Global existence for the $p$-Sobolev flow

Tuomo Kuusi*, Masashi Misawa†, and Kenta Nakamura∗

*Department of Mathematics and Statistics, University of Helsinki, PL 68 (Pietari Kalmin katu 5) 00014, Finland  tuomo.kuusi@helsinki.fi
†Faculty of Advanced Science and Technology, Kumamoto University, Kumamoto 860-8555, Japan  mmisawa@kumamoto-u.ac.jp
∗Headquarters for Admissions and Education, Kumamoto University, Kumamoto 860-8555, Japan  kntkuma21@kumamoto-u.ac.jp

Keywords— $p$-Sobolev flow, Nonlinear Intrinsic Scaling Transformation, Expansion of Positivity.

MSC2010— Primary: 35B45, 35B65,  Secondary: 35D30, 35K61.

Abstract

In this paper, we study a doubly nonlinear parabolic equation arising from the gradient flow for $p$-Sobolev type inequality, referred as $p$-Sobolev flow. In the special case $p = 2$ our theory includes the classical Yamabe flow on a bounded domain in Euclidean space. Our main aim is to prove the global existence of the $p$-Sobolev flow together with its qualitative properties.

Contents

1 Introduction 2
2 Preliminaries 5
3 Prototype Doubly Nonlinear Equation 7 3.1 Extinction of Solutions 9
4 Nonlinear Intrinsic Scaling Transformation 11
5 Proof of Theorem 1.1 14
Appendices 18
1 Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with smooth boundary $\partial \Omega$. For any positive $T \leq \infty$, let $\Omega_T := \Omega \times (0, T)$ be the space-time cylinder. Throughout the paper we fix $p \in [2, n)$ and set $q := \frac{np}{n-p}$ is the Sobolev conjugate of $p$. We consider the following doubly nonlinear parabolic system

$$\begin{cases}
\partial_t (|u|^{q-1}u) - \Delta_p u = \lambda(t)|u|^{q-1}u & \text{in } \Omega_\infty \\
\|u(t)\|_{L^{q+1}(\Omega)} = 1 & \text{for all } t > 0 \\
u = 0 & \text{on } \partial \Omega \times (0, \infty) \\
u(\cdot, 0) = u_0(\cdot) & \text{in } \Omega.
\end{cases}$$

(1.1)

Here the unknown function $u = u(x, t)$ is a real-valued function defined for $(x, t) \in \Omega_\infty$, and the initial data $u_0$ is assumed to be in the Sobolev space $W_0^{1,p}(\Omega)$, positive, bounded in $\Omega$ and satisfy $\|u_0\|_{L^{q+1}(\Omega)} = 1$, as usual, $\partial_t := \partial/\partial t$ and $\partial_\alpha := \partial/\partial x_\alpha$, $\alpha = 1, \ldots, n$ are the partial derivatives on time and space, respectively, and $\nabla := (\partial_1, \ldots, \partial_n)$ is the gradient on space, and $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian. The condition imposed in the second line of (1.1) is called the volume constraint and $\lambda(t)$ is Lagrange multiplier stemming from this volume constraint. Indeed, multiplying (1.1) by $u$ and integrating by parts, we find by a formal computation that $\lambda(t) = \int_{\Omega} |\nabla u(x, t)|^p dx$ (See [21, Proposition 5.2] for its proof). We call the system (1.1) as $p$-Sobolev flow.

Our main result in this paper is the following theorem.

**Theorem 1.1** Assume that the initial value $u_0$ belongs to the Sobolev space $W_0^{1,p}$, positive, bounded in $\Omega$, and satisfies $\|u_0\|_{L^{q+1}(\Omega)} = 1$. Then there exists a global weak solution to the equation (1.1), which is positive and bounded in $\Omega_\infty$ and is, together with its spatial gradient, locally Hölder continuous in $\Omega_\infty$.

In our forthcoming work we will proceed further with the analysis of the $p$-Sobolev flow. Especially, we will classify the limits as time tends to infinity.

The doubly nonlinear equations have been considered by Vespri [34], Porzio and Vespri [24], and Ivanov [14, 15]. See also [10, 35, 19, 18]. The regularity proofs for
doubly nonlinear equations are based on the intrinsic scaling method, originally introduced by DiBenedetto, and they have to be arranged in some way depending on the particular form of the equation. In [21] we have already treated the very fast diffusive doubly nonlinear equation such as the $p$-Sobolev flow (1.1), and obtained the positivity, boundedness and regularity of weak solutions. In particular, the expansion of positivity for (1.1) is shown by the De Giorgi’s iteration based on local energy estimates in the intrinsic scaling setting (refer to [8]). The solution to (2.4) remains positive for all finite times by the volume constraint. This is here applied for the global existence of the $p$-Sobolev flow (1.1), as explained later.

In compact manifold setting with $p = 2$, our $p$-Sobolev flow (2.4) is exactly the classical Yamabe flow equation in the Euclidean space. The classical Yamabe flow was originally introduced by Hamilton in his study of the so-called Yamabe problem ([36, 2, 3]), asking the existence of a conformal metric of constant curvature on $n(\geq 3)$-dimensional closed Riemannian manifolds ([12]). Let $(\mathcal{M}, g_0)$ be a $n(\geq 3)$-dimensional smooth, closed Riemannian manifold with scalar curvature $R_0 = R_{g_0}$. The classical Yamabe flow is given by the heat flow equation

$$u_t = (s - R)u = u^{-\frac{4}{n-2}}(c_n \Delta_{g_0} u - R_0 u) + su, \quad (1.2)$$

where $u = u(t), t \geq 0$, is a positive smooth function on $\mathcal{M}$ such that $g(t) = u(t)^{\frac{4}{n-2}}g_0$ is a conformal change of a Riemannian metric $g_0$, with volume constraint $\text{Vol}(\mathcal{M}) = \int_{\mathcal{M}} d\text{vol}_g = \int_{\mathcal{M}} u^{\frac{2n}{n-2}}d\text{vol}_{g_0} = 1$, having total curvature

$$s := \int_{\mathcal{M}} (c_n |\nabla u|^2_{g_0} + R_0 u^2) d\text{vol}_{g_0} = \int_{\mathcal{M}} R d\text{vol}_g, \quad c_n := \frac{4(n-1)}{n-2}.$$ Note that the condition for volume above naturally corresponds to the volume constraint in (1.1). Hamilton ([12]) proved a convergence of the Yamabe flow as $t \to \infty$ under some geometric conditions. Under the assumption that $(\mathcal{M}, g_0)$ is of positive scalar curvature and locally conformal flat, Ye ([37]) showed the global existence of the Yamabe flow and its convergence as $t \to \infty$ to a metric of constant scalar curvature. Schwetlick and Struwe ([25]) established the asymptotic convergence of the Yamabe flow for an initial positive scalar curvature in the case $3 \leq n \leq 5$, under an appropriate condition of Yamabe invariance $Y(\mathcal{M}, g_0)$, which is given by infimum of Yamabe energy $E(u) = \int_{\mathcal{M}} (c_n |\nabla u|^2_{g_0} + R_0 u^2) d\text{vol}_{g_0}$ among all positive smooth function $u$ on $\mathcal{M}$ with $\text{Vol}(\mathcal{M}) = 1$. In Euclidean case, since $R_{g_0} = 0$ their curvature assumptions are not verified. In above outstanding results concerning the Yamabe flow, the equation is equivalently transformed to the scalar curvature equation, and this is crucial for obtaining many properties for the Yamabe flow. In contrast to their methods, we are forced to take a direct approach dictated by the structure of the $p$-Laplacian leading to the degenerate or singular parabolic equation of the $p$-Sobolev flow. Let us remark that our results cover those of the classical Yamabe flow in the Euclidian setting.
Our global existence result for the \( p \)-Sobolev flow (1.1) is established by applying a nonlinear intrinsic scaling transformation to the following prototype doubly nonlinear parabolic equation

\[
\begin{cases}
\partial_s (|v|^{q-1}v) - \Delta_p v = 0 & \text{in } \Omega_S \\
v = 0 & \text{on } \partial \Omega \times (0, S) \\
v(\cdot, 0) = v_0(\cdot) & \text{in } \Omega.
\end{cases}
\]  

Here \( 0 < S \leq \infty \), the unknown function \( v = v(x, s) \) is real-valued function defined for \( (x, s) \in \Omega_S \), and the given function \( v_0 \) is in the Sobolev space \( W_0^{1,p}(\Omega) \), nonnegative and bounded in \( \Omega \). In [23] the existence result is obtained from the backward difference quotient on time and Galerkin’s procedure. For the global existence we crucially use the expansion of positivity for the \( p \)-Sobolev flow, as stated before. Based on the positivity estimates in [21], here we present the refinement for the expansion of positivity for (1.1) with its precise proof (See Appendix A). Combining the nonlinear intrinsic scaling applied for (1.3), and the refined expansion of positivity, we establish the global existence of a regular weak solution to the \( p \)-Sobolev flow.

The structure of this paper is as follows. In Section 2, we prepare some notation and give the definition of a weak solution of (2.4). In Section 3, we recall the global existence and regularity estimates for (1.3) obtained in [23, 21]. Starting from positive initial data, the solution of (1.3) is positive up to a finite time, that is, the positivity expands and, furthermore, the solution vanishes at a finite time, that is verified by the comparison principle. In Section 4, we present the nonlinear intrinsic scaling transformation from (1.3) to (2.4), which is justified via mollifier argument in Appendix C. In Section 5 we give the proof of Theorem 1.1. Here the expansion of positivity by the volume constraint for the \( p \)-Sobolev flow (1.1) is crucially applied for extending the life span of the solution and yielding the global existence. In Appendix A, we present the refined expansion of positivity for the \( p \)-Sobolev flow type equation by use of a stretching time transformation with its precise proof, and also prove the key propositions used in the proof of Theorem 1.1. In Appendix B we give the elementary convergence result with its proof, which is used in the next appendix. In Appendix C, we demonstrate that the nonlinear intrinsic scaling rigorously works.

Acknowledgments

The nonlinear intrinsic scaling transformation in Section 4 was kindly suggested to us by Professor J. L. Vazquez ([33]) in 2013. T. Kuusi is supported by the Academy of Finland (grant 323099) and the European Research Council (ERC) under the European Union’s Horizon 2020 research and innovation programme (grant agreement No 818437).
The work by M. Misawa was partially supported by the Grant-in-Aid for Scientific Research (C) Grant number No.18K03375 at Japan Society for the Promotion of Science. K. Nakamura is supported by Foundation of Research Fellows, The Mathematical Society of Japan.

2 Preliminaries

We prepare some notation and fundamental tools, which are used throughout this paper.

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a bounded domain with smooth boundary $\partial \Omega$. Let us define the parabolic boundary of the space-time cylinder $\Omega_T := \Omega \times (0, T)$ by

$$\partial_p \Omega_T := \partial \Omega \times [0, T) \cup \Omega \times \{t = 0\}.$$ 

Let $B_R(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < R\}$ denote the open ball with radius $R > 0$ centered at some $x_0 \in \mathbb{R}^n$. We denote the positive part of $a \in \mathbb{R}$ by $a^+ := \max\{a, 0\}$.

In what follows, we denote by $C, C_1, C_2, \cdots$ different positive constants in a given context. Relevant dependencies on parameters will be emphasized using parentheses. For instance $C = C(n, p, \Omega, \cdots)$ means that $C$ depends on $n, p, \Omega, \cdots$. As customary, the equation number $(\cdot)_n$ denotes the $n$-th line of the Eq. $(\cdot)$.

We next prepare some function spaces, defined on space-time region. For two indices $1 \leq p, q \leq \infty$, $L^q(t_1, t_2; L^p(\Omega))$ denotes the space of measurable real-valued functions on a space-time region $\Omega \times (t_1, t_2)$ with a finite norm

$$\|v\|_{L^q(t_1, t_2; L^p(\Omega))} := \begin{cases} \left( \int_{t_1}^{t_2} \|v(t)\|_{L^p(\Omega)}^q \, dt \right)^{1/q} & (1 \leq q < \infty) \\ \text{ess sup}_{t_1 \leq t \leq t_2} \|v(t)\|_{L^p(\Omega)} & (q = \infty), \end{cases}$$

where

$$\|v(t)\|_p = \|v(t)\|_{L^p(\Omega)} := \begin{cases} \left( \int_{\Omega} |v(x, t)|^p \, dx \right)^{1/p} & (1 \leq p < \infty) \\ \text{ess sup}_{x \in \Omega} |v(x, t)| & (p = \infty). \end{cases}$$

When $p = q$, we write $L^p(\Omega \times (t_1, t_2)) = L^p(t_1, t_2; L^p(\Omega))$ for brevity. For $1 \leq p < \infty$ the Sobolev space $W^{1,p}(\Omega)$ consists of measurable real-valued functions in $\Omega$ which are
weakly differentiable and of which weak derivatives are \( p \)-th integrable on \( \Omega \), with the norm
\[
\|v\|_{W^{1,p}(\Omega)} := \left( \int_{\Omega} |v|^p + |\nabla v|^p \, dx \right)^{1/p}
\]
and let \( W^{1,p}_0(\Omega) \) be the closure of \( C^\infty_0(\Omega) \), the smooth functions with a compact support, with respect to the norm \( \| \cdot \|_{W^{1,p}} \). The space \( L^q(t_1, t_2; W^{1,p}_0(\Omega)) \) comprises all measurable real-valued functions on space-time domain with a finite norm
\[
\|v\|_{L^q(t_1, t_2; W^{1,p}_0(\Omega))} := \left( \int_{t_1}^{t_2} \|v(t)\|_{W^{1,p}_0(\Omega)}^q \, dt \right)^{1/q}.
\]

Additionally, for an interval \( I \subset \mathbb{R} \), the space \( C(I; L^1_\text{loc}(\Omega)) \) consists of all continuous functions \( I \ni t \mapsto u(t) \in L^1(\Omega) \). The function space \( C(I; W^{1,p}_0(\Omega)) \) is defined analogously to the above.

We next need the following fundamental algebraic inequality, associated with the \( p \)-Laplace operator (see [5, 7]).

