Modulation Spaces, Multipliers Associated with the Special Affine Fourier Transform

M. H. A. Biswas¹ · H. G. Feichtinger² · R. Ramakrishnan¹

Received: 19 February 2022 / Accepted: 6 July 2022 © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022

Abstract
We study some fundamental properties of the special affine Fourier transform (SAFT) in connection with the Fourier analysis and time-frequency analysis. We introduce the modulation space $M^r_A$ in connection with SAFT and prove that if a bounded linear operator between new modulation spaces commutes with $A$-translation, then it is a $A$-convolution operator. We also establish Hörmander multiplier theorem and Littlewood-Paley theorem associated with the SAFT.

Keywords Chirp modulation · Gelfand triple · Modulation space · Multiplier · Short time Fourier transform · Time-frequency shift

Mathematics Subject Classification Primary 42A38; Secondary 42B25 · 42B35

1 Introduction

The special affine Fourier transform (SAFT) was first studied by Abe and Sheridan in [1] in connection with optical wave functions. It is an integral transform acting on the...
optical wave function and is related to the special affine linear transform of the phase space
\[
\left( \begin{array}{c}
  t' \\
  \omega'
\end{array} \right) = \left( \begin{array}{cc}
  a & b \\
  c & d
\end{array} \right) \left( \begin{array}{c}
  t \\
  \omega
\end{array} \right) + \left( \begin{array}{c}
  p \\
  q
\end{array} \right),
\]
where all the parameters \(a, b, c, d, p, q\) are real and \(ad - bc = 1\). The integral representation of the wave function transformation connected with (1.1) defines the special affine Fourier transform given as follows. For \(f \in L^1(\mathbb{R})\),
\[
\mathcal{F}_A f(\omega) = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} f(t) e^{\pi i (at^2 + 2pt - 2tot + 2(bq - dp)\omega + d\omega^2)} dt, \quad \omega \in \mathbb{R},
\]
where \(A\) stands for the set of parameters \(\{a, b, c, d, p, q\}\), with \(b \neq 0\). When \(p = q = 0\), it is called the linear canonical transform. For \(A = \{\cos \theta, \sin \theta, -\sin \theta, \cos \theta, 0, 0\}\) it is called the fractional Fourier transform, and for \(A = \{1, b, 0, 1, 0, 0\}\), it is known as the Fresnel transform.

In [3], Bhandari and Zayed studied shift invariant spaces associated with the SAFT using chirp modulation. Recently in [17], Biswas et al defined a new translation operator \(T^A\) and studied the corresponding shift invariant spaces in the context of the SAFT. This new translation operator leads to \(A\)-convolution and is used to generalize Wendel’s theorem for multipliers associated with the SAFT. Recently in [6] Chen et al studied variants of the Hörmander multiplier theorem and of the Littlewood-Paley theorem associated with fractional Fourier transforms.

Modulation spaces were introduced by Feichtinger in [12] and published in [13]. Meanwhile they have become a universal tool for various branches of analysis, including pseudo-differential operators, harmonic analysis in general and specifically time-frequency and Gabor analysis. The well known spaces such as weighted \(L^2\) spaces, Bessel potential spaces, Shubin-Sobolev spaces are important examples of modulation spaces. (See [4, 22]). There are certain interesting embedding results among modulation spaces. We refer to ([5, 25, 26]) in this connection. Further, using convolution, it has been shown in [24] that \(M^{p_1, q_1} \ast M^{p_2, q_2} \subset M^{p_0, q_0}\), where \(1/p_1 + 1/p_2 = 1 + 1/p_0, 1/q_1 + 1/q_2 = 1/q_0\). For a study of modulation spaces in connection with time-frequency analysis, we refer to the books [8, 18] and for applications to pseudo-differential operators and partial differential equations, we refer to [2].

In this paper, we look at some fundamental results in Fourier analysis in connection with \(A\)-translation, \(A\)-convolution and the SAFT. We will show that the usual approximate identity in \(L^1(\mathbb{R})\) also act in an expected way on \(L^r(\mathbb{R})\) via \(A\)-convolution. Further, we obtain an analogue of the fundamental formula for the Fourier transform, Young’s inequality, and the Hausdorff-Young inequality for the SAFT. In the classical case, it is well known that \(\widehat{(f')}(\xi) = 2\pi i \xi \hat{f}(\xi)\). We obtain an analogous result for the SAFT using the differential operator \(B = \frac{d}{dt} + \frac{2\pi i a}{b} t\). We also show that the operator \(B\) commutes with \(A\)-translations. We further obtain the solution \(u\) of the heat equation associated with the operator \(B^*B\) in terms of \(A\)-convolution, namely \(u(x, t) = g \ast_A h_t(x)\), where \(u(x, 0) = g(x)\), as in the classical case. It is well known that a multiplicative linear functional \(h\) on \(L^1(\mathbb{R})\), \(\ast\) is of the form \(h(f) = \hat{f}(\xi_0)\), for
some \( \xi_0 \in \mathbb{R} \). On the other hand in [20], Jaming proved that if \( T : L^1(\mathbb{R}) \to C_0(\mathbb{R}) \) is a continuous linear operator which converts a convolution product into the pointwise product, then \( T \) turns out to be \( T(f)(\xi) = \mathcal{F}_E(\xi) \hat{f}(\phi(\xi)), \xi \in \mathbb{R} \), for some \( E \subset \mathbb{R} \) and a function \( \phi : \mathbb{R} \to \mathbb{R} \). We prove the analogue of both the results in this paper using SAFT.

In the next part of the paper we study some fundamental properties of the SAFT in connection with time-frequency analysis. In fact, we write the covariance property of the short time Fourier transform using \( A \)-translation and \( A \)-modulation. We also obtain an analogue of fundamental identity of time-frequency analysis using the SAFT. We prove that \( f \in M'_m \) will give \( Cs f \in M'_{v_\lambda} \), where

\[
Cs f(t) = e^{\pi ist^2} f(t)
\]

and \( v_\lambda(x, \omega) = m(x, \omega - sx) \) and if \( f \in M'_{v_\lambda} \) then \( \mathcal{F}_A(f) \in M'_{w_\ell} \), where \( v_\lambda(x, \omega) = (1 + x^2 + \omega^2)^{\ell/2} \) and \( w_\ell = (1 + (c^2 + d^2)x^2 + (a^2 + b^2)\omega^2 - 2(ac + bd)x\omega)^{\ell/2} \). As a consequence, we conclude that the modulation space \( M'_m \) is invariant under SAFT. Further, we prove that the special affine Fourier transform and the pointwise multiplication operators defined using the auxiliary functions \( \lambda_A, \rho_A \) and \( \eta_A \) are Banach Gelfand triple automorphisms on the Banach Gelfand triple \((S_0(\mathbb{R}), L^2(\mathbb{R}), S'_0(\mathbb{R}))\), as described in [7]. Here \( S_0(\mathbb{R}) \) coincides with the modulation space \( M^1(\mathbb{R}) \), and the dual space is just \( M^\infty(\mathbb{R}) \) (see [18]).

We introduce weighted modulation spaces \( M_{v_\lambda}^{s} \) in connection with the SAFT and study some fundamental properties. It is well known that the classical weighted modulation space \( M^{1}_v \) possesses certain minimality property, namely if a time-frequency shift invariant Banach space \( B \) of tempered distributions has non-zero intersection with \( M^1_v \), then \( M^1_v \) is continuously embedded in \( B \). We prove an analogous minimality property for the \( A \)-modulation space \( M^1_{A,v} \) under \( A \)-time-frequency shifts. Subsequently we study multipliers in connection with the SAFT and verify how some of the classical theorems in multiplier theory for Fourier transform can be transferred to this setting.

In [15], Feichtinger and Narimani proved that if \( T : M^{1_1,s_1} \to M^{1_2,s_2} \) is bounded, linear and satisfying \( TT_x = T_x T \) for all \( x \in \mathbb{R} \), then there exists a unique \( u \in S'_0(\mathbb{R}) \) such that \( Tf = u \ast f \), for \( f \in S_0(\mathbb{R}) \). In this paper, we prove that if \( T : M^{1_1,s_1}_A \to M^{1_2,s_2}_A \) is bounded, linear and satisfying \( TT_x^A = T_x^A T \) then there exists a unique \( u \in S'_0(\mathbb{R}) \) such that \( Tf = u \ast_A f \), for all \( f \in S_0(\mathbb{R}) \). We also establish Hörmander multiplier theorem and Littlewood-Paley theorem associated with the SAFT.

## 2 Notation and Preliminaries

Let \( C_c(\mathbb{R}) \) denote the space of all compactly supported continuous, complex valued functions on \( \mathbb{R} \), \( C_0(\mathbb{R}) \), the space of all complex valued continuous functions on \( \mathbb{R} \), \( S(\mathbb{R}) \), the Schwartz class of rapidly decreasing functions and its dual, \( S'(\mathbb{R}) \), the space of tempered distributions. For a complex valued function \( f \) defined on \( \mathbb{R} \), we write \( f^\vee(x) = f(-x) \). We also make use of the following standard notations
\[ T_s f(t) = f(t - s), \quad M_s f(t) = e^{2\pi ist} f(t), \quad D_s f(t) = \frac{1}{\sqrt{|s|}} f(t/s), \]

for the translation, modulation, dilation operators respectively. A particular role in our presentation is taken by the chirp modulation operator, given by

\[ C_a b f(t) = e^{\pi i a b t^2} f(t). \]

**Definition 2.1** ([17]). For any \( s \in \mathbb{R} \) the \( A \)-translation of a measurable function \( f \) by \( s \), denoted by \( T_s^A f \), is defined as

\[ T_s^A f(t) = e^{-\frac{\pi i a s (t-s)}{b}} f(t). \quad (2.1) \]

Further, one has

\[ C_a b T_s^A f(t) = e^{\frac{\pi i a}{2} s^2 T_s (C_a b f)(t)}. \quad (2.2) \]

**Definition 2.2** ([17]). For \( f, g \in L^1(\mathbb{R}) \) the \( A \)-convolution of \( f \) and \( g \), is given by

\[ (f \ast_A g)(x) = \frac{1}{\sqrt{|b|}} \int_\mathbb{R} f(s) T_s^A g(x) ds. \quad (2.3) \]

It is compatible with chirp modulation operator as follows:

\[ C_a b (f \ast_A g) = \frac{1}{\sqrt{|b|}} (C_a b f \ast C_a b g). \quad (2.4) \]

It is interesting to note that for \( f, g \in C_c(\mathbb{R}) \) (2.3) can be written as a vector-valued integral

\[ \left( \int_\mathbb{R} T_s^A g f(s) ds \right)(x) = \delta_x \left( \int_\mathbb{R} T_s^A g d\mu_f(s) \right), \]

where \( \delta_x \) is the Dirac measure at \( x \) (as described in [14]). This can be viewed as an integrated action of \( f \) in \( L^1(\mathbb{R}) \) on \( g \).

We use the following auxiliary functions in this paper.

- \( \rho_A(t) = e^{\frac{n}{2 b^2} (at^2 + 2 pt)} \)
- \( \eta_A(\omega) = e^{\frac{n}{2 b^2} (d\omega^2 + 2\Omega \omega)}, \quad \Omega = bq - dp \)
- \( \lambda_A(t) = e^{\frac{n}{2 b^2} a t^2} \)

Notice that \( \lambda_A, \rho_A, \eta_A \) are continuous functions of absolute value 1. Hence these auxiliary functions induce unitary pointwise multiplication operators on \( L^2(\mathbb{R}) \) and isometry on \( L^r(\mathbb{R}) \) for \( 1 \leq r \leq \infty \).

They allow to express the SAFT with the help of the classical Fourier transform as follows:

\[ \mathcal{F}_A(f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (\rho_A f)\check{\check{\gamma}}(\omega/b) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (C_a b f)\check{\check{\gamma}}(\frac{\omega - p}{b}). \quad (2.5) \]
Definition 2.3 For any \( s \in \mathbb{R} \) the \( A \)-modulation of a measurable function \( f \) is defined as:

\[
M^A_s f(t) = \rho_A(-s)M^A_s f(t) = e^{\frac{\pi i}{B}(as^2 - 2ps + 2st)} f(t). \tag{2.6}
\]

The modified version of translation, modulation and Fourier transform are compatible in the expected way, i.e. one has

\[
\mathcal{F}_A(T^A_s f)(\omega) = M^A_{-s} \mathcal{F}_A(f)(\omega). \tag{2.7}
\]

Definition 2.4 [23] Let \( \phi \in S(\mathbb{R}) \) and \( \Lambda \in S'(\mathbb{R}) \). Then \( \Lambda \ast \phi \) is defined as

\[
(\Lambda \ast \phi)(x) = \Lambda(T_x \phi''), \quad x \in \mathbb{R}.
\]

Recall that a weight function \( v(x, \omega) \) on \( \mathbb{R}^2 \) is said to be \textit{submultiplicative} if \( v(x_1 + x_2, \omega_1 + \omega_2) \leq v(x_1, \omega_1) v(x_2, \omega_2) \). A weight function \( m(x, \omega) \) is said to be \( v \) \textit{moderate} if there exists \( C > 0 \) such that \( m(x_1 + x_2, \omega_1 + \omega_2) \leq C m(x_1, \omega_1) v(x_2, \omega_2) \). A weight function \( m \) is called \textit{moderate} if it is moderate with respect to some submultiplicative weight. A weight \( m \) is said to have polynomial growth if there exists \( C > 0 \) such that \( m(x, \omega) \leq C (1 + x + \omega)^s \), for some \( s \geq 0 \). In this paper, we consider moderate weights of polynomial growth.

