SU(ν) Generalization of Twisted Haldane-Shastry Model

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Abstract

The SU(ν) generalized Haldane-Shastry spin chain with 1/r² interaction is studied with twisted boundary conditions. The exact wavefunctions of Jastrow type are obtained for every rational value of the twist angle in unit of 2π. The spectral flow of the ground state is then discussed as a function of the twist angle. By resorting to the motif picture in the Bethe ansatz method, we show that the period of the spectral flow is ν, which is determined by the statistical interaction in exclusion statistics.

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I. INTRODUCTION

The Haldane-Shastry spin chain \( \frac{1}{r^2} \) is a one-dimensional integrable lattice model with \( 1/r^2 \) interaction \( \frac{3}{4} \), which has provided a number of valuable notions in low-dimensional quantum systems. Among others, (fractional) exclusion statistics proposed by Haldane \( \frac{3}{4} \) has received considerable attention. From the viewpoint of exclusion statistics, the Haldane-Shastry model is regarded as an idealized model which is completely free from any irrelevant perturbations and describes the fixed point Hamiltonian for massless spin chain models. In this idealized circumstance, it may be possible to observe characteristic features inherent in exclusion statistics.

Motivated by this, we have recently extended the Haldane-Shastry model to include twisted boundary conditions, and have shown that exclusion statistics can be explicitly observed in the period of the spectral flow \( \frac{3}{4} \). The strategy to probe exclusion statistics has been based on the idea that the response to twisted boundary conditions (or equivalently to external gauge fields) enables us to extract the knowledge on various properties for interacting particle systems \( \frac{4}{4} \). As mentioned above, the Haldane-Shastry model is free from any irrelevant perturbations, allowing us to observe exclusion statistics clearly in the spectral flow. It has been indeed shown \( \frac{3}{4} \) that the period of the ground state is solely determined by the statistical interaction in exclusion statistics.

In this paper, we generalize the twisted Haldane-Shastry model to the case with SU(\( \nu \)) spin symmetry, and solve it exactly. Such a generalization to the multicomponent models may be interesting in its own right \( \frac{12}{12} \) and also its close relationship to the fractional quantum Hall effect. We show that the period of the spectral flow in the SU(\( \nu \)) model is indeed determined by the matrix of the statistical interaction in exclusion statistics, i.e. the SU(\( \nu \)) Cartan matrix.

In the next section, we introduce the Hamiltonian of the SU(\( \nu \)) Haldane-Shastry model with twisted boundary conditions. We explain how the twisted boundary condition is imposed consistently with the long-range nature of the \( 1/r^2 \) interaction. In section III, the
exact solution of the twisted SU(\(\nu\)) model is derived by exploiting the Jastrow-type ansatz. In section IV, the exact spectrum thus obtained is shown to be reproduced by the Bethe ansatz. We discuss the spectral flow of the energy spectrum as a function of the twist angle in section V. By resorting to the notion of the motif \([14]\) in the Bethe ansatz, we give an interpretation for the period of the spectral flow in terms of exclusion statistics. Summary and discussions are given in section VI.

II. MODEL HAMILTONIAN

The Haldane-Shastry model has long-range \(1/r^2\) interaction, so that it is not trivial to impose twisted boundary conditions without loss of integrability. Recently, we have proposed one of the ideas for this purpose \([6]\). We first recall that the Haldane-Shastry model \([1,2]\) was originally introduced by imposing the periodic boundary condition and by summing up the pair-wise \(1/r^2\) interaction around the ring infinite times \([4]\). From this point of view, we start with the following Hamiltonian,

\[
H = \sum_{n<n'} \sum_{m=-\infty}^{\infty} \frac{1}{(n - n' - m\mathcal{N})^2} \frac{1}{2} P_{n,n+m\mathcal{N}}
\]

\[
= \sum_{n<n'} \sum_{m=-\infty}^{\infty} \frac{1}{(n - n' - m\mathcal{N})^2} \times \left\{ \frac{1}{2} \sum_{\alpha \in \Phi_+} \left( E_n^\alpha E_{n'+m\mathcal{N}}^{-\alpha} + E_{n'}^{-\alpha} E_n^{\alpha} \right) + \left( \sum_{k=1}^{\nu-1} H_n^k H_{n'+m\mathcal{N}}^k + \frac{1}{2\nu} \right) \right\}, \tag{2.1}
\]

where \(n\) denotes the index for sites \(n = 1, 2, \cdots, \mathcal{N}\) and \(P_{n,n'} \equiv 2 \sum_{a} T_n^a T_{n'}^a + 1/\nu\) is the exchange operator in SU(\(\nu\)) algebra. In the second line, \(\Phi_+\) is the set of the positive roots of SU(\(\nu\)) algebra, the generator \(E_n^\alpha\) is the so-called step operator associated with the root \(\alpha\) at site \(n\), and \(H_n^k\) with \(k = 1, 2, \cdots, \nu-1\) is the hermitian generator in Cartan subalgebra. We fix the normalization of the generators as \(\text{tr}(T^a T^b) = \frac{1}{2} \delta_{ab}\). The Hamiltonian (2.1) was already solved in the case of the periodic boundary condition \([12,13]\), where the summation over \(m\) can be explicitly carried out to give the well-known sine-inverse-square interaction \([4]\). However, once we impose twisted boundary conditions, it becomes a non-trivial solvable
model compatible with twisted boundary conditions. The form of the Hamiltonian (2.1) implies that the Cartan basis may be a natural basis to treat the system with twisted boundary conditions. By this reason, we shall span the basis of the wave function in terms of the Cartan basis, which is different from that in [12], though both bases give the same results in the case of the periodic boundary condition.

