C*-bundle dynamical systems

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February 28, 2022

Abstract

C*-bundle dynamical systems are introduced and their rôle within the theory of C*-subalgebras and Fell bundles is investigated. A C*-bundle dynamical system involves an action of a 1-parameter group of “spatial automorphisms” of the C*-bundle together with a notion of covariance with respect to the diffeomorphisms of the base manifold, and turns out to define a class of examples of Arveson’s \( A \)-dynamical systems. An embedding invariant for non-commutative C*-subalgebras (equivalent to a groupoid 2-cocycle by construction) emerges and has presentation analogous to a partition function towards a potential algebraic formulation of quantum gravity.

1 Introduction

Many physicists including Heisenberg, have highlighted the relevance of non-commutative configuration C*-algebras \( A \) as subalgebras in a C*-algebra of observables \( B \), so it is important to establish a good characterisation of the embeddings of not necessarily abelian C*-algebras \( A \) in ambient C*-algebras \( B \). Fell bundles as groupoid extensions to C*-bundles can provide a rich context for addressing problems in non-commutative geometry and physics.

This paper is a byproduct of a program of research motivated by attempts at translating certain ideas from spin foam quantum gravity into C*-algebraic language. Some of the new definitions emerged in part thanks to discussions with Paolo Bertozzini, Roberto Conti and Pedro Resende. There is no single main theorem, instead we present 5 smaller puzzle pieces theorems, plus a proposition and 2 lemmas, which all serve to address and illustrate the rôle of C*-bundle dynamical systems within the theory of C*-subalgebras and Fell bundles over groupoids, always in the light of their applications to physics.

We open with a review of Banach bundles, C*-bundles (or C*-algebra bundles), Fell bundles over groupoids with examples for the theoretical physicist unfamiliar with these topics. We follow those preliminary ideas with a study of a suitable notion of self-map for a C*-bundle \((E^0, \pi, X)\) over a locally compact space \( X \) and relate them to unitary representations of \( \text{Homeo}(X) \) on \( E^0 \). Next, we show how those notions relate to Kumjian’s normalisers for embeddings of \( C^*(E^0) \) as a C*-subalgebra in \( C^*(E) \). \((E^0, \pi, X)\) is the restriction to \( X = G^0 \) of a Fell bundle \((E, \pi, G)\) over a groupoid given by \( G = X \times X \). For coherence and for

*Research supported by Fundação para as Ciências e a Tecnologia (FCT) including programs POCI 2010/FEDER and SFRH/BPD/32331/2006. Email: rmartins-at-math.ist.utl.pt
the convenience of the reader we recall a few background details about C*-subalgebra theory as developed by Kumjian, Renault and Exel and discuss generalisations to non-commutative C*-subalgebras. Then we introduce C*-dynamical systems and give a precise comparison with Arveson’s A-dynamical systems (and the more general notion of C*-dynamical system) discussing the advantages to physics of C*-bundle dynamical systems. We construct an “embedding invariant” Φ to be equivalent to a groupoid 2-cocycle ω and it has an alternative presentation in a form which is analogous to a partition function for quantum gravity. We presume that this is the first appearance of such a partition function towards an algebraic formulation of quantum gravity. Finally, we show how a C*-bundle dynamical system can provide a bridge or enveloping structure between (E, π, G) for a principal groupoid G and (A, B) for A a diagonal C*-subalgebra in B.

Additional potential applications to physics may include: (i) The (strongly continuous) action taking part in a C*-bundle dynamical system is a lifting of a 1-parameter group of diffeomorphisms of the base differential manifold M. Moreover, the construction is similar to ideas about modular quantum gravity first appearing in BCL3. (ii) The covariance condition that applies to a C*-bundle dynamical system may have an application to the Poincaré covariance axiom from algebraic quantum field theory. (iii) Although in this paper we only work with the reversible case, irreversible C*-bundle dynamical systems should have applications to decoherence theories, entanglement and to topological inverse semigroups such as those describing Penrose tilings. (iv) The generator of the 1-parameter group of C*-bundle spatial automorphisms (over a simply connected compact manifold) should lead to an interpretation of a Dirac operator in a spectral triple as a generalised connection on a C*-bundle. There should also be a clear relation between the A-bimodules of 1-forms Ω1D(A) in a finite spectral triple (A, H, D), and both Arveson paths and Exel slices. We leave these geometrical considerations for further work.

2 Fell bundles and C*-bundles

In this section we aim to include a comprehensive set of preliminary definitions for completeness and accessibility, above all to aid physicists who may be unfamiliar with these topics but are interested in the applications of the results presented below.

For Dixmier, Fell and many others, Banach bundles and C*-bundles are not equivalent to vector bundles with additional structure:

**Definition 2.1.** [FD] A Banach bundle \((E, π, X)\) is a surjective continuous open map \(π : E → X\) such that \(∀x ∈ X\) the fibre \(E_x := π^{-1}(x)\) is a complex Banach space and satisfies the following additional conditions:

- the operation of addition + : \(E × E → E\) is continuous on the set \(E × E := \{(e_1, e_2) ∈ E × E \mid π(e_1) = π(e_2)\}\),
- the operation of multiplication by scalars: \(C × E → E\) is continuous,
- the norm \(\| \cdot \| : E → \mathbb{R}\) is continuous,

1These C*-subalgebras in C*-algebras are commonly denoted by \(B\) for the subalgebra and \(A\) for the ambient algebra. We choose the opposite notation convention here: \(A ⊂ B\), because of the connections to be made later to other topics also involving C*-subalgebras. We hope that this does not cause an inconvenience to readers familiar with the \(B ⊂ A\) convention.
• for all \( x_0 \in X \), the family \( U_{x_0} = \{ e \in E \mid \| e \| < \epsilon, \pi(e) \in O \} \) where \( O \subset X \) is an open set containing \( x_0 \in X \) and \( \epsilon > 0 \) is a fundamental system of neighbourhoods of \( 0 \in E_{x_0} \).

For a Hilbert bundle we require that for all \( x \in X \), the fibre \( E_x \) is a Hilbert space.

Modulo our notation, the current paragraph of remarks is taken from [FD]. Let us say that a Banach bundle \( E \) is locally trivial if for every \( x \in X \) there is a neighbourhood \( U \) of \( x \) such that the restricted Banach bundle \( E_U \) is isomorphic to a trivial bundle. Of course not all Banach bundles are locally trivial. It can be shown that a Banach bundle over a locally compact Hausdorff space, whose fibres are all of the same finite dimension, is necessarily locally trivial.

**Definition 2.2.** A morphism of Banach bundles \((f, f_0) : (E, \pi, X) \to (E', \pi', X)\) consists of a (norm-decreasing) continuous linear map \( f : E \to E' \) and a continuous linear map \( f_0 : X \to X \) such that

- \( \pi' \circ f = f_0 \circ \pi \);
- each induced fibrewise map \( f_x : E_x \to E'_{f_0(x)} \) is continuous.

If \( f \) and \( f_0 \) are invertible and \( \| e \|_E = \| f(e) \|_{E'} \), then \((f, f_0)\) define a Banach bundle isometric isomorphism.

For equivalent definitions of this isometric isomorphism we refer to the books [FD] and [D]. Also in [D], Dixmier gives an alternative definition of Banach bundle as a continuous field of Banach spaces over a topological space.

**Definition 2.3.** [FD] Let \( X \) be a locally compact Hausdorff space. By a bundle of C*-algebras (or C*-bundle) we mean a Banach bundle \((E^0, \pi, X)\) together with a multiplication \( \cdot \) and involution \(*\) in each fibre \( E^0_x \) of \( E^0 \), such that

1. For each \( x \in X \), \( E^0_x \) is a C*-algebra under the linear operations and norms of \( E^0 \) and the operations \( \cdot \) and \(*\);
2. the multiplication is continuous on \( \{ <a, b> \in E^0 \times E^0 : \pi(a) = \pi(b) \} \) (1) to \( E^0 \); and
3. the involution \(*\) is continuous on \( E^0 \) to \( E^0 \).

We associate a C*-algebra \( A \) to a C*-bundle \( E^0 \) as follows. The algebra of compactly supported continuous sections \( C_c(E^0) \) completed in the operator norm is a concrete C*-algebra (operating on \( H = L^2(E^0) \), the inner product norm completion of \( C_c(E^0) \)), which we call the enveloping algebra \( A \) of \( E^0 \). We also use the symbol \( C^*(E^0) \) when it is helpful to emphasise that \( A \) arises from a C*-bundle \( E^0 \) in this way.

(A Fell bundle over a groupoid is a generalisation of a Fell bundle over a topological group [FD], and also a generalisation of a C*-bundle),

**Definition 2.4.** [K1] A Banach bundle over a groupoid \( p : E \to G \) is said to be a Fell bundle if there is a continuous multiplication \( E^2 \to E \), where

\[ E^2 = \{(e_1, e_2) \in E \times E \mid (p(e_1), p(e_2)) \in G^2 \} \]

(\( G^2 \) denotes the space of composable pairs of elements of \( G \)) and an involution \( e \mapsto e^* \) that satisfy the following axioms.
1. \( p(e_1e_2) = p(e_1)p(e_2) \quad \forall (e_1, e_2) \in E^2; \)

2. The induced map \( E_{g_1} \times E_{g_2} \rightarrow E_{g_1g_2} \), \( (e_1,e_2) \mapsto e_1e_2 \) is bilinear \( \forall (g_1, g_2) \in G^2; \)

3. \( (e_1e_2)e_3 = e_1(e_2e_3) \) whenever the multiplication is defined;

4. \( \| e_1e_2 \| \leq \| e_1 \| \cdot \| e_2 \| \), \( \forall (e_1, e_2) \in E^2; \)

5. \( p(e^*) = p(e)^* \), \( \forall e \in E; \)

6. The induced map \( E_g \rightarrow E_{g^*}, \quad e \mapsto e^* \) is conjugate linear for all \( g \in G; \)

7. \( e^{**} = e \), \( \forall e \in E; \)

8. \( (e_1e_2)^* = e_2^*e_1^* \), \( \forall (e_1, e_2) \in E^2; \)

9. \( \| e^*e \| = \| e \|^2 \), \( \forall e \in E; \)

10. \( \forall e \in E, \ e^*e \geq 0 \) as element of the \( C^* \)-algebra \( E_{p(e^*)}. \)

The information in the following two paragraphs is also recalled from [K1]. The fibres \( \{E_x\}_{x \in G_0} \) over the unit space \( G_0 \) of \( G \) are \( C^* \)-algebras. A unital Fell bundle is one in which each of these \( C^* \)-algebras has an identity element. A Fell bundle is said to be saturated if \( E_{g_1g_2} \) is the closed linear span of \( E_{g_1}E_{g_2} \) for all \( (g_1, g_2) \in G^2. \) In this case, \( E_g \otimes E_{g^*} \cong E_{gg^*} \) and \( E_g \otimes E_{g^*} \cong E_{g^*g} \) for all \( g, g^* \in G, \) (including \( gg^*, g^*g \in G_0 \)) and the fibres of \( E \) are called imprimitivity bimodules or Morita-Rieffel equivalence bimodules, or in other words, the \( C^* \)-algebras \( E_{gg^*} \) and \( E_{g^*g} \) are strongly Morita equivalent or Morita-Rieffel equivalent. All Fell line bundles over groupoids (that is, a Fell bundle \( (E, \pi, G) \) with fibre \( C \)) are saturated since in one dimension, two algebras are Morita equivalent exactly when they are isomorphic.

