When does the F-signature exist?*

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Abstract. We show that the $F$-signature of an $F$-finite local ring $R$ of characteristic $p > 0$ exists when $R$ is either the localization of an $N$-graded ring at its irrelevant ideal or $Q$-Gorenstein on its punctured spectrum. This extends results by Huneke, Leuschke, Yao and Singh and proves the existence of the $F$-signature in the cases where weak $F$-regularity is known to be equivalent to strong $F$-regularity.

Résumé. Nous prouvons dans cet article l’existence de la F-signature d’un anneau local $F$-fini $R$, de caractéristique positive $p$, quand $R$ est la localisation à l’unique idéal homogène maximal d’un anneau $N$-gradué ou quand $R$ est $Q$-Gorenstein sur son spectre épointé. Ceci généralise les résultats de Huneke, Leuschke, Yao et Singh et prouve l’existence de la $F$-signature dans les cas où faible et forte $F$-régularité sont équivalentes.

1 A sufficient condition for the existence of the F-signature

Let $(R, \mathfrak{m}, k)$ be a reduced, local $F$-finite ring of positive characteristic $p > 0$ and Krull dimension $d$. Let

$$R^{1/q} = R^{a_q} \oplus M_q$$

be a direct sum decomposition of $R^{1/q}$ such that $M_q$ has no free direct summands. If $R$ is complete, such a decomposition is unique up to isomorphism. Recent research has focused on the asymptotic growth rate of the numbers $a_q$ as $q \to \infty$. In particular, the $F$-signature (defined below) is studied in [7] and [3], and more generally the Frobenius splitting ratio is studied in [2].

For a local ring $(R, \mathfrak{m}, k)$, we set $\alpha(R) = \log_p[k_R : k_R^p]$. It is easy to see that, for an $\mathfrak{m}$-primary ideal $I$ of $R$, $\lambda(R^{1/q}/IR^{1/q}) = \lambda(R/I[\overline{q}])/q^{\alpha(R)}$, where $\lambda(-)$ represents the length function over $R$.

We would like to first define the notion of $F$-signature as it appears in [3] and [7].

Definition 1.1. The $F$-signature of $R$ is $s(R) = \lim_{q \to \infty} \frac{a_q}{q^{d+\alpha(R)}}$, if it exists.
The following result, due to Aberbach and Leuschke [3], holds:

**Theorem 1.2.** Let \((R, m, k)\) be a reduced Noetherian ring of positive characteristic \(p\). Then
\[
\liminf_{q \to \infty} a_q / q^{d+\alpha(R)} > 0 \iff \limsup_{q \to \infty} a_q / q^{d+\alpha(R)} > 0 \iff R \text{ is strongly } F\text{-regular}.
\]

The question of whether or not, in a strongly \(F\)-regular ring, \(s(R)\) exists, is open. We show in this paper that its existence is closely connected to the question of whether or not weak and strong \(F\)-signature are equivalent.

Smith and Van den Bergh ([10]) have shown that the \(F\)-signature of \(R\) exists when \(R\) has finite Frobenius representation type (FFRT) type, that is, if only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules occur as direct summands of \(R^{1/q}\) for any \(q = p^e\). Yao has proven that, under mild conditions, tight closure commutes with localization in a ring of FFRT type, [11]. Moreover, Huneke and Leuschke proved that if \(R\) is also Gorenstein, then the \(F\)-signature exists, [7]. Yao has recently extended this result to rings that are Gorenstein on their punctured spectrum, [12]. Singh has also shown that the \(F\)-signature exists for monomial rings, [9].

Let \((R, m)\) be an approximately Gorenstein ring. This means that \(R\) has a sequence of \(m\)-primary irreducible ideals \(\{I_t\}_t\) cofinal with the powers of \(m\). By taking a subsequence, we may assume that \(I_t \supseteq I_{t+1}\). For each \(t\), let \(u_t\) be an element of \(R\) which represents a socle element modulo \(I_t\). Then there is, for each \(t\), a homomorphism \(R/I_t \to R/I_{t+1}\) such that \(u_t + I_t \mapsto u_{t+1} + I_{t+1}\). The direct limit of the system will be the injective hull \(E = E_R(R/m)\) and each \(u_t\) will map to the socle element of \(E\), which we will denote by \(u\). Hochster has shown that every excellent, reduced local ring is approximately Gorenstein ([5]).

