Extinction of decomposable branching processes

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Abstract

The asymptotic behavior, as \( n \to \infty \) of the conditional distribution of the number of particles in a decomposable critical branching process \( \mathbf{Z}(m) = (Z_1(m), ..., Z_N(m)) \), with \( N \) types of particles at moment \( m = n - k \), \( k = o(n) \), is investigated given that the extinction moment of the process is \( n \).

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1 Introduction

We consider a Galton-Watson branching process with \( N \) types of particles labelled 1, 2, ..., \( N \) in which a type \( i \) parent particle may produce children of types \( j \geq i \) only. Let \( \eta_{i,j} \) be the number of type \( j \) children produced by a type \( i \) parent particle. According to our assumption \( \eta_{i,j} = 0 \) if \( i > j \).

In what follows we rather often use the offspring generating function of type \( N \) particles. For this reason, to simplify notation we put

\[
h(s) = \mathbf{E} [s^{\eta_{N,N}}].
\]

Denote by \( \mathbf{e}_i \) the \( N \)-dimensional vector whose \( i \)-th component is equal to one while the remaining are equal to zero and let \( \mathbf{0} = (0, ..., 0) \) be an \( N \)-dimensional vector all whose components are equal to zero.

Let

\[
\mathbf{Z}(n) = (Z_1(n), ..., Z_N(n))
\]

be the population size at moment \( n \in \mathbb{Z}_+ = \{0, 1, ...\} \). We denote by

\[
m_{i,j}(n) = \mathbf{E} [Z_j(n)|\mathbf{Z}(0) = \mathbf{e}_i]
\]

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the expectations of the components of the vector \( Z(n) \). Let
\[
m_{i,j} = m_{i,j}(1) = E[\eta_{i,j}]
\]
be the average number of the direct descendants of type \( j \) generated by a type \( i \) particle.

We say that Hypothesis A is valid if the decomposable branching process with \( N \) types of particles is strongly critical, i.e. (see \([5]\))
\[
m_{i,i} = E[\eta_{i,i}] = 1, \quad i = 1, 2, ..., N
\]
and, in addition,
\[
m_{i,i+1} = E[\eta_{i,i+1}] \in (0, \infty), \quad i = 1, 2, ..., N - 1,
\]
\[
E[\eta_{i,j}\eta_{i,k}] < \infty, \quad i = 1, ..., N; \quad k, j = i, i + 1, ..., N,
\]
with
\[
b_i = \frac{1}{2} \text{Var} [\eta_{i,i}] \in (0, \infty), \quad i = 1, 2, ..., N.
\]
Thus, a particle of the process is able to produce the direct descendants of its own type, of the next in the order type, and (not necessarily, as direct descendants) of all the remaining in the order types, but not any preceding ones.

In the sequel we assume (if otherwise is not stated) that \( Z(0) = e_1 \), i.e. we suppose that the branching process in question is initiated at time \( n = 0 \) by a single particle of type 1.

Denote by \( T_N \) the extinction moment of the process. The aim of the present paper is to investigate the asymptotic behavior, as \( n \to \infty \) of the conditional distribution of the number of particles in the decomposable critical branching process \( Z(m) = (Z_1(m), ..., Z_N(m)) \) with \( N \) types of particles at moment \( m = n - k, k = o(n) \) given \( T_N = n \).

Decomposable branching processes of different structure have been investigated in a number of papers.

We mention in this connection paper \([2]\) dealing with the structure of the two-type decomposable critical Galton-Watson branching processes in which the total number of type 1 particles is fixed, and articles \([4]\)–\([14]\) in which, for \( N \)-type decomposable critical Markov branching processes asymptotic representations for the probability of the event \( \{T_N > n\} \) are found and Yaglom-type limit theorems are proved describing (under various restrictions) the distribution of the number of particles in these processes (and their reduced analogues) under the condition \( T_N > n \).

