N=2 Supersymmetry, Painleve III and Exact Scaling Functions in 2D Polymers

P. Fendley* and H. Saleur†

*Department of Physics
Boston University
590 Commonwealth Avenue
Boston, MA 02215

†Department of Physics
Yale University
New Haven, CT 06519

We discuss in this paper various aspects of the off-critical $O(n)$ model in two dimensions. We find the ground-state energy conjectured by Zamolodchikov for the unitary minimal models, and extend the result to some non-unitary minimal cases. We apply our results to the discussion of scaling functions for polymers on a cylinder. We show, using the underlying $N=2$ supersymmetry, that the scaling function for one non-contractible polymer loop around the cylinder is simply related to the solution of the Painleve III differential equation. We also find the ground-state energy for a single polymer on the cylinder. We check these results by numerically simulating the polymer system. We also analyze numerically the flow to the dense polymer phase. We find there surprising results, with a $c_{\text{eff}}$ function that is not monotonous and seems to have a roaming behavior, getting very close to the values $81/70$ and $7/10$ between its UV and IR values of $1$.

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Since the seminal paper on conformal field theory [1], there has been steady progress in understanding two-dimensional critical systems (see [2]). Properties at the critical point are by now (almost) fully controlled, and new developments have dealt with the vicinity of the critical point, where the ultimate goal is to compute all scaling functions. Most of the interesting models turn out to be integrable. Factorizability then allows one to find the $S$-matrix exactly [3], and use it to calculate the exact ground-state energy [4]. Although many scaling functions are known in principle, their explicit computation is a formidable task [5]. This is in contrast with the beautiful and much older case of the Ising model, where it was shown [6] that spin correlation functions follow from solutions of the Painleve III transcendent equation and its generalizations. Recently however, similar equations have been met when studying ground-state properties of $N=2$ supersymmetric field theories [7]. Using the underlying $N=2$ supersymmetry in polymers [8] we are able here to make contact with these results and to exhibit a scaling function in the polymer problem that is given by Painleve III. Remarkably, this property is derived directly from continuum considerations, and awaits a lattice derivation.

We start by considering the $O(n)$ model. Although results are expected to be universal we refer for concreteness to its realization on the hexagonal lattice [9]. Although the model is defined originally for $n$ integer, the high-temperature expansion allows analytic continuation to arbitrary $n$. The partition function reads

$$Z = \sum K^{\# \text{monomers}} n^{\# \text{loops}},$$

where the sum is taken over all configurations of self-avoiding and mutually-avoiding loops that can be drawn on the lattice, and each elementary edge that belongs to a loop is referred to as a monomer. Although this geometrical definition involves global properties, locality can be restored if one considers instead oriented loops with a weight $\exp(+i\alpha/6)$ for each left turn, and $\exp(-i\alpha/6)$ for each right turn. When $n = 2\cos \alpha$, the relation $|\# \text{left turns} - \# \text{right turns}| = 6$ for a closed loop on a planar hexagonal lattice gives the correct weight for each loop. (We discuss later what happens on a cylinder.)

The model has interesting phase transitions for $n$ between -2 and 2. The standard (also called dilute) critical point occurs for $K_c = \left[ 2 + (2 - n)^{\frac{1}{2}} \right]^{-\frac{1}{2}}$. Parametrizing

$$n = 2 \cos \frac{\pi}{1},$$

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with $2 < t < \infty$, the continuum theory at $K = K_c$ has central charge

$$c = 1 - 6/t(t + 1).$$

The phase $K < K_c$ is disordered. The phase $K > K_c$ is massless and is referred to as dense phase, because the polymers cover a non-vanishing fraction of space even in the continuum limit. The continuum limit does not depend on the value of $K$, and is a conformal field theory with central charge

$$c = 1 - 6/(t - 1)t.$$

In the dilute critical theory the thermal operator (i.e. the one coupled to $K$) is described as $\Phi_{13}$ with conformal weight $h = \frac{t - 1}{t + 1}$. As discussed in [11] this perturbation is integrable. We discuss mainly the case $K > K_c$ in the following. The corresponding S-matrix has been studied in [11]; the integrability requires that the $N$-body S-matrix elements factorize into a product of two-body ones and thus obey the Yang-Baxter equation. One starts by considering the $O(n)$ model for $n$ integer and assumes that there is a particle for each color, whose trajectories may be thought of intuitively as the loops in [11]. The $O(n)$ symmetry allows the S-matrix to be written in terms of a set of invariant tensors, where these tensors obey an algebra depending on $n$. Using this algebra, the Yang-Baxter equation reduces to a single functional equation with scalar quantities, where $n$ appears as a parameter. Analytic continuation becomes possible, although for $n$ non-integer it does not define a bonafide S-matrix. However, quantities we calculate using this S-matrix should be analytic in $n$. Therefore, the form-factors calculated in [11] and the ground-state energy levels derived here are valid for any $n$.

