SPHERICAL METRICS WITH CONICAL SINGULARITIES ON 2-Spheres

SUBHADIP DEY

Abstract. Suppose that $\theta_1, \theta_2, \ldots, \theta_n$ are positive numbers and $n \geq 3$. Does there exist a sphere with a spherical metric with $n$ conical singularities of angles $2\pi \theta_1, 2\pi \theta_2, \ldots, 2\pi \theta_n$? A sufficient condition was obtained by Gabriele Mondello and Dmitri Panov [5]. We show that it is also necessary when we assume that $\theta_1, \theta_2, \ldots, \theta_n \notin \mathbb{N}$.

1. Introduction

A classical question in the theory of Riemann surfaces asks which real functions $f$ on a Riemann surface $S$ are equal to the curvature of a pointwise conformal metric. For simplicity, unless otherwise mentioned, all the surfaces considered in this section are orientable closed 2-manifolds. If $S$ has genus $\geq 2$, then Berger [1] showed that any smooth negative function is the curvature of a unique conformal metric. In the case of a Riemann surface of genus 1, i.e. when $S$ is a torus with a flat metric $g$, Kazdan and Warner [3] proved that a function $f(\neq 0) : S \to \mathbb{R}$ is the curvature of a metric in the conformal class of $g$ if and only if $f$ changes sign and satisfies $\int_S f dA < 0$, where $dA$ is the area form $g$.

These results have been generalized by Marc Troyanov in the case of surfaces with singularities of a special type, called conical singularities as defined below. Let $S$ be a surface. A real divisor $\beta$ on $S$ is a formal sum

$$\beta = \beta_1 x_1 + \cdots + \beta_n x_n,$$

where $x_i \in S$ are pairwise distinct and $\beta_i$ are real numbers. For the pair $(S, \beta)$, define

$$\chi(S, \beta) = \sum_{i=1}^{n} \beta_i + \chi(S),$$

called the Euler characteristic of $(S, \beta)$.

Suppose that $\beta = \beta_1 x_1 + \cdots + \beta_n x_n$ is a real divisor on $S$ such that $\beta_i > -1$. Let $g$ be a Riemannian metric on $S$ defined away from $x_1, \ldots, x_n$ such that each point $x_i$ has a neighborhood $U_i$ in $S$ with coordinate $z_i$ satisfying $z_i(x_i) = 0$ on which $g$ has the following form,

$$ds^2 = e^{2u_i} |z_i|^{2\beta_i} |dz_i|^2.$$

Here $u_i : U_i \to \mathbb{R}$ is a continuous function such that $u_i|_{U_i - \{x_i\}}$ is differentiable (of class at least $C^2$). The point $x_i$ is called a conical singularity of the metric $g$ of angle $2\pi (\beta_i + 1)$. We refer to this type of metrics $g$ as (Riemannian) metrics with conical singularities. We say that the metric $g$ (with conical singularities) represents the divisor $\beta$.

In this terminology, Troyanov [7] proved the following two theorems.

Theorem 1.1 ([7]). Let $S$ be a Riemann surface with a real divisor $\beta = \beta_1 x_1 + \cdots + \beta_n x_n$ such that $\beta_i > -1$, $i = 1, \ldots, n$. If $\chi(S, \beta) < 0$, then any smooth negative function on $S$ is the curvature of a unique conformal metric which represents $\beta$.

Theorem 1.2 ([7]). Let $S$ be a Riemann surface with a real divisor $\beta = \beta_1 x_1 + \cdots + \beta_n x_n$ such that $\beta_i > -1$, $i = 1, \ldots, n$. If $\chi(S, \beta) = 0$, then any function $f(\neq 0) : S \to \mathbb{R}$ is the curvature of a conformal metric representing $\beta$ if and only if $f$ changes sign and satisfies $\int_S f dA < 0$, where $dA$ is the area element of a conformally flat metric on $S$ with singularities.
Theorem 1.1 generalizes Berger’s result and Theorem 1.2 generalizes Kazdan and Warner’s result.

Restricting our attention to only constant curvature metrics with conical singularities, we can formulate the following question.

**Question 1.3.** Let $S$ be a Riemann surface with a real divisor $\beta = \beta_1 x_1 + \cdots + \beta_n x_n$ such that $\beta_i > -1$, $i = 1, \ldots, n$. Can $\beta$ be represented by a conformal metric of constant curvature?

The case when $\beta = 0$ is completely understood due to the classical uniformization theorems. If $\chi(S) \geq 0$, then any conformal class has a representative of constant curvature. When $\chi(S) < 0$, existence and uniqueness of such a conformal metric is provided by classical uniformization theorems of Koebe and Poincaré.

