y_i\}$ be a pro-system representing a geometric point $\bar{y} \in \text{Sm}_S$ and $E \in F(S)$ with $i_{S_0}^*(E) \simeq 0$ for all $\bar{x}$. Put $E' = E/p$ if we are working at a prime $p$, or $E' = E$ else. We shall show that $(\ast)$ colim$_i E'(Y_i) \simeq 0$. Since this holds for all stalks, we conclude that $0 \simeq \Omega\infty E' \in \mathcal{P}(\text{Sm}_S)$. Since this also applies to all shifts (and twists if $F = \mathcal{SH}_{S,\text{ét}}$) of $E'$, the result follows.

It remains to prove $(\ast)$. Let $Y = \text{lim} Y_i$. Let $\bar{x}$ be the geometric point of $S$ under $\bar{y}$. Then $Y \to S$ factors through $S_x \to S$, hence $E'_x \simeq 0$. Since $Y$ is a henselization, we may assume that $Y_0$ (and in fact each $Y_i$) is qcqs. Since also $Y_0$ (and in fact each $Y_i$) is étale finite, Proposition 5.10 applies: we have $(p)$-continuity for $\{Y_i\}$. Hence $0 \simeq E'_x(1) \simeq \text{colim}_i E'(Y_i)$. The result follows. □

Corollary 5.12. Let $S$ be locally étale finite and of finite dimension. Then the family of functors $i_{S_0}^* : \mathcal{SH}_{S_0,\text{ét}}(S) \to \mathcal{SH}_{S_0,\text{ét}}(\bar{x})$ is conservative, where $\bar{x}$ runs through geometric points of $S$. The same is true for $\mathcal{SH}^S$. If instead $S$ is locally $p$-étale finite of finite dimension, then the same conclusions hold for the $p$-complete categories.

Proof. We prove this by induction on the dimension of $S$. Let $E = E' \in \mathcal{SH}_{S_0,\text{ét}}(S)$ (respectively $E \in \mathcal{SH}_{S,\text{ét}}(S)^\wedge, E' = E/p$), with $E_S \simeq 0$ for all geometric points $\bar{x}$ of $S$. By Lemma 2.15 $S_0$ is $(p)$-étale finite. The result thus holds for $U_{\bar{y}} := S_0 \setminus \bar{y}$, by induction. Hence $E'_x \simeq 0$ (by assumption) and $E'_{S_x} \simeq 0$. By localization (Theorem 5.1), $E'_S \simeq 0$. Since $\bar{y}$ was arbitrary, the result follows from Corollary 5.11. The proof for $\mathcal{SH}^S$ is the same. □

6. MAIN RESULTS

Lemma 6.1. Let $S$ be a scheme. The functor $e^* : \mathcal{SH}(S_0^\wedge) \to \mathcal{SH}(\text{Sm}_{S,\text{ét}}^\wedge)$ is fully faithful. Similarly for $\mathcal{SH}(S_0^\wedge) \to \mathcal{SH}(\text{Sm}_{S,\text{ét}}^\wedge)$

Proof. We need to prove that $e_* e^* \simeq id$. Since stabilization is natural for functors preserving finite limits, the claim for spectra immediately follows from the claim for sheaves. If $X \in \text{Sm}_{S,\text{ét}}$, then $e_* e^* X \simeq X$, since the étale topology is sub-canonical. Since representable sheaves generate $\mathcal{SH}(S_0^\wedge)$ under colimits, it suffices to show that $e_*$ preserves colimits. This is Corollary 5.5. □

Let $1/p \in S$. We construct a map $\sigma : G_m \to \mathcal{I}_{p,0}(1)[1] := e^* (\mathcal{I}_{p,0}(1)[1]) \in \mathcal{SH}(\text{Sm}_{S_0,\text{ét}}^\wedge)$ as follows. Since $G_{m,n}$ is the colimit of $i_0 : \mathcal{O}^0 \to (A^1 \setminus 0)_\mathbb{Z}$, constructing $\sigma$ is the same as constructing a map $1 \to p^* e^* \mathcal{I}_{p,0}(1)[1]$, where $p : (A^1 \setminus 0)_\mathbb{Z} \to S$ is the canonical map, such that the composite $1 \to \Sigma\infty (A^1 \setminus 0)_\mathbb{Z} \to \mathcal{I}_{p,0}(1)[1]$ is trivialized. Since $p^* e^* \simeq e^* p^*$, for this we can use $e^*$ of the map constructed at the end of Section 3.

