Determination of $\Lambda_{\overline{MS}}$ at Five Loops from Holographic QCD

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Abstract

The recent determination of the $\beta$–function of the QCD running coupling $\alpha_{\overline{MS}}(Q^2)$ to 5-loops, provides a verification of the convergence of a novel method for determining the fundamental QCD parameter $\Lambda_s$ based on the Light-Front Holographic approach to nonperturbative QCD. The new 5-loop analysis, together with improvements in determining the holographic QCD nonperturbative scale parameter $\kappa$ from hadronic spectroscopy, leads to an improved precision of the value of $\Lambda_s$ in the $\overline{MS}$ scheme close to a factor of two; we find $\Lambda_{\overline{MS}}^{(3)} = 0.339 \pm 0.019$ GeV for $n_f = 3$, in excellent agreement with the world average, $\Lambda_{\overline{MS}}^{(3)} = 0.332 \pm 0.017$ GeV. We also discuss the constraints imposed on the scale dependence of the strong coupling in the nonperturbative domain by superconformal quantum mechanics and its holographic embedding in anti-de Sitter space.

The strong coupling $\alpha_s$ is a central quantity for the study of Quantum Chromodynamics (QCD), the gauge theory of the strong interactions [1]. Traditionally, $\alpha_s$ –or equivalently, the perturbative QCD (pQCD) scale parameter $\Lambda_s$– has been determined...
from measurements of high momentum processes or from Lattice Gauge Theory. More recently, $\Lambda$ has also been determined from nonperturbative dynamics using light-front holographic QCD (LFHQCD) [2], an approach to color confinement that successfully describes both the hadronic spectrum and the bound-state light-front wave functions that control hadronic processes [3].

This new approach to hadron physics is based on superconformal quantum mechanics [4, 5, 6] and its light-front (LF) holographic embedding in a higher dimensional gravitational theory [7, 8, 9]. The result is a semiclassical effective theory which incorporates fundamental aspects of nonperturbative QCD that are not apparent from its classical Lagrangian, such as the emergence of a mass scale and confinement [10], the existence of a zero-mass bound state [6], the appearance of universal Regge trajectories and the breaking of chiral symmetry [6, 11]. In addition, it gives remarkable connections between the light meson and nucleon spectra [6]. Only one mass parameter appears – the confinement scale $\kappa$, which is constrained to better than 5% by measurements of hadron masses and other hadronic observables [12].

It has been recognized before the advent of QCD, that the linearity of the Regge trajectories implies oscillatory modes of constituent quarks within the hadron [13, 14]. The subsequent exploration of covariant two-particle Hamiltonians in the null plane lead uniquely to relativistic harmonic confinement if the wave equations are local differential equations [15]. These general results were extended to the case of spin-$\frac{1}{2}$ constituents in Ref. [16].

As shown in a remarkable article by de Alfaro, Fubini and Furlan (dAFF) [17], it is possible to generate a mass scale $\kappa$ and a confinement potential while maintaining the conformal symmetry of the action. In [17] dAFF write the quantum mechanical evolution operator as a superposition of the generators of the conformal group $Conf (R^4)$: The generator of time translation $H$, the generator of dilatations $D$, and the generator of special conformal transformations $K$. Since the generators of $Conf (R^4)$ have different dimensions, a mass scale is introduced which in the present context plays a fundamental role, as initially conjectured in Ref. [17]. The dAFF mechanism was extended to superconformal quantum mechanics in Refs. [4, 18]. One can reduce the LF Hamiltonian equations in QCD for massless quarks to a single-variable LF Schrodinger equation in $\zeta$, identical to the equations derived from AdS$_5$ in the variable $z$. The color confining potential is unique using the dAFF procedure. It has the form of a harmonic oscillator $\kappa^4 \zeta^2$. In LFHQCD, the soft-wall dilaton, which encodes the breaking of conformal symmetry in the higher dimensional anti-de Sitter AdS$_5$ space must thus have the form $e^{\kappa^2 z^2}$. 

