Generalized $\kappa$-Deformations and Deformed Relativistic Scalar Fields on Noncommutative Minkowski Space

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May 26, 2014

Abstract

We describe the generalized $\kappa$-deformations of $D = 4$ relativistic symmetries with finite masslike deformation parameter $\kappa$ and an arbitrary direction in $\kappa$-deformed Minkowski space being noncommutative. The corresponding bicovariant differential calculi on $\kappa$-deformed Minkowski spaces are considered. Two distinguished cases are discussed: 5D noncommutative differential calculus ($\kappa$-deformation in time-like or space-like direction), and 4D noncommutative differential calculus having the classical dimension (noncommutative $\kappa$-deformation in light-like direction). We introduce also left and right vector fields acting on functions of noncommutative Minkowski coordinates, and describe the noncommutative differential realizations of $\kappa$-deformed Poincaré algebra. The $\kappa$-deformed Klein-Gordon field on noncommutative Minkowski space with noncommutative time (standard $\kappa$-deformation) as well as noncommutative null line (light-like $\kappa$-deformation) are discussed. Following our earlier proposal (see [1,2]) we introduce an equivalent framework replacing the local noncommutative field theory by the nonlocal commutative description with suitable nonlocal star product multiplication rules. The modification of Pauli–Jordan commutator function is described and the $\kappa$-dependence of its light-cone behaviour in coordinate space is explicitly given. The problem with the $\kappa$-deformed energy-momentum conservation law is recalled.

*Supported by KBN grant 5P03B05620
1 Introduction

Recently the noncommutative framework has been studied in dynamical theories along the following lines:

i) The commutative classical Minkowski coordinates $x_\mu$ one replaces by the noncommutative ones (see e.g. [3]–[5])

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}(\hat{x}). \quad (1.1)$$

In particular it has been extensively studied the case with constant $\theta_{\mu\nu}(\hat{x}) \equiv \theta_{\mu\nu}$. In such a simple case the relativistic symmetries remain classical, only the constant tensor $\theta_{\mu\nu}$ implies explicit breaking of Lorentz symmetries.

ii) One can start the considerations from the generalization of classical symmetries with commutative parameters replaced by noncommutative ones. In order to include in one step the deformations of infinitesimal symmetries (Lie-algebraic framework) and deformed global symmetries (Lie groups approach) the Hopf-algebraic description should be used, providing quantum groups which are dual to quantum Lie algebras.

The Hopf algebra framework of quantum deformations [6]–[8] has been extensively applied to the description of modified $D = 4$ space-time symmetries in 1991–97 (see e.g. [9]–[30]). There were studied mostly\(^1\) in some detail the quantum deformations with mass-like parameters, in particular the $\kappa$-deformed $D = 4$ Poincaré algebra $\mathcal{U}_\kappa(\mathcal{P})$ written in different basis (standard [9, 11, 12], bicrossproduct [17] and classical one [23]), the $\kappa$-deformed Poincaré group $\mathcal{P}_\kappa$ [16, 17] as well as $D = 4$ $\kappa$-deformed AdS and conformal symmetries [27, 31].

The $\kappa$-deformed Minkowski space $\mathcal{M}_\kappa$, described by the translation sector of the $\kappa$-deformed Poincaré group $\mathcal{P}_\kappa$, is given by the following Hopf algebra [16, 17]

$$[x^\mu, x^\nu] = \frac{i}{\kappa}(\delta^\mu_0 x^\nu - \delta^\nu_0 x^\mu) \quad (1.2a)$$

with classical primitive coproduct

$$\Delta x^\mu = x^\mu \otimes 1 + 1 \otimes x^\mu \quad (1.2b)$$

as well as classical antipode ($S(x^\mu) = -x^\mu$) and classical counit ($\epsilon(x^\mu) = 0$). We see from [17, 21] that the space-time coordinate which is “quantized” by the deformation procedure is the time coordinate $x_0$, and the nonrelativistic $O(3)$ rotations remain unchanged. By considering different contraction schemes there were proposed also the $\kappa$-deformations along one of the space axes, for example $x_3$ (this is so-called tachonic $\kappa$-deformation [30] with $O(2, 1)$ classical subalgebra). Other interesting $\kappa$-deformation is the null-plane quantum Poincaré algebra [21]\(^2\), with the “quantized” light-cone coordinate $x_+ = x_0 + x_3$ and classical $E(2)$ subalgebra. Such a $\kappa$-deformation of Poincaré symmetry in light-like direction has the following two remarkable properties:

\(^1\)Since 1993 due to Majid and Woronowicz it is known that the Drinfeld-Jimbo deformation of Lorentz algebra with dimensionless parameter $q$ can not be extended to $q$-deformation of Poincaré algebra without introducing braided tensor products (see e.g. [14]).

\(^2\)We shall further call this algebra null-plane $\kappa$-deformed Poincaré algebra. The deformation presented in [21] is the particular case of generalized $\kappa$-deformations of Poincaré algebra, considered in [24, 26, 30].
i) The infinitesimal deformations of null-plane $\kappa$-deformed algebra are described by classical $r$-matrix satisfying the classical Yang-Baxter equation (CYBE). As a consequence, this deformation can be extended to larger $D = 4$ conformal symmetries \[27, 31\].

ii) For null-plane $\kappa$-deformed Minkowski space the bicovariant differential calculus is four-dimensional, with one-forms spanned by the standard differentials $dx_\mu$ (see \[26\]) and subsequently the differentials are not coboundary, similarly as in the classical case (see Sect. 3). We see therefore that the “null-plane” $\kappa$-deformation provides an example of 4D differential calculus on quantum Minkowski spaces considered by Podleś \[29\].

The generalized $\kappa$-deformation of $D = 4$ Poincaré symmetries were proposed in \[25\]. They are obtained by introducing an arbitrary symmetric Lorentzian metric $g_{\mu\nu}$ with signature $(+,−,−,−)$. Let us observe that the change of the linear basis in standard Minkowski space with Lorentz metric tensor $\eta_{\mu\nu} = \text{diag}(1,−1,−1,−1)$ implies the following replacement of the Lorentzian metric

$$x_\mu \rightarrow y_\mu = R_{\mu}^\nu x_\nu$$ (1.3)

implies the following replacement of the Lorentzian metric

$$\eta^{\mu\nu} \rightarrow g^{\mu\nu} = R^\rho_{\mu}R^\rho_{\nu}R^\nu_{\rho} = (R^\mu_{\rho})^T.$$ (1.4)

In \[25\] there was proposed a deformation of $D = 4$ Poincaré group with arbitrary Lorentzian metric $g^{\mu\nu}$. In such a case the deformed, “quantum” direction in standard Minkowski space is described by the coordinate $y_0 = R_0^\nu x_\nu$, where $R_0^\nu$ is chosen in such a way that the relation \[1.4\] is valid. In particular one can choose

i) For tachyonic $\kappa$-deformation ($\kappa$-deformed $x_3$-direction)

$$R = \begin{pmatrix}
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \rightarrow
g = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$ (1.5a)

ii) For null-plane $\kappa$-deformation ($\kappa$-deformed $x_+ = x_0 − x_3$)

$$R = \begin{pmatrix}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} \rightarrow
g = \begin{pmatrix}
0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}$$ (1.5b)

The plan of our presentation is the following:

In Sect. 2, following \[25\] we shall present the $\kappa$-deformation $U_\kappa(\mathcal{P}(g_{\mu\nu}))$ of the Poincaré algebra $\mathcal{P}(g_{\mu\nu})$ describing the group of motions in space-time with arbitrary constant metric $g_{\mu\nu}$ with signature $(+,−,−,−)$. The generators $M^{\mu\nu}$, $P_\mu$ ($P^\mu = g^{\mu\nu}P_\nu$) satisfy the following algebra:

$$[M^{\mu\nu}, M^{\rho\tau}] = i(g^{\mu\tau}M^{\nu\rho} − g^{\nu\tau}M^{\mu\rho} + g^{\nu\rho}M^{\mu\tau} − g^{\mu\rho}M^{\nu\tau}),$$ (1.6a)
\[ [M^{\mu\nu}, P_\rho] = i(\delta^\rho_\mu P^{\mu} - \delta^\mu_\rho P^{\nu}), \quad (1.6b) \]
\[ [P_\mu, P_\rho] = 0. \quad (1.6c) \]

where \( g^{\mu\nu} \) is the symmetric metric with the Lorentz signature. The deformed algebra will take the form of bicrossproduct Hopf algebra \([12, 32]\) which permits by duality the description of corresponding \( \kappa \)-deformed quantum Poincaré group \( \mathcal{P}(g_{\mu\nu}) \). In Sect. 2 we shall also describe the deformation map and its inverse, generalizing the results of ref. \([23]\) to the case of arbitrary metric \( g_{\mu\nu} \) (see also ref. \([33]\)). In Sect. 3 we shall describe the bicovariant differential calculus on \( \kappa \)-deformed Minkowski space; in particular, it will be explained that in the case \( g_{00} = 0 \) the calculus has a classical dimension and the Podleś condition \( F = 0 \) selecting the fourdimensional differential calculi in \( D = 4 \) \([29]\) is satisfied. Further, in Sect. 4, we shall introduce left and right vector fields on noncommutative \( \kappa \)-Minkowski space and, subsequently, write down the realizations of \( \kappa \)-deformed Poincaré algebra on \( \kappa \)-deformed Minkowski space. Exploiting duality we shall explain the relation between the realizations on noncommutative \( \kappa \)-Minkowski space and known realizations (see \([11, 12, 34, 35]\)) on commutative fourmomentum space. In Sect. 5 we shall describe the noncommutative plane wave decomposition of the Klein-Gordon (KG) equation on \( \kappa \)-Minkowski space using normally ordered exponentials \([17, 36]\) for the standard \( \kappa \)-deformation as well as for the light-cone \( \kappa \)-deformation. We shall show the relation of this approach to the technique using nonordered exponentials, proposed by Podleś in \([29]\). Further, in Sect. 6, we shall discuss an equivalent nonlocal K–G action on classical Minkowski space. In this commutative framework we shall calculate the deformation of Pauli–Jordan function, describing \( \kappa \)-deformed second-quantized free KG field and describe the \( \kappa \)-deformed behaviour around the light-cone. In Sect. 7 we present final remarks.

One should observe that recently the algebraic framework of \( \kappa \)-deformed symmetries as well as some elements of \( \kappa \)-deformed differential calculus were employed for the description of so–called doubly special relativistic (DSR) theories (see e.g. \([37]\)–\([44]\)). One of the aims of this paper is to provide some theoretical background for these more phenomenologically - oriented considerations.

## 2 \( \kappa \)-deformed Poincaré Algebra \( \mathcal{U}_\kappa(\hat{\mathcal{P}}(g_{\mu\nu})) \) and \( \kappa \)-deformed Poincaré Group \( \mathcal{P}_\kappa(g_{\mu\nu}) \) in Arbitrary Basis

The \( \kappa \)-deformation of the classical algebra \((1.6a)-(1.6c)\) is generated by the following \( r \)-matrix \([25]\)\(^3\)
\[ r = \frac{i}{\kappa} M_{0\mu} \wedge P^\mu = \frac{i}{\kappa} g^{\mu\nu} M_{0\mu} \wedge P^\nu. \quad (2.1) \]

The relations \((1.6a)\) and \((1.6c)\) remain unchanged. The cross-product relation \((1.6b)\) is deformed in the following way (we denote the \( \kappa \)-deformed generators by \( \mathcal{M}^{\mu\nu} = (\mathcal{M}^{ij}, \mathcal{M}^{i0}); \) \( \mathcal{P}_\mu = (\mathcal{P}_i, \mathcal{P}_0); \) \( i, j = 1, 2, 3)\):
\[ [\mathcal{M}^{ij}, \mathcal{P}_0] = 0, \quad (2.2a) \]

