Blow-Up Phenomena in Reaction-Diffusion Problems with Nonlocal and Gradient Terms

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This paper considers the blow-up phenomena for the following reaction-diffusion problem with nonlocal and gradient terms:

\[
\begin{align*}
    u_t &= \Delta u^m + \int_{\Omega} u' \, dx - |Vu|^s, & \text{in } \Omega \times (0, t^*), \\
    \frac{\partial u}{\partial \nu} &= g(u), & \text{on } \partial \Omega \times (0, t^*), \\
    u(x, 0) &= u_0(x) \geq 0, & \text{in } \Omega,
\end{align*}
\]

Here \(m > 1\), and \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\) is a bounded and convex domain with smooth boundary. Applying a Sobolev inequality and a differential inequality technique, lower bounds for blow-up time when blow-up occurs are given. Moreover, two examples are given as applications to illustrate the abstract results obtained in this paper.

1. Introduction

Since the 1960s, the blow-up phenomena of reaction-diffusion problems have received considerable interest. From then on, questions concerning finite-time blow-up of solutions to reaction-diffusion problems, as well as other classes of problems, have attracted more attention. A number of studies appeared on the topic of blow-up in time (see [1–4]) or global existence and boundedness of solutions (see [5]). Moreover, qualitative properties were investigated such as the blow-up set, rate, and profile of the blow-up [6].

From a practical point of view, apart from considering the above problems, one would like to know whether the solutions blow up, and if so, at what time blow-up occurs. However, when the solution does blow up at some finite time, the blow-up time can seldom be determined explicitly, and much effort has been devoted to the calculations of bounds for the blow-up time. As we know, the studies of many papers are lead to upper bounds for the blow-up time when blow-up does occur (see [7–10]). For the practical situations, lower bounds for the blow-up time are more important, which can be used to predict the unsteady state of the systems more accurately.

In 2006, Payne and Schaefer [11] used the differential inequality technique to derive the lower bound for the blow-up time when \(\Omega \subset \mathbb{R}^3\). Based on their work, many scholars tend to seek lower bounds for quantity of reaction-diffusion problems when \(\Omega \subset \mathbb{R}^3\) (see [12] and the reference therein) and when \(\Omega \subset \mathbb{R}^N\) \((N \geq 3)\) (see [13, 14]). In order to expand the underpinning theory of the mathematical analysis of problem, we aim to derive the results that have been extended to more general reaction-diffusion problems when \(\Omega \subset \mathbb{R}^N\) \((N \geq 2)\). In addition, aiming to be closer to more realistic models, in this paper, we deal with the following reaction-diffusion problem with nonlocal and gradient terms:

\[
\begin{align*}
    u_t &= \Delta u^m + \int_{\Omega} u' \, dx - |Vu|^s, & \text{in } \Omega \times (0, t^*), \\
    \frac{\partial u}{\partial \nu} &= g(u), & \text{on } \partial \Omega \times (0, t^*), \\
    u(x, 0) &= u_0(x) \geq 0, & \text{in } \Omega,
\end{align*}
\]
where $m > 1$, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded and convex domain with smooth boundary, $\partial u/\partial v$ denotes the outward normal derivative on $\partial \Omega$, and $t^*$ is the blow-up time if blow-up occurs. Throughout this paper, we suppose $g$ are non-negative $C(\mathbb{R}_+)$ functions and $u_0$ are nonnegative $C^1(\mathbb{R}^N)$ functions satisfying compatibility conditions. Problem (1) appears in the mathematical models for gas or fluid flow in porous media (see [15]); it can also be used to describe the evolution of some biological population $u$ (cells, bacteria, etc.), which live in a certain domain and whose growth is influenced by the law $+\int_{\Omega} u' \, dx - |\nabla u|^q$. Nonlocal terms $+\int_{\Omega} u' \, dx$ represent the births of the species, and $-|\nabla u|^q$ represents the accidental deaths of the species. For other related references, readers can refer to [16, 17].

In order to achieve our goal, we mainly pay our attention to the following works [18, 19]. Marras et al. in [18] investigated the following problem:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + k_1(t)u^p \, dx - k_2(t)|\nabla u|^q \quad \text{in} \quad \Omega \times (0, t^*), \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on} \quad \partial \Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},
\end{align*}
$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded and convex domain with smooth boundary. They obtained a lower bound for the blow-up time when $\Omega \subset \mathbb{R}^3$. Song in [19] considered the following reaction-diffusion problem:

$$
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + a + \int_{\Omega} u^p \, dx - ku^l \quad \text{in} \quad \Omega \times (0, t^*), \\
\frac{\partial u}{\partial v} &= 0 \quad \text{on} \quad \partial \Omega \times (0, t^*), \\
u(x, 0) &= u_0(x) \geq 0 \quad \text{in} \quad \overline{\Omega},
\end{align*}
$$

where $\Omega \subset \mathbb{R}^3$ is a bounded and convex domain with smooth boundary. The authors obtained a lower bound for the blow-up time when blow-up occurs.

