Solution of the System of Two Coupled First-Order ODEs with Second-Degree Polynomial Right-Hand Sides

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Abstract
The explicit solution $x_n(t)$, $n = 1, 2$, of the initial-values problem is exhibited of a subclass of the autonomous system of 2 coupled first-order ODEs with second-degree polynomial right-hand sides, hence featuring 12 a priori arbitrary (time-independent) coefficients:

$$\dot{x}_n = c_{n1} (x_1)^2 + c_{n2} x_1 x_2 + c_{n3} (x_2)^2 + c_{n4} x_1 + c_{n5} x_2 + c_{n6} , \quad n = 1, 2 .$$

The solution is explicitly provided if the 12 coefficients $c_{nj}$ ($n = 1, 2; \quad j = 1, 2, 3, 4, 5, 6$) are expressed by explicitly provided formulas in terms of 10 a priori arbitrary parameters; the inverse problem to express these 10 parameters in terms of the 12 coefficients $c_{nj}$ is also explicitly solved, but it is found to imply—as it were, a posteriori—that the 12 coefficients $c_{nj}$ must then satisfy 4 algebraic constraints, which are explicitly exhibited. Special subcases are also identified the general solutions of which are completely periodic with a period independent of the initial data ("isochrony"), or are characterized by additional restrictions on the coefficients $c_{nj}$ which identify particularly interesting models.

Keywords Solvable dynamical systems · Solvable systems of nonlinearly-coupled ordinary differential equations

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1 Introduction

It is well known that the investigation of the properties of the solutions of autonomous systems of nonlinearly coupled ODEs is one of the fundamental problems of pure mathematics, applied mathematics and mathematical physics. It is of course impossible to provide a review of these developments, which involve over time an enormous number of contributions, both papers and books. We limit ourselves here to quote only the ground-breaking paper [1] on the type of dynamical systems we treat in this paper and two recent books on these systems [2, 3].

In this paper we focus on one of the simplest—yet far from trivial—problems of this kind: the system characterized by just 2 nonlinearly-coupled ODEs with quadratic polynomial right-hand sides, hence reading

\[ \dot{x}_n(t) = c_{n1} [x_1(t)]^2 + c_{n2} x_1(t) x_2(t) + c_{n3} [x_2(t)]^2 + c_{n4} x_1(t) + c_{n5} x_2(t) + c_{n6}, \quad n = 1, 2, \]

namely

\[ \dot{x}_1(t) = c_{11} [x_1(t)]^2 + c_{12} x_1(t) x_2(t) + c_{13} [x_2(t)]^2 + c_{14} x_1(t) + c_{15} x_2(t) + c_{16}, \]

\[ \dot{x}_2(t) = c_{21} [x_1(t)]^2 + c_{22} x_1(t) x_2(t) + c_{23} [x_2(t)]^2 + c_{24} x_1(t) + c_{25} x_2(t) + c_{26}. \]

This model—which can be considered a prototypical example of autonomous dynamical systems—features the 12 a priori arbitrary time-independent parameters \( c_{nj} \) (\( n = 1, 2; j = 1, 2, 3, 4, 5, 6 \)). We previously investigated the special case with homogeneous right-hand sides (i.e. the system (1) with \( c_{nj} = 0 \) for \( n = 1, 2 \) and \( j > 3 \)) [4], and several analogous problems [5]; these papers have motivated this research, but of course the findings reported below are, to the best of our knowledge, new; while their applicative potential—as well of course as its limitations—shall be quite obvious to any practitioner of the applications of this kind of mathematics, be it to chemical reactions, population dynamics (including epidemiology), economics, electrical grids, many-body problems, you name it.

The main finding of the present paper is to show that the initial-values problem of the dynamical system (1) can be explicitly solved, provided the 12 coefficients \( c_{nj} \) (\( n = 1, 2; j = 1, 2, 3, 4, 5, 6 \)) satisfy 4 simple constraints: for a neat version of these constraints see below Section 4 where the solution of the initial-values problem of the system (1) is displayed.

**Notation** The 2 (possibly complex) numbers \( x_n \equiv x_n(t), n = 1, 2 \), are the dependent variables; \( t \) is the independent variable (“time”; but the treatment remains valid when \( t \) is considered a complex number); superimposed dots denote \( t \) -differentiations; the 12 time-independent (possibly complex) numbers \( c_{nj}, n = 1, 2, j = 1, 2, 3, 4, 5, 6 \), are parameters. Hereafter the time-dependence of variables is often not explicitly indicated, when this omission is unlikely to cause misunderstandings. The indices \( n, m, j, k, \ell \)—as indeed generally clear from the context—run respectively over the integers from 1 to 2 (\( n = 1, 2 \)), from 1 to 2 (\( m = 1, 2 \)), from 1 to 6 (\( j = 1, 2, 3, 4, 5, 6 \)), from 1 to 3 (\( k = 1, 2, 3 \)) and from 2 to 0 (\( \ell = 2, 1, 0 \)).
Remark 1.1 The system (1) is clearly invariant under the symmetry transformation

\[ \begin{align*}
&c_{11} \Leftrightarrow c_{23}, \\
&c_{12} \Leftrightarrow c_{22}, \\
&c_{13} \Leftrightarrow c_{21}, \\
&c_{14} \Leftrightarrow c_{25}, \\
&c_{15} \Leftrightarrow c_{24}, \\
&c_{16} \Leftrightarrow c_{26}; \\
&x_1(t) \Leftrightarrow x_2(t).
\end{align*} \] (2)

In Section 2 the technique used in this paper to solve the initial-values problem of system (1) is described: it involves the introduction of 10 parameters \(A_{nm}\) and \(a_{n\ell}\), in terms of which the 12 parameters \(c_{nj}\) are expressed; the inverse problem to express these 10 parameters \(A_{nm}\) and \(a_{n\ell}\) in terms of the 12 \textit{a priori arbitrary} parameters \(c_{nj}\) is then solved in Section 3, and it is shown there that this entails—as it were, \textit{a posteriori}—that the 12 parameters \(c_{nj}\) are thereby required to satisfy 4 rather simple constraints, which are explicitly exhibited. The reader who is only interested in the main results may jump over these 2 sections and go directly to Section 4, where a summary of the main results of this paper is presented. The subsequent Section 5 is devoted to two special cases of the system (1) which deserve a separate treatment; and an extremely terse Section 6 completes the main body of the paper; which also includes the 3 short Appendices A, B and C.

2 The Technique to Solve the System (1)

First position:

\[ x_n(t) = A_{n1}y_1(t) + A_{n2}y_2(t), \] (3a)

namely

\[ x_1(t) = A_{11}y_1(t) + A_{12}y_2(t), \] (3b)

\[ x_2(t) = A_{21}y_1(t) + A_{22}y_2(t). \] (3c)

This assignment implies the introduction of the 4, \textit{a priori arbitrary}, time-independent parameters \(A_{nm}\) \((n = 1, 2; m = 1, 2)\) and of the 2 \textit{auxiliary} variables \(y_1(t)\) and \(y_2(t)\).

Remark 2.1 This assignment is clearly invariant under the transformation

\[ x_1(t) \leftrightarrow x_2(t), \ y_1(t) \leftrightarrow y_2(t), \ A_{11} \leftrightarrow A_{22}, \ A_{12} \leftrightarrow A_{21}. \] (4)

Remark 2.2 In the following (except in Section 5.1) we generally assume that none of the 4 parameters \(A_{nm}\) vanishes, \(A_{nm} \neq 0\).