Let \( m(x, \omega) \) be a moderate weight of polynomial growth on \( \mathbb{R}^2 \), \( g \in S(\mathbb{R}) \) and \( 1 \leq r, s < \infty \). Then the \textit{modulation space} is defined as follows.

\[
M^r_m = \{ f \in S'(\mathbb{R}) : \| f \|_{M^r_m} < \infty \},
\]

where

\[
\| f \|_{M^r_m} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \ast M(x, \omega)g(x)|^r m(x, \omega)^r dx \right)^\frac{s}{r} d\omega \right)^\frac{1}{s}.
\]

If \( r = s \), then the modulation space \( M^r_m \) is denoted by \( M^r_m \) and if \( m(x, \omega) = 1 \), then we write \( M^r, M^r \) for \( M^r_m, M^r_m \) respectively.

The modulation space \( \hat{M}^1 \) is popularly known as \textit{Feichtinger Segal algebra}, denoted by \( S_0(\mathbb{R}) \) and its dual is denoted by \( S_0'(\mathbb{R}) \).

One can define \( \Lambda \ast \phi \) for \( \phi \in S_0(\mathbb{R}) \) and \( \Lambda \in S_0'(\mathbb{R}) \) as in Definition 2.4.

It is well known that modulation spaces are invariant under translation, modulation and dilation.

Definition 2.5 A linear functional \( h \) on a Banach algebra \( \mathcal{B} \) is said to be a \textit{multiplicative linear functional} if \( h(xy) = h(x)h(y) \) for all \( x, y \in \mathcal{B} \).

Definition 2.6 A \textit{Banach Gelfand triple} consists of a Banach space \( (\mathcal{B}, \| \cdot \|_\mathcal{B}) \) which is continuously and densely embedded into some Hilbert space \( \mathcal{H} \), which in turn is \textit{weak*}-continuously and densely embedded into the dual Banach space \( (\mathcal{B}', \| \cdot \|_{\mathcal{B}'}) \).
The well known examples are \((S_0(\mathbb{R}), L^2(\mathbb{R}), S_0'(\mathbb{R}))\), \((\mathcal{H}(\mathbb{R}), L^2(\mathbb{R}), \mathcal{H}'(\mathbb{R}))\), where the Sobolev space \(\mathcal{H}_s(\mathbb{R})\) is defined by
\[
\mathcal{H}_s(\mathbb{R}) = \{ f : (1 + | \cdot |^2)^{s/2} \hat{f} \in L^2(\mathbb{R}) \}.
\]

**Definition 2.7** [9] If \((B_1, \mathcal{H}_1, B'_1)\) and \((B_2, \mathcal{H}_2, B'_2)\) are Banach Gelfand triples then an operator \(T\) is called a [unitary] Gelfand triple isomorphism if
(i) \(T\) is an isomorphism between \(B_1\) and \(B_2\).
(ii) \(T\) is a [unitary operator resp.] isomorphism from \(\mathcal{H}_1\) to \(\mathcal{H}_2\).
(iii) \(T\) extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between \(B'_1\) and \(B'_2\).

**Lemma 2.8** [16] Let \(T : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\) be a unitary operator. Then the operator \(T\) extends to a Banach Gelfand triple isomorphism between \((S_0(\mathbb{R}), L^2(\mathbb{R}), S_0'(\mathbb{R}))\) and \((S_0(\mathbb{R}), L^2(\mathbb{R}), S_0'(\mathbb{R}))\) if and only if the restriction \(T|_{S_0(\mathbb{R})}\) defines a bounded bijective linear mapping of \(S_0(\mathbb{R})\) onto itself.

### 3 Fourier Analysis and the Special Affine Fourier Transform

We consider \(C_{\frac{-a}{b}} X_{[0,1]}\) and look at its SAFT. For \(\omega \neq p\) we have
\[
\mathcal{F}_A(C_{\frac{-a}{b}} X_{[0,1]})(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} X_{[0,1]} \left( \frac{\omega - p}{b} \right) = \frac{\eta_A(\omega)}{\sqrt{|b|}} \frac{1 - e^{-2\pi i \frac{\omega - p}{b}}}{2\pi i \frac{\omega - p}{b}}.
\]
and \(\mathcal{F}_A(C_{\frac{-a}{b}} X_{[0,1]})(p) = \frac{\eta_A(p)}{\sqrt{|b|}}\).

Similarly we have \(\mathcal{F}_A(C_{\frac{-a}{b}} X_{[-\frac{1}{2}, \frac{1}{2}]})\), where
\[
sinc(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.
\]

In particular, for the fractional Fourier transform, we take \(A_\theta = \{ \cos \theta, \sin \theta, -\sin \theta, \cos \theta, 0, 0 \}\). Then
\[
\mathcal{F}_{A_\theta}(C_{\frac{-a}{b}} X_{[-\frac{1}{2}, \frac{1}{2}]}) = \frac{1}{\sqrt{|\sin \theta|}} e^{\pi i \omega \cot \theta} \text{sinc}(\omega/\sin \theta).
\]

**Proposition 3.1** The special affine Fourier transform \(\mathcal{F}_A\) is a bijection on \(S(\mathbb{R})\).

**Proof** We have
\[
\mathcal{F}_A(f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} \mathcal{F}(\rho_A f)(\omega/b) = e^{\frac{\pi i}{2}(d\omega^2 + 2(bq - dp)\omega)} D_b \mathcal{F}(\rho_A f)(\omega)
\]
\[
= C_{\frac{q}{b}} M_{\frac{bq - dp}{b}} D_b \mathcal{F}(C_{\frac{a}{b}} M_{\frac{p}{b}} f)(\omega),
\]
where \( \mathcal{F} \) denotes the classical Fourier transform. In other words, \( \mathcal{F}_A \) is a composition of \( C_s \), modulation, dilation and the classical Fourier transform. It is easy to see that \( C_s \), modulation and dilation are bijections on \( S(\mathbb{R}) \). Hence the result follows from the fact that \( \mathcal{F} : S(\mathbb{R}) \to S(\mathbb{R}) \) is a bijection. \( \square \)

**Theorem 3.2** Let \( f, g \in S(\mathbb{R}) \). Then we have the variants of the fundamental relation of Fourier analysis for the SAFT.

1. \( \int_{\mathbb{R}} \eta_A(\omega) \mathcal{F}_A(\rho_A f)(\omega)g(\omega)d\omega = \int_{\mathbb{R}} \eta_A(\omega) f(\omega)\mathcal{F}_A(\rho_A g)(\omega)d\omega. \)
2. \( \int_{\mathbb{R}} \eta_A(\omega) \mathcal{F}_A f(\omega)\rho_A g(\omega)d\omega = \int_{\mathbb{R}} \eta_A(\omega)\rho_A f(\omega)\mathcal{F}_A g(\omega)d\omega. \)
3. If \( a = d \) and \( p = q = 0 \), then \( \int_{\mathbb{R}} \mathcal{F}_A(f)(\omega)g(\omega)d\omega = \int_{\mathbb{R}} f(\omega)\mathcal{F}_A(g)(\omega)d\omega. \)

**Proof** We have

\[
\mathcal{F}_A f(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (\rho_A f)^{\sim}(\omega/b)
\]

and

\[
\mathcal{F}_A(\rho_A f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} \hat{f}(\omega/b).
\]

(i) Consider

\[
\int_{\mathbb{R}} \eta_A(\omega) \mathcal{F}_A(\rho_A f)(\omega)g(\omega)d\omega = \int_{\mathbb{R}} \frac{1}{\sqrt{|b|}} \hat{f}(\omega/b)g(\omega)d\omega = \int_{\mathbb{R}} D_b \hat{f}(\omega)g(\omega)d\omega
\]

\[
= \int_{\mathbb{R}} (D_{1/b} f)^{\sim}(\omega)g(\omega)d\omega = \int_{\mathbb{R}} D_{1/b} f(\omega)\hat{g}(\omega)d\omega
\]

\[
= \sqrt{|b|} \int_{\mathbb{R}} f(b\omega)\hat{g}(\omega)d\omega = \int_{\mathbb{R}} \frac{1}{\sqrt{|b|}} \frac{1}{\sqrt{|b|}} f(\omega)\hat{g}(\omega/b)d\omega
\]

\[
= \int_{\mathbb{R}} \eta_A(\omega) f(\omega)\mathcal{F}_A(\rho_A g)(\omega)d\omega,
\]

using the fundamental formula for the Fourier transform and applying a change of variables.

Similarly we can prove (ii).

(iii) Consider

\[
\int_{\mathbb{R}} \mathcal{F}_A(f)(\omega)g(\omega)d\omega = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} g(\omega) \int_{\mathbb{R}} e^{\frac{\pi i}{b} (at^2 + a\omega^2 - 2\omega t)} f(t)dt d\omega
\]
\[ \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} \frac{1}{|b|} e^{\frac{\pi i}{2b} (at^2 + a\omega^2 - 2a\omega t)} g(\omega) d\omega dt = \int_{\mathbb{R}} f(t) \mathcal{F}_A(g)(t) dt, \]

using Fubini's theorem.

**Remark 3.3** We can take \( f, g \in S_0(\mathbb{R}) \) in the fundamental formula, either by observing that all the integrals are well defined (even as Riemann integrals).

**Remark 3.4** Theorem 3.2 (iii) leads to the fundamental formula for the fractional Fourier transform and the Fresnel transform. Let \( A_\theta = \{ \cos \theta, \sin \theta, -\sin \theta, \cos \theta, 0, 0 \} \) and \( A_\lambda = \{ 1, \lambda, 0, 1, 0, 0 \} \). Then the fundamental formula for the fractional Fourier transform and the Fresnel transform read

\[ \int_{\mathbb{R}} \mathcal{F}_{A_\theta}(f)(\omega) g(\omega) d\omega = \int_{\mathbb{R}} f(\omega) \mathcal{F}_{A_\theta}(g)(\omega) d\omega, \quad (3.1) \]

and

\[ \int_{\mathbb{R}} \mathcal{F}_{A_\lambda}(f)(\omega) g(\omega) d\omega = \int_{\mathbb{R}} f(\omega) \mathcal{F}_{A_\lambda}(g)(\omega) d\omega \quad (3.2) \]

respectively.

Let \( B \) be the differential operator defined by \( \frac{d^2}{dt^2} + \frac{2\pi ia}{b} t \). Then \( B^* = -B \). Further \( BB^* = B^*B = -(\frac{d^2}{dt^2} + \frac{4\pi i a}{b} t \frac{d}{dt} - \frac{4\pi^2 a^2}{b^2} t^2 + \frac{2\pi i a}{b} I) \), where \( I \) is the identity operator.

**Proposition 3.5** If \( f \in S(\mathbb{R}) \), then the following statements hold.

(i) \( \mathcal{F}_{A}(Bf)(\omega) = 2\pi i \frac{a\omega - p}{b} \mathcal{F}_{A}(f)(\omega) \).

(ii) \( B(\mathcal{F}_{A}f)(\omega) = 2\pi i \frac{a\omega + d\omega + \Omega}{b} \mathcal{F}_{A}(f)(\omega) - \frac{2\pi i}{b} \mathcal{F}_{A}(tf(t))(\omega) \), where \( \Omega = bq - dp \).

