Some remarks on uniform boundary Harnack Principles

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Abstract

We prove two versions of a boundary Harnack principle in which the constants do not depend on the domain.

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1 Introduction

The boundary Harnack principle (BHP) gives a result of the following general type. Let $D$ be a domain in $\mathbb{R}^d$, and $\xi \in \partial D$, satisfying suitable properties. Let $r > 0$, $A_0 \geq 2$, $B_1 = B(\xi, A_0 r)$ and $B_2 = B(\xi, r)$; here $B(\ldots)$ denote the usual Euclidean balls. Then there exists a constant $C_D$ such that if $u, v$ are positive harmonic functions on $B_1 \cap D$ vanishing on $\partial D \cap B_1$, one has

$$\frac{u(x)}{v(x)} \leq C_D \frac{u(y)}{v(y)} \text{ for } x, y \in D \cap B_2.$$  \hfill (1.1)

A BHP of this kind is called in [1] a uniform BHP, and in [13] a scale-invariant BHP. Here ‘uniform’ or ‘scale-invariant’ refers to the fact that the constant $C_D$ does not depend on $r$. For Lipschitz domains $D$ the scale invariant BHP was proved independently by Ancona, Dahlberg and Wu in [4, 10, 17]. This was extended to NTA domains by Jerison and Kenig [12]. Bass, Burdzy and Banuelos [7, 8] used probabilistic methods to obtain a BHP for Hölder domains, but their BHP is not uniform. In [1] a scale invariant BHP is proved for inner uniform domains in $\mathbb{R}^d$. See the papers [1, 2, 13] for a further discussion on history of the BHP, and the various different kinds of BHP.

In the above ‘harmonic function’ refers to functions which are harmonic with respect to the usual Laplacian operator in $\mathbb{R}^d$. (Thus these functions are harmonic with respect to

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the infinitesimal generator of the semigroup of standard Brownian motion in \( \mathbb{R}^d \). Recent papers have studied functions which are harmonic with respect to the semigroups of more general diffusion type processes – see [13, 14].

In all these results the constant \( C_D \) depends on the domain \( D \). For the standard Laplacian it is clear that such dependence is necessary, since the BHP does not hold for all domains \( D \subseteq \mathbb{R}^d \). (See however [9] where a result of BHP type with constants independent of the domain is proved for harmonic functions with respect to fractional Laplacians.)

This paper originates in the work of Masson [15], where a certain kind of BHP with constant \( C_D \) not depending on \( D \) was needed – see [15, Proposition 3.5]. Masson’s work was in the context of discrete potential theory for \( \mathbb{Z}^2 \). Let \( S^x = (S_k^x, k \geq 0) \) be the simple random walk on \( \mathbb{Z}^2 \), started at \( x \), and write \( S = S^0 \). Write \( \mathbb{Z}^2_0 = \{(x_1, x_2) \in \mathbb{Z}^2 : x_1 \leq 0\} \), and let \( Q(x, n) = \{y \in \mathbb{Z}^2 : |x - y| \leq n\} \). Let \( N \geq 1 \), \( K \subset Q(0, N) \cap \mathbb{Z}^2 \), and \( D = Q(0, N) - K \). (The case of interest is when \( 0 \in K \).) Let \( \tau^+ = \min\{k \geq 1 : S_k \not\subset D\} \), and \( F = \{S_{\tau^+} \subset Z^2 - K\} \), so that \( F \) is the event that \( S \) leaves \( Q(0, N) \) before hitting \( K \). Let \( W = \{(x_1, x_2) : 0 \leq |x_2| \leq x_1\} \), so that \( W \) is a cone with vertex \((0,0)\) and angle \( \pi/4 \). Masson’s theorem is that there exists \( p_0 > 0 \), independent of \( N \) and \( K \), such that

\[
\mathbb{P}(S_{\tau^+} \in W|F) \geq p_0. \tag{1.2}
\]

This result extends to give also \( \mathbb{P}(S^x_{\tau^+} \in W|F) \geq p_1 \) for \( x = (x_1, x_2) \in Q(0, N/16) \) with \( x_1 \geq 0 \). The fact that \( p_i \) do not depend on the structure of \( K \) is essential in the context of [15], since \( K \) is a random path (actually a loop erased random walk), and the estimate (1.2) was needed for all possible \( K \).

Although the connection with BHP is not made in [15], this result is clearly of BHP type. For \( x \in \mathbb{Z}^2 \) let \( \tau = \min\{k \geq 0 : S_k \not\subset D\} \), and define the functions

\[
v(x) = \mathbb{P}(S_{\tau^+} \subset K^c), \quad u(x) = \mathbb{P}(S_{\tau^+} \in K^c \cap W).
\]

These are (discrete) harmonic in \( D \), and \( \mathbb{P}(S^x_{\tau^+} \in W|F) = u(x)/v(x) \). Since \( u \leq v \) it is immediate that

\[
\frac{u(x)/v(x)}{u(y)/v(y)} \leq p_1^{-1}, \text{ for } x, y \in Q(0, N/16) \cap (\mathbb{Z}^2 - \mathbb{Z}^2_0). \tag{1.3}
\]

Thus we have a BHP in which the constant \( C_D = p_1^{-1} \) does not depend on \( K \); the price is that the inequality only holds for \( x \) close to 0 with \( x_1 \geq 0 \).

In this paper we present two further results on this kind. In Section 2 we extend Masson’s result to Euclidean space, and prove it for \( d \geq 2 \). (The result is trivial for \( d = 1 \).) In Section 3 we prove a BHP for \( d = 2 \) where the constant \( C_D \) does not depend locally on \( K \), but does require some regularity of \( K \) away from 0. A counterexample shows this result does not extend to \( d \geq 3 \).

