ANALYSIS AND GEOMETRY ON WORM DOMAINS

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Abstract. In this primarily expository paper we study the analysis of the Diederich-Fornæss worm domain in complex Euclidean space. We review its importance as a domain with nontrivial Nebenhülle, and as a counterexample to a number of basic questions in complex geometric analysis. Then we discuss its more recent significance in the theory of partial differential equations: the worm is the first smoothly bounded, pseudoconvex domain to exhibit global non-regularity for the ∂̅-Neumann problem. We take this opportunity to prove a few new facts. Next we turn to specific properties of the Bergman kernel for the worm domain. An asymptotic expansion for this kernel is considered, and applications to function theory and analysis on the worm are provided.

1. Introduction

The concept of “domain of holomorphy” is central to the function theory of several complex variables. The celebrated solution of the Levi problem tells us that a connected open set (a domain) is a domain of holomorphy if and only if it is pseudoconvex. For us, in the present paper, pseudoconvexity is Levi pseudoconvexity; this is defined in terms of the positive semi-definiteness of the Levi form. This notion requires the boundary of the domain to be at least $C^2$. When the boundary is not $C^2$ we can still define a notion of pseudoconvexity that coincides with the Levi pseudoconvexity in the $C^2$-case. When the Levi form is positive definite then we say that the domain is strongly pseudoconvex. The geometry of pseudoconvex domains has become an integral part of the study of several complex variables. (See [Kr1] for basic ideas about analysis in several complex variables.)

Consider a pseudoconvex domain $\Omega \subseteq \mathbb{C}^n$. Any such domain has an exhaustion $U_1 \subset \subset U_2 \subset \subset U_3 \subset \subset \cdots \subset \Omega$ with $\bigcup U_j = \Omega$ by smoothly bounded, strongly pseudoconvex domains. This information was fundamental to the solution of the Levi problem (see [Bers] for this classical approach), and is an important part of the geometric foundations of the theory of pseudoconvex domains.

It is natural to ask whether there is a dual result for the exterior of $\Omega$. Specifically, given a pseudoconvex domain $\Omega$, are there smoothly bounded, pseudoconvex domains $W_1 \supset \supset W_2 \supset \supset W_3 \supset \cdots \supset \supset \overline{\Omega}$ such that $\bigcap W_j = \overline{\Omega}$? A domain having this property is said to have a Stein neighborhood basis. A domain failing this property is said to have nontrivial Nebenhülle.

Early on, F. Hartogs in 1906 produced the following counterexample (which has come to be known as the Hartogs triangle): Let $\Omega = \{ (z_1, z_2) \in \mathbb{C}^2 : 0 < |z_1| < |z_2| < 1 \}$. Theorem 1.1. Any function holomorphic on a neighborhood of $\overline{\Omega}$ actually continues analytically to $D^2(0, 1) \equiv D \times D$. Thus $\overline{\Omega}$ cannot have a neighborhood basis of pseudoconvex domains. Instead it has a nontrivial Nebenhülle.

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Proof. Let $U$ be a neighborhood of $\overline{\Omega}$. For $|z_1| < 1$, the analytic discs
$$\zeta \mapsto (z_1, \zeta \cdot |z_1|)$$
have boundary lying in $U$. But, for $|z_1|$ sufficiently small, the entire disc lies in $U$. Thus a standard argument (as in the proof of the Hartogs extension phenomenon—see [Kr1]), sliding the discs for increasing $|z_1|$, shows that a holomorphic function on $U$ will analytically continue to $D(0,1) \times D(0,1)$. That proves the result. 

It was, however, believed for many years that the Hartogs example worked only because the boundary of $\Omega$ is not smooth (it is only Lipschitz). Thus, for over seventy years, mathematicians sought a proof that a smoothly bounded pseudoconvex domain will have a Stein neighborhood basis. It came as quite a surprise in 1977 when Diederich and Forneß [DFo1] produced a smoothly bounded domain—now known as the *worm*—which is pseudoconvex and which does not have a Stein neighborhood basis. In fact the Diederich-Forneß example is the following.

**Definition 1.2.** Let $W$ denote the domain
$$W = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 - e^{i \log |z_2|^2} \right|^2 < 1 - \eta(\log |z_2|^2) \right\},$$
where

1. $\eta \geq 0$, $\eta$ is even, $\eta$ is convex;
2. $\eta^{-1}(0) = I_\mu = [-\mu, \mu]$;
3. there exists a number $a > 0$ such that $\eta(x) > 1$ if $|x| > a$;
4. $\eta'(x) \neq 0$ if $\eta(x) = 1$.

Notice that the slices of $W$ for $z_2$ fixed are discs centered on the unit circle with centers that wind $\mu/\pi$ times about that circle as $|z_2|$ traverses the range of values for which $\eta(\log |z_2|^2) < 1$.

It is worth commenting here on the parameter $\mu$ in the definition of $W$. The number $\mu$ in some contexts is selected to be greater than $\pi/2$. The number $\nu = \pi/2\mu$ is half the reciprocal of the number of times that the centers of the circles that make up the worm traverse their circular path.

Many authors use the original choice of parameter $\beta$, where $\mu = \beta - \pi/2$ (see [Ba3, CheS, KrPe] e.g.). Here, we have preferred to use the notation $\mu$, in accord with the sources [Chr1, Chr2].

**Proposition 1.3.** The domain $W$ is smoothly bounded and pseudoconvex. Moreover, its boundary is strongly pseudoconvex except at the boundary points $(0, z_2)$ for $|\log |z_2|^2| \leq \mu$. These points constitute an annulus in $\partial W$.

**Proposition 1.4.** The smooth worm domain $W$ has nontrivial Nebenhülle.

The proofs of these propositions are deferred to Section 2.

As Diederich and Forneß [DFo1] showed, the worm provides a counterexample to a number of interesting questions in the geometric function theory of several complex variables. As an instance, the worm gives an example of a smoothly bounded, pseudoconvex domain which lacks a global plurisubharmonic defining function. It also provides counterexamples in holomorphic approximation theory. Clearly the worm showed considerable potential for a central role in the function theory of several complex variables. But in point of fact the subject of the worm lay dormant for nearly fifteen years after the appearance of [DFo1]. It was the remarkable paper of Kiselman [Ki] that re-established the importance and centrality of the worm.

In order to put Kiselman’s work into context, we must provide a digression on the subject of biholomorphic mappings of pseudoconvex domains. In the present discussion, all domains $\Omega$ are smoothly bounded. We are interested in one-to-one, onto, invertible mappings (i.e., biholomorphic mappings or biholomorphisms) of domains
$$\Phi : \Omega_1 \to \Omega_2.$$
Thanks to a classical theorem of Liouville (see [KrPa]), there are no conformal mappings, other than trivial ones, in higher dimensional complex Euclidean space. Thus biholomorphic mappings are studied instead. It is well known that the Riemann mapping theorem fails in several complex variables (see [Kr1, GKr1, GKr2, IKr]). It is thus a matter of considerable interest to find means to classify domains up to biholomorphic equivalence.

Poincaré’s program for such a classification consisted of two steps: (1) to prove that a biholomorphic mapping of smoothly bounded pseudoconvex domains extends smoothly to a diffeomorphism of the closures of the domains and (2) to then calculate biholomorphic differential invariants on the boundary. His program was stymied for more than sixty years because the machinery did not exist to tackle step (1) The breakthrough came in 1974 with Fefferman’s seminal paper [Fe]. In it he used remarkable techniques of differential geometry and partial differential equations to prove that a biholomorphic mapping of smoothly bounded, strongly pseudoconvex domains will extend to a diffeomorphism of the closures.

Fefferman’s proof was quite long and difficult, and left open the question of (a) whether there was a more accessible and more natural approach to the question and (b) whether there were techniques that could be applied to a more general class of domains. Steven Bell [Bel1] as well as Bell and Ewa Ligocka [BelLi] provided a compelling answer.

Let $\Omega$ be a fixed, bounded domain in $\mathbb{C}^n$. Let $A^2(\Omega)$ be the square integrable holomorphic functions on $\Omega$. Then $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$. The Hilbert space projection $P : L^2(\Omega) \rightarrow A^2(\Omega)$
can be represented by an integration formula

$$P f(z) = \int_\Omega K(z, \zeta) f(\zeta) \, dV(\zeta).$$

The kernel $K(z, \zeta) = K_\Omega(z, \zeta)$ is called the Bergman kernel. It is an important biholomorphic invariant. See [Bers, CheS, Kr1] for all the basic ideas concerning the Bergman kernel.

Clearly the Bergman projection $P$ is bounded on $L^2(\Omega)$. Notice that, if $\Omega$ is smoothly bounded, then $C^\infty(\Omega)$ is dense in $L^2(\Omega)$. In fact more is true: If $\Omega$ is Levi pseudoconvex and smoothly bounded, then $C^\infty(\Omega) \cap \{\text{holomorphic functions}\}$ is dense in $A^2(\Omega)$ (see [Cat3]).

Bell [Bel1] has formulated the notion of Condition $R$ for the domain $\Omega$. We say that $\Omega$ satisfies Condition $R$ if $P : C^\infty(\Omega) \rightarrow C^\infty(\Omega)$. It is known, thanks to the theory of the $\overline{\partial}$-Neumann problem (see Section 5), that strongly pseudoconvex domains satisfy Condition $R$. Deep work of Diederich-Fornæss [DFo2] and Catlin [Cat1, Cat2] shows that domains with real analytic boundary, and also finite type domains, satisfy Condition $R$. An important formula of Kohn, which we shall discuss in Section 5 relates the $\overline{\partial}$-Neumann operator to the Bergman projection in a useful way (see also [Kr2]). The fundamental result of Bell and Bell/Ligocka is as follows.

**Theorem 1.5.** Let $\Omega_j \subset \mathbb{C}^n$ be smoothly bounded, Levi pseudoconvex domains. Suppose that one of the two domains satisfies Condition $R$. If $\Phi : \Omega_1 \rightarrow \Omega_2$ is a biholomorphic mapping then $\Phi$ extends to be a $C^\infty$ diffeomorphism of $\overline{\Omega_1}$ to $\overline{\Omega_2}$.

This result established the centrality of Condition $R$. The techniques of proof are so natural and accessible that it seems that Condition $R$ is certainly the “right” approach to questions of boundary regularity of biholomorphic mappings. Work of Boas/Straube in [BoS2] shows that Condition $R$ is virtually equivalent to natural regularity conditions on the $\overline{\partial}$-Neumann operator.

For later reference, and for its importance in its own right, we mention here that the above Theorem 1.5 can be “localized”. To be precise, we say that a given smoothly bounded domain $\Omega$

\footnote{Here the $\overline{\partial}$-Neumann operator $N$ is the natural right inverse to the $\overline{\partial}\overline{\partial}^* = \overline{\partial} + \overline{\partial}^*$; see Section 5.}
satisfies the Local Condition R at a point \( p_0 \in \partial \Omega \) if there exists a neighborhood \( U \) of \( p_0 \) such that \( P : C^\infty(\overline{\Omega}) \to L^2(\Omega) \cap C^\infty(\overline{\Omega} \cap U) \). Then Bell’s local result is as follows; see [Bel2].

**Theorem 1.6.** Let \( \Omega_j \subset \mathbb{C}^n \) be smoothly bounded, pseudoconvex domains, \( j = 1, 2 \). Suppose that \( \Omega_1 \) satisfies Local Condition R at \( p_0 \in \partial \Omega_1 \). If \( \Phi : \Omega_1 \to \Omega_2 \) is a biholomorphic mapping then there exists a neighborhood \( U \) of \( p_0 \) such that \( \Phi \) extends to be a \( C^\infty \) diffeomorphism of \( \Omega_1 \cap U \) onto its image.

We might mention, as important background information, a result of David Barrett [Ba1] from 1984. This considerably predates the work on which the present paper concentrates. It does not concern the worm, but it does concern the regularity of the Bergman projection.

**Theorem 1.7.** There exists a smoothly bounded, non-pseudoconvex domain \( \Omega \subseteq \mathbb{C}^2 \) on which Condition R fails.

Although Barrett’s result is not on a pseudoconvex domain, it provides some insight into the trouble that can be caused by rapidly varying normals to the boundary. See [Ba2] for some pioneering work on this idea.

As indicated above, it was Kiselman [Ki] who established an important connection between the worm domain and Condition R. He proved that, for a certain non-smooth version of the worm (see below), a form of Condition R fails.

For \( s > 0 \), let \( H^s(\Omega) \) denote the usual Sobolev space on the domain \( \Omega \) (see, for instance, [Hor1], [Kt2]). Building on Kiselman’s idea, Barrett [Ba3] used an exhaustion argument to show that the Bergman projection fails to preserve the Sobolev spaces of sufficiently high order on the smooth worm.

**Theorem 1.8.** For \( \mu > 0 \), let \( \mathcal{W} \) be the smooth worm, defined as in Definition 1.2, and let \( \nu = \pi/2\mu \). Then the Bergman projection \( P \) on \( \mathcal{W} \) does not map \( H^s(\mathcal{W}) \) to \( H^s(\mathcal{W}) \) when \( s \geq \nu \).

The capstone of results, up until 1996, concerning analysis on the worm domain is the seminal paper of M. Christ. Christ finally showed that Condition R fails on the smooth worm. Precisely, his result is the following.

**Theorem 1.9.** Let \( \mathcal{W} \) be the smooth worm. Then there is a function \( f \in C^\infty(\overline{\mathcal{W}}) \) such that its Bergman projection \( Pf \) is not in \( C^\infty(\overline{\mathcal{W}}) \).

We note explicitly that the result of this theorem is closely tied to, indeed is virtually equivalent to, the assertion that the \( \overline{\partial} \)-Neumann problem is not hypoelliptic on the smooth worm, [BoS2].

