REDUCTION OF A FAMILY OF IDEALS

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Abstract

In the paper we prove that there exists a simultaneous reduction of one-parameter family of \( m_n \)-primary ideals in the ring of germs of holomorphic functions. Moreover, we generalize the result of A. Płoski [8] on the semicontinuity of the Łojasiewicz exponent in a multiplicity-constant deformation.

1. Introduction

Let \( R \) be a ring and \( I \) an ideal. We say that an ideal \( J \) is a reduction of \( I \) if it satisfies the following condition:

\[
J \subseteq I, \quad \text{and for some } r > 0 \text{ we have } I^{r+1} = JI^r.
\]

The notion of reduction is closely related to the notions of \textit{Hilbert-Samuel multiplicity} and \textit{integral closure} of an ideal.

Recall that if \((R, m)\) is a Noetherian local ring of dimension \( n \) and \( I \) is an \( m \)-primary ideal of \( R \), then the \textit{Hilbert-Samuel multiplicity} of \( I \) is given by the formula

\[
e(I) = \lim_{k \to \infty} \frac{n!}{k^n} \text{length}_R R/I^k.
\]

For the multiplicity theory in local rings see for example [7] or [4].

Let \( I \) be an ideal in a ring \( R \). An element \( x \in R \) is said to be \textit{integral over} \( I \) if there exists an integer \( n \) and elements \( a_k \in I^k \), \( k = 1, \ldots, n \), such that

\[
x^n + a_1 x^{n-1} + \cdots + a_n = 0.
\]

The set of all elements of \( R \) that are integral over \( I \) is called the \textit{integral closure} of \( I \), and is denoted \( \overline{I} \). If \( I = \overline{I} \) then \( I \) is called \textit{integrally closed}. It is well known that \( \overline{I} \) is an ideal.
The relationship between the above notions is given in the following Theorem due to D. Rees:

**Theorem 1** (Rees, [4, Cor. 1.2.5, Thm. 11.3.1]). Let \((R, m)\) be a formally equidimensional Noetherian local ring and let \(J \subset I\) be two \(m\)-primary ideals. Then the following conditions are equivalent:

1. \(J\) is a reduction of \(I\);
2. \(e(I) = e(J)\);
3. \(I = J\).

If \(R/m\) is infinite, \(\dim R = d\) and \(I\) is an \(m\)-primary ideal of \(R\) then any \(d\) “sufficiently general” elements of \(I\) form a reduction of \(I\). More precisely we have the following result

**Theorem 2** ([7, Theorem 14.14]). Let \((R, m)\) be a \(d\)-dimensional Noetherian local ring, and suppose that \(k = R/m\) is an infinite field; let \(I = (u_1, \ldots, u_t)\) be an \(m\)-primary ideal. Then there exist a finite number of polynomials \(D_a \in k[Z_{ij}]\), \(1 \leq i \leq d, 1 \leq j \leq s\), \(1 \leq \alpha \leq v\) such that if \(y_j = \sum a_{ij} u_j, i = 1, \ldots, d\) and at least one of \(D_a(\{a_{ij}; 1 \leq i \leq d, 1 \leq j \leq s\}) \neq 0\), then the ideal \((y_1, \ldots, y_d)R\) is a reduction of \(I\) and \(\{y_1, \ldots, y_d\}\) is a system of parameters of \(R\).

In fact, the above theorem could be generalized to arbitrary ideals in \(R\). Recall that if \((R, m)\) is a Noetherian local ring, the analytic spread of \(I\) (denoted \(\ell(I)\)) is the Krull dimension of the fiber cone of \(I\):

\[
\frac{R[I]}{mR[I]} \cong \frac{R}{m} \oplus \frac{I}{mI} \oplus \frac{I^2}{mI^2} \oplus \cdots
\]

where \(t\) is a variable over \(R\).

