INTERPOLATION ANALOGUES OF SCHUR Q-FUNCTIONS

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Abstract. We introduce interpolation analogues of Schur Q-functions — the multiparameter Schur Q-functions. We obtain for them several results: a combinatorial formula, generating functions for one-row and two-rows functions, vanishing and characterization properties, a Pieri-type formula, a Nimmo-type formula (a relation of two Pfaffians), a Giambelli-Schur-type Pfaffian formula, a determinantal formula for the transition coefficients between multiparameter Schur Q-functions with different parameters. We write an explicit Pfaffian expression for the dimension of skew shifted Young diagram. This paper is a continuation of author’s paper math.CO/0303169 and is a partial projective analogue of the paper by A. Okounkov and G. Olshanski q-alg/9605042 and of the paper by G. Olshanski, A. Regev and A. Vershik math.CO/0110077.

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1. Introduction

This text is an announcement of the results. In most cases we give only the main ideas of the proofs. The detailed proofs will be presented later.
In 1911 in the paper [Sc] I. Schur introduced the Q-functions in order to study the projective representations of the symmetric groups. The projective representations of the symmetric group \(S(n)\) are linearized by the spin-symmetric group \(\tilde{S}(n)\) (one of two possible central \(\mathbb{Z}_2\)-extensions of the group \(S(n)\)). The Schur Q-functions \(Q_\lambda\), where \(\lambda\) runs over partitions of a number \(n\) on distinct parts, play the same role in the representation theory of the spin-symmetric groups ([Sc], [HH], [SL], [Jo]) as the conventional Schur S-functions in the representation theory of the symmetric groups ([Ma, Ch.I, §7]). A "projective" analogue of the Schur-Weyl duality for \(gl(N)\) and \(S(n)\) ([We]) was obtained by A. Sergeev in [Se1]. A. Sergeev introduced a new group (now it is called the Sergeev group) instead of \(S(n)\) and replaced the Lie algebra \(gl(N)\) by the Lie superalgebra \(Q(N)\). Thus the Schur Q-functions, up to scalar factors, are equal to the characters of the irreducible representations of the Lie superalgebra \(Q(N)\), see also [Ya]. Besides the Schur Q-functions it is convenient to use Schur P-functions \(P_\lambda\), which differ from the Q-functions \(Q_\lambda\) by simple scalar factors (Definition 2.13). The Schur P-functions are \(t = -1\) specializations of the Hall-Littlewood polynomials and constitute a linear basis in the algebra of supersymmetric functions \(\Gamma\), see [Ma §8], [Pr], [HH] for more details. For a partition with distinct parts \(\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l > 0)\) the Schur P-function \(P_\lambda\) is the symmetric polynomial (suppose the number of variables \(n\) is greater than \(l\))

\[
P_\lambda(x_1, \ldots, x_n) = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} \prod_{i=1}^{l} x_{\omega(i)}^{\lambda_i} \prod_{i \leq l, i \leq j \leq n} \frac{x_{\omega(i)} + x_{\omega(j)}}{x_{\omega(i)} - x_{\omega(j)}}, \quad (1.1)
\]

\[
Q_\lambda(x_1, \ldots, x_n) = 2^l P_\lambda(x_1, \ldots, x_n), \quad (1.2)
\]

see Definition 2.8 for the details.

We replace the ordinary powers \(x^k\) by the generalized powers \((x - a_1)(x - a_2)\ldots(x - a_k)\) (we suppose \(a_1 = 0\)) in these definitions of the Schur P- and Q-functions and we obtain (Definition 2.10) new supersymmetric functions \(P_{\lambda; a}\) and \(Q_{\lambda; a}\). We call them the multiparameter Schur P- and Q-functions. The classical Schur Q-function \(Q_\lambda\) is the leading homogenous term of the function \(Q_{\lambda; a}\). As the ordinary Schur Q-functions \(Q_\lambda\) the multiparameter Schur Q-functions \(Q_{\lambda; a}\) constitute a linear basis in the algebra of supersymmetric functions. When all \(a_j\) are distinct, the function \(Q_{\lambda; a}\) can be viewed as a solution of some multivariate interpolation problem, see Section 3. For this reason we call these functions the interpolation analogues of the Schur Q-functions.
Earlier interpolation analogues for many other symmetric functions were found and studied: for analogues of the classical Schur S-functions, see \[BL\], \[Ok1\], \[OO1\], \[OO2\], and references in \[OO1\]; in the case of the supersymmetric Schur S-functions, see \[Mo\], \[ORV\]; in the case of the Jack functions, see \[KS\], \[KOO\], \[OO3\]; the case of the Macdonald polynomials, excepting the Hall-Littlewood polynomials, was considered in \[Kn\], \[Sa\], \[Ok2\], \[Ok3\], and references therein.

We obtain the following main results about the multiparameter Schur Q-functions:

- A Nimmo-type formula expressing \(P_{\lambda;a}\) as a relation of two Pfaffians (Section 3).
- A combinatorial formula expressing \(Q_{\lambda;a}\) in terms of marked shifted tableaux (Section 4). This formula has an additional symmetry as compared to its counterpart for the conventional Schur Q-functions. Also we rewrite our formula in terms of unmarked shifted tableaux.
- Generating functions for the one-row \(Q_{(r);a}\) and the two-rows \(P_{(r,s);a}\) multiparameter functions (Section 5). (Note that even in the ordinary case the formula for the generating function for the two-rows \(P_{(r,s)}\) seems to be new)
- A Giambelli-Schur-type Pfaffian formula that expresses an arbitrary multiparameter Schur Q-function \(Q_{\lambda;a}\) as a Pfaffian of the two-rows multiparameter Schur Q-functions (Section 9).

