Wigner transform and quasicrystals

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Abstract

Quasicrystals, defined as in [7], are tempered distributions $\mu$ which satisfy symmetric conditions on $\mu$ and $\hat{\mu}$. This suggests that techniques from time-frequency analysis could possibly be useful tools in the study of such structures. In this paper we explore this direction considering quasicrystals type conditions on time-frequency representations instead of separately on the distribution and its Fourier transform. More precisely we prove that a tempered distribution $\mu$ on $\mathbb{R}^d$ whose Wigner transform, $W(\mu)$, is supported on a product of two uniformly discrete sets in $\mathbb{R}^d$ is a quasicrystal. This result is partially extended to a generalization of the Wigner transform, called matrix-Wigner transform which is defined in terms of the Wigner transform and a linear map $T$ on $\mathbb{R}^{2d}$.

1 Introduction

By a Fourier quasicrystal we mean a tempered distribution $\mu \in \mathcal{S}'(\mathbb{R}^d)$ of the form $\mu = \sum_{\lambda \in \Lambda} a_\lambda \delta_\lambda$ for which $\hat{\mu} = \sum_{s \in S} b_s \delta_s$, where $\delta_\xi$ is the mass point at $\xi$, $\Lambda$ and $S$ are discrete subsets of $\mathbb{R}^d$. $\Lambda$ and $S$ are called respectively the support and the spectrum of $\mu$.

The basic examples of Fourier quasicrystals follow from the Poisson summation formula. N. Lev and A. Olevskii [7] proved that if a measure $\mu$ on $\mathbb{R}^d$ (which is assumed to be positive in the case $d > 1$) is a Fourier quasicrystal and both the support and the spectrum of $\mu$ are uniformly discrete (see Section 2) then there are a lattice $L$ on $\mathbb{R}^d$, vectors $\theta_j \in \mathbb{R}^d$ and trigonometric polynomials $P_j$ ($1 \leq j \leq N$) such that

$$\mu = \sum_{j=1}^N P_j(x) \sum_{\lambda \in L + \theta_j} \delta_\lambda.$$
We refer to [5, 8] for examples of quasicrystals with other structures. See also [3, 10].

The Wigner transform \( W(\mu) \) of a tempered distribution \( \mu \) gives a description of the time-frequency content of \( \mu \). Hence, it is reasonable to study which information about the structure of \( \mu \) follows from the knowledge that \( W(\mu) \) is a measure supported on a uniformly discrete set.

The answer to this question does not follow directly from [7]. Let us assume that \( \mu \in S'(\mathbb{R}^d) \) is an even distribution whose Wigner transform \( W(\mu) \) is a measure on \( \mathbb{R}^{2d} \) supported on a uniformly discrete set. The relation \( W(f)(-\frac{x}{2}, \frac{\omega}{2}) = 2^d W(f)(x, \omega) \) when \( f \) is an even square integrable function (see [4, 4.3, 4.2]) can be appropriately extended to even tempered distributions, implying that also the Fourier transform \( \hat{W}(\mu) \) is a measure supported on a uniformly discrete set. However, since the Wigner distribution is almost never non negative (see [4, Theorem 4.4.1]) we cannot apply [7, Theorem 2] to obtain a precise description of \( W(\mu) \).

Furthermore, from the fact that \( W(\mu) \) is supported on a uniformly discrete set, we cannot even deduce that the support of \( \mu \) is discrete. This is due to the interaction between the Wigner distribution and the metaplectic operators. More precisely, to every symplectic matrix \( A \in \text{Sp}(2, \mathbb{R}) \) one can associate a unitary operator \( T_A \) acting on \( L^2(\mathbb{R}) \) (denoted \( \mu(A) \) in [2]) such that

\[
W(T_A f, T_A g)(z) = W(f, g)(A^* z) \quad \forall z = (x, \omega) \in \mathbb{R}^2.
\]

Moreover, \( T_A \) extends to an isomorphism on \( S'(\mathbb{R}) \) and (1) holds for \( f, g \in S'(\mathbb{R}) \). We refer to [2], Propositions 4.27 and 4.28. Let us now consider the Dirac comb \( \mu = \sum_{n \in \mathbb{Z}} \delta_n \). When \( A = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \) \( \theta \in (-\pi, \pi) \)

\( T_A \mu \) is the fractional Fourier transform \( F^{\alpha} \mu \), where \( \alpha = \frac{2}{\pi} \theta \in (-2, 2) \). In this case \( W(T_A \mu) \) is a rotation of \( W(\mu) \), hence it is supported on a uniformly discrete set of \( \mathbb{R}^2 \). However, \( \theta \) can be chosen such that supp \( T_A \mu = \mathbb{R} \) (see [11, Theorem 1.2]).

Our objective will be to obtain information about \( \mu \in S'(\mathbb{R}^d) \) from the fact that the Wigner transform \( W(\mu) \) is a measure supported on the product of two uniformly discrete subsets of \( \mathbb{R}^d \). The main result of the paper is as follows.

**Theorem 1.** Let \( \mu \in S'(\mathbb{R}^d) \) satisfy \( W(\mu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)} \) where \( A, B \) are uniformly discrete sets. Then \( \mu \) and \( \tilde{\mu} \) are measures with supports contained in \( A \) and \( B \) respectively.
The proof is contained in section 3. To facilitate the reading, we have preferred to include the complete proof in dimension \( d = 1 \) and then indicate the necessary modifications to obtain the result in arbitrary dimension.

The usefulness of time-frequency representations in the study of quasicrystals is not limited to the Wigner transform, actually in section 4 we consider a generalization of the Wigner transform, called matrix-Wigner transform which is defined in terms of the Wigner transform and a linear map \( T \) on \( \mathbb{R}^{2d} \) and contains for particular choices of \( T \) most of the classic time-frequency representations. First we obtain some results that relate the support of two distributions \( \mu, \nu \) (or \( \hat{\mu}, \hat{\nu} \)) with that of the cross matrix-Wigner transform \( W_T(\mu, \nu) \), thus generalizing the information contained in Theorem 1 relative to the supports. Then, we focus on the one-dimensional case and obtain a version of Theorem 1 for the cross matrix-Wigner transform. We remark moreover that a particular choice of \( T \) connects our framework to that of Lev and Olevskii [7], which essentially corresponds to the case of the Rihaczek representation.

We suppose that the link between quasicrystals and time-frequency analysis presented in this paper could lead to further developments, both in view of recent results e.g. as in [9], as well as in the direction of using specific time-frequency representations to enlighten particular features of quasicrystals structures.

2 Notation

We use brackets \( \langle \mu, g \rangle \) to denote the extension to \( \mathcal{S}'(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) of the inner product \( \langle f, g \rangle = \int_{\mathbb{R}^d} f(t)\overline{g(t)}dt \) on \( L^2(\mathbb{R}^d) \). We will write \( \langle g, \mu \rangle \) instead of \( \langle \mu, g \rangle \).

