Boundary condition for Staggered Fermion
in Lattice Schrödinger Functional of QCD

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Abstract

The fermionic part of the Schrödinger functional of QCD is formulated in the lattice regularization with the staggered fermion. The boundary condition imposed on the staggered fermion field is examined in terms of the four-component Dirac spinor. The boundary terms are different from those of the Symanzik’s theory in the flavor structure due to the species doubling. It is argued that, in the case of the homogeneous Dirichlet boundary condition, surface divergence does not occur if the link variables of gauge field are introduced on the original lattice, not on the blocked one. Its application to the numerical calculation of the running coupling constant in QCD is discussed.

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Introduction

Numerical calculation of the strong coupling constant $\alpha_s$ in QCD has been recently attempted by several groups. These works involve various physical and technical ingredients, namely, charmonium spectrum [1], static quark-antiquark force [2]–[4] and the Schrödinger functional of SU(3) Yang-Mills theory in finite volume [5]–[7]. To fix the scale, low energy physical quantities were measured in the unit of the lattice spacing or in the unit of the linear extent of the finite volume. The scale was related to the running coupling constant in the (modified) minimal-subtraction scheme by the improved lattice perturbation theory [8] or through the nonperturbatively defined running coupling constant. The basic Monte Carlo simulations were all performed in the quenched approximation (the pure Yang-Mills theory). As to the charmonium spectrum, however, the potential model has allowed one to estimate systematic errors for the approximation [9].

The desired next step is surely the calculation of $\alpha_s$ based on the simulation with dynamical fermion. In this respect, the sea quark effect on the charmonium system has been observed by Onogi et al. [10] [11] and their result is consistent with the procedure adopted by Elkhadra et al. Quite recently, a new determination using the $\Upsilon$ spectrum and including the effect of dynamical quarks was reported by Davies et al. [12]. These works actually suggest that the precise determination of $\alpha_s$ through the quarkonium spectrum is promising.

On the other hand, the finite size scaling technique adopted in the works of Lüscher et al. can be also applied to QCD. The Schrödinger functional of QCD in the lattice regularization has been recently formulated by Sint [13]. He examined the boundary condition of fermion field in a finite extent of the time direction for the case of the Wilson fermion [14]. He derived the boundary terms and argued that the resulted boundary terms are not lattice artifacts and meaningful at the continuum limit. He also examined the spectrum of the Dirac operator of the free fermion and its squared. He has observed that the boundary condition forces a minimal frequency in the system and gives a chance to simulate QCD with light quarks.
Generally speaking, there exists the arbitrariness of choice for the boundary condition of the fermion field. Actually, in his original paper on the Schrödinger functional \[13\], Symanzik adopted a different boundary condition from that is derived from the Wilson fermion. Concerning the lattice regularization, another type of lattice fermion, the staggered fermion \[16\], is known to work. Then a different boundary condition is expected for it.

As to the staggered fermion, however, the four-fold degeneracy of flavor may cause us both good and harm in calculating \(\alpha_s\). It may be useful and economical in the simulation for the ideal case of four degenerate massless quarks. However, as a final goal, we need to put the realistic mass to each quark: a few MeV for the up quark on one side, but about one GeV for the charm quark on the other side. The possibility of giving the non-degenerate mass term to the staggered fermion has been discussed by several authors \[17\]– \[20\]. But it is yet an open question to give such large mass-difference to the staggered fermion in actual numerical computations. More technically, it may need a different treatment for the staggered fermion of the boundary effect due to the finite extent of the space-time.

In this paper, we will formulate the fermionic part of the Schrödinger Functional of QCD with the staggered fermion through its transfer matrix, which is formulated by Thun et al. \[21\], and work out its boundary condition. Then we will discuss the possibility of getting along with the staggered fermion in the calculation of \(\alpha_s\) in QCD.

**Transfer Matrix for the Staggered Fermion**

The Schrödinger Functional in lattice QCD can be naturally formulated by the transfer matrix as shown by Sint for the case of the Wilson fermion \[13\]. The transfer matrix for the staggered fermion has been given by H. S. Sharatchandra, H. J. Thun and P. Weisz \[21\]. Firstly we will review their results and fix the notation that we use. The lattice spacing \(a\) is chosen for unity, whereas the lattice sites are labeled by four integers \(n_\mu = (n_1, n_2, n_3, n_4) = (n, n_4)\). \(\hat{\mu}\) stands for a unit vector of the \(\mu\)-th direction. Since the mass terms are found not to be important when one constructs the transfer matrix of the staggered fermion, they are
omitted here. Then the action of free massless staggered fermion is given by

\[ S_F = \frac{1}{2} \sum_{n, \mu} \eta_\mu(n) \bar{\chi}(n) \{ \chi(n + \hat{\mu}) - \chi(n - \hat{\mu}) \} , \] (1)

where \( \chi(n) \) and \( \bar{\chi}(n) \) are one-component Grassmann variables \[22\] \[23\] \[21\]. \( \eta_\mu(n) \) is defined by

\[ \eta_\mu(n) = (-1)^{n_1 + \cdots + n_{\mu - 1}} , \quad \eta_1 = 1 . \] (2)

It is related to the Euclidean Dirac gamma matrices satisfying

\[ \{ \gamma^\mu, \gamma^\nu \} = 2 \delta^\mu\nu , \quad \gamma^\mu \dagger = \gamma^\mu , \] (3)

by the equation

\[ T(n) \gamma_\mu T^\dagger(n + \hat{\mu}) = \eta_\mu(n) I , \] (4)

where

\[ T(n) = \gamma_1^{n_1} \gamma_2^{n_2} \gamma_3^{n_3} \gamma_4^{n_4} . \] (5)

First of all, we rescale and denote the variables \( \chi(n) \) and \( \bar{\chi}(n) \) such as

\[ \chi_{n_4}(n) \equiv \chi(n) , \quad \bar{\chi}_{n_4}(n) \equiv \frac{1}{2} \bar{\chi}(n) \eta_4(n) . \] (6)

Since \( \eta_\mu(n) \) does not depend on \( n_4 \), we can define

\[ \eta_k'(n) \equiv \eta_k(n) \eta_4(n) , \] (7)

and it follows that

\[ \eta_k'(n + \hat{k}) = -\eta_k'(n) . \] (8)

