Entropy and its relationship to allometry

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Researchers have found that the metabolisms of organisms appear to scale proportionally to a 3/4 power of their mass. Mathematics in this article suggests that the capacity of isotropic energy distribution scales up by a 4/3 power as size, and therefore the degrees of freedom of its circulatory system, increases. Cellular metabolism must scale inversely by a 3/4 power, likely to prevent the 4/3 scaling up of the energy supply overheating the cells. The same 4/3 power scaling of energy distribution may explain cosmological dark energy.

Keywords: allometry — cosmological parameters — dark energy — entropy — The 4/3 Ratio of Degrees of Freedom Theorem — The 4/3 Ratio of Lengths Theorem — inflation

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I. INTRODUCTION AND BACKGROUND

The universe is not only expanding (Hubble 1929), it is accelerating (Riess et al. 1998; Perlmutter et al. 1999). The unknown cause of the acceleration has been called dark energy (Turner 1998). This article proposes that the distance light travels in a radiation reference frame appears to be $4/3$ greater when measured in a space reference frame. The $4/3$ stretching of radial length in a space reference frame relative to a radiation reference frame is due, according to the mathematics developed in this article, to the $4/3 : 1$ ratio of the scaling of radiation energy distribution relative to the scaling of a corresponding space.

The same $4/3 : 1$ ratio of radiation scaling relative to space scaling plays a role in allometry. Allometry is the study of biological scaling. Studies suggest that metabolism scales by a $3/4$ power of the size of an organism’s mass. Metabolism scaling proportionate to a $3/4$ power of an organism’s mass can be explained by the organism’s energy distribution system — its circulatory system — scaling up proportionate to a $4/3$ power of an organism’s mass.

Thus, the mathematics in this article suggests that laws of nature general in scope may explain both dark energy and metabolic scaling as well as other phenomena, such as the fractal dimension of Brownian motion.

This article:

- Begins with an overview of ideas leading to a mathematical model applicable to allometry and dark energy.
- Describes development of the idea that radiated energy distribution involves two reference frames, namely a radiation reference frame and a space reference frame.
- Discusses how the idea of degrees of freedom of the base of a scale factor arises.
- Derives The $4/3$ Ratio of Degrees of Freedom Theorem (The $4/3$ RDFT).
- Shows how The $4/3$ RDFT might account for $3/4$ metabolic scaling.
- Shows how The $4/3$ RDFT implies The $4/3$ Ratio of Lengths Theorem (The $4/3$ RLT).
- Connects The $4/3$ RLT to dark energy.
- Connects ideas related to $3/4$ metabolic scaling to dark energy.
- Gives other possible manifestations of the two $4/3$ theories and their mathematics.

The ideas in this article arose indirectly out of a question about IQs (which itself arose out of earlier questions): does constant improvement in society’s abstractions and ideas explain the rate of increase in average IQs. In September 2005, I found that the average English lexical growth rate from 1657 to 1899 is close to a measured 3.3% per decade rate of increase in average IQs. I wondered whether society’s lexical growth rate could be modelled using a scale factor.

At the end of May 2007 I found that a network’s mean path length as the base of any scale factor determined the number of degrees of freedom log $\mu(n)$ in a network relative to its mean path length. The mean path length as the base of a scale factor formed a network whose mean path length was $4/3$ times greater when measured in a space reference frame than in a radiation reference frame. The $4/3$ stretching of radial length in a space reference frame relative to a radiation reference frame is due, according to the mathematics developed in this article, to the $4/3 : 1$ ratio of the scaling of radiation energy distribution relative to the scaling of a corresponding space.
In March 2008, I adapted mean path length scaling to 3/4 metabolic scaling. Both mean path length scaling and 3/4 metabolic scaling involve degrees of freedom. Expecting that the algebra would output 3/4 scaling, the mathematics instead implied that circulatory system capacity scaled by a 4/3 power at every level of the circulatory system. I found in April 2008 that Stefan’s Law implied that similar 4/3 scaling applied to black body radiation. In September 2012, it appeared that 4/3 scaling described the ratio of radiation energy density to matter energy density in cosmology based on the distance scale factor $a(t)$.

Applying mean path length scaling to a variety of phenomena since 2007 helped test, explain, correct, verify, improve, refine and simplify the mathematical model. Part of this process is exhibited by the sequence of versions of articles I submitted to arXiv.org.

The Overview sketches the mathematical elements that lead to an explanation for 4/3 scaling, which give an explanation of 3/4 metabolic scaling and lead to a mathematical explanation for dark energy. The Overview has the advantage of simplicity and the disadvantage that it omits the development of the ideas leading to its focus on degrees of freedom. The Overview is intended to be a self-contained summary; the part of the article following it elaborates and discusses ideas in it.

II. OVERVIEW

A. Degrees of freedom of scale factors

Consider a length $\ell$ that at each step has $\beta$ choices of path for the next increment of $\ell$. For $k$ added increments, there are $\beta^k$ possible paths and $\log_\beta(\beta^k)\deg = k \deg$ where $\deg$ is the number of degrees of freedom in scale factor $\beta^k$, and $\beta$ is the base of the scale factor $\beta$.

Use $Deg$ as a function that outputs the exponent of the base of a scale factor. $Deg$ can be used to compare different scale factors with a common base. $Deg$ and $Deg^*$ are abbreviations defined below.

Suppose that for length $\ell_k$ we have $\ell_{k+1} = \beta \ell_k = \beta^k \ell_1$. Define $Deg$, $Deg^*$ and $Deg^{**}$ as follows:

$$Deg^*(\ell_k) \equiv Deg(\ell_{k+1}/\ell_k) = Deg(\beta \ell_k/\ell_k) = \log_\beta(\beta) deg = 1 deg. \quad (1)$$

$$Deg^{**}(\ell_k) \equiv Deg(\ell_{k+1}/\ell_1) = Deg(\beta^k \ell_1/\ell_1) = k deg. \quad (2)$$

Thus in this example $\ell_{k+1}$ relative to $\ell_1$ scales with one degree of freedom, and relative to $\ell_1$ scales with $k$ degrees of freedom.

$Deg^{**}(\beta^k)$ measures the capacity of the system to form paths of length $k \ell$. Thus degrees of freedom is a measure of the capacity to form a cumulative length, and is related to the number of available choices.

B. Reference frames for scaling

Consider $n$ particles (for example, gas molecules) all moving within a cubic volume $\Theta$. Divide $\Theta$ into a $n$ equally sized sub-volumes. Then the mean path length $L$ of the moving particles is 3/4 the distance $\ell$ between the centers of the sub-volumes (based on Clausius 1858 and 1860). Clausius used trigonometry to obtain his result. A distance of $3\ell$ with $3$ degrees of freedom would have $4$ degrees of freedom based on $L = (3/4)\ell$. This article uses degrees of freedom in what follows to obtain a general theorem, of which Clausius’s 1858 result is a particular instance.

Treat radiation volume $V$ with characteristic length $L$ and space $\Theta$ with characteristic length $\ell$ as distinct reference frames. $L$ is a measuring stick for the radiation frame of reference. $\ell$ is a measuring stick for the space frame of reference.

Suppose a quantum of energy $\epsilon$ is proportional to $L$ in the radiation reference frame. Suppose the number of characteristic lengths travelled is preserved as between radiation and space. Then one unit of energy results in length $L$ in the radiation reference frame and in length $\ell = (4/3)L$ when measured in the space reference frame. The relationship between $L$ and $\ell$ is the subject of The Ratio of Lengths Theorem, discussed below.

Uniformly moving particles on a stationary surface (Brownian motion) have radiation and space reference frames; for $n$ suspended particles divide the surface into $n$ equally sized areas. The moving particles are in the radiation reference frame; the stationary surface is the space reference frame. For black body radiation, consider motion within a cavity divided into sub-volumes. For an information network, such as a social, lexical, or economic network, information moves over mean path lengths between nodes that are stationary relative to moving information.

C. Of a static volume, $Deg$, $Deg^*$, $Deg^{**}$ and $deg$

In (Euclidean) space, let characteristic length be $\ell_1$, characteristic Area $A_1$ be $(\ell_1)^2$ and characteristic volume $\Theta_1$ be $(\ell_1)^3$.

A motion has one degree of freedom in a length, two in an area, three in a volume. Suppose length $\ell_k$ scales by a scale factor where $\beta$ is the base of the scale factor, so that $\ell_{k+1} = \beta \ell_k = \beta^k \ell_1$ for all $k$. Then

$$Deg(\ell_{k+1}/\ell_k) = Deg(\beta \ell_k/\ell_k) = Deg^*(\ell_k) = \log_\beta(\beta) deg = 1 deg. \quad (3)$$
In space, $\text{Deg}^*(A_k)$ and $\text{Deg}^*(\Theta_k)$ are based on on length $\ell_k$:

$$\text{Deg}^*(A_k) = \text{Deg}((\beta \ell_k)^2/(\ell_k)^2)$$

$$= \text{Deg}(\beta^2)$$

$$= 2 \text{ deg}. \quad (4)$$

$$\text{Deg}^*(\Theta_k)$$

$$= \text{Deg}((\beta \ell_k)^3/(\ell_k)^3)$$

$$= \text{Deg}(\beta^3)$$

$$= 3 \text{ deg}. \quad (5)$$

$\text{Deg}$ relationships for dimensions of space do not change if space stretches homogeneously and isotropically, that is, if lengths scale in all directions by the same scale factor. Let $a$ be the scale factor for square $A_k$ and $\theta$ be the scale factor for cube $\Theta_k$. Then

$$A_{k+1} = aA_k = a^kA_1 \quad (6)$$

$$= (\beta \ell_k)^2 = \beta^2(\ell_k)^2 = \beta^2 A_1 \text{ and}$$

$$\Theta_{k+1} = \theta\Theta_k = \theta^3\Theta_1 \quad (7)$$

$$= (\beta \ell_k)^3 = \beta^3(\ell_k)^3 = \beta^3 \Theta_1. \quad (8)$$

A result below shows that for a radiation volume, $V_k$,

$$\text{Deg}^*(V_k) = \text{Deg}(A_{k+1}/A_k) = \text{Deg}(\ell_{k+1}/\ell_k) = 1 \text{ deg}. \quad (9)$$

Unlike space where for a space volume, $\Theta_k$,

$$\text{Deg}^*(\Theta_k) = (2/3)\text{Deg}^*(A_k). \quad (10)$$

From equation (5), for radiation at an instant,

$$\text{Deg}^*(L_k) = (1/3)\text{Deg}^*(V_k). \quad (11)$$

\section{Degrees of freedom of radiation}

\subsection{Outline of inferences}

Model radiation distribution by considering a single radiation cone. Radiation cone $G_k = V_1 + V_2 + \ldots + V_k$ has radial length $kL$. Each radiation cone volume increment $V_k$ has radial length $L$. The goal of section II D is to find $x$ for $V_k$’s scale factor $v_k$ such that $v_{k+1} = v_k^x V_k$.

Let $A_k$ be the average cross-sectional area of $V_k = A_kL$. Assume radiation cone length grows $L$ per radiative event, and radiative events occur at a constant rate.

In section II D, the chain of inference for radiation is:

1. Constant $L_k = L$ per radiative event implies a radiation property: $\text{Deg}^*(L_k) = 0 \text{ deg}$.  

2. $\text{Deg}^*(L_k) = 0 \text{ deg}$ implies $\text{Deg}^*(V_k) = \text{Deg}^*(A_k)$.  

3. $\text{Deg}^*(V_k) = \text{Deg}^*(A_k)$ and $\text{Deg}^*(L_k) = (1/3)\text{Deg}^*(V_k)$ imply $\text{Deg}(V_{k+1}/V_k) = \text{Deg}(A_{k+1}/A_k) = \text{Deg}(L_{k+1}/L_k) = (4/3) \text{ deg}$.  

2. $\text{On } \text{Deg}^*(L_k) = 0 \text{ deg}$

Adopt a scale factor $s_k$ applied to the $k^{th}$ increment of $L, L_k$, such that $s_kL_k = L_{k+1} = L$. It is necessary to use a subscript for the scale factor $s_k$ since $s_k = (v_k)^{1/3}$ and $v_k$ changes from one radiation cone volume increment to the next.

Use the idea in section II C of degrees of freedom of the base of a scale factor. Then

$$\text{Deg}(L_{k+1}/L_k) = \text{Deg}(s_kL_k/L_k)$$

$$= \text{Deg}(L/L) = \log_{s_k}((s_k)^0) = 0 \text{ deg}. \quad (13)$$

$\text{Deg}^*(L_k) = 0 \text{ deg}$ for all $k$. Volume and area play no role.

$\text{Deg}^*(L_k) = 0 \text{ deg}$ for radiation length because $L_{k+1} = L_k$, unlike $\text{Deg}^*(\ell_k) = 1 \text{ deg}$ for space length, as in equation (2).

Let $A_k$ (scripted $A$) be $G_k$’s average cross-sectional area.

For $G_1$ the radius at radial distance $L_1 = L$ is $r$. The radius of $G_k$ at $kL$ is $kr$ by the similarity of the $G_k$. Since the average cross-sectional area $A_k$ (unscripted $A$) of the radiation cone volume increment $V_k = A_kL$ increases as $k$ increases, there exist scale factors $\alpha_k > 1$ such that $\alpha_kA_k = A_{k+1}$ and $v_k > 1$ such that $v_kV_k = V_{k+1}$.

As $k$ increases, the $V_k$ grow. $L_k = L$ being constant, $L_k$ scales down relative to the $V_k$: the height of a $V_k$ becomes a smaller proportion of the corresponding width. The shape of a radiation cone volume increment, unlike a sphere, is not the same at all scales.

3. $\text{On } \text{Deg}^*(V_k) = \text{Deg}^*(A_k)$

Compare radiation’s $\text{Deg}^*$ for $A_k$ and $V_k$ having regard for $L$’s radiation property $\text{Deg}^*(L_k) = 0$. Suppose a single radiative event occurs. $G_k$ extends its radial length $kL$ by $L$ to
become $G_{k+1}$ with radial length $(k + 1)L$. Then

$$v_k = \frac{V_{k+1}}{V_k} = \frac{A_{k+1}L_{k+1}}{A_kL_k} = \frac{a_kA_L}{A_kL} = a_k; \quad (14)$$

$v_k = a_k$ because a $V_k$’s radial height is a constant $L$. $L$’s radiation property, $\text{Deg}^*(L_k) = 0 \text{ deg}$ equivalently implies:

$$\text{Deg}^*(V_k) = \frac{\text{Deg}^*(A_k)L_k}{\text{Deg}^*(A_k)} = \text{Deg}^*(A_k) + \text{Deg}^*(L_k) = \text{Deg}^*(A_k) + 0 \text{ deg} = \text{Deg}^*(A_k); \quad (15)$$

$$\text{Deg}^*(V_k) = \text{Deg}^*(A_k).$$

Using $L$ and not $s_kL$ in equation (14) focuses on motion, ignoring $s_k$’s instantaneous equality to $(v_k)^{1/3}$. Equation (15) contrasts with stationary space volume $\Theta_k$ which has $\text{Deg}^*(A_k) = (2/3)\text{Deg}^*(\Theta_k)$. Equations (14) and (15) show that the entropy of $V_k$ equals the entropy of $A_k$, since their entropies are the logarithms of their respective scale factors.

4. On radiation’s (4/3) deg

Using section II D 3 find $x$ such that $(v_k)^x V_k = V_{k+1}$. Suppose a single radiation event has occurred. Since we want to find how $V_k$ scales to become $V_{k+1}$ with reference only to $v_k$, use $a_k = v_k$ from section II D 3 (derived from $L$’s radiation property), $s_k = (v_k)^{1/3}$ from section II C, and $V_k = A_kL_k$.

$$V_{k+1} = A_{k+1}L_{k+1} = a_kA_k s_kL_k = a_k s_k (A_kL_k) = v_k(v_k)^{1/3}V_k = (v_k)^{4/3}V_k; \quad (16)$$

in $\text{Deg}$ notation,

$$\text{Deg}(V_{k+1}/V_k) = (4/3) \text{ deg}. \quad (17)$$

If we simplify by considering $V_k$ a cube, for radiation $L_{k+1} = (V_{k+1})^{1/3} = (v^{4/3}V_k)^{1/3}$ implies

$$L_{k+1} = (v^{4/3})^{1/3}/(V_k)^{1/3} = (v^{1/3})^{4/3}L_k = (s_k)^{4/3}L_k. \quad (18)$$

Hence,

$$\text{Deg}(L_{k+1}/L_k) = (4/3) \text{ deg}. \quad (19)$$

Similarly, $\text{Deg}(A_{k+1}/A_k) = (4/3) \text{ deg}$. The 4/3 deg per change in unit area is pressure that causes volumes $V_k$ to grow. It seems to be a vacuum pressure (or energy) or cosmological constant.

Radiation cone volume increments at every scale, per length, area and volume have (4/3) deg: $\text{Deg}^*(V_k) = \text{Deg}^*(A_k) = \text{Deg}^*(L_k) = (4/3) \text{ deg}$.

E. The 4/3 Ratio of Degrees of Freedom Theorem (The 4/3 RDFT)

Having regard to equations (9) and (19),

$$\frac{\text{Deg}(L_{k+1}/L_k)}{\text{Deg}(L_{k+1}/L_k)} = \frac{4/3 \text{ deg}}{4/3 \text{ deg}} = 1, \quad (20)$$

or $\text{Deg}^*(L_k) = (4/3)\text{Deg}^*(L_k)$. Equation (20) represents The 4/3 Ratio of Degrees of Freedom Theorem (The 4/3 RDFT).

The $\text{Deg}^*(L_k) = (4/3) \text{ deg}$ of equation (19) describes how radiation length $L$ scales in radiation’s reference frame from one instantaneous position to the next. Radiation’s 4/3 deg per scaling induces a 4/3 : 1 ratio of scaling for radiation compared to space.

Equation (20) on the other hand compares scaling $L$ in radiation’s reference frame to scaling $\ell$ in space’s reference frame. The ratio in equation (20) can be treated as dimensionless, but it is probably better to retain $\text{deg}$ in both numerator and denominator as a mnemonic. The different scaling of radiation compared to space compels treating radiation and space as different reference frames.

Inversibility corollary: If $\text{Deg}(\Theta/\ell)$ is, for example, 3 deg, then $\text{Deg}((1/\Theta)/(1/\ell)) = 3 \text{ deg}$:

$$\text{Deg}\left(\frac{1/(\beta^3 e^3)}{1/\ell^3}\right) = \text{Deg}(1/\beta^3) = \log_{1/\beta}(1/\beta^3) \text{ deg} = 3 \text{ deg}. \quad (21)$$

Energy density corollary: If for a volume $\Theta$, $\text{Deg}(\Theta/\ell) = 3 \text{ deg}$, then for an amount of energy $E$,

$$\text{Deg}\left(\frac{E/\Theta}{1/\ell}\right) = 3 \text{ deg}. \quad (22)$$

If for a volume $V$, $\text{Deg}(V/L) = 4 \text{ deg}$, then for an amount of energy $E$,

$$\text{Deg}\left(\frac{E/V}{1/L}\right) = 4 \text{ deg}. \quad (23)$$

F. Evidence of The 4/3 RDFT

This section presents evidence that The 4/3 RDFT is a law of nature and therefore can be validly used to derive The 4/3 Ratio of Lengths Theorem used in the proposed mathematical model of dark energy.

Bekenstein (1972) observed a connection between a black hole horizon and entropy. Using Bekenstein’s observation Hawking (1975) inferred that the entropy of the volume of a black hole is proportional to the entropy of 1/4 of its surface area, a result similar to or the same as $\text{Deg}^*(V_k) = \text{Deg}^*(A_k)$ in section II D 3.

Lawler, Schramm and Werner (2001), on assumptions equivalent to those leading to The 4/3 RDFT, found that the fractal dimension of Brownian motion is 4/3.

Metabolism increases by a 3/4 power of an organism’s mass (Kleiber 1932). The cross-sectional area of a pipe at each
level of a circulatory system (compare West, Brown & Enquist 1997) corresponds to the average cross-sectional area of a radiation cone volume increment. The 4/3 RDFT implies that the capacity to supply energy through a circulatory system increases for larger organisms by a 4/3 power. Intracellular Brownian motion (Bressloff & Newby 2013) also has a 4/3 power for moving particles relative to stationary intracellular fluid. The energy for the intracellular Brownian motion is therefore a 4/3 power greater than required by the cell. In response, cellular metabolism scales down by a 3/4 power for larger organisms, so that the organism’s cells do not overheat. For a circulatory system $V_{k+1}$ larger than $V_k$, and for metabolism $n_{k+1}$ slower than $n_k$,

$$\text{Deg}^*(V_k)\text{Deg}^*(n_k) = (4/3)\ deg \times (3/4)\ deg = 1\ deg^2,$$  

that is, at all scales (for all $k$). Hence cell temperature should be relatively invariant at all scales for similar environments. Metabolic scaling involves two applications of The 4/3 RDFT, one in the supply of energy via the circulatory system’s hierarchically scaled physical structure, and the other in the supply of energy within the cell via intracellular Brownian motion.

The derivation of Stefan’s Law involves a step where $4/3$ power for moving particles relative to stationary intracellular fluid is conserved between reference frames; $L$ and $\ell$ are different measuring sticks for the same amount of energy.

Define $\phi$ by

$$\ell = \phi L.$$  

Using $\text{Deg}^*(L_k) = (4/3)\text{Deg}^*(\ell_k)$ based on equation (20),

$$kL = \text{Deg}^*(L_k)L = (4/3)\text{Deg}^*(\ell_k)(\ell/\phi) = (4/3)(1/\phi)\text{Deg}^*(\ell_k)\ell = (4/3)(1/\phi)k\ell = k\ell.$$  

Since $kL = k\ell$ energy units, $4/3 \times 1/\phi$ in equation (30) must equal 1; hence $\phi = 4/3$. For $k$ scalings,

$$\phi(kL) = (k\phi)L = k(4/3)L = k\ell,$$  

Radial distance in space is 4/3 the corresponding radiation distance. The ‘distances’ in terms of energy are the same but the measuring sticks in radiation and space differ.

The fractality of the relationship $\ell_k = (4/3)L_k$ at all scales is shown by

$$\sum_{i=1}^{n} \ell_i = \sum_{i=1}^{n} (4/3)L_i = (4/3) \sum_{i=1}^{n} L_i.$$  

The 4/3 ratio of lengths applies at every scale and globally.

The 4/3 RDFT and The 4/3 RLT are equivalent. Assume The 4/3 RLT. A radiation reference frame characteristic length $L$ per scaling is 3/4 of a space reference frame characteristic length $\ell$. Accordingly, for the same amount of energy – for the same scaling event – a length in a radiation reference frame has $4\ deg$ compared to $3\ deg$ in a corresponding space reference frame (and see section II.B).

If there were only one reference frame, then for the scale factor for distance to change, as cosmology currently assumes, $\phi$ itself would have to be a function of time. But with two reference frames, $\phi$ is unchanged, and it is $k\phi$ in equation (30) that is a function of time. The invariance of $\phi$, which is possible if there are two reference frames, would help solve dark energy.

The existence of the two reference frames is implied by metabolic scaling and Brownian motion. Perception of space as a single reference frame makes detecting the two reference frames difficult.

If there were only one reference frame, it would not be possible to know whether one had measured distance $d_1$ or $(4/3)d_1$ for a Type 1A supernova. Stretching is detectable if $d_1$ that light travels appears stretched by 4/3 in space’s different reference frame.

Distinct radiation and space reference frames simplifies solving the problem of dark energy. A constant scale factor $\phi$ also avoids theoretical and observational problems (such as...
those raised by the fluid equation) that arise if the scale factor changes over time (p. 52, Ryden 2003). A constant scale factor \( \phi \) might provide a common theoretical explanation for both inflation and acceleration of the universe.

H. Evidence of The 4/3 RLT

Clausius in his 1858 paper (p. 140 in Brush) remarks about molecules in a gas:

The mean lengths of path for the two cases (1) where the remaining molecules move with the same velocity as the one watched, and (2) where they are at rest, bear the proportion of \( \frac{1}{4} \) to 1. It would not be difficult to prove the correctness of this relation; it is, however, unnecessary for us to devote our time to it.

He gave a geometrical proof in 1860 (p. 434). His assumptions and result are equivalent to those of The 4/3 RLT.

Since The 4/3 RDT and The 4/3 RLT are equivalent, Clausius’s observation along with those in subsection II C support the validity of the two 4/3 theorems.

A constant scale factor \( \phi = 4/3 \) would also enable calculation of Hubble time, \( 1/H(t) \) where \( R(t)/R(t) = H(t) \). For \( t \) time and \( d \) distance in the radiation reference frame:

\[
\frac{R(t)}{R(t)} = \frac{(4/3)(d/t)}{(4/3)d} = \frac{1}{t} = H(t),
\]

consistent with observation, whereas the usual estimate \( t_0 = 2/3 \times 1/H_0 \) leads to an under-estimate (p. 84, Coles & Lucchin 2002).

I. The 4/3 RLT and space areas and volumes

For space, a characteristic area = \( \ell^2 \) and a characteristic volume = \( \ell^3 \) are calculated based on the space length \( \ell = (4/3)L \). In terms of \( L \)

\[
\ell^2 = [(4/3)L]^2 = (4^2/3^3)L^2 \quad \text{and} \quad (34)
\]

\[
\ell^3 = [(4/3)L]^3 = (4^3/3^3)L^3. \quad (35)
\]

J. Dark energy

Suppose light from a standard candle \( SC \) with intrinsic brightness \( B \) travels from \( SC_1 \) to Earth distance \( d_1 = kL \) in radiation’s reference frame. Suppose \( SC_2 \) with the same intrinsic brightness \( B \) travels in radiation’s reference frame \( 2d_1 \) to Earth. In the space reference frame, The 4/3 RLT predicts that \( SC_2 \) is

\[
\frac{B}{(4/3)d_1} = \frac{(3/4)B}{d_1}, \quad (36)
\]

3/4 as bright, or 25% fainter, relative to \( SC_1 \), than it would be in the radiation reference frame. The 1998 observations of Type Ia supernovae found that the supernovae appeared “about 25% fainter, that is, farther away than expected” (Dark Energy Survey, 2013; p. 259, Cheng 2010), as The 4/3 RLT predicts.

Construct a cube with side \( \ell = (4/3)L \) containing energy \( E \). The ratio of the energy densities using radiation characteristic length \( L \) in the numerator and space characteristic length \( \ell \) in the denominator is:

\[
\frac{E/L^3}{E/\ell^3} = \frac{64}{27} = \frac{0.7033}{0.2967} : 1. \quad (37)
\]

In this view, dark energy density results from a ratio based on energy in a cube, but with the cube measured in different reference frames each having its own characteristic length, such that \( \ell = (4/3)L \). In this view the energy density of dark energy is the ratio on the right side of equation (37).

Cosmology assumes that \( \rho_L + \rho_M = \rho_c \), the critical density. Dividing through by \( \rho_c \), with \( \Omega_X = \rho_X/\rho_c \), \( \Omega_L + \Omega_M + \Omega_M = 1 \). Since the measured value \( 8.4 \times 10^{-2} \) of \( \rho_c \) is very small compared to \( \rho_L \) and \( \rho_M \), \( \Omega_L + \Omega_M \approx 1 \). Then,

\[
\frac{\Omega_L}{\Omega_M} \approx \frac{0.7033}{0.2967} : 1. \quad (38)
\]

The mathematics predicts that the same amount of energy has a different energy density depending on in which reference frame it is measured, that the ratio of \( \Omega_L \) to \( \Omega_M \) should be about 0.7033/0.2967 : 1, or 0.7033 : 0.2967.

