Galactic rotation curves in brane world models

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ABSTRACT
In the brane world scenario the four-dimensional effective Einstein equation has extra source terms, which arise from the embedding of the three-brane in the bulk. These non-local effects, generated by the free gravitational field of the bulk, may provide an explanation for the dynamics of the neutral hydrogen clouds at large distances from the galactic centre, which is usually explained by postulating the existence of the dark matter. In the present paper we consider the asymptotic behaviour of the galactic rotation curves in the brane world models, and compare the theoretical results with observations of both high surface brightness and low surface brightness galaxies. For the chosen sample of galaxies we determine first the baryonic parameters by fitting the photometric data to the adopted galaxy model; then we test the hypothesis of the Weyl fluid acting as dark matter on the chosen sample of spiral galaxies by fitting the tangential velocity equation of the combined baryonic-Weyl model to the rotation curves. We give an analytical expression for the rotational velocity of a test particle on a stable circular orbit in the exterior region to a galaxy, with Weyl fluid contributions included. The model parameter ranges for which the $\chi^2$ test provides agreement (within 1σ confidence level) with observations on the velocity fields of the chosen galaxy sample are then determined. There is a good agreement between the theoretical predictions and observations, showing that extra-dimensional models can be effectively used as a viable alternative to the standard dark matter paradigm.

Key words: galaxies: haloes – dark matter – gravitation: relativistic processes.

1 INTRODUCTION
Gravitational effects which require more matter than what is visible can be explained in terms of a mysterious dark matter, the nature of which remains a longstanding problem in modern astrophysics. Two important observational issues, the behaviour of the galactic rotation curves and the mass discrepancy in clusters of galaxies, led to the necessity of considering the existence of dark matter both at galactic and at extra-galactic scales.

The rotation curves of spiral galaxies (Binney & Tremaine 1987; Persic, Salucci & Stel 1996; Boriello & Salucci 2001) are among the best evidence showing the problems Newtonian gravity and/or standard general relativity have to face on the galactic/intergalactic scale. In these galaxies neutral hydrogen clouds are observed at large distances from the centre, much beyond the extent of the luminous matter. Since these clouds move in circular orbits with velocity $v_{tg}(r)$, the orbits are maintained by the balance between the centrifugal acceleration $v_{tg}^2/r$ and the gravitational attraction $GM(r)/r^2$ of the total mass $M(r)$ contained within the orbit. This allows the expression of the mass profile of the galaxy in the form $M(r) = rv_{tg}^2/G$.

Observations show that the rotational velocities increase near the centre of the galaxy, in agreement with the theory, but then remain nearly constant at a value of $v_{tg} \approx 200–300$ km s$^{-1}$ (Binney & Tremaine 1987), which leads to a mass profile $M(r) = rv_{tg}^2/G$. Consequently, the mass within a distance $r$ from the centre of the galaxy increases linearly with $r$, even at large distances where very little luminous matter has been detected.

The second astrophysical evidence for dark matter comes from the study of the clusters of galaxies. The total mass of a cluster can be estimated in two ways. Knowing the motions of its member galaxies (Giovanelli et al. 1994), the virial theorem gives one estimate, say $M_{VT}$. The second is obtained by separately estimating the mass of the individual members, and summing up these masses, to give a total
baryonic mass $M_B$. Almost without exception it is found that $M_{VT}$ is considerably larger than $M_B$, $M_{VT} > M_B$, typical values of $M_{VT}/M_B$ being about 20–30 (Binney & Tremaine 1987).

This behaviour of the galactic rotation curves and of the virial mass of galaxy clusters is usually explained by postulating the existence of some dark (invisible) matter, distributed in a spherical halo around the galaxies. The dark matter is assumed to be a cold, pressureless medium. There are many possible candidates for dark matter; the most popular ones being the weakly interacting massive particles (WIMP) [for a review of the particle physics aspects of the dark matter see Overduin & Wesson (2004)]. Their interaction cross-sections with normal baryonic matter, while extremely small, are expected to be non-zero, and we may expect to detect them directly. Models based on right-handed (sterile) neutrinos (Biermann & Kusenko 2006; Munyanzea & Biermann 2006) as a form of warm dark matter were also advanced.

It has also been suggested that the dark matter in the Universe might be composed of super-heavy particles, with mass $\geq 10^{10}$ GeV. But observational results show that the dark matter can be composed of super-heavy particles only if these interact weakly with normal matter, or if their mass is above $10^{13}$ GeV (Albuquerque & Baudis 2003). Scalar fields or other long-range coherent fields coupled to gravity have also intensively used to model galactic dark matter (Fuchs & Mielke 2004; Hernandez et al. 2004; Giannios 2005; Guzmán & Ureña-López 2006; Bernal & Guzman 2006; Brissette 2011; Harko 2011).

However, up to now no non-gravitational evidence for dark matter has been found, and no direct evidence or annihilation radiation from it has been observed yet.

Therefore, it seems that the possibility that Einstein’s (and the Newtonian) theory of gravity breaks down at the scale of galaxies cannot be excluded a priori. Several theoretical models, based on a modification of Newton’s law of or of general relativity, have been proposed to explain the behaviour of the galactic rotation curves (Milgrom 1983; Sanders 1984; Moffat & Sokolov 1996; Mannheim 1997; Bekenstein 2004; Roberts 2004; Bronstein & Moffat 2006; Böhmer & Harko 2007a,b; Bertolami et al. 2007; Böhmer, Harko & Lobo 2008).

The idea of embedding our Universe in a higher dimensional space has attracted a considerable interest recently, due to the proposal by Randall and Sundrum (Randall & Sundrum 1999a,b) that our four-dimensional (4D) space–time is a three-brane, embedded in a 5D space–time (the bulk). According to the brane world scenario, the physical fields (electromagnetic, Yang-Mills, etc.) in our 4D Universe are confined to the three-brane. Only gravity can freely propagate in both the brane and bulk space–time, with the gravitational self-couplings not significantly modified. Even if the fifth dimension is uncompactified, standard 4D gravity is reproduced on the brane in the appropriate limit. Hence, this model allows the presence of large, or even infinite non-compact extra dimensions. Our brane is identified to a domain wall in a 5D anti-de Sitter space–time. For a review of the dynamics and geometry of brane universes, see e.g. Maartens & Koyama (2010). In the brane world scenario, the fundamental scale of gravity is not the Planck scale, but another scale which may be at the TeV level. The gravitons propagating through the bulk space give rise to a Kaluza–Klein tower of massive gravitons on the brane. These gravitons couple to the energy-momentum term of the standard model fields, and could be produced under the appropriate circumstances as real or virtual particles.

Due to the correction terms coming from the extra dimensions, significant deviations from the standard Einstein theory occur in brane world models at very high energies (Sasaki, Shiroimizu & Maeda 2000; Shiromizu, Maeda & Sasaki 2000; Maeda, Mizuno & Torii 2003; Gergely 2003, 2008, 2009). Gravity is largely modified at the electroweak scale of about 1 TeV. The cosmological and astrophysical implications of the brane world theories have been extensively investigated in the physical literature.

The static vacuum gravitational field equations on the brane depend on the generally unknown Weyl stresses, which can be expressed in terms of two functions, called the dark radiation $U$ and the dark pressure $P$ terms (the projections of the Weyl curvature of the bulk, generating non-local brane stresses) (Dadhich et al. 2000; Germani & Maartens 2001; Gergely 2006; Böhmer & Harko 2007c; Maartens & Koyama 2010). Generally, the vacuum field equations on the brane can be reduced to a system of two ordinary differential equations, which describe all the geometric properties of the vacuum as functions of the dark pressure and dark radiation terms (Harko & Mak 2003). In order to close the system of vacuum field equations on the brane a functional relation between these two quantities is necessary.

The full five-dimensional (5D) Einstein equations have been solved numerically for static, spherically symmetric matter localized on the brane in Wiseman (2002), yielding regular geometries in the bulk with axial symmetry. The same data that specify stars in 4D gravity, uniquely construct a 5D solution. An upper mass limit is observed for these small stars, and the geometry shows no global pathologies. The intrinsic geometry of large stars, with radius several times the AdS length, is described by 4D general relativity. The results obtained in Wiseman (2002) show that the Randall–Sundrum gravity, with localized brane matter, reproduces relativistic astrophysical solutions, such as neutron stars and massive black holes, in a way which is consistent with observations.

Several classes of spherically symmetric solutions of the static gravitational field equations in the vacuum on the brane have been obtained in Harko & Mak (2003, 2005), Mak & Harko (2004), Böhmer & Harko (2007c), Gergely (2007) and Horváth, Gergely & Hobill (2003). As a possible physical application of these solutions the behaviour of the angular velocity $v_{\phi}$ of the test particles in stable circular orbits has been considered (Mak & Harko 2004; Harko & Cheng 2006; Böhmer & Harko 2007c; Rahaman et al. 2008). The general form of the solution, together with two constants of integration, uniquely determines the rotational velocity of the particle. In the limit of large radial distances, and for a particular set of values of the integration constants, the angular velocity tends to a constant value. This behaviour is typical for massive particles (hydrogen clouds) outside galaxies (Binney & Tremaine 1987), and is usually explained by postulating the existence of the dark matter.

Thus, the rotational galactic curves can be naturally explained in brane world models, without introducing any additional hypothesis (Mak & Harko 2004; Harko & Cheng 2006; Böhmer & Harko 2007c). The galaxy is embedded in a modified, spherically symmetric geometry, generated by the non-zero contribution of the Weyl tensor from the bulk. The extra terms, which can be described in terms of the dark radiation term $U$ and the dark pressure term $P$, act as a ‘matter’ distribution outside the galaxy. The particles moving in this geometry feel the gravitational effects of $U$, which can be expressed in terms of an equivalent mass (the dark mass) $M_U$. The dark mass is linearly increasing with the distance, and proportional to the baryonic mass of the galaxy, $M_U(r) \approx M_\Omega (r/r_\Omega)$ (Mak &
2 THE FIELD EQUATIONS AND THE TANGENTIAL VELOCITY OF TEST PARTICLES FOR STATIC, SPHERICALLY SYMMETRIC VACUUM BRANES

In the present section, we present the field equations for static, spherically symmetric vacuum branes, and obtain the velocity of the test particles in stable circular orbits around the galactic centre.