**Theorem 3** ([4, Theorem 8.6.6]). Let \((R, m)\) be a Noetherian local ring with infinite residue field and \(l\) an ideal of analytic spread at most \(l\). There exists a non-empty Zariski-open subset \(U\) of \((I/mI)^l\) such that whenever \(x_1, \ldots, x_l \in I\) with \((x_1 + mI, \ldots, x_l + mI) \in U\), then \((x_1, \ldots, x_l)R\) is a reduction of \(I\).

Now, let \((\mathcal{O}_n, m_n)\) be the ring of germs of holomorphic functions \((\mathbb{C}^n, 0) \to \mathbb{C}\). From Theorem 3 we see that, if \(I = (f_1, \ldots, f_m)\mathcal{O}_n\) and \(l\) denotes the integer \(\ell(I)\), then the ideal

\[
\left(\sum a_{ij} f_j, \ldots, \sum a_{ij} f_j\right)\mathcal{O}_n,
\]

is a reduction of \(I\) for generic coefficients \(a_{ij} \in \mathbb{C}\).

If \(g = (g_1, \ldots, g_m) : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)\) is an analytic map germ, then we denote by \((g)\mathcal{O}_n\) the ideal of \(\mathcal{O}_n\) generated by \(g_1, \ldots, g_m\). The aim of this note is to study the following
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**Question 4.** Let $I := (F)\mathcal{O}_{n+k}$ and $I_t := (F_t)\mathcal{O}_n \subset m_n$ be a family of ideals given by a holomorphic map $F = F_t(x) = F(x, t) : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^m, 0)$. Assume that the analytic spread of $I_t$ is constant in some neighbourhood of $0 \in \mathbb{C}^k$. Denote this constant value $l$. Does there exist a linear map $\pi : \mathbb{C}^m \to \mathbb{C}^l$ such that $J_t := (\pi \circ F_t)\mathcal{O}_n$ is a reduction of $I_t$ for $t$ close to $0 \in \mathbb{C}^k$?

By Theorem 3, the answer is immediate if $\ell(I) = \ell(I_t)$ in some neighbourhood of $0 \in \mathbb{C}^k$. It turns out that the above condition is fulfilled in multiplicity-constant families of $m_n$-primary ideals. This fact is implicitly stated in the proof of the principle of specialization of integral dependence given by B. Teissier in [11]. We will here recall this argument.

**Proposition 5.** Let $I$ be as in Question 4. If $I_t$ are $m_n$-primary ideals and the function $t \mapsto e(I_t)$ is constant, then $\ell(I) = \ell(I_t) = n$ in some neighbourhood of $0 \in \mathbb{C}^k$. In particular, the answer to Question 4 is positive in this case.

**Proof.** By [4, Corollary 8.3.9] we have $ht(I) \leq \ell(I) \leq \dim R$ for any ideal $I$ in Noetherian local ring $(R, m)$. Thus $\ell(I_t) = n$ since $I_t$ are $m_n$-primary. Let $\pi : \mathbb{C}^m \to \mathbb{C}^n$ be a linear map such that $(\pi \circ F_0)\mathcal{O}_n$ is a reduction of $I_0$. Put $J_t := (\pi \circ F_t)\mathcal{O}_n$, for all $t$. We have $e(J_t) \leq e(J_0)$, since the multiplicity $e(\cdot)$ is upper semicontinuous. Moreover $e(I_t) \leq e(J_t)$, because $J_t \subset I_t$. Summing up

$$e(I_0) = e(J_0) \geq e(J_t) \geq e(I_t) = e(I_0).$$

Therefore $\overline{J}_t = \overline{J}_t$ by Rees theorem. From this and [11, Corollaire I.2, p. 132] we deduce that $\overline{J} = \overline{I}$ and consequently $\ell(I) = n$. \hfill $\Box$

In the next section we get as a corollary that in a multiplicity-constant family of ideals the Łojasiewicz exponent is a lower semicontinuous function. A. Płoski proved this result under the additional restriction $m = n$.