**Proof** (i) Consider

\[
\begin{align*}
\mathcal{F}_{A}(f')(\omega) &= \frac{\eta A(\omega)}{\sqrt{|b|}} \int_{\mathbb{R}} e^{\frac{\pi i}{2b} (at^2 + 2pt - 2a\omega t)} f'(t) dt \\
&= \frac{\eta A(\omega)}{\sqrt{|b|}} \int_{\mathbb{R}} e^{-2\pi i \frac{a\omega - p}{b} t} C_{\frac{2\pi}{b}}(f')(t) dt \\
&= \frac{\eta A(\omega)}{\sqrt{|b|}} \int_{\mathbb{R}} e^{-2\pi i \frac{a\omega - p}{b} t} (C_{\frac{2\pi}{b}} f)'(t) dt - \frac{2\pi i a}{b} C_{\frac{2\pi}{b}}(tf(t)) dt \\
&= \frac{\eta A(\omega)}{\sqrt{|b|}} \left( (C_{\frac{2\pi}{b}} f)'(t) \right) - \frac{2\pi i a}{b} \frac{\eta A(\omega)}{\sqrt{|b|}} \int_{\mathbb{R}} e^{-2\pi i \frac{a\omega - p}{b} t} C_{\frac{2\pi}{b}}(tf(t)) dt \\
&= \frac{\eta A(\omega)}{\sqrt{|b|}} 2\pi i \frac{a \omega - p}{b} \frac{\eta A(\omega)}{\sqrt{|b|}} \left( (C_{\frac{2\pi}{b}} f)'(t) \right) - \frac{2\pi i a}{b} \frac{\eta A(\omega)}{\sqrt{|b|}} \int_{\mathbb{R}} e^{-2\pi i \frac{a\omega - p}{b} t} C_{\frac{2\pi}{b}}(tf(t)) dt \\
&= \frac{\eta A(\omega)}{\sqrt{|b|}} 2\pi i \frac{a \omega - p}{b} \frac{\eta A(\omega)}{\sqrt{|b|}} \left( (C_{\frac{2\pi}{b}} f)'(t) \right) - \frac{2\pi i a}{b} \eta A(\omega) \int_{\mathbb{R}} e^{-2\pi i \frac{a\omega - p}{b} t} C_{\frac{2\pi}{b}}(tf(t)) dt.
\end{align*}
\]
\[
\times \int_{\mathbb{R}} e^{\pi i (at^2 + 2pt - 2\omega t)} t f(t) dt
= 2\pi i \frac{\omega - p}{b} \mathcal{F}_A(f)(\omega) - \frac{2\pi i a}{b} \mathcal{F}_A(tf(t))(\omega),
\]

using \( C_{\frac{a}{b}}(f')(t) = (C_{\frac{a}{b}} f)'(t) - \frac{2\pi i a}{b} C_{\frac{a}{b}} (tf(t))(t) \), where \( f' = \frac{df}{dt} \). Thus,

\[
\mathcal{F}_A(Bf)(\omega) = 2\pi i \frac{\omega - p}{b} \mathcal{F}_A(f)(\omega).
\]

(ii) Consider

\[
\frac{d}{d\omega}(\mathcal{F}_A f)(\omega) = \frac{1}{\sqrt{|b|}} \frac{d\eta_A}{d\omega} \int_{\mathbb{R}} e^{\pi i (at^2 + 2pt - 2\omega t)} f(t) dt
+ \eta_A(\omega) \frac{d}{d\omega} \int_{\mathbb{R}} e^{\pi i (at^2 + 2pt - 2\omega t)} f(t) dt
= \frac{2\pi i}{b} (d\omega + \Omega) \eta_A(\omega) \frac{d}{d\omega} \left( C_{\frac{a}{b}} f(\omega) - \frac{2\pi i}{b} \frac{\eta_A(\omega)}{\sqrt{|b|}} (C_{\frac{a}{b}} (tf(t)))^\wedge (\omega - p) \right)
= \frac{2\pi i}{b} (d\omega + \Omega) \mathcal{F}_A(f)(\omega) - \frac{2\pi i}{b} \mathcal{F}_A(f(\omega)) \mathcal{F}_A(tf(t))(\omega).
\]

Thus \( B(\mathcal{F}_A f)(\omega) = \frac{2\pi i}{b} (d\omega + \omega + \Omega) \mathcal{F}_A(f)(\omega) - \frac{2\pi i}{b} \mathcal{F}_A(f(\omega)) \mathcal{F}_A(tf(t))(\omega) \). \( \square \)

It is interesting to note that \( B \) commutes with \( A \)-translations. In fact,

\[
\mathcal{F}_A(BT^A_x f)(\omega) = 2\pi i \frac{\omega - p}{b} \mathcal{F}_A(T^A_x f)(\omega)
= 2\pi i \frac{\omega - p}{b} e^{\pi i (ax^2 + 2px - 2xt)} \mathcal{F}_A(f)(\omega)
= e^{\pi i (ax^2 + 2px - 2xt)} \mathcal{F}_A(Bf)(\omega)
= \mathcal{F}_A(T^A_x Bf)(\omega).
\]

Then it follows from uniqueness of the SAFT that \( BT^A_x = T^A_x B \).

Consider the heat equation associated with the operator \( B^* B \), given by,

\[
\frac{\partial u}{\partial t}(x, t) = -B^* Bu(x, t), \quad (3.3)
\]

with initial condition \( u(x, 0) = g(x), \ x \in \mathbb{R} \) and \( t > 0 \). We shall obtain the solution of (3.3).
By (3.3),
\[ \frac{\partial}{\partial t} \mathcal{F}_A u(\omega, t) = -(2\pi \frac{\omega - p}{b})^2 \mathcal{F}_A u(\omega, t), \]
using Proposition 3.5 (i). Thus
\[ \mathcal{F}_A u(\omega, t) = \mathcal{F}_A u(\omega, 0) e^{- (2\pi \frac{\omega - p}{b})^2 t} = \mathcal{F}_A (g) e^{- (2\pi \frac{\omega - p}{b})^2 t}. \]

Let \( g_t(x) = e^{-\frac{x^2}{4t}} \). Then
\[ \mathcal{F}_A (C_{-\frac{a}{b}} g_t)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} g_t(\frac{\omega - p}{b}) = \frac{\eta_A(\omega)}{\sqrt{|b|}} e^{-4\pi^2 t (\frac{\omega - p}{b})^2}. \]

Hence
\[ \mathcal{F}_A u(\omega, t) = \sqrt{\frac{|b|}{4\pi t}} \mathcal{F}_A (g) \mathcal{F}_A (C_{-\frac{a}{b}} g_t)(\omega) = \sqrt{\frac{|b|}{4\pi t}} \mathcal{F}_A (g \ast A C_{-\frac{a}{b}} g_t)(\omega). \]

It follows that
\[ u(x, t) = \sqrt{\frac{|b|}{4\pi t}} (h_t \ast A g)(x), \]
where \( h_t(x) = C_{-\frac{a}{b}} g_t(x) \) given in terms of \( A \)-convolution. On the other hand,
\[ (C_{-\frac{a}{b}} g_t \ast A g)(x) = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} T_y^A (C_{-\frac{a}{b}} g_t)(x) g(y) dy \]
\[ = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} e^{-\frac{\pi i a}{b} y(x-y)} e^{-\frac{\pi i a}{b} (x-y)^2} g_t(x-y) g(y) dy. \]

Upon simplification we get
\[ u(x, t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} e^{-\frac{\pi i a}{b} (x^2-y^2)} e^{-\frac{1}{\pi}(x-y)^2} g(y) dy. \]

Next we describe the action of the usual (bounded) approximate identities from \( L^1(\mathbb{R}) \) on \( L^r(\mathbb{R}) \) through \( A \)-convolution.

**Theorem 3.6** Given \( r \) with \( 1 \leq r < \infty \) and \( \phi \in L^1(\mathbb{R}) \) with \( \int_{\mathbb{R}} \phi(x) dx = 1 \), then for \( \phi_\epsilon(x) = \frac{1}{\epsilon} \phi(x/\epsilon) \) one has
\[ \lim_{\epsilon \to 0} \| f \ast A \phi_\epsilon - f \|_r = 0, \quad f \in L^r(\mathbb{R}). \]

In order to prove this theorem, we observe the following
Proposition 3.7 For \( f \in L^r(\mathbb{R}) \), \( 1 \leq r < \infty \) one has \( \| T_h^A f - f \|_r \to 0 \) as \( h \to 0 \).

**Proof** Consider

\[
\| T_h^A f - f \|_r = \int_{\mathbb{R}} |e^{-\frac{2\pi ia}{b} h(t-h)} f(t-h) - f(t)|^r dt
\]

\[
\leq \int_{\mathbb{R}} |e^{-\frac{2\pi ia}{b} h(t-h)} f(t-h) - e^{-\frac{2\pi ia}{b} h(t-h)} f(t)|^r dt
\]

\[
+ \int_{\mathbb{R}} |f(t)|^r |e^{-\frac{2\pi ia}{b} h(t-h)} - 1|^r dt
\]

\[
= \| T_h f - f \|_r + \int_{\mathbb{R}} |f(t)|^r |e^{-\frac{2\pi ia}{b} h(t-h)} - 1|^r dt.
\]

Since \( f \in L^r(\mathbb{R}) \), using dominated convergence theorem, we can show that the second term on the right hand side tends to 0 as \( h \to 0 \). Thus the result follows from the fact that \( \| T_h f - f \|_r \to 0 \) as \( h \to 0 \). \( \square \)

**Proof of Theorem 3.6** Consider

\[
\| f \ast_A \phi_\epsilon - f \|_r = \left( \int_{\mathbb{R}} \left| (f \ast_A \phi_\epsilon)(x) - f(x) \right|^r dx \right)^{1/r}
\]

\[
= \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} T_t^A f(x) \phi_\epsilon(t) dt - f(x) \right|^r dx \right)^{1/r}
\]

\[
= \left( \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (T_t^A f(x) - f(x)) \phi_\epsilon(t) dt \right|^r dx \right)^{1/r}
\]

\[
\leq \int_{\mathbb{R}} |\phi_\epsilon(t)| \| T_t^A f - f \|_r dt
\]

\[
= \int_{\mathbb{R}} |\phi(t)| \| T_\epsilon^A f - f \|_r dt,
\]

using Minkowski’s integral inequality. Now an application of Proposition 3.7 and Lebesgue dominated convergence theorem gives \( \lim_{\epsilon \to 0} \| f \ast_A \phi_\epsilon - f \|_r = 0 \). \( \square \)

**Theorem 3.8** (Riemann-Lebesgue lemma) If \( f \in L^1(\mathbb{R}) \) then \( \mathcal{F}_A(f) \in C_0(\mathbb{R}) \).

**Proof** We know that

\[
\mathcal{F}_A(f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (\rho_A f)^\wedge (\omega/b).
\]

Since \( f \in L^1(\mathbb{R}) \), \( \rho_A f \in L^1(\mathbb{R}) \). Applying classical Riemann-Lebesgue lemma, we obtain \( \omega \mapsto (\rho_A f)^\wedge (\omega) \) is continuous. This implies that \( \omega \mapsto (\rho_A f)^\wedge (\omega/b) \) is continuous. Further, \( \omega \mapsto \eta_A(\omega) \) is continuous, from which it follows that \( \mathcal{F}_A(f) \) is continuous. Moreover

\[
|\mathcal{F}_A(f)(\omega)| = \frac{1}{\sqrt{|b|}} |(\rho_A f)^\wedge (\omega/b)|
\]
and it follows from classical Riemann-Lebesgue lemma that \( \mathcal{F}_A(f)(\omega) \to 0 \) as \( |\omega| \to \infty \).

**Theorem 3.9** (Hausdorff-Young) Let \( 1 \leq r \leq 2 \) and \( 1/r + 1/r' = 1 \). Then \( \mathcal{F}_A : L^r(\mathbb{R}) \to L^{r'}(\mathbb{R}) \) is a bounded linear operator with \( \| \mathcal{F}_A \|_{L^r \to L^{r'}} \leq \frac{1}{\sqrt{|b|}} \| \mathcal{F} \|_{L^1 \to L^\infty}^{2/r-1} \), where \( \mathcal{F} \) is the classical Fourier transform.