In the work of Kiselman, Barrett, and Christ, the geometry of the boundary of the worm plays a fundamental role in the analysis. In particular, the fact that for large \( \mu \) the normal rotates quite rapidly is fundamental to all of the negative results. It is of interest to develop a deeper understanding of the geometric analysis of the worm domain, because it will clearly play a seminal role in future work in the analysis of several complex variables.

We conclude this discussion of biholomorphic mappings with a consideration of biholomorphic mappings of the worm. It is at this time unknown whether a biholomorphic mapping of the smooth worm \( \mathcal{W} \) to another smoothly bounded, pseudoconvex domain will extend to a diffeomorphism of the closures. Of course the worm does not satisfy Condition R, so the obvious tools for addressing this question are not available. As a partial result, So-Chin Chen has classified the biholomorphic self-maps of the worm \( \mathcal{W} \). His result implies that all biholomorphic self-maps of the worm \( \mathcal{W} \) do extend to diffeomorphisms of the boundary (see Section 6).

This article is organized as follows:

Section 2 gives particulars of the Diederich-Fornæss worm. Specifically, we prove that the worm is Levi pseudoconvex, and we establish that there is no global plurisubharmonic defining function.
We also examine the Diederich-Fornaess bounded plurisubharmonic exhaustion function on the smooth worm.

Section 3 considers non-smooth versions of the worm (these originated with Kiselman). We outline some of Kiselman’s results.

Section 4 discusses the irregularity of the Bergman projection on the worm. In particular, we reproduce some of Kiselman’s and Barrett’s analysis.

Section 5 discusses the failure of Condition R on the worm domains.

Section 6 treats the automorphism group of the smooth worm.

Section 7 engages in detailed analysis of the non-smooth worms $D_\beta$ and $D'_\beta$. Particularly, we treat $L^p$ boundedness properties of the Bergman projection, and we study the pathology of the Bergman kernel on the boundary off the diagonal.

Section 8 treats irregularity properties of the Bergman kernel on worm domains.

2. The Diederich-Fornaess Worm Domain

We now present the details of the first basic properties of the Diederich-Fornaess worm domain $\mathcal{W}$. Recall that $\mathcal{W}$ is defined in Definition 1.2. Some material of this section can also be found in the excellent monograph [CheS].

We begin by proving Proposition 1.3.

Proof of Proposition 1.3 Property (iii) of the worm shows immediately that the worm domain is bounded. Let

$$
\rho(z_1, z_2) = \left| z_1 + e^{i\log|z_2|^2} \right|^2 - 1 + \eta (\log |z_2|^2) .
$$

(1)

Then $\rho$ is (potentially) a defining function for $\mathcal{W}$. If we can show that $\nabla \rho \neq 0$ at each point of $\partial \mathcal{W}$ then the implicit function theorem guarantees that $\partial \mathcal{W}$ is smooth.

If it happens that $\partial \rho/\partial z_1(p) = 0$ at some boundary point $p = (p_1, p_2)$, then we find that

$$
\frac{\partial \rho}{\partial z_1}(p) = \overline{p}_1 + e^{-i\log |p_2|^2} = 0 .
$$

(2)

Now let us look at $\partial \rho/\partial z_2$ at the point $p$. Because of (2), the first factor in $\rho$ differentiates to 0 and we find that

$$
\frac{\partial \rho}{\partial z_2}(p) = \eta' (\log |p_2|^2) \cdot \frac{\overline{p}_2}{|p_2|^2} .
$$

Since $\rho(p) = 0$, we have that $\eta'(\log |p_2|^2) = 1$. Hence, by property (iv), $\eta'(\log |p_2|^2) \neq 0$. It follows that $\partial \rho/\partial z_2(p) \neq 0$. We conclude that $\nabla \rho(z) \neq 0$ for every boundary point $z$.

For the pseudoconvexity, we write

$$
\rho(z) = |z_1|^2 + 2\text{Re} \left( z_1 e^{-i\log |z_2|^2} \right) + \eta (\log |z_2|^2) .
$$

Multiplying through by $e^{\arg z_2^2}$, we have that locally $\mathcal{W}$ is given by

$$
|z_1|^2 e^{\arg z_2^2} + 2\text{Re} \left( z_1 e^{-i\log z_2^2} \right) + \eta (\log |z_2|^2) e^{\arg z_2^2} < 0 .
$$

The function $e^{-i\arg z_2^2}$ is locally well defined and holomorphic, and its modulus is $e^{\arg z_2^2}$. Thus the first two terms are plurisubharmonic. Therefore we must check that the last term is plurisubharmonic. Since it only depends on $z_2$, we merely have to calculate its Laplacian. We have, arguing as before, that

$$
\Delta \left( \eta (\log |z_2|^2) e^{\arg z_2^2} \right) = \left( \Delta \eta (\log |z_2|^2) \right) e^{\arg z_2^2} + \eta (\log |z_2|^2) \Delta e^{\arg z_2^2} \geq 0 .
$$

Because $\eta$ is convex and nonnegative (property (i)), the nonnegativity of this last expression follows. This shows that $\mathcal{W}$ is smoothly bounded and pseudoconvex.
In order to describe the locus of weakly pseudoconvex points, we consider again the local defining function
\[ \rho(z_1, z_2) = \left| z_1 \right|^2 e^{\arg z_2^2} + 2 \text{Re} \left( z_1 e^{-i \log z_2^2} \right) + \eta(\log |z_2|^2) e^{\arg z_2^2}. \]
This function is strictly plurisubharmonic at all points \((z_1, z_2)\) with \(z_1 \neq 0\) because of the first two terms, or where \(|\log |z_2|^2| > \mu\), because of the last term. Thus consider the annulus \(A \subset \partial W\) given by
\[ A = \{ (z_1, z_2) \in \partial W : z_1 = 0 \text{ and } |\log |z_2|^2| \leq \mu \}. \]
A direct calculation shows that the complex Hessian for \(\rho\) at a point \(z \in A\) acting on \(v = (v_1, v_2) \in \mathbb{C}^2\) is given by
\[ |v_1|^2 + 2 \text{Re} \left( v_1 \bar{v}_2 \frac{e^{i \log |z_2|^2}}{z_2} \right). \]
By pseudoconvexity, such an expression must be non-negative for all complex tangential vectors \(v\) at \(z\). But such vectors are of the form \(v = (0, v_2)\), so that the Levi form \(L_\rho \equiv 0\) on \(A\). This proves the result \(\square\).

It is appropriate now to give the proof of Diederich and Fornæss that the worm has nontrivial Nebenhülle. What is of interest here, and what distinguishes the worm from the older example of the Hartogs triangle, is that the worm is a bounded, pseudoconvex domain with smooth boundary.

We now show that \(\overline{W}\) does not have a Stein neighborhood basis.

**Proof of Proposition** \(T.4\) What we actually show is that if \(U\) is any neighborhood of \(\overline{W}\), then \(U\) will contain
\[ K = \{ (0, z_2) : -\pi \leq \log |z_2|^2 \leq \pi \} \cup \{ (z_1, z_2) : \log |z_2|^2 = \pi \text{ or } -\pi \text{ and } |z_1 - 1| < 1 \}. \]
In fact this assertion is immediate by inspection.

By the usual Hartogs extension phenomenon argument, it then follows immediately that if \(U\) is pseudoconvex then \(U\) must contain
\[ \tilde{K} = \{ (0, z_2) : -\pi \leq \log |z_2|^2 \leq \pi \text{ and } |z_1 - 1| < 1 \}. \]
Thus there can be no Stein neighborhood basis \(\square\).

We now turn to a few properties of the smooth worm \(W\) connected with potential theory. The significance of the next result stems from the paper [BoS3]. In that paper, Boas and Straube established the following.

**Theorem 2.1.** Let \(\Omega\) be a smoothly bounded pseudoconvex domain that admits a defining function that is plurisubharmonic on the boundary. Then, for every \(s > 0\),
\[ P : H^s(\Omega) \to H^s(\Omega) \]
is bounded. In particular, \(\Omega\) satisfies Condition \(R\).

For sake of completeness we mention here that, if the Bergman projection \(P\) on a domain \(\Omega\) is such that \(P : C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})\) is bounded (i.e. \(\Omega\) satisfies Condition \(R\)) \(P\) is said to be regular, while if \(P : H^s(\Omega) \to H^s(\Omega)\) for every \(s > 0\) (and hence \(\Omega\) satisfies Condition \(R\) a fortiori) \(P\) is said to be exactly regular.

Thanks to the result of Christ [Chr1], we now know that \(W\) does not satisfy Condition \(R\), hence a fortiori it cannot admit a defining function which is plurisubharmonic on the boundary. However, it is simpler to give a direct proof of this fact.

**Proposition 2.2.** There exists no defining function \(\tilde{\rho}\) for \(W\) that is plurisubharmonic on the entire boundary.
Proof. Suppose that such a defining $\tilde{\rho}$ exist. Then, there exists a smooth positive function $h$ such that $\tilde{\rho} = h\rho$. A direct calculation shows that the complex Hessian for $\tilde{\rho}$ at a point $z \in \mathcal{A}$ acting on $v = (v_1, v_2) \in \mathbb{C}^2$ is given by

$$\mathcal{L}_{\tilde{\rho}}(z; (v_1, v_2)) = 2\Re \left[ \bar{v}_1 v_2 \left( \frac{i\bar{h}}{z_2} + \partial_{z_2} h \right) e^{i\log |z_2|^2} \right] + \left[ h + 2\Re \left( \partial_{z_2} h \cdot e^{i\log |z_2|^2} \right) \right] |v_1|^2.$$

Since this expression is assumed to be always non-negative, we must have

$$\left( \frac{i\bar{h}}{z_2} + \partial_{z_2} h \right) e^{i\log |z_2|^2} = \partial_{z_2} (h e^{-i\log |z_2|^2}) \equiv 0,$$

on $\mathcal{A}$. Therefore, the function $g(z_2) = h(0, z_2)e^{-i\log |z_2|^2}$ is a holomorphic function on $\mathcal{A}$. Hence $g(z_2)e^{i\log |z_2|^2} = h(0, z_2)e^{2\arg z_2}$ is locally a holomorphic function. Thus it must be locally a constant, hence a constant $c$ on all of $\mathcal{A}$.

Therefore, on $\mathcal{A}$,

$$h(0, z_2) = ce^{-2\arg z_2}$$

which is impossible. This proves the result. □

We conclude this section with another important result about the Diederich-Fornæss worm domain $\mathcal{W}$. This result is part of potential theory, and is related to the negative result Proposition 2.2. In what follows, we say that $\lambda$ is a bounded plurisubharmonic exhaustion function for a domain $\Omega$ if

(a) $\lambda$ is continuous on $\overline{\Omega}$;
(b) $\lambda$ is strictly plurisubharmonic on $\Omega$;
(c) $\lambda = 0$ on $\partial \Omega$;
(d) $\lambda < 0$ on $\Omega$;
(e) For any $c < 0$, the set $\Omega_c = \{ z \in \Omega : \lambda(z) < c \}$ is relatively compact in $\Omega$.

A bounded plurisubharmonic exhaustion function carries important geometric information about the domain $\Omega$.

Now Diederich-Fornæss have proved the following [DF92].

**Theorem 2.3.** Let $\Omega$ be any smoothly bounded pseudoconvex domain, $\Omega = \{ z \in \mathbb{C} : \varrho(z) < 0 \}$. Then there exists $\delta, 0 < \delta \leq 1$, and a defining function $\tilde{\varrho}$ for $\Omega$ such that $-(-\tilde{\varrho})^\delta$ is a bounded strictly plurisubharmonic exhaustion function for $\Omega$.

The importance of this result in the setting of the regularity of the Bergman projection appears in the following related result, proved by Berndtsson-Charpentier [BeCh] and Kohn [Ko2], respectively.

**Theorem 2.4.** Let $\Omega$ be a smoothly bounded pseudoconvex domain and let $P$ denotes its Bergman projection. Let $\tilde{\rho}$ be a smooth defining function for $\Omega$ such that $-(-\tilde{\rho})^\delta$ is strictly plurisubharmonic. Then there exists $s_0 = s_0(\Omega, \delta)$ such that

$$P : H^s(\Omega) \to H^s(\Omega)$$

is continuous for all $0 \leq s < s_0$.

**Remark.** The sharp value of $s_0$ is not known, and most likely the exact determination of such a value might prove a very difficult task. The two sources [BeCh] and [Ko2] present completely different approaches and descriptions of $s_0$, that is of the range $[0, s_0)$ for which $P$ is bounded on $H^s$, with $s \in [0, s_0)$. In [BeCh] it is proved that such a range is at least $[0, \delta/2)$, i.e. they show that $s_0 \geq \delta/2$, while in [Ko2] the parameter $s_0$ is not so explicit, but it tends to infinity as $\delta \to 1$. The value found in [BeCh] has the advantage of providing an explicit lower bound for the regularity of the Bergman projection on a given domain, while the value given in [Ko2] is sharp in the sense given by Boas and Straube’s result Theorem 2.1.
Theorem 2.5. Let $\delta_0 > 0$ be fixed. Then there exists $\mu_0 > 0$ such that for all $\mu \geq \mu_0$ the following holds. If $\tilde{\rho}$ is a defining function for $\mathcal{W} = \mathcal{W}_\mu$, with $\mu \geq \mu_0$ and $\delta > 0$ is such that $-(-\tilde{\rho})^\delta$ is a bounded plurisubharmonic exhaustion function for $\mathcal{W}$, then $\delta < \delta_0$.

More precisely, we show that, in the notation above, $\delta < \nu = \pi/2\mu$.

