Improved sum rules for light mesons and thermal hadronic threshold

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Abstract. The thermal properties of light vector and scalar mesons are investigated in the framework of QCD sum rules. The phenomenological side of the correlation function can be calculated using either the quark-hadron duality approach or in terms of two-pion continuum contributions. In the quark-hadron duality approach, a free parameter (hadronic threshold) arises in the phenomenological part and it is necessary to know the temperature dependence of the hadronic threshold. A comparison of above mentioned approaches gives us additional information about the temperature dependence of the hadronic threshold. Taking into account the thermal spectral density and additional operators appearing at finite temperature the thermal QCD sum rules are improved. The decay constants of $\rho$ and $\sigma$ mesons are calculated and our investigations show that the above mentioned methods give us the same results.

1. Introduction

The nature of light mesons is a long-standing problem in hadron physics for several decades. In spite of the striking success of QCD theory for strong interaction, the underlying structure of the light mesons is still not fully understood. Especially the light scalar mesons have the same quantum numbers $J^{PC} = 0^{++}$ as the vacuum and so their structure is major issue in order to understand non-perturbative properties of the QCD vacuum.

In the literature there are various approaches and models which predict the thermal behavior of hadrons. The QCD sum rules method is one of the powerful methods for investigating the properties of hadrons [1]. QCD sum rules, which is based on the operator product expansion (OPE), QCD Lagrangian and quark-hadron duality, was later extended to the finite temperature case [2]. However, extension of sum rules to finite temperature case have some complications [3-5].

The leptonic decay constants of $\rho$ and $\sigma$ mesons are widely investigated in the studies [6,7]. In these studies, two-pion continuum contributions are used to obtain decay constants. But the quark-hadron duality approach requires knowledge on the temperature dependence of hadronic threshold, which generally is expressed by quark condensate $\langle \bar{\psi}\psi \rangle$ in the following form [8]:

$$s_0(T) = s_0 \frac{\langle \bar{\psi}\psi \rangle}{\langle 0|\bar{\psi}\psi|0 \rangle} \left( 1 - \frac{4m^2}{s_0} \right) + 4m^2.$$  \hspace{1cm} (1)

where $s_0$ is zero temperature hadronic threshold and $m$ is quark mass.

In this study we reanalyze the decay constants of $\rho$ and $\sigma$ mesons and compare our results with the results existing in the literature [6,7]. We take into account the thermal spectral density...
and perturbative two-loop order corrections to the correlation function and improve thermal QCD sum rules obtained in [6,7]. The leptonic decay constants are calculated using the values of energy density, quark and gluon condensates obtained via chiral perturbation theory [9] and lattice QCD theory [10,11]. Our calculations show that obtained results using both two pion continuum approach and quark-hadron duality approach are compatible with each other.

2. Improved QCD sum rules and thermal hadronic threshold

In this section we reanalyze the thermal sum rules for \( \rho \) and \( \sigma \) mesons. For this aim we start with two-point thermal correlation function,

\[
\Pi(q,T) = i \int d^4x \ e^{iqx} T \rho (J(x)J(0)),
\]

where \( \rho = e^{-\beta H}/Tr e^{-\beta H} \) is the thermal density matrix of QCD at temperature \( T = 1/\beta \), \( T \) indicates the time ordered product and \( J(x) =: \bar{Q}(x)\Gamma Q(x) : \) is interpolating current. In interpolating current \( Q(x) \) is quark field and \( \Gamma = I \) or \( \gamma_\mu \) for scalar and vector particles, respectively. The thermal spectral densities in lowest order of perturbation theory are calculated in the papers [2,12] and they are given by

\[
\rho(s,T) = \frac{s}{8\pi^2} v(s) f(s) \left( 1 - 2 n \left( \frac{\sqrt{s}}{2} \right) \right),
\]

where \( v(s) = \sqrt{1 - 4m^2/s} \) and \( f(s) = 3v^2(s) \) or \( f(s) = 3 - v^2(s) \) for \( \sigma \) and \( \rho \) mesons, respectively.

In our calculations, we also take into account the perturbative two-loop order \( \alpha_s \) correction to the spectral densities. These corrections at zero temperature can be written as [13]:

\[
\rho_{\alpha_s}(s,T) = \frac{11s}{8\pi^3} \alpha_s, \quad \text{and} \quad \rho_{\alpha_s}(s,T) = \frac{s}{4\pi^3} \alpha_s,
\]

for \( \sigma \) and \( \rho \) mesons, respectively. In Eqs. (4) we replace the strong coupling \( \alpha_s \) with its temperature dependent lattice improved expression \( \alpha(T) = 2.095(82)g^2(T)/4\pi \) (for more details see [12]).