To be more specific in our notations, let us denote the simple roots by \( \alpha_{(i)} \) with \( i = 1, 2, \ldots, \nu - 1 \). Then the positive roots can be expressed as \( \alpha_{(k)} + \cdots + \alpha_{(l)} \) with \( 1 \leq k \leq l \leq \nu - 1 \). Step operators associated with the positive roots are denoted simply as \( E^k = E^{\alpha_{(1)} + \cdots + \alpha_{(k)}} \) for \( k = 1, \ldots, \nu - 1 \) and \( E^{(k,l)}_n \equiv E^{\alpha_{(k+1)} + \cdots + \alpha_{(l)}}_n \) for \( 1 \leq k < l \leq \nu - 1 \). We label the state at each site as \( |\mu_{(i)}\rangle \), which is specified by the \( \nu \) weight vectors \( \mu_{(i)} \) with \( i = 0, 1, \ldots, \nu - 1 \) in the fundamental representation. As we have fixed the normalization of the generators, the weight vectors satisfy

\[
\mu_{(i)} \cdot \mu_{(j)} = \begin{cases} 
\frac{1}{2} - \frac{1}{2\nu} & \text{for } i = j \\
- \frac{1}{2\nu} & \text{for } i \neq j.
\end{cases}
\]  

The highest weight state is assumed to be \( |\mu_{(0)}\rangle \), and other descendant states are created by the lowering operators \( E^{-k} \) such that \( |\mu_{(k)}\rangle = E^{-k}|\mu_{(0)}\rangle \). Therefore, in this representation, \( E^k \) connects directly the highest weight state \( |\mu_{(0)}\rangle \) with a descendant state \( |\mu_{(k)}\rangle \), while \( E^{(k,l)} \) connects two descendant states \( |\mu_{(k)}\rangle \) and \( |\mu_{(l)}\rangle \).

The boundary condition we impose on the Hamiltonian (2.1) in this paper is

\[
E^{\pm k}_{n+mN} = e^{\pm 2\pi i \phi_k m} E^\pm_n, \\
E^{\pm (k,l)}_{n+mN} = e^{\pm 2\pi i (\phi_l - \phi_k) m} E^{\pm (k,l)}_n, \\
H^k_{n+mN} = H^k_n,
\]  

(2.3)

where the first line is the definition of the \( \nu - 1 \) twist angles \( \phi_k \) and the second line naturally follows from the commutation relation of the step operators. In what follows, we refer to the angle \( \phi_k \) defined in unit of \( 2\pi \) as the twist angle. Note that we can impose twisted boundary conditions independently on each species.
In order to obtain the exact solution, it is convenient to introduce the periodic operators by the following gauge transformations

\[ E_{n}^{\pm k} \rightarrow e^{\pm 2\pi i \phi_k n/N} E_{n}^{\pm k}, \]
\[ E_{n}^{\pm (k,l)} \rightarrow e^{\pm 2\pi i (\phi_k - \phi_l) n/N} E_{n}^{\pm (k,l)}. \] (2.4)

In the remainder of the paper, we always use the step operators in the gauge transformed form. Now we restrict ourselves to rational values

\[ \phi_k = \frac{p_k}{q_k} \] (2.5)

with \( p_k \) and \( q_k \) being integers. This constraint is, at present, essential to solve the twisted model. Then the summation over \( n \) can be done explicitly, and the Hamiltonian (2.1) is now written as

\[ H = \left( \frac{\pi}{N} \right)^2 \frac{1}{2} (T + H_{\text{int}}), \] (2.6)

where \( T \) and \( H_{\text{int}} \) are the hopping and interaction terms, respectively. The former is defined by

\[ T = \sum_k \sum_{n \neq n'} J_{\phi_k}(n - n') E_{n}^{k} E_{n'}^{-k} + \sum_{k < l} \sum_{n \neq n'} J_{\phi_l - \phi_k}(n - n') E_{n}^{(kl)} E_{n'}^{-(kl)}. \] (2.7)

Here \( J_{\phi}(n) \) is the \( 1/r^2 \) coupling compatible with the boundary condition (2.3),

\[ J_{\phi}(n) \equiv \frac{1}{q^2} \sum_{m=0}^{q-1} e^{2\pi i p(n + mN)/qN} \sin^{-2} \left[ \frac{\pi (n + mN)}{qN} \right], \quad \text{for } \phi = \frac{p}{q}. \] (2.8)

For later convenience, we rewrite the above expression as follows:

\[ T = \sum_k T_k(\phi_k) + \frac{1}{2} \sum_{k \neq l} T_{kl}(\phi_{kl}), \] (2.9)

where \( \phi_{kl} = \phi_k - \phi_l \) and

\[ T_k(\phi) = \sum_{n \neq n'} J_{\phi}(n - n') E_{n}^{k} E_{n'}^{-k}, \quad T_{kl}(\phi) = U_l T_k U_l^\dagger. \] (2.10)

Here \( U_l \) is the unitary operator of the \( \pi \) rotation around \( \alpha_{(1)} + \cdots + \alpha_{(k)} \), i.e., \( U_k = \exp\{i \frac{\pi}{2}(E_k + E^{-k})\} \). The interaction term is
\[ H_{\text{int}} = \sum_{n \neq n'} J_0(n - n') \left( \sum_{k=1}^{\nu-1} H_n^k H_{n'}^k + \frac{1}{2\nu} \right). \] 

(2.11)

Note that \( J_0(n) \) is of the usual sine-inverse-square form \( J_0(n) = \sin^{-2} \frac{\pi n}{N} \). This completes the introduction of the twisted SU(\( \nu \)) Haldane-Shastry model. We can see that after the gauge transformation (2.4) the effect of twisted boundary conditions is incorporated in effective hopping, and hence we can solve the Hamiltonian with periodic boundary conditions.

III. EXACT SOLUTION

In this section we construct the exact eigenstates for the model (2.6) and obtain the corresponding eigenenergies. In the previous paper [6], we have solved the simplest case, i.e. the twisted SU(2) model, by using the Jastrow-ansatz wave function. In that case, we have demonstrated that the Jastrow wave function can still be an eigenfunction of the twisted Haldane-Shastry model if we convert the operators into the periodic ones via the gauge transformation. In what follows, we will show that this is also the case for the SU(\( \nu \)) model, and then obtain its exact solution.

Before starting the calculation, let us fix some notations. There are \( \nu \) different states at each site. We take the highest weight state \(|\mu(0)\rangle\) as the reference state (background), and regard other descendant states \(|\mu(k)\rangle\) \((k = 1, 2, \cdots \nu - 1)\) as particle states. Since the number of states in \(|\mu(k)\rangle\) is conserved, let us denote it as \( M_k \). Then the relation \( \sum_{k=0}^{\nu-1} M_k = N \) holds. Let the subset of the sites denoted by \( \{n^{(k)}_{\alpha_k}\} \) with \( \alpha_k = 1, \cdots, M_k \) be the positions of the state \(|\mu(k)\rangle\).

