An efficient sum of squares nonnegativity certificate for quaterniony quartic
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Let \( f \in \mathbb{R}[X_0, \ldots, X_n] \) be homogeneous of degree \( 2d \) and nonnegative on \( \mathbb{R}^{n+1} \). While \( f \) can be represented as a sum of squares of rational functions, only for \( d = 1 \), or \( n = 2 \), reasonably good bounds on degrees of such functions are known, see e.g. [6]. A polynomial \( g \) is called sos polynomial if \( g = \sum_{k=1}^{N} h_k^2 \), for some polynomials \( h_k \).

The aim of this note is to show the following.

**Theorem.** Let \( f \) be a homogeneous 4-variate nonnegative polynomial of degree 4. Then \( f = p/q \), where \( p \) and \( q \) are homogeneous sos polynomials of degrees 8 and 4, respectively. The identity \( f = p/q \) holds almost everywhere in the usual measure-theoretic sense.

In what follows, we denote \( X := (X_1, \ldots, X_n) \). First, we recall a well-known lemma (e.g. similar observations are made in [4]).

**Lemma 1.** Let \( f \in \mathbb{R}[X_0, \ldots, X_n] \) be homogeneous of degree \( 2d \) and nonnegative on \( \mathbb{R}^{n+1} \), and \( S := \{ x \in \mathbb{R}^{n+1} \mid \sum_{k=0}^{n} x_k^2 = 1 \} \) be the unit sphere in \( \mathbb{R}^{n+1} \).

(i) Let \( f \) have a zero on \( S \). An orthogonal change of coordinates bringing the zero to \((1 : 0 : \ldots : 0)\) transforms \( f \) into the form

\[
f(X_0, X) = \sum_{k=2}^{2d} f_k(X)X_0^{2d-k}, \quad f_k \in \mathbb{R}[X_1, \ldots, X_n]. \tag{1}
\]

(ii) Let \( f \) be strictly positive on \( S \), with a minimum on \( S \) reached at \( x^* \in S \). An orthogonal change of coordinates bringing \( x^* \) to \((1 : 0 : \ldots : 0)\) transforms \( f \) into the form

\[
f(X_0, X) = X_0^{2d} + \sum_{k=2}^{2d} f_k(X)X_0^{2d-k}, \quad f_k \in \mathbb{R}[X_1, \ldots, X_n], \tag{2}
\]

so that \( f(X_0, X) - X_0^{2d} \) is nonnegative.

(iii) Polynomials \( f_2 \) and \( f_{2d} \) are nonnegative. Moreover, \( f_2 \) is an sos of linear forms.

**Proof.** (i). After the transformation, \( f \) cannot have a term \( X_0^{2d} \), as otherwise it cannot vanish on \((1 : 0 : \ldots : 0)\). It cannot have a term \( f_1(X)X_0^{2d-1} \), as it is nonnegative. Thus we have the required form.

(ii). We may assume without loss of generality that after the transformation we have

\[
f(X_0, X) = X_0^{2d} + g(X_0, X), \quad g(X_0, X) = \sum_{k=1}^{2d} f_k(X)X_0^{2d-k}.
\]

Note that on \( S \) one has \( f(X_0, X) \geq 1 \), as the minimum of \( f \) on \( S \) equals 1. Thus on \( S \) one has \( g(X_0, X) = f(X_0, X) - X_0^{2d} \geq f(X_0, X) - 1 \geq 0 \). Hence \( g(X_0, X) \) is (globally) nonnegative, as it is homogeneous. Moreover, it has a zero, \((1 : 0 : \ldots : 0)\), on \( S \), and this (i) applies, ensuring \( f_1 = 0 \), as required.

(iii). Nonnegativity of \( f_2 \) and \( f_{2d} \) follows from nonnegativity of \( f \) in the case (i), and nonnegativity of \( f(X_0, X) - X_0^{2d} \) in the case (ii). As \( f_2 \) is quadratic, it is an sos.