\(^3\)In implicit form the relation \((2.1)\) is present in \([20]\).
\[ [\mathcal{M}_{ij}, \mathcal{P}_k] = i\kappa (\delta^i_k g^{0i} - \delta^i_k g^{0j})(1 - e^{\frac{\mathcal{P}_0}{\kappa}}) + i(\delta^i_j \mathcal{P}^i - \delta^i_k \mathcal{P}^j), \quad (2.2b) \]

\[ [\mathcal{M}_{i0}, \mathcal{P}_0] = i\kappa g^{i0}(1 - e^{-\frac{\mathcal{P}_0}{\kappa}}) + i\mathcal{P}^i, \quad (2.2c) \]

\[ [\mathcal{M}_{i0}, \mathcal{P}_k] = -\frac{i\kappa}{2} g^{00} \delta^i_k (1 - e^{-2\mathcal{P}_0}) - i\delta^i_k g^{0s} \mathcal{P}_s e^{-\frac{\mathcal{P}_0}{\kappa}} + ig^{ai} \mathcal{P}_k (e^{-\frac{\mathcal{P}_0}{\kappa}} - 1) - \frac{i}{2\kappa} \delta^i_k \mathcal{P}_j \mathcal{P}^j - \frac{i}{\kappa} \mathcal{P}^i \mathcal{P}_k, \quad (2.2d) \]

where \( \mathcal{P}^k \equiv g^{kl} \mathcal{P}_l \) \((k, l) = 1, 2, 3\). The coproducts are the following:

\[ \Delta \mathcal{P}_0 = \mathcal{P}_0 \otimes 1 + 1 \otimes \mathcal{P}_0, \quad (2.3a) \]

\[ \Delta \mathcal{P}_i = \mathcal{P}_i \otimes e^{-\frac{\mathcal{P}_0}{\kappa}} + 1 \otimes \mathcal{P}_i, \]

\[ \Delta \mathcal{M}_{ij} = \mathcal{M}_{ij} \otimes 1 + 1 \otimes \mathcal{M}^{ij}, \quad (2.3b) \]

\[ \Delta \mathcal{M}_{i0} = \mathcal{M}_{i0} \otimes e^{-\frac{\mathcal{P}_0}{\kappa}} + 1 \otimes \mathcal{M}^{i0} + \frac{1}{\kappa} \mathcal{M}^{ij} \otimes \mathcal{P}_j, \quad (2.3c) \]

The antipodes and counits are

\[ S(\mathcal{P}_0) = -\mathcal{P}_0, \]

\[ S(\mathcal{P}_i) = e^{-\frac{\mathcal{P}_0}{\kappa}} \mathcal{P}_i, \quad (2.4a) \]

\[ S(\mathcal{M}_{ij}) = -\mathcal{M}^{ij}, \]

\[ S(\mathcal{M}_{i0}) = -e^{-\frac{\mathcal{P}_0}{\kappa}} (\mathcal{M}^{i0} + \frac{1}{\kappa} \mathcal{M}^{ij} \mathcal{P}_j), \quad (2.4b) \]

\[ \epsilon(\mathcal{M}^{\mu\nu}) = \epsilon(\mathcal{P}^\mu) = 0. \quad (2.5) \]

The Schouten bracket \([\cdot, \cdot]\) (see e.g. [32]) describing the modification of CYBE for the classical \( r \)-matrix \((2.1)\) is the following [25]

\[ [r, r] = i \frac{g^{00}}{\kappa} \mathcal{M}_{\mu\nu} \wedge \mathcal{P}^\mu \wedge \mathcal{P}^\nu. \quad (2.6) \]

The relation \((2.6)\) explains why the null-plane \( \kappa \)-deformation (see \((1.5b)\)) with \( g_{00} = 0 \) is described by CYBE.

