Besse conjecture with positive isotropic curvature

Seungsu Hwang · Gabjin Yun

Received: 3 May 2021 / Accepted: 20 June 2022 / Published online: 11 July 2022
© The Author(s), under exclusive licence to Springer Nature B.V. 2022

Abstract
The critical point equation arises as a critical point of the total scalar curvature functional defined on the space of constant scalar curvature metrics of a unit volume on a compact manifold. In this equation, there exists a function $f$ on the manifold that satisfies the following

$$(1 + f)\text{Ric} = Ddf + \frac{nf + n - 1}{n(n - 1)}sg.$$

It has been conjectured that if $(g, f)$ is a solution of the critical point equation, then $g$ is Einstein and so $(M, g)$ is isometric to a standard sphere. In this paper, we show that this conjecture is true if the given Riemannian metric has positive isotropic curvature.

Keywords Besse conjecture · Critical point equation · Einstein metric · Positive isotropic curvature

Mathematics Subject Classification 53C25 · 53C20

1 Introduction

In this paper, we consider closed smooth manifolds admitting Riemannian metrics as the critical points of a curvature functional. Let $M$ be a closed $n$-dimensional smooth manifold and $\mathcal{M}_1$ be the set of all smooth Riemannian metrics of a unit volume on $M$. The total scalar curvature functional $\mathcal{S}$ on $\mathcal{M}_1$ is given by

$$\mathcal{S}(g) = \int_M s_g \, dv_g,$$
where $s_g$ is the scalar curvature of the metric $g \in \mathcal{M}_1$ and $dv_g$ denotes the volume form of $g$. Critical points of $S$ on $\mathcal{M}_1$ are known to be Einstein (cf. [2]). Introducing a subset $\mathcal{C}$ of $\mathcal{M}_1$ consisting of metrics with a constant scalar curvature, the Euler–Lagrange equation of $S$ restricted to $\mathcal{C}$ can be written in the following

$$z_g = s_g^{\ast}(f),$$

which is called the critical point equation (CPE in short). Here, $z_g$ is the traceless Ricci tensor defined by $z_g := \text{Ric}_g - \frac{s_g}{n}g$, $\text{Ric}_g$ is the Ricci tensor, and $s_g^{\ast}(f)$ is given by

$$s_g^{\ast}(f) = D_gdf - (\Delta_g f)g - f\text{Ric}_g,$$

where $D_gdf$ and $\Delta_g f$ are the Hessian and (negative) Laplacian of $f$, respectively. From the variational problem for curvature functional, the Besse conjecture [2] describes that a nontrivial solution of the CPE

$$z_g = D_gdf - (\Delta_g f)g - f\text{Ric}_g,$$  \hspace{1cm} (1.1)

should be Einstein. By taking the trace of (1.1), we have

$$\Delta_g f = -\frac{s_g}{n-1}f,$$  \hspace{1cm} (1.2)

and therefore, using $z_g = \text{Ric}_g - \frac{s_g}{n}g$, (1.1) can be rewritten as

$$(1 + f)z_g = Ddf + \frac{s_gf}{n(n-1)}g.$$  \hspace{1cm} (1.3)

Since $g$ is clearly Einstein when $f = 0$, going forward, we assume that $f$ is not trivial. Besse conjecture describes that such a nontrivial solution $(g, f)$ of (1.1) is Einstein. Notably, some progress has been made to this conjecture. For example, if the Riemannian manifold is locally conformally flat, then the metric is Einstein [12]. It is also known that Besse conjecture holds if the manifold has a harmonic curvature [23, 24], or the metric is Bach-flat [18]. We say that $(M, g)$ has a harmonic curvature if the divergence of the Riemannian curvature tensor $R$ vanishes, in other words, $\delta R = 0$, and $(M, g)$ is said to be Bach-flat when the Bach tensor vanishes. Very recently, it was shown [1] that the Besse conjecture holds if the complete divergence of the Weyl curvature tensor $W$ is free, $\text{div}^4W = 0$ and the radially Weyl curvature vanishes. If $\text{min}_M f \geq -1$, it is clear that $(M, g)$ is Einstein. In fact, if we let $i_{\nabla f}z_g = z_g(\nabla f, \cdot)$, the divergence of $i_{\nabla f}z_g$ can be computed as $\text{div} \left(i_{\nabla f}z_g\right) = (1 + f)|z_g|^2$. Hence, by integrating it over $M$, we have $z_g = 0$ from the divergence theorem. As a 1-form or a vector field, the quantity $i_{\nabla f}z_g$ has a crucial structural meaning in CPE. In [23], we show that $i_{\nabla f}z_g$ is parallel to $\nabla f$ when $(M, g)$ has a harmonic curvature, and this property plays an important role in proving the main theorem.

First, we prove the Besse conjecture when $z_g$ is vanishing in the direction $\nabla f$ which means $z_g(\nabla f, X) = 0$ for any vector field $X$ orthogonal to $\nabla f$. In [23], we showed any nontrivial solution $(M, g, f)$ to the CPE having harmonic curvature satisfies this vanishing property on $z_g$. Thus, our first result can be considered as a generalization of the main result in [23].

**Theorem 1.1** Let $(g, f)$ be a nontrivial solution of (1.1). If $z(\nabla f, X) = 0$ for any vector field $X$ orthogonal to $\nabla f$, then $(M, g)$ is isometric to a standard sphere.

The second objective of this paper is to prove the Besse conjecture under the condition of positive isotropic curvature on $M$. Let $(M^n, g)$ be an $n$-dimensional Riemannian manifold.
with \( n \geq 4 \). The Riemannian metric \( g = \langle \cdot, \cdot \rangle \) can be extended either to a complex bilinear form \( \langle \cdot, \cdot \rangle \) or a Hermitian inner product \( \langle \cdot, \cdot \rangle \) on each complexified tangent space \( T_p M \otimes \mathbb{C} \) for \( p \in M \). A complex 2-plane \( \sigma \subset T_p M \otimes \mathbb{C} \) is totally isotropic if \( (Z, Z) = 0 \) for any \( Z \in \sigma \). For any 2-plane \( \sigma \subset T_p M \otimes \mathbb{C} \), we can define the complex sectional curvature of \( \sigma \) with respect to \( \langle \cdot, \cdot \rangle \) by

\[
K_\mathbb{C}(\sigma) = \langle \langle Z \wedge W, Z \wedge W \rangle \rangle, \tag{1.4}
\]

where \( R : \Lambda^2 T_p M \to \Lambda^2 T_p M \) is the curvature operator and \( \{Z, W\} \) is a unitary basis for \( \sigma \) with respect to \( \langle \cdot, \cdot \rangle \).

A Riemannian \( n \)-manifold \((M^n, g)\) is said to have a positive isotropic curvature (PIC in short) if the complex sectional curvature on the isotropic planes is positive, that is, for any totally isotropic 2-plane \( \sigma \subset T_p M \otimes \mathbb{C} \),

\[
K_\mathbb{C}(\sigma) > 0. \tag{1.5}
\]

If \((M, g)\) has a positive curvature operator, then it has a PIC [14]. Thus, a standard sphere \((S^n, g_0)\) has a PIC. Additionally, if the sectional curvature of \((M, g)\) is pointwise strictly quarter-pinched, then \((M, g)\) has a PIC [14]. It is well known that the product metric on \( S^{n-1} \times S^1 \) also has a PIC and the connected sum of manifolds with a PIC admits a PIC metric [15]. The existence of Riemannian metrics with a PIC on compact manifolds which fiber over the circle is discussed in [11]. One of the main results on manifolds with a positive isotropic curvature is that a simply connected compact \( n \)-dimensional Riemannian manifold with positive isotropic curvature is homeomorphic to a sphere [14]. In another major result relating the topology of a positive isotropic curvature manifold, it is proved [14] that the homotopy groups \( \pi_i(M) = 0 \) for \( 2 \leq i \leq \lfloor \frac{n}{2} \rfloor \) when \( \dim(M) = n \), and the fundamental group \( \pi_1(M) \) cannot contain any subgroup isomorphic to the fundamental group of a closed orientable surface [7, 8]. For even-dimensional manifolds, it was proved [15, 19] that a PIC implies the vanishing of the second Betti number. On the contrary, a PIC implies that \( g \) has a positive scalar curvature [15]. More details about the PIC are provided in [5] or [20], and the references are also presented therein.

Our second main result can be stated as follows.

**Theorem 1.2** Let \( M \) be a compact \( n \)-dimensional smooth manifold with \( n \geq 4 \). If \((g, f)\) is a nontrivial solution of (1.1) and \((M, g)\) has positive isotropic curvature, then \((M, g)\) is isometric to a standard sphere.

Due to [23, 24], it suffices to prove that \((M, g)\) has harmonic curvature in Theorem 1.2. Then, \( g \) should be Einstein, which implies that \((M, g)\) is isometric to a standard sphere due to Obata [17]. To show that \((M, g)\) has a harmonic curvature, we introduce a 2-form, \( df \wedge i_{\nabla f} z_g \), consisting of the total differential, \( df \), of the potential function \( f \) and the traceless Ricci tensor \( z_g \) and prove that it vanishes when \((M, g)\) has positive isotropic curvature.

**Notations and convention** Hereafter, for convenience and simplicity, we denote the Ricci curvature \( \text{Ric}_g \), the traceless Ricci curvature \( z_g \), the scalar curvature \( s_g \), and the Hessian and Laplacian of \( f \), \( D_g df, \Delta_g \) by \( r, z, s \), and \( Ddf, \Delta \), respectively, if there is no ambiguity. We also use the notation \( \langle \cdot, \cdot \rangle \) for metric \( g \) or inner product induced by \( g \) on tensor spaces. We follow convention on curvature tensors and operators as in [2] except only the Laplace operator. We consider the Laplace operator as a negative operator, which means \( \Delta = - (\delta df + d\delta) \) and so \( \Delta f = f'' \) on \( \mathbb{R} \).
2 Some preliminaries and tensors

As a preliminary as well as for subsequent use, we recall the Cotton tensor and briefly describe the properties of the Cotton tensor. Additionally, we introduce a structural tensor $T$, which plays a key role in proving our main theorems. (For the definition of $T$, refer (2.4).) This structural tensor $T$ has deep relations to the critical point equation (1.1) and the Cotton tensor (cf. [9]).