We now propose the following Jastrow-ansatz state as a candidate for the exact eigenstate of the twisted SU(\( \nu \)) model,

\[ |\Psi\rangle = \sum_{k=1}^{\nu-1} \sum_{n^{(k)}_1 < \cdots < n^{(k)}_{M_k}} \psi(\cdots, n^{(k)}_1, \cdots; J_k, \cdots) \prod_{k=1}^{\nu-1} \prod_{\alpha_k=1}^{M_k} E_{n^{(k)}_{\beta_k}}^{-k} |\mu_0, \mu_0, \cdots, \mu_0\rangle, \]

(3.1)

where \( \psi \) is a Jastrow-type wave function

\[ \psi = \prod_{k=1}^{\nu-1} \prod_{\alpha_k=1}^{M_k} z^{J_k n^{(k)}_{\alpha_k}} \prod_{k=1}^{\nu-1} \prod_{\alpha_k < \beta_k} d(n^{(k)}_{\alpha_k} - n^{(k)}_{\beta_k})^2 \prod_{k<l} \prod_{\alpha_k, \alpha_l} d(n^{(k)}_{\alpha_k} - n^{(l)}_{\alpha_l}). \]

(3.2)
Here, \( d(n) = \sin(\pi n/N) \), \( z = \exp(2\pi i/N) \), and the current for each species is defined by
\[
J_k = \frac{N - M_k - M_0}{2} \mod 1. \tag{3.3}
\]

We wish to prove below that this is indeed an eigenstate of the Hamiltonian.

Let us start by evaluating the action of the hopping terms on the wavefunction. We have introduced two kinds of the hopping Hamiltonians \( T_k(\phi_k) \) and \( T_{kl}(\phi_{kl}) \) in eq.(2.10). The former exchanges the pairs of \( \{n(0)\} \) and \( \{n(k)\} \), while the latter exchanges the pairs of \( \{n(k)\} \) and \( \{n(l)\} \). In what follows, we first evaluate the action of \( T_k \) on the wave function, which can be calculated directly. Although the direct manipulation of \( T_{kl} \) turns out to be difficult, there is a remarkable trick to simplify the calculation, found by Wang, Liu and Coleman [15] for the supersymmetric \( t \)-\( J \) model. By generalizing this trick to the \( \text{SU}(\nu) \) model, we then evaluate the action of \( T_{kl} \) by the use of the results obtained for the action of \( T_k \).

Although the wavefunction (3.2) contains the coordinates of various species, we note that \( T_k \) acts only on \( |\mu(0)\rangle \) and \( |\mu(k)\rangle \). We thus classify the set of total sites into three subsets \( \{n(0)\} \), \( \{n(k)\} \) and the remainder \( \{n\} - \{n(0)\} - \{n(k)\} \). To simplify the expressions, we introduce the following notations for the elements of each subset,
\[
x_\alpha \in \{n(k)\}, \quad \alpha = 1, 2, \cdots, M_k
\]
\[
y_j \in \{n\} - \{n(k)\} - \{n(0)\}, \quad j = 1, 2, \cdots, N - M_k - M_0. \tag{3.4}
\]

Then, the problem reduces essentially to that of two species with the background, and one finds that the action of \( T_k \) on (3.2) is analogous to that of the spin-hopping term for the supersymmetric \( t \)-\( J \) model with \( 1/r^2 \) interaction [16]. Therefore, by exploiting the techniques developed there, we have
\[
\frac{T_k(\phi_k)|\psi\rangle}{|\psi\rangle} = \sum_{n=1}^{N-1} J_{\phi_k}(n) z^{J_k n} \sum_{\alpha=1}^{M_k} \prod_{\beta(\neq \alpha)=1}^{M_k} B_{\alpha\beta}^{(n)} \prod_{j=1}^{N-M_k-M_0} F_{\alpha j}^{(n)}, \tag{3.5}
\]
where
\[
B_{\alpha\beta}^{(n)} = 1 - g_{\alpha\beta}^{(n)}, \quad g_{\alpha\beta}^{(n)} = \frac{(1 - z^n) z_\alpha^2 + (1 - z^{-n}) z_\beta^2}{(z_\alpha - z_\beta)^2} \tag{3.6}
\]
with \( z_\alpha = z^{x_\alpha} \) and

\[
F^{(n)}_{\alpha j} = \cos \frac{\pi n}{N} + \sin \frac{\pi n}{N} \cot \Theta_{\alpha j}, \quad \Theta_{\alpha j} = \frac{\pi (x_\alpha - y_j)}{N}.
\] (3.7)

The last factor \( F \) in eq.(3.5) describes the interaction between different species. Note that such interaction terms automatically disappear in the case of SU(2). Expanding the above formula in power series of \( z^n \) and \( z^{-n} \), and using the original notation for sites, we finally find the expression for the action of \( T_k \),

\[
\frac{T_k(\phi_k)\psi}{\psi} = \sum_{i=1}^{4} W_i^{(k)},
\] (3.8)

where

\[
W_1^{(k)} = 2M_k\tilde{\varepsilon}(J_k + \phi_k) + \frac{1}{3}M_k(N^2 - 1) + \frac{1}{6}M_k(M_k^2 - 1) + \frac{1}{2}M_kM_0^2,
\] (3.9)

\[
W_2^{(k)} = \frac{1}{2} \sum_{\alpha_k \neq \beta_k} J_0(n^{(k)}_{\alpha_k} - n^{(k)}_{\beta_k}) - \frac{1}{2} \sum_{\alpha_k, \alpha_0} J_0(n^{(k)}_{\alpha_k} - n^{(0)}_{\alpha_0}) + \frac{1}{2} \sum_{m=1}^{\nu-1} \sum_{\alpha_m, \alpha_k} J_0(n^{(k)}_{\alpha_k} - n^{(m)}_{\alpha_m}),
\] (3.10)

\[
W_3^{(k)} = 2i \left( J_k + \phi_k - \frac{N}{2} \right) \sum_{\alpha_k, \alpha_0} \cot \Theta_{\alpha_k \alpha_0},
\] (3.11)

\[
W_4^{(k)} = \sum_{\alpha_k \neq \beta_k, \alpha_0} \cot \Theta_{\alpha_k \beta_k} \cot \Theta_{\alpha_k \alpha_0} + \sum_{\alpha_0 \neq \beta_0, \alpha_k} \cot \Theta_{\alpha_0 \beta_0} \cot \Theta_{\alpha_0 \alpha_k}.
\] (3.12)

The function \( \tilde{\varepsilon}(J + \phi) \) is

\[
\tilde{\varepsilon}(J + \phi) = \begin{cases} 
\varepsilon(J + \phi), & \text{for integer } J, \\
\frac{1}{2} \left( \varepsilon(J + \phi - \frac{1}{2}) + \varepsilon(J + \phi + \frac{1}{2}) - \frac{1}{2} \right), & \text{for half-integer } J,
\end{cases}
\] (3.13)

where \( \varepsilon(k) \) is defined for every rational \( k \) as

\[
\varepsilon(k) = (2[k] - N + 1)k - [k] ([k] + 1).
\] (3.14)

The expression (3.8) holds for the currents which satisfy

\[
\frac{N + M_k - M_0}{2} - 1 \leq J_k + \phi_k \leq N - \left( \frac{N + M_k - M_0}{2} \right) - 1.
\] (3.15)

The derivation of the above equations is outlined in Appendix.

