On global strong solutions to the 3D MHD flows with density-temperature-dependent viscosities

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Abstract

In this paper, we establish the global existence of strong solutions for the 3D viscous, compressible, and heat conducting magnetohydrodynamic (MHD) flows with density-temperature-dependent viscosities in a bounded domain. We essentially show that for the initial boundary value problem with initial density allowed to vanish, the strong solution exists globally under some suitable small conditions. As a byproduct, we obtain the nonlinear exponential stability of the solution.

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1 Introduction

In this paper, we are concerned with the following 3D viscous, compressible, and heat conducting magnetohydrodynamic (MHD) equation, which is a combination of the compressible Navier–Stokes equation of fluid dynamics and Maxwell equation of electromagnetism (see [15, 18]):

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(2\mu D(u)) - \nabla(\lambda \text{div} u) + \nabla P &= (\text{curl } B) \times B, \\
\kappa \Delta \theta + P \text{div } u &= 2\mu |D(u)|^2 + \lambda (\text{div } u)^2 + \nu |\text{curl } B|^2, \\
B_t - \text{curl}(u \times B) &= \nu \Delta B, \quad \text{div } B = 0,
\end{align*}
\]

(1.1)

where \( t \geq 0 \) is the time, \( x \in \Omega \subset \mathbb{R}^3 \) is a smooth bounded domain, \( \rho, u = (u^1, u^2, u^3)^T, \theta, B = (b^1, b^2, b^3)^T, \) and \( P = \rho \theta R (R > 0) \) denote the fluid density, velocity, absolute temperature, magnetic field, and pressure, respectively, and \( D(u) \) is the deformation tensor given by

\[
D(u) = \frac{1}{2} [\nabla u + (\nabla u)^T].
\]
The viscosity coefficients \( \mu = \mu(\rho, \theta) \in C^1(\mathbb{R}^2) \) and \( \lambda = \lambda(\rho, \theta) \in C^1(\mathbb{R}^2) \) satisfy the physical restrictions

\[
\mu(\rho, \theta) > 0, \quad 2\mu(\rho, \theta) + 3\lambda(\rho, \theta) \geq 0.
\] (1.2)

The positive constants \( c_v, \kappa, \) and \( \nu \) are the heat capacity, heat conductivity, and magnetic diffusion coefficient, respectively.

As initial and boundary conditions, we consider

\[
(\rho, u, \theta, B)\big|_{t=0} = (\rho_0, u_0, \theta_0, B_0)(x), \quad x \in \Omega,
\] (1.3)

and

\[
(u, \theta, B)(x, t) = 0 \quad \text{on } \partial \Omega \times (0, T).
\] (1.4)

The compressible MHD system (1.1) describes the relationship between the compressible Navier–Stokes equation of fluid dynamics and Maxwell equation of electromagnetism, which has been studied by many papers \([3, 4, 9, 11, 12]\) and the references therein.

When there is no electromagnetic effect, that is, \( B = 0 \), system (1.1) reduces to the full compressible Navier–Stokes system

\[
\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) - \text{div}(2\mu D(u)) - \nabla(\lambda \text{div} u) + \nabla P = 0,
\end{cases}
\] (1.5)

Because of the important physical phenomenon and mathematical challenges of the full compressible Navier–Stokes system (1.5), there is a wide literature investigating the complexity and rich phenomena of system (1.5). In the case of strictly positive initial density and temperature, Nash [22] and Serrin [23] obtained the local existence and uniqueness of classical solutions, respectively. Matsumura and Nishida [19–21] first obtained the global classical solutions when the initial data are close to a nonvacuum equilibrium in \( H^s(\mathbb{R}^3) \). Valli and Zajączkowski [24] established the existence and stability of the periodic solution in a bounded domain. Hoff [7, 8] proved the global existence of weak solutions when the initial data are discontinuous.

When the initial density can contain vacuum, Feireisl [6] investigated the full compressible Navier–Stokes equations with temperature-dependent heat conductivity coefficient and obtained the existence of “variational” weak solutions for large initial data with vacuum. Bresch and Desjardins [1] studied the Cauchy problem of system (1.5) with density-dependent viscosities and obtained the global stability of weak solutions. Recently, Yu and Zhang [25] considered the three-dimensional full compressible Navier–Stokes equations with density-temperature-dependent viscosities and proved the existence of global strong solutions in a bounded domain in \( \mathbb{R}^3 \).

For the compressible MHD system (1.1), Chen and Wang in [2] investigated the nonlinear MHD equations with general initial data and obtained the global existence and uniqueness of solutions with large initial data. Hu and Wang [10] investigated the compactness of
weaksolution of 3D full compressible MHD equations with density-dependent-heat conductivity and the magnetic coefficient with vacuum. Later, Huang and Li [13] studied the mechanism of blowup and structure of possible singularities of strong solutions to system (1.1) and obtained a blowup criterion, which is analogous to the well-known Serrin blowup criterion for the Cauchy problem and the initial boundary value one of system (1.1). Due to the physical importance, complexity, and mathematical challenges, our main aim in this paper is to investigate the global existence of strong solutions to the 3D MHD flows with density-temperature-dependent viscosities in a bounded domain.

Before stating our main result, we define
\[
\frac{1}{3 + \beta} = \frac{1}{12} + \frac{1}{q}
\]
(1.6)
for some \(0 < \beta \leq 1\), and thus \(q \in (4, 6]\). For simplicity, we denote
\[
\| (f, g) \|_{L^p} \triangleq \| f \|_{L^p} + \| g \|_{L^p}.
\]

Now we are in a position to formulate our main results.

**Theorem 1.1** Assume that the initial data \((\rho_0, u_0, \theta_0, B_0)\) satisfy
\[
0 \leq \rho_0 \leq \bar{\rho}, \quad \rho_0 \in W^{1,q}, \quad \theta_0 \geq 0, \quad (u_0, \theta_0, B_0) \in H_0^1 \cap H^2
\]
(1.7)
for some positive constant \(\bar{\rho}\) and that the following compatibility conditions hold:
\[
\begin{aligned}
&-\text{div}(2\mu(\rho_0, \theta_0)D(u_0)) - \nabla(\lambda(\rho_0, \theta_0) \text{div} u_0) - \nabla P(\rho_0, \theta_0) - (\text{curl} B_0) \times B_0 \\
&= \rho_0^{1/2} g_1,
\end{aligned}
\]
\[
-\kappa \Delta \theta_0 + 2\mu(\rho_0, \theta_0)|D(u_0)|^2 + \lambda(\rho_0, \theta_0)(\text{div} u_0)^2 + \nu |\text{curl} B_0|^2 = \rho_0^{1/2} g_2
\]
(1.8)
for some \(g_1, g_2 \in L^2\). Then there exists a positive constant \(\varepsilon\), depending only on \(\Omega, g_1, g_2, \kappa, v, R, c_o, \) and \(\rho_0\), such that if
\[
C_0 \triangleq \|(\nabla u_0, \nabla \theta_0, \nabla B_0)\|_{L^2}^2 \leq \varepsilon,
\]
then the initial boundary value problem (1.1)–(1.4) has a global strong solution \((\rho, u, \theta, B)\) on \([0, 1] \times [0, \infty)\) satisfying
\[
\begin{aligned}
0 \leq \rho(x, t) \leq 2\rho_0, \quad \theta(x, t) \geq 0, \quad (x, t) \in \Omega \times [0, \infty), \\
\rho \in C([0, \infty); W^{1,q}), \quad \rho_t \in C([0, \infty); L^q), \\
(u, \theta, B) \in C([0, \infty); H_0^1 \cap H^2) \cap L^2(0, \infty; W^{2,q}), \\
(\rho^{1/2} u, \rho^{1/2} \theta, B) \in L^\infty(0, \infty; L^2), \quad (u, \theta, B) \in L^2(0, \infty; H_0^3).
\end{aligned}
\]
Moreover, for any \(t \geq 0\), we have that
\[
\|(\rho^{1/2} u, \rho^{1/2} \theta, B)(t)\|_{L^2} \leq C e^{-c t}
\]
and
\[
\| (\rho^{1/2}u, \rho^{1/2}\theta, B)(t) \|_{L^2} + \|(\nabla u, \nabla \theta, \nabla B)(t) \|_{H^1} \leq Ct^{-1}
\]
with positive constant C depending only on \( \Omega, \rho_0, \kappa, v, R, c, \gamma, g_1, \) and \( g_2. \)

Now we make some comments on the analysis of this paper. Note that for the Cauchy problem with constant viscosities satisfying (1.7)–(1.8), the local existence of strong solutions to the compressible MHD equations (1.1) with large initial data has been recently established [5]. Thus, to extend the strong solutions globally in time, we need global a priori estimates on smooth solutions for \((\rho, u, \theta, B)\). Some of the main new difficulties are due to the appearance of the density-temperature-dependent viscosities and the bounded domain. It turns out that the key issue of this paper is to derive the time-uniform upper bounds for the gradient of the density to bound \( \|\nabla^2 u\|_{L^q} \) and \( \|\nabla^2 B\|_{L^q} \). We start with the a priori hypothesis on \( \|\nabla \rho\|_{L^q} \) and initial layer analysis and succeed in deriving an estimate of \( \|\nabla u\|_{L^1(0,T;L^2)} \) and time-weighted estimates on the gradient of \( u_t, \theta_t, \) and \( B_t \). Another difficulty caused by the bounded domain can be overcome by the energy method.

The rest of the paper is organized as follows. In Sect. 2, we establish estimates of the global strong solutions, which are independent of time \( t \), to the initial boundary value problem (1.1)–(1.4). With the help of global (uniform) estimates at hand, in Sect. 3, we prove Theorem 1.1. In Sect. 4, we give some declarations of this paper.

## 2 Preliminaries

In this section, we recall some known facts and elementary inequalities. Before stating the results, we denote
\[
\int f \; dx = \int_{\Omega} f \; dx.
\]
We first begin with the following local existence result of the initial-boundary value problem (1.1)–(1.4), which is obtained on a small time interval in [5].

**Proposition 2.1** Assume that the initial data \((\rho_0 \geq 0, u_0, \theta_0, B_0)\) satisfy (1.7) and (1.8). Then there exist a small time \( T_\ast > 0 \) and a strong solution \((\rho, u, \theta, B)\) to the initial boundary value problem (1.1)–(1.4) on \( \Omega \times (0, T_\ast] \).

Next, we give the Korn inequality, which can be found in [14].

