THE CLASSIFICATION OF PURELY NON-SYMPLECTIC AUTOMORPHISMS OF HIGH ORDER ON K3 SURFACES

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Abstract. An automorphism of order n of a K3 surface is called purely non-symplectic if it multiplies the holomorphic symplectic form by a primitive n-th root of unity. We give the classification of purely non-symplectic automorphisms with \( \varphi(n) \geq 12 \) where \( \varphi \) denotes the Euler totient function.

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1. INTRODUCTION

A complex K3 surface is a smooth, compact, complex surface with vanishing irregularity and trivial canonical bundle. It is not necessarily algebraic.

An automorphism \( f \) of a K3 surface \( X \) is called symplectic if it acts trivially on the global holomorphic 2-forms, \( f^*H^0(X, \Omega^2_X) = \text{id} \), and non-symplectic otherwise. Furthermore we call \( f \) purely non-symplectic if all non-trivial powers are non-symplectic. Note that K3 surfaces admitting a non-symplectic automorphism of finite order are always algebraic [33, 3.1]. Finite symplectic group actions on K3 surfaces are fully classified in [18] (which builds on [33, 31, 21, 49]). However, for non-symplectic automorphisms beyond the case of prime order [4] most results are partial and there is only a classification of automorphisms of fixed order with extra conditions on their fixed locus or their action on divisors.
Let $X_i$ be K3 surfaces and $G_i \subseteq \text{Aut}(X_i)$ groups of automorphisms for $i \in \{1, 2\}$. We say that the pair $(X_1, G_1)$ is equivalent to $(X_2, G_2)$ if there is an isomorphism $F : X_1 \to X_2$ such that $FG_1 F^{-1} := \{F \circ g \circ F^{-1}\} = G_2$. The main result of this note is the following classification.

**Theorem 1.1.** Let $X$ be a K3 surface and $\mathbb{Z}/n\mathbb{Z} \cong G \subseteq \text{Aut}(X)$ a purely non-symplectic subgroup with $\varphi(n) \geq 12$, where $\varphi$ is the Euler totient function. All pairs $(X, G)$ up to isomorphism are listed in Table 3 where $G$ is generated by $f$.

Given two explicit K3 surfaces it can be hard to decide whether they are isomorphic or not, as it may be difficult to write down an isomorphism. A notable exception consists in singular K3 surfaces, i.e. with Picard number 20, which correspond up isomorphism to their oriented transcendental lattices as can be shown with the strong Torelli theorem for K3 surfaces.

The second main result can be seen as an analogue of this for K3 surfaces with a non-symplectic automorphism. It allows to decide effectively if two K3 surfaces (satisfying the hypothesis of the theorem) are abstractly isomorphic without having to give the isomorphism. To formulate it we need a little more terminology. Recall that the intersection pairing turns the Neron-Severi group $\text{NS}(X)$ into a lattice. Its dual lattice is denoted by $\text{NS}(X)^\vee$ and the finite quotient group $\text{NS}(X)^\vee/\text{NS}(X)$ is the discriminant group. The transcendental lattice of $X$ is denoted by $T(X)$.

**Theorem 1.2.** Let $X_i, i = 1, 2$ be complex K3 surfaces and $f_i \in \text{Aut}(X_i)$ automorphisms with $f_i^*(\omega_i) = \zeta_n \omega_i$ where $\zeta_n = \exp(2\pi i/n)$, $n \in \mathbb{N}$ and $\omega_i = H^0(X_i, \Omega^2_{X_i})$ such that $\text{rk} T(X_i) = \varphi(n)$. Let $I_i$ be the kernel of the natural map given by

$$\mathbb{Z}[\zeta_n] \to O(\text{NS}(X_i)^\vee/\text{NS}(X_i)), \quad \zeta_n \mapsto f_i^*.$$

If $f_i$ is of finite order, then $X_1 \cong X_2$ if and only if $I_1 = I_2$.

As an application we obtain that the three K3 surfaces with a purely non-symplectic automorphism of order 21 (see Table 3) are in fact abstractly isomorphic even though the three automorphisms (and the equations describing the K3 surfaces) are quite different.

2. Complex K3 surfaces

In this section we recall some basic facts about complex K3 surfaces and non-symplectic automorphisms. The main reference is [7, VIII §11].

Let $X$ be a complex K3 surface. Its second singular cohomology $H^2(X, \mathbb{Z})$ equipped with the cup product is an even unimodular lattice of signature $(3, 19)$. Such a lattice is unique up to isometry. We call it the K3 lattice and denote it by $L_{K3}$. Definitions and properties of lattices are given in the next section. By the Hodge decomposition

$$H^2(X, \mathbb{Z}) \otimes \mathbb{C} \cong H^2(X, \mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

where $H^{i,j}(X) \cong H^j(X, \Omega^i_{X})$, $H^{i,j}(X) = \overline{H^{j,i}(X)}$ and $H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^\perp$ has signature $(1, 19)$.

Conversely, the Hodge structure of a K3 surface determines it up to isomorphism as is reflected by the Torelli theorems.
Theorem 2.1. [7 VIII 11.1] Let $X, Y$ be complex K3 surfaces and

$$f : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$$

an isometry of lattices whose $\mathbb{C}$-linear extension maps $H^{2,0}(X)$ to $H^{2,0}(Y)$. Then $X \cong Y$. If moreover $f$ maps ample classes on $X$ to ample classes on $Y$, then $f = F^*$ for a unique isomorphism $F : Y \to X$.

By Lefschetz’ Theorem on $(1,1)$-classes we can recover the Néron-Severi group from the Hodge structure as

$$\text{NS}(X) = H^{2,0}(X)^\perp \cap H^2(X, \mathbb{Z}).$$

Its rank $\rho$ is called the Picard number of $X$. The transcendental lattice is defined as the smallest primitive sublattice $T(X) \subseteq H^2(X, \mathbb{Z})$ whose complexification contains $H^{2,0}(X) \subseteq T(X) \otimes \mathbb{C}$. The surface $X$ is projective if and only if $\text{NS}(X)$ has signature $(1, \rho - 1)$. In this case $T(X) = \text{NS}(X)^\perp$. If we consider just a single K3 surface $X$, we will usually omit the $(X)$ from notation and just write $H^{1,1}$, $\text{NS}$, $T$. From now on all K3 surfaces are assumed to be projective.

If $\delta \in \text{NS}$ has self intersection $-2$, then $\delta$ or $-\delta$ is effective. The set of roots is

$$\Delta_{\text{NS}} = \{ \delta \in \text{NS} | \delta^2 = -2 \}.$$

The positive cone $P$ is defined as connected component of the light cone $\{ x \in L \otimes \mathbb{R} | x^2 > 0 \}$ that contains the ample cone. The connected components of the set

$$P \setminus \bigcup_{\delta \in \Delta} \delta^\perp$$

are called the chambers of $L$. One of the chambers is the ample cone. Each root $\delta$ defines a reflection $r_\delta \in O(L)$ by $r_\delta(x) = x + (x, \delta)\delta$ along the hyperplane $\delta^\perp$. The Weyl Group $W(\text{NS})$ is the subgroup of $O(\text{NS})$ generated by reflections. It acts transitively on the set of chambers. Denote by $O^+(\text{NS})$ the subgroup of $O(\text{NS})$ preserving the positive cone and by $\Gamma(\text{NS})$ the subgroup preserving the ample cone. Then $\Gamma(\text{NS})$ is isomorphic to $O^+(L)/W(\text{NS})$ and the strong Torelli theorem provides that this group is up to finite index the automorphism group of $X$. We define the same notions in an analogous way for any hyperbolic lattice $L$ in place of $\text{NS}$ - after the choice of a chamber in place of the ample cone.

3. Lattices

In this section we recall basic facts about lattices and introduce the necessary notation. Our main references are [34, 13].

A lattice is a finitely generated free abelian group $L \cong \mathbb{Z}^n$ together with a non-degenerate bilinear pairing

$$L \times L \to \mathbb{Z}, \quad (x, y) \mapsto x.y$$

It is called even if $x^2 := x.x \in 2\mathbb{Z}$ for all $x \in L$. The pairing induces an isomorphism

$$\text{Hom}_\mathbb{Z}(L, \mathbb{Z}) \cong L^\vee := \{ x \in L \otimes \mathbb{Q} \mid x.L \subseteq \mathbb{Z} \}$$

with the dual lattice $L^\vee$. The quotient $L^\vee/L =: D_L$ is finite, abelian and called discriminant group. Its order is equal to $| \det L |$. If $D$ is a finite abelian group, the minimum number of generators of $D$ is called the length $l(D)$ of $D$. Note that $l(D_L) \leq \text{rk} L$. If $D_L = 0$, or equivalently $| \det L | = 1$, we call the lattice $L$
unimodular, and if \( nD_L = 0 \) for \( n \in \mathbb{N} \), we call \( L \) \( n \)-elementary. The discriminant group is equipped with a fractional form
\[
b : D_L \times D_L \to \mathbb{Q}/\mathbb{Z}, \quad (\mathbf{x}, \mathbf{y}) \mapsto x \cdot y + \mathbb{Z}.
\]
On an even lattice there is the discriminant form \( q \) given by
\[
q : D_L \to \mathbb{Q}/2\mathbb{Z}, \quad \mathbf{x} \mapsto x^2 + 2\mathbb{Z}.
\]
The discriminant group decomposes as an orthogonal direct sum of \( p \)-groups
\[
D_L \cong \bigoplus_p (D_L)_p.
\]
Note that, by polarization, \( b \mid (D_L)_p^2 \) and \( q \mid (D_L)_p \) carry the same information for odd primes \( p \neq 2 \). If \( (D_L)_p \) is an \( \mathbb{F}_p \)-vector space, then \( q \mid (D_L)_p \) takes values in \( \frac{\mathbb{Z}}{p\mathbb{Z}} \).

We say two lattices \( M, N \) are in the same genus if \( N \otimes \mathbb{Z}_p \cong M \otimes \mathbb{Z}_p \) are isometric over the \( p \)-adic integers for all primes \( p \) and \( N \otimes \mathbb{R} \cong M \otimes \mathbb{R} \) over the real numbers. We use the Conway Sloane [13, Chapter 15] genus symbols to describe a genus.

**Theorem 3.1.** [34, 1.9.4] The signature \((n_+, n_-)\) and discriminant form \( q \) determine the genus of an even lattice and vice versa.

**Notation.** We denote by \( A_n, D_n, E_i, i = 6, 7, 8 \) the negative definite root lattices with the respective Dynkin diagram. The unique unimodular lattice of signature \((1, 1)\) is called a hyperbolic plane and denoted by \( U \). The orthogonal direct sum of two lattices \( L_1, L_2 \) is denoted by \( L_1 \oplus L_2 \) and the lattice \( L \) with bilinear form rescaled by \( a \in \mathbb{Z} \) by \( L(a) \). The group of isometries of a lattice \( L \) is denoted by \( O(L) \) and the group of isometries of the discriminant form \( q_L \) of \( L \) by \( O(q_L) \). If \( N \subseteq L \) is a subset, then \( N^\perp \) is its orthogonal complement in \( L \). If \( f \) is an endomorphism of a free abelian group or vector space \( L \) we denote by \( \chi_f \) its characteristic polynomial, by \( \mu_f \) its minimal polynomial and for a subset \( N \subseteq L \) by \( fN := f(N) \) its image under \( f \).

3.1. **Primitive embeddings.** In this section we review the theory of primitive embeddings of lattices in the language of discriminant forms as developed in [34].

**Definition 3.2.** An embedding of lattices \( i : M \hookrightarrow L \) is called primitive if the cokernel is free.

**Example 3.3.** By Lefschetz theorem on (1, 1)-classes \( \text{NS}(X) \hookrightarrow H^2(X, \mathbb{Z}) \) is a primitive embedding.

We call two primitive embeddings \( i, j : M \hookrightarrow L \) isomorphic if there is a commutative diagram with \( f \in O(L) \).

\[
\begin{array}{ccc}
M & \xrightarrow{i} & L \\
\downarrow{j} & & \downarrow{f} \\
& L & 
\end{array}
\]

We shall say that \( S \) embeds uniquely into \( L \) if all primitive embeddings are isomorphic. A (weakened) criterion for this to happen is given in the next theorem.
Theorem 3.4. [34, Theorem 1.14.4][30, 2.8] Let $M$ be an even lattice of signature $(m_+, m_-)$ and $L$ an even unimodular lattice of signature $(l_+, l_-)$. If $l(D_M) + 2 \leq \operatorname{rk} L - \operatorname{rk} M$ and $l_+ > m_+, l_- > m_-$, then there is a unique primitive embedding of $M$ into $L$.

We mention the related

Theorem 3.5. [34, Theorem 1.14.2] Let $M$ be an even, indefinite lattice such that $\operatorname{rk} M \geq 2 + l(D_M)$, then the genus of $M$ contains only one class, and the homomorphism $O(M) \to O(q_M)$ is surjective.

