The Base Measure Problem and its Solution

ALEXEY RADUL, Google Research
BORIS ALEXEEV, Google Research

Probabilistic programming systems generally compute with probability density functions, leaving the base measure of each such function implicit. This usually works, but creates problems in situations where densities with respect to different base measures are accidentally combined or compared. We motivate and clarify the problem in the context of a composable library of probability distributions and bijective transformations. We also propose to solve the problem by standardizing on Hausdorff measure as a base, and by deriving a formula and software architecture for updating densities with respect to Hausdorff measure under diffeomorphic transformations. We hope that by adopting our solution, probabilistic programming systems can become more robust and general, and make a broader class of models accessible to practitioners.

1 INTRODUCTION

Suppose we are designing a composable library of software to represent probability distributions and transformations thereof which, following [12], we will call “bijectors”. TensorFlow Probability [12] and PyTorch Distributions [10] are such libraries in their own right, targeting the probabilistic machine learning space. General-purpose probabilistic programming languages such as Stan [3], BLOG [6], Anglican [13], or Venture [7] must of necessity also include such libraries, to implement their primitive distributions and deterministic functions.

Suppose furthermore that we are designing this library to operate on explicitly vector-valued probability distributions.¹ This is the case for TensorFlow Probability and PyTorch Distributions, for instance, to take advantage of vectorized hardware; and the treatment of the general case is instructive even for a scalar design.

In this setting, it’s conventional to represent a probability distribution \( P \) as a function \( p : \mathbb{R}^n \rightarrow \mathbb{R} \) computing the probability density of \( P \) at points in \( \mathbb{R}^n \), together with a sampler \( x \sim s() \) drawing random variates \( x \in \mathbb{R}^n \) distributed according to \( P \). Given a map \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \), the pushforward \( fP \) is then sampled as \( x' \sim f(s()) \), and its density is computed as

\[
fp(x') = \frac{1}{|\det(f_{x'})|} p(x), \quad \text{for} \ x = f^{-1}(x').
\]

The Jacobian-determinant correction \( 1/|\det(f_{x'})| \) accounts for the possibility that \( f \) changes the volume of an infinitesimal volume element near \( x \). The Jacobian determinant can be computed by forming the Jacobian of \( f \), for example with automatic differentiation; but for many functions \( f \) of interest, it’s available more efficiently. Thus a conventional choice is to package such \( f \), together with their inverse \( f^{-1} \), in a Bijector class with a method for computing said Jacobian determinant.

This conventional architecture admits a serious bug. We name this bug the Base Measure Problem, because it consists of neglecting the base measure with respect to which we are computing densities. Our contributions answer these questions:

- What’s the problem? We give a clear and intuitive example of the Base Measure Problem in Section 2;
- How common is it? We briefly survey several areas where the Base Measure Problem recurs in different guises in Section 3;

¹No distinction need be made for our purposes between vectors, matrices, tuples, or other structures, as long as joint distributions over non-trivial powers of \( \mathbb{R} \) are in scope.

Authors’ addresses: Alexey Radul, Google Research, axch@google.com; Boris Alexeev, Google Research, alexeev@google.com.
• What’s the right answer? We propose a more nuanced standard base measure in Section 4, and derive a complete density correction formula for that choice using standard results; and
• How do we fix our software? We detail common special cases of our correction formula in Section 5, as well as how to arrange a software library to optimize them.

We stress that while we do propose to explicitly represent information about measures, all the information we will need will be local to a single point (the point itself and various directional derivatives thereat), and thus require no symbolic algebra to compute with.

2 MOTIVATING EXAMPLE

Consider the uniform distribution \( P \) on the unit circle in \( \mathbb{R}^2 \). The natural probability density function to write down for this is

\[
p(x, y) = \begin{cases} 
\frac{1}{2\pi} & \text{when } x^2 + y^2 = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Of course, if we were sticklers we would note that the base measure implied by the type of the samples is Lebesgue measure on \( \mathbb{R}^2 \); and with respect to this base measure the density of \( P \) is \( +\infty \) on points on the unit circle. But that’s clearly less helpful to our users than \( 1/2\pi \), and \( 1/2\pi \) is a density for \( P \), just with respect to\(^2\) Lebesgue measure along the circle. So let’s wing it and go with that.

