MORE PROPERTIES OF YETTER-DRINFELD MODULES OVER QUASI-HOPF ALGEBRAS

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Abstract. We generalize various properties of Yetter-Drinfeld modules over Hopf algebras to quasi-Hopf algebras. The dual of a finite dimensional Yetter-Drinfeld module is again a Yetter-Drinfeld module. The algebra $H_0$ in the category of Yetter-Drinfeld modules that can be obtained by modifying the multiplication in a proper way is quantum commutative. We give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules, and deduce the existence and uniqueness of integrals from it.

1. Introduction

The motivation for studying Yetter-Drinfeld modules over quasi-Hopf algebras is the same as for Hopf algebras. It is well known that for any finite dimensional Hopf algebra $H$ the category of Yetter-Drinfeld modules $H\mathcal{YD}^H$ is isomorphic to the category of modules over the quantum double $D(H)$. From a categorical point of view, the quantum double $D(H)$ arises by considering the center $\mathcal{Z}(H\mathcal{M})$ of the monoidal category of left $H$-modules. More precisely, one has $\mathcal{Z}(H\mathcal{M}) \simeq D(H)\mathcal{M}$ if $H$ is finite dimensional. Actually, the category of Yetter-Drinfeld modules appears as an intermediate step in the proof of this isomorphism: one first proves that $\mathcal{Z}(H\mathcal{M}) \simeq H\mathcal{YD}^H$, and then $H\mathcal{YD}^H \simeq D(H)\mathcal{M}$, where the finite dimensionality is not needed in the proof of the first isomorphism, see [16] for full detail.

Quasi-bialgebras and quasi-Hopf algebras were introduced by Drinfeld [13]; a categorical interpretation is the following: a quasi-bialgebra $H$ is an algebra with the additional structure that is needed to make the category of left $H$-modules, with the tensor product over $k$ as tensor product and $k$ as unit object into a monoidal category. The difference with a usual bialgebra is that we do not require that the associativity isomorphism coincides with the associativity in the category of vector spaces. A quasi-Hopf algebra is a quasi-bialgebra with additional structure making the category of finite dimensional $H$-modules into a monoidal category with duality.

The center construction $\mathcal{Z}(\mathcal{C})$ can be applied to any monoidal category $\mathcal{C}$. Majid [19] computed the center of the category of left modules over a quasi-Hopf algebra $H$, and introduced the category of Yetter-Drinfeld modules over $H$. Hausser and Nill [14], [15] constructed the quantum double $D(H)$ of a finite dimensional quasi-Hopf algebra $H$, and proved that $H\mathcal{YD}^H \simeq D(H)\mathcal{M}$. Recently, Schauenburg [22]...

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gave the equivalence between the category of Yetter-Drinfeld modules \( \mathcal{YD}_H \) and the category \( \mathcal{M}_{H \otimes H}^H \) of Hopf bimodules. In \cite{5}, the relation between Yetter-Drinfeld modules and Radford’s biproduct is studied. In \cite{4}, the rigidity of the category of Yetter-Drinfeld modules is investigated, as well as the relations between left, left-right, right-left and right Yetter-Drinfeld modules.

In this paper, which can be seen as a sequel to \cite{4}, we continue our investigations of properties of Yetter-Drinfeld modules. In Section 3, we show that the linear dual of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module.

It was shown in \cite{7}, \cite{5} that the multiplication on \( H \) can be modified in such a way that we obtain an algebra in the category of left Yetter-Drinfeld modules. The main result of Section 4 is that \( H_0 \) is quantum commutative.

In Section 5, we will generalize Doi’s results \cite{12} about Hopf modules in the category of Yetter-Drinfeld modules to our situation: we give a Structure Theorem for Hopf modules in the category of Yetter-Drinfeld modules over a quasi-Hopf algebras, and we use this result to obtain the existence and uniqueness of integrals for a finite dimensional braided Hopf algebra in \( \mathcal{YD}_H \). We apply this to the braided Hopf algebra considered in Section 4 in the case where \( H \) is finite dimensional and quasitriangular.

2. Preliminary results

2.1. Quasi-Hopf algebras. We work over a commutative field \( k \). All algebras, linear spaces etc. will be over \( k \); unadorned \( \otimes \) means \( \otimes_k \). Following Drinfeld \cite{13}, a quasi-bialgebra is a fourtuple \((H, \Delta, \varepsilon, \Phi)\), where \( H \) is an associative algebra with unit, \( \Phi \) is an invertible element in \( H \otimes H \otimes H \), and \( \varepsilon : H \rightarrow H \otimes H \) and \( \Delta : H \rightarrow H \otimes H \) are algebra homomorphisms satisfying the identities

\[
(1) \quad (id \otimes \Delta)(\Delta(h)) = \Phi(\Delta \otimes id)(\Delta(h))\Phi^{-1},
\]

\[
(2) \quad (id \otimes \varepsilon)(\Delta(h)) = h \otimes 1, \quad (\varepsilon \otimes id)(\Delta(h)) = 1 \otimes h,
\]

for all \( h \in H \), and \( \Phi \) has to be a normalized 3-cocycle, in the sense that

\[
(3) \quad (1 \otimes \Phi)(id \otimes \Delta \otimes id)(\Phi)(\Phi \otimes 1) = (id \otimes id \otimes \Delta)(\Phi)(\Delta \otimes id \otimes id)(\Phi),
\]

\[
(4) \quad (id \otimes \varepsilon \otimes id)(\Phi) = 1 \otimes 1 \otimes 1.
\]

The map \( \Delta \) is called the coproduct or the comultiplication, \( \varepsilon \) the counit and \( \Phi \) the reassociator. As for Hopf algebras \cite{23} we use the notation \( \Delta(h) = \sum h_1 \otimes h_2 \).

Since \( \Delta \) is only quasi-coassociative we adopt the further notation

\[
(\Delta \otimes id)(\Delta(h)) = \sum h_{(1,1)} \otimes h_{(1,2)} \otimes h_2, \quad (id \otimes \Delta)(\Delta(h)) = \sum h_1 \otimes h_{(2,1)} \otimes h_{(2,2)},
\]

for all \( h \in H \). We will denote the tensor components of \( \Phi \) by capital letters, and the ones of \( \Phi^{-1} \) by small letters, namely

\[
\Phi = \sum X^1 \otimes X^2 \otimes X^3 = \sum T^1 \otimes T^2 \otimes T^3 = \sum V^1 \otimes V^2 \otimes V^3 = \cdots
\]

\[
\Phi^{-1} = \sum x^1 \otimes x^2 \otimes x^3 = \sum t^1 \otimes t^2 \otimes t^3 = \sum u^1 \otimes u^2 \otimes u^3 = \cdots
\]
A quasi-bialgebra $H$ is called a quasi-Hopf algebra if there exists an anti-automorphism $S$ of the algebra $H$ and $\alpha, \beta \in H$ such that:

\[
\begin{align*}
(5) \quad & \sum S(h_1)\alpha h_2 = \varepsilon(h)\alpha \quad \text{and} \quad \sum h_1\beta S(h_2) = \varepsilon(h)\beta, \\
(6) \quad & \sum X^1\beta S(X^2)\alpha X^3 = 1 \quad \text{and} \quad \sum S(x^1)\alpha x^2\beta S(x^3) = 1,
\end{align*}
\]

for all $h \in H$. It is shown in [2] that the condition that the antipode is bijective follows automatically from the other axioms in the case where $H$ is finite dimensional. Observe that the antipode of a quasi-Hopf algebra is determined uniquely up to a transformation $\alpha \mapsto U\alpha, \beta \mapsto \beta U^{-1}$, $S(h) \mapsto US(h)U^{-1}$, where $U \in H$ is invertible. The axioms for a quasi-Hopf algebra imply that $\varepsilon(\alpha)\varepsilon(\beta) = 1$, so, by rescaling $\alpha$ and $\beta$, we may assume without loss of generality that $\varepsilon(\alpha) = \varepsilon(\beta) = 1$ and $\varepsilon \circ S = \varepsilon$. The identities (2-4) also imply that

\[
(7) \quad (\varepsilon \otimes id \otimes id)(\Phi) = (id \otimes id \otimes \varepsilon)(\Phi) = 1 \otimes 1 \otimes 1.
\]

Together with a quasi-Hopf algebra $H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta)$ we also have $H^{op}$, $H^{cop}$ and $H^{op,cop}$ as quasi-Hopf algebras, where “op” means opposite multiplication and “cop” means opposite comultiplication. The reassociators of these three quasi-Hopf algebras are $\Phi_{op} = \Phi^{-1}$, $\Phi_{cop} = (\Phi^{-1})^{321}$, $\Phi_{op,cop} = \Phi^{321}$, the antipodes are $S_{op} = S_{cop} = (S_{op,cop})^{-1} = S^{-1}$, and the elements $\alpha, \beta$ are $\alpha_{op} = S^{-1}(\beta)$, $\beta_{op} = S^{-1}(\alpha)$, $\alpha_{cop} = S^{-1}(\alpha)$, $\beta_{cop} = S^{-1}(\beta)$, $\alpha_{op,cop} = \beta$ and $\beta_{op,cop} = \alpha$.