Aberbach and Leuschke have shown that, for every \(q\), there exists \(t_0(q)\), such that
\[
a_q / (q^{d+\alpha(R)}) = \lambda(R/(I_t^{[q]} : u_t^q)) / q^d,
\]
for all \(t \geq t_0(q)\) (see [3], p. 55).

The situation when \(t_0(q)\) can be chosen independently of \(q\) is of special interest.

**Definition 1.3.** We say that \(R\) satisfies Condition \((A)\), if there exist a sequence of irreducible \(m\)-primary ideals \(\{I_t\}\) and a \(t_0\) such that, for all \(t \geq t_0\) and all \(q\)
\[
(I_t^{[q]} : u_t^q) = (I_{t_0}^{[q]} : u_{t_0}^q).
\]

**Proposition 1.4.** Let \((R, m, k)\) be a local reduced \(F\)-finite ring. If \(R\) satisfies Condition A, then the \(F\)-signature exists.

**Proof.** We know that \(R\) is approximately Gorenstein and hence we will use the notation fixed in the paragraph above.

As explained above, Condition \((A)\) implies that there exists \(t_0\), independent of \(q\), such that
\[
a_q / (q^{d+\alpha(R)}) = \lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q)) / q^d,
\]
for all \(q\).

But \(\lambda(R/(I_{t_0}^{[q]} : u_{t_0}^q)) = \lambda(R/I_{t_0}^{[q]}) - \lambda(R/(I_{t_0} + u_{t_0}R)^{[q]})\). Dividing by \(q^d\) and taking the limit as \(q \to \infty\) yields \(s(R) = e_{HK}(I_{t_0}, R) - e_{HK}(I_{t_0} + u_{t_0}R, R)\).
Now we would like to concentrate on another condition, Condition (B), that appeared first in the work of Yao. First we need to introduce some notation.

Assume that $E$ is the injective hull of the residue field $k$. By $R^{(e)}$ we denote the $R$-bialgebra whose underlying abelian group equals $R$ and the left and right $R$-multiplication is given by $a \cdot r \cdot b = arb^\theta$, for $a, b \in R, r \in R^{(e)}$.

Let $k = Ru \to E$ be the natural inclusion and consider the natural induced map $\phi_e : R^{(e)} \otimes_R E \to R^{(e)} \otimes_R (E/k)$. Then $a_q/q^{\alpha(R)} = \lambda(\ker(\phi_e))$ (by Aberbach-Enescu, Corollary 2.8 in [2], see also Yao’s work [12]).

One can in fact see that

$$\lambda(\ker(\phi_e)) = \lambda(R/(c \in R : c \otimes u = 0 \in R^{(e)} \otimes_R E)) = \lambda(R/ \cup_t (I_t^{[q]} : u_t^q)).$$

**Definition 1.5.** We say that $R$ satisfies Condition (B) if there exists a finite length submodule $E' \subset E$ such that, if $\psi_e : R^{(e)} \otimes_R E' \to R^{(e)} \otimes_R E'/k$, then $\lambda(\ker(\phi_e)) = \lambda(\ker(\psi_e))$, for all $e$.

Yao [12] has shown that Condition (B) implies that the $F$-signature of $R$ exists.

**Proposition 1.6.** Let $(R, m, k)$ be a local reduced $F$-finite ring. Then Conditions (A) and (B) are equivalent.

**Proof.** Assume that Condition (A) holds. Then one can take $E' = R/I_{t_0}$ and then Condition (B) follows.

If Condition (B) holds, then take $t_0$ large enough such that $E' \subset \text{Im}(R/I_{t_0} \to E)$.

As noted above, one can compute the length of the kernel of $\psi_e$ as the colength of $\{c \in R : c \otimes u = 0 \in R^{(e)} \otimes_R E'\}$. Since $R/I_{t_0}$ injects into $E$ we see that $\{c \in R : c \otimes u = 0 \in R^{(e)} \otimes_R R/I_{t_0}\} = (I_{t_0}^{[q]} : u_{t_0}^q)$.

Since $(I_{t_0}^{[q]} : u_{t_0}^q) \subset (I_t^{[q]} : u_t^q)$ for all $t \geq t_0$, we see that Condition (B) implies that $(I_{t_0}^{[q]} : u_{t_0}^q) = (I_t^{[q]} : u_t^q)$ for all $t \geq t_0$, which is Condition (A).

\[\square\]

## 2 N-Graded Rings

Let $(R, m)$ be a Noetherian $\mathbb{N}$-graded ring $R = \oplus_{n \geq 0} R_n$, where $R_0 = k$ is an $F$-finite field of characteristic $p > 0$.

