EXAMPLES ON A CONJECTURE ABOUT MAKAR-LIMANOV INVARIANTS OF AFFINE UNIQUE FACTORIZATION DOMAINS

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ABSTRACT. The author introduces a conjecture about Makar-Limanov invariants of affine unique factorization domains over a field of characteristic zero. Then the author finds that the conjecture does not always hold when \( \mathbb{k} \) is not algebraically closed and gives some examples where the conjecture holds.

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1. Introduction

Throughout the work, $A^{[n]}$ denotes the polynomial ring over a given ring $A$ with $n$ variables; $\mathbb{k}$ denotes a field of characteristic zero; $\mathbb{A}^n$ denotes the affine space over $\mathbb{k}$ of dimension $n$. An affine $\mathbb{k}$-variety is an irreducible algebraic set over $\mathbb{k}$; given an affine variety $V \subseteq \mathbb{A}^n$, define $I(V) := \{ f \in \mathbb{k}^{[n]} : f(a) = 0, \forall a \in V \}$; given a subring $I$ of $\mathbb{k}^{[n]}$, define $\mathcal{Z}(I) := \{ a \in \mathbb{A}^n : f(a) = 0, \forall f \in I \}$.

In paper [14], Makar-Limanov provides several conjectures about Makar-Limanov invariant of an affine unique factorization domain. One of his conjectures is that

Conjecture 1. If $A$ is an affine UFD over $\mathbb{C}$, then $\mathcal{ML}(A) = \mathcal{ML}(A^{[1]})$.

Here we are going to discuss a generalized version of this conjecture which originally put forward on complex field $\mathbb{C}$.

Conjecture 2. If $A$ is an affine UFD over $\mathbb{k}$, then $\mathcal{ML}(A) = \mathcal{ML}(A^{[1]})$.

Based on following parts, one can see that $\mathcal{ML}(A^{[1]}) \subseteq \mathcal{ML}(A)$ and $\mathbb{k} \subseteq \mathcal{ML}(A)$. Therefore, we only need to focus on $\mathcal{ML}(A^{[1]}) \supseteq \mathcal{ML}(A)$ when $\mathcal{ML}(A) \neq \mathbb{k}$.

In Section 2, the author introduces the definition of an affine unique factorization domain and some methods to identify affine unique factorization domains. Also, the author reviews the definition of a locally nilpotent derivation over a $\mathbb{k}$-algebra and then introduce the Makar-Limanov invariant of a $\mathbb{k}$-algebra.

In Section 3, the author lists some important results about Conjecture 2. Based on those results, we only need to consider affine UFD $A$ over $\mathbb{k}$ with $\mathcal{ML}(A) \neq A$ and $\mathcal{ML}(A) \neq \mathbb{k}$. Then, the author proves that Conjecture 2 holds for Danielewski domains when $\mathbb{k}$ is algebraically closed and is false when $\mathbb{k} = \mathbb{R}$. Also, the author proves that Conjecture 2 is always true for Koras-Russell threefolds. Moreover, the author checks this conjecture on affine UFDs constructed by Finston and Maubach. In this paper, one of such affine UFDs is called a Finston-Maubach domain.

In section 4, the author makes several comments on this paper as well as the research topic.
2. Preliminaries

2.1. Affine unique factorization domains.

Definition 2.1.1. If a nonzero ring (with multiplicative identity) \( R \) has no nonzero zero divisors, then it is a domain. If the domain \( R \) is also commutative, then it is an integral domain.

Example 2.1.2. The ring of integers \( \mathbb{Z} \) is an integral domain while the Hurwitz integer \( \mathbb{H} \) is a domain but not an integral domain.

Definition 2.1.3. Given a \( \mathbb{K} \)-vector space \( A \) and a binary operation \( \cdot : A \times A \to A \), denote the product of \( x \) and \( y \) in \( \mathbb{K} \) by \( xy \) here and the addition in \( A \) by +, if

1. \( (a + b) \cdot c = a \cdot c + b \cdot c; \)
2. \( c \cdot (a + b) = c \cdot a + c \cdot b; \)
3. \( (x) = x(\cdot) \)

holds for all \( a, b, c \in A \) and \( x, y \in \mathbb{K} \), then the pair \( (A, \cdot) \) is an algebra over \( \mathbb{K} \), or simply a \( \mathbb{K} \)-algebra, also denoted by \( A \).

Definition 2.1.4. If an integral domain \( A \) containing \( \mathbb{K} \) is finitely generated as a \( \mathbb{K} \)-algebra, then \( A \) is an affine domain over \( \mathbb{K} \).

Theorem 2.1.5. (1) Given an affine \( \mathbb{K} \)-variety \( V \), its coordinate ring \( \mathbb{K}[V] := \mathbb{A}^{[n]} / \mathbb{I}(V) \) is an affine domain over \( \mathbb{K} \).

(2) If \( \mathbb{K} \) is algebraically closed, and let \( A \) be an affine domain over \( \mathbb{K} \). Then there exist an affine \( \mathbb{K} \)-variety, such that \( A \cong \mathbb{K}[V] \).

Proof. (1) We know that \( I := \mathbb{I}(V) \) is prime since \( V \) is irreducible, therefore \( \mathbb{K}[V] := \mathbb{A}^{[n]} / I \) is an integral domain. Note that \( \mathbb{K}[V] \) is clearly a finitely generated \( \mathbb{K} \)-algebra, it is an affine domain over \( \mathbb{K} \).

(2) Choose generators \( a_1, \ldots, a_n \) of \( A \), the surjective homomorphism \( F : \mathbb{K}^{[n]} \to A \) sending \( f \) to \( f(a_1, \ldots, a_n) \) yields \( A \cong \mathbb{K}^{[n]} / I \) where \( I = \text{Ker}(F) \). Since \( A \) is an integral domain, \( I \) is a prime ideal. Let \( V := \mathbb{Z}(I) \), we know \( I = \mathbb{I}(V) \) from the Hilbert’s Nullstellensatz, so \( A \cong \mathbb{K}[V] \).

Remark 2.1.6. Some very details of this proof can be found in book \[9\]. One can see that if \( \mathbb{K} \) is algebraically closed, then there is a bijection between the set of all affine domains over \( \mathbb{K} \) and the set of all affine varieties over \( \mathbb{K} \).

Definition 2.1.7. If each element \( x \neq 0 \) of an integral domain \( A \) is a product of a unit in \( A \) and prime elements \( p_i \) in \( A \), that is

\[ x = u \cdot p_1 \cdots p_m \]

then \( A \) is a unique factorization domain, abbreviated to a UFD. If a UFD is also an affine domain over \( \mathbb{K} \), then it is a affine unique factorization domain over \( \mathbb{K} \), abbreviated to an affine UFD over \( \mathbb{K} \). In addition, an affine \( \mathbb{K} \)-variety \( X \) is called factorial if its coordinate ring \( \mathbb{K}[X] \) is an affine UFD.