A different perspective on primitive embeddings is given by the concept of primitive extensions. Let $i : M \hookrightarrow L$ be a primitive embedding. Then $N := M^\perp$ is also a primitive sublattice and we call $M \oplus N \hookrightarrow L$ a primitive extension. A glue map is a map $\phi$ defined on certain subgroups

$$D_M \supseteq G_M \supseteq G_N \subseteq D_N,$$ 

with the extra condition that $q_M = -q_N \circ \phi$.

Theorem 3.6. [34, Prop 1.15.1] There is a one to one correspondence

$$\left\{ \text{Primitive extensions } M \oplus N \hookrightarrow L \right\} \longleftrightarrow \left\{ \text{Glue maps } D_M \supseteq G_M \supseteq G_N \subseteq D_N \right\}.$$ 

This correspondence arises as follows. Given a glue map $\phi$, we define the glue

$$G_\phi := \{ x + \phi(x) \mid x \in G_M \} \subseteq D_M \oplus D_N$$

as the graph of $\phi$. By construction, $G_\phi$ is a totally isotropic subspace of $D_M \oplus D_N$. Hence, we can define a lattice $L = M \oplus N$ via

$$L/(M \oplus N) = G_\phi.$$ 

The reader may check that $M \oplus N \hookrightarrow L$ is indeed a primitive extension.

Conversely, given a primitive extension as above, we interpret the totally isotropic subspace $L/(M \oplus N)$ as the defining graph $G_\phi$ of a glue map $\phi$. Explicitly this is given by the isomorphisms

$$G_M \cong L/(M \oplus N) = G_\phi \cong G_N.$$ 

The glue map is defined on the spaces $G_M := p_M(L)/M$ and $G_N := p_N(L)/N$ where

$$p_M : M^\vee \oplus N^\vee \to M^\vee$$

and $p_N : M^\vee \oplus N^\vee \to N^\vee$

are the orthogonal projections.

With the glue map $\phi$ understood, we will often drop it from notation and simply denote by $G = L/(M \oplus N)$ the glue of a primitive extension. In this situation we say that $M$ and $N$ are glued along $|G|$.

There is the following constraint on the size of the glue:

Lemma 3.7.

$$|D_N/G_N| \cdot |D_M/G_M| = \det L.$$
Proof. We divide the standard formula

$$\det M \det N = [L : M \oplus N]^2 \det L$$

by $[L : M \oplus N]^2$. The isomorphisms $[1]$ provide $[L : M \oplus N] = |G_N|$. The lemma follows with $|\det M| = D_M$ and the same reasoning for $N$. □

**Lemma 3.8.** Let $N \hookrightarrow L$ be a primitive embedding. Then there is a surjection $D_L \twoheadrightarrow D_N/G_N$.

**Proof.** We have the following induced diagram with exact rows

$$
\begin{array}{ccccccc}
0 & \rightarrow & L & \rightarrow & L\vee & \rightarrow & L\vee/L = D_L & \rightarrow & 0 \\
| & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & p_N(L) & \rightarrow & N\vee & \rightarrow & N\vee/p_N(L) = D_N/G_N & \rightarrow & 0
\end{array}
$$

where the primitivity of $N \hookrightarrow L$ gives the surjectivity of the central vertical arrow. The commutativity of the diagram then implies the desired surjection. □

**3.2. Extending isometries.** In this section we investigate the interplay between primitive extensions and isometries.

**Example 3.9.** If $X$ is a complex K3 surface, then

$$\text{NS} \oplus T \hookrightarrow H^2(X, \mathbb{Z})$$

is a primitive extension. Let $f$ be an automorphism of $X$. It acts (by pullback) on $\text{NS}$, $T$, $H^2(X, \mathbb{Z})$, $D_{\text{NS}} = G_{\text{NS}} \cong G \cong G_T = D_T$ in a compatible way. If confusion is unlikely, we will denote all these actions by $f$ or for example by $f|D_T$.

Clearly, an isometry $f = f_M \oplus f_N$ defined on $M \oplus N$ extends to a primitive extension $L$ if and only if $f(L/(M \oplus N)) = L/(M \oplus N)$, i.e. $f(G_f) = G_f$. In other words $f_M|D_M$ preserves $G_M$, $f_N|D_N$ preserves $G_N$ and $\phi \circ f_M = f_N \circ \phi$.

$$
\left\{ \text{Primitive extensions } M \oplus N \hookrightarrow L \text{ such that } f_M \oplus f_N \text{ extends to } L \right\} \xrightarrow{1:1} \left\{ \text{Glue maps } \phi : G_M \xrightarrow{\sim} G_N \text{ satisfying } \phi \circ f_M = f_N \circ \phi \right\}
$$

The following theorem uses the equivariance of the glue map to impose compatibility conditions on the minimal polynomials of the two actions.

**Theorem 3.10.** Let $M \oplus N \hookrightarrow L$ be a primitive extension and $f_M, f_N$ be endomorphisms of $M$ and $N$ with minimal polynomials $m(x)$ and $n(x)$. Suppose that $f_M \oplus f_N$ extends to $f : L \rightarrow L$. Then

$$dL \subseteq M \oplus N$$

where $d\mathbb{Z} = (m(x)\mathbb{Z}[x] + n(x)\mathbb{Z}[x]) \cap \mathbb{Z}$.

**Proof.** If $m(x)$ and $n(x)$ are not coprime in $\mathbb{Q}[x]$, then $d = 0$, and the theorem holds. Assume that they are coprime in $\mathbb{Q}[x]$. By the definition of $d$ we can find polynomials $u, v \in \mathbb{Z}[x]$ such that

$$d = u(x)n(x) + v(x)m(x).$$

Then, inserting $f$ for $x$, we obtain $d \cdot \text{id}_L = u(f)n(f) + v(f)m(f)$ as endomorphism of $L$. In the following we estimate the image of this endomorphism. Recall that
\( fL := f(L) \) denotes the image of \( f \). We note that since \( n(x) \) and \( m(x) \) are coprime and \( M \) is primitive in \( L \), \( n(f)L = \ker m(f) = M \) and similarly for \( N \).

\[
\begin{align*}
dL &= (uf(n)f + v(f)m(f)L) \\
    &\subseteq uf(n)fL + v(f)m(f)L \\
    &\subseteq \ker m(f) + \ker n(f) \\
    &= M \oplus N.
\end{align*}
\]

\[ \square \]

**Corollary 3.11.** If \( L \) is \( r \)-elementary, i.e. \( rD_L = 0 \), then \( M \) is \( dr \)-elementary, i.e.,

\[
drD_M = 0.
\]

In particular,

\[
d \mid (dr)^{rk M}.
\]

**Proof.** We take the chain of inclusions

\[
M \oplus N \subseteq L \subseteq L^\vee \subseteq M^\vee \oplus N^\vee
\]

From Theorem 3.10 we get that \( dL \subseteq M \oplus N \), and projecting this orthogonally to \( M^\vee \) gives

\[
dp_M(L) = p_M(dL) \subseteq p_M(M \oplus N) = M.
\]

Together with Lemma 3.8 this implies

\[
drD_M \subseteq dG_M = dp_M(L)/M = 0
\]

as desired. \[ \square \]

**Example 3.12.** As an example consider a non-symplectic automorphism \( f \) of a K3 surface of order 4. Set \( L = H^2(X, \mathbb{Z}) \), \( m(x) = x^2 + 1 \) and \( n(x) = x^2 - 1 \). Since \( L \) is unimodular, \( r = 1 \). One calculates \( d = 2 \), and thus that \( M = \ker f^2 + 1 \) is 2-elementary.

Note that \( d \) divides the resultant \( \text{res}(m(x), n(x)) \) and both have the same prime factors. But usually \( d < \text{res}(m(x), n(x)) \). For example \( \text{res}(x^2 + 1, x^2 - 1) = 4 \). We deduce the following corollary. It was originally stated in [26, Theorem 4.3] for unimodular primitive extensions.

**Corollary 3.13.** Let \( M, N, L \) be lattices and \( M \oplus N \hookrightarrow L \)

a primitive extension with glue \( G_M \cong G \cong G_N \). Let \( f_M, f_N \) be isometries of \( M \) and \( N \) with characteristic polynomials \( \chi_M \) and \( \chi_N \). If \( f_M \oplus f_N \) extends to \( L \), then any prime dividing \( |G| \) also divides the resultant \( \text{res}(\chi_M, \chi_N) \).

Since we are concerned with isometries of finite order, we give formulas for \( d \) and the resultants in the case of cyclotomic polynomials.

**Theorem 3.14.** [11, Thm. 1][16, Thm. 1] The resultant of two cyclotomic polynomials \( c_m, c_n \) with \( m < n \) is given by

\[
\text{res}(c_n, c_m) = \begin{cases} 
p^{\phi(m)} & \text{if } n/m = p^e \text{ is a prime power}, \\
1 & \text{otherwise}. \end{cases}
\]
Theorem 3.15. [16] Thm. 2
\[(\mathbb{Z}[x]c_n + \mathbb{Z}[x]c_m) \cap \mathbb{Z} = \begin{cases} p & \text{if } n/m = p^e \text{ is a prime power}, \\ 1 & \text{otherwise.} \end{cases}\]

3.3. Real orthogonal transformations and the sign invariant. In this section we review how to classify conjugacy classes of real orthogonal transformations. The classification works in terms of the so called sign invariant. Proofs and details can be found in [17 §2].

We denote by \(\mathbb{R}^{p,q}\) the vector space \(\mathbb{R}^{p+q}\) equipped with the quadratic form
\[x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2.\]

Let \(SO_{p,q}(\mathbb{R}) = SO(\mathbb{R}^{p,q})\) be the Lie group of real orthogonal transformations of determinant one, preserving the quadratic form. If the characteristic polynomial \(\chi(x)\) of \(F \in SO_{p,q}(\mathbb{R})\) is of even degree \(2n = p + q\) and separable, then it is reciprocal, i.e., \(x^{2n}\chi(x) = \chi(x^{-1})\). It has a trace polynomial \(r(x)\) defined by
\[\chi(x) = x^nr(x + x^{-1}).\]

Its roots are real of the form \(\lambda + \lambda^{-1}\) where \(\lambda\) is a root of \(\chi(x)\). Call \(T\) the set of roots of \(r(x)\) in the interval \((-2, 2)\). They correspond to conjugate pairs of roots \(\lambda + \overline{\lambda}\) of \(\chi(x)\) on the unit circle. We have an orthogonal direct sum decomposition
\[\mathbb{R}^{p,q} = \bigoplus_{\tau \in \mathbb{R}} E_{\tau}, \quad E_{\tau} := \ker(F + F^{-1} - \tau I).\]

On \(E_{\tau}, \tau \in T, F\) acts by rotation by angle \(\theta = \arccos(\tau/2)\). Hence \(E_{\tau}\) is either positive or negative definite. For \(\tau \in T\) this is encoded in the sign invariant.

\[\epsilon_F(\tau) = \begin{cases} +1 & \text{if } E_{\tau} \text{ has signature } (2, 0), \\ -1 & \text{if } E_{\tau} \text{ has signature } (0, 2). \end{cases}\]

Denote by \(2t\) the number of roots of \(\chi(x)\) outside the unit circle. We can recover the signature via
\[(p, q) = (t, t) + \sum_{\tau \in T} \begin{cases} (2, 0) & \text{if } \epsilon_F(\tau) = +1 \\ (0, 2) & \text{if } \epsilon_F(\tau) = -1 \end{cases} \]

Two isometries \(F, G \in SO_{p,q}(\mathbb{R})\) with the same characteristic polynomial are conjugate in \(O_{p,q}(\mathbb{R})\) iff \(\epsilon_F = \epsilon_G\).

3.4. Lattices in number fields. In this section we review the theory of lattice isometries associated to certain reciprocal polynomials as exploited in [20 §5]. For further reading consider [8][9][10].

Definition 3.16. A pair \((L, f)\) where \(L\) is a lattice and \(f \in O(L)\) an isometry with characteristic polynomial \(p(x)\), is called a \(p(x)\)-lattice.

We call two \(p(x)\)-lattices \((L, f)\) and \((N, g)\) isomorphic if there is an isometry \(\alpha : L \to N\) with \(\alpha \circ f = g \circ \alpha\).

Example 3.17. Let \(X\) be a complex K3 surface and \(f\) an automorphism of \(X\) acting by multiplication with an \(n\)-th root of unity on \(H^0(X, \Omega_X^2)\). If \(\text{rk} T(X) = \varphi(n)\), then
Theorem 3.20. \cite[5.2]{26} Let $p(x)$ be a simple reciprocal polynomial, then every $p(x)$-lattice is isomorphic to a twist $(L_0(a), f_0)$ of the principal $p(x)$-lattice.