In \([3]\) the decomposable critical branching processes obeying the conditions of Hypothesis A were investigated under the assumption \( T_N = n \). In the mentioned paper the conditional limit distributions are found for the properly scaled components of the vector
\[
Z(m) = (Z_1(m), ..., Z_N(m))
\]
given that the parameter $m = m(n)$ varies in such a way that

$$\lim_{n \to \infty} \frac{m}{n} = x \in [0, 1).$$

The following statement established in [3] is of a particular interest:

**Theorem 1** If Hypothesis A is valid and $m = m(n) \to \infty$ as $n \to \infty$ in such a way that $m \sim xn, x \in (0, 1)$, then, for any $s_i \in [0, 1], i = 1, 2, ..., N - 1$ and $\lambda_N > 0$

$$\lim_{n \to \infty} E \left[ s_1^{Z_1(m)} \cdots s_{N-1}^{Z_{N-1}(m)} \exp \left\{ -\lambda_N \frac{Z_N(m)}{b_{NN}} \right\} \mid T_N = n \right] = \left( \frac{1 + \lambda_N (1 - x)}{1 + \lambda_N x (1 - x)} \right)^{-1+1/2^{N-1}} \frac{1}{(1 + \lambda_N x (1 - x))^2}.$$  

It follows from Theorem 1 that if $T_N = n$ and the parameter $m = m(n)$ varies within the specified range, then the population consists (in the limit) of type $N$ particles only.

In the present paper, complimenting paper [3], we concentrate on the case

$$\lim_{n \to \infty} \frac{m}{n} = 1.$$  

**Theorem 2** If Hypothesis A is valid and $k = k(n) = n - m \to \infty$ as $n \to \infty$ in such a way that $k = o(n)$, then

$$\lim_{n \to \infty} E \left[ s_1^{Z_1(m)} \cdots s_{N-1}^{Z_{N-1}(m)} \exp \left\{ -\lambda_N \frac{Z_N(m)}{b_{NN}} \right\} \mid T_N = n \right] = \frac{1}{(1 + \lambda_N)^2}.$$  

Denote by $h_n(s)$ the $n$–th iteration of the probability generating function $h(s)$. It is known (see, for instance, [1], page 93, formula (16)), that

$$\lim_{n \to \infty} b_{NN}^2 (h_n(s) - h_n(0)) = U(s)$$  

exists for any fixed $s \in [0, 1)$, where $U(s)$ is the generating function of the so-called harmonic measure. In addition,

$$U(h(s)) = U(s) + 1, \quad s \in [0, 1).$$  

The following theorem, complimenting Theorem 2, describes the final stage of the development of the process given its extinction at a distant moment $n$.

**Theorem 3** If Hypothesis A is valid and $k = n - m = \text{const}$ as $n \to \infty$, then

$$\lim_{n \to \infty} E \left[ s_1^{Z_1(m)} \cdots s_{N-1}^{Z_{N-1}(m)} s_N^{Z_N(m)} \mid T_N = n \right] = s_N \left( U(sNh_{k+1}(0)) - U(sNh_k(0)) \right).$$  

**Remark.** We know by (7) that

$$U(h_{k+1}(0)) - U(h_k(0)) = 1$$  

and, therefore, the limit distribution we have found in Theorem 3 is proper.
2 Auxiliary results

We use the symbols $P_i$ and $E_i$ to denote the probability and expectation calculated under the condition that a branching process is initiated at moment $n = 0$ by a single particle of type $i$. Sometimes we write $P$ and $E$ for $P_1$ and $E_1$, respectively.

Introduce the constants

$$c_{N,N} = 1 / b_N, \quad c_{i,N} = \left( \frac{1}{b_N} \right)^{1/2} N^{1/2} \prod_{j=i}^{N-1} \left( \frac{m_{j,i+1}}{b_j} \right)^{1/2} i < N. \quad (8)$$

and

$$D_i = (b_i m_{i,i+1})^{1/2} c_{1,i}, i = 1, 2, ..., N. \quad (9)$$

It is not difficult to check that

$$c_{1,N} = D_{N-1} \left( \frac{1}{b_N} \right)^{1/2} = D_{N-1} (c_{N,N})^{1/2}. \quad (10)$$

Denote

$$T_{ki} = \min \{ n \geq 1 : Z_k(n) + Z_{k+1}(n) + ... + Z_i(n) = 0 | Z(0) = e_k \}$$

the extinction moment of the population consisting of particles of types $k, k + 1, ..., i$, given that the process was initiated at time $n = 0$ by a single particle of type $k$. To simplify formulas we set $T_i = T_{1i}$.