As was also observed in [12] in a slightly different language, the main property of the S-matrix of [11] is that the invariant tensors obey the Temperley-Lieb algebra [13] (with parameter $n$). As in lattice models, many interesting properties do not depend on the particular representation of this algebra. Thus we can switch from the representation used in [11], defined only for $n$ integer, to the 6-vertex model representation [14], defined for any $n$. In the present context, this amounts to considering the sine-Gordon S-matrix [15]

$$S = \frac{Z(\theta)}{\sinh \left( \frac{\theta - i\pi}{t} \right)} \left[ \sinh \left( \frac{\theta - i\pi}{t} \right) - \sinh \left( \frac{\theta}{t} \right) e \right],$$

where

$$e = q^{-1}E_{11} + qE_{22} - E_{12} - E_{21},$$
\( q = e^{i\pi/t}, (E_{ij})_{kl} = \delta_{ik}\delta_{jl} \). In this new picture the massive particles are the sine-Gordon soliton-antisoliton pair whose trajectories can be thought of as the oriented loops in the local formulation of the \( O(n) \) model. The S-matrix (5) differs from the usual one by a gauge transformation that ensures proper algebraic properties [16,17], but our calculation is invariant under this transformation. There are no bound states in the spectrum for the sine-Gordon couplings we will study.

We shall derive in the following the ground-state energies \( E(R) \) of the \( O(n) \) model on a cylinder of radius \( R \) and large length \( L \). We will calculate these energies for a variety of boundary conditions around the cylinder. It is well known [18] that when one switches from the \( O(n) \) S-matrix to the sine-Gordon S-matrix, boundary conditions must be treated very carefully. This is similar to the situation in the lattice model when one switches from the geometrical definition to the local one involving oriented loops. On a cylinder, there are non-contractible loops in the \( O(n) \) model, and for these the number of left and right turns are equal [19]. In the local formulation they therefore get a weight 2 instead of the desired value \( n \). This boundary condition does not exactly reproduce the \( O(n) \) model; it amounts to introducing at infinity an operator of conformal dimension \( h = -1/4t(t+1) \) [20].

A useful check on the results is to take the ultraviolet (mass \( \to 0 \)) limit of \( E(R) \), because a general result of conformal field theory [21,22] gives

\[
E_{UV} = -\frac{\pi}{6R}(c - 12\bar{h} - 12\bar{h}), \tag{7}
\]

where \( h \) and \( \bar{h} \) are the conformal dimensions of the operator creating this state. The ground state is that created by the lowest-dimension operator; only in unitary theories does this have \( h = \bar{h} = 0 \). In non-unitary theories one often defines an effective central charge \( c_{\text{eff}} \) as \(-6RE(R)/\pi\). Thus in the case where non-contractible loops have weight 2, \( c_{\text{eff}} = 1 \) in the UV. We will show that, as expected, the behavior of \( E(R) \) under thermal perturbation is given by that for the sine-Gordon model with unmodified boundary conditions, which indeed has \( c_{\text{UV}} = 1 \).

The technique we will use to calculate the ground-state energies from the S-matrix is known as the thermodynamic Bethe ansatz (TBA) [4]. The TBA gives the free energy \( f = -T \ln Z \) of a particle gas on a line of large length \( L \) at a temperature \( T \). This is equivalent to considering a spacetime of a cylinder of radius \( R = 1/T \) and length \( L \). If we change our point-of-view and think of the spatial direction as (Euclidean) time, only
the ground state contributes to $Z$, since only the lowest-energy state can propagate over the large distance $L$. Thus $Z = \exp(-LE(R))$, where $E(R)$ is the ground-state (Casimir) energy with space a circle of radius $R$. The equivalence between the two approaches (often called modular invariance) means that $E(R) = -\ln Z/L = Rf/L$. Introducing chemical potentials allows one to change the boundary conditions in the $R$ direction [23]. The TBA equations for arbitrary chemical potential [24] give the ground-state energy

$$E_{\{\lambda\}}(R) = -\sum_a \frac{m_a}{2\pi} \int d\theta \cosh \theta \ln(1 + \lambda_a e^{-\epsilon_a(\theta)}),$$  \hspace{1cm} (8)

where the $\epsilon_a$ obey the integral equations

$$\epsilon_a(\theta) = m_a R \cosh(\theta) - \sum_b \int \frac{d\theta'}{2\pi} \phi_{ab}(\theta - \theta') \ln(1 + \lambda_b e^{-\epsilon_b(\theta')}).$$  \hspace{1cm} (9)

The index $a$ labels the particle species, $\lambda_a$ are fugacities, $m_a$ are the particle masses and the $\phi_{ab}$ depend on the S-matrix. If the S-matrix is not diagonal, some of the particles in (9) are zero-mass “pseudo-particles”, which result from the diagonalization of the S-matrix.