Remark 2.3 The addition of 2 \textit{a priori arbitrary} additional parameters—say \(A_1\) respectively \(A_2\)—to the right-hand sides of the 2 eqs. (3) would only complicate the following developments without providing any significant additional generality to our treatment.
Evolution of the auxiliary variables \( y_n(t) \). Let us hereafter assume that \( y_1(t) \) and \( y_2(t) \) evolve according to the following system of 2 decoupled ODEs:

\[
\dot{y}_n(t) = a_{n2} [y_n(t)]^2 + a_{n1} y_n(t) + a_{n0},
\]

namely

\[
\dot{y}_1(t) = a_{12} [y_1(t)]^2 + a_{11} y_1(t) + a_{10},
\]

\[
\dot{y}_2(t) = a_{22} [y_2(t)]^2 + a_{21} y_2(t) + a_{20}.
\]

The initial-values problem of this system of 2 (decoupled) ODEs—which involve the 6 a priori arbitrary time-independent parameters \( a_{n\ell} \) \( (n = 1, 2; \ell = 2, 1, 0) \)—is easily seen to be explicitly solvable (see Appendix A):

\[
y_n(t) = \frac{y_n(0)[y_{n+} - y_{n-} \exp(\beta_n t)] - y_{n+} y_{n-} [1 - \exp(\beta_n t)]}{y_{n+} \exp(\beta_n t) - y_{n-} + y_n(0) [1 - \exp(\beta_n t)]}, \quad n = 1, 2,
\]

with the 6 (time-independent) parameters \( y_{n\pm} \) and \( \beta_n \) defined (above and hereafter) in terms of the 6 parameters \( a_{n\ell} \) as follows:

\[
y_{n\pm} = (-a_{n1} \pm \beta_n) / (2a_{n2}), \quad \beta_n = \sqrt{(a_{n1})^2 - 4a_{n0} a_{n2}}, \quad n = 1, 2.
\]

Remark 2.4 The system of 2 decoupled ODEs (5) is clearly invariant under the transformation

\[
y_1(t) \Leftrightarrow y_2(t), \quad a_{1\ell} \Leftrightarrow a_{2\ell}, \quad \ell = 2, 1, 0.
\]

Hence the combination of this invariance property with that reported above—see Remark 2.1—clearly entails the overall invariance property of the 2 systems of ODEs (1) and (5), as well as the change of variables (3), under the following transformations:

\[
x_1(t) \Leftrightarrow x_2(t), \quad y_1(t) \Leftrightarrow y_2(t), \quad A_{11} \Leftrightarrow A_{22}, \quad A_{12} \Leftrightarrow A_{21},
\]

\[
a_{11} \Leftrightarrow a_{21}, \quad a_{12} \Leftrightarrow a_{22}, \quad a_{10} \Leftrightarrow a_{20}.
\]

Let us emphasize that, while the solutions (6) are rather simple, their behaviors as functions of \( t \)—even for real \( t \) (“time”—can be fairly complicated if the parameters \( \beta_n \) are themselves not real; as may well be the case even if all the parameters \( a_{n\ell} \) are real numbers: see (6b). On the other hand if both \( \beta_n = i\rho_n \omega \) with \( n = 1, 2, \omega \) an arbitrary (nonvanishing) real number and with both parameters \( \rho_n \) rational (nonvanishing) real numbers, then clearly (or, if need be, see for instance [6, 7]) the evolution of both the pairs \( y_n(t) \) and \( x_n(t) \)—as functions of real \( t \) (“time”—is completely periodic with a period independent of the respective initial data, which is an integer multiple of the basic period \( 2\pi / |\omega| \) (“isochrony”).

Let us also note that \( y_{n\pm} \) (see (6b)) are the equilibrium positions of the system (5), while the asymptotic behavior of \( y_n(t) \) as the (real) time \( t \) diverges is always rather simple: indeed if \( \text{Re} [\beta_n] < 0 \), then

\[
\lim_{t \to +\infty} [y_n(t)] = y_{n+}.
\]
while if $\text{Re}[\beta_n] > 0$ then
\[
\lim_{t \to +\infty} [y_n(t)] = y_n. \tag{9b}
\]

And clearly the corresponding *equilibrium* configurations and *asymptotic* behaviors of the variables $x_n(t)$ are as well rather simple (see (3)): but note that the asymptotic behavior of these variables $x_n(t)$ is *asymptotically isochronous* (see [6, 8]), when one and only one of the 2 quantities $\beta_n$ is *purely imaginary*.

It is of course evident that the *coupled* system of 2 ODEs implied by the relations (3) and by the 2 ODEs (5) satisfied by the 2 variables $y_n(t)$ is *identical* to the system of 2 ODEs (1) satisfied by the 2 variables $x_n(t)$, of course provided the 12 parameters $c_{nj}$ are appropriately expressed in terms of the 10 $= 4 + 6$ parameters $A_{nm}$ in (3) and $a_{n\ell}$ in (5). The corresponding computation of the explicit expressions of the 12 parameters $c_{nj}$ in terms of the 10 parameters $A_{nm}$, $a_{n\ell}$ is a standard—if tedious—task (see Appendix B), yielding (for the 6 parameters $c_{1j}$) the following results:

\[
c_{11} = \left[a_{12}A_{11} (A_{22})^2 + a_{22}A_{12} (A_{21})^2\right]/D^2, \tag{10a}
\]

\[
c_{12} = -2A_{11}A_{12} [a_{12}A_{22} + a_{22}A_{21}]/D^2, \tag{10b}
\]

\[
c_{13} = A_{11}A_{12} [a_{12}A_{12} + a_{22}A_{11}]/D^2, \tag{10c}
\]

\[
c_{14} = (a_{11}A_{11}A_{22} - a_{21}A_{12}A_{21})/D, \tag{10d}
\]

\[
c_{15} = -(a_{11} - a_{21}) A_{11}A_{12}/D, \tag{10e}
\]

\[
c_{16} = a_{10}A_{11} + a_{20}A_{12}, \tag{10f}
\]

where the quantity $D$ is defined, above and hereafter, as follows:

\[
D = A_{11}A_{22} - A_{12}A_{21}. \tag{11}
\]

The analogous formulas for the 6 parameters $c_{2j}$ can be obtained from those written above via the transformations (see Remarks 1.1, 2.1 and 2.2)

\[
c_{11} \Leftrightarrow c_{23} , \quad c_{12} \Leftrightarrow c_{22} , \quad c_{13} \Leftrightarrow c_{21} , \quad c_{14} \Leftrightarrow c_{25} , \quad c_{15} \Leftrightarrow c_{24} , \quad c_{16} \Leftrightarrow c_{26} ;
\]

\[
A_{11} \Leftrightarrow A_{22} , \quad A_{12} \Leftrightarrow A_{21} ; \quad a_{1\ell} \Leftrightarrow a_{2\ell} , \tag{12}
\]

under which the quantity $D$, see (11), is clearly invariant.

Hence they read as follows:

\[
c_{23} = \left[a_{22}A_{22} (A_{11})^2 + a_{12}A_{21} (A_{12})^2\right]/D^2, \tag{13a}
\]

\[
c_{22} = -2A_{22}A_{21} [a_{22}A_{11} + a_{12}A_{12}]/D^2, \tag{13b}
\]

\[
c_{21} = A_{22}A_{21} [a_{22}A_{21} + a_{12}A_{22}]/D^2, \tag{13c}
\]

\[
c_{25} = (a_{21}A_{11}A_{22} - a_{11}A_{12}A_{21})/D, \tag{13d}
\]

\[
c_{24} = -(a_{21} - a_{11}) A_{22}A_{21}/D, \tag{13e}
\]

\[
c_{26} = a_{20}A_{22} + a_{10}A_{21}. \tag{13f}
\]

These findings of course imply that the solution of the *initial-values* problem for the system (1) is provided via the formulas (3) from the *explicit* solutions (6) of the *initial-values* problem for the system (5), hence essentially via rather simple, quite *explicit*, *algebraic* operations; this can be done for any assignment of the 12 parameters $c_{nj}$, such that the *explicit* formulas (10) and (13)—expressing the 12 parameters...
$c_{nj}$ in terms of the 10 parameters $A_{nm}$ and $a_{n\ell}$—can be inverted: in the following Section 3 we show how this can be done, provided the 12 parameters $c_{nj}$ satisfy 4 constraints.

3 The Inverse Problem: Expressing the 10 Parameters $A_{nm}$ and $a_{n\ell}$ in Terms of the 12 Coefficients $c_{nj}$

Our main task in this Section 3 is to discuss the inversion of the 12 relations obtained above—see (10) and (13)—expressing the 12 coefficients $c_{nj}$ in terms of the 10 parameters $A_{nm}$ and $a_{n\ell}$; namely to show how, given 12 a priori arbitrary coefficients $c_{nj}$, the 10 corresponding parameters $A_{nm}$ and $a_{n\ell}$ can be computed. We show below how this task can be explicitly accomplished; but that it entails that the 12 coefficients $c_{nj}$ ($n = 1, 2; j = 1, 2, 3, 4, 5, 6$) must—as it were, a posteriori—satisfy 4 algebraic conditions (explicitly obtained below).

An obvious route to achieve our main task is to try and solve the 12 algebraic equations (10) and (13) for the 10 parameters $A_{nm}$ and $a_{n\ell}$; but given the fairly large number of these algebraic relations and their nonlinear character this is a nontrivial job (beyond the power of standard algebraic manipulation packages such as Mathematica, used on a modern PC). More progress in this direction can be made via the following alternative procedure.

