Quantization of the Linearized Kepler Problem

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Abstract.

The linearized Kepler problem is considered, as obtained from the Kustaanheimo-Stiefel (K-S) transformation, both for negative and positive energies. The symmetry group for the Kepler problem turns out to be SU(2, 2). For negative energies, the Hamiltonian of Kepler problem can be realized as the sum of the energies of four harmonic oscillator with the same frequency, with a certain constrains. For positive energies, it can be realized as the sum of the energies of four repulsive oscillator with the same (imaginary) frequency, with the same constrains. The quantization for the two cases, negative and positive energies is considered, using group theoretical techniques and constrains. The case of zero energy is also discussed.

1. KS Regularization of the Kepler problem.

In this work we affront the task of the quantization of the Kepler problem, given by the Hamiltonian defined on $\mathbb{R}^3 \times \mathbb{R}^3$, $\mathcal{H} = \frac{\mathbf{p}^2}{2m} - \frac{1}{r}$, where $r = \sqrt{x^2 + y^2}$, $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$. For this purpose we shall use the linearization provided by the KS regularization, as introduced by P. Kustaanheimo and E. Stiefel, in the spinorial version due to Jost (see [1]). The KS transformation regularizes the Kepler problem and linearizes it, showing that the dynamical group of the Kepler problem is SU(2, 2). The linearization means that the Kepler problem, for the case of negative energy, can be seen as a system of 4 harmonic oscillators in resonance subject to constrains. For the case of positive energy, it turns to be a system of 4 repulsive harmonic oscillators in resonance subject to constrains. Finally, as we shall see, the singular case of zero energy can be expressed as 4 free particles subject to constrains.

The key point in the KS transformation is the commutativity of the diagram (see [1]):

$$
\begin{array}{ccc}
(z, w) & \xrightarrow{\pi^{-1}} & (\eta, \zeta) \\
\downarrow \pi & & \downarrow \pi^{-1} \\
(x, y) & \xrightarrow{\nu^{-1}} & (q, p) \in T^3 S^3
\end{array}
$$

(1)

In this diagram $\nu$ is Moser transformation (see [1]), which allows us to see $T^3 S^3$ as an embedded manifold in $\mathbb{R}^3 \times \mathbb{R}^3$, where $T^3 S^3 = \{ (q, p) \in \mathbb{R}^3, ||q|| = 1, <p, q> = 0, p \neq 0 \}$, is named Kepler manifold.

The KS transformation is the map $\pi$, which can be seen as a symplectic lift of the Hopf fibration, $\pi_0 : \mathbb{C}_0^3 \to \mathbb{R}_0^3, z = (z_1, z_2) \mapsto \pi_0(z) := z, \bar{\sigma}z >, (\bar{\sigma} \text{ are Pauli matrices})$:

$$
\pi : T^* \mathbb{C}_0^3 \to T^* \mathbb{R}_0^3, (z, w) \mapsto (\bar{x} = \pi_0(z), \bar{y} = \text{Im} <w, \bar{\sigma}z > / <z, z>)
$$

such that, $\pi^* \theta_{\mathbb{R}_0^3} = \theta_{\mathbb{C}_0^3}(I^{-1}(0))' = 2\text{Im} <w, dz> (= \theta_{\mathbb{R}_0^3} = \text{Im} <\eta, d\eta > = <\zeta, d\zeta >)$

up to a total differential) and $\theta_{\mathbb{R}_0^3}$ is the canonical potential form restricted to $\mathbb{R}_0^3$. The map

$$
\mathcal{C} = \frac{1}{\sqrt{2}} \begin{pmatrix}
\sigma_0 & \sigma_0 \\
\sigma_0 & -\sigma_0
\end{pmatrix}
$$

provides the injection of collision states. The function $I = \frac{1}{2}(<\eta, \eta> - <\zeta, \zeta>)$ defines the regularized space $I^{-1}(0)$ ($(I^{-1}(0))'$ doesn’t contain collision states), which is diffeomorphic to $\mathbb{C}_0^3 \times S^3$ while $I^{-1}(0)/U(1)$ is diffeomorphic to $\mathbb{R}_0^3 \times S^3$. 


