PRICING WITH COHERENT RISK

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Abstract. This paper deals with applications of coherent risk measures to pricing in incomplete markets. Namely, we study the No Good Deals pricing technique based on coherent risk. Two forms of this technique are presented: one defines a good deal as a trade with negative risk; the other one defines a good deal as a trade with unusually high RAROC. For each technique, the fundamental theorem of asset pricing and the form of the fair price interval are presented. The model considered includes static as well as dynamic models, models with an infinite number of assets, models with transaction costs, and models with portfolio constraints. In particular, we prove that in a model with proportional transaction costs the fair price interval converges to the fair price interval in a frictionless model as the coefficient of transaction costs tends to zero.

Moreover, we study some problems in the “pure” theory of risk measures: we present a simple geometric solution of the capital allocation problem and apply it to define the coherent risk contribution.

The mathematical tools employed are probability theory, functional analysis, and finite-dimensional convex analysis.

Key words and phrases: Capital allocation, coherent risk measures, extreme measures, generating set, No Good Deals, RAROC, risk contribution, risk-neutral measures, support function, Tail V@R, transaction costs, Weighted V@R.

1 Introduction

1. Overview. The three basic pillars of finance are:
   - optimal investment;
   - pricing and hedging;
   - risk measurement and management.

The most well-known financial theories related to the first pillar are the Markowitz mean-variance analysis and Sharpe’s CAPM, which are often termed the “first revolution in finance”. The most well-known result related to the second pillar is the Black–Scholes–Merton formula, which is often termed the “second revolution in finance”. Recently a very important innovation has appeared in connection with the third pillar. In 1997, Artzner, Delbaen, Eber and Heath [4], [5] introduced the concept of a coherent risk measure as a new way of measuring risk. Since 1997, the theory of coherent risk measures has rapidly been evolving and is already termed in some sources the “third revolution in finance” (see [52]). Let us mention, in particular, the papers [1], [3], [20], [31], [32], [38], [40], [53].
and the reviews [21], [33; Ch. 4], [47]. Currently, one of the major tasks is the problem of proper risk measurement in the dynamic setting; see, in particular, [12], [25], [36], [45], and [46].

The theory of coherent risk measures is important not only for risk measurement. Indeed, risk (≈ uncertainty) is at the very basis of the whole finance, and therefore, a new way of looking at risk yields new approaches to other problems of finance, in particular, to those related to the first and the second pillars. Nowadays, more and more research is aimed at applications of coherent risk measures to other problems of finance.

One of the major goals of modern financial mathematics is providing adequate price bounds for derivative contracts in incomplete markets. It is known that No Arbitrage price bounds in incomplete markets are typically unacceptably wide, and fundamentally new ideas are required to narrow these bounds. Recently, a promising approach to this problem termed No Good Deals (NGD) pricing has been proposed in [6], [17]. Let us illustrate its idea by an example. Consider a contract that with probability 1/2 yields nothing and with probability 1/2 yields 1000 USD. The No Arbitrage (NA) price interval for this contract is (0, 1000). But if the price of the contract is, for instance, 15 USD, then everyone would be willing to buy it, and the demand would not match the supply. Thus, 15 USD is an unrealistic price because it yields a good deal, i.e. a trade that is attractive to most market participants. The technique of the NGD pricing is based on the assumption that good deals do not exist.

A problem that arises immediately is how to define a good deal. There is no canonical answer, and several approaches have been proposed in the literature. Cochrane and Saá-Requejo [17] defined a good deal as a trade with unusually high Sharpe ratio, Bernardo and Ledoit [6] based their definition on another gain to loss ratio, while Černý and Hodges [11] proposed a generalization of both definitions (see also the paper [7] by Bjork and Slinko, which extends the results of [17]).

The technique of the NGD pricing can also be motivated as follows. When a trader sells a contract, he/she would charge for it a price, with which he/she will be able to superreplicate the contract. In theory the superreplication is typically understood almost surely, but in practice an agent looks for an offsetting position such that the risk of his/her overall portfolio would stay within the limits prescribed by his/her management (the almost sure superreplication is virtually impossible in practice). These considerations lead to the NGD pricing with a good deal defined as a trade with negative risk. The corresponding pricing technique has already been considered in several papers. Carr, Geman, and Madan [9] (see also the review paper [10]) studied this technique in a probabilistic framework (although they do not use the term “good deal”), while Jaschke and Küchler [35] studied this technique in a topological space framework in the spirit of Harrison and Kreps [34] (see also the paper [51] by Staum, which extends the results of [35]). Furthermore, Larsen, Pirvu, Shreve, and Tütüncü [41] considered pricing based on convex risk measures instead of coherent ones (convex risk measures were introduced by Föllmer and Schied [31]). Roorda, Schumacher, and Engwerda [46] studied pricing in the multiperiod model using as a basis dynamic coherent risk measures instead of static ones.

2. **Goal of the paper.** This is the first of a series of papers dealing with applications of coherent risk measures to the basic problems of finance (the other paper in the series is [15]). The basic idea behind the series is:
In this paper, we study applications to pricing in incomplete markets. Our approach is similar to that of [9], but [9] assumes an unrealistic world of a finite state space and a finite set of probabilistic scenarios defining a coherent risk measure (most natural coherent risk measures are defined through an infinite set of probabilistic scenarios; see Subsection 2.1). Our model is general in the sense that we consider an arbitrary \( \Omega \) and a general class of coherent risk measures (satisfying only a sort of compactness condition). Moreover, our approach applies to dynamic models, to models with an infinite number of assets, to models with transaction costs, and to models with convex portfolio constraints. Within this general model, we prove the Fundamental Theorem of Asset Pricing (Theorem 3.4) and provide the form of the fair price interval of a contingent claim (Corollary 3.6). We confine ourselves to static risk measures.

A problem that has attracted attention in several papers is as follows. Consider a model with proportional transaction costs. Is it true that the upper (resp., lower) price of a contingent claim in this model tends to the upper (resp., lower) price of this claim in the frictionless model as the coefficient of transaction costs tends to zero? It was shown in [14], [19], [42], and [50] that, for NA prices, the answer to this question is negative already in the Black-Scholes model (the contingent claim considered in these papers is a European call option). This result might be interpreted as follows: the NA technique is useless in continuous-time models with transaction costs. In this paper (Theorem 3.17), we prove that, for NGD prices, the answer to the above question is positive. This is done within a framework of a general model (the price follows an arbitrary process) with an infinite number of assets and an arbitrary contingent claim (satisfying only some integrability condition). The advantage of the NGD pricing is not only that this result is true, but also that its proof is very short.

Furthermore, we introduce a new variant of pricing based on coherent risk, which we call the RAROC-based NGD pricing. The idea is to define a good deal as a trade with unusually high Risk-Adjusted Return on Capital (RAROC), where RAROC is defined through coherent risk. On the mathematical side, this technique is reduced to the standard NGD pricing (with the original risk measure replaced by another one).

Although this series of papers deals primarily with applications of coherent risk measures to problems of finance, we also establish some results and give several definitions related to “pure” risk measures (these are needed for applications). In particular, we introduce the notion of an extreme measure. The results of this paper and [15] show that this notion is very convenient and important; it appears in the outcomes of several pricing techniques proposed in [15] and in considerations of the equilibrium problem in [15]. In the present paper, we provide a solution of the capital allocation problem in terms of extreme measures (Theorem 2.12). Let us remark that this problem was considered in [21], [24], [28], [39], [44], and [53].

Parallel with the measurement of outstanding risks, a very important problem is measuring the risk contribution of a subportfolio to a “big” portfolio. Based on our solution of the capital allocation problem, we propose several equivalent definitions of the coherent risk contribution.

Another notion we introduce is the notion of a generator. It establishes a bridge between coherent risks and convex analysis, opening the way for geometry. In particular, we provide (see Figure 1) a geometric solution of the capital allocation problem (thus there are two solutions: a probabilistic one is given in terms of extreme measures, while a geometric one is given in terms of generators). We also provide a geometric solution of the pricing and hedging problem (Proposition 3.21) for a model with a finite number
of assets. Furthermore, we provide in geometric solutions of several optimization problems, optimality pricing problems, and the equilibrium problem. In fact, for most problems considered in this series of papers, we provide two sorts of results:

- a geometric result applicable to a model with a finite number of assets is given in terms of generators;
- a probabilistic result applicable to a general model is typically given in terms of extreme measures.

3. Structure of the paper. Section 2 deals with “pure” risk measures rather than with their applications. Subsection 2.1 recalls some basic definitions related to coherent risks. In Subsection 2.2 we introduce the $L^1$-spaces associated with a coherent risk measure (these are employed in the technical conditions in theorems below). Subsection 2.3 presents the definition of an extreme measure. In Subsection 2.4 we provide a solution of the capital allocation problem. Subsection 2.5 deals with equivalent definitions of risk contribution.

Section 3 is related to the NGD pricing. In Subsections 3.1 and 3.2, we study the ordinary and the RAROC-based forms of this technique, respectively. The model considered is a general one, and in Subsections 3.3–3.5 we consider some particular cases of this model: a static model with a finite number of assets (for which fair price intervals admit a simple geometric description; see Figure 3), a continuous-time dynamic model, and a continuous-time dynamic model with transaction costs. Furthermore, in Subsection 3.6 we provide a geometric solution of the hedging problem for a static model with a finite number of assets (see Figure 4).

Acknowledgement. I am thankful to D.B. Madan for valuable discussions and important advice.

2 Coherent Risk Measures

2.1 Basic Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. The following definition was introduced in [4], [5]. These papers considered only a finite $\Omega$, in which case the continuity axiom (e) is not needed. It was added for a general $\Omega$ by Delbaen [20].

**Definition 2.1.** A coherent utility function on $L^\infty$ is a map $u : L^\infty \to \mathbb{R}$ with the properties:

(a) (Superadditivity) $u(X + Y) \geq u(X) + u(Y)$;
(b) (Monotonicity) If $X \leq Y$, then $u(X) \leq u(Y)$;
(c) (Positive homogeneity) $u(\lambda X) = \lambda u(X)$ for $\lambda \in \mathbb{R}_+$;
(d) (Translation invariance) $u(X + m) = u(X) + m$ for $m \in \mathbb{R}$;
(e) (Fatou property) If $|X_n| \leq 1$, $X_n \mathbb{P} \Rightarrow X$, then $u(X) \geq \limsup_n u(X_n)$.

The corresponding coherent risk measure is $\rho(X) = -u(X)$.

**Remark.** Typically, a coherent risk measure is defined only via conditions (a)–(d), and then one speaks about coherent risk measures with the Fatou property. However, only such risk measures are useful, and for this reason we find it more convenient to add (e) as a basic axiom.
The theorem below was established in [5] for the case of a finite \( \Omega \) (in this case the axiom (e) is not needed) and in [20] for the general case. We denote by \( \mathcal{P} \) the set of probability measures on \( \mathcal{F} \) that are absolutely continuous with respect to \( \mathcal{P} \). Throughout the paper, we identify measures from \( \mathcal{P} \) (these are typically denoted by \( \mathcal{Q} \)) with their densities with respect to \( \mathcal{P} \) (these are typically denoted by \( Z \)).

**Theorem 2.2 (Basic representation theorem).** A function \( u \) satisfies conditions (a)–(e) if and only if there exists a nonempty set \( D \subseteq \mathcal{P} \) such that

\[
\inf_{Q \in D} \mathbb{E}_Q X, \quad X \in L^\infty.
\]

So far, a coherent risk measure has been defined on bounded random variables. Let us ask ourselves the following question: Are “financial” random variables like the increment of a price of some asset indeed bounded? The right way to address this question is to split it into two parts:

- Are “financial” random variables bounded in practice?
- Are “financial” random variables bounded in theory?