The Seven Year Wilkinson Microwave Anisotropy Probe (WMAP) (Jarosik et al. 2010), measured (p. 2) dark energy density

\[
\Omega_L = 0.728^{+0.015}_{-0.016}, \quad (39)
\]

with a 68% confidence limit (also Komatsu et al. 2011).

The Planck satellite in March 2013 (p. 11) measured dark energy density

\[
\Omega_L = 0.686 \pm 0.020, \quad (40)
\]

with a 68% confidence limit.

The average of the WMAP and Planck measurements for \( \Omega_L, 0.707 \), is about one half per cent different than the 0.7033 in equation (37) predicted by The 4/3 RLT. The WMAP and Planck measurements differ more with each other than they do with the value for \( \Omega_L \) predicted by The 4/3 RLT.

It is unlikely that the role of 4/3 in the 4/3 distance ratio of Type 1A supernovae, the (4/3)\(^3\) ratio for \( \Omega_L \), the 4/3 ratio in equation (28), possible solution of Hubble time, the mean path length attributes of gas molecular motion found by Clausius, the 4/3 fractal dimension of Brownian motion, and 3/4 metabolic scaling in consequence of the 4/3 scaling of an organism’s energy supply, is mere coincidence.

So what is dark energy? The foregoing observations suggest that dark energy is the pressure arising from the degrees of freedom of a radiation length, area or volume being 4/3 the degrees of freedom of a corresponding space length, area or volume. Dark energy would then be the degrees of freedom ratio itself of radiation compared to space: 4/3 deg : 1 deg.
III. BACKGROUND OF IDEAS LEADING TO TWO REFERENCE FRAMES

The Overview makes observations about degrees of freedom of space compared to radiation. It distinguishes between $\text{Deg}'(L_k) = 0 \ deg$ for radiation (that describes how a radiation length scales relative to an immediately preceding radiation length) and $\text{Deg}'(L_k) = (1/3)\text{Deg}'(V_k)$ (that describes how at an instant radiation length scales relative to how radiation volume scales). Radiation has $(4/3) \ deg$ change per radiation length $L_k$, radiation area $A_k$ and radiation volume $V_k$, whereas space has $1 \ deg$ change per space length, area and volume.

Identification of assumptions logically necessary for a mathematical model can help isolate the model’s necessary logical conditions. In the development of the ideas leading to the $4/3$ RLT, two related approaches have been used. One approach is to suppose the existence of two systems, one an energy distribution system $S$ for Supply and the other an energy receiving system $R$ (for Receipt). A second approach is to observe that there are two reference frames, one for the distribution of radiated units, and other a passive or stationary system that receives the radiated units. In the reference frame approach, the same amount of energy results in a characteristic length $L$ in the radiation volume reference frame and a characteristic length $\ell$ in the space volume reference frame, with $L = (3/4)\ell$.

This section sketches the development of these two points of view and their related concepts. The reference frame point of view has advantages of simplicity. Each point of view highlights different issues.

A. Increasing average IQs

The role of scaling in the ideas leading to a mathematical model of dark energy began with a seemingly unrelated question: why do average IQs increase?

A history of mathematics shows that its ideas improve. Likewise, studies in historical linguistics show that language becomes more energy efficient over time. Words and phrases are truncated and condensed, short forms and contractions are adopted. Newly coined words stand for new concepts and technologies.

The linguist Otto Jespersen observed (p. 324, Jespersen 1922) that language develops with regard to energetics: speakers of a language seek the most efficient means for transmitting and receiving information. Zipf (1949) explored Jespersen’s observation about language efficiency at some length.

The modern development of software data compression suggests an analogy to language. Just as software engineers improve the compression of information transmission, so too society collectively improves the compression of information transmission by improving the energy efficiency of spoken and written language.

These observations suggest an explanation for increasing average IQs. Since about 1970 average IQs have been observed to increase at about the rate of 3.3% per decade. I hypothesized in 2005 that this was due to language increasing the compression of information represented by words, phrases, and so on, in all the hierarchies of a language’s structure. That would enable human beings as problem solvers to deploy — to wield — more compressed, and therefore more efficient, language tools and concepts in solving problems.

To test this hypothesis required comparing the rate of increasing average IQs to some collective rate involving words. A proxy for word inventiveness is the size of a lexicon. I found in August 2005 that the size of the English lexicon since about 1657 had increased by an average of about 3.3% per decade. Thus there was some basis for supposing that improving average IQs might be attributable to improvements in the efficiency of language.

One way to test the hypothesized connection between increasing average IQs and increasing information compression in language is to find other collective problem solving rates. Over the course of the next 4 years or so, I found various collective problem solving rates close to 3.3% per decade, such as, in terms of its labor cost, the rate of increase in lighting efficiency (p. 33, Cowles Paper, Nordhaus 1997).

Another way is, if possible, to compare 3.3% per decade to other measures of lexical growth. Glottochronology is such a measure.

Glottochronology was devised by Morris Swadesh (Swadesh 1972) to measure the rate at which two daughter languages diverged from a common mother tongue. By studying the historical written usage of words, he could estimate the rate at which daughter languages diverged from their mother tongue. Around 1966 he calculated that the rate of divergence of Indo-European languages from each other was not more than 14% per thousand years. More recent work suggests a divergence rate of about 11.2% per thousand years (Gray & Atkinson 2003), which is equivalent to each language diverging from its ancestral language at the rate of 5.6% per thousand years.

A third approach is to attempt to fashion a formula that measures the rate of lexical growth and test whether it is consistent with observation. Since glottochronology and lexicons both exhibit a rate of change, some formula might connect them.

The 5.6% per thousand years rate of change of each daughter language differs from the 3.3% per decade average rate of English lexical growth. Might the two rates be related by a formula for lexical growth? Is the 5.6% per thousand years divergence rate a kind of ‘fossil rate’ embedded in the 3.3% per decade average rate of English lexical growth? Glottochronology helps date the age of the mother tongue. Might a ‘fossil rate’ be used to date the beginning of language? One can estimate what the rate of growth in the size of a lexicon might be by supposing that language grew from one word to the 616,500 words of the Oxford English Dictionary in 1989. These questions arose beginning in September 2005.
B. Lexical scaling, a logarithmic formula

Naive reasoning suggests that a formula for lexical growth is logarithmic: parents in each generation transmit information to their children. If information was solely acquired in this way, the accumulation of information would be based on the number of generations, \( \log_2(2^k) = k \), where the log is calculated using base 2. The base of the logarithm, and of the scale factor, would be 2. But information is acquired both from contemporary peers and others. What, if the formula is logarithmic, is the appropriate base of the logarithm and of scale factors? The quest for a number for the base of the logarithm began about September 2005.

At the end of May 2007 calculations suggested that the mean path length \( \mu \) of a social network worked as the base of the logarithm. For calculations for real networks, \( \log_\mu(n) \) is multiplied by the network’s clustering coefficient, \( C \) to give a formula \( C \log_\mu(n) \). \( C \) measures the average connectedness of nodes to adjacent nodes (Watts & Strogatz 1998). The appearance of the formula \( C \log_\mu(n) \) resembles the appearance of the log formula for entropy.

C. The Network Rate Theorem (The NRT)

The mean path length of a social network (of actors) was measured in 1998 as 3.65 (Watts and Strogatz 1998). A network’s clustering coefficient, \( C \), is the average proportion of nodes one step away from given nodes that connect to those given nodes. If all adjacent nodes connect, then \( C = 1 \). Since in a social network on average nodes do not connect to all adjacent nodes, \( 0 < C < 1 \).

The resulting formula \( C \log_\mu(n) \equiv \eta \) leads to a rate theorem, here called The Network Rate Theorem,

\[
\eta = C \log_\mu(n) \mu_1, \tag{41}
\]

that calculates the multiplicative effect of networking on the rate of individual problem solving capacity without regard for the additional effect provided by language. From another perspective, it allows one to calculate collective capacity when individual capacity is known, and vice versa. In equation (41), \( \eta \) is the collective rate of problem solving, \( C \) is the network’s clustering coefficient, \( \log_\mu(n) \) is the log of the number \( n \) of nodes in the network base \( \mu \), \( \eta \) is the network’s mean path length, and \( \mu_1 \) is the average problem solving capacity of an individual (a node) in the network.

The problem solving capacity of a human society \( \mu_{(P-S)} \) which uses a language for communication is

\[
\mu_{(P-S)} = C_{soc} \quad \log_\mu(n_{soc}) \mu_1 \times C_{Lex} \quad \log_\mu(n_{Lex}). \tag{42}
\]

In equation (42) the degrees of freedom of social interaction \( C_{soc} \log_\mu(n_{soc}) = \eta_{soc} \) (the Greek letter eta standing for entropy) is multiplied by the degrees of problem solving freedom afforded by words (concepts), \( C_{Lex} \log_\mu(n_{Lex}) = \eta_{Lex} \), giving the total problem solving degrees of freedom available to the society. Both \( \eta_{soc} \) and \( \eta_{Lex} \) can be calculated using data on sizes and measured values of \( \mu \) and \( C \) for populations and for lexicons. Then for \( \mu_{(P-S)} \) and \( \mu_1 \), if one is known the other can be calculated.

Since there are measurements for \( C \) and \( \mu \) for social networks and for the English language, it is possible to calculate the individual rate of problem solving based on a collective problem solving rate of about 3.3% per decade. The calculated rate for \( \mu_1 \) is almost exactly 5.6% per thousand years. With hindsight, this follows from Swadesh’s approach. If as a result of collective problem solving each of two languages grow by 5.6% per thousand years, then their rate of divergence should be twice that, 11.2% per thousand years.

Applying 5.6% per thousand years to the 616,500 words of the OED gives an estimate of the start date for language of about 150,000 years ago. Experts in language have expressed doubt about glottochronology’s reliability in estimating the era for Indo-European, about 8,700 years ago, because spoken language leaves no physical artifacts. So an estimate that goes much further back in time to the start of language seems even more doubtful. On the other hand, a formula that validly estimates the beginning of spoken language suggests it could be a powerful mathematical characterization of network change.

Having regard for reservations in June 2007 about the validity of the formula in equation (41), a convincing physical model leading to the formula would increase its plausibility.

D. A physical model for The NRT

The average amount of information per mean path length \( \mu \) is proportional to \( \mu \), to the mean energy \( E(\mu) \) used to create or transmit that information, and to the mean time \( t(\mu) \) it takes to travel the mean path length. The proportionality of average information, length, energy, and time explains why \( \mu \) works as a scale factor. Scaling \( n \) by \( \mu \) is equivalent to scaling the network’s energy \( E(\mu) \) per time unit by \( E(\mu) \).

A network of size \( n \) with mean path length \( \mu \) can be spanned, for some \( k \), by \( \log_\mu(n) = k \) lengths each \( \mu \) steps long. This implies that \( n = \mu^k \). Since \( \mu \) is the mean path length, the mean connection distance between nodes is \( \mu \) steps. If the rate per \( \mu \), or per \( \mu^k \) for \( k = 1 \) is known, then the rate per single degree of freedom of \( \mu^k \) nodes or units of energy, length, or time is known. It takes \( k \) steps (or degrees of freedom) of length \( \mu \) to span the network, \( k \) units of time where each unit of time spans a single \( \mu \), and \( k \) units of energy where each unit of energy spans a single \( \mu \). In other words, if each generation is \( \mu \) times bigger than the preceding one (starting with a generation of size \( \mu \)), there are \( \log_\mu(\mu^k) = k \) such generations.

E. \( \eta \) and nested clusters

The function \( \log_\mu(n) \) can be thought of as the entropy \( \eta \) of a system, and the function \( C \log_\mu(n) \) can be thought of as a network’s entropy \( \eta \).

Attempting to fashion a model for equation (41) and for \( \eta \) raises various problems, including:
• The counting problem: if each of the \( n \) nodes in a network has \( \mu + \mu^2 + \cdots + \mu^k \) sources of information where \( \mu^k = n \) (that is, \( \mu^k \) sources in generations 1 to \( k \)) there would be more information sources than nodes.

• The \( n-1 \) problem: how can one out of \( n \) nodes receive a multiplicative capacity benefit with \( n \) as the argument in \( \log_p(n) \); that seems to require a node transmitting information to itself.

• A spanning problem: it seems that \( \log_p(n) \times \mu \) is required to span the network, consistent with Jensen’s inequality, but a mean path length \( \mu \) is sufficient to span the network by itself. It cannot be that both \( \log_p(n) \times \mu \) and \( \mu \) are the most efficient ways to span the network.

These problems are resolved if scaling occurs in a nested hierarchy. For example if there is a single cluster of \( \mu^k \) nodes, one scaling of the single cluster by \( \mu \) results in \( \mu \) clusters each with \( \mu^{k-1} \) nodes, and so on. So if there are \( \mu^k \) energy units, that cluster of energy divide into clusters each with \( 1/\mu \) as much energy. If there are \( k \) scalings, then each node can be considered to be simultaneously in \( k \) differently sized clusters, and within all \( k \) generations. The counting problem is resolved because clusters of size \( \mu^k \) reside in a next larger size cluster of size \( \mu^{k+1} \). The \( n-1 \) problem is resolved because a node resides in a cluster; the network benefit the node receives is a result of a node being in \( k \) levels of clusters or having \( k \) degrees of freedom. The spanning problem is resolved because \( \log_p(n) = k \) measures degrees of freedom, not a cumulative distance.

If nested scaling validly models \( \log_p(n) \), there should be physical systems that manifest this nested hierarchical structure. Each of the \( k \) different clusters in which a node can appear is one of the node’s degrees of freedom. Nested scaling of a network is one way to visualize its degrees of freedom.

To better understand the model, it is helpful to identify its key components and therefore to consider a possible set of postulates. In 2008, those initial postulates were:

• A network \( N \) exists. Its nodes require energy, are connectable and can transmit and receive benefits.

• Every node in \( N \) can respond to its environment and will minimize its use of energy for the acquisition of each unit of benefit received from the environment or from another node in \( N \), and will maximize the benefits it receives for each unit of energy it expends.

These two postulates assume that the network isotropically distributes information, which is proportional to energy.

Another reason to look for postulates from which the features of the model can be derived is as an aid in identifying material underlying physical conditions. That can help understand what makes the system work, and permit the observer to discard data irrelevant to the material conditions.

F. Jensen’s inequality

Shannon (1949) observed in his information theory monograph that entropy would be maximal if the probability \( p_i \) were equal for every scaling. The same applies for \( \log(n) \); it is maximal for a network if the base of the logarithm is an average, in the case of a network, the mean path length. If every node has the same scale factor, then the entropy is maximal; energy distribution is isotropic. Hence, Shannon’s observation is related to Jensen’s inequalities about convex and concave curves (Jensen 1905). Jaynes (1957) called Shannon’s observation the maximum entropy principle.

The maximum entropy principle is related to the evolution of organisms: an organism that is more efficient through its maximization of entropy has an energy efficiency advantage compared an organism that does not maximize entropy.

G. The Network Rate Theorem (The NRT)

For \( d(\Theta_k) \) representing a cumulation of \( \ell_i \) lengths,

\[
d(\Theta_k) = \ell_1 + \beta \ell_1 + \ldots + \beta^{k-1} \ell_1 = k \ell_1
\]

\[
= \log_p(\beta^k \ell_1) = \text{Deg}^* (\Theta_k) \ell_1. \tag{43}
\]

This is also discussed in connection with equation \([104]\) below.

The most efficient spanning of a system \( \Theta_k \) in units equal to the mean path length (or its proportional mean amount of energy or time) is expressed by equation \([43]\). If we use a unit \( X \) larger than the mean path length \( \ell = \mu \), then \( kX \) will be longer than \( k\ell \). If we use a unit shorter than \( \mu \), the number of scalings, relative to that unit, will increase. Jensen’s inequality (Jensen 1905) tells us that the most efficient way to span the distance with given units \( d(\Theta_k) \) is \( k \times \mu \), that is, to traverse the networks in units equal to the mean (or average) path length. We might also guess from these comments that there is something special about the number of scalings. In The Overview above, and later in this article, observations about dark energy suggest that the number of scalings is preserved between reference frames.

For a continuous convex function \( \phi \) (a different use of The Greek letter \( \phi \) than in \( \ell = \phi(L) \), Jensen’s inequality \([58]\) is

\[
\phi \left( \frac{\sum p_i x_i}{\sum p_i} \right) \leq \frac{\sum p_i \phi(x_i)}{\sum p_i}, \tag{44}
\]

where \( \sum p = 1 \). If the \( p_i \) are equal, \( \phi \)'s value on the left side of equation \([44]\) equals \( \phi \)'s value on the right side and so is maximal. If the function \( \phi = \log \), then the formula in Jensen’s inequality is the same as that for entropy. Stipulating that the \( p_i \) are equal is equivalent to requiring homogeneous scaling (that is, that \( \Theta_k \) be scaled by its mean path length) and to maximizing entropy.

It is this aspect of Jensen’s inequality that leads to the apparently paradoxical, but resolvable, observation mentioned above in section \([111]\) in connection with traversing a network. Jensen’s inequality and equation \([43]\) imply that the most efficient way to traverse \( d(\Theta_k) \) is by \( k \) lengths (or iterations) of \( \mu \).
But $\mu$ as the mean path length traverses (on average) the entire length of $d(\Theta_k)$ and is much shorter than $d(\Theta_k)$ in equation (4). These can’t be both true. The resolution is to think of $k\mu$ not as $k$ lengths, but as $k$ degrees of freedom of $\mu$. Each node has $\text{Deg}^{\infty}(\Theta_k)$ degrees of freedom.

In the early stages of The NRT the situation was characterized not by using degrees of freedom but by uniformly nested and scaled clusters; uniform scaling meant each node was scaled by the same scale factor. $\text{Deg}$ was characterized as entropy. Degrees of freedom is in some ways a better characterization; among other things, it leads to The 4/3 RDFT.

If real world phenomena can be modelled by degrees of freedom manifested as nested scaling, one would expect there to be many instances of hierarchical nested uniform scaling. Hierarchical nested scaling occurs in a wide variety of phenomena, as has long been recognized (for example (Simon 1962). Degrees of freedom, or nestedness, may apply to Everett’s 1957 many worlds hypothesis (Everett 1957) and to the problems raised by the ergodic hypothesis (p. 442, Ma 2000).

Now for an example applying $C \log_\mu(n)$.

Suppose a society has $n$ problem solvers with an average individual problem solving rate of $r_1$ per brain. Then the collective problem solving capacity of the society’s brains is $r_n = C \log_\mu(n)r_1$. $C \log_\mu(n)$ gives the multiplier effect of networking for problem solving brain capacity. The mean path length and clustering coefficient have been measured for various networks of known size so the effect of networking can be calculated by inserting $C$, $\mu$ and $n$ into the formula in equation (4). If the collective rate $r_n$ is known, then the individual rate $r_1$ can be calculated.

The problem solving capacity of an individual brain is increased by language. Neurons or synapses network in an individual brain. Individual brains can networked via language and other signs. The significance of language appears to be that it multiplies a society’s and an individual’s problem solving degrees of freedom. If a society has a collective problem solving capacity or rate of $r_\infty$ without language, then we multiply $r_\infty$ by the degrees of freedom afforded by ideas or, as representative of ideas, by the degrees of freedom of the lexicon. In this view, the enormous enhancement of human problem solving capacity is achieved mostly as a result of collective effort; language collectively invented by societies multiply problem solving degrees of freedom attributable to neurons. Written language enables a larger portion of previously solved problems to be preserved for posterity, and as a way of increasing posterity’s problem solving capacities.

One of the most striking results of using The NRT is the connection between glottochronology and the average rate of increase in IQs, mentioned above. This connection encouraged persistence in efforts to validate the formula $C \log_\mu(n)$ from June of 2007 to August of 2009, when an explanation of the connection came into view.

Not only does nature manage a 4/3 stretching of space using an algebra of degrees of freedom, using the same principles it creates the basis for a way to mathematically model emergence.

In this way, The NRT can be derived based on the ideas of the algebra of degrees of freedom connected to The 4/3 RDFT.

II. A natural logarithm theorem

The role of the mean path length $\mu$ in a network as a scale factor for a cluster of size $\mu^k$ implies that a cluster of size $\mu$ to a power changes at a rate proportional to $\mu$. In a network the cumulative length $k\mu$ is determined by $\log_\mu(\mu^k) = k$; $k\mu$ spans the network’s $k$ generations.

That suggests that for a network, $\mu$ is proportional to the natural logarithm. Since the mean path length $\mu$ was the base of the logarithmic formula that multiplied the capacity of an individual network node, that suggests $\mu$ is a scale factor for a cluster of size $\mu^k$ for $k$, including $k = 1$. This raises a new problem: is there a way to show the role of the natural logarithm in a network without assuming the mean path length as the base of a logarithmic formula representing the benefit of networking?

This natural logarithm theorem, that for mean path $\mu$, $d\mu/dt = \mu$, supports the validity of the ideas relating scaling, entropy, scale factors and capacity; it is an unanticipated and plausible inference based on the mathematics. The emergence of a natural logarithm theorem suggests the mathematics is on the right path. Trying to determine how scaling relates to increasing average IQs leads to a theorem of the natural logarithm.

I. Allometry and 3/4 metabolic scaling

1. Metabolic scaling as a test of the lexical scaling model

In September 2005 I began to look for the base of a logarithm that would work in a logarithmic formula describing the multiplicative effect of networking on lexical growth. In June 2007, a formula, $C \log_\mu(n)$, where the base $\mu$ of the log was the network’s mean path length, seemed to work. By the end of February 2008, a nested hierarchy of scaled energy clusters seemed to provide a physical model of the $\log_\mu(n)$ part of the formula (Shour 2008, lexical growth). There was good agreement between the model and the rate of lexical growth. The rate of lexical growth appeared to have some special correspondence to half the rate of lexical divergence of daughter languages under the suppositions of glottochronology. The lexical growth rate corresponded to the rate of increase in average IQs and also seemed to work in other settings.

The mathematical model for $C \log_\mu(n)$ seems to assist in answering otherwise difficult problems. That success is reason to be skeptical of its validity or its novelty, or both; the formula for the entropy of a network was not mentioned in the literature and one would suppose that it was unlikely that it had been overlooked. In light of that unlikelihood, I looked for other scaling phenomena to test the validity of the formula.

If the mathematical model of lexical scaling could explain other phenomena, that would increase confidence in the mathematical model. Moreover, the role of the mean path length as a scale factor was still unclear then (and for quite some time thereafter). The observation of mean path length scaling in connection with other phenomena might assist in understanding mean path length scaling. With that in mind, seeking other
scaling phenomena to which mean path length scaling ideas might apply seemed to be an appropriate next step.

The likely connection between thermodynamics and energy scaling applicable to lexical growth suggested that biological scaling might supply ways to test the mathematical model. In that case, a good place to start learning about biological scaling is Whitfield (2006). Whitfield recounts scientific investigation of the relationship between thermodynamics and biology. In particular, he discusses allometry and metabolic scaling.

2. Metabolic scaling and the surface law

The word allometry was coined by Huxley & Teissier (1936). Allometry is the study of quantitative relationships of organisms (Gayon 2000).

Historically, the surface law for metabolism was a proposed solution for $b$ in the formula

$$Y = aM^b$$

where $Y$ represents the whole organism’s metabolism, $a$ is a constant for a species, and $M$ is the organism’s mass (Whitfield 2006). The surface law $b = 2/3$ was probably first formulated in 1839 by Robiquet and Tillaye: $b = 2/3$ was proposed because an organism’s surface area dispersing body heat grows by a power of 2, while its mass that supplies the heat grows by a power of 3.

Max Rubner (1883) published statistics consistent with $b = 2/3$ (and see Rubner 1902, translated in 1982 into English). Max Kleiber (1932) analyzed the data and arguments in favour of the surface law, and concluded that a weight-power law should supplant the surface law, and that researcher’s measurements favoured $b = 3/4$.

3. WBE 1997

West, Brown and Enquist gave a geometrical explanation (1997) (hereafter WBE, sometimes in reference to the article and sometimes in reference to its authors) for $3/4$ metabolic scaling. They choose an organism’s circulatory system as a demonstration of their mathematical model of transport of materials to all parts of an organism through a linear branched network. WBE notes that, at that time in relation to equation (45) and other relationships with 4 in the exponent’s denominator, ‘No general theory explains the origin of these laws.’ Some (White 2003, 2005) still claim metabolic scaling is 2/3. The debate on this and other issues (Etienne 2006; Price 2007) continues.

A synopsis of WBE’s 1997 derivation of $3/4$ metabolic scaling is relevant here for these reasons: (1) To set the stage for showing how WBE’s scaling ideas were modified by scaling ideas used to model lexical growth (Shour 2008, on lexical growth); (2) To distinguish between WBE’s derivation and this article’s derivation of $3/4$ metabolic scaling; (3) To show why the $4/3$ energy scaling related to metabolism supports characterizing The 4/3 RDFT as a general law which provides a possible mathematical model of dark energy.

WBE makes three assumptions to derive the power $b$ in equation (45):

1. Fractal-like branching supplies the organism.
2. Capillaries for differently sized animals are size invariant.
3. Energy used for blood distribution is minimized.

WBE’s second assumption is necessary for WBE’s mathematical derivation of $3/4$ scaling, but it is not necessary for a derivation using the degrees of freedom ($\text{Deg}$) of scale factors, as discussed later in this article.

WBE’s third assumption is equivalent to Jaynes’s maximum entropy principle in this way: for a given amount of energy, maximize the output per unit energy. This is in turn related to Jensen’s inequality: entropy is maximized when the probabilities for different possible paths are equal, that is, when there is uniform scaling. If fractal-like branching is based on uniform scaling (in contrast to non-uniform scaling), the organism’s energy distribution system — its circulatory system — will have maximum entropy or degrees of freedom. For a given organism, the third assumption therefore implies the circulatory system consists of scaled nested tube clusters or, put another way, a uniformly scaled hierarchical system.

WBE’s second assumption implies capillary volume $V_c$, length $l_c$, and radius $r_c$ are independent of body size.

In WBE terminology and reasoning, there are $N$ branches from the aorta’s volume $V_0$ at level 0 to the capillaries. Capillaries each have volume $V_c$. $B$ is the metabolism of the whole organism, and $B \propto M^a$, for mass $M$ and some $a$, or $B = B_0M^a$ for some appropriate proportionality factor $B_0$.

The blood flowing through each $k^{th}$ level of the circulatory system has the same volume. WBE reasons that an organism’s blood volume and mass is proportional to the number of capillaries, so that the total number of capillaries

$$N_c \propto M^a$$

for $a < 1$. The validity of this inference may be doubted since it leads to an inconsistency noted by Kozlowski and Konarzewski (Kozlowski 2004). If instead the number of capillaries $N_c \propto M$ then the scaling down of metabolism might occur some way other than a scaled down number of capillaries. Equation (Eq-Nc-eq-M-power-a) figures prominently in the WBE derivation of $3/4$ scaling, as described below.

In the terminology of The Overview earlier in this article, the aorta (or perhaps better, the heart) would be considered $V_1$ instead of $V_0$ as in WBE, since level 0 describes the external supply of energy, namely food; the subdivision of energy as it moves away from the source is consistent with a radiation model of energy distribution.