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The holographic variable $z$ in the 5-dimensional classical gravity theory is identified with the invariant transverse separation $\zeta$ between the hadron constituents in the light-front quantization scheme [7, 19, 20]. The harmonic form of the confining light-front potential is equivalent to the familiar linear heavy quark $Q\bar{Q}$ potential in the instant form [21] and has been successful in reproducing essential nonperturbative QCD features, such as Regge trajectories and the $Q^2$-dependence of hadronic form factors [3].

In Quantum Field Theory, couplings acquire a scale-dependence due to short-distance quantum effects which are included in their definition. In particular for $\alpha_s$, the running is determined by pQCD and its renormalization group equation [22]. Likewise, the scale dependence of $\alpha_s$ in the nonperturbative domain can be obtained from the large-distance confining potential and, in the LFHQCD framework, follows from the specific embedding of light-front dynamics in anti-de Sitter (AdS) space [23]. Its specific form is obtained from the dilaton profile which breaks conformal invariance in the AdS$_5$ action: It is uniquely determined from the constraints imposed by the superconformal algebraic structure [5, 6, 11]. The matching of the short- and large-distance regimes of the strong coupling $\alpha_s$ determines the QCD perturbative scale $\Lambda_s$ in any renormalization scheme in terms of the physical hadronic scale $\kappa$ [24]. The procedure also sets the scale separating perturbative and nonperturbative hadron dynamics. We remark that since $\kappa$ is a physical parameter, it cannot depend on the choice of renormalization scheme, contrary to $\Lambda_s$. In fact, perturbative renormalization or evolution is not relevant to LFHQCD and $\kappa$. It is worth mentioning that some nonperturbative approaches, such as Lattice Gauge Theory, do become scheme-dependent because they are matched to perturbative results in order to fix parameters, but this is not the case of LFHQCD where $\kappa$ is fixed by observables, i.e., scheme-independent quantities. In our procedure which uses a scheme-independent nonperturbative formalism, the scheme-dependence of $\Lambda_s$ emerges from the infrared fixed-point value of $\alpha_s$, which is RS-dependent (the running coupling is not an observable, and it is thus scheme-dependent) and is not predicted by LFHQCD. LFHQCD predicts only the scale-dependence of $\alpha_s$. The infrared fixed-point value is determined in a particular scheme (the $g_1$ scheme) using a sum rule [23]. The values in other schemes, e.g., $\overline{MS}$, are then obtained using “Commensurate Scale Relations” [25], which are strict predictions of pQCD.

The method used to derive $\Lambda_s$ from LFHQCD uses the effective charge $\alpha_{g_1}$, defined from the Bjorken sum rule [26]. It has the analytic form [23]:

$$
\frac{\alpha_{g_1}^{IR} (Q^2)}{\pi} = \exp \left( -\frac{Q^2}{4\kappa^2} \right),
$$

(1)
in the infrared (IR) nonperturbative regime. Here $Q$ is the momentum transfer in the spin-dependent nucleon structure functions appearing in the Bjorken sum rule, and $\kappa$ is the fundamental LFHQCD scale parameter determined from the light hadron spectrum. This prediction for $\alpha_{g1}^{IR}(Q^2)$ agrees remarkably well with experimental data for $\alpha_{g1}(Q^2)$ in the domain $Q^2 \leq 1 \text{ GeV}^2$ [27] where LFHQCD is applicable, and it displays an infrared fixed point. In the nonperturbative domain, the relations between $\alpha_{g1}(Q^2)$ and the strong couplings $\alpha_s(Q^2)$ in other renormalization schemes, such as the $\overline{MS}$, MOM, or V schemes are given in Ref. [24]. Such relations are obtained by first assuming that $\alpha_s$ always has an infrared fixed point regardless of the scheme it is expressed in. Then, $\alpha_s(Q^2 = 0)$ is left as a free parameter to be determined by the matching procedure described below, but with the perturbative scale $\Lambda_s$ determined by the world data.

