TRAVELING WAVES AND THEIR STABILITY FOR A PUBLIC GOODS GAME MODEL

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Abstract. We study the traveling wave solutions to a reaction diffusion system modeling the public goods game with altruistic behaviors. The existence of the waves is derived through monotone iteration of a pair of classical upper- and lower solutions. The waves are shown to be unique and strictly monotonic. A similar KPP wave like asymptotic behaviors are obtained by comparison principle and exponential dichotomy. The stability of the traveling waves with non-critical speed is investigated by spectral analysis in the weighted Banach spaces.

1. Introduction

Due to their widespread applications in biology, physics and chemistry, there has been an increasing interest in the traveling wave solutions to reaction diffusion systems. A fruitful methods have been developed in deriving the traveling wave solutions, among which the monotone iteration method is proved to be rather effective. Such method reduces the existence problem to that of an ordered pair of upper and lower solutions. However, it is not an easy task to construct the upper and lower solution pairs in the classical sense, therefore, [4, 19, 25, 26] turn to search for the generalized upper and lower solution pairs and have achieved the desired results. Other methods such as geometric singular perturbation [2, 8], phase plane analysis [1, 6, 11, 13, 14, 15, 18, 21, 22, 23, 28] and the fixed point Theorem [19] are also successfully applied to various reaction diffusion systems to obtain the wave solutions.

One interesting yet elusive question is the long term behavior of the traveling wave solutions. In fact, even for the relatively well studied Lotka Volterra systems less results (see [1], [12]) are known on the asymptotic decay (growth) rates of the waves and their stability. This question seems to be related to environmental selection in a given ecosystem. Roughly speaking, traveling wave is formed by intersecting the unstable subspace from one equilibrium and stable subspace from another. The stable and unstable spaces are generally multi-dimensional, therefore, the traveling wave selects an unstable direction from the unstable equilibrium to emanate, and then a stable direction from the stable equilibrium to enter. The selection affects the asymptotic behaviors and stability of the wave solutions.

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We try to understand those problems through the study of the traveling wave solutions of the following reaction diffusion system:

\[
\begin{aligned}
\frac{\partial \hat{u}}{\partial t} &= \frac{\partial^2 \hat{u}}{\partial x^2} + \hat{u}(r_1 - \frac{\hat{u} + \hat{v}}{K(\hat{u})} - \alpha), \\
\frac{\partial \hat{v}}{\partial t} &= \frac{\partial^2 \hat{v}}{\partial x^2} + \hat{v}(r_2 - \frac{\hat{v} + \hat{u}}{K(\hat{u})}), \\
\text{where } \hat{u} &= \hat{u}(x, t), \quad \hat{v} = \hat{v}(x, t) \quad \text{are two competing populations and } r_1, r_2, \alpha \text{ are positive constants. The system is a continuous spatial-temporal version of a public goods games \([24]\) and describes the interaction between the populations } \hat{u}(x, t) \text{ and } \hat{v}(x, t). \text{ In the game, the population } \hat{u} \text{ employs an altruistic strategy in order to ensure the survival of both populations; while the population } \hat{v} \text{'s strategy is to maximize its own gain. The function } K(\hat{u}) = k_0 + k\hat{u} \text{ with } k, k_0 > 0 \text{ represents the public goods, contributed by the population } \hat{u} \text{ and shared by the population } \hat{v}. \text{ In this sense, we call population } \hat{u} \text{ the cooperators and population } \hat{v} \text{ the defectors. The cooperators have to pay certain penalty measured by } \alpha \hat{u} \text{ for such altruistic strategy. As the outcome of the game, the population } \hat{v} \text{ eventually wins while the population } \hat{u} \text{ loses. So two equilibrium states are of interest to us - the one dominated by } \hat{u} \text{ and the one dominated by } \hat{v}. \text{ The game is played by finding a passage between the two equilibrium states.}

After rescaling the density functions (see \([9]\)), we may further assume that } r_1 = r_2 = k_0 = 1. \text{ The rescaled system has two unstable constant equilibrium states } (0, 0), (K^*, 0) \text{ with } K^* = \frac{1 - \alpha}{1 - k + \alpha k} > 0 \text{ and one asymptotically stable state } (0, 1). \text{ Throughout the rest of the paper we make the following assumption:}

\textbf{H: } 0 < \alpha, k < 1.

The transformations } \hat{u} = K^* - u, \hat{v} = v \text{ change } (1.1) \text{ into a local monotone type. Disregarding the initial conditions, the traveling wave solution to } (1.1) \text{ has the form } (u(\xi), v(\xi))^T := (u(x + ct), v(x + ct))^T, c > 0 \text{ and solves the following system}

\[
\begin{aligned}
u'' - cu' - (K^* - u)(1 - \frac{K^* - u + v}{1 + k(K^* - u) - \alpha}) = 0, \\
v'' - cu' + v(1 - \frac{K^* - u + v}{1 + k(K^* - u)}) = 0,
\end{aligned}
\]

\[\text{(1.2)}\]

\text{where } T \text{ represents the vector transposition and the prime the derivative with respect to } \xi.

In our derivation of the traveling wave solutions, the classical results on KPP (Kolmogorov-Petrovskii-Piskunov, \([18]\)) equation and the monotone structure of \([1,2]\) play a central role. Noting that if } u \text{ is replaced by zero then the second equation in } (1.2) \text{ is a KPP equation. The construction of the upper- and lower solution pairs is based on this simple observation. Cares also be taken on the asymptotic decay rates of the upper and lower solution pairs to ensure that they are ordered and have the same decay rate at negative infinity. The existence of the traveling wave solutions for system } (1.2) \text{ then follows from the monotone iteration scheme by } [1] \text{ and } [20], \text{ and the asymptotics of the traveling wave will be obtained from the exponential dichotomy and comparison.

The stability of the traveling wave solutions with non-critical speed is investigated by spectral analysis. Since the wave solutions are essentially unstable in the}
unweighted spaces such as $L^\infty$ or $C_{\text{unif}}$, similar to the stability issues of KPP waves \cite{10, 21, 27, 29}, we first localize the initial perturbations to certain exponentially weighted Banach spaces. Then following the standard treatment of such problem as in \cite{2, 17, 20, 21}, we show that the linearized operator around the wave solution, with noncritical wave speeds, does not have eigenvalues with positive or zero real part. Several new ideas are incorporated to locate the eigenvalues of the linearized operator. This, along with the fact that the linearized operator is sectorial leads to the nonlinear stability of the wave solutions. The stability of the wave solution implies that a sufficiently small initial disturbance will not change the outcome of the game. We remark that though the paper is written for a particular system (1.1), the method presents here can be easily adapted to other reaction diffusion systems with monostable structure. We further remark that the stability of the waves with critical wave speed has a complete different nature and requires a different treatment, so we leave it open for now.

The paper is organized as follows: in Section 2 we show the existence, uniqueness, monotonicity, and derive the asymptotic decay (growth) rates of the traveling wave solutions. In Section 3 we study the stability of the traveling wave solutions in weighted Banach spaces.

2. Existence and Asymptotic Decay Rates

In this section, we establish the existence of traveling wave solutions and derive their asymptotic decay rates through monotone iteration of a pair of smooth upper and lower solutions. The definition of the upper and lower solutions is standard. For the rest of the paper the inequality between two vectors is component-wise.

**Definition 1.** A smooth function $(\bar{u}(\xi), \bar{v}(\xi))^T$, $\xi \in \mathbb{R}$ is an upper solution of (1.2) if its derivatives $(\bar{u}', \bar{v}')^T$ and $(\bar{u}'', \bar{v}'')^T$ are continuous on $\mathbb{R}$, and $(\bar{u}, \bar{v})^T$ satisfies

\begin{align*}
\begin{cases}
  u'' - cu' - (K^*-u)(1 - \frac{K^*-u+v}{1+k(K^*-u)}) - \alpha \leq 0,
  \\
  v'' - cv' + v(1 - \frac{K^*-u+v}{1+k(K^*-u)}) \leq 0,
\end{cases}
\end{align*}

(2.1)

with the boundary conditions

\begin{align*}
\begin{pmatrix}
u \\
u
\end{pmatrix}(-\infty) = \begin{pmatrix}0 \\
0
\end{pmatrix}, \\
\begin{pmatrix}u \\
u
\end{pmatrix}(+\infty) \geq \begin{pmatrix}K^* \\
1
\end{pmatrix}.
\end{align*}

(2.2)

A lower solution of (1.2) is defined in a similar way by reversing the inequalities in (2.1) and (2.2).

