Lattice QCD calculation of hadronic light-by-light scattering

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We perform a lattice QCD calculation of the hadronic light-by-light scattering amplitude in a broad kinematical range. At forward kinematics, the results are compared to a phenomenological analysis based on dispersive sum rules for light-by-light scattering. The size of the pion pole contribution is investigated for momenta of typical hadronic size. The presented numerical methods can be used to compute the hadronic light-by-light contribution to the anomalous magnetic moment of the muon. Our calculations are carried out in two-flavor QCD with the pion mass in the range of 270 to 450 MeV, and contain so far only the diagrams with fully connected quark lines.

I. INTRODUCTION

Light-by-light scattering, the elastic scattering of two photons, is a striking prediction of Quantum Electrodynamics (QED). The light-by-light (LbL) interaction appears prominently in corrections to the anomalous magnetic moment \((g-2)\) of the electron and muon. The muon \((g-2)\) exhibits a 3σ discrepancy between experiment and the Standard Model calculations [1]. While the current theory and experimental errors are comparable in size, a new \((g-2)_\mu\) experiment [2] aiming to reduce the theoretical error on \((g-2)\) by a factor of four is in preparation at Fermilab.

The theory error on \((g-2)\) is dominated by hadronic contributions, namely the hadronic vacuum polarization (HVP) and hadronic light-by-light (HLbL) scattering. Using unitarity and causality, the HVP contribution is expressed in terms of the total \(e^+e^-\to\) hadrons cross section, and hence its precision can systematically be improved by collider experiments alone. By contrast, the HLbL contribution cannot be expressed entirely in terms of cross sections for \(\gamma\gamma\)-fusion into hadrons; see [3-5] for dispersive approaches to the problem. A direct ab initio calculation within Quantum Chromodynamics (QCD) is very challenging due to its non-perturbative nature. In this work we address the problem using lattice QCD.

A first lattice QCD+QED calculation of the HLbL contribution to \((g-2)\) has recently been performed by Blum et al. [6]. We envisage a different method where the four-point function for LbL scattering is computed in lattice QCD and integrated over to yield the HLbL contribution. In this Letter we present the four-point function calculation and check it against the available phenomenology. Exploiting unitarity and causality, the forward HLbL amplitude can be expressed as a dispersive integral over the \(\gamma^*\gamma^*\to\) hadrons cross section [7, 8]. A parametrization of the latter allows us to confront the lattice calculation with phenomenology in a fairly straightforward manner. As the neutral pion \((\pi^0)\) pole dominates the HLbL contribution to \((g-2)\), in phenomenological calculations [1], we study its relative size both at forward and off-forward kinematics.

II. THEORY BACKGROUND

The Lehmann-Symanzik-Zimmermann reduction formula for the HLbL scattering amplitude implies [9]

\[
\mathcal{M}_{\mu_1\mu_2\mu_3\mu_4}(p_1, p_4 \to p_2, p_3) = e^4 \left( -\Pi_{\mu_1\mu_2\mu_3\mu_4}(p_2, p_3) \right)
\]

(1)

where \(p_3 = p_1 + p_4 - p_2\) and

\[
\Pi_{\mu_1\mu_2\mu_3\mu_4}(p_3; p_1, p_2) \equiv \int d^4x_1 d^4x_2 d^4x_4 e^{+i\sum_{j=x_1,x_2,x_4} p_j \cdot x_j} T\{J_{\mu_1}(x_1)J_{\mu_2}(x_2)J_{\mu_3}(0)J_{\mu_4}(x_4)\} |0\rangle
\]

(2)

is the Minkowski-space time-ordered correlator of the conserved vector current \(j_{\mu_k} = \frac{1}{2} \bar{\psi} \gamma_{\mu_k} \gamma^5 \psi - \frac{1}{4} \bar{\psi} \gamma_{\mu_k} \gamma^5 \bar{\psi} d + \ldots\). The index \(a\) takes the values 1, 2 and 4. The components of the current \(j_{\mu_k}\) used in the Euclidean theory [10] are related to their Minkowskian counterparts by \(J_0 = J_0\), \(J_k = i j_k\). The analytic continuation then yields the following relation to the Euclidean correlation function,

\[
\Pi^{\infty}_{\mu_1\mu_2\mu_3\mu_4}(p_3; p_1, p_2) \equiv \int d^4X_1 d^4X_2 d^4X_4 e^{4\sum_{a=0}^{\infty} P_{a} \cdot X_{a} \left\{ J_{\mu_1}(X_1)J_{\mu_2}(X_2)J_{\mu_3}(0)J_{\mu_4}(X_4) \right\}_E}
\]

(4)

where \(n_0\) is the number of temporal indices carried by the vector currents in the correlator.