Remark 3.21. If we drop the condition that $|p(1)p(-1)|$ is square-free, we have to allow twists in $r'(x+x^{-1})/D_K \cap \mathcal{O}_k = 1/(x-x^{-1})\mathcal{O}_k$, where $D_K = (p'(x))\mathcal{O}_K$ is the different of $K$. If $K/k$ ramifies over 2, these need not be even in general. Dropping the condition on the class number leads to so called ideal lattices surveyed in \cite{10}.

If $\mathbb{Z}[f] \cong \mathbb{Z}[x]/p(x)$ is the full ring of integers $\mathcal{O}_K$ of $K$ and $|p(1)p(-1)|$ is square-free. In this case $p(x)$ is called a simple reciprocal polynomial and we get the following theorem.

Lemma 3.22. Let $p(x)$ be a simple reciprocal polynomial. Then there is an element $b \in \mathcal{O}_K$ of absolute norm $|p(1)p(-1)|$ such that $L_0'(a) = \frac{1}{b} \mathcal{O}_K$. If $a \in \mathcal{O}_k$ is a twist, then $L_0(a)^\circ/L_0(a) \cong \mathcal{O}_K/ab\mathcal{O}_K$ as $\mathcal{O}_K$-modules.
Proof. Since $L_0' \subseteq K$ is a finitely generated $O_K$-module, it is a fractional ideal. By simplicity of $p(x) O_K$ is a PID and fractional ideals are of the form $\frac{1}{b} O_K$, for some $b \in O_K$. Then $L_0(a)^{\vee} = \frac{1}{a} L_0^{\vee} = \frac{1}{ab} L_0$ and $D_{L_0(a)} \cong O_K/abO_K$. □

Given a unit $u \in O_K^\times$ and $a \in O_K \setminus \{0\}$ the twist $L_0(ua^x a)$ is isomorphic to $L_0(a)$ via $x \mapsto ux$ as $p(x)$-lattice. Conversely, if $v \in O_k$ and $L_0(va) \cong L_0(a)$ as $p(x)$-lattices, then, by non-degeneracy of the trace map, we can find $u \in O_k$ with $v = uu^\sigma$. Since the cokernel of the norm map $N : O_K^\times \to O_k^\times$ is finite, the associates of $a \in O_k$ give only finitely many non-isomorphic twists.

By Lemma 3.22 the prime decomposition of $a \in O_k$ in $O_K$ determines the $O_K$-module structure of the discriminant, while twisting by a unit may change the signature (and discriminant form).

Let $T$ denote the set of real roots $\tau$ of $r(y)$ in the interval $(-2, 2)$. Each root $\tau \in T$ corresponds to a real place of $k = \mathbb{Q}[y]/r(y)$ that becomes complex in $K$. It is given by the embedding $\nu_\tau : k \to \mathbb{R}$ defined by $y \mapsto \tau$. The sign map is the homomorphism

$$\text{sign} : O_k^\times \to \{\pm 1\}^T$$

defined by $\text{sign}(u\tau) = \text{sign}(\nu_\tau(u))$. Using the sign map, we can compute the sign invariant $\epsilon_f$ of the isometry $f_0$ of a twist $(L_0(a), f_0)$, $a \in O_k$ of the principal lattice

$$\epsilon_f(\tau) = \text{sign}(a/r'(y))$$

(cf. [17, 4.2]).

All in all we have reviewed the classification of $p(x)$-lattices for $p(x)$ a simple reciprocal polynomial. In a concrete situation the $p(x)$-lattices (of given determinant) can be enumerated by a simple computer program.

4. Small cyclotomic fields

Motivated by the action of a non-symplectic automorphism on the transcendental lattice of a K3 surface, we study $c_n(x)$-lattices more closely. In order to do this we review some of the general theory on cyclotomic fields. Our main reference is [23].

For $n \in \mathbb{N}$, we denote by $K = \mathbb{Q}(\zeta_n)$ the $n$-th cyclotomic field and by $c_n(x)$ the $n$-th cyclotomic polynomial. The Euler totient function $\varphi(n)$ records the degree of $c_n(x)$.

The maximal real subfield of $K$ is $k = \mathbb{Q}[\zeta_n + \overline{\zeta_n}]$. The rings of integers of these two fields are

$$O_K = \mathbb{Z}[\zeta_n] \quad \text{and} \quad O_k = \mathbb{Z}[\zeta_n + \overline{\zeta_n}].$$

As before denote by $T$ the set of real places of $k$ that become complex in $K$. We have that $|T| = \varphi(n)/2$.

Lemma 4.1. The cyclotomic polynomials $c_n(x)$ are simple reciprocal polynomials for $2 \leq \varphi(n) \leq 21$, $n \neq 2^d$.

Proof. The only non-trivial part is that the class numbers are one. This is stated in [24, main thm.]. □
Note that even though \(|c_{2d}(1)c_{2d}(-1)| = 4\) is not square-free, every even \(c_{2d}\)-lattice \((2 \leq d \leq 5)\) is a twist of the principal \(c_{2d}\)-lattice (cf. Remark 3.21).

**Lemma 4.2.** [48 Prop. 2.8] If \(n \in \mathbb{N}\), has two distinct prime factors, then \(1 - \zeta_n\) is a unit in \(O_K\).

The kernel \(O_k^{\times+}\) of the sign map

\[
\text{sign} : O_k^* \to \{\pm 1\}^T
\]

is the set of totally positive units of \(O_k\).

**Proposition 4.3.** [42 A.2] If the relative class number \(h^{-}(K) = h(K)/h(k)\) is odd, then \(O_k^{\times+} = N_L O_{K}^{\times+}\).

**Corollary 4.4.** Let \(n \in \mathbb{N}\) with \(\varphi(n) \leq 20\), and \(K := \mathbb{Q}(\zeta_n)\) be the \(n\)-th cyclotomic field. Then the sign map

\[
\text{sign} : O_k^* / N(O_K) \to \{\pm 1\}^T
\]

is injective.

**Proof.** As \(\mathbb{Q}[\zeta_n]\) is a PID for \(\varphi(n) \leq 20\), the relative class number is one and we may apply Proposition 4.3. \(\square\)

The first cases where the relative class number is even is for \(n = 39, 56, 29\). There \(h^{-}(\mathbb{Q}[\zeta_n]) = 2, 2, 2^3\) (cf. [48 §3]), and the sign map has a kernel of order \(2, 2, 2^3\) as well.

**Proposition 4.5.** The isomorphism class of a \(c_n(x)\)-lattice \((L, f)\), with \(2 \leq \text{rk} L = \varphi(n) \leq 20\) is given by the kernel of

\[
\mathbb{Z}[x]/c_n(x) \to O(L^\vee/L), \quad x \mapsto f|L^\vee/L
\]

and the sign invariant of \(f\).

**Proof.** By Theorem 4.20 \((L, f)\) is isomorphic to a twist \((L_0(a), f_0)\) of the principal \(c_n(x)\)-lattice where \(a \in O_k\). Lemma 4.22 shows that the \(O_K\)-module structure of the discriminant determines the prime decomposition of \(a\). Thus we can write \(a = ub\) for some fixed \(b\) with a unit \(u \in O_k^*\). Now, the isomorphism class of \((L_0(a), f_0)\) depends on \(u\) only modulo the image of the norm map \(N : O_k^* \to O_k^*\). But by Corollary 4.4 this is captured by the sign map \(\text{sign}(u)\) which in turn computes the sign invariant \(\epsilon_{f_0}\) of \(f_0\) by (2). \(\square\)

5. **Transcendental Cycles and Uniqueness of \(X\) up to Isomorphism.**

In this section we prepare the proof of Theorem 1.2.

**Proposition 5.1.** Let \(X_i, i \in \{1, 2\}\) be two complex \(K3\) surfaces and \(f_i \in \text{Aut}(X_i)\) automorphisms with \(f_i^*(\omega_i) = \zeta_n \omega_i\) on \(\mathbb{C}\omega_i = H^0(X_i, \Omega_{X_i}^2)\) where \(\zeta_n = e^{2\pi i/n}\). Suppose that

1. \(\text{rk} T(X_i) = \varphi(n)\) and
2. \(T(X_i) \to L_{K3}\) uniquely.

Then there exists an isomorphism \(X_1 \cong X_2\) if and only if \(I_1 = I_2\) where \(I_i\) is the kernel of

\[
\mathbb{Z}[x]/c_n(x) \to O(T(X_i)^\vee/T(X_i)), \quad x \mapsto f_i|T(X_i)^\vee/T(X_i).
\]
Proof. Let $X_i$, $i \in \{1, 2\}$ be two K3 surfaces and $f_i$ be automorphisms as in the theorem. Let $T_i = T(X_i)$. Set $\tau = \zeta_n + \zeta_n^{-1}$ and $E_{\tau,i} = \ker(f_i|_{T_i} + f_i|_{T_i}^{-1} - \tau id_{T_i})$. Looking at $\omega_i, T_i \in E_{\tau,i} \otimes \mathbb{C}$ with $\omega_i, T_i > 0$, we see that $E_{\tau,i}$ has signature $(2, 0)$. Since the signature of $T_i$ is $(2, \varphi(n) - 2)$, this determines the sign invariants of $(T_1, f_1)$ and $(T_2, f_2)$ which is recorded by the complex $n$-th root of unity $\zeta_n$. By assumption the discriminants of $(T_1, f_1)$ have the same $O_K$-module structure and Proposition 4.5 implies that $(T_1, f_1) \cong (T_2, f_2)$ as $c_n(x)$-lattices.

Hence, we can find an isometry $\psi_T : T_1 \rightarrow T_2$ such that $f_2 \circ \psi_T = \psi_T \circ f_1$. The latter condition assures that $\psi_T$ is compatible with the eigenspaces of $f_1$ and $f_2$. Since $\text{rk} \ T_i = \varphi(n)$, the eigenspaces of $f_i|_{T_i}$ are $H^{2,0}(X_i)$. In particular,

\[
\psi_T(H^{2,0}(X_1)) = H^{2,0}(X_2).
\]

Now, choose markings $\phi_i$ of $X_i$. They provide us with two embeddings $\phi_1$ and $\phi_2 \circ \psi$ of $T_1$ into $L_{K3}$. By assumption (2) any two embeddings are isomorphic. That is, we can find $\psi \in O(L_{K3})$ such that the following diagram commutes.

\[
\begin{array}{ccc}
T_1 & \xrightarrow{f_1} & T_1 \subseteq H^2(X_1, \mathbb{Z}) \\
\downarrow{\psi_T} & & \downarrow{\exists \psi} \\
T_2 & \xrightarrow{f_2} & T_2 \subseteq H^2(X_2, \mathbb{Z})
\end{array}
\]

By construction $\phi_2^{-1} \circ \psi \circ \phi_1$ is a Hodge isometry. By the weak Torelli Theorem $X_1$ and $X_2$ are isomorphic. Conversely, let $f_1, f_2 \in \text{Aut}(X)$ with $f_1\omega = f_2\omega = \zeta_n\omega$. Note that $f_1 \circ f_2^{-1}$ is symplectic. Then $(f_1 \circ f_2^{-1})|T = id_T$, i.e., $f_1|T = f_2|T$ and in particular $f_1 = f_2$. \hfill \Box

Remark 5.2. Replacing $f$ by a power $f^k$ with $k$ coprime to $n$, we can fix the action on the 2-forms. This corresponds to the Galois action $\zeta_n \mapsto \zeta_n^k$ on $\mathbb{Q}(\zeta_n)$. In case the embedding is not unique, one can fix the isometry class of $\text{NS}$. Then isomorphism classes of primitive embeddings with $T^\perp = \text{NS}$ are given by glue maps $\phi : T^{\perp}/\text{NS} \to T^{\perp}/\text{NS}$ with $-q\phi = q\text{NS} \circ \phi$ modulo the action of $O(\text{NS})$ on the left. We can also allow for an action of the centralizer of $f|T$ in $O(T)$ on the right.

Let $n = \prod p_i^{e_i}$ be a natural number given in prime factorization. We call $\text{rad}(n) = \prod p_i$ the radical of $n$.

Proposition 5.3. Let $f$ be a non-symplectic automorphism acting with order $n$ on the global 2-forms of a complex K3 surface $X$ with $\varphi(n) = \text{rk} T(X)$. Then

\[
\det T(X) | \text{res}(c_n, \mu)
\]

where $\mu(x)$ is the minimal polynomial of $f|\text{NS}(X)$. If $\text{res}(c_n, \mu) \neq 0$ and $f$ is purely non-symplectic, then $T(X)$ is $\text{rad}(n)$-elementary, i.e. $\text{rad}(n)D_{T(X)} = 0$.