Remark 2.1.8. Since proving an integral domain to be a UFD is hard in general, it is natural to find special methods to tell an affine UFD over \( \mathbb{K} \) from an affine \( \mathbb{K} \)-domain. That topic is also interesting but not what we would mainly discuss in this note. In this case, I will list several important results without proofs. Proofs of those results and relevant definitions can be found in \[16\], \[9\] and \[8\].
Lemma 2.1.9. Given a noetherian domain $A$, following statements are equivalent:

1. $A$ is a UFD;
2. every height one prime ideal of $A$ is principal;
3. $A$ is integrally closed and the Weil divisor class group $\text{Cl}(\text{Spec}(A))$ is trivial.

Remark 2.1.10. Since each affine $k$-domain is noetherian, so Lemma 2.1.9 gives us a way to test affine UFDs. For example, if $V$ is a smooth $k$-variety, then from Proposition 6.15 in [9] we know that $\text{Pic}(k[V]) \cong \text{Cl}(k[V])$. In this case, $k[V]$ is an affine UFD implies that the Picard group $\text{Pic}(k[V])$ should be trivial.

Example 2.1.11. This two examples show some power of this lemma.

1. $A_1 = \mathbb{R}[x, y]/(x^2 + y^2 - 1)$ is not an affine UFD over $\mathbb{R}$;
2. $A_2 = \mathbb{C}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2 - 1)$ is not an affine UFD over $\mathbb{C}$ for each $n > 2$.

Proof. In fact, we could obtain that $\text{Pic}(A_1) = \mathbb{Z}/2\mathbb{Z}$ and $\text{Pic}(A_2) = \mathbb{Z}$. □

Proposition 2.1.12. Given a smooth affine curve $C$ over $k$, then its coordinate ring $k[C]$ is integrally closed.

Example 2.1.13. Note that $x^2 + y^2 = 1$ is a smooth affine curve in $\mathbb{A}^2$, the affine $k$-domain $A = \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ is an affine UFD over $\mathbb{C}$ since $\text{Pic}(A)$ is trivial.

Proposition 2.1.14. Given an affine $k$-domain $A$ and a multiplicatively closed set $S \subseteq A$ generated by a set of prime elements, then the localization $S^{-1}A$ is a UFD implies that $A$ is an affine UFD over $k$.

Example 2.1.15. Given $n > 2$, $\mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2 - 1)$ is an affine UFD over $\mathbb{R}$.

Proof. Denote this affine $\mathbb{R}$-domain by $A$, one can see that since

$$A/(x_n - 1) \cong \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_{n-1}^2 + x_n^2 - 1, x_n - 1)$$
$$\cong \mathbb{R}[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_{n-1}^2, x_n - 1)$$
$$\cong \mathbb{R}[x_1, \ldots, x_{n-1}]/(x_1^2 + \cdots + x_{n-1}^2)$$

is an integral domain, $x_n - 1$ is prime in $A$. Let $t = (x_n - 1)^{-1}$, one has

$$x_1^2 + \cdots + x_{n-1}^2 - 1 = 0 \Rightarrow x_1^2 + \cdots + x_{n-1}^2 + \frac{1}{t^2} + \frac{2}{t} = 0$$

which implies that $t \in \mathbb{R}[tx_1, \ldots, tx_{n-1}]$. Therefore

$$A_{x_n-1} = A[s]/(1 - s(z - 1)) = A[t] = \mathbb{R}[tx_1, \ldots, tx_{n-1}, \frac{1}{t}]$$

is a UFD and then $A$ is an affine UFD by Proposition 2.1.14. □

Remark 2.1.16. What’s more, Proposition 2.1.14 is actually a corollary of a famous theorem of Masayoshi Nagata (see Theorem 6.3 in [16]). One can then prove the following well-known theorem given by Masayoshi Nagata and Abraham A. Klein. I fail to find direct sources and give an adapted one according to Theorem 8.2 in [16].

Theorem 2.1.17. Given $n \geq 5$, if $f(x_1, \ldots, x_n)$ is a non-degenerate quadratic form, then affine $k$-domain $A = k[x_1, \ldots, x_n]/(f)$ is an affine UFD.
2.2. Nilpotent derivations and Makar-Limanov invariants.

**Definition 2.2.1.** Let $A$ be an algebra over $\mathbb{k}$, if a homomorphism of $\mathbb{k}$-algebras $D : A \to A$ satisfies the Leibniz rule

$$D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$, then it is a $\mathbb{k}$-derivation of $A$. $\text{Der}_\mathbb{k}(A)$ denotes the set of all $\mathbb{k}$-derivations of $A$.

**Proposition 2.2.2.** Given an algebra $A$ over $\mathbb{k}$ and a $\mathbb{k}$-derivation of $A$, then

1. $D(a) = 0$ for all $a \in \mathbb{k}$;
2. $\text{Ker}(D) := \{ a \in A : D(a) = 0 \}$ is a subalgebra of $A$;
3. $\text{Nil}(D) := \{ a \in A : \exists n_a \in \mathbb{N}_+, D^n(a) = 0 \}$ is a subalgebra of $A$.

**Proof.**

(1) Let $a = b = 1$, one has

$$D(1) = 1D(1) + D(1)1 = 2D(1)$$

which implies that $D(1) = 0$, then $D(a) = D(a \cdot 1) = aD(1) = 0$ for all $a \in \mathbb{k}$.

(2) From (1), we know that $\mathbb{k} \subseteq \text{Ker}(D)$, so $\text{Ker}(D)$ can succeed the algebra structure of $A$. Then it is sufficient to prove that $\text{Ker}(D)$ is a vector space over $\mathbb{k}$, which is not hard by the definition of vector spaces.

(3) From (1), we know that $\mathbb{k} \subseteq \text{Nil}(D)$, so $\text{Nil}(D)$ can succeed the algebra structure of $A$. Then it is sufficient to prove that $\text{Nil}(D)$ is a vector space over $\mathbb{k}$, which is not hard by the definition of vector spaces. □

**Remark 2.2.3.** In (2), one can also prove $\text{Ker}(D)$ to be a vector space over $\mathbb{k}$ by the isomorphism theorem for vector spaces (modules over a field). One can also see that $\text{Ker}(D) \subseteq \text{Nil}(D)$ as a subalgebra.

**Definition 2.2.4.** Given an algebra $A$ over $\mathbb{k}$ and a $\mathbb{k}$-derivation of $A$, if $\text{Nil}(D) = A$, then $D$ is locally nilpotent or is a locally nilpotent $\mathbb{k}$-derivation. The set of all locally nilpotent $\mathbb{k}$-derivations of $A$ is denoted by $\text{LND}_\mathbb{k}(A)$ or just $\text{LND}(A)$ when with no hazards to rise confusion.

**Definition 2.2.5.** Given a locally nilpotent $\mathbb{k}$-derivation over $\mathbb{k}$-algebra $A$ and a non zero element $a \in A$, we define

$$\deg_D(a) := \max_{n \in \mathbb{N}} \{ D^n(a) \neq 0 \}$$

as the degree of $a$ by $D$. Specifically, we define $\deg_D(0) = -\infty$.

