Properties of superconductor - Luttinger liquid hybrid systems

Rosario Fazio(1,5), F.W.J. Hekking(2), A.A. Odintsov(3), and R. Raimondi(4,5)

(1) Istituto di Fisica, Universit`a di Catania, viale A. Doria 6, 95129 Catania, Italy
(2) Theoretische Physik III, Ruhr-Universit`at Bochum, 44780 Bochum, Germany
(3) Faculteit der Technische Natuurkunde, TU Delft, 2628 CJ Delft, The Netherlands
(4) Dip. di Fisica "E. Amaldi", Universit`a di Roma3, Via della Vasca Navale 84, 00146 Roma, Italy
(5) Istituto Nazionale di Fisica della Materia (INFM), Italy

(March 24, 2022)

In this paper we review some recent results concerning the physics of superconductor - Luttinger liquid proximity systems. We discuss both equilibrium (the pair amplitude, Josephson current, and the local density of states) and nonequilibrium (the subgap current) properties.

I. INTRODUCTION

The properties of a normal metal (N) which is in good electric contact with a superconductor (S) are strongly modified, a phenomenon known as the proximity effect [1]. This effect is due to the presence in N of Cooper pairs leaking from S, thereby giving rise to a nonvanishing local pair amplitude. The microscopic origin for charge transfer across the N-S interface is the phenomenon of Andreev reflection [2]. The distance over which the presence of the superconductor is felt in N is determined by the length $\xi_N$ over which the two electrons forming the pair remain correlated. This length decreases with increasing temperature $T$. In particular, for clean normal metals, $\xi_N = v_F/T$, where $v_F$ is the Fermi velocity. The proximity effect manifests itself in various ways. An example is the Josephson effect, occurring in S-N-S sandwiches as long as the thickness of the N-layer does not exceed $\xi_N$ [3]. Another example is the local single particle density of states (DOS) of the metal, which acquires an energy dependence similar to the well-known BCS-DOS of a superconductor up to distances of the order of $\xi_N$ away from the N-S interface [4].

Due to the recent development of superconductor-semiconductor (S-Sc) integration technology, a revived interest arose in the properties of clean proximity systems. Present-day high-quality Sc heterostructures combine a number of attractive low-temperature electronic properties. The elastic mean free path can be as long as the thickness of the N-layer does not exceed $\xi_N$ [4]. In typical low-density systems, the Fermi wavelength $\lambda_F$ is of the order of 50nm. These lengths are much larger than the corresponding ones in an ordinary metal. In addition, the dephasing length $L_\phi$, over which the phase coherence of a single electron is maintained, can easily be of the order of 40nm. Thus, quantum ballistic electron propagation dominates in small-scale Sc systems. The interplay between phase-coherent electron propagation in Sc and macroscopic phase coherence in S gives rise to interesting new physics [5,6].

During the past years, an increasing interest developed in the effects of electron-electron interactions on the properties of low-density Sc nanostructures. The key point is that the Coulomb energy can become comparable to the kinetic energy, i.e., the parameter $e^2/v_F \sim \lambda_F/a_B$, where $e$ is the electron charge and $a_B$ is the Bohr radius, is no longer small. Moreover, screening becomes less effective as the system dimensionality is decreased. As a result a non-perturbative, microscopic treatment of interactions is required. For one-dimensional (1D) systems this can be done in the framework of the Luttinger model [7]. In a 1D interacting electron system, also referred to as a Luttinger liquid (LL), there are no fermionic quasiparticle excitations. Instead, the low energy excitations of the system consist of independent long-wavelength oscillations of the charge and spin density which propagate with different velocities. The properties of such a system therefore are strikingly different from those of a non-interacting 1D system. The Luttinger model is believed to be relevant for a description of transport in a number of physical systems like Sc quantum wires and edge states in the quantum Hall effect [8]. We should also mention the recent rapid advances in controlled fabrication of single-wall nanotubes [9]. Luttinger liquid behavior is expected in metallic nanotubes with two gapless 1D modes of excitations [10].

At present, relatively little is known about the influence of electron-electron interactions on the proximity effect in clean mesoscopic N-S systems. In view of the above one may conclude that S-Sc heterostructures are good candidates to study such effects. Of particular interest are 1D quantum wires, connected to a superconductor (S-LL systems). Various techniques are available nowadays to confine electrons in a semiconductor to a long and narrow channel. In combination with state-of-the-art S-Sc integration technology, this will make systematic studies of interaction effects in S-LL systems feasible in the near future. In this paper we review various properties of S-LL proximity systems. The paper is organized as follows: In Section II we define the spin-1/2 Luttinger model. In Section [11] we discuss two possi-
ble ways to couple the LL to a superconductor: via a highly transmissive and via a tunneling interface. The remainder of the review will be divided into two parts. In Sections II, III, and VII we discuss equilibrium properties of S-LL heterostructures: the pair amplitude, the Josephson effect, and the local density of states. In Sections IV, V, and VI we will focus on transport phenomena and calculate the subgap current through a S-LL interface.

II. THE SPIN-1/2 LUTTINGER LIQUID

In this Section we set the notation which is needed in the rest of the paper. For a detailed discussion of Luttinger liquids we refer to existing reviews on the topic [10].

The long-wavelength Hamiltonian of a 1D interacting electron system of length $L$ can be expressed as that of a harmonic fluid for the charge ($j = \rho$) and spin ($j = \sigma$) degrees of freedom

$$\hat{H}_L = \int \frac{dx}{\pi} \sum_{j=\rho,\sigma} v_j \left[ \frac{g_j}{2} (\nabla \phi_j)^2 + \frac{2}{g_j} (\nabla \theta_j)^2 \right].$$  

The parameters $g_j$ are related to the interaction strength ($g_j = 2$ for non-interacting electrons), and $v_j = 2v_F/g_j$ are the velocities of spin and charge excitations. The commutation relation between the Bose fields $\theta(x)$ and $\phi(x)$ for each spin sector (spin up $s = +1$ and down $s = -1$) is $[\phi_s(x), \theta_{s'}(x')] = (i\pi/2) \text{sign}(x'-x) \delta_{s,s'}$, where $\phi_s = \phi_\rho + s\phi_\sigma$, and $\theta_s = \theta_\rho + s\theta_\sigma$.