We now move to the action of \( T_{kl} \). As already mentioned, \( T_{kl} \) exchanges the pairs of \( \{n^{(k)}\} \) and \( \{n^{(l)}\} \), making its action on the wave function (3.2) rather complicated. However, we note the following identity
\[
\psi(\cdots, \{n^{(k)}\}, \cdots; J_1, \cdots, J_k, \cdots J_{\nu-1}) \\
= A\psi(\cdots, \{n^{(0)}\}, \cdots; J_1 - J_k + N/2, \cdots, N - J_k, \cdots, J_{\nu-1} - J_k + N/2),
\]

(3.16)

where \(A\) is a constant independent of coordinates. Let us denote the r.h.s. of the above equation as \(A\psi(k)\). Combining these two relations, we can evaluate the action of \(T_{kl}\) quite similarly to that of \(T_k\). The key formula is

\[
\frac{T_{kl}(\phi_{kl})\psi}{\psi} = \frac{T_k(\phi_{kl})\psi(l)}{\psi(l)},
\]

(3.17)

which was previously found for the supersymmetric \(t\)-\(J\) model [15]. The relation implies that what we have to do is to substitute the following replacements to eq.(3.8),

\[
\{n^{(0)}\} \leftrightarrow \{n^{(l)}\}, \quad (M_0 \leftrightarrow M_l), \quad J_k \rightarrow \begin{cases} J_{kl} + \frac{N}{2} & \text{for } k \neq l \\ N - J_k & \text{for } k = l, \end{cases}
\]

(3.18)

where \(J_{kl} = J_k - J_l\). The results are

\[
\frac{T_{kl}(\phi_{kl})\psi}{\psi} = \sum_{i=1}^{4} W_i^{(kl)},
\]

(3.19)

where

\[
W_1^{(kl)} = 2M_k\epsilon(J_{kl} + N/2 + \phi_{kl}) + \frac{1}{3}M_k(N^2 - 1) + \frac{1}{6}M_k(M_k^2 - 1) + \frac{1}{2}M_kM_l^2,
\]

(3.20)

\[
W_2^{(kl)} = \frac{1}{2} \sum_{\alpha_k \neq \beta_k} J_0(n^{(k)}_{\alpha_k} - n^{(k)}_{\beta_k}) - \frac{1}{2} \sum_{\alpha_k, \alpha_l} J_0(n^{(k)}_{\alpha_k} - n^{(l)}_{\alpha_l}) + \frac{1}{2} \sum_{m=0}^{\nu-1} \sum_{\alpha_m, \alpha_k} J_0(n^{(k)}_{\alpha_k} - n^{(m)}_{\alpha_m}),
\]

(3.21)

\[
W_3^{(kl)} = 2i(J_{kl} + \phi_{kl}) \sum_{\alpha_k, \alpha_l} \cot \Theta_{\alpha_k, \alpha_l},
\]

(3.22)

\[
W_4^{(kl)} = \sum_{\alpha_k \neq \beta_k, \alpha_l} \cot \Theta_{\alpha_k, \beta_k} \cot \Theta_{\alpha_l, \alpha_l} + \sum_{\alpha_l \neq \beta_l, \alpha_k} \cot \Theta_{\alpha_l, \beta_l} \cot \Theta_{\alpha_k, \alpha_k},
\]

(3.23)

provided that the currents satisfy the condition

\[
\frac{M_k - M_l}{2} - 1 \leq J_{kl} + \phi_{kl} \leq 1 - \frac{M_k - M_l}{2}.
\]

(3.24)

Consequently, from eqs.\((3.8)\) and \((3.19)\), we have the action of the total hopping Hamiltonian \((2.9)\) on the wave function,
\[
\frac{T\psi}{\psi} = \sum_{k=1}^{4} W_i, \tag{3.25}
\]

where
\[
W_i = \sum_{k=1}^{\nu-1} W_i^{(k)} + \frac{1}{2} \sum_{\substack{k,l=1 \atop k\neq l}}^{\nu-1} W_i^{(kl)}, \quad \text{for } i = 1, 2, 3, 4. \tag{3.26}
\]

In this expression, there still remain unwanted two- and three-body terms. In what follows, we show how these unwanted terms indeed vanish.

First, we can easily confirm that the two-body terms depending on the currents, \( W_3 \), vanish. The other two-body terms \( W_2 \) are reduced to
\[
W_2 = \frac{1}{2} \sum_{k=1}^{\nu-1} \sum_{\alpha_k \neq \beta_k} J_0(n_{\alpha_k}^{(k)} - n_{\beta_k}^{(k)}) - \frac{1}{2} \sum_{k=1}^{\nu-1} \sum_{\alpha_k, \alpha_0} J_0(n_{\alpha_k}^{(k)} - n_{\alpha_0}^{(0)}) + \frac{1}{4} (\nu - 2) \sum_{k=1}^{\nu-1} \sum_{\alpha_k, n} J_0(n_{\alpha_k}^{(k)} - n) = \frac{1}{2} \nu \sum_{k=0}^{\nu-1} \sum_{\alpha_k \neq \beta_k} J_0(n_{\alpha_k}^{(k)} - n_{\beta_k}^{(k)}) - \frac{1}{6} M_0(N^2 - 1) + \frac{1}{12} (\nu - 2) \sum_{k=1}^{\nu-1} M_k(N^2 - 1). \tag{3.27}
\]