Then the action (1) is rewritten as

\[
S_F = \sum_{n_4, n} \{ \chi_{n_4}^\dagger(n) \chi_{n_4+1}(n) + \chi_{n_4}(n) \chi_{n_4+1}^\dagger(n) \} \\
+ \sum_{n_4, n} \sum_k \{ \chi_{n_4}^\dagger(n) \eta_k'(n) \chi_{n_4}(n + \hat{k}) + \chi_{n_4}^\dagger(n + \hat{k}) \eta_k'(n) \chi_{n_4}(n) \} 
\] (9)
The next step is the characteristic one for the staggered fermion: we introduce a set of auxiliary fields $\varphi$ and $\varphi^\dagger$ using the $\delta$ function for Grassmann variable. Namely, we use

$$I = \int \prod_{n_{4}} [d\varphi_{n_{4}} d\varphi_{n_{4}+1}] \prod_{n,n_{4}} \delta(\varphi_{n_{4}}(n) - \chi_{n_{4}}(n))\delta(\varphi_{n_{4}}^\dagger(n) - \chi_{n_{4}}(n)), \quad (10)$$

where

$$\delta(\varphi_{n_{4}}^\dagger(n) - \chi_{n_{4}}(n)) \equiv (\varphi_{n_{4}}^\dagger(n) - \chi_{n_{4}}(n)). \quad (11)$$

Then the first two terms in the action (9) can be written as

$$\sum_{n_{4}} \sum_{n} \{\chi_{n_{4}}(n)\chi_{n_{4}+1}(n) + \varphi_{n_{4}}^\dagger(n)\varphi_{n_{4}+1}(n)\}. \quad (12)$$

Accordingly, we introduce two independent sets of operators, $\hat{\chi}$, $\hat{\chi}^\dagger$ and $\hat{\varphi}$, $\hat{\varphi}^\dagger$ satisfying the anti-commutation relations as

$$\{\hat{\chi}(n), \hat{\chi}^\dagger(n')\} = \delta_{n,n'}, \quad \{\hat{\varphi}(n), \hat{\varphi}^\dagger(n')\} = \delta_{n,n'}, \quad (13)$$

and otherwise zero. On the Fock space spanned by these operators, we can consider the coherent states given by

$$|\chi_{n_{4}+1}, \varphi_{n_{4}+1}\rangle \equiv \exp \sum_{n} \{\chi_{n_{4}}^\dagger(n)\chi_{n_{4}+1}(n) + \varphi_{n_{4}}^\dagger(n)\varphi_{n_{4}+1}(n)\} |0\rangle, \quad (14)$$

$$\langle \chi_{n_{4}}, \varphi_{n_{4}}^\dagger | \equiv \langle 0 | \exp \sum_{n} \{\chi_{n_{4}}^\dagger(n)\chi_{n_{4}}(n) + \varphi_{n_{4}}^\dagger(n)\varphi_{n_{4}}(n)\}. \quad (15)$$

It can be shown that these coherent states satisfy the following completeness relation.

$$I = \int [d\chi_{n_{4}}^\dagger d\chi_{n_{4}+1}] [d\varphi_{n_{4}}^\dagger d\varphi_{n_{4}+1}] \times \exp \left\{-\sum_{n}[\chi_{n_{4}}^\dagger(n)\chi_{n_{4}+1}(n) + \varphi_{n_{4}}^\dagger(n)\varphi_{n_{4}+1}(n)]\right\} |\chi_{n_{4}+1}, \varphi_{n_{4}+1}\rangle \langle \chi_{n_{4}}, \varphi_{n_{4}}^\dagger |. \quad (16)$$

We can see that the weight appearing in Eq. (16) can be identified with the kinetic term in the $n_{4}$-direction in the form of Eq. (12). The remaining terms in the exponential of the
action can be arranged into the product of the representatives of the transfer matrices on
the coherent state basis. The transfer matrix acting on the Fock space is then found to be
\[ \hat{T}_F = \exp \left\{ -\frac{1}{2} \sum_n \sum_{k=1}^3 \left[ \hat{\chi}^\dagger(n + \hat{k}) \eta_k(n) \hat{\phi}^\dagger(n) + \hat{\phi}^\dagger(n) \eta_k'(n) \hat{\chi}^\dagger(n + \hat{k}) \right] \right\} \]
\[ \times \prod_n : \left( \hat{\varphi}^\dagger(n) - \hat{\chi}(n) \right) \left( \hat{\varphi}(n) - \hat{\chi}^\dagger(n) \right) : \]
\[ \times \exp \left\{ -\frac{1}{2} \sum_n \sum_{k=1}^3 \left[ \hat{\varphi}(n + \hat{k}) \eta_k(n) \hat{\chi}(n) + \hat{\chi}(n) \eta_k'(n) \hat{\varphi}(n + \hat{k}) \right] \right\} . \]  
(17)
\[ \hat{T}_F \] is hermitian, but not positive definite. The Hamiltonian follows \((\hat{T}_F)^2\) and it is positive
definite.

**Schrödinger Functional**

Hereafter we will consider the staggered fermion in a three-dimensional Euclidean lattice
with finite volume \((2l)^3\) and its time development in a finite time interval \(2m\). The entire
four dimensional Euclidean lattice can be represented by
\[ \Gamma = \{ n_\mu = (n, n_4) \mid 0 \leq n_k < 2l \ (k = 1, 2, 3), \ 1 \leq n_4 \leq 2m \} . \]  
(18)
We assume that the fermionic variables \(\chi, \chi^\dagger\) and \(\varphi, \varphi^\dagger\) obey spatially periodic boundary
condition. Temporarily, certain configurations in terms of Grassmann variables are assumed
on initial and final equitime surfaces. Since the Dirac equation is of first order, the boundary
condition should fix only half of the components of the field variable on each initial or final
equitime surface. The choice of the half components is intimately related to the way of
constructing the Schrödinger Functional.

The wave functional in the Schrödinger representation is naturally introduced as the
representative of a state vector in the coherent-state basis,
\[ \Psi[\chi^\dagger, \phi^\dagger] \equiv \langle \chi^\dagger, \phi^\dagger | \Psi \rangle . \]  
(19)
Its dynamics is described by the kernel which acts on an initial wave functional at \(n_4 = 1\)
and results in a final wave functional at \(n_4 = 2m\). For the staggered fermion, it is given by
It is this kernel which is referred to the Schrödinger functional in the literature.