The construction of the smooth upper and lower solution pairs is based on the solutions of the KPP equation:

\begin{align*}
\begin{cases}
  w'' - cw' + f(w) = 0, \\
  w(-\infty) = 0, \\
  w(+\infty) = b,
\end{cases}
\end{align*}

(2.3)

where $f \in C^2([0, b])$ and $f > 0$ on the open interval $(0, b)$ with $f(0) = f(b) = 0$, $f'(0) = a_1 > 0$ and $f'(b) = -b_1 < 0$. We first recall the following result \cite{21}:

**Lemma 2.** Corresponding to every $c \geq 2\sqrt{a_1}$, system (2.3) has a unique (up to a translation of the origin) monotonically increasing traveling wave solution $w(\xi)$ for $\xi \in \mathbb{R}$. The traveling wave solution $w$ has the following asymptotic behaviors:

For the wave solution with non-critical speed $c > 2\sqrt{a_1}$, we have

\begin{align*}
w(\xi) = a_\infty e^{\frac{c\sqrt{c^2 - 4a_1}}{2}} - \xi + o(e^{\frac{-c\sqrt{c^2 - 4a_1}}{2}}) \text{ as } \xi \rightarrow -\infty,
\end{align*}

(2.4)
For the wave with critical speed \( c = 2\sqrt{\alpha_1} \), we have
\[
w(\xi) = (a_c + d_c \xi) e^{\sqrt{\alpha_1} \xi} + o(e^{\sqrt{\alpha_1} \xi}) \text{ as } \xi \to -\infty,
\]
where \( a_c \) and \( b_c \) are positive constants.

Lemma 3. For each fixed \( c \geq 2\sqrt{\alpha} \), we begin with the following form of KPP system:
\[
\begin{cases}
\tilde{v}'' - c\tilde{v}' + \frac{\alpha}{(1-k + \alpha k)(1 + kK^*(1 - \tilde{v}))} \tilde{v}(1 - \tilde{v}) = 0, \\
\tilde{v}(-\infty) = 0, \quad \tilde{v}(+\infty) = 1,
\end{cases}
\]
where relating to (2.3), \( f(\tilde{v}) = \frac{\alpha}{(1-k + \alpha k)(1 + kK^*(1 - \tilde{v}))} \tilde{v}(1 - \tilde{v}) > 0 \) for \( \tilde{v} \in (0, 1) \).

\( f(0) = f(1) = 0, \ f'(0) = \alpha > 0 \) and \( f'(1) = -\alpha/(1-k + \alpha k) < 0 \). According to Lemma 2, for each fixed \( c \geq 2\sqrt{\alpha} \), system (2.6) has a unique (up to a translation of the origin) traveling wave solution \( \tilde{v}(\xi) \) satisfying the given boundary conditions. Define
\[
(\bar{u}(\xi), \bar{v}(\xi)) = \left( K^* \tilde{v}(\xi), \tilde{v}(\xi) \right), \quad \xi \in \mathbb{R},
\]
then we have the following result,

**Lemma 3.** For each fixed \( c \geq 2\sqrt{\alpha} \), (2.7) is a smooth upper solution for system (1.2).

**Proof.** On the boundary, one has \((\bar{u}, \bar{v})^T(-\infty) = (0, 0)^T, \ (\bar{u}, \bar{v})^T(+\infty) = (K^*, 1)^T\).
As for the \( u \) component, we have
\[
\bar{u}'' - c\bar{u}' - (K^* - \bar{u})(1 - \alpha - \frac{K^* - K^*\bar{v} + \bar{v}}{1 + kK^*(1 - \bar{u})})
\]
\[
= K^*[\tilde{v}'' - c\tilde{v}' - (1 - \tilde{v})(1 - \alpha - \frac{K^* - K^*\tilde{v} + \tilde{v}}{1 + kK^*(1 - \tilde{v})})]
\]
\[
= -(1 - \tilde{v})[\frac{\alpha}{1-k+\alpha k} \tilde{v} - K^* \tilde{v} - \tilde{v} + 1 + kK^*(1 - \tilde{v})] - \alpha
\]
\[
= -(1 - \tilde{v}) \frac{\alpha k K^* \tilde{v}}{1 + kK^*(1 - \tilde{v})} \leq 0.
\]
We then verify the second component in (1.2),
\[
\bar{v}'' - c\bar{v}' + \bar{v}[1 - \frac{K^* - K^*\bar{v} + \bar{v}}{1 + kK^*(1 - \bar{u})}]
\]
\[
= -\frac{\alpha}{(1 + kK^*(1 - \bar{v}))(1-k + \alpha k)} \bar{v}(1 - \bar{v}) + \bar{v}[1 - \frac{K^* - K^*\bar{v} + \bar{v}}{1 + kK^*(1 - \bar{v})}]
\]
\[
= \bar{v}[\frac{\alpha}{1-k+\alpha k} \bar{v} - \bar{v}(1 + kK^* - K^*) - \frac{\alpha}{1-k+\alpha k} (1 - \bar{v})]
\]
\[
= 0.
\]
since $1 + kK^* - K^* = \frac{\alpha}{1 - k + \alpha k}$.

Thus $(\bar{u}, \bar{v})$ forms a smooth upper-solution for (1.2). \qed

We next construct the lower solution pair for system (1.2). For a small but fixed number $l$ with $0 < l < 1 - k + k\alpha$, we begin with yet another KPP system:

$$\begin{aligned}
\begin{cases}
\bar{v}'' - c\bar{v}' + \frac{\alpha}{(1 - k + \alpha k)(1 + kK^*(1 - l\bar{v}))}\bar{v}(1 - \frac{1 + kK^* - lK^*}{1 + kK^* - K^*}\bar{v}) = 0, \\
\bar{v}(-\infty) = 0, \quad \bar{v}(+\infty) = \frac{1 + kK^* - K^*}{1 + kK^* - lK^*} < 1.
\end{cases}
\end{aligned}$$

(2.8)

Corresponding to the notions in Lemma 2,

$$f(\bar{v}) = \frac{\alpha}{(1 - k + \alpha k)(1 + kK^*(1 - l\bar{v}))}\bar{v}(1 - \frac{1 + kK^* - lK^*}{1 + kK^* - K^*}\bar{v}) > 0$$

for $\bar{v} \in (0, \frac{1 + kK^* - K^*}{1 + kK^* - lK^*})$. $f(0) = f\left(\frac{1 + kK^* - K^*}{1 + kK^* - lK^*}\right) = 0$, $f'(0) = \alpha > 0$, and $f'\left(\frac{1 + kK^* - K^*}{1 + kK^* - lK^*}\right) = -\frac{1}{1 - l + l\alpha(1 - k + \alpha k)} < 0$.

For each fixed $c \geq 2\sqrt{\alpha}$, define

$$\begin{aligned}
\begin{pmatrix}
u(\xi) \\ \nu'(\xi)
\end{pmatrix} = \begin{pmatrix}
K^*\nu(\xi) \\ \nu(\xi)
\end{pmatrix}, \quad \xi \in \mathbb{R}
\end{aligned}$$

(2.9)

with $\nu(\xi)$ a solution of (2.8).

**Lemma 4.** For each $c \geq 2\sqrt{\alpha}$, (2.9) is a smooth lower solution of system (1.2).

**Proof.** On the boundary, one has

$$\begin{pmatrix}
u(-\infty) \\ \nu'(-\infty)
\end{pmatrix} = \begin{pmatrix}0 \\ 0\end{pmatrix},$$

and

$$\begin{pmatrix}
u(+\infty) \\ \nu(+\infty)
\end{pmatrix} = \begin{pmatrix}K^*\frac{1 + kK^* - K^*}{1 + kK^* - lK^*} \\ \frac{1 + kK^* - K^*}{1 + kK^* - lK^*} < 1\end{pmatrix}. $$
Furthermore,
\[
\frac{\partial^2 u}{\partial t^2} - c\frac{\partial u}{\partial t} - (K^* - u)(1 - \alpha - \frac{K^* - u + v}{1 + k(K^* - u)}) = 0
\]

Therefore, the conclusion of the lemma follows. \(\square\)
Lemma 5. Let \( c \geq 2\sqrt{\alpha} \) be fixed and \((\bar{u}, \bar{v})^T, (\underline{u}, \underline{v})^T\) be respectively the upper and lower solutions defined in (2.17) and (2.18), then there exists a number \( r \geq 0 \), such that \((\bar{u}, \bar{v})^T(\xi + r) \geq (\underline{u}, \underline{v})^T(\xi)\) for \( \xi \in \mathbb{R} \).

Proof. The proof is only for \( c > 2\sqrt{\alpha} \) since the one for \( c = 2\sqrt{\alpha} \) is similar. We first derive the asymptotic behaviors of the upper- and lower-solutions at infinities. By Lemma 2

\[
(2.10) \quad \left( \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right) (\xi) = \left( \begin{array}{c} K^* A_1 \\ A_1 \end{array} \right) e^{\sqrt{\frac{\alpha}{2}} \xi} + o(e^{\sqrt{\frac{\alpha}{2}} \xi})
\]

and

\[
(2.11) \quad \left( \begin{array}{c} u \\ v \end{array} \right) (\xi) = \left( \begin{array}{c} K^* B_1 \\ B_1 \end{array} \right) e^{\sqrt{\frac{\alpha}{2}} \xi} + o(e^{\sqrt{\frac{\alpha}{2}} \xi})
\]

as \( \xi \to -\infty \); and

\[
(2.12) \quad \left( \begin{array}{c} \bar{u} \\ \bar{v} \end{array} \right) (\xi) = \left( \begin{array}{c} K^* \\ 1 \end{array} \right) - \left( \begin{array}{c} K^* A_2 \\ A_2 \end{array} \right) e^{\sqrt{\frac{\alpha}{2}} \xi} + o(e^{\sqrt{\frac{\alpha}{2}} \xi})
\]

and

\[
(2.13) \quad \left( \begin{array}{c} u \\ v \end{array} \right) (\xi) = \left( \begin{array}{c} K^* l \\ 1 \end{array} \right) \frac{1 + kK^* - K^*}{1 + kK^* - 1K^*} \left( \begin{array}{c} K^* B_2 \\ B_2 \end{array} \right) e^{\mu \xi} + o(e^{\mu \xi})
\]

as \( \xi \to +\infty \), where \( \mu = \frac{1}{2}(c - \sqrt{c^2 + \frac{1}{1+\alpha}}) < 0 \) and \( A_1, A_2, B_1, B_2 \) are positive constants.

