Bender–Dunne Orthogonal Polynomials
And Quasi-Exact Solvability

Agnieszka Krajewska\textsuperscript{1}, Alexander Ushveridze\textsuperscript{2} and
Zbigniew Walczak\textsuperscript{3}
Department of Theoretical Physics, University of Lodz,
Pomorska 149/153, 90-236 Lodz, Poland

Abstract

The paper is devoted to the further study of the remarkable classes of orthogonal polynomials recently discovered by Bender and Dunne. We show that these polynomials can be generated by solutions of arbitrary quasi-exactly solvable problems of quantum mechanics both one-dimensional and multi-dimensional. A general and model-independent method for building and studying Bender-Dunne polynomials is proposed. The method enables one to compute the weight functions for the polynomials and shows that they are the orthogonal polynomials in a discrete variable $E_k$ which takes its values in the set of exactly computable energy levels of the corresponding quasi-exactly solvable model. It is also demonstrated that in an important particular case, the Bender-Dunne polynomials exactly coincide with orthogonal polynomials appearing in Lanczos tridiagonalization procedure.

\textsuperscript{1}akraj@mvii.uni.lodz.pl
\textsuperscript{2}alexush@mvii.uni.lodz.pl
\textsuperscript{3}walczak@mvii.uni.lodz.pl
1 Introduction

Recently Bender and Dunne \[1\] showed that the solution $\psi(x, E)$ of the Schrödinger equation $H \psi(x, E) = E \psi(x, E)$ for the one-dimensional quasi-exactly solvable model with Hamiltonian $H = -\frac{\partial^2}{\partial x^2} + c(c - 1)x^{-2} + (4M + 2c + 3)x^2 + x^6$ is the generating function for the set of polynomials $P_n(E)$ in the energy variable $E$. These polynomials, defined as the coefficients of the expansion of function $x^c \exp(x^4/4) \psi(x, E)$ in powers of $x^2$, satisfy the three-term recursion relations and therefore form an orthogonal set with respect to some weight function $\omega(E)$. Bender and Dunne demonstrated that a) for non-negative integer values of $M$ for which the model $H$ is quasi-exactly solvable, the norms of polynomials $P_n(E)$ vanish for $n \geq M + 1$; b) the zeros of the critical polynomial $P_{M+1}(E)$ coincide with the exactly calculable energy levels in model $H$; c) the polynomials $P_n(E)$ of degrees higher than $M + 1$ factor into a product of two polynomials, one of which is $P_{M+1}(E)$: $P_{M+1+n}(E) = P_{M+1}(E)Q_n(E)$; d) the factor-polynomials $Q_n(E)$ also form an orthogonal set.

These and other remarkable properties of polynomials $P_n(E)$ discussed in paper \[1\] suggest that they can be considered as an alternative mathematical language for describing the phenomenon of quasi-exact solvability. This phenomenon has been discovered several years ago (see e.g. the original papers \[2\] \[3\] \[4\] \[5\] and books \[6\] \[7\]). In their paper, Bender and Dunne did not present any proof of the existence of polynomials $P_n(E)$ with similar properties for other quasi-exactly solvable models. Many important questions, such as the problem of construction of some analogues of Bender – Dunne polynomials for multi-dimensional quasi-exactly solvable models, the problem of the computation of weight functions associated with polynomials $P_n(E)$ and the corresponding factor-polynomials $Q_n(E)$, also remained to be open.

The aim of the present paper is threefold. First of all, we intend to propose a model independent way of introducing Bender – Dunne polynomials for quasi-exactly solvable systems of quantum mechanics. We show that the polynomials $P_n(E)$ (which we call the Bender – Dunne polynomials of genus one) and the corresponding factor-polynomials $Q_n(E)$ (which we call the Bender – Dunne polynomials of genus two) naturally appear as the coefficients of the expansion of solutions $\psi(x, E)$ of Schrödinger equation in the bases in which the Hamiltonian $H$ has a tridiagonal form \[1\]. Second, we present a simple way for building the weight functions $\omega(E)$ and $\rho(E)$ associated with polynomials $P_n(E)$ and $Q_n(E)$ and show that these polynomials are orthogonal polynomials of a discrete variable $E_k$. In the case of $P_n(E)$ polynomials, the variable $E_k$ takes its values in a finite set of the exactly calculable energy levels in model $H$, and, in the case of $Q_n(E)$ polynomials, it runs an infinite set of all the remaining (exactly non-calculable) levels. Third, we demonstrate that the Bender – Dunne polynomials exactly coincide with orthogonal polynomials in an operator variable $H$ arising in the so-called Lanczos tridiagonalization procedure \[8\] \[9\] \[10\]. Since this procedure is applicable to any quasi-exactly solvable models (both one-dimensional and multi-dimensional), this enables one to make a conclusion of the universality of Bender – Dunne construction.

