$R$-matrices and the Yang-Baxter equation on GNS representations of C*-bialgebras (revised)

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Abstract

A new method of construction of $R$-matrix is given. Let $A$ be a C*-bialgebra with a comultiplication $\Delta$. For two states $\omega$ and $\psi$ of $A$ which satisfy certain conditions, we construct a unitary $R$-matrix $R(\omega, \psi)$ of the C*-bialgebra $(A, \Delta)$ on the tensor product of GNS representation spaces associated with $\omega$ and $\psi$. The set $\{R(\omega, \psi) : \omega, \psi\}$ satisfies a kind of Yang-Baxter equation. Furthermore, we show a nontrivial example of such $R$-matrices.

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1 Introduction

We have studied C*-bialgebras and tensor products of representations induced from the comultiplication. We proved that a certain C*-bialgebra has no universal $R$-matrix in [16]. In this paper, we construct (non-universal) $R$-matrices of the C*-bialgebra, which satisfy the Yang-Baxter equation. For this purpose, we show a theorem for general C*-bialgebras and states. In this section, we show our motivation, definitions of C*-bialgebras and the main theorem.

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1.1 Motivation

In this subsection, we roughly explain our motivation and the background of this study. Explicit mathematical definitions will be shown after § 1.2.

Define the $C^*$-algebra $\mathcal{O}_*$ as the direct sum of all Cuntz algebras except $\mathcal{O}_\infty$:

$$\mathcal{O}_* = \mathcal{O}_1 \oplus \mathcal{O}_2 \oplus \mathcal{O}_3 \oplus \mathcal{O}_4 \oplus \cdots,$$

(1.1)

where $\mathcal{O}_1$ denotes the 1-dimensional $C^*$-algebra $C$ for convenience. In [13], we constructed a non-cocommutative comultiplication $\Delta_\varphi$ of $\mathcal{O}_*$. Furthermore, we proved that $\mathcal{O}_*$ has no universal $R$-matrix in [16]. For quasi-cocommutative $C^*$-bialgebras, see [17, 18].

On the other hand, in the theory of quantum groups, $R$-matrix and the Yang-Baxter equation are fruitful subjects as relations with mathematical physics and topology [4, 6, 7, 10, 19, 23] for quasi-cocommutative bialgebras. They were also considered by Van Daele and Van Keer [22] for Hopf $*$-algebras.

As a construction method of $R$-matrix, the quantum double is well-known [4, 10, 19]. In this paper, we show a new method. We construct $R$-matrices from states of a $C^*$-bialgebra under some conditions:

states of a $C^*$-bialgebra $\Rightarrow$ $R$-matrices

Furthermore, we show a non-trivial example of this construction.

1.2 Local $R$-matrix of $C^*$-bialgebra

In this subsection, we recall definitions of $C^*$-bialgebra, and introduce local $R$-matrix of a $C^*$-bialgebra.

At first, we prepare terminologies about $C^*$-bialgebra according to [20, 21]. For two $C^*$-algebras $A$ and $B$, let $\text{Hom}(A, B)$ and $A \otimes B$ denote the set of all $*$-homomorphisms from $A$ to $B$ and the minimal $C^*$-tensor product of $A$ and $B$, respectively. Let $M(A)$ denote the multiplier algebra of a $C^*$-algebra $A$. We state that $f \in \text{Hom}(A, M(B))$ is nondegenerate if $f(A)B$ is dense in a $C^*$-algebra $B$. A pair $(A, \Delta)$ is a $C^*$-bialgebra if $A$ is a $C^*$-algebra and $\Delta \in \text{Hom}(A, M(A \otimes A))$ such that $\Delta$ is nondegenerate and the following holds:

$$(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.$$

(1.2)

We call $\Delta$ the comultiplication of $A$. Remark that $A$ has no unit in general for a $C^*$-bialgebra $(A, \Delta)$. Define the extended flip $\tilde{\tau}_{A,A}$ from $M(A \otimes A)$ to $M(A \otimes A)$ as $\tilde{\tau}_{A,A}(X)(x \otimes y) \equiv \tau_{A,A}(X(y \otimes x))$ for $X \in M(A \otimes A)$, $x, y \in A$
where $\tau_{A,A}$ denotes the flip of $A \otimes A$. The map $\Delta^{op}$ from $A$ to $M(A \otimes A)$ defined as $\Delta^{op} \equiv \tilde{\tau}_{A,A} \circ \Delta$ is called the \textit{opposite comultiplication} of $\Delta$. A $C^*$-bialgebra $(A, \Delta)$ is \textit{cocommutative} if $\Delta = \Delta^{op}$.