The transformation, $\vec{x} = \frac{1}{\sqrt{mk}} \vec{\gamma}$, $\vec{y} = k \sqrt{m} \vec{\gamma}$, with $\rho = \sqrt{\vec{x}^2}$, relates the variables in the Kepler problem to the variables used in $\nu$. The map $\hat{\nu}$ is a symplectomorphism between $I^{-1}(0)/U(1)$ and $T^+ S^3$, with the symplectic structures restricted to the corresponding spaces. The Kepler Hamiltonian for negative energy is associated with the momentum map associated with its action on the Kepler problem to the variables used in $I$.

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We can proceed in the same manner for the case of positive energies changing the Kepler Hamiltonian is associated with its action on the Kepler problem to the variables used in $I$ when acting on wave functions are given by $|\xi, \theta \rangle = (\sum_{n=0}^{\infty} \frac{C_n}{\sqrt{n!}} |n\rangle)$. The potential 1-form $\theta_{\eta, \zeta}$ is left invariant by the Lie group $U(2, 2)$, which also leaves invariant the constrain $I$ when acting on $\mathbb{C}^4_0$. This is thus the dynamical group for the Kepler problem. A convenient basis for the Lie algebra $u(2, 2)$ is given by the components of the momentum map associated with its action on $\mathbb{C}^4_0$ (here $I$ is central):

$$I, \quad J, \quad \hat{\mathcal{M}} = -\frac{1}{2} <\eta, \hat{\sigma} \eta>, \quad \hat{\mathcal{N}} = \frac{1}{2} <\zeta, \hat{\sigma} \zeta>, \quad Q = (-Im <\eta, \zeta>, Re <\eta, \hat{\sigma} \zeta>, \quad P = (Re <\eta, \zeta>, Im <\eta, \hat{\sigma} \zeta>).$$

Table I: KS regularization with physical constants

| $\vec{X}$ | $\vec{Y}$ |
|-----------|-----------|
| $\frac{\tilde{x}}{\sqrt{m}} (\vec{P} - \vec{R})$, $\theta_{\eta, \zeta}$ | $k \sqrt{m} \vec{P}$ |

$$\mathcal{H} = \frac{\tilde{x}^2}{2m} - \frac{\gamma}{||\vec{x}||} = \frac{k}{2(||\vec{P}|| + \hat{P}_0)} (k(||\vec{P}|| - \hat{P}_0) - 2\sqrt{m}), \quad \mathcal{A}\mathcal{M} = \hat{L} = \vec{X} \times \vec{Y} = \hat{M} + \hat{N},$$

$$\vec{R}' = \hat{M} - \hat{N}, \quad \mathcal{R} \mathcal{L} = \frac{\tilde{x} \times \tilde{E}}{m} - \frac{\gamma}{||\vec{x}||} = \frac{\sqrt{||\vec{x}||}}{\sqrt{m}} (kP_0 + \gamma \sqrt{m}) + \tilde{Q}(||\vec{P}|| - \gamma \sqrt{m}).$$

2. Quantization of Kepler problem: $E < 0$.

The KS transformation reveals that the Kepler problem for negative energies can be seen as the Hamiltonian system $(\mathbb{C}^4, \theta_{\eta, \zeta}, \mathcal{J})$ restricted to $I^{-1}(0)$. Defining $\mathbb{C} = (\mathbb{C}_1, \mathbb{C}_2) = (\eta, \zeta^+)$. The Hamiltonian $\mathcal{J}$ adopts the form $\mathcal{H}_{\text{bar}} = \omega \mathbb{C} \mathbb{C}^+$ which corresponds to four harmonic oscillators. The quantization of this system can be obtained from the group law of the corresponding symmetry group (a central extension of it by $U(1)$, rather, see [3]):

$$\lambda'' = \lambda' + \lambda, \quad \mathbb{C}' = \mathbb{C}' e^{-i\lambda} + \mathbb{C}, \quad \mathbb{C}'' + \mathbb{C}' = \mathbb{C}' + e^{i\lambda} + \mathbb{C} + \mathbb{C}^+,$$

$$\zeta'' = \zeta' \exp \left[ \frac{1}{2} \left( i \mathbb{C}' \mathbb{C}^{-i\lambda} - i \mathbb{C}' + \mathbb{C} e^{i\lambda} \right) \right].$$

(4)

where $\mathbb{C}, \mathbb{C}^+ \in \mathbb{C}^4$, $\zeta \in U(1)$ and $\lambda = \omega t \in \mathbb{R}$. We can obtain the quantum version of this system using any geometrical (like Geometric Quantization, see [2]) or group-theoretical method, like Group Approach to Quantization (GAQ, see [3]), the one used here.