2.1 The field equations in the brane world models

We start by considering 5D space–time (the bulk), with a single 4D brane, on which ordinary matter is confined, only gravity can probe the extra dimensions. The 4D brane world \([S, g_{ab}]\) is located at a hypersurface \([B(X^4) = 0]\) in the 5D bulk space–time \([S, g_{5IJ}]\), of which coordinates are described by \(X^4, A = 0, 1, \ldots, 4\). The 4D coordinates on the brane are \(x^a, a = 0, 1, 2, 3\). All tensors \(T_{AB}\) and vectors \(V_A\) satisfy \(T_{a5} = T_{ab} \delta^5_5 \delta_b^a\) and \(V_a = V_a \delta^5_5\) at the hypersurface.

The action of the system is given by (Sasaki et al. 2000)

\[
S = S_{\text{bulk}} + S_{\text{brane}},
\]

where

\[
S_{\text{bulk}} = \int_{S_{\text{5-brane}}} \sqrt{-g_5} \left[ \frac{1}{2k_5^2} R + (\delta L)_{\text{brane}} + \Lambda_5 \right] d^5x,
\]

and

\[
S_{\text{brane}} = \int_{S_{\text{brane}}} \sqrt{-g_4} \left[ \frac{1}{k_5^2} K^\pm + L_{\text{brane}}(\delta \nu, \psi) + \lambda_b \right] d^4x,
\]

where \(k_5^2 = 8\pi G_5\) is the 5D gravitational constant, \((5)R\) and \((5)L_{\text{brane}}\) are the 5D scalar curvature and the matter Lagrangian in the bulk, representing non-standard model fields, \(L_{\text{brane}}(g_{ab}, \psi)\) is the 4D Lagrangian, which is given by a generic functional of the brane metric \(g_{ab}\), and of the matter fields \(\psi, K^\pm\) is the trace of the extrinsic curvature on either side of the brane, and \(\Delta_5\) and \(\lambda_b\) (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively.

The Einstein field equations in the bulk can be obtained as (Sasaki et al. 2000)

\[
(5)G_{IJ} = k_5^2(5)T_{IJ},
\]

where

\[
(5)T_{IJ} = -2\frac{\delta (5)L_{\text{brane}}}{\delta g_{IJ}} + (5)g_{IJ}(5)L_{\text{brane}}
\]

is the energy-momentum tensor of bulk matter fields. The energy-momentum tensor of the matter localized on the brane, \(T_{\mu\nu}\), is defined by

\[
T_{ab} = -\frac{2}{k_5^2} \frac{\delta L_{\text{brane}}}{\delta g^{ab}} + g_{ab} L_{\text{brane}}.
\]

(The minus sign in the above definitions appears because the variation is done with respect to \(g^{ab}\) rather than \(g_{ab}\), and there are two terms because the Lagrangian is varied, rather than the Lagrangian density.)

The delta function \(\delta(B)\) denotes the localization of brane contribution. In the 5D space–time a brane is a fixed point of the \(Z_2\) symmetry. The basic equations on the brane are obtained by projections on to the brane world. The induced 4D metric is \(g_{ab} = (5)g_{ab} - n_I n_J\), where \(n_I\) is the space-like unit vector field normal to the brane hypersurface \((5)\). In the following we assume \((5)L_{\text{brane}} = 0\).
Assuming a metric of the form $dx^2 = (n_i T^i + g_{ij}) dx^i dx^j$, with $n_i dx^i = dX$ the unit normal to the $X =$ constant hypersurfaces and $g_{ij}$ the induced metric on $X =$ constant hypersurfaces, the effective 4D gravitational equation on the brane (the effective Einstein equation) takes the form (Sasaki et al. 2000):

$$G_{ab} = k_2^2 T_{ab} + k_4^2 S_{ab} - E_{ab},$$  

(7)

where $S_{ab}$ is the local quadratic energy-momentum correction

$$S_{ab} = \frac{1}{12} TT_{ab} - \frac{1}{4} T_{a}^c T_{b}^c + \frac{1}{24} k_2^2 (3T^{cd} T_{cd} - T^2) ,$$  

(8)

and $E_{ab}$ is the non-local effect from the free bulk gravitational field, the transmitted projection of the bulk Weyl tensor $C_{ijkl}$, $E_{ij} = C_{ijkl} r^i r^j$, with the property $E_{ij} \rightarrow E_{ab} k_2^2 \delta_{ab}$ as $X \rightarrow 0$. We have also denoted $k_2^2 = 8\pi G$, with $G$ the usual 4D gravitational constant. In the limit $k_2^2 \rightarrow 0$ we recover standard general relativity (Sasaki et al. 2000).

The Einstein equation in the bulk and the Codazzi equation also imply the conservation of the energy-momentum tensor of the matter on the brane, $\nabla_a T^a = 0$, where $\nabla_a$ denotes the brane covariant derivative. Moreover, from the contracted Bianchi identities on the brane it follows that the projected Weyl tensor obeys the constraint $\nabla_a E_{ab} = k_2^2 \nabla_a S_{bc}$ (for the brane it follows that the projected Weyl tensor obeys the constraint $\nabla_a E_{ab} = k_2^2 \nabla_a S_{bc}$) (Shiromizu et al. 2000).

The generic traceless $E_{ab}$ can be decomposed irreducibly with respect to a chosen four-velocity field $u^a$ as (Maartens & Koyama 2010)

$$E_{ab} = -k_4^2 \left[ U (u_a u_b + \frac{1}{3} h_{ab}) + P_a + 2 Q_a u_b \right] ,$$  

(9)

where the induced metric $h_{ab} = g_{ab} + u_a u_b$ projects orthogonal to $u^a$, the ‘dark radiation’ term $U = -k_4^2 E_{ab} u^a u^b$ is a scalar, $Q_a = k_4^2 h^c_d E_{ab} u^d$ is a spatial vector and finally $P_a = -k_4^2 (U h_{ab} - \frac{1}{2} h_{ab} u^d u^d) E_{cd}$ is a spatial, symmetric and trace-free tensor. In the following we neglect the effect of the cosmological constant on the geometry and dynamics of the galactic particles. In the case of the vacuum state we have $\rho = \rho = 0$, $T_{\mu\nu} = 0$ and consequently $S_{ab} = 0$. Therefore, the field equation describing a static brane takes the form

$$R_{ab} = -E_{ab},$$  

(10)

with the trace $R$ of the Ricci tensor $R_{ab}$ satisfying the condition $R = R^a_a = 0$.

In the vacuum case $E_{ab}$ also satisfies the constraint $\nabla_a E^a = 0$. In an inertial frame at any point on the brane we have $u^a = \delta^a_0$ and $h_{ab} = \delta(0, 1, 1, 1)$. In a static vacuum $Q_a = 0$ and the constraint for $E_{ab}$ takes the form (German & Maartens 2001)

$$\frac{1}{3} D_a U + \frac{4}{3} U A_a + D^a P_a + A^b P_b = 0,$$  

(11)

where $A_a = u^b \nabla_b u_a$ is the four-acceleration and $D_a$ denotes the covariant derivative associated with the metric $h_{ab}$. In the static spherically symmetric case we may choose $A_a = A(r) r_a$ and $P_a = P(r) (r_a r_b - \frac{1}{2} h_{ab})$, where $A(r)$ and $P(r)$ (the ‘dark pressure’ although the name dark anisotropic stress might be more appropriate) are some scalar functions of the radial distance $r$, and $r_a$ is a unit radial vector (Dadhich et al. 2000). Thus the expression (9) simplifies to

$$E_{ab} = -k_4^2 \left[ U (u_a u_b + \frac{1}{3} h_{ab}) + P_a + Pr_r r_b \right] ,$$  

(12)

2.2 The motion of particles in stable circular orbits on the brane

In brane world models test particles are confined to the brane. Mathematically, this means that the equations governing the motion are the standard 4D geodesic equations (Maartens & Koyama 2010). However, the bulk has an effect on the motion of the test particles on the brane via the metric. Since the projected Weyl tensor effectively serves as an additional matter source, the metric is affected by these bulk effects, and so are the geodesic equations. This has to be contrasted with Kaluza–Klein theories, where matter travels on 5D geodesics.

In order to obtain results which are relevant to the galactic dynamics, in the following we will restrict our study to the static and spherically symmetric metric given by

$$ds^2 = -e^{\nu(r)} dr^2 + e^{\pi(r)} dr^2 + r^2 d\Omega^2 ,$$  

(13)

where $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$.

The Lagrangian $L$ for a massive test particle travelling on the brane reads

$$L = \frac{1}{2} \left( -e^{\nu(r)} \dot{r}^2 + e^{\nu(r)} \dot{\phi}^2 + r^2 \dot{\Omega}^2 \right) ,$$  

(14)

where the dot means differentiation with respect to the affine parameter.

