A Topological Obstruction to Almost Global Synchronization on Riemannian Manifolds

Johan Markdahl

Abstract

Multi-agent systems on nonlinear spaces sometimes fail to reach consensus. This is usually attributed to the initial configuration of the agents being too spread out, the graph topology having certain undesired symmetries, or both. Besides nonlinearity, the role played by the geometry and topology of the manifold is often overlooked. This paper concerns gradient descent flows of quadratic disagreement functions on general Riemannian manifolds. By requiring that the flow converges to the consensus manifold $C$ for almost all initial conditions on any connected graph, we turn manifolds into the objects of study. We establish necessary conditions for almost global synchronization. If a Riemannian manifold contains a closed curve of locally minimum length, then there is a connected graph and a dense set of initial conditions from which the system fails to reach consensus. In particular, this holds if the manifold is closed but not simply connected. There is a class of extrinsic consensus protocols on the special orthogonal group $SO(n)$ that appears in the Kuramoto model over complex networks, rigid-body attitude synchronization, and the Lohe model of quantum synchronization. Unlike the corresponding system on the $n$-sphere for $n \in \mathbb{N}\{1\}$, the system on $SO(n)$ fails to converge to $C$ for a dense subset of initial values. This is because the $n$-sphere is simply connected for $n \in \mathbb{N}\{1\}$, whereas $SO(n)$ is not.

Key words: Consensus, Riemannian manifold, Simply connected, Multi-agent systems, Networked control system, Decentralization.

1 Introduction

Networked and multi-agent systems on nonlinear spaces sometimes suffer failures. Examples include power outages, diseases such as cancer, or an animal being separated from its herd. One common scenario is that an event has perturbed the system from its nominal operating condition to a configuration where the default control algorithm fails to stabilize it [Mottet, 2013]. While fail-safe control design may be able to restore system performance [Cornelius et al., 2013], it is equally important to design reliable systems that minimize the risk of failure to begin with. Scalability, reliability, and redundancy are key elements of distributed control systems that must be accounted for at the control design stage to ensure that a certain level of performance is guaranteed. On a technical level, for a specific problem like multi-agent consensus, we may require that a consensus protocol is based on local information, that the consensus set is asymptotically stable, that we are able to estimate the region of attraction, and that this holds for any connected graph topology. This paper explores the influence that the geometry and topology of the underlying nonlinear space has on our ability to design such consensus protocols with the strong convergence property of almost global synchronization.

In terms of scalability, the literature on multi-agent consensus has mainly focused on control solutions that require low computational complexity on behalf of each agent and low energy consumption in terms of sensing and communication. The question of how the region of attraction scales with the number of agents has received comparatively less attention. The literature contains a number of results that pertain to the case when all agents belong to a geodesically convex set [Afsari, 2011; Tron et al., 2013; Hartley et al., 2013; Chen et al., 2014a]. For example, guaranteed convergence to the consensus manifold of the $n$-sphere from any hemisphere [Zhu, 2013; Lageman and Sun, 2016; Thunberg et al., 2018a; Zhang et al., 2018]. Such guarantees do not scale well with the number of agents. The probability of randomly placing all agents on a hemisphere while drawing from a uniform distribution decreases as $O(2^{-N})$. By contrast, almost globally synchronizing consensus protocols yields systems for which the region of attraction of the consensus manifold has probability measure 1 regardless of $N$. 

* Corresponding author Johan Markdahl. Tel. +352 46 66 44 5085. Fax +352 46 66 44 35085. Email address: markdahl@kth.se (Johan Markdahl).
1.1 Literature Review

Consensus protocols on linear spaces can often be designed to render the consensus set globally asymptotically stable. For systems that evolve on compact manifolds there are certain restrictions. For example, to find all equilibria is an NP-hard problem \cite{Sarlette2011}. Global stability cannot be achieved by means of continuous, time-invariant feedback \cite{Bat2000}. Almost global convergence can be achieved using continuous, time-invariant feedback, but only for certain graphs \cite{Sepulchre2011}. An example is the Kuramoto model on complex networks, for which the problem of characterizing all graphs that yield almost global consensus is open \cite{Dorfler2014}. From an engineering perspective, convergence to a synchronized state is most often desired whereupon various control solutions has been proposed. The survey paper \cite{Sepulchre2011} discusses three options: (i) careful design of the potential function underlying a gradient descent flow \cite{Sarlette2009,Tron2012}, (ii) discrete-time gossip algorithms that converge in a probabilistic sense \cite{Sarlette2009,Mazzarella2013,Mishra2017}, and (iii) communication of auxiliary variables \cite{Scardovi2007,Sarlette2009,Lee2017,Thunberg2018b}.

It was recently shown that the high-dimensional Kuramoto model in complex networks on the Stiefel manifold $S^{p}(p,n)$ converges almost globally for all connected graphs provided $p \leq \frac{n}{2} - 1$ \cite{Markdahl2018b}. The feedback that results in those systems do not make use any of the techniques (i)-(iii) discussed in \cite{Sepulchre2011}. Rather, it turns out that for some nonlinear manifolds including the higher-dimensional spheres and most Stiefel manifolds, almost global synchronization is not graph dependent. In particular, there is a striking difference between the circle $S^1$ and all higher dimensional spheres. This discovery is important since the circle is the manifold underlying the Kuramoto model, which has been one of the most influential models in the field of synchronization. To find out that the Kuramoto model is, in a certain sense, a pathological case among spheres hence changes our expectations as to what can be achieved on other manifolds. It also reveals that consensus protocols for reduced rigid-body attitude synchronization on $S^2$ are more well-behaved than those for planar attitudes on $S^1$ and full attitudes on $SO(3)$.

The study of intrinsic consensus on general Riemannian manifolds was initiated by \cite{Tron2013}. Further work include both intrinsic \cite{Chen2013,Tron2014b,Aydoud2017} and extrinsic approaches \cite{Montenbruck2015,Montenbruck2016,Montenbruck2017,Aydoud2017,Fiori2018}. In a wider context of related works, there are plenty of results that pertain to synchronization on specific nonlinear Riemannian manifolds (or classes thereof \cite{Ha2018,Deville2018,Markdahl2018b}) and (non-distributed) averaging over nonlinear Riemannian manifolds. In particular, the circle $S^1$, sphere $S^2$, and special orthogonal group $SO(n)$ have received attention due to their relevance in a number of applications. This includes the Kuramoto model, rigid-body attitude synchronization \cite{Sarlette2009,Sarlette2009}, and the Lohe model of quantum synchronization \cite{Lohe2010}. For $S^1$, \textit{i.e.}, the Kuramoto model and its many variations, we refer to the survey papers \cite{Dorfler2014,Rodrigues2014} for a brief literature review of consensus on $S^1$. Results on averaging over Riemannian manifolds are given by \cite{Afsari2013}, although not from a distributed control perspective.