Now, a user looking at a density computed for a sample has no way to know which base measure we meant, so we have created an instance of the base measure problem. To see how it bites us, let \( f \) be a somewhat contrived non-isotropic scaling of \( \mathbb{R}^2 \), given by \( f(x, y) = (2x, 20y) \). What happens when we try to compose our uniform distribution \( P \) with our non-isotropic scaling \( f \)? The sampler is fine, but the conventional density rule (1) gives

\[
f_{P_{\text{bad}}}(x', y') = \frac{1}{40} p(f^{-1}(x', y')) = \begin{cases} 
\frac{1}{80\pi} & \text{when } (x'/2)^2 + (y'/20)^2 = 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Now we have definitely made a mistake. First of all, \( f_{P_{\text{bad}}} \) doesn’t integrate to 1, because the perimeter of the ellipse \((x'/2)^2 + (y'/20)^2 = 1\) is approximately 81.28, which is considerably less than \( 80\pi \approx 251.32 \). Second, as we can see by drawing a few samples and plotting them in Figure 1, the true distribution \( fP \) isn’t uniform! It’s clearly denser near \((0, 20)\) than \((2, 0)\).\(^3\)

\(^2\)Technically, the density function is the Radon–Nikodym derivative of our probability measure with respect to the base measure.

\(^3\)With respect to arc length as the base measure.
2.1 What went wrong?
The problem is that when computing the Jacobian correction induced by the change of variables $f$, we forgot that the base measure for the unit circle density wasn’t Lebesgue on $\mathbb{R}^2$. It’s actually Lebesgue along the circle, and $f$ changes arc length differently at different points. Indeed, locally near the point $(1, 0)$, the circle is the vertical line $x = 1$. Here $f$ acts to shift the line to $x = 2$, which has no effect on the length of line segments, and to stretch the line by 20 times in the $y$ direction. So the infinitesimal arc segment near $(1, 0)$ becomes twenty times larger under $f$, and the density $fp(2, 0)$ should be $1/40\pi$. Analogously, the arc segment near $(0, 1)$ is shifted to $(0, 20)$ and stretched by a factor of 2, so the density $fp(0, 20)$ should be $1/4\pi$.

3 HOW COMMON IS THIS PROBLEM?
While we chose to introduce the base measure problem on a continuous example, it actually occurs most often when transforming discrete distributions embedded in $\mathbb{R}^n$. The density function of, say, the Bernoulli distribution is $1/2$ at 0 or 1 and 0 elsewhere, but this is a density with respect to counting measure, not Lebesgue measure on $\mathbb{R}$. Therefore, when transforming this distribution with a bijector, we should not apply any Jacobian correction, because all bijections are counting-volume-preserving.

The same problem also shows up when dealing with distributions on symmetric matrices, such as the Wishart or LKJ distributions. The $k \times k$ symmetric matrices occupy a lower-dimensional sub-manifold of $\mathbb{R}^{k\times k}$, and the density of the Wishart distribution is with respect to Lebesgue measure on that submanifold rather than all of $\mathbb{R}^{k\times k}$. Bijections will in general change the volume under that measure differently than the volume under Lebesgue measure on $\mathbb{R}^{k\times k}$.

The same problem can also show up without any changes of variables at all. It suffices to compare the density of different points under the same distribution, or the density of the same point under different distributions. For instance, the Indian GPA problem discussed in e.g. [15] boils down to treating a density under counting measure as comparable to a density under Lebesgue measure on $\mathbb{R}$, even though the former represents an infinitely larger mass of probability.

4 WHAT’S THE RIGHT ANSWER?
The root of the base measure problem is that we didn’t want to compute with measures directly, but lost the base measure when representing probability distributions with densities. It’s not actually possible to infer the correct base measure from the data type representing the sample: a point on the unit circle in $\mathbb{R}^2$ is represented with two floating-point numbers, but using Lebesgue measure on $\mathbb{R}^2$ as the base is not helpful.