**Proof** Consider

\[
\| \mathcal{F}_A \|_{L^1 \to L^\infty} = \sup_{\| f \|_1 = 1} \sup_{\omega \in \mathbb{R}} | \mathcal{F}_A(f)(\omega) | = \frac{1}{\sqrt{|b|}} \sup_{\| f \|_1 = 1} \sup_{\omega \in \mathbb{R}} | \mathcal{F}(\rho_A f)(\omega/b) | = \frac{1}{\sqrt{|b|}} \sup_{\| \rho_A f \|_1 = 1} \sup_{\omega \in \mathbb{R}} | \mathcal{F}(\rho_A f)(\omega) | = \frac{1}{\sqrt{|b|}} \| \mathcal{F} \|_{L^1 \to L^\infty},
\]

using (2.5). Further \( \| \mathcal{F}_A \|_{L^2 \to L^2} = 1 \). Now applying Riesz-Thorin convexity theorem we get \( \mathcal{F}_A : L^r \to L^{r'} \) is bounded for \( 1 \leq r \leq 2 \) and \( \| \mathcal{F}_A \|_{L^r \to L^{r'}} \leq \frac{1}{\sqrt{|b|}} \| \mathcal{F} \|_{L^1 \to L^\infty}^{2/r-1} \), where \( t \) is given by \( 1/r = 1/2 + t/2 \). In other words, \( \| \mathcal{F}_A \|_{L^r \to L^{r'}} \leq \frac{1}{\sqrt{|b|}} \| \mathcal{F} \|_{L^1 \to L^\infty} \) for \( 1 \leq r \leq 2 \). \( \square \)

**Remark 3.10** The proof will show that even the more detailed behaviour of the Fourier transform as expressed by the corresponding theorem in [19] can be derived. A proof, which in some sense is closer to the spirit of Wiener amalgam spaces, can be derived from Theorem 3.2 of [10]. It implies among others that the SAFT maps \( W(L^p, l^q)(\mathbb{R}) \) boundedly into \( W(L^{q'}, l^{p'}), \) for \( 1 \leq p, q \leq 2 \). In this sense local properties of \( f \) imply global properties of the transform and vice versa.

**Theorem 3.11** (Young) If \( f \in L^r(\mathbb{R}), g \in L^s(\mathbb{R}) \) and \( 1/r + 1/s = 1 + 1/t \) for \( 1 \leq r, s, t \leq \infty \), then

\( f \star_A g \in L^t(\mathbb{R}). \)

**Proof** From (2.4) we have \( C_{\hat{b}}(f \star_A g) = \frac{1}{\sqrt{|b|}} (C_{\hat{b}} f \star C_{\hat{b}} g) \). This leads to

\[
| f \star_A g | = \frac{1}{\sqrt{|b|}} | C_{\hat{b}} f \star C_{\hat{b}} g |.
\]

Hence,

\[
\| f \star_A g \|_t = \frac{1}{\sqrt{|b|}} \| C_{\hat{b}} f \star C_{\hat{b}} g \|_t \leq \frac{1}{\sqrt{|b|}} \| C_{\hat{b}} f \|_r \| C_{\hat{b}} g \|_s \leq \frac{1}{\sqrt{|b|}} \| f \|_r \| g \|_s,
\]

using classical Young’s inequality and the fact that \( C_{\hat{b}} \) is an isometry on \( L^r(\mathbb{R}) \). \( \square \)
Theorem 3.12  

(i) Any multiplicative linear functional $h$ on $(L^1(\mathbb{R}), \star_A)$ is of the form

$$h(f) = \overline{\eta}_A(\omega_0) \mathcal{F}_A(f)(\omega_0), \; f \in L^1(\mathbb{R}),$$

for some $\omega_0 \in \mathbb{R}$.

(ii) Let $T : L^1(\mathbb{R}) \to C_0(\mathbb{R})$ be a continuous linear operator satisfying

$$T(f \star_A g)(\omega) = \overline{\eta}_A(\omega) T(f)(\omega) T(g)(\omega)$$

for all $f, g \in L^1(\mathbb{R})$. Then there exist $E \subset \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$T(f)(\omega) = \chi_E(\omega) \mathcal{F}_A(f)(\phi(\omega)), \; f \in L^1(\mathbb{R}).$$

Proof  

(i) Consider

$$h(f)h(g) = h(f \star_A g) = h(C^{-1}_a f \star_C g) = \frac{1}{\sqrt{|b|}} h(C^{-1}_a f \star_C g),$$

using (2.4). This in turn leads to

$$h(C^{-1}_a f)h(C^{-1}_a g) = \frac{1}{\sqrt{|b|}} h(C^{-1}_a f \star_C g).$$

Let $\tilde{h}(f) = \sqrt{|b|} h(C^{-1}_a f)$. Then $\tilde{h}(f \star g) = \tilde{h}(f) \tilde{h}(g)$. In other words, $\tilde{h}$ is a multiplicative linear functional on $(L^1(\mathbb{R}), \star)$. Hence there exists $\omega_0 \in \mathbb{R}$ such that $\tilde{h}(f) = f\left(\frac{\omega_0 - p}{b}\right)$. (See Chapter VIII, Theorem 2.10 in [21]). Therefore

$$\sqrt{|b|} h(C^{-1}_a f) = f\left(\frac{\omega_0 - p}{b}\right),$$

which implies the final assertion:

$$h(f) = \frac{1}{\sqrt{|b|}} (C^{-1}_a f)\left(\frac{\omega_0 - p}{b}\right).$$

(ii) As in (i), we can show that

$$\frac{1}{\sqrt{|b|}} T(C^{-1}_a (f \star g))(\omega) = \overline{\eta}_A(\omega) T(C^{-1}_a f)(\omega) T(C^{-1}_a g)(\omega).$$

Expressed differently we have for $f, g \in L^1(\mathbb{R})$:

$$\sqrt{|b|} \overline{\eta}_A(\omega) T(C^{-1}_a (f \star g))(\omega) = \sqrt{|b|} \overline{\eta}_A(\omega) T(C^{-1}_a f)(\omega) \sqrt{|b|} \overline{\eta}_A(\omega) T(C^{-1}_a g)(\omega).$$
Let $\tilde{T}(f)(\omega) = \sqrt{|b|} \eta_A(\omega) T(C_b^{-1} f)(\omega)$, or $\tilde{T}(f \ast g) = \tilde{T}(f) \tilde{T}(g)$. It then follows from Theorem 3.1 in [20], that there exist $E \subset \mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ such that $\tilde{T}(f)(\omega) = \chi_E(\omega) f(\frac{\phi(\omega) - p}{b})$. Hence

$$T(f)(\omega) = \chi_E(\omega) \eta_A(\omega) \sqrt{|b|} (C_a b f) \hat{\phi}(\frac{\phi(\omega) - p}{b}),$$

proving our assertion. \(\square\)

Now we shall define $A$-convolution of a tempered distribution and a Schwartz class function and establish a relation with corresponding classical convolution. In order to do so, first, we extend the definition of $C_s$ to the space of tempered distributions.

**Definition 3.13** Let $\Lambda \in S'(\mathbb{R})$. Then $C_s$ is defined on $S'(\mathbb{R})$ as $C_s \Lambda(\phi) = \Lambda(C_s \phi)$, $\phi \in S(\mathbb{R})$.

Observe that $C_s \Lambda \in S'(\mathbb{R})$, whenever $\Lambda \in S'(\mathbb{R})$. In order to define $A$-convolution between a tempered distribution and a Schwartz class function, we observe the following. For $f \in L'(\mathbb{R})$ and $\phi \in S(\mathbb{R})$, consider

$$(f \ast_A \phi)(x) = \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} f(y) T_y A \phi(x) dy$$

$$= \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} f(y) e^{2\pi ia y} e^{-2\pi ia x} e^{-\frac{2\pi ia}{b} y^2} \phi(y - x) dy$$

$$= \frac{1}{\sqrt{|b|}} \int_{\mathbb{R}} f(y) e^{2\pi ia y^2} e^{-\frac{2\pi ia}{b} y^2} \phi(y - x) dy$$

$$= \frac{e^{-\frac{2\pi ia}{b} x^2}}{\sqrt{|b|}} \int_{\mathbb{R}} f(y) e^{2\pi ia y^2} T_x A \phi^\vee(y) dy$$

$$= \frac{e^{-\frac{2\pi ia}{b} x^2}}{\sqrt{|b|}} \int_{\mathbb{R}} f(y) C_{\frac{2\pi}{b}} T_x A \phi^\vee(y) dy.$$

Thus we give the following

**Definition 3.14** For $\phi \in S(\mathbb{R})$ and $\Lambda \in S'(\mathbb{R})$, $\Lambda \ast_A \phi$ is defined as

$$(\Lambda \ast_A \phi)(x) = e^{-\frac{2\pi ia}{b} x^2} \frac{1}{\sqrt{|b|}} \Lambda(C_{\frac{2\pi}{b}} T_x A \phi^\vee).$$

Now we establish a relation between $A$-convolution and the corresponding classical convolution of a Schwartz class function with a tempered distribution.

**Proposition 3.15** For $\Lambda \in S'(\mathbb{R})$ we get

$$C_{\frac{a}{b}} (\Lambda \ast_A \phi) = \frac{1}{\sqrt{|b|}} (C_{\frac{a}{b}} \Lambda C_{\frac{a}{b}} \phi), \quad \phi \in S(\mathbb{R}).$$ (3.4)
Proof Consider
\[
(\Lambda \ast_A \phi)(x) = \frac{e^{-\frac{\pi i a x^2}{|b|}} \Lambda(C_{\frac{a}{b}} T_x^A \phi^\vee)}{\sqrt{|b|}} = \frac{e^{-\frac{\pi i a x^2}{|b|}} \Lambda(C_{\frac{a}{b}} C_{\frac{a}{b}} T_x^A \phi^\vee)}{\sqrt{|b|}} = \frac{e^{-\frac{\pi i a x^2}{|b|}} \Lambda(e^{\frac{\pi i a x^2}{|b|} T_x C_{\frac{a}{b}} \phi^\vee})}{\sqrt{|b|}} = \frac{e^{-\frac{\pi i a x^2}{|b|}} \Lambda(C_{\frac{a}{b}} \Phi^{\vee})}{\sqrt{|b|}}(x),
\]
using (2.2) and \(C_{\frac{a}{b}} (\phi^\vee) = (C_{\frac{a}{b}} \phi)^\vee\). Hence the result follows. \(\square\)

Remark 3.16 Similar to the arguments above, one can extend the definition of \(C_s\) on \(S'_0(\mathbb{R})\) and \(A\)-convolution of \(\phi \in S_0(\mathbb{R})\), \(\Lambda \in S'_0(\mathbb{R})\). Hence an analogue of Proposition 3.15 can be obtained by replacing \(S(\mathbb{R})\) with \(S_0(\mathbb{R})\).

4 Time-Frequency Analysis and the SAFT

Recall that a modulation space can also be defined using short time Fourier transform. The short time Fourier transform of \(f\) with respect to a window \(g\) is defined by
\[
V_g f(x, \omega) = \int_{\mathbb{R}} f(t) g(t - x) e^{-2\pi i t \omega} dt, \quad (x, \omega) \in \mathbb{R}^2.
\]
Using this, the modulation space \(M_{r,s}^{\nu}\) is defined to be \(\{f : V_g f \in L_{r,s}^{\nu}\}\) for a moderate weight \(m\). Here the space \(L_{r,s}^{\nu}\) is defined to be \(\{f : (\int_{\mathbb{R}} (\int_{\mathbb{R}} |f(x, y)|^r m(x, y)^r dx)^\frac{s}{r} dy)^{\frac{1}{s}} < \infty\}\).

Now we intend to look at the short time Fourier transform of the chirp modulation of a function, from which we show that whenever \(f \in M_{r,s}^{\nu}\), \(C_s f \in M_{r,s}^{\nu}\), where \(\nu_s(x, \omega) = m(x, \omega - sx)\).

Proposition 4.1 (i) If \(f \in S_0(\mathbb{R})\), \(g \in S'_0(\mathbb{R})\), then
\[
V_{(C_s g)} C_s f(x, \omega) = e^{-\pi i sx^2} V_g f(x, \omega - sx).
\]
(ii) \(f \in M_{r,s}^{\nu}\) if and only if \(C_s f \in M_{r,s}^{\nu}\), where \(\nu_s(x, \omega) = m(x, \omega - sx)\).

Proof (i) Writing short time Fourier transform using the duality relation between \(S_0(\mathbb{R})\) and \(S'_0(\mathbb{R})\) we get
\[
V_{(C_s g)} C_s f(x, \omega) = \langle C_s f, M_{\omega} T_x C_s g \rangle = \langle f, C_{-s} M_{\omega} T_x C_s g \rangle = e^{-\pi i sx^2} \langle f, M_{\omega - sx} T_x g \rangle = e^{-\pi i sx^2} V_g f(x, \omega - sx).
\]
(ii) Using (i) we get
\[ \| f \|_{M^r}^r = \| V_g \, f \|_{L^r_m}^r = \int_{\mathbb{R}} \int_{\mathbb{R}} |V_g \, f(x, \omega)|^r \, m(x, \omega)^r \, dx \, d\omega \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} |V(C_s \, g) \, C_s \, f(x, \omega + sx)|^r \, m(x, \omega)^r \, dx \, d\omega \]
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} |V(C_s \, g) \, C_s \, f(x, \omega)|^r \, m(x, \omega - sx)^r \, dx \, d\omega. \]

But there exist \( c_1, c_2 > 0 \) such that
\[
c_1 \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |V(C_s \, g) \, C_s \, f(x, \omega)|^r \, \nu_s(x, \omega)^r \, dx \, d\omega \right)^{1/r} \leq \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |V_g \, C_s \, f(x, \omega)|^r \, \nu_s(x, \omega)^r \, dx \, d\omega \right)^{1/r} \leq c_2 \left( \int_{\mathbb{R}} \int_{\mathbb{R}} |V(C_s \, g) \, C_s \, f(x, \omega)|^r \, \nu_s(x, \omega)^r \, dx \, d\omega \right)^{1/r}.
\]

Hence the result follows. \( \square \)

**Remark 4.2** If \( m(x, \omega) = 1 \), then \( M^r \) is invariant under the operator \( C_s \).

**Remark 4.3** The operator \( C_s : M^r \to M^r \) is bounded uniformly for \( s \in \mathbb{R} \) and bijective. In particular, by choosing \( r = 1 \), \( C_s \) is a bounded and bijective operator on \( S_0(\mathbb{R}) \). In fact, the later case was proved by Feichtinger already in [11].