Since the results of this paper are negative, the proofs are by means of counter-examples. The counter-examples concern $N$ agents connected by a graph consisting of a single cycle, \textit{i.e.}, a cycle or circular graph. The configuration of agents can hence be associated with the closed, piece-wise geodesic curve (broken geodesic) that interpolates the agents’ positions. This leads us to see the consensus seeking system as a form of curve shortening flow, a topic that has been studied in mathematics \cite{White2002}. The curve shortening flow has been applied to general Riemannian manifolds \cite{Grayson1989}. It has also been connected to consensus via the related problem of polygon shortening flow \cite{Smith2007}. Metaphorically speaking, we can imagine the agents as beads on a string. If the manifold is simple connected, then a continuous shortening of the string to a point results in consensus. If not, then the agents must be threaded to one end of the string. This requires two neighboring agents to move away from each other, which goes against the basic design principle of consensus protocols. This idea offers an alternative geometric interpretation of consensus seeking systems which can complement the idea of shrinking the convex hull of the agents \cite{Moreau2005} in the case when the agents do not belong to a (geodesically) convex set.

1.2 Contribution

This paper builds on previous work on almost global consensus on nonlinear spaces initiated by \cite{Scardovi2007,Sarlette2009}. The author has previously shown that the requirements for achieving almost global consensus on the $n$-sphere is weaker than on $S^1$ and $SO(3)$ \cite{Markdahl2018b}. However, it was not clear why this was the case. This paper shows that it the difference can be explained in terms of the topology of the manifold; it is property of simple connectedness which distinguishes the $n$-sphere from $S^1$ and $SO(3)$. Moreover, this paper also extends the work \cite{Tron2012,Tron2013} on intrinsic consensus on Riemannian manifolds. Previously, local stability had been established. Here, we show that if a Riemannian manifold contains a closed curve of locally minimum length then the system is multislab; it contains at least one other stable equilibrium set aside from consensus. In particular, the existence of such an equilibrium set shows that the consensus manifold is not almost globally asymptotically stable.
2 Problem Description

2.1 Preliminaries

Consider a network of $N$ interacting agents. The interaction topology is modeled by a graph $G = (\mathcal{V}, \mathcal{E})$ where the nodes $\mathcal{V} = \{1, \ldots, N\}$ represent agents and an edge $e = \{i, j\} \in \mathcal{E}$ indicates that agent $i$ and $j$ can communicate. Assume that the graph is connected, whereby there is at least an indirect path of communication between any two agents. In this paper, we focus on the cycle graph $\mathcal{H}_N = (\mathcal{V}, \mathcal{E}) = (\{1, \ldots, N\}, \{\{i, i+1\} | i \in \mathcal{V}\})$. (1)

For notational convenience we let $\mathcal{H}_N$, i.e., $\{1, N\} \in \mathcal{E}$.

Let $(\mathcal{M}, g)$ be a Riemannian manifold. The set $\mathcal{M}$ is a real, smooth manifold and the metric tensor $g_x$ is an inner product on the tangent space $T_x \mathcal{M}$ at $x$. We write $\mathcal{M}$ if the choice of $g$ is implicitly understand or irrelevant. The map $x \mapsto g_x$ is smooth for any two differentiable vector fields $X, Y$ on $\mathcal{M}$. The shortest curve $\gamma : [a, b] \to \mathcal{M}$ such that $\gamma(a) = x, \gamma(b) = y$ is a geodesic from $x$ to $y$ (up to parametrization). The length of a curve $\gamma : [a, b] \to \mathcal{M}$ is

$$l(\gamma) = \int_a^b g_x(\gamma', \gamma') \, dt.$$ (2)

Let $\Gamma$ denote the set of smooth curves on $\mathcal{M}$. The length of a geodesic curve is the geodesic distance

$$d_g(x, y) = \inf \{l(\gamma) : \gamma \in \Gamma, \gamma(a) = x, \gamma(b) = y\}.$$

For manifolds embedded in the ambient space $\mathbb{R}^{n \times m}$ we also use the Frobenius norm $\| \cdot \| : X \mapsto (\text{tr} X^T X)^{\frac{1}{2}}$ and the chordal distance $d_c(X, Y) = \| X - Y \|$. There is a Hilbert manifold of maps from the circle $S^1$ to $\mathcal{M}$ [Klingenberg, 1978]. That framework allows for a theory of closed geodesics on Riemannian manifolds. We will not detail it here, but provide the following result, adapted to the context of this paper:

**Theorem 1 (Klingenberg, 1978)** Assume that the Riemannian manifold $\mathcal{M}$ is closed but not simply connected. Then $\mathcal{M}$ contains a closed curve that is a local minimizer of the curve length function $l$ given by (2).

The property of simple connectedness refers to a path connected manifold on which each closed curve can be continuously deformed to a point. In this paper, we focus on manifolds $\mathcal{M}$ that are path connected since almost global consensus would be impossible otherwise. If closedness is omitted from the requirements of Theorem 1, then a counterexample is given by the punctured plane $\mathbb{R}^2 \backslash \{0\}$. This manifold is not simply connected, yet it does not contain a curve of minimum length.

**Example 2** The torus is connected, but not simply connected. A curve that wraps around the torus tube cannot be continuously deformed to a point, see Fig. 1 (left). A circle around the tube of the torus is a curve of minimum length. The sphere $S^2$ is simply connected. The closed geodesics on $S^2$ are great circles, e.g., the equator. The equator is not a curve of minimal length since there are closed curves of constant latitude arbitrarily close to the equator that are shorter than it, see Fig. 1 (middle). Take a one-sheeted hyperboloid, cut it twice so that it becomes compact, see Fig. 1 (right). To each cut we can graft a cap to obtain an hourglass shaped manifold that can be arbitrarily smooth. This manifold is simply connected, yet it has a curve of strictly minimum length. The condition of simple connectedness is hence necessary to rule out the existence of a curve of minimum length on a closed manifold, but it is not sufficient. Note that compactness is not essential (although closedness is); a single cut would have sufficed.

Fig. 1. A torus contains infinitely many circles of minimum length around its tube, and one of strictly minimum length along its toroidal direction (left). A great circle on a sphere is not a curve of minimum length (middle). A one-sheeted hyperboloid contains a closed curve of strictly minimum length around its waist (right).