Also we find explicitly the transition coefficients between two bases \(\{Q_{\lambda;a}\}\) and \(\{Q_{\lambda,b}\}\) for two sequences \((a_k)\) and \((b_k)\) (Section 10). In particular we may express each of the multiparameter Schur Q-functions \(Q_{\lambda;a}\) as a linear combination of the ordinary Schur Q-functions \(Q_{\mu}\).

Suppose \(a_k = k - 1\); then we obtain an important particular case of the multiparameter Schur Q-functions. We call them the factorial Schur Q-functions. In Section 7 we calculate the number of shifted standard tableaux of a given skew shape in terms of the factorial P-functions (called a dimension of a skew shifted Young diagram). We obtain for this dimension a simple Pfaffian expression (Theorem 7.5). This dimension may be also rewritten in terms of the multiplicity of the restriction of irreducible representations of the spin-symmetric group or the Sergeev group to the smaller subgroup, for the details see, for example, \[HH\], \[Iv3\]. Using this formula with the factorial P-functions in \[Iv1\] we obtain a new proof of Nazarov’s theorems about the characters of the infinite spin-symmetric group (\[Na1\]). Note also that the factorial Schur P-functions possibly may be obtained from the Schubert polynomials (\[La1\], \[La2\], \[LP\], \[FK\], \[BH\]) but we did not verify
this fact. Certain problems related to the factorial Schur Q-functions are considered in [Na2], [Iv3].

The author is very grateful to G. Olshanski for his constant attention to this work and many valuable remarks, to A. Okounkov for the main Definition 2.9 to A. Borodin for the statement of Theorem S4.1 (The generating function for the two-rows multiparameter P-functions).

2. Notation and definitions

Definition 2.1. A polynomial \( f(x_1, \ldots, x_n) \) is called \textit{supersymmetric} if the following conditions hold:

1. \( f(x_1, \ldots, x_n) \) is symmetric in variables \( x_1, \ldots, x_n \);
2. for all \( 1 \leq i < j \leq n \) the polynomial

\[
  f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_{j-1}, -t, x_{j+1}, \ldots, x_n)
\]

does not depend on \( t \).

Supersymmetric polynomials in \( n \) variables form an algebra. Denote this algebra by \( \Gamma(n) \). The algebra \( \Gamma(n) \) is graded by degree of polynomials. The specialization \( x_{n+1} = 0 \) is a morphism of the graded algebras

\[
  \Gamma(n + 1) \to \Gamma(n).
\]

Definition 2.2. The algebra \( \Gamma \) of supersymmetric functions is the projective limit

\[
  \Gamma = \lim_{\leftarrow} \Gamma(n), \quad n \to \infty,
\]

in the category of graded algebras, taken with respect to morphisms (2.1). In other words, an element \( f \in \Gamma \) is a sequence \( (f_n)_{n \geq 1} \) such that:

1. \( f_n \in \Gamma(n), \, n = 1, 2, \ldots, \)
2. \( f_{n+1}(x_1, \ldots, x_n, 0) = f_n(x_1, \ldots, x_n) \) (the stability condition),
3. \( \sup_n \deg f_n < \infty. \)

Definition 2.3. A finite sequence \( \lambda = (\lambda_1 \geq \ldots \geq \lambda_k \geq 0) \) of decreasing positive integers is called a \textit{partition}. The number of non-zero parts of \( \lambda \) is called the \textit{length of the partition} \( \lambda \) and is denoted by \( \ell(\lambda) \). Let us denote by \( |\lambda| \) the sum of the parts of the partition \( \lambda \):

\[
  |\lambda| = \sum_{k=1}^{\ell(\lambda)} \lambda_k.
\]
If $|\lambda| = n$ we shall also write $\lambda \vdash n$. Also put $m_i(\lambda) = \# \{ k \mid \lambda_k = i \}$ for $i \geq 1$. Obviously, we have

$$\ell(\lambda) = \sum_{i \geq 1} m_i(\lambda), \quad |\lambda| = \sum_{i \geq 1} im_i(\lambda).$$

**Definition 2.4.** A partition is called *strict* if any two non-zero parts of it are distinct. We shall denote by $DP_n$ the set of the strict partitions of the number $n$. A partition is called *odd* if all non-zero parts of it are odd. We shall denote by $OP_n$ the set of the odd partitions of the number $n$. Also put

$$DP = \bigcup_{k \geq 0} DP_k, \quad OP = \bigcup_{k \geq 0} OP_k. \quad (2.2)$$

**Definition 2.5.** For the strict partition one may define the shifted Young diagram besides the simple Young diagram [Ma][Ch.I, §1]. If $\lambda$ is a strict partition, then the set

$$\{(i, j) \in \mathbb{Z}^2 \mid i \leq j \leq \lambda_i + i - 1, 1 \leq i \leq \ell(\lambda)\}$$

is called the *shifted Young diagram of the partition* $\lambda$ and is denoted by $D(\lambda)'$. It is useful to draw unit squares instead of points of $\mathbb{Z}^2$. We assume that the first coordinate axis is directed downwards, the second coordinate axis is directed to the right and the point $(0, 0)$ is in the left upper corner of the figure. For example, if $\lambda = (7, 4, 3, 1)$, then $D(\lambda)' =$

![Diagram](attachment:image.png)