Since \((\bar{u}, \bar{v})^T(\xi + r)\) is also a solution of (2.10) for any \( \tilde{r} \in \mathbb{R} \). It then follows that \((\bar{u}, \bar{v})^T(\xi)\) is an upper solution for system \((1.2)\). For the asymptotic behavior of \((\bar{u}, \bar{v})^T(\xi)\) at \( -\infty \), we can simply replace \((K^* A_1, A_1)^T\) by \((K^* A_1, A_1)^T e^{\sqrt{\frac{\alpha}{2}} \tilde{r}}\) in (2.10). Now we choose \( \tilde{r} > 0 \) large enough such that

\[(K^* A_1, A_1)^T e^{\sqrt{\frac{\alpha}{2}} \tilde{r}} > (K^* A_1, A_1)^T.\]

Then there exists a sufficiently large \( N_1 > 0 \) such that

\[
(2.14) \quad \left( \begin{array}{c} \bar{u}(\xi) \\ \bar{v}(\xi) \end{array} \right) > \left( \begin{array}{c} u(\xi) \\ v(\xi) \end{array} \right) \quad \text{for } \xi \in (-\infty, -N_1].
\]

On the other hand, the boundary conditions of the upper and lower solutions at \( +\infty \) also imply that there exists a number \( N_2 > 0 \) such that

\[
(2.15) \quad \left( \begin{array}{c} \bar{u}(\xi) \\ \bar{v}(\xi) \end{array} \right) > \left( \begin{array}{c} u(\xi) \\ v(\xi) \end{array} \right) \quad \text{for } \xi \in [N_2, +\infty).
\]

We next show that the inequalities (2.14) and (2.15) also hold on the interval \([-N_1, N_2]\). There are two possible cases to deal with:
Case 1. If we already have

\[
(\bar{u}^0(\xi), \bar{v}^0(\xi)) \geq (u(\xi), v(\xi)) \quad \text{on } [-N_1, N_2],
\]

we then let $\bar{r} = r$ and obtain the Lemma.

Case 2. There exists a point $\xi_0 \in (-N_1, N_2)$ such that

\[
\left(\begin{array}{c}
\bar{u}^0(\xi_0) \\
\bar{v}^0(\xi_0)
\end{array}\right) \leq \left(\begin{array}{c}
u(\xi_0) \\
v(\xi_0)
\end{array}\right)
\]

with strict inequality holding for at least one of the two components.

In this case, we will use the the Sliding Domain method. We first shift $(\bar{u}, \bar{v})$ to the left by increasing $\bar{r}$ until we can find a $r_1 > \bar{r} > 0$ such that $(\bar{u}^r(\xi), \bar{v}^r(\xi))^T > (u(\xi), v(\xi))^T$ on the interval $[-N_1, N_2 - (r_1 - \bar{r})]$. We then shift $(\bar{u}^r(\xi), \bar{v}^r(\xi))^T$ back to the right by decreasing $r_1$ to some $r_2 > \bar{r}$ such that one of the branches of the upper solution touches its counterpart of the lower solution at some point $\xi_2$ in the interval $(-N_1 + r_2, N_2 - (r_1 - \bar{r}))$. On the endpoints of the interval $(-N_1 + r_2, N_2 - (r_1 - \bar{r}))$, we still have $(\bar{u}^r(\xi), \bar{v}^r(\xi))^T > (u(\xi), v(\xi))^T$. In summary, we now have $\bar{u}^r(\xi_2) = u(\xi_2)$ and $\bar{v}^r(\xi_2) > u(\xi_2)$, $\bar{v}^r(\xi) > u(\xi)$ for $\xi \in (-N_1 + r_2, N_2 - (r_1 - \bar{r}))$.

Letting $W(\xi) := (\bar{u}^r, \bar{v}^r)^T(\xi) - (\bar{\mu}, \bar{\nu})^T(\xi)$ and $F = (F_1, F_2)^T = (-K^* - \bar{u})(1 - \alpha - \frac{K^* - \bar{u}}{1 + (K^* - \bar{u})})^T$. For $\xi \in (-N_1 + r_2, N_2 - (r_1 - \bar{r}))$ we have

\[
W'' - cw' + \begin{bmatrix}
\frac{\partial F_1}{\partial \mu}(\bar{u} + \zeta_1 w_1, \bar{v}^r), & \frac{\partial F_1}{\partial \nu}(\bar{u}^r, \bar{v} + \zeta_2 w_2) \\
\frac{\partial F_2}{\partial \mu}(\bar{u} + \zeta_3 w_1, \bar{v}^r), & \frac{\partial F_2}{\partial \nu}(\bar{u}^r, \bar{v} + \zeta_4 w_2)
\end{bmatrix} W = 0,
\]

for some $\zeta_i \in [0, 1], i = 1, 2, 3, 4$. Since the above system is monotone and the cube $[0, 0, (K^*, 1)]$ is convex, we can readily deduce by Maximum Principle that $W(\xi) > 0$ for $\xi \in [-N_1 + r_2, N_2 - (r_1 - \bar{r})]$. This is a contradiction, i.e., such $\xi_2$ does not exist and we can further decrease $r_2$ to $\bar{r}$. This shows that $\xi_0$ does not exist either. We therefore have $(\bar{u}^r, \bar{v}^r)^T(\xi) \geq (\bar{u}, \bar{v})^T(\xi)$ for $\xi \in \mathbb{R}$.

To ease the burden of notations, we still use $(\bar{u}, \bar{v})^T$ to denote the shifted upper solution as given in lemma [5]. With such constructed ordered upper and lower solution pairs, we now have,

**Theorem 6.** For every $c \geq 2\sqrt{\alpha}$, system (1.2) has correspondingly a unique (up to a translation of the origin) traveling wave solution. The traveling wave solution is strictly increasing and has the following asymptotic properties:

1. Corresponding to the wave speed $c > 2\sqrt{\alpha}$,

\[
\begin{pmatrix}
u(\xi) \\ v(\xi)
\end{pmatrix} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\frac{-\sqrt{c^2-4\alpha}}{2} \xi} + o(e^{\frac{-\sqrt{c^2-4\alpha}}{2} \xi})
\]

as $\xi \to -\infty$; and

\[
\begin{pmatrix}
u(\xi) \\ v(\xi)
\end{pmatrix} = \begin{pmatrix} K^* \\ 1 \end{pmatrix} - \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{\frac{-\sqrt{c^2+4\alpha}}{2} \xi} + o(e^{\frac{-\sqrt{c^2+4\alpha}}{2} \xi})
\]

as $\xi \to +\infty$, where $A_1, A_2, A_1, A_2$ are positive constants.

2. Corresponding to the wave speed $c_{\text{critical}} = 2\sqrt{\alpha}$, we have
\begin{equation}
\begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} = \begin{pmatrix}
(A_{11c} + A_{12c}\xi) \\
(A_{21c} + A_{22c}\xi)
\end{pmatrix} e^{\sqrt{\alpha}\xi} + o(e^{\sqrt{\alpha}\xi})
\end{equation}

as \( \xi \to -\infty \); and

\begin{equation}
\begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} = \begin{pmatrix}
K^* \\
1
\end{pmatrix} - \begin{pmatrix}
\bar{A}_{11} \\
\bar{A}_{22}
\end{pmatrix} e^{(1-\sqrt{2})\sqrt{\alpha}\xi} + o(e^{(1-\sqrt{2})\sqrt{\alpha}\xi})
\end{equation}

as \( \xi \to +\infty \), where \( A_{12c}, A_{22c} < 0, A_{11c}, A_{21c} \in \mathbb{R} \) and \( \bar{A}_{11}, \bar{A}_{22} > 0 \).

Proof. Starting from the upper and lower solution pairs obtained in Lemma 3-4 and using the monotone iteration scheme given in [4, 26], we obtain the existence of the solution \((u(\xi),v(\xi))^T\) to the first two equations in (1.2) for every \( c \geq 2\sqrt{\alpha} \), which satisfies

\begin{equation}
\begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} \leq \begin{pmatrix}
u(\xi) \\
v(\xi)
\end{pmatrix} \leq \begin{pmatrix}\bar{u}(\xi) \\
\bar{v}(\xi)\end{pmatrix}.
\end{equation}

On the boundary, it is easy to see that the solution tends to \((0,0)^T\) as \( \xi \to -\infty \), and \((K^*,1)^T\) as \( \xi \to +\infty \) according to the last inequality.