**Lemma 2.1 (Algebraic inequality)** For every \( p \in (1, \infty) \) there exist positive constants \( C_1(p, n) \) and \( C_2(p, n) \) such that for all \( \xi, \eta \in \mathbb{R}^n \)
\[
||\xi||^{p-2}\xi - ||\eta||^{p-2}\eta| \leq C_1(||\xi|| + ||\eta||)^{p-2}||\xi - \eta||
\]
and
\[
(||\xi||^{p-2}\xi - ||\eta||^{p-2}\eta) \cdot (\xi - \eta) \geq C_2(||\xi|| + ||\eta||)^{p-2}||\xi - \eta||^2,
\]
where \( \cdot \) denotes the inner product in \( \mathbb{R}^n \). In particular, if \( p \geq 2 \), then
\[
(||\xi||^{p-2}\xi - ||\eta||^{p-2}\eta) \cdot (\xi - \eta) \geq C_2||\xi - \eta||^p.
\]

Following [21, Definition 3.2], we present the definition of a weak solution to the \( p \)-Sobolev flow equation (1.1).

**Definition 2.2 (weak solution of the \( p \)-Sobolev flow)** Let \( 0 < T \leq \infty \). A measurable function \( u \) defined on \( \Omega_T \) is called a weak solution of (1.1) if the following (D1)-(D4) are satisfied.

\( \text{(D1)} \) \( u \in L^\infty(0, T; W^{1,p}(\Omega)); \quad \partial_t (|u|^{q-1}u) \in L^2(\Omega_T). \)

\( \text{(D2)} \) There exists a function \( \lambda(t) \in L^1_{\text{loc}}(0, T) \) such that, for every \( \varphi \in C^\infty_0(\Omega_T) \),
\[
- \int_{\Omega_T} |u|^{q-1}u \varphi_t \, dz + \int_{\Omega_T} |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, dz = \int_{\Omega_T} \lambda(t)|u|^{q-1}u \varphi \, dz.
\]
(D3) \( \|u(t)\|_{L^{r+1}(\Omega)} = 1 \) for all positive \( t < T \).

(D4) \( u = 0 \) on \( \partial \Omega \times (0, T) \) and \( u(0) = u_0 \) in \( \Omega \) in the trace sense:
\[
\|u(t) - u_0\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as} \quad t \searrow 0.
\]

**Proposition 2.3** Let \( u \) be a weak solution of (1.1). If the initial value \( u_0 \) is nonnegative and bounded in \( \Omega \) then so is \( u \). Furthermore, the local \( L^1 \)-function \( \lambda(t) \) in the first line of (1.1) is given by
\[
\lambda(t) = \int_{\Omega} |\nabla u(t)|^p \, dx.
\]

**Proof.** See [21, Proposition 5.1, Proposition 5.2] for more details. \( \square \)

In what follows, under the assumption that the initial data \( u_0 \) is in the Sobolev space \( W_0^{1,p}(\Omega) \), positive and bounded in \( \Omega \) and satisfies \( \|u_0\|_{L^{r+1}(\Omega)} = 1 \), we address the following equation (2.4) in place of (1.1):
\[
\begin{align*}
\partial_t (u^q) - \Delta_p u &= \lambda(t)u^q \quad \text{in} \quad \Omega_{\infty} \\
\|u(t)\|_{L^{r+1}(\Omega)} &= 1 \quad \text{for all} \quad t \geq 0 \\
u &= 0 \quad \text{on} \quad \partial \Omega \times (0, \infty) \\
u(\cdot, 0) &= u_0(\cdot) \quad \text{in} \quad \Omega.
\end{align*}
\] (2.4)

### 3 Prototype Doubly Nonlinear Equation

In this section, we study the nonlinear scaling for the following doubly nonlinear equation (1.3). Firstly, we recall the definition and the global existence result of weak solutions to (1.3):

**Definition 3.1** Let \( 0 < S \leq \infty \). A measurable function \( v \), defined on \( \Omega_S := \Omega \times (0, S) \), is a weak supersolution (subsolution) of (1.3) if the following conditions are satisfied:

(i) \( v \in L^\infty(0, S; W_0^{1,p}(\Omega)) \), \( \partial_s(|v|^{q-1}v) \in L^2(\Omega_S) \).

(ii) For every nonnegative \( \eta \in C_0^\infty(\Omega_S) \)
\[
-\int_{\Omega_S} |v|^{q-1}v \cdot \partial_s \eta \, dxds + \int_{\Omega_S} |\nabla v|^{p-2}\nabla v \cdot \nabla \eta \, dxds \geq (\leq)0.
\]

(iii) \( v = 0 \) on \( \partial \Omega \times (0, S) \) and \( v(0) = v_0 \) in \( \Omega \) in the trace sense:
\( v(s) \in W_0^{1,p}(\Omega) \) for almost every \( s \in (0, S) \);
\[
\|v(s) - v_0\|_{W^{1,p}(\Omega)} \to 0 \quad \text{as} \quad s \searrow 0.
\]
Furthermore, a measurable function $v$ defined on $\Omega_S$ is called a \textit{weak solution} to (1.3) if it is simultaneously a weak super and subsolution, i.e.,
\[- \int_{\Omega_S} |v|^{q-1} v \cdot \partial_s \eta \, dxds + \int_{\Omega_S} |\nabla v|^{p-2} \nabla v \cdot \nabla \eta \, dxds = 0.\]
holds for every $\eta \in C_0^\infty(\Omega_S)$.

In [23, Theorem 1.1] and [21], we proved the global existence of a weak solution to (1.3) and its regularity estimates as follows:

\textbf{Theorem 3.2 (Global existence of (1.3) [21, 23])} Assume the initial value $v_0$ be in $W_0^{1,p}(\Omega)$, nonnegative and bounded in $\Omega$. Then there exists a global in time weak solution $v$ of (1.3), which is nonnegative and bounded in $\Omega_\infty$ that is,
\[0 \leq v \leq \|v_0\|_\infty \quad \text{in } \Omega_\infty.\] (3.1)

In addition, $v$ satisfies the following energy equality, for $0 \leq s_1 < s_2 < \infty$,
\[\|v(s_2)\|_{q+1}^{q+1} + \frac{q+1}{q} \int_{s_1}^{s_2} \|\nabla v(s)\|_p \, ds = \|v(s_1)\|_{q+1}^{q+1} \quad (3.2)\]
and, the integral inequalities hold true, for any nonnegative $s < \infty$,
\[\|v(s)\|_{q+1} \leq \|v_0\|_{q+1}, \quad \|\nabla v(s)\|_p \leq \|\nabla v_0\|_p, \quad \|\partial_s v^q\|_2^2 \leq C\|v_0\|_{\infty}^{q-1}\|\nabla v_0\|_p^p, \quad (3.3)\]
where $C = C(n, p)$ is a positive constant, and the $L^p$-norm on space of $v$ is denoted by $\|v\|_p := \|v\|_{L^p(\Omega)}$ for brevity.

\textit{Proof.} Eq. (3.1) follows from [21, Propositions 3.4, 3.5]. Using a similar argument to [21, Appendix B], one can prove (3.2) and from this, (3.3) immediately follows. By the same way as in [23, Lemma 3.2, (3.7); Lemma 4.1; Proof of Theorem 1.1], (3.4) is actually verified. Finally, (3.5) is proved via [21, Lemmas 3.4, 4.1].

\textit{Remark 3.3} The existence of a weak solution $v$ to (1.3) in Theorem 3.2 is proved by a time discretization and weak convergence of bounded approximating solutions in a reflexive Banach space. In our preceding work [23], we showed the existence on $[0, S]$ of a weak solution for any positive $S < \infty$. As seen from the proof [23, Section 5, pp.167–168], we can choose $S = +\infty$. 

8
3.1 Extinction of Solutions

We study the finite-time extinction of a solution to (1.3). Firstly, the extinction time is defined as follows:

**Definition 3.4** Let $v$ be a nonnegative weak solution to (1.3) in $\Omega_\infty$. We call a positive number $S^*$ as the *extinction time* of $v$ provided that the following conditions hold:

(i) $v(x,s)$ is nonnegative and not identically zero on $\Omega \times (0, S^*)$

(ii) $v(x,s) = 0$ for any $x \in \Omega$ and all $s \geq S^*$.

Now we will show a finite time extinction of a solution of (1.3). For this purpose we apply the comparison theorem [21, Theorem 3.6], which is originally provided by Alt-Luckhaus ([1, Theorem 2.2, p.325]).

**Theorem 3.5 (Comparison theorem)** Let $0 < S \leq \infty$ and, let $v_1$ and $v_2$ be a weak supersolution and subsolution to (1.3) in $\Omega_S$, respectively. Suppose that $v_1 \geq v_2$ on $\partial_p \Omega_S$. Then

$$v_1 \geq v_2 \text{ in } \Omega_S.$$ 

For the construction of an appropriate comparison function we use a special solution to the elliptic type equation associated with (1.3). This special solution is called *Talenti function* [28], defined as

$$Y_{a,b,y}(x) := \left( a + b|x - y|^{\frac{p}{p-1}} \right)^{-\frac{n-p}{p}}, \quad x, y \in \mathbb{R}^n, \quad (3.6)$$

where $a$ and $b$ are positive numbers. In his seminal paper [28], Talenti showed that this function realizes the best constant in the Sobolev inequality. Moreover, a direct computation shows that $Y_{a,b}$ solves the equation

$$-\Delta_p Y_{a,b,z} = n \left( \frac{n-p}{p-1} \right)^{p-1} ab^{p-1} Y_{a,b,y}^q \text{ in } \mathbb{R}^n.$$ 

As Sciunzi showed in [26], there is a one-parameter family of functions classifying the solutions, up to translations, of $-\Delta_p Y_{\lambda} = Y_{\lambda}^q$. Indeed, one chooses $a$ and $b$ so that $n \left( \frac{n-p}{p-1} \right)^{p-1} ab^{p-1} = 1$ and then, by [26], the solution of $-\Delta_p Y = Y^q$ is necessarily of the form

$$Y(x) = Y_{\lambda,y}(x) := \frac{1}{\lambda} \left( n \left( \frac{n-p}{p-1} \right)^{p-1} \right)^{\frac{1}{p}} \left( 1 + \left( \frac{|x - y|}{\lambda} \right)^{\frac{p}{p-1}} \right)^{-\frac{n-p}{p}} \quad (3.7)$$

with a parameter $\lambda > 0$. We show that a solution of (1.3) vanishes in finite time.
Proposition 3.6 (Finite time extinction of solutions) Let \( v \) be a nonnegative weak solution to (1.3) in \( \Omega_\infty \). Then there exists a extinction time \( S^* > 0 \) for \( v \), which is bounded as follows :

\[
S^* \leq \frac{q}{q + 1 - p} \left( \frac{\max_{\Omega} u_0}{\min_{\Omega} Y} \right)^{q+1-p},
\]

where \( Y \) is Talenti’s function defined by (3.7).

Proof. By translation, we may assume the origin \( 0 \in \Omega \). Let \( v = v(x, s) \) be a nonnegative weak solution to (1.3). By the nonnegativity \( v \) is a weak solution of

\[
\partial_s v^q - \Delta_p v = 0 \quad \text{in} \quad \Omega_\infty.
\]

Next, let \( W(x, s) = X(x)Z(s) \) be a nonnegative separable solution of

\[
\partial_s W^q - \Delta_p W = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty).
\]

Then \( X(x)Z(s) \) satisfies

\[
\begin{cases}
(Z(s)^q)' = \mu Z(s)^p-1 & \text{in } (0, \infty) \\
\Delta_p X = \mu X^q & \text{in } \mathbb{R}^n,
\end{cases}
\]

(3.8)

where \( \mu \) is a separation constant. By an integration by parts we see that \( \mu < 0 \). Set \( X := (-\mu)^{-\frac{1}{q+1-p}}Y \) to obtain

\[
-\Delta_p Y = Y^q \quad \text{in } \mathbb{R}^n.
\]

(3.9)

As discussed before the proof, an energy-finite solution to (3.9) is given by (3.7). By a straightforward computation, we find that

\[
Z(s) = Z(0) \left( 1 + \frac{q + 1 - p}{q} Z(0)^{p-(q+1)}s \right)^{\frac{1}{q+1-p}}
\]

solves the first equation in (3.8), where \( Z(0) \) is the initial data. Thus the vanishing time \( Z_0 \) of \( Z(s) \) is given by

\[
Z_0 = (-\mu)^{-1} \frac{q}{q + 1 - p} Z(0)^{q+1-p}.
\]

Let \( V(x, s) \) be \( (-\mu)^{-\frac{1}{q+1-p}}Y(x)Z(s) \). Then

\[
0 = v(x, s) \leq V(x, s) \quad \text{on } \partial \Omega \times [0, \infty)
\]

We choose the initial data for the ODE in (3.8) as

\[
Z(0) = \frac{\max_{\Omega} u_0}{\min_{\Omega} Y} (-\mu)^{\frac{1}{q+1-p}}
\]

(3.10)
and therefore, we find that
\[
u_0(x) \leq V(x, 0) \quad \text{in} \quad \Omega.
\]
According to the comparison theorem [21, Theorem 3.6], we have
\[
v(x, s) \leq V(x, s) \quad \text{in} \quad \Omega_S \quad \text{for any positive} \quad S < \infty
\]
and thus, the vanishing time \(S^*\) of \(v(x, s)\) is estimated as
\[
S^* \leq S_0 = \frac{q}{q + 1 - p} \left( \frac{\max_{\Omega} u_0}{\min_{\Omega} Y} \right)^{q+1-p},
\]
where (3.10) is used. The proof is complete.

4 Nonlinear Intrinsic Scaling Transformation

In this section we will introduce a scaling transformation, which transforms the prototype equation (1.3) into the \(p\)-Sobolev equation (2.4). Hereafter we choose the initial data \(v_0\) in (1.3) as \(u_0\) in (2.4). We suppose that the initial data \(u_0\) is in \(W_0^{1,p}(\Omega)\), positive and bounded in \(\Omega\), and \(\|u_0\|_{q+1} = 1\). As in Theorem 3.2, by [21, Proposition 3.4], the solution \(v\) of (1.3) must be nonnegative and thus, we can consider (1.3) as
\[
\begin{cases}
\partial_s (v^q) - \Delta_p v = 0 & \text{in} \quad \Omega_S \\
v = 0 & \text{on} \quad \partial \Omega \times (0, S) \\
v(\cdot, 0) = u_0(\cdot) & \text{in} \quad \Omega.
\end{cases}
\quad (4.1)
\]
From now on, we will mainly consider (4.1) instead of (1.3).

