Kähler quantization of $H^*(\mathbb{T}^2, \mathbb{R})$ and modular forms

Farhang Loran

Department of Physics, Isfahan University of Technology (IUT)
Isfahan, Iran
E-mail: loran@cc.iut.ac.ir

Abstract: Kähler quantization of $H^1(\mathbb{T}^2, \mathbb{R})$ is studied. It is shown that this theory corresponds to a fermionic $\sigma$-model targeting a noncommutative space. By solving the complex-structure moduli independence conditions, the quantum background independent wave function is obtained. We study the transformation of the wave function under modular transformation. It is shown that the transformation rule is characteristic to the operator ordering. Similar results are obtained for Kähler quantization of $H^2(\mathbb{T}^2, \mathbb{R})$.

Keywords: Topological Field Theories, Topological Strings
1. Introduction

Recently in [1] it is shown that the topological string partition function in real polarization is a holomorphic function but not a modular form in the usual sense. The partition functions of the topological B-model on a Calabi-Yau threefold satisfy the holomorphic anomaly equation [2] which can be obtained by Kähler quantization of $H^3(CY_3, \mathbb{R})$ [3, 4, 5]. In this paper, we study Kähler quantization of $H^1(T^2, \mathbb{R})$. This theory is equivalent to a fermionic $\sigma$-model $\mathbb{R} \to \mathbb{P}_2$ where $\mathbb{P}_2 = \mathbb{R}^2/\text{SL}(2, \mathbb{Z})$ is the two dimensional space of periods of $\Omega \in H^1(T^2, \mathbb{R})$. The quantum background independent wave function can be obtained by solving the complex-structure moduli independence conditions [5]. We show that in real polarization, the wave function is a quasi-modular form of weight one\(^1\),

$$\psi(\tau) \to |c\tau + d| \psi(\tau), \quad (1.3)$$

under the modular transformation,

$$\tau \to \tau' = \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}). \quad (1.4)$$

Similar results are obtained for Kähler quantization of $H^2(T^2, \mathbb{R})$. The weight of the wave function is characteristic to the operator ordering. By operator ordering one means

\(^1\)By definition a modular form of weight $k$, is a holomorphic function $f(\tau)$ which transforms as

$$f(\tau) \to (c\tau + d)^k f(\tau), \quad (1.1)$$

under modular transformation. Here we define a quasi-modular form of weight $k$ by transformation rule,

$$\psi(\tau) \to |c\tau + d|^k \psi(\tau), \quad (1.2)$$
the ordering of the operators e.g. $\hat{x}$ and $\hat{p}$, corresponding to the coordinate $x$ and the conjugate momentum $p$ in e.g. the expression $xp$. If one uses the standard convention $xp \to \frac{1}{2}[\hat{x}, \hat{p}]$ where $[\hat{x}, \hat{p}] = i\hbar$ is assumed, then the wave function in the Kähler quantization of $H^*(T^2, \mathbb{R})$ is a quasi-modular form of weight one as is given in Eq. (1.3). But if for example one uses the unconventional operator ordering $xp \to \hat{x}\hat{p}$ while assuming $[\hat{x}, \hat{p}] = i\hbar$, it can be shown that the wave function is a quasi-modular form of weight zero, i.e. it is invariant under the modular group SL(2, Z).

The organization of the paper is as follows. In section 2 our notation, conventions and some basic calculations are given. In section 3, Kähler quantization of $H^1(T^2, \mathbb{R})$ is studied and the moduli-independence conditions are solved for the quantum wave function. In section 3.1, Kähler quantization of $H^2(T^2, \mathbb{R})$ is studied. In section 4 we show that the cohomology $H^1(T^2, \mathbb{R})$ corresponds to the fermionic $\sigma$-model $\mathbb{R} \to \mathcal{P}_2$. Furthermore it is shown that $\mathcal{P}_2$ is a noncommutative space. In section 5 the dependence of the weight of the wave function on the operator ordering is examined in the unconventional ordering $xp \to \hat{x}\hat{p}$.