Throughout this paper we write \( X = (X_t, t \in [0, \infty), \mathbb{P}^x, x \in \mathbb{R}^d) \) for Brownian motion in \( \mathbb{R}^d \); \( \mathbb{P}^x \) is the law of \( X \) started at \( x \). For a set \( A \subset \mathbb{R}^d \) we define

\[
T_A = \inf\{t \geq 0 : X_t \in A\}, \quad \tau_A = T_{A^c} = \inf\{t \geq 0 : X_t \not\in A\}.
\]
We write $B(x, r) = \{ y : |x - y| < r \}$ for Euclidean balls. When we use notation such as $C = C(\alpha, d)$ this will mean that the (positive) constant $C$ depends only on the parameters $\alpha$ and $d$.

## 2 Uniform BHP for a harmonic functions associated with cones

In this section we will prove the Boundary Harnack Principle for two fundamental harmonic functions.

Let $\mathbb{H}_- = \{ x = (x_1, \ldots, x_d) \in \mathbb{R}^d : x_1 < 0 \}, \quad B^- = B(0, 1) \cap \mathbb{H}_-$, and define $\mathbb{H}_+, B^+$ analogously. We write $B = B(0, 1)$. Let $K \subset \overline{\mathbb{H}_-}$ be closed and connected. Set $D = B(0, 1) - K$.

Let $\pi_1 : \mathbb{R}^d \to \mathbb{R}^d$ be projection onto the $x_1$-axis, so $\pi_1((x_1, x_2, \ldots, x_d)) = (x_1, 0, \ldots, 0)$. Let $W_\alpha$ be the cone $W_\alpha = \{ z \in \mathbb{R}^d : |z - \pi_1(z)| < z_1 \tan(\alpha) \}$. Set $W_\alpha(r) = B(0, r) \cap W_\alpha$.

Write $\tau = \tau_D$ for the exit time of $X$ from $D$, and define the functions

$$v(x) = \mathbb{P}^x(X_\tau \in \partial D \cap K^c), \quad u(x) = \mathbb{P}^x(X_\tau \in \partial D \cap W_\alpha). \quad (2.1)$$

Thus $u, v$ are bounded, positive harmonic functions which vanish on $\partial K$, and have boundary values 1 on $\partial D \cap K^c$ and $\partial D \cap W_\alpha$ respectively. It is clear that $u \leq v$ on $D$. Both $u$ and $v$ are bounded, positive and harmonic inside the domain $D$. Hence they satisfy the usual Harnack Inequality, [6, Theorem II.1.19], in balls which are far enough from the boundary of $D$. The main result of this section is that these two functions satisfy a BHP with constant which depends only on $d$. (Note that since the geometry of the boundary of $K$ is not specified, classical results on the Boundary Harnack Principle such as [6, Theorem III.1.2], do not apply.)

The main result of this section is the following Theorem.

**Theorem 2.1.** Let $\alpha \in (0, \pi/2]$. There is a constant $C = C(\alpha, d) > 0$ depending only on $\alpha$ and $d$, and independent of $K$, such that for any $x, y \in B(0, 1/2) \cap \mathbb{H}_+$,

$$\frac{u(x)/u(y)}{v(x)/v(y)} \leq C. \quad (2.2)$$

**Remark 2.2.** (1) The result does not require $0 \in K$, though this is the most delicate case.
(2) Let \( x = (1/4, 0) \). Then the usual Harnack inequality gives that \( u(x) \geq C'(\alpha, d) \), and so, since \( v \leq 1 \), \( (2.2) \) implies that

\[
C'(\alpha, d) \leq u(y)/v(y) \leq 1 \quad \text{for } y \in B(0, 1/2) \cap \mathbb{H}_+.
\]

(3) Suppose that \( (2.2) \) (and therefore \( (2.3) \)) holds for some \( \beta \in (0, \pi/2) \), and let \( \alpha \in (\beta, \pi/2] \). Writing \( u_\alpha, u_\beta \) for the functions given by \( (2.1) \) we have \( u_\beta \leq u_\alpha \leq v \leq 1 \). So \( (2.3) \) for \( u_\beta \) implies \( (2.3) \), and therefore Theorem 2.1 for \( u_\alpha \). It is therefore sufficient to prove Theorem 2.1 for \( \beta \in (0, \alpha_d) \), where the angle \( \alpha_d \) depends on \( d \).

Let \( \kappa_d = [4(d + 2)]^{-1/2} \), and define \( \alpha_d = \arcsin(\kappa_d) \) for \( d \geq 2 \). Set

\[
A_{\alpha_d} = W_{\alpha_d} \cap \partial B(0, 1).
\]

Note that the proportion of this surface \( A_{\alpha_d} \) to the surface of the unit ball decreases as \( d \) grows. Similarly, let \( \beta \in (0, \alpha_d) \) and define \( \kappa_\beta = \sin(\beta) \) and

\[
A_\beta = W_\beta \cap \partial B(0, 1).
\]

First we observe that by Lemma 2.3 below it is enough to prove this statement for \( \pi_1(x) \) and \( \pi_1(y) \).

**Lemma 2.3.** If \( x \in W_{\alpha_d}(1/2) \) then \( \pi_1(x) \in W_{\alpha_d}(1/2) \), and there is \( c = c(\alpha_d) \) such that \( c^{-1}f(\pi_1(x)) \leq f(x) \leq cf(\pi_1(x)) \) for any non-negative bounded harmonic function \( f \) in \( D \).