We fix $N \geq 2$ and use, when it is needed, the notation

$$\gamma_0 = 0, \quad \gamma_i = \gamma_i(N) = 2^{-(N-i)}, \quad i = 1, 2, ..., N.$$

The starting point of our arguments is the following theorem proved in [4] (see also [5]):

**Theorem 4** Let $Z(n), n = 0, 1, ..., be a decomposable branching process meeting conditions (2), (3) and (4). Then, as $n \to \infty$

$$P_i(Z(n) \neq 0) \sim c_{i,N} n^{-1/2} i < N,$$

where $c_{i,N}$ are the same as in (8).

This result was complemented in [3] by the following two statements the first of which is a local limit theorem.

**Theorem 5** (see [3]) If Hypothesis A is valid, then, as $n \to \infty$

$$P_i(T_{iN} = n) \sim \frac{g_{i,N}}{n^{1+\gamma_i}}, \quad i = 1, 2, ..., N,$$

where

$$g_{i,N} = \gamma_i c_{i,N}.$$
Corollary 6 (see [3]) If \( \sqrt{n} \ll l \ll n \), then
\[
\lim_{n \to \infty} P(Z_1(l) + \cdots + Z_{N-1}(l) > 0 | T_N = n) = 0. \tag{14}
\]

Let \( \eta_{r,j}(k,l) \) be the number of type \( j \) daughter particles of the \( l \)-th particle of type \( r \) belonging to the \( k \)-th generation and let
\[
W_N = \sum_{r=1}^{N-1} \sum_{k=0}^{T_r} \sum_{q=1}^{Z_r(k)} \eta_{r,N}(k,q)
\]
be the total number of type \( N \) daughter particles generated by all the particles of types 1, 2, ..., \( N-1 \) ever born in the process given that the process is initiated at time \( n = 0 \) by a single particle of type 1.

Asymptotic properties of the tail distribution of the random variable \( W_N \) are described in the next lemma.

Lemma 7 (see [11], Lemma 1). Let Hypothesis A be valid. Then, as \( \theta \downarrow 0 \)
\[
1 - E[e^{-\theta W_N | Z(0) = e_1}] \sim D_{N-1} \theta^{\gamma_1}. \tag{15}
\]

Proving the main results of the paper we will rely on the following statement.

Lemma 8 If \( m_{N,N} = 1, b_N \in (0, \infty) \) and \( k = k(n) = n - m \to \infty \) as \( n \to \infty \) in such a way that \( k = o(n) \), and
\[
s = \exp \left\{ -\frac{\lambda}{b_N k} \right\}, \lambda > 0,
\]
then
\[
\lim_{n \to \infty} \frac{b_N \lambda n^2}{k} (h_m(s) - h_m(0)) = 1.
\]

Proof. Let \( q = q(k) \) be a positive integer such that
\[
h_q(0) \leq s \leq h_{q+1}(0).
\]

Then
\[
h_{m+q}(0) - h_m(0) \leq h_m(s) - h_m(0) \leq h_{m+q+1}(0) - h_m(0).
\]

Clearly, \( q(k) \to \infty \) as \( k \to \infty \), and in view of the representation
\[
1 - h_q(0) \sim (b_N q)^{-1}, \quad 1 - s \sim \lambda (b_N k)^{-1}
\]
and
\[
\lim_{h \to \infty} \frac{1 - h_q(0)}{1 - h_{q+1}(0)} = 1,
\]
we have \( q \sim k \lambda^{-1} \), \( k \to \infty \). This and the local limit theorem for the critical Galton-Watson processes (see, for instance, [1], Corollary 1.9.I, p. 23) imply that, as \( 1 \ll k \ll m \sim n \to \infty \)

\[
\begin{align*}
  h_m(s) - h_m(0) & \leq h_{m+q+1}(0) - h_m(0) = \sum_{j=1}^{q} (h_{m+j+1}(0) - h_{m+j}(0)) \\
  & \sim \sum_{j=1}^{q} \frac{1}{b_N (m+j)^2} \sim \frac{q}{b_N n^2} \sim \frac{k}{b_N \lambda n^2}.
  