The enveloping algebra \( C^*_e(E) \) of a Fell bundle \( (E, \pi, G) \) is the algebra of compactly supported continuous sections \( C_0(E) \) of \( (E, \pi, G), \) completed in the operator norm: \( C^*_e(E) \subset B(L^2(E)) \), where \( L^2(E) \) is the Hilbert space obtained from the completion of \( C_0(E) \) in the inner product norm. We also denote the enveloping algebra of \( E \) as \( C^*(E). \) Consider the restriction of the image of the surjection \( \pi : E \rightarrow G, \) to \( G_0, \) the object space or unit space of \( G, \) and let \( (E^0, \pi, G_0) \) denote the \( C^* \)-bundle corresponding to this restriction. Put \( P : C^*(E) \rightarrow C^*(E^0) \) for the restriction of the enveloping algebras. \( A = C^*(E^0) \) is also referred to as the diagonal algebra of the Fell bundle.

Here are some examples of \( C^* \)-bundles (or \( C^* \)-algebra bundles), followed by examples of Fell bundles:

**Examples 2.5.** 1. Recall that every arbitrary finite dimensional \( C^* \)-algebra takes the form \( A = \bigoplus_{i=1}^m M_{n_i} \), up to a canonical isomorphism. \( A \) is the enveloping algebra of a \( C^* \)-bundle \( E^0 \) where \( X \) is a discrete space with \( m \) points \( \{x\} \) indexed by \( i \) and each fibre \( E^0_{\{x\}} \) of \( E^0 \) is given by a simple matrix algebra \( M_{n_i}. \)

2. A fundamental example arises from treating \( C_0(X) \) (the algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space \( X \)) as the algebra of continuous sections (vanishing at infinity) of a complex line bundle over \( X. \)

3. A tensor product of the \( C^* \)-algebra \( C_0(X) \) with a \( C^* \)-algebra \( E^0_2. \) (For example, a minimal \( C^* \)-tensor product of \( C^* \)-algebras \( C \) and \( D \) acting on Hilbert spaces \( H \) and \( K, \) was defined by Tomiyama in [T] to be the closure of the algebraic tensor product \( C \otimes D \) by the operator norm of the \( C^* \)-algebra of bounded linear operators on \( H \otimes K. \))
4. A continuous field of elementary C*-algebras, usually satisfying Fell’s condition. (This is how the continuous trace-class C*-algebras arise.)

5. A Banach bundle whose fibre is a complex simple C*-algebra isomorphic to a Clifford algebra.

6. Let a C*-bundle $(E^0, \pi, X)$ be constructed from a separable C*-algebra $A$ as follows. Since $A$ is separable, we identify $A$ with $\bigoplus_{m} \pi_{m}$, the direct sum over all irreducible representations $\pi_{m}$ of $A$. Define the space of fibres of $E^0$ to be given by $\{E^0_x\}_{x \in X} = \{\pi_{m}\}$, where $X$ is identified with the pure state space $X(A)$, which is locally compact in the weak *-topology. Then the enveloping algebra $C^*(E^0)$ of $E^0$ is identified with $A$ and operates on $\mathcal{H} = L^2(E^0)$.

From [KL], any saturated Fell bundle is equivalent to a semidirect product arising from the action of a locally compact groupoid on a C*-bundle as follows.

**Example 2.6 (Semidirect product Fell bundle).** Following [KL] one constructs a saturated Fell bundle $E$ over a topological groupoid $(E, \pi, G)$ from a semidirect product structure as follows. Let $E^0$ be a C*-bundle over $G_0$ and let $r$ and $d$ denote the domain and range maps of $G$. Let the product of elements $e_1 = (g, a) \in E$ and $e_2 = (h, b) \in E$, (with $a, b \in E^0$ such that $\pi(a) = d(g), \pi(b) = d(h)$), for each pair $(g, h)$ such that $r(g) = d(h)$, be given by:

$$e_1 e_2 = (gh, \alpha_g(a)b)$$

where $\alpha_g$ is an isometric *-isomorphism of fibres $\alpha_g : E^0_{d(g)} \to E^0_{r(g)}$ defined by $\alpha_g(a) = uau^*$ with unitary elements $u \in E_g$, $u^* \in E_{g^*}$. The involution on $E$ is given by $e_1^* = (g, a)^* = (g^*, \alpha_g(a^*))$. We denote the resulting Fell bundle by $E = G \ltimes E^0$. This semidirect product structure induces an action of $G$ on the enveloping algebra $A = C^*(E^0)$ of the C*-bundle $E^0$, such that $C^*(E) = G \ltimes C^*(E^0)$.

(a) In the case that $E$ is a Fell line bundle over a locally compact étale groupoid $G$, then $C^*_r(E)$ is identified with the twisted convolution C*-algebra of $G$ (see for example [Ren]) and for the trivial action $\alpha = 1$, then $C^*_r(E)$ is identified with the ordinary or untwisted group convolution algebra $C^*_r(G)$ of $G$.

(b) Let $E$ be a locally trivial Fell bundle with non-commutative fibre over a pair groupoid over a locally compact simply connected (possibly discrete) manifold $G = M \times M$ such that the fibre of the corresponding C*-bundle $(E^0, \pi, M)$ is a simple C*-algebra.

**Example 2.7 (Imprimitivity Fell bundle).** Let $E$ be a unital saturated Fell bundle over a discrete groupoid $G$ whose induced C*-bundle over the discrete space $G_0$ is given by $E^0 = \bigoplus_{n} M_n(\mathbb{C})$. (Let $G_0$ have $i$ points and the fibres of $E^0$ be simple matrix algebras of varying dimension $n$.) The fibres of $E$ are $M_{n(i)}(\mathbb{C})$-bimodules. In other notation, the fibres $E^0_g$ are given by imprimitivity $E_{d(g)} - E_{r(g)}$-bimodules. Since $G_0$ is discrete, $E$ is locally trivial as Banach bundle, although its fibres are not in general topologically equivalent. In the case that $G = X \times X$, then $E$ defines a full C*-category [BCL1]. For C*-categories see [GLR].

### 3 Spatial automorphisms and ambient Fell bundles

#### 3.1 Spatial automorphisms

An automorphism of a C*-bundle with enveloping algebra $A$ consists of a Banach bundle isometric isomorphism preserving the structure of the bundle, extending to an isometric *-
isomorphism $\alpha : A \to A$. In examples there may exist automorphisms that are not implemented by unitary operators, that is, that are not spatial, but we will only make use of spatial automorphisms.

**Definition 3.1.** Let $(E^0, \pi, X)$ be a C*-bundle over a locally compact space $X$ with enveloping C*-algebra $A = C^*(E^0)$ represented on a separable Hilbert space $\mathcal{H}$. A spatial automorphism of the C*-bundle consists of invertible continuous linear maps $f_0$ and $f$ with commuting diagram:

\[
\begin{array}{c}
E^0 \\
\downarrow \pi \\
X \\
\end{array} 
\quad \xrightarrow{f} \quad  
\begin{array}{c}
E^0 \\
\downarrow \pi \\
X \\
\end{array}
\]

such that each induced fibrewise linear map $f_x : E^0_x \to E^0_{f_0(x)}$ is invertible continuous and such that $f$ extends to an isometric *-isomorphism $\hat{f} : A \to A$ of the form $\hat{f}(a) = UaU^*$ where $U$ is a unitary linear map on $\mathcal{H}$.

The previous definition can be thought of as a special case of a Fell bundle morphism from [BCL2].

Suppressing the circumflex, we will denote C*-bundle spatial automorphisms by pairs $(f_0, f)$ or sometimes more explicitly, by triples $(f_0, f, U)$. We will refer to the bundle structure preservation condition

\[\pi \circ f = f_0 \circ \pi\]

as the **covariance condition**.[2]

In our related work, we will see that [3] is one of the main features distinguishing C*-bundle dynamical systems from other C*-dynamical systems.

Clearly, these “covariant” automorphisms $(f_0, f)$ define a subgroup $\text{SpatialAut}_\pi(E^0)$ of the group $\text{Aut}(A)$ of all automorphisms of the C*-algebra $A$.

In the case that the fibres of $E^0$ are of varying dimension or are topologically inequivalent i.e. if $E^0$ is not locally trivial, then if $(f_0, f)$ is a spatial automorphism of $E^0$, then $f_0$ is the identity homeomorphism.

**Remark 3.2.** It was kindly pointed out by Roberto Conti that if $A$ is a C*-algebra $K(\mathcal{H})$ consisting of all compact operators on a Hilbert space $\mathcal{H}$ (considering the case of a C*-bundle with only one fibre, given by $K(\mathcal{H})$), then all automorphisms $\phi$ of $A$ are implemented by unitaries operators $U$ on $\mathcal{H}$ such that $U \not\in A$, so that in this case $\phi$ is never inner.

For each C*-bundle $(E^0, \pi, X)$ over a locally compact Hausdorff space, the homeomorphisms $f_0 : X \to X$ form a group, $\text{Homeo}(X)$.

We emphasise that the group of global bisections $\text{Bis}(G)$ of the pair groupoid $G = X \times X$ over a topological space $X$ is identified with the group $\text{Homeo}(X)$ of homeomorphisms of $X$, and if $M$ is a smooth manifold, then $\text{Bis}(G)$ for $G = M \times M$, is identified with the group $\text{Diff}(M)$ of diffeomorphisms. Let $\text{Bis}_{\text{herm}}(G)$ denote the abelian subgroup of self-adjoint diffeomorphisms,

\[g = g^* \in \text{Bis}_{\text{herm}}(G) \subset \text{Bis}(G).
\]

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2Observe that $(f_0, f)$ is permuting and transforming the fibres. In finite dimensions, $U$ is given by a matrix with one non-zero block in each row and each column of blocks.
Recall that the general linear groupoid \( GL(V) \) of a vector bundle \( V \) over a space \( X \) is the canonical groupoid of linear isomorphisms between the fibres of \( V \) and that a representation \( \rho_G \) of a groupoid \( G \) on a vector bundle \( V \) is a groupoid homomorphism \( \rho_G : G \to GL(V) \) into the general linear groupoid of \( V \).

**Definition 3.3.** The general linear groupoid \( GL(E^0) \) of a C*-bundle \((E^0, \pi, X)\) with isomorphic fibres, is the set of all isometric *-isomorphisms \( \phi_{(x,y)} \) between each pair of C*-algebras:

\[
GL(E^0) = \{ \phi_{(x,y)} : E^0_x \to E^0_y \mid x, y \in X \},
\]

together with the canonical composition of isomorphisms, inverses and units \( \iota_x : E^0_x \to E^0_x \).

**Lemma 3.4.** Let \((E^0, \pi, X)\) be a C*-bundle. \( \text{SpatialAut}_\pi(E^0) \) is a subgroup of the group \( \text{Bis}(GL(E^0)) \) of global bisections of the groupoid \( GL(E^0) \).

**Proof.** The global bisections \( \alpha \in \text{Bis}(GL(E^0)) \) of \( GL(E^0) \) satisfy the covariance condition \( \pi \circ \alpha = f_0 \circ \pi \) where \( f_0 \) is a bisection of the groupoid \( G = X \times X \). \( \square \)

Let \( \mathcal{U}(\mathcal{H}) \) denote the group of unitary linear maps on a Hilbert space \( \mathcal{H} \) and let \( \psi \in \mathcal{H} \). Recall that a strongly continuous unitary representation of a group \( \mathcal{G} \) is given by a group homomorphism \( g \mapsto U_g \in \mathcal{U}(\mathcal{H}) \) such that \( g \mapsto U_g \psi \) defines a continuous map from \( G \) to \( \mathcal{H} \).