**Proof.** By definition of the cone, it is clear that \( \pi_1(x) \in W_{\alpha_d}(1/2) \). If \( d \geq 3 \) then \( x \in B(\pi_1(x), x_1 \tan(\alpha_d)) \subset B(\pi_1(x), x_1) \subset B^+ \), and applying Harnack’s inequality, the result follows. If \( d = 2 \) then it is enough to consider the Harnack chain with two balls and to apply the Harnack inequality twice. \( \Box \)

We now consider the exit distribution of Brownian motion from the unit ball \( B \), without considering the set \( K \). Set

\[
h_\beta(x) = \mathbb{P}^x(X_{\tau_B} \in W_\beta).
\]

This is a harmonic function on \( B \), and potential analysis provides a density function which is the \( d \)-dimensional Poisson kernel

\[
P_d(x, y) = w_{d-1}^{-1} \frac{1 - |x|^2}{|x - y|^2};
\]

here \( w_{d-1} \) is the surface area of the unit sphere with respect to the surface measure \( \sigma_d(dy) \) on the \( d \)-dimensional unit sphere. Hence \( h_\beta \) admits an integral representation given by

\[
h_\beta(x) = \int_{W_\beta \cap \partial B} P_d(x, y)\sigma_d(dy).
\]

(See [6, Theorem II.1.17]).

Our argument is based on two main steps: comparison of the values of \( h_\beta \) inside the ball and the connection of \( h_\beta \) with two functions \( u \) and \( v \). For this purpose, we compare first the values of \( h_\beta \) on the left-half of the ball, \( B^- \), with the values on the positive axes \( \{ (x_1, 0, \ldots, 0) : 0 < x_1 < 1 \} \).
Lemma 2.4. For any $x = (x_1, 0, \ldots, 0)$ with $x_1 \in (0, 1)$

$$h_\beta(0) \leq h_\beta(x).$$

Proof. Define the distance function

$$d(z, E) = \inf \{|z - y| : y \in E\}.$$

Set $A_1 = \partial B \cap \partial W_\beta$; thus $A_1$ is a $d-2$ dimensional sphere. Let $r_1 = d(x, A_1)$; by symmetry $x$ is the same distance from all points in $A_1$. Set

$$B^x = B(x, r_1).$$

This is the ball with center at $x$ whose surface crosses the surface of the unit ball through $A_1$. Then $A_\beta = \partial B \cap B^x$, and writing $\sigma'_d(\cdot)$ is the surface measure on $\partial B^x$,

$$\frac{\sigma'_d(\partial B^x - B)}{\sigma'_d(\partial B^x)} \geq \frac{\sigma_d(A_\beta)}{\sigma_d(\partial B)}.$$

Note that a Brownian motion $X$ started at $x$ and leaving the ball $B^x$ through $\partial B^x - B$ must leave the unit ball through $A_\beta$. So

$$h_\beta(x) = \mathbb{P}^x(X_{\tau_B} \in A_\beta) \geq \mathbb{P}^x(X_{\tau_B} \in \partial B^x - B) = \frac{\sigma'_d(\partial B^x - B)}{\sigma'_d(\partial B^x)} \geq \frac{\sigma_d(A_\beta)}{\sigma_d(\partial B)} = \mathbb{P}^0(X_{\tau_B} \in A_\beta) = h_\beta(0).$$

\[\Box\]

The final piece of argument is based on the comparison of the values of $h_\beta$. For this purpose, we need the following two technical lemmas.

Lemma 2.5. Let $d \geq 2$. There is a dimensional constant $\kappa_d = \frac{1}{2\sqrt{d+2}}$ so that for any $x \in [0, \kappa_d]$

$$\frac{\kappa_d - x}{\kappa_d + x} \leq \left[1 + \frac{x^2 - 2x\kappa_d}{1 + x^2 + 2x\kappa_d}\right]^{1 + \frac{d}{2}}.$$

Proof. Denote the functions on the left hand side and the right hand side of the inequality by $f(x)$ and $g(x)$, respectively. It is clear that $f(0) = g(0)$. Hence it is enough to show that for every $x \in [0, \kappa_d]$ the derivatives satisfy $f'(x) \leq g'(x)$. Here

$$f'(x) = -\frac{2\kappa_d}{(\kappa_d + x)^2};$$

$$g'(x) = -\frac{2\kappa_d(d + 2)(1 - x^2)(1 + x^2 - 2x\kappa_d)^{d/2}}{(1 + x^2 + 2x\kappa_d)^{2+d/2}}.$$ 

Now using $0 \leq x \leq \kappa_d$, it is not difficult to see that

$$-g'(x) \leq 2\kappa_d(d + 2) = \frac{2\kappa_d}{(2\kappa_d)^2} \leq \frac{2\kappa_d}{(\kappa_d + x)^2} = -f'(x).$$

This inequality together with the Mean Value Theorem leads to our result. \[\Box\]
Lemma 2.6. Suppose $\kappa_d$ is as defined before. Then for any $x, z \in [0, \kappa_d]$

$$f(x, z) := \frac{1 - x^2}{(1 + x^2 - 2xz)^{d/2}} + \frac{1 - x^2}{(1 + x^2 + 2xz)^{d/2}} \leq 2.$$ 

Proof. The result follows immediately for $x = 0$. First we fix $x \in (0, \kappa_d]$ and consider the change in the direction of $z$. It is easy to show that $\frac{\partial}{\partial z} f(x, z) \geq 0$. Therefore the derivative is negative and the function $f(\cdot, \kappa_d)$ reaches its maximum at $x = 0$ where $f(0, \kappa_d) \leq 2$. So $f(x, z) \leq f(x, \kappa_d) \leq f(0, \kappa_d) \leq 2$ for any $x, z \in [0, \kappa_d]$. \hfill \Box

The previous lemma helps us to prove the following statement.

Lemma 2.7. Let $d \geq 2$. Then for any $x \in B^-$, $h_\beta(x) \leq h_\beta(0)$.

Proof. Since $h_\beta$ is harmonic in $B$ and continuous on $\overline{B}$, it reaches its maximum on $\partial B^- = (\partial B \cap \partial B^-) \cup \{(x_1, x_2, \ldots, x_d) : x_1 = 0\} \cap \partial B^-$. Since $h_\beta$ is zero on $(\partial B \cap \partial B^-)$, it is easy to find the maximum of $h_\beta$ on the set $\{(x_1, x_2, \ldots, x_d) : x_1 = 0\} \cap \partial B$ and to show that this maximum value is bounded above by $h_\beta(0)$.