Inspired by the aforementioned research studies, we consider the blow-up phenomena of problem (1). The highlight of this paper is considering both gradient term and nonlocal term sources, which make the problem more closer to the reality. In addition, there is little research on the blow-up phenomenon of the solution of problem (1) and even less research on the lower bound for the blow-up time. The main difficulty in studying (1) is to build suitable auxiliary functions. Since auxiliary functions defined in problems (2) and (3) are no longer applicable for our study, it is necessary to construct new auxiliary functions and use Sobolev inequalities to accomplish our research.

The paper is organized as follows. In Section 2, when $\Omega \subset \mathbb{R}^N (N \geq 3)$, we obtain a criterion for blow-up of the solution of problem (1) and give a lower bound for blow-up time. In Section 3, when $\Omega \subset \mathbb{R}^2$, a lower bound for blow-up time is derived. In Section 4, we present two examples to illustrate the applications of the abstract results obtained in this paper.

2. Lower Bound for Blow-Up Time

When $\Omega \subset \mathbb{R}^N (N \geq 3)$

In this section, our aim is to determine a lower bound for blow-up time $t^*$ when $\Omega \subset \mathbb{R}^N (N \geq 3)$. We now assume

$$
0 < g(u) \leq au',
$$

with constants $a > 0, l > 1$. Moreover, we suppose constants $r > 1, s > 2$, and

$$
s > m > 1, \quad m + 2l > r + 2.
$$

Let us define the following auxiliary function:

$$
J(t) = \int_{\Omega} u^s \, dx,
$$

where

$$
\alpha > \max \left\{ 1, 3 - m, 3 - m - \frac{2(m - 2)}{s - 2}, N(l - 1) \right\}.
$$

It is known from Corollary 9.14 in [20] that

$$
W^{1,2}(\Omega) \longrightarrow L^{2N/(N-2)}(\Omega), \quad N \geq 3,
$$

which implies

$$
(\int_{\Omega} w^{2N/(N-2)} \, dx)^{(N-2)/2N} \leq C(\int_{\Omega} w^2 \, dx + \int_{\Omega} |\nabla w|^2 \, dx)^{(1/2)}.
$$

Here $w \in W^{1,2}(\Omega)$ and $C = C(N, \Omega)$ is a Sobolev embedding constant depending on $N$ and $\Omega$. In this section, we need to use Sobolev inequality (9). The main result is formulated next.

**Theorem 1.** Let $u$ be a nonnegative classical solution of problem (1). Suppose (4)–(7) hold. If the solution $u$ blows up in the measure $J(t)$ at some finite time $t^*$, then $t^*$ is bounded by

$$
t^* \geq \frac{\int_{\Omega} C_1}{A_1 + A_2 \eta} \left[ \frac{\int_{\Omega} su^{s+1} \, dx}{A_1 + A_2 \eta} \right]^{\frac{1}{s+1}}
$$

where

$$
\int_{\Omega} C_1
$$
It follows from the Holder inequality that

\[ |\nabla u|^2 \leq \frac{2^2}{\sqrt{2}} \left( \frac{2}{\sqrt{2}} \right) \frac{a(m + l - 2)}{\min_{\Omega} |\Omega|} + \frac{a(m + l - 2)}{\rho_0 (|\Omega| + s)} \frac{2a(s - 2) + 2a(m - 2)}{s(a + m + 2l - 3)} \frac{s}{2} \left( \frac{s}{2} \right)^{-2(s - 2)} |\Omega|, \]

where \(|\Omega|\) is the measure of the bounded and convex domain \(\Omega\), \(\rho_0 = \min_{\Omega} (x \cdot \nu) > 0\), and \(d = \max_{\Omega} |x|\).

Proof. We compute by using (4), (7), and the divergence theorem.

\[
j' (t) = a \int_{\Omega} u^{a - 1} u_t dx = a \int_{\Omega} u^{a - 1} (\Delta u^m + \int_{\Omega} u^d dx - |\nabla u|^2) dx
\]

\[
= a \int_{\Omega} u^{a - 1} \Delta u^m dx + a \int_{\Omega} u^{a - 1} dx \int_{\Omega} u^d dx - a \int_{\Omega} u^{a - 1} |\nabla u|^2 dx
\]

\[
= -ma (a - 1) \int_{\Omega} u^{a - 1} |\nabla u|^2 dx + a \int_{\partial \Omega} u^{a - 2} \frac{\partial u}{\partial n} dS + a \int_{\Omega} u^{a - 1} dx \int_{\Omega} u^d dx
\]