Since the metric tensor coefficients do not explicitly depend on $t$ and $\Omega$, the Lagrangian (14) yields the following conserved quantities (generalized momenta):

$$-e^{\nu(r)} = E, \quad r^2 \Omega = L,$$  

(15)

where $E$ is related to the total energy of the particle and $L$ to the total angular momentum. With the use of conserved quantities we obtain from equation (14) the geodesic equation for massive particles (for which $2L = -1$ holds) in the form

$$e^{\nu(r)} \dot{r}^2 + e^{\nu(r)} \dot{\phi}^2 + r^2 \dot{\Omega}^2 = E^2. $$  

(16)

The second term of the left-hand side can, in some cases, be interpreted as an effective potential. For instance, for the Schwarzschild space–time, where $e^{\nu(r)} = 1$, the kinetic term is position independent. In that case the notion of an effective potential is appropriate. In other cases, even one can still compute the turning points of the kinetic term; however, the effective potential interpretation is lost.

For particles in circular and stable orbits, the following conditions must be satisfied: (a) $\dot{r} = 0$ (circular motion), (b) $\partial V_{eff}/\partial r = 0$ (extreme motion) and (c) $\partial^2 V_{eff}/\partial r^2 \leq 0$ (stable orbit), respectively. Conditions (a) and (b) immediately give the conserved quantities as

$$E^2 = e^{\nu(r)} \left( 1 + \frac{L^2}{r^2} \right) ,$$  

(17)

and

$$\frac{L^2}{r^2} = \frac{r^2}{2} e^{-\nu(r)} E^2 ,$$  

(18)

respectively. Equivalently, these two equations can be rewritten as

$$E^2 = e^{\nu(r)} \left( 1 - \frac{r^2}{2} \right) , \quad \frac{L^2}{r^2} = \frac{r^2}{2} - \frac{1}{r^2} .$$  

(19)

We define the tangential velocity $v_{\Omega}$ of a test particle on the brane, as measured in terms of the proper time, that is, by an observer located at the given point, as (Landau & Lifshitz 1975)

$$v_{\Omega}^2 = \frac{1}{1 - \frac{r^2}{2}} \left( 1 - \frac{r^2}{2} \right) ,$$  

(20)
In the second equality we have employed the second equation (15). By eliminating $L$ with equation (18) and subsequently $E$ with the first equation (15), we obtain the expression of the tangential velocity of a test particle in a stable circular orbit on the brane as (Matos, Guzman & Nunez 2000; Nucamendi, Salgado & Sudarsky 2001)

$$v_{\text{tg}}^{2} = \frac{r v'}{2}. \quad (21)$$

Let us emphasize again that the function $v'$ is obtained by solving the field equations containing the bulk effects as additional matter terms; we consider this in Section 3.

### 2.3 The gravitational field equations for a static spherically symmetric brane

For the metric given by equation (13) the gravitational field equations and the effective energy-momentum tensor conservation equation in the vacuum take the form (Harko & Mak 2003; Mak & Harko 2004)

$$-e^{-\lambda} \left( \frac{1}{r} - \frac{\lambda'}{r} \right) + \frac{1}{r^2} = 3\alpha_b U, \quad (22)$$

$$e^{-\lambda} \left( \frac{v'}{r} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \alpha_b (U + 2P), \quad (23)$$

$$e^{-\lambda} \left( \frac{1}{2} \left( \frac{v'}{r} + \frac{v'^2}{2} + \frac{v' - \lambda'}{2} - \frac{v'\lambda'}{2} \right) \right) = \alpha_b (U - P), \quad (24)$$

$$v' = -\frac{U' + 2P'}{2U + P} - \frac{6P}{r(2U + P)}, \quad (25)$$

where $'=d/dr$, and we have denoted $\alpha_b = k B/3$. Note that equation (25) is a consequence of equations (22), (23) and (24), respectively.

As for the motion of the test particle on the brane we assume that they follow stable circular orbits, with tangential velocities given by equation (21). Thus, the rotational velocity of the test body is determined by the metric coefficient $\exp(\nu)$ only.

The field equations (22)–(23) yield the following effective energy density $\rho_{\text{eff}}$, radial pressure $P_{\text{eff}}$ and orthogonal pressure $P_{\perp}$, respectively.

$$\rho_{\text{eff}} = 3\alpha_b U, \quad (26)$$

$$P_{\text{eff}} = \alpha_b (U + 2P), \quad (27)$$

$$P_{\perp} = \alpha_b (U - P), \quad (28)$$

which obey $\rho_{\text{eff}} - P_{\text{eff}} - 2P_{\perp} = 0$. This is expected for the ‘radiation’ like source, the projection of the bulk Weyl tensor, which is trace-less, $E_{\alpha}^\alpha = 0$.

### 3 STRUCTURE EQUATIONS OF THE VACUUM IN THE BRANE WORLD MODELS

Equation (22) can immediately be integrated to give

$$e^{-\lambda} = 1 - \frac{C_b}{r} - \frac{GM_b(r)}{r}, \quad (29)$$

where $C_b$ is an arbitrary constant of integration, and we denoted

$$GM_b(r) = 3\alpha_b \int_0^r U(r)r^2 \, dr. \quad (30)$$

The function $M_b$ is the gravitational mass corresponding to the dark radiation term (the dark mass). For $U = 0$ the metric coefficient given by equation (29) must tend to the standard general relativistic Schwarzschild metric coefficient, which gives $C_b = 2GM$, where $M = \text{constant}$ is the baryonic (usual) mass of the gravitating system.

By substituting $v'$ given by equation (25) into equation (23) and with the use of equation (29) we obtain the following system of differential equations satisfied by the dark radiation term $U$, the dark pressure $P$ and the dark mass $M_b$, describing the vacuum gravitational field, exterior to a massive body, in the brane world model (Harko & Mak 2003):

$$\frac{dU}{dr} = -\frac{2v_{\text{tg}}^2 (2U + P)}{r} - 2\frac{dP}{dr} - \frac{6P}{r}, \quad (31)$$

$$\frac{dM_b}{dr} = \frac{3\alpha_b}{G} r^2 U, \quad (32)$$

with the tangential velocity given as

$$v_{\text{tg}}^2 = \frac{1}{2} \left( \frac{2GM + GM_b + \alpha_b (U + 2P) r^3}{r (1 - \frac{2GM}{r} - \frac{GM_b}{r})} \right). \quad (33)$$

In order to close the system a supplementary functional relation between one of the unknowns $U$, $P$, $M_b$ and $v_{\text{tg}}$ is needed. Once this relation is known, equations (31)–(33) give a full description of the geometrical properties and of the motion of the particles on the brane.

The system of equations (31) and (32) can be transformed to an autonomous system of differential equations by means of the transformations

$$\theta = \ln r, \quad q = 1 - e^{-\lambda} = \frac{2GM}{r} + \frac{GM_b}{r}, \quad (34)$$

$$\mu = 3\alpha_b r^2 U, \quad p = 3\alpha_b r^2 P. \quad (35)$$

We shall call $\mu$ and $p$ the ‘reduced’ dark radiation and pressure, respectively.

With the use of the new variables given by equations (34), equations (32) and (31) become

$$\frac{dq}{d\theta} = \mu - q, \quad (36)$$

$$\frac{d\mu}{d\theta} = 2(\mu + p) \frac{[q + \frac{1}{2}(\mu + 2p)] - 2 \frac{dP}{d\theta} + 2\mu - 2p. \quad (37)$$

Equations (31) and (32), or, equivalently, (36) and (37), may be called the structure equations of the vacuum on the brane. In order to close this system an ‘equation of state’, relating the reduced dark radiation and the dark pressure terms, is needed. Generally, this equation of state is given in the form $P = PU$.

In the new variables the tangential velocity of a particle in a stable circular orbit on the brane is given by

$$v_{\text{tg}}^2 = \frac{1}{2} \frac{q + \frac{1}{2}(\mu + 2p)}{1 - q}. \quad (38)$$

By using the expression of the tangential velocity, equation (37) can be rewritten as

$$\frac{d}{d\theta} (\mu + 2p) = -2(\mu + p) v_{\text{tg}}^2 + 2\mu - 2p. \quad (39)$$

Equations (38) and (39) allow the easy check of the physical consistency of some simple equations of state for the dark pressure. The equation of state $\mu + 2p = 0$ immediately gives $v_{\text{tg}}^2 = 1$ and $q = 2\mu$, respectively, implying that all test particles in stable...
circular motion on the brane move at the speed of light. This result
contradicts the assumption that the test particles are time-like, as
well as the observations on galactic scale. Therefore the equation
of state $\mu + 2p = 0$ is not consistent with the rotation curves. The
equation of state $2\mu + p = 0$ gives $\mu = \mu_0/r^2$, where $\mu_0$ is
constant is an arbitrary integration constant, $U = \mu_0/3\sigma_0 r^4$ and $GMU = -$ $\mu_0/3r^3$, respectively. In the limit of large $r$, the tangential velocity $v_{th}^2$ tends to zero, $v_{th} \rightarrow 0$. Therefore, this model seems to be ruled out by observations. The case $\mu = p$ gives $\mu(\theta) = \mu_0 \exp [-2 \int v_{th}(\theta) d\theta]$, $\mu_0$ is constant and $\eta(\theta) = (2v_{th}^2 - \mu)(1 + 2v_{th}^2)$.

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FOR A LINEAR EQUATION OF STATE
OF THE WEYL FLUID

4.1 Linear equation of state for the Weyl fluid

In order to close the system of equations (36) and (37), we need to
specify the equation of state relating the dark matter to the dark
radiation. In the full 5D approach of Wiseman (2002) the Einstein
field equations were solved numerically for static, spherically
symmetric matter localized on the brane, yielding regular geometries
in a bulk with axial symmetry. For this a density profile, taken as a
deformed top hat function, was imposed.