The answer to the first question is positive (clearly, everything is bounded by the number of the atoms in the universe). The answer to the second question is negative because most distributions used in theory (like the lognormal one) are unbounded. So, as we are dealing with theory, we need to extend coherent risk measures to the space \( L^0 \) of all random variables. It is hopeless to axiomatize the notion of a risk measure on \( L^0 \) and then to obtain the corresponding representation theorem. Instead, we take representation (2.1) as the basis and extend it to \( L^0 \).

**Definition 2.3.** A coherent utility function on \( L^0 \) is a map \( u : L^0 \rightarrow [\infty, \infty] \) defined as

\[
u(X) = \inf_{Q \in D} \mathbb{E}_Q X, \quad X \in L^0,
\]

where \( D \subseteq \mathcal{P} \) and \( \mathbb{E}_Q X \) is understood as \( \mathbb{E}_Q X^+ - \mathbb{E}_Q X^- \) with the convention \( \infty - \infty = -\infty \). The corresponding coherent risk measure is \( \rho(X) = -u(X) \).

Clearly, a set \( D \), for which representations (2.1) and (2.2) are true, is not unique. However, there exists the largest such set given by \( \{ Q \in \mathcal{P} : \mathbb{E}_Q X \geq u(X) \text{ for any } X \} \). We introduce the following definition.

**Definition 2.4.** We will call the largest set, for which (2.1) (resp., (2.2)) is true, the determining set of \( u \).

**Remark.** Clearly, the determining set is convex. For coherent utility functions on \( L^\infty \), it is also \( L^1 \)-closed. However, for coherent utility functions on \( L^0 \), it is not necessarily \( L^1 \)-closed. As an example, take a positive unbounded random variable \( X_0 \) such that \( \mathbb{P}(X_0 = 0) > 0 \) and consider \( D_0 = \{ Q \in \mathcal{P} : \mathbb{E}_Q X_0 = 1 \} \). Clearly, the determining set \( D \) of the coherent utility function \( u(X) = \inf_{Q \in D_0} \mathbb{E}_Q X \) satisfies \( D_0 \subseteq D \subseteq \{ Q \in \mathcal{P} : \mathbb{E}_Q X_0 \geq 1 \} \). On the other hand, the \( L^1 \)-closure of \( D_0 \) contains a measure \( Q_0 \) concentrated on \( \{ X_0 = 0 \} \).

**Important Remark.** Let \( D \) be an \( L^1 \)-closed convex subset of \( \mathcal{P} \). (Let us note that a particularly important case is where \( D \) is \( L^1 \)-closed, convex, and uniformly integrable; this condition will be needed in a number of places below). Define a coherent utility function \( u \)
by \( (2.2) \). Then \( \mathcal{D} \) is the determining set of \( u \). Indeed, assume that the determining set \( \tilde{\mathcal{D}} \) is greater than \( \mathcal{D} \), i.e. there exists \( Q_0 \in \tilde{\mathcal{D}} \setminus \mathcal{D} \). Then, by the Hahn-Banach theorem, we can find \( X_0 \in L^\infty \) such that \( \mathbb{E}_{Q_0} X_0 < \inf_{Q \in \mathcal{D}} \mathbb{E}_Q X \), which is a contradiction. The same argument shows that \( \mathcal{D} \) is also the determining set of the restriction of \( u \) to \( L^\infty \).

In what follows, we will always consider coherent utility functions on \( L^0 \).

**Example 2.5.** (i) Tail \( \text{VaR} \) (the terms Average \( \text{VaR} \), Conditional \( \text{VaR} \), and Expected Shortfall are also used) is the risk measure corresponding to the coherent utility function

\[
u_\lambda(X) = \inf_{Q \in \mathcal{D}_\lambda} \mathbb{E}_Q X,
\]

where \( \lambda \in [0, 1] \) and

\[
\mathcal{D}_\lambda = \left\{ Q \in \mathcal{P} : \frac{dQ}{dP} \leq \lambda^{-1} \right\}.
\]

In particular, if \( \lambda = 0 \), then the corresponding coherent utility function has the form \( u(X) = \text{essinf}_\omega X(\omega) \). For more information on Tail \( \text{VaR} \), see \[3\], \[20; Sect. 6\], \[21; Sect. 7\], \[33; Sect. 4.4\], \[47; Sect. 1.3\].

(ii) Weighted \( \text{VaR} \) on \( L^\infty \) (the term spectral risk measure is also used) is the risk measure corresponding to the coherent utility function

\[
u_\mu(X) = \int_{[0,1]} u_\lambda(X) \mu(d\lambda), \quad X \in L^\infty,
\]

where \( \mu \) is a probability measure on \([0,1]\).

Weighted \( \text{VaR} \) on \( L^0 \) is the risk measure corresponding to the coherent utility function

\[
u_\mu(X) = \inf_{Q \in \mathcal{D}_\mu} \mathbb{E}_Q X, \quad X \in L^0,
\]

where \( \mathcal{D}_\mu \) is the determining set of \( u_\mu \) on \( L^\infty \).

Let us remark that, under some regularity conditions on \( \mu \), Weighted \( \text{VaR} \) possesses some nice properties that are not shared by Tail \( \text{VaR} \). In a sense, it is “smoother” than Tail \( \text{VaR} \). We consider Weighted \( \text{VaR} \) as one of the most important classes (or maybe the most important class) of coherent risk measures. For a detailed study of this risk measure, see \[1\], \[2\], \[26\], \[40\] as well as the paper \[16\], which is in some sense the continuation of the present paper.

\[\square\]

### 2.2 Spaces \( L^1_w \) and \( L^1_s \)

For a subset \( \mathcal{D} \) of \( \mathcal{P} \), we introduce the weak and strong \( L^1 \)-spaces

\[
L^1_w(\mathcal{D}) = \left\{ X \in L^0 : u(X) > -\infty, u(-X) > -\infty \right\},
\]

\[
L^1_s(\mathcal{D}) = \left\{ X \in L^0 : \lim_{n \to \infty} \sup_{Q \in \mathcal{D}} \mathbb{E}_Q |X| I(|X| > n) = 0 \right\}.
\]

Clearly, \( L^1_s(\mathcal{D}) \subseteq L^1_w(\mathcal{D}) \). If \( \mathcal{D} = \{ Q \} \) is a singleton, then \( L^1_w(\mathcal{D}) = L^1_s(\mathcal{D}) = L^1(Q) \), which motivates the notation.

In general, \( L^1_s(\mathcal{D}) \) might be strictly smaller than \( L^1_w(\mathcal{D}) \). Indeed, let \( X_0 \) be a positive unbounded random variable with \( \mathbb{P}(X_0 = 0) > 0 \) and let \( \mathcal{D} = \{ Q \in \mathcal{P} : \mathbb{E}_Q X_0 = 1 \} \). Then \( X_0 \in L^1_w(\mathcal{D}) \), but \( X_0 \notin L^1_s(\mathcal{D}) \). (One can also construct a similar counterexample with an \( L^1 \)-closed set \( \mathcal{D} \); see Example 2.11). However, as shown by the proposition below, in most natural situations weak and strong \( L^1 \)-spaces coincide.
Proposition 2.6. (i) If $\mathcal{D}_\lambda$ is the determining set of Tail VaR (see Example 2.5 (i)) with $\lambda \in (0, 1]$, then $L^1_w(\mathcal{D}_\lambda) = L^1_s(\mathcal{D}_\lambda)$.

(ii) If $\mathcal{D}_\mu$ is the determining set of Weighted VaR (see Example 2.5 (ii)) with $\mu$ concentrated on $(0, 1]$, then $L^1_w(\mathcal{D}_\mu) = L^1_s(\mathcal{D}_\mu)$.

(iii) If all the densities from $\mathcal{D}$ are bounded by a single constant and $P \in \mathcal{D}$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.

(iv) If $\mathcal{D}$ is a convex combination $\sum_{n=1}^N a_n \mathcal{D}_n$, where $\mathcal{D}_1, \ldots, \mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n) = L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.

(v) If $\mathcal{D} = \text{conv}(\mathcal{D}_1, \ldots, \mathcal{D}_N)$, where $\mathcal{D}_1, \ldots, \mathcal{D}_N$ are such that $L^1_w(\mathcal{D}_n) = L^1_s(\mathcal{D}_n)$, then $L^1_w(\mathcal{D}) = L^1_s(\mathcal{D})$.

Lemma 2.7. If $\mu$ is a convex combination $\sum_{n=1}^\infty a_n \delta_{\lambda_n}$, where $\lambda_n \in (0, 1]$, then the determining set $\mathcal{D}_\mu$ of Weighted VaR corresponding to $\mu$ has the form $\sum_{n=1}^\infty a_n \mathcal{D}_{\lambda_n}$, where $\mathcal{D}_\lambda$ is given by (2.3).

Proof. Denote $\sum_n a_n \mathcal{D}_{\lambda_n}$ by $\mathcal{D}$. Clearly, $\mathcal{D}$ is convex. Fix $X \in L^\infty$. It is easy to see that, for any $n$, the minimum of expectations of $\mathbb{E}XZ$ over $Z \in \mathcal{D}_{\lambda_n}$ is attained (for more details, see [15] Prop. 2.7). Hence, the minimum of expectations $\mathbb{E}P XZ$ over $Z \in \mathcal{D}$ is attained. By the James theorem (see [20]), $\mathcal{D}$ is weakly compact. As it is convex, an application of the Hahn-Banach theorem shows that it is $L^1$-closed.

Obviously, $u_\mu(X) = \inf_{Q \in \mathcal{D}} \mathbb{E}_Q X$ for any $X \in L^\infty$. Taking into account the Important Remark following Definition 2.3, we get $\mathcal{D}_\mu = \mathcal{D}$.

Proof of Proposition 2.6. The only nontrivial statement is (ii). In order to prove it, consider the measures $\tilde{\mu} = \sum_{k=1}^\infty a_k \delta_{2^{-k}}, \quad \bar{\mu} = \sum_{k=1}^\infty a_k \delta_{2^{-k+1}}$, where $a_k = \mu((2^{-k}, 2^{-k+1}))$. As $u_{\tilde{\mu}} \leq u_\mu \leq u_{\bar{\mu}}$, we have $\mathcal{D}_{\tilde{\mu}} \supseteq \mathcal{D}_\mu \supseteq \mathcal{D}_{\bar{\mu}}$. By Lemma 2.7,

$$\mathcal{D}_{\tilde{\mu}} = \left\{ \sum_{k=1}^\infty a_k Z_k : Z_k \in \mathcal{D}_{2^{-k}} \right\}, \quad \mathcal{D}_{\bar{\mu}} = \left\{ \sum_{k=1}^\infty a_k Z_k : Z_k \in \mathcal{D}_{2^{-k+1}} \right\}.$$

Take $X \in L^1_w(\mathcal{D}_{\tilde{\mu}})$. Consider $Z_k = 2^{k-1}I(X < q_k) + c_k I(X = q_k)$, where $q_k$ is the $2^{-k+1}$-quantile of $X$ and $c_k$ is chosen in such a way that $\mathbb{E}P Z_k = 1$. Then

$$\mathbb{E}P Z_k X = \min_{Z \in \mathcal{D}_{2^{-k+1}}} \mathbb{E}P Z X.$$ 

The density $Z_0 = \sum_{k=1}^\infty a_k Z_k$ belongs to $\mathcal{D}_{\bar{\mu}}$ and

$$\mathbb{E}P Z_0 X = \min_{Z \in \mathcal{D}_{\bar{\mu}}} \mathbb{E}P Z X.$$ 

In view of the inclusion $X \in L^1_w(\mathcal{D}_{\mu}) \subseteq L^1_w(\mathcal{D}_{\tilde{\mu}})$, the latter quantity is finite. Thus,

$$\sum_{k=1}^\infty a_k \min_{Z \in \mathcal{D}_{2^{-k+1}}} \mathbb{E}P Z X > -\infty,$$

which implies that

$$\sum_{k=1}^\infty a_k \min_{Z \in \mathcal{D}_{2^{-k+1}}} \mathbb{E}P Z(-X^-) > -\infty.$$ 