**Definition 3.5.** Let \( f_0, g \in \text{Homeo}(X) \). A unitary representation \( \rho_U \) of \( \text{Homeo}(X) \) on \( E^0 \) is a group homomorphism,

\[
\rho_U : \text{Homeo}(X) \to \text{SpatialAut}_\pi(E^0),
\]

\[
g \mapsto U_g a U_g^*,
\]

or \( f_0 \mapsto (f_0, f, U_{f_0}) \).

A unitary representation \( \rho_U(\text{Homeo}(X)) \) is said to be **strongly continuous** if the map from \( \text{Homeo}(X) \) to \( \mathcal{H} = L^2(E^0) \) defined by

\[
g \mapsto \alpha_g(a) \psi, \quad \psi \in \mathcal{H},
\]

is continuous for each \( a \in A \).

Of course, one may extend the previous definition to unitary representations of subgroups \( \mathcal{G} \subset \text{Homeo}(X) \) and moreover, if \( M \) is a smooth manifold and \( \mathcal{G} \) is a subgroup of \( \text{Diff}(M) \), one defines a unitary representation \( \rho_U \) of \( \mathcal{G} \) on a C*-bundle \((E^0, \pi, M)\) as a group homomorphism \( \rho_U : \mathcal{G} \to \text{SpatialAut}_\pi(E^0) \).

If \( \rho_U(\mathcal{G}) = \text{Bis}(\rho_G) \) for some subgroup \( \mathcal{G} \) of \( \text{Homeo}(X) \) or \( \text{Diff}(M) \), where \( \rho_G : G \to GL(E^0) \) is a groupoid homomorphism, then we say that \( \rho_U \) induces a groupoid representation \( \rho_G \) on the C*-bundle \( E^0 \).

Note that each unitary representation \( \rho_U \) on \( E^0 \) of a group \( \mathcal{G} \) provides an action \( \alpha \) of the group \( \mathcal{G} \) on the C*-algebra \( A \).

See III for unitary representations of groupoids on Hilbert bundles.

### 3.2 Covariance group

Let \( M \) be a locally compact Hausdorff space admitting a differential structure.
Definition 3.6. Let \((E^0, \pi, M)\) be a C*-bundle over a differentiable manifold \(M\) and let \(\{g(\lambda)\}_{\lambda \in \mathbb{R}}\) be a 1-parameter subgroup of \(\text{Diff}(M)\). A strongly continuous 1-parameter covariance group \(G_\sigma \subset \text{SpatialAut}(E^0)\) is the image of a strongly continuous representation of \(\{g(\lambda)\}_{\lambda \in \mathbb{R}}\) on \(E^0\).

Let \((E^0, \pi, M)\) be a C*-bundle as above in K3. Each \(G_\sigma\) is a subgroup of a unitary representation \(\rho_U\) of \(\text{Diff}(M)\) such that \(\rho_U = \text{Bis}(\rho_G)\). If \(\{g(\lambda)\}_{\lambda \in \mathbb{R}}\) has a minimal flow (having a dense orbit) on \(M\), we say that \(G_\sigma\) has a minimal flow across the fibres of \(E^0\). The significance of the minimal flow for this paper is that \(\{y : g(x) = y, \ x, y \in M, \ g \in \{g(\lambda)\}_{\lambda \in \mathbb{R}}\}\) is dense in \(M \times M\).

We will leave details on the infinitesimal generator \(\sigma\) of \(G_\sigma\) for another paper. Applications are expected to arise in the context of modular quantum gravity [BC L3].

3.3 The ambient Fell bundle

Crossed product C*-algebras arise in C*-dynamical systems and are by definition regular (see below). Kumjian observed that the structure of these C*-algebras can be illuminated by studying the way in which C*-subalgebras embed in them, using the set of normalisers:

Definition 3.7. K3 Suppose that \(A\) is a C*-subalgebra of a C*-algebra \(B\). An element \(b \in B\) is said to normalise \(A\) if

- \(b^* Ab \subset A\)
- \(bAb^* \subset A\)

The collection of all such normalisers is denoted \(N(A)\). Evidently, \(A \subset N(A)\); further, \(N(A)\) is closed under multiplication and taking adjoints. A normaliser, \(b \in N(A)\) is said to be free if \(a^2 = 0\). The collection of free normalisers is denoted \(N_f(A)\).

A C*-subalgebra \(A\) in \(B\) is said to be regular in \(B\) if the normalising set \(N(A)\) in \(B\) generates \(B\), that is, \(\text{clspan } N(A) = B\), where \(\text{clspan}\) denotes closed linear span.

Proposition 3.8. Let \((E, \pi, G)\) be a unital Fell bundle from which we obtain C*-algebras \(A \subset B\) such that \(A = C^*(E^0)\) and \(B = C^*(E)\). If \(E\) is saturated then \(A\) is regular in \(B\).

Proof. For each \(g, g^* \in G\), \(E_g\) is a bimodule over \(E_{gg^*}\) and \(E_g \otimes E_{g^*} \subset E_{gg^*}\). We may define a map \(E_{d(g)} \rightarrow E_{d(g)}\) by \(a \mapsto bab^*\) for \(b \in E_g\) (observe that \(b\) determines a free normaliser). Since \(E\) is saturated, \(E_{g} \otimes E_{g^*} \cong E_{gg^*}\), therefore \(b\) can chosen so that for any \(a' \in E_{d(g)}\) there is an \(a \in A\) such that \(\alpha_\pi(a) = a'\). We can repeat this process for all fibres \(E_g\) of \(E\), and since any \(\alpha_{\pi}(a) = a'\) (observe that \(b\) determines a free normaliser), it follows that \(\text{clspan } N(A) = B\).

Examples 3.9. 1. All complex line bundles over groupoids are saturated since in one dimension, two algebras are Morita equivalent exactly when they are isomorphic.

2. From [K1], any saturated Fell bundle is equivalent to a semi-direct product arising from the action of a locally compact groupoid on a C*-bundle.

3. A Fell bundle with the structure of a crossed product of the form \(C^*(E) = \mathcal{G} \rtimes A\) for a locally compact or discrete group \(\mathcal{G}\) is saturated. A Fell bundle over a topological group is called an algebraic bundle.
4. Let $E^0 = \bigoplus M_{n_i}(\mathbb{C})$ over a discrete space of $i$ points so that the fibres of $E^0$ are the simple matrix algebra summands of varying dimension $n$. The fibres of $E$ are $M_{n_i}(\mathbb{C}) + M_{m_j}(\mathbb{C})$ imprimitivity or Morita equivalence bimodules. Note that even though the algebra $C^*(E^0)$ is regular in $C^*(E)$, $C^*(E)$ is not a crossed product algebra arising from the action of a groupoid (or group).

5. Not all regular Fell bundles are saturated. An example of a Fell bundle that is not saturated is given by a Fell bundle $E$ over inverse semigroup $S$ if the multiplication of sections is given by $(s_1, a_1)(s_2, a_2) = (s_1s_2, (a_1)a_2)$ then $\alpha_{s_1}$ does not induce an isomorphism of fibres $E_1 \to E_2$.

The following result demonstrates the relationship between a $C^*$-bundle spatial automorphism group $\text{SpatialAut}_\pi(E^0)$ and the unitary normalisers of $C^*(E^0)$ in $C^*(E)$ that arises due to the covariance condition (3.8). (See also comment 6.3).

Note that the subgroup of spatial automorphisms $(f, f_0)$ implemented by unitary elements $U$ of the ambient algebra $C^*(E)$, form a subgroup $S$ of the group of inner automorphisms $\text{InnAut}(C^*(E))$ of $C^*(E)$.

**Theorem 3.10.** Let $(E^0, \pi, X)$ be a $C^*$-bundle with enveloping $C^*$-algebra $A$ and let $(E, \pi, G)$ be a saturated Fell bundle over the pair groupoid $G$ over a locally compact space $X$ such that $E^0$ is the restriction of $E$ over $G_0$ and such that all fibres of $E^0 \cong$ of equal dimension. Let $\text{Bis}(G)$ denote the group of global bisections of $G$ and let $B$ denote the enveloping $C^*$-algebra $C^*(E)$ of $E$. The following groups are isomorphic:

1. the group $S := \text{SpatialAut}_\pi(E^0) \cap \text{InnAut}(C^*(E))$

2. the group of unitary normalisers $N_U(A)$ of $A$ in $B$.

**Proof.** Each $U \in N_U(A)$ implements an element of $\text{Bis}(GL(E^0))$ and gives an isometric *-isomorphism $f : A \to A$ as follows. For $U \in N(A) \subset B$, we have $UaU^* \in A$. If $U$ were a section not supported on a bisection $g \in \text{Bis}(G)$ then $UaU^* \notin A$. Therefore $U$ is supported on $g \in \text{Bis}(G)$. Observe that if $U \in B$ implements some map $f(a) := UaU^* \in B \forall a \in A$, then the condition $\pi(U) = g = f_0$ for some $g \in \text{Bis}(G)$ is equivalent to the condition on $(f, f_0)$ that $\pi \circ f = g \circ \pi$, (3.8). This condition is satisfied for $U \in N_U(A)$. Since $f$ defines an isometric *-isomorphism $A \to A$, then $(f, f_0) \in \text{SpatialAut}_\pi(E^0)$ and so $(f, f_0) \in S$.

Conversely, every $(f, f_0) \in S$ can be implemented by some unitary normaliser $U \in N_U(A)$, because the homeomorphism $f_0$ involves a permutation transformation of the fibre set of $E^0$. In other words each $g \in \text{Bis}(G)$ designates a full permutation of the fibre set of $E^0$. Fix a $g \in \text{Bis}(G)$ and let $\text{Perm}$ be the corresponding set consisting of pairs of fibres $(E^0_x, E^0_y)$, where $x$ may be equal to $y$. Since $E$ is saturated, $A = C^*(E^0)$ is regular in $B = C^*(E)$ (3.8). For each pair $(E^0_x, E^0_y)$ in $\text{Perm}$, the free normalisers can provide isometric *-isomorphisms of the form,

$$\alpha_{xy} : E^0_x \to E^0_y, \quad \alpha_{xy}(a_x) = u_{xy}a_xu_{yx}^* = a_y \in E^0_y.$$  

(10)

with $u_{xy}$ a unitary element of the fibre $E_{xy}$ of $E$ and $u_{yx}^* \in E_{yx}^*$. Note that each $U \in N_U(A)$ is a full rank operator on $H$. A unitary normaliser $U \in N_U(A)$ can be approximated from a set of finite linear combinations of normalisers in $N(A)$. In this way arises a $U_0 \in N_U(A)$ and a map $f : E^0 \to E^0$, defined by $f(a) = U_0aU_0$, where each finite linear combination is a sum of maps $\alpha_{xy}$ parametrised by elements of the set $\text{Perm}$. This results in an assignment $(f_0 =) g \mapsto U_g \in N_U(A)$. 

9
4 Non-abelian C*-subalgebras

In view of the previous two sections, one might ask if C*-bundles and their spatial auto-
morphisms ought to be included or embedded in a larger algebraic context, seeing as the set
\( N_U(A) \notin A \). (See later section 6 on embedding invariants.) This section contains some back-
ground information on C*-subalgebras for convenience and coherence and also includes some
modifications to include non-commutative C*-subalgebras.