An alternative approach for closing the system of field equations
can be obtained from the 3+1 covariant approach in brane worlds,
developed in Keresztes & Gergely (2010a,b). First, we impose: (i)
cosmological vacuum in 5D space–time, (ii) the brane embedding is symmetrical, (iii) fine-tuning on the brane (the brane
cosmological constant vanishes), (iv) no matter on the brane, (v)
the brane is static and spherical symmetric, with metric given by
equation (13). The set of gravito-electro-magnetic quantities in the
3+1 covariant approach is presented in Appendix A. Due to
assumptions (ii) and (iv) the Lanczos equations impose that the
tensorial and vectorial projections of the extrinsic brane curvature
along the brane normal $n^I$ vanish: $\tilde{\sigma}_{ab} = 0 = K_n$. Therefore
$E_{ab} = \tilde{E}_{ab} - k_a^2 P_{ab}/2$ and $\tilde{\eta}_{ab} = H_{ab}$, respectively.

From the above assumptions we also find that the only non-zero
kinematical quantity related to the normal $n^I$ of the $r =$ const.
hypersurfaces is its acceleration $A_n = n^I D_n n^I$ (here $D_n$ is the covariant
derivative on the brane). The kinematical quantities of the vector
congruences $\omega^a$ and $\kappa^a$ are enlisted in Tables 1 and 2.

Due to the spherical symmetry, assumption (v), it is convenient
to decompose the brane space-time into a 2+1+1 form. Then, the
gravito-electro-magnetic quantities appearing in the brane equations
further reduce to $H_{ab} = 0$, $\tilde{E}_{ab} = \tilde{E}(r) (r_a r_b - h_{ab}/3)$, $k_a^2 U(r)$,
$Q_a = 0$ and $k_a^2 P_{ab} = k_a^2 P(r)(r_a r_b - h_{ab}/3)$, respectively, while $A_n = A(r) n^a$. Therefore under the assumption of spherical symmetry the

| Table 1. Kinematical scalars of the congruence $\omega^a$ in the 3+1 decomposition of $\nabla_a n_b$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Acceleration    | Expansion       | Shear           | Vorticity       |                  |
| $A$             | $0$             | $0$             | $0$             |                  |

| Table 2. Kinematical scalars of the congruence $\kappa^a$ in the 2+1 decomposition of $D_a n_b$. |
|-----------------|-----------------|-----------------|-----------------|-----------------|
| Acceleration    | Expansion       | Shear           | Vorticity       |                  |
| $0$             | $\tilde{\omega}$ | $0$             | $0$             |                  |

Weyl fluid is characterized by the set $U(r)$, $P(r)$; the electric part
of 4D Weyl tensor by $\tilde{E}(r)$ and the acceleration of the time-like
normal to the three-space by $A(r)$.

We also introduce the covariant derivative $D_a$ associated with
the three-metric $h_{ab}$ and the expansion $\Theta(r)$ of the radial geodesics
in the local 3D space [obeying $(3)D_a r_a = 0$] by the formula
$(3)D_r r_a = \tilde{\Theta}(r)$. The four independent field equations for the
variables $U$, $P$, $\tilde{E}$, $A$ and $\Theta$, arising in the 2+1+1 brane formalism,
are given in Appendix B. There are two algebraic equations (equations
B5 and B9) determining $U$ and $P$ as a function of $\tilde{E}$, $A$ and $\Theta$:

\[
k_a^2 U = \tilde{\Theta} \left( A - \frac{\tilde{\Theta}}{4} \right) + \frac{4\tilde{E}}{3} + \frac{1}{r^2},
\]

\[
k_a^2 P = \tilde{\Theta} \left( A + \frac{\tilde{\Theta}}{2} \right) - \frac{2\tilde{E}}{3} - \frac{2}{r^2}.
\]

and two first-order non-linear ordinary differential equations (equations
B11 and B12) for the variables $A$, $\Theta$ (also containing $E$):

\[
\frac{r}{2} \tilde{\Theta} + \frac{\tilde{\Theta}^2}{2} + \frac{4\tilde{E}}{3} + A\tilde{\Theta} = 0,
\]

\[
\frac{r}{2} \tilde{\Theta} A + \frac{\tilde{\Theta}^2}{4} - \frac{4\tilde{E}}{3} - \frac{1}{r^2} = 0.
\]

The last two equations contain only quantities defined and deter-
mined by the brane dynamics. In order to obtain a special solution
of the system (42)–(43), we need a third relation between the brane
kinematic quantities $A$, $\Theta$ and electric Weyl brane curvature $\tilde{E}$.
In order to establish this, first we remark that in the Schwarzschild
case these quantities obey

\[
\frac{2\tilde{E}}{3} + \tilde{\Theta} A = \Theta \left( \tilde{\Theta}^4 + A \right) - \frac{1}{r^2} = 0.
\]

By virtue of these the two equations, equations (42) and (43), are
found to coincide; thus $A$, $\Theta$ and $\tilde{E}$ are still defined by a set
of three equations. For a spherically symmetric solution on the brane
we allow for a slightly modified identity as compared to the first
(equation 44):

\[
\frac{2\tilde{E}}{3} + \tilde{\Theta} A = \Theta \left( \tilde{\Theta}^4 + A \right) - B \tilde{\Theta} \frac{1}{r^2},
\]

with $A$ and $B$ two constants, both reducing to 1 in the Schwarzschild
case. By employing equations (40) and (41), this equation can be
rewritten as a simple equation of state for the Weyl fluid:

\[
P = (a - 2) U - \frac{B}{k_a^2 r^2}.
\]

Here, we have introduced the new constants $a$, $B$ by redefining
$A = a/(2a - 3)$, $B = (a - B)/(2a - 3)$. We also remark that in
the variables given by equation (34) equation (46) takes the simple
linear form

\[
p(\mu) = (a - 2) \mu - B.
\]

Equations (42), (43) and (45), being a first-order ordinary system
of differential equations, determine the variables $A$, $\Theta$ and $\tilde{E}$.
Their existence is assured by the Cauchy–Peano theorem. When rewritten
in metric variables, the solution valid in the region where rotation
curve data are available will be obtained in the following section.

One can be rightly worried about the compatibility of equation
(46) with the full 5D gravitational dynamics; however this is but
4.2 The metric and tangential velocity on the brane

The dark radiation and the dark pressure can be obtained as a function of the tangential velocity in a closed analytical form for the equation of state given by equation (47). The reduced dark radiation can be obtained as

\[ \mu(\theta) = \theta^{2(3-a)/(2a-3)} \exp \left[ -\frac{2a}{2a-3} \int v^2_q(\theta) \, d\theta \right] \]

\[ \times \left( C_0 - \frac{3B}{2a-3} \int \left[ 1 + v^2_q(\theta) \right] \theta^{-2(3-a)/(2a-3)} \right) \]

\[ \times \exp \left[ \frac{2a}{2a-3} \int v^2_q(\theta) \, d\theta \right] \],

where \( C_0 \) is an arbitrary integration constant. Hence, if the velocity profile of a test particle in stable circular motion is known, one can obtain all the relevant physical parameters for a static spherically symmetric system on the brane.

By using the linear equation of state of the dark pressure equation (37) takes the form

\[ (2a-3) \frac{d\mu}{d\theta} = -\left( \frac{\mu-a}{2a-3} \right) [q + (2a-3) \mu/3 - 2B/3] \]

\[ = 1 - q + 2(3-a) \mu + 2B, \]

where we have neglected the possible effect of the cosmological constant on the structure of the cluster. Due to the mathematical structure of equation (49) there are two cases that can be considered separately, \( a = 3/2 \) and \( a \neq 3/2 \), respectively.

For a galactic dark matter halo with mass of the order of \( M = 10^{12} \, M_\odot \) and radius \( R = 100 \, \text{kpc} \), the quantity \( 2GM/R \) is of the order of \( 9.6 \times 10^{-7} \), which is much smaller than one. Since observations show that inside the galaxy the mass is a linearly increasing function of the radius \( r \), the value of this ratio is roughly the same at all points in the galaxy. Therefore, from its definition it follows that generally \( \eta \approx 1 \), and \( 1 - q \approx 1. \) Moreover, the quantities \( q^2 \) and \( q dq/d\theta \) are also very small as compared to \( q \). Equation (36) gives \( \mu = q + \frac{dq}{d\theta}, \) \( d\mu/d\theta = dq/d\theta + q^2 dq/d\theta^2 \).

4.2.1 The case \( a \neq 3/2 \)

Hence, by neglecting the second-order terms and assuming \( a \neq 3/2 \), we obtain for \( q \) the following differential equation:

\[ \frac{d^2 q}{d\theta^2} + m \frac{dq}{d\theta} - nq = b, \]

where we have denoted

\[ m = 1 - B \frac{2a}{3} - \frac{2a(B - 3) + 9}{2a - 3}, \quad a \neq \frac{3}{2}, \]

\[ n = \frac{2a(B - 3) + 9}{2a - 3}, \quad a \neq \frac{3}{2}, \]

and

\[ b = \frac{2B(B - 3)}{3 - 2a}, \quad a \neq \frac{3}{2}, \]

respectively. The general solution of equation (50) is given by

\[ q(\theta) = v_0 + C_1 e^{\eta \theta} + C_2 e^{\eta' \theta}, \quad a \neq \frac{3}{2}, \]

where \( C_1 \) and \( C_2 \) are arbitrary constants of integration, and we denoted

\[ v_0 = -\frac{b}{n} = \frac{B(B - 3)}{a(2B - 3) + 9} \]

and

\[ l_{1,2} = \frac{-m \pm \sqrt{m^2 + 4n}}{2}, \]

the reduced dark radiation term is given by

\[ \mu(\theta) = v_0 + C_1 (1 + l_1) e^{\eta \theta} + C_2 (1 + l_2) e^{\eta' \theta}. \]

The tangential velocity of a test particle in the ‘dark matter’ dominated region is given by

\[ v^2_q \approx \frac{1}{2} \left[ q + \frac{1}{3} (\mu + 2p) \right] \approx \frac{1}{2} \left[ q + \frac{(2a - 3) \mu - 2B}{3} \right], \]

or

\[ v^2_q(\theta) \approx v^2_{q_{\infty}} + \gamma e^{\eta \theta} + \eta e^{\eta' \theta}, \]

where we have denoted

\[ v^2_{q_{\infty}} = \frac{1}{3} (av_0 - B), \]

\[ \gamma = \frac{1}{2} \left[ \frac{2a - 3}{3} (1 + l_1) + 1 \right] C_1, \]

\[ \eta = \frac{1}{2} \left[ \frac{2a - 3}{3} (1 + l_2) + 1 \right] C_2. \]

In the initial radial coordinate \( r \) the tangential velocity can be expressed as

\[ v^2_q(r) \approx v^2_{q_{\infty}} + \gamma r^{l_1} + \eta r^{l_2}. \]

In order to have an asymptotically constant \( v^2_q(r) \), both \( l_1 \) and \( l_2 \) should be negative numbers, \( l_1 < 0 \) and \( l_2 < 0 \), respectively. This can be achieved if \( m > 0, \eta < 0 \) hold simultaneously.