**Proposition 4.4** Let \( m \) be a \( v \)-moderate weight. Then the modulation space \( M^r_{m} \) is invariant under A-time-frequency shifts and
\[
\| M^{A}_{\omega} T^{A}_{x} \, f \|_{M^r_{m}} \leq v \left( x, \frac{\omega - ax}{b} \right) \| f \|_{M^r_{m}}.
\]

**Proof** Consider
\[
\| M^{A}_{\omega} T^{A}_{x} \, f \|_{M^r_{m}} = \| \rho_A(-\omega) \, M^{\omega}_{\omega} \, T^{A}_{x} \, M^{-ax}_{b} \, f \|_{M^r_{m}} \leq \| e^{2\pi i ax/b} \, T^{A}_{x} \, M^{\omega}_{\omega} \, M^{-ax}_{b} \, f \|_{M^r_{m}} \leq \| T^{A}_{x} \, M^{\omega-ax}_{b} \, f \|_{M^r_{m}} \leq \| T^{A}_{x} \, M^{\omega}_{\omega} \, f \|_{M^r_{m}} \leq v \left( x, \frac{\omega - ax}{b} \right) \| f \|_{M^r_{m}},
\]
using \( \| T^{A}_{x} \, M^{\omega}_{\omega} \, f \|_{M^r_{m}} \leq v(x, \omega) \| f \|_{M^r_{m}} \). (See Theorem 11.3.5 in [18]). \( \square \)

The following result states that the SAFT maps \( M^r \) into \( M^{r'} \) for \( 1 \leq r \leq 2 \).

**Theorem 4.5** We have \( \mathcal{F}_A(M^r) \subseteq M^{r'} \) for \( 1 \leq r \leq 2 \) and \( 1/r + 1/r' = 1 \).
Proof We have
\[ \mathcal{F}_A(f)(\omega) = \frac{\eta_A(\omega)}{\sqrt{|b|}} (C_{\frac{a}{b}} f)(\omega - \frac{p}{b}) = \eta_A(\omega) T_p D_b (C_{\frac{a}{b}} f)(\omega). \]
Let \( f \in M^r \). Then \( (C_{\frac{a}{b}} f) \in M^r \), which in turn implies that \( (C_{\frac{a}{b}} f)(\omega) \in M^r \). But one can write \( \eta_A(\omega) T_p D_b (C_{\frac{a}{b}} f)(\omega) = C_{\frac{d}{b} M q - dp/b} T_p D_b (C_{\frac{a}{b}} f)(\omega) \). Since the modulation space \( M^r \) is invariant under translation, modulation, dilation and chirp modulation, it follows that \( \mathcal{F}_A(f) \in M^r \).

Let \( f \in M^r \). Then \( (C_{\frac{a}{b}} f) \in M^r \), which in turn implies that \( (C_{\frac{a}{b}} f)(\omega) \in M^r \). But one can write \( \eta_A(\omega) T_p D_b (C_{\frac{a}{b}} f)(\omega) = C_{\frac{d}{b} M q - dp/b} T_p D_b (C_{\frac{a}{b}} f)(\omega) \). Since the modulation space \( M^r \) is invariant under translation, modulation, dilation and chirp modulation, it follows that \( \mathcal{F}_A(f) \in M^r \).

We write the covariance property for short time Fourier transform using A-translation \( T^{A}_{\xi} \) and A-modulation \( M^{A}_{\eta} \).

Proposition 4.6 We have
\[ V_{g}(T^{A}_{\xi} M^{A}_{\eta} f)(x, \omega) = \rho_A(-\eta) e^{-2\pi i \xi \omega} V_{g} f \left( x - \xi, \omega + \frac{a \xi - \eta}{b} \right). \]
In particular,
\[ |V_{g}(T^{A}_{\xi} M^{A}_{\eta} f)(x, \omega)| = \left| V_{g} f \left( x - \xi, \omega + \frac{a \xi - \eta}{b} \right) \right|. \quad (4.1) \]

Proof Consider
\[ V_{g}(T^{A}_{\xi} M^{A}_{\eta} f)(x, \omega) = \int_{\mathbb{R}} T^{A}_{\xi} M^{A}_{\eta} f(t) g(t-x) e^{-2\pi i t \xi} dt \]
\[ = \int_{\mathbb{R}} e^{-\frac{2\pi a}{b} \xi(t-\xi)} e^{\frac{\pi i}{b} (\alpha \eta^2 - 2p \eta + 2 \eta(t-\xi))} f(t-\xi) g(t-x) e^{-2\pi i t \xi} dt \]
\[ = e^{\frac{2\pi a}{b} \xi^2} \rho_A(-\eta) e^{-\frac{2\pi i}{\eta} \xi^2} \]
\[ \times \int_{\mathbb{R}} e^{-2\pi i \xi \omega} e^{\frac{2\pi i}{\eta} \xi t} f(t-\xi) g(t-x) e^{-2\pi i \xi \omega} dt \]
\[ = e^{\frac{2\pi a}{b} \xi^2} \rho_A(-\eta) e^{-\frac{2\pi i}{\eta} \xi^2} \int_{\mathbb{R}} f(t) g(t+\xi-x) e^{-2\pi i (t+\xi)(\omega + \frac{a \xi - \eta}{b})} dt \]
\[ = \rho_A(-\eta) e^{-2\pi i \xi \omega} \int_{\mathbb{R}} f(t) g(t+\xi-x) e^{-2\pi i (\omega + \frac{a \xi - \eta}{b})} dt \]
\[ = \rho_A(-\eta) e^{-2\pi i \xi \omega} V_{g} f(x - \xi, \omega + \frac{a \xi - \eta}{b}), \]
applying change of variables.

It is well known that
\[ V_{g} f(x, \omega) = e^{-2\pi i x \omega} V_{g} \hat{f}(\omega, -x), \quad (4.2) \]
which is popularly known as fundamental identity of time-frequency analysis. (See (3.10) in [18]). We obtain an analogous result using the SAFT.
Theorem 4.7 We have

\[ V(\mathcal{F}_A) \mathcal{F}_A f(x, \omega) = \eta_A(-x)\overline{\lambda}_A(dx - b\omega)e^{2\pi ip\frac{dx}{b}} e^{-2\pi i(x+p)\omega} V_g f(dx - b\omega, a\omega - cx). \]

In particular,

\[ |V(\mathcal{F}_A) \mathcal{F}_A f(x, \omega)| = |V_g f(dx - b\omega, a\omega - cx)|. \quad (4.3) \]

Proof First we observe that

\[ V_{(T_yg)} T_y f(x, \omega) = e^{-2\pi iy\omega} V_g f(x, \omega), \]

\[ V_{(D_yg)} D_y f(x, \omega) = V_g f(x/y, y\omega). \quad (4.5) \]

Now consider

\[
V(\mathcal{F}_A) \mathcal{F}_A f(x, \omega) = \int_{\mathbb{R}} \eta_A(t)\overline{\eta}_A(t - x) T_p D_b(C_{\frac{a}{b}} f)^\wedge(t)\overline{T_p D_b(C_{\frac{a}{b}} g)^\wedge}(t - x)e^{-2\pi it\omega} dt
\]

\[
= \overline{\eta}_A(-x) \int_{\mathbb{R}} T_p D_b(C_{\frac{a}{b}} f)^\wedge(t)\overline{T_p D_b(C_{\frac{a}{b}} g)^\wedge}(t - x)e^{-2\pi it(\omega - \frac{dx}{b})} dt
\]

\[
= \overline{\eta}_A(-x) V_T D_b(C_{\frac{a}{b}} g)^\wedge T_p D_b(C_{\frac{a}{b}} f)^\wedge(x, \omega - \frac{dx}{b})
\]

\[
= \overline{\eta}_A(-x)e^{-2\pi ip(\omega - \frac{dx}{b})} V_{D_b(C_{\frac{a}{b}} g)^\wedge} D_b(C_{\frac{a}{b}} f)^\wedge(x, \omega - \frac{dx}{b})
\]

\[
= \overline{\eta}_A(-x)e^{-2\pi ip(\omega - \frac{dx}{b})} V_{C_{\frac{a}{b}} g^\wedge} (C_{\frac{a}{b}} f)^\wedge(\frac{x}{b}, b\omega - dx),
\]

using (4.4), (4.5). Then appealing to (4.2) we get

\[
V(\mathcal{F}_A) \mathcal{F}_A f(x, \omega) = \overline{\eta}_A(-x)e^{-2\pi ip(\omega - \frac{dx}{b})} e^{-2\pi i\frac{x}{b}(b\omega - dx)} V_{C_{\frac{a}{b}} g} C_{\frac{a}{b}} f(dx - b\omega, \frac{x}{b})
\]

\[
= \eta_A(-x)\overline{\lambda}_A(dx - b\omega)e^{2\pi ip\frac{dx}{b}} e^{-2\pi i(x+p)\omega}
\]

\[
\times V_g f(dx - b\omega, \frac{x}{b} - \frac{a}{b}(dx - b\omega)),
\]

using Proposition 4.1. But

\[ V_g f(dx - b\omega, -\frac{a}{b}(dx - b\omega)) = V_g f(dx - b\omega, \frac{c}{a} - \frac{c}{b}(dx - b\omega)) \]

using \( ad - bc = 1 \), from which the result follows. \( \square \)

Theorem 4.8 Let \( f \in M_{w_\ell}^r \). Then \( \mathcal{F}_A f \in M_{w_\ell}^r \), where \( v_\ell = (1 + x^2 + \omega^2)^\frac{\ell}{2} \) and \( w_\ell(x, \omega) = [1 + (c^2 + d^2)x^2 + (a^2 + b^2)\omega^2 - 2(ac + bd)\omega]^{\frac{\ell}{2}} \).
Proof By (4.3), we have \( |V_{(\mathcal{F}_A)} f(x, \omega)| = |V_g f(dx - b\omega, a\omega - cx)| \), which implies that \( |V_g f(u, v)| = |V_{(\mathcal{F}_A)} f(au + bv, cu + dv)| \). Now

\[
\|f\|_{M_{v,\ell}^\ell} = \int_{\mathbb{R}} \int_{\mathbb{R}} |V_g f(u, v)|^\ell v(\ell)(u, v)^\ell du dv \\
= \int_{\mathbb{R}} \int_{\mathbb{R}} |V_{(\mathcal{F}_A)} f(au + bv, cu + dv)|^\ell v(\ell)(u, v)^\ell du dv.
\]

Using again the transformation \( x = au + bv, \omega = cu + dv \), we get \( \mathcal{F}_A f \in M_{v,\ell}^\ell \). \( \square \)

**Corollary 4.9** The modulation space \( M_{v,\ell}^\ell \) is invariant under the special affine Fourier transform \( \mathcal{F}_A \).

**Proof** It is enough to show that the weights \( v(\ell)(x, \omega) = (1 + x^2 + \omega^2)^{\ell/2} \) and \( w(\ell)(x, \omega) = (1 + (c^2 + d^2)x^2 + (a^2 + b^2)\omega^2 - 2(ac + bd)x\omega)^{\ell/2} \) are equivalent. For

\[
M = \begin{pmatrix} 1 & 0 & 0 \\ 0 & d & -b \\ 0 & -c & a \end{pmatrix},
\]

one has \( \det(M) = \det(A) = ad - bc = 1 \). Thus \( M \) is invertible and there exist \( c_1, c_2 > 0 \) such that

\[
c_1 \| \begin{pmatrix} x \\ \omega \end{pmatrix} \|_2^2 \leq \| M \begin{pmatrix} x \\ \omega \end{pmatrix} \|_2^2 \leq c_2 \| \begin{pmatrix} x \\ \omega \end{pmatrix} \|_2^2.
\]

In other words,

\[
c_1(1 + x^2 + \omega^2) \leq (1 + (a\omega - cx)^2 + (dx - b\omega)^2) \leq c_2(1 + x^2 + \omega^2).
\]

The equivalence is finally established via the estimate

\[
c_1^{\ell/2}(1 + x^2 + \omega^2)^{\ell/2} \leq (1 + (a\omega - cx)^2 + (dx - b\omega)^2)^{\ell/2} \leq c_2^{\ell/2}(1 + x^2 + \omega^2)^{\ell/2}.
\]

\( \square \)

**Theorem 4.10** (i) The special affine Fourier transform \( \mathcal{F}_A \) is a Gelfand triple automorphism on the Banach Gelfand triple \( (S_0(\mathbb{R}), L^2(\mathbb{R}), S'_0(\mathbb{R})) \).

