Finite market size as a source of extreme wealth inequality and market instability

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Abstract

We study the finite-size effects in some scaling systems, and show that the finite number of agents $N$ leads to a cut-off in the upper value of the Pareto law for the relative individual wealth. The exponent $\alpha$ of the Pareto law obtained in stochastic multiplicative market models is crucially affected by the fact that $N$ is always finite in real systems. We show that any finite value of $N$ leads to properties which can differ crucially from the naive theoretical results obtained by assuming an infinite $N$. In particular, finite $N$ may cause in the absence of an appropriate social policy extreme wealth inequality $\alpha < 1$ and market instability.

Key words: Power law; Multiplicative process; Cut-off; Finite-size effect.

Power laws appear ubiquitously as probability distributions in a wide range of systems. They have been studied for a long time both theoretically and practically. For instance, the relative number of people having a wealth between $w$ and $w + dw$ has been found already 100 years ago to fulfill the Pareto power law [1]:

$$P(w)dw \sim w^{-1-\alpha}dw.$$ (1)

From the very beginning it turned out that especially for small values of exponents ($\alpha < 2$) which imply slow decay of the distribution at infinity, this distribution form presents quite a few surprises, paradoxes and unexpected

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properties. In particular, the limit of an infinite number of individuals \( N \to \infty \) is not defined for many of the measurements.

In the present paper we study some dynamical processes where the naive theoretical results obtained for \( N = \infty \) fail for any arbitrarily large finite \( N \). One of the main sources of this non-uniform limit is the upper cut-off effect which any finite value of \( N \) implies. We mean by "the upper cut-off effect" the following simple fact: Suppose the wealth of individuals in a particular system is distributed by a power law with \( \alpha < 2 \). Then, for any value \( w \), there is a finite probability that an individual \( i \) possesses a wealth \( w_i \) larger than \( w \). For a finite system, if one defines by \( x_i = w_i/W \) the relative wealth of the individual \( i \), with \( W \) the total wealth, one has obviously probability 0 that any \( x_i \) will take a value larger than 1. Therefore, obviously, the distribution of the relative wealth \( x_i \) cannot be a power law and it has an upper cut-off. Moreover, when this cut-off is taken into account the very value of the exponent \( \alpha \) (in the region of \( x_i \) values in which the power law does apply) is affected.

This discontinuous behavior of the limit \( N \to \infty \) is important in practical applications where the samples are always finite. Indeed, it means that certain theoretical results obtained within the assumption of \( N = \infty \) are invalid. The problem of finite-size effects in stock market models has been investigated [2], especially for the price changes. Here we concentrate mainly on the effect of finite-size cut-off on the power-law of the wealth distribution.

In this paper we highlight the qualitative difference between the behaviors of the finite \( N \) system and the theoretical results for \( N = \infty \). We consider first the example of a pure multiplicative stochastic process that leads for \( N = \infty \) to the log-normal distribution. The log-normal distribution implies an effective exponent which at asymptotically large values diverges to \( \alpha \to \infty \). This is compared with the actual behavior for any finite \( N \) which converges towards \( \alpha = 0 \). Then, we study the multiplicative stochastic processes with a lower bound, which lead theoretically for \( N = \infty \) to power laws with \( \alpha > 1 \)[3,4]. We contrast it with the finite \( N \) systems where values \( \alpha < 1 \) can take place.

In particular in financial systems, our results imply that in the absence of a social security policy which takes appropriately into account the effects of a finite population \( N \), the Pareto exponent \( \alpha \) may fall to values \( \alpha < 1 \). This in turn means the concentration of an \( O(1) \) fraction of the total wealth in the hands of just a few individuals even if the market size is asymptotically large. We show that beyond the moral and social problems which such an extreme unequal distribution of wealth implies, \( \alpha < 1 \) also means that the financial markets themselves become unstable as their fluctuations become macroscopic.
Let us start with a multiplicative random walk process involving \(N\) independent variables (or individual wealths in financial system) \(w_i, i = 1, \ldots, N\), which undergo each the updating process:

\[
  w_i(t + 1) = \lambda_i(t)w_i(t),
\]

where the random numbers \(\lambda_i\) are independent but extracted with the same time-fixed and \(i\)-independent probability distribution \(\Pi(\lambda)\).