\[\text{Note that the Hamiltonians of all quasi-exactly solvable models are explicitly tridiagonalizable (for details see section 7 – 9)}\]
2 Bender – Dunne polynomials of genus one

Consider a hamiltonian $H$ acting in Hilbert space $W$ in which it has an infinite and discrete spectrum $E_k$, $k = 0, 1, \ldots, \infty$. Let $\phi_n$, $n = 0, 1, \ldots, \infty$ be a certain (not necessarily orthogonal) basis in $W$ in which the hamiltonian takes an explicit tridiagonal form. This means that

$$H \phi_n = A_n \phi_{n-1} + B_n \phi_n + C_n \phi_{n+1}, \quad n = 0, 1, \ldots, \infty \quad (2.1)$$

where $A_n$, $B_n$ and $C_n$ are some algebraically computable coefficients. Note that values of these coefficients depend also on the normalization of basis vectors $\phi_n$.

Assume that there exists a certain non-negative integer $M$ for which $C_M = 0$, $C_n \neq 0$, for $n \neq M$. \quad (2.2)

Then, the model with hamiltonian $H$ becomes quasi-exactly solvable. Indeed, from formulas (2.1) and (2.2) it follows that a linear hull of the basis vectors $\phi_0, \ldots, \phi_M$ forms a $(M + 1)$-dimensional invariant subspace $W_M \subset W$ for the hamiltonian $H$. For this reason, the spectral problem for $H$ in $W$ breaks up into two independent spectral problems, one of which is formulated in the $(M + 1)$-dimensional space $W_M$. This enables one to determine at least $M + 1$ eigenvalues $E_0, E_1, \ldots, E_M$ of hamiltonian $H$ in a purely algebraic way. The remaining eigenvalues $E_{M+1}, E_{M+2}, \ldots$ cannot be determined algebraically and therefore the model under consideration is quasi-exactly solvable (see ref. [7]). Hereafter we shall call the levels $E_0, \ldots, E_M$ exactly calculable and the levels $E_{M+1}, E_{M+2}, \ldots$ exactly non-calculable.

Let us now consider the Schrödinger equation

$$H \psi(E) = E \psi(E) \quad (2.3)$$

in which $E$ is an arbitrary parameter. We look for its solution in the form of the formal expansion

$$\psi(E) = \sum_{n=0}^{\infty} \phi_n P_n(E) \quad (2.4)$$

in which $P_n(E)$ denote certain functions of $E$.

Note that the expansion (2.4) diverges for $E \neq E_k$ and converges for $E = E_k$. If the model is quasi-exactly solvable, this is, if the condition (2.2) is satisfied, then the series (2.4) becomes finite for all $E = E_k$, $k = 0, \ldots, M$. This means that for these values of $E$ the values of functions $P_n(E)$ should vanish for $n \geq M + 1$:

$$P_n(E_k) = 0, \quad n \geq M + 1, \quad k = 0, 1, \ldots, M. \quad (2.5)$$

In order to determine the form of functions $P_n(E)$ it is sufficient to substitute the expansion (2.4) into equation (2.3) and use the tridiagonality condition (2.1). This gives the recurrence relations for $P_n(E)$,

$$EP_n(E) = A_{n+1} P_{n+1}(E) + B_n P_n(E) + C_{n-1} P_{n-1}(E), \quad n = 0, 1, \ldots, \infty, \quad (2.6)$$

which can be supplemented by the initial conditions

$$P_{-1}(E) = 0, \quad P_0(E) = 1. \quad (2.7)$$
From formulas (2.6) and (2.7) it immediately follows that functions $P_n(E)$ are polynomials of degree $n$. From the general theory of orthogonal polynomials (see e.g. [11]) we know that $P_n(E)$ are orthogonal polynomials. This means that there exists a certain weight function $\omega(E)$, which can be normalized as

$$\int \omega(E) dE = 1,$$

(2.8)

for which

$$\int P_n(E) P_m(E) \omega(E) dE = p_n p_m \delta_{nm},$$

(2.9)

where $p_n$ denote the norms of polynomials $P_n(E)$. These norms can be found from the recurrence relations (2.6) after multiplying them by $E^{n-1}\omega(E)$ and taking integral over $E$. The result is

$$p_n = \left| \prod_{m=1}^{n} \frac{C_{m-1}}{A_m} \right|^{1/2}.$$  

(2.10)

If the model is quasi-exactly solvable, then from the condition (2.2) and formula (2.10) it follows that

$$p_n = 0, \quad n \geq M + 1,$$

(2.11)

i.e. the norms of all polynomials $P_n(E)$ with $n \geq M + 1$ vanish. Hereafter, we shall call the polynomials $P_n(E)$ the Bender – Dunne polynomials of genus one.