The Schrödinger functional can be represented by functional integral as follows. Inserting the completeness relation in-between the transfer matrices, and integrating over the auxiliary fields \( \varphi_{n+1}, \varphi_n \) \( (n_4 = 1, \cdots, 2m - 1) \) and \( \chi_1, \chi_{2m} \) to eliminate the \( \delta \)-functions, we obtain

\[
Z_F[\chi_{2m}, \varphi_{2m}; \chi_1, \varphi_1] = \int \prod_{n_4=1}^{2m-1} \left[ d\chi_{n_4} d\varphi_{n_4+1} d\varphi_{n_4} \right] \prod_{n_4=1}^{2m} \left\{ - \sum_{n,n_4=1}^{2m-1} \left[ \chi_{n_4}(n) \chi_{n_4+1}(n) + \varphi_{n_4}(n) \varphi_{n_4+1}(n) \right] \right\} \]

\[
= \int \prod_{n_4=2}^{2m-1} \left[ d\chi_{n_4} d\chi_1 \right] \exp \left\{ - S_F^{tot} \right\}.
\]

\( S_F^{tot} \) consists of three parts,

\[
S_F^{tot} = S^{(T)}_B + S_F + S^{(0)}_B.
\]

If we use the original conjugate variables \( \chi(n, n_4) \) for \( 1 \leq n_4 \leq 2m - 1 \) and \( \tilde{\chi}(n, n_4) \) for \( 2 \leq n_4 \leq 2m \), which are rescaled back from \( \chi_{n_4}(n) \) and \( \chi_{n_4}(n) \) by Eq. (6), and we also redefine

\[
\tilde{\chi}_1(n) = 2\eta_1(n) \varphi_1(n), \quad \chi_{2m}(n) = \varphi_{2m}(n),
\]

then, these three parts read

\[
S_F = \sum_{n_4=1}^{2m-1} \sum_n \sum_{\mu=1}^4 \frac{1}{2} \eta_{\mu}(n) \tilde{\chi}(n) \left[ \chi(n + \hat{\mu}) - \chi(n - \hat{\mu}) \right],
\]

\[
S^{(T)}_B = \sum_n \sum_{k=1}^3 \frac{1}{2} \eta_k(n, 2m) \tilde{\chi}(n, 2m) \left[ \chi(n + \hat{k}, 2m) - \chi(n - \hat{k}, 2m) \right] - \sum_n \frac{1}{2} \eta_1(n) \tilde{\chi}(n, 2m) \chi(n, 2m - 1),
\]

\[
S^{(0)}_B = \sum_n \sum_{k=1}^3 \frac{1}{2} \eta_k(n, 1) \tilde{\chi}(n, 1) \left[ \chi(n + \hat{k}, 1) - \chi(n - \hat{k}, 1) \right] + \sum_n \frac{1}{2} \eta_1(n) \tilde{\chi}(n, 1) \chi(n, 2).
\]
Boundary Condition in terms of Four-component Spinor

Starting from the transfer matrix of the staggered fermion, we have obtained the functional integral form of the Schrödinger Functional for the free quarks. The first part of the action, $S_F$, given by Eq. (24), is the usual action for the staggered fermion (in finite volume). $S_B^{(0)}$ and $S_B^{(T)}$ are boundary terms involving the boundary values of the fermion field, $\chi_1$, $\bar{\chi}_1$ and $\chi_{2m}$, $\bar{\chi}_{2m}$, respectively. In order to examine these boundary terms more closely and to communicate with the counterparts in the continuum theory, we will express them in terms of the four-flavored four-component Dirac spinor [24] [25]. We follow here the formulation given by H. Kluberg-stern et al. [25].

The original index of the lattice $n_\mu$ may be written in the form of

$$ n_\mu = 2x_\mu + \rho_\mu, \quad (27) $$

where $x_\mu$ is an integer four-vector and $\rho_\mu$ is a four-vector whose components are either one or zero. $x_\mu$ labels the site on the sublattice with the spacing twice as large as the original one. $\rho_\mu$ points one of the sixteen variables $\chi(2x + \rho)$ near to $x_\mu$ making them associated to the site on the sublattice. Accordingly, we adopt the notation such as

$$ \chi_\rho(x) = \chi(2x + \rho), \quad \bar{\chi}_\rho(x) = \bar{\chi}(2x + \rho). \quad (28) $$

It follows immediately that

$$ \chi(2x + \rho + \hat{\mu}) = \sum_{\rho'} [\chi_{\rho'}(x)\delta_{\rho',\rho+\hat{\mu}} + \chi_{\rho'}(x + \hat{\mu})\delta_{\rho',\rho-\hat{\mu}}], \quad (29) $$

$$ \chi(2x + \rho - \hat{\mu}) = \sum_{\rho'} [\chi_{\rho'}(x)\delta_{\rho',\rho-\hat{\mu}} + \chi_{\rho'}(x - \hat{\mu})\delta_{\rho',\rho+\hat{\mu}}]. \quad (30) $$

Then, noting that $\eta_\mu(2x + \rho) = \eta_\mu(\rho)$, we can rewrite the action $S_F$ as

$$ S_F = \frac{1}{4} \sum_{x=1}^{m-1} \sum_{\rho\rho'} \sum_{\mu} \left\{ \bar{\chi}_\rho(x)(A_\mu)_{\rho\rho'} [\chi_{\rho'}(x + \hat{\mu}) - \chi_{\rho'}(x - \hat{\mu})] ight. $$

$$ + \left. \bar{\chi}_\rho(x)(A_\mu^5)_{\rho\rho'} [\chi_{\rho'}(x + \hat{\mu}) + \chi_{\rho'}(x - \hat{\mu}) - 2\chi_{\rho'}(x)] \right\}, \quad (31) $$

where
These $\Lambda_\mu$'s can be expressed by the traces over the products of gamma matrices. First we introduce matrices

$$R_\rho = (R_\rho)_a^\alpha \equiv \frac{1}{2}T(2x + \rho) = \frac{1}{2} \gamma_1^\rho_1 \gamma_2^\rho_2 \gamma_3^\rho_3 \gamma_4^\rho_4,$$

satisfying unitary conditions

$$\sum_{\alpha a}(R_\rho^\dagger)_a^\alpha (R_\rho)_a^\alpha = \text{Tr} \{R_\rho^\dagger R_\rho\} = \delta_{\rho\rho'},$$
$$\sum_\rho (R_\rho)_a^\alpha (R_\rho^\dagger)_b^\beta = \delta^\beta_\alpha \delta^a_b.$$  

Then we can show the following relations.

$$(\Lambda_\mu)_{\rho\rho'} = \text{Tr} \{R_\rho^\dagger \gamma^\mu R_\rho'\} ,$$  
$$(\Lambda^5_\mu)_{\rho\rho'} = \text{Tr} \{R_\rho^\dagger \gamma^5 R_\rho' \gamma^5 \gamma^\mu\} ,$$
$$\sum_{\rho\rho'} (R_\rho)_a^\alpha (\Lambda_\mu)_{\rho\rho'} (R_\rho^\dagger)_b^\beta = (\gamma^\mu)_a^\beta \otimes 1^a_b \equiv (\Gamma_\mu)_a^\beta b,$$
$$\sum_{\rho\rho'} (R_\rho)_a^\alpha (\Lambda^5_\mu)_{\rho\rho'} (R_\rho^\dagger)_b^\beta = (\gamma^5)_a^\beta \otimes (\gamma^*_\mu \gamma^*_5)_a^\beta b \equiv (\Gamma^5_\mu)_a^\beta b.$$  