We denote by $\sigma$ also its image in $\mathcal{SH}^S(S)^\wedge$ and $\mathcal{SH}_{S,\text{ét}}(S)^\wedge$. We denote by $\mathcal{SH}_{S,\text{ét}}(S)^\wedge_{\sigma}$ the monoidal localisation of $\mathcal{SH}^S(S)^\wedge$ at the map $\sigma$; in other words this is the localization at all the maps $id_{\Sigma \infty X} \wedge \sigma$ for $X \in \text{Sm}_S$.

Corollary 6.2. Let $S$ be locally $p$-étale finite, and assume that $1/p \in S$. Then the canonical functor $\mathcal{SH}(S_0^\wedge) \to \mathcal{SH}_{S,\text{ét}}(S)^\wedge_{\sigma}$ is fully faithful.
Proof. By Lemma 6.1, it suffices to prove that if $E \in \mathcal{SH}^s(S^\to_{et})^\beta_p$, then $e^*(E) \in \mathcal{SH}^s(S^\to_{et})^\beta_p$ is $A^1$-local and $\sigma$-local. In other words for $X \in S^\to_{et}$, the maps $e^*(E)(X) \to e^*(E)(\mathbb{A}^1_k)$ and $e^*(E)(X+\mathbb{A}^1_m(1)) \to e^*(E)(X+\mathbb{A}^1_m(1))$ are equivalences. By the projection formula (and since $\mathbb{A}^1_p(1)$ is stable under base change), we may replace $S$ by $X$ and so assume that $S = X$. Note that $e^*(E)(\mathbb{A}^1_k) \simeq \text{map}(1,q^*E) \simeq \text{map}(1,q^*E) \simeq \mathcal{SH}_{et}(\mathbb{A}^1_k)$, which is well-known to induce an equivalence after étale localization.

Let $k$ be a field, say of finite étale cohomological dimension. Recall the étale motive functor $M : \mathcal{SH}_{et}(\mathbb{A}^1_k) \to \mathcal{DM}_{et}(k,\mathbb{Z}/p)$; e.g. in [Bac18b] the functor $\mathcal{SH}(k) \to \mathcal{DM}(k) \to \mathcal{DM}(k,\mathbb{Z}/p)$ is defined; upon localization and $p$-completion this induces $\mathcal{SH}_{et}(\mathbb{A}^1_k) \to \mathcal{DM}_{et}(k,\mathbb{Z}/p)_\sigma \simeq \mathcal{DM}_{et}(k,\mathbb{Z}/p)$ since $\mathcal{DM}_{et}(k,\mathbb{Z}/p)$ is already $p$-complete (all objects being $p$-torsion).

**Lemma 6.3.** The map $M(\sigma) : \mathbb{1}(1)[1] \to \mathbb{A}^\infty_p(1)[1] \in \mathcal{DM}_{et}(k,\mathbb{Z}/p)$ is an equivalence.

Proof. Lemma 3.10 together with [CD16, Theorem 4.5.2] implies that $M(\mathbb{A}^\infty_p(1)[1]) \simeq \mu_p[1]$, and hence $M(\sigma)$ classifies a $\mu_p$-torsor on $\mathbb{A}^1 \setminus \{0\}$ (which is trivial over $1$). As in the proof of Proposition 4.5, tracing through the definitions, we find that this torsor is essentially $C_1$, so in particular non-trivial. Since $\mathbb{1}(1) \simeq \mu_p \in \mathcal{DM}_{et}(k,\mathbb{Z}/p)$ [CD16, Proposition 3.2.2], we find that $M(\sigma)$ is an equivalence (since it is non-zero).

**Lemma 6.4.** Let $k$ be a separably closed field, $1/p \in k$. There is a map $\tau : \mathbb{1}[1] \to G_m \in \mathcal{SH}^s(S^\to_{et})^\beta_p$ such that $M(\tau) : \mathbb{1}[1] \to M(G_m) \simeq \mathbb{1}[1] \in \mathcal{DM}_{et}(k,\mathbb{Z}/p)$ is an equivalence.