We can reduce the set of interest to an even further subset. The rotational invariance of Brownian motion together with the symmetry of the domain give that $h_\beta(x) = h_\beta(y)$ for any $x = (0, x_2, \ldots, x_d)$ and $y = (0, y_2, \ldots, y_d)$ with $|x| = |y|$. Hence we only need to consider the points of the form $x = (0, x_2, 0, \ldots, 0)$ with $x_2 \in [0, 1)$.

First note that for any $z = (z_1, \ldots, z_d) \in A_\beta$, $|z_2| < \kappa_\beta < \kappa_d$. So if $\kappa_\beta \leq x_2 < 1$ then

$$\frac{\partial h_\beta}{\partial x_2} = \frac{1}{\omega_{d-1}} \int_{A_{\kappa_d}} \frac{-2x_2|x-z|^2 - d(x_2 - z_2)(1 - x_2^2)}{|x-z|^{d+2}} \sigma_d(dz) \leq 0.$$ 

Hence $h_\beta$ is a decreasing function in $x_2$ whenever $x_2 \in (\kappa_\beta, 1)$ and

$$h_\beta((0, x_2, 0, \ldots, 0)) \leq h_\beta((0, \kappa_\beta, 0, \ldots, 0))$$ \hfill (2.4)

for $\kappa_\beta \leq x_2 < 1$.

Assume that $0 \leq x_2 < \kappa_\beta < \kappa_d$. Split the surface $A_\beta$ into $2n$ parts as follows: Take $n + 1$ non-negative numbers $\{\zeta(i)\}_{i=0}^n$ such that $0 = \zeta(0) < \zeta(1) < \zeta(2) < \cdots < \zeta(n) = \kappa_\beta$ and define strips $\{S_i\}_{i=1}^n$ in a way that

$$S_i = \{z = (z_1, \ldots, z_d) \in A_\beta : z_2 \in [\zeta(i - 1), \zeta(i))\}$$
and the measure of each strip, \( \sigma_d(S_i) \), equals each other. Similarly, define \( n \) negative numbers \( \{ \zeta(i) \}_{i=-n}^{-1} \) by \( \zeta(-i) = -\zeta(i) \) for \( i = 1, \ldots, n \). Also define the strips, \( \{ S_{-i} \}_{i=0}^{n-1} \), on the lower-half of \( A_\beta \) the same way as above

\[
S_{-i} = \{ z = (z_1, \ldots, z_d) \in A_\beta : z_2 \in [\zeta(-i-1), \zeta(-i)] \}.
\]

Then

\[
A_\beta = \bigcup_{-n+1 \leq i \leq n} S_i \quad \text{and} \quad \sigma_d(S_i) = \sigma_d(S_j) = \frac{\sigma_d(A_\beta)}{2n} \quad i, j \in \{-n+1, \ldots, n\}.
\]

Note that if \( z = (z_1, \ldots, z_d) \in S_i \) then

\[
|x-z|^2 = 1 + x_2^2 - 2x_2z_2 \geq 1 + x_2^2 - 2x_2\zeta(i).
\]

Using this partition

\[
h_\beta(x) = \int_{A_\beta} P_d(x, z) \sigma(dz) = \sum_{i=-n+1}^{n} \int_{S_i} P_d(x, z) \sigma(dz)
\]

\[
= \int_{S_0} P_d(x, z) \sigma(dz) + \int_{S_0} P_d(x, z) \sigma(dz) + \sum_{i=1}^{n-1} \left[ \int_{S_i} P_d(x, z) \sigma(dz) + \int_{S_{-i}} P_d(x, z) \sigma(dz) \right]
\]

\[
\leq \frac{1}{\omega_{d-1}} \left[ \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\kappa_d)^{d/2}} \sigma_d(S_0) + \frac{1 - x_2^2}{(1 + x_2^2)^{d/2}} \sigma_d(S_0) \right]
\]

\[
+ \frac{1}{\omega_{d-1}} \sum_{i=1}^{n-1} \left[ \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^d/2} \sigma_d(S_i) + \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(-i))^d/2} \sigma_d(S_{-i}) \right]
\]

\[
\leq \frac{\sigma_d(A_\beta)}{\omega_{d-1} \cdot 2n} \left[ \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\kappa_d)^{d/2}} + \frac{1 - x_2^2}{(1 + x_2^2)^{d/2}} \right]
\]

\[
+ \frac{\sigma_d(A_\beta)}{\omega_{d-1} \cdot 2n} \sum_{i=1}^{n-1} \left[ \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^d/2} + \frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(-i))^d/2} \right]
\]

By the restriction \( 0 \leq x_2 < \kappa_\beta < \kappa_d \), the first term is bounded by \( c_d/2n \) where \( c_d = 2\sigma_d(A_{\kappa_d})/\omega_{d-1}(1 - 2\kappa_d^2)^{d/2} \). For the term inside the second bracket, Lemma 2.6 provides an upper bound and hence

\[
\frac{1 - x_2^2}{(1 + x_2^2 - 2x_2\zeta(i))^{d/2}} + \frac{1 - x_2^2}{(1 + x_2^2 + 2x_2\zeta(i))^{d/2}} \leq 2.
\]

Hence for any \( n \in \mathbb{Z}^+ \) we obtain

\[
h_\beta(x) \leq \frac{c_d}{2n} + \frac{\sigma_d(A_\beta)}{\omega_{d-1} \cdot n} \cdot n - 1.
\]
Finally, if we take the limit as \( n \to \infty \) then
\[
h_\beta(x) \leq \frac{\sigma_d(A_\beta)}{\omega_{d-1}} = h_\beta(0).
\] (2.5)
for any \( x = (0, x_2, 0, \ldots, 0) \) with \( x_2 \in [0, \kappa_\beta) \).

