Notes on the equality in SSA of entropy on CAR algebra

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Abstract
We prove a necessary and sufficient condition for the states which satisfy strong subadditivity of von Neumann entropy with equality on CAR algebra and we show an example when the equality holds but the state is not separable.

1 Introduction
A remarkable property of the von Neumann entropy
\[ S(D) = - \text{Tr} D \log D \] (1)
of a density operator \( D > 0 , \text{Tr} D = 1 \) on a Hilbert space \( \mathcal{H} \) is the strong subadditivity (SSA) which was proved by Lieb and Ruskai \[ S(D_{12}) + S(D_{23}) \geq S(D_{13}) + S(D_{2}). \] (2)
with a tripartite state \( D_{123} \) on the system \( \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \). It is interesting to find the states which saturate the SSA inequality because they are exactly the markovian states for tensor product systems. In an explicit characterisation of such quantum states was given.

For CAR algebras (we summarize their properties in the next section) the SSA inequality of entropy was showed by Araki and Moriya in \[ \text{I} \], but the question of the case of the equality is still open. We give an equivalent condition for the states which is already well-known for tensor product. We show a separable and a non-separable class of states which saturate the SSA inequality.

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2 CAR

In this section we summarize known properties of the algebra of the canonical anticommutation relation. The paper of Araki and Moriya [1] contains all what we need.

Assume that the unital C*-algebra $A_I$ is generated by the elements $\{ a_i : i \in I := \{1, 2, \ldots, n\} \}$ which satisfy the relations

\[
a_i a_j + a_j a_i = 0 \quad a_i a_j^* + a_j^* a_i = \delta_{i,j}
\]

for $i, j \in I$. It is easy to see that $A_I$ is the linear span of the identity and monomials of the form

\[
A_i^{(1)} A_i^{(2)} \cdots A_i^{(k)}
\]

where $1 \leq i(1) < i(2) < \cdots < i(k) \leq n$ and each factor $A_i^{(j)}$ is one of the four operators $a_i^{(j)}, a_i^{(j)} a_i^{(j)} a_i^{(j)} a_i^{(j)}, a_i^{(j)} a_i^{(j)} , a_i^{(j)} a_i^{(j)} .$

It is known that $A_I$ is isomorphic to a matrix algebra $M_{2n}(\mathbb{C}) \simeq M_2(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$. Namely, the relations

\[
e_{11}^{(i)} := a_i a_i^* \quad e_{12}^{(i)} := V_{i-1} a_i \quad e_{21}^{(i)} := V_{i-1}^* a_i^* \quad e_{22}^{(i)} := a_i^* a_i
\]

\[
V_i := \prod_{j=1}^{i} (I - 2a_j^* a_j)
\]

determine a family of mutually commuting $2 \times 2$ matrix units for $i \in I$. Since

\[
a_i = \prod_{j=1}^{i-1} \left( e_{11}^{(j)} - e_{22}^{(j)} \right) e_{12}^{(i)}
\]

the above matrix units generate $A_I$ and give an isomorphism between $A_I$ and $M_{2n}(\mathbb{C}) \otimes \cdots \otimes M_2(\mathbb{C})$:

\[
e_{i_1 j_1}^{(1)} e_{i_2 j_2}^{(2)} \cdots e_{i_n j_n}^{(n)} \leftrightarrow e_{i_1 j_1} \otimes e_{i_2 j_2} \otimes \cdots \otimes e_{i_n j_n}.
\]

(Here $e_{ij}$ stand for the standard matrix units in $M_2(\mathbb{C})$.) It follows from this isomorphism that $A_I$ has a unique tracial state $\tau$.

Let $J \subset I$. There exists a unique automorphism $\Theta_J$ of $A_I$ such that

\[
\Theta_J(a_i) = -a_i \quad \text{and} \quad \Theta_J(a_i^*) = -a_i^* \quad (i \in I) \quad \Theta_J(a_i) = a_i \quad \text{and} \quad \Theta_J(a_i^*) = a_i^* \quad (i \notin I).
\]

