Triviality results for quasi $k$-Yamabe solitons

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Abstract. In this paper, we show that any compact quasi $k$-Yamabe gradient soliton must have constant $\sigma_k$-curvature. Moreover, we provide a certain condition for a compact quasi $k$-Yamabe soliton to be gradient, and for noncompact solitons, we present a geometric rigidity under a decay condition on the norm of the soliton field.

Mathematics Subject Classification. 53C21, 53C24, 53C25.

Keywords. $\sigma_k$-curvature, Quasi $k$-Yamabe solitons, Yamabe solitons, Invariant solutions, Scalar curvature.

1. Introduction and main results. A Riemannian manifold $(M^n, g)$, $n \geq 3$, is an Einstein-type manifold if there exists a vector field $X \in \mathfrak{X}(M)$ and a smooth function $\lambda : M \to \mathbb{R}$ such that

$$\alpha \text{Ric} + \frac{\beta}{2} \mathcal{L}_X g + \mu X^g \otimes X^g = (\rho R + \lambda)g$$

for some constants $\alpha, \beta, \mu, \rho \in \mathbb{R}$ with $(\alpha, \beta, \mu) \neq (0, 0, 0)$. Here $\mathcal{L}_X g$, $X^g$, and $R$ stand, respectively, for the Lie derivative of $g$ in the direction of $X$, the 1-form metrically dual to the vector field $X$, and the scalar curvature of $M$. The concept of Einstein-type manifold was introduced recently by Catino et al. [9]. It is worth noting that in terms of Equation (1), an Einstein-type manifold unifies various particular cases well studied in the literature, such as gradient Ricci solitons, gradient almost Ricci solitons, Yamabe solitons, quasi Yamabe solitons, conformal gradient solitons, quasi Einstein manifolds, and $\rho$-Einstein solitons. Each of them has a particular importance.

Our purpose is to study some cases of the Einstein-type manifolds which were not addressed in [9]. More precisely, we focus our analysis on the class $\alpha = 0$, $\beta = 1$, $\mu = -\frac{1}{m}$, and $\lambda = \rho R - \sigma_k + c$, $c \in \mathbb{R}$, where $\sigma_k$ is the $\sigma_k$-curvature of $g$. We recall that, if we denote by $\lambda_1, \lambda_2, \ldots, \lambda_n$ the eigenvalues of
the symmetric endomorphism $g^{-1}A_g$, where $A_g$ is the Schouten tensor defined by

\[ A_g = \frac{1}{n-2} \left[ Ric_g - \frac{R}{2(n-1)}g \right], \]

then the $\sigma_k$-curvature of $g$ is defined as the $k$-th symmetric elementary function of $\lambda_1, \ldots, \lambda_n$, namely

\[ \sigma_k = \sigma_k(g^{-1}A_g) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for } 1 \leq k \leq n. \]

In this sense, we introduce the following class of manifolds.

**Definition 1.1.** A Riemannian manifold $(M^n, g)$, $n \geq 3$, is a quasi $k$-Yamabe soliton if there exists a vector field $X \in \mathfrak{X}(M)$ and two constants $m, \lambda$ (where $m$ is not zero) such that

\[ \frac{1}{2} LXg - \frac{1}{m} X^b \otimes X^b = (\sigma_k - \lambda)g. \]  

(2)

We will write the soliton in (2) as $(M^n, g, X, \lambda)$ for the sake of simplicity. If $X = \nabla f$ for some smooth function $f : M \to \mathbb{R}$, we say that $(M^n, g, f, \lambda)$ is a quasi $k$-Yamabe gradient soliton. In this case, Equation (2) can be rewritten as

\[ \nabla^2 f - \frac{1}{m} df \otimes df = (\sigma_k - \lambda)g. \]  

(3)

Moreover, when $f$ is a constant or $X = 0$, the soliton is called trivial.

It is worth noting that quasi $k$-Yamabe solitons correspond to a large class of manifolds well studied in the literature. For instance, if $m \to \infty$, then quasi $k$-Yamabe solitons reduce to $k$-Yamabe solitons [2, 4, 8, 23]. Also, since $\sigma_1 = \frac{R}{2(n-1)}$, 1-Yamabe solitons naturally correspond to Yamabe solitons [5–7, 10–13, 15, 17, 22], and quasi 1-Yamabe solitons correspond to quasi Yamabe solitons studied in [16, 25]. When $X = \nabla f$ is a gradient vector field, quasi 1-Yamabe gradient solitons correspond to an $f$-almost Yamabe soliton [27].

In recent years, many efforts have been devoted to study the geometry of Yamabe solitons and their generalizations. For instance, Hsu in [15] has shown that any compact gradient 1-Yamabe soliton is trivial. For $k > 1$, the extension of the previous result was recently investigated. Catino et al. [8] proved that any compact gradient $k$-Yamabe soliton with nonnegative Ricci curvature is trivial. Bo et al. [4] also proved that any compact gradient $k$-Yamabe soliton with negative constant scalar curvature necessarily has constant $\sigma_k$-curvature. In [23], it was shown that any compact gradient $k$-Yamabe soliton must be trivial. For $m \neq \infty$, Huang and Li [16] proved that any compact quasi 1-Yamabe gradient soliton is trivial. Our first theorem extends all these results at once.

