MONOPOLES AND THE SEN CONJECTURE

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Abstract. We describe compactifications of the moduli spaces of SU(2) monopoles on $\mathbb{R}^3$ as manifolds with corners, with respect to which the hyperKähler metrics admit asymptotic expansions up to each boundary face. The boundary faces encode monopoles of charge $k$ decomposing into widely separated monopoles of lower charge, and the leading order asymptotic of the metric generalizes the one obtained by Gibbons, Manton and Bielawski in the case of complete decomposition into monopoles of unit charge. From the structure of the compactifications, we prove part of Sen’s conjecture for the $L^2$ cohomology of the strongly centered moduli spaces by adapting an argument of Segal and Selby.

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Part I. A Description of the Compactification

1. Introduction

The moduli space $\mathcal{M}_k$ of non-abelian magnetic monopoles of charge $k$ (gauge group SU(2)) has received much attention from both mathematicians and mathematical physicists. It is well known that for each positive integer $k$, $\mathcal{M}_k$ is a complete, non-compact hyperKähler manifold of dimension $4k$. The question of the asymptotic behaviour of the metric $g_k$, say, on $\mathcal{M}_k$ has also been studied in various special cases [5, 6, 7, 12, 3, 16]. Apart from the intrinsic interest in understanding the asymptotic behaviour of this metric completely, it is also essential for the study of the $L^2$ harmonic forms on the monopole moduli spaces, which are the subject of the Sen Conjecture [23, 22].

In this paper we shall give a complete description of the asymptotic behaviour of $g_k$, the complete proof of which will appear in Part II, and we shall combine our results with an argument of Segal and Selby [22] to prove the ‘coprime case’ of the Sen Conjecture. This argument is quite ‘soft’ and exploits some very specific features of the asymptotic geometry of the monopole moduli spaces. In particular, detailed analysis of the Hodge-de Rham operator $d + d^*$ on $\mathcal{M}_k$ is not required. It would seem that such an analysis will, however, be needed to prove the other cases of the Sen Conjecture; we hope to return to this in the near future.

In order to state the Sen Conjecture, recall first the definition of the strongly centred space $\mathcal{M}_k^0$ of monopoles of charge $k$. This is the universal cover of the quotient $\mathcal{M}_k / \mathbb{R}^3 \times S^1$, where $\mathbb{R}^3$ acts by translations and $S^1$ by rotations of the framing (or ‘large’ gauge transformations). The quotient has fundamental group $\mathbb{Z}_k$, which we identify with the group of complex $k$-th roots of unity [3, 22]. Since $\mathbb{R}^3 \times S^1$ acts isometrically on $\mathcal{M}_k$, the quotient and its universal cover inherit a natural metric from that of $\mathcal{M}_k$. If $\zeta \in \mathbb{Z}_k$, denote by $\alpha_\zeta$ the corresponding deck transformation, which will be an isometry of $\mathcal{M}_k^0$.

The Sen Conjecture predicts the dimension of the space $\mathcal{H}^i(\mathcal{M}_k^0)$, the space of $L^2$ harmonic $i$-forms on $\mathcal{M}_k^0$. More precisely, let

$$\mathcal{H}_{k, \ell} = \{ u \in \mathcal{H}^i(\mathcal{M}_k^0) : \alpha_\zeta^* u = \zeta^\ell u \}, \quad \ell = 0, 1, \ldots, k - 1.$$  (1.1)

(The physics interpretation of $\ell$ is as the electric charge of the quantum state $u$.)

Conjecture 1.1 (Sen Conjecture). [23, 22]

(S.1) If $k$ and $\ell$ are coprime, then $\mathcal{H}_{k, \ell}^{2k-2} \cong \mathbb{C}$, while $\mathcal{H}_{k, \ell}^i = 0$ for $i \neq 2k - 2$;

(S.2) if $k$ and $\ell$ are not coprime, then $\mathcal{H}_{k, \ell}^i = 0$ for all $i$.

We shall prove the ‘coprime case’ (S.1) of the conjecture, along the lines suggested in [22].

Theorem 1.2. Statement (S.1) of the Sen Conjecture holds true.

1.1. A Metric Compactification of $\mathcal{M}_k$. Denote by $\mathcal{M}_k$ the quotient $\mathcal{M}_k / \mathbb{R}^3$ of the moduli space by translations. It is almost equivalent to think of $\mathcal{M}_k$ as the space of monopoles centred at the origin in $\mathbb{R}^3$. If $k \geq 2$, $\mathcal{M}_k$ is still non-compact and it is really the non-compactness of the translation-group $\mathbb{R}^3$ which underlies the non-compactness $\mathcal{M}_k$ itself. To explain this, let $m' \in \mathcal{M}_k$ be a divergent sequence. Then to paraphrase [3, Proposition 3.8], a subsequence of $m'$ consists of ‘widely separated monopoles of type $a$’ for some proper partition $a = (k_1, \ldots, k_n)$ of $k$. (Proper means $n \geq 2$ and all $k_j \geq 1$). The condition of wide separation means that there is a configuration of points $(p_1', \ldots, p_n')$ such that

$$\varepsilon^{-1} := \min_{i<j} |p_i' - p_j'| \gg 1$$  (1.2)

and a collection of centred monopoles

$$(m_1, \ldots, m_n) \in \mathcal{M}_{k_1} \times \cdots \mathcal{M}_{k_n}$$  (1.3)

In fact thinking of $\mathcal{M}_k$ as the space of centred monopoles, $\mathcal{M}_k / S^1$ can be identified as an $S^1$-hyperKähler quotient of $\mathcal{M}_k$. In particular, $\mathcal{M}_k / S^1$ inherits a hyperKähler metric from $\mathcal{M}_k$. 

such that for each \( j \), \( m^\nu(z - p_j^\nu) \) converges to \( m_j(z) \) on any fixed ball \( \{ |z| < R \} \). In other words for large \( \nu \), \( m \) looks like an approximate superposition of the translated monopoles \( m_j(z + p_j^\nu) \).

(NB: given \( a \), there is a subgroup of the symmetric group \( \Sigma_n \) which acts on configurations of type \( a \), consisting of those permutations \( \sigma \) of \( \{1, \ldots, n\} \) with \( k_{\sigma(i)} = k_i \), for all \( i \). A widely separated configuration of monopoles really involves unordered configurations of points (and monopoles) where we factor out by this group action. See \( \S \S 2.3 \) for more detail.)

Remark 1.3. This classification of divergent sequences according to ‘type’ strongly suggests that \( \mathcal{M}_k \) should have asymptotic regions which correspond to the different types of divergent sequences in \( \mathcal{M}_k \). This intuition is supported by intuition coming from the identification of \( \mathcal{N}_k \) with \( \text{Rat}_k \), the space of based rational functions of degree \( k \) [11]. The basic idea is that if \( f_1 \) and \( f_2 \) are rational functions respectively of degrees \( k_1 \) and \( k_2 \), then generically their sum \( f_1 + f_2 \) will be a rational function of degree \( k_1 + k_2 \). There are, however, subtleties in using this to try to describe the asymptotic regions of \( \mathcal{N}_k \) because (for example) the identification \( \mathcal{N}_k = \text{Rat}_k \) breaks the symmetry of \( \mathcal{N}_k \) by singling out a direction in \( \mathbb{R}^3 \). However, the idea is largely captured by the cover of \( \mathcal{M}_k \) by open sets corresponding to ‘decomposable monopoles’ which appears in \( 13 \) of this paper.

We should also note that the case \( a = (1, \ldots, 1) \) is well understood [14, 6, 12, 3] and that \( L^2 \) harmonic forms on the Atiyah-Hitchin manifold \( \mathcal{M}_2/\mathbb{T} \) have been studied in [13, 7.1.2]; furthermore the results of [16] give a partial description of such regions.

We shall introduce a compactification \( \bar{\mathcal{M}}_k \) of \( \mathcal{M}_k \), which will be a manifold with corners (MWC). This provides a convenient and powerful way to deal with the complexities of the asymptotic geometry of \( \mathcal{M}_k \), including good definitions of the various asymptotic regions, their intersections, and the behaviour of the \( L^2 \) metric in each region. A similar approach, using MWCs to study complete Calabi–Yau metrics with complicated asymptotic behaviour, can be found in [8]. Manifolds with corners are convenient for the study of many other non-compact and singular problems in geometric analysis, see for example [2, 1] [9, 20, 21]. Vasy’s approach via MWCs to many-body geometry [23] underlies our discussion of ‘ideal configurations’ of points in a euclidean space and is an essential ingredient in our construction.

As a compact manifold with corners, \( \bar{\mathcal{M}}_k \) has a finite number of boundary hypersurfaces; these are indexed (at least for the moment) by proper partitions \( a \) of \( k \) and denoted \( N_a \). To say that \( \bar{\mathcal{M}}_k \) is a compactification of \( \mathcal{M}_k \) means that the interior of \( \bar{\mathcal{M}}_k \) is \( \mathcal{M}_k \),

\[
\bar{\mathcal{M}}_k \setminus \bigcup_a N_a = \mathcal{M}_k.
\]

Part of the definition of MWC is that the boundary hypersurfaces are embedded. In particular, we may choose a boundary defining function \( \rho_a > 0 \) for each \( N_a \) and for sufficiently small \( \delta > 0 \), the sublevel set \( U_a = \{ \rho_a < \delta \} \) will be diffeomorphic to the product \( [0, \delta) \times N_a \). The (hitherto ill-defined) asymptotic regions of \( \bar{\mathcal{M}}_k \) can now be defined precisely as the interiors of the \( U_a \); these are diffeomorphic to products \( (0, \delta) \times (N_a)^c \), where of course \( (N_a)^c \) is the interior of the MWC \( N_a \).

One of the advantages of \( \bar{\mathcal{M}}_k \) is that the corners structure encodes the intersection properties of the different asymptotic regions. Recall that for partitions \( a \) and \( b \) of \( k \), \( a \) is finer than \( b \), written \( a \preceq b \), if \( b \) is obtained from \( a \) by bracketing terms in \( a \). The boundary hypersurfaces \( N_a \) and \( N_b \) will intersect if and only if the corresponding partitions \( a \) and \( b \) of \( k \) are comparable, that is \( a \preceq b \) or \( b \preceq a \). For example, when \( k = 3 \), we have the two proper partitions \( a = (1, 1, 1) \) and \( b = (1, 2) \) of \( k \), and \( a \) is a refinement of (or simply finer than) \( b \). There are two asymptotic regions of \( \mathcal{M}_3 \) and their intersection consists of widely separated monopoles of type \( (1, 1, 1) \) with centres at \( (p_1, p_2, p_3) \) such that the distance \( |p_1 - p_2| \) is large but much smaller than the distances \( |p_1 - p_3| \) and \( |p_2 - p_3| \). Our compactification handles these configurations through the parameters

\[
\rho_1 = \frac{1}{|p_1 - p_2|}, \quad \rho_2 = \frac{|p_1 - p_2|}{|p_1 - p_3|}.
\]
which turn out to be local boundary defining functions for the two boundary hypersurfaces. To see why, notice that if \( p_1 \to 0 \) with \( p_2 > 0 \) fixed, then \(|p_1 - p_2|, |p_1 - p_3|, |p_2 - p_3|\) all tend to \( \infty \) and the ratios between them are all bounded; on the other hand, if \( p_1 > 0 \) is fixed and \( p_2 \to 0 \), then \(|p_1 - p_2|\) remains bounded while \(|p_1 - p_3|\) and \(|p_2 - p_3|\) both go to \( \infty \). The two parameters \((p_1, p_2)\) can thus be used to describe diverging triples \((p_1^s, p_2^s, p_3^s)\) where \( 1 \ll |p_1^s - p_2^s| \ll |p_1^s - p_3^s|\).

More generally, \( N_a \cap N_b \) will have a number of disconnected components, corresponding to inequivalent ways of bracketing the terms in \( a \) to produce \( b \). The simplest example occurs for \( k = 5 \):

\[
1 + 1 + 1 + 2 = (1 + 1 + 1) + 2, \quad 1 + 1 + 1 + 2 = (1 + 1) + (1 + 2) \tag{1.6}
\]

both of which display the partition \( 1 + 1 + 1 + 2 \) as a refinement of \( 2 + 3 \). This is not a mere technicality as the two ways of bracketing terms correspond to different intersections of asymptotic regions of \( \mathcal{M}_5 \). For this example, we have a 2-monopole and a 3-monopole, widely separated. In the first case, the 3-monopole is in the \((1, 1, 1)\) asymptotic region of \( \mathcal{M}_3 \) while the 2-monopole remains in a bounded subset of \( \mathcal{M}_2 \); in the second, the 3-monopole is in the \((2, 1)\) asymptotic region of \( \mathcal{M}_3 \) and the 2-monopole is in the asymptotic \((1, 1)\) region of \( \mathcal{M}_2 \).

The situation is best described in terms of partitions \( \lambda \) of the set \( k = \{1, \ldots, k\} \). Such \( \lambda \) has a type \( a = [\lambda] \), by taking the sizes of the blocks of \( \lambda \). Equivalently, \( \Sigma_k \) acts on the set of partitions \( \{ \lambda \} \), and the set of orbits is precisely the set of partitions \( \{ a \} \) of \( k \). Refinement of partitions \( \lambda \) (where \( \lambda \leq \mu \) if every block of \( \mu \) is a union of blocks of \( \lambda \)) goes over to refinement of integer partitions. The point illustrated by the above example is that the set of \( \Sigma_k \)-orbits of length-2 chains \( \lambda < \mu \) is not the same as the length-2 chains \( a < b \) in the set of integer partitions: it is the former, not the latter, that labels the codimension-2 hypersurfaces of \( \mathcal{M}_k \). More generally, the codimension-\( d \) corners of \( \mathcal{M}_k \) are labelled by the \( \Sigma_k \)-orbits of length-\( d \) chains \( \lambda_1 < \lambda_2 < \cdots < \lambda_d \) in the set of partitions of \( k \).

One should think of \( N_a \) as the (compactified) moduli space of \textit{ideal} monopoles of type \( a \)—ideal in the sense of ‘infinitely separated’ configurations of points. This will be made precise in the next section: it is noteworthy that \( N_a \) has a natural definition as a manifold with corners, whereas to define ‘asymptotic regions’ requires arbitrary choices.

Let us now explain how asymptotic behaviour of the metric on \( \mathcal{M}_k \) is captured by the compactification \( \bar{\mathcal{M}}_k \). The metric behaviour reflects the additional structure of a fibration

\[
\phi_a : N_a \to B_a, \tag{1.7}
\]

of each boundary hypersurface, where base and fibre are compact MWC. The fibrations enjoy compatibility conditions at the non-empty intersections \( N_a \cap N_b \), giving \( \bar{\mathcal{M}}_k \) an \textit{iterated boundary fibration (IBF) structure}\(^3\)\(^2\)\(^1\)\(^9\)\(^8\) which we shall recall in \( \S 3 \) below. Generally, if \( M \) is a compact MWC with an iterated boundary fibration structure, there is a smooth vector bundle which we shall denote\(^3\)\(^2\)\(^1\)\(^9\)\(^8\) by \( \Phi TM \), whose restriction to the interior \( M^o \) is canonically isomorphic to \( TM^o \), but whose sections have particular decay properties at the boundary (see \( \S 3 \)).

A smooth metric on \( \Phi TM \) (smooth up to and including all boundary hypersurfaces) will automatically define a complete metric on \( M^o \) and the smoothness, as a metric on \( \Phi TM \), captures precise asymptotic behaviour near each boundary hypersurface. A metric arising in this way will be called a \( \Phi \)-metric. Such metrics were first introduced in \( \S 10 \)\(^8\), where they are referred to as ‘QAC’ or ‘QFB’ metrics.

Then our main theorem about the metric structure of \( \bar{\mathcal{M}}_k \) is as follows, a more precise version of which will be given in \( \S 4.6 \).

\begin{theorem}
\textbf{Theorem 1.4.} The moduli space \( \mathcal{M}_k \) has a compactification \( \bar{\mathcal{M}}_k \) as a compact MWC with iterated boundary fibration, and the \( L^2 \) metric \( g_k \) extends to a smooth \( \Phi \)-metric on \( \bar{\mathcal{M}}_k \), which we denote by \( g_k \) again. Moreover, there is an isometric \( \mathcal{T} \)-action on \( \bar{\mathcal{M}}_k \) whose restriction to the interior is the triholomorphic \( \mathcal{T} \)-action on \( \mathcal{M}_k \), and whose orbits are of bounded length with respect to \( g_k \).
\end{theorem}

\(^2\)Essentially the same structure appears in \( \S 2 \)\(^1\)\(^9\)\(^8\) but there is unfortunately no agreement on terminology.

\(^3\)Again, there is no agreement on terminology.
Thus among the various possible compactifications of $\mathcal{M}_k$ which arise from the different descriptions of the moduli space, our compactification $\bar{\mathcal{M}}_k$ is ‘metrically natural’. This result, combined with a slight refinement of the argument in [22, Sect. 3], leads to a quick proof of the coprime case of the Sen Conjecture.