According to [4, 10, 22], we introduce unitary $R$-matrix, the quasi-cocommutativity and local $R$-matrix for $C^*$-bialgebra as follows.

\textbf{Definition 1.1} (i) An element $R$ in $M(A \otimes A)$ is called a (unitary) universal $R$-matrix of $(A, \Delta)$ if $R$ is a unitary and

$$R \Delta(x) R^* = \Delta^{op}(x) \quad (x \in A).$$  \hfill (1.3)

In this case, we state that $(A, \Delta)$ is quasi-cocommutative (or almost cocommutative [2]).

(ii) Let $(H_i, \pi_i)$ be a representation of $A$ for $i = 1, 2$. A unitary operator $R_{\pi_1, \pi_2}$ on $H_1 \otimes H_2$ is called a local $R$-matrix of $(A, \Delta)$ on $H_1 \otimes H_2$ if it satisfies

$$R_{\pi_1, \pi_2}(\pi_1 \otimes \pi_2)(\Delta(x)) R_{\pi_1, \pi_2}^* = (\pi_1 \otimes \pi_2)(\Delta^{op}(x)) \quad (x \in A).$$  \hfill (1.4)

In Definition 1.1(ii), if $\pi_1 = \pi_2$, then the flip of $H_1 \otimes H_2$ is a (trivial) local $R$-matrix of $(A, \Delta)$ on $H_1 \otimes H_2$. If $\pi_1 \neq \pi_2$, then a local $R$-matrix does not always exist when $(A, \Delta)$ is not cocommutative.

In addition, we introduce a new notion for $C^*$-bialgebra. Let $A \otimes B$ denote the algebraic tensor product of $C^*$-algebras $A$ and $B$.

\textbf{Definition 1.2} A $C^*$-bialgebra $(A, \Delta)$ is \textit{algebraic} if there exists a dense $*$-subalgebra $A_0$ of the $C^*$-algebra $A$ such that $\Delta(A_0) \subset A_0 \otimes A_0$. We call $A_0$ an algebraic part of $(A, \Delta)$.

If a $C^*$-bialgebra $(A, \Delta)$ is algebraic with an algebraic part $A_0$, then $\Delta(x)$ is written as follows for any element $x \in A_0$: There exist $1 \leq m < \infty$ and $x_1, \ldots, x_m, x_1'', \ldots, x_m''$ in $A_0$ such that

$$\Delta(x) = x_1' \otimes x_1'' + \cdots + x_m' \otimes x_m''.$$  \hfill (1.5)

\textbf{1.3 Main theorem}

In this subsection, we show the main theorem. For a $C^*$-algebra $A$, let $S(A)$ and $\text{Rep}A$ denote the set of all states and the class of all nondegenerate representations of $A$, respectively.
Definition 1.3 \cite{26,27,28} For $\omega \in \mathcal{S}(A)$ of a $C^*$-algebra $A$ with the Gel’fand-Naimark-Segal (=GNS) triple $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$, the map $\Lambda_\omega$ from $A$ to $\mathcal{H}_\omega$ defined by

$$\Lambda_\omega(x) \equiv \pi_\omega(x)\Omega_\omega \quad (x \in A),$$

(1.6)
is called the GNS map associated with $\omega$.

Then our main theorem is stated as follows.

Theorem 1.4 Let $(A, \Delta)$ be an algebraic $C^*$-bialgebra. For $\pi_1, \pi_2 \in \text{Rep} A$ and $\omega_1, \omega_2 \in \mathcal{S}(A)$, we write

$$\pi_1 \ast \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \Delta, \quad \omega_1 \ast \omega_2 \equiv (\omega_1 \otimes \omega_2) \circ \Delta$$

(1.7)

where $(\omega_1 \otimes \omega_2)(x \otimes y) \equiv \omega_1(x)\omega_2(y)$ for $x, y \in A$ \cite{22}, p.847). Assume that a subset $S_0$ of $\mathcal{S}(A)$ satisfies the following conditions:

(a) The set $S_0$ is closed with respect to the operation $\ast$ in (1.7) and the semigroup $(S_0, \ast)$ is abelian.

(b) For any $\omega_1, \omega_2 \in S_0$, $\Omega_{\omega_1} \otimes \Omega_{\omega_2}$ is a cyclic vector for the representation $(\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}, \pi_{\omega_1} \ast \pi_{\omega_2})$ of $A$.