\[
- a \int_{\Omega} u^{a - 1} |\nabla u|^2 dx
\]

\[
= -ma (a - 1) \int_{\Omega} u^{a - 1} |\nabla u|^2 dx + a m \int_{\Omega} u^{a - 1} g(u) dS + a \int_{\Omega} u^{a - 1} dx \int_{\Omega} u^d dx
\]

\[
- a \int_{\Omega} u^{a - 1} |\nabla u|^2 dx
\]

\[
\leq -ma (a - 1) \int_{\Omega} u^{a - 1} |\nabla u|^2 dx + a m a \int_{\partial \Omega} u^{a - 1} |\nabla u|^2 dS + a \int_{\Omega} u^{a - 1} dx \int_{\Omega} u^d dx
\]

\[
- a \int_{\Omega} u^{a - 1} |\nabla u|^2 dx.
\]
\[
\int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx \\
\leq \left( \int_{\Omega} u^{a-1}|\nabla u|^2 \, dx \right)^{(2/s)} \left( \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx \right)^{(s-2)/s} \\
\leq \left( \frac{2}{s} \int_{\Omega} u^{a-1}|\nabla u|^2 \, dx \right)^{(2/s)} \left[ \left( \frac{s}{2} \right)^{-2(s-2)} \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx \right]^{(s-2)/s} \\
\leq \int_{\Omega} u^{a-1}|\nabla u|^2 \, dx + \frac{s-2}{s} \left( \frac{s}{2} \right)^{-2(s-2)} \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx,
\]

which is equivalent to

\[
-a \int_{\Omega} u^{a-1}|\nabla u|^2 \, dx \leq -a \int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx + \frac{a(s-2)}{s} \left( \frac{s}{2} \right)^{-2(s-2)} \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx.
\]

Inserting (18) into (16), we derive

\[
J'(t) \leq -ma(\alpha - 1) \int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx + ma \int_{\partial \Omega} u^{a+m+l-2} \, dS \\
\quad + a \int_{\Omega} u^{a-1} \int_{\Omega} u^{l} \, dx - a \int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx + \frac{a(s-2)}{s} \left( \frac{s}{2} \right)^{-2(s-2)} \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx \\
\quad \leq \frac{4a[m(\alpha - 1) + 1]}{(a + m - 1)^2} \int_{\Omega} |\nabla u|^{a+m-1/2} \, dx + ma \int_{\partial \Omega} u^{a+m+l-2} \, dS \\
\quad + a \int_{\Omega} u^{a-1} \int_{\Omega} u^{l} \, dx + \frac{a(s-2)}{s} \left( \frac{s}{2} \right)^{-2(s-2)} \int_{\Omega} u^{a+m-3+2(m-2)/(s-2)} \, dx.
\]

In view of the lemma in [21], we have

\[
\int_{\partial \Omega} u^{a+m+l-2} \, dS \leq \frac{N}{\rho_0} \int_{\Omega} u^{a+m+l-2} \, dx + \frac{(a + m + l - 2)d}{\rho_0} \int_{\Omega} u^{a+m-3}|\nabla u|\, dx.
\]

For the second term on the right side of (20), we apply the Hölder inequality and the Young inequality to get

\[
\int_{\Omega} u^{a+m-3}|\nabla u|\, dx \leq \left( \int_{\Omega} u^{a+m+2l-3} \, dx \right)^{(1/2)} \left( \int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx \right)^{(1/2)} \\
\leq \frac{1}{2\epsilon_1} \int_{\Omega} u^{a+m+2l-3} \, dx + \frac{\epsilon_1}{2} \int_{\Omega} u^{a+m-3}|\nabla u|^2 \, dx \\
= \frac{1}{2\epsilon_1} \int_{\Omega} u^{a+m+2l-3} \, dx + \frac{2\epsilon_1}{(a + m - 1)^2} \int_{\Omega} |\nabla u|^{a+m-1/2} \, dx,
\]
where \( \varepsilon_1 \) is given in (15). Substituting (20) and (21) into (19), we derive

\[
J'(t) \leq -\frac{4a[m(a-1)+1]}{(a+m-1)^2} \int_{\Omega} \left| \nabla u^{(a+m-1)/2} \right|^2 dx \\
+ \frac{m a a d}{\rho_0} \int_{\Omega} u^{a+m-1} dx + \frac{(a+m+l-2)d}{\rho_0} \int_{\Omega} u^{a+m-3} \left| \nabla u \right| dx \\
+ \int_{\Omega} u^{a-1} dx \int_{\Omega} u' dx + \frac{a(s-2)(s/2)^{-2(s-2)}}{\rho_0} \int_{\Omega} u^{a+m-3}/(2(m-2)/(s-2)) dx
\]

where

\[
\begin{align*}
\Omega & \leq \frac{2a[m(a-1)+1]}{(a+m-1)^2} \int_{\Omega} \left| \nabla u^{(a+m-1)/2} \right|^2 dx + \frac{m a a d}{\rho_0} \int_{\Omega} u^{a+m-1} dx \\
& + \int_{\Omega} u^{a-1} dx \int_{\Omega} u' dx + \frac{a(s-2)(s/2)^{-2(s-2)}}{\rho_0} \int_{\Omega} u^{a+m-3}/(2(m-2)/(s-2)) dx
\end{align*}
\]