Proof. We may assume that $\tilde{\rho} = h\rho$, where $\rho = \rho_\mu$ is defined in (1) and $h$ is a smooth positive function on $\mathcal{W}$. Then, by hypothesis $-h^\delta(-\rho)^\delta$ is strictly plurisubharmonic on $\mathcal{W}$.

Let
\[
\sigma(z_1, z_2) = -\frac{1}{2\pi} \int_0^{2\pi} h^\delta(z_1, e^{i\theta}z_2) (-\rho(z_1, e^{i\theta}z_2))^\delta \, d\theta
\]
\[
= -\frac{1}{2\pi} \int_0^{2\pi} h^\delta(z_1, e^{i\theta}z_2) \, d\theta (-\rho(z_1, z_2))^\delta
\]
\[
= -\tilde{h}(z_1, z_2)(-\rho(z_1, z_2))^\delta.
\]

Obviously, $\sigma$ is also strictly plurisubharmonic on $\mathcal{W}$, and $\tilde{h}$ is strictly positive and smooth on $\mathcal{W}$.

We can also write $\tilde{h}(z_1, z_2) = h^\#(z_1, |z_2|^2)$, where $h^\#$ is defined for $(z_1, t) \in \mathbb{C} \times \mathbb{R}^+$ such that if $|z_2|^2 = t$ then $(z_1, z_2) \in \mathcal{W}$. For simplicity of notation, we rename such a function $h$ again.

Thus we have that
\[
\sigma(z_1, z_2) = -h(z_1, |z_2|^2)(-\rho(z_1, z_2))^\delta
\]
is strictly plurisubharmonic on $\mathcal{W}$.

Now consider the points in $\mathcal{W}$ of the form $p = (z_1, z_2) = (\varepsilon e^{i\log|z_2|^2}, z_2)$ with $e^{-\mu/2} \leq |z_2| \leq e^{\mu/2}$.

For these points one has that
\[
\partial \rho(p) = ((1 - \varepsilon)e^{i\log|z_2|^2}, 0).
\]

A straightforward computation shows that, at such points $p = (\varepsilon e^{i\log|z_2|^2}, z_2)$ the Levi form $\mathcal{L}_\sigma$ of $\sigma$ calculated at vectors $v = (v_1, v_2) \in \mathbb{C}^2$ equals (all the functions are evaluated at the points $p$ and we write $\zeta$ in place of $e^{i\log|z_2|^2}$)
\[
\mathcal{L}_\sigma(p; (v_1, v_2)) = e^{\delta - 2}(2 - \varepsilon)^{\delta - 2} \left\{(2 - \varepsilon)\left(-\varepsilon^2(2 - \varepsilon)\partial^2_{z_1 \bar{z}_1} h + 2\delta \varepsilon (1 - \varepsilon)\Re(\zeta \partial_{z_1} h) + \delta \varepsilon h
\right.
\right.
\]
\[
+ \delta (1 - \delta)\frac{(1 - \varepsilon)^2}{2 - \varepsilon} h |v_1|^2
\right.
\]
\[
+ 2\varepsilon(2 - \varepsilon)\Re\left[(-\varepsilon(2 - \varepsilon)\partial^2_{z_1 \bar{z}_2} h + \delta (1 - \varepsilon)\partial_{z_2} h + \delta \frac{i\zeta}{z_2} h)v_1 v_2\right]
\left.
\right.
\]
\[
+ \delta(2 - \delta)(-2(2 - \delta)\partial^2_{z_2 \bar{z}_2} h + 2\delta \frac{2\delta}{|z_2|^2} h |v_2|^2\right\}.
\]

Next, we evaluate the above Levi form at vectors of the form $(v_1, v_2) = (u_1, \varepsilon u_2)$. Making the obvious simplification, we see that the necessary condition in order for $\sigma$ to be strictly plurisubharmonic is that and $0 < \varepsilon < 1$ and, for all $(u_1, u_2) \in \mathbb{C}^2$,
\[
\left(-\varepsilon^2(2 - \varepsilon)\partial^2_{z_1 \bar{z}_1} h + 2\delta \varepsilon (1 - \varepsilon)\Re(\zeta \partial_{z_1} h) + \delta \varepsilon h + \delta (1 - \delta)\frac{(1 - \varepsilon)^2}{2 - \varepsilon} h |u_1|^2
\right.
\]
\[
+ 2\varepsilon(2 - \varepsilon)\Re\left[(-\varepsilon(2 - \varepsilon)\partial^2_{z_1 \bar{z}_2} h + \delta (1 - \varepsilon)\partial_{z_2} h + \delta \frac{i\zeta}{z_2} h)u_1 \bar{u}_2\right] + \left(-(2 - \varepsilon)\partial^2_{z_2 \bar{z}_2} h + 2\delta \frac{2\delta}{|z_2|^2} h |u_2|^2\right) \geq 0.
\]
Since $h \in C^\infty(\overline{\mathbb{W}})$ this inequality must hold also for $\varepsilon = 0$ and $(0, z_2) \in \mathcal{A}$. Then we have
\[
\left(\frac{1}{2} \delta(1 - \delta)\bar{h}\right)|u_1|^2 + 2\text{Re} \left[ (\delta \zeta \partial_{z_2} \bar{h} + \delta \frac{i}{z_2} \bar{h})\bar{u}_1u_2 \right] + \left( -(2 - \varepsilon)\partial^2_{z_2 \bar{z}_2} \bar{h} + \frac{2\delta}{|z_2|^2} |u_2|^2 \right) \geq 0 .
\] (4)

Next, we substitute for $h$ the function $\tilde{h}$ defined on $\mathbb{C} \times \mathbb{R}^+$ such that $h(z_1, z_2) = \tilde{h}(z_1, |z_2|^2)$. Then
\[
\partial_{z_2} \tilde{h}(0, z_2) = \bar{z}_2 \partial_{\bar{z}} \tilde{h}(0, |z_2|^2) \quad \text{and} \quad \partial^2_{z_2 \bar{z}_2} \tilde{h}(0, z_2) = |z_2|^2 \partial^2_{t \bar{t}} \tilde{h}(0, |z_2|^2) + \partial_{\bar{z}} \tilde{h}(0, |z_2|^2) .
\]
Plugging these into (4) we then obtain the differential inequality for the function $\tilde{h}$:
\[
\frac{1}{2} \delta(1 - \delta)\bar{h}|u_1|^2 + 2\text{Re} \left[ \delta \zeta (\partial_t \tilde{h} + \frac{i}{|z_2|^2} \bar{h})u_1\bar{u}_2 \right] + \left( -2|z_2|^2 \partial^2_t \tilde{h} - 2\partial_t \tilde{h} + \frac{2\delta}{|z_2|^2} \bar{h} \right) |u_2|^2 \geq 0
\]
for all $(u_1, u_2) \in \mathbb{C}^2$, $e^{-\mu/2} \leq |z_2| \leq e^{\mu/2}$ (and the function $\tilde{h}$ being evaluated at the points $(0, |z_2|^2)$). Now if we choose $(u_1, u_2)$ of the form $(2e^{i\theta}/|z_2|, 1)$ in such a way that the second term in the above display becomes non-positive, we obtain that the function $\sigma$ is plurisubharmonic only if
\[
\frac{\delta(1 - \delta)}{|z_2|^2} \bar{h} - 2\delta( (\partial_t \tilde{h})^2 + \frac{\bar{h}^2}{|z_2|^2}) \geq 0
\]
for all points $(0, |z_2|^2)$ with $e^{-\mu/2} \leq |z_2| \leq e^{\mu/2}$.

We now set $g(s) = \tilde{h}(0, e^{s\delta})$ for $s \in [-\mu, \mu]$. Notice that $|z_2|^2 = e^s$ and that
\[
g' = e^s \partial_t \tilde{h} \quad \text{and} \quad g'' = e^{2s} \partial^2_t \tilde{h} .
\]
From (5) we obtain the differential inequality
\[
g'' + \delta^2 g \leq 0 ,
\]
for $s \in [-\mu, \mu]$, where $g$ is a smooth strictly positive function. From the strict positivity of $g$ it follows that, for all $0 < \delta' < \delta$, it must be that
\[
g'' + \delta'^2 g < 0 ,
\]
again for all $s \in [-\mu, \mu]$. Setting $\overline{g}(s) = g(s/\delta')$ the differential inequality above can be re-written as
\[
\overline{g}'' + \overline{g} < 0
\]
for all $s \in [-\mu \delta', \mu \delta']$. Finally, by translation (calling the new function $g$ again), i.e. setting $g(s) = \overline{g}(s + \mu \delta')$, we obtain that
\[
g'' + g < 0
\]
for a smooth strictly positive function $g$, for all $s \in [0, 2\mu \delta']$.

We now claim that there exists a smooth strictly positive function $\varphi$ such that
\[
\varphi'' + \varphi < 0 \quad \text{and} \quad \varphi' < 0
\]
for all $s \in [0, \mu \delta']$. For notice that if $g$ as above is such that $g'(a) < 0$, then $g'(s) < 0$ for $s \in [a, 2\mu \delta']$, while, if instead $g'(a) \geq 0$, then $g'(s) > 0$ for $s \in [0, a)$, since $g'' < 0$ on $[0, 2\mu \delta']$. In this latter case, making the substitution $s \mapsto 2\mu \delta' - s$ that preserves (6), we obtain a function with negative derivative on $[a, 2\mu \delta']$. By the arbitrariness of $\delta' < \delta$ we establish the claim.

Now, the argument at the end of the proof of Theorem 6 in [DF01] shows that the differential inequalities (7) above are possible only if $\mu \delta' < \pi/2$, i.e.
\[
\mu \delta' < \frac{\pi}{2\mu} = \nu .
\]
This proves the result. ☐
3. Non-Smooth Versions of the Worm Domain

In order to perform certain analyses on \( W \) some simplifications of the domain turn out to be particularly useful.

In the first instance, one can simplify the expression of the defining function \( \rho \) for \( W \) by taking \( \eta \) to be the characteristic function of the interval \([−\mu, \mu]\). This has the effect of truncating the two caps and destroying in part the smoothness of the boundary. Precisely, we can define

\[
W' = \left\{(z_1, z_2) \in C^2 : |z_1 - e^{i \log |z_2|^2}| < 1, \; |\log |z_2|^2| < \mu \right\}. \tag{8}
\]

We remark that \( W' \) is a bounded, pseudoconvex domain with boundary that is \( C^\infty \) except at points that satisfy

(i) \( |z_2| = e^{\mu/2} \) and \( |z_1 - e^{-i \log |z_2|^2}| = 1; \)
(ii) \( |z_2| = e^{-\mu/2} \) and \( |z - e^{-i \log |z_2|^2}| = 1. \)

Of interest are also two non-smooth, unbounded worms. Here, in order to be consistent with the results obtained in [KrPe], we change the notation a bit. (In practice, we set \( \mu = \beta - \pi/2 \).

For \( \beta > \pi/2 \) we define

\[
D_\beta = \left\{ \zeta \in C^2 : \Re (\zeta_1 e^{-i \log |\zeta_2|^2}) > 0, \; |\log |\zeta_2|^2| < \beta - \frac{\pi}{2} \right\} \tag{9}
\]
and

\[
D'_\beta = \left\{ z \in C^2 : |\Im z_1 - \log |z_2|^2| < \frac{\pi}{2}, \; |\log |z_2|^2| < \beta - \frac{\pi}{2} \right\} \tag{10}
\]

It should be noted that these latter two domains are biholomorphically equivalent via the mapping

\[
(z_1, z_2) \ni D'_\beta \leftrightarrow (e^{z_1}, z_2) \ni D_\beta. \tag{11}
\]

Neither of these domains is bounded. Moreover, these domains are not smoothly bounded. Each boundary is only Lipschitz, and, in particular, their boundaries are Levi flat.

We notice in passing that the slices of \( D_\beta \), for each fixed \( \zeta_2 \), are halfplanes in the variable \( \zeta_1 \). Likewise the slices of \( D'_\beta \), for each fixed \( \zeta_2 \), are strips in the variable \( \zeta_1 \).

The geometries of these domains are rather different from that of the smooth worm \( W \), which has smooth boundary and all boundary points, except those on a singular annulus \((0, e^{i \log |z_2|^2})\) in the boundary, are strongly pseudoconvex. However our worm domain \( D_\beta \) is actually a model for the smoothly bounded \( W \) (see, for instance, [Ba3]), and it can be expected that phenomena that are true on \( D_\beta \) or \( D'_\beta \) will in fact hold on \( W \) as well. We will say more about this symbiotic relationship below.

We now illustrate a first application of these non-smooth domains in the analysis of \( W \). We begin with the main result of [Kl].

**Theorem 3.1.** Let \( W' \) be as above. Then, there is a function \( f \in C^\infty(\overline{W'}) \) such that its Bergman projection \( Pf \) is not Hölder continuous of any positive order on \( \overline{W'} \).

Following Kiselman [Kl], we now describe an outline of the proof of this theorem. The steps are as follows:

(a) We construct a subspace \( C^+(W') \) of \( L^2(W') \) which contains all the Hölder continuous functions on \( \overline{W'} \).

(b) We construct a linear functional \( T \) whose values are obtained as holomorphic extensions of inner products \( \langle f, g_\alpha \rangle \) for certain elements \( g_\alpha \) of the Bergman space. That is to say, for a fixed \( f \in C^+(W') \), we define a holomorphic function of the complex variable \( \alpha \) by \( \Phi(\alpha) = \langle f, g_\alpha \rangle \); here \( \Re \alpha > -1 \). We set \( F(f) = \Phi(-2) \).