Proof. If $c_n$ and $\mu$ have a common factor, then $\text{res}(c_n, \mu) = 0$, and the statement is certainly true. We may assume that $\gcd(c_n, \mu) = 1$. Then we know that

\[
T(X) = \ker c_n(f|H^2(X, \mathbb{Z}))
\]

and we can view $(T(X), f)$ as a $c_n(x)$-lattice. Then $D_{T(X)} \cong O_K/I$, $K = \mathbb{Q}(\zeta_n)$, for some ideal $I < O_K$. The isomorphisms $D_{\text{NS}(X)} \cong D_{T(X)} \cong O_K/I$ are compatible with $f$. In particular $\mu(f|\text{NS}(X)) = 0$ implies that $\mu(f|D_{T(X)}) = 0$, i.e., $\mu(\zeta_n) \in I$. By definition of the norm $N$ and resultant

\[
|\det T(X)| = |O_K/I| = N(I) | N(\mu(\zeta_n)) = \text{res}(c_n, \mu).
\]
It remains to prove that $nD_{T,X} = 0$. By Corollary 3.11 it is enough to show that
\[
\text{rad}(n) = \mathbb{Z} \subseteq (\mathbb{Z}[x]c_n + \mathbb{Z}[x]\mu) \cap \mathbb{Z}.
\]
Since $f$ is assumed to be purely non-symplectic, we may replace $\mu$ by $\prod_{d|\mu, d \neq n} d = \frac{\mathbb{Z}}{c_n(x)}$. The equation can now be checked with the help of a computer algebra system for any $n$ with $\varphi(n) \leq 20$. We used SageMath [14] for this.

**Corollary 5.4.** Suppose that $\text{rk} T(X) = \varphi(n)$ and $f$ is purely non-symplectic with $\text{gcd}(\mu_f|\text{NS}(X), c_n) = 1$. Set $\mu = \mu_f|\text{NS}(X)$. Then we have the following restrictions on $T(X)$:

(1) $T(X)$ has signature $(2, \varphi(n) - 2)$;

(2) $2 \leq \varphi(n) \leq 20$;

(3) $(x - 1) | \mu$ and $\mu | \prod_{k<n} c_k$;

(4) $\deg \mu \leq 22 - \varphi(n)$ and $\det T(X) | \text{res}(c_n, \mu)$;

(5) $\exists b \in \mathcal{O}_k$ such that $T(X) \cong L_0(b)$ is a twist of the principal $c_n(x)$-lattice.

The resulting determinants are listed in Table 7.

**Proof.** (1) and (2) are clear. (3) Since $f$ preserves an ample divisor $\mu$ is divided by $(x - 1)$. For the second part note that $f$ is of order $n$ and $\mu$ not divided by $c_n$. (4) This is Proposition 5.3. (5) By assumption $T(X)$ is a $c_n(x)$-lattice of rank $\deg c_n$. For $2 \leq \varphi(n) \leq 21$ all cyclotomic polynomials $c_n(x)$ are simple and Theorem 3.20 provides the claim. It remains to compute the values of Table 7. This is easily done with the help of a computer algebra system. The author used SageMath [14] for this purpose. To illustrate the computation we do it for $n = 28$. By Theorem 3.14 a factor $c_k(x)$ of $\mu(x)$ will contribute to the resultant if and only if $n/k$ is a prime power. Hence, the only possibilities are $c_1, c_7, c_{14}, c_4c_7, c_4c_{14}$ which are of degree $2, 6, 6, 8, 8$ and give resultants $7^2, 2^6, 2^67^2, 2^67^2$. The principal $c_{28}(x)$-lattice is unimodular. Hence $|\det T(X)| = |\det L_0(b)| = |N_\mathcal{O}_K^\mathcal{O}_K(b)|$. We investigate the prime factorization of $b$. The unique prime dividing 2 in $\mathcal{O}_k$ has norm $2^6$ when viewed in $\mathcal{O}_K$. Similarly, the unique prime above 7 is of norm $7^2$. These are the only possible prime factors of $b$ which results in the 4 possible determinants $1, 2^6, 7^2$ or $2^67^2$. We can exclude $7^2$ and $2^67^2$ since a computation shows that there is no twist $b$ of the right signature $(2, 10)$. This leaves us with determinants 1 and $2^6$.

**Lemma 5.5.** Let $2 \leq \varphi(n) \leq 20$. For each prime $p \mid n$, there is a unique prime ideal in $\mathcal{O}_k$ dividing $p$.

**Proof.** Since we need the statement only for finitely many $n$, it can be checked with the help of a computer algebra system.

**Lemma 5.6.** The $c_n(x)$-lattices $(T, f)$ of Table 7 are determined up to isomorphism by their determinants and sign invariant. They admit a unique primitive embedding into $L_{k3}$, except $(n, \det T) = (32, 2^6)$ which does not embed into $L_{k3}$.

**Proof.** The $c_n$-lattices are twists of the principal $c_n$-lattice by elements in $\mathcal{O}_k$. The determinant of $T$ gives the norm of the twist which by Lemma gives the prime factorization of the twist up to a unit. The unit is determined by the sign invariant (compare 1.5) and thus the isomorphism class of $(T, f)$.

Now one can explicitly compute all $c_n(x)$-lattices in the table and check that $\varphi(n) + l(T^\vee/T) \leq 20$ for all pairs $(n, d)$ except $(25, 5)$, $(27, 3^3)$ and $(32, 2^6)$. Then
Table 1. Possible determinants of the transcendental lattice

| n  | ϕ(n) | det T          | n  | ϕ(n) | det T          |
|----|------|----------------|----|------|----------------|
| 3, 6 | 2    | 3              | 20 | 8    | 2^4, 2^45^2   |
| 4   | 2    | 2^2            | 21 | 12   | 1, 7^2        |
| 5, 10 | 4    | 5              | 24 | 8    | 2^2, 2^6, 2^23^4, 2^63^4 |
| 7, 14 | 6    | 7              | 25 | 20   | 5             |
| 8   | 4    | 2^2, 2^4       | 27, 18  | 18 | 3, 3^3         |
| 9, 18 | 6    | 3, 3^3         | 28 | 12   | 1, 2^6        |
| 11 | 10   | 11             | 32 | 16   | 2^2, 2^4, 2^6 |
| 12 | 4    | 1, 2^23^4, 2^4 | 33, 66  | 20 | 1             |
| 13, 26 | 12 | 13             | 36 | 12   | 1, 3^4, 2^63^2 |
| 15, 30 | 8    | 5^2, 3^4      | 40 | 16   | 2^4           |
| 16 | 8    | 2^2, 2^4, 2^6, 2^8 | 44 | 20   | 1             |
| 17, 34 | 16  | 17             | 48 | 16   | 2^2           |
| 19, 38 | 18  | 19             | 60 | 16   | –             |

Theorem 1.2 provides uniqueness (and existence) of a primitive embedding outside those 3 cases.

We have to check in case n = 25 that T embeds uniquely into the K3-lattice. It has rank 20 and determinant 5. Its orthogonal complement NS is an indefinite binary quadratic form of determinant 5. It is unique in its genus and the canonical map \( O(NS) \to O(NS^*/NS) \) is surjective since both groups are generated by \(-id\). By [34, 1.14.1] the embedding of T into \( L_{K3} \) is unique.

For the case (27, 3^3), we need more theory not explained here, see e.g. [29]. By [29] VII 7.6 NS is 3-semiregular and p-regular for \( p \neq 3 \). Now [29] VIII 7.5 provides surjectivity of \( O(NS) \to O(NS^*/NS) \) and uniqueness in its genus (alternatively cf. [27, 28]). Uniqueness of the embedding follows again with [34, 1.14.1].

It remains to check that (32, 2^6) does not embed into the K3-lattice. Suppose that it does. Then its orthogonal complement is isomorphic to \( A_1(-1) \oplus 5A_1 \) which is the only lattice of signature (1, 5) and discriminant group \( \mathbb{F}_2^6 \) (cf. [13] Tbl. 15.5]). Its discriminant form takes half integral values. Up to sign it is isomorphic to the discriminant form of its orthogonal complement \( T \cong U(2) \oplus U(2) \oplus D_4 \oplus E_8 \) which takes integral values, contradicting the existence of a primitive embedding.

**Proposition 5.7.** Let X be a complex K3 surface and \( f \in \text{Aut}(X) \) be an automorphism of finite order with \( f^*\omega = \zeta_n\omega \) on \( \bigwedge^2 H^0(X, \Omega^2_X) \). Suppose that \( \text{rk} T_X = \phi(n) \). Then there is a unique primitive embedding \( T_X \hookrightarrow L_{K3} \).

**Proof.** If \( \phi(n) \leq 10 \), then \( \text{rk} T_X + l(D_{T(X)}) \leq 2 \text{rk} T_X = 2\phi(n) \leq 20 \) and Theorem 3.4 provides uniqueness of the embedding. If \( \phi(n) > 10 \), then \( \phi(n) \geq 12 \) and \( \zeta_n \) is not an eigenvalue of \( f|NS \). By Corollary 5.4 there are only finitely many possibilities.
of $T_X$ up to isometry. Uniqueness of the embedding is checked individually in Lemma 5.5.

**Proof of Theorem 1.2.** The difference between Theorem 1.2 and Proposition 5.4 is that the automorphism $f_1$ is of finite order and we do not require the uniqueness of $T_i \mapsto L_{K3}$. Since $f_1$ is of finite order $T(X_1)$ has a unique embedding into the $K3$ lattice by Proposition 5.7. Since $I_1 = I_2$ and the sign invariants agree, we get $T(X_1) \cong T(X_2)$ and thus uniqueness of the embedding for $T(X_2)$ as well. 

**Lemma 5.8.** The pair $(54, 3^3)$ is not realized by a $K3$ surface.

**Proof.** Suppose that it is realized. A computation of the $c_{54}(x)$-lattice of determinant $3^3$ shows that $D_T \cong \mathbb{F}_3^\perp$ and that the action of $f$ on $D_T$ is of order 6. Hence, $\text{NS} \cong U(3) \oplus \mathbb{A}_2$ and the action on $D_{NS} \cong D_T$ is of order 6 as well. Using Vinberg’s algorithm (see Lemma 6.11) one explicitly computes the exact sequence

$$1 \to W(\text{NS}) \to O^+(\text{NS}) \to O(q_{\text{NS}}) \to \{\pm 1\} \to 1$$

and the image of $O^+(\text{NS})$ in $O(q_{\text{NS}})$. However, an enumeration of its 24 elements shows that it does not contain an element of order six. But $f|\text{NS}$ is an element $O^+(\text{NS})$, giving the contradiction.

The following is a generalization of Vorontsov’s theorem [17].

**Theorem 5.9.** Let $X$ be a $K3$ surface and $f$ a purely non-symplectic automorphism of order $n$ such that $\text{rk}T = \varphi(n)$ and $\zeta_n$ is not an eigenvalue of $f|\text{NS} \otimes \mathbb{C}$.

Set $d = |\det \text{NS}|$, then $X$ is determined up to isomorphism by the pair $(n, d)$. Conversely, all possible pairs $(n, d)$ and equations for $X$ and (some) $f$ are given in Tables 2, 3.

**Proof.** Comparing Tables 1 and 2, 3 we have to exclude the pairs $(16, 2^8)$ and $(54, 3^3)$. For $(16, 2^8)$, this is done in [40, 4.1]. The pair $(54, 3^3)$ is ruled out in Lemma 5.8.

By Lemma 5.6, the transcendental lattice is uniquely determined by $(n, d)$ and embeds uniquely into $L_{K3}$. By Theorem 1.2, $X$ is determined up to isomorphism by $(\zeta_n, I)$, where $I$ is the kernel of

$$Z[x]/c_n(x) \to O(T^\vee/T), \quad x \mapsto f|T^\vee/T$$

and $f^*\omega_X = \zeta_n\omega_X$ for $\mathbb{C}\omega_X = H^0(X, \Omega_X^2)$. By Lemma 5.3, $f$ is determined uniquely by $(n, d)$. Replacing $f$ with $f^k$, $(n, k) = 1$, does not affect $(n, d)$, hence $I$. However, in this way we can fix a primitive $n$-th root of unity $\zeta_n$.

It remains to compute the Néron-Severi group of the examples in Tables 2, 3 not found in the literature. In most cases this can be done by collecting singular fibers of an elliptic fibration (see [13, p. 365]) or determining the fixed lattice $S(f^k) = H^2(X, \mathbb{Z})^{f^k}$ of a suitable power of the automorphism $f$ via its fixed points. The corresponding tables of fixed lattices are collected in [31, 2, 36, Figure 2].