2.1 Cotton tensor

Let $(M^n, g)$ be a Riemannian manifold of dimension $n$ with the Levi–Civita connection $D$. We begin with a differential operator acting on the space of symmetric 2-tensors. Let $b$ be a symmetric 2-tensor on $M$. The differential $D b$ can be defined as follows:

$$d D b(X, Y, Z) = DX b(Y, Z) - DY b(X, Z)$$

for any vectors $X, Y,$ and $Z$. The *Cotton tensor* $C \in \Gamma(\Lambda^2 M \otimes T^*M)$ is defined by

$$C = d D \left( r - \frac{s}{2(n-1)}g \right) = d D r - \frac{1}{2(n-1)} ds \wedge g. \quad (2.1)$$

It is known that, for $n = 3$, $C = 0$ if and only if $(M^3, g)$ is locally conformally flat. Moreover, for $n \geq 4$, the vanishing of the Weyl tensor $W$ implies the vanishing of the Cotton tensor $C$, while $C = 0$ corresponds to the Weyl tensor being harmonic, i.e., $\delta W = 0$ due to the identity [2]:

$$\delta W = -\frac{n-3}{n-2} d D \left( r - \frac{s}{2(n-1)}g \right) = -\frac{n-3}{n-2} C \quad (2.2)$$

under the following identification

$$\Gamma(T^* M \otimes \Lambda^2 M) \equiv \Gamma(\Lambda^2 M \otimes T^* M).$$

Let $\{E_i\}$ be a local frame with $C(E_i, E_j, E_k) = C_{ijk}$. Then, it follows from (2.1) that

$$C_{ijk} = R_{jk;i} - R_{ik;j} - \frac{1}{2(n-1)} S_i \delta_{jk} + \frac{1}{2(n-1)} S_j \delta_{ik},$$

where $r(E_i, E_j) = R_{ij}, \; R_{ij;k} = D_{E_k} r(E_i, E_j)$ and $ds(E_i) = S_i$. The first direct observation on $C$ is that the cyclic summation of $C_{ijk}$ is vanishing. The second observation on $C$ is that $C_{ijk}$ is skew symmetric for the first two components, which implies that $C(X, X, \cdot) = 0$ for any vector $X$, and trace-free in any two indices. Since $r = z + \frac{s}{n} g$, equation (2.1) can be rewritten as

$$C = d D z + \frac{n-2}{2n(n-1)} ds \wedge g.$$

In the local coordinates, we have

$$C_{ijk} = z_{jk;i} - z_{ik;j} + \frac{n-2}{2n(n-1)} (S_i \delta_{jk} - S_j \delta_{ik}).$$

In the CPE, since the scalar curvature $s$ is constant, we have

$$C = d D r = d D z$$
and
\[ C_{ijk} z_{jk;}^i = |C|^2 + C_{ijk} z_{ik;}^j = |C|^2 - C_{ijk} z_{ik;}^j. \]

Here, we follow the Einstein convention for indices, i.e.,

\[ \frac{1}{2} |C|^2 = C_{ijk} z_{jk;}^i = C(E_i, E_j, E_k) D_{E_i} z(E_j, E_k). \]

Moreover,

\[ \frac{1}{2} |C|^2 = C_{ijk} z_{jk;}^i = (C_{ijk} z_{jk;}^i) - C_{ijk} z_{jk} \]
\[ = (C_{ijk} z_{jk;}^i) + \langle \delta C, z \rangle, \tag{2.3} \]

where \( \delta C(X, Y) = -\text{div} C(X, Y) = -D_{E_i} C(E_i, X, Y). \)

### 2.2 The tensor \( T \)

We now define a 3-tensor \( T \) by

\[ T = \frac{1}{n-2} df \wedge z + \frac{1}{(n-1)(n-2)} i_{\nabla f \! z} \wedge g. \tag{2.4} \]

Recall that \( i_{\nabla f \! z} \) denotes the (usual) interior product to the first factor defined by \( i_{\nabla f \! z} W(Y, Z, U) = W(Y, Z, U, X) \) for any vectors \( Y, Z, \) and \( U \).

**Proof** Taking the differential \( d^D \) in (1.1), we have

\[ (1 + f) d^D z = i_{\nabla f \! z} W - \frac{n-1}{n-2} df \wedge z - \frac{1}{n-2} i_{\nabla f \! z} \wedge g. \]

(For more details about this, refer [23].) From the definition of \( T \) together with \( C = d^D z \), we obtain

\[ (1 + f) C = i_{\nabla f \! z} W - (n-1)T. \]
Now, we derive certain expressions relating $T$ and $C$ to their divergences. For a symmetric 2-tensor $b$ on a Riemannian manifold $(M^n, g)$, we define $\hat{W}(b)$ by

$$\hat{W}(b)(X, Y) = \sum_{i=1}^{n} b(\nabla X, E_i)Y = \sum_{i,j=1}^{n} W(X, E_i, Y, E_j) b(E_i, E_j)$$

for any local frame $\{E_i\}$. For the Riemannian curvature tensor $R$, $\hat{R}(b)$ is similarly defined.

**Lemma 2.2** Let $(g, f)$ be a solution of the CPE. Then,

$$\delta(\tilde{i}_f W)(X, Y) = -\frac{n-3}{n-2} \hat{C} + (1 + f)\hat{W}(z),$$

where $\hat{C}$ is a 2-tensor defined as

$$\hat{C}(X, Y) = C(Y, \nabla f, X)$$

for any vectors $X, Y$.

**Proof** Let $\{E_i\}$ be a local frame normal at a point in $M$. At the point, it follows from definition together with the CPE that

$$\delta(\tilde{i}_f W)(X, Y) = \delta W(X, Y, \nabla f) - Ddf(E_i, E_j)W(E_i, X, Y, E_j).$$

Since the trace of $W$ in the first and fourth components vanishes, applying (2.2), we obtain

$$\delta W(X, Y, \nabla f) - Ddf(E_i, E_j)W(E_i, X, Y, E_j) = -\frac{n-3}{n-2} C(Y, \nabla f, X) + (1 + f)\hat{W}(z)(X, Y).$$

Moreover, the divergence of the tensor $\hat{C}$ has the following form.

**Lemma 2.3** For any vector $X$, we have

$$\delta \hat{C}(X) = -(1 + f)\langle i_X C, z \rangle.$$

**Proof** Let $\{E_i\}$ be a local geodesic frame around a point $p \in M$. Using the fact that $\delta C$ is symmetric and that cyclic summation of $C$ is vanishing, we can show

$$\delta \hat{C}(E_j) = -(1 + f)z(E_i, E_k)C(E_j, E_k, E_i) = -(1 + f)\langle i_{E_j} C, z \rangle.$$

We are now ready to prove that $T = 0$ implies the CPE conjecture.

**Theorem 2.4** Let $(g, f)$ be a nontrivial solution of the CPE. If $T = 0$, then the Besse conjecture holds.

**Proof** It suffices to prove that $C = 0$. First, on each level hypersurface $f^{-1}(t)$ for a regular value $t$ of $f$, it follows from the definition of $T$ that
\[ i\nabla f T = \frac{[\nabla f]^2}{n-2} \left( z + \frac{\alpha}{n-1} g \right). \]

Since \( T = 0 \), we have
\[ z(E_i, E_j) = -\frac{\alpha}{n-1} \delta_{ij} \quad (2 \leq i, j \leq n) \tag{2.5} \]
for a local orthonormal frame \( \{ E_1 = N, \cdots, E_{n-1}, E_n \} \) with \( N = \frac{\nabla f}{|\nabla f|} \). Also, from
\[ 0 = (n-2)i\nabla f T(E_j, \nabla f) = \frac{n-2}{n-1}|\nabla f|^2 z(E_j, \nabla f) \]
for \( 2 \leq j \leq n \), we have
\[ z(\nabla f, E_j) = 0. \tag{2.6} \]

Therefore,
\[ \langle i\nabla f C, z \rangle = -\frac{\alpha}{n-1} \sum_{i=2}^{n} C(\nabla f, E_i, E_i) = \frac{\alpha}{n-1} C(\nabla f, N, N) = 0. \tag{2.7} \]

On the other hand, from Lemma 2.1, we have
\[ (1 + f)C = \tilde{i}\nabla f \mathcal{W} \tag{2.8} \]
and hence,
\[ C(X, Y, \nabla f) = 0 \]
for any vector fields \( X \) and \( Y \). Since the cyclic summation of \( C \) vanishes, we have
\[ C(Y, \nabla f, X) + C(\nabla f, X, Y) = 0. \]

Taking the divergence \( \delta \) of (2.8) and applying Lemma 2.2, we obtain
\[ -i\nabla f C + (1 + f)\delta C = \frac{n-3}{n-2} i\nabla f C + (1 + f)\mathcal{W}(z). \]
Thus,
\[ (1 + f)\delta C - (1 + f)\mathcal{W}z = \frac{2n-5}{n-2} i\nabla f C. \tag{2.9} \]

Note that, by (2.8) and definition of \( \mathcal{W}(z) \), the following
\[ \mathcal{W}(z)(\nabla f, X) = -(1 + f)\langle iX C, z \rangle \]
holds for any vector field \( X \). In particular, by (2.7), we have
\[ \mathcal{W}(z)(\nabla f, \nabla f) = 0. \]