Assume that the manifold is geodesically complete which implies that there exists at least one geodesic path between any two points $x, y \in \mathcal{M}$. Moreover, assume that for some neighborhood of $x$, $B_\varepsilon(x) = \{z \in \mathcal{M} | d_g(x, z) < \varepsilon\}$, there exists a unique geodesic from $x$ to each $y \in B_\varepsilon(x)$. The largest value $\varepsilon > 0$ for which this holds is the injectivity radius $r(x)$. We assume that $\inf_{x \in \mathcal{M}} r(x) > 0$. The results of this paper concern the local behaviour of a multi-agent system where the distance $d_g(x_i, x_j)$ between any pair of interacting agents can be made arbitrarily small by starting with sufficiently many agents $N$. As such, we can (almost) always assume we are working on a subset of the manifold where all geodesics are uniquely defined.

Given a point $x \in \mathcal{M}$ and a tangent vector $v \in T_x \mathcal{M}$, the exponential map $\exp_x : T_x \mathcal{M} \to \mathcal{M}$ gives the point $y \in \mathcal{M}$ that lies at a distance $d_g(v, w)$ from $x$ along the geodesic that passes through $x$ with $v$ as a tangent vector. Let $T_x \mathcal{M}$ be an open set on which $\exp_x$ is a diffeomorphism, e.g., a ball with radius $r(x)$. Define the logarithm map $\log_x : \mathcal{M} \to T_x \mathcal{M}$ as the inverse of the exponential map $\log_x = \exp_x^{-1}$.

The directional derivative of a smooth function $f : \mathcal{M} \to \mathbb{R}$ at $x \in \mathcal{M}$ along $v \in T_x \mathcal{M}$ is given by $\frac{d}{dt} f(\gamma(t))|_{t=0}$, where
\( \gamma \in \Gamma \) satisfies \( \gamma(0) = x, \dot{\gamma}(0) = v \). The intrinsic gradient of \( f \) is uniquely defined as the vector \( \nabla_x f(x) \in T_x \mathcal{M} \) which satisfies

\[
g_x(\nabla_x f(x), v) = \frac{d}{dt} f(\gamma(t))|_{t=0}.
\]

In particular, it holds that \( \nabla_x f^2(x, y) = -\log_x(y) \).

The main challenge for control on a nonlinear manifold is to achieve good performance on a global level. It is not possible to achieve global asymptotic stability on a compact manifold by means of continuous, time-invariant feedback. It is however possible to render an equilibrium or equilibrium set almost globally asymptotically stable (AGAS):

**Definition 3** An equilibrium set \( \mathcal{Q} \) of a dynamical system \( \Sigma \) on a Riemannian manifold \((\mathcal{M}, g)\) is referred to as almost globally asymptotically stable (AGAS) if it is stable and the flow \( \Phi(t, x_0) \) of \( \Sigma \) satisfies \( \lim_{t \to \infty} d_{\mathcal{Q}}(\Phi(t, x_0)) = 0 \) for all \( x_0 \in \mathcal{M} \setminus \mathcal{N} \), where the Riemannian measure of \( \mathcal{N} \subset \mathcal{M} \) is zero.

### 2.2 Distributed Control Design on Riemannian Manifolds

The consensus manifold \( \mathcal{C} \) of a Riemannian manifold \((\mathcal{M}, g)\) is the set

\[
\mathcal{C} = \{(x)_i^{N} \in \mathcal{M}^{N} \}.
\]

The consensus set is a Riemannian manifold; in fact, it is diffeomorphic to \( \mathcal{M} \) by the map \((x)_i^{N} \mapsto x \). For any connected graph \( \mathcal{G} \), an equivalent definition is

\[
\mathcal{C} = \{(x)_i^{N} \in \mathcal{M}^{N} \mid x_i = x_j, \forall \{i, j\} \in \mathcal{E} \}. \tag{3}
\]

Given a graph \((\mathcal{V}, \mathcal{E})\), define the disagreement function \( V : \mathcal{M}^N \to \mathbb{R} \) by

\[
V = \frac{1}{2} \sum_{\{i, j\} \in \mathcal{E}} w_{ij} d_g^2(x_i, x_j),
\]

where \( w_{ij} \in (0, \infty) \) and \( w_{ji} = w_{ij} \) for all \( \{i, j\} \in \mathcal{E} \). The consensus seeking system on \( \mathcal{M} \) obtained from \( U \) is the gradient descent flow

\[
\dot{x} = (\dot{x})_i^{N} = -\nabla V = (-\nabla_i V)^{N} , \tag{4}
\]

where \( \nabla_i \) denotes the gradient with respect to \( x_i \) and \( x_i(0) \in \mathcal{M} \) for all \( i \in \mathcal{V} \). If \( \mathcal{G} \) is connected, then by (3), \( x \in \mathcal{C} \) if and only if \( V = 0 \).

Agent \( i \) does not have access to \( V \), but can calculate

\[
V_i = \frac{1}{2} \sum_{j \in \mathcal{N}_i} w_{ij} d_g^2(x_i, x_j)
\]

at its current position. Symmetry of \( d_g \) gives \( V = \frac{1}{2} \sum_{i \in \mathcal{V}} V_i \) whereby it follows that \( \nabla_i V_i = \nabla_i V \). From a control design perspective, we can assume that the dynamics of each agent takes the form \( \ddot{x}_i = u_i \) with \( u_i \in T_x \mathcal{M} \). Since agent \( i \) can evaluate \( V_i \) at its current position, it is reasonable to assume that it can also calculate \( u_i = -\nabla_i V_i \).

**Algorithm 4** The closed-loop system gradient descent flow of \( V \) for \( \dot{x}_i = u_i \) under the feedback \( u_i = -\nabla_i V_i \) is

\[
\dot{x}_i = \sum_{j \in \mathcal{N}_i} w_{ij} \log_{x_i}(x_j), \forall i \in \mathcal{V}. \tag{5}
\]

The intrinsic discrete-time equivalent of Algorithm 4 is introduced by [Tron et al. 2013].

### 2.3 Extrinsic Consensus in an Ambient Euclidean Space

Let the manifold \( \mathcal{M} \) be embedded in an ambient Euclidean space \( \mathbb{R}^{n \times m} \). Denote the system state by \( X = (X_i)_i^{N} \in (\mathbb{R}^{n \times m})^N \). Introduce an extrinsic disagreement function \( U : \mathcal{M} \to [0, \infty) \) given by

\[
U = \frac{1}{2} \sum_{\{i, j\} \in \mathcal{E}} w_{ij} \|X_i - X_j\|^2. \tag{6}
\]

Let \( U = \sum_{i \in \mathcal{V}} U_i \), where \( U_i = \frac{1}{2} \sum_{j \in \mathcal{M}} w_{ij} \|X_i - X_j\|^2 \).