We propose standardizing on Hausdorff measure as a universal base measure for probability densities. Specifically, for $P$ a distribution on a $d$-dimensional manifold $M$ embedded in $\mathbb{R}^n$, let’s define the density to be with respect to the $d$-dimensional Hausdorff measure $H^d$. We propose $d$-Hausdorff measure because it provides a coherent definition for “$d$-dimensional volume” of a surface embedded in $\mathbb{R}^n$, even when the surface is curved. We scale $H^d$ to agree with the standard volume of straight surfaces, and thus with Lebesgue measure when $d = n$. We present just fixed-dimension distributions in the main text for simplicity, and briefly address mixed-dimension distributions in Appendix B.

\footnote{We propose standardizing at all in order to minimize the amount of information carried by the base measure, so it can be as implicit as possible. The only degree of freedom in Hausdorff measure is the dimension, and that is unavoidable.}
Fig. 2. Change of local 2-volume under a transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$. On the left, the small parallelogram with sides $\delta v_0$ and $\delta v_1$ is tangent to $M$ at $x$ and forms a volume element. On the right, $f$ takes $M$ to $fM$, $x$ to $f(x)$, and the volume element to the small parallelogram tangent to $fM$ at $f(x)$ with sides $\delta v'_0$ and $\delta v'_1$. When computing the density at $f(x)$ of the pushforward distribution $fP$ supported on $fM$, we have to correct for the change in volume of this parallelogram, but we have to disregard any stretching or compression $f$ may do in the perpendicular direction. Theorem 1 formalizes this for arbitrary dimensions.

4.1 Manifolds of dimension $d$

What do we need to represent about a sample $x$ besides its density $p(x)$ in order to compute in this scheme? Clearly we need to represent $d$; but, as the example of the unit circle shows, that’s not enough to transform densities correctly. Recall that our scaling transformation had a constant Jacobian when viewed as a function from $\mathbb{R}^2$ to $\mathbb{R}^2$, but we have to apply a position-dependent correction to account for its effect on the unit circle density. The additional thing we need to represent is the tangent space of $M$ at the sample point $x$. Notably, we do not need to computationally represent any non-local information about $M$ or $P$—just the tangent space at $x$, its dimensionality $d$, and the density $p(x)$.

Let us now formalize what we are actually trying to compute and how we can compute it.

**Definition 1.** Given a probability distribution $P$ over any space $\Omega$, and any function $f : \Omega \to \Omega'$, the pushforward distribution $fP$ is the measure

$$fP(S) = P(f^{-1}(S))$$

for all events $S \subset \Omega'$ for which $f^{-1}(S)$ is measurable in $\Omega$.

The sampler for $fP$ is just to apply $f$ to a sample from $P$. We seek a tractable density function for $fP$, and fortunately the following theorem gives it to us for nice $f$.

**Theorem 1.** Consider

- a probability distribution $P$ over a $d$-dimensional manifold $M$ in $\mathbb{R}^n$,
- with density $p$ with respect to $d$-dimensional Hausdorff measure $H^d$;
- a diffeomorphism $f : \mathbb{R}^n \to (f\mathbb{R}^n \subseteq \mathbb{R}^m)$; and
- a point $x \in M$.

Let $v_i$ be an arbitrary basis for the tangent space to $M$ at $x$, and pack it into a $d$-by-$n$ matrix $V$.

Then

1. The pushforward $fP$ is supported on the pushforward manifold $fM$ in $\mathbb{R}^m$.
2. The directional derivatives $v'_i = df(x + \varepsilon v_i)/d\varepsilon$ are a basis for the tangent space to $fM$ at $f(x)$.
   Let us pack them into a $d$-by-$m$ matrix $V'$.
3. The density $fp$ of $fP$ with respect to $H^d$ at $x' = f(x)$ is

