Genus-zero modular functors and intertwining operator algebras

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Abstract

In [H5] and [H7], the author introduced the notion of intertwining operator algebra, a nonmeromorphic generalization of the notion of vertex operator algebra involving monodromies. The problem of constructing intertwining operator algebras from representations of suitable vertex operator algebras were solved implicitly earlier in [H3]. In the present paper, we generalize the geometric and operadic formulation of the notion of vertex operator algebra given in [H1], [H2], [HL1], [HL2] and [H6] to the notion of intertwining operator algebra. We show that the category of intertwining operator algebras of central charge $c \in \mathbb{C}$ is isomorphic to the category of algebras over rational genus-zero modular functors (certain analytic partial operads) of central charge $c$ satisfying certain generalized meromorphicity. This result is one main step in the construction of genus-zero conformal field theories from representations of vertex operator algebras announced in [H5]. One byproduct of the proof of the present isomorphism theorem is a geometric construction of (framed) braid group representations from intertwining operator algebras and thus from representations of suitable vertex operator algebras.

0 Introduction

In this paper, we show that the category of intertwining operator algebras of central charge $c \in \mathbb{C}$ is isomorphic to the category of algebras over rational genus-zero modular functors of central charge $c$ satisfying a certain generalized meromorphicity. This result is one main step in the construction of genus-zero conformal field theories from representations of vertex operator algebras announced in [H5].
The geometric and operadic formulation of the notion of vertex operator algebra ([H1], [H2], [HL1], [HL2], [H6]) establishes a direct connection between vertex operator algebras and the geometry of genus-zero Riemann surfaces with extra structures. In particular, this formulation gave a simple and conceptual definition of vertex operator algebra: A vertex operator algebra is a meromorphic algebra over a vertex partial operad. The equivalence between this formulation and the algebraic formulation of the notion of vertex operator algebra ([B], [FLM]) provides a practical way to construct genus-zero holomorphic conformal field theories. See [H6] for a presentation of this theory.

On the other hand, vertex operator algebras are not enough for either the study of conformal field theories or mathematical problems related to vertex operator algebras. Most conformal field theories have monodromies which are described by multivalued operator-valued functions. In the study of the moonshine module vertex operator algebra constructed Frenkel, Lepowsky and Meurman [FLM] and related problems, it is often necessary to study intertwining operators. The study of these multivalued operator-valued functions and intertwining operators leads us naturally to the notion of intertwining operator algebra introduced in [H5] and [H7]. A construction of these algebras from representations of suitable vertex operator algebras were given implicitly in [H3]. In fact, even for a problem whose statement involves only vertex operator algebras, the solution often involves intertwining operator algebras. One example is the conceptual construction of the vertex operator algebra structure on the moonshine module given in [H4].

In this paper, we generalize the geometric and operadic formulation of the notion of vertex operator algebra to a geometric and operadic formulation of the notion of intertwining operator algebra. We construct a rational genus-zero modular functor (a certain analytic partial operad) from an intertwining operator algebra and show that the intertwining operator algebra gives an algebra satisfying a certain generalized meromorphicity over this partial operad. Since a generalized-meromorphic algebra over a rational genus-zero modular functor is a genus-zero weakly holomorphic conformal field theory (see [S1], [S2] and [F]), we obtain a construction of genus-zero weakly holomorphic conformal field theories from intertwining operator algebras. In particular, we obtain a geometric construction of (framed) braid group representations from vertex operator algebras and modules. Conversely, we also show that an algebra satisfying the generalized meromorphicity over such a partial operad (a generalized-meromorphic algebra over a rational genus-zero
modular functor) gives an intertwining operator algebra. The main result of the present paper is that the category of intertwining operator algebras of central charge $c \in \mathbb{C}$ is isomorphic to the category of canonically-generalized-meromorphic algebras over rational genus-zero modular functors of central charge $c$ (a subcategory of the category of generalized-meromorphic algebras over rational genus-zero modular functors of central charge $c$). In particular, we obtain a simple and conceptual definition of intertwining operator algebra: An intertwining operator algebra is a (canonically-)generalized-meromorphic algebra over a rational genus-zero modular functor.

The method used in the present paper is almost the same as in [H6], except that the duality properties for vertex operator algebras are replaced by the more complicated duality properties for intertwining operators. In fact, the constructions and the proofs in the present paper are adaptations or modifications of those in [H6]. Thus, in many steps, instead of giving detailed constructions or proofs, we shall indicate the differences between the discussions in the present paper and those in [H6], and refer the reader to the corresponding constructions or proofs in [H6] for more details.

We assume that the reader is familiar with the material in [H6]. In Section 1, we give definitions of genus-zero modular functor and rational genus-zero modular functor. We also construct flat connections on holomorphic vector bundles underlying rational genus-zero modular functors and show that rational genus-zero modular functors give representations of the (framed) braid groups in this section. A definition of intertwining operator algebra is given in Section 2. In Section 3, we construct a rational genus-zero modular functor and a generalized-meromorphic algebra over it from an intertwining operator algebra. The main result of the present paper is given in Section 4.

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1 Genus-zero modular functors

We first give definitions of genus-zero modular functor and rational genus-zero modular functor in terms of the language of partial operads. Modular functors were first introduced by G. Segal in his geometric formulation of
the notion of conformal field theory (see [S1] and [S2]). See also [H5] for the genus-zero case. The definitions given in this section are slightly different from the one in [S1] and [S2] (restricted to the genus-zero case) because we are using them to prove precise theorems. We formulate the definition of modular functor in the genus-zero case more generally to include the irrational case, and more precisely for our study of intertwining operator algebras. We have added Axioms 3 and 5 in Definitions 1.1 and and 1.2, respectively, and we have also modified Axioms 2 and 6. These modifications are necessary for formulating the precise geometric definition of intertwining operator algebra and our main theorem.

Definition 1.1 A genus-zero modular functor is an analytic partial operad $M$ together with a morphism $\pi : M \rightarrow K$ of analytic partial operads a finite-rank holomorphic vector bundle structure on the triple $(M(j), K(j), \pi)$ for any $j \in \mathbb{N}$ satisfying the following axioms:

1. For any $Q \in M(k), Q_1 \in M(j_1), \ldots, Q_k \in M(j_k), k, j_1, \ldots, j_k \in \mathbb{N}$, the substitution $\gamma_M(Q; Q_1, \ldots, Q_k)$ in $M$ exists if (and only if)
   $$\gamma_K(\pi(Q); \pi(Q_1), \ldots, \pi(Q_k))$$
   exists.

2. Let $Q \in K(k), Q_1 \in K(j_1), \ldots, Q_k \in K(j_k), k, j_1, \ldots, j_k \in \mathbb{N}$, such that $\gamma(Q; Q_1, \ldots, Q_k)$ exists. The map from the Cartesian product of the fibers over $Q, Q_1, \ldots, Q_k$ to the fiber over $\gamma_K(Q; Q_1, \ldots, Q_k)$ induced from the substitution map of $M$ is multilinear and surjective.

3. The rank of $M(0)$ is 1.

The simplest examples of genus-zero modular functors are $\mathbb{C}$-extensions of $K$ discussed in Section 6.9 of [H6]. In fact any genus-zero modular functor $L$ such that $L(j)$ for any $j \in \mathbb{N}$ is a line bundle is a $\mathbb{C}$-extension of $K$.

Given a genus-zero modular functor $M$, let $M^*(j)$ be the dual vector bundle of $M(j)$ for $j \in \mathbb{N}$. Then it is clear that $M^* = \{M^*(j)\}_{j \in \mathbb{N}}$ is a genus-zero modular functor. We call $M^*$ the dual of $M$. Given genus-zero modular functors $M_1$ and $M_2$, Let $(M_1 \otimes M_2)(j)$ be the tensor product bundle $M_1(j) \otimes M_2(j)$ of $M_1(j)$ and $M_2(j)$. Then it is also clear that $M_1 \otimes M_2 = \{(M_1 \otimes M_2)(j)\}_{j \in \mathbb{N}}$ is a genus-zero modular functor. We call $M_1 \otimes M_2$ the tensor product of $M_1$ and $M_2$. 