The cross-Wigner distribution of \( f, g \in L^2(\mathbb{R}^d) \) is

\[
W(f,g)(x,\omega) = \int_{\mathbb{R}^d} f(x + \frac{t}{2})\overline{g(x - \frac{t}{2})}e^{-2\pi i \omega t}dt, \quad x, \omega \in \mathbb{R}^d.
\]

It happens that \( W(f,g) \in L^2(\mathbb{R}^{2d}) \). Moreover, the cross-Wigner distribution maps \( \mathcal{S}(\mathbb{R}^d) \times \mathcal{S}(\mathbb{R}^d) \) into \( \mathcal{S}(\mathbb{R}^{2d}) \). The Wigner distribution of \( f \in L^2(\mathbb{R}^d) \) is \( W(f) := W(f,f) \). It is a quadratic representation of the signal \( f \) both in time and frequency and it is covariant, which means that

\[
W(T_\alpha M_\beta f)(x,\omega) = W(f)(x - \alpha, \omega - \beta).
\]
Here, $T_\alpha$ and $M_\beta$ are the translation and modulation operators, defined by
\[ M_\omega f(t) = e^{2\pi i \omega t} f(t) \quad \text{and} \quad T_x f(t) = f(t - x). \]
The cross Wigner distribution can be extended as a continuous map from $S'(\mathbb{R}^d) \times S'(\mathbb{R}^d)$ into $S'(\mathbb{R}^{2d})$ as follows \[4, 4.3.3\]
\[ \langle W(\mu, \nu), \phi \rangle = \langle \mu \otimes \mathcal{T}_s, \mathcal{T}_s^{-1} \mathcal{F}_2^{-1} \phi \rangle \]
for any $\phi \in S(\mathbb{R}^{2d})$, where $\mathcal{F}_2$ denotes the partial Fourier transform
\[ \mathcal{F}_2 F(x, \omega) = \int_{\mathbb{R}^d} F(x, t) e^{-2\pi i \omega t} \, dt, \quad x, \omega \in \mathbb{R}^d \]
and $\mathcal{T}_s$ is the symmetric coordinate change defined by
\[ \mathcal{T}_s F(x, t) = F(x + \frac{t}{2}, x - \frac{t}{2}), \quad x, t \in \mathbb{R}^d. \] (2)
This extension satisfies Moyal’s formula, that is, for any functions $\phi, \psi \in S(\mathbb{R}^d)$ one has (see for instance \[4, 4.3.2\])
\[ \langle W(\mu, \nu), W(\phi, \psi) \rangle = \langle \mu, \phi \rangle \cdot \langle \psi, \nu \rangle. \]
A set $A \subset \mathbb{R}^d$ is said to be uniformly discrete (u.d. from now on) if there is $\delta > 0$ such that $|r - s| \geq \delta$ whenever $s, r \in A, s \neq r$.

3 The proof of Theorem 1

The next result is well-known and will play a role in the proof of Theorem 1. We include a proof for the convenience of the reader. As usual, for a multiindex $\alpha \in \mathbb{N}_0^d$ we denote its length by $|\alpha| = \alpha_1 + \ldots + \alpha_d$. $B_\varepsilon$ stands for the ball with radius $\varepsilon$ centered at the origin.

Lemma 2. Let $\mu \in S'(\mathbb{R}^d)$ be a tempered distribution with u.d. support $A$. Then there are $N \in \mathbb{N}_0$ and complex numbers $\{b_\alpha^r : 0 \leq |\alpha| \leq N, r \in A\}$ such that
\[ \mu = \sum_{r \in A} \sum_{|\alpha| \leq N} b_\alpha^r \delta^{(\alpha)} \]
on $S(\mathbb{R}^d)$. Moreover
\[ \sup_{|\alpha| \leq N} \sup_{r \in A} |b_\alpha^r| (1 + |r|)^{-N} < \infty. \]
Proof. Let $N \in \mathbb{N}_0$ and $C > 0$ satisfy
\[|\mu(f)| \leq C \sup_{x \in \mathbb{R}^d} \sup_{|\alpha| \leq N} (1 + |x|)^N \left|f^{(\alpha)}(x)\right|\]
for every $f \in \mathcal{S}(\mathbb{R}^d)$. Since $\mu$ has u.d. support we find complex numbers $\{b_\alpha^r : |\alpha| \leq N, r \in A\}$ such that
\[\mu(\varphi) = \sum_{r \in A} \sum_{|\alpha| \leq N} b_\alpha^r \delta^{(\alpha)}_r(\varphi)\] (3)
for every $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Let us check that the right hand side on (3) defines a tempered distribution. We take $0 < \varepsilon < \inf \{|r - r' : r, r' \in A\}$. For $|\alpha| \leq N$ we take $\varphi(\alpha) \in \mathcal{D}(B_\varepsilon)$ such that $\varphi(\alpha)(0) = 1$ and $\varphi(\beta)(0) = 0$ for $|\beta| \leq N, \beta \neq \alpha$. Then
\[|b_\alpha^r| = \left|\mu(T_r \varphi(\alpha))\right| \leq \tilde{C} (1 + |r|)^N\]
where the constant $\tilde{C}$ does not depend on $\alpha$ or $r \in A$. Therefore, the right hand side in (3) defines a tempered distribution. Finally, the density of $\mathcal{D}(\mathbb{R}^d)$ on $\mathcal{S}(\mathbb{R}^d)$ gives the conclusion. \qed

Lemma 3. Let $\mu \in \mathcal{S}'(\mathbb{R}^d)$ satisfy $W(\mu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta^{(r,s)}$ where $A, B$ are u.d. sets. Then $\text{supp } \mu \subset A$ and $\text{supp } \hat{\mu} \subset B$. Moreover, $\frac{r_1 + r_2}{2} \in A$ for any $r_1, r_2 \in \text{supp } \mu$.

Proof. The inclusions $\text{supp } \mu \subset A$ and $\text{supp } \hat{\mu} \subset B$ follow from standard properties of the Wigner transform. Let us now fix $r_1, r_2 \in \text{supp } \mu$ and consider $\nu := T_{-r_1} \mu$, so that $0, r_0 := r_2 - r_1 \in \text{supp } \nu$. From the covariance property of the Wigner transform we obtain a representation
\[W(\nu) = \sum_{(r,s) \in A_1 \times B} b_{r,s} \delta^{(r,s)}\]
where $A_1 = A - r_1$. Since $\nu$ is a tempered distribution with u.d. support contained in $A_1$ we have
\[\nu = \sum_{r \in A_1} \sum_{|\alpha| \leq N} b_{\alpha}^r \delta^{(\alpha)}_r\]
for some $N \in \mathbb{N}$ and $b_\alpha^r \in \mathbb{C}$. We aim to prove that $\frac{r_1 + r_2}{2} \in A_1$, which means $\frac{r_1 + r_2}{2} \in A$ as desired. Proceeding by contradiction, let us assume that $\frac{r_1 + r_2}{2} \notin A_1$. We choose $0 < \varepsilon < \text{dist} \left(\frac{r_1 + r_2}{2}, A_1\right)$. Since $0 \in \text{supp } \nu$ we can find $g \in \mathcal{D}(B_\varepsilon)$ real-valued and satisfying $\langle \nu, g \rangle \neq 0$. For every $\alpha \in \mathbb{N}_0^N$ with $|\alpha| \leq N$ let
$f_\alpha \in D(r_0 + B_\varepsilon)$ be a real-valued function such that $f^{(\alpha)}_\alpha(r_0) = (-1)^{|\alpha|}$ but $f^{(\beta)}_\beta(r_0) = 0$ for any $\beta \neq \alpha$. Now we observe that
\begin{align*}
\langle g, \nu \rangle b^\alpha_{r_0} &= \langle g, \nu \rangle \cdot \langle \nu, f_\alpha \rangle = \langle W(\nu), W(f_\alpha, g) \rangle \\
&= \sum_{(r,s) \in A_1 \times B} b_{r,s} \int_{\mathbb{R}^d} e^{2\pi i t s} f_\alpha(r + \frac{t}{2}) g(r - \frac{t}{2}) \, dt.
\end{align*}
Now,
\[ f_\alpha(r + \frac{t}{2}) g(r - \frac{t}{2}) \neq 0 \]
implies that
\[ r + \frac{t}{2} \in r_0 + B_\varepsilon \text{ and } r - \frac{t}{2} \in B_\varepsilon, \]
and hence
\[ r \in (r_0 + B_\varepsilon) \cap A = \emptyset. \]
We conclude $b^\alpha_{r_0} = 0$ for every $|\alpha| \leq N$, which is a contradiction since $r_0 \in \text{supp } \nu$. \hfill \square

**Remark.** Under the hypothesis of Lemma 3 the set
\[ \frac{\text{supp } \mu + \text{supp } \mu}{2} \]
is u.d.. We note however that there are u.d. sets $A$ such that $\frac{A + A}{2}$ has accumulation points. As an example in dimension $d = 1$ we can consider $A = \{n + \frac{1}{|n|} : n \in \mathbb{Z} \setminus \{0\}\}$, for which 0 is an accumulation point of $\frac{A + A}{2}$. \hfill \square

The next elementary result will be used in the proof of Theorem 1. We omit the proof.