(4, 2$^2$) We see two fibers of type II$^*$ over $t = 0, \infty$ and two fibers of type $I_2$ over $t = \pm 1$. Then $\text{NS} \cong U \oplus 2E_8 \oplus 2\mathbb{A}_1$ as expected. The two form is given in local coordinates by $dx \wedge dt/2y$, and $f^*(dx \wedge dt/2y) = -dx \wedge -dt/(2y_2) = \zeta_4^2dx \wedge dt/2y$. Hence the action is non-symplectic. The fixed lattice is $U \oplus 2E_8$ while the $I_2$ fibers.
Hence, its fixed lattice is of rank $18$. We get fibers of type $II, II^*$, $2 \times I_0^*$ which results in the lattice NS $= U \oplus E_8 \oplus 2D_4$. 

$(8, 2^4)$ The fourfold cover of $\mathbb{P}^2$ is a special member of a family in [3, Ex. 5.3]. It has five $A_3$ singularities. The fixed locus of the non-symplectic involution $f^4$ consists of 8 rational curves, where each $A_3$ configuration contains 1 fixed curve. Hence, its fixed lattice is of rank 18 and determinant $-2^4$. It equals NS.

$(12, 2^4)$ We get fibers of type $1 \times II, 1 \times II^*, 2 \times I_0^*$ which results in the lattice NS $= U \oplus E_8 \oplus 2A_2 \oplus D_4$.

$(12, 2^33^2)$ This time the zero section and the fibers span the lattice NS $= U \oplus E_8 \oplus 2A_2 \oplus D_4$. 

\[
\begin{array}{|c|c|c|c|}
\hline
n & \det T & X & f \\
\hline
3, 6 & 3 & y^2 = x^3 - t^6(t - 1)^5(t + 1)^2 & (\zeta_8x, \pm y, t) \quad [20, (7.9)] \\
4 & 2^2 & y^2 = x^3 + 3t^2x + t^5(t^2 - 1) & (-x, \zeta_4y, -t) \\
5, 10 & 5 & y^2 = x^3 + t^3x + t^7 & (\zeta_8^2x, \pm \zeta_4^2y, \zeta_4^2t) \quad [20, (7.6)] \\
8 & 2^2 & y^2 = x^3 + tx^2 + t^7 & (\zeta_8^5x, \zeta_8y, \zeta_4^2y) \\
2^4 & t^4 = (x_0^2 - x_1^2)(x_0^2 + x_1^2) & (\zeta_4t; x_1 : x_0 : x_2) \\
12 & 1 & y^2 = x^3 + t^6(t^2 - 1) & (-\zeta_3x, \zeta_4y, -t) \quad [20, (3.4)] \\
2^23^2 & y^2 = x^3 + t^3(t^2 - 1)^2 & (-\zeta_3x, \zeta_4y, t) \\
2^4 & y^2 = x^3 + t^3(t^2 - 1)^3 & (-\zeta_3x, \zeta_4y, -t) \\
7, 14 & 7 & y^2 = x^3 + t^3x + t^8 & (\zeta_7^3x, \pm \zeta_7^4y, \zeta_7^2t) \quad [20, (7.5)] \\
9, 18 & 3 & y^2 = x^3 + t^3(t^3 - 1) & (\zeta_9^7x, \pm \zeta_9^3y, \zeta_9^2t) \quad [20, (7.8)] \\
3^3 & y^2 = x^3 + t^3(t^3 - 1)^2 & (\zeta_3^7x, \pm \zeta_3^3y, \zeta_3^3t) \\
16 & 2^2 & y^2 = x^3 + t^2x + t^7 & (\zeta_2_{16}^7x, \zeta_{16}^{11}y, \zeta_{16}^{10}t) \quad [40, 4.2] \\
2^4 & y^2 = x^3 + t^3(t^4 - 1)x & (\zeta_{16}^{12}x, \zeta_{16}^{5}y, \zeta_{16}^{4}t) \quad [15, 4.1] \\
2^6 & y^2 = x^3 + x + t^8 & (x, iy, \zeta_{16}t) \quad [40, 2.2] \\
20 & 2^4 & y^2 = x^3 + (t^5 - 1)x & (-x, \zeta_4y, \zeta_5t) \\
2^45^2 & y^2 = x^3 + 4t^2(t^5 + 1)x & (-x, \zeta_4y, \zeta_5t) \\
2^6 & y^2 = x^3 + t^6(t^4 + 1) & (\zeta_8^6x, \zeta_8y, \zeta_8^2t) \\
2^62^4 & y^2 = x^3 + t^6(t^4 - 1)^2 & (\zeta_8^6x, \zeta_8y, \zeta_8^2t) \\
2^63^4 & y^2 = x^3 + x + t^{12} & (-x, \zeta_{24}^6y, \zeta_{24}t) \\
15, 30 & 5^2 & y^2 = x^3 + 4t^5(t^5 - 1) & (\zeta_5x, \pm \zeta_5y, \zeta_5t) \\
3^4 & y^2 = x^3 + t^5x + t^2 & (\zeta_5^2x, \pm \zeta_5^2y, \zeta_5^2t) \quad [20, (7.4)] \\
11, 22 & 11 & y^2 = x^3 + t^5x + t^2 & (\zeta_7^3x, \pm \zeta_7^2y, \zeta_7^2t) \\
\hline
\end{array}
\]


| $n$ | $\det T$ | $X$ | $f$ |
|-----|------------|-----|-----|
| 13, 26 | 13 | $y^2 = x^3 + t^2 x + t$ | $(\zeta_{13}^0 x, \pm \zeta_{13} y, \zeta_{13}^7 t)$ |
| 26 | 13 | $y^2 = x^3 + t^7 x + t^4$ | $(\zeta_{13}^0 x, -\zeta_{13}^7 y, \zeta_{13} t)$ |
| 26 | 13 | $w^2 = x_1^4 y_1^4 + x_1^4 y_1^2 y_2 + x_0 x_1^4 y_1^4$ | $((\zeta_{13}^0 x_0 : x_1), (\zeta_{13}^0 y_0 : y_2), -\zeta_{13}^7 w)$ |
| 21, 42 | 1 | $y^2 = x^3 + t^2 (t^2 - 1)$ | $(\zeta_{42}^2 x, \zeta_{42}^2 y, \zeta_{42}^5 t)$ |
| 21, 42 | 2 | $y^2 = x^3 + t^2 (t^2 + 1)$ | $(\zeta_{3}^2 x, \pm \zeta_{2}^7 y, \zeta_{2} t)$ |
| 21 | 7 | $x_0 x_1^4 + x_0^4 x_2 + x_0 x_2^3 - x_0 x_3^2$ | $(\zeta_{2} x_0 : \zeta_{2} x_1 : x_2 : \zeta_{2} x_3)$ |
| 28 | 1 | $y^2 = x^3 + x + t^7$ | $(-x, \zeta_{4 y}, -\zeta_{7 t})$ |
| 28 | 2 | $y^2 = x^3 + (t^7 + 1) x$ | $(-x, \zeta_{4 y}, \zeta_{7 t})$ |
| 17, 34 | 17 | $y^2 = x^3 + t^7 x + t^2$ | $(\zeta_{17}^2 y, \pm \zeta_{17} y, \zeta_{17}^2 t)$ |
| 34 | 17 | $x_0 x_1^4 + x_0^4 x_2 + x_0^4 x_2^3 x_3 = y^2$ | $(-y_0 : x_1 : \zeta_{17} x_1, \zeta_{17}^2 y_2, \zeta_{17}^2 x_2)$ |
| 32 | 2 | $y^2 = x^3 + t^3 x + t^{11}$ | $(\zeta_{32}^3 x, \zeta_{32}^3 y, \zeta_{32}^7 t)$ |
| 2 | 2 | $y^2 = x_0 (x_1^5 + x_0 x_2 + x_1 x_2^4)$ | $(\zeta_{2} x_0 : \zeta_{2} x_0 : x_1 : \zeta_{2}^2 x_2)$ |
| 36 | 1 | $y^2 = x^3 - t^5 (t^5 - 1)$ | $(\zeta_{36}^5 x, \zeta_{36}^5 y, \zeta_{36}^9 t)$ |
| 34 | 1 | $y^2 = x^3 + x + t^9$ | $(-x, \zeta_{4 y}, -\zeta_{9 t})$ |
| 28 $^{32}$ | 20 | $x_0 x_3^3 + x_0^3 x_1 + x_1^4 + x_2^2$ | $(x_0 : \zeta_{3}^3 x_1 : \zeta_{3}^2 x_2 : x_0 x_3^3)$ |
| 40 | 2 | $z^2 = x_0 (x_0^4 x_1 + x_0^3 + x_0^2 x_2)$ | $(x_0 : \zeta_{20} x_0 : \zeta_{4} x_1 : \zeta_{8} x_2)$ |
| 48 | 2 | $y^2 = x^3 + t (t^8 - 1)$ | $(\zeta_{48}^2 x, \zeta_{48}^2 y, \zeta_{48}^5 t)$ |
| 19, 38 | 19 | $y^2 = x^3 + t^7 x + t$ | $(\zeta_{19}^2 t, \pm \zeta_{19} y, \zeta_{19}^2 t)$ |
| 38 | 19 | $y^2 = x_0^2 x_1 + x_0 x_1^2 x_2 + x_2^2$ | $(x_0 : \zeta_{19} x_1 : \zeta_{19} x_2 : -\zeta_{19}^9 y)$ |
| 27, 54 | 2 | $y^2 = x^3 + t (t^9 - 1)$ | $(\zeta_{54}^2 x, \zeta_{54}^2 y, \zeta_{54}^7 t)$ |
| 27 | 3 | $x_0 x_3^3 + x_0^3 x_1 + x_2 (x_3^3 - x_2^3)$ | $(x_0 : \zeta_{27}^3 x_1 : \zeta_{27}^2 x_2 : \zeta_{27} x_3)$ |
| 25, 50 | 5 | $z^2 = (x_0^2 + x_0^2 x_1^2 x_2 + x_2^2)$ | $(z : x_0 : \zeta_{25}^2 x_1 : \zeta_{25}^2 x_2)$ |
| 33, 66 | 1 | $y^2 = x^3 + t (t^{11} - 1)$ | $(\zeta_{66}^2 x, \zeta_{66}^2 y, \zeta_{66}^5 t)$ |
| 13 | 1 | $y^2 = x^3 + x + t^{11}$ | $(-x, \zeta_{4 y}, \zeta_{11} t)$ |

(15, 52) This elliptic K3 surface arises as a degree 5 base change from the rational surface $Y: y^2 = x^4 + 4t(t + 1)$. We see the section $(x, y) = (\zeta_5^k, 1 + 2t)$ of $Y$ and then $(x, y) = (\zeta_5^k, 1 + 2t^5)$ generating the Mordell-Weil group of $X_{15,5^2}$. Alternatively one can compute that NS is the fixed lattice of $f^3$.

(15, 34) The 5-th power $f^5$ of $f$ is a non-symplectic automorphism of order 3 acting trivially on NS. It has 2 fixed curves of genus 0 lying in the $E_7$ fiber and 6 isolated fixed points over $t = 0$ and $t = \infty$. The classification of the fixed lattices of non-symplectic automorphisms of order 3 provides the fixed lattice of $f^5$ which equals NS. In order to get explicit generators of the Mordell-Weil group we can base change with $t \mapsto t^5$ from the rational surface $y^2 = 2x^3 + tx + t^4$ with sections
(26, 13) The third case is the minimal resolution of a double cover of \( \mathbb{P}^1 \times \mathbb{P}^1 \) branched over the curve \( x_0^4y_0^4 + x_1^4y_1^4 + x_0x_1^4y_0^4 \) of bi-degree \((4, 4)\). The double cover has an \( E_6 \) singular point above \(((1 : 0, 0 : 1))\). The branch curve is fixed by the covering involution. Its strict transform is a smooth curve of genus 6. The involution fixes a single smooth rational curve coming from the resolution of the \( E_6 \) singularity. Therefore the fixed lattice of the involution has rank 6 and determinant 2^4.

(34, 17) In the first case the fixed locus of \( g_1^{17} \) consists of a curve \((y = 0)\) of genus 8 and a rational curve - the zero section. This leads to the fixed lattice \( U \oplus (−2) \oplus (−2) \).

Since the fixed locus of \( g_2^{17} \) is a curve of genus 8, \( S(g_1^{17}) \cong (2) \oplus (−2) \oplus (−2) \). Note that there is an \( A_4 \) singularity at zero. Since the fixed lattices of the two automorphisms are different, the actions are distinct as well.

(20, 2^4) The elliptic fibration has 5 fibers of type \( III \) and a single fiber of type \( III^* \). This results in the lattice \( U \oplus E_7 \oplus 5A_1 \) spanned by fiber components and the zero section. It has determinant 2^8. Since there is also a 2-torsion section, \( \det \text{NS} = 2^4 \).