**Proposition 2.6.** Let $r(x_1, \ldots, x_l)$ be a polynomial in variables $x_1, \ldots, x_l$. For arbitrary $n \geq l$ we put

$$R_n(x_1, \ldots, x_n) = r(x_1, \ldots, x_l) \prod_{i \leq l, i < j \leq n} \frac{x_i + x_j}{x_i - x_j} \quad (2.3)$$

and

$$\tilde{R}_n(x_1, \ldots, x_n) = \sum_{\omega \in S(n)} R_n(x_{\omega(1)}, \ldots, x_{\omega(n)}). \quad (2.4)$$
Then we have

a) $\tilde{R}_n$ is the polynomial in $x_1, \ldots, x_n$ and
\[ \deg \tilde{R}_n \leq \deg r; \]
b) $\tilde{R}_n$ is the supersymmetric polynomial;
c) if for some $i \neq j$ the polynomial $r(x_1, \ldots, x_l)$ is symmetric in $x_i, x_j$, then $\tilde{R}_n = 0$;
d) if $r(x_1, \ldots, x_l)$ is divisible by $x_1x_2 \ldots x_l$, then
\[ \tilde{R}_{n+1}(x_1, \ldots, x_n, 0) = (n + 1 - l)\tilde{R}_n(x_1, \ldots, x_n). \]

Proof. See [Iv1, Proposition 1.1].

Next we consider three special cases of Proposition 2.6 (Definitions 2.8, 2.9, and 2.10). First let us introduce generalizations of the ordinary power $x^n$.

**Definition 2.7.** Put
\[ (x \downarrow n) = \prod_{k=1}^{n} (x - k + 1) \]
for $n \geq 1$. Also put $(x \downarrow 0) = 1$. Suppose $(a_k)_{k \geq 1}$ is an arbitrary sequence of complex numbers. Put
\[ (x | a)^n = \prod_{k=1}^{n} (x - a_k), \quad (x | a)^0 = 1. \]

**Definition 2.8.** Suppose $\lambda$ is a partition, $\ell(\lambda) = l < n$. Put
\[ r(x_1, \ldots, x_l) = \frac{\prod_{i=1}^{l} x_i^{\lambda_i}}{(n - l)!}, \]
in Proposition 2.6. Namely, put
\[ P_{\lambda|n}(x_1, \ldots, x_n) = \frac{1}{(n - l)!} \sum_{\omega \in S(n)} \prod_{i=1}^{l} x_{\lambda_i} \prod_{i \leq l, i \leq j \leq n} \frac{x_{\omega(i) + \omega(j)}}{x_{\omega(i)} - x_{\omega(j)}}. \]

If $\lambda \in DP$ then the polynomials $P_{\lambda|n}$ are the specializations of the Hall-Littlewood polynomials when the parameter $t = -1$, see [Ma, Ch.III, §2, §8]. If $\ell(\lambda) > n$ then we put $P_{\lambda|n} = 0$. From Proposition 2.6 it follows that the sequence $(P_{\lambda|n})_{n \geq 1}$ defines the supersymmetric function, which is denoted by $P_{\lambda}$. The functions $P_{\lambda}$ are called the Schur $P$-functions. Note that if $\nu$ is not a strict partition then from Proposition 2.6 it follows that $P_{\nu} = 0$. So our definition differs at this point from the traditional definition of the Schur P-functions, see [Ma, Ch.III, §2, (2.2)].
Next definition is due to A. Okounkov.

**Definition 2.9.** Suppose \( l = \ell(\lambda) \leq n, \lambda \in DP \). Put

\[
F_{\lambda \mid n}(x_1, \ldots, x_n) = \prod_{i=1}^{l} (x_i \downarrow \lambda_i) \prod_{i \leq l, i < j \leq n} \frac{x_i + x_j}{x_i - x_j}.
\]

By definition, we put

\[
P_{\lambda \mid n}^* = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} F_{\lambda \mid n}(x_{\omega(1)}, \ldots, x_{\omega(n)}).
\]

It corresponds to the case

\[
r(x_1, \ldots, x_l) = \frac{\prod_{i=1}^{l} (x_i \downarrow \lambda_i)}{(n-l)!}
\]
in Proposition 2.6. If \( \ell(\lambda) > n \) then, by definition, put \( P_{\lambda \mid n}^* = 0 \). From Proposition 2.6 it follows that the sequence \( (P_{\lambda \mid n}^*)_{n \geq 1} \) defines the supersymmetric function \( P_{\lambda}^* \). We call these functions the factorial Schur P-functions.

Next definition generalizes both Definition 2.8 and Definition 2.9.

**Definition 2.10.** Suppose \( a = (a_k)_{k \geq 1} \) is an arbitrary sequence of complex numbers, \( \lambda \in DP, \ell(\lambda) = l < n \). Then put

\[
P_{\lambda \mid n}^a(x_1, \ldots, x_n) = \frac{1}{(n-l)!} \sum_{\omega \in S(n)} \prod_{i=1}^{l} (x_{\omega(i)} \mid a)^{\lambda_i} \prod_{i \leq l, i < j \leq n} \frac{x_{\omega(i)} + x_{\omega(j)}}{x_{\omega(i)} - x_{\omega(j)}}.
\]

If \( \ell(\lambda) > n \) then we put \( P_{\lambda \mid n}^a = 0 \). From Proposition 2.6 it follows that if \( a_1 = 0 \) then the sequence \( (P_{\lambda \mid n}^a)_{n \geq 1} \) defines the supersymmetric function, which is denoted by \( P_{\lambda \mid n}^a \). The functions \( P_{\lambda \mid n}^a \) are called the multiparameter Schur P-functions. Further in the text we suppose \( a_1 = 0 \).

**Proposition 2.11.** Suppose \( \lambda \in DP; \ell(\lambda) \leq n \), then

\[
P_{\lambda \mid n}(x_1, \ldots, x_n) = P_{\lambda}(x_1, \ldots, x_n) + g(x_1, \ldots, x_n),
\]

where \( g(x_1, \ldots, x_n) \) is a supersymmetric polynomial such that \( \deg g < |\lambda| \).