For a circulatory system, $N_k$ is level $k$’s number of tubes, $r_k$ is level $k$’s tube radius and $l_k$ is level $k$’s tube length. $N_0 = 1$ in WBE’s terminology, where $N_0$ represents the value associated with the aorta. Each tube at the $k^{th}$ level branches (is scaled
WBE’s scale factors are as follows:

\[
\frac{N_{k+1}}{N_k} = n_k, \quad (47)
\]
\[
\frac{r_{k+1}}{r_k} = \beta_k, \quad (48)
\]
\[
\frac{l_{k+1}}{l_k} = \gamma_k. \quad (49)
\]

At a \(k\)th branching into smaller tubes, tube radius decreases by scale factor \(\beta_k\), tube length decreases by scale factor \(\gamma_k\), and the number of tubes increases by scale factor \(n_k\).

Compare radiation cone volume increments to circulatory system tubes. Radiation cone volume increments increase in size for a given amount of energy as the radiated energy moves away from the source. For radiation cone volume increments it is not the number of tubes that increases but the volume. This difference does not affect the algebra relating to the scaling of scale factors.

WBE states that it can be shown that for all \(k\) the scale factors are constant:

\[
\frac{N_{k+1}}{N_k} = n, \quad (50)
\]
\[
\frac{r_{k+1}}{r_k} = \beta, \quad (51)
\]
\[
\frac{l_{k+1}}{l_k} = \gamma. \quad (52)
\]

WBE reasoning is roughly as follows.

\(n_k = n\) because the circulatory system is assumed to be a self-similar fractal. This is an approximation and an idealization. The optimality of uniform scaling can be shown by Jensen’s inequality which is discussed in section III. \(\gamma_k = \gamma\) because for the fluid in the circulatory system to reach all parts of the organism, the fluid must be distributed by space-filling fractal structures. The ‘service volume’ reached by a capillary is assumed to be a sphere with radius equal to \(l_c/2\). WBE assumes that this observation about service volumes applies at all \(k\) levels, an assumption which follows from their assumption of fractality.

For \(N\) large and with \(r_k << l_c\), the sum of all spherical service volumes reaches all parts of the organism.

On this reasoning, the total service volume equals the organism’s total volume, which is proportional to the organism’s total mass. Since WBE assumes that capillaries for all animals are size invariant, the number of capillaries would be proportional to the service volume, \(N_c \propto M\), inconsistent with equation (46). The derivation of \(4/3\) scaling in this article assumes that \(N_c \propto M\), which is to say, the circulatory system volume grows in proportion to \(M\).

For WBE’s analysis, one must overlook overlapping spheres or gaps between spheres. The sum of the service volume spheres at the \(k\)th level is then:

\[
(4/3)\pi (l_c/2)^3 N_k, \quad (53)
\]

where \(N_k\) is the number of spheres at the \(k\)th level. Then

\[
(\gamma_k)^3 \equiv \left(\frac{l_{k+1}}{l_k}\right)^3 \approx \frac{N_k}{N_{k+1}} = \frac{1}{n}, \quad (54)
\]

showing that

\[
\gamma_k = n^{-1/3}. \quad (55)
\]

(More precisely, \(\gamma_k \approx n^{-1/3}\).) In this way, if we accept that \(N_k/N_{k+1}\) scales in a uniform way, we find that the scale factor \(\gamma\) does not depend on \(k\).

Similar reasoning applies to obtain \(\beta_k \equiv \beta\). Since the blood flowing through the circulatory system is incompressible, it must be that the cross-sectional areas \(A_k = \pi r_k^2\) of every \(k\)th level is the same. In particular, the \(k+1\) tubes that are smaller than a \(k\) level tube of which they are branches have a radius such that

\[
r_k = nr_{k+1}. \quad (56)
\]

Dividing both sides of equation (56) by \(r_k\) gives

\[
\beta_k \equiv \frac{r_{k+1}}{r_k} = n^{-1/2}, \quad (57)
\]

and since \(n\) is not dependent on \(k\) neither is \(\beta\).

To show that \(n, \gamma, \beta\) are constants in an energy distribution system, we could instead follow the ideas in the Overview above, and compare the exponents of the scale factors via Degree.

WBE notes that \(N_c = n^N\); on their assumptions, the number of capillaries \(N_c\) along one branch from the aorta scales by \(n\) at each level \(k\) for \(N\) levels. This calculation shows why the aorta is assigned level 0 by WBE; the \(N\) branches occur after the \(0\)th level. This implies that

\[
\ln(N_c) = \ln(n^N) = N \ln(n). \quad (58)
\]

Using WBE’s point at equation (46) that \(N_c \propto M^\alpha\),

\[
\ln(N_c) = \ln(M^\alpha/M_0) = \alpha \ln(M/M_0) \quad (59)
\]

where \(M_0\) is used to permit the formation of an equality based on the proportionality \(N_c \propto M^\alpha\).

Equating \(\ln(N_c)\) in equations (58) and (59) gives the equality

\[
N \ln(n) = \alpha \ln(M/M_0), \quad (60)
\]

that is,

\[
N = \frac{\alpha \ln(M/M_0)}{\ln(n)}, \quad (61)
\]

which is the result in equation (3) of WBE’s paper.
Using the scale factors defined in WBE,

$$\frac{\pi(r_k)^2l_kn^k}{\pi(r_{k+1})^2l_{k+1}n^{k+1}} = \frac{\pi(r_k)^2l_kn^k}{\pi(b_{r_k})^2(y_{l_k})m} = \frac{1}{n^{\gamma \beta}}.$$  \hspace{1cm} (62)

The terms on the left side of the first line of equation (62) are shown on the right side with their scale factors. The right side is then simplified to yield the second line in equation (62). The factor $(n^{\gamma \beta})^{-1}$ is used as the $r$ term appearing in a geometric series $S_n = 1 + r + r^2 + \ldots + s^k$ to arrive at the second line in equation (63), also using the fact that each $k^{th}$ level has volume $n^kV_c$.

WBE calculates the total blood volume $V_b$ in the circulatory system as

$$V_b = \sum_{k=0}^{N} N_k V_k = \sum_{k=0}^{N} \pi(r_k)^2l_kn^k = \left(\frac{(n^{\gamma \beta})^{-(N+1)} - 1}{(n^{\gamma \beta})^{-1} - 1}\right)n^kV_c.$$  \hspace{1cm} (63)

In the first line of equation (63), there are $N_k$ tubes of volume $V_k$ at each $k^{th}$ level in the idealized circulatory system, leading to the expression $N_k V_k$, and $n^k$ tubes of volume $\pi(r_k)^2l_kn^k$ leading to the expression $\pi(r_k)^2l_kn^k$.

WBE states that $n^{\gamma \beta} < 1$ and $N \gg 1$. This observation is used to justify treating the exponent $-(N + 1)$ of the first expression in the numerator in the second line of equation (63) as approximately $-N$.

WBE finds, based on equation (63) that

$$V_b = \frac{V_0}{1 - n^{\gamma \beta}} = \frac{V_c(n^{\gamma \beta})^{-N}}{1 - n^{\gamma \beta}}.$$  \hspace{1cm} (64)

The first line in equation (64) is found by substituting $V_0$ for $n^kV_c$, as in the second line of equation (63). $n^kV_c = V_0$ since every level has the same amount of fluid. The second line in equation (64) is because $(n^{\gamma \beta})^{-N} - 1 \approx (n^{\gamma \beta})^{-N} - 1$. Then the $n^k$ term in $(n^{\gamma \beta})^{-N}$ cancels the $n^k$ term in $n^kV_c$ in equation (63).

With these steps, and assuming that capillaries are invariant for all organisms, WBE arrives at

$$(\gamma \beta)^{-N} \propto M.$$  \hspace{1cm} (65)

This conclusion critically relies on $V_c$ in the second line of equation (64) performing the role of a constant. Without the assumption of the invariance of capillaries, equation (65) would not follow from equation (64). This also relies on WBE’s assumption that the capillaries irrigate the whole organism which has mass $M$.

Using $M_0$ as in equation (59), and taking the log of each side of equation (65) gives

$$-N \ln(n^{\gamma \beta}) = \ln(M/M_0),$$  \hspace{1cm} (66)

and so

$$N = -\frac{\ln(M/M_0)}{\ln(n^{\gamma \beta})}.$$  \hspace{1cm} (67)

The right sides of equations (61) and (67) both equal $N$. Equating them, and substituting for $\gamma$ and $\beta$ their values in terms of $n$

$$a = -\frac{\ln(n)}{\ln(n^{\gamma \beta})} = -\frac{\ln(n)}{\ln(n^{-1/3}(-1/2)^2)} = \frac{\ln(n)}{\ln(n^{-4/3})} = -\ln(n)$$  \hspace{1cm} (68)

and so $a = 3/4$. The last two steps in equation (68) are omitted in WBE. Those last two steps involve only algebraic manipulation. In the mathematical model of dark energy the last two steps in equation (65) are significant. The last two steps are not only algebraical steps; they hint at the 4/3 geometrical scaling of an energy distribution system.

That concludes a summary of the portions of WBE 1997 relevant to this paper.

4. Kozlowski and Konarzewski on WBE 1997

About WBE 1997 Kozlowski and Konarzewski note (Kozlowski 2004): “The assumption that the final branch is size-invariant causes the number of levels to be a function of body size (or vice versa): more levels are required to fill a larger volume with the same density of final vessels.” They argue (Kozlowski 2004, 2005) that the assumption of size invariant terminal supplying vessels leads to an inconsistency: “for large animals the volume of blood vessels would exceed body volume” because “the number of capillaries is, according to their model, proportional to body size” (p. 284, Kozlowski 2004), contradicting the model’s assumption that the number of vessels equals $M^a$.

Kozlowski and Konarzewski note (Kozlowski 2004) that it cannot be both that $N_c \propto M$ and $N_c \propto M^n$, unless $a = 1$.

WBE respond (Brown 2005): “the service volumes of tissue supplied by each capillary are free to vary …”. This is unlikely if the biochemistry and bio-mechanics of energy distribution and tissue are similar for different animals as may be supposed (p. 2249, Gillooly 2001).

In the derivation of 3/4 metabolic scaling set out later in this article, the number of levels $k$ is proportional to mass $M$. The idea that the number of levels $k$ is proportional to mass $M$ is consistent with the supposition that an organism’s circulatory system is uniformly scaled (or branching) and hierarchical. It is also consistent with assigning an important role to the number of degrees of freedom $k$ of an organism’s circulatory system, and with the mathematical model of scaling applicable to lexical scaling described earlier in this article.
5. Adapting lexical scaling to metabolic scaling

Metabolic scaling offers a possible way to test the mathematical model of lexical scaling. WBE 1997 terminology suggests analogies between WBE’s model of 3/4 metabolic scaling and the mean path length scaling of lexical growth. In 2008, listing similarities and differences between WBE 1997 and mean path length scaling was a starting point.

Mean path length scaling and WBE’s mathematics used to explain 3/4 metabolic scaling have these similarities:

1. Both involve networks, one a circulatory system, and the other networked people and networked words.
2. Capillaries are assumed size invariant. Network nodes are fundamental units.
3. Energy to distribute resources is minimized.
4. The number of circulatory system branch generations scales by \( n \). The network’s number of cluster generations scales by the average energy required to connect two nodes.
5. Circulatory system fluid volume \( V_k \) in the \( k^{th} \) branch generation of the circulatory system is the same for all \( k \). Total network capacity is \( k \) times the capacity of a single cluster generation for a network for a network with \( k \) scaled generations.

Mean path length scaling and WBE’s mathematics used to explain 3/4 metabolic scaling have these differences:

1. Metabolism \( Y \) in equation (45) measures the rate of energy use by all the organism’s cells, not the average energy cost, proportional to the mean path length \( \mu \) between nodes.
2. The circulatory system is a sub-network of the organism. Lexical scaling is about the whole network.
3. Circulatory system fluid is distributed in one direction. An information network has a bidirectional exchange of information.
4. The circulatory system is a network of pipes physically different in scale from one generation to the next. In an information network, the nodes in different cluster generations are the same nodes.
5. Metabolic scaling remains the same through 21 orders of magnitude. Network entropy for lexical scaling increases as the number of nodes increases.

Suppose the same mathematical model can account for lexical and 3/4 metabolic scaling. If so, then there must be a way to resolve the apparent differences listed above. In April 2008 the differences resolved due to two observations.

First, in networked lexical scaling, directly or indirectly, each node can receive information from any other node, and can transmit information to any other node. For an organism, a cell in the circulatory system can benefit a cell in, for example, the lungs, or receive a benefit from a cell in the lungs, if not directly through networked cells, then at least indirectly from a system within the organism. In an organism, directly or indirectly, each cell can receive benefits from any other cell, and can transmit benefits to any other cell.

Second, an idealized social or lexical network has the same \( n \) nodes in every cluster generation, while a circulatory system levels have tubes of different sizes. In lexical growth scaling though energy scaling leads to a flattened hierarchy (nested scaling); a physically observable hierarchy may manifest itself in networks of cells in organisms. To analogize a network to a circulatory system, the circulatory system must have only \( n \) nodes in all generations or levels. This is so if each node corresponds to a single scaling energy cluster per unit time from heart to capillary. The perceived ‘different’ tube sizes of a circulatory system compare to distinct energy cluster generations in a network. In other words, treat the circulatory system as having one energy cluster (at the heart level) that scales at each subsequent level or generation, just as with lexical scaling.

Infer that an organism’s tissue is arranged to maximize entropy, that is, to maximize the efficiency of energy distribution.

Following WBE 1997, each capillary irrigates a corresponding volume in an organism. Therefore, the rate of energy use by the organism’s mass equals the rate of energy supply from the organism’s circulatory system \( \text{Circ} \). Assume that the organism’s energy use is proportional to both its mass and its volume. All this suggests that the volume \( V_{\text{Circ}} \) of the circulatory system is in proportion to the volume of the organism and also in proportion to its mass \( M \). One volume unit of the organism receiving energy distributed by the circulatory system contains one mass unit that requires one energy unit per time unit. What \( Y \propto M^\alpha \) says is that circulatory system only needs to use \( (V_{\text{Circ}})^\beta \propto M^\alpha \) energy units to irrigate all \( M \) mass units.

If per unit time, \((V_{\text{Circ}})^\beta \) energy units are sufficient to supply energy to \( M \) mass units, then \( [(V_{\text{Circ}})^\beta]^\frac{1}{\beta} = V_{\text{Circ}} \propto M^{\frac{1}{\alpha}} \) is sufficient to supply energy to a mass of \( M^{\frac{1}{\alpha}} \) mass units. This amounts to observing that \( V_{\text{Circ}} \) has too much capacity to distribute energy unless that capacity is scaled down by \( a < 1 \). Put another way: \( Y \propto M^\alpha \) is not really a statement about how metabolism scales; it is a consequence of how the capacity of the circulatory system to deliver energy scales up with size.

It follows that if, observationally, \( a = 3/4 \), then the capacity of the circulatory system to distribute energy is scaling with size by a \( 4/3 \) power.

Instead of the short inference in equation (68) about \( 1/\alpha \), I used the following observation about scaling. If \( r_1 = r_2 \), then \( \beta r_2 = \beta r_1 = \beta^2 r_1 \), and in general \( r_{k+1} = \beta r_k = \beta^k r_1 \). For \( r_{k+1} \) there are \( k = \log_\beta (\beta^k) \) scalings. If each scaling adds to a cumulative length with units each \( r_1 \) long, then we can think of \( k \) as the entropy of a cumulative radius \( kr \) or as \( k \) degrees of freedom relative to the scale factor \( \beta \), as in the terminology adopted in The Overview above.

In 2008, the observation that \( r_{k+1} = \beta r_k = \beta^k r_1 \) seemed a mathematically advantageous way of characterizing scaling,
because it simplifies calculation. By 2012 or so, it began to seem that regarding the exponent of a scale factor as its degrees of freedom was of more consequence that realized in 2008.

In 2008 these Three Assumptions,

1. $\beta r_k = \beta^k r_1$,

2. the capacity of a system equals the number of scalings (degrees of freedom) times the capacity of single generation, and

3. $(V_{\text{circ}})\pi^a \propto M^a$ energy units are sufficient to irrigate $M$ mass units,

permitted a shorter, simpler derivation of 3/4 metabolic scaling. Most of the steps in WBE represented in the present article by equations (58) to (68) can then be omitted, and the assumption that all animals have the same size capillaries can be dispensed with.

Suppose a $k^{th}$ generation ‘tube’ has radius $r_k$, length $l_k$, and that for the number $N_k$ of $k^{th}$ generation circulatory system tubes $N_{k+1} = n N_k$.

As in WBE 1997, let

- $r_{k+1}/r_k \equiv \beta$,
- $l_{k+1}/l_k \equiv \gamma$.

In general for the $(k + 1)^{st}$ generation of tubes, a circulatory system tube has cross-sectional area $\pi \beta^{2k}(r_k)^2$ and the volume it irrigates is $(4/3)\pi \gamma^{k}(l_k/2)^3$. As in WBE 1997, $\beta = n^{-1/2}$ and $\gamma = n^{-1/3}$.

Suppose that the energy content of blood is proportional to its volume. Then the supply volume of a $(k + 1)^{st}$ level of a circulatory system expressed in terms of tube volume, $\pi(r_k)^2 l_k$, is the same as $V_k$. A representative volume of a tube at the $(k + 1)^{st}$ level of the circulatory system is

$$\pi \beta^{2k}(r_k)^2 \gamma^k l_1$$

or $n^k V_1$.

Now equate the volume after $k$ scalings. Scale factors $\beta$ and $\gamma$ are on the left side and the scale factor $n$ is on the right side. With $k$ scalings there are $k + 1$ tube levels. Using the Three Assumptions just above, for the cumulative volume of $k$ levels of the circulatory system for one scaling tube (that is, totaling the volumes along all generations following a single path)

$$(k + 1)\pi \beta^{2k}(r_1)^2 \gamma^k l_1 = ((k + 1)n^k V_1)^{1/a}.$$  (70)

Since equation (70) holds for all sizes of mass, suppose its fractality results in it also holding at all tube levels. This simplifies the mathematics. Then equation (70) would hold in particular for $k = 1$. With $k = 1$ equation (70) becomes

$$2\pi \beta^2 (r_1)^2 \gamma l_1 = (2nV_1)^{1/a}$$

which is equivalent to

$$\beta^2 \gamma n^{1/a} = (2V_1)^{1/a} (2\pi r_1)^2 l_1.$$  (72)

For a given animal, the right side of equation (72) is a number because all the components of the numerator and denominator are numbers (2, $\pi$) or have numeric values ($V_1$, $r_1$, $l_1$, 1/a). We are interested in the scaling relationship pertaining to only the left side of equation (72) which are not affected by constants that multiply a scale factor, so for scaling purposes treat the right side of equation (72) as 1. We then exploit the scaling relationships among $\beta^2$, $\gamma$ and $n$ on the left side of equation (72) to solve for $a$. Multiply the top and bottom of the left side of equation (72) by $n^{-1/a}$.

$$\beta^2 \gamma = n^{-1/a},$$  (73)

and so $(n^{-1/2})^2 n^{-1/3} = n^{-1/6}$ which gives $-4/3 = -1/a$, so $a = 3/4$.

It is possible to abbreviate derivation of the scaling of supply or circulatory system tubes still further, with $\beta = n^{-1/2}$ and $\gamma = n^{-1/3}$. This derivation is of particular interest because of its brevity. Its brevity facilitates compact derivations and proofs. After $k$ scalings (that is, at the $(k + 1)^{st}$ level), by collecting powers on the left side to give the right side in the following equation:

$$\pi n^{-k}(r_1)^2 n^{-k/3} l_1 = n^{k(-4/3)\pi r_1 l_1}.$$  (74)

The corresponding spherical recipient volume irrigated by a circulatory system tube is equivalent to

$$(4/3)\pi \gamma^{-3k}(l_1/2)^3 = \theta^{-1k}\Theta_1,$$  (75)

for a sphere $\Theta_1$ which has $\theta$ as the base of its scale factor. Spheres scale by unity relative to other spheres since they are symmetric to their origin. In this perspective, circulatory system tubes scale per generation by a 4/3 power (the 4/3 power of $n^{-3k(4/3)}$) while the corresponding recipient spherical volume scales by a power of 1 (the 1 power of $-1k$) as on the right side of equation (75).

Thinking of the exponents of the scale factors as representing entropy (or degrees of freedom), the exponent of circulatory system supply is 4/3 and the exponent of the corresponding receipt is 1. This applies for every $k^{th}$ generation. It is not necessary to assume that capillaries are the same size for all animals. Since the energy distribution system scales by a 4/3 power and the energy receipt system scales by a 1 power, this suggests treating them as distinct systems; they scale differently. Designate the components of the distribution system differently than the components of the receiving system.

6. Two systems

There are two systems in an organism, one distributing energy — the subsystem that is the circulatory system — and the other — the entire mass of the organism — receiving that energy. There are also two systems in an information or social network.

An information network has a system that distributes information and a system that receives it. But the members of society who distribute information are the same people who,
the analogous requirement for an information network is obscured because the two systems have identical membership.

The exponent 4/3 of the base of the scale factor for the circulatory system found in April 2008 was initially perplexing. The exponent sought was 3/4, not 4/3. Despite redoing the calculation many times, whether the capillaries were generation 0 or generation N, the answer was 4/3. The provenance of 4/3 resolved upon realizing that the metabolic rate scales down by a 3/4 power to offset the capacity of the circulatory system to distribute energy scaling up by a 4/3 power.

The entropy of a distribution network increases the network’s energy capacity. The entropy (the exponent of the base of the scale factor for energy distribution) of the circulatory system’s capacity to distribute energy scales by 4/3 with size. This observation is consistent with the idea that the entropy of a network using a base equal to its mean path length measures the increased collective capacity of the network relative to the capacity of a single node; that is, that capacity is proportional to degrees of freedom relative to the appropriate base of a scale factor. This observation is likely more significant than the idea that the mean path length scales a network, which initially and for some time seemed to be the important principle.

The initial attempt to model metabolic scaling contained technical and algebraic mistakes. It left unanswered the question of why distribution by the capillaries to a spherical volume as supposed in WBE 1997, as opposed to some other shape of volume, worked.

J. Stefan-Boltzmann Law

In April 2008 the article on metabolic scaling was submitted to arXiv. After it appeared, that same month, while reading Allen and Maxwell’s A Text-book of Heat (1939), page 742 described Stefan’s law. Stefan’s law is so called because Stefan first conjectured it. It is also sometimes called the Stefan-Boltzmann law, because Boltzmann later gave a mathematical explanation of it.

Stefan’s Law is, in Allen and Maxwell’s notation,

$$E \propto T^4$$

where $E$ is energy density and $T$ is temperature in degrees Kelvin. This applies to an enclosed space with perfectly reflecting walls, what is sometimes described as the setting for black body radiation.

In their presentation of Boltzmann’s proof, based on an intermediate step, they have this inference:

$$\frac{\partial S}{\partial v} = \frac{4 E}{3 T}.$$  \hspace{1cm} (77)

In equation (77), $S$ on the left side is entropy, $\partial S$ is the change in entropy, and entropy’s change is per volume $\partial v$. This is the same as in the case of metabolic scaling, where 4/3 scaling is relative to the volume of a circulatory system tube. Since $S$ can be represented by a logarithmic function, so too in the case of a circulatory system tube scaling, the scaling is represented by the exponent of a scale factor.

These considerations suggest that equation (77) represents the same kind of 4/3 power scaling derived in connection with 3/4 metabolic scaling and that the scaling method used for 3/4 metabolic scaling can be adapted to radiation.

Why a spherical volume based on a one half a circulatory system tube’s length worked is a puzzle arising out of the derivation of 3/4 metabolic scaling in WBE 1997. The intermediate step in equation (77) in the derivation of Stefan’s Law provides a clue connecting metabolic scaling and Stefan’s Law.

The sphere receiving fluid from capillaries connects to the isotropy of black body radiation and of the cosmic background radiation, which is isotropic to one part in 100,000 (Fixsen 1996). The sphere scales in the same way for all axes from its center, and so models the isotropic distribution of energy. This suggests that the 4/3 power scaling of energy distribution connects to the 4/3 power scaling of a circulatory system, black body radiation, and cosmic background radiation; one mathematical model for life, energy and light.

K. Adapting circulatory system scaling to radiation scaling

1. Analogies

Radiation from a point source can be modeled as energy distribution via radiation cones. A scaled circulatory system is analogous to a radiation cone. A radiation cone scales uniformly as it grows larger; it is in the nature of a cone. Mark fixed lengths radially along the radiation cone, with each length defining a radiation cone increment. Analogize the radiation cone to a circulatory system tube length. Analogize the average radius of a radiation cone increment to the fixed radius of a circulatory system tube. These analogies permit us to adapt derivation of 4/3 power scaling for an organism’s circulatory system to derive scaling of light or energy radiated distribution from a point.

The scale factor for the average radius of radiation cone increments from one generation to the next can be set as $\beta$ as with circulatory system tubes. The radial length of a radiation cone increment would be constant, not scaling up or down, if the rate or speed of radiation is constant as it would be for light.

The space receiving isotropically distributed radiation can be described by a spherical volume, with a radius equal to one half the length of the radiation cone increment. As with the circulatory system, we can use $\gamma$ as the scale factor for the radial length; here $\gamma$ helps keep track of the effect of scaling, since the radial length itself is constant. (This article uses $s$ or $s_k$ to scale radiation length $L$.) Now apply the same method used for the circulatory system to the radiation system to derive 4/3 power scaling.
2. Differences and similarities

An organism’s circulatory system distributing energy has its tubes *scale down* in size. As radiation is distributed from a point source, the volume of the radiation cone increments, each with the same radial length, *scale up* in size. At any instant, summing the volumes from larger to smaller as with the circulatory system, or from smaller to larger as with radiation, does not affect the scaling relationship between length and volume (except perhaps for a plus or minus sign in exponents) which is per generation.

Theory suggests denoting the start of circulatory system energy distribution at the heart, since that is where energy distribution for the circulatory system begins. We might instead think of capillaries scaling up in size until circulatory system tubes reach the heart. Since the number of branchings for a circulatory system is finite, the direction in which we add does not affect the mathematics. As an energy distribution system, though, the energy is being subdivided as it courses through circulatory system; as such it is better to consider scaling of circulatory system tubes to begin at the heart.

The radiation case mathematically suggests, perhaps requires, that the counting of levels must begin at the first radiation cone increment next to the point source. Assigning numbers to radiation cone increment levels is mathematically consistent if each additional radiation cone increment farther from the point source increases the level number. Counting does not entail shifting an entire array farther from the beginning to relabel level 1 as level 2 and start with a new number 1 closer to the point source. If our usual notion of counting reflects an underlying physical reality to the structure of the universe (a not unlikely inference), then counting must begin at the heart for circulatory systems and at the first radiation cone increment (nearest the point source) for radiation.

Mathematical consistency therefore suggests denoting the heart and the first radiation cone increment closest to the point source as level 1. An argument can be made to treat the heart as level 1 for an organism’s circulatory system, since the circulatory system can be considered to a set of nested energy clusters manifested as a branched linear system.

If distributed energy is being scaled for each generation or at each level, and if each level has been scaled, then the source of energy must be external to the energy distribution system. For the circulatory system, food and oxygen that are transported are external to the organism. This observation suggests designating the external source of energy as level 0. This would support the heart as level 1 for an organism’s circulatory system.

The amount of energy per circulatory system level is constant. As the energy is scaled into smaller tubes, the energy per fluid volume remains the same. The amount of energy per radiation level is constant. The energy density per volume of the radiation system scales down.

For the circulatory system, tube size scales down and energy density is constant. For radiation, the volume of the radiation cone increments increases as the distance from the source increases, and energy density scales down. From this it appears that for radiation energy density scaling down and single constant radiation cone are respectively analogous to, for an organism’s circulatory system, the scaling of the number of tubes and the constancy of energy density per unit of fluid volume.

