NESTED POLYHEDRA AND INDICES OF ORBITS OF
COXETER GROUPS OF NON-CRYSTALLOGRAPHIC TYPE

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ABSTRACT. The indices of order 2, 4, 6 together with other higher even degree indices, as well as the anomaly numbers (the indices of degree 3), are known invariants of finite dimensional representations of simple Lie algebras. The definition and properties of such indices adapted to individual orbits of the non-crystallographic reflection groups are presented. It is shown that the representation-orbit replacement yields the following generalizations: the indices, the anomaly numbers and the embedding indices of irreducible representations become those of orbits.

1. INTRODUCTION

Finite dimensional representations of simple Lie algebras have many applications in numerous problems in science. During the past decades, it has been convenient to characterize representations by their dimensions [7, 10, 20]. Generally, the formula for the dimensions of irreducible representations of simple Lie algebras is well-known, although its difficulty in practical exploitation rapidly increases together with the rank of the corresponding Lie algebra. Some years ago E.B. Dynkin introduced the index of the irreducible representation of a Lie algebra, known as the Dynkin index [3, 17]. It can be calculated for any finite-dimensional representation of any simple Lie algebra. Such an index turns out to be a useful tool in the classification of semi-simple Lie subalgebras of semi-simple Lie algebras.

The higher order indices of finite irreducible representations of simple Lie algebras were defined for the first time in [19]. Here we consider the characteristics that are analogues to such indices by replacing irreducible representations of simple Lie algebras by orbits of the finite reflection groups. Such a replacement yields several advantages as the size of an orbit is always finite, and the product of orbits is decomposable.

In this paper, we focus only on the non-crystallographic groups $H_2$, $H_3$ and $H_4$. Unlike the crystallographic groups, those one can relate any orbit to a finite-dimensional representation, the non-crystallographic ones do not have an underlying Lie algebra. The non-crystallographic groups found a great number of applications in solid state physics, in particular in crystallography, as well as in biophysics and chemistry [5, 14]. While $H_2$ and $H_4$ groups play an essential role in the construction of quasicrystals [12], the icosahedral symmetry of the $H_3$-group reveals the structure of wide diversity of molecules [4, 22]. Moreover, during the past decade, $H_3$ has gained a lot of interest from the point of view of mathematical

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virology, as it serves as tool for better understanding of architecture and assembly of viruses [2, 8, 21, 23].

Any orbit $O_\lambda(G)$ of any finite reflection group $G$ is conveniently characterized by its dominant point (or a seed point) $\lambda \in \mathbb{R}^n$. Such a point is unique for each orbit. In general, the coordinates of a seed point are presented in $\omega$-basis, and they take values of any positive integers. However, here the choice of coordinates is not restricted to integers, and a dominant point can be represented by any real numbers (as long as they stay non-negative). For example, viewing any orbit of $H_3$ as a geometric object (spherical or Euclidean polytope) provides a deeper knowledge of a chosen seed point, as the variation of its coordinates scales the lower-dimensional faces, represented as edges (arcs) and polygons (tiles on a sphere).

In this work we continue along this logical thread with several tasks. First, we determine the indices of the orbits of the finite reflection groups that were not previously defined in [19]. Secondly, we want to explore and generalize the properties of such indices by examining individual orbits of the non-crystallographic groups. We demonstrate that the extended definition of indices Coxeter groups of non-crystallographic type contains most of the properties of the indices of the irreducible representations of simple Lie algebras.

The even-degree indices of the orbits as well as the odd-degree indices are defined in Section 2 and 3. The latter ones are also known in physics literature as anomaly numbers [15, 18]. An application of anomaly numbers often only requires the decision whether the application is zero or not. Such a decision can be made for irreducible representations as well as for orbits of any finite reflection groups.

The Dynkin index of a semi-simple subalgebra of a simple Lie algebra remains a valid invariant only if a single orbit of the finite reflection group is involved in its definition. The embedding index (Section 4) becomes an analogue of the Dynkin index, and it has the same value for every orbit of a finite reflection group. However, the generalization of such indices is not obvious. The calculations of the embedding index proceed whenever the branching rule for the finite reflection group and its subgroup ($G' \subset G$) is known. These rules are well-established, and they were calculated for the crystallographic groups (for the rank up to $n = 8$) (for example, in [11] and references therein). Recently, they have been obtained for the non-crystallographic groups as well [6].

In addition, as we are restricted to orbits of the non-crystallographic groups, we introduce the algorithm for the search for lower orbits, i.e. the orbits of smaller radii that may appear inside of an initial one (Section 5). Choosing a seed point with its coordinates and proceeding with the subtraction of simple roots provides the dominant points of lower orbits. As it turns out, this method coincides with the root-subtraction for orbits of crystallographic type. Such a procedure forms a weight-system that is analogues to a weight-system of a representation of a simple Lie algebra even if there is no such algebra underlying the non-crystallographic cases. In a geometrical interpretation, the obtained set of orbits of different radii results in the structure of nested polyhedra (Fig. 6). Such a set of polytopes is rather unusual as it differs from the sets obtained for crystallographic cases. For the latter ones, the vertices of a polytope of a bigger radius are found in the middle of the edges of a polytope of a smaller one. However, the nested polyhedra of the non-crystallographic groups does not have this property.
2. Even degree indices for orbits

The even-degree indices of representations are found in several papers [16, 19, 20]. Such indices of irreducible representations have the same property as the decomposition of products does. However, such a property is limited to the indices of degree 2, 4 and, for some groups, degree 6. If the weight system of an irreducible representation is replaced by vertices of a single orbit of a Coxeter group, its even-degree anomalies can be calculated explicitly for any degree.