To do the TBA calculation, we need to diagonalize the sine-Gordon S-matrix, meaning that we find the eigenvalues $\Lambda(\theta_i | \theta_1, \theta_2, \ldots, \theta_N)$ for bringing the $i$th particle of momentum $p_i = m \sinh \theta_i$ through all the others [23,24]. These eigenvalues are defined precisely in the Appendix. Diagonalizing the sine-Gordon transfer matrix is formally equivalent to diagonalizing the XXZ model transfer matrix, so doing our thermodynamics is very similar to doing the thermodynamics of the XXZ model, which was considered in [27]. Putting periodic boundary conditions in the $L$ direction quantizes the $i$-th momentum via

$$e^{i p_i L} \Lambda(\theta_i | \theta_1, \theta_2, \ldots, \theta_N) = 1.$$  \hspace{1cm} (10)

This enables us to define the density of states $P_0(\theta)$ for the particles, whether solitons or antisolitons (since the scattering is elastic, the total number of particles is conserved). Taking the logarithm of (10) gives

$$2n\pi = m \sinh \theta_i + \text{Im} \ln \Lambda(\theta_i | \theta_1, \ldots, \theta_N)$$

where $n$ is an integer. Then

$$2\pi P_0(\theta_i) = 2\pi \frac{dn}{d\theta_i} = mL \cosh \theta_i + \frac{d}{d\theta_i} \text{Im} \ln \Lambda(\theta_i | \theta_1, \ldots, \theta_N).$$  \hspace{1cm} (11)
To find the partition function, one takes the thermodynamic trace over all the eigenstates, which is equivalent to summing over all soliton and antisoliton configurations. The difficulty of this study depends on arithmetic properties of the parameter $t$.

We shall first consider the case $t$ integer. In this case, the eigenvalues for the XXZ transfer matrix can be written as functionals of densities of pseudoparticles called $a$-strings with $a = 1, 2, \ldots, t - 1$ and an antipseudoparticle $(1^-)$. Call $P_a^+$ the associated densities, $P_a^-$ the corresponding densities of holes, and $P_a = P_a^+ + P_a^-$. Call also $P_0^+$ the density of real particles—the states which are actually occupied by solitons or antisolitons; the density of corresponding holes $P_0^- = P_0 - P_0^+$. As derived in [26] in the case $t = 2$ and in the general case in the Appendix, the densities obey the equations

\begin{equation}
2\pi P_0(\theta) = mL \cosh \theta + \phi_{12} \ast \rho_1, \tag{12}
\end{equation}

and

\begin{equation}
2\pi P_a = \sum_b \phi_{ab} \ast \rho_b, \tag{13}
\end{equation}

where $\phi_{ab}(\theta) = l_{ab}(\cosh(\theta))^{-1}$, $\rho_0 = P_0^+$, $\rho_1^- = P_1^+$, and $\rho_a = P_a^-$ otherwise; $\ast$ denotes convolution. The sum is over all particles, real and pseudo; $l_{ab}$ is the incidence matrix for the diagram

Each open node represents a string or the antipseudoparticle, while the node labelled $\otimes$ represents the real particle.

After minimizing the free energy with respect to $P_a^+$ subject to the constraints (12) and (13) and defining

\begin{equation}
\frac{\rho_a(\theta)}{P_a(\theta)} = \frac{\lambda_a e^{-\epsilon_a(\theta)}}{1 + \lambda_a e^{-\epsilon_a(\theta)}}, \tag{14}
\end{equation}

one finds a system of the form (9) where each node in the above diagram is a particle species (the open nodes have $m_a = 0$ while the node labeled $\otimes$ is massive), and $\phi_{ab}(\theta)$ is same as above.
Taking the UV limit of (8) involves a few tricks [4]. We define $x_a = \exp(-\epsilon_a(0))$, and $y_a = \exp(-\epsilon_a(\infty))$. One finds that

$$\lim_{cR \to 0} c_{\text{UV}} = \frac{6}{\pi} E(mR \to 0) = \frac{6}{\pi^2} \sum_a (\mathcal{L}_{\lambda_a}(x_a) - \mathcal{L}_{\lambda_a}(y_a)),$$

where

$$\mathcal{L}_\lambda(x) = \frac{1}{2} \int_C d\epsilon \left[ \frac{\epsilon \lambda e^{-\epsilon}}{1 + \lambda e^{-\epsilon}} + \ln(1 + \lambda e^{-\epsilon}) \right],$$

with the contour $C$ going from $x$ to infinity. A change of variables shows that $\mathcal{L}_1(x) = L(x/(1+x))$, and $\mathcal{L}_{-1}(x) = -L(x)$, where $L(x)$ is the Rogers dilogarithm function

$$L(x) = -\frac{1}{2} \int_0^x dy \left[ \frac{\ln y}{(1-y)} + \frac{\ln(1-y)}{y} \right],$$

It follows from (3) that the constants $x_a$ are the solutions to the equations

$$x_a = \prod_b (1 + \lambda_b x_b)^{N_{ab}},$$

where $N_{ab} = \frac{1}{2\pi} \int d\theta \phi_{ab}(\theta)$. The constants $y_a$ in (13) are nonzero only for those species $a'$ with $m_{a'} = 0$, where they are the solutions to

$$y_{a'} = \prod_{b'} (1 + \lambda_{b'} y_{b'})^{N_{a'b'}},$$

where $b'$ also runs only over massless species. These equations for the UV limit are true only when $\phi_{ab} = \phi_{ba}$.