**Lemma 4.** Let $A \subset \mathbb{R}^d$ be a u.d. set. Then for every $\alpha > 0$ there exists $\beta > 0$ such that
\[ \sum_{r \in A} \sum_{s \in A} (1 + |r|)^{\alpha} (1 + |s|)^{\alpha} (1 + |r + s|)^{-\beta} (1 + |r - s|)^{-\beta} < \infty. \]

**Proof of Theorem 1 in dimension $d = 1$:**
Let us write \( D \) for the support of \( \mu \), which is contained in \( A \). According to Lemma 2 we can put
\[
\mu = \sum_{r \in D} \sum_{j=0}^{N} a^j_r \delta^{(j)},
\]
with \( a^j_r \in \mathbb{C} \). We now assume \( N \geq 1 \) and show that \( a^N_r = 0 \) for all \( r \in D \).

For any real-valued functions \( \phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}) \) we have, for \( \phi = \phi_1 \otimes \phi_2 \),
\[
(T^{-1}F^{-1}_2 \phi)(u, v) = (F^{-1}_2 \phi) \left( \frac{u + v}{2}, u - v \right) = \phi_1 \left( \frac{u + v}{2} \right) \hat{\phi}_2(v - u),
\]
hence
\[
\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \langle \mu_u, \langle \mu_v, \phi_1 \left( \frac{u + v}{2} \right) \hat{\phi}_2(v - u) \rangle \rangle.
\]

A simple calculation gives
\[
\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \sum_{j=0}^{N} \sum_{k=0}^{j} \sum_{\ell=0}^{k} \sum_{m=0}^{\ell} \frac{(-1)^{j+k} \lambda^\ell_m(j,k)}{\lambda^\ell_m(j,k)} \sum_{r, s \in D} a^i_r \overline{a^i_s} \phi_1^{(\ell+m)} \left( \frac{r + s}{2} \right) \hat{\phi}_2 (r-s),
\]
where
\[
\lambda^\ell_m(j,k) = \binom{j}{\ell} \binom{k}{m} (-1)^{k-m} \frac{1}{2^{m+\ell}}.
\]

Since
\[
\sup_{r \in D} |a^i_r| (1 + |r|)^{-N} < \infty
\]
for every \( 0 \leq j \leq N \), an application of Lemma 4 permits to conclude that the double series in the right hand side of (4) is absolutely convergent. We take
\[
0 < \varepsilon < \delta(A) := \inf \{|r - r'| : r, r' \in A, r \neq r'\}.
\]

By Lemma 3 \( r \mp s \in A \) for any \( r, s \in D \), therefore \( \{ \mp s : r, s \in D \} \) has no accumulation points. Fix \( r_0 \in D \) and choose \( \phi_1 \in \mathcal{D}(r_0 - \varepsilon, r_0 + \varepsilon) \) such that \( \phi_1^{(n)}(r_0) = 0 \) for \( n = 0, \ldots, 2N - 1 \) whereas \( \phi_1^{(2N)}(r_0) \neq 0 \). Then, for \( \phi_2 \in \mathcal{S}(\mathbb{R}) \) (real valued), we have
\[
\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \frac{1}{2^{2N}} \phi^{(2N)} \left( r_0 \right) \sum_{r, s \in D(r_0)} a^N_r \overline{a^N_s} \phi_2(r-s).
\]
Here $D(r_0) := \{(r, s) : r, s \in D; r + s = 2r_0\}$. Hence, if $\phi_2$ has also compact support,

$$\left| \phi_1^{(2N)}(r_0) \sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} \phi_2(r - s) \right| = 2^{2N} \left| \langle W(\mu), \phi_1 \otimes \phi_2 \rangle \right| \leq C \| \phi_1 \|_{\infty} \| \phi_2 \|_{\infty},$$

where the constant $C$ depends on the (compact) support of $\phi_1 \otimes \phi_2$.

Next, we fix $\psi \in D((-\varepsilon, \varepsilon))$ such that $\psi^{(n)}(0) = 0$ for $n = 0, \ldots, 2N - 1$ and $\psi^{(2N)}(0) = 1$, and for each $t \geq 1$, let us consider $\psi^t(x) := \psi(tx)$ and $\phi_1^t(x) = \psi^t(x - r_0)$. Hence, as the supports of the $\phi_1^t$’s shrink as $t$ increases, we have, for every $t \geq 1$,

$$t^{2N} \left| \sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} \phi_2(r - s) \right| \leq C \| \psi \|_{\infty} \| \phi_2 \|_{\infty},$$

where $C$ depends on the support of $\phi_1^t \otimes \phi_2$. Taking limits as $t$ goes to infinity we conclude that

$$\sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} \phi_2(r - s) = \sum_{r \in D_0} a_r^N a_{2r_0 - r}^N \phi_2(2(r - r_0)) = 0 \quad (6)$$

for every $\phi_2 \in D(\mathbb{R})$, where

$$D_0 = \{ r \in D : \text{there exists } s \in D \text{ with } r + s = 2r_0 \}.$$

From (5) it follows that the map

$$\phi \mapsto \sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} \phi(r - s)$$

defines a tempered distribution, which coincides with

$$\sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} e^{-2\pi i (r-s)x} = \mathcal{F} \left( \sum_{(r, s) \in D(r_0)} \overline{a_r^N a_s^N} \delta_{r-s} \right)$$

the series being convergent in $\mathcal{S}'(\mathbb{R})$. Hence, by density, equation (6) holds for every $\phi_2 \in \mathcal{S}(\mathbb{R})$. Now we consider $\phi_2 \in \mathcal{S}(\mathbb{R})$ such that $\text{supp} \hat{\phi}_2$ is a so small compact set that it does not contain other points of $(D - r_0)$ other
than possibly 0, a fact which is possible since $D \subset A$ has no accumulation points. Then equation (6) reduces to $a^N_{r_0} a^N_{s_0} = 0$, i.e. $a^N_{r_0} = 0$.

Proceeding by recurrence, we finally get that $a^j_r = 0$ for all $r \in D$ whenever $j \geq 1$. This proves that $\mu$ is a measure, as desired.

The conclusion for $\hat{\mu}$ now follows from $W(\hat{\mu})(x, \omega) = W(\mu)(-\omega, x)$ (see [4, 4.3.2]). □

Let $A, B$ u.d. subsets of $\mathbb{R}$. It is well-known that $T := \sum\limits_{(r, s) \in A \times B} c_{r,s} \delta_{(r,s)}$ defines a tempered distribution if, and only if, for some $N \geq 0$
\[ \sup_{(r, s) \in A \times B} |c_{r,s}| (1 + |r| + |s|)^{-N} < +\infty. \]

If $T$ is obtained as the Wigner transform of a tempered distribution then a more restrictive condition on the coefficients is satisfied.

