On the moduli space of quadruples of points in the boundary of complex hyperbolic space

Heleno Cunha and Nikolay Gusevskii *
Departamento de Matemática
Universidade Federal de Minas Gerais
30123-970 Belo Horizonte - MG - Brazil

Abstract
We consider the space $M$ of ordered quadruples of distinct points in the boundary of complex hyperbolic $n$-space, $H^n_C$, up to its holomorphic isometry group $PU(n, 1)$. One of the important problems in complex hyperbolic geometry is to construct and describe a moduli space for $M$. For $n = 2$, this problem was considered by Falbel, Parker, and Platis. The main purpose of this paper is to construct a moduli space for $M$ for any dimension $n \geq 1$. The major innovation in our paper is the use of the Gram matrix instead of a standard position of points.

MSC: 32H20; 20H10; 22E40; 57S30; 32G07; 32C16
Keywords: Complex hyperbolic space; Cross-ratio; Cartan’s invariant; Gram matrix.

Introduction
An important problem in complex hyperbolic geometry is to classify ordered $m$-tuples of distinct points in complex hyperbolic $n$-space, $H^n_C$, or in its boundary, $\partial H^n_C$, up to congruence in the holomorphic isometry group $PU(n, 1)$ of $H^n_C$. This problem is trivial for $m = 1, 2$, and it was completely solved by Cartan and Brehm for triangles. If the vertices of a triangle are in the boundary of complex hyperbolic space, then Cartan’s angular invariant $\lambda$ associated to this triangle describes its congruence class in $PU(n, 1)$, Cartan [4], see also Goldman [9]. So, we can describe the moduli space of triangles in the boundary of complex hyperbolic space as the closed interval of real numbers $[-\pi/2, \pi/2]$. If the vertices of a triangle are in $H^n_C$, then its congruence class in $PU(n, 1)$ is described by the side lengths and Brehm’s shape invariant, see Brehm [1]. If $m > 3$, the problem becomes more difficult. It was shown by Brehm and Et-Taoui [2] that for points in $H^n_C$, the congruence class of an $m$-tuple in $PU(n, 1)$, $m > 3$, is determined by the congruence classes of all of its triangles. Easy examples show that this is not true for $m$-tuples in the boundary of complex hyperbolic space. This implies that Cartan’s invariants of all of the triangles of an ordered $m$-tuple in the boundary of complex hyperbolic space in the case $m > 3$ and $n > 1$ do not determine its congruence class in $PU(n, 1)$, and, therefore, we need something else to construct a moduli space of such configurations. For instance, one can try to use some invariants of four points.

For $n = 2$ and $m = 4$, this problem was considered by Falbel [7], see also Falbel-Platis [8], and Parker-Platis [10, 17]. For the convenience of the reader, we describe briefly their results. To do it, we recall the definition of the Korányi-Reimann complex cross-ratio.

In [15], the classical cross-ratio was generalized by Korányi and Reimann to complex hyperbolic space, see also Goldman [9]. Let $p = (p_1, p_2, p_3, p_4)$ be an ordered quadruple of distinct points of $\partial H^n_C$. Following

*Corresponding author. Supported by CNPq and FAPEMIG.
Goldman \cite{goldman2}, we define the Korányi-Reimann complex cross-ratio, or simply the complex cross-ratio, of $p$, as follows

$$X(p) = X(p_1, p_2, p_3, p_4) = \frac{\langle P_3, P_2 \rangle \langle P_1, P_3 \rangle}{\langle P_1, P_2 \rangle \langle P_3, P_2 \rangle},$$

where $\langle X, Y \rangle$ is the Hermitian product in $\mathbb{C}^{n+1}$ of signature $(n, 1)$, and $P_i \in \mathbb{C}^{n+1}$ is a null lift of $p_i$, see \cite{goldman2}. This number is independent of the chosen lifts $P_i$. Moreover, $X(p)$ is invariant with respect to the diagonal action of $\mathrm{PU}(n, 1)$. Basic properties of the complex cross-ratio may be found in Goldman \cite{goldman2}.

Let $\mathcal{M}$ be the configuration space of ordered quadruples of distinct points in the boundary of the complex hyperbolic plane, that is, the quotient of the set of ordered quadruples of distinct points of $\partial \mathbb{H}^2_\mathbb{C}$ with respect to the diagonal action of $\mathrm{PU}(2, 1)$ equipped with the quotient topology.

Falbel’s cross-ratio variety $\mathcal{X}$ is the subset of $\mathbb{C}^3$ defined by the following equations:

- $|\omega_0| |\omega_1| |\omega_2| - 1 = 0$,
- $|\omega_0 - 1|^2 - 1 + |\omega_0|^2 (|\omega_1 - 1|^2 - 1) + |\omega_0|^2 |\omega_1|^2 (|\omega_2 - 1|^2 - 1) = 0$,

see \cite{falbel1, falbel2}. It is easy to see that $\mathcal{X}$ is a real algebraic variety of dimension four in $\mathbb{C}^3$.

Let $m(p) \in \mathcal{M}$ be the point represented by $p = (p_1, p_2, p_3, p_4)$. Falbel \cite{falbel1, falbel2} defines the map $\pi : \mathcal{M} \rightarrow \mathcal{X}$ by the following formula

$$\pi : m(p) \mapsto (\omega_0, \omega_1, \omega_2) = (X(p_1, p_2, p_3, p_4), X(p_1, p_4, p_2, p_3), X(p_1, p_3, p_4, p_2)).$$

The main result of \cite{falbel1} is that this map is a bijection, see Proposition 2.4 there. In other words, this means that the $\mathrm{PU}(2, 1)$-congruence class of ordered quadruples of distinct points in the boundary of the complex hyperbolic plane is completely determined by these three complex cross-ratios satisfying these two real equations, that is, Falbel’s cross-ratio variety could serve as a moduli space for the configuration space $\mathcal{M}$.

In \cite{parker}, Parker and Platis used a slightly different construction from that of Falbel to describe $\mathcal{M}$. They define the map $\pi' : \mathcal{M} \rightarrow \mathbb{C}^3$ by the following formula

$$\pi' : m(p) \mapsto (X_1 = X(p_1, p_2, p_3, p_4), X_2 = X(p_1, p_3, p_2, p_4), X_3 = X(p_2, p_3, p_1, p_4)).$$

They proved that $X_1, X_2, X_3$ satisfy the following relations

- $|X_2| = |X_1||X_3|$,
- $2|X_1|^2 \Re(X_3) = |X_1|^2 + |X_2|^2 + 1 - 2 \Re(X_1 + X_2)$.

The Parker-Platis cross-ratio variety $\mathcal{X}'$ is the subset of $\mathbb{C}^3$, where these relations are satisfied. The main result of \cite{parker} is that the map $\pi' : \mathcal{M} \rightarrow \mathcal{X}'$ defined above is a bijection, see Proposition 5.5 and Proposition 5.10 in \cite{parker}. Again, this implies that the Parker-Platis cross-ratio variety could be considered as a moduli space for the configuration space $\mathcal{M}$. It is easy to see that $\mathcal{X}$ is homeomorphic to $\mathcal{X}'$, \cite{parker}, \cite{parker2}.

These results of Falbel and Parker-Platis have been used in a series of recent papers, see, \cite{platis}, \cite{parker}, \cite{will}, and others. For instance, in \cite{parker}, Parker and Platis used points of $\mathcal{X}'$ in their generalization of Fenchel-Nielsen coordinates to the complex hyperbolic setting, see also Will, \cite{will} for related topics. Falbel and Platis \cite{platis} used Falbel’s cross-ratio variety to describe some geometric properties of the configuration space $\mathcal{M}$.
In the present article, we show that, unfortunately, these results of Falbel and Parker-Platis on the description of the configuration space $\mathcal{M}$, in spite of their important contribution to the solution to this problem, are not correct.

The reason is that the maps $\pi : \mathcal{M} \to \mathcal{X}$ and $\pi' : \mathcal{M} \to \mathcal{X}'$ are not injective. The reader can find explicit examples that show this for the Parker-Platis variety in Section 3.3 of the paper. In fact, we show that there exists a 1-parameter family of pairs of distinct points in $\mathcal{M}$, $m(t)$ and $m^*(t)$, corresponding to $\mathbb{C}$-plane quadruples such that $\pi'(m(t)) = \pi'(m^*(t))$ (a quadruple of points in $\partial \mathbb{H}_c^2$ is $\mathbb{C}$-plane if all its points lie in the boundary of a complex geodesic, or equivalently, in a $\mathbb{C}$-circle). We will also prove that the above examples describe all the points of $\mathcal{M}$, where the map $\pi' : \mathcal{M} \to \mathcal{X}'$ is not injective. Of course, the same examples serve for Falbel’s variety.

All this implies that both Falbel’s and the Parker-Platis cross-ratio varieties cannot serve as moduli spaces for the configuration space $\mathcal{M}$.

We now state our results. The main purpose of our paper is to present a correct description of the configuration space $\mathcal{M}$. Also, we construct a moduli space of ordered quadruples of distinct points in the boundary of complex hyperbolic space $\mathcal{H}^2_c$ for any dimension $n \geq 1$. The major innovation in our paper is the use of the Gram matrix instead of a standard position of points used in [7], [8], [16].

Let $p = (p_1, p_2, p_3, p_4)$ be an ordered quadruple of distinct points of $\partial \mathbb{H}_c^2$, and let $m(p) \in \mathcal{M}$ be the point represented by $p = (p_1, p_2, p_3, p_4)$. We define the map

$$\tau : \mathcal{M} \to \mathbb{C}^2 \times \mathbb{R}$$

by the following formula:

$$\tau : m(p) \mapsto (X_1 = X(p_1, p_2, p_3, p_4), \ X_2 = X(p_1, p_3, p_2, p_4), \ \Lambda = \Lambda(p_1, p_2, p_3)),$$

where $\Lambda = \Lambda(p_1, p_2, p_3)$ is the Cartan invariant of the triple $(p_1, p_2, p_3)$.