**Lemma 2.1** Let condition (1.2) be satisfied. Assume that \( \Omega \) is a smooth bounded domain, and let \( u \in H^1, \mu(x), \lambda(x) \in C(\Omega) \). Then there exist a positive constant \( \mu \equiv \mu(\Omega, \mu, \lambda) \) such that
\[
\int (2\mu |D(u)|^2 + \lambda (\text{div } u)^2) \; dx \geq \mu \int |\nabla u|^2 \; dx.
\]

Consider the elliptic system
\[
\begin{cases}
\sum_{j=1}^3 \sum_{i=1}^3 D_i (G_{ij}^{\partial \Omega} (x) D_j u^i) = f_i, & i = 1, 2, 3, \\
u = 0, & x \in \partial \Omega,
\end{cases}
\tag{2.1}
\]
with smooth bounded domain \( \Omega \subset \mathbb{R}^3 \), \( \mathbf{u} = (u^1, u^2, u^3) \), and

\[
G^{\alpha\beta}_{ij}(x)\xi^i_x \xi^j_x \geq \sigma |\xi|^2 \quad \forall \xi \in \mathbb{R}^3, \sigma > 0, A^{\alpha\beta}_{ij}(x) \in L^\infty.
\]

**Lemma 2.2** ([17]) For \( p \in [2, \infty) \), assume that \( \nabla G^{\alpha\beta}_{ij}(x) |\nabla \mathbf{u}|f_i \in L^p \). Then the solution \( \mathbf{u} \) of system (2.1) satisfies \( \mathbf{u} \in W^{2,p} \) with

\[
\|\nabla^2 \mathbf{u}\|_{L^p} \leq C (\|\nabla G^{\alpha\beta}_{ij}(x) |\nabla \mathbf{u}| \|_{L^p} + \|f_i\|_{L^p}).
\]

**Lemma 2.3** (Gagliardo–Nirenberg [16]) For \( p \in [2, 6], q \in (1, \infty), \) and \( r \in (3, \infty) \), there exists a generic constant \( C > 0 \), depending only on \( q, r, \) and \( \Omega \), such that

(i) if \( f \in H^1_0 \) or \( f \in H^1 \), \( \tilde{f} = 0 \), and \( g \in W^{1,r}_0 \) or \( g \in W^{1,r}, \tilde{g} = 0 \), then

\[
\|f\|_{L^p} \leq C \|f\|_{L^2}^{\frac{6-p}{2}} \|\nabla f\|_{L^2}^{\frac{3p-6}{2}},
\]

\[
\|g\|_{L^\infty} \leq C \|g\|_{L^q}^{\frac{q(r-3)}{3}} \|\nabla g\|_{L^r}^{\frac{3p-6}{2}},
\]

where \( \tilde{f} \triangleq \frac{1}{|\Omega|} \int_{\Omega} f \, dx \) and \( \tilde{g} \triangleq \frac{1}{|\Omega|} \int_{\Omega} g \, dx \).

(ii) if \( f \in H^1 \), then

\[
\|f\|_{L^p} \leq C \|f\|_{L^2}^{\frac{6-p}{2}} \|f\|_{H^1}^{\frac{3p-6}{2}}.
\]

**3 A priori estimates**

In this section, we establish the uniform a priori estimates of solutions to the initial boundary value problem (1.1)–(1.4) to extend the local strong solution guaranteed by Proposition 2.1. Then we assume that \((\rho, \mathbf{u}, \theta, B)\) is a smooth solution to (1.1)–(1.4) on \( \Omega \times (0, T) \) for some positive time \( T > 0 \) with smooth initial data \((\rho_0, \mathbf{u}_0, \theta_0, B_0)\) satisfying (1.7) and (1.8). Define

\[
A_1(T) \triangleq \sup_{0 \leq t \leq T} \left( \|\nabla \mathbf{u}\|_{L^2}^2 + \int_0^T \|\rho^{1/2} \mathbf{u}_t, \nabla^2 \mathbf{B}, B_t\|_{L^2}^2 \, dt \right),
\]

\[
A_2(T) \triangleq \sup_{0 \leq t \leq T} \left( \|\nabla \theta\|_{L^2}^2 + \int_0^T \|\rho^{1/2} \theta_t\|_{L^2}^2 \, dt \right),
\]

\[
A_3(T) \triangleq \sup_{0 \leq t \leq T} \left( \|\rho^{1/2} \mathbf{u}_t, \nabla^2 \mathbf{B}\|_{L^2}^2 + \int_0^T \|\nabla \mathbf{u}_t, \nabla \mathbf{B}_{t}\|_{L^2}^2 \, dt \right),
\]

\[
A_4(T) \triangleq \sup_{0 \leq t \leq T} \left( \|\rho^{1/2} \theta_t\|_{L^2}^2 + \int_0^T \|\nabla \theta_t\|_{L^2}^2 \, dt \right),
\]

\[
A_5(T) \triangleq \sup_{0 \leq t \leq T} \|\nabla \rho\|_{L^4},
\]

\[
A_6(T) \triangleq \sup_{0 \leq t \leq T} \left( \|\nabla \mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{H^1}^{1/2} + \|\nabla \mathbf{u}\|_{L^2}^{1/4} \|\nabla \mathbf{u}\|_{H^1}^{3/4} \right),
\]

\[
A_7(T) \triangleq \sup_{0 \leq t \leq T} \|\nabla G^{\alpha\beta}_{ij}(\rho, \theta)\|_{L^2}^{\frac{3}{2}} \|\nabla \mathbf{u}\|_{L^2}^{\frac{1}{2}}, \quad \text{(see } G^{\alpha\beta}_{ij} \text{ in (2.1))}.
\]
and

\[ K_1 \triangleq \left\| \rho^{1/2} u, B \right\|_{L^2}^2 \bigg|_{t=0}, \quad K_2 \triangleq \left\| \rho^{1/2} \theta_t \right\|_{L^2}^2 \bigg|_{t=0}. \]

We have the following key a priori estimates on \((\rho, u, \theta, B)\).

**Proposition 3.1** For a constant \(\tilde{\rho} > 0\) and \(q\) satisfying (1.6), assume that \((\rho_0, u_0, \theta_0, B_0)\) satisfies (1.7) and (1.8). Let \((\rho, u, \theta, B)\) be a smooth solution of (1.1)–(1.4) on \(\Omega \times (0, T]\) satisfying

\[
\begin{align*}
0 \leq \rho(x, t) \leq 3\tilde{\rho} & \quad \text{for all } (x, t) \in \Omega \times [0, T], \\
A_1(T) \leq 2C_0^{1/4}, \quad A_2(T) \leq 2C_0^{1/2}, \quad A_3(T) \leq 3K_1, \\
A_4(T) \leq 3K_2, \quad A_5(T) \leq 4\|\nabla \rho_0\|_{L^6}, \quad A_6(T) + A_7(T) \leq 2.
\end{align*}
\] (3.1)

Then there exists a constant \(\varepsilon > 0\) such that

\[
\begin{align*}
0 \leq \rho(x, t) \leq 2\tilde{\rho} & \quad \text{for all } (x, t) \in \Omega \times [0, T], \\
A_1(T) \leq C_0^{1/4}, \quad A_2(T) \leq C_0^{1/2}, \quad A_3(T) \leq 2K_1, \\
A_4(T) \leq 2K_2, \quad A_5(T) \leq 3\|\nabla \rho_0\|_{L^6}, \quad A_6(T) + A_7(T) \leq 1,
\end{align*}
\] provided that \(C_0 \leq \varepsilon\).

**Proof** The proof of Proposition 3.1 will be done by a series of lemmas below. \(\square\)

Throughout this paper, we denote by \(C\) and \(C_i\) \((i = 1, 2, \ldots)\) generic positive constants, which may depend on \(\Omega, \tilde{\rho}, \|\rho_0\|_{L^2}, \kappa, \nu, R, C_0, g_1, g_2\). We start with the following uniform estimates for \((\rho, u, \theta, B)\) under conditions (3.1).

**Lemma 3.1** Under condition (3.1), we have

\[
\sup_{t \in [0, T]} \left( \|\nabla \theta\|_{H^1} + \|\theta\|_{L^\infty} + \|\nabla \theta\|_{L^6} \right) \leq C. \tag{3.2}
\]

**Proof** Equation (1.1)3, together with (3.1), (2.2), and (2.4), yields that

\[
\|\nabla^2 \theta\|_{L^2} \leq C(\|\rho\|_{L^2} + \|\rho u\cdot \nabla \theta\|_{L^2} + \|\rho \theta \text{ div } u\|_{L^2} + \|\nabla u\|_{L^4}^2 + \|\nabla B\|_{L^4}^2)
\]
\[
\leq C(\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\theta\|_{L^6} \|\nabla u\|_{L^2} + \|\nabla u\|_{L^4}^2 + \|\nabla B\|_{L^4}^2)
\]
\[
\leq C(\|\rho^{1/2} \theta_t\|_{L^2} + A_1^{1/2}(T) \|\nabla \theta\|_{L^2}^{1/2} \|\nabla \theta\|_{H^1(t)}^{1/2} + A_6(T) \|\nabla \theta\|_{L^2})
\]
\[
+ CA_6(T)(\|\nabla u\|_{L^2}^{1/4} + \|\nabla B\|_{L^2}^{1/4})
\]
\[
\leq \frac{1}{2} \|\nabla \theta\|_{L^2}^2 + C\left( \|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2}^{1/4} + \|\nabla B\|_{L^2}^{1/4} \right),
\]
and thus

\[
\|\nabla \theta\|_{H^1} \leq C(\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2}^{1/4} + \|\nabla B\|_{L^2}^{1/4} ), \tag{3.3}
\]
On the other hand, from the Sobolev and Poincaré inequalities it follows that

\[ \|\theta\|_{L^\infty} \leq C\|\nabla \theta\|_{L^6} \leq C\|\nabla \theta\|_{H^1}, \]

which, together with (3.3), leads to (3.2). The proof of the lemma is therefore completed. \(\square\)

**Lemma 3.2** Under condition (3.1),

\[
\sup_{t \in [0,T]} \left( \|\nabla \rho\|_{L^r} + \|(\nabla u, \nabla B)\|_{L^r} \right) \leq C, \quad r \in [2, q], \tag{3.4}
\]

where \(q\) is defined as in (1.6).

**Proof** Thanks to the bounded domain \(\Omega\), we obtain

\[ \|\nabla \rho\|_{L^r} \leq C\|\nabla \rho\|_{L^q} \leq CA_5(T) \leq C \quad \text{for } r \in [2, q]. \]

On the other hand, it follows from Lemma 2.2, (2.2), (2.4), and (3.1) that

\[
\|\nabla^2 u\|_{L^2} \leq C(\|\rho u\|_{L^2} + \|\rho \cdot \nabla u\|_{L^2} + \|\nabla G_{ij}^\rho (\rho, \theta) \nabla u\|_{L^2})
\]

\[
+ C(\|\nabla P\|_{L^2} + \|(\text{curl } B) \times B\|_{L^2})
\]

\[
\leq C(\|\rho^{1/2} u\|_{L^2} + \|u\|_{L^6} \|\nabla u\|_{L^2} + \|\nabla G_{ij}^\rho (\rho, \theta) \|_{L^6} \|\nabla u\|_{L^2})
\]

\[
+ C(\|\theta\|_{L^6} \|\nabla \theta\|_{L^2} + \|\nabla \theta\|_{L^2} + \|B\|_{L^6} \|\nabla B\|_{L^2})
\]

\[
\leq \frac{1}{2} \|\nabla^2 u\|_{L^2} + \frac{1}{4} \|\nabla^2 B\|_{L^2} + C(\|\rho^{1/2} u\|_{L^2} + \|\nabla u\|_{L^2}^2 + \|\nabla B\|_{L^2}^2)
\]

\[
+ C \|\nabla G_{ij}^\rho (\rho, \theta) \|_{L^6} \|\nabla u\|_{L^2} + C(1 + \|\nabla \rho\|_{L^3}) \|\nabla \theta\|_{L^2}
\]

\[
\leq \frac{1}{2} \|\nabla^2 u\|_{L^2} + \frac{1}{4} \|\nabla^2 B\|_{L^2} + C\left( \|\rho^{1/2} u\|_{L^2} + A_1^{5/4}(T) \left( \|\nabla u\|_{L^2}^{1/2} + \|\nabla B\|_{L^2}^{1/2} \right) \right)
\]

\[
+ CA_7(T) \|\nabla u\|_{L^2}^{1/2} + C \|\nabla \theta\|_{L^2} + CA_5(T) \|\nabla \theta\|_{L^2},
\]

and thus

\[
\|\nabla^2 u\|_{L^2} \leq \frac{1}{2} \|\nabla^2 B\|_{L^2} + C\left( \|\rho^{1/2} u\|_{L^2} + \|\nabla u\|_{L^2}^{1/2} + \|\nabla B\|_{L^2}^{1/2} + \|\nabla \theta\|_{L^2} \right)
\]

\[
\leq C(A_3^{1/2}(T) + A_1^{1/2}(T) + A_1^{5/4}(T)).
\]