Recall next that the definition of a quasi-Hopf algebra is “twist coinvariant”, in the following sense. An invertible element $F \in H \otimes H$ is called a gauge transformation or twist if $(\varepsilon \otimes id)(F) = (id \otimes \varepsilon)(F) = 1$. If $H$ is a quasi-Hopf algebra and $F = \sum F^1 \otimes F^2 \in H \otimes H$ is a gauge transformation with inverse $F^{-1} = \sum G^1 \otimes G^2$, then we can define a new quasi-Hopf algebra $H_F$ by keeping the multiplication, unit, counit and antipode of $H$ and replacing the comultiplication, antipode and the elements $\alpha$ and $\beta$ by

\[
\begin{align*}
(8) \quad & \Delta_F(h) = F\Delta(h)F^{-1}, \\
(9) \quad & \Phi_F = (1 \otimes F)(id \otimes \Delta)(F)\Phi(\Delta \otimes id)(F^{-1})(F^{-1} \otimes 1), \\
(10) \quad & \alpha_F = \sum S(G^1)\alpha G^2, \quad \beta_F = \sum F^1\beta S(F^2).
\end{align*}
\]

It is well-known that the antipode of a Hopf algebra is an anti-coalgebra morphism. The corresponding statement for a quasi-Hopf algebra is the following: there exists a gauge transformation $f \in H \otimes H$ such that

\[
\begin{align*}
\begin{align*}
(11) \quad & f\Delta(S(h))f^{-1} = \sum (S \otimes S)(\Delta^{cop}(h)), \\
\end{align*}
\end{align*}
\]

for all $h \in H$, where $\Delta^{cop}(h) = \sum h_2 \otimes h_1$. The element $f$ can be computed explicitly. First set

\[
\begin{align*}
(12) \quad & \sum A^1 \otimes A^2 \otimes A^3 \otimes A^4 = (\Phi \otimes 1)(\Delta \otimes id \otimes id)(\Phi^{-1}), \\
(13) \quad & \sum B^1 \otimes B^2 \otimes B^3 \otimes B^4 = (\Delta \otimes id \otimes id)(\Phi)(\Phi^{-1} \otimes 1)
\end{align*}
\]

and then define $\gamma, \delta \in H \otimes H$ by

\[
\begin{align*}
(14) \quad & \gamma = \sum S(A^2)\alpha A^3 \otimes S(A^1)\alpha A^4 \quad \text{and} \quad \delta = \sum B^1\beta S(B^4) \otimes B^2\beta S(B^3).
\end{align*}
\]
Then $f$ and $f^{-1}$ are given by the formulas
\begin{align}
(15) \quad f &= \sum (S \otimes S)(\Delta^{op}(x^1))\gamma\Delta(x^2\beta S(x^3)), \\
(16) \quad f^{-1} &= \sum \Delta(S(x^1)\alpha x^2)\delta(S \otimes S)(\Delta^{op}(x^3)).
\end{align}
Moreover, $f$ satisfies the following relations:
\begin{align}
(17) \quad f\Delta(\alpha) &= \gamma, \quad \Delta(\beta)f^{-1} = \delta.
\end{align}
Furthermore the corresponding twisted reassociator (see (9)) is given by
\begin{align}
(18) \quad \Phi_f &= \sum (S \otimes S \otimes S)(X^3 \otimes X^2 \otimes X^1).
\end{align}
In a Hopf algebra $H$, we obviously have the identity
\begin{align}
\sum h_1 \otimes h_2 S(h_3) = h \otimes 1, \text{ for all } h \in H.
\end{align}
We will need the generalization of this formula to the quasi-Hopf algebra setting.

Following [14, 15], we define
\begin{align}
(19) \quad p_R &= \sum p_R^1 \otimes p_R^2 = \sum x^1 \otimes x^2 \beta S(x^3), \\
(20) \quad q_R &= \sum q_R^1 \otimes q_R^2 = \sum X^1 \otimes S^{-1}(\alpha X^3)X^2, \\
(21) \quad p_L &= \sum p_L^1 \otimes p_L^2 = \sum X^2 S^{-1}(X^1 \beta) \otimes X^3, \\
(22) \quad q_L &= \sum q_L^1 \otimes q_L^2 = \sum S(x^1)\alpha x^2 \otimes x^3.
\end{align}
We then have, for all $h \in H$,
\begin{align}
(23) \quad \sum \Delta(h_1)p_R[1 \otimes S(h_2)] &= p_R(h \otimes 1), \\
(24) \quad \sum [1 \otimes S^{-1}(h_2)]q_R\Delta(h_1) &= (h \otimes 1)q_R, \\
(25) \quad \sum \Delta(h_2)p_L[S^{-1}(h_1) \otimes 1] &= p_L(1 \otimes h), \\
(26) \quad \sum [S(h_1) \otimes 1]q_L\Delta(h_2) &= (1 \otimes h)q_L,
\end{align}
and
\begin{align}
(27) \quad (q_R \otimes 1)(\Delta \otimes id)(q_R)\Phi^{-1} &= \sum [1 \otimes S^{-1}(X^3) \otimes S^{-1}(X^2)]

\begin{align}
(28) \quad (\Delta \otimes id)(R) &= \sum \Phi_{312}R_{13}\Phi_{132}^{-1}R_{23}\Phi, \\
(29) \quad (id \otimes \Delta)(R) &= \sum \Phi_{231}^{-1}R_{13}\Phi_{213}R_{12}\Phi^{-1}, \\
(30) \quad \Delta^{cop}(h)R &= R\Delta(h), \text{ for all } h \in H, \\
(31) \quad (\varepsilon \otimes id)(R) &= (id \otimes \varepsilon)(R) = 1.
\end{align}
Here we used the following notation: if $\sigma$ is a permutation of $\{1, 2, 3\}$, then we write $\Phi_{\sigma(1)\sigma(2)\sigma(3)} = \sum X^{\sigma^{-1}(1)} \otimes X^{\sigma^{-1}(2)} \otimes X^{\sigma^{-1}(3)}$; $R_{ij}$ means $R$ acting non-trivially
on the $i$-th and $j$-th tensor factors of $H \otimes H \otimes H$.
It is shown in [10] that $R$ is invertible. Furthermore, the element
\begin{equation}
(32) \quad u = \sum S(R^2 p^2) \alpha R^1 p^1,
\end{equation}
with $p_R = \sum p^1 \otimes p^2$ defined as in [10], is invertible in $H$, and
\begin{equation}
(33) \quad u^{-1} = \sum X^1 R^2 p^2 S(S(X^2 R^1 p^1) \alpha X^3),
\end{equation}
\begin{equation}
(34) \quad \varepsilon(u) = 1 \text{ and } S^2(h) = uhu^{-1},
\end{equation}
for all $h \in H$. Consequently the antipode $S$ is bijective, so, as in the Hopf algebra
case, the assumptions about invertibility of $R$ and bijectivity of $S$ can be dropped.
Moreover, the $R$-matrix $R = \sum R^1 \otimes R^2$ satisfies the identity (see [11, 15, 10]):
\begin{equation}
(35) \quad f_{21} R f^{-1} = (S \otimes S)(R)
\end{equation}
where $f = \sum f^1 \otimes f^2$ is the twist defined in [15], and $f_{21} = \sum f^2 \otimes f^1$.

2.2. Monoidal categories. A monoidal or tensor category is a sixtuple $(\mathcal{C}, \otimes, 1, a, \ell, r)$,
where $\mathcal{C}$ is a category, $\otimes$ is a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ (called the tensor product),
$1$ is an object of $\mathcal{C}$, and
\begin{align*}
a_{U,V,W} : (U \otimes V) \otimes W &\to U \otimes (V \otimes W) \\
l_V : V \cong V \otimes 1 &\; ; \; r_V : V \cong 1 \otimes V
\end{align*}
are natural isomorphisms satisfying certain coherence conditions, see for example
[16, 18, 20]. An object $V$ of a monoidal category $\mathcal{C}$ has a left dual if there exists an
object $V^*$ and morphisms $ev_V : V^* \otimes V \to 1$, $coev_V : 1 \to V \otimes V^*$ in $\mathcal{C}$ such that
\begin{align*}
(36) \quad l_V^{-1} \circ (id_V \otimes ev_V) \circ a_{V,V^*,V} \circ (coev_V \otimes id_V) \circ r_V &= id_V, \\
(37) \quad r_V^{-1} \circ (ev_V \otimes id_{V^*}) \circ a_{V^*,V,V} \circ (id_{V^*} \otimes coev_V) \circ l_V &= id_{V^*}.
\end{align*}
$\mathcal{C}$ is called a rigid monoidal category if every object of $\mathcal{C}$ has a dual.
A braided monoidal category is a monoidal category equipped with a commutativity
natural isomorphism $c_{U,V} : U \otimes V \to V \otimes U$, compatible with the unit and the
associativity.
In a braided monoidal category, we can define algebras, coalgebras, bialgebras and
Hopf algebras. For example, a bialgebra $(B, m, \eta, \Delta, \varepsilon)$ consists of $B \in \mathcal{C}$, a multi-
plication $m : B \otimes B \to B$ which is associative up to the natural isomorphism $\alpha$, and
a unit $\eta : 1 \to B$ such that $m \circ (\eta \otimes id) = m \circ (id \otimes \eta) = id$. The properties of the
comultiplication $\Delta$ and the counit $\varepsilon$ are similar. In addition, $\Delta : B \to B \otimes B$ has to be an algebra morphism, where $B \otimes B$ is an algebra with multiplication $m_{B \otimes B}$,
defined as the composition
\begin{equation}
(38) \quad (B \otimes B) \otimes (B \otimes B) \xrightarrow{a^{-1}} B \otimes ((B \otimes B) \otimes B) \\
\xrightarrow{id \otimes \alpha^{-1}} B \otimes ((B \otimes B) \otimes B) \\
\xrightarrow{id \otimes c \otimes id} B \otimes ((B \otimes B) \otimes B) \\
\xrightarrow{id \otimes a} B \otimes (B \otimes (B \otimes B)) \\
\xrightarrow{a^{-1}} (B \otimes B) \otimes (B \otimes B) \\
\xrightarrow{m \circ m} B \otimes B
\end{equation}
A Hopf algebra $B$ is a bialgebra with a morphism $S : B \to B$ in $\mathcal{C}$ (the antipode)
satisfying the usual axioms $m \circ (S \otimes id) \circ \Delta = \eta \circ \varepsilon = m \circ (id \otimes S) \circ \Delta$. It is known, see
e.g. [21], that the antipode \( S \) of a Hopf algebra \( B \) in a braided monoidal category \( \mathcal{C} \) is an antialgebra and anticoalgebra morphism, in the sense that
\[
S \circ m = m \circ (S \otimes S) \circ c_{B,B} \quad \text{and} \quad \Delta \circ S = c_{B,B} \circ (S \otimes S) \circ \Delta.
\]
Recall also that an algebra \( A \) in a braided monoidal category \( \mathcal{C} \) is called quantum commutative if \( m \circ c_{A,A} = m \).

Assume that \( (H, \Delta, \varepsilon, \Phi) \) is a quasi-bialgebra, and let \( U, V, W \) be left \( H \)-modules. We define a left \( H \)-action on \( U \otimes V \) by
\[
h \cdot (u \otimes v) = \sum h_1 \cdot u \otimes h_2 \cdot v.
\]
We have isomorphisms \( a_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W) \) in \( H \mathcal{M} \) given by
\[
a_{U,V,W}((u \otimes v) \otimes w) = \Phi \cdot (u \otimes (v \otimes w)).
\]
The counit \( \varepsilon : H \rightarrow k \) makes \( k \in H \mathcal{M} \), and the natural isomorphisms \( \lambda : k \otimes H \rightarrow H \) and \( \rho : H \otimes k \rightarrow H \) are in \( H \mathcal{M} \). With these structures, \( (H \mathcal{M}, \otimes, k, a, \lambda, \rho) \) is a monoidal category. If \( H \) is a quasi-Hopf algebra then the category of finite dimensional left \( H \)-modules is rigid; the left dual of \( V \) is \( V^* \) with the \( H \)-module structure given by \( (h \cdot \varphi)(v) = \varphi(S(h) \cdot v) \), for all \( v \in V, \varphi \in V^*, h \in H \) and with
\[
ev_V(\varphi \otimes v) = \varphi(\alpha \cdot v), \quad \text{coev}_V(1) = \sum \beta_i \cdot v_i \otimes v^i,
\]
where \( \{v_i\} \) is a basis in \( V \) with dual basis \( \{v^i\} \).

