REMARKS ON $A^{(1)}_n$ FACE WEIGHTS

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Abstract

Elementary proofs are presented for the factorization of the elliptic Boltzmann weights of the $A^{(1)}_n$ face model, and for the sum-to-1 property in the trigonometric limit, at a special point of the spectral parameter. They generalize recent results obtained in the context of the corresponding trigonometric vertex model.

1. Introduction

In the recent work [8], the quantum $R$ matrix for the symmetric tensor representation of the Drinfeld-Jimbo quantum affine algebra $U_q(A^{(1)}_n)$ was revisited. A new factorized formula at a special value of the spectral parameter and a certain sum rule called sum-to-1 were established. These properties have led to vertex models that can be interpreted as integrable Markov processes on one-dimensional lattice including several examples studied earlier [2, Fig.1,2]. In this note we report analogous properties of the Boltzmann weights for yet another class of solvable lattice models known as IRF (interaction round face) models [2] or face models for short. More specifically, we consider the elliptic fusion $A^{(1)}_n$ face model corresponding to the symmetric tensor representation [6, 5]. For $n=1$ it reduces to [1] and [4] when the fusion degree is 1 and general, respectively. There are restricted and unrestricted versions of the model. The trigonometric case of the latter reduces to $A_1$ and [4] when the fusion degree is 1 and general, respectively. There are restricted and unrestricted versions of the model. The trigonometric case of the latter reduces to the $U_q(A^{(1)}_n)$ vertex model when the site variables tend to infinity. See Proposition 5. In this sense Theorem 1 and Proposition 5 given below, which are concerned with the unrestricted version, provide generalizations of [8, Th.2] and [8, eq.(30)] so as to include finite site variables (and also to the elliptic case in the former). In Section 3 we will also comment on the restricted version and difficulties to associate integrable stochastic models.

2. Results

Let $\theta_1(u) = \theta_1(u,p) = 2p^\tau \sin \pi u \prod_{k=1}^{\infty} (1 - 2p^{2k} \cos 2\pi u + p^{4k})(1 - p^{2k})$ be one of the Jacobi theta function $(|p| < 1)$ enjoying the quasi-periodicity

$$\theta_1(u + 1; e^{\pi i \tau}) = -\theta_1(u; e^{\pi i \tau}), \quad \theta_1(u + \tau; e^{\pi i \tau}) = -e^{-\pi i \tau - 2\pi i u} \theta_1(u; e^{\pi i \tau}),$$

where $\text{Im} \tau > 0$. We set

$$[u] = \theta_1(u/L; p), \quad [u]_k = [u][u-1] \cdots [u-k+1], \quad \begin{bmatrix} u \\ k \end{bmatrix} = \frac{[u]_k}{[k]_k} \quad (k \in \mathbb{Z}_{\geq 0}),$$

with a nonzero parameter $L$. These are elliptic analogue of the $q$-factorial and the $q$-binomial:

$$(z)_m = (z; q)_m = \prod_{i=0}^{m-1} (1 - zq^i), \quad \binom{m}{l}_q = \frac{(q)_m}{(q)_l(q)_{m-l}}.$$ 

For $\alpha = (\alpha_1, \ldots, \alpha_k)$ with any $k$ we write $|\alpha| = \alpha_1 + \cdots + \alpha_k$. The relation $\beta \geq \gamma$ or equivalently $\gamma \leq \beta$ means $\beta_i \geq \gamma_i$ for all $i$.

We take the set of local states as $\mathcal{P} = \eta + \mathbb{Z}^{n+1}$ with a generic $\eta \in \mathbb{C}^{n+1}$. Given positive integers $l$ and $m$, let $a, b, c, d \in \mathcal{P}$ be the elements such that

$$\alpha = d - a \in B_1, \quad \beta = c - d \in B_m, \quad \gamma = c - b \in B_l, \quad \delta = b - a \in B_m,$$

where $B_m$ is defined by

$$B_m = \{ \alpha = (\alpha_1, \ldots, \alpha_{n+1}) \in \mathbb{Z}^{n+1}_{\geq 0} \mid |\alpha| = m \}.$$ 