In particular, we write $\Theta$ instead of $\Theta_I$. The odd and even parts of $A_I$ are defined as

\[
A_J^+ := \{ a \in A_I : \Theta(a) = a \}, \quad A_J^- := \{ a \in A_I : \Theta(a) = -a \}.
\]
\( A_I^+ \) is a subalgebra but \( A_I^- \) is not. The graded commutation relation for CAR algebras is well known: if \( A \in A(K) \) and \( B \in A(L) \) where \( K \cap L = \emptyset \), then \( AB = \epsilon(A, B)BA \) where

\[
\epsilon(A, B) = \begin{cases} 
-1 & \text{if } A \text{ and } B \text{ are odd} \\
+1 & \text{otherwise.} 
\end{cases}
\]  

(6)

The map

\[
A \mapsto \frac{1}{2} (A + \Theta(A))
\]

(7)

is a conditional expectation of \( A_I \) onto \( A_I^+ \). We have \( A_I = A_I^+ + A_I^- \) and \( \Theta \) leaves the trace \( \tau \) invariant.

Let \( J \subset I \). Then \( A_J \subset A_I \) and there exists a unique conditional expectation \( E_I^J : A_I \to A_J \) which preserves the trace. This fact follows from generalities about conditional expectations or the isomorphism (4). Inspite of these, it is useful to have a construction for \( E_I^J \). The C*-algebra generated by the commuting subalgebras \( A_J \) and \( A_I^+ \setminus J \) is isomorphic to their tensor product. We have a conditional expectation

\[
F_1 : A_I \to A_J \otimes A_I^+ \setminus J, \quad F_1(A) = \frac{1}{2} (A + \Theta_I \setminus J(A))
\]

(8)

and another

\[
F_2 : A_J \otimes A_I^+ \setminus J \to A_J, \quad F_2(A \otimes B) = \tau(B) A.
\]

(9)

The composition \( F_2 \circ F_1 \) is \( E_I^J \). To have an example, assume that \( I = [1, 4] \), \( J = [1, 2] \) and consider the action of the above conditional expectations on terms like \( b_1b_2b_3b_4 \). \( F_1 \) keeps \( a_1a_2^*a_3^*a_4^*a_1a_2a_3a_4 \) fixed and \( F_2 \) sends it to \( a_1a_2^*a_2a_3a_4 \). Moreover, \( E_I^J \) sends \( a_1a_2^*a_2a_3a_4a_1a_2a_3a_4a_1a_2^*a_2a_3a_4a_1a_2a_3a_4a_1 \) to \( a_1a_2^*a_2a_3a_4 \). We make here two remarks. First, we have benefitted from the product property of the trace: If \( A \in A_{J_1}, B \in A_{J_2} \) for disjoint subsets \( J_1 \) and \( J_2 \) of \( I \), then

\[
\tau(AB) = \tau(A)\tau(B).
\]

(10)

Moreover, for arbitrary subsets \( J_1, J_2 \subset I \)

\[
E_I^J |_{A_{J_1}} \cdot A_{J_2} = E_I^{J_1 \cap J_2}_{J_1}
\]

(11)

holds. This means that we have a commuting square:

\[
\begin{array}{ccc}
A_I & \xrightarrow{E_I^J} & A_J \\
\downarrow{E_I^{J_1}} & & \downarrow{E_I^{J_2}} \\
A_{J_1} & & A_{J_2} \\
\end{array}
\]

\[
\begin{array}{ccc}
E_I^{J_1} & \xrightarrow{E_I^{J_2}} & E_I^{J_1 \cap J_2} \\
\downarrow{E_I^{J_1}} & & \downarrow{E_I^{J_2}} \\
A_{J_1 \cap J_2} & & A_{J_1 \cap J_2} \\
\end{array}
\]
3 Equality in SSA

Araki and Moriya proved that the strong subadditive property of the von Neumann entropy also holds for CAR systems [1].

**Theorem 1** For finite subsets $I$ and $J$ of $\mathbb{Z}^n$, the strong subadditivity (SSA) of $S$ holds for any state $\psi$ of $\mathcal{A}$:

$$S(\psi_{I\cup J}) - S(\psi_I) - S(\psi_J) + S(\psi_{I\cap J}) \leq 0$$

where $\psi_K$ denotes the restriction of $\psi$ to $\mathcal{A}(K)$, and $S$ is the von Neumann entropy.

To investigate the condition of the equality we need the following theorem, which can be found as Theorem 3.8 in the monograph of Ohya and Petz [2].