The parameters $g_j$ can be determined once one defines an appropriate microscopic Hamiltonian. For a Sc quantum wire with spin-independent interactions, one may take $g_\rho = 2$ and $g_\sigma = 2/\sqrt{1 + 2V_0/v_F}$, where $V_0$ is the zero-momentum Fourier component of the interaction potential [14]. Spin and charge excitations propagate with different velocities (spin-charge separation). In general there will be additional nonlinear terms appearing in Eq. (1) due to backscattering or Umklapp processes. Usually, in quantum wires away from half filling and in the ballistic regime these terms can be ignored; therefore we will not discuss them in the following. The electron field operator $\hat{\Psi}$ is expressed in terms of the boson field in terms of the spin and charge degrees of freedom

$$\hat{\Psi}_s(x) \sim \sqrt{\rho_0} \sum_{\delta = \pm} e^{i(k_F x + \xi)} e^{i(\theta_s - \delta \phi_s)}$$  

where $k_F$ is the Fermi wave vector and $\rho_0 = k_F/\pi$ the electron density.

III. SUPERCONDUCTOR-LUTTINGER LIQUID INTERFACES

A superconductor can contact a quantum wire at the end (edge contact) or at some internal point or segment of the wire (lateral contact). Both edge and lateral contacting have been already implemented in experiments with S-Sc structures [13]. The quality of S-LL interfaces can be characterized by the transparency of the barrier at the interface. We will consider below edge contacts with both high and poor transparency of the barriers as well as poorly transmitting lateral point contacts.

Perfectly transmitting interfaces - Maslov et al. [14] and Takane and Koyama [15] recently developed a bosonization scheme to treat clean S-LL interfaces. Following Ref. [14] we consider the case of two superconductors, kept at a phase difference $\chi$, and adiabatically connected to a quantum wire of length $L$ (the results for a single interface can be obtained by taking the limit $L \to \infty$). The bosonization scheme can be carried out once the boundary conditions for the right and left moving fields at the S-LL interface are determined.

The mode expansions for the fields $\theta_j$ and $\phi_j (j = \rho, \sigma)$, such that the Fermi operators (2) satisfy the proper boundary conditions at the interface, are

$$\theta_{\rho}(x) = \varphi_{\rho} + \sqrt{\frac{g_{\rho}}{2}} \sum_{q > 0} \gamma_{q}^+ \hat{b}_{\rho,q},$$  

$$\theta_{\sigma}(x) = \frac{\pi x}{4L} M_{\sigma} + \sqrt{\frac{g_{\sigma}}{2}} \sum_{q > 0} \gamma_{q}^+ \hat{b}_{\sigma,q},$$  

$$\phi_{\rho}(x) = \frac{\pi x}{4L} (1 + \frac{J_{\rho}}{2} + \frac{\chi}{\pi}) + \sqrt{\frac{2}{g_{\rho}}} \sum_{q > 0} \gamma_{q}^- \hat{b}_{\rho,q},$$  

$$\phi_{\sigma}(x) = \varphi_{\sigma} + \sqrt{\frac{2}{g_{\sigma}}} \sum_{q > 0} \gamma_{q}^+ \hat{b}_{\sigma,q}.$$  

where $q = n\pi L / 2$, $\gamma_{q}^+ = (\pi/2qL)^{1/2} \cos(qx)$, and $\gamma_{q}^- = i(\pi/2qL)^{1/2} \sin(qx)$. The system is characterized by the two topological numbers $J_{\rho}$ and $M_{\sigma}$; the topological numbers $N_{\sigma} \equiv (M_{\sigma} \pm J_{\rho})/2$ must be odd integers [16].

Substituting the mode expansions (3) – (6) into the phase Hamiltonian (1), we obtain a Hamiltonian of the form

$$H = \frac{\pi}{4L} \left[ \frac{g_{\rho} v_{\rho}}{2} \left( \frac{J_{\rho}}{2} + \frac{\chi}{\pi} + 1 \right)^2 + 2v_{\sigma} \left( \frac{M_{\sigma}}{2} \right)^2 \right]$$  

$$+ \sum_{j=\rho,\sigma} \sum_{q > 0} v_j q \hat{b}_{j,q} \hat{b}_{j,q}.$$  

Takane and Koyama [17] extended this bosonization scheme to include the energy dependence of the phase shift related to Andreev reflection, thereby showing that it may be important if a sufficiently strong potential barrier is placed at the S-LL interface. In the following we will neglect this energy dependence.

Poorly transmitting interfaces - If the quantum wire is weakly coupled to a superconductor at $x = 0$, the system can be treated in terms of the tunnel Hamiltonian
formalism. The Hamiltonian of the whole system \[ H = H_S + H_L + H_T, \]
contains the BCS-Hamiltonian \( H_S \)
describing the bulk superconductor, the Hamiltonian \( H_L \)
of the quantum wire, and the tunnel Hamiltonian,
\[ H_T = \sum_s t_0 \hat{\Psi}_{S,s}^\dagger(x = 0)\hat{\Psi}_{L,s}(x = 0) + \text{(h.c.)}. \]  
(8)
The tunnel matrix elements \( t_0 \) can be related to the tunnel conductances \( G_T \) of the junctions, \( G_T = 4\pi e^2 N_L(0)N_S(0)|t_0|^2 \), where \( N_L(0) = 1/\pi v_F, \) and \( N_S(0) \) is the normal state density of states at the Fermi level of the superconductor.

If two superconductors are attached to the ends of a quantum wire one should use the recently developed bosonization technique for finite 1D systems with open boundaries \[13\]. The presence of lateral tunnel contacts does not impose any additional boundary conditions on the Fermi fields. We will neglect possible inhomogeneities of the quantum wire near the contacts, which might be the source of electron backscattering. For a discussion of possible modifications due to the presence of a barrier potential, see Ref. \[19\].