When $\beta \neq 0$, the Theorems 1.1 and 1.2 give complete understanding in the case when $\chi(S, \beta) \leq 0$. See also the work of McOwen [4] for the $\chi(S, \beta) < 0$ case.

Here we focus on the particular case when $S$ is the Riemann sphere, which has been least understood. Given a real divisor $\beta = \beta_1 x_1 + \cdots + \beta_n x_n$ with $\beta_i > -1$, $i = 1, \ldots, n$, does there exist a conformal metric $g$ with conical singularities on $S$ which represents $\beta$? If we further assume that $g$ has constant curvature 1, then Gauss-Bonnet Theorem gives a restriction on $\beta$, namely

$$\chi(S, \beta) > 0.$$  

But further restrictions on $\beta$ were found in addition to this in the case $n = 2, 3$. When $n = 2$, we simply need to require that $\beta_1 = \beta_2$ by the work of Troyanov [6]. When $n = 3$, by Eremenko’s result in [2] a conformal metric with conical singularities exists if and only if some inequalities on the numbers $\beta_1, \beta_2, \beta_3$ are satisfied. Moreover, in this case, there exists a spherical triangle with angles $\pi(\beta_1 + 1), \pi(\beta_2 + 1), \pi(\beta_3 + 1)$. Here a spherical triangle means a topological closed disk with a metric of constant curvature 1 such that the boundary is a piecewise geodesic loop with three singular points. Examples include geodesic triangles immersed in ordinary sphere $S^2$. A sphere with three conical singularities is the double of such a triangle. When $n > 3$, the situation becomes more complicated.

Therefore, before answering the Question 1.3 for higher $n$’s, perhaps one needs to have a complete list real divisors $\beta$ which may be represented by a spherical metric $g$ on a sphere with conical singularities. Here spherical means that the metric has constant curvature 1. Note that the divisor $\beta$ can be represented by a spherical metric with conical singularities if and only if there exists a spherical metric with $n$ conical singularities of angles $2\pi(\beta_1 + 1), \ldots, 2\pi(\beta_n + 1)$.

Let $\mathbb{R}^n_+$ be the set of all points in $\mathbb{R}^n$ with positive coordinates. A point $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n_+$ is called admissible if there exists a sphere $S$ with a spherical metric $g$ with $n$ conical points $x_1, \ldots, x_n$ of angles $2\pi\theta_1, \ldots, 2\pi\theta_n$ respectively. In this case, the real divisor $\beta = (\theta_1 - 1)x_1 + \cdots + (\theta_n - 1)x_n$ is represented by $g$. By abuse of notation, we write $\theta - 1 = \beta$, where $1 = (1, \ldots, 1)$.

**Question 1.4.** Which points $\theta \in \mathbb{R}^n_+$ are admissible?

A major progress was done to answer this question by Mondello and Panov [5], as stated in the next two theorems.

**Theorem 1.5 ([5]).** If $\theta \in \mathbb{R}^n_+$ is admissible, then

(P) \hspace{1cm} \chi(S, \theta - 1) > 0, \\
(H) \hspace{1cm} d_1(\mathbb{Z}^n_+, \theta - 1) \geq 1,

where $d_1$ is the standard $\ell^1$ distance on $\mathbb{R}^n$ and $\mathbb{Z}^n_+$ is the set of all points $v \in \mathbb{R}^n$ with integer coordinates such that $d_1(v, 0)$ is odd. Moreover, if the equality (H) is attained, then the holonomy of a metric which corresponds to $\theta$ is coaxial.
Here we clarify what we mean by saying that the holonomy of a metric with conical singularities is coaxial. Note that a metric \( g \) on \( S \) with conical singularities is actually defined as a Riemannian metric only on \( S - \{ \text{conical singularities} \} \). Thus we have a holonomy representation of the metric \( g \) which is a homomorphism \( \varphi : \pi_1(S - \{ \text{conical singularities} \}) \rightarrow SO(3) \) from the fundamental group of \( S - \{ \text{conical singularities} \} \) to the group of rotations of the standard sphere \( S^2 \). The metric \( g \) is said to have co axial holonomy if the image of \( \varphi \) is contained in an one-parameter subgroup of \( SO(3) \).

The following is a partial converse to Theorem 1.5.

**Theorem 1.6** ([5]). If \( \theta \in \mathbb{R}_+^n \) satisfies the positivity constraints (P) and the holonomy constraints (H) strictly, then \( \theta \) is admissible. Moreover, each metric corresponding to \( \theta \) has non-coaxial holonomy.

Theorems 1.5 and 1.6 classified all the admissible points which do not satisfy (P) and (H’) simultaneously, where (H’) is the equality in (H),

\[
d_1(\mathbb{Z}_n^0, \theta - 1) = 1,
\]

but did not provide answer in the remaining case. We formulate this open case in the following question.