Proof. We use the following facts, which hold for any object $E$ in a presentable stable ∞-category

1. There is an inverse system $E \to \cdots \to E/p^3 \to E/p^2 \to E/p \to E/\mathbb{1}.$
2. The induced map $E \to \lim_n E/p^n = E^\to_p$ is a $p$-equivalence.
3. For each $n$ there are distinguished triangles $E/p \to E/p^{n+1} \to E/p^n.$

The Milnor sequence [GJ09, Proposition VI.2.15] then shows that there is for any other object $F$ a surjection $[F, E^\to_p] \to \lim_n [F, E^\to_p].$

We apply this to the category $\mathcal{SH}^s(k).$ Note that there is no étale localization! Then we know that $[\mathbb{1}, G_m]_{\mathcal{SH}^s(k)} = K^M(k)$ and $[\mathbb{1}[-1], G_m]_{\mathcal{SH}^s(k)} = 0$ [Mor12, Corollary 6.43]. Choose a primitive $p$-th root of unity $\zeta \in k^\times$. Since $-1$ is a square in $k$, we have $(-1) = 1 \in GW(k)$ and hence $p[\zeta] = p[\zeta] = [\zeta] = [\zeta] = 1 \in K^M(k)$ [Mor12, Lemma 3.14]. Consequently $[\zeta] \in \ker ([\mathbb{1}, G_m] \to [\mathbb{1}, G_m])$. Let $\tau_1 \in [\mathbb{1}, G_m/p]$ lift $\zeta$. We shall construct a sequence of elements $\{\tau_n \in [\mathbb{1}, G_m/p^n]\}$ with $r_n(\tau_{n+1}) = \tau_n.$ This implies that the $\{\tau_n\}$ define an element of $\lim_n [\mathbb{1}, G_m/p^n]$ and hence lift to an element $\tau \in [\mathbb{1}, G_m/p]$. We construct the $\{\tau_n\}$ inductively. Suppose $\tau_n$ is constructed. Using (3), $\tau_{n+1}$ exists if and only if the image of $\tau_n$ under the boundary map $[\mathbb{1}[1], G_m/p^n] \to [\mathbb{1}[1], G_m/p[1]]$ is zero. It is thus enough to show that $[\mathbb{1}[1], G_m/p[1]] = 0$.

We define the map $\tau$ required for this lemma to be the étale localization of the map $\tau$ we constructed. It remains to show that $M(\tau)$ is an equivalence. Let $M' : \mathcal{SH}^s(k) \to \mathcal{DM}(k,\mathbb{Z}/p)$ denote the “associated Nisnevich motive” functor. Then $M'(\tau) : \mathbb{1}[1] \to \mathcal{DM}(k,\mathbb{Z}/p)$ corresponds to a map $\tau' : \mathbb{1}[1] \to G_m \to HZ/p \in \mathcal{SH}(k)$, where $HZ/p$ denotes the motivic Eilenberg-MacLane spectrum. By construction, $\tau'$ is homotopic to the composite $[\mathbb{1}[1] \to G_m/p \to G_m/p \simeq G_m \simeq HZ/p.$ It follows that $\tau'$ corresponds precisely to the map $\beta : \mathbb{1} \to \mathbb{1}[1] \in \mathcal{DM}(k,\mathbb{Z}/p)$ as constructed for example in [HH05, Section 2], which is well-known to induce an equivalence after étale localization.

**Theorem 6.5.** Let $S$ be a scheme, $1/p \in S$. Then $\sigma : G_m \to \mathbb{A}^\infty_p(1)[1] \in \mathcal{SH}_{et}(S^\to)^\beta_p$ is an equivalence.

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3Points (1) and (3) follow from the same statement in $\mathcal{SH}$, using that $E/p^n = \mathbb{Z}/p^n \wedge E$. The distinguished triangle in (3) can be constructed using the octahedral axiom. To see that $E \to E^\to_p$ is a $p$-equivalence, note that $E^\to_p \simeq \text{map}(1/p^\infty[-1], E)$ where $1/p^\infty = \text{colim}(1/p^n, 1/p^n).$ It is thus enough to show that $1/p^\infty/p \simeq 1/p$, which again follows from the same statement in $\mathcal{SH}$. For a treatment in the case of $\mathcal{SH}$, see e.g. [Bou79, Section 2].
Proof. It suffices to treat the case $S = \text{Spec}(\mathbb{Z}[1/p])$. By Example 5.9, $S$ is now locally étale finite. By Corollary 5.12 (and stability of $\sigma$ under base change), we may thus reduce to $S = \text{Spec}(k)$, where $k$ is a separably closed field with $1/p \in k$. Consider the composite $\sigma' : \mathbb{I}[1] \to \mathbb{I}_p(1)[1] \simeq \mathbb{I}[1] \in \mathcal{SH}^S_{\et}(k)^\vee$. Then $M(\sigma')$ is an equivalence, by Lemmas 6.3 and 6.4. Corollary 6.2 implies that $M : [\mathbb{I}[1], \mathbb{I}[1]]_{\mathcal{SH}^S_{\et}(k)} \to [\mathbb{I}[1], \mathbb{I}[1]]_{\mathcal{DM}(k, \mathbb{Z}/p)}$ is the reduction map $\mathbb{Z}_p \to \mathbb{Z}/p$. It follows that $\sigma'$ is an equivalence. Hence $G_m[-1]$ splits off a copy of $\mathbb{I}$ in $\mathcal{SH}^S_{\et}(k)^\vee$, and hence also in $\mathcal{SH}^S_{\et}(k)^\wedge$. It now follows from [Bac18a, Lemma 30] that $\sigma$ is a $G_m$-stable equivalence (and so is $\tau$). This concludes the proof.