To derive the asymptotic decay rate of the traveling wave solutions at \( \pm\infty \), we let \( c \geq 2\sqrt{\alpha} \) and

\begin{equation}
U(\xi) := (u(\xi),v(\xi))^T \text{ for } -\infty < \xi < \infty
\end{equation}

be the corresponding traveling wave solution of (1.2) generated from the monotone iteration. We differentiate (1.2) with respect to \( \xi \), and note that \( U'(\xi) := (w_1, w_2)^T(\xi) \) satisfies

\begin{equation}
w_1'' - cw_1' + A_{11}(u,v)w_1 + A_{12}(u,v)w_2 = 0,
\end{equation}

\begin{equation}
w_2'' - cw_2' + A_{21}(u,v)w_1 + A_{22}(u,v)w_2 = 0,
\end{equation}

where

\begin{align*}
A_{11}(u,v) &= 1 - \frac{K^* - u + v}{1 + k(K^* - u)} - \alpha \\
&\quad + \left( K^* - u \right) \frac{kv - 1}{(1 + k(K^* - u))^2}, \\
A_{12}(u,v) &= \frac{K^* - u}{1 + k(K^* - u)}, \\
A_{21}(u,v) &= -\frac{kv - 1}{(1 + k(K^* - u))^2}v, \\
A_{22}(u,v) &= 1 - \frac{K^* - u + v}{1 + k(K^* - u)} \\
&\quad - \frac{1}{1 + k(K^* - u)}v.
\end{align*}

Lemma 2 implies that the upper- and the lower-solutions as derived in Lemma 3 and Lemma 4 have the same asymptotic rates at \( -\infty \). (2.19) and (2.21) then follow from Lemma 5.
We next study the exponential decay rates of the traveling wave solution \(U(\xi)\) at \(+\infty\). The asymptotic system of (2.24) and (2.25) as \(\xi \to +\infty\) is

\[
\begin{align*}
\psi''_1 - cw_1' - \alpha \psi_1 &= 0, \\
\psi''_2 - cw_2' + (1 - k)\psi_1 - \psi_2 &= 0.
\end{align*}
\]

(2.26)

It is easy to see that system (2.26) admits exponential dichotomy [5]. Since the traveling wave solution \((u(\xi), v(\xi))^T\) converges monotonically to a constant limit as \(\xi \to \pm \infty\), the derivative of the traveling wave solution satisfies \((w_1(\pm \infty), w_2(\pm \infty)) = (0, 0)\) ([26], p658 Lemma 3.2). Hence we are only interested in finding bounded solutions of (2.26) at \(+\infty\).

Since the first equation in (2.26) is decoupled, we can write its general solution as

\[
\psi_1 = A^1 e^{\sqrt{\frac{c^2+4k}{2}}\xi} + B^1 e^{-\sqrt{\frac{c^2+4k}{2}}\xi}
\]

for some constants \(A^1\) and \(B^1\). Since \(w_1 \to 0\) as \(\xi \to +\infty\), one immediately has \(A^1 = 0\).

We then study the second equation of (2.26), rewriting the equation as

\[
(\psi_2)^'' - c(\psi_2)' - \psi_2 = -(1 - k)\psi_1,
\]

(2.27)

since \(\psi_2(\xi) \to 0\) as \(\xi \to +\infty\), then \(\psi_2(\pm \infty, 0\) for \(\xi \to +\infty\), we have \(B_1 \neq 0\). Also noticing that (2.27) is non-homogeneous, we have \(B_1 \neq 0\). By roughness of the exponential dichotomy (5) and comparing with the upper solution, we obtain the asymptotic decay rate of the traveling wave solutions at \(+\infty\) given in (2.26) and (2.23).

We next show the strict monotonicity of the traveling wave solutions, which will be a key ingredient in locating the eigenvalues of the linearized operator in the next section. By the monotone iteration process [26], the traveling wave solution \(U(\xi)\) is increasing for \(\xi \in \mathbb{R}\), it then follows that \((w_1(\xi), w_2(\xi))^T = U(\xi) \geq 0\) and satisfies

\[
w_1'' - cw_1' + \frac{\partial F_1}{\partial u}(u, v)w_1 + \frac{\partial F_1}{\partial v}(u, v)w_2 = 0,
\]

(2.29)

\[
w_2'' - cw_2' + \frac{\partial F_2}{\partial u}(u, v)w_1 + \frac{\partial F_2}{\partial v}(u, v)w_2 = 0,
\]

(2.30)

and

\[
(w_1(\xi), w_2(\xi))^T \geq 0, \ (w_1, w_2)^T(\pm \infty) = 0.
\]

(2.31)

The strong Maximum Principle implies that \((w_1, w_2)^T(\xi) > 0\) for \(\xi \in \mathbb{R}\). This concludes the strict monotonicity of the traveling wave solutions.

On the uniqueness of the traveling wave solution for every \(c \geq 2\sqrt{\alpha}\), we only prove the conclusion for the traveling wave solutions with asymptotics (2.19) and (2.20), since other case can be proved similarly. Let \(U_1(\xi) = (u_1, v_1)^T\) and \(U_2(\xi) = (u_2, v_2)^T\) be two traveling wave solutions of system (1.1) with the same speed \(c > 2\sqrt{\alpha}\). There exist positive constants \(A_i, B_i, i = 1, 2, 3, 4\) and a large number \(N > 0\) such that
for $\xi < -N$, 

\begin{equation}
U_1(\xi) = \begin{pmatrix} A_1 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi} \\
A_2 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi} \end{pmatrix} + o(e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi})
\end{equation}

\begin{equation}
U_2(\xi) = \begin{pmatrix} A_3 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi} \\
A_4 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi} \end{pmatrix} + o(e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi})
\end{equation}

and for $\xi > N$,

\begin{equation}
U_1(\xi) = \begin{pmatrix} K^* - B_1 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi} \\
1 - B_2 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi} \end{pmatrix} + o(e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi})
\end{equation}

\begin{equation}
U_2(\xi) = \begin{pmatrix} K^* - B_3 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi} \\
1 - B_4 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi} \end{pmatrix} + o(e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi})
\end{equation}

The traveling wave solutions of system (1.2) are translation invariant, thus for any $\theta > 0$, $U^\theta_1(\xi) := U_1(\xi + \theta)$ is also a traveling wave solution of (1.2). By (2.32) and (2.34), the solution $U_1(\xi + \theta)$ has the asymptotics

\begin{equation}
U^\theta_1(\xi) = \begin{pmatrix} A_1 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \theta e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi}} \\
A_2 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \theta e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi}} \end{pmatrix} + o(e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \xi})
\end{equation}

for $\xi \leq -N$;

\begin{equation}
U^\theta_1(\xi) = \begin{pmatrix} K^* - B_1 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \theta e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi}} \\
1 - B_2 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \theta e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi}} \end{pmatrix} + o(e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \xi})
\end{equation}

for $\xi \geq N$.

Choosing $\theta > 0$ large enough such that

\begin{equation}
A_1 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \theta} > A_3,
\end{equation}

\begin{equation}
A_2 e^{\frac{c+\sqrt{c^2-4\alpha^2}}{2} \theta} > A_4,
\end{equation}

\begin{equation}
B_1 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \theta} < B_3,
\end{equation}

\begin{equation}
B_2 e^{\frac{c-\sqrt{c^2+4\alpha^2}}{2} \theta} < B_4,
\end{equation}

then one has

\begin{equation}
U^\theta_1(\xi) > U_2(\xi)
\end{equation}
for $\xi \in (-\infty, -N] \cup [N, +\infty)$. We now consider system (1.2) on $[-N, +N]$. First, suppose $U^0(\xi) \geq U_2(\xi)$ on $[-N, +N]$, then the function $W(\xi) = (u_1(\xi), w_2(\xi))^T := U^0(\xi) - U_2(\xi) \geq 0$ and satisfies for some $\zeta_i \in (0, 1)$, $i = 1, 2, 3, 4$,
\begin{equation}
W'' - cW' + \begin{bmatrix}
\frac{\partial F}{\partial u}(u_2 + \zeta_3w_1, v_1), & \frac{\partial F}{\partial v}(u_1, v_2 + \zeta_2w_2) \\
\frac{\partial F}{\partial u}(u_2 + \zeta_4w_1, v_1), & \frac{\partial F}{\partial v}(u_1, v_2 + \zeta_1w_2)
\end{bmatrix} W = 0, \quad \xi \in (-N, N),
\end{equation}
(2.43)
$W(-N) > 0$, $W(+N) > 0$.

Since the above system is monotone, we can readily deduce by the Maximum Principle that $W(\xi) > 0$ on $[-N, N]$. Consequently, we have $U^0_1(\xi) > U_2(\xi)$ on $\mathbb{R}$ in this case.