Now let \( H \) be a quasitriangular quasi-Hopf algebra, with \( R \)-matrix \( R = \sum R^1 \otimes R^2 \).

For two left \( H \)-modules \( U \) and \( V \), we define
\[
c_{U,V} : U \otimes V \rightarrow V \otimes U
\]
by
\[
c_{U,V}(u \otimes v) = \sum R^2 \cdot v \otimes R^1 \cdot u
\]
and then \( (H \mathcal{M}, \otimes, k, a, \lambda, \rho, c) \) is a braided monoidal category (cf. [19] or [20]).

3. Yetter-Drinfeld modules and the quasi-Yang-Baxter equation

From [19], we recall the notion of Yetter-Drinfeld module over a quasi-bialgebra.

**Definition 3.1.** Let \( H \) be a quasi-bialgebra with reassociator \( \Phi. A \) left \( H \)-module \( M \) together with a left \( H \)-coaction
\[
\lambda_M : M \rightarrow H \otimes M, \quad \lambda_M(m) = \sum m_{(-1)} \otimes m_{(0)}
\]
is called a left Yetter-Drinfeld module if the following equalities hold, for all \( h \in H \) and \( m \in M \):
\[
\sum X^1 \cdot m_{(0)} = \sum Y^1 \cdot m_{(0)} - h_1 \cdot m_{(0)} = \sum (h_1 \cdot m_{(-1)} \otimes h_2 \cdot m_{(0)}).
\]
The category of left Yetter-Drinfeld \( H \)-modules and \( k \)-linear maps that intertwine
the \( H \)-action and \( H \)-coaction is denoted by \( \mathcal{H} \mathcal{D}^H \). In [19] it is shown that \( \mathcal{H} \mathcal{D}^H \) is
a prebraided monoidal category. The forgetful functor \( \mathcal{H} \mathcal{D}^H \to \mathcal{M} \) is monoidal,
and the coaction on the tensor product \( M \otimes N \) of two Yetter-Drinfeld modules \( M \) and \( N \) is given by
\begin{equation}
\lambda_{M \otimes N}(m \otimes n) = \sum X^1(x^1Y^1 \cdot m)(-1)x^2(Y^2 \cdot n)(-1)Y^3
\end{equation}
\begin{equation}
\otimes X^2 \cdot (x^1Y^1 \cdot m)(0) \otimes X^3x^3 \cdot (Y^2 \cdot n)(0).
\end{equation}
The braiding is given by
\begin{equation}
c_{M,N}(m \otimes n) = \sum m(-1) \cdot n \otimes m(0).
\end{equation}
This braiding is invertible if \( H \) is a quasi-Hopf algebra [5], and its inverse is then
given by
\begin{equation}
c_{M,N}^{-1}(n \otimes m) = \sum y_1^3X^2 \cdot (x^1 \cdot m)(0)
\end{equation}
\begin{equation}
\otimes S^{-1}(S(y^1)\beta x^1(x^1 \cdot m)(-1)x^2\beta S(y^3x^3)) \cdot n.
\end{equation}
Let \((H, R)\) be a quasitriangular quasi-bialgebra. It is well-known (see for example
[16]) that \( R \) satisfies the so-called quasi-Yang-Baxter equation in \( H \otimes H \otimes H \):
\[ R_{12}R_{312}R_{13}R_{132}^{-1}R_{23} \Phi = \Phi_{321}R_{23} \Phi_{231}^{-1}R_{123} \Phi_{213}R_{123} \Phi \]
On the other hand, if \( H \) is a bialgebra and \( M \) is a left-right Yetter-Drinfeld module
over \( H \), with structures
\[ H \otimes M \to M, \quad h \otimes m \mapsto h \cdot m; \]
\[ M \otimes H \to M, \quad m \mapsto \sum m(0) \otimes m(1), \]
then the map \( R_M : M \otimes M \to M \otimes M \), \( R_M(m \otimes n) = \sum n(-1) \cdot m \otimes n(0) \) is a solution
in \( \text{End}(M \otimes M) \) of the quantum Yang-Baxter equation
\[ R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}, \]
see for instance [17].
We will show a similar result for quasi-bialgebras; first we define left-right Yetter-
Drinfeld modules over quasi-bialgebras as follows
\[ \mathcal{H} \mathcal{D}^H = \mathcal{H} \mathcal{D}^H. \]
This is stated more explicitly in the next definition.

**Definition 3.2.** Let \( H \) be a quasi-bialgebra. A \( k \)-linear space \( M \) with a left \( H \)-
action \( h \otimes m \mapsto h \cdot m \), and a right \( H \)-coaction \( M \to M \otimes H \), \( m \mapsto \sum m(0) \otimes m(1) \)
is called a left-right Yetter-Drinfeld module if the following relations hold, for all
\( m \in M \) and \( h \in H \):
\begin{equation}
\sum (x^2 \cdot m(0))(0) \otimes (x^2 \cdot m(0))(1)x^1 \otimes x^3m(1)
\end{equation}
\begin{equation}
= \sum x^1 \cdot (y^3 \cdot m)(0) \otimes x^2(y^3 \cdot m)(1)y^1 \otimes x^3(y^3 \cdot m)(1)y^2
\end{equation}
\begin{equation}
\sum \varepsilon(m(1))m(0) = m
\end{equation}
\begin{equation}
\sum h_1 \cdot m(0) \otimes h_2m(1) = \sum (h_2 \cdot m)(0) \otimes (h_2 \cdot m)(1)h_1.
\end{equation}
Proposition 3.3. Let \( H \) be a quasi-bialgebra and \( M \in \mathcal{HYD}^H \). The map \( R = R_M : M \otimes M \rightarrow M \otimes M, R(m \otimes n) = \sum n_{(1)} \cdot m \otimes n_{(0)} \), is a solution of the quasi-Yang-Baxter equation

\[
R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi = \Phi_{321} R_{23} \Phi_{231}^{-1} \Phi_{213} R_{12}
\]

on \( \text{End}(M \otimes M \otimes M) \).

We considered \( R_{12}, \Phi_{312}, \text{etc.} \) as elements in \( \text{End}(M \otimes M \otimes M) \) by left multiplication, for example \( R_{12}(l \otimes m \otimes n) = \sum R^1_1 \cdot l \otimes R^2_2 \cdot m \otimes n, \Phi_{312}(l \otimes m \otimes n) = \sum X^2 \cdot l \otimes X^3 \cdot m \otimes X^1 \cdot n \) etc.

Proof. \( \mathcal{HYD}^H \) is a prebraided category, hence the result is a consequence of the fact (see [17]) that the braiding satisfies the categorical version of the Yang-Baxter equation. A direct proof is also possible. For all \( l, m, n \in M \), we compute that

\[
R_{12} \Phi_{312} R_{13} \Phi_{132}^{-1} R_{23} \Phi(l \otimes m \otimes n)
= \sum (Y^3 \cdot x^3(X^3 \cdot n)_{(1)}) X^2 \cdot m_{(1)} \otimes Y^2 \cdot (X^3 \cdot n)_{(0)} \cdot X^1 \cdot l \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)}
\]

\[
= \sum (Y^3 \cdot x^3(y^3 X^3 \cdot n)_{(1)}) y^2 X^2 \cdot m_{(1)} \otimes Y^2 x^2(y^3 X^3 \cdot n)_{(1)} \cdot y^1 X^1 \cdot l \otimes Y^1 x^1 \cdot (y^3 X^3 \cdot n)_{(0)}
\]

\[
= \sum (n_{(1)} \cdot m_{(1)}) n_{(1)} \cdot l \otimes n_{(1)} \cdot m_{(0)} \otimes n_{(0)}
\]

and

\[
\Phi_{321} R_{23} \Phi_{231}^{-1} R_{213} R_{12}(l \otimes m \otimes n)
= \sum Y^3 \cdot x^3(X^3 \cdot n)_{(1)} X^2 \cdot m_{(1)} \otimes Y^2(x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)} X^1 \cdot m_{(0)} \otimes Y^1 \cdot (x^2 \cdot (X^3 \cdot n)_{(0)})_{(0)}
\]

\[
= \sum Y^3 \cdot x^3(y^3 X^3 \cdot n)_{(1)} y^2 X^2 \cdot m_{(1)} \otimes Y^2 x^2(y^3 X^3 \cdot n)_{(1)} y^1 X^1 \cdot m_{(0)} \otimes Y^1 x^1 \cdot (y^3 X^3 \cdot n)_{(0)}
\]

\[
= \sum n_{(1)} \cdot m_{(1)} \cdot l \otimes n_{(1)} \cdot m_{(0)} \otimes n_{(0)}
\]

and (52) follows. \( \square \)

We will now present a generalization of [17] Prop. 4.4.2], stating that the dual \( M^* \) of a finite dimensional right-left Yetter-Drinfeld module is a left-right Yetter-Drinfeld module and that \( R_{M^*} = R_M^* \).