Let us note that the relation (77a) in Appendix B implies that the variable $y_1(t)$ clearly satisfies the ODE

$$\dot{y}_1 = (A_{22}\dot{x}_1 - A_{12}\dot{x}_2)/D, \quad (14a)$$

hence, via (1),

$$\dot{y}_1 = A_{22}\left[ c_{11} (x_1)^2 + c_{12}x_1x_2 + c_{13} (x_2)^2 + c_{14}x_1 + c_{15}x_2 + c_{16} \right]$$

$$- A_{12}\left[ c_{21} (x_1)^2 + c_{22}x_1x_2 + c_{23} (x_2)^2 + c_{24}x_1 + c_{25}x_2 + c_{26} \right]/D, \quad (14b)$$

yielding, via (3) and some trivial if tedious algebra,

$$\dot{y}_1 = b_{11} (y_1)^2 + b_{12}y_1y_2 + b_{13} (y_2)^2 + b_{14}y_1 + b_{15}y_2 + b_{16}, \quad (15)$$

with

$$b_{11} = \left[ (A_{11})^2(-A_{12}c_{21} + A_{22}c_{11}) + A_{11}A_{21}(-A_{12}c_{22} + A_{22}c_{12}) \right. \left. + (A_{21})^2(-A_{12}c_{23} + A_{22}c_{13}) \right]/D, \quad (16a)$$

$$b_{12} = A_{11} \left[ -2(A_{12})^2c_{21} + A_{12}A_{22}(2c_{11} - c_{22}) + (A_{22})^2c_{12} \right]$$

$$+ A_{21} \left[ -(A_{12})^2c_{22} + A_{12}A_{22}(c_{12} - 2c_{23}) + 2(A_{22})^2c_{13} \right]/D, \quad (16b)$$
\[ b_{13} = \left\{ (A_{12})^2 [-A_{12}c_{21} + A_{22}(c_{11} - c_{22})] \\
+ (A_{22})^2 [A_{12}(c_{12} - c_{23}) + A_{22}c_{13}] \right\} / D, \quad (16c) \]
\[ b_{14} = [A_{11} (-A_{12}c_{24} + A_{22}c_{14}) + A_{21} (-A_{12}c_{25} + A_{22}c_{15})] / D, \]
\[ b_{15} = \left[ - (A_{12})^2 c_{24} + A_{12}A_{22}(c_{14} - c_{25}) + (A_{22})^2 c_{15} \right] / D, \quad (16d) \]
\[ b_{16} = (-A_{12}c_{26} + A_{22}c_{16}) / D. \quad (16e) \]

And now a comparison of (5b) with (15) yields, via (16), the following 6 relations:

\[ a_{12} = \left[ (A_{11})^2 (-A_{12}c_{21} + A_{22}c_{11}) + A_{11}A_{21} (-A_{12}c_{22} + A_{22}c_{12}) \\
+ (A_{21})^2 (-A_{12}c_{23} + A_{22}c_{13}) \right] / D, \quad (17a) \]
\[ a_{11} = [A_{11} (-A_{12}c_{24} + A_{22}c_{14}) + A_{21} (-A_{12}c_{25} + A_{22}c_{15})] / D, \quad (17b) \]
\[ a_{10} = (-A_{12}c_{26} + A_{22}c_{16}) / D; \quad (17c) \]

\[ A_{11} \left[ -2 (A_{12})^2 c_{21} + A_{12}A_{22}(2c_{11} - c_{22}) + (A_{22})^2 c_{12} \right] \\
+ A_{21} \left[ -(A_{12})^2 c_{22} + A_{12}A_{22}(c_{12} - 2c_{23}) + 2 (A_{22})^2 c_{13} \right] = 0, \quad (18a) \]

\[ (A_{12})^2 [-A_{12}c_{21} + A_{22}(c_{11} - c_{22})] \\
+ (A_{22})^2 [A_{12}(c_{12} - c_{23}) + A_{22}c_{13}] = 0, \quad (18b) \]

\[ - (A_{12})^2 c_{24} + A_{12}A_{22}(c_{14} - c_{25}) + (A_{22})^2 c_{15} = 0. \quad (18c) \]

By a completely analogous development, based on the ODE (5c) satisfied by \( y_2(t) \) rather than (5b) satisfied by \( y_1(t) \)—or, more easily, via the symmetry properties associated to the transformation (12)—one gets the following 6 additional relations:

\[ a_{22} = \left[ (A_{22})^2 (-A_{21}c_{13} + A_{11}c_{23}) + A_{22}A_{12} (-A_{21}c_{12} + A_{11}c_{22}) \\
+ (A_{12})^2 (-A_{21}c_{11} + A_{11}c_{21}) \right] / D, \quad (19a) \]
\[ a_{21} = [A_{22} (-A_{21}c_{15} + A_{11}c_{25}) + A_{12} (-A_{21}c_{14} + A_{11}c_{24})] / D, \quad (19b) \]
\[ a_{20} = (-A_{21}c_{16} + A_{11}c_{26}) / D; \quad (19c) \]

\[ A_{22} \left[ -2 (A_{21})^2 c_{13} + A_{21}A_{11}(2c_{23} - c_{12}) + (A_{11})^2 c_{22} \right] \\
+ A_{12} \left[ -(A_{21})^2 c_{12} + A_{21}A_{11}(c_{22} - 2c_{11}) + 2 (A_{11})^2 c_{21} \right] = 0, \quad (20a) \]

\[ (A_{21})^2 [-A_{21}c_{13} + A_{11}(c_{23} - c_{12})] \\
+ (A_{11})^2 [A_{21}(c_{22} - c_{11}) + A_{11}c_{21}] = 0, \quad (20b) \]

\[ - (A_{21})^2 c_{15} + A_{21}A_{11}(c_{25} - c_{14}) + (A_{11})^2 c_{24} = 0. \quad (20c) \]

It is thus seen that the 6 parameters \( c_{n\ell} (n = 1, 2; \ell = 2, 1, 0) \) are given explicitly by the 6 formulas (17) and (19) in terms of the 12 coefficients \( c_{nj} \) and the 4 parameters \( A_{nm} \).
The remaining task is to extract the expressions of the 4 parameters $A_{nm}$ in terms of the 6 parameters $c_{nk}$ ($n = 1, 2; k = 1, 2, 3$), the only ones featured in the remaining 6 algebraic equations (18) and (20).

To this end, let us now introduce the 2 auxiliary parameters $z_1$ and $z_2$:

$$z_1 = A_{11}/A_{21}, \quad z_2 = A_{12}/A_{22};$$  \hspace{1cm} (21)

it is then easily seen that the 2 eqs. (18b) and (20b) yield—recall Remark 2.2—the same cubic equation for these 2 quantities:

$$c_{21} (z_n)^3 + (c_{22} - c_{11}) (z_n)^2 + (c_{23} - c_{12}) z_n - c_{13} = 0, \quad n = 1, 2,$$  \hspace{1cm} (22a)

namely

$$c_{21} (z_1)^3 + (c_{22} - c_{11}) (z_1)^2 + (c_{23} - c_{12}) z_1 - c_{13} = 0,$$  \hspace{1cm} (22b)

$$c_{21} (z_2)^3 + (c_{22} - c_{11}) (z_2)^2 + (c_{23} - c_{12}) z_2 - c_{13} = 0.$$  \hspace{1cm} (22c)

Remark 3.1 Note that, consistently with the transformations (2) and (8), the corresponding transformations of the 2 auxiliary parameters $z_n$ are $z_1 \leftrightarrow 1/z_2$ and (of course) $z_2 \leftrightarrow 1/z_1$, implying the invariance under all these transformations of the eqs. (22).

The 2 algebraic equations (22) allow to compute (explicitly, via the Cardano formulas) the 2 quantities $z_n$ in terms of the 6 coefficients $c_{nk}$; of course, they do not imply that the 2 quantities $z_1$ and $z_2$ coincide, indeed we exclude hereafter this possibility because it would imply the vanishing of $D$ (see (11) and (21)).

