Survey for Double Barrier Backward Doubly Stochastic Differential Equations

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Abstract

In this paper, we tackle the problem of the existence and uniqueness of the solution to double barrier backward doubly stochastic differential equations, by means of the penalization method.

1 Introduction

Backward Stochastic Differential Equations (BSDEs, for short) were introduced by Peng and Pardoux [14]. The interest in BSDEs comes from their connections with different mathematical fields, such as mathematical finance, stochastic control and partial differential equations. In these fields, numerous results have been obtained, for example, [5], [8], [9], [13], [15], [17], [18], [19] and [20]. Then we are interested in further extension of BSDEs, that is, Double barrier backward doubly stochastic differential equations (DB-BDSDEs, for short).

DB-BDSDEs with the data \((f,g,\xi,L,U)\) are equations with two different directions of stochastic integrals, i.e., the equations involve both a standard (forward) stochastic integral \(dW_t\) and a “backward” stochastic integral \(dB_t\), for \(t \in [0,T]\),

\[
Y_t = \xi + \int_0^T f(s,Y_s,Z_s)ds + \int_0^T g(s,Y_s,Z_s)dW_s - t \int_0^T Z_s dW_s + (K_T^+ - K_t^+) - (K_T^- - K_t^-),
\]

(1.1)

\[L_t \leq Y_t \leq U_t,\]

\[\int_0^T (Y_t - L_t) dK^+_t = \int_0^T (U_t - Y_t)dK^-_t = 0.\]

where \(\xi\) is a random variable. The functions \(f: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) and \(g: \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}\) are two jointly measurable processes, where \(W\) and \(B\) are two mutually independent standard Brownian motion, with values in \(\mathbb{R}\).

This kind of equations is a joint version of backward doubly stochastic differential equations (BDSDEs, for short) and double barrier backward stochastic differential equations (DB-BSDEs, for short). The former has been introduced by Pardoux and Peng [16]. They have proved the connection with a class of systems of quasilinear SPDEs and an existence and uniqueness result of such PDEs. The latter has been done by Bahlali et al. [3], Cvitanic et al. [4] and Hamadene et al. [10]. A solution of DB-BDSDEs is a quadruple \((Y,Z,K^+,K^-)\) with values in \(\mathbb{R} \times \mathbb{R} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}\) which satisfies (1.1). In this paper, we have proved an existence and uniqueness of solution under appropriate conditions by using the penalization method, so-called. Here, Karouf [7] has proved that via the other method.

The paper is organized as follows. In section 2, we give some settings, or notations, definitions and assumptions. In section 3, we refer to the result of single barrier backward doubly stochastic differential equations. In section 4, we show the main result. Finally, we mention our conclusion in section 5.

2 Settings

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space, and \(T > 0\) be fixed throughout this paper. Let \(\{W_t, 0 \leq t \leq T\}\) and \(\{B_t, 0 \leq t \leq T\}\) be two mutually independent standard Brownian motion processes, with values in \(\mathbb{R}\), defined on \((\Omega, \mathcal{F}, \mathbb{P})\). Let \(\mathcal{N}\) denote the class of \(\mathbb{P}\)-null sets of \(\mathcal{F}\). For each \(t \in [0,T]\), we define \(\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^B\). For any process \(\{\eta_t; t \in [0,T]\}\) and any \(0 \leq s \leq t \leq T\), \(\mathcal{F}_{s,t} = \sigma(\eta_r; s \leq r \leq t) \vee \mathcal{N}\) and \(\mathcal{F}_t^\eta = \mathcal{F}_{0,t}^\eta\). \(\mathcal{N}\) denotes the class of \(\mathbb{P}\)-null sets of \(\mathcal{F}\).

