NONCOHERENCE OF THE MULTIPLIER ALGEBRA OF THE DRURY-ARVESON SPACE $H^2_n$ FOR $n \geq 3$

AMOL SASANE

Abstract. Let $H^2_n$ denote the Drury-Arveson Hilbert space on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$, and let $\mathcal{M}(H^2_n)$ be its multiplier algebra. We show that for $n \geq 3$, the ring $\mathcal{M}(H^2_n)$ is not coherent.

1. Introduction

The aim of this article is to investigate a certain algebraic property of rings, called coherence, which is a generalization of the property of being Noetherian, for a particular algebra of holomorphic functions in the unit ball in $\mathbb{C}^n$.

Definition 1.1 (Coherent ring). Let $R$ be a unital commutative ring, and for an $n \in \mathbb{N} := \{1, 2, 3, \cdots \}$, let $R^n = R \times \cdots \times R$ ($n$ times). If $f \in R^n$, say $f = (f_1, \cdots, f_n)$, then a relation $g$ on $f$, written $g \in f^\perp$, is an $n$-tuple $g = (g_1, \cdots , g_n) \in R^n$ such that $g_1 f_1 + \cdots + g_n f_n = 0$. The ring $R$ is said to be coherent if for each $n$ and each $f \in R^n$, the $R$-module $f^\perp$ is finitely generated.

A property which is equivalent to coherence is that the intersection of any two finitely generated ideals in $R$ is finitely generated, and the annihilator of any element is finitely generated [4]. We refer the reader to the monograph [7] for the relevance of the property of coherence in homological algebra. All Noetherian rings are coherent, but not all coherent rings are Noetherian. For example, the polynomial ring $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is not Noetherian (because the sequence of ideals $\langle x_1 \rangle \subset \langle x_1, x_2 \rangle \subset \langle x_1, x_2, x_3 \rangle \subset \cdots$ is ascending and not stationary), but $\mathbb{C}[x_1, x_2, x_3, \cdots]$ is coherent [7, Corollary 2.3.4].

For algebras of holomorphic functions in the unit disc

$$\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$$

in $\mathbb{C}$, it is known that the Hardy algebra $H^\infty(\mathbb{D})$, consisting of all bounded and holomorphic functions on $\mathbb{D}$ with pointwise operations, is coherent, while the disc algebra $A(\mathbb{D})$ (of all functions in $H^\infty(\mathbb{D})$ that admit a continuous extension to the closure of $\mathbb{D}$ in $\mathbb{C}$) is not coherent [8]. For $n \geq 3$, Amar

1991 Mathematics Subject Classification. Primary 16S15; Secondary 46E22, 47B32, 46J15, 13J99.

Key words and phrases. coherent ring, Drury-Arveson space, multiplier algebra.
showed that the Hardy algebra $H^\infty(\mathbb{B}_n)$, consisting of all bounded and holomorphic functions in the unit ball

$$\mathbb{B}_n := \{ z = (z_1, \cdots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1 \},$$

is not coherent. Related results about some other subalgebras of holomorphic functions in the ball and the polydisc were also obtained in [1]. Whether or not the Hardy algebra $H^\infty(D^2)$ (of the bidisc $\mathbb{D}^2$) and $H^\infty(\mathbb{B}^2)$ are coherent does not seem to be known.

The aim of this article is to prove the noncoherence of the multiplier algebra of the Drury-Arveson space in $\mathbb{C}^n$ with $n \geq 3$, and our main result is the following.

**Theorem 1.2.** For $n \geq 3$, $\mathcal{M}(H^2_n)$ is not coherent.

We give the pertinent definitions and notation below.