2. Preliminaries

In this section we derive necessary equations used in the subsequent sections and give our notations and conventions.

Take a complex plane $\mathbb{C}$ and define a lattice $L(\omega_1, \omega_2) = \{\omega_1m + \omega_2n | m, n \in \mathbb{Z}\}$ where $\omega_1$ and $\omega_2$ are nonvanishing complex numbers such that $\omega_2/\omega_1 \notin \mathbb{R}$. A two-dimensional torus $T^2 \cong \mathbb{C}/L(\omega_1, \omega_2)$ is obtained by identifying the points $z_1, z_2 \in \mathbb{C}$ such that $z_1 - z_2 = \omega_1m + \omega_2n$ for some $m, n \in \mathbb{Z}$. The complex structure on $T^2$ is defined by the pair of complex numbers $(\omega_1, \omega_2)$ modulo a constant factor and PSL(2, Z) [6]. Consequently, one can take 1 and the modular parameter $\tau = \omega_2/\omega_1$ to be the generators of the lattice. The complex structure of $T^2$ is thus specified by $\tau$. Furthermore, $\tau$ and $\tau' = (a\tau + b)/(c\tau + d)$ define the same complex structure if,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}).$$

To any point $(x, y) \in L(1, \tau)$ we assign coordinates $(\alpha, \beta) \in T^2$ (see Fig. 1),

$$\alpha = x - y \cot \phi, \quad \beta = \frac{y}{\tau \sin \phi},$$

in which $\tau$ and $\phi$ are moduli defined in terms of the modular parameter $\tau = \tau e^{i\phi}$. By definition,

$$\int_A d\alpha = 1, \quad \int_B d\alpha = 0,$$

$$\int_A d\beta = 0, \quad \int_B d\beta = 1.$$  (2.3)

In real polarization, a one-form $\Omega \in H^1(T^2, \mathbb{R})$ is given in terms of its periods,

$$h_r = \int_A \Omega, \quad h_i = -\int_B \Omega.$$  (2.4)
Figure 1: $T^2$: the length of $A$-cycle is 1 and the length of $B$-cycle is $\tau$.

Thus,

$$\Omega = h_r d\alpha - h_i d\beta.$$  (2.5)

Under an infinitesimal moduli variation $\delta\tau$ and $\delta\phi$,

$$\delta\alpha = \frac{\tau \beta}{\sin \phi} \delta\phi,$$

$$\delta\beta = -\left(\frac{1}{\tau} \delta\tau + \cot \phi \delta\phi\right) \beta.$$  (2.6)

Since $\Omega$ is independent of moduli, i.e. $\delta\Omega = 0$, one obtains,

$$\delta h_r = 0,$$

$$\delta h_i = \frac{h_r \tau}{\sin \phi} \delta\phi + h_i \left(\frac{\delta\tau}{\tau} + \cot \phi \delta\phi\right).$$  (2.7)

On the other hand, under modular transformation (1.4),

$$\tau \cos \phi \rightarrow \tau' \cos \phi' = \frac{\tau \cos \phi + \frac{act^2 + 2bc \tau \cos \phi + bd}{|c\tau + d|^2}}{|c\tau + d|^2},$$

$$\tau \sin \phi \rightarrow \tau' \sin \phi' = \frac{\tau \sin \phi}{|c\tau + d|^2},$$  (2.8)

thus,

$$\alpha \rightarrow \alpha' = \alpha - \beta \left(\frac{act^2 + 2bc \tau \cos \phi + bd}{|c\tau + d|^2}\right),$$

$$\beta \rightarrow \beta' = |c\tau + d|^2 \beta.$$  (2.9)

