We construct a neutrino model using the $S_4$ flavour group under the type-1 see-saw mechanism. The flavon vacuum alignments in the model lead to TM$_1$ mixing with $\sin \theta_{13} = \frac{1}{\sqrt{3}} \sin \frac{\pi}{12}$. The mixing also exhibits $\mu-\tau$ symmetry. By fitting the eigenvalues of the effective see-saw mass matrix with the observed neutrino mass-squared differences, we predict the individual light neutrino masses. The vacuum alignment of the $S_4$ triplet appearing in the Majorana mass term plays a key role in obtaining the aforementioned TM$_1$ scenario. Since the symmetries of the $S_4$ group are not sufficient to determine this alignment, we apply the recently proposed framework of the auxiliary group in the model. Using this framework the $S_4$ triplet is obtained by coupling together several multiplets which transform under the $S_4$ group as well as an auxiliary group which we call $Y_{24}$. The vacuum alignments of these multiplets are uniquely defined in terms of their residual symmetries under $S_4 \times Y_{24}$ which helps us to uniquely define the $S_4$ triplet as well.

Several models have been proposed which generate TM$_1$ mixing. In Refs. [4–7], the discrete group $S_4$ is used to implement this mixing. In Ref. [8], a modified Friedberg–Lee symmetry is utilised. A TM$_1$ scenario with one texture zero is discussed in Ref. [9]. TM$_1$ scenarios with $\mu-\tau$ symmetry and CP symmetry are discussed in Ref. [5]. A model with two highly degenerate right-handed neutrinos resulting in resonant leptogenesis and TM$_1$ mixing is constructed in Ref. [10]. Constrained sequential dominance (CSD) models often predict TM$_1$ mixing along with the lightest neutrino mass equal to zero. We have several such CSD models implemented using $A_4$ [11–13] and $S_4$ [14, 15] symmetries.

Using the parametrisation, Eq. (2), we obtain

$$\sin^2 \theta_{13} = \frac{\sin^2 \theta}{3},$$

(3)

$$\sin^2 \theta_{12} = 1 - \frac{2}{3 - \sin^2 \theta},$$

(4)

$$\sin^2 \theta_{23} = \frac{1}{2} \left(1 + \frac{\sqrt{6} \sin 2\theta \cos \zeta}{3 - \sin^2 \theta}\right),$$

(5)

$$J = \frac{\sin 2\theta \sin \zeta}{6\sqrt{6}},$$

(6)

where

$$J = \text{Im}(U_{\mu 3} U_{e 3}^* U_{e 2} U_{\mu 2}^*),$$

(7)

is the Jarlskog’s rephasing invariant [16–20].

Global fit [21] of neutrino oscillation data gives

$$\sin^2 \theta_{13} = 0.02240^{+0.00065}_{-0.00066},$$

(9)

$$\sin^2 \theta_{12} = 0.310^{+0.013}_{-0.012}.$$  

(10)

This leads to $|U_{12}|^2 = \sin^2 \theta_{12} \cos^2 \theta_{13} = 0.303^{+0.012}_{-0.012}$ and $|U_{12}|^2 + |U_{13}|^2 = \sin^2 \theta_{12} \cos^2 \theta_{13} + \sin^2 \theta_{13} = 0.325^{+0.013}_{-0.012}$. The TBM prediction of $|U_{12}|^2 = \frac{1}{4}$ is disfavoured while the TM$_1$ constraint $|U_{12}|^2 + |U_{13}|^2 = \frac{1}{3}$ is consistent with
the experimental fit at 1σ level. Therefore, we may argue that the most promising among TM1 (TM′) mixings is perhaps TM1.

Discrete groups have been used extensively to model the observed flavour symmetries [22–25]. The three families of fermions are often assumed to transform as a triplet under a given discrete group. Flavour models also include scalar fields called flavons which transform as various multiplets under the group. The Yukawa couplings and thus the fermion mass matrices in the Standard Model (SM) are obtained in terms of the Vacuum Expectation Values (VEVs) of these flavon fields. The VEVs emerge through Spontaneous Symmetry Breaking (SSB) of the flavon potentials. Therefore, the features of the mass matrices have their origin in the properties of the flavour group such as the types of its irreducible multiplets, the Clebsch-Gordan coefficients appearing in their tensor products, the alignment of the flavon VEVs etc.

This paper can be broadly divided into two parts. In the first part a model is constructed using the discrete group $S_4$. The SM is extended by the addition of three right-handed neutrinos transforming as a triplet under $S_4$. Several flavon $S_4$ multiplets are also introduced. The symmetries and the features of $S_4$ as well as the alignments of flavon VEVs ultimately result in the specific phenomenology predicted in this paper. In the second part, we discuss the problem of uniquely defining the vacuum alignments on the basis of discrete symmetries. In this context, we introduce additional symmetries in our model with the help of the framework of the auxiliary group [26]. The alignment of the $S_4$ triplet in the model, which plays a crucial role in determining $\sin \theta_{13} = \frac{1}{\sqrt{3}} \sin \frac{\pi}{7}$, is uniquely obtained in this framework.

The paper is organised as follows. In Section II, we briefly list the essential features of the $S_4$ group and its representations. In Section III, we introduce the flavon content of the model and we construct the mass terms using these flavons, the SM fields and the right-handed neutrinos. We also construct the flavon potential terms and obtain specific flavon VEVs through the mechanism of Spontaneous Symmetry Breaking. The charged-lepton and the neutrino mass matrices are obtained in terms of these VEVs. Section IV covers the phenomenology where we extract the PMNS parameters as well as the light neutrino masses. We compare the results with the experimental data. In Section V, we study the alignments of the triplets of $S_4$ which can be uniquely defined in terms of their residual symmetries. We note that the vacuum alignment of an $S_4$ triplet in the model can not be uniquely defined using the symmetries of $S_4$. In this context, we briefly describe the framework of the auxiliary group with which additional symmetries can be incorporated to obtain a richer choice of vacuum alignments. In Section VI, we construct a discrete group having properties suitable for being an auxiliary group in our model. In Section VII, we recast the model using this auxiliary group to uniquely define the vacuum alignment of the $S_4$ triplet. Finally, we conclude in Section VIII.

### II. THE GROUP $S_4$

$S_4$ is the group of permutations of four objects. It has the presentation,

$$\langle P, Q, R \mid P^2 = Q^3 = R^4 = PQR = I \rangle.$$  (11)

Therefore, $S_4$ is the von Dyck group with parameters (2,3,4). As can be inferred from its presentation, only two of its generators are independent, i.e.

$$P = QR, \quad Q = PR^3, \quad R = Q^2P.$$  (12)

The conjugacy classes and the irreducible representations of $S_4$ are listed in Table I.

The defining triplet representation, 3, can be generated using the matrices,

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$  (13)

This triplet representation denotes the rotational symmetries of a cube, Figure 1(a). In the figure, the unit vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ correspond to the basis states $(1,0,0)^T$, $(0,1,0)^T$ and $(0,0,1)^T$ respectively. $P$, $Q$ and $R$ are rotations by angles $\pi$, $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ about the axes aligned

| $S_4$ |  $\langle 0 \rangle$ |  $\langle 12 \rangle$ |  $\langle 34 \rangle$ |  $\langle 1234 \rangle$ |  $\langle 123 \rangle$ |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $1$   | $1$             | $1$             | $1$             | $1$             | $1$             |
| $1'$  | $1$             | $1$             | $1$             | $1$             | $1$             |
| $2$   | $-1$            | $-1$            | $-1$            | $-1$            | $-1$            |
| $3$   | $-1$            | $-1$            | $-1$            | $-1$            | $-1$            |
| $3'$  | $-1$            | $-1$            | $-1$            | $-1$            | $-1$            |

TABLE I: The character table of the $S_4$ group.
along the directions $i + k$, $i + j + k$ and $i$ respectively. $S_4$ has 24 elements which fall under five conjugacy classes, Table I. The class (123) consists of the elements conjugate to $Q$. They represent rotations by angles $\frac{2\pi}{3}$ and $\frac{4\pi}{3}$ about the body diagonals of the cube. There are 8 elements in this conjugacy class. The elements in the class (1234) are those which are conjugate to $P$. They represent $\pi$-rotations about the axes passing though the centres of the opposite edges of the cube. We have 6 such rotations. The conjugacy class (12) also has 6 elements. They are conjugate to $R$ and they represent rotations by angles $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ about the axes aligned along the basis states. On the other hand $\pi$-rotations about these axes constitute the conjugacy class (12)(34). We have 3 such rotations and they are conjugate to $R^2$.

In model building, a commonly adopted complex basis for $3$ involves the generators $S$, $T$ and $U$,

$$S = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & -1 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

$$U = - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where $\omega = e^{i\frac{2\pi}{3}}$ and $\bar{\omega} = e^{-i\frac{2\pi}{3}}$ are the complex cube roots of unity. The two sets of bases, Eqs.(13, 14), are related by

$$S = U_\omega R^2 U_\omega^\dagger, \quad T = U_\omega Q U_\omega^\dagger, \quad U = U_\omega PRQP U_\omega^\dagger = PRQP,$$

where

$$U_\omega = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \bar{\omega} \\ 1 & \bar{\omega} & \omega \end{pmatrix}$$

is the $3 \times 3$ trimaximal matrix. Unlike $P$, $Q$ and $R$, the generators $S$, $T$ and $U$ are independent. $S_4$ can not be generated from any two of them. In this paper, we adopt the basis, Eqs.(13), so as to provide a geometric interpretation to our results.