Consequently, by (2.9), we obtain
\[ (1 + f)\delta C(N, N) = 0. \tag{2.10} \]
Finally, it follows from (2.5) and (2.6) together with
\[ \sum_{j=2}^{n} C_{ijj;i} = -C_{i11;i} \]
that
\[ \langle \delta C, z \rangle = -C_{ijk;izjk} = \frac{\alpha}{n-1} \sum_{i=1}^{n} \sum_{j=2}^{n} C_{ijj;i} - \sum_{i=1}^{n} C_{i11;i} \]
\[ = -\frac{n\alpha}{n-1} \sum_{i=1}^{n} C_{i11;i} = \frac{n\alpha}{n-1} \delta C(N, N) = 0. \]

Hence, integrating (2.3) over \( M \), we have \( C = dDz = 0 \), which implies that \((M, g)\) has a harmonic curvature. Therefore, it follows from [23, 24] that \((M, g)\) is Einstein. \( \square \)

**Remark 2.5** Introducing the Bach tensor on a Riemannian manifold \((M^n, g)\), we can directly show that \( C = 0 \) in Theorem 2.4 without using the divergence theorem in a higher-dimensional case \( n \geq 5 \). The Bach tensor \( B \) is defined as
\[ B = \frac{1}{n-3} \delta \delta W + \frac{1}{n-2} \hat{W}(r). \]

When the scalar curvature \( s \) is constant, we have \( \hat{W}(r) = \hat{W}(z) \). Therefore, from (2.2), we have
\[ B = \frac{1}{n-2} \left( -\delta C + \hat{W}(z) \right). \]

Furthermore, the following property holds in general (cf. [4]) for \( n \geq 4 \): for any vector field \( X \),
\[ (n-2)\delta B(X) = -\frac{n-4}{n-2} \langle i_X C, z \rangle. \]

The complete divergence of the Bach tensor has the following form:
\[ \delta \delta B = \frac{n-4}{(n-2)^2} \left( \frac{1}{2} |C|^2 - \langle \delta C, z \rangle \right). \]

Using these identities on the Bach tensor and taking the divergence of \( T \), the following identities can be obtained:
(i) \((n-2)(1 + f)B = -i\nabla f C + \frac{n-3}{n-2} \hat{C} + (n-1)\delta T.\)
(ii) For any vector field \( X \),
\[ \delta \delta T(X) = \frac{1}{n-2} (1 + f) \langle i_X C, z \rangle + \langle i_X T, z \rangle. \]

Therefore, we obtain the following: Let \((g, f)\) be a nontrivial solution of the CPE and \( T = 0 \). Then, \( \langle i_X C, z \rangle = 0 \) for any vector field \( X \) by (ii), and hence,
\[ \frac{1}{2} |C|^2 = \langle \delta C, z \rangle = 0. \]

Before closing this section, we would like to mention some of the identities on the square norm and the divergence of \( T \), which will be used later in the proof of our main theorem.

**Lemma 2.6** ([25]) Let \((g, f)\) be a solution of the CPE. Then,
\[ |T|^2 = \frac{2}{(n-2)^2} |\nabla f|^2 \left( |z|^2 - \frac{n}{n-1} |i_N z|^2 \right). \]
Lemma 2.7 Let \((g, f)\) be a solution of the CPE, Then,
\[
(n-1)\delta T = \frac{sf}{n-1} z - D_{\nabla f} z + \frac{1}{n-2} \widehat{C} + \frac{n}{n-2} (1+f) z \circ z - \frac{1}{n-2} (1+f) |z|^2 g.
\]
Here, \(z \circ z\) is defined by
\[
z \circ z (X, Y) = \sum_{i=1}^{n} z(X, E_i) z(Y, E_i)
\]
for a local frame \(\{E_i\}_{i=1}^{n}\) of \(M\).

Proof By using (1.2) and (1.3), we can compute
\[
\delta (df \wedge z) = \frac{sf}{n} z - D_{\nabla f} z + (1+f) z \circ z.
\]
Recall that since the scalar curvature \(s\) is constant, we have \(\delta z = \delta r = -\frac{1}{2} ds = 0\). From this fact and using (1.3) again, we can obtain
\[
\delta (i_{\nabla f} z \wedge g) = D_{\nabla f} z + \widehat{C} - (1+f) |z|^2 g + (1+f) z \circ z - \frac{sf}{n(n-1)} z.
\]
By combining these identities, the proof follows from the definition of the tensor \(T\). \(\square\)

3 CPE with \(z(\nabla f, X) = 0\)

For the potential function \(f\) of the CPE, we let
\[
\alpha := z(N, N)
\]
for convenience. Here, recall that \(N = \frac{\nabla f}{|\nabla f|}\). Note that the function \(\alpha\) is well defined only on the set \(M \setminus \text{Crit}(f)\), where \(\text{Crit}(f)\) is the set of all critical points of \(f\). However, \(|\alpha| \leq |z|\), \(\alpha\) can be extended to a \(C^0\) function on the whole \(M\). Refer [23] for more details.

First, it is easy to compute that
\[
N(|\nabla f|) = (1+f) \alpha - \frac{sf}{n(n-1)} \quad (3.1)
\]
and thus,
\[
N \left( \frac{1}{|\nabla f|} \right) = -\frac{1}{|\nabla f|^2} \left[ (1+f) \alpha - \frac{sf}{n(n-1)} \right]. \quad (3.2)
\]
In particular, we have the following lemma.

Lemma 3.1 Assume \(z(\nabla f, X) = 0\) for any vector field \(X\) orthogonal to \(\nabla f\). Then \(i_{\nabla f} z = \alpha \nabla f\) as a vector field, and \(D_N N = 0\).

Proof It is obvious that \(i_{\nabla f} z = \alpha \nabla f\). From this, by combining (1.3) and (3.2), it is easy to see that \(D_N N = 0\). \(\square\)

From now, throughout this section, we assume that \((M^n, g, f)\) is a nontrivial solution of the CPE satisfying
\[
z(\nabla f, X) = 0 \quad (3.3)
\]
Lemma 3.3. The functions $\alpha$ and $|\nabla f|^2$ are all constants on each level set $f^{-1}(c)$ of $f$.

Proof. We may assume that $c$ is a regular value of $f$ by the Sard’s theorem. Let $X$ be a tangent vector on $f^{-1}(c)$. Then, from (1.3), we have

$$\frac{1}{2} X(|\nabla f|^2) = Ddf(\nabla f, X) = (1 + f)\alpha = -\frac{sf}{n-1} g(\nabla f, X) = 0.$$

Now, let $X$ be a (local) vector field orthogonal to $\nabla f$ such that $z(X, \nabla f) = 0$. Since $D_N N = 0$ by Lemma 3.1, we have $g(D_N X, N) = -g(X, D_N N) = 0$ and $D_N \nabla f = -|\nabla f| N \left( \frac{1}{|\nabla f|} \right) \nabla f$. So,

$$D_N z(X, \nabla f) = -z(D_N X, \nabla f) = z(X, D_N \nabla f) = 0. \tag{3.6}$$

Since $\tilde{i}_f f C = 0$ by Lemma 3.2, we have

$$0 = C(X, N, \nabla f) = D_X z(N, \nabla f) - D_N z(X, \nabla f) = D_X z(N, \nabla f) = |\nabla f| X(\alpha),$$

implying that $\alpha$ is a constant on $f^{-1}(c)$.
The property, $D_N N = 0$, also implies $[X, N]$ is orthogonal to $\nabla f$, and therefore,

$$X(N(\alpha)) = N(X(\alpha)) - [X, N](\alpha) = 0.$$  

Since $\nabla \alpha = N(\alpha)N$, it shows $Dd\alpha(X, \nabla f) = 0$, and from (3.5), we have $X(|z|^2) = 0$. Consequently, we conclude that $|z|^2$ is a constant along each level set $f^{-1}(c)$ of $f$. \hfill $\square$

**Lemma 3.4** $\alpha$ is nonpositive on $M$.

**Proof** First assume that each level set $f^{-1}(t)$ is connected. Suppose that $\alpha > 0$ on a level set $f^{-1}(c)$. If $c \leq -1$, the divergence theorem shows that

$$0 \geq \int_{f \leq c} (1 + f)|z|^2 = \int_{f \leq c} \text{div}(i\nabla f z) = \int_{f = c} \alpha|\nabla f|,$$

which is impossible as $\alpha$ is constant on each level set. Note that $-1$ is a regular value of $f$ (see “Appendix”). If $c > -1$, the divergence theorem states that

$$0 \leq \int_{f \geq c} (1 + f)|z|^2 = -\int_{f = c} \alpha|\nabla f|,$$

which is also impossible. In any case, we get a contradiction. For general case, i.e., if level sets may not be connected, refer [23]. (In the proof of Lemma 5.2 there, we used only the CPE and the condition $z(\nabla f, X) = 0$ for $X \perp \nabla f$.) \hfill $\square$

**Lemma 3.5** Let $(g, f)$ be a nontrivial solution of (1.1) on an $n$-dimensional compact manifold $M$ with $z(\nabla f, X) = 0$ for $X \perp \nabla f$. Then, there are no critical points of $f$ except at its maximum and minimum points unless $g$ is Einstein.