The effective charge $\alpha_{g1}$ can be expressed at high momentum transfer as a perturbative expansion in the perturbative coupling $\alpha_{MS}(Q^2)$, as defined by the $\overline{MS}$ renormalization scheme [27]:

$$\alpha_{g1}(Q^2) = \pi \left[ \frac{\alpha_{MS}(Q^2)}{\pi} + a_1 \left( \frac{\alpha_{MS}(Q^2)}{\pi} \right)^2 + a_2 \left( \frac{\alpha_{MS}(Q^2)}{\pi} \right)^3 + \cdots \right],$$

(2)

with the coefficients $a_i$ known up to $a_4$ [28] and $a_5$ having been only estimated [29]. The normalization and evolution of $\alpha_{g1}$ is then determined in the $\overline{MS}$ renormalization scheme by the QCD $\beta_{\overline{MS}}$-function and the mass scale $\Lambda_{\overline{MS}}$ [22]. Global hadron-parton duality [30] predicts that the nonperturbative description for $\alpha_{g1}(Q^2)$ overlaps with the pQCD expression at intermediate values of $Q^2$. Matching the LFHQCD and pQCD expressions of $\alpha_{g1}(Q^2)$ and their derivatives then allows us to determine $\Lambda_{\overline{MS}}$ and the scale $Q_0$ characterizing the transition between the perturbative and nonperturbative descriptions. The comparison between $\Lambda_{\overline{MS}}$ obtained from light-front holographic QCD and the world data provides a key test of this novel approach to nonperturbative QCD.

It is usually argued that one determines the proton mass and other aspects of the QCD mass scale starting from a measurement of $\Lambda_s$ in the pQCD domain. This ansatz is difficult to justify since $\Lambda_s$ is renormalization scheme dependent, whereas masses or other physical observables are not. In fact, the procedure outlined above is the opposite: $\Lambda_s$ is determined in any scheme starting from the fundamental –scheme independent– confinement scale $\kappa$ of nonperturbative QCD. Since the QCD Lagrangian has no mass parameter in the limit where the quark masses are neglected, the magnitude of the mass parameter $\kappa$ cannot be determined in fixed units by QCD itself. Actually, the units normally used for mass, GeV, are a convention. The key predictions are thus
ratios such as $\Lambda_s/\kappa$. The value of $\kappa$ determines all other mass scales in the chiral limit. Indeed, holographic QCD predicts the ratios of masses and mass times radius, etc. For example, it predicts $m_p/\Lambda_s$ [2], $m_\rho/m_p$, $m_p \times R_p$ [3], etc. Thus $\kappa$ is in a sense a “holding parameter”, a scale which arises from color confinement and the breaking of conformal symmetry, but it cannot be determined in absolute units by QCD. In fact, while the emergence of the QCD mass scale is attributed in LFHQCD to the dAFF symmetry breaking procedure, its value is essentially unknown, since the vacuum state in the dAFF construction is chosen \textit{ab initio}. A specific value for $\kappa$ is not determined by QCD alone. The scale only becomes fixed when we make a measurement such as the pion decay constant or the $\rho$ mass. Thus QCD with massless quarks can only predict ratios such as $m_p/m_\rho = \sqrt{2}$. The dAFF mechanism also differs from spontaneous symmetry breaking or explicit symmetry breaking by adding mass terms to the Lagrangian.

Since our initial LFHQCD determination of $\Lambda_{\overline{\text{MS}}}$ reported in Ref. [2], several new developments have occurred which allow us to efficiently test the convergence of our determination as well as significantly improve the comparison between light-front holographic QCD and the world data: 1) The LFHQCD scale parameter $\kappa$ has been determined with greater accuracy from a systematic analysis of the light-quark excitation spectra [12] in the context of the semiclassical superconformal approach unifying mesons and baryons [6]; 2) The running of $\alpha_{\overline{\text{MS}}} (Q^2)$ has been computed to five loops [31], that is the $\beta$–function is now known up to order $\beta_4$ in the $\overline{\text{MS}}$ renormalization scheme; and 3) the average world data for $\Lambda_{\overline{\text{MS}}}$ has been updated [32].