**Theorem 1.2.** Any compact quasi $k$-Yamabe gradient soliton $(M^n, g, f, \lambda)$ is trivial, i.e., has constant $\sigma_k$-curvature.
The Hodge–de Rham decomposition theorem shows that any vector field $X$ on a compact oriented Riemannian manifold $M^n$ can be decomposed as follows:

$$X = \nabla h + Y,$$

where $h: M \to \mathbb{R}$ is a smooth function and $Y$ is a divergence free vector field on $M^n$. In fact, by the Hodge–de Rham theorem [26], we have that $X^\flat$ takes the following form:

$$X^\flat = d\alpha + \delta \beta + \gamma.$$

Taking $Y = (\delta \beta + \gamma)^\sharp$ and $(d\alpha)^\sharp = \nabla h$, we arrive at the desired result.

Our next result provides a necessary and sufficient condition for a compact quasi $k$-Yamabe soliton to be gradient.

**Theorem 1.3.** The compact quasi $k$-Yamabe soliton $(M^n, g, X, \lambda)$ is gradient if and only if

$$\int_{M^n} \left[ Ric(\nabla h, Y) + \frac{1}{m} g(\nabla^2 h, X^\flat \otimes X^\flat) - \frac{2}{m} g(\nabla^2 h, dh \otimes dh) + \frac{1}{m^2} |dh \otimes dh|^2 
+ \frac{n}{2m} (\sigma_k - \lambda) |X|^2 + \frac{2}{m} (\sigma_k - \lambda) |\nabla h|^2 + \frac{3}{2m^2} |X|^4 \right] dv_g \leq 0,$$

where $h$ and $Y$ are the Hodge–de Rham decomposition components of $X$.

Note that when $m \to \infty$, then Theorem 1.3 corresponds to [23, Theorem 1.3]. On the other hand, combining Theorem 1.2 and Theorem 1.3, we have the following result.

**Corollary 1.4.** Any compact quasi $k$-Yamabe soliton $(M^n, g, X, \lambda)$ satisfying (5) is trivial, i.e., has constant $\sigma_k$-curvature.

Now we notice that the same result obtained in [1] for compact Ricci solitons also works for compact quasi $k$-Yamabe solitons. More precisely, we have the following theorem.

**Theorem 1.5.** Let $(M^n, g, X, \lambda)$ be a compact quasi $k$-Yamabe soliton and $X = \nabla h + Y$ the Hodge–de Rham decomposition of $X$. If

$$\int_{M^n} g(\nabla h, X) dv_g \leq 0,$$

then $(M^n, g)$ is trivial, i.e., has constant $\sigma_k$-curvature.

The next theorem provides a rigidity result in the scope of noncompact quasi $k$-Yamabe gradient solitons.

In [18], Ma and Miguel study a Liouville type theorem of harmonic functions with finite weighted Dirichlet integral and use it to prove a rigidity result for gradient Yamabe solitons with nonnegative Ricci curvature. We also provide a rigidity result assuming that the weighted integral of the soliton vector field is finite.
Theorem 1.6. Let \((M^n, g, f, \lambda)\) be a complete and noncompact quasi \(k\)-Yamabe gradient soliton satisfying
\[
\int_{M^n \setminus B(r)} d(x, x_0)^{-1} |\nabla f| d\mu < \infty,
\]
where \(d\) is the distance function with respect to \(g\), \(B(r)\) is the open ball of radius \(r > 0\) centered at \(x_0\), and \(d\mu = e^{-\frac{f}{m}} dv_g\). Thus, if
\[
\Delta f - \frac{1}{m} |\nabla f|^2 \geq 0,
\]
then \((M^n, g)\) has constant \(\sigma_k\)-curvature.

Before we present the proof of main results, let’s take a look at some examples. For others examples, see Section 3.

Example 1.7. The identities
\[
\text{Ric}_{g^n} = (n - 1)g^n, \quad \text{R}_{g^n} = n(n - 1), \quad \text{A}_{g^n} = \frac{1}{2}g^n
\]
rule the Ricci tensor, scalar curvature, and Schouten tensor, respectively, of the Euclidean sphere \((\mathbb{S}^n, g^n)\). Therefore, we have that
\[
\sigma_k(g^n_1 A_{g^n}) = \frac{1}{2k} \binom{n}{k}, \quad 1 \leq k \leq n.
\]
Then \((\mathbb{S}^n, g^n)\) is a trivial quasi \(k\)-Yamabe gradient soliton with \(\sigma_k = \lambda\).

Note that, according to Theorem 1.2, the Euclidean sphere does not admit any non-trivial structure of quasi \(k\)-Yamabe gradient soliton.