### 3 Factorization property

The $P_n(E)$ polynomials of degrees $n \geq M + 1$ have a remarkable factorization property first noted by Bender and Dunne [1]. Indeed, from formula (2.5) it follows that the $M + 1$ exactly calculable eigenvalues $E_0, E_1, \ldots, E_M$ of hamiltonian $H$ are the roots of these polynomials. This enables one to write

$$P_{M+1}(E) \sim \prod_{n=0}^{M} (E - E_n)$$

(3.1)

and

$$P_{M+1+n}(E) = P_{M+1}(E) Q_n(E),$$

(3.2)

where $Q_n(E)$ are certain polynomials of degrees $n = 0, 1, \ldots, \infty$.

The first formula (3.1) gives a practical way of determining the exactly calculable eigenvalues of the hamiltonian $H$ from the equation

$$P_{M+1}(E) = 0.$$  

(3.3)

The second formula (3.2) expresses an important factorizability property of the polynomials $P_n(E)$ with $n \geq M + 1$. The factor-polynomials $Q_n(E)$ we shall call the Bender – Dunne polynomials of genus two.
4 Bender – Dunne polynomials of genus two

Substituting (3.2) into (2.6), one can obtain the recurrence relations immediately for the factor-polynomials $Q_n(E)$:

$$EQ_n(E) = A_{M+2+n}Q_{n+1}(E) + B_{M+1+n}Q_n(E) + C_{M+n}Q_{n-1}(E), \quad n = 0, 1, \ldots, \infty. \quad (4.1)$$

Since $C_M = 0$, it is sufficient to supplement the relations (4.1) by the initial conditions

$$Q_0(E) = 1. \quad (4.2)$$

We see that $Q_n(E)$ are again the orthogonal polynomials. Denoting the corresponding weight function by $\rho(E)$ and normalizing it as

$$\int \rho(E) dE = 1, \quad (4.3)$$

we can write

$$\int Q_n(E)Q_m(E)\rho(E) dE = q_n q_m \delta_{nm}. \quad (4.4)$$

The norms of polynomials $Q_n(E)$ computed from the recurrence relations (4.1) are

$$q_n = \prod_{m=1}^{n} \frac{C_{M+m}}{A_{M+m+1}} \frac{1}{2}. \quad (4.5)$$

According to (2.2), the norms of $Q_n(E)$ polynomials do not vanish.

5 Weight functions

In this section we compute the weight functions $\omega(E)$ and $\rho(E)$ for Bender – Dunne orthogonal polynomials of genus one and two.

**Weight function for** $P_n(E)$ **polynomials.** From formulas (2.7) – (2.9) it follows that

$$\int P_n(E)\omega(E) dE = \delta_{n0}. \quad (5.1)$$

Multiplying the relation (2.4) by $\omega(E)$, taking the integral over $E$ and using (5.1), we obtain

$$\int \psi(E)\omega(E) dE = \phi_0. \quad (5.2)$$

The basis element $\phi_0$ can obviously be represented as a linear combination of eigenvectors of hamiltonian $H$ corresponding to exactly calculable energy levels:

$$\phi_0 = \sum_{k=0}^{M} \omega_k \psi(E_k). \quad (5.3)$$

Substituting (5.3) into (5.2), we obtain

$$\int \psi(E')\omega(E') dE' = \sum_{k=0}^{M} \omega_k \psi(E_k). \quad (5.4)$$
Let us now act on both hand sides of the relation (5.4) by the operator function $\delta(H - E)$. Using (2.3), we obtain

$$
\psi(E)\omega(E) = \sum_{k=0}^{M} \omega_k \delta(E - E_k) \psi(E_k).
$$

(5.5)

Dividing both hand sides of (5.5) by $\psi(E)$ and taking into account the projection property of delta-function, we obtain the final expression for the weight function $\omega(E)$:

$$
\omega(E) = \sum_{k=0}^{M} \omega_k \delta(E - E_k).
$$

(5.6)

Thus, we have arrived at interesting result: the Bender – Dunne polynomials of genus one are orthogonal polynomials in a discrete variable $E_k$, $k = 0, 1, \ldots, M$ which takes its values in the finite set of the exactly calculable eigenvalues of hamiltonian $H$. The orthogonality property for these polynomials can now be written in the form

$$
\sum_{k=0}^{M} \omega_k P_n(E_k) P_m(E_k) = \delta_{nm}.
$$

(5.7)

The coefficients $\omega_k$ can be computed algebraically. Indeed, substituting into (5.3) the expansions (2.4), we obtain the system of algebraic equations for numbers $\omega_k$:

$$
\sum_{k=0}^{M} P_n(E_k) \omega_k = \delta_{n0}, \quad n = 0, 1, \ldots, M.
$$

(5.8)

The numerical analysis of this system for various models shows that the numbers $\omega_k$, playing the role of a discrete weight function are always positive (see e.g. section 6). It would be an interesting task to proof this in general case.