**Theorem A** Let $X_1, X_2, \Lambda$ be the numbers defined by $\tau$. Then they satisfy the following relation:

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X_2} e^{-i\Lambda}) + |X_1|^2 + |X_2|^2 + 1 = 0.$$

Our main result is the following theorem.

**Theorem B** The configuration space $\mathcal{M}$ is homeomorphic to the set of points $X = (X_1, X_2, A)$ in $\mathbb{C}^2 \times \mathbb{R}$ defined by

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X_2} e^{-iA}) + |X_1|^2 + |X_2|^2 + 1 = 0,$$

subject to the following restrictions:

$$-\pi/2 \leq A \leq \pi/2, \ \text{Re}(X_1 e^{-iA}) \geq 0,$$

where $\mathbb{C}_s = \mathbb{C} \setminus \{0\}$.

We denote this set by $\mathbb{M}$ and call $\mathbb{M}$ the moduli space for $\mathcal{M}$. The map $\tau : \mathcal{M} \to \mathbb{M}$ is a homeomorphism provided that $\mathbb{M}$ is equipped with the topology induced from $\mathbb{C}^2 \times \mathbb{R}$.

Finally, we get a generalization of Theorem B for any dimension. Let $\mathcal{M}(n, 4)$ be the configuration space of ordered quadruples of distinct points in the boundary of complex hyperbolic space of dimension $n \geq 2$.

**Theorem C** $\mathcal{M}(n, 4)$ is homeomorphic to the set of points $X = (X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R}$ defined by

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X_2} e^{-iA}) + |X_1|^2 + |X_2|^2 + 1 \leq 0,$$

$$-\pi/2 \leq A \leq \pi/2, \ \text{Re}(X_1 e^{-iA}) \geq 0,$$

where $\mathbb{C}_s = \mathbb{C} \setminus \{0\}$.
The equality in the first inequality happens if and only if the quadruples are in the boundary of a complex hyperbolic 2-space.

It should be noticed here that the problem of the construction of the moduli space for the configuration space of ordered \(m\)-tuples of distinct points in \(\partial \mathbb{H}^2\), in the case \(m = n + 1\), was considered by Hakim-Sandler, see [14]. The invariants they used are too complicated to be described here.

The paper is organized as follows. In Section 1, we review some basic facts in complex hyperbolic geometry. In Section 2, we obtain the principal formulae we need. Section 3.1 is devoted to the construction of the moduli space for the configuration space of ordered quadruples of points in the boundary of complex hyperbolic space of any dimension. Finally, in Section 3.4, we describe the moduli space for the configuration space of ordered quadruples of points in the boundary of complex hyperbolic space of any dimension.

1 Complex hyperbolic space and its boundary

Let \(V^{n,1}\) be a \((n+1)\)-dimensional \(\mathbb{C}\)-vector space equipped with a Hermitian form \(\langle -,- \rangle\) of signature of \((n,1)\). The projective model of complex hyperbolic space \(\mathbb{H}^2\) is the set of negative points in the projective space \(\mathbb{P}\mathbb{C}^n\). It is well known that \(\mathbb{H}^2\) can be identified with the unit open ball in \(\mathbb{C}^n\). We will consider \(\mathbb{H}^2\) equipped with the Bergman metric, see [9]. Then \(\mathbb{H}^2\) is a complete Kähler manifold of constant holomorphic sectional curvature \(-1\). The boundary of \(\mathbb{H}^2\), denoted by \(\partial \mathbb{H}^2\), is the \((2n-1)\)-sphere formed by all isotropic points. Let \(U(n,1)\) be the unitary group corresponding to this Hermitian form. The holomorphic isometry group of \(\mathbb{H}^2\) is the projective unitary group \(PU(n,1)\), and the full isometry group \(Isom(\mathbb{H}^2)\) is generated by \(PU(n,1)\) and complex conjugation.

For the purposes of our paper it is convenient to work with coordinates in \(V^{n,1}\) in which the Hermitian product is represented by:

\[
\langle Z,W \rangle = z_1w_{n+1} + z_2w_2 + \cdots + z_nw_n + z_{n+1}w_1,
\]

where

\[
Z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ z_{n+1} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} w_1 \\ \vdots \\ w_n \\ w_{n+1} \end{bmatrix}.
\]

These coordinates were used by Burns-Shnider [3], Epstein [6] (the "second Hermitian form"), Parker-Platis [16], and also in [12], [13], [5], among others. In what follows, we denote by \(\mathbb{C}^{n,1}\) the Hermitian vector space \(V^{n,1}\) equipped with these coordinates. In these coordinates, the boundary of complex hyperbolic space is identified with the one point compactification of the boundary of the Siegel domain \(S^n\). Following Goldman-Parker [10], we give the Siegel domain horospherical coordinates. Horospherical coordinates are defined as follows: The height \(u \in \mathbb{R}_+\) of a point \(q = F(Z)\) in \(S^n\) is defined by \(u = -\langle Z,Z \rangle /2\). The locus of points where the height is constant is called a horosphere and naturally carries the structure of the Heisenberg group \(\mathcal{H} = \{(z,t) : z \in \mathbb{C}^{n-1}, t \in \mathbb{R}\} = \mathbb{C}^{n-1} \times \mathbb{R}\). The horospherical coordinates of a point in the Siegel domain are just \((z,t,u) \in \mathcal{H} \times \mathbb{R}_+\). We define the relation between horospherical coordinates on the Siegel domain and vectors in \(\mathbb{C}^{n,1}\) by the following map:
where $\langle z, z' \rangle$ is the standard Hermitian product on $\mathbb{C}^{n-1}$, and $q_\infty = \infty$ is a distinguished point in the boundary of $H^n_\mathbb{C}$. We call the vectors $\psi(z, t, u)$ and $\psi(q_\infty)$ the standard lifts of $(z, t, u)$ and $q_\infty$.

This shows that the boundary of $H^n_\mathbb{C}$, $\partial H^n_\mathbb{C}$, may be thought of as the one point compactification of the horosphere of height $u = 0$.

There are two types of totally geodesic submanifolds of $H^n_\mathbb{C}$ of real dimension two:

- **Complex geodesics** (copies of $H^1_\mathbb{C}$) have constant sectional curvature $-1$.

- **Totally real geodesic 2-planes** (copies of $H^2_\mathbb{R}$) have constant sectional curvature $-1/4$.

The intersection of a complex geodesic $L$ with $\partial H^n_\mathbb{C}$ is called a chain $C = \partial L$. Chains passing through $q_\infty$ are called vertical or infinite. A chain which does not contain $q_\infty$ is called finite. If $L$ is a complex geodesic, then there is a unique inversion in $PU(n, 1)$ whose fixed-point set is $L$. It acts on $\partial H^n_\mathbb{C}$ fixing $C = \partial L$.

The intersection of a totally real geodesic 2-subspace with $\partial H^n_\mathbb{C}$ is called an $\mathbb{R}$-circle. Just as for chains, an $\mathbb{R}$-circle in $H$ is one of two types, depending on whether or not it passes through $q_\infty$. An $\mathbb{R}$-circle is called infinite if it contains $q_\infty$, otherwise, it is called finite. If $R$ is a totally real geodesic 2-subspace, then there is a unique inversion (anti-holomorphic automorphism of $H^2_\mathbb{C}$) whose fixed-point set is $R$. It acts on $\partial H^n_\mathbb{C}$ fixing $\partial R$.

## 2 Gram matrix, Cartan’s invariant, Cross-ratio

### 2.1 The Gram matrix

Let $p = (p_1, \cdots, p_m)$ be an ordered $m$-tuple of distinct points in $\partial H^n_\mathbb{C}$. Then we consider the Hermitian $m \times m$-matrix

$$G = G(p) = (g_{ij}) = (\langle P_i, P_j \rangle),$$

where $P_i$ is a lift of $p_i$. We call $G$ a Gram matrix associated to the $m$-tuple $p$. Of course, $G$ depends on the chosen lifts $P_i$. When replacing $P_i$ by $\lambda_i P_i$, where $\lambda_i \in \mathbb{C}_*$, $\mathbb{C}_* = \mathbb{C} \setminus \{0\}$, we get $\tilde{G} = D^* GD$, where $D$ is the diagonal matrix, $D = (\lambda_i \delta_{ij})$, with $\delta_{ij} = 0$, if $i \neq j$, and $\delta_{ij} = 1$, if $i = j$.

We say that two Hermitian $m \times m$-matrices $H$ and $\tilde{H}$ are equivalent if there exists a diagonal matrix $D$, $D = (\lambda_i \delta_{ij})$, $\lambda_i \in \mathbb{C}_*$, such that $\tilde{H} = D^* HD$.

Thus, to each $m$-tuple $p$ of points in $\partial H^n_\mathbb{C}$ is associated an equivalence class of Hermitian $m \times m$-matrices with $0$'s on the diagonal. We remark that for any two Gram matrices $G$ and $\tilde{G}$ associated to an $m$-tuple $p$ the equality $\det \tilde{G} = \lambda \det G$ holds, where $\lambda > 0$. This implies that the sign of $\det G$ does not depend on the chosen lifts $P_i$.

Now we consider the case $m = 4$.

**Proposition 2.1** Let $p = (p_1, p_2, p_3, p_4)$ be an ordered quadruple of distinct points of $\partial H^2_\mathbb{C}$. Then the equivalence class of Gram matrices associated to $p$ contains a unique matrix $G = (g_{ij})$ with $g_{ii} = 0$, $g_{12} = g_{23} = g_{34} = 1$, $|g_{13}| = 1$. 

5
Remark: It follows from Sylvester’s Criterion that all these determinants are negative or vanish. Also, it is easy to see that \( \det G \) is negative if and only if \( p \) is in the boundary of a complex hyperbolic 2-space and \( \det G(i, j, k) = 0 \) if and only if the corresponding triple lies on a chain.