**Corollary 5.** Let $\mu \in \mathcal{S}'(\mathbb{R})$ satisfy $W(\mu) = \sum\limits_{(r, s) \in A \times B} c_{r,s} \delta_{(r,s)}$ where $A, B$ are u.d. sets. Then
\[ \sup_{(r, s) \in A \times B} |c_{r,s}| < +\infty. \]

**Proof.** According to Theorem [4] we can apply [7, Theorems 1,3] to conclude that there are $a > 0$ and $N \in \mathbb{N}$ such that $\mu = \sum_{j=1}^{N} \mu_j$, where each $\mu_j$ is a finite linear combination of time-frequency shifts of $\sum_{n \in \mathbb{Z}} \delta_{na}$. The conclusion follows from the properties of the cross-Wigner distribution (see for instance [4, 4.3.2(c)] in the $L^2$ setting) and the fact that, for $\Lambda = a\mathbb{Z}$,
\[
W\left(\sum_{\lambda \in \Lambda} \delta_{\lambda}\right) = \frac{1}{2a} \sum_{\lambda \in \Lambda} \delta_{\lambda} \otimes \sum_{\lambda' \in \Lambda} \delta_{\lambda'} + \frac{1}{2a} \sum_{\lambda \in \Lambda} \delta_{\lambda + \frac{1}{2}} \otimes \sum_{\lambda' \in \Lambda'} e^{-\pi i \lambda' x} \delta_{\lambda'},
\]
where $\Lambda' = \frac{1}{a} \mathbb{Z}$ is the dual lattice. □

Our next goal is to adapt the previous arguments to the case of arbitrary dimension $d$. We first need a technical lemma. For any $\gamma \in \mathbb{N}_0^d$ we denote
\[
F^d_\gamma = \left\{ (\alpha, \beta) \in \mathbb{N}_0^d \times \mathbb{N}_0^d : |\alpha| = |\beta| = |\gamma|, \ alpha + beta = 2\gamma \right\}.
\]

**Lemma 6.** Let $N \geq 1$ and let us assume that the family of complex numbers \{a^\gamma\} indexed by $\gamma \in \mathbb{N}_0^d$ satisfies
\[ \sum_{(\alpha, \beta) \in F^d_\gamma} a^\alpha \overline{a^\beta} = 0 \quad (7) \]
for every $\gamma \in \mathbb{N}_0^d$ with $|\gamma| = N$. Then $a^\gamma = 0$ whenever $|\gamma| = N$. 9
Proof. We proceed by induction on $d$. For $d = 1$ the lemma is obvious since condition (7) means $|a^\gamma|^2 = 0$ for $\gamma = N$. Let us now assume that the lemma holds in dimension $d - 1$ ($d \geq 2$) for every $N \geq 1$ and let $\{a^\gamma : \gamma \in \mathbb{N}_0^d\} \subset \mathbb{C}$ be given such that condition (7) is satisfied. Now we proceed by induction on $k$ to prove that $a^\gamma = 0$ whenever $|\gamma| = N$ and at least one component of $\gamma$ equals $k$. Let us first assume that $k = 0$, so $\gamma$ has at most $d - 1$ non-null components. Without loss of generality, we can assume $\gamma_d = 0$. For every $\delta' \in \mathbb{N}_0^{d-1}$ we define
\[
b^{\delta'} = a^{(\delta',0)}.
\]
For any $\delta' \in \mathbb{N}_0^{d-1}$ with $|\delta'| = N$ we put $\delta = (\delta',0)$. Then
\[
\sum_{(\alpha',\beta') \in F_{d-1}^{\delta'}} b^{\alpha'} \overline{b^{\beta'}} = \sum_{(\alpha,\beta) \in F_d^{\delta}} a^{\alpha} \overline{a^{\beta}} = 0.
\]
Our hypothesis on the validity of the lemma in dimension $d - 1$ permits to conclude that $a^\gamma = 0$. Let us now assume that $a^\gamma = 0$ whenever $|\gamma| = N$ and at least one component of $\gamma$ is less than $k$ ($k \geq 1$) and let us fix $\gamma \in \mathbb{N}_0^d$ such that $|\gamma| = N$ and at least one component equals $k$. We can assume $\gamma_d = k$ and $\gamma_j \geq k$ for every $1 \leq j \leq d - 1$ (otherwise $a^\gamma = 0$). This implies $1 \leq k < N$. For every $\delta' \in \mathbb{N}_0^{d-1}$ we define
\[
b^{\delta'} = a^{(\delta',k)}
\]and we put $\delta = (\delta',k)$. Then, for every $\delta' \in \mathbb{N}_0^{d-1}$ with $|\delta'| = N - k$,
\[
\sum_{(\alpha',\beta') \in F_{d-1}^{\delta'}} b^{\alpha'} \overline{b^{\beta'}} = \sum_{(\alpha,\beta) \in F_d^{\delta}} a^{\alpha} \overline{a^{\beta}} = 0.
\]
We observe that $|\delta| = N$ and condition $(\alpha,\beta) \in F_d^{\delta}$ implies $\alpha_d + \beta_d = 2k$, hence we can assume $\alpha_d = \beta_d = k$. Otherwise some of the coefficients $\alpha_d, \beta_d$ is less than $k$, from where it follows $a^{\alpha} \overline{a^{\beta}} = 0$. Our hypothesis on the validity of the lemma in dimension $d - 1$ (applied to $N - k$ instead of $N$) permits to conclude that $b^{\delta'} = 0$ for any $\delta' \in \mathbb{N}_0^{d-1}$ such that $|\delta'| = N - k$. In particular, $a^\gamma = b^{\gamma'} = 0$. Here $\gamma' = (\gamma_1, \ldots, \gamma_{d-1})$. The proof is finished.

Proof of Theorem 1 for arbitrary $d$:
As in the case $d = 1$ we only need to check the statement concerning $\mu$. Let us write $S_\mu$ for the support of $\mu$, which is contained in $A$. According to Lemma 2 we put
\[
\mu = \sum_{r \in S_\mu} \sum_{|\alpha| \leq N} a^r_\alpha \delta_r^{(\alpha)},
\]
with \( a_\alpha^r \in \mathbb{C} \). Our aim is to show that \( a_\alpha^r = 0 \) for all \( r \in S_\mu \) and \( |\alpha| \geq 1 \).