According to the generalized central limit theorem, if the distribution \(\Pi(\lambda)\) is characterized by the mean

\[
  \langle \ln \lambda \rangle = v,
\]

and variance

\[
  \langle (\ln \lambda)^2 \rangle - \langle \ln \lambda \rangle^2 = D,
\]

the asymptotic time dependence of the \(w\) probability distribution is:

\[
  P(w, t)dw = \frac{1}{\sqrt{2\pi Dt}} \exp \left( -\frac{1}{2Dt}(\ln w - vt)^2 \right) \frac{1}{w} dw.
\]

In particular, it was pointed out that this log-normal distribution can be regarded as an effective power law (1) with exponent \(\alpha\) changing with \(w\) [5]. For small \(w\), \(\alpha \sim 0\) which corresponds to \(1/w\) distribution, while in the large \(w\) tail, this distribution (5) behaves as a power law with exponent diverging to infinity [6].

To see this, let us put the distribution in the form [6]:

\[
  P(w, t) = \frac{1}{\sqrt{2\pi Dt}} \frac{\exp(\alpha(w)vt)}{w^{1+\alpha(w)}},
\]

where the effective \(w\)-dependent power exponent \(\alpha(w)\) is defined by

\[
  \alpha(w) = \frac{1}{2Dt} \ln \left( \frac{w}{\exp(vt)} \right).
\]

Then formally \(\alpha(w \to \infty) = \infty\), i.e., the tail of the distribution has an effective power law with exponent infinity.

We will show however that this tail is completely irrelevant for any finite system at large enough times: there is simply no variable \(w\) with such large
value within the system. As the time goes on and the distribution spreads
over larger intervals, the initial number of variables $N$ becomes too sparse
to sample the far away tail. Thus, in reality, the behavior of the theoretical
$N = \infty$ distribution at asymptotic times is quite different for any system
containing a finite number $N$ of variables.

An upper bound for the values which the $w_i$’s can take is given by the total
wealth in the system, and obviously none of the individuals can have a larger
value, that is,

$$w_i < W,$$

(8)

with the total wealth $W$ defined as

$$W(t) = \sum_{i=1}^{N} w_i(t) = N\bar{w}(t).$$

(9)

Thus, from Eqs. (7) and (8) we have

$$\alpha < \frac{1}{2Dt} \ln \left( \frac{W(t)}{\exp(vt)} \right).$$

(10)

In the limit $N = \infty$, $\bar{w}$ in Eq. (9) is proportional to the average value

$$\langle w \rangle = \int wP(w)dw,$$

(11)

and for the multiplicative system (2) it is [7]

$$\langle w \rangle = \langle \lambda \rangle^t = \exp(\ln(\lambda)t).$$

(12)

However, the result is very different for finite $N$. There, the observed mean
wealth $\bar{w} = \sum_i w_i/N$ is much less than the $N = \infty$ average $\langle w \rangle$, since the
average values (and other statistical properties) of the multiplicative system
are determined by the extreme events and the most extreme events can appear
only for large enough $N$ [8]. It has been shown in Ref. [8] that the mean $\bar{w}$
approaches the value of $\langle w \rangle$ (Eq. (12)) when the system size $N > \exp(\mu t)$ (with
$\mu$ a constant and decided by the details of the multiplicative process), which at
asymptotic times is far beyond the size of simulations or real (e.g., financial)
systems. In fact, for finite-size multiplicative systems the total wealth $W(t) = \sum_i w_i(t)$ at large enough times mainly depends on the largest value among $w_i$
[7,9], that is,

$$W(t) \sim w_{\max} = \max_{i=1}^{N} w_i(t).$$

(13)
The value of $w_{\text{max}}$ can be obtained analytically for $N \ll \exp(Dt)$, (i.e., for finite $N$ and asymptotic times) in the case in which the random numbers $\ln \lambda$ are extracted from a Gaussian distribution of zero mean $v$ and finite variance $D$ [7,10]:

$$w_{\text{max}} \simeq \exp \left( \sqrt{2Dt \ln N} \right).$$

Substituting Eqs. (13) and (14) into (10) one obtains that

$$\alpha < \sqrt{\frac{\ln N}{2Dt}}.$$  \hspace{1cm} (15)

Thus, Eq. (15) implies that for finite size $N$ and large enough time $t$, the effective $\alpha$ value is far from being infinite even for the large $w$ tail. In fact, the exponent $\alpha$ decreases as time takes larger and larger values.