From now on, consider the case \( d = 2 \). By Lemma 1, we have

\[
g(X_0, X) := f_2(X)X_0^2 + f_3(X)X_0 + f_4(X) \geq 0, \quad f_2(X) \geq 0, \quad f_4(X) \geq 0.
\]

Note that the nonnegativity of \( g \) implies that \( 4f_2(X)f_4(X) - f_4(X)^2 \geq 0 \). Write, omitting \( X \) for brevity:

\[
4f_2g(X_0) = 4f_2^2X_0^2 + 4f_2f_3X_0 + 4f_2f_4 = (2f_2X_0 + f_3)^2 - f_3^2 + 4f_2f_4.
\]
The latter is the sum of a square and a nonnegative homogenous degree 6 polynomial $h(X) := 4f_2(X)f_4(X) − f_3(X)^2$ in $n$ variables.

In particular, for $n = 3$, by [3] (or by a recent [1, Sect. 5-6]), $h(X) = u(X)/q(X)$, with $u$ and $q$ sos polynomials of degrees 8 and 2, respectively. To summarise, we state

**Lemma 2.** Let $g(X_0, X) := f_2(X)X_0^2 + f_3(X)X_0 + f_4(X)$ be a nonnegative degree 4 homogeneous polynomial in $X_0, X$, with $X = (X_1, X_2, X_3)$. Then there exists a homogeneous degree 2 sos polynomial $q(X)$ such that

$$q(X)f_2(X)g(X_0, X) = \sum_{k=1}^N r_k(X_0, X)^2,$$

i.e. it is an sos polynomial.

As an orthogonal change of coordinates (e.g. the inverse $G$ of the one in Lemma 1 (i)) respects sos decompositions, and as $g(1, 0, 0, 0) = 0$, we obtain, applying $G$ to the both sizes of (3), the following.

**Lemma 3.** Let $f$ be a nonnegative degree 4 homogeneous polynomial in $X_0, . . . , X_3$ with a zero on $S$. Then there are degree 2 homogeneous sos polynomials $q_t(X_0, . . . , X_3)$, $(t = 1, 2)$ such that $q_1q_2f = s$ is an sos polynomial, and $f = s/(q_1q_2)$.

To complete the proof of the Theorem, it remains to observe that in the case of strictly positive $f(X_0) = X_1^4 + f_2X_0^2 + f_3X_0 + f_4$ in the form (2), by Lemma 2 and Lemma 1 (ii) we have $f(X_0) = X_0^4 + \sum_{k=1}^N r_k(X)^2$, i.e.

$$q(X)f_2(X)f(X_0, X) = q(X)f_2(X)X_0^4 + \sum_{k=1}^N r_k(X)^2.$$

The 1st term on the RHS of the latter is an sos, and the argument used to establish Lemma 3 applies here, too.

**Remarks**

(a) It can be seen from the proof that $q$ in Theorem is the product of two quadrics, each of which is a sum of at most 3 squares.

(b) The decomposition $qf = p$ can be used to apply semidefinite programming to compute $q$ and $p$ in terms of their Gram matrices (cf. e.g. [5]), following the approach described in [2].

(c) The degree of $q$ is 4. Is this the best possible?

(d) What can be said about $f_2$ in Lemma 1? For instance, one can see that if $f_2$ is a square, $f_2 = \ell^2$, then $f$ is sos. Indeed, in this case $4f_2f_4 − f_3^2$ factors: $4f_2f_4 − f_3^2 = (2\ell)^2f_4 − f_3^2 \geq 0$ shows that $\ell$ divides $f_3$. This implies that $f$ is sos polynomial (cf. [4, Sect. 2.2]).

(f) It is easy to show, using the decomposition of $qf$ into a sum of squares, that $f$ can be written as a sum of at most 8 squares of of rational functions; this is the same number of terms as given by the classical result of Pfister.

**References**

[1] E. de Klerk and D. V. Pasechnik. Products of positive forms, linear matrix inequalities, and Hilbert 17th problem for ternary forms. *European J. Oper. Res.*, 157(1):39–45, 2004.

[2] A. Pfister and C. Scheiderer. An elementary proof of Hilbert’s theorem on ternary quartics. *J. Algebra*, 371:1–25, 2012.

[3] B. Reznick. Some concrete aspects of Hilbert’s 17th Problem. In *Real algebraic geometry and ordered structures (Baton Rouge, LA, 1996)*, volume 253 of *Contemp. Math.*, pages 251–272. Amer. Math. Soc., Providence, RI, 2000.

[4] C. Scheiderer. Positivity and sums of squares: a guide to recent results. In *Emerging applications of algebraic geometry*, volume 149 of *IMA Vol. Math. Appl.*, pages 271–324. Springer, New York, 2009.