\end{align*}
\]

Thus,

\[
\limsup_{n \to \infty} \frac{b_N \lambda n^2}{k} (h_m(s) - h_m(0)) \leq 1.
\]

Similar arguments show that

\[
\liminf_{n \to \infty} \frac{b_N \lambda n^2}{k} (h_m(s) - h_m(0)) \geq 1.
\]

The lemma is proved.

**Lemma 9** If Hypothesis A is valid, then, for any \( \lambda > 0 \)

\[
\lim_{n \to \infty} \frac{1}{n^{1-\gamma_1}} E\left[ W_N \exp\left\{ -\lambda \frac{W_N}{b_N n} \right\} \right] = \frac{\gamma_1 b_N c_{1,N}}{\lambda^{1-\gamma_1}} = \frac{b_N g_{1,N}}{\lambda^{1-\gamma_1}}.
\]

**Proof.** Let

\[
W_N(k) = W_N^{(1)} + W_N^{(2)} + ... + W_N^{(k)},
\]

where the summands are independent random variables and \( W_N^{(i)} \overset{d}{=} W_N \), \( i = 1, ..., k \). Setting \( \theta = \lambda (b_N n)^{-1} \) in Lemma 7 and recalling (10), we obtain

\[
\lim_{n \to \infty} n^{\gamma_1} \eta_{c_{1,N}\lambda^{\gamma_1}} = D_{N-1} \left( \frac{\lambda}{b_N} \right)^{\gamma_1} = c_{1,N} \lambda^{\gamma_1}.
\]

Hence it follows that, for all \( \lambda > 0 \)

\[
\lim_{n \to \infty} E\left[ \exp\left\{ -\frac{\lambda}{b_N n} W_N ([n^{\gamma_1}]) \right\} | Z(0) = [n^{\gamma_1}] e_1 \right] = \lim_{n \to \infty} E^{[n^{\gamma_1}]} \left[ \exp\left\{ -\frac{\lambda}{b_N n} W_N \right\} | Z(0) = e_1 \right] = \exp \left\{ -c_{1,N} \lambda^{\gamma_1} \right\}. (16)
\]

Since the sequence of functions under the limit consists of analytical and uniformly bounded functions in the domain \( \{ \text{Re} \lambda > 0 \} \):

\[
\left| E^{[n^{\gamma_1}]} \left[ \exp\left\{ -\frac{\lambda}{b_N n} W_N \right\} | Z(0) = e_1 \right] \right| \leq 1,
\]

it follows from the Montel theorem (see [6], Ch. VI, Section 7) that this sequence is compact. Moreover, since this sequence converges for real \( \lambda > 0 \), the Vitali
theorem (see [6], Ch. VI, Section 8) and the uniqueness theorem for analytical
functions imply convergence in (16) for all \( \lambda \) satisfying the condition \( \{\Re \lambda > 0 \} \) .
Moreover, according to the Weierstrass theorem (see [6], Ch. VI, Section 6) the
derivatives of the prelimiting functions converge to the derivative of the limiting
function in the specified domain. Whence, on account of the equality
\[
\lim_{n \to \infty} \mathbb{E} \left[ e^{-\frac{\lambda}{b_N n} W_N} \right] |Z(0) = e_1| = 1
\]
it follows that
\[
\lim_{n \to \infty} \frac{1}{b_N n^{1-\gamma_1}} \mathbb{E} \left[ W_N \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} \mathbb{E}^{\left( n^{\gamma_1} \right)} \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} |Z(0) = e_1| \right] = -\frac{\partial}{\partial \lambda} \exp \{-c_{1,N} \lambda^{\gamma_1}\}
\]
or, in view of (16) and (13)
\[
\lim_{n \to \infty} \frac{1}{n^{1-\gamma_1}} \mathbb{E} \left[ W_N \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} I_{N-1}(n^{2/3}) \right] = \gamma_1 b_N c_{1,N} \frac{\lambda^{1-\gamma_1}}{\lambda^{1-\gamma_1}} = \frac{b_N g_{1,N}}{\lambda^{1-\gamma_1}}.
\]
The lemma is proved.
Let
\[ I_k(m) = I\{Z_1(m) + \cdots + Z_k(m) = 0\} \]
be the indicator of the event that there are no particles in of types 1, 2, ..., \( k \) in
the population at time \( m \). We also agree to consider that \( I_0(m) = 1 \).