**Example 6.** Let $F : (\mathbb{C}^2 \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be given by $F(x, y, t) := (x^5, y^5, t^5xy)$. By the main result of [2] we have $\ell((F)\mathcal{O}_3) = 3$. However, if $\pi : \mathbb{C}^3 \to \mathbb{C}^2$ is given by $\pi(u, v, w) = (u + w, v + w)$ then $\pi \circ F_t$ generate a reduction of $(F_t)\mathcal{O}_2$ for any $t$. Indeed, if we put $I_t := (F_t)\mathcal{O}_2$, $J_t := (\pi \circ F_t)\mathcal{O}_2$ then $I_t^2 = J_t I_t$.

Observe that the family $(F_t)\mathcal{O}_2$ is not multiplicity-constant. We have

$$e((F_t)\mathcal{O}_2) = \begin{cases} 10 & t \neq 0, \\ 25 & t = 0. \end{cases}$$

Our main result is given in the next theorem. It is a positive answer for Question 4 in case of one-parameter families of $m_n$-primary ideals.

**Theorem 7.** Let $F = F_t(x) = F(x, t) : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^m, 0)$ be a holomorphic map. Assume that $(F_t)\mathcal{O}_n$ is an $m_n$-primary ideal for all $t$. Then there exists...
a complex linear map \( \pi : \mathbb{C}^m \to \mathbb{C}^n \) such that for all \( t \) the ideal \((\pi \circ F_t)\mathcal{O}_n\) is a reduction of \((F_t)\mathcal{O}_n\).

We give the proof of Theorem 7 in Section 5. It is based on some geometric property of Hilbert-Samuel multiplicity given in Section 3. In Section 4 we recall the notion of elementary blowing-up.

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2. Semicontinuity of the Łojasiewicz exponent

Let \((R, \mathfrak{m})\) be a local ring and let \( I \) be an \( \mathfrak{m} \)-primary ideal. By the Łojasiewicz exponent \( \mathcal{L}(I) \) of \( I \) we define the infimum of

\[
\left\{ \frac{p}{q} : \mathfrak{m}^p \subset I^q \right\}.
\]

It was proved in [5] that if \( F : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0) \) is a holomorphic map with an isolated zero at the origin and \( I := (F)\mathcal{O}_n \), then \( \mathcal{L}(I) \) is the optimal exponent \( \nu \) in the inequality

\[
|F(x)| \geq C|x|^{\nu},
\]

where \( C \) is some positive constant and \( x \) runs through sufficiently small neighbourhood of \( 0 \in \mathbb{C}^n \).

**Lemma 8.** Let \((R, \mathfrak{m})\) be a Noetherian local ring. If \( I \) is an \( \mathfrak{m} \)-primary ideal of \( R \) and \( J \) is a reduction of \( I \) then \( \mathcal{L}(I) = \mathcal{L}(J) \).

**Proof.** Obviously \( \mathcal{L}(I) \leq \mathcal{L}(J) \). Assume that \( \mathfrak{m}^p \subset \mathcal{I}^q \). Since \( J \) is a reduction of \( I \), then also \( J^q \) is a reduction of \( I^q \) [4, Prop. 8.1.5]. Thus \( \mathcal{I}^q = \mathcal{J}^q \) by Theorem 1, which gives \( \mathfrak{m}^p \subset \mathcal{J}^q \). This proves the inequality \( \mathcal{L}(J) \leq \mathcal{L}(I) \) and ends the proof. \( \square \)

**Corollary 9 (A. Płoski for \( m = n \), [8]).** Let \( F : (\mathbb{C}^n \times \mathbb{C}^k, 0) \to (\mathbb{C}^m, 0) \) be a holomorphic map. Put \( I_t := (F_t)\mathcal{O}_n \). If the function \( t \mapsto \mathcal{L}(I_t) \) is constant and finite then the function \( t \mapsto \mathcal{L}(I_t) \) is lower semicontinuous.