The same estimate is true for $X^+$, and therefore,

$$\sum_{k=1}^\infty a_k \sup_{Z \in \mathcal{D}_{2^{-k}}} \mathbb{E}P Z|X| \leq 2 \sum_{k=1}^\infty a_k \sup_{Z \in \mathcal{D}_{2^{-k+1}}} \mathbb{E}P Z|X| < \infty. \quad (2.4)$$
It is clear that $X \in L^1$, and thus, for each $k$,
\[
\sup_{Z \in \mathcal{D}_{2^{-k}}} \mathbb{E}_P Z |X| I(|X| > n) \leq 2^k \mathbb{E}_P |X| I(|X| > n) \xrightarrow{n \to \infty} 0.
\]
This, combined with (2.4), yields
\[
\sup_{Z \in \mathcal{D}_n} \mathbb{E}_P Z |X| I(|X| > n) \leq \sup_{Z \in \mathcal{D}_n} \mathbb{E}_P Z |X| I(|X| > n)
= \sum_{k=1}^\infty a_k \sup_{Z \in \mathcal{D}_{2^{-k}}} \mathbb{E}_P Z |X| I(|X| > n) \xrightarrow{n \to \infty} 0.
\]

### 2.3 Extreme Measures

**Definition 2.8.** Let $u$ be a coherent utility function with the determining set $\mathcal{D}$. Let $X \in L^0$. We will call a measure $Q \in \mathcal{D}$ an *extreme measure* for $X$ if $\mathbb{E}_Q X = u(X) \in (-\infty, \infty)$.

The set of extreme measures will be denoted by $\mathcal{X}_D(X)$.

Let us recall some general facts related to the weak topology on $L^1$. The *weak topology* on $L^1$ is induced by the duality between $L^1$ and $L^\infty$ and is usually denoted as $\sigma(L^1, L^\infty)$. The Dunford-Pettis criterion states that a set $\mathcal{D} \subseteq \mathcal{P}$ is weakly compact if and only if it is weakly closed and uniformly integrable. Furthermore, an application of the Hahn-Banach theorem shows that a convex set $\mathcal{D} \subseteq \mathcal{P}$ is weakly compact if and only if it is $L^1$-closed.

**Proposition 2.9.** If the determining set $\mathcal{D}$ is weakly compact and $X \in L^1_\delta(\mathcal{D})$, then $\mathcal{X}_D(X) \neq \emptyset$.

**Proof.** It is clear that $u(X) \in (-\infty, \infty)$. Find a sequence $Z_n \in \mathcal{D}$ such that $\mathbb{E}_P Z_n X \to u(X)$. This sequence has a weak limit point $Z_\infty \in \mathcal{D}$. Clearly, the map $\mathcal{D} \ni Z \mapsto \mathbb{E}_P Z X$ is weakly continuous. Hence, $\mathbb{E}_P Z_\infty X = u(X)$, which means that $Z_\infty \in \mathcal{X}_D(X)$. \hfill \square

**Example 2.10.** (i) If $u$ corresponds to Tail VaR of order $\lambda \in (0,1]$ (see Example 2.5 (i)) and $X$ has a continuous distribution, then it is easy to see that $\mathcal{X}_D(X)$ consists of a unique density $\lambda^{-1} I(X \leq q_\lambda)$, where $q_\lambda$ is a $\lambda$-quantile of $X$.

(ii) If $u$ corresponds to Weighted VaR with the weighting measure $\mu$ (see Example 2.5 (ii)) and $X$ has a continuous distribution, then $\mathcal{X}_D(X)$ consists of a unique density $g(X)$, where $g(x) = \int_{[F(x),1]} \lambda^{-1} \mu(d\lambda)$ and $F$ is the distribution function of $X$ (see [16] Sect. 6)). Note that this density reflects the risk aversion of an agent possessing a portfolio that produces the P&L (Profit&Loss) $X$. \hfill \square

The condition that $\mathcal{D}$ should be weakly compact is very mild and is satisfied for the determining sets of most natural coherent risk measures. For example, the determining set $\mathcal{D}_\lambda$ of Tail VaR is weakly compact provided that $\lambda \in (0,1]$. The determining set $\mathcal{D}_\mu$ of Weighted VaR is weakly compact provided that $\mu$ is concentrated on $(0,1]$; this follows from the explicit representation of this set provided in [8] (the proof can also be found in [33] Th. 4.73 or [47] Th. 1.53); this can also be seen from the representation of $\mathcal{D}_\mu$ provided in [16].

The following example shows that the condition $X \in L^1_\delta(\mathcal{D})$ in Proposition 2.8 cannot be replaced by the condition $X \in L^1_{\text{w}}(\mathcal{D})$. 


Example 2.11. Let $\Omega = [0, 1]$ be endowed with the Lebesgue measure. Consider $Z_n = \sqrt{n}I_{[0,1/n]} + 1 - 1/\sqrt{n}$, $n \in \mathbb{N}$. Then $Y_n := Z_n - 1 \xrightarrow{L^1} 0$, and therefore, the set

$$D = \left\{ 1 + \sum_{n=1}^{\infty} a_n Y_n : a_n \geq 0, \sum_{n=1}^{\infty} a_n \leq 1 \right\}$$

is convex, $L^1$-closed, and uniformly integrable. Thus, $D$ is weakly compact. Now, consider $X(\omega) = -1/\sqrt{n}$. Then $E_P Z_n X = -4 + 2/\sqrt{n}$. Thus, $\inf_{Q \in D} E_Q X = -4$, while there exists no $Q \in D$ such that $E_Q X = -4$. \hfill \Box

2.4 Capital Allocation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $u$ be a coherent utility function with the determining set $D$, and let $X^1, \ldots, X^d \in L_w^1(D)$ be the discounted P&Ls produced by different components of a firm (P&L means the Profit&Loss, i.e. the difference between the terminal wealth and the initial wealth). We will use the notation $X = (X^1, \ldots, X^d)$.

Informally, the capital allocation problem is the following. How is the total risk of the firm does not exceed the risk carried by that part.

Let us introduce the notation $G = \text{cl}\{E_Q X : Q \in D\}$, where “cl” denotes the closure. Note that $G$ is convex and compact. We will call it the generating set or simply the generator for $X$ and $u$. This term is justified by the line

$$u(\langle h, X \rangle) = \inf_{Q \in D} E_Q \langle h, X \rangle = \inf_{Q \in D} \langle h, E_Q X \rangle = \min_{x \in G} \langle h, x \rangle, \quad h \in \mathbb{R}^d. \quad (2.7)$$

Note that the last expression is a classical object of convex analysis known as the support function of the convex set $G$.

**Theorem 2.12.** The set $U$ of utility allocations between $X^1, \ldots, X^d$ has the form

$$U = \arg\min_{x \in G} \langle e, x \rangle, \quad (2.8)$$
where \( e = (1, \ldots, 1) \). Furthermore, for any utility allocation \( x \), we have

\[
\forall h^1, \ldots, h^d \in \mathbb{R}, \quad \sum_{i=1}^{d} h^i x^i \geq u\left(\sum_{i=1}^{d} h^i X^i\right)
\]  

(2.9)

If moreover \( X^1, \ldots, X^d \in L_1^d(D) \) and \( D \) is weakly compact, then

\[
U = \left\{ \mathbb{E}_Q X : Q \in \mathcal{X}_D \left( \sum_{i=1}^{d} X^i \right) \right\}.
\]  

(2.10)

Proof. (The proof is illustrated by Figure 1.) For \( h \in \mathbb{R}^d \), we set

\[
L(h) = \left\{ x \in \mathbb{R}^d : \langle h, x \rangle = \min_{y \in G} \langle h, y \rangle \right\},
\]

\[
M(h) = \left\{ x \in \mathbb{R}^d : \langle h, x \rangle \geq \min_{y \in G} \langle h, y \rangle \right\}.
\]

It is seen from (2.7) that the set of points \( x \in \mathbb{R}^d \) that satisfy (2.5) is \( L(e) \). The set of points \( x \) that satisfy (2.6) is \( \bigcap_{h \in \mathbb{R}^d} M(h) = G + \mathbb{R}^d_+ \). The set of points \( x \) that satisfy (2.9) is \( \bigcap_{h \in \mathbb{R}^d} M(h) = G \). This proves (2.8) and (2.9). Furthermore, the set \( \{ \mathbb{E}_Q X : Q \in D \} \) is closed (the proof is similar to the proof of Proposition 2.9). Now, equality (2.10) follows immediately from (2.8) and the definition of \( \mathcal{X}_D \). \( \square \)

![Figure 1. Solution of the capital allocation problem](image)

If \( G \) is strictly convex (i.e. its interior is nonempty and its border contains no interval), then a utility allocation is unique. However, in general it is not unique as shown by the example below.

**Example 2.13.** Let \( d = 2 \) and \( X^2 = -X^1 \). Then \( G \) is the interval with the endpoints \((u(X^1), -u(X^1))\) and \((-u(-X^1), u(-X^1))\). In this example, \( U = G \). \( \square \)

Let us now find the solution of the capital allocation problem in the Gaussian case.

**Example 2.14.** Let \( X \) have Gaussian distribution with mean \( a \) and covariance matrix \( C \). Let \( u \) be a law invariant coherent utility function, i.e. \( u(X) \) depends only on the distribution of \( X \); we also assume that \( u \) is finite on Gaussian random variables.
Then there exists $\gamma > 0$ such that, for a Gaussian random variable $\xi$ with mean $m$ and variance $\sigma^2$, we have $u(\xi) = m - \gamma \sigma$. Let $L$ denote the image of $\mathbb{R}^d$ under the map $x \mapsto Cx$. Then the inverse $C^{-1}: L \to L$ is correctly defined. It is easy to see that

$$G = a + \{C^{1/2}x : \|x\| \leq \gamma\} = a + \{y \in L : \langle y, C^{-1}y \rangle \leq \gamma^2\}.$$

Let $e = (1, \ldots, 1)$ and assume first that $Ce \neq 0$. In this case the utility allocation $x_0$ between $X^1, \ldots, X^d$ is determined uniquely. In order to find it, note that, for any $y \in L$ such that

$$\frac{d}{d\varepsilon} \Big|_{\varepsilon=0} \langle x_0 - a + \varepsilon y, C^{-1}(x_0 - a + \varepsilon y) \rangle = 0,$$

we have $\langle e, y \rangle = 0$. This implies that $C^{-1}(x_0 - a) = \alpha \text{pr}_L e$ with some constant $\alpha$ ($\text{pr}_L$ denotes the orthogonal projection on $L$). Thus, $x_0 = a + \alpha Ce$. As $x_0$ should belong to the relative border of $G$ (i.e. the border in the relative topology of $a + L$), we have $\langle x_0 - a, C^{-1}(x_0 - a) \rangle = \gamma^2$, i.e. $\alpha = -\gamma \langle e, Ce \rangle^{-1/2}$. As a result, the utility allocation between $X^1, \ldots, X^d$ is $a - \gamma \langle e, Ce \rangle^{-1/2}Ce$.

Assume now that $Ce = 0$. This means that $e$ is orthogonal to $L$, and then the set of utility allocations between $X^1, \ldots, X^d$ is $G$.