The relations (3) imply $\alpha + \beta = \gamma + \delta$. The situation is summarized as
To the above configuration round a face we assign a function of the spectral parameter $u$ called Boltzmann weight. Its unnormalized version, denoted by $\mathcal{W}_{l,m}(a\ b\ d\ c\ |\ u)$, is constructed from the $l = 1$ case as follows:

$$
\mathcal{W}_{l,m}(a\ b\ d\ c\ |\ u) = \sum_{i=0}^{l-1} \prod_{i} \mathcal{W}_{1,m}(a^{(i)}\ b^{(i)}\ d^{(i+1)}\ c^{(i+1)}\ |\ u - i),
$$

$$
\mathcal{W}_{1,m}(a\ b\ d\ c\ |\ u) = \frac{[u + b_{\mu} - a_{\mu}] \prod_{j=1 \atop j \neq \mu}^{n+1} [b_{\nu} - a_{j} + 1]}{\prod_{j=1}^{n+1} [c_{\nu} - b_{j}]} \quad (d = a + e_{\mu}, \ c = b + e_{\nu}),
$$

where $e_{i} = (0, \ldots, 0, 1, 0, \ldots, 0)$. In (5), $a^{(0)}, \ldots, a^{(l)} \in \mathcal{P}$ is a path form $a^{(0)} = a$ to $a^{(l)} = d$ such that $a^{(i+1)} - a^{(i)} \in B_{1} (0 \leq i < l)$. The sum is taken over $b^{(1)}, \ldots, b^{(l-1)} \in \mathcal{P}$ satisfying the conditions $b^{(i+1)} - b^{(i)} \in B_{1} (0 \leq i < l)$ with $b^{(0)} = b$ and $b^{(l)} = c$. It is independent of the choice of $a^{(1)}, \ldots, a^{(l)}$ (cf. [4] Fig.2.4). We understand that $\mathcal{W}_{l,m}(a\ b\ d\ c\ |\ u) = 0$ unless (3) is satisfied for some $\alpha, \beta, \gamma, \delta$.

The normalized weight is defined by

$$
W_{l,m}(a\ b\ d\ c\ |\ u) = \mathcal{W}_{l,m}(a\ b\ d\ c\ |\ u) \frac{[1]^{l} [m]^{-1}}{[l]^{1}},
$$

It satisfies the (unrestricted) star-triangle relation (or dynamical Yang-Baxter equation) [2]:

$$
\sum_{g} W_{k,m}(a\ b\ f\ g\ |\ u) W_{l,m}(f\ g\ d\ c\ |\ u - v) W_{k,l}(b\ c\ d\ e\ |\ u) = \sum_{g} W_{k,l}(a\ b\ e\ f\ |\ u - v) W_{l,m}(a\ b\ d\ c\ |\ u) W_{k,m}(g\ c\ e\ d\ |\ u),
$$

where the sum extends over $g \in \mathcal{P}$ giving nonzero weights. Under the same setting as in (6), we introduce the product

$$
S_{l,m}(a\ b\ d\ c) = \prod_{1 \leq i,j \leq n+1} \frac{[c_{i} - d_{j}]}{[c_{i} - b_{j}]}.
$$

Note that $S_{l,m}(a\ b\ d\ c) = 0$ unless $d \leq b$ because of the factor $\prod_{i=1}^{n+1} [c_{i} - d_{j}]$. The following result giving an explicit factorized formula of the weight $W_{l,m}$ at special value of the spectral parameter is the elliptic face model analogue of [8] Th.2.

**Theorem 1.** If $l \leq m$, the following equality is valid:

$$
W_{l,m}(a\ b\ d\ c\ |\ u = 0) = S_{l,m}(a\ b\ d\ c).
$$

**Proof.** We are to show

$$
\overline{W}_{l,m}(a\ b\ d\ c\ |\ 0) = \prod_{\nu} \overline{W}_{l,m}(a\ b\ d'\ c'\ |\ 0) \overline{W}_{1,m}(d'\ c'\ |\ -1)
$$