**Theorem 2** Let $\mathcal{A}$ be a finite quantum system, and $A, B, C \in \mathcal{A}$. Then the following inequality holds:

$$\text{Tr} e^{CT} \exp(-A)(e^B) \geq \text{Tr} e^{A+B+C}$$

where $T_A(K) = \int_0^\infty (t+A)^{-1} K(t+A)^{-1} \, dt$.

Now we can prove the following

**Proposition 3** Equality holds in SSA iff

$$\log D + \log D_{I\cap J} = \log D_I + \log D_J$$

where $D$ is the density matrix of $\omega$ and $D_K$ is its restriction by the conditional expectation onto the subalgebra $\mathcal{A}(K)$.

**Proof.** From the proof of the SSA we can observe that the equality holds if and only if

$$S(\omega, \omega \circ E_I^{I\cup J}) = S(\omega \circ E_J^{I\cup J}, \omega \circ E_I^{I\cup J})$$

where $\omega$ is a faithful normal state on $\mathcal{A}(I \cup J)$. We show that this condition is equivalent to the above mentioned one.

For the sufficiency let us consider a $D$ density matrix on $\mathcal{A}(I \cup J)$, ie. $D \in \mathcal{A}(I \cup J)$, $D > 0$ and $\tau(D) = 1$.

$$S(\omega, \omega \circ E_I^{I\cup J}) - S(\omega \circ E_J^{I\cup J}, \omega \circ E_I^{I\cup J}) =$$

$$\omega(\log D - \log E_I^{I\cup J}(D)) - \omega \circ E_J^{I\cup J}(\log E_I^{I\cup J}(D) - \log E_I^{I\cup J}(D)) =$$

$$\omega(\log D - \log E_I^{I\cup J}(D)) - \omega(\log E_I^{I\cup J}(D) - \log E_I^{I\cup J}(D)) =$$

$$\omega(\log D - \log D_I + \log D_J + \log D_{I\cap J}) = 0$$

where we used that $E_I^{I\cup J}(D)$ and $E_I^{I\cup J}(D)$ are elements of $\mathcal{A}_{I\cup J}$. This gives us the sufficiency. For the necessity we have
\[-S(D) + S(D_I) - S(D_{I\cap J}) + S(D_J) = \]
\[\tau(D(\log D - (\log D_I - \log D_{I\cap J} + \log D_J)) \geq \]
\[\tau(D \exp(\log D_I - \log D_{I\cap J} + \log D_J))\]

where we used the Klein inequality \(\tau(A(\log A - \log B)) \geq \tau(A-B)\) and equality holds if and only if \(A = B\).

By using the theorem above with \(A = -\log D_{I\cap J}, B = \log D_J\), and \(C = \log D_I\) we get the following inequality:

\[\tau(\exp(\log D_I - \log D_{I\cap J} + \log D_J)) \leq \tau \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} D_J(tI + D_{I\cap J})^{-1} \, dt \right)\]

(15)

By using that the conditional expectation is trace preserving i.e. \(\tau = \tau \circ E\) and the property \(E_{N}^M(ABC) = AE_{N}^M(B)C\) where \(A, C \in N\) and \(B \in M\) we get

\[\tau \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} D_J(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \circ E_{I\cup J}^I \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} D_J(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} E_{I\cup J}^I(D_J)(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} D_{I\cap J}(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \circ E_{J\cup J}^J \left( \int_0^\infty D_I(tI + D_{I\cap J})^{-1} D_{I\cap J}(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \left( \int_0^\infty E_{J\cup J}^J(D_I)(tI + D_{I\cap J})^{-1} D_{I\cap J}(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \left( \int_0^\infty D_{I\cap J}(tI + D_{I\cap J})^{-1} D_{I\cap J}(tI + D_{I\cap J})^{-1} \, dt \right) = \]
\[\tau \left( \int_0^\infty (D_{I\cap J})^2 (tI + D_{I\cap J})^{-2} \, dt \right) = \tau(D_{I\cap J})\]

In the last step we used the fact

\[\int_0^\infty \frac{\lambda^2}{(t + \lambda)^2} \, dt = \lambda\]

(16)

Substituted this result into the inequality above, we have

\[\tau(D - \exp(\log D_I - \log D_{I\cap J} + \log D_J)) \geq \tau(D) - \tau(D_{I\cap J}) = 0\]

(17)
Using the necessary and sufficient condition for the equality in the Klein inequality, we get

$$D = \exp(\log D_I - \log D_{I \cap J} + \log D_J)$$  \hspace{1cm} (18)

but $D = \exp(\log D)$ which give us the necessity and our proof is complete.