But more progress is possible. Indeed, let us take advantage of the definitions (21) to rewrite the 2 eqs. (18a) and (20a), getting thereby (again recalling Remark 2.2)

$$\left[-2c_{21} (z_{n+1})^2 + (2c_{11} - c_{22}) z_{n+1} + c_{12}\right] z_n$$

$$-c_{22} (z_{n+1})^2 + (c_{12} - 2c_{23}) z_{n+1} + 2c_{13} = 0, \quad n = 1, 2 \text{ mod } [2],$$  \hspace{1cm} (23a)

namely (also dividing by 2)

$$\left[-c_{21} (z_2)^2 + (c_{11} - c_{22}/2) z_2 + c_{12}/2\right] z_1$$

$$-c_{22} (z_2)^2/2 + (c_{12}/2 - c_{23}) z_2 + c_{13} = 0,$$  \hspace{1cm} (23b)

$$\left[-c_{21} (z_1)^2 + (c_{11} - c_{22}/2) z_1 + c_{12}/2\right] z_2$$

$$-c_{22} (z_1)^2/2 + (c_{12}/2 - c_{23}) z_1 + c_{13} = 0.$$  \hspace{1cm} (23c)

Let us now sum the 2 eqs. (22b) and (23c), and likewise the 2 eqs. (22c) and (23b). We thus obtain (using the fact that $z_1 - z_2 \neq 0$; see above) the same quadratic equation for the 2 quantities $z_1$ and $z_2$:

$$2c_{21} (z_n)^2 - (2c_{11} - c_{22}) z_n - c_{12} = 0, \quad n = 1, 2.$$  \hspace{1cm} (24a)
implying
\[
z_n = \left[ 2c_{11} - c_{22} + (-1)^n \sqrt{(2c_{11} - c_{22})^2 + 8c_{12}c_{21}} \right] / (4c_{21}) , \ n = 1, 2 . \tag{24b}
\]

These formulas (24b) feature of course only square-roots—rather than the cubic-roots that would be featured by the Cardano solutions of the cubic equations (22)—and moreover they yield the explicit expressions (24b) of the 2 auxiliary parameters \( z_n \) in terms of (only!) the 4 parameters \( c_{nm} (n = 1, 2; m = 1, 2) \). Hence by inserting these expressions of \( z_1 \) and \( z_2 \) in any 2 of the 4 eqs. (22) and (23), we get a system of 2 algebraic equations satisfied by the 6 parameters \( c_{nk} (n = 1, 2; k = 1, 2, 3) \) which features the 2 parameters \( c_{13} \) and \( c_{23} \) (only!) linearly and therefore allows to express both these coefficients explicitly in terms of the other 4 coefficients \( c_{nm} (n = 1, 2; m = 1, 2) \). For instance the 2 eqs. (23b) and (23c) yield (by subtracting the second multiplied by \( z_2 \) from the first multiplied by \( z_1 \), and by subtracting the second from the first)
\[
\begin{align*}
c_{13} &= -c_{11}z_1z_2 - c_{12} (z_1 + z_2) / 2 , \tag{25a} \\
c_{23} &= -c_{21}z_1z_2 - c_{22} (z_1 + z_2) / 2 , \tag{25b}
\end{align*}
\]

hence, via (24b),
\[
\begin{align*}
4c_{13}c_{21} - c_{12}c_{22} &= 0 , \tag{26a} \\
2 (-c_{12} + 2c_{23}) c_{21} + (2c_{11} - c_{22}) c_{22} &= 0 . \tag{26b}
\end{align*}
\]

**Remark 3.2** Since throughout our treatment we have assumed that \( z_1 \) is different from \( z_2 \), clearly the formula (24b) implies that the 4 parameters \( c_{11}, c_{12}, c_{21}, c_{22} \) must satisfy the inequality
\[
(2c_{11} - c_{22})^2 + 8c_{12}c_{21} \neq 0 . \tag{27}
\]

Two additional relations can be obtained by inserting in the 2 eqs. (18c) and (20c) the expressions
\[
A_{11} = z_1A_{21} , \ \ A_{12} = z_2A_{22} , \tag{28}
\]

implied by (21), obtaining thereby (again) 2 identical second-degree equations for the 2 parameters \( z_1 \) and \( z_2 \):
\[
\begin{align*}
c_{24} (z_n)^2 + (c_{25} - c_{14}) z_n - c_{15} &= 0 , \ n = 1, 2 . \tag{29a}
\end{align*}
\]

implying of course
\[
z_n = \left[ c_{14} - c_{25} + (-1)^n \sqrt{(c_{14} - c_{25})^2 + 4c_{15}c_{24}} \right] / (2c_{24}) , \ n = 1, 2 . \tag{29b}
\]

**Remark 3.3** Note that these equations (29) only involve the 4 parameters \( c_{14}, c_{24}, c_{15}, c_{25} \); and that the condition \( z_1 \neq z_2 \) entails the inequality
\[
(c_{25} - c_{14})^2 + 4c_{15}c_{24} \neq 0 . \tag{30}
\]

Since from the 4 eqs. (18a), (18b), (20a), (20b) we extracted the 2 constraints (26), clearly from these 4 equations we can only obtain 2 additional relations constraining the parameters \( c_{nj} \). A convenient way to get such relations is to subtract the
eq. (29a) multiplied by 2$c_{21}$ from the eq. (24a) itself multiplied by $c_{24}$, getting thereby the following 2 identical first-degree equations for the parameters $z_1$ and $z_2$:  

\[-[c_{24} (2c_{11} - c_{22}) + 2c_{21} (c_{25} - c_{14})] z_n = c_{12}c_{24} - 2c_{15}c_{21}, \quad n = 1, 2.\]  

(31)

But these 2 first-degree equations seem to imply that $z_1 = z_2$, while we know that this is not the case (at least, provided the two inequalities (27) and (30) hold true; which is generally the case for any generic assignment of 12 parameters $c_{nj}$). Hence these first-degree eqs. (31) satisfied by $z_1$ and $z_2$ must have the property to be identically satisfied for any arbitrary value of $z_1$ and $z_2$, which is of course the case provided the 8 parameters $c_{11}, c_{12}, c_{14}, c_{15}, c_{21}, c_{22}, c_{24}, c_{25}$ satisfy the following 2 constraints:  

\[c_{24} (2c_{11} - c_{22}) + 2c_{21} (c_{25} - c_{14}) = 0,\]  

(32a)

\[c_{12}c_{24} - 2c_{15}c_{21} = 0\]  

(32b)

(to obtain the first of these 2 equations we assumed $c_{24} \neq 0$, consistently with our assumption that the 12 parameters $c_{nj}$ have generic values).

In conclusion, we have obtained 4 constraints on the 10 parameters $c_{np}$ ($n = 1, 2; p = 1, 2, 3, 4, 5$): see the 2 eqs. (26) and the 2 eqs. (32); note that the 2 parameters $c_{n6}$ are not involved at all in these constraints.

There remain to compute the 4 parameters $A_{nm}$.

To compute the 4 parameters $A_{nm}$, rather than using the 6 eqs. (18) and (20)—out of which we already extracted the 4 constraints (26) and (32); so that we can expect to be only able to compute only 2 of the 4 parameters $A_{nm}$ in terms of the other 2 (and of course the coefficients $c_{nj}$)—the simplest way is to use the relations implied by the definitions (21):  

\[A_{11} = z_1 A_{21}, \quad A_{12} = z_2 A_{22};\]  

(33)

here of course $z_1$ and $z_2$ are defined by their expressions (24b) or, equivalently, (29b), and the 2 parameters $A_{21}$ and $A_{22}$ can be considered as free parameters; so that these relations can be rewritten as follows:  

\[A_{21} = \lambda_1, \quad A_{22} = \lambda_2, \quad A_{11} = z_1 \lambda_1, \quad A_{12} = z_2 \lambda_2,\]  

(34)

with $\lambda_1$ and $\lambda_2$ two arbitrary (nonvanishing) parameters.

This concludes both the identification of the subclass of the dynamical system (1) which is treated in this paper and the solution of its initial-values problem; except for the further step of inserting in all the relevant formulas—in addition to the expression (34)—the following rather simple expressions, say, of $c_{13}$ and $c_{23}$,

\[c_{13} = c_{12} c_{22} / (4 c_{21}),\]  

(35a)

\[c_{23} = [2 c_{12} c_{21} - c_{22} (2 c_{11} - c_{22})] / (4 c_{21})\]  

(35b)

implied by (26), and likewise, say, of $c_{24}$ and $c_{25}$,

\[c_{24} = 2 c_{15} c_{21} / c_{12},\]  

(36a)

\[c_{25} = c_{14} - c_{24} (2 c_{11} - c_{22}) / (2 c_{21})\]  

(36b)

implied by (32) (reminder: we always ignore some obvious nongeneric cases, for instance those with $c_{12} = 0$ or with $c_{24} = 0$; except of course below in Section 5, which is indeed devoted to the treatment of some special cases).
An essential compendium of all the relevant formulas is displayed in the following Section 4, for the convenience of the reader who is more interested in using these findings than in following their derivation.

Remark 3.4 A final observation. The reader who has followed our derivation up to this point might justifiably be puzzled by the fact that our solution seems to feature the 2 free parameters $\lambda_1$ and $\lambda_2$. But in fact these 2 free parameters are not present at all in the solution $x_n(t)$ ($n = 1, 2$) of the dynamical system (1) obtained above. This is proven in Appendix C; the development reported there are also useful to get the final formulas reported in the following Section 4 (which indeed do not feature the 2 parameters $\lambda_n$).