**Weight function for $Q_n(E)$ polynomials.** Using formulas (4.2) – (4.4) we obtain

$$
\int Q_n(E) \rho(E) dE = \delta_{n0}.
$$

(5.9)

Using the factorization property (3.2), let us represent the expression (2.4) in the form

$$
\psi(E) = \sum_{n=0}^{M} \phi_n P_n(E) + P_{M+1}(E) \sum_{n=0}^{\infty} \phi_{M+1+n} Q_n(E).
$$

(5.10)

Multiplying (5.10) by $\rho(E) P_{M+1}^{-1}(E)$, taking the integral over $E$ and using (5.9), we obtain

$$
\int \frac{\psi(E')}{P_{M+1}(E')} \rho(E') dE' = \sum_{m=0}^{M} \phi_m \int \frac{P_m(E')}{P_{M+1}(E')} \rho(E') dE' + \phi_{M+1}.
$$

(5.11)

Expressing vectors $\phi_n$, $n = 0, 1, \ldots, M$ and $\phi_{M+1}$ via the orthonormalized eigenvectors of hamiltonian $H$, we can rewrite (5.11) as

$$
\int \frac{\psi(E')}{P_{M+1}(E')} \rho(E') dE' = \sum_{k=0}^{M} \mu_k \psi(E_k) + \sum_{k=M+1}^{\infty} \nu_k \psi(E_k),
$$

(5.12)
where \( \mu_k \) and \( \nu_k \) are certain, in general, algebraically non-computable coefficients. Note that
\[
\nu_k = \langle \phi_{M+1}, \psi(E_k) \rangle, \quad k = M + 1, \ldots, \infty,
\]  
(5.13)
where \( \langle , \rangle \) denotes the scalar product in Hilbert space \( W \). Let us now act on both hand sides of the relation (5.12) by the operator function \( \delta(H - E) \). Using (2.3), we obtain
\[
\frac{\psi(E)}{P_{M+1}(E)} \rho(E) = \sum_{k=0}^{M} \mu_k \delta(E - E_k) \psi(E_k) + \sum_{k=M+1}^{\infty} \nu_k \delta(E - E_k) \psi(E_k).
\]  
(5.14)
Dividing both hand sides of (5.14) by \( \psi(E) P_{M+1}(E) \) and taking into account the projection property of delta-function and formula (2.5), we obtain the final expression for the weight function \( \rho(E) \):
\[
\rho(E) = \sum_{k=M+1}^{\infty} \rho_k \delta(E - E_k),
\]  
(5.15)
where
\[
\rho_k = P_{M+1}(E_k) \nu_k = P_{M+1}(E_k) \langle \phi_{M+1}, \psi(E_k) \rangle, \quad k = M + 1, \ldots, \infty.
\]  
(5.16)
Thus, we see that the Bender – Dunne polynomials of genus two are the orthogonal polynomials in a discrete variable \( E_k \), \( k = M + 1, \ldots, \infty \) whose values are nothing else than the exactly non-calculable eigenvalues of hamiltonian \( H \). The orthogonality property for these polynomials can now be written in the form
\[
\sum_{k=M+1}^{\infty} \rho_k Q_n(E_k) Q_m(E_k) = q_n q_m \delta_{nm}.
\]  
(5.17)
Note that the numbers \( \rho_k \) cannot be found by means of algebraic methods.

6 An example

In this section we construct the discrete weight function \( \omega_k \) for Bender – Dunne polynomials of genus one associated with the simplest quasi-exactly solvable models with polynomial potentials. The hamiltonians of these models, which were first discussed in ref. [3], have the form
\[
H = -\frac{\partial^2}{\partial x^2} + x^6 + 2bx^4 + [b^2 - 4M - 3]x^2,
\]  
(6.1)
and for any given non-negative integer \( M \) admit \( M + 1 \) explicit solutions corresponding to energy levels \( E_0, E_1, \ldots, E_M \) describing the first \( M + 1 \) even states with ordinal numbers 0, 2, \ldots, 2M. It is easily seen that in the basis
\[
\phi_n(x) = \frac{(-x^2)^n}{(2n)!} \exp \left[ -\frac{x^4}{4} - \frac{bx^2}{2} \right],
\]  
(6.2)
the hamiltonian \( H \) takes an explicit tridiagonal form. The corresponding coefficients \( A_n, B_n \) and \( C_n \) are
\[
A_n = 1, \quad B_n = b(4n + 1), \quad C_n = 8a(n + 1)(2n + 1)(n - M).
\]  
(6.3)
The general formulas of sections 2 – 5 allow one to reconstruct the polynomials $P_n(E)$ and compute the corresponding discrete weight functions. Below we consider two examples of such calculations.