In the original radial variable \( r \) we obtain for the dark radiation and the mass distribution inside the cluster the expressions

\[ 3aU(r) = \frac{v_0}{r^2} + C_1 (1 + l_1) r^{l_1-2} + C_2 (1 + l_2) r^{l_2-2} \]

and

\[ GM(r) = r \left( v_0 + C_1 r^{l_1} + C_2 r^{l_2} \right) - 2GM, \]
5. FURTHER SIMPLIFICATION FROM OBSERVATIONAL CONSTRAINTS

In Section 4, the assumption $q \ll 1$ was made in order to derive the analytical solution. As it can be seen from its definition, the quantity $q$ is basically a post-Newtonian parameter and as such the assumption is justified by galactic rotation curves. Further, as we are studying bounded motions in the limit of large $r$, the condition $r \ll 1$ is valid.

Focusing on the galactic rotation curve, we have

$$\ddot{v} \approx -\frac{GM}{r^2},$$

where $C$ is the galactic constant. In the limit of large distances $r$, the equation of motion can be approximated as

$$\ddot{v} \approx -\frac{GM}{r^2},$$

and the solution obtained in Section 4 is valid only for $r > r_0$. However, the solution must satisfy the condition $M_\text{tot} > 0$. Therefore, the solution obtained in the present section is valid only for $r > r_0$ and by assuming that the coefficient $a$ is positive, the solution obtained in Section 4 is valid only for $r > r_0$.

Secondly, we approximate the dark mass by $M_\text{tot} = C/\rho$, where $C$ is a universal constant.

Finally, we obtain

$$\ddot{v} \approx -\frac{GM_\text{tot}}{r^2},$$

and the solution obtained in Section 4 is valid only for $r > r_0$.

6.6. THE BARYONIC SECTOR: THE BULGE-DISC DECOMPOSITION

We model the distribution of baryonic mass in a galaxy by a sum of disc and bulge components. We estimate the disc parameters from the surface brightness of the galaxy. The surface brightness (specific intensity) of the spheroidal bulge is given by a generalized Sersic function of the form

$$I(r) = I_0 \exp\left(\frac{r}{b}\right)^n, \quad n > 0, \quad b > 0, \quad I_0 > 0.$$

The surface brightness of the bulge is given by a generalized Sersic function of the form

$$I(r) = I_0 \exp\left(\frac{r}{b}\right)^n, \quad n > 0, \quad b > 0, \quad I_0 > 0.$$

and

$$I(r) = I_0 \exp\left(\frac{r}{b}\right)^n, \quad n > 0, \quad b > 0, \quad I_0 > 0.$$
(Sérsic 1968):

\[ I_b(r) = I_{0,b} \exp \left( -\frac{r}{r_0} \right), \]

\[ I_d(r) = I_{0,d} \exp \left( -\frac{h}{r} \right), \]

where \( I_{0,b} \) is the central surface brightness of the bulge, \( r_0 \) is its characteristic radius and \( n \) is the shape parameter of the magnitude–radius curve. As a rule, early-type spiral galaxy bulges have \( n > 1 \), while late-type spiral galaxy bulges are characterized by \( n < 1 \). The radius of the bulge \( r_b \) is defined by the condition of the surface brightness being equal to \( \mu_0 = 2, 64 \times 10^{-4} \) mJy arcsec\(^{-2} \) in the I-band images.

In a spiral galaxy, the radial surface brightness profile of the disc exponentially decreases with the radius (Freeman 1970)

\[ I_d(r) = I_{0,d} \exp \left( -\frac{h}{r} \right), \]

where \( I_{0,d} \) is the disc central surface brightness and \( h \) is a characteristic disc length-scale. In order to measure the light and mass distribution in the disc, the model image of the bulge is subtracted from the original I-band image. All remaining light in these images is then assumed to originate from the disc component.

The (bolometric) luminosity of any of the galaxy components is an integral over surface, solid angle and frequency of \( I_b \) and \( I_d \), respectively. The respective mass over luminosity is the mass-to-light ratios \( n \) and \( r_b \) will be given in units of \( \gamma_\odot \) (solar units). We will also give the masses in units of the solar mass \( M_\odot = 1.98892 \times 10^{30} \) kg. Assuming that the mass distribution of a spiral galaxy follows the de-projected surface brightness distribution (Portinari et al. 2004; Kannappan & Gawiser 2007), corrected with respect to the inclination (cf. Palunas & Williams 2000), we have derived the best-fitting values of \( I_{0,b}, n, r_0, r_b, I_{0,d} \) and \( h \) from the photometric data as well as the mass-to-light ratio of each component \( \sigma \) and \( r_b \) by fitting equation (82) to the data on rotation curves represented in fig. 1 of Palunas & Williams (2000). These are all collected in Table 3.

### 6.2 Combined baryonic and Weyl model

We assume that within the bulge radius \( r_b \) the contribution of the Weyl fluid can be neglected and the part of the observed rotation curves lying below \( r_b \) could be explained with baryonic matter alone. Outside this radius we switch on the Weyl fluid and take into account the exponential disc as a perturbation that does not affect the geometry. We add, however, its contribution to the rotation velocity,

\[ v_\beta^2(r) = \frac{G [M_b(r) + M_d(r)]}{r} + c^2 \left[ \beta + C \left( \frac{r_b}{r} \right)^{1-\alpha} \right] H_b(r_b), \]

where \( H_b(r_b) \) is some function smoothly approaching the Heaviside step function:

\[ H_b(r_b) = \lim_{k \to \infty} H_b(r_b) = \begin{cases} 0, & r < r_b \\ 1, & r \geq r_b \end{cases} \]

The baryonic matter at any \( r \) is given by

\[ M_{\text{baryons}}(r) = M_b(r) + M_d(r). \]

In Palunas & Williams (2000), a maximum disc mass model was presented, and the bulge-disc decomposition was performed for a sample of 74 HSB spiral galaxies, for which the I-band surface brightness profiles and Hz velocity profiles were also given. From this set we have left out those galaxies, which have either bars or rings, which would both contradict the assumption of spherical symmetry. In addition we have left out the galaxies for which the above baryonic model cannot be applied, since they have non-vanishing bulge surface brightness values, but vanishing bulge mass (Palunas & Williams 2000). Finally we have also omitted the galaxies where the observed data sequence exhibits a wavy pattern instead of a plateau, suggesting a disc structure strongly contradicting our assumption for rotational symmetry about the galaxy centre. Such patterns are illustrated in Fig. 1. As a result of this selection process we end up with nine galaxies. Their rotation curves are represented in Fig. 2.

As for the bulge parameters of the nine selected galaxies, we have derived the best-fitting values of \( I_{0,b}, n, r_0, r_b, I_{0,d} \) and \( h \) from the photometric data as well as the mass-to-light ratio of each component \( \sigma \) and \( r_b \) by fitting equation (82) to the data on rotation curves represented in fig. 1 of Palunas & Williams (2000). These are all collected in Table 3.
Figure 2. Best-fitting curves for the chosen HSB galaxies compatible with the spherical symmetry assumption, the baryonic model and having sufficiently accurate rotation curve and photometric data available. In the baryonic matter dominated radial range $r < r_b$ the model curves bend as determined by the photometric data, shown in Table 3, together with the Weyl parameters of the model. The discontinuity in the first derivative of the model curves indicates the bulge radius $r_b$.

