Is f(x) unique? Prospective teachers’ conceptual and procedural knowledge on a definite integral problem

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Abstract. This is a descriptive-explorative research which investigates the extent to which prospective teachers who completed calculus courses involve their procedural or conceptual knowledge in relation to the concept of definite integral through a problem-solving task. Data were collected from 30 prospective teachers’ work on a problem-solving task related to Integral. Data were analysed by categorizing the prospective teachers’ work into whether they considered f(x), as an integrand, meet the property of uniqueness or not. Results on initial test point out that more than a half of prospective teachers (n = 17) answered that the integrand of f(x) is unique, which means there is only one f(x) which satisfy the equation, while the remaining 13 prospective teachers answered f(x) is not unique. The reasons of those who answered f(x) is unique were found around procedural errors related to technique of integration and misunderstanding of using fundamental theorem of calculus. More specifically, there was no prospective teachers who related their performance with the concept of definite integral as area-relation and limit of Riemann sums. Results on the test after reflection activity points out that the prospective teachers seemed aware of their mistakes in finding f(x), proven by the increasing number of responses arguing that f(x) is unique as well as increasing number of variety of functions that can be possibly generated.

1. Introduction

According to Hiebert and Lefevre [1], the main concepts of integral theory are that the definite integral of a function is the limit of Riemann sums, the integral-area relation and the fundamental theorem of calculus. However, research has found learners’ limited understanding of underlying concepts related to those three concepts. For example, while learners could perform routine procedures for finding the area under a curve, the learners rarely could explain their procedures [2]. Also, many learners were found to be able to perform procedures and basic techniques of integration, but they failed to understand what behind the procedures [3]. Furthermore, research also found that learners could compute the area under a given curve and could apply the Fundamental Theorem of Calculus to evaluate a definite integral, but they do not know why area under a curve is represented by the definite integral or why the theorem yields the expected result [4]. Along the same lines, Sealey [3] found that some learners in his study understood that the given word problem should be solved using integral concept which utilize the representation of area under curve, but they experienced difficulties on graphing appropriate curve to get the correct representation. In relation to the relationship between symbolic notation and its contextual meaning, Mahir reported that while learners show manipulation of concepts on variable which is limited to algebraic symbol, they do not concern on the possible contextual meaning of such algebraic manipulation [2]. Thus, regarding knowledge of integration, all the preceding integral-related findings indicates learners’ conceptual knowledge, i.e. knowledge of concepts, which are abstract and general
principles [5], is not as good as learners’ procedural knowledge, i.e. knowledge of procedures, i.e. a series of steps, or actions done to accomplish a goal [5].

There is a consensus empirically proven by evidence regarding mathematics learning arguing that conceptual knowledge supports procedural knowledge, as well as vice versa [6]. In this regard, Mahir found that the learners who have conceptual understanding on integral theory, indicated by successfully solving problem related to the concepts of Fundamental theorem of calculus, integral-area relation, and integral of function as the algebraic sum of areas simultaneuously, could also have good procedural performance, indicated by succesfully solving problems related to integral techniques. However, it does not mean that learners always use conceptual approach when doing conceptual task in character, as found by Engelbrecht et al [7]. Rather, they often use procedural approach which lead to the failure of obtaining solution or the longer steps written to get solution.

Findings on the lack of understandings become warrants for not only students studying at school levels and undergraduate level, but also for those who are prepared to be future teachers. This is because many findings also indicate prospective teachers’ limited knowledge regarding undergraduate mathematics topics, such as mathematical proof [8], derivatives [9], concept of limit [10] and of course integral calculus [4-5]. Accordingly, capturing the prospective teachers’ understanding on the integral-related concepts within their studies is required as one of tools of evaluation for the improvement of courses programs related to calculus offered in the curriculum at university level.

The aim of this study is to investigate the extent to which prospective teachers who completed calculus course involve their procedural or conceptual knowledge in relation to the concept of definite integral through a problem-solving task.