$$fp(x') = p(x) \frac{\sqrt{\text{det}(VV')}}{\sqrt{\text{det}(V'V')}},$$

\[\text{det}(V'V')\]
Claims 1 and 2 are standard, and give us the rule for propagating a basis for the tangent space at \( x \) to \( x' \). Claim 3 amounts to a restatement in probability terms of standard notions of change of volume. The intuition is that the relevant volume is volume in the tangent space to \( M \) at \( x \) (see Figure 2). We choose as a volume element a small parallelepiped: centered at \( x \) and with sides \( \delta \mathbf{v}_i \) for \( \delta \to 0 \). The volume of this element is \( \delta^d \sqrt{\det(VV^T)} \). The function \( f \) transforms this volume element to the parallelepiped with center \( x' \) and sides \( \delta \mathbf{v}_i' \). Therefore, the volume correction we are looking for is \( \frac{\sqrt{\det(VV'^T)}}{\sqrt{\det(VV^T)}} \). We reproduce a formal proof of Claim 3 in Appendix A.

Observe that the inputs required for Theorem 1 are all local: the point \( x \), a basis \( \mathbf{v}_i \) for the tangent space to \( M \) at \( x \), the density \( p(x) \) of \( P \) at \( x \), and the ability to compute the value and directional derivatives of \( f \) at \( x \). We do not need any non-local information about \( P, M \), or \( f \), except that \( f \) is invertible. Observe also that the outputs given by Theorem 1 give enough information about \( fP \) at \( x' = f(x) \) to apply the same theorem again with a new diffeomorphism \( g: \mathbb{R}^m \to (g\mathbb{R}^m \subseteq \mathbb{R}^k) \). We invite the interested reader to verify that pushing distributions forward by Theorem 1 commutes with function composition, that is, the double pushforward \( g(fP) \) computes the same densities as a single pushforward by the composition \( (g \circ f)P \). This equivalence lends itself to modularity in software, permitting, for example, the distribution \( g(fP) \) to be represented taking advantage of special structure in \( f \) or \( g \), without requiring the user to derive that special structure for \( g \circ f \).

### 4.2 Comparisons

When doing MCMC or SMC, we find ourselves computing densities of different points, or densities of the same point under different distributions. In MCMC this happens when computing the acceptance ratio, and in SMC this happens when summing the particle weights, and during resampling. In order to get these comparisons right, in addition to the densities themselves, we need to know the dimension \( d \) of the Hausdorff measure with respect to which our numbers are densities. We compare dimensionality-major: any finite density with respect to a higher-dimensional measure is always infinitely smaller than any non-zero density with respect to a lower-dimensional one.

### 5 HOW DO WE FIX THE SOFTWARE?

We know from Section 4 that to compute the density of a pushforward \( fP \) at a point \( x' \) with respect to \( d \)-Hausdorff measure it suffices to compute

- The preimage point \( x = f^{-1}(x') \);
- The density \( p(x) \) of \( P \) at \( x \);
- A basis \( \mathbf{v}_i \) for the tangent space to the support \( M \) of \( P \) at \( x \);
- The directional derivatives \( \mathbf{v}_i' \) of \( f \) at \( x \) in directions \( \mathbf{v}_i \), forming a basis for the tangent space at \( x' \) to the support of \( fP \); and
- The correction term \( \frac{\sqrt{\det(VV^T)}}{\sqrt{\det(VV'^T)}} \) from (2).

Fortunately, there are several special cases we can take advantage of to save work, and to recover the performance of the conventional architecture for cases where it gives the correct answer:

- If our distribution \( P \) is actually discrete, the tangent space is 0-dimensional, the matrix \( VV^T \) is the 0x0 matrix, and its determinant is vacuously 1.
- If \( P \) is supported on all of \( \mathbb{R}^n \), the tangent space is also \( \mathbb{R}^n \), and we may take \( \mathbf{v}_i \) to be the standard basis, so the \( \sqrt{\det(VV^T)} \) term is unity and we need not explicitly compute it.
- If in addition \( f \) maps \( \mathbb{R}^n \) to \( \mathbb{R}^n \) and not to \( \mathbb{R}^m \) for \( m > n \), then the matrix \( V' \) is the Jacobian of \( f \) at \( x \) and itself has a determinant. We can save a matrix multiply and a square root

---

5Since the \( \mathbf{v}_i \) are vectors in the tangent space to \( M \) at \( x \), it would be more natural to let \( x \) be the vertex of the parallelepiped. However, we want to end up with a neighborhood of \( x \), so we shift the parallelepiped to have \( x \) in the center.
because in this case $\sqrt{\det(V'V^T)} = |\det V'|$. This is where we recover the conventional Jacobian-determinant correction to probability densities.