By (2.4) and (2.5), we conclude that \( h_\beta(x) \leq h_\beta(0) \) for any \( x \in B^- \).

**Proof of Theorem 2.1.** Let \( r \in (0, 1] \), \( 0 < \beta \leq \alpha_d \), and write \( h \) for \( h_\beta \). Let \( u \) be the function defined in (2.1) with angle \( \beta \).

First, let \( x = (x_1, 0, \ldots, 0) \), with \( 0 < x_1 \leq 1/2 \). By the Markov property,
\[
\mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] = \mathbb{E}^{X_{\tau_K}}[h(X_{\tau_B})] = h(X_{T_K}) \leq \sup_{y \in K} h(y).
\]
By Lemma 2.4 and Lemma 2.7 we have \( h(y) \leq h(0) \leq h(x) \) for any \( y \in K \). Thus
\[
\mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] \leq h(x).
\]
So
\[
h(x) = \mathbb{E}^x[h(X_{\tau_B})]
\]
\[
= \mathbb{E}^x[h(X_{\tau_B})|T_K \leq \tau_B] \cdot \mathbb{P}^x[T_K \leq \tau_B] + \mathbb{E}^x[h(X_{\tau_B}); \tau_B < T_K]
\]
\[
\leq h(x)(1 - v(x)) + u(x).
\]
Hence
\[
c_\beta = h(0) \leq h(x) \leq \frac{u(x)}{v(x)} = \mathbb{P}^x[X_{\tau_B} \in A_\beta|\tau_B < T_K].
\] (2.6)
This proves (2.2) for \( x \) on the \( x_1 \)-axis, and by Lemma 2.3 it then follows for \( x \in W_\beta \).

Now let \( x \in B(0, 1/4) \cap H_+ \), and set \( x' = x - \pi_1(x) \). Let \( W' = x' + W_\beta \), \( A' = W' \cap \partial B(x', 1/4) \), and write \( \tau' = \tau_{B(x', 1/4)} \). Then applying (2.6) to the ball \( B(x', 1/4) - K \), we obtain
\[
c_\beta \leq \mathbb{P}^x[X_{\tau'} \in A'|\tau' < T_K] = \frac{\mathbb{P}^x[X_{\tau'} \in A'; \tau' < T_K]}{\mathbb{P}^x[\tau' < T_K]}.
\] (2.7)
Since \( \tau' < \tau = \tau_B \), this implies
\[
\mathbb{P}^x[X_{\tau'} \in A'; \tau' < T_K] \geq c_\beta v(x).
\]
Now by the standard Harnack inequality,
\[
\mathbb{P}^y[X_{\tau} \in A_\beta, \tau < T_K] \geq c_1 \text{ for } y \in A'.
\] (2.8)
Then
\[
u(x) = \mathbb{P}^x[X_{\tau} \in A_\beta, \tau < T_K]
\]
\[
\geq \mathbb{P}^x[X_{\tau} \in A_\beta, X_{\tau'} \in A', \tau' < T_K, \tau < T_K]
\]
\[
= \mathbb{E}^x[1_{(\tau' < T_K, X_{\tau'} \in A')} \mathbb{P}^{X_{\tau'}}[X_{\tau} \in A_\beta, \tau < T_K]]
\]
\[
\geq c_1 \mathbb{P}^x[\tau' < T_K, X_{\tau'} \in A'] \geq c_1 c_\beta v(x).
\]
Since we have \( u(x) \leq v(x) \) everywhere, it follows that

\[
C(\beta, d) \leq \frac{u(x)}{v(x)} \leq 1 \text{ for } x \in \mathbb{H}_+ \cap B(0, 1/2). \tag{2.9}
\]

This proves (2.3) for \( \beta \), and Theorem 2.1 then follows from Remark 2.2.

\[\square\]

3 General Case: BHP for positive harmonic functions in \( d = 2 \)

In the previous section, we proved a uniform Boundary Harnack Principle for a particular pair of harmonic functions. In this section, we give a second result of this type, where the constant in the BHP does not depend on the structure of \( \partial D \) near the boundary point 0. The result is proved for \( d = 2 \) only, and a simple counterexample which is given at the end of the section shows it does not hold for higher dimensions.

We define the quasihyperbolic metric \( k_\Omega(x, y) \) for a bounded domain \( \Omega \subset \mathbb{R}^2 \) by

\[
k_\Omega(x, y) = \inf_\gamma \int_\gamma \frac{ds(z)}{d(z, \partial \Omega)}
\]

where the infimum is taken over all rectifiable curves \( \gamma \) connecting \( x \) to \( y \) in \( \Omega \), and \( d(x, \partial \Omega) \) denotes the distance of \( x \) to the boundary of the domain \( \Omega \). It should be noted that the metric \( k_\Omega \) is closely related with Harnack inequality for positive harmonic functions. To see this, we say \( \{B(x_i, r_i)\}_{i=1}^n \) forms a Harnack chain connecting \( x \in \Omega \) to \( y \in \Omega \) if \( B(x_i, 2r_i) \subset \Omega \), \( x \in B(x_1, r_1) \), \( y \in B(x_n, r_n) \) and \( B(x_i, r_i) \cap B(x_{i+1}, r_{i+1}) \neq \emptyset \) for \( i = 1, \ldots, n-1 \). The shortest length of a Harnack chain connecting \( x \) to \( y \) is comparable to \( k_\Omega(x, y) + 1 \). Thus we can rewrite the Harnack inequality for a positive bounded harmonic function \( h \) on the domain \( \Omega \) as

\[
\exp(-c(k_\Omega(x, y) + 1)) \leq \frac{h(x)}{h(y)} \leq \exp(-c(k_\Omega(x, y) + 1)), \quad x, y \in \Omega.
\]

**Definition 3.1.** We say that \( \Omega \) satisfies the quasihyperbolic boundary condition if there exists \( x_0 \in \Omega \) and constants \( C_1 \) and \( C_2 \) such that

\[
k_\Omega(x, x_0) \leq C_1 \log(d(x_0, \partial \Omega)/d(x, \partial \Omega)) + C_2, \quad x \in \Omega. \tag{3.1}
\]

We will make use of the following two lemmas concerning quasihyperbolic metrics.

