THE CLOSURE OF THE SPACE OF SPECTRAL CURVES
OF CONSTANT MEAN CURVATURE TORI IN $\mathbb{R}^3$ WITH SPECTRAL GENUS 2

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ABSTRACT. Where $H^2$ is the space of spectral curves of real solutions of the sinh-Gordon equation of spectral genus 2, and $S^2$ is the closure in $H^2$ of the space of spectral curves of constant mean curvature tori in $\mathbb{R}^3$ of spectral genus 2, we study the boundary of $S^2$ at first in the space of polynomials of degree $\leq 4$, and then also the boundary where coefficients of the spectral polynomial go to $\infty$. We also prove that the Wente family, i.e. the family of members of $S^2$ described by polynomials with real coefficients, is 1-dimensional, smooth and connected.

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1. Introduction and Definitions

The spectral curves corresponding to finite type real solutions of the sinh-Gordon equation with spectral genus $g$ are hyperelliptic complex curves $\Sigma = \Sigma_a$ described by the equation $\nu^2 = \lambda a(\lambda)$, where $a(\lambda)$ is a polynomial of degree $2g$. By compactification, we regard $\Sigma$ as a compact, hyperelliptic surface above the Riemann sphere $\mathbb{P}^1$, and denote the part of $\Sigma$ that is above $\mathbb{C}^\times \ni \lambda$ by $\Sigma^o = \{ (\lambda, \nu) \in \mathbb{C}^\times \times \mathbb{C} \mid \nu^2 = \lambda a(\lambda) \}$. By rescaling $\lambda$ and $\nu$, the polynomial $a(\lambda)$ can be chosen to have the following properties:

1. Reality condition: $a \in P^g_\mathbb{R}$, where for $d \in \mathbb{N}$ we define $P^d_\mathbb{R}$ as the space of polynomials $p$ of degree at most $d$ which satisfy the reality condition
   \[ \lambda^d \cdot p(\bar{\lambda}^{-1}) = p(\lambda) . \]
2. Positivity condition: $\lambda^{-g} \cdot a(\lambda) > 0$ for $\lambda \in S^1$
3. Non-degeneracy: The roots of $a$ are all pairwise distinct.
4. Normalisation: The highest coefficient of $a$ has absolute value 1.

The condition that the roots of $a$ are pairwise distinct ensures that the corresponding spectral curve is smooth. We denote the space of polynomials $a$ which satisfy these conditions by $H^g$. We regard $H^g$ and its subsets which are defined below as topological subspaces of the space $\mathbb{C}^{2g}[\lambda]$ of complex polynomials in $\lambda$ of degree $\leq 2g$. In the following, all topological closures of such sets are taken in $\mathbb{C}^{2g}[\lambda]$.

Let $a \in H^g$ and $\Sigma$ be the corresponding spectral curve, i.e. the hyperelliptic complex curve defined by the equation $\nu^2 = \lambda a(\lambda)$. $\Sigma$ is equipped with the holomorphic involution

\[ \sigma : \Sigma \to \Sigma, \quad (\lambda, \nu) \mapsto (\lambda, -\nu) , \]

whose existence expresses the fact that $\Sigma$ is hyperelliptic. Moreover, due to the reality condition for $a$, $\Sigma$ is equipped with the anti-holomorphic involution

\[ \rho : \Sigma \to \Sigma, \quad (\lambda, \nu) \mapsto (\bar{\lambda}^{-1}, \bar{\lambda}^{-(g+1)} \bar{\nu}) . \]

Note that all points $(\lambda, \nu) \in \Sigma$ with $\lambda \in S^1$ are fixed points of $\rho$. 

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For any \( b \in P^{g+1}_R \) we define the differential form on \( \Sigma \)
\[
\Theta_b = \frac{b(\lambda)}{\nu} \frac{d\lambda}{\lambda}.
\]
Notice that
\[
\sigma^* \Theta_b = -\Theta_b
\]
holds, and that the reality condition on \( b \) is equivalent to
\[
\rho^* \Theta_b = -\Theta_b.
\]
We let \( B_a \) be the \( \mathbb{R} \)-linear space of polynomials \( b \in P^{g+1}_R \), such that the corresponding differential form \( \Theta_b \) has purely imaginary periods. This space \( B_a \) has real dimension 2. Moreover, any \( b \in B_a \) is uniquely determined by the value of \( b(0) \), in other words, the \( \mathbb{R} \)-linear map \( B_a \to \mathbb{C} \), \( b \mapsto b(0) \) is an isomorphism of \( \mathbb{R} \)-linear spaces. It follows that there exists a unique basis \( (b_1, b_2) \) of \( B_a \) with \( b_1(0) = 1 \) and \( b_2(0) = i \). We call this basis the normalised basis of \( B_a \).

The importance of the differentials \( \Theta_b \) where \( b \in B_a \) comes from the fact that these differentials correspond to the translational flows on the isospectral set corresponding to the spectral curve \( \Sigma \). Consequently, any solution of the sinh-Gordon equation that corresponds to a constant mean curvature torus in \( \mathbb{R}^3 \) (i.e. is doubly periodic and satisfies the closing conditions) corresponds to a spectral curve \( \Sigma \) for whose associated polynomial \( a(\lambda) \in \mathcal{H}^g \) the following holds: There exists a Sym point \( \lambda \in S^1 \), a basis \((b_1, b_2)\) of \( B_a \) and functions \( \mu_1, \mu_2 \) on \( \Sigma \) such that the following conditions hold:

1. For \( k \in \{1, 2\} \), the function \( \mu_k \) is the logarithmic primitive function of \( \Theta_{b_k} \), i.e. \( d \log \mu_k = \Theta_{b_k} \). The multivalued function \( \log \mu_k \) is holomorphic on \( \Sigma^0 \).
2. We have \( b_1(\lambda_0) = b_2(\lambda_0) = 0 \) and \( \mu_1(\lambda_0) = \mu_2(\lambda_0) = \pm 1 \).

We now define for \( \lambda_0 \in S^1 \)
\[
S^g_{\lambda_0} := \{ a \in \mathcal{H}^g \mid b(\lambda_0) = 0 \text{ for all } b \in B_a \}
\]
and
\[
S^g = \bigcup_{\lambda_0 \in S^1} S^g_{\lambda_0}.
\]
By the above characterisation, it is clear that the set \( \mathcal{P}^g \) of polynomials \( a(\lambda) \in \mathcal{H}^g \) which correspond to constant mean curvature tori in \( \mathbb{R}^3 \) is contained in \( S^g \). Indeed, due to the fact that the unit circle is closed in \( \mathbb{C} \), the closure of \( \mathcal{P}^g \) in \( \mathcal{H}^g \) is also contained in \( S^g \). In this paper we are interested in the case \( g = 2 \). In [C-S-2] it was shown that that for \( g = 2 \), the closure of \( \mathcal{P}^2 \) in \( \mathcal{H}^2 \) is equal to \( S^2 \). Indeed this follows from [C-S-2] Theorem 5.8] together with the following statement:

**Lemma 1.1.** \( S^2 \) and \( S^2_1 \) are smooth submanifolds of \( \mathcal{H}^2 \) of dimension 3 and 2 respectively.

**Proof.** Due to [C-S-2] Theorem 3.2], \( \deg(\text{gcd}(B_a)) \leq 1 \) for all \( a \in \mathcal{H}^2 \) with equality for \( a \in S^2_1 \). Therefore [C-S-2] Theorem 5.5(i),(ii)] show that \( S^2_1 \) and \( S^2 \) are smooth submanifolds of \( \mathcal{H}^2 \) of dimension 2 and 3 respectively. \( \square \)

It is the objective of the present paper to investigate the boundary of \( S^2 \). In Section 2 we first give a description of the closure of \( S^2 \) in the space of polynomials \( \mathbb{C}^4[\lambda] \), see Proposition 2.2. This closure turns out not to be compact, and therefore we also consider the “boundary” where coefficients of the polynomial \( a(\lambda) \in S^2 \) go to infinity. For this purpose, we construct a blow-up of the spectral parameter \( \lambda \) and an associated embedding \( \iota \) of \( \mathbb{H}^2 \) into a blow-up space \( A \). We then discuss which points in the exceptional fibre of \( A \) correspond to boundary points of \( S^2 \), yielding the result of Theorem 2.12. In this manner we construct essentially a “compactification” of \( S^2 \) in the following sense: With respect to any given sequence \( (a_n) \) in \( S^2 \), there are two
possibilities: One possibility is that the coefficients of the \( a_n \) are bounded for \( n \to \infty \), and then there is a subsequence of the \( a_n \) that converges in the usual sense to a boundary point described in Proposition 2.2. Otherwise there exists a subsequence \( (a_{n_k}) \) of the \( a_n \) for which the coefficients tend to infinity as \( n \to \infty \), and such that \( \iota(a_{n_k}) \) converges to a boundary point of \( \iota[S^2] \) in the exceptional fibre of \( \mathcal{A} \) as described in Theorem 2.12.

The interpretation of the condition for the boundary points of \( \iota[S^2] \) as given in Theorem 2.12 is not obvious. We therefore study this condition further in Section 3. In particular we there prove that this condition corresponds to an unbounded, closed, connected, 1-dimensional submanifold in the moduli space of elliptic curves.

Finally, in Section 4 we study the Wente family \( \mathcal{W} \) in \( S^2 \), i.e. the family of polynomials \( a \in S^2 \) with real coefficients, see [Ab]. We prove that \( \mathcal{W} \) is a connected, 1-dimensional submanifold of \( S^2 \), in fact it is the image of a maximal integral curve of a suitable Whitham flow on \( S^2 \). Moreover we give an explicit description of \( \mathcal{W} \) in terms of a single transcendental number \( \delta_0 \). We conjecture that \( S^2 \) and \( S^2 \) are also connected.

For the description of the boundary of \( S^2 \), the meromorphic function \( f = \frac{b_2}{b_1} \), where \( (b_1, b_2) \) is any basis of \( B_a \), will turn out to be important. This function depends on the choice of the basis \( (b_1, b_2) \) only by a real Möbius transformation in the range. Now suppose that \( (b_1, b_2) \) is the normalised basis of \( B_a \). Then the corresponding function \( f = \frac{b_2}{b_1} \) maps the unit circle onto the real line; the composition

\[
(1.3) \quad \tilde{f} = \frac{f - i}{f + 1} = \frac{b_1 + ib_2}{b_1 - ib_2}
\]

of \( f \) with the Cayley transform maps the unit circle onto itself. We define the winding number \( n(a) \) to be the winding number of \( \tilde{f}|_{S^1} \), see [C-S-2, Section 3]; \( n(a) \) depends only on \( a \). We observe that the meromorphic differentials \( \Theta_{b_1} + i\Theta_{b_2} \) and \( \Theta_{b_1} - i\Theta_{b_2} \) each have only one pole, namely at \( \lambda = 0 \) resp. at \( \lambda = \infty \).

As in [C-S-2], we define for any \( g \geq 0 \) and \( j \in \mathbb{Z} \)

\[
V_j = \{ a \in \mathcal{H}^g \setminus S^g \mid n(a) = j \}.
\]

As special cases of [C-S-2, Theorem 3.5] we then have

\[
(1.4) \quad \mathcal{H}^0 = V_1, \quad \mathcal{H}^1 = V_0 \quad \text{and} \quad \mathcal{H}^2 \setminus S^2 = V_1 \cup V_{-1}.
\]

The sets \( V_j \) mentioned in (1.4) are non-empty, open and in the case of the last equation, disjoint.

**Lemma 1.2.** We have \( S^2 = V_1 \cap V_{-1} \cap \mathcal{H}^2 \).

*Proof.* This statement follows from Lemma 1.1 and [C-S-2, Theorem 5.5(iii), (C) \( \Rightarrow \) (B)]. \( \square \)

At the end of this introductory section, we note the following fact for further use:

**Lemma 1.3.** If \( a \) in the closure of \( \mathcal{H}^2 \) has a double root, then all \( b \in B_a \) vanish at that point. If \( a \) has a root of order 4, then all \( b \in B_a \) vanish to second order.

*Proof.* See the proof of [K-PH-S, Lemma 3.4]. \( \square \)

2. The closure of \( S^2 \)

In this section we first determine the closure of \( S^2 \) in \( \mathbb{C}^4[\lambda] \) in Proposition 2.2. As explained in the Introduction, we then construct a blow-up to compactify the closure of \( S^2 \). The description of the closure in this blowup is achieved in Theorem 2.12.
Lemma 2.1. For \( a_0 = \frac{(\lambda - \lambda_0)^2(\lambda - \alpha)(\bar{\alpha} \lambda - 1)}{\lambda_0 |\alpha|} \) with \( \lambda_0 \in S^1 \) and \( 0 < |\alpha| < 1 \), the following statements are equivalent:

1. \( a_0 \) belongs to the closure of \( S^2_{\lambda_0} \).
2. \( a_0 \) belongs to the closure of \( S^2 \).
3. \( df(\lambda_0) = 0 \), where \( f = \frac{b_2}{b_1} \) for a basis \( b_1, b_2 \) of \( \mathcal{B}_{a_0} \).

Note that the condition \( df(\lambda_0) = 0 \) is invariant under Möbius transformations, so that it is well-defined even if \( f(\lambda_0) = \infty \).

Proof. Note that
\[
\overline{\mathcal{H}^2} = \left\{ a \in \mathbb{C}^4[\lambda] \left| a(0) = 1, 4a_1(1/\lambda) = a(\lambda), \lambda^{-2}a(\lambda) \geq 0 \right. \text{ for } \lambda \in S^1 \right\}.
\]

If some polynomial \( a \in \overline{\mathcal{H}^2} \) has roots of higher order, then there exists a unique polynomial \( \tilde{a} \in \mathcal{H}^1 \cup \mathcal{H}^0 \) which has simple roots at the roots of \( a \) of odd order and no others. The quotient \( a/\tilde{a} \) is the square of a polynomial \( p \), which is unique up to sign. Therefore \( \Sigma_\tilde{a} \) is the normalisation of \( \Sigma_a \). It follows from [K-PH-S Lemma 3.4] that this polynomial \( p \) divides \( \text{gcd}(\mathcal{B}_a) \), and that the map \( b \mapsto p \cdot b \) is an isomorphism \( \mathcal{B}_\tilde{a} \to \mathcal{B}_a \). This implies that \( a \) and \( \tilde{a} \) define the same function \( f \), and therefore \( n(a) = n(\tilde{a}) \) holds.

For \( a_0 = \frac{(\lambda - \lambda_0)^2(\lambda - \alpha)(\bar{\alpha} \lambda - 1)}{\lambda_0 |\alpha|} \), we have \( \tilde{a}_0 = \frac{(\lambda - \alpha)(\bar{\alpha} \lambda - 1)}{|\alpha|} \in \mathcal{H}^1 \) and therefore \( n(a_0) = n(\tilde{a}_0) = 0 \) by Equation (1.3). Now suppose that \( a_0 \) belongs to the closure of \( S^2_{\lambda_0} \). By the implication (D) \( \Rightarrow \) (B) in [C-S-2 Theorem 5.8], every neighbourhood \( O \) of \( a_0 \) in \( \mathbb{C}^4[\lambda] \) has non-empty intersection with \( V_1 \) and with \( V_{-1} \).

Assume for a contradiction that \( df(\lambda_0) \neq 0 \). This assumption is equivalent to \( df(\lambda_0) \neq 0 \), where \( \tilde{f} \) is as in Equation (1.3) with the normalised basis \( (b_1, b_2) \) of \( \mathcal{B}_a \). We define \( \text{sign}(df(\lambda_0)) \) to be either \( +1 \) or \( -1 \), according to whether the tangent map \( df(\lambda_0) \) preserves or reverses the orientation of \( S^1 \). By the last formula in the proof of [C-S-2 Theorem 3.5] there exists a neighbourhood \( O \) of \( a_0 \) in \( \mathbb{C}^4[\lambda] \) so that
\[
(2.1) \quad n(a) = n(a_0) - \text{sign}(df(\lambda_0))
\]
for every \( a \in O \cap (\mathcal{H}^2 \setminus S^2) \). Therefore we have
\[
O \cap (\mathcal{H}^2 \setminus S^2) \subset V_{-1} \text{ for } \text{sign}(df(\lambda_0)) > 0, \\
O \cap (\mathcal{H}^2 \setminus S^2) \subset V_1 \text{ for } \text{sign}(df(\lambda_0)) < 0.
\]

This implies \( df(\lambda_0) = 0 \).

Conversely, let \( df(\lambda_0) = 0 \). We now show that \( \overline{\mathcal{H}^2} \) is near \( a_0 \) a manifold with boundary. Its elements near \( a_0 \) are of the form
\[
\frac{(\lambda - \beta)(\bar{\beta} \lambda - 1)}{|\beta|}, \frac{(\lambda - \alpha')(\bar{\alpha}' \lambda - 1)}{|\alpha'|}
\]
with \( \beta \in B(\lambda_0, \epsilon) \) and \( \alpha' \in B(\alpha, \epsilon) \) for some \( \epsilon > 0 \). Writing \( \beta = re^{i\varphi} \) with \( r > 0 \) and \( \varphi \in \mathbb{R} \), we have
\[
\frac{(\lambda - \beta)(\bar{\beta} \lambda - 1)}{|\beta|} = \frac{\beta}{|\beta|} \lambda^2 - \frac{1 + \beta \bar{\beta}}{|\beta|} \lambda + \frac{\beta}{|\beta|} = e^{-i\varphi} \lambda^2 - (r + r^{-1}) \lambda + e^{i\varphi}. \]

So \( (r + r^{-1}, \varphi, \alpha') \in [2, \infty) \times \mathbb{R} \times \mathbb{C} \) are local coordinates near \( a_0 \) of the manifold with boundary \( \mathcal{H}_2 \) and the boundary near \( a_0 \) is given by
\[
\partial \overline{\mathcal{H}^2} = \left\{ a \in \overline{\mathcal{H}^2} \left| a \text{ has a double zero on } S^1 \right. \right\}. 
\]
Thus every neighborhood of $a_0 \in \partial \mathcal{H}^2$ contains an open neighborhood $O$ of $a_0$ such that $O \cap \mathcal{H}^2$ is connected.

![Diagram](image-url)

Since $b_1$ and $b_2$ have a common root at $\lambda = \lambda_0$, $\deg(f)$ is either 1 or 2. But [CS2] Theorem 3.2 excludes $\deg(f) = 1$, so $\deg(f) = 2$ holds. By the Riemann-Hurwitz formula, $df$ has two roots. These must be distinct, because a double root would require the covering map $f$ to have at least three sheets. Therefore the root of $df$ at $\lambda = \lambda_0$ is simple. By Equation (2.1), we have $O \cap V_1 \neq \emptyset$ and $O \cap V_{-1} \neq \emptyset$. The sets $V_1$ and $V_{-1}$ are open and disjoint, but $O$ is connected, so $(O \cap V_1) \cup (O \cap V_{-1}) \subseteq O$. This implies $O \cap \mathcal{S}^2 \neq \emptyset$ and $a_0 \in \partial \mathcal{S}^2$.

Let $(a_k)$ be a sequence in $\mathcal{S}^2$ that converges to $a_0$. Due to [K-PH-S] Lemma 3.4 the sequence $(\lambda_k)$ of common roots of the elements of $\mathcal{B}_{a_k}$ depends continuously on $a_k$ and converges to a common root of the elements of $\mathcal{B}_{a_0}$. By Equation (1.4), $\deg \gcd \mathcal{B}_{a_0} = 0$. Therefore $\lim \lambda_k = \lambda_0$.

The rotation $\lambda \mapsto \lambda \cdot \lambda_0 \cdot \lambda_k^{-1}$ transforms an element $a_k \in \mathcal{S}^2$ into an element $\tilde{a}_k$ of $\mathcal{S}_{a_0}^2$. Because of $\lim \lambda_k = \lambda_0$, also the sequence $(\tilde{a}_k)$ converges to $a_0$. Hence $a_0$ is in the closure of $\mathcal{S}_{a_0}^2$.

**Proposition 2.2.** The boundary of $\mathcal{S}_{a_0}^2$ in $\mathbb{C}^4[\lambda]$ is

$$\left\{ \frac{(\lambda - \lambda_0)^2(\lambda - \alpha)(\tilde{a} \lambda - 1)}{\lambda_0 |\alpha|} \right\} \quad 0 < |\alpha| < 1, df(\lambda_0) = 0 \right\} \cup \left\{ \frac{(\lambda - \lambda_0)^4}{\lambda_0^4} \right\}.