**Proof.** It follows from Proposition 2.6, Definition 2.8 and Definition 2.10.

**Proposition 2.12.**

a) The set \( \{P_{\lambda} \mid \lambda \in DP\} \) is a linear basis of the algebra \( \Gamma \).
b) The set \( \{ P_{\lambda,a} \mid \lambda \in DP \} \) is a linear basis of the algebra \( \Gamma \). In particular, the set \( \{ P^{*}_{\lambda} \mid \lambda \in DP \} \) is a linear basis of the algebra \( \Gamma \).

Proof. a) It follows from [Pr, Theorem 2.11].

b) It follows from the assertion a) and Proposition 2.11. \( \square \)

Suppose the partition \( \nu \notin DP \); then from Proposition 2.6 it follows that \( P_{\nu,a} \equiv P^{*}_{\nu} \equiv 0 \).

Definition 2.13. For arbitrary partition \( \lambda \) we put

\[
Q_{\lambda} = 2^{\ell(\lambda)} P_{\lambda}, \quad Q^{*}_{\lambda} = 2^{\ell(\lambda)} P^{*}_{\lambda}, \quad Q_{\lambda;a} = 2^{\ell(\lambda)} P_{\lambda;a}.
\]

The supersymmetric functions \( Q_{\lambda} \) were introduced by I. Schur in [Sc]. They are called the Schur Q-functions. The supersymmetric functions \( Q^{*}_{\lambda} (Q_{\lambda;a}) \) are called the factorial (multiparameter) Schur Q-functions.

3. Nimmo-type formula

In this section we obtain the formula for \( P_{\lambda;a} \), which is an analogue of the formula for the ordinary Schur P-functions obtained by Nimmo ([Ni]).

Recall the definition of the Pfaffian of a skew-symmetric matrix.

Definition 3.1. Suppose \( A \) is a skew-symmetric matrix \( 2n \times 2n \). Put

\[
Pf(A) = \sum_{\omega \in \tilde{S}(2n)} \text{sgn}(\omega) \prod_{i=1}^{n} a_{\omega(2i-1)\omega(2i)},
\]

where the sum is over \( \omega \in \tilde{S}(2n) \subset S(2n) \) such that

\[
\omega(2i-1) < \omega(2i) \quad \text{and} \quad \omega(1) < \omega(3) < \cdots < \omega(2n-3) < \omega(2n-1).
\]

Theorem 3.2. Suppose \( n \geq \ell(\lambda) = l, \lambda \in DP, a = (a_k)_{k \geq 1} \) is an arbitrary sequence of complex numbers, \( a_1 = 0 \). Let \( A_0(x_1, \ldots, x_n) \) denote the skew-symmetric \( n \times n \) matrix

\[
\begin{pmatrix}
\frac{x_i - x_j}{x_i + x_j}
\end{pmatrix}_{1 \leq i,j \leq n}
\]

and let \( B_{\lambda} \) denote the \( n \times l \) matrix

\[
\left( (x_i a^{\lambda_i+1-j})_{i \leq n, j \leq l} \right).
\]

Let \( A_{\lambda}(x_1, \ldots, x_n) \) be the skew-symmetric \((n + l) \times (n + l)\) matrix

\[
\begin{pmatrix}
A_0(x_1, \ldots, x_n) & B_{\lambda}(x_1, \ldots, x_n) \\
-B_{\lambda}(x_1, \ldots, x_n)^t & 0
\end{pmatrix}.
\]
Put

\[ \text{Pf}_0(x_1, \ldots, x_n) = \begin{cases} \text{Pf}(A_0(x_1, \ldots, x_n)) & \text{if } n \text{ is even;} \\ \text{Pf}(A_0(x_1, \ldots, x_n, 0)) & \text{if } n \text{ is odd.} \end{cases} \]

Also put

\[ \text{Pf}_\lambda(x_1, \ldots, x_n) = \begin{cases} \text{Pf}(A_\lambda(x_1, \ldots, x_n)) & \text{if } n + l \text{ is even;} \\ \text{Pf}(A_\lambda(x_1, \ldots, x_n, 0)) & \text{if } n + l \text{ is odd.} \end{cases} \]

Then

\[ P_{\lambda,0}(x_1, \ldots, x_n) = \frac{\text{Pf}_\lambda(x_1, \ldots, x_n)}{\text{Pf}_0(x_1, \ldots, x_n)}. \]

**Proof.** The proof follows Nimmo’s method, see [Ni] or [Ma, Ch.III, §8, Example 13]. □

### 4. Combinatorial Formulas

**Definition 4.1.** Consider the ordered alphabet \( \mathbb{P}_n = \{1' < 1 < 2' < 2 < \cdots < n' < n\} \). By definition, put

\[ |k'| = |k| = k, \quad \text{sgn}(k) = -\text{sgn}(k') = 1 \]

for an arbitrary natural number \( k \).

**Definition 4.2.** Let \( \lambda \) be an arbitrary strict partition. A *marked shifted Young tableaux of shape \( \lambda \) and length \( n \) is a mapping \( T : D'_\lambda \rightarrow \mathbb{P}_n \) such that the following conditions hold:

1. \( T(i, j) \leq T(i + 1, j) \), \( T(i, j) \leq T(i, j + 1) \);
2. for each natural number \( k \) there is at most one \( k \) in the image of each column and at most one \( k' \) in the image of each row of \( D'_\lambda \).

Let us denote by \( \text{MSTab}(\lambda, n) \) the set of all shifted marked Young tableaux of shape \( \lambda \) and length \( n \).