A multivariable analogue of the classical Hardy space on $D$ in $\mathbb{C}$ is the Drury-Arveson space $H^2_n$ on the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ [2], [5]. The space $H^2_n$ is a Hilbert function space that has a natural $n$-tuple of operators acting on it, giving it the structure of a Hilbert module, and has been the object of intensive study in the last decade or so owing to its relation to multivariable operator theory (for example the von Neumann inequality for commuting row contractions [3]) and multivariable function theory (for instance Nevanlinna-Pick interpolation [3]).

**Definition 1.3** (The Drury-Arveson space $H^2_n$). The Drury-Arveson space $H^2_n$ is a reproducing kernel Hilbert space of holomorphic functions on $\mathbb{B}_n$ with the kernel

$$K(z, w) = \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}_n.$$ We will use the standard multi-index notation: For $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+ := \{0, 1, 2, 3, \cdots\}$,

$$\alpha! := \alpha_1! \alpha_2! \cdots \alpha_n!, \quad |\alpha| := \alpha_1 + \cdots + \alpha_n, \quad \zeta^\alpha := \zeta_1^{\alpha_1} \cdots \zeta_n^{\alpha_n}.$$

**Definition 1.4** (The multiplier algebra $\mathcal{M}(H^2_n)$). A holomorphic function $f$ on $\mathbb{B}_n$ is called a multiplier for $H^2_n$ if $f \cdot H^2_n \subset H^2_n$.

$\mathcal{M}(H^2_n)$ is the ring of all multipliers on $H^2_n$ with pointwise operations. If $f$ is a multiplier, then the multiplication operator $M_f : H^2_n \rightarrow H^2_n$ corresponding to $f$ defined by

$$M_f(g) := fg, \quad g \in H^2_n,$$

is necessarily bounded on $H^2_n$ [2], and the multiplier norm of $f$ in $\mathcal{M}(H^2_n)$ is defined to be the operator norm of $M_f$. Then $\mathcal{M}(H^2_n)$ is a strict sub-algebra of $H^\infty(\mathbb{B}_n)$ if $n \geq 2$ [2]. If $n = 1$, then $H^2_n = H^2_1$ is the usual Hardy space of the disc, and $\mathcal{M}(H^2_1) = H^\infty(\mathbb{D})$, the Hardy algebra on the disc $\mathbb{D}$.

The proof of our main result, Theorem 1.2, is an adaption to the case of $\mathcal{M}(H^2_n)$ of the proof given in Amar [1] for showing the noncoherence of $H^\infty(D^n)$, $n \geq 3$. 
2. Preliminaries

The following result is shown along the same lines as the calculation done in [6, Lemma 2.3], where it was shown that

\[
\frac{z_2}{1 - sz_1} \in \mathcal{M}(\mathbb{H}_n^2)
\]

for all real \(s \in (0, 1)\).

Lemma 2.1. Let \(\alpha \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}\). The function \(G_\alpha : \mathbb{B}_n \to \mathbb{C}\), given by

\[
G_\alpha(z) = \frac{z_2}{(1 - \alpha z_1^2)^{1/4}}, \quad z = (z_1, \cdots, z_n) \in \mathbb{B}_n,
\]

belongs to \(\mathcal{M}(\mathbb{H}_n^2)\).

Before proving this result, we need some preliminaries from [6, Section 2], reproduced here for the convenience of the reader as they will play an essential role in the justification of Lemma 2.1. Let

\[
\mathcal{B} := \{(0, \beta_2, \cdots, \beta_n) : \beta_2, \cdots, \beta_n \in \mathbb{Z}_+\} \subset \mathbb{Z}^n_+.
\]

We will denote as before, the components of \(z\) by \(z_1, \cdots, z_n\). For each \(\beta \in \mathcal{B}\), define the closed linear subspace

\[
H_\beta = \text{span}\{z_1^k z_\beta : k \geq 0\}
\]

of \(\mathbb{H}_n^2\). Then we have the orthogonal decomposition

\[
\Pi_n^2 = \bigoplus_{\beta \in \mathcal{B}} H_\beta.
\]

For each \(\beta \in \mathcal{B}\), we have an orthonormal basis \(\{e_{k,\beta} : k \geq 0\}\) for \(H_\beta\), where

\[
e_{k,\beta}(z) = \sqrt{(k + |\beta|)! / k!|\beta|!} z_1^k z_\beta.
\]  

(2.1)

Then \(H_0 = H_1^2\), the Hardy space of the unit disc \(\mathbb{D}\). For the proof of Lemma 2.1, we need to identify each \(H_\beta, \beta \neq 0\), as a weighted Bergman space on the unit disc.