**Proof.** The boundary of $\mathcal{S}_{a_0}^2$ is contained in $\partial \mathcal{H}^2 = \overline{\mathcal{H}^2} \setminus \mathcal{H}^2$. For a general $a_0 \in \partial \mathcal{H}^2$, the condition $|a_0(0)| = 1$ excludes roots at $\lambda = 0$. Note that $a_0$ either has a higher order root at some $\lambda \in \mathbb{C}^x$, or $\lambda^{-2}a_0(\lambda) = 0$ holds for some $\lambda \in \mathbb{S}^1$. From $\lambda^{-2}a_0(\lambda) \geq 0$ for $|\lambda| = 1$ it follows that any unimodular root of $a_0$ has even order. Consequently $a_0$ has either a double root on $\mathbb{S}^1$, or two double roots on $\mathbb{S}^1$ (which may coincide), or two double roots away from $\mathbb{S}^1$ interchanged by $\lambda \mapsto \lambda^{-1}$.

We now show for $a_0 \in \partial \mathcal{S}_{a_0}^2$ that $a_0$ has an even order root at $\lambda = \lambda_0$. Thereby we exclude the possibility that $a_0$ has two even order roots on $\mathbb{S}^1 \setminus \{\lambda_0\}$, or pairs of double roots off $\mathbb{S}^1$. Let $(b_1, b_2)$ be the normalised basis of $\mathcal{B}_{a_0}$. By [K-PH-S] Lemma 3.4, $b_1$ and $b_2$ define continuous functions of $a_0 \in \overline{\mathcal{H}^2}$. For $a_0 \in \partial \mathcal{S}_{a_0}^2$ with higher order roots, we write $a_0(\lambda) = p^2(\lambda) \cdot \tilde{a}_0(\lambda)$ with $\tilde{a}_0 \in \mathcal{H}^{2-deg p}$. The proof of [K-PH-S] Lemma 3.4 shows that $p$ divides both $b_1$ and $b_2$, and $b_1/p, b_2/p \in \mathcal{B}_{a_0}$. In particular, if $p(\lambda_0) \neq 0$, then $\tilde{a}_0 \in \mathcal{S}^{2-deg p}$. Since $\mathcal{S}^2 = \emptyset$ for $g \in \{0, 1\}$ by [CS2], $p(\lambda_0) = 0$ follows.

We next exclude the case that $a_0$ has a double root at $\lambda = \lambda_0$ and another double root at some $\beta \in \mathbb{S}^1 \setminus \{\lambda_0\}$. Assume that this case occurs for some $a_0 \in \partial \mathcal{S}_{a_0}^2$. Then $f$ has degree 1, so $df$ has no zeros. We now use the notation of the proof of Lemma 2.1. Let $\tilde{a}_0 = \beta (\lambda - \beta)^2 \tilde{a}_0$. Then by [CS1] Lemma 8 and [CS2] Theorem 3.2 there exists a neighbourhood $\tilde{O}$ of $\tilde{a}_0$ in $\mathbb{C}^2[\lambda]$ such that for $\tilde{a} \in \tilde{O} \cap \mathcal{H}^1$, $df(\lambda_0)$ is non-zero and $\pi(\tilde{f}(\tilde{a}_0(\lambda_0))) = \pi(\tilde{f}(\tilde{a}_0(\lambda_0))) = 1$. By Equation (1.4) we have $n(\tilde{a}) = 0$. We now choose a neighbourhood $O$ of $a_0$ in $\mathbb{C}^4[\lambda]$ whose...
pre-image with respect to the map $\widehat{a} \mapsto \lambda_0(\lambda - \lambda_0)^2 \widehat{a}$ is contained in $\widehat{O}$. Equation (2.1) applies to $a \in O \cap H^2$ and gives $n(a) = n(\widehat{a}) - \text{sign}(df_{\widehat{a}}(\lambda_0)) = -1$. So $O \cap V_1 = \varnothing$. This contradicts Lemma 1.2. Now Lemma 2.1 shows that $\partial S^2_{\lambda_0}$ is contained in the set (2.2).

We finally show that conversely the set (2.2) is contained in $\partial S^2_{\lambda_0}$. For the first set of the union, this is shown in Lemma 2.1. Now we show that the third equality follows.

Proof. Proposition 2.2] the corresponding $\lambda$ is up to M"obius transformations equal to $\frac{(\lambda - \beta_k)(\lambda - \beta_{k}^{-1})}{\lambda_k}$ with $\beta_k \in (k,1)$. Then $df$ has roots at $\frac{2\beta_k + (\beta_k^2 - 1)}{\beta_k^2 + 1} \in S^1$. In the limit $k \to 1$ we have $\beta_k \to 1$. For $a_k = k \cdot \lambda_0 \cdot \left(\frac{2\beta_k + (\beta_k^2 - 1)}{\beta_k^2 + 1}\right)^{-1}$, we have $a_k := \frac{(\lambda - \lambda_0)^2(\lambda - \alpha_k)(\lambda - \beta_k)}{\lambda_0|\alpha_k|} \in S^2_{\lambda_0}$ by Lemma 2.1. In the limit $k \to 1$, $a_k$ converges to $\lambda_0^2 \cdot (\lambda - \lambda_0)^4$, therefore $\lambda_0^2 \cdot (\lambda - \lambda_0)^4$ is in $S^2_{\lambda_0}$. □

Lemma 2.3. We have

$$V_1 \cap V_1^{-1} = \overline{S^2} = S^2 \cup \partial S^2 \quad \text{and} \quad \partial S^2 = \bigcup_{\lambda_0 \in S^1} \partial S^2_{\lambda_0},$$

where $\partial S^2_{\lambda_0}$ is described in Proposition 2.2.

Proof. The second equality is obvious. For a sequence in $S^2$ with limit in $\partial S^2$, the sequence of the corresponding $\lambda_0 \in S^1$ has a convergent subsequence, therefore $\bigcup_{\lambda_0 \in S^1} \partial S^2_{\lambda_0}$ is closed, whence the third equality follows.

Due to Lemma 1.2 we have $\overline{S^2} \subset V_1 \cap V_1^{-1} \subset H^2$. To prove the first equality, it therefore suffices to show that the points $a \in \partial H^2 \setminus \partial S^2$ do not belong to $V_1 \cap V_1^{-1}$. Any such $a$ either has two double roots off $S^1$ or two different double roots on $S^1$. In the first case, $a \mapsto f_0$ has a continuous extension to $\overline{H}^2$ near $a$ by $K$-PIT-S Lemma 3.4, so by $C$-S $\overline{H}^2$, Equation (2), the winding number $n(a)$ is locally constant on that neighbourhood. In the proof of Proposition 2.2 we showed that in the second case, the winding number $n(a)$ is also locally constant near $a$. In either case, this implies $a \notin V_1 \cap V_1^{-1}$. □

A sequence $(a_n)_{n \in \mathbb{N}}$ in $H^2$ for which at least one root converges to zero has no accumulation point in $C^4[\lambda]$. We will now construct a model $\mathcal{A}$ with an embedding $\iota : H^2 \to \mathcal{A}$ such that the boundary of $\iota(H^2)$ contains a two-dimensional subset consisting of limit points of such sequences. We will construct the model $\mathcal{A}$ by considering a blow-up of the variable $\lambda$ near $\lambda = 0$ or $\lambda = \infty$. For these blow-ups, we use the spectral parameter $\lambda^+_t := \frac{t}{\lambda}$ near $\lambda = 0$, and the spectral parameter $\lambda^-_t := t\lambda$ near $\lambda = \infty$, where $t \in \mathbb{R}^\times$. Note that due to the reality condition for $a \in H^2$, the behaviour of $a(\lambda)$ under these two blow-ups is equivalent. In the construction of the model $\mathcal{A}$ and the embedding $\iota$ we will only use the parameter $\lambda^-_t$ and therefore consider the behaviour of $a(\lambda)$ only outside the unit disk.

Let $C^{2 \times} := C^2 \setminus \{(0,0)\}$ and $\tilde{A} = \mathbb{C} \times C^{2 \times}$. Consider $a \in H^2$. Due to the positivity condition for $a$, $a(\lambda)$ has no roots on $S^1$, and due to the reality condition, the set of roots of $a(\lambda)$ is invariant under $\lambda \mapsto \lambda^{-1}$. Hence there are exactly two roots $\alpha_1, \alpha_2$ of $a(\lambda)$ outside the unit disk (counted with multiplicity), and the two roots in the unit disk are $\tilde{\alpha}_1^{-1}$ and $\tilde{\alpha}_2^{-1}$. Let $\tilde{\alpha}(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2) = \lambda^2 + a_1 \lambda + a_2$ with $a_1 = \alpha_1 + \alpha_2$ and $a_2 = \alpha_1 \alpha_2$ be the unique normalised polynomial of degree 2 with roots $\alpha_1$ and $\alpha_2$. Now let $t \in \mathbb{R}^\times$ be given, and consider the following blow-up of $\tilde{a}(\lambda)$:

$$(2.3) \quad a^+_t(\lambda^-_t) := t^2 \tilde{a}(t^{-1}\lambda^-_t) = (\lambda^-_t)^2 + t a_1 \lambda^-_t + t^2 a_2 .$$

In this setting we define $\tilde{\iota} : \mathbb{R}^\times \times H^2 \to \tilde{A}, (t,a) \mapsto (t,a_1,a_2)$. Note that for $t = 1$, $\lambda^-_1 = \lambda$ and $a^-_1(\lambda^-_1) = \tilde{a}(\lambda)$ holds.
We now define a $\mathbb{C}^\times$-action on $\tilde{A}$ by
\[ s \cdot (t, a_1, a_2) = (ts, a_1 s, a_2 s^2) \quad \text{for} \quad s \in \mathbb{C}^\times, \; (t, a_1, a_2) \in \tilde{A} \]
and let $A$ be the quotient space of $\tilde{A}$ by this $\mathbb{C}^\times$-action. $A$ is a 2-dimensional complex space, which can be regarded as a blow-up of $\mathbb{C}^2 \ni (a_1, a_2)$ at infinity. The compact subset
\[ \mathcal{E} := \{ [(0, a_1, a_2)] \mid (a_1, a_2) \in \mathbb{C}^2 \} \]
is the exceptional fibre of this blow-up. For any $a \in H^2$, the images under $\tilde{\iota}$ of $(t, a)$ and of $(t', a)$ with $t, t' \in \mathbb{R}^\times$ are in the same equivalence class in $A$ due to Equation (2.3), and therefore there exists a unique map $\iota : H^2 \to A$ so that the following diagram commutes:
\[
\begin{array}{ccc}
\mathbb{R}^\times \times H^2 & \xrightarrow{\tilde{\iota}} & \tilde{A} \\
pr_{H^2} \downarrow & & \downarrow \pi \\
H^2 & \xrightarrow{\iota} & A.
\end{array}
\]
Because any $a \in H^2$ is uniquely determined by the roots $\alpha_1, \alpha_2$, the map $\iota$ is an embedding.

The construction of $A$ and $\iota$ is based on a decomposition of polynomials $a \in H^2$ into factors with roots inside and outside the unit disk, respectively, which are then expressed in terms of the appropriate blowup parameter $\lambda_t^+$ or $\lambda_t^-$. By varying $t$, one obtains families of polynomials as factors of the decomposition. Because we will need the full decomposition for both $a \in H^2$ and the corresponding $b \in B_a$, we now describe the decomposition we need for general $p \in P^d_{\mathbb{R}}$. In the following lemma, we decompose such a $p$ into three factors $p^+, p^0$, and $p^-$, which contain the roots of $p$ in $B(0,1) \setminus S^1$ and $\mathbb{C} \setminus \overline{B(0,1)}$, respectively. Thus we consider the decomposition
\[
(2.4) \quad p(\lambda) = (\lambda_t^+) - \deg(p^+) \cdot p^+(\lambda_t^+) \cdot p^0(\lambda) \cdot p^-(\lambda_t^-),
\]
where $p^\pm(\lambda_t^\pm)$ are polynomials of the same degree $d'$ with roots outside of $\overline{B(0,t)}$ and the roots of the polynomial $p^0(\lambda)$ are all on $S^1$.

**Lemma 2.4.** Take $\varphi \in \mathbb{R}$. Each $p \in P^d_{\mathbb{R}}$ has a unique decomposition (2.4) with the following normalisations:

1. $|p^0(0)| = 1$ and $p^0 \in P^d_0$, where $d^0$ is the number of roots of $p$ on $S^1$.
2. The coefficients $p^\pm_k$ of $p^\pm(\lambda_t^\pm) = \sum_k p^\pm_k (\lambda_t^\pm)^k$ obey $p^- = p^+$.
3. $p^\pm_d \in e^{k^2} \cdot \mathbb{R}_+$.

**Proof.** The existence of polynomials $p^\pm$ with roots outside of $\overline{B(0,t)}$ and $p^0$ with roots only on $S^1$ so that Equation (2.4) holds, is obvious. Because the roots of $p$ are invariant under $\lambda \mapsto 1/\lambda$, we have $\deg(p^+) = \deg(p^-) = d'$. This involution preserves the roots of $p^0$, so $p^0$ can be chosen in $P_{\mathbb{R}}$. The condition $|p^0(0)| = 1$ determines $p^0$ uniquely up to sign. We have
\[
p^\pm(\lambda_t^\pm) = p^\pm_d \cdot \prod_{k=1}^{d'} (\lambda_t^\pm - \varrho^\pm_{t,k}),
\]
where the $\varrho^\pm_{t,k}$ are the roots of $p^\pm$.

The highest resp. the lowest coefficient of the right-hand side of Equation (2.4) is
\[
p^\pm_d \cdot p^\pm_d \cdot p^0 (\prod_{k=1}^{d'} (-\varrho^\pm_{t,k}) \quad \text{resp.} \quad p^\pm_d \cdot p^\pm_d \cdot p^0 (\prod_{k=1}^{d'} (-\varrho^\pm_{t,k})).
\]

The value of $\lambda$ at $\varrho^\pm_{t,k}$ is $t/\varrho^\pm_{t,k}$, and the value of $\lambda$ at $\varrho^\pm_{t,k}$ is $\varrho^\pm_{t}/t$. The involution $\lambda \mapsto 1/\lambda$ maps the roots of $p^*$ to the roots of $p^-$, therefore the $\varrho^\pm_{t,k}$ can be numbered in such a way that
Lemma 2.5. \( E \) is contained in the closure of \( \iota(S^2_{\varphi}) \) in \( A \).

Proof. Let \((a_1, a_2) \in \mathbb{C}^2 \times \) be given. Choose a sequence \((a_{1,n}, a_{2,n})_{n \in \mathbb{N}} \) in \( \mathbb{C}^2 \times \) with limit \((a_1, a_2)\) so that \( a_{2,n} \neq 0 \) and \( a_{1,n}^2 - 4a_{2,n} \neq 0 \) for all \( n \in \mathbb{N} \). If \( a_2 = 0 \), we further require that \( |n \cdot a_{2,n}| \to \infty \) as \( n \to \infty \). The roots of the polynomial \( \lambda^2 + \lambda a_{1,n} + a_{2,n} \) converge to the roots of \( \lambda^2 + \lambda a_1 + a_2 \), and at least one of the latter roots is non-zero. The roots of \( \lambda^2 + \lambda n a_{1,n} + n^2 a_{2,n} \) are equal to \( n \) times the roots of \( \lambda^2 + \lambda a_1 + a_2 \). For sufficiently large \( n \) these roots belong to \( \mathbb{C} \setminus B(0,1) \) because of the condition \( |n \cdot a_{2,n}| \to \infty \). Therefore \([\varphi, a_{1,n}, a_{2,n}] = [1, n a_{1,n}, n^2 a_{2,n}]\) then belongs to \( \iota(H^2) \), and this sequence converges to \([0, a_1, a_2] \in \mathcal{E} \).

Our next goal is to prove the analogue of Proposition 2.2 for \( \iota(S^2_{\varphi}) \cap \mathcal{E} \). For any \([1, a_1, a_2] \in \iota(H^2)\) we define the family \((a_t)_{t \in [-1,1]} \setminus \{0\}\) in \( H^2 \) by the property that

\[
\iota(a_t) = [t, a_1, a_2]
\]

holds for all \( t \in [-1,1] \setminus \{0\} \). In fact, for such \( t \),

\[
a_t(\lambda) = |a_2|^{-1} \cdot (\lambda^t)^2 + a_1 \lambda^t + a_2 = \lambda^2 + t^{-1} a_1 \lambda + t^{-2} a_2
\]

then only has simple roots on \( \mathbb{C} \setminus B(0,1) \). The constant \( c > 0 \) is independent of \( t \) and equal to \(|a_2|\), and therefore we have

\[
a_t(\lambda) = |a_2|^{-1} \cdot (\lambda^t)^2 + a_1 \lambda^t + a_2 = \lambda^2 + t^{-1} a_1 \lambda + t^{-2} a_2
\]

For \( \beta \in \mathbb{C}^\times \), we let \( b_{\beta,t} \) be the unique element of \( B_{a_t} \) with \( b_{\beta,t}(0) = \beta \). Because of the reality condition, the highest coefficient of \( b_{\beta,t} \) is equal to \( \overline{\beta} \). We apply Lemma 2.4 to \( b_{\beta,t} \) with \( \varphi = 0 \). In this way, we obtain a unique decomposition

\[
b_{\beta,t}(\lambda) = (\lambda^t)^{-\deg(b_{\beta,t})} \cdot b_{\beta,t}(\lambda^t) = b_{\beta,t}(\lambda^t)
\]

of the kind described in Lemma 2.4.

Lemma 2.6. Let \( X \) be a compact Riemann surface and \( \psi \) a Mittag-Leffler distribution on \( X \). Then there exists a unique abelian differential \( \omega \) of the second kind with purely imaginary periods so that \( \omega - d\psi \) is holomorphic.
Proof. Let $K$ be the canonical divisor of $X$. For any positive divisor $D$ on $X$, $H^1(X, \mathcal{O}_{K+D}) = 0$ by Serre duality. Therefore $\dim(H^0(X, \mathcal{O}_{K+D})/H^0(X, \mathcal{O}_K)) = \deg(D)$ by Riemann-Roch. This shows that for every Mittag-Leffler distribution $\psi$ on $X$ there exists an abelian differential of the second kind $\omega$ so that $\omega - d\psi$ is holomorphic.

Due to Riemann’s bilinear relations, any holomorphic differential with real periods vanishes. So any choice of real periods occurs as the real parts of the periods of a holomorphic differential. By adding an appropriate holomorphic differential to $\omega$, all periods become purely imaginary. The uniqueness of $\omega$ again follows from Riemann’s bilinear relations. □

Lemma 2.7. Let $X$ be a hyperelliptic Riemann surface with hyperelliptic involution $\sigma$ and quotient $X/\sigma \cong \mathbb{P}^1$. Let $\omega$ be an abelian differential of the second kind with purely imaginary periods and $\sigma^*\omega = -\omega$. Then there exists a unique harmonic function $h$ on the complement of the poles of $\omega$ with $dh = \text{Re}(\omega)$ and $\sigma^*h = -h$. The equation $h = 0$ defines curves on $X/\sigma$ which are smooth away from the zeros of $\omega$ and the branch points of $X$. A point with $h = 0$ which is a zero of $\omega$ of order $n > 0$ but not a branch point of $X$ is the transversal intersection point of $n+1$ such curves. A point with $h = 0$ which is a branch point of $X$ is the intersection of $n+1$ curves ending in this point, where $n \geq 0$ is the order of root of $\omega$ at this point. □

Proof. The differential $\omega$ has purely imaginary periods, hence its real part is exact. Therefore there exists a harmonic function $h$ away from the poles of $\omega$ so that $dh$ is the real part of $\omega$. Because of the symmetry $\sigma^*\omega = -\omega$, the integral of $\text{Re}(\omega)$ from one branch point to another is one-half of a period of $\text{Re}(\omega)$ and therefore equal to zero. This shows that the constant of integration for $h$ can be uniquely chosen so that $\sigma^*h = -h$ holds.