(ii) The pointwise multiplication operators defined using the auxiliary functions \( \lambda_A, \rho_A, \eta_A \) are Gelfand triple automorphisms on \( (S_0(\mathbb{R}), L^2(\mathbb{R}), S'_0(\mathbb{R})) \).

**Proof** Due to Lemma 2.8, it is enough to prove that the operators are unitary on \( L^2(\mathbb{R}) \) and their restrictions on \( S_0(\mathbb{R}) \) define bounded and bijective mappings on \( S_0(\mathbb{R}) \) onto itself.

(i) we know that \( \mathcal{F}_A : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}) \) is a unitary operator. Further, one can write \( \mathcal{F}_A(f)(\omega) = C_d/b M_{d - dp/b T_p D_b(C_a f)}(\omega) \). Then using Remark 4.3 it follows that...
$C_{\frac{g}{b}}$ is bounded and bijective on $S_0(\mathbb{R})$. Since the Fourier transform is bounded and bijective on $S_0(\mathbb{R})$, $f \mapsto \hat{f}$ is bounded and bijective on $S_0(\mathbb{R})$. Further, we can easily show that

$$V_{(M_y g)} M_y f(x, \omega) = e^{2\pi i x y} V_g f(x, \omega). \quad (4.6)$$

Then using (4.4), (4.5), (4.6) and proceeding as in Proposition 4.1 one can show that $T_y, M_y, D_y$ are bounded and bijective on $S_0(\mathbb{R})$. Hence $\mathcal{F}_A : S_0(\mathbb{R}) \to S_0(\mathbb{R})$ is bounded and invertible. Thus the result follows from Lemma 2.8.

(ii) Since $|\lambda_A(\omega)| = |\rho_A(\omega)| = |\eta_A(\omega)| = 1$, the pointwise multiplication operators defined by $\lambda_A, \rho_A, \eta_A$ are unitary on $L^2(\mathbb{R})$. Further,

$$\begin{align*}
\lambda_A(x)f(x) &= C_{\frac{g}{b}} f(x) \\
\rho_A(x)f(x) &= C_{\frac{g}{b}} M_{\frac{g}{b}} f(x) \\
\eta_A(x)f(x) &= C_{\frac{g}{b}} M_{\frac{g}{b}} f(x).
\end{align*}$$

This means that the multiplication operators defined by $\lambda_A, \rho_A, \eta_A$ are composition of $C_s$ and $M_y$. In part (i), we have already shown that $M_y$ is bounded and bijective on $S_0(\mathbb{R})$. Further, by Remark 4.3, $C_s$ is bounded and bijective on $S_0(\mathbb{R})$. Thus our assertion follows from Lemma 2.8.

Proposition 4.11 The map $s \mapsto T^A_s$ is a strongly continuous and isometric projective representation on $S_0(\mathbb{R})$.

Proof It is easy to see that

$$T^A_x \circ T^A_y = e^{-\frac{2\pi i a}{b} xy} T^A_{x+y}.$$ 

Moreover, an easy computation shows that

$$\begin{align*}
V_g T_s f(x, \omega) &= e^{-2\pi i s \omega} V_g f(x - s, \omega) \quad (4.7) \\
V_g M_s f(x, \omega) &= V_g f(x, \omega - s). \quad (4.8)
\end{align*}$$

Thus

$$\|T_s f\|_{S_0(\mathbb{R})} = \|V_g T_s f\|_{L^1(\mathbb{R}^2)} = \|V_g f\|_{L^1(\mathbb{R}^2)} = \|f\|_{S_0(\mathbb{R})},$$

using (4.7). Similarly, using (4.8) we can show that $\|M_s f\|_{S_0(\mathbb{R})} = \|f\|_{S_0(\mathbb{R})}$. Consequently, $\|T^A_s f\|_{S_0(\mathbb{R})} = \|T_s M_{-\frac{a}{b}} f\|_{S_0(\mathbb{R})} = \|f\|_{S_0(\mathbb{R})}$. Now consider

$$\begin{align*}
\|T^A_s f - f\|_{S_0(\mathbb{R})} &= \|V_g T^A_s f - V_g f\|_{L^1(\mathbb{R}^2)} \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |V_g T^A_s f(x, \omega) - V_g f(x, \omega)|\,dx\,d\omega \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| e^{-\frac{2\pi i a}{b} x(t-s)} f(t-s) g(t-x) e^{-2\pi i \omega t} dt - V_g f(x, \omega) \right| dx\,d\omega.
\end{align*}$$
\[
\begin{align*}
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{-2\pi i g s t} f(t) g(t + s - x) e^{-2\pi i \omega(t+s)} dt - V_g f(x, \omega) dx d\omega \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |e^{-2\pi i s t} V_g f(x - s, \omega) - V_g f(x, \omega)| dx d\omega \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} |e^{-2\pi i s t} V_g f(x - s, \omega) - e^{-2\pi i s t} V_g f(x, \omega)| dx d\omega \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} |e^{-2\pi i s t} - 1| |V_g f(x, \omega)| dx d\omega \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} |T(s, \alpha) V_g f(x, \omega) - V_g f(x, \omega)| dx d\omega \\
&+ \int_{\mathbb{R}} \int_{\mathbb{R}} |e^{-2\pi i s t} - 1| |V_g f(x, \omega)| dx d\omega,
\end{align*}
\]

by applying change of variables, where \( T(h, k) f(x, y) = f(x - h, y - k) \). Finally, using \( \|T(h, k) f - f\|_{L^1(\mathbb{R}^2)} \to 0 \) as \( \|(h, k)\| \to 0 \) and Lebesgue dominated convergence theorem we obtain \( \|T_A f - f\|_{S_0(\mathbb{R})} \to 0 \) as \( s \to 0 \). In other words, the projective representation \( s \mapsto T_A^s \) is strongly continuous. \( \square \)

5 A-modulation Spaces

In this section, we wish to define the modulation spaces associated with the SAFT. We follow the classical definition of a modulation space as provided in [13].

**Definition 5.1** Let \( m(x, \omega) \) be a moderate weight of polynomial growth on \( \mathbb{R}^2 \) and \( 1 \leq r, s < \infty \). Then, for \( g \in S(\mathbb{R}) \), the modulation space associated with the SAFT, called \( A \)-modulation space, is defined as follows.

\[
M_{A,m}^{r,s} = \{ f \in S'(\mathbb{R}) : \| f \|_{M_{A,m}^{r,s}} < \infty \},
\]

where

\[
\| f \|_{M_{A,m}^{r,s}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \ast_A M_{\omega}^{A} g(x)|^r m(x, \omega)^r dx \right)^{\frac{1}{r}} d\omega \right)^{\frac{1}{r}}.
\]

If \( r = s \), then the modulation space \( M_{A,m}^{r,r} \) is denoted by \( M_{A,m}^{r} \). If \( m(x, \omega) = 1 \), then we write \( M_{A}^{r,s} \), \( M_{A}^{r} \) for \( M_{A,m}^{r,s} \), \( M_{A,m}^{r} \) and so on.

We can rewrite \( f \ast_A M_{\omega}^{A} g \) in terms of ordinary convolution using \( C_{\frac{\omega}{b}} \).

**Proposition 5.2** Let \( f \in S'(\mathbb{R}) \) and \( g \in S(\mathbb{R}) \). Then

\[
|f \ast_A M_{\omega}^{A} g(x)| = \frac{1}{\sqrt{|b|}} |C_{\frac{\omega}{b}} f \ast M_{\omega}^{A} C_{\frac{\omega}{b}} g(x)|.
\] (5.1)
Proof Consider

\[
\sqrt{|b|}(f \star A M^A_{\omega} g)(x) = \int_{\mathbb{R}} f(y) T^A_y M^A_{\omega} g(x) dy
\]

\[
= \int_{\mathbb{R}} f(y) e^{-\frac{2\pi i a}{b} y(x-y)} M^A_{\omega} g(x-y) dy
\]

\[
= \int_{\mathbb{R}} f(y) e^{-\frac{2\pi i a}{b} y(x-y)} e^{\frac{\pi i}{b}(a\omega^2 - 2p\omega + 2\omega(x-y))} g(x-y) dy
\]

\[
= \rho_{\lambda}(-\omega) \int_{\mathbb{R}} f(y) e^{-\frac{2\pi i a}{b} y(x-y)} M^\omega_{\frac{\omega}{b}} g(x-y) dy
\]

\[
= \rho_{\lambda}(-\omega) \int_{\mathbb{R}} f(y) T^A_y M^\omega_{\frac{\omega}{b}} g(x) dy
\]

\[
= \rho_{\lambda}(-\omega) \sqrt{|b|}(f \star A M^A_{\omega} g)(x).
\]

Thus, using (2.4) and the fact that \(M^\omega_{\frac{\omega}{b}} C_a^b = C_a^b M^\omega_{\frac{\omega}{b}}\), we get

\[
|(f \star A M^A_{\omega} g)(x)| = |f \star A M^\omega_{\frac{\omega}{b}} g(x)|
\]

\[
= |C_a^b (f \star A M^\omega_{\frac{\omega}{b}} g)(x)|
\]

\[
= \frac{1}{\sqrt{|b|}} |C_a^b f \star M^\omega_{\frac{\omega}{b}} C_a^b g(x)|,
\]

proving our assertion. \(\Box\)

Now we give a relation between the new modulation space and the classical modulation space.

**Theorem 5.3** Let \(1 \leq r, s < \infty\). Then \(f \in M^{r,s}_{A,m}\) if and only if \(C_a^b f \in M^{r,s}_{m_b}\), where \(m_b(x, \omega) = m(x, b\omega)\).

**Proof** Consider

\[
\|f\|_{M^{r,s}_{A,m}}^s = \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \star A M^A_{\omega} g(x)|^r m(x, \omega)^s dx \right)^{\frac{s}{r}} d\omega
\]

\[
= \frac{1}{|b|^{s/2}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |C_a^b f \star M^\omega_{\frac{\omega}{b}} C_a^b g(x)|^r m(x, \omega)^s dx \right)^{\frac{s}{r}} d\omega,
\]

using (5.1). Then

\[
\|f\|_{M^{r,s}_{A,m}}^s = |b|^{1-s/2} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |C_a^b f \star M^\omega_{\omega} C_a^b g(x)|^r m(x, b\omega)^s dx \right)^{\frac{s}{r}} d\omega,
\]
applying change of variables. But

\[
c_1 \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |C_{\frac{b}{R}} f \ast M_{\frac{b}{R}} g(x)|^r m(x, b_\omega)^r dx \right)^{\frac{s}{r}} d\omega \right)^{\frac{1}{s}} \leq \|f\|_{M_{mb}^{r,s}}
\]

\[
\leq c_2 \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |C_{\frac{b}{R}} f \ast M_{\frac{b}{R}} g(x)|^r m(x, b_\omega)^r dx \right)^{\frac{s}{r}} d\omega \right)^{\frac{1}{s}},
\]

for some \(c_1, c_2 > 0\), from which the result follows. \(\square\)

**Corollary 5.4** We have the following inclusion between A-modulation spaces.

\[
M_{A}^{r_1,s_1} \subseteq M_{A}^{r_2,s_2}, \text{ for } r_1 \leq r_2, s_1 \leq s_2.
\]

In particular, \(S_0(\mathbb{R}) \subseteq M_{A}^{r,s} \) for \(r, s \geq 1\).

**Proof** Let \(f \in M_{A}^{r_1,s_1}\). Then \(C_{\frac{b}{R}} f \in M_{A}^{r_1,s_1}\). This implies that \(C_{\frac{b}{R}} f \in M_{A}^{r_2,s_2}\), from which it follows that \(f \in M_{A}^{r_2,s_2}\). \(\square\)

**Corollary 5.5** We have

\[
M_{A}^{r_1,s_1} \ast_A M_{A}^{r_2,s_2} \subseteq M_{A}^{r_0,s_0},
\]

where \(\frac{1}{r_1} + \frac{1}{r_2} = 1 + \frac{1}{r_0}, \frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s_0}\).