Recall that \( \{ \frac{r+s}{2} : r, s \in S_\mu \} \) has no accumulation points. Fix \( r_0 \in S_\mu \) and \( \gamma \in \mathbb{N}_0^d \) with \( |\gamma| = N \) and choose a smooth function \( \phi_1 \) supported on a sufficiently small neighborhood of \( r_0 \) and satisfying \( \phi_1^{(2\gamma)}(r_0) \neq 0 \) while \( \phi_1^{(\alpha)}(r_0) = 0 \) for each \( \alpha \neq 2\gamma \). Then, for every \( \phi_2 \in \mathcal{S}(\mathbb{R}^d) \) (real valued), we have

\[
\langle W(\mu), \phi_1 \otimes \phi_2 \rangle = \phi_1^{(2\gamma)}(r_0) \left( \frac{1}{2} \right)^{2|\gamma|} \sum_{(r,s) \in D(r_0)} \left( \sum_{(\alpha,\beta) \in \mathbb{F}_d} a_\alpha^r \bar{a}_\beta^s \right) \cdot \bar{\phi}_2(r - s),
\]

where \( D(r_0) := \{(r,s) : r, s \in S_\mu, r + s = 2r_0 \} \). Proceeding as in the case \( d = 1 \) we obtain

\[
\sum_{(\alpha,\beta) \in \mathbb{F}_d} a_\alpha^r \bar{a}_\beta^r = 0.
\]

An application of Lemma 6 gives \( a_\gamma^r = 0 \) whenever \( |\gamma| = N \) and \( r \in S_\mu \). Now a recurrence argument shows that \( a_\gamma^r = 0 \) for every \( r \in S_\mu \) and \( \gamma \in \mathbb{N}_0^d \) with \( |\gamma| \geq 1 \). The conclusion follows. \( \square \)

### 4 The matrix-Wigner transform

A natural generalization of the hypothesis of Theorem 1 is the case where different input functions or distributions \( \mu \) and \( \nu \) are considered for the Wigner transform. This situation is more involved but still some results can be obtained. Furthermore we shall consider a generalization of the Wigner transform, called matrix-Wigner transform, see [1], which, using a composition with linear maps, will yield a unifying framework connecting our results to those of [7], [8]. We need some preliminaries.

We begin by recalling the following notations. For a set \( E \in \mathbb{R}^{2d} = \mathbb{R}_x^d \times \mathbb{R}_\omega^d \) we indicate the projections on the \( x \) and \( \omega \)-coordinates as:

\[
\Pi_1(E) = \{ x \in \mathbb{R}_x^d : \exists \omega \in \mathbb{R}_\omega^d \text{ such that } (x, \omega) \in E \}, \\
\Pi_2(E) = \{ \omega \in \mathbb{R}_\omega^d : \exists x \in \mathbb{R}_x^d \text{ such that } (x, \omega) \in E \}.
\]

When it is clear from the context we shall however omit the subscripts \( x \) and \( \omega \) in \( \mathbb{R}_x^d \) and \( \mathbb{R}_\omega^d \). We recall without proof the following well-known property which will be used later.

**Proposition 7.** Setting \( \mathcal{F}_1 F(\nu, t) = \int_{\mathbb{R}^d} F(x, t)e^{-2\pi i \nu x} \, dx \) and \( \mathcal{F}_2 F(x, \omega) = \int_{\mathbb{R}^d} F(x, t)e^{-2\pi i \omega t} \, dt \) for \( F \in \mathcal{S}(\mathbb{R}^{2d}) \), the usual extensions to \( \mathcal{S}'(\mathbb{R}^{2d}) \), the partial Fourier transforms \( \mathcal{F}_1, \mathcal{F}_2 \) are bicontinuous isomorphisms from \( \mathcal{S}(\mathbb{R}^{2d}) \) to \( \mathcal{S}(\mathbb{R}^{2d}) \) and from \( \mathcal{S}'(\mathbb{R}^{2d}) \) to \( \mathcal{S}'(\mathbb{R}^{2d}) \).
We need now to discuss some properties concerning supports.

**Lemma 8.** Suppose that $\Psi \in S'(\mathbb{R}^{2d})$. If $\Pi_1 \text{supp }\Psi$ or $\Pi_1 \text{supp } \mathcal{F}_2\Psi$ are u.d. sets in $\mathbb{R}^d$, then $\Pi_1 \text{supp }\Psi = \Pi_1 \text{supp } \mathcal{F}_2\Psi$ (and therefore both are u.d.).

**Proof.** Let $I$ be an interval in $\mathbb{R}^d$ (i.e. the cartesian product of $d$ open intervals of $\mathbb{R}$). Preliminary we observe that a distribution $\Psi \in S'(\mathbb{R}^{2d})$ vanishes on the strip $I \times \mathbb{R}^d$ if and only if the restriction of $\mathcal{F}_2\Psi$ to the same strip also vanishes. Indeed if the distribution $\Psi$ vanishes on $I \times \mathbb{R}^d$, then for every $\phi \in S(\mathbb{R}^{2d})$ with $\text{supp }\phi \subset I \times \mathbb{R}^d$, we have

$$0 = \langle \Psi, \phi \rangle = \langle \mathcal{F}_2\Psi, \mathcal{F}_2\phi \rangle,$$

which means that also $\mathcal{F}_2\Psi$ vanishes on $I \times \mathbb{R}^d$ because $\mathcal{F}_2 : S(\mathbb{R}^{2d}) \rightarrow S(\mathbb{R}^{2d})$ is a bijection which preserves the inclusion of the supports in $I \times \mathbb{R}^d$. The converse is analogous.

Let us suppose now that $\Pi_1 \text{supp }\Psi$ is a u.d. set and $x_0 \notin \Pi_1 \text{supp }\Psi$, then there exists an open interval $I \subset \mathbb{R}^d$ containing $x_0$ such that $\Psi$ vanishes on $I \times \mathbb{R}^d$. From the first part of this proof we know that also $\mathcal{F}_2\Psi$ vanishes on $I \times \mathbb{R}^d$ and therefore $x_0 \notin \Pi_1 \text{supp } \mathcal{F}_2\Psi$. This proves the inclusion

$$\Pi_1 \text{supp } \mathcal{F}_2\Psi \subseteq \Pi_1 \text{supp }\Psi,$$

which in particular shows that also $\Pi_1 \text{supp } \mathcal{F}_2\Psi$ is u.d.

The opposite inclusion $\Pi_1 \text{supp }\Psi \subseteq \Pi_1 \text{supp } \mathcal{F}_2\Psi$, is proved by the same argument and we have therefore

$$\Pi_1 \text{supp }\Psi = \Pi_1 \text{supp } \mathcal{F}_2\Psi.$$

Finally the case where $\Pi_1 \text{supp } \mathcal{F}_2\Psi$ is a u.d. set can be proved in analogous way.

**Remark.** We observe that the hypothesis of u.d.ness of either $\Pi_1 \text{supp }\Psi$ or $\Pi_1 \text{supp } \mathcal{F}_2\Psi$ in the previous proposition is essential. Consider for example

$$\Psi = \sum_{n \in \mathbb{N}} \delta_{1/n}(x) \otimes \delta_n(\omega).$$

Then 0 belongs to $\Pi_1 \text{supp } \mathcal{F}_2\Psi$ but not to $\Pi_1 \text{supp }\Psi$.

In order to treat the matrix-Wigner transform, to be defined later, we need to consider some properties of linear maps in connection with supports.

**Proposition 9.** Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear bijective map. Then for every $\Psi \in S'(\mathbb{R}^n)$ we have $\text{supp } (\Psi \circ T) = T^{-1}(\text{supp }\Psi)$. 

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Proof. As $T$ is a diffeomorphism, by definition of the composition with a distribution, we have for $\phi \in \mathcal{S}(\mathbb{R}^n)$:

$$\langle \Psi \circ T, \phi \rangle = |\det T^{-1}| \langle \Psi, \phi \circ T^{-1} \rangle$$

As $\Psi \in \mathcal{S}'(\mathbb{R}^n) \subseteq \mathcal{D}'(\mathbb{R}^n)$ we have that $x \in \text{supp } \Psi$ implies that for every neighborhood $U$ of the origin there exists a function $\phi \in C_c^\infty(x + U)$ such that $\langle \Psi, \phi \rangle \neq 0$.