Let us consider the following nonlinear intrinsic scaling transforming (4.1) to (2.4).

**Proposition 4.1 (Nonlinear intrinsic scaling)** Let \(v\) be a nonnegative weak solution to the equation (4.1) in \(\Omega_\infty\) and let \(S^* < +\infty\) be a finite extinction time of \(v\). There exist unique \(\Lambda \in C^1[0, \infty)\) solving
\[
\begin{cases}
\Lambda'(\tau) = (S^*)^{-1} \left( \int_{\Omega} v^{q+1} \left( x, S^* \left( 1 - e^{-\Lambda(\tau)} \right) \right) \, dx \right)^\frac{p}{n} \\
\Lambda(0) = 0
\end{cases}
\quad (4.2)
\]
and, subsequently, \(g \in C^1[0, \infty)\) solving
\[
\begin{cases}
g'(t) = e^{\Lambda(g(t))} \\
g(0) = 0
\end{cases}
\quad (4.3)
\]
such that the following is valid: Let
\[ s(t) = S^* \left( 1 - e^{-\Lambda(g(t))} \right) \]
and set
\[ u(x,t) := \frac{v(x,s(t))}{\gamma(t)}, \quad \gamma(t) := \left( \int_{\Omega} v^{q+1}(x,s(t)) \, dx \right)^{\frac{1}{q+1}}. \]

Then \( u \) is a nonnegative weak solution of the \( p \)-Sobolev flow \((2.4)\) on \( \Omega_\infty \). More precisely, \( u \) satisfies the conditions \((D1)-(D4)\) of Definition 2.2, where \( \lambda(t) := -\frac{q}{\gamma(t)} = \int_{\Omega} |\nabla u(x,t)|^p \, dx \).

**Proof of Proposition 4.1.** Here, we will make a formal computation and show the relevance of intrinsic scaling above to the \( p \)-Sobolev flow. The rigorous argument will be given in Appendix C.

Firstly, let us verify that \( u \) satisfies \((2.4)_1\). Noticing \((4.4)\)
\[ s(t) = S^*(1 - e^{-\Lambda(g(t))}) \iff \Lambda(g(t)) = \log \left( \frac{S^*}{S^* - s(t)} \right), \]
we compute as
\[ \frac{d}{dt} \Lambda(g(t)) = \Lambda'(g(t))g'(t) \]
\[ = (S^*)^{-1} e^{\Lambda(g(t))} \left( \int_{\Omega} v^{q+1}(x,s(t)) \, dx \right)^{\frac{q}{q+1}} \]
\[ = (S^*)^{-1} e^{\Lambda(g(t))} \gamma(t)^{(q+1)\frac{q}{q+1}} \]
and thus,
\[ s_t = \frac{ds}{dt} = S^* e^{-\Lambda(g(t))} \frac{d}{dt} \Lambda(g(t)) = \gamma(t)^{(q+1)\frac{q}{q+1}}. \]

By \((3.2)\) and \((3.4)\) in Theorem 3.2, \( s \mapsto \left( \int_{\Omega} v^{q+1}(x,s) \, dx \right)^{\frac{q}{q+1}} \) is Lipschitz continuous.

This together with \((4.6)\) provides
\[ \partial_t u^q = \partial_x v^q s_t \gamma^{-q} + v^q(-q)\gamma^{-q-1} \gamma'(t) \]
\[ = \partial_x v^q \gamma^{-q+(q+1)\frac{q}{q+1}} - qv^q \gamma^{-q-1} \gamma'(t). \]

Multiplying \((2.4)_1\) by \( v \) and integration by parts give
\[ \frac{q}{q+1} \frac{d}{ds} \int_{\Omega} v^{q+1}(s) \, dx + \int_{\Omega} |\nabla v(s)|^p \, dx = 0, \]
that is the same reasoning as (3.2) in Theorem 3.2. Furthermore

$$\gamma'(t) = \frac{1}{q + 1} \left( \int_{\Omega} v^{q+1}(x, s(t)) \, dx \right)^{\frac{q}{q+1}} \frac{d}{ds} \int_{\Omega} v^{q+1}(s) \, dx \bigg|_{s=s(t)}$$

$$= \frac{1}{q} \gamma^{1-(q+1)} \int_{\Omega} |\nabla v(s(t))|^p \, dx \cdot s_t$$

$$\overset{(4.8)}{=} -\frac{1}{q} \gamma^{1-p} \int_{\Omega} |\nabla u(t)|^p \, dx$$

(4.9)

and

$$\Delta_p u = \gamma^{1-p} \Delta_p v.$$  (4.10)

From (4.7) and (4.9) it follows that

$$\partial_t u^q = \partial_k v^q \gamma^{1-p} + \left( \int_{\Omega} |\nabla u(t)|^p \, dx \right) u^q$$

$$= \left[ \partial_k v^q + \left( \int_{\Omega} |\nabla v(s(t))|^p \, dx \right) v^q \gamma^{-(q+1)} \right] \gamma^{1-p}.$$  (4.12)

Eq. (4.11) together with (4.10) and Eq. (2.4) yield that

$$\partial_t u^q - \Delta_p u = \left( \int_{\Omega} |\nabla u(t)|^p \, dx \right) u^q,$$

which is exactly (2.4).1.

We will verify that $u$ satisfies the condition (D1) in Definition 2.2. Let $t_0 < \infty$ be any positive number and set $s_0 = S^*(1 - e^{-\Lambda(g(t_0)))}$. We shall notice the fact: As shown later in (C.6) in Lemma C.2, we find that there is a positive number $c_0$ such that $c_0 := \min_{0 \leq s \leq s_0} \|v(s)\|_{q+1} > 0$. From $\gamma(t) \geq c_0 > 0$ and (3.4), it follows that

$$\int_{\Omega} |\nabla u(t)|^p \, dx = \frac{1}{\gamma(t)} \int_{\Omega} |\nabla v(s(t))|^p \, dx$$

$$\leq c_0^{-p} \left\|\nabla v_0\right\|_{p} < \infty$$  (4.13)

and thus, $u \in L^\infty(0, t_0; W^{1,p}(\Omega))$. By changing of variable $s = S^*(1 - e^{-\Lambda(g(t)))}$ and (4.6), and merging (4.12),(C.6) in Lemma C.2, (3.1), (3.3), (3.4) and (3.5) we get

$$\int_{0}^{t_0} \int_{\Omega} (\partial_t u^q)^2 \, dxdt = \int_{0}^{s_0} \int_{\Omega} (\partial_t u^q)^2 \gamma^{-(q+1)} \, dxds$$
In addition, \( T \) is (D3) with \( g \) because \( \Lambda(\tau) = 0 \) by (4.14) and (4.15), \( g(0) = 0 \) by (4.3) and (4.2). By the Minkowski and Sobolev inequalities, we get, as \( t \to 0 \),

\[
|\gamma(t) - \gamma(0)| \leq \|v(s(t))\|_{q+1} - \|u(0)\|_{q+1} \leq C \|v(s(t)) - u_0\|_{W^{1,p}(\Omega)} \to 0
\]

since \( \|v(s) - u_0\|_{W^{1,p}(\Omega)} \to 0 \) as \( s \to 0 \). Merging \( \gamma(t) \geq c_0 > 0 \), (4.14) and (4.15), we obtain \( \|u(t) - u_0\|_{W^{1,p}(\Omega)} \) as \( t \to 0 \), that gives (D4).

Therefore we finish the proof. \( \square \)

5 Proof of Theorem 1.1

In this section, we shall prove our main theorem, Theorem 1.1.

The scheme of our proof is the following: Firstly, by Theorem 3.2, we will solve the prototype equation (1.3) with the initial data \( u_0 \) and then, by Proposition 4.1, we transform the solution \( v \) to the desired solution \( u \) of the \( p \)-Sobolev flow (2.4), that can be possible up to any finite time, since the extinction time of solution \( v \) is converted to the infinity. Here, the expansion of positivity of the solution of (2.4) on the domain is
used. In particular, the time-length of expansion of positivity is estimated only by the volume, \( \|u_0\|_{L^{r+1}(\Omega)} = 1 \), the boundedness, and the positive lower bound of the initial data \( u_0 \) in the interior of domain. See Proposition 5.4 for details. The solution \( u \) is actually bounded at any finite time as in (5.1) of Proposition 5.1. In this way, we have the global existence of solution of the \( p \)-Sobolev flow (2.4).

We have the boundedness of weak solutions of \( p \)-Sobolev flow (2.4). Here we use by the fact that by Proposition 2.3 we have that \( \lambda(t) = \|\nabla u(t)\|^p_p \) in (2.4).

Proposition 5.1 \textbf{(Boundedness of the \( p \)-Sobolev flow)} Let \( u \) be a nonnegative weak solution of (2.4) in \( \Omega_T \). Then \( u \) is bounded from above in \( \Omega_T \) and

\[
\|u(t)\|_\infty \leq e^{\frac{2}{15}} \int_0^T \|\nabla u(t)\|^p_p dt \|u_0\|_\infty \quad \text{for every} \quad 0 \leq t < T. \tag{5.1}
\]

In [21] we proved the expansion of positivity of a solution of the doubly nonlinear equation such as (2.4) and (1.3). In particular, the convexity of domain is not needed by virtue of the so-called Harnack chain argument. See [21, Theorem 4.7, Corollary 4.8] for detail and its proof. We are going to deduce the refined assertion of them.

Before stating, we set the notation as below. Let \( \Omega' \) be a subdomain contained compactly in \( \Omega \). Let \( \rho \) be any positive number satisfying \( \rho \leq \frac{1}{16} \text{dist}(\Omega', \partial \Omega) \).

Now, we state the refinement of expansion of positivity with a waiting time (cf. [21, Theorem 4.9]).

Theorem 5.2 \textbf{(Expansion of positivity with a waiting time)} Let \( u \) be a nonnegative weak solution of (2.4) in \( \Omega_T \). Let \( \Omega' \) be a subdomain contained compactly in \( \Omega \). Let \( \rho \) be any positive number satisfying \( \rho \leq \text{dist}(\Omega', \partial \Omega)/16 \). Let \( t_0 \in (0, T] \). Suppose that

\[
|\Omega' \cap \{u(t_0) \geq L\}| \geq \alpha |\Omega'|
\]

holds for some \( L > 0 \) and \( \alpha \in (0, 1] \). Then there exist positive integer \( N = N(\Omega', \rho) \), positive real number families \( \delta_0, \delta_N, \eta_N, \eta_{N+1}, \sigma_N \in (0, 1) \), \( J_N, I_N \in \mathbb{N} \) depending on \( \alpha, N, n, p \) and independent of \( L \), and a time \( t_N > t_0 \) such that

\[
u \geq \eta_{N+1}L
\]
almost everywhere in
\[ \Omega' \times (t_N + (1 - \sigma_N)\delta_N(\eta_N L)^{q+1-p} \rho^p, t_N + \delta_N(\eta_N L)^{q+1-p} \rho^p), \]
where \( \sigma_N = e^{-(\tau_N + 2e^{-\tau_N})} \), \( e^{\tau_N} = C2^{l_N + \lambda_N} \) with \( C = C(n, p) > 0 \), and \( t_N \) is written as
\[ t_N = t_0 + (\delta_0 - \delta_N \rho^p) L^{q+1-p} \rho^p. \]

and thus, the terminal time of the time interval above is
\[ t_0 + \delta_0 L^{q+1-p} \rho^p. \]

Proof. The proof of this theorem is postponed, and will be given in Appendix A.2.

We also state the refined expansion of positivity without a waiting time (cf. [21, Corollary 4.10]).

**Proposition 5.3 (Expansion of positivity without a waiting time)** Let \( u \) be a nonnegative weak solution of (2.4) in \( \Omega_T \). Let \( \Omega' \) be a subdomain contained compactly in \( \Omega \). Let \( \rho \) be any positive number satisfying \( \rho \leq \text{dist}(\Omega', \partial \Omega)/16 \). Suppose that \( u(t_0) > 0 \) in \( \Omega \) for some \( t_0 \in [0, T) \). Then there exist positive numbers \( \eta_0 \) and \( \tau_0 \) such that
\[ u \geq \eta_0 \text{ a.e. in } \Omega' \times (t_0, t_0 + \tau_0). \]

We also give the proof of this proposition in Appendix A.2.

Applying Theorem 5.2 and Proposition 5.3 with \( t_0 = 0 \), we have another refinement of the interior positivity by the volume constraint (cf. [21, Proposition 5.4]).

**Proposition 5.4 (Interior positivity by the volume constraint)** Let the initial data \( u_0 \in W_{0,\Omega}^{1,\rho}(\Omega) \) be positive, bounded in \( \Omega \) and satisfy \( \|u_0\|_{q+1} = 1 \). Let \( u \) be a nonnegative weak solution of (2.4) in \( \Omega_T \) with \( T > 0 \). Put \( M := e^{\frac{1}{4} \|\nabla u_0\|_{p}^{2}} \|u_0\|_{\infty} \) and let \( \Omega' \) be a subdomain compactly contained in \( \Omega \) satisfying \( \|\Omega \setminus \Omega'\| \leq \frac{1}{4M^{q+1}}. \) Then there exists a positive constant \( \eta \) such that
\[ u(x, t) \geq \eta L \text{ in } \Omega' \times [0, T]. \]

Here \( 0 < L \leq \min \left\{ \left( \frac{1}{\|u_0\|_{\Omega'}} \right)^{\frac{1}{q+1}}, \inf_{\Omega'} u_0 \right\} \), where \( \Omega' \) is compactly contained in \( \Omega \) and compactly containing \( \Omega' \), and the positive constant \( \eta \) depends only on \( p, n, \Omega', M \) and \( N \), where \( N \) is the number of chain balls of \( \Omega' \). The constant \( \eta \) is given as a non-increasing positive function in \( T \) and \( \|u_0\|_{\infty} \).

We present the proof of this proposition in Appendix A.3.