Let us remark that in this example the solution of the capital allocation problem depends on $u$ rather weakly, i.e. it depends only on $\gamma$. \hfill \qed

### 2.5 Risk Contribution

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $u$ be a coherent utility function with the determining set $\mathcal{D}$, $X \in L^0$ be the discounted P&L produced by a component of some firm, and $Y \in L^0$ be the discounted P&L produced by the whole firm.

From the financial point of view, such a firm assesses the risk of $X$ not as $\rho(X)$ but rather as $\rho(W+X) - \rho(X)$. Below we define a risk contribution $\rho^c(X;W)$ in such a way that it is a coherent risk measure as a function of $X$ and $\rho^c(X;W) \approx \rho(W+X) - \rho(W)$ provided that $X$ is small as compared to $W$ (the precise statement is Theorem 2.16).

**Definition 2.15.** The utility contribution of $X$ to $Y$ is

$$u^c(X;W) = \inf_{Q \in \mathcal{X}_{\mathcal{D}}(Y)} E_Q X.$$

The risk contribution of $X$ to $Y$ is defined as $\rho^c(X;Y) = -u^c(X;Y)$.

The utility contribution is a coherent utility function provided that $\mathcal{X}_\mathcal{D}(Y) \neq \emptyset$.

If $\mathcal{D}$ is weakly compact and $X,Y \in L^1_s(\mathcal{D})$ then, by Theorem 2.12

$$u^c(X;Y) = \inf\{x^1 : (x^1, x^2) \text{ is a utility allocation between } X, Y - X\}.$$

This formula enables one to define risk contribution under a weaker assumption $X, Y \in L^1_w(\mathcal{D})$.

If $\mathcal{D}$ is weakly compact, $X^1, \ldots, X^d \in L^1_s(\mathcal{D})$, and $\mathcal{X}_\mathcal{D}\left(\sum_i X^i\right)$ is a singleton, then (in view of Theorem 2.12) the utility allocation between $X^1, \ldots, X^d$ is unique and has the form

$$\begin{pmatrix}
  u^c\left(X^1; \sum_{i=1}^d X^i\right), \\
  \vdots \\
  u^c\left(X^d; \sum_{i=1}^d X^i\right)
\end{pmatrix}.$$

This shows the relevance of the given definition. Another argument supporting this definition is the statement below.
Theorem 2.16. If $\mathcal{D}$ is weakly compact and $X,Y \in L^1_s(\mathcal{D})$, then

$$u^c(X;Y) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (u(Y + \varepsilon X) - u(Y)).$$

Proof. (The proof is illustrated by Figure 2.) Consider the generator $G = \text{cl}\{E_Q(X,Y) : Q \in \mathcal{D}\}$ and set $b = \inf\{y : \exists x : (x,y) \in G\}$, $I = \{x : (x,b) \in G\}$, $J = \{x : \exists y : (x,y) \in G\}$, $a = \inf\{x : x \in I\}$. Note that $u^c(X;Y) = a$. The minimum $\min_{(x,y) \in G} \langle \varepsilon,1 \rangle \langle (x,y) \rangle$ is attained at a point $(a(\varepsilon),b(\varepsilon))$. We have $a(\varepsilon) \leq a$, $b(\varepsilon) \geq b$, and $(a(\varepsilon),b(\varepsilon)) \rightarrow (a,b)$. Furthermore, $\varepsilon a(\varepsilon) + b(\varepsilon) \leq \varepsilon a + b$, which implies that $0 \leq b(\varepsilon) - b \leq \varepsilon (a - a(\varepsilon))$. As a result,

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (u(Y + \varepsilon X) - u(Y)) = \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (\varepsilon a(\varepsilon) + b(\varepsilon) - b) = a + \lim_{\varepsilon \downarrow 0} \varepsilon^{-1} (b(\varepsilon) - b) = a = u^c(X;Y).$$

Example 2.17. (i) Let $Y$ be a constant. In this case $\mathcal{X}_D(Y) = \mathcal{D}$, so that $u^c(X;Y) = u(X)$.

(ii) Let $X = \alpha Y$ with $\alpha \in \mathbb{R}_+$. Then $u^c(X;Y) = \alpha u(Y)$.

(iii) Let $X, Y$ have a jointly Gaussian distribution with mean $(EX, EY)$ and covariance matrix $C$. Let $u$ be a law invariant coherent utility function that is finite on Gaussian random variables. Then there exists $\gamma > 0$ such that, for a Gaussian random variable $\xi$ with mean $m$ and variance $\sigma^2$, we have $u(\xi) = m - \gamma \sigma$. Assume that $X$ and $Y$ are not degenerate and $\text{corr}(X,Y) \neq \pm 1$. It follows from Example 2.14 that

$$u^c(X;Y) = EX - \gamma \langle e_2, Ce_2 \rangle^{-1/2} Ce_2
\quad = EX - \gamma \frac{\text{cov}(X,Y)}{\text{var} Y^{1/2}}
\quad = EX + (u(X) - EX) \text{corr}(X,Y),$$

where $e_2 = (0,1)$. In particular, if $EX = EY = 0$, then

$$\frac{u^c(X;Y)}{u(X)} = \text{corr}(X,Y) = \frac{V@R^c(X;Y)}{V@R(X)},$$

where $\text{var}$ denotes the variance and $V@R^c$ denotes the $V@R$ contribution (for the definition, see [43 Sect. 7]).
3 Good Deals Pricing

3.1 Utility-Based Good Deals Pricing

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, \(u\) be a coherent utility function with the weakly compact determining set \(\mathcal{D}\), and \(A\) be a convex subset of \(L^0\). From the financial point of view, \(A\) is the set of various discounted P&Ls that can be obtained in the model under consideration by employing various trading strategies (examples are given in Subsections [3.3–3.5]). It will be called the set of attainable P&Ls under consideration by employing various trading strategies (examples are given in Subsections [3.3–3.5]). It will be called the set of attainable P&Ls under consideration by employing various trading strategies (examples are given in Subsections [3.3–3.5]).

First, we give the definition of a risk-neutral measure. Of course, this notion is a classical object of financial mathematics, but the particular definition we need is taken from [13] (it is adapted to the \(L^0\)-case).

**Definition 3.1.** A risk-neutral measure is a measure \(Q \in \mathcal{P}\) such that \(E_Q X \leq 0\) for any \(X \in A\) (we use the convention \(EX = EX^+ - EX^-\), \(-\infty - \infty = -\infty\)).

The set of risk-neutral measures will be denoted by \(\mathcal{R}\) or by \(\mathcal{R}(A)\) if there is a risk of ambiguity.

**Definition 3.2.** We will say that \(A\) is \(\mathcal{D}\)-consistent if there exists a set \(A' \subseteq A \cap L^1(\mathcal{D})\) such that \(\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A')\).

**Definition 3.3.** A model satisfies the utility-based NGD condition if there exists no \(X \in A\) such that \(u(X) > 0\).

**Theorem 3.4 (Fundamental Theorem of Asset Pricing).** A model satisfies the NGD condition if and only if \(\mathcal{D} \cap \mathcal{R} \neq \emptyset\).

**Proof.** The “if” part is obvious. Let us prove the “only if” part.

Fix \(X_1, \ldots, X_M \in A'\). It follows from the weak continuity of the maps \(\mathcal{D} \ni Q \mapsto E_Q X_m\) that the set \(G = \{E_Q(X_1, \ldots, X_M) : Q \in \mathcal{D}\}\) is compact. Clearly, \(G\) is convex. Suppose that \(G \cap (-\infty, 0]^M = \emptyset\). Then there exist \(h \in \mathbb{R}^M\) and \(\varepsilon > 0\) such that \(\langle h, x \rangle \geq \varepsilon\) for any \(x \in G\) and \(\langle h, x \rangle \leq 0\) for any \(x \in (-\infty, 0]^M\). Hence, \(h \in \mathbb{R}^M_+\). Without loss of generality, \(\sum_m h_m = 1\). Then \(X = \sum_m h_m X_m \in A\) and \(E_Q X \geq \varepsilon\) for any \(Q \in \mathcal{D}\), so that \(u(X) > 0\).

The obtained contradiction shows that, for any \(X_1, \ldots, X_M \in A'\), the set

\[
B(X_1, \ldots, X_M) = \{Q \in \mathcal{D} : E_Q X_m \leq 0\ \text{for any} \ m = 1, \ldots, M\}
\]

is nonempty. As \(X_m \in L^1(\mathcal{D})\), the map \(\mathcal{D} \ni Q \mapsto E_Q X_m\) is weakly continuous, and therefore, \(B(X_1, \ldots, X_M)\) is weakly closed. Furthermore, any finite intersection of sets of this form is nonempty. Consequently, there exists a measure \(Q\) that belongs to each \(B\). Then \(E_Q X \leq 0\) for any \(X \in A'\), which means that \(Q \in \mathcal{D} \cap \mathcal{R}(A')\). As \(A\) is \(\mathcal{D}\)-consistent, \(Q \in \mathcal{D} \cap \mathcal{R}\).

**Remarks.** (i) As opposed to the fundamental theorems of asset pricing dealing with the NA condition and its strengthenings (see [13, 22, 23]), here we need not take any closure of \(A\) when defining the NGD. Essentially, this is the compactness of \(\mathcal{D}\) that yields the fundamental theorem of asset pricing.

(ii) If \(\mathcal{D} = \mathcal{P}\), then the NGD condition means that there exists no \(X \in A\) with \(\text{essinf}_X X(\omega) > 0\). This is very close to the NA condition. However, in this case \(\mathcal{D}\) is not
uniformly integrable and Theorem 3.4 might be violated. Indeed, let $A = \{hX : h \in \mathbb{R}\}$, where $X$ has uniform distribution on $[0, 1]$. Then the NGD is satisfied, while $R = \emptyset$.

Now, let $F \in L^0$ be the discounted payoff of a contingent claim.

**Definition 3.5.** A utility-based NGD price of $F$ is a real number $x$ such that the extended model $(\Omega, \mathcal{F}, P, \mathcal{D}, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGD condition.

The set of the NGD prices will be denoted by $I_{NGD}(F)$.

**Corollary 3.6 (Fair price interval).** For $F \in L^1(\mathcal{D})$,

$$I_{NGD}(F) = \{E_Q F : Q \in \mathcal{D} \cap R\}.$$

**Proof.** Denote $\{h(F - x) : h \in \mathbb{R}\}$ by $A(x)$. Clearly, $A + A(x)$ is $\mathcal{D}$-consistent (in order to prove this, it is sufficient to consider $A' + A(x)$). It follows from Theorem 3.4 that $x \in I_{NGD}(F)$ if and only if $\mathcal{D} \cap R(A + A(x)) \neq \emptyset$. It is easy to check that $Q \in R(A + A(x))$ if and only if $Q \in R$ and $E_Q F = x$. This completes the proof.

**Remark.** As opposed to the NA price intervals, the NGD price intervals are closed (this follows from the weak continuity of the map $\mathcal{D} \cap R \mapsto E_Q F$).

To conclude the subsection, we will discuss the origin of $\mathcal{D}$. First of all, $\mathcal{D}$ might be the determining set of a coherent utility function like Tail V@R or Weighted V@R. The set $\mathcal{D}$ might also correspond to a weighted average or the minimum of several coherent utility functions. It is also possible that $\mathcal{D}$ originates from the classical utility maximization as described by the example below.