Here is an example of the shifted Young tableaux of shape \((7, 4, 3, 1)\) and length 6:

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 3' & 4' & 4 & 6' \\
2' & 3' & 3 & 4' \\
3 & 4' & 4 \\
4' \\
\end{array}
\]
**Theorem 4.3.** (The combinatorial formula for the multiparameter Schur Q-functions)

\[ Q_{\lambda;a}(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{MST}_{ab}(\lambda, n)} \prod_{(i,j) \in D^{\prime \prime}_{\lambda}} (x_{|T(i,j)|} - \text{sgn}(T(i,j))a_{j-i+1}). \]

**Proof.** The combinatorial formula for the factorial Schur Q-functions (see Corollary 4.4 below) is proved in [Iv2]. We may use the same method in the case of the multiparameter Schur Q-functions with evident changes. □

**Corollary 4.4.** (The combinatorial formula for the factorial Schur Q-functions)

\[ Q^{\ast}_{\lambda}(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{MST}_{ab}(\lambda, n)} \prod_{(i,j) \in D^{\prime \prime}_{\lambda}} (x_{|T(i,j)|} - \text{sgn}(T(i,j))(j-i)). \]

The combinatorial formulas for the other interpolation analogues of the symmetric functions were obtained earlier in [OO1 §11], [Ok2], [Mo], [ORV].

We may reformulate Theorem 4.3 in terms of unmarked shifted tableaux.

**Definition 4.5.** Suppose \( \mu \) and \( \lambda \) are strict partitions, \( D(\mu)^{\prime} \subseteq D(\lambda)^{\prime} \); then we write \( \mu \subset \lambda \). Let us consider in this case the skew shifted diagram \( D(\lambda)^{\prime} \setminus D(\mu)^{\prime} \). For each box \( b \in D(\lambda)^{\prime} \setminus D(\mu)^{\prime} \) let us denote by \( i(b) \) and \( j(b) \) its first and second coordinates respectively. The skew shifted diagram \( \nu = D(\lambda)^{\prime} \setminus D(\mu)^{\prime} \) is called a border strip if the following conditions hold:

1. The diagram \( \nu \) has no \( 2 \times 2 \) block of squares.
2. The set \( I(\nu) = \{ j(b) - i(b) \mid b \in \nu \} \subset \mathbb{Z}_+ \cup \{0\} \) is an interval of integers, i.e. \( I(\nu) = [\min I(\nu), \max I(\nu)] \cap \mathbb{Z} \). When this condition is satisfied we say that \( \nu \) is connected.

Following [ORV §4] consider interior sides of the squares of the shape \( \nu \): an interior side is adjacent to two squares of \( \nu \); the total number of the interior sides is equal to \(|\nu| - 1\). To each interior side \( s \) we attach the coordinates \((\varepsilon, \delta)\) of its midpoint and we write \( s = (\varepsilon, \delta) \). Note that one of the coordinates is always half-integer while another coordinate is integer.

Suppose \( a = (a_n)_{n \geq 1} \) is a sequence of complex numbers, \( \nu \) is a border strip; then put

\[ f(\nu; a; x) = 2x \prod_{\varepsilon} (x + a_{\delta-\varepsilon+1/2}) \prod_{\delta} (x - a_{\delta-\varepsilon+1/2}). \]
where $\prod'$ is taken over the horizontal interior sides $s = (\varepsilon, \delta)$ of $\nu$, and $\prod''$ is taken over the vertical interior sides $s = (\varepsilon, \delta)$ of $\nu$.

Suppose $\nu = D(\lambda)' \setminus D(\mu)'$ has no $2 \times 2$ block of squares; then we may represent $\nu$ as a disjoint union of the minimal number of border stripes $\nu = \bigcup_{j=1}^{k} \nu_j$, i.e., such that $\forall i, j$ the set $\nu_i \cup \nu_j$ is not connected in the sense of the condition 2. In this case we put

$$f(\nu; a; x) = \prod_{j=1}^{k} f(\nu_j; a; x).$$

A (unmarked) shifted tableau $T$ of shape $\lambda$ and length $n$ is a sequence of strict partitions $\varnothing = \lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \lambda^{(n)} = \lambda$ such that each $D(\lambda^{(j)})' \setminus D(\lambda^{(j-1)})'$ has no $2 \times 2$ block of squares. In this case we put

$$f(T; a; x_1, \ldots, x_n) = \prod_{j=1}^{n} f((D(\lambda^{(j)})' \setminus D(\lambda^{(j-1)})'); a; x_j).$$

**Theorem 4.6.** As always we suppose that $a = (a_n)_{n \geq 1}$ is a sequence of complex numbers such that $a_1 = 0$; then we have

$$Q_{\lambda; a}(x_1, \ldots, x_n) = \sum_{T} f(T; a; x_1, \ldots, x_n)$$

summed over all shifted tableaux of shape $\lambda$ and length $n$.

**Proof.** We may easily deduce this theorem from Theorem 4.3. \qed

5. **Characterisation properties**

**Definition 5.1.** Suppose $\mu$ is a strict partition, $a = (a_n)_{n \geq 1}$ is a sequence of complex numbers; then, by definition, we put

$$H_a(\mu) = \prod_{k=1}^{\ell(\mu)} (a_{\mu_k} + 1 | a)_{\mu_k} \prod_{i<j} \frac{a_{\mu_i} + 1 - a_{\mu_j} + 1}{a_{\mu_i} + 1 - a_{\mu_j} + 1} =$$

$$\prod_{i<j} \left( a_{\mu_i} + 1 + a_{\mu_j} + 1 \prod_{k=\mu_j+2}^{\mu_i} (a_{\mu_i} + 1 - a_k) \right).$$

(5.1)

In the "factorial" case, i.e., when $a_k = k - 1$ the expression $H_a(\mu)$ becomes the "shifted product of hook lengths" (see Ma Ch.III, §7, examples): $H(\mu) = \prod_{t=1}^{\ell(\mu)} \mu_t! \prod_{i<j} \frac{\mu_i + \mu_j}{\mu_i - \mu_j}.$
**Definition 5.2.** Suppose $\lambda$ is a strict partition, $a = (a_n)_{n \geq 1}$ is a sequence of complex numbers; then let us define the collection of variables $x(\lambda)$ by means of formula
\[
x(\lambda)_i = a_{\lambda_i+1}.
\]

**Theorem 5.3.** *(Vanishing property)* Assume that the numbers $a_j$ are pairwise distinct and $a_1 = 0$.

a) If $\mu, \lambda \in DP$, $\mu \nsubseteq \lambda$ then $Q_{\mu,a}(x(\lambda)) = P_{\mu,a}(x(\lambda)) = 0$.

b) $P_{\mu,a}(x(\mu)) = H_a(\mu)$.