The numerical simulations of the system for various times and $N$ values confirm this prediction. One way to analyze the simulation data and to validate the power law is to use a Zipf plot [11]. In order to make a Zipf plot one re-labels the various values of the individual wealths $w_i$ existing in the system at a given time in descending order as follows. One denotes by $w(1)$ the wealth $w_{\text{max}}$ of the richest individual, $w(2)$ the wealth of the wealthiest among the remaining individuals, and so on, until one reaches the poorest individual whose wealth is denoted by $w(N)$. With these notations, the power law Eq. (1) leads to the Zipf law:

$$w(n) \sim n^{-1/\alpha}.$$  \hspace{1cm} (16)

Thus, in the log-log plot of $w(n)$ (the Zipf plot) the power law (1) corresponds to a straight line with slope $-1/\alpha$. Visually, steeper slopes represent smaller exponents $\alpha$. E.g., the $1/w$ distribution (corresponding to $\alpha = 0$) leads to an infinite slope in the Zipf plot, and conversely, a flat and almost horizontal Zipf plot corresponds to $\alpha \to \infty$.

Figs. 1 and 2 exhibit the Zipf plots for the multiplicative system (2) with $\ln \lambda$ distributed by a Gaussian with $v = 0$ and $D = 0.01$. As expected from Eq. (15), all the slopes for finite $N$ are larger than 0, corresponding to power laws with effective exponents $\alpha$ much smaller than the theoretical $N = \infty$ prediction $\alpha = \infty$. In fact, in contrast to the $N = \infty$ result, as time increases $\alpha \to 0$. This is clearly seen in Fig. 1 where the slope $-1/\alpha$ becomes steeper with increasing time. In accordance with the same Eq. (15) the Zipf plot (Fig. 2) gives steeper and steeper slopes ($\alpha \to 0$) as $N$ decreases from 5000 to 50.

The same results can be obtained for different distributions of the random
factor $\ln \lambda$. Fig. 3 shows the Zipf plots for $\ln \lambda$ uniformly distributing in a range $(-0.1, 0.1)$. The above phenomena caused by the upper cut-off (8) are still present.

This upper cut-off effect holds for any finite value of $N$ for large enough times, and is not limited to the log-normal distribution. A similar argument insures that other slow-decaying distributions have sharp truncations and/or departures from the $N = \infty$ results. In order to uncover these effects one has to use appropriate plots for each quantity. For instance, the upper cut-off does not affect the straight line shape of the Zipf plot for a system obeying the power-law distribution (1). However, this cut-off is made explicit by first normalizing each variable (wealth) to the total wealth and averaging over many realizations to obtain an estimation of the probability Pareto distribution $P(w/W)$. By plotting then $\log P(w/W)$ against $\log(w/W)$, obviously one cannot expect a straight line since at least for $w/W > 1$ one has $\log P(w/W) = -\infty$. In the actual plots, indeed a dramatic bent is shown in the graph when the variable $w$ approaches the total capital $W$.

Thus, we encounter an unexpected subtle effect related with the special properties of the power laws. The effect originates in the fact that the density of probability $P(w)$ has to be 0 at $w = W$ due to Eq. (8), in spite of the naive $N = \infty$ expectation [12] that the distribution Eq. (1) will continue to infinity.

A particular model where these observations are relevant is the multiplicative stochastic model with lower bound introduced in Refs. [4,13]. In this model, the process (2) is supplemented by a lower bound:

$$w_i(t + 1) \geq c\bar{w}(t), \quad (17)$$

(i.e., whenever according to (2) $w_i(t + 1)$ becomes less than $c\bar{w}(t)$, the actual updated value $w_i(t + 1)$ is taken as $c\bar{w}(t)$). Such a lower bound represents in the model the economic effects of subsidies, securities, and services.