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Definition 1.2 A rational genus-zero modular functor is a genus-zero modular functor $\mathcal{M}$ and a finite set $\mathcal{A}$ satisfying the following axioms:

4. For any $j \in \mathbb{N}$ and $\alpha_0, \alpha_1, \ldots, \alpha_j \in \mathcal{A}$, there are finite-rank holomorphic vector bundles $\mathcal{M}_{\alpha_1 \ldots \alpha_j}^{\alpha_0}(j)$ over $K(j)$ such that

$$\mathcal{M}(j) = \bigoplus_{\alpha_0, \alpha_1, \ldots, \alpha_j \in \mathcal{A}} \mathcal{M}_{\alpha_1 \ldots \alpha_j}^{\alpha_0}(j)$$

where $\bigoplus$ denotes the direct sum operation for vector bundles.

5. There exists $\epsilon \in \mathcal{A}$ such that the rank of $\mathcal{M}_{\epsilon}^{\alpha_0}(0)$ is 1 if $\alpha_0 = \epsilon$ and is 0 if $\alpha_0 \neq \epsilon$. For any $\alpha_0, \alpha_1 \in \mathcal{A}$, the rank of $\mathcal{M}_{\alpha_0 \alpha_1}(1)$ is 1 if $\alpha_0 = \alpha_1$ and is 0 if $\alpha_0 \neq \alpha_1$.

6. For any $j \in \mathbb{N}$ and $\beta_0, \beta_1, \ldots, \beta_j \in \mathcal{A}$, let $P_{\beta_0}^{\beta_1}(j)$ be the projection from $\mathcal{M}(j)$ to $\bigoplus_{\alpha_1, \ldots, \alpha_j \in \mathcal{A}} \mathcal{M}_{\beta_1 \ldots \beta_j}^{\alpha_0 \alpha_1}(j)$ and $P_{\beta_1 \ldots \beta_j}^{\beta_0}(j)$ the projection from $\mathcal{M}(j)$ to $\bigoplus_{\alpha_0 \alpha_1 \ldots \alpha_j \in \mathcal{A}} \mathcal{M}_{\beta_1 \ldots \beta_j}^{\alpha_0}(j)$. Then for any $k \in \mathbb{Z}_+$, $j_1, \ldots, j_k \in \mathbb{N},$

$$\gamma_{\mathcal{M}} = \sum_{\beta_1, \ldots, \beta_k \in \mathcal{A}} \gamma_{\mathcal{M}} \circ (P_{\beta_1 \ldots \beta_k}^{\beta_0}(k) \times P_{\beta_1}^{\beta_1}(j_1) \times \cdots \times P_{\beta_k}^{\beta_k}(j_k)),$$

and for any $Q \in K(k)$, $Q_1 \in K(j_1), \ldots, Q_k \in K(j_k), k \in \mathbb{Z}_+, j_1, \ldots, j_k \in \mathbb{N}$, such that $\gamma(Q; Q_1, \ldots, Q_k)$ exists and any $\beta_0, \alpha_l^{(i)} \in \mathcal{A}, l = 1, \ldots, j_i, i = 1, \ldots, k,$

$$\bigoplus_{\beta_1, \ldots, \beta_k \in \mathcal{A}} \mathcal{M}_{\beta_1 \ldots \beta_k}^{\beta_0 \alpha_1^{(1)} \ldots \alpha_{j_1}^{(1)}}|Q_1 \otimes \cdots \otimes \mathcal{M}_{\alpha_1^{(k)} \ldots \alpha_{j_k}^{(k)}}^{\beta_k}|Q_k$$

(where we use $\mathcal{M}|_Q$ to denote the fiber of a vector bundle $\mathcal{M}$ at an element $Q$ of the base manifold) to $\mathcal{M}|_{\gamma_{\mathcal{M}}(Q; Q_1, \ldots, Q_k)}$, induced from the map $\gamma_{\mathcal{M}} \circ (P_{\beta_1 \ldots \beta_k}^{\beta_0}(k) \times P_{\beta_1}^{\beta_1}(j_1) \times \cdots \times P_{\beta_k}^{\beta_k}(j_k))$ is an isomorphism.

Remark 1.3 Recall the notions of rescaling group of a partial operad and rescalable partial operad in [HL1], [HL2] and Appendix C of [H6]. In [H6], it was shown that $\mathbb{C}^\times$ can be identified with a rescaling group of $K(1)$ and that $K$ is a $\mathbb{C}^\times$-rescalable partial operad. From Axioms 4 and 5 in the definitions above, we see that the set $\mathcal{M}^\times(1)|_{\mathbb{C}^\times}$ of nonzero elements of the fibers over elements of the rescaling group of $K$ is a rescaling group of $\mathcal{M}$ and $\mathcal{M}$ is an $\mathcal{M}^\times(1)|_{\mathbb{C}^\times}$-rescalable partial operad.
We now construct a canonical holomorphic flat connection $\nabla(j)$ on $M(j)$ for each $j \in \mathbb{N}$. Recall the partial suboperad $\hat{K}$ and $K$ discussed in [H6]: For any $j \in \mathbb{N}$, $\hat{K}(j)$ is the subset of $K(j)$ consisting of elements of the form

$$(z_1, \ldots, z_{j-1}; A(z_j; 1), (a_0^{(1)}, 0), \ldots, (a_0^{(j)}, 0)),$$

where $z_1, \ldots, z_{j-1} \in \mathbb{C}^\times$ satisfying $z_k \neq z_l$ for $k \neq l$, $z_j \in \mathbb{C}$, $a_0^{(1)}, \ldots, a_0^{(j)} \in \mathbb{C}^\times$, and $A(z_j, 1)$ is the sequence of complex numbers whose first component is 1 and whose other components are 0. Then

$$\hat{K} = \{\hat{K}(j)\}_{j \in \mathbb{N}}$$

is a partial suboperad of $K$. For any $j \in \mathbb{N}$, $K(j)$ is the subset of $\hat{K}(j)$ consisting of elements of the form

$$(z_1, \ldots, z_{j-1}; A(z_j; 1), (1, 0), \ldots, (1, 0)).$$

Then $\overline{K} = \{\overline{K}(j)\}_{j \in \mathbb{N}}$ is also a partial suboperad of $K$. Note that for any $j \in \mathbb{Z}_+$, $K(j)$ can be identified with the configuration space

$$\{(z_1, \ldots, z_j) \in \mathbb{C}^j \mid z_k \neq z_l, k \neq l\}$$

by the map

$$(z_1, \ldots, z_{j-1}; A(z_j; 1), (1, 0), \ldots, (1, 0)) \mapsto (z_1 - z_j, \ldots, z_{j-1} - z_j, -z_j).$$

We consider the restrictions $\hat{M}(j)$ of the holomorphic vector bundles $M(j)$ to $\hat{K}(j)$ for $j \in \mathbb{N}$. Then $\hat{M} = \{\hat{M}(j)\}_{j \in \mathbb{N}}$ is a partial suboperad of $M$. We first construct holomorphic flat connections on $\hat{M}(j)$ for $j \in \mathbb{N}$.

Let $U$ be a simply connected open subset of $\mathbb{C}^\times$ containing 1. We first construct for any $\alpha \in A$ a holomorphic flat connection on the restriction $M_\alpha(1)|_U$ of $M_\alpha(1)$ to $U \subset \mathbb{C}^\times \subset \hat{M}(1)$. The composition map for the partial operad $M$ gives a holomorphic bundle map from $M_\alpha(1)|_U \times M_\alpha(1)|_U$ to $M_\alpha(1)|_{U \cdot U}$ where $U \cdot U$ is the image of $U \times U$ under the multiplication of complex numbers. Since $U$ is simply connected, we can always find a holomorphic trivialization of $M_\alpha(1)|_U$ such that any pair of pullbacks of the flat sections of the product bundle over $U$ under the trivialization is mapped to a section of $M_\alpha(1)|_{U \cdot U}$ whose restriction to $U$ is the pullback of a flat connection of the product bundle over $U$. This trivialization gives a holomorphic
flat connection on $\mathcal{M}_0^\alpha(1)|_U$. These holomorphic flat connections together give a holomorphic connection on $\mathcal{M}(1)|_U$.