The circulatory system has a finite number of branchings. Radiation constantly adds radiation cone increments.

3. Radiation scaling as a general law

Suppose scaling by a $4/3$ power is a valid attribute or model of radiation. Radiation in the universe preceded life. One can infer then that the $4/3$ scaling of the capacity of the circulatory system of organisms is modeled on, or represents nature adapting a physical law governing isotropic energy distribution to living organisms.

From a different perspective, one might observe that the same physical principles that govern the distribution of energy in the universe and that apply to black body radiation (and so apply at a quantum level) also apply to the distribution of energy in our own bodies. Suppose the distribution of energy in the universe via light is maximally efficient, perhaps perfectly efficient. Then subsystems within the universe, including organisms, might emulate the universe’s maximally efficient energy distribution as they evolve over time.

If the analogy of energy distribution by radiation to energy distribution by the circulatory system is valid, then energy scaling applies at all scales from the scale of light up to the scale of a circulatory system, and presumably at all scales in between. Generalizing, if energy scaling applies to energy radiation, since radiation is a fundamental mechanism for energy distribution and since every array of energy is built up from energy clusters eventually comprised of light energy, then energy scaling applies at all scales. If energy scaling applies at all scales, then the lexicon as a system of networked concepts is a particular instance of energy scaling. These observations suggest that the scaling ideas here outlined have universal applicability at all scales.

One might ask: did the way the universe emerged dictate our mathematics, or does mathematics constrain the universe’s choices about how to maximize the efficiency of energy distribution? It is, perhaps, more likely that the mathematics we have developed reflects and conforms to the manner of the universe’s emergence. If so, then the mathematical ideas and structures that have emerged and evolved due to our contemplation of the universe, including its earth-bound and starry features, indirectly model the emergence of the universe.

4. The problem of the extra $1/3$ power

A mistaken way of examining $4/3$ scaling for radiation helps illustrate the problem of the extra $1/3$ power in the $4/3$ power of energy distribution scaling.

Suppose a radiation cone volume increment $V_k$ scales by scale factor $v$ such that
\[ vV_k = V_{k+1}, \] (78)
and the average radius \( r_k \) of a radiation cone increment \( V_k \) scales by \( \beta \) such that

\[
\beta r_k = r_{k+1}.
\]  
(79)

The relationship between scale factors \( \beta \) and \( v \) can be shown as

\[
V_{k+1}/V_k = v = \pi(\beta r_k)^2 sL_k/\pi(r_k)^2 L_k = \beta^2
\]  
(80)

so that \( \beta = v^{1/2} \).

Suppose the radial length \( L \) of a radiation cone increment has scale factor \( s \) such that

\[
v V_k = (sL_k)^3 = s^3(L_k)^3
\]  
(81)

giving

\[
s = v^{1/3}.
\]  
(82)

Using the preceding equations (78) to (82), and in particular \( v^{1/2} = \beta \) and \( v^{1/3} = s \), describe the scaling that occurs from \( V_k \) to \( V_{k+1} \) as follows:

\[
V_{k+1} = v V_k = \pi(\beta r_k)^2 sL_k = \pi \beta^2 (r_k)^2 s L_k = \pi v (r_k)^2 (v^{1/3}) L_k = \pi v (r_k)^2 (v^{1/3}) L_k = v^{4/3} \pi (r_k)^2 L_k = v^{4/3} V_k.
\]

We used the substitutions \( v^{1/2} = \beta \) and \( v^{1/3} = s \) in the fourth line of equation (83).

Observations and problems arising from equation (83) include:

- With respect to the scale factor \( \beta \) of the average radius \( r_k \), first we find that \( \beta = v^{1/2} \) in equation (80). Then in equation (83), and in the derivations relating to 3/4 metabolic scaling, we square \( \beta = v^{1/2} \), ending with \( \beta^2 = v \) in equation (83). \( \beta \) is used to scale the radius or average radius, but only \( \beta^2 \) plays a role in the derivation. Infer that we can dispense with the scaling of the average radius, and need only have regard for the scaling of the average cross-sectional area \( A \) of a radiation cone increment. Scaling the average radius is an unnecessary additional step. This partly explains why only the scaling of the average cross-sectional area \( A \) of a radiation cone increment plays a role in The Overview, and why it is unnecessary to use scaling of the average radius. Another reason may be that it only matters how much radiation is passing through the cross-sectional area \( A \).

- The foregoing observations and problems about equation (83) must somehow relate to the requirement that a source system is distinct from a receipt system. That is to say, the problem of the two different scalings, \( vV_k \) and \( v^{4/3} \), may be resolved by recognizing the existence of two reference frames.

The result of the foregoing list of observations and problems is that if we can resolve these problems — which is what The Overview attempts to do — we may be able to explain dark energy.

5. Two systems, two reference frames

An energy distribution system scales by a 4/3 power, and the space into which the energy is distributed scales by the power 1. This suggests characterizing them as distinct systems. To distinguish volumes in the two systems, designate a radiation volume as \( V \) (for volume) and a space volume as \( \Theta \) (the shape of the Greek letter \( \Theta \), almost circular, connotes isotropy). Since they scale differently, denote their scale factors differently, using corresponding small letters for their scale factors.

The problematic equation (83) has radiation cone increment volumes scale by a 4/3 power in the last line, whereas a spherical isotropic static space scales by power 1. This, by reason of the 4/3 : 1 ratio, suggests that radiation causes space to expand, enlarges space, and perhaps creates space.

This exploration of ideas would be assisted by finding postulates on which to base the mathematics.

One approach is to suppose that there are two systems in the universe. One, \( S \), can be considered the Source or Supply of energy or radiation, the other, \( R \), can be considered the Receipt of the energy or radiation. \( S \) grows at a constant rate, at least in its own reference frame. \( R \) is static.

In 2011, my article on Isotropy, entropy and energy scaling used a descriptive notation \( \text{Deg} \) to describe the relationship between \( S \) and \( R \): \( \text{Deg}(S) = (4/3)\text{Deg}(R) \), which is probably incorrect. This way of expressing the relationship between \( S \) and \( R \) has the advantage of economy but has the
disadvantage of not revealing the fractal nature of the relationship; the 4/3 : 1 ratio occurs at every scale. Another problem with Deg(S) = (4/3)Deg(R) is that it is probably wrong, as discussed in section [B]. The correct description would be Deg(S + R) = (4/3)Deg(R). In 2012, I decided to try using Deg notation to focus on degrees of freedom to help identify how the extra 1/3 power for the scaling of radiation cone volume increments arises.

An advantage of Deg(S + R) = (4/3)Deg(R) apart from economy is that this form of expression leads to questions about cosmogenesis: what starts things going? how does S create R? what are the fundamental attributes of each? Focusing on degrees of freedom via the notation Deg, Deg*, Deg** and deg avoids these cosmogenesis questions; perhaps that is a superior approach.

Notwithstanding the shortcomings of using the S and R terminology, a different approach to the same problem — increasing the degrees of problem-solving freedom — may improve the odds of learning something about it. Because of that possibility, I have included reference to S and R in what follows.

The Overview on the other hand mostly adopts the degrees of freedom point of view. That approach has many advantages for presentation of these ideas. In the Overview, the two systems are treated as two reference frames. The two reference frames, radiation and space, are connected: Deg*(Vr) = (4/3)Deg*(θr).

IV. EQUVALENCE OF THE NRT AND THE 4/3 RDFT; THE NRT AS A SPECIAL CASE

The 4/3 RDFT and the related mathematics of scaling helps to explain 3/4 metabolic scaling and in so doing also provides evidence of the validity of the 4/3 RDFT. Adapting mean path length scaling used to model lexical growth and the NRT to metabolic scaling led to the 4/3 RDFT. Now proceed in the reverse direction, using the 4/3 RDFT to find mean path length scaling.

If mean path length scaling can imply the 4/3 RDFT, and if the 4/3 RDFT can imply mean path length scaling, then each implies the other; they are equivalent or are based on the same principles. If so, then evidence in support of the NRT is also evidence in favour of the 4/3 RDFT.

The principle common to the NRT and the 4/3 RDFT is that the degrees of freedom of the scale factor of a homogeneously scaled system multiplies the capacity of the system. The 4/3 RDFT connects to capacity by showing that a receipt receiving radiation from a source having 4/3 the receipt's degrees of freedom must grow; the receipt cannot receive 4/3 of the receipt's intrinsic degrees of freedom without growing. This suggests that the degrees of freedom affects receipt's capacity to accept energy, or for an information system, to accept information. In fact, there is a linear relationship between a system's degrees of freedom and its system-wide capacity.

Consider the mathematics pertaining to degrees of freedom in the context of a network R with n nodes, mean path length µ, an average individual rate of information transmission between notes of r1, and a collective rate of transmission r_n. R's collective rate r_n is related to r_1 as follows:

\[ r_n = \text{Deg}^*(n)r_1. \]  

What is Deg*(n)? Since the mean path length of R is µ, then

\[ \text{Deg}^*(n) = \log_\mu(n), \]

since that is the number of degrees of freedom available to R to form paths with a measuring stick µ units long.

In 2007 the mean path length as the scale factor for lexical scaling seemed to be the governing principle. But the mean path length is the scale factor only in the special case that occurs at a particular instant in the life of a network. The general principle is that the degrees of freedom of the scale factor is linearly related to a system's capacity or cumulative energy.

Until 2011 or so, 4/3 scaling seemed an attribute of energy distribution incidental to mean path length scaling. Having found mean path length scaling (in connection with networks), it appeared (incorrectly) to embody a principle more fundamental than 4/3 power scaling. It was instead a vital clue to 4/3 power scaling, a more fundamental principle.

Since the NRT describes an emergent phenomenon, the possibility of actually deducing the NRT based on the ideas used to derive the 4/3 RDFT is probably close to nil. By supposing a log formula might exist that relates society's collective knowledge to average individual knowledge and finding that the mean path length µ works as a scale factor led to the NRT. Mean path length scaling reveals the role of degrees of freedom in calculating a collective network capacity and therefore in a system's capacity generally.

V. AN ALGEBRA OF DEGREES OF FREEDOM

A. Postulate of two systems versus reference frames

1. Systems and reference frames assumptions

To arrive at a mathematical model of radiated energy distribution, one approach is that set out in the Overview: regard radiative distribution and stationary space, related by two 4/3 theorems, as occurring in distinct reference frames. Another approach is to postulate two systems, one that distributes energy homogeneously and isotropically, and another that receives that energy. The approach in the Overview is simpler, requires fewer assumptions and avoids problems that arise in trying to formulate postulates for two systems.

Postulating two systems provides a different point of view and raise some different issues, including some related to the origins of the universe. I currently prefer the approach in the Overview, mainly because it seems to simplify the task of presentation. It is possible that if the postulational approach can be improved, it might be a more effective way of presenting these ideas than the two reference frames approach.

Bearing that in mind, let’s try to devise some postulates that describe two systems, one radiating energy, the other receiving it.
1. A system $S$ — a supply of radiation — exists. Its initial point source, a singularity, is designated $[0]$. Whether as a matter of logic $[0]$ should or should not belong to $S$ seems to make no difference to the mathematics in this article, so I do not suppose one or the other. But, logically, once $[0]$ has ‘scaled’, it no longer exists. Once $[0]$ has been scaled, any subsequent scaling must relate to a generation later than the $[0]$ generation. $[0]$ can only be scaled once. $[0]$ is not part of the distribution system. Radiation itself has no volume in $S$.

The most important feature of $S$ is its radial distribution from every point of $S$. We require that feature because from the discussion to this point, uniform scaling is necessary to the mathematical model, and uniform scaling is a feature of homogeneity and isotropy. For our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system. Borrowing from our purposes, any system that distributes energy in that way is considered a radiation system.

In view of the application of the mathematics to various systems, including Brownian motion and the circulatory systems of organisms, it seems the this postulate might seem to make no difference.

2. A Receipt or Space $\mathcal{R}$ exists. It is three dimensional and isotropic along all axes for a spherical volume. An unconstrained particle in $\mathcal{R}$ has three independent degrees of freedom. $\mathcal{R}$ itself has no radiation. Based on the mathematics that follows, it seems that $S$ somehow creates $\mathcal{R}$, but that supposition plays no express role in the inferences that follow. No radiation originates in $\mathcal{R}$. An important aspect of $\mathcal{R}$ or space is that it is a stationary environment. Measurements within that system occur in a space reference frame. A paradigm instance would be the stationary fluid in which moving particles are suspended, attributes of Brownian motion.

3. The energy of each radiation pulse that originates in $S + \mathcal{R}$ is entirely received without loss of energy by a corresponding part of $\mathcal{R}$. This is an idealization and simplification that models light transmission. Every iteration or pulse of radiationhomogeneously radiates with a constant length $L_k = L$ into a corresponding spatial volume $\Theta_k$ in $\mathcal{R}$.

This last assumption appears to be equivalent to the law of conservation of energy: an amount of energy measured in a volume in the radiation reference frame is the same as in a corresponding volume in the space reference frame.

It follows that a radiation volume has both radiation and volume and so is within, and has features of, both $S + \mathcal{R}$. The $S$ feature of $S + \mathcal{R}$ results in radial distribution and motion. The $\mathcal{R}$ feature of $S + \mathcal{R}$ is that, at an instant, $\text{Deg}^*(L_k) = (1/3)\text{Deg}^*(V_k)$.

B. The role of $S + \mathcal{R}$

The volumes in a hierarchical energy distribution system, such as that represented by a circulatory system, scale by a $4/3$ power, compared to the $1$ power of the corresponding energy Receipt space volumes. This different scaling suggests that there are two distinct systems involved in energy distribution. Since volumes in a radiation system $S + \mathcal{R}$ scale by $4/3$ and corresponding volumes in the Receipt $\mathcal{R}$ scale by $1$, it might seem that it is appropriate to describe the relationship as $\text{Deg}(S) = (4/3)\text{Deg}(\mathcal{R})$. This section discusses why the correct characterization of the relationship is more likely to be $\text{Deg}(S + \mathcal{R}) = (4/3)\text{Deg}(\mathcal{R})$. The reasons are based on dimension, observations, logic, and the mathematical derivation of The 4/3 RDFT.

Using the idea of dimension, apply the postulate about $S$. For $S$, there is distribution from every point. If we add the requirement of homogeneous scaling, then for a characteristic radiation length $L$, we have $sL_k = L_{k+1}$, and after $k$ scalings that starts with $L_1$ there is a cumulative length of $L_1 + \log_k(s^3)L_1 = (k + 1)L$. On this view, radiation is a one dimensional system; it resembles a line. Space by postulate has $3$ dimensions. Together radiation and space $S + \mathcal{R}$ have $4$ dimensions.

Observations support the notion that the radiation system $S$ manifests itself or is perceived as part of space $\mathcal{R}$ in $S + \mathcal{R}$. A circulatory system is part of an organism. The sun and stars as sources of radiation appear in space.

The derivation of The 4/3 RDFT compares how radiation cone volume increments scale compared to how space volumes scale. Volume is a feature of space, not of radiation. A radiation cone volume increment is a hybrid thing: it distributes radiation, but is also has volume, so it must be in both $S$ and $\mathcal{R}$, here denoted $S + \mathcal{R}$.

Thus $\text{Deg}(S + \mathcal{R}) = 4$ whereas $\text{Deg}(\mathcal{R}) = 3$, and it follows that

$$\text{Deg}(S + \mathcal{R}) = (4/3)\text{Deg}(\mathcal{R}).$$

The following observations apply to equation (86).

The 4 dimensions of $S + \mathcal{R}$ resemble the 4 dimensions of space-time. If $S + \mathcal{R}$ can be identified with space-time, then we may infer that the rate at which radiation occurs, or the scaling of the characteristic radial length, is proportional to time. Since the cumulative effect of scaling the characteristic radial length can be expressed as a logarithm $\log_k(s^3)$, we may infer that, cumulative time in $S$ (with the proviso that it is unclear that time can be defined for $S$) is proportional to cumulative distance in $S + \mathcal{R}$, which is the entropy of the energy distribution. From this one might suppose that entropy and the expansion of the universe are in proportion at least in the $S + \mathcal{R}$ reference frame, and that radiation is responsible for the expansion of the universe.

Equation (86) also describes the difficulty observers face in inferring the existence of the distinct systems $S$ and $\mathcal{R}$. We do not perceive separate systems $S$ and $\mathcal{R}$; we perceive one system, $S + \mathcal{R}$. To reconcile our observations of two systems that we perceive as one requires work arounds such as special rel-
activity (with inertial reference frames relative to each other), and makes deciphering dark energy difficult.

We may therefore characterize $S$ as $(S + R) - R$, and characterize $R$ as $(S + R) - S$.

The different scaling of an organism’s circulatory system compared to its mass provides a clue to the existence of separate systems $S$ and $R$. It is difficult, if not impossible, to imagine how one might guess that a clue to a cosmological problem might be found in the scaling of organisms (or gas molecular motion or Brownian motion). The likelier path of discovery is to observe a relationship and subsequently connect it to dark energy.

If the foregoing observations have any value at all, that would make considering the 4/3 theorems from the point of view of two systems $S$ and $R$ potentially helpful.

C. $S$ and $R$ and making inferences about them

The salient mathematical feature of $S + R$ is homogeneous radiation into (stationary) $R$ at a constant length $L$ per radiation iteration or pulse. The salient mathematical features of $R$ are, at all scales, its three degrees of freedom relative to a length, its passive receipt of radiation and its isotropy. Passive receipt is intended to mean that the recipient system is unmoving relative to the homogeneously scaling transmitting system. The salient attributes connecting $S$ and $R$ are that radiation volumes reside in $S + R$ and that each transmission pulse of a radiation volume in $S + R$ has a corresponding recipient volume in the Receipt $R$. The connection between $S$ and $R$ might be better described as $\ell = (4/3)L$, for $\ell$ the characteristic length for $R$ and $L$ the characteristic length for $S + R$.

No postulates assign any other physical attributes to $S$ and $R$. The physical nature of $S$ and $R$ are not otherwise supposed. How physical counterparts of these abstractions $S$ and $R$ (if there are such) came into existence, why they have the analogs of these attributes, and why $\{0\}$ begins to radiate are outside the domain of this article.

The advantage of the postulates is simplicity: logical inferences arising from the postulates are spared the clutter of extraneous assumptions. (The Overview is even simpler.) The mathematical reasoning leading to The 4/3 Ratio of Degrees of Freedom Theorem (The 4/3 RDFT) that arises out of these postulates ignores everything else. Observing the universe leads to mathematical abstractions and relationships used to describe the universe. By idealizing the abstractions as far as possible, and so divorcing them from reality, the possibility of identifying the underlying principle — The 4/3 RDFT — is, ironically, increased.

Modelling radiation space and space using scaling does not require that the model correspond to what physically occurs in reality. All we may infer is that by using scaling we are able to construct models that help us to infer outcomes based on logic that correspond to observable outcomes. In other words, we cannot be sure that scaling as a means to model reality describes physical events, only that it can model them.

A general mathematical principle is supported by particular instances of it. Conversely, a mathematical treatment of a particular instance, generalized, proves the general case. This aspect of generalization and particularization will be utilized repeatedly in what follows.

A mathematical model is constrained by the assumption that the universe is self-consistent. An observation consistent with a model of a feature of the universe supports it. If a model leads to inconsistency with accurate observation or physical law, then the model’s inferences or assumptions must be wrong. (On the other hand, sometimes theory can create doubt about observations or previously accepted theories.) If mutually inconsistent implications are compatible with the model, then the model itself can not be right; the universe is not self-contradictory. A mathematical inconsistency in a theory tells you: try another path. Inconsistencies encountered on applying precursors of these ideas to a variety of physical phenomena suggested corrections and refinements.

To be emphatic: The part of this article following the Overview develops mathematical relationships of abstractions, including those relating to energy distribution system $S + R$ and Receipt $R$. Those mathematical relationships reveal mathematical patterns. If the mathematical patterns correspond to measurements of physical systems, infer that the abstract mathematical relationships are analogs of physical relationships and that they provide an approximate description of those physical systems. (Similar considerations make computer simulations useful.) One then may hypothesize that the approximation is a general law, and subject that hypothesis to other tests.

The utility of this approach arises if the chosen mathematical abstractions usefully idealize physical attributes of the universe, such as those considered in the following section.

D. Physical correspondences to the postulates

The mathematical abstraction $S$, manifested in $S + R$ an idealized radiation cone, grows at the constant rate of length $L$ per radiation in every direction. At any instant, how $R$ receives radiation is therefore undifferentiated as to place — $R$ is homogeneous.

From another perspective, radiation distribution at the same rate everywhere can be considered a radiation reference frame. Similar to $R$ as homogeneous radiation Receipt, our physical universe is isotropic to one part in $100,000$ (Fixsen 1996) and homogeneous.

The observed homogeneity and isotropy of our universe, analogous to an energy Receipt, imply an initial system $\{0\}$, a primeval atom (Lemaître 1950), a ‘big bang’ theory, similar to the postulated singularity $\{0\}$ in section $\nabla A$. The idea of a singularity $\{0\}$ is analogous to the inference that the singularity $\{0\}$ level for a circulatory system must be the source of energy — food — external to the circulatory system.

The speed of light is the same in all reference frames, similar to the abstract $S + R$ radiating at a constant $L$ per radiative event in what may be considered the radiation reference frame. Our universe is made up of energy quanta. Similarly,
there is some quantum attribute in the almost featureless abst-
raction of $S$ leading to a volume in $S + R$ growing by a con-
stant — quantum — $L$ per radiative event.

We perceive space to have three dimensions, like $R$. The cum-
ulative number of radiative events generated from a ra-
diation point of the abstraction $S + R$ is counted by $1L$, $2L$, $3L$, and so on. That resembles counting time and, in physical space, measuring distance.

Postulate $5$ connects $S$ and $R$. $S + R$ is an idealization of the role of an organism’s circulatory system as an energy dis-
tribution system — tubes scaling in number and size from the heart, with each increment of energy supplied by a level of the circulatory system entirely received by a corresponding three dimensional mass (in $R$) that is part of the organism. Treating $S + R$ and $R$ differently is required because they scale differ-
ently. On the other hand, an organism’s circulatory system is part of the organism, and consists of tissues and tubes with volumes. So the circulatory system should be considered to be an analog of $S + R$. I have inferred that the distribution of energy by the circulatory system to an organism it is part of is an instance of a principle pertaining to the distribution of energy in the universe because the same mathematical model appears to apply to the distribution of energy both within an organism and within the universe, namely scaling with a 4/3 exponent.

$S$ as an idealization of what results in constant motion of energy distributed through an organism’s circulatory system arises from the circumstance that in April 2008 I found a 4/3 exponent for the scaling of energy supply via an organism’s circulatory system tubes, and later that month noticed the sim-
ilarity of that to an intermediate step in the proof of Stefan’s postulate might lay the foundation for the mathematics that ensues.

Physical attributes of our universe resemble the features of abstract $S$ and $R$, with radiation volumes manifested in a con-
joined $S + R$. The abstractions are based on the postulates of section $\text{VE}$, mathematical inferences based on those post-
ulates may model attributes of the physical universe. I use $S + R$ in this article to denote the union of $S$ and $R$, inhabited by volumes that radiate.

### Volumes, Lengths, and Scale Factors

The purpose of this section $\text{VE}$ is mainly to develop the degrees of freedom notation used in this article, and at times to comment on the notation and related concepts.

1. $S$ initiating $R$

By Postulate $[10]$ initiates homogeneous radiation — at a constant rate of $L$ per radiation or pulse in $S + R$. After (or concurrent with) this, $S$ comes into existence as the set of radiations that have already occurred as of a point in time.

The initial radiation also brings $R$ (perhaps it would be bet-
ter to say, $S + R$) into existence. $R$ (or $S + R$) appears to grow in volume everywhere; at least, that is how I interpret radiative events occurring at every part of $S + R$ (homogeneity). It may be incorrect to say $S$ has parts.

After the initial radiation, $S$ is within a system $S + R$ com-
bined with the Receipt $R$ and yet maintains its own radiative character. $S$ has only one degree of freedom per pulse relative to the preceding cumulative radiation; which is to say, radiation is constrained in $S + R$ to travel along what we perceive as a straight line.

I infer that, in part, $S$ causes space to grow everywhere be-
cause radiation volumes within $S + R$ scale by a 4/3 power, and then scaled parts scale, and so on. The proviso ‘in part’ applies because The 4/3 RDFT plays a role in causing $R$ to grow relative to $S + R$.

There is a push-pull question in regards to $S$ and $R$. Does $S$ compel $S + R$ to expand, because $R$, with only one degree of freedom per scaling, does not have enough ‘room’ to accom-
modate 4/3 degrees of freedom? Or is it the case that $S$ (or $S$ as it is manifested within $S + R$) is compelled to continue to scale by the expansion of $R$? Does the push-pull question make sense, and if it does, is it a question that physics can answer?

The Overview notes that, due to its constant rate or speed, $L_k = L$ has a radiation property $\text{Deg}^k(L_k) = 0$ deg and a prop-
erty $\text{Deg}^k(L_k) = (1/3)\text{Deg}^k(V_k)$.

Using the concept of degrees of freedom as applied to motion tells us that radiation itself has one degree of freedom. This characterizes radiation, namely that in $S + R$ radiation travels radially in a straight line. The rate of speed of radi-
ation in units of $L$ per radiation event has a one-to-one relation-
ship with time in radiation’s reference frame. Time is also linear, and each radiation event adds one unit of time and one radiation length $L$ to the existing cumulative radial radiation length of a radiation cone $G_k$. This one dimensional aspect of radiation appears to correspond, as noted above, to the one dimensional aspect of time in connection with space-time.

The Overview shows that hypothesizing how $S$ leads to $R$ is unnecessary to the mathematics used there. Mathematics, however, raises the question: what physical conditions lead to the mathematical model. This section considers what postu-
lates might lay the foundation for the mathematics that ensues.

Due to $R$’s three dimensions, an unconstrained point mov-
ing in $R$ has three independent degrees of freedom. Therefore, a spatial point cannot be within $S$ because at each point of $S$, for a cumulative radiation length, only one degree of freedom of linear motion per radiation is conferred.

Looked at in another way, radiation occurs in time, is a dy-
namic event, involves motion only, not space. There is a one-
to-one relationship between lengths $L$, added by $S$ in $S + R$, and time. Space is static. A radiation event moves (or joins) one generation of space to the next; or we might say, each generation of space is embedded in the expanded space that comes into existence in the next generation.

A radiation volume that radiates at the rate of $L$ per radia-
tion event must be in both $S$ (as radiation) and $R$ (being con-
tained in a volume), here denoted $S + R$. A radiation volume has four degrees of freedom, three arising from the three di-
ensions of $R$ and one arising from $S$’s one degree of free-
dom with respect to radiation (or with respect to radial motion
in \( S + \mathcal{R} \).

The first radiation generation creates new radiation or energy clusters, which in turn radiate or scale. When there have been two generations of radiation from \([0]\) there has been only one generation of radiation from the new sub-clusters arising from the first generation radiation after \([0]\). I suppose this to be the case since scaling (radiation) in every generation originates from every part of \( S + \mathcal{R} \).

There are three systems to consider. \( S \) results in a constant stream of radiative events. In \( S + \mathcal{R} \) radiation from a radiation volume has one degree of freedom; the cumulative radial length of a radiation cone increases linearly. \( L \) plays a role in the concepts to follow as a measuring stick for distances in \( S + \mathcal{R} \). \( \mathcal{R} \) is a three dimensional space that receives radiation, but itself does not radiate. \( S \) and \( \mathcal{R} \) reside in \( S + \mathcal{R} \). A radiation volume is not wholly in \( \mathcal{R} \) because \( \mathcal{R} \) itself has no radiation, and is not wholly in \( S \) because \( S \) is only radiation without volume.