As for the intrinsic algorithm, it holds that \( \nabla_i U = \nabla_i U_i \). To calculate the gradient, take a (any) smooth extension \( W : (\mathbb{R}^{n \times m})^N \to \mathbb{R} \) of \( U \), i.e., \( W|_{\mathcal{M}} = U \), and utilize that

\[
\nabla_i U = \Pi_i \frac{\partial}{\partial X_i} W_i,
\]

where \( \nabla_i \) denotes the intrinsic gradient with respect to \( X_i \), \( \Pi_i : \mathbb{R}^{n \times m} \to T_{X_i} \mathcal{M} \) is an orthogonal projection map, \( \frac{\partial}{\partial X_i} \) is the extrinsic gradient in the ambient Euclidean space with respect to \( X_i \), and \( W = \sum_{i \in \mathcal{V}} W_i \) with \( W_i : \mathbb{R}^{n \times m} \to \mathbb{R} \) being a (any) smooth extension of \( U_i \).

A gradient descent flow of \( U \) is given by:

**Algorithm 5** The closed-loop system gradient descent flow of \( U \) for \( \dot{X}_i = U_i \) under the feedback \( u_i = -\nabla_i U_i \) is

\[
\dot{X}_i = -\Pi_i \sum_{j \in \mathcal{N}_i} w_{ij} (X_i - X_j). \tag{7}
\]

We note that if all \( X \in \mathcal{M} \) have constant norm, then \( \Pi X = 0 \), whereby the dynamics (7) simplify to

\[
\dot{X}_i = \Pi_i \sum_{j \in \mathcal{N}_i} w_{ij} X_j.
\]
This implies that the system can be thought of as evolving in the ambient space of a high-dimensional sphere. Examples of such manifolds are the Stiefel manifolds, including the orthogonal group $O(n)$ and its submanifold $SO(n)$.

2.4 Problem Statement

The aim of this paper is to show that the consensus manifold $C$ is not an AGAS equilibrium set of (i) the intrinsic gradient descent flow (5) generated by Algorithm 4 if $M$ contains a closed curve $\gamma$ which is a local minimizer of $l$ given by (2) and (ii) the extrinsic gradient descent flow (7) generated by Algorithm 5 if $M$ is not simply connected. Note that the condition of objective (i) is satisfied by any manifold that is path connected but not simply connected by Theorem 1.

2.5 Optimization on Riemannian Manifolds

Since the system (5) is a gradient descent flow, it will be useful to study it from an optimization perspective.

Definition 6 A graph $G$ is $M$-synchronizing if all minimizers of $V$ belong to $C$.

The concept of $S^1$-synchronizing graphs was introduced to study the Kuramoto model in complex networks [E.A. Canale and Monzón, 2007]. The concept of $M$-synchronizing graphs is a direct generalization thereof.

It is not immediate that $G$ being $M$-synchronizing implies that $C$ is an AGAS equilibrium of (4) or (7). Since (4) is a gradient descent of $V$, $x$ cannot converge to a maximum of $V$ (likewise for the extrinsic system (7)). Moreover, any saddle point of $V$ is unstable. However, a set of saddle points may still have a region of attraction with positive measure, in which case $C$ cannot be AGAS. For specific manifolds such as the $n$-sphere and the Stiefel manifold, it can be shown that $G$ being $M$-synchronizing implies $C$ is AGAS. For the purpose of this paper we only require the inverse implication:

Proposition 7 If a manifold $M$ is not $G$-synchronizing, then the consensus manifold $C$ is not AGAS for the network $G$.

PROOF. Since the manifold is not $V$-synchronizing, there exists a minimizer $y = (y_i)_{i=1}^N$ of $V$ which does not belong to $C$. Because the system is an analytic gradient descent flow, $y$ is stable [Absil and Kurdyka, 2006]. Since $y$ is stable, for some $\varepsilon > 0$ there is an open ball $B_\varepsilon(y)$ such that if $x(0) \in B_\varepsilon(y)$ then $x(t) \in B_\varepsilon(y)$ for all $t \in (0, \infty)$. Because $d_g(y, C) > 0$ we can choose $\varepsilon$ small enough that $B_\varepsilon(y) \cap C = \emptyset$. The probability measure of $B_\varepsilon(y)$ is positive and $B_\varepsilon(y) \cap R(C) = \emptyset$. The probability measure of $R(C)$ is hence strictly less than 1. □

3 Main Results

3.1 Intrinsic Consensus

Consider the configuration where the agents are distributed equidistantly over a curve of minimum length. This configuration is stable when all weights are equal if $N \geq 3$. This is not the case for $N = 2$ since the closed curve collapses into a non-closed curve. Generalizing to the case of unequal weights, we have the following result:

Theorem 8 Suppose $M$ contains a curve $c$ of minimum length $L = l(c)$. The set

$$\{(x_i)_{i=1}^N \in M^N \mid d_g(x_i, x_{i+k}) = \frac{L}{3} \sum_{j=1}^{i+k-1} W^{-1} w_{i1} w_{j1}^{-1}, N \geq 3, \quad \forall \, k \in \mathbb{N}, \, k \leq \left\lfloor \frac{N}{2} \right\rfloor, \forall i \in V\},$$

is stable under Algorithm 4 for the cycle graph $H_N$ defined by (1).

PROOF. It suffices to show that the elements of the set are locally optimal solutions to

$$\min_{i=1}^N \sum_{i=1}^N w_{ii+1} d_g(x_i, x_{i+1})^2$$

$$x_i \in M, \forall i \in V,$$

where $w_{ii+1} \in (0, \infty)$ [Absil and Kurdyka, 2006].

Consider an initial agent configuration in the vicinity of $c$. The constraint

$$\sum_{i=1}^N d_g(x_i, x_{i+1}) \geq L$$

holds locally. First consider the case when the constraint is active, i.e., the optimization problem

$$\min_{i=1}^N \sum_{i=1}^N w_{ii+1} d_g(x_i, x_{i+1})^2,$$

$$\sum_{i=1}^N d_g(x_i, x_{i+1}) = L.$$  (10)

We require that $N \geq 3$, since at least three points are needed to ensure that the curve does not degenerate into two geodesics that overlap everywhere. In the rather trivial case of $N = 2$, $C$ contains all minimizers of $V$.

We may consider $d_g(x_i, x_{i+1}) \in [0, \infty)$ to be the variables of the problem (10). The problem (10) requires us to distribute the agents over $c$ with positions specified in terms of
weighted arc length from some arbitrary reference point on the curve. Essentially, we relax the problem (8) restricted to the curve c by disregarding the geometry of the Riemannian manifold. This is possible since (10) only depends on the geodesic distances.