- Distributions on structured-sparse matrices, such as lower-triangular matrices, are supported on manifolds with axis-aligned tangent spaces. In such a case, we need only keep track of a mask representing the present dimensions, taking the basis $v_i$ as the subset of the standard basis for that subspace. The $\sqrt{\det(VV^T)}$ term again disappears, though if $f$ does not preserve the sparsity, the $\sqrt{\det(V'V^T)}$ term may need to be computed.

- If in addition $f$ is a univariate transformation applied coordinate-wise, the sparse structure will be preserved, and the correction will just be the product of partial derivatives of $f$,$\sqrt{\det(V'V^T)} = \prod_i \frac{\partial}{\partial x_i} f(x)$, with $i$ here ranging over the dimensions present in the manifold but not the others.

- Finally, simplices and symmetric matrices may also merit special treatment, because the basis $v_i$ can again be implicit.

This list of special cases suggests ad-hoc polymorphism as a representation strategy. Specifically, our system probably already has a Bijector class (hierarchy) for representing transformations $f$. We can add a TangentSpace class hierarchy for the tangent space to $\mathcal{M}$ at $x$. To take advantage of special cases based both on the tangent space and the bijector, we can define the density correction function with two-argument dispatch, or a Visitor-like pattern [5] to emulate it in languages where only single-argument dispatch is available.

When working with distributions on high-dimensional spaces, or with large batches of distributions, we expect the cost of the dispatch to be small compared with the cost of the linear algebra the dispatch is trying to avoid. Furthermore, on a tracing platform like JAX [2], TensorFlow [1], or TorchScript [4], the dispatch cost is only paid once during tracing, avoiding linear algebra that would have happened potentially many times during execution.

6 RELATED WORK

Broadly, all general-purpose probabilistic programming systems are related to the present work, in that they must either produce incorrect results when the base measure problem arises [11, 14], avoid the problem entirely, or somehow solve it. Two specific systems are worth mentioning:

BLOG discusses addressing the base measure problem (not by that name) in [15]. Relative to the present work, that treatment is simpler in two ways: they only discuss scalar random variables, where every tangent space is either $\mathbb{R}$ or the zero vector space; and they do not explicitly discuss transformations of random variables.

Some cases of the base measure problem are addressed by Hakaru’s disintegration transform [9], which generalizes density and also needs correct base measures. Hakaru defines a restricted language of base measures with respect to which it can symbolically compute (unnormalized) disintegrations of s-finite measures. As this restricted set includes discrete-continuous mixtures, it is sufficient to correctly handle the Indian GPA problem; but for the present example of the unit circle, it would need a richer notion of constraints and their tangent or normal spaces [8].

7 CONCLUSION

We have named the Base Measure Problem and provided a solution to it. Implementing the solution in probabilistic programming systems should cause negligible loss of performance for cases that were already correctly handled, and expand the set of models in which the system can compute correct probability densities. Implementation does carry a code complexity cost, but that cost is minimized by using two-argument dispatch, or emulating it with a Visitor pattern. Despite correctly accounting for measures, no non-local information is required.
8 ACKNOWLEDGMENTS

The authors thank Srinivas Vasudevan, Wynn Vonnegut, and Praveen Narayanan for enlightening discussion.

REFERENCES

[1] Martín Abadi, Ashish Agarwal, Paul Barham, Eugene Brevdo, Zhifeng Chen, Craig Citro, Greg S. Corrado, Andy Davis, Jeffrey Dean, Matthieu Devin, Sanjay Ghemawat, Ian Goodfellow, Andrew Harp, Geoffrey Irving, Michael Isard, Yangqing Jia, Rafal Jozefowicz, Lukasz Kaiser, Manjunath Kudlur, Josh Levenberg, Dandelion Mané, Rajat Monga, Sherry Moore, Derek Murray, Chris Olah, Mike Schuster, Jonathon Shlens, Benoit Steiner, Ilya Sutskever, Kunal Talwar, Paul Tucker, Vincent Vanhoucke, Vijay Vasudevan, Fernanda Viégas, Oriol Vinyals, Pete Warden, Martin Wattenberg, Martin Wicke, Yuan Yu, and Xiaoqiang Zheng. 2015. TensorFlow: Large-Scale Machine Learning on Heterogeneous Systems. https://www.tensorflow.org/ Software available from tensorflow.org.