**Corollary 10** If Hypothesis \( A \) is valid, then, for any \( \lambda > 0 \)
\[
\lim_{n \to \infty} \frac{1}{n^{1-\gamma_1}} \mathbb{E} \left[ W_N \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} I_{N-1}(n^{2/3}) \right] = \frac{b_N g_{1,N}}{\lambda^{1-\gamma_1}}.
\]

**Proof.** To check the validity of the statement of the lemma it is sufficient
to note that, by virtue of (11)
\[
P \left( Z_1(n^{2/3}) + \cdots + Z_{N-1}(n^{2/3}) > 0 \right) = O \left( \left( n^{2/3} \right)^{-1} \right) = o \left( n^{-1/2} \right) = o \left( n^{-\gamma_1} \right)
\]
to make use of the equalities
\[
\mathbb{E} \left[ \left( 1 - \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} \right) I_{N-1}(n^{2/3}) \right] = \mathbb{E} \left[ 1 - \exp \left\{ -\frac{\lambda}{b_N n} W_N I_{N-1}(n^{2/3}) \right\} \right] = \mathbb{E} \left[ 1 - \exp \left\{ -\frac{\lambda}{b_N n} W_N \right\} \right] - P \left( Z_1(n^{2/3}) + \cdots + Z_{N-1}(n^{2/3}) > 0 \right),
\]
and, replacing \( W_N \) by \( W_N I_{N-1}(n^{2/3}) \), to repeat the arguments we have applied
to prove Lemma [9].

7
Lemma 11 If Hypothesis A is valid, then, for any \( \lambda > 0 \) and \( 1 \ll k \ll n \)

\[
\lim_{n \to \infty} \frac{1}{n^{1+\gamma_1} k^2} \mathbb{E} \left[ Z_N(m) \exp \left\{ -\frac{\lambda}{b_N} \frac{Z_N(m)}{b_N k} \right\} I_{N-1}(n^{2/3}) \right] = \frac{b_N g_{1,N}}{\lambda^2}.
\]

Proof. We consider the difference

\[
\Delta(m, k; \lambda) = \mathbb{E} \left[ \exp \left\{ -\frac{\lambda}{b_N} \frac{Z_N(m)}{b_N k} \right\} I_{N-1}(n^{2/3}) \right] - \mathbb{E} \left[ I \left\{ Z_N(m) = 0 \right\} I_{N-1}(n^{2/3}) \right].
\]

Clearly,

\[
\frac{\partial \Delta(m, k; \lambda)}{\partial \lambda} = -\frac{1}{b_N k} \mathbb{E} \left[ Z_N(m) \exp \left\{ -\frac{\lambda}{b_N} \frac{Z_N(m)}{b_N k} \right\} I_{N-1}(n^{2/3}) \right].
\]

Introduce, for \( m \geq T_{N-1} \) the quantity

\[
\mathcal{H}_m(s) = \prod_{r=1}^{N-1} \prod_{k=0}^{T_r} \prod_{l=1}^{Z_r(k)} (h_{m-k}(s))^{W_{r,N}(k,l)}.
\]

Then, for \( n^{2/3} < m < n \)

\[
\Delta(m, k; \lambda) = \mathbb{E} \left[ (\mathcal{H}_m(s) - \mathcal{H}_m(0)) I_{N-1}(n^{2/3}) \right],
\]

where

\[
s = \exp \left\{ -\frac{\lambda}{b_N} \right\}.
\]

Observe that, by the criticality condition

\[
h_{l+1}(s) = h(h_i(s)) \geq h_i(s)
\]

and

\[
h_{l+1}(s) - h_{l+1}(0) = h(h_i(s)) - h(h_i(0)) \leq h_i(s) - h(0)
\]

for any fixed \( s \in [0, 1] \). This yields

\[
(h_{m-T_{N-1}}(s))^{W_N} \leq \mathcal{H}_m(s) \leq (h_m(s))^{W_N}.
\]

It is not difficult to check that for any \( s \in [0, 1] \) and \( n^{2/3} < m < n \)

\[
\mathbb{E} \left[ s^{Z_N(m)} I_{N-1}(n^{2/3}) \right] = \mathbb{E} \left[ I \left\{ Z_N(m) = 0 \right\} I_{N-1}(n^{2/3}) \right].
\]