Let \(dA\) be the area measure on \(\mathbb{D}\) with the normalization \(A(\mathbb{D}) = 1\). For each integer \(m \geq 0\), let

\[
\mathcal{B}^{(m)} := L_\alpha^2\left(\mathbb{D}, (1 - |\zeta|^2)^m dA(\zeta)\right),
\]

the usual weighted Bergman space of weight \(m\). Then

\[
\{e_k^{(m)} : k \in \mathbb{Z}_+\}
\]

is the standard orthonormal basis for \(\mathcal{B}^{(m)}\), where

\[
e_k^{(m)}(\zeta) = \sqrt{(k + m + 1)! / k!m!} \zeta^k.
\]  

(2.2)
For each \( \beta \in B \setminus \{0\} \), define the unitary operator \( W_\beta : H_\beta \to B^{(|\beta|-1)} \) by
\[
W_\beta e_k,\beta = e_{(|\beta|-1)k}, \quad k \in \mathbb{Z}_+.
\] (2.3)

It follows from (2.1) and (2.2) that the weighted shift \( M_{z_1} |H_\beta \) is unitarily equivalent to \( M_\zeta \) on \( B^{(|\beta|-1)} \). Thus if \( \beta \in B \setminus \{0\} \), then
\[
W_\beta M_{z_1} h_\beta = M_\zeta W_\beta h_\beta \quad \text{for all } h_\beta \in H_\beta.
\]

Note that \( M_{z_1} |H_0 \) is the unilateral shift.

We will also need the following fact.

**Lemma 2.2.** \(|1 - \zeta^2|^{-1/2}dA(\zeta)\) is a Carleson measure for the Hardy space \( H^2_1 \) of the unit disc \( \mathbb{D} \).

**Proof.** For \( z = e^{i\varphi} \), where \( \varphi \in (-\pi, \pi] \), let
\[
S_\theta(z) := \{re^{it} : 1 - \theta \leq r < 1, |t - \varphi| \leq \theta\}.
\]

Then we have
\[
\int \int_{S_\theta(z)} |1 - \zeta^2|^{-1/2}dA(\zeta) = \int_{\varphi - \theta}^{\varphi + \theta} \int_{1 - \theta}^{1} \frac{1}{|1 - (re^{it})^2|^{1/2}} rdrdt
\]
\[
= \int_{\varphi - \theta}^{\varphi + \theta} \int_{1 - \theta}^{1} \frac{1}{\sqrt{4 - 2r^2 \cos(2t) + r^4}} rdrdt
\]
\[
\leq \int_{\varphi - \theta}^{\varphi + \theta} \int_{1 - \theta}^{1} \frac{1}{\sqrt{4 - 2r^2 + r^4}} rdrdt
\]
\[
= \int_{\varphi - \theta}^{\varphi + \theta} \int_{1 - \theta}^{1} \frac{1}{\sqrt{1 - r^2}} rdrdt
\]
\[
= \int_{\varphi - \theta}^{\varphi + \theta} \int_{0}^{1 - (1 - \theta)^2} \frac{1}{2\sqrt{u}} du dt \quad \text{(with } u = 1 - r^2\}
\]
\[
= \int_{\varphi - \theta}^{\varphi + \theta} \sqrt{u} \bigg|_{0}^{1 - (1 - \theta)^2} dt
\]
\[
= \int_{\varphi - \theta}^{\varphi + \theta} \sqrt{1 - (1 - \theta)^2} dt
\]
\[
\leq \int_{\varphi - \theta}^{\varphi + \theta} 1 dt = 2\theta.
\]