(c) We show that if \( f \) and \( Pf \) both belong to \( C^+(W') \) then \( T(Pf) = T(f) \); in particular, \( f - Pf \) is orthogonal to \( O^2(W') \), hence \( f - Pf \) is orthogonal to the \( g_\alpha \).
(d) We show that $T(f) = 0$ if $f$ is in $C^+(W')$ and holomorphic.

(e) We show that $T$ is not identically zero on $C^+(W')$. Specifically, $C^+(W')$ contains $C^\infty(W')$ and $T$ is not zero on $C^\infty(W')$.

(f) We finish the proof by taking an $f \in C^\infty(W')$ with $T(f) \neq 0$. If $Pf$ belongs to $C^+(W')$, then steps (a) and (c) tell us that $T(Pf) = T(f) \neq 0$. That contradicts (b).

Kiselman’s work was pioneering in that it put the worm domain at the forefront for examples that bear on Condition R and the regularity of the $\bar{\partial}$ problem.

The “worm” that Kiselman studies does not have smooth boundary. Yum-Tong Siu [Siu] later proved a version of Kiselman’s theorem on the smooth worm. His result is:

**Theorem 3.2.** For a suitable version of the smooth worm $W$, there is a function $f \in C^\infty(W')$ such that the Bergman projection $Pf$ is not Hölder continuous of any positive order on $W$.

Siu’s proof is quite intricate, and involves an argument with de Rham cohomology to show that caps may be added to Kiselman’s domain to make it into a smooth worm.

### 4. Irregularity of the Bergman Projection

We begin this section by discussing the proof of Barrett’s result Theorem [Ba1]. Now let us describe these ideas in some detail. We begin with some of Kiselman’s main ideas.

Let the Bergman space $H = A^2$ be the collection of holomorphic functions that are square integrable with respect to Lebesgue volume measure $dV$ on a fixed domain. Following Kiselman [Ki] and Barrett [Ba2], using the rotational invariance in the $z_2$-variable, we decompose the Bergman space for the domains $D_\beta$ and $D'_\beta$ as follows. Using the rotational invariance in $z_2$ and elementary Fourier series, each $f \in H$ can be written as

$$f = \sum_{j=-\infty}^{\infty} f_j,$$

where each $f_j$ is holomorphic and satisfies $f_j(z_1, e^{i\theta}z_2) = e^{ij\theta}f(z_1, z_2)$ for $\theta$ real. In fact such an $f_j$ must have the form

$$f_j(z_1, z_2) = g_j(z_1, |z_2|)z_j^2,$$

where $g_j$ is holomorphic in $z_1$ and locally constant in $z_2$.

Therefore

$$H = \bigoplus_{j \in \mathbb{Z}} H^j,$$

where

$$H^j = \{ f \in L^2 : f \text{ is holomorphic and } f(w_1, e^{i\theta}w_2) = e^{ij\theta}f(w_1, w_2) \}.$$

If $K$ is the Bergman kernel for $H$ and $K_j$ the Bergman kernel for $H^j$, then we may write

$$K = \sum_{j=-\infty}^{\infty} K_j.$$

Notice that, by the invariance property of $H^j$, with $z = (z_1, z_2)$ and $w = (w_1, w_2)$, we have that

$$K_j(z, w) = H_j(z_1, w_1)z_j^2/|z_2|^2.$$

Our job, then, is to calculate each $H_j$, and thereby each $K_j$. The first step of this calculation is already done in [Ba2]. We outline the calculation here for the sake of completeness.
Proposition 4.1. Let \( \beta > \pi/2 \). Then

\[
H_j(z_1, w_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(z_1 - \bar{w}_1)\xi} (\xi - \frac{i+1}{2})}{\sinh(\pi\xi) \sinh \left((2\beta - \pi) \left(\xi - \frac{i+1}{2}\right)\right)} \, d\xi.
\] (12)

The papers [Ki] and [Ba2] calculate and analyze only the Bergman kernel for \( \mathcal{H}^{-1} \) (i.e., the Hilbert subspace with index \( j = -1 \)). This is attractive to do because certain “resonances” cause cancellations that make the calculations tractable when \( j = -1 \). One of the main thrusts of the work [KrPe] is to perform the more difficult calculations for all \( j \) and then to sum them over \( j \).

Proof. We begin by following the calculations in [Ki] and [Ba2] in order to get our hands on the Bergman kernels of the \( \mathcal{H}^j \). Let \( f_j \in \mathcal{H}^j \) and fix \( w_2 \). Then \( f_j(w_1, w_2) = h_j(w_1)w_2^j \) (where we of course take into account the local independence of \( h_j \) from \( w_2 \)). Now, writing \( w_1 = x + iy \), \( w_2 = re^{i\theta} \), and then making the change of variables \( \log r^2 = s \), we have

\[
\|f_j\|_{\mathcal{H}}^2 = \int_{D_\beta} |h_j(w_1)|^2 |w_2|^{2j} dV(w)
= \int_0^\infty \int_{|y| < \beta} |h_j(x + iy)|^2 \int_{|\log r^2| < \beta - \frac{\pi}{2}} r^{2j+1} ds \, dy \, dx
= \pi \int_{\mathbb{R}} \int_{|y| < \beta} |h_j(x + iy)|^2 \int_{|s| < \beta - \frac{\pi}{2}} e^{s(j+1)} ds \, dy \, dx
= \int_{S_\beta} \int_{\mathbb{R}} |h_j(w_1)|^2 \left( \chi_{\pi/2} * \left[e^{(j+1)(\cdot)} \chi_{\beta-\pi/2}(\cdot)\right] \right)(y) \, dy \, dx;
\] (13)

here we have set

\[
S_\beta = \{ x + iy \in \mathbb{C} : |y| < \beta \}
\]

and used the notation

\[
\chi_\alpha(y) = \begin{cases} 1 & \text{if } |y| < \alpha, \\ 0 & \text{if } |y| \geq \alpha. \end{cases}
\]

For \( \beta > \frac{\pi}{2} \), we now set

\[
\lambda_j(y) = \left( \chi_{\pi/2} * \left[e^{(j+1)(\cdot)} \chi_{\beta-\pi/2}(\cdot)\right] \right)(y).
\]

So line (13) equals

\[
\int_{S_\beta} \int_{\mathbb{R}} |h_j(w_1)|^2 \lambda_j(y) \, dx \, dy.
\]

Thus we have shown that, if \( f_j \in \mathcal{H}^j \), \( f_j = h_j(w_1)w_2^j \), then

\[
\|f_j\|_{\mathcal{H}}^2 = \int_{S_\beta} \int_{\mathbb{R}} |h_j(w_1)|^2 \lambda_j(y) \, dx \, dy.
\]

Now let \( \varphi \in A^2(S_\beta, \lambda_j \, dA) \). That is, \( \varphi \) is square-integrable on \( S_\beta \) with respect to the measure \( \lambda_j \, dA \) (here \( dA = dx \, dy \) is two-dimensional area measure). Note that \( \lambda_j \) depends only on the single variable \( y \). Let \( \hat{\varphi} \) denote the partial Fourier transform of \( \varphi(x + iy) \) in the \( x \)-variable. Then (by standard Littlewood-Paley theory)

\[
\hat{\varphi}(\xi, y) = \int \varphi(x + iy) e^{-ix\xi} \, dx = e^{-y\xi} \hat{\varphi}_0(\xi),
\]

where \( \hat{\varphi}_0 \) is the Fourier transform of \( \varphi \) with respect to the variable \( y \).
where \( \varphi_0(x) = \varphi(x + i0) \). Therefore, denoting by \( B_\beta = B_\beta^{(j)} \) the Bergman kernel for the strip \( S_\beta \) with respect to the weight \( \lambda_j \) and writing \( \omega = s + it \) and denoting by \( \xi \) the variable dual to \( s \), we have
\[
\int_{\mathbb{R}} \tilde{\varphi}_0(\xi)e^{i\xi t} \, d\xi = 2\pi \varphi(\xi) = 2\pi \int_{S_\beta} \varphi(\omega) B_\beta(\zeta, \omega) \lambda_j(\text{Im} \, \omega) \, dA(\omega)
\]
\[
= \int_{-\beta}^{\beta} \int_{\mathbb{R}} \tilde{\varphi}(\zeta, t) B_\beta(\zeta, t, \xi) \lambda_j(t) \, d\xi \, dt
\]
\[
= \int_{\mathbb{R}} \tilde{B}_\beta(\zeta, t) \int_{-\beta}^{\beta} \tilde{\varphi}_0(\xi)e^{-2\xi t} \lambda_j(t) \, dt \, d\xi.
\]
Notice that there is a factor of \( e^{-\xi t} \) from each of the Fourier transform functions in the integrand.

This gives a formula for \( B_\beta \):
\[
\tilde{B}_\beta(\zeta, (\xi, 0)) = \frac{e^{i\xi t}}{\int_{-\beta}^{\beta} e^{-2\xi t} \lambda_j(t) \, dt} = \frac{e^{i\xi t}}{\lambda_j(-2i\xi)}.
\]

Amalgamating all our notation, and using the fact that the (Hermitian) diagonal in \( \mathbb{C}^2 \) is a set of determinacy, we find that
\[
B_\beta(z, w) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i(z - \bar{z} + t)\xi}}{\lambda_j(-2i\xi)} \, d\xi.
\]

But of course \( (\chi_{\pi/2})_\beta(\xi) = (e^{i\pi/2} - e^{-i\pi/2})/\xi \), so that
\[
(\chi_{\pi/2})_\beta(-2i\xi) = \frac{1}{\xi} \sinh(\pi \xi).
\]

Furthermore,
\[
(e^{(j+1)s})\chi_{j} \beta - \frac{s}{2}(s) = \frac{\sinh((2\beta - \pi)(\xi - \frac{j+1}{2}))}{\xi - \frac{j+1}{2}}.
\]

Thus
\[
\tilde{\lambda}_j(-2i\xi) = \frac{\sinh(\pi \xi) \sinh((2\beta - \pi)(\xi - \frac{j+1}{2}))}{\xi(\xi - \frac{j+1}{2})}
\]
and
\[
\frac{1}{\lambda_j(-2i\xi)} = \frac{\xi(\xi - \frac{j+1}{2})}{\sinh(\pi \xi) \sinh((2\beta - \pi)(\xi - \frac{j+1}{2}))}.
\]

In conclusion,
\[
H_j(z_1, w_1) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i(w_1 - \bar{w}_1)\xi}}{\sinh(\pi \xi) \sinh((2\beta - \pi)(\xi - \frac{j+1}{2}))} \, d\xi,
\]
thus proving (12). □

At this point we sketch the proof of the main result of Barrett in [Ba3].

**Sketch of the proof of Theorem 1.8.** The proof starts from the observation that the Bergman projection \( P \) on \( W \) preserves each \( \mathcal{H}^j \). Therefore in order to show that \( P \) is not continuous on \( H^s \), for some \( s \), it suffices to show that \( P \) fails to be continuous when restricted to some \( \mathcal{H}^j \).

The first step is to calculate the asymptotic expression for the kernel when \( j = -1 \). Recall that we are working on the non-smooth domain \( D'_\beta \). Using the method of contour integral it is not difficult to obtain that
\[
K'_{-1}(z, w) = (e^{-\nu |z_1 - \bar{w}_1|} + O(e^{-\nu |\text{Re} \, z_1 - \text{Re} \, w_1|})) \cdot (z_2 \bar{w}_2)^{-1}
\]
as \( \text{Re} \, z_1 - \text{Re} \, w_1 \) → +∞, uniformly in all closed strips \( \{ |\text{Im} \, z_1|, |\text{Im} \, w_1| \leq \ldots \} \), with \( \nu > \nu_0 \).
By applying the biholomorphic transformation (11) one obtains an asymptotic expression for the kernel $K_{-1}$ relative to the domain $D_\beta$:

$$K_{-1}(\zeta, \omega) = \left( \frac{|\zeta_1||\omega_1|}{|\zeta_1|} + O(|\omega_1|^{\nu_\beta}/|\zeta_1|^{\nu_\beta}) \right)^{-\nu_\beta} \cdot (\zeta_2 \bar{\omega}_2)^{-1},$$

with $\nu > \nu_\beta$, as $|\zeta_1| - |\omega_1| \to 0^+$. The proof of these two assertions can be found in [Ba3] (or see [CheS]).

The next step is a direct calculation to show that $K_{-1}(\cdot, w) \notin H^s(D_\beta)$ for $s \geq \nu_\beta$. This assertion is proved by using the characterization of Sobolev norms for holomorphic functions on a domain $\Omega$: For $-1/2 < t < 1/2$, a non-negative integer, the norm

$$\sum_{|\alpha| \leq m} \left\| \rho^t \partial^\alpha \bar{h} \right\|_{L^2(\Omega)}$$

is equivalent to the $H^{m-t}$-norm of the holomorphic function $h$. The proof of such a characterization can be found in [Lig2].

Next, one notices that the reproducing kernel $K_{-1}(\cdot, w)$ can be written as the projection of a radially symmetric smooth cut-off function $\chi$, translated at $w$. That is, if we denote by $P_{-1}$ the projection relative to the subspace $H^{-1}$, then

$$K_{-1}(\cdot, w) = P_{-1} \left( \chi(\cdot - w) \right).$$

Therefore, since $K_{-1}(\cdot, w) \notin H^s(D_\beta)$ for $s \geq \nu_\beta$, then $P_{-1}$, and therefore $P_{D_\beta}$ is not continuous on $H^s(D_\beta)$.

The final step of the proof is to transfer this negative result from $D_\beta$ to $W$. This is achieved by an exhaustion argument. We adapt this kind of argument to obtain a negative result in the $L^p$-norm in the proof of Theorem 7.6 and we do not repeat the argument here.