$\Omega$ is invariant under modular transformation. Thus under modular transformation,

$$h_r \rightarrow h_r',$$

$$h_i \rightarrow h_i' = \frac{h_i}{|c\tau + d|^2} - h_r \frac{act^2 + 2bc \tau \cos \phi + bd}{|c\tau + d|^2}.$$  (2.10)
A two form \( \Xi \in H^2(\mathbb{T}^2, \mathbb{R}) \) in real polarization is given by \( \Xi = \xi d\alpha \wedge d\beta \). Assuming invariance of \( \Xi \) under infinitesimal moduli variation, it is easy to show that,

\[
\delta \xi = \left( \frac{\delta \tau}{\tau} + \cot \phi \delta \phi \right) \xi.
\]  

(2.11)

Furthermore under modular transformation,

\[
\xi \rightarrow \xi' = \frac{\xi}{e^{\tau + d}}.
\]  

(2.12)

3. Kähler quantization of \( H^1(\mathbb{T}^2, \mathbb{R}) \)

Using Eq.(2.7), the classical generators of translation along moduli directions can be defined as follows,

\[
H_\tau = \frac{1}{\tau} h_i \pi_i,
\]

\[
H_\phi = \left( \frac{h_{r\tau}}{\sin \phi} + h_i \cot \phi \right) \pi_i,
\]  

(3.1)

in which \( \pi_i \) is the momentum conjugate to \( h_i \) satisfying the Poisson algebra \( \{ h_i, \pi_i \} = 1 \).

For quantization, one replaces the classical fields with the corresponding operators and uses the standard commutation relation,

\[
[\hat{h}_i, \hat{\pi}_i] = i.
\]  

(3.2)

Henceforth by e.g. \( h_i \) we mean the operator \( \hat{h}_i \).

Using the standard operator ordering the quantum mechanical generators of translation along moduli directions are defined as,

\[
H_\tau = \frac{1}{\tau} \left[ h_i, \pi_i \right]_+ = -i \left( h_i \frac{\partial}{\partial h_i} + \frac{1}{2} \right),
\]

\[
H_\phi = \frac{h_{r\tau}}{\sin \phi} \pi_i + \frac{\left[ h_i, \pi_i \right]_+}{2} \cot \phi = -i \frac{h_{r\tau}}{\sin \phi} \frac{\partial}{\partial h_i} - i \cot \phi \left( h_i \frac{\partial}{\partial h_i} + \frac{1}{2} \right),
\]  

(3.3)

The background independent wave function \( \psi \) is defined by the following relations \[10\],

\[
i \frac{\partial}{\partial \tau} \psi = H_\tau \psi,
\]

\[
i \frac{\partial}{\partial \phi} \psi = H_\phi \psi.
\]  

(3.4)

Solving for \( \psi \) one obtains,

\[
\psi = \begin{cases} 
\frac{1}{\sqrt{h_i}} f \left( \frac{h_i}{\tau \sin \phi} \right), & h_r = 0, \\
c_1 \frac{\sqrt{\tau \sin \phi}}{h_i + h_{r\tau} \cos \phi} + c_2 \frac{\sqrt{\tau \sin \phi}}{h_i}, & h_r \neq 0,
\end{cases}
\]  

(3.5)

in which \( f \) is an arbitrary \( C^1 \) function and \( c_1 \) and \( c_2 \) are two arbitrary constants. Using Eqs.(2.9) and (2.10) one can simply show that the wave function \( \psi \) is a quasi-modular form of weight one. See Eq.(1.3).
3.1 Kähler quantization of $H^2(T^2, \mathbb{R})$

Using Eq. (2.11) one can obtain the quantum mechanical generators of translation along moduli directions,

$$
H_{\xi\tau} = -\frac{i}{\tau} \left( \xi \frac{\partial}{\partial \xi} + \frac{1}{2} \right),
$$

$$
H_{\xi\phi} = -i \cot \phi \left( \xi \frac{\partial}{\partial \xi} + \frac{1}{2} \right).
$$

(3.6)

The corresponding quantum background independent wave function satisfies,

$$
i \frac{\partial}{\partial \tau} \psi_\xi = H_{\xi\tau} \psi_\xi,
$$

$$
i \frac{\partial}{\partial \phi} \psi_\xi = H_{\xi\phi} \psi_\xi.
$$

(3.7)

Thus,

$$
\psi_\xi = \frac{1}{\sqrt{\xi}} f_\xi \left( \frac{\xi}{\tau \sin \phi} \right),
$$

(3.8)

where $f_\xi$ is an arbitrary $C^1$ function. Using Eqs. (2.9) and (2.12), one can show that $\psi_\xi$ is a quasi-modular form of weight one.