Similarly,

\[
\|\nabla^2 B\|_{L^2} \leq C(\|B\|_{L^2} + \|u\|_{L^6} \|\nabla B\|_{L^3} + \|B\|_{L^\infty} \|\nabla u\|_{L^2})
\]

\[
\leq \frac{1}{2} \|\nabla^2 B\|_{L^2} + C(\|B\|_{L^2} + \|\nabla u\|_{L^2} \|\nabla B\|_{L^2})
\]

\[
\leq \frac{1}{2} \|\nabla^2 B\|_{L^2} + C\|B\|_{L^2} + C \|\nabla u\|_{L^2}^{1/2},
\]
and thus

\[
\|\nabla^2 u\|_{L^2} + \|\nabla^2 B\|_{L^2} \leq C\left(\|
abla u\|_{L^2}^{1/2} + \|\nabla B\|_{L^2}^{1/2} + \|\nabla \theta\|_{L^2}ight)
\]

\[
\leq C(A_3^{1/2}(T) + A_2^{1/2}(T) + A_1^{1/4}(T)),
\]

(3.5)

which gives (3.4). The proof of the lemma is therefore completed. \(\blacksquare\)

**Lemma 3.3** There exist a constant \(\varepsilon_1 > 0\) such that

\[ A_6(T) + A_7(T) \leq 1, \]

provided that \(C_0 \leq \varepsilon_1.\)

**Proof** From (3.1), (3.2), and (3.4) we obtain

\[
A_6(T) + A_7(T) \leq C\left(\|\nabla \rho\|_{L^q} + \|\nabla \theta\|_{L^q}\right)^{\frac{q}{q-3}} \|\nabla u\|_{L^2}^{1/2}
\]

\[
+ \left(\|\nabla u\|_{L^2}^{1/4}\|\nabla u\|_{H^1} + \|\nabla u\|_{H^1}\right)\|\nabla u\|_{L^2}^{1/4}
\]

\[
+ \left(\|\nabla B\|_{L^2}^{1/4}\|\nabla B\|_{H^1} + \|\nabla B\|_{H^1}\right)\|\nabla B\|_{L^2}^{1/4}
\]

\[
\leq C_1 C_0^{-1/32} \leq 1,
\]

provided that \(C_0 \leq \varepsilon_1 \equiv \min\{1, C_1^{-32}\}.\) The proof of the lemma is therefore completed. \(\blacksquare\)

**Lemma 3.4** Let \((\rho, u, \theta, B)\) be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exist positive constants \(C\) and \(\varepsilon_2\), both depending only on \(\kappa, v, R, c_1, \mu, \bar{\rho}, \Omega, g_1,\) and \(g_2\) such that

\[
\sup_{t \in [0,T]} t\left\|\left(\rho^{1/2} u, \rho^{1/2} \theta, B\right)\right\|_{L^2}^2 + \int_0^T t\left\|\left(\nabla u, \nabla \theta, \nabla B\right)\right\|_{L^2}^2 dt \leq CC_0
\]

(3.6)

for \(i = 0, 1, \ldots, 32\), provided that \(C_0 \leq \varepsilon_2.\)

**Proof** Multiplying \((1.1)_2\) by \(u\) in \(L^2\), from (2.2) and (3.4) we have that

\[
\frac{1}{2} \frac{d}{dt} \left(\rho^{1/2} u \right)_{L^2}^2 + \int [2\mu (\rho, \theta) |D(u)|^2 + \lambda(\rho, \theta) \text{div} u]^2 dx
\]

\[
= \int R\rho \theta \text{div} u dx - \int B \cdot \nabla u \cdot B dx + \frac{1}{2} \int |B|^2 \text{div} u dx
\]

\[
\leq \frac{1}{2} \mu \|\nabla u\|_{L^2}^2 + C \|\rho\|_{L^3}^2 \|\theta\|_{L^6}^2 + C \|B\|_{L^2}^2 \|B\|_{L^6}^2
\]

\[
\leq \frac{1}{2} \mu \|\nabla u\|_{L^2}^2 + C \|\nabla \theta\|_{L^2}^2 + C \|\nabla B\|_{L^2}^2,
\]

and thus by Lemma 2.1 we have

\[
\frac{d}{dt} \left(\rho^{1/2} u \right)_{L^2}^2 + \mu \|\nabla u\|_{L^2}^2 \leq C_2 \|\nabla \theta\|_{L^2}^2 + C_3 \|\nabla B\|_{L^2}^2.
\]

(3.7)
Choosing $C_0 \leq \varepsilon_{2,1} \doteq \min\{\varepsilon_1, (\frac{\kappa}{4\pi})^8\}$, we obtain
\[
\frac{d}{dt} \|\rho^{1/2} \theta\|_{L^2}^2 + \kappa \|\nabla \theta\|_{L^2}^2 \leq C C_0^{1/8} \|\nabla B\|_{L^2}^2 + C C_0^{1/8} \|\nabla u\|_{L^2}^2. \tag{3.8}
\]

Multiplying (1.1)_4 by $B$ in $L^2$, we obtain from (2.2), (3.1), and (3.4) that
\[
\frac{1}{2} \frac{d}{dt} \|B\|_{L^2}^2 + \nu \|\nabla B\|_{L^2}^2 = -\frac{1}{2} \int B^2 \text{div} u \, dx + \int B \cdot \nabla u \cdot B \, dx \leq C \|\nabla B\|_{L^2} \|B\|_{L^2} \|\nabla u\|_{L^2} \leq C \|\nabla B\|_{L^2}^2 \|\nabla u\|_{L^2}^2 \leq C \|\nabla B\|_{L^2}^2,
\]
where we have used the Poincaré inequality and the estimates
\[
\|B\|_{L^2} \leq C \|B\|_{L^2}^{1/2} \|\nabla B\|_{L^2}^{1/2} \leq C \|\nabla B\|_{L^2}.
\]

Thus
\[
\frac{d}{dt} \|B\|_{L^2}^2 + \nu \|\nabla B\|_{L^2}^2 \leq 0, \tag{3.9}
\]
provided that $C_0 \leq \varepsilon_{2,2} \doteq \min\{1, (\frac{\nu}{\kappa})^8\}$. Calculating (3.7) + $2C_2 \times (3.8) + \frac{2C_3 \times (3.9)}{\nu}$ yields
\[
\frac{d}{dt} \left( \|\rho^{1/2} u\|_{L^2}^2 + \frac{2C_2}{\kappa} \|\rho^{1/2} \theta\|_{L^2}^2 + \frac{2C_3}{\nu} \|\nabla B\|_{L^2}^2 \right) + \mu \|\nabla u\|_{L^2}^2 + C_2 \|\nabla \theta\|_{L^2}^2 + C_3 \|\nabla B\|_{L^2}^2 \leq C_6 C_0^{1/8} \|\nabla B\|_{L^2}^2 + C_7 C_0^{1/8} \|\nabla u\|_{L^2}^2,
\]
and thus
\[
\frac{d}{dt} \left( \rho^{1/2} u, \rho^{1/2} \theta, B \right) \|_{L^2}^2 + \left\| \left( \nabla u, \nabla \theta, \nabla B \right) \right\|_{L^2}^2 \leq 0,
\]

(3.10)

provided that \(C_0 \leq \varepsilon_2 \triangleq \min\{\varepsilon_{2,1}, \varepsilon_{2,2}, (\frac{C_2}{\mu_2})^8, (\frac{\mu}{\mu_2})^8\}\).

Integrating (3.10) over \([0, T]\) and using the Poincaré inequality, we get
\[
\sup_{t \in [0, T]} \left\| \left( \rho^{1/2} u, \rho^{1/2} \theta, B \right) \right\|_{L^2}^2 + \int_0^T \left\| \left( \nabla u, \nabla \theta, \nabla B \right) \right\|_{L^2}^2 \, dt \leq C \left\| \left( \nabla u_0, \nabla \theta_0, \nabla B_0 \right) \right\|_{L^2}^2 \leq CC_0.
\]

(3.11)

Multiplying (3.10) by \(t\), integrating the result over \([0, T]\), and using the Poincaré inequality again, we have
\[
\sup_{t \in [0, T]} t \left\| \left( \rho^{1/2} u, \rho^{1/2} \theta, B \right) \right\|_{L^2}^2 + \int_0^T t \left\| \left( \nabla u, \nabla \theta, \nabla B \right) \right\|_{L^2}^2 \, dt \leq C \left\| \left( \nabla u_0, \nabla \theta_0, \nabla B_0 \right) \right\|_{L^2}^2 \leq CC_0.
\]

(3.12)

which, together with (3.11), leads to (3.6) for \(i = 0, 1\). Similarly to the proof of (3.11) and (3.12), we can obtain (3.6) for \(i = 3, 4, \ldots, 32\). The proof of the lemma is therefore completed. \(\square\)

Before stating the following lemma, we define
\[\sigma(T) \triangleq \min\{1, T\}.\]

Then we be establish a uniform upper bound for \(A_1(T)\).