**Example 3.7.** Let $P_1, \ldots, P_N$ be a family of probability measures, $u_1, \ldots, u_N$ be a family of classical utility functions (i.e. smooth concave increasing functions $\mathbb{R} \to \mathbb{R}$), and $W_1, \ldots, W_N$ be a family of random variables. From the financial point of view, $P_n$, $u_n$, and $W_n$ are the subjective probability, the utility function, and the future wealth of the $n$-th market participant, respectively. Consider a measure $Q_n = c_n u'_n(W_n)P_n$, where $c_n$ is the normalizing constant. Then, for any trading opportunity $X \in L^0$, we have

$$\frac{d}{d\epsilon} \bigg|_{\epsilon=0} u_n(W_n + \epsilon X) = E_{P_n} u'_n(W_n)X = E_{Q_n} c_n^{-1}X$$

(3.1)

(we assume that all the expectations exist and integration is interchangeable with differentiation). Thus, an opportunity $\epsilon X$ with a small $\epsilon > 0$ is attractive to the $n$-th participant if and only if $E_{Q_n} X > 0$, so that $Q_n$ might be called the valuation measure of the $n$-th participant. Take $\mathcal{D} = \text{conv}(Q_1, \ldots, Q_N)$ and consider the corresponding coherent utility function $u$. Then $u(X) > 0$ if and only if $E_Q X > 0$ for any $n$. In view of (3.1), this means that $\epsilon X$ with some $\epsilon > 0$ is attractive to any market participant (this is similar to the notion of a strictly acceptable opportunity introduced in [9]). Thus, in this example the NGD means the absence of a trading opportunity that is attractive to every agent. \hfill \Box

### 3.2 RAROC-Based Good Deals Pricing

Let $(\Omega, \mathcal{F}, P)$ be a probability space, $\mathcal{RD} \subset \mathcal{P}$ be a convex weakly compact set, $\mathcal{PD}$ be an $L^1$-closed convex subset of $\mathcal{RD}$, and $A$ be a convex subset of $L^0$. We will call $\mathcal{PD}$ the profit-determining set. Thus, the profit of a position that yields a P&L $X$ is $\inf_{Q \in \mathcal{PD}} E_Q X$. We will call $\mathcal{RD}$ the risk-determining set, so that the risk of a position that yields a P&L $X$ is $-\inf_{Q \in \mathcal{RD}} E_Q X$. A canonical example is: $\mathcal{PD} = \{P\}$ and $\mathcal{RD}$ is the determining set of a coherent utility function. We will assume that $A$ is $\mathcal{RD}$-consistent. Finally, we fix a positive number $R$ meaning the upper limit on a possible RAROC.
Definition 3.8. The Risk-Adjusted Return on Capital (RAROC) for $X \in L^0$ is defined as

$$\text{RAROC}(X) = \begin{cases} +\infty & \text{if } \inf_{Q \in PD} E_Q X > 0 \text{ and } \inf_{Q \in RD} E_Q X \geq 0, \\ \inf_{Q \in PD} E_Q X - \inf_{Q \in RD} E_Q X & \text{otherwise} \end{cases}$$

with the convention $\frac{0}{0} = 0, \frac{\infty}{\infty} = 0$.

Definition 3.9. A model satisfies the RAROC-based NGD condition if there exists no $X \in A$ such that $\text{RAROC}(X) > R$.

Theorem 3.10 (Fundamental Theorem of Asset Pricing). A model satisfies the NGD condition if and only if

$$\left(\frac{1}{1 + R} PD + \frac{R}{1 + R} RD\right) \cap \mathcal{R} \neq \emptyset.$$  

Proof. Let us first consider the case $R > 0$. Then, for any $X \in L^0$,

$$\text{RAROC}(X) > R \iff \inf_{Q \in PD} E_Q X + R \inf_{Q \in RD} E_Q X > 0 \iff \inf_{Q \in D} E_Q X > 0,$$

where $D = \left(\frac{1}{1 + R} PD + \frac{R}{1 + R} RD\right)$. Clearly, $D$ is weakly compact (note that $D \subseteq RD$, while $L^1_s(D) = L^1_s(RD)$) and $A$ is $D$-consistent. Now, the statement follows from Theorem 3.4.

Let us now consider the case $R = 0$. Then the “if” part is obvious, and we should check the “only if” part. Take $A' \subseteq A \cap L^1_s(RD)$ such that $RD \cap \mathcal{R} = RD \cap R(A')$. For any $X \in \text{conv} A'$, $\inf_{Q \in PD} E_Q X \leq 0$. Repeating the arguments from the proof of Theorem 3.10, we get $PD \cap \mathcal{R} \neq \emptyset$. $\square$

Definition 3.11. A RAROC-based NGD price of a contingent claim $F$ is a real number $x$ such that the extended model $(\Omega, F, P, PD, RD, A + \{h(F - x) : h \in \mathbb{R}\})$ satisfies the NGD condition.

The set of the NGD prices will be denoted by $I_{NGD}(F)$.

Corollary 3.12 (Fair price interval). For $F \in L^1_s(D)$,

$$I_{NGD}(F) = \left\{ E_Q F : Q \in \left(\frac{1}{1 + R} PD + \frac{R}{1 + R} RD\right) \cap \mathcal{R} \right\}.$$ 

This statement follows from Theorem 3.10.

### 3.3 Static Model with a Finite Number of Assets

We consider the model of the previous subsection with $A = \{\langle h, S_1 - S_0 \rangle : h \in \mathbb{R}^d\}$, where $S_0 \in \mathbb{R}^d$ and $S_1^1, \ldots, S_1^d \in L^1(RD)$. From the financial point of view, $S_n^i$ is the discounted price of the $i$-th asset at time $n$. Clearly, in this model $A$ is $RD$-consistent and $RD \cap \mathcal{R} = RD \cap M$, where $M$ is the set of martingale measures:

$$M = \{Q \in \mathcal{P} : E_Q|S_1| < \infty \text{ and } E_Q S_1 = S_0\}.$$ 

Remark. We have $M \subseteq \mathcal{R}$, but the reverse inclusion might be violated. Indeed, let $d = 1$ and let $S_1$ be such that $E_P S_1^+ = E_P S_1^- = \infty$. Then $P \in \mathcal{R}$, while $P \notin M$.  

15
Let us now provide a geometric interpretation of Theorems 3.4 and 3.10. For this, we only assume that $\mathcal{P} D \subseteq \mathcal{R} D \subseteq \mathcal{P}$ are convex sets and $S_1 \in L^1_w(\mathcal{R} D)$. Let us introduce the notation (see Figure 3)

\begin{align*}
E &= \text{cl}\{E_Q S_1 : Q \in \mathcal{P} D\}, \\
G &= \text{cl}\{E_Q S_1 : Q \in \mathcal{R} D\}, \\
G_R &= \frac{1}{1 + R} E + \frac{R}{1 + R} G, \\
D &= \text{conv supp Law}_P S_1,
\end{align*}

where “supp” denotes the support, and let $D^o$ denote the relative interior of $D$ (i.e. the interior in the relative topology of the smallest affine subspace containing $D$). It is easy to see from the equalities

\begin{align*}
\inf_{Q \in \mathcal{P} D} E_Q \langle h, S_1 - S_0 \rangle &= \inf_{x \in E} \langle h, x - S_0 \rangle, \\
\inf_{Q \in \mathcal{R} D} E_Q \langle h, S_1 - S_0 \rangle &= \inf_{x \in G} \langle h, x - S_0 \rangle
\end{align*}

that the following equivalences are true:

\begin{align*}
\text{RAROC-based NGD} &\iff S_0 \in G_R, \\
\text{utility-based NGD corresponding to } u &\iff S_0 \in G, \\
\text{NA} &\iff S_0 \in D^o
\end{align*}

(the last equivalence is a well-known result of arbitrage pricing; see [49 Ch. V, §2e]).

Now, let $F \in L^1_w(\mathcal{R} D)$ be the discounted payoff of a contingent claim. Let $\tilde{E}$, $\tilde{G}$, $\tilde{G}_R$, $\tilde{D}$, and $\tilde{D}^o$ denote the versions of the sets $E$, $G$, $G_R$, $D$, and $D^o$ defined for $\tilde{S}_1 = (S_1^1, \ldots, S_1^d, F)$ instead of $S_1$. Let $I_{\text{NGD}(R)}(F)$ denote the RAROC-based NGD price interval, $I_{\text{NGD}}(F)$ denote the utility-based NGD price interval (corresponding to $u$), and $I_{\text{NA}}(F)$ denote the NA price interval. Then

\begin{align*}
I_{\text{NGD}(R)}(F) &= \{ x : (S_0, x) \in \tilde{G}_R \}, \\
I_{\text{NGD}}(F) &= \{ x : (S_0, x) \in \tilde{G} \}, \\
I_{\text{NA}}(F) &= \{ x : (S_0, x) \in \tilde{D}^o \}.
\end{align*}

**Example 3.13.** Let $S_1$ have Gaussian distribution with mean $a$ and covariance matrix $C$. Let $\mathcal{P} D = \{P\}$ and $\mathcal{R} D$ be the determining set of a law invariant coherent utility function $u$ that is finite on Gaussian random variables. Let $F$ be such that the vector $(S_1^1, \ldots, S_1^d, F)$ is Gaussian. Denote $c = \text{cov}(S_1, F)$ (we use the vector form of notation).

There exists $b \in \mathbb{R}^d$ such that $Cb = c$. We can write $F = \langle b, S_1 - a \rangle + EF + \tilde{F}$. Then $E \tilde{F} = 0$ and $\text{cov}(\tilde{F}, S_1) = 0$, so that $\tilde{F}$ is independent of $S_1$. Note that

\[ \sigma^2 := \text{var} \tilde{F} = \text{var} F - \text{var} \langle b, S_1 - a \rangle = \text{var} F - \langle b, Cb \rangle = \text{var} F - \langle b, c \rangle. \]

Clearly, if $\sigma^2 = 0$, then

\[ I_{\text{NGD}(R)}(F) = I_{\text{NGD}}(F) = I_{\text{NA}}(F) = \{ \langle b, S_0 - a \rangle + EF \}. \]

Let us now assume that $\sigma^2 > 0$. 

16
Obviously, \( I_{NA}(F) = \mathbb{R} \).

In order to find \( I_{NGD}(F) \), note that \( I_{NGD}(F) = \langle b, S_0 - a \rangle + EF + I_{NGD}(\tilde{F}) \). Let \( L \) denote the image of \( \mathbb{R}^d \) under the map \( x \mapsto Cx \). Then the inverse \( C^{-1} : L \rightarrow L \) is correctly defined. As \( u \) is law invariant, there exists \( \gamma > 0 \) such that, for a Gaussian random variable \( \xi \) with mean \( m \) and variance \( \sigma^2 \), we have \( u(\xi) = m - \gamma \sigma \). From this, it is easy to see that the set \( \widetilde{G} := \{ \mathbb{E}Q(S_1, \tilde{F}) : Q \in \mathcal{RD} \} \) has the form

\[
\widetilde{G} = (a, 0) + \{(x, y) : x \in L, \ y \in \mathbb{R} : \langle x, C^{-1} x \rangle + \sigma^{-2}y^2 \leq \gamma^2 \}.
\]

Consequently,

\[
I_{NGD}(F) = [\langle b, S_0 - a \rangle + EF - \alpha, \langle b, S_0 - a \rangle + EF + \alpha],
\]

where \( \alpha = (\sigma^2 \gamma - \sigma^2 \langle S_0 - a, C^{-1}(S_0 - a) \rangle)^{1/2} \). (In particular, the NGD is satisfied if and only if \( \langle S_0 - a, C^{-1}(S_0 - a) \rangle \leq \gamma^2 \).)

Similar arguments show that

\[
I_{NGD(R)}(F) = [\langle b, S_0 - a \rangle + EF - \alpha(R), \langle b, S_0 - a \rangle + EF + \alpha(R)],
\]

where \( \alpha(R) = (\frac{\sigma^2 \gamma^2 R^2}{(1+R)^2} - \sigma^2 \langle S_0 - a, C^{-1}(S_0 - a) \rangle)^{1/2} \). (In particular, the NGD(R) condition is satisfied if and only if \( \langle S_0 - a, C^{-1}(S_0 - a) \rangle \leq \frac{\gamma^2 R^2}{(1+R)^2} \).)