The problem (10) together with the constraint $d_g(x_i, x_{i+1}) \in [0, \infty)$ for all $i \in V$ is a special case of the continuous quadratic knapsack problem [Robinson et al., 1992]. If we relax the positivity constraint on $d_g(x_i, x_{i+1})$, it becomes a quadratic program. It can be solved using the Lagrange conditions for optimality [Nocedal and Wright, 1999].

$$\begin{bmatrix} H & a^\top \\ a & 0 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} -c \\ b \end{bmatrix},$$

where $H$ is the Hessian matrix, $a$ is the constraint vector, $x$ are the variables, $\lambda$ is the Lagrange multiplier, $c$ is the coefficients of the linear term in the objective function and $b$ is the right-hand side of the constraints. For the problem (10) this becomes

$$\begin{bmatrix} W^{1\top} \\ 1 \end{bmatrix} \begin{bmatrix} d \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ L \end{bmatrix},$$

where $d \in \mathbb{R}^N$ is given by $d_i = d_g(x_i, x_{i+1})$, $1 = \{1 \ldots 1\} \in \mathbb{R}^N$, and $W$ with $W_{ii} = w_{ii+1}$ is diagonal.

Denote

$$A = \begin{bmatrix} W^{1\top} \\ 1 \end{bmatrix}, \quad M = W^{-1}. $$

It can be shown that

$$A^{-1} = \frac{1}{1M^{1\top}} \begin{bmatrix} (1M^{1\top})M - M^{1\top}1M & 1M \\ 1M & -1 \end{bmatrix},$$

from which it follows

$$\begin{bmatrix} d \\ \lambda \end{bmatrix} = \frac{1}{1M^{1\top}} \begin{bmatrix} (1M^{1\top})M - M^{1\top}1M & 1M \\ 1M & -1 \end{bmatrix} \begin{bmatrix} 0 \\ L \end{bmatrix} = \frac{L}{1M^{1\top}} \begin{bmatrix} M^{1\top} \\ -1 \end{bmatrix}.$$  

(11)

The objective value of (10) is

$$\sum_{i=1}^N w_{ii+1}d_g(x_i, x_{i+1}) = d^\top Wd = \left( \frac{L}{1M^{1\top}} \right)^2 1M^{1\top}WM1^\top.$$

The optimal solution to (10) does not belong to the consensus manifold $C$. It cannot, because the constraint (9) makes $C$ infeasible. Hence we are able to extract the optimal distribution of agents on $c$. Suppose that the agents are moved off of $c$ to some $(z_i)_{i=1}^N \in \mathcal{M}$ by a small perturbation. Then they lie on another curve, a broken closed geodesic $c'$, which satisfies $l(c') = L' \geq L$. If $l(c') = L$, then the state still belongs to the set of curves of locally minimum length. The perturbed state may or may not be optimal on $c'$. However, if $L' > L$, then the optimization problem (10) restricted to $c'$ instead of $c$ has optimal value

$$\frac{L'^2}{\sum_{i=1}^N w_{ij}^{-1}} > \frac{L^2}{\sum_{i=1}^N w_{ij}^{-1}}.$$

The value of the objective function at $(z_i)_{i=1}^N$ is larger or equal to the minimum over $c'$. This is illustrated in Fig. 2.

The solution (11) yields a local optimum to the problem

$$\min_{i=1}^N \sum w_{ii+1}d_g(x_i, x_{i+1})^2,$$

(13)

$$\sum_{i=1}^N d_g(x_i, x_{i+1}) \geq L.$$

Introducing the constraint (9) to the original problem (8) does not result in any loss of generality since it holds for all closed curves in an open neighborhood of $\gamma$.  

Fig. 2. The agents are perturbed from an optimal distribution on the inner circle of the torus (×) to a closed, broken geodesic which interpolates the perturbed positions (∨). The resulting configuration is (most likely) suboptimal, given that the agents are restricted to this curve. There is an optimal distribution of agents on this curve (∨). The objective value of that distribution (∨) is higher than that on the inner circle (×). As such, the distribution (∨) cannot be better than (×).

Proposition 8 leads to the main result of this paper, which states that the non-existence of a curve of minimum length is a necessary condition for $C$ to be an AGAS equilibrium of (5) regardless of graph topology.
Corollary 9  Consider Algorithm 1 on a Riemannian manifold \((\mathbb{M}, g)\). Suppose that \(\mathbb{M}\) contains a closed curve which is a local minimizer of the length function \(l_p\). The closed-loop system has a stable equilibrium set \(\mathcal{Q}\) such that \(\mathcal{Q} \cap \mathcal{C} = \emptyset\) for at least one connected graph \(\mathcal{G}\). Whether the consensus manifold \(\mathcal{C}\) is AGAS or not depends on \(\mathcal{G}\).

**PROOF.** This is a direct consequence of Proposition 7 and Proposition 8.

3.2 Extrinsic Synchronization in the Ambient Space \(\mathbb{R}^{n \times m}\)

Extrinsic consensus algorithms on Stiefel manifolds \(\text{St}(p, n)\) are known to converge to \(\mathcal{C}\) almost globally for any connected graph provided \(p \leq \frac{2}{3} n - 1\). However, convergence does not hold on \(S^1 \simeq \text{St}(1, 2), \text{SO}(3) \simeq \text{St}(2, 3)\). Here we show that these negative findings extend to any closed manifold which is not simply connected.

**Theorem 10** Let \(\mathbb{M}\) be a closed, geodesically complete Riemannian manifold which is not simply connected. Then there exists a dense set of initial conditions, a graph \(\mathcal{G}\), and a \(N \in \mathbb{N}\) such that the closed loop system (7) generated by Algorithm 5 does not converge to the consensus manifold \(\mathcal{C}\). Whether \(\mathcal{C}\) is AGAS or not depends on \(\mathcal{G}\).

**PROOF.** For all \(N \in \mathbb{N}\), distribute \(N\) agents \((X_i^N)_{i=1}^N\) approximately equidistantly (by the chordal distance in the ambient space \(\mathbb{R}^{n \times m}\)) over a closed curve \(\gamma \subset \mathbb{M}\) which is not homeomorphic to a point (see Fig. 3). Such a curve exists by virtue of the manifold not being simply connected. Denote, \(L_N = \sum_{i=1}^N |X_{i+1} - X_i|\). In particular, \(L_N \rightarrow l(\gamma)\) as \(N \rightarrow \infty\). We have chosen the agent positions such that \(|X_{i+1} - X_i| \in \mathcal{O}(N^{-1})\) for large \(N\). It follows that \(U = \sum_{i=1}^N u_{i+1}^T (X_{i+1} - X_i)^2 \in \mathcal{O}(N^{-1})\) for large \(N\) and that \(U \rightarrow 0\) as \(N \rightarrow \infty\).