The forward scattering case is obtained in Eq. (1) by setting \(p_2 = p_1\). Renaming the momenta to match the conventional notation, we have

\[
\mathcal{M}^{\text{forward}}_{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2) \equiv \mathcal{M}_{\mu_1\mu_2\mu_3\mu_4}(q_1, q_2 \to q_1, q_2)
\]

(5)

\[
e^4 \left( -i\Pi_{\mu_1\mu_2\mu_3\mu_4}(q_2; -q_1, -q_1) \right)
\]
The forward scattering amplitude can be decomposed into eight Lorentz-invariant amplitudes [11]. They are functions of the virtualities \(q_1^2\) and \(q_2^2\) of the photons, as well as of the variable \(\nu \equiv q_1 \cdot q_2\). Using the projector \(R_{\mu\nu}\) onto the subspace orthogonal to \(q_1\) and \(q_2\), we focus here on the amplitude [12]

\[
\mathcal{M}_{TT}(q_1^2, q_2^2, \nu) = \frac{1}{4} R_{\mu_1 \nu_1} R_{\mu_2 \nu_2} \mathcal{M}_{TT}^{\text{forward}}(q_1, q_2). \tag{6}
\]

Combining Eqs. (5) and (3), we can access the amplitude \(\mathcal{M}_{TT}\) from the Euclidean correlator,

\[
\mathcal{M}_{TT}(-Q_1^2, Q_2^2, -Q_1 \cdot Q_2) = \frac{e^4}{4} R_{\mu_1 \nu_1} R_{\mu_2 \nu_2} \Pi^{E}_{\mu_1 \nu_1 \mu_2 \nu_2} (-Q_2^2; -Q_1, Q_1),
\]

\[
R_{\mu \nu}^{E} \equiv \frac{\delta_{\mu \nu} - \frac{1}{(Q_1 \cdot Q_2)^2 - Q_1^2 Q_2^2}}{Q_1 \cdot Q_2}.
\tag{7}
\]

The largest value of \(|\nu|\) that can be reached with Euclidean kinematics is limited by the virtualities of the photons [13], \(|\nu| \leq (Q_1^2 Q_2^2)^{1/2} \leq 1/2 (Q_1^2 + Q_2^2) \equiv v_0\), while the nearest singularity is the s-channel \(\pi^0\) pole located at \(v_\pi = \frac{1}{2}(m_\pi^2 + Q_1^2 + Q_2^2)\). A technical issue arises when \(Q_1\) and \(Q_2\) are collinear: the projector \(R_{\mu \nu}^{E}\) becomes ambiguous. To resolve the issue, we note that \(R_{\mu \nu}^{E} = \bar{R}_{\mu \nu} - U_1 \mu U_{1 \nu}\), where \(\bar{R}_{\mu \nu} \equiv \delta_{\mu \nu} - Q_1 \mu Q_{1 \nu} / Q_1^2\) and \(U_1\) is the unit vector parallel to the projection of \(Q_2\) onto the subspace orthogonal to \(Q_1\). The average of the applied projector over the directions of \(U_1\) in that subspace yields

\[
\langle (R_{\mu_1 \nu_1} R_{\mu_2 \nu_2}) \rangle U_1 = \frac{2}{5} \bar{R}_{\mu_1 \nu_1} \bar{R}_{\mu_2 \nu_2} + \frac{1}{25} \bar{R}_{\mu_1 \mu_2} \bar{R}_{\mu_3 \nu_3}.
\tag{8}
\]

We use this averaged projector in Eq. (7) when \(Q_1\) and \(Q_2\) are collinear.