Replacing irreducible representations of simple Lie algebras with orbits of finite reflection groups has several advantages. First, the size of an orbit of any Coxeter group is always limited. Secondly, the points of such an orbit have just real numbers as their coordinates. Moreover, the product of several orbits can be always decomposed into a sum of orbits of smaller sizes.

Definition 1. Let $G$ be a finite reflection group, $O_\lambda(G)$ denotes the elements of orbit with dominant point $\lambda$. The numbers defined by

$$I^{2p}_\lambda(G) = \sum_{\mu \in O_\lambda(G)} \langle \mu, \mu \rangle^p,$$

where $p \in \mathbb{Z}$, are called indices of order $2p$ of orbits. The summation extends over all elements of the orbit $O_\lambda(G)$, and $\langle \mu, \mu \rangle^p$ is the scalar product in the weight-space of $G$.

The elements of any orbit are equidistant from the origin so we have the following remark.

Remark. The formula for even degree indices has the form:

$$I^{2p}_\lambda(G) = |O_\lambda(G)| \langle \lambda, \lambda \rangle^p,$$  \hspace{1cm} (1)

where $|O_\lambda(G)|$ denotes the size of orbit generated from a seed point $\lambda$. The sizes of orbits of any non-crystallographic group are presented in Tab. 1

| $\lambda$ | $O_\lambda(H_2)$ | $\lambda$ | $O_\lambda(H_3)$ | $\lambda$ | $O_\lambda(H_4)$ |
|---------|----------------|---------|----------------|---------|----------------|
| $(a, 0)$ | 5              | $(a, 0, 0)$ | 12             | $(a, 0, c)$ | 60             |
| $(0, b)$ | 5              | $(0, b, 0)$ | 30             | $(0, b, c)$ | 60             |
| $(a, b)$ | 10             | $(0, 0, c)$ | 20             | $(a, b, c)$ | 120            |
|         |                | $(a, b, 0)$ |                | $(a, b, 0)$ | 60             |

| $\lambda$ | $O_\lambda(H_4)$ | $\lambda$ | $O_\lambda(H_4)$ | $\lambda$ | $O_\lambda(H_4)$ |
|---------|----------------|---------|----------------|---------|----------------|
| $(a, 0, 0, 0)$ | 120       | $(a, 0, c, 0)$ | 3600          | $(a, b, c, 0)$ | 7200          |
| $(0, b, 0, 0)$ | 720       | $(a, 0, 0, d)$ | 2400          | $(a, b, 0, d)$ | 7200          |
| $(0, 0, c, 0)$ | 1200      | $(0, b, c, 0)$ | 3600          | $(a, 0, c, d)$ | 7200          |
| $(0, 0, 0, d)$ | 600      | $(0, b, 0, d)$ | 3600          | $(0, b, c, d)$ | 7200          |
| $(a, b, 0, 0)$ | 1440     | $(0, 0, c, d)$ | 2400          | $(a, b, c, d)$ | 14400         |

Table 1. Number of elements of an orbit $O_\lambda(H_k)$ of the non-crystallographic groups $H_k, k = 2, 3, 4, a, b, c, d \neq 0$ for each type of the dominant point $\lambda$.

The general formulas for indices of order $2p$ for the non-crystallographic groups are the following ones:
Remark. Denoted it as: dominant points

In this case, the highest weight is \( \lambda \) as follows:

\[
(3-\tau)pI_{(a,b)}^2(H_2) = |O_{(a,b)}(H_2)| \cdot [2(a^2+\tau ab+b^2)]^p,
\]

\[
(4-2\tau)pI_{(a,b,c)}^2(H_3) = |O_{(a,b,c)}(H_3)| \cdot [(3-\tau)a^2+4b^2+3c^2+4ab+2\tau ac+4\tau bc]^p,
\]

\[
(5-3\tau)pI_{(a,b,c,d)}(H_4) = |O_{(a,b,c,d)}(H_4)| \cdot [2((2-\tau)a^2+(3-\tau)b^2+3c^2+2d^2+(3-\tau)ab
+2ac+\tau ad+4bc+2\tau bd+3\tau cd)]^p,
\]

where \( \tau = \frac{1+\sqrt{5}}{2} = 1.618\ldots \) is one of the solutions of the quadratic equation \( x^2 = x + 1 \), and it is called a golden ratio.