4 A Summary of the Solution of the System (1)

In this Section 4 we summarize the main results obtained in this paper so far. For the convenience of the reader who is only interested in these results and not in their derivation, we report these findings in a self-consistent fashion, even at the cost of the repetition of some key formulas already displayed above and in the Appendices A, B and C.

Let us recall that our focus is on the system of 2 nonlinearly-coupled first-order ODEs (1), i.e.

\[ \dot{x}_n = c_{n1} (x_1)^2 + c_{n2} x_1 x_2 + c_{n3} (x_2)^2 + c_{n4} x_1 + c_{n5} x_2 + c_{n6}, \quad n = 1, 2. \] (37)

Our main finding is the solution in explicit form of the initial-values problem for this system, which is however achieved only provided its 12 a priori arbitrary parameters $c_{nj}$ ($n = 1, 2$; $j = 1, 2, 3, 4, 5, 6$) do satisfy—as it were, a posteriori—the following 4 algebraic constraints:

\begin{align}
4c_{13}c_{21} - c_{12} c_{22} & = 0, \quad \text{(38a)} \\
2 (-c_{12} + 2c_{23}) c_{21} + (2c_{11} - c_{22}) c_{22} & = 0, \quad \text{(38b)} \\
c_{24} (2c_{11} - c_{22}) + 2c_{21} (c_{25} - c_{14}) & = 0, \quad \text{(38c)} \\
c_{12} c_{24} - 2c_{15} c_{21} & = 0. \quad \text{(38d)}
\end{align}

Note that the first 2 of these 4 constraints only involve the 6 parameters $c_{nk}$ ($n = 1, 2$; $k = 1, 2, 3$), and the last 2 only involve the 8 parameters $c_{11}, c_{12}, c_{14}, c_{15}, c_{21}, c_{22}, c_{24}, c_{25}$; while the 2 parameters $c_{n6}$ are unconstrained and only influence (see below) the 2 parameters $\alpha_{n0}$.

The explicit solution of the initial-values problem for the system (37) with (38) is then provided by the following formulas, for whose derivation the interested reader should go through the developments reported in the rest of this paper (for some guidance see below Remark 4.1):

\begin{align}
& x_1(t) = z_1 w_1(t) + z_2 w_2(t), \quad \text{(39a)} \\
& x_2(t) = w_1(t) + w_2(t); \quad \text{(39b)} \\
& z_n = \left[ 2c_{11} - c_{22} + (-1)^n \sqrt{(2c_{11} - c_{22})^2 + 8c_{12}c_{21}} \right] / (4c_{21}), \quad n = 1, 2. \quad \text{(40)}
\end{align}
\[ w_n(t) \equiv w_n(C, t) , \]
\[ w_n(0) = w_{n+} - w_{n-} \exp(\beta_n t) - w_{n+} w_{n-} \left[ 1 - \exp(\beta_n t) \right], \]
\[ n = 1, 2; \quad (41) \]

\[ w_1(0) \equiv w_1(C, 0) = \left[ x_1(0) - z_2 x_2(0) \right]/(z_1 - z_2) , \quad (42a) \]
\[ w_2(0) \equiv w_2(C, 0) = -\left[ x_1(0) - z_1 x_2(0) \right]/(z_1 - z_2) ; \quad (42b) \]

\[ w_{n\pm} = (-\alpha_{n1} \pm \beta_n)/(2\alpha_{n2}) , \quad \beta_n = \sqrt{(\alpha_{n1})^2 - 4\alpha_{n0}\alpha_{n2}} , \quad n = 1, 2 ; \quad (43) \]

\[ \alpha_{12} = \left[ (z_1)^2 (c_{11} - z_2 c_{21}) + z_1 (c_{12} - z_2 c_{22}) + c_{13} - z_2 c_{23} \right]/(z_1 - z_2) , \quad (44a) \]
\[ \alpha_{11} = \left[ z_1 (c_{14} - z_2 c_{24}) + c_{15} - z_2 c_{25} \right]/(z_1 - z_2) , \quad (44b) \]
\[ \alpha_{10} = (c_{16} - z_2 c_{26})/(z_1 - z_2) ; \quad (44c) \]
\[ \alpha_{22} = \left[ (-c_{13} + z_1 c_{23}) + z_2 (-c_{12} + z_1 c_{22}) 
+ (z_2)^2 (-c_{11} + z_1 c_{21}) \right]/(z_1 - z_2) , \quad (44d) \]
\[ \alpha_{21} = \left[ -c_{15} + z_1 c_{25} + z_2 (-c_{14} + z_1 c_{24}) \right]/(z_1 - z_2) , \quad (44e) \]
\[ \alpha_{20} = \left[ -c_{16} + z_1 c_{26} \right]/(z_1 - z_2) . \quad (44f) \]

**Remark 4.1** The 4 constraints (38) coincide with the formulas (26) and (32); the formulas (39) come from the eqs. (3), (34), and (80); the formulas (40) coincide with the eqs. (24b); the formulas (41) come from the eqs. (80), (79b) and (6a); the formulas (42) come from the eqs. (80), (77) and (34); the formulas (43) come from the eqs. (80), (79), (6b) and (34); the 6 formulas (44) come from the eqs. (17), (19), (79a), and (34). (Reminder: we always assume that all parameters take generic values; of course except for the restrictions implied by the constraints they are require to satisfy).

**Remark 4.2** The special subcases of this system which feature the remarkable property to be isochronous are clearly those characterized by the requirement that the 2 parameters \( \beta_n \)—see (43) with (44) and (40)—be both rational multiples of the same imaginary number:

\[ \beta_n = i\rho_n\omega , \quad n = 1, 2 , \quad (45) \]

where \( i\omega \) is an arbitrary imaginary number and \( \rho_n \) are 2 real (positive or negative) rational numbers (this is rather obvious, but in case of doubt see, for instance, [6]). While, if one of the 2 parameters \( \beta_n \) is an arbitrary purely imaginary number and the other is not a purely imaginary number, then the system (37) is asymptotically isochronous (see [6, 8]).
5 Two Special Cases of the System (1)

The treatment reported up to this point has assumed that the 12 coefficients $c_{nj}$ in (1) take generic values (except, of course, for satisfying the 4 constraints (38)). However in several contexts this is not the case; for instance the subclass of systems (1) characterized by the restrictions $c_{13} = c_{15} = c_{21} = c_{24} = 0$ is relevant in many applicable contexts; and the system (1) with homogeneous second-degree polynomial right-hand sides—i.e., with $c_{nj} = 0$ for $n = 1, 2$ and $j = 4, 5, 6$—also deserves a special treatment, in order to compare the findings presented in the present paper with those reported in the recent paper [4]. These 2 special cases are treated in the following 2 subsections of this Section 5.

5.1 The Subcase of (1) with $c_{13} = c_{15} = c_{21} = c_{24} = 0$

In many applicable contexts it is unreasonable to assume that the time-evolution of the dependent variable $x_n(t)$ is influenced by agents (represented by terms in the right-hand sides of the ODEs (1)) which depend only on the other variable $x_{n+1}(t)$ (with $n = 1, 2 mod[2]$). Hence the subcase of the system (1) characterized by the restrictions

$$c_{13} = c_{15} = c_{21} = c_{24} = 0 \quad (46a)$$

deserves special attention, featuring indeed in many applicable contexts. For this reason in this Section 5.1 we focus on the special case of (1) characterized by these limitations (46a), introducing moreover—for notational simplicity—the following new notation for the 8 remaining coefficients $c_{nj}$ in (1):

$$c_{11} = f_{11}, \quad c_{12} = f_{12}, \quad c_{14} = g_{1}, \quad c_{16} = h_{1},$$
$$c_{22} = f_{21}, \quad c_{23} = f_{22}, \quad c_{25} = g_{2}, \quad c_{26} = h_{2}; \quad (46b)$$

so that the system (1) reads hereafter (in this Section 5.1) as follows:

$$\dot{x}_n = x_n \left( f_{n1} x_1 + f_{n2} x_2 + g_n \right) + h_n, \quad n = 1, 2 \quad (47a)$$

namely

$$\dot{x}_1 = x_1 \left( f_{11} x_1 + f_{12} x_2 + g_1 \right) + h_1, \quad (47b)$$
$$\dot{x}_2 = x_2 \left( f_{21} x_1 + f_{22} x_2 + g_2 \right) + h_2. \quad (47c)$$

Remark 5.1 Clearly this system of 2 coupled nonlinear ODEs is invariant under the following transformation:

$$x_1(t) \leftrightarrow x_2(t), \quad f_{11} \leftrightarrow f_{22}, \quad f_{12} \leftrightarrow f_{21}, \quad g_1 \leftrightarrow g_2, \quad h_1 \leftrightarrow h_2; \quad (48)$$

which clearly replaces the analogous transformation reported in Remark 1.2.