**Proof.** By Proposition 5 and Theorem 2 there exists a linear map \( \pi : \mathbb{C}^m \to \mathbb{C}^n \) such that \( J_t := (\pi \circ F_t)\mathcal{O}_n \) is a reduction of \( I_t \) for all \( t \). Thus \( \mathcal{L}(J_t) = \mathcal{L}(I_t) \) and \( e(J_t) = e(I_t) \) by Theorem 1 and Lemma 8. Consequently \( t \mapsto e(J_t) \) is constant and finite and the assertion follows from the case \( m = n \) proved by A. Płoski. \( \square \)

More direct proof of this result we will give in our forthcoming paper [10].
3. Improper intersection multiplicity

Let $V, Z$ be a pair of analytic sets defined in some neighbourhood of $p \in \mathbb{C}^n$ and assume that $p$ is an isolated point of $V \cap Z$. If $\dim_p V + \dim_p Z = N$ then it is well known how to define the intersection index $i(V \cdot Z, p)$ of $V$ and $Z$ at $p$ (see e.g. [3]). Now, assume that $\dim_p V + \dim_p Z < N$ and $Z$ is smooth at $p$. In this case the intersection index of $V$ and $Z$ at $p$ was defined in [1] by the formula $i(V \cdot Z; p) = \min_W i_p(V \cdot W; p)$, where $W$ goes over all analytic sets defined in some neighbourhood of $p$ such that

- $Z \cap U \subset W \cap U$ for some neighbourhood $U$ of $p$,
- $p$ is an isolated point of $V \cap W$,
- $\dim_p V + \dim_p W = N$.

Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$ be a holomorphic map with an isolated zero. Using the above definition one may define (see [9]) the so-called improper intersection multiplicity of $f$ by the formula

$$i_0(f) := i(\text{graph } f \cdot (\mathbb{C}^n \times \{0\}^m); (0, 0)).$$

We recall one more definition. Let $V$ be a germ of an analytic set at $p \in \mathbb{C}^n$. Then the (Whitney) tangent cone of $V$ is the set of all $v \in \mathbb{C}^n$ such that there exist $\{p_n\} \subset V$, $\{t_n\} \subset \mathbb{C}$ with $p_n \to p$ and $t_n(p_n - p) \to v$. For the map $f$ as above by $C_f$ we will denote the tangent cone of the germ of the image of $f$ at the origin.

The following observation is due to S. Spodzieja.

**Theorem 10 ([9]).** If $\pi : \mathbb{C}^m \to \mathbb{C}^l$ is a linear map such that $\ker \pi \cap C_f = \{0\}$, then $\pi \circ f$ has an isolated zero in the origin and $i_0(f) = i_0(\pi \circ f)$. If additionally $l = n$ then $i_0(f) = e((\pi \circ f) \cap \mathcal{O}_n)$. Moreover, the number $i_0(f)$ depends only on the ideal generated by the components of $f$.

In what follows we will write $i_0(I) := i_0(f)$, where $f = (f_1, \ldots, f_m)$ are any generators of an $m_n$-primary ideal $I$ in $\mathcal{O}_n$.

**Corollary 11.** If $I$ is an $m_n$-primary ideal in $\mathcal{O}_n$, then $i_0(I) = e(I)$.

**Proof.** Let $I = (f_1, \ldots, f_m)\mathcal{O}_n$. By Theorems 2 and 10 there exist linear combinations $g_i = \sum a_{ij} f_j$, $i = 1, \ldots, n$ such that $J = (g_1, \ldots, g_n)\mathcal{O}_n$ is a reduction of $I$, $\{g_1, \ldots, g_n\}$ is a system of parameters of $\mathcal{O}_n$ and $i_0(I) = i_0(J) = e(J)$. From Theorem 1 we get $e(I) = e(J)$. This ends the proof.

**Corollary 12.** If $\pi : \mathbb{C}^m \to \mathbb{C}^l$ is a linear map such that $\ker \pi \cap C_f = \{0\}$, then the ideal $J$ generated by $\pi \circ f$ is a reduction of $I$.