Note that \(\{\mathcal{F}_t^W\}\) is an increasing filtration and \(\{\mathcal{F}_t^B\}\) is a decreasing filtration, and the collection \(\{\mathcal{F}_t; t \in [0,T]\}\) is neither increasing nor decreasing so it is not a filtration. Now, let us introduce some spaces:

We denote by \(L^2\) the space of \(\mathbb{R}\)-valued and \(\mathcal{F}_T\)-measurable random variable \(\xi\) such that

\[||\xi||^2 = \mathbb{E}[||\xi||^2] < +\infty,\]

by \(S^2\) the space of \(\mathbb{R}\)-valued and \(\mathcal{F}_T\)-progressively measurable random variable \(Y\) such that

\[||Y||^2 = \mathbb{E}\left[\sup_{0 \leq t \leq T} Y_t^2\right] < +\infty,\]

and by \(H^2\) the space of \(\mathbb{R}\)-valued and \(\mathcal{F}_t\)-progressively measurable random variable \(Z\) such that

\[||Z||^2 = \mathbb{E}\left[\sup_{0 \leq t \leq T} Z_t^2\right] < +\infty,\]
measurable random variable $Z$ such that
\[ ||Z||^2 = \mathbb{E} \left[ \int_0^T |Z_t|^2 dt \right] < +\infty. \]

Next, we consider the following conditions:
\begin{enumerate}[label=(A\arabic*)]
  \item The terminal condition $\xi \in \mathbb{L}^2$.
  \item The functions $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are two jointly measurable processes, such that for any $(y, z) \in \mathbb{R} \times \mathbb{R},$
\[ f(\cdot, y, z), \ g(\cdot, y, z) \in \mathbb{L}^2. \]
Moreover, we assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0, T], (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R},$
\[ |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + |z_1 - z_2|^2), \]
\[ |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2, \]
and for any $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}, \mathbb{P}$-a.s.,
\[ |f(t, y, z)| \leq C(1 + |y| + |z|). \]
\item The two reflecting barriers $L$ and $U$ are $\mathcal{F}_t$-adapted and continuous real-valued processes which satisfy
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |L_t^+|^2 \right] + \mathbb{E} \left[ \sup_{0 \leq t \leq T} |U_t^−|^2 \right] < \infty, \]
where $L^+$ and $U^−$ are positive and negative parts of $L$ and $U$, respectively.
\item The barriers $L$ and $U$ satisfy $L_t \leq U_t,$ $L_T \leq \xi \leq U_T.$
\item The barrier $U$ is regular i.e., there exists a sequence of $\{U^n\}_{n \geq 1}$ such that
\begin{enumerate}[label=(ii)\arabic*]
  \item $\forall t \leq T$, $U^n \geq U^{n+1}$ and $\mathbb{P}$-a.s.,
\[ \lim_{n \to \infty} U^n_t = U_t, \]
  \item $\forall n \geq 1, \forall t \leq T,$
\[ U^n_t = U^n_0 + \int_0^t a^n_s ds + \int_0^t b^n_s dW_s, \]
where $U^n_0 \in \mathbb{L}^2$, $a^n$ and $b^n$ are $\mathcal{F}_t$-adapted and continuous processes such that
\[ \sup_{n \geq 1} \sup_{0 \leq t \leq T} |a^n_t| \leq C^*, \]
\[ \mathbb{E} \left[ \int_0^T |b^n_t|^2 dt \right] ^{\frac{1}{2}} < +\infty. \]
\end{enumerate}
\end{enumerate}

\section{Existence and Uniqueness of a solution to Single Barrier Backward Doubly Stochastic Differential Equations}

In this section, we present a result of the existence and uniqueness of a solution to single barrier backward doubly stochastic differential equations (SB-BDSDEs) with the data $(f, g, \xi, L)$.

Now, we set the following assumptions:
\begin{enumerate}[label=(B\arabic*)]
  \item The terminal condition $\xi \in \mathbb{L}^2$.
  \item The functions $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are two jointly measurable processes, such that for any $(y, z) \in \mathbb{R} \times \mathbb{R},$
\[ f(\cdot, y, z), \ g(\cdot, y, z) \in \mathbb{L}^2. \]
Moreover, we assume that there exist constants $C > 0$ and $0 < \alpha < 1$ such that for any $(\omega, t) \in \Omega \times [0, T], (y_1, z_1), (y_2, z_2) \in \mathbb{R} \times \mathbb{R},$
\[ |f(t, y_1, z_1) - f(t, y_2, z_2)|^2 \leq C(|y_1 - y_2|^2 + |z_1 - z_2|^2), \]
\[ |g(t, y_1, z_1) - g(t, y_2, z_2)|^2 \leq C|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2, \]
\[ |f(t, y_1, z_1)| \leq C(1 + |y| + |z|). \]
\item The reflecting barrier $L$ is $\mathcal{F}_t$-adapted and continuous real-valued processes which satisfy
\[ \mathbb{E} \left[ \sup_{0 \leq t \leq T} |L_t^+|^2 \right] < \infty, \]
where $L^+$ is positive part of $L$.
\end{enumerate}

The following result established by Aman et al. \cite{Aman2016} via the penalization method, which is approximation of the SB-BDSDE by a sequence of BDSDEs without reflection.