Example 1.8. Consider the hyperbolic space \(\mathbb{H}^{n+1} = \mathbb{R} \times e^t \mathbb{R}^n\) furnished with the warped product metric [20]:
\[
g = dt^2 + e^{2t} g_{\mathbb{R}^n}^2.
\]
It is well known that the horospheres of the hyperbolic space are totally umbilical hypersurfaces isometric to \(\mathbb{R}^n\) and correspond to slices \(\{t_0\} \times \mathbb{R}^n, t_0 \in \mathbb{R}\).

Hence, taking the inclusion
\[
i : \{t_0\} \times \mathbb{R}^n \to \mathbb{R} \times \mathbb{R}^n, \quad (t_0, x) \mapsto (t_0, x),
\]
we deduce that the height function from the immersion satisfies \(f(x) = t_0\), and then the standard Euclidean space \(\{t_0\} \times \mathbb{R}^n\) is a trivial quasi \(k\)-Yamabe gradient soliton immersed into hyperbolic space with potential \(f(x) = t_0\) and \(\lambda = 0\).

Example 1.9. Consider \(M^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_n > 0\}\), \(g_{ij} = (k_0 x_n)^{-2} \delta_{ij}\), \(k_0 \in (0, \infty)\), and the potential function
\[
f(x_1, \ldots, x_n) = -m \log \left(\frac{k_1}{x_n}\right), \quad k_1 \in (0, \infty).
\]
By a direct computation, we deduce that
\[
\sigma_k = -\frac{n!}{k!(n - k)!} (-1)^{k-1} \left(\frac{k_0^2}{2}\right)^k, \quad \nabla^2 f - \frac{1}{m} df \otimes df = -mk_0^2 g.
\]
Then \((M^n, g)\) is a quasi \(k\)-Yamabe gradient soliton with 
\[
\lambda = -\frac{n!}{k!(n-k)!}(-1)^{k-1}\left(\frac{k^2}{2}\right)^k + mk_0^2.
\]

**Example 1.10.** Consider the Euclidean subset \(M^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 + \cdots + x_n > 0\}\) furnished with the metric tensor \(g_{ij} = \delta_{ij}\) and potential function given by 
\[
f(x_1, \ldots, x_n) = -m \log (x_1 + \cdots + x_n).
\]
Since \(\text{Ric}_g = 0, R_g = 0\), we have that \(A_g = 0\). Consequently \(\sigma_k = 0\) for all \(k \in \{1, \ldots, n\}\). On the other hand, from the potential function \(f\), we deduce that \(\nabla^2 f = \frac{1}{m} df \otimes df\). So, \((M^n, g)\) is a quasi \(k\)-Yamabe gradient soliton with \(\lambda = 0\).

Example 1.10 shows that for a noncompact quasi \(k\)-Yamabe gradient soliton, the condition of constant \(\sigma_k\)-curvature does not imply that the potential function \(f\) is constant.

**Example 1.11.** Consider \(M^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_1 + \cdots + x_n > 0\}, g_{ij} = (x_1 + \cdots + x_n)^{-2}\delta_{ij}\), and the potential function 
\[
f(x_1, \ldots, x_n) = -m \log \left(\frac{1}{x_1 + \cdots + x_n}\right).
\]
By a direct computation, we deduce that 
\[
\sigma_k = -\frac{n!}{k!(n-k)!}(-1)^{k-1}\left(\frac{n}{2}\right)^k, \quad \nabla^2 f - \frac{1}{m} df \otimes df = -mng.
\]
Then \((M^n, g)\) is a quasi \(k\)-Yamabe gradient soliton with 
\[
\lambda = -\frac{n!}{k!(n-k)!}(-1)^{k-1}\left(\frac{n}{2}\right)^k + mn.
\]

Examples 1.9, 1.10, and 1.11 show that the compactness of the manifold can not be discarded in the proof of Theorem 1.2.

2. **Proofs.**

**Proof of Theorem 1.2.** If \(k = 1\), then \((M^n, g)\) is a quasi Yamabe gradient soliton and the result is well known from [16]. Now, consider \(k \geq 2\) and suppose by contradiction that \(f\) is nonconstant. Set \(u = e^{-\frac{f}{m}}\). Then 
\[
\nabla u = -\frac{u}{m} \nabla f, \quad \nabla^2 u = \frac{u}{m^2} df \otimes df - \frac{u}{m} \nabla^2 f,
\]
and (3) can be rewritten as follows: 
\[
\nabla^2 u = -\frac{u}{m} (\sigma_k - \lambda) g.
\] (6)
Since \(f\) is nonconstant, \(u\) is also nonconstant and so we get from the third item of [8, Theorem 1.1] that \((M^n, g)\) is rotationally symmetric and \(M^n \setminus \{N, S\}\) is locally conformally flat. Here \(N, S\) correspond to the extremal points of \(u\) in
M. From (6), we know that $\nabla u$ is a conformal Killing vector field; hence, we can apply [24, Theorem 5.2] (see also [14, Theorem 1]) to deduce

$$0 = \int_{M^n \setminus \{N,S\}} g(\nabla\sigma_k, \nabla u) dv_g = \int_{M^n} g(\nabla\sigma_k, \nabla u) dv_g = \frac{n}{m} \int_{M^n} u\sigma_k(\sigma_k - \lambda) dv_g,$$