2. Procedural and conceptual knowledge on definite integral

Literature has much indicated that learners often evaluate definite integrals by applying Fundamental Theorem of Calculus (FTC). FTC could not only connect the definite and indefinite integral, but also providing an efficient method for evaluating definite integrals [11].

Conceptually, Thompson and Silverman [12] asserts that students should not only conceive definite integral as the area under a curve, but also must interpret it as the quantities being accumulated as being created by accruing incremental bits formed multiplicatively. In the context of area under curve, this describe Riemann sum. In this case, Sealey [4] provides a conceptual knowledge need to be understood by a learner. Through his research related to uncovering students’ understanding of definite integral from Riemann sum, a learner should figure out that such knowledge has some layers, namely the Product layer, summation layer, limit layer, and function layer to thoroughly understand the meaning of \( f(b) = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x \). Radmehr and Drake [13] have conceptualised four types of knowledge regarding integral, two of which are conceptual knowledge and procedural knowledge. Conceptual knowldge consist of knowledge of classifications and categories, knowledge of principles and generalizations, and knowledge of theories, models, and structures. Regarding definite integral, this is related the knowledge about the relationship between definite integral and Riemann sums, integral-area relationship, and FTC. Meanwhile, procedural knowledge consist of knowledge of subject-specific skills and algorithms, knowledge of subject-specific techniques and methods, and knowledge of criteria for determining when to use appropriate procedures. In relation to definite integral, this is related to the knowledge of classifying an integrand according to its form, knowledge of different methods for finding areas (e.g. with respect to \( x \) and \( y \)), knowing that it is easier to find the definite integral using FTC rather than sums when antiderivative can be found in terms of elementary functions.

In this study, the concern is around prospective teachers’ understanding on definite integral related to the first part of FTC: “If \( f \) is continues at every point in \([a, b]\) and \( F \) is any antiderivative of \( f \) on \([a, b]\), then \( \int_{a}^{b} f(x)dx = F(b) - F(a) \)” (Thomas, p.277) [14], the uniqueness of definite integral resulted from the left part of the equation within such theorem, and the extent to which they relate their responses to the concept of integral-area relationship or even the limit of riemann sums.
3. Methods

3.1. Participants and Instruments
Participants of this study is 30 prospective teachers who were studying at bachelor of mathematics education program at Universitas Negeri Surabaya, Surabaya, Indonesia, having completed some basic mathematics courses such as calculus and advanced calculus. Aside the researchers themselves, the main instrument is a problem-solving task involving the concept of calculus integral. The indicator of the problem-solving task is to find out the possible integrand of \( f(x) \) if both the value of definite integral associated with such \( f(x) \) and the interval of \( x \) are given. See Figure 1.

![Figure 1. The problem-solving task of definite integral](image)

The prospective teachers were then provided a week after solving such task to reflect on whether they need to revise their work and the extent to which they carried out any changes. This activity is to clarify the problem-solving processes they undertook when solving the task in the initial test.

3.2. Analysis on prospective teachers’ procedural and conceptual knowledge
Data were analysed by coding each of the responses into an appropriate procedural error and conceptual errors, as well as procedural and conceptual knowledge correctly performed by prospective teachers. Also, the responses were also categorized into whether \( f(x) \) is unique or not. Thus, they were summarised into a table. To describe, we exemplify the responses of student teachers’ proof schemes on those two tasks by presenting examples of each of the levels. To maintain the validity and reliability of the analysis, the coding was first carried out by the second author, then it was checked and clarified by the first author. Furthermore, the authors discussed the possible explanation of each of the codings to be presented by comparing the findings with theoretical review discussing conceptual and procedural knowledge on definite integral, as discussed in the preceding section.

4. Results and discussion

4.1. Variety of functions of \( f(x) \): Conceptual and Procedural Performance

| Performances | Constant function | Linear function | Quadratic function | Unknown |
|--------------|-------------------|-----------------|--------------------|---------|
|               | Initial test | After reflection* | Initial test | After reflection* | Initial test | After reflection* | Initial test | After reflection* |
| Conceptual error | 5      | 2                | 0         | 0              | 0         | 0              | 4         | 1                |
| Procedural error  | 1      | 2                | 0         | 0              | 0         | 0              | 5         | 3                |
| Correct performance | 12     | 21               | 2         | 16             | 1         | 6              | 0         | 0                |

*The number of response in total for each of the results of initial test and after reflection is more than 30 because at after reflection, particularly, some participants give more than one form of functions.