This completes the proof. \( \square \)

We are now ready to prove Lemma 2.1

**Proof of Lemma 2.1.** It is enough to consider the case when \( \alpha = 1 \). Let \( h_\beta \in H_\beta \), where \( \beta = (0, \beta_2, \cdots, \beta_n) \). Then
\[
h_\beta(z) = \sum_{k=0}^{\infty} c_k z_1^k z^{\beta}.
\]
First we assume that $\beta \neq 0$. By (2.3),

$$(W_\beta h_\beta)(\zeta) = \sqrt{\frac{\beta!}{(|\beta| - 1)!}} \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D}.$$  

Then $W_\beta h_\beta \in B^{[|\beta| - 1]}$. Denote $e_2 = (0, 1, \cdots, 0)$. Since $z_2 z^{\beta} = z^{\beta + e_2}$, we have

$$(W_{\beta + e_2} z_2 h_\beta)(\zeta) = \sqrt{\frac{(\beta + e_2)!}{|\beta|!}} \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D},$$  

and $W_{\beta + e_2} z_2 h_\beta \in B^{[\beta]}$. Now suppose that

$$h_\beta(z) = (1 - z^2_1)^{-1/4} f_\beta(z),$$  

where

$$f_\beta(z) = \sum_{k=0}^{\infty} a_k z_1^k z^{\beta}.$$  

For $\zeta \in \mathbb{D}$, we have $|1 - \zeta^2| \geq 1 - |\zeta|^2$, and so

$$|1 - \zeta^2|^{1/2} \geq (1 - |\zeta|^2)^{1/2} \geq 1 - |\zeta|^2.$$  

(2.4)

We have

$$\|z_2 (1 - z^2_1)^{-1/4} f_\beta\|_{H_n^2}^2$$

$$= \left\| z_2 h_\beta \right\|_{H_n^2}^2 = \left\| W_\beta + e_2 \zeta_2 h_\beta \right\|_{B^{[|\beta|]}}^2$$

$$= \frac{(\beta + e_2)!}{|\beta|!} \int_{\mathbb{D}} \left| \sum_{k=0}^{\infty} c_k \zeta^k \right|^2 (1 - |\zeta|^2)^{|\beta|} dA(\zeta)$$

$$= \frac{(\beta + e_2)!}{|\beta|!} \int_{\mathbb{D}} \left| \frac{1}{(1 - \zeta^2)^{1/4}} \sum_{k=0}^{\infty} \frac{1}{(1 - \zeta^2)^{1/2}} a_k \zeta^k \right|^2 (1 - |\zeta|^2)^{|\beta|} dA(\zeta)$$

$$= \frac{(\beta + e_2)!}{|\beta|!} \int_{\mathbb{D}} \sum_{k=0}^{\infty} a_k \zeta^k \left| (1 - |\zeta|^2)^{|\beta|} \right| dA(\zeta)$$

$$\leq \frac{(\beta + e_2)!}{|\beta|!} \int_{\mathbb{D}} \left( 1 - |\zeta|^2 \right)^{|\beta| - 1} dA(\zeta)$$

(2.4)

$$= \frac{\beta_2 + 1}{|\beta|} \frac{\beta!}{(|\beta| - 1)!} \int_{\mathbb{D}} \sum_{k=0}^{\infty} a_k \zeta^k \left| (1 - |\zeta|^2)^{|\beta| - 1} \right| dA(\zeta)$$

$$= \frac{\beta_2 + 1}{|\beta|} \left\| W_\beta f_\beta \right\|_{B^{[|\beta| - 1]}}^2 \leq \frac{\beta_2 + 1}{|\beta|} \left\| f_\beta \right\|_{H_n^2}^2 \leq 2 \left\| f_\beta \right\|_{H_n^2}^2.$$  