Table 3. The baryonic parameters ($D, I_0, n, r_0, r_b, I_0, h, k$) of the nine HSB galaxy sample. The additional baryonic parameters $\sigma, \tau_b$ and the Weyl parameter $\alpha$ are determined by $\chi^2$ fitting. The best-fitting parameters ($\sigma = M_d/M_b, \tau_b = M_d/M_b, \alpha$) and the minimum value of the $\chi^2$ statistic $\chi^2_{\text{min}}$ are also given. All $\chi^2_{\text{min}}$ are within $1\sigma$.

| Galaxy      | $D$  | $I_0$ | $n$  | $r_0$ | $r_b$ | $I_0$ | $h$  | $k$  | $\sigma$ | $\tau_b$ | $\alpha$ | $\beta$ | $\chi^2_{\text{min}}$ |
|-------------|------|-------|------|-------|-------|-------|------|------|-----------|-----------|-----------|---------|---------------------|
| ESO215G39   | 61.29| 0.1171| 0.6609| 0.78  | 2.58  | 0.0339| 4.11 | 26.28| 0.04      | 2.1       | 0.67      | 2.37 $\times 10^{-7}$| 28.29   |
| ESO322G76   | 64.28| 0.2383| 0.8344| 0.91  | 4.50  | 0.0251| 5.28 | 15.35| 0.47      | 3.28      | 0.76      | 3.21 $\times 10^{-7}$| 38.69   |
| ESO322G77   | 38.19| 0.1949| 0.7552| 0.33  | 1.37  | 0.0744| 2.20 | 50.32| 1.27      | 2.7       | 0.57      | 3.92 $\times 10^{-7}$| 10.15   |
| ESO323G25   | 59.76| 0.1113| 0.4626| 0.43  | 0.99  | 0.0825| 3.47 | 34.58| 1.85      | 0.69      | 5.25 $\times 10^{-7}$| 34.77   |
| ESO383G02   | 85.40| 0.6479| 0.7408| 0.42  | 1.94  | 0.5118| 3.82 | 17.79| 0.47      | 0.22      | 0.70      | 3.72 $\times 10^{-7}$| 21.51   |
| ESO445G19   | 66.05| 0.1702| 0.6133| 0.57  | 1.79  | 0.0478| 4.27 | 38.59| 1.39      | 0.46      | 3.64 $\times 10^{-7}$| 29.26   |
| ESO446G01   | 98.34| 0.2093| 0.8427| 1.28  | 6.33  | 0.0357| 5.25 | 10.90| 0.93      | 2.82      | 0.54      | 4.64 $\times 10^{-7}$| 43.35   |
| ESO509G80   | 92.86| 0.2090| 0.7621| 1.10  | 4.69  | 0.0176| 11.03| 14.75| 0.61      | 5.50      | 0.87      | 6.3 $\times 10^{-7}$  | 25.71   |
| ESO569G17   | 57.77| 0.2452| 0.4985| 0.45  | 1.18  | 0.1348| 2.06 | 58.74| 1.4       | 0.65      | 3.18 $\times 10^{-7}$| 7.24    |

The value of $k$ gives the sharpness of the transition. We determine the parameter $k$ by imposing the conditions $H_i(0.95r_b) = 0.001$, which also implies $H_i(1.05r_b) = 0.999$ and $H_i(r_b) = 0.5$ and its values are given in Table 3 for each of the nine selected galaxies (Palunas & Williams 2000).

The baryonic parameters $\tau_b, \sigma$ and the Weyl parameter $\alpha$ have to be determined by a $\chi^2$-minimization fitting of equations (77)–(85), with the rotation curve data for each individual galaxy. The parameter $\beta$ of the Weyl sector gives the asymptotic rotation velocity. We identify $\beta$ with the average of the points of the rotation curves for $r > r_b$ (these are on the plateau).

At the end of this section we prove that the ‘truncation’ of the Weyl fluid at $r_b$ does not induce any distributional source layer at $r_b$; thus it is consistent with the junction conditions across the sphere with radius $r_b$. For this we first remark that at a formal level the truncation can be imposed in all equations by the replacements...
\[
\beta \rightarrow \beta H_0(r_0). \text{ By also keeping in mind that } C = C_G c^{-2} r_c^{-1} = -\beta \text{ was chosen, equation (72) shows that } U \text{ smooths to zero with } H_0(r_0) \text{ across } r_b. \text{ The equation of state (47) and } B = (3\beta/2)(1/\alpha - 1) \text{ then guarantee that } P \text{ has the same property. As both inside and outside } r_b \text{ we have the same spherically symmetric metric and the energy-momentum tensor changes smoothly across } r_b, \text{ there is no discontinuity in either the metric or its first derivative; therefore the even stronger Lichnerowicz continuity conditions are obeyed. The Israel–Lanczos–Darmois junction conditions of the continuity of induced metric and extrinsic curvature thus follow. Our approach is different from that used by Wiseman (2002), in which a given density profile is considered, together with the condition of the isotropy on the brane. Therefore, while in Wiseman (2002) the matching conditions impose these requirements as boundary conditions, in our approach the matching conditions are automatically satisfied.}
\]

6.3 HSB galaxy rotation curves

Despite the differences in the surface brightness profiles and rotation curves for the chosen HSB galaxies, the combined baryonic+Weyl model equation (83) fits well the sample.

A restriction on the parameter space emerges as \( \beta = -C \) from the condition that the Weyl contribution to \( v^3(r) \) vanishes at \( r_0 \) (by switching on the Weyl fluid at \( r = r_0 \)). Therefore equation (83) simplifies to

\[
v^3(r) = \frac{GM_0}{r} \left(1 - \left(\frac{r_0}{r}\right)^{1-\alpha}\right) H_0(r_0).
\]

The rotation curves themselves together with the data are shown in Fig. 2.

7 LOW SURFACE BRIGHTNESS GALAXIES

7.1 LSB galaxy rotation curves

A typical LSB galaxy resembles a normal late-type spiral, usually with some ill-defined spiral arms. They usually have H I masses of a few times \( 10^8 M_\odot \). Thuan, Gott & Schneider (1987) and Bothun et al. (1993) have found that, although LSB galaxies follow the spatial distribution of HSB galaxies, they tend to be more isolated from their nearest neighbours than HSB galaxies. LSB galaxies can be distinguished from the galaxies defining the Hubble sequence by their LSB, rather than small size (LSB galaxies are not necessarily dwarfs). The central surface brightness of LSB galaxies is much lower than \( \mu_0(0) = 21.65 \pm 0.3 \) mag arcsec\(^{-2} \) – the typical B-band value for HSB galaxies, established by the Freeman law (Freeman 1970; van der Hulst et al. 1993).

The mass density distribution of LSB galaxies at small radii is dominated by a nearly constant density core with a total mass \( M_0 \), and a radius \( r_c \) of only a few kpc, as established by de Blok et al. (2001). Hence, we ignore the mass contributions of the stellar and gas components. At low radii \( r < r_c \), the constant density implies \( v^2(r) = GpVr/r \), where \( V \) is the volume of a sphere with radius \( r \), and \( p = 3M_0/4r_c^3 \pi, \text{ i.e. } v \sim r. \) The Weyl fluid can reproduce this behaviour if we choose the equation of state \( p(\mu) = 0 \) (i.e. \( c_2 = 0 \) and \( B = 0 \) in equation 47). Then the dark anisotropic stress/pressure also vanishes (\( P = 0 \)), and the remaining part of the Weyl fluid energy-momentum tensor (as \( Q_{\mu} = 0 \)) then represents radiation. Using equations (51)–(52), (55)–(56) and (60)–(62), for \( C_G = 0 \) the tangential velocity, given by equation (63), yields to \( v^3 \approx \gamma r^2 \),

the desired behaviour. For \( r > r_c \) we use equation (76), without baryonic matter (\( M_b^e = 0 \)), and apply a new notation: \( M^e_0 = M_0 \). Requiring the continuity of the rotation velocity through \( r = r_c \) we have

\[
v^3(r) = \frac{GM_0}{r} \left(1 - \left(\frac{r_0}{r}\right)^{1-\alpha}\right) H_0(r_0),
\]

where \( H_0(r_c) \) is the logistic function given by equation (85).

We test the model with a sample of nine LSB galaxies, extracted from a larger sample from de Blok & Bosma (2002) as typical galaxies exhibiting the plateau region. We fit equation (87) with the rotation curve data taken from combined HI and H\,\alpha measurements. The fitted curves are represented in Fig. 3. In all cases we find remarkably good agreement between the model and observations. Outside the core radius the parameters of the equation of state \( \alpha \) and \( B \) are determined by fitting through \( \alpha \) and \( \beta \) (see Section 5).

From a \( \chi^2 \)-test we determine \( M_0 \) and \( r_c \), and also the Weyl parameters \( \alpha \) and \( \beta \), shown in Table 4 and Fig. 4.

8 DISCUSSIONS AND FINAL REMARKS

The shape of the observed rotation curves strongly indicates the need for dark matter or equivalent modifications of gravity on galactic scale and above. However, there are no universally accepted candidates explaining the whole amount of dark matter needed for agreement with observations. Because dark matter does not interact with ordinary matter, except gravitationally, the question is that whether dark matter is rather an effect of modified gravity. We have investigated whether extra dimensions could provide this type of modification in the simplest, co-dimensional one-brane-world scenario. We have derived an analytical expression for the rotational velocity of a test particle on a stable circular orbit in the exterior region to a galaxy, with Weyl fluid contributions (the dark pressure and dark radiation) included. For this we have assumed a linear equation of state for the Weyl fluid and we have argued for the rightness of this as follows.

In Section 4.1, we have derived a system of two first-order differential equations for three-brane variables with straightforward geometrical meaning. These variables are: (1) the acceleration of the normal congruence to the \( t = \text{const.} \) hypersurfaces; (2) the expansion of the normal congruences to the \( r = \text{const spheres in the } t = \text{const hypersurfaces} \) and (3) the scalar characterizing the electric part of the Weyl curvature of the brane (due to its construction, this is defined on the \( t = \text{const hypersurfaces} \). The dark pressure and dark radiation do not enter these equations.

We also derived additional algebraic relations for these three quantities valid in Schwarzschild space–time. As the brane-world spherically symmetric space–time is a modification of the Schwarzschild solution, we have slightly modified one of these relations (by introducing two continuous deformation parameters, which take the value 1 for Schwarzschild), closing in this way the system of equations for these variables. The Cauchy–Peano theorem then proves the existence of a solution for this system.

We have shown then how the dark pressure and dark radiation are determined algebraically in terms of the above mentioned two kinematical and one brane Weyl quantities. We find that the above-mentioned choice induces a linear relation between them, the desired equation of state.

There are two types of evolutions, which in principle could destroy this solution. No problem is posed by the temporal evolution,
Figure 3. Best-fitting curves for the LSB galaxy sample. The parameters of the model velocity curves are given in Table 4.