**Proposition 2.1** Let \( G \) be the normalized Gram matrix of \( p \). Then

\[
\det G = -2\text{Re}(g_{14}) - 2\text{Re}(g_{13}g_{24}) - 2\text{Re}(g_{13}\bar{g}_{14}g_{24}) + |g_{14}|^2 + |g_{24}|^2 + 1.
\]

**Proposition 2.2** Let \( p = (p_1, p_2, p_3, p_4) \) and \( p' = (p_1', p_2', p_3', p_4') \) be two ordered quadruples of distinct points of \( \partial H^2_n \). Then \( p \) and \( p' \) are congruent in \( \text{PU}(n, 1) \) if and only if their associated Gram matrices are equivalent.

**Corollary 2.1** Let \( p \) and \( p' \) be two ordered quadruples of distinct points of \( \partial H^2_n \), and let \( G \) and \( G' \) be their normalized Gram matrices. Then \( p \) and \( p' \) are congruent in \( \text{PU}(n, 1) \) if and only if \( G = G' \).

**Corollary 2.2** Let \( p \) and \( p' \) be two ordered quadruples of distinct points of \( \partial H^2_n \). Then \( p \) and \( p' \) are congruent with respect to an anti-holomorphic isometry of \( H^2_n \) if and only if their normalized Gram matrices are conjugate.

By direct computation we obtain the following formulae.

**Proposition 2.3** Let \( G \) be the normalized Gram matrix of \( p \). Then

\[
\det G = -2\text{Re}g_{14} - 2\text{Re}g_{13}\bar{g}_{24} - 2\text{Re}g_{13}\bar{g}_{14}g_{24} + |g_{14}|^2 + |g_{24}|^2 + 1.
\]

**Proposition 2.4** Let \( G(i, j, k) \) be the submatrix of \( G \) corresponding to the triple \( (p_i, p_j, p_k) \). Then

\[
\det G(1, 2, 3) = 2\text{Re}g_{13}, \quad \det G(1, 2, 4) = 2\text{Re}g_{24}\bar{g}_{14}, \quad \det G(1, 3, 4) = 2\text{Re}g_{13}\bar{g}_{14}, \quad \det G(2, 3, 4) = 2\text{Re}g_{24}\bar{g}_{24}.
\]

**Remark** It follows from Sylvester’s Criterion that all these determinants are negative or vanish. Also, it is easy to see that \( \det G = 0 \) if and only if \( p \) is in the boundary of a complex hyperbolic 2-space and \( \det G(i, j, k) = 0 \) if and only if the corresponding triple lies on a chain.
2.2 Cartan’s angular invariant

Let \( p = (p_1, p_2, p_3) \) be an ordered triple of distinct points in the boundary \( \partial \mathbb{H}_n^\mathbb{C} \) of complex hyperbolic \( n \)-space. Then Cartan’s angular invariant \( \mathcal{A}(p) \) of \( p \) is defined to be

\[
\mathcal{A}(p) = \arg(-\langle p_1, p_2, p_3 \rangle),
\]

where \( p_i \in V^{n,1} \) are corresponding lifts of \( p_i \), and

\[
\langle p_1, p_2, p_3 \rangle = \langle p_1, p_2 \rangle \langle p_2, p_3 \rangle \langle p_3, p_1 \rangle \in \mathbb{C}
\]

is the Hermitian triple product. It is verified that \( \mathcal{A}(p) \) is independent of the chosen lifts and satisfies

\[-\pi/2 \leq \mathcal{A}(p) \leq \pi/2.\]

The Cartan invariant is the only invariant of a triple of points: \( p \) and \( p' \) are congruent in \( \text{PU}(n, 1) \) if and only if \( \mathcal{A}(p) = \mathcal{A}(p') \). Basic properties of the Cartan invariant can be found in Goldman [9].

The following proposition shows a relation between Cartan’s invariant and the Gram determinants in Proposition 2.4.

**Proposition 2.5** Let \( p = (p_1, p_2, p_3, p_4) \) be an ordered quadruple of distinct points of \( \partial \mathbb{H}_n^\mathbb{C} \) and \( G = (g_{ij}) \) be the normalized Gram matrix of \( p \). Then

\[
\mathcal{A}(p_1, p_2, p_3) = \arg(-\bar{g}_{13}), \quad \mathcal{A}(p_1, p_2, p_4) = \arg(-g_{24}\bar{g}_{14}),
\]

\[
\mathcal{A}(p_1, p_3, p_4) = \arg(-g_{13}\bar{g}_{14}), \quad \mathcal{A}(p_2, p_3, p_4) = \arg(-\bar{g}_{24}).
\]

One may use this to see again that the Cartan invariant lies in the interval \([-\pi/2, \pi/2]\).

2.3 The complex cross-ratio

In [15], Kőnáyi and Reimann defined a complex-valued invariant associated to an ordered quadruple of distinct points of \( \partial \mathbb{H}_n^\mathbb{C} \). This invariant generalizes the usual cross-ratio of a quadruple of complex numbers. Let \( p = (p_1, p_2, p_3, p_4) \) be an ordered quadruple of distinct points of \( \partial \mathbb{H}_n^\mathbb{C} \). Following Goldman [9], the Kőnáyi - Reimann complex cross-ratio is defined to be

\[
X = X(p) = \frac{\langle p_3, p_1 \rangle \langle p_4, p_2 \rangle}{\langle p_4, p_1 \rangle \langle p_3, p_2 \rangle},
\]

where \( p_i \in V^{n,1} \) are corresponding lifts of \( p_i \). It is verified that the complex cross-ratio is independent of the chosen lifts \( p_i \) and is invariant with respect to the diagonal action of \( \text{PU}(n, 1) \). Since the points \( p_i \) are distinct, \( X \) is finite and non-zero. More properties of the complex cross-ratio may be found in Goldman [9].

Let \( p = (p_1, p_2, p_3, p_4) \) be an ordered quadruple of distinct points of \( \partial \mathbb{H}_n^\mathbb{C} \). We define

\[
X_1 = X(p_1, p_2, p_3, p_4), \quad X_2 = X(p_1, p_3, p_2, p_4), \quad X_3 = X(p_2, p_3, p_1, p_4).
\]

**Remark** These complex cross-ratios were considered by Parker-Platis [16].

Easy computations give the following.
Proposition 2.6 Let $p = (p_1, p_2, p_3, p_4)$ be an ordered quadruple of distinct points of $\partial \mathbb{H}^2_\mathbb{C}$ and $G = (g_{ij})$ be the normalized Gram matrix of $p$. Then

$$X_1 = \frac{\bar{g}_{13} g_{24}}{g_{14}}, \quad X_2 = \frac{1}{g_{14}}, \quad X_3 = \frac{1}{g_{13} g_{24}},$$

and

$$g_{13} = -e^{-i\mathcal{A}}, \quad g_{14} = \frac{1}{X_2}, \quad g_{24} = -\frac{\bar{X}_1}{X_2} e^{i\mathcal{A}},$$

where $\mathcal{A}$ is the Cartan invariant of the triple $(p_1, p_2, p_3)$.

Using these formulae and applying Corollary 2.1, we get the following important result.

Corollary 2.3 The numbers $X_1$, $X_2$, and $\mathcal{A}$ define uniquely the congruence class of $p$ in $PU(n, 1)$.

Remark It is easy to see that it is impossible to express uniquely $g_{ij}$ in terms of $X_1$, $X_2$, $X_3$. In fact, we have that

$$(g_{13})^2 = \frac{X_2}{X_1 X_3}, \quad (g_{24})^2 = \frac{\bar{X}_1}{X_2 X_3}.$$

Therefore, the congruence class of $p$ in $PU(n, 1)$ is not defined uniquely by $X_1$, $X_2$, $X_3$.

Substituting $g_{ij}$ from the expressions in Proposition 2.6 into the formulae in Proposition 2.3 and Proposition 2.4 and rearranging, we get the formulae for the Gram determinants in terms of Cartan’s invariant and the complex cross-ratios.

Proposition 2.7 The determinant of the normalized Gram matrix of $p$ is given by

$$\det G = \frac{1}{|X_2|^2} \left[ -2\text{Re}(X_1 + X_2) - 2\text{Re}(X_1 \bar{X}_2 e^{-i\mathcal{A}}) + |X_1|^2 + |X_2|^2 + 1 \right].$$

Proposition 2.8 The determinants of the submatrices $G(i, j, k)$ of $G$ are given by

$$\det G(1, 2, 3) = -2\text{Re}(e^{i\mathcal{A}}), \quad \det G(1, 2, 4) = -2\text{Re}\left(\frac{X_1}{|X_2|^2} e^{i\mathcal{A}}\right),$$

$$\det G(1, 3, 4) = -2\text{Re}\left(\frac{\bar{X}_2}{|X_2|^2} e^{-i\mathcal{A}}\right), \quad \det G(2, 3, 4) = -2\text{Re}\left(\frac{X_1 \bar{X}_2}{|X_2|^2} e^{-i\mathcal{A}}\right).$$

3 The configuration space of ordered quadruples in the boundary of complex hyperbolic space and its moduli space

3.1 The moduli space of ordered quadruples in the boundary of the complex hyperbolic plane

Let $\mathcal{M}$ be the configuration space of ordered quadruples of distinct points in the boundary of the complex hyperbolic plane, that is, the quotient of the set of ordered quadruples of distinct points of $\partial \mathbb{H}^2_\mathbb{C}$ with respect to the diagonal action of $PU(2, 1)$ equipped with the quotient topology. In this section, we construct a moduli space for $\mathcal{M}$.

The following proposition is the crucial result in our construction of the moduli space for $\mathcal{M}$.
Proposition 3.1 Let \( G = (g_{ij}) \) be an Hermitian \( 4 \times 4 \)-matrix such that \( g_{ii} = 0 \), \( g_{12} = g_{23} = g_{34} = 1 \), \( |g_{13}| = 1 \), \( g_{14} \neq 0 \), \( g_{24} \neq 0 \). Then \( G \) is the normalized Gram matrix for some ordered quadruple \( p = (p_1, p_2, p_3, p_4) \) of distinct points of \( \partial \mathbb{H}_C^2 \) if and only if \( \text{Re}(g_{13}) \leq 0 \), \( \text{Re}(g_{24} g_{14}) \leq 0 \), and \( \det G = 0 \).