**Proof** Let \(f \in M_{A}^{r_1,s_1}\), \(g \in M_{A}^{r_2,s_2}\). Then \(C_{\frac{b}{R}} f \in M_{A}^{r_1,s_1}\) and \(C_{\frac{b}{R}} g \in M_{A}^{r_2,s_2}\). Now using the fact that \(M_{A}^{r_1,s_1} \ast_A M_{A}^{r_2,s_2} \subseteq M_{A}^{r_0,s_0}\), we get \(C_{\frac{b}{R}} f \ast_C g \in M_{A}^{r_0,s_0}\). But \(C_{\frac{b}{R}} f \ast_C g = (f \ast_C g)\). Hence \(C_{\frac{b}{R}} (f \ast_C g) \in M_{A}^{r_0,s_0}\). In other words, \(f \ast_C g \in M_{A}^{r_0,s_0}\). \(\square\)

**Corollary 5.6** If \(1 \leq r, s < \infty\), then \(M_{A}^{r,s} \subseteq S_0'(\mathbb{R})\).

**Proof** Let \(f \in M_{A}^{r,s}\). Then \(C_{\frac{b}{R}} f \in M_{A}^{r,s}\). Now using the fact that \(M_{A}^{r,s} \subseteq S_0'(\mathbb{R})\), we get \(C_{\frac{b}{R}} f \in S_0'(\mathbb{R})\). Thus it is enough to prove that \(S_0'(\mathbb{R})\) is invariant under the operator \(C_{\frac{b}{R}}\). Let \(\Lambda \in S_0'(\mathbb{R})\). Then for \(\phi \in S(\mathbb{R})\), we get

\[
|C_{\frac{b}{R}} \Lambda(\phi)| = |\Lambda(C_{\frac{b}{R}} \phi)| \leq \|\Lambda\| \|C_{\frac{b}{R}} \phi\|_{S_0}(\mathbb{R}) \leq c' \|\Lambda\| \|\phi\|_{S_0}(\mathbb{R}), \text{ for some } c' > 0.
\]

Hence the result follows. \(\square\)

**Corollary 5.7** The A-modulation space \(M_{A}^{r}\) coincides with the classical modulation space \(M_{r}\) in the sense of equivalent norm.

**Proof** The proof follows from Proposition \(4.1\) and Theorem \(5.3\). \(\square\)
Remark 5.8 If \( m(x, \omega) = 1 \), then the operator \( C_\omega^b : M^r,s_A \to M^r,s \) is bounded and invertible. The inverse is given by \( C_\omega^{-1} = C_{-\omega}^b \). Moreover, using the fact that modulation is an isometry on \( M^r,s \), we get the invertibility of \( M^r,s_A C_\omega^b = \rho_A : M^r,s_A \to M^r,s \).

As in the case of classical modulation spaces, we obtain the following

Theorem 5.9 (i) \( M^r,s_{A,m} \) is a Banach space for \( 1 \leq r, s < \infty \).

(ii) The definition of modulation spaces is independent of the choice of window \((0 \neq) g \in S(\mathbb{R})\): different windows define the same space and equivalent norms.

(iii) \( S(\mathbb{R}) \) is dense in \( M^r,s_A \), for \( 1 \leq r, s < \infty \).

We omit the proof as it follows from the corresponding properties of the classical modulation spaces.

Proposition 5.10 Let \( m \) be a \( v \)-moderate weight. Then the A-modulation space \( M^r,s_A \) is invariant under A-time-frequency shifts and

\[
\| M^A_{\omega} T^A_x f \|_{M^r,s_{A,m}} \leq v(x, \omega) \| f \|_{M^r,s_{A,m}}.
\]

Proof Consider

\[
\sqrt{|b|} M^A_{\omega} T^A_x f * A M^A_{\omega_0} g(x_1) = \int_{\mathbb{R}} T^A_y M^A_{\omega} T^A_x f(x_1) M_{\omega_0} g(y) dy \\
= \int_{\mathbb{R}} \rho_A(-\omega) e^{-\frac{2\pi i a}{b} y (x_1-y)} e^{\frac{2\pi i a}{b} (x_1-y)} e^{-\frac{2\pi i a}{b} x (x_1-x-y)} \\
\times f(x_1-x-y) \rho_A(-\omega_0) e^{\frac{2\pi i a}{b} y (y) dy} \\
= \rho_A(-\omega) \rho_A(-\omega_0) e^{\frac{2\pi i}{b} (\omega x_1-axx_1+ax^2)} \\
\times \int_{\mathbb{R}} e^{-\frac{2\pi i a}{b} y (x_1-x-y)} f(x_1-x-y) M_{\omega_1-\omega_0} g(y) dy \\
= \rho_A(-\omega) \rho_A(-\omega_0) e^{\frac{2\pi i}{b} (\omega x_1-axx_1+ax^2)} \\
\times \int_{\mathbb{R}} T^A_y f(x_1-x) \rho_A(\omega-\omega_0) M_{\omega_1-\omega} g(y) dy \\
e^{\frac{2\pi i}{b} (\omega x_1-axx_1+ax^2+a_0(x_1-x)^2) - 2p_0} \int_{\mathbb{R}} T^A_y f(x_1-x) M^A_{\omega_1-\omega} g(y) dy \\
= e^{\frac{2\pi i}{b} (\omega x_1-axx_1+ax^2+a_0(x_1-x)^2) - 2p_0} \sqrt{|b|} f * A M^A_{\omega_1-\omega} g(x_1 - x).
\]

Thus

\[
|M^A_{\omega} T^A_x f * A M^A_{\omega_0} g(x_1)| = |f * A M^A_{\omega_1-\omega_0} g(x_1 - x)|.
\]

Then

\[
\| M^A_{\omega} T^A_x f \|_{M^r,s_{A,m}} = \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |M^A_{\omega} T^A_x f * A M^A_{\omega_0} g(x_1)|^r m(x_1, \omega_0) dx_1 \right)^{s/r} d\omega \right)^{1/s}
\]

\[
= \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f * A M^A_{\omega_1-\omega_0} g(x_1 - x)|^r m(x_1, \omega_0) dx_1 \right)^{s/r} d\omega \right)^{1/s}.
\]
by applying a change of variables.

\[
\begin{align*}
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \star A M_{\omega_1} g(x_1 - x)|^r m(x_1, \omega_1)^r \, dx_1 \right)^{s/r} \, d\omega_1 \right)^{1/s} &= \\
\left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \star A M_{\omega_1} g(x_1 + x, \omega_1 + \omega)|^r m(x_1, \omega_1 + \omega)^r \, dx_1 \right)^{s/r} \, d\omega_1 \right)^{1/s} \\
&\leq v(x, \omega) \left( \int_{\mathbb{R}} \left( \int_{\mathbb{R}} |f \star A M_{\omega_1} g(x_1)|^r m(x_1, \omega_1)^r \, dx_1 \right)^{s/r} \, d\omega_1 \right)^{1/s} \\
&= v(x, \omega) \| f \|_{M_{A,m}^{r,s}}.
\end{align*}
\]

**Corollary 5.11** The \(A\)-modulation space \(M_{A,v}^{r,s}\) is invariant under classical time-frequency shifts and

\[
\| M_{\omega} T_{x} f \|_{M_{A,v}^{r,s}} \leq v(x, ax + b\omega) \| f \|_{M_{A,v}^{r,s}}.
\]

**Proof** Consider

\[
\| M_{\omega} T_{x} f \|_{M_{A,v}^{r,s}} = \| M_{ax + b\omega}^{A} T_{x} f \|_{M_{A,v}^{r,s}} \leq v(x, ax + b\omega) \| f \|_{M_{A,v}^{r,s}},
\]

using \(M_{\omega} T_{x} = e^{-2\pi i a b \omega} A^{A}_{ax + b\omega} T_{x}\).

**Theorem 5.12** Let \(B\) be a Banach space continuously embedded into the space of tempered distributions with the following properties:

(i) \(B\) is invariant under \(A\)-time-frequency shifts and \(\| M_{\omega}^{A} T_{x} f \|_{B} \leq v(x, \omega) \| f \|_{B}\), for all \(f \in B\).

(ii) \(M_{A,v}^{1} \cap B \neq \{0\}\).

Then \(M_{A,v}^{1}\) is continuously embedded in \(B\).

**Proof** Choose a non-zero \(g \in M_{A,v}^{1} \cap B\). Then \(C_{b} g \in M_{vb}^{1}\), where \(v_{b}(x, \omega) = v(x, b\omega)\). Let \(f \in M_{A,v}^{1}\). Then \(C_{b} f \in M_{vb}^{1}\). Now using Theorem 12.1.8 in [18], we can express \(C_{b} f\) as a non-uniform Gabor expansion of time-frequency shifts of \(C_{b} g\). In other words,

\[
C_{b} f = \sum_{n=1}^{\infty} c_{n} T_{x_{n}} M_{\omega_{n}}^{A} C_{b} g,
\]

with \(\| C_{b} f \|_{M_{vb}^{1}} = \inf_{n=1}^{\infty} \sum_{n=1}^{\infty} |c_{n}| v_{b}(x_{n}, \omega_{n}) = \inf_{n=1}^{\infty} \sum_{n=1}^{\infty} |c_{n}| v(x_{n}, \omega_{n})\), where the infimum is taken over all such representations of \(C_{b} f\). Using (5.2) \(f\) can be expressed as

\[
f = \sum_{n=1}^{\infty} c_{n} C_{b}^{-1} T_{x_{n}} M_{\omega_{n}}^{A} C_{b} g.
\]
Consider
\[ C_{a/b}^{-1} T_{x_n} M_{\omega_n/b} C_{a/b} g(t) = e^{-\frac{\pi i a t^2}{b^2}} e^{2\pi i \frac{a t}{b}(t-x_n)} e^{-\frac{\pi i a}{b}(t-x_n)^2} g(t-x_n) \]
\[ = e^{-\frac{\pi i a t^2}{b^2}} e^{2\pi i \frac{a t}{b}(t-x_n)} e^{-\frac{2\pi i a}{b} x_n(t-x_n)} g(t-x_n) \]
\[ = e^{-\frac{\pi i a t^2}{b^2}} e^{-\frac{2\pi i a}{b} x_n(\omega_n) T_A^{\omega_n} x_n g(t)} \]
\[ = \lambda(x_n + \omega_n) e^{-\frac{2\pi i a}{b} p_{\omega_n} M^{A_{\omega_n}} T_{x_n} g(t)}. \]

Thus
\[ f = \sum_{n=1}^{\infty} c_n \lambda(x_n + \omega_n) e^{-\frac{2\pi i a}{b} p_{\omega_n} M^{A_{\omega_n}} T_{x_n} g(t)}. \]

Then
\[ \|f\|_{B} \leq \sum_{n=1}^{\infty} |c_n| \|M^{A_{\omega_n}} T_{x_n} g\|_{B} \leq \sum_{n=1}^{\infty} |c_n| v(x_n, \omega_n) \|g\|_{B} < \infty. \quad (5.3) \]

This shows that \( M_{1,v}^{A} \subseteq B \). Taking infimum over all representations of \( C_{a/b} f \), (5.3) turns out to be
\[ \|f\|_{B} \leq \|C_{a/b} f\|_{M_{1,v}^{A}} \|g\|_{B} \leq C \|f\|_{M_{1,v}^{A}}, \quad \text{for some } C > 0, \]

using Theorem 5.3. Hence the inclusion is continuous. \( \square \)

6 Multipliers and Littlewood-Paley Theorem Associated with SAFT

First we prove an analogue of Theorem 10 in [15].

**Theorem 6.1** Given a bounded linear operator \( T : \mathcal{M}_{A}^{r_1,s_1} \rightarrow \mathcal{M}_{A}^{r_2,s_2} \), one has the following.

(i) If \( TT_{x}^{A} = T_{x}^{A} T \) for all \( x \in \mathbb{R} \), then there exists a unique \( u \in S_{0}'(\mathbb{R}) \) such that \( T f = u \ast f \), for all \( f \in S_{0}(\mathbb{R}) \).

(ii) If \( TT_{x} = T_{x} T \) for all \( x \in \mathbb{R} \), then there exists a unique \( u \in S_{0}'(\mathbb{R}) \) such that \( T f = u \ast f \), for all \( f \in S_{0}(\mathbb{R}) \).