The formulae \((2.2a) - (2.3c)\) describe the \( \kappa \)-deformation in bicrossproduct basis.

Further we would like to make the following comments:

i) The fourmomentum Hopf algebra described by \((1.6c)\) and \((2.3a)\) does not depend on the metric \( g^{\mu\nu} \). The Hopf algebra \((1.2a) - (1.2b)\) describing the translation sector of \( \hat{\mathcal{P}}(g_{\mu\nu}) \) as well as \( \kappa \)-deformed Minkowski space is dual to the fourmomentum Hopf algebra and is also metric-independent.

ii) The Lorentz sector of the \( \kappa \)-deformed Poincaré group is classical for any metric \( g_{\mu\nu} \)

\[ [\Lambda_\mu^\nu, \Lambda_\rho^\tau] = 0, \quad \Lambda_\mu^\nu \Lambda_\rho^\tau = 69^\tau, \quad (2.7) \]
\[ \Delta(\Lambda_{\mu}^{\nu}) = \Lambda_{\mu}^{\rho} \otimes \Lambda_{\rho}^{\nu}. \]  

(2.8)

The metric-dependent term occur only in the cross relation between the Lorentz group and translation generators. Using duality relations one obtains [25]:

\[ [\Lambda_{\rho}^{\tau}, a^{\mu}] = -\frac{i}{\kappa} \{ (\Lambda^{\rho} g_{\nu} - \delta^{\rho}_{\nu}) \Lambda_{\tau}^{\mu} + (\Lambda^{\mu}_{\nu} g_{\rho} - g_{\nu} g_{\rho} g^{\rho}_{\mu} \}. \]  

(2.9)

iii) In order to introduce the physical space-time basis, with standard Lorentz metric, one should introduce the inverse formulae to the ones given by (1.3). In such a case one obtains e.g. for null-plane basis given by (1.5b) the light cone coordinates usually used for the description of null plane relativistic kinematics (see e.g. [45]):

\[ y_0 = \frac{1}{\sqrt{2}}(x_0 - x_3), \quad x_0 = \frac{1}{\sqrt{2}}(y_0 - y_3), \]
\[ y_3 = -\frac{1}{\sqrt{2}}(x_0 + x_3), \quad x_3 = -\frac{1}{\sqrt{2}}(y_0 + y_3), \]
\[ y_2 = x_2, \quad y_1 = x_1. \]  

(2.10)

Similarly one can write the relation between the \( M_{\mu\nu} \) generators and the physical Lorentz generators \( M_{\mu\nu} \):

\[ M_{\mu\nu} = R_{\mu\rho} R_{\nu\sigma} M_{\rho\sigma}. \]  

(2.11)

iv) The mass Casimir for arbitrary metric \( g_{\mu\nu} \) is given by the following formula [26]

\[ \mathcal{M}^2(P_\mu) = g^{00}(2\kappa \sinh P_0 P_0) \quad [2.12] \]

Substituting the metric (1.5b) in (2.10) one gets the mass Casimir for null-plane \( \kappa \)-Poincaré algebra, obtained firstly in [21].

v) One can extend the deformation maps, written in [23] for standard Lorentz metric \( g^{\mu\nu} = \eta^{\mu\nu} \) and express the fourmomenta generators \( P_\mu, M_{\mu\nu} \) satisfying (2.2a)-(2.2d) in terms of the classical Poincaré generators \( P_\mu, M_{\mu\nu} \) satisfying (1.6a)-(1.6c) (deformation map) and write down the inverse formulae. We put \( M_{\mu\nu} = M^{\mu\nu} \), and for the fourmomenta sector the generalization of the formulae [23] for general \( g^{\mu\nu} \) looks as follows (see also [33]):

a) Deformation map

\[ P_0 = \kappa \ln \left( \frac{P_0 + C}{C - g_{00} A} \right), \]  

(2.13a)

\[ P_i = \frac{\kappa P_i}{P_0 + C} + \frac{\kappa A}{P_0 + C} g_{i0}, \]  