Some of these provisional ideas probably are not necessary for the mathematics leading to the 4/3 theorems developed below. The physics of how all this might occur plays no role in this section; only the mathematical machinery for deriving the degrees of freedom of a volume of radiation in \( S + \mathcal{R} \) relative to a corresponding volume in abstract three dimensional space \( \mathcal{R} \) per radiation pulse matters.

A radial distance in \( S + \mathcal{R} \) is of necessity (it may be argued) a radial distance in \( \mathcal{R} \), since the spatial and dimensional characteristics of \( S + \mathcal{R} \) are aspects of, or inherited by, \( \mathcal{R} \). Length, area and volume are concepts that conceptually reside in three dimensional space; area and volume can be defined relative to length. We also rely on the derivation of 4/3 scaling of radiation cone volume increments. Radiation cone volume increments are in \( S + \mathcal{R} \). Radial space lengths that are of length \( \ell \) are measuring sticks for \( \mathcal{R} \)’s reference frame.

It may not matter, but it seems more logically sound to describe a radial radiation length \( L \) as being in the \( S + \mathcal{R} \) reference frame, rather than in the \( S \) reference frame. Perhaps it is best to think of \( \ell \) as the space or \( \mathcal{R} \) measuring stick and \( L \) as the radiation or \( S + \mathcal{R} \) measuring stick.

What distinguishes \( S + \mathcal{R} \) and \( \mathcal{R} \) is that radial lengths are differently measured by \( L \) and \( \ell \), and that lengths are related by \( \ell = (4/3)L \). Or we may think of \( \ell \) and \( L \) as different lengths used to measure, indirectly, the same amount of energy.

Designate as \( \ell \) the increment of length in \( \mathcal{R} \)’s reference frame that corresponds to \( L \) in \( (S + \mathcal{R}) \)’s reference frame.

The fraction \( \phi \) formed by the ratio of \( \ell \) to \( L \), to be determined, is:

\[
\phi = \frac{\ell_k}{L_k}
\]

Equation \( (87) \) is a consequence of the correspondence of lengths between \( S + \mathcal{R} \) and \( \mathcal{R} \), perhaps better described as the correspondence of lengths between \( V_k \) and \( \Theta_k \). From equation \( (87) \)

\[
L_k = \frac{\ell_k}{\phi}
\]

The dark energy question: This article’s modelling of dark energy reduces to: what is \( \phi \)?

2. Degrees of freedom of a scale factor

The idea of degrees of freedom arises in relation to three dimensional space. A particle has 3 degrees of freedom of motion in a 3 dimensional space. Implicit in this usage is the idea that the degrees of freedom are independent of each other, related to the orthogonality of the three (Cartesian) axes that can be used model three dimensional space.

Also implicit in this usage is the idea that degrees of freedom of a system is relative to degrees of freedom along a single axis or line: an area for example affords two degrees of freedom because points within it can be located by two orthogonal (linearly independent) axes. If there are two particles each with 3 degrees of freedom we say that the 2 particle system located in 3 dimensional space has \( 2 \times 3 = 6 \) degrees of freedom, and so on. These observations are also considered in section \( V E 3 \).

Given the conventional deployment of the term ‘degrees of freedom’, to characterize scaling as exhibiting ‘degrees of freedom of a scale factor’ is a generalization. Adopting this extended or generalized usage is helpful in modelling and solving various scaling problems. That does not require that degrees of freedom of a scale factor be a faithful description of reality, only that as a mathematical technique its mathematical relationships mirror physical relationships. This section explores degrees of freedom of a scale factor in more detail than does The Overview above.

By postulate, Receipt (or space) \( \mathcal{R} \) has three dimensions. A radiation volume increment \( V_k \) in \( S + \mathcal{R} \) produces a corresponding spatial sphere \( \Theta_k \) in \( \mathcal{R} \). Spheres scale in relation to each other isotropically along their three orthogonal axes. A sphere is used to model space in relation to scaling because a sphere is isotropic at all scales and space has been measured to be isotropic to one part in 100,000, very nearly isotropic (Fixsen 1996). A sphere is therefore used to model space in the context of scaling because it closely models observed attributes of space.

With each pulse of radiation that increases radial radiation distance (the radial length of a radiation cone \( G_k \) by \( L \), the radius \( d(\Theta_k) \) of a spatial sphere \( G_k \) corresponding to increases by a radial increment \( \ell = \phi L \).

Consider \( \ell \) in its setting in \( \mathcal{R} \). Let \( \Theta_{k+1} = \theta_k \Theta_k > \Theta_k \). The scale factor \( \theta_k > 1 \) since radiation distance cumulatively growing by radial increments of \( L \) causes the spatial volume corresponding to a radiation cone volume to grow. Denote a scale factor \( \beta_k \) for \( \ell \) such that \( \ell_{k+1} = \beta_k \ell_k = \ell \) for all \( k \). Here \( \beta_k \) is the scale factor for the radial increment \( \ell \) which is constant, and so \( \beta_k = 1 \). But \( \beta_k \) relative to the scale factor \( \theta_k \) for the entire sphere (not just its incremental increase in volume) changes as \( k \) changes; since the cumulative spatial volume \( \Theta_k \) grows with each scaling, \( \ell \) relative to distance \( d(\Theta_k) \) from the center of \( \Theta_k \) becomes a smaller proportion of \( d(\Theta_k) \) with each
scaling:

\[ \Theta_{k+1} = \frac{4}{3} \pi (\ell_{k+1})^3 \]
\[ = \frac{4}{3} \pi (\beta_k \ell_k)^3 \]
\[ = \frac{4}{3} \pi (\beta_k^3 (\ell_k)^3) \]
\[ = (\beta_k)^3 \left( \frac{4}{3} \pi (\ell_k)^3 \right) \]
\[ = \theta_k \Theta_k. \]

Hence, \((\beta_k)^3 = \theta_k\) and \(\beta_k = (\theta_k)^{1/3}\). The same relationship, \(\beta_k = (\theta_k)^{1/3}\) that applies to the relationship between a length in a sphere, applies when the scaling of an increment of any spatial volume is compared to the scaling of an increment of spatial length \(\ell\). Both reflect \(\ell\) having one dimension and space having 3 dimensions relative to a spatial length.

Think of \(\beta\) in the preceding example as the base of a scale factor, terminology analogous to the idea of the base of a logarithm. The exponent of the base of a scale factor is its degrees of freedom. For example, \(\beta\) is the base of the scale factor \(\beta^k\). To measure scaling we take the exponent of the ratio of scale factors with a common base, as for example in

\[ \text{Deg}(\beta^m \ell / \beta^n \ell) = \log (\beta^{m-n}) \text{ deg} = (m-n) \text{ deg}. \]  

(90)

The dimensionality of \(R\) can compute degrees of freedom to the base \(\beta\) of the scale factor \(\beta^m\) of \(\ell\), for example, if we describe a volume \(\Theta\) as \(\beta^n \ell^3\). The degrees of freedom of the scale factor of a length in a three dimensional volume compared to the scale factor of that volume is 1/3, which we obtain by comparing \(\beta_k\) to \((\beta_k)^3\).

A space volume \(\Theta_k\) incrementally adding a constant \(\ell\) to its radius is a different circumstance than the situation described in the Overview, in which the side of a cube increases by a constant scale factor \(\beta > 1\); in that case the scaling of the cube side from one generation to the next is always \(\beta\), and the scaling of \(\Theta_k\) is always by \(\theta = \beta^3\). But the \(\text{Deg}\) relationship between space volume and a length in space is the same regardless of whether \(\ell\) is constant or scales by \(\beta > 1\):

\[ \text{Deg}'(\ell_k) = (1/3) \text{Deg}'(\Theta_k). \]

3. Degrees of freedom notation

By May 2012, it was apparent that degrees of freedom played an important role in mathematically modelling 3/4 metabolic scaling. With this in mind, infer that notation that compresses ideas relating to degrees of freedom of a scale factor makes it easier to isolate degrees of freedom concepts relevant to the 4/3 scaling of homogeneous energy distribution systems. With easier to manipulate degrees of freedom concepts, one can better focus on the relationship of those concepts. An appropriate notation will free the observer and reader to focus on salient principles, without having to translate similar words and expressions into understandable conceptual chunks repeatedly.

The 4/3 Degrees of Freedom Theorem implies that a fractional 4/3 degrees of freedom is a sensible notion. Fractional degrees of freedom arises due to the comparing of scale factors. Degrees of freedom of a motion within a length, area and volume are described with whole numbers. A particle in a volume has three degrees of freedom relative to its motion along a length. If in space each gas molecule has 3 degrees of freedom of motion, then \(n\) gas molecules have \(3n\) degrees of freedom.

We are not as interested in degrees of freedom within one, two or three dimensions as in the degrees of freedom of a scale factor relative to another scale factor. As mentioned above, the relativity of degrees of freedom is implicit in the usual definition of dimension: a volume has three dimensions relative to a length.

Before v. 15 of this article, the \(\text{Deg}\) notation went through various permutations. To devise an appropriate notation, observe that to compare systems that scale, the ratio of scale factors to a common base is effective when the length, area or volume belong to the same system. The comparison of energy densities involving dark energy led to the idea of a \(\text{deg}\) unit, in order to be able to apply dimensional analysis to calculations.

Let \(\text{Deg}\) be a function that compares degrees of freedom of a scale factor for different things; the output is the exponent of the base of the scale factor. The \(\text{deg}\) unit applies to the output, to track the appropriateness of the ratio. There are different contexts to which \(\text{Deg}\) applies.

- Compare the \(\text{Deg}\) relationships between length, area and volume; the context is dimensionality, an aspect of space or \(R\).
- Compare how a part of a radiation distribution system scales when it grows; this is a radiation context.
- Compare how a radiation cone volume increment scales to how space scales. The ratio thus obtained, unlike the preceding two cases, can be dimensionless since \(\text{deg}\) appears in both numerator and denominator.

The \(\text{Deg}\) function’s use is in evaluating ratios. The \(\text{deg}\) unit helps to illustrate that. So we have

\[ \text{Deg}(X_{k+1}/X_k) = \text{Deg}(\beta^k X_1/X_1) \]
\[ = \log (\beta^k) \text{ deg}. \]  

(91)

On the other hand if we wrote

\[ \text{Deg}(X_{k+1}) = \text{Deg}(\beta^k X_1) \]
\[ = \log (\beta^k) \text{ deg} \]  

(92)

we would have \(X_1\) somehow vanishing from the argument of \(\text{Deg}\) on the right side in equation (92). The usage in equation (91) is superior to that of equation (92). The notation \(\text{deg}\) helps ensure that an equation obeys the rules of dimensional analysis.

We can define a short form notation \(\text{Deg}'\) for comparing the
$k^{th}$ to the $(k + 1)^{th}$ generation of a scaled system:

$$\text{Deg}^\ast(X_k) \equiv \text{Deg}(X_{k+1}/X_k) = \text{Deg}(\beta X_k/X_k) = \log_b(\beta) \text{ deg} = 1 \text{ deg.} \tag{93}$$

To evaluate the degrees of freedom of a system relative to another let the first scale while the other does not. Then take the logarithm, with the logarithm’s base a common scale factor, of the fraction formed of the first over the second.

For example, find the effect of a cube $\Theta_k$ scaling by $\theta_k > 1$ relative to a stationary version of itself.

$$\text{Deg} \left( \frac{\Theta_{k+1}}{\Theta_k} \right) = \text{Deg} \left( \frac{\theta_k \Theta_k}{\Theta_k} \right) = \log_b(\theta_k) \text{ deg} = 1 \text{ deg}. \tag{94}$$

Compare degrees of freedom for a cube $\Theta_k$ scaling relative to the side $\ell_k$ which scales by $\beta_k$. As in equation (93), $\text{Deg}^\ast(\Theta_k) = 3 \text{Deg}^\ast(\ell_k)$, and $\text{Deg}^\ast(\ell_k) = (1/3) \text{Deg}^\ast(\Theta_k)$.

Suppose that in $\mathbb{R}$ the area $A_k$ of the side of a cube $\Theta_k$ scales by $a > 1$. All relative to $\ell_k$, for $\Theta_k$

$$\text{Deg}^\ast(\Theta_k) = \text{Deg}^\ast(A_k) + \text{Deg}^\ast(\ell_k) = 3 \text{Deg}^\ast(\ell_k) = 3 \text{ deg.} \tag{95}$$

and $\text{Deg}^\ast(A_k) = (2/3) \text{Deg}^\ast(\Theta_k)$.

A system can have a fractional number of degrees of freedom relative to another system. In a growth context, a radiation cone scales differently relative to uniformly scaling volumes such as cubes and spheres, as section V E 4 shows.

Since we want to be able to compare the scaling of level 1 relative to level $k + 1$, we use the abbreviation $\text{Deg}^{**}$, as in

$$\text{Deg}^{**}(\ell_k) = \text{Deg}(\ell^k/\ell_1) = \log_b(\ell^k) \text{ deg} = k \text{ deg.} \tag{96}$$

which makes explicit the implicit aspect of $\text{Deg}$ that degrees of freedom is determined relative to some other system attribute. It also has the result that the unit involved, such as $\Theta_k$ in the first line of equation (93), naturally cancels out, rather than having to mandate that $\text{Deg}$ is the exponent of a scale factor.

The notation distinguishes between $\text{Deg}(\ell_{k+1}/\ell_k) \equiv \text{Deg}(\ell_k) = 1 \text{ deg}$ and $\text{Deg}(\ell_{k+1}/\ell_1) = \text{Deg}(\ell^k/\ell_1) = k \text{ deg} \equiv \text{Deg}^{**}(\ell_k) = k \text{ deg}$.\mathbf{4. Scaling lengths and volumes in } S + \mathbb{R}

In $S + \mathbb{R}$ designate the $(k + 1)^{th}$ radial increment of length for each pulse of radiation as $L_{k+1} = L_k = L$ for all $k \geq 1$.

In a growing sphere, a given length scales from its center in the same way along each of its three axes. That is not the case for radiation.

In a radiation volume, a length that is radial from the source grows by a constant $L$ from one radiation volume increment to the next while the diameter of the radiation cross-section orthogonal to the radial length increases in size. A uniformly scaling spherical volume cannot be used to model relative axial scaling (the scaling along the three axes) of radiation because scaling is not uniform for each of radiation volume’s three axes in the way that scaling is for a uniformly scaling spherical volume. A uniformly scaling spherical volume can be used to model a homogeneously scaling isotropic space.

Let $r_k$ be the average radius of a radiation cone volume increment $V_k$ between two points on the radial axis that are at distances $kL$ and $(k + 1)L$ from the source. Let the scale factor for $r_k$ be $\beta_k$ and for $V_k$, $\beta_k > 1$ since the cross-sectional area (and diameter, as above noted) of the radiation cone gets bigger as the radial distance from the source increases. The radial radiation length increment $L$ is constant. To discriminate between the scaling up of the radius of the average cross-sectional area of radiation as compared to the constancy of the radial increment $L$, the scaling of radiation, in contrast to a uniformly scaling spherical space, must be modelled using a radiation cone or a volume that permits one to distinguish between the scaling of the cross-sectional area and a length orthogonal to that area.

The radial length of radiation cone $G_k$ is $kL$. Suppose the radius at the end of $G_k$ farthest from its apex is $r$. The radial length increases linearly with $k$. Since a radiation cone $G_k$ is in profile a triangle, the radius of the far end of a radiation cone volume increases in proportion: the radius at the far end of radiation cone is $kr$. Here we ignore the curvature of the base of the far end of a radiation cone.

Since for $G_k$, radial length $kL$ and far end radius $kr$ increase linearly, scaling would not at first instance appear capable of having a portion of a radiation cone volume grow by a power greater than one for volume increments, but it is capable of that; that is the result of $4/3$ power scaling for radiation cone volume increments.

The scaling from one radiation cone volume increment to the next is $(k + 1)/k$. Scaling of a radiation cone volume increment must play a role, because we want to compare the scaling of a volume in $\mathbb{R}$ to the way a radiation cone volume increment scales in $S + \mathbb{R}$ per generation. To derive The 4/3 RDFT it is necessary to compare scaling of radiation volume increment $V_k$, which is $\text{Deg}^\ast(V_k)$, to the scaling of its corresponding space volume $\Theta_k$, which is $\text{Deg}^\ast(\Theta)$. To accomplish that we start by defining a scale factor $s_k$ for an increment of radiation length $L_k = L$.

For the scale factor $s_k$ of $L$ we have: $s_k = (v_k)^{1/3}$. This is an instance of the general case that a scale factor for a length has $1/3$ the degrees of freedom relative to the scale factor for a volume. Since radiation radial length $L$ is constant, $s_k = (v_k)^{1/3}$ implies that as $k$ increases the radiation cone volume increment $V_k$ takes on a skinnier aspect; $L$’s proportion of the width of $V_k$’s cross-section — its average diameter — gets smaller.
Each radiation cone volume increment $V_k$ transmits radiation to a corresponding sphere $\Theta_k$ that is part of $\mathcal{R}$. All the energy of $V_k$ goes into $\Theta_k$ by Postulate \[ \text{(5)} \]

Compare the scale factor $\beta_k$ for $r_k$ to the scale factor $v_k$ for $V_k$, disregarding how radiation length $L_k$ scales relative to $V_k$.

$$V_{k+1} = v_kv_k = \pi(\beta_k r_k)^2 L_k = (\beta_k)^2 \pi(r_k)^2 L = (\beta_k)^2 V_k$$ \hspace{1cm} (97)

Hence, $(\beta_k)^2 = v_k$ and $\beta_k = (v_k)^{1/2}$. As well, if $a_k = (\beta_k)^2$ is the scale factor for the average cross-sectional area $A_k$, then $a_k = v_k$.

It follows that, disregarding how radiation length $L_k$ scales relative to $V_k$:

$$\text{Deg}'(V_k) = \text{Deg}'(A_k),$$ \hspace{1cm} (98)

where $A_k$ is the average cross-sectional area of $V_k$. That is so because $\text{Deg}'(L_k) = 0 \text{ deg} \ (L's \ radiation \ property)$ since $L_k$ is the same for all $k$, and so $a_k = v_k$.

Consider the $\text{Deg}$ relationship of $V_k$ as follows:

$$\text{Deg}'(V_k) = \text{Deg}'(V_{k+1}/V_k) = \text{Deg}(A_{k+1}/A_kL) = \text{Deg}'(A_k).$$ \hspace{1cm} (99)

Equation (99) omits $L$'s scale factor $s_k$ which only plays a role in comparing how length scales to how volume scales. It follows that, for radiation, $\text{Deg}'(A_k) = \text{Deg}'(V_k)$.

The $\text{Deg}'$ relationship for a radiation volume and its average cross-sectional area contrasts with the $\text{Deg}'$ relationship for a uniformly scaling spherical volume in space $\mathcal{R}$ in equation \[ \text{Deg}' \mathcal{R} \text{ (equation 95)}, \] where the scale factor along each radial axis is $\theta^{1/3}$.

For a uniformly scaling spherical volume in space $\mathcal{R}$:

$$\text{Deg}'(\mathcal{A}_k) = (2/3)\text{Deg}'(\Theta_k),$$ \hspace{1cm} (100)

unlike the radiation volume increment relationship in \[ \text{(98)}. \]

Radial length for radiation scales by 1/3 the power of the scaling of a radiation volume and, since radiation is at a steady $L$ per scaling, radial length relative to the width of the average cross-section of a radiation cone volume increment is getting smaller. This, as the sections to follow show, results in spatial radial length $\ell$ being $4/3$ as long as the corresponding radial radiation length $L$ in $S + \mathcal{R}$ for the same amount of energy.

F. The average incremental scale factor of a cumulation

Relative to the entire radiation cone $G_k$, a scale factor for cumulative radial length $kL$ resulting in increased radial length $[(k+1)/k]L$ would be $(k+1)/k = \sigma$. Here distinguish the scale factor $s$ for radial length $L$ from one radiation cone volume increment to the next from the scale factor for the cumulative radial length of $G_k$.

Over an interval of $k$ scalings, with the interval beginning from about $k - 1/2$ to about $k + 1/2$ scalings, it will be helpful to think of the base of $L$'s average scale factor $\sigma_{av}$ over that interval, with respect to a portion of the radial length of the radiation cone $G_k$.

To characterize the linearity of $G_k$'s cumulative radial length $kL$ as a scaling relationship seems odd. This odd approach is required because there is a scaling relationship per generation between radiation radial length $L_k$ and $V_k$ that is an outcome of the way $G_k$'s cumulative radial length $kL$ grows.

$L_k$ is added linearly to result in a cumulative radial length for $G_k$, but has a scaling relationship to a radiation cone volume increment $V_k$. The scaling relationship between $L_k$'s scale factor $s_k$ and $V_k$'s scale factor $v_k$, namely $s_k = (v_k)^{1/3}$, indirectly incorporates the linearity of $G_k$'s cumulating radial length $kL$, which is to say, reflects radial motion.

Combined, $\text{Deg}'(L_k) = 0 \text{ deg}$ and $s_k = (v_k)^{1/3}$ compress a lot of information about radiation.

The cross-sectional area of the radiation cone volume increment $V_k$ scales by the square, $(\beta_k)^2 = [(k + 1)/k]^2$, of the base of the scale factor $\beta$ of its radius $r_k$. $V_k$'s scale factor $v_k$ scales in proportion to a cubed scale factor, $[(k + 1)/k]^3$, where the base of the scale factor is the scale factor for a length. $v_k$ changes from generation to generation, from pulse to pulse. Since the scale factor per generation changes with $k$, it is necessary to add the subscript $k$ to the scale factors.

Suppose that for some system a cumulative length or amount of energy $D_k$ scales by a scale factor $\sigma_k$ that might vary from one scaling to scaling next succeeding, such that $D_{k+1} = \sigma_k D_k$.

Take $D_{av}$ to be the mean cumulative length (or energy) over the interval from $k - (1/2)k + 1$ to $k + (1/2)k$, that is, over an interval of $k$ scalings. Find the average scale factor $\sigma_{av}$ for the interval of $k$ scalings. An incremental scale factor, that is from the $i^{th}$ scaling to the $(i + 1)^{st}$ scaling is $(i + 1)/i$, and the average scale factor over the interval of $k$ scalings is

$$\sigma_{av} = \frac{\Sigma(i + 1)^n/i}{k}$$ \hspace{1cm} (101)

chosen so that the average scale factor $\sigma_{av} = (1 + k)/k = 1 + 1/k$. Then we can say

$$D_{k+1} = (\sigma_{av})^k D_{av} = \left(1 + \frac{1}{k}\right)^k D_{av}$$ \hspace{1cm} (102)

over the interval of $k$ scalings. This observation connects the natural logarithm to scaling as discussed in section \[ \text{XTE} \]

1. Cumulative length and cumulative scaling

In the $S + \mathcal{R}$ reference frame the distance of the end of the cumulative radial length of the radiation cone $G_k$ from its
source is, recalling that over an interval the average scale factor is $\sigma_{av}$, 

$$d(G_k) = (\sigma_{av})^k L_1 + \ldots + (\sigma_{av})^k L_1 = kL_1$$

$$= \log_{\sigma_{av}}((\sigma_{av})^k) L_1$$

$$= \text{Deg}(G_{k+1}/G_1) L_1$$

$$= \text{Deg}^\ast(G_k) L_1$$

$$= kL_1$$

(103)

using $d$ as a measure of the cumulation of any parameter (here suppressing $\text{deg}$); here the parameter is length. This measure of a cumulation works where the scale factor, $\sigma_{av}$ in this instance, is constant or where an average scale factor over an interval is used. In other words, there has to be a relationship leading to $\log_{\sigma_{av}}(\sigma^k) = k$.

For the cumulative radius of a corresponding spherical spatial volume $\Theta_k$

$$d(\Theta_k) = (\beta_{av})^k \ell_1 + \ldots + (\beta_{av})^k \ell_1 = k\ell_1$$

$$= \log_{\beta_{av}}((\beta_{av})^k) \ell_1$$

$$= \text{Deg}^\ast(\Theta_k) \ell_1.$$ 

(104)

Over an interval with $k$ scalings, the ratio of the cumulative distances in $\mathcal{R}$ compared to the corresponding distance in $S + \mathcal{R}$ is

$$\frac{k\ell_1}{kL_1} = \frac{\ell_1}{L_1} = \phi$$

(105)

or for constant $\ell$ and $L$

$$\frac{k\ell}{kL} = \frac{\ell}{L} = \phi.$$  

(106)

having regard for the definition of $\phi$ in equation (107).

Thus $\phi$ measures the amount by which a distance in space $\mathcal{R}$ scales radially (with hindsight, stretches) — that is, in every direction — relative to radiation distance in $S + \mathcal{R}$.

2. Cumulative scaling and entropy

In equation (104), suppose that every length $\ell$ is proportional to an amount of energy of $\ell$ units, and that the equation expresses a relationship between the cumulative energy $d(\Theta_k)$ and the energy of its $k$ components each having $\ell$ energy units or a proportional equivalent.

Now take equation (104) and divide each of the equivalent cumulative energies $d(\Theta_k)$ and $\log_{\beta_{av}}((\beta_{av})^k) \ell$ by $\ell$:

$$\frac{d(\Theta_k)}{\ell} = \log_{\beta_{av}}((\beta_{av})^k).$$

(107)

The right side of equation (107) is the log formula for entropy. The left side, continuing to treat the units as units of energy, is a cumulative energy divided by the unit of energy being scaled. Think of $d(\Theta_k)$ as the $dQ$ of thermodynamics, and think of $\ell$ as proportional to an amount of heat $T$ represented by one degree Kelvin in the particular system being considered. Then generalizing, based on these observations about equation (107), one can say entropy is the number of degrees of freedom of an amount of energy relative to a reference amount of energy that is being scaled.

The observations in this section pertain to Stefan’s Law discussed in section [VIII.D].

G. Terminology of $\Theta_k$

In this article, $\Theta_k$ represents a spherical volume in $\mathcal{R}$. In relation to the radiation volume increment $V_k$, $\Theta_k$ represents the spherical spatial volume corresponding to $V_k$, or which receives $V_k$’s energy. When used in relation to the entire radiation cone $G_k$, $\Theta_k$ represents the spherical spatial volume corresponding to $G_k$. When we compare $\Theta_{k+1}$ to $\Theta_k$, the context should make it obvious whether we are comparing $\Theta_k$ to a radiation cone $G_k$ or to a radiation cone volume increment $V_k$.

When each radiation cone volume increment $V_k$ corresponds to a spherical spatial volume $\Theta_k$, it is implicit that space grows everywhere; every radiation cone volume increment $V_k$ of every radiation cone creates a corresponding $\Theta_k$.

H. Radiation and space, $\text{Deg}$ and length

To compare the degrees of freedom of the base of the scale factor for a radiation cone volume increment $V_k$ in $S + \mathcal{R}$ to the degrees of freedom of the base of the scale factor for a corresponding spherical spatial volume $\Theta_k$ in $\mathcal{R}$, it is necessary to model a radiation volume as a radiation cone.

One might guess: To compare length in $S + \mathcal{R}$ to length in $\mathcal{R}$, as opposed to comparing scaling, it is necessary to model a radiation volume as a sphere since the length of the radius from a point is then the salient homogeneous attribute; per scaling, the distance from the center changes by the same (constant) length $L$. This guess is verified in section [VIII.D].

The reason why helps reveal some aspects of homogeneity in relation to scaling and degrees of freedom on one hand, and length on the other hand.