First we define right-left Yetter-Drinfeld modules for quasi-bialgebras as follows:

\[ \mathcal{HYD}^H = \mathcal{H}^{\text{op, cop}} \mathcal{HYD}^{\text{op, cop}} \]

More explicitly:

Definition 3.4. Let \( H \) be a quasi-bialgebra. A \( k \)-linear space \( M \) with a right \( H \)-action \( m \otimes h \mapsto m \cdot h \), and a left \( H \)-coaction \( M \rightarrow H \otimes M, m \mapsto \sum m_{(-1)} \otimes m_{(0)} \) is called a right-left Yetter-Drinfeld module if the following relations hold, for all
\( m \in M \) and \( h \in H \):
\[
\sum m_{(-1)}x^1 \otimes x^3 (m_{(0)} \cdot x^2)_{(-1)} \otimes (m_{(0)} \cdot x^2)_{(0)}
\]
\[
\quad = \sum y^2 (m \cdot y^1)_{(-1)} x^1 \otimes y^3 (m \cdot y^1)_{(-1)} x^2 \otimes (m \cdot y^1)_{(0)} \cdot x^3
\]
\[
\sum \varepsilon(m_{(-1)}) m_{(0)} = m
\]
\[
\sum m_{(-1)} h_1 \otimes m_{(0)} \cdot h_2 = \sum h_2 (m \cdot h_1)_{(-1)} \otimes (m \cdot h_1)_{(0)}.
\]
For \( M \in H^\mathcal{YD}_H \), we consider the map
\[
R_M : M \otimes M \to M \otimes M, \quad R_M(m \otimes n) = \sum m \cdot n_{(-1)} \otimes n_{(0)}.
\]
If we consider \( M \) as an object in \( H_{\text{op},\text{cop}} \mathcal{YD}^{H_{\text{op},\text{cop}}} \), then we obtain the same map \( R_M \), so \( R_M \) is also a solution of the corresponding quasi-Yang-Baxter equation, which is obtained after replacing \( \Phi \) by \( \Phi_{\text{op},\text{cop}} = \Phi^{321} \).
Now let \( M \) be a finite dimensional right-left Yetter-Drinfeld module. Then \( M^* \) is a left \( H \)-module, with action given by \( (h \cdot m^*)(m) = m^*(m \cdot h) \), for all \( h \in H, m \in M, m^* \in M^* \). We also define a \( k \)-linear map \( M^* \to M^* \otimes H, m^* \mapsto \sum m^*_{(0)} \otimes m^*_{(1)} \), by the condition
\[
\sum m^*_{(0)}(m)m^*_{(1)} = \sum m^*(m_{(0)})m_{(-1)}
\]
for all \( m \in M \). We can prove now the following result.

**Proposition 3.5.** Let \( H \) be a quasi-bialgebra, \( M \) a finite dimensional right-left Yetter-Drinfeld module. Then
\begin{itemize}
  \item[(i)] \( M^* \in_H \mathcal{YD}^H \);
  \item[(ii)] \( R_{M^*} = R_M^{*} \).
\end{itemize}

**Proof.** (i) We prove that \( 50, 51, 52 \) are satisfied. For \( m^* \in M^* \) and \( m \in M \), we compute:
\[
\sum (x^2 \cdot m^*_{(0)}(m)(x^2 \cdot m^*_{(0)}(m))_{(1)}x^1 \otimes x^3 m^*_{(1)}
\]
\[
\quad = \sum (x^2 \cdot m^*_{(0)})(m_{(0)})(m_{(-1)}x^1 \otimes x^3 m^*_{(1)}
\]
\[
\quad = \sum m^*_{(0)}(m_{(0)} \cdot x^2)m_{(-1)}x^1 \otimes x^3 m^*_{(1)}
\]
\[
\quad = \sum m^*((m \cdot y^1)_{(0)} \cdot x^3)y^2(m \cdot y^1)_{(-1)}x^1 \otimes y^3(m \cdot y^1)_{(-1)}x^2
\]
\[
\quad = \sum (x^3 \cdot m^*)((m \cdot y^1)_{(0)}y^2(m \cdot y^1)_{(-1)}x^1 \otimes y^3(m \cdot y^1)_{(-1)}x^2
\]
\[
\quad = \sum (x^3 \cdot m^*)(m \cdot y^1)y^2(x^3 \cdot m^*)_{(1)}x^1 \otimes y^3(x^3 \cdot m^*)_{(1)}x^2
\]
\[
\quad = \sum (y^1 \cdot (x^3 \cdot m^*)_{(0)}(m)y^2(x^3 \cdot m^*)_{(1)}x^1 \otimes y^3(x^3 \cdot m^*)_{(1)}x^2
\]
so obtain \( 53 \). Now we compute:
\[
\sum \varepsilon(m^*_{(1)}) m^*_{(0)}(m) = \sum \varepsilon(m^*_{(0)}(m)m^*_{(1)})
\]
\[
\quad = \sum \varepsilon(m^*(m_{(0)})m_{(-1)}) = \sum m^*(\varepsilon(m_{(-1)})m_{(0)}) = m^*(m),
\]
using (51) at the last step. Thus (51) holds. For \( h \in H \), we compute:

\[
\sum (h_1 \cdot m_{(0)}(m)h_2m^*_{(1)}) = \sum m_{(0)}^*(m \cdot h_1)h_2m^*_{(1)}
\]

\[
\sum m^*((m \cdot h_1)h_2(m \cdot h_1)_{(-1)}) = \sum m^*(m(0) \cdot h_2)m_{(-1)}h_1
\]

\[
\sum (h_2 \cdot m^*)(m(0))m_{(-1)}h_1 = \sum (h_2 \cdot m^*)(0)(m)(h_2 \cdot m^*)_{(1)}h_1
\]

and (52) follows.

(ii) We identify \((M \otimes M)^* = M^* \otimes M^*\), and we prove that \(R_M^*\) and \(R_M^*\) coincide as maps \(M^* \otimes M^* \rightarrow M^* \otimes M^*\). For \( m, n \in M \) and \( m^*, n^* \in M^*\), we compute:

\[
R_{M^*}(m^* \otimes n^*)(m \otimes n) = \sum (n^*_{(1)} \cdot m^*)(m)n^*_{(0)}(n)
\]

\[
= \sum m^*(m \cdot n^*_{(1)})n^*_{(0)}(n)
\]

\[
= \sum m^*(m \cdot n_{(-1)})n^*(n(0))
\]

\[
= (m^* \otimes n^*)(R_M(m \otimes n))
\]

\[
= R_M^*(m^* \otimes n^*)(m \otimes n),
\]

as needed.

\[\square\]

4. The quantum commutativity of \( H_0 \)

Let \( H \) be a Hopf algebra. It is well-known that \( H \) is an algebra in the monoidal category \( H^H \mathcal{YD} \), with left action and coaction given by

\[
h \triangleright h' = \sum h_1h'S(h_2), \quad \lambda(h) = \sum h_1 \otimes h_2.
\]

Moreover, \( H \) is quantum commutative as an algebra in \( H^H \mathcal{YD} \), see for example [11].

We will now prove a similar result for quasi-Hopf algebras. Let \( H \) be a quasi-Hopf algebra. In [11], a new multiplication on \( H \) was introduced; this multiplication is given by the formula

\[
h \circ h' = \sum X^1hS(x^1x^2)S^2h'S(x^3x^2_2)
\]

for all \( h, h' \in H \). \( \beta \) is a unit for this multiplication \( \circ \). Let \( H_0 \) be the \( k \)-linear space \( H \), with multiplication \( \circ \), and left \( H \)-action given by

\[
h \triangleright h' = \sum h_1h'S(h_2).
\]

Then \( H_0 \) is a left \( H \)-module algebra. In \( H_0 \), we also define a left \( H \)-coaction, as follows

\[
\lambda_{H_0}(h) = \sum h_{(-1)} \otimes h_{(0)}
\]

\[
= \sum X^1Y^1_1h_{1}g_{1}S(q_{2}Y^2_2)Y^3 \otimes X^2Y^2_1h_{2}g_{2}S(Q_{2}Y^3_1),
\]

where \( f^{-1} = \sum q_1 \otimes q^2 \) and \( q_R = \sum q_1 \otimes q^2 \) are the elements defined by (10) and (19). Then \( H_0 \) is an algebra in \( H^H \mathcal{YD} \), see [3] for details. In Proposition 12 we will show that \( H_0 \) is quantum commutative. But first we need the following formulas, which are of independent interest. Recall that \( q_{2} = \sum q_1 \otimes q^2 \), \( q_L = \sum f_1 \otimes f^2 \), and \( f^{-1} = \sum g_1 \otimes g^2 \) are defined by (20), (22), (15) and (16).
Lemma 4.1. Let $H$ be a quasi-Hopf algebra. Then we have

\begin{align*}
(61) & \quad \sum q^1 y^1 \otimes S(q^2 y^2) y^3 = 1 \otimes \alpha, \\
(62) & \quad \Phi(\Delta \otimes id)(f^{-1}) = \sum q^1 S(X^3) f^1 \otimes g_2^1 G S(X^2) f^2 \otimes g_2^2 G S(X^1), \\
(63) & \quad \sum S(g^1) \alpha g^2 = S(\beta), \quad \sum f^1 \beta S(f^2) = S(\alpha), \\
(64) & \quad \sum S(q^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2) = (id \otimes S)(q_2),
\end{align*}

Proof. (61) and (62) are a direct consequence of \[6\] and \[10\]. (63) has been proved in \[6\] Lemma 2.6 and \[10\] Lemma 2.5. We are left to prove (64). Using (27), we obtain:

\[(id \otimes \Delta)(q) = \sum (1 \otimes S^{-1}(x^3 g^2) \otimes S^{-1}(x^2 g^1))(q \otimes 1)(\Delta \otimes id)(q) \Phi^{-1}(id \otimes \Delta)(\Delta(x^1))\]

and, using the formula (see \[5\])

\[(\Delta \otimes id)(q) \Phi^{-1} = \sum Y^1 \otimes q^1 Y^2_1 \otimes S^{-1}(Y^3) q^2 Y^2_2,\]

we obtain

\[(id \otimes \Delta)(q) = \sum Q^1 Y^1 x^1_1 \otimes S^{-1}(x^3 g^2) Q^2 q^1 Y^2_1 x^1_1 \otimes S^{-1}(Y^3 x^2 g^1) q^2 Y^2_2 x^1_2,\]

where $q_{12} = \sum q^1 \otimes q^2 = \sum Q^1 \otimes Q^2$. Now we compute

\[
\sum S(q^2 X^3) f^1 \otimes S(q^1 X^1 \beta S(q_1^2 X^2) f^2)
\]

as needed. \hfill \Box

We can prove now the main result of this Section.