**Proof** Note that $|\nabla f|^2$ is constant on each level set of $f$. From the Bochner–Weitzenböck formula together with (1.3), we have

$$\frac{1}{2} \Delta |\nabla f|^2 - \frac{1}{2} \nabla f(\nabla f^2) + \frac{s}{n(n - 1)} \frac{1}{1 + f} |\nabla f|^2 = |Dd f|^2.$$  

By maximum principle, the function $|\nabla f|^2$ cannot have its local maximum in $M_0 := \{x \in M : f(x) < -1\}$. Let $p$ be a critical point of $f$ in $M$ other than the minimum or maximum points of $f$. From the argument above, we can see that $p$ cannot be in $M_0 := \{x \in M : f(x) < -1\}$ unless $f$ is a constant. In fact, if $p \in M_0$ and $p$ is not a minimum point, then there should be a local maximum point of $|\nabla f|^2$ in $\{x \in M : f < f(p)\}$, which is impossible. We also claim that $p$ cannot be in $M^0 := \{x \in M : f(x) > -1\}$. Recall that we have $f(p) \neq -1$ and

$$\nabla f(\nabla f^2) = \frac{2s}{n(n - 1)} |\nabla f|^2 > 0$$

on the set $f = -1$. This shows that $|\nabla f|^2$ might have its maximum at a point, say $q$, in $M^0$. However, since

$$0 = \frac{1}{2} \nabla f(\nabla f^2) = (1 + f)\alpha|\nabla f|^2 - \frac{sf}{n(n - 1)} |\nabla f|^2$$

at the point $q$ and $(1 + f)\alpha|\nabla f|^2 \leq 0$, $q$ should lie in $\Omega := \{x \in M : -1 < f(x) < 0\}$. This implies that there are no critical points of $f$ in $M^0 \setminus \Omega$. Now, if $|z|^2(p) \neq 0$ with $p \in \Omega$, by Lemma 3.9 of [23], the point $p$ has to be a local maximum point of $f$, which is not possible because

$$\Delta f = -\frac{s}{n - 1} f > 0$$
on $\Omega$. Thus, we have $|z|^2(p) = 0$ with $p \in \Omega$. In particular, if we let $f(p) = c$ with $-1 < c < 0$, then we have $\alpha = 0$ on the level set $f^{-1}(c)$. Finally, from the divergence theorem, we have

$$0 \leq \int_{f > c} (1 + f)|z|^2 = -\int_{f = c} \alpha|\nabla f| = 0,$$

which means $z = 0$ on the set $f \geq c$. Then, by the analyticity of $g$ and $f$, $z$ must be vanishing on the set $M^0$ and consequently on the entire $M$. \hfill $\Box$

Furthermore, we can show that the potential function $f$ has only one maximum point and only one minimum point by investigating the topology of minimum set and maximum set of $f$ and using PDE theory related to the stability operator.

**Theorem 3.6** Let $(g, f)$ be a nontrivial solution of the CPE on an $n$-dimensional compact manifold $M$ with $z(\nabla f, X) = 0$ for $X \perp \nabla f$. Then, there are only two isolated critical points of $f$; in other words, only one maximum point and only one minimum point of $f$ on $M$ exist.

**Proof** Let $\min_{M} f = a$ and suppose $f^{-1}(a)$ is not discrete. By Lemma 3.5, all the connected components of the level hypersurface $f^{-1}(t)$ for any regular value $t$ of $f$ have the same topological type. In particular, since $M$ is smooth and $f^{-1}(a)$ is not discrete, $f^{-1}(a)$ must be a hypersurface and it also has the same topological type as any connected component of $f^{-1}(t)$ for any regular value $t$ of $f$. Moreover, for a sufficiently small $\epsilon > 0$, $f^{-1}(a + \epsilon)$ has two connected components, say $\Sigma^+_1$, $\Sigma^-_1$. Note that $\Sigma^+_1 = \Sigma^-_1 = f^{-1}(a)$.

Let $v$ be a unit normal vector field on $\Sigma := f^{-1}(a)$. Then, $v$ can be extended smoothly to a vector field $\mathcal{E}$ defined on a tubular neighborhood of $f^{-1}(a)$ such that $\mathcal{E}|_{f^{-1}(a)} = v$ and $\mathcal{E}|_{\Sigma^+} = N = \frac{\nabla f}{|\nabla f|}$, $\mathcal{E}|_{\Sigma^-} = -N = -\frac{\nabla f}{|\nabla f|}$. Note that

$$\lim_{\epsilon \to 0^+} N = \lim_{\epsilon \to 0^-} (-N) = v.$$

On the hypersurface $f^{-1}(a + \epsilon)$ near $f^{-1}(a)$, the Laplacian of $f$ is given by

$$\Delta f = Ddf(N, N) + m|\nabla f|,$$

where $m$ denotes the mean curvature of $f^{-1}(a + \epsilon)$. In particular, by letting $\epsilon \to 0$, we have

$$\Delta f = Ddf(v, v)$$

on $f^{-1}(a)$. Note that the mean curvature $m$ does not blow up on $f^{-1}(a)$. Therefore, from the CPE $-\frac{sf}{n-1} = (1 + f)z(v, v) - \frac{sf}{n(n-1)}$, we have

$$z(v, v) = -\frac{sf}{n(1 + f)} = -\frac{sa}{n(1 + a)} \tag{3.7}$$

on the set $\Sigma = f^{-1}(a)$. In particular, we have $z_p(v, v) < 0$, since we may assume $a < -1$ as mentioned in Introduction.

Now, since $a$ is the minimum value of $f$, for each point $p \in f^{-1}(a)$, the index of $Ddf_p$ is zero, i.e., for any vector $v$ at $p$, we have $Ddf_p(v, v) \geq 0$. Choosing an orthonormal basis $\{e_1 = v(p), e_2, \cdots, e_n\}$ on $T_pM$, we obtain

$$Ddf_p(v, v) = (1 + f)z_p(v, v) - \frac{sf}{n(n-1)} \geq 0$$

\copyright Springer
and for all $2 \leq i \leq n$,

$$Ddf_p(e_i, e_i) = (1 + f)z_p(e_i, e_i) - \frac{sf}{n(n-1)} \geq 0.$$  

In particular, from (3.7),

$$z_p(e_i, e_i) \leq \frac{s}{n(n-1)} \cdot \frac{a}{1+a} = -\frac{1}{n-1} z_p(v, v).$$

Summing up $i = 2, \ldots, n$, we can see

$$z_p(e_i, e_i) = -\frac{1}{n-1} z_p(v, v) = \frac{s}{n(n-1)} \cdot \frac{a}{1+a} > 0 \quad (3.8)$$

for each $i = 2, \ldots, n$. Thus, for each $i = 2, \ldots, n$,

$$Ddf_p(e_i, e_i) = (1 + f)z_p(e_i, e_i) - \frac{sf}{n(n-1)} = 0. \quad (3.9)$$

Now, we claim that the minimum set $\Sigma = f^{-1}(a)$ is totally geodesic and in particular, the mean curvature is vanishing, i.e., $m = 0$ on $\Sigma$. In fact, fix $i$ for $i = 2, 3, \ldots, n$ and let $\gamma : [0, l] \to M$ be a unit speed geodesic such that $\gamma(0) = p \in \Sigma$ and $\gamma'(0) = e_i \in T_p \Sigma$. Then, we have

$$D\gamma'N = \langle \gamma', N \rangle N \left( \frac{1}{|\nabla f|} \right) \nabla f + \frac{1}{|\nabla f|} \left[ (1 + f)z(\gamma', \cdot) - \frac{sf}{n(n-1)} \gamma' \right]. \quad (3.10)$$

Recall that, by (3.2)

$$N \left( \frac{1}{|\nabla f|} \right) = -\frac{1}{|\nabla f|^2} \left( (1 + f)\alpha - \frac{sf}{n(n-1)} \right).$$

Let

$$\gamma' = \langle \gamma', N \rangle N + (\gamma')^\top,$$

where $(\gamma')(^\top)$ is the tangential component of $\gamma'(t)$ to $f^{-1}(\gamma(t))$, and substituting these into (3.10), we obtain

$$|\nabla f|D\gamma'N = (1 + f)z((\gamma')^\top, \cdot) - \frac{sf}{n(n-1)}(\gamma')^\top.$$  

Taking the covariant derivative in the direction $N$, we have

$$Ddf(N, N)D\gamma'N + |\nabla f|D_N D\gamma'N$$

$$= |\nabla f|z((\gamma')^\top, \cdot) + (1 + f)D_N[z((\gamma')^\top, \cdot)]$$

$$- \frac{s}{n(n-1)} |\nabla f|(\gamma')^\top - \frac{sf}{n(n-1)} D_N \gamma'^\top.$$  

Letting $t \to 0^+$, we obtain

$$- \frac{sa}{n-1} D_{e_i} v = (1 + a)D_v[z((\gamma')^\top, \cdot)] \bigg|_p - \frac{sa}{n(n-1)} D_v(\gamma')^\top \bigg|_p \quad (3.11)$$

because the covariant derivative depends only on the point $p$ and initial vector $e_i$. Now since $z(N, X) = 0$ for $X \perp N$, we may assume that $\{e_i\}_{i=2}^n$ diagonalizes $z$ at the point $p$. Then,
\[ z(D_v(y')^\top, e_i) \bigg|_p = 0. \] (3.12)

**Assertion:** \( \nu(z(y'(t), y'(t)))|_{t=0} = 0. \)

Defining \( \varphi(t) := f \circ y(t) \), we have \( \varphi'(0) = 0 \) and also \( \varphi''(0) = Ddf(e_i, e_i) = 0 \) by (3.9). Since \( \Sigma \) is the minimum set of \( f \), \( \varphi'(t) \) is nondecreasing when \( \varphi(t) \) is sufficiently close to \( a = \min f \) and so \( \varphi''(t) \geq 0 \) for sufficiently small \( t \). So

\[ \varphi''(t) = Ddf(y'(t), y'(t)) = [1 + \varphi(t)]z(y'(t), y'(t)) - \frac{s}{n(n-1)} \varphi(t) \geq 0 \] (3.13)

for sufficiently small \( t > 0 \). However, by (3.8), we have \( z(y'(t), y'(t)) > 0 \) for sufficiently small \( t > 0 \). So,

\[ 0 < z(y'(t), y'(t)) \leq \frac{s}{n(n-1)} \frac{\varphi(t)}{1 + \varphi(t)} \]

for sufficiently small \( t > 0 \). Defining \( \xi(t) = z(y'(t), y'(t)) - \frac{s}{n(n-1)} \frac{\varphi(t)}{1 + \varphi(t)} \), we have \( \xi(t) \leq 0 \) and \( \xi(0) = 0 \) by (3.9). Thus,

\[ \xi'(0) = \frac{d}{dt} \bigg|_{t=0} z(y'(t), y'(t)) = 0. \]

Now considering an extension of \( y \) to an interval \((-\epsilon, 0]\), we can see that the derivative \( \frac{d}{dt}|_{t=0} z(y'(t), y'(t)) \) cannot be negative since \( \Sigma \) is the minimum set of \( f \). In other words, we must have

\[ \frac{d}{dt}|_{t=0} z(y'(t), y'(t)) = 0, \]

which completes the Assertion.