(7)

where in the last equality we have used the divergence theorem together with (6). On the other hand, again from the divergence theorem, we get

$$0 = \int_{M^n} \Delta u dv_g = -\frac{n}{m} \int_{M^n} u(\sigma_k - \lambda) dv_g.$$

(8)

Combining Eqs. (7) and (8), we arrive at

$$\frac{n}{m} \int_{M^n} u(\sigma_k - \lambda)^2 dv_g = 0,$$

which implies that $\sigma_k = \lambda$ and $u$ is harmonic. Since $M^n$ is compact, $u$ is a constant, which leads to a contradiction. This proves that $f$ is constant. □

Proof of Theorem 1.3. From the Hodge–de Rham decomposition $X = \nabla h + Y$, we deduce that

$$\frac{1}{2} \mathcal{L}_Y g = \frac{1}{2} \mathcal{L}_X g - \frac{1}{2} \mathcal{L}_{\nabla h} g,$$

(9)

and now using Equation (2), we arrive at

$$T_m := \frac{1}{2} \mathcal{L}_Y g - \frac{1}{m} X^\flat \otimes Y^\flat - \frac{1}{m} Y^\flat \otimes dh = (\sigma_k - \lambda) g - \nabla^2 h + \frac{1}{m} dh \otimes dh.$$

(10)

Therefore, to prove that $(M^n, g)$ admits a quasi $k$-Yamabe gradient soliton structure, it is necessary and sufficient to show that $T_m = 0$. Since $M^n$ is a compact manifold, we get from Equations (10) and (2) that

$$\int_{M^n} |T_m|^2 dv_g = \int_{M^n} \left[ n(\sigma_k - \lambda)^2 - 2(\sigma_k - \lambda)\Delta h + 2(\sigma_k - \lambda) \left| \nabla h \right|^2 \right] \frac{1}{m} + \left| \nabla^2 h \right|^2$$

$$- \frac{2}{m} g(\nabla^2 h, dh \otimes dh) + \frac{1}{m^2} |dh \otimes dh|^2 \right] dv_g$$

$$= \int_{M^n} \left[ \left| \nabla^2 h \right|^2 - n(\sigma_k - \lambda)^2 + 2(\sigma_k - \lambda) \left| \nabla h \right|^2 - \left| X \right|^2 \right] \frac{1}{m}$$

$$- \frac{2}{m} g(\nabla^2 h, dh \otimes dh) + \frac{1}{m^2} |dh \otimes dh|^2 \right] dv_g.$$

(11)

We are going to compute the right-hand side of the above equation using the following integral identity:

$$\int_{M^n} 2\text{Ric}(\nabla h, Y) dv_g = \int_{M^n} \left[ \text{Ric}(X, X) - \text{Ric}(\nabla h, \nabla h) - \text{Ric}(Y, Y) \right] dv_g.$$

(12)
Taking the divergence of (9), we get
\[
\frac{1}{2} \text{div}(\mathcal{L}_Y g)(Y) = \frac{1}{2} \text{div}(\mathcal{L}_X g)(Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y) \\
= \text{div}(\sigma_k - \lambda)(Y) + \frac{1}{m} \text{div}(X^\flat \otimes X^\flat)(Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y) \\
= g(\nabla \sigma_k, Y) + \frac{1}{m} \text{div}(X^\flat \otimes X^\flat)(Y) - \frac{1}{2} \text{div}(\mathcal{L}_{\nabla h} g)(Y). \tag{13}
\]

From Bochner’s formula (see [21, Lemma 2.1]) and \( \text{div}(Y) = 0 \), we can write (13) like
\[
\frac{1}{2} \Delta |Y|^2 - |\nabla Y|^2 + \text{Ric}(Y, Y) \\
= \text{div}(\mathcal{L}_Y g)(Y) \\
= 2g(\nabla \sigma_k, Y) + \frac{2}{m} \text{div}(X^\flat \otimes X^\flat)(Y) - \text{div}(\mathcal{L}_{\nabla h} g)(Y) \\
= 2g(\nabla \sigma_k, Y) + \frac{2}{m} \text{div}(X^\flat \otimes X^\flat)(Y) - 2\text{Ric}(Y, \nabla h) - 2g(\nabla \Delta h, Y).
\]

Since \( M^n \) is compact, we get that
\[
\int_{M^n} \text{Ric}(Y, Y) dv_g = \int_{M^n} \left[ \frac{2}{m} \text{div}(X^\flat \otimes X^\flat)(Y) - 2\text{Ric}(Y, \nabla h) + |\nabla Y|^2 \right] dv_g. \tag{14}
\]

On the other hand, the same argument as above shows that
\[
\frac{1}{2} \Delta |X|^2 - |\nabla X|^2 + \text{Ric}(X, X) \\
= \text{div}(\mathcal{L}_X g)(X) - \nabla_X \text{div}(X) \\
= -(n - 2)g(\nabla \sigma_k, X) - \frac{1}{m} \nabla_X |X|^2 + \frac{2}{m} \text{div}(X^\flat \otimes X^\flat)(X), \tag{15}
\]

where we have used the fact that
\[
\text{div}(X) = n(\sigma_k - \lambda) + \frac{1}{m} |X|^2. \tag{16}
\]