**Example 1.** Let us take $b = 0$ and $M = 8$. Then Bender – Dunne polynomials of genus one reconstructed from recurrence relations read

$$
\begin{align*}
P_0(E) &= 1, \\
P_1(E) &= E, \\
P_2(E) &= -64 + E^2, \\
P_3(E) &= -400 E + E^3, \\
P_4(E) &= 46080 - 1120 E^2 + E^4, \\
P_5(E) &= 494080 E - 2240 E^3 + E^5, \\
P_6(E) &= -66355200 + 2106880 E^2 - 3680 E^4 + E^6, \\
P_7(E) &= -848977920 E + 5655040 E^3 - 5264 E^5 + E^7, \\
P_8(E) &= 96613171200 - 3916595200 E^2 + 11013120 E^4 - 6720 E^6 + E^8, \\
P_9(E) &= 911631974400 E - 9345433600 E^3 + 16066560 E^5 - 7680 E^7 + E^9. 
\end{align*}
$$

The energy levels coinciding with the roots of the last polynomial $P_9(E)$ are

$$
\begin{align*}
E_0 &= -68.1568, \\
E_1 &= -46.5609, \\
E_2 &= -27.2997, \\
E_3 &= -11.021, \\
E_4 &= 0, \\
E_5 &= 11.021, \\
E_6 &= 27.2997, \\
E_7 &= 46.5609, \\
E_8 &= 68.1568, 
\end{align*}
$$

and the components of the corresponding weight function $\omega_k$ computed by means of formula (5.8) are

$$
\begin{align*}
\omega_0 &= 0.0000117487, \\
\omega_1 &= 0.00058638, \\
\omega_2 &= 0.013045, \\
\omega_3 &= 0.172498, \\
\omega_4 &= 0.627717, \\
\omega_5 &= 0.172498, \\
\omega_6 &= 0.013045, \\
\omega_7 &= 0.00058638, \\
\omega_8 &= 0.0000117487.
\end{align*}
$$

We see that all these numbers are positive. Note also that these numbers are distributed symmetrically with respect to the reflection $k \to M + 1 - k$, while the energy levels are
distributed anti-symmetrically. This is a trivial consequence of the self duality property of the model (6.1) with $b = 0$ noted in ref. [12].

**Example 2.** Let us now take $b = 3$ and $M = 8$. Then Bender – Dunne polynomials of genus one reconstructed from recurrence relations read

\[
\begin{align*}
P_0(E) &= 1, \\
P_1(E) &= -3 + E, \\
P_2(E) &= -19 - 18 E + E^2, \\
P_3(E) &= 1521 + 131 E - 45 E^2 + E^3, \\
P_4(E) &= -45639 + 9372 E + 1166 E^2 - 84 E^3 + E^4, \\
P_5(E) &= 624069 - 670331 E + 306 E^2 + 4330 E^3 - 135 E^4 + E^5, \\
P_6(E) &= 26403813 + 29359242 E - 2368649 E^2 - 151524 E^3 + 11395 E^4 + \\
&
-198 E^5 + E^6, \\
P_7(E) &= -2968811271 - 1113735033 E + 206523213 E^2 + 2136931 E^3 + \\
&-792309 E^4 + 24661 E^5 - 273 E^6 + E^7, \\
P_8(E) &= 219842628849 + 51179080248 E - 15632501620 E^2 + 241229160 E^3 + \\
&+54476694 E^4 - 2649528 E^5 + 46956 E^6 - 360 E^7 + E^8, \\
P_9(E) &= -18914361435891 - 3777700684023 E + 1400534456148 E^2 + \\
&-41565642220 E^3 - 4391346906 E^4 + 293105406 E^5 - 7036092 E^6 + \\
&+81636 E^7 - 459 E^8 + E^9. \\
\end{align*}
\] (6.6)

For the energy levels we have

\[
\begin{align*}
E_0 &= -14.4717, \\
E_1 &= -2.52391, \\
E_2 &= 6.59852, \\
E_3 &= 21.606, \\
E_4 &= 40.6664, \\
E_5 &= 62.6983, \\
E_6 &= 87.2765, \\
E_7 &= 114.122, \\
E_8 &= 143.028, \\
\end{align*}
\] (6.7)

and the components of the weight function are

\[
\begin{align*}
\omega_0 &= 0.026295, \\
\omega_1 &= 0.475485, \\
\omega_2 &= 0.423177, \\
\omega_3 &= 0.067025, \\
\omega_4 &= 0.00741457, \\
\omega_5 &= 0.000573463, \\
\omega_6 &= 0.0000291886, \\
\end{align*}
\]
\[ \omega_7 = 8.80537 \times 10^{-7}, \]
\[ \omega_8 = 1.1954 \times 10^{-8}. \] (6.8)

We see that the discrete weight function is again positive. At the same time, we have no symmetry of its components and no anti-symmetry of the energy levels, because for \( b \neq 0 \) the model (6.1) is not self-dual \([2]\).