**Proof** (i) Define \( \tilde{T} : \mathcal{M}_{r_1-s_1}^{A} \rightarrow \mathcal{M}_{r_2-s_2}^{A} \) by \( \tilde{T} = C_{a/b} T C_{a/b}^{-1} \). Then \( \tilde{T} \) is bounded and linear. Further, using \( TT_{x}^{A} f = T_{x}^{A} T f \) and (2.2), we get
\[ Te^{-\frac{\pi i a t^2}{b^2}} C_{a/b}^{-1} T_{x} C_{a/b} f = e^{-\frac{\pi i a t^2}{b^2}} C_{a/b}^{-1} T_{x} C_{a/b} T f. \]
This is equivalent to

\[ C_a^{\frac{1}{p}} T C_a^{-1} T_x C_a^{\frac{1}{p}} f = T_x C_a^{\frac{1}{p}} T C_a^{-1} C_a^{\frac{1}{p}} f. \]

In other words, \( \tilde{T} T_x C_a^{\frac{1}{p}} f = T_x \tilde{T} C_a^{\frac{1}{p}} f \), for \( f \in M_{r,s}^{r,s} \), from which it follows that \( \tilde{T} \) commutes with classical translations. Then by Theorem 10 in [15], there exists a unique \( u \in S_0'(\mathbb{R}) \) such that \( \tilde{T} f = u \ast f \), for all \( f \in S_0(\mathbb{R}) \). This means that

\[ C_a^{\frac{1}{p}} T C_a^{-1} f = C_a^{\frac{1}{p}} C_a^{\frac{1}{p}} u \ast C_a^{\frac{1}{p}} C_a^{-1} f = \sqrt{|b|} C_a^{\frac{1}{p}} (C_a^{-1} u \ast A C_a^{-1} f), \]

using Proposition 3.15 and Remark 3.16. Define \( u_1 = \sqrt{|b|} C_a^{-1} u \), which belongs to \( S_0'(\mathbb{R}) \). Thus

\[ C_a^{\frac{1}{p}} T C_a^{-1} f = C_a^{\frac{1}{p}} (u_1 \ast A C_a^{-1} f), \]

from which our assertion follows.

(ii) We omit the proof as it is using the same ideas as in the proof of Theorem 10 in [15], with \( M_{r,s}^{r,s} \) in place of \( M_{r,s}^{r,s} \).

Recall that \( m \in L^\infty(\mathbb{R}) \) is called a Fourier multiplier for \( L^r(\mathbb{R}), 1 \leq r < \infty \), if the operator defined by \( (T_m f)^\wedge(\xi) = m(\xi) \hat{f}(\xi) \) on \( L^2(\mathbb{R}) \cap L^r(\mathbb{R}) \), extends to a bounded linear operator on \( L^r(\mathbb{R}) \).

**Definition 6.2** Let \( m \in L^\infty(\mathbb{R}) \). Then \( m \) is called a SAFT multiplier for \( L^r(\mathbb{R}), 1 \leq r < \infty \) if the operator \( T_{A,m} \), defined by

\[ \mathcal{F}_A(T_{A,m} f)(\omega) = m(\omega) \mathcal{F}_A(f)(\omega), \quad f \in L^2(\mathbb{R}) \cap L^r(\mathbb{R}). \]

extends to a bounded linear operator on \( L^r(\mathbb{R}) \).

**Theorem 6.3** (Hörmander) Let \( m \in C^1(\mathbb{R} \setminus \{0\}) \) satisfy \( |m'(x)| \leq C|x|^{-1} \) for some \( C > 0 \). Then for any \( 1 < r < \infty \), \( m \) is a SAFT multiplier for \( L^r(\mathbb{R}) \).

**Proof** We first observe that

\[ \mathcal{F}_A(T_{A,m} f)(\omega) = m(\omega) \mathcal{F}_A(f)(\omega) \]

is equivalent to

\[ \eta_A(\omega) \sqrt{|b|} (\rho_A T_{A,m} f)(\omega/b) = \eta_A(\omega) \sqrt{|b|} m(\omega) (\rho_A f)(\omega/b). \]

This in turn implies that

\[ (\rho_A T_{A,m} \tilde{\rho}_A f)(\omega) = m(b \omega) \tilde{f}(\omega). \]
Further, let \( m_1(x) = m(bx) \). Then \( m'_1(x) = bm'(bx) \). Hence

\[
|m'_1(x)| \leq |b|C|bx|^{-1} = C|x|^{-1}.
\]

Now by applying classical Hörmander theorem to \( m_1 \), it follows that \( m_1 \) is a Fourier multiplier for \( L^r(\mathbb{R}) \). Thus we can find \( C' > 0 \) such that

\[
\|\rho_A T_{A,m}\hat{\rho}_A f\|_r \leq C'\|f\|_r.
\]

In other words,

\[
\|T_{A,m}\hat{\rho}_A f\|_r \leq C'\|\hat{\rho}_A f\|_r,
\]

which leads to \( \|T_{A,m}f\|_r \leq C'\|f\|_r \), proving our assertion.  \( \square \)

**Theorem 6.4 (Littlewood-Paley)** For any \( 1 < r < \infty \) there exist \( m_r, \quad M_r > 0 \) such that

\[
m_r\|f\|_r \leq \left\| \left( \sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_r \leq M_r\|f\|_r, \quad f \in L^r(\mathbb{R}),
\]

with \( \mathcal{F}_A(S_j f)(\omega) = \chi_{\Delta_j}(\omega)\mathcal{F}_A(f)(\omega), \) for \( \Delta_j = [-2^{j+1}, -2^j] \cup [2^j, 2^{j+1}] \).

**Proof** First we observe that

\[
\frac{1}{b}\Delta_j = \left\{ \begin{array}{ll}
\left( -\frac{2^{j+1}}{b}, -\frac{2^j}{b} \right] \cup \left[ \frac{2^j}{b}, \frac{2^{j+1}}{b} \right), & b > 0 \\
\left[ -\frac{2^j}{b}, -\frac{2^{j+1}}{b} \right) \cup \left( \frac{2^j}{b}, \frac{2^{j+1}}{b} \right], & b < 0
\end{array} \right.
\]

Further we can see that

\[
\mathcal{F}_A(S_j f)(\omega) = \chi_{\Delta_j}(\omega)\mathcal{F}_A(f)(\omega),
\]

which is equivalent to

\[
\frac{\eta_A(\omega)}{\sqrt{|b|}}(\rho_A S_j f)(\omega/b) = \chi_{\Delta_j}(b\omega)\frac{\eta_A(\omega)}{\sqrt{|b|}}(\rho_A f)(\omega/b).
\]

In other words,

\[
(\rho_A S_j f)(\omega) = \chi_{\Delta_j}(b\omega)(\rho_A f)(\omega),
\]

which leads to

\[
(\rho_A S_j \hat{\rho}_A f)(\omega) = \chi_{\frac{1}{b}\Delta_j}(\omega)\hat{f}(\omega).
\]
Let $\tilde{S}_j = \rho_A S_j \bar{\rho}_A$. Then $(\tilde{S}_j f) \hat{}(\omega) = \chi_{\frac{1}{\Delta_j}}(\omega) \hat{f}(\omega)$. Now by applying classical Littlewood-Paley theorem to the sequence of intervals $\{\frac{1}{\Delta_j} : j \in \mathbb{Z}\}$ we can find $m_r, M_r > 0$ such that

$$m_r \|f\|_r \leq \left( \sum_{j \in \mathbb{Z}} |\tilde{S}_j f|^2 \right)^{1/2} \|f\|_r \leq M_r \|f\|_r.$$ 

In other words,

$$m_r \|\bar{\rho}_A f\|_r \leq \left( \sum_{j \in \mathbb{Z}} |\rho_A S_j \bar{\rho}_A f|^2 \right)^{1/2} \|f\|_r \leq M_r \|\bar{\rho}_A f\|_r,$$

which in turn implies that

$$m_r \|f\|_r \leq \left( \sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \|f\|_r \leq M_r \|f\|_r.$$ 

\[\square\]

Declarations

Data availability statement The manuscript has no associated data.

References

1. Abe, S., Sheridan, J.T.: Optical operations on wave functions as the Abelian subgroups of the special affine Fourier transformation. Opt. Lett. \textbf{19}(22), 1801–1803 (1994)
2. Bényi, Á., Okoudjou, K.A.: Modulation Spaces: With Applications to Pseudodifferential Operators and Nonlinear Schrödinger Equations. Springer Nature (2020)
3. Bhandari, A., Zayed, A.I.: Shift-invariant and sampling spaces associated with the special affine Fourier transform. Appl. Comput. Harmon. Anal. \textbf{47}(1), 30–52 (2019)
4. Boggiatto, P., Cordero, E., Gröchenig, K.: Generalized anti-Wick operators with symbols in distributional Sobolev spaces. Integral Equations Operator Theory \textbf{48}(4), 427–442 (2004)
5. Boggiatto, P., Toft, J.: Embeddings and compactness for generalized Sobolev-Shubin spaces and modulation spaces. Appl. Anal. \textbf{84}(3), 269–282 (2005)
6. Chen, W., Fu, Z., Grafakos, L., Wu, Y.: Fractional Fourier transforms on $L^p$ and applications. Appl. Comput. Harmon. Anal. \textbf{55}, 71–96 (2021)
7. Cordero, E., Feichtinger, H.G., Luef, F.: Banach Gelfand triples for Gabor analysis. Pseudo-differential Operators, Lect. Notes Math., vol. 1949, Springer, pp. 1–33 (2008)
8. Cordero, E., Rodino, L.: Time-Frequency Analysis of Operators. De Gruyter Studies in Mathematics, vol. 75, De Gruyter, Berlin, [2020] © 2020
9. Feichtinger, H., Luef, F., Cordero, E.: Banach Gelfand triples for Gabor analysis, Pseudo-differential Operators. Lecture Notes in Math., vol. 1949, Springer, Berlin, pp. 1–33 (2008)
10. Feichtinger, H.G.: Banach spaces of distributions of Wiener’s type and interpolation. Proc. Conf. Functional Analysis and Approximation, Oberwolfach August 1980. In: Butzer, P., Nagy, S.B., Görlich, E. (eds.) Internat. Ser. Numer. Math., no. 69, Birkhäuser Boston, pp. 153–165 (1981)
11. Feichtinger, H.G.: On a new Segal algebra. Monatsh. Math. 92(4), 269–289 (1981)
12. Feichtinger, H.G.: Modulation spaces on locally compact Abelian groups, Tech. report, University of Vienna (January 1983)
13. Feichtinger, H.G.: Modulation spaces on locally compact abelian groups. In: Krishna, M., Radha, R., Thangavelu, S. (eds.) Wavelets and their Applications, Chennai, India, pp. 99–140. Allied Publishers, New Delhi (2003)
14. Feichtinger, H.G.: Homogeneous Banach spaces as Banach convolution modules over $M(G)$. Mathematics 10(3), 1–22 (2022)
15. Feichtinger, H.G., Narimani, G.: Fourier multipliers of classical modulation spaces. Appl. Comput. Harmon. Anal. 21(3), 349–359 (2006)
16. Feichtinger, H.G., Zimmermann, G.: A Banach space of test functions for Gabor analysis, Gabor Analysis and Algorithms, Appl. Numer. Harmon. Anal., pp. 123–170. Birkhäuser Boston, Boston, MA (1998)
17. Biswas, M.H.A., Filbir, F., Ramakrishnan, R.: New translations associated with the special ane Fourier transform and shift invariant spaces, communicated for publication, 2022.
18. Gröchenig, K.: Foundations of Time-Frequency Analysis, Applied and Numerical Harmonic Analysis. Birkhäuser Boston Inc., Boston, MA (2001)
19. Holland, F.: Harmonic analysis on amalgams of $L^p$ and $\ell^q$. J. London Math. Soc. 10, 295–305 (1975)
20. Jaming, P.: A characterization of Fourier transforms. Colloq. Math. 118(2), 569–580 (2010)
21. Katznelson, Y.: An Introduction to Harmonic Analysis, 3rd edn. Cambridge Mathematical Library, Cambridge University Press, Cambridge (2004)
22. Luef, F., Rahbani, Z.: On pseudodifferential operators with symbols in generalized Shubin classes and an application to Landau-Weyl operators. Banach J. Math. Anal. 5(2), 59–72 (2011)
23. Rudin, W.: Functional Analysis. second ed., International Series in Pure and Applied Mathematics, McGraw-Hill, Inc., New York (1991)
24. Toft, J.: Continuity properties for modulation spaces, with applications to pseudo-differential calculus. I. J. Funct. Anal. 207(2), 399–429 (2004)
25. Toft, J.: Convolutions and embeddings for weighted modulation spaces, Advances in Pseudo-differential Operators. Oper. Theory Adv. Appl., vol. 155, Birkhäuser, Basel, pp. 165–186 (2004)
26. Toft, J., Wahlberg, P.: Embeddings of $\alpha$-modulation spaces. Pliska Stud. Math. Bulgar. 21, 25–46 (2012)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.