Let \( \{N, E_2, \cdots, E_n\} \) be a local frame around \( p \) such that \( E_i(p) = e_i \) for \( 2 \leq i \). Then, for \( i \geq 2 \),

\[ z(E_i, \cdot) = \sum_{j=2}^{n} z(E_i, E_j)E_j \]

and

\[ D_N[z(y'^\top, \cdot)] = D_N \left[ z(y'^\top, E_j)E_j \right] = N \left( z(y'^\top, E_j) \right) E_j + z(y'^\top, E_j)D_NE_j. \]

So,

\[ \langle D_N[z(y'^\top, \cdot), E_i] \rangle = N \left( z(y'^\top, E_i) \right) + z(y'^\top, E_j)\langle D_NE_j, E_i \rangle. \]

Note that \( z(y', y') = \|y'^\top\|z(y'^\top, E_i) + \langle y', N \rangle^2 \alpha \) with \( \alpha = z(N, N) \). Since \( \|y'^\top\| \) attains its maximum at \( p \) and \( \langle y', N \rangle^2 \) attains its minimum 0 at \( p \), we have

\[ \nu[z(y', y')] \bigg|_p = \|y'^\top\|\nu[z(y'^\top, E_i)] + \langle y', N \rangle^2 \nu(\alpha) = \nu[z(y'^\top, E_i)] \bigg|_p, \]

which shows that

\[ \nu[z(y'^\top, E_i)] \bigg|_p = 0 \] (3.14)
by Assertion. Letting $t \to 0$, and applying (3.12) and (3.14), we obtain
\[
\langle D_{\nu} [z(y^{\top}, \cdot)], E_i \rangle|_p = \nu \left( z(y^{\top}, E_i) \right)|_p + z(e_i, e_i)\langle D_{\nu} E_i, E_i \rangle|_p = 0
\]
Thus, by (3.11) and (3.12) again, we have
\[
-\frac{sa}{n-1} \langle D_{e_i} v, e_i \rangle = (1+a)\langle D_{\nu} [z(y^{\top}, \cdot)], E_i \rangle|_p - \frac{sa}{n(n-1)}(D_{e_i} E_i, E_i)|_p = 0.
\]
That is,
\[
\langle D_{e_i} v, e_i \rangle = 0.
\]
Hence, the square norm of the second fundamental form $A$ is given by
\[
|A|^2(p) = \sum_{i=2}^n \left( (D_{e_i} e_i)^\perp \right)^2 = \sum_{i=2}^n \langle D_{e_i} v, e_i \rangle^2 = 0,
\]
which shows $\Sigma = f^{-1}(a)$ is totally geodesic.

Finally, from (3.7), we have
\[
\text{Ric}(v, v) = \frac{s}{n} \cdot \frac{1}{1+a} < 0.
\]
Furthermore, the stability operator for hypersurfaces with a vanishing second fundamental form clearly becomes
\[
\int_{\Sigma} \left[ |\nabla \varphi|^2 - \text{Ric}(v, v)\varphi^2 \right] \geq 0
\]
for any function $\varphi$ on $\Sigma$. By Fredholm alternative (cf. [6], Theorem 1), there exists a positive function $\varphi > 0$ on $\Sigma$ satisfying
\[
\Delta_{\Sigma} \varphi + \text{Ric}(v, v)\varphi = 0.
\]
However, it follows from the maximum principle that $\varphi$ must be a constant since $\Sigma$ is compact, which is impossible. A similar argument shows that $f$ has only one maximum point.

4 Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Let $(M^n, g, f)$ be a nontrivial solution of the CPE satisfying $z(\nabla f, X) = 0$ for any vector field $X$ which is orthogonal to $\nabla f$. From Theorem 2.4, it suffices to prove that $T = 0$ or $(M, g)$ has harmonic curvature. To this end, we introduce a warped product metric involving $\frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}$ as a fiber metric on each level set $f^{-1}(c)$. Consider a warped product metric $\tilde{g}$ on $M$ by
\[
\tilde{g} = \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + |\nabla f|^2 g_{\Sigma},
\]
where $g_{\Sigma}$ is the restriction of $g$ to $\Sigma := f^{-1}(-1)$. Note that, from Theorem 3.6, the metric $\tilde{g}$ is smooth on $M$ except, possibly at two points, the maximum and minimum points of $f$. Furthermore, applying Morse theory [16] together with Theorem 3.6, we can see that $M$ is.
homeomorphic to $S^n$, and fiber $f^{-1}(t)$ is topologically $S^{n-1}$ except the two critical points of $f$.

The following lemma shows that $\nabla f$ is a conformal Killing vector field with respect to the metric $\tilde{g}$.

**Lemma 4.1** Let $(g, f)$ be a nontrivial solution of the CPE on an $n$-dimensional compact manifold $M$ with $z(\nabla f, X) = 0$ for $X \perp \nabla f$. Then,

$$\frac{1}{2} \mathcal{L}_{\nabla f} \tilde{g} = N(|\nabla f|)\tilde{g} = \frac{1}{n}(\tilde{\Delta} f) \tilde{g}.$$

Here, $\mathcal{L}$ denotes the Lie derivative.

**Proof** Note that, by (1.3) we have,

$$\frac{1}{2} \mathcal{L}_{\nabla f} g = D_\phi df = (1 + f)z - \frac{sf}{n(n - 1)} g.$$

Let $X$ and $Y$ be two vector fields with $X \perp \nabla f$ and $Y \perp \nabla f$. By the definition of Lie derivative,

$$\frac{1}{2} \mathcal{L}_{\nabla f}(df \otimes df)(X, Y) = Ddf(X, \nabla f)df(Y) + df(X)Ddf(Y, \nabla f) = 2 \left((1 + f)\alpha - \frac{sf}{n(n - 1)}\right) df \otimes df(X, Y).$$

Therefore, from (3.1),

$$\frac{1}{2} \mathcal{L}_{\nabla f} \left( \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right) = N(|\nabla f|) df \otimes df(|\nabla f|). \quad (4.1)$$

Since

$$\frac{1}{2} \mathcal{L}_{\nabla f}(|\nabla f|^2 g \Sigma) = \frac{1}{2} \nabla f(|\nabla f|^2)g \Sigma = Ddf(\nabla f, \nabla f)g \Sigma = N(|\nabla f|)|\nabla f|^2 g \Sigma,$$

we conclude that

$$\frac{1}{2} \mathcal{L}_{\nabla f} \tilde{g} = \tilde{D}df = N(|\nabla f|)\tilde{g}.$$

In particular, we have $\tilde{\Delta} f = nN(|\nabla f|)$.

**Lemma 4.2** Let $(g, f)$ be a nontrivial solution of the CPE on an $n$-dimensional compact manifold $M$ with $z(\nabla f, X) = 0$ for $X \perp \nabla f$. Then, $T = 0$ on $M$.

**Proof** Let $p, q \in M$ be two points such that $f(p) = \min_M f$ and $f(q) = \max_M f$, respectively, and let $\bar{M} = M\setminus\{p, q\}$. Due to Theorem 3.6 and Lemma 4.1, we can apply Tashiro’s result [21] and can see that $(\bar{M}, \tilde{g})$ is conformally equivalent to $S^n \setminus \{\tilde{p}, \tilde{q}\}$, where $\tilde{p}$ and $\tilde{q}$ are the points in $S^n$ corresponding to $p$ and $q$, respectively. In particular, by Theorem 1 in [3], the fiber space $(\Sigma, g|\Sigma)$ is a space of constant curvature. Thus,

$$(\Sigma, g|\Sigma) \equiv (S^{n-1}, r \cdot g_{S^{n-1}}),$$

where $r > 0$ is a positive constant and $g_{S^{n-1}}$ is a round metric.

Now, replacing $\Sigma = f^{-1}(-1)$ by $\Sigma_t := f^{-1}(t)$ in (4.1), it can be easily concluded that the warped product metric $\tilde{g}_t$, also satisfies Lemma 4.1, and hence, the same argument mentioned above shows that, for any level hypersurface $\Sigma_t := f^{-1}(t)$,

$$(\Sigma_t, g|\Sigma_t) \equiv (S^{n-1}, r(t) \cdot g_{S^{n-1}}).$$
Therefore, the original metric \( g \) can also be written as

\[
g = \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + b(f)^2 g_\Sigma, \tag{4.2}
\]

where \( b(f) > 0 \) is a positive function depending only on \( f \). From (4.1) and the following identity

\[
\frac{1}{2} \mathcal{L}_{\nabla f} (b^2 g_\Sigma) = b(\nabla f, \nabla b) g_\Sigma = b|\nabla f|^2 \frac{db}{df} g_\Sigma.
\]

we obtain

\[
\frac{1}{2} \mathcal{L}_{\nabla f} g = N(|\nabla f|) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + b|\nabla f|^2 \frac{db}{df} g_\Sigma. \tag{4.3}
\]

On the contrary, from (1.1) together with (3.1) and (4.2), we have

\[
\frac{1}{2} \mathcal{L}_{\nabla f} g = Ddf = (1 + f)z - sf \frac{b}{n(n-1)} g
\]

\[
= N(|\nabla f|) \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} + (1 + f)z - (1 + f)\alpha \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}
\]

\[
- sf \frac{b}{n(n-1)} b^2 g_\Sigma.
\]