Define the unitary $R(\omega_1, \omega_2)$ from $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ by

$$R(\omega_1, \omega_2)\Lambda_{\omega_1,\omega_2}(\Delta(x)) \equiv \Lambda_{\omega_1,\omega_2}(\Delta^{\text{op}}(x)) \quad (x \in A)$$

(1.8)

where $\Lambda_{\omega_1,\omega_2} \equiv \Lambda_{\omega_1} \otimes \Lambda_{\omega_2}$. Then $R(\omega_1, \omega_2)$ is well-defined and the following holds:

(i) For each $\omega_1, \omega_2 \in S_0$,

$$R(\omega_1, \omega_2)(\pi_{\omega_1} \otimes \pi_{\omega_2})(\Delta(x))(R(\omega_1, \omega_2))^* = (\pi_{\omega_1} \otimes \pi_{\omega_2})(\Delta^{\text{op}}(x))$$

(1.9)

for $x \in A$ on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ and

$$R(\omega_1, \omega_2)\tau_{\omega_2,\omega_1}R(\omega_2, \omega_1)\tau_{\omega_1,\omega_2} = I_{\omega_1} \otimes I_{\omega_2}$$

(1.10)

where $\tau_{\omega_1,\omega_2}$ denotes the flip of $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$.

(ii) $(\pi_{\omega_1} \otimes \pi_{\omega_2}) \circ \Delta$ and $(\pi_{\omega_2} \otimes \pi_{\omega_1}) \circ \Delta$ are unitarily equivalent.

(iii) For each $\omega_1, \omega_2, \omega_3 \in S_0$,

$$R_{12}(\omega_1, \omega_2)R_{13}(\omega_1, \omega_3)R_{23}(\omega_2, \omega_3) = R_{23}(\omega_2, \omega_3)R_{13}(\omega_1, \omega_3)R_{12}(\omega_1, \omega_2)$$

(1.11)
on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2} \otimes \mathcal{H}_{\omega_3}$ where we use the leg numbering notation in \cite{22}. For example, $R_{12}(\omega_1, \omega_2) \equiv R(\omega_1, \omega_2) \otimes I_{\omega_3}$. 

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From (1.9), $R(\omega_1, \omega_2)$ is a local $R$-matrix of $(A, \Delta)$ on $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$. The equation (1.11) is regarded as a kind of Yang-Baxter equation.

**Remark 1.5**

(i) If $\omega_1 = \omega_2$, then $R(\omega_1, \omega_2) = I_{\omega_1} \otimes I_{\omega_2}$. If not, then this does not hold in general. If $(S_0, *)$ is not abelian, then (1.9) does not hold in general. We will show these examples in §3.3.

(ii) We explain conditions in Theorem 1.4. The condition (b) does not always hold even if the condition (a) holds. For $\pi \in \text{Rep}(A)$, let $[\pi]$ denote the unitary equivalence class of $\pi$. For $\pi_1, \pi_2 \in \text{Rep}(A)$, the new product $[\pi_1] * [\pi_2] \equiv [\pi_1 \ast \pi_2]$ is well-defined. Let $R(A)$ denote the set of unitary equivalence classes of all representations of $A$. Then we obtain the map $f$ from $S_0$ to $R(A)$ by $f(\omega) \equiv [\pi_{\omega}]$ for $\omega \in S_0$. Then the condition in Theorem 1.4 is interpreted as follows:

$$f(\omega \ast \psi) = f(\omega) \ast f(\psi) \quad (\omega, \psi \in S_0).$$

That is, $f$ is a homomorphism between two semigroups $(S_0, *)$ and $(R(A), \ast)$. Such examples will be shown in §3.2.

(iii) The GNS map in Definition 1.3 was used in constructions of Kac-Takesaki operators [5, 20, 21] and an analogue of multiplicative isometry [15].

In §2 we will prove Theorem 1.4. In §3, we will treat $(O_{\ast}, \Delta_{\phi})$ and certain states of $O_{\ast}$ as an example of Theorem 1.4.

2 Proof of main theorem

In this section, we prove Theorem 1.4.

2.1 Lemma for GNS maps

In order to prove Theorem 1.4, we prepare an elementary lemma for GNS maps in (1.6) in this subsection.

**Lemma 2.1** Let $\Lambda_{\omega}$ be as in (1.6) and let $A, B, C$ be $C^*$-algebras.

(i) For $\phi \in \text{Hom}(A, B)$ and a state $\omega$ of $B$, define

$$U \Lambda_{\omega \phi}(x) \equiv \Lambda_{\omega}(\phi(x)) \quad (x \in A).$$

Then $U$ is an isometry from $\mathcal{H}_{\omega \phi}$ to $\mathcal{H}_{\omega}$ such that $U^* \pi_{\omega}(\phi(x))U = \pi_{\omega \phi}(x)$ for $x \in A$. 5
(ii) In addition to (i), if $\Omega_\omega$ is a cyclic vector for $(\mathcal{H}_\omega, \pi_\omega \circ \phi)$, then $U$ is a unitary.