For any graded $R$-module $M$ one can define a natural grading on $R^{(e)} \otimes M$: the degree of any tensor monomial $r \otimes m$ equals $\deg(r) + q \deg(m)$.

In what follows we will need the following important Lemma by Lyubeznik and Smith ([8], Theorem 3.2):

**Lemma 2.1.** Let $R$ be an $\mathbb{N}$-graded ring and $M, N$ two graded $R$-modules. Then there exists an integer $t$ depending only on $R$ such that whenever

$$M \to N$$
is a degree preserving map which is bijective in degrees greater than $s$, then the induced map

$$R^{(e)} \otimes M \to R^{(e)} \otimes N$$

is bijective in degrees greater than $p^e(s + t)$.

Let $E$ be the injective hull of $R_m$. In fact, $E$ is also the injective hull of $R/m$ over $R$ and as a result is naturally graded with socle in degree 0. We can write $E = \oplus_{n \leq 0} E_n$.

Let $t$ be as in the Lemma 2.1, and let $s \leq -t - 1$. Obviously the map $E' = \oplus_{s \leq n \leq 0} E_n \to E = \oplus_{n \leq 0} E_n$ is bijective in degrees greater than $s$. So by Lemma 2.1, the map $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $p^e(s + t) \leq -p^e$.

**Theorem 2.2.** Let $R$ be an $\mathbb{N}$-graded reduced ring over an $F$-finite field $k$ of positive characteristic. Then Condition $(B)$ is satisfied by $R$ and hence the $F$-signature of $R$ exists.

**Proof.** Let $E$ be the injective hull of $k = R/m$ over $R_m$. As above, $E = \oplus_{n \leq 0} E_n$, where 0 is the degree of the socle generator $u$ of $E$.

Using the notation introduced above, we will let $s = -t - 1$ and $E' = \oplus_{s \leq n \leq 0} E_n \to E$. So, $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $-p^e$. In particular it is bijective in degrees greater or equal to 0.

We have the following exact sequences:

$$0 \to k = Ru \to E \to E/k \to 0$$

and

$$0 \to k = Ru \to E' \to E'/k \to 0.$$

After tensoring with $R^{(e)}$, we get the exact sequences

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \to R^{(e)} \otimes E \xrightarrow{\phi} R^{(e)} \otimes E/k \to 0$$

and

$$R^{(e)} \otimes k = R^{(e)} \otimes Ru \to R^{(e)} \otimes E' \xrightarrow{\psi} R^{(e)} \otimes E'/k \to 0.$$

One can easily see that ker($\phi_e$) and ker($\psi_e$) are the submodules generated by $1 \otimes u$ in $R^{(e)} \otimes E$ and $R^{(e)} \otimes E'$, respectively.

The degree of $1 \otimes u$ is $q \cdot 0 = 0$ and we have noted that the natural map $R^{(e)} \otimes E' \to R^{(e)} \otimes E$ is bijective in degrees greater than $-p^e$. This shows that ker($\phi_e$) \simeq ker($\psi_e$) and hence Condition $(B)$ is satisfied.

\[\Box\]

### 3 $\mathbb{Q}$-Gorenstein Rings

We turn now to showing that Condition $(A)$ holds in strongly $F$-regular local rings which are $\mathbb{Q}$-Gorenstein on the punctured spectrum. Let $(R, m, k)$ be such a ring of dimension
and assume that $R$ has a canonical module (e.g. $R$ is complete). In this case $R$ has an unmixed ideal of height 1, say $J \subseteq R$, which is a canonical ideal. We may pick an element $a \in J$ which generates $J$ at all minimal primes of $J$, and then an element $x_2 \in \mathfrak{m}$ which is a parameter on $R/J$ such that $x_2 J \subseteq aR$. It is easy to see that then $x^n R/J \subseteq a^n R$ for all $n \geq 1$ (where $J^{(n)}$ is the height one component of $J^n$). The condition that $R$ is $\mathbb{Q}$-Gorenstein on the punctured spectrum implies that there is an integer $h$ and two sequences of elements $x_3, \ldots, x_d \in \mathfrak{m}$ and $a_3, \ldots, a_d \in J^{(h)}$ such that $x_i J^{(h)} \subseteq a_i R$ for $3 \leq i \leq d$, and $x_2, \ldots, x_d$ is a s.o.p. on $R/J$. We may then pick $x_1 \in J$ such that $x_1, \ldots, x_d$ is an s.o.p. for $R$. See [1], section 2.2 for more detail. Then by [1], Lemma 2.2.3 we have that for any $N \geq 0$ and any $n \geq 0$,