Comparing this to (4.3), we obtain

\[
\left( b|\nabla f|^2 \frac{db}{df} + sf \frac{b}{n(n-1)} b^2 \right) g_\Sigma = (1 + f) \left( z - \alpha \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|} \right). \tag{4.4}
\]

Now, let \( \{E_1, E_2, \cdots, E_n\} \) be a local frame with \( E_1 = N \). Then, we have

\[
b|\nabla f|^2 \frac{db}{df} = (1 + f)z(E_i, E_i) - sf \frac{b}{n(n-1)} b^2
\]

for each \( 2 \leq j \leq n \). Summing up these, we obtain

\[
(n-1)b|\nabla f|^2 \frac{db}{df} = -(1 + f)\alpha - \frac{sf}{n} b^2.
\]

Substituting this into (4.4), we get

\[
- \frac{\alpha}{n-1} g_\Sigma = z - \alpha \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}. \tag{4.5}
\]

Replacing \((\Sigma, g_\Sigma)\) by \((\Sigma_i, g_{\Sigma_i})\), we can see that the argument mentioned above is also valid. Thus, (4.5) shows that, on each level hypersurface \( f^{-1}(t) \), we have

\[
z(E_i, E_j) = - \frac{\alpha}{n-1}
\]

for \( 2 \leq j \leq n \). Hence,

\[
|z|^2 = a^2 + \frac{\alpha^2}{n-1} = \frac{n}{n-1} a^2 = \frac{n}{n-1}|i_N z|^2,
\]

since \( z(N, E_i) = 0 \) for \( i \geq 2 \). As a result, it follows from Lemma 2.6 that \( T = 0 \). \( \square \)
Remark 4.3 Let \((g, f)\) be a nontrivial solution of the CPE with \(z(\nabla f, X) = 0\) for \(X \perp \nabla f\). In “Appendix,” we show the following result.

\[
\frac{s}{n(n-1)} g = R_N + z + \frac{1 + f}{|\nabla f|^2} i_{\nabla f} C - \left( \alpha - \frac{s}{n(n-1)} \right) \frac{df}{|df|} \otimes \frac{df}{|df|}. \tag{4.6}
\]

Here, \(R_N\) is defined as follows:
\[
R_N(X, Y) = R(X, N, Y, N)
\]
for any vector field \(X\) and \(Y\). Let \(\epsilon := \frac{s}{n(n-1)}\) and
\[
h := R_N + \left( z - \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} \right) + \frac{1 + f}{|\nabla f|^2} i_{\nabla f} C.
\]

We can then rewrite the metric \(g\) as
\[
g = \frac{1}{\epsilon} h + \frac{df}{|df|} \otimes \frac{df}{|df|}. \tag{4.7}
\]

Then, the following can be proved on the set \(f^{-1}(-1):\)
\[
\mathcal{L}_{\nabla f} \left( \frac{h}{|\nabla f|^2} \right) = 0. \tag{4.8}
\]

Therefore, we can conclude that \(g\) can be expressed as a warped product metric and that it is, in fact, equal to the metric \(\tilde{g}\) defined at the beginning of Sect. 4. Refer to “Appendix” for the detailed proofs of (4.6) and (4.8).

Combining Theorem 2.4 and Lemma 4.2, we obtain the following theorem.

Theorem 4.4 Let \((g, f)\) be a nontrivial solution of the CPE on an \(n\)-dimensional compact manifold \(M\) with \(z(\nabla f, X) = 0\) for \(X \perp \nabla f\). Then, \(M\) is isometric to a standard sphere \(S^n\).

5 CPE with positive isotropic curvature

In this section, we will prove that if \((g, f)\) is a nontrivial solution of the CPE with positive isotropic curvature, then \(M\) is isometric to a standard sphere. In view of Theorem 1.1, it suffices to show that,
\[
z(\nabla f, X) = 0
\]
for any \(X \perp \nabla f\). To do this, we introduce a 2-form \(\omega\) on \(M\) defined as
\[
\omega := df \wedge i_{\nabla f} z
\]
by considering \(i_{\nabla f} z\) as a 1-form.

In this section, the dimension of the manifold \(M\) is assumed to be \(n \geq 4\). First, we have the following.

Lemma 5.1 We have
\[
\omega = (n-1)i_{\nabla f} T = -(1 + f)i_{\nabla f} C. \tag{5.1}
\]
**Proof** As in the proof of Lemma 3.2, it follows from the definition of $T$ that
\[(n - 2)T(X, Y, \nabla f) = \frac{n - 2}{n - 1} df \wedge i_{\nabla f} z(X, Y) = \frac{n - 2}{n - 1} \omega(X, Y)\]
for vectors $X$ and $Y$. The second equality follows from Lemma 2.1.\[\square\]

**Lemma 5.2** Let $\{E_1, E_2, \ldots, E_n\}$ be a local frame with $E_1 = N = \frac{\nabla f}{|\nabla f|}$. Then,
\[\omega = 0 \text{ if and only if } \tilde{i}_{\nabla f} C(N, E_j) = 0 \quad (j \geq 2).\]

**Proof** It follows from the definition of $\omega$ that
\[\omega(E_j, E_k) = 0 \quad \text{for all } j, k \geq 2,\]
which shows, by Lemma 5.1,
\[\tilde{i}_{\nabla f} C(E_j, E_k) = 0\]
for $2 \leq j, k \leq n$. Therefore, it is easy to see that
\[\omega = 0 \text{ if and only if } \tilde{i}_{\nabla f} C(N, E_j) = 0\]
for $2 \leq j \leq n$.\[\square\]

We would like to mention that the local frame $\{E_1, E_2, \ldots, E_n\}$ in Lemma 5.2 is well defined only on the set $M \setminus \text{Crit}(f)$. However, we can see that it suffices to show $\omega = 0$ on $M \setminus \text{Crit}(f)$ by continuity. In fact, since $\Delta f = -\frac{s}{n-1} f$, by elliptic PDE theory, the set $\text{Crit}(f)$ cannot be an open subset of $M$ with nonzero measure.

Next, we prove that $\omega$ is closed, and when $(M, g)$ has positive isotropic curvature, $\omega$ is vanishing.

**Lemma 5.3** As a 2-form, we have the following
\[\tilde{i}_{\nabla f} C = d i_{\nabla f} z.\]

**Proof** Choose a local frame $\{E_i\}$ which is normal at a point $p \in M$, and let $\{\theta^i\}$ be its dual coframe so that $d\theta^i | p = 0$. Since $i_{\nabla f} z = \sum_{l,k=1}^{n} f_l z_{lk} \theta^k$ with $E_i(f) = f_i$ and $z(E_l, E_k) = z_{lk}$, by (1.3), we have
\[
d i_{\nabla f} z = \sum_{j,k} \sum_l (f_{lj} z_{lk} + f_l z_{lk;j}) \theta^j \wedge \theta^k = \sum_{j<k} \sum_l \left\{ (f_{lj} z_{lk} - f_{lk} z_{lj}) + f_l (z_{lk;j} - z_{lj;k}) \right\} \theta^j \wedge \theta^k
\]
\[= \sum_{j<k} \sum_l \left[ \left\{ (1 + f) z_{lj} - \frac{s f \delta_{lj}}{n(n - 1)} \right\} z_{lk} - \left\{ (1 + f) z_{lk} - \frac{s f \delta_{lk}}{n(n - 1)} \right\} z_{lj} \right]\]
\[\times \theta^j \wedge \theta^k + \sum_{j<k} \sum_l f_l C_{jkl} \theta^j \wedge \theta^k
\]
\[= \sum_{j<k} \sum_l f_l C_{jkl} \theta^j \wedge \theta^k = \tilde{i}_{\nabla f} C.\]
\[\square\]
Lemma 5.4 \( \omega \) is a closed 2-form, i.e., \( d\omega = 0 \).

Proof Choose a local frame \( \{ E_i \} \) with \( E_1 = N = \nabla f/|\nabla f| \), and let \( \{ \theta^i \} \) be its dual coframe. Then, by Lemma 5.2 and Lemma 5.3

\[
div_f z = \sum_{j<k} \sum_l f_l C_{jkl} \theta^j \wedge \theta^k = \sum_{j<k} \nabla f |C_{jkl} \theta^j \wedge \theta^k = |\nabla f| \sum_{k=2}^n C_{1k1} \theta^1 \wedge \theta^k.
\]

Thus, by taking the exterior derivative of \( \omega \) in (5.1), we have

\[
d\omega = -df \wedge div_f z = -|\nabla f| \theta^1 \wedge \left( |\nabla f| \sum_{k=2}^n C_{1k1} \theta^1 \wedge \theta^k \right) = 0.
\]

\( \square \)

Now, let \( \Omega = \{ p \in M \mid \omega_p \neq 0 \text{ on } T_p M \} \). Then, \( \Omega \) is an open subset of \( M \). We start with the following observation.

Lemma 5.5 Suppose that \( \omega_p \neq 0 \) at \( p \in M \). Then,

\[
|D\omega|^2(p) \geq |\delta\omega|^2(p). \tag{5.3}
\]

Proof First of all, since \( \omega_p \neq 0 \), we have \( df_p \neq 0 \) and \( f(p) \neq -1 \) by definition of \( \omega \) and Lemma 5.1. Define \( A : T_p M \to T_p M \) by \( g(Au, v) = \omega(u, v) \) for any \( u, v \in T_p M \).

Assertion 1: \( \nabla f(p) \notin \ker A \).

Let \( \{ e_1, e_2, \cdots, e_n \} \) be an orthonormal basis on \( T_p M \) with \( e_1 = N(p) \). If \( \nabla f(p) \in \ker A \), then,

\[
0 = \langle Ae_1, e_j \rangle = \omega(e_1, e_j) = -(1 + f) \tilde{\nabla}_f C(e_1, e_j)
\]

for \( j \geq 2 \). This implies that \( \omega_p = 0 \) from Lemma 5.2, a contradiction.