(20, 2^45^2) In this case \( X_{(20, 2^45^2)} \) has a single fiber of type \( I_0^* \) and 6 fibers of type \( III \). This results in the lattice \( U \oplus D_4 \oplus 6A_1 \) of rank 12 and determinant 2^8. Again there is 2-torsion. We reach a lattice of determinant 2^8. We get \( X \) by a degree 5 base change from \( y^2 = x^3 + 4t^2(t + 1)x \) with section \((x, y) = (t^2, t^3 + 2t^2)\). We get the sections \((x, y) = (t^5, t^9 + 2t^4)\) and \((−x(t), iy(t))\) generating the Mordell-Weil lattice \( A_1^* (5) \oplus 2 \) of \( X \).

(21, 7^2) We can base change this elliptic fibration from \( y^2 = x^3 + 4t^4(t − 1) \) to get the sections \((x, y) = (c_3^k t^6, 2t^2 + t^3)\) which generate the Mordell-Weil lattice of \( X \). Using the height pairing one can then compute NS. Alternatively note that \( f^3 \) is an order 7 non-symplectic automorphism acting trivially on \( NS \) and not fixing a curve of genus 0 point-wise. There is only a single possible fixed lattice of rank 10, namely \( U(7) \oplus E_8 \). For the other possible actions see Lemma 6.22.

(24, 2^2) The fibration has 4 type \( II \) fibers, one type \( I_0^* \) and an \( II^* \) fiber. We get \( \text{NS} = U \oplus D_4 \oplus E_8 \).

(24, 2^6) In this case the fixed locus of \( f^{12} \) consists of 4 rational curves and a curve of genus 1. This leads to a fixed lattice of rank 14 and determinant 2^6 as expected.

(24, 2^33^4) The trivial lattice is \( U \oplus D_4 \oplus 4A_2 \). It equals \( NS \) for absence of torsion sections.

(24, 2^63^4) Since \( X \) has a purely non-symplectic automorphism of order 24, the rank of \( NS \) is either 6 or 14. Since \( \text{Fix} (f^{12}) \) consists of 2 smooth curves of genus 0, its fixed lattice \( S(f^{12}) \) is 2-elementary of rank 10 and determinant \( −2^8 \). Hence, we see...
that \( \text{rk} \text{NS} = 14 \). Since the orthogonal complement of \( S = S(f^{12}) \) in \( \text{NS} \) is of rank 4, the glue \( G_S \) is an at most 4 dimensional subspace. Then \( 2^4 \leq |D_S/G_S| \leq |D_{\text{NS}}| \) by Lemma 5.9. Hence \( 2^4 \mid \text{det} \text{NS} \). Note that \( S(f^{12}) = \ker f^{12} - 1 \) and then \( \ker c_24 c_8(f) = S(f^{12}) \perp \) is of rank 12. This shows that the characteristic polynomial of \( f \mid \text{NS} \) is divided by \( c_8 \) but not by \( c_{24} \). We are in the situation of Theorem 5.9 As \( 2^4 \mid \text{det} \text{NS} \), it is either \( -2^6 \) or \( -2^6 3^4 \). We show that \( 3 \mid \text{det} \text{NS} \). A direct computation reveals that \( \text{Fix}(f^8) \) consists of a smooth curve of genus 1 and 3 isolated fixed points. This leads to a fixed lattice

\[
S(f^8) = \ker(c_8 c_4 c_2 c_1)(f) \cong U(3) \oplus 3A_2
\]

of rank 8 and determinant \( -3^5 \). Now we view \( S(f^8) \) as a primitive extension of \( \ker c_8(f) \oplus \ker c_4 c_2 c_1(f) \). The rank of both summands is 4, while the length of the discriminant group of \( S(f^8) \) is 5. Then each summand must contribute to the discriminant group. We see that \( 3 \mid \text{det} \ker c_8(f) \). However,

\[
3 \nmid \text{res}(c_8, c_12 c_6 c_4 c_3 c_2 c_1) = 2^4.
\]

In particular the 3 part of \( \text{Dker} c_8(f) \) cannot be glued inside \( \text{NS} \). Then \( 3 \mid \text{det} \text{NS} \).

(27, 3^3) The action of \( f^9 \) has an isolated fixed point and a fixed curve of genus 3. We see that the fixed lattice of \( f^9 \) is \( U(3) \oplus A_2 = \text{NS} \). It is spanned by the 4 lines at \( x_3 = 0 \). Note that \( f^3 \) acts trivially on \( \text{NS} \) while \( f \) does not.

(28, 2^6) This fibration has 8 fibers of type \( \text{III} \) and a 2-torsion section. Together they generate the Néron-Severi group.

(32, 2^2) The elliptic fibration has a singular fiber of type \( I_0^* \), of type \( \text{II} \) and 16 of type \( I_1 \). Thus \( \text{NS} \cong U \oplus D_4 \). Here \( f \) has 6 isolated fixed points.

(32, 2^4) The fixed locus of \( S(f^{16}) \) is the strict transform of \( y = 0 \) which is the disjoint union of a rational curve and a curve of genus 5. Thus \( \text{det NS} = 2^4 \). Note that \( f \) has 4 isolated fixed points.

(36, 3^4) The fixed curves of \( f^{12} \) are a smooth of genus 0 over \( t = 0 \) and the central rational curve in the \( D_4 \) fiber. This leads to the fixed lattice \( U \oplus 4A_2 = \text{NS} \).

(36, 2^6 3^2) If we can show that \( 2 \mid \text{det} \text{NS} \), then the only possibility is \( \text{det} \text{NS} = -2^6 3^2 \). The action of \( f^{18} \) fixes a smooth curve of genus 3 (it is a plane curve of degree 4 in the hypersurface defined by \( x_2 = 0 \)) and nothing else. Hence, its fixed lattice \( S \) is 2-elementary of rank 8 and determinant \( 2^8 \). Denote by \( C = S^\perp \subseteq \text{NS} \) the orthogonal complement of \( S \) inside \( \text{NS} \). It has rank 2. Assume that \( 2 \mid \text{det} \text{NS} \). Then \( (D_S)_2 \cong (D_C)_2 \) which is impossible, since \( (D_S)_2 \) has dimension 8, while \( (D_C)_2 \) is generated by at most 2 elements.

Remark 5.10. The pair \((21, 7^2)\) contradicts [19, 2.1]. There it is claimed that a purely non-symplectic automorphism of order 21 acts trivially on \( \text{NS} \). As a consequence it is claimed that there is only a single K3 surface of order 21. However there are two. The pair \((28, 2^6)\) and its uniqueness are probably known to J. Jang independently.
The pair $(32, 2^4)$ contradicts [43 Thm. 1.2]. There the uniqueness of $(X, \langle g \rangle)$ where $g$ is a purely non-symplectic automorphism of order 32 is claimed.

In [44, 1.8, 4.8] non symplectic automorphisms of 3-power order acting trivially on NS are classified. The author is missing a case. It is claimed that if $\text{NS}(X) = U(3) \oplus A_2$ then there is no purely non-symplectic automorphism of order 9 acting trivially on NS. The pair $(27, 3^3)$ contradicts this result - there the automorphism acts with order 3 on NS. It is a special member of the family

$$x_0 x_3^3 + x_0 x_1 + x_2 (x_1 - x_2)(x_1 - ax_2)(x_1 - bx_2)$$

with automorphism given by $(x_0 : x_1 : x_2 : x_3) \mapsto (x_0 : \zeta_3^3 x_1 : \zeta_3^3 x_2 : \zeta_0 x_3)$ and generically trivial action on NS as a fixed point argument shows. It was found by first computing the action of $f$ on cohomology through gluing, thus proving its existence. Then one specializes a family with automorphism of order 3 given in [24.49].

6. **CLASSIFICATION OF NON-SYMPLECTIC AUTOMORPHISMS OF HIGH ORDER**

Let $X_i, i = 1, 2$, be K3 surfaces and $G_i \subset \text{Aut}(X_i)$ subsets of their automorphism groups. Recall that the pairs $(X_1, G_1), (X_2, G_2)$ are said to be equivalent if there is an isomorphism $F : X_1 \rightarrow X_2$ with $G_2 = FG_1 F^{-1} = \{ F \circ g F^{-1} | g \in G_1 \}$.

**Theorem 6.1.** Let $X_{(n,d)}$ be as in Tables 5.3.

1. For $(n, d) = (66, 1), (48, 2^2), (44, 1), (50, 5), (42, 1), (28, 1), (36, 1), (32, 2^2), (32, 2^4), (40, 2^4), (54, 3), (27, 3^3), (24, 2^2), (16, 2^2)$, we have

   $$\text{Aut}(X_{(n,d)}) = \langle g_{(n,d)} \rangle \cong \mathbb{Z}/n\mathbb{Z}.$$  

2. For $(n, d) = (28, 2^6), (12, 1), (16, 2^4), (20, 2^4)$ we have

   $$\text{Aut}(X) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}.$$  

**Remark 6.2.** In [23 Thm. 1] (1) is proven for $n = 66, 44, 50$ and 40 by a different method. All other entries of Tables 2 and 3 have infinite automorphism group. To see this, one checks if the Néron Severi lattice is 2-reflective, which by definition means that the Weyl group is of finite index in its orthogonal group. One can look up the classification of 2-reflective lattices in [35 Prop 0.2.1, Thm 0.2.2].

Before proving the Theorems 5.1 and 6.1 we refine our terminology. A lattice with isometry is a pair $(A, a)$, where $A$ is a lattice and $a \in O(A)$ an isometry. A morphism $\phi : (A, a) \rightarrow (B, b)$ of lattices with isometry is an isometry $\phi : A \rightarrow B$ such that $\phi \circ a = b \circ \phi$. A primitive extension of lattices with isometry is a morphism $\phi : (A, a) \oplus (B, b) \rightarrow (C, c)$ such that the embedding of $A$ is primitive and $\phi(B) = \phi(A)^\perp$. Note that since $\phi$ is a morphism, $a \oplus b = c|_{A \oplus B}$.

**Definition 6.3.** We call two primitive extensions of lattices with isometry

$$(A, a_i) \oplus (B_i, b_i) \hookrightarrow (C_i, c_i), \quad i = 1, 2$$

isomorphic if there is a commutative diagram

$$
\begin{array}{ccc}
(A_1, a_1) & \oplus & (B_1, b_1) \\
\downarrow & & \downarrow \\
(A_2, a_2) & \oplus & (B_2, b_2) \\
\downarrow & & \downarrow \\
(C_1, c_1) & & (C_2, c_2)
\end{array}
$$

where the vertical arrows are isomorphisms.
Proposition 6.4. There is a one to one correspondence between isomorphism classes of primitive extensions of \((A, a) \oplus (B, b)\) and the double coset

\[
\text{Aut}(B, b) \setminus \left\{ \text{Glue maps } \phi : G_A \xrightarrow{\sim} G_B \right\} / \text{Aut}(A, a).
\]

Proof. Let

\[
\begin{array}{ccc}
(A, a) \oplus (B, b) & \xrightarrow{\psi_A} & (C_1, c_1) \\
(A, a) \oplus (B, b) & \xrightarrow{\psi_B} & (C_2, c_2)
\end{array}
\]

be an isomorphism of primitive extensions. We can view \(C_1\) and \(C_2\) as overlattices of \(A \oplus B\). On the level of glue groups this gives a commutative diagram

\[
\begin{array}{ccc}
(G_A, a) & \xrightarrow{\phi_1} & (G_B, b) \\
(G'_A, a) & \xrightarrow{\phi_2} & (G'_B, b)
\end{array}
\]

Then \(\phi_2 = \psi_B^{-1} \circ \phi_1 \circ \psi_A\). Conversely, if \(\phi_2 = g \phi_1 h\) with \(g \in \text{Aut}(B, b)\) and \(h \in \text{Aut}(A, a)\) is another glue map, then one can check that \(g^{-1} \circ h\) extends to an equivariant isomorphism of the overlattices defined by \(\phi_1\) and \(\phi_2\). \(\square\)

Definition 6.5. We say that a lattice with isometry \((A, a)\) has simple glue if

\[
\text{Aut}(A, a) \rightarrow \text{Aut}(q_A, a) = \{ g \in O(q_A) \mid g \circ a = a \circ g \}
\]

is surjective.

Example 6.6. Let \((A, a)\) be a lattice with isometry such that \(D_A \cong \mathbb{F}_p\). Then \(\text{Aut}(D_A, a) = \{ \pm \text{id}_{D_A} \}\), and we see that \((A, a)\) has simple glue.

Lemma 6.7. Let \((A, a)\) be a simple \(c_n(x)\)-lattice. Then \(\text{Aut}(A, a) = \langle \pm a \rangle\).