**IV. PAIR AMPLITUDE**

In this section we will evaluate the pair amplitude induced into a Luttinger liquid which is connected to a superconductor at \( x = 0 \). We calculate the pair amplitude at a distance \( x \) from the contact. The pair amplitude is defined as the anomalous time-ordered expectation value \( \Xi(x, \tau) \equiv -\langle T_{\tau} \psi_\uparrow(x, \tau^+)\psi_\downarrow(x, \tau) \rangle \), where \( \tau^+ \) tends to \( \tau \) from above. Throughout this section, we will be interested in the smooth spatial variation of \( \Xi \) (we ignore spatial variations on the scale of \( \Lambda_F \)). We will evaluate \( \Xi \) both for a highly transmissive and for a tunneling interface.

**Perfectly transmitting interfaces** - The pair amplitude for an adiabatic interface between S and LL has been obtained by Maslov et al. \[13\]: \( \Xi(x) = -2\rho_0 \langle \exp 2i\phi_\rho \rangle \cos 2\theta_\rho \). Since Maslov’s bosonization procedure has been developed for Andreev scattering at energies much smaller than the superconducting gap \( \Delta \), we shall evaluate \( \Xi \) at large distances from the interface, i.e., with a short wavelength cut-off \( \alpha \) such that \( x \gg \alpha \sim \xi_S \equiv v_F/\Delta \). Performing the averages with respect to \( \| \), we obtain
\[ \Xi(x) = -2\rho_0 \left[ \frac{\pi\alpha/\beta v_F}{\sinh 2\pi x/\beta v_F} \right]^{1/2} \left[ \frac{\pi\alpha/\beta v_F}{\sinh 2\pi x/\beta v_F} \right]^{1/g_\rho}, \]  
(9)
where \( \beta = 1/T \). The result consists of a product of a spin (first term in brackets) and a charge contribution (last term in brackets); only the latter is sensitive to interactions.

At zero temperature, the pair amplitude for non-interacting electrons decays slowly away from the junction: \( \Xi_{T=0}(x) \sim 1/x \). Nothing prevents the superconducting correlations from being present arbitrarily deep in the quantum wire. In the presence of repulsive interactions, the two electrons can be scattered out of their time-reversed state and therefore \( \Xi \) decays faster with increasing distance \( x \) from the interface,
\[ \Xi_{T=0} \sim \rho_0 \left[ \frac{\alpha}{2\pi} \right]^{1/2+1/g_\rho}. \]  
(10)

At finite temperatures, the coherence length in the LL becomes finite, and, correspondingly, the pair amplitude should decay faster. This can be seen by calculating the ratio \( \Xi_T/\Xi_{T=0} \). At low temperatures \( T \ll v_F/x \), this ratio behaves as:
\[ \frac{\Xi_T(x)}{\Xi_{T=0}(x)} \simeq 1 - \frac{1}{6} \left( \frac{\pi x T}{v_F} \right)^2, \]  
(11)
independent of the interaction strength. At higher temperatures \( T \gg v_F/x \), the suppression will be exponential:
\[ \frac{\Xi_T(x)}{\Xi_{T=0}(x)} \simeq \left( g_\rho/2 \right)^{1/g_\rho} \left( \frac{2\pi\alpha T}{v_F} \right)^{\frac{1}{2}+\frac{1}{g_\rho}} e^{-2\pi x T/v_F}. \]  
(12)

Here, interactions determine the pre-exponential temperature dependence.

**Poorly transmitting interfaces** - In order to evaluate the pair amplitude for a tunnel interface between S and LL, we use standard perturbation theory in the tunnel Hamiltonian \( H_T \), Eq. \[3\]. In the tunneling regime the usual bosonization scheme can be used, with short wavelength cut-off \( \alpha \sim \Lambda_F \). The lowest order non-vanishing contribution to the pair potential in the LL is given by the following expression
\[ \Xi(x, \tau) = t_0^2 \int_0^\beta d\tau_1 d\tau_2 \Pi_{+, -, -, +}(x, \tau^+; x, \tau; 0, \tau_1; 0, \tau_2) \times F_+, -, 0, \tau_2; 0, \tau_1. \]  
(13)
Here \( F_+, -, 0, \tau_2; 0, \tau_1 \) is the usual anomalous Green’s function for the superconductor, and we introduced the four-point correlator \( \Pi_{1, 2, 3, 4}(1, 2, 3, 4) \equiv \langle T_{\tau_2} \psi_{s1}(1)\psi_{s2}(2)\psi_{s3}^\dagger(3)\psi_{s4}^\dagger(4) \rangle \). The relevant process consists in the tunneling of two electrons from S into the LL. This process is of second order in the tunneling. First, an electron tunnels into the LL, leaving behind a quasiparticle excitation in S. Then, the second electron tunnels from S into LL, annihilating the quasiparticle in S. Thus, two time scales play a role in this tunneling process: the lifetime \( |\tau_1 - \tau_2| \sim 1/\Delta \) of the intermediate state with one quasiparticle in S, and the time \( |\tau - \tau_1|, |\tau - \tau_2| \sim |x|/v_F \) during which each electron propagates in the LL to the
position $x$. The behavior of the pair amplitude will depend on the relative magnitude $|x|\Delta/v_F$ of these time scales for propagation and tunneling.

**Pair amplitude close to the junction** - For distances close to the junction, $\xi_S \gg |x| \gg \alpha \sim \lambda_F$, the time needed for the pair to traverse the LL is negligible compared to the quasiparticle lifetime $1/\Delta$. Hence, the main contribution to $\Xi$ stems from electrons that propagate fast and independently through the LL. In this case, we can approximate the four-point correlator by

$$F_{\text{G}}(x, g_\rho),$$

where

$$\Xi^{0} \equiv 2\rho_0 \frac{G_T}{(4\pi^2/\pi)} \frac{\alpha}{v_F} \sinh^{-1}(\omega_D/\Delta).$$

Expressions for the function $F_{\text{G}}(x, g_\rho)$ are given in Appendix A.