Consider the level set \(\{(Y_i^N)^{i=1}_N | U \leq \varepsilon\}\). For any \(\varepsilon > 0\) there is a \(N\) such that \((X_j^N)_{i=1}^N \in \{(Y_i^N)^{i=1}_N | U \leq \varepsilon\}\). Let \(L_\varepsilon\) denote the connected component of the level set which contains \((X_j^N)_{i=1}^N\). Note that for \((X_i(0))_{i=1}^N \in L_\varepsilon\) it holds that \(|X_{i+1}(t) - X_i(t)| \leq (u_{i+1}^T \varepsilon)^{1/2}\) for all \(t \in [0, \infty)\).

Recall that \(r > 0\) denotes the injectivity radius of \(\mathbb{M}\). Consider the ball \(\mathcal{B} = \{Y \in \mathbb{M} | d(Y, X_i^N) < r, \forall j \in N_i, \forall i \in \mathcal{V}\}\). There is an open ball \(\mathcal{B}\) dense in the ambient space \(\mathbb{R}^{n \times m}\), such that \(\mathcal{B} \cap \mathbb{M} \subset \mathcal{D}\). Suppose this is false, then there is a sequence \((Z_i)_{i=1}^\infty \subset \mathbb{M}\), such that \(\lim_{i \rightarrow \infty} Z_i = X_i\) but \(\lim_{i \rightarrow \infty} d(Z_i, X_i) \geq r\). That contradicts the continuity of \(d\). Hence we may chose \(N\) large enough (i.e., \(\varepsilon\) small enough) that the geodesic distance \(d(X_j^N(t), X_i(t)) < r\), for all \(j \in N_i\) and all \(t \in [0, \infty)\).

The geodesic between two agents can be constructed from the exponential map. For \(d(X_j^N, X_i^N) < r\), the exponential map is a diffeomorphism. The closed broken geodesic \(\gamma(t)\) that interpolates the agent positions (in numerical order) can therefore not change discontinuously. The system evolution hence corresponds to a continuous deformation of \(\gamma(t)\). If the system converges to the consensus manifold, then \(\lim_{t \rightarrow \infty} \gamma(t) \in \mathcal{C}\). This contradicts \(\gamma = \gamma(0)\) not being homeomorphic to a point.

Fig. 3. The agents (\(\times\)) are equidistantly distributed on a plane with a hole \(\mathbb{R}^2 \setminus \{x \in \mathbb{R}^2 | \|x\|_2 < 1\}\) (left). Their positions are interpolated by a closed, broken geodesic (\(\bigcirc\)). The injectivity radius \(r\) of the manifold is half of the hole’s circumference. In order to pass over the hole, two neighbors need to be separated by a chordal distance greater than or equal to the hole’s diameter \(d\) (middle). From such an initial condition the agents may be able to approach the consensus manifold (right). That outcome is not possible if the agents start out in a configuration where \(U = \sum_{i=1}^N \|X_{i+1} - X_i\|^2 \in \mathcal{O}(N^{-1})\) for large \(N\) and that \(U \rightarrow 0\) as \(N \rightarrow \infty\).

4 Simulations

Let us explore the question if a manifold being simply connected is not just necessary but also sufficient for the consensus manifold \(\mathcal{C}\) of the closed loop system (7) generated by Algorithm 5 to be AGAS by means of simulations.

Consider the canonical embedding of the Stiefel manifold \(\text{St}(p, n)\) as a matrix manifold in \(\mathbb{R}^{n \times p}\) given by

\[
\text{St}(p, n) = \{S \in \mathbb{R}^{n \times p} | S^T S = I\}.
\]

We restate Algorithm 5 in the case of \(\mathbb{M} = \text{St}(p, n)\):

**Algorithm 11** The input \(U_i \in T, \text{St}(p, n)\) is the negative gradient of the disagreement function, i.e., \(U_i = -\nabla_i U\). The closed-loop system is a gradient descent flow given by

\[
(S_i)_{i=1}^N = -\nabla U = (-\nabla_i U)_{i=1}^N = (-\Pi_{i=1}^N \frac{d}{ds} U)_{i=1}^N,
\]

\[
S_i = S_i \text{ skew } S_i^T \sum_{j \in N_i} S_j + (I - S_i S_i^T) \sum_{j \in N_i} S_j,
\]

\[
S_i(0) \in \text{St}(p, n).
\]

Note that the path on \(\text{St}(n-1, n)\) generated by Algorithm 11 (i.e., Algorithm 5 on \(\text{St}(n-1, n)\)) is different from the
corresponding path generated by Algorithm 5 on $O(n)$ restricted to $SO(n) \subset O(n)$ (see Appendix A).

The consensus manifold $C$ on $St(p, n)$ under the dynamics (14) is AGAS for any connected graph if $p \leq \frac{n}{2} - 1$ \cite{Markdahl2018}. What about other values of $p$? The results of this paper show that $C$ cannot be AGAS if the manifold is not simply connected. The manifold $St(p, n)$ is simply connected if $p \leq n - 2$ \cite{James1976}. The only exceptions are $St(n - 1, n) \simeq SO(n)$, which is path connected but not simply connected, and $St(n, n) \simeq O(n)$, which is separated by the function $\det : O(n) \rightarrow \{-1, 1\}$. What about the cases of $\frac{n}{2} - 1 < p \leq n - 2$? We conjecture that $C$ is AGAS for all such $p$, i.e., the that condition $p \leq n - 2$ is both necessary and sufficient for $C$ to be AGAS for all connected graphs.

Let $\Phi : \mathbb{R} \times St(p, n)^N \rightarrow St(p, n)^N$ denote the flow of (14), $\Phi(t, (S_i(t)_{i=1}^N)) = (S_i(t)_{i=1}^N)$ given that $(S_i(0)_{i=1}^N) = (S_i, 0)_{i=1}^N \in St(p, n)^N$. Denote $\Phi = (\Phi_i)_{i=1}^N$. Let $\mathcal{R}$ denote the region of attraction of $C$,

$$\mathcal{R} = \{(S_i)_{i=1}^N \in St(p, n)^N | \lim_{t \rightarrow \infty} \Phi(t, (S_i)_{i=1}^N) \in C\}.$$

The probability measure $\mu(\mathcal{R})$ of $\mathcal{R}$ is the fraction of initial conditions that ultimately yield a consensus (i.e., the size of the sync basin \cite{Wiley2006}).