We now construct the flat connection on $\mathcal{M}(j)$ for any $j \in \mathbb{N}$. Let

$$Q = (z_1, \ldots, z_{j-1}; A(z_j, 1), (a_0^{(1)}(t), 0), \ldots, (a_0^{(j)}(t), 0))$$

be an element of $K(j)$ and

$$\ell = \{ Q(t) = (z_1(t), \ldots, z_{j-1}(t); A(z_j(t), 1), (a_0^{(1)}(t), 0), \ldots, (a_0^{(j)}(t), 0)) \in \tilde{K}(j) \mid t \in [0, 1]\}$$

be a smooth path beginning at $Q(0) = Q$. Then for sufficiently small $t$, $(a_0^{(i)})^{-1}a_0^{(i)}(t) \in U$, $i = 1, \ldots, j$, and

$$Q(t) = ((\cdots (Q_1 \circ \cdots (A(z_1(t) - z_1; 1), ((a_0^{(1)})^{-1}a_0^{(1)}(t), 0)_{2 \infty} \cdots j_{-1 \infty}\circ (A(z_{j-1}(t) - z_{j-1}; 1), ((a_0^{(j-1)})^{-1}a_0^{(j-1)}(t), 0))_{\infty}, (A(z_j(t); 1), ((a_0^{(j)})^{-1}a_0^{(j)}(t), 0))).$$

From the definition of genus-zero modular functor, the substitution maps for $\mathcal{M}$ give an isomorphism from the tensor product of the fibers of $\tilde{\mathcal{M}}(j)$, $\tilde{\mathcal{M}}(1), \ldots, \tilde{\mathcal{M}}(1)$ over $Q$, $(A(z_1(t) - z_1; 1), (a_0^{(1)}(t), 0)), \ldots, (A(z_{j-1}(t) - z_{j-1}; 1), (a_0^{(j-1)}(t), 0))$, $(A(z_j(t); 1), (a_0^{(j)}(t), 0))$, respectively, to the fiber of $\tilde{\mathcal{M}}(j)$ over $Q(t)$. On the other hand, the holomorphic flat connection $\mathcal{M}(1)|_U$ gives isomorphisms from the fibers of $\mathcal{M}(1)|_U$ over

$$(A(z_i(t) - z_i; 1), ((a_0^{(i)})^{-1}a_0^{(i)}(t), 0)), i = 1, \ldots, j - 1,$$

and

$$(A(z_j(t); 1), (a_0^{(j)}(t), 0))$$

to the fiber of $\mathcal{M}(1)|_U$ over $(0, (1, 0))$. Thus we obtain an isomorphism from the fiber of $\tilde{\mathcal{M}}(j)$ over $Q$ to the fiber of $\tilde{\mathcal{M}}(j)$ over $Q(t)$. Repeating the procedure above starting from points in the path other than $Q$, we obtain for any $t \in [0, 1]$ such an isomorphism from the fiber of $\tilde{\mathcal{M}}(j)$ over $Q$ to the fiber of $\tilde{\mathcal{M}}(j)$ over $Q(t)$. Given any element of the fiber of $\tilde{\mathcal{M}}(j)$ over $Q$, these isomorphisms give a path in $\tilde{\mathcal{M}}(j)$ beginning at this element such that its projection image in $\tilde{K}(j)$ is the given path $\ell$. Since $Q$ and $\ell$ are arbitrary, we obtain a connection on $\tilde{\mathcal{M}}(j)$. From the construction, we see that this connection is flat and holomorphic.
Since \( K(j) \) is homotopically equivalent to \( \hat{K}(j) \), we can extend the holomorphic vector bundle \( \hat{M}(j) \) over \( \hat{K}(j) \) and the flat holomorphic connection just constructed trivially to a holomorphic vector bundle \( M^\sim(j) \) over \( K(j) \) and a holomorphic flat connection on \( M^\sim(j) \), respectively. We can also extend the partial operad structure on \( \hat{M} \) to the sequence \( \{M^\sim(j)\}_{j \in \mathbb{N}} \), so that \( M^\sim \) becomes a genus-zero modular functor.

Consider the genus-zero modular functor \( M \otimes (M^\sim)^* \). Since the fibers of the restriction of \( (M \otimes (M^\sim)^*)(j) \) to \( \hat{K}(j) \) for any \( j \in \mathbb{N} \) are tensor products of vector spaces and their dual spaces, we have a canonical isomorphism from this restriction to a trivial line bundle over \( \hat{K}(j) \). Since \( K(j) \) is homotopically equivalent to \( \hat{K}(j) \), this canonical isomorphism can be extended to a canonical isomorphism from \( (M \otimes (M^\sim)^*)(j) \) to a trivial line bundle \( L(j) \) over \( K(j) \). The genus-zero modular functor structure on \( (M \otimes (M^\sim)^*)(j) \) induces a genus-zero modular functor structure on \( L = \{L(j)\}_{j \in \mathbb{N}} \) and the canonical isomorphisms constructed above give a morphism of genus-zero modular functors. So \( M \) is isomorphic to \( M^\sim \otimes L \). Since \( L(j) \), \( j \in \mathbb{N} \), are all line bundles, \( L \) is a \( \mathbb{C} \)-extension of \( K \). By Theorem D.6.3 in [H6], there exists a complex number \( c \) such that \( L \) is isomorphic to the \( c \)-th power \( \hat{K}^c \) of \( \hat{K} \). Thus \( M \) is isomorphic to \( M^\sim \otimes \hat{K}^c \). Since for any \( j \in \mathbb{N} \), there are canonical holomorphic flat connections on \( M^\sim \) and \( \hat{K}(j) \), we obtain a holomorphic flat connection \( \nabla(j) \) on \( M(j) \).

In the construction above, we obtain a complex number \( c \). We call this complex number the central charge of the genus-zero modular functor \( M \).

We now consider the restrictions \( \hat{M} \) and \( M \) to the partial suboperad \( \hat{K} \) and \( K \), respectively. Recall that for any \( j \in \mathbb{N} \) \( K(j) \) can identified with the configuration space \( F(j) \). So we can view \( M(j) \) as a vector bundle over \( F(j) \) with a holomorphic flat connection. Since by the definition of genus-zero modular functor, the action of \( S_j \) on \( M \) are given by isomorphisms of holomorphic vector bundles covering the action of \( S_j \) on \( F(j) \), the holomorphic vector bundle \( \hat{M}(j) \) on \( F(j) \) and its holomorphic flat connection induce a holomorphic vector bundle on \( F(j)/S_j \) and a holomorphic flat connection on this induced bundle. In fact this holomorphic vector bundle and the holomorphic flat connection on it is the restriction of a holomorphic vector bundle on \( \hat{M}/S_j \) and a holomorphic flat connection on this vector bundle induced from the holomorphic vector bundle \( \hat{M}(j) \) on \( F(j) \) and its holomorphic flat connection. We know that a flat connection on a vector bundle over a connected manifold gives a structure of a representation of the fundamental group of the base manifold on the fiber.
of the vector bundle over any point on the manifold. We also know that the braid group $B_j$ on $n$ strings is by definition the fundamental group of $F(j)/S_j$ and the framed braid group on $j$ strings is the fundamental group $\tilde{M}/S_j$. So we obtain:

**Theorem 1.4** Let $\mathcal{M}$ be a genus-zero modular functor, $j \in \mathbb{Z}_+$ and $\mathcal{V}$ the fiber over any point in $F(j) \subset K(j)$ of the vector bundle $\mathcal{M}(j)$. Then $\mathcal{V}$ has a natural structure of a representation of the framed braid group on $j$ strings. In particular, $\mathcal{V}$ has a natural structure of a representation of the braid group $B_j$ on $j$ strings.

2 Intertwining operator algebras

The following definition of intertwining operator algebra was first given in [H5]:

**Definition 2.1** An intertwining operator algebra of central charge $c \in \mathbb{C}$ consists of the following data:

1. A finite-dimensional commutative associative algebra $A$ and a basis $\mathcal{A}$ of $A$ containing the identity $\epsilon \in A$ such that all the structure constants $N_{\alpha_1 \alpha_2}^{\alpha_3}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{A}$, are in $\mathbb{N}$.

2. A vector space

$$W = \bigoplus_{\alpha \in \mathcal{A}, n \in \mathbb{R}} W_{(n)}^\alpha, \text{ for } w \in W_{(n)}^\alpha, \ n = \text{wt } w, \ \alpha = \text{clr } w$$

doubly graded by $\mathbb{R}$ and $\mathcal{A}$ (graded by weight and by color, respectively).