We have set up the TBA formalism. We only need to specify the boundary conditions in the $R$ direction by fixing the fugacities. If we set all $\lambda_a = 1$, then this corresponds to taking periodic boundary conditions: the modified $O(n)$ model where non-contractible loops have a weight of 2. It is easy to find from (15)–(18) (and some dilogarithm identities [28]) one obtains $c_{\text{UV}} = 1$ as expected.

To get the actual $O(n)$ model TBA there are two possible approaches. The first one is to notice that in the lattice model, non-contractible loops on the cylinder can be given a weight $n$ if one introduces a seam [21]. Using the analogy between particle trajectories and the loops, this amounts to computing $\text{Tr} e^{-RH} e^{2\pi i\gamma F}$ with $\gamma = 1/t$, where $F = \frac{1}{2} (\# \text{solitons} - \# \text{antisolitons})$. More formally, this relation follows from the fact that $\text{tr} e = n$ for the original $O(n)$ representation while $\text{tr} e = 2$ in the sine-Gordon
replacement, even though both obey the same Temperley-Lieb algebra. From the Bethe Ansatz calculation outlined in the appendix, one finds \[F = \frac{t}{2} \int d\theta (\rho_{1-}(\theta) - \rho_{(t-1)}(\theta)),\] (19)
The appropriate trace is therefore reproduced by choosing the fugacities
\[\lambda_1 = e^{-i\pi\gamma t}, \lambda_{(t-1)} = e^{i\pi\gamma t},\] (20)
while \(\lambda = 1\) for the other species. For general \(\gamma\), this amounts to giving non-contractible loops a weight of \(2\cos \gamma \pi\).

In the UV limit, one finds, using (15)–(18) (we end up with dilogarithms at complex argument) and taking the second derivative of \(c_{UV}\) with respect to \(\gamma\)
\[c_{UV}(\gamma) = 1 - 6 \frac{(\gamma t)^2}{t(t+1)}.\] (21)
For the \(O(n)\) model ground state, \(\gamma = 1/t\) so we recover (3). Notice that this choice corresponds to \(\lambda_{(t-1)} = \lambda_1 = -1\), while \(\lambda = 1\) for the rest. This ends up removing the nodes \(t-2, t-1\) and \(1^-\). Indeed, for these nodes, the solutions of (17) are the same as (18): \(x_{(t-1)} = y_{(t-1)} = x_1 = y_1 = 1\) for the two end ones, and \(x_{(t-2)} = y_{(t-2)} = 0\) for the node joining them. When \(x_a = y_a, \epsilon_a(0) = \epsilon_a(\infty)\), which in fact means that \(\epsilon_a(\theta)\) does not depend on \(\theta\). Thus \(\epsilon_{(t-2)}(\theta) = \infty\) for all \(\theta\), it no longer contributes to the equation for \(\epsilon_{(t-3)}\), and we see that these three nodes decouple from the rest of the system without contributing to anything. Thus the \(O(n)\) TBA system at \(n = 2\cos(\pi/t)\) is described by
\[
\begin{array}{ccccccccc}
0 & 1 & 2 & t-4 & t-3 \\
\times & \circ & \circ & \circ & \circ
\end{array}
\]
where all the fugacities \(\lambda_a = 1\).

Although the \(O(n)\) model at \(t\) integer is not identical to the \(t\)-th perturbed unitary minimal model [19], their ground states coincide. Thus we have proven that the minimal-model TBA is described by the above diagram, as conjectured (and proven for \(t = 3, 4\)) in [25]. Another way of deriving the TBA for the unitary minimal models would be to consider a scattering theory that uses the RSOS S-matrix [30,16] instead of the sine-Gordon S-matrix [25]. Again, as was discussed in the context of lattice models [18] or integrable
perturbations, the S-matrix obeys another representation of the Temperley-Lieb algebra, so the thermodynamics deduced from these two cases are equivalent provided one considers the sine-Gordon or XXZ system with twisted boundary conditions. The mere effect of this is to forbid the $1^-$ antipseudoparticle and the $(t-1)$ string. The $(t-2)$ string decouples too and we are back to the above discussion.

The mass $m$ in TBA depends on the $O(n)$ lattice parameters as

$$m = \left(\frac{K_c - K_s K_c}{s R_c}\right)^{(t+1)/4},$$

where $s$ is a non-universal scale. Notice that for $\gamma = 0$ the same TBA equations would describe the ground state energy of say the 8-vertex model. In that case however, if $\delta$ is the Boltzmann weight of the additional vertices that do not conserve $U(1)$ charge, the mass $m$ would scale as $\delta^{t/2}$, i.e. with a different exponent.