The probability measure $\mu(\mathcal{R})$ can be calculated by means of Monte Carlo integration:

$$\frac{1}{M}\sum_{k=1}^{M} \mathbb{1}_{C}(\lim_{t \rightarrow \infty} \Phi(t, (S_i(t)_{i=1}^N))) \xrightarrow{n \rightarrow \infty} \mu(\mathcal{R}) \text{ as } M \rightarrow \infty,$$

where $1 : St(p, n)^N \rightarrow \{0, 1\}$ is the indicator function and $(S_i)_{i=1}^N$ for each $k \in \{1, \ldots, M\}$ are samples drawn from the uniform distribution on $St(p, n)^N$. To draw a sample $S$ from the uniform distribution on $St(p, n)$, draw an $X \in \mathbb{R}^{n \times p}$ such that each element of is independent and identically normally distributed $N(0, 1)$ and form $S = X \cdot X^T$ \cite{Chikuse2012}.

A stop criteria is needed to (approximately) calculate $\lim_{t \rightarrow \infty} \Phi(t, (S_i(t)_{i=1}^N))$. The simulation stops if

$$\max_{(j,k) \in E} \frac{1}{2}||\Phi_j(T, (S_i)_{i=1}^N) - \Phi_k(T, (S_i)_{i=1}^N)|| < \varepsilon$$

for a threshold value $\varepsilon \in (0, \infty)$ at a fixed time $T$. The tensor $\Phi(T, (S_i(t)_{i=1}^N))$ is obtained by integrating (14) using the function ode45 in MATLAB. If (16) is satisfied, we count this as a case of convergence to $C$. If (16) is not satisfied the simulation time is extended, with a maximum duration of $S \gg T$ (the purpose of the check at $t = T$ is simply efficiency). There are potential issues with long simulation times causing numerical errors to accumulate so that either the system leaves the Stiefel manifold or is perturbed from $\mathcal{R}$ into $\mathcal{R}$. We did not detect any such issues. This approach is biased towards underestimating the value of $\mu(\mathcal{R})$ due to $S < \infty$, but the effect is small for large $S$.

The results are displayed in Table 1. Note that they are in agreement with our conjecture that $p \leq n - 2$ guarantees convergence to the consensus manifold $C$. The pairs $(p, n)$ that satisfies $\frac{n}{2} - 1 < p \leq n - 2$ all have $\mu(\mathcal{R}) = 1$ (marked in bold). The $(p, n)$ pairs with 1s in Table 1 which are not marked in bold satisfy the inequality $p \leq \frac{n}{2} - 1$ whereby $C$ is AGAS \cite{Markdahl2018}. Failures to reach consensus occur when $p \in \{n - 1, n\}$, i.e., for the special orthogonal group and the orthogonal group. For the case of $O(n)$, the probability that all agents belong to the same connected component is $2^{-n} \approx 0.06$, which explains the numbers on the diagonal where $p = n$.

### Table 1

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|-----|---|---|---|---|---|---|---|---|
| 1   | .95 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2   | .05 | .92 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3   | .06 | .92 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4   | .05 | .91 | 1 | 1 | 1 | 1 | 1 | 1 |
| $p$ 5 | .06 | .89 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6   | .05 | .90 | 1 | 1 | 1 | 1 | 1 | 1 |
| 7   | .06 | .90 | 1 | 1 | 1 | 1 | 1 | 1 |
| 8   | .06 | .90 | 1 | 1 | 1 | 1 | 1 | 1 |
| 9   | .06 | | | | | | | |

Results that guarantee consensus on a geodesically convex set, e.g., hemispheres of the $n$-sphere \cite{Zhu2013, Lageman2016, Thunberg2018, Zhang2018}, do not scale well with the number of agents. However, even though a guarantee does not scale well, actual system performance may still do. This is the case on $S^n$ for $n \in N \setminus \{1\}$ since $C$ is AGAS \cite{Markdahl2018}. It is of interest to investigate how $\mu(\mathcal{R})$ scales with $N$ when the consensus manifold is not AGAS. Figure 4 reveals that on $St(1, 2)$, $\mu(\mathcal{R})$ decreases with $N$ (see also \cite{Wiley2006}). On $St(2, 3)$, $St(3, 4)$, and $St(4, 5)$ it appears that $\lim_{N \rightarrow \infty} \mu(\mathcal{R}) \approx 0.5$. Note however that there is not any immediate reason to expect that $\lim_{N \rightarrow \infty} \mu(\mathcal{R})$ even exists. For each $N = 5k$, where $k \in N$, there is a new
equilibrium set on $\text{St}(n-1,n)$. It corresponds to wrapping the closed curve that interpolate the agents’ states $k$ times around an equilibrium configuration of 5 agents. Presumably, this set has sufficiently small inter-agent distances to be stable. However, there does not appear to be any large difference between $\mu(R)$ when $N = 5k$ for $k \in \mathbb{N}$ compared to $N - 1$ except for the case of $N = 5$. The size of $\text{St}(p,n)^N$ grows with $N$. The average time required to satisfy (16) conditional on $S_1(0) \in R$ may hence increase with $N$. This would introduce a dependence on $N$ in the bias of our estimate of $\mu(R)$, but the effect is small for large $S$.

5 Conclusions

Previous research on almost global synchronization has focused on the graph topology and its influence on convergence [Sepulchre, 2011, Dörfler and Bullo, 2014]. It has been discovered that for a number of matrix manifolds, almost global synchronization for any connected graph topology results by the application of a canonical, extrinsic consensus protocol [Markdahl et al., 2018, 2019]. However, it was not properly understood why certain manifolds have such strong convergence properties whereas others do not. This paper shows that the convergence can be tied to a topological property of the manifold: simple connectedness. If the manifold is not simply connected, as is e.g., the case for the circle $S^1$ and $SO(n)$, then there exists a topological obstruction to almost global synchronization. Overcoming this obstruction requires ad hoc control design [Sarlette, 2009, Sarlette and Sepulchre, 2003, Sepulchre, 2011]. However, in the case of a simply connected manifold, such advanced techniques are not always needed. Results from simulations indicate that on the Stiefel manifold, simple connectedness may be sufficient for the consensus manifold to be AGAS for all connected graphs. These findings explain why synchronization of the Kuramoto model on $S^1$ and rigid-body attitude synchronization on $SO(3)$ are, in a sense, more difficult to achieve than synchronization on the $n$-sphere for $n \in \mathbb{N}\backslash \{1\}$. As such they also encourage further studies of synchronization on the $n$-sphere for $n \in \mathbb{N}\backslash \{1\}$.