Proof: Let us assume that \( G \) is the normalized Gram matrix associated to an ordered quadruple \( p = (p_1, p_2, p_3, p_4) \) of distinct points of \( \partial \mathbb{H}_C^2 \). Then it follows from Proposition 2.4 that

\[
\det G(1, 2, 3) = 2 \text{Re}(\overline{g}_{13}), \quad \det G(1, 2, 4) = 2 \text{Re}(g_{24} \overline{g}_{14}).
\]

By Sylvester’s Criterion, these determinants are negative or vanish. Since \( p \) is in the boundary of the complex hyperbolic space of dimension 2, any vectors \( P_1, P_2, P_3, P_4 \) representing \( p_1, p_2, p_3, p_4 \) are linearly dependent. This implies that \( \det G = 0 \).

Now let \( G = (g_{ij}) \) be an Hermitian \( 4 \times 4 \)-matrix such that \( g_{ii} = 0 \), \( g_{12} = g_{23} = g_{34} = 1 \), \( |g_{13}| = 1 \), \( \text{Re}(g_{13}) \leq 0 \), \( \text{Re}(g_{24} g_{14}) \leq 0 \), and \( \det G = 0 \). We are going to show that there exist four null (isotropic) vectors \( P_1, P_2, P_3, P_4, P_1 \in \mathbb{C}^2 \), whose Gram matrix is equal to \( G \).

We will look for these vectors in the following form:

\[
P_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ 1 \end{bmatrix},
\]

where \( z_i, w_i \) are complex numbers, and \( |z_1| = 1 \).

Then we have

\[
\langle P_1, P_2 \rangle = 1, \quad \langle P_1, P_3 \rangle = \overline{z}_1, \quad \langle P_1, P_4 \rangle = \overline{w}_1, \quad \langle P_2, P_3 \rangle = 1, \quad \langle P_2, P_4 \rangle = \overline{w}_3, \quad \langle P_3, P_4 \rangle = z_1 \overline{w}_3 + z_2 \overline{w}_2 + \overline{w}_1.
\]

Since we need \( P_3 \) and \( P_4 \) to be null vectors, we have the following equations:

\[
z_1 + |z_2|^2 + \overline{z}_1 = 0, \quad w_1 \overline{w}_3 + |w_2|^2 + \overline{w}_1 w_3 = 0.
\]

From the definition of the Gram matrix, we have

\[
g_{12} = 1, \quad g_{13} = \overline{z}_1, \quad g_{14} = \overline{w}_1, \quad g_{23} = 1, \quad g_{24} = \overline{w}_3,
\]

and

\[
g_{34} = z_1 \overline{w}_3 + z_2 \overline{w}_2 + \overline{w}_1 = 1.
\]

This implies that we have already found \( z_1, w_3 \) in terms of \( g_{ij} \). Therefore, we need to find \( z_2 \) and \( w_2 \).

We consider the following system of equations

\[
(1) \quad z_1 + |z_2|^2 + \overline{z}_1 = 0, \quad (2) \quad w_1 \overline{w}_3 + |w_2|^2 + \overline{w}_1 w_3 = 0, \quad (3) \quad z_1 \overline{w}_3 + z_2 \overline{w}_2 + \overline{w}_1 = 1,
\]

and show that it has a solution under the conditions of the proposition. We write the first two equations in the following form:

\[
|z_2|^2 = -2 \text{Re}(z_1) = -2 \text{Re}(g_{13}), \quad |w_2|^2 = -2 \text{Re}(w_1 \overline{w}_3) = -2 \text{Re}(g_{24} \overline{g}_{14}).
\]

We immediately get that there exist solutions for \( |z_2| \) and \( |w_2| \) under our conditions.

The third equation may be written as

\[
z_2 \overline{w}_2 = 1 - z_1 \overline{w}_3 - \overline{w}_1.
\]
Let us assume that the equation \(|z_2 \bar{w}_2| = |1 - z_1 \bar{w}_3 - \bar{w}_1|\) holds for some \(z_2\) and \(w_2\) satisfying equations (1) and (2), and let \((z_2, w_2)\) be any solution to this equation. First, we suppose that \(z_2 \neq 0, w_2 \neq 0\). In this case, we put \(z'_2 = e^{i\alpha} z_2\). Then \(z'_2 \bar{w}_2 = e^{i\alpha} (z_2 \bar{w}_2)\). Since two complex numbers have the same norm if and only if there exists a rotation which sends one number to another, we have that there exists \(\alpha\) such that \(z'_2 \bar{w}_2 = 1 - z_1 \bar{w}_3 - \bar{w}_1\). Therefore, the system above has solutions if and only if

\[
A = |1 - z_1 \bar{w}_3 - \bar{w}_1|^2 - |z_2 \bar{w}_2|^2 = 0.
\]

Substituting \(|z_2|\) and \(|w_2|\) from equations (1) and (2) into \(A\), and then rearranging, we have that

\[
A = |1 - z_1 \bar{w}_3 - \bar{w}_1|^2 - |z_2 \bar{w}_2|^2 = (1 - z_1 \bar{w}_3 - \bar{w}_1) (1 - \bar{z}_1 w_3 - w_1) - (z_1 + \bar{z}_1) (w_1 \bar{w}_3 + w_3 \bar{w}_1) = 1 + |z_1|^2 |w_3|^2 + |w_1|^2 - 2 \Re(w_1) - 2 \Re(z_1 \bar{w}_3) - 2 \Re(z_1 w_3 \bar{w}_1).
\]

Since \(g_{13} = \bar{z}_1\), \(g_{14} = \bar{w}_1\), \(g_{24} = \bar{w}_3\), and \(|g_{13}| = 1\), we get the following expression for \(A\) in terms of \(g_{ij}\):

\[
A = -2 \Re(g_{14}) - 2 \Re(g_{13} \bar{g}_{24}) - 2 \Re(g_{13} \bar{g}_{14} \bar{g}_{24}) + |g_{14}|^2 + |g_{24}|^2 + 1.
\]

By applying Proposition 2.3, we see that \(A = \det G\). Since, by our hypothesis \(\det G = 0\), we get the result we need in the case \(z_2\) and \(w_2\) are not equal to zero. It is easy to see that there exists a solution to the system (1) – (3) when one of the numbers \(z_2\) or \(w_2\) is equal to zero, since in this case the third equation is satisfied automatically provided that \(A = 0\). Finally, one sees that if \(g_{14} \neq 0, g_{24} \neq 0\), then the points \(p_i\) defined by the vectors \(P_i\) are distinct. This proves the proposition.

**Corollary 3.1** Let \(G = (g_{ij})\) be an Hermitian \(4 \times 4\)-matrix such that \(g_{ii} = 0, g_{12} = g_{23} = g_{34} = 1, |g_{13}| = 1, g_{14} \neq 0, g_{24} \neq 0, \) and \(\det G = 0\). Then the inequalities \(\Re(g_{13}) \leq 0\) and \(\Re(g_{24} \bar{g}_{14}) \leq 0\) imply the inequalities \(\Re(g_{13} \bar{g}_{14}) \leq 0\) and \(\Re(\bar{g}_{24}) \leq 0\).

**Proof:** It follows from Proposition 3.1 that under the conditions above \(G\) is the normalized Gram matrix for some ordered quadruple of distinct points of \(\partial H_2^2\). Then the result follows from Proposition 2.4.

**Corollary 3.2** Let \(G = (g_{ij})\) be an Hermitian \(4 \times 4\)-matrix satisfying all the conditions of Proposition 3.1. Then any two quadruples of points \(p = (p_1, p_2, p_3, p_4)\) and \(p' = (p'_1, p'_2, p'_3, p'_4)\) of \(\partial H_2^2\) defined by the quadruples of null vectors \(P = (P_1, P_2, P_3, P_4)\) and \(P' = (P'_1, P'_2, P'_3, P'_4)\) that correspond to solutions with \((z_2, w_2)\) and \((z'_2, w'_2)\) respectively are congruent in \(\PU(2,1)\).

**Proof:** It follows from the proof of Proposition 3.1 that the quadruples \(p = (p_1, p_2, p_3, p_4)\) corresponding to the vectors \(P_1, P_2, P_3, P_4\) defined there have the same normalized Gram matrix. By applying Proposition 2.2, we get the result we want.

Let \(p = (p_1, p_2, p_3, p_4)\) be an ordered quadruple of distinct points of \(\partial H_2^2\), and let \(m(p) \in \mathcal{M}\) be the point represented by \(p = (p_1, p_2, p_3, p_4)\). We define the map

\[
\tau : \mathcal{M} \rightarrow \mathbb{C}^2 \times \mathbb{R}
\]

by the following formula:

\[
\tau : m(p) \mapsto (X_1 = X(p_1, p_2, p_3, p_4), X_2 = X(p_1, p_3, p_2, p_4), \mathcal{A} = \mathcal{A}(p_1, p_2, p_3)),
\]

where \(\mathcal{A} = \mathcal{A}(p_1, p_2, p_3)\) is the Cartan invariant of the triple \((p_1, p_2, p_3)\).

**Proposition 3.2** Let \(X_1, X_2, \mathcal{A}\) be the numbers defined by \(\tau\). Then they satisfy the following relation:

\[
-2 \Re(X_1 + X_2) - 2 \Re(X_1 \overline{X}_2 e^{-i2\mathcal{A}}) + |X_1|^2 + |X_2|^2 + 1 = 0.
\]

10
we construct an Hermitian $4 \times 3$. 2.