So we have shown that for $\beta \neq 0$, the norm of the restriction of the operator of multiplication by $z_2 (1 - z^2_1)^{-1/4}$ to $H_\beta$ does not exceed $\sqrt{2}$.
Next we consider the case when $\beta = 0$. We know that $H_0 = H^2_1$, the Hardy space on $\mathbb{D}$. Let $h \in H_0$. Then
\[
h(z) = \sum_{k=0}^{\infty} c_k z_1^k.
\]
we have
\[
(W_{e_2} z_2 h)(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^k, \quad \zeta \in \mathbb{D}
\]
and $W_{e_2} z_2 h$ belongs to the Bergman space $B(0)$. Now suppose
\[
h(z) = (1 - z_1^2)^{-1/4} f(z),
\]
for some
\[
f(z) = \sum_{k=0}^{\infty} a_k z_1^k.
\]
Then
\[
\| z_2 (1 - z_1^2)^{-1/4} f \|_{H^2_n}^2 = \| W_{e_2} z_2 h \|_{B(0)}^2
\]
\[
= \int_{\mathbb{D}} \left| \sum_{k=0}^{\infty} c_k \zeta^k \right|^2 dA(\zeta)
\]
\[
= \int_{\mathbb{D}} \left| \frac{1}{(1 - \zeta^2)^{1/4}} \sum_{k=0}^{\infty} a_k \zeta^k \right|^2 dA(\zeta)
\]
\[
= \int_{\mathbb{D}} \left| \sum_{k=0}^{\infty} a_k \zeta^k \right|^2 |1 - \zeta^2|^{-1/2} dA(\zeta)
\]
\[
\leq C \| f \|_{H^2_n}^2,
\]
where the last inequality follows from the fact that $|1 - \zeta^2|^{-1/2} dA(\zeta)$ is a Carleson measure for $H^2_1$ (Lemma 2.2 above). So we have shown that the norm of the restriction of the operator of multiplication by $z_2 (1 - z_1^2)^{-1/4}$ to $H_0$ does not exceed $\sqrt{C}$.

If $\beta \neq \beta'$, $f_\beta \in H_\beta$, and $f_{\beta'} \in H_{\beta'}$, then
\[
\frac{z_2}{(1 - z_1^2)^{1/4}} f_\beta \perp \frac{z_2}{(1 - z_1^2)^{1/4}} f_{\beta'}.
\]
Thus it follows from the two paragraphs above that the multiplication operator $M_{G_\alpha}$ corresponding to
\[
G_\alpha = \frac{z_2}{(1 - z_1^2)^{1/4}}
\]
is a continuous linear map on $H^2_n$, that is, $G_\alpha \in \mathcal{M}(H^2_n)$. This completes the proof. \qed
3. Noncoherence of $\mathcal{M}(\mathbb{H}_n^2)$

Proof of Theorem 1.2. We will prove the claim by contradiction. Suppose that $\mathcal{M}(\mathbb{H}_n^2)$ is a coherent ring. Let $f = (f_1, f_2) \in (\mathcal{M}(\mathbb{H}_n^2))^2$, where $f_1 := z_1$ and $f_2 := z_2$. As $\mathcal{M}(\mathbb{H}_n^2)$ is coherent, $f^\perp$ will be finitely generated, say by $h_1, \ldots, h_k$ in $(\mathcal{M}(\mathbb{H}_n^2))^2$. For $\alpha \in T$, define $g_\alpha = (g_{1,\alpha}, g_{2,\alpha})$ by

$$
    g_{1,\alpha}(z) := \frac{z_2}{(1 - \alpha z_2^2)^{1/4}},
    g_{2,\alpha}(z) := \frac{-z_1}{(1 - \alpha z_2^2)^{1/4}},
$$

for $z = (z_1, \ldots, z_n) \in \mathbb{B}_n$. Note that by Lemma 2.1, we know that $g_\alpha$ is in $\mathcal{M}(\mathbb{H}_n^2)$ for each $\alpha \in T$.