In the noninteracting case $g_\rho = 2$ the pair amplitude becomes space-independent at zero temperature, $\Xi_{g_\rho=2}(x) \to \Xi^0$. This reflects the fact the time-reversed electrons from $S$ reach points in the LL within a distance $\sim \xi_S$ from the junction instantaneously. In the presence of repulsive interactions, propagation through the LL becomes faster, $v_\rho > v_F$, and one naively would expect the pair amplitude to remain space independent. However, for $g_\rho < 2$ we find at zero temperature,

$$\Xi^{0}_{T=0}(x) = \Xi^{0} \left[ \frac{\alpha}{x} \right]^{1/g_\rho+g_\rho/4-1} F_{\text{G}}^2(g_\rho).$$

Due to repulsive interactions, the pair amplitude becomes $x$-dependent: it decays algebraically away from the junction. This suppression of the pair amplitude is related to the fact that the effective tunneling amplitude of Cooper pairs is renormalized at low energies by the interactions, similar to the tunneling amplitude of single electrons into a LL [11].

At finite temperatures, we therefore expect a competition between two effects: (i) enhancement of the tunneling due to thermal fluctuations leading to an enhancement of the pair amplitude as a function of temperature; (ii) decreasing coherence length, leading to a suppression of the pair amplitude with temperature. As a result of the above competition, the pair amplitude shows a maximum as a function of temperature if $g_\rho < 2$. Analytical results can be obtained in the case of weak interactions $(1 - g_\rho/2)\pi x/v_F \ll 1$. At low temperatures $\beta v_F \gg x$

$$\Xi^{T=0}_{\Xi} \approx 1 + (2 - g_\rho) \frac{x}{\beta v_F} - 2 \frac{\Gamma(\nu+1/2)}{\sqrt{\pi} \Gamma(\nu+1)} \left(\frac{x}{\beta v_F}\right)^{2\nu}$$

with $\nu = 1/4g_\rho + g_\rho/16 + 1/4$. Note that the first term, describing the thermal enhancement of the pair amplitude, vanishes in the noninteracting case $g_\rho = 2$, as it should.

**Pair amplitude away from the junction** - If we are interested in the pair amplitude at large distances from the interface, $|x| \gg \xi_S$, we can neglect the time $1/\Delta$ spent in the virtual state in comparison with the long time $x/v_F$ needed to traverse the LL. We thus approximate the anomalous Green’s function in $S$ with the help of a $\delta$-function in time as

$$F_{\text{G}}(0, \tau; v_F, 0, T) = N_S(0)\Delta \sinh^{-1}(\omega_D/\Delta),$$

where the Debye-frequency $\omega_D$ is a high energy cut-off in $S$. We thus arrive at

$$\Xi(x) = \Xi^{0} \left[ \frac{\alpha}{x} \right]^{1/g_\rho+g_\rho/4-1} F_{\text{G}}^2(g_\rho),$$

where

$$\Xi^{0} \equiv 2\rho_0 \frac{G_T}{(4\pi^2/\pi)} \frac{\alpha}{v_F} \sinh^{-1}(\omega_D/\Delta).$$

The function $\Xi_C(x, \tau)$ is defined in Appendix B. In the noninteracting case $g_\rho = 2$, the propagator $\Xi_C$ can be integrated analytically. In the zero temperature limit $\beta \to \infty$, the pair amplitude is given by

$$\Xi_{g_\rho=2}(x) = 2\rho_0 \frac{G_T}{(4\pi^2/\pi)} \frac{\alpha}{x}.$$

We see that $\Xi$ decays slowly away from the junction. With increasing temperature, the distance $v_F/T$ over which two electrons maintain their relative phase coherence decreases, and in the limit $\beta v_F/|x| \ll 1$ the pair amplitude decays exponentially with $x$:

$$\Xi_{g_\rho=2}(x) \sim 4\rho_0 \frac{G_T}{(4\pi^2/\pi)} \frac{2\pi\alpha}{\beta v_F} \exp(-2\pi x/\beta v_F).$$

We now turn to the case of repulsively interacting electrons, $g_\rho < 2$. In the zero temperature limit we obtain

$$\Xi(x) = 2\rho_0 \frac{G_T}{(4\pi^2/\pi)} \left[ \frac{\alpha}{x} \right]^{2/g_\rho} F_{\text{C}}(g_\rho),$$

where $F_{\text{C}}(g_\rho)$ is given in Appendix B. We find a power law decay of $\Xi$ with $x$: compared to the noninteracting case, Eq. (21), the pair amplitude decays faster away from the junction. In the high temperature limit $\beta v_F/|x| \ll 1$, the decay becomes exponential,
\[ \Xi(x) \approx 4\rho_0 \frac{G_T}{(4e^2/\pi)^2} \left[ \frac{2\pi\alpha}{\beta v_F} \right]^{2/g_\rho} \exp(-2\pi x/\beta v_F). \]  

(23)

Note, however, the algebraic temperature dependence of \( \Xi \) through the pre-exponential factor.

V. JOSEPHSON EFFECT

The Josephson effect is an observable consequence of the penetration of the pair amplitude into LL [13, 16, 21]. The calculation of the Josephson current through S-LL-S system is analogous to the one for the pair amplitude (see Section V). For perfectly transmitting interfaces [14] the critical current is equal to its value for non-interacting electrons independent of the actual interaction in LL. This result stems from the fact that features of LL are observable in transport experiments only in the presence of backscattering, both in normal [11] and superconducting systems.

The case of poorly transmitting interfaces has been treated in Refs. [13] within the tunnel Hamiltonian formalism. An infinite quantum wire coupled to two superconductors by tunnel junctions at a distance \( x \) was considered. In the lowest order in \( H_F \) [8], the DC Josephson current \( I_J = I_c \sin \chi \) shows standard dependence on the phase difference \( \chi \) between the superconductors. The critical current \( I_c \) can be estimated as,

\[ I_c \sim \frac{G_{T1}G_{T2}}{(2e^2/\pi)^2} \epsilon \delta \Xi_c, \]  

(24)

where the characteristic energy scale \( \epsilon \) and the scaling factor \( \Xi_c \) characterizing the decay of the Cooper pair density in the LL are given by \( \delta = \Delta, \Xi_c = (\alpha/x)^{2/4+1/g_\rho}, \) for \( x < \xi_S \), and \( \delta = v_F/x, \Xi_c = (\alpha/x)^{2/4-1}, \) for \( x \gg \xi_S \) (cf. Eqs. (21), (22)). Similarly to the pair amplitude, the critical current shows maximum as a function of temperature for both \( T < \xi_S \) and \( T > \xi_S \). At high temperatures \( T > v_F/x \) the Josephson current is suppressed exponentially (cf. Eq. (24)). Numerical estimates [13] show that for typical experimental parameters the critical current [24] is in a few nanoamp range.