The interested reader will easily verify that a direct adaptation to the present case of the final findings reported above (see Section 4) is a priori unjustified, because the conditions (46) render illegitimate some of the steps performed in that section—where the treatment was indeed based on the assumption that the coefficients $c_{nj}$ in (1) have generic values (except for satisfying the constraints (38)). So below (in this
Section 5.1) we review the above treatment, adapting it to the new situation. On the other hand, for the convenience of the reader who is only interested in the solution of the dynamical system (47) and not in the details of how that solution has been obtained, we report in the following Section 5.1.1—in analogy to what we did in Section 4 for the dynamical system (1)—the explicit solution of the system (47); even at the cost of some repetitions.

The 12 equations (10) and (13) read now (via (46a) and (46b)) as follows:

\[ f_{11} = \left[ a_{12} A_{11} (A_{22})^2 + a_{22} A_{12} (A_{21})^2 \right] / D^2 , \quad (49a) \]
\[ f_{12} = -2 A_{11} A_{12} [a_{12} A_{22} + a_{22} A_{21}] / D^2 , \quad (49b) \]
\[ A_{11} A_{12} (a_{12} A_{22} + a_{22} A_{11}) = 0 , \quad (49c) \]
\[ g_1 = (a_{11} A_{11} A_{22} - a_{21} A_{12} A_{21}) / D , \quad (49d) \]
\[ a_{11} = a_{21} , \quad (49e) \]
\[ h_1 = a_{10} A_{11} + a_{20} A_{12} ; \quad (49f) \]
\[ f_{22} = \left[ a_{22} A_{22} (A_{11})^2 + a_{12} A_{21} (A_{12})^2 \right] / D^2 , \quad (49g) \]
\[ f_{21} = -2 A_{22} A_{21} (a_{22} A_{11} + a_{12} A_{12}) / D^2 , \quad (49h) \]
\[ A_{22} A_{21} (a_{22} A_{21} + a_{12} A_{22}) = 0 , \quad (49i) \]
\[ g_2 = (a_{21} A_{11} A_{22} - a_{11} A_{12} A_{21}) / D , \quad (49j) \]
\[ h_2 = a_{10} A_{21} + a_{20} A_{22} , \quad (49k) \]

of course with \( D \) defined as above, see (11).

Remark 5.2 In this Section 5.1, as mentioned above, we assume that the parameter \( D \) does not vanish, but we do not exclude the possibility that one of the 4 parameters \( A_{nm} \) vanish (in contrast with Remark 2.2).

It is now again convenient to introduce the 2 auxiliary parameters \( z_1 = A_{11} / A_{21} \) and \( z_2 = A_{12} / A_{22} \) (see (21)). Then the 2 eqs. (18b) and (18a) yield (via (46))

\[ [(f_{21} - f_{11}) z_n + f_{22} - f_{12}] z_n = 0 , \quad n = 1, 2 \]

implying

\[ z_1 = 0 , \quad z_2 = \frac{f_{22} - f_{12}}{f_{11} - f_{21}} , \quad (51a) \]

or

\[ z_2 = 0 , \quad z_1 = \frac{f_{22} - f_{12}}{f_{11} - f_{21}} , \quad (51b) \]

since (see Remark 5.2) we exclude the solution \( z_1 = z_2 \) which implies \( D = 0 \) (see (11) and (21)). Note that—via (21)—\( z_1 = 0 \) implies \( A_{11} = 0 \) and likewise \( z_2 = 0 \) implies \( A_{12} = 0 \), each of these 2 equalities reducing eq. (49c) to the identity \( 0 = 0 \).

Let us now see how the remaining 11 eqs. (49) simplify in the \( z_1 = 0 \) case, when clearly

\[ A_{11} = 0 \quad (52a) \]

implying \( D = -A_{12} A_{21} \); for analogous results in the \( z_2 = 0 \) case see below Remark 5.3.
As already mentioned above, this eq. (52a) implies that eq. (49c) holds identically \((0 = 0)\); hence only the following 10 equations remain (note that they are reported below in a somewhat different order than in (49)):

\[
\begin{align*}
a_{22}A_{21} + a_{12}A_{22} &= 0, \\
a_{11} &= a_{21}, \\
f_{11} &= a_{22}/A_{12}, \\
f_{12} &= 0, \\
f_{21} &= -2a_{12}A_{22}/(A_{12}A_{21}), \\
f_{22} &= a_{12}/A_{21}, \\
g_1 &= a_{21}, \\
g_2 &= a_{11}, \\
h_1 &= a_{20}A_{12}, \\
h_2 &= a_{10}A_{21} + a_{20}A_{22}.
\end{align*}
\]

Clearly the first 3 of these 11 equations (52) provide 3 constraints on the 8 parameters \(A_{nm}\) and \(a_{nm}\); while the last 8 of these 11 eqs. (52) express explicitly—in terms of the 8 parameters \(A_{nm}\) and \(a_{nm}\)—the 8 parameters \(f_{nm}, g_n, h_n\) which characterize the system of ODEs (47).

The next task is to invert the last 8 formulas (52), namely to express in terms of the 8 parameters \(f_{nm}, g_n, h_n\) featured by the system (47), the 8 parameters \(A_{nm}\), and \(a_{nm}\) which characterize the explicit solution of this system (47) via the formulas of Section 1 complemented by the restrictions and redefinitions (46); and as well to identify—most importantly—the constraints implied by our treatment on the 8 parameters \(f_{nm}, g_n, h_n\).

These findings can be obtained by appropriately specializing the formulas obtained in the preceding Section 3, i.e. by inserting in them the formulas (46) as well as the findings reported above in this Section 5.1.

In this manner from the 3 eqs. (17) we get (using the constraint \(f_{12} = 0\) already obtained above, see (52e))

\[
\begin{align*}
a_{12} &= A_{21}f_{22}, \\
a_{11} &= g_2, \\
a_{10} &= (A_{12}h_2 - A_{22}h_1)/(A_{12}A_{21});
\end{align*}
\]

and likewise from the 3 eqs. (19) we get

\[
\begin{align*}
a_{22} &= A_{12}f_{11}, \\
a_{21} &= g_1, \\
a_{20} &= h_1/A_{12}.
\end{align*}
\]

Next, let us look at the 6 eqs. (18) and (20), using again the constraint \(f_{12} = 0\) (see (52e)) to simplify some of them.
The 3 eqs. (18) read then as follows:

\[ A_{12}f_{21} + 2A_{22}f_{22} = 0 , \]  

\[ A_{12}(f_{11} - f_{21}) - A_{22}f_{22} = 0 , \]  

\[ g_1 - g_2 = 0 . \]  

(55a)  
(55b)  
(55c)

It is now easily seen that the first 2 of these 3 equations imply (since we assume that \( A_{12} \) and \( A_{22} \) do not vanish) the following second constraint on the 2 parameters \( f_{11} \) and \( f_{21} \):

\[ f_{21} = 2f_{11} ; \]  

(56)

and we moreover find from eq. (55c) the following third constraint,

\[ g_1 = g_2 . \]  

(57)

On the other hand the 3 eqs. (20) are identically satisfied—i.e., \( 0 = 0 \)—thanks to (52a), to the conditions (46) and, again, to the constraint \( f_{12} = 0 \), see (52e).

So, let us summarize the findings in this \( z_1 = 0 \) case. There are the 2 constraints \( f_{12} = 0 \) (see (52e)) and \( f_{21} = 2f_{11} \) (see (56)) on the parameters \( f_{nm} \) of the system (47), and the third constraint \( g_1 = g_2 \); note that the first 2 of these 3 constraints imply \( z_2 = -f_{22}/f_{11} \). Provided these 3 constraints are satisfied, the explicit solution of the initial-values problem for this system (47) is provided by the treatment of Section 2, of course with the parameters \( c_{nj} \) replaced by the parameters \( f_{nm}, g_n, h_n \) as implied by the relations (46), and with the parameters \( a_{n\ell} \) expressed by the formulas (53) and (54) in terms of the 8 parameters \( f_{nm}, g_n, h_n \), and also of the 4 parameters \( A_{nm} \). As for these latter parameters, they are themselves determined in terms of the parameters \( f_{nm} \) as follows (the last of these formulas is of course implied by \( A_{12} = z_2 A_{22} \) with \( z_2 = -f_{22}/f_{11} \), see above):

\[ A_{21} = \lambda_1 , \quad A_{22} = \lambda_2 , \quad A_{11} = 0 , \quad A_{12} = -(f_{22}/f_{11}) \lambda_2 . \]  

(58)

Here \( \lambda_1 \) and \( \lambda_2 \) are again 2 arbitrary (nonvanishing) parameters, which can be freely assigned because their values do not influence the solution of the problem (as explained at the end of Section 3 and in Appendix C in the context of the more general case of the system (1); and see also Section 5.1.1).