It follows from (5)–(7), the Hölder inequality, and the Young inequality that

\[
\begin{align*}
\int_{\Omega} u^{a-1} dx \int_{\Omega} u' dx \leq & \int_{\Omega} u^{a+r-1} dx |\Omega| \\
\leq & \left( \int_{\Omega} u^{a+m+2l-3} dx \right)^{(a+r-1)/(a+m+2l-3)} \left( \int_{\Omega} u^{m+2l-r-2}/(a+m+2l-3) dx \right)^{(a+m+2l-3)/(a+m+2l-3)} |\Omega| \\
= & \frac{a + r - 1}{a + m + 2l - 3} |\Omega| \int_{\Omega} u^{a+m+2l-3} dx + \frac{m + 2l - r - 2}{a + m + 2l - 3} |\Omega|,
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega} u^{a+m-3} = (2(m-2)/(s-2)) dx \leq & \left( \int_{\Omega} u^{a+m+2l-3} dx \right)^{(s-2)(a+m-3)+2(m-2)/(s-2)(a+m+2l-3))} |\Omega| \int_{\Omega} u^{a+m+2l-3} dx \\
\leq & \frac{(s-2)(a+m-3)+2(m-2)}{(s-2)(a+m+2l-3)} |\Omega| \int_{\Omega} u^{a+m+2l-3} dx + \frac{2l(s-2)-2(m-2)}{(s-2)(a+m+2l-3)} |\Omega|,
\end{align*}
\]

\[
\begin{align*}
\int_{\Omega} u^{a+m+2l-3} dx \leq & \left( \int_{\Omega} u^{a+m+2l-3} dx \right)^{(a+m+2l-3)/(a+m+2l-3)} |\Omega| \int_{\Omega} u^{a+m+2l-3} dx \\
\leq & \frac{a + m + l - 2}{a + m + 2l - 3} |\Omega| \int_{\Omega} u^{a+m+2l-3} dx + \frac{l - 1}{a + m + 2l - 3} |\Omega|.
\end{align*}
\]
Substituting (23)–(25) into (22) yields

\[
J'(t) \leq -\frac{2\alpha[m(a-1)+1]}{(\alpha+m-1)^2}\int_\Omega |\nabla u|^{(4m+1)/2} \, dx + A + M\int_\Omega u^{a+m+2l-3} \, dx, \tag{26}
\]

where \(A, M\) are defined in (13) and (14), respectively. Applying (7) and the Sobolev inequality (9), we have

\[
\int_\Omega u^{a+m+2l-3} \, dx \\
\leq \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} \left( \int_\Omega \left( u^{(a+m-1)/2} \right)^{2N/(N-2)} \, dx \right)^{(m+2l-3)(N-2)/(2a+N(m-1))} \\
\leq \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} C^{-2N/(N-2)} \left( \int_\Omega u^{a+m-1} \, dx + \int_\Omega |\nabla u|^{(4m+1)/2} \, dx \right)^{N/(N-2)} \left( \int_\Omega \left( u^{(a+m-1)/2} \right)^{2N/(N-2)} \, dx \right)^{(m+2l-3)(N-2)/(2a+N(m-1))} \\
= C^{2N(m+2l-3)/(2a+N(m-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} \left( \int_\Omega u^{a+m-1} \, dx + \int_\Omega |\nabla u|^{(4m+1)/2} \, dx \right)^{N/(N-2)} \left( \int_\Omega \left( u^{(a+m-1)/2} \right)^{2N/(N-2)} \, dx \right)^{(m+2l-3)(N-2)/(2a+N(m-1))}. \tag{27}
\]

Using the inequality

\[
(i_1 + i_2)^j \leq i_1^j + i_2^j, \quad i_1, i_2 > 0, \quad 0 \leq j < 1, \tag{28}
\]

we rewrite inequality (27) as

\[
\int_\Omega u^{a+m+2l-3} \, dx \\
\leq C^{2N(m+2l-3)/(2a+N(m-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} \left( \int_\Omega u^{a+m-1} \, dx \right)^{N(m+2l-3)/(2a+N(m-1))} \\
\quad + C^{2N(m+2l-3)/(2a+N(m-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} \left( \int_\Omega |\nabla u|^{(4m+1)/2} \, dx \right)^{N(m+2l-3)/(2a+N(m-1))}. \tag{29}
\]