Besides the defining triplet $3$, $S_4$ has another triplet $3'$. $3'$ differs from $3$ by an additional space inversion (multiplication by $-1$) along with the rotations for the elements in the conjugacy classes (12) and (1234). We have

$$P(3') = -P, \quad Q(3') = Q, \quad R(3') = -R.$$  

Like $3$, $3'$ is also a faithful representation. The singlet $1'$ involves multiplication with $-1$ corresponding to the space inversion in $3'$, i.e.

$$P(1') = -1, \quad Q(1') = 1, \quad R(1') = -1.$$  

### TABLE II: The fields in the model as the multiplets of $S_4$.

| | $L_{\mu R}$ | $\mu_R$ | $\tau_R$ | $N$ | $H$ | $\chi$ | $s$ | $\phi$ | $\eta$ |
|---|---|---|---|---|---|---|---|---|---|
| $S_4$ | $3'$ | $1$ | $1$ | $3$ | $1$ | $1$ | $3'$ | $2$ |
| $C_4 \times C_3$ | $i$ | $i$ | $i\omega$ | $i\bar{\omega}$ | $i$ | $\omega$ | $-1$ | $-1$ | $1$ |

The generators of the doublet, $2$, are

$$P(2) = \begin{pmatrix} -\frac{1}{2} & -\sqrt{3} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}, \quad Q(2) = \begin{pmatrix} -\frac{1}{2} & \sqrt{3} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R(2) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

$2$ forms the symmetry group of an equilateral triangle, Figure 1(b). $P(2)$ and $R(2)$ are the reflections about the directions $\hat{u}$ and $\frac{-1}{\sqrt{2}}\hat{u} + \frac{1}{\sqrt{2}}\hat{v}$ respectively where $\hat{u}$ and $\hat{v}$ are the basis states $(1, 0)^T$ and $(0, 1)^T$. $Q(2)$ is the rotation by an angle $\frac{2\pi}{3}$ in the $\hat{u}$-$\hat{v}$ plane.

The tensor product expansion of two triplets ($3s$) is given by

$$3 \times 3 = 1 + 2 + 3' + 3.$$  

With $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$ transforming as $3s$, we obtain

$$(ab)_1 = a_1 b_1 + a_2 b_2 + a_3 b_3,$$  

$$(ab)_2 = (2a_1 b_1 - a_2 b_2 - a_3 b_3, \sqrt{3}(a_2 b_2 - a_3 b_3))^T,$$  

$$(ab)_3' = (a_2 b_1 + a_3 b_2, a_3 b_1 + a_1 b_3, a_1 b_2 + a_2 b_1)^T,$$  

$$(ab)_3 = (a_2 b_3 - a_3 b_2, a_3 b_3 - a_1 b_2, a_1 b_3 - a_2 b_1)^T.$$  

where $(ab)_1$, $(ab)_2$, $(ab)_3'$, $(ab)_3$ denote the irreducible parts in the tensor product of $a$ and $b$ which transform as $1$, $2$, $3'$, $3$ respectively. The tensor product of two doublets, $p = (p_1, p_2)$ and $q = (q_1, q_2)$, leads to

$$2 \times 2 = 1 + 1' + 2,$$  

with

$$(pq)_1 = p_1 q_1 + p_2 q_2,$$  

$$(pq)_1' = p_1 q_2 - p_2 q_1,$$  

$$(pq)_2 = (-p_1 q_1 + p_2 q_2, p_1 q_2 + p_2 q_1)^T.$$  

### III. THE MODEL USING $S_4$

The field content of the model is given in Table II. The three families of the left-handed weak-isospin lepton doublets and the three right-handed heavy neutrinos form $S_4$ triplets, $L$ and $N$, respectively. $H$ is the SM Higgs. The
flavons, $s$, $\chi$, $\phi$ and $\eta$, are scalar fields and are gauge invariants. Using this field content and their symmetry properties, we obtain the following Lagrangian:

$$\mathcal{L} = y_s \bar{L}_3 \frac{\chi}{\Lambda} \tau R H + y_\phi \mu R H + y_\chi \bar{L}_3 \frac{(\chi^* \chi)}{\Lambda^2} e R H + y_\nu \bar{N}_c \frac{(N^c N)}{\Lambda} + \mathcal{M}_s (\bar{N}^c N) \frac{s}{\Lambda} + \mathcal{M}_\phi \frac{(N^c N) \phi \phi}{\Lambda},$$

(31)

where $y_s$ are the coupling constants. $(\cdot)_1$, $(\cdot)_3$ and $(\cdot)_3'$ denote the tensor products which transform as $1$, $3$ and $3'$ respectively under $S_4$. The flavon $\chi$ transforms as $3$ under $S_4$ and $\omega$ under $C_3$. We have introduced the $C_3$ group so that $\chi$, $\chi^*$ and $(\chi^* \chi)_3$ couple with $\tau R$, $\mu R$ and $e$ respectively along with the $S_4$ triplet $L$. They form the mass terms in the charged-lepton sector. The next term in the Lagrangian, $y_\nu \bar{N} \frac{N H}{\Lambda}$, is the Dirac mass term for the neutrinos. The flavons, $s$ and $\phi$, transform as $1$ and $3'$ and couple with $(N^c N)_1$ and $(N^c N)_3'$ respectively. They constitute the Majorana mass term for the neutrinos. We have introduced the $C_3$ group to prevent unwanted couplings. $\Lambda$ is the cut-off scale of the theory. $\mathcal{M}_s$ and $\mathcal{M}_\phi$ are constants of mass dimension one which have their origin at a very high energy scale. Through the type-I see-saw mechanism, the light-neutrino masses get suppressed by this scale.

The Higgs acquires the VEV,

$$\langle H \rangle = (0, v)^T,$$

(32)

through Spontaneous Symmetry Breaking (SSB). The VEV corresponds to a minimum point in the Higgs potential. The flavon fields also acquire VEVs through SSB,

$$\langle s \rangle = v_s,$$

(33)

$$\langle \chi \rangle = v_\chi (1, \omega, \bar{\omega})^T,$$

(34)

$$\langle \eta \rangle = v_\eta \left( -\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right)^T,$$

(35)

$$\langle \phi \rangle = v_\phi \left( -\frac{1}{2\sqrt{2}}, -\sqrt{3}, \frac{1}{2\sqrt{2}} \right)^T.$$ (36)

In the following section, we construct the flavon potentials that lead to these VEVs. Note that the flavon $\eta$ does not couple to any of the fermions in the Lagrangian at the lowest order and hence it does not contribute directly to the fermion mass matrices. We use it as a driving field [27] in the construction of the potential of the flavon $\phi$.

A. The flavon potentials

The charged-lepton sector ($\chi$)

With $\chi = (\chi_1, \chi_2, \chi_3)$ and using Eqs. (23-25), we construct the following quadratic expressions:

$$(\chi^* \chi)_1 = \chi_1^* \chi_1 + \chi_2^* \chi_2 + \chi_3^* \chi_3,$$

(37)

$$(\chi \chi)_1 = \chi_1^2 + \chi_2^2 + \chi_3^2,$$

(38)

$$(\chi \chi)_2 = \left(2 \chi_1^2 - \chi_2^2 - \chi_3^2, \sqrt{3}(\chi_2^2 - \chi_3^2) \right)^T,$$

(39)

$$(\chi \chi)_3' = 2(\chi_2 \chi_3, \chi_3 \chi_1, \chi_1 \chi_2)^T.$$ (40)

The antisymmetric expression $(\chi \chi)_3$ vanishes. The only quadratic invariant term is

$$\mathcal{T}(\chi^2) = (\chi^* \chi)_1.$$ (41)

At the cubic order, we do not have any invariant. At the quartic order, we obtain the following invariant terms:

$$\mathcal{T}_1(\chi^4) = (\chi \chi)_1^* (\chi \chi)_1,$$ (42)

$$\mathcal{T}_2(\chi^4) = (\chi \chi)_2^* (\chi \chi)_2,$$ (43)

$$\mathcal{T}_3(\chi^4) = (\chi \chi)_3'^* (\chi \chi)_3'.$$ (44)

$\mathcal{T}(\chi^2)^2$ is related to the above invariants through the relation,

$$6(\mathcal{T}(\chi^2))^2 = 2\mathcal{T}_1(\chi^4) + 2\mathcal{T}_2(\chi^4) + 3\mathcal{T}_3(\chi^4).$$ (45)

Hence we have only three independent invariants at the quartic order. We use them to construct the flavon potential,

$$\mathcal{V}_\chi = \kappa_1 \mathcal{T}_1(\chi^4) + \kappa_2 \mathcal{T}_2(\chi^4) + \kappa_3 \mathcal{T}_3(\chi^4) - 4(\kappa_2^2 + 2\kappa_3) v_\chi^4 \mathcal{T}(\chi^2),$$ (46)

where $\kappa_1, \kappa_2, \kappa_3$ are dimensionless real parameters and $v_\chi$ is a real parameter of mass dimension one. By calculating the first derivative of this potential with respect to the components of $\chi$, we can show that it has a set of extremum points,

$$\chi = e^{i\theta} g_i v_\chi (1, \omega, \bar{\omega})^T,$$ (47)

where $g_i$ are the elements of the group $S_4 \times C_3$ and $e^{i\theta}$ is an arbitrary phase. Through SSB, the flavon acquires one among these extrema as its VEV.

$$\langle \chi \rangle = v_\chi (1, \omega, \bar{\omega})^T.$$ (48)

The neutrino sector ($s, \eta, \phi$)

For the singlet flavon $s$, we write the potential,

$$\mathcal{V}_s = k_s s^2 - 2k_\eta v_\chi^2 s^2,$$ (49)

1. The phase, $e^{i\theta}$, corresponds to the continuous $U(1)$ symmetry of the potential, $\chi \rightarrow e^{i\theta} \chi$. This $U(1)$ symmetry is accidental and will be broken by the higher-order non-renormalisable terms. Since this phase does not have any phenomenological relevance to our model, we ignore it.