Next, the three-body terms \( W_4 \) are calculated as
\[
W_4 = \sum_{\substack{k,l=0 \atop k \neq l}}^{\nu-1} \cot \Theta_{\alpha_k, \beta_k} \cot \Theta_{\alpha_k, \alpha_l} \cot \Theta_{\alpha_k, \gamma_k} \left( \sum_{n \neq n_{\alpha_k}^{(k)}}^{\nu-1} \cot \Theta_{\alpha_k, n} - \sum_{\gamma_k (\neq \alpha_k)} \cot \Theta_{\alpha_k, \gamma_k} \right) = - \sum_{k=0}^{\nu-1} \sum_{\alpha_k \neq \beta_k} J_0(n_{\alpha_k}^{(k)} - n_{\beta_k}^{(k)}) + \frac{1}{3} \sum_{k=0}^{\nu-1} M_k(M_k^2 - 1). \tag{3.28}
\]

Combining these formulae, we end up with
\[
\frac{T\psi}{\psi} = 2E - \frac{1}{2} \sum_{k=0}^{\nu-1} J_0(n_{\alpha_k}^{(k)} - n_{\beta_k}^{(k)}). \tag{3.29}
\]

Now, it is seen that the two-body terms are exactly canceled out with \( H_{\text{int}} \). Consequently, we can prove that \( \psi \) is an eigenfunction of the Hamiltonian, and the corresponding eigenenergy, defined by \( H\psi = (\frac{\pi}{N})^2 E\psi \), is given by
\[ E = \sum_{k=1}^{\nu-1} M_k \bar{\varepsilon}(J_k + \phi_k) + \frac{1}{2} \sum_{k,l=1 \atop k \neq l}^{\nu-1} M_k \bar{\varepsilon}(J_{kl} + N/2 + \phi_{kl}) \]

\[ + \frac{1}{24}(\nu + 4) \sum_{k=1}^{\nu-1} M_k (M_k^2 - 1) + \frac{1}{8} \sum_{k=1}^{\nu-1} (N - M_k - M_0) M_k^2 \]

\[ + \frac{1}{24}(3\nu - 2)(N - M_0)(N^2 - 1) + \frac{1}{4}(N - M_0) M_0^2 + \frac{1}{6} M_0 (M_0^2 - 1) - \frac{1}{12} M_0 (N^2 - 1), \quad (3.30) \]

where the energy depends explicitly on \( \{\phi\} \) via the first line of the expression. This is the exact eigenenergy for the SU(\( \nu \)) twisted Haldane-Shastry model, which is the main result in this section.

It is now instructive to write down the ground state energy. If we restrict ourselves to \( N = \nu M \) with an even integer \( M \) and \( 0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_{\nu-1} \leq 1 \), the ground state is realized by the choice \( M_0 = M_1 = \cdots = M_{\nu-1} \) and \( J_1 = J_2 = \cdots = J_{\nu-1} \). Then the ground state energy as a function of the twisted angle reads

\[ E_{g.s.} = M \sum_{k=1}^{\nu-1} (2k - \nu + 1) \phi_k - \frac{1}{12} \nu^2 (\nu - 2) M^3 - \frac{1}{12} \nu (2\nu - 1) M. \quad (3.31) \]

Note that this is the ground state energy for restricted twist angles, from which it is difficult to get full information on the spectral flow beyond the restriction, though it is possible in principle. The main difficulty comes from the expression of the spectrum (3.30) which contains the Gauss symbol (see the definition of \( \varepsilon(k) \) in (3.14)). Namely, whenever one of \( \{\phi_k\} \) jumps over an integer or other \( \phi_j \)'s, the energy has a singular cusp structure (see also Fig.1). It is hence not easy to pass through these cusps and trace the spectral flow. In order to overcome the difficulty and to discuss the spectral flow correctly in the full region of the twist angles, it is necessary to resort to an alternative method, i.e. the Bethe ansatz, which will be described in the following section.

**IV. BETHE ANSATZ APPROACH**

Before discussing the spectral flow, we first show that the exact spectrum obtained in the previous section can be reproduced by the Bethe ansatz (sometimes called the asymptotic
The advantage to exploit the Bethe-ansatz description is, as will be shown below, that the spectral flow is naturally determined by appropriately choosing the quantum numbers in the Bethe equations. Also, by using the motif picture \cite{14}, which can visualize how the quantum numbers in the Bethe ansatz are determined, we can naturally interpret the period of the spectral flow in terms of fractional exclusion statistics.

In order to formulate the Bethe equations for the SU($\nu$) model, let us introduce the $\nu - 1$ kinds of rapidities, denoted here as $k_{a_i}^{(i)}$ with $i = 1, 2, \cdots, \nu - 1$, and $a_i = 1, 2, \cdots, \sum_{\alpha=1}^{\nu-1} M_\alpha$. Following a standard procedure in the nested Bethe ansatz \cite{17}, we can formally write down the Bethe equations as

\begin{align}
  k_{a_1}^{(1)} &= I_{a_1}^{(1)} + \phi_1 - \frac{1}{2} \sum_{a_2} \text{sgn}(k_{a_1}^{(1)} - k_{a_2}^{(2)}) + \frac{1}{2} \sum_{b_1(\neq a_1)} \text{sgn}(k_{a_1}^{(1)} - k_{b_1}^{(1)}) \\
  & \quad \vdots \\
  \frac{1}{2} \sum_{b_i(\neq a_i)} \text{sgn}(k_{a_i}^{(i)} - k_{b_i}^{(i)}) + I_{a_i}^{(i)} + \phi_i - \phi_{i-1} &= \frac{1}{2} \sum_{a_{i-1}} \text{sgn}(k_{a_i}^{(i)} - k_{a_{i-1}}^{(i-1)}) + \frac{1}{2} \sum_{a_{i+1}} \text{sgn}(k_{a_i}^{(i)} - k_{a_{i+1}}^{(i+1)}) \\
  & \quad \vdots \\
  \frac{1}{2} \sum_{b_{\nu-1}(\neq a_{\nu-1})} \text{sgn}(k_{a_{\nu-1}}^{(\nu-1)} - k_{b_{\nu-1}}^{(\nu-1)}) + I_{a_{\nu-1}}^{(\nu-1)} + \phi_{\nu-1} - \phi_{\nu-2} &= \frac{1}{2} \sum_{a_{\nu-1}} \text{sgn}(k_{a_{\nu-1}}^{(\nu-1)} - k_{a_{\nu-2}}^{(\nu-2)}) \quad (4.1)
\end{align}