Using the inequalities

\[
\sum_{i=1}^{J} r_i \frac{(a_i - b_i)}{b_i} b^{\sum_{j=1}^{J} r_j} \leq \sum_{i=1}^{J} r_i \frac{(a_i - b_i)}{b_i} \prod_{j=1}^{J} b^{r_j}_{j} \leq \prod_{j=1}^{J} a^{r_j}_{j} - \prod_{j=1}^{J} b^{r_j}_{j} \leq \sum_{i=1}^{J} r_i \frac{(a_i - b_i)}{a_i} \prod_{j=1}^{J} a^{r_j}_{j} \leq \sum_{i=1}^{J} r_i \frac{(a_i - b_i)}{a_i} \prod_{j=1}^{J} a^{r_j}_{j} - 1,
\]

8
being valid for $0 < b_j \leq a_j \leq a \leq 1$ and nonnegative integers $r_j$, we obtain

$$W_N \left( h_{m-TN-1}(0) \right) W_N \left( h_m(s) - h_m(0) \right)$$

$$\leq \sum_{r=1}^{N-1} \sum_{k=0}^{T_r} \sum_{l=1}^{Z_r(k)} \eta_{r,N} (k, l) \frac{(h_{m-k}(s) - h_{m-k}(0))}{h_{m-k}(0)} \mathcal{H}_m (0)$$

$$\leq \mathcal{H}_m (s) - \mathcal{H}_m (0)$$

$$\leq \sum_{r=1}^{N-1} \sum_{k=0}^{T_r} \sum_{l=1}^{Z_r(k)} \eta_{r,N} (k, l) \frac{(h_{m-k}(s) - h_{m-k}(0))}{h_{m-k}(s)} \mathcal{H}_m (s)$$

$$\leq W_N \left( h_m(s) \right)^{W_N-1} \left( h_{m-TN-1}(s) - h_{m-TN-1}(0) \right).$$

Hence, on account of the condition $T_{N-1} \leq n^{2/3}$ we conclude that

$$(h_m(s) - h_m(0)) E \left[ W_N \left( h_{m-n^{2/3}}(0) \right) W_N I_{N-1}(n^{2/3}) \right]$$

$$\leq E \left[ (\mathcal{H}_m (s) - \mathcal{H}_m (0)) I_{N-1}(n^{2/3}) \right]$$

$$\leq (h_{m-n^{2/3}}(s) - h_{m-n^{2/3}}(0)) E \left[ W_N \left( h_m(s) \right)^{W_N-1} I_{N-1}(n^{2/3}) \right].$$

Observe that if the parameter $s$ has form (17), then for $1 \ll k \ll m \sim n \rightarrow \infty$

$$h_m(s) - h_m(0) \sim h_{m-n^{2/3}}(s) - h_{m-n^{2/3}}(0) \sim \frac{k}{b_N \lambda n^{2}}$$

in view of Lemma 10 and

$$1 - h_m(s) \sim 1 - h_m(0) \sim \frac{1}{b_N n}$$

by the criticality condition. This, combined with Corollary 11 yields

$$\Delta(m, k; \lambda) \sim (h_m(s) - h_m(0)) E \left[ W_N \left( h_m(s) \right)^{W_N} I_{N-1}(n^{2/3}) \right]$$

$$\sim \frac{k}{b_N \lambda n^{2}} b_N g_{1,N} n^{1-\gamma_1} = \frac{kg_{1,N}}{\lambda n^{1+\gamma_1}}.$$ 

Thus, for $1 \ll k \ll m \sim n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} \frac{n^{1+\gamma_1}}{k} \Delta(m, k; \lambda) = \frac{g_{1,N}}{\lambda}.$$ 

Let now $\lambda$ be a complex variable and

$$s = \exp \left\{ -\frac{\lambda}{b_N k} \right\}, \ Re \lambda > 0.$$ 

It is not difficult to check that for $Re \lambda \geq \lambda_0 > 0$ and $k \ll m \sim n \rightarrow \infty$ there exists a constant $C_1(\lambda_0) > 0$ such that

$$\frac{n^2}{k} \left| h_{m-n^{2/3}}(s) - h_{m-n^{2/3}}(0) \right| \leq \frac{n^2}{k} \left( h_{m-n^{2/3}}(s) - h_{m-n^{2/3}}(0) \right) \leq \frac{C_1(\lambda_0)}{\lambda_0}.$$
and, in addition, by Corollary 10 there exists a constant $C_2(\lambda_0) > 0$ such that

$$n^{-1} \left| E \left[ W_N (h_m(s))^{W_N} I_{N-1}(n^{2/3}) \right] \right| \leq n^{-1} E \left[ W_N (h_m(|s|))^{W_N} I_{N-1}(n^{2/3}) \right] \leq C_2(\lambda_0).$$