Assertion 2: \( \ker A \subset (\nabla f)^\perp \).

Let \( u \in \ker A \) so that \( \langle Au, v \rangle = 0 \) for any \( v \in T_p M \). Let \( \{ e_1, e_2, \cdots, e_n \} \) be an orthonormal basis on \( T_p M \) with \( e_1 = N(p) \). Then, by Lemma 5.1 together with (5.2), we have

\[
0 = \langle Au, e_k \rangle = \omega(u, e_k) = -(1 + f) \tilde{\nabla}_f C(u, e_k)
\]

\[
= -(1 + f) \sum_{j=1}^n \langle u, e_j \rangle \tilde{\nabla}_f C(e_j, e_k) = -(1 + f) \langle u, e_1 \rangle \tilde{\nabla}_f C(e_1, e_k)
\]

for any \( 1 \leq k \leq n \). Since \( \omega_p \neq 0 \), we have \( \tilde{\nabla}_f C(e_1, e_k) \neq 0 \) for some \( k \geq 2 \) by Lemma 5.2. So, \( \langle u, e_1 \rangle = 0 \), which implies that \( \nabla f(p) \notin (\ker A)^\perp \).

Assertion 3: Let \( \{ e_1, \ldots, e_n \} \) be an orthonormal basis on \( T_p M \) with \( e_1 = N(p) \). Then,

\[
||Ae_1|| = \sup_{u \in (\ker A)^\perp \atop ||u|| = 1} ||Au||.
\]

First, by Assertion 1 and Assertion 2, \( Ae_1 \neq 0 \) and \( e_1 \in (\ker A)^\perp \). Since \( g(Ae_1, e_1) = \omega(e_1, e_1) = 0 \), \( Ae_1 \) is orthogonal to \( e_1 \), we may assume that \( e_2 = Ae_1/||Ae_1|| \). In particular, we have \( e_2 \in (\ker A)^\perp \) by the skew-symmetry of \( A \).

Let \( u \in (\ker A)^\perp \) with \( ||u|| = 1 \). Since \( \langle Ae_1, e_j \rangle = \omega(e_1, e_j) = 0 \) for \( i, j \geq 2 \) by (5.2), we have
\[ Au = \sum_{j=1}^{n} (Au, e_j)e_j = \sum_{j=1}^{n} \sum_{i=1}^{n} (u, e_i) \langle Ae_i, e_j \rangle e_j \]

\[ = \langle u, e_1 \rangle Ae_1 - \sum_{i=1}^{n} (u, e_i) \langle e_i, Ae_1 \rangle e_1 \]

\[ = \langle u, e_1 \rangle Ae_1 - \langle u, Ae_1 \rangle e_1 \]

\[ = ||Ae_1||\langle u, e_1 \rangle e_2 - ||Ae_1||\langle u, e_2 \rangle e_1. \]

Since \( ||u||^2 = \sum_j \langle u, e_j \rangle^2 = 1 \), we have

\[ ||Au||^2 = ||Ae_1||^2 (\langle u, e_1 \rangle^2 + \langle u, e_2 \rangle^2) \leq ||Ae_1|| \]

and hence Assertion 3 is satisfied.

Note that, for \( u = e_j \) with \( j \geq 2 \), we also have

\[ Ae_j = -\langle e_j, Ae_1 \rangle e_1 \quad \text{and so} \quad ||Ae_j|| \leq ||Ae_1||. \]

Now, let us show inequality (5.3). Applying the argument in the proof of Lemma 2.2 from [26], we may conclude that there exists a local orthonormal frame \( \{E_1, \ldots, E_{2m}, \ldots, E_n\} \) around a point \( p \in M \) such that

\[ \omega = \sum_{i=1}^{m} a_i \theta^{2i-1} \wedge \theta^{2i}, \]

where \( \{\theta^1, \ldots, \theta^n\} \) is the dual coframe of \( \{E_i\}_{i=1}^{n} \). In particular, since \( \nabla f (p) \notin \ker A \), we may choose \( E_1 = N = \nabla f / ||\nabla f|| \). Since \( \omega(E_j, E_k) = 0 \) for \( j, k \geq 2 \) by Lemma 5.2, we have

\[ \omega = u \theta^1 \wedge \theta^2 \]

for a local smooth function \( u \). Thus, it is easy to see (cf. [13], p.25) that

\[ \delta \omega = E_2(u) \theta^1 - E_1(u) \theta^2 \]

and so,

\[ |\delta \omega|^2 = (E_1(u))^2 + (E_2(u))^2. \]

On the contrary, from \( \omega = df \wedge \iiota f z = u \theta^1 \wedge \theta^2 \), we have

\[ \omega(E_1, E_j) = |\nabla f| z (\nabla f, E_j) = 0 \quad (j \geq 3). \]

Thus,

\[ D_{E_1} \omega(E_1, E_2) = E_1(\omega(E_1, E_2)) - \omega(D_{E_1} E_1, E_2) - \omega(E_1, D_{E_1} E_2) \]

\[ = E_1(u) - \sum_{j=3}^{n} \langle D_{E_1} E_1, E_j \rangle \omega(E_j, E_2) - \sum_{j=3}^{n} \langle D_{E_1} E_2, E_j \rangle \omega(E_1, E_j) \]

\[ = E_1(u). \]

Similarly, since \( D_{E_1} \omega(E_1, E_2) = E_2(u) \), we may conclude that

\[ |\delta \omega|^2 = (E_1(u))^2 + (E_2(u))^2 = |D_{E_1} \omega(E_1, E_2)|^2 + |D_{E_2} \omega(E_1, E_2)|^2 \leq |D \omega|^2. \]
Using Lemma 5.5 and Bochner–Weitzenböck formula for 2-forms, we can prove the following structural property for the CPE with positive isotropic curvature.

**Theorem 5.6** Let \((g, f)\) be a nontrivial solution of (1.1) on a compact manifold \(M\) of dimension \(n \geq 4\). If \((M, g)\) has positive isotropic curvature, then the 2-form \(\omega = df \wedge i \nabla f \) is vanishing.

**Proof** It suffices to prove that \(\Omega = \emptyset\), where \(\Omega = \{ p \in M \mid \omega_p \neq 0 \text{ on } T_p M \}\). Suppose, on the contrary, \(\Omega \neq \emptyset\). For \(p \in \Omega\), let \(\Omega_0\) be a connected component of \(\Omega\) containing \(p\). Note that \(\Delta \omega = -d \delta \omega\) by Lemma 5.4. It follows from the Bochner–Weitzenböck formula for 2-forms (cf. [13, 22]) that

\[
\frac{1}{2} \Delta |\omega|^2 = \langle \Delta \omega, \omega \rangle + |D \omega|^2 + \langle E(\omega), \omega \rangle,
\]

where \(E(\omega)\) is a (local) 2-form containing isotropic curvature terms as its coefficients. In particular, if \((M, g)\) has positive isotropic curvature, following Proposition 2.3 in [26] (cf. [15], [19]) we have

\[
\langle E(\omega), \omega \rangle > 0.
\]

Therefore, integrating (5.4) over \(\Omega_0\), we obtain

\[
\frac{1}{2} \int_{\Omega_0} \Delta |\omega|^2 = \int_{\Omega_0} \langle \Delta \omega, \omega \rangle + |D \omega|^2 + \int_{\Omega_0} \langle E(\omega), \omega \rangle.
\]

Since \(\omega = 0\) on the boundary \(\partial \Omega_0\) and \(\omega\) is a closed form by Lemma 5.4, we have

\[0 = \int_{\Omega_0} |D \omega|^2 - |\delta \omega|^2 + \langle E(\omega), \omega \rangle.\]

However, by Lemma 5.5 and inequality (5.5), the above equation is impossible if \(\omega\) is nontrivial. Hence, we may conclude that \(\omega = 0\), or \(\Omega_0 = \emptyset\).

**Theorem 5.7** Let \(M\) be an \(n\)-dimensional compact smooth manifold with \(n \geq 4\). If \((g, f)\) is a nontrivial solution of (1.1) and \((M, g)\) has positive isotropic curvature, then \((M, g)\) is isometric to a standard sphere.

**6 Appendix**

In this Appendix, we first claim that \(-1\) is a regular value of the potential function \(f\) when \(\text{Ric}(\nabla f, X) = 0\) for any vector field \(X\) orthogonal to \(\nabla f\) unless \((M, g)\) is Einstein. Second, we prove (4.6) and (4.8) in Remark 4.3.

**6.1 Regularity of \(f\) on the set \(f^{-1}(-1)\)**

Define

\[
\varphi := \frac{1}{2} |\nabla f|^2 + \frac{s}{2n(n-1)} f^2.
\]

Then, we have,

\[
N(\varphi) = (1 + f) \alpha |\nabla f|.
\]

\(\square\) Springer
and
\[ \Delta \varphi = (1 + f)^2 |z|^2 + |\nabla f|^2 \alpha. \]
Therefore,
\[ \Delta \varphi - \frac{|\nabla f|}{1 + f} N(\varphi) = (1 + f)^2 |z|^2 \geq 0. \]  \hfill (6.1)
By the maximum principle,
\[ \max_{f \leq -1 - \varepsilon} \varphi = \max_{f = -1 - \varepsilon} \varphi \]
and
\[ \max_{f \geq -1 + \varepsilon} \varphi = \max_{f = -1 + \varepsilon} \varphi \]
for sufficiently small \( \varepsilon > 0 \). So, letting \( \varepsilon \to 0 \), we have
\[ \max_{M} \varphi = \max_{f = -1} \varphi. \]  \hfill (6.2)
Now, as mentioned in Introduction, if \( \min_{M} f \geq -1 \), then \((M, g)\) is Einstein, and so we may assume that \( \min_{M} f < -1 \). Let \( \min_{x \in M} f(x) = f(x_0) \). Then, for any point \( p \in f^{-1}(-1) \), we have
\[ \varphi(p) = \frac{1}{2} |\nabla f|^2(p) + \frac{s}{2n(n - 1)} \geq \varphi(x_0) = \frac{s}{2n(n - 1)} [f(x_0)]^2 > \frac{s}{2n(n - 1)}, \]
which proves our claim.