Second, suppose there is some point in $(-N, N)$ such that one of the components, say the $j$-th component, satisfies $(U^0_1(\xi))_j < (U_2(\xi))_j$ at that point, $j = 1$ or 2. We then increase $\theta$, that is shift $U^0(\xi)$ further left, so that $U^0_1(\xi) > U_2(\xi)$, $U^0_1(N) > U_2(N)$. By the monotonicity of $U^0_1$ and $U_2$, we can find a $\theta \in (0, 2N)$ such that in the interval $(-N, N)$, we have $U^0(\xi + \theta) > U_2(\xi)$. Shifting $U^0(\xi + \theta)$ back until one component of $U^0_1(\xi + \theta)$ first touches its counterpart of $U_2(\xi)$ at some point $\xi \in [-N, N]$. Since $U^0_1(\xi + \theta)$ and $U_2(\xi)$ are strictly increasing, $\xi \in (-N, N)$, while at $\xi = \pm N$, we still have $U^0_1(\xi + \theta) > U_2(\xi)$. However, by Maximum Principle for that component again, we find that component of $U^0_1$ and $U_2$ are identically equal for all $\xi \in [-N, N]$ for a larger $\theta$ than the original one for which (2.42) holds. This is a contradiction. Therefore, we must have
\[ U^0_1(\xi) > U_2(\xi) \]
for all $\xi \in \mathbb{R}$, where $\theta$ is the one chosen by means of (2.38)-(2.41) as described above.

Now, decrease $\theta$ until one of the following situations happens.

1. There exists a $\bar{\theta} \geq 0$, such that $U^0_1(\xi) \equiv U_2(\xi)$. In this case we have finished the proof.

2. There exists a $\bar{\theta} \geq 0$ and $\xi_1 \in \mathbb{R}$, such that one of the components of $U^\bar{\theta}$ and $U_2$ are equal there; and for all $\xi \in \mathbb{R}$, we have $U^\bar{\theta}_1(\xi) \geq U_2(\xi)$. On applying the Maximum Principle on $\mathbb{R}$ for that component, we find $U^\bar{\theta}_1$ and $U_2$ must be identical on that component. To fix ideas, we suppose that the component is the first component. Then $U^\bar{\theta}_1 - U_2$ satisfies (2.28) and (2.29). Plugging $w_1 \equiv 0$ into (2.29) again we find that there is at least one $\xi_\bar{\theta}$ such that $W_2(\xi_\bar{\theta}) = 0$. Then by applying maximum principle to (2.30), we have $w_2(\xi) \equiv 0$ for $\xi \in \mathbb{R}$. We have then returned to case 1.

Consequently, in either situation, there exists a $\bar{\theta} \geq 0$, such that
\[ U^\bar{\theta}_1(\xi) \equiv U_2(\xi) \]
for all $\xi \in \mathbb{R}$.

The next Theorem shows that the lower bound $2\sqrt{\alpha}$ for the wave speed $c$ is optimal, hence it is the critical minimal wave speed.

**Theorem 7.** There is no monotone traveling wave solution of (1.2) for any $0 < c < 2\sqrt{\alpha}$.

**Proof.** Suppose there is a constant $c$ with $0 < c < 2\sqrt{\alpha}$ and a solution $V(\xi) = (v_1, v_2)^T(\xi)$ of (1.2) corresponding to it. Similar to the proof of Theorem 6, the
asymptotic behaviors of $V(\xi)$ at $-\infty$ are described by
\[
\begin{pmatrix}
v_1(\xi) \\
v_2(\xi)
\end{pmatrix} = 
\begin{pmatrix} A_s \\ B_s
\end{pmatrix} e^{\frac{-\sqrt{c^2 - 4\alpha^2}}{2}\xi} + 
\begin{pmatrix} A_s \\ B_s
\end{pmatrix} e^{\frac{\sqrt{c^2 - 4\alpha^2}}{2}\xi} + h.o.t,
\]
where $(A_s, B_s)^T$ and $(\tilde{A}_s, \tilde{B}_s)$ cannot be both zero due to the non-homogeneity of the limit system of (2.29) and (2.30) at $-\infty$. The condition $0 < c < 2\sqrt{\alpha}$ implies that $V(\xi)$ is oscillating. This concludes that any solution of (1.2) with $c < 2\sqrt{\alpha}$ is not strictly monotone. □

**Remark 8.** An interesting implication of Theorem 6 is that the populations $u$ and $v$ have the same exponential rate determined by $\alpha$, the measurement of the penalty for altruism that $u$ receives. The bigger the penalty, the faster the population $u$ decreases and population $v$ increases. Given the same population growth rate of $u$ and $v$, according to the model, the cooperators always loses the game.

### 3. Stability of the Traveling Waves

In this section, we investigate the stability of the traveling waves with non-critical speed. We first show that the traveling wave solution obtained in Theorem 6 with non-critical speed is unstable in the continuous function spaces $C(\mathbb{R}) \times C(\mathbb{R})$. This motivates us to investigate the spectrum of the linearized operator (see (3.3)) in the exponentially weighted Banach spaces.

Retaining all the transformations and rescalings in section 2, we consider system (1.2) in the following equivalent form, with initial conditions as given in (1.1)
\[
\begin{aligned}
U_t(x, t) &= U_{xx} + F(U), \\
U(x, 0) &= \bar{U}(x).
\end{aligned}
\]
where $U = (u, v)^T$ and $F(U) = (- (K^* - u)(1 - \alpha - \frac{K^*-u+v}{1 + K^*-u}), v(1 - \frac{K^*-u+v}{1 + K^*-u}))^T$, $\bar{U}(x)$ is the initial function.

Rewriting the above equivalent system in terms of $(\xi, t)$ variable, with moving coordinates $\xi = x + ct$, we have
\[
\begin{aligned}
U_t(\xi, t) &= U_{\xi\xi} - cU_{\xi} + F(U), \\
U(\xi, 0) &= \bar{U}(\xi).
\end{aligned}
\]

For a fixed wave speed $c^* > 2\sqrt{\alpha}$ and $\xi = x + c^*t$, $U^*(\xi) = (u^*(\xi), v^*(\xi))^T$ obtained in Theorem 6 is then a steady state of (3.1). Let $U(\xi, t) = U^*(\xi) + V(\xi, t)$ be a solution of (3.1). We then obtain the following system for the perturbation function $V(\xi, t)$:
\[
\begin{aligned}
V_t &= \mathcal{L}V + \mathcal{R}(V, U^*), \\
V(\xi, 0) &= \bar{U}(\xi) - U^*(\xi),
\end{aligned}
\]
where
\[
\mathcal{L}V = V_{\xi\xi} - c^*V_{\xi} + \frac{\partial F}{\partial U}(U^*)V
\]
is a linear operator, and
\[
\mathcal{R}(V, U^*) = F(U^* + V) - F(U^*) - \frac{\partial F}{\partial U}(U^*)V
\]
is a nonlinear operator.
The stability of the traveling wave solution $U^*(\xi)$ in a specific Banach space depends critically on the location of the spectrum $\sigma(\mathcal{L})$ of $\mathcal{L}$. Since $\mathcal{L}$ is defined on $\mathbb{R}$, it has point spectrum (eigenvalue)

$$\sigma_p(\mathcal{L}) = \{ \lambda \in \sigma(\mathcal{L}) \mid \lambda \text{ is an isolated eigenvalue of finite multiplicity} \}$$

as well as essential spectrum $\sigma_e(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \sigma_p(\mathcal{L})$ (2 [14], 23 [21]). Let $C(\mathbb{R})$ be the space of all continuous functions on $\mathbb{R}$ and $C_0$ be a subspace of $C(\mathbb{R})$,

$$C_0 = \{ U(\xi) \mid U(\xi) \in C(\mathbb{R}) \times C(\mathbb{R}), \lim_{|\xi| \to \infty} \| U(\xi) \| = 0 \}$$

where

$$\| U \|_{C_0} = \sup_{\xi \in \mathbb{R}} \| U(\xi) \|.$$

**Theorem 9.** The traveling wave solution $U^*(\xi)$ of system (3.1) with non-critical wave speed $c^* > 2\sqrt{\alpha}$, obtained in Theorem 6, is unstable with initial functions in $C_0$.

**Proof.** We prove the trivial solution of (3.2) is unstable. Thus, it suffices to show that in the space $C_0$, the operator $\mathcal{L}$ in (3.3) has essential spectrum with positive real parts. It turns out that (7, 23) the essential spectrum of the operator $\mathcal{L}$ is bounded by the spectrum of $\mathcal{L}$ at $\pm\infty$, which are denoted by $\mathcal{L}^+$ and $\mathcal{L}^-$ respectively. More precisely, we have

\[
\mathcal{L}^+ V = V_{\xi \xi} - c^* V_\xi + \frac{\partial F}{\partial U}(U^*_+) V = V_{\xi \xi} - c^* V_\xi + \begin{bmatrix} -\alpha & 0 \\ 1 - k & -1 \end{bmatrix} V,
\]

\[
\mathcal{L}^- V = V_{\xi \xi} - c^* V_\xi + \frac{\partial F}{\partial U}(U^*_-) V
\]

where $U^*_\pm$ denotes the limit of $U^*(\xi)$ as $\xi \to \pm\infty$.