The topology of the moduli space

3.2 Some interesting subsets of $M$ define

**Proposition 2.7.**

**Remark** We denote this set by $S_g$. We call this set $S$. Let also

**Theorem 3.1** The configuration space $M$ is homeomorphic to the set of points $X = (X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R}$ defined by

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X}_2 e^{-A}) + |X_1|^2 + |X_2|^2 + 1 = 0,$$

subject to the following conditions

$$-\pi/2 \leq A \leq \pi/2, \quad \text{Re}(X_1 e^{-iA}) \geq 0.$$ 

**Remark** We denote this set by $M$ and call $M$ the moduli space for $M$.

**Proof:** It follows from Proposition 3.2 and the formulae in Proposition 2.8 that the map $\tau : M \rightarrow \mathbb{R}$ is surjective. Given $X = (X_1, X_2, A) \in M$, we define $g_{13}, g_{14}, g_{24}$ in terms of $X_1, X_2, A$, that is, we put

$$g_{13} = -e^{-iA}, \quad g_{14} = \frac{1}{X_2}, \quad g_{24} = -\frac{X_1}{X_2} e^{iA}.$$ 

Also we put $g_{ii} = 0, g_{12} = g_{23} = g_{34} = 1$. This defines $G$ completely. Comparing the formulae in Propositions 2.3 and 2.4 with those in Propositions 2.7 and 2.8, we see that $G$ satisfies all the conditions in Proposition 3.1. By applying Proposition 3.1, we get that $G$ is the normalized Gram matrix for some ordered quadruple $p = (p_1, p_2, p_3, p_4)$ of distinct points of $\partial \mathbb{H}^2_\mathbb{C}$. It is readily seen from the formulae in Proposition 2.6 that $\tau(m(p)) = (X_1, X_2, A)$. This proves that $\tau$ is surjective. On the other hand, it follows from Corollary 2.1 and Corollary 2.2 that $\tau$ is injective. It is clear that $\tau : M \rightarrow \mathbb{R}$ is a homeomorphism provided that $M$ is equipped with the topology induced from $\mathbb{C}^2 \times \mathbb{R}$. This completes the proof of the theorem.

**Remark** In Section 3.2.5, we will show that the equation

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X}_2 e^{-A}) + |X_1|^2 + |X_2|^2 + 1 = 0,$$

and the strong inequalities $-\pi/2 < A < \pi/2$ imply the inequality $\text{Re}(X_1 e^{-iA}) \geq 0$.

### 3.2 Some interesting subsets of $M$ and the topology of the moduli space

#### 3.2.1 Singular set of $M$ and $\mathbb{C}$-plane configurations

The topology of the moduli space $M$ is quite interesting. To describe it, we define the following sets.

Let $S$ be the set of points in $\mathbb{C}^2 \times \mathbb{R}$, where the equation

$$-2 \text{Re}(X_1 + X_2) - 2 \text{Re}(X_1 \overline{X}_2 e^{-A}) + |X_1|^2 + |X_2|^2 + 1 = 0$$

is satisfied. We call this set $S$ the basic variety.

Let also

- $S_{123} = \{(X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R} : \text{Re}(e^{iA}) = 0\}$,
- $S_{124} = \{(X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R} : \text{Re}(\overline{X}_1 e^{iA}) = 0\}$,
\begin{itemize}
  \item \(S_{134} = \{(X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R} : \text{Re}(X_2 e^{-iA}) = 0\}\),
  \item \(S_{234} = \{(X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R} : \text{Re}(X_1 X_2 e^{-iA}) = 0\}\).
\end{itemize}

It is easy to see that all the varieties \(S_{ijk}\) are not singular. In fact, \(S_{ijk}\) is diffeomorphic to the disjoint union of 4-planes in \(\mathbb{C}^2 \times \mathbb{R}\). For instance,
\[S_{123} = \{(X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R} : A = \pm \pi/2 + 2k\pi, \ k \in \mathbb{Z}\}.
\]

We call \(S_{ijk}\) the Cartan varieties.

Let \(p = (p_1, p_2, p_3, p_4)\) be an ordered quadruple of distinct points of \(\partial H^2\). We call \(p\) a tetrahedron with vertices \((p_1, p_2, p_3, p_4)\), and also we call the triples \((p_i, p_j, p_k)\) the faces of \(p\).

We say that a tetrahedron \(p\) is almost \(\mathbb{C}\)-plane if there exists a face of \(p\) which lies on a chain (we call such a face \(\mathbb{C}\)-plane). Also, we define a tetrahedron \(p\) to be \(\mathbb{C}\)-plane if all of its vertices are in a chain.

Given tetrahedron \(p\), let \(m(p) \in \mathcal{M}\) be the point represented by \(p\). Then we have the following description of almost \(\mathbb{C}\)-plane and \(\mathbb{C}\)-plane tetrahedra.

**Proposition 3.3** A tetrahedron \(p\) is almost \(\mathbb{C}\)-plane if and only if \(\tau(m(p))\) belongs to some Cartan’s variety. Moreover, a tetrahedron \(p\) is \(\mathbb{C}\)-plane if and only if \(\tau(m(p))\) belongs to the intersection of at least two Cartan’s varieties.

**Proof:** The first assertion is obvious because of Proposition 2.8. The second one follows from the fact that if two chains intersect in at least two points then they coincide. In particular, a tetrahedron is \(\mathbb{C}\)-plane if and only if it has at least two \(\mathbb{C}\)-plane faces.

Let \(\mathcal{C}\) be the set of the points in \(\mathcal{M}\) which represent \(\mathbb{C}\)-plane tetrahedra. In what follows, we will describe the singular set of \(\mathcal{S}\) when \(-\pi/2 \leq A \leq \pi/2\). Also, we will find a relation between \(\mathcal{C}\) and the singular set of \(\mathcal{M}\).

**Proposition 3.4** The basic variety \(\mathcal{S}\) has no singular points for \(-\pi/2 < A < \pi/2\) and \(X_1 \neq 0, \ X_2 \neq 0\).

**Proof:** We write \(X_1 = a + bi, \ X_2 = c + di, \ e^{-i2A} = \cos 2A - i \sin 2A\). Then easy calculations show that \(\mathcal{S}\) in terms of \(a, b, c, d, A\) is given by the following equation
\[F(a, b, c, d, A) = -2(a + c) - 2[(ac + bd) \cos 2A + (bc - ad) \sin 2A] + a^2 + b^2 + c^2 + d^2 + 1 = 0.
\]

Computing the partial derivatives of \(F\), we get the following system for finding singular points of \(\mathcal{S}\).
\begin{align*}
1\: \frac{\partial F}{\partial a} &= 0 \iff -1 - c \cos 2A + d \sin 2A + a = 0, \\
2\: \frac{\partial F}{\partial b} &= 0 \iff -1 - a \cos 2A - b \sin 2A + c = 0, \\
3\: \frac{\partial F}{\partial c} &= 0 \iff -d \cos 2A - c \sin 2A + b = 0, \\
4\: \frac{\partial F}{\partial d} &= 0 \iff -b \cos 2A + a \sin 2A + d = 0, \\
5\: \frac{\partial F}{\partial A} &= 0 \iff (ac + bd) \sin 2A - (bc - ad) \cos 2A = 0.
\end{align*}

Since \(X_1 \neq 0\) and \(X_2 \neq 0\), we have that \(a^2 + b^2 \neq 0\) and \(c^2 + d^2 \neq 0\). We consider the following cases: \(a = 0, b \neq 0\); \(a \neq 0, b = 0\); \(a \neq 0, b \neq 0\). In the two first cases, the equations become very simple, and a straightforward verification, which is left to the reader, shows that the system above has no solutions for \(-\pi/2 < A < \pi/2\). So, in these cases \(\mathcal{S}\) has no singular points.

12
Let us assume now that \(a \neq 0, b \neq 0\). By multiplying the first equation by \(b\) and the third equation by \((-a)\), and then summing the resulting equations, we get that \((ac + bd) \sin 2A - (bc - ad) \cos 2A - b = 0\). Now, if equation (5) does not hold, then the gradient of \(F\) is not the null vector. If equation (5) holds, then the above implies \(b = 0\), that contradicts our assumption. This completes the proof of the proposition.

Next we show that \(S\) has singular points for \(A = \pm \pi/2\) and describe the singular set of \(M\).

**Proposition 3.5** The basic variety \(S\) has singular points for \(A = \pm \pi/2\).

**Proof:** Substituting \(A = \pm \pi/2\) into equations (1) – (5), it is readily seen that the system has solutions if and only if \(a + c = 1\) and \(b = d = 0\). On the other hand, using the equation from Proposition 3.4, we have that \(S\) is given for \(A = \pm \pi/2\) by the following equation

\[
F(a, b, c, d) = -2(a + c) + 2(ac + bd) + a^2 + b^2 + c^2 + d^2 + 1 = 0.
\]

Rearranging this gives the equation

\[
(a + c - 1)^2 + (b + d)^2 = 0.
\]

Thus, for \(A = \pm \pi/2\), \(S\) is defined by the following system of equations

\[
a + c = 1, \quad b + d = 0.
\]

All this implies that in all the points of \(S\), where \(A = \pm \pi/2, a + c = 1\) and \(b = d = 0\), the gradient of \(F\) is the null vector, and, therefore, all these points are singular points of \(S\).

As a corollary of the proof of Proposition 3.5, we obtain the following

**Proposition 3.6** The intersection of the Cartan variety \(S_{123}\) with the basic variety \(S\) is not transversal. This intersection is the union of 2-dimensional real analytic varieties. In the coordinates above, it is given by the equations

\[
a + c = 1, \quad b + d = 0, \quad A = \pm \pi/2 + 2k\pi.
\]

Using calculations similar to those in Proposition 3.5, we get the following

**Proposition 3.7** The intersection of the Cartan variety \(S_{124}\) with the basic variety \(S\) is not transversal. This intersection is the union of 2-dimensional real analytic varieties. In the coordinates above, it is given by the equations

\[
a + b \tan A = 0, \quad a + c = 1, \quad b - d = 0,
\]

provided that \(A \neq \pm \pi/2 + 2k\pi\), and by the equations

\[
a + c = 1, \quad b = 0, \quad d = 0,
\]

provided that \(A = \pm \pi/2 + 2k\pi\).