**Example 2.2.6.** For each $\mathbb{k}$-algebra $A$, the zero endomorphism $\theta_A : A \to A$ which sends all elements to 0 is a locally nilpotent $\mathbb{k}$-derivation, called zero derivation.

**Example 2.2.7.** All partial derivations are locally nilpotent $\mathbb{k}$-derivation of $\mathbb{k}^{[n]}$.

**Definition 2.2.8.** Given an algebra $A$ over $\mathbb{k}$, we define set

$$\mathcal{ML}(A) := \bigcap_{D \in \text{LND}_\mathbb{k}(A)} \text{Ker}(D)$$

as Makar-Limanov invariant of $\mathbb{k}$-algebra $A$.

**Example 2.2.9.** The partial derivation $\frac{\partial}{\partial x} : \mathbb{R}[x, y] \to \mathbb{R}[x, y]$ is a locally derivation over the field of real numbers $\mathbb{R}$ and one can see $\deg_{\frac{\partial}{\partial x}}(x^3) = 3$.

**Example 2.2.10.** Given a positive integer $n$, $\mathcal{ML}(\mathbb{k}^{[n]}) = \mathbb{k}$.

**Proof.** First, we know that $\mathbb{k} \subseteq \mathcal{ML}(\mathbb{k}^{[n]})$ from Proposition 2.2.2. Then since all partial derivations are locally nilpotent derivations over $\mathbb{k}$, so $\mathbb{k} \supseteq \mathcal{ML}(\mathbb{k}^{[n]})$. □
Example 2.2.11. Given an affine UFD $A := \mathbb{C}[x, y]/(x^2 + y^2 - 1)$ over the complex field $\mathbb{C}$, one has $\mathcal{ML}(A) = A$.

Proof. It is sufficient to prove that the only element in $\text{LND}_\mathbb{C}(A)$ is zero derivation. Given a nonzero $D \in \text{LND}(A)$, one has

$$0 = D(1) = D(x^2 + y^2) = 2x D(x) + 2y D(y)$$

from which we could get that

$$D(x) = yp(x, y) \quad \text{and} \quad D(y) = -xp(x, y)$$

for a nonzero $p(x, y) \in A$. Then one has

$$\deg_D(x) - 1 = \deg_D(yp(x, y)) = \deg_D(y) + \deg_D(p(x, y))$$

but also

$$\deg_D(y) - 1 = \deg_D(-xp(x, y)) = \deg_D(x) + \deg_D(p(x, y))$$

which implies that $\deg_D(p(x, y)) = -1$, impossible. \hfill \Box

Example 2.2.12. ([12], Example 11) Given affine domain $A := \mathbb{C}[x, y]/(x^3 - y^2)$ over the complex field $\mathbb{C}$ (by the way, $A$ is not a UFD), then $\mathcal{ML}(A) = A$.

Proof. It is sufficient to prove that the only element in $\text{LND}_\mathbb{C}(A)$ is zero derivation. If $D \in \text{LND}(A)$ is not zero, then one has $D(x) \neq c$ for all $c \in \mathbb{C} - \{0\}$; otherwise

$$2yD(y) = D(y^2) = D(x^3) = 3x^2D(x) = 3cx^2$$

which leads to $D(y) = \frac{3x^2}{2y} \notin A$, a contradiction.

Therefore, $D(x)$ is nonconstant in $A$. Since $D(y) = \frac{3x^2}{2y} D(x) \in A$, $x|D(x)$ and then one can set $D(x) = yp(x) + xq(x) \in A$. Let $m := \deg_D(x), n := \deg_D(y)$, one has

$$n = \frac{1}{2} \deg_D(y^2) = \frac{1}{2} \deg_D(x^3) = \frac{3}{2} m$$

so $n > m$. Notice that

$$0 = D^{m+1}(x) = D^m(yp(x) + xq(x)) = D^m(yp(x)) + D^m(xq(x))$$

and

$$D^{m+1}(xq(x)) = \frac{d^{m+1}xq(x)}{dx} D^{m+1}(x) = 0$$

we could obtain $D^{m+1}(yp(x)) = 0$. In this case, one can see

$$n + \deg_D(p(x)) = \deg_D(y) + \deg_D(p(x)) = \deg_D(yp(x)) \leq m$$

This is impossible, hence the only element in $\text{LND}_\mathbb{C}(A)$ is zero derivation. \hfill \Box

Remark 2.2.13. Personally, I do not really agree with Makar-Limanov’s way to compute $\mathcal{ML}(A)$ in this example because he uses the undefined $\deg_D(t)$ where $t$ is a parameter for $A \cong \mathbb{C}[t^2, t^3]$. 
3. On the Makar-Limanov’s Conjecture

3.1. Several existed results. Here we list some results about Conjecture 2. From now on, transcendence degree of an affine \( \mathbb{k} \)-domain \( A \) is denoted by \( \text{tr.deg}_\mathbb{k} A \).

**Theorem 3.1.1.** ([12], Lemma 21) 
Given an affine domain \( A \) over \( \mathbb{k} \). If \( \mathcal{ML}(A) = A \), then \( \mathcal{ML}(A^{[1]}) = A \).

**Remark 3.1.2.** In this case, one can see \( \mathcal{ML}(A) = \mathcal{ML}(A^{[1]}) = A \) for the affine domain \( A \) in Example 2.2.11.

**Theorem 3.1.3.** ([3], Lemma 2.3) 
Given an affine domain \( A \) over \( \mathbb{k} \) with \( \text{tr.deg}_\mathbb{k} A = 1 \). Then
(1) \( A \cong \mathbb{k}^{[1]} \) if and only if \( \mathcal{ML}(A) = \mathbb{k} \);
(2) \( A \neq \mathbb{k}^{[1]} \) if and only if \( \mathcal{ML}(A) = A \).

**Remark 3.1.4.** Here we could get that \( \mathcal{ML}(A) = \mathcal{ML}(A^{[1]}) \) holds for each affine domain \( A \) over \( \mathbb{k} \) with \( \text{tr.deg}_\mathbb{k} A = 1 \).

**Theorem 3.1.5.** ([7], Theorem 9.12) 
Given an affine unique factorization domain \( A \) over an algebraically closed field \( \mathbb{k} \) with \( \text{tr.deg}_\mathbb{k} A = 2 \). Then \( \mathcal{ML}(A) = \mathbb{k} \) is equivalent to \( A = \mathbb{k}^{[2]} \).

**Remark 3.1.6.** Example 3.1.7 shows that \( \text{tr.deg}_\mathbb{k} A = 2 \) is necessary for this theorem.

**Example 3.1.7.** Consider the affine \( \mathbb{C} \)-domain \( A = \mathbb{C}[x, y, z, w]/(xz - yw - 1) \), one has \( A \) is an affine UFD over \( \mathbb{C} \) and \( \mathcal{ML}(A) = \mathbb{C} \).