With this unitary matrices $R_\rho$, we can transform $\chi_\rho(x)$ into the four-flavored Dirac spinor as

$$\psi_a^\alpha(x) = \frac{1}{\sqrt{2}} \sum_\rho (R_\rho)_a^\alpha \chi_\rho(x),$$
$$\bar{\psi}_a^\alpha(x) = \frac{1}{\sqrt{2}} \sum_\rho \bar{\chi}_\rho(x)(R_\rho^\dagger)_a^\alpha ,$$

where Greek and Latin suffixes can be taken to denote spinor and flavor indices, respectively. In terms of the four-component spinor thus defined, we obtain

$$2 \text{ The normalization in Eq. (11) is determined so as to scale all dimension-full quantities in the four component spinor representation by the new lattice spacing } b(=2a) \text{ twice as large as } a.$$
\[ S_F = \sum_{x_4=1}^{m-1} \sum_{x} \tilde{\psi}(x) \left[ (\gamma_\mu \otimes 1) \tilde{\nabla}_\mu + \frac{1}{2} (\gamma_5 \otimes \gamma_\mu^* \gamma_5^*) \Delta_\mu \right] \psi(x), \]  

where

\[ \tilde{\nabla}_\mu \psi(x) \equiv \frac{1}{2} [\psi(x + \hat{\mu}) - \psi(x - \hat{\mu})], \]

\[ \Delta_\mu \psi(x) \equiv \psi(x + \hat{\mu}) + \psi(x - \hat{\mu}) - 2\psi(x). \]

Now we consider the surface term, \( S_B^{(T)} \), at \( x_4 = m \equiv T/b \). It can be rewritten in the form of

\[ S_B^{(T)} = - \sum_x \sum_{\rho \rho'} \frac{1}{2} \chi_{(\rho,0)}(x, m) (\Lambda_{\mu} - \Lambda_{\mu}^5)_{(\rho,0)(\rho',1)} \chi_{(\rho',1)}(x, m - 1) \]

\[ + \sum_x \sum_{\rho \rho'} \sum_k 1 \chi_{(\rho,0)}(x, m) \left[ (\Lambda_k)_{(\rho,0)(\rho',1)} \tilde{\nabla}_k + \frac{1}{2} (\Lambda_{\mu}^5)_{(\rho,0)(\rho',1)\Delta_k} \right] \chi_{(\rho,0)}(x, m). \]

Here we denoted the four-vector \( \rho \) by a set of a three-vector and a definite forth component as \( (\rho, 0) \) or \( (\rho, 1) \). It is useful to introduce projectors which act on the space of the four-vectors \( \{\rho\} \) and project those elements with a definite value for the forth component \( \rho_4 \). We define

\[ (\tilde{P}_0)_{\rho \rho'} \equiv \delta_{\rho + \hat{\mu}, \rho' + \hat{\mu}}, \quad (\tilde{P}_1)_{\rho \rho'} \equiv \delta_{\rho - \hat{\mu}, \rho' - \hat{\mu}}. \]

The relations below follow the definitions of Eqs. (33) and (34):

\[ \sum_\sigma (\Lambda_\mu)_{\rho \sigma} (\Lambda_\mu)_{\sigma \rho'} = \delta_{\rho + \hat{\mu}, \rho' + \hat{\mu}} + \delta_{\rho - \hat{\mu}, \rho' - \hat{\mu}} = \delta_{\rho, \rho'} \quad (\mu = 1, \ldots, 4), \]

\[ \sum_\sigma (\Lambda_{\mu}^5)_{\rho \sigma} (\Lambda_{\mu}^5)_{\sigma \rho'} = \delta_{\rho + \hat{\mu}, \rho' + \hat{\mu}} - \delta_{\rho - \hat{\mu}, \rho' - \hat{\mu}} \quad (\mu = 1, \ldots, 4). \]

By using these relations, the above projectors can be expressed by \( \Lambda_4 \) and \( \Lambda_4^5 \) as follows.

\[ (\tilde{P}_0)_{\rho \rho'} = \frac{1}{2} \sum_\sigma (\Lambda_4)_{\rho \sigma} (\Lambda_4 + \Lambda_4^5)_{\sigma \rho'}, \]

\[ (\tilde{P}_1)_{\rho \rho'} = \frac{1}{2} \sum_\sigma (\Lambda_4)_{\rho \sigma} (\Lambda_4 - \Lambda_4^5)_{\sigma \rho'}. \]

Making use of these projectors, Eq. (45) can be written as
\[ S_B^{(T)} = -\frac{1}{4} \sum_x \sum_{\rho, \rho'} \bar{\chi}_\rho(x, m) \left[ \bar{P}_0 (\Lambda_4 - \Lambda_4^5) P_1 \right]_{\rho \rho'} \chi_{\rho'}(x, m - 1) \]
\[ + \frac{1}{2} \sum_x \sum_{\rho \rho'} \sum_{k=1}^3 \bar{\chi}_\rho(x, m) \left[ \bar{P}_0 (\Lambda_k \nabla_k + \frac{1}{2} \Lambda_k^5 \Delta_k) P_0 \right]_{\rho \rho'} \chi_{\rho'}(x, m) . \]  

(51)

With Eqs. (39), (40) and (41), we can again rewrite it in terms of the four-component spinor.

\[ S_B^{(T)} = -\sum_x \bar{\psi}(x, m) P_0 \Gamma_4 \psi(x, m - 1) \]
\[ + \sum_x \sum_{k=1}^3 \bar{\psi}(x, m) P_0 (\Gamma_k \nabla_k + \frac{1}{2} \Gamma_k^5 \Delta_k) \psi(x, m) , \]  

(52)

where \( P_0 \) and \( P_1 \) are the projectors acting on the spinor and flavor spaces, which correspond to \( \bar{P}_0 \) and \( \bar{P}_1 \), respectively: the unitary transformation by \( R \) leads to the following expressions.

\[ P_0 = \frac{1}{2} \Gamma_4 (\Gamma_4 + \Gamma_4^5) = \frac{1}{2} (1 \otimes 1 + \gamma_4 \gamma_5 \otimes \gamma_4^* \gamma_5^*) , \]  

(53)

\[ P_1 = \frac{1}{2} \Gamma_4 (\Gamma_4 - \Gamma_4^5) = \frac{1}{2} (1 \otimes 1 - \gamma_4 \gamma_5 \otimes \gamma_4^* \gamma_5^*) . \]  

(54)

Here we have used the anti-commutation relations of \( \Gamma \).

\[ \{ \Gamma_\mu , \Gamma_\nu \} = 2 \delta_{\mu \nu} 1 \otimes 1 , \]  