Table 1 presents the changes of responses regarding the forms of function at initial test and the test after reflection distributed into whether they are correct or conceptually/procedurally incorrect. In general, most of the responses at initial test in relation to the type of functions are recorded as constant function and unknown function, which means the prospective teachers only concerned on a simple polynomial function that can represent the \( f(x) \). After reflection, the prospective teachers seemed aware of their mistakes in finding \( f(x) \), proven by the increasing number of correct answer. With regard to prospective
teachers’ performances, it was found that a half of them found correct \( f(x) \) and the remaining found incorrect \( f(x) \) at initial test. Conceptual errors were found around two types: (1) the assumption that \( f(x) \) is only a constant function, and (2) the assumption that \( f(x) \) is the result of definite integral. In detail, the first conceptual error was found from 5 prospective teachers and the second error was found from 4 prospective teachers. Figure 1 shows these two errors.

![Figure 1: Conceptual error in finding \( f(x) \)](image1)

The prospective teacher whose response is shown at figure 2(a) seemed did not consider the various possible of \( f(x) \). Instead, he assumes that \( f(x) \) is a constant function such that it can follow one of theorems in integral, which is if \( k \) constant, then \( \int k \, dx = kx + c \). Meanwhile, figure 2(b) shows a prospective teacher who unable to differentiate between the result of an integral operation and the integrand so that \(-6\) becomes a constant function that can changes become \(-6x\) after an integration process. In conclusion, figure 2(a) and 2(b) shows that the prospective teachers was unable to use appropriate application of FTC conceptually.

Procedural errors, on the other hand, were found around three types: (1) result of integral is assumed as the part of integrand, (2) inappropriate use of distributive law in operating integrand, and (3) mistakes in writing up the integrand: no brackets for more than one terms in the integrand. Figure 2(a), 2(b), and 2(c) respectively point out these three errors.

![Figure 2: Procedural error in finding \( f(x) \)](image2)

Procedurally, response on figure 3(a) indicates an error since the prospective teacher at this figure moved the result on integral in the right side to the left side as many learners do when carrying out an algebraic procedure, then assume it as part of the integrand that can be then integrated with a formula of elementary integral. Figure 3(b) shows that \( F(x) \) is not viewed as a unity of symbol representing an integrated function of \( f(x) \). Instead, it is viewed as \( F \) times \( x \) so that the prospective teacher having this response factorize the form of \( F(5) - F(2) \) becomes \( F(5 - 2) \). Meanwhile, figure 3(c) indicates an
incorrect writing of terms in an integrand, which leads the integral does not make sense symbolically. The response should have been written as \[ \int_2^5 \left(-\frac{2}{5}x - \frac{3}{5}\right) dx \], instead of \[ \int_2^5 \left(-\frac{2}{5}x + \frac{3}{5}\right) dx \].

Correct responses are also performed by the prospective teachers in this study. The responses ranges from \( f(x) \) in the form of constant function to quadratic function. Figure 3(a) and 3(b) respectively shows these two types of responses.

![4(a)](image1)

![4(b)](image2)

**Figure 4.** Correct performance of finding \( f(x) \)

The responses on figure 4(a) and 4(b) show that the linear function and the quadratic function is respectively supposed as \( ax + b \) and \( ax^2 + bx + c \). Then, these are integrated by applying a formula of elementary integral within FTC. In the end, each variable of \( a, b, \) and \( c \) is found by selecting integers which can substitute \( a, b, \) and \( c \) to satisfy the integral result equation.