**Lemma 3.2** (Smith-Stegenga [16]). Let \( x_0 \in \Omega \) and \( \Omega \subset B(0, 2) \) satisfy the quasihyperbolic boundary condition with constants \( C_1, C_2 \). Then there exist constants \( \tau > 0 \) and \( c_1 \), depending only on \( d, C_1 \) and \( C_2 \), such that

\[
\int_\Omega \exp(\tau k_\Omega(x, x_0)) dx \leq c_1. \tag{3.2}
\]
In [16] it is only stated that the integral in (3.2) is finite; however the proof there gives that it is bounded by a constant depending only on the constants $C_1, C_2$ in (3.1), the dimension $d$, and the diameter of $\Omega$.

**Lemma 3.3.** Let $u$ be a non-negative subharmonic function on a bounded domain $\Omega \subset \mathbb{R}^2$. Suppose there is $\delta > 0$ such that

$$I_\delta := \int_{\Omega} (\log^+ u(x))^{1+\delta} \, dx < \infty.$$ 

Then

$$u(x) \leq \exp(2 + c I_\delta^{1/\delta} d(x, \partial \Omega)^{-2/\delta}),$$

where $c > 0$ is a constant depending only on $\delta$.

While the main argument of this lemma is due to Domar [11], the proof of this lemma as stated here can be found in [3, Lemma 3.1].

For the simplicity of notation, let us denote the unit disk and the disk with the center at $(1,0)$ and radius 1 by $B_0$ and $B_1$, respectively. Consider a compact and connected set $K \subset B_0 - B_1$. For $r \in (0,1)$ and $\epsilon \in (0,1)$ we define two hitting angles

$$\theta_1(r, \epsilon, K) = \inf \{ \theta \in (0, 2\pi) : \overline{B(re^{i\theta}, r\epsilon)} \cap K \neq \emptyset \},$$

$$\theta_2(r, \epsilon, K) = \sup \{ \theta \in (-2\pi, 0) : \overline{B(re^{i\theta}, r\epsilon)} \cap K \neq \emptyset \},$$

and a tube around the arc from $\theta_2$ to $\theta_1$

$$F(r, \epsilon, K) = \bigcup_{\theta_2(r, \epsilon, K) \leq \theta \leq \theta_1(r, \epsilon, K)} \overline{B(re^{i\theta}, r\epsilon)}$$

assuming there exists at least one $\theta \in (0, 2\pi)$ such that $\overline{B(re^{i\theta}, r\epsilon)} \cap K \neq \emptyset$.

**Definition 3.4.** We call a set $K \subset B_0 - B_1$ as $(r, \epsilon)$-good if there are two points $x_1 \in (\partial K) \cap \overline{B(re^{i\theta_1}, r\epsilon)}$ and $x_2 \in (\partial K) \cap \overline{B(re^{i\theta_2}, r\epsilon)}$ such that $B(x_1, r\epsilon) - K$ and $B(x_2, r\epsilon) - K$ satisfy the quasihyperbolic boundary condition. These two points $x_1$ and $x_2$ will be called roots of $K$.

In this section, we consider on an $(r, \epsilon)$-good set $K$ with roots $x_1$ and $x_2$, and with constants $C_1$ and $C_2$ used in (3.1). Unless otherwise stated all constants $c_i$ will depend only on $C_1, C_2, d$ and $\epsilon$.

The main result of this section is the following theorem.

**Theorem 3.5.** Suppose $K \subset B_0 - B_1$ is a compact and connected, $(r, \epsilon)$-good set for some $\frac{1}{4} < r < \frac{3}{4}$ and $\epsilon < \frac{1}{8}$. Suppose that $u$ and $v$ are positive, bounded and harmonic on $D = B_0 - K$, continuous on $\overline{D}$ and vanishing on $(\partial D) \cap B_0$. Then there is a positive constant $c = c(\epsilon, d, C_1, C_2)$ such that

$$\frac{u(x)/v(x)}{u(y)/v(y)} < c, \quad \text{for } x, y \in B(0, 1/32) \cap D.$$
We begin by proving a rooted local Carleson estimate.

**Lemma 3.6 (Carleson estimate).** Suppose $u$ and $K$ are as in Theorem 3.5 and $x_1$, $x_2$ are two roots of $K$ corresponding to $\theta_1$ and $\theta_2$, respectively. Let $w_0 = (r, 0) \in B_0$. Then

$$u(x) \leq c_{\epsilon, c_1, c_2} u(w_0), \quad x \in B(x_i, \epsilon/2) - K, \quad i = 1, 2.$$ 

**Proof.** We will prove this for $x_1$; the same argument also applies to $x_2$. First denote $B(re^{i\theta_1}, \epsilon) \cup B(x_1, \epsilon)$ by $\tilde{B}$. By our assumption, $u$ is a positive and bounded harmonic function in $\tilde{B} - K$ vanishing on $(\partial K) \cap \tilde{B}$. Extend $u$ to $\tilde{B}$ by setting it equal to zero on $K \cap \tilde{B}$ and take its upper-semicontinuous regularization. Without loss of generality we denote this extension by $u$ as well. This function $u$ is non-negative sub-harmonic in $\tilde{B}$ and harmonic in $\tilde{B} - K$. Then

$$I_1 = \int_{\tilde{B}} \left( \log^+ \left( \frac{u(x)}{u(re^{i\theta_1})} \right) \right)^2 \, dx = \int_{\tilde{B} - K} \left( \log^+ \left( \frac{u(x)}{u(re^{i\theta_1})} \right) \right)^2 \, dx.$$ 