First compare radiation and uniform space in relation to length. Radiation adds a constant unchanging radial length $L$ to the cumulative radial length already existing. For a radiation cone, the radius of the average cross-sectional area of a radiation cone volume increment increases, unlike the unchanging radial length increments.

The homogeneity or constancy of radiation manifests itself as a constant radial length increment. In contrast, with respect to length, the homogeneity of space manifests itself as a constant rate of radial increase in all directions. For radiation scaling in the radial direction is distinguished from the scaling and growth of the two axes — of the cross-section — orthogonal to the radial direction.

To model a comparison between how radiation and space grow differently, therefore use a cone for one and a corre-
sponding sphere for the other. Two ways to proceed are as follows. Have a radiation cone with constant radial radiation length $L$ and compare the scaling of the uniformly scaling sphere to the scaling of the radiation cone volume increments. Another way is to have a radiation sphere’s radius growing by an unchanging length being transmitted into a spatial cone, where the length $\ell$ will be $4/3$ that of $L$.

Using the different shapes of a cone and a sphere affords a way to model the different way radiation changes, either from the point of view of scaling or of length, compared to space. Cones and spheres do not therefore necessarily provide an intrinsic difference between radiation and space; they just provide a way to model the difference in radiation radial length compared to a corresponding space radial length.

I.  

Deg as a conceptual reference frame

Observations about the exponent of the base of a scale factor in relation to cone and sphere volumes helped derive the 4/3 RDFT, as set out in The Overview.

The cumulative length of a radiation cone is $kL$, and the cumulative length scales as $(k + 1)/k$ at the $k^{th}$ level. This compels the use of subscripts $k$ for the scale factor $v_k$ for $V_k$ and the scale factor $a_k$ for $A_k$. It would be conceptually and notationally simpler to drop the subscripts for the scale factors. It may be possible to mathematically justify such an approach. Having used geometry and algebra as a scaffolding to arrive at The 4/3 RDFT, it may be possible to proceed as if the scaffolding is no longer necessary.

Simplified scaling would help in the proof of the natural logarithm theorem based on the limit of scaled intervals, as outlined in XI E.

J. Using the algebra of degrees of freedom

Ideas in this section will be deployed in the first proof of The 4/3 Ratio of Degrees of Freedom Theorem (The 4/3 RDFT) in the next section. The 4/3 Ratio of Degrees of Freedom Theorem leads to The 4/3 Ratio of Lengths Theorem, which in turn suggests an explanation for astronomical observations related to dark energy. The 4/3 RDFT itself suggests an explanation of vacuum pressure or dark energy.

VI. THE 4/3 RATIO OF DEGREES OF FREEDOM THEOREM (THE 4/3 RDFT)

In section VIA the 4/3 RDFT is first proved in section VIA using degrees of freedom observations. The proofs that follow it are treated as proofs or examples of particular instances of the general proposition that The 4/3 RDFT represents.

The 4/3 RDFT: A radiation volume in $S + \mathcal{R}$ scales homogeneously and isotropically transmitting degrees of freedom or energy into Receipt $\mathcal{R}$. Relative to their respective scale factors, for radiation volume $G_k$ in $S + \mathcal{R}$ and its corresponding spatial volume $\Theta_k$ in $\mathcal{R}$, $\deg'(G_k) = (4/3)\deg'(\Theta_k)$ and it follows, $\deg''(G_k) = (4/3)\deg''(\Theta_k)$.

A. Proof based on section VIA

Proof. $\deg''(\Theta_k) = 3\deg''(v_k)$ in $\mathcal{R}$ by Postulate II

Let $V_k$ be a radiation cone volume increment, part of a radiation cone $G_k$ in $S + \mathcal{R}$ growing radially by $L$ per pulse. $V_k$ radiates into a spherical spatial volume $\Theta_k$ in $\mathcal{R}$. Per pulse, $\Theta_k$ grows radially by $\ell$ per pulse when $G_k$ grows by $L$.

$V_k$ scales by $v_k$ without regard to how $L_k$ scales relative to $V_k$. $V_k$’s average cross-sectional area $a_k$ scales by $a_k = (\beta_k)^2$ where $\beta_k$ is the base of the scale factor of the average radius $r_k$ of $V_k$. $\Theta_k$ scales by $\theta_k$ as in section VIA. By stipulating that we can consider the scaling of $V_k$ without regard for how $L_k$ scales relative to $V_k$, we in effect require that we ignore radiation motion in relation to $V_k$, or that we ignore the steady increase in the cumulative radial length of $G_k$.

The relationship between $v_k$ and $a_k$ without regard to how $L_k$ scales relative to $V_k$ is revealed by

$$
\deg\left(\frac{V_{k+1}}{V_k}\right) = v_k = \deg\left(\frac{A_{k+1}L}{A_kL}\right) = \deg\left(\frac{a_kA_kL}{A_kL}\right) = a_k,
$$

in which case $v_k = a_k$.

To calculate how $V_{k+1}$ scales relative to $V_k$ in $S + \mathcal{R}$ taking into account how $L_k$ scales relative to $V_k$, compare scale factors relative to $v_k$:

$$
\deg\left(\frac{V_{k+1}}{V_k}\right) = \deg\left(\frac{a_kA_k 	imes s_kL_k}{A_kL_k}\right) = \deg(a_k 	imes s_k) = \deg(v_k 	imes (v_k)^{4/3}) = \deg\left((v_k)^{4/3}\right) = (4/3)\deg.
$$

On the other hand,

$$
\deg\left(\frac{\Theta_{k+1}}{\Theta_k}\right) = \deg\left(\frac{\theta_k\Theta_k}{\Theta_k}\right) = \log_{\theta_k}(\theta_k)\deg = 1\deg
$$

so relative to their respective scale factors, $\deg(V_k) = (4/3)\deg(\Theta_k)$. Put another way

$$
\frac{\deg(V_{k+1}/V_k)}{\deg(\Theta_{k+1}/\Theta_k)} = \frac{(4/3)\deg}{1\deg} = \frac{4}{3} : 1.
$$

Scaling by a 4/3 power applies to the scale factor of each volume increment $V_k$ of a radiation cone relative to the corresponding $\Theta_k$ in the Receipt. \qed
The 4/3 RDFT may also be described this way: the fractal dimension of $G_k$ in $S + R_k$ relative to a corresponding $G_k$ in $R$ is 4/3. This characterization of The 4/3 RDFT has a connection to Brownian motion discussed in section VI C below.

Assuming that $V_k$ as a distribution system supplies energy to itself is wrong, because the $V_k$ are the means by which the radiated quantity is distributed. They are not the radiation source; they form a radiation distribution system.

The 4/3 RDFT does not apply to the whole radiation cone since the cumulative length $kL$ of a radiation cone $G_k$ scales by $(k + 1)/k$ from one generation to the next. Cumulative length of a radiation cone scales differently than the way $L_{k+1}$ scales relative to $L_k$. To arrive at The 4/3 RDFT it is critical to have $Deg'(L_k) = 0$ which requires that $L_k = L$ for all $k$.

Discriminating between the radiation and volume properties of $L$ is required. The radiation property of $L_k$ leads to $Deg'(L_k) = 0$. The volume property leads to $Deg'(L_k) = (1/3)Deg'(V_k)$. If the radiation property and the volume properties are not distinguished, then one would have both $V_{k+1} = v_1 V_k$ (without regard to the scaling of $L_k$ relative to volume $V_k$) and also $V_{k+1} = v_1^{3/4} V_k$ (which does take into account the scaling of $L_k$ relative to the volume of $V_k$), which cannot both be true in the same context or conceptual reference frame. The mathematics requires two contexts or conceptual reference frames.

One of nature’s subtleties in relation to radiation is that, for $L_k$’s volume property, $L_k$ scales relative to a radiation cone volume increment $V_k$ but $L_k$ itself is constant. If person $A$ is an unchanging 5 feet tall, and person $B$ is increasing in height, $A$ is scaling relative to $B$; $A$’s height is becoming a smaller fraction of $B$’s height. $L_k$ is constant in reference frame $S + R_k$ but scales down by a 1/3 power relative to the increase in volume of a radiation cone volume increment from $V_k$ to $V_{k+1}$.

$G_k$’s radial length $kL_k$ increasing in a linear way is another subtlety; cumulative radial length growing linearly by a constant added increment $L_k = L$ per radiation event seems to preclude scaling of $L_k$ having a role, but scaling of $L_k$ relative to volume does play a role in measuring the scaling of radiation cone volume increments.

By $L_k = L$ not scaling — being constant — radially in the case of a radiation cone, compared to radius uniformly scaling in the case of a spherical volume in space $R_k$, might one expect $\Theta$’s degrees of freedom to be bigger than $V_k$? Section VIII shows that it is the radial length in $\Theta$ that is larger than the corresponding radial length in $V_k$, that is, $L_k = (4/3)L_k$. The lesser scaling in $S + R_k$ — its truncated scaling in the radial direction compared to the scaling of the radius of its cross-sectional area — compared to scaling in $R_k$ leads to space distance stretching by 4/3 relative to radiation distance.

This degrees of freedom proof has the advantage of involving a length corresponding to a scale factor $s$ of the radiation radial length increment $L_k$. This aspect is helpful later in this article in connecting the arrow of time to the expansion of the universe.

### B. Proof based on dimensions

**Proof.** A cumulative number of radiation events, like cumulative time, is linear, and so $Deg(S) = 1$ deg. For space $R_k$, $Deg(R_k) = 3$ deg. Therefore

$$
\frac{Deg(S + R_k)}{Deg(R_k)} = \frac{1\text{ deg} + 3\text{ deg}}{3\text{ deg}} = \frac{4\text{ deg}}{3\text{ deg}} = \frac{4/3\text{ deg}}{1\text{ deg}}
$$

□

The necessary explicit assumption implicit in the foregoing short proof is that distinct reference frames $S$ and $R_k$ exist, and that $S$ appears to us as part of $S + R_k$.

From The 4/3 RDFT and the assumption that $Deg(R_k) = 3$ deg, $Deg(S + R_k) = (4/3)Deg(R_k) = (4/3) \times 3\text{ deg} = 4\text{ deg}$; if $R_k$ has 3 dimensions, $S + R_k$ has 4 dimensions. The 4 dimensional aspect of $S + R_k$ is more simply revealed by the two systems approach than the two reference frames approach.

The 4/3 RDFT is implicit in the perception that time is one dimensional and space three dimensional, and thus in the four dimensional characterization of space-time.

### C. Proof based on the 4/3 fractal dimension of Brownian motion

Benoit Mandelbrot in his 2012 memoir (p. 247, Mandelbrot 2012) mentions “My 4/3 conjecture about Brownian motion was chosen in 1998 [for the Mittag-Leffler Institute], when its difficulty had become obvious.”

In 2001, Lawler, Schramm and Werner (Lawler 2001) found that the fractal dimension of Brownian motion is 4/3. Their 2001 article sketches proofs that appeared in a sequence of four earlier articles which involved the mathematics of stochastic Löwner evolution. The mathematics is lengthy, intricate and specialized — A Guide to Stochastic Löwner Evolution and its Applications by Kager and Wouter (Kager 2004) is over 75 pages long — and entirely different than the algebra of degrees of freedom used to prove The 4/3 RDFT.

Lawler (p. 11, Lawler 2004) sets out the assumptions that apply to the stochastic Löwner (or Loewner) evolution process as including

- Independent increments.
- Identically distributed increments.
- Symmetry about the origin.

These assumptions, if we consider average scale factors as corresponding to the scale factors involved in The 4/3 RDFT, match those leading to the proof of The 4/3 RDFT using the relative degrees of freedom of scale factors. Lawler, Schramm and Werner’s result is a particular instance of the general case.
The 4/3 fractal dimension of Brownian motion taken as a specific instance of the general case provides another proof of The 4/3 RDFT.

D. Proof based on Stefan’s Law

Stefan’s Law (also known as the Stefan–Boltzmann Law) concerns black body energy radiation. In black body radiation both the radiation energy distribution system \( S + \mathcal{R} \) and the Receive \( \mathcal{R} \) are contained within the same cavity, which reaches an equilibrium temperature. The cavity appears to represent one system, but in fact two distinct reference frames apply.

An intermediate step in the proof of Stefan’s Law, first derived by Boltzmann in 1884 (Boltzmann 1884), is described by Planck (p. 62, Planck 1914) as (using Planck’s notation)

$$\left( \frac{\partial S}{\partial V} \right)_T = \frac{4u}{3T} \quad (113)$$

(and see p. 742, Allen & Maxwell 1939). In equation (113), \( \partial S \) represents change in entropy, \( \partial V \) represents change in volume, and the subscript \( T \) on the left side of the equation indicates that the equation applies at a given temperature. In equation (113), the expression \( u/T \) gives the number of degrees of freedom in volume density of radiation \( u \) (p. 61, Planck 1914) relative to a degree of absolute temperature.

A single degree \( T \) of absolute temperature is proportional to the mean math length \( \mu \) (not radiation volume density \( u \) in the numerator in equation (113) on the right hand side) as a scale factor of the total energy in a system. The cumulative temperature of a system is proportional to the degrees of freedom in the system, as reflected in the thermodynamic equation,

$$\frac{Q}{T} = S \quad (114)$$

for \( Q \) a quantity of heat and \( S \) entropy.

If the temperature is \( nT = \mu^2T \), then \( \log(\mu^2) = k \). The remarks in section V.F.2 apply. Equation (113) says that in a cavity, when a radiation volume \( G_t \) grows, the change in its entropy — degrees of freedom — (the \( \partial S \)) relative to the change in volume (the \( \partial V \)) scales by 4/3 of \( \mathcal{R} \)’s energy density \( u \) relative to energy’s scale factor \( T \), another instance of The 4/3 RDFT.

On average, within the cavity holding the black body radiation, there is homogeneous radiation or scaling. The setting is analogous to that described by the postulates in section V.A. Proof of Stefan’s Law is proof of a particular instance of The 4/3 RDFT.

E. Proof by comparing volume scale factors for \( S + \mathcal{R} \) and \( \mathcal{R} \)

All of the energy transmitted by a radiation cone \( V_k \) is received by a corresponding space \( \Theta_k \). Suppose energy is proportional to volume, and treat volume units as energy units. On a scaling of \( V_k \) and \( \Theta_k \), from section V it follows that \((v_k)^{4/3}V_k = \theta_k\Theta_k^5\). Since \( V_k = \Theta_k \) energy units, \( v_k \) cannot be equal to \( \theta_k \). This justifies choosing \( V_k \)’s scale factor to be represented by a symbol different from \( \Theta_k \)’s in section V. It also must be that for the respective scale factors \((v_k)^{4/3} = \theta_k\).

That implies that \( Deg^*(V_k) = (4/3)Deg^*(\Theta_k) \).

F. Proving The 4/3 RDFT with a result in cosmology

This section relies on some cosmology. In Wang’s notation (p. 17, Wang 2010; p. 64, Ryden 2003), the cosmic scale factor stretching distance is \( a(t) \), which in this article is denoted \( \phi \). (Treatment of \( a \) as a function of time is at odds with The 4/3 RDFT which assumes that the scale factor ratio for \( S + \mathcal{R} \) compared to \( \mathcal{R} \) is 4/3 \( deg : 1 \) degree at all scales. Let’s pass over that difference for now.)

Wang writes (her notation) that cosmology reasons that for the energy density of matter, \( \rho_m \),

$$\rho_m \propto \frac{1}{a^4} \quad (115)$$

and for the energy density of radiation, \( \rho_r \),

$$\rho_r \propto \frac{1}{a^4} \quad (116)$$

Identify radiation with the scaling of \( S \) in \( S + \mathcal{R} \) and matter as being included in space \( \mathcal{R} \). Suppose the same energy \( E = \rho_r \times V_k = \rho_m \times \Theta_k \).

From the Energy density corollary in section II.E in The Overview, equations (115) and (116) imply The 4/3 RDFT. Note that the degrees of freedom of the distance scale factor for radiation in \( S \) compared to the degrees of freedom of the distance scale factor for \( \mathcal{R} \) is 4 \( deg : 3 \) deg, whereas for \( S + \mathcal{R} \) compared to \( \mathcal{R} \) it is 4/3 \( deg : 1 \) deg. The two ratios appear equivalent which adds to the difficulty of deriving the appropriate mathematics.

G. A possible proof based on a MIMO network

In a multiple input multiple output (MIMO) antenna system (such as a cell phone network), if the number of antennas at each node are \( M > 1 \), if ‘channel’ matrices are non-degenerate then the precise degrees of freedom \( \eta_k = \frac{4}{3}M \) (Jafar and Shamai 2008). The set up Jafar and Shamai describe seems analogous to homogeneous radiation from an antenna system to recipient antennas. Their proof is entirely dissimilar to those above. If it is a special instance of The 4/3 RDFT, it is another proof of it. Is it a special instance?

H. Network degrees of freedom and energy rate use

Let \( \xi = r(\Theta_{k+1})/r(\Theta_k) \) be a ratio of rates \( r(\Theta_k) \) of energy use by \( \Theta_k \). We can consider \( \xi \) to be the base of a scale factor \( \xi^\phi \) such that \( r(\Theta_{k+1}) = \xi^\phi r(\Theta_k) \). Then

$$r(\Theta_{k+1}) = \xi^\phi r(\Theta_k) \quad (117)$$
and $\text{Deg}^*(r(\Theta_k)) = k$ based on $\xi$ as the base of scale factor for $r(\Theta_1)$.

Postulate stipulates that per scaling the amount of energy transmitted by a volume in $S + \mathcal{R}$ equals the amount of energy received by a corresponding volume in $\mathcal{R}$. That implies that

$$V_k \times r(V_k) = \Theta_k \times r(\Theta_k) \quad (118)$$

energy units. Are there phenomena where for equation (117) $\xi < 1$? See section VII on allometry.

I. The NRT based on The 4/3 RDFT

In the case of length $\ell$, area $A$ and volume $\Theta$ in space, determination of the degrees of freedom $\text{deg}$ of motion is relative to the number of dimensions in which the motion occurs. The uniform scaling length for a cube in space, as in the Overview, induces the $\text{Deg}$ relationships pertaining to length $\ell$, area $A$ and volume $\Theta$ for their respective scale factors, relative to the scale factor for length. $\text{Deg}$ applied to space measures how length, area and volume scale relative to $\ell$'s scale factor $\beta$:

1. $\text{Deg}(\ell_{k+1}/\ell_k) = \text{Deg}(\beta_{k}/\ell_k) = \log_\beta(\beta^1) \deg = 1 \deg$;
2. $\text{Deg}(A_{k+1}/A_k) = \text{Deg}((\beta_{k}^2)/(\ell_k)^2) = \log_\beta(\beta^2) \deg = 2 \deg$;
3. $\text{Deg}(\Theta_{k+1}/\Theta_k) = \text{Deg}((\beta_{k}^3)/(\ell_k)^3) = \log_\beta(\beta^3) \deg = 3 \deg$.

A different approach applies to the degrees of freedom of constant radiation for a radiation cone. $\text{Deg}$ for the radiation property of radial length $L$, compares not length to area and volume, but how $L$ changes relative to a preceding $L$, for $L_{k+1}$ compared to $L_k$, or to be precise, how $L_k = L$ does not change. For $L_k$'s scale factor $s = 1$:

$$\text{Deg}^*(L_k) = \text{Deg}(L_{k+1}/L_k) = \text{Deg}(sL_k/L_k) = \text{Deg}(L/L)$$

$$= \log_\beta(s) \deg = 0 \deg. \quad (119)$$

Using the idea of degrees of freedom in a network has the special result that the base of the network's scale factors is the network's mean path length $\mu$. For a network with $n = \mu^k$ nodes there are $k$ generations based on $\mu$ as the base of the network's scale factor.

- There are $\log_\mu(\mu^k) = k$ generations or levels of scaled clusters based on the mean path length $\mu$.

A cluster of units (of energy or information for example) divides into $\mu$ clusters each with $1/\mu$ as many units.

In the particular case of a static network, this double role of $\mu$ presents quandaries: is $\mu$ a scale factor or is it a mean path length; how is it possible for a length such as the mean path length to also be a scale factor? how can it be that the average distance between network nodes is $\mu$ and yet, according to Jensen’s inequality, $\log_\mu(\mu^k) = k\mu$ is the most efficient way of spanning the network?

The answers to these questions occur in the preceding sections.

For the first and second questions, $\mu$ is the base of the scale factor $\mu^k$. $\log_\mu(\mu^k) = k$ gives the number of lengths of size $\mu$ required to span a scaled path through a network’s nested hierarchy of clusters scaled by $\mu$. Or, in other words, there are $k$ degrees of freedom associated with $\mu$ if we treat $\mu$ as the base of the scale factors for the network. Thus $\mu$ in this special case is both the base of the scale factor and the characteristic length for a network. In this role, $\mu$ as the base of scale factors plays the same role as the base $s$ for scale factors $s^k$ for radiation length $L_k$, and as a length plays the same role as radiation length $L_k$.

For the third question concerning spanning the network, $\mu$ as the base of a scale factor applicable to the network, tells us, by reference to the exponent applied to it, the number of degrees of freedom in the network relative to $\mu$.

If one supposes that the role of the mean path length $\mu$ is the conceptual fulcrum on which all else turns, as I did for a few years, theory remains cloudy. If one supposes, as I now do, that the conceptual fulcrum is degrees of freedom of a scale factor, some of the clouds part.

The degrees of freedom of a scale factor as the more fundamental idea is revealed by applying scaling to radiation distributed by a radiation cone without bound. The length per scaling is a constant $L_k = L$. The exponent of the average base of $L$'s scale factor for a cumulative length over an interval changes. In a network at any given point in time, the number of degrees of freedom $k$ relative to its mean path length $\mu$ is unchanging. For radiation, change is part of its essence.

At a given point in time, the average rate per network or per $\mu$ may be calculated if appropriate information is available.

J. A theory of emergence

The 4/3 RDFT applied to homogeneous scaling may help explain the initiation and expansion of a system. The other concept needed is the linear relationship between a network rate and the network’s degrees of freedom relative to a characteristic length.

The 4/3 RDFT causes the system to grow, perhaps initiates growth. As the number of scalings increases due to the pressure of radiation distribution having 4/3 the degrees of freedom of the recipient space , the cumulative degrees of freedom in the recipient system increase. As the degrees of freedom of the recipient system increases, its collective rate increases, as The NRT indicates.

These observations raise the push-pull question mentioned in section VII.

For a system receiving energy, such as a system of organisms, as an environment ages the amount of stored energy resources (or accumulated store of solved problems) in the environment inhabited by the organisms increases. As the stored or accumulated energy or information resources increases, available degrees of freedom increase, and the collective rate
of change increases. The combined effect of The 4/3 RDFT and The NRT may predispose systems to emerge.

These observations about emergence when they are applied to networks suggest why there are economies of scale. Larger and more developed societies, economies, languages and sciences have more degrees of freedom. Increasing opportunities to choose among beneficial degrees of freedom might promote emergence and growth.

VII. ALLOMETRY AND 3/4 METABOLIC SCALING: A USE OF THE 4/3 RDFT

A. Application of The 4/3 RDFT to metabolic scaling

Some background to the problem of 3/4 metabolic scaling is discussed in III in this article. More is contained in Whitfield (2006). Now apply the idea of degrees of freedom of a scale factor in an attempt to account for 3/4 metabolic scaling.

Treat the tubes of an organism’s circulatory system as a uniformly scaled energy distribution system.

Assume that the energy per unit volume of blood is constant. The same amount of energy flows in each level or generation of tubing. The average cross-sectional areas of a radiation cone’s volume increments scale the same way as the cross-sectional areas of a circulatory system’s tubes scale from one generation to the next and, likewise, have constant radiation per scaling or tube level. That is, the amount of blood flowing through the area of the cross-section at every level of an organism’s circulatory system is the same; this fact is analogous to a constant rate of radiation.

Denote as (Circ)k the circulatory system of organism k. For (Circ)k, level 1 is the heart as theory suggests. (Circ)k has k scalings (or branchings) so that there are k + 1 levels including all levels from level 1 to level k + 1, the level of the capillaries. Let Vc be the volume of a capillary, and Θc be the spherical volume irrigated by a capillary volume Vc.

Let τk+1 represent the rate of metabolism of organism k. For the base of a scale factor ξ let

\[ \tau_{k+1} = \xi^{3/4} \tau_k \]  

Then

\[ \text{Deg}^*(\tau_k) = \text{Deg} \left( \frac{\tau_{k+1}}{\tau_k} \right) = \text{Deg} \left( \frac{\xi^{3/4} \tau_k}{\tau_k} \right) = \log_{\xi} (\xi^{3/4}) \text{ deg} = (3/4) \text{ deg} \]  

and

\[ \text{Deg}^*(\tau_k) = \text{Deg} \left( \frac{\tau_{k+1}}{\tau_k} \right) = \text{Deg} \left( \frac{\xi^{3/4} \tau_k}{\tau_k} \right) = \log_{\xi} (\xi^{3/4}) \text{ deg} = (3/4) \text{ deg} \]  

The energy supply capacity of the circulatory system for organism k is increased by 4/3 power scaling. So (Circ)k with energy E per time unit has capacity to supply energy to its capillaries of \( v^{(4/3)k} E \) per time unit, by reason of The 4/3 RDFT.

Two observations based on the foregoing would account for 3/4 metabolic scaling.

First, \( \text{Deg}^*((\text{Circ})_k) = (4/3) \text{ deg} \). Then

\[ (\text{Deg}^*((\text{Circ})_k) \times \text{Deg}^*(\tau_k) = (4/3) \text{ deg} \times (3/4) \text{ deg} \]

\[ = 1 \text{ deg}^2. \]

Equation (123) implies that for organisms the product of the \( \text{Deg} \) of energy supply capacity times the \( \text{Deg} \) of the rate of metabolism is invariant for all k, that is, for all sizes of organism.

The likeliest explanation for this invariance involves intracellular chemistry. Cellular temperature affects rates of biochemical reactions (Gillooly 2001). If the biochemistry of cells is uniform or invariant for differently sized animals, which would be a conservative way for evolution to adjust for different body sizes, then the temperature of the cells in organisms with similar biochemistries should lie in a narrow range. If the individual cell temperatures for similar organisms are the same, then larger organisms must have a reduced rate of metabolism, in line of equation (123).

If cellular metabolism did not scale down by a 3/4 power when an animal’s size scaled up, the extra 1/3 power (4/3 compared to 1) of energy supplied by the circulatory system compared to that of the cells receiving energy would overheat the organism’s cells. Adaptation in response to the 4/3 scaling of the energy distribution system, lowering the rate of cellular metabolism of organism k + 1 compared to organism k, avoids cellular damage caused by a body temperature too high and, as well, the extra effort required to obtain food for unneeded energy. The 4/3 RDFT provides a new, plausible explanation of 3/4 metabolic scaling and the economies of organism scale.

The second observation relates to \( \tau_k \). Equation (122) shows that if \( k \) increases, then metabolism scales down by a 3/4 power. If the number of levels in an organism’s circulatory system is proportional to its mass, then as organisms increase in size, metabolism slows by a 3/4 power of mass. So, based on equation (122), we have

\[ \text{Deg}^*(\tau_{k+1}) = \text{Deg} \left( \frac{\tau_{k+2}}{\tau_{k+1}} \right) = \text{Deg} \left( \frac{\xi^{3/4}(k+1) \tau_{k+1}}{\tau_k} \right) = \log_{\xi} (\xi^{3/4} (k+1)) \text{ deg} = (3/4)(k + 1) \text{ deg} \]
and so organism \( k + 1 \) has a metabolism that is a
\[
(3/4)((k + 1)/k) \approx 3/4
\] (125)
power compared to that of organism \( k \). The larger \( k \) is the more closely \((k + 1)/k\) approaches \( 1 \) and the more closely the scaling approaches a \( 3/4 \) power relative to the mass size. The factor \((k + 1)/k\) arises because we are comparing cumulative metabolisms for entire organisms, not metabolism per tube level.