Renault provided a notion of a Cartan subalgebra \( A \subset B \) for the context of C*-algebras,
called Cartan pair \((A, B)\), with \( A \) a maximal \textit{abelian} subalgebra (masa) \[\text{[Ren]}\]. Previously,
Kumjian defined (abelian) C*-diagonals, a similar but less general notion \[\text{[K3]}\]. For more
information, see \[\text{[7]}\] below. Kumjian showed that an invariant of C*-diagonals is a Fell line
bundle over a \textit{principal} groupoid, and following those techniques, Renault showed that a Ca-
rtn pair invariant is given by a Fell bundle over a (locally compact Hausdorff) \textit{essentially principal}
groupoid, thus widening the scope of the examples. (An essentially principal groupoid \( G_{ep} \)
is defined as an étale groupoid in which the interior of the isotropy group bundle is equal to the
object space \( G_{ep}^0 \) of \( G_{ep} \). Equivalently, the set of points of \( G_{ep}^0 \) with trivial isotropy is dense.
Every essentially principal groupoid is isomorphic to the groupoid of germs of a pseudogroup
of locally defined homeomorphisms between the open sets of some topological space \( X \) \[\text{[Ren]}\].)

Introducing a notion of non-commutative Cartan pair, Exel addresses an even wider class
of examples of regular C*-algebras, by constructing a Fell bundle over a topological inverse
semigroup for each generalised Cartan pair \[\text{[E]}\].

4.1 Pure states and faithful conditional expectations

**Definition 4.1.** \[\text{[P]}\] Let \( B \) be a C*-algebra and let \( A \subseteq B \) be a C*-subalgebra. Then we call
\( P : B \to A \) a conditional expectation of \( B \) onto \( A \) if it satisfies the following three properties:

1. \( P(a) = a, \forall a \in A \);
2. \( P(a_1 b a_2) = a_1 P(b) a_2 \forall b \in B, \forall a_1, a_2 \in A \);
3. \( b \geq 0 \implies P(b) \geq 0 \forall b \in B \).

We say that \( P \) is a faithful conditional expectation if, in addition, \( P(b^* b) \neq 0 \) for all non-zero
\( b \in B \).

Note that in definition 4.1 there is no declaration that \( A \) should be commutative. Let \( p, p_i \) be central projections of \( B \) for some Cartan pair \( (A, B) \) \textit{such that} \( A \) \text{is not necessarily commutative}. If \( A \) is non-commutative, then the projections \( p_i \) may vary in rank. The following
are examples of faithful conditional expectations of \( B \) onto \( A \).

**Examples 4.2.** \[\text{[P, K3]}\]

1. \( P(b) = pbp + (1 - p)b(1 - p), b \in B \).
2. \( P(b) = \sum_i p_i bp_i \) where \((p_i, ..., p_n)\) is an \( n \)-tuple of pairwise orthogonal projections of
possibly varying rank.

\[\text{[3]}\] Exel seems to have made this choice perhaps because he was not interested in examples of Fell bundles over
groupoids with non-commutative fibres over \( G^0 \).
3. Note that in this example, $A$ is abelian.

Let $B = M_n(C)$, the algebra of complex $n$ by $n$ matrices. Choose a set of matrix units, \( \{ e_{ij} : 1 \leq i, j \leq n \} \) (one has $e_{ik} = e_{ij} e_{jk}$ and $e_{ij}^* = e_{ji}$), and let $A$ denote the diagonal subalgebra (viz, $A$ is spanned by the $e_{ii}$s). Then $a = \Sigma \lambda_{ij} e_{ij}$ normalises $A$ if and only if for each $i$, $\lambda_{ij} \neq 0$ for at most one $j$, and for each $j$, $\lambda_{ij} \neq 0$ for at most one $i$ (i.e., at most one entry is non-zero in each row and column). If $i \neq j$, $e_{ij} \in N_f(A)$. Let $P : B \to A$ be given by:

$$P(b) = \sum e_{ii} b e_{ii}. \quad (11)$$

This defines a faithful conditional expectation for which:

$$\ker P = \operatorname{clspan} N_f(A). \quad (12)$$

where $\operatorname{clspan} N_f(A)$ denotes the closed linear span of the free normalisers of $A$ in $B$.

**Example 4.3.** Let $\alpha$ be an action specifying a Fell line bundle $(E, \pi, G)$ as described in 2.6, where $G$ is an essentially principal groupoid over a locally compact space $G_0 = X$. Recall that $(A = C^*(E^0), B = C^*(E))$ is a Cartan pair. (Note that $\alpha g^* g(b) = \alpha x(b)$ is not defined if $b$ is not in $A$.) Then,

$$P(b) = \int_X \alpha x(b) dx \quad (13)$$

is the unique faithful conditional expectation from $B$ onto $A$.

Let $A$ be a (resp. unital) C*-algebra. Recall that the space of pure states $X(A)$ of $A$ is (resp. locally) compact in the weak *-topology. (We identify $X(A)$ with its image $X$).

**Definition 4.4.** [K3] If $(A, B)$ is a Cartan pair with abelian C*-subalgebra $A$, the ambient algebra $B$ is said to have the extension property if the pure states of $B$ restrict to the set $X(A)$, that is, for $x \in X(A)$, if $x \circ P(B) = X(A)$. Equivalently, $B = A + [B, A]$.

The extension property ensures the existence of a conditional expectation $P$ such that $x \circ P(B) = X(A)$. Let $A$ be abelian. If $B$ has the extension property, it follows that the abelian C*-subalgebra $A$ is maximal.

Following [K3], for $n \in N(A)$, put

$$s(n) = \{ x \in X : x(n^* n) > 0 \}, \quad (14)$$

$$I(n) = \{ a \in A : x(a) \neq 0 \implies x \in s(n) \}. \quad (15)$$

Note that $s(n)$ is open in $X$ and $I(n)$ is an ideal in $A$. Each $n \in N(A)$ defines a partial homeomorphism of $X$,

$$f_{0,n} : s(n) \to s(n^*). \quad (16)$$

Finally, recall that the span of a possibly infinite dimensional vector space $V$, is the set of all finite linear combinations of vectors $v \in V$. Similarly, since $B = \operatorname{clspan} N(A)$ where span denotes closed linear span, any element of $B$ can be approximated by a set of finite linear combinations of normalisers (this was referred to in [K3] and will be useful to us below).
4.2 Cartan pairs and C*-diagonals

Definition 4.5. [Ren] Let $A$ and $B$ be C*-algebras and let $A \subset B$. $A$ is said to be a Cartan subalgebra if:

1. $A$ contains an approximate unit of $B$;
2. $A$ is maximal abelian;
3. $A$ is regular in $B$;
4. There is a faithful conditional expectation of $B$ onto $A$.

If in addition, $\ker P = \text{clspan}_f(A)$, then $A$ is said to be diagonal in $B$ ([K3]).

In [Ren], Renault shows uniqueness of the faithful conditional expectation $P : B \to A$ for Cartan pairs $(A,B)$ and in [Sc2], Exel warns that for non-commutative algebras, there may be many inequivalent probability measures. For this reason, he imposes a condition to ensure that the faithful conditional expectation associated to a given non-commutative Cartan pair must be unique. Moreover, in some of the examples we give below, the set (we define) of projections is a feature of the algebra itself. Below we impose uniqueness of $P$ as an axiom.

Our non-commutative generalisation is :-

Definition 4.6. Let $A$ and $B$ be C*-algebras $A \subset B$, where $A$ is not necessarily commutative and let $A$ contain the unit of $B$ or an approximate unit for $B$. $(A,B)$ is said to be a (not necessarily commutative) Cartan pair if $A$ is regular in $B$ and if there is a unique faithful conditional expectation $P : B \to A$.

If in addition, $\ker P = \text{clspan}_f(A)$, then $A$ is said to be diagonal in $B$.

Lemma 4.7. Let $A$ be a non-commutative C*-diagonal in $B$. Then $B$ satisfies the criterion for the extension property relative to $A$, namely, that $B = A + [B,A]$. (The overline denotes closed linear span.)

Proof. This follows from:

- $A$ is regular in $B$;
- $\ker P + A = B$;
- $B$ is an $A$-bimodule.

Examples 4.8. Here are some simple examples of non-commutative Cartan pairs to illustrate the previous definition.

1. Consider example 2.7 (imprimitivity Fell bundle), where $B = C^*(E)$ and $A = C^*(E^0)$, building on example 3.5(2). Since $E$ is saturated, $A$ is regular in $B$. Identify the pure state space $X$ of $A$ with the (discrete) base space $X$ of the C*-bundle $E^0$. There is a unique faithful conditional expectation of $B$ onto $A$, given by,

$$P(b) = \sum_{i} p_i b p_i$$

(17)
where \( (p_1, \ldots, p_m) \) is an \( m \)-tuple of pairwise orthogonal projections, as in example (2).

The rank of each projection \( (p_1, \ldots, p_m) \) is given by the dimension of each fibre,

\[
E^0_{(x)} = p(x)BP(x).
\]

In other words, we define the set \( (p_1, \ldots, p_m) \) to be the maximal projections in the C*-algebras \( E^0_i \). Clearly, \( P : B \to A \) is unique.

2. Consider a locally trivial saturated Fell bundle \( (E, \pi, G) \) (as in (2)), where \( B = C^*(E) \) and \( A = C^*(E^0) \) and let \( X \) be a discrete space. Let \( \{p_x\}_{x \in X} \) be an \( n \)-tuple of pairwise orthogonal projections in \( A \) such that

\[
\{E^0_x\}_{x \in X} = \{p_x E^0 p_x \mid \text{rank } p_x = \dim E^0_x, \forall x \in X\}
\]

One identifies the fibres \( E^0_x = p_x E^0 p_x = \text{clspan}_{a \in A} \{p_x ap_x\} \) of \( E^0 \) with the unitary equivalence classes of irreducible representations, or pure states, of \( A \). Since \( E^0 \) is a trivial C*-bundle over \( X \), the projections \( p_x \) are of constant rank, \( \text{rank } p_x = \dim E^0_x, \forall x \in X \).

From this we interpret the probability measure to be unique on \( X \). Now one makes the generalisation for locally trivial C*-bundles \( E^0 \) over locally compact spaces \( X \), because \( X \) has unique probability measure: the direct generalisation of equation (13) is,

\[
P(b) = \int_X \alpha_x(b)dx
\]

so that \( P \) is the unique faithful conditional expectation from \( B \) onto \( A \). (Note that \( \alpha \) determines an action of the groupoid \( G \) on \( B \), in which \( \alpha_{g^{-1}}(b) := 0 \) when \( b \) is not a member of \( A \).)

## 5 C*-bundle dynamical systems

In this section, C*-bundle dynamical systems are introduced and their relationship to other C*-dynamical systems is studied, especially Arveson’s \( A \)-dynamical systems. The key differences that distinguish C*-bundle dynamical systems from more general C*-dynamical systems are, (i) the covariance condition and (ii) additional geometrical data encoded in the action of a 1-parameter group of diffeomorphisms lifted to the C*-bundle. We include a discussion of the advantages to physics.

Orientability of Fell bundles (and spectral triples) is commented on and also connections between Exel’s slices and Arveson’s paths, are discussed below.

A more direct application of C*-bundle dynamical systems to C*-subalgebra theory is developed later in the final section.