**Question 1.7.** For \( n \geq 4 \), which points in \( \mathbb{R}_+^n \) satisfying (P) and (H’) together are admissible?

In Theorem 2.1, we give a partial answer to this question, which gives an improvement to the results obtained by Mondello and Panov [5]. In particular, we prove that when \( \theta_1, \ldots, \theta_n \not\in \mathbb{N} \), the sufficient conditions in Theorem 1.6 are actually necessary.

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## 2. Main Result

The aim of this short paper is to prove the following theorem.

**Theorem 2.1.** Suppose that \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}_+^n \) satisfies (P) and (H’) and \( n \geq 3 \). If \( \theta_i \not\in \mathbb{N} \) for all \( 1 \leq i \leq n \) then \( \theta \) is not admissible.

The case \( n = 3 \) has already been treated in [2]. Theorem 2.1 follows from a more general result, which we actually prove below.

**Theorem 2.2.** Suppose that \( S \) is a spherical surface with \( n \geq 2 \) conical singularities \( x_1, \ldots, x_n \) such that \( d_S(x_i, x_j) \geq \pi \), for all distinct singular points \( x_i, x_j \), where \( d_S \) is the distance function on \( S \) induced by the metric. Then \( n = 2 \).

Combining Theorem 2.1 with Theorems 1.5 and 1.6, we get necessary and sufficient conditions on \( (\theta_1, \ldots, \theta_n) \), provided \( \theta_i \not\in \mathbb{N} \), for which there exists a sphere \( S \) with \( n \geq 3 \) conical singularities of angles \( 2\pi \theta_1, \ldots, 2\pi \theta_n \). The analogous case when \( n \leq 2 \) is known from [6], as discussed above.

**Theorem 2.3.** Suppose that \( \theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}_+^n \) where \( n \geq 3 \). If \( \theta_i \not\in \mathbb{N}, 1 \leq i \leq n \), then \( \theta \) is admissible if and only if

\[
\chi(S, \theta - 1) > 0,
\]

\[
d_1(\mathbb{Z}_n^0, \theta - 1) > 1,
\]

where \( d_1 \) is the standard \( \ell^1 \) distance on \( \mathbb{R}^n \) and \( \mathbb{Z}_n^0 \) is the set of all points \( v \in \mathbb{R}^n \) with integer coordinates such that \( d_1(v, 0) \) is odd.
Let $S$ be a sphere with a spherical metric $g$ with $n$ conical singularities $x_1, \ldots, x_n$ of positive angles $2\pi\theta_1, \ldots, 2\pi\theta_n$ respectively, where $\theta_i$'s are non-integer and the tuple $(\theta_1, \ldots, \theta_n)$ satisfies (H'). The metric induces the notion of distance between two points in $S$, which is the minimum over the length of all piecewise smooth paths connecting these two points. Given two points $x, y \in S$, we denote the distance between $x$ and $y$ by $d_S(x, y)$.

An important feature of this metric $g$ on $S$ is that the holonomy representation is coaxial, which follows from Theorem 1.5.

**Theorem 2.4.** Let $S$ be a sphere with a spherical metric with conical singularities with coaxial holonomy. Suppose that $\gamma$ a geodesic arc $\gamma$ from $x_1$ to $x_2$ in $S$, where $x_1$ and $x_2$ are conical singularities of angles $\notin 2\pi\mathbb{N}$. Then the length of $\gamma$ is an integral multiple of $\pi$. In particular, $d_S(x_1, x_2) \geq \pi$.

This can be derived from Lemma 2.10 of [5], which says that if there exists a geodesic arc of length $l \notin \pi\mathbb{N}$ between $x_1$ and $x_2$, then the holonomy of the metric is non-coaxial.

**Remark.** When the some of the numbers $\theta_i$'s are integer, which still satisfy (H'), then Theorem 2.4 is no longer true. The simplest counterexample can be found when we graft together two standard spheres $S^2$ along a small geodesic arc.

### 3. Decomposition of $S$

In this section, we assume the hypothesis in Theorem 2.2. Given a singular point $x \in \{x_1, \ldots, x_n\}$ on $S$, we consider the closed metric ball $B(x)$ of radius $\pi/2$ in $S$ centered at $x$. By a **geodesic arc** (or, **geodesic segment**), we mean a path $\gamma : [a, b] \to S$ such that the image $\gamma((a, b)) \subset S - \{x_1, \ldots, x_n\}$ and is locally geodesic. In the following, we shall discuss the possible geometrical shapes of $B(x)$.