with $i = 2, \cdots, \nu - 2$. Here we note that the two-body phase shift takes the step-like form, which is inherent in the $1/r^2$ systems \cite{4}. This special form of the phase shift is essential for the model to be ideal in the sense of exclusion statistics \cite{5}. The total energy of the model, defined in the unit of $\left(\frac{\pi}{N}\right)^2$ as before, is given by

\begin{equation}
  E = \frac{1}{12} N(N^2 - 1) + \sum_{a_1=1}^{M_1 + \cdots + M_{\nu-1}} \varepsilon(k_{a_1}^{(1)}) \quad (4.2)
\end{equation}

with $\varepsilon(k)$ defined in (3.14). The energy thus seems to be solely determined by the rapidity $k^{(1)}$, but we should remember that its configuration is affected by the other rapidities via the nested equations. Therefore, when we follow the spectrum as a function of the twist angle, we must specify the precise configuration including $k^{(2)}, \cdots, k^{(\nu-1)}$, whose behavior should be quite different from each other.
To see clearly how the rapidities are nested with each other, we take the simplest but non-trivial example, i.e. the SU(3) case. It is straightforward, though a little bit complicated, to generalize the following discussions to the SU(\(\nu\)) models. To be specific, we use the simplified notations, \(k^{(1)}_{a_1} = k_\alpha\) and \(k^{(2)}_{a_2} = \lambda_i\), and write down the Bethe equations as

\[
k_\alpha = I^{(1)}_\alpha + \phi_1 - \frac{1}{2} \sum_{\beta \neq \alpha} \text{sgn}(k_\alpha - k_\beta) + \frac{1}{2} \sum_j \text{sgn}(k_\alpha - \lambda_j),
\]

(4.3)

\[
\frac{1}{2} \sum_{j(\neq i)} \text{sgn}(\lambda_i - \lambda_j) + I^{(2)}_i + \phi_{21} = \frac{1}{2} \sum_{\beta} \text{sgn}(\lambda_i - k_\beta),
\]

(4.4)

where \(\phi_{21} = \phi_2 - \phi_1\).

Here we notice that eq.(4.4), which has been formally deduced by the nested Bethe ansatz, does not seem to hold for a fractional value of \(\phi_{21}\) at a first glance. This problem stems from the fact that the two-body phase shift in the present system is of the step-like form. We now wish to explain how we can deal with eq.(4.4) correctly. To this end, let us consider the following configuration at \(\phi_1 = \phi_2 = 0\) as an example,

\[
\cdots < k_{\alpha-1} < \lambda_i < k_\alpha < \cdots.
\]

(4.5)

If we put \(\phi_{21} = 1\), we see from eq.(4.4) that \(I^{(2)}_i\) is shifted to \(I^{(2)}_i + 1\) and the above configuration should be changed to

\[
\cdots < k_{\alpha-1} < k_\alpha < \lambda_i < \cdots.
\]

(4.6)

Namely, the position of \(\lambda_i\) is exchanged with that of \(k_\alpha\) sitting on its right neighbor. The fractional \(\phi_{21}\) between 0 and 1 should smoothly interpolate these two configurations. One readily notices that this is done by introducing an infinitesimal width \(\eta\) in the step-like phase shift and then taking the limit of \(\eta \to 0\). This observation naturally leads us to divide the l.h.s. of eq.(4.4) as \(I^{(2)}_i + \phi_{21} \equiv (I^{(2)}_i + [\phi_{21}]) + ([\phi_{21}] - \phi_{21})\). Then the fractional part \([\phi_{21}] - \phi_{21}\) can be absorbed into \(\text{sgn}(\lambda_i - k_\alpha) \equiv \text{sgn}(0)\) in the case of \(\lambda_i = k_\alpha\) for \(0 < \phi_{21} < 1\). Consequently, we get the simple result from eq.(4.4)

\[
\frac{1}{2} \text{sgn}(\lambda_i - k_\alpha) = \phi_{21} - \frac{1}{2}
\]

(4.7)
for $0 < \phi_{21} < 1$. By substituting the above formula into (4.3), we finally have

$$k_\alpha = \begin{cases} 
  k_0 + \phi_1, & \text{for } \cdots < k_{\alpha-1} < k_\alpha < \cdots, \\
  k_0 + \phi_2, & \text{for } \cdots < \lambda_i < k_\alpha < \cdots,
\end{cases}$$

(4.8)

for $0 \leq \phi_{21} < 1$, where $k_0$ is the rapidity for $\{\phi\} = 0$. From this equation we can see that $k$ with (without) $\lambda$ in its left neighbor labels the rapidities for species-1 (-2) degree of freedom. Up to now, we have restricted ourselves to the case $\phi_{21} \leq 0$. For the other case $\phi_{21} \leq 0$, $\lambda$ exchanges the position with $k$ sitting on its left when $\phi_{21}$ decreases. Namely, the role of “left” and “right” in the previous discussion exchanges.

The above interpretation of the two-body phase shift can be generalized to the SU($\nu$) case. As a result, it is easily confirmed that by suitably taking the quantum numbers $I^{(j)}_\alpha$, the Bethe equations (4.1) indeed reproduce the exact energy (3.30) for the SU($\nu$) case obtained in the previous section.

V. SPECTRAL FLOW AND EXCLUSION STATISTICS

We now discuss the spectral flow with the aid of the Bethe ansatz description. For this purpose, the notion of the motif is particularly useful, allowing us to trace the spectral flow correctly. Let us first recall that the solution of the Bethe equations (4.1) can be specified graphically by the sequence of 0 and 1 denoting respectively empty and occupied states of the momentum $k_j$ [3]. Note that the unoccupied momentum, 0, is introduced to represent the repulsion effect of the two-body phase shift. This is the essence of the motif. For example, as was shown in ref. [3], the motif for the SU(2) case reads 01010101010, which implies that the two-body phase shift enlarges the spacing of rapidities twice as large as the free fermion case. For the SU($N$) case, one can find from the Bethe equations (4.1) that the rapidity $k_j^{(1)}$ is subject to the constraint that any configurations with more than $\nu - 1$ consecutive 1’s are prohibited. For example, the ground state of the SU(3) chain described by eqs.(4.3) and (4.4) is characterized by the motif for $k_\alpha$

$$011011 \cdots 0110,$$

(5.1)
where 1 denotes the occupied $k_\alpha$. In order to include the solution of the auxiliary rapidity $\lambda_i$, the motif should read more precisely

$$01\circ10\circ1\cdots01\circ10,$$

(5.2)

where we have introduced $\circ$ which denotes the position of the occupied $\lambda_i$. We need this type of the motif when we discuss the spectral flow for the SU($\nu$) case ($\nu \geq 3$) with several different twist angles $\phi_i$.