We now discuss in more detail the case $n = 0$, which corresponds to $t = 2$. As is well known this describes the physics of polymers: for small $n$ we the partition function allows only a single loop, the polymer. In that case the algebraic properties of the S-matrix in can also be obtained by acting in a $\mathbb{Z}_2$ graded space, the loop cancellation occurring here due to boson-fermion symmetry. The minimal such S-matrix is equivalent, up to another gauge transformation, to the above sine-Gordon S-matrix.

As discussed in the continuum limit of polymers or $O(n = 0)$ model is described by (twisted) $N=2$ supersymmetry. This is true only when $n$ is exactly 0, so only properties which can be described precisely at this point are related to supersymmetry. The thermal perturbation $\Phi_{13}$ corresponds in the $N = 2$ language to the top component of the chiral primary field $X$, hence it preserves the $N=2$ supersymmetry. Indeed, the S-matrix for this model is precisely the sine-Gordon S-matrix for $t = 2$ discussed above.

The diagram describing TBA in this case has only 3 nodes, one of which is massive. With $\gamma = 0$ the ground state energy is minus the logarithm of the largest eigenvalue of a polymer transfer matrix that describes non-contractible loops only, with fugacity $2^3$. The values of $c_{\text{eff}}(mR)$ can be extracted from numerical solution of the TBA equations. They can also be measured on the lattice by numerical diagonalization of the polymer transfer matrix. We considered the honeycomb lattice with cylinders of radius up to

1. This is true in the regime we are interested in, but not always: see
2. The S-matrices we refer to here differ from the one in by action of the permutation operator
3. Because the lattice bulk free energy vanishes in the polymer case, we can directly identify $E$ in the numerical study of $c_{\text{eff}}$ and $d$
$R = 6$ honeycomb faces. Agreement between TBA and lattice computations was found to be excellent, even though $s$ must be determined numerically before most comparisons can be made. The corresponding curve is given in fig. 1 for $R = 6$. The main feature of this curve is the inflexion point at which the value of $c_{\text{eff}}$ extracted from TBA is 0.7714. Lattice estimates of this value are

| $R$ | 2   | 3   | 4   | 5   | 6   |
|-----|-----|-----|-----|-----|-----|
|     | 0.732 | 0.748 | 0.754 | 0.758 | 0.762 |

which converge nicely to their expected limit. Although $c_{\text{eff}}$ is not very interesting as a geometrical object, we think this numerical check is useful. In particular the fact that thermal perturbation in polymers does not break $N=2$ supersymmetry, although reasonable, is a little difficult to derive from lattice analysis (due in particular to twisting) and is indirectly checked here.

In the perturbed $N=2$ theory, a very interesting quantity is $\text{Tr} \ e^{-RHF(-)F}$. It can of course be computed from the above by introducing a fugacity $\gamma$ close to $1/2$ and taking a derivative with respect to $\gamma$, hence appearing as a solution of an integral equation. What is remarkable however is that, as discussed in [38], it is a kind of supersymmetric index, and in this case is related to the solution of a simple differential equation of Painleve III type, the radial part of the sinh-Gordon equation. Introduce the function $u$, the solution of

$$u_{zz} + z^{-1}u_z = \sinh u$$  \hspace{1cm} (22)

where $u$ has no singularity on the real axis and has asymptotic behavior as $z \to 0$

$$u(z) \sim -2/3 \ln(z) + \text{constant}$$  \hspace{1cm} (23)

Introduce on the other hand the generating function

$$Z_{nc} = \sum K^{\# \text{monomers}}$$  \hspace{1cm} (24)

where the sum is taken over all configurations of a single non-contractible loop on a cylinder of length $L$ and circumference $R$. One sees from (1) and the identification of fermion numbers for the $O(n)$ model and the sine-Gordon theory already discussed that this is precisely $\text{Tr} e^{-RHF(-)F}$. The results of [38] then give

$$Z_{nc}(z = mR) \to -\frac{L}{2R^2}z \frac{d}{dz}u(z) \quad \text{as} \quad L/R \to \infty.$$  \hspace{1cm} (25)
In particular, at the critical point one has $Z_{nc} \sim \frac{L}{3R}$. The differential equation (22) is easy to solve numerically, and for the lattice quantity it can be evaluated using a simple modification of the transfer matrix used in the previous problem. Introduce

$$d = \lim_{L/R \to \infty} \frac{6R}{\pi L} Z_{nc};$$

(26)

d goes to $2/\pi$ in the UV limit and goes to zero in the IR. Lattice measurements for $R = 5$ are represented in fig. 1. Once again, agreement with (25) is very good. For instance, the value of $d$ at the inflexion point is $.5104$. Lattice estimates are