Definition 2. Let \( G \) be a finite reflection group. The direct sum of orbits is presented as follows:

\[
O_{\lambda_1} \oplus \ldots \oplus O_{\lambda_k}(G) = \bigcup_{\mu_i \in O_{\mu} (G)} \mu_i = O_{\lambda_1}(G) \cup \ldots \cup O_{\lambda_k}(G).
\]

The number of element of such a sum is equal to:

\[
|O_{\lambda_1} \oplus \ldots \oplus O_{\lambda_k}(G)| = |O_{\lambda_1}(G)| + \ldots + |O_{\lambda_k}(G)|.
\]

Definition 3. Let \( G \) be a finite reflection group. The product of orbits of \( G \) with dominant points \( \lambda_1, \ldots, \lambda_k \), where \( k \geq 2 \) and the coordinates are non-negative, is a summation of the elements of each orbit with the elements of other orbits. We denoted it as:

\[
O_{\lambda_1} \otimes \ldots \otimes O_{\lambda_k}(G) = \bigcup_{\mu_i \in O_{\mu_1} (G), \ldots, \mu_k \in O_{\mu_k} (G)} (\mu_1 + \ldots + \mu_k).
\]

The number of element of such a product is equal to:

\[
|O_{\lambda_1} \otimes \ldots \otimes O_{\lambda_k}(G)| = |O_{\lambda_1}(G)| \cdot \ldots \cdot |O_{\lambda_k}(G)|.
\]

Remark. The product of \( k \)-orbits of \( G \) decomposes into a union of several orbits. In this case, the highest weight is \( \lambda_1 + \ldots + \lambda_k \), and the product of orbits decomposes as follows:

\[
\lambda_1 \otimes \ldots \otimes \lambda_k = (\lambda_1 + \ldots + \lambda_k) \cup \ldots \cup \text{the other lower-order orbits}.
\]

Example 1. Let us consider two orbits \( O_{(1,0)}(H_2) \) and \( O_{(0,\tau)}(H_2) \). The summation and the product of orbits are written as:

\[
O_{(1,0) \oplus (0,\tau)}(H_2) = \{(1,0), (-1, \tau), (\tau, -\tau), (-\tau, 1), (0, -1),
\]

\[
(0, \tau), (\tau + 1, -\tau), (-\tau - 1, \tau + 1), (1, -\tau - 1), (-\tau, 0)\}
\]

\[
O_{(1,0) \otimes (0,\tau)}(H_2) = \{(1, \tau), (\tau + 2, -\tau), (-\tau, \tau + 1), (\tau + 1, -\tau - 1), (1 - \tau, 0), (-1, 2\tau),
\]

\[
(\tau, 0), (-\tau - 2, 2\tau + 1), (\tau - 1, -1), (-\tau - 1, \tau), (\tau, 0), (2\tau + 1, -2\tau),
\]

\[
(-1, 1), (2\tau - 2\tau - 1), (0, -\tau), (-\tau, \tau + 1), (1, 1 - \tau), (-2\tau - 1, \tau + 2),
\]

\[
(0, -\tau), (-2\tau, 1), (0, -\tau - 1), (\tau + 1, -\tau - 1), (-\tau - 1, \tau), (\tau, -\tau - 2), (-\tau, -1)\}
\]

The product of two orbits decomposes into the union of orbits, namely:
\[(1, 0) \otimes (0, \tau) = (1, \tau) \cup 2(\tau, 0) \cup (0, \tau - 1) .\]

Here each number that is attached to the dominant point of each orbit of \(H_2\)-group indicates the size of the orbit. The number of elements of the product of orbits is equal to the number of elements after the decomposition is done.

**Proposition 2.1.** Let \(G\) be a finite reflection group. The formulas for lower order indices of a product of \(k\)-orbits of \(G\) are given by:

\[
I^2_{\lambda_1 \otimes \cdots \otimes \lambda_k}(G) = \sum_{j=1}^{k} \left( I^2_{\lambda_j}(G) \prod_{i=1, i \neq j}^{k} I^0_{\lambda_i}(G) \right),
\]

\[
I^4_{\lambda_1 \otimes \cdots \otimes \lambda_k}(G) = \sum_{j=1}^{k} \left( I^4_{\lambda_j}(G) \prod_{i=1, i \neq j}^{k} I^0_{\lambda_i}(G) \right) + \frac{2(r+2)}{r} \sum_{j=1}^{k} \sum_{j \neq l=1}^{k} \left( I^2_{\lambda_j}(G) I^2_{\lambda_l}(G) \prod_{i=1, i \neq j, l}^{k} I^0_{\lambda_i}(G) \right),
\]

where \(k \in \mathbb{N}^\geq 2\), \(|O_\lambda(G)|\) denotes the size of an orbit of \(\lambda\) of a group \(G\), and \(r\) corresponds to the rank of \(G\).

**Remark.** In general, the indices of \(k\)-th product of orbits of a group \(G\) are defined recursively as follows:

\[
I^{2p}_{\lambda_1 \otimes \cdots \otimes \lambda_k}(G) = I^{2p}_{\lambda_1 \otimes (\lambda_2 \otimes \cdots \otimes \lambda_k)}(G), \quad k \in \mathbb{N}^\geq 2.
\]

**Proposition 2.2.** Let \(G = G_1 \times \ldots \times G_k\) be a finite reflection group. The formula for indices of any \(2p\) order of a product of \(k\) orbits of \(G_1, \ldots, G_k\) is given by:

\[
I^{2p}_{\lambda_1 \otimes \cdots \otimes \lambda_k}(G) = \prod_{i=1}^{k} |O_{\lambda_i}(G_i)| \cdot \sum_{j=1}^{k} \langle \lambda_j, \lambda_j \rangle^p,
\]

where \(|O_\lambda(G)|\) denotes the size of an orbit of \(\lambda\) of a group \(G\).