(55)

\[ \{ \Gamma_5^\mu , \Gamma_5^\nu \} = -2 \delta_{\mu \nu} 1 \otimes 1 , \]  

(56)

\[ \{ \Gamma_\mu , \Gamma_5^\nu \} = 0 . \]  

(57)

By a similar consideration, \( S_B^{(0)} \) can be expressed in the form of

\[ S_B^{(0)} = \sum_x \bar{\psi}(x, 0) P_1 \Gamma_4 \psi(x, 1) \]
\[ + \sum_x \sum_{k=1}^3 \bar{\psi}(x, 0) P_1 (\Gamma_k \nabla_k + \frac{1}{2} \Gamma_k^5 \Delta_k) \psi(x, 0) . \]  

(58)

Now we summarize the total action \( (22) \) in terms of the four-flavored four-component spinor:

\[ S_F^{tot} = -\sum_x \bar{\psi}(x, m) P_0 \Gamma_4 \psi(x, m - 1) \]
\[ S_F^\text{tot} = \int_0^T dy_4 \int_0^L d^3 y \bar{\Psi}(y) \left[ (\gamma_\mu \otimes 1) \partial_\mu \right] \Psi(y) \Big|_{\text{H.B.C.}} \]
\[ - \int_0^L d^3 y \left\{ \bar{\Psi}(y, +0) \Gamma_4 P_1 \eta_1(y) - \bar{\eta}_1(y) \Gamma_4 P_0 \Psi(y, +0) \right\} \]
\[ - \int_0^L d^3 y \left\{ \bar{\eta}_T(y) \Gamma_4 P_1 \Psi(y, T - 0) - \bar{\Psi}(y, T - 0) \Gamma_4 P_0 \eta_T(y) \right\} , \quad (62) \]

where H.B.C. stands for the homogeneous boundary condition with \( \eta(y) \) and \( \bar{\eta}(y) \) vanishing.
\[ P_1 \Psi(y, 0) = 0, \quad \bar{\Psi}(y, 0) P_1 = 0. \]
\[ P_0 \Psi(y, T) = 0, \quad \bar{\Psi}(y, T) P_0 = 0. \quad (63) \]

It is also clear in the lattice formulation that, in the viewpoint of the path integral, the integral variables are \( \Psi(y) \) and \( \bar{\Psi}(y) \) supplemented with the homogeneous boundary condition (63), and \( \eta(y) \) and \( \bar{\eta}(y) \) which are localized on the boundary surfaces can be regarded as external sources.

A few remarks are in order here. It is interesting to note that the boundary condition for the staggered fermion (60) and the projectors (53) and (54), are very similar to those Symanzik adopted for fermion field in his paper on the Schrödinger representation of the renormalizable field theory [15]. According to Symanzik’s prescription, the boundary terms of the fermion field in a renormalizable theory can be constructed in the Minkowski space as follows. We first find out the operator which generates the time reversal transformation for fermion. Such an operator may be chosen as

\[
B = i \int d^3 y \bar{\Psi}(y) \gamma_0 (-i \gamma_0 \gamma_5) \Psi(y) = i \int d^3 y \left\{ \bar{\Psi}(y) \gamma_0 P_- \Psi(y) - \bar{\Psi}(y) \gamma_0 P_+ \Psi(y) \right\}, \quad (64)
\]

where \( P_\pm \equiv (1 \pm i \gamma_0 \gamma_5)/2 \), and the transformation is given by

\[
i [B, P_- \Psi(y)] = + P_- \Psi(y), \quad i [B, P_+ \Psi(y)] = - P_+ \Psi(y). \quad (65)
\]

The component without a change of the sign is referred to “Dirichlet component” and the component with a change of the sign is “Neumann component”. First consider imposing the homogeneous Dirichlet boundary condition;

\[
P_- \Psi(y, +0) = 0, \quad \bar{\Psi}(y, +0) P_- = 0 \quad (66)
\]

at \( y_0 = 0 \). (Here the region \( y_0 > 0 \) is referred to “Dirichlet side”. ) It can be achieved by adding the following boundary term to the action as an interaction.

\[
B_0 = i \int d^3 y \left\{ \bar{\Psi}(y, +0) \gamma_0 P_- \Psi(y, -0) - \bar{\Psi}(y, -0) \gamma_0 P_+ \Psi(y, +0) \right\}. \quad (67)
\]

To implement the inhomogeneous Dirichlet boundary condition,
the Dirichlet component valued functions, $\eta(y)$ and $\bar{\eta}(y)$, are introduced as the sources which are localized at the boundary surface and the Neumann components in the Dirichlet side are made to couple with the sources. Namely, the following source term is added to the action:

$$B_\eta = i \int d^3 y \left\{ \bar{\Psi}(y, +0) \gamma_0 \eta(y) - \bar{\eta}(y) \gamma_0 \Psi(y, +0) \right\}.$$ (69)

We can now clearly see the correspondence between the boundary condition for the staggered fermion field and the one that is imposed on the continuum fermion field in the Symanzik’s theory: the time reversal transformation can be associated to the matrix $\pm (\gamma_4 \gamma_5 \otimes \gamma_4^* \gamma_5^*)$, and accordingly, the projectors $P_{0,1} = (1 \pm \gamma_4 \gamma_5 \otimes \gamma_4^* \gamma_5^*)/2$ enter the boundary terms. Except for the structure in the flavor space due to the species doubling, the boundary condition is essentially the same.

**Gauge Interaction**

The gauge field part of the Schrödinger Functional of QCD can be formulated just as that of the pure Yang-Mills theory given by Lüscher et al. [5] and that of QCD with the Wilson fermion given by Sint [13]. In the case of the staggered fermion $\chi(n)$, one can introduce the gauge link variables on the lattice of the spacing $a$. In this case, there is no essential difference in constructing the transfer matrix and the Schrödinger Functional from the case of the free staggered fermion [21].