| $R$  | 2   | 3   | 4   | 5   |
|------|-----|-----|-----|-----|
|      | .492| .498| .502| .504|

The knowledge of (24) gives access to very non-trivial geometrical information. Call $\mu = 1/K_c$ the effective connectivity constant for a given lattice. Then the number of configurations of a non-contractible loop made of $N$ monomers on a cylinder of circumference $R$ and length $L$ is expected to scale as

$$\omega_N(R) \sim LR^{-7/3} \mu^N s f(Ns/R^{4/3})$$

(27)

where $f$ is a universal function and $s$ is the lattice-dependent scale factor. From the above one finds

$$d(mR) = \frac{6}{\pi} \int e^{(mR)^{4/3} f(y)} dy$$

(28)

where $m = \left(\frac{K_c - K}{sK_c}\right)^{3/4}$. The scaling function $f$ is just simply obtained by the Laplace transform of the solution of the Painleve equation.

Another interesting geometrical quantity is the generating function $Z$ defined in a way similar to (24) but summing also over configurations where the loop is contractible. To obtain the asymptotic behavior of this function is considerably more involved. Indeed, while non-contractible loops have a behavior fully determined by the $N=2$ theory, contractible ones depend on properties of the $O(n)$ model for $n$ close but not strictly equal to 0. We therefore have to discuss how to generalize the above analysis to the case where $t$ is close to 2. We shall restrict to

$$t = 2 + \frac{1}{M}.$$  

(29)

The analysis is easier when $t$ is rational, so we take $M$ to be an integer. This corresponds to the non-unitary minimal models $(2M + 1, 3M + 1)$. Relying on the analysis of [27] we find that the free energy is as in (8), and the integral equations are of the form (9).
There is one massive particle, which we label by 0 as before. There are $M + 2$ massless pseudoparticles. The ones labelled $a = 1, \ldots, M, M + 1$ are $|2a - 1|$-strings (antistrings) if $a$ is odd (even), while $M + 1$ is a 2-string. It turns out to be simpler to write out the TBA integral equations (9) in a form where $\phi_{12}$ is not symmetric. Then we have

$$
\phi_{01}(\theta) = \frac{1}{\cosh \theta} \\
\phi_{11}(\theta) = \int dke^{ik\theta} \frac{\cosh((1 - \frac{1}{M}) \frac{\pi k}{2})}{\cosh \frac{\pi k}{2} \cosh \frac{\pi k}{2M}} \\
\phi_{12}(\theta) = -\phi_{21}(\theta) = \frac{M}{\cosh M\theta} \\
\phi_{j,j+1}(\theta) = \phi_{M,M+1}(\theta) = \frac{M}{\cosh M\theta}
$$

for $j = 2, \ldots, M$. All the $\phi_{ab}$ other than $\phi_{12}$ are symmetric, and all the remaining ones are zero. This system has an incidence diagram similar to that at $t$ integer, although here particle 1 couples to itself.

The system with all $\lambda = 1$ gives of course $c_{UV} = 1$ for all $M$, since it corresponds to sine-Gordon at different couplings.

To get the actual ground state energy (which is not the central charge) of the non-unitary minimal models, we change the fugacity for the two end nodes, just like in the unitary minimal models, by setting $\lambda_{M+1} = \lambda_{M+1} = -1$, while the rest remain 1. This again has the effect of merely removing the three end nodes $(M, M + 1, M + 1)$. We have checked that the correct value $c_{UV} = 1 - 6/(2M + 1)(3M + 1)$ is obtained in that case.

To observe the actual central charge of the minimal model one needs to project out the negative-dimension ground state. To do this, we set also $\lambda_1 = -1$. As before the three end nodes decouple. Let us give some details of the UV calculation. Our choice of $\phi_{ab}$ is not symmetric; symmetric equations are found by rewriting the equations in terms of $-\epsilon_2, -\epsilon_3, \ldots, -\epsilon_{M-1}$. (We do not change the sign on $\epsilon_0$ or $\epsilon_1$.) The UV free energy is then proportional to

$$
L \left( \frac{x_0}{1 + x_0} \right) - [L(x_1) - L(y_1)] + \sum_{j=2}^{M-1} \left[ L \left( \frac{1}{1 + x_j} \right) - L \left( \frac{1}{1 + y_j} \right) \right].
$$

(31)
The solutions of (17) and (18) are

\[ x_0 = \frac{\sin \frac{\pi}{3M+1}}{\sin \frac{M\pi}{3M+1}} \]

\[ 1 - x_1 = \left( \frac{\sin \frac{3M+1}{M\pi}}{\sin \frac{M\pi}{3M+1}} \right)^2 \]

\[ 1 + x_j = \left( \frac{\sin \frac{(M+1-j)\pi}{3M+1}}{\sin \frac{\pi}{3M+1}} \right)^2 \]

\[ 1 - y_1 = \left( \frac{\sin \frac{\pi}{2M+1}}{\sin \frac{M\pi}{2M+1}} \right)^2 \]

\[ 1 + y_j = \left( \frac{\sin \frac{(M+1-j)\pi}{2M+1}}{\sin \frac{\pi}{2M+1}} \right)^2 \]

(32)

We checked numerically that these give the correct answer \( c_{UV} = 1 - 6M^2/(2M+1)(3M+1) \). In the \( M \to \infty \) limit, the contribution from \( x_0 \) is order \( 1/M \), while the contributions of the others are individually \( 1/M^2 \). However, there are order \( M \) such contributions, so these also must be included. By replacing the sum over \( j/M \) with an integral one checks directly that \( c_{UV} \sim 5/6M \) as \( M \to \infty \) and therefore \( Z \sim \frac{5L}{18R} \).