7 One-dimensional quasi-exactly solvable models

The hamiltonians \( H = -\partial^2/\partial x^2 + V(x) \) of one-dimensional quasi-exactly solvable models introduced in paper \([3]\) are distinguished by the fact that for them it is possible to find two functions \( \lambda(x) \) and \( \rho(x) \) for which the ansatz
\[ \psi(x, E) = f(\lambda(x), E)\rho(x) \] (7.1)
reduces the corresponding Schrödinger equation \( H\psi(x, E) = E\psi(x, E) \) to the form
\[ \left[ R_3(\lambda) \frac{\partial^2}{\partial \lambda^2} + R_2(\lambda) \frac{\partial}{\partial \lambda} + R_1(\lambda) \right] f(\lambda, E) = Ef(\lambda, E) \] (7.2)
with some polynomials \( R_1(\lambda), R_2(\lambda) \) and \( R_3(\lambda) \) of degrees \( m_1 = 1, m_2 = 2 \) and \( m_3 \leq 3 \), respectively. The phenomenon of quasi-exact solvability appears when
\[ M(M - 1) \lim_{\lambda \to \infty} \lambda^{-3} R_3(\lambda) + M \lim_{\lambda \to \infty} \lambda^{-2} R_2(\lambda) + \lim_{\lambda \to \infty} \lambda^{-1} R_1(\lambda) = 0, \] (7.3)
where \( M \) is a certain non-negative integer. It is easily seen that, for any \( M \), there exist \( M + 1 \) values of \( E = E_k \) for which the equation (7.3) has solutions \( f(\lambda, E) = f(\lambda, E_k) \) represented by certain polynomials of degree \( M \). From (7.3) it immediately follows that in the basis
\[ \phi_n(x) = [\lambda(x) - r]^n \rho(x), \quad n = 0, 1, \ldots, \infty \] (7.4)
in which \( r \) denotes any of the roots of the polynomial \( R_3(\lambda) \), the hamiltonian of the model (irrespective of whether the condition of quasi-exact solvability is satisfied or not) takes an explicit tridiagonal form (7.4) with trivially computable coefficients \( A_n, B_n \) and \( C_n \). Indeed, introducing the new variable \( t = \lambda - r \) and taking \( f(\lambda, E) \equiv g(t, E) \), we obtain instead of (7.2):
\[ \left[ (a_3 t^3 + b_3 t^2 + c_3 t) \frac{\partial^2}{\partial t^2} + (a_2 t^2 + b_2 t + c_2) \frac{\partial}{\partial t} + (a_1 t + b_1) \right] g(t, E) = Eg(t, E). \] (7.5)

It is easily seen that the operator \( L \) standing in the left-hand of the equation (7.5) is a linear combination of three operators \( L^+, L^0, L^- \) acting on the monomials \( t^n \) as \( L^+ t^n \sim t^{n+1}, L^0 t^n \sim t^n \) and \( L^- t^n \sim t^{n-1} \). But this means that this \( L \)-operator is tridiagonal in the basis \( t^n, n = 0, \ldots, \infty \). This, in turn, results in the tridiagonality of \( H \) in basis (7.4). According to general prescriptions given above, this means that any one-dimensional quasi-exactly solvable model from the list given in ref. \([4]\) trivially generates \( m_3 \leq 3 \) Bender – Dunne polynomials of genus one and two. An explicit construction of these polynomials and computation of the corresponding weight functions will be given in a separate publication.
8 Lanczos tridiagonalization procedure

There are infinitely many ways of reducing a given hamiltonian $H$ to a tridiagonal form. One of the simplest ways is based on the use of the so-called Lanczos tridiagonalization procedure (see e.g. refs.[8, 9, 10]). In terms of the Lanczos procedure, the properties of Bender – Dunne polynomials become especially transparent and simple.

Let $H$ be a hamiltonian and $\phi \in W$ be an arbitrary vector of Hilbert space. Consider the sequence of vectors

$$\phi_n = \bar{P}_n(H)\phi, \quad n = 0, 1, \ldots, \infty,$$

(8.1)
in which $\bar{P}_n(H)$ denote certain polynomials in the operator variable $H$. Let us choose these polynomials from the condition of orthonormalizability of vectors

$$\langle \phi_n, \phi_m \rangle = \delta_{nm}$$

(8.2)

with respect to the scalar product in the space $W$. If the hamiltonian is given and the scalar product is explicitly defined, then the orthogonalization is a purely algebraic procedure. The sequence $\phi_n$ can be considered as an orthogonal basis in Hilbert space. Let $\psi(E_k)$ be the orthonormalized eigenvectors of the hamiltonian $H$,

$$\langle \psi(E_k), \psi(E_l) \rangle = \delta_{kl}.$$ 

(8.3)