2. For a large region of the parameter space $(\kappa_1, \kappa_2, \kappa_3, v_\chi)$, the extremum points correspond to minima. For the stability of the VEV, we assume that the parameters fall in this region.
where \( k_s \) is a dimensionless real parameter and \( v_s \) is a real parameter of mass dimension one. For \( k_s > 0 \), this potential has two points of minima,

\[
  s = \pm v_s. \tag{50}
\]

Through SSB, we obtain one of these minima as the VEV,

\[
  \langle s \rangle = v_s. \tag{51}
\]

We use Eqs. (24) to obtain the tensor product of two flavon doublets, \( \eta = (\eta_1, \eta_2) \), resulting in the quadratic invariant,

\[
  T(\eta^2) = (\eta \eta)_1 = \eta_1^2 + \eta_2^2, \tag{52}
\]

and the doublet,

\[
  (\eta \eta)_2 = (-\eta_1^2 + \eta_2^2, 2\eta_1 \eta_2)^T. \tag{53}
\]

The antisymmetric expression \((\eta \eta)_1\) vanishes. Using \((\eta \eta)_2\), we obtain the cubic invariant,

\[
  T(\eta^3) = \eta^3 (\eta \eta)_2 = -\eta_1^3 + 3\eta_2 \eta. \tag{54}
\]

At the quartic order, we have

\[
  T(\eta^4) = (\eta \eta)_2 \cdot (\eta \eta)_2 = (\eta_1^2 + \eta_2^2)^2. \tag{55}
\]

Note that \( T(\eta^4) = T(\eta^2)^2 \).

Using the flavon \( \phi \), we obtain the following quadratic multipoles:

\[
  (\phi \phi)_1 = \phi_1^2 + \phi_2^2 + \phi_3^2, \tag{56}
\]

\[
  (\phi \phi)_2 = (2\phi_1^2 - \phi_2^2 - \phi_3^2, \sqrt{3}(\phi_2^2 - \phi_3^2))^T, \tag{57}
\]

\[
  (\phi \phi)_2' = (2(\phi_2 \phi_3, \phi_3 \phi_1, \phi_1 \phi_2))^T. \tag{58}
\]

At the quadratic order we have the invariant,

\[
  T(\phi^2) = (\phi \phi)_1. \tag{59}
\]

At the quartic order, we obtain the invariants,

\[
  T_1(\phi^4) = (\phi \phi)_1^2, \tag{60}
\]

\[
  T_2(\phi^4) = (\phi \phi)_2^T (\phi \phi)_2, \tag{61}
\]

\[
  T_3(\phi^4) = (\phi \phi)_2^T (\phi \phi)_2. \tag{62}
\]

These invariants are related by

\[
  4T_1(\phi^4) = T_2(\phi^4) + 3T_3(\phi^4), \tag{63}
\]

so only two of them are independent.

We can also couple \( \eta \) and \( \phi \) to obtain the following invariants:

\[
  T(\eta \phi^2) = \eta^T (\phi \phi)_2, \tag{64}
\]

\[
  T(\eta^2 \phi^2) = (\eta \eta)_2 (\phi \phi)_2. \tag{65}
\]

As a result, we obtain a total of eight independent invariant terms involving \( \eta \) and \( \phi \): \( T(\eta^2), T(\eta^3), T(\eta^4), T(\phi^2), T_1(\phi^4), T_2(\phi^4), T(\eta \phi^2), T(\eta^2 \phi^2) \).

Using these terms, we construct the potential,

\[
  V_{\eta \phi} = c_1 T(\eta^2) + c_2 T(\eta^3) + c_3 T(\eta^4) + c_4 T(\phi^2)
  + c_5 T_1(\phi^4) + c_6 T_2(\phi^4) + c_7 T(\eta \phi^2) + c_8 T(\eta^2 \phi^2). \tag{66}
\]

These terms can be rearranged to obtain

\[
  V_{\eta \phi} = k_1 (T(\eta^2) - v_\eta^2)^2 + k_2 (3T(\phi^2) - (k_3 v_\eta^2 + 4 v_\eta^2))^2
  + k_3 (v_\eta \eta + (\eta \eta)_2)^T (v_\eta \eta + (\eta \eta)_2)
  + k_4 (k_5 v_\eta \eta - (\phi \phi)_2)^T (k_5 v_\eta \eta - (\phi \phi)_2)
  + k_5 (k_6 \eta \eta + (\phi \phi)_2)^T (k_6 \eta \eta + (\phi \phi)_2)
  - k_7 v_\eta^2 - k_8 (k_6 v_\eta^2 + 4 v_\eta^2)^2, \tag{67}
\]

where \( v_\eta, v_\phi \) are parameters with mass dimension one. The parameters \( k_1, ..., k_8 \) are dimensionless and are assumed to be positive. The eight parameters, \( v_\eta, v_\phi, k_1, ..., k_5, k_6 \) are related to \( c_1, ..., c_8 \) through the equations,

\[
  c_1 = (-2k_1 + k_3 + k_3 k_5^2) v_\eta^2, \quad c_5 = 9k_2, \tag{68}
\]

\[
  c_2 = 2k_3 v_\eta, \quad c_6 = k_4 + k_5, \tag{69}
\]

\[
  c_3 = k_1 + k_3 + k_5 k_6^2, \quad c_7 = -2k_5 k_6 v_\eta, \tag{70}
\]

\[
  c_4 = -6k_2 (k_6 v_\eta^2 + 4 v_\eta^2), \quad c_8 = 2k_5 k_6. \tag{71}
\]

The potential, Eq. (67), consists of five terms with the coefficients \( k_1 \) to \( k_5 \). Each of them is positive semidefinite and hence they should all vanish when minimised. The constant factor, \(-k_1 v_\eta^2 - k_2 (k_6 v_\eta^2 + 4 v_\eta^2)^2\), does not play any role in the minimisation of the potential; it is added only for equating Eq. (66) with Eq. (67). The first term, \((T(\eta^2) - v_\eta^2)^2\), vanishes when \( |\eta|^2 = v_\eta^2\); its minimum corresponds to a continuous set of points. The third term, \((\eta \eta)_2 + v_\eta \eta)^T (\eta \eta)_2 + v_\eta \eta\), breaks this continuous symmetry. This term vanishes for

\[
  \eta = g_i v_\eta \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)^T, \tag{72}
\]

where \( g_i \) are the elements of the doublet representation of group \( S_4 \).

The second term, \((3T(\phi^2) - (k_6 v_\eta^2 + 4 v_\eta^2))^2\), is invariant under the three dimensional orthogonal transformation, \( O(3) \), of the flavon \( \phi \) and this term vanishes when \( 3|\phi|^2 = k_6 v_\eta^2 + 4 v_\eta^2 \). This minimum represents a continuous set of points corresponding to the \( O(3) \) symmetry. The forth and the fifth terms couple \( \phi \) with \( \eta \) and they break this continuous symmetry to \( S_4 \). These terms vanish when

\[
  \phi = g_i v_\phi \left( \frac{1}{\sqrt{2}} \sin \alpha, -2 \cos \alpha, -\frac{1}{\sqrt{2}} \sin \alpha \right)^T, \tag{73}
\]
\[
\sin \alpha = \frac{1}{3} \sqrt{8 - k_c \frac{v^2}{\langle \chi \rangle}}, \quad \cos \alpha = \frac{1}{3} \sqrt{1 + k_c \frac{v^2}{\langle \chi \rangle}} \tag{74}
\]

and \(q_i\) are the elements of the triplet representation \((3')\) of \(S_4\). As a result, we obtain a discrete set of minima for the potential.

Through SSB, the flavons acquire one among the above mentioned minima as their VEVs,

\[
\langle \eta \rangle = v_\eta \left( \frac{1}{2} \sqrt{3} \right)^T, \tag{75}
\]

\[
\langle \phi \rangle = v_\phi \left( \frac{1}{\sqrt{2}} \sin \alpha, -2 \cos \alpha, -\frac{1}{\sqrt{2}} \sin \alpha \right)^T. \tag{76}
\]

The VEV assumed in our model, Eq. (34), corresponds to Eq. (76) with \(\alpha = -\frac{\pi}{6}\). We may assign specific values to the parameters in the potential, \(v_\eta, v_\phi, k_c\), so that, using Eq. (74), we obtain \(\alpha = -\frac{\pi}{6}\) and thus obtain the desired VEV. However, tuning the parameters in the potential to obtain the desired VEV is not justified in the context of discrete symmetry. On the other hand, it may point towards additional symmetries. This question is addressed in the second part of this paper where we use the framework of the auxiliary group to uniquely define the VEV in terms of discrete symmetries.