If a DC voltage \( eV \ll 2\Delta \) is applied between the superconductors, the Josephson phase difference \( \chi \) becomes time-dependent, \( \chi = \omega_J t = 2eVt \), and the Josephson current oscillates with the frequency \( \omega_J \) (AC-Josephson effect).

For non-interacting electrons the Josephson current \( I_J(t) \propto I_{ac} \sin(\omega_J t - \chi_0) \) acquires additional phase shift \( \chi_0 = eVx/v_F \) due to the propagation of electrons between the contacts. In the interacting case also the amplitude \( I_{ac} \) of AC current becomes voltage dependent. Namely, for moderate interaction the amplitude \( I_{ac}(V) \) shows pronounced oscillations with the period \( eV_0 = 2\pi/|x/v_F - x/v_F| \) corresponding to 2\( \pi \) difference between the phases of charge \( \rho \) and spin \( \sigma \) excitations [18]. Therefore, the AC Josephson effect can be used as a tool for the observation of the spin-charge separation in LL.

A different geometry - finite quantum wire of length \( x \gg \xi_S \) contacting the superconductors at the ends - was considered in Refs. [10, 24]. The result by Maslov et. al. [10] \( (3_c = (\alpha/x)^{2/4+1/g_\rho-1}) \) was revised by Takane [24], who found stronger decay of the critical current, \( 3_c = (\alpha/x)^{2/4+1/g_\rho-1} \), due to a pinning of spin fluctuations by superconductors.

VI. DENSITY OF STATES

The local DOS is defined through the retarded one-electron Green’s function of the LL \( G_R(x,x';\omega) \equiv -i\langle \{\psi(x,t),\psi(x',0)\}\rangle \theta(t) \) as

\[ N(x,\omega) = -\frac{1}{\pi} \text{Im} \int_{-\infty}^{\infty} dt e^{i\omega t} G_R(x,x,t). \]  

(25)

In the case of an infinite LL it is straightforward to compute this quantity and get \( N(\omega) \sim \omega^{g_\rho+1/4-1/g_\rho}. \) Contrary to Fermi liquids, whose quasiparticle residue is finite, Luttinger liquids have a density of states which vanishes at the Fermi energy as a power law, both for repulsive \( (g_\rho < 2) \) and attractive \( (g_\rho > 2) \) interactions.

What are the modifications of the local DOS due to proximity effect? Here we discuss the space and frequency dependence of the DOS of a LL contacted at \( x = 0 \) with a superconductor [21], which corresponds to the limit \( L \to \infty \) in the mode expansion given by Eqs. (21) – (24). In this case only the non-zero modes contribute to the local DOS. The correlation function \( \langle \psi^\dagger(x,t)\psi(x',0) \rangle \) can be evaluated using the boson representation (the correlator is not translationally invariant due to the presence of the interface). At small energies the DOS behaves as

\[ N_{S-LL}(\omega) \sim \omega^{g_\rho+1/4-1/2}. \]  

(26)

The exponent of the DOS is negative \( (g_\rho < 2) \) and, hence, there is a strong enhancement at low energies whereas in the absence of S the LL would show a vanishing DOS at the Fermi energy. The enhancement of the DOS occurs regardless of the distance \( x \) from the interface. The scale of the enhancement is set by a space dependent high frequency cutoff \( \omega_c = v_F/x \). The induced pair amplitude in the LL, which is characteristic of the presence of the superconductor, decays as a power [10] of the distance \( x \) (see Section V). This profound difference in the space dependence demonstrates that the DOS provides different information compared to the proximity effect. The reason why the DOS does not approach the well-known behaviour of an Luttinger liquid far from the superconducting contact is in part related to the fact that we are
considering a clean wire. In this case the states in the LL are extended and the DOS enhancement does not depend on $x$.

So far we discussed only the case in which the interface between the superconductor and the Luttinger liquid has a high transparency. Let us shortly comment on the opposite limit, in which the Luttinger liquid is connected to the superconductor by a tunnel junction. At low energies, we find for the DOS close to the junction $N_{S-LL} \sim \omega^{(g_\rho/2-1)+(1/2g_\rho-g_\rho/8)}$. Although the exponent is different from the one appearing in Eq. (24), the DOS is clearly enhanced. Moreover, also in this case the enhancement is found regardless of the distance from the junction.

VII. TWO-ELECTRON TUNNELING INTO A LUTTINGER LIQUID

In this Section, we will calculate the subgap conductance for a LL, connected to S via a lateral tunnel junction [22]. The tunnel current is calculated in the standard way as

$$I(t) = -e\langle \tilde{N}_L(t) \rangle = -ie\langle [H_T(t), N_L(t)] \rangle.$$  

(27)

Here, $N_L = \sum_s \int dx \psi_{L,s}^\dagger(x)\psi_{L,s}(x)$; the time-dependent tunnel Hamiltonian is given by

$$H_T(t) = \sum_s e^{ieVt}t_0\psi_{L,s}^\dagger(0,t)\psi_{S,s}(0,t) + \text{h.c.},$$

(28)

where $eV$ is the applied bias voltage between S and LL. The time dependent field operators are defined as

$$\psi_{i,s}(x,t) = e^{i(H_i-\mu_i)\tau} \psi_{i,s}(x)e^{-i(H_i-\mu_i)\tau},$$

where $\mu_i$ is the chemical potential. We then perform an expansion in the tunneling Hamiltonian (which is switched on adiabatically at $t = -\infty$), and obtain

$$I(t) = -ie \int_{-\infty}^{\infty} dt_1 dt_2 dt_3 G_R(t_1,t_2,t_3),$$

(29)

where we introduced the retarded Green’s function

$$G_R(t_1,t_2,t_3) = i\theta(t_1-t_2)\theta(t_3-t_2)\theta(t_2-t_3) \times \langle [[[H_T(t), N_L(t)], H_T(t_1)], H_T(t_2)], H_T(t_3)] \rangle.$$  