The plan of the rest of this paper is as follows. In Part I, we shall give our main results about the moduli spaces, deferring the proof of the main theorem, Theorem 1.4 to Part II. We start in the next section with compactifications of configuration spaces of points in a euclidean space inspired by Vasy’s resolved many-body spaces [25]. In §3 we recall the definition of iterated boundary fibrations and then in §4 we describe the compactification $\bar{\mathcal{M}}_k$ and its iterated boundary fibration structure. In §5, we explain, using the a priori estimates of Taubes, why any divergent sequence of monopoles in $\mathcal{M}_k$ has a limit point in $\bar{\mathcal{M}}_k$. Finally, in §6 we prove the coprime case of the Sen Conjecture, following an idea of Segal–Selby. Sections 4 and 6 are independent of each other and can be read in either order.

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2. Geometric Preliminaries

Before coming to the compactification of $\mathcal{M}_k$, we devote this section to the compactification of a simpler family of so-called ‘many-body spaces’, in particular the ‘reduced configuration spaces’ $\mathbb{R}^{3n}/\mathbb{R}^3$ of $n \leq k$ points in $\mathbb{R}^3$ up to translation. This interlude serves several purposes: first, these spaces serve as simplified models for the monopole moduli spaces themselves (indeed, these reduced configuration spaces, modulo action by the symmetric group, are essentially equivalent to the moduli spaces of abelian U(1) monopoles with appropriate framing); second, the appropriate compactifications of many-body spaces are quite easy to construct, yet still illustrate the essential combinatorial and geometric structure of our compactification $\bar{\mathcal{M}}_k$ of $\mathcal{M}_k$, the construction of which is significantly more difficult; finally, the compactified many-body space machinery plays a key technical role in our actual construction of $\bar{\mathcal{M}}_k$ in Part II. The forbearance of the reader is appreciated as we proceed to introduce a certain amount of notation.

We assume familiarity with the basic notions of manifolds with corners as presented, for example in [18] or [19]. Other useful references are [25, 2, 1, 9, 8].

2.1. Euclidean Many-Body Spaces and their Resolutions. Let $V$ be a real euclidean vector space of dimension $N$. Let $\mathcal{W}$ be a finite family of linear subspaces of $V$ satisfying

\[0, V \in \mathcal{W}\]  \tag{2.1}

\[W, W' \in \mathcal{W} \Rightarrow W \cap W' \in \mathcal{W}.\]  \tag{2.2}

We refer to the second condition as ‘intersection closure’ and to a such a family $\mathcal{W}$ as a linear many-body structure. The set $\mathcal{W}$ is partially ordered by inclusion. Let $\overrightarrow{\mathcal{W}}$ denote the radial compactification of $V$, and $\partial V$ its boundary. It is best to think of $\partial V$ as the quotient $(V \setminus 0)/\mathbb{R}^+$. For $W \in \mathcal{W}$ we denote by $\overrightarrow{W}$ and $\partial W$ the corresponding radial compactifications and boundaries. Note that $\partial \{0\} = \emptyset$. It is also convenient to put

\[\partial \mathcal{W} = \{\partial W : W \in \mathcal{W}\}\]  \tag{2.3}

so that $\partial \mathcal{W}$ is a set of submanifolds of $\partial V$.

Definition 2.1. If $\mathcal{W}$ satisfies (2.1) and (2.2), set

\[M(V, \mathcal{W}) := [\overrightarrow{V}; \partial \mathcal{W}^*],\]  \tag{2.4}
and
\[ B(V, \mathcal{W}) := [\partial V; \partial \mathcal{W}^*] \] (2.5)
where \( \mathcal{W}^* := \mathcal{W} \setminus \{V\} \). We call \( M(V, \mathcal{W}) \) the many-body compactification of \( V \) with respect to \( \mathcal{W} \) and \( B(V, \mathcal{W}) \) the free boundary of \( M(V, \mathcal{W}) \).

**Remark 2.2.** When it is clear what family \( \mathcal{W} \) is under consideration, and there is no risk of confusion, we shall abbreviate the notation to \( M(V) \), and \( B(V) \).

**Remark 2.3.** We shall see below that the lift to \( M(V, \mathcal{W}) \) of \( \partial V \) is \( B(V, \mathcal{W}) \), which is the reason for the terminology ‘free boundary’.

This space is a natural compactification of the set of all ‘ideal points’ of \( V \) which do not lie on any of the subspaces in \( \mathcal{W}^* \).

**Remark 2.4.** It is to be understood that the blow-ups are performed in size order. The intersection-closure means that the blow-ups are well-defined; after \( j \) blow-ups, the lifts of the remaining submanifolds are \( p \)-submanifolds, and those that intersect in the original family are disjoint when their intersection is blown up \([25, 15]\).

The important for example for us comes from the family of diagonals in \( E^k \), where \( E \) is a (finite-dimensional) euclidean space.

**Example 2.5** (Diagonals and configurations). Let \( E = \mathbb{R}^m \), and \( V = E^k \) (\( k \)-fold product). The set \( \mathcal{D} \) of all diagonals of \( E^k \) satisfies conditions (2.1) and (2.2) provided that we regard \( 0 \) and \( E^k \) itself as diagonals. In this case \( M(E^k, \mathcal{D}) \) is a natural compactification of the configuration space of \( k \) points in \( \mathbb{R}^m \).

**Example 2.6** (Reduced Configuration Spaces). Continuing the previous example, note that all true diagonals (i.e. we exclude the subspace \( 0 \)) contain the minimal diagonal
\[ D_k = \{ p_1 = \cdots = p_k \}. \]
Thus there is a quotient family
\[ \mathcal{D}' = \{ D/D_k : D \in \mathcal{D}, D \neq 0 \} \] (2.6)
of linear subspaces of \( E^k/E \) which is again intersection-closed. Then \( M(E^k/E, \mathcal{D}') \) is the compactification of the space of configurations of \( k \) points mod translation, and \( B(E^k/E, \mathcal{D}') \) is the space of ideal configurations mod translation. We also refer to these spaces as (compactified) reduced configuration spaces.

If \( V \) is a euclidean space with many-body structure \( \mathcal{W} \) and \( A \in \mathcal{W} \), then both \( A \) and \( V/A \) inherit many-body structures. The many-body structure on \( A \) is just the set of \( W \in \mathcal{W} \) with \( W \subset A \); that on \( V/A \) is the set of quotients \( W/A \), with \( W \supset A \). We shall write \( M(A), B(A) \) for the many-body compactification and free-boundary of \( A \) with this induced many-body structure and similarly for \( V/A \). Given the euclidean structure of \( V \) we can of course replace the quotient \( V/A \) by \( A^\perp \); then the many-body structure is the set \( \{ W^\perp : W \in \mathcal{W}, W \supset A \} \).

These sub- and quotient-many-body structures appear when describing the boundary faces of \( M(V, \mathcal{W}) \). From the definition, the boundary hypersurfaces of \( M(V, \mathcal{W}) \) are labelled precisely by the non-zero elements of \( \mathcal{W} \). The free boundary \( B(V, \mathcal{W}) \) corresponds to the element \( V \in \mathcal{W} \) and is the lift to the blow-up of the boundary of \( V \). Similarly, the boundary hypersurfaces of \( B(V, \mathcal{W}) \) are in one-one correspondence with the non-zero elements of \( \mathcal{W}^* \).

**Theorem 2.7** ([15], Theorem 5.1). Let \( 0 \neq A \in \mathcal{W} \). Then:
(a) The boundary hypersurface \( N \) in \( M(V, \mathcal{W}) \) corresponding to \( A \) is the compact MWC
\[ N = M(V/A) \times B(A). \] (2.7)
(b) The boundary hypersurfaces \( N_1 \) and \( N_2 \) corresponding to \( A_1, A_2 \in \mathcal{W} \) meet in \( M(V) \) if and only if \( A_1 \) and \( A_2 \) are comparable \((A_1 \subseteq A_2 \) or vice versa\). More generally the non-empty \( d \)-fold intersections of boundary hypersurfaces of \( M(V) \) correspond precisely to length-\( d \) chains
\[ A_d \subset \cdots \subset A_2 \subset A_1 \] (2.8)
of elements of $\mathcal{W}$; every codimension-$d$ boundary face is a connected component of such an intersection. With $N_j$ the boundary hypersurface corresponding to $A_j$, we have

$$N_1 \cap N_2 \cap \cdots \cap N_d = M(V/A_1) \times B(A_1/A_2) \times \cdots \times B(A_d).$$

(2.9)

**Remark 2.8.** Here of course the many-body structure on $A_j/A_{j+1}$ is understood to be the family of subspaces $\{W/A_{j+1} : A_{j+1} \subset W \subset A_j\}$.

We refer to [L3] for the proof. The first part of this theorem is proved by observing first that only those submanifolds that are commensurable with $A$ can enter in the lift of $A$ to the corresponding blow-up. Blowing up those that are contained in $A$ produces in each case the first factor. Upon blow-up of $A$ itself, we get either $V/A$ in the first case or $\partial V/A$ in the second. The lift to this of $\partial W$ where $W \supset A$ is just $\partial W/A$, and this is where the second factor comes from. The second part follows by induction.

Returning to Example 2.6, with which we shall be concerned from now on, it follows that the boundary hypersurfaces of the reduced configuration space $M(E^k/E)$ are in bijection with the quotients

$$D_{\lambda k} := D_\lambda / D_k,$$

where $\lambda$ is a partition of the set $k = \{1, \ldots, k\}$ and $D_\lambda$ is the diagonal in which $p_i = p_j$ whenever $i$ and $j$ lie in the same block. It is convenient to denote by 0 the minimal partition of $k$ into $k$ singletons and by $k$ the maximal partition of $k$ as a single set. Then $D_0 = E^k$, $D_k$ is consistent with the earlier definition as the minimal diagonal, and $D_{0,k} = D_0/D_k$ is the quotient configuration space $E^k/E$.

We denote the corresponding boundary hypersurface by

$$N_\lambda \cong M(D_{0\lambda}) \times B(D_{\lambda k}),$$

and we note in passing the identification

$$D_{0\lambda} = E^k / D_\lambda \cong E^{k_1} / E \times \cdots \times E^{k_n} / E$$

where $k_1, \ldots, k_n$ are the sizes of the blocks in the partition $\lambda$, while $D_{\lambda k} \cong E^n / E$.

From Theorem 2.7 and the fact that diagonal $D_\lambda$ is contained in $D_\mu$ if and only if $\mu$ refines $\lambda$ (in case we write $\mu \leq \lambda$), the codimension $d$ boundary faces of $M(E^k/E)$ have the form

$$N_{\lambda_1} \cap \cdots \cap N_{\lambda_d} \cong M(D_{0\lambda_1}) \times B(D_{\lambda_1\lambda_2}) \times \cdots \times B(D_{\lambda_{d-1}\lambda_d})$$

(2.10)

for totally ordered chains $0 \leq \lambda_1 < \cdots < \lambda_d < k$. Here the many body structure on $D_{\lambda\mu} = D_\lambda / D_{\mu}$ is understood to be the set $\{D_{\lambda\mu} : \mu \leq k \leq \lambda\}$.

In particular, the free boundary $B(E^k/E) = N_0$ has as its boundary hypersurfaces the spaces $N_0 \cap N_\lambda \cong B(D_{0\lambda})$.

### 2.2. Divergent Sequences of Configurations

The motivation for compactifying $E^k$ this way comes from the discussion of divergent sequences of monopoles in the Introduction. There we noted that divergence of a sequence in $\mathcal{M}_k$ always corresponds to divergent configurations of points, which are essentially the centers of monopoles of lower charge. Let us consider the role of compactifications of $E^k/E$ in handling such divergent sequences of configurations in the case $k = 3$.

If $((p_{1s}, p_{2s}, p_{3s}) : s \in \mathbb{N})$ is a divergent sequence in $E^3/E$, then after passing to a subsequence it has a limit in the interior of some boundary face of $M(E^3/E)$, which we suppose for simplicity has codimension one. In the case that the mutual separations $|p_{1s} - p_{2s}|$ all diverge as $s \to \infty$, then the limit lies on the free boundary $B(E^3/E)$, the interior of which is identified with the sphere $\partial E^3/E$. This limit encodes the relative displacements $\lim_{s \to \infty} \frac{p_{1s}' - p_{2s}'}{|p_{1s}' - p_{2s}'|}$ up to overall translation. (This can be done explicitly, for example, by using translation freedom to set $p_{3s}' = 0$.)

On the other hand, if one of the separations remains bounded, say $|p_{1s}' - p_{2s}'| < \infty$, then the sequence remains in a neighbourhood of the diagonal $D_{\lambda k}$, where $\lambda$ is the partition $\{\{1, 2\}, \{3\}\}$, and the limit lies on the boundary hypersurface $N_\lambda = M(D_{0\lambda}) \times B(D_{\lambda k})$. We may think of this as the 2 particle cluster $(p_{1s}', p_{2s}')$ diverging from the third particle $p_{3s}'$. In this situation, the
sequence of relative configurations \( w^s = (w^s_0, w^s_1) = (\frac{1}{2}(p^s_1 + p^s_2), p^s_3) \in E^2/E \) of their centers of mass converges to a limit in \( B(D_{\lambda k}) \), under the identification of \( D_{\lambda k} \) with \( E^k/E \), while the (recentered) cluster \( p^s = (p^s_1 - w^s_0, p^s_2 - w^s_0) \) converges to a limit \( p \in E^2/E \cong D_{0\lambda} \), the interior of \( M(D_{0\lambda}) \). Thus
\[
(p^s_1, p^s_2, p^s_3) \equiv (w^s_0, w^s_1, p^s) \to (w, p) \in B(D_{\lambda k}) \times M(D_{0\lambda}) = N_\lambda,
\]
where \( w = \lim_{s \to \infty} \frac{w^s_0 - w^s_1}{|w^s_0 - w^s_1|} \).

Note that were we to use the radial compactification only, we would only retain the limit \( w \) on the boundary of \( D_{\lambda k} \) inside \( \partial E^3/E \). The information about the relative limiting configuration \( p \) of the 2 particle cluster would be lost.

In our compactification of \( \mathcal{M}_k \), the relative configurations \( w^s = (w^s_1, \ldots, w^s_n) \) of points in \( E^n/E \) will be retained, but the role of an \( n \)-particle cluster in the preceding discussion will be replaced by a charge \( n \) monopole.

2.3. Unordered Configuration Spaces. The symmetric group \( \Sigma_k \) acts on \( E^k \) and \( E^k/E \) by permutation of the factors. The quotient spaces are singular, and we shall not consider them directly. However, the action is free on sufficiently small collar neighbourhoods \( U \) of \( B(E^k/E) \) and we need to understand \( U/\Sigma_k \), essentially because the configurations of points that emerge from divergent sequences of monopoles are unordered. More precisely, if a monopole of charge \( k_j \) is attached to \( p_j \), then points \( p_j \) carrying monopoles of the same charge must be regarded as indistinguishable.

Recall that the boundary hypersurfaces of \( B := B(E^k/E) \) are labelled by partitions \( \lambda \), \( 0 < \lambda < k \). For such \( \lambda \) we have the subgroup \( \Sigma_\lambda \),
\[
\sigma \in \Sigma_\lambda \Leftrightarrow \sigma \text{ leaves } D_\lambda \text{ invariant.} \tag{2.11}
\]
Then \( \Sigma_\lambda \) is the natural group of symmetries acting on \( (E^k/E)/D_\lambda \) and the stabilizer of the generic point is just \( \{1\} \). Combinatorially,
\[
\Sigma_\lambda = \{ \sigma \in \Sigma_k : i \sim_\lambda j \Leftrightarrow \sigma(i) \sim_\lambda \sigma(j) \}, \quad \text{Stab}_{\Sigma_k}(D_\lambda) = \{ \sigma \in \Sigma_k : \sigma(i) \sim_\lambda i \}.
\]

Here we have written \( i \sim_\lambda j \) to mean that \( i \) and \( j \) are in the same block of \( \lambda \). The group
\[
\Sigma_\lambda = \Sigma_k / \text{Stab}_{\Sigma_k}(D_\lambda) \tag{2.12}
\]
is the group of symmetries of the diagonal \( D_\lambda \). Informally, we think of this as the group of symmetries of configurations of type \( \lambda \): if \( r(\lambda) = n \), then \( \Sigma_n \) acts on the \( n \) blocks of \( \lambda \) and \( \Sigma_\lambda \) is the subgroup in which two blocks can be switched only if they have the same size. From this description, it is clear that if \( n_j \) denotes the number of blocks of size \( j \), then \( \Sigma_\lambda \) is the product of symmetric groups \( \Sigma_{n_1} \times \cdots \times \Sigma_{n_k} \). In particular \( \Sigma_{00} = \Sigma_k \) and \( \Sigma_{k0} = \{1\} \).

Note that the action of \( \Sigma_k \) upon \( B(E^k/E) \) is free, and the quotient provides a good definition of unordered ideal configurations of \( k \) points in \( E \). We may choose a \( \Sigma_k \)-invariant product neighbourhood \( U \) of \( B(E^k/E) \) on which the action is still free (it suffices to stay away from all diagonals). Then \( U/\Sigma_k \) is a space of widely-separated (and ideal) unordered configurations of \( k \) points in \( E \).