For infinite $N$, this process leads to the precise power law with the exponent

$$\alpha = 1/(1 - c). \quad (18)$$

In reality, that is, for finite $N$, one obtains a different result [14]: for $c \ll 1/\ln N$, the exponent is

$$\alpha \approx \ln N/(\ln N - \ln c) < 1, \quad (19)$$

while for $c \gg 1/\ln N$ the relation (18) is recovered. (The large $w_i$ scaling properties of this system are representative of a larger class of systems [4,12–
of the form:

\[ w_i(t + 1) = \lambda_i(t)w_i(t) + a\bar{w}(t) + b(\bar{w}, t)w_i(t), \]

where \( b \) is an arbitrary function [18].

Numerical results probing the properties of this multiplicative stochastic model with lower bound are shown in Figs. 4, 5 and 6. (We used here the realistic asynchronous dynamics, i.e., different wealths are updated in successive random order: at each time \( t \) only one wealth is randomly chosen for updating. However, similar results are obtained for the synchronous dynamics). In the simulations the random factor \( \lambda \) is distributed with uniform probability between 0.9 and 1.1.

As explained above, while the Zipf plots of the individual wealth configurations do not show any deviation from the precise power-law behavior (see Fig. 4), the Pareto distribution of the normalized \( w/W \), obtained by averaging over many runs, shows a sharp bent to \( \log P(w) = -\infty \) for \( w/W \rightarrow 1 \) (Figs. 5 and 6).

In Fig. 5, we have \( c > 1/\ln N \) and then the formula (18) holds especially for large \( N \). However, when there is no significant social security policy, that is, \( c < 1/\ln N \) in the lower bound Eq. (17), one obtains the low values of exponent \( \alpha < 1 \), as predicted by Eq. (19). This is clearly seen in Fig. 6.

For the power-law distribution (1), the ratio between the largest wealth and the total wealth is \( w_{\text{max}}/W \sim N^{1/\alpha - 1} \). Thus, when \( \alpha < 1 \), the richest person has an \( O(1) \) fraction of the total wealth even if the market size is asymptotically large. This extreme financial inequality is questionable from moral point of view and unacceptable from social point of view [15,16].

However, here we make the observation that such an inequality is equally dangerous even from the narrower point of view of financial stability. To understand the implications of the \( \alpha < 1 \) on the market fluctuations, recall [4,17] that the market index fluctuations are expected to also obey (modulo the cut-off tails) a power-law behavior with an exponent close to or larger than that of the individual wealth Pareto distribution. Telegraphically, the argument in Ref. [4] consisted of the following steps:

- The market index is identified with the total wealth \( W \) in the market, i.e. with the sum of the individual invested wealths \( w_i \);
- The market fluctuations over a certain time period are therefore the sum of the variations of the individual wealths \( w_i \), with \( i \) randomly selected for updating during that period;
- The individual variations of the \( w_i \)'s are, according to Eq. (2), stochastically proportional to the \( w_i \)'s themselves, which are distributed by a power law
with exponent $\alpha$;

- Thus, the market fluctuations are the sums of steps with sizes distributed by a power law with exponent $\alpha$.

Therefore, a wealth distribution with $\alpha < 1$ implies market fluctuations distributed by a power law with a little larger but still small effective exponent (modulo the truncation effects, the details of which are outside the scope of the present paper and were studied in Ref. [17]. They fit well the experimentally observed cross-over to larger power-law exponents and to exponential-type behavior [19,20] for extremely large fluctuations.) In turn this small exponent implies the emergence with finite probability of macroscopically large market fluctuations, which means that the financial markets in a society with a Pareto wealth distribution of $\alpha < 1$ are open to a significant risk of macroscopic market crashes. The conclusion is that a responsible social policy which insures the smallest wealth $w_{\text{min}} \gg \bar{w}/\ln N$ is not just a humane duty but also a vital interest of the capital markets.