(iii) Let $\phi \in \text{Hom}(A, B \otimes C)$ and let $\omega_1$ and $\omega_2$ be states of $B$ and $C$, respectively. Let $\omega = (\omega_1 \otimes \omega_2) \circ \phi$. If $\Omega_{\omega_1} \otimes \Omega_{\omega_2}$ is a cyclic vector for $(\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}, (\pi_{\omega_1} \otimes \pi_{\omega_2}) \circ \phi)$, then

$$U(\omega_1, \omega_2; \phi) \Lambda_{\omega_1, \omega_2}(\phi(x)) = \Lambda_{\omega_1, \omega_2}(\phi(x))$$

for $x \in A$ (2.2)

defines a unitary $U(\omega_1, \omega_2; \phi)$ from $\mathcal{H}_{\omega_1}$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ such that

$$(U(\omega_1, \omega_2; \phi))^*(\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi(x))U(\omega_1, \omega_2; \phi) = \pi_\omega(x)$$

for $x \in A$. (2.3)

(iv) In addition to the assumption in (iii), assume that $B = C$ and

$$(\omega_1 \otimes \omega_2) \circ \phi = (\omega_2 \otimes \omega_1) \circ \phi.$$ (2.4)

Define the unitary $R(\omega_1, \omega_2; \phi)$ from $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ to $\mathcal{H}_{\omega_1} \otimes \mathcal{H}_{\omega_2}$ by

$$R(\omega_1, \omega_2; \phi) \Lambda_{\omega_1, \omega_2}(\phi(x)) = \Lambda_{\omega_1, \omega_2}(\phi^{\text{op}}(x))$$

for $x \in A$ (2.5)

where $\phi^{\text{op}} \equiv \tau \circ \phi$ and $\tau$ denotes the flip of $B \otimes B$. Then $R(\omega_1, \omega_2; \phi)$ is well-defined and the following holds:

$$R(\omega_1, \omega_2; \phi)(\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi(x))(R(\omega_1, \omega_2; \phi))^* = (\pi_{\omega_1} \otimes \pi_{\omega_2})(\phi^{\text{op}}(x))$$

for $x \in A$. (2.6)

**Proof.** From the uniqueness of the GNS representation, (i) and (ii) hold. For example, see Proposition 4.5.3 in [8].

(iii) The statement holds from (i) and (ii).

(iv) By (2.4), we obtain $\mathcal{H}_{(\omega_1 \otimes \omega_2) \circ \phi^{\text{op}}} = \mathcal{H}_{(\omega_2 \otimes \omega_1) \circ \phi} = \mathcal{H}_{(\omega_1 \otimes \omega_2) \circ \phi}$. From this and the definition of $R(\omega_1, \omega_2; \phi)$, we see that

$$R(\omega_1, \omega_2; \phi) = U(\omega_1, \omega_2; \phi^{\text{op}})(U(\omega_1, \omega_2; \phi))^*.$$ (2.7)

Hence the statement holds from (iii).

We illustrate Lemma 2.1(iv) in Figure 2.2.
Remark 2.3 Lemma 2.1 holds for any pair of a $*$-homomorphism $\phi$ and a set of states which satisfy assumptions of the cyclicity of the GNS vector and (2.4) without the assumption that $\phi$ is a comultiplication. Since the logic of the proof is very elementary, the assumption about states is essential.

2.2 Proof of Theorem 1.4

We prove Theorem 1.4 in this subsection. Let $(A, \Delta)$ be as in Theorem 1.4. Applying Lemma 2.1(iv) to the case $B = C = A$ and $\phi = \Delta$, we see that $R(\omega_1, \omega_2) = R(\omega_1, \omega_2; \Delta)$. Hence $R(\omega_1, \omega_2)$ is well-defined.

(i) From Lemma 2.1(iv), (1.9) holds. We can verify that (1.10) holds on $\Lambda(\omega_1, \omega_2; \Delta)$. Hence (1.10) holds because $\{\Lambda(\omega_1, \omega_2; \Delta) : x \in A\}$ is dense in $H_{\omega_1} \otimes H_{\omega_2}$.

(ii) This follows from (i).

(iii) Define $F_R \equiv (id \otimes \Delta) \circ \Delta$, $F_L \equiv (\Delta \otimes id) \circ \Delta$. (2.8)

From (1.2), $F_R = F_L$. Let $A_0$ be an algebraic part of $(A, \Delta)$. By assumption, we can write as follows for $x \in A_0$:

$$F_R(x) = \sum_i x'_i \otimes (x''_i)' \otimes (x''_i)'', \quad (2.9)$$

$$F_L(x) = \sum_j (y'_j)' \otimes (y''_j)' \otimes y''_j \quad (2.10)$$

where R.H.S.s of (2.9) and (2.10) are finite sums.