Here and in what follows unless otherwise stated, the sums and products are taken always over $1, \ldots, n + 1$ under the condition (if any) written explicitly. We invoke the induction on $l$. It is straightforward to check (10) for $l = 1$. By the definition (10) the $l + 1$ case is expressed as

$$
\overline{W}_{l+1,m}(a\ b\ d\ c\ |\ 0) = \sum_{\nu} \overline{W}_{l,m}(a\ b\ d'\ c'\ |\ 0) \overline{W}_{1,m}(d'\ c'\ |\ -1) \quad (d' = d - e_{\mu}, c' = c - e_{\nu}).
$$
for some fixed $\mu \in [1, n+1]$. Due to the induction hypothesis on $\Pi_{l,m}$, the equality to be shown becomes

$$\sum_{\nu} \frac{[l]_t}{[1]_t} \left( \prod_{i,j} \frac{[c_i - d_j][c_i' - b_i]}{[c_i - b_i][c_i' - d_j]} \right) \frac{[-l + c_{i
u} - d_{i\mu}] \prod_{k \neq \mu} [c_{i
u} - d_{i\mu} + 1]}{\prod_{k} [c_{i
u} - c_k]}$$

$$= \frac{[l+1]_t c_i - d_j}[1]_t \prod_{i,j} \frac{[c_i - d_j][c_i' - b_i]}{[c_i - b_i][c_i' - d_j]}$$

(11)

After removing common factors using $c_i' = c_i - \delta_{i\mu}$, $d_i' = d_i - \delta_{i\mu}$, one finds that (11) is equivalent to

$$\sum_{\nu} [c_{i\nu} - d_{i\mu} - l] \prod_{i,j} \frac{[c_i - d_{i\mu} + 1]}{[c_i - c_j]} \prod_{j} [c_{i\nu} - b_j] = [l+1] \prod_{i} [b_i - d_{i\mu} + 1]$$

with $l$ determined by $l+1 = \sum_j (c_j - b_j)$. One can eliminate $d_{i\mu}$ and rescale the variables by $(b_j, c_j) \to (Lb_j + d_{i\mu}, Lc_j + d_{i\mu})$ for all $j$. The resulting equality follows from Lemma 2.

Lemma 2. Let $b_1, \ldots, b_n, c_1, \ldots, c_n \in \mathbb{C}$ be generic and set $s = \sum_{i=1}^{n} (c_i - b_i)$. Then for any $n \in \mathbb{Z}_{\geq 1}$ the following identity holds:

$$\sum_{i=1}^{n} \theta_1(z + c_i - s) \prod_{j=1}^{n} \left[ \frac{\theta_1(z + c_j)}{\theta_1(c_i - c_j)} \right] \prod_{j=1}^{n} \theta_1(c_j - b_j) = \sum_{i=1}^{n} \theta_1(z + b_i).$$

Proof. Denote the LHS – RHS by $f(z)$. From (11) we see that $f(z)$ satisfies (12) with $B = \frac{N}{2}$, $A_1 = \frac{n(n+1)}{2} + \sum_{j=1}^{n} b_j$ and $A_2 = n$. Moreover it is easily checked that $f(z)$ possesses zeros at $z = -c_1, \ldots, -c_n$. Therefore Lemma 3 claims $-(c_1 + \cdots + c_n) - (B \tau + \frac{1}{2}(A_2 - A_1)) \equiv 0 \mod \mathbb{Z} + \mathbb{Z} \tau$. But this gives $s \equiv 0$ which is a contradiction since $b_j, c_j$ can be arbitrary. Therefore $f(z)$ must vanish identically.

Lemma 3. Let $\text{Im} \tau > 0$. Suppose an entire function $f(z) \not\equiv 0$ satisfies the quasi-periodicity

$$f(z + 1) = e^{-2\pi i B} f(z), \quad f(z + \tau) = e^{-2\pi i (A_1 + A_2 z)} f(z).$$

(12)

Then $A_2 \in \mathbb{Z}_{\geq 0}$ holds and $f(z)$ has exactly $A_2$ zeros $z_1, \ldots, z_{A_2} \mod \mathbb{Z} + \mathbb{Z} \tau$. Moreover $z_1 + \cdots + z_{A_2} \equiv B \tau + \frac{1}{2} A_2 - A_1 \mod \mathbb{Z} + \mathbb{Z} \tau$ holds.

Proof. Let $C$ be a period rectangle $(\xi, \xi + 1, \xi + 1 + \tau, \xi + \tau)$ on which there is no zero of $f(z)$. From the Cauchy theorem the number of zeros of $f(z)$ in $C$ is equal to $\int_{C} \frac{f(z)}{f(z)} \frac{dz}{2\pi i}$. Calculating the integral by using (12) one gets $A_2$. The latter assertion can be shown similarly by considering the integral $\int_{C} \frac{f(z)}{f(z)} \frac{dz}{2\pi i}$.