(30)

Using imaginary time techniques, $G_R$ can be expressed as a product of time-ordered correlation functions for LL and S. As long as the relevant energies ($eV, T$) are small compared to the gap of the superconductor, the time-dependence of the anomalous correlations in the superconductor can be approximated with the help of $\delta$-functions, see Eq. (18). In the zero temperature limit we get

$$I = 24e^2(\rho_0\alpha)^2V \sum_{\alpha} \left[ \frac{(g_\rho/2)\alpha_1(\alpha eV/v_F)\alpha_1-2}{\Gamma(\alpha_1)} \right] + \left( \frac{(g_\rho/2)\alpha_2(\alpha eV/v_F)\alpha_2-1}{\Gamma(\alpha_2+1)} \right),$$

(31)

where $\alpha_1 = 2/g_\rho + g_\rho/2$ and $\alpha_2 = 2/g_\rho$. Note that the current depends on the applied bias in a power law fashion, which is common for transport through an interacting 1D system [11]. For noninteracting electrons $I \sim V$ as one expects. For repulsive interactions $g_\rho < 2$, and the dominant contribution to current at low voltages reads $I \sim V^2/g_\rho$.

At finite temperatures we have

$$G = \frac{1}{4} G_R^2 R_K \left[ c_1 \left( \frac{T}{E_F} \right)^{\frac{\alpha_1}{2} + \frac{\alpha_2}{2} - 2} + c_2 \left( \frac{T}{E_F} \right)^{\frac{\alpha_2}{2} - 1} \right].$$

(32)

Here, $E_F$ is the Fermi energy; $c_1, c_2$ are dimensionless constants, which can be determined numerically (some examples are $c_1 = c_2 = 1.0$ for $g_\rho = 2.0$, $c_1 = 1.549$, $c_2 = 2.467$ for $g_\rho = 1.0$, and $c_1 = 1.153$ and $c_2 = 0.765$ for $g_\rho = 3.0$).

In the case of a chiral LL the subgap conductance has been studied by Fisher [23].

VIII. RENORMALIZATION GROUP FOR THE SUBGAP CONDUCTANCE

In this last section we discuss the subgap current by means of a poor man’s renormalization group [17,24,25] previously developed for the case of an impurity in a Luttinger liquid [1]. To be specific we consider a semi-infinite normal metal for $x < 0$ and a superconductor for $x > 0$. At $x = 0$ there is an insulating barrier which is modeled by a delta-like potential $U(x) = U_0\delta(x)$. In the absence of electron-electron interaction in the normal metal, the scattering states, due to the presence of the interface with the superconductor, are

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} \left( \begin{array}{c} e^{ikx} + r_0 e^{-ikx} \\ -ir_a e^{ikx} \end{array} \right), \quad x < 0$$

$$\phi_{-k^*}(x) = \frac{1}{\sqrt{2\pi}} \left( \begin{array}{c} e^{-ik^*x} + r_0 e^{ik^*x} \\ ir_a e^{-ik^*x} \end{array} \right), \quad x < 0.$$  

(33)

The states $\phi_k$ and $\phi_{-k^*}$ correspond to incoming particles and holes; $r_0$ and $-ir_a$ correspond to the amplitudes for normal and Andreev scattering. For the sake of simplicity, we make the Andreev approximation: in the absence of a potential barrier (i.e., $U_0 = 0$) at the interface, one has $r_0 = 0$ and $r_a = 1$. For our present case of a delta-like potential
\[ r_0 = -\frac{2zz^* + i(z + z^*)}{(1 + iz)(1 - iz^*) + zz^*}, \]

\[ r_a = \frac{1}{(1 + iz)(1 - iz^*) + zz^*}, \]

where \( z = mU_0/k; \) furthermore \( k = \sqrt{2m(\mu + i\omega_n)}, \) and \( \omega_n = 2\pi(n + 1/2)T. \) We work with Matsubara frequencies. To get the proper retarded Green function eventually \( i\omega_n \rightarrow \omega + i\delta \) as usual. The effect of the interaction is analyzed first in the Born approximation. Consider for simplicity the Hartree contribution in first order in perturbation theory. The scattering states of Eq. (33) generate an oscillating electron density, which in turn generates a potential \( V_H(x). \) The correction to the wave function can be written in the form

\[ \delta\psi_{k,\alpha}(x) = \int_{-\infty}^{0} dy G_{k,\alpha\beta}(x, y)V_H(y)\phi_{k,\beta}(y), \]

where \( \alpha \) and \( \beta \) take the values 1 and 2 corresponding to the particle and hole degree of freedom, respectively. The matrix elements of the Green’s function take up the values

\[ G_{k,11}(x, y) = \frac{a}{m} \left( e^{ik|x-y|} + r_0 e^{-ik(x+y)} \right), \]

\[ G_{k,12}(x, y) = \frac{a}{m} \frac{ir_0}{k} e^{-ik(x-y)}, \]

\[ G_{k,21}(x, y) = -\frac{a}{m} \frac{ir_0}{k} e^{ik(x-y)}, \]

\[ G_{k,22}(x, y) = -\frac{a}{m} \left( e^{-ik|x-y|} + r_0 e^{ik(x+y)} \right). \]

The correction to the Andreev scattering coefficient is obtained by setting \( \alpha = 2 \) and \( \beta = 1, \) and reads

\[ \delta\psi_{k,2}(x) = \int_{-\infty}^{0} dy \left[ G_{k,21}(x, y)V_H(y)\phi_{k,1}(y) + G_{k,22}(x, y)V_H(y)\phi_{k,2}(y) \right]. \]

By inserting the expression for the Green’s function \( G_{k,\alpha\beta}(x, y) \) and the non-interacting scattering states \( \phi_{k,\beta}(x) \) one gets \( \delta r_a = \delta r_a^{(1)} + \delta r_a^{(2)}, \) with

\[ \delta r_a^{(1)} = \frac{mr_0r_0}{4\pi k} \int_{-\infty}^{0} dq V(q) e^{-iky}, \]

\[ \delta r_a^{(2)} = \frac{mr_0r_0}{2\pi k} \int_{-\infty}^{0} dq V(q) e^{iky}. \]