We have then

$$0 \neq \langle \Psi, \phi \rangle = |\det T| \langle \Psi \circ T, \phi \circ T \rangle$$

where $\phi \circ T \in C_c^\infty(T^{-1}x + T^{-1}U)$, which implies $T^{-1}x \in \text{supp } (\Psi \circ T)$. This proves the inclusion $T^{-1}(\text{supp } \Psi) \subseteq \text{supp } (\Psi \circ T)$. Clearly $\Psi = (\Psi \circ T) \circ T^{-1}$, so the same argument with $T^{-1}$ instead of $T$ proves the opposite inclusion. \(\square\)

**Remark.** $\Psi \circ T$ is the pull-back $T^* \Psi$ of $\Psi$, here for analogy with the case where $\Psi$ is a function we shall write for short $T(\Psi)$.

In particular for $n = 2d$ and $\mu, \nu \in \mathcal{S}'(\mathbb{R}^d)$, we have that $x_1 \in \text{supp } \mu$, $x_2 \in \text{supp } \nu$ implies $(\frac{x_1 + x_2}{2}, x_1 - x_2) \in \text{supp } (T_s(\mu \otimes \nu))$, where $T_s$ is the symmetric coordinate change operator defined in [2].

More generally let $T : (x, y) \in \mathbb{R}^{2d} \rightarrow (u, v) = T(x, y) \in \mathbb{R}^{2d}$ be an invertible linear transformation, with abuse of notation we shall still indicate with $T$ the matrix associated with the transformation and the change of coordinate operator given by

$$T : F(x, y) \rightarrow (TF)(x, y) = F \circ T(x, y).$$

with natural extension to distributions. We have then the following:

**Proposition 10.** Suppose that $\mu, \nu \in \mathcal{S}'(\mathbb{R}^{2d})$, and $T : \mathbb{R}^{2d} \rightarrow \mathbb{R}^{2d}$ is a linear invertible transformation. Let us write the inverse matrix of $T$ as:

$$T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

with $A, B, C, D$ submatrices of dimension $d \times d$. Then the following hold:

(i) $\Pi_1 \text{supp } T(\mu \otimes \nu)$ is an u.d. set, $\det A \neq 0 \implies \text{supp } \mu$ is a u.d. set

(ii) $\Pi_1 \text{supp } T(\mu \otimes \nu)$ is an u.d. set, $\det B \neq 0 \implies \text{supp } \nu$ is a u.d. set

(iii) $\Pi_2 \text{supp } T(\mu \otimes \nu)$ is an u.d. set, $\det C \neq 0 \implies \text{supp } \mu$ is a u.d. set

(iv) $\Pi_2 \text{supp } T(\mu \otimes \nu)$ is an u.d. set, $\det D \neq 0 \implies \text{supp } \nu$ is a u.d. set
Proof. We prove (i), the others are analogous. By contradiction suppose that 
\( \text{supp} \mu \) is not u.d.. Then for every \( \epsilon > 0 \) there exist \( x, y \in \text{supp} \mu \) such that \( 0 < \|x - y\| < \epsilon \).

Let \( z \in \text{supp} \nu \), then \( (x, z) \) and \( (y, z) \) belong to \( \text{supp} \mu \otimes \nu \).

Let \( P = T^{-1}(x, z) \in \mathbb{R}^{2d} \) and \( Q = T^{-1}(y, z) \in \mathbb{R}^{2d} \), i.e.

\[
P = \begin{pmatrix} Ax + Bz \\ Cx + Dz \end{pmatrix}, \quad Q = \begin{pmatrix} Ay + Bz \\ Cy + Dz \end{pmatrix},
\]

then, by Proposition 9, \( P \) and \( Q \) belong to \( \text{supp} T(\mu \otimes \nu) \).

As \( \det A \neq 0 \) and \( x - y \neq 0 \), we have

\[
0 < \|A(x - y)\| = \|\Pi_1 P - \Pi_1 Q\| \leq \|P - Q\| \leq \|T^{-1}\| \| (x, z) - (y, z) \| = \|T^{-1}\| \|x - y\| < \|T^{-1}\| \epsilon.
\]

Then \( \Pi_1 P \) and \( \Pi_1 Q \) are distinct points of \( \Pi_1 \text{supp} T(\mu \otimes \nu) \) with arbitrary small distance, i.e. \( \Pi_1 \text{supp} T(\mu \otimes \nu) \) is not a u.d. set.

As mentioned above, following [1], we introduce next a Matrix-Wigner transform which is a natural generalization of the Wigner transform \( W(\mu, \nu) = F_2(T_\pi(\mu \otimes \nu)) \) where the change of coordinates \( T_\pi \) have been replaced by a general bijective linear map \( T \). This sesquilinear transform turns out to be a quite comprehensive tool including most of the basic time-frequency representations, we refer to [1] for details and properties.

Definition 11. Let \( T \) and \( A, B, C, D \) be as before, then the Matrix-Wigner transform of \( \mu, \nu \in S'(\mathbb{R}^d) \) is defined as:

\[
W_T(\mu, \nu) = F_2(T(\mu \otimes \nu)).
\]

As usual we shall write \( W_T(\mu) \) for \( W_T(\mu, \mu) \).

In connection with our previous discussion we have the following property.

Proposition 12. If \( \Pi_1 \text{supp} W_T(\mu, \nu) \) is a u.d. set, then

\[
\det A \neq 0 \implies \text{supp} \mu \text{ is u.d.}, \quad \det B \neq 0 \implies \text{supp} \nu \text{ is u.d.}
\]

In particular for the classical Wigner transform we have

\[
T^{-1} = \begin{pmatrix} \frac{1}{2} \text{Id} & \frac{1}{2} \text{Id} \\ \frac{1}{2} \text{Id} & -\text{Id} \end{pmatrix}
\]

where \( \text{Id} \) is the identity, therefore, as all subdeterminants are non zero, we have that \( \Pi_1 \text{supp} W(\mu, \nu) \) u.d. implies that both \( \text{supp} \mu \) and \( \text{supp} \nu \) are u.d.
Proof. From Lemma 8 we have $\Pi_1 \text{supp} W_T(\mu, \nu) = \Pi_1 \text{supp} (T(\mu \otimes \nu))$. Then $\Pi_1 \text{supp} (T(\mu \otimes \nu))$ is u.d. and an application of Proposition 10 in the cases (i) or (ii) yields the thesis.

Our next aim is to obtain an analogous of Proposition 12 for the case of the Fourier transforms of $\mu$ and $\nu$. Before giving the statement we need the following remark on block matrices.