**Proof.** Generally, the matrix of the inner product can be written in a diagonal form:

\[
(\langle \omega, \omega \rangle)_G = \begin{pmatrix}
\langle (\omega, \omega) \rangle_{G_1} & \cdots & 0 \\
0 & \ddots & \vdots \\
0 & \cdots & \langle (\omega, \omega) \rangle_{G_k}
\end{pmatrix}.
\]

It is easy to see that formula (2) holds. \(\square\)

3. **Odd-order indices for orbits**

Representations of simple Lie algebras underly possible models in particle physics. In this case, a value of odd-order index that is defined for a representation serves as a limitation imposed on a particular model [15]. Such indices can be understood as a generalisation of the triangular anomaly number \(A_\lambda\) of a representation \(\lambda\) of a Lie algebra [18, 25]. Anomaly numbers were defined for the group \(SU(n)\) as a sum ...
of cubes of the components of weights corresponding to the $U(1)$ subgroup in the reduction $SU(n) \supset U(1) \times SU(n - 1)$.

The crucial part in obtaining the anomaly number is to find a vector (a direction) passing through the origin of the weight-space. After the projection of orbits of a lower subgroup (that are orthogonal to a chosen direction) and summation of the distances between the projected points are done, one should verify if such a sum yields zero or not. It is a non-trivial problem to identify such a direction for any orbit, in order to get a non-zero result. A unitary group $U(1)$ is usually considered. However, any other direction can be explored, as long as it gives a non-zero result.

Non-zero anomaly numbers exist only for the groups with a symmetric Coxeter-Dynkin diagram. For the non-crystallographic groups the simple roots are all of the same length. From the Coxeter-Dynkin diagrams for the non-crystallographic groups (Fig. 1), one can notice that only the anomaly number of the $H_2$-group is non-zero. The groups $H_3$ and $H_4$ are anomaly-free groups, as their diagrams are not symmetric.

![Figure 1. The Coxeter-Dynkin diagrams of the non-crystallographic groups $H_2$, $H_3$ and $H_4$. The nodes correspond to the simple roots $\alpha_k$, $k = 1, \ldots, 4$.](image)

**Definition 4.** Let $G$ be a finite reflection group with a subgroup of the form $U(1) \times G'$. The number defined by

$$A_\lambda^{2p-1}(G) = \sum_{\mu \in O_\lambda(G')} \langle \mu, v \rangle^{2p-1},$$

where $p \in \mathbb{N}$, and $v$ is the unit vector of the one-dimensional subspace of the weight-space corresponding to the subgroup $U(1)$ of $G$, is called the anomaly number of an orbit, or the index of order $2p - 1$ of an orbit.

**Remark.** An orbit $O_\lambda(G)$ is decomposed into a union of orbits of $G'$. All the orbits of a subgroup $G'$ of a group $G$ are parallel to each other and lying on the surfaces orthogonal to the line spanned by vector $v$.

**Example 2.** Consider the non-crystallographic group $H_2$ with the dominant point $\lambda = (a, b)$. Using the branching rule described in [6] we can decompose such an orbit into several orbits of $A_1$. In this case, the projection matrix is $(\tau, \tau)$. Therefore, we have the following decomposition:

$$(a, b) \supset (a \tau + b \tau) + (a + b \tau) + (a \tau + b) + (a) + (b).$$

In Fig. 2 we present the decomposition of the orbit $O_{(a,b)}(H_2)$ into five orbits of the crystallographic group $A_1$. All the orbits are parallel to each other, they are orthogonal to the affine mirror $r_\xi$, where $\xi$ stand for the highest weigh. We choose the vector $v = (-\tau, \tau)$ to be a direction in a weight-space. The calculations of the anomaly numbers yield:
\[ A^{2p-1}_{(a,b)}(H_2) = 2 \left( \frac{\tau}{2+\tau} \right)^{2p-1} \{(a+b)^{2p-1} + (a+a\tau+b)^{2p-1} - (a+b+b\tau)^{2p-1} \} \]

**Remark.** The anomaly numbers \( A^1_{(a,b)}(H_2) = A^3_{(a,b)}(H_2) = 0 \) for any \( a, b \in \mathbb{R} \). The odd-order indices \( A^{2p-1}_{(a,b)}(H_2) \neq 0 \) for \( a \neq b \) for \( p > 2 \).

**Figure 2.** The root system of the Coxeter group \( H_2 \) is shown. Dashed lines \( r_\alpha, r_\beta \), are the mirrors orthogonal to the simple roots \( \alpha \) and \( \beta \), respectively. The root \( \xi \) stands for the highest root of \( H_2 \). The mirror \( r_\xi \) is orthogonal to \( \xi \) and it’s passing by \( \tau \xi / 2 \). The orbit with a dominant point \( \lambda = (a,b) \) of \( H_2 \) is shown. The shaded triangle represents the fundamental region \( F \) of \( H_2 \). The green segments correspond to the orbits of \( A_1 \)-group.