The most basic topological property of $B(x)$ is that it is a closed subset in $S$. Also, by our assumption $d_S(x_i, x_j) \geq \pi$, for all $i \neq j$, the interiors of $B(x)_i$'s are disjoint,

$$\phi = \text{interior}(B(x_i)) \cap \text{interior}(B(x_j)), \quad \text{for } i \neq j.$$  

The set of all boundary points of $B(x)$ in $S$ will be denoted by $\partial B(x)$.

Given a point $y \in \partial B(x)$ and a geodesic arc $\gamma$ from $x$ to $y$ of length $\pi/2$ and an $\epsilon > 0$, a **tubular $\epsilon$-neighborhood** of $\gamma$ is the set of all points that are $\epsilon$-close to $\gamma$ (Figure 1).

**Figure 1.** A tubular $\epsilon$-neighborhood of $\gamma$, when $\epsilon$ is very small.

By a **spherical polygon** (or, **polygon**) we mean a piecewise geodesic simple closed curve on $S - \{x_1, \ldots, x_n\}$. In the following, we shall prove that the components of $\partial B(x_i)$ are spherical polygons.

**Proposition 3.1.** Let $x \in \{x_1, \ldots, x_n\}$. The boundary $\partial B(x)$ of $B(x)$ is the union of finite number of disjoint spherical polygons. Moreover, all of these polygons have internal angles strictly bigger than $\pi$.

The following theorem about the metric spaces will be used repeatedly in the proof of Proposition 3.1.

**Theorem 3.2** (Arzelá-Ascoli). If $X$ is a compact metric space and $Y$ is a separable metric space, then every equicontinuous sequence of maps $f_i : Y \to X$ has a subsequence that converges uniformly on compacts in $Y$ to a continuous map $f : Y \to X$. 
Lemma 3.3. Let $x \in \{x_1, \ldots, x_n\}$ and $y$ be a boundary point of $B(x)$. Then there exists finitely many minimizing geodesic arcs from $x$ to $y$.

Proof. Suppose that there exists infinitely many distinct minimizing geodesic arcs $\gamma_i : [0, \pi/2] \to S$, $i \in \mathbb{N}$, such that $\gamma_i(0) = x$ and $\gamma_i(\pi/2) = y$, and for each $i \in \mathbb{N}$, $\gamma_i$ is an isometry, i.e. for $a, b \in [0, \pi/2]$, $d_S(\gamma_i(a), \gamma_i(b)) = |a - b|$.

Thus, the family $(\gamma_i)$ is an equicontinuous family. Therefore, by Arzelá-Ascoli Theorem 3.2, after passing to a subsequence, we may assume that $(\gamma_i)$ converges uniformly to a continuous map $\gamma : [0, \pi/2] \to S$, $\gamma_i \to \gamma$.

This $\gamma$ must also be an isometry with $\gamma(0) = x$ and $\gamma(\pi/2) = y$.

For sufficiently small $\epsilon$, a simply-connected tubular $\epsilon$-neighborhood $U$ of $\gamma$ in $S$ (Figure 1) will contain no singular points other than $x$. Since $(\gamma_i)$ converges uniformly to $\gamma$, there exists $N \in \mathbb{N}$ such that for all $i \geq N$, the geodesic $\gamma_i$ will lie in $U$. The geodesics $\gamma$ and $\gamma_i$ both start at $x$ and end at $y$. After making a slit on $U$ along a geodesic segment $\alpha$ from $x$ to the boundary of $U$ and disjoint from $\gamma$, $\gamma_i$, $i \geq N$, we have a developing map $\text{dev} : U \to \text{image}(\alpha) \to S^2$ on the standard sphere, which is a local diffeomorphism near $y$. This map $\text{dev}$ extends continuously on $x$. But then, $\text{dev} \circ \gamma_i \to \text{dev} \circ \gamma$, for all $i \geq N$, as these curves are geodesics on $S^2$ of length $\pi/2$ connecting $\text{dev}(x)$ and $\text{dev}(y)$. Consequently, the velocity vectors satisfy $(\text{dev} \circ \gamma/i)'(\pi/2) = (\text{dev} \circ \gamma_i)'(\pi/2)$, which implies that $\gamma'(\pi/2) = \gamma_i'(\pi/2)$, for all $i \geq N$. Since a geodesic $\delta : [0, \pi/2] \to S - \{x_1, \ldots, x_n\}$ is uniquely determined by its final point $\delta(\pi/2)$, its final velocity vector $\delta'(\pi/2)$, $\gamma = \gamma_i$, for $i \geq N$.

This makes $\gamma_N = \gamma_{N+1} = \ldots$, i.e. $(\gamma_i)$ can not be a sequence of distinct geodesic arcs, which leads to a contradiction. \qed

The geometry of $B(x)$ at a boundary point $y$ which is connected by a unique minimizing geodesic to $x$ will be different from the geometry at the boundary points which are connected by more than one minimizing geodesic, as shown in the pictures below (Figures 2 and 3). In particular, the later points will form corners at the boundary of $B(x)$. This is the content of the next lemma.