### 5.1 Three families of C*-dynamical system

To begin with, recall some fundamental definitions from (non-relativistic) non-commutative C*-dynamical systems following the work of Raeburn [Ra], Williams, Arveson and many others and then we modify them for the case of an action on a C*-bundle:

**Definition 5.1.** A C*-dynamical system is a triple \((B, G, \alpha)\) where \( \alpha \) is a strongly continuous action of a locally compact group \( G \) on a C*-algebra \( B \).
For the action, we make a choice of a representation of $G$, 
\[ \alpha_g(b) = U_g b U_g^* \] 
for each $b \in B$, where:

\[ U_g U_h = \omega(g, h) U_{gh} \]  
(21)

giving an action $\alpha$ of $G$ on $B$:

\[ (g_1, b_1) (g_2, b_2) = (g_1 g_2, \alpha_{g_1}(b_1) b_2) \in G \ltimes B \]  
(23)

Note that $\omega$ measures the departure of the assignment $g \mapsto U_g$ from being a group homomorphism i.e. representation of $G$. The elements of $H^2(G, T)$ consist of equivalence classes of 2-cocycles $[\omega]$ (equivalent if they only differ by a coboundary). For details see [Ra].

In the previous definition the dynamical system is reversible. (A reversible dynamical system is characterised by an action of a group and an irreversible system by an action of an inverse semigroup, exactly because a group has inverses and an inverse semigroup only has quasi-inverses.)

Next we study Arveson’s $A$-dynamical systems, which take into account the structure of C*-algebras in terms of their C*-subalgebras. Since we are using the symbol $B$ for the algebra instead of $A$, we can think of $A$ as standing for Arveson, rather than change to $B$-dynamical system.

**Definition 5.2.** [A1] Arveson’s $A$-dynamical system is a triple $(\iota, B, \alpha)$ consisting of a semi-group $\alpha = \{ \alpha_t : t \geq 0 \}$ of *-endomorphisms acting on a C*-algebra $B$ and an injective *-homomorphism $\iota : A \rightarrow B$, such that $B$ is generated by $\bigcup_{t \geq 0} \alpha_t(\iota(A))$, where $A$ is a C*-subalgebra of $B$.

In Arveson’s definition there is no continuity requirement of the semigroup with respect to $t$. Of course in the case of a reversible $A$-dynamical system the $\alpha_t$ will form a group.

$B$ is the norm-closed linear span of finite products,

\[ B = \text{clspan}\{\alpha_{t_1}(a_1)\alpha_{t_2}(a_2)...\alpha_{t_k}(a_k)\} \]  
(24)

where $t_1, ..., t_k \geq 0$, $a_1, ..., a_k \in A$, $k = 1, 2, ...$. For different times $t_1 \neq t_2$, the C*-algebras $\alpha_{t_1}(A)$ and $\alpha_{t_2}(A)$ do not commute with each other.

Let $M$ be a smooth (or possibly discrete) manifold. Our modification to C*-bundles:-

**Definition 5.3.** A reversible C*-bundle dynamical system $(E^0, G_\sigma)$ is given by a C*-bundle $(E^0, \pi, M)$ over a differentiable manifold $M$ and a strongly continuous 1-parameter covariance group $G_\sigma$ of C*-bundle spatial automorphisms. ($G_\sigma$ was defined earlier in 3.6.)

**Theorem 5.4.** Reversible C*-bundle dynamical systems provide a class of examples of reversible A-dynamical systems.

**Proof.** Let $(E^0, G_\sigma)$ be a C*-bundle dynamical system with enveloping algebra $A = C^*(E^0)$ where $E^0$ is a C*-bundle over the pure state space $X$ of $A$, and let $G = X \times X$ (we assume that $X$ is a locally compact Hausdorff space $M$ admitting a differentiable structure). The ambient algebra $B$ for the A-dynamical system is given by $B = G \ltimes A$ where the action $\alpha$ of the group $G = \text{Bis}(G)$ on $A$ is determined by the unitary representation $\rho_U$ on $E^0$ associated to $G_\sigma$ as follows. See also the earlier explanation of these terms in section 3. Since $G_\sigma$ acts minimally
(is densely transitive) on the space of fibres \( \{E^0_x\}_{x \in M} \) of \( E^0 \), we can associate to it a (strongly continuous) unitary representation \( \rho_U : \text{Bis}(G) \to \text{SpatialAut}_x(E^0) \), where \( G = M \times M \) and \( \text{Bis}(G) \) is identified with \( \text{Diff}(M) \).

The slices \( \alpha_t(a) \) are then obtained from the representation \( \rho_U \), by restricting \( \rho_U \) to \( g \in \text{Bis}_{\text{term}}(G) \) (definition 4) and then forming the closed linear span, \( \mathcal{M} = \text{clspan}_{a \in A} aU_g \) for \( U_g = \rho_U |_{g \in \text{Bis}_{\text{term}}} \). The action \( \alpha \) extends to an action \( \alpha = \{ \alpha_t : t \geq 0 \} \) on \( \mathcal{B} \) such that \( \cup_{t \geq 0} \alpha_t(\iota(A)) \) generates \( \mathcal{B} \) with \( \iota : A \to \mathcal{B} \) an injective \( * \)-homomorphism.

Note that in a C*-bundle dynamical system, there is a clear distinction between the configuration algebra \( A \) and the observable algebra \( B \), whereas in an A-dynamical system, the \( \alpha_t \) are interpreted as alternative configuration spaces.

In an irreversible C*-bundle dynamical system, the covariance group \( G_\sigma \) should be replaced by a inverse semigroup of \( * \)-endomorphisms of \( E^0 \) which preserve the bundle structure of \( E^0 \). The ambient Fell bundle \( E \) will then be regular but not saturated and the fibres of \( E^0 \) will not usually be isomorphic. We begin a discussion on the generalisation to the irreversible case in an additional section, with which we close this paper.

\section{5.2 A remark on Arveson paths and Exel slices}

Let \( A \) be a regular C*-subalgebra in a C*-algebra \( B \),

**Definition 5.5.** \( E \) A slice is any closed linear space \( \mathcal{M} \subseteq N(A) \subset B \) such that both \( \mathcal{A} \mathcal{M} \) and \( \mathcal{M} \mathcal{A} \) are contained in \( \mathcal{M} \).

Observe that the \( \alpha_t(a) \) in definition 5.2 are closed linear subspaces \( \mathcal{M} \) of \( N(A) \) such that \( \mathcal{A} \mathcal{M} = \mathcal{M} \) and \( \mathcal{M} \mathcal{A} = \mathcal{M} \). Since \( \mathcal{M}^* \mathcal{M} = A \) and \( \mathcal{M} \mathcal{M}^* = A \), each \( \mathcal{M} \) is a Hilbert \( \mathcal{A} \)-bimodule. This means that for each \( \mathcal{A} \), \( \alpha_t(a) \) satisfies the definition of a slice \( \mathcal{M} \).

The unital saturated Fell bundle over a discrete groupoid of example 2.7 describes a C*-category of Morita equivalence (or imprimitivity) bimodules. The space of fibres of the Fell bundle \( (E, \pi, G) \) (as opposed to the diagonal bundle \( E^0 \)) therefore forms a structure that obeys a set of rules equivalent to the defining axioms for groupoids but up to isomorphism. (Including \( E_g \otimes E_{g'}^* \cong E_{gg'}^*, E_g \otimes E_{g2} \cong E_{g1g2} \forall (g1, g2) \in G^2, E_g \otimes E_{g'g} \cong E_{g'}, E_{gg'} \otimes E_g \cong E_g \).) Finally observe that the Arveson “path” (see [A2]) by a closed linear space \( \alpha_t(a) \), is exactly a self-adjoint bisection of this “weak” groupoid.

Let \( \sigma \) denote the generator of \( G_\sigma \). Note that \( \sigma = \rho_U(g(\lambda_0)) \). In the case that \( \mathcal{M} \) is discrete and \( \sigma \) is a bounded linear operator, the slices \( \mathcal{M} \) or Arveson paths are given by \( \alpha_t(a) = \text{clspan}_{a \in A} [\sigma, a] \). (We leave the association to the non-commutative differential calculus for finite spectral triples \( (A, \mathcal{H}, D = \sigma) \) for another chapter.)

\section{6 Embeddings}

Following Kumjian’s observation that the normalisers \( N(A) \) of \( A \) in \( B \) characterise the way \( A \) embeds in \( B \) ([K3]), we define an “embedding invariant” \( \Phi_{\infty} \) to be equivalent to a groupoid 2-cocycle \( \omega \), and involving \( N(A) \). This characterises the crossed product algebras of the form \( B = G \rtimes A \) for \( G = X \times X \), the pair groupoid over the state space \( X \) of a possibly non-commutative C*-algebra \( A \) such that \( A = C^*(E^0) \) for some C*-bundle \( E^0 \) with fibre \( M_n(\mathbb{C}) \). C*-bundles over discrete spaces in which the dimension of the fibres is not constant, are also included.
6.1 Embedding invariant

Since $\Phi_{\rightarrow}$ is to be constructed from a groupoid 2-cocycle $\omega$, we first explain the relevance of the latter.

Renault obtained a characterisation of Cartan pairs $(A, B)$ (definition 4.5) in the form of a groupoid 2-cocycle,

$$\omega : G_{ep} \times G_{ep} \to \mathbb{T} \quad (25)$$

where $G_{ep}$ is an essentially principal groupoid and $\mathbb{T}$ denotes the 1-dimensional unitary matrices. It follows that for each example of a Cartan pair $(A, B)$ (definition 4.5) one can define the multiplication in $B$ from this 2-cocycle data in terms of a semidirect product of $G_{ep}$ with $A$, as demonstrated in example 2.6 (saturated Fell bundle). So a Fell line bundle over an essentially principal groupoid $G_{ep}$, defined with an action $\alpha$ is equivalent to an essentially principal groupoid $G_{ep}$ together with a 2-cocycle $\omega$.

More in detail, a twisted action $(\alpha, \omega)$ is given by an assignment of unitaries $g \mapsto u_g \in N(A)$ to each $g \in G_{ep}$, which does not define a representation of $G_{ep}$, because

$$u_g u_h = \omega(g, h) u_{gh} \quad (26)$$

where $u_{gh}$ is the composition of two unitaries $u_g \circ u_h$ implementing the action $\alpha$ of $G_{ep}$ on $A$, or equivalently, the representation of $G_{ep}$ on $E^0$. Therefore, an embedding invariant $\Phi_{\rightarrow}$ equivalent to a 2-cocycle mapping $\omega : G_{ep} \times G_{ep} \to \mathbb{T}$, can be presented as a continuous assignment of unitaries $g \mapsto u_g \in N(A)$ satisfying (26).

In the special case where the groupoid $G$ is principal, then $A$ is diagonal in $B$. Note that since given any C*-diagonal $A$ in $B$, a Fell line bundle $E$ over $G$ (as shown in 2.6) can be found such that $A = C^*(E^0)$ and $B = C^*(E)$, the action $\alpha$ of $G$ on $A$, which controls the multiplication in $E$, corresponds to a representation $\rho_G$ of $G$ on $E^0$. Secondly, from $\rho_G$ one obtains a unitary representation $\rho_U : \text{Bis}(\rho_G) \to \text{SpatialAut}_x(E^0)$ implemented by a mapping $g \mapsto U_g \in N(A)$ for each $g \in \text{Bis}(G)$, (where the mapping $g \mapsto U_g$ is induced directly from the previous mapping $g \mapsto u_g$ for each $g \in G$).

Let $G$ be a groupoid and let $\mathcal{G}$ be the group $\mathcal{G} = \text{Bis}(G)$. Note that it may be the case that a group 2-cocycle for $\mathcal{G}$ may be a coboundary (that is, defines a representation of $G$, with $U_g U_h = U_{gh}$) while a groupoid 2-cocycle is not, that is, it satisfies $u_g u_h = \omega(g, h) u_{gh}$ for non-trivial $\omega$.