For the first term on the right side of (29), we make use of the Hölder inequality and the Young inequality to get

\[
C^{2N(m+2l-3)/(2a+N(m-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(m-1))} \left( \int_\Omega u^{a+m-1} \, dx \right)^{N(m+2l-3)/(2a+N(m-1))} \\
= \left[ \left( \frac{2C^2 N (m + 2l - 3)}{2\alpha + N (m - 1)} \right)^{N(m+2l-3)/(2a+N(m-1))} \left( \int_\Omega u^a \, dx \right)^{(2a+N(m-1)-(m+2l-3)(N-2))/(2a+N(1-l))} \left( \int_\Omega \left( u^{(a+m-1)/2} \right)^{2N/(N-2)} \, dx \right)^{(m+2l-3)(N-2)/(2a+N(1-l))} \right]^{(2a+2N(1-l))/(2a+N(m-1))}.
\]
Moreover, it follows from the Hölder inequality and the Young inequality that

\[
\int_{\Omega} u^{\alpha+m-3} \text{d} x \leq \left( \int_{\Omega} u^{(\alpha+m-1)/2} \text{d} x \right)^2 N(m+2l-3)/(2\alpha+N(m-1)) \left( \int_{\Omega} |\nabla u|^{(\alpha+m-1)/2} \text{d} x \right) N(m+2l-3)/(2\alpha+N(m-1))
\]

\[
\leq \left( C^2 \varepsilon_2^{-1} N(m+2l-3)/(2\alpha+N(m-1)) \left( \int_{\Omega} u^{\alpha+m-1} \text{d} x \right)^{2(\alpha+N(m-1)-(m+2l-3)(N-2))/(2\alpha+N(m-1))} \right) \left( \int_{\Omega} |\nabla u|^{(\alpha+m-1)/2} \text{d} x \right) N(m+2l-3)/(2\alpha+N(m-1))
\]

\[
+ \frac{N(m+2l-3)}{2\alpha+N(m-1)} \varepsilon_2 \int_{\Omega} |\nabla u|^{(\alpha+m-1)/2} \text{d} x.
\]

Inserting (31)–(33) into (29), we have

\[
\int_{\Omega} u^{\alpha+m-1} \text{d} x \leq \frac{4 [\alpha + N (m - 1)] C^{N(m+2l-3)/(\alpha+N(1-l))} N(m+2l-3)/(2\alpha+N(1-l))}{2\alpha+N(m-1)} \left( \int_{\Omega} u^{\alpha+m-1} \text{d} x \right)^{-N(m+2l-3)/(2\alpha+N(1-l))} + \varepsilon_2^{-N(m+2l-3)/(2\alpha+N(1-l))} + \frac{4 (l - 1)}{\alpha + m + 2l - 3} \left( \int_{\Omega} u^\alpha \text{d} x \right)^{2(\alpha+N(m-1)-(m+2l-3)(N-2))/(2\alpha+N(1-l))} \left( \int_{\Omega} |\nabla u|^{(\alpha+m-1)/2} \text{d} x \right)^{-(\alpha+m-1)/2(1-l)}
\]

\[
|\Omega| + \frac{2N(m+2l-3)}{2\alpha+N(m-1)} \varepsilon_2 \int_{\Omega} |\nabla u|^{(\alpha+m-1)/2} \text{d} x.
\]
Combining (33) and (26), we derive

$$f'(t) \leq A_1 + A_2 \left( \int_{\Omega} u^\delta \, dx \right),$$

(34)

where $A_1$, $A_2$ are given in (11) and (12). Integrating between 0 and $t^*$, we arrive at

$$t^* \geq \int_{0}^{\infty} \frac{d\eta}{A_1 + A_2 \eta^{\frac{(2\alpha_1 N(m-1) - (m+2l-3)(N-2))(2\alpha_2 N(1-\delta))}{(2\alpha_2 N(1-\delta))}}},$$

(35)

3. Lower Bound for Blow-Up Time

When $\Omega \subset \mathbb{R}^2$

In this section, we will give a lower bound for the blow-up time when $\Omega \subset \mathbb{R}^2$. Here we still suppose that conditions (4) and (5) hold. Since the embedding theorem in (9) is no longer available when $N = 2$, before proving our main theorem, we note that the following Sobolev embedding:

$$W^{1,2}(\Omega) \rightarrow L^3(\Omega),$$

(36)

implies that

$$\bar{A}_1 = \bar{A} + \bar{M} \begin{cases} 4(l - 1) \frac{\beta + m + 2l - 3}{\beta + m + 2l - 3} (\beta + m + 2l - 3) \left( \frac{1}{\beta + m - 1} \right)^{(\beta + m - 1)/2(l-1)} |\Omega|, \end{cases}$$