Since
\[
\int_{M^n} |\nabla X|^2 dv_g = \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 + 2g(\nabla \nabla h, \nabla Y) \right] dv_g \\
= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 + 2g(\nabla \Delta h + \text{Ric}(\nabla h), Y) \right] dv_g \\
= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 - 2\text{Ric}(\nabla h, Y) \right] dv_g,
\]
we may integrate (15) over $M^n$ to conclude that

$$
\int_{M^n} Ric(X, X) dv_g \\
= \int_{M^n} \left[ |\nabla X|^2 + \frac{2}{m} div(X^b \otimes X^b)(X) - (n - 2)g(\nabla \sigma_k, X) - \frac{1}{m} \nabla_X |X|^2 \right] dv_g \\
= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 - 2Ric(\nabla h, Y) + \frac{2}{m} div(X^b \otimes X^b)(X) \right. \\
\left. + (n - 2)\sigma_k \left[ (\sigma_k - \lambda)n + \frac{1}{m}|X|^2 \right] - \frac{1}{m} \nabla_X |X|^2 \right] dv_g \\
= \int_{M^n} \left[ |\nabla^2 h|^2 + |\nabla Y|^2 - 2Ric(\nabla h, Y) + \frac{2}{m} div(X^b \otimes X^b)(X) \right. \\
\left. + n(n - 2)(\sigma_k - \lambda)^2 - (n - 2)\lambda \frac{1}{m}|X|^2 \\
+ \frac{(n - 2)}{m}\sigma_k |X|^2 - \frac{1}{m} \nabla_X |X|^2 \right] dv_g,$$

where we have used (16) and integration by parts with the consequence that

$$0 = \int_{M^n} div(X) dv_g = n \int_{M^n} (\sigma_k - \lambda) dv_g + \frac{1}{m} \int_{M^n} |X|^2 dv_g.$$

Once again applying [21, Lemma 2.1] together with $div(X) = \Delta h$ and (16), we arrive at

$$
\int_{M^n} Ric(\nabla h, \nabla h) dv_g = \int_{M^n} \left[ n^2(\sigma_k - \lambda)^2 \\
+ \frac{2n}{m}(\sigma_k - \lambda)|X|^2 + \frac{1}{m^2}|X|^4 - |\nabla^2 h|^2 \right] dv_g. \quad (18)
$$

Replacing (14), (17), and (18) back into (12), we get

$$
\int_{M^n} 2Ric(\nabla h, Y) dv_g = \int_{M^n} \left[ 2|\nabla^2 h|^2 + \frac{2}{m} div(X^b \otimes X^b)(X - Y) - 2n(\sigma_k - \lambda)^2 \right. \\
\left. - \frac{(n + 2)}{m}(\sigma_k - \lambda)|X|^2 - \frac{1}{m} \nabla_X |X|^2 - \frac{1}{m^2}|X|^4 \right] dv_g.
$$

(19)

Since

$$div((X^b \otimes X^b)(\nabla h)) = div(X^b \otimes X^b)(\nabla h) + g(\nabla^2 h, X^b \otimes X^b),$$

we know that (19) can be rewritten like this
\[
\int_{M^n} 2\text{Ric}(\nabla h, Y) dv_g = \int_{M^n} [2|\nabla^2 h|^2 - \frac{2}{m} g(\nabla^2 h, X^b \otimes X^b) - 2n(\sigma_k - \lambda)^2 \\
- \frac{(n+2)}{m} (\sigma_k - \lambda)|X|^2 - \frac{1}{m} |\nabla X|^2 - \frac{1}{m^2} |X|^4] dv_g.
\]  
(20)

Combining (2), (5), (11), and (20) with
\[
\langle \nabla X X, X \rangle = (\sigma_k - \lambda)|X|^2 + \frac{1}{m} |X|^4,
\]
we produce the desired result. \qed