Here we write elements in $S_0$ as $a, b, c, \ldots$ for simplicity of description. From (2.3) and the assumption for $\Omega_a \otimes \Omega_b$, $\pi_a \ast \pi_b$ is unitarily equivalent to $\pi_{a \ast b}$ for each $a, b \in S_0$. From this, $(\pi_a \ast \pi_b) \ast \pi_c$ is unitarily equivalent to $\pi_{a \ast b} \ast \pi_c$. By assumption, $\Omega_{a \ast b} \otimes \Omega_c$ is a cyclic vector for $(H_{a \ast b} \otimes H_b, \pi_{a \ast b} \ast \pi_c)$. Hence $\Omega_a \otimes \Omega_b \otimes \Omega_c$ is a also cyclic vector for $(\pi_a \ast \pi_b) \ast \pi_c$. From this, $\Omega_a \otimes \Omega_b \otimes \Omega_c$ is a cyclic vector for $(\pi_a \otimes \pi_b \otimes \pi_c) \circ F_R$ $(= (\pi_a \otimes \pi_b \otimes \pi_c) \circ F_L)$ on $H_a \otimes H_b \otimes H_c$. Therefore $\{\Lambda_{a,b,c}(F_R(x)) : x \in A_0\}$
is dense in $\mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c$ where $\Lambda_{a,b,c} \equiv \Lambda_a \otimes \Lambda_b \otimes \Lambda_c$. Hence it is sufficient to show (1.11) on $\{\Lambda_{a,b,c}(F_R(x)) : x \in A_0\}$. The following holds:

$$
R_{12}(a,b)R_{13}(a,c)R_{23}(b,c)\Lambda_{a,b,c}(F_R(x)) \\
= R_{13}(a,c)R_{23}(b,c)\Lambda_{a,b,c}((id \otimes \Delta^{op})(\Delta(x))) \\
= \sum \ R_{12}(a,b)R_{13}(a,c)\Lambda_{a,b,c}(x_i' \otimes (x_i'') \otimes (x_i')) \\
= \sum \ R_{12}(a,b)R_{13}(a,c)\Lambda_{a,b,c}((y_j' \otimes y_j'' \otimes (y_j)'')) \quad (by \ F_R(x) = F_L(x)) \\
= \sum \ R_{12}(a,b)\Lambda_{a,b,c}((y_j'') \otimes y_j'' \otimes (y_j')) \\
= \sum \ R_{12}(a,b)\Lambda_{a,b,c}((x_i')' \otimes (x_i'') \otimes x_i') \quad (by \ F_R(x) = F_L(x)) \\
= \sum \ \Lambda_{a,b,c}((x_i')'' \otimes (x_i')'' \otimes x_i') \\
= \Lambda_{a,b,c}((\Delta^{op} \otimes id)(\Delta^{op}(x))).
$$

Remark that every "$\sum$" is a finite sum. As the same token,

$$
R_{23}(b,c)R_{13}(a,c)R_{12}(a,b)\Lambda_{a,b,c}(F_L(x)) = \Lambda_{a,b,c}((id \otimes \Delta^{op})(\Delta^{op}(x))). \quad (2.11)
$$

Hence the statement holds from the coassociativity of $\Delta^{op}$.

### 3 C*-bialgebra defined as the direct sum of Cuntz algebras

We show a set of states of the C*-algebra $\mathcal{O}_s$ in (1.1) which satisfies assumptions in Theorem 1.4 in this section.

#### 3.1 C*-bialgebra $(\mathcal{O}_s, \Delta_\varphi)$

In this subsection, we recall the C*-bialgebra $(\mathcal{O}_s, \Delta_\varphi)$ in [13]. For $n \geq 2$, consider the Cuntz algebra $\mathcal{O}_n$ [3], that is, a C*-algebra which is universally generated by generators $s_1, \ldots, s_n$ satisfying $s_i^*s_j = \delta_{ij}I$ for $i,j = 1, \ldots, n$ and $\sum_{i=1}^n s_is_i^* = I$ where $I$ denotes the unit of $\mathcal{O}_n$. Since $\mathcal{O}_n$ is simple, that is, there is no non-trivial closed two-sided ideal, any *-homomorphism
from $\mathcal{O}_n$ to a C$^*$-algebra is injective. If $t_1, \ldots, t_n$ are elements of a unital C$^*$-algebra $A$ such that $t_1, \ldots, t_n$ satisfy the relations of canonical generators of $\mathcal{O}_n$, then the correspondence $s_i \mapsto t_i$ for $i = 1, \ldots, n$ is uniquely extended to a $*$-embedding of $\mathcal{O}_n$ into $A$ from the uniqueness of $\mathcal{O}_n$. Therefore we simply call such a correspondence among generators by an embedding of $\mathcal{O}_n$ into $A$.