In this article, we improve our determination of $\Lambda_{\overline{\text{MS}}}$ from the light-front holographic QCD framework [2] utilizing these new developments. We also study the convergence of this determination. The pQCD approximants are asymptotic Poincaré series that converge up to an optimal order $\sim 1/a$, where $a = \alpha_s^{\text{QCD}}/\pi$ is the expansion parameter of the series. Indeed, we have shown in Ref. [24] that the transition between the LFHQCD description of $\alpha_s(Q^2)$ and its pQCD description occurs at $Q_0^2 = 0.75 \pm 0.07$ GeV$^2$ in the $\overline{\text{MS}}$ scheme: The optimal order in the Poincaré series is thus $1/a(Q_0^2) \sim 8$. Consequently, it is advantageous to use $\alpha_{\overline{\text{MS}}}^{\text{PQCD}} (Q^2 > Q_0^2)$ evaluated at five loops to obtain an accurate value of $\Lambda_{\overline{\text{MS}}}$ following the matching procedure with the nonperturbative regime described above.
1 Result for $\Lambda_{\overline{MS}}$

The perturbative series of the $\beta$ function

$$Q^2 \frac{\partial}{\partial Q^2} \frac{\alpha_s}{4\pi} = \beta(\alpha_s) = -\left(\frac{\alpha_s}{4\pi}\right)^2 \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{4\pi}\right)^n \beta_n,$$  \hspace{1cm} (3)

calculated up to order $\beta_4$ yields the five-loop expression of $\alpha_{\overline{MS}}^{pQCD}$ [33]:

$$\alpha_{\overline{MS}}^{pQCD}(Q^2) = \frac{4\pi}{\beta_0 t} \left[ 1 - \frac{\beta_1}{\beta_0} \ln(t) \right] + \frac{\beta_1^2}{\beta_0^2 t^2} \left( \ln^2(t) - \ln(t) - 1 \right) \left( \beta_2 / \beta_1 \right)$$

$$+ \frac{\beta_1^3}{\beta_0^3 t^3} \left( -\ln^3(t) + \frac{5}{2} \ln^2(t) + 2 \ln(t) - \frac{1}{2} - 3 \frac{\beta_2}{\beta_1} \ln(t) + \frac{\beta_3 / \beta_1^2}{2} \right)$$

$$+ \frac{\beta_1^4}{\beta_0^4 t^4} \left( \ln^4(t) - \frac{13}{3} \ln^3(t) - \frac{3}{2} \ln^2(t) + 4 \ln(t) + \frac{7}{6} + \frac{3 \beta_2}{\beta_1} \ln(t) - \frac{1}{2} \right)$$

$$+ \frac{\beta_3 / \beta_1^3}{2} \left( 2 \ln(t) + \frac{1}{6} \right) + \frac{5 \beta_2^2 / \beta_1^2}{3} \left( \beta_4 / \beta_1^2 \right) + O \left( \frac{\ln(t)^6}{t} \right),$$  \hspace{1cm} (4)

with $t = \ln(Q^2/\Lambda_{\overline{MS}}^2)$ and

$$\beta_0 = 11 - \frac{2}{3} n_f,$$  \hspace{1cm} (5)

$$\beta_1 = 102 - \frac{38}{3} n_f,$$  \hspace{1cm} (6)

$$\beta_2 = \frac{2857}{2} - \frac{5033}{18} n_f + \frac{325}{54} n_f^2,$$  \hspace{1cm} (7)

$$\beta_3 = \left( \frac{149753}{6} + 3564 \xi(3) \right) - \left( \frac{1078361}{162} + \frac{6508}{27} \xi(3) \right) n_f + \left( \frac{50065}{162} + \frac{6472}{81} \xi(3) \right) n_f^2$$