Proof of Theorem 1.5. Since the Hodge–de Rham decomposition is orthogonal in $L^2(M)$, we get
\[
\int_{M^n} g(\nabla h, X) dv_g = \int_{M^n} g(\nabla h, \nabla h + Y) dv_g = \int_{M^n} |\nabla h|^2 dv_g.
\]
Therefore, if
\[
\int_{M^n} g(\nabla h, X) dv_g \leq 0,
\]
we obtain that $\nabla h = 0$ and, consequently, $X = Y$. Now, from (2) and the divergence freeness of $Y$, we deduce that
\[
0 = \text{div} Y = \text{div} X = n(\sigma_k - \lambda) + \frac{1}{m} |X|^2,
\]
implying that
\[
n(\sigma_k - \lambda) = -\frac{1}{m} |X|^2.
\]
On the other hand, from the fundamental equation (2), we have
\[
\langle \nabla X X, X \rangle = (\sigma_k - \lambda)|X|^2 + \frac{1}{m} |X|^4 = -\frac{1}{nm} |X|^4 + \frac{1}{m} |X|^4 = \frac{n-1}{nm} |X|^4.
\]
Note that
\[
\text{div}(|X|^2 X) = |X|^2 \text{div} X + \langle \nabla |X|^2, X \rangle \\
= |X|^2 \text{div} X + \nabla X |X|^2 \\
= |X|^2 \text{div} X + 2\langle \nabla X X, X \rangle \\
= 2\langle \nabla X X, X \rangle.
\]
Hence
\[
0 = \int_{M^n} \text{div}(|X|^2 X) dv_g = 2 \int_{M^n} \frac{n-1}{nm} |X|^4 dv_g.
\]
Therefore $X = 0$ and $\sigma_k = \lambda$. \qed
Proof of Theorem 1.6. As we already know, the fundamental equation
\[ \nabla^2 f - \frac{1}{m} df \otimes df = (\sigma_k - \lambda) g \]
leads to
\[ L(f) := e^{\frac{f}{m}} \text{div}(e^{-\frac{f}{m}} \nabla f) = \Delta f - \frac{1}{m} |\nabla f|^2 = n(\sigma_k - \lambda), \quad (21) \]
and because we suppose that \( L(f) \geq 0 \), we must then admit that \( \sigma_k - \lambda \geq 0 \).
So, if we now take a cut-off function \( \psi : M \rightarrow \mathbb{R} \) satisfying
\[ 0 \leq \psi \leq 1 \text{ on } M, \quad \psi \equiv 1 \text{ in } B(r), \quad \text{supp}(\psi) \subset B(2r), \quad \text{and} \quad |\nabla \psi| \leq \frac{K}{r}, \]
where \( K > 0 \) is a real constant, we are in place to conclude that
\[ n \int_{B(r)} (\sigma_k - \lambda) d\mu \]
\[ = \int_{B(r)} n\psi (\sigma_k - \lambda) d\mu \leq \int_{B(2r)} n\psi (\sigma_k - \lambda) d\mu = \int_{B(2r)} \psi L(f) d\mu \]
\[ = - \int_{B(2r)} g(\nabla \psi, \nabla f) d\mu \leq \int_{B(2r) \setminus B(r)} |\nabla \psi||\nabla f| d\mu \]
\[ \leq K \int_{B(2r) \setminus B(r)} \frac{|\nabla f|}{r} d\mu \leq 2K \int_{M^n \setminus B(r)} \frac{|\nabla f|}{d(x, x_0)} d\mu, \]
from what it follows that
\[ 0 \leq \int_{M^n} (\sigma_k - \lambda) d\mu = \lim_{r \to \infty} \int_{B(r)} (\sigma_k - \lambda) d\mu \]
\[ \leq \frac{2K}{n} \lim_{r \to \infty} \int_{M^n \setminus B(r)} \frac{|\nabla f|}{d(x, x_0)} d\mu = 0. \]
Henceforth, we have that \( L(f) = \sigma_k - \lambda = 0 \) which proves the theorem. \( \square \)

3. Translation invariant examples of quasi \( k \)-Yamabe solitons. In this section, we provide a method to construct examples of conformally flat quasi \( k \)-Yamabe gradient solitons. We focus our attention on solitons whose solutions are invariant under the action of the translation group. More precisely, we consider the Riemannian metric
\[ \delta = \sum_{i=1}^{n} dx_i \otimes dx_i, \]
in coordinates \( x = (x_1, \ldots, x_n) \) of \( \mathbb{R}^n \), where \( n \geq 3 \). For an arbitrary choice of a nonzero vector \( \alpha = (\alpha_1, \ldots, \alpha_n) \), we define the translation function \( \xi : \mathbb{R}^n \rightarrow \mathbb{R} \) by
\[ \xi(x_1, \ldots, x_n) = \alpha_1 x_1 + \cdots + \alpha_n x_n. \]
Next, we consider that $\mathbb{R}^n$ admits a group of symmetries consisting of translations $[19]$ and we then look for smooth functions $\varphi, f : (a, b) \subset \mathbb{R} \to \mathbb{R}$, with $\varphi > 0$, such that the compositions

$$f = f \circ \xi : \xi^{-1}(a, b) \longrightarrow \mathbb{R}, \quad \varphi = \varphi \circ \xi : \xi^{-1}(a, b) \longrightarrow \mathbb{R},$$

satisfy

$$\nabla^2 f - \frac{1}{m} df \otimes df = (\sigma_k - \lambda) g \quad \text{with} \quad g = \varphi^{-2}\delta. \quad (22)$$

What has been said above is summed up in the next result.