From Theorem 1 and 7 it follows that $S_{l,m} \left( \frac{a}{d} \frac{b}{c} \right)$ also satisfies the (unrestricted) star-triangle relation $\text{K}$ without spectral parameter. The discrepancy of the factorizing points $u = 0$ in $\text{K}$ and “$u = l - m$” in $\text{K}$ Th.2) is merely due to a conventional difference in defining the face and the vertex weights.

Since $\text{K}$ and $\text{K}$ are homogeneous of degree 0 in the symbol $\ldots$, the trigonometric limit $p \to 0$ may be understood as replacing $\text{K}$ by $|u| = q^{n/2} - q^{-n/2}$ with generic $q = \exp \frac{2\pi i}{L}$. Under this prescription the elliptic binomial $\left( \frac{m}{l} \right)$ from $\text{K}$ is replaced by $q^{(l-m)/2} \left( \frac{m}{l} \right)_q$, therefore the trigonometric limit of $\text{K}$ becomes

$$S_{l,m} \left( \frac{a}{d} \frac{b}{c} \right)_{\text{trig}} = \left( \frac{m}{l} \right)^{-1} \prod_{1 \leq i,j \leq n+1} \frac{(q^{b_i - b_j} - 1)_{c_i - b_i}}{(q^{c_i - d_j} - 1)_{c_i - b_i}}.$$ 

(13)

The following result is a trigonometric face model analogue of $\text{K}$ Th.6).

Theorem 4. Suppose $l \leq m$. Then the sum-to-1 holds in the trigonometric case:

$$\sum_b S_{l,m} \left( \frac{a}{d} \frac{b}{c} \right)_{\text{trig}} = 1,$$

(14)

where the sum runs over those $b$ satisfying $c - d \in B_m$ and $d - a \in B_l$. 

Proof. The relation (14) is equivalent to
\[
\left(\begin{array}{l}
m \\
l
\end{array}\right) = \sum_{\gamma \in B_1, 1 \leq \beta \leq \gamma, 1 \leq i, j \leq n+1} \prod_{i,j} \frac{(q^{c_{ij} - \gamma_i + \beta_j + 1})^{\gamma_i}}{(q^{c_{ij} - \gamma_i + \beta_j + 1})^{\gamma_i}} (c_{ij} = c_i - c_j)
\] (15)
for any fixed \( \beta = (\beta_1, \ldots, \beta_{n+1}) \in B_n, l \leq m \) and the parameters \( c_1, \ldots, c_{n+1} \), where the sum is taken over \( \gamma \in B_1 \) under the constraint \( \gamma \leq \beta \). In fact we are going to show
\[
(w_1^{-1} \cdots w_n^{-1} q^{-l+1})_l \prod_{|\gamma|=l} \frac{(q^{-\gamma_1 + 1} z_1/(z_j w_j))^{\gamma_i}}{(q^{-\gamma_1 + 1} z_1/z_j)^{\gamma_i}}
\] (l \( \in \mathbb{Z}_{\geq 0} \)),
(16)