$$+ \frac{1093}{729} n_f^3, \hspace{1cm} (8)$$
\[ \beta_4 = \frac{8157455}{16} + \frac{621885}{2} \xi_3 - \frac{88209}{2} \xi_4 - 288090 \xi_5 + \left( -\frac{336460813}{1944} - \frac{4811164}{81} \xi_3 + \frac{33935}{6} \xi_4 + \frac{1358995}{27} \xi_5 \right) n_f + \left( -\frac{25960913}{1944} + \frac{698531}{81} \xi_3 - \frac{10526}{9} \xi_4 - \frac{381760}{81} \xi_5 \right) n_f^2 + \left( -\frac{1205}{2916} - \frac{152}{81} \xi_3 \right) n_f^3, \] (9)

with \( \xi_n \) the Riemann zeta function [31]. The coefficients \( \beta_0 \) and \( \beta_1 \) are scheme independent and the higher order coefficients are given in the \( \overline{MS} \) renormalization scheme. Here, we will set \( n_f = 3 \) and use the updated value of the holographic QCD scale parameter, \( \kappa = 0.523 \pm 0.024 \) GeV determined from the excitation spectra of all light mesons and baryons [12]. This value characterizes the mass scale of light-quark hadron spectroscopy and is compatible with the fit to the Bjorken sum data at low \( Q^2 \) [34] in the holographic QCD validity domain, which yields \( \kappa = 0.496 \pm 0.007 \) GeV [1]. The updated value of \( \kappa \) is lower than –but compatible with– the value we used in [2]: \( \kappa = m_\rho / \sqrt{2} = 0.548 \) GeV [3], with \( m_\rho \) the \( \rho \)-meson mass. This value is also used in the study of hadronic form factors, which are expressed in terms of \( \rho \) mass poles and its radial recurrences [3, 35].

As in Ref. [2], we compute \( \alpha_{g_1}^{pQCD}(Q^2) \) using the Bjorken sum rule [26] up to 5th order in \( \alpha_{g_1}^{pQCD} \) [28]. At \( \beta_4 \) and \( \left( \alpha_{g_1}^{pQCD} \right)^4 \) orders, we obtain \( \Lambda_{\overline{MS}} = 0.339 \pm 0.019 \) GeV and \( Q_0^2 = 1.14 \pm 0.12 \) GeV by matching the nonperturbative and perturbative expressions for the couplings, Eqs. (1) and (2) respectively. This value of \( \Lambda_{\overline{MS}} \) is to be compared to the present world data, \( \Lambda_{\overline{PDG}} = 0.332 \pm 0.017 \) GeV for \( n_f = 3 \) [32]. (The value of \( Q_0 \) is given in the \( g_1 \) scheme and is higher than the corresponding value in the \( \overline{MS} \) scheme [24].)

The uncertainties entering our determination stem from the uncertainty on \( \kappa (\pm 0.016 \) GeV), the uncertainty from the chiral limit approximation (\( \pm 0.003 \) GeV) and the truncation uncertainty on the Bjorken and \( \alpha_{g_1}^{pQCD} \) series, Eqs. (2) and (4), respectively (\( \pm 0.010 \) GeV). This uncertainty is taken, for order \( n \), as the difference between the results at orders \( n \) and \( n + 1 \), the uncertainty at the highest order being taken equal to that of the preceding order. The first two contributions to the total uncertainty reflect
the consequence of approximations necessary to make the LFHQCD approach tractable. One could systematically improve LFHQCD towards exact QCD by diagonalizing the true QCD LF Hamiltonian on an orthonormal basis constructed from the AdS/QCD solutions. This is a method called BLFQ (Basis Light-Front Quantization) [36].

The total uncertainty has significantly improved compared to our previous determination, $\Lambda_{\overline{MS}} = 0.341 \pm 0.032$ GeV [2]. The updated prediction of the running coupling is shown in Fig. 1, together with the previous determination [2] and experimental data [27].