Expanding the vector $\phi$ in eigenvectors $\psi(E_k)$,

$$\phi = \sum_{k=0}^{\infty} c_k \psi(E_k),$$

(8.4)

and substituting this expansion into (8.1) we obtain

$$\phi_n = \sum_{k=0}^{\infty} c_k \bar{P}_n(E_k) \psi(E_k).$$

(8.5)

The substitution of (8.5) into the orthonormalizability condition (8.2) gives the relation

$$\sum_{k=0}^{\infty} c_k^2 \bar{P}_n(E_k) \bar{P}_m(E_k) = \delta_{nm},$$

(8.6)

which shows that $\bar{P}_n(E)$ are orthogonal polynomials in a discrete variable $E_k$, $k = 0, 1, \ldots, \infty$, which takes its values in the set of eigenvalues of hamiltonian $H$ (see ref.[10]). The corresponding discrete weight function $c_k^2$ is obviously positive. Hereafter, we shall call $\bar{P}_n(H)$ the Lanczos polynomials. Because of the orthogonality, the polynomials $\bar{P}_n(H)$ obey the recurrence relation

$$H \bar{P}_n(H) = A_n \bar{P}_{n-1}(H) + B_n \bar{P}_n(H) + C_n \bar{P}_{n+1}(H), \quad n = 0, 1, \ldots, \infty,$$

(8.7)
in which $A_n$, $B_n$ and $C_n$ are certain algebraically computable coefficients. Acting by the operator equality (8.7) on $\phi$ we obtain

$$H \phi_n = A_n \phi_{n-1} + B_n \phi_n + C_n \phi_{n+1}, \quad n = 0, 1, \ldots, \infty,$$

(8.8)
which means that the operator $H$ is tridiagonal in the basis $\phi_n$, $n = 0,1,\ldots,\infty$. Since $H$ is a hermitian operator and the basis $\phi_n$ is orthogonal, the upper and lower diagonals of $H$ should coincide. This gives us the conditions

$$C_n = A_{n+1}$$

(8.9)

which reduce the recurrence relations (8.7) to the form

$$H \bar{P}_n(H) = C_{n-1} \bar{P}_{n-1}(H) + B_n \bar{P}_n(H) + C_n \bar{P}_{n+1}(H), \quad n = 0,1,\ldots,\infty.$$  

(8.10)

Let us now consider the equation (2.3) for the hamiltonian $H$ lying outside the space $W$. This procedure enables one to determine the polynomials $P_n(H)$ associated with the expansion (8.8) satisfy the recurrence relation

$$EP_n(E) = C_{n-1} P_{n-1}(E) + B_n P_n(E) + C_n P_{n+1}(E), \quad n = 0,1,\ldots,\infty.$$  

(8.11)

From the coincidence of the recurrence relations (8.10) and (8.11) it follows that the Bender – Dunne polynomials $P_n$ are nothing else that the Lanczos polynomials $\bar{P}_n(E)$. Hereafter, we shall not distinguish between the Lanczos and Bender – Dunne polynomials and for both of them use the same notation $P_n(E)$.

Up to now, we implicitly assumed that all the coefficients $c_k$ in the expansion (8.4) differ from zero. Assume now that this expansion has only a finite number of terms

$$\phi = \sum_{k=0}^M c_k \psi(E_k).$$  

(8.12)

Let us denote by $W_M$ the linear span of functions $\psi(E_k)$, $k = 0,1,\ldots,M$. The space $W_M$ is a $(M+1)$-dimensional invariant subspace for the hamiltonian $H$. It is quite obvious that, in this case, the Lanczos procedure becomes finite and consists only of $M+1$ essential steps. Indeed, assume that the Lanczos polynomials $P_n(E)$ with $0 \leq n \leq M$ are already known and we want to construct the next polynomials $P_{M+1+n}(H)$. According to general prescriptions of Lanczos theory, the vectors $\phi_{M+1+n}$ should have the form $\phi_{M+1+n} = P_{M+1+n}(H)\phi$ and thus should belong to the space $W_M$. On the other hand, they should be orthogonal to all the linearly independent basis vectors $\phi_0, \phi_1,\ldots,\phi_M$ of the space $W_M$, which means that $\phi_{M+1+n} = P_{M+1+n}(H)\phi = 0$. This is possible only if $P_{M+1+n}(H) = \prod_{k=0}^M (H - E_k)Q_n(H)$, where $E_0, E_1,\ldots,E_M$ are the eigenvalues of the hamiltonian $H$ in the space $W_M$ and $Q_n(H)$ are the arbitrary polynomials of degrees $n = 0,1,\ldots,\infty$. The fact that the functions $\phi_{M+1+n}$ are non-normalizable, prevents one from determining the polynomials $Q_n(H)$ uniquely.