Because $\omega$ preserves multiplication with $i$ at every point away from its poles, $\text{Re}(\omega) = dh$ can vanish only at the roots of $\omega$. At an $n$-th order root ($n \geq 0$) of $\omega$ there exists a local coordinate $z$ of $X$ with $\omega = dz^{n+1}$. Therefore the level set $h = 0$ is away from the roots a 1-dimensional submanifold of $X$ and an $n$-th order root is the transversal intersection point of $n$ branches of this level set. This level set is invariant under $\sigma$. The quotient of the level set by $\sigma$ therefore defines a set on $X/\sigma$ with all branch points as boundary points. At a branch point, $\sigma$ corresponds to $z \mapsto -z$ because of $\sigma^*\omega = -\omega$, and therefore $\omega$ can have only a root of even order at a branch point. Thus the $n+1$ level curves intersecting in this branch point project to $n+1$ curves in $X/\sigma$ which end all at this branch point. □

Lemma 2.8. Let $\Theta_{b,t} := b_t \frac{d\lambda}{\sqrt{|a_2|}}$ on the family of spectral curves $\Sigma_t = \{(\lambda, \nu_t) \in \mathbb{C}^2 \mid \nu_t^2 = \lambda \cdot a_t(\lambda)\}$ defined by $a_t$. For fixed $t$ we then have in the limit $\lambda \to \infty$

$$\Theta_{b,t} = \left(\beta \cdot \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \cdot \lambda^{-1/2} + O(\lambda^{-3/2})\right) d\lambda = \left(t^{-1/2}\beta \cdot \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \cdot (\lambda_t^{-})^{-1/2} + O((\lambda_t^{-})^{-3/2})\right) d\lambda_t^{-}$$

and in the limit $\lambda \to 0$

$$\Theta_{b,t} = \left(\beta \cdot \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \cdot \lambda^{-1/2} + O(\lambda^{1/2})\right) d\lambda = - \left(t^{1/2}\beta \cdot \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \cdot (\lambda_t^{+})^{-1/2} + O((\lambda_t^{+})^{-3/2})\right) d\lambda_t^{+}.$$

Proof. For $\lambda \to \infty$ we have

$$b_t = \beta \lambda^3 + O(\lambda^2), \quad a_t = \frac{\alpha_t}{|a_2|}\lambda^4 + O(\lambda^3)$$

and thus

$$\nu_t = \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \lambda^{5/2} + O(\lambda^{3/2}).$$

Thus we obtain the first equality. The second equality follows because of $\lambda = \lambda_t^{-}/t$.

For $\lambda \to 0$ we have

$$b_t = \beta + O(\lambda), \quad a_t = \frac{\alpha_t}{|a_2|} + O(\lambda)$$

and thus

$$\nu_t = \frac{\sqrt{|a_2|}}{\sqrt{|a_2|}} \lambda^{1/2} + O(\lambda^{3/2}).$$

Thus we obtain the first equality. □
In order to describe $\mathfrak{t}(S^\mathfrak{t}) \cap \mathcal{E}$, we wish to find a natural analogue of the condition $df(1) = 0$ which appeared in Lemma 2.1. This analogous statement will be expressed as a condition on the limiting curve $\Sigma_0$ of the spectral curves $\Sigma_t$.

We now introduce a singular curve $\Sigma_0$ which we will show to arise as a certain limit of the spectral curves $\Sigma_t$ as $t \to 0$. The normalisation of $\Sigma_0$ has three connected components $\Sigma_0^+$, $\Sigma_0^0$ and $\Sigma_0^-$, and each connected component is hyperelliptic. Altogether $\Sigma_0$ is a two-sheeted covering over three copies $\mathbb{P}^1_{\lambda^+}$, $\mathbb{P}^1_{\lambda}$ and $\mathbb{P}^1_{\lambda^-}$ of $\mathbb{P}^1$ which are joined by two double points, where the subscripts denote the parameter we use for the respective copy. In the limit $t \to 0$, the equations

$$\lambda \cdot \lambda^+ = t \quad \text{and} \quad \lambda^- \cdot \lambda^{-1} = t$$

respectively describe the double point $(\lambda, \lambda^+) = (0, 0)$ which joins $\mathbb{P}^1_{\lambda^+}$ to $\mathbb{P}^1_{\lambda}$ and the double point $(\lambda, \lambda^-) = (\infty, 0)$ which joins $\mathbb{P}^1_{\lambda}$ to $\mathbb{P}^1_{\lambda^-}$. These double points are described by the equations

$$\lambda \cdot \lambda^+ = 0 \quad \text{and} \quad \lambda^- \cdot \lambda^{-1} = 0 .$$

Let $\Sigma_0^+$, $\Sigma_0^0$ and $\Sigma_0^-$ be the hyperelliptic curves which are the one-point compactifications of

$$\{(\lambda^+, \nu^+) \in \mathbb{C}^2 \mid (\nu^+)^2 = \lambda^+ ((\lambda^+)^2 + a_1 \lambda^+ + a_2)\} \quad \text{at} \quad \lambda^+ = \infty$$

$$\{(\lambda, \nu^0) \in \mathbb{C}^2 \mid (\nu^0)^2 = \lambda\} \quad \text{at} \quad \lambda = \infty$$

$$\{(\lambda^-, \nu^-) \in \mathbb{C}^2 \mid (\nu^-)^2 = \lambda^- ((\lambda^-)^2 + a_1 \lambda^- + a_2)\} \quad \text{at} \quad \lambda^- = \infty ,$$

respectively. The curve $\Sigma_0$ is obtained by identifying $(\lambda^+, \nu^+) = (0, 0) \in \Sigma_0^+$ with $(\lambda, \nu^0) = (0, 0) \in \Sigma_0^0$ to form an ordinary double point, and similarly identifying $(\lambda, \nu^0) = (\infty, \infty) \in \Sigma_0^0$ with $(\lambda^-, \nu^-) = (0, 0) \in \Sigma_0^-$ to form another ordinary double point.

We now explain in what sense $\Sigma_0$ is the limit of the spectral curves $\Sigma_t$ as $t \to 0$. For any compact set $K \subset \mathbb{C}^\times$ we define

$$\Sigma_{t,K}^+ := \{ (\lambda, \nu) \in \Sigma_t \mid \lambda^+ = t/\lambda \in K \} \quad \Sigma_{0,K}^+ := \{ (\lambda^+, \nu^+) \in \Sigma_0^+ \mid \lambda^+ \in K \}$$

$$\Sigma_{t,K}^0 := \{ (\lambda, \nu) \in \Sigma_t \mid \lambda \in K \} \quad \Sigma_{0,K}^0 := \{ (\lambda, \nu^0) \in \Sigma_0^0 \mid \lambda \in K \}$$

$$\Sigma_{t,K}^- := \{ (\lambda, \nu) \in \Sigma_t \mid \lambda^- = t\lambda \in K \} \quad \Sigma_{0,K}^- := \{ (\lambda^-, \nu^-) \in \Sigma_0^- \mid \lambda^- \in K \} .$$

For every $K$ there exists $\epsilon > 0$ so that for $t \in (-\epsilon, \epsilon) \setminus \{0\}$ the hyperelliptic Riemann surfaces $\Sigma_{t,K}^+$, $\Sigma_{t,K}^0$ and $\Sigma_{t,K}^-$ have the same branch points as $\Sigma_{0,K}^+$, $\Sigma_{0,K}^0$ and $\Sigma_{0,K}^-$, respectively. Therefore there exist biholomorphic maps

$$\Phi_t^+ : \Sigma_{0,K}^+ \to \Sigma_{t,K}^+ \quad \Phi_t^0 : \Sigma_{0,K}^0 \to \Sigma_{t,K}^0 \quad \Phi_t^- : \Sigma_{0,K}^- \to \Sigma_{t,K}^-$$

with

$$\Phi_t^+ \ast \lambda = \lambda^+ \quad \Phi_t^0 \ast \lambda = \lambda \quad \Phi_t^- \ast \lambda = \lambda^- .$$
Let $\eta$ be the anti-holomorphic involution of $\Sigma_t$ without fixed points which covers $\lambda \mapsto \bar{\lambda}^{-1}$. It gives rise to an anti-holomorphic involution on $\Sigma_0$ described by
\[
\lambda^+ \mapsto \bar{\lambda}^- , \quad \lambda \mapsto \bar{\lambda}^{-1} \quad \text{and} \quad \lambda^- \mapsto \bar{\lambda}^+ ,
\]
which we also denote by $\eta$.

In the next lemma, we will show that $\Theta_{\beta,t}$ converges for $t \to 0$ to a differential on $\Sigma_0$ in the following sense: There exists a differential $\Theta_{\beta,0}$ on $\Sigma_0$ so that for every compact $K \subset \mathbb{C}^\times$, the pullbacks $(\Phi_+^*)^* \Theta_{\beta,t}$, $(\Phi_-^*)^* \Theta_{\beta,t}$ and $(\Phi^*)^* \Theta_{\beta,t}$ converge uniformly on $\Sigma_{0,K}^+$, $\Sigma_{0,K}^0$ resp. $\Sigma_{0,K}^-$ to $\Theta_{\beta,0}$.

Given complex polynomials $b^\pm$ and $b^0$ of degree 1 we define the meromorphic differentials
\[
\Theta_{b^\pm}^0 = \frac{b^+(\lambda^\pm)}{\nu^\pm} d\lambda^\pm \quad \text{and} \quad \Theta_{b^0}^0 = \frac{b^0(\lambda)}{\nu^0} \frac{d\lambda}{\lambda}
\]
on $\Sigma_0^+$ and on $\Sigma_0^0$ respectively. For any $\beta \in \mathbb{C}^\times$ we let
\[
b^\pm_\beta(\lambda^+) = \beta \frac{\sqrt{a_2}}{|a_2|} \lambda^+ + b^\pm_\beta(0) , \quad b^0_\beta(\lambda) = b^\pm_\beta(0) \lambda + b^\pm_\beta(0) \quad \text{and} \quad b^-_\beta(\lambda^-) = \bar{\beta} \frac{\sqrt{a_2}}{|a_2|} \lambda^- + b^\pm_\beta(0)
\]
where $b^\pm_\beta(0) \in \mathbb{C}$ is chosen such that $\Theta_{b_\beta}^\pm$ has purely imaginary periods. The leading coefficients of these polynomials are chosen here such that they describe the limits of $b^\pm_{\beta,t}$ and $b^0_{\beta,t}$ for $t \to 0$, compare Lemma 2.10 below.

Any choice of $\mathbb{R}$-linearly independent $\beta_1, \beta_2 \in \mathbb{C}^\times$ defines rational functions
\[
f^\pm(\lambda^\pm) = \frac{b^\pm_{\beta_2}(\lambda^\pm)}{b^\pm_{\beta_1}(\lambda^\pm)} \quad \text{and} \quad f^0(\lambda) = \frac{b^0_{\beta_2}(\lambda)}{b^0_{\beta_1}(\lambda)}
\]
Each function is unique up to real Möbius transformations. We will see that the condition characterising $a \in \iota(\mathbb{S}^1) \cap \mathcal{E}$ is $f^- (\lambda^- = 0) \in \mathbb{R}$. This provides an analogue of the condition $df(\lambda_0) \neq 0$ of Lemma 2.1. The requirement $f^- (\lambda^- = 0) \in \mathbb{R}$ is equivalent to $f^+(\lambda^+ = 0) \in \mathbb{R}$ because the coefficients of $b^\pm_\beta$ are the complex conjugates of the coefficients of $b^-_\beta$.

**Lemma 2.9.** Let $K \subset \mathbb{C}^\times \subset \mathbb{P}^1$ be compact. As $t \to 0$, the following limits are uniform:
\[
a_t(t/\lambda^+) \to \frac{a_2}{|a_2|} \cdot (\lambda^+)^2 \cdot a^+(\lambda^+) \quad \text{for} \quad \lambda^+ \in K
\]
\[
t^2 \cdot a_t(\lambda) \to |a_2| \cdot |\lambda|^2 \quad \text{for} \quad \lambda \in K
\]
\[
t^4 \cdot a_t(\lambda^-/t) \to \frac{a_2}{|a_2|} \cdot (\lambda^-)^2 \cdot a^-(\lambda^-) \quad \text{for} \quad \lambda^- \in K.
\]

**Proof.** For the first limit, we substitute $\lambda = t/\lambda^+$ and $\lambda^- = t^2/\lambda^+_t$ into the decomposition formula (2.5) and then take the limit as $t \to 0$. For the second limit, we similarly substitute $\lambda^+_t = t/\lambda$ and $\lambda^-_t = t\lambda$, and for the third limit we substitute $\lambda^+_t = t^2/\lambda^-_t$ and $\lambda = \lambda^-_t/t$.

The following lemma shows that for $f^- (\lambda^- = 0) \in \mathbb{C} \setminus \mathbb{R}$, as $t \to 0$, one of the three roots of $b_{\beta,t}(\lambda)$ tends to $\lambda = 0$, one root remains bounded in $\mathbb{C}^\times$ and the last root tends to $\lambda = \infty$. Moreover, the roots that tend to $\lambda = 0$ or $\lambda = \infty$ do so at the same rate as the roots of $a_t(\lambda)$ that have the same limit.

**Lemma 2.10.** Suppose that $f^- (\lambda^- = 0) \in \mathbb{C} \setminus \mathbb{R}$ holds. Let $K \subset \mathbb{C}^\times \subset \mathbb{P}^1$ be compact. For sufficiently small $t$, the polynomials $b^\pm_{\beta,t}$ and $b^0_{\beta,t}$ have degree 1, and as $t \to 0$, the following limits are uniform:
In this way, we get

\[ b_{\beta, t}(t/\lambda^+) \to (\lambda^+)^{-1} b_\beta^+ (\lambda^+) \leftarrow \frac{\beta}{|\beta|} \frac{b_\beta(0)}{|\beta|} (\lambda^+)^{-1} \cdot b_\beta^+ (\lambda^+) \quad \text{for } \lambda^+ \in K \]

\[ t \cdot b_{\beta, t}(\lambda) \to \lambda \cdot b_\beta^0 (\lambda) \leftarrow |b_\beta(0)| \lambda \cdot b_{\beta, t}^0 (\lambda) \quad \text{for } \lambda \in K \]

\[ i^3 \cdot b_{\beta, t}(\lambda^- / t) \to (\lambda^-)^2 \cdot b_\beta^- (\lambda^-) \leftarrow \frac{\beta}{|\beta|} \frac{b_\beta(0)}{|\beta|} (\lambda^-)^2 \cdot b_{\beta, t}^- (\lambda^-) \quad \text{for } \lambda^- \in K . \]

**Proof.** It follows from [K-PH-S, Lemma 3.4] that the coefficients of $b_{\beta, t}$ depend continuously on $t \in (0, \varepsilon)$. This implies that the coefficients of $b_{\beta, t}^0 (\lambda_+^\pm)$ and $b_{\beta, t}^0 (\lambda)$ also depend continuously on $t \in (0, \varepsilon)$, as long as the degrees of these polynomials do not change.

We now show continuity of $b_{\beta, t}^- (\lambda^-)$ at $t = 0$; the corresponding statement for $b_{\beta, t}^+ (\lambda_+^\pm)$ follows because the complex coefficients of $b_{\beta, t}^0$ resp. $b_\beta^+$ are the complex conjugates of the coefficients of $b_{\beta, t}^-$ resp. $b_\beta^-$. We let a sequence $(t_n)_{n \in \mathbb{N}}$ of non-zero real numbers with $t_n \to 0$ be given. We put $a_n := a_{t_n}$, $b_n^\pm := b_{\beta, t_n}^\pm$, $b_n^0 := b_{\beta, t_n}^0$ and $\Theta_n^- := \Theta_{\beta, t_n}^-$. Since $b_n^0(0) \in S^1$, after passing to a suitable subsequence of $(t_n)$, $b_n^0(0)$ converges to $\delta \in S^1$.

As $n \to \infty$, $\lambda_n^+ = t_n^2/\lambda_n^-$ and $\lambda_n^+ = n \to \lambda_n^- \converge$ uniformly to 0 on $K$. Because of Lemma 2.9, we therefore have

\[ \lim_{n \to \infty} \frac{t_n^{5/2} \cdot (\Phi_{t_n})^{*} \nu_{t_n}}{\nu_2} = \lambda^- \cdot \nu^- \cdot \frac{\sqrt{a_2}}{|a_2|} \]

uniformly on $\Sigma_{t_n, K}$, where we choose the signs of the square roots appropriately.

Next we prove by contradiction that $\deg(b_n^-) = 1$. So we assume that $\deg(b_n^-) = 0$ after replacing $t_n$ by a subsequence. Then $b_n^\pm$ are constant polynomials, so $|b_n^\pm| \in \mathbb{R}$. We have by Equation (2.6)

\[ b_n(\lambda) = b_n^0 b_n^0(\lambda) b_n^- = |b_n|^2 b_n^0(\lambda) = |b_n|^2 b_n^0 (\lambda_n) . \]

The highest coefficient of $b_n^0$ converges to $\delta$. All coefficients of the third-order polynomial $b_n^0$ are bounded, because the highest coefficient and all the roots of $b_n^0$ are unimodular. Thus the sequence $(t_n/\lambda_n)^3 \cdot b_n^0(\lambda_n)$ converges uniformly on $K$ to $\delta$. We now compare asymptotics of $\lim_{n \to \infty} (t_n)^{1/2} (\Phi_{t_n})^{*} \Theta_{\beta, t_n}$ for $\lambda \to \infty$ (via Lemma 2.8) and for $\lambda^- \to \infty$ (via Equation 2.11). In this way, we get

\[ \left( \frac{\beta \cdot \sqrt{a_2}}{|a_2|} \frac{1}{\sqrt{\lambda^-}} + O((\lambda^-)^{-3/2}) \right) \frac{d\lambda^-}{\lambda^-} = \lim_{n \to \infty} (t_n)^{1/2} (\Phi_{t_n})^{*} \Theta_{\beta, t_n} = \lim_{n \to \infty} \frac{|b_n|^2 \cdot \delta \cdot (\lambda^-)^2 \cdot \sqrt{a_2}}{\lambda^- \cdot \nu^- \cdot \sqrt{|a_2|}} \frac{d\lambda^-}{\lambda^-} . \]

Since $\nu^- = (\lambda^-)^{3/2} + O((\lambda^-)^{1/2})$ for $\lambda \to \infty$, we first see that $\lim_{n \to \infty} |b_n^-|^2 \neq 0$. Because of $\delta \in S^1$, it follows that $\delta = |\beta| \cdot \rho$, and therefore

\[ \lim_{n \to \infty} |b_n^-|^2 = |\beta| . \]

By our choice $\varphi = 0$, we have $b_n^- > 0$, and thus we obtain

\[ \lim_{n \to \infty} b_n^- = \sqrt{|\beta|} . \]

Furthermore, the limit

\[ \lim_{n \to \infty} t_n^{1/2} (\Phi_{t_n})^{*} \Theta_{\beta, t_n} = \frac{\beta \cdot \sqrt{a_2}}{\sqrt{|a_2|}} \cdot \frac{\lambda^-}{\nu^-} d\lambda^- \]

is uniform on $\Sigma_{0, K}$ and defines a meromorphic differential of the second kind on $\Sigma^-$, with a zero at $\lambda^- = 0$. By choosing all cycles of $\Sigma_{t_n}$ to lie in $\Sigma_{t_n, K}$ for sufficiently large $n$, we see that
all periods of this differential are purely imaginary. Therefore \( b^-(\lambda^-) = \tilde{\beta} \sqrt{\frac{a_2}{|a_2|}} \cdot \lambda^- \), and hence
\( b^-(\lambda^- = 0) = 0 \). This implies \( f^- (\lambda^- = 0) \in \mathbb{R} \). This contradiction forces \( \text{deg}(b^\pm_n) = \text{deg}(b^n_0) = 1 \) for sufficiently large \( n \).

We now show that the sequence \( b^*_n \) converges to \( b^- \).