In the following section, we obtain the mass matrices in the charged-lepton and the neutrino sectors in terms of the assigned VEVs.

### B. The charged-lepton mass matrix

The flavon \(\chi\) couples in the charged-lepton mass term,

\[
y_\tau \frac{\chi}{\Lambda} \tau_R H + y_\mu \frac{\chi^*}{\Lambda} \mu_R H + y_\nu \frac{\chi^*}{\Lambda} \eta^* \epsilon_R H. \tag{77}
\]

Due to the \(C_3\) assignments in Table II, we can see that \(\chi\) couples with \(\tau_R\), \(\chi^*\) couples with \(\mu_R\) and \((\chi^* \chi)_3\) couples with \(\epsilon_R\) at the lowest order. Using Eq. (26), we obtain

\[
(\chi^* \chi)_3 = (\chi_2^* \chi_3 - \chi_5^* \chi_2, \chi_3^* \chi_1, -\chi_1^* \chi_3, \chi_1^* \chi_2 - \chi_2^* \chi_1). \tag{78}
\]

The VEV, Eq. (34),

\[
\langle \chi \rangle = v_\chi (1, \omega, \bar{\omega}), \tag{79}
\]

couples with \(\tau_R\) and its conjugate

\[
\langle \chi^* \rangle = v_\chi (1, \bar{\omega}, \omega), \tag{80}
\]

couples with \(\mu_R\). Using Eqs. (78, 79, 80), we obtain

\[
(\chi^* \chi)_3 = i \sqrt{3} v_\chi^2 (1, 1, 1), \tag{81}
\]

which couples with \(\epsilon_R\). Substituting the VEVs, Eqs. (79, 80, 81) and the Higgs VEV, Eq. (32), in Eq. (77), we obtain the charged-lepton mass term after SSB,

\[
\bar{l}_L M_L l_R, \tag{82}
\]

where

\[
l_L = \begin{pmatrix} e_L \\ \mu_L \\ \tau_L \end{pmatrix}, \quad l_R = \begin{pmatrix} e_R \\ \mu_R \\ \tau_R \end{pmatrix}, \tag{83}
\]

and

\[
M_l = i \sqrt{3} v_\chi \frac{v^2}{\Lambda^2} \begin{pmatrix} y_\nu y_\mu & y_\tau \omega y_\mu & y_\tau \omega y_\mu \\ 0 & y_\mu y_\tau & y_\nu y_\tau \end{pmatrix}. \tag{84}
\]

\(M_l\) is the charged-lepton mass matrix.

### C. The neutrino mass matrices

The Dirac mass term for the neutrinos is

\[
y_\nu \bar{L} N \bar{H}. \tag{85}
\]

The Dirac term does not contain flavon fields at the lower order. This term leads to

\[
\bar{\nu}_L M_D N, \tag{86}
\]

where

\[
\nu_L = \begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix}, \quad N = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}, \tag{87}
\]

\[
M_D = v_y \nu_I. \tag{88}
\]

The charged-lepton mass matrix, \(M_D\), is proportional to the identity, \(I\), at the lowest order. Substituting the VEVs, Eqs. (33, 36), in the Majorana mass terms in the Lagrangian, Eq. (31), we obtain

\[
\bar{N}^c M_M N \tag{89}
\]

where

\[
M_M = \frac{v_s}{\Lambda} M_s I + \frac{v_\phi}{\Lambda} M_\phi \Theta \tag{90}
\]

is the Majorana mass matrix with \(\Theta\) being the off-diagonal matrix,

\[
\Theta = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix}. \tag{91}
\]

The effective see-saw mass matrix is given by

\[
M_{ss} = -M_D M_M^{-1} M_D^T. \tag{92}
\]

Using Eqs. (88, 92), we obtain

\[
M_{ss} = -v^2 y_\nu^2 M_D^{-1}. \tag{93}
\]
IV. PHENOMENOLOGY

We diagonalise the charged-lepton mass matrix, $M_l$, Eq. (84), using the $3 \times 3$ trimaximal matrix, $U_\omega$, Eq. (18),

$$U_\omega M_l \text{diag}(-i, 1, 1) = \text{diag}(m_e, m_, m_\tau),$$  

where $m_e = \sqrt{3} g_e v_e v_\tau$, $m_\tau = g_\tau v_e v_\tau$ and $m_\mu = g_\mu v_e v_\tau$ are the charged-lepton masses.

To diagonalise the effective see-saw mass matrix, we study the diagonalisation of $\Theta$, Eq. (91). Applying the (13)-bimaximal matrix,

$$U_{BM} = \begin{pmatrix} 1 & \sqrt{3} & 0 \\ 1 & 1 & 0 \\ \sqrt{3} & 0 & 1 \end{pmatrix},$$  

on $\Theta$ we obtain,

$$U_{BM}^T \Theta U_{BM} = \begin{pmatrix} -3 \sqrt{3} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \sqrt{3} \\ 0 & -\frac{1}{2} \sqrt{3} & 0 \end{pmatrix} = \frac{\sqrt{3}}{2} I + \begin{pmatrix} -3 \sqrt{3} & 0 & 0 \\ 0 & -\cos \alpha \sin \alpha & 0 \\ 0 & \sin \alpha \cos \alpha & 0 \end{pmatrix}$$  

where $\alpha = -\frac{\pi}{6}$. The above matrix is diagonalised as follows:

$$U_{BM}^T U_{BM} = \text{diag}(-\sqrt{3}, \sqrt{3} \frac{2}{3}, 1),$$  

where

$$U_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \text{ with } \theta = -\frac{\pi}{12},$$  

With the help of the above result, we diagonalise the effective see-saw mass matrix,

$$U_{BM}^T M_{ss} U_{BM} U_\theta = -v^2 g^2 U_{BM}^T M_{BM}^{-1} U_{BM} U_\theta = \text{diag}(m_1, m_2, m_3),$$  

where $m_1, m_2, m_3$ are the light neutrino masses,

$$m_1 = (\sqrt{3} - r)^{-1} m, \quad m_2 = (-\frac{\sqrt{3}}{2} + 1 - r)^{-1} m, \quad m_3 = (-\frac{\sqrt{3}}{2} - 1 - r)^{-1} m,$$  

with

$$r = \frac{v_s M_s}{v_\phi M_\phi}, \quad m = v^2 g^2 \frac{\Lambda}{v_\phi M_\phi}.$$  

Here, we have used the fact that a real symmetric matrix as well as its inverse are diagonalised by the same orthogonal matrix. The eigenvalues of the inverse matrix are simply the inverse of the eigenvalues of the original matrix.

The PMNS matrix is obtained by multiplying the diagonalising matrices of the charged-lepton mass matrix and the effective see-saw mass matrix,

$$U = U_\omega U_{BM} U_\theta.$$  

The trimaximal matrix, Eq. (1), can be obtained in terms of $U_\omega$ and $U_{BM}$,

$$U_{BM} = \text{diag}(1, \omega, \bar{\omega}) U_{TBM} \text{diag}(1, 1, i).$$  

Therefore, we obtain

$$U = \text{diag}(1, \omega, \bar{\omega}) U_{TBM} \text{diag}(1, 1, i) U_\theta.$$  

Comparing this equation with Eq. (2), we get

$$U = \text{diag}(1, \omega, \bar{\omega}) U_{TBM} (\frac{\sqrt{2}}{2} \frac{i}{2} \bar{\omega}) \text{diag}(1, 1, i).$$  