**Proof.** (1) First, since 
\[
A/(w) \cong \mathbb{C}[x, y, z]/(xy - 1) = (\mathbb{C}[x, y]/(xy - 1))[z] \cong (\mathbb{C}[x][1/z])[z] \cong (\mathbb{C}[x])_x[z]
\]
is an integral domain (actually an affine UFD over \( \mathbb{C} \)), \( w \) is prime in \( A \).

Let \( x' = \frac{x}{w}, y' = y, z' = z \), then \( x', y', z' \) are algebraically independent and 
\[
A_w \cong A[1/w] = \mathbb{C}[x', y', z'][1/w] = \mathbb{C}[x', y', z']
\]
is a UFD over \( \mathbb{C} \). Therefore, by Proposition 2.1.14, \( A \) is an affine UFD over \( \mathbb{C} \).

(2) One can check that the derivations \( D_1, D_2, D_3 \) and \( D_4 \) on \( A \) generated by 
\[
D_1(x) = 0, \quad D_1(y) = z, \quad D_1(z) = 0, \quad D_1(w) = x
\]
\[
D_2(x) = 0, \quad D_2(y) = w, \quad D_2(z) = x, \quad D_2(w) = 0
\]
\[
D_3(x) = z, \quad D_3(y) = 0, \quad D_3(z) = 0, \quad D_3(w) = y
\]
\[
D_4(x) = w, \quad D_4(y) = 0, \quad D_4(z) = y, \quad D_4(w) = 0
\]
are locally nilpotent. Here one can see \( \text{Ker}(D_1) \) and \( \mathbb{C}[x, z] \) are both algebraically closed in \( A \). What’s more, one has \( \mathbb{C}[x, z] \subseteq \text{Ker}(D_1) \subseteq A \) and 
\[
\text{tr.deg}_\mathbb{C} \text{Ker}(D_1) = \text{tr.deg}_\mathbb{C} \mathbb{C}[x, z] = 2
\]
Hence \( \text{Ker}(D_1) = \mathbb{C}[x, z] \). Similarly, we could obtain \( \text{Ker}(D_2) = \mathbb{C}[x, w], \text{Ker}(D_3) = \mathbb{C}[y, z] \) and \( \text{Ker}(D_4) = \mathbb{C}[y, w] \). In this case, 
\[
\mathbb{C} = \bigcap_{i=1}^{4} \text{Ker}(D_i) \supseteq \mathcal{ML}(A) \supseteq \mathbb{C}
\]
and thus \( \mathcal{ML}(A) = \mathbb{C} \). Moreover, since \( \mathbb{C} \subseteq \mathcal{ML}(A^{[1]}) \subseteq \mathcal{ML}(A) = \mathbb{C} \), it is clear that \( \mathcal{ML}(A^{[1]}) = \mathcal{ML}(A) = \mathbb{C} \). \( \square \)
3.2. On Danielewski domains.

**Definition 3.2.1.** Given integer \( n \geq 0 \) and a polynomial \( p \in S \) where
\[
S := \bigcup_{d \geq 1} \{ p \in \mathbb{k}[x, y] \mid d = \deg_x p = \deg_y p(0, y) \}
\]
we define the subring \( \mathcal{D}_{(n, p)} \) of \( \mathbb{k}[x, x^{-1}, y] \) by
\[
\mathcal{D}_{(n, p)} := \mathbb{k}[x, y, z]/(x^n z - p(x, y))
\]
to be a **Danielewski domain** over \( \mathbb{k} \).

**Remark 3.2.2.** Here we should notice that we could assume \( \deg_x p(x, y) < n \). If a monomial \( x^s y^t \) appears in \( p \) with \( s \geq n \), then one has \( x^n (z - x^{s-n} y^t) = p'(x, y) \) for \( p'(x, y) = p(x, y) - x^s y^t \). Therefore, one has \( \mathcal{D}_{(n, p)} \cong \mathcal{D}_{(n, p')} \). Repeating this, we could eventually obtain a \( p_0(x, y) \in S \) with \( \deg_x p_0(x, y) < n \) and \( \mathcal{D}_{(n, p)} \cong \mathcal{D}_{(n, p_0)} \).

Danielewski domains over \( \mathbb{k} \) are clearly affine \( \mathbb{k} \)-domains and we want to know when one such domain would become an affine UFD over \( \mathbb{k} \). Firstly, we introduce the following simple and well-known result.

**Proposition 3.2.3.** The Danielewski domain
\[
\mathcal{D}_{(1, p)} = \mathbb{k}[x, y, z]/(xz - p(y))
\]
is an affine UFD if and only if \( p \in \mathbb{k}[y] - \mathbb{k} \) is irreducible.

**Proof.** If nonconstant polynomial \( p(y) \in \mathbb{k}[y] \) is irreducible, then the ideal \( (p(y)) \) is prime in both \( \mathbb{k}[y] \) and \( \mathbb{k}[y, z] \). Therefore, the quotient ring
\[
\mathcal{D}_{(1, p)}/(x) = \mathbb{k}[y, z]/(p(y))
\]
is an integral domain which implies that \( x \) is prime in \( \mathcal{D}_{(1, p)} \). Note that
\[
(\mathcal{D}_{(1, p)})_x \cong \mathcal{D}_{(1, p)}[\frac{1}{x}] = \mathbb{k}[x, y, \frac{1}{x}]
\]
is a UFD over \( \mathbb{k} \), \( \mathcal{D}_{(1, p)} \) is an affine UFD over \( \mathbb{k} \) according to Proposition 2.1.14.

On the other hand, a reducible \( p \in \mathbb{k}[y] - \mathbb{k} \) can be decomposed as a product of nonconstant polynomials in \( \mathbb{k} \). In this case, \( \mathcal{D}_{(1, p)} \) cannot be UFD or each irreducible element in \( \mathcal{D}_{(1, p)} \) is prime, which implies a contradiction form \( xz = p(y) \).

**Example 3.2.4.** Consider affine \( \mathbb{k} \)-domain \( A = \mathbb{k}[x, y, z]/(xz - y^2 - 1) \), one can see from Proposition 3.2.2 that \( A \) is an affine UFD when \( \mathbb{k} = \mathbb{R} \) and is not when \( \mathbb{k} = \mathbb{C} \).

**Proposition 3.2.5.** If Danielewski domain \( \mathcal{D}_{(1, p)} \) is a UFD, then \( \mathcal{ML}(\mathcal{D}_{(1, p)}) = \mathbb{k} \).