The idea of the Brownian ratchet as a way to supply energy to cellular machinery seems to fit nicely with The 4/3RDFT. The exact mechanism and implementation of the Brownian ratchet in connection with The 4/3 RDFT may be subject to experimentation and further theoretical investigation. Is it the case that the 4/3 power increase in energy supply for a larger animal is constrained by a ratchet and pawl mechanism? Perhaps the cells in a larger organism can only take up so much of the (4/3)th power larger energy supply.

Suppose a larger organism utilizes part of its extra 1/3 degree of freedom of energy for growth. Then the ratio of the degrees of freedom of the scale factors for metabolism will be less than 4/3 : 1, and the exponent in \( M^k \) will be higher than 3/4 (since \( b \) is inverse to 4/3), which has been found to be so (p. 871, Hulbert 2004).

On the other hand, suppose that an organism cannot obtain the full benefit of the extra 1/3 degrees of freedom of energy (entropy) of its energy distribution system. Then the degrees of freedom ratio may be somewhat greater than 4/3 : 1, and the degrees of freedom ratio for that organism’s metabolism will be somewhat less than 3/4 : 1, approaching 2/3 : 1 (which might explain why some of the biological data misleadingly implies 2/3 scaling). The factor \((k + 1)/k\) may also explain some observations.

For an organism, the source \( \{0\} \) of the energy supply — food — is external to the organism. Treating \( S \) (instead of \( S + \mathcal{R} \)) and \( \mathcal{R} \) as distinct reference frames still leads to a 4/3 ratio per generation implying The 4/3 RDFT but creates a logical gap: the circulatory system (but not the energy supply source \( \{0\} \)) is part of the organism, not separate from it. Since the circulatory system as the energy distribution system is both an energy distribution system and also has the attributes of volume and mass, treat the circulatory system as analogous to a radiation distribution system \( S + \mathcal{R} \).

The metabolic scale factor 3/4 as the sum of an infinite series as supposed in WBE 1997 is incompatible with equation (125) which is scale invariant. No summing or limiting process is required; the 4/3 deg : 1 deg ratio applies to each level or generation of the organism’s circulatory system compared to its corresponding mass.

WBE’s comparing of scale factors combined with mean path length scaling, degrees of freedom of a scale factor, nested uniform scaling and the connection to Stefan’s Law led to the derivation of The 4/3 RDFT. Metabolic scaling is one of the phenomena that, possibly, provides a clue to solving dark energy.

Thus it may be that a feature of the circulatory system that resides in a human body, an object of biological study, models energy distribution of light in the universe in a way that provides a clue to dark energy, an object of astrophysical study.

B. Lexical scaling and allometry issues

Seeking a way to test the validity of mean path length scaling as a model of lexical growth led to considering scaling in biology (Whitfield 2006), metabolic scaling, the ideas in WBE 1997, and investigating whether 3/4 scaling would emerge from algebraic manipulation of mean path length scaling.

A first difficulty is that in 3/4 metabolic scaling the exponent of mass \( M \) is fixed as the mass increases, whereas in mean path length scaling for a system of size \( n = \mu^k \) for some \( k \), the exponent \( k \) increases as \( n \) increases. Resolving this difference requires recognizing that a single arterial path from the aorta could be thought of as a single set of nested tubes.

This is the approach taken in the first few versions of this arXiv article in 2008. In April 2008, the scaling approach adapted from lexical scaling gave 4/3 energy supply scaling instead of 3/4 metabolic scaling.

The discrepancy between measured 3/4 metabolic scaling and 4/3 energy supply scaling is resolved by supposing that as energy supply capacity scales up by 4/3, the metabolic rate scales down by 3/4. The organism’s circulatory system volume has a 4/3 entropy (and also energy capacity per unit time and fractal dimension) compared to the capacity of the organism’s mass to use energy supplied to it. Since energy supply scales up by a 4/3 power, metabolism must slow by a 3/4 power to maintain the existing temperature of the organism’s cells.

For the mathematics to work, energy distribution and energy recipient systems and their respective scale factors must be distinct.

An intermediate step of Stefan’s Law also has a 4/3 entropy relationship. That suggests that the mathematics leading to the 4/3 entropy of metabolic scaling also applies to black body radiation, and to radiation distribution in general, and that 4/3 scaling for a radiated distribution compared to its recipient system reflects a general law.

WBE 1997 (West 1997) supposed that capillaries distributed blood to a spherical volume. It was not clear why WBE 1997 used a spherical volume. The similarity of Stefan’s law to 4/3 scaling and the involvement of radiant energy distribution suggests that spherical volumes work because they scale homogeneously and isotropically, just as the cosmic background radiation is approximately homogeneous and isotropic. The connections between metabolic scaling and radiation suggest that 4/3 power scaling applies to cosmology.

Since radiation can in principle increase continually, that implies that a model of metabolic scaling should not treat the initial generation as the capillaries. The reason is that energy that is scaling into smaller clusters would not result in clusters growing from small clusters to larger ones; and distance from the source would not increase by adding generations at the source, but by new generations being added at the points farthest from the source, just as counting adds to an existing
cumulation. Thus the heart should be considered the first generation in the energy distribution system.

Several corrections and refinements occurred after these issues were addressed. So it is that this article that concerns dark energy began with problems seemingly remote from cosmology.

VIII. THE 4/3 RATIO OF LENGTHS THEOREM (THE 4/3 RLT)

The 4/3 RLT: Assume a radiation distribution system \( S + \mathcal{R} \) scales homogeneously and isotropically into a corresponding Receipt \( \mathcal{R} \). Then for radiation length \( L \) relative to its corresponding Receipt (or spatial) length \( \ell \), \( \ell = (4/3)L \). (In equation [106], \( \phi = 4/3 \).

A. Proofs of The 4/3 RLT based on The 4/3 RDFT

Proof. The idea here is to compare, for homogeneous radiation, corresponding lengths \( kL \) and \( k\ell \) that represent the same amount of energy use. This assumes that length \( kL \) in \( S + \mathcal{R} \) becomes length \( k\ell \) in \( \mathcal{R} \).

Use the definition of \( \phi \) in equation (106), \( \ell/L = \phi \).

From equation (103) the radial length of a radiation cone is \( d(G_k) = kL = \text{Deg}^{**}(G_k)L \). From equation (104), the corresponding radial distance in a spatial sphere \( \Theta \) in \( \mathcal{R} \) is \( d(\Theta_k) = k\ell = \text{Deg}^{**}(\Theta_k)\ell \). The energy of a radiation length \( kL \) leads to a spatial length \( k\ell \). Measured in energy units \( kL = k\ell \).

Suppose that it takes a unit \( \ell \) of energy for radiation to travel \( L \), and hence \( k\ell \) to cover the distance \( kL \). Since \( L \) in \( S + \mathcal{R} \) projects into \( \ell \) in space \( \mathcal{R} \), it follows that \( ke = kL = k\ell \). The 4/3 RDFT shows that \( \text{Deg}^{*}(G_k) = (4/3)\text{Deg}^{*}(\Theta_k) \) and \( \text{Deg}^{**}(G_k) = (4/3)\text{Deg}^{**}(\Theta_k) \). Then in energy units

\[
d(G_k) = \text{Deg}^{**}(G_k)L = \left(\frac{4}{3}\right)\text{Deg}^{**}(\Theta_k)\left(\frac{\ell}{\phi}\right) = \left(\frac{4}{3}\right)\left(\frac{1}{\phi}\right)\text{Deg}^{**}(\Theta_k)\ell = \left(\frac{4}{3}\right)\left(\frac{1}{\phi}\right)d(\Theta_k) = d(\Theta_k),
\]

so it must be that in the second last line of equation [126] \( 4/3 \times 1/\phi = 1 \) since its first and last lines are equal. Hence \( \phi = 4/3 \).

Or consider \( L \) and \( \ell \) as measuring sticks for the same amount of energy. For the same amount of energy, \( L \) turns over 4 times compared to \( \ell \)'s 3 times, because the ratio of degrees of freedom for a length \( L \) in \( S + \mathcal{R} \) compared to a length \( \ell \) in \( \mathcal{R} \) is \( 4/3 : 1 \). That degrees of freedom ratio requires that the corresponding length ratio is \( L : \ell = 3 : 4 \), so \( \ell = (4/3)L \).

Base another perspective on the volume property of \( L \) applied to \( \text{Deg}(V_{k+1}/V_k) = (4/3)\deg \) from based on section [VIII]. For the scale factor \( s_k \), a scaling of \( L \) from \( k \) to \( k + 1 \) is

\[
s_k = (4^{1/3})^{1/3}.
\]

Thus the radiation length travelled is

\[
\log_4 (4^{1/3}L) = 4/3 \times 1/3 \times L.
\]

The scale factor for a scaling of \( \ell \) from \( k \) to \( k + 1 \) is

\[
\beta_k = (\theta^4)^{1/3}
\]

and for the space length scale factor \( \beta_k \) the distance travelled is

\[
\log_4(\theta^4 \times 1/3)\ell = 1/3 \times \ell.
\]

Since the energy of a radiation length is fully transmitted into its spatial length, equating the right sides of equations [128] and [130], \( (4/3)L = \ell \); and therefore \( \phi = 4/3 \).

Space — \( \mathcal{R} \) — stretches radially by 4/3 relative to a radial radiation distance in \( S + \mathcal{R} \).

The approach of taking a 1/3 power with respect to length suggests taking a 2/3 power with respect to average radiation cross-sectional area \( A_s \), so that \( \text{Deg}^*(V_k) = \text{Deg}^*(A_k) = \text{Deg}^*(L_k) = 4/3 \deg \), a generalization of The 4/3 RLT. The relationship between 4/3 deg radiation pressure and energy (dark energy) expanding space implies that 4/3 deg is connected to energy, perhaps at all scales.

Every increment of radiation in \( S + \mathcal{R} \) results in radial length in \( \mathcal{R} \) increasing by a constant \( \ell \); there is no stretching within \( \mathcal{R} \)'s reference frame relative to \( \mathcal{R} \) itself. The 4/3 stretching of space only represents radial stretching in \( \mathcal{R} \) relative to radial radiation distance in \( S + \mathcal{R} \). Characterizing the stretching of space requires the observer to recognize that distance in \( L \) units equals the distance in \( \ell \) units; the stretching arises because \( \ell = (4/3)L \).

We can conceive of \( L \) and \( \ell \) as different measuring sticks for the same distance, but we perceive a distance measured with measuring sticks \( L \) as shorter than the same distance measured with measuring sticks \( \ell \). If this difference between measuring sticks \( L \) and \( \ell \) accounts for dark energy, among other observed phenomena, that implies the existence of different reference frames, in which the same light distance is perceived differently. If these observations are valid, hopefully a simpler perspective can be found.

The ratio for degrees of freedom of radiation length to a corresponding space is inverse to the ratio of their lengths. For degrees of freedom of scale factors, for \( S + \mathcal{R} \) relative to \( \mathcal{R} \), the scaling is \( 4/3 = \phi \). For length, for \( S + \mathcal{R} \) relative to \( \mathcal{R} \), the scaling is \( 3/4 = 1/\phi \). If there is a lossless transmission of energy from \( S + \mathcal{R} \) to \( \mathcal{R} \), then the product of the degrees of freedom ratio \( 4/3 \) and the corresponding length ratio \( 3/4 \) equals 1, that is, is invariant. Does that invariance have some physical or other significance?
B. Equivalence of The 4/3 RDFT and The 4/3 RLT

Proving The 4/3 RDFT for scale factors implies scaling for lengths in The 4/3 RLT is inverse to that for degrees of freedom. The 4/3 RDFT and The 4/3 RLT imply each other. Since The 4/3 RDFT and The NRT imply each other, evidence for any one of The 4/3 RDFT, The 4/3 RLT, and The NRT is also evidence for the other two.

From another perspective, suppose the rate of energy use \( \rho(E) = \rho(V_k) = \rho(\Theta_k) \). Then like equation (126),

\[
\rho(E) = \rho(V_k) = \rho(\Theta_k) = \rho(\Theta_k) \times \ell = \rho(\Theta_k) \times \ell = \rho(\Theta_k) = \rho(\Theta_k).
\]

The first and last lines of equation (131) are equal by assumption. \((4/3) \times (1/\phi) = 1\), so \( \lambda = 4/3 \). The relations in equation (131) also work in the reverse direction.

C. Proving The 4/3 RLT: maximum volume cone inscribed in a sphere

Fermat posed this problem (p. 238, Fermat Oeuvres): ‘pour trouver le cône de surface maxima qui peut être inscrit dans une sphère donnée’, that is, to find the maximum surface area of a cone inscribed in a given sphere. (Cone surface excludes the area of the base of the cone). Lockhart in his recent book (p. 351–356, Lockhart 2012) poses and presents the solution of an equivalent problem: what inscribed cone has maximum volume for a given sphere? If the sphere has radius \( r \), then the requisite cone has height \( (4/3)r \), which can be proved by using trigonometry and by using calculus.

Reorient the inscribed maximum volume cone so that its tip is at the center of the sphere. The radial length of the cone is \( 4/3 \) the radius of the sphere according to Fermat’s proof.

Consider a sphere of homogeneous radiation from a point; its radius grows uniformly by \( L \) per pulse. Characterize radiation by its homogeneous additions of length \( L \) along every radial line from the point; this requires a radiation sphere. Since we want to compare radial radiation length to radial spatial length, we must use a spatial cone to reveal the difference between radiation length and spatial length.

Suppose that all the energy from the portion of the sphere that has the same volume as the inscribed cone is transmitted. Then a radiation length of 1 in the radiation sphere corresponds to a spatial length in the spatial cone that is 4/3 as great, as in The 4/3 RLT. In other words, a portion of a radiation sphere having the same amount of energy as a maximal inscribed spatial cone gives an inscribed cone that has a height equal to 4/3 the radius of the radiation sphere.

The mathematical relationship between the ratio of the height of a cone with maximal volume inscribed in a sphere to the sphere’s radius can be considered to be another instance of The 4/3 RLT.

D. Proving The 4/3 RLT: Equal volume sphere and cone

The ratio described by The 4/3 RLT also applies to a cone and sphere with equal volumes (as opposed to considering only part of radiation sphere volume, as in section V.E.3).

The volume of a sphere is

\[
\frac{4}{3}\pi r^3.
\]

The volume of a cylinder with height \( h \) is

\[
\pi r^2 h.
\]

Equate equations (132) and (133), letting \( h = \phi r \). Then

\[
\frac{4}{3}\pi r^3 = \pi r^2 \phi r,
\]

which implies \( \phi = 4/3 \), and so \( h = (4/3)r \). For the cylinder, now substitute a cone with average radius \( r \) over its height. Then the space distance \( h = \ell = (4/3)r = (4/3)L \), which is \( 4/3 \) of a radiation radial distance, as in The 4/3 RLT.

This geometric relationship of cone and volume raises the question of how The 4/3 RDFT relates to geometry, and whether the geometry of space arises out of The 4/3 RDFT.

E. Proving The 4/3 RLT using Clausius’s mean path length ratio theorem

Clausius in his 1858 paper (p. 140 in Brush’s translation) remarks:

The mean lengths of path for the two cases (1) where the remaining molecules move with the same velocity as the one watched, and (2) where they are at rest, bear the proportion of \( 4 \) to \( 1 \). It would not be difficult to prove the correctness of this relation; it is, however, unnecessary for us to devote our time to it.

Gas molecules all moving with the same average velocity correspond to a homogeneous, isotropically radiating energy distribution by \( S + \mathcal{R} \). Stationary gas molecules correspond to the Receipt, \( \mathcal{R} \). Clausius’s mean path length ratio theorem, generalized, is another proof of The 4/3 RLT.

Clausius’s device of having a configuration of gas molecules in which only one molecule moves compared to all gas molecules moving randomly is analogous to finding relative degrees of freedom by having one system scale (all parts in motion) while the other does not (only one moving gas molecule), as in section V.E.3. I adapted Clausius’s device for section V.E.3’s definition of the relative degrees of freedom of scale factors.
Clausius’s 1860 proof (p. 434, Clausius 1860) is not widely reproduced because early on his result was considered incomplete; it is not included in Brush’s compilation of readings on Kinetic Theory (1948). To obtain the result Clausius started by finding that for two particles \( \mu \) and \( m \) and the angle \( \theta \) between them “the relative velocity between \( \mu \) and \( m \)” is

\[
\sqrt{u^2 + v^2 - 2uv \cos \theta}
\]

and then calculating an integral formed of the product in equation (135) multiplied by the number of particles \( N \), with the result divided by \( N \) to obtain a mean velocity.

The editor of Maxwell’s works adds a note (p. 387, Vol. 1, Maxwell, 1890) that Clausius had assumed that the mean relative velocity for the two cases was of the same form, whereas Maxwell assumed the moving particles obeyed a statistical distribution. Longair (p. 7, Longair 2013) observes that Clausius’s calculation did not agree with the ratio of the specific heat capacities for a fixed pressure \( (C_p) \) and constant volume \( (C_v) \). Clausius’s 1857 paper predicted \( C_p/C_v \) would be 1.67 whereas experiments revealed it was about 1.4. Maxwell’s innovative statistical approach led to an accurate estimate \( C_p/C_v = \sqrt{2} \) and laid the foundation for statistical mechanics.

In order to model the ratio \( C_p/C_v \), the amount of energy in each specific heat capacity is required. It is necessary to take into account the distribution of velocities. Clausius was aware that actual velocities varied: “we must assume that the velocities of the several molecules deviate within wide limits on both sides of this mean value” (p. 113, Clausius 1860). Clausius’s result in relation to gas molecules gives a ratio of lengths which does not accord with the measured values for the ratio \( C_p/C_v \). It does, however, prove a particular instance of The 4/3 RLT, and, if The 4/3 RLT is valid and connects to dark energy, is on that account remarkable, a noteworthy addition to his contributions to the science of thermodynamics.

IX. A PROPOSED EXPLANATION OF DARK ENERGY

The foregoing suggests that the 4/3 degrees of freedom of \( S + R \), for volumes, areas and lengths, relative to \( R \), is a vacuum pressure — the pressure of empty space. Cosmologists designate vacuum pressure as \( \Lambda \), the cosmological constant, and as dark energy. They note that pressure, as in vacuum pressure, and energy have the same units. Since 4/3 \( \text{deg} : 1 \text{deg} \) arises from radiation having more degrees of freedom than space, the situation can be described this way: vacuum pressure equals radiation pressure equals the ratio of degrees of freedom of \( S + R \) relative to \( R \) for corresponding volumes.

This point of view also suggests a relationship between (4/3) deg and energy. Since The 4/3 RLT applies at all scales, that suggests that the relationship between (4/3) deg and energy also applies at all scales.

The following explanation for dark energy is based mainly on The 4/3 RLT.

A. Theory

Suppose \( \Omega \) corresponds to what we conventionally think of as the space part of space-time (or space-radiation) \( S + R \). Matter resides in \( R \). Suppose \( S \) corresponds to what we perceive as radiation in the radiation part of space-time (or space-radiation) and that a radiation volume in \( S + R \) creates a vacuum, or degrees of freedom, pressure relative to a spatial volume in \( R \).

Cosmology assumes that the energy density of space is the sum \( \rho_\Lambda + \rho_\gamma + \rho_M = \rho_c \), where \( \rho_c \) is the critical density. Cosmology defines

- \( \Omega_\Lambda \equiv \rho_\Lambda/\rho_c \),
- \( \Omega_\gamma \equiv \rho_\gamma/\rho_c \), and
- \( \Omega_M \equiv \rho_M/\rho_c \).

Then \( \Omega_\Lambda + \Omega_M + \Omega_\gamma = 1 \). \( \Omega_\Lambda \) is the energy density of dark energy relative to \( \rho_c \).

The currently measured \( \Omega_{0\gamma} \) (the 0 in \( \Omega \)’s subscript γ, 0 denotes the present) is only \( 8.4 \times 10^{-5} \) (p. 67 Ryden 2003), much smaller than the energy densities of matter and dark energy. Accordingly, \( \Omega_\Lambda + \Omega_M \approx 1 \), disregarding \( \Omega_\gamma \) since it is so small compared to measured values for \( \Omega_\Lambda \) and \( \Omega_M \).

Then,

\[
\frac{\Omega_\Lambda}{\Omega_M} = \frac{\rho_\Lambda/\rho_c}{\rho_M/\rho_c} = \frac{\rho_\Lambda}{\rho_M}.
\]

Construct a spatial cube in \( R \) (in which matter resides) with side \( \ell \) and measure the same cube in \( S + R \) but using as a measuring stick \( L = (3/4)\ell \). Regardless of whether we use \( L \) or \( \ell \) to measure the side of the cube, the amount of energy in the cube is the same.

The vacuum’s energy density \( \rho_\Lambda \) is the ratio of the energy density \( E/L^3 = E/[(3/4)\ell]^3 = \rho_\gamma \) of the cube measured in \( S + R \) using characteristic length \( L \) compared to the energy density \( E/\ell^3 = \rho_M \) of the same cube measured in \( R \) using characteristic length \( \ell \) which is:

\[
\rho_\Lambda \equiv \frac{\rho_\gamma}{\rho_M} = \frac{\rho_\gamma}{\rho_M} \frac{\Omega_\gamma}{\Omega_M} = \frac{E/L^3}{E/\ell^3} \left(\frac{3}{4}\right)^3 = \frac{64}{27} \approx 0.7033:0.2967
\]

The mathematics predicts that the ratio of \( \Omega_\Lambda \) to \( \Omega_M \) should be about 0.7033 : 0.2967.

The cube containing energy \( E \) is measured in two ways, one way using the characteristic measuring stick for radiation \( L \) and the second way using the characteristic measuring stick for space \( \ell \). This may take some getting used to.

\( R \) does not stretch relative to itself in terms of the number of characteristic lengths \( k\ell \) but only relative to the number of corresponding radiation characteristic lengths \( kL \) in \( S + R \).
The distance is the same, but the measurement of it changes depending whether we measure the distance relative to light distance or space distance.

The vacuum pressure arises from the ratio $\text{Deg}^*(V_k)/\text{Deg}^*(\Theta_k) = (4/3) \text{ deg}/1 \text{ deg}$. Similarly, vacuum energy density $\rho_{\lambda}$ is the ratio of relative energy densities described in equation \textcolor{red}{[37]}, namely $\rho_{\lambda}/\rho_M$. A hurdle raised by the nature of vacuum pressure is the difficulty in distinguishing between the ratios $(4/3) \text{ deg} : 1 \text{ deg}$ and $4 : 3$. It is the ratio $(4/3) \text{ deg} : 1 \text{ deg}$ that characterizes dark energy. Dark energy is a consequence of the different degrees of freedom that radiation has compared to space. Clausius may have been the first physicist to observe the shadow this ratio casts when he compared the mean path lengths of gas molecules all moving to gas molecules that were stationary.

The 4/3 RLT makes another prediction.

Suppose light from a kind of standard candle $C$ with intrinsic brightness $B$ travels from object $SC_1$ distance $d_1 = 1$ distance benchmark to Earth. Suppose light from another such object $SC_2$ with the same intrinsic brightness $B$ travels distance $2d_1$ to Earth. Distance $d_1$ is the light distance in $S + \mathcal{R}$, but radial distance in $\mathcal{R}$ stretches by $4/3$ when evaluated in terms of characteristic length $R$. The light from $SC_2$ takes $4/3$ as long to reach Earth because of the $4/3$ radial stretching of space in $\mathcal{R}$, so $SC_2$ appears

$$\frac{B}{(4/3)d_1} = (3/4)B,$$  

(138)

$3/4$ as bright, or $25\%$ fainter, relative to $SC_1$ compared to what would be expected if the ratio of corresponding distances between $\mathcal{R}$ compared to $S + \mathcal{R}$ was 1 : 1 instead of $4/3 : 1$.

A third prediction. Let $R(t)$ be a speed such as 50 miles per hour (mph), $t$ an elapsed time such as 2 hours and $R(t)$ as a distance. Then the distance $R(t)$ travelled in 2 hours would be $t \times R(t) = 2 \text{ hours} \times 50 \text{ mph} = 100 \text{ miles}$. If we know the speed and distance, the time required to travel a distance is distance divided by speed equals trip duration,

$$\frac{R(t)}{R(t)} = t,$$  

(139)

or, in other words, to travel 100 miles at 50 miles per hour takes (100/50) (miles/miles per hour) = 2 hours.

Apply the same approach to degrees of freedom per time unit. If $R(t)kL = \delta kL = (4/3)kL$ then $R(t) = 4/3$ would be the rate of scaling. Since the ticking of each of the $k$ scalings is constant, we can think of $k$ time units as proportional to the radiation distance $kL$. Inverting the left side of \textcolor{red}{[139]}

$$\frac{R(t)}{R(t)} = \frac{1}{t},$$  

(140)

The 4/3 RLT therefore implies

$$\frac{R(t)}{R(t)} = \frac{4/3}{4/3t} = \frac{1}{t} = H(t),$$  

(141)

where $H(t)$ is Hubble’s constant. Equation \textcolor{red}{[141]} conforms to the usual relationship between speed, time and distance, as described by equation \textcolor{red}{[139]}.

In the next section we compare the predictions in this section to observations.

B. Observation and the Benchmark Model

If vacuum pressure results from the 4/3 degrees of freedom of radiation relative to space then vacuum pressure is, for a corresponding volumes,

$$\frac{\text{Deg}(S + \mathcal{R})}{\text{Deg}(\mathcal{R})} = \frac{(4/3) \text{ deg}}{1 \text{ deg}},$$  

(142)

where $\mathcal{R}$ is space and $S + \mathcal{R}$ is radiation plus space.

The Seven Year Wilkinson Microwave Anisotropy Probe (WMAP) (Jarosik et al.) measured (p. 2) dark energy density $\Omega_{\lambda} = 0.728^{+0.015}_{-0.016}$ with a 68% confidence limit (also Komatsu 2011). The Planck satellite March 2013 (p. 11, Planck Collaboration 2013) measured dark energy density $\Omega_{\lambda} = 0.686 \pm 0.020$, with a 68% confidence limit.

The 4/3 RLT predicts $\Omega_{\lambda} = 0.7033$ compared to $\Omega_{\lambda}$ which lies close to (only 0.0037 — one half per cent — different than) the midpoint, 0.707, of the Planck and WMAP measurements. The difference in the Planck and WMAP measurements is 0.042, eleven times greater than 0.0037. Since the measured isotropy of the cosmic microwave background radiation CMB is good to one part in 100,000, the match between theory and observation should be close. Current measurements of $\Omega_{\lambda}$ can not tell us that The 4/3 RLT explanation of dark energy is wrong; the midpoint of the WMAP and Planck measurements, too close to be a mere coincidence, suggests that, on the contrary, The 4/3 RLT is valid. If The 4/3 RLT is valid, then so is the equivalent The 4/3 RDFT.

Observations provide a range of estimates for $\Omega_{M}$ (Sec. 1.6, Weinberg 2008), but the realistic case based on the data is $\Omega_{M} = 0.3$ (p. 50, 51, Weinberg 2008). Ryden (p. 67, Ryden 2003) calls the theory based on data from observations the Benchmark Model. (In her book, Ryden uses $\epsilon$ for energy density instead of $\rho$ and $m$ for matter instead of $M$.)