The rest of the proof is the same, mutatis mutandis, as the proof given in [1, Section 1, pages 69-71]. We repeat it here making sure that the implicit but straightforward changes needed in that proof to adapt it to our different situation, are made explicit here for the convenience of the reader.

Moreover,

$$
    f_1 g_{\alpha,1} + f_2 g_{\alpha,2} = z_1 \cdot \frac{z_2}{(1 - \alpha z_2^2)^{1/4}} + z_2 \cdot \frac{-z_1}{(1 - \alpha z_2^2)^{1/4}} = 0,
$$

and so $g_\alpha = (g_{1,\alpha}, g_{2,\alpha}) \in f^\perp$. Thus there exist $\gamma_{\alpha,i} \in \mathcal{M}(\mathbb{H}_n^2)$ such that

$$
    g_\alpha = \sum_{i=1}^k \gamma_{\alpha,i} h_i. \quad (3.1)
$$

If $h_i =: (r_i, s_i) \in (\mathcal{M}(\mathbb{H}_n^2))^2$, then we have

$$
    z_1 r_i + z_2 s_i = 0.
$$

So if $z_2 = 0$, then $z_1 r_i = 0$. Thus $r_i = 0$ on

$$
    \{ z = (z_1, \ldots, z_n) \in \mathbb{B}_n : z_2 = 0 \}.
$$

Hence there exist $t_i$, holomorphic in $\mathbb{B}_n$ such that

$$
    r_i(z) = z_2 t_i(z), \quad i = 1, \ldots, k, \quad z \in \mathbb{B}_n.
$$

So it now follows from (3.1) that

$$
    \frac{z_2}{(1 - \alpha z_2^2)^{1/4}} = \sum_{i=1}^k \gamma_{\alpha,i} z_2 t_i(z),
$$

that is,

$$
    \varepsilon_\alpha(z) := \frac{1}{(1 - \alpha z_2^2)^{1/4}} = \sum_{i=1}^k \gamma_{\alpha,i} t_i(z), \quad \alpha \in T, \quad z \in \mathbb{B}_n.
$$

Let $\alpha_1, \ldots, \alpha_k, \alpha_*$ be $k + 1$ distinct points on $T$. We interpret

$$
    \varepsilon_\alpha(z) = \sum_{i=1}^k \gamma_{\alpha,i} t_i(z) \quad (3.2)
$$
for these $k+1$ choices of $\alpha$ as a system of $k+1$ linear equations in $k$ unknowns, the $t_i(z)$'s:

$$
\begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} & \varepsilon_{\alpha_1} \\
\vdots & \vdots & \vdots & \\
\gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} & \varepsilon_{\alpha_k} \\
\gamma_{\alpha_+,1} & \cdots & \gamma_{\alpha_+,k} & \varepsilon_{\alpha_+}
\end{bmatrix}
\begin{bmatrix}
t_1 \\
\vdots \\
t_k \\
-1
\end{bmatrix}
= 0.
$$

(3.3)

Since (3.3) is solvable, we must have

$$
\det
\begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} & \varepsilon_{\alpha_1} \\
\vdots & \vdots & \vdots & \\
\gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} & \varepsilon_{\alpha_k} \\
\gamma_{\alpha_+,1} & \cdots & \gamma_{\alpha_+,k} & \varepsilon_{\alpha_+}
\end{bmatrix}
= 0.
$$

Expanding the determinant along the last column gives

$$
\det
\begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} \\
\vdots & \vdots & \\
\gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k} \\
\gamma_{\alpha_+,1} & \cdots & \gamma_{\alpha_+,k}
\end{bmatrix}
\cdot \varepsilon_{\alpha_+} = \sum_{i=1}^k \Lambda_{\alpha_+,i} \cdot \varepsilon_{\alpha_i},
$$