By expanding the Hartree potential in a Fourier series, one obtains an expression for the scattering coefficients

\[ \delta r_a^{(1)} = -\frac{mr_0r_0}{k} \int_{-\infty}^{0} dq V(q)\delta n(q) \frac{1}{q - 2k}, \]

\[ \delta r_a^{(2)} = \frac{mr_0r_0}{k} \int_{-\infty}^{0} dq V(q)\delta n(q) \frac{1}{q + 2k}, \]

where

\[ \delta n(q) = \int_{-\infty}^{0} dq e^{-iqy}\delta n(x). \]

Notice that the convergence of the integral is automatically controlled by the imaginary parts of the momenta \( k \) and \( k^*. \) The density \( \delta n(x) \) of the electron is evaluated in terms of the non-interacting scattering states, and can be written as

\[ n(x) = n_0 + \delta n(x) = -\frac{1}{2} \left( T \sum_{\omega_n} G_{11}(x, x) + T \sum_{\omega_n} G_{22}(x, x) \right). \]

For the Fourier transform \( \delta n(q) \) we then get

\[ \delta n(q) = \lim_{\delta \to 0^+} \frac{1}{2} \left( i\frac{r_0}{2\pi} \ln \left( \frac{2k_F + q + i\delta}{q + i\delta} \right) - i\frac{r_0}{2\pi} \ln \left( \frac{q - 2k_F + i\delta}{q + i\delta} \right) \right). \]

Hence

\[ \delta r_a^{(1)} = -\frac{mr_0r_0}{4\pi k} \int_{-\infty}^{0} dq V(q) \frac{1}{q - 2k}, \]

\[ \times \left( r_0 \ln \left( \frac{2k_F + q + i\delta}{q + i\delta} \right) - r_0^* \ln \left( \frac{q - 2k_F + i\delta}{q + i\delta} \right) \right), \]

and

\[ \delta r_a^{(2)} = \frac{mr_0r_0}{4\pi k} \int_{-\infty}^{0} dq V(q) \frac{1}{q + 2k}, \]

\[ \times \left( r_0 \ln \left( \frac{2k_F + q + i\delta}{q + i\delta} \right) - r_0^* \ln \left( \frac{q - 2k_F + i\delta}{q + i\delta} \right) \right). \]

Using the Cauchy theorem, we find

\[ \delta r_a^{(1)} = \frac{mr_0r_0}{4\pi k} \left( r_0 V(2k) \ln \left( \frac{2k_F + 2k + i\delta}{2k + i\delta} \right) - r_0^* V(2k) \ln \left( \frac{2k - 2k_F + i\delta}{2k + i\delta} \right) \right), \]

and

\[ \delta r_a^{(2)} = -\frac{mr_0r_0}{4\pi k} \left( r_0 V(-2k^*) \ln \left( \frac{-2k^* + 2k + i\delta}{-2k^* + i\delta} \right) - r_0^* V(-2k^*) \ln \left( \frac{-2k^* - 2k + i\delta}{-2k^* + i\delta} \right) \right). \]

Keeping only the divergent terms, we are finally left with

\[ \delta r_a = \frac{r_0r_0}{2\pi v} V(2k_F) \ln \left( \frac{E_F}{e} \right). \]

As it is usual in 1D systems, perturbative corrections are logarithmically divergent, signalling an instability of the Fermi liquid ground state and the emergence of the Luttinger liquid behavior. One can then sum the leading logarithmic singularity by deriving and solving the
renormalization group equation. Here we derive an equation for the Andreev reflection coefficient \( R_a = |r_a|^2 \). The subgap conductance is obtained via the formula \( G = 4\pi e^2 R_a \), where the condition \( R_a + R_0 = 1 \) has been used. By defining \( \alpha = \frac{V(2k_F)}{2\pi v} \) one obtains the equation

\[
\frac{dR_a}{d\eta} = 2R_a(1 - R_0),
\]

where \( \eta = \ln \Delta/e \) with initial condition \( \eta = 0 \) (\( \epsilon = \Delta \)). The solution of Eq. (47) gives

\[
R_a(\epsilon) = \frac{R_a(\Delta/e)^{2\alpha}}{R_0 + R_a(\Delta/e)^{2\alpha}}.
\]

(48)

Here \( R_a \) and \( R_0 \) are the initial values of the scattering coefficients. Our analysis has neglected the cutoff dependence of the interaction coupling, which a more complete calculation should take into account, as well as the Fock terms. In Ref. [20] this calculation has been carried out and the result (48) remains unchanged provided \( \alpha \rightarrow \alpha - 2\beta \) with \( \beta = \frac{V(0)}{2\pi v} \).

**APPENDIX A: ONE-PARTICLE CORRELATION FUNCTION**

In this Appendix, we present some explicit expressions for the integrated single particle Green’s function of an infinitely long LL, defined as \( G(x) = \int \frac{d\tau}{-\beta/2} G_{x,x}(x,\tau) \), where \( G_{x,x'}(x,\tau) = \langle x|\hat{\psi}_s(x,\tau)\hat{\psi}^\dagger_{s'}(0,0)\rangle \). The latter quantity is readily evaluated [10], using the Bose representation [2]. Upon integration over \( \tau \) one obtains in the long wavelength limit

\[
G(x) = \frac{2}{v_F} \sin(k_Fx) \left( \frac{\alpha}{x} \right)^{1/2g_\rho + g_\rho/8 - 1/2} F_C(x, g_\rho).
\]

(A1)

In the zero temperature limit, \( F_C \) is \( x \)-independent,

\[
F_C(x, g_\rho) \rightarrow \int_{-\infty}^{\infty} \frac{dz}{\pi} \sin \left[ \arctan(1/z) + \arctan(g_\rho/2z) \right] \times \left[ \frac{1}{1 + z^2} \right]^{1/4} \left[ \frac{1}{1 + (2z/g_\rho)^2} \right]^{1/4g_\rho + g_\rho/16} \equiv F_C(g_\rho).
\]

(A2)

Note that \( F_C(2) = 1 \). For noninteracting electrons, \( F_C(x, g_\rho) \) can be found explicitly,

\[
F_C(x, 2) = 2/\pi \arctan[1/\sinh(\pi x/\beta v_F)].
\]

(A3)

In the case of weak repulsive interactions, \( (2 - g_\rho)\pi x/\beta v_F \ll 1 \) and low temperatures \( \pi x/2\beta v_F \ll 1 \), the function \( F_C(x, g_\rho) \) can be approximated as

\[
F_C(x, g_\rho) \approx F_C(g_\rho) \left[ 1 - \frac{2x}{\beta v_F} + \frac{2 - g_\rho}{2} \frac{\pi x}{\beta v_F} \right].
\]

(A4)

Here we dropped terms \( O((2 - g_\rho)(\pi x/\beta v_F)^2) \).