Redefine the C$^*$-algebra $\mathcal{O}_n$ in (1.1) as the direct sum of the set $\{\mathcal{O}_n : n \in \mathbb{N}\}$ of Cuntz algebras:

$$\mathcal{O}_n \equiv \bigoplus_{n \in \mathbb{N}} \mathcal{O}_n = \{(x_n) : \|(x_n)\| \to 0 \text{ as } n \to \infty\}$$

(3.1)

where $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathcal{O}_1$ denotes the 1-dimensional C$^*$-algebra for convenience. For $n \in \mathbb{N}$, let $I_n$ denote the unit of $\mathcal{O}_n$ and let $s_1^{(n)}, \ldots, s_n^{(n)}$ denote canonical generators of $\mathcal{O}_n$ where $s_1^{(1)} \equiv I_1$. For $n, m \in \mathbb{N}$, define $\varphi_{n,m} \in \text{Hom}(\mathcal{O}_{nm}, \mathcal{O}_n \otimes \mathcal{O}_m)$ by

$$\varphi_{n,m}(s_{m(i-1)+j}^{(nm)}) \equiv s_i^{(n)} \otimes s_j^{(m)} \quad (i = 1, \ldots, n, j = 1, \ldots, m).$$

(3.2)

Theorem 3.1 ([13], Theorem 1.1) For the set $\varphi \equiv \{\varphi_{n,m} : n, m \in \mathbb{N}\}$ in (3.2), define the $*$-homomorphism $\Delta_\varphi$ from $\mathcal{O}_n \otimes \mathcal{O}_m$ into $\mathcal{O}_n$ by

$$\Delta_\varphi \equiv \bigoplus \{\Delta_{\varphi}^{(n)} : n \in \mathbb{N}\},$$

(3.3)

$$\Delta_{\varphi}^{(n)}(x) \equiv \sum_{(m,l) \in \mathbb{N}^2, ml=n} \varphi_{m,l}(x) \quad (x \in \mathcal{O}_n, n \in \mathbb{N}).$$

(3.4)

Then $(\mathcal{O}_n, \Delta_\varphi)$ is an algebraic C$^*$-bialgebra such that $\Delta_\varphi(\mathcal{O}_n) \subset \mathcal{O}_n \otimes \mathcal{O}_n$.

About properties of $(\mathcal{O}_n, \Delta_\varphi)$, see [13, 15]. About a generalization of $(\mathcal{O}_n, \Delta_\varphi)$, see [14].

Let $\text{Rep}\mathcal{O}_n$ denote the class of all $*$-representations of $\mathcal{O}_n$. For $\pi_1, \pi_2 \in \text{Rep}\mathcal{O}_n$, we define the relation $\pi_1 \sim \pi_2$ if $\pi_1$ and $\pi_2$ are unitarily equivalent. Then the following holds.

Lemma 3.2 ([12], Lemma 1.2) For $\varphi_{n,m}$ in (3.2), $\pi_1 \in \text{Rep}\mathcal{O}_n$ and $\pi_2 \in \text{Rep}\mathcal{O}_m$, define $\pi_1 \otimes_\varphi \pi_2 \in \text{Rep}\mathcal{O}_{nm}$ by

$$\pi_1 \otimes_\varphi \pi_2 \equiv (\pi_1 \otimes \pi_2) \circ \varphi_{n,m}.$$  

(3.5)

Then the following holds for $\pi_1, \pi_1' \in \text{Rep}\mathcal{O}_n$, $\pi_2, \pi_2' \in \text{Rep}\mathcal{O}_m$ and $\pi_3 \in \text{Rep}\mathcal{O}_l$: 

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(i) If $\pi_1 \sim \pi'_1$ and $\pi_2 \sim \pi'_2$, then $\pi_1 \otimes \varphi \pi_2 \sim \pi'_1 \otimes \varphi \pi'_2$.

(ii) $\pi_1 \otimes \varphi (\pi_2 \oplus \pi'_2) = \pi_1 \otimes \varphi \pi_2 \oplus \pi_1 \otimes \varphi \pi'_2$.