**Proof.** One can check that the derivations \( D_1 \) and \( D_2 \) on \( \mathcal{D}_{(1, p)} \) generated by
\[
\begin{align*}
D_1(x) &= 0, \quad D_1(y) = x, \quad D_1(z) = p'(y) \\
D_2(x) &= p'(y), \quad D_2(y) = z, \quad D_2(z) = 0
\end{align*}
\]
are locally nilpotent. Here one can see \( \text{Ker}(D_1) \) and \( \mathbb{k}[x] \) are both algebraically closed in \( \mathcal{D}_{(1, p)} \). What’s more, one has \( \mathbb{k}[x] \subseteq \text{Ker}(D_1) \subset \mathcal{D}_{(1, p)} \) and
\[
\text{tr.deg}_\mathbb{k} \text{Ker}(D_1) = \text{tr.deg}_\mathbb{k} \mathbb{k}[x] = 1
\]
Hence \( \text{Ker}(D_1) = \mathbb{k}[x] \). Also, we could obtain that \( \text{Ker}(D_2) = \mathbb{k}[z] \). In this case,
\[
\mathbb{k} = \text{Ker}(D_1) \cap \text{Ker}(D_2) \supseteq \mathcal{ML}(\mathcal{D}_{(1, p)}) \supseteq \mathbb{k}
\]
and thus \( \mathcal{ML}(\mathcal{D}_{(1, p)}) = \mathbb{k} \), which implies that \( \mathcal{ML}(\mathcal{D}_{(1, p)})^{[1]} = \mathbb{k} \). \( \square \)
This proposition implies that Makar-Limanov’s conjecture holds for $\mathbb{D}_{(1,p)}$ and we want to look into other cases. Furthermore, Alhajjar proved the following theorem in his Ph.D. thesis [1] based on some works of Makar-Limanov and Freudenburg listed this result in his book [7] (Theorem 9.2).

**Theorem 3.2.6.** ([1], Proposition 6.16) If $n, d \geq 2$ and $\deg_y p(x, y) = d$ for $p \in \mathcal{S}$, then $\mathcal{ML}(\mathbb{D}_{(n,p)}) = \mathbb{k}[x]$.

Note that if $\mathbb{D}_{(n,p)}$ is an affine UFD with $n, d \geq 2$ and $\deg_y p(x, y) = d$, then $x$ should be prime and the quotient ring

$$\mathbb{D}_{(n,p)}/(x) = \mathbb{k}[y, z]/(p_y(0, y))$$

is an integral domain. Therefore, $p(0, y) \in \mathbb{k}[y]$ is irreducible. However, it is impossible when $\mathbb{k}$ is algebraically closed because $\deg_y p(0, y) = \deg_y p(x, y) = d > 1$. So here we only need to consider potential counterexamples to Makar-Limanov’s conjecture with a non-algebraically closed field.

**Example 3.2.7.** Consider the Danielewski domain $A = \mathbb{R}[x, y, z]/(x^2 z - (y^2 + 1))$, we could prove the following statements.

1. $A$ is an affine UFD;
2. $\mathcal{ML}(A^{[1]}) = \mathbb{R}$.

**Proof.** (1) First, since $A/(x) = \mathbb{R}[y, z]/(y^2 + 1)$ is an integral domain, $x$ is prime in affine $\mathbb{R}$-domain $A$. Then from $z = \frac{y^2 + 1}{x^2}$, one can see

$$A_x \cong A[\frac{1}{x}] = \mathbb{R}[x, y, \frac{1}{x}]$$

is a UFD. Therefore, $A$ is an affine UFD by Proposition 2.1.14.

(2) Consider the Danielewski domain $B = \mathbb{R}[x, y, z]/(xz - (y^2 + 1))$, we know that it is an affine UFD by Example 3.2.4 and then

$$\mathcal{ML}(B^{[1]}) = \mathcal{ML}(B) = \mathbb{R}$$

from Proposition 3.2.5. Moreover, according to Theorem 10.1 in [7], one has

$$\mathcal{ML}(A^{[1]}) = \mathcal{ML}(B^{[1]}) = \mathbb{R}$$

so we are done. □

**Remark 3.2.8.** This example implies that when $\mathbb{k}$ is not algebraically closed, it is possible that $\mathcal{ML}(A^{[1]}) \neq \mathcal{ML}(A)$ for certain affine UFD over $\mathbb{k}$. Therefore, we will only consider the conjecture on algebraically closed fields in the following part.

Now we are moving to Danielewski domains $\mathbb{D}_{(n,p)}$ with $\deg_y p(x, y) = 1$ and $n \geq 2$. In this case, the polynomial $p(x, y) = a(x)y + b(x)$ where $a, b \in \mathbb{k}[x]$ and $a(0) \neq 0$. Note that $\gcd(x^n, a(x)) = 1$, then one can see

$$\mathbb{k}[x, y, z]/(x^n z - a(x)y - b(x)) \cong \mathbb{k}[x, y, z]/(z) \cong \mathbb{k}[x, y]$$

from $\mathbb{k}[x, y, z] \cong \mathbb{k}[x, y, x^n z - a(x)y - b(x)]$. Therefore, the Makar-Limanov invariant of one such Danielewski domain is $\mathbb{k}$.

Based on the previous discussions in this part, we clearly have the following result as a conclusion.

**Theorem 3.2.9.** If $\mathbb{k}$ is algebraically closed, then $\mathcal{ML}(\mathbb{D}_{(n,p)}) = \mathcal{ML}(\mathbb{D}_{(n,p)}^{[1]})$ for each Danielewski domain $\mathbb{D}_{(n,p)}$ over $\mathbb{k}$.
3.3. On Koras-Russell threefolds.

Definition 3.3.1. A Koras-Russell threefold of the first kind is defined by

\[ R = \mathbb{k}[x, y, z, w]/(x + x^d y + z^u + w^v) \]

where \( d, u, v \geq 2 \) and \( \gcd(u, v) = 1 \).

In this part, we are going to investigate the conjecture on Koras-Russell threefolds which are clearly affine domains. First of all, we want to know when a Koras-Russell threefold \( R \) is an affine UFD. Note that

\[ x + x^d y + z^u + w^v = 0 \Rightarrow y = \frac{-1}{x^{d-1}} - \frac{z^u}{x^d} - \frac{w^v}{x^d} \in \mathbb{k}[x, z, w][\frac{1}{x}] \]

then one can see

\[ R_x \cong R[\frac{1}{x}] = \mathbb{k}[x, z, w][\frac{1}{x}] \]

Hence \( R_x \) is a localization of a polynomial ring which is a UFD. It is not hard to check that the element \( x \) is prime in \( R \). Therefore, we are able to see the following result according to Proposition 2.1.14.

Proposition 3.3.2. Each Koras-Russell threefold \( R \) is an affine UFD.

Then we wonder if \( \mathcal{ML}(R) = \mathcal{ML}(R[1]) \) also always holds. In order to figure out this problem, we need to introduce a fundamental result at first.

As a concise version of Makar-Limanov’s original proof in [13], Daigle, Freudenburg and Moser-Jauslin proved a feature of Koras-Russell threefolds in [5]. Based on their result, Freudenburg computed Makar-Limanov invariants of those threefolds in new edition of his book [7].