The resulting wave functions (defined on the group) are $\psi = \zeta e^{-\frac{i}{2} \mathbb{C} \mathbb{C}^+} \phi(\mathbb{C}^+, \lambda)$, and Schrödinger equation for this system is $i \frac{d\phi}{dx} = i [\mathbb{C}^+, \frac{d\phi}{d\mathbb{C}^+}]$. In this formalism, quantum operators are constructed from the right-invariant vector fields on the group [4], and in this case creation and annihilation operators are given by $\hat{\mathbb{C}}^+ = X_{\mathbb{C}}^R$ and $\hat{\mathbb{C}} = X_{\mathbb{C}^+}^R$, respectively. Since the momentum map [3] is expressed as quadratic functions on $\mathbb{C}$ and $\mathbb{C}^+$, we can resort to Weyl prescription to obtain the quantization of these functions on the (right) enveloping algebra of the group [4]. In this way we obtain a Lie algebra of quantum operators isomorphic to the one satisfied by the momentum map [3] with the Poisson bracket associated with $\theta_{\eta, \zeta}$. The Hamiltonian operator and the quantum version of the constrains, when acting on wave functions are given by $(\mathcal{W} = \zeta e^{-\frac{i}{2} \mathbb{C} \mathbb{C}^+})$: $\hat{\mathcal{W}} = -\frac{1}{2} \mathcal{W} (2 + \mathbb{C}^+ + \frac{d\mathbb{C}^+}{d\mathbb{C}^+}) \phi$ and $\hat{\mathcal{W}} = -\frac{1}{2} \mathcal{W} (2 + \mathbb{C} + \frac{d\mathbb{C}}{d\mathbb{C}^+}) \phi$. To obtain the quantum version of the Kepler manifold.
(that is, the Hilbert space of states of the Hydrogen atom for $E < 0$), we must impose the constrain $\hat{I}\psi = 0$. This means that the energy of the first two oscillators must equal the energy of the other two. It is easy to check that the operators in the (right) enveloping algebra of the group \([\mathfrak{g}]\) preserving the constrain (see \([5,6]\) for a characterization of these operators) is the algebra $su(2,2)$ of the quantum version of the momentum map \([\mathfrak{g}]\). These operators act irreducibly on the constrained Hilbert space, as can be checked computing the Casimirs of $su(2,2)$, which are constant. The quantum operators commuting with the Hamiltonian (and providing the degeneracy of the spectrum) are $\hat{\mathcal{M}}$ and $\hat{\mathcal{N}}$. They define two commuting $su(2)$ algebras in the same representation $((\hat{\mathcal{M}})^2 = (\hat{\mathcal{N}})^2 = \frac{1}{4}(\hat{\mathcal{J}})^2 - \frac{1}{4})$, and linear combinations of them provide us with the angular momentum and the Runge-Lenz vector (see Table I).

The relation between the Kepler Hamiltonian $\mathcal{H}$ and the Hamiltonian $\mathcal{J}$ is $\mathcal{H} = -\frac{m^2}{2\mathcal{J}}$. If we act on eigenstates of the number operator for each oscillator, $\psi_{n_1,n_2,n_3,n_4} \approx (C_1^{+})^{n_1}(C_2^{+})^{n_2}(C_3^{+})^{n_3}(C_4^{+})^{n_4}$, and taking into account that: $\hat{\mathcal{E}}\psi_{n_1,n_2,n_3,n_4} = \hat{\mathcal{J}}\psi_{n_1,n_2,n_3,n_4} = \frac{1}{2}(2 + \sum n_i)\psi_{n_1,n_2,n_3,n_4}$, we recover the spectrum of the Hydrogen atom, $E_n = -\frac{m^2 n^2}{2\mathcal{J}}$, $n = 1 + n_1 + n_2$. The degeneracy is provided by the dimension of the representations of the algebra $su(2) \times su(2)$, which turn to be $n^2$ (if spin 1/2 is considered, the degeneracy is doubled).