Let us remark that \( I_{NGD}(F) \) and \( I_{NGD(R)}(F) \) depend on \( u \) rather weakly, i.e. they depend only on \( \gamma \).

\[\square\]

### 3.4 Dynamic Model with an Infinite Number of Assets

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,T]}, \mathbb{P})\) be a filtered probability space. We assume that \( \mathcal{F}_0 \) is trivial. Let \( \mathcal{D} \subseteq \mathcal{P} \) be a convex weakly compact set (in the framework of Subsection 3.1, \( \mathcal{D} \) is the determining set of \( u \); in the framework of Subsection 3.2, \( \mathcal{D} = \frac{1}{1+R} \mathcal{P} \mathcal{D} + \frac{R}{1+R} \mathcal{R} \mathcal{D} \)). Let \((S^i)_{i \in I}\) be a family of \((\mathcal{F}_t)\)-adapted c\'adl\'ag processes (the set \( I \) is arbitrary and we impose no assumptions on the probabilistic structure of \( S^i \) like the assumption that \( S^i \) is...
a semimartingale). From the financial point of view, $S^i_t$ is the discounted price process of the $i$-th asset. We assume that $S^i_t \in L^1(\mathcal{D})$ for any $t \in [0, T], i \in I$. The set of P&Ls an agent can obtain by piecewise constant trading strategies (and only such strategies can be employed in practice) is naturally defined as

$$A = \left\{ \sum_{n=1}^{N} \sum_{i \in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}) : N \in \mathbb{N}, u_0 \leq \cdots \leq u_N, \text{ are } (\mathcal{F}_t)\text{-stopping times,} \right\}.$$  

(3.4)

**Lemma 3.14.** We have $\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{M}$, where

$$A' = \{ H(S^i_t - S^i_u) : u \leq v \in [0, T], i \in I, \text{ } H \text{ is } \mathcal{F}_u \text{-measurable and bounded}, \}$$

$$\mathcal{M} = \{ Q \in \mathcal{P} : \text{ for any } i \in I, S^i \text{ is an } (\mathcal{F}_t, Q)\text{-martingale} \}.$$ 

**Proof.** The inclusions $\mathcal{D} \cap \mathcal{R} \subseteq \mathcal{D} \cap \mathcal{R}(A') \subseteq \mathcal{D} \cap \mathcal{M}$ are clear. So, it is sufficient to prove the inclusion $\mathcal{D} \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{R}$. Let $Q \in \mathcal{D} \cap \mathcal{M}$. Take $X = \sum_{n=1}^{N} \sum_{i \in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}) \in A$. The process

$$M_k = \sum_{n=1}^{k} \sum_{i \in I} H^i_n(S^i_{u_n} - S^i_{u_{n-1}}), \quad k = 0, \ldots, N$$

is an $(\mathcal{F}_{u_k}, Q)$-local martingale. Suppose that $E_Q X^- < \infty$ (otherwise, $E_Q X = -\infty$). Then $M$ is a martingale (see [19] Ch. II, § 1c), and hence, $E_Q X = E_Q M_N = 0$. Thus, in any case, $E_Q X \leq 0$, which proves that $Q \in \mathcal{R}$. 

**Example 3.15.** Let us consider the Black-Scholes model in the framework of the RAROC-based pricing. Thus, $S_t = S_0 \exp(\mu t + \sigma B_t)$, where $B$ is a Brownian motion; we are given a risk-determining set $\mathcal{RD}$, and we take $\mathcal{PD} = \{ \mathcal{P} \}$. Surprisingly enough, in this model $\sup_{X \in A} \text{RAROC}(X) = \infty$. Indeed, the set $\mathcal{M}$ consists of a unique measure $Q_0$ and $\frac{dQ_0}{dP}$ is not bounded away from zero, so that condition (3.2) is violated for any $R > 0$.

Let us construct explicitly a sequence $X_n \in A$ with $\text{RAROC}(X_n) \to \infty$. Consider $\mathcal{D}_n = \{ \frac{dQ_0}{dP} < n^{-1} \}$ and set $X_n = a_n I(D_n) - I(\Omega \setminus D_n)$, where $a_n$ is chosen in such a way that $E_{Q_0} X_n = 0$. Then $E_P X_n \to \infty$, while $\inf_{Q \in \mathcal{RD}} E_Q X \geq -1$, so that $\text{RAROC}(X_n) \to \infty$. Actually, $X_n \notin A$, but, for each $n$, there exists a sequence $(Y^m_n) \in A$ such that $-2 \leq Y^m_n \leq a_n + 1$ and $Y^m_n \xrightarrow{m \to \infty} X_n$ (we leave this to the reader as an exercise). Then $\text{RAROC}(Y^m_n) \xrightarrow{m \to \infty} \text{RAROC}(X_n)$, so that $\text{RAROC}(Y^m_n) \to \infty$ for some subsequence $m(n)$.

This example shows that complete models are typically inconsistent with the RAROC-based NGD pricing. But this technique is primarily aimed at incomplete models because in complete ones the NA price intervals are already exact.

Let us also remark that the utility-based NGD condition might be naturally satisfied in the Black-Scholes model.

**3.5 Dynamic Model with Transaction Costs**

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, T]}, \mathcal{P})$ be a filtered probability space. We assume that $\mathcal{F}_0$ is trivial and $(\mathcal{F}_t)$ is right-continuous. Let $\mathcal{D} \subseteq \mathcal{P}$ be a convex weakly compact set. Let $S^{ai}, S^{bi}, i \in I$
be two families of \((\mathcal{F}_t)\)-adapted càdlàg processes. From the financial point of view, \(S^{ai}\) (resp., \(S^{bi}\)) is the discounted ask (resp., bid) price process of the \(i\)-th asset, so that \(S^a \geq S^b\) componentwise. We assume that \(S^{ai}_t, S^{bi}_t \in L^1(\mathcal{D})\) for any \(t \in [0, T], i \in I\). The set of P&Ls that can be obtained in this model is naturally defined as

\[
A = \left\{ \sum_{n=0}^{N} \sum_{i \in I} \left[ -H^i_n I(H^i_n > 0) S^{ai}_{u_n} - H^i_n I(H^i_n < 0) S^{bi}_{u_n} \right] : \\
N \in \mathbb{N}, \ u_0 \leq \cdots \leq u_N \text{ are } (\mathcal{F}_t)\text{-stopping times, } H^i_n = \mathcal{F}_{u_n}\text{-measurable,} \\
H^i_n = 0 \text{ for all } i, \text{ except for a finite set, and } \sum_{n=0}^{N} H^i_n = 0 \text{ for any } i \right\}.
\]

Here \(H^i_n\) means the amount of the \(i\)-th asset that is bought at time \(u_n\) (so that \(\sum_{k=0}^{n} H^i_k\) is the total amount of the \(i\)-th asset held at time \(u_n\)). Note that if there are no transaction costs, i.e. \(S^{ai} = S^{bi} = S^i\) for each \(i\), then the set of attainable P&Ls coincides with the set given by \((3.1)\).

**Lemma 3.16.** We have \(\mathcal{D} \cap \mathcal{R} = \mathcal{D} \cap \mathcal{R}(A') = \mathcal{D} \cap \mathcal{M}\), where

\[
A' = \{ G(S^{bi}_v - S^{ai}_u) + H(-S^{ai}_v + S^{bi}_u) : i \in I, \ u \leq v \text{ are simple } (\mathcal{F}_t)\text{-stopping times, } G, H \text{ are positive, bounded, } \mathcal{F}_u\text{-measurable}, \}
\]

\[
\mathcal{M} = \{ Q \in \mathcal{P} : \text{for any } i, \text{ there exists an } (\mathcal{F}_t, Q)\text{-martingale } M^i \text{ such that } S^{bi}_t \leq M^i \leq S^{ai}_t \}.
\]

(A stopping time is simple if it takes on a finite number of values.)

**Proof.** The inclusion \(\mathcal{D} \cap \mathcal{R} \subseteq \mathcal{D} \cap \mathcal{R}(A')\) is obvious.

Let us prove the inclusion \(\mathcal{D} \cap \mathcal{R}(A') \subseteq \mathcal{D} \cap \mathcal{M}\). Take \(Q \in \mathcal{D} \cap \mathcal{R}(A')\). Fix \(i \in I\). For any simple stopping times \(u \leq v\), we have \(S^{ai}_u, S^{bi}_u, S^{ai}_v, S^{bi}_v \in L^1(\mathcal{D})\) and

\[
\mathbb{E}_Q(S^{ai}_v | \mathcal{F}_u) \geq S^{bi}_u, \quad \mathbb{E}_Q(S^{bi}_v | \mathcal{F}_u) \leq S^{ai}_u. \quad (3.5)
\]

Consider the Snell envelopes

\[
X_t = \operatorname{esssup}_{\tau \in \mathcal{T}_t} \mathbb{E}_Q(S^{bi}_{\tau} | \mathcal{F}_t), \quad t \in [0, T],
\]

\[
Y_t = \operatorname{essinf}_{\tau \in \mathcal{T}_t} \mathbb{E}_Q(S^{ai}_{\tau} | \mathcal{F}_t), \quad t \in [0, T],
\]

where \(\mathcal{T}_t\) denotes the set of simple \((\mathcal{F}_t)\)-stopping times such that \(\tau \geq t\). (Recall that \(\operatorname{esssup}_x \xi_x\) is a random variable \(\xi\) such that, for any \(\alpha, \xi \geq \xi_\alpha\) a.s. and for any other random variable \(\xi'\) with this property, we have \(\xi \leq \xi'\) a.s.) Then \(X\) is an \((\mathcal{F}_t)\)-supermartingale, while \(Y\) is an \((\mathcal{F}_t, Q)\)-submartingale (see \(\S 27\) Th. 2.12.1).

Let us prove that, for any \(t \in [0, T]\), \(X_t \leq Y_t\) Q-a.s. Assume that there exists \(t\) such that \(\mathbb{P}(X_t > Y_t) > 0\). Then there exist \(\tau, \sigma \in \mathcal{T}_t\) such that

\[
\mathbb{Q}(\mathbb{E}_Q(S^{bi}_{\tau} | \mathcal{F}_t) > \mathbb{E}_Q(S^{ai}_{\sigma} | \mathcal{F}_t)) > 0.
\]

This implies that \(\mathbb{Q}(\xi > \eta) > 0\), where \(\xi = \mathbb{E}_Q(S^{bi}_{\tau} | \mathcal{F}_{\tau \land \sigma})\) and \(\eta = \mathbb{E}_Q(S^{ai}_{\sigma} | \mathcal{F}_{\tau \land \sigma})\). Assume first that \(\mathbb{Q}(\{\xi > \eta\} \cap \{\tau \leq \sigma\}) > 0\). On the set \(\{\tau \leq \sigma\}\) we have

\[
\xi = S^{bi}_{\tau \land \sigma}, \quad \eta = \mathbb{E}_Q(S^{ai}_{\sigma} | \mathcal{F}_{\tau \land \sigma}) = \mathbb{E}_Q(S^{ai}_{\tau \land \sigma} | \mathcal{F}_{\tau \land \sigma}),
\]
and we obtain a contradiction with (3.5). In a similar way we get a contradiction if we assume that $Q(\{\xi > \eta \} \cap \{\tau \geq \sigma \}) > 0$. As a result, $X_i \leq Y_i$ Q-a.s. Now, it follows from [37] Lem. 3] that there exists an $(\mathcal{F}_t, Q)$-martingale $M$ such that $X \leq M \leq Y$. As a result, $Q \in \mathcal{M}$.