Also, we have

**Proposition 3.8** The intersection of the Cartan variety \(S_{134}\) with the basic variety \(S\) is not transversal. This intersection is the union of 2-dimensional real analytic varieties. In the coordinates above, it is given by the equations

\[
c - d \tan A = 0, \quad a + c = 1, \quad b - d = 0,
\]

provided that \(A \neq \pm \pi/2 + 2k\pi\), and by the equations

\[
a + c = 1, \quad b = 0, \quad d = 0,
\]

provided that \(A = \pm \pi/2 + 2k\pi\).
**Proposition 3.9** The intersection of the Cartan variety $S_{234}$ with the basic variety $S$ is not transversal. This intersection is the union of 2-dimensional real analytic varieties. In the coordinates above, it is given by the equations

$$(ac + bd) + (bc - ad) \tan A = 0, \quad a + c = 1, \quad b + d = 0,$$

provided that $A \neq \pm \pi/2 + 2k\pi$, and by the equations

$$a + c = 1, \quad b = 0, \quad d = 0,$$

provided that $A = \pm \pi/2 + 2k\pi$.

Now let

$$S_1 = S \cap S_{123}, \quad S_2 = S \cap S_{124}, \quad S_3 = S \cap S_{134}, \quad S_4 = S \cap S_{234}.$$

Thus, we get that $S_i$ is a 2-dimensional real analytic variety, and, moreover, we have the following

**Corollary 3.3** The intersection of all $S_i$ is exactly the singular set of the basic variety $S$.

**Corollary 3.4** The set $C$ is equal to the singular set of the moduli space $M$.

### 3.2.2 $\mathbb{C}$-plane configurations

In this section, we describe the set of $\mathbb{C}$-plane tetrahedra in terms of our coordinates $X_1, X_2, A$.

**Theorem 3.2** Let $p$ be a tetrahedron, and let $\tau(m(p)) = (X_1, X_2, A)$. Then $p$ is $\mathbb{C}$-plane if and only if $A = \pm \pi/2$ and $X_1, X_2$ are real numbers satisfying the relation $X_1 + X_2 = 1$.

**Proof:** If $p$ is $\mathbb{C}$-plane then it follows immediately from the proof of Proposition 3.5 that $X_1, X_2, A$ satisfy all the conditions of the theorem. Assume now that $X_1, X_2, A$ satisfy the conditions of the theorem. Then it is easy to see that they satisfy all the conditions of Theorem 3.1. Therefore, there exists a tetrahedron $p = (p_1, p_2, p_3, p_4)$ such that $\tau(m(p)) = (X_1, X_2, A)$. Since $A = \pm \pi/2$, the points $p_1, p_2, p_3$ are in a chain. Moreover, it follows from the proof of Proposition 3.5 that the points $p_1, p_2, p_4$ are also in a chain. This implies that the points $p_1, p_2, p_3, p_4$ must be in the same chain. $\blacksquare$

### 3.2.3 $\mathbb{R}$-plane configurations

Let $p = (p_1, p_2, p_3, p_4)$ be a tetrahedron of distinct points of $\partial H_2^2$. We say that $p$ is $\mathbb{R}$-plane if all of its vertices are in an $\mathbb{R}$-circle.

**Theorem 3.3** Let $p$ be a tetrahedron, and let $\tau(m(p)) = (X_1, X_2, A)$. Then $p$ is $\mathbb{R}$-plane if and only if $A = 0$ and $X_1, X_2$ are positive real numbers satisfying the relation

$$-2(X_1 + X_2) - 2X_1X_2 + X_1^2 + X_2^2 + 1 = 0.$$

**Proof:** It follows immediately from Theorem 3.1 that if $p$ is $\mathbb{R}$-plane, then $A = 0$ and $X_1, X_2$ are real positive numbers satisfying the relation in the theorem. Let us assume now that $A = 0$ and $X_1, X_2$ are real numbers satisfying the conditions of the theorem. Then it is readily seen that they satisfy all the conditions of Theorem 3.1. Therefore, there exists a tetrahedron $p = (p_1, p_2, p_3, p_4)$ such that $\tau(m(p)) = (X_1, X_2, 0)$. Since $X_1, X_2$ are real and $A = 0$, by applying the formulae in Proposition 2.6, we get that the normalized Gram matrix associated to $p$ has real coefficients. Then it follows from Corollary 2.2 that there exists an anti-holomorphic involution which fixes every point $p_i$, $i = 1, 2, 3, 4$. So, these points must be in the same $\mathbb{R}$-circle. $\blacksquare$
3.2.4 Real slice of the basic variety \( \mathbb{S} \)

We call the subset \( \mathcal{R} \) of the basic variety \( \mathbb{S} \) the real slice of \( \mathbb{S} \) if and only if \( X_1 \) and \( X_2 \) are real numbers.

**Theorem 3.4** Any point in the real slice \( \mathcal{R} \) satisfies the inequality \( \text{Re}(X_1 e^{-iA}) \geq 0 \) for \(-\pi/2 \leq A \leq \pi/2\).

**Proof:** We write \( X_1 = a + bi, X_2 = c + di, e^{-i2A} = \cos 2A - i \sin 2A \). It is readily seen that \( \mathbb{S} \) in terms of \( a, b, c, d, A \) is given by the following equation

\[
F(a, b, c, d, A) = F(a, c, A) = -2(a + c) - 2(ac) \cos 2A + a^2 + c^2 + 1 = 0,
\]

provided that \( b = d = 0 \). We consider this equation as an equation in \( a \) and \( c \), where \( A \) is a parameter. Thus, we have a family of conics. Easy calculations show that the discriminant \( D \) of any conic in this family is equal to \( D = -4 \sin^2 2A \). Therefore, these conics are of elliptic or parabolic type. Moreover, easy considerations show that any conic in this family is either ellipse or parabola for any \(-\pi/2 < A < \pi/2\) (which are tangent to the axes at the points \((a, c) = (1, 0)\) and \((a, c) = (0, 1)\), or the double line \((a + c - 1)^2 = 0\) for \( A = \pm \pi/2 \) (compare this with the equations in Propositions 3.4 - 3.5). This implies that if \((X_1, X_2, A)\) lies in the real slice \( \mathcal{R} \) then \( a > 0 \) or \( A = \pm \pi/2 \). So, in this case, \( \text{Re}(X_1 e^{-iA}) = a \cos A \geq 0 \). This proves the theorem.

**Corollary 3.5** Any point in the real slice \( \mathcal{R} \) represents a point in the configurations space \( \mathcal{M} \) provided that \(-\pi/2 \leq A \leq \pi/2\). The \( \mathbb{C} \)-plane and \( \mathbb{R} \)-plane tetrahedra define a subset lying in the real slice.

3.2.5 Topological picture of the moduli space \( \mathbb{M} \)

In this section, we describe the topology of the moduli space \( \mathbb{M} \).

It follows from Theorem 3.1 that not all points of \( \mathbb{S}_1 \) with \( A = \pm \pi/2 \) belong to the moduli space \( \mathbb{M} \): the singular set \( \mathcal{C} \) divides this set into two parts, and only the part defined by the restrictions in Theorem 3.1 belongs to \( \mathbb{M} \). We call such a part *positive*. Using notations as in Theorem 3.1, it is easy to see that the positive part is defined by the inequality \( b \geq 0 \), when \( A = \pi/2 \), and by the inequality \( b \leq 0 \), when \( A = -\pi/2 \).

Another important observation which may help to understand the topology of \( \mathbb{M} \) is the following.

**Proposition 3.10** Any point \((X_1, X_2, A)\) in the basic variety \( \mathbb{S} \) satisfies the inequality \( \text{Re}(X_1 e^{-iA}) \geq 0 \) for \(-\pi/2 < A < \pi/2\).

**Proof:** Let us write the equation for the basic variety \( \mathbb{S} \) as in the proof of Proposition 3.4, that is,

\[
F(a, b, c, d, A) = -2(a + c) - 2[(ac + bd) \cos 2A + (bc - ad) \sin 2A] + a^2 + b^2 + c^2 + d^2 + 1 = 0.
\]

We consider this equation as an equation in \( a \) and \( b \), where \( c, d, A \) are considered as parameters. Thus, we have a family of conics. It is easy to see that the discriminant \( D \) of any conic in this family is equal to \( D = -1 \). Therefore, all these conics are of elliptic type. We are going to determine the relative position of any conic in this family and the line \( L \) given by the equation \( \text{Re}(X_1 e^{-iA}) = a \cos A + b \sin A = 0 \) for \(-\pi/2 < A < \pi/2\). Let \( K \) be a conic in this family. We assume first that \( K \) is not degenerate, that is, \( K \) is an ellipse. Using Proposition 3.8, we see that \( K \) and \( L \) are tangent, and, therefore, \( K \) intersects \( L \) in only one point which is a unique point of tangency of \( K \) and \( L \). Hence, \( K \) lies entirely in the closure of a component of the complement of \( L \). Then easy arguments show that \( K \) lies in the component where \( a \cos A + b \sin A \geq 0 \). On the other hand, since \( \mathbb{S} \) has no singular points for \(-\pi/2 < A < \pi/2\), see Proposition 3.8, it readily seen that \( K \) is not degenerate for \(-\pi/2 < A < \pi/2\). All this implies that for \(-\pi/2 < A < \pi/2\) the basic variety \( \mathbb{S} \) lies in the component of the complement of the Cartan variety \( \mathbb{S}_{134} \), where \( \text{Re}(X_1 e^{-iA}) \geq 0 \). This proves the proposition.
Resuming all the above, we get the following topological picture for $\mathcal{M}$: the moduli space $\mathcal{M}$ looks like a real analytic 4-dimensional variety in $\mathbb{C}^2 \times \mathbb{R}$ truncated by a 4-dimensional analytic variety intersecting this variety non-transversally along a 2-dimensional analytic subvariety. The truncated part contains the singular set of $\mathcal{M}$ which is a 1-dimensional real analytic subvariety.

This implies that the moduli space $\mathcal{M}$ cannot be described as a real analytic (algebraic) variety.