Lemma 3.4. Let $x \in \{x_1, \ldots, x_n\}$ and $y \in \partial B(x)$.

1. If $y$ is connected to $x$ by a unique minimizing geodesic arc, then there exists a small circular disk $D$ centered at $y$ such that $D \cap B(x)$ is a half disk filled in one side of a geodesic diameter (Figure 2).

2. Otherwise, there exists a small circular disk $D$ centered at $y$ such that $D \cap B(x)$ is a region bounded an boundary arc $\zeta$ of $\partial D$ and two radial geodesic rays $\lambda_1, \lambda_2$ drawn from the center $y$ to the endpoints of $\zeta$. Moreover, the angle formed by $\lambda_1$ and $\lambda_2$ on $B(x)$ is greater than $\pi$ (Figure 3).

Proof. (1) Suppose that there exists a unique minimizing geodesic arc $\gamma$ from $x$ to $y$ (i.e. of length $\pi/2$). Consider a thin $\epsilon$-neighborhood $U_{\epsilon}$ of $\gamma$ as depicted in Figure 1. Let $\lambda$ be a geodesic arc passing through $y$ perpendicularly to $\gamma$, which extends to the outside of the tubular neighborhood $U_{\epsilon}$. In the next paragraph, we show that for some $0 < \epsilon' < \epsilon$, $\lambda_{\epsilon'} = \lambda \cap U_{\epsilon'}$ is contained in the boundary $\partial B(x)$.

Suppose that this is not true, i.e. for all (small) $\epsilon' > 0$, $\lambda_{\epsilon'} \not\subset \partial B(x)$. Then there exists a converging sequence of distinct points $y_k \to y$ on $\lambda$ such that $y_k \in \lambda_{1/k}$ ($\lambda_{1/k}$ is always defined for large $k$’s) and there exists a minimizing geodesic arc $\gamma_k$ from $x$ to $y_k$ of length $l_k < \pi/2$. Then, $\gamma_k$ is an isometry,
\(\gamma_k : [0, l_k] \to S\). Define \(\tau_k : [0, \pi/2] \to S\) by \(\tau_k(r) = \gamma_k(r)\) when \(r \leq l_k\) and \(\tau_k(r) = y_k\) when \(r > l_k\). Clearly, \(\tau_k\) are 1-Lipschitz functions,

\[
d_S(\tau_k(r_1), \tau_k(r_2)) \leq |r_1 - r_2| \quad \forall r_1, r_2 \in [0, \pi/2].
\]

So, \((\tau_k)\) is an equicontinuous family of maps from \([0, \pi/2]\) to \(S\). By Arzelá-Ascoli Theorem 3.2, after extraction, \((\tau_k)\) converges to a map \(\tau : [0, \pi/2] \to S\), which is also a 1-Lipschitz function and satisfies \(\tau(0) = x, \tau(\pi/2) = y\). Therefore, by our uniqueness assumption, \(\tau = \gamma\). Hence, for any small \(\epsilon' > 0\), there exists \(k_0 \in \mathbb{N}\) such that \(\tau_k \subset U_{\epsilon'}\) for all \(k \geq k_0\). After making an appropriate slit along a geodesic arc \(\alpha\) from \(x\) as in the proof of Lemma 3.3, we can develop \(\text{dev} : (U_{\epsilon'} - \alpha) \cup \{x\} \to S^2\) on the standard sphere. Using \(\text{dev} \circ \gamma\), we note that \(d_{S^2}(\text{dev}(y_k), \text{dev}(x)) = \pi/2\). For \(k \geq k_0\), the point \(\text{dev}(y_k) = \text{dev} \circ \tau_k(l_k)\) lies on the geodesic \(\text{dev} \circ \lambda\) orthogonal to \(\text{dev} \circ \gamma\) at \(y\). Therefore, \(d_{S^2}(\text{dev}(y_k), \text{dev}(x)) = \pi/2\). However, for \(k \geq k_0\), \(\text{dev} \circ \gamma_k : [0, l_k] \to S\) is a 1-Lipschitz function with \(\text{dev} \circ \gamma_k(0) = x, \text{dev} \circ \gamma_k(l_k) = y\) and \(l_k < \pi/2\). This gives a contradiction.