**Theorem 3.1.** Let $(\mathbb{R}^n, \delta)$ and $f = f \circ \xi, \varphi = \varphi \circ \xi$ as above. Then $\delta_{ij}\varphi^{-2}$ is a quasi $k$-Yamabe gradient soliton metric with potential function $f = -m \log u$ if and only if

$$u'' + 2\frac{u'\varphi'}{\varphi} = 0, \quad (23)$$

$$b_{n,k} \left[ k\varphi\varphi'' - \frac{n}{2}(\varphi')^2 \right] (\varphi')^2 (k-1)\alpha^2 - m\varphi'\frac{u'}{u} = \lambda, \quad (24)$$

where

$$b_{n,k} = \frac{(n-1)!}{k!(n-k)!} (-1)^{k-1} \frac{1}{2^{k-1}}.$$

**Proof of Theorem 3.1.** It is well known that for the conformal metric $\tilde{g} = \varphi^{-2}\delta$, the Ricci curvature is given by $[3]$:

$$\text{Ric}_{\tilde{g}} = \frac{1}{\varphi^2} \left\{ (n-2)\varphi \nabla^2 \varphi + [\varphi \Delta_{\delta} \varphi - (n-1)|\nabla_{\delta} \varphi|^2] \delta \right\}, \quad (25)$$

and consequently, the scalar curvature $R_{\tilde{g}}$ on the conformal metric is given by

$$R_{\tilde{g}} = (n-1)(2\varphi \Delta_{\delta} \varphi - n|\nabla_{\delta} \varphi|^2). \quad (26)$$

In order to compute the Schouten tensor on the conformal geometry $A_{\tilde{g}}$, we evoke the expression

$$A_{\tilde{g}} = \frac{1}{n-2} \left[ \text{Ric}_{\tilde{g}} - \frac{R_{\tilde{g}}}{2(n-1)} \tilde{g} \right].$$

Therefore, from (25) and (26), we deduce that

$$A_{\tilde{g}} = \frac{\nabla^2 \varphi}{\varphi} - \frac{|\nabla_{\delta} \varphi|^2}{2\varphi^2} \delta.$$

Denote by $\varphi_{x_i}, \varphi_{x_i x_j}$ the derivative of $\varphi$ with respect the variables $x_i$ and $x_i x_j$, respectively. That being said, since we are assuming that $\varphi(\xi)$ and $f(\xi)$ are functions of $\xi = \alpha_1 x_1 + \cdots + \alpha_n x_n$, we get

$$\varphi_{x_i} = \varphi' \alpha_i, \quad f_{x_i} = f' \alpha_i, \quad \varphi_{x_i x_j} = \varphi'' \alpha_i \alpha_j, \quad f_{x_i x_j} = f'' \alpha_i \alpha_j.$$

Hence

$$(\tilde{g}^{-1}A_{\tilde{g}})_{ij} = \varphi \varphi'' \alpha_i \alpha_j - \frac{1}{2} (\varphi')^2 \alpha^2 \delta_{ij}.$$
The eigenvalues of $\bar{g}^{-1}A\bar{g}$ are $\theta = -\frac{1}{2}(\varphi')^2\|\alpha\|^2$ with multiplicity $(n-1)$, and $\mu = (\varphi'' - \frac{1}{2}(\varphi')^2\|\alpha\|^2$ with multiplicity 1. The formula for $\sigma_k$ can be found easily by the binomial expansion of $(x - \theta)^{n-1}(x - \mu)$

$$\sigma_k = \frac{(n-1)!}{k!(n-k)!} [(n-k)\theta + k\mu] \theta^{k-1}$$

$$= \frac{(n-1)!}{k!(n-k)!} (-1)^{k-1} \frac{1}{2k-1} \left[k\varphi\varphi'' - \frac{n}{2}(\varphi')^2\right] (\varphi')^{2(k-1)}\|\alpha\|^{2k}. \quad (27)$$

Now, in order to compute the Hessian of $u$ relatively to $\bar{g}$, we evoke the expression

$$(\nabla^2_{\bar{g}}u)_{ij} = u_{x_i,x_j} - \sum_{k=1}^{n} \Gamma_{ij}^k u_{x_k},$$

where the Christoffel symbols $\Gamma_{ij}^k$ for distinct $i, j, k$ are given by

$$\Gamma_{ij}^k = 0, \quad \Gamma_{ij}^i = -\frac{\varphi x_j}{\varphi}, \quad \Gamma_{ii}^k = \frac{\varphi x_k}{\varphi}, \quad \text{and} \quad \Gamma_{ii}^i = -\frac{\varphi x_i}{\varphi}.$$ 

Therefore,

$$(\nabla^2_{\bar{g}}u)_{ij} = u_{x_i,x_j} + \varphi^{-1}(\varphi_{x_i} u_{x_j} + \varphi_{x_j} u_{x_i}) - \delta_{ij} \sum_k \varphi^{-1}\varphi_{x_k} u_{x_k}$$

$$= \alpha_i \alpha_j u'' + (2\alpha_i \alpha_j - \delta_{ij}\|\alpha\|^2)\varphi^{-1}\varphi' u'. \quad (28)$$

If we make the change $u = e^{-\frac{f}{m}}$, then the fundamental soliton equation in (22) can be rewritten as follows:

$$\nabla^2 u = -\frac{m}{u} (\sigma_k - \lambda) g. \quad (29)$$

Substituting (27) and (28) into (29) and considering $i \neq j$, we obtain

$$\alpha_i \alpha_j \left(u'' + 2\frac{\varphi' u'}{\varphi}\right) = 0.$$ 

If there exist $i, j$, such that $\alpha_i \alpha_j \neq 0$, then we get

$$u'' + 2\frac{u'\varphi'}{\varphi} = 0,$$

which provides Equation (23). For $i = j$, substituting (27) and (28) into (29), we obtain (24).