### 3.3 The configuration space of ordered quadruples in the boundary of the complex hyperbolic plane and the Falbel-Parker-Platis cross-ratio varieties

In this section, we find a relation between our moduli space $\mathcal{M}$ and the cross-ratio varieties constructed by Falbel and Parker-Platis. We show that their varieties cannot serve as moduli spaces for the configuration space $\mathcal{M}$. We consider only the Parker-Platis variety which is more easy to describe. We remark that Falbel’ variety is homeomorphic to the Parker-Platis variety, see [17].

In [16], Parker and Platis define the map $\pi' : \mathcal{M} \to \mathbb{C}^3$ by the following formula:

$$\pi'(p) \to (X_1 = X(p_1, p_2, p_3, p_4), \ X_2 = X(p_1, p_3, p_2, p_4), \ X_3 = X(p_2, p_3, p_1, p_4)).$$

They proved that $X_1, X_2, X_3$ satisfy the following relations:

- $|X_2| = |X_1| |X_3|$,
- $2 |X_1|^2 \text{Re}(X_3) = |X_1|^2 + |X_2|^2 + 2 \text{Re}(X_1 + X_2)$.

The Parker-Platis cross-ratio variety $\mathcal{X}'$ is the subset of $\mathbb{C}^3$, where these relations are satisfied. The main result of [16] is that the map $\pi' : \mathcal{M} \to \mathcal{X}'$ defined above is a bijection, see Proposition 5.5 and Proposition 5.10 in [16]. Next we show that this map is not injective.

We define the map $\theta : \mathcal{M} \to \mathcal{X}'$ by the formula

$$\theta : (X_1, X_2, A) \to (X_1, X_2, (X_2/X_1) e^{2iA}).$$

It follows from Proposition 2.6 that $X_3 = (X_2/X_1) e^{2iA}$. Also, easy calculations show that for every $(X_1, X_2, A) \in \mathcal{M}$ the point $\theta(X_1, X_2, A)$ belongs to $\mathcal{X}'$. In fact, the first equation in the definition of the Parker-Platis cross-ratio variety is exactly the property of the Korányi-Reimann complex cross-ratios which relates $X_1, X_2, X_3$, see p.225 in Goldman [9]. The second relation is equivalent to the equation $\det G = 0$, where $G$ is a Gram matrix associated to $p$, and the Gram determinant is expressed in terms of $X_1, X_2, X_3$, see Proposition 2.6. Thus, $\theta$ defines a map $\mathcal{M} \to \mathcal{X}'$.

It follows from the definition that $\theta$ is injective for $-\pi/2 < A < \pi/2$. On the other hand, it is easy to see that $\theta$ is not injective being restricted to the subset of $\mathcal{M}$, where $A = \pm \pi/2$. Using the results of Section 3.2.1, we see that $\theta$ is not injective exactly on the singular set $\mathcal{C}$, that is,

$$\theta(X_1, X_2, \pi/2) = \theta(X_1, X_2, -\pi/2)$$

if and only if $X_1, X_2$ satisfy the conditions of Theorem 3.1. Thus, $\theta$ glues the points of $\mathcal{M}$ corresponding to the $\mathbb{C}$-plane tetrahedra whose (123) -faces have Cartan’s invariants with opposite signs. This implies that the map $\pi' : \mathcal{M} \to \mathcal{X}'$ is not injective.

For those readers, who prefer to use the standard position of points, see, for instance, Falbel [7], here is an explicit example which illustrates the above situation. To describe this example, we use horospherical coordinates $(z, v)$, $z \in \mathbb{C}$, $v \in \mathbb{R}$, on $\partial \mathbb{H}^2$, see Section 1.
We define the following quadruples

\[ p(t) = (p_1, p_2, p_3, p_4) = ((0, 0), \infty, (0, 1), (0, t)), \]

and

\[ p^*(t) = (p^*_1, p^*_2, p^*_3, p^*_4) = ((0, 0), \infty, (0, -1), (0, -t)), \]

where \( t > 0, t \neq 1. \)

It is easy to see that the tetrahedra \( p(t) \) and \( p^*(t) \) lie in the vertical line (in horospherical coordinates, this line represents a \( \mathbb{C} \)-circle, see [9]). The standard lifts for \( p_i \) and \( p^*_i \), see Section 1, are given by the following vectors

\[
P_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} i \\ 0 \\ 0 \end{bmatrix}, \quad P_4 = \begin{bmatrix} it \\ 0 \\ 0 \end{bmatrix},
\]

\[
P^*_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad P^*_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad P^*_3 = \begin{bmatrix} -i \\ 0 \\ 0 \end{bmatrix}, \quad P^*_4 = \begin{bmatrix} -it \\ 0 \\ 0 \end{bmatrix},
\]

We compute the Hermitian products

\[
\langle P_1, P_2 \rangle = 1, \quad \langle P_1, P_3 \rangle = -i, \quad \langle P_1, P_4 \rangle = -it, \quad \langle P_2, P_3 \rangle = 1, \quad \langle P_2, P_4 \rangle = 1, \quad \langle P_3, P_4 \rangle = i(1 - t),
\]

\[
\langle P^*_1, P^*_2 \rangle = 1, \quad \langle P^*_1, P^*_3 \rangle = i, \quad \langle P^*_1, P^*_4 \rangle = it, \quad \langle P^*_2, P^*_3 \rangle = 1, \quad \langle P^*_2, P^*_4 \rangle = 1, \quad \langle P^*_3, P^*_4 \rangle = i(t - 1).
\]

Then by definition of the cross-ratios, we have

\[
X_1(p(t)) = X_1(p^*(t)) = 1/t, \quad X_2(p(t)) = X_2(p^*(t)) = (t - 1)/t, \quad X_3(p(t)) = X_3(p^*(t)) = 1 - t.
\]

Also, we compute Cartan’s invariants

\[
\mathcal{A}(p_1, p_2, p_3) = -\pi/2, \quad \mathcal{A}(p^*_1, p^*_2, p^*_3) = \pi/2.
\]

Thus, the tetrahedra \( p(t) \) and \( p^*(t) \) have the same Parker-Platis coordinates. Moreover, it is clear that \( p(t) \) and \( p^*(t) \) are not congruent with respect to the diagonal action of \( \text{PU}(2, 1) \) since

\[
\mathcal{A}(p_1, p_2, p_3) = -\mathcal{A}(p^*_1, p^*_2, p^*_3).
\]

Let \( m(t) \) and \( m^*(t) \) be the points of \( \mathcal{M} \) corresponding to \( p(t) \) and \( p^*(t) \). We see that \( m(t) \neq m^*(t) \), but \( \pi'(m(t)) = \pi'(m^*(t)) \). So, we have another proof of the fact that the map \( \pi' : \mathcal{M} \to \mathcal{X}' \) is not injective.

**Remark** In Parker-Platis [10], the proof of the fact that the map \( \pi' : \mathcal{M} \to \mathcal{X}' \) is injective is based on Proposition 5.10. Our example shows that this proposition is not correct.

**Remark** It is interesting to note that \( p(t) \) and \( p^*(t) \) are congruent with respect the diagonal action of the whole isometry group of complex hyperbolic space: there exists an anti-holomorphic isometry (for instance, the reflection in the real axes) which sends \( p(t) \) to \( p^*(t) \).
3.4 The moduli space for the configuration space of ordered quadruples in the boundary of complex hyperbolic $n$-space

Let $\mathcal{M}(4, n)$ be the configuration space of ordered quadruples of distinct points in the boundary of complex hyperbolic $n$-space, that is, the quotient of the set of ordered quadruples of distinct points of $\partial \mathbb{H}_n^2$ with respect to the diagonal action of $\text{PU}(n, 1)$ equipped with the quotient topology.

In this section, we construct a moduli space for $\mathcal{M}(4, n)$.

**Theorem 3.5** $\mathcal{M}(n, 4)$ is homeomorphic to the set of points $X = (X_1, X_2, A) \in \mathbb{C}^2 \times \mathbb{R}$ defined by

$$-2\text{Re}(X_1 + X_2) - 2\text{Re}(X_1 \overline{X}_2 e^{-i2A}) + |X_1|^2 + |X_2|^2 + 1 \leq 0,$$

$$-\pi/2 \leq A \leq \pi/2, \quad \text{Re}(X_1 e^{-iA}) \geq 0.$$

The equality in the first inequality happens if and only if the quadruples are in the boundary of a complex hyperbolic 2-space.

**Proof:** The proof of this theorem is a slight modification of the proof of Theorem 3.1. The only thing we need is the following proposition which substitutes Proposition 3.1.

**Proposition 3.11** Let $G = (g_{ij})$ be an Hermitian $4 \times 4$-matrix such that $g_{ii} = 0$, $g_{12} = g_{23} = g_{34} = 1$, $|g_{13}| = 1$, $g_{14} \neq 0$, $g_{24} \neq 0$. Then $G$ is the normalized Gram matrix for some ordered quadruple $p = (p_1, p_2, p_3, p_4)$ of distinct points of $\partial \mathbb{H}_n^2$ if and only if $\text{Re}(g_{13}) \leq 0$, $\text{Re}(g_{24}g_{14}) \leq 0$, and $\det G \leq 0$.

The determinant $\det G = 0$ if and only if $p$ is in the boundary of a complex hyperbolic 2-space.

**Proof:** Let us assume that $G$ is the normalized Gram matrix associated to an ordered quadruple $p = (p_1, p_2, p_3, p_4)$ of distinct points of $\partial \mathbb{H}_n^2$. Then it follows from Proposition 2.4 that

$$\det G(1, 2, 3) = 2\text{Re}(g_{13}), \quad \det G(1, 2, 4) = 2\text{Re}(g_{24}g_{14}).$$

Since the Hermitian form in the definition of complex hyperbolic $n$-space has signature $(n, 1)$, it follows from Sylvester’s Criterion that these determinants are negative or vanish. Moreover, this also implies that $\det G \leq 0$. If $p$ is in the boundary of a complex hyperbolic 2-space, then $\det G = 0$, since in this case any vectors $P_1, P_2, P_3, P_4$ representing $p_1, p_2, p_3, p_4$ are linearly dependent.