As \( \text{deg}(b^*_n) = 1 \), we write
\[
\frac{b^*_n(\lambda^-)}{b^-} = b^*_{n, 1} \lambda^- + b^*_{n, 0} \quad \text{and} \quad b^*_{n, 0} = b^*_{n, 1} + b^*_{n, 0}.
\]

Therefore the decomposition of \( b_n \) is
\[
b_n(\lambda) = \frac{\lambda^-}{\bar{t}_n} b^*_n(\lambda^-/t_n) b^*_n(\lambda^-/t_n) b^-_n(\lambda^-).
\]

We also have \( \text{deg}(b^*_n) = 1 \), and therefore, after passing to a subsequence of \( (t_n) \), the sequence
\[
b^*_n = \lim_{n \to \infty} \frac{1}{|b_n| |b_{n, 1}|} b_n \lambda^- b^-_n(\lambda^-)
\]
converges to a polynomial of degree 1 with respect to \( \lambda^- \). Because of Equations (2.11) and (2.12), this means that
\[
\lim_{n \to \infty} \frac{t_n^{\lambda^-/t_n} \Theta_{b^*_n}}{b^*_n} = \lim_{n \to \infty} \frac{t_n^{\lambda^-/t_n} \Theta_{b^-_n}}{b^-_n}.
\]

This implies that
\[
\lim_{n \to \infty} \frac{1}{b^*_n b_{n, 1}} \Theta_{b^*_n} = \lim_{n \to \infty} \frac{1}{b^-_n b_{n, 1}} \Theta_{b^-_n}.
\]

and therefore, again by Lemma 2.8
\[
\tilde{\beta} = \lim_{n \to \infty} \frac{b^-_n b_{n, 1}}{b^*_n b_{n, 1}}.
\]
This shows in particular that all the summands of \( b_n^+ b_n^- \) in (2.13) converge separately. By choosing all the cycles of \( \Sigma_{t_n} \) inside \( \Sigma_{t_n,K} \) for sufficiently large \( n \), \( t_n^{1/2} (\Phi_{t_n})^* \Theta_{b_n,t_n} \) converges uniformly on \( \Sigma_{0,K} \) to \( \Theta_{b_n}^- \). Because of \( f^-(\lambda^- = 0) \not\in \mathbb{R} \), we have \( b_n^- (0) \neq 0 \) as before. Therefore we obtain

\[
\lim_{n \to \infty} b_{n,0} b_{n,1} = \delta \cdot \bar{\beta}, \quad \lim_{n \to \infty} (b_{n,1} b_{n,1} t_n^2 + b_{n,0} b_{n,0}) = \delta \cdot b_n^- (0).
\]

If \( \lim_{n \to \infty} b_{n,1} b_{n,1} t_n^2 \neq 0 \), then \( |b_{n,1}| = O(t_n^{-1}) \) and \( |b_{n,0}| = O(t_n) \). This would imply that the roots of \( b_n^- \) are of order \( O(t_n^2) \) with respect to \( \lambda^- \), and therefore of order \( O(t_n) \) with respect to \( \lambda \). This is a contradiction to the definition of \( b_n^- \). Thus we have \( \lim_{n \to \infty} b_{n,1} b_{n,1} t_n^2 = 0 \). Therefore we have \( \lim_{n \to \infty} b_{n,0} b_{n,1} = \delta \cdot b_n^- (0) \neq 0 \). This implies

\[
(2.14) \quad \delta = \frac{b_n^- (0)}{|b_n^- (0)|}.
\]

By our choice of \( \varphi = 0, b_{n,1} \in \mathbb{R}^+ \). Therefore we obtain

\[
(2.15) \quad \lim_{n \to \infty} b_{n,0} = \frac{\beta \cdot b_n^- (0)}{|\beta| \cdot \sqrt{|b_n^- (0)|}} \quad \text{and} \quad \lim_{n \to \infty} b_{n,1} = \frac{|\beta|}{\sqrt{|b_n^- (0)|}}.
\]

Therefore

\[
\lim_{n \to \infty} b_n^- (\lambda^-) = \frac{\beta}{|\beta| \cdot \sqrt{|b_n^- (0)|}} \cdot \left( \frac{\beta}{|\beta|} \cdot \lambda^- + b_n^- (0) \right) = \frac{\beta}{|\beta| \cdot \sqrt{|b_n^- (0)|}} \cdot b_n^- (\lambda^-).
\]

Hence \( b_{n,0} \) and \( b_{n,1} \) converge, and the limits are uniquely determined by \( \beta \). This completes the proof that \( b_n^- \) and \( b_n^+ \) converge for \( \lambda_{t_n}^- \in K \) as claimed in (2.10). Because this holds for a subsequence of any sequence \( t_n \) of non-zero real numbers with \( t_n \to 0 \), and the limit is unique, this proves both limits in (2.10).

For (2.8) we note that the coefficients of \( b_{\beta,t}^+ \) and \( b_{\beta}^+ \) are the complex conjugates of the coefficients of \( b_{\beta,t}^- \) and \( b_{\beta}^- \), respectively. By (2.15), the limit on the right of (2.8) follows. For the limit on the left, we insert \( \lambda_t^+ = \lambda^+ = t/\lambda^+ \) and \( \lambda_t^- = t^2/\lambda^+ \) into the decomposition (2.6) to obtain

\[
\lim_{t \to 0} b_{\beta,t}^+ t(\lambda^+) = \lim_{t \to 0} (\lambda^+)^{-1} b_{\beta,t}^+ (\lambda^+) t_0^0 b_{\beta,t}^- (\frac{t^2}{\lambda^+}) = (\lambda^+)^{-1} \cdot \lim_{t \to 0} b_{\beta,t}^+ (\lambda^+) \cdot \lim_{t \to 0} b_{\beta,t}^0 (0) \cdot \lim_{t \to 0} b_{\beta,t}^- (0).
\]

By the limit on the right of (2.8) and Equations (2.14) and (2.15), the limit on the left of (2.8) follows.

By definition of \( \delta \), we have \( \lim_{t \to 0} b_{\beta,t}^0 (\lambda) = \delta \lambda + \delta \), and by Equation (2.11), the limit on the right of (2.9) follows. For the limit on the left of (2.9), we insert \( \lambda_t^+ = t/\lambda \) and \( \lambda_t^- = t \lambda \) into the decomposition (2.6) to obtain

\[
\lim_{t \to 0} t \cdot b_{\beta,t} (\lambda) = \lim_{t \to 0} \lambda b_{\beta,t}^+ (\frac{t}{\lambda}) b_{\beta,t}^0 (\lambda) b_{\beta,t}^- (t \lambda) = \lambda \cdot \lim_{t \to 0} b_{\beta,t}^+ (0) \cdot \lim_{t \to 0} b_{\beta,t}^0 (\lambda) \cdot \lim_{t \to 0} b_{\beta,t}^- (0).
\]

By Equation (2.15) and the limit on the right of (2.9), the limit on the left of (2.9) follows. \( \square \)

**Lemma 2.11.** Suppose that \( f^- (\lambda^- = 0) \in \mathbb{C} \setminus \mathbb{R} \) holds. Then there exists \( \varepsilon > 0 \) such that for \( t \in (-\varepsilon, 0) \cup (0, \varepsilon) \) the winding number \( n(f_t) \) equals \( -\text{sign} \ \text{Im}(f^- (\lambda^- = 0)) \), where \( f_t = b_{\beta,t} \frac{b_{\beta,t}^-}{b_{\beta,t}^+} \) and \( f^- = \frac{b_{\beta,t}^-}{b_{\beta,t}^+} \).
Proof. For small $|t|$ the function $\hat{f}_t$ is a M"{o}bius transformation by Lemma \[2.10\] Hence the winding number $n(f_t)$ is $\pm 1$ in any case, and it is $+1$ if and only if $\hat{f}_t$ maps $\mathbb{C} \setminus B(0, 1)$ onto itself. Because of Equation \[1.3\] this is equivalent to $\text{Im} f_t(\infty) < 0$. Hence

$$n(f_t) = -\text{sign Im } f_t(\lambda = \infty) = -\text{sign Im } \left( \frac{b_{\beta=1}^{-}(0)}{b_{\beta=1}^{-}(0)} \right) = -\text{sign Im } f^{-}(\lambda^{-} = 0).$$

\[\square\]

We now arrive at the promised description of $\iota(S^2_1) \cap \mathcal{E}$:

**Theorem 2.12.** \([0, E_1, E_2] \in \mathcal{E}\) belongs to the closure of $\iota(S^2_1)$ if and only if \([0, E_1, E_2]\) belongs to the closure of

\[\{[0, E_1, E_2] \in \mathcal{E} \mid f^{-}(\lambda^{-} = 0) \in \mathbb{R}, E^2_1 - 4E_2 \neq 0, E_2 \neq 0 \\}\]

in $\mathcal{A}$.

**Remark 2.13.** In the following section we will investigate the set \[2.16\]. Within the set

\[\{[0, E_1, E_2] \in \mathcal{E} \mid f^{-}(\lambda^{-} = 0) \in \mathbb{R} \\},\]

the points where either $E^2_1 - 4E_2 = 0$ or $E_2 = 0$ correspond to the cases where the spectral curve described by $a^{-}(\lambda^{-})$ becomes singular, and this corresponds to $\tau \to \infty$ for the conformal class $\tau$ of the elliptic curve that corresponds to this spectral curve.

**Proof of Theorem 2.12.** If \([0, E_1, E_2] \in \mathcal{E}\) belongs to the closure of $\iota(S^2_1)$, then there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $S^2_1$ so that $\iota(a_n)$ converges to \([0, E_1, E_2]\). Due to \[\text{[C.S.2] Theorem 3.2,}\] \[\deg(\gcd(B_n)) = 1\] for all $a \in S^2_1$. Because of \[\text{[C.S.2] Theorem 5.5(i)},\] $S^2_1$ is a smooth, 2-dimensional submanifold of $\mathcal{H}^2$. By the implication (D) $\Rightarrow$ (B) in \[\text{[C.S.2] Theorem 5.8},\] it now follows that each $a_n$ belongs to the closure of $V_{-1}$ and of $V_1$ (in $\mathcal{H}^2 \setminus S^2$). Therefore there exist sequences $(a^n_n)_{n \in \mathbb{N}}$ in $V_1$ and $(a^n_n)_{n \in \mathbb{N}}$ in $V_{-1}$ with $\lim a^n_n = a$. For $f^{-}(\lambda^{-}) \not\in \mathbb{R}$, Lemma \[2.11\] implies that there exists a neighborhood of \([0, E_1, E_2]\) in $\mathcal{A}$ such that

$$O \cap \iota(\mathcal{H}^2 \setminus S^2) \subset V_1 \text{ for } \text{Im}(f^{-}(\lambda^{-} = 0)) > 0,$$

$$O \cap \iota(\mathcal{H}^2 \setminus S^2) \subset V_{-1} \text{ for } \text{Im}(f^{-}(\lambda^{-} = 0)) < 0.$$ 

This implies $f^{-}(\lambda^{-} = 0) \in \mathbb{R}$. Hence \([0, E_1, E_2]\) belongs to the set \[2.16\], and thus also to the closure of the set \[2.16\].

Conversely, let $f^{-}(\lambda^{-} = 0) \in \mathbb{R}$. We only need to consider the case where $E^2_1 - 4E_2 \neq 0$ and $E_2 \neq 0$. We then show that every neighborhood of \([0, E_1, E_2]\) contains an open neighborhood of \([0, E_1, E_2]\) such that $O \cap \iota(\mathcal{H}^2)$ is connected. Such an $O$ can indeed be chosen in the form $\{[s, E_1, 1] \mid s \in \hat{0}_1, \ E_1 \in \hat{0}_1\}$ with suitable connected neighborhoods $\hat{0}_0$ of 0 and $\hat{0}_1$ of $E_1/\sqrt{E_2}$ in $\mathbb{C}$.

Then by analogous arguments to those in the proof of Lemma \[2.5\] we have

$$O \cap \iota(\mathcal{H}^2) = \{[s, E_1, 1] \mid s \in \hat{0}_0 \setminus \{0\}, \ E_1 \in \hat{0}_1\}.$$ 

This set is connected.

The map $f^{-}(\lambda^{-} = 0)$ is holomorphic on $O$ and non-constant by the proof of Lemma \[3.4\] and hence open. This means that $O$ contains points where $\text{Im}(f^{-}(\lambda^{-} = 0)) > 0$ and points where $\text{Im}(f^{-}(\lambda^{-} = 0)) < 0$. By Lemma \[2.11\] we therefore have $O \cap \iota(V_1) \neq \emptyset$ and $O \cap \iota(V_{-1}) \neq \emptyset$. The sets $V_1$ and $V_{-1}$ are open and disjoint, and therefore $\iota(V_1)$ and $\iota(V_{-1})$ are also open and disjoint, but $O$ is connected, so $(O \cap \iota(V_1)) \cup (O \cap \iota(V_{-1})) \subseteq O$. This implies that $O \cap \iota(S^2) \neq \emptyset$. Since $O$ can be chosen arbitrarily small, there exists a sequence $(a_n)_{n \in \mathbb{N}}$ in $S^2$ with $\lim \iota(a_n) = [0, E_1, E_2]$. 


Because of \( a_n \in S^2 \) there exists a root \( \lambda_n \in S^1 \) of \( \gcd(B_n) \). After passing to a sub-sequence of the \( a_n \), the sequence \( (\lambda_n) \) converges to some \( \lambda_* \in S^1 \). Let \( \tilde{a}_n(\lambda) := \lambda_*^2 \cdot a_n(\lambda_*^{-1} \lambda) \). Then for some \( \lambda_* \in S^2 \) and \( \lim \tilde{a}_n = [0, \lambda_*^{-1} E_1, \lambda_*^{-2} E_2] = [0, E_1, E_2] \). Hence \( [0, E_1, E_2] \) is in the boundary of \( \iota(S^2_1) \).

\[ \square \]

3. The condition \( f(0) \in \mathbb{R} \)

As we saw in Theorem 2.12 a point in the exceptional fibre \( E \) belongs to the closure of \( \iota(S^2_1) \) if and only if \( f^- (\lambda^- = 0) \in \mathbb{R} \) (and the closure of two “dense” conditions) holds, where \( f^- \) is a certain meromorphic function on the complex curve \( \Sigma^- \) of genus 1. In this section we would like to investigate the meaning of this condition more closely. For this purpose we study genus 1 spectral curves, which we here denote simply by \( \Sigma \) (rather than \( \Sigma^- \)).

Let

\[
(3.1) \quad \mathcal{M}_1 := \{ \tau \in \mathbb{C} \mid |\tau| \geq 1, \text{Im}(\tau) > 0, -\frac{1}{2} \leq \text{Re}(\tau) \leq \frac{1}{2} \} / \sim ,
\]

where the equivalence relation \( \sim \) identifies the points \( \tau \) on the boundary of the above region with \( -\overline{\tau} \). This is the standard fundamental domain of the action of \( \text{SL}(2, \mathbb{Z}) \) on the upper half plane.

For given \( a_1, a_2 \), we let \( \tau(a_1, a_2) \in \mathcal{M}_1 \) be the unique element so that the elliptic curve \( \Sigma = \{(\lambda, \nu) \in \mathbb{C} \times \mathbb{C} \mid \nu^2 = \lambda (\lambda^2 + a_1 \lambda + a_2)\} \) is biholomorphic to \( \mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau) \). The corresponding biholomorphic map becomes unique by requiring that the marked point \( (\lambda, \nu) = (\infty, \infty) \) is mapped onto the marked point \( z = 0 \).

We define \( f(\lambda) = f(a_1, a_2)(\lambda) = \frac{b_2(\lambda)}{b_1(\lambda)} \), where \( (b_1, b_2) \) is any basis of \( B_a \), and

\[
\mathcal{M}_{1,=0} := \{ \tau(a_1, a_2) \in \mathcal{M}_1 \mid \text{Im}(f(a_1, a_2)(0)) = 0 \}
\]
\[
\mathcal{M}_{1,>0} := \{ \tau(a_1, a_2) \in \mathcal{M}_1 \mid \text{Im}(f(a_1, a_2)(0)) > 0 \}
\]
\[
\mathcal{M}_{1,<0} := \{ \tau(a_1, a_2) \in \mathcal{M}_1 \mid \text{Im}(f(a_1, a_2)(0)) < 0 \}.
\]

**Theorem 3.1.** \( \mathcal{M}_{1,=0} = \partial \mathcal{M}_{1,>0} \cap \partial \mathcal{M}_{1,<0} \) and is an unbounded, closed, connected 1-dimensional submanifold of \( \mathcal{M}_1 \). \( \mathcal{M}_{1,>0} \) and \( \mathcal{M}_{1,<0} \) are connected and open subsets of \( \mathcal{M}_1 \). Moreover \( \mathcal{M}_{1,=0} \cap \{ |\tau| = 1 \} = \{ e^{i\vartheta}, -e^{-i\vartheta} \} / \sim \) for some \( \vartheta \in (\frac{\pi}{4}, \frac{3\pi}{4}) \) and \( \mathcal{M}_{1,=0} \cap \{ \text{Re}(\tau) \in \{0, \pm \frac{1}{2}\} \} = \emptyset \).
The proof is contained in the following four lemmata.

**Lemma 3.2.** $\mathcal{M}_{1,=0} \cap i\mathbb{R} = \emptyset$.

**Proof.** Suppose that $\tau := \tau(a_1,a_2) \in i\mathbb{R}$. Then the elliptic curve $\Sigma$ is biholomorphic to $\mathbb{C}/(\mathbb{Z}1 \oplus \mathbb{Z}\tau)$ which has the anti-holomorphic involutions $z \mapsto \pm \bar{z}$. The fixed point set of each anti-holomorphic involution has two connected components:

$$\text{Fix}(z \mapsto \bar{z}) = \{z \mid z - \bar{z} \in \{0, \tau\}\} \quad \text{and} \quad \text{Fix}(z \mapsto -\bar{z}) = \{z \mid z + \bar{z} \in \{0, 1\}\}.$$ 

The product of these two anti-holomorphic involutions is the hyperelliptic involution which fixes the marked point $z = 0$. After an appropriate M"obius transformation, the anti-holomorphic involutions therefore correspond to $(\lambda, \nu) \mapsto (\bar{\lambda}, \bar{\nu})$ and $(\lambda, \nu) \mapsto (\bar{\lambda}, -\bar{\nu})$. This shows that $a_1, a_2 \in \mathbb{R}$. Because the fixed point sets of these anti-holomorphic involutions have two connected components contained in $\{(\lambda, \nu) \mid \lambda \in \mathbb{R}\}$, it follows that $\lambda a(\lambda) := \lambda \cdot (\lambda^2 + a_1 \lambda + a_2)$ changes sign three times along the real line. Therefore $\lambda a(\lambda)$ has three real roots. These roots divide the real line into four intervals, of which two are unbounded and two are bounded. Let $A$ and $B$ be the two bounded intervals, where $A$ is to the left-hand side of $B$. Then $\lambda a(\lambda)$ is non-negative on $A$ and non-positive on $B$. So the bounded connected component of the fixed point set of $(\lambda, \nu) \mapsto (\bar{\lambda}, \bar{\nu})$ and of $(\lambda, \nu) \mapsto (\bar{\lambda}, -\bar{\nu})$ covers $A$ and $B$, respectively. Therefore there exist unique $\beta_1 \in A$ and $\beta_2 \in B$ so that $\int_A \frac{\lambda - \beta_1}{\nu} \, d\lambda = \int_B \frac{i(\lambda - \beta_2)}{\nu} \, d\lambda = 0$. Then $\int_B \frac{\lambda - \beta_1}{\nu} \, d\lambda$ and $\int_A \frac{i(\lambda - \beta_2)}{\nu} \, d\lambda$ are purely imaginary. Thus $f(\lambda) = -\frac{i(\lambda - \beta_2)}{\lambda - \beta_1}$. Hence $f(\lambda) \in \mathbb{C} \setminus \mathbb{R}$ at any zero $\lambda$ of $\lambda a(\lambda)$. □

**Lemma 3.3.** The intersection of $\mathcal{M}_{1,=0}$ with the boundary of the domain in Equation (3.1) is equal to $\{e^{i\vartheta}, -e^{-i\vartheta}\}/\sim$ for some $\vartheta \in (\frac{\pi}{3}, \frac{\pi}{2})$. 

Proof. We first show that the boundary of the domain in Equation (3.1) consists of those $\tau(a_1, a_2)$ where $a_1, a_2 \in \mathbb{R}$ and $a(\lambda) := \lambda^2 + a_1 \lambda + a_2$ has a pair of complex conjugate non-real zeros. In fact, we obtain a pair of anti-holomorphic involutions on $\mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z} \tau)$ by $z \mapsto \pm (\bar{z} - 1)$ when $\tau = \frac{1}{2} + \ii t$ with $t \geq \frac{\sqrt{2}}{2}$, and by $z \mapsto \pm \tau \bar{z}$ when $\tau = e^{i\varphi}$ with $\varphi \in [\frac{\pi}{4}, \frac{3\pi}{4}]$. In either case, the fixed point sets of these involutions are non-empty and connected. As in the proof of Lemma 3.2 we implement these involutions by $(\lambda, \nu) \mapsto (\lambda, \pm \nu)$. Then it follows that $a_1, a_2 \in \mathbb{R}$ and that $a(\lambda)$ has non-real complex conjugate roots. We note that $\tau = 1$ belongs both to the situation considered here and to that of Lemma 3.2 and is represented in both situations by different polynomials $a(\lambda) = \lambda^2 + c$, where $c > 0$ here (giving complex conjugate roots for $a$) whereas $c < 0$ in Lemma 3.2 (giving two real roots for $a$). The first situation is transformed into the second by $\lambda \mapsto 1\lambda$.