Proposition 4.2. Let $H$ be a quasi-Hopf algebra. Then $H_0$ is quantum commutative as an algebra in $H^hYD$, that is, for all $h, h' \in H$:

\[h \circ h' = \sum (h(-1) \circ h') \circ h(0).\]
Proof. For all \( h, h' \in H \) we compute:

\[
\sum (h_{(-1)} \triangleright h') \circ h_{(0)} = \sum (X^1 Y^1 h_1 g_1 S(q^2 Y^2) Y^3 \triangleright h') \circ X^2 Y^2 h_2 g_2 S(X^3 q^1 Y^1)
\]

\[
\sum Z^1 X^1 Y^1 h_{(1,1)} g_1 S(q^2 Y^2) Y^1 h'
\]

\[
S(x^1 Z^2 X^2 Y_{(1,2)} h_{(1,2)} g_2 S(q^2 Y^2) Y^3)
\]

\[
\alpha x^2 Z^1 X^2 Y^2 h_2 g_2 S(x^3 Z^2 X^3 q^1 Y^1)
\]

\[
\sum Z^1 Y^1 h_{(1,1)} g_1 S(q^2 Y^2) Y^3 h'
\]

\[
S(Z^2 Y^1 h_{(1,2)} g_2 S(q^2 Y^2) Y^3)
\]

\[
\alpha Z^3 Y^2 h_2 g_2 S(q^1 Y^1)
\]

\[
\sum Z^1 [Y^1 h S(Y^2)]_{(1,1)} g_1 S(q^2) Y^3 h'
\]

\[
S(Z^2 [Y^1 h S(Y^2)]_{(1,2)} g_2 S(q^2) Y^3)
\]

\[
\alpha Z^3 [Y^1 h S(Y^2)]_{2} g_2 S(q)
\]

\[
\sum Y^1 h S(Y^2) Z^1 g_1 S(q^2) Y^3 h' S(Z^2 g_2 S(q^2) Y^3) \alpha Z^3 g^2 S(q^1)
\]

\[
\sum Y^1 h S(Y^2) g_1 S(X^3) f^1 S(q^3) Y^3 h'
\]

\[
S(g^1 G^1 S(X^3) f^2 S(q^2) Y^3) \alpha g^2 G^2 S(q^1 X^1)
\]

\[
\sum Y^1 h S(X^3 Y^2) f^1 S(q^2) Y^3 h' S(q^1 X^2) S(q^2) Y^3)
\]

\[
\sum Y^1 h S(g^2 X^3 Y^2) f^2 Y^3 h' S(q^1 X^2) f^2 Y^3)
\]

\[
\sum Y^1 h S(x^1 Y^2) \alpha x^2 Y^1 h' S(x^3 Y^2)
\]

\[
\sum Y^1 h S(x^1 Y^2) \alpha x^2 Y^1 h' S(x^3 Y^2)
\]

\[
h \circ h'.
\]

\[\square\]

5. Hopf Modules in \( \mathcal{H} \mathcal{Y} \mathcal{D} \). Integrals

Let \( H \) be a quasi-Hopf algebra. The aim of this Section is to define the space of integrals of a finite dimensional braided Hopf algebra in \( \mathcal{H} \mathcal{Y} \mathcal{D} \), and to prove, following \[24\], \[12\], that it is an object of \( \mathcal{H} \mathcal{Y} \mathcal{D} \), and that it has dimension 1. We will apply our results to the braided Hopf algebra associated to \( H \), in the case where \( H \) is a quasitriangular quasi-Hopf algebra.

Let \( A \) be an algebra in a monoidal category \( \mathcal{C} \). Recall that a right \( A \)-module \( M \) is an object \( M \in \mathcal{C} \) together with a morphism \( \omega_M : M \otimes A \rightarrow M \) in \( \mathcal{C} \) such that \( \omega_M \circ (id_M \otimes \eta) = l_M^{-1} \) and the following diagram is commutative:

\[
\begin{array}{ccc}
(M \otimes A) \otimes A & \xrightarrow{id_A \otimes id_A} & M \otimes A \\
\downarrow a_{M,A,A} & & \downarrow \omega_M \\
M \otimes (A \otimes A) & \xrightarrow{id_M \otimes m} & M \otimes A.
\end{array}
\]
Clearly $A$ itself is a right $A$-module, by right multiplication. Right comodules over a coalgebra $C$ in $\mathcal{C}$ can be defined in a similar way: we need $N \in \mathcal{C}$ together with a morphism $\rho_N : N \to N \otimes C$ in $\mathcal{C}$ such that $(id_N \otimes \varepsilon) \circ \rho_N = l_N$ and the following diagram is commutative:

\[
\begin{array}{cccc}
N & \xrightarrow{\rho_N} & N \otimes C & \xrightarrow{\rho_N \otimes id_C} & (N \otimes C) \otimes C \\
& & \downarrow & & \downarrow a_{N,C,C} \\
N \otimes C & & id_N \otimes \Delta & & N \otimes (C \otimes C).
\end{array}
\]

$C$ itself is a right $C$-comodule via the comultiplication $\Delta$.

From [3, 21, 24], we recall the following.

**Definition 5.1.** Let $B$ be a bialgebra in a braided category $\mathcal{C}$. A right $B$-Hopf module is a triple $(M, \omega_M, \rho_M)$, where $(M, \omega_M)$ is a right $B$-module and $(M, \rho_M)$ is a right $B$-comodule such that $\rho_M : M \to M \otimes B$ is right $B$-linear. The $B$-module structure $\omega_{M \otimes B} : (M \otimes B) \otimes B \to M \otimes B$ on $M \otimes B$ is given by the following composition:

\[
(M \otimes B) \otimes B \xrightarrow{id_{M \otimes B} \otimes \Delta} (M \otimes B) \otimes (B \otimes B) \xrightarrow{\alpha_{M,B,B}} M \otimes ((B \otimes B) \otimes B) \\
(M \otimes B) \otimes B \xrightarrow{id_{M \otimes B} \otimes \Delta^{-1}} M \otimes ((B \otimes B) \otimes B) \\
\vdots \\
(M \otimes B) \otimes B \xrightarrow{\omega_M \otimes m} M \otimes B
\]

(66)

$\mathcal{M}_B^H$ will denote the category of right $B$-Hopf modules and morphisms in $\mathcal{C}$ preserving the $B$-action and the corresponding $B$-coaction.

We can consider algebras, coalgebras, bialgebras and Hopf algebras in the braided category $H^H \mathcal{YD}$ over a quasi-Hopf algebra $H$. More precisely, an algebra $B$ in $H^H \mathcal{YD}$ is an object $B \in H^H \mathcal{YD}$ such that

- $B$ is a left $H$-module algebra, i.e. $B$ has a multiplication $m$ and a usual unit $1_B$ satisfying the following conditions:

\[
(ab)c = \sum (X^1 \cdot a)((X^2 \cdot b)(X^3 \cdot c)),
\]

(67)

\[
h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b), \quad h \cdot 1_B = \varepsilon(h)1_B,
\]

(68)

for all $a, b, c \in B$ and $h \in H$.

- $B$ is a quasi-comodule algebra, that is, the multiplication $m$ and the unit $\eta$ of $B$ intertwine the $H$-coaction $\lambda_B$. By [17] this means:

\[
\lambda_B(bb') = \sum X^1(x^1Y^1 \cdot b)(-1)x^2(Y^2 \cdot b')(\cdot(-1))Y^3
\]

\[
\otimes[X^2 \cdot (x^3Y^1 \cdot b)(0)]X^3x^3 \cdot (Y^2 \cdot b')(0)],
\]

(69)
for all $b, b' \in B$, and
\begin{equation}
\lambda_B(1_B) = 1_H \otimes 1_B.
\end{equation}

$M \in \mathcal{H} \mathcal{Y}D$ is a right $B$-module if there exists a morphism $\omega_M : M \otimes B \to M$ in $\mathcal{H} \mathcal{Y}D$ (we will denote $\omega_M(m \otimes b) := m \leftarrow b$) such that
\begin{equation}
m \leftarrow 1_B = m, \ (m \leftarrow b) \leftarrow b' = \sum (X^1 \cdot m) \leftarrow [(X^2 \cdot b)(X^3 \cdot b')]
\end{equation}
for all $m \in M, b, b' \in B$. The fact that $\omega_M$ is a morphism in $\mathcal{H} \mathcal{Y}D$ means (see (47))
\begin{equation}
h \cdot (m \leftarrow b) = \sum (h_1 \cdot m) \leftarrow (h_2 \cdot b), \quad \lambda_M(m \leftarrow b) = \sum X^1(x^1 Y^1 \cdot m)(-1)x^2(Y^2 \cdot b)(-1)Y^3 \otimes [X^2 \cdot (x^1 Y^1 \cdot m)(0)] \leftarrow [X^3 x^3 \cdot (Y^2 \cdot b)(0)]
\end{equation}
for all $m \in M, b \in B$.

Similarly, $B \in \mathcal{H} \mathcal{Y}D$ is a coalgebra if
- $B$ is a left $H$-module coalgebra, i.e. $B$ has a comultiplication $\Delta_B : B \to B \otimes B$ (we will denote $\Delta_B(b) = \sum b_1 \otimes b_2$) and a usual counit $\epsilon_B$ such that:
\begin{equation}
\sum X^1 \cdot b_{(-1)} \otimes X^2 \cdot b_{(0)} \otimes X^3 \cdot b_2 = \sum b_1 \otimes b_2 \cdot b_3,
\end{equation}
\begin{equation}
\Delta_B(h \cdot b) = \sum h_1 \cdot b_1 \otimes h_2 \cdot b_2, \quad \epsilon_B(h \cdot b) = \epsilon(h)\epsilon_B(b),
\end{equation}
for all $h \in H, b \in B$, where we use the same notation for the quasi-coassociativity of $\Delta_B$ as in Section 2.
- $B$ is a quasi-comodule coalgebra, i.e. the comultiplication $\Delta_B$ and the counit $\epsilon_B$ intertwine the $H$-coaction $\lambda_B$. Explicitly, for all $b \in B$ we must have that:
\begin{equation}
\sum b_{(-1)} \otimes b_{(0)} \cdot b_{(0)} \otimes b_{(2)} \cdot b_2 = \sum X^1(x^1 Y^1 \cdot b_2)(-1)x^2(Y^2 \cdot b_2)(-1)Y^3 \otimes X^2 \cdot (x^1 Y^1 \cdot b_1)(0) \otimes X^3 x^3 \cdot (Y^2 \cdot b_2)(0),
\end{equation}
and
\begin{equation}
\sum \epsilon_B(b_{(0)})b_{(-1)} = \epsilon_B(b)1.
\end{equation}

A right $B$-comodule in $\mathcal{H} \mathcal{Y}D$ is an object $M \in \mathcal{H} \mathcal{Y}D$ together with a morphism $\rho_M : M \to M \otimes B$ in $\mathcal{H} \mathcal{Y}D$ (we will denote $\rho_M(m) = \sum m_{(0)} \otimes m_{(1)}$ for all $m \in M$) such that the following relations hold, for all $m \in M$:
\begin{equation}
\sum X^1 \cdot m_{(0)} \otimes X^2 \cdot m_{(0)} \otimes X^3 \cdot m_{(1)} = \sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)},
\end{equation}
\begin{equation}
\sum \epsilon(m_{(1)})m_{(0)} = m,
\end{equation}
where we will denote
\begin{equation}
(\rho_M \otimes id_B)(\rho_M(m)) = \sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \text{ etc.}
\end{equation}

The fact that $\rho_M$ is a morphism in $\mathcal{H} \mathcal{Y}D$ means that (see (47))
\begin{equation}
\rho_M(h \cdot m) = \sum h_1 \cdot m_{(0)} \otimes h_2 \cdot m_{(1)},
\end{equation}
\begin{equation}
\sum \epsilon_M(m_{(1)})m_{(0)} = m,
\end{equation}
where we will denote
\begin{equation}
(\rho_M \otimes id_B)(\rho_M(m)) = \sum m_{(0)} \otimes m_{(1)} \otimes m_{(2)} \text{ etc.}
\end{equation}
and
\[ \sum m_{(-1)} \otimes m_{(0)} \otimes m_{(0)} = \sum X^1 (x^1 Y^1 \cdot m_{(0)}) (-1) x^2 (Y^2 \cdot m_{(1)}) (-1) Y^3 \]
\[ \otimes X^2 \cdot (x^1 Y^1 \cdot m_{(0)}) \otimes X^3 x^3 \cdot (Y^2 \cdot m_{(1)}) (0), \]
for all \( h \in H \) and \( m \in M \).