5. Failure of Global Hypoellipticity and Condition $R$

In order to discuss the failure of Condition $R$ on the Diederich-Fornaess worm domain, we recall the basic facts about the $\bar{\partial}$-Neumann problem.

Let $\Omega \subset \mathbb{C}^n$ be a bounded domain with smooth boundary and let $\rho$ be a smooth defining function for $\Omega$. The $\bar{\partial}$-Neumann problem on $\Omega$ is a boundary value problem for the elliptic partial differential operator

$$\Box = \bar{\partial} \partial^* + \partial^* \bar{\partial}. $$

Here $\bar{\partial}$ denotes the $L^2$-Hilbert space adjoint of the (unbounded) operators $\partial$. In order to apply $\Box$ to a form or current $u$ one needs to require that $u, \partial u \in \text{dom}(\bar{\partial})$. These conditions translate into two differential equations on the boundary for $u$ and the are called the two $\bar{\partial}$-Neumann boundary conditions (see [FoKo] or [H]). These equations are

$$u \cdot \bar{\partial} \rho = 0, \quad \text{and} \quad \bar{\partial} u \cdot \partial \rho = 0, \quad \text{on} \ \partial \Omega. \quad (14)$$

Thus the equation $\Box u = f$ becomes a boundary value problem.

$$\begin{cases}
\Box u = f & \text{on} \ \Omega \\
u \cdot \bar{\partial} \rho, \partial u \cdot \partial \rho = 0 & \text{on} \ \partial \Omega.
\end{cases} \quad (15)$$

This is an equation defined on forms. The significant problem is for $(0,1)$-forms, and we restrict to this case in the present discussion.

It follows from Hörmander’s original paper on the solution of the $\bar{\partial}$-equation [H02] that the $\bar{\partial}$-Neumann problem is always solvable on a smoothly bounded pseudoconvex domain $\Omega$ in $\mathbb{C}^n$. 
for any data \( f \in L^2(\Omega) \). We denote by \( N \)—the Neumann operator—such a solution operator. Moreover, \( N \) turns out to be continuous in the \( L^2 \)-topology:
\[
\|Nu\|_{L^2} \leq c\|u\|_{L^2}.
\]

An important formula of Kohn says that
\[
P = I - \overline{\nabla} N \nabla.
\]

The proof of this is a formal calculation—see [Kr2]. Important work by Boas and Straube [BoS2] essentially established that the Neumann operator \( N \) has a certain regularity (that is, it maps some Sobolev space \( H^s \) to itself, for instance) if and only if \( P \) will have the same regularity property. In particular if \( N \) is continuous on a Sobolev space \( H^s \) for some \( s > 0 \) (of \( (0,1) \)-forms), then the Bergman projection \( P \) is continuous on the same Sobolev space \( H^s \) (of functions).

Such regularity is well known to hold on strongly pseudoconvex domains ([FoKo, Kr2]). In addition, Catlin proved a similar regularity result on finite type domains (see [Cat1], [Cat2], [Kr1]).

Michael Christ’s milestone result [Chr1] has proved to be of central importance for the field. It demonstrates concretely the seminal role of the worm, and points to future directions for research. Certainly the research program being described here, including the calculations in [KrPe], is inspired by Christ’s work.

Christ’s work is primarily concerned with global regularity, or global hypoellipticity. A partial differential operator \( L \) is said to be \textit{globally hypoelliptic} if, whenever \( Lu = f \) and \( f \) is globally \( C^\infty \), then \( u \) is globally \( C^\infty \). We measure regularity, here and in what follows, using the standard Sobolev spaces \( H^s \), \( 0 < s < \infty \) see [Kr2], [Hör1]).

Christ’s proof of the failure of global hypoellipticity is a highly complex and recondite calculation with pseudodifferential operators. We cannot replicate it here. But the ideas are so important that we feel it worthwhile to outline his argument. We owe a debt to the elegant and informative paper [Chr2] for these ideas.

As a boundary value problem for an elliptic operator, the \( \overline{\partial} \)-Neumann problem may be treated by Calédon’s method of reduction to a pseudodifferential equation on \( \partial \Omega \). The sources [Hör3] and [Chr2] give full explanations of the classical ideas about this reduction. In the more modern reference [CNS], Chang, Nagel and Stein elaborate the specific application of these ideas to the \( \overline{\partial} \)-Neumann problem in \( C^2 \). (Thus, in the remaining part of this discussion, \( \Omega \) will denote a smoothly bounded pseudoconvex domain in \( C^2 \).) The upshot is that one reduces the solution of the equation \( \square u = f \) to the solution of an equation \( \square v = g \) on the boundary. Here \( u \) and \( f \) are \((0,1)\)-forms, while \( v \) and \( g \) are sections of a certain complex line bundle on \( \Omega \). (The fact that this bundle is 1-dimensional is a consequence of the inclusion \( \Omega \subset C^2 \).)

To be more explicit, the solution \( u \) of (15) can be written as \( u = Gf + \mathcal{R}v \), where \( G \) is the \textit{Green operator} and \( \mathcal{R} \) is the \textit{Poisson operator}2 for the operator \( \square \) and \( v \) is chosen in such a way to satisfy the boundary conditions. In fact,
\[
\square (Gf + \mathcal{R}v) = f + 0 = f \quad \text{on} \quad \Omega
\]
\[
(Gf + \mathcal{R}v) \overline{\partial} \rho = v \overline{\partial} \rho = 0 \quad \text{on} \quad \partial \Omega
\]
\[
\overline{\partial} (Gf + \mathcal{R}v) \overline{\partial} \rho = \overline{\partial} Gf \overline{\partial} \rho + \overline{\partial} v \overline{\partial} \rho = 0 \quad \text{on} \quad \partial \Omega.
\]

The section \( v \) has two components, but one of these vanishes because of the first \( \overline{\partial} \)-Neumann boundary condition. The second \( \overline{\partial} \)-Neumann boundary condition may be written as an equation \( \square ^+ v = g \) on \( \partial \Omega \), where \( \square ^+ \) is a pseudodifferential operator of order 1. Also we note that \( g = (\overline{\partial} Gf \overline{\partial} \rho) \) restricted to \( \partial \Omega \).

2Thus \( \mathcal{G} \) is the solution operator for the elliptic boundary value problem \( \square (\mathcal{G}f) = f \) on \( \Omega \) and \( \mathcal{G}f = 0 \) on \( \partial \Omega \), while \( \mathcal{R} \) is the solution operator for the elliptic boundary value problem \( \square (\mathcal{R}v) = 0 \) on \( \Omega \) and \( \mathcal{R}v = v \) on \( \partial \Omega \).
Christ’s argument begins with a real-variable model for the $\overline{\partial}$-Neumann problem that meshes well with the geometry of the boundary of the worm domain $\mathcal{W}$.

Let $M$ be the 2-torus $\mathbb{T}^2$ and let $X, Y$ two smooth real vector fields on $M$. Fix a coordinate patch $V_0$ in $M$ and suppose that $V_0$ has been identified with $\{(x, t) \in (-2, 2) \times (-2\delta, -2\delta)\} \subset \mathbb{R}^2$. Let $J = [-1, 1] \times \{0\} \subset V_0$.

Call a piecewise smooth path $\gamma$ on $M$ admissible if every tangent to $\gamma$ is in the span of $X, Y$. Assume that

(i) The vector fields $X, Y, [X, Y]$ span the tangent space to $M$ at every point of $M \setminus J$.
(ii) In $V_0$, $X \equiv \partial_x$ and $Y \equiv b(x, t) \partial_t$.
(iii) For all $|x| \leq 1$ and $|t| \leq \delta$, we have that $b(x, t) = \alpha(x)t + \mathcal{O}(t^2)$, where $\alpha(x)$ is nowhere vanishing.

It follows then that every pair $x, y \in M$ is connected by an admissible path.

**Theorem 5.1.** With $X, Y, M$ as above, let $L$ be any partial differential operator on $M$ of the form $L = -x^2 - Y^2 + a$, where $a \in C^\infty(M)$ and

$$\|u\|^2 \leq C(Lu, u) \quad (16)$$

for all $u \in C^2(M)$. Then $L$ is not globally regular in $C^\infty$.

We note that our hypotheses, particularly inequality (16), imply that $L$ has a well-defined inverse $L^{-1}$ which is a bounded linear operator on $L^2(M)$.

The following theorem gives a more complete, and quantitative, version of this result:

**Theorem 5.2.** Let $X, Y, M, L$ be as above. Then $L$ has the following global properties:

(a) There is a positive number $s_0$ such that, for every $0 < s < s_0$, $L^{-1}$ preserves $H^s(M)$;
(b) For each $s > s_0$, $L^{-1}$ fails to map $C^\infty(M)$ to $H^s(M)$;
(c) There is a sequence of values $s < r$ tending to infinity such that if $u \in H^s(M)$ satisfies $Lu \in H^r(M)$ then $u \in H^r$;
(d) There are arbitrarily large values of $s$ with a constant $C = C_s$ such that if $u \in H^s(M)$ is such that $Lu \in H^s(M)$ then

$$\|u\|_{H^s} \leq C\|Lu\|_{H^s}. \quad (17)$$

(e) For each value of $s$ as in part (d), $\{f \in H^s(M) : L^{-1}f \in H^s(M)\}$ is a closed subspace of $H^s$ with finite codimension.

The proof of Theorem 5.1 breaks into two parts. The first part consists of proving the a priori inequality (16). The second part, following ideas of Barrett in [Ba3], shows that, for any $s \geq s_0$, the operator $L$ cannot be exactly regular on $H^s(M)$. We refer the reader to [Chr1] for the details. Section 8 of [Chr2] also provides a nice outline of the analysis.

The next step is to reduce the analysis of the worm domain, as defined in our Sections 2 and 3, to the study of the manifold $M$ as above. With this idea in mind we set $\mathcal{T} = \overline{\partial}_b$ and $L$ its complex conjugate. The characteristic variety $\Sigma$ of $L$ is a real line bundle $\Sigma$ that splits smoothly as two rays: $\Sigma = \Sigma^+ \cup \Sigma^-$. The principal symbol of $\square^+$ vanishes only on $\Sigma^+$ that is half the characteristic variety We may compose $\square^+$ with an elliptic pseudodifferential operator of order $+1$ to change $\square^+$ to the form

$$L = \mathcal{T}L + B_1\mathcal{T} + B_2L + B_3 \quad (18)$$

microlocally in a conical neighborhood of $\Sigma^+$, where each $B_j$ is a pseudodifferential operator with order not exceeding 0. Since $\square^+$ is elliptic on the complement of $\Sigma^+$, our analysis may thus be microlocalized to a small conical neighborhood of $\Sigma^+$.

---

(3) The characteristic variety of a pseudodifferential operator is the conic subset of the cotangent bundle on which its principal symbol vanishes.
Theorem 6.1. Let \( W \) be the worm. Then there is a discrete subset \( S \subset \mathbb{R}^+ \) such that, for each \( s \notin S \) and each \( j \in \mathbb{Z} \), there is a constant \( C = C(s, j) < \infty \) such that, for each \((0, 1)\) form \( u \in \mathcal{H}_d^1 \cap C^\infty(\overline{W}) \) such that \( Nu \in C^\infty \), it holds that
\[
\|Nu\|_{\mathcal{H}_d^r(W)} \leq C \cdot \|u\|_{\mathcal{H}_d^r(W)}.
\]

The operators \( \mathcal{L}, \mathcal{T}, L, B_j \) in (17) may be constructed so as to commute with the circle action in the second variable, hence they will preserve each \( \mathcal{H}^j \). In summary, for each \( j \), the action of \( \mathcal{L} \) on \( \mathcal{H}^j(\partial W) \) may be identified with the action of an operator \( \mathcal{L}_j \) on \( L^2(\partial W/S^1) \).

Of course \( \partial W \) is 3-dimensional, hence \( \partial W/S^1 \) is a real 2-dimensional manifold. It is convenient to take coordinates \((x, \theta, t)\) on \( \partial W \) so that
\[
\begin{align*}
z_2 &= \exp(x + i\theta) \quad \text{and} \quad z_1 = \exp(i2x)(e^{it} - 1); \\
\end{align*}
\]
here \(|\log|z_2|^2| \leq r \) and \( \mathcal{L}_j \) takes the form \( \mathcal{L} = B_1 \mathcal{L} + B_2 L + B_3 \) (just as in (18)\!). In this last formula, \( \mathcal{L} \) is a complex vector field which has the form \( \mathcal{L} = \partial_x + \alpha(t)\partial_t \), where \(|x| \leq r/2, \alpha(0) \neq 0 \), and each \( B_j \) is a classical pseudodifferential operator of order not exceeding 0—depending on \( j \) in a non-uniform manner.

We set \( J = \{(x, t) : |x| \leq r/2, t = 0\} \) and write \( \mathcal{L} = X + iY; \) then the vector fields \( X, Y, [X, Y] \) span the tangent space to \( \partial W/S^1 \) at each point of the complement of \( J \), and are tangent to \( J \) at every point of \( J \). We conclude that the operator \( \mathcal{L}_j \) on \( \partial W/S^1 \) is quite similar to the two-dimensional model that we discussed above.

There are two complications which we must note (and which are not entirely trivial): (1) There are pseudodifferential factors, and the reduction of the \( J \)-Neumann problem to \( \mathcal{L} \), and thereafter to \( \mathcal{L}_j \), requires only a microlocal \( a \) priori estimate for \( \mathcal{L}_j \) in a conic subset of phase space; (2) The lower order terms \( B_j, B_3 \) are not negligible, indeed they determine the values of the exceptional Sobolev exponents, but the analysis can be carried out for these terms as well.