Suppose that $\tau(a_1, a_2) \in \mathcal{M}_{1=0}$ holds for such $(a_1, a_2)$. Because of $f(0) = 0$, there exists a real linear combination of $b_1$ and $b_2$ which vanishes at $\lambda = 0$, and hence there exists $\psi \in \mathbb{R}$ so that $e^{\psi} \lambda \dd \lambda$ has purely imaginary periods. The involution $(\lambda, \nu) \mapsto (\lambda, \bar{\nu})$ maps this differential form onto a multiple of itself, which shows that $e^{\psi} \lambda$ is either real or purely imaginary. For imaginary $e^{\psi}$, we replace $\lambda$ by $-\lambda$ and thus $(a_1, a_2)$ by $(-a_1, a_2)$ resp. $\tau(a_1, a_2)$ by $-\tau(a_1, a_2)$. Therefore we may suppose that $e^{\psi} = 1$. This shows that for a spectral curve $\Sigma$ whose conformal class is on the boundary of $\mathcal{M}_{1=0}$, we have $\Im(f(0)) = 0$ if and only if $\frac{1}{\nu} \dd \lambda$ has purely imaginary periods for one of the two values of $\tau$ on the boundary of the domain in Equation (3.1) which represent the conformal class of $\Sigma$.

We now construct a Whitham deformation (see, for example, [C-S2] Section 4) on the boundary of the domain in Equation (3.1). We thus consider spectral curves defined by $\nu^2 = \lambda a(\lambda)$ where the quadratic polynomial $a(\lambda) = (\lambda - \alpha)(\lambda - \bar{\alpha}) = \lambda^2 - 2 \Re(\alpha) \lambda + 1$ has the complex-conjugate zeros $\alpha = e^{2\pi} \in \mathbb{S}^1 \setminus \mathbb{R}$ and $\bar{\alpha}$. We use the deformation to calculate the polynomials $b(\lambda) = \beta_0(\lambda - \beta_1)$ with $\beta_0, \beta_1 \in \mathbb{R}$ so that $\Theta := \frac{b(\lambda)}{\nu} \dd \lambda$ has purely imaginary periods, see Lemma 2.6. Let $q$ denote the anti-derivative of $\Theta$ as a multi-valued meromorphic function on $\Sigma$.

To describe the deformation, we denote its parameter by $t$, and for any function $f$ depending on $t$, we let $\dot{f}$ be the derivative with respect to $t$. Along the deformation, the periods of $\Theta$ will be constant, and $\dot{q}$ will be a global, meromorphic function on $\Sigma$ with $\sigma \dot{q} = -\dot{q}$, which has at most first order poles at the non-zero roots of $a$ and no other singularities. Therefore $\dot{q}$ is of the form $\frac{\lambda c(\lambda)}{\nu}$ with the first order polynomial $c(\lambda) = c_0 \lambda + c_1$ and $c_0, c_1 \in \mathbb{R}$. Note that $\nu' = \frac{a + \lambda a'}{2\nu}$ and $\dot{\nu} = \frac{\dot{a}}{\nu}$ holds. Thus we have

\[
(3.2) \quad \frac{d}{d\lambda} \frac{\lambda c(\lambda)}{\nu} = \frac{d}{dt} \frac{b(\lambda)}{\nu} \iff 2a\dot{b} - \dot{a}b = (a - \lambda a')c + 2\lambda ac'.
\]

We shall see that the choice

\[
(3.3) \quad c_0 := \beta_1 - \cos(\varphi)
\]

yields a vector field without zeros or poles.

By comparing the coefficients of highest order in $\lambda$ in Equation (3.2), we then obtain

\[
(3.4) \quad \beta_0 = \frac{1}{2} c_0 = \frac{\beta_1 - \cos(\varphi)}{2}.
\]

Moreover, by inserting $\lambda = \alpha$ into Equation (3.2) we get

\[
(3.5) \quad \dot{\alpha} = \frac{c(\alpha)}{b(\alpha)} \alpha.
\]

Because we want our deformation to move $\alpha$ on $\mathbb{S}^1$, we require $\dot{\alpha}/\alpha$ to be purely imaginary. From Equation (3.5) we thus obtain the condition (where we recall $\alpha \in \mathbb{S}^1$ and $\beta_k, c_k \in \mathbb{R}$)

\[
0 = \Re \left( \frac{c(\alpha)}{b(\alpha)} \right) = \Re \left( \frac{c_0 \alpha + c_1}{\beta_0(\alpha - \beta_1)} \right) = \Re \left( \frac{(c_0 \alpha + c_1)(\alpha - \beta_1)}{\beta_0|\alpha - \beta_1|^2} \right) = \frac{(1 - \beta_1 \cos(\varphi))c_0 + (\cos(\varphi) - \beta_1)c_1}{\beta_0|\alpha - \beta_1|^2}.
\]
With our choice \( c_0 \) of \( c_0 \), this condition is satisfied if and only if
\[
(3.6) \quad c_1 = 1 - \beta_1 \cos(\varphi) .
\]
By inserting Equations (3.5) and (3.6) into Equation (3.5) we obtain
\[
(3.7) \quad \dot{\alpha} = \frac{\alpha (\beta_1 - \bar{\alpha})}{\beta_0 a(\beta_1)}((\beta_1 - \cos(\varphi)) \alpha + (1 - \beta_1 \cos(\varphi))) .
\]
Finally, by inserting \( \lambda = \beta_1 \) into Equation (3.2) we obtain
\[
(3.8) \quad 2 a(\beta_1) \dot{b}(\beta_1) = (a(\beta_1) - \beta_1 a'(\beta_1))\beta_1 + 2\beta_1 a(\beta_1) \dot{c}(\beta_1) .
\]
We now note that \( \dot{b}(\beta_1) = -\beta_0 \beta_1 \), that \( a(\beta_1) - \beta_1 a'(\beta_1) = 1 - \beta_1^2 \), that (by Equations (3.5) and (3.6))
\[
c(\beta_1) = c_0 \beta_1 + c_1 = \beta_1^2 - 2 \cos(\varphi) \beta_1 + 1 = a(\beta_1) ,
\]
and that \( c'(\beta_1) = c_0 - \beta_1 - \cos(\varphi) \) holds. By inserting these formulae into Equation (3.8) and dividing by \( a(\beta_1) \), we obtain
\[
-2 \beta_0 \dot{\beta}_1 = (1 - \beta_1^2) + 2 \beta_1 (1 - \cos^2(\varphi)) = 1 + \beta_1^2 - 2 \beta_1 \cos(\varphi) = a(\beta_1)
\]
and thus
\[
(3.9) \quad \dot{\beta}_1 = -\frac{a(\beta_1)}{2\beta_0} .
\]
Let \( X \) be the vector field defined by Equations (3.7), (3.4) and (3.9) on the domain \( D := \{ \alpha = e^{i \varphi} \in S^1 \setminus \mathbb{R}, \beta_0 \in \mathbb{R}^+, \beta_1 \in \mathbb{R} \} \). Note that on \( D \), \( \beta_0 \neq 0 \) and \( a(\beta_1) \neq 0 \) holds, and therefore \( X \) has neither zeros nor poles on \( D \).

We now consider a maximal integral curve \( \gamma = (\alpha, \beta_0, \beta_1) \) of the vector field \( X \) in \( D \). By construction, \( \alpha \) corresponds to points on the boundary of the domain in Equation (3.1). We now show that \( \alpha \) covers all of this boundary. The expression \( (\beta_1 - \cos(\varphi)) \alpha + (1 - \beta_1 \cos(\varphi)) \) with \( \beta_1 \in \mathbb{R} \) can vanish only for real \( \alpha = e^{i \varphi} \), and therefore \( \dot{\alpha} \) is non-zero by Equation (3.7). This shows that the curve on the boundary of \( \mathcal{M}_1 \) that corresponds to \( \alpha \) is nowhere stationary. At the ends of the domain of \( \gamma \), \( \gamma \) leaves every compact subset of \( D \), and therefore one of the following cases occurs: \( \alpha \) becomes real, or \( \beta_0 \) or \( \beta_1 \) goes to infinity, or \( \beta_0 \) goes to zero. In any event, because \( \alpha \) is contained in the compact set \( S^1 \), there exists a subsequence of \( \gamma \), so that the corresponding values of \( \alpha \) converge to some \( \alpha_s \in S^1 \). If \( \alpha_s \neq \pm 1 \), it follows from the fact that the periods of \( \Theta \) are constant along the Whitham flow that there exist unique \( \beta_{0,s}, \beta_{1,s} \in \mathbb{R} \) so that the differential form \( \frac{\beta_{0,s}(\lambda - \beta_{1,s})}{\nu} \) on the spectral curve corresponding to \( \alpha_s \) has these periods. Because at least one of the periods is non-zero, we have \( \beta_{0,s} \neq 0 \), and for reasons of continuity, \( \beta_k \) converges to \( \beta_{k,s} \) for \( k \in \{0,1\} \). Therefore \( \gamma \) approaches the boundary of \( D \) in such a way that \( \alpha \to \pm 1 \) holds (where the two ends of the interval correspond to different signs here). This shows that \( \gamma \) covers the entire boundary of \( \mathcal{M}_1 \).

Moreover, \( \beta_1 \to \alpha \) as \( \alpha \to \pm 1 \): Consider any sequence \( (t_n) \) with \( \alpha(t_n) \to 1 \). Then consider the differential \( \tilde{\Theta}_n = \left| \beta_0(t_n) \right|^{-1} \tilde{\Theta}(t_n) \) and the harmonic map \( \tilde{h}_n \) with \( d\tilde{h}_n = \text{Re}(\tilde{\Theta}_n) \). For \( n \to \infty \), \( \tilde{h}_n \) remains bounded on any compact subset of \( \{ (\lambda, \nu) \in \Sigma : |\lambda| \neq 1, \infty \} \). By the Maximum Principle it follows that the limit of \( \tilde{h}_n \) is also bounded at \( \lambda = 1 \), and hence converges to a harmonic function on the curve \( \Sigma^0 \) corresponding to \( \alpha = 1 \). Therefore \( \beta_1(t_n) \to 1 \). The case \( \alpha \to -1 \) is analogous.

It follows from Equation (3.9) that \( \dot{\beta}_1 \) is non-zero. Therefore, along the maximal integral curve, as \( \alpha \) passes from 1 to \( -1 \) along \( S^1 \), \( \beta_1 \) passes strictly monotonically from 1 to \( -1 \) in the open interval \( (-1,1) \). In particular, \( |\beta_1| < 1 \) holds on the entire integral curve, and there is exactly one zero of \( \beta_1 \).
It remains to show that the $\alpha$ for which $\beta_1$ is zero corresponds to a $\tau = e^{i\vartheta}$ with $\vartheta \in (\frac{\pi}{6}, \frac{\pi}{2})$. It will turn out that this corresponds to $\varphi \in (\frac{\pi}{6}, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{5\pi}{6})$. We therefore determine the sign of $\beta_1$ for $\varphi \in (\frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6})$. For this purpose we suppose that the maximal integral curve $\gamma$ has been chosen such that $\beta_0 = 1$ for $\varphi = \frac{\pi}{2}$ (corresponding to $\tau = i$).

For $\varphi = \frac{\pi}{2}$ we have $\alpha(\lambda) = \lambda^2 + 1$ and therefore $\nu^2 = \lambda(\lambda^2 + 1) = \lambda^3(1 + \frac{1}{\lambda^2})$. Because the periods of $\Theta = \frac{\lambda - \beta_1}{\nu} \mathrm{d}\lambda$ are purely imaginary, there exists a unique harmonic function $g$ which is anti-symmetric with respect to the hyperelliptic involution and whose differential is $\mathrm{Re}(\Theta)$. The derivative of $g(\lambda, \nu) - 2\mathrm{Re}(\lambda^{1/2})$ (where the sign of $\lambda^{1/2}$ is chosen to match the sign of $\nu$) is a harmonic function at $\lambda = \infty$ which is anti-symmetric with respect to the hyperelliptic involution, and therefore vanishes at $\lambda = \infty$.

The cycle $A$ is defined by moving from $\lambda = 0$ to some $\lambda = \lambda_0 > 0$ along the real line in the leaf of $\Sigma$ where $\nu > 0$, then moving anti-clockwise along the circle $C_{\lambda_0}: |\lambda| = |\lambda_0|$ into the other leaf of $\Sigma$, and then moving back to $\lambda = 0$ along the real line in the leaf of $\Sigma$ where $\nu < 0$. Because $\Theta$ is real for $\lambda > 0$, we have

$$0 = \int_A \mathrm{Re}(\Theta) = 2 \int_0^{\lambda_0} \Theta + \int_{C_{\lambda_0}} \mathrm{Re}(\Theta)$$

and therefore

$$0 = \lim_{\lambda_0 \to \infty} \left( 2 \int_0^{\lambda_0} \Theta - 4 \lambda_0^{1/2} \right).$$

On the other hand, we have for $\lambda_0 > 0$

$$\int_0^{\lambda_0} \Theta - 2\lambda_0^{1/2} = \int_0^{\lambda_0} \frac{\lambda}{\nu} \mathrm{d}\lambda - 2\lambda_0^{1/2} - \beta_1 \int_0^{\lambda_0} \frac{1}{\nu} \mathrm{d}\lambda$$

and therefore

$$\beta_1 = \lim_{\lambda_0 \to \infty} \left( \frac{\int_0^{\lambda_0} \frac{\lambda}{\nu} \mathrm{d}\lambda - 2\lambda_0^{1/2}}{\int_0^{\lambda_0} \frac{1}{\nu} \mathrm{d}\lambda} \right) < 0.$$

$\beta_1$ strictly decreases as $\varphi$ increases. It follows that at $\varphi = \frac{5\pi}{6} > \frac{\pi}{2}$ (corresponding to $\tau = \frac{\pi}{2}$), we have $-1 < \beta_1(\varphi = \frac{5\pi}{6}) < 0$. Let $C_1$ be the circle in the $\lambda$-plane which passes through the three branch points $0, \alpha = e^{5\pi i/6}$ and $\alpha$. Its center is at $\lambda = -\frac{1}{\sqrt{3}}$. Because of $2\lambda_0 < -1 < \beta_1(\varphi = \frac{5\pi}{6}) < 0$, the circle $C_2$ with center $\lambda_0$ which passes through $\beta_1$ is contained in the interior of $C_1$. Let $\beta_2(\varphi = \frac{5\pi}{6}) < 0$ be the second intersection of $C_2$ with the real line. We claim that the differential $\frac{i(\lambda - \beta_2(\varphi = \frac{5\pi}{6}))}{\nu} \mathrm{d}\lambda$ has purely imaginary periods. Indeed, with $\tilde{\lambda} = \lambda - \lambda_0$, the spectral curve corresponding to $\alpha = e^{5\pi i/6}$ is given by the equation $\nu^2 = \tilde{\lambda}^3 - 1$ and has the automorphism $\rho: (\tilde{\lambda}, \nu) \mapsto (q\tilde{\lambda}, q\nu)$ of order 3, where $q = e^{2\pi i/3}$. Note that in these coordinates, $\beta_2 = -\beta_1$ holds. Because the differential $\Theta = \frac{\lambda - \beta_2}{\nu}$ has purely imaginary periods, the differential

$$\frac{1}{\sqrt{3}}((\rho^2)^*\Theta - \rho^*\Theta) = \frac{1}{\sqrt{3}} \left( \frac{q^2 \tilde{\lambda} - \beta_1}{\nu} q^2 \mathrm{d}\tilde{\lambda} - \frac{q \tilde{\lambda} - \beta_1}{\nu} q \mathrm{d}\tilde{\lambda} \right) = \frac{i(\tilde{\lambda} + \beta_1)}{\nu} \mathrm{d}\tilde{\lambda} = \frac{i(\tilde{\lambda} - \beta_2)}{\nu} \mathrm{d}\tilde{\lambda}$$

also has purely imaginary periods.

We now consider besides the spectral curve for $\varphi = \frac{5\pi}{6}$ also the spectral curve for $\varphi = \frac{\pi}{2}$ (the other representant of $\tau = \frac{\pi}{2}$). $\chi: (\lambda, \nu) \mapsto (-\lambda, i\nu)$ is a biholomorphic map between these two spectral curves. Because the differential $\frac{i(\lambda - \beta_2(\varphi = \frac{5\pi}{6}))}{\nu} \mathrm{d}\lambda$ has purely imaginary periods, the differential

$$\chi^* \frac{i(\lambda - \beta_2(\varphi = \frac{5\pi}{6}))}{\nu} \mathrm{d}\lambda = \frac{\lambda + \beta_2(\varphi = \frac{5\pi}{6})}{\nu} \mathrm{d}\lambda$$
also has purely imaginary periods. This shows that \( \beta_1(\varphi = \frac{\pi}{6}) = -\beta_2(\varphi = \frac{5\pi}{6}) > 0 \) holds.

Because of \( \beta_1(\varphi = \frac{\pi}{6}) > 0 \), \( \beta_1(\varphi = \frac{5\pi}{6}) < 0 \), there exists a \( \varphi \in \left( \frac{\pi}{6}, \frac{\pi}{2} \right) \) with \( \beta_1(\varphi) = 0 \). The corresponding conformal class \( \tau \) is in \( \mathcal{M}_{1,=0} \).

Lemma 3.4. \( \mathcal{M}_{1,=0} = \partial \mathcal{M}_{1,>0} \cap \partial \mathcal{M}_{1,<0} \) holds. Moreover \( \mathcal{M}_{1,=0} \) is a smooth, closed, 1-dimensional submanifold of \( \mathcal{M}_1 \).

Proof. To prove this Lemma, we will use the implicit function theorem; here we will use another Whitham flow which moves transversally to \( \mathcal{M}_{1,=0} \) to show that \( \tau \mapsto \text{Im}(f(0)) \) is submersive on \( \mathcal{M}_{1,=0} \). Note that the two orbifold points \( \tau = e^{i}\pi/2 \) and \( \tau = e^{i3\pi/2} \) of \( \mathcal{M}_1 \) are not contained in \( \mathcal{M}_{1,=0} \) by Lemma 3.3.

To describe this flow, we consider spectral curves given by the equation \( \nu^2 = \lambda (\lambda - 1) (\lambda - \alpha) \) with \( \alpha \in \mathbb{C} \setminus \{0, 1\} \); we denote the corresponding conformal class by \( \tau(\alpha) = \tau(a_1 = -(\alpha + 1), a_2 = \alpha) \in \mathcal{M}_1 \). Moreover we consider a basis \( b_k(\lambda) = \delta_k (\lambda - \beta_k) \) (where \( k \in \{1, 2\} \)) of \( \mathcal{B}_a \). We will construct the flow in such a way that all periods of the \( \Theta_k := \Theta_{b_k} \) are preserved. Again we let \( \dot{q}_k \) be the anti-derivative of \( \Theta_k \) as a multi-valued meromorphic function on \( \Sigma \). Then \( \dot{q}_k \) will be a global meromorphic function on \( \Sigma \). Because \( \dot{q}_k \) vanishes at the branch points \( \lambda = 0 \) and \( \lambda = 1 \) which are fixed under the deformation, we have \( \dot{q}_k = \frac{\gamma_k \lambda (\lambda - 1)}{\nu} \) with some \( \gamma_k \in \mathbb{C} \). Then the integrability condition

\[
\frac{d}{dt} b_k(\lambda) = \frac{d}{d\lambda} q_k
\]

yields the equation

\[
2 (\lambda - \alpha) \dot{b}_k + b_k \dot{\alpha} = \gamma_k \cdot (\lambda^2 - 2\alpha\lambda + \alpha).
\]

By inserting \( \lambda = \alpha \) in this equation, we obtain

\[
\dot{\alpha} = -\frac{\gamma_1 \alpha (\alpha - 1)}{b_1(\alpha)} = -\frac{\gamma_2 \alpha (\alpha - 1)}{b_2(\alpha)}
\]

and therefore

\[
\gamma_1 b_2 - \gamma_2 b_1 = Q \cdot (\lambda - \alpha)
\]
with \( Q \in \mathbb{C} \). On the other hand, for any given value \( Q \in \mathbb{C} \), Equation (3.12) gives unique values for \( \gamma_1, \gamma_2 \in \mathbb{C} \) because the roots of \( b_1 \) and \( b_2 \) are different, and then Equation (3.11) gives a unique value of \( \alpha \) and Equation (3.10) gives unique values for the coefficients of \( b_k \).

