REMARK ON THE PAPER "ASYMPTOTIC BEHAVIOR OF POLYNOMIALS ORTHONORMAL ON A HOMOGENEOUS SET"

F. PEHERSTORFER, AND P. YUDITSKII

Abstract. Minor modifications are given to prove the Main Theorem under the Blaschke (instead of Carleson) condition as well as a small historical comment.

Because of the reference [1] on our paper [4] a certain historical comment is needed.

In [4] we generalized H. Widom’s Theorem [5] based on an absolutely new idea, dealing with one dimensional perturbation of a given Jacobi matrix. As it is well known if a Jacobi matrix $J$ has the compact $E = [b_0, a_0] \cup \cup_{j \geq 1} (a_j, b_j)$ as spectral set, then its one dimensional perturbation may have in addition to $E$ spectral points in the gaps (one point in each gap). Since in our case, the set $E$ is possibly a Cantor type set, an infinite number of spectral points $X$ has to be added to the support of the spectral measure. A homogeneous set $E$ possesses the following very nice property [2]. Let $B(z, z_0) = B(z, z_0; \Omega)$ be the Blaschke factor in the domain $\Omega = \mathbb{C} \setminus E$ with zero at $z_0 \in \Omega$. From each interval $(a_j, b_j)$ let us pick arbitrarily exactly one $x_j$. Then

$$\inf_k \prod_{j \neq k} |B(x_k, x_j)| > 0.$$  \hfill (1)

Note that the convergence of the product $\prod_j |B(z_0, x_j)| > 0$ (the Blaschke condition) corresponds to the so called Widom property of the domain $\Omega$. So, the domain with a homogeneous boundary has even a better property: the more restrictive condition (1), the so called Carleson condition, holds for the given Blaschke product.

Thus to use our idea on a one dimensional perturbation, we were enforced to work with spectral measures supported on a homogeneous set $E$ but also having, possibly an infinite set of mass points supported on an arbitrary (real) set $X$, satisfying (similar to (1)) the Carleson condition

$$\inf_{x \in X} \prod_{y \in X, y \neq x} |B(x, y)| > 0.$$  \hfill (2)

Date: March 30, 2022.
When the results of [4] were presented and the manuscript was submitted for publication, we recognized that the math community is much more interested in an infinite number of mass point than in a Cantor type spectral sets. We wrote [3] considering \( E \) just as a single interval. It became, almost immediately clear that the Carleson condition was too restrictive and a simple trick [3, (2.9)] allows to use only Blaschke condition.

Unfortunately, the first paper was finally published later than the second one (in this way we have, formally, a more fresh paper with the more restrictive condition [4, (6.1)] on the set \( X \)).

To prove the Main Theorem under the Blaschke condition one has to estimate [4, (6.9)] in the way [3, (2.9)], of course, following to the standard strategy in this paper (Lemma 4.2, Lemma 5.3, etc):

1. For a fixed \( \varepsilon > 0 \) use the finite covering of \( \Gamma^* \) (see [4, p.139]),

\[
\Gamma^* = \bigcup_{j=1}^{j(\varepsilon)} \{ \beta : \text{dist}(\beta, \beta_j) \leq \eta(\varepsilon) \}.
\] (3)

2. For the exhaustion \( \{X_N\} \) of \( X \) by finite sets [4, p.138], let \( M_N \) be the character of the Blaschke product \( B_N \) with zeros at \( X \setminus X_N \). Since \( B_N(0) \to 1 \) and \( M_N \to \Gamma^* \) we can choose \( N \) so big that

\[
1 - B_N(0) \leq \varepsilon, \quad 1 - \Delta M_N^{-1}(0) \leq \varepsilon
\] (4)

(for the definition of \( \Delta^a \) see [4, p.125]).

3. Having a finite number of the reproducing kernels (characters \( \beta_j \)) and a finite number of points \( (X_N = \{\text{zeros of } B/B_N\}) \), choose \( n \) due to the estimation

\[
\left| \sum_{X_N} \left\{ b^{n+1} K^\beta_j \left( \frac{z'}{\psi'} \right) P_n \right\} \right| 
\leq (\sup_{X_N} |b|)^n \sqrt{\sum_{X_N} |P_n|^2} \sigma_l \sqrt{\sum_{X_N} |K^\beta_j b / \psi|^2} \sigma_l.
\] (5)

Let us mention that the key Lemmas 1.1, 2.2, 2.4, 5.2 were proved under the Blaschke condition [4, (2.5)].

REFERENCES

1. D. Damanik, R. Killip, and B. Simon Perturbation of orthogonal polynomials with periodic recursion coefficients, Preprint.
2. P. W. Jones and D. E. Marshall, Critical points of Green’s function, harmonic measure, and the corona problem, Ark. Mat. 23, 281–314 (1985).
3. F. Peherstorfer and P. Yuditskii, Asymptotics of orthonormal polynomials in the presence of a denumerable set of mass points, Proc. Amer. Math. Soc. 129 (2001), 3213–3220.
4. F. Peherstorfer and P. Yuditskii, Asymptotic behavior of polynomials orthonormal on a homogeneous set, J. Analyse Math. 89, 113–154 (2003).
5. H. Widom, Extremal polynomials associated with a system of curves in the complex plane, Adv. Math. 3 (1969), 127–232.