3. For each triple $(\alpha_1, \alpha_2, \alpha_3) \in \mathcal{A} \times \mathcal{A} \times \mathcal{A}$, an $N_{\alpha_1 \alpha_2}^{\alpha_3}$-dimensional subspace $\mathcal{Y}_{\alpha_1 \alpha_2}^{\alpha_3}$ of the vector space of all linear maps $W_{\alpha_1}^{\alpha_2} \otimes W_{\alpha_2}^{\alpha_3} \to W_{\alpha_3}^{\alpha_3}\{x\}$, or equivalently, an $N_{\alpha_1 \alpha_2}^{\alpha_3}$-dimensional vector space $\mathcal{Y}_{\alpha_1 \alpha_2}^{\alpha_3}$ whose elements are linear maps

$$\mathcal{Y} : W_{\alpha_1}^{\alpha_2} \to \text{Hom}(W_{\alpha_2}^{\alpha_2}, W_{\alpha_3}^{\alpha_3})\{x\}$$

$$w \mapsto \mathcal{Y}(w, x) = \sum_{n \in \mathbb{R}} \mathcal{Y}_n(w)x^{-n-1} \ (\text{where } \mathcal{Y}_n(w) \in \text{End } W).$$
4. Two distinguished vectors $1 \in W^\epsilon$ (the vacuum) and $\omega \in W^\epsilon$ (the Virasoro element).

These data satisfy the following axioms for $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathcal{A}$, $w_{(\alpha)} \in W^\alpha_i$, $i = 1, 2, 3$, and $w'_{(\alpha_4)} \in W'_{\alpha_4}$:

1. The weight-grading-restriction conditions: For any $n \in \mathbb{Z}$ and $\alpha \in \mathcal{A}$,

$$\dim W^\alpha_{(n)} < \infty$$

and

$$W^\alpha_{(n)} = 0$$

for $n$ sufficiently small.

2. Axioms for intertwining operators:

(a) The single-valuedness condition: for any $\mathcal{Y} \in \mathcal{V}_{\epsilon \alpha_1}$,

$$\mathcal{Y}(w_{(\alpha_1)}, x) \in \text{Hom}(W^\alpha, W^\alpha[[x, x^{-1}]].$$

(b) The lower-truncation property for vertex operators: for any $\mathcal{Y} \in \mathcal{V}_{\alpha_1 \alpha_2}$, $\mathcal{Y}_n(w_{(\alpha_1)})w_{(\alpha_2)} = 0$ for $n$ sufficiently large.

3. Axioms for the vacuum:

(a) The identity property: for any $\mathcal{Y} \in \mathcal{V}_{\epsilon \alpha_1}$, there is $\lambda_{\mathcal{Y}} \in \mathbb{C}$ such that $\mathcal{Y}(1, x) = \lambda_{\mathcal{Y}} I_{W^\alpha_1}$, where $I_{W^\alpha_1}$ on the right is the identity operator on $W^\alpha_1$.

(b) The creation property: for any $\mathcal{Y} \in \mathcal{V}_{\alpha_1 \epsilon}$, there is $\xi_{\mathcal{Y}} \in \mathbb{C}$ such that $\mathcal{Y}(w_{(\alpha_1)}, x)1 \in W[[x]]$ and $\lim_{x \to 0} \mathcal{Y}(w_{(\alpha_1)}, x)1 = \xi_{\mathcal{Y}} w_{(\alpha_1)}$ (that is, $\mathcal{Y}(w_{(\alpha_1)}, x)1$ involves only nonnegative integral powers of $x$ and the constant term is $\xi_{\mathcal{Y}} w_{(\alpha_1)}$).

4. Axioms for products and iterates of intertwining operators:

(a) The convergence properties: for any $m \in \mathbb{Z}_+, \alpha_i, \beta_i, \mu_i \in \mathcal{A}$, $w_{(\alpha)} \in W^\alpha_i$, $\mathcal{Y}_i \in V_{\epsilon \alpha_i \beta_i + 1}$, $i = 1, \ldots, m$, $w'_{(\mu)} \in (W^\mu)^\gamma$ and $w_{(\beta)} \in W^\beta_m$, the series

$$\langle w'_{(\mu_1)}, \mathcal{Y}_1(w_{(\alpha_1)}, x_1) \cdots \mathcal{Y}_m(w_{(\alpha_m)}, x_m)w_{(\beta)} \rangle_{W^\mu_1} \mid x_i = e^{a \log z_i}, i = 1, \ldots, m, n \in \mathbb{R}$$

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is absolutely convergent when $|z_1| > \cdots > |z_m| > 0$, and for any $\mathcal{V}_1 \in \mathcal{V}_{\alpha_1 \alpha_2}^{\alpha_3}$ and $\mathcal{V}_2 \in \mathcal{V}_{\alpha_3}^{\alpha_4}$, the series
\[
\langle w'_1(\omega), \mathcal{Y}_1(w(\omega_1), x_0)w(\omega_2), x_2)w(\omega_3) \rangle w^4 \bigg|_{x_0 = e^{\sum a(n \log (z_1 - z_2))}, x_2 = e^{|\sum a(n \log z_2)}, n \in \mathbb{R}}
\]
is absolutely convergent when $|z_2| > |z_1 - z_2| > 0$.

(b) The associativity: for any $\mathcal{V}_1 \in \mathcal{V}_{\alpha_1 \alpha_5}^{\alpha_4}$ and $\mathcal{V}_{\alpha_2}^{\alpha_3}$, there exist $\mathcal{V}_3^\alpha \in \mathcal{V}_{\alpha_1 \alpha_2}^{\alpha_4}$ and $\mathcal{V}_4^\alpha \in \mathcal{V}_{\alpha_3}^{\alpha_4}$ for all $\alpha \in \mathcal{A}$ such that the (multivalued) analytic function
\[
\langle w'_1(\omega), \mathcal{Y}_1(w(\omega_1), x_1)\mathcal{Y}_2(w(\omega_2), x_2)w(\omega_3) \rangle w \bigg|_{x_1 = z_1, x_2 = z_2}
\]
on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1| > |z_2| > 0\}$ and the (multivalued) analytic function
\[
\sum_{\alpha \in \mathcal{A}} \langle w'_1(\omega), \mathcal{Y}_4^\alpha(w(\omega_1), x_0)w(\omega_2), x_2)w(\omega_3) \rangle w^4 \bigg|_{x_0 = z_1 - z_2, x_2 = z_2}
\]
on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_2| > |z_1 - z_2| > 0\}$ are equal on $\{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \mid |z_1| > |z_2| > |z_1 - z_2| > 0\}$.

5. Axioms for the Virasoro element:

(a) The Virasoro algebra relations: Let $Y$ be the element of $\mathcal{V}^{\alpha_1}_{\alpha_2}$ such that $Y(1, x) = I_{w^{\alpha_1}}$ and let $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$. Then
\[
[L(m), L(n)] = (m - n)L(m + n) + \frac{c}{12}(m^3 - m)\delta_{m+n,0}
\]
for $m, n \in \mathbb{Z}$.

(b) The $L(0)$-grading property: $L(0)w = nw = (wt)w$ for $n \in \mathbb{R}$ and $w \in W(n)$.

(c) The $L(-1)$-derivative property: For any $\mathcal{V} \in \mathcal{V}^{\alpha_3}_{\alpha_1 \alpha_2}$,
\[
\frac{d}{dx}\mathcal{V}(w(\alpha_1), x) = \mathcal{V}(L(-1)w(\alpha_1), x).
\]

(d) The skew-symmetry: There is a linear map $\Omega$ from $\mathcal{V}_{\alpha_1 \alpha_2}^{\alpha_3}$ to $\mathcal{V}_{\alpha_2 \alpha_1}^{\alpha_3}$ such that for any $\mathcal{V} \in \mathcal{V}_{\alpha_1 \alpha_2}^{\alpha_3}$,
\[
\mathcal{V}(w(\alpha_1), x)w(\alpha_2) = e^{xL(-1)}(\Omega(\mathcal{V}))(w(\alpha_2), y)w(\alpha_1) \bigg|_{y^n = e^{i\pi n}x^n}.
\]
To make our study slightly easier, we shall assume in the present paper that intertwining operator algebras also satisfy the following additional condition: For any \( \alpha \in A \), there exists \( h_\alpha \in \mathbb{R} \) such that \( (W^\alpha)_{(n)} = 0 \) for \( n \not\in h_\alpha + \mathbb{Z} \). But this is in fact a very minor restriction and in addition, all the results on intertwining operator algebras satisfying this additional property can be generalized easily to intertwining operator algebras not satisfying this condition.

The intertwining operator algebra defined above is denoted by

\[
(W, A, \{V^\alpha_{\alpha_1 \alpha_2}\}, 1, \omega)
\]

or simply by \( W \). The commutative associative algebra \( A \) is called the Verlinde algebra or the fusion algebra of \( W \). The linear maps in \( V^\alpha_{\alpha_1 \alpha_2} \) are called intertwining operators of type \( (\alpha_{\alpha_1 \alpha_2}) \).

**Remark 2.2** There are also other equivalent definitions of intertwining operator algebras using only formal variables. See [H7] and [H8] for details.