(41)

$$\bar{A}_2 = \bar{M} \begin{cases} \frac{4}{\beta + m + 2l - 3} (\beta + m + 2l - 3) \left( \frac{1}{\beta + m - 1} \right)^{(\beta + m - 1)/2(l-1)} \left( \frac{1}{\beta + m + 2l - 3} \right)^{(\beta + m - 1)/2(l-1)} \\ \frac{4}{\beta + m + 2l - 3} (\beta + m + 2l - 3) \left( \frac{1}{\beta + m - 1} \right)^{(\beta + m - 1)/2(l-1)} \left( \frac{1}{\beta + m + 2l - 3} \right)^{(\beta + m - 1)/2(l-1)} \\ \frac{4}{\beta + m + 2l - 3} (\beta + m + 2l - 3) \left( \frac{1}{\beta + m - 1} \right)^{(\beta + m - 1)/2(l-1)} \left( \frac{1}{\beta + m + 2l - 3} \right)^{(\beta + m - 1)/2(l-1)} \end{cases}$$

(42)

$$\bar{A} = - \frac{2m\alpha_1 (l - 1)}{\rho_0 (\beta + m + 2l - 3)} |\Omega| + \frac{\beta (m + 2l - r - 2)}{\beta + m + 2l - 3} |\Omega|^2 + \frac{2\beta (s - 2)(s - 2) - (m - 2)}{s (\beta + m + 2l - 3)} \left( \frac{1}{s} \right)^{2(s - 2)} |\Omega|,$$

(43)

$$\bar{M} = \frac{2m\alpha_1 (\beta + m + l - 2)}{\rho_0 (\beta + m + 2l - 3)} + \frac{m\alpha_1 (\beta + m + l - 2)}{\beta + m + 2l - 3} |\Omega| + \frac{2s^2}{s (\beta + m + 2l - 3)} \left( \frac{1}{s} \right)^{2(s - 2)} ,$$

(44)

$$\bar{\xi}_1 = \rho_0 ^2 (\beta + m + l - 2),$$

$$\bar{\xi}_2 = \frac{\beta (m (\beta - 1) + 1) (\beta (\delta - 2) + \delta (m - 1))}{M \delta (m + 2l - 3) (\delta + m - 1)^2} ,$$

(45)

where $|\Omega|$ is the measure of the bounded and convex domain $\Omega$, $\rho_0 = \min_{\Omega} (x \cdot \nabla) > 0$, and $d = \max \{ |x| \}$. $\eta$

Proof. Repeating the calculations in (16)–(25), we have

$$H'(t) \leq \frac{2\beta [m (\beta - 1) + 1]}{(\beta + m - 1)^2} \int_{\Omega} |\nabla u^{(\beta + m - 1)/2}|^2 \, dx$$

+ $\bar{A} + \bar{M} \int_{\Omega} u^{\beta + m + 2l - 3} \, dx,$

(46)
where $\bar{A}, \bar{M}$ are defined in (43) and (44). Now, we pay our attention to the term $\int_\Omega \beta \varphi^{\beta + m + 2\delta - 3} \varphi_{x} \, dx$. It follows from the Hölder inequality and Sobolev inequality (37) that

$$
\int_\Omega \beta \varphi^{\beta + m + 2\delta - 3} \varphi_{x} \, dx \\
\leq \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{1 - (2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1)))} \left( \int_\Omega \left( \beta \varphi^{\beta + m - 1} \right)^{2} \varphi_{x} \, dx \right)^{2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))}
\leq \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{1 - (2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1)))} \left[ C^\delta \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx + \int_\Omega \left| \nabla \beta \varphi^{\beta + m - 1} \right|^{2} \varphi_{x} \, dx \right)^{\delta/(2\delta)} \right]^{2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))}
\leq C^{2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))} \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{1 - (2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1)))} \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx + \int_\Omega \left| \nabla \beta \varphi^{\beta + m - 1} \right|^{2} \varphi_{x} \, dx \right)^{2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))}.
\tag{47}
$$

Again using inequality (28), we rewrite (38) as

$$
\int_\Omega \beta \varphi^{\beta + m + 2\delta - 3} \varphi_{x} \, dx \\
\leq C^{2\delta(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))} \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{1 - (2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1)))} \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx \right)^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))}
\tag{48}
$$