**Example 3.** Let \( G \) be the icosahedral group \( H_3 \) acting in \( E_3 \). Its subgroup \( U(1) \times H_2 \) is chosen in such a way that it defines the direction of \( U(1) \). Let us consider the orbit \( O_{(1,1,0)}(H_3) \). It can be decomposed into a union of several orbits of \( H_2 \)-group, namely: \( (1,0), (1,\tau), (2,1), (2,0), (0,2), (1,2), (\tau,1), (0,1) \). The ”pancake” structure of such a decomposition is presented in Fig. 3. Therefore, the anomaly number is calculated as follows:

\[ A^{2p-1}_{(1,1,0)}(H_3) = 5 \left( 2 + \frac{3}{2}\tau \right)^{2p-1} + 5 \left( 1+\frac{3}{2} \right)^{2p-1} + 10 \left( \frac{3}{2} \right)^{2p-1} + 10 \left( \frac{1}{2} \right)^{2p-1} \]

\[ + 10 \left( -\frac{1}{2} \right)^{2p-1} + 10 \left( -\frac{3}{2} \right)^{2p-1} + 5 \left( -1-\frac{3}{2} \right)^{2p-1} + 5 \left( -2-\frac{3}{2} \right)^{2p-1} = 0 \]
Figure 3. On the left, the orbits $O_{(1,1,0)}(H_3)$ viewed in the direction perpendicular to $H_2$-plane. On the right, ”the pancake” decomposition for $O_{(1,1,0)}(H_3)$ is presented.

4. Embedding index

The embedding index for the subalgebras of simple Lie algebras was defined by E.B. Dynkin in [3]. Such indices were introduced only for the maximal subalgebras. In general, given a Lie algebra and its subalgebra one should calculate a branching rule. It is a well-known problem that was solved for a large number of irreducible representations of simple Lie algebras [13].

In this section, we define the index that is analogous to the Dynkin index. Applying a branching rule to an orbit of a Coxeter group, any chosen orbit is reduced to a sum of several orbits. Such a decomposition of orbits corresponds to subgroups of a chosen Coxeter group. Division of the size of an orbit of any Coxeter group by the size of its reduced orbit provides a specific ratio. We call this ratio the embedding index.

Such an index depends only on the rank of a finite reflection group and regardless of the branching rule, the index is the same for all orbits. The index takes the same value for all the orbits of any branching rule being considered. Given that the embedding index can be obtained for any orbit of any crystallographic group by using Dynkin’s formula, are able to demonstrate that such a property applies to the non-crystallographic groups as well.

**Definition 5.** Let $G$ be a reflection group of order $n$, $G_1 \times \ldots \times G_k$, $k \leq n$ be a maximal subgroup of $G$. The index of order 2 of the embedding $G \leftrightarrow G_1 \times \ldots \times G_k$ is given by the formula

$$\gamma = \frac{I^2(G)}{I^2(G_1 \times \ldots \times G_k)}.$$ 

**Remark.** The formula for the embedding index is general for any parameter $k$. In this paper we focus only on the non-crystallographic cases with $k \in \{0, \ldots, 4\}$.

In Tab. 2 we present the embedding index $\gamma$ for any Coxeter group of non-crystallographic type and its maximal subgroup.

**Theorem 4.1.** The embedding index $\gamma$ for any Coxeter group $G$ of non-crystallographic type is a fraction of ranks, group $G$ and its subgroup $G'$, namely:

$$\gamma = \frac{\text{rank } G}{\text{rank } G'}.$$
\[ G \quad G' \quad \gamma \quad G \quad G' \quad \gamma \]

\begin{array}{|c|c|c|c|c|c|}
\hline
G & G' & \gamma & G & G' & \gamma \\
\hline
H_2 & A_1 & 2 & H_4 & A_2 \times A_2 & 1 \\
H_3 & A_1 \times A_1 \times A_1 & 1 & H_4 & A_1 \times A_1 \times A_1 \times A_1 & 1 \\
H_3 & A_2 & 3/2 & H_4 & H_3 \times A_1 & 1 \\
H_3 & H_2 & 3/2 & H_4 & A_4 & 1 \\
& & & H_4 & D_4 & 1 \\
\hline
\end{array}

Table 2. The embedding index for the non-crystallographic groups.

Proof. We consider two cases: (i) rank \( G = \text{rank} \ G' \), and (ii) rank \( G > \text{rank} \ G' \).

(i) The elements of any orbit \( O_\lambda \) of group \( G \) lie on the sphere of a radius \( r \). Applying the branching rule method to \( \lambda \), we get several orbits of a subgroup \( G' \) of a group \( G \). All the elements of orbits of \( G' \) lie on the sphere of the same radius \( r \) as rank \( G' = \text{rank} \ G \). Since the index of second order is a summation over squared distances between the elements of orbits and the origin, then \( I^2(G) = I^2(G') \). In such a case, the index \( \gamma \) is equal 1.