The corresponding solution of the initial-values problem for the system (47) is reported in the following Section 5.1.1.

Finally, let us conclude this Section 5.1 by emphasizing that the 3 constraints (52e), (56) and (57) entail a significant limitation on the generality of the system (47) treated in this section; for instance, they exclude models of the Lotka-Volterra type, which require (at least!) an arbitrary (nonvanishing) assignment of the parameter \( f_{12} \). Nevertheless the fact that the initial-values problem for the system (47) can be explicitly solved provided only the 3 restrictions indicated above hold seems a significant new finding.

### 5.1.1 Solution of the Initial-Values Problem for the System (47)

In this Section 5.1.1 we report the solution of the initial-values problem for the system characterized by the following 2 equations of motion:

\[ \dot{x}_1 = x_1 (f_1 x_1 + g) + h_1 , \]  

(59a)
which correspond to the system (47) treated in this Section 5.1 by taking into account the following constraints and simplified notation:

\begin{align*}
  c_{12} &= c_{13} = c_{15} = c_{21} = c_{24} = 0, \\
  c_{11} &= f_1 = f_1, \quad c_{14} = c_{25} = g_1 = g_2 = g, \quad c_{16} = h_1, \\
  c_{22} &= f_2 = 2f_1, \quad c_{23} = f_{22} = f_2, \quad c_{23} = f_{22} = f_2; \quad c_{26} = h_2,
\end{align*}

implied by our treatment, see above.

The solution is then provided by the following formulas (obtained via (60) from the corresponding solution reported in Section 4):

\begin{align*}
  x_1 (t) &= - \frac{f_2}{f_1} \xi_2 (t), \quad x_2 (t) = \xi_1 (t) + \xi_2 (t), \\
  \xi_n (t) &= \frac{\xi_n (0) \left[ \xi_{n+} - \xi_{n-} \exp (\gamma_n t) \right] - \xi_{n+} \xi_{n-} \left[ 1 - \exp (\gamma_n t) \right]}{\xi_{n+} \exp (\gamma_n t) - \xi_{n-} \xi_n (0) \left[ 1 - \exp (\gamma_n t) \right]}, \\
  n &= 1, 2; \\
  \xi_1 (0) &= x_2 (0) + (f_1/f_2) x_1 (0), \quad \xi_2 (0) = - (f_1/f_2) x_1 (0), \\
  \xi \pm_n &= (-\eta_{n1} \pm \gamma_n) / (2\eta_{n2}), \quad \gamma_n = \sqrt{(\eta_{n1})^2 - 4\eta_{n0}\eta_{n2}}, \quad n = 1, 2; \\
  \eta_{12} &= f_2, \quad \eta_{11} = g, \quad \eta_{10} = (f_1/f_2) h_1 + h_2, \\
  \eta_{22} &= -f_2, \quad \eta_{21} = g, \quad \eta_{20} = - (f_1/f_2) h_1;
\end{align*}

implying

\begin{align*}
  \gamma_1 &= \sqrt{g^2 - 4f_2\eta_{10}}, \quad \gamma_2 = \sqrt{g^2 + 4f_2\eta_{20}}.
\end{align*}

Remark 5.3 The diligent reader will check that the same result is obtained in the alternative case with \( z_2 = 0 \) (i.e. with (51b) replacing (51a)); of course provided all the corresponding notational changes are made, consisting essentially to an exchange of the roles of the auxiliary variables \( \xi_1 (t) \) and \( \xi_2 (t) \)—themselves corresponding, of course up to an appropriate rescaling, to the variables \( y_1 (t) \) and \( y_2 (t) \), in analogy to the treatment detailed, for the more general system (1), in Appendix C.

And let us reiterate our general reminder that these findings are valid for generic values of the parameters they feature (for instance trivial cases such as those with \( f_1 = 0 \) or \( f_2 = 0 \) are obviously excluded).

5.2 The Subcase of (1) with Homogeneous Second-Degree Polynomial Right-Hand Sides, i.e. \( c_{nj} = 0 \) for \( n = 1, 2 \) and \( j = 4, 5, 6 \)

The special case of the dynamical system (1) treated in this Section 5.2 is characterized by the following ODEs:

\begin{align*}
  \dot{x}_n (t) &= c_{n1} [x_1 (t)]^2 + c_{n2} x_1 (t) x_2 (t) + c_{n3} [x_2 (t)]^2, \quad n = 1, 2,
\end{align*}
namely
\[
\dot{x}_1 (t) = c_{11} [x_1 (t)]^2 + c_{12} x_1 (t) x_2 (t) + c_{13} [x_2 (t)]^2 , \quad (65b)
\]
\[
\dot{x}_2 (t) = c_{21} [x_1 (t)]^2 + c_{22} x_1 (t) x_2 (t) + c_{23} [x_2 (t)]^2 . \quad (65c)
\]
A large class of subcases of this system—identified by explicit restrictions on the 6 parameters \( c_{nk} \) \((n = 1, 2; k = 1, 2, 3)\) and characterized by the property to be \textit{solvable} by algebraic operations—has been identified in the recent paper [4]. In this Section 5.2 we compare the results obtained in this paper [4] with those obtained in the present paper. The main conclusion of this comparison is that there is, of course, a certain overlap among the cases treated in the present paper and those treated in [4]; however subcases of (1) identified as \textit{solvable} in [4] are not included in the treatment provided in the present paper; and likewise subcases of (1) identified as \textit{solvable} in the present paper are not included in the treatment provided by [4]. Hence the 2 approaches—with their similarities and their differences—are in some sense \textit{complementary}. This is explained in detail in the following 2 Sections 5.1.1 and 5.1.2.

Remark 5.4 There is a significant difference among the treatments of [4] and the present paper. In [4] the systems identified are \textit{algebraically solvable} in the following sense: the computation of their \( t \)-evolution is reduced to the evaluation of the zeros of a \( t \)-dependent polynomial \( P_N (x; t) \) of (finite) order \( N \) in its argument \( x \) (\( N \) being a \textit{positive integer} whose value depends on the particular model under consideration), the \( t \) -evolution of which is \textit{explicitly} known (while the expressions of the solutions \( x_n (t) \) of the system (65) can of course only be obtained \textit{explicitly} if \( N \leq 4 \)). In the present paper the systems identified as \textit{solvable} allow the \textit{explicit} display of the \( t \)-evolution \( x_n (t) \) of the solutions of their initial-values problem, as reported in Section 4.

Remark 5.5 Another significant difference among the class of systems treated in [4] and in the present paper is that—while the system (1) features the 12 \textit{a priori arbitrary} coefficients \( c_{nj} \)—the solutions obtained in [4] generally feature 9 \((9 = 2 + 6 + 1: \text{see below Section 5.2.2})\) \textit{freely assigned} parameters, while those obtained in the present paper feature 8 \((8 = 12 - 4: \text{see above Section 4})\) \textit{freely assigned} parameters (in addition of course, in both cases, to the \( 2 \) initial data \( x_n (0) \)). But, as shown in the following 2 Sections 5.1.1 and 5.1.2 (see their titles), neither one of these two subclasses of the system (1)—that treated in [4] and that treated in the present paper—includes the other subclass: a confirmation that these 2 papers are actually \textit{complementary}.

5.2.1 Subcases of (1) shown to be \textit{solvable} in [4] which are \textit{not} included among those shown to be \textit{solvable} in the present paper

In [4] it is noted that essentially the entire class of systems (65) can be reduced to the following simpler system (see eq. (6) of [4]):
\[
\dot{x}_1 = x_1 x_2 , \quad \dot{x}_2 = A [(x_1)^2 + (x_2)^2] + B x_1 x_2 , \quad (66)
\]
and that this system can be solved by algebraic operations if the 2 parameters $A$ and $B$ are suitably restricted, for instance sufficient conditions are that

$$A = \frac{n + q - 1}{n + q}, \quad B = \pm \frac{n - q}{n + q} \sqrt{\frac{n + q - 1}{nq}}$$

(67)

with $n$ an arbitrary positive integer, and $q$ an arbitrary, possibly complex, rational number (see eqs. (17) of [4]; there also are other possibilities, see eqs. (18-20) of [4], but we do not need to evoke them to make our point).