**Proof.** a) This claim follows from Definition 2.10.

b) This formula can be readily obtained by a direct computation. \qed

Next we write explicitly Theorem 5.3 in the "factorial" case.

**Corollary 5.4.**

a) If $\mu, \lambda \in DP$, $\mu \nsubseteq \lambda$ then $Q^*_{\mu}(x(\lambda)) = P^*_{\mu}(x(\lambda)) = 0$.

b) $P^*_{\mu}(x(\mu)) = H_a(\mu)$.

Following the method of [Ok1], [OO1] we may obtain characterization theorems for the multiparameter Schur P-functions.

**Theorem 5.5.** *(Characterization Theorem I)* Assume that the numbers $a_j$ are pairwise distinct. Suppose $f \in \Gamma$ satisfies the following conditions:

1. The top degree homogenous component of $f$ is equal to
\[
\sum_{\mu \in DP_n} c_{\mu} P_{\mu}.
\]

2. $\forall \lambda \in DP$ such that $|\lambda| < n$ we have $f(x(\lambda)) = 0$.

Then
\[
f \equiv \sum_{\mu \in DP_n} c_{\mu} P_{\mu,a}.
\]

**Proof.** We may use Proposition 2.11 and Theorem 5.3. \qed

**Theorem 5.6.** *(Characterization Theorem II)* Assume that the numbers $a_j$ are pairwise distinct. Suppose $f \in \Gamma$ satisfies the following conditions for some strict partition $\mu$:

1. $f(x(\mu)) = H_a(\mu)$.

2. $\deg f \leq |\mu|$.

3. $\forall \lambda \in DP$ such that $|\lambda| \leq |\mu|$ and $\lambda \neq \mu$ we have $f(x(\lambda)) = 0$.

Then $f \equiv P_{\mu,a}$. \qed
Proof. Again we use Proposition 2.11 and Theorem 5.3

Note that these characterization theorems may be easy reformulated in the particular "factorial" case.

Theorem 5.5 and Theorem 5.6 show that the multiparameter Schur P- and Q-functions can be considered as the solutions of the multivariate interpolation problems. Earlier such interpolation analogues for other classical symmetric functions were obtained, see [Ok1], [OO1], [OO2], [Mc], [ORV], [KS], [OO3], [Kn], [Ok2], [Sa]. The general scheme of such interpolation for the symmetric polynomials was considered in [Ok3].

6. Pieri-type formula

Definition 6.1. For \( \mu, \lambda \in DP \) we will write \( \mu \nearrow \lambda \), if \(|\mu| + 1 = |\lambda|\) and \( \mu \subset \lambda \) (Definition 4.5).

Recall that
\[
P_{(1)a}(x) = P^*_a(x) = P_{(1)}(x) = \sum_j x^j.
\]

Theorem 6.2. As usual we assume \( a_1 = 0 \). Then we have for an arbitrary \( \mu \in DP \)
\[
P_{\mu,a}(P_{(1)} - \ell(\mu)) = \sum_{\mu \nearrow \lambda} P_{\lambda,a}.
\]

Proof. First we consider the case when all \( a_j \) are pairwise distinct. Then (6.1) may be easily deduced from Theorem 5.3. In the general case we use the continuity of the both sides of (6.1). □

Corollary 6.3. For arbitrary \( \mu \in DP \) we get
\[
P^*_\mu(P_{(1)} - |\mu|) = \sum_{\mu \nearrow \lambda} P^*_\lambda.
\]

7. Dimensions of skew shifted Young diagrams

Definition 7.1. Suppose \( \mu \) and \( \lambda \) are strict partitions and \( \mu \subset \lambda \) (Definition 4.5). A standard shifted Young tableau \( T \) of shape \( \lambda/\mu \) is a bijection \( T : D(\lambda)' \setminus D(\mu)' \to \{1,2,\ldots,|\lambda| - |\mu|\} \) such that \( T(i,j) < T(i+1,j) ; T(i,j) < T(i,j+1) \). The number of the standard shifted tableaux of shape \( \lambda/\mu \) is called a dimension of skew shifted Young diagram \( \lambda/\mu \) and is denoted by \( g_{\lambda/\mu} \).

Now we give another description of this dimension. First we define a Schur graph (first considered in [Bo], [BO]).
Definition 7.2. The set of vertices of the Schur graph is labelled by the set $DP$. The Schur graph is an acyclic directed graph; there is an edge directed from $\mu \in DP$ to $\lambda \in DP$ if and only if $\mu \not\succ \lambda$ (Definition 6.1).

Proposition 7.3. There exists a directed path in the Schur graph from $\mu \in DP$ to $\lambda \in DP$ if and only if $\mu \subset \lambda$. In this case the number of these directed paths is equal to $g_{\lambda/\mu}$.

Proof. The proof is trivial.