Theorem 3.3.3. ([7], Theorem 9.9) Let \( R \) be a Koras-Russell threefold

\[ R = \mathbb{k}[x, y, z, w]/(x + x^d y + z^u + w^v) \]

where \( d, u, v \geq 2 \) and \( \gcd(u, v) = 1 \). Then \( \mathcal{ML}(R) = \mathbb{k}[x] \).

In their arguments and computations, some propositions and lemmas are important. We are going to introduce some less-known definitions relevant to those propositions and lemmas at first and then give the propositions and lemmas.

Definition 3.3.4. Let \( G \) be an abelian group, and let \( B \) be a ring. A \( G \)-grading of \( B \) is a family \( \{B_a\}_{a \in G} \) of subgroups of \((B, +)\) such that

\[ B = \bigoplus_{a \in G} B_a \text{ and } B_a B_b \subseteq B_{a+b} \text{ for all } a, b \in G \]

and a ring (or domain/integral domain/affine \( \mathbb{k} \)-domain) \( B \) with a \( G \)-grading is called a \( G \)-grading ring (or domain/integral domain/affine \( \mathbb{k} \)-domain).

Definition 3.3.5. Given \( B = \bigoplus_{a \in G} B_a \) a \( G \)-grading ring, a nonzero \( f \in B \) is called \( G \)-homogeneous if \( f \in B_a \) for a unique \( a \in G \). Here we say that \( f \) is of degree \( a \) and write \( \deg_G f = a \).

Definition 3.3.6. Given \( B = \bigoplus_{a \in G} B_a \) a \( G \)-grading ring and a nonzero \( f \in B \). \( \overline{f} \) denotes the highest-degree homogeneous summand of \( f \). In case \( f = 0 \), we define \( \overline{0} = 0 \). Here one can see \( \overline{x} = x \) if \( x \) is \( G \)-homogeneous.

Definition 3.3.7. Given \( B = \bigoplus_{a \in G} B_a \) a \( G \)-grading \( \mathbb{k} \)-domain and a \( \mathbb{k} \)-algebra \( R \subseteq B \), we define the associated \( G \)-graded domain \( \overline{R} \) to be the \( \mathbb{k} \)-subalgebra of \( B \) generated by the set \( \{ \overline{r} \mid r \in R, r \neq 0 \} \).
Definition 3.3.8. Given \( B = \oplus_{a \in G} B_a \) a \( G \)-grading \( k \)-domain and \( D \in \text{Der}_k B \), if
\[
\deg_G D := \max \{ \deg_G (Df) - \deg_G (f) : f \neq 0, f \in B \}
\]
exists, then we define \( \overline{D} : \overline{B} \to \overline{B} \) as
\[
\overline{D}_{x} = \begin{cases} 
\overline{D}_{x}, & \deg_G (Df) - \deg_G (f) = \deg_G D \\
0, & \deg_G (Df) - \deg_G (f) < \deg_G D
\end{cases}
\]
Moreover, one can see \( \deg_G (\overline{D}) = \deg_G D \) and \( \ker (D) \subseteq \ker (\overline{D}) \).

Lemma 3.3.9. \((5), \text{Lemma 3.7}\) Let \((G, \leq, +, 0)\) be a totally ordered abelian group and \( B = \oplus_{a \in G} B_a \) a \( G \)-graded integral domain. Let \( A = \oplus_{a \leq 0} B_a, x \in B, \) and \( R = A[x] \). Then \( \overline{R} = A[\overline{x}] \).

Proposition 3.3.10. \((5), \text{Theorem 3.8}\) Let \((G, \leq, +, 0)\) be a totally ordered abelian group, \( B \) a \( G \)-graded \( k \)-affine domain, and \( R \subseteq B \) a \( k \)-subalgebra such that \( B \) is a localization of \( R \). Let \( \deg_G : R \to G \cup \{-\infty\} \) be the restriction of the degree function on \( B \) determined by the grading. Then \( \deg (D) \) is defined for every \( D \in \text{Der}_{k}(R) \).

Proposition 3.3.11. \((5), \text{Corollary 6.3}\) Let \((G, \leq, +, 0)\) be a totally ordered abelian group, \( B = \oplus_{a \in G} B_a \) a \( G \)-graded integral domain containing \( \mathbb{Z} \), where \( B \) is finitely generated as a \( A \)-algebra. Then one can write \( B = A[x_1, \ldots, x_n] \) where \( x_i \neq 0 \) is homogeneous of degree \( d_i \neq 0 \) for each \( i \). Let \( H_i = \langle d_1, \ldots, d_i, \ldots, d_n \rangle \) for each \( i \in [n] \). Then for every \( G \)-homogeneous \( D \in \text{LND}_{\mathbb{Z}}(B) \) the following conditions hold.
(1) For each \( i \in [n] \) such that \( H_i \neq G(B), D^2 x_i = 0 \).
(2) For every choice of distinct \( i, j \in [n] \) such that \( H_i \neq G(B) \) and \( H_j \neq G(B) \), one has \( D x_i = 0 \) or \( D x_j = 0 \).

In line with their works, we are able to make the following computation.

Theorem 3.3.12. Let \( R = \mathbb{K}[x, y, z, w] / (x + x^d y + z^u + w^v) \) where \( d, u, v \geq 2 \) and \( \gcd (u, v) = 1 \). Then \( \mathcal{ML}(R[t]) = \mathbb{K}[x] \).

Proof. Let group \( G = \mathbb{Z}^2 \) and define a total order \( \preceq \) on \( G \) by lexicographical ordering. Consider a \( G \)-grading on \( B = \mathbb{K}[x, x^{-1}, z, w, t] \) with \( x, z, w, t \) homogeneous and
\[
\deg_G (x, z, w, t) = ((-1, 0, 0), (0, -v, 0), (0, -u, 0), (0, 0, -1))
\]
Note that \( \gcd (u, v) = 1 \), one has
\[
\{ f \in B | \deg (f) \leq (0, 0, 0) \} = \mathbb{K}[x, z, w, t] \subseteq R[t]
\]
We set \( A = \{ f \in B | \deg (f) \leq (0, 0, 0) \} = \mathbb{K}[x, z, w, t] \). The degree function \( \deg_G \) on \( B \) restricts to affine \( k \)-domain \( R[t] \), where \( \deg_G y = (d, -uv, 0) \). According to Lemma 3.3.9, one has \( \overline{R}[t] = A[\overline{y}] = \mathbb{K}[x, z, w, t, \overline{y}] \). Since \( y = -x^{-d}(x + z^u + w^v) \) in \( B \), one can see that \( \overline{y} = -x^{-d}(z^u + w^v) \) and then \( x^d \overline{y} + z^u + w^v = 0 \).