**Lemma 3.5** Let \((\rho, u, \theta, B)\) be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(\varepsilon_3\), depending only on \(k, v, R, c_0, \mu, \bar{\rho}, \Omega, g_1, \) and \(g_2\), such that
\[
\sup_{t \in [0, T]} \left\| \left( \nabla u, \nabla B \right) \right\|_{L^2}^2 + \int_0^T \left\| \left( \rho^{1/2} u, B, \nabla^2 B \right) \right\|_{L^2}^2 \, dt \leq C_0^{1/4}
\]

(3.13)

and
\[
\sup_{t \in [0, T]} t \left\| \left( \nabla u, \nabla B \right) \right\|_{L^2}^2 + \int_0^T t \left\| \left( \rho^{1/2} u, B, \nabla^2 B \right) \right\|_{L^2}^2 \, dt \leq CC_0^{5/16}
\]

(3.14)

for \(i = 1, 2, \ldots, 8\), provided that \(C_0 \leq \varepsilon_3\).
Proof. We multiply (1.1)_2 by \( u_t \) and integrate the result over \( \Omega \):

\[
\frac{d}{dt} \int \left[ \mu(\rho, \theta) |D(u)|^2 + \frac{\lambda(\rho, \theta)}{2} (\text{div} \ u)^2 \right] \ dx + \| \rho^{1/2} u_t \|_{L^2}^2 \leq \frac{\lambda_t}{2} \| \rho_t \|_{L^2}^2 \| \text{div} \ u \|_{L^2}^2 + \frac{\lambda_t}{2} \| \rho_t \|_{L^2}^2 \| \text{div} \ u \|_{L^2}^2.
\]

(3.15)

The right-hand side terms of (3.15) can be estimated as follows. By (3.4) and the fact that \( \mu(\rho, \theta), \lambda(\rho, \theta) \in C^1(\mathbb{R}^2) \) we have that

\[
\left| \int \left[ \mu_t \rho \frac{|D(u)|^2}{\rho} \ dx \right] \leq C \int \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^2} + \| \rho \|_{L^2} \| \nabla u \|_{L^2} \ dx \right| \leq \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2},
\]

and

\[
\left| \int \left[ \mu_t \theta_t \frac{|D(u)|^2}{\theta_t} \ dx \right] \leq \| \theta_t \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2} \right| \leq \| \nabla \rho \|_{L^2} \| \nabla u \|_{L^2} + C \| \nabla u \|_{L^2}.
\]

By (3.4) and the Poincaré inequality we have

\[
N_2 = - \int \rho u \cdot \nabla \cdot u_t \ dx \leq \frac{1}{2} \| \rho^{1/2} u_t \|_{L^2}^2 + C \| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2,
\]

\[
N_3 = \int R \rho \theta \ \text{div} \ u_t \ dx \leq C \| \theta \|_{L^2} \| \nabla u_t \|_{L^2} \leq C \| \nabla \theta \|_{L^2} \| \nabla u_t \|_{L^2},
\]

and thus

\[
N_1 \leq C \| \nabla u \|_{L^2}^2 + C \| \nabla \theta_t \|_{L^2} \| \nabla u \|_{L^2} \leq \frac{1}{2} \| \rho^{1/2} u_t \|_{L^2}^2 + C \| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2,
\]

\[
N_2 \leq \frac{1}{2} \| \rho^{1/2} u_t \|_{L^2}^2 + C \| u \|_{L^2}^2 \| \nabla u \|_{L^2}^2,
\]

\[
N_3 \leq \frac{1}{2} \| \rho^{1/2} u_t \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2.
\]
and
\[
N_4 = - \int \mathbf{B} \cdot \nabla \mathbf{u}_t \cdot \mathbf{B} \, dx + \int \mathbf{B}^2 \, \text{div} \mathbf{u}_t \, dx
\]
\[
\leq C\|\mathbf{B}\|_{L^6} \|\mathbf{B}\|_{L^3} \|\nabla \mathbf{u}_t\|_{L^2}^2
\]
\[
\leq C\|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{u}_t\|_{L^2}^2.
\]

Plugging \(N_j \ (j = 1, 2, 3, 4)\) into (3.15) yields
\[
\frac{d}{dt} \left[ \mu(\rho, \theta)|D(\mathbf{u})|^2 + \frac{\lambda(\rho, \theta)}{2}(\text{div} \mathbf{u})^2 \right] + \rho^{1/2} \|\mathbf{u}_t\|_{L^2}^2
\]
\[
\leq C\|\nabla \mathbf{u}\|_{L^2} + C\|\nabla \theta_t\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^{1/2} \|\nabla \mathbf{u}\|_{H^1}^{1/2}
\]
\[
+ C\|\nabla \theta\|_{L^2} \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2.
\]

Multiplying (1.1) by \(\mathbf{B}_t\) in \(L^2\), we have from (2.3) and the Poincaré inequality that
\[
\frac{\nu}{2} \frac{d}{dt} \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2
\]
\[
= - \int \mathbf{u} \cdot \nabla \mathbf{B} \cdot \mathbf{B}_t \, dx + \int \mathbf{B} \cdot \nabla \mathbf{u} \cdot \mathbf{B}_t \, dx - \int \mathbf{B} \cdot \mathbf{B}_t \, \text{div} \mathbf{u} \, dx
\]
\[
\leq C\|\mathbf{u}\|_{L^6} \|\nabla \mathbf{B}\|_{L^3} \|\mathbf{B}_t\|_{L^2} + C\|\nabla \mathbf{u}\|_{L^2} \|\mathbf{B}\|_{L^\infty} \|\mathbf{B}_t\|_{L^2}
\]
\[
\leq \frac{1}{2} \|\mathbf{B}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{B}\|_{H^1}^2,
\]
and thus from (3.1) and (3.4) it follows that
\[
\frac{d}{dt} \|\nabla \mathbf{B}\|_{L^2}^2 + \|\mathbf{B}_t\|_{L^2}^2 \leq C\|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{B}\|_{H^1}^2 \leq C\|\nabla \mathbf{u}\|_{L^2}^2.
\]

On the other hand, from (1.1) it follows that
\[
\|\nabla^2 \mathbf{B}\|_{L^2}^2 \leq C(\|\mathbf{B}_t\|_{L^2}^2 + \|\mathbf{u}\|_{L^6}^2 \|\nabla \mathbf{B}\|_{L^3}^2 + \|\mathbf{B}\|_{L^\infty}^2 \|\nabla \mathbf{u}\|_{L^2}^2)
\]
\[
\leq \frac{1}{2} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + C(\|\mathbf{B}_t\|_{L^2}^2 + \|\nabla \mathbf{u}\|_{L^2}^2 \|\nabla \mathbf{B}\|_{L^2}^2)
\]
\[
\leq \frac{1}{2} \|\nabla^2 \mathbf{B}\|_{L^2}^2 + C\|\mathbf{B}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}\|_{L^2}^2.\]

Combining (3.16), (3.17), and (3.18), we have
\[
\frac{d}{dt} \left[ \mu(\rho, \theta)|D(\mathbf{u})|^2 + \frac{\lambda(\rho, \theta)}{2}(\text{div} \mathbf{u})^2 + |\nabla \mathbf{B}|^2 \right] dx + \|\rho^{1/2} \mathbf{u}_t, \mathbf{B}_t, \nabla^2 \mathbf{B}\|_{L^2}^2
\]
\[
\leq C\|\nabla \mathbf{u}\|_{L^2} + C\|\nabla \theta\|_{L^2} \|\nabla \mathbf{u}\|_{H^1} \|\nabla \mathbf{u}\|_{H^1}^{1/2}
\]
\[
+ C\|\nabla \theta\|_{L^2} \|\nabla \mathbf{u}_t\|_{L^2} + C\|\nabla \mathbf{B}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2}^2.
\]
The Hölder inequality, together with (3.1) and (3.6), yields that for \( i = 1, 2, \ldots, 8, \)

\[
\int_0^T t^i \|
abla u \|_{L^2} dt \leq \left( \int_0^T t^{2i} \|
abla u \|_{L^2}^2 dt \right)^{1/2} \left( \int_0^T dt \right)^{1/2} + \left( \int_0^T t^{2i-2} \|
abla u \|_{L^2}^2 dt \right)^{1/2} \left( \int_0^T t^{-2} dt \right)^{1/2}
\]

(3.20)

By (3.1), (3.5), (3.6), and (3.20) we have that

\[
\int_0^T t^i \|
abla \theta \|_{L^2} \|
abla u \|_{L^2} \|
abla u \|_{H^1} dt \leq \left( \int_0^T \|
abla \theta \|_{L^2}^2 dt \right)^{1/2} \left( \int_0^T \|
abla u \|_{H^1}^2 dt \right)^{1/2} \left( \int_0^T t^i \|
abla u \|_{L^2}^2 dt \right)^{1/4} \left( \int_0^T \|
abla \theta \|_{L^2}^2 dt \right)^{1/4}
\]

(3.21)

and

\[
\int_0^T t^i \|
abla \theta \|_{L^2} \|
abla u \|_{L^2} dt \leq \left( \int_0^T \|
abla \theta \|_{L^2}^2 dt \right)^{1/2} \left( \int_0^T \|
abla u \|_{L^2}^2 dt \right)^{1/2} \leq C A_1^{1/2} C_0^{1/2}
\]

(3.22)

Integrating (3.19) over \([0, T]\) and using Lemma 2.1, (3.1), (3.6), and (3.20)–(3.22) for \( i = 0, \)

we obtain

\[
\sup_{t \in [0, T]} \| (\nabla u, \nabla B) \|_{L^2}^2 dx + \int_0^T \| (\rho^{1/2} u, B, \nabla^2 B) \|_{L^2}^2 dt \\
\leq C \| (\nabla u_0, \nabla B_0) \|_{L^2}^2 + C C_0^{1/2} + C C_0^{5/16}
\]

(3.23)

provided that \( C_0 \leq \varepsilon_3 \triangleq \min \{ \varepsilon_2, C_8^{-16} \} \). Then we immediately get (3.13).

Next, multiplying (3.19) by \( t \) and integrating the result over \([0, T]\), it follows from (3.6) and (3.20)–(3.22) that

\[
\sup_{t \in [0, T]} t \| (\nabla u, \nabla B) \|_{L^2}^2 dx + \int_0^T t \| (\rho^{1/2} u, B, \nabla^2 B) \|_{L^2}^2 dt \leq C C_0^{5/16}.
\]

(3.24)

Similarly to the proof of (3.23) and (3.24), we can obtain (3.14) for \( i = 2, 3, \ldots, 8 \). The proof of the lemma is therefore completed. \( \Box \)
Lemma 3.6 Let \((\rho, u, \theta, B)\) be a smooth solution of \((1.1)-(1.4)\) satisfying \((3.1)\). Then there exists a positive constant \(\varepsilon_4\), depending only on \(\kappa, \nu, R, c_{\nu}, \bar{\mu}, \bar{\rho}, \Omega, \gamma_1\), and \(g_2\), such that

\[
\sup_{t \in [0, T]} \left\| (\rho^{1/2} u, B, \nabla^2 B) \right\|_{L^2}^2 + \int_0^T \left\| (\nabla u, \nabla B) \right\|_{L^2}^2 \, dt \leq 2K_1, \tag{3.25}
\]

provided that \(C_0 \leq \varepsilon_4\).