The boundary hypersurfaces of \( M(E^k/E)/\Sigma_k \) and \( B(E^k/E)/\Sigma_k \) are labelled by the types \([\lambda]\) of partitions of \( k \). In order to describe the boundary hypersurface \( N_\lambda \) of \( M(E^k/E)/\Sigma_k \) (which are both orbifolds, though their singularities will be of no concern here), pick \( \lambda \) such that \([\lambda] = a \). The subgroup \( \Sigma_\lambda \) then acts on the boundary hypersurface \( N_\lambda \) and \( N_0 = N_\lambda/\Sigma_\lambda \). This quotienting can be carried out in two stages, corresponding to the exact sequence
\[
\{1\} \to \text{Stab}_{\Sigma_k}(D_\lambda) \to \Sigma_\lambda \to \text{Sym}_\lambda \to \{1\} \tag{2.13}
\]
and the fibred structure of \( N_\lambda \)
\[
N_\lambda = M(D_{0\lambda}) \times B(D_{\lambda k}) \to B(D_{\lambda k}), \tag{2.14}
\]
(see Theorem 2.7). It is clear that the subgroup \( \text{Stab}_{\Sigma_k}(D_\lambda) \) of \( \Sigma_\lambda \) consists precisely of those permutations which cover the identity on the base \( B(D_{\lambda k}) \) and the quotient is
\[
(M(D_{0\lambda})/\text{Stab}_{\Sigma_k}(D_\lambda)) \times B(D_{\lambda k}). \tag{2.15}
\]
Remark 3.3. Iterated boundary fibrations arise naturally in resolving smooth group actions on vary slightly between these references. Unfortunately, the notational conventions and terminology

body compactifications of Vasy [25] are also highly relevant examples, though the above formal

QAC spaces [8]. It is the latter applications that are the most relevant here. The resolved many-

many-manifolds [2], resolving stratified pseudomanifolds, [1, 9] and in compactification of QALE and

boundary hypersurface of \(B\) and so it would be too much to assume in part (iii) of the definition that

\[\Sigma_{\lambda} \subset \Sigma_{\lambda}\]

invariant, and denote by \(\text{Sym}_{\lambda}\nu\) the quotient \(\Sigma_{\lambda}/\text{Stab}_{\Sigma}(D_{\lambda})\), where \(\text{Stab}_{\Sigma}(D_{\lambda})\) is the stabilizer of the flag. When we divide \(M(\partial_{\lambda}/\partial_{\nu})\) by \(\Sigma_{\nu}\), the boundary hypersurfaces are labelled by the \(\Sigma_{\nu}\)-orbits of partitions \(\lambda < \nu\). Then \(\Sigma_{\lambda}\nu\) acts on the hypersurface \(N_{\lambda\mu} = M(\partial_{\lambda}/\partial_{\lambda}) \times B(\partial_{\lambda}/\partial_{\nu})\), with quotient the boundary hypersurface

\[N_{(\lambda\nu)} = N_{\lambda\nu}/\Sigma_{\lambda}\nu\]

Again, this quotient can be performed in two stages, dividing first by \(\text{Stab}_{\Sigma}(D_{\lambda})\) and then by \(\text{Sym}_\nu\). The construction can be generalized in straightforward fashion to corners of higher codimension in these many-body spaces.

3. IBF Structures and Compatible \(\Phi\)-Metrics

In this section we recall some definitions and terminology which provide a general framework within which we shall describe the structure of the compactification \(\mathcal{M}_k\) of \(\mathcal{M}_k\).

The fibred structure of the boundary hypersurfaces of \(\mathcal{M}_k\) (and the way the fibrations fit together at the corners) is an example of an iterated boundary fibration structure [2]. Here is the definition:

**Definition 3.1.** We say that \(M\) has an iterated boundary fibration (IBF) if

(i) \(M\) is a manifold with corners, with boundary hypersurfaces denoted \(N_{\lambda}\), for \(\lambda\) in some index set \(I\);

(ii) Every boundary hypersurface \(N_{\lambda}\) of \(M\) is equipped with a fibration \(\phi_{\lambda} : N_{\lambda} \to B_{\lambda}\), where \(B_{\lambda}\) and the fibre \(F_{\lambda}\) of \(\phi_{\lambda}\) are manifolds with corners;

(iii) If \(N_{\lambda} \cap N_{\mu} \neq \emptyset\), then \(\dim B_{\lambda} \neq \dim B_{\mu}\). If without loss of generality \(\dim B_{\lambda} > \dim B_{\mu}\), then \(\phi_{\lambda}(N_{\lambda} \cap N_{\mu})\) is a disjoint union of boundary hypersurfaces of \(B_{\lambda}\) with full fibre \(F_{\mu}\), and \(\phi_{\mu}\) maps \(N_{\lambda} \cap N_{\mu}\) surjectively to the base \(B_{\mu}\), with fibre a boundary hypersurface (or disjoint union thereof) of \(F_{\mu}\). Finally there is a fibration \(\phi_{\lambda\mu} : \phi_{\lambda}(N_{\lambda} \cap N_{\mu}) \to B_{\mu}\) which satisfies the compatibility condition

\[
\begin{array}{ccc}
N_{\lambda} \cap N_{\mu} & \xrightarrow{\phi_{\lambda}} & \phi_{\lambda}(N_{\lambda} \cap N_{\mu}) \\
\phi_{\mu} & \downarrow & \\
B_{\mu} & \xrightarrow{\phi_{\lambda\mu}} & \\
\end{array}
\]

**Remark 3.2.** We have phrased the definition slightly differently from [2 Definition 3.3] by avoiding the notion of ‘collective boundary hypersurface’. Recall that by definition a boundary hypersurface of a manifold with corners is connected. However, \(N_{\lambda} \cap N_{\mu}\) need not be connected and so it would be too much to assume in part (iii) of the definition that \(\phi_{\lambda}(N_{\lambda} \cap N_{\mu})\) is a single boundary hypersurface of \(B_{\lambda}\).

**Remark 3.3.** Iterated boundary fibrations arise naturally in resolving smooth group actions on manifolds [2], resolving stratified pseudomanifolds, [1, 9] and in compactification of QALE and QAC spaces [8]. It is the latter applications that are the most relevant here. The resolved many-body compactifications of Vasy [26] are also highly relevant examples, though the above formal definition was not discussed there. Unfortunately, the notational conventions and terminology vary slightly between these references.

\[\text{In [2] the inequality is equivalently expressed in terms of the dimension of the fibres rather than the dimension of the base}\]
If \( M \) has an IBF structure, then there is a natural partial order on \( I \) defined by the condition:

\[
\lambda < \mu \iff N_\lambda \cap N_\mu \neq \emptyset \quad \text{and} \quad \dim F_\lambda < \dim F_\mu.
\]

(3.1)

This ordering gives a notion of ‘depth’, where \( N_\mu \) is of greater depth than \( N_\lambda \) if \( \lambda < \mu \). Every corner of \( M \) of codimension \( m \) is then a connected component of an intersection of \( m \) boundary hypersurfaces, and such intersections correspond precisely to a chain of length \( m \) in the partially ordered set \( I \).

**Remark 3.4.** From the definition of IBF, if \( N_\mu \) is any boundary hypersurface, then its boundary hypersurfaces are of two kinds: Firstly, those that are connected components of \( N_\lambda \cap N_\mu \) with \( \lambda < \mu \) fibre over \( B_\mu \) and fit into the picture

\[
\begin{array}{ccc}
F_{\lambda\mu} & \longrightarrow & N_\lambda \cap N_\mu \\
\downarrow \phi_\mu & & \downarrow \phi_\mu \\
B_\mu & & B_\mu
\end{array}
\]

(3.2)

obtained by restricting \( \phi_\mu \) to a connected component of \( N_\lambda \cap N_\mu \) and where \( F_{\lambda\mu} \) is a disjoint union of boundary hypersurfaces of \( F_\mu \). The other kind is a connected component of \( N_\mu \cap N_\nu \), where \( \mu < \nu \) and is the total space of a connected component of the fibration

\[
\begin{array}{ccc}
F_\mu & \longrightarrow & N_\mu \cap N_\nu \\
\downarrow \phi_\mu & & \downarrow \phi_\mu \\
B_\mu & & B_\mu
\end{array}
\]

(3.3)

again obtained by restricting \( \phi_\mu \), where this time \( B_{\mu\nu} \) is a disjoint union of boundary hypersurfaces of \( B_\mu \). Thus the boundary hypersurfaces of \( F_\mu \) are (connected components of) the \( F_{\lambda\mu} \) with \( \lambda < \mu \) and the boundary hypersurfaces of \( B_\mu \) are (connected components of) the \( B_{\mu\nu} \) with \( \mu < \nu \), and together they give the boundary hypersurfaces of \( N_\mu \). We shall stick to this notation in what follows.

We also observe that Definition 3.1 induces an IBF structure on each fibre \( F_\lambda \) and base space \( B_\lambda \), the boundary hypersurfaces of which are indexed by \( \{ \mu \in I : \mu < \lambda \} \) and \( \{ \mu \in I : \mu > \lambda \} \), respectively.

**Example 3.5 (Many-body Spaces).** As an instructive example, let us consider how the many-body compactification \( M(V^k / V) \) is endowed with an iterated boundary fibration. As noted above, the boundary hypersurfaces \( N_\lambda \) of \( M(V^k / V) \) are indexed by partitions \( \lambda \) and have the form

\[
N_\lambda = B(D_{\lambda k}) \times M(D_{0\lambda})
\]

Either of the two factors work as the bases of the boundary fibrations, as long as we make a consistent choice; in light of the construction of \( M_k \) (and as suggested by the notation), we take \( \phi_\lambda \) to be the projection onto \( B_\lambda := B(D_{\lambda k}) \), with fibre \( F_\lambda := M(D_{0\lambda}) \).

If \( \lambda < \mu \), then

\[
N_\lambda \cap N_\mu = B_\mu \times X_{\lambda\mu} \times F_\lambda := B(D_{\mu k}) \times B(D_{\lambda\mu}) \times M(D_{0\lambda}).
\]

The compatibility condition between the fibrations \( \phi_\mu \) and \( \phi_\lambda \) reads

\[
\begin{array}{ccc}
N_\lambda \cap N_\mu & \longrightarrow & B_\mu \times X_{\lambda\mu} \\
\downarrow \phi_\mu & & \downarrow \phi_\mu \\
\phi_\lambda N_\lambda \cap N_\mu & = & \phi_\lambda N_\lambda \cap N_\mu
\end{array}
\]

and we identify \( B_\mu \times X_{\lambda\mu} \) as a boundary face of \( B_\lambda \), and likewise \( X_{\lambda\mu} \times F_\lambda \) as a boundary face of \( F_\mu \).
3.1. Product Structures and Construction of an Adapted Cover.

**Definition 3.6.** Let $M$ be a compact manifold with IBF and boundary hypersurfaces indexed as above. Then a **boundary product structure** consists of the following data. For each boundary hypersurface $N_\lambda$, an open neighbourhood $U_\lambda$, a smooth boundary-defining function $\rho_\lambda$ and a smooth vector field $v_\lambda$ defined in $U_\lambda$, such that, for any pair $\lambda \neq \mu$,

$$v_\lambda \rho_\lambda = 1 \text{ but } v_\lambda \rho_\mu = 0 \text{ in } U_\lambda \cap U_\mu$$  \hspace{1cm} (3.4)

and

$$[v_\lambda, v_\mu] = 0 \text{ in } U_\lambda \cap U_\mu.$$  \hspace{1cm} (3.5)

Such a boundary product structure is said to be **compatible** with the IBF if for each pair $N_\lambda, N_\mu$ of intersecting hypersurfaces with $\lambda < \mu$,

$$\rho_\mu | N_\lambda \in \phi_\lambda \mathcal{C}^\infty(B_\lambda) \text{ near } N_\mu$$  \hspace{1cm} (3.6)

$$v_\mu | N_\lambda \text{ is } \phi_\lambda\text{-related to a vector field on } B_\lambda \text{ near } N_\mu$$  \hspace{1cm} (3.7)

$$v_\lambda | N_\mu \text{ is tangent to the fibres of } \phi_\mu.$$  \hspace{1cm} (3.8)

It is shown in [2, Prop. 1.2, 3.7] that such compatible boundary product structures always exist. The argument is inductive, a key point being that if $M$ is a manifold with IBF, then in the fibration $\phi_\lambda : N_\lambda \rightarrow B_\lambda$ the base $B_\lambda$ (and the fibre $F_\lambda$) inherit an IBF structure by virtue of the definition.

Notice that the flow of the vector field $v_\lambda$ defines a retraction of $U_\lambda$ to $N_\lambda$ and so a diffeomorphism from a set of the form $\{\rho_\lambda < \delta\}$ onto $N_\lambda \times [0, \delta]$.

Let us construct a similar type of compatible boundary structure where the covering sets $W_\lambda$ are not product neighbourhoods of the $N_\lambda$ but still of a very useful form:

**Proposition 3.7.** Let $M$ be a compact MWC with IBF and compatible boundary product structure. There is a cover $\{W_\lambda\}$ of a neighbourhood of $\partial M$ so that the restriction of $\phi_\lambda$ to $W_\lambda \cap N_\lambda$ is surjective, each fibre of $\phi_\lambda$ meets $W_\lambda$ in a relatively compact subset of $F_\lambda^0$ and $W_\lambda \cap W_\mu = \emptyset$ if $\lambda$ and $\mu$ are not comparable in $I$.

**Proof.** Minimal elements in $I$ need not be unique, but the corresponding boundary hypersurfaces must be disjoint. If $\lambda$ is a minimal element of $I$ then the fibre $F_\lambda$ of $\phi_\lambda$ must be a compact boundaryless manifold by Remark 3.4. For such $\lambda$, we take $W_\lambda$ to be a product neighbourhood $U_\lambda$ of $N_\lambda$, as per the above definition. And naturally we may assume (and will do so) that these $W_\lambda$ are disjoint.

Now let $\Lambda \subset I$ be some subset with the property

$$\mu \in \Lambda, \lambda \prec \mu \Rightarrow \lambda \in \Lambda.$$

Suppose that the $W_\lambda$ have been constructed for all $\lambda \in \Lambda$. We can now construct $W_{\mu_0}$ for any minimal element $\mu_0$ of $I \setminus \Lambda$ as follows. The minimality condition means that if $\lambda \prec \mu_0$, then $\lambda \in \Lambda$. Hence the intersections $W_\lambda \cap N_{\mu_0}$ for $\lambda \prec \mu$ cover the union of the $N_\lambda \cap N_{\mu_0}$ for $\lambda \prec \mu_0$ and in particular the entire boundary of each fibre $F_{\mu_0}$, again by Remark 3.4. Because the base is compact, we can choose an open subset $W_{\mu_0}^0$ of $N_{\mu_0}$ which meets each fibre in a relatively compact open subset such that the union of $W_{\mu_0}^0$ with all the intersections $W_\lambda \cap N_{\mu_0}$ covers $N_{\mu_0}$. Now use $v_{\mu_0}$ to push $W_{\mu_0}^0$ out into $M$ and denote this product neighbourhood of $W_{\mu_0}^0$ by $W_{\mu_0}$. Provided we don’t push out too far, this set will not intersect any $W_\lambda$ with $\lambda$ not comparable to $W_{\mu_0}$. This completes the inductive step. \hfill $\Box$

3.2. $\Phi$-Tangent Bundle. Let $M$ be a compact manifold with an IBF. Suppose that $\{\rho_\lambda\}$ is a collection of compatible boundary-defining functions and let $\rho$, the total boundary-defining function, be the product of the $\rho_\lambda$.

Recall that on any MWC $M$, $\text{Vect}_b(M)$ is the space of all smooth vector fields which are tangent to all boundary hypersurfaces. There is a vector bundle $bTM \rightarrow M$, whose restriction to the interior is canonically isomorphic to $TM^0$, and such that $\mathcal{C}^\infty(M, bTM) = \text{Vect}_b(M)$.

**Definition 3.8.** The vector field $v \in \text{Vect}_b(M)$ is called a **$\Phi$-vector field** if
• $v|N$ is tangent to the fibres of $\phi_N$, or equivalently $(\phi_N)_*v = 0$ for every boundary hypersurface $N$;
• $v(\rho) \in \rho^2C^\infty(M)$.

Roughly speaking, $v$ is bounded in the fibre directions and scales in an asymptotically conic fashion in the base directions. The space of all $\Phi$-vector fields will be denoted by $\text{Vect}_\Phi(M)$ and as in the case of $b$-vector fields there is a $C^\infty$ vector bundle $\Phi TM$ with the property that $C^\infty(M, \Phi TM) = \text{Vect}_\Phi(M)$.

**Remark 3.9.** The first item does not depend upon the choice of $\rho$, but the second one does. $\text{Vect}_\Phi(M)$ is the same as the algebra of vector fields defined in [8], we have given a different but equivalent definition. In [9] there is a closely related definition which is, however, not equivalent.

We have chosen to call these $\Phi$-vector fields over ‘QFB’ vector fields because the present notion is the natural generalization of the initial use of ‘$\Phi$’ in Mazzeo–Melrose [17] to manifolds with fibred boundary. When $M$ has a single boundary hypersurface $N$, we recover the original notion.