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A Consensus Protocol on $\text{St}(n-1,n)$

Consider the canonical embedding of $\text{O}(n)$ as a matrix manifold in the ambient space $\mathbb{R}^{n \times n}$. Let $d_c : \text{O}(n) \times \text{O}(n) \to [0, \infty)$ denote the chordal distance on $\mathbb{R}^{n \times n}$ given by

$$d_c : (\mathbf{A}, \mathbf{B}) \mapsto \left\| \mathbf{A} - \mathbf{B} \right\|.$$  

Let $(\mathbf{Q}_i)_{i=1}^N$ denote the state of a multi-agent system on $\text{O}(n)$. The disagreement function $U$ can be simplified as

$$U = \sum_{e \in E} d_c(Q_i, Q_j) = \sum_{e \in E} \left\| Q_i - Q_j \right\|^2$$

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where a factor of 2 has been introduced for notational convenience. Algorithm 5 in the special case of \( M = O(n) \) is:

**Algorithm 12** The input \( \mathbf{U}_i \in T_i O(n) \) is the negative gradient of the disagreement function, i.e., \( \mathbf{U}_i = -\nabla_i \mathbf{U} \). The closed-loop system is a gradient descent flow given by

\[
(\mathbf{Q}_i)_{i=1}^N = -\nabla \mathbf{U} = (\nabla_i \mathbf{U})_{i=1}^N = (-\Pi_i \mathbf{U}^{\mathbf{Q}}_{i=1} \mathbf{U})_{i=1}^N
\]

\[
\dot{\mathbf{Q}}_i = 2\mathbf{Q}_i \text{ skew} \sum_{j \in \mathcal{N}_i} \mathbf{Q}_j
\]

\[
= \sum_{j \in \mathcal{N}_i} \mathbf{Q}_j - \mathbf{Q}_i \mathbf{Q}_j^\top \mathbf{Q}_i.
\]  

(A.1)

For the restriction to \( \text{SO}(n) \subset O(n) \) and a circulant graphs \( \mathcal{G} \), there are stable equilibria \( (\mathbf{R}_i)_{i=1}^N \notin \mathcal{C} \) [DeVille, 2018].

Consider the relation between Algorithm 12 on \( \text{SO}(n) \) and Algorithm 11 on \( \text{St}(p, n) \) in the case of \( p = n - 1 \) for which \( \text{St}(p, n) \simeq \text{SO}(n) \).

Let \( \mathbf{T}_i, \mathbf{t}_i \) be such that \( \mathbf{R}_i = [\mathbf{T}_i \mathbf{t}_i] \in \text{SO}(n) \) where \( \mathbf{R}_i \) follow the dynamics generated by Algorithm 12 on \( \text{SO}(n) \). Then

\[
\dot{\mathbf{R}}_i = [\dot{\mathbf{T}}_i \dot{\mathbf{t}}_i] = \sum_{j \in \mathcal{N}_i} [\mathbf{T}_j \mathbf{t}_j] - [\mathbf{T}_i \mathbf{t}_i] \begin{bmatrix} \mathbf{T}_j^\top \\ \mathbf{t}_j \end{bmatrix} [\mathbf{T}_i \mathbf{t}_i]
\]

\[
= \sum_{j \in \mathcal{N}_i} [\mathbf{T}_j \mathbf{t}_j] - (\mathbf{T}_i \mathbf{T}_j^\top + \mathbf{t}_i \mathbf{t}_j^\top)[\mathbf{T}_i \mathbf{t}_i]
\]

which yields

\[
\dot{\mathbf{T}}_i = \sum_{j \in \mathcal{N}_i} \mathbf{T}_j - \mathbf{T}_i \mathbf{T}_j^\top \mathbf{T}_i - \mathbf{t}_i \mathbf{t}_j^\top \mathbf{T}_i
\]

\[
= 2\mathbf{T}_i \text{ skew} \sum_{j \in \mathcal{N}_i} \mathbf{T}_j + 2\mathbf{t}_i \mathbf{t}_i^\top \sum_{j \in \mathcal{N}_i} (\mathbf{T}_j - \frac{1}{2} \mathbf{t}_j \mathbf{t}_j^\top),
\]

\[
\dot{\mathbf{t}}_i = \sum_{j \in \mathcal{N}_i} \mathbf{t}_j - \mathbf{T}_i \mathbf{T}_j^\top \mathbf{t}_i - \mathbf{t}_i \mathbf{t}_j^\top \mathbf{t}_i
\]

\[
= 2 \text{ skew} \left( \sum_{j \in \mathcal{N}_i} \mathbf{T}_j \mathbf{T}_j^\top \right) \mathbf{t}_i + \sum_{j \in \mathcal{N}_i} \mathbf{t}_j - \langle \mathbf{t}_j, \mathbf{t}_i \rangle \mathbf{t}_i. \quad \text{(A.2)}
\]

Let \( \mathbf{S}_i \in \text{St}(n - 1, n) \) follow the dynamics of Algorithm 11. Let \( \mathbf{s}_i \in \mathbb{R}^n \) be such that \( [\mathbf{S}_i \mathbf{s}_i] \in \text{SO}(n) \) for all \( i \in \mathcal{V} \).

Then \( \mathbf{S}_i^\top \mathbf{s}_i = 0 \), i.e., \( \mathbf{S}_i^\top \mathbf{S}_i + \mathbf{s}_i^\top \mathbf{s}_i = 0 \). This yields \( \mathbf{S}_i^\top \dot{\mathbf{s}}_i = -\sum_{j \in \mathcal{N}_i} \mathbf{S}_j^\top \mathbf{s}_i \). Since \( \mathbf{S}_i \mathbf{S}_i^\top + \mathbf{s}_i \mathbf{s}_i^\top = \mathbf{I} \) and \( \mathbf{s}_i \mathbf{s}_i^\top = 0 \), we obtain \( \dot{\mathbf{s}}_i = -\mathbf{S}_i \sum_{j \in \mathcal{N}_i} \mathbf{S}_j^\top \mathbf{s}_i = \text{skew} \left( \sum_{j \in \mathcal{N}_i} \mathbf{S}_j^\top \right) \mathbf{s}_i \).

The system (14) on \( \text{St}(n - 1, n) \) hence yields the following system on \( \text{SO}(n) \):

\[
\dot{\mathbf{S}}_i = \mathbf{S}_i \text{ skew} \sum_{j \in \mathcal{N}_i} \mathbf{S}_j + \mathbf{s}_i \mathbf{s}_i^\top \sum_{j \in \mathcal{N}_i} \mathbf{S}_j, \quad \text{(A.3)}
\]

\[
\dot{\mathbf{s}}_i = \text{skew} \left( \sum_{j \in \mathcal{N}_i} \mathbf{S}_j \mathbf{S}_j^\top \right) \mathbf{s}_i.
\]

The systems (A.2) and (A.3) yield different paths on \( \text{SO}(n) \) (and not just different trajectories, i.e., different time-parameterizations of a single path), as is clear by inspection.