Now we modify proceedings to the non-commutative case. Let $A$ be a not necessarily commutative diagonal C*-subalgebra in a C*-algebra $B$. Making use of the normalising set $N(A) \subset B$, below we define the embedding invariant $\Phi_{\rightarrow}$ to characterise such pairs $(A, B)$. In the case that $A = C^*(E^0)$ for $E^0$ a C*-bundle with isomorphic fibres, we define $\Phi_{\rightarrow}$ to be equivalent to a principal groupoid 2-cocycle,

$$\omega : G \times G \to \bigoplus_{i=1}^n \mathbb{T} \quad (27)$$

where $n$ is the dimension of the fibre of $E^0$.

**Definition 6.1.** Let $A$ be (a possibly non-commutative) C*-diagonal in a C*-algebra $B$ where both $A$ and $B$ operate on a separable Hilbert space $\mathcal{H}$. Since $\mathcal{H}$ is separable, it has a countable orthonormal basis $\{p_i\}_{i \in X}$ whose linear span is dense in $\mathcal{H}$. 

Let \( u_{(i,j) \neq (i,i)} \in N_f(A) \) and \( u_{(i,i)} \in A \) satisfy:

\[
p_k u_{(i,j)} p_l = \begin{cases} 
  u_{ij}, & \text{if } (k,l) = (i,j), \\
  0, & \text{otherwise}.
\end{cases}
\] (28)

where each \( u_{ij} \) is a partial isometry, \( u_{ij} \mathcal{H}_j \to \mathcal{H}_i \) where \( \mathcal{H}_i = p_i \mathcal{H}, \mathcal{H}_j = p_j \mathcal{H} \). (In the case that the rank of the projections \( p_i \) is constant, each \( u_{ij} : \mathcal{H}_j \to \mathcal{H}_i \) is an isometry.)

Given a choice of action \( \alpha \), the diagonal embedding \( \Phi_\to \) of \( A \) in \( B \) is fixed by an approximation of a maximal rank element of \( B \):

\[
\Phi_\to = \left\{ \sum_{(i,j)} u_{(i,j)} \mid (i,j) \in Y \times Y \subset X \times X \right\} \in B
\] (29)

for all subsets \( Y \times Y \subset X \times X \).

The above definition may be generalised to Cartan pairs by replacing the pair groupoid \( \{ p_i \} \times \{ p_i \} \), with the effective pair groupoid, which can be constructed from \( N(A) \) using (16): \( G_{ep} \) is the groupoid of germs of the partial homeomorphisms defined by (16). In this case, even if \( \mathcal{H} \) is finite dimensional, then \( \Phi_\to \) can no longer be presented as matrix. We include this generalisation as an Appendix.

Examples 6.2. 1. Let \( A \) be a non-commutative finite dimensional C*-algebra, which is diagonal in a C*-algebra \( B \). This automatically provides a C*-bundle over \( X \), as described in 3.5(2) and where \( X \) is interpreted as the space of pure states of \( A \).

\[
\Phi_\to = \sum_{(x,y) \in X \times X} u_{(x,y)} \quad (x,y) \in X \times X \quad \in B
\] (30)

with

\[
u_{xy} = p_x u_{(x,y)} p_y \quad \forall (x,y) \in X \times X.
\] (31)

where each \( u_{xy} \) is a partial isometry \( u_{xy} \in M_{rs}(\mathbb{C}) \) where \( r = \text{rank } p_x \) and \( s = \text{rank } p_y \).

In this case, \( \Phi_\to \) is a state-transition matrix for algebraic quantum gravity.

2. Let \( E^0 \) be a locally trivial C*-bundle over \( X \) with non-commutative fibres and let \( E \) be a Fell bundle over \( G = X \times X \) as in example 2.6(b) (Saturated Fell bundle). \( \Phi_\to \) as in equation (29) where each \( u_{ij} \) is a unitary matrix whose rank and dimension is equal to the dimension of the fibres of \( E^0 \).

3. Let \( E^0 \) be a 1-dimensional C*-bundle over a compact space \( X \) where \( B = C^*(E) \) and \( A = C^*(E^0) \). Identify \( X \) with the image of \( X(A) \). Each \( u_{ij} \in T \) where \( T \) denotes the 1-dimensional unitary matrices.

Comment 6.3. This notion of embedding for the context of C*-algebras is motivated by the fact that in topology, an embedding is a homeomorphism onto its image. In analogy, the way a C*-algebra \( A \) embeds in a C*-algebra \( B \) is determined by inner automorphisms (non-commutative homeomorphisms) implemented by the unitary normalisers of \( A \) in \( B \).

Remark 6.4. In summary, the embedding invariant \( \Phi_\to \) provides all the information about the Fell bundle and the way in which \( A \) embeds in \( B \). It might be helpful to keep in mind its form as a transition matrix. The following data can be read-off directly from \( \Phi_\to \): \( \alpha, \omega, (E, \pi, G), (E^0, \pi, G_0), A \subset B, N(A), N_U(A), P : B \to A, \rho_U \) and \( \rho(G) \).
Given a possibly non-commutative diagonal C*-subalgebra $A$ in $B$, the embedding invariant $\Phi_\rightarrow$ can be generated by a covariance subgroup $G_\sigma$ of the associated unitary representation $\rho_U$, as follows.

**Theorem 6.5.** Let $(E^0, G_\sigma)$ be a C*-bundle dynamical system and let $(E, \pi, G)$ be a saturated Fell bundle over a principal groupoid over a (possibly discrete) differentiable manifold $M$. The embedding invariant $\Phi_\rightarrow$ associated to $(A, B)$ can be constructed from the abelian subgroups $G_\sigma \subset \rho_U$.

**Proof.** (i) Consider first a discrete space $M$ with $n$ points $x \in M$. Then by inspection of definition 6.1,

$$\Phi_\rightarrow = \sum_{m=1}^{n} \prod_{i=1}^{m} U_{g_i}, \quad i = 1, \ldots, m, \quad m = 1, \ldots, n. \quad (32)$$

because since $\text{Diff}(M)$ is generated by a finite diffeomorphism $g$ or $g(\lambda_0)$, a 1-parameter group of diffeomorphisms is a finite product group, that is, $\{g(\lambda)\}_{\lambda \in \mathbb{Z}}$ and since $\sigma = \rho_U(g(\lambda_0))$, elements $(f_0, f, U_{g_0})$ or $(g(\lambda), f, U_{g(\lambda)})$ of $G_\sigma$ are obtained by self-multiplications of $\rho_U(g(\lambda_0))$.

More generally, since $A$ is regular in $B$, we use the fact that $B = \text{clspan} N(A)$ and approximate $\Phi_\rightarrow$ in $B$ by sets of finite linear combinations of finite products of normalisers $U_{g_i}$.

**Remark 6.6.** Note that $G_\sigma$ and $\Phi_\rightarrow$ are embedding invariants equivalent to a groupoid 2-cocycle $\omega$.

**Remark 6.7.** In quantum gravity, a partition function is a discretisation of the path integral in quantum field theory. The algebra invariant $\Phi_\rightarrow$ is a topological invariant and plays the role of a state-transition matrix. When diagonalised, it is analogous to a partition function with constant weight for a spin foam on a discretised manifold, since $\Phi_\rightarrow$ is a sum over geometrical states (or irreducible representations of the algebra $A$) and the product is over parallel transports, where each $(f_0, f, U_{g_0}) \in \text{SpatialAut}_\pi(E^0)$ can be said to provide a system of parallel transports in the C*-bundle since it provides a set of isometric *-isomorphisms between the fibres of $E^0$.

**Example 6.8.** For illustrative purposes, consider a discrete space $M$ with $n = 4$ points. Let $G$ denote the pair groupoid over this discrete space and $\text{Bis}(G)$ the group of global bisections. The diagrams illustrate how we generate $\Phi_\rightarrow$ from actions of subgroups of $\text{Bis}(G)$ on a finite dimensional C*-bundle $E^0$ over $M$. Let $g$ be the following transitive\footnote{In the discrete situation, there is no distinction between a minimal and a transitive flow.} element of $\text{Bis}(G)$,

$$g = \begin{array}{c} \text{\vdots} \end{array} \begin{array}{c} \vdots \end{array} \begin{array}{c} \vdots \end{array} \begin{array}{c} \vdots \end{array}, \quad (33)$$

$$g \mapsto U_g, \quad (34)$$

$$U_g = \begin{pmatrix} 0 & u_{12} & 0 & 0 \\ 0 & 0 & u_{23} & 0 \\ 0 & 0 & 0 & u_{34} \\ u_{41} & 0 & 0 & 0 \end{pmatrix} \quad (35)$$

18
\[ g \circ g = \begin{array}{c}
\circ \\
\circ 
\end{array} \] (36)

\[ g \circ g \mapsto U_{g \circ g} \] (37)

\[ U_{g \circ g} = \begin{pmatrix}
0 & 0 & u_{13} & 0 \\
0 & 0 & 0 & u_{24} \\
u_{41} & 0 & 0 & 0 \\
0 & u_{42} & 0 & 0
\end{pmatrix} \] (38)

where for example, \( u_{12}u_{23} = \omega(12, 23)u_{13} \).

The self-adjoint elements of \( \text{Bis}(G) \) include:

\[ \begin{array}{c}
\circ \\
\circ
\end{array} \] (39)

\[ \begin{array}{c}
\circ \\
\circ
\end{array} \] (40)

\[ \begin{array}{c}
\circ \\
\circ
\end{array} \] (41)

**Comment 6.9 (Orientability).** We have been implicitly assuming that \( E^0 \) is orientable since we have been interpreting \( \Phi \mapsto \) as a nowhere vanishing global section. If \( E^0 \) is not orientable then \( \Phi \mapsto \) is not an element of \( B \). In that case, to treat \( \Phi \mapsto \) with a topological twisting of the bundle, we will have to insert \( u_g u_h = \omega(1 - u_{gh}) \) for a pair of elements \( g, h \in G \). If \( (C^*(E^0), \mathcal{H}, \sigma, \gamma) \) is an even spectral triple \( (A, \mathcal{H}, D, \gamma) \), then the orientability condition for spectral triples \( D\gamma = \gamma D \) (which is automatically satisfied for \( D = \sigma \) \([M2]\)) might be replaced by the condition that \( E^0 \) is orientable: that \( \Phi \mapsto \) is an element of \( B \). (See \([52]\) for discussions on the orientability condition for spectral triples.) In the context of the non-commutative standard model and Fell bundles \( ([M2]) \), a transition matrix (to counterpart \( \Phi \mapsto \)) for a non-orientable Fell bundle might indicate a topological defect in the vacuum manifold.

## 7 C*-bundle dynamical systems and Fell bundles

Earlier in section 4, we recalled some important results by Kumjian, Renault and Exel on C*-subalgebras. More specifically, Kumjian established the correspondence between pairs of C*-algebras \( (A, B) \) such that \( A \) is a C*-diagonal in \( B \) and twisted Fell line bundles \( E \) over principal groupoids \( G \). \([K3]\) (although Kumjian defined Fell bundles over groupoids later in \([K1]\)). Following some of Kumjian’s techniques, Renault generalised the situation to Fell line bundles over (locally compact Hausdorff) essentially principal groupoids \( G_{\text{ep}} \), with \( C^*_r(E) \cong C^*_r(G_{\text{ep}}) \), where \( C^*_r(G_{\text{ep}}) \) is the reduced C*-algebra completion of the groupoid convolution algebra. He introduced a notion of Cartan pair \( (A, B) \) for the context of C*-algebras and then established a correspondence between them and essentially principal groupoids together with
a groupoid 2-cocycle $\omega$, which he obtained by the twisted groupoid convolution product $[\text{Ren}]$. The crossed product algebras $B$ involved in each of these works, which are by construction regular, take the form $B = G \ltimes A$ ($[K3]$) or $B = G_{ep} \ltimes A$ ($[\text{Ren}]$).