Table 4. The best-fitting parameters of the nine LSB galaxy sample ($M_0$, $r_c$, $\alpha$, $\beta$). The minimum values of the $\chi^2$ statistic $\chi^2_{\text{min}}$ are also given. All $\chi^2_{\text{min}}$ are within 1σ.

| Galaxy  | $k$ (kpc$^{-1}$) | $M_0$ (☉) | $r_c$ (kpc) | $\alpha$ | $\beta$ | $\chi^2_{\text{min}}$ |
|---------|-----------------|-----------|-------------|-----------|--------|------------------|
| DDO 189 | 57.5            | $4.05 \times 10^8$ | 1.25 | 0.3 | $6.43 \times 10^{-8}$ | 0.742 |
| NGC 2366 | 46.0            | $1.05 \times 10^9$ | 1.47 | 0.8 | $1.12 \times 10^{-7}$ | 2.538 |
| NGC 3274 | 138.1           | $4.38 \times 10^8$ | 0.69 | $-0.4$ | $6.73 \times 10^{-8}$ | 18.099 |
| NGC 4395 | 30.0            | $2.37 \times 10^8$ | 0.71 | 0.9 | $3.43 \times 10^{-7}$ | 27.98 |
| NGC 4455 | 99.7            | $2.26 \times 10^8$ | 1.03 | 0.9 | $2.72 \times 10^{-7}$ | 7.129 |
| NGC 5023 | 86.3            | $2.69 \times 10^8$ | 0.74 | 0.9 | $4.53 \times 10^{-7}$ | 10.614 |
| UGC 10310 | 36.4           | $1.28 \times 10^8$ | 2.6 | 0.4 | $1.12 \times 10^{-7}$ | 0.729 |
| UGC 1230 | 15.3            | $3.87 \times 10^8$ | 3.22 | $-1.7$ | $1.12 \times 10^{-7}$ | 0.539 |
| UGC 3137 | 34.5            | $5.32 \times 10^8$ | 3.87 | $-0.5$ | $1.23 \times 10^{-7}$ | 4.877 |

The model has several free parameters, which in principle could be fixed in such a way to explain the observed galactic rotation curve behaviour.

In order to much closely test this assumption, we have considered a sample of nine HSB and nine LSB galaxies with well-measured combined H I and Hα rotation curves. Since LSB galaxies are known to be dark matter dominated (McGaugh et al. 2001), reproducing their rotation curves without the need of any dark matter component would be a major achievement. Fitting the model to rotation curve data and photometric measurements allowed us to constrain the Weyl parameters $\alpha$ and $\beta = -C$, to also determine the mass-to-light ratios ($M/L$) of the baryonic components in HSB galaxies, and the total core mass $M_0$ and radius $r_c$ of LSB galaxies. The fit was in all cases within 1σ confidence level, which supports the choice of the equation of state (46) from a physical point of view. In all cases we found $\beta \ll 1$ and in most cases $\alpha \in (0.4, 0.9)$, which for the constants $A = \frac{1}{2} (1 + \alpha)$, $B = A (1 - \beta) + \alpha \beta$ introduced in Section 4.1, gives $A \approx B \in (0.7, 0.95)$. These express the difference of our model from a Schwarzschild solution, for which $A = B = 1$.

With the parameters determined from the fit the theoretical rotation curves will have an almost flat (slightly increasing) asymptotic behaviour at larger radii than the available observations of HI and Hα velocities for these galaxies. This tendency is somewhat contradictory with the shape of the universal rotation curve, scaling only with the virial mass (Salucci et al. 2007), at least for the well-fitting model parameters. The asymptotic shape of the universal rotation curve shows a decreasing tendency, as has been found from N-body simulations, which assume the Navarro–Frenk–White cold...
dark matter model (Navarro, Frenk & White 1996). We note though that parameters of the Weyl fluid reproducing the asymptotics of the universal rotation curve can be found, but for these the fit with the H I and Hα velocity data falls outside the 3σ confidence level.

Observationally, the galactic rotation curves remain flat to the farthest distances that can be observed. On the other hand, there is a simple way to estimate an upper bound for the cut-off of the constancy of the tangential velocities. The idea is to consider the point at which the decaying density profile of the dark radiation associated with the galaxy becomes smaller than the average energy density of the Universe. Let the value of the coordinate radius at the point where the two densities are equal be $R_{U}^{\text{max}}$. Then at this point $3\alpha_{b}U(R_{U}^{\text{max}}) = (8\pi G(c^{2})\rho_{\text{univ}}$, where $\rho_{\text{univ}}c^{2}$ is the mean energy density of the universe. In the limit of large $r$, with the use of equation (72), we can approximate the dark radiation term as $3\alpha_{b}U \approx v_{0}/r^{2}$. Hence for $R_{U}$ we obtain

$$R_{U}^{\text{max}} = c \sqrt{\frac{v_{0}}{8\pi G\rho_{\text{univ}}}}.$$  

(88)

The mean density of the universe is given by $\rho_{\text{univ}} = \rho_{\text{crit}} = 3H_{0}^{2}/8\pi G$, where $H_{0}$ is the Hubble constant, given by $H_{0} = 100h$ km s$^{-1}$ Mpc, $1/2 \leq h \leq 1$. Therefore

$$R_{U}^{\text{max}} \approx cH_{0}^{-1} \sqrt{\frac{v_{0}}{3}} \approx 3 \times 10^{3} \times \sqrt{\frac{v_{0}}{3}} h^{-1} \text{ Mpc}. \quad (89)$$

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A numerical evaluation of $R_{U}^{\text{max}}$ requires the knowledge of the constant $v_{0}$ in the flat region, and of the basic fundamental cosmological parameters. In the case of the HSB galaxies $v_{0} \in (0.237 \times 10^{-6}, 0.63 \times 10^{-6})$, while in the case of the LSB galaxies, $v_{0} \in (0.316 \times 10^{-7}, 0.316 \times 10^{-6})$. This gives for $R_{U}^{\text{max}}$ a range of $R_{U}^{\text{max}} \in (0.84 h^{-1} \text{ Mpc}, 1.37 h^{-1} \text{ Mpc})$ for the HSB galaxies, and $R_{U}^{\text{max}} \in (0.307 h^{-1} \text{ Mpc}, 0.97 h^{-1} \text{ Mpc})$ for the LSB galaxies, respectively. The measured flat regions are about $R \approx 2 \times R_{\text{opt}}$, where $R_{\text{opt}}$ is the radius encompassing 83 per cent of the total integrated light of the galaxy (Binney & Tremaine 1987; Persic et al. 1996; Boriello & Salucci 2001). If we take as a typical value $R \approx 30$ kpc, then it follows that $R \ll R_{U}^{\text{max}}$. However, according to our model, the flat rotation curve region should extend far beyond the present measured range.

An alternative estimation of $R_{U}^{\text{max}}$ can be obtained from the observational requirement that at the cosmological level the energy density of the dark matter represents a fraction $\Omega_{m} \approx 0.3$ of the total energy density of the universe $\Omega = 1$. Therefore, the dark matter contribution inside a radius $R_{U}^{\text{max}}$ is given by $4\pi\Omega_{m}(R_{U}^{\text{max}})^{3} \rho_{\text{crit}}/3$, which gives

$$R_{U}^{\text{max}} \approx cH_{0}^{-1} \sqrt{\frac{v_{0}}{3}} \approx 3 \times 10^{3} \times \sqrt{\frac{v_{0}}{3}} h^{-1} \text{ Mpc}. \quad (90)$$
Therefore, by assuming that the dark radiation contribution to the total energy density of the Universe is of the order of $\Omega_m \approx 0.3$ we have $R_{\text{vir}}^{\text{Weyl}} \in (388 \, h^{-1}, 3881 \, h^{-1}) \, \text{kpc}$ for $U_g \in (10^{-4}, 10^{-3})$.

The limiting radius at which the effects of the extra-dimensions extend, far away from the baryonic matter distribution, is given in the present model by equations (89) or (90). In the standard dark matter models this radius is called the truncation parameter $s$, and it describes the extent of the dark matter haloes. Values of the truncation parameter by weak lensing have been obtained for several fiducial galaxies by Hoekstra, Yee & Gladders (2004). In the following we compare our results with the observational values of $s$ obtained by fitting the observed values with the truncated isothermal sphere model, as discussed in some detail in Hoekstra et al. (2004). The truncation parameter $s$ is related to $R_{\text{vir}}^{\text{Weyl}}$ by the relation $s = R_{\text{vir}}^{\text{Weyl}}/2\pi$ (see equation 4 in Hoekstra et al. 2004). Therefore, generally $s$ can be obtained from the relation

$$s \approx \frac{\sigma}{\sqrt{6\pi}} \left( \frac{1}{2\Omega_m} H_0^{-1}, \right) \tag{91}$$

where $\sigma$ is the velocity dispersion, expressed in km s$^{-1}$. Hence the truncation parameter is a simple function of the velocity dispersion and of the cosmological parameters only. For a velocity dispersion of $\sigma = 146$ km s$^{-1}$ and with $\Omega_m = 0.3$, equation (91) gives $s \approx 245 h^{-1}$ kpc, while the truncation size obtained observationally in Hoekstra et al. (2004) is $s = 213 h^{-1}$ kpc. For $\sigma = 110$ km s$^{-1}$ we obtain $s \approx 184 h^{-1}$ kpc, while $\sigma = 136$ km s$^{-1}$ gives $s \approx 228 h^{-1}$ kpc. All these values are consistent with the observational results reported in Hoekstra et al. (2004), the error between prediction and observation being of the order of 20 per cent. We have also to mention that the observational values of the truncation parameter depend on the scaling relation between the velocity dispersion and the fiducial luminosity of the galaxy. Two cases have been considered in Hoekstra et al. (2004), the case in which the luminosity $L_B$ does not evolve with the redshift $z$ and the case in which $L_B$ scales with $z$ as $L_B \propto (1 + z)$. Depending on the scaling relation slightly different values of the velocity dispersion and truncation parameter are obtained.