It should be noted that a special feature of the worm is that the rotational symmetry in \( z_2 \) makes possible (as we have noted) a reduction to a 2-dimensional analysis, and this in turn produces a certain convenient ellipticity. There is no uniformity of estimates with respect to \( j \), but the analysis can be performed for each fixed \( j \).

6. The Automorphism Group of the Worm Domain

It is of interest to know whether a biholomorphic mapping of the smooth worm \( W \) to any other smoothly bounded pseudoconvex domain will extend to a diffeomorphism of the closures. Of course the worm does not satisfy Condition \( R \), so the obvious tools for addressing this question are not available. As a partial result, So-Chin Chen [Che1] has shown that the automorphism group of \( W \) reduces to the rotations in the \( z_2 \)-variable; hence all biholomorphic self-maps of \( W \) do extend smoothly to the boundary. His result is this:

**Theorem 6.1.** Let \( W \) be the smooth worm. Then any automorphism (i.e., biholomorphic self-map) of \( W \) must be a rotation in the \( z_2 \)-variable. In particular, the automorphism must extend to a diffeomorphism of the closure.

**Proof.** This is an interesting calculation. First recall that, by Proposition 5.3, the boundary of the smooth worm \( W \) is strongly pseudoconvex except on the annulus \( \mathcal{A} \) of the points \((0, e^t \log|z_2|^2)\) for \(|\log|z_2|^2| \leq \mu \).
Let now \( g = (g_1, g_2) \) be an automorphism of the worm \( W \). Then, by the fundamental result of Bell [Bel2], \( g \) can be extended smoothly to all the strongly pseudoconvex points of the boundary. In other words, \( g \) extends to a \( C^\infty \)-diffeomorphism of \( W \setminus A \) onto itself.

Consider now, for \( e^{-\mu/2} < a < e^{\mu/2} \), the set

\[
T_a = \{ (z_1, z_2) \in \partial W : e^{-\mu/2} < |z_2| = a < e^{\mu/2}, \; z_1 \neq 0 \}.
\]

Notice that \( T_a \) is a deleted torus, made of points of strong pseudoconvexity in \( \partial W \). Then, \( g|_{T_a} \) is a \( C^\infty \)-diffeomorphism having image contained in \( \partial W \setminus \{ (z_1, z_2) \in \partial W : \; z_1 \neq 0, \; |\log|z_2|| < \mu \} \).

Then, if \( \rho(z_1, z_2) = |z_1 - e^{i \log |z_2|^2} - (1 - \eta(\log|z_2|^2)) \) is the obvious defining function for \( W \), we see that the conditions

\[
|z_2| = a \quad \text{and} \quad |g_1(z_1, z_2) - e^{i \log |g_2(z_1, z_2)|^2}| = 1
\]

define \( T_a \). This implies that \( \log |g_2(z_1, e^{i\theta}a)|^2 = \log a^2 \) for all \( \theta \) real. Thus \( |g_2(z_1, z_2)|^2 = |z_2|^2 \cdot e^{2k \pi} \) for some integer \( k \). By considering points \( (z_1, z_2) \in T_a \) with \( a \) close to either \( e^{-\mu/2} \) or \( e^{\mu/2} \), we may conclude that \( k = 0 \) and \( |g_2(z_1, z_2)| = |z_2| \) for \( (z_1, z_2) \in T_a \), \( e^{-\mu/2} < a < e^{\mu/2} \).

Now, with \( a \) fixed as before, let \( \tilde{z}_1 \) be a point with \( |\tilde{z}_1 - e^{i \log |a|^2}| < 1 \) and so that \( \tilde{z}_1 \) lies in a small open neighborhood of \( 2e^{i \log |a|^2} \).

Consider the set given by

\[
\{(\tilde{z}_1, z_2) \in \mathbb{C}^2 \cap W \}.
\]

It is not difficult to see that such a set is the union of (finitely many) concentric annuli. Let \( A_{\tilde{z}_1} \) denote one of these annuli, the inner boundary of \( A_{\tilde{z}_1} \) being some circle \( C_\alpha = \{ (\tilde{z}_1, z_2) \in \partial W : |z_2| = \alpha \} \) and the outer boundary being another circle \( C_\beta = \{ (\tilde{z}_1, z_2) \in \partial W : |z_2| = \beta \} \) with \( \alpha < a < \beta \). Then \( A_{\tilde{z}_1} \) is easily identified with the planar annulus \( A = \{ z_2 \in \mathbb{C} : \alpha < |z_2| < \beta \} \).

We therefore obtain (using this identification)

\[
g_2(\tilde{z}_1, C_\alpha) = C_\alpha \quad \text{and} \quad g_2(\tilde{z}_1, C_\beta) = C_\beta.
\]

Note that \( g_2(\tilde{z}_1, \cdot) \) can be extended to an entire function on all of \( \mathbb{C} \) just by using the Schwarz reflection principle.

Now line (19) tells us that

\[
g_2(z_1, z_2) = e^{i \theta(z_1)} \cdot z_2
\]

for some real function \( \theta(z_1) \). But \( g_2 \) is holomorphic in \( z_1 \), we may conclude that \( \theta \) is a real constant \( \theta_0 \). Thus

\[
g_2(z_1, z_2) = e^{i \theta_0} \cdot z_2 \quad \text{for} \; (z_1, z_2) \in W.
\]

As a consequence, since \( g : W \to W \) we have that

\[
\rho(g_1(z_1, z_2), g_2(z_1, z_2)) = \left| g_1(z_1, z_2) - e^{i \log |g_2(z_1, z_2)|^2} \right|^2 < 1 - \eta(\log |g_2(z_1, z_2)|^2);
\]

that is,

\[
\left| g_1(z_1, z_2) - e^{i \log |z_2|^2} \right|^2 < 1 - \eta(\log |z_2|^2) \quad (21)
\]

Now examine the open, solid torus \( \Pi_a \) given by

\[
\Pi_a = \{ (z_1, z_2) \in W : e^{-\mu/2} < |z_2| = a < e^{\mu/2}, \; |z_1 - e^{i \log |z_2|^2}| < 1 \}.
\]

Set, for \( \theta \) real,

\[
\Delta_{a, \theta} = \{ (z_1, a e^{i \theta}) \in W : |z_1 - e^{i \log |a|^2}| < 1 \}.
\]

By (21) it follows that the restriction of \( g_1 \) to \( \Delta_{a, \theta_1} \) must map \( \Delta_{a, \theta_1} \), biholomorphically onto \( \Delta_{a, \theta_2} \) for some \( \theta_2 \). Thus the restriction of \( g_1 \) to \( \Delta_{a, \theta_1} \) can be extended smoothly to \( \Delta_{a, \theta_1} \). We know
that \( g_1(0, ae^{i\theta_1}) = 0 \), so it follows that \( g_1(z_1, z_2) \) can be expressed by way of the well-known automorphisms of the unit disc (see [GKr3]). We see then that

\[
g(z_1, ae^{i\theta}) = e^{i\theta_0} b - \left( z_1 - e^{i\log|a|^2} \right) + e^{i\log|a|^2}
\]

for some \( \theta \) real and \( b = (ae^{i\log|a|^2}) \) with \( |b| < 1 \). Using the fact that \( g_1(0, ae^{i\theta_1}) = 0 \), we calculate \( \theta_0 \) and we obtain that

\[
g_1(z_1, z_2) = e^{i\log|z_2|^2} \left( 1 + \frac{b(z_2)e^{i\log|z_2|^2}}{1 + b(z_2)e^{i\log|z_2|^2}} \right) \frac{b(z_2) + e^{i\log|z_2|^2} - z_1}{1 - b(z_2)(z_1 - e^{i\log|z_2|^2})} - e^{i\log|z_2|^2}; \tag{22}
\]

here \( b(z_2) \) is a real analytic function satisfying \( |b(z_2)| < 1 \) for \( e^{-\mu/2} < |z_2| < e^{\mu/2} \). Equation 22 shows that there is a small \( \varepsilon > 0 \) such that \( g_1(z_1, z_2) \) is real analytic on \( \Delta(0, \varepsilon) \times A_\delta \), where \( A_\delta = \{ z_2 \in \mathbb{C} : e^{\delta + \mu/2} < |z_2| < e^{-\delta + \mu/2} \} \), for some small \( \delta > 0 \). Thus we see that \( g_1(z_1, z_2) \) is holomorphic on \( \Delta(0, \varepsilon) \times A_\delta \).

As a consequence, one can write

\[
g_1(z_1, z_2) = \sum_{j=1}^{\infty} a_j(z_2) z_1^j,
\]

with \( a_j(z_2) \) holomorphic on \( A_\delta \) for all \( j \geq 1 \). Direct calculation yields that

\[
a_1(z_2) = \frac{\partial g_1}{\partial z_1}(0, z_2) = \frac{1 - |b(z_2)|^2}{|1 + b(z_2)e^{i\log|z_2|^2}|^2}.
\]

Thus \( a_1(z_2) \) is a positive real constant, i.e., \( a_1(z_2) = c > 0 \).

Next we turn to the computation of \( a_2(z_2) \). Now

\[
a_2(z_2) = \frac{1}{2} \cdot \frac{\partial^2 g_1}{\partial z_1^2}(0, z_2) = c \cdot \frac{b(z_2)}{1 + b(z_2)e^{i\log|z_2|^2}}. \tag{23}
\]

We assert that \( a_2 \equiv 0 \). Now set

\[
h(z_2) = \frac{a_2(z_2)}{c} = \frac{b(z_2)}{1 + b(z_2)e^{i\log|z_2|^2}},
\]

which is holomorphic on \( A_\delta \). We see that

\[
c = \frac{1 - |b(z_2)|^2}{|1 + b(z_2)e^{i\log|z_2|^2}|^2}
\]

\[
= \frac{|1 + h(z_2)e^{i\log|z_2|^2} - |h(z_2)|^2}{1 + 2\text{Re}(h(z_2)e^{i\log|z_2|^2})}.
\]

In conclusion, we may write

\[
h(z_2)e^{i\log|z_2|^2} = c_0 + iI(z_2),
\]

where \( c_0 = \frac{1}{2}(c - 1) \) and \( I(z_2) \) is a smooth, real-valued function on \( A_\delta \). Thus we have

\[
h(z_2) = c_0 e^{-i\log|z_2|^2} + iI(z_2)e^{-i\log|z_2|^2}, \tag{24}
\]

which is holomorphic on \( A_\delta \). Locally we may multiply equation (22) by \( e^{2i\log z_2} \) to obtain a new holomorphic function, and we find that

\[
h(z_2)e^{2i\log z_2} = c_0 e^{-2\arg z_2} + iI(z_2)e^{-2\arg z_2}.
\]
Of course the real part of \( g(z_2) e^{2i \log z_2} \) is a harmonic function. We write \( z_2 = u + iv \) as usual. By direct computation we find that
\[
\delta_{z_2}(c_0 e^{-2\text{arg} z_2}) = c_0 \delta_{z_2}(e^{-2\tan^{-1} v/u}) = \frac{4c_0}{u^2 + v^2} e^{-2\tan^{-1} v/u} \equiv 0.
\]
This entails \( c_0 = 0 \) so that \( c = 1 \). Thus (24) reduces to
\[
-ih(z_2) = I(z_2) e^{-i\log |z_2|^2},
\]
which is holomorphic on \( A_\delta \). Repeating the very same argument, we find that
\[
-ih(z_2) e^{2i \log z_2} = I(z_2) e^{-2\text{arg} z_2} = c_1,
\]
where \( c_1 \) is a constant. Thus
\[
I(z_2) = c_1 e^{2\text{arg} z_2}
\]
is a well-defined function on \( A_\delta \). This forces \( c_1 = 0 \). As a result, \( h(z_2) \equiv 0 \). We see in sum that \( a_2(z_2) = 0 \) as claimed.

Now we may conclude from (23) that \( b(z_2) \equiv 0 \) on \( A_\delta \). Therefore equation (22) simplifies to
\[
\delta_1(z_1, z_2) \equiv z_1 \quad \text{on } W.
\]
The result now follows from (20) and (25). □

7. Analysis on \( D_\beta \) and \( D'_\beta \)

We now summarize our main results about the non-smooth worm domains \( D_\beta \) and \( D'_\beta \). Details appear in [KrPe].

It is of interest, in its own right and as a model for the smooth case, to study the behavior of the Bergman kernel and projection on the non-smooth worm domain \( D_\beta \) and its biholomorphic copy \( D'_\beta \). An important transformation rule for the Bergman kernel (see [Kr1]) says that if \( \Phi : \Omega \to \Omega' \) is biholomorphic then
\[
K_{\Omega}(z, \zeta) = \det \text{Jac}_z(\Phi(z)) \cdot K_{\Omega'}(\Phi(z), \Phi(\zeta)) \cdot \det \text{Jac}_\zeta(\zeta).
\]
It is obvious from this transformation rule that it suffices to obtain the expression for the kernel in just one of the two domains (see [Kr1]). However, the \( L^p \)-mapping properties of the Bergman projections of the two domains turn out to be substantially different (just because \( L^p \) spaces of holomorphic functions do not transform canonically under biholomorphic maps when \( p \neq 2 \), due to the presence of the Jacobian factor). We shall explore this result in the next Theorems 7.3 and 7.4.

We shall discuss here the results (contained in detail in [KrPe]) concerning the explicit expression of the Bergman kernels for \( D_\beta \) and \( D'_\beta \). Once these are available we study the \( L^p \)-mapping properties of the corresponding Bergman projections. More precisely we prove the following theorems. There are two principal results of [KrPe]. Of course the proofs, which are quite technical, must be omitted. But the applications that we provide give a sense of the meaning and significance of these two theorems.