The unobservable phases, $\text{diag}(1, \omega, \bar{\omega})$, can be ignored. The Majorana phases, $\text{diag}(1, 1, i)$, are potentially observable in the neutrinoless double-beta decay experiments. Substituting the values of $\theta = \frac{\pi}{12}$ and $\zeta = \frac{\pi}{2}$ in Eqs. (3, 4, 5, 6), we obtain

$$\sin^2 \theta_{13} = \frac{1}{3} \sin^2 \frac{-\pi}{12} = 0.0223,$$  

$$\sin^2 \theta_{12} = 1 - \frac{2}{3 - \sin^2 \frac{-\pi}{12}} = 0.318,$$  

$$\sin^2 \theta_{23} = \frac{1}{2},$$  

$$J = -\frac{1}{12\sqrt{6}}.$$  

Comparing the values of $J$ and the mixing angles with Eq. (7), we obtain

$$\delta_{CP} = \frac{\pi}{2}.$$  

Our predictions, $\sin^2 \theta_{13} = 0.0223$ and $\sin^2 \theta_{12} = 0.318$, are within the $1\sigma$ ranges of the global fit, Eqs. (9, 10). From Eq. (2), it is clear that $TM_1 (\zeta = \pm \frac{\pi}{2})$ results in the $\mu$ and the $\tau$ rows of the mixing matrix being conjugate to each other. This property, termed as the $\mu$-$\tau$ symmetry [28-31], leads to maximal atmospheric mixing, i.e. $\sin^2 \theta_{23} = \frac{1}{2}$ as predicted by the model. There are indications that the atmospheric mixing may not be maximal, which gives rise to the problem of the octant degeneracy. However, at the $3\sigma$ level, the allowed range is $0.428 < \sin^2 \theta_{23} < 0.624$. Therefore, $\sin^2 \theta_{23} = \frac{1}{2}$ is not ruled out yet. CP violation in the lepton sector is still an open problem and we do not have an accurate measurement of $\delta_{CP}$. At the $3\sigma$ level, we have a large range, $135^\circ < \delta_{CP} < 366^\circ$, and the maximal value of $\delta_{CP} = -\frac{\pi}{2}$ lies with in this range.
The neutrino masses, Eqs. (100, 101, 102) are given in terms of two free parameters, \( r \) and \( m \). Using the known mass-squared differences,
\[
\Delta m_{21}^2 = 67.9 \rightarrow 80.1 \text{ meV}^2, \\
\Delta m_{31}^2 = 2431 \rightarrow 2622 \text{ meV}^2,
\]
we can fit these parameters and predict the individual light neutrino masses. The allowed parameter space is shown in Figure 2. The resulting values of the individual neutrino masses are
\[
m_1 = 5.1 \rightarrow 5.6 \text{ meV}, \\
m_2 = 8.5 \rightarrow 9.3 \text{ meV}, \\
m_3 = 49.4 \rightarrow 51.3 \text{ meV},
\]
and their sum is
\[
\Sigma m_i = 63.1 \rightarrow 66.1 \text{ meV}.
\]

The sum of the neutrino masses is constrained by cosmological observations. The upper bounds provided by these observations are a few hundreds of meV \([32, 33]\), the most stringent bound being \( \Sigma m_i < 78 \text{ meV} \) \([32]\). Our predicted value, Eq. (118), is below this bound.

The neutrinoless double-beta decay experiments seek to determine if the neutrinos are Majorana particles. If the neutrinoless double-beta decay is observed, the measurement of half life leads to the determination of the effective mass,
\[
m_{\beta\beta} = U_{e1}^2 m_1 + U_{e2}^2 m_2 + U_{e3}^2 m_3,
\]
where \( U_{e1} \) are the elements of the first row of the PMNS matrix. Using Eq. (107), we get
\[
U_{e1}^2 = \frac{2}{3}, \quad U_{e2}^2 = \frac{1}{12} \left(2 + \sqrt{3}\right), \quad U_{e3}^2 = \frac{1}{12} \left(2 - \sqrt{3}\right)
\]
and as a result we obtain
\[
m_{\beta\beta} = \frac{2}{3} m_1 + \frac{1}{12} \left(2 + \sqrt{3}\right) m_2 + \frac{1}{12} \left(2 - \sqrt{3}\right) m_3.
\]

Given our predicted masses, Eqs. (115, 116, 117), we obtain
\[
m_{\beta\beta} = 7.2 \rightarrow 7.7 \text{ meV}.
\]
This range is well below the upper bounds set by the recent 0ν\(\beta\beta\) experiments \([34, 35]\). Figure 3 shows the ranges of \( \Sigma m_i \) and \( m_{\beta\beta} \) predicted by the model along with the best fit point.

\section*{V. UNIQUELY DEFINING THE VEVS}
\subsection*{A. Orbits and stabilisers}

In this section, we describe the concepts of orbits and stabilisers by using the action of the triplet representations of \( S_4 \) as examples. We utilise these concepts later in
the paper. Various representations of a group act on the corresponding vector spaces. The triplet representations of $S_4$ act on the three dimensional real space. Consider the set of all unit vectors forming a sphere in the 3-D real space. The action of an element of $3$ of $S_4$ on any unit vector rotates it from one point on the sphere to another. Since $3$ has 24 distinct elements, the group action on a given point leads to a set of 24 points on the sphere in the most general case, as shown in Figure 4(a). This set of points form the orbit of the given point. The group action on any point within the orbit produces the same orbit; therefore, the orbit remains closed under the group action.

It need not be the case that all the orbits have their cardinality equal to the number of elements in the representation. Group action on $x = \frac{1}{\sqrt{3}}(1, 1, 1)$ will produce the points,

$$\frac{1}{\sqrt{3}} \{(\pm 1, \pm 1, \pm 1), (\pm 1, \pm 1, \mp 1),$$

$$(\pm 1, \mp 1, \pm 1), (\pm 1, \mp 1, \mp 1)\}, \quad (123)$$

which form the vertices of a cube, Figure 4(b). We have only 8 points rather than 24, because each point on the cube remains invariant under the action of certain elements of the group. The set of such elements form a subgroup and it is termed as the stabiliser of the point. The stabiliser of $\frac{1}{\sqrt{3}}(1, 1, 1)$ is $\{Q, Q^2, I\}$ which forms a $C_3$ subgroup of $S_4$. We have four such $C_3$ subgroups, corresponding to the four pairs of opposite vertices of the cube. The orbit-stabiliser theorem states that

$$|\text{Orb}(x)| = \frac{|G|}{|\text{Stab}(x)|} \quad (124)$$

where $|$ denotes the cardinality and $\text{Orb}(x)$ and $\text{Stab}(x)$ denote the orbit and the stabiliser respectively of a point $x$. In the case of the cube, $|\text{Stab}(x)| = 3$; therefore, we have

$$|\text{Orb}(x)| = \frac{24}{3} = 8, \quad (125)$$

which is consistent with the number of vertices of the cube.

The group action on $x = (1, 0, 0)^T$ produces an octahedron,

$$\{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}, \quad (126)$$

Figure 4(c). In this case, the stabiliser is the $C_4$ subgroup, $\{R, R^2, R^3, I\}$. Using the orbit-stabiliser theorem, we obtain the cardinality of the orbit (the number of vertices of the octahedron) to be $24/4=6$. The group action on $x = \frac{1}{\sqrt{2}}(1, 0, 1)$ produces a cuboctahedron,

$$\frac{1}{\sqrt{2}} \{(\pm 1, 0, \pm 1), (\pm 1, \pm 1, 0), (0, \pm 1, \pm),$$

$$(\pm 1, 0, \mp 1), (\pm 1, \mp 1, 0), (0, \pm 1, \mp)\}, \quad (127)$$

which forms the vertices of the tetrahedron, Figure 4(d). In this case, the stabiliser is the $C_2$ subgroup, $\{P, I\}$ and we obtain the number of vertices of the cuboctahedron to be $24/2=12$. The cube, Eq. (123), the octahedron, Eq. (126), and the cuboctahedron, Eq. (127), are the only ‘unique’ orbits that can be constructed in the case of the representation $3$. For any other orbit of $3$, the stabiliser of a point will be the trivial group and the orbit will have the cardinality of 24.

Now let us analyse the representation $3'$. We have $P(3') = -P$, $Q(3') = Q$, $R(3') = -R$, Eqs. (19). This representation is the symmetry group of a tetrahedron which includes 12 proper rotations and 12 improper rotations. The action of $3'$ on the point $\frac{1}{\sqrt{3}}(1, 1, 1)$ results in the orbit

$$\frac{1}{\sqrt{3}} \{(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)\}, \quad (128)$$

which forms the vertices of the tetrahedron, Figure 5(a). The point $\frac{1}{\sqrt{3}}(1, 1, 1)$ has the stabiliser $\{Q, Q^2, -QPQ^2R^3, -QPQ^2R^2Q, -QPQ^2R^3Q, -Q^2PQ^3R^2Q\}$ which forms the subgroup $D_6$. The cardinality of the orbit (the number of vertices of the tetrahedron) is $24/6 = 4$. We have a second tetrahedron,

$$\frac{1}{\sqrt{3}} \{(-1, -1, -1), (-1, 1, 1), (1, -1, 1), (1, 1, -1)\}, \quad (129)$$

which also forms an orbit. This tetrahedron is the space inversion of Figure 5(a). Combining the two tetrahedra,
we obtain a cube.

The group action on the point \((1,0,0)\) produces the orbit,

\[ \{ (\pm 1,0,0), (0,\pm 1,0), (0,0,\pm 1) \} , \tag{130} \]

which forms an octahedron, Figure 5(b). The point \((1,0,0)^T\) has the stabiliser \(\{ R^2, -QPQ^2, -QPQ^2R^2, I \} \) which forms the subgroup \(C_2 \times C_2\). The cardinality of the orbit is 24/(2 \times 2) = 6. The tetrahedra, Eqs.(128,129) and the octahedron, Eq. (130) are the only unique orbits that can be constructed in the case of the representation \(3'\).

The action of \(3'\) on a random point produces an orbit with 24 elements. This situation is shown in Figure 5(c). Here we have a trivial stabiliser. For \(3'\), a point having a non-trivial stabiliser is not the sufficient condition for its orbit to be unique. Consider a point \((\omega,\chi,\eta)\) has the stabiliser \(\{ -P, I \} \) which is a non-trivial stabiliser. The orbit of this point has 24/2 = 12 elements and is shown in Figure 5(d). It is clear that this orbit is not unique, rather it depends on the arbitrary parameter \(\alpha\).