![Figure 1: Running of $\alpha_{g_1}(Q)$ for $\kappa = 0.523$ GeV, $\Lambda_{\overline{MS}} = 0.339$ GeV (red line). Also shown are experimental data [27] and the earlier determination of $\alpha_{g_1}(Q)$ for $\kappa = 0.548$ GeV, $\Lambda_{\overline{MS}} = 0.341$ GeV (black line) [2]. The arrow marks the transition scale $Q_0$ to the perturbative regime.](image)

The result using the Bjorken sum rule coefficient $a_5$ in Eq. (2), which is assessed in Ref. [29], is $\Lambda_{\overline{MS}} = 0.317 \pm 0.019$ GeV. The uncertainty stems from the uncertainty on $\kappa$ ($\pm 0.015$ GeV), the uncertainty from the chiral limit approximation ($\pm 0.003$ GeV), the truncation uncertainty on the Bjorken and $\alpha_{pQCD_{\overline{MS}}}$ series, Eqs. (2) and (4), respectively, ($\pm 0.010$ GeV), and an estimate on the $a_5$ uncertainty ($\pm 0.005$ GeV). This latest contribution is assessed by rescaling $a_5$ by the factor 175.7/130 and obtaining $\Lambda_{\overline{MS}}$ with this
rescaled value. Indeed, the estimate of the Bjorken sum rule coefficient $a_4$ in Ref. [29] was 130 while the recent exact calculation yields $a_4 = 175.7$ [28]. The ratio 175.7/130 thus provides an indication of the uncertainty on $a_5$. We will not quote $\Lambda_{\overline{MS}}$ at fifth order, since the coefficient $a_5$ in Eq. 2 has only been estimated rather than computed.

The present uncertainty on $\Lambda_{\overline{MS}}$ has improved by close to a factor of 2 compared to the result reported in Ref. [2]. Another way to quantify the improvement between our previous determination and the present result is to inspect the residual between $\alpha_{g_1}(Q)$ obtained on the full $Q$-range using our matching procedure and the experimental data. We show such residuals in Fig. 2. The matching procedure does not involve any fit to the experimental data and has no free parameter: $\kappa$ is fixed and $\Lambda_{\overline{MS}}$ is obtained from the matching, without influence from data. Thus, the departure from zero of the residual and the $\chi^2$ of its averaged value quantify the agreement between two determinations of $\alpha_{g_1}$ –from experiments, and from the matching procedure of LFHQCD and pQCD described here— that are fully independent. The averaged residual for the present result is $5.7 \times 10^{-4} \pm 9.2 \times 10^{-3}$ (exp.) $\pm 6.4 \times 10^{-2}$ (theo.) with $\chi^2 = 7.2$. The result from Ref. [2] yields an averaged residual of $5.7 \times 10^{-2} \pm 9.2 \times 10^{-3}$ (exp.) $\pm 1.8 \times 10^{-1}$ (theo.) with $\chi^2 = 8.3$. The “experimental” uncertainty reflects for the gaussian deviation of the experimental data from the average value of the residual. It is evidently the same in both cases, since they use the same experimental data. The theoretical uncertainty reflects the uncertainty of the theoretical prediction for $\alpha_{g_1}(Q)$ obtained with our matching procedure, that is the uncertainty on $\kappa$ and $\Lambda_{\overline{MS}}$. The size of the theoretical uncertainty improved significantly, by a factor of 3. It largely dominates the “experimental” uncertainty. In addition, the residual value for the present result is much closer to zero compared to the result obtained in Ref. [2], although both of them are compatible with zero. The $\chi^2$ is also improved, albeit marginally. In all, these comparisons quantify the significant improvement of our determination of $\Lambda_{\overline{MS}}$ and consequently of $\alpha_{g_1}(Q)$ compared to our earlier result.