Now, a bialgebra \( B \in \mathcal{H} \) is an algebra and a coalgebra in \( \mathcal{H} \) such that \( \Delta_B \) is an algebra morphism, i.e. \( \Delta_B(1_B) = 1_B \otimes 1_B \) and, by (83) and (85), for all \( b, b' \in B \) we have that:
\[ \Delta_B(bb') = \sum [y^1 X^1 \cdot b_1] [y^2 Y^1 (x^1 X^2 \cdot b_2)] (-1) x^2 X^3 \cdot b'_1 \]
\[ \otimes [y^3 Y^2 \cdot (x^1 X^2 \cdot b_2)(0)] [y^2 Y^3 x^3 X^3 \cdot b'_2]. \]

If \( B \in \mathcal{H} \) is a bialgebra then \( M \in \mathcal{H} \) is a right \( B \)-Hopf module if \( M \) is a right \( B \)-module (as above, we will denote \( \omega_M (m \otimes b) = m \leftarrow b \)) and a right \( B \)-comodule such that the right \( B \)-coaction on \( M \), \( \rho_M : M \to M \otimes B \), is right \( B \)-linear, which means that the following relation holds, for all \( m \in M \) and \( b' \in B \) (see (63)):
\[ \rho_M (m \leftarrow b) = \sum (y^1 X^1 \cdot m_{(0)}) \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m_{(1)})(-1) x^2 X^3 \cdot b_1] \]
\[ \otimes [y^3 Y^2 \cdot (x^1 X^2 \cdot m_{(0)})(0)] [y^2 Y^3 x^3 X^3 \cdot b'_2]. \]

Finally, a bialgebra \( B \) in \( \mathcal{H} \) is a braided Hopf algebra if there exists a morphism \( S : B \to B \) in \( \mathcal{H} \) such that \( \sum S(b_1) b_2 = \sum b_1 S(b_2) = \varepsilon(b) 1_B \), for all \( b \in B \).

Since \( S \) is a morphism in \( \mathcal{H} \), we have that:
\[ S(h \cdot b) = h \cdot S(b) \quad \text{and} \quad \sum S(b_{(-1)}) \otimes S(b_{(0)}) = \sum b_{(-1)} \otimes S(b_{(0)}), \]
for all \( h \in H \), \( b \in B \). Also, by (83) and (85) we obtain that:
\[ S(bb') = \sum [b_{(-1)} \cdot S(b') S(b_{(0)})] \quad \text{and} \quad \Delta(S(b)) = \sum b_{(-1)} \cdot S(b_2) \otimes S(b_{(1)}), \]
for all \( b, b' \in B \).

The first step to prove the existence and uniqueness of integrals in a finite dimensional braided Hopf algebra is the structure theorem for Hopf modules. To this end we need first the following result.

**Lemma 5.2.** Let \( H \) be a quasi-bialgebra, \( B \) a bialgebra in \( \mathcal{H} \) and \( N \in \mathcal{H} \).

Then \( N \otimes B \in \mathcal{H} \) with following action \( \omega_{N \otimes B} : (N \otimes B) \otimes B \to N \otimes B \) and coaction \( \rho_{N \otimes B} : N \otimes B \to (N \otimes B) \otimes B \) given by
\[ (n \otimes b) \leftarrow b' = \sum X^1 \cdot n \otimes [(X^2 \cdot b)(X^3 \cdot b')], \]
\[ \rho_{N \otimes B}(n \otimes b) := \sum x^1 \cdot n \otimes x^2 \cdot b_1 \otimes x^3 \cdot b_2. \]
for all \( n \in N \) and \( b, b' \in B \).

**Proof.** \( \mathcal{H} \) is a braided category, so \( N \otimes B \in \mathcal{H} \). It is not hard to see that (11) and (14) imply that \( \omega_{N \otimes B} \) is left \( H \)-linear. It intertwines also the corresponding \( H \)-coaction. Indeed, by (14), the left \( H \)-coaction on \( (N \otimes B) \otimes B \) is given by
\[ \lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b') = \sum Z^1 X^1 (x^1 Y^1 y_{11} T_{11} \cdot n)(-1) x^2 (Y^2 y_{21} T_{21} \cdot b)(-1) Y^3 y_{31} (T^2 \cdot b')(0) (-1) T^3 \]
\[ \otimes Z^2 X^2 \cdot (x^1 Y^1 y_{12} T_{12} \cdot n)(0) \otimes Z^3 x^3 \cdot (Y^2 y_{22} T_{22} \cdot b)(0) \otimes Z^4 y^3 \cdot (T^2 \cdot b')(0), \]
for all \( n \in N, b, b' \in B \). Therefore:

\[
(id_B \otimes \omega_{N \otimes B}) \circ \lambda_{(N \otimes B) \otimes B}((n \otimes b) \otimes b')
\]

\[
= \sum Z^1 X^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 (Y^2 y_2^1 T_2^1 \cdot b)_{(-1)} Y^3 y^2 (T^2 \cdot b')_{(-1)} T^3 \otimes W^1 Z^2 X^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
\otimes [(W^2 Z^2 X^2 x^3 \cdot (Y^2 y_2^1 T_2^1 \cdot b)_{(0)}) ([W^3 Z^2 Y^2 x^3 \cdot (T^2 \cdot b')_{(0)}])]
\]

\[
= \sum Z^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 X^1 (y^1 Y^2 T_2^1 \cdot b)_{(-1)} Y^2 (Y^2 y_2^1 T_2^1 \cdot b')_{(-1)} T^3 \otimes Z^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
\otimes Z^3 \cdot [(X^2 \cdot (y^1 Y^2 T_2^1 \cdot b)_{(0)}) (X^3 y^3 \cdot (Y^2 T_2^1 \cdot b')_{(0)})]]
\]

\[
= \sum Z^1 (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(-1)} x^2 X^1 (y^1 Y^2 T_2^1 \cdot b)_{(-1)} Y^2 (Y^2 y_2^1 T_2^1 \cdot b')_{(-1)} T^3 \otimes Z^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)} \\
\otimes Z^3 \cdot [(Y^2 T^2 \cdot b)(Y^2 T^3 \cdot b')_{(-1)} Y^2 \cdot (x^1 Y^1 y_1^1 T_1^1 \cdot n)_{(0)}] \\
\otimes Z^2 \cdot [(Y^2 T^2 \cdot b)(Y^2 T^3 \cdot b')_{(-1)} Y^2]_{(0)}
\]

\[
= \lambda_{N \otimes B} (T^1 \cdot n \otimes (T^2 \cdot b)(T^3 \cdot b'))
\]

\[
= \lambda_{N \otimes B} \circ \omega_{N \otimes B}((n \otimes b) \otimes b')
\]

for all \( n \in N \) and \( b, b' \in B \). In a similar way, it can be proved that the map \( \lambda_{N \otimes B} \) is a morphism in \( Y \mathcal{Y} D \), we leave it to the reader to verify the details.

Using (67) and (3), it easily follows that \( N \otimes B \) is a right \( B \)-module. Also, it is not hard to see that (74), (75) and (3) imply that \( N \otimes B \) is a right \( B \)-comodule. It remains only to show that \( \lambda_{N \otimes B} \) is right \( B \)-linear. By (66), we have that the right \( B \)-module structure of \((N \otimes B) \otimes B \) is given by

\[
[(n \otimes b) \otimes b'] \cdot b'' = \sum [Z^1 y_1^1 X^1_1 \cdot n \otimes (Z^2 y_2^1 X^2_1 \cdot b)(Z^3 y^2 Y^1 (x^1 X^2 \cdot b')_{(-1)} x^2 X^2_1 \cdot b'')]_{(0)} \\
\otimes [y_2^1 Y^2 \cdot (x^1 X^2 \cdot b')_{(0)}] [y_2^3 Y^2 x^3 X^3_2 \cdot b''],
\]

for all \( n \in N \) and \( b, b', b'' \in B \). This allows us to compute, for any \( n \in N \) and \( b, b' \in B \), that:

\[
\lambda_{N \otimes B}(n \otimes b) \cdot b' = \sum [(z^1 \cdot n \otimes z^2 \cdot b_1) \otimes z^3 \cdot b_2] \cdot b'
\]

\[
= \sum [Z^1 y_1^1 X^1_1 z^1 \cdot n \\
\otimes (Z^2 y_2^1 X^2_2 z^2 \cdot b_1)(Z^3 y^2 Y^1 (x^1 X^2 \cdot b_2)_{(-1)} x^2 X^2_1 \cdot b'')]_{(0)} \\
\otimes [y_2^1 Y^2 \cdot (x^1 X^2 z^3 \cdot b_2)_{(0)}] [y_2^3 Y^2 x^3 X^3_2 \cdot b'']
\]
Theorem 5.3. Let $\rho \in M$ be a quasi-Hopf algebra, $B$ a Hopf algebra in $\mathcal{H}_H\mathcal{YD}$, and $\mathcal{M} \in \mathcal{M}_B$. Then, we have

(i) $M^{coB} = \{m \in M \mid \rho_M(m) = m \otimes 1_B\} \subseteq \mathcal{H}_H\mathcal{YD}$.

(ii) For all $m \in M$, we have that $P(m) = \sum m_{(0)} \leftarrow S(m_{(1)}) \in M^{coB}$.

(iii) $\rho_M(n \mapsto b) = \sum (x^1 \cdot n) \mapsto (x^2 \cdot b) \otimes x^3 \cdot b_2$ and $P(n \mapsto b) = x(b)n$, for all $n \in M^{coB}$ and $b \in B$.