We shall also need the generalized rationality and commutativity for intertwining operator algebras. Here we state a theorem which gives these properties together with the associativity:

**Theorem 2.3** Let

\[
(W, A, \{V^\alpha_{\alpha_1 \alpha_2}\}, 1, \omega)
\]

be an intertwining operator algebra. Then for any \( \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5 \in A \), any \( w_{(\alpha_1)} \in W_{\alpha_1}, \ w_{(\alpha_2)} \in W_{\alpha_2}, \ w_{(\alpha_3)} \in W_{\alpha_3}, \ w_{(\alpha_4)}' \in (W_{\alpha_4})' \), and any intertwining operators \( \chi_1 \) and \( \chi_2 \) of type \( (W_{\alpha_1})_{(W_{\alpha_2})} \) and \( (W_{\alpha_4})_{(W_{\alpha_5})} \), respectively, there exist real numbers \( r_j, s_j, t_j, \ j = 1, \ldots, p \), polynomial functions \( g_j(z_1, z_2) \), \( j = 1, \ldots, p \), in \( z_1 \) and \( z_2 \), intertwining operators \( \chi_3^\alpha, \chi_4^\alpha, \chi_5^\alpha, \chi_6^\alpha, \alpha \in A, \) of type \( (W_{\alpha_1})_{(W_{\alpha_2}), (W_{\alpha_3}), (W_{\alpha_4}), (W_{\alpha_5})} \), respectively, such that

\[
\langle w_{(\alpha_4)}, \chi_1(w_{(\alpha_1)}, x_1)\chi_2(w_{(\alpha_2)}, x_2)w_{(\alpha_3)} \rangle_W|_{x_1=z_1, x_2=z_2},
\]

\[
\sum_{\alpha \in A} \langle w_{(\alpha_4)}, \chi_1^\alpha(\chi_3(w_{(\alpha_1)}; x_0)w_{(\alpha_2)}, x_2)w_{(\alpha_3)} \rangle_{W_{\alpha_4}}|_{x_0=z_1, x_2=z_2}
\]

and

\[
\sum_{\alpha \in A} \langle w_{(\alpha_4)}, \chi_1^\alpha(w_{(\alpha_1)}, x_1)\chi_2^\alpha(w_{(\alpha_2)}, x_2)w_{(\alpha_3)} \rangle_W|_{x_1=z_1, x_2=z_2}
\]
are the restrictions to the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1 - z_2| > 0$ and $|z_2| > |z_1| > 0$, respectively, of the (multivalued) analytic functions

$$\sum_{j=1}^{p} z_1^{-r_j} z_2^{-s_j} (z_1 - z_2)^{-t_j} g_j(z_1, z_2).$$

For a proof of this result, see Section 2 of [H7].

The theorem above can be generalized easily to more than two intertwining operators. Here we state the generalized rationality part of this generalization:

**Theorem 2.4** Let

$$(W, \mathcal{A}, \{V_{\alpha_1 \alpha_2}^m\}, 1, \omega)$$

be an intertwining operator algebra. Then for any $m \in \mathbb{Z}_+$, $\alpha_i, \beta_i, \mu_i \in \mathcal{A}$, $w(\alpha_i) \in W^{\alpha_i}$, $V_i \in V_{\alpha_i \beta_{i+1}}^{m}$, $i = 1, \ldots, m$, $w'(\mu_i) \in (W^{\mu_i})'$ and $w(\beta_m) \in W^{\beta_m}$, there exist real numbers $r_j^{(k)}$, $s_j^{(kl)}$, $j = 1, \ldots, p$, $k, l = 1, \ldots, m$, $k > l$, and polynomial functions $g_j(z_1, \ldots, z_m)$, $i = 1, \ldots, m$, in $z_1$ and $z_2$, such that

$$\langle w'(\mu_1), V_1(w(\alpha_1), x_1) \cdots V_m(w(\alpha_m), x_m)w(\beta_m) \rangle_{W^{\mu_1}}|_{x_i = e^{n \log z_i}, i = 1, \ldots, m, n \in \mathbb{R}}$$

defined on the region $|z_1| > \cdots > |z_m| > 0$ can be analytically extended to the generalized rational function

$$\sum_{j=1}^{p} g_j(z_1, \ldots, z_m) \prod_{k=1}^{m} z_k^{-r_j^{(k)}} \prod_{k>l} (z_1 - z_2)^{-s_j^{(kl)}}.$$

We omit the proof of this theorem since it is completely analogous to the proof of Theorem 2.3.

For constructions and further study of intertwining operator algebras, see [H3], [H5], [H7] and [H8].

### 3 Genus-zero modular functors and algebras over them from intertwining operator algebras

We construct in this section a genus-zero modular functor from an intertwining operator algebra and prove that the intertwining operator algebra
gives an algebra over the partial operad underlying this genus-zero modular functor, satisfying some additional conditions, including a certain generalized meromorphicity.

Let \((W, A, \{\mathcal{V}^{\alpha_3}_{\alpha_1\alpha_2}\}, 1, \omega)\) be an intertwining operator algebra of central charge \(c\). We first construct holomorphic vector bundles \(\mathcal{M}^{W;\alpha_0}_\alpha(j)\) over \(K(j)\) for \(j \in \mathbb{N}\) and \(\alpha_0, \ldots, \alpha_j \in A\). For \(j = 0\) and \(\alpha_0 \in A\), we take \(\mathcal{M}^{W;\alpha_0}(0)\) over \(K(0)\) to be the trivial product line bundle \(K(0) \times \mathbb{C}\) over \(K(0)\) if \(\alpha_0 = \epsilon\), and to be the rank-0 bundle over \(K(0)\) if \(\alpha_0 \neq \epsilon\).

In the case of \(j = 1\), for any \(\alpha_0 \in A\), we have a holomorphic multivalued operator-valued function \(e^{(\alpha_0)}\) on \(K(1)\) valued in \(\text{Hom}(W^{\alpha_0}, W^{(1)})\): The values of \(e^{(\alpha_0)}\) at \((A^{(0)}, (a^{(1)}_0, A^{(1)})) \in K(1)\)

\[
e^{(\alpha_0)}_{(A^{(0)}, (a^{(1)}_0, A^{(1)})]} = e^{-L^-_{A^{(0)}}} e^{-L^+_{A^{(1)}}(a^{(1)}_0) - L(0)}
\]

where for a sequence \(A = \{A_j\}_{j \in \mathbb{Z}_+}\) of complex numbers,

\[
L^-_A = \sum_{j \in \mathbb{Z}_+} A_j L(-j),
\]

\[
L^+_A = \sum_{j \in \mathbb{Z}_+} A_j L(j).
\]

This multivalued operator-valued function gives a holomorphic line bundle \(\mathcal{M}^{W;\alpha_0}_\alpha(1)\) over \(K(1)\).

Explicitly, this holomorphic bundle is constructed as follows: Given any simply connected open subset of \(K(1)\), the restriction of the holomorphic multivalued operator-valued function above have single-valued branches and any two such branches are different from each other by a constant factor. These single-valued branches gives a line bundle over the open subset together with a holomorphic trivialization. Now consider the intersection of two such open subsets. The connected components of such an intersection must be simply connected. We consider one of such component. Through such a component, a single-valued branch on one of the open subset discussed above has a unique analytic extension to a single-valued branch on the other open subset. Together with the trivializations of the line bundles over the two open subsets given by the the single-valued branches on these two open subsets, we obtain a transition function defined on this component. Combining the transition functions on all the components of the intersection, we obtain a transition function defined on the intersection. From the definition, we see...
that these transition functions satisfy the required properties and thus give a holomorphic line bundle $\mathcal{M}^{W_{\alpha}}(1)$. For any $\alpha_{0}, \alpha_{1} \in A$ such that $\alpha_{0} \neq \alpha_{1}$, we take the vector bundle $\mathcal{M}^{W_{\alpha_{0},\alpha_{1}}}(1)$ to be the rank-0 bundle over $K(1)$.