Finally, we give the positivity near the boundary for the solutions of p-Sobolev flow (2.4). See [21, Propositions 5.5, 4.9] for details.
Proposition 5.5 (Positivity around the boundary) Suppose that \( u_0 > 0 \) in \( \Omega \). Let \( u \) be a nonnegative weak solution \( u \) to (2.4) in \( \Omega_T \). Then \( u \) is positive in \( \Omega_T \) near the boundary.

Under the above preliminaries, we now prove Theorem 1.1.

Proof of Theorem 1.1. We divide the proof in three steps.

Step 1: We choose the initial data \( u_0 \) as that of the \( p \)-Sobolev flow (2.4) and solve the prototype equation (1.3) with the initial data \( v_0 = u_0 \). Let \( u \) be a nonnegative weak solution of (1.3) in \( \Omega \) with the initial data \( v_0 = u_0 \). Let \( S^* \) be the extinction time of \( u \).

Step 2: Here we will verify the positivity for the \( p \)-Sobolev flow (2.4). From Propositions C.5 and 5.5, for any positive \( T < \infty \)

\[
0 < \tilde{c} \leq u \leq M =: e^{\frac{1}{q} \int_0^T \| \nabla u(t) \|_p^p dt} \| u_0 \|_{\infty} \quad \text{in} \quad \Omega' \times [0, T].
\]

From the global existence result for the prototype doubly nonlinear equation (1.3) in Theorem 3.2 and the nonlinear intrinsic scaling transformation in Proposition 4.1, we plainly get the global existence for the \( p \)-Sobolev flow (2.4).

Step 3: Finally, we will show the local Hölder regularity for the \( p \)-Sobolev flow (2.4).

Following [21, Section 5.2], we recall the result of Hölder and gradient Hölder continuity of the solution to \( p \)-Sobolev flow (2.4) with respect to space-time variable.

Suppose the initial value \( u_0 > 0 \) in \( \Omega \). Then by Propositions 5.4 and 5.1, for any \( \Omega' \) compactly contained in \( \Omega \) and \( T \in (0, \infty) \), we can take a positive constant \( \tilde{c} \) such that

\[
0 < \tilde{c} \leq u \leq M =: e^{\frac{1}{q} \int_0^T \| \nabla u(t) \|_p^p dt} \| u_0 \|_{\infty} \quad \text{in} \quad \Omega' \times [0, T].
\]

As discussed in [21, Section 5.2], by (5.4), we can rewrite the first equation of (2.4) as follows: Set \( v := u^q \), which is equivalent to \( u = v^{\frac{1}{q}} \) and put \( g := \frac{1}{q} v^{1/q-1} \) and then, it is easily seen that the first equation of (2.4) is equivalent to

\[
\partial_t v - \text{div} ( |\nabla v|^{p-2} g^{p-1} \nabla v ) = \lambda(t)v \quad \text{in} \quad \Omega' \times [0, T]
\]

and thus, \( v \) is a positive and bounded weak solution of the evolutionary \( p \)-Laplacian equation (5.5). By (5.4) \( g \) is uniformly elliptic and bounded in \( \Omega_T' \).

The following Hölder continuity is proved via the local energy inequality for a local weak solution \( v \) to (5.5) ([21, Lemma C.1]) and standard iterative real analysis methods. See also[7, Chapter III] or [30, Section 4.4, pp.44–47] for more details.
Theorem 5.6 (Hölder continuity [21, Theorem 5.6]) Let \( v \) be a positive and bounded weak solution to \((5.5)\). Then \( v \) is locally Hölder continuous in \( \Omega_T \) with a Hölder exponent \( \beta \in (0,1) \) on a parabolic metric \( |x| + |t|^{1/p} \).

By a positivity and boundedness as in \((5.4)\) and a Hölder continuity in Theorem 5.6, we see that the coefficient \( g^{p-1} \) is Hölder continuous and thus, obtain a Hölder continuity of its spatial gradient.

Theorem 5.7 (Gradient Hölder continuity [21, Theorem 5.7]) Let \( v \) be a positive and bounded weak solution to \((5.5)\). Then, there exist a positive constant \( C \) depending only on \( n, p, \tilde{c}, M, \lambda(0), \beta, \|\nabla v\|_{L^p(\Omega_T)}, [g]_{\beta, \Omega_T}, [v]_{\beta, \Omega_T} \) and a positive exponent \( \alpha < 1 \) depending only on \( n, p \) and \( \beta \) such that \( \nabla v \) is locally Hölder continuous in \( \Omega_T \) with an exponent \( \alpha \) on the usual parabolic metric. Furthermore, its Hölder constant is bounded above by \( C \), where \([f]_{\beta}\) denote the Hölder semi-norm of a Hölder continuous function \( f \) with a Hölder exponent \( \beta \).

By using an elementary algebraic estimate and an interior positivity, boundedness and a Hölder regularity of \( v \) and its gradient \( \nabla v \) in Theorems 5.6 and 5.7, we also obtain a local Hölder regularity of the weak solution \( u \) to \((2.4)\) and its gradient \( \nabla u \), which gives our final assertion in Theorem 1.1.

Theorem 5.8 (Hölder regularity for the \( p \)-Sobolev flow [21, Theorem 5.7]) Let \( u \) be a positive and bounded weak solution to \((2.4)\). Then, there exist a positive exponent \( \gamma < 1 \) depending only on \( n, p, \tilde{c}, M, \lambda(0), \beta, \alpha \) and a positive constant \( C \) depending only on \( n, p, \tilde{c}, M, \lambda(0), \beta, \alpha, \|\nabla u\|_{L^p(\Omega_T)}, [g]_{\beta, \Omega_T}, [v]_{\beta, \Omega_T} \) such that, both \( u \) and \( \nabla u \) are locally Hölder continuous in \( \Omega_T \) with an exponent \( \gamma \) on a parabolic metric \( |x| + |t|^{1/p} \) and on the parabolic one, respectively. The Hölder constants are bounded above by \( C \), where \([f]_{\beta}\) denote the Hölder semi-norm of a Hölder continuous function \( f \) with a Hölder exponent \( \beta \).

From Steps 1 to 3, the proof of Theorem 1.1 is concluded.

□

Appendix A  Refined Expansion of Positivity

This section is devoted to the refinement of the expansion of positivity which is proven in [21, Section 4]. Firstly, we give the transformation stretching the time-interval, which is needed for the proof of Theorem 5.2 and Proposition 5.3.

In this section, following [21, Sections 3 and 4], we consider the doubly nonlinear equations of \( p \)-Sobolev flow type:

\[
\begin{aligned}
    \partial_t u^q - \Delta_p u & = cu^q \quad \text{in } \Omega_T \\
    0 \leq u \leq M & \quad \text{on } \partial_p \Omega_T,
\end{aligned}
\]  

\((pST)\)
where $T \in (0, \infty)$, $u = u(x,t) : \Omega_T \rightarrow \mathbb{R}$ be a nonnegative real valued function, and $c$ and $M$ are nonnegative constant and positive one, respectively. Here the initial value $u_0$ is in the Sobolev space $W^{1,p}_0(\Omega)$, positive and bounded in $\Omega$.

As mentioned in [21, Remark 3.3], a nonnegative weak solution of $p$-Sobolev flow (2.4) is a weak supersolution of $(pST)$ with $c = 0$.

Here we recall the fundamental positivity results, proved in [21, Section 4], which are referred later.

**Proposition A.1 ([21, Proposition 4.1])** Let $u$ be a nonnegative weak supersolution of $(pST)$. Let $B_\rho(x_0) \subset \Omega$ with center $x_0 \in \Omega$ and radius $\rho > 0$, and $t_0 \in (0,T)$. Suppose that

$$|B_\rho(x_0) \cap \{u(t_0) \geq L\}| \geq \alpha |B_\rho|$$

(A.1)

holds for some $L > 0$ and $\alpha \in (0,1]$. Then there exist positive numbers $\delta_0, \varepsilon_0 \in (0,1)$ depending only on $p,n$ and $\alpha$ and independent of $L$ such that

$$|B_\rho(x_0) \cap \{u(t) \geq \varepsilon L\}| \geq \frac{\alpha}{2} |B_\rho|$$

(A.2)

holds for any positive $\delta \leq \delta_0$, any positive $\varepsilon \leq \varepsilon_0$ and all $t \in [t_0, t_0 + \delta L^{q+1-p} \rho^p]$. Here, if $t_0$ is very close to $T$, then $\delta > 0$ is chosen so small that $\delta L^{q+1-p} \rho^p = T - t_0$.

**Proof.** This proposition is proved by combination of the De Giorgi iteration method and the following Caccioppoli type estimate. See [21, Proposition 4.1] for detailed proof. \hfill \Box

**Proposition A.2 ([21, Proposition 3.8])** Let $k \geq 0$. Let $u$ be a nonnegative weak supersolution of $(pST)$. Let $K$ be a subset compactly contained in $\Omega$, and $0 < t_1 < t_2 \leq T$. Here we use the notation $K_{t_1,t_2} = K \times (t_1, t_2)$. Let $\zeta$ be a Lipschitz function such that $\zeta = 0$ outside $K_{t_1,t_2}$. Then, there exists a positive constant $C$ depending only on $p,n$ such that

$$\text{ess sup}_{t_1 < t < t_2} \int_{K \times \{t\}} (k - u)^{q+1}_+ \zeta^p dx + \int_{K_{t_1,t_2}} |\nabla (k - u)_+ + \zeta|^p dx dt$$

$$\leq C \int_{K \times \{t_1\}} k^{q-1}(k - u)^{2}_+ \zeta^p dx + C \int_{K_{t_1,t_2}} (k - u)^p_+ \zeta^p dx dt$$

$$+ C \int_{K_{t_1,t_2}} k^{q-1}(k - u)^2_+ |\zeta_t| dx dt.$$  

(A.3)

We further recall the following crucial lemma.

**Lemma A.3 ([21, Lemma 4.2])** Let $u$ be a nonnegative weak supersolution of $(pST)$. Suppose further (A.1). Let $Q_{4\rho}(z_0) := B_{4\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p} \rho^p) \subset \Omega_T$, where $\delta$ is selected in Proposition A.1. Then for any $\nu \in (0,1)$ there exists a positive number $\varepsilon_{\nu}$ depending only on $p,n,\alpha,\delta,\nu$ such that

$$|Q_{4\rho}(z_0) \cap \{u < \varepsilon_{\nu} L\}| < \nu |Q_{4\rho}|.$$
Proof. This lemma is also shown by the above Caccioppoli type estimate and De Giorgi’s inequality. See [21, Lemma 4.2] for detailed proof.

As a corollary of [21, Theorem 4.4], if a solution is positive at some time \( t_0 \), its positivity expands in space-time without "waiting time". This corollary is used in the proof of Proposition 5.3. See Appendix A.2.

**Proposition A.4 ([21, Corollary 4.6])** Let \( u \) be a nonnegative weak supersolution of (pST). Assume that \( u(t_0) > 0 \) in \( B_{4\rho}(x_0) \subset \Omega \). Then there exist positive numbers \( \eta_0 \) and \( \tau_0 \) such that

\[
u \geq \eta_0 \quad \text{a.e. in} \quad B_{2\rho}(x_0) \times (t_0, t_0 + \tau_0).
\]

Proof. The proof is done by combination of letting \( L = \inf_{B_{4\rho}(x_0)} u(t_0) \) in Proposition A.1 and De Giorgi’s iteration method. See [21, Corollary 4.6] for detailed proof.

### A.1 Expansion of Positivity via Transformation Stretching the Time-interval

We study the expansion of interior positivity under a changing of variables stretching the time-interval. We choose \( \rho > 0 \) such that

\[
Q_{16\rho}(z_0) := B_{16\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p}\rho^p) \subset \Omega_T,
\]

where the a positive \( \delta \leq \delta_0 \) is selected in Proposition A.1. By translation, we may assume \( z_0 = (x_0, t_0) = (0, 0) \). Following [8, Section 5.1, pp.73–78], we consider the following changing of variables stretching the time-interval:

\[
y = \frac{x}{\rho}, \quad -e^{-\tau} = \frac{t - \delta L^{q+1-p}\rho^p}{\delta L^{q+1-p}\rho^p}.
\]

Figure A.1: Stretching the time-interval

Note that this transformation \( (x, t) \mapsto (y, \tau) \) maps \( Q_{16\rho}(0) \) to \( B_{16} \times (0, \infty) \). For a nonnegative weak solution \( u \) to (pST) we set

\[
v(y, \tau) := \frac{1}{L} u(x, t)e^{\frac{-\tau}{q+1-p}}.
\]
and thus, by simple calculation, we find that \( v \) satisfies the following equation in the weak sense:

\[
\partial_\tau v^q = \delta \Delta_p v + \left( c\delta L^{q+1-p} r^p e^{-\tau} + \frac{q}{q + 1 - p} \right) v^q. \tag{A.6}
\]

Using (A.5), we write the conclusion (A.2) in Proposition A.1 as

\[
\left| B_1 \cap \{ v(\tau) \geq \varepsilon e^{\frac{\tau}{q+1-p}} \} \right| \geq \frac{1}{2} \alpha |B_1| \tag{A.7}
\]

for every \( \tau \in (0, \infty) \). Letting

\[
k_0 := \varepsilon e^{\frac{\tau_0}{q+1-p}}, \quad k_j := \frac{k_0}{2^j}, \quad j = 0, 1, \ldots
\]

with the parameter \( \tau_0 > 0 \) determined later, we obtain from (A.7) that

\[
\left| B_8 \cap \{ v(\tau_0) \geq k_j \} \right| \geq \frac{\alpha}{2} 8^{-n} |B_8|. \tag{A.8}
\]

We define the following two space-time cylinders:

\[
Q := B_8 \times (\tau_0 + k_0^{q+1-p}, \tau_0 + 2k_0^{q+1-p}),
\]

\[
Q' := B_{16} \times (\tau_0, \tau_0 + 2k_0^{q+1-p}).
\]