By inserting \( \lambda = 0 \) into Equation (3.10) we obtain

\[
(3.13) \quad b_k(0) = \frac{1}{2} \left( \frac{b_k(0) \dot{\alpha}}{\alpha} - \gamma_k \right).
\]

We now suppose that \( \tau(\alpha) \in M_{1,=0} \) holds. This is equivalent to a real linear combination of \( b_1 \) and \( b_2 \) vanishing at \( \lambda = 0 \), or alternatively to \( b_1(0) \) and \( b_2(0) \) being linearly dependent over \( \mathbb{R} \).

In this situation the flow infinitesimally stays in \( M_{1,=0} \) and the flow is tangential to \( x, y \) curves of the flow pass through the unique point \( Q \).

Lemma 3.5.

Proof. We will use the same Whitham flow as in the proof of Lemma 3.4 but now choose \( Q \) so that the flow is tangential to \( M_{1,=0} \). Note that we have shown in the proof of Lemma 3.4 that the two lines \( \mathbb{R} b_2^2(0)/\alpha \) are equal unless \( b_k(0) = 0 \) holds for either \( k \in \{1, 2\} \). Hence for all \( \alpha \in \mathbb{C} \setminus \{1, 0\} \) with \( \tau(\alpha) \in M_{1,=0} \) there exists a unique \( Q \in \mathbb{R} b_2^2(0)/\alpha \cup \mathbb{R} b_2^2(0)/\alpha \) with \( |Q| = 1 \) so that \( \text{Im}(\lambda = 0) \) increases along the transversal flow induced by \( iQ \). We will then show that all maximal integral curves of the flow pass through the unique point \( e^{i\theta} \in M_{1,=0} \) described in Lemma 3.3 and that \( \tau \) is unbounded at both ends of the maximal integral curve.

For every \( \tau(\alpha) \in M_{1,=0} \), a real linear combination \( b = x \cdot b_1 + y \cdot b_2 \) vanishes at \( \lambda = 0 \) for a unique \((x, y) \in \mathbb{R}P^1\). The corresponding \( \Theta_b \) defines a unique harmonic function \( h \) which satisfies \( \text{d} h = \text{Re}(\Theta_b) \) and \( \sigma^* h = -h \), see Lemma 2.7. The projection of \( \{ h = 0 \} \) to the \( \lambda \)-plane is the union of three paths from \( \lambda = 0 \) to each of the other branch points \( \lambda = 1, \lambda = \alpha \) and \( \lambda = \infty \), which intersect only at \( \lambda = 0 \). In particular, both the zero set of \( h \) and its projection to the \( \lambda \)-plane are connected. The three paths are projections of non-trivial smooth cycles \( \Gamma_1, \Gamma_\alpha \) and \( \Gamma_\infty \) on \( \Sigma \). Due to the anti-symmetry of \( \Theta_b \) with respect to \( \sigma \) the purely imaginary values of the integrals \( \int_{\Gamma_1} \Theta_b \) and \( \int_{\Gamma_\alpha} \Theta_b \) are twice the values of the corresponding integrals from \( \lambda = 0 \) to \( \lambda = 1 \) and \( \lambda = \alpha \), respectively. Since \( \Theta_b \) has no roots along these paths these integrals do not vanish. We now show that the quantity

\[
\kappa := \frac{\int_{\Gamma_\alpha} \Theta_b}{\int_{\Gamma_1} \Theta_b}
\]

is strictly monotonic along the Whitham flow. Equation (3.13) and \( x \cdot b_1(0) + y \cdot b_2(0) = 0 \) implies

\[
0 = \frac{d}{dt} \left( x \cdot b_1(0) + y \cdot b_2(0) \right) = \dot{x} \cdot b_1(0) + \dot{y} \cdot b_2(0) - \frac{1}{2} (x \gamma_1 + y \gamma_2)
\]

For \( Q \alpha \neq 0 \) Equation (3.12) shows that \( (\gamma_1, \gamma_2) \) and \( (b_1(0), b_2(0)) \) are linearly independent. Therefore \( x \gamma_1 + y \gamma_2 \neq 0 \) and the vectors \((\dot{x}, \dot{y})\) and \((x, y)\) are \( \mathbb{R} \)-linearly independent. Since both cycles
Γ₁ and Γₐ have intersection number ±1 they generate \( H₁(Σ, \mathbb{Z}) \). The vectors \( (\int_{Γ₁} Θ₁, \int_{Γ₀} Θ₁) \) and \( (\int_{Γ₁} Θ₂, \int_{Γ₀} Θ₂) \) are linearly independent and preserved along the integral curves. This implies that

\[
\frac{d}{dt} \left( \int_{Γ₁} Θ₁, \int_{Γ₀} Θ₁ \right) = \frac{d}{dt} \left( x \cdot \int_{Γ₁} Θ₁ + y \cdot \int_{Γ₀} Θ₂, x \cdot \int_{Γ₁} Θ₁ + y \cdot \int_{Γ₀} Θ₂ \right)
\]

\[
= \dot{x} \cdot \left( \int_{Γ₁} Θ₁, \int_{Γ₀} Θ₁ \right) + \dot{y} \cdot \left( \int_{Γ₁} Θ₂, \int_{Γ₀} Θ₂ \right)
\]

and \( (\int_{Γ₁} Θ₁, \int_{Γ₀} Θ₁) = x \cdot (\int_{Γ₁} Θ₁, \int_{Γ₀} Θ₁) + y \cdot (\int_{Γ₁} Θ₂, \int_{Γ₀} Θ₂) \) are linearly independent. \( 〈κ \rangle \) is zero if and only if

\[
\left( \frac{d}{dt} \int_{Γ₀} Θ₁ \right) \int_{Γ₁} Θ₁ - \int_{Γ₀} Θ₁ \left( \frac{d}{dt} \int_{Γ₁} Θ₁ \right) = 0
\]

holds; by expanding that expression, one sees that \( 〈κ \rangle \neq 0 \).

The argument in the proof of Lemma 3.3 that \( 〈κ \rangle = ±1 \) can be applied here. Therefore at the ends of the maximal integral curve \( α \rightarrow 0 \) we have \( 〈κ \rangle \rightarrow 0 \). Next we show that \( 〈κ \rangle \) cannot converge to 1. For any sequence \( αₙ \) with \( τ(αₙ) \in \mathcal{M}_0 \) with limit \( α \rightarrow 1 \) there exists a sequence \( δₙ = βₙ λ \) such that \( Θ_{δₙ} \) has purely imaginary periods. If \( |βₙ| = 1 \), a subsequence of the corresponding harmonic functions \( hₙ \) with \( dhₙ = Re(Θ_{δₙ}) \) and \( σ^*hₙ = -hₙ \) converges to a harmonic function with a pole at \( λ = 1 \). This contradicts the Maximum Principle in a similar way as in the proof of Lemma 3.3.

Now we show that at the end of a maximal integral curve where \( α \rightarrow 0 \), we have \( 〈κ \rangle \rightarrow 0 \). In this situation, the roots of \( b₁ \) and \( b₂ \) go to zero by an analogous argument as the one showing \( βₙ \rightarrow α \) in the proof of Lemma 3.3. After renormalising \( b \), we see that \( 〈κ \rangle \rightarrow 0 \). For \( α \rightarrow ∞ \) we may replace \( λ \) by \( 〈λ \rangle = α⁻¹ λ \) such that the two branchpoints \( λ = 0 \) and \( λ = 1 \) are replaced by the coalescing branch points \( 〈λ \rangle = 0 \) and \( 〈λ \rangle = α⁻¹ 〈\lambda \rangle \). Then \( 〈κ \rangle \rightarrow ∞ \) analogously as before. Therefore every maximal integral curve yields a path between \( 〈κ \rangle = 0 \) and \( 〈κ \rangle = ∞ \) and a path between \( 〈α \rangle = 0 \) and \( 〈α \rangle = ∞ \).

In particular, every connected component of \( \mathcal{M}_0 \) contains an element with \( 〈κ \rangle = 1 \) and because at the ends of the integral curve two branchpoints coalesce, each connected component has two unbounded ends.

Finally we show that for \( 〈κ \rangle = 1 \), we have \( |α| = 1 \) and that the corresponding elliptic curve \( Σ \) then is endowed with an anti-holomorphic involution \( (λ, ν) \mapsto (αλ, α³/₂ν) \) interchanging \( (λ, ν) = (1, 0) \) and \( (λ, ν) = (α, 0) \). We remove from \( Σ \) the smooth cycles \( Γ₁ \) and \( Γₐ \). The resulting Riemann surface \( Σ \setminus (Γ₁ ∪ Γₐ) \) is simply connected and has four boundary cycles. \( σ \) lifts to a unique holomorphic involution on \( Σ \setminus (Γ₁ ∪ Γₐ) \) with the unique fixed point \( λ = ∞ \). There exists a unique meromorphic function \( q \) on \( Σ \setminus (Γ₁ ∪ Γₐ) \) with \( dq = Θₜ \) and \( σ^*q = -q \). This function is purely imaginary on the boundary. The cycle \( Γ∞ \) is a path in \( Σ \setminus (Γ₁ ∪ Γₐ) \), which connects the intersection point of two boundary components with another intersection point of two boundary components. The point \( λ = ∞ \) divides \( Γ∞ \) into two halves, so that the range of \( q \) on one half of \( Γ∞ \) is the negative of the range of \( q \) on the other half. \( Σ \setminus (Γ₁ ∪ Γₐ ∪ Γ∞) \) decomposes into two triangles, each of which has three boundary components \( Γ₁ \), \( Γₐ \) and \( Γ∞ \). Because \( q \) takes purely imaginary values only on the boundary, the imaginary part of \( q \) is strictly monotonic along the boundary components of both triangles with the exception of the first order pole \( λ = ∞ \) of \( q \). Indeed this statement requires proof only at the roots of \( Θₜ \), which correspond to the vertices of both triangles. At such a point, \( q \) is represented by \( z \mapsto z^3 \) in a suitable local coordinate \( z \), and therefore the two edges of the triangle are locally near that point mapped onto an interval on the real line. This shows monotonicity at that point. For \( 〈κ \rangle = 1 \), on each of the two triangles the range of \( q \) on \( Γₐ \) is the negative of the range of \( q \) on \( Γ₁ \). By Schwarz’s Reflection Principle, the map \( q \mapsto −q \) induces a unique anti-holomorphic involution of \( Σ \setminus (Γ₁ ∪ Γₐ) \), with the fixed point set \( Γ∞ \). 
This involution induces an involution of $\Sigma$ which interchanges $(\lambda, \nu) = (1, 0)$ and $(\lambda, \nu) = (\alpha, 0)$ and fixes $(\lambda, \nu) = (0, 0)$ and $(\lambda, \nu) = (\infty, \infty)$. This involution acts on $\lambda$ as $\lambda \mapsto \alpha \bar{\lambda}$. Because this map is an involution, we have $|\alpha| = 1$. In particular, for $\kappa = 1$ the conformal class $\tau(\alpha)$ is the unique $\tau = e^{i\vartheta} \in \mathcal{M}_{1,=0}$ described in Lemma 3.3. Therefore $\mathcal{M}_{1,=0}$ is connected. Since every open proper subset of $\mathcal{M}_1$ has non-empty boundary, the two open subsets $\mathcal{M}_{1,>0}$ and $\mathcal{M}_{1,<0}$ are also connected, and because of $\mathcal{M}_{1,=0} = \partial \mathcal{M}_{1,>0} \cap \partial \mathcal{M}_{1,<0}$ (see Lemma 3.4) the sets $\mathcal{M}_{1,>0}$ and $\mathcal{M}_{1,<0}$ are also unbounded. $\square$

4. The Wente family

In the present section we study what we call the Wente family

$$\mathcal{W} := \{ a \in S^2_1 \mid a(\bar{\lambda}) = a(\lambda) \}$$

in $S^2_1$. A cmc torus in $\mathbb{R}^3$ of spectral genus 2 is called a Wente torus [Ab] if the corresponding polynomial $a \in S^2$ describing the spectral curve is a member of $\mathcal{W}$. The following theorem in particular again proves the existence of Wente tori. For any $a \in S^2_1$ we define a basis of the homology of the corresponding spectral curve $\Sigma$ as follows: We label the zeros of $a$ inside the unit circle as $\alpha_1, \alpha_2 \in B(0, 1)$. Then we let $A_k$ be the cycle that encircles $\alpha_k$ and $\bar{\alpha}_k^{-1}$ once, and no other branch points of $\Sigma$. We have $\rho(A_k) \simeq -A_k$ because these cycles intersect the fixed point set $S^1$ of the anti-holomorphic involution $\rho$ of Equation (1.2) twice. For any $b \in B_a$ we thus have by definition of $B_a$

$$\int_{A_k} \Theta_b = -\int_{A_k} \Theta_b = -\int_{A_k} \Theta_b = \int_{A_k} \rho^* \Theta_b = \int_{\rho(A_k)} \Theta_b = -\int_{A_k} \Theta_b$$

and hence $\int_{A_k} \Theta_b = 0$.

Moreover let $B_1$ be the cycle that circles the branch points $\lambda = 0$ and $\alpha_1$, and let $B_2$ be the cycle that circles the branch points $\bar{\alpha}_2^{-1}$ and $\lambda = \infty$. Then the $A$-cycles and the $B$-cycles together form a canonical basis of the homology of $\Sigma$. Because the anti-holomorphic involution $\rho$ reverses both the orientation and the intersection number of cycles, it follows that $\rho(B_k)$ is homological to $B_k$ up to a linear combination of $A$-cycles.
For \( a \in W \) we choose \( b \in B_a \) so that \( \int_{B_1} \Theta_b = \int_{B_2} \Theta_b = 2\pi \) holds. Then there exists a global, single-valued meromorphic function \( \mu \) on \( \Sigma \) so that \( d \ln \mu = \Theta_b \) holds. We choose the integration in \( \mu \) such that \( \sigma^* \mu = \mu^{-1} \) holds; by this choice \( \mu \) becomes unique up to an integer multiple of \( \pi \). In this way we obtain the map

\[
\Phi : W \to \mathbb{R}, \quad a \mapsto -\frac{1}{2} \ln \mu|_{\lambda=1}.
\]

Via the Whitham flow adapted to \( W \) it is easy to see that \( \Phi \) is a local parameterisation of \( W \), compare Lemma 4.5 below.

**Theorem 4.1.** \( W \) is the image of a maximal integral curve of a suitable Whitham flow on \( S^2_1 \), in particular it is a connected, 1-dimensional submanifold of \( S^2_1 \). \( \Phi \) is a diffeomorphism that maps \( W \) bijectively onto an open interval of length \( \pi \).

\( W \) is given explicitly as

\[
W = \{ \lambda^4 - (\frac{4}{5}\varphi + 4)\lambda^3 + (\varphi^2 + \frac{8}{5}\varphi + 6)\lambda^2 - (\frac{4}{5}\varphi + 4)\lambda + 1 \mid \varphi > 0 \}
\]

in terms of a transcendental constant \( \delta_0 \) with \(-8 < \delta_0 < 8 \). The polynomials \( a \in W \) have no zeros on \( \mathbb{R} \cup S^1 \).

**Remark 4.2.** The number \( \delta_0 \) can be described in terms of elliptic functions. Numerically, one obtains \( \delta_0 \approx 5.2178 \ldots \).

The objective of the remainder of the section is the proof of this theorem. The proof is composed of several lemmata. We begin with a general construction of the Whitham flows on \( S^g_1 \).

**Lemma 4.3.** Let \( g \geq 1 \), \( a \in S^g_1 \) with \( \deg \gcd(B_a) = 1 \) and \((b_1, b_2)\) a basis of \( B_a \). Then the Whitham equations have for given values \((c_1(1),c_2(1))\in \mathbb{R}^2\)

\[
2a b_k \dot{a} - a b_k = 2i\lambda a c_k' - ic_k (a + \lambda a') \quad \text{for} \ k = 1,2
\]

\[
b_2 c_1 - b_1 c_2 = Q a
\]

a unique solution \((c_1,c_2,Q,\dot{a},b_1,b_2)\in P^{g+1}_R \times P^{g+1}_R \times P^2_R \times T_a S^g_1 \times P^{g+1}_R \times P^{g+1}_R\).
For every root \(a\), we then have \(c_2(1)\) conjugates, respectively. So for given \((\alpha, a, c_1(1), c_2(1))\) by Equation (4.5) the transformation replaces both sides of the first two equations by the negatives of their complex conjugates and by (4.8), (4.9) both sides of the third equations by their complex conjugates, respectively. So for given \((\alpha, a, c_1(1), c_2(1))\) \(\in \mathbb{R}^2\) this transformation preserves the unique solution \((c_1, c_2, Q)\) of Equations (4.4)-(4.7), which shows that \(c_k(1)\) is the prescribed value and \(c_k'(1)\) vanishes and \(Q\) does not depend on \(c_k(1)\). In this case \(Q\) alone does not uniquely determine the solutions \(c_1\) and \(c_2\) of (4.3)-(4.5).

We next show \(c_k \in P_{\mathbb{R}}^{g+1}\). For \(p \in C^m[\lambda]\) we have

\[
m \lambda^m p(\lambda^{-1}) - 2 \lambda \frac{d}{d\lambda} \left( \lambda^m p(\lambda^{-1}) \right) = m \lambda^m p(\lambda^{-1}) - 2m \lambda^m p(\lambda^{-1}) + \lambda^{m+1} \frac{d}{d\lambda} \lambda^m p(\lambda^{-1}) = -\lambda^m \left( m p(\lambda^{-1}) - 2 \lambda^{-1} p'(\lambda^{-1}) \right).
\]

For \(m(mp - 2\lambda p') - 2\lambda \frac{d}{d\lambda} (dp - 2\lambda p') = m^2 p - 4\lambda(m - 1)p' + 4\lambda^2 p''\) we obtain

\[
m^2 \lambda^m p(\lambda^{-1}) - 4\lambda(m - 1) \frac{d}{d\lambda} \left( \lambda^m p(\lambda^{-1}) \right) + 4\lambda^2 \frac{d^2}{d\lambda^2} \left( \lambda^m p(\lambda^{-1}) \right) = \lambda^m \left( m^2 p(\lambda^{-1}) - 4\lambda^{-1}(m - 1)p'(\lambda^{-1}) + 4\lambda^2 p''(\lambda^{-1}) \right).
\]

In order to show the invariance of the Equations (4.5)-(4.7) with respect to the transformation \((a, c_1, c_2, Q) \mapsto (\lambda^2 a, c_1(1), \alpha^g c_2(1), \lambda^g Q(1))\), we rewrite these equations as

\[
2a(1) \left( (g + 1)c_k(1) - 2c'_k(1) \right) = (2ga(1) - 2a'(1)) c(1),
\]

\[
2a(1)Q'(1) = a(1)(2Q(1) - 2Q(1)) = (3b_1(1) - 2b'_1(1))c(1) - (3b_2(1) - 2b'_2(1))c(1),
\]

\[
2(4a(1)(4Q''(1) + 4Q(1)) + 2(2a'(1) - 2a(1))(2Q'(1) - 2Q(1)) = (4b'_1(1) - 4gb'_2(1) + (g + 1)^2 b_2(1)c(1) - (4b'_1(1) - 4gb'_1(1))(1) + (g + 1)\lambda^g b_2(1)c(1)\).
\]

By Equation (4.8), this transformation replaces both sides of the first two equations by the negatives of their complex conjugates and by (4.8)-(4.9) both sides of the third equations by their complex conjugates, respectively. So for given \((c_1(1), c_2(1))\) \(\in \mathbb{R}^2\) this transformation preserves the unique solution \((c_1, c_2, Q)\) of Equations (4.4)-(4.7), which shows \(c_k \in P_{\mathbb{R}}^{g+1}\) and \(Q \in P_{\mathbb{R}}^{2^g}\). For every root \(a\) there exists \(k \in \{1, 2\}\) so that \(b_k(\alpha) \neq 0\). The corresponding equation in (4.3) prescribes a value of \(\dot{a}(\alpha)\), and in the case of \(b_1(\alpha), b_2(\alpha) \neq 0\), Equation (4.4) ensures that the two prescribed values for \(\dot{a}(\alpha)\) are equal. The right hand side of (4.3) may be rewritten as

\[
2i \lambda a c_k' - i c_k (\alpha + \lambda a') = i (2\lambda a c_k' - (g + 1)c_k) - (\lambda a' - ga)c_k),
\]

and belongs by Equation (4.8) to \(P_{\mathbb{R}}^{2g+1}\). So \(b_k \in P_{\mathbb{R}}^{2g+1}\) implies \(\alpha^{2g} \dot{a}(\alpha^{-1}) = \dot{a}(\alpha)\). Because the highest coefficient of \(a \in \mathcal{H}^g\) is the square root of the product of all roots of \(a\), which is unital, there exists a unique \(\dot{a} \in T_a \mathcal{H}_a\) taking all these values \(\dot{a}(\alpha)\) at the roots of \(a\). By Equation (4.3) we then have

\[
b_k = \frac{2i \lambda a c_k' - i c_k (\alpha + \lambda a') + \dot{a} c_k}{2a}.
\]
The numerator vanishes at every root $\alpha$ of $a$ by the choice of $\dot{a}(\alpha)$ if $b_k(\alpha) \neq 0$, and due to Equation (4.4) if $b_k(\alpha) = 0$. So this $b_k$ belongs to $\mathbb{C}^{g+1}[\lambda]$. Furthermore, it belongs to $P_{\mathbb{R}}^{g+1}$ since the numerator belongs to $P_{\mathbb{R}}^{3g+1}$ and $a \in H^g \subset P_{\mathbb{R}}^{2g}$. In total the value $(c_1(1), c_2(1)) \in \mathbb{R}^2$ and (4.3)–(4.7) uniquely determine $(\dot{a}, \dot{b}_1, \dot{b}_2) \in T_a S^g_1 \times P_{\mathbb{R}}^{g+1} \times P_{\mathbb{R}}^{g+1}$ for given $(a, b_1, b_2) \in S^g \times B_a \times B_a$. □

The following Lemma shows that a certain type of Whitham flow flows inside $W$.