Consequently for any scheme $S$ the canonical functor $\mathcal{SH}^S_{\et}(S)^\wedge_p \to \mathcal{SH}^S_{\et}(S)^\wedge_p$ factors through $\mathcal{SH}^S_{\et,\sigma}(S)^\wedge_p$. Since $G_m$ is invertible in $\mathcal{SH}^S_{\et,\sigma}(S)^\wedge_p$, we find that the induced functor $\mathcal{SH}^S_{\et,\sigma}(S)^\wedge_p \to \mathcal{SH}^S_{\et}(S)^\wedge_p$ is an equivalence: inverting $G_m$ commutes with localizations (like $p$-completion), see e.g. [Bac18a, Lemma 26], and inverting an already invertible object does not do anything, as is clear from the universal property.

Theorem 6.6. Let $S$ be locally $p$-étale finite, and $1/p \in S$. Then the canonical functor $e : \mathcal{SH}(S^\wedge_{\et})^\wedge_p \to \mathcal{SH}(S^\wedge_{\et})^\wedge_p$ is an equivalence.

Proof. We use the argument from [CD16, Theorem 4.5.2]: the functor $e$ is fully faithful and preserves colimits, by the above remarks and Corollary 6.2. It hence identifies $\mathcal{SH}(S^\wedge_{\et})^\wedge_p$ with a localizing subcategory of $\mathcal{SH}^S_{\et}(S)^\wedge_p$. We need to show it is essentially surjective. The category $\mathcal{SH}^S_{\et}(S)$ is generated by objects of the form $q_* (1 \wedge G_m^n) \simeq q_* (1 \wedge G_m^n)$, where $q : T \to S$ is projective [CD09, Proposition 4.2.13]; hence the same holds for $\mathcal{SH}^S_{\et}(S)^\wedge_p$. Since $G_m \simeq \mathbb{Z}_p(1)[1] \in \mathcal{SH}^S_{\et}(S)^\wedge_p$, and $\mathbb{Z}_p(1)$ (and all its positive or negative powers) are in the essential image of $e$, it thus suffices to show that $e$ commutes with $q_*$. This is proved exactly as in [CD16, Proposition 4.4.3]: it boils down to the fact that both sides satisfy proper base change (see Corollary 4.4 and Theorem 5.1).

7. Applications

Theorem 7.1. Let $1/p \in S$ and assume that $S$ is locally étale finite. There is a symmetric monoidal "étale realization" functor $\mathcal{SH}^S_{\et}(S) \to \mathcal{SH}(S^\wedge_{\et})^\wedge_p$.

Proof. Take the composite of the symmetric monoidal localization $\mathcal{SH}^S_{\et}(S) \to \mathcal{SH}^S_{\et}(S)^\wedge_p$ with the symmetric monoidal inverse of the symmetric monoidal equivalence $\mathcal{SH}(S^\wedge_{\et})^\wedge_p \to \mathcal{SH}^S_{\et}(S)^\wedge_p$ from Theorem 6.6.

We can also determine the endomorphisms of the unit in $\mathcal{SH}^S_{\et}(S)$ without $p$-completion, at least away from primes non-invertible in $S$, and under some more restrictive hypotheses on $S$.

Theorem 7.2. Let $X$ be locally étale finite. Let $S$ be the set of primes not invertible on $X$. Consider the functor $e : \mathcal{SH}(X^\wedge_{\et})^\wedge_1[S] \to \mathcal{SH}(X)^\wedge_1[S]$.

(1) If $X$ is regular, noetherian and finite dimensional, then $e$ is fully faithful when restricted to the localizing subcategory generated by $1$.

(2) If $X$ is the spectrum of a field, then $e$ is fully faithful.