Proof By \((1.1)\) and \((1.1)\) we have that

\[
\rho u_t + \rho u \cdot \nabla u_t - \text{div} \left(2\mu \nabla u + \lambda \text{div} u \right) = \text{div}(\rho u(u + \nabla u)) - \text{div}(2\mu \rho, \rho_t + 2\mu_0 \theta_t) D(u) + \text{div}[\rho u(u + \nabla u)] + \nabla \left(\lambda \rho + \lambda_\theta \theta \right) \text{div} u \right) - \text{div} P_t + B \cdot \nabla B_t + B_t \cdot \nabla B - \nabla B_t \cdot B - \nabla B \cdot B_t. \tag{3.26}
\]

Multiplying \((3.26)\) by \(u_t\) and integrating the resulting equation over \([0, 1]\), we get

\[
\frac{1}{2} \frac{d}{dt} \left\| \rho^{1/2} u \right\|_{L^2}^2 + \int [2\mu(\rho, \theta)]^2 + \lambda(\rho, \theta)(\text{div} u)^2 \right\| dx
\]

\[
= \int \text{div}(\rho u)(u_t + u \cdot \nabla u) \cdot u_t \, dx - \int \rho u_t \cdot \nabla u \cdot u_t \, dx + \int P_t \, dx - \int \mu(\rho, \rho_t + \mu_0 \theta_t) [D(u)]^2 \, dx - \frac{1}{2} \int (\lambda, \rho + \lambda_\theta \theta_t) [D(u) u_t] \, dx
\]

\[
+ \int (B \cdot \nabla B_t + B_t \cdot \nabla B - \nabla B_t \cdot B - \nabla B \cdot B_t) \cdot u_t \, dx
\]

\[
= \sum_{j=1}^6 M_j. \tag{3.27}
\]

Now we estimate \(M_j (j = 1, 2, \ldots, 6)\). It follows from \((2.2)-(2.4)\) and \((3.4)\) that

\[
M_1 \leq C \int \rho \text{div} u + u \cdot \nabla \rho \left| (u_t)^2 + |u| \nabla u \cdot u_t \right| \, dx
\]

\[
\leq C(\| \rho \|_{L^6} \| \nabla u \|_{L^2}^2 + \| u \|_{L^6} \| \nabla \rho \|_{L^2} \| u \|_{L^6})^2 + C(\| \rho \|_{L^6} \| \nabla u \|_{L^6} + \| u \|_{L^\infty} \| \nabla \rho \|_{L^2}) \| u \|_{L^\infty} \| \nabla u \|_{L^2} \| u \|_{L^6}
\]

\[
\leq C \| \nabla u \|_{L^2}^2 \| u_t \|_{L^6}^2 + C \| \nabla u \|_{L^6}^{3/2} \| \nabla u \|_{L^2}^{3/2} \| \nabla u \|_{L^2}
\]

\[
\leq CA_1^{1/2} \| \nabla u \|_{L^6}^2 + C \| \nabla u \|_{L^2}^2
\]

\[
\leq C_9 C_0^{1/8} \| \nabla u \|_{L^2}^2 + C \| \nabla u \|_{L^2}^2, 
\]

\[
M_2 \leq \left| \int \rho u_t \cdot \nabla u \cdot u_t \, dx \right| \leq C \| \rho \|_{L^6} \| \nabla u \|_{L^2} \| u_t \|_{L^6}^2
\]

\[
\leq C_9 C_0^{1/8} \| \nabla u \|_{L^2}^2.
\]
and
\[
M_3 \leq \left| \int P_t \div \mathbf{u}_t \, dx \right| = R \left| \int (\rho_0 \theta + \rho \theta_t) \div \mathbf{u}_t \, dx \right|
\leq C \int \left( (\rho \div \mathbf{u} + \mathbf{u} \cdot \nabla \rho) \theta |\nabla \mathbf{u}_t| + \rho |\theta_t| |\nabla \mathbf{u}_t| \right) \, dx
\leq C(\|\rho\|_{L^6} |\nabla \mathbf{u}|_{L^6} + \|\mathbf{u}\|_{L^\infty} |\nabla \rho|_{L^3}) \|\theta_t\|_{L^6} \|\nabla \mathbf{u}_t\|_{L^2}
+ C\|\rho^{1/2} \theta_t\|_{L^2} \|\nabla \mathbf{u}_t\|_{L^2}
\leq \frac{1}{8} \mu \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\rho^{1/2} \theta_t\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2.
\]

By (1.1) and (3.4) we have
\[
M_4 \leq \left| \int (\mu \rho \rho_t + \mu_0 \theta_t) \left[ |D(\mathbf{u})|^2 \right] \, dx \right|
\leq C \int \left( |\mu_0| |\rho \div \mathbf{u} + \mathbf{u} \cdot \nabla \rho| |\nabla \mathbf{u}_t| + |\mu_0| \|\theta_t\| |\nabla \mathbf{u}_t| \right) \, dx
\leq C|\|\rho\|_{L^\infty} |\nabla \mathbf{u}_t|_{L^3} + \|\mathbf{u}\|_{L^\infty} |\nabla \rho|_{L^3}| \|\nabla \mathbf{u}_t\|_{L^2}
+ C\|\theta_t\|_{L^6} \|\nabla \mathbf{u}_t\|_{L^2}
\leq C\|\nabla \mathbf{u}_t\|^{1/2}_{L^2} \|\nabla \mathbf{u}_t\|_{L^2} + C\|\nabla \mathbf{u}_t\|^{1/2}_{L^2} \|\nabla \theta_t\|_{L^2} \|\nabla \mathbf{u}_t\|_{L^2}
\leq \frac{1}{8} \mu \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}_t\|_{L^2} + C\|\nabla \mathbf{u}_t\| \|\nabla \theta_t\|_{L^2}^2,
\]
\[
M_5 \leq \left| \int (\lambda \rho \rho_t + \lambda_0 \theta_t) \left[ (\div \mathbf{u})^2 \right] \, dx \right|
\leq \frac{1}{8} \mu \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}_t\|_{L^2} + C\|\nabla \mathbf{u}_t\| \|\nabla \theta_t\|_{L^2}^2,
\]
and
\[
M_6 \leq \left| \int (\mathbf{B} \cdot \nabla \mathbf{B}_t + \mathbf{B}_t \cdot \nabla \mathbf{B} - \nabla \mathbf{B}_t \cdot \mathbf{B} - \nabla \mathbf{B} \cdot \mathbf{B}_t) \cdot \mathbf{u}_t \, dx \right|
\leq C\|\mathbf{B}\|_{L^3} \|\nabla \mathbf{B}_t\|_{L^2} \|\mathbf{u}_t\|_{L^6} + C\|\mathbf{B}\|_{L^3} \|\nabla \mathbf{u}_t\|_{L^2} \|\mathbf{B}_t\|_{L^6}
\leq \frac{1}{8} \mu \|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\nabla \mathbf{B}_t\|_{L^2}^2.
\]

Substituting $M_1 - M_6$ into (3.27) and using (3.1) and Lemma 2.1, we obtain
\[
\frac{d}{dt} \left( \|\rho^{1/2} \theta_t\|_{L^2}^2 + \|\nabla \mathbf{u}_t\|_{L^2}^2 \right)
\leq C_{11} \|\nabla \mathbf{B}_t\|_{L^2}^2 + C\|\nabla \mathbf{u}_t\|_{L^2}^2 + C\|\rho^{1/2} \theta_t\|_{L^2}^2 + C\|\nabla \theta\|_{L^2}^2 + C\|\nabla \mathbf{B}_t\|_{L^2}^2,
\]
provided that $C_0 \leq \epsilon_{4,1} \triangleq \min(1, (\frac{\mu}{\rho c_p})^8, (\frac{\mu}{\rho c_{10}})^8)$.

Differentiating (1.1) with respect to $t$ yields
\[
\mathbf{B}_{tt} - \nu \Delta \mathbf{B}_t = -\mathbf{u}_t \cdot \nabla \mathbf{B} - \mathbf{u} \cdot \nabla \mathbf{B}_t + \mathbf{B}_t \cdot \nabla \mathbf{u} + \mathbf{B} \cdot \nabla \mathbf{u}_t - \mathbf{B}_t \div \mathbf{u} - \mathbf{B} \div \mathbf{u}_t.
\]

(3.29)
Multiplying (3.29) by $B_t$ in $L^2$ and integrating by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \|B_t\|^2_{L^2} + v \|\nabla B_t\|^2_{L^2} \leq C(\|\nabla u_t\|_{L^2} \|B_t\|_{L^2} + \|u_t\|_{L^2} \|B_t\|_{L^2} + \|\nabla B_t\|^2_{L^2})$$

$$+ C(\|u_t\|_{L^2} \|\nabla B_t\|_{L^2} + C(\|B_t\|_{L^2} \|\nabla u_t\|_{L^2} + \|B_t\|_{L^2} \|\nabla u_t\|_{L^2}))$$

$$\leq \frac{v}{4} \|\nabla B_t\|^2_{L^2} + C(\|B_t\|^2_{L^2} \|\nabla u_t\|_{L^2} + \|B_t\|^2_{L^2} \|\nabla u_t\|_{L^2})$$

$$\leq \left(\frac{v}{4} + C \|\nabla u_t\|^2_{L^2} + C \|\nabla B_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2}\right) \|\nabla B_t\|^2_{L^2} + C C^{1/4} \|\nabla u_t\|^2_{L^2},$$

where we have used (2.2) and the Poincaré inequality. Thus

$$\frac{d}{dt} \|B_t\|^2_{L^2} + \|\nabla B_t\|^2_{L^2} \leq C_1 C_0^{1/4} \|\nabla u_t\|^2_{L^2}, \tag{3.30}$$

provided that $C_0 \leq \epsilon_{4,2} \triangleq \min\{1, (\frac{v}{4C_{11}})^4\}$.

Calculating (3.28)+(3.30) $\times 2C_{11}$ gives

$$\frac{d}{dt} \left(\|\rho^{1/2} u_t\|^2_{L^2} + \|B_t\|^2_{L^2} + \|\nabla u_t\|^2_{L^2} + \|\nabla B_t\|^2_{L^2}\right)$$

$$\leq C \|\nabla u_t\|^2_{L^2} + C \|\rho^{1/2} \theta_t\|^2_{L^2} + C \|\nabla \theta\|^2_{L^2} + C C_0^{1/8} \|\nabla \theta_t\|^2_{L^2}, \tag{3.31}$$

provided that $C_0 \leq \epsilon_{4,3} \triangleq \min\{1, (\frac{1}{4C_{11} C_{12}})^4\}$. Integrating (3.31) over $[0, T]$ and using (3.18), we have from (3.1), (3.6), and (3.20) that

$$\sup_{t \in [0, T]} \left(\|\rho^{1/2} u_t\|^2_{L^2} + \|B_t\|^2_{L^2} + \|\nabla^2 B_t\|^2_{L^2}\right) + \int_0^T \left(\|\nabla u_t\|^2_{L^2} + \|\nabla B_t\|^2_{L^2}\right) dt \leq K_1 + C \|\nabla u\|^2_{L^2} + C C_0^{1/8} A_4(T) + C A_2(T) + C C_0^{1/2}$$

$$\leq K_1 + C A_1^{1/2}(T) + C C_0^{1/8} A_4(T) + C A_2(T) + C C_0^{1/2}$$

$$\leq K_1 + C_1 C_0^{1/8} \leq 2K_1,$$

provided that $C_0 \leq \epsilon_{4} \triangleq \min\{\epsilon_{4,1}, \epsilon_{4,2}, \epsilon_{4,3}, (\frac{K_1}{C_{14}})^8\}$. Thus we immediately obtain (3.25). The proof of the lemma is therefore completed. \(\square\)

**Lemma 3.7** Let $(\rho, u, \theta, B)$ be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant $\epsilon_5$, depending only on $k, v, R, c_{0, f}, \bar{\rho}, \bar{\Omega}, \Omega_0, g_1$, and $g_2$, such that

$$\sup_{t \in [0, T]} \|\nabla \theta\|^2_{L^2} + \int_0^T \|\rho^{1/2} \theta_t\|^2_{L^2} dt \leq C_0^{1/2} \tag{3.32}$$
and

\[
\sup_{t \in [0, T]} t^2 \| \nabla \theta \|_{L^2}^2 + \int_0^T t^2 \| \rho^{1/2} \theta_t \|_{L^2}^2 \, dt \leq CC_0^{9/16},
\]  

(3.33)

provided that \( C_0 \leq \varepsilon_5 \).