Now consider the equation

$$\frac{\partial V}{\partial t} = \mathcal{L}^+ V.$$

Following [23], we replace $V$ by $e^{(\lambda t + i\zeta \xi)}I$, where $I$ is the identity matrix. We then have

\[
e^{(\lambda t + i\zeta \xi)}(-\zeta^2 I - c^* \zeta i I + \frac{\partial F}{\partial U}(U^*_+) - \lambda I) = 0.
\]

The spectrum of the operator $\mathcal{L}^-$ consists of curves

\[
\det(-\zeta^2 I - c^* \zeta i I + \frac{\partial F}{\partial U}(U^*_-) - \lambda I) = 0.
\]

Therefore we have

\[
-\zeta^2 - c^* \zeta - \alpha - \lambda = 0,
\]

or

\[
-\zeta^2 - c^* \zeta - 1 - \lambda = 0.
\]
Let $\lambda = x + yi$, then by (3.9) we have

$$(3.11) \quad x = -\frac{y^2}{(c^*)^2} - \alpha,$$

or by (3.10),

$$(3.12) \quad x = -\frac{y^2}{(c^*)^2} - 1.$$  

Similarly, the spectrum of $L^+$ consists of curves:

$$(3.13) \quad x = -\frac{y^2}{(c^*)^2} - (1 - \alpha)(1 - k + \alpha k),$$

or

$$(3.14) \quad x = -\frac{y^2}{(c^*)^2} + \alpha$$

in the complex plane. Consequently, we have

$$\max \Re \sigma_e(L) = \max \{-\alpha, -1, -1 - (1 - \alpha)(1 - k + \alpha k), \alpha\} = \alpha > 0.$$  

Hence, by [7, 20], the steady state $U^*(\xi)$ of (3.1) is unstable in $C_0$.

□

Theorem 9 says that the $C_0$ norm for the initial perturbation is too large to stabilize the traveling wave solutions. To have further control on the exponential rates of functions in $C_0$, we introduce the following weighted Banach space. Let $\sigma_1, \sigma_2$ be two non-negative numbers and the space $C_{\sigma_1,\sigma_2}$ be defined as:

$$C_{\sigma_1,\sigma_2} = \{U(\xi) \mid U(\xi)(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}) \in C_0\},$$

with the norm

$$\| U \|_{C_{\sigma_1,\sigma_2}} = \sup_{\xi \in \mathbb{R}} \| U(\xi)(e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}) \|.$$  

We can also similarly define the Banach space $C^{(2)}_{\sigma_1,\sigma_2}$ as:

$$C^{(2)}_{\sigma_1,\sigma_2} = \{U \mid U(\cdot), U'(\cdot), U''(\cdot) \in C_{\sigma_1,\sigma_2}\}$$

with norm

$$\| U \|_{C^{(2)}_{\sigma_1,\sigma_2}} = \sup_{\xi \in \mathbb{R}} \Sigma_{i=0}^2 \left\| (e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}) \frac{d^i U(\xi)}{d\xi^i} \right\|.$$  

It can be easily verified that $C_{\sigma_1,\sigma_2}$ and $C^{(2)}_{\sigma_1,\sigma_2}$ are both Banach spaces.

We now restrict the initial conditions and the operator $L$ to the newly defined Banach space $C_{\sigma_1,\sigma_2}$ with $\sigma_1 \geq 0, \sigma_2 \geq 0$ and $\sigma_1^2 + \sigma_2^2 \neq 0$. To relate the operator $L$ in $C_0$ to an equivalent operator in $C_{\sigma_1,\sigma_2}$, we introduce the following translation operator $T : C_{\sigma_1,\sigma_2} \to C_0$ defined as

$$TV := (e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})V.$$  

The operator $T$ is thus linear, bounded and has a bounded inverse $T^{-1} : C_0 \to C_{\sigma_1,\sigma_2}$ with $T^{-1}V = (e^{\sigma_1 \xi} + e^{-\sigma_2 \xi})^{-1}V$.

Consider operator

$$(3.15) \quad \hat{L}V = TL^{-1}V.$$
One can easily see that $\hat{L}$ is a linear operator with domain $C^{(2)}(\mathbb{R}) \times C^{(2)}(\mathbb{R})$. By relation (3.15), considering $\mathcal{L}$ in $C_{\sigma_1, \sigma_2}$ is equivalent to considering $\hat{L}$ in $C_0$. The operator $\hat{L}$ can be explicitly written as

$$\hat{L} V = V_{\xi \xi} - (2g_1 + c^*) V_{\xi} + (2g_1^2 - g_2 + c^* g_1 + \frac{\partial F}{\partial U}(U^*)) V,$$

where $(2g_1 + c^*)$ and $(2g_1^2 - g_2 + c^* g_1)$ in (3.16) represent the matrices $(2g_1 + c^*)I$ and $(2g_1^2 - g_2 + c^* g_1)I$ with

$$g_1(\xi) = \frac{\sigma_1 e^{\sigma_1 \xi} - \sigma_2 e^{-\sigma_2 \xi}}{e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}},$$

$$g_2(\xi) = \frac{\sigma_1^2 e^{\sigma_1 \xi} + \sigma_2^2 e^{-\sigma_2 \xi}}{e^{\sigma_1 \xi} + e^{-\sigma_2 \xi}},$$

and further properties

$$\lim_{\xi \to \infty} g_1(\xi) = \sigma_1, \quad \lim_{\xi \to -\infty} g_1(\xi) = -\sigma_2;$$

$$\lim_{\xi \to \infty} g_2(\xi) = \sigma_2^2, \quad \lim_{\xi \to -\infty} g_2(\xi) = \sigma_2^2.$$ 

For the convenience of later use let $M(\xi) = 2g_1^2 - g_2 + c^* g_1 + \frac{\partial F}{\partial U}(U^*)$. We now locate the essential spectrum of the operator $\hat{L}$ in the space $C_0$.

**Lemma 10.** Let $c^* > 2\sqrt{\alpha}$ and if $\sigma_1$ and $\sigma_2$ satisfy

$$0 \leq \sigma_1 < -\sqrt{c^* + \sqrt{c^*^2 + 4\alpha}}, \quad c^* - \sqrt{c^*^2 + 4\alpha} < \sigma_2 < \frac{c^* + \sqrt{c^*^2 - 4\alpha}}{2},$$

then the essential spectrum of the operator $\hat{L}$ in the space $C_0(\mathbb{R}) \times C_0(\mathbb{R})$ is contained in a closed sector in the left half complex plane with vertex on the horizontal axis left to the origin. Outside this sector, there are only eigenvalues and the resolvent of $\hat{L}$.

**Proof.** Same as in the proof of Theorem 9, we first study the operator $\hat{L}$ at infinities. We have

$$\hat{L}^+ V = V_{\xi \xi} - (2\sigma_1 + c^*) V_{\xi} + (\sigma_1^2 + c^* \sigma_1 + \frac{\partial F}{\partial U}(U^*)) V,$$

$$\hat{L}^- V = V_{\xi \xi} - (-2\sigma_2 + c^*) V_{\xi} + (\sigma_2^2 + c^* \sigma_2 + \frac{\partial F}{\partial U}(U^*)) V,$$

where $\sigma_1^2 + c^* \sigma_1 + \frac{\partial F}{\partial U}(U^*)$ and $\sigma_2^2 + c^* \sigma_2 + \frac{\partial F}{\partial U}(U^*)$ are, respectively, the constant matrices

$$M^+ = \begin{bmatrix} \sigma_1^2 + c^* \sigma_1 - \alpha & 0 \\ 1 - k & \sigma_1^2 + c^* \sigma_1 - 1 \end{bmatrix},$$

and

$$M^- = \begin{bmatrix} \sigma_2^2 - c^* \sigma_2 - (1 - \alpha)(1 - k + k\alpha) & 1 - \alpha \\ 0 & \sigma_2^2 - c^* \sigma_2 + \alpha \end{bmatrix}.$$ 

The essential spectrum of the operator $\hat{L}^+$ consists of curves:

$$\det(-\zeta^2 I - i(2\sigma_1 + c^*) I \zeta + M^+) = 0.$$ 

Therefore,

$$-\zeta^2 - i(2\sigma_1 + c^*) \zeta + \sigma_1^2 + c^* \sigma_1 - \alpha - \lambda = 0,$$
or
\[ -\zeta^2 - i(2\sigma_1 + c^*)\zeta + \sigma_1^2 + c^*\sigma_1 + \epsilon_1 - 1 - \lambda = 0. \]

Equivalently, one has
\[ x = \frac{-y^2}{(2\sigma_1 + c^*)^2} + \sigma_1^2 + c^*\sigma_1 - \alpha \]
or
\[ x = \frac{-y^2}{(-2\sigma_2 + c^*)^2} + \sigma_2^2 - c^*\sigma_2 - (1 - \alpha)(1 - k + k\alpha), \]

Similarly, the essential spectrum of \( L^- \) consists of curves:
\[ x = \frac{-y^2}{(-2\sigma_2 + c^*)^2} + \sigma_2^2 - c^*\sigma_2 + \alpha. \]

Consequently, we have
\[
\text{max}\{\text{Re}\, \sigma_e(\mathcal{L})\} = \max\{\sigma_1^2 + c^*\sigma_1 - \alpha, \sigma_1^2 + c^*\sigma_1 - 1, \sigma_2^2 - c^*\sigma_2 + (1 - \alpha)(1 - k + k\alpha), \sigma_2^2 - c^*\sigma_2 - \alpha\}
\]

A simple calculation shows the validity of the Lemma.