Given a nonzero \( D \in \text{LND}_{\mathbb{Z}}(R[t]) \). By Proposition 3.3.10, the induced \( G \)-homogeneous derivation \( \overline{D} \) of \( R[t] \) is nonzero and locally nilpotent. Since
\[
\langle \deg_G \overline{y}, \deg_G z, \deg_G w, \deg_G t \rangle = d\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}
\]
\[
\langle \deg_G x, \deg_G \overline{y}, \deg_G w, \deg_G t \rangle = \mathbb{Z} \times u\mathbb{Z} \times \mathbb{Z}
\]
\[
\langle \deg_G x, \deg_G \overline{y}, \deg_G z, \deg_G t \rangle = \mathbb{Z} \times v\mathbb{Z} \times \mathbb{Z}
\]
are proper subgroups of $G(R[t]) = \mathbb{Z}^3$, we know that at least two of $Dx, Dz, Dw$ must be zero according to Proposition 3.3.11.

When $Dz = Dw = 0$, one has $D(x^d\overline{y}) = 0$ and then $Dx = D\overline{y} = 0$. In this case, the only possible is $Dt \neq 0$.

When $Dx = Dz = 0$ or $Dx = Dz = 0$, one can see that either $D\overline{y} \neq 0$ or $Dx = Dz = Dw = 0, Dt \neq 0$.

Therefore, either $\text{Ker} D \subset A$ or $\text{Ker} D = \mathbb{k}[x, y, z, w]$.

Now suppose that $Dx \neq 0$. Choose $f, g \in \text{Ker} D$ which are algebraically independent.

Let $f_1, g_1 \in \mathbb{k}[x, y, z, w, t]$ and $f_2, g_2 \in \mathbb{k}[y, z, w, t]$ be such that

$$f = xf_1 + f_2 \quad \text{and} \quad g = xg_1 + g_2$$

Here $f_2$ and $g_2$ should be algebraically independent in $R$. Otherwise, there exists $P \in \mathbb{k}^2$ with $P(f_2, g_2) = 0$. But then $P(f, g) \in xR[t]$, which implies that $Dx = 0$, a contradiction. In addition, since $\deg_G(xf_1) \prec \deg_G f_2$, one has $\deg_G f = \deg_G f_2$.

Similarity, $\deg_G g = \deg_G g_2$. Now one has $f, g \in \text{Ker} D$ where

$$\overline{f} = f_2 \quad \text{and} \quad \overline{g} = g_2$$

If $\text{Ker} D \subset A$, then $D\overline{y} \neq 0$ and $f_2, g_2 \in \mathbb{k}[z, w, t]$. This case, $\mathbb{k}[z, w, t]$ is the algebraic closure of $\mathbb{k}[f, \overline{y}]$ and thus $\mathbb{k}[z, w, t] \subset \text{Ker} D$. However, we could obtain $0 = D(x^d\overline{y})$ and then $D = 0$, a contradiction.

If $\text{Ker} D = \mathbb{k}[x, \overline{y}, z, w]$, then $D$ restricts to be a zero derivation in $R$. So $D$ restricts to a zero derivation in $R$, a contradiction.

Therefore, we always get a contradiction with $Dx \neq 0$, which implies that $Dx = 0$. So one has $\mathbb{k}[x] \subseteq \mathcal{ML}(R[t])$. From Theorem 3.3.3, we know that

$$\mathcal{ML}(R[t]) \subseteq \mathcal{ML}(R) = \mathbb{k}[x]$$

and thus $\mathcal{ML}(R[t]) = \mathbb{k}[x]$. \hfill \qed

Here one has proved that the Conjecture 2 holds for all Koras-Russell threefolds. However, methods applied in this proof rely on specific structures of Koras-Russell threefolds and are hard to be generalized.

### 3.4. On Finston-Maubach domains.

In paper [6], Finston and Maubach construct a series of affine UFDs (called Finston-Maubach domains in this paper) with non-trivial Makar-Limanov invariant based on Brieskorn-Catalan-Fermat rings. We are going to check Conjecture 2 on Finston-Maubach domains in the following part.

**Definition 3.4.1.** Given $n \geq 3$, we define $F \in \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$ as

$$F := x_1^{d_1} + \cdots + x_n^{d_n} + (x_2y_1 - x_1y_2)^{e_2} + \cdots + (x_ny_1 - x_1y_n)^{e_n}$$

where

$$\frac{1}{d_1} + \cdots + \frac{1}{d_n} + \frac{1}{e_2} + \cdots + \frac{1}{e_n} \leq \frac{1}{2n-3}$$

Then the affine domain $R$ given by

$$R := \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(F)$$

is called a **Finston-Maubach domain** of order $n$.

**Proposition 3.4.2.** Each Finston-Maubach domain $R$ is an affine UFD.
Proof. It is clear that $R$ is an affine domain. Moreover, Lemma 3.5 in paper [6] tells us that $R$ is a UFD. One can also try to prove it independently by Proposition 2.1.14 and the factoriality of Brieskorn-Catalan-Fermat rings with $n \geq 5$. \hfill \square

Then we are going to compute the Makar-Limanov invariant of Finston-Maubach domains based on Finston and Maubach’s results. For that, one has to introduce a lemma at first.

**Lemma 3.4.3.** ([6], Lemma 2.7) Let $A$ be an affine $\mathbb{Q}$-domain. Consider a subset $\mathcal{F} = \{f_1, f_2, \ldots, f_n\}$ of $A$ and positive integers $d_1, \ldots, d_n$ satisfying:
- $f := f_1^{d_1} + f_2^{d_2} + \cdots + f_n^{d_n}$ is a prime element of $A$.
- Non nontrivial sub-sum of $f_1^{d_1}, f_2^{d_2}, \ldots, f_n^{d_n}$ lies in $(f)$.
Additionally, assume that
\[
\frac{1}{d_1} + \frac{1}{d_2} + \cdots + \frac{1}{d_n} \leq \frac{1}{n-2}
\]
Set $R := A/(f)$ and let $D \in \text{LND}(R)$, one has $D(f_i) = 0$ for all $1 \leq i \leq n$.

**Proposition 3.4.4.** $\mathcal{ML}(R) = \mathbb{C}^{|n|}$ for a Finston-Maubach domain $R$ of order $n$.

Proof. Firstly, it is not hard to check that $D_0 \in \text{LND}(R)$ where $D_0$ is generated by $D_0(x_i) = 0$ and $D_0(y_i) = x_i$. Given $D \in \text{LND}(R)$, by Lemma 3.4.3, one has $D(x_i) = 0$ and $D(l_i) = 0$ where $l_i = x_i y_1 - x_1 y_i$. Therefore, one has
\[
x_1 D(y_i) = y_1 D(x_i), \quad \forall i \in \{2, 3, \ldots, n\}
\]
Since $R$ is a UFD, $D(y_i) = \alpha x_i$ for some $\alpha \in R$. Then one can see $D = \alpha D_0$. Notice that $D$ is nonzero if and only if $\alpha$ is nonzero, one can see $\text{Ker}(D) = \mathbb{C}[x_1, \ldots, x_n]$ for each nonzero $D \in \text{LND}(R)$. Hence one has $\mathcal{ML}(R) = \mathbb{C}[x_1, \ldots, x_n] = \mathbb{C}^{|n|}$. \hfill \square

Now we are going to compute $\mathcal{ML}(R[t])$ for a Finston-Maubach domain $R$ of order $n$ and we also need to introduce several lemmas in [2] and [6].