We can recover the \( t = 2 \) case (\( M = \infty \)) by noticing that except for \( \phi_{01} \), all the non-zero \( \phi_{ab} = \pi \delta(\theta) \) at \( M = \infty \). Then all the equations (4) except for \( a = 0, 1 \) are not integral equations any more, but difference equations in \( M \). As detailed in [27], solving these gives the \( t = 2 \) result from above. The leading correction involves an infinite number of coupled integral equations, which we have not been able to simplify noticeably. Numerical solution for finite \( M \) and extrapolation to \( M \) infinite produces results in good agreement with the lattice computation. Results in the latter case for

\[ dd = \text{Lim}_{L/R \to \infty} \frac{6L}{\pi R} Z \]

(33)

are represented in fig. 1 for a cylinder of \( R = 5 \) honeycomb faces.

The flow from the critical to the dense \( O(n) \) model should also present interesting features, but the scattering theory is not so well-understood and we postpone most of its study to a subsequent publication. We shall content ourselves with the following observation. For unitary theories, one can define a \( c \)-function, which is the central charge at the conformal point and which never increases under renormalization group flows (increasing \( mR \)) [39]. It has been conjectured several times that the ground-state energy on a cylinder plays the role of an alternate \( c \)-function, even for non-unitary theories. The dilute- and dense-polymer theories are non-unitary. Moreover, in the sector where non-contractible

\[ ^4 \text{In this case a non-universal bulk free energy term must be subtracted to get } E. \]
loops get a weight 2, they both have $c_{\text{eff}} = 1$ in the UV. If a $c_{\text{eff}}$-theorem held, $c_{\text{eff}}(mR)$ here should be a constant in the flow from dilute to dense. We have estimated this quantity numerically using our polymer simulation. As always, in the dense region the convergence is a little worse than in the dilute region. But the main features of the resulting curve proved very stable, with uncertainties of only around 1% for the asymptotic quantities. The curve for $R = 6$ hexagonal faces is given in fig. 2. It violates the $c_{\text{eff}} = 1$ theorem in a spectacular fashion. Moreover, the maximum and minimum lie respectively very close to 81/70 and 7/10. These are the two values of the central charge neighboring $c = 1$ in the minimal $N = 1$ series. Recall that the $N=2$ theory related to polymers belongs also to the $N=1$ series. The flow between dilute and dense polymers seems therefore to be an example of a “roaming trajectory” as introduced in [11].

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Appendix A. Derivation of sine-Gordon TBA equations

In this appendix, we derive the equations (12) and (13) for the density of states of the sine-Gordon model at integer $t$. The allowed $\Lambda(\theta_i|\theta_1, \theta_2 \ldots \theta_N)$ in (11) are the eigenvalues of the sine-Gordon “transfer matrix” $T_{ab}(\theta_i|\theta_1, \theta_2 \ldots \theta_N)$ for bringing the $i$-th particle (with rapidity $\theta_i$) through the other $N-1$ particles and ending up with itself. The components of $T_{ab}(\theta_i)$ can be written in terms of the S-matrix elements as

$$
(T_{ab}(\theta_i))_{\{d_j\}}^{\{c_j\}} \equiv \sum_{\{k_j\}=0,1} S_{k_{Nc1}}^{d_1k_1}(\theta - \theta_1)S_{k_{1c2}}^{d_2k_2}(\theta - \theta_2) \ldots S_{k_{N-1cN}}^{d_Nk_N}(\theta - \theta_N), \quad \text{(A.1)}
$$

where the $c_j$ ($d_j$) label whether each of the $N$ particles before (after) the scattering process is a soliton or antisoliton; the $k_j$ are the intermediate particles. The sine-Gordon S-matrix is given in (3), where

$$
Z(\theta) = \exp \left( \frac{i}{2} \int_{-\infty}^{\infty} \frac{dk}{k} \sin(\theta k) \frac{\sin(t - 1) \pi k}{\sin t \pi k \cosh \frac{\pi k}{2}} \right), \quad \text{(A.2)}
$$
Formally the sine-Gordon “transfer matrix” is equivalent to the transfer matrix of the six-vertex model or of the XXZ spin chain. To diagonalize it, we can rely on the well-discussed technique of the Bethe ansatz [41]. The only additional ingredient here is the prefactor \( Z(\theta) \); by the definition (A.1) we see that its presence causes the eigenvalues \( \Lambda(\theta) \) to be multiplied by \( \prod_{j=1}^{N} Z(\theta_i - \theta_j) \). The result for integer \( t \) (for example, see [27] or [29], or [26] for the case \( t = 2 \)), is that