For the first term on the right side of inequality (48), we make use of the Hölder inequality and the Young inequality to obtain

$$
\begin{align*}
C^\delta &\left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{1 - (2(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1)))} \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx \right)^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))} \leq \left[ \frac{2C^2 \delta(m + 2\delta - 3)}{\beta(\delta - 2) + \delta(m - 1)} \right]^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) - 2\delta(l - 1))} \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{\frac{\beta(\delta - 2) + \delta(m - 1)}{2\delta(m + 2\delta - 3)}} \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx \right)^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) + \delta(m - 1))} \\
&\leq \frac{\beta(\delta - 2) - 2\delta(l - 1)}{\beta(\delta - 2) + \delta(m - 1)} \left( \frac{2C^2 \delta(m + 2\delta - 3)}{\beta(\delta - 2) - 2\delta(l - 1)} \right)^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) - 2\delta(l - 1))} \left( \int_\Omega \beta \varphi^{\beta} \, dx \right)^{\frac{\beta(\delta - 2) + \delta(m - 1)}{2\delta(m + 2\delta - 3)}} \left( \int_\Omega \beta \varphi^{\beta + m - 1} \, dx \right)^{\delta(m + 2\delta - 3)/(\beta(\delta - 2) - 2\delta(l - 1))}
\end{align*}
\tag{49}
$$
Moreover,

\[
\int_{\Omega} u^{\delta_{m+1}} \, dx \\
\leq \left( \int_{\Omega} u^{\beta+m+2l-3} \, dx \right)^{(\beta+m-1)/(\beta+m+2l-3)} \left( \int_{\Omega} |\nabla u|^{2(l-1)/(\beta+m+2l-3)} \right)^{(2l-1)/(\beta+m+2l-3)} \\
= \left( \frac{\beta + m + 2l - 3}{2(\beta + m - 1)} \int_{\Omega} u^{\beta+m+2l-3} \, dx \right)^{(\beta+m-1)/(\beta+m+2l-3)} \left( \frac{\beta + m + 2l - 3}{2(\beta + m - 1)} \right)^{(2l-1)/(\beta+m+2l-3)} \Omega \\
\leq \frac{1}{2} \int_{\Omega} u^{\beta+m+2l-3} \, dx + \frac{2(l-1)}{\beta + m + 2l - 3} \left( \frac{\beta + m + 2l - 3}{2(\beta + m - 1)} \right)^{-(2l-1)/(\beta+m+2l-3)} \Omega.
\]  

(50)

Similar to the computation process in (50), we get

\[
\mathcal{C}^{2\delta(m+2l-3)/(\beta(\delta-2)+\delta(m-1))} \left( \int_{\Omega} u^{\beta} \, dx \right)^{1 - (2(m+2l-3)/(\beta(\delta-2)+\delta(m-1)))} \left( \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx \right)^{(\beta+m+2l-3)/(\beta(\delta-2)+\delta(m-1))} \\
\leq \left[ \mathcal{C}^{2 - 1} \right]^{\delta(m+2l-3)/(\beta(\delta-2)+\delta(m-1))} \left( \int_{\Omega} u^{\beta} \, dx \right)^{(\beta(\delta-2)+\delta(m-1)-2(m+2l-3))/(\beta(\delta-2)+2\delta(l-1))} \left( \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx \right)^{(\beta-2)+\delta(m-1)-2(m+2l-3))/(\beta(\delta-2)+2\delta(l-1))} \\
\cdot \left( \frac{2\delta(m+2l-3)}{\beta(\delta-2)+\delta(m-1)} \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx \right)^{2\delta(m+2l-3)/(\beta(\delta-2)+2\delta(l-1))} \\
\leq \left( \frac{2\delta(m+2l-3)}{\beta(\delta-2)+\delta(m-1)} \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx \right)^{2\delta(m+2l-3)/(\beta(\delta-2)+2\delta(l-1))} \\
+ \frac{\delta(m+2l-3)}{\beta}(\delta-2)+\delta(m-1)-2(l-1) \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx,
\]

(51)

where \( \mathcal{C} \) is given in (45). It follows from (48)–(51) that

\[
\int_{\Omega} u^{\beta+m+2l-3} \, dx \\
\leq \frac{2\delta(\delta-2) - 4\delta(l-1)}{\beta}(\delta-2)+\delta(m-1) \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx \left( \frac{2\delta(m+2l-3)}{\beta}(\delta-2)+\delta(m-1)-2(l-1) \right)^{-(\delta(m+2l-3)/(\beta(\delta-2)-2\delta(l-1)))} \\
\cdot \left( \int_{\Omega} u^{\beta} \, dx \right)^{(\beta(\delta-2)+\delta(m-1)-2(m+2l-3))/(\beta(\delta-2)-2\delta(l-1)))} \\
+ \frac{4(l-1)}{\beta+m+2l-3} \left( \frac{\beta+m+2l-3}{2(\beta+m-1)} \right)^{-\delta(m+2l-3)/(\beta(\delta-2)-2\delta(l-1))} \Omega \\
+ \frac{2\delta(m+2l-3)}{\beta}(\delta-2)+\delta(m-1) \int_{\Omega} |\nabla u|^{(\beta+m-1)/2} \, dx.
\]

(52)

Substituting (52) into (46), we have

\[
H^\prime(t) \leq \tilde{A}_1 + \tilde{A}_2 H^{\beta(\delta-2)+\delta(m-1)-2(m+2l-3)/(\beta(\delta-2)-2\delta(l-1))}(t),
\]

(53)
where $\tilde{A}_1, \tilde{A}_2$ are defined in (41) and (42). Integrating (53) between 0 and $t^*$, we deduce that
\[
t^* \geq \int_{H(0)}^\infty \frac{d\eta}{A_1 + A_2 \eta}\]
(54)

\section{Applications}

In this section, two examples are given to illustrate the abstract results of Theorems 1 and 2.