We now construct $\mathcal{M}^{W_{\alpha_{0},...\alpha_{j}}}(j)$ for $j \geq 2$ and $\alpha_{0}, ..., \alpha_{j} \in A$. We need the notion of generalized-meromorphic function on $K(m)$, $m \in \mathbb{N}$. For $m \in \mathbb{N}$, a generalized-meromorphic function on $K(m)$ is a multivalued map from $K(m)$ to $\mathbb{C}$ which can be expressed as a finite sum whose summands are polynomials in $z_{1}, \ldots, z_{m-1}, a^{(1)}_{0}, \ldots, a^{(m)}_{0}, A^{(0)}_{j}, \ldots, A^{(m)}_{j}$, $j \in \mathbb{Z}_{+}$, divided by real powers of $z_{1}, \ldots, z_{m-1}, a^{(1)}_{0}, \ldots, a^{(m)}_{0}$ and $z_{k} - z_{i}$, $k \neq l$. The points $z_{i} = 0, \infty$, $z_{k} = z_{i}$, $a^{(i)}_{0} = 0, \infty$, $A^{(i)}_{j} = \infty$ are called the singularities of this generalized-meromorphic function on $K(m)$. The real powers of $z_{k} - z_{i}$ appearing in the generalized-meromorphic function such that the corresponding term are not zero are called the orders of the singularity $z_{k} = z_{i}$. By definition, a meromorphic function on $K(m)$ defined in $\mathbb{H}$ is a generalized-meromorphic function.

By the definition of intertwining operator algebra, we know that for any $\alpha_{0}, \ldots, \alpha_{j}, \beta_{1}, \ldots, \beta_{j-2} \in A$, $\mathcal{Y}_{1} \in V_{\alpha_{0}}^{\beta_{0}}, \mathcal{Y}_{i} \in V_{\alpha_{i}}^{\beta_{i}}, i = 2, \ldots, j - 2$, $\mathcal{Y}_{j-1} \in V_{\alpha_{j-1}}^{\beta_{j-2}}$, any $w(\alpha_{i}) \in W^{\alpha_{i}}, i = 1, \ldots, j$, $w(\alpha_{0}) \in (W^{\alpha_{0}})'$, the series

$$
\langle e^{-L^{+}_{A(0)}(0)}w(\alpha_{0}), \mathcal{Y}_{1}(e^{-L^{+}_{A(1)}(0)}w(\alpha_{1}), x_{1}) \cdots \mathcal{Y}_{j-1}(e^{-L^{+}_{A(j-1)}(0)}w(\alpha_{j-1}), x_{j-1}) \cdots e^{-L^{+}_{A(j)}(0)}w(\alpha_{j}) \rangle_{x_{i} = z_{i}, i = \ldots, j-1}
$$

is absolutely convergent when $|z_{1}| > \cdots > |z_{j-1}| > 0$. Using the associativity, commutativity and generalized rationality for intertwining operators (Theorems 2.3 and 2.4) and the weight-grading-restriction condition, it is easy to see that this multivalued analytic function defined on the region determined by $|z_{1}| > \cdots > |z_{j-1}| > 0$ can be analytically extended to a generalized-meromorphic function on $K(j)$. Thus for fixed $\alpha_{0}, \ldots, \alpha_{j}, \beta_{1}, \ldots, \beta_{j-2} \in A$, $\mathcal{Y}_{1} \in V_{\alpha_{0}}^{\beta_{0}}, \mathcal{Y}_{i} \in V_{\alpha_{i}}^{\beta_{i}}, i = 2, \ldots, j - 2$, $\mathcal{Y}_{j-1} \in V_{\alpha_{j-1}}^{\beta_{j-2}}$, we obtain an multivalued operator-valued function $m_{\mathcal{Y}_{1}, \ldots, \mathcal{Y}_{j-1}}$ on $K(j)$ valued in $\text{Hom}(W^{\alpha_{1}} \otimes \cdots \otimes W^{\alpha_{j}}, W^{\alpha_{0}})$ defined by

$$
m_{\mathcal{Y}_{1}, \ldots, \mathcal{Y}_{j-1}}(Q(j, z, a, A)) = e^{-L_{A(0)}(0)}\mathcal{Y}_{1}(e^{-L_{A(1)}(0)}w(\alpha_{1}), x_{1}) \cdots \mathcal{Y}_{j-1}(e^{-L_{A(j-1)}(0)}w(\alpha_{j-1}), x_{j-1})e^{-L_{A(j)}(0)}w(\alpha_{j})
$$
for

\[ Q(j, z, a, A) = (z_1, \ldots, z_{j-1}; A(z_1), (a^{(1)}_0), \ldots, (a^{(j)}_0)) \in K(j). \]

For fixed \( \alpha_0, \ldots, \alpha_j \in A \), these multivalued operator-valued functions determine a holomorphic vector bundle \( M^{W; \alpha_0}_{\alpha_1, \ldots, \alpha_j}(j) \) of finite rank as in the case of \( j = 1 \) above.

Recall from [H6] that for any \( j \in \mathbb{N} \), the \( c/2 \)-power \( \tilde{K}^c(j) \) of the determinant line bundle over \( K(j) \) is the trivial product line bundle. Thus the tensor product of \( \tilde{K}^c(j) \) and any holomorphic vector bundle over \( K(j) \) is canonically isomorphic to the holomorphic vector bundle itself. Because of this fact, we shall often identify \( M^{W; \alpha_0}_{\alpha_1, \ldots, \alpha_j}(j) \) with \( M^{W; \alpha_0}_{\alpha_1, \ldots, \alpha_j}(j) \otimes \tilde{K}^c(j) \).

Taking direct sums of the holomorphic vector bundles we just constructed, we obtain a sequence \( M^W \) of finite-rank holomorphic vector bundles

\[ M^W(j) = \oplus_{\alpha_0, \ldots, \alpha_j \in A} M^{W; \alpha_0}_{\alpha_1, \ldots, \alpha_j}(j), \quad j \in \mathbb{N}. \]

Next we define the composition maps for \( M^W \). As usual, we use \( (Q, F) \) to denote an element of \( M(j) \) for any \( j \in \mathbb{N} \) where \( Q \in K(j) \) and \( F \) is in the fiber over \( Q \). Let \( (Q_1, F_1) \in M^W(j) \) and \( (Q_2, F_2) \in M^W(k) \). Then by definition, there are multivalued operator-valued functions which are linear combinations of \( m_{Y^{(1)}_1, \ldots, Y^{(1)}_{j-1}} \) and \( m_{Y^{(2)}_1, \ldots, Y^{(2)}_{k-1}} \) such that \( F_1 \) and \( F_2 \) are values of these functions at \( Q_1 \) and \( Q_2 \), respectively. To define the composition maps, we need only consider the case in which \( F_1 \) and \( F_2 \) are values of \( m_{Y^{(1)}_1, \ldots, Y^{(1)}_{j-1}} \) and \( m_{Y^{(2)}_1, \ldots, Y^{(2)}_{k-1}} \) at \( Q_1 \) and \( Q_2 \), respectively. Assume that for \( 1 \leq i \leq j \), \( Q_1 \) does not contain \( Q_2 \). Choose simply connected open subsets \( U_1 \) and \( U_2 \) of \( K(j) \) and \( K(k) \) containing \( Q_1 \) and \( Q_2 \), respectively, such that the image of the direct product of these open subsets under the sewing operation is a simply connected open subset \( U_3 \) of \( K(j+k-1) \) containing \( Q_1 \cdot Q_2 \). Using the associativity, commutativity and skew-symmetry for intertwining operators instead of the corresponding properties for vertex operators associated to a vertex operator algebra, we can adapt the proof of Proposition 5.4.1 in [H6] to show that if we use \( (m_{Y^{(1)}_1, \ldots, Y^{(1)}_{j-1}})(Q_1) \) and \( (m_{Y^{(2)}_1, \ldots, Y^{(2)}_{k-1}})(Q_2) \) to denote the set of all values of \( m_{Y^{(1)}_1, \ldots, Y^{(1)}_{j-1}} \) and \( m_{Y^{(2)}_1, \ldots, Y^{(2)}_{k-1}} \) at \( Q_1 \) and \( Q_2 \), respectively, then the set of the contractions

\[ (m_{Y^{(1)}_1, \ldots, Y^{(1)}_{j-1}})(Q_1) \ast_0 (m_{Y^{(2)}_1, \ldots, Y^{(2)}_{k-1}})(Q_2) \]
is in fact the set of values of a linear combination of the multivalued operator-valued functions of the form \( m_{\gamma_1, \ldots, \gamma_{j+2}} \) at \( Q_1 \circ \circ \circ Q_2 \). (Note that the central charge \( c \) of the intertwining operator algebra \( W \) enters in the coefficients of the linear combination.)