The only way to construct the polynomials $Q_n(H)$ is to choose a new vector

$$\phi_{M+1} = P_{M+1}(H) \sum_{k=M+1}^\infty c_k \psi(E_k)$$  

(8.13)

lying outside the space $W_M$ and apply to it the Lanczos tridiagonalization procedure. This vector is, obviously, normalizable, and starting with it, we can construct the set of orthonormalized vectors $\phi_{M+1+n} = Q_n(H)\phi_{M+1}$ belonging to an orthogonal complement of the space $W_M$. This procedure enables one to determine the polynomials $Q_n(H)$ uniquely.

It is easily seen that the orthonormalizability condition for functions $\phi_n$, $n = 0,1,\ldots,M$ reads

$$\sum_{k=0}^M c_k^2 P_n(E_k)P_m(E_k) = \delta_{nm}$$  

(8.14)
and is nothing else than the orthonormalizability condition for Bender–Dunne polynomials of genus one. Note the positive definiteness of the corresponding weight function \( \omega_k = c_k^2 \). At the same time, the orthonormalizability condition for functions \( \phi_{M+1+n}, \ n = 0, 1, \ldots, \infty \) takes the form

\[
\sum_{k=0}^{M} c_k^2 P_{M+1}^2(E_k)Q_n(E_k)Q_m(E_k) = \delta_{nm} \quad (8.15)
\]

and becomes that for Bender–Dunne polynomials of genus two. The corresponding weight function \( \rho_k = c_k^2 P_{M+1}^2(E_k) \) is also positively defined.

9 Conclusion

In conclusion note that the Lanczos tridiagonalization procedure can be applied to any hermitian operators in Hilbert space. This means that \( H \) may be the Hamiltonian of an arbitrary one or multi-dimensional model. However, if we want to construct the Bender–Dunne polynomials of genus one, one should take care that the trial function \( \phi \) belongs to a finite-dimensional invariant subspace \( W_M \). This can always be done for quasi-exactly solvable models of quantum mechanics. Indeed, consider a quantum model whose Hamiltonian can be represented in the form

\[
H = \sum_{\alpha,\beta} C_{\alpha\beta} J_\alpha(h)J_\beta(h) + \sum_\alpha C_\alpha J_\alpha(h) \quad (9.1)
\]

where \( C_{\alpha\beta} \) and \( C_\alpha \) are some numbers and \( J_\alpha(h) \) are generators of a certain Lie algebra realising a representation with highest weight \( h \) and having the form of first-order differential operators. In this case, it is natural to identify the vector \( \phi \) with the highest weight vector \( |h\rangle \). If the representation is infinite-dimensional and irreducible, then all the terms of the Lanczos sequence \( \phi_n = P_n(H)|h\rangle \) are linearly independent and this sequence is infinite. Otherwise, the operator \( H \) would have a finite-dimensional invariant subspace, which would contradict the condition of irreducibility of the representation. Assume now that the representation is finite-dimensional, so that the model (9.1) is quasi-exactly solvable. Since the highest weight vector \( |h\rangle \) belongs, by definition, to the representation space, the Lanczos sequence will contain in this case only a finite number of linearly independent terms. According to general reasonings given above this naturally leads us to Bender–Dunne polynomials of genus one.

References

[1] C.M. Bender and G.V. Dunne, Preprint hep-th/9511138, 20 November, 1995
[2] O.V. Zaslavsky and V.V. Ulyanov, Sov. Phys. - JETP 60 991 (1984)
[3] A.V. Turbiner and A.G. Ushveridze, Phys. Lett. 126A, 181 (1987)
[4] A.G. Ushveridze, Sov. Phys. - Lebedev Inst. Rep. 2 50, 54 (1988)
[5] A.V. Turbiner, Comm. Math. Phys. 118, 467 (1988)
[6] N. Kamran and P. Olver (eds), "Lie Algebras, Cohomologies and New Findings in Quantum Mechanics", Contemporary Mathematics, AMS 160 (1994)

[7] A.G. Ushveridze, "Quasi-Exactly Solvable Problems in Quantum Mechanics", IOP publishing, Bristol (1994)

[8] C. Lanczos, J. Res. NBS, 45, 255 (1950)

[9] J.H. Wilkinson, "The Algebraic Eigenvalue Problem", Clarendon, Oxford (1965)

[10] A.G. Ushveridze, J. Phys. A, 20, 5145 (1987)

[11] G. Szegö, "Orthogonal Polynomials", AMS, New York (1939)

[12] A. Krajewska, A. Ushveridze and Z. Walczak, Preprint hep-th 9601025, 8 January (1996)