\section{Existence and Uniqueness of a solution to Double Barrier Backward Doubly Stochastic Differential Equations}

In this section, under the assumptions on $f, g, \xi, L$ and $U$ as above, we show the existence and uniqueness
of a solution to the DB-BDSDE (1.1) via the penalization method, which is approximation of the DB-BDSDE by a sequence of SB-BDSDEs without upper-reflection. (see Essaky, Harraj and Ouknine [6], Hamadene, Lepeltier and Matoussi [10], Hamadene and Ouknine [11] and Marzougue and Otmani [12]). Our main claim is the following.

**Theorem 4.1.** Under the assumptions (A1) - (A5), the DB-BDSDE (1.1) has a unique solution \((Y, Z, K^+, K^-)\).

Here, as we mention in the introduction, Karouf [7] has proved this result by assuming the Mokobodskii condition (MC) which is key one without (A5):

\[(MC)\] There exist two non-negative supermartingales \(h = \{h_t\}_{0 \leq t \leq T}\) and \(h' = \{h'_t\}_{0 \leq t \leq T}\) in \(\mathbb{S}^2\), such that \(L_t \leq h_t - h'_t \leq U_t\), \(0 \leq t \leq T\).

(Outline of the proof for Theorem 4.1)

We introduce the following to use the penalization method.

Let \((Y^n, Z^n, K^{n+})\) be the solution of the SB-BDSDE associated with \((f(t, y, z) - n(y - U_t)^+, g(t, y, z), \xi, L)\): \[Y^n_t = \xi + \int_t^T f(s, Y^n_s, Z^n_s)ds - n \int_t^T (Y^n_s - U_s)^+ ds + \int_t^T g(s, Y^n_s, Z^n_s)dB_s - \int_t^T Z^n_sdW_s + K^{n+}_T - K^{n+}_t,\]

\(L_t \leq Y^n_t\) and \(\int_0^T (Y^n_t - L_t)dK^{n+}_t = 0\).

**Step 1** There exists a positive constant \(C\) such that \[\mathbb{E}\left[ \sup_{0 \leq t \leq T} |Y^n_t|^2 + \int_0^T |Z^n_t|^2 ds + |K^{n+}_T|^2 \right] \leq C.\]

Moreover, there exist two \(\mathcal{F}_t\)-adapted processes \(\{Y_t\}_{t \leq T}\) and \(\{K^+_t\}_{t \leq T}\) such that \(Y^n \rightarrow Y, K^{n+} \rightarrow K^+\) and

\[\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T |Y^n_t - Y_t|^2 ds \right] = 0,\]

\[\lim_{n \to \infty} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |K^{n+}_t - K^+_t|^2 \right] = 0.\]

**Step 2** It holds that

\[\lim_{n \to \infty} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |(Y^n_t - U_t)^+|^2 \right] = 0.\]

**Step 3** There exist two \(\mathcal{F}_t\)-adapted processes \(\{Z_t\}_{t \leq T}\) and \(\{K^-_t\}_{t \leq T}\) such that

\[\lim_{n \to \infty} \mathbb{E}\left[ \int_0^T |Z^n_t - Z_t|^2 dt \right] = 0,\]

\[\lim_{n \to \infty} \mathbb{E}\left[ \sup_{0 \leq t \leq T} |Y^n_t - Y_t|^2 + \sup_{0 \leq t \leq T} |K^{n-}_t - K^-_t|^2 \right] = 0.\]

5 Conclusions

We have researched about the existence and uniqueness of the solution to BSDEs, BDSDEs and SB-BDSDEs. Taking into account the facts as above, we have the outlook with that of a solution to DB-BDSDEs via the penalization method.

6 Disclaimer

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