We take a particular value of this linear combination as follows: Consider the single-valued branches of the restrictions of \( m_{\gamma_1^{(1)}, \ldots, \gamma_{j-1}^{(1)}} \) and \( m_{\gamma_1^{(2)}, \ldots, \gamma_{k-1}^{(2)}} \) to \( U_1 \) and \( U_2 \), respectively, such that \( F_1 \) and \( F_2 \) are the restrictions of these branches at \( Q_1 \) and \( Q_2 \), respectively. The contraction of these branches gives a single-valued branch of the restriction of the linear combination above to \( U_3 \). The value of this single-valued branch at \( Q_1 \circ \circ \circ Q_2 \) is the particular value we want. Denote this particular value by \( F_3 \). We define \( (Q_1, F_1) \circ \circ \circ (Q_2, F_2) \) to be \( (Q_1 \circ \circ \circ Q_2, F_3) \). Using this operation, we can define the substitution map \( \gamma_{M^W} \) for \( M^W \) similarly to the substitution map for the vertex partial operads \( K^c \), \( c \in \mathbb{C} \), in [H6].

In \( M^W(1) \), we have an identity \( I_{M^W} \) which corresponds to the operator whose restriction to \( W^\alpha \) for \( \alpha \in A \) is the identity operator \( I_{W^\alpha} : W^\alpha \to W^\alpha \). Also there is an obvious action of \( S_j \) on \( M^W(j) \) for \( j \in \mathbb{N} \).

We have:

**Theorem 3.1** The sequence \( M_W = \{M_W(j)\}_{j \in \mathbb{N}} \) together with the substitution \( \gamma_{M^W} \), the identity \( I_{M^W} \) and the actions of \( S_j \), \( j \in \mathbb{N} \), is a rational genus-zero modular functor.

**Proof.** From the construction of \( M_W \) above, we see that the only axiom need to be verified is the operad-associativity. Let \( (Q_1, F_1) \in M^W(j) \), \( (Q_2, F_2) \in M^W(k) \) and \( (Q_3, F_3) \in M^W(l) \). Assume that for \( 1 \leq i_1 \leq j \) and \( 1 \leq i_2 \leq i_1 + k - 1 \), \( 1 \leq i_2 \leq i_1 + k - 1 \), both \( (Q_1, \circ \circ \circ Q_2) \circ \circ \circ Q_3 \) and \( Q_1 \circ \circ \circ (Q_2 \circ \circ \circ Q_3) \) exist. Choose simply connected open subsets \( U_1 \subset K(j) \), \( U_2 \subset K(k) \) and \( U_3 \subset K(l) \) such that for any elements \( Q_1' \in U_1 \), \( Q_2' \in U_2 \) and \( Q_3' \in U_3 \), both \( (Q_1' \circ \circ \circ Q_2') \circ \circ \circ Q_3' \) and \( Q_1' \circ \circ \circ (Q_2' \circ \circ \circ Q_3') \) exist and such that the open subsets

\[
(U_{1,1} \circ \circ \circ U_2)_{12} \circ \circ \circ U_3 = U_{1,1} \circ \circ \circ (U_{2,1} \circ \circ \circ U_{1,1-1} \circ \circ \circ U_3)
\]

(3.1)

of \( K(j+k+l-2) \) are simply connected. Then using the definition of the composition for \( M^W \) and commutativity and skew-symmetry for intertwining operators, and adapting the arguments in the proof of Proposition 5.4.1 in [H6], we can show that both

\[
((Q_1, F_1) \circ \circ \circ (Q_2, F_2))_{12} \circ \circ \circ (Q_3, F_3)
\]

(3.2)
are restrictions at \((Q_1, F_1)_{i_1 \infty_0} \circ \circ \circ (Q_3, F_3)\)
and\((Q_1, F_1)_{i_2 \infty_0} \circ \circ \circ (Q_3, F_3)\), respectively, of a single-valued branch on \((3.2)\) of a linear combination of the multivalued operator-valued functions of the form $m_{Y_1}, \ldots, Y_j + k + l - 3$. Since
\[(Q_1, F_1)_{i_1 \infty_0} \circ \circ \circ (Q_3, F_3) = (Q_1, F_1)_{i_2 \circ \circ \circ (Q_3, F_3)}\],
(3.2) and (3.3) are equal, proving the operad-associativity.

In fact the constructions and proof above also give an algebra over the\((M W^\times(1)|_{C \times})\)-rescalable partial operad \(M W^\times\) satisfying certain additional properties (recall the definition of algebra over a rescalable partial operad in [HL1], [HL2] and [H6]). For simplicity, by an algebra over a rational genus-zero modular functor \(M\), we shall mean an algebra over the underlying \(M^\times(1)|_{C \times}\)-rescalable partial operad of \(M\). To describe such an algebra, we first need discuss a certain class of irreducible modules for \(M^\times(1)|_{C \times}\).

Since \(M(1)|_{C \times}\) is in fact a line bundle over \(C^\times\), the restriction \(M^\times(1)|_{R^+}\) to the positive real numbers can be identified, after choosing a suitable section, with the space of pairs \((\alpha, \mu), \alpha \in R^+, \mu \in C^\times\). Note that \(M^\times(1)|_{R^+}\) is a subgroup of \(M^\times(1)|_{C \times}\) and the product is given by
\[
\gamma_M((\alpha_1, \mu_1); (\alpha_2, \mu_2)) = (\alpha_1 \alpha_2, \mu_1 \mu_2)
\]
for \(\alpha_1, \alpha_2 \in R^+\) and \(\mu_1, \mu_2 \in C^\times\). In particular, the subset of \(M^\times(1)|_{R^+}\) consisting of pairs \((\alpha, 1), \alpha \in R^+,\) is a subgroup of \(M^\times(1)|_{R^+}\) isomorphic to the group \(R^+_+\). Let \(W\) be an irreducible module for \(R^+_+\). Then \(W\) must be one-dimensional and there is a complex number \(n\) such that the action of \(R^+_+\) on \(W\) is given by
\[
R^+_+ \rightarrow \text{End } W
\]
\[
\alpha \mapsto \alpha^n.
\]

We shall call \(n\) the weight of \(W\). Given any irreducible module \(W\) for \(R^+_+\) of weight \(n \in C\) and any flat holomorphic connection on \(M(1)|_{C \times}\), we define an irreducible \(M(1)|_{C \times}\)-module structure on \(W\) as follows: Using the flat holomorphic connection on \(M(1)|_{C \times}\), we identify \(M(1)|_{C \times}\) with the set of pairs of the form \((l, \mu)\) where \(l \in C\) satisfying $0 \leq \Im(l) < 2\pi$, and \(\mu \in C\).

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Then the $\mathcal{M}(1)|_{\mathbb{C}^\times}$-module structure is defined by

$$
\mathcal{M}(1)|_{\mathbb{C}^\times} \to \text{End } W
(l, \mu) \mapsto \mu e^{\alpha l}.
$$

We call such an irreducible $\mathcal{M}(1)|_{\mathbb{C}^\times}$-module an *irreducible $\mathcal{M}(1)|_{\mathbb{C}^\times}$-module induced from an irreducible $\mathbb{R}_+^+$-module of weight $n$.*

Recall that the underlying vector space of an algebra over an $\mathcal{M}(1)|_{\mathbb{C}^\times}$-rescalable partial operad is a direct sum of irreducible $\mathcal{M}(1)|_{\mathbb{C}^\times}$-modules (see [HL1], [HL2] and Appendix C in [H6]). We have the following notion:

**Definition 3.2** An *generalized-meromorphic algebra over a genus-zero modular functor $\mathcal{M}$* is an algebra $(W, V, \Upsilon)$ over $\mathcal{M}$ satisfying the following conditions:

1. $W = \coprod_{n \in \mathbb{R}} W_{(n)}$ where for any $n \in \mathbb{R}$, $W_{(n)}$ is an irreducible $\mathcal{M}(1)|_{\mathbb{C}^\times}$-module induced from an irreducible $\mathbb{R}_+^+$-module of weight $n$ (note that we have a flat holomorphic connection on $\mathcal{M}(1)|_{\mathbb{C}^\times}$).

2. $W_{(n)} = 0$ for $n$ sufficiently small.