Cartan subalgebras that are not C*-diagonals:

Let $G_{ep}$ be an essentially groupoid over a locally compact space $X$. Let $(E, \pi, X)$ be a Fell bundle over $G_{ep}$. Let $(A, B)$ be a Cartan pair with faithful conditional expectation $P : B \to A$. Recall that $A$ is diagonal in $B$ if and only if $\ker P = \text{clspan}\, N_f(A)$. Observe that in this case of a Fell bundle over an essentially principal groupoid as opposed to a principal groupoid, the kernel of $P : C^*(E) \to C^*(E^0)$ is a larger set than the set $\text{clspan}\, N_f(C^*(E^0))$. It follows that $A := C^*(E^0)$ is a Cartan subalgebra of $B := C^*(E)$, and not a C*-diagonal in $C^*(E)$. This is why the scope of examples treated in $[\text{Ren}]$ is larger, and in turn, why the classifications of C*-subalgebras defined by each of Kumjian and Renault, are not equivalent. Two families of these “extra” examples that have regular masas and are not C*-diagonals, include the graph C*-algebras and the Cuntz algebras $[\text{Ren}]$. For an extensive work on the automorphisms of Cuntz algebras, see $[CS]$ and see $[K2]$ for a survey on graph C*-algebras.

Later, in Exel’s work $[E]$, a correspondence between crossed product C*-algebras of the form $B = S \ltimes A$ and Fell bundles over inverse semigroups $S$, with enveloping algebra $C^*(E) = B$ is established, where $A = C^*(E^0)$ is a non-commutative C*-algebra, arising from the twisted convolution algebra of $S$. Exel underlines his motivation that N. Siebens notion of Fell bundles over inverse semigroups should be thought of as twisted étale groupoids with non-commutative unit space.

In this section we treat examples of locally trivial Fell bundles over groupoids that are not necessarily one dimensional, and so their enveloping algebras $C^*(E)$ are not convolution algebras, neither of groupoids nor of inverse semigroups. (Refer to examples 2.5.) These algebras $C^*(E)$ can be thought of as tensor product algebras $B = C^*_r(G) \otimes M_n(\C)$. The advantage of these examples to physics includes example 2.5(5), and in the case that $G$ is discrete, we refer to arguments that a realistic notion of space-time manifold $M$ is unlikely to be either continuous or commutative ($[I], [Cr1], [Cr3]$). Another example of a non-commutative diagonal pair is from $[BeCo]$:

Let $\Sigma = (B, G, \alpha, \omega)$ be a unital discrete twisted C*-dynamical system (as studied in $[BeCo]$) such that $B = A \rtimes^\alpha G$ where $A$ is a simple C*-algebra and where $G$ is a discrete subgroup of $\text{Bis}(G)$ for $G = M \times M$, then $(A, B)$ is a not necessarily commutative diagonal pair.

### 7.1 Groupoid 2-cocycles as Cartan pair invariants

The following generalises Renault’s result that a one dimensional Fell bundle over an essentially principal groupoid $G_{ep}$, specified by a groupoid 2-cocycle, is equivalent to his definition of a Cartan pair $(A, B)$ of C*-algebras. In this generalised case, we consider locally trivial saturated Fell bundles $(E, \pi, G_{ep})$ (from $[K1]$), saturated Fell bundles over locally compact groupoids are equivalent to semidirect product example 2.6. Since these algebras $C^*(E)$ can be thought of as tensor product algebras $B = C^*_r(G) \otimes M_n(\C)$ it should be possible to produce the following result using only Renault’s construction but we provide these alternative techniques because they have the potential to be generalised to non-saturated Fell bundles over inverse semigroups and because the result leads into the next result which involves C*-dynamical systems and the unitary representations as discussed in section 3.

**Theorem 7.1.** (a) Each locally trivial Fell bundle $(E, \pi, G_{ep})$ over an essentially principal groupoid $G_{ep}$ over a locally compact Hausdorff space $X$, gives rise to a not necessarily abelian Cartan pair $(A, B)$ of separable C*-algebras (as in definition 4.6) such that $B = G_{ep} \ltimes^\alpha A$, where $A$ is a Cartan subalgebra and $B = C^*_r(G) \otimes M_n(\C)$. The crossed product algebras $B$ involved in each of these works, which are by construction regular, take the form $B = G \ltimes A$ ($[K3]$) or $B = G_{ep} \ltimes A$ ($[\text{Ren}]$).
where $G_{ep}$ is an essentially principal groupoid over $X(A)$. (b) All such Cartan pairs, (with $B = G_{ep} \ltimes \alpha A$ for some essentially principal groupoid $G_{ep}$ and some separable C*-algebra $A$), arise in this way.

**Proof.** (a) First of all, the restriction of $E$ to the unit space $G_{ep}^0$ of $G_{ep}$ provides a C*-bundle $E^0$, whose enveloping C*-algebra $C^*(E^0)$, as detailed above in section 2, will provide the C*-subalgebra $A$ in the pair $(A,B)$ that we are constructing, $A := C^*(E^0)$. The particular embedding of $A$ in $B$ for the Cartan pair $(A,B)$, is specified by the semidirect product structure as in examples [2,3] above, providing the ambient algebra $B = G_{ep} \ltimes \alpha A$. There is a unique faithful conditional expectation $P : B \to A$, given by the restriction map $P : C^*(E) \to C^*(E^0)$.

(b) Let $(A,B)$ be a not necessarily commutative Cartan pair with unique faithful conditional expectation $P : B \to A$. A C*-bundle $(E^0, \pi, X)$ is constructed from the C*-subalgebra $A$ as follows. Since $A$ is separable, (and assuming that $A$ is faithfully represented on the associated Hilbert space $\mathcal{H}$), we identify $A$ with $\bigoplus_{m} \pi_m$, the direct sum over all irreducible representations $\pi_m$ of $A$. Define the space of fibres of $E^0$ to be given by $\{E^0_x\}_{x \in X} = \{\pi_m\}$, where $X$ is identified with the pure state space $X(A)$ which is locally compact in the weak *-topology. Then the enveloping algebra $C^*(E^0)$ of $E^0$ is identified with $A$.

To construct a Fell bundle $(E, \pi, G_{ep})$ from $(A,B)$ one defines an essentially principal groupoid $G_{ep}$ as the groupoid of germs of the locally defined homeomorphisms on $X$ (see [13]) and the Fell bundle $E$ over $G_{ep}$ is given by $E = G_{ep} \ltimes A_0$ such that $B = C^*(E)$, with $P : C^*(E) \to C^*(E^0)$ and where the representation of $A$ extends to a faithful representation on $\mathcal{H} = L^2(E)$. To see that all such Cartan pairs $(A,B)$ arise in this way, one constructs the groupoid 2-cocycle from the interaction between $B$ and $G_{ep}$ in order to specify $E$. Recall the earlier result [8,11] and note that

$$\text{clspan}(A N_U(A)) = \text{clspan}(A \text{Bis}(\rho_{G_{ep}})) \quad (42)$$

for a certain representation $\rho_{G_{ep}} : G_{ep} \to \text{SpatialAut}_{\pi}(E^0)$ of $G_{ep}$ on $E^0$. (In the case that $G_{ep}$ is principal $G$ we have a representation $\rho_G : G \to GL(E^0)$ inducing a representation Bis$(\rho_G) = \rho_U : \text{Bis}(G) \to \text{SpatialAut}_{\pi}(E^0)$.) Since $A$ is regular in $B$,

$$B = \text{clspan}(N(A)) = \text{clspan}(A \rho_{G_{ep}}) \quad (43)$$

which illustrates that the data for the representation $\rho_{G_{ep}}$ and the action $\alpha$ come from the multiplication rule in $B$, and in turn the action $\alpha$ gives the mapping $\omega : G_{ep} \times G_{ep} \to \bigoplus_{i=1}^n \mathbb{T}$ (where $n$ is the dimension of the fibre of $E^0$) and a Fell bundle is found such that $B = C^*(E)$. Finally, note that $E^0$ is locally trivial and the map $P : B \to A$ (as in [13]) gives the restriction of enveloping algebras $P : C^*(E) \to C^*(E^0)$.

### 7.1.1 Bridge theorem

Let $(E^0, G_e)$ be a reversible C*-bundle dynamical system (definition [5,3]). In the following theorem we make use of several constructions set out previously in this paper, to show that $(E^0, G_e)$ provides an enveloping structure capturing non-commutative Cartan pairs together with their respective Fell bundles, where the embedding invariant $\Phi_{\omega\rho}$ creates a bridge between the two descriptions. The Fell bundles in question are not necessarily 1-dimensional and we use new techniques provided by unitary representations of groups and groupoids as discussed in [8].

We assume that in all examples, the state space $X$ of $A$ is a locally compact Hausdorff space (or topological manifold) admitting a smooth structure.
Theorem 7.2 (Bridge). (a) Let \( M \) be a locally compact simply connected manifold. Each reversible \( C^* \)-bundle dynamical system, \((E^0, \mathcal{G}_\sigma)\) gives rise to a locally trivial Fell bundle over a connected principal groupoid and its uniquely associated (not necessarily abelian) Cartan pair. (b) All such Fell bundles over connected principal groupoids \( G \) together with their uniquely associated Cartan pair, arise in this way.

Proof. (a) In order to identify the Cartan pair \((A, B)\) and Fell bundle \((E, \pi, G_{\rho_\sigma})\) that arises from each \((E^0, \mathcal{G}_{\rho_{ep, \sigma}})\), one constructs an embedding invariant \( \Phi_{\rightarrow} \) from \((E^0, \mathcal{G}_{\rho_{ep, \sigma}})\) and then one reads-off all the required data from \( \Phi_{\rightarrow} \).

Treat first the case where \( \mathcal{H} \) is a finite dimensional Hilbert space and \( M \) is a discrete manifold with \( n \) points. Let \( G \) be the pair groupoid over \( M \) and let \( g(\lambda_0) \) be the generating diffeomorphism of the finite product group \( \{ g(\lambda) \}_{\lambda} \subset \text{Bis}(G_{\rho_{ep}}) \) such that

\[
\mathcal{G}_\sigma : \{ g(\lambda) \} \rightarrow \text{SpatialAut}_\sigma(E^0)
\]

Let \( U_g \) denote the operators implementing the unitary representation given by \( \mathcal{G}_\sigma \). Since \( \mathcal{G}_\sigma \) is minimal (over the space of fibres of \( E^0 \)) one writes:

\[
\Phi_{\rightarrow} = \prod_{m=1}^{n} \bigcap_{i=1}^{m} U_{g_i}, \quad i = 1, ..., m, \quad m = 1, ..., n.
\]

as in (12).