Basing on these estimates and using the inequalities

$$\left| \prod_{j=1}^{J} a_j^{r_j} - \prod_{j=1}^{J} b_j^{r_j} \right| = \left| \sum_{l=1}^{J} (a_l^{r_l} - b_l^{r_l}) \prod_{j=l+1}^{J} a_j^{r_j} \prod_{k=1}^{l-1} b_k^{r_k} \right|$$

$$\leq \sum_{l=1}^{J} r_l |a_l - b_l| |a_l|^{-1} \left| \prod_{j=l+1}^{J} a_j^{r_j} \prod_{k=1}^{l-1} b_k^{r_k} \right|$$

$$\leq \sum_{l=1}^{J} r_l |a_l - b_l| a^{-J} r^{-1},$$

being valid for $0 < |b_j| \leq |a_j| \leq a \leq 1$ and nonnegative integers $r_j$, we conclude that for $Re\lambda \geq \lambda_0 > 0$

$$\frac{n^{1+\gamma_1}}{k} |\Delta(m, k; \lambda)| \leq \frac{n^2}{k} (h_{m-n^{2/3}(|s|)} - h_{m-n^{2/3}(0)}) n^{-1} E \left[ W_N (h_m(|s|))^{W_N} I_{N-1}(n^{2/3}) \right]$$

$$\leq C_3(\lambda_0)$$

for some constant $C_3(\lambda_0) > 0$. This fact allows us, the same as in the proof of Lemma 9 to apply the Montel, Vitali and Weierstrass theorems and to conclude that for $1 \ll k \ll n$

$$\lim_{n \to \infty} \frac{n^{1+\gamma_1}}{k^2} E \left[ Z_N(m) \exp \left\{ -\lambda Z_N(m) \right\} I_{N-1}(n^{2/3}) \right]$$

$$= -\frac{\partial}{\partial \lambda} \lim_{n \to \infty} \frac{b_N n^{1+\gamma_1}}{k} \Delta(m, k; \lambda) = \frac{b_N \alpha_1 N}{\lambda^2}.$$

The lemma is proved.

### 3 Proofs of Theorems 2 and 3

**Proof of Theorem 2** By virtue of (14) it is sufficient to show that, for $n \sim m \gg k \to \infty$ and $k = n - m$

$$E \left[ \exp \left\{ -\lambda Z_N(m) b_N k \right\} I_{N-1}(n^{2/3}) | T_N = n \right] \to \frac{1}{(\lambda N + 1)^2}.$$

Put

$$s = \exp \left\{ -\lambda Z_N(m) b_N k \right\}.$$

Clearly, for $m > n^{2/3}$

$$E \left[ s Z_N(m) I_{N-1}(n^{2/3}); T_N = n \right] = E \left[ \left( (sh_{k+1}(0))^{Z_N(m)} - (sh_k(0))^{Z_N(m)} \right) I_{N-1}(n^{2/3}) \right]$$
and, therefore, as $k \to \infty$

\[
s (h_{k+1}(0) - h_k(0)) E \left[ Z_N(m) (sh_k(0))^{Z_N(m)} I_{N-1(n^2/3)} \right] \\
\leq E \left[ s^{Z_N(m)} I_{N-1(n^2/3)}; T_N = n \right] \\
\leq s (h_{k+1}(0) - h_k(0)) E \left[ Z_N(m) (sh_{k+1}(0))^{Z_N(m)-1} I_{N-1(n^2/3)} \right].
\]

By the local limit theorem for the critical Galton-Watson processes

\[
h_{k+1}(0) - h_k(0) \sim \frac{1}{b_N k^2}, \quad k \to \infty.
\]