To see how well the motif picture works for our problem, let us discuss the spectral flow by taking the SU(3) model as an example. In Fig.1, we have shown the numerical diagonalization results of the spectrum for the finite system. These numerical results may be complementary to the exact results obtained in section III, since the latter provides a specific series of the eigenstates. We can see several characteristic properties in the spectral flow, e.g. the linear $\phi$-dependence on the twist angle, cusp structures, etc. These behaviors seem rather peculiar, because the smooth $\phi^2$ dependence for small $\phi$ is expected for ordinary spin models with short-range interaction. We find that the above characteristic behaviors are closely related to high symmetry of the system which is essential for the $1/r^2$ model to be an idealized model without any irrelevant perturbations. We shall discuss later that the linear $\phi$-dependence indeed results from high symmetry of the $1/r^2$ model.

Now, let us analyze the spectral flow by exploiting the motif picture. As mentioned above the motif for the SU(3) singlet ground state is given by $0110110\cdots$. When the boundary is twisted as $\phi_1 = \phi_2 = \phi$, the motif develops with the increase of the twist angle as

$$0110110\cdots0110 \rightarrow 1011011\cdots1010 \rightarrow 1101101\cdots1100 \rightarrow 0110110\cdots0110,$$

(5.3)

where we have omitted $\circ$ for simplicity, because 1’s move in parallel, and $\circ$ is always sandwiched by the same 1’s. Here, the arrow means that unit $\phi$ is added; $\delta\phi = 1$. The corresponding spectral flow in Fig.1 is: (a) $\rightarrow$ (b) $\rightarrow$ (c). From this interpretation with the motif, we can say that the period of the above spectral flow is 3. There may be another type of the
spectral flow, since there are two independent twist angles $\phi_1$ and $\phi_2$. For example, when $\phi_1 = 0$ and $\phi_2 = \phi$, the motif changes like,

$$01\circ101\circ10\cdots01\circ10 = 01\circ101\circ10\circ10\circ10 = \cdots = 01\circ101\circ10\circ10,$$

as the twist angle $\phi$ is increased. The corresponding spectral flow thus has the period 3. In both cases, we can see that the period of the spectral flow in the SU(3) model is 3. Similarly we can analyze the spectral flow for more general SU($N$) cases based on the motif. For example, the period of the ground state turns out to be $\nu$ for $\phi_1 = \phi_2 = \cdots = \phi_{\nu-1} = \phi$. In this way, we can determine the period of the spectral flow in terms of the motif picture.

We now wish to discuss how the above description is related to the notion of exclusion statistics. According to Haldane [5], the statistical interaction $g$ in exclusion statistics is defined as

$$\frac{\partial D_{\mu}}{\partial N_{\nu}} = -g_{\mu\nu},$$

where $D_{\mu}$ and $N_{\mu}$ are the number of hole- and particle-states in species $\mu$, respectively. The cases $g_{\mu\nu} = g\delta_{\mu\nu}$ with $g = 0$ and $g = 1$ correspond, respectively, to free bosons and fermions. A remarkable point is that the statistical interaction $g$ is uniquely determined by the two-body phase shift in the Bethe equations. For instance, in the one-component system such as the SU(2) model, the statistical interaction $g$ is given by the two-body phase shift via the relation $\frac{1}{2}(g-1)\Sigma_{j}(\neq i)\text{sgn}(k_i-k_j)$. Therefore, we can deduce from the Bethe equations (4.1) for $\nu = 2$ that the statistical interaction for the SU(2) Haldane-Shastry model is $g = 2$. Since the effect of the two-body phase shift is represented schematically by the motif, one can see that the above analysis based on the motif is directly related to exclusion statistics. Namely, the period for the spectral flow is determined by the statistical interaction $g$. 

\[1\text{If we include the Ising-anisotropy } \Delta = g(g-1)/2 \text{ for the SU}(2) \text{ model, the period may become } g, \text{ as was indeed shown for } g = 4 \text{ in } [10].\]
For the SU(\(\nu\)) Haldane-Shastry model, it is known that the statistical interaction is given by the \((\nu - 1) \times (\nu - 1)\) Cartan matrix of SU(\(\nu\)) algebra \([13]\),

\[
\tilde{g} = \begin{pmatrix}
2 & -1 & 0 \\
-1 & 2 & -1 & 0 \\
& & \ddots \\
0 & -1 & 2 & -1 \\
& & & 0 & -1 & 2
\end{pmatrix},
\tag{5.6}
\]

which directly reflects the structure of the nested Bethe equations \([4.1]\). We can easily generalize the above analysis to the SU(\(\nu\)) model. For example, the period of the flow for \(\phi_1 = \phi_2 = \cdots = \phi_{\nu-1} = \phi\) is \(\nu\), which is related with

\[
1 - (\tilde{g}^{-1})_{11} = \frac{1}{\nu}.
\tag{5.7}
\]

In this way, the period of the spectral flow in the SU(\(\nu\)) model is determined by the matrix of the statistical interaction. Note that the matrix \(\tilde{g}\) corresponds to the spin sector of the topological-order matrix characterizing internal structure of the quantum Hall system \([18]\).

VI. SUMMARY AND DISCUSSIONS

Summarizing, we have proposed the SU(\(\nu\)) Haldane-Shastry model compatible with twisted boundary conditions, and have solved it exactly. The spectrum thus obtained can be correctly reproduced by the Bethe ansatz solution. We have then discussed the spectral flow of the model in terms of the motif in the Bethe ansatz, and found that the period of the ground state is \(\nu\), reflecting fractional exclusion statistics.