Let us study the VEVs used in our model in terms of their residual symmetries and check whether these VEVs can be uniquely defined. Consider the VEV of the doublet flavon, \(\langle \eta \rangle \propto \left( \frac{1}{\sqrt{2}} \sin \alpha, -2 \cos \alpha, -\frac{1}{\sqrt{2}} \sin \alpha \right)\), Eq. (76). This point remains invariant under the action of \(\{ -P, I \} \) which is a non-trivial stabiliser. The orbit of this point has 24/2 = 12 elements and is shown in Figure 5(d). It is clear that this orbit is not unique, rather it depends on the arbitrary parameter \(\alpha\).

The flavon \(\chi\) transforms as 3 under \(S_4\). It is also assigned a \(C_3\) symmetry \((\omega)\), Table II, and hence it has complex degrees of freedom. Its VEV, \(\langle \chi \rangle \propto (1, \omega, \bar{\omega})\), Eq. (34), is uniquely defined in terms of its residual symmetry,

\[ \omega Q^2 \langle \chi \rangle = \langle \chi \rangle . \tag{132} \]

\(\omega Q^2\) generates the \(C_3\) residual symmetry of \(\langle \chi \rangle\).

The flavon \(\phi\) transforms as \(3'\) under \(S_4\). Its VEV \(\langle \phi \rangle \propto \left( -\frac{1}{2\sqrt{3}}, -\sqrt{3}, \frac{1}{2\sqrt{3}} \right)\), Eq. (36), has the following \(C_2\) residual symmetry:

\[ P(3') \langle \phi \rangle = -P \langle \phi \rangle = \langle \phi \rangle . \tag{133} \]

This symmetry ensures that the third component of the VEV is negative of the first component. However, it does not determine the value of the second component in relation to the others. In fact, this value can not be determined by any of the symmetries of \(S_4\). Rather we obtained it in terms of \(\alpha\), Eq. (76), which is a continuous function of the parameters, Eq. (74), appearing in the flavon potential, Eq. (67). This potential results in a set of minima which forms the orbit shown in Figure 5(d). We tuned them so that the second component of the VEV is negative of the first component.

The action of 3 leading to \(-\frac{\pi}{2}\). This tuning of the parameters changes the orbit (resizes the four small triangles in Figure 5(d)) as a continuous function and enables us to choose the VEV from among an infinite set of possible alignments. We argue that such a dependence of the VEV on the parameters of the potential goes against the spirit of using discrete symmetries in model building. We can resolve this problem by incorporating additional symmetries in the model. A way to achieve this is by utilizing the recently proposed framework of the auxiliary group [26].

### B. The Framework of the Auxiliary Group

In this framework the flavour group \((G_f)\) is obtained as the direct product of a subgroup of \(U(3)\) \((G_{U(3)})\) and the auxiliary group \((G_X)\),

\[ G_f = G_{U(3)} \times G_X . \tag{134} \]

The fields in the Standard Model transform as multiplets under \(G_{U(3)}\) while the flavons transform under both \(G_{U(3)}\) and \(G_X\). Unlike \(G_{U(3)}\), \(G_X\) need not be a subgroup of \(U(3)\). The purpose of \(G_X\) is to incorporate symmetries in addition to those originating from \(G_{U(3)}\).

An effective multiplet of a given irreducible representation of \(G_{U(3)}\) can be obtained by coupling together several flavons transforming as irreducible multiplets under \(G_{U(3)}\times G_X\). The vacuum alignments of these individual irreducible multiplets are constructed in such a way that they are invariants under the action of various subgroups of \(G_{U(3)}\times G_X\), i.e. these subgroups form the residual symmetries of the VEVs of the respective multiplets. In other words, we uniquely define the VEVs of the multiplets of \(G_{U(3)}\times G_X\) in terms of their residual symmetries. As a result, the VEV of the effective multiplet of \(G_{U(3)}\) also becomes uniquely defined even though it lives in the space of \(G_{U(3)}\) and its VEV may not be uniquely defined in terms of its residual symmetries under \(G_{U(3)}\). In this framework the vacuum alignments and the resulting phenomenology do not depend on the tuning of the parameters in the flavon potential. Therefore, we may even avoid constructing the potential altogether.

### VI. THE GROUP \(Y_{24}\)

The auxiliary group used in our model is named \(Y_{24}\). In this section we construct this group and briefly study its structure. Let us consider the group generated by

\[
A = \begin{pmatrix}
0 & 0 & 0 & -\bar{\tau} & 0 \\
0 & 0 & 0 & 0 & i\bar{\omega} \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -i\omega & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{pmatrix}, \quad
B = \begin{pmatrix}
0 & \bar{\tau} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \tau \\
0 & 0 & 0 & 0 & -i\omega \\
0 & 0 & i\bar{\omega} & 0 & 0 \\
\end{pmatrix} ,
\]

(135)
where $\tau = e^{i\frac{\pi}{3}}$ and $\bar{\tau} = e^{-i\frac{\pi}{3}}$ are the complex eighth roots of unity. Using $A$ and $B$, we obtain the group elements $D = (AB)^3$ and $E = (AB)^2$ which are given by

$$
D = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}, \quad E = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.
$$

We also obtain the following diagonal group elements consisting of complex phases,

$$
C_{\omega_3} = (DA)^8 = \text{diag}(1, 1, 1, 1, \omega, \omega), \quad C_{\omega_2} = E(DA)^8E^2 = \text{diag}(1, 1, 1, \bar{\omega}, \bar{\omega}, \omega),
$$

$$
C_{\omega_3} = DC_{\omega_3}D = \text{diag}(1, \omega, \omega, 1, 1, 1), \quad C_{\omega_4} = DC_{\omega_4}D = \text{diag}(\bar{\omega}, 1, \omega, 1, 1, 1),
$$

$$
C_{\tau_1} = (DA)^9 = \text{diag}(\tau, 1, 1, \bar{\tau}, \tau, 1), \quad C_{\tau_2} = E(DA)^9E^9 = \text{diag}(1, \tau, 1, -i, \bar{\tau}, i),
$$

$$
C_{\tau_3} = E^2(DA)^9E = \text{diag}(1, 1, 1, 1, \bar{\tau}, \bar{\tau}).
$$

We obtain the structure of the group generated by $A$ and $B$ with the help of the above mentioned group elements, $D$, $E$ and $C_1$. We have

$$
D^2 = I, \quad E^3 = I, \quad DE = ED.
$$

$D$ and $E$ form the cyclic groups $C_2$ and $C_3$ respectively and they commute with each other. Therefore, the two of them form the group $C_6$. The diagonal group elements $C_{\omega_1}$, $C_{\omega_2}$, $C_{\omega_3}$ and $C_{\omega_4}$ generate four separate $C_3$ groups. Similarly the elements $C_{\tau_1}$, $C_{\tau_2}$, $C_{\tau_3}$ generate three separate $C_3$ groups. We have

$$
DC_{\omega_1} = C_{\omega_3}D, \quad EC_{\omega_1} = C_{\omega_2}E,
$$

$$
DC_{\omega_2} = C_{\omega_4}D, \quad EC_{\omega_2} = C_{\omega_1}^2C_{\omega_2}E,
$$

$$
DC_{\omega_3} = C_{\omega_1}D, \quad EC_{\omega_3} = C_{\omega_4}E,
$$

$$
DC_{\omega_4} = C_{\omega_2}D, \quad EC_{\omega_4} = C_{\omega_1}^2C_{\omega_4}E,
$$

$$
DC_{\tau_1} = C_{\tau_1}^7C_{\tau_2}^2C_{\tau_3}^6D, \quad EC_{\tau_1} = C_{\tau_2}E,
$$

$$
DC_{\tau_2} = C_{\tau_1}^2C_{\tau_2}^3C_{\tau_3}D, \quad EC_{\tau_2} = C_{\tau_3}E,
$$

$$
DC_{\tau_3} = C_{\tau_1}^5C_{\tau_2}^2C_{\tau_3}^6D, \quad EC_{\tau_3} = C_{\tau_1}E.
$$

Also, the diagonal group elements, $C_1$, commute with each other. Therefore, $D$, $E$ and $C_1$ form a closed group. The generators $A$ and $B$ can be expressed in terms of $D$, $E$ and $C_1$.

$$
A = C_{\omega_2}C_{\tau_1}^5C_{\tau_2}^2C_{\tau_3}^6D, \quad B = C_{\omega_1}C_{\tau_1}^{15}E^2.
$$