[2] James Bradbury, Roy Frostig, Peter Hawkins, Matthew Johnson, Christopher Leary, Dougal Maclaurin, and Skye Wanderman-Milne. 2017–2019. JAX. https://github.com/google/jax Specifically the vmap functionality.

[3] Bob Carpenter, Andrew Gelman, Matthew D Hoffman, Daniel Lee, Ben Goodrich, Michael Betancourt, Marcus Brubaker, Jiqiang Guo, Peter Li, and Allen Riddell. 2017. Stan: A Probabilistic Programming Language. Journal of statistical software 76, 1 (2017).

[4] PyTorch Contributors. 2018. Torch script. https://pytorch.org/docs/master/jit.html

[5] Erich Gamma, Richard Helm, Ralph Johnson, and John Vlissides. 1995. Design Patterns: Elements of Reusable Object-Oriented Software. Addison Wesley.

[6] Lei Li and Stuart J Russell. 2013. The BLOG Language Reference. tech. rep., Technical Report UCB/EECS-2013–51 (2013).

[7] Vikash Mansinghka, Daniel Selsam, and Yura Perov. 2014. Venture: a Higher-Order Probabilistic Programming Platform with Programmable Inference. arXiv preprint arXiv:1404.0099 (2014).

[8] Praveen Narayanan. 2020. Personal communication.

[9] PyTorch Contributors. 2018. Torch script.

[10] The Venture Team. 2017. Venture suffers from the base measure problem. Personal communication.

[11] The TFP Team. 2018–2019. TensorFlow Probability. https://github.com/tensorflow/probability

[12] David Tolpin, Jan-Willem van de Meent, Hongseok Yang, and Frank Wood. 2016. Design and Implementation of Probabilistic Programming Language Anglican. In Proceedings of the 28th Symposium on the Implementation and Application of Functional Programming Languages (IFL 2016). Association for Computing Machinery, New York, NY, USA, Article 6, 12 pages. https://doi.org/10.1145/3064899.3064910

[13] Richard Warfield. 2020. TensorFlow Probability Issue #761: Incorrect probabilities after transforming a QuantizedDistribution. https://github.com/tensorflow/probability/issues/761

[14] Yi Wu, Siddharth Srivastava, Nicholas Hay, Simon S. Du, and Stuart Russell. 2018. Discrete-Continuous Mixtures in Probabilistic Programming: Generalized Semantics and Inference Algorithms. In International Conference on Machine Learning. https://arxiv.org/pdf/1806.02027.pdf

A PROOF OF THEOREM 1, CLAIM 3

To prove Theorem 1, Claim 3 formally, we go through the measures $P$ and $fP$, starting with the following lemma. The only technical trick is to enlarge our parallelepipeds to open sets in $\mathbb{R}^n$ and $\mathbb{R}^m$, so that measuring them with $P$ and $fP$ captures all the mass near $x$ and $x'$, respectively, despite any curvature of $M$ or $fM$. Then we will note that in the limit the only thing we care about is the projections of those open sets back to the respective tangent spaces, which are the parallelepipeds we started with.