One of the most straightforward evidence for dark matter comes from the radial Tully–Fisher relation (Yegorova & Salucci 2007): at a given galactocentric distance (measured in unit of the optical radius) there is a relation between rotation velocities and the absolute magnitudes of the galaxies. From this relation we can extract information about the mass distribution of spiral galaxies. We have studied whether the Weyl model satisfies this relation. The plot in Fig. 5 is in good agreement with the result of Yegorova & Salucci (2007).

Table 5. The calculated halo masses for the investigated sample of HSB galaxies.

| Galaxy       | $M_{\text{Weyl}}^{10^{12} M_\odot}$ | $M_{\text{simulation}}^{10^{12} M_\odot}$ |
|--------------|--------------------------------------|------------------------------------------|
| ESO215G39    | 1.27                                 | 1.89                                     |
| ESO322G76    | 2.43                                 | 8.53                                     |
| ESO322G77    | 2.19                                 | 1.487                                    |
| ESO323G25    | 3.75                                 | 7.36                                     |
| ESO383G02    | 2.35                                 | 3.12                                     |
| ESO445G19    | 2.33                                 | 1.97                                     |
| ESO446G01    | 4.69                                 | 9.7                                      |
| ESO509G80    | 10.7                                 | 205                                      |
| ESO569G17    | 1.55                                 | 1.17                                     |

Figure 5. The radial Tully–Fisher relation of our HSB and LSB galaxy samples: absolute magnitude of the galaxies as a function of the rotational velocities predicted by the Weyl model at 0.2 (red), 0.4 (yellow), 0.6 (green), 0.8 (cyan), 1 (navy), 1.2 (lilac) optical radii.

Some tension between the Weyl fluid model and the numerical simulations.

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\[ C_1(A6) \approx = \tilde{C}_{ABC}p_{AB} = \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u, \]

\[
\begin{align}
-k_4^4 U &= \tilde{C}_{ABC}p_{AB} = \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u, \\
-k_4^4 Q_K &= \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u, \\
-k_4^4 P_{AB} &= \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u
\end{align}
\]

\[ \text{APPENDIX A: 3+1+1 DECOMPOSITION OF THE 5D WEYL CURVATURE} \]

With the brane normal \( n^4 \) and the temporal normal \( u^\theta \) to the spatial hypersurfaces singled out, the 5D Weyl tensor \( \tilde{C}_{ABCD} \) admits a 3+1+1 decomposition (Keresztes & Gerlady 2010a), generalizing the corresponding decomposition of the 4D Weyl tensor \( C_{abcd} \).

The projections

\[ \begin{align}
\mathcal{E}_{KL} &= \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u, \\
\mathcal{H}_{KL} &= \frac{1}{2} \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u
\end{align} \]

are related to the electric and magnetic parts of the 4D Weyl tensor on the brane, \( \tilde{E}_{ab} = \tilde{C}_{abcd}u^{b}u^{d} \) and \( \tilde{H}_{ab} = \frac{1}{2} \tilde{C}_{abcd}u^{b}c^{d} \), as

\[ \begin{align}
\mathcal{E}_{ab} &= E_{ab} - \frac{k_4^4}{2} P_{ab} - \frac{1}{3} \left( \tilde{K} + \tilde{\Theta} \right) \mathcal{E}_{ab}, \\
&+ \frac{1}{2} \left( \tilde{K} + \tilde{\Theta} \right) \mathcal{E}_{ab}, \\
\mathcal{H}_{ab} &= H_{ab} - \epsilon_{ab} \mathcal{E}_{ab} \tilde{K}.
\end{align} \]

Here \( \tilde{K} = u^\theta c^{L} \nabla_{CN}B, \tilde{K}A = h_{k}^{\gamma} u^{\gamma} c^{L} \nabla_{CN}B, \tilde{\Theta} = h^{AB} \tilde{\nabla}_{AB}B, \mathcal{E}_{ab} = c_{ABCD}u^{b}c^{d} \), and \( \mathcal{H}_{ab} = c_{ABCD}u^{b}c^{d} \). These are kinematic quantities emerging as various projections of the extrinsic curvature of the brane (\( \Theta, \mathcal{E}_{ab} \)), being the expansion and shear of the brane normal, \( \tilde{\nabla}_{A} \) is the covariant derivative in the 5D space–time.

The rest of the components

\[ \begin{align}
\mathcal{E}_{K} &= \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u, \\
\mathcal{F}_{KL} &= \tilde{C}_{ABC}h_{k}^{\gamma} u_{\gamma}^n c^u
\end{align} \]

do not enter the equations on the brane.
APPENDIX B: DYNAMICS ON THE BRANE

In the 2+1+1 formalism on a brane with spherical symmetry the covariant derivative of any scalar field \( f(r) \), taken along the integral curves of the radial vector field \( r^a \), is defined (Keresztes & Gergely 2010a) as \( f^* = r^a D_a f \). The non-trivial dynamical equations are four ordinary differential equations:

\[
\begin{align*}
\ddot{\Theta}^* &+ \frac{\dot{\Theta}^2}{2} + \frac{2}{3} \ddot{E} + \frac{k^4}{3} (2U + P) = 0, \\
(U + 2P) \dddot{\Theta}^* + 4AU + (2A + 3\dot{\Theta}) P &= 0, \\
A^* + A(\ddot{\Theta} + A) - k^2 U &= 0, \\
A^* + A \left( A - \frac{\dot{\Theta}}{2} \right) - \ddot{E} + \frac{k^4}{2} P &= 0,
\end{align*}
\]

(B1) \hspace{2cm} (B2) \hspace{2cm} (B3) \hspace{2cm} (B4)

for the five variables \( \Theta, A, \dot{\Theta}, U \) and \( P \). Equation (B2) is equivalent with the constraint equation (11). The difference of equations (B3) and (B4) gives a simple algebraic relation between the dark radiation \( U \), dark pressure \( P \), acceleration \( A \) of temporal normals, expansion of radial geodesics \( \ddot{\Theta} \) and electric part of the 4D Weyl tensor \( \ddot{E} \), respectively:

\[
k^4 (2U + P) = 3A \ddot{\Theta} + 2\ddot{E}.
\]

(B5)

This enables us to eliminate the Weyl contributions from equation (B3), obtaining

\[
\dddot{\Theta}^* + \frac{\dot{\Theta}^2}{2} + \frac{4}{3} \ddot{E} + A \ddot{\Theta} = 0.
\]

(B6)

Another algebraic relation emerges as follows. From (i) the relation between the Ricci scalar of the sphere and the Riemann tensor of the three-space:

\[
\begin{align*}
2\mathcal{R} &= \left( h^{ac} - r^a r^c \right) \left( h^{bd} - r^b r^d \right) \\
&\quad \times \left[ \frac{3}{3} \mathcal{R}_{abcd} + \left( D_a r_c \right) \left( D_b r_d \right) - \left( D_a r_d \right) \left( D_b r_c \right) \right] \\
&= 2 \left( k^4 (U - P) - 2\ddot{E} \right) + \frac{\dot{\Theta}^2}{2},
\end{align*}
\]

(B7)

where we have used \( D_a r_b = \ddot{\Theta} (h_{ab} - r_a r_b) / 2 \), (ii) the 3D Riemann tensor given in Keresztes & Gergely (2010a),\(^1\) which in the spherically symmetric case simplifies to

\[
\begin{align*}
\mathcal{R}_{abcd} &= 2k^4 \left( U_{a|b|d} - 2 \left( \ddot{E}_{|b|d} - \ddot{E}_{a|b|d} \right) \\
&\quad - k^4 \left( P_{a|b|d} - P_{d|a|b} \right) \right),
\end{align*}
\]

(B8)

and (iii) the Gaussian curvature of the 2D space-like group orbits orthogonal to \( r^a \) and \( r^a \) being \( \dddot{r}^a = 2/r^2 \), we obtain

\[
\dddot{E} = k^4 \left( U - P \right) + \frac{3}{8} \dddot{\Theta}^2 - \frac{3}{2r^2}.
\]

(B9)

We need to relate the newly introduced curvature coordinate to the \( * \)-derivative. For this we take the \( * \)-derivative of equation (B9), employ equations (B6), (B2), (B3); also the \( * \)-derivative of equation (B5) for eliminating \( \dddot{E} \), finding

\[
(\ln r^2)^* = \ddot{\Theta}.
\]

(B10)

This relation allows to replace the \( * \)-derivative in all equations by \( r \)-derivatives, denoted by a prime. Thus equations (B6), (B2), (B3) can be rewritten as

\[
\begin{align*}
\frac{r}{2} \dddot{\Theta} + \frac{\dot{\Theta}^2}{2} + \frac{4}{3} \ddot{E} + A \ddot{\Theta} &= 0, \\
\frac{r}{2} \dddot{\Theta} (U + 2P)^* + 4A \frac{U + 2P}{U} &= 0, \\
\frac{r}{2} \dddot{\Theta} (U + 2P)^* + 4A \frac{U + 2P}{U} &= 0,
\end{align*}
\]

(B11) \hspace{2cm} (B12) \hspace{2cm} (B13)

where the prime denotes the derivative with respect to \( r \).

In summary, the variables \( U, P, A, \dot{\Theta}, \ddot{\Theta} \) are constrained by two independent first-order ordinary differential equations (B11) and (B13) and two algebraic equations (B5) and (B9). Equation (B12) follows from these.

\(^1\) We note that there is a missing factor 2 in front of \( \mathcal{E} \) in equation (23) of Keresztes & Gergely (2010a). [The conversion of notations to the notations of the present paper is: \( \mathcal{E}, \dddot{E}_{ab}, \dddot{E}_{ab}, Q_{ab}, k^4 P_{ab}, \dddot{E}_{ab} \).]

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