**Proof.** We multiply (1.1) \( \delta \) by \( \theta_t \) in \( L^2 \). Then from (2.2)–(2.4), (3.1), (3.2), and (3.4) it follows that

\[
\begin{align*}
\kappa \frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + cv \| \rho^{1/2} \theta_t \|_{L^2}^2 &= \int \left[ -c_v \rho \mathbf{u} \cdot \nabla \theta - P \text{div} \mathbf{u} + 2\mu |\nabla \mathbf{u}|^2 + \lambda (\text{div} \mathbf{u})^2 + v|\text{curl} \mathbf{B}|^2 \right] \theta_t \, dx \\
&\leq \frac{c_v}{2} \| \rho^{1/2} \theta_t \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{H^1} \| \nabla \theta \|_{L^2}^2 + C \| \nabla \theta \|_{H^1} \| \nabla \mathbf{u} \|_{L^2}^2 \\
&\quad + C \| \nabla \mathbf{u} \|_{L^2} \| \nabla \mathbf{u} \|_{L^2} \| \theta_t \|_{L^2} + C \| \nabla \mathbf{B} \|_{L^2} \| \theta_t \|_{L^2}^2 \\
&\leq \frac{c_v}{2} \| \rho^{1/2} \theta_t \|_{L^2}^2 + C \| \nabla \theta \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{L^2}^2 + CA_{1/4}(T)(\| \nabla \mathbf{u} \|_{L^2} + \| \nabla \mathbf{B} \|_{L^2}) \| \nabla \theta_t \|_{L^2},
\end{align*}
\]

and thus

\[
\frac{d}{dt} \| \nabla \theta \|_{L^2}^2 + \| \rho^{1/2} \theta_t \|_{L^2}^2 \leq C \| \nabla \theta \|_{L^2}^2 + C \| \nabla \mathbf{u} \|_{L^2}^2 + CC_0^{1/16}(\| \nabla \mathbf{u} \|_{L^2} + \| \nabla \mathbf{B} \|_{L^2}) \| \nabla \theta_t \|_{L^2}.
\]  

(3.34)

Integrating (3.34) over \([0, T]\), from (3.1) and (3.6) we have that

\[
\sup_{t \in [0, T]} \| \nabla \theta \|_{L^2}^2 + \int_0^T \| \rho^{1/2} \theta_t \|_{L^2}^2 \, dt \leq C_0 + CC_0^{9/16}A_{1/4}(T)
\]

\[
\leq C_{15} C_0 + C_{16} C_0^{9/16} A_{1/4}(T) \leq C_0^{1/2},
\]

provided that \( C_0 \leq \varepsilon_{5.1} \equiv \min\{\varepsilon_4, \frac{c_v^{1/2}}{3c_\delta^{1/2}}, (\frac{C_0^{1/2}}{C_\delta^{1/2}})^{16/9}\} \), which leads to (3.32).

Next, multiplying (3.34) by \( t^2 \) and integrating the result over \([0, T]\), from (3.1) and (3.6) we have that

\[
\sup_{t \in [0, T]} t^2 \| \nabla \theta \|_{L^2}^2 + \int_0^T t^2 \| \rho^{1/2} \theta_t \|_{L^2}^2 \, dt \leq C_{17} C_0 + C_0^{9/16} A_{1/4}(T) \leq CC_0^{9/16},
\]

provided that \( C_0 \leq \varepsilon_{5.2} \equiv \min\{1, \frac{C_0^{9/16}}{C_{17}^{1/2}}\} \). Thus we immediately obtain (3.33). Choosing \( C_0 \leq \varepsilon_5 \equiv \min\{\varepsilon_{5.1}, \varepsilon_{5.2}\} \), we complete the proof of the lemma. \( \square \)
Lemma 3.8 Let \((\rho, \mathbf{u}, \theta, \mathbf{B})\) be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(\varepsilon_6\), depending only on \(\kappa, \nu, R, c_v, \mu, \bar{\rho}, \Omega, g_1\), and \(g_2\), such that

\[
\sup_{t \in [0, T]} t^2 \left\| \rho^{1/2} \mathbf{u}_t \right\|_{L^2}^2 + \int_0^T t^2 \left\| \nabla \mathbf{u}_t \right\|_{L^2}^2 dt \leq C C_0^{5/16},
\]

provided that \(C_0 \leq \varepsilon_6\).

Proof Multiplying (3.31) by \(t^2\) and integrating the resulting equation over \([0, T]\), from (3.6), (3.14), (3.18), (3.20), and (3.33) it follows that

\[
\sup_{t \in [0, T]} t^2 \left( \left\| \rho^{1/2} \mathbf{u}_t \right\|_{L^2}^2 + \left\| \mathbf{B}_t \right\|_{L^2}^2 + \left\| \nabla^2 \mathbf{B} \right\|_{L^2}^2 \right) + \int_0^T t^2 \left( \left\| \nabla \mathbf{u}_t \right\|_{L^2}^2 + \left\| \nabla \mathbf{B}_t \right\|_{L^2}^2 \right) dt
\]

\[
\leq C \int_0^T t \left( \left\| \rho^{1/2} \mathbf{u}_t \right\|_{L^2}^2 + \left\| \mathbf{B}_t \right\|_{L^2}^2 + \left\| \nabla^2 \mathbf{B} \right\|_{L^2}^2 \right) dt
\]

\[
+ C \int_0^T t^2 \left( \left\| \nabla \mathbf{u}_t \right\|_{L^2}^2 + \left\| \rho^{1/2} \theta_t \right\|_{L^2}^2 + \left\| \nabla \theta_t \right\|_{L^2}^2 + C \left\| \nabla \mathbf{u}_t \right\|_{L^2} \left\| \nabla \theta_t \right\|_{L^2} \right) dt
\]

\[
\leq C C_0^{5/16} + C C_0^{1/2} + C C_0^{9/16} + C_0
\]

\[
\leq C C_0^{5/16},
\]

provided that \(C_0 \leq \varepsilon_6 \triangleq \min\{1, \varepsilon_5\}\). Thus we immediately obtain (3.35). The proof of the lemma is therefore completed. \(\square\)

Lemma 3.9 Let \((\rho, \mathbf{u}, \theta, \mathbf{B})\) be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(\varepsilon_7\), depending only on \(\kappa, \nu, R, c_v, \mu, \bar{\rho}, \Omega, g_1\), and \(g_2\), such that

\[
\sup_{t \in [0, T]} \left\| \rho^{1/2} \theta_t \right\|_{L^2}^2 + \int_0^T \left\| \nabla \theta_t \right\|_{L^2}^2 dt \leq 2K_2,
\]

provided that \(C_0 \leq \varepsilon_7\).

Proof By (1.1) \(_1\) and (1.1) \(_3\) we have that

\[
c_v [\rho \theta_t + \rho \mathbf{u} \cdot \nabla \theta] - \kappa \Delta \theta_t
\]

\[
= \text{div}(\rho \mathbf{u})(c_v \theta_t + c_v \mathbf{u} \cdot \nabla \theta + R \theta \text{ div } \mathbf{u}) - \rho (c_v \mathbf{u} \cdot \nabla \theta + R \theta \text{ div } \mathbf{u}) - R \rho \theta \text{ div } \mathbf{u}_t
\]

\[
+ (2\mu \rho_t + 2\mu \theta_t) |\mathbf{D}(\mathbf{u})|^2 + 4\mu (\rho, \theta) \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{u}_t)
\]

\[
+ (\lambda \rho_t + 2\lambda \theta_t)(\text{div } \mathbf{u})^2 + 2\lambda (\rho, \theta) \text{ div } \mathbf{u} \text{ div } \mathbf{u}_t + 2\nu \text{ curl } \mathbf{B} \cdot \text{ curl } \mathbf{B}_t.
\]

Multiplying (3.37) by \(\theta_t\) and integrating the resulting equation on \(\Omega\), we get

\[
\frac{1}{2} \frac{d}{dt} \left\| \rho^{1/2} \theta_t \right\|_{L^2}^2 + \kappa \left\| \nabla \theta_t \right\|_{L^2}^2
\]

\[
= \int [\text{div}(\rho \mathbf{u})(c_v \theta_t + c_v \mathbf{u} \cdot \nabla \theta + R \theta \text{ div } \mathbf{u})] \theta_t \, dx
\]
\[- \int [\rho(\mathbf{c}, \mathbf{u}) \cdot \nabla \theta + R \theta_t \div \mathbf{u}] \theta_t \, dx - \int R \rho \theta \div \mathbf{u} \theta_t \, dx \]

\[+ \int 4 \mu(\rho, \theta) \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{u}) \theta_t \, dx \]

\[+ \int (2 \mu \rho \theta_t + 2 \mu_0 \theta_t) |\mathcal{D}(\mathbf{u})|^2 \theta_t \, dx + \int (\lambda \rho \theta_t + \lambda_0 \theta_t)(\div \mathbf{u})^2 \theta_t \, dx \]

\[+ 2 \int \lambda(\rho, \theta) \div \mathbf{u} \div \mathbf{u} \theta_t \, dx + 2 \int \nu \text{curl} \mathbf{B} \cdot \text{curl} \mathbf{B}_t \, dx \]

\[\sum_{i=1}^{6} \int I_i. \]

The right-hand side of (3.38) can be estimated as follows. By (2.2)–(2.4), (3.1), (3.2), and (3.4) we have

\[I_1 = \int \left[ (\mathbf{u} \cdot \nabla \rho + \rho \div \mathbf{u})(c_2 \theta_t + c_1 \mathbf{u} \cdot \nabla \theta + R \theta_t \div \mathbf{u}) \right] \theta_t \, dx \]

\[\leq C \left( \|\mathbf{u}\|_{L^6} \|\nabla \rho\|_{L^6} + \|\rho\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \right) \|\theta_t\|_{L^6}^2 + C \|\mathbf{u}\|_{L^2}^2 \|\nabla \rho\|_{L^2} \|\nabla \theta_t\|_{L^6} \]

\[+ C \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^3} \|\nabla \theta_t\|_{L^6} + C \|\mathbf{u}\|_{L^6} \|\theta_t\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^6} \]

\[+ C \|\mathbf{u}\|_{L^6} \|\theta_t\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^6} \]

\[\leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \rho\|_{L^2} \|\nabla \theta_t\|_{L^2} + C \|\mathbf{u}\|_{L^6} \|\theta_t\|_{L^2} \left( \|\theta_t\|_{L^2} + \|\nabla \rho\|_{L^3} + 1 \right) \|\nabla \theta_t\|_{L^2} \|\nabla \theta_t\|_{L^2} \]

\[\leq \left[ CC_{0}^{1/8} \frac{\kappa}{16} \right] \|\nabla \theta_t\|_{L^2}^2 + C \|\nabla \theta_t\|_{L^2}^2, \]

\[I_2 = - \int \left[ \rho(\mathbf{c}, \mathbf{u} \cdot \nabla \theta + R \theta_t \div \mathbf{u}) \right] \theta_t \, dx \]