From (A.6) and similar calculation as [21, Proposition 3.8], we obtain the Caccioppoli type inequality of \( v \)

\[
\sup_{\tau_1 < \tau < \tau_2} \int_{K \times \{ \tau \}} (k - v)^{q+1} \zeta^p dy + \int_{K_{\tau_1, \tau_2}} |\nabla (k - v) + \zeta|^p dyd\tau \leq \frac{C}{\delta} \int_{K \times \{ \tau \}} k^{q-1} (k - v)^2 \zeta dxd\tau + C \int_{K_{\tau_1, \tau_2}} (k - v)^p |\nabla \zeta|^p dyd\tau + \frac{C}{\delta} \int_{K_{\tau_1, \tau_2}} k^{q-1} (k - v)^2 |\zeta| dyd\tau, \tag{A.9}
\]

where \( k \geq 0, K_{\tau_1, \tau_2} = K \times (\tau_1, \tau_2) \) for a compact set \( K \subset B_{16} \) and \( \tau_2 > \tau_1 > 0 \), and \( \zeta \) is a smooth function such that \( 0 \leq \zeta \leq 1 \) and \( \zeta = 0 \) outside \( K_{\tau_1, \tau_2} \). By the Caccioppoli type inequality (A.9) and the very similar argument as the proof of [21, Lemma 4.2], for every \( \nu > 0 \), there exists a natural number \( J \geq \left( \frac{C(n,p)}{\nu \alpha \delta^p} \right)^{\frac{1}{q-1}} \) such that

\[
\left| Q \cap \{ v < k_J \} \right| < \nu |Q|. \tag{A.10}
\]

As mentioned in [21, Remark 4.3], we can choose \( \varepsilon \) such that

\[
\varepsilon = \left( \frac{\delta}{2h} \right)^{\frac{1}{q+1-p}}. \tag{A.11}
\]
for some large natural number $I_1$ depending only on $n, p$ and $\alpha$. We also choose $k_j$ as

$$k_j = \left(\frac{\delta e^{\tau_0}}{2^j + 1}\right)^{\frac{1}{q+1-p}} \quad \text{for } j = 0, 1, \ldots, J. \quad (A.12)$$

Under such choice as above we note that $k_0^{q+1-p}/k_j^{q+1-p} = 2^j$ is a positive integer. We divide $Q$ along time direction into parabolic cylinders of number $2^J$ with each time-length $k_j^{q+1-p}$, and set

$$Q^{(\ell)} := B_8 \times \left(\tau_0 + k_0^{q+1-p} + \ell k_j^{q+1-p}, \tau_0 + k_0^{q+1-p} + (\ell + 1)k_j^{q+1-p}\right)$$

for $\ell = 0, 1, \ldots, 2^J - 1$. By (A.10) there is a $Q^{(\ell)}$ such that

$$|Q^{(\ell)} \cap \{v < k_j\}| < \nu |Q^{(\ell)}|. \quad (A.13)$$

As a result, we obtain the expansion of interior positivity of time-stretched solution $v$

![Figure A.2: Image of $Q^{(\ell)}$](image)

as follows.

**Proposition A.5** Let $v$ be a nonnegative weak solution of (A.6) in $B_{16} \times (0, \infty)$ defined by (A.5). Then

$$v \geq \frac{k_j}{2} \quad \text{a.e. in } B_4 \times \left(\tau_0 + k_0^{q+1-p} + \ell k_j^{q+1-p}, \tau_0 + k_0^{q+1-p} + (\ell + 1)k_j^{q+1-p}\right).$$

**Proof.** The proof is done by the same argument as [21, Theorem 4.4].

Under preliminaries as above, we are in position to state the main theorem in this subsection.

**Theorem A.6 (Expansion of interior positivity up to end time)** Let $u$ be a nonnegative weak solution to (pST). Let $B_{16\rho}(x_0) \times (t_0, t_0 + \delta L^{q+1-p}\rho^p) \subset \Omega_T$ with $x_0 \in \Omega, \rho > 0$, and $t_0 \in (0, T]$. Suppose (A.1). Then there exist positive numbers $\eta < 1$ and $\sigma < 1$ depending only on $n, p, \alpha$ and independent of $L$ such that

$$u \geq \eta L \quad \text{a.e. in } B_{2\rho}(x_0) \times \left(t_0 + (1 - \sigma)\delta L^{q+1-p}\rho^p, t_0 + \delta L^{q+1-p}\rho^p\right), \quad (A.14)$$

where the $\delta = \delta(n, p, \alpha) > 0$ is selected in Proposition A.1.
Proof. The assertion is verified by Proposition A.5, the scaling back \((x, t) \leftrightarrow (y, \tau)\) and De Giorgi’s iteration method as in [21, Theorem 4.4]. Again, by translation, we may consider \((x_0, t_0) = (0, 0)\). It follows from Proposition A.5 and (A.12) that for almost every time \(\tau_1, \tau_0 + k_0^{q+1-p} + \ell k_j^{q+1-p} < \tau_1 < \tau_0 + k_0^{q+1-p} + (\ell + 1)k_j^{q+1-p}\)

\[
v(y, \tau_1) \geq \frac{1}{2} \left( \frac{\delta e^{\tau_0}}{2^{l+J}} \right)^\frac{1}{q+1-p} \quad \text{a.e. in } B_4,
\]

which, letting \(t_1\) by (A.4) with \(\tau_1\) as

\[
-e^{-\tau_1} = \frac{t_1 - \delta L_0^{q+1-p} \rho^p}{\delta L_0^{q+1-p} \rho^p},
\]

leads to

\[
u(x, t_1) \geq \frac{1}{2} \left( \frac{\delta e^{-(\tau_1-\tau_0)}}{2^{l+J}} \right)^\frac{1}{q+1-p} \quad \text{L a.e. in } B_{4\rho}.
\]

(A.16)

For brevity, we set

\[
L_0 := \frac{1}{2} \left( \frac{\delta e^{-(\tau_1-\tau_0)}}{2^{l+J}} \right)^\frac{1}{q+1-p} \quad \text{L}.
\]

(A.17)

Since it follows from (A.16) that \(|B_{4\rho} \cap \{u(t \geq L_0) \geq \frac{1}{2} |B_{4\rho}|\} \geq \frac{1}{2} |B_{4\rho}|\)

(A.18)

holds for every \(t \in [t_1, t_1 + \tilde{\delta}L_0^{q+1-p} \rho^p]\). For a positive \(\theta \leq \tilde{\delta}L_0^{q+1-p}\) let \(Q_{4\rho}^\theta := B_{4\rho} \times (t_1, t_1 + \theta \rho^p)\). By Lemma A.3, for every \(\tilde{\nu} \in (0, 1)\) there exists a positive \(\tilde{\epsilon}\rho\) such that

\[
|Q_{4\rho}^\theta \cap \{u < \tilde{\epsilon}\rho L_0 \} < \tilde{\nu} |Q_{4\rho}^\theta|.
\]

(A.19)

As in the proof of [21, Theorem 4.4], let

\[
\rho_m := \left( 2 + \frac{1}{2m-1} \right) \rho, \quad Q_m := B_{\rho_m} \times (t_1, t_1 + \theta \rho^p),
\]

\[
\theta := \tilde{\delta}L_0^{q+1-p}, \quad \kappa_m := \left( \frac{1}{2} + \frac{1}{2m+1} \right) \tilde{k}_{I_2}.
\]

where \(\tilde{k}_{I_2} := \frac{\tilde{\epsilon}\rho L_0}{2^{l+J}}\) with \(\tilde{\epsilon}\rho\) in (A.19) and a natural number \(I_2\) satisfying \(I_2 \geq C_{(n, p)}\)

\[
\left( \frac{\rho_0}{\rho_\infty} \right)^{\frac{p}{q+1-p}}. \quad \text{It then plainly holds that}
\]

\[
\begin{cases}
4\rho = \rho_0 \geq \rho_m \geq \rho_\infty = 2\rho \\
Q_{4\rho}^\theta = Q_0 \supset Q_m \supset Q_\infty = B_{2\rho} \times (t_1, t_1 + \theta \rho^p) \\
\tilde{k}_{I_2} \geq \kappa_m \geq \kappa_\infty = \tilde{k}_{I_2}/2.
\end{cases}
\]
Letting \( Y_m := \int_{Q_m} \chi_{\{\kappa_m - u > 0\}} \, dz \) and using the Caccioppoli type inequality (A.3) with the cut-off function \( \zeta = \zeta(x) \) in \( Q_m \) as the proof of [21, Corollary 4.6], we obtain
\[
Y_{m+1} \leq C b^m Y_m^{1 + \frac{p}{n}}, \quad m = 0, 1, \ldots, 
\]
where \( b := 2^{(1 + \frac{2}{p}) + \frac{2 + p + 1}{n}} > 1 \). By the fast geometric convergence lemma (see [21, Lemma 2.3] and also [7, Lemma 4.1, p12]), if
\[
Y_0 \leq C b^{n\nu} =: \tilde{\nu}_0, 
\]
then
\[
Y_m \to 0 \quad \text{as} \quad m \to \infty. 
\]
Eq. (A.20) follows from (A.19) with \( \tilde{\nu} = \tilde{\nu}_0 \) and thus, (A.21) gives that
\[
u \geq \tilde{\nu}_0 \quad \text{in} \quad B_{2\rho} \times (t_1, t_1 + \theta \rho^p),
\]
that is,
\[
u \geq \frac{\tilde{\nu}_0}{4} \left( \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{l+J}} \right)^{\frac{1}{q+1-p}} L \quad \text{in} \quad B_{2\rho} \times (t_1, t_1 + \delta L_0^{q+1-p} \rho^p),
\]
where we put \( I := I_1 + I_2 \in \mathbb{N} \), which depends only on \( n, p, \alpha, \varepsilon \) and \( \delta \) and independent of \( L \) and \( \tilde{\nu} = \tilde{\nu}_0 \). Here we note that
\[
\delta L_0^{q+1-p} = \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{l+J}} \left( \frac{L}{2} \right)^{q+1-p}.
\]
Thus, the inequality (A.22) is written as
\[
u(x, t) \geq \frac{\tilde{\nu}_0}{4} \left( \frac{\delta e^{-(\tau_1 - \tau_0)}}{2^{l+J}} \right)^{\frac{1}{q+1-p}} L \quad \text{in} \quad B_{2\rho}
\]
for all times \( t_1 < t < t_1 + \delta L_0^{q+1-p} \rho^p \). The transformed \( \tau_0 \) is still free of choice, and it will be selected as follows. By the change of variables (A.4) with (A.15), \( \tau_0 \) is chosen as
\[
\delta L_0^{q+1-p} \rho^p - t_1 = \frac{\delta \delta L_0^{q+1-p}}{2^{l+J}} \left( \frac{L}{2} \right)^{q+1-p} \rho^p
\]
\[
\iff \tau_0 = \log \left( \frac{2^{l+J+q+1-p}}{\delta} \right). 
\]
This \( \tau_0 \) depends only on \( n, p, \varepsilon, \delta \) and \( \alpha \) because \( \delta \) and \( \tilde{\nu} \) depend only on \( n, p \). Therefore, (A.23) holds for all times
\[
t_1 = \delta L_0^{q+1-p} \rho^p - \frac{\delta \delta L_0^{q+1-p}}{2^{l+J}} \left( \frac{L}{2} \right)^{q+1-p} \rho^p < t \leq \delta L_0^{q+1-p} \rho^p.
\]
Lastly, we will estimate the above "left edge time" $t_1$. Since

$$
\tau_1 \leq \tau_0 + 2k_0^{q+1-p} \iff e^{-\tau_1} \geq e^{-(\tau_0 + 2k_0^{q+1-p})},
$$

$$
k_0^{q+1-p} = \varepsilon^{q+1-p} \varepsilon^{\tau_0} \leq e^{\tau_0} \iff e^{-2k_0^{q+1-p}} \geq e^{-2\varepsilon^{\tau_0}}, \tag{A.26}
$$

it follows from (A.24) and (A.26) that

$$
t_1 = \delta L^{q+1-p} \rho^p - \frac{\delta e^{-\tau_1}}{2^{q+1-p}} \left( \frac{L}{2} \right)^{q+1-p} \rho^p
= (1 - e^{-\tau_1}) \delta L^{q+1-p} \rho^p
\leq (1 - e^{-(\tau_0 + 2k_0^{q+1-p})}) \delta L^{q+1-p} \rho^p
\leq (1 - e^{-(\tau_0 + 2\varepsilon^{\tau_0})}) \delta L^{q+1-p} \rho^p.
$$

Here we set $\sigma := e^{-(\tau_0 + 2\varepsilon^{\tau_0})} \in (0, 1)$. This together with (A.23) implies that

$$
u \geq \frac{\tilde{\varepsilon}}{4} \left( \frac{\delta e^{-2\varepsilon^{\tau_0}}}{2^{q+1-p}} \right)^{\frac{1}{q+1-p}} L \text{ in } B_{2\rho} \times ((1 - \sigma)\delta L^{q+1-p}, \delta L^{q+1-p} \rho^p)
$$

and thus, letting $\eta := \frac{\tilde{\varepsilon}}{4} \left( \frac{\delta e^{-2\varepsilon^{\tau_0}}}{2^{q+1-p}} \right)^{\frac{1}{q+1-p}}$, we complete the proof. \hfill $\square$

### A.2 Proof of Theorem 5.2 and Proposition 5.3

This subsection is devoted to the detailed proof of Theorem 5.2 and Proposition 5.3.

We shall prove Theorem 5.2 by using Theorem A.6 and a method of chain of finitely many balls as used in Harnack’s inequality for harmonic functions, which is so-called **Harnack chain** (see [9, Theorem 11, pp.32–33] and [4, 17] in the $p$-parabolic setting). Here we use the special choice of parameters, as explained in Theorem A.6 (see also [21, Theorem 4.4]).

#### Proof of Theorem 5.2

We follow a similar argument as [21, Section 4.3].

We will prove the assertion in four steps.

**Step 1**: Since $\Omega'$ is compact, it is covered by finitely many balls $\{B_{\rho}(x_j)\}_{j=1}^N$ ($x_j \in \Omega'$, $j = 1, 2, \ldots, N$) with $N = N(\Omega', \rho)$, such that

$$
\Omega' \subset \bigcup_{j=1}^N B_{\rho}(x_j), \quad \rho < |x_i - x_{i+1}| < 2\rho, \quad B_{16\rho}(x_i) \subset \Omega, \text{ for all } 1 \leq i \leq N,
$$

where we put $x_{N+1} = x_1$. For brevity we denote $B_{\rho}(x_j)$ by $B_j$ for each $j = 1, 2, \ldots, N$. 