**Definition 3.10.** Given a compact manifold with IBF, a $\Phi$-metric is a smooth (up to all boundary hypersurfaces) metric on the bundle $\Phi TM$.

**Remark 3.11.** It follows easily from the definition that the restriction of a $\Phi$ vector field (resp. $\Phi$-metric) to a fibre $F_\lambda$ of a boundary hypersurface of $M$ is again a $\Phi$ vector field (resp. $\Phi$-metric) on $F_\lambda$, with respect to its IBF structure induced from that on $M$.

**Proposition 3.12.** Suppose that $M$ is a compact manifold with IBF and let $g$ be a compatible $\Phi$-metric on $M$. Then, denoting the boundary fibrations etc. as before, there exists a finite open cover $\{V_0\} \cup \{V_\lambda\}_{\lambda \in I}$ of $M$ and a partition of unity $\chi_\lambda$ subordinate to this cover so that, for $\lambda \neq 0$, $\phi_\lambda(V_\lambda \cap \partial M) = B_\lambda$, $V_\lambda \cap \phi_\lambda^{-1}\{p\} \subset F_\lambda^0$ is relatively compact, and such that all $|\nabla \chi_\lambda|_g$ are uniformly bounded.

We have already constructed a cover $\{W_\lambda\}$ of $\partial M$ with the some of the required properties, cf. Proposition [8]. Thus we only need to extent this to cover all of $M$ and show that any partition of unity subordinate to such a cover has uniformly bounded derivatives. But this does not depend on the specific choice of cover and is true on more general grounds:

**Proof.** So let $M$ be a compact MWC with IBF, compatible $\Phi$-metric $g$ and boundary fibrations denoted as before, i.e., $\phi_\lambda : N_\lambda \to B_\lambda$ with fibre $F_\lambda$ where $\lambda \in I$ indexes the boundary hypersurfaces of $M$. Suppose $\{W_\lambda\}$ is the cover of $\partial M$ constructed in Section [8]. Then, the sets $W_\lambda$ already satisfy $\phi_\lambda(W_\lambda \cap \partial M) = B_\lambda$, and the sets $\phi_\lambda^{-1}\{p\} \cap W_\lambda$ are relatively compact in $F_\lambda^0$. Thus, the cover is global in the base and local in the fibres. Finally, let $V_\lambda = W_\lambda$, for $\lambda \in I$ and $V_0$ be an open neighbourhood of $M \setminus \bigcup_{\lambda \in I} W_\lambda$.

Now let $\chi_\lambda$ be a smooth partition of unity subordinate to $\{V_0\} \cup \{V_\lambda\}_{\lambda \in I}$. Since the $\chi_\lambda$ are smooth, so too are the 1-forms $d\chi_\lambda$. Any smooth 1-form is also a smooth section of $\Phi^*TM$, hence the functions $|d\chi_\lambda|^2_g : M \to \mathbb{R}$ are smooth, compactly supported and there are only finitely many of them. Thus, they are uniformly bounded on $M$. □

The significance of this result is that such good covers with controlled partitions of unity are required in the proof of the coprime case of the Sen Conjecture presented in [6]. In particular, they allow us to use a Mayer-Vietoris argument to compute the smooth-$L^2$-de Rham cohomology of $M$.

**Remark 3.13.** Using the specific structure of the cover $\{V_\lambda\}$ and of $\Phi^*TM$, we could in fact deduce more precise information about the behaviour of $|d\chi_\lambda|^2_g$ near $\partial M$. But this amount of detail will not be needed in what is to follow.

4. The Moduli Space Compactification

4.1. Monopoles. We recall the precise definition of the moduli space of framed euclidean monopoles of charge $k$, from a point of view that makes subsequent generalizations natural.
First of all, we shall use \( \mathcal{C} \) for the space of ‘monopole data’ on a radially compactified euclidean 3-dimensional euclidean space \( E = \mathbb{R}^3 \): thus \( \mathcal{C} \) consists of pairs \( (A, \Phi) \) where \( A \) is an \( SU(2) \) connection on a bundle \( P \to E \) and \( \Phi \) is a smooth section of the adjoint bundle. Here smooth means ‘up to and including the boundary’: to say \( A \) is smooth is to say that in any smooth local trivialization of \( P \) the connection 1-form is smooth (including neighbourhoods of boundary points of \( E \)). We shall use obvious variations of the notation such as \( \mathcal{C}(E, P) \) if we need to make the base space or bundle explicit. The gauge group \( \mathcal{G} \) consists of automorphisms of \( P \) which again are smooth over \( E \). This group is the natural infinite-dimensional symmetry group of \( \mathcal{C} \): it acts by pull-back.

The (euclidean) Bogomolny equations on \( \mathcal{C} \) are

\[
\mathcal{B}(A, \Phi) = \nabla_A \Phi - *F_A,
\]

where \( F_A \) is the curvature of \( A \) and \( * \) is the euclidean Hodge star operator. The equations are gauge-invariant and a naïve definition of the monopole moduli space would be to divide the zeros of \( \mathcal{B} \) by the action of \( \mathcal{G} \).

The framed moduli space \([\mathcal{C}][16]\) is a refinement in which we fix \((A, \Phi)\) up to and including \( O(\rho) \) terms, where \( \rho \) is the standard defining function of the boundary of the radia-compactification (reciprocal of distance from 0). We denote by \((A_b, \Phi_b)\) these fixed data defined near the boundary and define

\[
\mathcal{C}_{Fr} = \{(A, \Phi) \in \mathcal{C} : (A - A_b, \Phi - \Phi_b) = O(\rho^2)\}
\]

where we use the euclidean metric to measure the length of the 1-form \( A - A_b \). There is a corresponding framed gauge group \( \mathcal{G}_{Fr} \) consisting of those gauge transformations which preserve \( \mathcal{C}_{Fr} \) and equal to the identity on the boundary. The framing data \((A_b, \Phi_b)\) are essentially given by an abelian monopole, and hence there is an associated topological charge \( k \), the winding number of \( \Phi_b \). We shall denote a framing of charge \( k \) by \( Fr_k \).

Consider the 1-parameter subgroup \( \gamma_1 = \exp(i\Phi/2) \) of gauge transformations. Then \( \gamma_1 \in \mathcal{C}_{Fr} \).

Thus \( g_{2\pi} \) acts as the identity on the framed moduli space. So with the factor of 2, we have an action of \( T = \mathbb{R}/2\pi\mathbb{Z} \) on the framed moduli space (cf. \([3]\)).

**Definition 4.1.** The framed moduli space of euclidean charge-\( k \) monopoles is defined as follows

\[
\mathcal{M}_k = \{(A, \Phi) \in \mathcal{C}(E, P; Fr_k) : \mathcal{B}(A, \Phi) = 0\}/\mathcal{G}_{Fr}.
\]

**Remark 4.2.** Any two choices of framing are gauge-equivalent (though not by an element of \( \mathcal{G}_{Fr} \)). Thus different choices of framing in the definition of \( \mathcal{M}_k \) lead to diffeomorphic framed moduli spaces.

4.1.1. Properties of \( \mathcal{M}_k \). It is known that \( \mathcal{M}_k \) is a smooth manifold of dimension \( 4k \). It carries a natural \( L^2 \) metric, \( g_k \), say, which is complete and hyperKähler. The reader is referred to \([3]\) and references therein for these standard facts.

On \( \mathcal{M}_k \) there are important isometries: those induced by translations of \( \mathbb{R}^3 \) and the action of \( T \) by ‘frame rotation’, induced by the 1-parameter group \( t \mapsto \gamma_t \) described above. Accordingly there is a reduced moduli space \( \mathcal{M}_k = \mathcal{M}_k/\mathbb{R}^3 \) of dimension \( 4k - 3 \). Factoring out by the translations is essentially the same as restricting to monopoles with centre at \( 0 \in \mathbb{R}^3 \). One definition of the centre (of mass) of a monopole is in terms of the \( T \)-action on \( \mathcal{M}_k \). This is a triholomorphic isometry with a hyperKähler moment map \( c : \mathcal{M}_k \to \mathbb{R}^3 \). This map is a submersion and \( c^{-1}(0) \) is the submanifold of monopoles centred at \( 0 \). If \( m \in \mathcal{M}_k \), there is a unique \( p \in \mathbb{R}^3 \) so that \( m(-p) \in c^{-1}(0) \). Thus the quotient \( \mathcal{M}_k \) and \( c^{-1}(0) \subset \mathcal{M}_k \) are essentially interchangeable.

This second point of view shows that the reduced moduli space

\[
\mathcal{M}_k^0 = \mathcal{M}_k/\mathbb{T} = c^{-1}(0)/\mathbb{T}
\]

is again hyperKähler, of dimension \( 4k - 4 \). We can equally define \( \mathcal{M}_k^0 = \mathcal{M}_k/(\mathbb{R}^3 \times \mathbb{T}) \), for the two actions of \( \mathbb{R}^3 \) and \( T \) on \( \mathcal{M}_k \) commute with each other. It is known that \( \pi_1(\mathcal{M}_k^0) = \mathbb{Z}_k \) (cf.
It is natural to think of $I$-integer $k$ are a compactification $\bar{\mathcal{M}}$ type is $a$ covers the action of $\text{Sym}^\nu_1 \ldots \text{Sym}^\nu_d$ and a certain rank-$r(\nu_1, \ldots, \nu_d)$ structure of the compactification $\bar{\mathcal{M}}$. The condition that $\phi$ and $\psi$ have no common factors is equivalent to the non-vanishing of the resultant

$R(\phi, \psi) = \prod_j \phi(\beta_j)$

where the $\beta_j$ are the roots of $\psi$. $\mathcal{R}$ is homogeneous of degree $k$ in the coefficients $(a_0, \ldots, a_{k-1})$. With this description, $\mathcal{M}$ is the subspace of rational maps with

$b_{k-1} = 0, |R(\phi, \psi)| = 1.$

The $T$-action is just $\phi \mapsto \lambda \phi, |\lambda| = 1$. The strongly centred space $\mathcal{M}_k$ is then given by the conditions

$b_{k-1} = 0, R(\phi, \psi) = 1.$

The subgroup $\mathbb{Z}_k \subset T$ preserves these conditions because of the observation about the homogeneity of $\mathcal{R}$ in the coefficients of $\phi$. This $\mathbb{Z}_k$ is the group of deck transformations, and factoring out by it gives the space $\mathcal{M}_k^0/\mathbb{Z}_k = \mathcal{M}_k^0 = \mathcal{M}_k/T$.

Note that the metric is not easy to describe in this picture. As observed by Atiyah and Hitchin in [3], p. 19, the description of $\mathcal{M}_k^0$ in terms of rational maps exhibits it as a dense open subset of $\mathbb{C}P^{k-1} \times \mathbb{C}P^{k-1}$. This observation certainly gives a compactification of $\mathcal{M}_k^0$, but this will be different from ours and is unlikely to have good properties with respect to the monopole metric.

### 4.2. Ideal Monopoles and the Boundary Hypersurfaces of $\mathcal{M}_k$.

We now describe the structure of the compactification $\mathcal{M}_k$ of $\mathcal{M}_k$ in detail. For each partition $a = (k_1, \ldots, k_n)$ of the integer $k$, there is a boundary hypersurface $\mathcal{I}_a$, and $\mathcal{I}_a$ meets $\mathcal{I}_b$ if and only if the partitions $a$ and $b$ are comparable. In order to describe $\mathcal{I}_a$ it is best to choose a partition $\nu$ of $k$ whose type is $a$ (so that the sizes of the blocks of $\nu$ are the integers $k_1, \ldots, k_n$). Then we have

$$\mathcal{I}_a = \mathcal{I}_\nu / \text{Sym}_\nu,$$  \hspace{1cm} (4.5)

It is natural to think of $\mathcal{I}_\nu$ as an ‘ordered version’ of $\mathcal{I}_a$. The ingredients needed to define $\mathcal{I}_\nu$ are a compactification $\mathcal{M}_\nu$ of the product

$$\mathcal{M}_\nu = \mathcal{M}_{k_1} \times \mathcal{M}_{k_2} \times \cdots \times \mathcal{M}_{k_n},$$  \hspace{1cm} (4.6)

and a certain rank-$r(\nu)$ torus-bundle

$$\mathcal{T}_\nu \hookrightarrow B_{\nu k},$$  \hspace{1cm} (4.7)

where $B_{\nu k}$ is the boundary of the free region of $\mathcal{M}(D_{\nu}/D_k)$ as before. This torus-bundle has the property that it can be chosen to admit an action of $\text{Sym}_{\nu}$ which permutes the factors and covers the action of $\text{Sym}_{\nu}$ on $B_{\nu k}$. Similarly, the $T^{r(\nu)}$- and permutation-action of $\text{Sym}_{\nu}$ on $\mathcal{M}_\nu$ extend smoothly to $\mathcal{M}_\nu$. Using the $T^{r(\nu)}$-action, we define

$$\mathcal{I}_\nu = \mathcal{T}_{\nu k} \times_{T^n} \mathcal{M}_\nu,$$  \hspace{1cm} (4.8)

this space inherits an action of $\text{Sym}_{\nu}$, allowing us to define $\mathcal{I}_a$ as the quotient $[4,5]$. The torus-bundle $\mathcal{T}_{\nu k}$ appeared in [10] and in Bielawski’s work and is called a (generalized) Gibbons–Manton bundle; the version for the ‘free’ partition $(1, \ldots, 1)$ appears in the original paper [12] in the description of this asymptotic region of $\mathcal{M}_k$. We shall recall the definition in a moment, but
pause first to note that (4.8) is clearly incomplete without a definition of the compactification $\tilde{M}_\nu$. This, however, has a description very analogous to that of $\tilde{M}_k$ itself, where the boundary hypersurfaces are now finite quotients of spaces

$$\tilde{\mathcal{I}}_{\lambda \nu} = \tilde{M}_\lambda \times_{\nabla^{\nu}(\lambda)} \tilde{\mathcal{I}}_{\lambda \nu}$$

where now $0 \leq \lambda < \nu$ and $\tilde{\mathcal{I}}_{\lambda \nu}$ is a Gibbons–Manton torus bundle of rank $r(\nu)$ over $B_{\lambda \nu}$, the boundary of the free region of $M_{\lambda \nu} = M(D_{\lambda}/D_\nu)$. Since (4.9) only involves the spaces $\tilde{M}_\lambda$ with $\lambda < \mu$, these spaces can be built up inductively starting from the free partition

$$\tilde{M}_0 = \tilde{M}_0 = \tilde{M}_1 \times \cdots \times \tilde{M}_4 = (\mathbb{S}^1)^k$$

and ending with $\tilde{M}_k$.

In the remainder of this section, we fill in the details, summarise the properties of the space $\tilde{M}_\mu$ that are needed to make the induction work. We also verify that the collection of manifolds $\{\tilde{\mathcal{I}}_{\lambda \mu} : 0 \leq \lambda < \mu\}$ satisfy the compatibility conditions needed for $\tilde{M}_\mu$ to be a compact MWC with IBF.

### 4.3. Gibbons–Manton Bundles

Let $\lambda < \nu$ be two partitions of the set $k$. The Gibbons–Manton bundle $\tilde{\mathcal{I}}_{\lambda \nu}$ is defined initially over the space

$$E_{\lambda \nu} = D_\lambda/D_\nu \setminus \bigcup_{\lambda < \nu \leq \nu} D_\nu/D_\nu.$$  

Recall that $D_\lambda \subset (E^0)^k$, where $E$ is our fixed radially compactified 3-dimensional euclidean space. For each $1 \leq i, j \leq k$, define the difference map

$$\pi_{ij} = p_i - p_j$$

so $\pi_{ij} : (E^0)^k \rightarrow E^0$.

Notice that $\pi_{ij}$ vanishes on $D_\nu$ if and only if $i \sim_\nu j$. Therefore,

**Lemma 4.3.** The difference map $\pi_{ij}$ is inducs a map on $E_{\lambda \nu}$ if and only if $i \sim_\nu j$, to be denoted by the same symbol. This induced map is non-zero on $E_{\lambda \nu}$ if and only if $i \not\sim_\nu j$ (but $i \sim_\nu j$).

Now denote by $\omega$ the $\SO(3)$-invariant closed 2-form on $E^0 \setminus 0$ whose de Rham class $[\omega]$ generates $H^2(E^0 \setminus 0, \mathbb{Z})$. For $1 \leq i \leq k$, define

$$\omega_i = 2 \sum_{i \sim_\nu j, \; i \not\sim_\nu j} \pi_{ij}^*(\omega).$$

Then $\omega_i$ is a closed integral 2-form on $E_{\lambda \nu}$ and one can find a circle-bundle $Q_i$ with connection $\alpha_i$ such that $d\alpha_i = 2\pi\sqrt{-1}\omega_i$. Since $E_{\lambda \nu}$ is simply connected, $(Q_i, \alpha_i)$ is unique up to isomorphism. From the definition of $\omega_i$, it is clear that $\omega_i = \omega_i'$ if $i \sim_\nu i'$, for the induced maps $\pi_{ij}$ and $\pi_{i'j}$ on $E_{\lambda \nu}$ are then equal. So pick a set of indices $i_1, \ldots, i_\ell (\ell = r(\lambda))$ such that $i_j$ is in the $j$-th block of $\lambda$ and define

$$\tilde{\mathcal{I}}_{\lambda \nu} = Q_{i_1} \times \cdots \times Q_{i_\ell}.$$  

This is the (generalized) Gibbons–Manton bundle (of type $\lambda \nu$). Note that its rank is equal to $r(\lambda)$. We note the following:

**Lemma 4.4.** As defined above, $\tilde{\mathcal{I}}_{\lambda \nu}$ extends uniquely from $E_{\lambda \nu}$ to the free region of $M_{\lambda \nu}$ and in particular to $B_{\lambda \nu}$.