\[\leq C \|\rho\|_{L^6} \|\mathbf{u}\|_{L^6} \|\nabla \theta_t\|_{L^2} \|\theta_t\|_{L^6} + C \|\rho\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^6} \]

\[\leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \theta_t\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^2} + C \|\nabla \rho\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^2} \]

\[\leq \left[ CC_{0}^{1/8} \frac{\kappa}{16} \right] \|\nabla \theta_t\|_{L^2}^2 + CC_{0}^{1/8} \|\nabla \mathbf{u}\|_{L^2}^2, \]

\[I_3 = - R \int \rho \div \mathbf{u} \theta_t \, dx \]

\[\leq \|\rho\|_{L^6} \|\theta\|_{L^6} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^6} \]

\[\leq \|\nabla \rho\|_{L^2} \|\nabla \theta_t\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^2} \]

\[\leq CC_{0}^{1/8} \|\theta_t\|_{L^2}^2 + CC_{0}^{1/8} \|\nabla \mathbf{u}\|_{L^2}^2 \]

and

\[I_4 = \int 4 \mu(\rho, \theta) \mathcal{D}(\mathbf{u}) : \mathcal{D}(\mathbf{u}) \theta_t \, dx + \int (2 \mu \rho \theta_t + 2 \mu_0 \theta_t) |\mathcal{D}(\mathbf{u})|^2 \theta_t \, dx \]

\[\leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\theta_t\|_{L^6} + C \|\nabla \mathbf{u}\|_{L^2}^2 \|\theta_t\|_{L^6} \]

\[+ \left( \|\mathbf{u}\|_{L^6} \|\nabla \rho\|_{L^6} + \|\nabla \mathbf{u}\|_{L^2} \right) \|\nabla \mathbf{u}\|_{L^6} \|\theta_t\|_{L^6} \]

\[\leq C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\nabla \theta_t\|_{L^2} + C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \|\nabla \theta_t\|_{L^2} \]

\[+ \left( \|\nabla \mathbf{u}\|_{L^2} \|\nabla \mathbf{u}\|_{L^2} \right) \|\nabla \mathbf{u}\|_{L^6} \|\nabla \theta_t\|_{L^2} \]

\[\leq \frac{\kappa}{8} \|\nabla \theta_t\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2} \|\nabla \theta_t\|_{L^2}^2 + C \|\nabla \mathbf{u}\|_{L^2}^2 \]
exists a positive constant $\kappa$. Proof

Lemma 2.2, together with (1.1)2, (1.6), (3.1), (3.2), (3.4), and the Hölder inequality, gives

$$\left\| \nabla^2 \mathbf{u} \right\|_{L^2} \leq C \left( \left\| \rho \mathbf{u} \right\|_{L^2} + \left\| \mathbf{u} \cdot \nabla \mathbf{u} \right\|_{L^2} + \left\| (\nabla \rho + \nabla \theta) \mathbf{u} \right\|_{L^2} \right).$$

and thus

$$I_6 = 2 \int \nu \text{curl} \mathbf{B} \cdot \text{curl} \mathbf{B} \rho dx \leq C \left\| \nabla \mathbf{B} \right\|_{L^2} \left\| \nabla \mathbf{B} \right\|_{L^2} \left\| \theta_t \right\|_{L^2}.$$
where we have used the estimates

\[ \|\rho u\|_{L^{3+\beta}} \leq C \left( \int |\rho u|^{(3+\beta)\times \frac{1}{3+\beta}} \, dx \right)^{\frac{1}{3+\beta}} \left( \int 1^{\frac{1}{3+\beta}} \, dx \right)^{\frac{1}{3+\beta}} \leq C \|\rho u\|_{L^4} \]

and

\[ \|\nabla u\|_{L^2} = \|\nabla u\|_{L^4}^{1/2} \leq C \|\nabla u\|_{H^1}^{1/2}. \]

Thus

\[ \|\nabla^2 u\|_{L^{3+\beta}} \leq \frac{1}{4} \|\nabla^2 u\|_{L^{3+\beta}} + C \|\nabla^2 u, \nabla \theta, \nabla B\|_{H^1} + C \|\rho^{1/2} u\|_{L^2} \|\nabla u\|_{L^2}^{3/4}. \tag{3.41} \]

On the other hand, from (1.1) and (3.4) it follows that

\[ \|\nabla^2 B\|_{L^{3+\beta}} \leq C \left( \|B_t\|_{L^{3+\beta}} + \|u\cdot \nabla B\|_{L^{3+\beta}} + \|B\cdot \nabla u\|_{L^{3+\beta}} + \|B \text{ div } u\|_{L^{3+\beta}} \right) \]

\[ \leq C \|B_t\|_{L^4} + C \|u\|_{L^6} \|\nabla B\|_{L^2} + C \|\nabla u\|_{L^6} \|B\|_{L^{12}} \]

\[ \leq C \|B_t\|_{L^2} \|\nabla B\|_{L^2}^{3/4} + C \|\nabla B\|_{L^2} + C \|\nabla B\|_{L^{2(3+\beta)/(1+\beta)}} \|\nabla^2 B\|_{L^{3+\beta}}^{1/2} \]

\[ \leq \frac{1}{2} \|\nabla^2 B\|_{L^{3+\beta}} + C \|B_t\|_{L^2} \|\nabla B\|_{L^2}^{3/4} + C \|\nabla B\|_{H^1}, \]

and thus

\[ \|\nabla^2 B\|_{L^{3+\beta}} \leq C \|B_t\|_{L^2} \|\nabla B\|_{L^2}^{3/4} + C \|\nabla B\|_{H^1}. \tag{3.42} \]

Due to (3.1), (3.3), and (3.5), we have

\[ \|\nabla u, \nabla \theta, \nabla B\|_{H^1} \leq C \left( \|\rho^{1/2} u_t, \rho^{1/2} \theta_t, B_t\|_{L^2} + C \|\nabla \theta\|_{L^2} + C \|\nabla u, \nabla B\|_{L^2} \right). \tag{3.43} \]
Combining (3.41) and (3.42) with (3.43), yields

$$
\| \nabla^2 \mathbf{u} \|_{L^2} + \| \nabla^2 \mathbf{B} \|_{L^2} \\
\leq C \| \rho^{1/2} \mathbf{u} \|_{L^2}^{1/4} \| \nabla \mathbf{u} \|_{L^2}^{3/4} + C \| \mathbf{B} \|_{L^2}^{1/4} \| \nabla \mathbf{B} \|_{L^2}^{3/4} \\
+ C \left( \| \rho^{1/2} \mathbf{u}, \rho^{1/2} \mathbf{B} \|_{L^2} + C \| \nabla \theta \|_{L^2} + C \| (\nabla \mathbf{u}, \nabla \mathbf{B}) \|_{L^2} \right). 
$$

(3.44)

Integrating (3.44) over $[0, T]$ gives

$$
\int_0^T \left( \| \nabla^2 \mathbf{u} \|_{L^2} + \| \nabla^2 \mathbf{B} \|_{L^2} \right) dt \\
\leq C \int_0^T \| \rho^{1/2} \mathbf{u} \|_{L^2}^{1/4} \| \nabla \mathbf{u} \|_{L^2}^{3/4} dt + C \int_0^T \| \mathbf{B} \|_{L^2}^{1/4} \| \nabla \mathbf{B} \|_{L^2}^{3/4} dt \\
+ C \int_0^T \left( \| \rho^{1/2} \mathbf{u}, \rho^{1/2} \mathbf{B} \|_{L^2} dt + C \int_0^T \| \nabla \theta \|_{L^2} dt \\
+ C \int_0^T \| (\nabla \mathbf{u}, \nabla \mathbf{B}) \|_{L^2}^{1/4} dt \right) \triangleq \sum_{i=1}^5 J_i. 
$$

(3.45)

The right-hand side of (3.45) can be estimated as follows. By (3.13), (3.14), and (3.25) we have

$$
J_1 \leq C \left( \int_0^T \| \rho^{1/2} \mathbf{u} \|_{L^2}^{2/5} dt \right)^{5/8} \left( \int_0^T \| \nabla \mathbf{u} \|_{L^2}^{2/5} dt \right)^{3/8} \\
\leq C \left( \int_0^{\sigma(T)} \| \rho^{1/2} \mathbf{u} \|_{L^2}^{2/5} dt \right)^{5/8} + C \left( \int_0^{\sigma(T)} \| \rho^{1/2} \mathbf{u} \|_{L^2}^{2/5} dt \right)^{5/8} \\
\leq C \left( \int_0^{\sigma(T)} \| \rho^{1/2} \mathbf{u} \|_{L^2}^{2/5} dt \right)^{1/2} \left( \int_0^{\sigma(T)} dt \right)^{1/2} \\
+ C \left( \int_0^{\sigma(T)} t^{8} \| \rho^{1/2} \mathbf{u} \|_{L^2}^{2} dt \right)^{1/8} \left( \int_0^{\sigma(T)} t^{-2} dt \right)^{1/2} \leq C C_0^{1/32}. 
$$

Similarly,

$$
J_2 \leq C \left( \int_0^{\sigma(T)} \| \mathbf{B} \|_{L^2}^{2} dt \right)^{1/8} \left( \int_0^{\sigma(T)} dt \right)^{1/2} \\
+ C \left( \int_0^{\sigma(T)} t^{8} \| \nabla \mathbf{B} \|_{L^2}^{2} dt \right)^{1/8} \left( \int_0^{\sigma(T)} t^{-2} dt \right)^{1/2} \leq C C_0^{1/32}. 
$$

By (3.13), (3.14), (3.32), and (3.33) we have

$$
J_3 \leq C \left( \int_0^{\sigma(T)} \| \rho^{1/2} \mathbf{u}, \rho^{1/2} \mathbf{B} \|_{L^2}^{2} dt \right)^{1/2} \left( \int_0^{\sigma(T)} dt \right)^{1/2} \\
+ C \left( \int_0^{\sigma(T)} t^{2} \| \rho^{1/2} \mathbf{u}, \rho^{1/2} \mathbf{B} \|_{L^2}^{2} dt \right)^{1/2} \left( \int_0^{\sigma(T)} t^{-2} dt \right)^{1/2} \leq C C_0^{1/8}, 
$$
\[ J_4 \leq C \left( \int_0^{\sigma(T)} \| \nabla \theta \|_{L^2}^2 \, dt \right)^{1/2} \left( \int_0^{\sigma(T)} \, dt \right)^{1/2} + C \left( \int_{\sigma(T)}^T t^2 \| \nabla \theta \|_{L^2}^2 \, dt \right)^{1/2} \left( \int_{\sigma(T)}^T t^{-2} \, dt \right)^{1/2} \leq CC_0^{1/2}, \]

and

\[ J_5 \leq C \left( \int_0^{\sigma(T)} \left( \nabla u, \nabla B \right) \| \nabla \theta \|_{L^2} \, dt \right)^{1/8} \left( \int_0^{\sigma(T)} \, dt \right)^{7/8} + C \left( \int_{\sigma(T)}^T t^8 \left( \nabla u, \nabla B \right) \| \nabla \theta \|_{L^2} \, dt \right)^{1/8} \left( \int_{\sigma(T)}^T t^{-8/7} \, dt \right)^{7/8} \leq CC_0^{1/8}. \]