After having constructed \( \Phi_{\rightarrow} \) for \((E^0, \mathcal{G}_\sigma)\), we read-off from \( \Phi_{\rightarrow} \) all the required information about the \( C^* \)-subalgebra \( A = C^*(E^0) \) and the way in which it embeds in the ambient \( C^* \)-algebra \( B = C^*(E) \) as determined by \( \alpha, \omega \) and \( P \). More in detail, denoting \( u_{(i,j)} \) by \( u_{(p_i,p_j)} \), we approximate the elements of the algebras \( A \) and \( B \) from the following sets of finite sums,

\[
a = \{ \sum a_i u_{(p_i, p_i)} \mid (p_i, p_i) \in (p_1, ..., p_k) \times (p_1, ..., p_k) \} \in A
\]

\[
b = \{ \sum a_i u_{(p_i, p_j)} \mid (p_i, p_j) \in (p_1, ..., p_k) \times (p_1, ..., p_k) \} \in B
\]

where \((p_1, ..., p_k)\) is an \( n \)-tuple of pairwise orthogonal projections with \((p_1, ..., p_k) \subset (p_1, ..., p_n)\) and where \( a_i \) denote arbitrary elements of \( p_i A \).

\[B = \text{clspan} N(A)\] where \( n \in N(A) \) is approximated by \( n = \sum_{(i,j)} a_i u_{(i,j)} \) for pairs \((p_i, p_j)\) and choices of \( a_i \in p_i A \).

The set \((p_1, ..., p_n)\) also provides the following.

\[\bigcup_{i} p_i \in A\] forms a unit or an approximate unit for \( B \).

There is a unique faithful conditional expectation, \( P : B \rightarrow A \), \( P(b) = \sum_i p_i bp_i \in A \) with \( \ker P = N_f(A) \).

(b) Conversely, one shows that each Fell bundle \((E, \pi, G)\) together with its uniquely associated non-commutative Cartan pair \((A = C^*(E^0), B = C^*(E))\), arises as above. Present the information specifying the action \( \alpha \) and the 2-cocycle \( \omega \) for \((A, B)\) and \((E, \pi, G_{\rho_\sigma})\), by associating an embedding invariant \( \Phi_{\rightarrow} \) to \((A, B)\) and \((E, \pi, G_{\rho_\sigma})\) with restriction map \( P : C^*(E) \rightarrow C^*(E^0) \). Then, from \( \Phi_{\rightarrow} \), one approximates a unitary representation \( \rho_G : G \rightarrow GL(E^0) \) inducing \( \rho_U = \text{Bis}(\rho_G) \), and then a covariance group \( \mathcal{G}_\sigma \subset \rho_U \) is obtained such that \( \mathcal{G}_\sigma \) has minimal flow by restricting \( \rho_U \) to a 1-parameter subgroup of diffeomorphisms \( \{ g(\lambda) \} \subset \text{Bis}(G) \). In the case that the principal groupoid is a pair groupoid, then \( \mathcal{G}_\sigma \) should have minimal (densely transitive) flow. \( \square \)
8 Pre-requisites for irreversible C*-bundle dynamical systems

Here is some additional material that will be required in order to define the more general notion of irreversible C*-bundle dynamical system. This material is not particularly new since Paterson (see the book \[Pa\]) already developed the theory of partial isometry representations of inverse semigroups.

Let $V$ be an operator on a Hilbert space $H$. Recall that $V$ is a partial isometry if there exists a unique operator $V^*$ (which we call quasi-inverse) such that $VV^*V = V$ and also $V^*VV^* = V^*$. Clearly, the invertible partial isometry operators are the isometries $U$, (satisfying $U^*U = 1$, $UU^* = 1$). The set of all partial isometries $V$ on $H$ form an inverse semigroup $\mathcal{V}(H)$, whereas the set of all unitary operators $U$ on $H$ form a group $\mathcal{U}(H)$.

Strictly, a groupoid and an essentially principal groupoid are special cases of inverse semigroups $S$, (a groupoid is an inverse category with a 0 element formally adjoined, in which the quasi-inverses are true inverses.)

**Definition 8.1.** Let $(E^0, \pi, X)$ be a C*-bundle over a locally compact (possibly discrete) space $X$, with enveloping algebra $A$. We define $\text{ISL}(E^0)$ the inverse category consisting of all involutive *-endomorphisms $\alpha_x$ between fibres of $E^0$ such that for each $\alpha_x$, there is a unique $\alpha_x^*$ satisfying, $\alpha_x \circ \alpha_x^* \circ \alpha_x = \alpha_x$ and $\alpha_x^* \circ \alpha_x \circ \alpha_x^* = \alpha_x^*$. Call $\alpha_x^*$ the quasi-inverse of $\alpha_x$.

Note that $\text{ISL}(E^0)$ has the structure of an inverse semigroup with 0 element formally adjoined.

**Definition 8.2.** Let $(E^0, \pi, X)$ be a C*-bundle over a locally compact space $X$, with enveloping algebra $A$, faithfully represented on a separable Hilbert space $H$. A C*-bundle spatial *-endomorphism consists of continuous maps $f$ and $f_0$ with commuting diagram:

\[
\begin{array}{ccc}
E^0 & \xrightarrow{f} & E^0 \\
\pi \downarrow & & \downarrow \pi \\
X & \xrightarrow{f_0} & X
\end{array}
\]

• such that each induced fibrewise map $f_x : E^0_x \to E^0_{f_0(x)}$ is continuous,

• and such that $f$ extends to a *-endomorphism $\hat{f} : A \to A$ of the form $\hat{f}(a) = V a V^*$ where $V$ is a partial isometry map on $H$.

Obviously, the invertible C*-bundle spatial *-endomorphisms are the C*-bundle spatial automorphisms $\text{SpatialAut}_\pi(E^0)$.

The set of inner *-endomorphisms of $E^0$ form an inverse category $\text{InnEnd}_\pi(E^0)$, or an inverse semigroup with 0 adjoined.

**Definition 8.3.** A partial isometry representation $\rho_V$ of an inverse semigroup $S$ on a C*-bundle $E^0$ is an inverse semigroup homomorphism: $\rho_V : S \to \text{InnEnd}_\pi(E^0)$ such that $V$ is a partial isometry on a separable Hilbert space $H$. 

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Obviously, the invertible partial isometry representations \( \rho_V \) are the unitary representations \( \rho_V, \) \(^{[33]}\).

**Example 8.4.** Let \((A, B)\) be a non-commutative Cartan pair such that \(B\) is a crossed product algebra: \(B = S \ltimes A\) where \(S\) is an inverse semigroup. We have already mentioned that \(A\) is regular in \(B\). The action of \(S\) on \(A\) corresponds to a partial isometry representation \(\rho_V\) of \(S\) on \(E^0\). This is the main family of examples considered in \([E]\).

9 Appendix

The following presentation of an embedding invariant \(\Phi_{\rightarrow}\) for non-commutative Cartan pairs of the form \(B = \text{G}_{ep} \ltimes_{\alpha} A\), is equivalent to a groupoid 2-cocycle \(\omega\) : \(\text{G}_{ep} \times \text{G}_{ep} \rightarrow \bigoplus_{i=1}^{n} \mathbb{T}\). This is useful as it allows one to readily switch between a Fell bundle specified by a 2-cocycle \(\omega\) and its associated Cartan pair \((A, B)\).

**Definition 9.1** (Embedding invariant, \(\Phi_{\rightarrow}\)). Let \((E, \pi, \text{G}_{ep})\) be an orientable, (not necessarily 1-dimensional) locally trivial Fell bundle, specified by 2-cocycle data \(\omega : \text{G}_{ep} \times \text{G}_{ep} \rightarrow ISL(E^0)\).

And let \((A, B)\) be the (possibly non-commutative) Cartan pair associated to \(E\). Both \(A\) and \(B\) operate on a separable Hilbert space, \(H = L^2(E)\).

Let \(\{p_i\}\) be a countable orthonormal basis for \(H\) which is dense in \(G^0_{ep}\), given by maximal central projections in \(B\) as follows. Define simple matrix units \(e_{xy} \in E_{(x,y)}\) for all \(x \in X(A)\) such that the rank of central projections \(e_{xy}e_{xy}^*\) in \(B\), satisfy: \(\text{rank } e_{xy}e_{xy}^* = \dim (E^0_l)\) for each \(x \in G^0_{ep}\). Unless \(G_{ep}\) is principal (that is, if it has trivial isotropy) then \(e_{xy}e_{xy}^* \in \text{InnAut}(E^0_x)\) is not identified with \(e_{xz}e_{xz}^* \in \text{InnAut}(E^0_z)\), which explains the need for the additional index \(i\).

We have, \(G_{ep} \cong \{e_{xy}, e_{xy}e_{xy}^*\}\) and \(\{p_i\} = \{e_{xy}e_{xy}^*\}\).

In the special case that the base space of \(E\) is a principal groupoid \(G\), then \(\ker P = \text{clspan}_{n \in N_f(A)} \{n, nn^*\}\)

\[ \ker P = \text{clspan}_{n \in N_f(A)} \{n, nn^*\} \] \hspace{1cm} (49)

Let \(u_{(i,j\neq i)} \in N_f(A)\) and \(u_{(i,i)} \in A\) satisfy:

\[ p_k u_{(i,j)} p_l = \begin{cases} u_{ij}, & \text{if } (k, l) = (i, j), \\ 0, & \text{otherwise}. \end{cases} \] \hspace{1cm} (50)

where each \(u_{ij}\) is an isometry, \(u_{ij} : \mathcal{H}_i \rightarrow \mathcal{H}_j\), where \(\mathcal{H}_i := p_i \mathcal{H}\).

Given the action \(\alpha\), we have an assignment,

\[ g \mapsto u_g = p_i u_{(i,j)} p_j \] \hspace{1cm} (51)

where \(u_g\) denotes the unitaries implementing the action \(\alpha\), which is in turn specified by the 2-cocycle \(\omega : \text{G}_{ep} \times \text{G}_{ep} \rightarrow \bigoplus_{i=1}^{n} \mathbb{T}\). (Note that the mapping \(g \mapsto u_g\) does not define a representation of \(G_{ep}\) except in the special case of \(E^0\) a trivial bundle over \(G_{ep}\).) Also we obtain directly from \(\alpha\), a representation \(\rho_{G_{ep}} : \text{G}_{ep} \rightarrow ISL(E^0)\) and \(\alpha\) also induces a representation \(\rho_V : \text{Bis}(G_{ep}) \rightarrow \text{InnEnd}(E^0)\).

The embedding invariant \(\Phi_{\rightarrow}\) of \(A\) in \(B\) is then fixed by an approximation of a maximal rank element of \(B\), formed by finite linear combinations:-
\[
\Phi_{\to} = \{ \sum_y p_x u(p_x,p_y)p_y \mid p_x, p_y \in Y \subset \{e_{xy},e_{xy}^*\} \cong G_{ep} \}
\] (52)

for all subsets \(Y \in \{e_{xy},e_{xy}e_{xy}^*\}\).

In general, the probability measure on the pure state space of a non-commutative algebra is not unique. However, in our case, since \(E^0\) is a locally trivial Banach bundle, the \(p_i\) are all equivalent, and so there is only one probability measure on \(X\). From this, it follows that the faithful conditional expectation \(P : B \to A\) is unique and is given by \(P(b) = \int_X \alpha_\phi(b) dx\) as in (5.2) where \(X = G_{ep}^0\) and where \(dx\) is the unique probability measure on \(X\) obtained from the basis of \(H = L^2(E)\), dense in \(X\), as defined above. In finite dimensions we have \(P(b) = \sum_i p_i b p_i\). Then \(P\) is identified with the unique restriction map \(P : C^*(E) \to C^*(E^0)\).

Also, \(\cup_{p_i} \in A\) forms a unit or an approximate unit for \(B\).

10 Acknowledgements

Many thanks to Paolo Bertozzini, Roberto Conti and Pedro Resende for their helpful insights (alphabetical order). See title page for affiliation.

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