Araki and Moriya proved the following theorem in [4].

**Theorem 4** Let $I_1, I_2, \ldots$ be an arbitrary number of mutually disjoint subsets of $\mathbb{N}$ and $\omega_i$ be given state of $\mathcal{A}(I_i)$ for each $i$. A product state extension of $\omega_i$, $i = 1, 2, \ldots$ exists if and only if all states $\omega_i$ except at most one are even. It is unique if it exists.

With the help of this theorem we can prove easily the following

**Proposition 5** If $\omega$ is a separable state for $\mathcal{A}(I \setminus J)$, $\mathcal{A}(I \cap J)$, $\mathcal{A}(J \setminus I)$ ie.

$$\omega = \sum_i \lambda_i \omega_{1,i} \circ \omega_{2,i} \circ \omega_{3,i}$$  \hspace{1cm} (19)

with $\lambda_i > 0$, $\sum_i \lambda_i = 1$ where $\omega_{1,i}$, $\omega_{2,i}$ and $\omega_{3,i}$ are states on $\mathcal{A}(I \setminus J)$, $\mathcal{A}(I \cap J)$ and $\mathcal{A}(J \setminus I)$ respectively with orthogonal supports, then $\omega$ satisfy the SSA of entropy with equality.

**Proof.** If the product state extension $\omega_{1,i} \circ \omega_{2,i} \circ \omega_{3,i}$ exists, then among the marginal states $\omega_{1,i}$, $\omega_{2,i}$, $\omega_{3,i}$ at least two must be even for all $i$. It means the same condition for their density matrices $D_{1,i}$, $D_{2,i}$, $D_{3,i}$ so they commute with each other by the graded commutation relation. The density matrix of $\omega$ is given by

$$D = \sum_i \lambda_i D_{1,i} D_{2,i} D_{3,i}$$  \hspace{1cm} (20)

and its restrictions to $I$, $J$ and $I \cap J$ are $D = \sum_i \lambda_i D_{1,i} D_{2,i}$, $D = \sum_i \lambda_i D_{2,i} D_{3,i}$ and $D = \sum_i \lambda_i D_{1,i}$, respectively. For their entropies we have

$$S(D) = S(\sum_i \lambda_i D_{1,i} D_{2,i} D_{3,i}) = H(\lambda_i) + \sum_i \lambda_i S(D_{1,i} D_{2,i} D_{3,i}) = H(\lambda_i) + \sum_i \lambda_i (S(D_{1,i}) + S(D_{2,i}) + S(D_{3,i}))$$

where $H(\lambda_i)$ is the Shannon entropy of the classical distribution $\lambda_i > 0$, $\sum_i \lambda_i = 1$ and in the last step we have used the fact that the marginal densities commute with each other. Similarly we have

$$S(D_I) = H(\lambda_i) + \sum_i \lambda_i (S(D_{1,i}) + S(D_{2,i}))$$  \hspace{1cm} (21)

$$S(D_J) = H(\lambda_i) + \sum_i \lambda_i (S(D_{2,i}) + S(D_{3,i}))$$  \hspace{1cm} (22)

$$S(D_{I \cap J}) = H(\lambda_i) + \sum_i \lambda_i S(D_{2,i})$$  \hspace{1cm} (23)
Substituting the entropies, the equality in SSA is hold. ■

Contrary to tensor product systems in CAR systems there are nonseparable states which satisfy the SSA of entropy with equality as the next result shows.

**Proposition 6** Let consider the set partition \( I \cap J = \bigcup_i K_i \) where \( K_i \cap K_j = \emptyset \) if \( i \neq j \). The following density matrix satisfies the SSA with equality

\[
D = \sum_i \alpha_i A_i B_i C^+ \tag{24}
\]

where \( A_i \in \mathcal{A}(I \setminus J) \), \( B_i \in \mathcal{A}(K_i) \) and \( C^+ \in \mathcal{A}(J \setminus I)^+ \) (independent on \( i \)) are monomials and \( \alpha_i \) are normalization constants. If \( B_i \) is odd and \( A_i \) is even, \( A_i \) must be a product of two disjoint odd elements, i.e. there exist \( L_i^1 \), \( L_i^2 \) disjoint sets with \( L_i^1 \cup L_i^2 = I \setminus J \) and there exist \( a_i^1 \in \mathcal{A}(L_i^1)^- \) and \( a_i^2 \in \mathcal{A}(L_i^2)^- \) such that \( A_i = a_i^1 a_i^2 \).