Therefore, there exists a small \(\epsilon' > 0\) such that \(\lambda_{\epsilon'} = \lambda \cap U_{\epsilon'}\) is a part of the boundary \(\partial B(x)\). If \(D_{\epsilon'}\) is the circular disk about \(y\) in \(U_{\epsilon'}\), of radius \(\epsilon'\), then \(\lambda_{\epsilon'}\) dissects \(D_{\epsilon'}\) into two halves, one of them, say \(D_{\epsilon'}^1\), embeds in \(B(x)\). We attach the boundary \(\lambda_{\epsilon'}\) to \(D_{\epsilon'}^1\). In the next paragraph, we show that there is a smaller number \(\epsilon'' < \epsilon'\) for which the complement of \(D_{\epsilon''}^1\), in \(D_{\epsilon''}\), denoted \(D_{\epsilon''}^2\), has an empty intersection with \(B(x)\). This will prove (1).

Suppose that it is not true. Then for each (small) \(\epsilon'' > 0\), there exists a point \(z \in D_{\epsilon''}^2 \cap B(x)\). So, we have a converging sequence \(z_k \to y\) such that \(z_k \in D_{1/k}^2\) (we may need to start this sequence from some big index \(k \gg 0\) to ensure that \(D_{1/k}^2\) is properly defined). For each \(z_k\) in the sequence, there exists a geodesic arc \(\zeta_k\) from \(x\) to \(z_k\) of length \(\leq \pi/2\). By Arzelá-Ascoli Theorem 3.2, for some \(k \gg 1/\epsilon', \zeta_k\) lies in the tubular \(\epsilon'\)-neighborhood \(U_{\epsilon'}\) of the arc \(\gamma\). But then, \(\zeta_k\) must intersect \(\lambda_{\epsilon'}\) at some point \(y'\), which gives a geodesic arc from \(x\) to \(z\) of length shorter than \(\pi/2\). This is a contradiction because we already have shown that \(d_S(y', x) = \pi/2\).

Below we prove the second part of the lemma. Most of the proof runs parallel to the proof of the part (1).

(2) Suppose that \(\gamma_1, \ldots, \gamma_m\) are all the minimizing geodesic arcs connecting \(x\) and \(y\). Let \(\lambda_1, \ldots, \lambda_m\) be small geodesic arcs passing through \(y\) perpendicularly to \(\gamma_1, \ldots, \gamma_m\). Then, there exists a small disk \(D_{\epsilon}\) of radius \(\epsilon > 0\) about \(y\) such that each \(\lambda_i\) splits the disk into two components. As before, when \(\epsilon\) is small enough, one component of \(D_{\epsilon} - \gamma_i\) is contained in \(B(x)\). We call this component \(D_{\epsilon, i}^1\). Note that interior of the complement of \(D_{\epsilon, i}^1\) in \(D_{\epsilon}\), denoted by \(D_{\epsilon, i}^2\), is not a subset of \(B(x)\), otherwise \(y\) would be an interior point of \(B(x)\).
Clearly, there exists two particular indices, say 1, 2 ∈ {1, ..., m}, such that \( D_{\epsilon, k} \subset D^1_{\epsilon, 1} \cup D^1_{\epsilon, 2} \) for any \( k \in \{1, \ldots, m\} \). The piecewise geodesic curve \( \lambda = (\lambda_1 - D^1_{\epsilon, 2}) \cup (\lambda_2 - D^1_{\epsilon, 1}) \) forms the boundary of \( D^1_{\epsilon, 1} \cup D^1_{\epsilon, 2} \) in \( D_\epsilon \). Moreover, \( \lambda \) has only one non-smooth point at \( y \).

We prove that when \( \epsilon \) is sufficiently small, \( \lambda \subset \partial B(x) \). Suppose that this is not true. Then, as in the part (1), we can find a sequence \( y_p \to y \) on \( \lambda \) and geodesic arcs \( \delta_p \), for each \( p \), of length < \( \pi/2 \) connecting \( x \) and \( y \). Without loss of generality, we may assume that \( \{y_p\} \subset (\lambda_i - D^1_{\epsilon, j}) \). By a similar application of the Arzelà-Ascoli Theorem 3.2 as in the proof of part (1), given \( \epsilon' > 0 \), we can find \( p \in \mathbb{N} \) and \( k \in \{1, \ldots, m\} \) such that \( \delta_p \subset U_{\epsilon', k} \), where \( U_{\epsilon', k} \) denotes the tubular \( \epsilon' \)-neighborhood of \( \gamma_k \). Now \( \delta_p \) has a segment connecting \( x \) to a point on \( \lambda_k \) of length < \( \pi/2 \). This gives a contradiction, as in part (1).

We also need to prove that when \( \epsilon \) is sufficiently small, the interior of the complement of \( D^1_{\epsilon} = D^1_{\epsilon, 1} \cup D^1_{\epsilon, 2} \) in \( D_\epsilon \), denoted by \( D^2_{\epsilon} \) (the unshaded part in the Figure 3), has the empty intersection with \( B(x) \). This can be proven in a very similar way as in the last part of the proof of part (1). Therefore, we skip the details.