Remark. Let $Z$ be an invertible $2d \times 2d$ matrix, and write

$$Z = \begin{pmatrix} Y & U \\ V & W \end{pmatrix}, \quad Z^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix},$$

where $Y, U, V, W, E, F, G, H$ are $d \times d$ matrices. From [6, Theorem 2.1] we have that

$$\det Y \neq 0 \implies \det H \neq 0, \quad \det W \neq 0 \implies \det E \neq 0. \quad (8)$$

Since

$$Z_1 = \begin{pmatrix} U & Y \\ W & V \end{pmatrix} \implies Z_1^{-1} = \begin{pmatrix} G & H \\ E & F \end{pmatrix},$$

applying (8) to $Z_1$ we also have

$$\det U \neq 0 \implies \det F \neq 0, \quad \det V \neq 0 \implies \det G \neq 0. \quad (9)$$

Proposition 13. If $\Pi_2 \text{supp} W_T(\mu, \nu)$ is a u.d. set and $T$ is an invertible matrix satisfying

$$T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad (10)$$

as in Proposition 12 then

$$\det A \neq 0 \implies \text{supp } \hat{\nu} \text{ is u.d.}, \quad \det B \neq 0 \implies \text{supp } \hat{\mu} \text{ is u.d.}$$

Proof. From [1, Proposition 4] we have that

$$W_R(\hat{\mu}, \hat{\nu})(x, \omega) = |\det R|^{-1} W_T(\mu, \nu)(-\omega, x) \quad (11)$$

where

$$T = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} (R^{-1})^t \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}. \quad (12)$$
Here and in the following, for convenience we write explicitly the names of the variables in the Matrix-Wigner transform (of course, when \( \mu \) and \( \nu \) are distributions, the notation \((−\omega, x)\) in (11) stands for the corresponding linear change of variables). A simple calculation shows that, from (11) and (12) we have

\[
W_T(\mu, \nu)(x, \omega) = \left|\det T\right|^{-1}W_R(\hat{\mu}, \hat{\nu})(\omega, -x) \tag{13}
\]

where

\[
R = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} (T^{-1})^t \begin{pmatrix} 0 & \text{Id} \\ \text{Id} & 0 \end{pmatrix}. \tag{14}
\]

Now, from (14) and (12) we get

\[
R = \begin{pmatrix} C & A \\ -D & -B \end{pmatrix},
\]

and \( R \) is invertible since \( T \) is invertible. Then, writing

\[
R^{-1} = \begin{pmatrix} E & F \\ G & H \end{pmatrix}, \tag{15}
\]

from (8) and (9) we obtain

\[
\begin{align*}
\det A \neq 0 & \implies \det F \neq 0, & \det B \neq 0 & \implies \det E \neq 0, \\
\det C \neq 0 & \implies \det H \neq 0, & \det D \neq 0 & \implies \det G \neq 0.
\end{align*} \tag{16}
\]

Now, from (13) we have

\[
\Pi_2\text{supp} W_T(\mu, \nu) = \Pi_1\text{supp} W_R(\hat{\mu}, \hat{\nu}),
\]

and so, by (15) and (16) the thesis follows by an application of Proposition 12 to \( \Pi_1\text{supp} W_R(\hat{\mu}, \hat{\nu}) \).

From Propositions 12 and 13 we immediately have the following corollary.

**Corollary 14.** Let \( T \) be an invertible matrix satisfying

\[
T^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.
\]

(i) Suppose that \( \det A \neq 0 \) and \( \det B \neq 0 \). If both \( \Pi_1\text{supp} W_T(\mu, \nu) \) and \( \Pi_2\text{supp} W_T(\mu, \nu) \) are u.d. sets, then \( \text{supp} \mu, \text{supp} \nu, \text{supp} \hat{\mu} \) and \( \text{supp} \hat{\nu} \) are u.d. sets.

(ii) Suppose that \( \det A \neq 0 \) or \( \det B \neq 0 \). If both \( \Pi_1\text{supp} W_T(\mu) \) and \( \Pi_2\text{supp} W_T(\mu) \) are u.d. sets, then \( \text{supp} \mu \) and \( \text{supp} \hat{\mu} \) are u.d. sets.
The basic connection between our setting and the hypothesis assumed in [7] is the following remark.

**Remark.** In the case

\[ T = \begin{pmatrix} \text{Id} & 0 \\ 0 & -\text{Id} \end{pmatrix} \]

the matrix-Wigner transform is given by

\[ W_T(\mu, \nu) = \mathcal{F}_2(\mu \otimes \nu)(x, \omega) = \mu(x)\tilde{\nu}(\omega), \]

therefore Lev-Olevskii hypothesis (see [7])

\[ \mu = \sum_{\alpha \in \Lambda} a_{\alpha} \delta_{\alpha}; \quad \tilde{\nu} = \sum_{\beta \in S} b_{\beta} \delta_{\beta} \]

with \( \Lambda, S \) u.d. sets, (and with \( \mu = \nu \) in [7]) is a particular case of the hypothesis

\[ W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)} \]

where \( A, B \) are u.d. sets.

The previous results, Propositions 12, 13 and Corollary 14, where we can obtain u.d.ness of the supports of signals from that of their matrix-Wigner transform, include many classical time-frequency transforms such as the STFT and the Rihaczek transforms. As an example we consider the *Ambiguity function*.

**Example 15.** The Ambiguity function is defined as

\[ A(\mu, \nu)(x, \omega) = \int_{\mathbb{R}^d} e^{-2\pi i x \omega} \mu(t + x/2)\overline{\nu(t - x/2)} \, dt \]

for \( \mu, \nu \in \mathcal{S}(\mathbb{R}^d) \), which is generalized to

\[ A(\mu, \nu) = W_T(\mu, \nu) \]

for \( T = \begin{pmatrix} \frac{1}{2} \text{Id} & \text{Id} \\ \frac{1}{2} \text{Id} & \text{Id} \end{pmatrix} \). Then \( \Pi_1 \supp A(\mu, \nu) \) u.d. implies \( \supp \mu \) and \( \supp \nu \) u.d. In fact, it suffices to apply Proposition 12.

Our next aim is to give a version of Theorem 1 for the case of \( W_T(\mu, \nu) \) for a \( 2 \times 2 \) matrix \( T \), giving then a general picture of the situation for the case \( \mu, \nu \in \mathcal{S}'(\mathbb{R}) \).

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be an invertible linear transformation with inverse

\[ T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \]

We assume from now on that \( ab \neq 0 \).
Theorem 16. Let \( \mu, \nu \in S'(\mathbb{R}) \setminus \{0\} \) satisfy \( W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)} \) where \( A, B \) are uniformly discrete sets. Then \( \mu \) and \( \nu \) are measures supported in u.d. sets.

Proof. We apply Proposition 12 to conclude that \( \mu \) and \( \nu \) have u.d. support, denoted \( S_\mu \) and \( S_\nu \) respectively. According to Lemma 2 we can put

\[
\mu = \sum_{r \in S_\mu} a_r^j \delta_r, \quad \nu = \sum_{s \in S_\nu} b_s^k \delta_s
\]

with \( a_r^j, b_s^k \in \mathbb{C} \). We now assume \( M \geq 1 \) and \( b_s^M \neq 0 \) for some \( s_0 \in S_\nu \) and show that \( \mu = 0 \), which is a contradiction from where we conclude that \( \nu \) is a measure.

For any real-valued functions \( \phi_1, \phi_2 \in S(\mathbb{R}) \) we have, for \( \phi = \phi_1 \otimes \phi_2 \),

\[
\langle W_T(\mu, \nu), \phi \rangle = \langle T(\mu \otimes \nu), F_2^{-1} \phi \rangle = |\text{det} T^{-1}| \langle \mu \otimes \nu, (F_2^{-1} \phi) \circ T^{-1} \rangle.
\]

Hence

\[
|\text{det} T| \langle W_T(\mu, \nu), \phi \rangle = \langle \mu_u, \langle \nu_v, \left( \phi_1 \otimes \overline{\phi_2} \right) \circ T^{-1}(u, v) \rangle \rangle.
\]