Therefore, the group generated by $A$ and $B$ is the same as the one generated by $D$, $E$ and $C_1$. The diagonal elements $C_{\omega_1}$, $C_{\omega_2}$, $C_{\omega_3}$, $C_{\omega_4}$, $C_{\tau_1}$, $C_{\tau_2}$, $C_{\tau_3}$ form the group $C_3 \times C_3 \times C_3 \times C_3 \times C_6 \times C_6 \times C_6$ which is equivalent to $C_3 \times C_24 \times C_24 \times C_24$. As mentioned earlier, $D$ and $E$ form the group $C_6$. Therefore, the group generated by $A$ and $B$ is simply the semidirect product,

$$
G(A, B) = (C_3 \times C_24 \times C_24 \times C_24) \rtimes C_6.
$$

We expand this group with an additional generator,

$$
C_1 = \text{diag}(1, 1, 1, 1, -1, -1).
$$

Using $C_1$, we obtain the group element,

$$
C_2 = E_1E^2 = \text{diag}(1, 1, 1, -1, -1, -1).
$$

The closure property of the expanded group is ensured by the following equations:

$$
DC_1 = C_1C_2^4C_3^4D, \quad EC_1 = C_2E,
$$

$$
DC_2 = C_2C_1^4C_3^4D, \quad EC_2 = C_1C_2E.
$$

$C_1$ and $C_2$ generate two $C_2$ groups. As a result we obtain

$$
G(A, B, C_1) = (C_2 \times C_2 \times C_3 \times C_24 \times C_24 \times C_24) \rtimes C_6.
$$

In this paper, we use $G(A, B, C_1)$ as the auxiliary group. For convenience we name this group as $Y_{24}$.

$$
Y_{24} = G(A, B, C_1) = (C_2 \times C_6 \times C_24 \times C_24 \times C_24) \rtimes C_6.
$$

Any element of $Y_{24}$ can be uniquely expressed as,

$$
g_i = C_1^{i_1}C_2^{j_2}C_3^{j_3}C_4^{j_4}C_5^{j_5}C_6^{j_6}D \bar{C}_1^{j_1}E^{j_2},
$$

where $i_x = 1, 2, j_x = 1, 2, 3$ and $k_x = 1, 2...8$. This group has a total of $2 \times 6 \times 24^3 \times 6 = 995328$ elements. We used the computational package GAP [36] to obtain the group with $A$, $B$ and $C_1$ as the generators. According to GAP, such a group has 995328 elements confirming our analysis.

The sextet representation of $Y_{24}$, Eqs. (135, 148), is named $6$. Its tensor product with itself leads to the following expansion,

$$
6 \times 6 = 6_0 + 6_1 + 6_2 + 3 + 6_1 + 6_2 + 3'.
$$

The first four multiplets in the RHS of the above equation form the symmetric part and the next three form the antisymmetric part of the tensor product. In this paper, besides $6$, we also utilise the representation $6_1$. This sextet is given by

$$
6_1 \equiv \{a_2, b_3\}, \{a_3, b_1\}, \{a_1, b_2\}, \{a_5, b_6\}, \{a_6, b_4\}, \{a_4, b_5\}
$$

with $a_i, b_j = a_i b_j + a_j b_i$.
| S_4 | Y_{24} | C_3 \times C_3 | C_2 |
|-----|------|---------------|------|
| 3 1 1 1 3 3 1 1 1 3 3 1 | 1 1 1 1 1 1 1 1 1 6 6 6 | i i \omega i \omega i 1 \omega 1 1 -1 1 1 -1 | 1 1 1 1 1 1 1 -1 -1 1 -1 -1 |

This completes our discussion of the structure of the group, \( Y_{24} \).

**VII. THE MODEL RECAST USING \( S_4 \times Y_{24} \)**

The field content of the model recast using the auxiliary group is given in Table III. The left-handed lepton field \( (L) \), the right-handed neutrino field \( (N) \) and the flavon field \( \chi \) are triplets \( (3) \) under \( S_4 \) and invariants under \( Y_{24} \). The flavons \( \Phi \) and \( \Phi \) are not only triplets \( (3) \) under \( S_4 \) triplet, but they also sextets \( (6) \) under \( Y_{24} \). The \( C_2 \) symmetries are added so that \( \tilde{S}, \tilde{S}, S \) as well as \( \Phi, \tilde{\Phi}, \Delta \) couple together in the Lagrangian. In addition to the symmetries listed in the table, we impose the following complex conjugation symmetry: \( \Phi \rightarrow \Phi^*, \tilde{\Phi} \rightarrow \tilde{\Phi}^*, \Delta \rightarrow \Delta^* \). Based on these symmetries, we construct the following Lagrangian:

\[
\mathcal{L} = y_{\tau} \bar{L} \frac{\chi}{\Lambda} \tau_R H + y_{\tilde{\nu}} \frac{\tilde{\nu}^*}{\Lambda} \mu_R H + y_e \frac{L (\chi^* \chi) \bar{\nu}}{\Lambda^2} e_R H + y_{\nu} \bar{L} N \tilde{H} + \mathcal{M}_S (\tilde{N} e N) \frac{\bar{S} \tilde{S} \bar{S}}{\Lambda^3} + \mathcal{M}_\Phi (\tilde{N} e N) \frac{\bar{\Phi} \tilde{\Phi} + (\Phi^* \Delta^* \Phi^*)}{\Lambda^3}.
\] (157)

In the Lagrangian, \( \Phi, \tilde{\Phi} \) and \( \Delta \) couple together resulting in the ‘effective’ triplet \( (3') \) of \( S_4 \):

\[
\left( \bar{\Phi} \Phi \right)_{3'} + \left( \Phi^* \Delta^* \Phi^* \right)_{3'}.
\] (158)

Our original Lagrangian, Eq. (31), is the same as Eq. (157), except for the fact that the triplet \( \phi \) is replaced with the ‘effective’ triplet and the singlet \( s \) is replaced with \( \bar{S} \tilde{S} \).

The flavons \( \Phi \) and \( \tilde{\Phi} \) can be expressed using two indices, \( \Phi_{\alpha i} \) \( (\tilde{\Phi}_{\alpha i}) \), where the Greek and the Latin letters correspond to \( Y_{24} \) and \( S_4 \) respectively. To obtain the effective triplet explicitly we use the tensor product expansions given in Eqs. (25, 155). The Clebsch-Gordon coefficients corresponding to Eq. (25) and Eq. (155) can be given by

\[
C_{lmn} = 1 \quad \forall \ l \neq m \neq n
\]
and

\[
C_{lmn} = 1 \quad \forall \ l \ mod \ 3 \neq m \ mod \ 3 \neq n \ mod \ 3 \quad \& \quad (l - 1) \ div \ 3 = (m - 1) \ div \ 3 = (n - 1) \ div \ 3 = 0 \quad \text{otherwise}
\]

respectively. Using these coefficients we obtain

\[
\left( \bar{\Phi} \Delta \Phi \right)_{3'} = \sum C_{lmn} C_{\alpha \beta \gamma} \bar{\Phi}_{\beta m} \Phi_{\gamma n} \Delta_{\alpha},
\] (161)

where the summation is over all the repeated indices. \( \left( \bar{\Phi}^* \Delta^* \Phi^* \right)_{3'} \) is simply the complex conjugate of \( \left( \bar{\Phi} \Delta \Phi \right)_{3'} \).

The flavons are assigned the following VEVs:

\[
\langle \Phi \rangle = v_\Phi \begin{pmatrix} 1 & 0 & 0 \\ 0 & \tau & 0 \\ -i \omega & 0 & 0 \end{pmatrix}, \quad \langle \Phi \rangle = v_\Phi \begin{pmatrix} i \omega & 0 & 0 \\ 0 & \tau & 0 \\ -1 & 0 & 0 \end{pmatrix},
\]

\[
\langle \Delta \rangle = v_\Delta \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \end{pmatrix}.
\] (162)

Here, \( \langle \Phi \rangle \) and \( \langle \Phi \rangle \) are given in matrix forms with the rows (columns) representing \( Y_{24} \) \( (S_4) \). Using Eqs. (161, 162, 163), we obtain

\[
\left( \left( \bar{\Phi} \Delta \Phi \right)_{3'} + \left( \Phi^* \Delta^* \Phi^* \right)_{3'} \right) = 2 v_\Delta^2 v_\Delta \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 2 \sqrt{2} & 0 \\ 0 & 0 & 2 \sqrt{2} \end{pmatrix} \right).
\] (164)

This alignment is proportional to the VEV of the triplet flavon which we originally proposed in Eq. (36). We also assign

\[
\langle \bar{S} \tilde{S} \rangle = v_S^3
\] (166)

which corresponds to the VEV of the singlet flavon, Eq. (33).

We did not include the \( S_4 \) effective singlet, \( \left( \bar{\Phi} \Delta \Phi \right) + \left( \Phi^* \Delta^* \Phi^* \right)_1 \) and the doublet, \( \left( \bar{\Phi} \Delta \Phi \right)_2 + \left( \Phi^* \Delta^* \Phi^* \right)_2 \), in the Lagrangian, Eq. (157), because they vanish for the given set of VEVs, Eqs. (162, 163).
A. The Residual Symmetries of $\langle \Phi \rangle$, $\langle \bar{\Phi} \rangle$ and $\langle \Delta \rangle$

In this section we show that each of the VEVs, $\langle \Phi \rangle$, $\langle \bar{\Phi} \rangle$, $\langle \Delta \rangle$, is an invariant under the action of certain elements of the flavour group. The VEV does not fully break the flavour group, rather it is broken into a specific subgroup which constitutes its residual symmetries. In this way, the alignment of each of the VEVs is uniquely determined.