It is also possible, following Kluberg-stern et al. [25], to define the four-component Dirac spinor in a gauge invariant manner. Although the action in terms of the four-component Dirac spinor can be obtained only in the expansion with respect to $a$, this is enough in order to read off the boundary condition for the fermion field in the classical continuum limit. The gauge invariant four-spinor can be defined by

$$\psi^a_\alpha(x) = \frac{1}{\sqrt{2}} \sum_\rho (R_\rho)^a_\alpha U_\rho(x) \chi_\rho(x),$$ (70)
where

\[ U_\rho(x) = U_1(2x)^{\rho_1} U_2(2x + \rho_1 \hat{1})^{\rho_2} U_3(2x + \rho_1 \hat{1} + \rho_2 \hat{2})^{\rho_3} U_4(2x + \rho_1 \hat{1} + \rho_2 \hat{2} + \rho_3 \hat{3})^{\rho_4}. \]  

(71)

The action reads in the classical continuum limit

\[ S_{\text{tot}}^F = \int_0^T dy_4 \int_0^L d^3y [\gamma_\mu \otimes 1] D_\mu] \Psi(y) \mid_{\text{H.B.C.}} \]

\[ - \int_0^L d^3y \left\{ \bar{\Psi}(y, +0) \Gamma_4 P_1 \eta_0(y) - \bar{n}_0(y) \Gamma_4 P_0 \Psi(y, +0) \right\} \]

\[ - \int_0^L d^3y \left\{ \bar{n}_T(y) \Gamma_4 P_1 \Psi(y, T - 0) - \bar{\Psi}(y, T - 0) \Gamma_4 P_0 \eta_T(y) \right\}, \]  

(72)

where

\[ D_\mu \equiv \partial_\mu + A_\mu(y). \]  

(73)

The boundary condition is the same as Eqs. (63).

**Boundary counterterms**

We will next discuss the possible appearance of divergence at the surfaces by interaction. According to Symanzik [15], since the effect of the boundary surface can be expressed by the local surface interaction as Eq. (67), it is possible to apply the ordinary prescription for the (perturbative) renormalization with the local counterterms, as far as the theory concerning is renormalizable in infinite volume. In QCD, if there would appear divergence at the surfaces, they could be subtracted by the boundary counterterms which consist of gauge invariant local operators of three or less dimensions. For fermions, such terms are only the bilinear operators of dimension three and are given in general as

\[ \Delta S_{\text{boundary}} = \int_0^L d^3y \Psi(y, T) Z_T \Psi(y, T) + \int_0^L d^3y \bar{\Psi}(y, 0) Z_0 \Psi(y, 0), \]  

(74)

where \( Z_t \ (t=0,T) \) are some matrices.

To determine \( Z_0 \) and \( Z_T \) in the lattice regularization with the staggered fermion, we should take into account of the symmetry of the staggered fermion in the finite volume space. We should consider the following three symmetry transformations:
(i) discrete spatial rotation on $(i, j)$-plane $(i, j=1, 2, 3)$ \[19\]

$$x \to A^{-1}x;$$

$$(A^{-1}x)_i = x_j, (A^{-1}x)_j = -x_i, \text{ rest unchange,} \quad (75)$$

$$\psi(x) \to S(A) \otimes R^T(A) \psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}(x) S(A)^{-1} \otimes [R^T(A)]^{-1}, \quad (76)$$

with

$$S(A) = \frac{1}{\sqrt{2}} (1 - \gamma_i \gamma_j), \quad R^T(A) = \frac{1}{\sqrt{2}} (\gamma^i_j - \gamma^j_i). \quad (77)$$

(ii) $(\gamma_4 \otimes \gamma_5^\dagger)$-parity transformation \[19\]

$$\psi(x, x_4) \to (\gamma_4 \otimes \gamma_5^\dagger) \psi(-x, x_4),$$

$$\bar{\psi}(x, x_4) \to \bar{\psi}(-x, x_4) (\gamma_4 \otimes \gamma_5^\dagger). \quad (78)$$

(iii) $(\gamma_5 \otimes \gamma_5^\dagger)$-chiral transformation (when massless) \[25\]

$$\psi(x) \to e^{i \alpha (\gamma^5 \otimes \gamma^5)} \psi(x),$$

$$\bar{\psi}(x) \to \bar{\psi}(x) e^{i \alpha (\gamma^5 \otimes \gamma^5)}. \quad (79)$$

By these symmetries, the general form of $Z$ is restricted as

$$Z = c_4 (\gamma_4 \otimes 1) + c_5 (\gamma_5 \otimes \gamma_4^\dagger \gamma_5^\dagger) + \bar{c}_4 (\gamma_4 \otimes \bar{\gamma}^\dagger \gamma_4^\dagger) + \bar{c}_5 (\gamma_5 \otimes \bar{\gamma}^\dagger \gamma_5^\dagger)$$

$$= c_0 (\gamma_4 \otimes 1) P_0 + c_1 (\gamma_4 \otimes 1) P_1 + \bar{c}_0 (\gamma_5 \otimes \bar{\gamma}^\dagger \gamma_5^\dagger) P_0 + \bar{c}_1 (\gamma_5 \otimes \bar{\gamma}^\dagger \gamma_5^\dagger) P_1. \quad (80)$$

where $\bar{\gamma} \equiv \sum_{i=1}^3 \gamma_i$ and $c$'s are arbitrary constants. Then, taking into account of the boundary condition \[14\], the boundary counterterms can be written as

$$\Delta S_{boundary} = \Delta S^{(0)} + \Delta S^{(T)}, \quad (81)$$

$$\Delta S^{(0)} = \int_0^L d^3y \left\{ c_0^{(0)} \bar{\eta}_0(y) (\gamma_4 \otimes 1) P_0 \psi(y, +0) + c_1^{(0)} \bar{\psi}(y, +0) (\gamma_4 \otimes 1) P_1 \eta_0(y) \right\}$$

$$+ \int_0^L d^3y \left\{ c_0^{(0)} \bar{\eta}(y, +0) (\gamma_5 \otimes \bar{\gamma}^\dagger \gamma_5^\dagger) P_0 \psi(y, +0) + c_1^{(0)} \bar{\eta}_0(y) (\gamma_5 \otimes \bar{\gamma}^\dagger \gamma_5^\dagger) P_1 \eta_0(y) \right\}, \quad (82)$$

16
\[
\Delta S^{(T)} = \int_0^L d^3y \left\{ c_0^{(T)} \bar{\Psi}(y, T-0)(\gamma_4 \otimes 1)P_0 \eta_T(y) + c_1^{(T)} \bar{\eta}_T(y)(\gamma_4 \otimes 1)P_1 \Psi(y, T-0) \right\} \\
+ \int_0^L d^3y \left\{ \bar{c}_0^{(T)} \bar{\eta}_T(y)(\gamma_5 \otimes \gamma_5^*)P_0 \eta_T(y) \\
+ \bar{c}_1^{(T)} \bar{\Psi}(y, T-0)(\gamma_5 \otimes \gamma_5^*)P_1 \Psi(y, T-0) \right\}.
\] (83)

These expressions give the general form of the boundary counterterms which may appear in the lattice regularization with the staggered fermion.