**Lemma 3.4.5.** ([2], Theorem 3.1) Let $f_1, f_2, \ldots, f_n \in K[s]$, where $K$ is an algebraically closed field containing $\mathbb{Q}$. Assume $f_1^{d_1} + f_2^{d_2} + \cdots + f_n^{d_n} = 0$. Additionally, assume that for every $1 \leq i_1 < i_2 < \cdots < i_s \leq n$,
\[
f_1^{d_{i_1}} + f_{i_2}^{d_{i_2}} + \cdots + f_{i_s}^{d_{i_s}} = 0 \Rightarrow \gcd\{f_{i_1}, f_{i_2}, \cdots, f_{i_s}\} = 1
\]
Then
\[
\sum_{i=1}^{n} \frac{1}{d_i} \leq \frac{1}{n-2}
\]
implies that all $f_i$ are constant.

**Lemma 3.4.6.** ([6], Lemma 2.2) Let $D$ be a locally nilpotent derivation on a domain $A$ containing $\mathbb{Q}$. Then $A$ embeds into $K[s]$, where $K$ is some algebraically closed field of characteristic zero, in such way $D = \partial_s$ on $K[s]$.

**Proposition 3.4.7.** $\mathcal{ML}(R[t]) = \mathbb{C}^{|n|}$ for a Finston-Maubach domain $R$ of order $n$.

Proof. Let Finston-Maubach domain $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]/(F)$ where
\[
F := x_1^{d_1} + \cdots + x_n^{d_n} + (x_2 y_1 - x_1 y_2)^2 + \cdots + (x_n y_1 - x_1 y_n)^2
\]
Suppose $D \in \text{LND}(R[t])$, where $D \neq 0$. By Lemma 3.4.6 with $K$ an algebraic closure of the quotient field of $(R[t])^D$, we realize $D$ as partial derivation $\partial_s$ on $K[s] \supseteq R[t]$. 

Notice that $F(s) = 0$ and there cannot be a subsum of $x_1^{d_1} + \cdots + x_n^{d_n} + l_2^{e_2} + \cdots + l_n^{e_n}$ to be zero, where $l_i = x_i y_1 - x_1 y_i$. Then by Lemma 3.4.5, one has $x_i, l_i$ to be constant in $K[s]$. In this case, one has $D(x_i) = \partial_s x_i = 0$ and $D(y_i) = \partial_s l_i = 0$ in $R[t]$. Therefore, one has $\mathcal{ML}(R[t]) \supseteq \mathbb{C}[x_1, \ldots, x_n]$. In other hands, we already know that $\mathcal{ML}(R[t]) \subseteq \mathcal{ML}(R) = \mathbb{C}[x_1, \ldots, x_n]$. So $\mathcal{ML}(R^{[1]}) = \mathcal{ML}(R) = \mathbb{C}[n]$. \qed

Here we know that Conjecture 2 is also valid as to Finston-Maubach domains. The proof in Proposition 3.4.7 is nothing new but to imitate the proof of Finston and Maubach for the almost rigidity of Finston-Maubach domains.

4. Comments

This work was originally started as an extension of my Bachelor thesis. At that time, I planned to study Conjecture 2 by observing examples. However, I found that most of existing methods of computing Makar-Limanov invariants rely on specific structure of an affine domain. So it may be hard to obtain a promising way to solve the conjecture by them. Additionally, my first semester for a master degree will begin soon and I will hardly have time for this work. In this case, I decide to stop it here.

In my opinion, the core of this conjecture is to build a connection between the UFD property and Makar-Limanov invariant. It is difficult because the UFD itself is a really profound area, which is too fundamental to be given a description by locally nilpotent derivations. If we want to use current ways to compute Makar-Limanov invariants, we need to focus on the structure of $(f)$ in an affine UFD $k[[n]]/(f)$. Or one can try to interpret Makar-Limanov invariants from the perspective of geometry, which may bring us some new ideas.

These three examples in this paper are computed by three different methods. The first one is simple and direct. The second one uses a famous technique in this area, called homo-generalization. Readers can find more information about this technique in [7], [11] and [12]. The third example has some connection with the famous ABC-theorem and one can find more details in [2] and [7].
References

[1] Bachar Alhajjar. *Locally Nilpotent Derivations of Integral Domains*. Ph.D. thesis, Université de Bourgogne, 2017.

[2] Michiel de Bondt. Another generalization of Mason’s ABC-theorem. Preprint, arXiv:0707.0434, 2009.

[3] Anthony Crachiola and Leonid Makar-Limanov. On the rigidity of small domains. *Journal of Algebra*, 284(1):1-12, 2005.

[4] Anthony Crachiola and Leonid Makar-Limanov. An algebraic proof of a cancellation theorem for surfaces. *Journal of Algebra*, 320(8):3113-3119, 2008.

[5] Daniel Daigle, Gene Freudenburg and Lucy Moser-Jauslin. Locally nilpotent derivations of rings graded by an abelian group. *Advanced Studies in Pure Mathematics*, 75:29-48, 2017.

[6] David Finston and Stefan Maubach. Constructing (almost) rigid rings and a UFD having infinitely generated Derksen and Makar-Limanov invariant. *Canadian Mathematics Bulletin*, 53(1):77-86, 2010.

[7] Gene Freudenburg. *Algebraic Theory of Locally Nilpotent Derivations (Second Edition)*. Springer, New York, The United States, 2017.

[8] Robin Hartshorne. *Algebraic Geometry*. Springer, New York, The United States, 1977.

[9] Gregor Kemper. *A Course in Commutative Algebra*. Springer, Berlin, Germany, 2009.

[10] Shulim Kaliman and Leonid Makar-Limanov. On the Russell-Koras contractible threefolds. *Journal of Algebraic Geometry*, 6(2):247-268, 1997.

[11] Shulim Kaliman and Leonid Makar-Limanov. AK-invariant of affine domains. In *Affine algebraic geometry*, pages 231-255. Osaka University Press, Osaka, Japan, 2007.

[12] Leonid Makar-Limanov. Locally nilpotent derivations, a new ring invariant and applications. Lecture notes, ResearchGate:265356937, 1998.

[13] Leonid Makar-Limanov. On the hypersurface $x + x^2 y + z^2 + t^3$ in $\mathbb{C}^4$ or a $\mathbb{C}^3$-like threefold which is not $\mathbb{C}^3$. *Israel Journal of Mathematics*, 96:419-429, 1996.

[14] Leonid Makar-Limanov. Some conjectures, examples and counterexamples. *Annales Polonici Mathematici*, 76(1-2):139-145, 2001.

[15] Leonid Makar-Limanov. On the group of automorphisms of a surface $x^ny = p(z)$. *Israel Journal of Mathematics*, 121:113-123, 2001.

[16] Pierre Samuel. *On unique factorization domains*. Lecture notes, AlgebraNote 30, 1998.

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