To calculate the phenomenological side of correlation function, we insert a complete set of physical intermediate state to Eq. (2). The decay constants \( f_\sigma \) and \( f_\rho \) are defined by the matrix element of the scalar current \( J(0) \) and vector current \( J_\mu(0) \) between the vacuum and hadronic states:

\[
\langle 0 | J(0) | \sigma \rangle = f_\sigma m_\sigma, \quad \text{and} \quad \langle 0 | J_\mu(0) | \rho \rangle = f_\rho m_\rho \epsilon^\mu_\rho,
\]

where \( \epsilon^\mu_\rho \) is polarization states of vector mesons. Finally matching the spectral representation and results of OPE and using quark-hadron duality, our improved sum rules both \( \sigma \) and \( \rho \)-mesons take the form:

\[
f^2(T) m_\sigma^2(T) \exp \left( -\frac{m_\sigma^2(T)}{M^2} \right) = \int_{4m^2}^{s_0(T)} ds \ [\rho_{\alpha_s}(s) + \rho_{\alpha_s}(s)] \exp \left( -\frac{s}{M^2} \right) + \hat{\Pi}_{\sigma}^\mu(q,T), \]

where \( M^2 \) is Borel mass parameter. In Eq. (6) the non-perturbative parts for \( \sigma \) and \( \rho \)-mesons are given by [6,7]

\[
\hat{\Pi}_{\sigma}^\mu(q,T) = 3m(\bar{\psi} \psi) + \frac{g^2}{32\pi^2} (G_{\mu\nu}^a G^{a\mu\nu}) - \frac{6}{11} \left[ (u\Theta u) + \lambda(M^2) \left( \frac{16}{3} \langle u\Theta f u \rangle - \langle u\Theta g u \rangle \right) \right],
\]

\[
\hat{\Pi}_{\rho}^\mu(q,T) = -2m(\bar{\psi} \psi) - \frac{g^2}{48\pi^2} (G_{\mu\nu}^a G^{a\mu\nu}) - \frac{4}{11} \left[ (u\Theta u) + \lambda(M^2) \left( \frac{16}{3} \langle u\Theta f u \rangle - \langle u\Theta g u \rangle \right) \right],
\]

2
where \( \lambda(M^2) = \left( \frac{1}{2} \frac{\mu^2}{M^2} \right) \ln \left( \frac{M^2}{\mu^2} \right) \), \( \Theta_{\mu\nu}^f \) and \( \Theta_{\mu\nu}^g \) are fermionic and gluonic parts of energy momentum tensor \( \Theta_{\mu\nu} \), respectively.

Our numerical analysis show that in the intervals \( s_0 = (1 - 1.5) \text{ GeV}^2 \) for the \( \sigma \) and \( s_0 = (1.5 - 2) \text{ GeV}^2 \) for the \( \rho \) channels, the obtained results weakly depend on these parameters. The working region for the Borel mass parameters are \( 0.3 \text{ GeV}^2 \leq M^2 \leq 0.6 \text{ GeV}^2 \) for \( \sigma \) and \( 0.5 \text{ GeV}^2 \leq M^2 \leq 1.2 \text{ GeV}^2 \) for \( \rho \) mesons, respectively. Taking into account the temperature dependences of continuum threshold, energy density, quark and gluon condensates (for more details see [9-12]) we found that the temperature dependences of decay constants for \( \sigma \) and \( \rho \) mesons can be expressed by the following fit function:

\[
f = A \exp(-T/\lambda) + B,
\]

where the \( A, \lambda \) and \( B \) are constant parameters and the obtained values of these parameters are given in Table 1.

|       | \( A \) (GeV) | \( \lambda \) (GeV) | \( B \) (GeV) |
|-------|--------------|---------------------|-------------|
| \( \sigma \) | -4.18646 \times 10^{-4} | -0.02988 | 0.20328 |
| \( \rho \) | -4.72423 \times 10^{-5} | -0.01877 | 0.20966 |

**Table 1.** The values of constant parameters in fit function.

We calculated the values of the decay constants of the \( \sigma \) and \( \rho \) mesons at \( T = 0 \) and we found that \( f_\sigma = 205.92 \text{ MeV} \) and \( f_\rho = 210.53 \text{ MeV} \). These results are in good consistency with the existing studies in the literature [14-16]. Our investigations show that thermal spectral density and two loop order contributions are significantly important, and two-pion continuum and quark-hadron duality approaches give the same results.

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