**Theorem 7.1.** Let \( c_0 \) be a positive fixed constant. Let \( \chi_1 \) be a smooth cut-off function on the real line, supported on \( \{x : |x| \leq 2c_0\} \), identically 1 for \( |x| < c_0 \). Set \( \chi_2 = 1 - \chi_1 \).

Let \( \beta > \pi \) and let \( \nu_\beta = \pi/[2\beta - \pi] \). Let \( h \) be fixed, with
\[
\nu_\beta < h < \min(1, 2\nu_\beta).
\]
Then there exist functions \( F_1, F_2, \ldots, F_8 \) and \( \bar{F}_1, \bar{F}_2, \ldots, \bar{F}_8 \), holomorphic in \( z \) and anti-holomorphic in \( w \), for \( z = (z_1, z_2), w = (w_1, w_2) \) varying in a neighborhood of \( D'_\beta \), and having size \( O(|\text{Re} z_1 - \text{Re} z_2|) \).
Re $w_1$), together with all their derivatives, for $z, w \in \overline{D_\beta}$, as $|\text{Re } z_1 - \text{Re } w_1| \to +\infty$. Moreover, there exist functions $E, \tilde{E} \in C^\infty(\overline{D_\beta} \times \overline{D_\beta})$ such that

$$D_{z_1}^\alpha D_{w_1}^\gamma E(z, w), D_{z_1}^\alpha D_{w_1}^\gamma \tilde{E}(z, w) = O(|\text{Re } z_1 - \text{Re } w_1|^{\alpha+\gamma}),$$

as $|\text{Re } z_1 - \text{Re } w_1| \to +\infty$. (Here, for $\lambda \in \mathbb{C}$, $D_\lambda$ denotes the partial derivative in $\lambda$ or $\bar{\lambda}$.)

Then the following holds. Set

$$K_b(z, w) = \frac{F_1(z, w)}{(i(z_1 - \overline{w_1}) + 2\beta)^2(e^{(\beta-\pi)/2} - z_2\overline{w_2})^2} + \frac{F_2(z, w)}{(i(z_1 - \overline{w_1}) + 2\beta)^2(z_2\overline{w_2} - e^{-i(z_1 - \overline{w_1})+\pi/2})^2} + \frac{F_3(z, w)}{(i(z_1 - \overline{w_1}) - 2\beta)^2((\pi-i(z_1 - \overline{w_1}))/2 - z_2\overline{w_2})^2} + \frac{F_4(z, w)}{(i(z_1 - \overline{w_1}) - 2\beta)^2(z_2\overline{w_2} - e^{-(\beta-\pi)/2})^2} + \frac{F_5(z, w)}{(i(z_1 - \overline{w_1}) + 2\beta)^2(e^{(\beta-\pi)/2} - z_2\overline{w_2})^2} + \frac{F_6(z, w)}{(i(z_1 - \overline{w_1}) - 2\beta)^2((\pi-i(z_1 - \overline{w_1}))/2 - z_2\overline{w_2})^2} + E(z, w) \equiv K_1(z, w) + \cdots + K_8(z, w) + E(z, w).$$

(27)

Define $K_b$ by replacing $F_1, \ldots, F_8$ and $E$ by $\tilde{F}_1, \ldots, \tilde{F}_8$ and $\tilde{E}$ and thus $K_1, \ldots, K_8$ by $\tilde{K}_1, \ldots, \tilde{K}_8$ respectively in formula (27).

Then there exist functions $\phi_1, \phi_2$ entire in $z$ and $\overline{w}$ (that is, anti-holomorphic in $w$), which are of size $O(|\text{Re } z_1 - \text{Re } w_1|)$, together with all their derivatives, uniformly in all closed strips $\{\text{Im } z_1 + \text{Im } w_1 \leq C\}$, such that the Bergman kernel $K_{D_\beta}$ on $D_\beta$ admits the asymptotic expansion

$$K_{D_\beta}(z, w) = \chi_1(\text{Re } z_1 - \text{Re } w_1) K_b(z, w) + \chi_2(\text{Re } z_1 - \text{Re } w_1) \left\{ e^{-h \text{sgn}(\text{Re } z_1 - \text{Re } w_1)}(\text{Re } z_1 - \text{Re } w_1) K_b(z, w) + e^{-h \text{sgn}(\text{Re } z_1 - \text{Re } w_1)}(\text{Re } z_1 - \text{Re } w_1) \left( \frac{\phi_1(z_1, w_1)}{(e^{(\pi-i(z_1 - \overline{w_1}))/2 - z_2\overline{w_2})^2} + \frac{\phi_2(z, w)}{(e^{-i(z_1 - \overline{w_1})+\pi/2 - z_2\overline{w_2})^2} \right) \right\}.$$

(28)

Here $h$ is specified as in (26) above.

**Theorem 7.2.** With the notation as in Theorem 7.1 there exist functions $g_1, g_2, G_1, G_2, \ldots, G_8$ and $G_1, G_2, \ldots, G_8$, holomorphic in $\zeta$ and anti-holomorphic in $\omega$, for $\zeta = (\zeta_1, \zeta_2)$, $\omega = (\omega_1, \omega_2)$ varying in $D_\beta \setminus \{(0, z_2)\}$, such that

$$\partial_\zeta^\alpha \partial_\omega^\beta G(\zeta, \omega) = O(|\zeta_1|^{-\alpha} |\omega_1|^{-\beta}) \quad \text{as} \quad |\zeta_1|, |\omega_1| \to 0,$$

where $G$ denotes any of the functions $g_j, G_j, \tilde{G}_j$. Moreover, there exist functions $E, \tilde{E} \in C^\infty(\overline{D_\beta} \times \overline{D_\beta} \setminus \{(0, z_2)\})$ such that

$$D_{\zeta_1}^\alpha D_{\zeta_2}^\gamma E(\zeta, \omega), D_{\zeta_1}^\alpha D_{\zeta_2}^\gamma \tilde{E}(\zeta, \omega) = O(|\zeta_1|^{-\alpha} |\omega_1|^{-\beta}) \quad \text{as} \quad |\zeta_1|, |\omega_1| \to 0.$$

(Here $D_\lambda$, for $\lambda \in \mathbb{C}$, $D_\lambda$ denotes the partial derivative in $\lambda$ or $\bar{\lambda}$.)
Then the following holds. Set
\[
H_b(\zeta, w) = \frac{G_1(\zeta, w)}{\left(\frac{1}{i \log(\zeta/\omega_1)} + 2\beta\right)^2 (e^{(\beta+\pi/2)} - \zeta_2 \omega_2)^2} \\
+ \frac{G_2(\zeta, w)}{\left(\frac{1}{i \log(\zeta/\omega_1)} + 2\beta\right)^2 (e^{(\beta+\pi/2)} + \zeta_2 \omega_2)^2} \\
+ \frac{G_3(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} - \zeta_2 \omega_2)^2} \\
+ \frac{G_4(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} + \zeta_2 \omega_2)^2} \\
+ \frac{G_5(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{-\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} - \zeta_2 \omega_2)^2} \\
+ \frac{G_6(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{-\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} + \zeta_2 \omega_2)^2} \\
+ \frac{G_7(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{-\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} - \zeta_2 \omega_2)^2} \\
+ \frac{G_8(\zeta, w)}{(\zeta_1/\omega_1)^{-i/2} e^{-\pi/2} - \zeta_2 \omega_2)^2 (e^{(\beta-\pi/2)} + \zeta_2 \omega_2)^2} + E(\zeta, w) \\
= H_1(\zeta, \omega) + \cdots + H_8(\zeta, \omega) + E(\zeta, \omega).
\] (29)

Define $H_b$ by replacing $G_1, \ldots, G_8$ and $E$ by $\tilde{G}_1, \ldots, \tilde{G}_8$ and $\tilde{E}$, and $H_1, \ldots, H_8$ by $\tilde{H}_1, \ldots, \tilde{H}_8$, respectively.

Then, setting $t = |\zeta_1| - |\omega_1|$, we have this asymptotic expansion for the Bergman kernel on $D_{\beta}$:

\[
K_{D_{\beta}}((\zeta_1, \zeta_2), (\omega_1, \omega_2)) \\
= \chi_1(t) \frac{H_b(\zeta, \omega)}{\zeta_1 \omega_1} + \chi_2(t) \left\{ \left(\frac{|\zeta_1|}{|\omega_1|}\right)^{-hsgnt} e^{-hsgnt(\arg\zeta_1 + \arg\omega_1)} \frac{H_b(\zeta, \omega)}{\zeta_1 \omega_1} \right\} \\
+ \left(\frac{|\zeta_1|}{|\omega_1|}\right)^{-v_{\beta}s} e^{-v_{\beta}s(\arg\zeta_1 + \arg\omega_1)} \left( \frac{g_1(\zeta_1, \omega_1)}{\zeta_1 \omega_1} \cdot \frac{1}{((\zeta_1/\omega_1)^{-1/2} e^{\pi/2} - \zeta_2 \omega_2)^2} \right) \\
+ \left(\frac{g_2(\zeta_1, \omega_1)}{\zeta_1 \omega_1} \cdot \frac{1}{((\zeta_1/\omega_1)^{-1/2} e^{-\pi/2} - \zeta_2 \omega_2)^2} \right),
\]

where $h$ is defined in (20).

The Bergman projection is trivially bounded on $L^2(\Omega)$. It is of some interest to ask about the mapping properties of $P$ on $L^p(\Omega)$ for $1 \leq p \leq \infty$. In general $P$ will be bounded on either $L^1$ or $L^\infty$. This assertion follows because $P$ is in the nature of a Hilbert integral, see [PhS1], [PhS2]. Details are provided in that source.

The Bergman projection $P_{D_{\beta}}$ on the domain $D_{\beta}$ is bounded on $L^p$ for $1 < p < \infty$. 

**Theorem 7.3.** The Bergman projection $P_{D_{\beta}}$ on the domain $D_{\beta}$ is bounded on $L^p$ for $1 < p < \infty$.

**Theorem 7.4.** Let $\beta > \pi$ and $\nu_{\beta} = \pi/[2\beta - \pi]$. The Bergman projection $P_{D_{\beta}}$ on the domain $D_{\beta}$ is only bounded on $L^p$ for $2/[1 + \nu_{\beta}] \leq p \leq 2/[1 - \nu_{\beta}]$. It is unbounded on $L^p$ for $p$ outside this range.
This situation is at first puzzling because the two domains $D'_\beta$ and $D_\beta$ are biholomorphic. But, whereas it is well known that $L^2$ transforms canonically under biholomorphic maps (see [KTI]), such is not the case for $L^p$ when $p \neq 2$.

We shall now describe the proof of the result on $D_\beta$, which is the most interesting case. In fact, we concentrate on the negative part of the result, since it bears some consequences on the unboundedness of the Bergman projection of $W$. The proof positive part is more direct, and it relies on an involved, systematic application of Schur’s Lemma.

The proof of the cognate result for $D'_\beta$ is somewhat more elementary, although too elaborate to present here.

We concentrate the on the negative result. Let $1 < p < \infty$ and assume that $P : L^p(D_\beta) \to L^p(D_\beta)$ is bounded. It follows that for any $\zeta \in D_\beta$ fixed, $K_{D_\beta}(\cdot, \zeta) \in L^{p'}(D_\beta)$, where $p' = p/(p-1)$ is the exponent conjugate to $p$.

For, if $P = P_{D_\beta}$ is bounded, then for all $f \in L^p(D_\beta)$ and all $\zeta \in D_\beta$, 
\[
|\langle f, K_{D_\beta}(\cdot, \zeta) \rangle| = |Pf(\zeta)| \leq c_\zeta \|Pf\|_{L^p} \leq C\|f\|_{L^p}.
\]

**Lemma 7.5.** For any $\zeta \in D_\beta$ it holds that $K_{D_\beta}(\cdot, \zeta) \in L^p(D_\beta)$ only if $2/(1+\nu_\beta) < p < 2/(1-\nu_\beta)$.

**Proof.** Fix $\zeta \in D_\beta$ and define 
\[
\Omega_\zeta = \{ \omega \in D_\beta : |\omega_1| < |\zeta_1|, 1/4 \leq |e^{\pi/2}(\zeta_1/\zeta_2)^{\pm i/2} - \zeta_2| \leq 1/2 \}.
\]
Recall the expansion for the kernel $K_{D_\beta}(\zeta, \omega)$ given in Theorem 7.2. Then, for $\omega \in \Omega_\zeta$, we have that 
\[
|H_\beta(\zeta, \omega)|, |H_\beta(\zeta, \omega)| \leq C_\zeta
\]
for some constant independent of $\omega$, so that 
\[
|K_{D_\beta}(\zeta, \omega)| \geq c_\zeta |\omega_1|^{\nu_\beta-1}
\]
for $\omega \in \Omega_\zeta$.

Therefore 
\[
\int_{D_\beta} |K_{D_\beta}(\cdot, \zeta)|^{p'} dV(\omega) \geq \int_{\Omega_\zeta} |K_{D_\beta}(\cdot, \zeta)|^{p'} dV(\omega) \geq c_\zeta \int_{\Omega_\zeta} \left(|\omega_1|^{\nu_\beta-1}\right)^{p'} dV(\omega) = c \int_0^{\left|\zeta_1\right|} r^{p'(\nu_\beta-1)+1} dr.
\]

Obviously for convergence we need $p'(\nu_\beta-1) + 1 > -1$, that is $p' < 2/[1+\nu_\beta]$. Hence if $p \geq 2/[1-\nu_\beta]$ then the integral diverges. The other result, for $p \leq 2/[1+\nu_\beta]$, follows by duality. This proves the lemma.

We now show that we can use Barrett’s exhaustion procedure (see [Ba2]) to obtain a negative result with the same indices on the smooth worm $W$.