Proof. Let \(h \in \text{Aut}(A, a)\). Then \(h\) is a \(\mathbb{Z}[a]\)-module homomorphism, i.e. \(h \in \mathbb{Z}[a]^{\times} \subseteq K^{\times}\). Since \(h\) is an isometry and \((A, a)\) is simple, we get that

\[
\text{Tr}^K_{\mathbb{Q}}(hh^*x) = \text{Tr}^K_{\mathbb{Q}}(x) \quad \forall x \in K.
\]

By non-degeneracy of the trace form, we get \(hh^* = 1\), i.e. \(|h| = 1\). By Kronecker’s theorem, \(h\) is a root of unity. \(\square\)

Proposition 6.8. Let \((L_0(a), f)\) be a twist of the principal, simple \(p(x)\)-lattice and \(I < \mathcal{O}_K\) such that \(D_{L_0(a)} \cong \mathcal{O}_K/I\). Then

\[
\text{Aut}(q_{L_0(a)}, f) = \{ [u] \in (\mathcal{O}_K/I)^{\times} \mid uu^\sigma \equiv 1 \pmod{I} \}.
\]

Proof. Let \(g \in \text{Aut}(q_{L_0(a)}, f)\). Under the usual identifications \(g = [u] \in (\mathcal{O}_K/I)^{\times}\).

Note that \((f - f^{-1})\mathcal{O}_K = D_{L_0(a)}^K\) is the relative different of \(K/k\), and set \(d = f - f^{-1}\).

Then \(L_0(a)^\vee = \frac{d}{ad} \mathcal{O}_K\) and \(I = ad\mathcal{O}_K\) (Lemma 3.22). Since \([u]\) preserves the discriminant form, we get that \(b(x, y) = b([u]x, [u]y)\) for all \(x, y \in \frac{1}{d_{ad}} \mathcal{O}_K\), i.e.

\[
\text{Tr}^K_{\mathbb{Q}} \left( \frac{1 - uu^\sigma}{v'(f + f^{-1})ad^2} \mathcal{O}_K \right) \subseteq \mathbb{Z}.
\]
By definition of the different, this is equivalent to
\[
1 - uu^a \equiv r'(f + f^{-1})ad^2 \in D_K^{-1},
\]
and consequently
\[
1 - uu^a \equiv ad^2r'(f + f^{-1})D_K^{-1}.
\]
Now, the different $D_K = p'(f)O_K = (f - f^{-1})r'(f + f^{-1})O_K$. Hence
\[
1 - uu^a \equiv adO_K = I
\]
as claimed. Conversely, let $u \equiv 1 \mod I$. A similar computation shows that the
discriminant quadratic form $q_{L(a)}$ is preserved if and only if
\[
1 - uu^a \equiv ad^2O_k = a(D_K^2)^2 \cap k.
\]
However, we already know $1 - uu^a \in aD_k^K \cap k$. By simplicity, we know that the
norm $N(D_K^2) = |p(1)p(-1)|$ is squarefree, and hence
\[
D_k^K \cap k = (D_k^2)^2 \cap k.
\]
\[\square\]

Remark 6.9. Instead of $|p(1)p(-1)|$ being squarefree, one may assume $K/k$ to be
tamely ramified and the proof works. However, $\mathbb{Q}((\zeta_2))$ is ramified at two and then $D_k^K \cap k$ is the prime ideal of $O_k$ above 2. In this case we need the condition $uu^a \equiv 1 \mod ID_k^K \cap k$.

Lemma 6.10. All entries in Table 4 except $24, 2^63^4$ have simple glue.

Proof. We do the calculation for $(27, 3^3)$. The other cases are similar. Set $\zeta = \zeta_27$.
Then $I = (1 - \zeta)^3$, and $O_K/I = \mathbb{Z}[\zeta]/(1 - \zeta)^3$ has 18 units. They are given by
\[
u_e = \epsilon_0 + \epsilon_1(1 - \zeta) + \epsilon_2(1 - \zeta)^2, \quad (\epsilon_0 \in \{1, 2\}, \epsilon_1, \epsilon_2 \in \{0, 1, 2\})
\]
We obtain the equation
\[
0 \equiv (1 - uu_e^2) \equiv \epsilon_0(\epsilon_1 + \epsilon_2)(2 - \zeta - \zeta^{-1}) \mod (1 - \zeta)^3.
\]
We get 6 distinct solutions for $u_e$. However $\pm \zeta^k$ for $k = 1, 2, 3$ are all distinct
modulo $(1 - \zeta)^3$. The claim follows. \[\square\]

For $n \in \mathbb{N}$ we denote by $S_n$ the symmetric group of $n$ elements and by $D_n$ the
dihedral group - the symmetry group of a regular polygon with $n$ sides.

Lemma 6.11. Let $L$ be a hyperbolic lattice. Fix a chamber of the positive cone and
denote by $O^+(L)/W(L) \cong \Gamma(L) \subseteq O^+(L)$ the subgroup generated by the isometries
preserving the chamber. Set
\[
\phi : \Gamma(L) \to O(q_L) \quad f \mapsto f|D_L
\]
then for $L \neq U(3) \oplus A_2$ in Table 4 $\phi$ is surjective. For $L = U(3) \oplus A_2$ the cokernel of
$\phi$ is generated by $-id$. It is injective as well for all $L$ in the table except $U(2) \oplus 2D_4$
and $U(2) \oplus D_4 \oplus E_8$ where its kernel is of order 2.
Table 4. Symmetry groups of a chamber

\[
\begin{array}{|c|c|}
\hline
L & \Gamma(L) \\
\hline
U \oplus A_2 & S_2 \\
U(3) \oplus A_2 & D_4 \\
U \oplus 4A_1 & S_4 \\
U(2) \oplus D_4 & S_5 \\
U \oplus E_8 & S_1 \\
U \oplus D_4 \oplus E_8 & S_3 \\
\hline
\end{array}
\begin{array}{|c|c|}
\hline
L & \Gamma(L) \\
\hline
U \oplus 2A_2 & D_4 \\
U(3) \oplus 2A_2 & S_6 \times S_2 \\
U \oplus D_4 & S_3 \\
U \oplus E_6 & S_2 \\
U(2) \oplus 2D_4 & S_8 \times S_2 \\
U(2) \oplus D_4 \oplus E_8 & S_5 \times S_2 \\
\hline
\end{array}
\]

Proof. In all cases we can compute a fundamental root system using Vinberg’s algorithm [46, §3]. An isometry preserves the chamber corresponding to the fundamental root system if and only if it preserves the fundamental root system. We get a sequence

\[
0 \to O^+(L)/W(L) \cong \Gamma(L) \to Sym(\Gamma) \to 0
\]

where Sym(\Gamma) denotes the symmetry group of the dual graph of a fundamental root system. Since the fundamental roots form a basis of \(L \otimes \mathbb{Q}\), the sequence is exact. The calculation of \(\ker \phi\) is done by computer. For \(L = U(2) \oplus 2D_4\) see also [22, 2.6].  

Proof of Theorem 1.1. Fix some pair \((X,G)\) as in the theorem and write \(G = \langle g \rangle\) for \(g \in G\) such that \(g^* \omega = \zeta_n \omega\). In order to prove the theorem, we have to show that the pair \((H^2(X, \mathbb{Z}), g)\) is unique up to isomorphism. We have seen that \(n\) and \(d = \det T\) determine \((T, g|_T)\) (and thus \(X\) by Thm. 1.2) up to isomorphism. By Lemma 6.11 \((T, g|_T)\) has simple glue. Hence the isomorphism class of \((H^2(X, \mathbb{Z}), g)\) is determined by the isomorphism class of \((\text{NS}, g|_{\text{NS}})\). What remains is to determine all possible isomorphism classes of \((\text{NS}, g|_{\text{NS}})\) for \((n, d)\) fixed. This is done in the following lemmas.  

Lemma 6.12. For \((p, p)\), \(p = 13, 17, 19\), \(g|_{\text{NS}}\) is the identity.  

Proof. Since the order of \(g\) on NS is strictly smaller than \(n = p\) in these cases, it can only be one.  

Proof of Theorem 6.1. (1) By Lemma 6.11 we have for these lattices that

\[O^+(\text{NS})/W(\text{NS}) \cong O(q_{\text{NS}}) \cong O(q_T)\]

Consequently, automorphisms are determined by their action on the transcendental lattice and this group is generated by \(g_{(n,d)}\). (2) In this case \(\phi : \Gamma(\text{NS}) \to O(q_{\text{NS}})\) has a kernel of order two and there are exactly two possibilities for \(g|_{\text{NS}}\). They differ by an element of the kernel corresponding to a symplectic automorphism of order two.  

We note the following theorem on the rational square class of a lattice with isometry for later use.
Figure 1. Dynkin diagrams of the fundamental root systems

Theorem 6.13. [32, 3.3.14 (ii)] Let \((L, f)\) be a \(c_n(x)^m\)-lattice and \(f\) of order \(n\).

\[
\det L \in \begin{cases} 
p^m \cdot (\mathbb{Q}^\times)^2, & \text{for } n = p^k, p \neq 2, \\
(\mathbb{Q}^\times)^2, & \text{else.}
\end{cases}
\]
Let $X$ be a K3 surface and $g \in \text{Aut}(X)$ of finite order $n$, and note that since $g|\text{NS}$ is of finite order $g|\text{NS}$ preserves a chamber of the positive cone if and only if
$$\ker \left( g^{n-1} + g^{n-2} \cdots + 1 \right)|_{\text{NS}}$$
is root free ([26] calls such roots cyclic). We turn this into a definition.

**Definition 6.14.** Let $N$ be a hyperbolic or positive definite lattice and $g \in O(N)$ an isometry of finite order $n$. We call $g$ **unobstructed** if
$$\ker \left( g^{n-1} + g^{n-2} \cdots + 1 \right)|_{\text{NS}}$$
is root free. Else we call $g$ **obstructed**.

In the following we set
$$C_i = \ker c_i(g|H^2(X, \mathbb{Z})), \quad C_iC_j = \ker c_i c_j(g|H^2(X, \mathbb{Z})).$$
Recall that for $n \in \mathbb{N}$ we say that two lattices glue along $n$ if $n$ is the order of the glue.

**Lemma 6.15.** For $(38, 19)$ there are two pairs $(X, g_1)$, $(X, g_2)$ up to isomorphism. Set
$$R_1 = \begin{pmatrix} -2 & -1 \\ -1 & -10 \end{pmatrix}, \quad R_2 = \begin{pmatrix} -8 & 2 \\ 2 & -10 \end{pmatrix}.$$
Then $\text{NS} \cong U \oplus R_1$ and $g_1|_{\text{NS}}$ is given by the gluing of
$$C_1 \cong U \oplus (-2) \quad \text{and} \quad C_2 \cong (-38)$$along 2. Further $g_2|_{\text{NS}}$ is given by the gluing of
$$C_1 \cong (2) \oplus (-2) \quad \text{and} \quad C_2 \cong R_2$$along $2^2$.

**Proof.** There are 3 cases
$$\chi_{g|_{\text{NS}}} = (x - 1)^r(x + 1)^{4-r}, \quad (r = 1, 2, 3).$$
Note that $C_1$ is 2-elementary and $\det C_1 | 2^m$ where $m = \min\{r, 4 - r\}$ (Thm 3.10).

$r = 1$: Here $C_1 = (2)$ and $C_2 = (-2) \oplus R_1$ which is the unique even, negative definite lattice of determinant $-38$.

$r = 2$: We see $\det C_2 | 2^{219}$ and there are 4 such lattices - $R_1, (2) \oplus (-38), R_1(2)$ and $R_2$. The first two lattices have roots and $R_1(2)$ has wrong 19 glue, since the Legendre symbol $\left( \frac{2}{19} \right) = -1$. We are left with $R_2$. The glue is easily seen to be unique.

$r = 3$: We have the single choice $C_2 = (-38)$ and $C_1 = U \oplus (-2)$. Indeed the gluing exists and is unique.

□

**Lemma 6.16.** (34, 17) There are two pairs $(X, g_1)$ and $(X, g_2)$ for (34, 17). The action of $g_1|_{\text{NS}}$ is given by the gluing of
$$C_1 = U \oplus (-2) \oplus (-2) \quad \text{and} \quad C_2 = -\begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}.$$along $2^2$. 
The action of $g_2|_{NS}$ is given by the gluing of
\[ C_1 = (2) \oplus (-2) \oplus (-2) \quad \text{and} \quad C_2 = -2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix} \]
along $2^3$.

Proof. Here we need a little more work. Note that $NS$ is in the genus $\Pi_{(1,5)}(17^{-1})$. There are the 5 cases
\[ \chi_g|_{NS} = (x - 1)^r (x + 1)^{6-r}, \quad r \in \{1, \ldots, 5\}. \]
In any case $17 \mid \det C_2 \mid 2^m 17$ where $m = \min\{r, 6 - r\}$. The 17 part of the genus symbol of $C_2$ is $17^{-1}$, and moreover $2(D_{C_2})_2 = 0$. Then the genus symbol of $C_2$ is $\Pi_{(0,6-r)}(2^r 17^{-1})$ where the asterisk is an unknown entry.