In [8], it was shown that the HLBQL amplitude \(\mathcal{M}_{TT}(\nu)\), for fixed spacelike photon virtualities, can be obtained from the following dispersive sum rule,

\[
\mathcal{M}_{TT}(q_1^2, q_2^2, \nu) - \mathcal{M}_{TT}(q_1^2, q_2^2, 0) = \frac{2\nu^2}{\pi} \int_{v_0}^{\infty} dw' \frac{\sqrt{w' - Q_1^2 Q_2^2}}{w'(w' - w_0^2 - i\epsilon)} (\sigma_0 + \sigma_2)(w'),
\tag{10}
\]

where \(\sigma_0\) and \(\sigma_2\) are the total cross sections \(\gamma^*(q_1^2)\gamma^*(q_2^2) \rightarrow \text{hadrons with total helicity 0 and 2}\) respectively. It can be shown [8] that \(\mathcal{M}_{TT}\) vanishes at \(\nu = 0\) if either of the photons is real. It is interesting to test the sum rule for the \(\pi^0\) pole contribution. Using the expression for \(\Pi_{\mu \nu \rho \sigma}\) given in [14] and Eqs. (5, 6), one finds

\[
\mathcal{M}_{TT}^{\pi^0}(-Q_1^2, -Q_2^2, \nu) = e^4 (\nu^2 - Q_1^2 Q_2^2) \\
F(-Q_1^2, -Q_2^2)^2 \frac{Q_1^2 + Q_2^2 + m_\pi^2}{(Q_1^2 + Q_2^2 + m_\pi^2)^2 - 4\nu^2}.
\tag{11}
\]

with \(F(q_1^2, q_2^2)\) the pion transition form factor as defined in [14]. For \(q_2^2 = 0\), the same result is obtained from the sum rule, using the expression for the \(\gamma^* \gamma \rightarrow \pi^0\) cross-section given in [8].

In summary, the amplitude \(\mathcal{M}_{TT}\) can be computed on the lattice via Eq. (7) and from \(\gamma^* e^-\) collider data via Eq. (10). In the following, we present a comparison of the two approaches.

### III. IMPLEMENTATION OF THE EUCLIDEAN FOUR-POINT FUNCTION IN LATTICE QCD

In numerical lattice QCD calculations of n-point functions, the quark path integral is evaluated analytically to yield a sum of contractions of quark propagators. For the four-point function of vector currents, these fall into five distinct topologies, illustrated in Fig. 1. In this work, we compute only the six contractions that are fully quark-connected.

We use a Wilson-type quark action, three lattice conserved currents \(J_\mu\) and one site-local current \(J^l_\mu\) (see for instance [15] for an explicit definition). Generically, we evaluate the fully-connected contribution to

\[
\Pi^\text{lat}_{\mu \nu \rho \sigma}(X_4; f_1 f_2) \equiv \sum_{X_1 X_2} f_1(X_1) f_2(X_2) \langle \bar{J}^\mu_{\mu_1}(X_1) J^l_\nu_{\nu_1}(X_2) J^\rho_{\rho_1}(0) J^\sigma_{\sigma_1}(X_4) \rangle + \text{contact terms},
\tag{12}
\]

for some fixed functions \(f_{1,2}\) and all values of \(\{\mu_0\}\) and \(X_4\). The contact terms are present when two or three lattice conserved currents coincide, and serve to ensure that the conserved-current relations hold, e.g., \(\Delta_{\mu}^{X_4} \Pi^\text{lat}_{\mu \nu \rho \sigma}(X_4; f_1 f_2) = 0\), where \(\Delta_{\mu}^{X_4}\) is the backward lattice derivative.

![FIG. 1. Four-point function quark contraction topologies. The vertices represent vector currents and the lines are quark propagators. In this work, we compute only the leftmost, fully-connected class of diagrams.](image1)

![FIG. 2. Fully-connected four-point function quark contractions. Each panel represents two contractions with opposite directions of quark flow. The solid quark lines are computed using a point-source propagator, the dashed lines using sequential propagators, and the dotted lines using double-sequential propagators.](image2)
The fully-connected contribution to Eq. (12) is evaluated using the method of sequential propagators. First, a point-source propagator is computed from $X_3$. Then, it is combined with the function $f_1$ or $f_2$ to form the source for a new (sequential) propagator. These sequential propagators are then used to form sources for double-sequential propagators that depend on both $f_1$ and $f_2$. Finally, the fully-connected contraction is formed using all three kinds of propagators; this is illustrated in Fig. 2. For generic complex $f_1$ and $f_2$, this requires one point-source, 16 sequential and 32 double-sequential propagators, although these counts can be reduced in various special cases. We have verified that in our implementation the four-point function matches the lattice perturbation theory calculation if the gauge link variables are set to the physical pion mass. We have also employed these counts in various special cases. We have verified that in our implementation the four-point function matches the lattice perturbation theory calculation if the gauge link variables are set to the physical pion mass.