**Corollary 11.** Assume that all the hypotheses in Lemma 10 hold, then the essential spectrum of the operator \( L \) in the space \( C_{\sigma_1, \sigma_2} \) lies in the left half complex plane.

**Proof.** The conclusion follows immediately from Lemma 10 and relation (3.15).

We now locate the eigenvalues of the operator \( L \) in the space \( C_{\sigma_1, \sigma_2} \). The next lemma shows that 0 is not an eigenvalue of the operator \( L \) in the space \( C_{\sigma_1, \sigma_2} \).

The idea of the proof is as follows. If there is an eigenfunction associated with the eigenvalue zero of the operator \( L \) in the space \( C_{\sigma_1, \sigma_2} \), then we can show the derivative of the traveling wave solution is larger than or equal to any multiple of this function by examining their asymptotics at infinities. But the boundedness of the derivative of the traveling wave solution makes this impossible.

**Lemma 12.** Assuming all the hypotheses in Lemma 10, then 0 is not an eigenvalue of the operator \( L \) in \( C_{\sigma_1, \sigma_2} \).

**Proof.** Let \( U^*(\xi) \) be a traveling wave solution of (3.1) found in Theorem 6. By taking Gateaux derivative we can verify that
\[
\mathcal{L}((U^*(\xi))') = 0.
\]
Suppose that there exists a nonzero function \( \tilde{V} \in C_{\sigma_1, \sigma_2} \) satisfying the equation
\[
\mathcal{L}\tilde{V} = 0.
\]
we then claim that the inequality \( |r\tilde{V}(\xi)| \leq (U^*)'(\xi) \) is consequently true for all \( r \in \mathbb{R} \) and \( \xi \in \mathbb{R} \). Writing \( S = \{ r \in \mathbb{R} | |r\tilde{V}(\xi)| \leq (U^*)'(\xi), \xi \in \mathbb{R} \} \), we verify the following properties of \( S \):

1. \( S \) is non-empty since \( 0 \in S \).
2. \( S \) is closed. Let \( r_i \in S, i = 1, 2, ... \) and \( r_i \to r \) as \( i \to +\infty \), then we will have \( |r_i\tilde{V}(\xi)| \leq (U^*)' \), which implies that \( |r\tilde{V}(\xi)| \leq (U^*)'(\xi) \), we therefore have \( r \in S \).
3. $S$ is open. Let $r \in S$, we will show that there exists a $\delta > 0$ such that $(r - \delta, r + \delta) \subset S$. We claim that $|rV(\xi)| \leq (U^*)'(\xi)$ implies $|rV(\xi)| < (U^*)'(\xi)$. In fact, let $W(\xi) = (U^*)'(\xi) - r\bar{V}(\xi)$ then $W(\xi) \geq 0$ for $\xi \in \mathbb{R}$ and it satisfies the following equation:

$$
\begin{align*}
&w''_1 - cw_1' + A_{11}w_1 + A_{12}w_2 = 0, \\
&w''_2 - cw_2' + A_{21}w_1 + A_{22}w_2 = 0,
\end{align*}
$$

(3.25)

where $A_{ij}$, $i, j = 1, 2$ are the entries of the Jacobian $\frac{\partial F}{\partial \xi}(U^*)$. The Maximum Principle implies that $W(\xi) = (w_1(\xi), w_2(\xi))^T > 0$ for $\xi \in \mathbb{R}$. This shows that $(U^*)'(\xi) - r\bar{V}(\xi) > 0$, $\xi \in \mathbb{R}$. Similarly we can show that $(U^*)'(\xi) + r\bar{V}(\xi) > 0$ for $\xi \in \mathbb{R}$. The claim then follows.

We next show that the claim further implies $|r\bar{V}(\xi)| < (U^*)'(\xi)$ as long as $\bar{r}$ is sufficiently close to $r$. According to condition (3.17) and the assumption that $\bar{V} \in C_{\sigma_1, \sigma_2}$ for any fixed $\bar{r} \in \mathbb{R}$, there exists a $N > 0$ sufficiently large such that $(e^{\sigma_1 t} + e^{-\sigma_2 t})((U^*)'(\xi) - r\bar{V}(\xi)) > 0$ for all $\xi \in (-\infty, -N]$, this implies that $(U^*)'(\xi) > r\bar{V}(\xi)$ also holds there. Furthermore, due to the claim and the boundedness of the functions $(U^*)'$ and $\bar{V}$, we can find a $\delta > 0$ such that for any $\bar{r} \in (-\delta + r, \delta + r)$, $(U^*)'(\xi) > r\bar{V}(\xi)$ on the finite interval $[-N, N]$. Now we fix $\bar{r} = \bar{r}$ and show $[(U^*)'(\xi) - r\bar{V}(\xi)] > 0$ for $\xi \in [N, +\infty)$.

Noting the entries of the diagonal of the matrix $\frac{\partial^2 F}{\partial \xi^2}(U^*(+\infty))$ are both negative, we can choose a vector $P_+ > 0$ such that (by increasing $N$ if necessary) $\frac{\partial^2 F}{\partial \xi^2}(U^*(\xi))P_+ < 0$ for $\xi \in [N, +\infty)$.

We have to deal with two cases:

Case A. If we already have $(U^*)'(\xi) - r\bar{V}(\xi) \geq 0$ for $\xi \geq N$, then the Maximum Principle implies that $(U^*)'(\xi) - r\bar{V}(\xi) > 0$ on $[N, +\infty)$. Analogly $(U^*)'(\xi) + r\bar{V}(\xi) > 0$ is also true for $\xi \in \mathbb{R}$. Consequently, $S$ is open.

Case B. If there is a point in the interval $(-\infty, -N)$ such that one of the components of vector $(U^*)'(\xi) - r\bar{V}(\xi)$ takes negative local minimum on this point, we consider function $W(\xi) = (U^*)'(\xi) - r\bar{V}(\xi) + \tau P_+$. The asymptotic rates of $(U^*)'$ and $\bar{V}$ at $+\infty$ imply that there is a sufficiently large $\tau > 0$ such that $W = (U^*)'(\xi) - r\bar{V}(\xi) + \tau P_+ \geq 0$ for $\xi \in [N, +\infty)$. We further assume that one of the components of $W(\xi)$, say $w_1$ for example, takes minimum at a finite point $\xi_2$ in $[N, +\infty)$. It is not hard to verify that there is a $\tau = \tau_2 > 0$ such that the corresponding $W(\xi)$ satisfying $w_1(\xi_2) = 0$ and $W(\xi) \geq 0$ for $\xi \in (-\infty, -N)$. For such $\tau_2$ on the one hand, we have

$$
\begin{align*}
\mathcal{L}W &= \bar{W}_{\xi_2} - c^*\bar{W}_\xi + \frac{\partial F}{\partial \xi}(U^*)\bar{W} \\
&= \tau_2 \frac{\partial F}{\partial \xi}(U^*)P_+ < 0,
\end{align*}
$$

(3.26)

on the other hand at $\xi = \xi_2$, the first component to the left hand side of (3.26) is larger than or equals zero. We then have a contradiction, and consequently $(U^*)'(\xi) - r\bar{V}(\xi) \geq 0$ for $\xi \in [N, +\infty)$. We are again in the situation described by case A. By a similar argument, we can show $(U^*)'(\xi) + r\bar{V}(\xi) \geq 0$ for $\xi \in [N, +\infty)$.

In summary, for any $\bar{r} \in (-\delta + r, \delta + r)$, we have $|r\bar{V}(\xi)| < (U^*)'(\xi)$, i.e., $S$ is open.

Now the set $S$ is a non-empty, open and closed subset of $\mathbb{R}$, then $S \equiv \mathbb{R}$, but this is impossible since $(U^*)'$ is bounded. Therefore the equation (3.24) does not have non-zero solution in $C_{\sigma_1, \sigma_2}$. □
The next lemma shows that there is no eigenvalue of $\mathcal{L}$ in $C_{\sigma_1, \sigma_2}$ with positive or zero real part.

**Lemma 13.** Let $C_0^\mathbb{C}$ be the complexified space of $C_0(\mathbb{R})$ and $\lambda$ be an eigenvalue of the operator $\mathcal{L}$ with $\underline{U} \in C_0^\mathbb{C}$ as its eigenfunction, then $\text{Re}\, \lambda < 0$.

**Proof.** Let $\lambda = \lambda_1 + \lambda_2i$ and $U(\xi) = U^1(\xi) + iU^2(\xi)$ for $\xi \in \mathbb{R}$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1^2 + \lambda_2^2 \neq 0$ and $U^1(\xi) \in C_0(\mathbb{R})$.