(3.4)

with $\Lambda_{\alpha_+,i} \in \mathcal{M}(\mathbb{H}_n^2) \subset \mathcal{H}^\infty(\mathbb{B}_n)$ (since the $\gamma_{\alpha_j,i} \in \mathcal{M}(\mathbb{H}_n^2)$). Now we consider the following two possible cases separately:

1° The determinant $\Delta$ is not identically 0 on the variety

$\mathcal{V} := \{z = (z_1, \ldots, z_n) \in \mathbb{B}_n : z_1 = z_2 = 0\}$.

2° $\Delta \equiv 0$ on $\mathcal{V}$.

Let us consider case 1° first. The map $z_3 \mapsto \Delta|_{\mathcal{V}}(0,0,z_3) : \mathbb{D} \to \mathbb{C}$ is holomorphic and bounded, independent of the $\alpha_+$. As $\Delta|_{\mathcal{V}}$ is not identically zero, there exists a point $\alpha_+ \in \mathbb{T}$, which is distinct from $\alpha_1, \ldots, \alpha_k$, such the radial limit of $\Delta|_{\mathcal{V}}(0,0,\cdot)$ is nonzero as $z_3 \to \overline{\alpha_+}^{-1/2}$. Then $z_3^2$ approaches $\overline{\alpha_+}$, and we see in (3.4) that the left hand side approaches $\infty$, while it is not the case that the right hand side approaches $\infty$ (because the $\Lambda_{\alpha_+,i}$ and the $\varepsilon_{\alpha_i}$, with $\alpha_j \neq \alpha_+$, stay bounded). This contradiction shows that this case can’t be possible.

So we now consider case 2°. Suppose that $\Delta = 0$ on $\mathcal{V}$ for every choice of $\alpha_1, \ldots, \alpha_k$ in $\mathbb{T}$. Let $\ell$ be the rank

$$
\ell := \text{rank}_\mathcal{V}
\begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,k} \\
\vdots & \vdots & \\
\gamma_{\alpha_k,1} & \cdots & \gamma_{\alpha_k,k}
\end{bmatrix}
:= \max_{z \in \mathcal{V}}\text{rank}_z
\begin{bmatrix}
\gamma_{\alpha_1,1}(z) & \cdots & \gamma_{\alpha_1,k}(z) \\
\vdots & \vdots & \\
\gamma_{\alpha_k,1}(z) & \cdots & \gamma_{\alpha_k,k}(z)
\end{bmatrix}.
$$
Thus \( \ell < k \) owing to the fact that \( \Delta = 0 \) on \( \mathcal{V} \). After a rearrangement (if necessary) of the \( \alpha_i \), we arrive at

\[
\det \begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} \\
\vdots & & \vdots \\
\gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell}
\end{bmatrix} \neq 0 \text{ on } \mathcal{V}.
\]

From (3.2), we can deduce that \( \ell \) can’t be zero. Indeed, otherwise all the \( \gamma_{\alpha_j,i} \equiv 0 \) on \( \mathcal{V} \) and by (3.2), we would have \( 1/(1-\alpha z^2)^{1/4} = 0, z \in \mathbb{D} \), which is clearly impossible. So we have that \( \ell \geq 1 \), and from the definition of the rank it follows that

\[
D_{ij} = \det \begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & \gamma_{\alpha_1,i} \\
\vdots & & \vdots & \vdots \\
\gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & \gamma_{\alpha_\ell,i} \\
\gamma_{\alpha_j,1} & \cdots & \gamma_{\alpha_j,\ell} & \gamma_{\alpha_j,i}
\end{bmatrix} \equiv 0 \text{ on } \mathcal{V} \text{ for all } i,j \in \{1, \cdots, k\}.
\]