**Lemma 1.** Let $P$, $M$, $p$, $x$, $v_i$, and $V$ be as in the statement of Theorem 1. Let $G_V$ be the $d$-parallelepiped with center $x$ and sides $v_i$, and let $G_{V+}$ be any $n$-parallelepiped centered at $x$ with section $G_{V}$. Let $G_V^\delta$ and $G_{V+}^\delta$ be $G_V$ and $G_{V+}$, respectively, scaled by $\delta$ about $x$. Then the density $p$ of $P$ with respect to
d-dimensional Hausdorff measure is given by

\[ p(x) = \frac{1}{\sqrt{\det(VV^T)}} \lim_{\delta \to 0} P(G_{V,+}^\delta) / \delta^d. \]

**Proof of Lemma.** The definition of \( p \) being a density for \( P \) with respect to \( H^d \) is that for all measurable subsets \( B \subset \mathbb{R}^n \),

\[ P(B) = \int_B p \, dH^d. \]

\( P \) is only supported on \( M \), so we may take the integral on the right to be over the intersection \( B \cap M \). We choose for \( B \) the neighborhood \( G_{V,+}^\delta \).

As \( G_{V,+}^\delta \) becomes sufficiently small, we may assume \( p \) is constant on \( G_{V,+}^\delta \cap M \), so

\[ P(G_{V,+}^\delta) \approx p(x)H^d(G_{V,+}^\delta \cap M). \]

As \( \delta \to 0 \), \( M \) near \( x \) approaches its tangent space, so \( G_{V,+}^\delta \cap M \) becomes \( G_V^\delta \). The right-hand term becomes the \( d \)-dimensional Hausdorff measure of \( G_V^\delta \), which given our choice of scaling for the Hausdorff measure is just the volume thereof, namely \( \delta^d \sqrt{\det(VV^T)} \). \( \square \)

**Proof of Claim 3.** Let \( G_{V'_+} \) be the \( d \)-parallelepiped in \( \mathbb{R}^m \) with center \( x' \) and sides \( v'_i \). Let \( G_{V'_+} \) be any \( m \)-parallelepiped in \( \mathbb{R}^m \) with section \( G_{V'_+} \) centered at \( x' \). Let \( G_{V'_+}^\delta \) be \( G_{V'_+} \) scaled by \( \delta \) about \( x' \). By the lemma, we have

\[ fp(x') = \frac{1}{\sqrt{\det(V'V'^T)}} \lim_{\delta \to 0} fP(G_{V'_+}^\delta) / \delta^d. \]

The preimage \( f^{-1}(G_{V'_+}^\delta) \) approaches an \( n \)-parallelepiped centered at \( x \) with section \( G_V^\delta \). This is because the basis vectors \( v'_i \) forming \( V' \) are the directional derivatives of \( f \) in the directions given by the basis vectors \( v_i \) forming \( V \). In directions normal to \( M \), we just need the derivatives of \( f \) to be finite. We may thus apply the lemma again to write

\[ \lim_{\delta \to 0} fP(G_{V'_+}^\delta) / \delta^d = \lim_{\delta \to 0} f(\delta^{-1}G_{V'_+}^\delta) / \delta^d = \sqrt{\det(VV^T)}p(x), \]

giving the desired result. \( \square \)

**B Mixed-Dimension Manifolds**

Theorem 1 has given us a formula for transforming densities of probability distributions on \( d \)-dimensional manifolds embedded in \( \mathbb{R}^n \), and justifies standardizing on \( d \)-dimensional Hausdorff measure as a universal base measure for densities of such distributions.

The results generalize directly to finite unions of manifolds, of potentially different dimensions. Namely, for \( P \) a sum of measures \( P_j \) on manifolds \( M_j \) of dimensions \( d_j \), we suggest standardizing on the sum of corresponding Hausdorff measures \( H^{d_j} \) on those manifolds as the base measure. Then the density \( p(x) \) representing \( P \) is given by the density \( p_j(x) \) of the \( P_j \) corresponding to the lowest-dimensional manifold \( M_j \) that \( x \) is on.

If the manifolds intersect, and \( x \) is on more than one of them, any densities \( p_j' \) corresponding to measures on higher-dimensional manifolds become zero with respect to \( H^{d_j} \). If \( x \) occurs on multiple supporting manifolds of minimal dimension, then we conjecture without proof that the right thing (or at least a consistent thing) is to sum those densities, and again interpret the result with respect to \( d_j \)-dimensional Hausdorff measure. It might also be consistent to average those densities, and interpret the result with respect to the sum of \( k \) copies of \( H^{d_j} \), where \( k \) is the number of \( d_j \)-dimensional manifolds that intersect at \( x \).