3. For any $j \in \mathbb{N}$, $\Upsilon_j$ is linear on any fiber of $\mathcal{M}(j)$.

4. Let $j \in \mathbb{N}$ and let $U$ be any simply connected open subset of $K(j)$. Then for any flat section $\psi : U \to \mathcal{M}(j)|_{U}$ of the restriction $\mathcal{M}(j)|_{U}$ of the vector bundle $\mathcal{M}(j)$ and any $w' \in W'$, $w_1, \ldots, w_j \in W$, the function

$$Q \to \langle w', (\Upsilon_j(\psi(Q)))(w_1 \otimes \cdots \otimes w_j) \rangle$$

on $U$ can be analytically extended to a generalized-meromorphic function on $K(j)$ satisfying the following property: If $z_k$ and $z_l$ are the $k$-th and $l$-th punctures of $Q \in K(j)$ respectively, then for any $w_k$ and $w_l$ in $W$ there exists a positive integer $N(w_k, w_l)$ such that for any $w' \in W'$, $w_i \in W$, $i \neq k, l$, the orders of the singularity $z_k = z_l$ (we use the convention $z_j = 0$) of

$$\langle w', (\Upsilon_j(\psi(Q)))(w_1 \otimes \cdots \otimes w_j) \rangle$$

is less than $N(w_k, w_l)$.  

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We shall denote the generalized-meromorphic algebra over \( M \) above by \((W, V, \Upsilon)\).

Note that from the generalized-meromorphic functions in Axiom 1 above, we can construct holomorphic vector bundles using the method in the construction of a modular functor from an intertwining operator algebra above. Clearly these holomorphic vector bundles are isomorphic to \( M(j), j \in \mathbb{N} \). But in general they are certainly not the same as \( M(j), j \in \mathbb{N} \). To formulate the isomorphism theorem (Theorem 1.2 in the next section precisely, we need the following notion:

**Definition 3.3** A canonically-generalized-meromorphic algebra over a rational genus-zero modular functor \( M \) is a generalized meromorphic algebra \((W, V, \Upsilon)\) over \( M \) such that the vector bundles on \( M(j), j \in \mathbb{N} \), is the same (not just isomorphic to) as the ones constructed from the generalized-meromorphic functions given by Axiom 1 in Definition 3.2.

*Homomorphisms* and *isomorphisms* between generalized-meromorphic algebras over genus-zero modular functors are defined in the obvious way.

**Remark 3.4** If the fourth axiom in the above definition is replaced by the holomorphicity of \( \Upsilon \), then we do not need the flat holomorphic connections \( \nabla(j) \) on \( M(j), j \in \mathbb{N} \). An algebra \((W, V, \Upsilon)\) over a genus-zero modular functor \( M \) satisfying the first three axioms in the definition above and the holomorphicity of \( \Upsilon \) is called a genus-zero weakly holomorphic conformal field theory over \( M \) (see [H5], [S1] and [S2]). The fourth axiom above clearly implies the holomorphicity of \( \Upsilon \). Thus the underlying algebra over \( M \) of a generalized-meromorphic algebra over \( M \) is a genus-zero weakly holomorphic conformal field theory over \( M \).

Let \((W, A, \{V^{\alpha_1}_{\alpha_2}\}, 1, \omega)\) be an intertwining operator algebra. Let \( V \) be the vertex operator subalgebra of \( W \) generated by \( 1 \) and \( \omega \). By Theorem 3.1, we have a rational genus-zero modular functor \( M^W \). We define a morphism of partial operad

\[
\Upsilon^W : M^W \to \mathcal{H}_{W,V}
\]

as follows: Given any element \((Q, F)\) of \( M^W \) for \( j \in \mathbb{N} \), by definition, \( F \) is a value of a linear combination of generalized-meromorphic functions on \( K(j) \) of the form \( m_{j_1,...,j_{j-1}} \). So \( F \) is an element of \( \mathcal{H}_{W,V}(j) \). We define \( \Upsilon^W_j ((Q, F)) = F \).

We have:
Theorem 3.5 The quadruple \((W, V, \Upsilon^W)\) is a generalized-meromorphic algebra over the genus-zero modular functor \(M^W\). In particular, \((W, V, \Upsilon^W)\) is a genus-zero weakly holomorphic conformal field theory over \(M^W\).

Proof. Using the duality properties for intertwining operator algebras instead of the corresponding properties for vertex operator algebras, we can adapt the argument in the proof of Proposition 5.4.1 of [H6] to prove that \((W, V, \Upsilon^W)\) is an algebra over \(M^W\). The generalized meromorphy is clear from the definition of \(\Upsilon\) and the constructions of the holomorphic flat connection on \(M^W(j), j \in \mathbb{N}\), in Section 1 and this section. The second conclusion follows from Remark 3.4. \(\Box\)

Combining Theorems 3.3 and 1.4, we obtain:

Corollary 3.6 Let \((W, A, \{V_{\alpha_1, \alpha_2}\}, 1, \omega)\) be an intertwining operator algebra and \(V_{W, j}\) the fiber over any point in \(F(j) \subset K(j)\) of the vector bundle \(M^W(j)\). Then \(V\) has a natural structure of a representation of the framed braid group on \(j\) strings. In particular, \(V\) has a natural structure of a representation of the braid group \(B_j\) on \(j\) strings.

4 The equivalence theorem

In this section, we first show that a generalized-meromorphic algebra over a rational genus-zero modular functor gives an intertwining operator algebra. The main result of the present paper follows easily.

Let \((W, V, \Upsilon)\) be a generalized-meromorphic algebra over a rational genus-zero modular functor \(M\). By the definition of rational genus-zero modular functor, we already have a finite set \(A\). Let \(\epsilon\) be the element of \(A\) such that the identity \(I_M\) is in \(M^\epsilon(1)\). For any \(a_1, a_2, a_3 \in A\), by the definition of generalized meromorphicity, the image under \(\Upsilon_2\) of any flat section of \(M_{a_1, a_2}^\epsilon\) over a simply connected open subset of \(M(2)\) is an element of \(H_{W, V}(2)\) such that its matrix elements can be analytically extended to generalized-meromorphic functions. Considering the subset \(M^1\) of \(K(2)\), we see that the image of such a flat section in particular gives an multivalued operator-valued function on \(M^1\), which can be identified with \(\mathbb{C}^\times\). Such an multivalued operator-valued function on \(\mathbb{C}^\times\) gives a map from \(W^{a_1} \otimes W^{a_2} \rightarrow W^{a_3}\{x\}\). We define \(V_{a_1, a_2}^{\alpha_3}\) to be the space of all maps obtained in this way. The vacuum \(1\)
and the Virasoro element $\omega$ are defined by
\[ 1 = \Upsilon_0((0;1)); \]
and
\[ \omega = \left. \frac{d}{d\epsilon} \nu((A(\epsilon;2);1)) \right|_{\epsilon=0}. \]

We have:

**Theorem 4.1** Let $(W,V,\Upsilon)$ be a generalized-meromorphic algebra over a rational genus-zero modular functor $M$ of central charge $c$. Then $(W,A,\nu_{a_3}^{a_1a_2},1,\omega)$ is an intertwining operator algebra of central charge $c$.

**Proof.** The proof of this result is completely analogous to the proof of Proposition 5.4.4 in [H6] except for the commutative associative structure on the vector space $A$ spanned by elements of $A$. This structure can be constructed as follows: For any $\alpha_1, \alpha_2, \alpha_3 \in A$, let $N_{\alpha_1\alpha_2}^{\alpha_3}$ be the rank of $M_{\alpha_1\alpha_2}^{\alpha_3}$. We define the product in $A$ by
\[ \alpha_1 \cdot \alpha_2 = \sum_{\alpha_3 \in A} N_{\alpha_1\alpha_2}^{\alpha_3} \alpha_3. \]

Then clearly this product is commutative and associative. From the axioms for rational genus-zero modular functors, it is easy to see that $N_{\alpha_1\alpha_2}^{\alpha_3}$ is 1 when $\alpha_2 = \alpha_3$ and is 0 when $\alpha_2 \neq \alpha_3$. So $A$ has an identity $\epsilon$. ⧫

Combining the constructions and results in the preceding and the present sections, we obtain the following main result of this paper:

**Theorem 4.2** The category of intertwining operator algebras of central charge $c$ and the category of canonically-generalized-meromorphic algebras over rational genus-zero modular functors of central charge $c$ are isomorphic.

**Proof.** In the preceding, we have already constructed a functor from the category of intertwining operator algebras of central charge $c$ to the category of canonically-generalized-meromorphic algebras over rational genus-zero modular functors of central charge $c$. Theorem 4.1 above gives a functor from the category of canonically-generalized-meromorphic algebras over rational genus-zero modular functors of central charge $c$ to the category of intertwining operator algebra of central charge $c$. The argument in the proof of the equivalence theorem for vertex operator algebras in Section 5.4 of [H6] can be adapted in the obvious way to show that the compositions of these two functors are the identity functors. ⧫
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