$r = 1$: Then $C_1 = (2)$ and $\det C_2 = -34$. Hence in order to glue above 2, $C_2$ must belong to $\Pi_{(0,5)}(2; 17^{-1})$ or $\Pi_{(0,5)}(2^{2} 17^{-1})$, but both genera are empty as they contradict the oddity formula [13, Chapter 15, (16)].

$r = 2$: Here $C_2$ is even of signature $(0, 4)$ and determinant $d = -17, -2 \cdot 17, -4 \cdot 17$. Looking at the tables in [37], we see that there are 1, 0, 7 such forms, and all of them contain roots.

$r = 3$: From the tables in [12], we extract the following. If $v = 0, \pm 2$ the respective genera are empty. If $v = 1$, there is a single genus, namely $\Pi_{(0,3)}(2; 17^{-1})$ containing two classes - both have maximum $-2$. However, for $v = 3$ there are 9 negative definite ternary forms of determinant $2^3 17$. Only a single one of them has the right 2-genus symbol and no roots. It is given by
\[ C_2 = -2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{pmatrix}. \]
Indeed, here $C_1 = (2) \oplus (-2) \oplus (-2)$ works just fine, and as $|O(qC_1)| = 2$ it is evident that the gluing is unique as well.

$r = 4$: Here $\det C_2 \mid 2^2 17$, and we get the possibilities
\[ - \begin{pmatrix} 2 & 0 & 34 \\ 0 & 4 & 2 \\ 34 & 2 & 18 \end{pmatrix}, - \begin{pmatrix} 6 & 2 \\ 2 & 12 \end{pmatrix}. \]
The first two have wrong 17 glue. We are left with the third one. It has
\[ (qR)_2 \cong (1/2) \oplus (1/2). \]
Then there is the single possibility $C_1 \cong U \oplus (-2) \oplus (-2)$. Surjectivity of $O(C_1) \to O(qC_1)$, hence uniqueness of the extension is provided by Theorem 3.5.

$r = 5$: Here $C_2 = (-34) \in \Pi_{(0,1)}(2^7 17)$ has wrong 17-glue.

$\square$
The holomorphic Lefschetz formula. For the next lemma we use the holomorphic (see [5, p.542] and [6, p.567]) and topological Lefschetz formula. We give a short account. See [44] for a similar application.

Recall that $g$ is a purely non-symplectic automorphism of the K3 surface $X$ with $g^*\omega = \zeta_n \omega$, where $0 \neq \omega \in H^0(X, \Omega^2_X)$. Let $x$ be a fixed point of $g$. Then the local action of $g$ at $x$ can be linearized and diagonalized (in the holomorphic category).

**Definition 6.17.** A fixed point is said to be of type $(i, j)$ if the action is locally of the form

$$\begin{pmatrix} \zeta_i^n & 0 \\ 0 & \zeta_j^n \end{pmatrix}.$$

This implies that the fixed point set $X^g$ is the disjoint union of isolated fixed points and smooth curves $C_1, \ldots, C_N$. Set

$$a_{ij} = \frac{1}{(1 - \zeta_i^n)(1 - \zeta_j^n)} \quad \text{and} \quad b(g) = \frac{(1 + \zeta_n)(1 - g)}{(1 - \zeta_n)^2}.$$

Denote by $m_{i,j}$ the number of isolated fixed points of type $(i, j)$, and set $g_l = g(C_l)$ the genus of the fixed curve $C_l$. The topological Lefschetz formula is

$$e(X^g) = \sum_{i=0}^{4} (-1)^i Tr(g^*|H^i(X, \mathbb{Z}))$$

which in our setting amounts to

$$M + \sum_{l=1}^{N} (2 - 2g(C_l)) = 2 + Tr(g^*|T) + Tr(g^*|\text{NS})$$

where $M = \sum_{1 \leq i \leq j < n} m_{i,j}$ is the number of isolated fixed points. The holomorphic Lefschetz formula is

$$1 + \zeta_n = \sum_{i=0}^{2} (-1)^i Tr(g^*|H^i(X, \mathcal{O}_X)) = \sum_{i+j=n+1 \atop 1 \leq i \leq j < n} a_{ij} m_{ij} + \sum_{l=1}^{N} b(g_l).$$

**Lemma 6.18.** For $(26, 13)$ there are three pairs $(X, f_1)$, $(X, f_2)$ and $(X, f_3)$. The action of $g_1|\text{NS}$ is given by the gluing of

$$C_1 = U \oplus D_4 \oplus A_1 \quad \text{and} \quad C_2 = -\begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 10 \end{pmatrix}$$

along $2^3$. The action of $g_2|\text{NS}$ is given by the gluing of

$$C_1 \cong (2) \oplus E_8 \quad \text{and} \quad C_2 \cong (-26)$$

along 2.

The action of $g_3|\text{NS}$ is given by the gluing of

$$C_1 = U(2) \oplus D_4 \quad \text{and} \quad C_2 = -\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 2 & 2 \\ 2 & 2 & 4 & 2 \\ 2 & 2 & 2 & 8 \end{pmatrix}$$

along $2^4$. 
Proof: We already know the uniqueness of \((X,g^2)\). One can check that \(g^2\) has 9 isolated fixed points and a (pointwise) fixed curve of genus 0. By \([4, 8.4]\) their local types are given by
\[
m_{2,12} = 3, m_{3,11} = 3, m_{4,10} = 2, m_{5,9} = 1.
\]
Since \(X^g \subseteq X^{g^2}\), either \(g\) fixes a curve of genus 0 and at most 9 isolated points, or \(g\) does not fix a curve and at most 11 points.

A calculation of the holomorphic Lefschetz formula yields the following possibilities: \(X^g\) fixes a curve of genus zero and 7 or 9 points, or \(X^g\) fixes 4, 5, 6 or 7 points and no curve. In any case the fixed locus has Euler characteristic \(\chi = 2 + \frac{1}{2}(X)\cdot c_1\).

Write
\[
\chi_{g|\text{NS}} = (x - 1)^r(x + 1)^{10 - r}
\]
for the characteristic polynomial of the action of \(g\) on \(\text{NS}\). Then the topological Lefschetz formula reads
\[
e(X^g) = 2 + Tr(g^*[T]) + Tr(g^*|\text{NS}) = 2 + 1 + r - (10 - r) = 2r - 7,
\]
and consequently \(6 \leq r \leq 9\) We view \(\text{NS}\) as a primitive extension of \(C_1 \oplus C_2\). Since \(\text{res}(c_1,c_{26}) = 1\), we see that \(13 \mid \det C_2\). Further, \(C_1\) is 2-elementary. We conclude that \(\det C_2 = 2^k 13\) where \(k \leq \min\{r, 10 - r\}\).

\(r = 6\): Looking at the tables in \([37]\), we see that for \(k = 0, 1, 2, 3\) all even forms of signature \((0, 4)\) and determinant \(2^k 13\) have roots. For \(k = 4\) there are three genera of determinant \(2^4 \cdot 13\). They give a total of 5 even forms. Exactly one of them has no roots. It results in \(g_3\).

\(r = 7\): Here we use the tables of \([12]\) to list even forms of signature \((0, 3)\) and determinant \(2^k 13\).
- For \(k = 0\) there is no lattice of this determinant.
- For \(k = 1\) there is a single class, but it is obstructed.
- For \(k = 2\) there are two genera of this determinant, but their 2 discriminant group is isomorphic to \(\mathbb{Z}/4\mathbb{Z}\).
- For \(k = 3\) there is a single genus with right 2 discriminant and 13 glue. It is
\[
\Pi_{(0,3)}(2_1^{-3} 13^{-1})
\]
and consists of the two classes
\[
\begin{pmatrix}
2 & 0 & 0 \\
0 & 2 & 0 \\
0 & 0 & 26
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
4 & 2 & 2 \\
2 & 4 & 2 \\
2 & 2 & 10
\end{pmatrix} \cong C_2
\]
one of which, \(C_2\), has no roots. Then
\[
C_1 \cong U \oplus D_4 \oplus A_1.
\]

\(r = 8\): There are no negative definite lattices of rank 2 and determinant 13 or 26. But there are two of determinant \(2^2 \cdot 13\):
\[
\begin{pmatrix}
-2 & 0 \\
0 & -26
\end{pmatrix}, \quad \text{and} \quad \begin{pmatrix}
-4 & 2 \\
2 & -14
\end{pmatrix}.
\]
The first one is obstructed while the second one has wrong 19-glue.

\(r = 9\): Here we can take \(C_2 \cong (-26)\) and \(C_1 \cong (2) \oplus E_8\). The lattices glue.

We have to check uniqueness of the gluings. This is provided by the surjectivity of
\[
O(C_1) \rightarrow O(q_{C_1})
\]
Lemma 6.19. For \((36, 2^6 3^2)\), the characteristic polynomial is
\[
\chi_g = c_{36} c_{18} c_4 c_2 c_1
\]
and the gluings are given by the following diagram. Edges correspond to glue maps between the respective sublattices and they are decorated with the order of the glue.

This determines \((\text{NS}, g|\text{NS})\) uniquely up to isomorphism.

Proof. The possible contributors to the resultant are \(c_9, c_{18}, c_4\) and \(c_{12}\). First the \(2^6\) contribution is coming from either \(c_9\) or \(c_{18}\) dividing \(\chi(g|\text{NS})\). Then there is no room for \(c_{12}\) left. Thus the \(3^2\) contribution is coming from \(c_4\). This leaves us with
\[
\chi_g = c_{36} c_{18} c_4 (x + 1)(x - 1) \quad \text{or} \quad c_{36} c_9 c_4 (x + 1)(x - 1).
\]
Since the principal \(c_4(x)\)-lattice has determinant \(2^2\), we have to glue it over \(2^2\). This determines the characteristic polynomial to be \(c_{36} c_{18} c_4 c_2 c_1\) or \(c_{36} c_9 c_4 c_2 c_1\). At this point we know \(C_{36} \cdot C_4 \cong (-6) \oplus (-6), C_{18} \cong E_6(2)\) and their gluings which exist by [25, Theorem 3.1]. Then
\[
(q_{C_1} c_2)_3 \cong (q_{E_6(2)})_3(-1) \cong (2/3).
\]
The case \(C_1 C_2 = C_1 \oplus C_2\) leads to \(C_2 = (\cdot - 2)\) which is obstructed or \(C_2 = (\cdot - 6)\) which has the wrong 3-glue. Thus we have to glue. Then \(C_1 \cong (4)\) and \(C_2 \cong (\cdot - 12)\) as \(C_1 \cong (\cdot 12)\) has wrong 3-glue. This gluing is unique since \((\mathbb{Z}/4\mathbb{Z})^\times = \{\pm 1\}\). Since \((D_{C_2})_3\) can be glued to \(C_{18}\) but not to \(C_9\), we have
\[
\chi_g = c_{36} c_{18} c_4 c_2 c_1.
\]
The only step at which we have non-trivial freedom in the choice of glue is between \(C_1 C_2\) and \(C_4\). This freedom is due to the action of \(g|D_{C_4}\). Thus it does not affect the isomorphism class of \((C_1 C_2 C_4, g_1 \oplus g_2 \oplus g_4)\) and uniqueness of \((\text{NS}, g)\) up to isomorphism follows. □

The proof of the following Lemmas is similar to what we have already seen. We refer the reader to the authors thesis [11] for the computational details.

Lemma 6.20. For \((36, 3^4)\) the action of \(g|\text{NS}\) is uniquely determined by the following gluing diagram.
Lemma 6.21. \( (21, 7^2), (42, 7^2) \) On the K3 surface \( X_{(21, 7^2)} \) there are 3 (resp. 2) conjugacy classes of purely non-symplectic automorphisms of order 21 (resp. 42). They are distinguished by their invariant lattices

1. \( C_1 \cong U \oplus E_6 \),
2. \( C_1 \cong U \oplus 2A_2 \),
3. \( C_1 \cong U(3) \oplus A_2 \).

Only (1) and (2) are the square of an automorphism of order 42. They

Lemma 6.22. Affine Weierstraß models for \( X_{(21, 7^2)} \) and the automorphisms of order 21 and 42 corresponding to the cases (1), (2) in Lemma 6.21 are given below. For case (3) there is a singular projective model.

1. \( y^2 = x^3 + t^4(7^2 + 1) \), \( (x, y, t) \mapsto (\zeta_7 x, \pm \zeta_7 y, \zeta_7 t) \);
2. \( y^2 = x^3 + t^3(7^2 + 1) \), \( (x, y, t) \mapsto (\zeta_7 x, \pm \zeta_7 y, \zeta_7 t) \);
3. \( x_0^3 + x_1^3 + x_2^3 - x_0 x_1^2 \), \( (x_0, x_1, x_2, x_3) \mapsto (\zeta_7 x_0, \zeta_7 x_1, x_2, \zeta_7 x_3) \).

Proof. One can identify the three cases by computing the fixed lattice of \( f^{14} \). \( \square \)

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