According to the above expressions, we note that even in the case of the homogeneous Dirichlet boundary condition there remain the surface counterterms, which give the coupling between the Neumann components.

\[
\Delta S_{surface} = \bar{c}_0^{(T)} \sum_x \bar{\psi}(x, +1)(\gamma_5 \otimes \gamma^*_5)P_0 \psi(x, +1) \\
+ \bar{c}_1^{(T)} \sum_x \bar{\psi}(x, m-1)(\gamma_5 \otimes \gamma^*_5)P_1 \psi(x, m-1) \\
= \sum_n \sum_{k=1}^3 \eta_k(n, 2) \bar{\chi}(n, 2)[\chi(n - \hat{k}, 2) - \chi(n + \hat{k}, 2)] \\
+ \frac{\bar{c}_0^{(T)} + \bar{c}_1^{(T)}}{2} \sum_n \sum_{k=1}^3 \eta_k(n, 2m-1) \bar{\chi}(n, 2m-1)[\chi(n - \hat{k}, 2m-1) - \chi(n + \hat{k}, 2m-1)].
\] (84)

The symmetry of the staggered fermion (in terms of the four-component spinor) does not seem to be able to exclude these counterterms. This means that the boundary condition (83) could not be upheld and the Schrödinger Functional of QCD in the lattice regularization with staggered fermion would not be well-defined.

This is not the case, however. To examine this problem, we consider the boundary counterterms at finite lattice spacing. Noting \( P_0 \psi(x, +1) = \lim_{b \to 0} \frac{1}{b^3} P_0 \psi(x, +1) \) and the similar equation for \( P_1 \Psi(y, T-0) \), we obtain the lattice counterparts of \( \Delta S_{surface} \) as

\[
\Delta S'_{surface} = \frac{\bar{c}_0^{(0)}}{2} \sum_n \sum_{k=1}^3 \eta_k(n, 2) \bar{\chi}(n, 2)[\chi(n - \hat{k}, 2) - \chi(n + \hat{k}, 2)] \\
+ \frac{\bar{c}_1^{(T)}}{2} \sum_n \sum_{k=1}^3 \eta_k(n, 2m-1) \bar{\chi}(n, 2m-1)[\chi(n - \hat{k}, 2m-1) - \chi(n + \hat{k}, 2m-1)].
\] (85)

From this expression, we can see the following nature of the surface counterterms. They are local and \( \mathcal{O}(1) \) in terms of the variables \( \psi(x) \) and \( \bar{\psi}(x) \) on the lattice with the spacing \( b \).
On the other hand, they are derivative terms and $O(a)$ in terms of the variables $\chi(n)$ and $\bar{\chi}(n)$ on the lattice with the spacing $a$. As a result, if the link variables of gauge field were introduced on the lattice with the spacing $b$, the surface divergence could occur and we would need the above counterterms. On the other hand, as far as they are introduced on the lattice with the spacing $a$, these counterterms may not be necessary.

It is worth noting that the similar thing happens to the self-energy of the staggered fermion. As shown in [21] [19] [20], if the link variables are defined on the finer lattice, the self-energy correction induced by the gauge interaction is proportional to the bare mass. Then the chiral limit is achieved when the bare mass vanishes. However, if they are defined on the blocked lattice, a linear-divergent mass term is actually induced and we need a fine tuning to obtain massless fermion.

Note also that the above result is consistent with the case in which the dimensional regularization is adopted for the continuum theory with the boundary condition (60). In this case, a kind of parity becomes symmetry of the theory (including the boundary condition). It is given by the following transformation,

$$
\psi(x, x_4) \rightarrow (\gamma_4 \otimes \gamma_4^*) \psi(-x, x_4), \\
\bar{\psi}(x, x_4) \rightarrow \bar{\psi}(-x, x_4)(\gamma_4 \otimes \gamma_4^*). \quad (86)
$$

By virtue of this symmetry, the surface counterterms are eliminated.

From these considerations, we can expect that as far as the gauge field is introduced on the lattice with the spacing $a$, the boundary counterterms are given in the following form in the continuum limit.

$$
\Delta S_{\text{boundary}} = \int_0^L d^3y \left\{ c_0(0) \bar{\eta}_0(y)(\gamma_4 \otimes 1)P_0 \Psi(y, +0) + c_1(0) \bar{\Psi}(y, +0)(\gamma_4 \otimes 1)P_1 \eta_0(y) \right\} \\
+ \int_0^L d^3y \left\{ c_0(T) \bar{\Psi}(y, T - 0)(\gamma_4 \otimes 1)P_0 \eta_T(y) + c_1(T) \bar{\eta}_T(y)(\gamma_4 \otimes 1)P_1 \Psi(y, T - 0) \right\}. \quad (87)
$$
Quark Masses

For the staggered fermion, which is equivalent to four-flavored quarks, the degenerate mass $M$ can be naturally introduced as

\[ S_M = \sum_{x_4=0}^{m} \sum_x M \bar{\psi}_a^\alpha(x) \psi_\alpha^a(x) = \sum_{n \in \Gamma} \frac{1}{2} M \bar{\chi}(n) \chi(n) . \]  

(88)

Such degenerate mass term for the staggered fermion does not cause any essential modification of the above formulation of the Schrödinger Functional. $S_M$ is simply added to $S_{F}^{\text{tot}}$.

Towards the more realistic QCD, we should lift the mass degeneracy of four flavor quarks. This possibility for the staggered fermion has been discussed by several authors \[17, 20\]. If we adopt the representation for the gamma matrices such as $\gamma_4 = \sigma_3 \otimes 1$ and $\gamma_i = \sigma_1 \otimes \sigma_i$ for $i = 1, 2, 3$, the non-degenerate diagonal mass terms can be expressed as

\[ M_a^{\delta_{ab}} = m \mathbf{1}_{ab} + m_4 (\gamma_4^2 \gamma_4^4)_{ab} + i m_{124} (\gamma_4^2 \gamma_4^4 \gamma_4^4)_{ab} . \]  

(89)

In terms of the one-component field $\chi(x)$, it reads

\[ S'_{M} = \sum_x \sum_{a} M_a^{\delta_{ab}} \bar{\psi}_a^\alpha(x) \psi_\alpha^a(x) \]  

(90)

\[ = \sum \left\{ \frac{1}{2} m \bar{\chi}(n) \chi(n) + \frac{1}{2} m_4 (-)^{n_1+n_2+n_3} \eta_4(n) \bar{\chi}(n) \chi(n) + \frac{1}{2} m_{12} (-)^{n_1+n_2} \eta_1(n) \eta_2(n) \bar{\chi}(n) \chi(n) + \frac{1}{2} m_{124} (-)^{n_1+n_3} \eta_1(n) \eta_2(n) \bar{\chi}(n) \chi(n) \right. \]  

\[ + i \frac{1}{2} m_{124} (-)^{n_1+n_3} \eta_1(n) \eta_2(n) \bar{\chi}(n) \chi(n) + \frac{1}{2} m_{124} (-)^{n_3} \eta_1(n) \eta_2(n) \bar{\chi}(n) \chi(n) \]  

\[ \times \bar{\chi}(n) \chi(n) + (-)^{n_1} \hat{1} + (-)^{n_2} \hat{2} + (-)^{n_4} \hat{4} \} . \]  

(91)

We can see that the general non-degenerate mass terms lead to the non-local terms extending to the next-to-nearest neighbors. They also break the cubic rotational symmetry in the four dimensional Euclidean lattice. If we require the spatial cubic symmetry and the locality in time ($n_4$) direction, the non-degeneracy is necessarily reduced to two \[19\]. Such mass is given by
\[ M_{ab} = m \, 1_{ab} + \frac{1}{\sqrt{3}} m' (\bar{\gamma}^*)_{ab}, \]  

(92)

where \( \bar{\gamma} \equiv \sum_{i=1}^{3} \gamma_i \). The two eigenvalues of \( M_{ab} \) are given by \( M_0 = m + m' \) and \( m_0 = m - m' \).