\[
\frac{d}{d\theta_i} \ln \Lambda(\theta_i) = \left( \frac{d}{d\theta_i} \text{Im} \ln Z \right) \ast P_0^+ - \sum_a p_a \ast P_a^+ + p_{1-} \ast P_{1-}^+, \tag{A.3}
\]

and

\[
2\pi P_a = p_a \ast P_0^+ + \sum_b \Theta_{ab} \ast P_b^+ - q_a \ast P_{1-}^+, \tag{A.4}
\]

where the sums run over the strings (but not the antipseudoparticle). The kernels in these equations are defined most easily in terms of their Fourier transforms:

\[
\tilde{\Theta}_{ab}(k) = \int \frac{d\theta}{2\pi} e^{ik\theta} \Theta_{ab}(\theta) = \delta_{ab} - 2 \frac{\cosh \frac{\pi k}{2} \sinh (t-a) \frac{\pi k}{2} \sinh b \frac{\pi k}{2}}{\sinh \frac{\pi k}{2} \sinh t \frac{\pi k}{2}}, \tag{A.5}
\]

for \( a \geq b \) with \( \Theta_{ab} = \Theta_{ba} \), and

\[
\tilde{p}_a(k) = \frac{\sinh (t-a) \frac{\pi k}{2}}{\sinh t \frac{\pi k}{2}}
\]

\[
\tilde{p}_{1-} = \tilde{p}_{(t-1)}
\]

\[
\tilde{q}_b = \tilde{\Theta}_{(t-1),b}
\]

To simplify these equations, it is much more convenient to work in Fourier space. We define \( \tilde{z}(k) \) as the Fourier transform of \( \frac{d}{d\theta_i} \text{Im} \ln Z \), so that

\[
2\tilde{z}(k) \cosh \frac{\pi k}{2} = \frac{\sinh (t-1) \frac{\pi k}{2}}{\sinh \frac{\pi k}{2} t} = \tilde{p}_1
\]

It is easy to prove the following trigonometric identities:

\[
2\tilde{\Theta}_{ab} \cosh \frac{\pi k}{2} = \tilde{\Theta}_{a+1,b} + \tilde{\Theta}_{a-1,b} - \delta_{a+1,b} - \delta_{a-1,b}
\]

\[
2\tilde{p}_a \cosh \frac{\pi k}{2} = \tilde{p}_{a-1} + \tilde{p}_{a+1} \tag{A.7}
\]

\[
2\tilde{q}_a \cosh \frac{\pi k}{2} = \tilde{q}_{a-1} + \tilde{q}_{a+1} - \delta_{a,t-2} - \delta_{a,t}.
\]
Using these in (A.4) and (A.3) gives

\[ 2 \tilde{P}_a \cosh \frac{\pi k}{2} - \tilde{P}_{a+1} - \tilde{P}_{a-1} = -\tilde{P}_{a+1}^{+} - \tilde{P}_{a-1}^{+} + \delta_{a,t-2} P_{1}^{+} \]
\[ 2 \tilde{P}_{(t-1)} \cosh \frac{\pi k}{2} - \tilde{P}_{(t-2)} = -\tilde{P}_{(t-2)}^{+} \]  \hspace{1cm} (A.8)
\[ 2 \tilde{l}(k) \cosh \frac{\pi k}{2} - \tilde{P}_{1} = -\tilde{P}_{1}^{+}. \]

where \( \tilde{l}(k) \) is the Fourier transform of \( \frac{d}{d\theta} Im \ln \Lambda(\theta) \), and \( a = 1, \ldots, t-2 \). Using \( P^{-} \equiv P - P^{+} \) and the fact that the Fourier transform of \( 1/(2 \cosh \frac{\pi k}{2}) \) is \( 1/\cosh \theta \) gives us (A.12) and (A.13).

From the Bethe ansatz one also finds that the number of solitons minus the number of antisolitons is

\[ \int d\theta \left( -P_{0}^{+}(\theta) + 2 \sum_{a} a P_{a}^{+}(\theta) + 2 P_{1}^{+}(\theta) \right) \]  \hspace{1cm} (A.9)

In the XXZ language, this is twice the value of the third component of the spin. Using the relation (A.4) for \( a = p - 1 \) gives the relation (A.19).
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Figure Captions

Fig. 1. Lattice measurements for the various scaled ground states $c_{\text{eff}}$: non-contractible polymers only, each with a weight 2. $d$: one non contractible polymer. $dd (c)$: one polymer, contractible or not.

Fig. 2. Flow of $c_{\text{eff}}$ to the dense phase.