Now let $G = (g_{ij})$ be an Hermitian $4 \times 4$-matrix such that $g_{ii} = 0$, $g_{12} = g_{23} = g_{34} = 1$, $|g_{13}| = 1$, $\text{Re}(g_{13}) \leq 0$, $\text{Re}(g_{24}g_{14}) \leq 0$, $\det G \leq 0$. We are going to show that there exist four null (isotropic) vectors $P_1, P_2, P_3, P_4, P_i \in \mathbb{C}^{n, 1}$, whose Gram matrix is equal to $G$.

We will look for these vectors in the following form:

$$P_1 = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad P_3 = \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ 1 \end{bmatrix}, \quad P_4 = \begin{bmatrix} w_1 \\ \vdots \\ w_n \\ w_{n+1} \end{bmatrix},$$

where $z_i, w_i$ are complex numbers, and $|z_1| = 1$.

Then we have

$$\langle P_1, P_2 \rangle = 1, \quad \langle P_1, P_3 \rangle = \bar{z}_1, \quad \langle P_1, P_4 \rangle = \bar{w}_1, \quad \langle P_2, P_3 \rangle = 1, \quad \langle P_2, P_4 \rangle = \bar{w}_{n+1},$$

$$\langle P_3, P_4 \rangle = z_1 w_{n+1} + z_2 \bar{w}_2 + \cdots + z_n \bar{w}_n + \bar{w}_1 = z_1 \bar{w}_{n+1} + \bar{w}_1 + \langle z, w \rangle,$$

where $z = (z_2, \ldots, z_n)$, $w = (w_2, \ldots, w_n)$, and $\langle z, w \rangle$ is the standard Hermitian product on $\mathbb{C}^{n-1}$.
Since we need $P_3$ and $P_4$ to be null vectors, we have the following equations:

\[ z_1 + |z|^2 + \bar{z}_1 = 0, \quad w_1 \bar{w}_{n+1} + |w|^2 + \bar{w}_1 w_{n+1} = 0, \]

where $|z|^2 = \langle z, z \rangle$, and $|w|^2 = \langle w, w \rangle$.

From the definition of the Gram matrix, we have

\[ g_{12} = 1, \quad g_{13} = \bar{z}_1, \quad g_{14} = \bar{w}_1, \quad g_{23} = 1, \quad g_{24} = \bar{w}_{n+1}, \]

and

\[ g_{34} = z_1 \bar{w}_{n+1} + \langle z, w \rangle + \bar{w}_1 = 1. \]

This implies that we have already found $z_1, w_1, w_{n+1}$ in terms of $g_{ij}$. Therefore, we need to find the vectors $z$ and $w$.

We consider the following system of equations

\begin{enumerate}
\item $z_1 + |z|^2 + \bar{z}_1 = 0,$
\item $w_1 \bar{w}_{n+1} + |w|^2 + \bar{w}_1 w_{n+1} = 0,$
\item $z_1 \bar{w}_{n+1} + \langle z, w \rangle + \bar{w}_1 = 1,$
\end{enumerate}

and show that it has a solution under the conditions of the proposition. We write the first two equations in the following form:

\[ |z|^2 = -2\text{Re}(z_1) = -2\text{Re}(g_{13}), \quad |w|^2 = -2\text{Re}(w_1 \bar{w}_{n+1}) = -2\text{Re}(g_{24} \bar{g}_{14}). \]

We immediately see that there exist solutions for $|z|$ and $|w|$ under our conditions. The third equation can be written as

\[ \langle z, w \rangle = 1 - z_1 \bar{w}_{n+1} - \bar{w}_1. \]

Let us write the Cauchy-Schwarz inequality for the vectors $z$ and $w$:

\[ |\langle z, w \rangle|^2 \leq |z|^2 |w|^2. \]

Substituting $|z|$ and $|w|$ from equations (1) and (2) and $\langle z, w \rangle$ from the third equation, we rewrite this inequality in the following form:

\[ |1 - z_1 \bar{w}_{n+1} - \bar{w}_1|^2 - (z_1 + \bar{z}_1) (w_1 \bar{w}_{n+1} + \bar{w}_1 w_{n+1}) \leq 0. \]

By computations similar to those in Proposition 3.1, we have

\[ |1 - z_1 \bar{w}_{n+1} - \bar{w}_1|^2 - (z_1 + \bar{z}_1) (w_1 \bar{w}_{n+1} + \bar{w}_1 w_{n+1}) = \det G. \]

This implies that if there exist solutions $(z, w)$ to the equation

\[ |\langle z, w \rangle| = |1 - z_1 \bar{w}_{n+1} - \bar{w}_1| \]

that satisfy equations (1) and (2) then necessarily $\det G \leq 0$. Let us assume now that $\det G \leq 0$. Then there exist vectors $z$ and $w$ satisfying equations (1) and (2) such that the inequality

\[ |1 - z_1 \bar{w}_{n+1} - \bar{w}_1|^2 \leq |z|^2 |w|^2 \]

holds. One verifies that if one of the vectors, $z$ or $w$, is the null vector, then the system above has a solution because the third equation is satisfied automatically in this case provided that $\det G \leq 0$. So, we may suppose that both vectors $z$ and $w$ are not null. Let $\mathbb{C}z$ and $\mathbb{C}w$ be the complex lines in the
underlying real vector space of $\mathbb{C}^{n-1}$ spanned by $z$ and $w$, and let $\angle(Cz, Cw)$ be the angle between $Cz$ and $Cw$, see [9]. Then it follows from the formula

$$|\langle \langle z, w \rangle \rangle| = |z||w|\cos(\angle(Cz, Cw))$$

proved in Lemma 2.2.2, [9], that by choosing an appropriate angle between $Cz$ and $Cw$ (without changing the norms of $z$ and $w$), we may assume that $z$ and $w$ satisfy the equality $|\langle \langle z, w \rangle \rangle| = |1 - z_1 \bar{w}_{n+1} - \bar{w}_1|$. Let $z' = e^{i\theta}z$. Then $\langle \langle z', w \rangle \rangle = e^{i\theta} \langle \langle z, w \rangle \rangle$. This implies that there exists $\theta$ such that $(z', w)$ is a solution to the equation $\langle \langle z, w \rangle \rangle = 1 - z_1 \bar{w}_{n+1} - \bar{w}_1$ (here we have used the fact that if two complex numbers have the same norm then there exists a rotation which sends one number to another). Finally, it is easy to see that if $g_{14} \neq 0$, $g_{24} \neq 0$, then the points $p_i$ defined by the vectors $P_i$ are distinct. This proves the statement of the proposition.

References

[1] U. Brehm, The shape invariant of triangles and trigonometry in two-point homogeneous spaces. Geom. Dedicata 33 (1990), no. 1, 59–76.
[2] U. Brehm, B. Et-Taoui, Congruence criteria for finite subsets of complex projective and complex hyperbolic spaces. Manuscripta Math. 96 (1998), no. 1, 81–95.
[3] D. Burns, S. Shnider, Spherical hypersurfaces in complex manifolds. Invent. Math. 33 (1976), no. 3, 223–246.
[4] E. Cartan, Sur le groupe de la géométrie hypersphérique. Comm. Math. Helv. 4 (1932), 158–171.
[5] F. Dutenhefner, N. Gusevskii, Complex hyperbolic Kleinian groups with limit set a wild knot. Topology 43 (2004), 677–696.
[6] D. Epstein, Complex hyperbolic geometry. Analytical and geometric aspects of hyperbolic space (Coventry/Durham, 1984), 93–111, London Math. Soc. Lecture Note Ser., 111, Cambridge Univ. Press, Cambridge, 1987.
[7] E. Falbel, Geometric structures associated to triangulations as fixed point sets of involutions. Topology Appl. 154 (2007), no. 6, 1041–1052.
[8] E. Falbel, I. Platis, The $PU(2,1)$ configuration space of four points in $S^3$ and the cross-ratio variety. Math. Ann. 340 (2008), no. 4, 935–962.
[9] W.M. Goldman, Complex hyperbolic geometry. Oxford Mathematical Monographs. Oxford Science Publications. The Clarendon Press, Oxford University Press, New York, 1999. xx+316 pp.
[10] W.M. Goldman, J.R. Parker, Dirichlet polyhedra for dihedral groups acting on complex hyperbolic space. J. Geom. Anal. 2 (1992), no. 6, 517–554.
[11] C. Grossi, PhD Thesis, Universidade Estadual de Campinas, 2006.
[12] N. Gusevskii, J.R. Parker, Representations of free Fuchsian groups in complex hyperbolic space. Topology 39 (2000), no. 1, 33–60.
[13] N. Gusevskii, J.R. Parker, Complex hyperbolic quasi-Fuchsian groups and Toledo’s invariant. Special volume dedicated to the memory of Hanna Miriam Sandler (1960–1999). Geom. Dedicata 97 (2003), 151–185.
[14] J. Hakim, H. Sandler, The moduli space of $n + 1$ Points in Complex Hyperbolic $n$-Space. Geom. Dedicata 97 (2003), 3-15.

[15] A. Korányi, H. M. Reimann, The complex cross ration on the Heisenberg group. Enseign. Math. 33 (1987), no.(3-5), 291-300.

[16] J.R. Parker, I. Platis, Complex hyperbolic Fenchel-Nielsen coordinates. Topology 47 (2008), no. 2, 101–135.

[17] J. R. Parker, I. Platis. Global, geometrical coordinates on Falbel’s cross-ratio variety. Canad. Math. Bull., to appear.

[18] W. Scharlau, Quadratic and Hermitian forms. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 270. Springer-Verlag, Berlin, 1985. x+421 pp.

[19] P. Will, Traces, Cross-ratios and 2-generator Subgroups of PU(2,1). Canad. J. Math., to appear.

E-mail addresses:  cunha@mat.ufmg.br  
    nikolay@mat.ufmg.br