Putting \( J_1 - J_5 \) into (3.45), we obtain

\[ \int_0^T \left( \| \nabla^2 u \|_{L^3} + \| \nabla^2 B \|_{L^3} \right) \, dt \leq CC_0^{1/32}. \]  

(3.46)

Thus, combining (2.2)–(2.4) with (3.46), we have

\[ \int_0^T \left( \| \nabla u \|_{L^\infty} + \| \nabla B \|_{L^\infty} \right) \, dt \leq C \int_0^T \left( \| \nabla u \|_{L^4} + \| \nabla B \|_{L^4} + \| \nabla^2 u \|_{L^{3,\beta}} + \| \nabla^2 B \|_{L^{3,\beta}} \right) \, dt \]

\[ \leq CC_0^{1/32}. \]  

(3.47)

By (1.1), we have that

\[ \frac{d}{dt} \rho(t, U(t, s, x)) = -\rho(t, U(t, s, x)) \, \text{div} u(t, U(t, s, x)), \]  

where

\[ \frac{d}{dt} \rho(t, U(t, s, x)) = \rho_t(t, U(t, s, x)) + u(t, U(t, s, x)) \cdot \nabla \rho(t, U(t, s, x)), \]

and \( U \in C([0, T] \times [0, T]) \times \Omega \) is the solution to the initial value problem

\[ \begin{align*}
\frac{d}{dt} U(t, s, x) &= u(t, U(t, s, x)), & t &\in [0, T], \\
U(t, s, x) &= x, & s &\in [0, T], x \in \Omega.
\end{align*} \]

With the help of (3.47) and (3.48), from the Gronwall inequality we get that

\[ \rho \leq \tilde{\rho} \exp \left\{ \int_0^T \| \nabla u \|_{L^\infty} \, dt \right\} \leq \tilde{\rho} \exp \left\{ C_{21} C_0^{1/32} \right\} \leq 2\tilde{\rho}, \]

provided that \( C_0 \leq \varepsilon_8 \doteq \min \{ \varepsilon_7, (\frac{C_3}{C_{21}})^{32} \} \). Thus we immediately obtain (3.40). The proof of the lemma is therefore completed. \( \square \)
Lemma 3.11 Let \((\rho, u, \theta, B)\) be a smooth solution of (1.1)–(1.4) satisfying (3.1). Then there exists a positive constant \(\varepsilon\), depending only on \(\kappa, \nu, \varepsilon, \tilde{\rho}, \Omega, g_1,\) and \(g_2\), such that

\[
\sup_{t \in [0,T]} \|\nabla \rho\|_{L^q} \leq 3 \|\nabla \rho\|_{L^q}, \tag{3.49}
\]

and

\[
\int_0^T \left( \|\nabla^2 u\|_{L^q} + \|\nabla^2 B\|_{L^q} \right) dt \leq C, \tag{3.50}
\]

provided that \(C_0 \leq \varepsilon\).

Proof Differentiating (1.1) with respect to \(\xi_i\) and multiplying the results by \(q|\partial_i \rho|^{q-2}\partial_i \rho\) give

\[
\begin{align*}
&\left|\nabla \rho\right|^{q} + \text{div}\left(|\nabla \rho|^q u\right) + (q - 1)|\nabla \rho|^q \text{div} u \\
&\quad + q|\nabla \rho|^{q-2}(\nabla \rho)^T \nabla u(\nabla \rho) + q\rho|\nabla \rho|^{q-2}\nabla \rho \cdot \nabla \text{div} u = 0. \tag{3.51}
\end{align*}
\]

Integrating (3.51) on \(\Omega\) yields

\[
\frac{d}{dt} \|\nabla \rho\|_{L^q} \leq C \|\nabla u\|_{L^\infty} \|\nabla \rho\|_{L^q} + C_{22} \|\nabla^2 u\|_{L^q}. \tag{3.52}
\]

With the help of (1.1)2, Lemma 2.2, (3.2)–(3.4), (3.40), we get

\[
\|\nabla^2 u\|_{L^q} \leq C(\|\rho u\|_{L^q} + \|\rho \cdot \nabla u\|_{L^q} + \|\nabla \rho + \nabla \theta\|_{L^q} \nabla u) \\
+ C(\|\nabla P\|_{L^q} + \|B \cdot \nabla B\|_{L^q}) \\
\leq C\left[\|\rho u\|_{L^q} + \|\nabla \rho\|_{L^q} + \|\nabla \theta\|_{L^q}\right] \|\nabla u\|_{L^\infty} \\
+ C\left[\|\nabla \rho\|_{L^\infty} \|\nabla \theta\|_{L^q} + \|\nabla \rho\|_{L^q} \|\nabla \theta\|_{L^\infty} + \|B\|_{L^q} \|\nabla B\|_{L^\infty}\right] \\
\leq C(\|\nabla u\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla \rho\|_{L^2} + \|B\|_{L^\infty}) \\
+ C(\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2}^{1/4} + \|\nabla B\|_{L^2}^{1/4}). \tag{3.53}
\]

Thus, similarly to (3.45),

\[
\int_0^T \|\nabla^2 u\|_{L^q} dt \leq C \int_0^T \|\nabla u\|_{L^2} dt + \int_0^T \left(\|\nabla u\|_{L^\infty} + \|\nabla B\|_{L^\infty}\right) dt \\
+ C \int_0^T \left(\|\rho^{1/2} \theta_t\|_{L^2} + \|\nabla \theta\|_{L^2} + \|\nabla u\|_{L^2}^{1/4} + \|\nabla B\|_{L^2}^{1/4}\right) dt \tag{3.54}
\]

\[
\leq C \int_0^T \|\nabla u\|_{L^2} dt + CC_0^{1/32}.
\]
On the other hand, from (3.25) and (3.35) it follows that
\[
\int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \leq C \left( \int_0^T \|\nabla u_t\|_{L^2}^2 \, dt \right)^{1/2} \left( \int_0^T \, dt \right)^{1/2} + C \left( \int_T^s \, t^2 \|\nabla u_t\|_{L^2}^2 \, dt \right)^{1/2} \left( \int_T^s \, t^{-2} \, dt \right)^{1/2}
\leq C s^{1/2} + C s^{-1/2} C_0^{5/32}, \quad \forall s \in (0, T],
\]
which, together with (3.52) and (3.54), yields
\[
C_{22} \int_0^T \|\nabla^2 u\|_{L^2} \, dt \leq C_{23} s^{1/2} + C_{24} s^{-1/2} C_0^{5/32} + C_{25} C_0^{1/32}, \quad \forall s \in (0, T].
\]
Fixing \( s \ll \sigma(T) \) such that \( C_{23} s^{1/2} \leq \frac{1}{6} \|\nabla \rho_0\|_{L^q} \), we have
\[
C_{22} \int_0^T \|\nabla^2 u\|_{L^2} \, dt \leq \frac{1}{2} \|\nabla \rho_0\|_{L^q}, \quad (3.55)
\]
provided that \( C_0 \leq \varepsilon_9 \equiv \min\{\varepsilon_9, (\frac{C_{1/2}}{6C_{24}})^{32/5}, (\frac{\|\nabla \rho_0\|_{L^q}}{6C_{25}})^{32}\} \).

Using (3.47), (3.52), and (3.55), from the Gronwall inequality we obtain that
\[
\sup_{t \in [0, T]} \|\nabla \rho\|_{L^q} \leq \exp \left( \int_0^T C \|\nabla u\|_{L^q} \, dt \right) \left[ C \|\nabla \rho_0\|_{L^q} + C_{22} \int_0^T \|\nabla^2 u\|_{L^q} \, dt \right]
\leq \frac{3}{2} \exp \left( C_{26} C_0^{1/32} \right) \|\nabla \rho_0\|_{L^q}
\leq 3 \|\nabla \rho_0\|_{L^q},
\]
provided that \( C_0 \leq \varepsilon_9 \equiv \min\{\varepsilon_9, (\frac{C_{1/2}}{6C_{24}})^{32}\} \). Thus we immediately obtain (3.49).

On the other hand, from (3.42) it follows that
\[
\|\nabla^2 B\|_{L^q} \leq C \left( \|B_t\|_{L^q} + \|u \cdot \nabla B\|_{L^q} + \|B \cdot \nabla u\|_{L^q} + \|\text{div} u\|_{L^q} \right)
\leq C \|B_t\|_{L^q} + C \|u\|_{L^q} \|\nabla B\|_{L^q} + C \|\nabla u\|_{L^q} \|B\|_{L^q}
\leq C \|\nabla B_t\|_{L^q} + C \|\nabla B\|_{L^q} + C \|\nabla u\|_{L^q},
\]
which, together with (3.53), gives
\[
\|\nabla^2 u\|_{L^q} + \|\nabla^2 B\|_{L^q}
\leq C \left( \|\nabla u\|_{L^q} + \|\nabla^2 u\|_{L^q} + \|\nabla u\|_{L^q} + \|\nabla B\|_{L^q} \right)
+ C \left( \|\rho^{1/2} \theta\|_{L^q} + \|
\theta\|_{L^q} + \|\nabla u\|_{L^q} + \|\nabla B\|_{L^q} \right),
\]
(3.56)
Similarly to (3.54), from (3.56) we can immediately obtain (3.50). The proof of the lemma is therefore completed. \( \square \)

4 Proof of Theorem 1.1

In this section, we prove Theorem 1.1 with the fundamental uniform-in-time estimates established in Sect. 3. By Proposition 2.1 we know that there exists a positive time \( T_* > 0 \)
such that system \((1.1)-(1.4)\) possesses a strong solution \((\rho, u, \theta, B)\) in \(\Omega \times (0, T^*)\). Next, using all the a priori estimates established in Sect. 3, we extend the local strong solution to the global one.

**Proof of global existence.** First, in view of the definitions of \(A_i(T) \ (i = 1, 2, \ldots, 7)\), from \((3.1)\) we easily deduce that

\[
\begin{align*}
A_1(0) &\leq C_0^{1/4}, & A_2(0) &\leq C_0^{1/2}, & A_3(0) &\leq 2K_1, & 0 \leq \rho \leq 2\bar{\rho}, \\
A_4(0) &\leq 2K_2, & A_5(0) &\leq 3\|\nabla \rho_0\|_{L^q}, & A_6(0) + A_7(0) &\leq 1,
\end{align*}
\]

since \(C_0 \leq \varepsilon\). Thus there exists \(T_1 \in (0, T^*)\) such that \((3.1)\) holds for \(T = T_1\).

Set

\[
T^* \triangleq \sup \{ T | (\rho, u, \theta, B) \text{ is a strong solution on } [0, T] \}
\]

and

\[
T^*_1 \triangleq \sup \{ T | (\rho, u, \theta, B) \text{ is a strong solution on } [0, T] \text{ satisfying } (3.1) \}.
\]

Thus \(T^*_1 \geq T_1 > 0\). By Proposition 3.1 we know that

\[
T^* = T^*_1,
\]

provided that \(C_0 \leq \varepsilon\).

Next, similarly to the proof of [25, Sect. 4], we can claim that \(T^* = \infty\). Thus the proof of the theorem is therefore complete.

**Conflict of interest**

The author declares that he has no conflict of interest.

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