---

25
By (5.2), there exists at least one \( B_j = B_\rho(x_j) \), denoted by \( x_1 = x_j \) and \( B_1 = B_j \), such that
\[
|B_1 \cap \{ u(t_0) \geq L \}| \geq \frac{\alpha}{2^n}|B_1|.
\]
Thus, by Theorem A.6, there exists positive numbers \( \delta_0, \varepsilon_0, \sigma_0, \eta_1 \in (0, 1) \) depending only on \( p, n \) and \( \alpha_0 = \alpha \) and independent of \( L \) such that
\[
u \geq \eta_1 L \quad \text{a.e. in} \quad B_1 \times (t_0 + (1 - \sigma_0)\delta_0 L^{q+1-p}\rho^p, t_0 + \delta_0 L^{q+1-p}\rho^p), \quad (A.27)
\]
where \( \sigma_0 := e^{-(\tau_0 + 2\varepsilon_0)} \), \( e^{\tau_0} = C(n, p)2^{I_0 + J_0} \) for some \( I_0, J_0 \in \mathbb{N} \) depending only on \( n, p \) and \( \alpha_0 \), and \( \eta_1 = \frac{\tilde{\varepsilon}_0}{4} \left( \frac{\delta_0 e^{-2\tau_0}}{2^{I_0 + J_0}} \right)^{\frac{q+1-p}{q+1-p}} \) for some \( \tilde{\varepsilon}_0 = \tilde{\varepsilon}_0(n, p) \in (0, 1) \). We put \( \eta_0 = 1 \) for later reference.

**Step 2:** By \( \rho < |x_1 - x_2| < 2\rho \),
\[
D_1 := B_1 \cap B_2 \neq \emptyset.
\]

Via (A.27), we have
\[
u \geq \eta_1 L \quad \text{a.e.} \quad D_1 \times I_0, \quad (A.28)
\]
where let \( I_0 := (t_0 + (1 - \sigma_0)\delta_0 L^{q+1-p}\rho^p, t_0 + \delta_0 L^{q+1-p}\rho^p) \). By (A.28), for any \( t_1 \in I_0 \),
\[
|D_1 \cap \{ u(t_1) \geq \eta_1 L \}| = |D_1|,
\]
which is, setting \( \alpha_1 := \frac{|D_1|}{|B_2|} \in (0, 1) \),

\[
|B_2 \cap \{ u(t_1) \geq \eta_1 L \}| \geq \alpha_1 |B_2|.
\]

By the very same argument as Step 1, there exist positive numbers \( \delta_1, \sigma_1, \eta_2 \in (0, 1) \) depending only on \( p, n \) and \( \alpha_1 \) and independent of \( L \) such that

\[
u \geq \eta_2 L \quad \text{a.e. in } B_2 \times \left( t_1 + (1 - \sigma_1)\delta_1(\eta_1 L)^{q+1-p} \rho^p, t_1 + \delta_1(\eta_1 L)^{q+1-p} \rho^p \right),
\]

(A.29)

where \( \sigma_1 := e^{-\tau_1}, \ e^{\tau_1} = C(n, p)^{\frac{1}{2}q+1}, \ \text{for some large } I_1, J_1 \in \mathbb{N} \) depending only on \( n, p \) and \( \alpha_1 \), and \( \eta_2 = \frac{\tilde{\varepsilon}_1}{4} \left( \frac{\delta_1 e^{-\tau_1}}{2^{I_1+J_1}} \right) \eta_1 \) for some \( \tilde{\varepsilon}_1 = \tilde{\varepsilon}_1(n, p) \in (0, 1) \). Here we choose \( t_1 \in \mathcal{I}_0 \) as

\[
t_1 := t_0 + (\delta_0 - \delta_1(\eta_1 L)^{q+1-p} L^{q+1-p} \rho^p) > 0
\]

\[ \iff t_1 + \delta_1(\eta_1 L)^{q+1-p} \rho^p = t_0 + \delta_0 L^{q+1-p} \rho^p.\]

Note that this choice of \( t_1 \in \mathcal{I}_0 \) is admissible by

\[
t_1 - (t_0 + (1 - \sigma_0)\delta_0 L^{q+1-p} \rho^p) = \left( \sigma_0 - \delta_1 \left( \frac{\tilde{\varepsilon}_0}{4} \right)^{q+1-p} e^{-2\tau_0} \frac{\delta_1}{2^{I_0+J_0}} \right) \delta_0 L^{q+1-p} \rho^p > 0
\]

and \( \delta_1 \) can be chosen as small.

**Step 3:** We will proceed by induction on \( m \). Suppose that for some \( m \in \{1, 2, \ldots, N\} \)

\[
u \geq \eta_m L \quad \text{a.e. in } B_m \times \mathcal{I}_{m-1}.
\]

(A.30)

Here let

\[
\mathcal{I}_{m-1} := \left( t_{m-1} + (1 - \sigma_{m-1})\delta_{m-1}(\eta_{m-1} L)^{q+1-p} \rho^p, t_{m-1} + \delta_{m-1}(\eta_{m-1} L)^{q+1-p} \rho^p \right)
\]

with \( t_{m-1} := t_0 + (\delta_0 - \delta_{m-1}(\eta_{m-1} L)^{q+1-p} L^{q+1-p} \rho^p) \), where \( \delta_{m-1}, \sigma_{m-1} \in (0, 1), \tau_{m-1} > 0 \) are determined inductively as follows: \( \sigma_{m-1} := e^{-\tau_{m-1}}, \ e^{\tau_{m-1}} = C(n, p)^{\frac{1}{2}q+1}, \ \text{for some large } I_{m-1}, J_{m-1} \in \mathbb{N} \) depending only on \( n, p \) and \( \alpha_{m-1} = \frac{|D_{m-1}|}{|B_m|} = \frac{|D_{m-1}|}{|B_2|} = \alpha_1 \) as before, and \( \eta_m = \frac{\tilde{\varepsilon}_{m-1}}{4} \left( \frac{\delta_{m-1} e^{-2\tau_{m-1}}}{2^{I_{m-1}+J_{m-1}}} \right) \eta_{m-1} \) for some \( \tilde{\varepsilon}_{m-1} = \tilde{\varepsilon}_{m-1}(n, p) \in (0, 1) \).

By \( \rho < |x_m - x_{m+1}| < 2\rho \) again,

\[
D_m := B_m \cap B_{m+1} \neq \emptyset
\]

and thus, (A.30) yields that

\[
u \geq \eta_m L \quad \text{a.e. in } D_m \times \mathcal{I}_{m-1}.
\]

(A.31)
By (A.31), for any \( t_m \in I_{m-1} \),
\[
|B_{m+1} \cap \{ u(t_m) \geq \eta_m L \}| \geq \alpha_m |B_{m+1}|,
\]
where let \( \alpha_m := \frac{|D_m|}{|B_{m+1}|} = \frac{|D_1|}{|B_2|} = \alpha_1 \in (0, 1) \). Again, similarly as in Steps 1 and 2, there exist positive numbers \( \delta_m, \sigma_m, \eta_{m+1} \in (0, 1) \) depending only on \( p, n \) and \( \alpha_1 \) and independent of \( L \) such that
\[
u \geq \eta_{m+1} L \quad \text{a.e. in } B_{m+1} \times (t_m + (1 - \sigma_m)(\delta_m L)^{q+1-p} \rho^p, t_m + (\delta_m L)^{q+1-p} \rho^p), \tag{A.32}
\]
where \( \sigma_m := e^{-(\tau_m + 2\tau_m^0)}, e^{\tau_m} = C(n, p)2^{I_m + J_m} \) for some large \( I_m, J_m \in \mathbb{N} \) depending only on \( n, p \) and \( \alpha_1 \), and \( \eta_{m+1} = \frac{\bar{\rho}_m}{4} \left( \frac{\delta_m e^{-2\tau_m}}{2^{I_m + J_m}} \right)^{q+1-p} \eta_m \) for some \( \bar{\rho}_m = \bar{\rho}_m(n, p) \in (0, 1) \). Again, we choose \( t_m \in I_{m-1} \) as
\[
t_m := t_0 + (\delta_0 - \delta_m \eta_{m+1} L)^{q+1-p} \rho^p > 0
\]
\[
\iff t_m + \delta_m (\eta_m L)^{q+1-p} \rho^p = t_0 + \delta_0 L^{q+1-p} \rho^p,
\]
because
\[
t_m - (t_{m-1} + (1 - \sigma_{m-1})\delta_{m-1}(\eta_{m-1} L)^{q+1-p} \rho^p) = \left( \sigma_{m-1} - \delta_m \left( \frac{\bar{\rho}_{m-1}}{4} \right)^{q+1-p} \frac{e^{-2\tau_{m-1}}}{2^{I_{m-1} + J_{m-1}}} \right) \delta_{m-1}(\eta_{m-1} L)^{q+1-p} \rho^p > 0
\]
and \( \delta_m \) can be taken to be small enough. Thus our induction on \( m \) is done.

**Step 4:** By Step 3, we have, for all \( m = 1, 2, \ldots, N \),
\[
u \geq \eta_{m+1} L \quad \text{a.e. in } B_{m+1} \times I_m, \tag{A.33}
\]
where let \( B_{N+1} := B_1 \). Since, by construction, \( \{ \eta_m \}_{m=1}^{N+1} \) is decreasing, it follows from (A.33) that, for all \( m = 1, 2, \ldots, N \),
\[
u \geq \eta_{N+1} L \quad \text{a.e. in } B_{m+1} \times I_N,
\]
where we set
\[
I_N := (t_N + (1 - \sigma_N)(\delta_N L)^{q+1-p} \rho^p, t_N + (\delta_N L)^{q+1-p} \rho^p)
\]
with \( t_N = t_0 + (\delta_0 - \delta_N \eta_N L)^{q+1-p} \rho^p \). Therefore we complete the proof. \( \square \)

Lastly, we will prove Proposition 5.3.
Proof of Proposition 5.3. Since $\Omega'$ is compact, it is covered by finitely many balls
\{B_{\rho}(x_j)\}_{j=1}^{N} (x_j \in \Omega', j = 1, 2, \ldots, N), where N = N(\Omega', \rho), such that
\[ \Omega' \subset \bigcup_{j=1}^{N} B_{\rho}(x_j), \quad \rho < |x_i - x_{i+1}| < 2\rho, \quad B_{16\rho}(x_i) \subset \Omega, \text{ for all } 1 \leq i \leq N, \]
where we put $x_{N+1} = x_1$. For brevity we denote $B_{\rho}(x_j)$ by $B_j$ for each $j = 1, 2, \ldots, N$ and let
$4B_j := B_{4\rho}(x_j)$. By assumption, $u(t_0) > 0$ in each ball $4B_j$, $j = 1, \ldots, N$. Let $L_1 = \inf_{4B_j} u(t_0)$. By Corollary A.4 with $L = L_1$ (see the proof of [21, Corollary 4.6]) there exist positive numbers $\eta_1$ and $\tau_1$ depending on $n$ and $p$ such that
\[ u \geq \eta_1 L_1 \quad \text{a.e. in } B_1 \times (t_0, t_0 + \tau_1 L_1^{q+1-p} \rho^p). \]
Letting $L_2 := \inf_{4B_2} u(t_0)$ and applying Corollary A.4 with $L = L_2$, there exists positive numbers $\eta_2$ and $\tau_2$ depending on $n$ and $p$ such that
\[ u \geq \eta_2 L_2 \quad \text{a.e. in } B_2 \times (t_0, t_0 + \tau_2 L_2^{q+1-p} \rho^p). \]
Repeating this argument finitely, there exist positive numbers $\eta_N$ and $\tau_N$ depending on $n$ and $p$ such that, letting $L_N = \inf_{4B_N} u(t_0)$,
\[ u \geq \eta_N L_N \quad \text{a.e. in } B_j \times (t_0, t_0 + \tau_N L_N^{q+1-p} \rho^p). \]
for all $j = 1, \ldots, N$.

Finally, we define the subdomain of $\Omega$ as $\Omega'' := \{ x \in \Omega : \text{dist}(x, \Omega') < 4\rho \}$, and put $L := \inf_{\Omega''} u(t_0) > 0$. Then $L \leq L_i$ for $i = 1, \ldots, N$. Thus, putting $\eta_0 := \min_{i=1,\ldots,N} \eta_i L$ and $\tau_0 := \min_{i=1,\ldots,N} \tau_i L_i^{q+1-p} \rho^p$, we complete the proof.

\[ \square \]

### A.3 Proof of Proposition 5.4

We will prove Proposition 5.4 here.

Proof of Proposition 5.4. Following [21, Proposition 5.4] with a minor change, we give the proof.