Let us prove the inclusion $\mathcal{D} \cap \mathcal{M} \subseteq \mathcal{D} \cap \mathcal{R}$. Take $Q \in \mathcal{D} \cap \mathcal{M}$, so that, for any $i$, there exists an $(\mathcal{F}_t, Q)$-martingale $M^i$ such that $S^{bi} \leq M^i \leq S^{ai}$. For any

$$X = \sum_{n=0}^{N} \sum_{i \in I} [-H^i_n I(H^i_n > 0)S^{a_i}_{u_n} - H^i_n I(H^i_n < 0)S^{b_i}_{u_n}] \in A,$$

we have

$$X \leq \sum_{n=0}^{N} \sum_{i \in I} [-H^i_n I(H^i_n > 0)M^i_{u_n} - H^i_n I(H^i_n < 0)M^i_{u_n}] = \sum_{n=1}^{N} \sum_{i \in I} \left(\sum_{k=0}^{n-1} H^i_k\right)(M^i_{u_n} - M^i_{u_{n-1}}).$$

Repeating the arguments used in the proof of Lemma [3.14 we get $E_Q X \leq 0$. As a result, $Q \in \mathcal{R}$. \hfill \Box

Consider now a model with proportional transaction costs, i.e. $S^{ai} = S^i$, $S^{bi} = (1 - \lambda^i)S^i$, where each $S^i$ is positive, $\lambda^i \in (0, 1)$. Denote the interval of the NGD prices in this model by $I_1(F)$ (the NGD pricing technique might be utility-based or RAROC-based as the latter one is reduced to the former one by considering $\mathcal{D} = \frac{1}{1+R} \mathcal{PD} + \frac{R}{1+R} \mathcal{RD})$. Let $(\lambda_n) = (\lambda_n; i \in I, n \in \mathbb{N})$ be a sequence such that $\lambda^i_n \rightarrow 0$ for any $i$.

**Theorem 3.17.** For $F \in L^1_\lambda(\mathcal{D})$, we have $I_{\lambda_n}(F) \rightarrow I_0(F)$ in the sense that the right (resp., left) endpoints of $I_{\lambda_n}(F)$ converge to the right (resp., left) endpoint of $I_0(F)$.

**Proof.** Let $r$ denote the right endpoint of $I_0(F)$. Suppose that the right endpoints of $I_{\lambda_n}(F)$ do not converge to $r$. Then there exists $r' > r$ such that, for each $n$ (possibly, after passing on to a subsequence), there exists $Q_n \in \mathcal{D} \cap \mathcal{R}_{\lambda_n}$ with the property: $E_Q F \geq r'$ ($\mathcal{R}_\lambda$ is the set of risk-neutral measures in the model corresponding to $\lambda$). The sequence $(Q_n)$ has a weak limit point $Q_\infty \in \mathcal{D}$. Fix $i \in I$, $u \leq v \in [0, T]$, and a positive bounded $\mathcal{F}_u$-measurable function $H$. For any $n$, we have $E_{Q_n} H((1 - \lambda^i_n)S^i_v - S^i_u) \leq 0$. As $S^i \in L^1_\lambda(\mathcal{D})$, we have $\sup_{Q \in \mathcal{D}} E_Q S^i_v < \infty$, and hence, $\limsup_{n} E_{Q_n} H(S^i_v - S^i_u) \leq 0$. As the map $\mathcal{D} \ni Q \mapsto E_Q H(S^i_v - S^i_u)$ is weakly continuous, we get $E_{Q_\infty} H(S^i_v - S^i_u) \leq 0$. In a similar way, we prove that $E_{Q_\infty} H(-S^i_v + S^i_u) \leq 0$. Thus, $S^i$ is an $(\mathcal{F}_t, Q_\infty)$-martingale, so that $Q_\infty \in \mathcal{D} \cap \mathcal{R}_0$. As the map $\mathcal{D} \ni Q \mapsto E_Q F$ is weakly continuous, we should have $E_{Q_\infty} F \geq r'$. But this is a contradiction. \hfill \Box

### 3.6 Hedging

Consider the model of Subsection 3.1.

**Definition 3.18.** The upper and lower NGD prices of a contingent claim $F$ are defined by

$$\overline{V}(F) = \inf \{x : \exists X \in A \text{ such that } u(X - F + x) \geq 0\},$$

$$\underline{V}(F) = \sup \{x : \exists X \in A \text{ such that } u(X + F - x) \geq 0\}.$$
The problem of finding $\bar{\nabla}(F)$ has some similarities with the superreplication problem considered by Cvitanić, Karatzas [13] and by Sekine [18], but the difference is that in those papers the risk is measured not as $\rho(X - F + x)$, but rather as $\rho((X - F + x)^-)$.  

**Proposition 3.19.** If $A$ is a cone and $F \in L^1_s(D)$, then  
\[
\nabla(F) = \sup \{\mathbb{E}_Q F : Q \in D \cap R\}, \\
\bar{\nabla}(F) = \inf \{\mathbb{E}_Q F : Q \in D \cap R\}.
\]

**Proof.** Take $x_0 \in \mathbb{R}$ and set $A(x_0) = A + \{h(x_0 - F) : h \in \mathbb{R}_+\}$. Using Theorem 3.3, we can write  
\[
\begin{align*}
\nabla(F) &\geq x_0 \iff \exists X \in A \text{ such that } u(X - F + x_0) > 0 \\
&\iff \exists X \in A(x_0) \text{ such that } u(X) > 0 \\
&\iff D \cap R(A(x_0)) \neq \emptyset \\
&\iff \exists Q \in D \cap R \text{ such that } \mathbb{E}_Q F \geq x_0.
\end{align*}
\]
This yields the formula for $\nabla(F)$. The representation of $\bar{\nabla}(F)$ is proved similarly. \hfill \square 

**Remarks.** (i) The above theorem is formally true if the NGD is violated. In this case $\nabla(F) = -\infty$ and $\bar{\nabla}(F) = \infty$.  
(ii) The above argument shows that there exist $\overline{Q}, \underline{Q} \in D \cap R$ such that $\mathbb{E}_{\overline{Q}} F = \nabla(F)$, $\mathbb{E}_{\underline{Q}} F = \bar{\nabla}(F)$. This is in contrast with the NA technique. \hfill \square 

(iii) Under the conditions of the above corollary, we have $I_{\text{NGD}}(F) = [\nabla(F), \bar{\nabla}(F)]$.  

Let us now study the sub- and super-replication problem for a particular case of a (frictionless) static model with a finite number of assets. Thus, we are given $S_0 \in \mathbb{R}^d$ and $S_1^1, \ldots, S_1^d \in L^1_w(D)$. From the financial point of view, $S_n^i$ is the discounted price of the $i$-th asset at time $n$.  

**Definition 3.20.** The superhedging and subhedging strategies are defined by  
\[
\begin{align*}
\overline{H}(F) &= \{h \in \mathbb{R}^d : u(h, S_1 - S_0) - F + \nabla(F)) \geq 0\}, \\
\underline{H}(F) &= \{h \in \mathbb{R}^d : u(h, S_1 - S_0) + F - \bar{\nabla}(F)) \geq 0\}.
\end{align*}
\]

Below we provide a simple geometric procedure to determine these quantities. Assume that $F \in L^1_w(D)$ and let us introduce the notation  
\[
\begin{align*}
G &= \text{cl} \{\mathbb{E}_Q(S_1, F) : Q \in D\}, \\
\overline{V} &= \sup \{x : (S_0, x) \in G\}, \\
\underline{V} &= \inf \{x : (S_0, x) \in G\}, \\
\overline{N} &= \{h \in \mathbb{R}^{d+1} \cap \mathbb{R}^d : \forall x \in G, (h, x - (S_0, \overline{V})) \geq 0\}, \\
\underline{N} &= \{h \in \mathbb{R}^{d+1} \cap \mathbb{R}^d : \forall x \in G, (h, x - (S_0, \underline{V})) \geq 0\},
\end{align*}
\]
i.e. $G$ is the generator for $(S_1, F)$ and $u; \overline{N}$ (resp., $\underline{N}$) is the set of inner normals to $G$ at the point $(S_0, \overline{V})$ (resp., $(S_0, \underline{V})$); see Figure 4.  

**Proposition 3.21.** We have  
\[
\begin{align*}
\nabla(F) &= \overline{V}, \\
\bar{\nabla}(F) &= \underline{V}, \\
\overline{H} &= \{h \in \overline{N} : h^{d+1} = -1\}, \\
\underline{H} &= \{h \in \underline{N} : h^{d+1} = 1\}.
\end{align*}
\]
Remark. The statement is true both in the case, where the NGD is satisfied, and in the case, where the NGD is not satisfied (in the latter case $S_0$ does not belong to the projection of $G$ on $\mathbb{R}^d$, $v = \nabla(F) = -\infty$, $v = \nabla(F) = \infty$, $\overline{N} = \overline{H} = \emptyset$, $\underline{N} = \underline{H} = \emptyset$).

Proof of Proposition 3.21. This is an easy consequence of the line

$$u((h, S_1 - S_0) \pm F \mp x) = \inf_{z \in G} \langle (h, \pm 1), z - (S_0, x) \rangle, \quad h \in \mathbb{R}^d.$$
Proposition 3.23. We have

\[ V(F) = \lambda^{-1} \int_0^{q_a} f(x)Q(dx) + \lambda^{-1} \int_{q_{1-b}}^{\infty} f(x)Q(dx), \]

\[ V(F) = \lambda^{-1} \int_{q_c}^{q_d} f(x)Q(dx), \]

\[ H(F) = \frac{f(q_{1-b}) - f(q_a)}{q_{1-b} - q_a}, \]

\[ H(F) = -\frac{f(q_d) - f(q_c)}{q_d - q_c}. \]

Proof. Let us first prove the representation for \( V(F) \) under an additional assumption that \( f \) is strictly convex. By Proposition 3.21,

\[ V(F) = \sup_{Z \in \mathcal{D}_\lambda: EZX = S_0} EZ f(X) \]

(\( \mathcal{D}_\lambda \) is given by (2.3)). Take

\[ Z_0 \in \arg\max_{Z \in \mathcal{D}_\lambda: EZX = S_0} EZ f(X) \]

(\( Z_0 \) exists by a compactness argument). Passing from \( Z_0 \) to \( E(Z_0 | X) \), we can assume that \( Z_0 \) is \( X \) measurable, i.e. \( Z_0 = \varphi(X) \). Let us prove that

\[ Z_0 = \lambda^{-1} I(X < q_a) + \lambda^{-1} I(X > q_{1-b}). \] (3.6)

Assume the contrary. Then there exist \( 0 < \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \) such that

\[ Q(\{\varphi < \lambda^{-1}\} \cap (\alpha_1, \alpha_2)) > 0, \]
\[ Q(\{\varphi > 0\} \cap (\alpha_2, \alpha_3)) > 0, \]
\[ Q(\{\varphi < \lambda^{-1}\} \cap (\alpha_3, \alpha_4)) > 0. \]

For \( h_1, h_2, h_3 \in [0, \lambda^{-1}] \), we set

\[ \tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \notin (\alpha_1, \alpha_4), \\ \varphi(x) \vee h_1, & x \in (\alpha_1, \alpha_2), \\ \varphi(x) \wedge h_2, & x \in (\alpha_2, \alpha_3), \\ \varphi(x) \vee h_3, & x \in (\alpha_3, \alpha_4). \end{cases} \]

We can find \( h_1, h_2, h_3 \) such that

\[ Q(\{\tilde{\varphi} > \varphi\} \cap (\alpha_1, \alpha_2)) > 0, \]
\[ Q(\{\tilde{\varphi} < \varphi\} \cap (\alpha_2, \alpha_3)) > 0, \]
\[ Q(\{\tilde{\varphi} > \varphi\} \cap (\alpha_3, \alpha_4)) > 0, \]

\[ \int_0^\infty x\tilde{\varphi}(x)Q(dx) = \int_0^\infty x\varphi(x)Q(dx) = S_0, \]
\[ \int_0^\infty \tilde{\varphi}(x)Q(dx) = \int_0^\infty \varphi(x)Q(dx) = 1. \]
Consider the affine function \( \tilde{f} \) that coincides with \( f \) at \( \alpha_2 \) and \( \alpha_3 \). Then

\[
\int_0^\infty (\tilde{\varphi}(x) - \varphi(x)) \tilde{f}(x)Q(dx) = 0.
\]

Furthermore, as \( f \) is strictly convex, \( \tilde{f} < f \) on \((\alpha_1, \alpha_2)\), \( \tilde{f} > f \) on \((\alpha_2, \alpha_3)\), and \( \tilde{f} > f \) on \((\alpha_3, \alpha_4)\). Consequently,

\[
\int_0^\infty (\tilde{\varphi}(x) - \varphi(x)) f(x)Q(dx) > 0.
\]

Thus, we have found \( \tilde{Z}_0 = \tilde{\varphi}(X) \in D_\lambda \) such that \( \mathbb{E}\tilde{Z}_0X = S_0 \) and \( \mathbb{E}\tilde{Z}_0f(X) > \mathbb{E}Z_0f(X) \), which contradicts the choice of \( Z_0 \). As a result, (3.6) is satisfied, which yields the desired representation of \( \overline{V}(F) \).