Now, we need to consider the case $\alpha_{k_0} = 1$, $\alpha_k = 0$ for $k \neq k_0$. In this case, substituting (28) into (29), we obtain

$$-\frac{m}{u} (\sigma_k - \lambda) \frac{1}{\varphi'} = -\frac{\varphi' u'}{\varphi} \quad (30)$$

for $i \neq k_0$, that is, $\alpha_i = 0$ with $i = j$, and

$$-\frac{m}{u} (\sigma_k - \lambda) \frac{1}{\varphi'} = u'' + (2 - 1) \frac{\varphi' u'}{\varphi} \quad (31)$$

for $i = k_0$, that is, $\alpha_{k_0} = 1$ with $i = j = k_0$. 
However, (30) and (31) are equivalent to equations (23) and (24). This completes the proof. \[\square\]

In what follows, we provide examples illustrating Theorem 3.1. We point out that Examples 1.9, 1.10, and 1.11 are contemplated in the next examples.

**Example 3.2.** In Theorem 3.1, consider an arbitrary direction \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\). Therefore the functions

\[
f(x_1, \ldots, x_n) = -m \log \left[ k_1 (\alpha_1 x_1 + \cdots + \alpha_n x_n) \right],
\]

\[
\varphi(x_1, \ldots, x_n) = k_0, \quad k_0, k_1 \in (0, \infty),
\]

provide a family of quasi \(k\)-Yamabe gradient solitons on \(\{(x_1, \ldots, x_n) \in \mathbb{R}^n; \alpha_1 x_1 + \cdots + \alpha_n x_n > 0\}\) with soliton constant \(\lambda = 0\).

**Example 3.3.** In Theorem 3.1, consider an arbitrary direction \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\). Therefore the functions

\[
f(x_1, \ldots, x_n) = -m \log \left( \frac{k_1}{\alpha_1 x_1 + \cdots + \alpha_n x_n} \right),
\]

\[
\varphi(x_1, \ldots, x_n) = k_0 (\alpha_1 x_1 + \cdots + \alpha_n x_n),
\]

where \(k_0, k_1 \in (0, \infty)\), provide a family of quasi \(k\)-Yamabe gradient solitons on the Euclidean subset \(\{(x_1, \ldots, x_n) \in \mathbb{R}^n; \alpha_1 x_1 + \cdots + \alpha_n x_n > 0\}\) with soliton constant

\[
\lambda = -\frac{n!}{k!(n-k)!} (-1)^{k-1} \left( \frac{k_0^2 \|\alpha\|^2}{2} \right)^k + mk_0^2 \|\alpha\|^2.
\]

**Example 3.4.** In Theorem 3.1, consider the direction \(\alpha = (0, 0, \ldots, 0, 1)\). Therefore the functions

\[
f(x_1, \ldots, x_n) = -m \log \left( \frac{k_1}{x_n} \right), \quad \varphi(x_1, \ldots, x_n) = k_0 x_n, \quad k_0, k_1 \in (0, \infty),
\]

provide a family of quasi \(k\)-Yamabe gradient solitons on \(\{(x_1, \ldots, x_n) \in \mathbb{R}^n; x_n > 0\}\) with soliton constant

\[
\lambda = -\frac{n!}{k!(n-k)!} (-1)^{k-1} \left( \frac{k_0^2}{2} \right)^k + mk_0^2.
\]

**Example 3.5.** In Theorem 3.1, consider an arbitrary direction \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_{2k})\). Therefore the functions

\[
f(x_1, \ldots, x_{2k}) = k_0, \quad \varphi(x_1, \ldots, x_{2k}) = k_1 e^{\alpha_1 x_1 + \cdots + \alpha_{2k} x_{2k}}, \quad k_0, k_1 \in (0, \infty),
\]

provide a family of quasi \(k\)-Yamabe gradient solitons on \(\mathbb{R}^{2k}\) with soliton constant \(\lambda = 0\).

**Example 3.6.** In Theorem 3.1, consider an arbitrary direction \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)\). Therefore the functions

\[
f(x_1, \ldots, x_n) = k_0,
\]

\[
\varphi(x_1, \ldots, x_n) = k_2 \{ k_1 k + (2k-n) [\alpha_1 x_1 + \cdots + \alpha_n x_n] \} \frac{2k}{2k-n}, \quad k_0, k_1, k_2 \in (0, \infty),
\]

provide a family of quasi \(k\)-Yamabe gradient solitons on \(\mathbb{R}^n\) with soliton constant \(\lambda = 0\).
Acknowledgements. The authors would like to thank the referee for his or her careful reading and valuable suggestions.

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Received: 30 January 2022

Revised: 15 July 2022

Accepted: 28 September 2022.