(ii) First, let us point out that for any orbit \( O_\lambda(H_3) \) written in \( \omega \)-basis, the same values occur the same number of times in each coordinate. The coordinates are affected by the non-crystallographic group \( H_3 \) containing the tetrahedral rotation group. The orbits of subgroups \( G' \) of rank 2 of \( H_3 \) are selected in the following way: after removing one of the coordinates (the first one, in case of \( H_2 \)-group, and the last one, in case of \( A_2 \)) of \( O_\lambda(H_3) \), the other two give us the union of orbits of a subgroup \( G' \). Since the values appear at each coordinate, the index \( I^2 \) of the subgroup \( G' \) of rank 2 of group \( H_3 \) is equal to \( \frac{2}{3}I^2(H_3) \). Therefore, the embedding index \( \gamma = \frac{2}{3} \). An analogous explanation can be made for \( H_2 \).

\( \square \)

5. Lower Orbits of \( H_3 \)

As the crystallographic groups have an underlying Lie algebra, it is essential to consider their irreducible representations. It has been shown that using the highest weight of an irreducible representation, one is able to determine its dominant weight by subtraction of simple roots [1]. The computational problem has the following steps: determination of the highest weight, subtraction of weights from the highest weight and an algorithm that describes a subtraction path. The appearance of multiplicities (grater than one) of dominant weights for the crystallographic cases is due to the non-commutativity of certain elements of a Lie algebra. In case of finite reflection groups, all reflections commute.

\[ C_{H_2} = \begin{pmatrix} 2 & -\tau \\ -\tau & 2 \end{pmatrix}, \quad C_{H_3} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -\tau \\ 0 & -\tau & 2 \end{pmatrix}, \quad C_{H_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -\tau \\ 0 & 0 & -\tau & 2 \end{pmatrix}. \]

Table 3. The Cartan matrices for the non-crystallographic groups \( H_2, H_3 \) and \( H_4 \).
Although, the non-crystallographic groups do not have corresponding Lie algebras, the similar procedure can be developed and properly applied to orbits of the non-crystallographic group. Nonetheless, here we only consider the $H_3$-group. The simple roots $\alpha_i$ are identified from the Cartan matrix (Tab. 3). The algorithm for the orbits consists of the following steps:

(i) determine a dominant point $\lambda = (l_1, l_2, l_3)$, $l_i = a_i + b_i \tau \in \mathbb{Z}[\tau] > 0$, $i = 1, 2, 3$;

(ii) establish the correspondence between the coordinates of a dominant point $\lambda$ and the index $i = \{1, 2, 3\}$ of a simple root $\alpha_i$: $i \rightarrow l_i$;

(iii) if at least one of $l_i > 0$, $i = 1, 2, 3$ then proceed the following subtraction:
- if $b_i = 0$ then $\mu_i = \lambda - j \cdot \alpha_i$, $j = \{1, \ldots, a_i\}$;
- if $b_i \geq 1$:
  - $a_i = 0$ then $\mu_i = \lambda - k \tau \cdot \alpha_i$, $k = \{1, \ldots, b_i\}$,
  - $a_i \geq 1$ then $\mu_i = \lambda - \frac{k}{\gcd(a_i, b_i)} \cdot \alpha_i$, $k = \{1, \ldots, \gcd(a_i, b_i)\}$;

(iv) replace the point $\lambda$ in (i) with $\mu_i$;
(v) repeat the steps (ii)-(iv) until at least one of the coordinates $\mu_i$ is greater than zero.

This recursive method provides a tree-diagram for any type of a dominant point of the group $H_3$ (Fig. 5). Such a method allows one to know the coordinates of dominant points of lower orbits that are sitting inside of an initial one. By geometrical construction of such orbits, one should be able to retain the sets of polytopes (nested polyhedra) \cite{9,24}.

Note that for such an algorithm, we only consider dominant points that have integer coefficients as their coordinates. The rules for resulting dominant points of lower orbits are presented in Tab. (4). Note, that in order to get such expressions, one only considers the coordinates of dominant points with equal "dynamic" coefficients (for example, for $(a, b, 0)$ it is necessary that $a = b$). However, if $a, b, c > 0$, the number of vertices of a corresponding polytope is $|O_{(a,b,c)}(H_3)| = 120$, which makes it difficult to generalize the coordinates of seed points of lower orbits. Therefore, this case is omitted. In order to generalize each case depending on the type of a dominant point, we only consider $a, b, c \in \{1, \ldots, 9\}$. Nevertheless, such a generalization can be adapted to any $a, b, c \in \mathbb{N}$.

The same procedure can be applied to finding the subtraction paths for the non-crystallographic groups $H_2$ and $H_4$. Although, due to a large number of elements of the latter one ($120^2$), the computational routine becomes laborious.

The multiple occurrences of the same dominant weight within one system necessarily involves the same multiple occurrences of dominant points of corresponding orbits. Although, the actual method of determining such multiplicities for the non-crystallographic cases has not yet been developed it is likely that it will prove to be related to the determination of multiplicities for the crystallographic cases.