(iii) $\pi_1 \otimes \varphi (\pi_2 \otimes \varphi \pi_3) = (\pi_1 \otimes \varphi \pi_2) \otimes \varphi \pi_3$.

From Lemma 3.2(i), we can define $[\pi_1] \otimes \varphi [\pi_2] \equiv [\pi_1 \otimes \varphi \pi_2]$ where $[\pi]$ denotes the unitary equivalence class of $\pi$.

Let $S_n$ denote the set of all states of $O_n$. For $(\omega, \omega') \in S_n \times S_m$, define

$$\omega \otimes \varphi \omega' \equiv (\omega \otimes \omega') \circ \varphi_{nm} \quad (3.6)$$

where $(\omega \otimes \omega')(x \otimes y) \equiv \omega(x)\omega'(y)$ for $x \in O_n$ and $y \in O_m$. Then we see that $\omega \otimes \varphi (\omega' \otimes \varphi \omega'') = (\omega \otimes \varphi \omega') \otimes \varphi \omega''$.

By identifying $O_n$ with a C*-subalgebra of $O_*$, any state and representation of $O_n$ are naturally identified with those of $O_*$. From this and the definition of $\Delta_\varphi$, the following holds:

$$\pi_1 \otimes \varphi \pi_2 = (\pi_1 \otimes \pi_2) \circ \Delta_\varphi, \quad \omega_1 \otimes \varphi \omega_2 = (\omega_1 \otimes \omega_2) \circ \Delta_\varphi. \quad (3.7)$$

### 3.2 Pure states of Cuntz algebras parametrized by unit vectors

In this subsection, we show examples of set of states which satisfies assumptions in Theorem 1.4. We recall states in 11 and show tensor product formulae among them. Let $S(C^n)$ denote the set of all unit vectors in $C^n$ for $n \geq 2$.

**Definition 3.3 (11, Proposition 3.1)** For $n \geq 2$, let $s_1, \ldots, s_n$ denote canonical generators of $O_n$. For $z = (z_1, \ldots, z_n) \in S(C^n)$, define the state $\varrho_z$ of $O_n$ by

$$\varrho_z(s_{j_1} \cdot \cdot \cdot s_{j_a} s_{k_b} \cdot \cdot \cdot s_{k_1}) \equiv \overline{z_{j_1}} \cdot \cdot \cdot \overline{z_{j_a}} z_{k_b} \cdot \cdot \cdot z_{k_1} \quad (3.8)$$

for each $j_1, \ldots, j_a, k_1, \ldots, k_b \in \{1, \ldots, n\}$ and $a, b \geq 1$.

The following results for $\varrho_z$ are known: For any $z$, $\varrho_z$ is pure when $n \geq 2$. Define the state $\varrho_1$ of $O_1 = C$ by $\varrho_1(x) = x$ for each $x \in O_1$. We define $S(C^1) = \{1\} \subset C$ for convenience. Then $\varrho_z$ in (3.8) makes sense for each $z \in \bigcup_{n \geq 1} S(C^n)$. If $z, y \in S(C^n)$ and $z \neq y$, then GNS representations associated with $\varrho_z$ and $\varrho_y$ are not unitarily equivalent.

Let $GP(z)$ denote the unitary equivalence class of the GNS representation associated with $\varrho_z$. If $\pi_1$ and $\pi_2$ are representatives of $GP(z)$ and $GP(y)$ for $z \in S(C^n)$ and $y \in S(C^m)$, respectively, then we write $GP(z) \otimes \varphi GP(y)$ as $[\pi_1] \otimes \varphi [\pi_2]$ for simplicity of description.
Theorem 3.4 (15, Theorem 3.2) For $\otimes \varphi$ in (3.7), the following holds for each $z \in S(C^n)$ and $y \in S(C^m)$:

(i) $\varrho_z \otimes \varphi \varrho_y = \varrho_z \varrho_y$,
(ii) $GP(z) \otimes \varphi GP(y) = GP(z \boxtimes y)$

where $z \boxtimes y \in S(C^{nm})$ is defined as

$$
(z \boxtimes y)_{m(i-1)+j} \equiv z_i y_j \quad (i = 1, \ldots, n, j = 1, \ldots, m). \quad (3.9)
$$

For $z \in S(C^n)$, let $(H_z, \pi_z, \Omega_z)$ denote the GNS triple associated with $\varrho_z$. From Theorem 3.4(ii), $(H_z \otimes H_y, \pi_z \otimes \varphi \pi_y)$ is irreducible for each $z, y \in \bigcup_{n \geq 1} S(C^n)$. From this, $\Omega_z \otimes \Omega_y$ is a cyclic vector for $(H_z \otimes H_y, \pi_z \otimes \varphi \pi_y)$.