The statement about the angle between \( \lambda_1 \) and \( \lambda_2 \) is clear from the Figure 3. The only thing to note here is that the velocity vectors \( \gamma'_1(\pi/2) \neq \pm \gamma'_2(\pi/2) \).

From Lemma 3.4, \( \partial B(x) \) is a union of pairwise disjoint piecewise smooth closed curves in \( S - \{x_1, \ldots, x_n\} \). From part (1) of the same lemma, \( \partial B(x) \) is locally geodesic in \( S \) at a non-singular point of \( \partial B(x) \), and from part (2), the non-singular points on \( \partial B(x) \) are isolated. Hence, \( \partial B(x) \) is disjoint union of geodesic polygons. Also, from the second part of the lemma, it follows that the internal angles of these polygons in \( B(x) \) are > \( \pi \). This completes the proof of Proposition 3.1.

Using Proposition 3.1, we can decompose \( S \) into ‘simple’ (closed) geometrical pieces, each of which are genus zero compact surface with boundary. By ‘simple’ we mean one of the following two geometrical shapes.

(1) If a geometrical piece contains a conical singularity \( x \), then it’s of the form \( B(x) \).

(2) If a geometrical piece does not contain a conical singularity, then it’s boundary components are spherical polygons with internal angles < \( \pi \).

The pieces in (2) are the connected components of \( S - \bigcup_{i=1}^n B(x_i) \). It is possible that some pieces \( Z \) of type (2) are topological disks. Then it is easy to see that the boundary \( \partial Z \) is a component of some \( \partial B(x_i) \). For otherwise, there will be a point \( w \in \partial Z \) such that \( w \in \partial B(x_i) \cap \partial B(x_j) \). But then, the angles \( \alpha, \alpha_i, \alpha_j \), which are the internal angles at \( w \) of \( \partial Z, \partial B(x_i) \) and \( \partial B(x_j) \) respectively, satisfy \( \alpha \leq 2\pi - \alpha_i - \alpha_j \leq 0 \), from Proposition 3.1. Therefore, we can attach the pieces \( Z \) to \( B(x_i) \) along
\( \partial Z \subset \partial B(x_i) \). The metric ball \( B(x_i) \) with all possible topological disks \( Z \)'s attached will be denoted by \( B'(x_i) \). Note that the boundary \( \partial B'(x_i) \) of \( B'(x_i) \) is a non-empty subset of \( \partial B(x_i) \).

Using \( B'(x_i) \), we have an ‘improved’ decomposition of \( S \). The surface \( S \) is now composed of the following geometrical pieces.

(1') If a geometrical piece contains a conical singularity \( x \), then it is of the form \( B'(x) \).

(2') If a geometrical piece does not contain a conical singularity, then it is non-simply connected genus 0 compact surface with boundary such that the boundary components are spherical polygons with internal angles < \( \pi \).

In the following, we prove that the pieces of type (2') do not occur.

**Proposition 3.5.** A surface of type (2') does not exist.

**Proof.** Suppose that \( \Sigma \) is a surface of type (2'). We obtain a closed surface with conical singularities by “doubling”, i.e. taking two copies of \( \Sigma \) and identifying the identical edges. The resulting surface \( D \) has genus \( g \) at least 1. Moreover, all the angles \( \alpha_1, \ldots, \alpha_m \) \((m \text{ can be zero})\) at the conical singularities are < \( 2\pi \). Then, using Gauss-Bonnet formula (see Proposition 2.1 of [7]), we get

\[
\sum_{i=1}^{m} (\alpha_i - 2\pi) + 2\pi \chi(D) = \int_D dA,
\]

where \( \chi(D) = 2 - 2g \) is the Euler's characteristic of \( D \) and \( dA \) is the area form of the metric. The right side of the equation is the area of \( D \) which must be positive, while the left side is non-positive. This gives a contradiction. \( \square \)

**Corollary 3.6.** \( S = \bigcup_{i=1}^{n} B'(x_i) \).

Now we prove that \( B'(x_i) \) are disks for all \( i \). First, we claim that \( B'(x_i) \) has only smooth boundary components.