Further,

\[
sh_{k+1}(0) = \exp \left\{ -\frac{\lambda_N}{b_N k} + \log h_{k+1}(0) \right\} = \exp \left\{ -\frac{\lambda_N + 1}{b_N k} (1 + o(1)) \right\}.
\]

Using Lemma 11 we conclude that for $n \sim m \gg k \to \infty$ and $k = n - m$

\[
E \left[ Z_N(m) \exp \left\{ -(\lambda_N + 1) (1 + o(1)) \frac{Z_N(m)}{b_N k} \right\} I_{N-1(n^2/3)} \right] \sim \frac{k^2 b_N g_{1,N}}{n^{1+\gamma_1} (\lambda_N + 1)^2}.
\]

Since, as $n \to \infty$

\[
P (T_N = n) \sim \frac{g_{1,N}}{n^{1+\gamma_1}}
\]

in view of (12), we have

\[
E \left[ s^{Z_N(m)} I_{N-1(n^2/3)}; T_N = n \right] \sim \frac{1}{b_N k^2 n^{1+\gamma_1} (\lambda_N + 1)^2} \\
= \frac{g_{1,N}}{n^{1+\gamma_1} (\lambda_N + 1)^2} \sim P (T_N = n) \frac{1}{(\lambda_N + 1)^2},
\]

as required.

**Proof of Theorem 3.** The same as in the proof of Theorem 2, it is sufficient to show that, for a fixed $k = n - m$ and $n \to \infty$

\[
E \left[ s^{Z_N(m)} I_{N-1(n^2/3)}; T_N = n \right] \to s \left( U (sh_{k+1}(0)) - U (sh_k(0)) \right).
\]

Put

\[
k^* = \left\lfloor \sqrt{n} \right\rfloor, \quad m^* = m - \left\lfloor \sqrt{n} \right\rfloor = m - k^*
\]

and write the representation

\[
E \left[ s^{Z_N(m)} I_{N-1(n^2/3)}; T_N = n \right] \\
= E \left[ ( (h_{k^*} (sh_{k+1}(0)))^{Z_N(m^*)} - (h_{k^*} (sh_k(0)))^{Z_N(m^*)} I_{N-1(n^2/3)} \right].
\]

11
Similarly to the estimates used in the proof of Theorem 2, we have

$$s(h^*_{k^*}(sh_{k^*+1}(0)) - h^*_{k^*}(sh_k(0))) E \left[ Z_N(m^*) (h^*_{k^*}(sh_{k^*+1}(0))) Z_N(m^*) I_{N-1}(n^{2/3}) \right]$$

$$\leq E \left[ s^2_Z N(m) I_{N-1}(n^{2/3}); T_N = n \right]$$

$$\leq s(h^*_{k^*}(sh_{k^*+1}(0)) - h^*_{k^*}(sh_k(0))) E \left[ Z_N(m^*) (h^*_{k^*}(sh_{k^*+1}(0))) Z_N(m^*)^{-1} I_{N-1}(n^{2/3}) \right].$$

According to (6)

$$\lim_{n \to \infty} (k^*)^2 (h^*_{k^*}(sh_{k^*+1}(0)) - h^*_{k^*}(sh_k(0))) = b_N^{-1} (U(sh_{k^*+1}(0)) - U(sh_k(0))).$$

Since

$$1 - h^*_{k^*}(sh_{k^*+1}(0)) \sim 1 - h^*_{k^*}(0) \sim \frac{1}{b_N k^*}$$

as \(n \to \infty\), and \(k^* \ll m^* \sim n\), then, according to Lemma 11

$$\lim_{n \to \infty} \frac{n^{1+\gamma_1}}{(k^*)^2} E \left[ Z_N(m^*) \exp \left\{ -\frac{Z_N(m^*)}{b_N k^*} \right\} I_{N-1}(n^{2/3}) \right] = b_N g_{1,N}$$

Thus, for any fixed \(k = n - m\) and \(n \to \infty\)

$$E \left[ s^2_Z N(m) I_{N-1}(n^{2/3}); T_N = n \right] \sim s(U(sh_{k^*+1}(0)) - U(sh_k(0))) b_N g_{1,N}(k^*)^2$$

$$\sim s(U(sh_{k^*+1}(0)) - U(sh_k(0))) P(T_N = n),$$

as required.

Theorem 3 is proved.

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