Note that a nonnegative weak solution $u$ of (2.4) is a weak supersolution to (pST) with $c = 0$. By the volume constraint together with the boundedness, letting $M := e^{\int_0^T \|\nabla u(t)\|_\infty^p \, dt} \|u_0\|_\infty$, we have, for any $t \in [0, T]$
\[ 1 = \int_{\Omega} u^{q+1}(t) \, dx = \int_{\Omega \cap \{ u(t) \geq L \}} u^{q+1}(t) \, dx + \int_{\Omega \cap \{ u(t) < L \}} u^{q+1}(t) \, dx + \int_{\Omega \setminus \Omega'} u^{q+1}(t) \, dx \leq M^{q+1} |\Omega \cap \{ u(t) \geq L \} | + L^{q+1} |\Omega' | + M^{q+1} |\Omega \setminus \Omega' |; \]
i.e.,
\[
\frac{1 - L^{q+1}|\Omega'| - M^{q+1}|\Omega \setminus \Omega'|}{M^{q+1}} \leq |\Omega' \cap \{u(t) \geq L\}|
\]
Under the choice of \(\Omega'\) and \(L\) in Proposition 5.4, we find that, for any \(t \in [0, T]\),
\[
\alpha|\Omega'| \leq |\Omega' \cap \{u(t) \geq L\}|
\]
where \(\alpha := \frac{1}{2M^{q+1}|\Omega'|}\). Let \(\rho > 0\) be arbitrarily taken and fixed, satisfying \(\rho \leq \text{dist}(\Omega', \partial \Omega)/16\). We choose subdomain \(\Omega''\) as \(\Omega'' = \{x \in \Omega : \text{dist}(x, \Omega') < 4\rho\}\). By Theorem 5.2, there are positive integer \(N = N(\Omega', \rho)\), positive real number families \(\delta_0, \delta_N, \eta_N, \eta_{N+1}, \sigma_N \in (0, 1)\), \(J_N, I_N \in \mathbb{N}\) depending on \(\alpha, N, n, p\) and independent of \(L\) and a time \(t_N > t_0\) such that
\[
u \geq \eta_{N+1}L \quad \text{a.e. in} \quad \Omega' \times I_N(t),
\]
where
\[
I_N(t) := \left( t_N + (1 - \sigma_N)\delta_N(\eta_N L)^{q+1-p} \rho, t_N + \delta_N(\eta_N L)^{q+1-p} \rho \right),
\]
with \(\sigma_N = e^{-(\tau N + 2e^{-\tau}N)}\), \(e^{\tau} = C(n, p)2^{N+J_N}\) and \(t_N\) given by
\[
t_N = t + (\delta_0 - \delta_N \eta_N^{q+1-p} L^{q+1-p} \rho).
\]
Notice that the terminal time of \(I_N(t)\) is \(t + \delta_0 L^{q+1-p} \rho^p\). Meanwhile, it follows from \(u_0 > 0\) in \(\Omega\) and Proposition 5.3 with \(t_0 = 0\) that, there exist positive number \(\eta\) and \(\tau\) depending only on \(N, n, p\) such that
\[
u \geq \eta L \quad \text{a.e. in} \quad \Omega' \times (0, \tau L^{q+1-p} \rho^p).
\]
Furthermore, if \(t\) is very close to \(T\), then we can choose \(\delta_0 > 0\) so small that \(t + \delta_0 L^{q+1-p} \rho^p = T\). Since \(t \in [0, T]\) is arbitrary, choosing \(\bar{\eta} := \min\{\eta_{N+1}, \eta\}\), we have that
\[
u(x, t) \geq \bar{\eta} L \quad \text{a.e. in} \quad \Omega' \times [0, T],
\]
which is our assertion of Proposition 5.4.

**Appendix B  Convergence result**

We will recall the fundamental convergence result, used in Appendix C. A weak convergent sequence in \(L^r\), \(1 < r < \infty\), satisfying the norm convergence is strong convergent in \(L^r\), that was originally proved by Clarkson ([6]) and Hanner ([13]).

**Lemma B.1** Let \(r \in (1, \infty)\). Let \(\{f_j\}_{j=1}^{\infty}\) be a sequence in \(L^r(\Omega)\). Assume that \(f_j \rightharpoonup f\) weakly in \(L^r(\Omega)\) and \(\|f_j\|_r \rightarrow \|f\|_r\). Then we have \(f_j \rightarrow f\) strongly in \(L^r(\Omega)\).

**Proof.** It is a consequence of the uniform convexity of \(L^p\)-space. We omit the details of proof. \(\square\)
Appendix C Proof of Proposition 4.1

In this appendix, we shall prove Proposition 4.1. From now on, we will show that the function \( u \) defined by (4.5) satisfies the conditions (D1)–(D4) in Definition 2.2. Firstly, we introduce the mollifier, which is used later. For a function \( f \in L^1(\mathbb{R}^{n+1}) \), we denote the mollifier of \( f \) by

\[
f_\varepsilon(z) := (f * \rho_\varepsilon)(z) = \int_{\mathbb{R}^{n+1}} f(w)\rho_\varepsilon(z - w) \, dw.
\]

Here, \( \varepsilon > 0 \) and \( z = (x, t), w = (y, s) \in \mathbb{R}^{n+1} \) are space-time points, and let \( \rho_\varepsilon(z) := \frac{1}{\varepsilon^{n+1}}\rho\left(\frac{z}{\varepsilon}\right) \), where \( \rho(z) \) is a smooth symmetric function in the following sense:

\[
\rho(x, t) = \rho(|x|, |t|) \quad \text{for} \quad z = (x, t) \in \mathbb{R}^{n+1}
\]

and satisfies

\[
\int_{\mathbb{R}^{n+1}} \rho(z) \, dz = 1, \quad \text{supp} \rho \subset \{(x, t) : |x| < 1, |t| < 1\}, \quad \rho \geq 0.
\]

In what follows, let \( v \) be a weak solution of (1.3) with the initial data \( v_0 = u_0 \), obtained from Theorem 3.2. We extend \( v \) as \( v = u_0 \) for \( t \leq 0 \), and \( v = 0 \) outside \( \Omega \) and then the extended function is also written by the same notation. Note that this extension is Lipschitz extension of \( v \). Let us denote the mollification of \( v \) by \( v_\varepsilon \).

**Lemma C.1** We have the following uniform convergences: As \( \varepsilon \searrow 0 \),

\[
v_\varepsilon \longrightarrow v \quad \text{locally uniformly in} \quad C([0, \infty); L^{q+1}(\Omega)) \tag{C.1}
\]

and

\[
(v^q)_\varepsilon \longrightarrow v^q \quad \text{locally uniformly in} \quad C([0, \infty); L^{\frac{q+1}{q}}(\Omega)). \tag{C.2}
\]

**Proof.** From the energy equality (3.2) and the continuity of \( x^\frac{1}{q+1} \) for \( x \geq 0 \), we find that \( \|v(s)\|_{q+1} \) is locally continuous in \( s \in [0, \infty) \), in fact, locally absolutely continuous on \( [0, \infty) \). This together with Lemma B.1 and (3.2) implies that \( v \in C([0, \infty); L^{q+1}(\Omega)) \). Via the fundamental property of mollifier, (C.1) immediately follows. By the same argument as above, (C.2) readily follows. \( \square \)

Let \( S^* \) be the extinction time of this solution \( v \) of (4.1). Define \( \Lambda(\tau) \) by a \( C^1 \)-solution to the following ODE

\[
\begin{cases}
\frac{d}{d\tau} \Lambda(\tau) = \left( \int_{\Omega} v^{q+1}(x, \Lambda(\tau)) \, dx \right)^{\frac{q}{q+1}} \\
\Lambda(0) = 0.
\end{cases} \tag{C.3}
\]
Let \( g(t) \) be a \( C^1 \)-solution of the ODE on \([0, \infty)\)

\[
\begin{align*}
g'(t) &= e^{\Lambda(g(t))} \\
g(0) &= 0.
\end{align*}
\]  

(C.4)

For unique \( \Lambda \in C^1(0, \infty) \) solving (C.3) and, subsequently, \( g \in C^1(0, \infty) \) solving (C.4), let

\[
s(t) = S^* \left( 1 - e^{-\Lambda(g(t))} \right)
\]

and set

\[
u(x, t) := \frac{v(x, s(t))}{\gamma(t)} \quad \text{and} \quad \gamma(t) := \left( \int_{\Omega} v^{q+1}(x, s(t)) \, dx \right)^{\frac{1}{q+1}}.
\]

(C.5)

The ODE (C.3) is actually solvable, since the integral of the right hand side of the ODE (4.2) is (locally absolutely) continuous on \( t \) in \([0, \infty)\), by the energy equality (3.2).

We further remark that the following relation holds:

\[
s \nearrow S^* \iff t \nearrow \infty.
\]

Let \( t_0 < \infty \) be any positive number and set \( s_0 := S^* \left( 1 - e^{-\Lambda(g(t_0))} \right) \). We now deduce the positivity of \( \|v(s)\|_{q+1} \) for any nonnegative \( s \leq s_0 \).

**Lemma C.2**  There exists a positive number \( c_0 \) such that, for every nonnegative \( s \leq s_0 \),

\[
\|v(s)\|_{q+1} \geq c_0.
\]

(C.6)

**Proof.** Since \( \|v(s)\|_{q+1} \) is continuous and positive on \([0, s_0]\) there exists a positive number \( c_0 \) satisfying \( c_0 := \min_{0 \leq s \leq s_0} \|v(s)\|_{q+1} > 0 \), yielding (C.6). The proof is done.

We have to verify the regularity of the scaled solution \( u \), defined by (4.2) and (C.5).

**Lemma C.3 (The regularity of a composite function and its chain rule)**  Let \( u \) be defined by (C.5). Then there holds that

\[
u \in C([0, \infty); L^{q+1}(\Omega)), \quad \nabla u \in L_{\text{loc}}^\infty([0, \infty); L^{p}(\Omega)).
\]

(C.7)

Furthermore, the function \( \gamma(t) \) is Lipschitz on \([0, t_0]\) and the weak derivative on time \( \partial_t v^q(x, S^* \left( 1 - e^{-\Lambda(g(t))} \right)) \) is in \( L^2(\Omega_{\infty}) \). In addition, there exists \( \partial_t u^q \in L^2(\Omega_{t_0}) \) for any \( t_0 < \infty \) such that

\[
\partial_t u^q = \partial_t v^q \cdot \gamma^{-q} - q v^q \gamma^{-q-1} \gamma'(t)
\]

(C.8)

in a weak sense.
Proof. The first part of this proof deals with \( u \in C([0, \infty); L^{q+1}(\Omega)) \). From (3.2) \( v(x, S^*(1 - e^{-\Lambda(g(t))})) \in C([0, \infty); L^{q+1}(\Omega)) \) and \( \gamma(t) \in C([0, \infty)) \) and thus, by the very definition (C.5), \( u(x, t) \in C([0, \infty); L^{q+1}(\Omega)) \).

Note that

\[
s(t) = S^* \left(1 - e^{-\Lambda(g(t))}\right) \quad \iff \quad t = (\Lambda \circ g)^{-1} \left(\log \left(\frac{S^*}{S^* - s}\right)\right)
\]

and, by (C.3) and (C.4),

\[
s_t = \frac{ds}{dt} = S^* e^{-\Lambda(g(t))} \frac{d}{dt} \Lambda(g(t)) = S^* e^{-\Lambda(g(t))} \Lambda'(g(t))g'(t) = \|v(s)\|_{q+1}\frac{q}{n}.
\]

Thus, by the changing of variable \( s = s(t) = S^* \left(1 - e^{-\Lambda(g(t))}\right) \) and integration by parts, one has for any \( \varphi \in C_0^\infty(\Omega_\infty) \),

\[
\int_0^\infty \int_\Omega u(x, t) \nabla \varphi(x, t) \, dx \, dt = \int_0^{S^*} \frac{v(x, s)}{\|v(s)\|_{q+1}} \nabla \varphi \left(x, (\Lambda \circ g)^{-1} \left(\log \left(\frac{S^*}{S^* - s}\right)\right)\right) \, dx \, \|v(s)\|_{q+1}^{-\frac{q+1}{n}} \, ds
\]

\[
= \int_0^{S^*} \|v(s)\|_{q+1}^{-\frac{1}{n}} \nabla v(x, s) \varphi \left(x, (\Lambda \circ g)^{-1} \left(\log \left(\frac{S^*}{S^* - s}\right)\right)\right) \, dx \, \|v(s)\|_{q+1}^{-\frac{q+1}{n}} \, ds,
\]

since \( \nabla v \in L^\infty((0, \infty); L^p(\Omega)) \) by (3.4) and \( \text{supp} \; \varphi \left(x, (\Lambda \circ g)^{-1} \left(\log \left(\frac{S^*}{S^* - s}\right)\right)\right) \subset \Omega_{S^*} \).

Again, the changing of variable \( t = (\Lambda \circ g)^{-1} \left(\log \left(\frac{S^*}{S^* - s}\right)\right) \) yields

\[
\int_0^\infty \int_\Omega u(x, t) \nabla \varphi(x, t) \, dx \, dt = -\int_0^\infty \nabla \varphi \left(x, \frac{s(t)}{\gamma(t)}\right) \, dx \, dt
\]

and thus, there exists a weak derivative \( \nabla u(x, t) \) such that, for any nonnegative \( t_0 < \infty \),

\[
\nabla u(x, t) = \frac{\nabla v(x, s(t))}{\gamma(t)} \in L^\infty([0, \infty); L^p(\Omega)).
\]
In order to prove (C.8) being valid in the weak sense, we verify that \( \gamma(t) \) and \( v^q(x,s(t)) \) are weak differentiable in \( t \) and their weak derivatives are integrable on \((0,t_0)\) and \( \Omega_\infty \), respectively.

Firstly, we show \( \gamma(t) \) is weak differentiable in \((0,\infty)\). Now, set \( f(s) := \|v(s)\|^{q+1}_{q+1} \).

From (3.2) \( f(s) \) is a locally absolutely continuous function on \( s \in [0,\infty) \) and for a.e. \( s \in [0,\infty) \),
\[
\frac{d}{ds}f(s) = \frac{d}{ds}\|v(s)\|^{q+1}_{q+1} = -\frac{q+1}{q}\|\nabla v(s)\|^p_p \tag{C.10}
\]
and \( \frac{d}{ds}f(s) \) is bounded on \((0,\infty)\) by (3.4) and thus, \( f(s) \) is actually Lipschitz function.