4.2. The uniqueness of integrand

In the initial test, regardless of whether the \( f(x) \) found by the participants is correct or not, there were 17 prospective teachers assuming that the integrand of \( f(x) \) is unique, which means there is only one \( f(x) \) which satisfies the equation, while the remaining 13 prospective teachers answered \( f(x) \) is not unique (more than one possible forms). Table 1 shows distribution of prospective teachers’ responses on the problem-solving task on initial test and after reflection.

| Uniqueness of integrand | Correct | Incorrect |
|-------------------------|---------|-----------|
| Initial test            | After reflection | Initial test | After reflection |
| Unique                  | 7       | 3         | 10          | 0           |
| Not unique              | 8       | 25        | 5           | 2           |

Table 2 shows that in initial test the number of responses of correct and incorrect responses are equally distributed, in which the responses of integrand as a unique function increases and those as not unique function decreases from initial test to the test after reflection. Thus, in general, the reflection activity made the prospective teachers become more aware of their errors they undertook during the initial test.

The following section describe some examples of findings examplifying each of entries in table 1.
4.2.1. Integrand as a unique function

Findings indicate some reasons for those assuming the integrand as a unique function. The reasons of those who answered \( f(x) \) is unique were found around procedural errors related to technique of integration and misunderstanding of using fundamental theorem of calculus. The latter finding, particularly, is related to the prospective teachers’s perceived assumption that since the result of a definite integral is unique, the integrand is unique as well. In other words, the converse of a statement that “for \( f(x) \) continuous function, if the integral result for the operation of \( \int_a^b f(x)dx \) is \( k \), then \( k \) is unique”, is assumed correct. The other finding is related to the failure of finding \( f(x) \) procedurally which leads prospective teacher conclude that \( f(x) \) is unique, of course without any supporting argument.

4.2.2. Integrand as not unique functions

For those who answered that \( f(x) \) is not unique, none of their answer was indicated to show possible various algebraic form of \( f(x) \) which represent the integrand. Instead, students were found to let \( f(x) \) as constant function, linear function, and quadratic function, instead polynomial function which has degree more than 2 or other types of form of functions. More specifically, there was no prospective teachers who related their performance with the concept of definite integral as area-relation. Thus, it indicate prospective teachers’ lack of conceptual understanding of definite integral. However, once the prospective teachers revised their work, most of them (28), primarily for those who answered \( f(x) \) is unique, changed their answer to the assumption that \( f(x) \) is not unique, and only 2 prospective teachers who kept their answers, i.e. \( f(x) \) is unique. Regarding problem-solving stages, prospective teachers experienced the stages of devising a plan as the most difficult steps to determine \( f(x) \). This is because the prospective teachers found difficulties in determining the possible form of \( f(x) \) which can be included in the integral calculation. Also, at ‘understanding the problem’ stages, prospective teachers were also found to have misunderstanding of the meaning of ‘unique’ for \( f(x) \).

The findings in this study reveals that no prospective teachers who considered the concepts of integral-area relationship or even Riemann sums as the approach in finding \( f(x) \). This is aligned with previous report arguing that learners tend to focus on the application of the first part of FTC, instead of such two approaches in solving a definite integral problem [3–4]. Therefore, the results of this study encourage future empirical research to further study about the impact of conceptual understanding on procedural fluency, particularly about integral concept, and vice versa. Also, the study also may consider further studying the performance of prospective teachers regarding the factual knowledge, procedural knowledge, conceptual knowledge, and metacognitive knowledge as suggested by Radmehr and Drake [8] and how its impact on their students. The latter suggestion is related to the mathematical knowledge for teaching a particular topic that a teacher should hold [15].

5. Conclusion

To highlight, this present study found that the number of prospective teachers who answered that the integrand of \( f(x) \) is unique, which means there is only one \( f(x) \) which satisfy the equation, is more that those who answered that \( f(x) \) is not unique for a given definite integral problem with the result of integral and the interval of the integral are both given. The classification of the preservice teachers’ work into conceptual error or procedural error for some of their responses, in this case, maybe somewhat superficial. This is due to the finding of both procedural and conceptual errors which appears, which lead to the questions of which one of those errors dominantly occurs. Thus, we need to further explore by interviewing the person having the responses to obtain some clarification.

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