4. Fermions on a line

The cohomology $H^1(T^2, \mathbb{R})$ has the structure of a phase space. Similar to the cohomology $H^3(CY_3, \mathbb{R})$ which describes the classical solutions for the 7 dimensional action \[4, 5\],

$$
S = \frac{1}{2} \int_{CY_3 \times \mathbb{R}} \gamma \wedge d\gamma,
$$

(4.1)

$H^1(T^2, \mathbb{R})$ describes the classical solutions for the 3 dimensional action,

$$
S \sim \int_{T^2 \times \mathbb{R}} \eta \wedge d\eta,
$$

(4.2)

in which $\eta$ is a real one-form on $T^2 \times \mathbb{R}$.

$$
\eta = \Omega + \rho dt
$$

(4.3)

where $\Omega$ is a real one form along $T^2$, $\rho$ is a zero-form and $t$ parameterizes $\mathbb{R}$. This claim can be verified by noting that the action (4.2) is equivalent to,

$$
S \sim \int_{\mathbb{R}} \int_{T^2} (-\Omega \wedge \partial_\tau \Omega - 2\rho d_{T^2} \Omega).
$$

(4.4)

Obviously, the equation of motion corresponding to $\rho$ implies that

$$
d_{T^2} \Omega = 0.
$$

(4.5)
Therefore, \( \Omega \) is a real closed one form which has been the subject of study in section 3.

Using Eqs. (2.4) and (2.5), the action (4.4) can be written as follows,

\[
S \sim \int_{\mathbb{R}} \left( h_r \partial_t h_i - h_i \partial_t h_r \right) dt. \tag{4.6}
\]

Thus defining the spinor \( h \),

\[
h = \begin{pmatrix} h_r \\ h_i \end{pmatrix}, \tag{4.7}
\]

and the gamma matrix,

\[
\gamma^0 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{4.8}
\]

it can be verified that action (4.6) describes free fermions \( h \) on a real line \( \mathbb{R} \),

\[
S \sim \int \left( -ih^t \gamma^0 \partial_t h \right) dt. \tag{4.9}
\]

It should be noted that the spinor field \( h \) is subject to the identification (2.10) under modular group \( \text{SL}(2, \mathbb{Z}) \). Consequently, the action (4.9) describes the fermionic \( \sigma \)-model \( \mathbb{R} \to \mathcal{P}_2 \) noted in section 1.

\( \mathcal{P}_2 \) is a noncommutative space since it is described by a first order Lagrangian. Indeed, using Eq. (4.6) one obtains,

\[
\pi_r = \frac{\delta S}{\delta \partial_t h_r} = -bh_i,
\]

\[
\pi_i = \frac{\delta S}{\delta \partial_t h_i} = bh_r, \tag{4.10}
\]

in which \( b \) is the unspecified constant coefficient in the definition of the action (4.6). Therefore, the action (4.6) is describing a constrained system of second class \( [7] \). The Dirac bracket algebra implies that,

\[
\{h_i, h_r\}_{\text{DB}} = \frac{1}{2b}, \tag{4.11}
\]

though the corresponding Poisson bracket vanishes. Consequently the target space \( \mathcal{P}_2 \) is equipped with a noncommutative structure. In [5], in a similar situation, the coefficient of the action (4.1) is determined by requiring that quantization of \( H^3(CY_3, \mathbb{R}) \) gives exactly the holomorphic anomaly equation in the topological string model [2]. Here, \( b \) remains undetermined.