**Proof.** This can be seen by noting that $\pi_{ij}$ extends to define a smooth map $M_{\lambda \nu} \rightarrow E$, which is non-zero on the free region, provided, of course, that

$$i \sim_\nu j, \; i \not\sim_\nu j.$$  

The form $\omega$ also extends smoothly to $E \setminus 0$. Thus $\omega_i$, as defined in (4.13), extends to define a smooth 2-form on the free region of $M_{\lambda \nu}$. Knowing this, it follows that the $Q_i$ admit smooth extensions to this free region as well. \end{proof}
For the compatibility conditions between the different $\mathcal{I}_{\lambda\nu}$ we shall need to understand the restriction of $\mathcal{I}_{\lambda\nu}$ to a boundary hypersurface of $B_{\lambda\nu}$. Recall that these boundary hypersurfaces are indexed by partitions $\mu$ with $0 \leq \mu < \nu$. The $\mu$-boundary hypersurface will be denoted by $\partial_\mu B_{\lambda\nu}$, and we have seen (Theorem 2.7) that

$$\partial_\mu B_{\lambda\nu} = B_{\lambda\mu} \times B_{\mu\nu}. \quad (4.15)$$

There are two natural torus-bundles over this space. One is the restriction of $\mathcal{I}_{\lambda\mu}$ and $\mathcal{I}_{\mu\nu}$, respectively of ranks $r(\lambda)$ and $r(\mu)$. Here $r(\mu) < r(\nu)$ and there is a canonical inclusion $T^{r(\mu)} \subset T^{r(\nu)}$ corresponding in the obvious way to the inclusion $D_\mu \hookrightarrow D_\lambda$. Thus $T^{r(\mu)}$ acts on both $\mathcal{I}_{\lambda\mu}$ and $\mathcal{I}_{\mu\nu}$ and we may define the rank-$r(\lambda)$ torus-bundle

$$\mathcal{I}_{\lambda\mu} \times T^{r(\nu)} \mathcal{I}_{\mu\nu} \cong B_{\lambda\mu} \times B_{\mu\nu}. \quad (4.16)$$

The key result is as follows:

**Lemma 4.5.** When $\partial_\mu B_{\lambda\nu}$ is identified with the product as in (4.15), we have

$$\mathcal{I}_{\lambda\nu} \mid \partial_\mu B_{\lambda\nu} \cong \mathcal{I}_{\lambda\mu} \times T^{r(\nu)} \mathcal{I}_{\mu\nu} \text{ over } B_{\lambda\mu} \times B_{\mu\nu}. \quad (4.17)$$

**Proof.** Starting from the identification $D_{\lambda\nu} \cong D_{\lambda\mu} \times D_{\mu\nu}$, we obtain the relation

$$E_{\lambda\nu} \cong E_{\lambda\mu} \times E_{\mu\nu}$$

between the free regions of $D_{\lambda\nu}$, $D_{\lambda\mu}$ and $D_{\mu\nu}$, since only diagonals of the form $D \times E_{\mu\nu}$ and $E_{\lambda\mu} \times D'$ are removed on the right-hand side.

Then, fixing a block of $\lambda$ with representative element $i$, and working complex line bundles $L_i$ instead of the circle-bundles $Q_i$, we have an

$$L_{i,\lambda\nu} \cong L_{i,\lambda\mu} \otimes L_{i,\mu\nu};$$

this follows at the level of Chern classes by splitting the sum (4.13) defining the LHS according as $j$ is or is not in the same $\mu$-block of $i$. As before, this extends to an isomorphism over the boundary hypersurface $\partial_\mu B_{\lambda\nu}$ as required. \hfill $\Box$

This completes our discussion of the generalized Gibbons–Manton bundles. Now that these are defined, (4.9) makes sense for any space $\mathcal{M}_\mu$ with a free $T^{r(\mu)}$-action.

### 4.4. Main Theorem

We are now nearly ready to state the main result, the proof of which will appear in Part II. First recall the definitions $\Sigma_{\lambda\nu}$, $\text{Sym}_\nu$ from (2.3). Define also, for any pair $\lambda < \nu$,

$$\text{Sym}^0_{\lambda\nu} = \frac{\text{Stab}_{\Sigma_{\lambda\nu}}(D_{\nu})}{\text{Stab}_{\Sigma_k}(D_\lambda)} \quad \text{Sym}_{\lambda\nu} = \frac{\Sigma_{\lambda\nu}}{\text{Stab}_{\Sigma_k}(D_{\lambda})} \quad (4.18)$$

so we have the exact sequence

$$\{1\} \rightarrow \text{Sym}^0_{\lambda\nu} \rightarrow \text{Sym}_{\lambda\nu} \rightarrow \frac{\Sigma_{\lambda\nu}}{\text{Stab}_{\Sigma_k}(D_{\nu})} \rightarrow \{1\}. \quad (4.19)$$

Both $\text{Sym}^0_{\lambda\nu}$ and $\text{Sym}_{\lambda\nu}$ are symmetry groups of the flag $D_{\nu} \subset D_\lambda$, the former being the subgroup of permutations equal to the identity on $D_{\nu}$.

We note that these groups act on the set of 2-forms $\{\omega_1, \ldots, \omega_k\}$, so that $\sigma^*(\omega_i) = \omega_{\sigma^{-1}(i)}$. There is a corresponding lift of these group actions from $B_{\lambda\nu}$ to the torus-bundle $\mathcal{I}_{\lambda\nu}$.

**Theorem 4.6.** Let $k > 1$ and let $\nu$ be a partition of $k$. Then there is a compactification $\bar{\mathcal{M}}_{\nu}$ of the product $\mathcal{M}_{\nu}$ as a manifold with iterated boundary fibration structure having the following properties:

1. The boundary hypersurfaces of $\bar{\mathcal{M}}_{\nu}$ are indexed by the $\text{Stab}_{\Sigma_k}(D_{\nu})$-orbits of partitions $\lambda$ with $0 \leq \lambda < \nu$. Given $\lambda < \nu$, the corresponding boundary hypersurface is

$$N_{\lambda\nu} = \mathcal{I}_{\lambda\nu} / \text{Sym}^0_{\lambda\nu}. \quad (4.20)$$

where

$$\mathcal{I}_{\lambda\nu} = \bar{\mathcal{M}}_{\lambda} \times T^{r(\nu)} \mathcal{I}_{\mu\nu}. \quad (4.21)$$
In particular, with \( \nu \) fixed, for each \( \lambda \) we have a fibration
\[
\phi_\lambda : N_{\lambda \nu} \to B^\prime_{\lambda \nu},
\]
where
\[
B^\prime_{\lambda \nu} = B_{\lambda \nu} / \text{Sym}_{\lambda \nu}^0.
\]
(2) If \( \lambda < \mu < \nu \), then the intersection \( N_{\lambda \nu} \cap N_{\mu \nu} \) is non-empty, and all such intersections arise in this way up to the action of \( \text{Sym}_{\lambda \nu}^0 \). In this case, there exists \( \phi_{\lambda \mu} \) giving the compatibility conditions of an IBF structure (cf. Definition 3.1).

(3) The \( T^{(\nu)} \)- and \( \text{Sym}_\nu \)-actions extend smoothly from \( \mathcal{M}_\nu \) to \( \bar{\mathcal{M}}_\nu \). The quotient \( \bar{\mathcal{M}}_{[\nu]} = \mathcal{M}_\nu / \text{Sym}_\nu \) has boundary hypersurfaces \( \mathcal{I}_{[\lambda \nu]} \), say, indexed by the \( \text{Sym}_\nu \)-orbits \([\lambda \nu]\) of partitions \( \lambda < \nu \), and
\[
\mathcal{I}_{[\lambda \nu]} = \mathcal{I}_{\lambda \nu} / \text{Sym}_{\lambda \nu}.
\]

This will not be proved here, but we shall carry out the consistency checks that are implied by it. In particular we shall check that the definitions of the boundary hypersurfaces are consistent with the points enumerated in the Theorem.

**Remark 4.7.** In the statement of this Theorem we abuse notation: we label the boundary hypersurfaces by partitions \( \lambda < \nu \), where really the labelling is by the \( \text{Stab}(D_\nu) \)-orbits of such partitions. The only difficulty with this is that we ‘overcount’ the boundary hypersurfaces this way: \( N_{\lambda \nu} \), and \( N_{\lambda \nu} \) are the same boundary hypersurface of \( \bar{\mathcal{M}}_\nu \) if and only if \( \lambda' = \sigma(\lambda) \) for some \( \sigma \in \text{Stab}_{\Sigma_k}(D_\nu) \).

Let us start with a check on dimensions. The dimension of \( \mathcal{M}_k \) is \( 4k - 3 \), so
\[
\dim \mathcal{M}_\lambda = 4k - 3r(\lambda).
\]
Similarly \( \dim D_\lambda = 3r(\lambda) \). Hence the dimension of \( B_{\lambda \nu} \), being a boundary hypersurface of \( M_{\lambda \nu} \), is \( 3(r(\lambda) - r(\nu)) - 1 \). Thus
\[
\dim \mathcal{I}_{\lambda \nu} = 4k - 3r(\lambda) + 3(r(\lambda) - r(\nu)) - 1 = 4k - 3r(\nu) - 1.
\]
Moreover, if \( \lambda < \mu \) then \( \dim B_{\lambda \nu} > \dim B_{\mu \nu} \) so our ordering conventions are consistent with those used in Definition 3.1.

We now come to the main point, the intersections of the boundary hypersurfaces of \( \bar{\mathcal{M}}_\nu \). Notice that \( N_{\lambda \nu} \) has two types of boundary hypersurface: from the inductive description, there are those corresponding to partitions \( \mu \) with
\[
D_\nu \subset D_\mu \subset D_\lambda \tag{4.22}
\]
and those corresponding to partitions \( \kappa \) with
\[
D_\mu \subset D_\lambda \subset D_\kappa. \tag{4.23}
\]
Fixing the chain \( \text{(4.22)} \), consider the \( \mu \) boundary hypersurface of \( N_{\lambda \nu} \) and the \( \lambda \)-boundary hypersurface of \( N_{\mu \nu} \). Now the \( \mu \)-boundary surface of \( \mathcal{I}_{\lambda \nu} \) is just the restriction of the fibration
\[
\bar{\mathcal{M}}_\lambda \times_{T^{(\lambda)}} \mathcal{I}_{\lambda \nu} \to B_{\lambda \nu}
\]
to the the \( \mu \)-boundary hypersurface of the base,
\[
\partial_\mu B_{\lambda \nu} = B_{\lambda \mu} \times B_{\mu \nu}. \tag{4.24}
\]
To take into account the group action, we must factor out by the subgroup \( G \), say, of \( \text{Sym}_{\lambda \nu}^0 \) which leaves \( \text{(4.22)} \) invariant. Thus
\[
G = \text{Stab}_{\Sigma_{\lambda \mu \nu}}(D_\nu) / \text{Stab}(D_\lambda) \tag{4.25}
\]
where \( \Sigma_{\lambda \mu \nu} \) is the group of all permutations in \( \Sigma_k \) which leave \( \text{(4.22)} \) invariant. Using Lemma 4.3 we obtain
\[
\partial_\mu N_{\lambda \nu} = (\bar{\mathcal{M}}_\lambda \times_{T^{(\lambda)}} \mathcal{I}_{\mu \nu} / \mathcal{I}_{\mu \nu}) / G \to (B_{\lambda \mu} \times B_{\mu \nu}) / G. \tag{4.26}
\]
On the other hand, the \( \partial_\lambda N_{\mu \nu} \) is obtained by restricting \( \phi_\mu \) to the \( \lambda \)-boundary in the fibres, that is
\[
N_{\lambda \mu} \times_{T^{(\nu)}} \mathcal{I}_{\mu \nu} / G'.
\]
Here $G'$ is the subgroup of $\text{Sym}^0_{\mu\nu}$ which leaves $D_\lambda$ invariant,

$$G' = \frac{\text{Stab}_{\Sigma_{\mu\nu}}(D_\nu)}{\text{Stab}_{\Sigma_{\mu\nu}}(D_\mu)} = \frac{\text{Stab}_{\Sigma_{\lambda\mu\nu}}(D_\nu)}{\text{Stab}_{\Sigma_{\lambda\mu\nu}}(D_\mu)}$$

By the inductive assumption,

$$N_{\lambda\mu} = \mathcal{M}_{\lambda} \times_{\mathcal{T}\times_{T(\lambda)}} \mathcal{T}_{\lambda\mu} / \text{Sym}^0_{\lambda\mu},$$

so

$$\partial_{\lambda} N_{\mu\nu} = \left( \left( \mathcal{M}_{\lambda} \times_{\mathcal{T}\times_{T(\lambda)}} \mathcal{T}_{\lambda\mu} \times_{\mathcal{T}\times_{T(\mu)}} \mathcal{T}_{\mu\nu} \right) / \text{Sym}^0_{\lambda\mu} \right) / G'.$$

If we ignore the group actions, we see that we have the same manifolds in (4.26) and (4.27), as both are equal to

$$\mathcal{M}_{\lambda} \times_{\mathcal{T}\times_{T(\lambda)}} \mathcal{T}_{\lambda\mu} \times_{\mathcal{T}\times_{T(\mu)}} \mathcal{T}_{\mu\nu} \to B_{\lambda\mu} \times B_{\mu\nu}$$

(4.28)

So it remains only to see that the successive quotients first by $\text{Sym}^0_{\lambda\mu}$ and then $G'$ in (4.27) are equivalent to factoring out by $G$ in (4.26). From the descriptions of $G$ and $G'$, however, we see that $G'$ is a quotient of $G$, and in fact

$$\{1\} \to \text{Sym}^0_{\lambda\mu} \to \text{Stab}(D_\lambda) \to G \to G' \to \{1\}.$$  (4.29)

From this it follows that (4.26) and (4.27) are naturally diffeomorphic.

Identifying the intersection of boundary hypersurfaces corresponding to the chain (4.22) with (4.26), the restriction of $\phi_\lambda$ is the projection map given there. The second projection $B_{\lambda\mu} \times B_{\mu\nu} \to B_{\mu\nu}$ induces a map

$$\phi_{\lambda\mu} : B_{\lambda\mu} \times B_{\mu\nu} / G \to B_{\mu\nu} / \text{Sym}^0_{\mu\nu},$$

and we clearly have $\phi_{\mu} = \phi_{\lambda\mu} \circ \phi_\lambda$ as required by Definition 3.1. Thus, although we haven’t yet proved that the compactification $\mathcal{M}_{\nu}$ of $\mathcal{M}_{\nu}$ exists, if we assume that all $\mathcal{M}_{\lambda}$, for $\lambda < \nu$ have been constructed with the above properties, then we can form a collection of MWCs, namely the $\mathcal{T}_{\lambda\nu}$ and their quotients the $N_{\lambda\nu}$, which fit together to form a ‘formal boundary’ with IBF structure.

We have not yet discussed the group actions, point (3) of the above. On the boundary hypersurface $N_{\lambda\nu}$, there is a $T(\lambda)$-action, by the inductive hypothesis that $\mathcal{M}_{\lambda}$ has a smooth $T(\lambda)$ action. As in the previous discussion of the Gibbons–Manton bundles, the inclusion $D_\nu \subset D_\lambda$ gives an inclusion of the corresponding tori $T^{(\nu)} \to T^{(\lambda)}$ and this inclusion gives the action of $T^{(\nu)}$ on $N_{\lambda\nu}$.

In order to make $\text{Sym}_{\nu}$ act on the set of boundary hypersurfaces, we must pick a lift $\sigma \to \Sigma_\nu$. This leads to a well-defined action on $\mathcal{M}_\nu$ with the claimed properties because the choice of lift is compensated for by dividing out by the groups $\text{Sym}^0_{\lambda\nu}$.

We note that Theorem 4.6 contains as a special case the compactification of $\mathcal{M}_k$ as a manifold with corners:

**Theorem 4.8.** The moduli space $\mathcal{M}_k$ of dimension $4k - 3$ has a compactification $\bar{\mathcal{M}}_k$, which is a compact manifold with corners and a natural IBF structure. The boundary hypersurfaces are indexed by the $\Sigma_k$-orbits of partitions of $k$. If $\lambda$ and $\mu$ are partitions, then the corresponding boundary hypersurfaces intersect if and only if $\lambda$ and $\mu$ are comparable up to the action of $\Sigma_k$. The boundary hypersurface corresponding to $\lambda$ is

$$\mathcal{I}_{[\lambda]} = \mathcal{I}_\lambda / \text{Sym}_\lambda$$

where

$$\mathcal{I}_\lambda = \mathcal{M}_\lambda \times_{T(\lambda)} \mathcal{T}_{\lambda\kappa} \to B_{\kappa},$$

and $\mathcal{T}_{\lambda\kappa}$ is the Gibbons–Manton bundle of type $\lambda$. 