Let us now prove the representation for \( \overline{V}(F) \) in the general case. Take \( Z_0 \) given by (3.6). Find a strictly convex function \( \tilde{f} \) of linear growth. Then the function \( f_\varepsilon = f + \varepsilon \tilde{f} \) is strictly convex and the result proved above shows that \( \mathbb{E}f_\varepsilon(X) \leq \mathbb{E}Z_0f_\varepsilon(X) \) for any \( Z \in D_\lambda \). Passing on to the limit as \( \varepsilon \downarrow 0 \), we get \( \mathbb{E}f(X) \leq \mathbb{E}Z_0f(X) \) for any \( Z \in D_\lambda \). This yields the desired representation of \( \overline{V}(F) \).

Let us now prove the representation for \( \overline{H}(F) \). Consider the function

\[
g(x) = \sup_{Z \in D_\lambda : EZX = x} \mathbb{E}f(X), \quad x \in [u(S_1), -u(-S_1)].
\]

It follows from the reasoning given above that \( g = g_1 \circ g_2^{-1} \), where

\[
g_1(x) = \lambda^{-1} \int_0^{q_1} f(y)Q(dy) + \lambda^{-1} \int_{q_1 - \lambda + x}^\infty f(y)Q(dy), \quad x \in [0, \lambda^{-1}],
\]

\[
g_2(x) = \lambda^{-1} \int_0^{q_1} yQ(dy) + \lambda^{-1} \int_{q_1 - \lambda + x}^\infty yQ(dy), \quad x \in [0, \lambda^{-1}].
\]

Applying Proposition 3.21, we get

\[
\overline{H}(F) = g'(S_0) \frac{f(q_1-b) - f(q_0)}{q_1-b - q_0}.
\]

The representations for \( \overline{V}(F) \) and \( \overline{H}(F) \) are proved in a similar way. \( \square \)
References

[1] C. Acerbi. Spectral measures of risk: a coherent representation of subjective risk aversion. Journal of Banking and Finance, 26 (2002), p. 1505–1518.

[2] C. Acerbi. Coherent representations of subjective risk aversion. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004, p. 147–207.

[3] C. Acerbi, D. Tasche. On the coherence of expected shortfall. Journal of Banking and Finance, 26 (2002), No. 7, p. 1487–1503.

[4] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath. Thinking coherently. Risk, 10 (1997), No. 11, p. 68–71.

[5] P. Artzner, F. Delbaen, J.-M. Eber, D. Heath. Coherent measures of risk. Mathematical Finance, 9 (1999), No. 3, p. 203–228.

[6] A. Bernardo, O. Ledoit. Gain, loss, and asset pricing. Journal of Political Economy, 108 (2000), No. 1, p. 144–172.

[7] T. Bjork, I. Slinko. Towards a general theory of good deal bounds. Preprint, available at: www.newton.cam.ac.uk/webseminars/pg+ws/2005/dqf.

[8] G. Carlier, R.A. Dana. Core of convex distortions of a probability. Journal of Economic Theory, 113 (2003), No. 2, p. 199–222.

[9] P. Carr, H. Geman, D. Madan. Pricing and hedging in incomplete markets. Journal of Financial Economics, 62 (2001), p. 131–167.

[10] P. Carr, H. Geman, D. Madan. Pricing in incomplete markets: from absence of good deals to acceptable risk. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004, p. 451–474.

[11] A. Černý, S. Hodges. The theory of good-deal pricing in incomplete markets. In: Mathematical Finance — Bachelier Congress 2000. H. Geman, D. Madan, S. Pliska, T. Vorst (Eds.). Springer, 2001, p. 175–202.

[12] P. Cheridito, F. Delbaen, M. Kupper. Dynamic monetary risk measures for bounded discrete-time processes. Article math.PR/0410453 on Mathematics ArXiv, http://arxiv.org.

[13] A.S. Cherny. General arbitrage pricing model: probability approach. To be published in Lecture Notes in Mathematics. Available at: http://mech.math.msu.su/~cherny.

[14] A.S. Cherny. General arbitrage pricing model: transaction costs. To be published in Lecture Notes in Mathematics. Available at: http://mech.math.msu.su/~cherny.

[15] A.S. Cherny. Equilibrium with coherent risk. Preprint, available at: http://mech.math.msu.su/~cherny.

[16] A.S. Cherny. Weighted V@R and its properties. To be published in Finance and Stochastics. Available at: http://mech.math.msu.su/~cherny.
[17] J.H. Cochrane, J. Saá-Requejo. Beyond arbitrage: good-deal asset price bounds in incomplete markets. Journal of Political Economy, 108 (2000), No. 1, p. 79–119.

[18] J. Cvitanić, I. Karatzas. On dynamic measures of risk. Finance and Stochastics, 3 (1999), p. 451–482.

[19] J. Cvitanić, H. Pham, N. Touzi. A closed-form solution to the problem of super-replication under transaction costs. Finance and Stochastics, 3 (1999), No. 1, p. 35–54.

[20] F. Delbaen. Coherent risk measures on general probability spaces. In: K. Sandmann, P. Schönbucher (Eds.). Advances in Finance and Stochastics. Essays in Honor of Dieter Sondermann. Springer, 2002, p. 1–37.

[21] F. Delbaen. Coherent monetary utility functions. Preprint, available at http://www.math.ethz.ch/~delbaen under the name ‘Pisa lecture notes’.

[22] F. Delbaen, W. Schachermayer. A general version of the fundamental theorem of asset pricing. Mathematische Annalen, 300 (1994), p. 463–520.

[23] F. Delbaen, W. Schachermayer. The fundamental theorem of asset pricing for unbounded stochastic processes. Mathematische Annalen, 312 (1998), p. 215–250.

[24] M. Denault. Coherent allocation of risk capital. Journal of Risk, 4 (2001), No. 1, p. 1–34.

[25] K. Detlefsen, G. Scandolo. Conditional and dynamic convex risk measures. Finance and Stochastics, 9 (2005), No. 4, p. 539–561.

[26] K. Dowd. Spectral risk measures. Financial Engineering News, electronic journal available at: http://www.fenews.com/fen42/risk-reward/risk-reward.htm.

[27] N. El Karoui. Les aspects probabilistes du contrôle stochastique. Lecture Notes in Mathematics, 876 (1981), p. 73–238.

[28] T. Fischer. Risk capital allocation by coherent risk measures based on one-sided moments. Insurance: Mathematics and Economics 32 (2003), No. 1, p. 135–146.

[29] K. Floret. Weakly compact sets. Lecture Notes in Mathematics, 801 (1980).

[30] H. Föllmer, P. Leukert. Quantile hedging. Finance and Stochastics, 3 (1999), p. 251–273.

[31] H. Föllmer, A. Schied. Convex measures of risk and trading constraints. Finance and Stochastics, 6 (2002), p. 429–447.

[32] H. Föllmer, A. Schied. Robust preferences and convex measures of risk. In: K. Sandmann, P. Schönbucher (Eds.). Advances in Finance and Stochastics, Essays in Honor of Dieter Sondermann. Springer, 2002, p. 39–56.

[33] H. Föllmer, A. Schied. Stochastic finance. An introduction in discrete time. 2nd Ed., Walter de Gruyter, 2004.

[34] J.M. Harrison, D.M. Kreps. Martingales and arbitrage in multiperiod securities markets. Journal of Economic Theory, 20 (1979), p. 381–408.
[35] S. Jaschke, U. Küchler. Coherent risk measures and good deal bounds. Finance and Stochastics, 5 (2001), p. 181–200.

[36] A. Jobert, L.C.G. Rogers. Pricing operators and dynamic convex risk measures. Preprint, available at: http://www.statslab.cam.ac.uk/~chris.

[37] E. Jouini, H. Kallal. Martingales and arbitrage in securities markets with transaction costs. Journal of Economic Theory, 66 (1995), No. 1, p. 178–197.

[38] E. Jouini, M. Meddeb, N. Touzi. Vector-valued coherent risk measures. Finance and Stochastics, 8 (2004), p. 531–552.

[39] M. Kalkbrenner. An axiomatic approach to capital allocation. Mathematical Finance, 15 (2005), No. 3, p. 425–437.

[40] S. Kusuoka. On law invariant coherent risk measures. Advances in Mathematical Economics, 3 (2001), p. 83–95.

[41] K. Larsen, T. Pirvu, S. Shreve, R. Tütüncü. Satisfying convex risk limits by trading. Finance and Stochastics, 9 (2004), p. 177–195.

[42] S. Leventhal, A. V. Skorokhod. On the possibility of hedging options in the presence of transaction costs. Annals of Applied Probability, 7 (1997), p. 410–443.

[43] C. Marrison. The fundamentals of risk measurement. McGraw Hill, 2002.

[44] L. Overbeck. Allocation of economic capital in loan portfolios. In: W. Härdle, G. Stahl (Eds.). Measuring risk in complex stochastic systems. Lecture Notes in Statistics, 147 (1999).

[45] F. Riedel. Dynamic coherent risk measures. Stochastic Processes and their Applications, 112 (2004), No. 2, p. 185–200.

[46] B. Roorda, J.M. Schumacher, J. Engwerda. Coherent acceptability measures in multiperiod models. Mathematical Finance, 15 (2005), No. 4, p. 589–612.

[47] A. Schied. Risk measures and robust optimization problems. Lecture notes of a minicourse held at the 8th symposium on probability and stochastic processes. Preprint.

[48] J. Sekine. Dynamic minimization of worst conditional expectation of shortfall. Mathematical Finance, 14 (2004), No. 4, p. 605–618.

[49] A.N. Shiryaev. Essentials of stochastic finance. World Scientific, 1999.

[50] H.M. Soner, S.E. Shreve, J. Cvitanić. There is no nontrivial hedging portfolio for option pricing with transaction costs. Annals of Applied Probability, 5 (1995), p. 327–355.

[51] J. Staum. Fundamental theorems of asset pricing for good deal bounds. Mathematical Finance, 14 (2004), No. 2, p. 141–161.

[52] G. Szegö. On the (non)-acceptance of innovations. In: G. Szegö (Ed.). Risk measures for the 21st century. Wiley, 2004, p. 1–9.

[53] D. Tasche. Expected shortfall and beyond. Journal of Banking and Finance, 26 (2002), p. 1519–1533.