**Lemma 3.7.** Each component of \( \partial B'(x_i) \) is a geodesic cycle.

**Proof.** Let \( P \subset \partial B'(x_i) \) be a component and \( y \) be a point on \( P \). Using Proposition 3.5, there exists \( j \neq i \) such that \( y \in \partial B'(x_j) \). Denote the component of \( \partial B'(x_j) \) containing \( y \) by \( P' \). Since the component \( P \) remains unchanged when we formed \( B'(x_i) \) from \( B(x_i) \) by attaching disks, \( P \subset \partial B(x_i) \) and the internal angle \( \alpha_i \) of \( P \) at \( y \) on \( B'(x_i) \) is still \( \geq \pi \), by Proposition 3.1. Similarly, the internal angle \( \alpha_j \) of \( P' \) at \( y \) on \( B'(x_j) \) is \( \geq \pi \). As \( y \) is a non-singular point of \( S \), the total angle at \( y \) is \( 2\pi \). So, we have

\[ \alpha_i \geq \pi, \quad \alpha_j \geq \pi \quad \text{and} \quad \alpha_i + \alpha_j \leq 2\pi. \]

This implies \( \alpha_i = \alpha_j = \pi \). Hence, \( y \) is a non-singular point in the smoothness structure of \( P \). \( \square \)

**Proposition 3.8.** \( B'(x_i) \) has only one boundary component. Since \( B'(x_i) \) is a compact surface of genus 0, this is equivalent to saying that \( B'(x_i) \) is topologically a disk. Moreover, \( B(x_i) = B'(x_i) \).

**Proof.** Let \( P \) be a boundary component of \( B'(x_i) \). By Lemma 3.7, \( P \) is a geodesic cycle. This together with Lemma 3.4 implies that any point \( y \) on \( P \) can be connected to \( x_i \) by a unique minimizing geodesic on \( B(x_i) \). Also, this geodesic varies continuously with \( y \). By compactness of \( P \), we can infer that there is a continuous map from the closed 2-dimensional disk \( D^2 \to B(x_i) \) which maps the boundary \( \partial D^2 \) homeomorphically onto \( P \).

In particular, the boundary component \( P \) is null-homologous in \( B(x_i) \), which happens only when \( B(x_i) \) is topologically a disk. \( \phi \neq \partial B'(x_i) \subset \partial B(x_i) \) implies \( B'(x_i) = B(x_i) \). \( \square \)

**Remark.** Another direct argument can be given by gluing two copies along the boundary, as in the proof of Proposition 3.5, to prove that \( B'(x_i) \) is a topological disk. Suppose that \( B'(x_i) \) had more
than one boundary component. Then we could glue two copies of $B(x_i)$ along their boundaries and we would get a surface of genus $\geq 1$ of constant curvature 1, which is impossible. This would have been enough to prove Theorem 2.1, but the following decomposition result is nicer to have.

**Corollary 3.9** (Decomposition of $S$). The surface $S$ can be decomposed into $n$ embedded disks $B(x_i)$’s. Moreover, the boundary of such a disk is a geodesic loop in $S - \{x_1, \ldots, x_n\}$.

**Proof.** Combining this proposition with Corollary 3.6, we get

$$S = \bigcup_{i=1}^{n} B(x_i).$$

\[\square\]

4. **Proof of Theorem 2.1**

Theorem 2.1 follows from Theorems 2.4 and 2.2 and the coaxiality criteria in Theorem 1.5.

**Proof of Theorem 2.2.** According to Corollary 3.9, we can decompose $S$ into disks $B(x_1), \ldots, B(x_n)$. Let $y \in \partial B(x_1)$. We can find an index $i \neq 1$, say $i = 2$, such that $y \in B(x_2)$. We will show that $\partial B(x_1) = \partial B(x_2)$.

The loops $\partial B(x_1)$ and $\partial B(x_2)$ are parametrized geodesic cycles $\lambda_1, \lambda_2 : [0, 1] \to S$ respectively, based at $y$, i.e. $y = \lambda_k(0) = \lambda_k(1)$, for $k = 1, 2$. We argue that the velocity vectors $\lambda'_1(0)$ and $\lambda'_2(0)$ are equal (up to a sign). This will prove that the images of $\lambda_1$ and $\lambda_2$ are equal. The loop $\lambda_1$ splits $S$ into two connected regions, one of them being the interior of $B(x_1)$. If $\lambda'_1(0) \neq \pm \lambda'_2(0)$, then $\lambda_1$ and $\lambda_2$ would transversally intersect each other at $y$. This means that $\lambda'_1$ intersects the interior of $B(x_1)$, say at $z \in B(x_1)$. But then

$$d_S(x_1, x_2) \leq d_S(z, x_1) + d_S(z, x_2) < \pi,$$

which violates the assumption $d_S(x_1, x_2) \geq \pi$. Therefore, $\lambda'_1(0) = \pm \lambda'_2(0)$.

Since $\partial B(x_1) = \partial B(x_2)$, we have an embedded sphere $B(x_1) \cup B(x_2)$ in $S$. Consequently, $B(x_1) \cup B(x_2) = S$ i.e. $n = 2$. \[\square\]

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