The Benchmark Model has $\Omega_{M,0} \approx 0.3$. Hence (in the notation of this article), and following Ryden

$$\frac{\rho_{\lambda,0}}{\rho_{M,0}} = \frac{\Omega_{\lambda,0}}{\Omega_{M,0}} \approx \frac{0.7}{0.3} \approx 2.3,$$  

(143)

In the past (Ryden notes), mass in the universe occupied a smaller volume scaling by a scale factor $a$; mass energy density changed with $\rho_{M,0}/a^3$.

$$\frac{\rho_{\lambda}(a)}{\rho_{M}(a)} = \frac{\rho_{\lambda,0}}{\rho_{M,0}/a^3} = \frac{\rho_{\lambda,0}}{\rho_{M,0}} a^3.$$  

(144)

The implicit assumption in equation \textcolor{red}{[144]} is that $a$ changes with time, consistent with a itself being scaled by a quantum scale factor. (The $a$ used by Ryden corresponds to $R(t)$, the scale factor for the R-W metric.) As noted above in section \textcolor{red}{[IG]} using the idea of two reference frames enables us to dispense with the inference that $a(t)$ changes over time. Stretching occurs on every radiative event, for $\mathcal{R}$ relative to $S + \mathcal{R}$ due to their different characteristic lengths (their different characteristic measuring sticks).

(We may suspect therefore that in the two slit experiment the appearance of waves in the case of the two slits is the
stretching of space; it is the background in motion, a background that moves at all scales, that results in what is observed.)

Ryden writes (p. 67), having regard to equation (144): “If the universe has been expanding from an initial very dense state, at some moment in the past, the energy density of matter and $\Lambda$ must have been equal.” This implies that the left side of equation (144) equals 1. Then solving for $a = a_{M, \Lambda}$ in that (p. 68, Ryden 2003),

$$a_{M, \Lambda} = \left(\frac{\Omega_{M,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \approx \left(\frac{0.3}{0.7}\right)^{1/3} \approx 0.75 = \frac{3}{4}. \quad (145)$$

Equation (145) matches the result in section XA. Equation (145), however, differs from The 4/3 RLT, which says that the ratio of energy densities has been the same at all epochs. Since equation (145) is true for all epochs, it was also true at the beginning. The reasoning that Ryden describes differs in that respect from The 4/3 RLT.

Second, the 1998 observations of Type Ia supernovae (Fig. 13, Riess 1998) used as standard candles found that these supernovae appeared “about 25% fainter, that is, farther away than expected” (Dark Energy Survey 2013); and Cheng (p. 259, Cheng 2010) as predicted by The 4/3 RLT.

Third, Hubble’s constant $H(t)$ is defined as

$$\frac{R(t)}{R(0)} = H(t) \quad (146)$$

(p. 49, Ryden 2003; p. 42, Bernstein 1995) for the scale factor $R$ in the R-W metric. ‘Classic’ cosmological solutions (p. 37, Liddle 2003) which apply a fluid equation suggest that the elapsed age of the universe $t_0$ for the early radiation dominated universe was

$$t_0 = 1/2 \times 1/H_0 \quad (147)$$

and for the current matter dominated universe is

$$t_0 = 2/3 \times 1/H_0 \quad (148)$$

(p. 37, Coles 2002; p. 40, Weinberg 2008). Equation (148) implies an age

$$t_0 = (6.5 - 10) \times 10^9 h^{-1} \text{ years} \quad (149)$$

(p. 84, Coles 2002), $h \equiv H_0/100 \text{km s}^{-1} \text{Mpc}^{-1}$. $h = 0.67$ (p. 11, Planck Collaboration 2013), which, using equation (149), leads to an underestimate compared to the currently estimated age of the universe, about 13.7 billion years (p. 84, Coles 2002). Equation (140) gives $H(t) = 1/t$, a result consistent with cosmological observation, and with the usual relationship, described by equation (140), of speed, time and distance.

The 4/3 RLT thus provides three cosmological explanations or predictions consistent with corresponding observations.

X. OTHER POSSIBLE CONNECTIONS TO COSMOLOGY

A. Time, time’s arrow, and the universe’s expansion

In $(S + \mathcal{R})$’s reference frame there is a constant increase in the cumulative radiation distance which increases at an invariant rate of $L$ per scaling. The increase in the cumulative radiation distance is unidirectional, that is, it only increases and does not decrease, which via The 4/3 RDT induces a unidirectional expansion of $\mathcal{R}$ relative to $S + \mathcal{R}$.

Both the expansion of space, induced in $\mathcal{R}$ by the scaling of $\mathcal{R}$, and time are unidirectional, thus possibly explaining time’s arrow. “This seems to suggest that this expansion was produced by the radiation itself” (Lemaître 1927), as Lemaître suggested many years ago. The 4/3 RDT is consistent with Lemaître’s suggestion.

Adapt the approach of section VII. Compare how $V_k$ scales relative to itself, to how $V_k$ scales relative to $L_k$.

$$\text{Deg} \left( \frac{V_{k+1}}{V_k} \right) = \text{Deg} \left( \frac{(V_k)^{4/3}}{(V_k)^{1/3}} \right) \quad (150)$$

Equation (150) shows that $S + \mathcal{R}$ has four degrees of freedom relative to a radiation length in $\mathcal{R}$. $\mathcal{R}$ — space — itself scales with three degrees of freedom. Equation (150) provides some justification for looking at space-time (what in this article might be better referred to as space-radiation) as four dimensional.

Constant radiation at the rate of $L$ per scaling in $S + \mathcal{R}$ (in proportion to the travel time for radiation) results in 4/3 degrees of freedom for a radiation volume in $S + \mathcal{R}$ relative to a corresponding spatial volume in $\mathcal{R}$, which leads to the acceleration of the universe. In other words, constant radiation causes the universe to expand and accelerate; constant radiation is also proportional to time in $S + \mathcal{R}$’s reference frame since $L$ is added at a constant rate in $S + \mathcal{R}$’s reference frame.

Years ago Thomas Gold speculated that “The large scale motion of the universe thus appears to be responsible for time’s arrow” which he thought might be deduced “from an observation of the small scale effects only” (Gold 1962). If The 4/3 DFT applies at all scales, then it would apply at the quantum level. At the same time, The 4/3 DFT would explain the expanding universe. This would make Gold prescient indeed.

B. Inflation and the horizon problem

Examine the horizon problem in cosmology — the problem of how remote parts of the universe connect — in $S + \mathcal{R}$’s reference frame.

The horizon problem arises because space expands with the result that over time light speed has a reducing ability to span space. In $S + \mathcal{R}$’s reference frame however, things are not too far apart for light to reach them. In $S + \mathcal{R}$’s reference frame, the largest radial distance accrued to date arises from the distance light has travelled since it began its journey.

Similarly, in quantum mechanics, entangled particles might be connected in $S + \mathcal{R}$’s reference frame and only appear to be disconnected when measured in $\mathcal{R}$’s reference frame.

The theory of inflation in cosmology attempts to account for the horizon problem and the flatness problem. The factor required for inflation ranges between $4 \times 10^2$ to $5 \times 10^{59}$ (p. 204,
If radial radiation length $L$ is proportional to Planck time or Planck length, the number of iterations of $L$ in the first few seconds of the universe might be sufficient to account for inflation.

If dark energy, the $(4/3)$ deg of $S + R$, compared to the 1 deg of $R$, is the same at all places and times, then the dark energy accelerating the universe today could have inflated the universe at the universe’s beginning. The same principles that explain dark energy may also explain the universe’s early inflation. It is simpler to suppose one mechanism accounts for inflation and dark energy than to suppose distinct mechanisms. It is simpler to suppose a constant relationship $\ell = (4/3)L$ at all times than to posit an initial inflation followed by deceleration in turn succeeded by currently observed acceleration.

### C. Special relativity and Bondi’s $k$ calculus

The physicist Hermann Bondi’s *Relativity and Common Sense* was published in 1962 (Bondi 1980). In it he explains special relativity using a $k$ calculus. He takes $k$ to be the ratio of the time interval for light transmission to the time interval for light reception (p. 72). Using $k$, he derives the Lorentz transformations that characterize special relativity (Ch. X). Bohm (Bohm 1996), Freund (Freund 2008), and d’Inverno (sec. 2.7-2.12, d’Inverno 1992) discuss the $k$ calculus in their books on relativity.

In this section, I will use $m$ instead of $k$ as an exponent of the base $s$ of scale factors $s^m$, to avoid confusion with Bondi’s $k$ calculus notation.

Bondi gives an example of two observers Alfred and David at rest relative to each other. Bondi shows that two inertial observers have the same scale factor $k$, so that a round trip of light flashes starting with Alfred takes $k \times kT = k^2T$ time units, for $T$ an interval of time. The exponent of $k$ in the $k$ calculus resembles degrees of freedom of the base of a scale factor. If the algebra of degrees of freedom is analogous to Bondi’s $k$ calculus the algebra of degrees of freedom may apply to special relativity.

A difference between the algebra of degrees of freedom and Bondi’s $k$ calculus is that $s$ is the base of a scale factor for radiation length $L$ in the algebra of degrees of freedom, whereas $k$ is a ratio of the time intervals for transmission relative to reception.

Suppose there is a length of time proportional to the base $s$ of scale factors for $L$, and that degrees of freedom apply to $s$ as in the algebra of degrees of freedom.

For observers $A$ and $B$ at rest relative to one another, each would be scaling by the same power of $s$, say $s^m$. A round trip of light flashes would scale as $(s^m) \times (s^m) = (s^m)^2$. If we set $k \equiv s^m$, this is the same result that Bondi obtains for a round trip light flash in these circumstances. If the two inertial observers are not at rest relative to each other, then the exponents of the base of the scale factor $s$ for each of them would differ.

By then following Bondi’s derivation of special relativity, one may infer that the algebra of degrees of freedom leads to a result equivalent to that obtained by Bondi.

### XI. NATURAL LOGARITHM THEOREMS

Since mathematics is often based on the idealization of natural phenomena, the following derivations of the natural logarithm suggest that there is a physical basis for the natural logarithm.

Most of these derivations of the natural logarithm seem to use some aspect of the idea of degrees of freedom of the base of a scale factor.

#### A. Scaling proof

**Natural logarithm theorem 1:** A homogeneously and isotropically radiating energy distribution system $S + R$ has the natural logarithm as the base of its scale factors.

**Proof:** Lengths in $S + R$ scale with one degree of freedom at any given time: over an interval, for (an average) base $s$ (or $s_m$) of a scale factor, $s^k$ itself scales by its base, $s$. The rate of change of the scale factor over the interval $s^k$ is $s$ for each $k$.

What applies for $k = 1$ applies for all intervals of size 1, i.e. from $k$ to $k + 1$:

$$\frac{ds}{dt} = s,$$

which implies that over that interval $s \approx e$, the natural logarithm. For a large number of scalings, $s \approx e$ very closely. $\square$

An observation similar to this in connection with lexical scaling led to this scaling proof which arrived unsought. This proof suggested looking for a second proof that might support the validity of the scaling proof. That led to the lexical benefit proof.

#### B. Lexical benefit proof

The idea of the lexical benefit proof is that the per node benefit of networking must be equal to the per node cost of networking, in terms of energy or information per node.

**Natural logarithm theorem 2a:** For a finite isotropic network $R$, the base of the logarithmic function describing how information scales when per node network benefit equals network cost is the natural logarithm.

**Proof:** The contribution of networking to the multiplication of capacity per transmitting node of $R$’s $n = \sigma^d$ nodes as a proportion of $\eta$ is

$$\frac{d\eta}{dn} = \frac{d[\log_\sigma \eta]}{d(\sigma^d)} = \frac{1}{\ln(\sigma) \sigma^d}.$$  

The per node reception of the increase in capacity $\eta$ due to networking as a proportion of $\eta$ is $1/n = 1/\sigma^d$. For an isotropic network, which is an idealized network of maximum efficiency having regard for Jensen’s inequality, the contribution
to the increase in capacity per transmitting node, as described in Equation (152), equals the increase in capacity per receiving node, so

\[
\frac{1}{\sigma^2} = \frac{1}{\ln(\sigma) \sigma^2} \Rightarrow \ln(\sigma) = 1 \Rightarrow \sigma = e. \quad (153)
\]

C. Average degrees of freedom proof

This average degrees of freedom proof is an updated perspective on the lexical benefit proof, employing the concepts of degrees of freedom and nestedness instead of the idea of benefit and cost per node. The NRT implies that the number of degrees of freedom for a whole system is the same as the number of degrees of freedom per node that is nested within the system. That leads to:

*Natural logarithm theorem 2b:* A network homogeneously scaled and its individual nodes have the same number of degrees of freedom, and are thus scaled by the natural logarithm measured in steps.

**Proof.** For a system, for each single scaling, the average number of scalings per node is

\[
\frac{d \log_\eta(n)}{dn} = \frac{1}{\ln(s) \times n} \quad (154)
\]

for average scale factor \( s \) over an interval.

The average number of degrees of freedom — scalings — per node per single scaling for a system of \( n \) nodes is also

\[
\frac{1}{n} \quad (155)
\]

Equating equations (154) and (155) implies \( s = e \). □

D. Nestedness proof

*Natural logarithm theorem 3:* For a network, the mean path length \( \mu = s \) as the base of a scale factor spans \( S \) in one generation \( d\eta \), which is equivalent to \( \eta \) segments each \( \mu \) units; that implies \( s = e \).

**Proof.** The observed equivalence reduces to

\[
\frac{ds}{d\eta} = \eta \times s \quad (156)
\]

which implies \( s = e^\eta \), which for \( \eta = 1 \) implies \( s = e \). □

Here \( ds \) is the distance proportional to the scale factor \( \mu \) and \( d\eta \) is the number of generations to be spanned, so the left side of equation (156) tells us the value of the mean path length per generation and the right side of the same equation gives us the distance measured in units of \( \mu \). This proof takes advantage of the apparent paradox that the left side and the right side of equation (156) are different ways of spanning the network.

E. Limit proof

*Natural logarithm theorem 4:* In equation (102)

\[
D_{h+1} = (\sigma_{av})^k D_{av} = \left(1 + \frac{1}{k}\right)^k D_{av}. \quad (157)
\]

Let \( k \) increase. Then

\[
\lim_{n \to \infty} \left(1 + \frac{1}{k}\right)^k = e. \quad (158)
\]

If for example \( k = 10^6 \), the scale factor of \( D_{av} \) will be very close to the value of the natural logarithm. The scale factor for a time comparable to Planck time — about \( 5.39 \times 10^{-44} \) second — would accommodate \( 10^6 \) scalings in a small fraction of a second. This may explain why the natural logarithm so commonly is the average scale factor for natural phenomena.

F. Measured mean path lengths

The natural logarithm theorems imply that the measured mean path lengths for isotropic networks should be close to the value of \( e \approx 2.71828 \). That observation combined with The 4/3 RLT might explain some measurements of mean path lengths for actual networks.

The mean path length for English words has been measured as 2.67 (Ferrer i Cancho 2001), for neurons in *C. elegans* as 2.65 (Watts & Strogatz 1998) and for the neurons of the human brain as 2.49 (Achard et al. 2006), all close to the value of \( e \), consistent with these networks being isotropic energy (or information) distribution systems.

A human network receiving information (a Receipt) should have a mean path length equal to

\[
\frac{4}{3} \times e = \frac{4}{3} \times 2.71828 \approx 3.624. \quad (159)
\]

The mean path length for a network of 225,226 actors (Watts & Strogatz 1998) was measured as 3.65, pretty close to 3.624, less than one per cent different. These measurements are consistent with The 4/3 RLT, where the mean path length \( \ell \) of the Receipt was 4/3 the mean path length of the information distribution system, so that \( \ell = (4/3)L \). Concepts that relate to expansion of the universe thus appear to connect to the way that information is distributed within societies.

G. Efficiency and the number of sub-systems

According to The NRT and Jensen’s Inequality, for maximum efficiency the natural logarithm should scale systems; the number of subsystems per generation should equal the natural logarithm. But an organism might have a circulatory system, digestive system, and respiratory system, for example. Three systems depart from the ideal scaling afforded by scaling by the natural logarithm, approximately 2.71828. Does
that mean that organisms, for example, cannot achieve max-
imal efficiency, because a whole number of their sub-systems
can not equal the natural logarithm?

Natural logarithm scaling can be achieved if the functions
of the sub-systems that are discretely named overlap. One
should find that for sub-systems such as those comprising an
organism there is frequently overlap in their different func-
tions; part of the reason may be that natural logarithm scaling
is more efficient than scaling by a whole number such as 3.

XII. ENTROPY

A. Transmitting degrees of freedom

All of the foregoing suggests that energy transmission can
be modelled by the transmission of degrees of freedom. Per
radiative event, 4/3 degrees of freedom of a scaling radiation
distribution system cannot fit into a space that scales with only
1 degree of freedom, unless space distance grows radially by
4/3 relative to a corresponding radiation length. For one de-
gree of freedom of some system to transmit exactly, the other
system must be of the same kind – spatial or radiating – and
must also have one degree of freedom, a matching principle.

If one system is a homogeneously radiating system and the
other an isotropic recipient system, then the product of the
ratio of their relative degrees of freedom and the ratio of their
relative lengths is one.

B. Definitions of entropy

The historical development of entropy began with Sadi
Carnot’s 1824 monograph (Carnot 1960) that described an
ideal heat engine modelled on a steam engine. In 1848,
William Thomson, Lord Kelvin (Kelvin 1848) devised an ab-
solute temperature scale; a degree Kelvin varies in proportion
to the volume of an ideal gas.

Equipped with the concept of absolute temperature, for an
ideal heat engine’s reversible heat cycle Clausius in 1865 (p.
331, Clausius 1865) derived, in effect, that

\[
\frac{dQ_1}{T_1} = dS_1 = dS_2 = \frac{dQ_2}{T_2},
\]

where \( S \) represents entropy, \( dS \) represents change in entropy,
\( Q \) represents heat, \( dQ \) represents change in heat, \( T \) is tem-
perature in degrees Kelvin, subscript 1 is for the portion of the
ideal heat cycle during which heat is removed, and subscript
2 is for the portion of the ideal heat cycle during which heat
is added. Clausius coined the term entropy (p. 357, Clausius
1865).

Boltzmann’s \( H \) Theory (Boltzmann 1872, 1898) led to
a probability and logarithm-based description of entropy. Planck
gives a derivation of the logarithm-based characteriza-
tion of entropy (Part III, Ch. I, Planck 1914).

The physical nature of entropy (Vol. 1, Ch. 44, Feynman
1963) is obscured by the use of the degree of absolute temper-
ature in equation (160).

The degree Kelvin is derived from the Centigrade scale that
assigns one degree between the freezing and boiling of water,
based on the invented concept of temperature.

Entropy as the number of scalings of a scale factor of a
designated quantum amount is simpler. A simpler character-
ization of a concept is easier to understand and to deploy as
a problem solving tool, and, if it better and more fundamen-
tally describes physical reality, improves the likelihood that
the concept will help reveal an underlying physical process.

C. Entropy as degrees of freedom

The matching principle described in section XII A coupled
with perfect efficiency (an assumption in Carnot’s ideal heat
engine) requires that

\[
\text{Deg} \left( \frac{dQ_1}{T_1} \right) = \text{Deg} \left( \frac{dQ_2}{T_2} \right).
\]

Consider equation (160). For a given ideal heat engine with
temperature \( T_1 \) higher than temperature \( T_2 \), with \( e \) a unit of
energy and \( n \) the base of the scale factors, let

\[
dQ_1 = \log_e(e^{s_1+n})e \quad \text{and} \quad T_1 = \log_e(e^{m+n})e
\]

\[
dQ_2 = \log_e(e^{s_2})e \quad \text{and} \quad T_2 = \log_e(e^m)e.
\]

Then

\[
\text{Deg} \left( \frac{dQ_1}{T_1} \right) = \text{Deg} \left( \frac{dQ_2}{T_2} \right) = x \text{ deg}
\]

relative to the base \( e \) of the scale factors. When temperature
\( T_2 = \log_e(e^m)e \) increases to \( T_1 = \log_e(e^{m+n})e \), equality of
the left hand side and the middle expression in equation (164)
must result since the heat engine’s working substance matches
the temperature of the heat reservoir it is brought into contact
with.

XIII. DISCUSSION

A. How to describe The 4/3 RDFT

Which is the correct or notationally best characterization of
The 4/3 RDFT:

- \( \text{Deg}(S) = (4/3)\text{Deg}(R) \),
- \( \text{Deg}_e(S + R) = (4/3)\text{Deg}_e(S + R) \), or
- \( \text{Deg}(S + R) = (4/3)\text{Deg}(R) ? \)
The first choice is likely wrong. The 4/3 RDFT involves the relative degrees of freedom of a radiation volume compared to a spatial volume, and the system $S$ cannot be characterized as a volume, because from the dimensional point of view it has (like time) only one degree of freedom. A radiation volume can only be in $S + R$, since it has a volume, and so is not in $S$ and has radiation, and so is not in $R$. Radiation, space and time occur together in our perception, inhabited by stars and space and planets. This observation favours using $S + R$ instead of $S$ to describe where radiation volumes reside. The model of dark energy requires a comparison of the degrees of freedom of a radiation volume relative to a corresponding space.

Astronomical observation also suggests the first choice is wrong. If the first choice were correct, then it would follow that $\rho_S / \rho_M$ would be 64 : 27, but $\rho_S$ has been measured to be much smaller than $\rho_M$.

The second choice, using a subscript for $\text{Deg}$ emphasizes that The 4/3 RDFT compares a radiation volume in $S + R$ to a spatial volume in $R$. $\text{Deg}_R(S + R) = \text{Deg}(R)$ so a subscript $R$ for $S + R$ unnecessary; just use $\text{Deg}(R)$. $S + R$ when compared to $R$ necessarily connotes a radiation volume; so a subscript $S$ is unnecessary.

The advantage of the third choice over the second is succinctness.

### B. Dark energy’s difficulty

If dark energy can be solved using existing theories of cosmology, it must be a hard problem because it has so far defied solution.

If dark energy can not be solved using the existing theories of cosmology, then trying to solve it within cosmology is currently impossible.

It is impossible to know whether or not there is a missing theory that will solve dark energy without having a theory to test. Suppose it could somehow be inferred that there is a missing theory. Since there are for all practical purposes a limitless supply of phenomena that might provide a vital analogical clue to the missing theory, the task of identifying such a phenomenon for a theory physics does not know about is impossible.

Hence, solving dark energy within existing theories of cosmology is likely either very difficult or impossible.

Even after finding a 4/3 relationship, it is difficult to find the means by which such scaling is achieved. 4/3 scaling per generation may be mathematically impossible if there is only one system that has scaling; likely it would have to be scaling over time. 4/3 scaling per generation probably could only occur if there are two systems, but to all appearances we live in one universe.

We can not detect $S$ and $R$ as separate frames of reference through direct perception, but not only through observation and inferences based on those observations.

Suppose that there does exist some hitherto unidentified principle that can solve dark energy. Then the only way for it to be located is for that principle to be mathematically inferred in some other context. Then by seeking to test its validity in a variety of other contexts, eventually it might be tested against dark energy.

The hypothesis of this article is that The 4/3 RDFT is the missing principle and appears to be a law of nature, a theory of physics that nature deploys but that is hard to observe.

The question of whether average IQs improve because language improves led to the question of a scale factor for lexical growth, led to 3/4 metabolic scaling, led to a connection to Stefan’s Law and black body radiation, led to a radiation proof of The 4/3 RDFT, and led eventually to a possible explanation of dark energy. The chance of following such an inferential path is probably slight.

Likely, other paths also lead to The 4/3 RDFT. The development of the 4/3 scaling of degrees of freedom began with observations about 3/4 metabolic scaling, hardly an obvious starting point for deciphering dark energy. Many degrees of freedom in reasoning might have to be explored to find an explanation for dark energy. Perhaps The 4/3 RDFT is one of the degrees of problem-solving freedom that explains dark energy.

### C. The fluid equation

The fluid equation used in cosmology suggests that the formula for Hubble’s parameter transitions over time, as described in section IXB. The results in this article suggest that $H(t) = 1 / t$. If The 4/3 RLT is valid, and if the fluid equation cannot be reconciled with it, then the fluid equation may be inapplicable to the determination of Hubble time.

### D. Is The 4/3 RLT proved?

The 4/3 RDFT and The 4/3 RLT are logically equivalent. The 4/3 RDFT and The NRT can be derived from each other. I have not in this article set out all the various results from The NRT, but there are several that suggest The NRT is valid. Considering the number of proofs for The 4/3 RDFT and The 4/3 RLT alone, there are strong grounds for thinking The 4/3 RLT is valid. The NRT bolsters the case, in its applications including the natural logarithm theorems. If The 4/3 RDFT is a general law of nature, then Brownian motion, Clausius’s ratio of mean path lengths, Stefan’s Law, the scaling of energy densities, metabolic scaling, and dark energy are all instances of the general law.

Scaling and related concepts has been considered in connection with a variety of phenomena in three other articles on arXiv (Shour, Lexical growth; Isotropy; Intelligence) as well as in earlier versions of this article. Scaling and degrees of freedom suggest connections to, and possible explanations for, various problems, such as the ergodic hypothesis and the many worlds thesis (Everett 1957), lexical growth, increasing average IQs, economic growth rates, special relativity, quantum mechanics.

That complexity arises from the manifestation of degrees of freedom available to a variety of systems seems not unlikely.
The algebra of scaling, though unfamiliar in light of its novelty, can be simply described. The manner in which these ideas developed plays a role in my view of their plausibility. The question in September 2007 involving scaling was whether increasing information compression of language could explain increasing average IQs. Solutions raised questions which in turn led to other solutions and more questions. The solutions belong to nature; observation of nature’s solutions increases the chance that there is something to the preceding ideas.

E. Some questions

Do the following questions make sense, and if so, what are their answers?

- Are there cosmological observations other than those described in this article that are consistent with The 4/3 RDFT and The 4/3 RLT?
- Are their ratios in nature relating to the square (4/3)^2 : 1^2?
- Is it possible to measure radiation pressure, such as the pressure generated by the solar wind, to test The 4/3 RDFT?
- What is S?
- Why is \( \text{Deg}(\mathcal{R}) = 3 \)? Is that implicit in the nature of S or in the emergence of a universe or peculiar to our universe? Could there exist a universe with \( \text{Deg}(\mathcal{R}) \neq 3 \)? If so, what would be its attributes?
- Does The 4/3 RDFT determine the geometry of our universe? How?
- Is there a reason why radiation cones and space spheres, and space cones and radiation spheres form a kind of dual relationship?
- Is it possible to test whether the cosmic scale factor \( a(t) \) is the same at all epochs?
- Is 4/3 in the formula for the volume of a sphere related to The 4/3 RDFT?
- Is the 4/3 ratio of degrees of freedom theory in MIMO an instance of The 4/3 RDFT?
- If the measured mean path length for neurons in the human brain is accurate, why is it less than \( e \)?
- Can general relativity be described in terms of The 4/3 RDFT?
- Is there a way to relate tensors to The 4/3 RLT?
- Is energy at all scales related to The 4/3 RDFT or The 4/3 RLT at all scales?
- Does the concept of degrees of freedom explain Everett’s (1957) many worlds hypothesis?
- Does the concept of degrees of freedom resolve the problems raised by the ergodic hypothesis?
- Is \( s \times i \) a scale factor in Schrödinger’s equation analogous to the role of the scale factor \( s \) relative to \( L \)?
- Are there other phenomena that instance The 4/3 RDFT and The 4/3 RLT?
- Are dark energy and dark matter related?

If the ideas sketched in this article are valid, these questions may be only a beginning.

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