An explicit formula for $g_{\lambda/\emptyset}$ is well-known ([Ma, Ch.III, §7, Examples]):

$$g_{\lambda/\emptyset} = \frac{|\lambda|!}{\prod_{k=1}^{\ell(\lambda)} \lambda_k!} \prod_{i<j} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j}.$$ 

In the next theorem we obtain an explicit expression for $g_{\lambda/\mu}$.

Theorem 7.4. Suppose $\mu, \lambda \in DP$, $\mu \subset \lambda$; then

$$g_{\lambda/\mu} = g_{\lambda/\emptyset} \frac{P^*_\mu(\lambda_1, \ldots, \lambda_{\ell(\lambda)})}{(|\lambda| - |\mu|)}.$$ (7.1)

Proof. This theorem is proved in [IV1] by a direct computation. Also following [OO1, §9] we may deduce this theorem from the Pieri-type formula (Corollary 6.3).

Earlier in [OO1] A. Okounkov and G. Olshanski studied the case of the Young graph which is connected with the Schur S-functions and the linear representations of the symmetric groups. Theorem 7.4 may be viewed as the projective analogue of [OO1] Theorem 8.1], see also [ORV]. The analogous result for the more general case of the Jack graph was obtained in [OO3, Section 5].

Now we rewrite the expression (7.1) in the Pfaffian’s form. In the ordinary case the analogous formula has a determinantal form, see, for example, [ORV] Proof of Proposition 1.2

Theorem 7.5. Suppose $\mu, \lambda \in DP$, $\mu \subset \lambda$. Let $X(\lambda_1, \ldots, \lambda_k)$ denote the skew-symmetric $k \times k$ matrix

$$\begin{pmatrix} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} & \frac{1}{\lambda_i!\lambda_j!} \\ \frac{1}{\lambda_i!\lambda_j!} & \frac{1}{\lambda_k!\lambda_j!} \end{pmatrix}_{1 \leq i,j \leq k}$$

and let $Y(\lambda_1, \ldots, \lambda_k; \mu)$ denote the $k \times \ell(\mu)$ matrix

$$\begin{pmatrix} \frac{1}{(\lambda_i - \mu_j)!} \\ \frac{1}{(\lambda_i - \mu_j)!} \end{pmatrix}_{\lambda_k, \lambda_j \leq \ell(\mu)}.$$
where we suppose $\frac{1}{m} = 0$ if $m < 0$. Put $s = \ell(\lambda) + \ell(\mu)$. If $s$ is even then let $A_{\lambda/\mu}$ be the skew-symmetric $s \times s$ matrix

$$
\begin{pmatrix}
X(\lambda_1, \ldots, \lambda_{\ell(\lambda)}) & Y(\lambda_1, \ldots, \lambda_{\ell(\lambda)}; \mu) \\
-Y(\lambda_1, \ldots, \lambda_{\ell(\lambda)}; \mu)^t & 0
\end{pmatrix}.
$$

If $s$ is odd then let $A_{\lambda/\mu}$ be the skew-symmetric $(s + 1) \times (s + 1)$ matrix

$$
\begin{pmatrix}
X(\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0) & Y(\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0; \mu) \\
-Y(\lambda_1, \ldots, \lambda_{\ell(\lambda)}, 0; \mu)^t & 0
\end{pmatrix}.
$$

Then

$$
g_{\lambda/\mu} = (|\lambda| - |\mu|)! \operatorname{Pf}(A_{\lambda/\mu}). \quad (7.2)
$$

Proof. The formula (7.2) can be easily deduced from Theorem 3.2 and Theorem 7.4. \qed

8. Generating functions

It is known (see e.g. [Ma, III, §2, (2.10)]) that

$$
\sum_{r=0}^{\infty} Q_r(x_1, x_2, \ldots) y^r = \prod_{j=1}^{\infty} \frac{1 + x_j y}{1 - x_j y}.
$$

Definition 8.1. Let $a = (a_n)_{n \geq 1}$ be a sequence of complex numbers, then denote by $(\tau a)$ the result of shifting the sequence $a$ to the left:

$$(\tau a)_n = a_{n+1}.$$

Thus,

$$(x|\tau a)^k = \prod_{j=2}^{k+1} (x - a_j).$$

Theorem 8.2.

$$
\sum_{r=0}^{\infty} \frac{Q_r(x_1, x_2, \ldots)}{(u|\tau a)^r} = \prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j}.
$$

Proof. By our assumption (Definition 2.10) $a_1 = 0$. Then the following identity holds:

$$
1 + 2 \sum_{r=1}^{\infty} \frac{(x|a)^r}{(u|\tau a)^r} = \frac{u + x}{u - x}.
$$

Then we may reason as in the proof of [Ma, III, §2,(2.10)]. \qed

Corollary 8.3.

$$
\sum_{r=0}^{\infty} \frac{Q^*_r(x_1, x_2, \ldots)}{(u \downarrow r)} = \prod_{i=1}^{\infty} \frac{u + 1 + x_i}{u + 1 - x_i}.
$$
Next formula is due to A. Borodin.

**Theorem 8.4.** If $0 \leq k \leq l$ then put $P_{(k,l)\alpha} \equiv -P_{(l,k)\alpha}$. Then we have

$$
4uv(u + v) \sum_{k,l \leq 0} \frac{P_{(k,l)\alpha}(x_1, x_2, \ldots)}{(u|\alpha)^{k+1}(v|\alpha)^{l+1}} =
\begin{align*}
&u \left( \prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j} + 1 \right) \left( -\prod_{j=1}^{\infty} \frac{v + x_j}{v - x_j} + 1 \right) - \\
&v \left( -\prod_{j=1}^{\infty} \frac{u + x_j}{u - x_j} + 1 \right) \left( \prod_{j=1}^{\infty} \frac{v + x_j}{v - x_j} + 1 \right). 
\end{align*}
(8.1)