Let us first study the symmetries of the alignment

$$\langle \Phi \rangle = |\langle \Phi \rangle| = |\langle \bar{\Phi} \rangle| = v_\Phi $$

This alignment remains invariant under the following group actions:

$$\mathcal{O}_1 \langle \Phi \rangle = C_1 DC_1 D \langle \Phi \rangle \ , \ \mathcal{Q}RQPQ = \langle \Phi \rangle, \hspace{1cm} (168)$$

$$\mathcal{O}_2 \langle \Phi \rangle = C_2 DC_2 D \langle \Phi \rangle \ , \ \mathcal{R}QPQ = \langle \Phi \rangle, \hspace{1cm} (169)$$

$$\mathcal{O}_3 \langle \Phi \rangle = D \langle \Phi \rangle I = \langle \Phi \rangle, \hspace{1cm} (170)$$

$$\mathcal{O}_4 \langle \Phi \rangle = E \langle \Phi \rangle Q = \langle \Phi \rangle, \hspace{1cm} (171)$$

where

$$C_1 DC_1 D = \text{diag}(1, -1, -1, 1, -1, -1), \hspace{1cm} (172)$$

$$QRQPQ = \text{diag}(1, 1, -1), \hspace{1cm} (173)$$

$$C_2 DC_2 D = \text{diag}(-1, 1, -1, -1, 1), \hspace{1cm} (174)$$

$$RQPQ = \text{diag}(-1, -1, 1). \hspace{1cm} (175)$$

The group actions $\mathcal{O}_1$ and $\mathcal{O}_2$, Eqs. (168, 169), multiply certain components of $\langle \Phi \rangle$ with $-1$. Invariance under this action ensures that those components vanish. Eqs. (168, 169) ensure that all the components marked with zeros in Eq. (167) vanish. $\mathcal{O}_3$ and $\mathcal{O}_4$ form a $C_2 \times C_2$ subgroup of the flavour group. $\mathcal{O}_3$ exchanges the first three rows of $\langle \Phi \rangle$ with the last three. Invariance under $\mathcal{O}_3$ ensures that the upper three rows are equal to the lower three. This condition is satisfied by $\langle \Phi \rangle$. $\mathcal{O}_3$ generates another $C_2$ subgroup. The group action $\mathcal{O}_4$ cycles the following sets of components: $\{1, 1\}$, $\{2, 2\}$, $\{3, 3\}$, $\{1, 3\}$, $\{2, 1\}$, $\{3, 2\}$, $\{1, 2\}$, $\{2, 3\}$, $\{3, 1\}$, $\{4, 1\}$, $\{5, 2\}$, $\{6, 3\}$, $\{4, 3\}$, $\{5, 1\}$, $\{6, 2\}$, $\{4, 2\}$, $\{5, 3\}$, $\{6, 1\}$. Invariance under this action ensures that the components within each set are equal. This condition is also satisfied by the VEV. $\mathcal{O}_4$ generates a $C_3$ subgroup. The conditions, Eqs. (168, 169, 170, 171) results in the vacuum alignment to be proportional to $\langle \Phi \rangle$ given in Eq. (167). It can be shown that the group generated by $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ and $\mathcal{O}_4$ has 96 elements and has the structure $C_2 \times ((C_2 \times C_2) \times C_2) \times C_3) = C_2 \times (C_2 \times C_3)$. This group represents the residual symmetries of $\langle \Phi \rangle$ and it uniquely defines $\langle \Phi \rangle$.

Now we show that $\langle \bar{\Phi} \rangle$ and $\langle \bar{\Phi} \rangle$ can also be defined in terms of specific $C_2 \times (C_2 \times C_3)$ subgroups of the flavour group. Consider the group elements

$$\tilde{g} = C_2 C_2 C_2 = \text{diag}(1, 1, -i \omega, \tau, i \omega) \hspace{1cm} (176)$$

$$\tilde{g} = C_2 D C_2 C_2 = \text{diag}(i \omega, \tau, -i \omega, -1, \bar{\tau}, -1) \hspace{1cm} (177)$$

Their action on $\langle \bar{\Phi} \rangle$ is

$$\tilde{g} \langle \bar{\Phi} \rangle = \langle \bar{\Phi} \rangle, \hspace{1cm} \tilde{g} \langle \bar{\Phi} \rangle = \langle \bar{\Phi} \rangle. \hspace{1cm} (178)$$

We define a new set of group elements,

$$\tilde{O}_i = \tilde{g} \tilde{O}_i \tilde{g}^{-1}, \hspace{1cm} \tilde{O}_i = \tilde{g} \tilde{O}_i \tilde{g}^{-1}, \hspace{1cm} (179)$$

where $i = 1, ..., 4$. These elements uniquely define $\langle \bar{\Phi} \rangle$ and $\langle \bar{\Phi} \rangle$ in terms of their residual symmetries,

$$\tilde{O}_i \langle \bar{\Phi} \rangle = \langle \bar{\Phi} \rangle, \hspace{1cm} \tilde{O}_i \langle \bar{\Phi} \rangle = \langle \bar{\Phi} \rangle. \hspace{1cm} (180)$$

The model $\Delta$ transforms as $\bar{6}_1$. The generators of $\bar{6}_1$ are given in Eqs. (148). Using these generators we can show that $D(\bar{6}_1) = D(\bar{6}) = D$ and $E(\bar{6}_1) = E(\bar{6}) = E$. The vacuum alignment, $\langle \Delta \rangle = (1, 1, 1, 1, 1, 1)$, Eq. (163), remains invariant under the following group actions:

$$D(\langle \Delta \rangle) = \langle \Delta \rangle, \hspace{1cm} E(\langle \Delta \rangle) = \langle \Delta \rangle. \hspace{1cm} (181)$$

$D$ interchanges the first three elements of $\langle \Delta \rangle$ with the last three. Invariance under this action ensures that the 1st, the 2nd and the 3rd components are equal to the 4th, the 5th and the 6th respectively. $E$ cycles the first three components as well as the last three. Invariance under $E$ ensures that the first three components are equal to one another and also the last three components are equal to one another. Invariance under both $D$ and $E$ implies that all the elements of the sextet are equal which is true for $\langle \Delta \rangle$. $D$ and $E$ generate $C_2$ and $C_3$ respectively and together they generate $C_6$ which forms the residual symmetry of $\langle \Delta \rangle$. This residual symmetry uniquely defines $\langle \Delta \rangle$.

VIII. CONCLUSION

TM1 mixing preserves the first column of the tribimaximal mixing. TM1 can be parametrised using an angle $\theta$ and a phase $\zeta$. In this parametrisation the reactor mixing angle is given by $\sin^2 \theta_{31} = \frac{1}{3} \sin^2 \theta$. In this paper, we construct a model based on the $S_4$ discrete group and obtain TM1 mixing with $\theta = -\frac{\pi}{12}$ and $\zeta = \frac{\pi}{4}$ consistent with the current experimental data. The mixing obtained exhibits $\mu - \tau$ symmetry so that we have $\theta_{32} = \frac{\pi}{4}$. We also obtain maximal CP violation with $\delta = -\frac{\pi}{2}$. The model is built in the type-1 see-saw framework and we obtain
Majorana neutrinos having normal hierarchical masses. The masses are functions of two model parameters and by fitting them with the experimental values of the mass-squared-differences we predict the individual masses. We also predict the effective neutrino mass applicable to the neutrinoless double-beta decay.

To construct the fermion mass terms, we introduce the flavons $\chi$, $s$, $\phi$ which transform as $3$, $1$ and $3'$ respectively under $S_4$. We construct the flavon potentials which when minimised lead to their vacuum expectation values. The flavon $\chi$ couples in the charged-lepton sector and its vacuum alignment leads to the trimaximal contribution to the neutrino mixing matrix. The flavons $s$ and $\phi$ couple in the Majorana sector. Their vacuum alignments constitute the Majorana mass matrix. The product of the trimaximal matrix and the diagonalising matrix from the Majorana sector generates the TM$_1$ mixing with $\theta = -\frac{\pi}{12}$ and $\zeta = \frac{\pi}{2}$. Obtaining the required vacuum alignment of $\phi$ entails tuning of the parameters in the flavon potential. This tuning implies that the symmetries of $S_4$ are not sufficient to uniquely determine the vacuum alignment of $\phi$. Therefore, we introduce additional symmetries using the framework of the auxiliary group.

We construct the group $Y_{24}$ which plays the role of the auxiliary group for our model. We introduce the flavons $\Phi$, $\bar{\Phi}$ and $\Delta$ which transform as $3 \times 6$, $3 \times 6$ and $1 \times \bar{6}$, respectively under $S_4 \times Y_{24}$. They couple together resulting in the effective triplet of $S_4$, $\left(\Phi \Delta \bar{\Phi}\right)_3$ and $\left(\Phi^* \Delta^* \bar{\Phi}^*\right)_3$.

We assign vacuum alignments to $\Phi$, $\bar{\Phi}$ and $\Delta$ which are uniquely defined in terms of their respective residual symmetries under $S_4 \times Y_{24}$. As a result, the vacuum alignment of the effective triplet of $S_4$ is uniquely obtained.

**ACKNOWLEDGMENTS**

I would like to thank Paul Harrison and Bill Scott for the stimulating discussions. I acknowledge the help and support from Ambar Ghosal and Debasish Majumdar. I am grateful to Sujatha Ramakrishnan for helping me with constructing the figures.

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