Since $u$ is harmonic in $\tilde{B} - K$, we obtain using the Harnack inequality

$$\frac{u(x)}{u(re^{i\theta_1})} \leq c \exp \left( c k_{\tilde{B} - K}(x, re^{i\theta_1}) \right). \tag{3.3}$$

Now we need the following elementary inequality: given positive constants $\alpha > 0$ and $\beta > 0$

$$\log^\alpha(x) \leq \left( \frac{x^{\beta/\alpha}}{\beta/\alpha} \right)^\alpha = \left( \frac{\alpha}{\beta} \right)^\alpha x^\beta, \quad x \geq 1.$$ 

To see this, it is enough to observe that derivative of $\log(x)$ is bounded above by the derivative of $x^{\beta/\alpha}/(\beta/\alpha)$ for $x \geq 1$. Making suitable choices for $\alpha$ and $\beta$, and using this inequality with (3.3) and Lemma 3.2 gives

$$I_1 \leq c \int_{\tilde{B} - K} \exp(\tau k_{\tilde{B} - K}(x, re^{i\theta_1})) \leq c_1.$$ 

Now if we use Lemma 3.3 and restrict $x$ to $B(x_1, \epsilon/2) - K$ then $d(x, \partial \tilde{B}) \geq \epsilon/2 > \epsilon/8$ and hence

$$\frac{u(x)}{u(re^{i\theta_1})} \leq c_{\epsilon, c_1, c_2}.$$ 

Finally we can find a Harnack chain in $F(r, \epsilon, K)$ connecting two points $re^{i\theta_1}$ and $w_0$ to show that $u(re^{i\theta_1}) \leq c_{\epsilon, c_1, c_2} u(w_0)$. The result now follows. \hfill \qed

**Proof of Theorem 3.5.** First we define the following three paths which enclose the region we will work on.

$$E_1(r) := \{t x_1 + (1 - t) re^{i\theta_1} : t \in (0, 1)\},$$
$$E_2(r) := \{t x_2 + (1 - t) re^{i\theta_2} : t \in (0, 1)\},$$
$$E_3(r) := \{re^{i\theta} : \theta_2(r, \epsilon, K) \leq \theta \leq \theta_1(r, \epsilon, K)\}.$$
Let $\Omega(r, \epsilon, K)$ be the domain enclosed by the curves $E_1(r), E_2(r), E_3(r)$ and the and the set $K$. Define $E_4(r) = \partial \Omega(r, \epsilon, K) - (E_1(r) \cup E_2(r) \cup E_3(r))$. Since $u$ is harmonic in $D$, for any $x \in \{B(0, 1/32) \cap D\} \subset \Omega(r, \epsilon, K)$

$$u(x) = \int_{\partial \Omega(r, \epsilon, K)} u(y) \mathbb{P}(X_{\tau_{\Omega(r, \epsilon, K)}} \in dy) = \int_{E_1(r) \cup E_2(r) \cup E_3(r)} u(y) \mathbb{P}(X_{\tau_{\Omega(r, \epsilon, K)}} \in dy).$$

By the Carleson estimate Lemma 3.6, $u$ is bounded above by $c_{\epsilon, c_1, c_2} u(w_0)$ on $E_1(r) \cup E_2(r)$. By using a Harnack chain as in the proof of Lemma 3.6 and the regular Harnack inequality, we also obtain that $u(y) \leq c_{\epsilon, c_1, c_2} u(w_0)$ for $y \in E_3(r)$. Hence

$$u(x) \leq c_{\epsilon, c_1, c_2} u(w_0) \mathbb{P}(X_{\tau_{\Omega(r, \epsilon, K)}} \in E_1(r) \cup E_2(r) \cup E_3(r)). \quad (3.4)$$

Second, we define a tube $\tilde{F}$ around $E_1(r) \cup E_2(r) \cup E_3(r)$.

$$\tilde{F} = \bigcup_{y \in E_1(r) \cup E_2(r) \cup E_3(r)} B(y, r\epsilon/4).$$

Let $\psi$ be smooth function with compact support in $\tilde{F}$ such that $\psi = 1$ on $E_1(r) \cup E_2(r) \cup E_3(r)$. We can choose $\psi$ so that $|\Delta \psi| \leq c_d$. Then for $x \in B(0, 1/32) \cap D$

$$\int_{\partial \Omega(r, \epsilon, K)} \psi(y) \mathbb{P}(X_{\tau_{\Omega(r, \epsilon, K)}} \in dy) = \psi(x) + \int_{\Omega(r, \epsilon, K) \cap \text{supp}(\psi)} \Delta \psi(y) G_{\Omega(r, \epsilon, K)}(x, y) dy.$$

(see [2] Lemma 1) for details). This equation yields to

$$\mathbb{P}(X_{\tau_{\Omega(r, \epsilon, K)}} \in E_1(r) \cup E_2(r) \cup E_3(r)) \leq \int_{\Omega(r, \epsilon, K) \cap \tilde{F}} |\Delta \psi(y)| G_D(x, y) dy. \quad (3.5)$$

$G_D(x, \cdot)$ is a positive harmonic function in the domain $D - B(0, 1/32)$ containing $\Omega(r, \epsilon, K) \cap \tilde{F}$. Moreover,

$$\partial(\Omega(r, \epsilon, K) \cap \tilde{F}) = [\partial(\Omega(r, \epsilon, K) \cap \tilde{F}) \cap \partial D] \cup [\partial(\Omega(r, \epsilon, K) \cap \tilde{F}) \cap D]$$

$$= [\partial\Omega(r, \epsilon, K) \cap \tilde{F}] \cap \partial D] \cup [\partial\Omega(r, \epsilon, K) \cap D] \cup [\partial\tilde{F} \cap \Omega(r, \epsilon, K)].$$