Finally, we wish to make some comments on an unusual linear \(\phi\)-dependence of the spectral flow. We show here that this behavior is related to high symmetry inherent in the present system. For this purpose let us first remember that the periodic Haldane-Shastry model has Yangian symmetry which is larger than SU(\(\nu\)). Twisting the model breaks Yangian symmetry as well as SU(\(\nu\)). However, a nontrivial conserved quantity associated
with Yangian still persists even for twisted systems of finite size \[6\], and we expect that this conserved quantity controls the linear dependence of the spectral flow. This feature can be clearly seen when we consider the conformal limit of the model. To be more specific, let us consider the SU(2) Haldane-Shastry model. For the periodic Haldane-Shastry model, we can take conformal limit explicitly as follows \[14\]:

\[
H_{\text{PHS}} = \sum_{n>0} n : J^a_n J^a_{-n} : ,
\]

(6.1)

where \(J^a_n\) is the current operator of the \(k = 1\) SU(2) WZW model. The level-1 Yangian generators are also given by \[14\]

\[
Q^a_1 = \sum_{n>0} \epsilon^{abc} : J^b_{-n} J^c_n : .
\]

(6.2)

For the twisted model, a similar calculation for the Hamiltonian (2.6) leads to

\[
H_{\text{THS}} = H_{\text{PHS}} + \phi Q^3_1.
\]

(6.3)

From this equation, we can clearly see that \(Q^3_1\) remains as a conserved quantity, reflecting high symmetry of the system, even when the boundary is twisted \[6\]. We can also see from (6.3) that the existence of this conserved quantity explains why we have encountered the linear \(\phi\)-dependence in the spectral flow.

We have been concerned with the model of rational twist angles in this paper. Within the present framework, we may apply the continued fraction approximation to discuss the case of irrational twist angles. It remains still an open question to construct the exact solution directly for the irrational cases, which is now under consideration. Also, the proof for integrability of the present model remains as an interesting issue to be solved in the future work.

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APPENDIX A:

In this appendix, we derive (3.8) in a bit detail. We concentrate on the case where $Q_k \equiv N - M_k - M_0$ is an even number and hence the current $J_k$ defined in eq.(3.3) is an integer. It is straightforward to apply a similar calculation to the other case ($Q_k = \text{odd number}$).

The key formula is

$$S^{\phi}_{st}(J) = \frac{1}{4} \sum_{n=1}^{N-1} J_\phi(n) z^J n (1 - z^n)^s (1 - z^{-n})^t$$

\[
\begin{aligned}
S^{\phi}_{st}(J) &= \begin{cases} 
(-)^s & \text{for } s + t = 2 \\
(-)^s(J + \phi - \frac{N}{2}) - \frac{1}{2} & \text{for } s + t = 1 \\
2z(J + \phi) + \frac{1}{3}(N^2 - 1) & \text{for } s = t = 0 \\
0 & \text{for others,}
\end{cases}
\end{aligned}
\]

(A1)

provided $t \leq J + \phi \leq N - s$, which was first introduced in [10], by extending the original one in [1]. Now let us expand eq.(3.8) in power series of $1 - z^n$ and $1 - z^{-n}$. It consists of finite polynomials of the form $(1 - z^n)^s(1 - z^{-n})^t$ with $0 \leq s + t \leq Q_k/2 + M_k - 1$. Therefore, provided that $Q_k/2 + M_k - 1 \leq J_k + \phi_k \leq N - (Q_k/2 + M_k - 1)$, terms with $3 \leq s + t$ should vanish in eq.(3.5). We can also find that such terms in the second order as $(1 - z^n)^2 - (1 - z^{-n})^2$ or $\{(1 - z^n)(1 - z^{-n})\}^2$ vanish, so that it is sufficient to consider in the expansion

$$\sum \prod_{\alpha \neq \beta} B_{\alpha \beta}^{(n)} \prod_{j=1}^{N-M_k-M_0} F_{\alpha j}^{(n)}$$

\[
= M_k - \sum_{\alpha \neq \beta} g_{\alpha \beta}^{(n)} + \frac{1}{2} \sum_{\alpha, \beta, \gamma} g_{\alpha \beta}^{(n)} g_{\alpha \gamma}^{(n)} \\
- \frac{Q_k}{8} (M_k - \sum_{\alpha \neq \beta} g_{\alpha \beta}^{(n)}) \{(1 - z^n) + (1 - z^{-n})\} \\
+ \frac{i}{4} \sum_{\alpha} (1 - \sum_{\beta \neq \alpha} g_{\alpha \beta}^{(n)}) \{(1 - z^n) - (1 - z^{-n})\} \sum_j \cot \Theta_{\alpha j} \\
+ \frac{1}{4} \sum_{\alpha} (1 - \sum_{\beta \neq \alpha} g_{\alpha \beta}^{(n)}) \{(1 - z^n) + (1 - z^{-n})\} \sum_{j \neq j'} \cot \Theta_{\alpha j} \cot \Theta_{\alpha j'}. \quad (A2)
\]

Applying the formula (A1), we then have
\[
\frac{T_k(\phi_k)}{\psi} = 2M_k \tilde{\varepsilon}(J_k + \phi_k) + \frac{1}{3}M_k(N^2 - 1) + \frac{2}{3}M_k(M_k^2 - 1) + \frac{1}{2}Q_kM_k \\
- \sum_{\alpha \neq \beta} J_0(x_\alpha - x_\beta) \\
- 2i(J_k + \phi_k - N/2) \sum_{\alpha, j} \cot \Theta_{\alpha j} \\
- 2 \sum_{\alpha \neq \beta, j} \cot \Theta_{\alpha \beta} \cot \Theta_{\alpha j} - \frac{1}{2} \sum_{i \neq j, \alpha} \cot \Theta_{i \alpha} \cot \Theta_{j \alpha}.
\]  
\text{(A3)}

So far we have used the simplified notations defined in eq.(3.4). The remaining task is to convert the expression into the original notation. For example, in the third line, 
\[
\sum_{\alpha, j} \cot \Theta_{\alpha j} = \sum_{\alpha_k} \sum_{l=1}^{\nu-1} \sum_{\alpha_l} \cot \Theta_{\alpha_k \alpha_l}.
\]
Consequently we end up with the results shown in the text.
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Note added in proofs

After submitting the paper, we have learned that Liu and Wang (cond-mat/9608026) solved the supersymmetric $t$-$J$ model with $1/r^2$ interaction for irrational twist angles. Their method can be directly applied to the present model.
FIG. 1. Exact spectral flow of the $N = 6$ SU(3) model with $M_0 = M_1 = M_2 = 2$ and $\phi = \phi_1 = \phi_2$. Lower 20 levels are plotted. The flow of the ground state (0110110) is described by $(a) \rightarrow (b) \rightarrow (c)$. 