**Theorem 7.6.** Let $\mathcal{P}$ denote the Bergman projection on the smooth, bounded worm $W = W_\beta$, with $\beta > \pi/2$. Then, if $\mathcal{P} : L^p(W_\beta) \to L^p(W_\beta)$ is bounded, necessarily $2/[1+\nu_\beta] < p < 2/[1-\nu_\beta]$.

**Proof.** Suppose $\mathcal{P} : L^p(W) \to L^p(W)$ is bounded for a given $p \neq 2$. Without loss of generality, we may assume that $p > 2$.

Let $\tau_R$ be defined as $\tau_R(z_1, z_2) = (Rz_1, z_2)$, for $R \geq 1$. Recall that 
\[
W' = \{ (z_1, z_2) : |z_1 - e^{i\log|z_2|^2}| < 1, |\log|z_2|| < \beta - \pi/2 \}
\]
is the truncated version of $W$. Then, 
\[
\tau_R(W) \supseteq \tau_R(W') = \{ (w_1, w_2) : |w_1|^2/R - 2Re(w_1 e^{-i\log|w_2|^2}) | < 0, \ |\log|w_2|| < \beta - \pi/2 \}\]
and
\[ \tau_R(W') \not\subset D_\beta \quad \text{as} \quad R \to +\infty. \]

Let \( T_R : L^p(\tau_R(W)) \to L^p(W) \) be defined as \( T_R(f) = f \circ \tau_R \). Notice that
\[ \|T_R(f)\|_{L^p(W)} = R^{-2}\|f\|_{L^p(\tau_R(W))} \]
for all \( f \in L^p(\tau_R(W)) \). Moreover, set \( P_R = T_R^{-1}PT_R \) and notice that \( P_R \) is the Bergman projection of \( \tau_R(W') \). Using the boundedness of \( P \) on \( L^p(W) \) it follows that
\[ \|P_R\varphi\|_{L^p(\tau_R(W'))} \leq C\|\varphi\|_{L^p(\tau_R(W'))} \quad (31) \]
for all continuous functions \( \varphi \) with compact support in \( \tau_R(W') \), where \( C \) is a constant independent of \( R \).

We now claim that \((P_R\varphi)^- \to (P_{D_\beta}\varphi)^-\) weakly in \( L^p(C^2) \), as \( R \to +\infty \). Here, by \( f^- \) we denote the extension of \( f \) to all of \( C^2 \) (defining \( f^- = 0 \) outside the natural domain of definition of \( f \)). Notice that, since \( \tau_R(W') \not\subset D_\beta \) as \( R \to +\infty \), if \( \varphi \) has compact support in \( D_\beta \) then there exists \( R_0 \) such that \( \varphi \) has compact support in \( \tau_R(W') \), for \( R \geq R_0 \).

Assume the claim for now, and we finish the proof. For all \( \varphi \) continuous with compact support in \( D_\beta \), by (31) and the claim it follows at once that
\[ \|P_{D_\beta}\varphi\|_{L^p(D_\beta)} \leq C\|\varphi\|_{L^p(D_\beta)} \cdot \]
The result now follows from Theorem \( \text{[74]} \).

Finally, we prove the claim. Since \( \|(P_R\varphi)^-\|_{L^p(C^2)} \leq C\|\varphi^-\|_{L^p(C^2)} \), there exists a subsequence of \( \{(P_R\varphi)^-\} \) that converges weakly to \( g \in L^p(C^2) \). Notice that \( g \) vanishes out \( D_\beta \), since \((P_R\varphi)^-\) does for all \( R \geq 1 \). Moreover, \( g \) is holomorphic on \( D_\beta \), since \( \tau_R(W') \not\subset D_\beta \) and, \((P_{R_2}\varphi)^- = (P_{R_1}\varphi)^-\) on \( \tau_{R_1}(W') \) when \( R_2 > R_1 \). Moreover notice that, since \( \|(P_R\varphi)^-\|_{L^2(C^2)} \leq \|\varphi^-\|_{L^2(C^2)} \), there exists a subsequence \( \{(P_{R_j}\varphi)^-\} \) that converges weakly also in \( L^2(C^2) \), to the same function \( g \). We wish to show that \( g = P_{D_\beta} \) on \( D_\beta \).

Now let \( h \in A^p \cap A^2(D_{\beta}) \) (where \( A^p \) denotes the Bergman space, and \( p' \) is the exponent conjugate to \( p \)). We have that
\[ \langle g - P_{D_\beta}\varphi, h \rangle_{D_\beta} = \lim_{j \to +\infty} \langle (P_{R_j}\varphi)^- - P_{D_\beta}\varphi, h \rangle_{D_\beta} \]
\[ = \lim_{j \to +\infty} \int_{\tau_{R_j}(W')} P_{R_j}\varphi \overline{h} - \int_{D_\beta} P_{D_\beta}\varphi \overline{h} \]
\[ = \lim_{j \to +\infty} \int_{D_\beta \setminus \tau_{R_j}(W')} \varphi \overline{h} = 0 \cdot \]

If we show that \( A^p \cap A^2(D_{\beta}) \) is dense in \( A^2(D_{\beta}) \), it would follow that \( g - P_{D_\beta}\varphi \perp A^2(D_{\beta}) \), that is \( g = P_{D_\beta} \), and we would be done.

Notice that for \( \delta > 0 \) we have
\[ \int_{D_\beta} |e^{-\delta z_1^2}|^q d\zeta_1 d\zeta_2 = \int_{D_{q'}} |e^{-\delta \log \zeta_1^2} \frac{1}{|\zeta_1|^q} d\zeta_1 d\zeta_2 \]
\[ \leq c \int_{|\log |\zeta_1|| < \mu} \int_{|\Im \zeta_1 - \log |\zeta_1|^2| < \pi/2} e^{-\delta \log^2 |\zeta_1|} \frac{1}{|\zeta_1|^q} d\zeta_1 d\zeta_2 \]
\[ \leq c \int_0^{+\infty} e^{-\delta \log^2 r} r^{-2} dr < \infty \cdot \]

that is \( e^{-\delta z_1^2} \in L^q(D_{\beta}) \) for all \( q \)'s. Now, for any \( h \in A^2(D_{\beta}) \), consider \( h_\delta = he^{-\delta z_1^2} \). Since \( e^{-\delta z_1^2} \) is bounded, \( h_\delta \in A^2(D_{\beta}) \). Moreover, since \( p \) is taken to be larger than 2, by Hölder inequality
we have $\|h_\delta\|_{L^{p'}} \leq c\|h\|_2^{1/p'} < \infty$. Thus $h_\delta \in A^2 \cap A^p(D_\beta)$ and clearly $h_\delta$ converges to $h$ in the $L^2$-norm. This completes the proof of the theorem.

□

8. Irregularity Properties of the Bergman Kernel

We now examine the boundary asymptotics for the Bergman kernel on the domains $D_\beta$ and $D'_\beta$ and determine various irregularity properties of the corresponding Bergman kernels.

Begin with the asymptotic formula in Theorem 7.1. We point out particularly that there are two kinds of behavior: one kind at the "finite portion of the boundary" and the other one as $|\text{Re } z_1 - \text{Re } 1| \to +\infty$. These two different behaviors are expressed by the first and second terms in (2.8), respectively. For the former type, we notice from (27) that the lead terms have expressions in the denominator of products of two terms like

$$(i(z_1 \pm \bar{w}_1) + 2\beta)^2, \quad (z_2 \bar{w}_2 - e^{\pm(i \beta - \pi/2)})^2, \quad \text{and} \quad (z_2 \bar{w}_2 - e^{-i(z_1 - \bar{w}_1) \pm \pi/2})^2.$$ 

These singularities are similar to the ones of a Bergman kernel of a domain in $\mathbb{C}^2$ which is essentially a product domain. It is important to observe that the kernel does not become singular only when $z, w$ tend to the same point on the boundary. For instance, it becomes singular as $((z_1 \pm \overline{w}_1) + 2\beta) \to 0$, while there is no restriction on the behavior of $z_2$ and $w_2$. We will be more detailed below in the case of the domain $D_\beta$. For the case of this domain, we finally notice that the main term at infinity, that is when $|\text{Re } z_1 - \text{Re } 1| \to +\infty$, behaves like $e^{-|\nu| |z_1 - \bar{w}_1|} \cdot (z_2 \bar{w}_2)^{-1}$.

Next we consider the case of $D'_\beta$. The mapping $(z_1, z_2) \in D'_\beta \mapsto (\xi_1, \xi_2) = (e^{z_1}, z_2) \in D_\beta$ sends the point at infinity (in $z_1$) into the origin (in $\xi_1$). Keeping into account the Jacobian factor, when $|\xi_1| - |\omega_1| \to 0^+$, the kernel on $D_\beta$ is asymptotic to

$$\frac{|\omega_1|^{p_2 - 1}}{|\xi_1|^{p_2 + 1}} \cdot (\xi_2 \bar{\omega}_2)^{-1}.$$ 

Recall the inequalities that define $D_\beta$:

$$D_\beta = \left\{ \xi \in \mathbb{C}^2 : \text{Re } \left( \xi_1 e^{-i \log |\xi_1|^2} \right) > 0, \right\} \left| \log |\xi_2|^2 \right| < \beta - \frac{\pi}{2}.$$

If we take $\zeta, \omega \in D_\beta$ and let $\omega_1$ tend to 0, then clearly $\omega \to \partial D_\beta$ and $\zeta_1, \zeta_2, \omega_2$ are unrestricted. Therefore, $K_{D_\beta}(\cdot, \omega) \notin C^\infty(D_\beta)$ for $\omega \in \{0, \omega_2\}$, with $|\log |\omega_2|^2| < \beta - \pi/2$.

Notice that this is in contrast, for instance, to the situation on the ball or, more generally, on a strongly pseudoconvex domain. On either of those types of domains $\Omega$, the kernel is known to be smooth on $\overline{\Omega} \times \overline{\Omega} \setminus (\Delta \cap [\partial \Omega \times \partial \Omega])$. See [Ker] and [Kr1].

By the same token (by almost the same calculation), it is easy to conclude that the Bergman projection on $D_\beta$ cannot map functions in $C^\infty(\overline{D_\beta})$ to functions in $C^\infty(\overline{D_\beta})$. This, of course, is the failure of Condition $R$ on these domains.

In Section 5 we have seen that $P_V : C^\infty(\overline{W}) \not\to C^\infty(\overline{W})$, that is, that $W$ does not satisfy Condition $R$. A philosophically related fact, due to Chen [Che2] and Ligocka [Lig1] independently, is that the Bergman kernel of $W$ cannot lie in $C^\infty(\overline{W} \times \overline{W} \setminus \Delta)$ (where $\Delta$ is the boundary diagonal). In fact, in [Che2] it is shown that this phenomenon is a consequence of the presence of a complex variety in the boundary of $W$. We shall also explore this singularity phenomenon in what follows.

The proof of the general result of So-Chin Chen follows a classical paradigm for establishing propagation of singularities for the $\overline{\partial}$-Neumann problem and similar phenomena.
Theorem 8.1. Let \( \Omega \subseteq \mathbb{C}^n \) be a smoothly bounded, pseudoconvex domain with \( n \geq 2 \). Assume that there is a complex variety \( V \), of complex dimension at least 1, in \( \partial \Omega \). Then
\[
K_\Omega(z, w) \not\in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta(\partial \Omega)),
\]
where \( \Delta(\partial \Omega) = \{(z, z) : z \in \partial \Omega \} \).

Proof. Let \( p \in V \) be a regular point. Let \( n_p \) be the unit outward normal vector at \( p \). Then there are small numbers \( \delta, \varepsilon_0 > 0 \) such that \( w - \varepsilon n_p \in \Omega \) for all \( w \in \partial \Omega \cap B(p, \delta) \) and all \( 0 < \varepsilon < \varepsilon_0 \). Let \( d \) be an analytic disc in \( \partial \Omega \cap B(p, \delta) \cap V \). We may assume that this disc is centered at \( p \). In other words, \( d \) is the image of the unit disc in the plane mapped into \( \mathbb{C}^n \) with the origin going to \( p \).

Seeking a contradiction, we assume that \( K_\Omega(z, w) \in C^\infty(\overline{\Omega} \times \overline{\Omega} \setminus \Delta(\partial \Omega)) \). Then we certainly have
\[
\sup_{w \in \partial d} |K_\Omega(p, w)| \leq M < +\infty \tag{22}
\]
for some positive, finite number \( M \). On the other hand, we know (see [BlPf]) that
\[
\lim_{\varepsilon \to 0} K_\Omega(p - \varepsilon n_p, p - \varepsilon n_p) = +\infty. \tag{23}
\]
By the maximum modulus principle we then obtain
\[
\sup_{w \in \partial d} |K_\Omega(p - \varepsilon n_p, w)| \geq K_\Omega(p - \varepsilon n_p, p - \varepsilon n_p),
\]
where \( d = d - \varepsilon n_p \subseteq \Omega \). We conclude that
\[
\sup_{w \in \partial d} |K_\Omega(p, w)| = \lim_{\varepsilon \to 0} \sup_{w \in \partial d} |K_\Omega(p - \varepsilon n_p, w)| = +\infty.
\]
This gives a contradiction, and the result is established. \( \square \)

9. Concluding Remarks

The worm domains are assuming an ever more prominent role in the function theory of several complex variables. Originally created for the study of elementary facts about the geometry of pseudoconvex domains, they are now playing an ever-more-prominent role in the hard analytic questions of these domains. It is becoming clear that understand the worms will help us to understand pseudoconvex domains in general. Particularly, it is now apparent that the right approach to the function theory of a domain is best formulated in the language of the invariant geometry of that domain. The subtleties of the worms will bring that geometry to the fore, and help us to push this program forward.

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