$$(J^{(nh)} x_2^N, \ldots, \widetilde{x_i}^N, \ldots, x_d^N) : x_i^\infty = (J^{(nh)} x_2^N, \ldots, \widetilde{x_i}^N, \ldots, x_d^N) : x_i^n.$$  \hspace{1cm} (3.1)$$

**Theorem 3.1.** Let $(R, \mathfrak{m}, k)$ be an $F$-finite strongly $F$-regular ring which is $\mathbb{Q}$-Gorenstein on the punctured spectrum. Then $R$ satisfies Condition (A). In particular the $F$-signature of $R$ exists.

**Proof.** If $R$ is not complete, we observe that, since $R$ is excellent, $\hat{R}$ is strongly $F$-regular and $\mathbb{Q}$-Gorenstein on the punctured spectrum. If $\{I_t\}$ is a sequence of ideal in $\hat{R}$ showing condition (A) in $\hat{R}$, then $\{I_t \cap R\}$ does so for $R$. Thus we will assume that $R$ is complete.

Let $J$, $h$, and $x_1, \ldots, x_d$ be as discussed above. Let $I_t = (x_1^{-1} J, x_2^t, \ldots, x_d^t)$. Since $x_1^n J \cong J$ as $R$-modules, the quotient $R/x_1^n J$ is Gorenstein. The hypothesis that $x_2, \ldots, x_d$ are parameters on $R/J$ and $R/x_1 R$ (hence on $R/x_1^n J$) then shows that $I_t$ is irreducible (see [4], Proposition 3.3.18). The sequence $\{I_t\}$ is then a sequence of $\mathfrak{m}$-primary irreducible ideals cofinal with the powers of $\mathfrak{m}$. If $u_1$ represents the socle element of $I_1$, then we may take $u_t = (x_1 \cdots x_d)^{t-1} u_1$ to represent the socle element of $I_t$. We will show that $t_0$ may be taken to be 3.

Suppose that $c \in I_t^{[q]} : u_t^q$ for some $q$. We will show that $c \in I_3^{[q]} : u_3^q$. Raising to the $q$'th power we have $c^q u_t^{q^2} = c^q ((x_1 \cdots x_d)^{t-1} u_1)^{q^2} \subseteq I_t^{[q^2]} = (x_1^{-1} J, x_2^t, \ldots, x_d^t)^{[q]}. Hence c^q ((x_2 \cdots x_d)^{t-1} u_1)^{q^2} \subseteq (x_2^t, \ldots, x_d^t)^{[q]} : x_1^{(t-1) q^2} + (J, x_2^t, \ldots, x_d^t)^{[q]} = (J, x_2^t, \ldots, x_d^t)^{[q]}.

Write $q q' = n q + r q'$ with $0 \leq r q' < h$. Repeated application of equation 3.1 (using 1 rather than $h$ for $x_2$) gives

$$c^q ((x_2 \cdots x_d) u_1)^{q q'} \subseteq (J^{(n q + h)}, x_2^{q q'}, \ldots, x_d^{q q'}).$$  \hspace{1cm} (3.2)$$

Let $d \in J^{(h)} \subseteq J^{(r q')}$. Multiplying by $x_2^{q q'}$ and using that $x_2^{q q'} J^{(q q')} \subseteq a^{q q'} R \subseteq J^{(q q')}$ we have $d c^q ((x_2 \cdots x_d)^2 u_1)^{q q'} \subseteq (J, x_2^{q q'}, \ldots, x_d^{q q'})^{[q q']}. Multiplying by $x_1^{q q'}$ shows that $d c^q u_3^{q q'} = d (cu_3^q)^{q'} \subseteq (I_3^{[q]} : u_3^q). Thus cu_3^q \in (I_3^{[q]})^* = I_3^{[q]}, as desired. \hfill \Box$

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