For evaluating the momentum-space correlator, we set the functions to be plane waves, $f_α(X) = e^{-iP_α X}$ and compute the Fourier modes with respect to $X_4$. Thus, $Π^{β1β2μ3μ4}_μ(P_4; P_1, P_2)$ can be evaluated efficiently at fixed $P_{1,2}$ for all $P_4$ available on the lattice.

### IV. RESULTS

We have used three lattice QCD ensembles with two degenerate flavors of non-perturbatively O($a$) improved Wilson quarks and a plaquette gauge action. The ensembles are at a single lattice spacing $a = 0.063$fm [16], correspond to pion masses $m_π = 451, 324$ and 277 MeV, and are respectively of spatial linear size 32, 48 and 48, the time direction being twice as long; see [17] for more details. Only the up and down quark contributions to the electromagnetic current are included. The local vector current $J_μ^γ$ is renormalized non-perturbatively [18]. The results shown here were obtained using fairly low statistics, with a maximum of 300 samples.

Due to the finite volume of the lattice, the momenta take discrete values. The subtracted forward scattering amplitude, $M_{TT}(-Q_1^2, -Q_2^2, ν) - M_{TT}(-Q_1^2, -Q_2^2, 0)$ (which is even in $ν$), is obtained by linearly interpolating the second term between the available $Q_2^2$ to match the first term. It is shown in Fig. 3 at fixed pion mass and fixed $Q_1^2$, and also in Fig. 4 with both photon virtualities fixed. For the latter, linear interpolation in $Q_1^2$ was also used in the first term, except for the points at maximal $ν$. At fixed $ν$, the amplitude tends to decrease as the virtualities are increased, at fixed virtualities it tends to increase with $|ν|$, and at fixed kinematics we do not find a significant dependence on the pion mass.

We compare the lattice data with results from the sum rule, Eq. (10), using a phenomenological model for the transverse $γ^*γ^* →$ hadrons cross section, $a_0 + a_2$, based on Ref. [19]. We include pseudoscalar, scalar, axial-vector, and tensor mesons, as well as the non-resonant $π^+π^−$ contribution (in scalar treelevel QED with pion electromagnetic form factors). The $γ^*γ^* →$ meson form factors have not been measured experimentally; they are assumed to factorize as $F(q_1^2, q_2^2) = F(q_1^2, 0)F(q_2^2, 0)/F(0, 0)$. For the pseudoscalar and axial-vector mesons, $F(q^2, 0) = F(0, q^2)$ is described based on experimental data as in Ref. [8] and, lacking guidance from experiment, we assume a monopole form factor for the scalar and tensor resonances with a pole mass set by hand to $Λ = 1.6$ GeV. The model is modified for unphys-
We have demonstrated that the fully connected contribution to the momentum-space four-point function of the electromagnetic current can be computed with moderate computational effort in lattice QCD if two of the three momenta are fixed. As an application, we computed one of the forward $\gamma\gamma^*$ scattering amplitudes in a broad kinematic range. Via a dispersive sum rule, it is related model-independently to $\gamma\gamma^* \rightarrow$ hadrons cross sections. Modelling the latter, we find the comparison of the lattice calculation with the phenomenological approach to be successful. The systematic uncertainties of the comparison are presently still large, mainly because our current calculations are performed at heavier quark masses than the physical ones, but this model dependence can be systematically reduced. Also, the not fully connected contraction topologies depicted in Fig. 1 could be important. We investigated the size of the pion pole contribution both in the forward and the off-forward amplitude. Both the lattice data and the model show that it is by no means dominant in a range of kinematic invariants of typical hadronic size.

The numerical methods presented can be applied to a direct lattice calculation of the HLLB contribution to $(g − 2)_{\mu}$; we are currently working on a position-space approach where the photon propagators are integrated out semi-analytically in infinite volume. The dominant systematic effects are likely to be quite different from those in the method of Blum et al. [6], allowing for useful cross-checks. Since phenomenological calculations indicate that the $\pi^0$ is dominant in the HLLB contribution to $(g − 2)_{\mu}$ [1], realistically light quark masses and large volumes will be required to treat this long-range contribution correctly. Lattice data on the HLLB amplitude itself can also help discriminate between phenomenological models used in the calculation of $(g − 2)_{\mu}$.

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