In consequence,

$$S_0 \equiv \bigcup_{n \geq 1} \{ \varrho_z : z \in S(C^n) \} \quad (3.10)$$

satisfies all assumptions in Theorem 1.4 with respect to $(O_\ast, \Delta_\varphi)$. Furthermore, if $\mathcal{M}$ is a subsemigroup of the semigroup $(S_0, \boxtimes)$, then $\{ \varrho_z : z \in \mathcal{M} \}$ also satisfies all assumptions in Theorem 1.4. Such subsemigroups are shown in the last paragraph of § 3 in [15].

3.3 Examples

In this subsection, we show two examples of Theorem 1.4.

(i) We show a nontrivial example of Theorem 1.4. Let $\{s_i^{(n)}\}_{i=1}^n$ denote the canonical generators of $O_n$ and let $l_2(\mathbb{N})$ denote the Hilbert space with the orthonormal basis $\{e_k : k \in \mathbb{N}\}$ where $\mathbb{N} = \{1, 2, 3, \ldots\}$. Define the representation $\pi_n$ of $O_n$ on $l_2(\mathbb{N})$ by

$$
\pi_n(s_i^{(n)})e_k \equiv e_{n(k-1)+i} \quad (i = 1, \ldots, n, k \in \mathbb{N}). \quad (3.11)
$$

Let $\omega_n$ denote the vector state of $O_n$ associated with the unit vector $e_1$. Then $\omega_n \ast \omega_m = \omega_{nm}$ for each $n, m \geq 2$. We identify $\pi_n$ with the GNS representation of $O_n$ by $\omega_n$. Here we consider cases $n = 2, 3$. Let $R \equiv R(\omega_2, \omega_3)$ be as in (1.8). Remark that

$$
e_1 \otimes e_3 = (\pi_2 \otimes \pi_3)(\Delta_\varphi(s_3^{(6)}))(e_1 \otimes e_1) = (\pi_2 \otimes \pi_3)(\varphi_{2,3}(s_3^{(6)}))(e_1 \otimes e_1). \quad (3.12)
$$
Then

\[ R(e_1 \otimes e_3) = (\pi_2 \otimes \pi_3) (\varphi_{3,2}^{op}(s_3^{(6)})) (e_1 \otimes e_1) = \pi_2(s_1^{(2)}) e_1 \otimes \pi_3(s_2^{(3)}) e_1 = e_1 \otimes e_2. \]

In this way, \( R \) is not the identity operator on \( l_2(N) \otimes l_2(N) \).

(ii) We show an example which does not satisfy (1.9) because of the non-commutativity of states. Let \( \omega_2 \) and \( \pi_2 \) be as in the previous example. Define \( \omega_2' \equiv \omega_2 \circ \alpha \) where \( \alpha \) denotes the flip automorphism of the canonical generators of \( \mathcal{O}_2 \) and let \( \pi_2' \) denote the GNS representation of \( \mathcal{O}_2 \) by \( \omega_2 \) which is identified with \( \pi_2 \circ \alpha \). Then \( \omega_2' \not= \omega_2 \).

Let \( v \equiv e_1 \otimes e_1 \). Then \( (\pi_2 \otimes \pi_2') (\Delta_{\varphi}(s_2^{(4)})) v = v \). Define the operator \( R \) on \( l_2(N) \otimes l_2(N) \) by

\[ R(\pi_2 \otimes \pi_2')(\Delta_{\varphi}(x)) v \equiv (\pi_2 \otimes \pi_2')(\Delta_{\varphi}^{op}(x)) v \quad (x \in \mathcal{O}_4). \] (3.13)

Then

\[ R(\pi_2 \otimes \pi_2')(\Delta_{\varphi}(s_2^{(4)})) R^* v = R(\pi_2 \otimes \pi_2')(\Delta_{\varphi}(s_2^{(4)})) v = R v = v. \] (3.14)

Since \( (\pi_2 \otimes \pi_2')(\Delta_{\varphi}^{op}(s_2^{(4)})) = \pi_2(s_2^{(2)}) \otimes \pi_2(s_1^{(2)}) \) and \( \langle v | (\pi_2(s_2^{(2)}) \otimes \pi_2(s_1^{(2)})) v \rangle = 0 \), we see that \( v \not= (\pi_2(s_2^{(2)}) \otimes \pi_2(s_1^{(2)})) v \). Hence \( R(\pi_2 \otimes \pi_2')(\Delta_{\varphi}(s_2^{(4)})) R^* \not= (\pi_2 \otimes \pi_2')(\Delta_{\varphi}^{op}(s_2^{(4)})). \)

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