**APPENDIX B: TWO-PARTICLE CORRELATION FUNCTION**

Next we consider the two-particle correlation function \( \Pi_C(x, \tau) = \Pi_{++,---}(x, \tau; x, \tau; \tau, 0, 0; 0, 0) \), which describes particle-particle propagation of a spin singlet in a LL over a distance \( x \) during a time \( \tau \). Integrated over imaginary time, this quantity can be written as

\[
\Pi_C(x) = \int_{-\beta/2}^{\beta/2} d\tau \Pi_C(x, \tau) = \rho_0 \frac{2}{v_F} \left( \frac{\alpha}{x} \right)^{2/g_\rho} F_C(x, g_\rho).
\]

(B1)

Note that \( F_C(2) = 1 \). In the noninteracting case \( F_C(x, g_\rho) \) can be calculated analytically also at finite temperatures,

\[
F_C(x, 2) = \frac{(2\pi x/\beta v_F)}{\sqrt{\cosh^2(2\pi x/\beta v_F) - 1}}.
\]

(B2)

In the limit of high temperatures, \( 2\pi x/(\beta v_F) \gg 1 \), \( F_C \) can be found for arbitrary interaction strength,

\[
F_C(x, g_\rho) \approx 2 \left( \frac{2\pi x}{\beta v_F} \right)^{2/g_\rho} \exp(-2\pi x/\beta v_F).
\]

(B3)

In the case of weak repulsive interactions and low temperatures an expansion similar to the one leading to (A4) enables us to approximate

\[
\frac{F_C(x, g_\rho)}{F_C(g_\rho)} \approx 1 - \frac{2}{3} \left( \frac{\pi x}{\beta v_F} \right)^2 + (2 - g_\rho) \left( \frac{\pi x}{\beta v_F} \right)^{2/g_\rho}.
\]

(B4)

[1] G. Deutscher and P. G. de Gennes, in *Superconductivity*, edited by R. D. Parks (Marcel Dekker, New York, 1965), Vol. II, p. 1005.

[2] A. F. Andreev, Zh. Eksp. Teor. Fiz. 46, 1823 (1964) [Sov. Phys. JETP 19, 1228 (1964)].
[3] L. G. Aslamazov, A. I. Larkin, and Yu. N. Ovchinnikov, Zh. Eksp.Teor. Fiz. 55, 323 (1968) [Sov. Phys. JETP 28, 171 (1969)].
[4] A. A. Golubov and M. Yu. Kupriyanov, J. Low Temp. Phys. 70, 83 (1988).
[5] In this paper we concentrate only on ballistic systems, for a comprehensive review of various properties of SN systems in the diffusive regime see Mesoscopic Electron Transport, edited by L. Kouwenhoven, L. Sohn and G. Schön, NATO ASI series E, Vol. 345, Kluwer (1997).
[6] Mesoscopic Superconductivity, edited by F.W.J. Hekking, G. Schön, and D. V. Averin, Physica B 203 No. 3 & 4 (1994).
[7] C.W.J. Beenakker, in Mesoscopic Quantum Physics , edited by E. Akkermans, G. Montambaux, J. -L. Pichard, and J. Zinn-Justin (North-Holland, Amsterdam, 1995).
[8] C.J. Lambert and R. Raimondi, J. Phys. Condens. Matter 10, 901 (1998).
[9] A.F. Volkov and V.V. Pavlovskii, cond-mat/9711255 (unpublished).
[10] See, e.g., J. Voit, Rep. Prog. Phys., 58, 977 (1995), and references therein.
[11] For a review, see M.P.A. Fisher and L.I. Glazman, in Ref. [3].
[12] A. Thess et al., Science 273, 483 (1996).
[13] R. Egger and A.O. Gogolin, Phys. Rev. Lett. 79, 5082 (1997); C. Kane, L. Balents and M. P. A. Fisher, ibid. p. 5086 (1997); H. Yoshioka and A.A. Odintsov, to be publ. in Phys. Rev. Lett., preprint cond-mat/9805106.
[14] It is assumed that the Coulomb potential is screened, e.g., by a back-gate, on distances larger than $\lambda_F$, but shorter than the long wavelengths of interest here.
[15] J. Nitta, H. Takayanagi, T. Akazaki, Jpn. J. Appl. Phys. 34, 6977 (1995).
[16] D. L. Maslov, M. Stone, P. M. Goldbart, and D. Loss, Phys. Rev. B 53, 1548 (1996).
[17] Y. Takane and Y. Koyama, J. Phys. Soc. Japan, 65, 3630 (1996).
[18] R. Fazio, F. W. J. Hekking, and A. A. Odintsov, Phys. Rev. Lett. 74, 1843 (1995); Phys. Rev. B 53, 6653 (1996).
[19] M. Fabrizio and A. Gogolin, Phys. Rev. B 51, 17827 (1995).
[20] Y. Takane, J. Phys. Soc. Japan, 66, 537 (1997).
[21] C. Winkelholz, R. Fazio, F. W. J. Hekking, and G. Schön, Phys. Rev. Lett. 77, 3200 (1996).
[22] R. Fazio, K. -H. Wagenblast, C. Winkelholz, and G. Schön, Physica B 222, 364 (1995).
[23] M. P. A. Fisher, Phys. Rev. B 49, 14550 (1994).
[24] Y. Takane and Y. Koyama, J. Phys. Soc. Japan 66, 419 (1996).
[25] R. Raimondi, unpublished.