In terms of the one-component field \( \chi(x) \), it looks like

\[ S''_M = \sum_{x_4=0}^{m} \sum_a M_{ab} \bar{\psi}_a \alpha \psi_b (x) \]

\[ = \sum_{n \in \Gamma} \left\{ \frac{1}{2} m \bar{\chi}(n) \chi(n) + \frac{1}{2\sqrt{3}} m' (-)^{n_1+n_2+n_3+n_4} \sum_i (-)^{n_i} \eta_i(n) \bar{\chi}(n) \chi(n + (-)^{n_i} \hat{i}) \right\}. \]  

(94)

Note that this mass term, like the degenerate mass (91), does not cause any essential modification of the above formulation of the Schrödinger Functional. \( S''_M \) is simply added to \( S^f_{tot} \).

The degenerate mass term is local and gauge invariant. For the non-degenerate case (94), the gauge invariance of the mass term can be assured by the suitable insertion of the link variables. As shown in \[21\] \[19\] \[20\], if the link variables are defined on the finer lattice with the lattice spacing \( a \), the self-energy correction induced by the gauge interaction is proportional to the bare mass. Then the chiral limit is achieved when the bare mass vanishes.

**Discussion**

In the classical continuum limit of the lattice QCD with the staggered fermion in finite volume, we obtained the action (92) supplemented by the boundary condition (63). We also determined the possible form of the boundary counterterms as (87) in the case that the gauge field is introduced on the lattice with the spacing \( a \). This way of introducing the gauge field also keeps the chiral property of the staggered fermion \[21\] \[19\] \[20\].

In the application to the determination of \( \alpha_s \) in QCD by the finite size scaling technique, it seems a practical choice to adopt the homogeneous boundary condition in the numerical calculation of \( \alpha_s \) \[13\]. For this boundary condition, the boundary counterterms (87) vanish. This means that the Schrödinger Functional of QCD with the four-flavored quarks which
is supplemented by the homogeneous boundary condition (63) can become finite by the renormalization of the gauge coupling constant and the quark mass. As to the quark mass, the degenerate mass $M$ will be the first practical choice. The extrapolation to the massless limit $M \to 0$ can reduce the number of parameters to be renormalized in the continuum limit. Towards the more realistic QCD, we should lift the mass degeneracy of the four-flavor quarks. It may be possible to lift the degeneracy between the two “light” quarks and the two “heavy” quarks by using the non-degenerate mass term (94). Even though, this is rather crude description of the real quark masses. This kind of problem of the nonzero quark mass is yet an open question in the practical numerical calculation for QCD in finite volume.

There are many works remaining to be done. Especially, as the next step, one should calculate the contribution of the quark loop to the finite part of the Schrödinger Functional, $\Gamma_1[B]$ in ref. [5], to the first nontrivial order in the weak gauge coupling expansion. The calculation in two different regularization schemes would verify the universality of the Schrödinger Functional of QCD.

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REFERENCES

[1] A. El-Khadra, G. Hockney, A. Kronfeld, and P. Mackenzie, Phys. Rev. Lett. 69 (1992) 729.

[2] C. Michael, Phys. Lett. B283 (1992) 103.

[3] G. S. Bali and K. Schilling, Phys. Rev. D47 (1993) 661.

[4] G. S. Bali and K. Schilling, Nucl. Phys. B (Proc. Suppl.) 30 (1993) 513.

[5] M. Lüscher, R. Narayanan, P. Weisz and U. Wolff, Nucl. Phys. B384 (1992) 168.

[6] M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B389 (1993) 247.

[7] M. Lüscher, R. Sommer, P. Weisz and U. Wolff, Nucl. Phys. B413 (1994) 481.

[8] G.P. Lepage and P.B. Mackenzie, Nucl. Phys. B (Proc. Suppl.) 20 (1991) 173. ; Phys. Rev D48 (1993) 2250.

[9] G. P. Lepage, Nucl. Phys. B (Proc. Suppl.) 26 (1992) 45. and references therein.

[10] T. Onogi, S. Aoki, M. Fukugita, S. Hashimoto, N. Ishizuka, H. Mino, M. Okawa, and A. Ukawa, Nucl. Phys. B (Proc. Suppl.) 34 (1994) 492.

[11] S. Aoki, M. Fukugita, S. Hashimoto, N. Ishizuka, H. Mino, M. Okawa, T. Onogi and A. Ukawa, UTHEP-280, July 1994.

[12] C. T. H. Davies, K. Hornbostel, G. P. Lepage, A. Lidsey, J. Shigemitsu and J. Sloan, OHSTPY-HEP-T-94-013, FSU-SCRI-94-79, August 1994.

[13] S. Sint, Nucl Phys. B421 (1994) 135.

[14] K. Wilson, New phenomena in subnuclear physics. (Plenum, New York, 1977).

[15] K. Symanzik, Nucl. Phys. B190[FS3] (1981) 1.

[16] L. Susskind, Phys. Rev. D16 (1977) 3031.

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[17] P. Mitra, Phys. Lett. 123B (1983) 77.

[18] P. Mitra, Nucl. Phys. B227 (1983) 349.

[19] P. Mitra and P. Weisz, Phys. Lett. 126B (1983) 355.

[20] M. Göckeler, Phys. Lett. 142B (1984) 197.

[21] H.J. Thun H.S. Sharatchandra and P. Weisz, Nucl. Phys. B192 (1981) 205.

[22] A. Chodos and J.B. Healy, Nucl. Phys. B127 (1977) 426.

[23] N. Kawamoto and J. Smit, Nucl. Phys. B192 (1981) 100.

[24] F. Gliozzi, Nucl. Phys. B204 (1982) 419.

[25] O. Napoly H. Kluberg-Stern, A. Morel and B. Petersson, Nucl. Phys. B220[FS8] (1983) 447.