5. Dependence of the weight of quasi-modular form on operator ordering

In this section we examine the dependence of the weight of quasi-modular form on the operator ordering by considering the unconventional ordering \( xp \to \hat{x}\hat{p} \). Under this ordering,
the quantum mechanical generators of translation along moduli directions \( H_\tau \) and \( H_\phi \) in the Kähler quantization of \( H^1(\mathbb{T}^2, \mathbb{R}) \) that can be obtained from Eq. (3.1) are,

\[
H_\tau = -i \frac{h_i}{\tau} \frac{\partial}{\partial h_i}, \\
H_\phi = -i \frac{h_i \tau}{\sin \phi} \frac{\partial}{\partial h_i} - i \cot \phi h_i \frac{\partial}{\partial h_i}.
\] (5.1)

The solution to the corresponding wave-equation is,

\[
\psi = \begin{cases} 
  f \left( \frac{h_i}{\tau \sin \phi} \right), & h_\tau = 0, \\
  \frac{h_i \tau \sin \phi}{h_i + h_\tau \tau \cos \phi} c_1 + c_2, & h_\tau \neq 0,
\end{cases}
\] (5.2)

This can be easily shown to be invariant under the modular transformation (2.8) and (2.10). Thus under the unconventional operator ordering \( xp \rightarrow \hat{x} \hat{p} \), the background independent wave function \( \psi \) in real polarization becomes a quasi-modular form of weight zero.

A similar result can be obtained in the Kähler quantization of \( H^2(\mathbb{T}^2, \mathbb{R}) \). Here,

\[
H_\xi \tau = -i \frac{\xi}{\tau} \frac{\partial}{\partial \xi}, \\
H_\xi \phi = -i \cot \phi \xi \frac{\partial}{\partial \xi}.
\] (5.3)

and thus,

\[
\psi_\xi = f_\xi \left( \frac{\xi}{\tau \sin \phi} \right),
\] (5.4)

which is invariant under the modular transformation.

Summary

In the Kähler quantization of \( H^3(CY_3, \mathbb{R}) \), the genus \( g \) partition function is shown to be an almost modular form \([1]\). Here we showed that in the Kähler quantization of \( H^*(\mathbb{T}^2, \mathbb{R}) \), the background independent wave function in real polarization is a quasi-modular form of weight one in the standard operator ordering \( xp \rightarrow \frac{1}{2}[\hat{x}, \hat{p}]_+ \), while it is a quasi-modular form of weight zero in the unconventional operator ordering \( xp \rightarrow \hat{x} \hat{p} \).

Acknowledgement

The financial support of Isfahan University of Technology (IUT) is acknowledged.

References

[1] M. Aganagic, V. Bouchard and A. Klemm, Topological Strings and (Almost) Modular Forms [hep-th/0607100].

[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes, Commun. Math. Phys. 165 (1994) 311 [hep-th/9309140].
[3] E. Witten, *Quantum background independence in string theory*, [hep-th/9306122].

[4] E. Verlinde, *Attractors and the Holomorphic Anomaly*, [hep-th/0412139];
A.A. Gerasimov and S.L. Shatashvili, *Towards integrability of topological strings, I. Three-forms on Calabi-Yau manifolds*, JHEP 11 (2004) 074 [hep-th/0409238];
C. Gomez and S. Montanez, *A Comment on Quantum Distribution Functions and the OSV Conjecture*, [hep-th/0608162].

[5] F. Loran, JHEP 0512 (2005) 004, [hep-th/0510163].

[6] N. Koblitz, *Introduction to elliptic curves and modular forms*, Springer-Verlag, New York, 1984.

[7] P.A.M. Dirac, *Generalized hamiltonian dynamics*, Can. J. Math. 2 (1950) 129; *Generalized hamiltonian dynamics*, Proc. R. Soc. London Ser. A 246 (1958) 326; *Lectures on quantum mechanics*, Yeshiva University Press, New York, 1964;
M. Henneaux and C. Teitelboim, *Quantization of gauge system*, Princeton University Press, Princeton, New Jersey, 1992;
J. Govaerts, *Hamiltonian quantisation and constrained dynamics*, Leuven Notes in *Theoretical and Mathematical Physics*, Leuven University Press, 1991.