Lemma 3.6 and the Harnack inequality apply to the function $G_D(x, \cdot)$ on $\partial\Omega(r, \epsilon, K) \cap D$ so that

$$G_D(x, y) \leq c_{\epsilon, c_1, c_2} G_D(x, w_0), \quad y \in \partial\Omega(r, \epsilon, K) \cap D. \quad (3.6)$$

as in the first part of the proof. Next,

$$\partial\tilde{F} \cap \Omega(r, \epsilon, K) = [\partial\tilde{F} \cap \Omega(r, \epsilon, K) \cap B(x_1, r\epsilon/2)]$$

$$\cup [\partial\tilde{F} \cap \Omega(r, \epsilon, K) \cap B(x_2, r\epsilon/2)]$$

$$\cup [\partial\tilde{F} \cap \Omega(r, \epsilon, K) - (B(x_1, r\epsilon/2) \cup B(x_2, r\epsilon/2))].$$
We note that any point on the third part is at least at a positive distance away from the boundary of $K$ and this distance is bounded below by a constant depending on $\epsilon$. On the first two parts, we can use Lemma 3.6 and on the third part we can use the Harnack inequality on a Harnack chain to prove that

$$G_D(x, y) \leq c_{\epsilon, c_1, c_2} G_D(x, w_0), \quad y \in \partial \tilde{F} \cap \Omega(r, \epsilon, K).$$

(3.7)

So by (3.6) and (3.7) we obtain

$$G_D(x, y) \leq c_{\epsilon, c_1, c_2} G_D(x, w_0), \quad y \in \partial(\tilde{F} \cap \Omega(r, \epsilon, K)),$$

and by maximum principle this inequality also holds inside the domain $\tilde{F} \cap \Omega(r, \epsilon, K)$. Then Inequality (3.5) together with the last inequality show

$$\mathbb{P}^x(X_{\tau_{\Omega(r,\epsilon,K)}} \in E_1(r) \cup E_2(r) \cup E_3(r)) \leq c_{\epsilon, c_1, c_2} c' G_D(x, w_0),$$

(3.8)

where $c' = \int_{\Omega(r,\epsilon,K) \cap \tilde{F}} |\Delta \psi(y)| \, dy$.

For the final part of the proof, denote the point $(r/2, 0)$ by $\tilde{w}_0$ and consider the circle $\partial B(\tilde{w}_0, r/8)$. By our assumption on the harmonic function $v$ and the Harnack inequality $v(z) \geq c v(\tilde{w}_0), \quad z \in \partial B(\tilde{w}_0, r/8)$.

Moreover

$$G_D(z, \tilde{w}_0) \leq c, \quad z \in \partial B(\tilde{w}_0, r/8)$$

and so

$$G_D(z, \tilde{w}_0) \leq c \frac{v(z)}{v(\tilde{w}_0)}, \quad z \in \partial (D - B(\tilde{w}_0, r/8))$$

since $v$ is positive. By the maximum principle, the last inequality holds inside the domain $D - B(\tilde{w}_0, r/8)$ which includes $B(0, 1/32) \cap D$. Using this inequality, (3.4) and (3.8), we obtain

$$u(x) \leq c_{\epsilon, c_1, c_2} u(w_0) \frac{v(x)}{v(\tilde{w}_0)} \leq c_{\epsilon, c_1, c_2} u(w_0) \frac{v(x)}{v(w_0)}$$

where we applied the Harnack inequality to compare $v(\tilde{w}_0)$ to $v(w_0)$.

Finally, if we switch the roles of $u$ and $v$ and the roles of $x$ and $y$ we also obtain

$$\frac{v(y)}{u(y)} \leq c_{\epsilon, c_1, c_2} \frac{v(w_0)}{u(w_0)}$$

which leads to the result

$$\frac{u(x)/v(x)}{u(y)/v(y)} \leq c_{\epsilon, c_1, c_2}.$$

\begin{flushright}
$\square$
\end{flushright}
We conclude this paper with an example which shows that one cannot expect a uniform BHP in higher dimensions, even if the set $K$ is contained in $\mathbb{H}_-$ and we restrict the inequality to $x, y$ in $\{ x : \pi_1(x) > 0 \}$. Let $B = B(0, 1)$, $K_0 = B \cap \mathbb{H}_0$, where $\mathbb{H}_0 = \{ x : \pi_1(x) = 0 \}$. Let $\epsilon, \delta$ be small and positive. Set

$$K = K_0 - B(0, \delta), \quad D = B(0, 1) - K.$$ 

Let $y$ be on the $x_1$ axis with $\pi_1(y) = 1/4$. Let $u_-$ and $u_+$ be the harmonic functions in $D$ with boundary condition 1 on $\partial B \cap \mathbb{H}_-$ and $\partial B \cap \mathbb{H}_+$ respectively, and zero boundary conditions elsewhere. Set $v = u_- + u_+$. So if $\tau = \tau_D$ then we can write

$$u_-(x) = \mathbb{P}^x(X_\tau \in \partial D \cap \mathbb{H}_-, \tau < T_K), \quad v(x) = \mathbb{P}^x(\tau < T_K).$$

By symmetry we have

$$\frac{u_-(0)}{v(0)} = 1/2.$$ 

On the other hand if $B' = B(0, \delta)$ then

$$\mathbb{P}^y(T_{B'} < \tau_D) \leq \mathbb{P}^y(T_{B'} < \tau_B) \leq c\delta^{d-2}.$$ 

So we have

$$v(y) \leq 1, \quad u_-(y) \leq c\delta^{d-2}.$$ 

Thus

$$\frac{u_-(0)/v(0)}{u_-(y)/v(y)} \geq c\delta^{2-d}. \quad (3.9)$$

By continuity this inequality will also hold if 0 is replaced by a point $x$ close to 0 with $\pi_1(x) > 0$.

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