**Lemma 4.4.** In the setting of Lemma 4.3 suppose that $a \in W$.

(a) $a(\lambda)$ can be written in the form

$$a(\lambda) = \lambda^4 + \frac{1}{2}(a_+ - a_-)\lambda^3 + \frac{1}{2}(a_+ + a_- - 4)\lambda^2 + \frac{1}{2}(a_+ - a_-)\lambda + 1,$$

where $a_\pm = a(\pm 1) > 0$. Moreover, there exists a basis $b_1, b_2$ of $B_a$ so that

$$b_1(\lambda) = b_1(\lambda) \quad \text{and} \quad b_2(\lambda) = -b_2(\lambda)$$

holds, and then there exist $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$ with

$$b_1(\lambda) = \beta_1 (\lambda - 1)^2 (\lambda + 1) \quad \text{and} \quad b_2(\lambda) = i\beta_2 (\lambda - 1) (\lambda^2 + (\beta_3 + 2)\lambda + 1).$$

(b) The Whitham vector field $(\dot{a}, \dot{b}_1, \dot{b}_2, c_1, c_2, Q)$ with $c_1(1) = 0$ and $c_2(1) \in \mathbb{R}$ then satisfies

$$\frac{\dot{a}(\lambda)}{c_1(\lambda)} = \dot{a}(\lambda), \quad \frac{\dot{b}_1(\lambda)}{c_1(\lambda)} = b_1(\lambda), \quad \frac{\dot{b}_2(\lambda)}{c_1(\lambda)} = -b_2(\lambda), \quad \frac{Q(\lambda)}{c_1(\lambda)} = Q(\lambda).$$

Such a Whitham flow is tangential to $W$.

(c) We represent

$$\dot{a}(\lambda) = \frac{1}{4}(\dot{a}_+ - \dot{a}_-)\lambda^3 + \frac{1}{2}(\dot{a}_+ + \dot{a}_-)\lambda^2 + \frac{1}{2}(\dot{a}_+ - \dot{a}_-)\lambda,$$

$$\dot{b}_1(\lambda) = \dot{\beta}_1 (\lambda - 1)^2 (\lambda + 1) \quad \text{and}$$

$$\dot{b}_2(\lambda) = i\dot{\beta}_2 (\lambda - 1) (\lambda^2 + (\beta_3 + 2)\lambda + 1) + i\dot{\beta}_2 \dot{\beta}_3 (\lambda - 1) \lambda$$

with $\dot{a}_\pm, \dot{\beta}_k \in \mathbb{R}$. For $c_1(1) = 0$ and $c_2(1) = -4a_+ \beta_2 \beta_3 \in \mathbb{R}$ we then have

$$\dot{a}_+ = 4a_+ a_-$$

$$\dot{a}_- = 2(a_+ + a_- - 16) a_-$$

$$\dot{\beta}_1 = -a_- \beta_1$$

$$\dot{\beta}_2 = -(a_- + 4\beta_3) \beta_2$$

$$\dot{\beta}_3 = 4\beta_3^2 + 2a_- \beta_3 + 4 a_-.$$

**Proof.** For (a) we note that because of $a \in W$, all coefficients of $a(\lambda)$ are real-valued. Thus the reality condition and the normalisation for $a \in H^2$ imply that the highest and the lowest coefficient of $a(\lambda)$ are equal to one, whereas the $\lambda^3$-coefficient and the $\lambda$-coefficient are equal. This implies Equation (4.10). The statement $a(\pm 1) > 0$ follows from the positivity condition for $a \in H^2$.

By its definition, the space $B_a$ is invariant under the map $b(\lambda) \mapsto \overline{b(\lambda)}$. Because $b \in B_a$ is uniquely determined by the value $b(0)$ it follows that

$$b(0) \in \mathbb{R} \iff \overline{b(\lambda)} = b(\lambda) \quad \text{and} \quad b(0) \in i\mathbb{R} \iff \overline{b(\lambda)} = -b(\lambda).$$

This shows the existence of $b_1$ and $b_2$. $b_1$ vanishes at $\lambda = 1$ by the definition of $S^g_1$ and at $\lambda = -1$ because it is both real and purely imaginary there. Therefore $b_1$ is of the form $b_1(\lambda) = (\lambda - 1) (\lambda + 1) p$ with $p \in iP_{\mathbb{R}}^1$ and $\overline{p(\lambda)} = p(\lambda)$. Thus $p(\lambda) = \beta_1 (\lambda - 1)$ with $\beta_1 \in \mathbb{R}$, and hence $b_1$ is as in Equation (4.12). $b_2$ is also zero at $\lambda = 1$, and therefore of the form
$b_2(\lambda) = i(\lambda - 1)p$ with $p \in P^2_\mathbb{R}$ and $p(\lambda) = p(\lambda)$. It follows that $p(\lambda) = \beta_2 (\lambda^2 + (\beta_3 + 2)\lambda + 1)$ holds with suitable $\beta_2, \beta_3 \in \mathbb{R}$, which shows (1.12) also with respect to $b_2$. For (b), if the $b_k$ satisfy the conditions of Equation (4.11), then the anti-derivative of $\Theta_{b_k}$ is real on the real line, and the anti-derivative of $\Theta_{b_{-\cdot}}$ is purely imaginary on the real line. If we want to preserve this property under the deformation, then the equations for $c_k$ in (1.13) must hold. In this case we have $c_1(1) \in \mathbb{R} \cap i\mathbb{R} = \{0\}$ and $c_2(1) \in \mathbb{R}$. If this is the case, then $Q'(1), Q''(1) \in \mathbb{R}$ by Equations (4.6) and (4.7). Therefore $Q$ satisfies (1.13). Because of the uniqueness of solutions of Lemma 4.3, the corresponding solutions satisfy (1.18).

In (c), the representations of $\dot{a}$ and $\dot{b}_k$ in (1.14) follow immediately from (a), because the flow in question is tangential to $W$. We have $c_1(0) = 0$ and therefore also $c'_1(0) = 0$ by Equation (4.5). It follows that $c_1(\lambda) = i(\lambda - 1)^2 p(\lambda)$ holds, where $p \in i\mathbb{P}^1_\mathbb{R}$ satisfies $\overline{p(\lambda)} = p(\lambda)$. Thus $\lambda - 1$ divides $p$, and hence $c_1(\lambda) = i\gamma_1 (\lambda - 1)^3$ holds with some $\gamma_1 \in \mathbb{R}$. $c_2$ vanishes at $\lambda = -1$ because it is both real and purely imaginary there, and thus is of the form $c_2(\lambda) = \gamma_2 (\lambda + 1) (\lambda^2 + (\gamma_3 + 2)\lambda + 1)$ with some $\gamma_2, \gamma_3 \in \mathbb{R}$. By Equations (4.6) and (4.7) we now obtain $Q'(1) = Q'(1) = 0$ and

$$Q''(1) = -\frac{b''_1(1) c_2(1)}{a_+} = -\frac{4\beta_1 \cdot 2\gamma_2 (\gamma_3 + 4)}{a_+},$$

and hence

$$Q(\lambda) = -\frac{4\beta_1 \gamma_2 (\gamma_3 + 4)}{a_+} (\lambda - 1)^2.$$

By inserting the representations of the polynomials $b_k, c_k, Q$ and $a$ into Equation (4.4) and collecting like powers of $\lambda$ we obtain a system of linear equations in $\gamma_k$. Under the additional condition $c_2(1) = -4a_+ \beta_2 \beta_3$, it has the unique solution

$$\gamma_1 = 2a_- \beta_1, \quad \gamma_2 = -2\beta_2 (a_- + 4\beta_3) \quad \text{and} \quad \gamma_3 = \frac{a_+ \beta_3}{a_- + 4\beta_3} - 4.$$

Inserting the representations of the various polynomials and the equations for $\gamma_k$ into Equation (4.3) for $k = 1$ and collecting like powers of $\lambda$ yields another system of linear equations in $\dot{a}_+, \dot{a}_-$ and $\dot{b}_1$, which has the unique solution given by Equations (4.15), (4.16) and (4.17). Treating Equation (4.13) for $k = 2$ in the same way, one obtains a system of linear equations in $\dot{a}_+, \dot{a}_-, \dot{b}_2$ and $\dot{b}_3$ with the unique solution given by Equations (4.15), (4.16), (4.18) and (4.19). \hfill \Box

**Lemma 4.5.** If $W \neq \emptyset$, then $W$ is a smooth, 1-dimensional submanifold of $S^2$. The connected components of $W$ are images of maximal integral curves of the Whitham flow described in Lemma 4.4. On every such integral curve, $a_+$ is strictly increasing, and we have $a_+ \to 0$ at the lower boundary and $a_+ \to \infty$ at the upper boundary of the curve. The map $\Phi : W \to \mathbb{R}$ (Equation (4.11)) is a local diffeomorphism that maps each connected component of $W$ bijectively onto an open interval of length $\pi$.

**Proof.** Recall that $S^2$ is a smooth, real-2-dimensional manifold, see Lemma 1.1. For $a \in S^2$ and a basis $(b_1, b_2)$ of $B_a$ we consider the Whitham flows constructed in Lemma 4.3. If $\dot{a} = 0$ holds for some values of $c_k(1)$, then Equation (4.3) implies that $c_k$ vanishes at all roots of $a$, and therefore $c_k = 0$ holds. This shows that the linear map $g : \mathbb{R}^2 \to T_a S^2$, which associates to $(t_1, t_2) \in \mathbb{R}^2$ the $\dot{a}$ defined in Lemma 4.3 with $c_k(1) = t_k$, is injective. Because of $\dim S^2 = 2$, $g$ is in fact an isomorphism of linear spaces. Because the Whitham vector fields defined in Lemma 4.3 for different values of $c_k(1)$ commute with each other, the flow of these vector fields defines a map $\tilde{g}$ of a neighborhood of $(0,0) \in \mathbb{R}^2$ into $S^2$ so that $\tilde{g}(0,0) = a$ and $d_{(0,0)} \tilde{g} = g$ holds. This map is a submersion.  

\footnote{As we will see in the proof of Theorem 4.1 below, $W$ is in fact non-empty and connected.}
Now suppose \( a \in \mathcal{W} \) and let the basis \( (b_1, b_2) \) of \( \mathcal{B}_a \) be chosen as in Lemma 4.4(a). Then \( g(1, 0) \) is transversal to \( \mathcal{W} \), whereas \( g(0, 1) \) is tangential to \( \mathcal{W} \). This shows that \( \mathcal{W} \) is near \( a \) a 1-dimensional submanifold of \( S^2_f \). It also follows that the connected components of \( \mathcal{W} \) are images of maximal integral curves of the Whitham flow defined by \( c_1(1) = 0, \ c_2(1) = 1 \).

On such an integral curve we have \( a_\pm = a(\pm 1) > 0 \) because of \( a \in S^2_f \), and therefore Equation (4.15) shows that \( a_+ \) is strictly increasing. At its boundary points, the integral curve either converges to a boundary point of \( S^2_f \), which implies that \( a_+ \to 0 \) by Proposition 2.2 or else \( a_+ \to \infty \). Because \( a_+ \) is strictly increasing, \( a_+ \to 0 \) occurs at the lower boundary and \( a_+ \to \infty \) occurs at the upper boundary of the curve.

Note that \( \int_{b_1} \Theta b_2 = \int_{b_2} \Theta b_2 \neq 0 \) holds. The polynomial \( b_2 \in \mathcal{B}_a \) is thus a non-zero multiple of the \( b \in \mathcal{B}_a \) used in the definition of \( \Phi \) in Equation (4.1). Let \( \mu_2 \) be a global, single-valued meromorphic function on \( \Sigma \) with \( d(\ln \mu_2) = \Theta b_2 \). Then \( \frac{d}{dt} \ln \mu_2|_{\lambda=1} = \frac{c_2(1)}{\nu(\pm 1)} \neq 0 \) holds. Therefore \( \Phi \) is a local diffeomorphism, and injective on every maximal integral curve in \( \mathcal{W} \). \( a_+ \to 0 \) means \( \mu_2|_{\lambda=1} \to 1 \) and hence \( \frac{1}{2} \ln \mu_2|_{\lambda=1} \to 2\pi k \) for some \( k \in \mathbb{Z} \). \( a_+ \to \infty \) means \( \mu_2|_{\lambda=1} \to -1 \) and hence \( \frac{1}{2} \ln \mu_2|_{\lambda=1} \to 2\pi k \pm \pi \). Therefore the image of the maximal integral curve under \( \Phi \) is an open interval of length \( \pi \). \( \square \)

In the sequel we consider the vector field defined by Equations (4.15)–(4.19) also as a vector field on \( (a_+, a_-, \beta_1, \beta_2, \beta_3) \in \mathbb{R}^5 \). It is a remarkable fact that the differential equations for \( a_+ \) and \( a_- \) do not depend on the \( \beta_k \) in our situation, and thus split off to give a vector field on \( (a_+, a_-) \in \mathbb{R}^2 \) defined by Equations (4.15), (4.16). Similarly the differential equations for \( (a_+, a_-, \beta_2, \beta_3) \) do not depend on \( \beta_1 \), and thus define a vector field on \( (a_+, a_-, \beta_2, \beta_3) \in \mathbb{R}^4 \). Any smooth integral curve of the latter differential equation can be “supplemented” by the function \( \beta_1 = \exp(-\int a_-(s) \, ds) \) to produce an integral curve of the full system of differential equations (4.15)–(4.19) with the same domain of definition as before.

Lemma 4.5 shows that the connected components of \( \mathcal{W} \) correspond to maximal integral curves of the vector field on \( (a_+, a_-) \in \mathbb{R}^2 \) given by Equations (4.15), (4.16) with \( a_+ > 0 \). Note that conversely however, not all integral curves of that vector field correspond to members of \( \mathcal{W} \); in fact we will see that only a single integral curve has this property. The reason is that whereas the periods of the differential \( \Theta b_1 \) with \( b_1(\lambda) \) given by Equation (4.12) on the spectral curve defined by \( \nu^2 = \lambda a(\lambda) \) with \( a(\lambda) \) given by Equation (4.10) are constant along integral curves, there is no reason in general why these periods should be purely imaginary. Hence the corresponding \( a(\lambda) \) will generally not be in \( S^2_f \) and in particular not in \( \mathcal{W} \).

**Lemma 4.6.** For every maximal integral curve of the vector field defined by Equations (4.15), (4.16) with \( a_+ > 0 \) there exists \( \delta \in \mathbb{R} \) so that \( a_- = a_+ + \delta \sqrt{a_+ + 16} \) holds.

**Proof.** We consider a maximal integral curve, and regard \( y = a_--16 \) as a function of \( x = a_+ \). Then it follows from Equations (4.15), (4.16) that \( y(x) \) satisfies the inhomogeneous linear differential equation

\[
\frac{dy}{dx} = \frac{a_-}{a_+} = \frac{1}{x} y + \frac{1}{2}.
\]

The general solution of the homogeneous equation \( \frac{dy_h}{dx} = \frac{1}{x} y \) is given by \( y_h(x) = C \sqrt{x} \) with a constant \( C \). By variation of the constant, it follows that the general solution of (4.20) is \( y(x) = f(x) \sqrt{x} \), where \( f'(x) = \frac{1}{2 \sqrt{x}} \). We thus have \( f(x) = \sqrt{x} + \delta \) with a constant \( \delta \in \mathbb{R} \) and hence \( y(x) = x + \delta \sqrt{x} \). \( \square \)

**Lemma 4.7.** For every maximal integral curve of the vector field on \( (a_+, a_-) \) given by Equations (4.15), (4.16) with \( a_+ > 0 \) and \( a_+ \to 0 \) at the lower boundary, there exists a unique
corresponding maximal integral curve of the vector field on \((a_+, a_-, \beta_1, \beta_2, \beta_3)\) given by Equations (4.15), (4.16) with the same lower boundary (but a possibly smaller upper boundary), such that \(\beta_1 \cdot a_+^{1/4} \rightarrow 1\), \(\beta_2 \rightarrow 1\) and \(\beta_3 \rightarrow -4\) at the lower boundary. For this integral curve, \(\Theta_{b_2}\) with \(b_2(\lambda)\) defined by Equation (4.12) has purely imaginary periods.

**Remark 4.8.** It is to be expected that \(\beta_1\) is unbounded, but \(\beta_2\), \(\beta_3\) are bounded in the situation of the Lemma. Indeed it was shown in [K-PH-S] the proof of Theorem 3.5] that for a limit of \(a(\lambda) \rightarrow (\lambda - 1)^4\) in \(S_1^2\), the coefficients of some corresponding \(b(\lambda) \in B_a\) remain bounded if and only if \(\int_{B_2} \Theta_b = \int_{B_2} \Theta_b\) holds. Also note that in \(W\), \(a_+ \rightarrow 0\) implies \(\beta_3 \rightarrow -4\).

**Proof of Lemma 4.7.** By Lemma 4.6 there exists \(\delta \in \mathbb{R}\) so that \(a_- = a_+ + \delta \sqrt{a_+} + 16\) holds. Thus \(a_+ \rightarrow 0\) implies \(a_- \rightarrow 16\). We now consider the vector field on \((a_+, a_-, \beta_2, \beta_3) \in \mathbb{R}^4\) given by Equations (4.15), (4.16), (4.18), (4.19). We show that this vector field has a suitable integral curve flowing out of \((a_+, a_-, \beta_2, \beta_3) = (0, 16, 1, -4)\). The vector field has a zero at the point \((a_+, a_-, \beta_2, \beta_3) = (0, 16, 1, -4)\), and one can show that its Jacobi matrix is non-hyperbolic there. Therefore we need to blow up this point in order to study the integral curves near it.