We have

\[
\det \begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & \varepsilon_{\alpha_1} \\
\vdots & & \vdots & \vdots \\
\gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & \varepsilon_{\alpha_\ell} \\
\gamma_{\alpha_j,1} & \cdots & \gamma_{\alpha_j,\ell} & \varepsilon_{\alpha_j}
\end{bmatrix} = \det \begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} & t_1 & \cdots & t_k & \gamma_{\alpha_1,k} \\
\vdots & & \vdots & \vdots & & \vdots & \vdots \\
\gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell} & t_1 \gamma_{\alpha_\ell,1} & \cdots & t_k \alpha_{\alpha_\ell,k} & \gamma_{\alpha_\ell,k}
\end{bmatrix} = \sum_{i=1}^{k} t_i D_{ij} \equiv 0 \text{ on } \mathcal{V} \text{ for all } j \in \{1, \cdots, k\}.
\]

By expanding the determinant on the left hand side along the last column, we obtain

\[
\det \begin{bmatrix}
\gamma_{\alpha_1,1} & \cdots & \gamma_{\alpha_1,\ell} \\
\vdots & & \vdots \\
\gamma_{\alpha_\ell,1} & \cdots & \gamma_{\alpha_\ell,\ell}
\end{bmatrix} \cdot \varepsilon_{\alpha_j} = \sum_{i=1}^{\ell} \lambda_{\alpha_j,i} \cdot \varepsilon_{\alpha_i} \text{ on } \mathcal{V},
\]

with \( \lambda_{\alpha,i} \in \mathcal{M}(H^2_\mathcal{V}) \subset H^\infty(\mathbb{B}_n) \). If it is not the case that \( \delta \equiv 0 \) on \( \mathcal{V} \), then we repeat the argument in 1° (replacing \( \alpha_i \) by \( \alpha_j \)), and arrive at a contradiction. So we conclude that \( \delta \equiv 0 \) on \( \mathcal{V} \), but this contradicts the definition of the \( \ell \). Hence case 2° is impossible too.

Consequently, \( f^\perp \) is not finitely generated, and so \( \mathcal{M}(H^2_\mathcal{V}) \) is not coherent. \( \square \)
Acknowledgement: I would like to thank Professor Jingbo Xia (State University of New York at Buffalo) for showing me an outline of the proof of Lemma 2.1 and for several useful discussions relating to it.

REFERENCES

[1] E. Amar. Non cohérence de certains anneaux de fonctions holomorphes. *Illinois Journal of Mathematics*, 25:68-73, no. 1, 1981.
[2] W. Arveson. Subalgebras of $C^*$-algebras. III. Multivariable operator theory. *Acta Mathematica*, 181:159-228, no. 2, 1998.
[3] J.A. Ball and V. Bolotnikov. Interpolation problems for Schur multipliers on the Drury-Arveson space: from Nevanlinna-Pick to abstract interpolation problem. *Integral Equations Operator Theory*, 62:301-349, no. 3, 2008.
[4] S.U. Chase. Direct products of modules. *Transactions of the American Mathematical Society*, 97:457-473, 1960.
[5] S.W. Drury. A generalization of von Neumann’s inequality to the complex ball. *Proceedings of the American Mathematical Society*, 68:300-304, no. 3, 1978.
[6] Q. Fang and J. Xia. Commutators and localization on the Drury-Arveson space. *Journal of Functional Analysis*, 260:639-673, no. 3, 2011.
[7] S. Glaz. *Commutative coherent rings*. Lecture Notes in Mathematics, 1371, Springer-Verlag, Berlin, 1989.
[8] W.S. McVoy and L.A. Rubel. Coherence of some rings of functions. *Journal of Functional Analysis*, 21:76-87, no. 1, 1976.
[9] K. Zhu. *Operator theory in function spaces*. Monographs and Textbooks in Pure and Applied Mathematics, 139, Marcel Dekker, New York, 1990.