In general, in order to retain dominant points of lower orbits, one is allowed to chose any real numbers as the coordinates of a seed point (as long as they are non-negative). As shown in the example below (Fig. 4), the values from the ring $\mathbb{Z}[\tau]$ also can be chosen as the coordinates of a dominant point. Such a choice does not affect the subtraction path.
| Orbit | Conditions | Type |
|-------|-------------|------|
| \((a,0,0)\): | \((a-2k,k,0)\), \(k \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) any \(a\) | odd \(a > 3\) |
| | \((0,\frac{a}{2}(\tau-1),0)\) even \(a\) | \((0,\frac{a}{2} - \left\lfloor \frac{a+2}{2} \right\rfloor,\tau)\) odd \(a > 3\) |
| \((0,a,0)\): | \((k,a-2k,k\tau),\ k \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) any \(a\) | \((0,0,0),(\frac{a}{2}(\tau-1),0,0),\ (a,\frac{a}{2}(\tau-1),0)\) even \(a\) | \((\left\lfloor \frac{a}{2} \right\rfloor,\tau-\left\lfloor \frac{a+2}{2} \right\rfloor,\tau+1,\left\lfloor \frac{a+2}{2} \right\rfloor-\tau)\), \((a,\left\lfloor \frac{a}{2} \right\rfloor,\tau-\left\lfloor \frac{a+2}{2} \right\rfloor,\tau)\) odd \(a > 3\) |
| \((0,0,a)\): | \((0,k\tau,a-2k),\ k \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) any \(a\), even \(a\) | \((0,0,0),(a,\frac{a}{2}(\tau-1),0),(\frac{a}{2}\tau,0,\frac{a}{2}(\tau-1))\) odd \(a > 3\) |
| \((\left\lfloor \frac{a}{2} \right\rfloor,\tau,\left\lfloor \frac{a}{2} \right\rfloor-\left\lfloor \frac{a+2}{2} \right\rfloor,0)\) | \((\left\lfloor \frac{a}{2} \right\rfloor,\tau,\left\lfloor \frac{a}{2} \right\rfloor-\left\lfloor \frac{a+2}{2} \right\rfloor,0)\) odd \(a > 3\) |
| \((a,a,0)\): | \((a,a,0),(0,0,a\tau),(a,a(\tau-1),0)\) any \(a\) | \((a-2k,a+k,0),(a+k,a-2k,k\tau)\), \(k \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) \(a > 1\) | \(\frac{a}{2}(2\tau-1,0,2-\tau),\ \frac{a}{2}(0,\tau-1,0)\), \(\frac{a}{2}(4,\tau-1,0)\), \(\frac{a}{2}(0,2-\tau,a)\) even \(a\) | \((a,a-1)\tau-(a+1),2\tau)\) \(a > 4\) |
| | \((2a,\left\lfloor \frac{a}{2} \right\rfloor,\tau-\left\lfloor \frac{a+1}{2} \right\rfloor,\tau),(0,\left\lfloor \frac{a}{2} \right\rfloor,\tau-\left\lfloor \frac{a+1}{2} \right\rfloor,\tau)\) odd \(a > 3\) | \((a,a-2)\tau-(a+2),4\tau)\) \(a > 8\) |
| \((a,0,a)\): | \((a,0,0),(a\tau,0,0)\), any \(a\) | \((a-2k,k,a),(a,k,a-2k)\), \(k \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) \(a > 1\) | \((0,0),(a-2k-1)\tau-\left\lfloor \frac{a+k+1}{2}\right\rfloor,(2k+1)(\tau+1))\), \(k \in \{0, \ldots, \left\lfloor \frac{a-2}{2} \right\rfloor \}\) \(a > 1\) | \(\frac{a}{2}(1,0,2-\tau),\ \frac{a}{2}(\tau-1,0,2\tau-1)\) even \(a\) | \((0,0),(a-2k)\tau-\frac{a}{2}-k,2k(\tau+1))\), \(k \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) \(a > 1\) |
| | \((\tau+2,\left\lfloor \frac{a}{2} \right\rfloor,\tau-1,0)\) odd \(a > 1\) | \((0,\left\lfloor \frac{a}{2} \right\rfloor,\tau,\left\lfloor \frac{a}{2} \right\rfloor,\tau-\left\lfloor \frac{a+1}{2} \right\rfloor,\tau+1,\tau+1,\left\lfloor \frac{a-1}{2} \right\rfloor,\tau-\left\lfloor \frac{a+1}{2} \right\rfloor)\) odd \(a > 3\) | \(2\tau+4,\ \left\lfloor \frac{a}{2} \right\rfloor-\tau-2,0)\) even \(a > 4\) |
| | \((3\tau+6,\left\lfloor \frac{a}{2} \right\rfloor-\tau-3,0)\) odd \(a > 5\) | | | |
| \((0,a,a)\): | | | | | |
| | \((0,a,0),(a,a(\tau+1),0,0)\) any \(a\) | \((k,a-2k,k\tau+a),(0,k\tau+a,a-2k)\), \(k \in \{1, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) \(a > 1\) | \((0,0,a),(\frac{a}{2}(2\tau-1,0,\tau),(0,\frac{a}{2}(\tau-1),0)\) even \(a\) | \((\left\lfloor \frac{a}{2} \right\rfloor,\tau+1,2k(\tau+1),(a-2k)\tau-\left\lfloor \frac{a}{2}+k \right\rfloor)\) even \(a\) | \((a,a-2k)\tau-\frac{a}{2}-k,2k(\tau+1))\), \(k \in \{0, \ldots, \left\lfloor \frac{a}{2} \right\rfloor \}\) \(a > 1\) |
| | \((\left\lfloor \frac{a}{2} \right\rfloor,\tau+1),(2k+1)(\tau+1),(a-2k-1)\tau-\left\lfloor \frac{a}{2}+k+1 \right\rfloor)\) odd \(a > 1\) | \((a,a-2k-1)\tau-\left\lfloor \frac{a}{2}+k+1 \right\rfloor,(2k+1)(\tau+1))\), \(k \in \{0, \ldots, \left\lfloor \frac{a-3}{2} \right\rfloor \}\) odd \(a > 3\) | \((a-1)\tau-\left\lfloor \frac{a}{2}+1 \right\rfloor,2\tau+1,\left\lfloor \frac{a}{2} \right\rfloor-\tau-1)\) odd \(a > 3\) | | |