2 Convergence

The convergence with respect to the $\beta$-order is shown in Fig. 3 for the Bjorken series calculated at order $\left(\alpha_{\overline{MS}}^{QCD}\right)^4$. This series oscillates but nevertheless converges well. The convergence with respect to the Bjorken series order is shown in Fig. 4 for $\alpha_{\overline{MS}}^{pQCD}$ calculated at order $\beta_4$. The overall convergence of our method is estimated with both the $\beta$- and the Bjorken series calculated at the same order. This is also shown in
**3 Conclusion**

We have updated the analysis initially reported in Ref. [2]. The improved prediction $\Lambda_{\overline{MS}} = 0.339 \pm 0.019$ GeV obtained from matching the light-front holographic QCD (LFHQCD) predictions, constrained by the superconformal algebraic structure, and the perturbative QCD five-loop computation, is in excellent agreement with the value from the present world data, $\Lambda_{\overline{MS}}^{PDG} = 0.332 \pm 0.017$ GeV. The GeV units conventionally used for mass involves physics external to QCD. QCD only predicts dimensionless ratios of masses such as $m_\rho/m_p$. We thus cast our main result as the ratio:

$$\frac{\Lambda_{\overline{MS}}}{m_p} = 0.361 \pm 0.020,$$

Fig. 4. The convergence is slightly faster than the case when the $\beta$-series is kept at order $\beta_4$. 

Figure 2: Residual between the LFHQCD and pQCD matched $\alpha_g(Q)$ and the experimental data. The top panel corresponds to the previous determination obtained in Ref. [2]. The bottom panel is our present result. In each panel, the broader band is the total uncertainty while the thinner and denser band inside represents the fit uncertainty only.
Figure 3: Convergence of our determination of $\Lambda_{\overline{MS}}$ (black squares) as a function of the $\beta$-series order for $n_f = 3$. The pQCD series for the Bjorken sum rule is computed at order $(\alpha_p^{\overline{MS}})^4$. The error bars reflect only the uncertainty from the truncation of the $\beta$-series. The blue band gives the latest world data.

Our method is applicable for setting the perturbative QCD scale $\Lambda_s$ in any renormalization scheme. We have used the $\overline{MS}$ scheme since this has been the conventional choice for pQCD analyses.

We performed a convergence analysis that validates and improves the method. This could not be done without the newly available 5-loop calculation, as can be seen by removing the last point of Fig. 4. The convergence of the method is satisfactory overall, for both the $\beta$-series and the pQCD prediction for the Bjorken sum rule. The largest uncertainty stems from the truncation of the Bjorken sum pQCD series. A calculation of its next term—presently only estimated—and the application of the Principle of Maximum Conformality (PMC) [37, 38] would be valuable for further improving the accuracy of the method discussed here. The uncertainty from the determination of the mass scale $\kappa$ from hadronic spectroscopy contributes similarly. Thus a reduction in the uncertainty of its value will provide an even more accurate holographic prediction for $\Lambda_{\overline{MS}}$.

The excellent agreement between the light-front holographic prediction and the world data validates with high accuracy the relevance of the gauge/gravity approach to nonper-
Figure 4: Convergence of our determination of $\Lambda^{(3)}_{\overline{MS}}$ as a function of the Bjorken series order (squares). $\alpha^{pQCD}_{\overline{MS}}$ is computed at order $\beta_4$ and for $n_f = 3$. The triangles represent the results when both the $\beta$- and the Bjorken series are computed at the same order. The error bars include only the uncertainty from the series truncation. The blue band gives the latest world data.

perturbative strong interaction phenomena and the constraints imposed by superconformal quantum mechanics. The LFHQCD approach is a remarkable advance for hadron physics since it provides a direct connection between the mass scale $\kappa$, underlying the masses of the proton and other hadrons, with the mass scale $\Lambda^{(3)}_{\overline{MS}}$ underlying perturbative QCD. It also leads to a description of the QCD running coupling at all scales and determines a transition scale between the nonperturbative and perturbative domains. These advances have been long-term goals of hadron physics.

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