Consider the Cauchy problem (\[\text{(3.27)}\]),

$$V_t = \tilde{\mathcal{L}}V - \lambda_1 V, \quad V(\xi, 0) = U^1(\xi).$$

It is easy to verify that $V(\xi, t) = U^1(\xi) \cos(\lambda_2 t) - U^2(\xi) \sin(\lambda_2 t)$ is bounded and solves (\[\text{(3.27)}\]) for $\xi \in \mathbb{R}$ and $t \geq 0$. We may assume that at least one of the components of $V$ takes positive value at some point of $\xi$ and $t$, otherwise we can consider $-V$. Since the vector $T(U^*)(\xi) \to +\infty$ as $\xi \to -\infty$, similar to the proof of Lemma 12, we can choose a large $\xi_0 > 0$ such that

(\[\text{(3.28)}\])

$$M(\xi) P^+ < 0$$

for $\xi \geq \xi_0$, and at the same time

(\[\text{(3.29)}\])

$$V(\xi, t) < T(U^*)(\xi)$$

for $\xi \leq -\xi_0$ and $t \geq 0$, and $M(\xi) = 2g_1^2 - g_2 + c^* g_1 + \frac{\partial F}{\partial \sigma}(U^*)$ in (\[\text{(3.28)}\]). Moreover, we can choose a small $\bar{\epsilon} > 0$ such that

(\[\text{(3.30)}\])

$$(\epsilon^2 I - \bar{\epsilon}(2g_1 + c^*)I + M(\xi)) P^+ < 0.$$}

for $\xi \geq \xi_0$. We can choose an even larger $\xi_0$ if necessary such that at least one of the positive values of $V$ occurs inside the interval $(-\xi_0, \xi_0)$. The positivity of $T(U^*)(\xi)$ for $\xi \in \mathbb{R}$ implies that there is a $r > 0$ such that

(\[\text{(3.31)}\])

$$V(\xi, t) \leq rT(U^*)(\xi) \quad \text{for} \quad \xi \in [-\xi_0, \xi_0], \quad t \geq 0.$$}

Suitably adjusting $r$ such that the equality in (\[\text{(3.31)}\]) holds on at least one component at a point $(\xi_1, t_1)$ with $\xi_1 \in [-\xi_0, \xi_0]$ and $t_1 \geq 0$. Now let $\lambda_1 \geq 0$, we have

**Claim:** $V(\xi, t) \leq rT(U^*)(\xi)$ for all $\xi \in \mathbb{R}, \quad t \geq 0$.

The claim is already true for $\xi \leq \xi_0$ by the discussion above. Suppose the assertion is not true for $\xi > \xi_0$, then we can find a $\bar{\xi} > \xi_0$, and a $t_1 \geq 0$ such that $V(\bar{\xi}, t_1) > rT(U^*)(\bar{\xi})$. Since $Q^+(\xi) \equiv e^{\epsilon \xi} P^+ \to +\infty$ as $\xi \to +\infty$, there is a $\tilde{r} > 0$ such that $V(\xi, t) \leq \tilde{r} T(U^*)(\xi) + \tilde{r} Q^+(\xi)$ for all $\xi \geq \xi_0$ and $t \geq 0$, and for at least one $j (j = 1, 2, \ldots)$ and one $\xi_j \geq \xi_0$ and one $t_2 > 0$, we have the equality for the $j$-th component:

$$V_j(\xi_2, t_2) = \tilde{r} T(U^j_*)(\xi_2) + \tilde{r} Q_j^+(\xi_2).$$

Let $Y(\xi, t) = rT(U^j_*)(\xi) + rQ_j^+(\xi) - V(\xi, t)$, then $Y_j$ has the following properties: $Y_j(\xi_2, t_2) = 0$, $Y_j(\xi_0, t) > 0$, $Y_j(\xi, t) \geq 0$ for all $\xi \geq \xi_0$, $t \geq 0$. Hence $Y_j(\xi_2, t_2) \leq 0$, $Y_j(\xi_2, t_2) = 0$ and $Y_j(\xi, t) \geq 0$, and $Y_j(\xi, t)$ satisfies
\begin{align}
Y_{j,t} &= -V_{j,t} \\
&= -(\mathcal{L}V - \lambda_1 V)_j \\
&> (\mathcal{L}V + \lambda_1 V + \mathcal{L}r T(U^*)' + \mathcal{L}\tilde{r} Q^+ - \lambda_1 (r T(U^*)' + \tilde{r} Q^+))_j \\
&= (\mathcal{L}Y - \lambda_1 Y)_j \\
&= Y_{j,\xi}\xi - (2g_1 + c^*)Y_{j,\xi} + M_{j,1}Y_1 + M_{j,2}Y_2 - \lambda_1 Y_j,
\end{align}

owing to (3.30), \(\mathcal{L}r T(U^*)' = 0\) and \(r T(U^*)' + \tilde{r} Q^+ \geq 0\). However, we find that at \((\xi_0, t_0)\) the left hand side of the inequality (3.32) is smaller than or equal to zero, while the right hand side is larger than zero. We then have a contradiction. Therefore the claim is proved.

We next show that the above is also true even for \(r = 0\). In fact, let \(S = \{ r \geq 0 | V(\xi, t) \leq r T(U^*)'(\xi) \text{ for } \xi \in \mathbb{R}, t \geq 0 \}\), by the claim \(S \neq \emptyset\), then \(r_0 \doteq \min S \geq 0\).

We need to show \(r_0 = 0\). Suppose on the contrary that we have \(r_0 > 0\), then
\begin{equation}
V(\xi, t) \leq r_0 T(U^*)'(\xi) \text{ for } \xi \in \mathbb{R}, t \geq 0.
\end{equation}

Since \(T(U^*)'(\xi) \to +\infty\) as \(\xi \to -\infty\), the strict inequality holds at \(\xi \approx -\infty\). We first assume that an equality occurs in (3.33) at one of the components (say, the \(i\)-th component) at a point \((\hat{\xi}, \hat{t})\) with \(\hat{\xi} \in \mathbb{R}\) and \(\hat{t} \geq 0\). Moreover from
\begin{align}
Y_{i,t} &\geq (\mathcal{L}Y - \lambda_1 Y)_i \\
&= Y_{i,\xi}\xi - (2g_1 + c^*)Y_{i,\xi} + M_{i,1}(\xi)Y_1 + M_{i,2}(\xi)Y_2 - \lambda_1 Y_i
\end{align}

with \(Y(\xi, t) = r(U^*)'(\xi) - V(\xi, t)\) and \((M_{i,1}, M_{i,2})\) the \(i\)-th row of the matrix function \(\mathcal{M}(\xi)\). The positivity Lemma (23) implies that \(Y_i(\xi, t) > 0\) for \(\xi \in \mathbb{R}\) and \(t > \hat{t}\), further by the periodicity of \(V\), we have that \(Y_i(\xi, t) > 0\) for \(\xi \in \mathbb{R}\) and \(t > 0\). This contradicts with the existence of \(\hat{\xi}\) and shows that
\[ V(\xi, t) < r_0 T(U^*)'(\xi) \text{ for } \xi \in \mathbb{R}, t > 0. \]

Similar to the justification of the claim in the proof of Lemma 12, we can show there exists a sufficiently small \(\delta\) with \(r_0 \gg \delta > 0\), such that
\[ V(\xi, t) < (r_0 - \delta) T(U^*)'(\xi) \text{ for } \xi \in \mathbb{R}, t > 0. \]

It then follows that \(r_0 - \delta \in S\) thus contradicts with the definition of \(r_0\). Then we conclude that \(r_0 = 0\) but this again is impossible since by assumption a component of \(V\) is positive at some point \(\xi\) and \(t\). Hence we have proved the Lemma. \(\square\)

**Lemma 14.** Under the hypotheses of Lemma 10, the operator \(\mathcal{L}\) in \(C_0\) is sectorial.

**Proof.** See [7]. \(\square\)

**Theorem 15.** Under the hypotheses of [14] the operator \(\mathcal{L}\) generates an analytical semigroup in \(C_{\sigma_1, \sigma_2}\).

**Proof.** The conclusion follows from Lemma 14 and [7]. \(\square\)

We now state the stability theorem,

**Theorem 16.** Assume the hypotheses of Lemma 10. The traveling wave solution \(U^*\) of (1.2), with wave speed \(c^* > 2\sqrt{\alpha}\), is asymptotically stable according to norm \(\| \cdot \| := \| \cdot \|_{C_{\sigma_1, \sigma_2}}\). That is, if the initial condition \(U(\xi, 0) = \tilde{U}(\xi) \in C\) with \((\tilde{U}(\xi) - U^*(\xi)) \in C_{\sigma_1, \sigma_2}\) and \(\| \tilde{U} - U^* \|\) sufficiently small, then the solution \(U(\xi, t)\) to (3.3) exists uniquely for all \(t > 0\) and satisfies
\begin{equation}
\| U(\xi, t) - U^*(x + ct) \| \leq Me^{-bt},
\end{equation}

\(\square\)
where the constants $M > 0$, $b > 0$ are independent of $t$ and $U$.

**Proof.** Corollary 11, Lemma 10, Lemma 12 and 13 show that the spectrum of the operator $\mathcal{L}$ in the space $C_{\sigma_1, \sigma_2}$ is contained in an angular region in the left complex plane. Thus, the trivial solution of system (5.1) is linearly asymptotically stable. Theorem 15 further implies the local nonlinear stability of the traveling wave solutions ([2, 17, 20]). $\square$

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