4.5. Low-Charge Examples Revisited. If \( k = 1 \), there is only the trivial partition \((1)\) of \( k \) and \( \mathcal{M}_1 \cong \mathbb{T} \) is already a compact space.

For \( k = 2 \), the only nontrivial partition of \( k \) is the ‘free’ partition \( 0 = (1, 1) \). Our compactification of \( \mathcal{M}_2 \) is as a manifold with boundary \( \partial \mathcal{M}_2 = \mathcal{I}_0 \), which fibers over \( B_{02} = S(\mathbb{R}^6/\mathbb{R}^3)/\Sigma_2 \cong \mathbb{R}P^2 \) with fiber \( \mathcal{M}(1,1) = \mathcal{M}_1 \times \mathcal{M}_1 \cong \mathbb{T}^2 \). The quotient of \( \mathcal{M}_2 \) by \( \mathbb{T} \) (which acts diagonally on the fibre \( \mathbb{T}^2 \) in the above) is the well-known Atiyah-Hitchin manifold of dimension 4, and we recover the known fact that it may be compactified by adding a boundary hypersurface which is a circle fibration over \( \mathbb{R}P^2 \).

The case \( k = 3 \) is more interesting. The non-trivial partitions \( 0 = (1, 1, 1) \) and \( a = (1, 2) \) lead to two boundary hypersurfaces: \( \mathcal{I}_0 \rightarrow B_{03} \) with fibre \( F_0 = \mathcal{M}_1 \times \mathcal{M}_1 \times \mathcal{M}_1 \cong \mathbb{T}^3 \) and \( \mathcal{I}_a \rightarrow B_{03} \) with fibre a compactification of \( \mathcal{M}_1 \times \mathcal{M}_2 \) (which in this case is simply \( \mathcal{M}_1 \times \mathcal{M}_2 \) since \( \mathcal{M}_1 \) is already compact). There is a single codimension two boundary face \( \mathcal{I}_0 \cap \mathcal{I}_a \).

For \( k = 4 \), the non-trivial partitions are \( 0 = (1, 1, 1, 1) \), \( a = (1, 2, 1) \), \( b = (1, 3) \) and \( c = (2, 2) \) and hence \( \mathcal{M}_4 \) has four boundary hypersurfaces.

The boundary faces of codimension \( \ell \) are enumerated by the \( \Sigma_4 \) orbits of chains of partitions of the set \( \{1, 2, 3, 4\} \) of length \( \ell \), which in this case are equivalent to length \( \ell \) chains of integer partitions; in other words these codimension \( \ell \) faces are just the \( \ell \)-fold intersections of the boundary hypersurfaces \( \mathcal{I}_a \) for \( \alpha \in \{0, a, b, c\} \). There are five corners of codimension 2: \( \mathcal{I}_0 \cap \mathcal{I}_a \), \( \mathcal{I}_0 \cap \mathcal{I}_b \), \( \mathcal{I}_0 \cap \mathcal{I}_c \), \( \mathcal{I}_a \cap \mathcal{I}_b \) and \( \mathcal{I}_a \cap \mathcal{I}_c \) and two corners of codimension 3: \( \mathcal{I}_0 \cap \mathcal{I}_a \cap \mathcal{I}_b \) and \( \mathcal{I}_0 \cap \mathcal{I}_a \cap \mathcal{I}_c \). There are no corners of higher codimension.

For \( k \geq 5 \), the intersections of the boundary hypersurfaces \( \mathcal{I}_a \) are no longer connected in general, since there is a distinction between the \( \Sigma_k \) orbits of chains of set partitions, which enumerate the boundary faces of a given codimension, and the chains of integer partitions, which correspond to intersections of the \( \mathcal{I}_a \). For example in \( k = 5 \) the intersection \( \mathcal{I}_a \cap \mathcal{I}_b \) of the two boundary hypersurfaces \( \mathcal{I}_a \), \( a = (1, 1, 1, 2) \) and \( \mathcal{I}_b \), \( b = (2, 3) \) is disconnected: among its components are the quotients by \( \Sigma_5 \) of \( \mathcal{I}_\lambda \cap \mathcal{I}_\nu \) and \( \mathcal{I}_\lambda \cap \mathcal{I}_\nu \), where \( \lambda = \{1\}, \{2\}, \{3\}, \{4, 5\}, \nu = \{1, 2, 3\}, \{4, 5\} \) and \( \nu' = \{1, 2\}, \{3, 4, 5\} \).

4.6. Asymptotic Metrics. Now that we have described the boundary hypersurfaces of our compactification in more detail, we can also give more information about the metric. We have already stated that it is a \( \Phi \) metric adapted to the IBF structure of the compactification \( \mathcal{M}_k \).

But there is also a relatively simple description in terms of the adapted covers constructed in Proposition 3.7. Let \( W_\lambda \) be the set corresponding to the boundary hypersurface labelled by the \( (\Sigma_k\text{-orbit of}) \lambda \).

Denote by \( T_{sc, \lambda} \) the lift to \( M_{\lambda, k} \) of the scattering tangent bundle of \( D_\lambda/D_k \). Denote by \( \eta_\lambda \) the lift to \( M_{\lambda, k} \) of the euclidean metric on \( D_\lambda/D_k \), so that \( \eta_\lambda \) is a smooth metric on \( T_{sc, \lambda} \). Denote by \( g_\lambda \) the riemannian product metric on \( \mathcal{M}_\lambda \). It is clear that these metrics descend to the quotients of these spaces by \( \text{Sym}_\lambda \).

**Theorem 4.9.** The boundary fibration \( \phi_\lambda : N_\lambda \cap W_\lambda \rightarrow B_\lambda/\text{Sym}_\lambda \) admits a smooth extension \( W_\lambda \rightarrow U_\lambda \), a product neighbourhood of \( B_\lambda \) in \( M_\lambda \), such that

- \( T_\Phi \) is isomorphic to \( \phi_\lambda^* T_{sc, \lambda} \oplus T.\mathcal{M}_\lambda \)
- relative to this decomposition, \( g_k \) is smooth and its restriction to \( N_\lambda \cap W_\lambda \) is the direct sum \( \eta_\lambda \oplus g_\lambda \).

5. Decomposable Monopoles and Clusters

Throughout, we follow the observation that asymptotically, monopoles decompose into clusters of lower charge monopoles, cf. Sections 1.1 and 2.2. For sequences of monopoles, this has been shown by Atiyah and Hitchin [3, Prop. 3.8]. As we wish to attach, in a consistent fashion (see Section 4.4), a collection of boundary hypersurfaces associated to such clusters to \( \mathcal{M}_k \), we need to identify not only limits of sequences in \( \mathcal{M}_k \) but asymptotic regions that can be associated to proper clusters. And we need to show that these regions cover \( \mathcal{M}_k \) up to a relatively compact subset.
Given \( k > 1 \), a type \( a \) of \( k \) and two parameters \( R, \varepsilon > 0 \), the first thought to be large and the second small, we will define open sets \( \mathcal{A}_d(R, \varepsilon) \subset \mathcal{M}_k \) of decomposable monopoles. \( R \) is a large radius of balls which we use to 'cover the components of the clusters' while \( \varepsilon \) is the reciprocal of a separation parameter giving a lower bound on the separation of these components. We then show that finitely many of such sets already suffice to cover \( \mathcal{M}_k \) and use this alongside Theorem 4.6 to show that any sequence in \( \mathcal{M}_k \) has a subsequence that either converges in \( \mathcal{M}_k \) or to an \textit{ideal configuration} in one of the boundary hypersurfaces \( \mathcal{I}_a \), proving that our ansatz indeed yields a compactification of \( \mathcal{M}_k \).

In the following, a pair \( (A, \Phi) \) will always refer to framed monopole data, \( (A, \Phi) \in \mathcal{C}_{Fr} \), solving the Bogomolny equation (4.1). By abuse of notation, we will call such a pair a (magnetic) monopoles as well.

5.1. Decomposable Monopoles. We start by reviewing some key results from [24]. There, it is shown that there are numbers \( \kappa(k) > 0, N(k) \in \mathbb{N}, R(k) > 0 \) and \( c_0(k) > 0 \) with the following significance: For any charge \( k \) monopole \( (A, \Phi) \), we let

\[
\mathcal{U}(A, \Phi) = \left\{ z \in \mathbb{R}^3 \mid \int_{B(z, 1)} |F_A(z')|^2 \, dz' > \frac{1}{2} \kappa(k) \right\} \tag{5.1}
\]

and define the \textit{strong-field region} of \( (A, \Phi) \) by

\[
\mathcal{U}_{s}(A, \Phi) = \left\{ z \in \mathbb{R}^3 \mid \text{dist}(z, \mathcal{U}(A, \Phi)) < 1 \right\}. \tag{5.2}
\]

Then, the set \( \mathcal{U}(A, \Phi) \) has \( N \) connected components, where \( 0 < N \leq N(k) \), and the centres of mass \( \zeta_1, \ldots, \zeta_N \in \mathbb{R}^3 \) of the connected components are uniformly bounded. We define the \textit{weak field region} to be the complement of the \( R(k) \)-neighbourhood of the strong-field region,

\[
\mathcal{U}_{w}(A, \Phi) = \left\{ z \in \mathbb{R}^3 \mid \text{dist}(z, \mathcal{U}(A, \Phi)) > R(k) \right\}. \tag{5.3}
\]

On \( \mathcal{U}_{w}(A, \Phi) \), Taubes proved the following estimates [24, C.1.4]:

\[
1 - |\Phi(z)| < \frac{1}{16} \quad , \quad |\nabla_A \Phi(z)| < \frac{1}{16} \quad , \quad |\nabla_A \nabla_A \Phi(z)| < \frac{1}{16}. \tag{5.4}
\]

Moreover, he proves multipole estimates for both \( \Phi \) and \( \nabla_A \Phi \) [24, C.2.1, C.3.1] on all of \( \mathbb{R}^3 \):

There exist numbers \( \alpha_1, \ldots, \alpha_N \in \mathbb{R} \) depending on \( (A, \Phi) \) and a constant \( c_0(k) \) depending on \( k \) only, so that

\[
|\Phi(z)| - 1 + \sum_{j=1}^{N} \frac{\alpha_j}{|z - \zeta_j|} \leq c_0(k) \sum_{j=1}^{N} \frac{1}{|z - \zeta_j|^2}
\]

\[
(\Phi(z), \nabla_A \Phi(z)) + \sum_{j=1}^{N} \alpha_j d|z - \zeta_j|^{-1} \leq c_0(k) \sum_{j=1}^{N} \frac{1}{|z - \zeta_j|^3}
\]

\[
|\Phi(z), \nabla_A \Phi(z)| \leq c_0(k) \sum_{j=1}^{N} e^{-\frac{1}{10}|z - \zeta_j|}.
\tag{5.5}
\]

Let \( W \) be the convex hull of a connected component of \( \mathbb{R}^3 \setminus \mathcal{U}_{w}(A, \Phi) \). As \( |\Phi| \) is non-zero on \( \partial W \), we can consider the mapping degree of \( \Phi|\Phi|^{-1} \) over \( \partial W \). This will of course be an integer, but not necessarily positive. Positivity is crucial, though, as we need to make sure that the individual components of a cluster have positive charge. A sufficient condition for positivity is obtained by comparing the size of \( W \) to its distance to other non-weak regions:

**Lemma 5.1.** There is \( R' = R(k) > 0 \) such that for \( R > R' \) the following holds: If there is \( p_0 \in \mathbb{R}^3 \) satisfying

\[
\text{dist} \left( \partial B(p_0, R), \mathcal{U}_{w}(A, \Phi) \right) > R'^{1/4} \quad \text{and} \quad B(p_0, R) \cap \mathcal{U}_{w}(A, \Phi) \neq \emptyset,
\tag{5.6}
\]


then the set $\mathcal{B}(p_0, R) \cap \mathcal{W}_{(A, \Phi)}$ has strictly positive topological charge for $\Phi$, i.e.
\[
\deg \Phi \bigg|_{\partial \mathcal{B}(p_0, R)} > 0.
\]  

Proof. First of all, we need to do a small calculation: If $V$ is an open set having the property that $|\Phi| > 0$ near $\partial V$, we may diagonalise $\Phi$ near $\partial V$ and obtain
\[
\int_V |F_A(z)|^2 \, dz = \frac{1}{2} \int_V \text{tr} (F_A \wedge * F_A) = \frac{1}{2} \int_V d_A (F_A \wedge \Phi) = \frac{1}{2} \int_{\partial V} \text{tr} (F_A \wedge \Phi) = \frac{1}{2} \int_{\partial V} iF_a \varphi
\]
\[
= 2\pi c_1(L|_{\partial V}) - i \int_{\partial V} F_a(1 - |\Phi|),
\]
where, near $\partial V$, $\Phi = \text{diag}(i\varphi, -i\varphi)$ and $F_A = \text{diag}(F_a, -F_a) + \{\text{off-diagonal terms}\}$, both with respect to the bundle decomposition $\text{ad}E = L \oplus L^{-1}$ into eigenbundles of $\Phi$, and where $c_1(L)$ denotes the Chern number of $L$. Now suppose $(5.6)$ is satisfied for some $p_0 \in \mathbb{R}^3$. Then, combining $(5.8)$ with $(5.2)$, we get
\[
\kappa(k) < 4\pi c_1(L|_{\partial \mathcal{B}(p_0, R)}) - 2i \int_{\partial \mathcal{B}(p_0, R)} F_a(1 - |\Phi|).
\]
The latter term is real and can be bounded using $(5.9)$:
\[
\left| 2i \int_{\partial \mathcal{B}(p_0, R)} F_a(1 - |\Phi|) \right| \leq 2 |\partial \mathcal{B}(p_0, R)| \max \left( |F_A| \right) \leq 8\pi R^2 \cdot c_0(k) R^{-9/4} = c R^{-1/4},
\]
where $c$ depends on $k$ only. Thus, we arrive at
\[
4\pi c_1(L|_{\partial \mathcal{B}(p_0, R)}) > \kappa(k) - c R^{-1/4}.
\]
Then, if we choose $R'(k) = \max \{R(k), (\kappa(k) / 4)^{1/4}\}$ and $R \geq R'(k)$, we obtain $c_1(L|_{\partial \mathcal{B}(p_0, R)}) > 0$ and consequently $(5.7)$.

Thus, whenever we can put some number of connected components of the strong-field region in a ball whose boundary is sufficiently far away from the strong-field region, this ball contains positive charge. In particular, any collection of components of the strong-field region which is widely separated from the rest carries positive charge.

Definition 5.2. The set of decomposable monopoles of type $a = (k_1, \ldots, k_n)$, size $R > 0$ and separation $\varepsilon > 0$ is the set $A_a(R, \varepsilon)$ of pairs $(A, \Phi)$ as above for which there are $p_1, \ldots, p_n \in \mathbb{R}^3$, so that:

1. $\mathbb{R}^3 \setminus \bigcup_{j=1}^n \mathcal{B}(p_j, R) \subset \mathcal{W}_{(A, \Phi)}$
2. $\deg \Phi \bigg|_{\partial \mathcal{B}(p_j, R)} = 2\pi k_j$
3. $\sum_{i<j} (p_i - p_j)^{-1} < \varepsilon$

The definition of the weak-field region $\mathcal{W}_{(A, \Phi)}$ lifts to $M_k$, as does the definition of $A_a(R, \varepsilon)$, resulting in sets of centered monopoles $\mathcal{A}_a(R, \varepsilon)$, which we call decomposable monopoles as well. Notice that if $a = k$ is the trivial type, $\mathcal{A}_a(R, \varepsilon)$ does not depend on the separation parameter $\varepsilon$ (as there is but a single $p_j$), we will sometimes denote this set by $\mathcal{A}_k(R)$.

Proposition 5.3. For each $k \in \mathbb{N}$, there are $M(k) \in \mathbb{N}$, $\varepsilon(k) > 0$ and radii $R_0(k), \ldots, R_{M(k)}(k)$, $R_j(k) > R'(k)$ with $R'(k)$ as in Lemma 5.7 so that
\[
M_k \subset \mathcal{A}_k(R_0(k)) \cup \bigcup_{a} \bigcup_{j=1}^{M(k)} \mathcal{A}_a(R_j(k), \varepsilon(k)),
\]
where a runs over all proper types of $k$.

Proof. The proof is constructive, to each monopole we associate one of the sets $\mathcal{A}_i(R_j(k), \epsilon(k))$. The principle is as follows: Given a monopole $m$, take its strong field region $\mathcal{U}_m$ and cover it by balls, checking whether the boundaries of these are sufficiently far away from the strong-field region. If not, consecutively enlarging the balls and grouping together more connected components of $\mathcal{U}_m$, we show that in the end we arrive at a cover as in Definition 5.2. Hereby, the estimates from Taubes and Proposition 5.1 will be essential. Then, we argue that a finite set of radii and one single $\epsilon$ is enough to achieve this.