On the other hand it is easily seen that the system (66), which corresponds to the system (1) only if all the 12 parameters $c_{nj}$ vanish except for the following 4 of them,

$$c_{12} = 1, \quad c_{21} = A, \quad c_{22} = B, \quad c_{23} = A,$$

(68)

entails, via the 4 constraints (38), either

$$A = B = 0,$$

(69)

which is also consistent with (67) (say, with $q = 1 - n$), or

$$B = 0, \quad A = 1/2,$$

(70)

which is consistent (say, with $n = q = 1$); both assignments, of course, much less general than (67) with $n$ an arbitrary positive integer, and $q$ an arbitrary, possibly complex, rational number.

This shows that there are some special subcases of (1) which are solvable both via the technique of [4] and via the technique of the present paper; and many more which are solvable via the technique of [4] but are not solvable via the technique of the present paper. Q. E. D.

5.2.2 Subcases of (1) shown to be solvable in the present paper which are not included among those shown to be solvable in [4]

Since the system (1)—even with the constraints (38)—is clearly more general than the system (65) treated in [4]—because of the additional 6 terms featuring the coefficients $c_{nj}$ with $n = 1, 2$ and $j = 4, 5, 6$, it might seem that what we want to prove in this Section 5.2.2 (see its title) is altogether obvious. But the subclass of the system (1) treated in [4] includes the additional possibility to perform a linear transformation—with arbitrary time-independent coefficients—of the dependent variables. So this argument is not cogent.

But such a transformation—which generally features 6 free parameters—cannot change the dependence on the independent variable $t$ from algebraic to exponential; while the dependence on the variable $t$ of the solutions reported in the present paper is indeed generally exponential (see for instance above eq. (41)).

However this argument is still not entirely conclusive because of the possibility to extend the system (65) via the simple invertible change of dependent and independent variables

$$x_n(t) = \exp(\lambda t) \zeta_n(\tau), \quad \tau = \left[\exp(\lambda t) - 1\right]/\lambda,$$

(71)
which transforms the following system for $\zeta_n(\tau)$, reading

$$d\zeta_n(\tau)/d\tau = c_n1[\zeta_1(\tau)]^2 + c_n2\zeta_1(\tau)\xi_2(\tau) + c_n3[\zeta_2(\tau)]^2, \quad n = 1, 2,$$

hence being included among those treated in [4], into the, also autonomous—and as well solvable (via (71))—system

$$\dot{x}_n(t) = \lambda x_n(t) + c_n1[x_1(t)]^2 + c_n2x_1(t)x_2(t) + c_n3[x_2(t)]^2, \quad n = 1, 2.$$  (73)

This new system features the additional free parameter $\lambda$ and—most importantly with respect to the previous argument—its solutions clearly feature now an exponential dependence on the independent variable $t$ (see (71)).

But this implies that the solutions of this model—even after a linear reshuffling of the dependent variables—can only feature a dependence on the single exponential $\exp(\lambda t)$; while the solutions obtained in the present paper feature the 2, generally different, exponentials $\exp(\beta_n t)$, $n = 1, 2$, see above eqs. (41) and (43).

It is thereby shown that there indeed are subcases of (1) shown to be solvable in the present paper which are not included among those shown to be solvable in [4].

Q. E. D.

6 Conclusions and Outlook

The prototypical system (1) and its subcases treated above in Section 5 have been investigated over time by top mathematicians and subtend an enormous number of applied mathematics models in several scientific fields. It is our hope to obtain analogous results for analogous models in the future; for one such result see [9].

Appendix A

In this Appendix A we tersely demonstrate the following elementary fact, which clearly implies the result (6): that the solution of the initial-values problem for the ODE

$$\dot{y}(t) = a_2[y(t)]^2 + a_1y(t) + a_0,$$  (74a)

is provided by the formula

$$y(t) = \frac{y_+[y(0) - y_-] - y_-[y(0) - y_+]\exp(\beta t)}{y(0) - y_- - [y(0) - y_+]\exp(\beta t)},$$  (74b)

with $y_\pm$ defined as follows:

$$y_\pm = (-a_1 \pm \beta)/(2a_2), \quad \beta = \sqrt{(a_1)^2 - 4a_0a_2}.$$  (74c)

Indeed the ODE (74a) can clearly be reformulated as follows:

$$\dot{y} = a_2(y - y_+)(y - y_-).$$  (75a)
with $y_\pm$ defined by (74c); and then (again, via (74c)) this ODE can be rewritten as follows:

$$\dot{y} \left[ (y - y_+) - 1 - (y - y_-) - 1 \right] = \beta.$$  \hfill (75b)

The integration of this ODE for the dependent variable $y (t')$ over the independent variable $t'$—from $t' = 0$ to $t' = t$—clearly yields

$$\ln \left[ \frac{y (t) - y_+}{y (0) - y_+} \right] - \ln \left[ \frac{y (t) - y_-}{y (0) - y_-} \right] = \beta t,$$  \hfill (76)

which coincides—after exponentiation—with (74b). Q. E. D.

Finally let us emphasize that the results reported above are valid for generic values of the parameters; the interested reader shall have no difficulty to figure out the results in special cases, for instance those with $a_2 = 0$ or $\beta = 0$.

**Appendix B**

In this Appendix B we tersely outline the derivation of the expressions (10) and (13) of the 12 parameters $c_{nj}$ in terms of the 10 parameters $A_{nm}$ and $a_{n\ell}$.

The first step is to invert the relations (3), getting

$$y_1 (t) = \frac{A_{22} x_1 (t) - A_{12} x_2 (t)}{D},$$  \hfill (77a)

$$y_2 (t) = \frac{A_{11} x_2 (t) - A_{21} x_1 (t)}{D},$$  \hfill (77b)

where the quantity $D$ is defined as above, see (11).

The second step is to note that the relations (3) imply

$$\dot{x}_n = A_{n1} \dot{y}_1 + A_{n2} \dot{y}_2, \quad n = 1, 2,$$  \hfill (78a)

hence, via the ODEs (5),

$$\dot{x}_n = A_{n1} \left[ a_{12} (y_1)^2 + a_{11} y_1 + a_{10} \right] + A_{n2} \left[ a_{22} (y_2)^2 + a_{21} y_2 + a_{20} \right], \quad n = 1, 2.$$  \hfill (78b)

The third and last step is to insert the expressions (77) of $y_1$ and $y_2$ in terms of $x_1$ and $x_2$ in the right-hand sides of these ODEs. Then, via a bit of trivial if tedious algebra, there obtains the system (1) with the definitions (10) and (13) of the 12 coefficients $c_{nj}$. Q. E. D.

**Appendix C**

In this Appendix C we show that the solutions $x_n (t)$ of the dynamical system (1)—as treated above, see Sections 2 and 3—do not depend on the free parameters $\lambda_n$ introduced via the positions (34).

To this end we insert the expressions (34) of the parameters $A_{nm}$ in terms of the free parameters $\lambda_n$ in the formulas (17) and (19) expressing the 6 parameters $a_{nk}$; in
order to display the very simple dependence of these 8 parameters from the 2 free parameters $\lambda_n$. We thus easily find the following formulas:

$$a_{n\ell} \equiv (\lambda_n)^{\ell-1} c_{n\ell} \; , \quad n = 1, 2, \quad \ell = 0, 1, 2, \quad (79a)$$

where the notation $C$ indicates—above and hereafter—the set of the 12 parameters $c_{nj}$, and the 6 functions $a_{n\ell} (C)$ are explicitly displayed in Section 4, see (44); of course to this end we also used the definitions (24b) of the 2 auxiliary parameters $z_n$ in terms of the 4 coefficients $c_{nm} (n = 1, 2; m = 1, 2)$.

The next step is to insert the formulas (79a) in the expressions (6), getting thereby

$$y_n \pm \equiv (\lambda_n)^{-1} w_{n\pm} \; , \quad \beta_n \equiv \beta_n (C) \; , \quad n = 1, 2, \quad (79b)$$

again with the functions $w_{n\pm} (C)$ and $\beta_n (C)$ explicitly displayed in Section 4, see (41) and (43).

The insertion of these formulas in the expressions (6a) of the solutions $y_n (t)$ of the auxiliary dynamical system (5) evidences the following, very simple, dependence of these functions from the free parameters $\lambda_n$:

$$y_n (t) \equiv (\lambda_n)^{-1} w_n (C, t) \; , \quad n = 1, 2, \quad (80)$$

where again the 2 functions $w_n (C, t)$ are explicitly displayed in Section 4, see (41).

And via the insertion in the expressions (3) of $x_n (t)$ of these formulas (80), together with the expressions (34) of $A_{nm}$, we conclude that the solutions $x_n (t)$ are independent of the free parameters $\lambda_n$; as indeed displayed in Section 4, see the set of eqs. from (39) to (44). Q. E. D.

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