Since

\[
\langle \nu_v, \left( \phi_1 \otimes \overline{\phi_2} \right) \circ T^{-1}(u, v) \rangle = \sum_{s \in S_\nu} \sum_{k=0}^{M} (-1)^k b_s^k \sum_{m=0}^{k} \binom{k}{m} \phi_1^{(m)}(au + bs)b^m \phi_2^{(k-m)}(cu + ds)d^{k-m}
\]

we finally obtain

\[
|\text{det} T| \langle W_T(\mu, \nu), \phi \rangle = \sum_{j=0}^{N} \sum_{\ell=0}^{j} \sum_{k=0}^{M} \sum_{m=0}^{k} \lambda_{j,k}^{\ell,m} \sum_{r \in S_\mu} \sum_{s \in S_\nu} a_r^j b_s^k \phi_1^{(m+\ell)}(ar + bs) \phi_2^{(k+j-m-\ell)}(cr + ds),
\]

where

\[
\lambda_{j,k}^{\ell,m} = (-1)^{k+j} \binom{k}{m} \binom{j}{\ell} a^\ell b^m c^{j-\ell} d^{k-m}.
\]
The functions $\phi_1^{(m+\ell)}$ and $\phi_2^{(k+j-m-\ell)}$ are in $S(\mathbb{R})$, therefore, for every $\beta > 0$ and suitable constant $C > 0$, we have
\[|\phi_1^{(m+\ell)}(ar + bs)| \leq C|ar + bs|^{-\beta}, \quad |\phi_2^{(k+j-m-\ell)}(cr + ds)| \leq C|cr + ds|^{-\beta}.
\]
Furthermore we remark that linear bijections on $\mathbb{R}^2$ are bi-continuous and therefore preserve u.d. sets, so that $T^{-1}(S_\mu \times S_\nu)$ is u.d. Supposing then that $T$ has matrix
\[T = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix},
\]
and indicating by $C$ generic (possibly different) suitable constants, we have:
\[
\sum_{r \in S_\mu, s \in S_\nu} |a'_r| |b'_s| |\phi_1^{(m+\ell)}(ar + bs)| |\phi_2^{(k+j-m-\ell)}(cr + ds)|
\leq C \sum_{r \in S_\mu, s \in S_\nu} (r)^\alpha (s)^\alpha |ar + bs|^{-\beta} |cr + ds|^{-\beta}
\leq C \sum_{(u,v) \in T^{-1}(S_\mu \times S_\nu)} (a'u + b'v)^\alpha (c'u + d'v)^\alpha |u|^{-\beta} |v|^{-\beta}
\leq C \sum_{(u,v) \in T^{-1}(S_\mu \times S_\nu)} (1 + |u|^2 + |v|^2)^{(2\alpha - \beta)},
\]
where $\beta$ can be chosen arbitrary large and $T^{-1}(S_\mu \times S_\nu)$ is a u.d. set, and therefore the last sum is convergent, i.e. (17) is absolutely convergent.

Let us take now
\[0 < \varepsilon < \delta(A) := \inf\{|r - r'| : r, r' \in A, r \neq r'\}.
\]
Fix $r_0 \in S_\mu, s_0 \in S_\nu$, put $x_0 = ar_0 + bs_0$ and choose $\phi_1 \in \mathcal{D}(x_0 - \varepsilon, x_0 + \varepsilon)$ such that $\phi_1^{(n)}(x_0) = 0$ for $n = 0, \ldots, N + M - 1$ whereas $\phi_1^{(N+M)}(x_0) \neq 0$. Then, for $\phi_2 \in S(\mathbb{R})$ (real valued), we have
\[
|\det T| (W_T(\mu, \nu), \phi) = a^N b^M \phi_1^{(N+M)}(x_0) \sum_{(r,s) \in D} a'_r b'_s \phi_2(cr + ds).
\]
Here $D = \{(r, s) \in S_{\mu} \times S_{\nu} : ar + bs = x_0\}$. Hence, if $\phi_2$ has also compact support,

$$\left| \phi_1^{(N+M)}(x_0) \sum_{(r,s) \in D} a_r^N b_s^M \hat{\phi}_2(cr + ds) \right| = |a|^{-N} |b|^{-M} |\det T| |\langle W_T(\mu, \nu), \phi_1 \otimes \phi_2 \rangle|$$

$$\leq C \|\phi_1\|_{\infty} \|\phi_2\|_{\infty},$$

where the constant $C$ depends on the (compact) support of $\phi_1 \otimes \phi_2$. Next, we fix $\psi \in D((-\varepsilon, \varepsilon))$ such that $\psi^{(n)}(0) = 0$ for $n = 0, \ldots, N+M-1$ and $\psi^{(N+M)}(0) = 1$, and for each $t \geq 1$, let us consider $\psi_t(x) := \psi(tx)$ and $\phi_t^1(x) = \psi_t(x - r_0)$. As in the proof of Theorem 1 we conclude that

$$\sum_{(r,s) \in D} a_r^N b_s^M \hat{\phi}_2(cr + ds) = 0 \quad (19)$$

for every real valued $\phi_2 \in D(\mathbb{R})$. To discuss the meaning of the obtained expression we need some notation. Denote

$$S = \{r \in S_{\mu} : (r, s) \in D \text{ for some } s \in S_{\nu}\}.$$

For $r \in S$ we denote by $s(r)$ the unique $s \in S_{\nu}$ such that $(r, s) \in D$. Observe that there are constants $\alpha, \beta$ such that $\alpha \neq 0$ and $cr + ds = \alpha r + \beta$ whenever $(r, s) \in D$. From (18) and the fact that $S$ is a u.d. set it follows that

$$\sum_{r \in S} a_r^N b_{s(r)}^M \delta_{\alpha r + \beta}$$

defines a tempered distribution. Condition (19) means that the Fourier transform of that distribution vanishes. Consequently

$$a_r^N b_{s(r)}^M = 0 \quad \forall r \in S.$$

Since $s(r_0) = s_0$ and $b_{s_0}^M \neq 0$ we conclude $a_{r_0}^N = 0$. Since $r_0 \in S_{\mu}$ is arbitrary we conclude $a_r^N = 0$ for every $r \in S_{\mu}$. Proceeding by recurrence on the order of $\mu$ we finally get that $a_r^j = 0$ for all $r \in S_{\mu}$ and $0 \leq j \leq N$, from where it follows $\mu = 0$. This contradiction proves that $M = 0$ and $\nu$ is a measure. The same argument but swapping the role of the distributions proves that also $\mu$ is a measure.

**Corollary 17.** Under the same hypothesis as in Theorem 16 $\hat{\mu}$ and $\hat{\nu}$ are measures supported on u.d. sets.
Proof. By [1, Proposition 4, (ii)]

\[
W_S(\hat{\mu}, \hat{\nu})(x, \omega) = |\text{det} S|^{-1} W_T(\mu, \nu)(-\omega, x)
\]

where

\[
S = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.
\]

Hence,

\[
S^{-1} = \begin{pmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{pmatrix}
\]

where, as \(ab \neq 0\), by direct calculation or from (8), (9), we have \(\tilde{a}\tilde{b} \neq 0\), and we can then apply Theorem 16 to conclude.

Corollary 18. Let \(\mu, \nu \in \mathcal{S}'(\mathbb{R}) \setminus \{0\}\) satisfy \(W_T(\mu, \nu) = \sum_{(r,s) \in A \times B} c_{r,s} \delta_{(r,s)}\) where \(A, B\) are u.d. sets. Then, there are \(a, b > 0\) such that \(\mu\) is a finite linear combination of time-frequency shifts of \(\sum_{n \in \mathbb{Z}} \delta_{an}\) and \(\nu\) is a finite linear combination of time-frequency shifts of \(\sum_{n \in \mathbb{Z}} \delta_{bn}\).

Proof. Theorem 16 and Corollary 17 imply that \(\mu, \nu, \hat{\mu}, \hat{\nu}\) are measures supported on u.d. sets in \(\mathbb{R}\). We now apply [7, Theorems 1,3] to conclude.

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