**Proof.** We have $J \subset I$ and $e(J) = e(I)$. This and Theorem 1 give the assertion.
4. Elementary blowing-up

Here we recall the notion of an elementary blowing-up after [6].

Let $U \subset \mathbb{C}^n$ be an open and connected neighbourhood of $0 \in \mathbb{C}^n$; let $f = (f_0, \ldots, f_m) \neq 0$ be a sequence of holomorphic functions on $U$. Put $S = \{x \in U : f(x) = 0\}$ and

$$E(f) = \{(x, u) \in U \times \mathbb{P}^m : f_i(x)u_j = f_j(x)u_i, \ i, j = 0, \ldots, m\},$$

where $u = [u_0 : \ldots : u_m] \in \mathbb{P}^m$.

Let $Y$ be the closure of $E(f) \setminus (S \times \mathbb{P}^m)$ in $U \times \mathbb{P}^m$. The natural projection

$$\pi : Y \to U$$

is called the (elementary) blowing-up of $U$ by means of $f_0, \ldots, f_m$. The analytic subset $S$ is called the centre of the blowing-up and its inverse image $\pi^{-1}(S) \subset Y$ is called the exceptional set of the blowing-up.

Proposition 13. Under above notations we have:

1. $Y$ is an analytic subset of $U \times \mathbb{P}^m$;
2. $\pi$ is proper, its range is $U$ and the restriction $\pi|_{Y, \pi^{-1}(S)}$ is a biholomorphism onto $U \setminus S$;
3. $Y$ is irreducible;
4. The exceptional set $\pi^{-1}(S)$ is analytic in $U \times \mathbb{P}^m$ and it is of pure dimension $n - 1$.

Proof. For items 1. and 2. see [6, VII.5.1]. Item 3. follows from our assumption that $U$ is connected. For the proof of 4., let us consider the analytic map

$$F : U \times \mathbb{P}^m \ni (x, u) \mapsto (f(x), u) \in \mathbb{C}^{m+1} \times \mathbb{P}^m.$$

Let $y_0, \ldots, y_m$ be coordinates in $\mathbb{C}^{m+1}$. If we denote by $\pi_{m+1} : \Pi_{m+1} \to \mathbb{C}^{m+1}$ the blowing-up of $\mathbb{C}^{m+1}$ by means of $y_0, \ldots, y_m$ then for the restriction $\tilde{f} = F|_Y$ we get the following commutative diagram of analytic maps:

$$\begin{array}{ccc}
Y & \xrightarrow{f} & \Pi_{m+1} \\
\downarrow{\pi} & & \downarrow{\pi_{m+1}} \\
U & \xrightarrow{\tilde{f}} & \mathbb{C}^{m+1}
\end{array}$$

Take $(x_0, u_0) \in \pi^{-1}(0)$. Let $\Omega \subset \Pi_{m+1}$ be a neighbourhood of $(0, u_0)$, $h : \Omega \to \mathbb{C}$ an analytic function such that

$$\pi_{m+1}^{-1}(0) \cap \Omega \subset \{ (y, u) \in \Omega : h(y, u) = 0 \}.$$ 

Let $\Omega \subset Y$ be a neighbourhood of $(x_0, u_0)$ such that $\tilde{f}(\Omega) \subset \Omega$. Since

$$\tilde{f}^{-1}(\pi_{m+1}^{-1}(0)) = \pi^{-1}(S)$$

we get

$$\pi^{-1}(S) \cap \Omega = \{ (x, u) \in \Omega : h \circ \tilde{f}(x, u) = 0 \}.$$
Thus there exists a neighbourhood $\Delta \subset U \times \mathbb{P}^m$ of $(x_0, u_0)$ and an analytic set $V \subset \Delta$ of pure dimension $n + m - 1$ such that
\[ \pi^{-1}(S) \cap \Delta = V \cap Y \cap \Delta. \]