\textbf{Example 1.} Let $u(x,t)$ be the nonnegative classical solution of the following problem:
\[
\begin{cases}
u_t = \Delta u^2 + \frac{1}{100}u^2 \quad \text{in } \Omega \times (0,t^*), \\
\frac{\partial u}{\partial \nu} = \frac{1}{100}u^2 \quad \text{on } \partial \Omega \times (0,t^*), \\
u(x,0) = 9.9995 \times 10^{-2} + 5 \times 10^{-4}|x|^2 \quad \text{in } \Omega,
\end{cases}
\]
(55)

where $\Omega = \{x = (x_1, x_2, x_3) | |x|^2 = x_1^2 + x_2^2 + x_3^2 < 1/100\}$ is a ball in $\mathbb{R}^3$. Here we choose $N = 3, m = 2, l = 2, a = 1, r = 2, s = 3, \rho_0 = d = 1/10, |\Omega| = 4\pi/3000$, and $u_0(x) = 9.9995 \times 10^{-2} + 5 \times 10^{-4}|x|^2$. It is easy to check that conditions (4)–(6) hold. Moreover, from Theorems 2.1 and 3.2 in [22], we have $C = 7.5931$. By inserting the above constants into (12)–(16), we get $\epsilon_1 = 84.7805, \epsilon_2 = 0.6034, M = 3.7204, A = 3.5353 \times 10^{-3}, A_1 = 3.5870 \times 10^{-2},$ and $A_2 = 3.3855 \times 10^7$. It follows from Theorem 1 that
\[
t^* \geq \int_{H(0)}^\infty \frac{d\eta}{A_1 + A_2 \eta^2} = \int_{4.1882 \times 10^{-10}}^{\infty} \frac{d\eta}{3.5870 \times 10^{-2} + 3.3855 \times 10^7 \eta^2} \approx 0.1425,
\]
(56)

with
\[
J(0) = \int_\Omega u_0^6 \mathrm{d}x = \int_\Omega (9.9995 \times 10^{-2} + 5 \times 10^{-4}|x|^2)^6 \mathrm{d}x = 4.1883 \times 10^{-9}.
\]
(57)

\textbf{Example 2.} Let $u(x,t)$ be the nonnegative classical solution of following problem:
\[
\begin{cases}
u_t = \Delta u^2 + \frac{1}{100}u^2 \quad \text{in } \Omega \times (0,t^*), \\
\frac{\partial u}{\partial \nu} = \frac{1}{100}u^2 \quad \text{on } \partial \Omega \times (0,t^*), \\
u(x,0) = 9.9995 \times 10^{-2} + 5 \times 10^{-4}|x|^2 \quad \text{in } \Omega,
\end{cases}
\]
(58)

Here $\Omega = \{x = (x_1, x_2) | |x|^2 = x_1^2 + x_2^2 < 1/100\}$ is a circular domain in $\mathbb{R}^2$. Select $\delta = 4, m = 2, l = 2, \beta = 7, r = 2, s = 3, \rho_0 = d = 1/10, |\Omega| = \pi/100$, and $u_0(x) = 9.9995 \times 10^{-2} + 5 \times 10^{-4}|x|^2$. Moreover, combining (41)–(45), we can compute $A_1 \approx 2.9720 \times 10^{-1}, A_2 \approx 1.1905 \times 10^3, A = 3.1030 \times 10^{-2}, \epsilon_1 \approx 72.2223, \epsilon_2 \approx 1.6412, and M \approx 3.3269.$ It is easy to verify that (4), (5), and (9) hold. According Theorem 2, we have the lower bound for the blow-up time:
\[
t^* \geq \int_{H(0)}^\infty \frac{d\eta}{A_1 + A_2 \eta^2} = \int_{4.1882 \times 10^{-10}}^{\infty} \frac{d\eta}{2.9720 \times 10^{-1} + 1.1905 \times 10^3 \eta^2} \approx 8.3590 \times 10^{-2},
\]
(59)

\section{Data Availability}

No data were used to support this study.

\section{Conflicts of Interest}

The authors declare that they have no conflicts of interest.

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