So let $(A, \Phi)$ be a charge $k$ monopole and $\mathcal{U}(A, \Phi)$ be its strong-field region. There are numbers $N = N(k)$ and $d = d(k)$ so that $\mathcal{U}(A, \Phi)$ has at most $N$ connected components, say $U_1, \ldots, U_N$, and each of them has diameter bounded above by $d$. (This follows readily from the definition of the strong-field region and the fact that the total curvature of $A$ is $2\pi k$.) We now describe an algorithm which gives centres and balls as in Definition 5.2. For definiteness, let us be detailed here. We will denote the different steps in the algorithm by a letter $t$ (used as superscripts). Moreover, the letter $\omega$ will denote auxiliary partitions not directly related to the partitions $\lambda$ or the types $a$.

(0) Define the following data:

\begin{align*}
\omega^0 &= \{1\}, \ldots, \{N\} \quad , & d^0 &= \max\{d, 15\} \quad , \\
R^0 &= d^0 + R'(k) + 1 \quad , & \gamma^0 &= 3R^0 - \frac{3}{4}d^0 \quad , \\
p_j^0 &= |U_j|^{-1} \int_{U_j} zdz \quad , & \mathcal{B}_j^0 &= \mathcal{B}(p_j^0, R^0) \\
\end{align*}

and set $t = 0$.

(1) Partition the set $I^t = \{1, \ldots, r(\omega^t)\}$ as follows: For each $j \in I^t$, let

\begin{align*}
I_j^t &= \{j': |p_j^0 - p_{j'}^0| \leq \gamma^t\} \\
\end{align*}

and then let $\omega^{t+1}$ be the finest partition of $I^t$ so that each $I_j^t$ is contained in a single block of $\omega^{t+1}$.

(2) If $\omega^{t+1}$ consists of singletons only, stop. Else continue with step (3).

(3) Define the next batch of data:

\begin{align*}
d^{t+1} &= \left(\max_{1 \leq j < r(\omega^{t+1})} |\omega_{j}^{t+1}| - 1 \right)\gamma^t + 2R^t \quad , \\
R^{t+1} &= d^{t+1} + 1 \quad , & \gamma^{t+1} &= 3R^{t+1} - \frac{3}{4}d^{t+1} \quad , \\
p_j^{t+1} &= |\omega_{j}^{t+1}|^{-1} \sum_{j' \in \omega_{j}^{t+1}} p_{j'}^t \quad , & \mathcal{B}_j^{t+1} &= \mathcal{B}(p_j^{t+1}, R^{t+1}) \\
\end{align*}

do t \mapsto t + 1$ and repeat from step (1).

Let us look closer at this algorithm. Step (0) is initialisation of data. Here, blocks of $\omega^0$ correspond to the connected components $U_1, \ldots, U_N$ whose separation we need to check, $R^0$ is already chosen sufficiently large so that we can apply Lemma 5.1. $\gamma^0$ is a separation threshold against which we check and the remaining data is either self-explanatory or auxiliary. In step (1), members of the same block in $\omega^t$ correspond to sets of connected components of $\mathcal{U}(A, \Phi)$ that cannot be widely separated in the sense of Lemma 5.1 while the distance between members of different blocks is sufficiently large. Hence, if $\omega^t$ consists of singletons only, this corresponds to a widely separated cluster, and we can halt in step (2). In step (3), we enlarge the radii (so as to be able to put all components that were not sufficiently far away from each other into single balls) and correspondingly increase the separation threshold $\gamma^t$.

In each run, we either obtain $r(\omega^{t+1}) < r(\omega^t)$ or else the algorithm stops in step (2). Since $r(\omega^0) = N \leq N(k)$ is uniformly bounded, after at most $N(k) - 1$ runs, the algorithm stops. Say it stops after $l_0$ runs and we end up with data $d^{l_0}$, $R^{l_0}$, $\gamma^{l_0}$, $p_0^{l_0}$, $\mathcal{B}_0$, $\mathcal{B}_n$. By choice of $d^0$, $p_0^0$ and $R^0$, the union of the balls $\mathcal{B}_j^0$ covers the $R(k)$-neighbourhood of $\mathcal{U}(A, \Phi)$ and...
consequently we have $\mathbb{R}^3 \setminus \bigcup_j B_j^0 \subset \mathcal{W}_{(A, \Phi)}$. Moreover, the construction of $\omega_{t+1}$ and the choice of $d^{t+1}$ and $R^{t+1}$ ensure that for each $j'$ there is $j$ with $B_{j'}^t \subset B_j^{t+1}$. Thus,

$$\mathbb{R}^3 \setminus \bigcup_{j=1}^n B_j^0 \subset \mathcal{W}_{(A, \Phi)}.$$ 

As we stop if and only if $\omega_{t+1}$ consists of singletons only, we have $\|p_{j'}^0 - p_i^0\| > \gamma^{t_0}$ for all $1 \leq i < j \leq n$ and since $\gamma^{t_0} - 2R^0 = R^0 - \frac{1}{2}d^0 = \frac{1}{2}d^0 + 1 > 0$, the balls $B_j^0$ are mutually disjoint.

Now, as $R^0 > 16$ and $R^{t+1} \geq R^t$, we have $1 - (R^0)^{-1/4} > \frac{1}{2}$ and jointly with $R^0 > d^0$ this implies $R^0 - \frac{1}{2}d^0 > (R^0)^{3/4}$. Then,

$$\text{dist} \left( \partial B_j^0, \mathcal{W}_{(A, \Phi)} \cap B_j^0 \right) > \text{dist} \left( \partial B_j^0, B(p_{j'}^0, \frac{1}{2}d^0) \right) = R^0 - \frac{1}{2}d^0 > (R^0)^{3/4}$$

(5.11)

and, for $i \neq j$,

$$\text{dist} \left( \partial B_j^0, \mathcal{W}_{(A, \Phi)} \cap B_i^0 \right) > \text{dist} \left( \partial B_j^0, \partial B_i^0 \right) > \gamma^{t_0} - 2R^0 = R^0 - \frac{1}{2}d^0 > (R^0)^{3/4}. \quad (5.12)$$

Which shows that $\text{dist} \left( B_j^0, \mathcal{W}_{(A, \Phi)} \right) > (R^0)^{3/4}$ and by Lemma 5.1 that the Higgs-field has positive degree $2\pi k_j$ around each $B_j^0$, for some $1 \leq k_j \leq k$. As the complement of the union of the balls is contained in the weak-field region, these add up to the total charge $k$ and we obtain a type $a = (k_1, \ldots, k_n)$ of $k$ of length $n$. Finally, we note that $|p_{j'}^0 - p_i^0| > \gamma^{t_0}$ implies $\sum_{i < j} |p_{j'}^0 - p_i^0|^{-1} < n^2(\gamma^{t_0})^{-1}$ and so we have $(A, \Phi) \in A_a(R^0, n^2(\gamma^{t_0})^{-1})$.

Apart from the centres $p_j^0$, the initial data is independent of the choice of $(A, \Phi)$ and in fact depends on $k$, only. (If $\omega^0$ depends on $(A, \Phi)$, but since $N \leq N(k)$ is uniformly bounded, this can be remedied by repeating one of the connected components $U_j$ sufficiently many times.)

Looking at step (3) which defines the next batch of data, we see that only a finite number $M(k)$ of different radii may arise, depending on the maximal length of the blocks of the $\omega^j$. These are the radii $R_1(k), \ldots, R_M(k)(k)$. $R_0(k)$ can be chosen as the maximum, $R_0(k) = \max_{1 \leq j \leq M(k)} R_j(k)$, since clearly $A_k(R, \varepsilon) \subset A_k(R', \varepsilon)$ for $R < R'$. Similarly, we can only encounter a finite number of $\gamma^{t_0}$’s and may take their minimum, say $\gamma$. (This is in fact $\gamma^0$ since $\gamma^t \leq \gamma^{t+1}$.) As we also have the trivial inclusion $A_k(R, \varepsilon') \subset A_k(R, \varepsilon)$ whenever $\varepsilon' < \varepsilon$, we can choose $\varepsilon(k) = \frac{k^2}{\gamma}$ to obtain sets $A_k(R_0(k))$ and $A_k(R_0(k), \varepsilon(k))$ whose union contains all possible $(A, \Phi)$. Factoring out by translations and the framed gauge group, we arrive at the claim. 

Given a charge $k$ monopole $(A, \Phi)$, we call the data $a$, $p_1, \ldots, p_n$ and $R$ obtained in this construction the decomposition data, i.e., $p_j = p_j^M$ and $R = R_M$. Since types are unordered, we need to consider the configuration $p_1, \ldots, p_n$ as being unordered as well.

**Remark 5.4.** Notice that by definition of the sets $\mathcal{A}_a(R, \varepsilon)$, there is a map

$$\mathcal{A}_a(R, \varepsilon) \longrightarrow (0, \varepsilon) \times B_{\lambda a}^t,$$

mapping a monopole to the unordered configuration $p_1, \ldots, p_n$: The left hand side of item (3) of Definition 5.2 defines a boundary defining function $\rho_{\lambda a}$ for $B_{\lambda a}^t$. Part of Theorem 4.6 is then that, at infinity, this map yields a fibration with base $B_{\lambda a}^t$. What we have shown so far is that there is a set $K = \mathcal{A}_0(R_0(k)) \subset \mathcal{M}_k$ and a well-defined map

$$\mathcal{M}_k \setminus K \longrightarrow \bigcup_n (0, \varepsilon) \times B_a,$$

(5.13)

associating a ‘cluster configuration’ (of a proper type) to each monopole outside of the core region. In the next section, we will see that this core region $K$ is in fact relatively compact.
5.2. Limits of Decomposable Sequences. Given any sequence \( m^s \in \mathcal{M}_k \), it is clear that there is a subsequence \( m^s \) and a set \( \mathcal{A}_a(R, \varepsilon) = \mathcal{A}_a(R_j(k), \varepsilon(k)) \) so that \( m^s \in \mathcal{A}_a(R, \varepsilon) \) for all \( s \in \mathbb{N} \). Let the respective decomposition data be \( a, p_1^s, \ldots, p_n^s, R \) and define a sequence in \( \mathbb{R}_+ \) by
\[
\varepsilon^s := \rho_{\lambda k}(p_1^s, \ldots, p_n^s) = \sum_{i<j} |p_i^s - p_j^s|^{-1} < \varepsilon.
\] (5.14)
Comparing the results of [3, Prop. 3.8] and of Proposition 5.3 it is not difficult to see that they necessarily lead to the same type of \( a \) and to sequences of bounded distances: \( |p_j^s - w_j^s| \leq c \) for all \( s \in \mathbb{N} \), where \( w_j^s \) denotes the set of sequences obtained from [3 Prop. 3.8]. The reason for this is that the sequences \( w_j^s \) of [3] arise as sequences of zeroes of the Higgs-fields and that Taubes’ estimates [3.3] show that a zero of the Higgs-field is contained in the complement of the weak-field region and thus, in our case, in one of the balls \( B(p_j^s, R) \). The same line of thought, applied to the case of \( a = (k) \), yields the following:

Lemma 5.5. Let \( m^s \) be a sequence of charge \( k \)-monopoles so that \( (\mathbb{R}^3 \setminus B(p, R)) \subset \mathcal{W}_m^s \) for all \( s \in \mathbb{N} \). Then, \( m^s \) has a convergent subsequence. In particular, the sets \( \mathcal{A}_a(R_0) \) are relatively compact.

Returning to our original sequence, we may use this to show the following dichotomy:

Proposition 5.6. Any sequence in \( \mathcal{M}_k \) has a subsequence \( m^s \) so that either
\begin{enumerate}
\item there is \( m \in \mathcal{M}_k \) so that \( m^s \rightarrow m \) uniformly on compact subsets, or
\item there is a set \( \mathcal{A}_a(R, \varepsilon) \) so that \( m^s \in \mathcal{A}_a(R, \varepsilon) \) for all \( s \in \mathbb{N} \) and \( \varepsilon^s \rightarrow 0 \), where \( \varepsilon^s \) is defined as in (5.14).
\end{enumerate}
In particular, if \( m^s \in \mathcal{M}_k \) is a sequence leaving any compact subset, then \( \varepsilon^s \rightarrow 0 \).

Proof. Suppose there is \( c > 0 \) so that \( \varepsilon^s \geq c \) for all \( s \in \mathbb{N} \). Then, \( |p_i^s - p_j^s| \leq c^{-1}(1 + \varepsilon) \) is uniformly bounded and we can find a single center of mass \( \bar{p} \in \mathbb{R}^3 \) and a single radius \( \bar{R} > R_0 \) so that \( B(p_j^s, R) \subset B(\bar{p}, \bar{R}) \) for all \( 1 \leq j \leq n \). Lemma 5.5 then shows that there is a subsequence converging in \( \mathcal{M}_k \). Thus, one and only one of items (1) or (2) holds.

This also shows that bounding \( \rho_{\lambda k} \) from below defines a relatively compact subset in \( \mathcal{M}_k \). Whence if we have a sequence that leaves any compact subset, we necessarily have a subsequence on which \( \varepsilon^s = \rho_{\lambda k}(p_1^s, \ldots, p_n^s) \rightarrow 0 \). \( \square \)

Thus, the ‘non-compactness’ of \( \mathcal{M}_k \) is completely described by families of monopoles for which \( \varepsilon^s \rightarrow 0 \). But these are precisely the families approaching one of the boundary hypersurfaces from Theorem 4.6.

Theorem 5.7. \( \mathcal{M}_k \) is compact.

Proof. Since its boundary hypersurfaces are compact, it clearly suffices to show that every sequence in \( \mathcal{M}_k \) has a subsequence that converges in \( \mathcal{M}_k \). By Proposition 5.6 on a subsequence \( m^s \) we either have convergence in \( \mathcal{M}_k \) or we have \( \varepsilon^s \rightarrow 0 \). So let us assume the latter is true and that we are in the situation of item (2) of Proposition 5.6.

Since \( \varepsilon^s \) is the value of the boundary defining function \( \rho_{\lambda k} \) for \( B'_{\lambda k} \) in its respective many-body compactification, we see that the sequence \( (p_1^s, \ldots, p_n^s) \) converges to an element \( \xi \in B'_{\lambda k} \). Moreover, there are \( m_j \in \mathcal{M}_k \), so that the translates of \( m^s \) by \( p_j^s \) converge to \( m_j \): \( T_j m^s \rightarrow m_j \) uniformly on compact subsets. But since the ideal data \( (\xi, m_1, \ldots, m_n) \) defines an element of \( \mathcal{A}_a \), (2) implies that there is a neighbourhood \( U_\alpha \) of \( \mathcal{A}_a \) and \( s_0 \in \mathbb{N} \) so that, for all \( s \geq s_0 \), we have \( m^s \in U_\alpha \). As any such neighbourhood is relatively compact in \( \mathcal{M}_k \), there is a further subsequence that converges in \( U_\alpha \subset \mathcal{M}_k \). \( \square \)

Let us compare this to earlier results: In Theorem 4.6 we claim that \( \mathcal{M}_k \) carries an iterated boundary fibration, cf. Section 4.1. Moreover, we have shown in Proposition 5.7 that for any compact MWC with IBF, there is a cover \( \{ W_\lambda \} \) of a neighbourhood of its boundary so that \( W_\lambda \cap W_\mu = \emptyset \) if \( \lambda \) and \( \mu \) are not comparable and so that the boundary fibrations \( \phi_\lambda : N_\lambda \rightarrow B_\lambda \)
restrict to be surjective with fibres contained in a relatively compact subset of the interior of the unrestricted fibres.

Since the sets \( \mathcal{A}_i(R, \varepsilon) \) cover \( \mathcal{M}_k \) up to a relatively compact subset, their closures in \( \mathcal{M}_k \) clearly cover a neighbourhood of \( \partial \mathcal{M}_k \). Looking at the definition of the sets \( \mathcal{A}_i(R, \varepsilon) \) it is clear that an intersection \( \mathcal{A}_a(R, \varepsilon) \cap \mathcal{A}_b(R', \varepsilon) \) is non-empty if and only if the types \( a \) and \( b \) are comparable: Assume \( R \leq R' \) and that the intersection is not empty. Since there are sets of balls \( \{B(p_j, R)\} \) and \( \{B(p'_j, R')\} \) covering the strong-field region of the same monopole, by looking at intersections we obtain a surjective map \( \{p_j\} \rightarrow \{p'_j\} \) and in this way \( a \leq b \).

For the moment, we will not show that the restrictions of the boundary fibrations are still surjective (this will be shown in Part II), but consider the fibres instead. If \( m^s \in \mathcal{A}_i(R, \varepsilon) \) for all \( s \), by not allowing the radius \( R \) of the balls \( B(p_j, R) \) to grow along the sequence, we exclude the possibility of the monopole ‘falling apart further’. Asymptotically speaking, in terms of fibres \( \mathcal{M}_{\Lambda k} \) and bases \( B'_{\Lambda k} \), the sequence ends up in and stays in the set
\[
\mathcal{A}_k(R) \times \cdots \times \mathcal{A}_k(R) \subset \mathcal{M}_k \times \cdots \times \mathcal{M}_n \subset \mathcal{M}_{\Lambda k},
\]
which by Lemma 5.5 and Theorem 4.6 is a relatively compact subset of the interior of the fibre \( \mathcal{M}_{\Lambda k} \). Thus, the cover \( \{5.10\} \) is a cover of the type constructed in Proposition 3.7.