**Proof.** We can observe that the density matrix above can contain the product of odd monomials living in disjoint subsets, so the necessary condition of the separability does not hold.

From the form of \( D \) we have the following reduced density matrices

\[
D_I = E_{i}^{I\cup J} = \sum_i \alpha_i A_i B_i \tau(C^+) \tag{25}
\]
\[
D_J = E_{j}^{I\cup J} = \sum_i \alpha_i \tau(A_i) B_i C^+ \tag{26}
\]
\[
D_{I\cap J} = E_{i}^{I\cap J} = \sum_i \alpha_i \tau(A_i) B_i \tau(C^+) \tag{27}
\]

We recall the graded commutation relation: if \( A \in \mathcal{A}(K) \) and \( B \in \mathcal{A}(L) \) where \( K \cap L = \emptyset \), then \( AB = \epsilon(A, B)BA \) where

\[
\epsilon(A, B) = \begin{cases}
-1 & \text{if } A \text{ and } B \text{ are odd} \\
+1 & \text{otherwise}
\end{cases}
\]

We show that \( D_I \) and \( D_J \) commute.

\[
D_ID_J = \left( \sum_i \alpha_i A_i B_i \tau(C^+) \right) \left( \sum_j \alpha_j \tau(A_j) B_j C^+ \right)
= \sum_{i,j} \alpha_i \alpha_j \tau(C^+) \tau(A_j) \epsilon(B_i, B_j) \epsilon(A_i, B_j) \epsilon(B_i, C^+) \epsilon(A_i, C^+) B_j C^+ A_i B_i
= \left( \sum_j \alpha_j \tau(A_j) B_j C^+ \right) \left( \sum_i \alpha_i A_i B_i \tau(C^+) \right) = D_J D_I
\]

where we used that \( \epsilon(B_i, C^+) = \epsilon(A_i, C^+) = 1 \) since \( C^+ \) is even. If \( B_j \) is even we have \( \epsilon(A_i, B_j) = \epsilon(B_i, B_j) = 1 \). If \( B_j \) is odd and \( A_j \) is even, our construction gives \( A_j = a_j^1 a_j^2 \), where \( a_j^1 \) and \( a_j^2 \) are odd elements living in disjoint algebras and by using the fact that \( \tau \) is an even product state we have \( \tau(A_j) = \tau(a_j^1) \tau(a_j^2) = 0 \). If \( A_j \) is odd we have \( \tau(A_j) = 0 \) immediately.
A similar computation shows that $D$ and $D_{I\cap J}$ also commute.

\[
DD_{I\cap J} = \left( \sum_i \alpha_i A_i B_i C^+ \right) \left( \sum_j \alpha_j \tau(A_j) B_j \tau(C^+) \right)
\]

\[
= \sum_{i,j} \alpha_i \alpha_j \tau(C^+) \tau(A_j) \epsilon(C^+, B_j) \epsilon(A_i, B_j) B_j A_i B_i C^+
\]

\[
= \left( \sum_j \alpha_j \tau(A_j) B_j \tau(C^+) \right) \left( \sum_i \alpha_i A_i B_i C^+ \right) = D_{I\cap J} D
\]

Since $\epsilon(C^+, B_j) = 1$ because $C^+$ is even. If $B_j$ is even $\epsilon(A_i, B_j) = \epsilon(B_i, B_j) = 1$.

If $B_j$ is odd and $A_j$ is even we have $\tau(A_j) = \tau(a_1^j) \tau(a_2^j) = 0$ by our construction.

If $A_j$ is odd $\tau(A_j) = 0$ is hold immediately.

Since $C^+$ and $B_j$ commute for all $j$ it is easy to see that

\[
DD_{I\cap J} = D_I D_J
\]

which gives

\[
\log(DD_{I\cap J}) = \log(D_I D_J).
\]

By using our commutation relations proved above, we have

\[
\log D + \log D_{I\cap J} = \log D_I + \log D_J
\]

which is equivalent condition to the equality in the SSA.

With this we gave an example for non-separable states which satisfy the SSA inequality of entropy with equality.

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