This gives
\[ \dim_{(x_0, u_0)} \pi^{-1}(S) \geq \dim_{(x_0, u_0)} Y - 1 = n - 1. \]

Since $Y$ is irreducible and $\pi^{-1}(S) \subseteq Y$ we get that $\dim_p \pi^{-1}(S) = n - 1$ for any $p \in \pi^{-1}(S)$. This ends the proof. \( \square \)

5. Proof of Theorem 7

**Lemma 14.** Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^{m+1}, 0)$, $m \geq n$ be a holomorphic map. Assume that 0 is an isolated point of $F_t^{-1}(0)$ for $|t| < \delta$. Then there exists $\delta > \varepsilon > 0$ and a complex line $V \subset \mathbb{C}^{m+1}$, such that $V \cap C_{F_t} = \{0\}$ for $|t| < \varepsilon$.

**Proof.** Let $F : U \to \mathbb{C}^{m+1}$, where $U \subset \mathbb{C}^n \times \mathbb{C}$ is a connected neighbourhood of the origin. Put $S = \{(z, t) \in U : F(z, t) = 0\}$ and let $\pi : U \times \mathbb{C}^m \to Y \to U$ be the elementary blowing-up of $U$ by $F$. By Proposition 13 its exceptional set $E := \pi^{-1}(S)$ is an analytic set of pure dimension $n$. Let $\mathcal{E}$ be the set of irreducible components $W$ of $E$ such that the origin in $\mathbb{C}^{m+1}$ is an accumulation point of $\pi(W) \cap (\{0\} \times \mathbb{C})$. Then $\mathcal{E}$ is finite. Denote by $\bar{C}_{F_t}$ the image of the cone $C_{F_t}$ in $\mathbb{P}^m$. Observe that
\[ \{(0, t)\} \times \bar{C}_{F_t} \subset \bigcup_{W \in \mathcal{E}} W, \quad |t| < \delta. \]

On the other hand, if $W \in \mathcal{E}$ then $W$ is $n$-dimensional irreducible set and $W \not\subset \{0\} \times \mathbb{P}^m$. Consequently
\[ \dim W \cap (\{0\} \times \mathbb{P}^m) \leq n - 1 < m. \]

Thus there exists $\varepsilon > 0$ and an open set $G \subset \mathbb{P}^m$ such that
\[ \{(0, t)\} \times G \cap \left( \bigcup_{W \in \mathcal{E}} W \right) = \emptyset, \quad 0 < |t| < \varepsilon. \]

As a result if $V$ is a line in $\mathbb{C}^{m+1}$ corresponding to some point in $G$ then $V \cap C_{F_t} = \{0\}$ for $0 < |t| < \varepsilon$. Since $G$ is not a subset of $C_{F_0}$ we get the assertion. \( \square \)

**Proof of Theorem 7.** Induction on $m$. In the case $m = n$ there is nothing to prove. Let us assume that the assertion is true for some $m \geq n$ and let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}^{m+1}, 0)$ be a holomorphic map such that the ideals $(F_t)\mathcal{O}_n$ are $m_n$-primary. By Lemma 14 there exists $\varepsilon > 0$ and a linear mapping $\pi' : \mathbb{C}^{m+1} \to \mathbb{C}^m$ such that $\ker \pi' \cap C_{F_t} = \{0\}$ for $|t| < \varepsilon$. Thus, by Corollary 12 the ideal $(\pi' \circ F_t)\mathcal{O}_n$ is a reduction of $(F_t)\mathcal{O}_n$, for all $t \in \mathbb{C}$ such that $|t| < \varepsilon$. On
the other hand, by induction hypothesis, there exists a linear map \( \pi'' : C^m \to C^n \)
such that \( (\pi'' \circ \pi' \circ F_t)^C_n \) is a reduction of \( (\pi' \circ F_t)^C_n \) for small \( t \). Thus if we put
\[ \pi := \pi'' \circ \pi' \]we get the assertion. 

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