where the sum is over \( \gamma \in \mathbb{Z}_{\geq 0} \) such that \( |\gamma| = l \), and \( w_1, \ldots, w_n, z_1, \ldots, z_n \) are arbitrary parameters. The relation (14) is deduced from (16) \( n \to n+1 \) by setting \( z_i = q^{c_i}, w_i = q^{-\beta_i} \) and specializing \( \beta_i \)'s to nonnegative integers. In particular, the constraint \( \gamma \leq \beta \) automatically arises from the \( i \) factor \( \prod_{i=1}^{n} (q^{-\gamma_i + 1 + \beta_i})^{\gamma_i} \) in the numerator. To show (16) we rewrite it slightly as
\[
q^2 (w_1 \cdots w_n)_l = \prod_{|\gamma|=l} \frac{(q^2 w_i)}{(q^{2n+1})^{\gamma_i}} \prod_{1 \leq i \neq j \leq n} \frac{(z_j w_j/z_i)^{\gamma_i}}{(q^{\gamma_i} z_j/z_i)^{\gamma_i}}.
\] (17)
Denote the RHS by \( F_n(w_1 \cdots w_n | z_1, \ldots, z_n) \). We will suppress a part of the arguments when they are kept unchanged in the formulas. It is easy to see
\[
F_n(w_1, w_2 | z_1, z_2) = F(w_2, w_1 | z_2, z_1) = F_n(z_2 w_2/z_1, z_1 w_1/z_2, z_1, z_2).
\]
Thus the coefficients in the expansion \( F_n(w_1, w_2 | z_1, z_2) = \sum_{0 \leq i, j \leq l} C_{i,j}(z_1, z_2) w_i w_j \) are rational functions in \( z_1, \ldots, z_n \) obeying \( C_{i,j}(z_1, z_2) = C_{j,i}(z_2, z_1) = (q^2 z_j/z_i)^{\gamma_i} \). On the other hand from the explicit formula (17), one also finds that any \( C_{i,j}(z_1, z_2) \) remains finite in the either limit \( \frac{z_j}{z_i} \to \infty \) or \( \frac{z_i}{z_j} \to \infty \) for \( i \geq 3 \). It follows that \( C_{i,j}(z_1, z_2) = 0 \) unless \( i = j \), hence \( F_n(w_1, w_2, \ldots, w_n | z_1, \ldots, z_n) = F_{n-1}(w_1 w_2, w_3, \ldots, w_{n-1}, z_1, \ldots, z_n) \). Moreover it is easily seen \( F_n(1, w_1, w_2, \ldots, w_n | z_1, \ldots, z_n) = F_{n-1}(w_1, w_2, \ldots, w_{n-1} | z_1, \ldots, z_n) \). Repeating this we reach \( F_1(w_1 \cdots w_n | z_n) \) giving the LHS of (17).
\[\square\]
We note that the sum-to-1 (13) does not hold in the elliptic case. Remember that our local states are taken from \( \mathcal{P} = \eta + \mathbb{Z}^n+1 \) with a generic \( \eta \in \mathbb{C}^{n+1} \). So we set \( a = \eta + \hat{a} \) with \( \hat{a} \in \mathbb{Z}^n+1 \) etc in (13), and assume that it is valid also for \( \hat{a}, \hat{b}, \hat{c}, \hat{d} \). It is easy to check