Table 4. Dominant points for lower orbits obtained by subtraction of simple roots for any type of a dominant point of an initial orbit: \((a,0,0),(0,a,0),(0,0,a),(a,a,0),(0,a,a),(a,0,a)\), where \(a \in \{1,2,\ldots,9\}\), and \(\left\lfloor \cdot \right\rfloor\) denotes the integer part of a number.
Example 4. Let one consider the orbit of $H_2$-group with at least one irrational co-ordinate, namely $(\tau, 1)$. In this case, subtraction of simple roots yields the following tree-diagram (Fig. 4).

\[
\begin{array}{c}
\tau_{a_1} \\
(\tau, 1) \\
(\tau + 2)a_2 \\
(2\tau + 1, -\tau - 2) \\
(2\tau + 1)a_1 \\
(-2\tau - 1, 2\tau) \\
\tau_{a_2} \\
(-\tau, 0) \\
(1-2\tau) \\
(-\tau - 2, \tau) \\
\end{array}
\]

**Figure 4.** A tree-diagram (a subtraction path) of the orbit $O_{(\tau,1)}(H_2)$ is presented. Dominant points are displayed in boxes. The points that do not belong to $O_{(\tau,1)}(H_2)$ are marked by gray color.

Example 5. Consider the orbits of $H_3$ with the dominant points $(1,0,0)$ and $(0,0,1)$. The coordinates of the vertices are obtained from the following tree-diagrams:

\[
\begin{array}{c}
\tau_{a_1} \\
(0,0,1) \\
(0,\tau,-1) \\
\tau_{a_2} \\
(\tau,-\tau,\tau) \\
\tau_{a_3} \\
(\tau,\tau,-1) \\
\tau_{a_4} \\
(-\tau,\tau,-1) \\
\tau_{a_5} \\
(-\tau,0,1) \\
\tau_{a_6} \\
(-\tau,\tau,\tau) \\
\tau_{a_7} \\
(-\tau,\tau,-1) \\
\tau_{a_8} \\
(0,1,-\tau) \\
\tau_{a_9} \\
(1,-1,0) \\
\tau_{a_{10}} \\
(-1,0,0) \\
\end{array}
\]

**Figure 5.** The tree-diagrams of the orbits $O_{(1,0,0)}(H_3)$ and $O_{(0,0,1)}(H_3)$. The coordinates of the vertices are presented. The repetitive coordinates and subtraction paths are marked by gray color.
Example 6. Consider the orbit of $H_3$ with the seed point $(2, 0, 0)$. As shown in the tree-diagram below, such an orbit has two lower orbits with the dominant points $(0, 1, 0)$ and $(0, -\tau', 0)$, where $\tau' = 1 - \tau$. The set of nested polytopes is generated as presented in Fig. 6.

\[
\begin{align*}
(2, 0, 0) \quad \vdots \quad (0, -\tau', 0) \\
\alpha_1 \quad \vdots \quad \alpha_2 \\
(-2, 0) \quad \vdots \quad (0, -2, 2\tau) \\
(-1, 0, \tau) \quad \vdots \quad (\tau + 1, 0, 2) \\
(0, 1, 0) \\
\end{align*}
\]

Figure 6. The tree-diagram for the orbit $O_{(2,0,0)}(H_3)$, and the set nested polytopes arising from it. The orbits $O_{(2,0,0)}(H_3)$, $O_{(0,1,0)}(H_3)$ and $O_{(0,-\tau',0)}(H_3)$ are presented in green, black and solid assorted colors, respectively.

Example 7. Consider the orbits $O_{(3,1,0)}(H_3)$ and $O_{(0,1,3)}(H_3)$. Due to the substitution of simple roots, obtained dominants points for such orbits are as follows:

\[
\begin{align*}
(3, 1, 0) : & \quad (3, 1, 0), (1, 2, 0), (2, 0, \tau), (0, 1, \tau); \\
(0, 1, 3) : & \quad (0, 1, 3), (0, \tau + \frac{1}{2}, 1), (\tau + \frac{1}{2}, 0, 2), (\tau + \frac{1}{2}, \tau - 1, 2\tau - 2).
\end{align*}
\]

\[
\begin{align*}
H_3(3, 1, 0) \\
H_3(0, 1, 3)
\end{align*}
\]

Figure 7. The nested polytopes provided by the algorithm of root-subtraction for the orbits $O_{(3,1,0)}(H_3)$ and $O_{(0,1,3)}(H_3)$.

Both of such nested polytopes contain four orbits of different radius as shown in Fig. 7. Depending on the radius of each orbit (that is descending form left to right), they are distinguished by cyan, blue, green and black colors.
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