This finite-size cut-off effect also occurs in systems other than the multiplicative ones, in particular for the critical phenomena. For the two- or three-dimensional percolation system, which has been well studied and understood [21], the number $n_s$ of clusters containing $s$ sites each decays for large $s$ as a power law

$$n_s \sim 1/s^\tau,$$

in an infinite lattice and at the percolation threshold $p_c$. The Fisher exponent $\tau$ is 2.05 for two dimensions ($d = 2$), while for three dimensions ($d = 3$) it is 2.19. In a finite lattice, the cut-off behavior similar to Eq. (8) exists, that is, no cluster can be larger than the whole lattice and then the cluster radius

$$R_s < L,$$

in a $d$-dimensional lattice of linear size $L$, where at $p_c$ this radius behaves as $R_s \sim s^{1/D}$, with $D = 1.9$ for $d = 2$ or 2.5 for $d = 3$. The size distributions for these largest clusters have been investigated [22].

Thus, things are more complicated for the cut-off effect of the finite systems. For the critical phenomena like the percolation, little is known about the extremely rare clusters at $p_c$ with size $s$ between $L^D$ and $L^d$ which are outside the scaling region. As in the case of the multiplicative processes such as Eqs. (2)-(17), more work is needed in order to investigate e.g. the margin of the wealth value obeying the power-law scaling as well as its dependence on the system size.

Returning to the financial applications, our results caution against the use of
the $N = \infty$ limit in estimating the parameters of the distributions for wealth and market fluctuations. In particular, while for $N = \infty$ some random multiplicative mean field models seem to insure $\alpha > 1$ [12], in reality for any finite number of individuals $N$, in the absence of an appropriate social policy which insures the smallest wealth $w_{\text{min}} \gg \bar{w}/\ln N$ one is led to a very unbalanced wealth distribution $\alpha < 1$. For such a wealth distribution, a large fraction of the total wealth remains in the hands of just a few richest individuals. This in turn is bound to lead not only to social unrest but also to unbearable financial markets fluctuations.

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Fig. 1. The Zipf plots obtained from the simulations of multiplicative process (2). $\ln \lambda$ is extracted from a Gaussian distribution with $v = 0$ and $D = 0.01$. The system size is $N = 500$, and the measurements are performed at 2 different times: $t = 10^6$ (+) and $10^7$ (*).

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Fig. 2. The Zipf plots obtained from the simulations of multiplicative process (2). \( \ln \lambda \) is extracted from a Gaussian distribution with \( v = 0 \) and \( D = 0.01 \). The time at which the measurement is performed is \( t = 10^7 \), and the system sizes are: \( N = 50 \) (+), 500 (*), and 5000 (×).

Fig. 3. The numerical Zipf plots of the multiplicative process (2). \( \ln \lambda \) is extracted from a uniform probability distribution between the values \(-0.1 \) and \( 0.1 \). The measurements are performed at time \( t = 1.1 \times 10^8 \). The system sizes are \( N = 50 \) (+), 500 (*), and 5000 (×). Inset: \( N = 500 \) and different times: \( t = 10^7 \) (+) and \( 1.1 \times 10^8 \) (*).
Fig. 4. The Zipf plots obtained from the numerical simulations of model (2) with the lower bound (17) and asynchronous dynamics, for late time \( t = 2 \times 10^5 N \) and different values of \( c \) and size \( N \). The random factor \( \lambda \) is extracted from an uniform probability distribution between 0.9 and 1.1.

Fig. 5. Log-log plot of the probability distribution for normalized wealth \( w/W \) of the model with lower bound: Eqs. (2) and (17), with simulation parameters: \( c = 0.3, N = 100, 1000, \) and 10000. \( \lambda \) is uniformly distributed in the range (0.9, 1.1). The straight line has the slope \( 1 + \alpha \), with exponent \( \alpha \) determined by Eq. (18).
Fig. 6. Similar to Fig. 5, but with parameters $c = 0.05$ and $N = 50, 100, \text{ and } 500$, so that $c < 1/\ln N$. The straight line has the slope 2, corresponding to $\alpha = 1$. 