**Proposition 5.** Assume \( l \leq m \) and \( |q| < 1 \). Then the following equality holds:
\[
\lim_{l \to \infty} S_{l,m} \left( \left\{ \begin{array}{c}
a & \beta \\
\eta & c
\end{array} \right\} \right)_{\text{trig}} = q^{\sum_{i<j} (\beta_i - \gamma_i) \gamma_i} \left( \begin{array}{c}
m \\
l
\end{array} \right)_q^{-1} \prod_{i=1}^{n+1} \frac{(\beta_i^{\gamma_i})}{(\gamma_i^{\gamma_i})},
\] (18)
where the limit means \( \eta_i - \eta_{i+1} \to \infty \) for all \( 1 \leq i \leq n \), and the RHS is zero unless \( 0 \leq \gamma_i \leq \beta_i \), \( \forall i \).

The limit reduces the unrestricted trigonometric \( A_n^{(1)} \) face model to the vertex model at a special value of the spectral parameter in the sense that the RHS of (18) \( |q| = q \) reproduces [8 eq.(23)] that was obtained as the special value of the quantum \( R \) matrix associated with the symmetric tensor representation of \( U_q(A_n^{(1)}) \).

3. Discussion

Since the weights \( W_{l,m} \left( \left\{ \begin{array}{c}
a & b \\
c & d
\end{array} \right\} |u \right) \) remain unchanged by shifting \( a, b, c, d \in \mathcal{P} \) by const \( \cdot (1, \ldots, 1) \), we regard them as elements from \( \mathcal{P} := \mathcal{P} / \mathbb{C}(1, \ldots, 1) \) in the sequel. Given \( l, m_1, \ldots, m_M \in \mathbb{Z}_{\geq 1} \) and \( u, w_1, \ldots, w_M \), the transfer matrix \( T_l(u) = T_l \left( \left\{ \begin{array}{c}
m_1, \ldots, m_M \\
w_1, \ldots, w_M
\end{array} \right\} \right) \) of the unrestricted \( A_n^{(1)} \) face model with periodic boundary condition is a linear map on the space of independent row configurations on length \( M \) row \( \bigoplus \mathbb{C}^{|a(1) \cdots a(M)|} \) where the sum is taken over \( a(1) \cdots a(M) \in \mathcal{P} \) such that \( a(i+1) = a(i) \in B_m, (a^{M+1}) = a(1) \). Its action is specified as \( T_l(u) |b^{(1)} \cdots b^{(M)}\rangle = \sum_{a^{(1)} \cdots a^{(M)}} T_l(w) |a^{(1)} \cdots a^{(M)}\rangle |b^{(1)} \cdots b^{(M)}\rangle \) in terms of the matrix elements
\[
T_l(u) a^{(1)} \cdots a^{(M)} |b^{(1)} \cdots b^{(M)}\rangle = \prod_{i=1}^{M} W_{l,m} \left( \left\{ \begin{array}{c}
a^{(i)} & a^{(i+1)} \\
b^{(i)} & b^{(i+1)}
\end{array} \right\} |u - w_i\right),
\] (a^{M+1}) = a(1), b^{M+1}) = b(1).
Theorem 1 tells that $S_I := T_I(u)|_{u = w_1 = \ldots = w_M}$ has a simple factorized matrix elements. We write its elements as $S_{l,b}^{a^{(1)},a^{(M)}}$. The star-triangle relation (7) implies the commutativity $[T_I(u), T_{I'}(u')] = [S_I, S_{I'}] = 0$.

Let us consider whether $X = T_I(u)$ or $S_I$ admits an interpretation as a Markov matrix of a discrete time stochastic process. The related issue was treated in [4] for $n = 1$ and mainly when $\min(l, m_1, \ldots, m_M) = 1$. One needs (i) sum-to-1 property $\sum_a a^{(1)}, a^{(M)} X_{b^{(1)},\ldots,b^{(M)}} = 1$ and (ii) non-negativity $\forall X_{a^{(1)},\ldots,a^{(M)}} \geq 0$. We concentrate on the trigonometric case in what follows. From Theorem 1 and the fact that $S_{l,m}^{a,b,c,d}_{\text{trig}}$ in (13) is independent of $a$, (i) indeed holds for $S_I$. On the other hand (13) also indicates that (ii) is not valid in general without confining the site variables in a certain range. A typical such prescription is restriction [1] [3] [5], where one takes $L = \ell + n + 1$ in [2] with some $\ell \in \mathbb{Z}_{\geq 1}$ and lets the site variables range over the finite set of level $\ell$ dominant integral weights $\{L + a_{n+1} - a_i - 1, \lambda_0 \mid \lambda_0 + \sum_{i=1}^n (a_i - a_{i+1} - 1) \lambda_i \mid L + a_{n+1} > a_1 > \cdots > a_{n+1} > a_i - a_j \in \mathbb{Z}\}$. They are to obey a stronger adjacency condition [5, (c-2)] than (3) which is actually the fusion rule of the WZW conformal field theory. (The formal limit $\ell \rightarrow \infty$ still works to restrict the site variables to the positive Weyl chamber and is called “classically restricted”.) Then the star-triangle relation remains valid by virtue of nontrivial cancellation of unwanted terms. However, discarding the contribution to the sum (13) from those $b$ not satisfying the adjacency condition spoils the sum-to-1 property. For example when $(n, l, m) = (2, 1, 2), a = (2, 1, 0), c = (4, 2, 0), d = (3, 1, 0)$ and $\ell$ is sufficiently large, the unrestricted sum (13) consists of two terms $S_{l,m}^{a,b,c,d}_{\text{trig}} = \left(\frac{2}{1}\right)_{q}^{-1} \left(\frac{q^2; q^{2\ell}}{q^2 q^2}\right)$ for $b = (4, 1, 0)$ and $S_{l,m}^{a,b',c,d}_{\text{trig}} = \left(\frac{2}{1}\right)_{q}^{-1} \left(\frac{q^2; q^{2\ell}}{q^2 q^2}\right)$ for $b' = (3, 2, 0)$ summing up to 1, but $b'$ must be discarded in the restricted case since $a \Rightarrow b'$ [5, (c-2)] does not hold. Thus we see that in order to satisfy (i) and (ii) simultaneously one needs to resort to a construction different from the restriction.

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