We introduce the new variable \(x = a_+^{1/4}\), which provides a local coordinate of the time domain because \(a_+ > 0\) is strictly monotonous, and \(a_+ = x^4\). Moreover, we introduce new variables \(y_1, y_2, y_3\) by the equations

\[
(4.21) \quad a_- - 16 = \delta x^2 + y_1 x^4, \quad \beta_2 - 1 = y_2 x^2 \quad \text{and} \quad \beta_3 + 4 = y_3 x^2.
\]

With respect to the variables \((x, y_1, y_2, y_3)\), the vector field defined by Equations (4.15), (4.16), (4.18), (4.19) is given by

\[
\begin{align*}
\dot{x} &= x \cdot (16 + \delta x^2 + y_1 x^4), \\
\dot{y}_1 &= -2 (y_1 - 1) \cdot (16 + \delta x^2 + y_1 x^4) \\
\dot{y}_2 &= -\delta - 32 y_2 - 4 y_3 - x^2 \cdot (y_1 + \delta y_2 + 4 y_2 y_3 + 3 y_1 y_2 x^2) \\
\dot{y}_3 &= -4 \delta - 32 y_3 - x^2 \cdot 4(y_1 - y_3^2).
\end{align*}
\]

We will regard \(x\) as the blow-up variable, by which we blow up the functions \(a_- - 16 - \delta x^2\), \(\beta_2 - 1\) and \(\beta_3 + 4\) to give the blown up functions \(y_1, y_2, y_3\). The exceptional fibre of this blow-up is \(\{x = 0\}\). The vector field given above has exactly one zero on the exceptional fibre, namely at \((x, y_1, y_2, y_3) = (0, 1, -\frac{1}{64}\delta, -\frac{1}{8}\delta)\). The Jacobi matrix of the vector field at this zero is

\[
\begin{pmatrix}
16 & 0 & 0 & 0 \\
0 & -32 & 0 & 0 \\
0 & 0 & -32 & -4 \\
0 & 0 & 0 & -32
\end{pmatrix}.
\]

This matrix is hyperbolic. Its eigenvalues are 16 and -32. The eigenvectors for the negative eigenvalue -32 are tangential to the exceptional fibre, but the eigenspace for the eigenvalue 16, which is spanned by \((1, 0, 0, 0)\), is transversal to the exceptional fibre. The Hartman-Grobman linearisation theorem therefore shows that there exists a unique integral curve of the blown up vector field that starts at \((x, y_1, y_2, y_3) = (0, 1, -\frac{1}{64}\delta, -\frac{1}{8}\delta)\) into the direction \(x > 0\). Because of \(y_1 = 1\) at the starting point, the \((x, y_1)\)-component of this integral curve has the same tangent vector as the correspondingly blown up originally given integral curve \((a_+, a_-)\) of (4.15), (4.16), and thus these two curves are equal on the intersection of their domains of definition. If we now take Equations (4.21) to define functions \(\beta_2\) and \(\beta_3\) in terms of \(x, y_1, y_2, y_3\), we obtain an integral curve \((a_+, a_-, \beta_2, \beta_3)\) of Equations (4.15), (4.16), (4.18), (4.19). Because of \(x \rightarrow 0\) at the lower boundary, we have \(\beta_2 \rightarrow 1\) and \(\beta_3 \rightarrow -4\) there. Thus we have \(a(\lambda) \rightarrow (\lambda - 1)^4\) and \(b_2(\lambda) \rightarrow i(\lambda - 1)^3\), and therefore \(\Theta_{b_2} \rightarrow i \frac{\lambda - 1}{\sqrt{2}} d\lambda =: \tilde{\Theta}\). The B-periods of \(\Theta_{b_2}\) converge under this limit to \(\mathbb{Z}\)-linear combinations of the residues of \(\tilde{\Theta}\) at \(\lambda = 0, \infty\). These residues are all purely
imaginary. Because the periods of $\Theta_{b_2}$ are constant along the Whitham flow, it follows that the periods of $\Theta_{b_2}$ are purely imaginary.

Finally we supplement with a function $\beta_1$ such that $(a_+, a_-, \beta_1, \beta_2, \beta_3)$ is an integral curve of the full system of differential equations (4.15)–(4.19). It follows from Equations (4.15) and (4.17) that a function $\beta_1$ has this property if and only if $\dot{\beta}_1/\dot{a}_+ = -\frac{1}{2} (1/a_+) \frac{C}{a_+^{1/4}}$ with any constant $C \neq 0$ satisfies the requirements. We may choose $C = 1$, and then $\beta_1 \cdot a_+^{1/4} \to 1$ holds.

We saw in the preceding lemma that on an integral curve of the Whitham flow that preserves $W$, the differential form $\Theta_{b_2}$ always has purely imaginary periods. For the polynomial $a$ to lie in $S$, also the other corresponding differential form $\Theta_{b_1}$ needs to have purely imaginary periods. The question of when this is the case is discussed in the following two lemmas.

**Lemma 4.9.** There exists a number $-8 < \delta_0 < 8$, further characterised in Lemma 4.10 below, with the following property:

For any maximal integral curve of the vector field given by Equations (4.15)–(4.19) with $a_+ > 0$ and $a_+ \to 0$, $a_- \to 16$, $\beta_1 \cdot a_+^{1/4} \to 1$, $\beta_2 \to 1$, $\beta_3 \to -4$ at a boundary of the curve, the corresponding differential form $\Theta_{b_1}$ has purely imaginary periods if and only if $\delta = \delta_0$ holds for the constant $\delta$ from Lemma 4.6.

**Proof.** We blow up the spectral parameter at the Sym point $\lambda = 1$ of the spectral curve. To facilitate this, we replace $\lambda$ with the parameter $\kappa = \frac{1}{\lambda + \frac{1}{2}}$ that maps the Sym point $\lambda = 1$ and its antipodal point $\lambda = -1$ to $\kappa = 0$ and $\kappa = \infty$, respectively. Moreover, this parameter maps $\lambda = 0$ and $\lambda = \infty$ to $\kappa = -1$ and $\kappa = 1$, respectively, and therefore the unit circle $\mathbb{S}^1 \ni \lambda$ to $\mathbb{R} \ni \kappa$.

Because of Lemma 4.6, the polynomial $a(\lambda)$ of Equation (4.10) is with respect to $\kappa$ given by

$$a(\kappa) = \frac{a_- \kappa^4 + (a_+ + a_- - 16) \kappa^2 + a_+}{(\kappa - i)^4} = \frac{a_- \kappa^4 + (2a_+ + \delta \sqrt{a_+}) \kappa^2 + a_+}{(\kappa - i)^4}$$

and on the spectral curve defined by the equation $\nu^2 = \lambda a(\lambda)$ we thus have

$$\nu^2 = -\frac{\kappa + 1}{\kappa - 1} a(\kappa).$$

Moreover we have $d\lambda = \frac{d\kappa}{d\kappa} d\kappa = \frac{2i}{(\kappa - i)^3} d\kappa$ and therefore the differential form $\Theta_{b_1}$ with $b_1(\lambda)$ given by (4.12) is

$$\Theta_{b_1} = -\frac{16\beta_1}{(\kappa - i)^4 (\kappa + i) \nu} \frac{\kappa^2}{d\kappa}.$$

Note that the degree of $\Theta_{b_1}/d\kappa$ in $\kappa$ is one lower than the degree of $\Theta_{b_1}/d\lambda$ in $\lambda$. The reason is that the zero $\lambda = -1$ of $b_1(\lambda)$ is moved to $\kappa = \infty$ by the transformation to the $\kappa$ coordinate.

We now blow up the parameter $\kappa$ near $\kappa = 0$. We again define the blow-up variable $x$ by $a_+ = x^4$, and then blow up $\kappa$ by defining the new variable $\tilde{\kappa}$ by $\kappa = x \cdot \tilde{\kappa}$. Under the limit $x \to 0$ we then have

$$x^{-4} \cdot a(\tilde{\kappa}) \to 16 \tilde{\kappa}^4 + \delta \tilde{\kappa}^2 + 1 =: \tilde{a}(\tilde{\kappa}).$$

We now consider the elliptic curve $\Sigma$ defined by the equation $\tilde{\nu}^2 = \tilde{a}(\tilde{\kappa})$. Then $x^{-2} \cdot \nu \to \tilde{\nu}$; in this sense we can regard $\Sigma$ as the blow up of the spectral curves $\Sigma$.

Under the limit $x \to 0$ we have $\beta_1 x \to 1$ by hypothesis, and therefore

$$\Theta_{b_1} \to 16i \frac{\tilde{\kappa}^2}{\tilde{\nu}} d\tilde{\kappa} =: \tilde{\Theta}.$$
Because the periods of $\Theta_{b_1}$ are constant along the Whitham integral curve, these periods are purely imaginary if and only if the differential form $\Theta$ on $\Sigma$ has purely imaginary periods. By the following Lemma 4.10 this is the case if and only if $\delta = \delta_0$ holds.

**Lemma 4.10.** Let $\delta \in \mathbb{R}$, $a(\sigma) = 16 \sigma^4 + \delta \sigma^2 + 1$ be a polynomial of degree 4 and

$$\Sigma = \{ (\sigma, \nu) \in \mathbb{C}^2 \mid \nu^2 = a(\kappa) \}$$

be the elliptic curve associated to $a(\sigma)$.

Then there exists a unique $\delta_0 \in \mathbb{R}$ so that the differential form $\Theta = \frac{i}{\nu} d \sigma$ has purely imaginary periods if and only if $\delta = \delta_0$ holds. We have $-8 < \delta_0 < 8$, and therefore $a(\sigma)$ has no zeros on $\mathbb{R} \cup i\mathbb{R}$ for $\delta = \delta_0$.

**Proof of Lemma 4.10.** The polynomial $a(\sigma)$ satisfies

$$a(-\sigma) = a(\sigma) \quad \text{and} \quad a(\bar{\sigma}) = a(\sigma).$$

Therefore the elliptic curve $\Sigma$ has the holomorphic involution $\vartheta : (\sigma, \nu) \mapsto (-\sigma, \nu)$ and the anti-holomorphic involution $\zeta : (\sigma, \nu) \mapsto (\bar{\sigma}, \bar{\nu})$. Due to the invariance of $\Sigma$ under $\vartheta$, there exist $\alpha_1, \alpha_2 \in \mathbb{C}^\times$, $\alpha_1 \neq \pm \alpha_2$ so that the branch points of $\Sigma$ are exactly $\sigma = \pm \alpha_1$ and $\sigma = \pm \alpha_2$.

If $\delta \leq -8$ holds, then $\alpha_1, \alpha_2 \in \mathbb{R}$. Let us suppose $0 < \alpha_1 < \alpha_2$. In this case every homology class on $\Sigma$ can be realised by a cycle in $\Sigma$ above $\{\kappa \in \mathbb{R}\}$. On $\{\kappa \in \mathbb{R}\}$, $a(\kappa)$ is real, and alternates its sign at its zeros. Consequently $\nu$ alternates between being real and being imaginary on the intervals between the zeros of $a(\sigma)$:

$$\begin{array}{cccccc}
\sigma > 0 & -\alpha_2 & \sigma < 0 & -\alpha_1 & \sigma < 0 & \alpha_2 \\
\nu \in \mathbb{R} & \nu \in i\mathbb{R} & \nu \in \mathbb{R} & \nu \in i\mathbb{R} & \nu \in \mathbb{R} & \quad \sigma \in \mathbb{R}
\end{array}$$

Thus also the periods of $\Theta$ between adjacent branch points of $\Sigma$ alternate between being real and being purely imaginary. Moreover, no such period can vanish. Therefore it is not possible for all periods of $\Theta$ to be purely imaginary for $\delta \leq -8$.

For $\delta \geq 8$ we have $\alpha_1, \alpha_2 \in i\mathbb{R}$, and an analogous argument as before shows that the periods of $\Theta$ cannot be purely imaginary.
We have $\rho^*\Theta = -\Theta$, and therefore
\[ \int_A \Theta = - \int_{\xi(A)} \Theta = - \int_A \xi^* \Theta = \int_B \Theta = \int_{\xi(B)} \Theta = - \int_B \Theta, \]
hence $\int_A \Theta \in \mathbb{R}$, $\int_B \Theta \in i\mathbb{R}$ holds. It follows that $\Theta$ has purely imaginary periods if and only if $\int_A \Theta$ vanishes. By a similar calculation we have $\int_A \frac{1}{\nu} d\kappa \in \mathbb{R}$ and $\int_B \frac{1}{\nu} d\kappa \in i\mathbb{R}$; moreover $\int_A \frac{1}{\nu} d\kappa \neq 0$ holds, because otherwise $\frac{1}{\nu} d\kappa$ would be a non-zero holomorphic differential form on $\Sigma$ with purely imaginary periods, which contradicts Riemann's bilinear relations. It follows that for every given value of $\delta$ there exists one and only one maximal integral curve of the vector field defined by Equations (4.15), (4.16) so that
\[ \Theta_b := \frac{b(\nu)}{\nu} d\kappa \quad \text{with} \quad b(\kappa) = b_0 i (\kappa^2 + \beta), \quad b_0 \in \mathbb{R} \]
has purely imaginary periods. In this situation $\Theta$ has purely imaginary periods if and only if $\beta = 0$ holds.

For $\delta \to -8$, the branch points of $\Sigma$ coalesce to double points on the real line at $\kappa = \pm \frac{1}{2}$. Therefore $b(\pm \frac{1}{2}) \to 0$ (see the proof of [K-PH-S, Lemma 3.4]), which shows $\beta \to -\frac{1}{4}$. Similarly, for $\delta \to 8$ we have $\beta \to \frac{1}{4}$. For reasons of continuity there thus exists some $\delta_0 \in (-8, 8)$ so that the corresponding value of $\beta$ is zero.

It remains to show that there exists only one such $\delta_0$. For this purpose we deform $(a,b)$ by a Whitham transformation. The requirement that the periods of $\Theta_b$ do not change under the transformation means that $\dot{\Theta}$ is the differential of a global, anti-symmetric, meromorphic function on $\Sigma$ which has at most first order poles at the zeros of $a(\kappa)$ and no other singularities, and hence is of the form $\frac{c(\kappa)}{\nu}$ with a polynomial $c(\kappa)$ of degree $\leq 3$. This polynomial should be odd and have purely imaginary coefficients so that the form of $a$ and $b$ is preserved. We then have
\begin{equation}
\dot{\Theta} = \frac{d}{d\kappa} \frac{c(\kappa)}{\nu} \quad \Longleftrightarrow \quad 2b a - b\dot{a} = 2c' a - c a'.
\end{equation}

We shall see that by choosing $c(\kappa) = i b_0 (\delta - 32 \beta) \kappa^3 + i \gamma \kappa$ with some $\gamma \in \mathbb{R}$ we obtain a vector field that has no zeros or poles with $-8 < \delta < 8$. In fact, by evaluating the right-hand side equation of (4.23) and collecting like powers of $\kappa$, we obtain $\gamma = b_0 (2 - \delta \beta)$ and with
\[ \dot{a}(\kappa) = \delta \kappa^2 \quad \text{and} \quad \dot{b}(\kappa) = i b_0 (\kappa^2 + \beta) + i b_0 \beta \]
we have
\[ \dot{\delta} = 2 (64 - \delta^2), \quad \dot{b}_0 = b_0 (\delta - 32 \beta), \quad \dot{\beta} = 2 (16 \beta^2 - \delta \beta + 1). \]
This vector field has no zeros on the domain $D := \{ (\delta, b_0, \beta) \in \mathbb{R}^3 \mid -8 < \delta < 8 \}$, so any maximal integral curve in $D$ tends to the boundary of $D$. The coefficients of $b$ remain bounded along any integral curve in $D$, and furthermore $\dot{\delta} > 0$ holds, so any maximal integral curve has $\delta \to -8$ at its lower boundary, $\delta \to 8$ at its upper boundary. It now follows from the preceding consideration that up to scaling of $b_0$, there exists one and only one maximal integral curve of the vector field such that $\Theta_b$ has purely imaginary periods. Due to $-8 < \delta < 8$, the polynomial $16 \beta^2 - \delta \beta + 1$ is positive for all real $\beta$, and thus $\dot{\beta} > 0$ holds. Therefore $\beta$ is strictly monotonic along that integral curve. It follows that there exists only one value $\delta_0$, $-8 < \delta_0 < 8$, so that $\beta = 0$ holds for $\delta = \delta_0$.

Proof of Theorem 4.1. We saw in Lemma 4.5 that $\mathcal{W}$ is the union of all those maximal integral curves of the Whitham flow from Lemma 4.4 which run in $\mathcal{W}$. But we do not yet know that there actually exists such an integral curve, nor that it is unique. To prove these two points, we begin by constructing an integral curve of the vector field given by Equations (4.15), (4.16) so that the
corresponding polynomials $a(\lambda)$ defined by Equation (4.10) are members of $\mathcal{W}$. Let $\delta_0$ be as in Lemma 4.10 and consider the autonomous differential equation

$$
\dot{\varphi} = 2\varphi^3 + 2\delta_0\varphi^2 + 32\varphi
$$

that is obtained by substituting $a_- = a_+ + \delta_0 \sqrt{a_+ + 16}$ and $a_+ = \varphi^2$ in Equation (4.15). We obtain a solution of this differential equation by separation of variables. Indeed, because of $|\delta| < 8$, we have $\frac{1}{2\varphi^3 + 2\delta_0\varphi^2 + 32\varphi} > 0$ and therefore the function

$$
\varphi \mapsto t = \int^\varphi \frac{1}{2\varphi^3 + 2\delta_0\varphi^2 + 32\varphi} \, d\varphi
$$

is strictly increasing. It tends to $-\infty$ for $\varphi \to 0$ and to some finite $t_0 > 0$ for $\varphi \to \infty$. Its inverse function $t \mapsto \varphi(t)$ is a solution of (4.24) that is defined for all $t < t_0$. Therefore

$$
a_+(t) = \varphi(t)^2 \quad \text{and} \quad a_-(t) = \varphi(t)^2 + \delta_0 \varphi(t) + 16
$$

defines a maximal integral curve of the vector field given by Equations (4.15), (4.16) with $a_+(t) \to 0$ for $t \to -\infty$ and $a_+(t) \to \infty$ for $t \to t_0$. By Lemma 4.7 there exist functions $\beta_k(t)$ ($k = 1, 2, 3$) defined at least for $t < t_1$ with some $t_1 \leq t_0$ so that $a_\pm(t)$ and $\beta_k(t)$ give a maximal integral curve of the vector field given by Equations (4.15)–(4.19) and with $\beta_1 \cdot a_+^{1/4} \rightarrow 1$, $\beta_2 \rightarrow 1$ and $\beta_3 \rightarrow -4$ for $t \to -\infty$. By Lemma 4.9 and Lemma 4.10 the periods of $\Theta_{b_1}$ and $\Theta_{\delta_0}$ are purely imaginary on this integral curve. Therefore the $a(\lambda)$ defined by Equation (4.10) for this integral curve are in $S^2_2$ and therefore in $\mathcal{W}$. Because the integral curve runs in $S^2_2$, the $\beta_k$ are in fact defined for all times $t < t_0$.

Conversely, if there were another maximal integral curve of the vector field given by Equations (4.15)–(4.19) (up to scaling of $\beta_1, \beta_2$) that runs in $S^2_2$, then the equation of Lemma 4.6 would hold with some $\delta \neq \delta_0$. But then the periods of $\Theta_{b_1}$ would not be purely imaginary by Lemma 4.9 which is a contradiction.

Therefore $\mathcal{W}$ is the image of the single maximal integral curve with $\delta = \delta_0$ constructed above. In particular $\mathcal{W} \neq \emptyset$ and $\mathcal{W}$ is connected. Lemma 4.5 hence implies that $\mathcal{W}$ is a 1-dimensional submanifold of $S^1_2$ and that $\Phi : \mathcal{W} \to \mathbb{R}$ is a diffeomorphism onto an open interval of length $\pi$. One obtains the explicit representation (4.22) of $\mathcal{W}$ by substituting (4.25) into Equation (4.10).

Finally we saw in Lemma 4.10 that the blown up spectral curve for $a_+ \to 0$ has no branch points on $\mathbb{R} \cup i\mathbb{R}$. Therefore the polynomial $a(\lambda)$ has no zeros where $\lambda = i\frac{\lambda - 1}{\lambda + 1} \in \mathbb{R} \cup i\mathbb{R}$, hence $\lambda \in S^1_1 \cup \mathbb{R}$, at least for $t$ near $-\infty$. Along the integral curve, zeros on $S^1$ or $\mathbb{R}$ can only occur when two roots of $a$ coalesce. However, the integral curve runs in $S^2_2$, so this cannot happen. Hence $a$ does not have any zeros on $\mathbb{R} \cup S^1$ for all times $t$.

We saw that as the integral curve covers all of $\mathcal{W}$, the function $\varphi$ covers $\mathbb{R}_+$. Equation (4.22) now follows from Equations (4.25) and (4.10). □

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