Proof. Write $e : \mathcal{SH}(X^\wedge_{\et}) \simeq \mathcal{SH}^S_{\et}(X) : e_*$ for the adjunction. Under our assumptions, $e$ preserves compact generators by Corollary 5.7 and Lemma 2.7. In particular $e_*$ preserves $S$-localizations, rationalizations and so on. We wish to prove that $\eta : \text{id} \Rightarrow e_* e$ is an isomorphism on $\mathcal{SH}(X^\wedge_{\et})^\wedge_1[S]$ (in case (2)) or the localizing subcategory thereof generated by the unit (in case (1)). Then $\mathcal{SH}(X^\wedge_{\et})^\wedge_1[S]$ is fully faithful on an appropriate subcategory. Thus in order to show that $\eta$ is an equivalence it suffices to show that it is a rational equivalence. We thus reduce to showing that $e_0 q : \mathcal{SH}^S_{\et}(X) \to \mathcal{SH}^S_{\et}(X)$ is fully faithful on an appropriate subcategory.

Recall that $\mathcal{SH}(X)^\wedge_1[S]\simeq D_{\et}(X^\wedge_1[S])\simeq D_{\et}(X, Q)^+ \times D_{\et}(X, Q)^-$, for essentially any $X$ [CD09, 16.2.1.6]. For $X$ noetherian, finite dimensional and geometrically unibranch (e.g. regular [Sta18, Tag 0BQ3]), we have $D_{\et}(X, Q)^+ \simeq D(M(X, Q)) \simeq D(M(X))$ [CD09, Theorems 16.1.4 and 16.2.13]. Moreover, all objects in $D(M(X))$ satisfy étale hyperdescent [CD09, 16.1.3] (note that in that reference, “descent” means “hyperdescent” [CD09, Definition 3.2.5]). Recall from Example 2.14 that étale-locally on any scheme, $-1$ is a sum of squares. It thus follows from [CD09, Corollary 16.2.14] that the identity endomorphism of the unit object in $D_{\et}(X, Q)^-$ vanishes étale-locally. This implies (via Lemma 5.3)
that the image of $1_{D_{et}(X,\mathbb{Q})}$ in $SH_{et}(X,\mathbb{Q})$ is zero, and hence the étale hyperlocalization of $D_{et}(X,\mathbb{Q})$ is zero. All in all we have obtained:

$$SH_{et}(X,\mathbb{Q}) \cong DM(X,\mathbb{Q}) \cong DM_{et}(X,\mathbb{Q}) \cong DM_{fl}(X).$$

It remains to prove then that $e: D(X^\text{et}_{1},\mathbb{Q}) \to DM(X,\mathbb{Q})$ is fully faithful on an appropriate subcategory. In case (2), the left hand side is compact-rigidly generated (by comparison with Galois cohomology). This implies that $e$ is fully faithful as soon as $e_*e(1) \cong 1$ [Bac18b, Lemma 22]. Hence in either case we are reduced to proving that $e_*e(1) \cong 1$. The problem is local on $X$, so we may assume that $X$ is affine. We have $[1, 1][1]_{D(X^\text{et}_{1},\mathbb{Q})} \cong H^0_{et}(X,\mathbb{Q})$, which $= 0$ for $i \neq 0$ (see e.g. [Den88, 2.1]), while $H^0_{et}(X,\mathbb{Q}) \cong H^0_{et}(X,\mathbb{Q})$ (e.g. by Lemma 3.8). We need to show that $[1, 1][1]_{DM_{et}(X)} = Gr^B_{i}K_{-i}(X,\mathbb{Q})$ [CD09, Corollary 14.2.14]. This is $= 0$ if $i > 0$ since $X$ is regular, and $= 0$ for $i < 0$ by [Wei13, sentence before Proposition IV.5.10]. For $i = 0$ we precisely get $H^0(X,\mathbb{Q})$ [Wei13, Theorem II.4.10(4)]. This concludes the proof. \qed

**Corollary 7.3.** With the notation and assumptions of Theorem 7.2, we have

$$[1, n][1]_{SH_{et}(X)[1/S]} \cong H^{-n}_{et}(X, [1/S]),$$

where the right hand side denotes étale hypercohomology with coefficients in the (classical) sphere spectrum.

**Proof.** This is just a restatement of the theorem: $H^{-n}_{et}(X, [1/S]) \cong [1, n][1]_{SH(X^\text{et}_{1})[1/S]}$, essentially by definition. \qed

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