3. Quantization of Kepler problem: $E > 0$.

The KS transformation, for the case of positive energies, maps the Kepler Hamiltonian to the function $-P_0$, with the constrains $I = 0$ and $-P_0 > 0$ and with the same potential 1-form $\theta(\gamma, \zeta)$. Performing the change of variables:

$$q_i = \frac{1}{2}(\alpha_{i+} + \nu_{i+}), \quad i = 0, 1, 2, 3, \quad p_i = \frac{1}{2}(\alpha_{i+} - \nu_{i+}), \quad i = 0, 2, \quad p_i = \frac{1}{2}(\nu_i - \alpha_i), \quad i = 1, 3$$

$$z_1 = q_0 + i q_1, \quad z_2 = q_2 + i q_3, \quad w_1 = p_0 + i p_1, \quad w_1 = p_2 + i p_3,$$

the new Hamiltonian $-P_0$ can be written as $\sim \alpha\nu$, which corresponds to four repulsive oscillators in resonance 1-1-1-1. The (extended) symmetry group for this system is given by (see \([4]\)):

$$\begin{align*}
\lambda'' &= \lambda' + \lambda, \\
\alpha'' &= \alpha' e^{\lambda} + \alpha, \\
\nu'' &= \nu' e^{-\lambda} + \nu, \\
\varsigma'' &= \varsigma' e^{\frac{1}{2}(\alpha\nu' e^{-\lambda} - \alpha'\nu e^{\lambda})}.
\end{align*}$$

(5)

where $\alpha, \nu \in \mathbb{R}^4$ and $\lambda := \omega t \in \mathbb{R}$. The Hamiltonian for this system is $\mathcal{H}_{rep} = -\omega \alpha\nu$. Applying GAQ we obtain that the wave functions are $\psi = \varsigma e^{-i\frac{1}{2}|\alpha\nu|} \phi(\nu, \lambda)$, and the Schrödinger equation is $i \frac{\partial \phi}{\partial \lambda} = \nu \frac{\partial \phi}{\partial \nu}$.

With the same procedure as in the case of negative energies, we construct a realization of the algebra \([\mathfrak{g}]\) resorting to the enveloping algebra of the group \([\mathfrak{g}]\). The quantum version of $P_0$ and the constrain are $\sim (\mathcal{W} = \varsigma e^{-i\alpha\nu})$. The representation defined by the quantum version of \([\mathfrak{g}]\) is irreducible, since the Casimirs are constant. The additional constrain $-\hat{P}_0 > 0$ restrict further the algebra of physical operators, being generated by the angular momentum $\hat{\mathcal{L}}$ and the Runge-Lenz vector $\hat{\mathcal{Q}}$. These operators close the Lorentz algebra, $[\hat{L}_i, \hat{Q}_j] = -2i\varepsilon_{ijk} \hat{Q}_k$, $[\hat{L}_i, \hat{L}_j] = -2i\varepsilon_{ijk} \hat{L}_k$ and $[\hat{Q}_i, \hat{Q}_j] = +2i\varepsilon_{ijk} \hat{L}_k$, as expected.

Again, using the relation between the Kepler Hamiltonian and $P_0$, $\mathcal{H} = \frac{m^2 \gamma^2}{4P_0^2}$, we obtain the spectrum of the Hydrogen atom for positive energies, $E_\tau = \frac{m^2 \gamma^2}{4\tau^2}$; $\tau = 2 + \tau_1 + \tau_2$. 

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4. Quantization of Kepler problem: $E = 0$.

For the case of zero energy, it is not clear which is the Kepler manifold, and some of the expressions obtained for negative and positive energies have no limit when the energy (related to $k$) goes to zero. We propose a candidate for the linearized zero energy Kepler problem, using a group-theoretical argument. The idea is to look, in the $su(2,2)$ algebra, for an appropriate Runge-Lenz vector $\vec{RL}$ (angular momentum doesn’t change, but Runge-Lenz vector depends on the energy). From Table I, we observe that $\mathcal{H} = -\frac{k^2}{2} \mapsto \vec{RL} = \frac{k}{\sqrt{m}} \vec{R}$ and $\mathcal{H} = \frac{k^2}{2} \mapsto \vec{RL} = \frac{k}{\sqrt{m}} \vec{Q}$. This suggests the choice, for zero energy (and for $k \neq 0$), $\vec{RL} = \frac{k}{2\sqrt{m}} (\vec{R} + \vec{Q})$, implying that the Hamiltonian is the sum of the two Hamiltonian (with equal opposite energies!), $E_0 = \frac{1}{2} (J - P_0) = \frac{1}{2} (|P| - P_0)$. This can be achieved if we impose the constrain (zero energy) $||P|| = P_0 = \frac{2\gamma\sqrt{m}}{k}$, derived from our choice of Runge-Lenz vector. The constrain $I = 0$ is also satisfied in this case, by construction. Now we perform the following change of variables:

$$b_0 = q_0, \quad a_0 = q_1, \quad B_0 = p_1, \quad A_0 = p_0, \quad b_1 = q_2, \quad a_1 = q_3, \quad B_1 = p_3, \quad A_1 = p_2,$$

obtaining that the Hamiltonian is written $E_0 = \frac{1}{2} \sum_i (A_i^2 + B_i^2)$, that is, a system of four free particles. It must be stressed that the variables $a_i, A_i, i = 0, 1$ satisfy commutation relations with opposite sign to that of $b_i, B_i, i = 0, 1$, as can be seen from the potential 1-form:

$$\theta = -B_0 db_0 + b_0 dB_0 - B_1 db_1 + b_1 dB_1 + A_0 da_0 - a_0 dA_0 + A_1 da_1 - a_1 dA_1. \quad (6)$$

The (extended) symmetry group for the system of four free particles is given by:

$$\lambda'' = \lambda' + \lambda, \quad a'' = a + a' + A'\lambda, \quad b'' = b + b' + B'\lambda, \quad A'' = A + A', \quad B'' = B + B'$$

$$\varsigma'' = \varsigma' \varsigma \exp\{i [\lambda' (\lambda + \frac{1}{2} A')]\} \exp\{i [b' B' + \frac{1}{2} B' B]\},$$

where the relative sign in the canonical structure of the $(a_i, A_i)$ and $(b_i, B_i)$ variables has been taken into account in the 2-cocycle. Repeating the procedure of the non-zero energy cases, we obtain a new realization of the algebra $su(2,2)$ on this space, which is irreducible.

The energy and constrain operators are ($\mathcal{W} = \varsigma e^{-ib B} e^{i(B/2-A^2)}\lambda$; $\hat{E}_0 \psi = -\frac{1}{2} \mathcal{W} \sum_i (A_i^2 + B_i^2)\psi$, and $\hat{I} \psi = i \mathcal{W} \sum_i (A_i \frac{\partial}{\partial B_i} - B_i \frac{\partial}{\partial A_i})\psi$). If we impose $E = 0$, we obtain the new constrain $\sum_i (A_i^2 + B_i^2) = \frac{2\gamma\sqrt{m}}{k}$. Therefore $k$ defines a foliation in spheres. The operators preserving this new constrain are again the angular momentum $\hat{L}$ and the Runge-Lenz vector $\hat{S} = \hat{R}' + \hat{Q}'$, closing the Euclidean algebra $e(3)$, as expected. This guarantees that we have made the correct choice of Hamiltonian and Runge-Lenz vector for $E = 0$.

An interesting application of these results is the fact that the linearization is preserved in some perturbed problems, such as the lunar problem or the Stark effect, see [7].

A similar study can be found in [8], where the quantization of the Kepler problem for $E \neq 0$ is considered in the Weyl-Wigner-Moyal formalism using the KS transformation.

References

[1] M. Kummer, Comm. Math. Phys. 84, 133 (1982).
[2] N.M.J. Woodhouse, Geometric Quantization (2nd ed.), Oxford University Press (1991).
[3] V. Aldaya and J.A. de Azcárraga, J. Math. Phys. 23, 1297 (1982).
[4] V. Aldaya, A. de Azcárraga and K.B. Wolf, J. Math. Phys. 25, 506 (1984).
[5] V. Aldaya, M. Calixto, J. Guerrero, Comm. Math. Phys. 178, 399 (1996).
[6] J. Guerrero, M. Calixto and V. Aldaya, J. Math. Phys. 40 3773 (1999).
[7] J. Guerrero and J.M. Pérez, in preparation.
[8] J.M. Gracia-Bonda, Phys. Rev. A30, 691 (1984).