6.2 Warped product metric

**Lemma 6.1** Suppose that \( z(\nabla f, X) = 0 \) for \( X \perp \nabla f \). Then,
(1) for vectors \( X, Y \) orthogonal to \( \nabla f \),
\[ i_{\nabla f} T(X, Y) = \frac{|\nabla f|^2}{n - 2} \left( z + \frac{\alpha}{n - 1} g \right)(X, Y). \]
(2) \( i_{\nabla f} T(\nabla f, X) = i_{\nabla f} T(X, \nabla f) = 0 \) for any vector \( X \).

**Proof** If \( \omega = 0 \), then \( i_{\nabla f} z = \alpha df \) and hence
\[ T = \frac{1}{n - 2} df \wedge \left( z + \frac{\alpha}{n - 1} g \right). \]
\[ \Box \]

It follows from Lemma 2.1 that
\[ -|\nabla f|^2 W_N = (1 + f)i_{\nabla f} C + (n - 1)i_{\nabla f} T. \]  \hfill (6.3)
For the Weyl curvature tensor \( W, W_N \) can be similarly defined as \( R_N \).

**Lemma 6.2** Let \((g, f)\) be a nontrivial solution of the CPE with \( \omega = 0 \). Then,
\[ \frac{s}{n(n - 1)} g = R_N + z + \frac{1 + f}{|\nabla f|^2} i_{\nabla f} C - \left( \alpha - \frac{s}{n(n - 1)} \right) \frac{df}{|df|} \otimes \frac{df}{|df|}. \]
\textbf{Proof} Let 
\[ \Phi := \frac{s}{n(n-1)} g - z - \frac{1+f}{|\nabla f|^2} i\nabla f C. \]
For vector fields \( X, Y \) with \( X \perp \nabla f \) and \( Y \perp \nabla f \), from the curvature decomposition 
\[ R = \frac{s}{2n(n-1)} g \otimes g + \frac{1}{n-2} z \otimes g + \mathcal{W}, \]
we obtain 
\[ R_N(X, Y) = \frac{s}{n(n-1)} g(X, Y) + \frac{1}{n-2} z(X, Y) + \frac{\alpha}{n-2} g(X, Y) + \mathcal{W}_N(X, Y). \]
Since, by Lemma 6.1 together with (6.3), 
\[ \mathcal{W}_N(X, Y) = - \frac{1+f}{|\nabla f|^2} i\nabla f C(X, Y) - \frac{n-1}{n-2} z(X, Y) - \frac{\alpha}{n-2} g(X, Y), \]
we have 
\[ R_N(X, Y) = \Phi(X, Y). \] (6.4)

Now, let \( X \) and \( Y \) be arbitrary tangent vector fields. Then, \( X \) and \( Y \) can be decomposed into 
\[ X = X_1 + \langle X, N \rangle N, \quad Y = Y_1 + \langle Y, N \rangle N \]
with \( \langle X_1, N \rangle = 0 = \langle Y_1, N \rangle \). Thus, 
\[ R_N(X, Y) = R_N(X_1, Y_1) = \Phi(X_1, Y_1) \]
\[ = \Phi(X, Y) - \langle X, N \rangle \langle Y, N \rangle \Phi(N, N) \]
\[ = \Phi(X, Y) + \left( \alpha - \frac{s}{n(n-1)} \right) \frac{df}{|df|} \otimes \frac{df}{|df|} (X, Y). \]
\hfill \Box

\textbf{Lemma 6.3} \textit{On the set } \( f^{-1}(-1) \), \textit{we have}
\[ \mathcal{L}_{\nabla f} \left( \frac{h}{|\nabla f|^2} \right) = 0. \]

Here, recall that \( h := R_N + \left( z - \alpha \frac{df}{|df|} \otimes \frac{df}{|df|} \right) + \frac{1+f}{|\nabla f|^2} i\nabla f C. \)

\textbf{Proof} First, from CPE, we have 
\[ \frac{1}{2} \mathcal{L}_{\nabla f} g = Ddf = (1+f)z - fh - \epsilon \frac{df}{|df|} \otimes \frac{df}{|df|} \]
with \( \epsilon = \frac{s}{n(n-1)} \). It follows from (3.1), (4.1), and (4.7) that 
\[ \frac{1}{2\epsilon} \mathcal{L}_{\nabla f} h = (1+f)z - fh - (1+f)\alpha \frac{df}{|\nabla f|} \otimes \frac{df}{|\nabla f|}. \]
Thus, 
\[ \frac{1}{2} \mathcal{L}_{\nabla f} \left( \frac{h}{|\nabla f|^2} \right) = \frac{1}{2} \nabla f (|\nabla f|^{-2}) h + \frac{1}{2|\nabla f|^2} \mathcal{L}_{\nabla f} h \]
\[ = -N(|\nabla f|) \frac{h}{|\nabla f|^2} + \frac{1}{|\nabla f|^2} \frac{1}{2} \mathcal{L}_{\nabla f} h. \]
In particular, on the set $f^{-1}(-1)$, we have

$$
\frac{1}{2} \mathcal{L}_{\nabla f} \left( \frac{h}{|\nabla f|^2} \right) = 0.
$$

\[\square\]

**Acknowledgements** The authors wish to thank the anonymous referee for pointing out typos and valuable remarks. The first author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2018R1D1A1B05042186) and the second and corresponding author by the Ministry of Education (NRF-2019R1A2C1004948).

**References**

1. Baltazar, H.: On critical point equation of compact manifolds with zero radial Weyl curvature. Geom. Dedicata **202**, 337–355 (2019)
2. Besse, A.L.: Einstein Manifolds. Springer, New York (1987)
3. Brozos-Vázquez, M., García-Río, E., Vázquez-Lorenzo, R.: Some remarks on locally conformally flat static space-times. J. Math. Phys. **46**(2), 11 (2005)
4. Cao, H.D., Chen, Q.: On Bach-flat gradient shrinking Ricci solitons. Duke Math. J. **162**(6), 1149–1169 (2013)
5. Chen, B.-L., Huang, X.-T.: Four-manifolds with positive isotropic curvature. Front. Math. China **11**(5), 1123–1149 (2016)
6. Fischer-Colbrie, D., Schoen, R.: The structure of complete stable minimal surfaces in 3-manifolds of non-negative scalar curvature. Commun. Pure App. Math. **33**, 199–211 (1980)
7. Fraser, A.: Fundamental groups of manifolds with positive isotropic curvature. Ann. Math. **158**(1), 345–354 (2003)
8. Fraser, A., Wolfson, J.: The fundamental group of manifolds of positive isotropic curvature and surface groups. Duke Math. J. **133**(2), 325–334 (2006)
9. Hwang, S., Yun, G.: Vacuum static spaces with vanishing of complete divergence of Weyl tensor. J. Geom. Anal. **31**(3), 3060–3084 (2021)
10. Kobayashi, O., Obata, M.: Conformally-flatness and static space-times. Manifolds Lie Groups Progr. Math. **14**, 197–206 (1981)
11. Labbi, M.-L.: On compact manifolds with positive isotropic curvature. Proc. Am. Math. Soc. **128**(5), 1467–1474 (1999)
12. Lafontaine, J.: Sur la géométrie d’une généralisation de l’équation différentielle d’Obata. J. Math. Pures Appl. **62**(1), 63–72 (1983)
13. Li, P.: Geometric Analysis, Cambridge Studies in Advanced Mathematics, 134. Cambridge University Press (2012)
14. Micallef, M., Moore, J.D.: Minimal two-spheres and the topology of manifolds with positive curvature on totally isotropic two-planes. Ann. Math. **127**(2), 199–227 (1988)
15. Micallef, M., Wang, M.Y.: Metrics with nonnegative isotropic curvature. Duke Math. J. **72**(3), 649–672 (1993)
16. Milnor, J.: Morse Theory. Princeton University Press, Princeton (1963)
17. Obata, M.: Certain conditions for a Riemannian manifold to be isometric with a sphere. J. Math. Soc. Jpn. **14**(3), 333–340 (1962)
18. Qing, J., Yuan, W.: A note on static spaces and related problems. J. Geom. Phys. **74**, 13–27 (2013)
19. Seaman, W.: On manifolds with nonnegative curvature on totally isotropic 2-planes. Trans. Am. Math. Soc. **338**(2), 843–855 (1993)
20. Seshadri, H.: Isotropic Curvature: A Survey, Séminaire de théorie spectrale et géométrie, Grenoble, **26**, 139–144 (2007–2008)
21. Tashiro, Y.: Complete Riemannian manifolds and some vector fields. Trans. Am. Math. Soc. **117**, 251–275 (1965)
22. Wu, H.: The Bochner Technique in Differential Geometry, in: Mathematical Reports, vol. 3, Pt 2, Harwood Academic Publishing, London (1987)
23. Yun, G., Chang, J., Hwang, S.: Total scalar curvature and harmonic curvature. Taiwanese J. Math. **18**(5), 1439–1458 (2014)
24. Yun, G., Chang, J., Hwang, S.: Erratum to: Total scalar curvature and harmonic curvature. Taiwanese J. Math. 20(3), 699–703 (2016)
25. Yun, G., Hwang, S.: Gap Theorems on critical point equation of the total scalar curvature with divergence-free Bach tensor. Taiwanese J. Math. 23(4), 841–855 (2019)
26. Zhu, P.: Harmonic two-forms on manifolds with non-negative isotropic curvature. Ann. Glob. Anal. Geom. 40, 427–434 (2011)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.