(iv) The map $F : M^{coB} \otimes B \to M$, $F(n \otimes b) = n \mapsto b$.

is an isomorphism of Hopf modules in $\mathcal{H}_H\mathcal{YD}$, with inverse $G$ given by

$$G(m) = \sum P(m_{(0)}) \otimes m_{(1)}.$$  

Proof. (i) If $n \in M^{coB}$, then $\rho_M(h \cdot n) = \sum h_1 \cdot n \otimes h_2 \cdot 1_B = h \cdot n \otimes 1_B$, by (72) and (74). This shows that $M^{coB}$ is an $H$-submodule of $M$. On the other hand, for any $n \in N$ we have

$$\sum n_{(-1)} \otimes n_{(0)} \otimes n_{(1)} = \sum X^1(x^1Y^1 \cdot n)_{(-1)}x^2(Y^2 \cdot 1_B)_{(-1)}Y^3 \otimes X^2 \cdot (x^1Y^1 \cdot n)_{(0)} \otimes X^3(x^3 \cdot (Y^2 \cdot 1_B)_{(0)}$$

by (81) twice, (70) = $\sum n_{(-1)} \otimes n_{(0)} \otimes 1_B$.

Thus, $\rho_M(n) = \sum n_{(-1)} \otimes n_{(0)} \in H \otimes M^{coB}$ which means that $M^{coB}$ is a left $H$-quasi-subcomodule of $M$. It follows from the above arguments that $M^{coB} \subseteq \mathcal{H}_H\mathcal{YD}$.
(ii) For any $m \in M$, we have that
\[
\rho_M(P(m)) = \sum \rho_M(m_{(0)} \leftarrow \mathcal{S}(m_{(1)}))
\]
\[= \sum (y^1 X^1 \cdot m_{(0)}) \leftarrow [y^2 Y^1 (x^1 X^2 \cdot m_{(0)}) (-1) x^2 X^3 \cdot \mathcal{S}(m_{(1)}))] - \sum (y^2 Y^2 \cdot (x^1 X^2 \cdot m_{(0)})) [y^3 Y^3 x^3 X^2 \cdot \mathcal{S}(m_{(1)}))]
\]
\[= \sum (y^1 \cdot m_{(0)}) \leftarrow [y^2 Y^1 (x^1 \cdot m_{(1)})] (-1) x^2 X^3 \cdot \mathcal{S}(m_{(2)})) (-1) \cdot (m_{(1)}) \mathcal{S}(m_{(2)}))]
\]
\[\otimes y^3 \cdot [(y^2 \cdot (x_1 \cdot m_{(1)})) (-1) x^2 X^3 \cdot \mathcal{S}(m_{(2)}))] (0)
\]
\[= \sum (y^1 \cdot m_{(0)}) \leftarrow (y^2 \cdot \mathcal{S}(m_{(1)})) \otimes y^3 \cdot 1_B = P(m) \otimes 1_B.
\]

(iii) For all $n \in N$ and $b \in B$, we compute, using (83),
\[
\rho_M(n \leftarrow b) = \sum (y^1 n \leftarrow [y^2 Y^1 (x^1 X^2 \cdot 1_B) (-1) x^2 X^3 \cdot b_1]) \otimes [y^2 Y^2 \cdot (x^1 X^2 \cdot 1_B)(0)] [y^3 Y^3 x^3 X^2 \cdot b_2]
\]
\[= \sum (y^1 n \leftarrow (y^2 \cdot b_1) \otimes y^3 \cdot b_2.
\]

For all $m \in M^{coB}$, we find
\[
P(n \leftarrow b) = \sum [(y^1 n \leftarrow (y^2 \cdot b_2)] \leftarrow \mathcal{S}(y^3 \cdot b_2)
\]
\[= \sum n \leftarrow b_2 \mathcal{S}(b_2) = \mathcal{S}(b)n \leftarrow 1_B = \mathcal{S}(b)n.
\]

(iv) By (i) and Lemma 3.7, we obtain that $M^{coB} \otimes B \in \mathcal{M}_B^B$. It follows from (72) that $F$ is left $H$-linear. It also intertwines the corresponding left $H$-coaction by (74) and (88). Now we will prove that $F$ and $G$ are inverses. For all $m \in M$, we have
\[
FG(m) = \sum P(m_{(0)}) \leftarrow m_{(1)}
\]
\[= \sum (X^1 \cdot m_{(0)}) \leftarrow [(X^2 \cdot \mathcal{S}(m_{(1)}))(X^3 \cdot m_{(1)}))]
\]
\[= \sum m_{(0)} \leftarrow \mathcal{S}(m_{(1)}) = m \leftarrow 1_B = m.
\]

Similarly, for any $n \in M^{coB}$ and $b \in B$, we compute
\[
GF(n \otimes b) = \sum P((n \leftarrow b_{(0)}) \otimes (n \leftarrow b_{(1)})
\]
\[= \sum P((x^1 \cdot n) \leftarrow (x^2 \cdot b_1) \otimes x^3 \cdot b_2)
\]
\[= \sum P(n) \otimes b = n \otimes b.
\]

We are left to show that $F$ is a morphism in $\mathcal{M}_B^B$. It is not hard to see that (86) and (71) imply that $F$ is right $B$-linear. Also, (iii) implies that $\rho_M \circ F(n \otimes b) = (F \otimes id_B) \circ \rho_{M^{coB} \otimes B}(n \otimes b) = \sum (x^1 \cdot n) \leftarrow (x^2 \cdot b_1) \otimes x^3 \cdot b_2$, for all $n \in N$ and $b \in B$, and this finishes the proof.  \[\square\]
Let $H$ be a quasi-Hopf algebra, and let $\mathcal{YD}^{fd}_H$ be the category of finite dimensional left Yetter-Drinfeld modules over $H$. If $M \in \mathcal{YD}^{fd}_H$, then $M^* \in \mathcal{YD}^{fd}_H$ (cf. [21]). The action and coaction are given by

\begin{equation}
(h \cdot m^*)(m) = m^*(S(h) \cdot m)
\end{equation}

\begin{equation}
\lambda_M \cdot (m^*) = \sum m^*_i \otimes m_i^0 = \sum_{i=1}^n \langle m^*, f^2 \cdot (g^1 \cdot i m)(0) \rangle
\end{equation}

\begin{equation}
S^{-1}(f^1(g^1 \cdot i m)(-1)g^2) \otimes i m
\end{equation}

for all $h \in H$, $m^* \in M^*$, $m \in M$. Here $\sum f^1 \otimes f^2$ is the twist defined in [15], $(i_{m})_{i=1}^{i=M}$ is a basis of $M$ and $(l_{m})_{i=1}^{i=M}$ its dual basis. Moreover, $\mathcal{YD}^{f.d.}_H$ is a rigid monoidal category. For each object $M \in \mathcal{YD}^{fd}_H$, the evaluation and coevaluation maps ($\text{ev}_M$ and $\text{coev}_M$, respectively) are given by [11].

In addition, if $B \in \mathcal{YD}^{fd}_H$ is a Hopf algebra, then $B^*$ is a Hopf algebra in $\mathcal{YD}^{f.d.}_H$. The structure is the following.

- The multiplication and unit are given by

\begin{equation}
(\varphi \ast \psi)(b) = \langle \varphi, f^2 \cdot 2^2 \cdot 2^3 \cdot 1 \cdot (q^1 \cdot 1 \cdot 2 \cdot (1 \cdot p^1) \cdot p^2) \cdot b \rangle
\end{equation}

\begin{equation}
\langle \psi, f^1 \cdot 2^1 \cdot (p^1 \cdot 1 \cdot 2 \cdot (0) \rangle ,
\end{equation}

\begin{equation}
1_{B^*} = \underline{1}
\end{equation}

for all $\varphi, \psi \in B^*$, $b \in B$, where $q_L = \sum q^1 \otimes q^2$ and $p_R = \sum p^1 \otimes p^2$ are the elements defined in [21] and [10].

- The comultiplication and counit are given by the formulas

\begin{equation}
\Delta_{B^*}(\varphi) = \sum_{i,j=1}^n \langle \varphi, [(g^1 \cdot j b)(-1)g^2 \cdot j b](g^1 \cdot j b)(0) \rangle b \otimes j b
\end{equation}

\begin{equation}
\varepsilon_{B^*}(\varphi) = \varphi(1_B),
\end{equation}

for any $\varphi \in B^*$, where $f^{-1} = \sum g^1 \otimes g^2$ was defined in [15], $(i_{b})_{i=1}^{i=M}$ is a basis of $B$ and $(l_{b})_{i=1}^{i=M}$ the corresponding dual basis of $B^*$.

- The antipode is given by

\begin{equation}
S_{B^*} = S^*, \text{ i.e. } S_{B^*}(\varphi) = \varphi \circ S
\end{equation}

for all $\varphi \in B^*$.

**Proposition 5.4.** Let $B \in \mathcal{YD}^{fd}_H$ a Hopf algebra. Then $B^*$ is a right $B$-Hopf module, with structure:

\begin{equation}
\langle \varphi \leftarrow b, b' \rangle = \sum \langle \varphi, [(1 \cdot b)(-1)2 \cdot b \cdot (1 \cdot b)(0) \rangle
\end{equation}

\begin{equation}
\rho_{B^*}(\varphi) = \sum_{i=1}^n \langle S(\tilde{p}^2) \cdot \tilde{b}, (\tilde{b} \otimes \tilde{b})(0) \rangle ,
\end{equation}

for all $\varphi \in B^*$, $b, b' \in B$, where

\begin{equation}
U = \sum U^1 \otimes U^2 := \sum q^1 S(q^2) \otimes S(q^1),
\end{equation}
By (21), (19) and (16), and \( \{b_i\}_{i=1,\cdots,n} \) is a basis of \( B \) with corresponding dual basis \( \{ \tilde{b}_i\}_{i=1,\cdots,n} \). Moreover,

\[
B^{\text{co}B} = \{ \Lambda \in B^* \mid \sum (p^1 \cdot \varphi) \ast (\tilde{p}^2 \cdot \Lambda) = \varphi(1_B) \Lambda \text{ for all } \varphi \in B^* \}.
\]

**Proof.** If \( B \) is a Hopf algebra in a braided rigid monoidal category \( C \), then \( B^* \) is a right Hopf \( B \)-module, as follows.