Proof. The equality (8.1) is equivalent to following two equalities:

\begin{align*}
P_{(k+1,l)\alpha} + P_{(k,l+1)\alpha} + (a_{k+1} + a_{l+1})P_{(k,l)\alpha} &= P_{(k)\alpha}P_{(l+1)\alpha} - P_{(k+1)\alpha}P_{(l)\alpha} + (a_{l+1} - a_{k+1})P_{(k)\alpha}P_{(l)\alpha}; \\
P_{(k+1)\alpha} + P_{(k)\alpha} + a_{k+1}P_{(k)\alpha} &= P_{(k)\alpha}P_{(l)\alpha}. 
\end{align*}
(8.2)

The equality (8.3) is a particular case of Theorem 6.2. Let us prove (8.2). From [Ma, Ch.III, §5, (5.7)] we get the following identity for the ordinary Schur P-functions

\begin{align*}
P_{(k+1,l)\alpha} + P_{(k,l+1)\alpha} + 2 \sum_{j=1}^{l-1} P_{(k+1,j,l-j)}.
\end{align*}

Consequently we have

\begin{align*}
P_{(k+1,l)} + P_{(k,l+1)} = P_{(k)\alpha}P_{(l+1)\alpha} - P_{(k+1)\alpha}P_{(l)\alpha}. 
\end{align*}
(8.4)

First we suppose that the numbers $a_j$ are pairwise distinct; then from (8.4) and Theorem 5.5 we get that

\begin{align*}
P_{(k+1,l)\alpha} + P_{(k,l+1)\alpha} + \alpha P_{(k,l)\alpha} &= P_{(k)\alpha}P_{(l+1)\alpha} - P_{(k+1)\alpha}P_{(l)\alpha} + (a_{l+1} - a_{k+1})P_{(k)\alpha}P_{(l)\alpha} 
\end{align*}
for some $\alpha$. Evaluating both sides of (8.5) at the point $(a_{k+1}, a_{l+1})$ we obtain that $\alpha = a_{k+1} + a_{l+1}$. Thus (8.2) is proved for sequences $(a_j)$ without repetitions. Using the continuity argument we have that (8.2) holds for all sequences $(a_j)$. This concludes the proof.

One may compare Theorem 8.4 with its analog for the multiparameter supersymmetric Schur functions ([ORV, Proposition 7.1]).
9. **Giambelli-Schur-type formula**

**Theorem 9.1.** As in Theorem 8.4, we put \( Q_{(k,l);a} \equiv -Q_{(l,k);a} \) for arbitrary integers \( l \geq k \geq 0 \). Also we will use the notation \( \lambda_{l(\lambda)+1} \), certainly, \( \lambda_{l(\lambda)+1} = 0 \). If \( \lambda \) is a strict partition then we have

\[
Q_{\lambda,a} = Pf \left( (Q_{(\lambda_i,\lambda_j);a})_{1 \leq i,j \leq 2[\lambda(\lambda)+1]} \right),
\]

where \([x]\) stands for the integral part of \( x \).

**Proof.** We use Theorem 4.3 and the method of Stembridge \( \text{St2} \) with simple modifications. \( \square \)

I. Schur (\( \text{Sc} \)) used the particular case of this formula \( (a_j \equiv 0) \) as the definition of the Q-functions. This formula may be viewed as the projective analogue of the Giambelli formula for the ordinary s-functions [Ma, Ch.I, §3, Examples]. We may compare Theorem 9.1 with its determinantal analogue for the multiparameter supersymmetric Schur functions [ORV].

10. **Transition coefficients**

In this paragraph we use the method of [ORV §2, §7]. From [ORV Lemma 2.5] we get the following identity for arbitrary sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\)

\[
\frac{1}{(u-b_1) \ldots (u-b_{r+1})} = \sum_{r=r'}^{\infty} \frac{h_{r-r'}(b_1, \ldots, b_{r'+1}; -a_1, \ldots, -a_r)}{(u-a_1) \ldots (u-a_{r+1})},
\]

where \( h_0 = 1, h_1, h_2, \ldots \) denote the conventional complete homogenous functions in the super realization of the algebra \( \Lambda \) of the symmetric functions [Ma Ch.I, §5, Examples]. From now in this paragraph suppose that the sequences \((a_n)_{n \geq 1}\) and \((b_n)_{n \geq 1}\) are fixed and \( a_1 = b_1 = 0 \). Then

\[
h_k(b_1, b_2, \ldots, b_{r'+1}; -a_1, -a_2, \ldots, -a_r) = h_k(b_2, \ldots, b_{r'+1}; -a_2, \ldots, -a_r).
\]

Denote by \( d_{r,r'} \) the value \( h_k(b_2, \ldots, b_{r'+1}; -a_2, \ldots, -a_r) \), where \( r \geq r' \). If \( r < r' \) then put \( d_{r,r'} = 0 \).

**Proposition 10.1.** Let \( r, s \geq 1 \). Then

\[
P_{(r,s);a} = \sum_{r'=1}^{r} \sum_{s'=1}^{s} d_{r,r'}(a, b) d_{s,s'}(a, b) P_{(r', s');b},
\]
Proof. The proof is based on (10.1) and Theorem 8.4. □

Theorem 10.2. Suppose $\mu$ is a strict partition; then

$$Q_{\mu;\alpha} = \sum_{\nu \in DP, \ell(\nu) = \ell(\mu), \nu \subset \mu} d_{\mu\nu} Q_{\nu;\beta},$$

where

$$d_{\mu\nu} = \det \left( (d_{\mu_i\nu_j})_{1 \leq i,j \leq \ell(\mu)} \right).$$

Proof. We use Theorem 9.1 and Proposition 10.1. □

Theorem 10.2 is the projective analogue of [ORV, Theorem 7.3].

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