Now, \( f_h(s) \) and \( \frac{d}{ds}f_h(s) \) denote the mollification with respect to time variable \( s \) of \( f(s) \) and \( \frac{d}{ds}f(s) \), respectively. According to the fundamental property of mollifier, we have, as \( h \to 0 \),
\[
f_h(s) \to f(s) \quad \text{locally uniformly on } [0,\infty) \tag{C.11}
\]
and, for all \( r \geq 1 \),
\[
\frac{d}{ds}f_h(s) \to \frac{d}{ds}f(s) \quad \text{strongly in } L^r(0,\infty). \tag{C.12}
\]

By integration by parts we see that, for every \( \phi(t) \in C^\infty_c((0,\infty)) \),
\[
\int_0^\infty \phi(t) \left[ f_h(s(t)) \right] ' dt = -\int_0^\infty \phi'(t) f_h(s(t)) dt, \tag{C.13}
\]
where \( ' := \frac{d}{dt} \). By changing of the variable \( s = s(t) = S^* (1 - e^{-\Lambda(g(t))}) \), the integration in the left hand side of (C.13) is computed as
\[
\int_0^\infty \phi(t) \left[ f_h(s(t)) \right] ' dt = \int_{0}^{\infty} \phi(t) \frac{d}{ds}f_h(s(t)) s_t dt \tag{C.14}
\]
which converges to
\[
\int_{0}^{\infty} \phi ((\Lambda \circ g)^{-1} (\log (\frac{S^*}{S^*/s})) \frac{d}{ds}f_h(s) ds, \tag{C.15}
\]
as \( h \to 0 \), by (C.12). On the other hand, the integration in the right hand side of (C.13) converges to
\[
-\int_0^\infty \phi'(t) f_h(s(t)) dt \to -\int_0^\infty \phi'(t) f(s(t)) dt \tag{C.15}
\]
as \( h \searrow 0 \), by (C.11). From (C.13), (C.14) and (C.15) we obtain
\[
\int_0^\infty \phi(t) s_i \frac{d}{ds} f(s(t)) \, dt = - \int_0^\infty \phi'(t) f(s(t)) \, dt
\]
and thus, there exists a weak derivative \( \frac{d}{dt} f(s(t)) \) in \((0, \infty)\) such that
\[
\frac{d}{dt} f(s(t)) = s_i \frac{d}{ds} f(s(t))
\]
and therefore, by (3.3) and (4.2), \( f(s(t)) \) is Lipschitz on \([0, \infty)\). From (C.16), \( \gamma(t) \) is weak differentiable in \((0, t_0)\) and
\[
\gamma'(t) = \frac{d}{dt} \|v(s(t))\|_{q+1}^{\frac{1}{q+1}}
\]
\[
= \frac{1}{q+1} \|v(s(t))\|^{-q}_{q+1} \frac{d}{dt} f(s(t))
\]
\[
= \frac{1}{q+1} \|v(s(t))\|^{-q}_{q+1} s_i \frac{d}{ds} f(s(t)),
\]
because \( \gamma^{\frac{1}{q+1}} \) is Lipschitz for \( \gamma \geq c_0 \). From (3.3), (3.4), (4.2), (C.6), (C.10) and (C.17), it follows that, for every \( t \in (0, t_0) \),
\[
|\gamma'(t)| \leq \frac{c_0}{q} \|u_0\|^{(q+1)\frac{p}{q}}_{q+1} \|\nabla u_0\|_p
\]
and thus, \( \gamma(t) \) is surely Lipschitz on \([0, t_0]\).

Next, we will verify that the weak derivative on time of \( v^q(x, S^* (1 - e^{-\Lambda(g(t)))}) \) is in \( L^2(\Omega_\infty) \). From Definition 3.1 it follows that \( \partial_s v^q(x, s) \in L^2(\Omega_\infty) \) and thus,
\[
\partial_s (v^q)_\epsilon \longrightarrow \partial_s v^q \text{ strongly in } L^2(\Omega_\infty).
\]
Again, by integration by parts we see that, for every \( \varphi \in C^\infty_0(\Omega_\infty) \),
\[
\int_0^\infty \int_\Omega \varphi \partial_t (v^q)_\epsilon \, dx \, dt = - \int_0^\infty \int_\Omega \partial_t \varphi (v^q)_\epsilon \, dx \, dt.
\]
By \( s = s(t) = S^* (1 - e^{-\Lambda(g(t)))} \) and (C.19), the integration in the left hand side of (C.20) is computed as
\[
\int_0^\infty \int_\Omega \varphi \partial_t (v^q)_\epsilon \, dx \, dt = \int_0^\infty \int_\Omega \varphi(x, t) \partial_s (v^q)_\epsilon (x, f(s(t)))) s_t \, dx \, dt
\]
35
\[
\int_0^S \int_0^\Omega \varphi(x, (\Lambda \circ g)^{-1} \left( \log \left( \frac{S^*}{S^{-x}} \right) \right)) \partial_s(v^q)(x, s) \, dx ds, 
\]
which converges to
\[
\int_0^S \int_0^\Omega \varphi(x, (\Lambda \circ g)^{-1} \left( \log \left( \frac{S^*}{S^{-x}} \right) \right)) \partial_s(v^q)(x, s) \, dx ds
\]
\[
= \int_0^\infty \varphi(x, t) \partial_s v^q(x, s(t)) s_t \, dx dt 
\]
(C.21) as \( \varepsilon \downarrow 0 \). The integration in the right hand side of (C.20) converges as
\[
- \int_0^\infty \partial_t \varphi(v^q)_\varepsilon \, dx dt \rightarrow - \int_0^\infty \partial_t \varphi v^q \, dx dt 
\]
(C.22) as \( \varepsilon \downarrow 0 \), by (C.2). By (C.20), (C.21) and (C.22) we have
\[
\int_0^\infty \varphi(x, t) \partial_s v^q(x, s(t)) s_t \, dx dt = - \int_0^\infty \partial_t \varphi v^q \, dx dt
\]
and thus, there exists a weak derivative \( \partial_t v^q(x, s(t)) \) in \( \Omega \times (0, \infty) \) such that
\[
\partial_t v^q = \partial_s v^q(x, s(t)) s_t. 
\]
(C.23)
By (3.3), (4.2) and (C.23)
\[
|\partial_t v^q| \leq \|u_0\|_{q+1}^\frac{q+1}{q} \left| \partial_s v^q(x, S^* \left( 1 - e^{-\Lambda(g(t))} \right)) \right| 
\]
(C.24)
and thus, the weak derivative on time of \( v^q(x, s(t)) \) is in \( L^2(\Omega_\infty) \).

Note that \( \gamma \mapsto \gamma^{-q} \) is locally Lipschitz on \( \{ \gamma \geq c_0 \} \) and \( \gamma(t)^{-q} \) is the composite function of \( \gamma^{-q} \) with \( \gamma(t) \). This together with (4.5), (C.17), (C.18), (C.23) and (C.24) yields that there exists \( \partial_t u^q \in L^2(\Omega_{t_0}) \) for any \( t_0 < \infty \) such that
\[
\partial_t u^q = \partial_t v^q \cdot \gamma^{-q} - qv^q \gamma^{-q-1} \gamma'(t)
\]
in a weak sense, which is the desired result (C.8).

**Remark C.4** Let \( t_0 < \infty \) be any positive number. By the very definition (4.5) of \( u \), one has \( \|u(t)\|_{q+1} = 1 \) for \( t \in [0, t_0] \) and \( 0 \leq u \leq c_0^{-1} \|u_0\|_{\infty} \) for every \( (x, t) \in \Omega \times [0, t_0] \) via (C.6), (3.1) and the nonnegativity of \( v \). Applying the same argument as the proof
of [21, Proposition 5.2], we readily get $\lambda(t) = \|\nabla u(t)\|_p^p$ for every $t \in [0, t_0]$ and thus, by (C.6), (C.9) and (3.4), $\lambda(t) \leq c_0^{-p}\|\nabla u_0\|_p^p < \infty$. Therefore the following equation holds true in a weak sense, and almost everywhere in $\Omega_{t_0}$:

$$\partial_t u^q - \Delta_p u = \lambda(t)u^q \quad \text{in} \quad \Omega_{t_0}$$

and thus, it plainly holds that $\Delta_p u \in L^2(\Omega_{t_0})$.

Here we obtain the interior positivity with $c_0$ in (C.6) by the volume constraint.

**Proposition C.5 (Interior positivity with $c_0$ in (C.6) by the volume constraint)**

Let the initial data $u_0 \in W^{1,p}_0(\Omega)$ be positive, bounded in $\Omega$ and satisfy $\|u_0\|_q+1 = 1$. Let $u$ be a nonnegative weak solution of (C.25) in $\Omega_{t_0}$ with any positive $t_0 < \infty$. Put $\tilde{M} := c_0^{-p}\|\nabla u_0\|^{p/q}/\|u_0\|_\infty$ and let $\Omega'$ be a subdomain compactly contained in $\Omega$ satisfying $|\Omega \setminus \Omega'| \leq \frac{1}{4M^{q+1}}$. Then there exists a positive constant $\tilde{\eta}$ such that

$$u(x, t) \geq \tilde{\eta} L \quad \text{in} \quad \Omega' \times [0, t_0].$$

Here $0 < L \leq \min \left\{ \left( \frac{1}{\|u_0\|_q+1} \right)^{\frac{1}{q+1}}, \inf_{\Omega'} u_0 \right\}$, where $\Omega''$ is compactly contained in $\Omega$ and compactly containing $\Omega'$, and the positive constant $\tilde{\eta}$ depends only on $p, n, \Omega', M$ and $N$, where $N$ is the number of chain balls of $\Omega'$. The constant $\tilde{\eta}$ also depends on the positive constant $c_0$.

**Proof.** The proof of this proposition is done by the same argument as Proposition 5.4. \hfill \Box

**References**

[1] H.W. Alt and S. Luckhaus, Quasilinear Elliptic-Parabolic Differential Equations, *Math. Z.* 183 (1983), 311–341.

[2] T. Aubin, Equations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, *J. Math. Pures Appl.* 55 (1976), 269–296.

[3] T. Aubin, *Some nonlinear problems in Riemannian geometry*, Springer Monographs in Mathematics, 1997.

[4] B. Avelin, T. Kuusi and K. Nyström: Boundary behavior of solutions to the parabolic $p$-Laplace equation, *Anal. PDE* 12(1) (2019), 1–42.

[5] J.W. Barrett and W.B. Liu, Finite element of approximation of the parabolic $p$-Laplacian, *SIAM J. Numer. Anal.*, Vol. 34(2) (1994), 413–428.

[6] J. A. Clarkson, Uniformly convex spaces, *Trans. Amer. Math. Soc.* 40 (1936), no. 3, 396–414.
[7] E. DiBenedetto, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[8] E. DiBenedetto, U. Gianazza and V. Vespri, *Harnack’s inequality for degenerate and singular parabolic equations*, Springer Monographs in Mathematics, 2012.

[9] L.C. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, 1998.

[10] U. Gianazza and V. Vespri, Parabolic De Giorgi classes of order $p$ and the Harnack inequality, *Calc. Var. Partial Differential Equations*, 26(3) (2006), 379–399.

[11] M. Giaquinta, *Introduction to regularity theory for nonlinear elliptic systems*, Birkhäuser Verlag (1993).

[12] R.S. Hamilton, *Lectures on geometric flows*, (1989) (unpublished)

[13] O. Hanner, On the uniform convexity of $L^p$ and $\ell^p$, *Ark. Mat.* 3 (1956), 239–244.

[14] A. V. Ivanov, Uniform Hölder estimates for generalized solutions of quasilinear parabolic equations that admit double degeneration, *Algebra i Analiz*, 3(2):139–179, 1991, Translation in *St. Petersburg Math. J.* 3 (1992), no. 2, 363–403.

[15] A. V. Ivanov. Hölder estimates for a natural class of equations of fast diffusion type Hölder estimates for equations of fast diffusion type. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 229 (11):29–322, 1995, Translation in *J. Math. Sci.* (New York) 89 (1998), no. 6, 1607–1630.

[16] C. Karim and M. Misawa, Gradient Hölder regularity for nonlinear parabolic systems of p-Laplacian type, *Differential Integral Equations* 29 (2016), no. 3-4, 201–228.

[17] T. Kuusi, G. Mingione and K. Nyström.: A boundary Harnack inequality for singular equations of p-parabolic type, *Proc. Amer. Math. Soc.* 142(8) (2014), 2705-2719.

[18] T. Kuusi, J. Siljander and J.M. Urbano, Local Hölder continuity for doubly nonlinear parabolic equations, *Indiana Univ., Math. J.* 61(1) (2012), 399–430.

[19] J. Kinnunen and T. Kuusi, Local behavior of solutions to doubly nonlinear parabolic equations. *Math. Ann.* 337(3) (2007), 705–728.

[20] T. Kuusi, Harnack estimates for weak supersolutions to nonlinear degenerate parabolic equations, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (V)* 7 (2008), 673-716.

[21] T. Kuusi, M. Misawa and K. Nakamura, Regularity estimates for the $p$-Sobolev flow, *Journal of Geometric Analysis*, ”*Perspectives of Geometric Analysis in PDEs*” (2019), 1–48.

[22] M. Misawa, Local Hölder regularity of gradients for evolutional $p$-Laplacian systems, *Ann. Mat. Pura Appl.* 181 (2002), 389–405.
[23] K. Nakamura and M. Misawa, Existence of a weak solution to the $p$-Sobolev flow, *Non. Anal. TMA* **175C** (2018), 157–172.

[24] M. M. Porzio and V. Vespri, Hölder estimates for local solutions of some doubly nonlinear degenerate parabolic equations, *J. Differential Equations*, **103** (1) (1993), 146–178.

[25] H. Schwetlick and M. Struwe, Convergence of the Yamabe flow for large energies, *J. Reine Angew. Math.* **562** (2003), 59–100.

[26] B. Sciunzi, Classification of positive $\mathcal{D}^{1,p}(\mathbb{R}^N)$-solution to the critical $p$-Laplace equation, *Adv. Math.* **291** (2016), 12–23.

[27] T. Suzuki and Y. Ueoka, Lecture on partial differential equations–a course in semi-linear elliptic equations, *Baifukan*, 2005 (Japanese).

[28] G. Talenti, Best constant in Sobolev inequality, *Ann. Mat. Pura Appl.*, (4)**110** (1976), 353–372.

[29] N.S. Trudinger, Pointwise estimates and quasilinear parabolic equations, *Comm. Pure Appl. Math.*, **21** (1968), 205–226.

[30] J.M. Urbano, *The method of intrinsic scaling*, Lecture Notes in Mathematics **1930**, Springer-Verlag, Berlin, 2008.

[31] J.L. Vazquez, *The porous medium equation*. Mathematical theory. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, Oxford, 2007.

[32] J.L. Vazquez, *Smoothing and decay estimates for nonlinear diffusion equations. Equations of porous medium type*. Oxford Lecture Series in Mathematics and its Applications, 33. Oxford University Press, Oxford, 2006.

[33] J.L. Vazquez, Personal communication, 2013.

[34] V. Vespri, On the local behavior of solutions of a certain class of doubly nonlinear parabolic equations, *Manuscripta Math.* **75** (1992), 65–80.

[35] V. Vespri, Harnack type inequalities for solutions of certain doubly nonlinear parabolic equations. *J. Math. Anal. Appl.*, **181**(1) (1994), 104–131.

[36] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, *Osaka Math. J.* **12** (1960), 21–37.

[37] R. Ye, Global existence and convergence of Yamabe flow, *J.Diff. Geom.*, **39** (1994), 35–50.