6. Proof of the Sen Conjecture, Coprime Case

Let us now use Theorem 4.6, Proposition 3.12 and results from [22] to prove the coprime case of the Sen Conjecture.

As in Section 4.1, let \( \mathcal{M}_k^\natural \) denote the space of strongly centred monopoles of charge \( k \), which is the universal cover of the hyperKähler quotient \( \mathcal{M}_k^0 \). The latter space has fundamental group isomorphic to \( \mathbb{Z}_k \) and thus the deck transformations of \( \mathcal{M}_k^\natural \) are given by the subgroup \( \mathbb{Z}_k \subset \mathbb{T} \).

Before proceeding we make two further remarks about the relation between the compactification \( \mathcal{M}_k^\natural \) and the universal cover. The first observation is that for any compact manifold with corners \( M \), with interior \( M^0 = M \setminus \partial M \), it is the case that \( M \) and \( M^0 \) have the same homotopy type. This can be seen by choosing a global boundary defining function \( \rho \), say. If
\[
M_\delta = M \setminus \{\rho < \delta\}
\]

it is easy to see that \( M_\delta \) is homotopy equivalent to both \( M \) and \( M^0 \). In particular, the homotopy types of \( \mathcal{M}_k^\natural / \mathbb{T} \) and \( \mathcal{M}_k / \mathbb{T} \) are the same and both have fundamental group \( \mathbb{Z}_k \).

The next observation is that if \( M \) is a compact MWC with finite fundamental group, then its universal cover \( \tilde{M} \) is, in a natural way, again a compact MWC. The proof is an adaptation of the proof that the universal cover of a smooth manifold is in a natural way again smooth.

Combining these two points, we see that the compactification \( \mathcal{M}_k^\natural \) of \( \mathcal{M}_k \) yields a compactification of \( \mathcal{M}_k^0 \), which we will call \( \mathcal{M}_k^\natural \) in this section, and the IBF and metric properties proved for \( \mathcal{M}_k^\natural \) hold also, simply by lifting, for the compactification \( \mathcal{M}_k^\natural \).

If \( \mathcal{H}^i_k \) denotes the space of \( L^2 \) harmonic forms of degree \( i \) on \( \mathcal{M}_k^\natural \), we can decompose \( \mathcal{H}^i_k \) according to the \( \mathbb{Z}_k \)-action and denote by \( \mathcal{H}^i_{k,\ell} \) the component in \( \mathcal{H}^i_k \) on which \( \zeta \in \mathbb{Z}_k \) acts by multiplication with \( \zeta^\ell \). Analogously, we write \( H^s(U)_\ell, H^s(U)_c^\ell \) and \( H^2(U)_c^\ell \) for the respective components of the de Rham cohomology, the de Rham cohomology with compact supports and the \( L^2 \)-cohomology of \( U \subset \mathcal{M}_k^\natural \). (The latter meaning the subcomplex of the de Rham complex consisting of smooth forms \( \alpha \) for which both \( \alpha \) and \( d\alpha \) are square-integrable on \( U \).) The integer \( \ell \) is also referred to as the electric charge.

The Sen Conjecture can be stated as follows (cf. [22, 23]):

(S.1) If \( k \) and \( \ell \) are coprime, then \( \mathcal{H}^{2k-2}_{k,\ell} \cong \mathbb{C} \) and \( \mathcal{H}^{i}_{k,\ell} = 0 \) for \( i \neq 2k - 2 \), and
(S.2) if \( k \) and \( \ell \) have a common factor, then \( \mathcal{H}^{i}_{k,\ell} = 0 \) for all \( i \).

We will prove (S.1), the coprime case, by adapting an argument in [22, §2] and proving the following.
Theorem 6.1. Let the integers \( k \) and \( \ell \) be coprime. Then the space \( \mathcal{H}_{k,\ell}^i \) of harmonic forms of degree \( i \) and electric charge \( \ell \) is canonically isomorphic to
\[
\text{Im} \left( H^i(\mathcal{M}_k^0)_{\ell} \rightarrow H^i(\mathcal{M}_k^0)_{\ell} \right).
\]
In particular, the coprime case of the Sen Conjecture holds true:
\[
\mathcal{H}_{k,\ell}^i \cong \begin{cases} 
\mathbb{C} & \text{if } i = 2k - 2, \\
0 & \text{else}.
\end{cases}
\]

Part of the proof will be showing the existence of a finite open cover \( \{V_i\} \) of the compactification \( \mathcal{M}_k^0 \) of \( \mathcal{M}_k \) with the following properties:

1. \( V_0 \) is relatively compact
2. For each \( i > 0 \) there is a proper partition \( \lambda_i \) of \( k \) so that \( V_i \) can be identified with an open set of ordered clusters of monopoles of type \( \lambda_i \)
3. \( T^{(\lambda_i)} \) acts on \( V_i \), extending the action of \( T \) on \( V_i \subset \mathcal{M}_k^0 \); this action is by near isometries and its orbits are of bounded size
4. There is a partition of unity \( \{\chi_i\} \) subordinate to the cover \( \{V_i\} \) so that each \( |d\chi_i| \) is bounded

Then, we will proceed as in [22] and use the \( T^{(\lambda_i)} \)-action on restrictions of the \( V_i \) to \( \mathcal{M}_k^0 \) in order to reduce \( H^2(\mathcal{M}_k^0)_{\ell} \rightarrow H^2(\mathcal{M}_k^0)_{\ell} \). This will lead to a proof of Theorem 6.1.

Lemma 6.2. There exists a finite open cover \( \{V_i\} \) of \( \mathcal{M}_k^0 \) satisfying conditions (1) – (4).

Proof. We will first show the existence of a cover of \( \mathcal{M}_k \) satisfying (1)–(3), then lift this cover to \( \mathcal{M}_k^0 \) and use Proposition 3.12 to obtain (4).

By Theorem 4.6 for each boundary hypersurface \( N \) of \( \mathcal{M}_k \) we have a \( \text{Stab}_{\Sigma_k} \langle D_k \rangle \)-orbit (i.e. a type) \( [\lambda] \) of a partition \( \lambda \) of \( k \) and a fibration \( \varphi_\lambda : N = N_{\lambda k} \rightarrow B'_{\lambda k} \), where \( B'_{\lambda k} \) is the ideal or free boundary of the space of unordered configurations of type \( [\lambda] \). \( B'_{\lambda k} \) is the quotient of the ideal boundary of the space of ordered configurations associated to \( \lambda \), by the action of the symmetry group \( \text{Sym}_{\lambda}^0 = \text{Sym}_{\lambda}^0 \), cf. (2.12). Let \( \pi_\lambda \) denote the canonical quotient map. Since \( B'_{\lambda k} \) is compact, we can choose an open cover by a finite number of connected open sets, say \( \tilde{O}_{j,\lambda} \), with the property that \( \pi_\lambda^{-1}(\tilde{O}_{j,\lambda}) \subset B_{\lambda k} \) is diffeomorphic to the union of \( \text{Sym}_{\lambda} \) disjoint copies of \( \tilde{O}_{j,\lambda} \). For each \( j \), we pick one of these folds and denote it by \( O_{j,\lambda} \).

Furthermore, since \( N_{\lambda k} \) is compact, we can pick a finite open cover \( \{\mathcal{U}_{i,\lambda}\} \) of \( N_{\lambda k} \) and, as the \( \pi_\lambda(O_{j,\lambda}) \) cover \( B_{\lambda k} \), refine this cover in such a way that each \( \phi_\lambda|_{\mathcal{U}_{i,\lambda}} \) is contained in one \( \pi_\lambda(O_{j,\lambda}) \). Then, the \( \mathcal{U}_{i,\lambda} \) constitute a finite open cover of \( N_{\lambda k} \) and each of the elements of this cover can be identified with an open set of ordered ideal configurations associated to the partition \( \lambda \).

Now choose a boundary product structure compatible to the IBF of \( \mathcal{M}_k \) as in Definition 3.6. Using the retraction \( v_\lambda \) (cf. the paragraphs after Definition 3.6), we may push out the sets \( \mathcal{U}_{i,\lambda} \) to obtain a cover of a product neighbourhood of \( N_{\lambda k} \). Doing this for all proper types \( [\lambda] \) of \( k \), we obtain a cover of a product neighbourhood of \( \partial \mathcal{M}_k \). Letting \( \mathcal{V}_0 \) be an open neighbourhood of \( \mathcal{M}_k \setminus \bigcup_{i,\lambda} \mathcal{U}_{i,\lambda} \), we obtain a cover of \( \mathcal{M}_k \).

If \( \tilde{\pi} : \mathcal{M}_k^0 \rightarrow \mathcal{M}_k^0 \) denotes the quotient map, we may refine the cover \( \{\mathcal{U}_{i,\lambda}\} \) of \( \mathcal{M}_k \) so that for each element of it, \( \tilde{\pi}^{-1}(\mathcal{U}_{i,\lambda}) \) consists of \( k \) disjoint open sets. These sets form a cover \( \{\tilde{\mathcal{U}}_{i,\lambda}\} \) of \( \mathcal{M}_k^0 \). Now let \( \{W_i\} \) be a cover for \( \mathcal{M}_k^0 \) obtained by Proposition 3.12 and take \( \{V_i\} \) to be a common refinement of \( \{W_i\} \) and \( \{\tilde{\mathcal{U}}_{i,\lambda}\} \).

Conditions (1) and (2) are clearly satisfied by construction, as is condition (3): Each \( V_i \) is identifiable with an open set of ordered clusters of monopoles of type \( \lambda_i \), in particular \( T^{(\lambda_i)} \) acts freely on \( V_i \) extending the action of \( T \) given on \( \mathcal{M}_k \). Moreover, \( T^{(\lambda_i)} \) acts fibre-wise by
isometries and, due to the form of the metric (cf. Proposition 3.12 and Theorem 4.16), acts by isometries with bounded orbits on \( V_i \). Lastly, condition (4) follows since the cover \( \{ V_i \} \) is a refinement of the cover from Proposition 3.12.

Now consider sets \( \tilde{V}_i = (V_i \setminus \partial \tilde{\mathcal{M}}_k^0) \subset \tilde{\mathcal{M}}_k^0 \). If the type corresponding to \( V_i \) (downstairs in \( \tilde{\mathcal{M}}_k \)) is \( a_i = [\lambda_i] = (k_1, \ldots, k_n) \), then

\[
G_{\lambda_i} = \{ (\zeta_1, \ldots, \zeta_n) \in \mathbb{T}^n \mid \prod_{j} \zeta_j^{k_j} = 1 \}
\]

acts on \( \tilde{V}_i \). The diagonal subgroup of \( G_{\lambda_i} \) is isomorphic to \( \mathbb{Z}_k \) and acts by deck transformations on \( \tilde{V}_i \) (cf. [22, p. 779]).

**Lemma 6.3.** If \( U \) is a \( G_{\lambda_i} \)-stable open submanifold of \( \tilde{V}_i \), where \( i > 0 \), and \( k \) and \( \ell \) are coprime, then \( H_2^U(U)_{\ell} = 0 \).

**Proof.** The argument is an extension of the proof of [22, Lemma 3.1]. If \( d \) is the greatest common divisor of the numbers \( k_1, \ldots, k_n \) in \( a_i \), then the vector \( (k_1, \ldots, k_n) \) is \( d \) times a primitive vector in \( \mathbb{Z}^n \). Thus we can find \((n - 1)\) vectors which, along with \((k_1, \ldots, k_n)\) span a lattice of index \( d \) in \( \mathbb{Z}^n \). Thus \( G_{\lambda_i} = \mathbb{T}^n \times \mathbb{Z}d \). If the action of \((\zeta_1, \ldots, \zeta)\) on \( \tilde{\mathcal{M}}_k^0 \) is denoted by \( A_{\zeta} \), then it follows that the diffeomorphism \( A_{\zeta}^d \) is in the identity component of \( G_{\lambda_i} \) and hence linked to it by a homotopy generated by a bounded vector field. Thus, if \( \alpha \) represents an element of \( H_2^U(U)_{\ell} \), we have \((A_{\zeta}^d)^*\alpha - \alpha = d\beta \) and from \((A_{\zeta}^d)^*\alpha = \zeta^{\ell\beta} \alpha \) and \( \zeta^\ell \neq 1 \), we obtain \( \alpha = d((\zeta^{\ell\beta} - 1)^{-1}\beta) \). □

**Lemma 6.4.** If \( k \) and \( \ell \) are coprime, the map \( H_1^c(\tilde{\mathcal{M}}^0_k)_{\ell} \to \mathcal{H}^c_{k,\ell} \) given by orthogonal projection is onto.

**Proof.** Let \( \{ V_i \} \) denote the cover from Lemma 6.2 and again write \( \tilde{V}_i = V_i \setminus \partial \tilde{\mathcal{M}}_k^0 \). Then, \( \tilde{V} = \bigcup_{i > 0} \tilde{V}_i \) is a finite union and by condition (4), there is a smooth partition of unity \( \{ \chi_i \} \) subordinate to the cover \( \{ \tilde{V}_i \} \) such that the differentials \( d\chi_i \) are all bounded. Hence we may use a Mayer-Vietoris argument to compute \( H_2^U(\tilde{V})_{\ell} \) from the \( H_2^U(\tilde{V}_i)_{\ell} \). Since any intersection \( \tilde{U} \) of sets \( \tilde{V}_i \) is compact, when \( i > 0 \), is \( G_{\lambda_i} \)-stable for some \( \lambda = \lambda_i \), we have \( H_2^U(\tilde{V})_{\ell} = 0 \) and consequently obtain \( H_2^U(\tilde{V})_{\ell} = 0 \) by iteration of the standard Mayer-Vietoris argument.

Now let \( 0 \neq \alpha \in \mathcal{H}^c_{k,\ell} \). As \( (\tilde{\mathcal{M}}^0_k, g) \) is complete, \( \alpha \) defines a non-zero element in \( H_1^c(\tilde{\mathcal{M}}^0_k)_{\ell} \). But then, \( \alpha|_{\tilde{V}} = d\gamma \) for some smooth and square-integrable form \( \gamma \) on \( \tilde{V} \), since \( H_2^U(\tilde{V})_{\ell} = 0 \). Using any smooth extension \( \tilde{\gamma} \) of \( \gamma \) to \( \tilde{\mathcal{M}}^0_k \), we see that \( \beta = \alpha - d\tilde{\gamma} \) is compactly supported and closed, thus defines an element in \( H_1^c(\tilde{\mathcal{M}}^0_k)_{\ell} \). Its projection onto \( \mathcal{H}^c_{k,\ell} \) is precisely \( \alpha \). □

With these preparations at hand, we can address the proof of Theorem 6.1.

**Proof of Thm. 6.1.** The obvious map \( H_1^c(\tilde{\mathcal{M}}_k^0) \to H^c(\tilde{\mathcal{M}}_k^0) \), induced by inclusion \( \Omega^c \to \Omega^* \), factors through \( \mathcal{H}^c_k \) as is shown in [22, 1.4] for instance. Thus we have maps

\[
H_1^c(\tilde{\mathcal{M}}_k^0)_{\ell} \xrightarrow{\sigma} \mathcal{H}^c_{k,\ell} \xrightarrow{\tau} H^c(\tilde{\mathcal{M}}_k^0)_{\ell}.
\]

We have just shown \( \sigma \) to be surjective, and \( \tau \) is injective since \( (\tilde{\mathcal{M}}_k^0, g) \) is complete and because \( H_1^c(\tilde{\mathcal{M}}_k^0)_{\ell} \) is a subcomplex of \( H^c(\tilde{\mathcal{M}}_k^0)_{\ell} \). This is to say that \( \mathcal{H}^c_k \) is canonically isomorphic to \( \text{Im}(\tau \circ \sigma) \), i.e., to \( (d_1 \cup \tau \circ \sigma)^* \). Using results of [22], this implies the coprime case of the Sen Conjecture, (S.1): There, it is shown that (6.2) holds for \( H^c(\tilde{\mathcal{M}}_k^0)_{\ell} \) and \( H_1^c(\tilde{\mathcal{M}}_k^0)_{\ell} \) and that Poincaré-duality gives \( H^c_{2k-2}(\tilde{\mathcal{M}}_k^0)_{\ell} \cong H^c_{2k-2}(\tilde{\mathcal{M}}_k^0)_{\ell} \). But then,

\[
\mathcal{H}^c_{k,\ell} \cong \text{Im}\left( H_1^c(\tilde{\mathcal{M}}_k^0)_{\ell} \to H^c(\tilde{\mathcal{M}}_k^0)_{\ell} \right) \cong \begin{cases} 
\mathbb{C} & \text{if } i = 2k - 2, \\
0 & \text{else}. 
\end{cases}
\]

□
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