- the right \( B \)-module structure \( \triangleright: B^* \otimes B \to B^* \) is the composition

\[
\begin{array}{c}
B^* \otimes B \\
\overset{1_{B^*}}{\longrightarrow} (B^* \otimes B) \otimes 1 \\
\overset{a_{B^*,B,B}^{-1} \otimes \text{coev}_B}{\longrightarrow} (B^* \otimes B) \otimes (B \otimes B^*) \\
\overset{\text{coev}_B \otimes id_{B^*}}{\longrightarrow} (B \otimes B^*) \otimes B^* \\
\overset{\varepsilon_{B,B^*} \otimes id_{B^*}}{\longrightarrow} B^* \otimes B.
\end{array}
\]

(98)

- the right \( B \)-comodule structure \( \mu_{B^*} : B^* \to B^* \otimes B \) on \( B^* \) is the composition

\[
\begin{array}{c}
B^* \\
\overset{a_{B,B^*}^{-1}}{\longrightarrow} B \otimes (B^* \otimes B^*) \\
\overset{\text{coev}_B \otimes id_{B^*}}{\longrightarrow} (B \otimes B^*) \otimes B^* \\
\overset{id_B \otimes \varepsilon_{B,B^*}}{\longrightarrow} B \otimes B^* \\
\overset{\varepsilon_{B,B^*}}{\longrightarrow} B^* \otimes B.
\end{array}
\]

(99)

Let \( \gamma = \sum \gamma^1 \otimes \gamma^2 \) and \( f^{-1} = \sum g^1 \otimes g^2 \) be the elements defined in (14) and (16). By (18), we have, for all \( \varphi \in B^* \) and \( b, b' \in B \):

\[
\langle \varphi \triangleright b, b' \rangle = \sum \langle \varphi, S(X^1p_1_1) \alpha \cdot [(X^2p_2_1 \cdot S(b))(-1)X^3p_2_2 \cdot b'](X^2p_2_1_1 \cdot S(b)(0)) 
\]

(19)

\[
= \sum \langle \varphi, [(S(X^1p_1_1)1_1 X^2p_2_1 \cdot S(b))(-1)S(X^1p_1_2 1_2 X^3p^3 \cdot b'] 
\]

(18)

\[
= \sum \langle \varphi, [(g^1S(X_2p_2_1_2)1_2 X^2p_2_1 \cdot S(b))(-1)g^2S(X_1p_1_1_1)1_1 X^3p_2_2 \cdot b'] 
\]

(14)

\[
= \sum \langle \varphi, [(g^1S(Y^2p_2_1_2)1_2 Y^3p^3 \cdot S(b))(-1)g^2S(Y^1p_1_1_1)1_1 Y^3p^3 \cdot b'] 
\]

(14)

\[
= \sum \langle \varphi, [(g^1S(Y^2)1_2 Y^3 \cdot S(b))(-1)g^2S(Y^1_1)1_1 b'] 
\]

(14)

\[
= \sum \langle \varphi, [(g^1S(Y^2)Y^3 \cdot S(b))(-1)g^2S(Y^1)1_1 b'] 
\]

(14)

\[
= \sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b']S(U^1 \cdot b)(0) 
\]

(14)

\[
= \sum \langle \varphi, [(U^1 \cdot b)(-1)U^2 \cdot b']S((U^1 \cdot b)(0))
\]
which is just (102), (103) follows easily by (101), the details are left to the reader. Finally, by (102) we have

\[ \Lambda \in B^{\ast \text{co}(B)} \iff \rho_B \ast (\Lambda) = \Lambda \otimes 1_B \]

\[ \iff \sum_{i=1}^n S(\hat{p}^1_i \cdot_i b \otimes i b \ast (\hat{p}^2 \cdot \Lambda)) = 1_B \otimes \Lambda \]

\[ \iff \sum (\hat{p}^1 \cdot \varphi) \ast (\hat{p}^2 \cdot \Lambda) = \varphi(1_B)\Lambda, \text{ for all } \varphi \in B^\ast. \]

\[ \square \]

We define the space of left integrals by \( I_l(B^\ast) = B^{\ast \text{co}(B)} \). From the Fundamental Theorem for Hopf modules, we then obtain.

**Corollary 5.5.** Let \( H \) be a quasi-Hopf algebra and \( B \) a finite dimensional Hopf algebra in \( \mathcal{H}_B \mathcal{Y}D \). Then \( I_l(B^\ast) \otimes B \approx B^\ast \) as right \( B \)-Hopf modules. In particular, \( \dim_k(I_l(B^\ast)) = 1 \).

Now, let \( H \) be a quasi-Hopf algebra and \( H_0 \) the \( H \)-module algebra described in Section 4. If \( (H, R) \) is quasitriangular, then \( H_0 \) is a Hopf algebra in \( \mathcal{H}_B \mathcal{Y}D \), see [5]. The additional structure is the following.

\[ \lambda_{H_0}(h) = \sum R^2 \otimes R^1 \triangleright h, \]

\[ \Delta(h) = \sum h_1 \otimes h_2 \]

\[ = \sum x^1 X^1 h_1 g^3 S(x^2 R^2 y^3 X^3_2) \otimes x^3 R^1 \triangleright y^1 X^2 h_2 g_2 S(y^2 X^3_1), \]

\[ \varrho(h) = \varrho(h), \]

\[ \bar{S}(h) = \sum X^1 R^2 p^2 S(q^1 (X^2 R^1 p^1 \triangleright h) S(q^2) X^3), \]

for all \( h \in H \), where \( R = \sum R^1 \otimes R^2 \) and \( f^{-1} = \sum g^1 \otimes g^2, p_R = \sum p^1 \otimes p^2 \) and \( q_R = \sum q^1 \otimes q^2 \) are the elements defined by (101), (102), and (103). By the above arguments, if \( H \) is a finite dimensional Hopf algebra, then \( H_0^\ast \) is also a Hopf algebra in \( \mathcal{H}_B \mathcal{Y}D \), with structure

\[ \langle \varphi \ast \Psi \rangle(h) = \sum \langle \varphi, f^2 \mathcal{R}^2 \triangleright h_2 \rangle \langle \Psi, f^1 \mathcal{R}^1 \triangleright h_2 \rangle \]

\[ = \sum \langle \varphi, f^2 \triangleright Y^2 \mathcal{R}^2 X^1 x_1 h_1 g^3 S(Y^3 x^3) \rangle \]

\[ \langle \Psi, f^1 Y^1 \mathcal{R}^1 \triangleright x_1 x_2 h_2 g_2 S(X^3 x^2) \rangle, \]

\[ 1_{H_0^\ast} = \varrho, \]

\[ \Delta_{H_0^\ast}(\varphi) = \sum_{i,j=1}^n \langle \varphi, (R^2 g^2 \triangleright_i e)(R^1 g^1 \triangleright_j e) \rangle_i e \otimes_j e, \]

\[ \varrho_{H_0^\ast}(\varphi) = \varphi(\beta), \]

\[ \bar{S}_{H_0^\ast}(\varphi) = v \circ \varrho, \]

for all \( h \in H \) and \( \varphi \in H^\ast \), where \( R^{-1} = \sum \mathcal{R}^1 \otimes \mathcal{R}^2, \{i e\}_{i=1,\ldots,m} \) is a basis of \( H \) and \( \{e_i\}_{i=1,\ldots,m} \) the corresponding dual basis of \( H^\ast \). In this particular case we have

\[ I_l(H_0^\ast) = \{\Lambda \in H^\ast : \sum \Lambda(S(\hat{p}^2) f^1 \mathcal{R}^1 \triangleright h_2) S(\hat{p}^1) f^2 \mathcal{R}^2 \triangleright h_1 = \Lambda(\beta), \text{ for all } h \in H\}. \]
References

[1] D. Altschuler and A. Coste, Quasi-quantum groups, knots, three-manifolds, and topological field theory, Comm. Math. Phys. 150 (1992), 83–107.
[2] N. Andruskiewitsch and M. Graña, Braided Hopf algebras over abelian finite groups, Bol. Acad. Ciencias (Córdoba) 63 (1999), 45–78.
[3] Y. Bespalov, T. Kerler and V. Lyubashenko, Integrals for braided Hopf algebras, J. Pure Appl. Algebra 148 (2000), 113–164.
[4] D. Bulacu, S. Caenepeel and F. Panaite, Yetter-Drinfeld categories over quasi-Hopf algebras, in preparation.
[5] D. Bulacu and E. Nauwelaerts, Radford’s biproduct for quasi-Hopf algebras and bosonization, J. Pure Appl. Algebra 174 (2002), 1–42.
[6] D. Bulacu and E. Nauwelaerts, Relative Hopf modules for (dual) quasi-Hopf algebras, J. Algebra 229 (2000), 632–659.
[7] D. Bulacu and S. Caenepeel, Quasi-Hopf algebras actions and smash products, Comm. Algebra 28 (2000), 631–651.
[8] D. Bulacu and S. Caenepeel, The quantum double for quasitriangular quasi-Hopf algebras, Comm. Algebra 31 (2003), 1403–1425.
[9] D. Bulacu and S. Caenepeel, Integrals for (dual) quasi-Hopf algebras. Applications, J. Algebra 266 (2003), 552–583.
[10] D. Bulacu and E. Nauwelaerts, Quasitriangular and ribbon quasi-Hopf algebras, Comm. Algebra 31 (2003), 1–16.
[11] S. Caenepeel, F. Van Oystaeyen and Y. H. Zhang, Quantum Yang-Baxter module algebras, K-Theory 8 (1994), 231–255.
[12] Y. Doi, Hopf Modules in Yetter-Drinfeld categories, Comm. Algebra 26 (1998), 3057–3070.
[13] V. G. Drinfeld, Quasi-Hopf algebras, Leningrad Math. J. 1 (1990), 1419–1457.
[14] F. Hausser and F. Nill, Diagonal crossed products by duals of quasi-quantum groups, Rev. Math. Phys. 11 (1999), 553–629.
[15] F. Hausser and F. Nill, Doubles of quasi-quantum groups, Comm. Math. Phys. 199 (1999), 547–589.
[16] C. Kassel, “Quantum Groups”, Graduate Texts Math. 155, Springer Verlag, Berlin, 1995.
[17] L. A. Lambe and D. E. Radford, Algebraic aspects of the quantum Yang-Baxter equation, J. Algebra 154 (1992), 228–288.
[18] S. Mac Lane, Categories for the working mathematician, second edition, Graduate Texts Math. 5, Springer Verlag, Berlin, 1997.
[19] S. Majid, Quantum double for quasi-Hopf algebras, Lett. Math. Phys. 45 (1998), 1–9.
[20] S. Majid, “Foundations of quantum group theory”, Cambridge Univ. Press, Cambridge, 1995.
[21] S. Majid, Algebras and Hopf algebras in braided categories, in “Advances in Hopf Algebras”, Lect. Notes Pure Appl. Math. 158, Dekker, New York, 1994, 55–105.
[22] P. Schauenburg, Hopf modules and the double of a quasi-Hopf algebra, preprint 2002.
[23] M. E. Sweedler, “Hopf algebras”, Benjamin, New York, 1969.
[24] M. Takeuchi, Finite Hopf algebras in braided tensor categories, J. Pure Appl. Algebra 138 (1999), 59–82.