(2.13b)

where

\[ g_{00} A^2 (M^2) - 2 A (M^2) C (M^2) + M^2 = 0 \]  

(2.14)

and

\[ M^2 = g^{\mu\nu} P_\mu P_\nu. \]  

(2.15)
One can calculate that
\[
\mathcal{M}^2 = 4\kappa^2 \frac{A^2}{M^2 - g_{00}A^2},
\]
\[
M^2 = A^2(\frac{4\kappa^2}{M^2} + g_{00}).
\] (2.16)

If \( g_{00} \neq 0 \) one can put the relation (2.14) in the form
\[
\tilde{A}^2 - \frac{1}{g_{00}} = M^2,
\] (2.17)
where
\[
\tilde{A} = (g_{00})^{-\frac{1}{2}}(C - g_{00}A).
\] (2.18)

In particular if we choose
\[
\tilde{A} = \kappa, \quad C = g_{00}\sqrt{M^2 + \kappa^2}
\] (2.19)
one gets
\[
\mathcal{M}^2 = \frac{2\kappa^2}{g_{00}} \left( -1 + \sqrt{1 + \frac{M^2}{\kappa^2}} \right),
\]
\[
M^2 = g_{00}\mathcal{M}^2(1 + g_{00}\frac{M^2}{4\kappa^2}),
\] (2.20)
and for \( g_{00} = 1 \) one obtains the formulae given in [23].

If \( g_{00} = 0 \) it follows from (2.14) that \( A = \frac{\mathcal{M}^2}{2C(M^2)} \); the simplest choice is provided by
\[
C = \kappa, \quad A = \frac{M^2}{2\kappa}
\] (2.21)
In such a case the relation (2.16) takes the simplest possible form
\[
\mathcal{M}^2 = M^2.
\] (2.22)

b) Inverse deformation map. One obtains
\[
P_0 = (C - g_{00}A)e^{\frac{P_0}{\kappa}} - C,
\] (2.23a)
\[
P_i = \frac{C - g_{00}A}{\kappa}e^{\frac{P_0}{\kappa}} P_i - g_{i0}A.
\] (2.23b)

In particular if \( g_{00} = 0 \) and we choose \( A \) and \( C \) as in the formulae (2.21), one gets the formulae
\[
P_0 = \kappa(e^{\frac{P_0}{\kappa}} - 1),
\] (2.24a)
\[
P_i = e^{\frac{P_0}{\kappa}} P_i - g_{i0}\frac{\mathcal{M}^2}{2\kappa}.
\] (2.24b)

vi) One can calculate the invariant volume element

\[
\mathcal{M} = \varepsilon_{\alpha\beta\sigma\tau} P_{\alpha} P_{\beta} P_{\sigma} P_{\tau}.
\]
\[ d^4\mathcal{P} = \det \left( \frac{\partial P}{\partial P} \right) d^4 P , \quad (2.25) \]

in the deformed four-momentum space. For simplicity we shall put \( g_{00} = 1 \) and \( g_{0i} = 0 \) in the formulae (2.13a), (2.13b). One gets from (2.13)

\[ \det \left( \frac{\partial P}{\partial P} \right) = \frac{(C - A)^4}{2\kappa^4|CA' - AC'|} e^{\frac{3\kappa_0}{\kappa}} , \quad (2.26) \]

where \( A' = \frac{dA}{dM^2} \) and \( C' = \frac{dC}{dM^2} \). Because the functions \( A, C \) are Lorentz-invariant, the invariant measure in deformed four-momentum space takes the form

\[ d^4\mu(P) = e^{\frac{3\kappa_0}{\kappa}} d^4 P . \quad (2.27) \]

It should be mentioned that the formula (2.27) can be also derived \[46\] from the Hopf algebraic scheme without employing the explicit formula for the deformation map.

### 3 Bicovariant Differential Calculus and Vector Fields on \( \kappa \)-deformed Minkowski Space

Because the \( \kappa \)-Minkowski space \( \mathcal{M}_\kappa \), described by the relations (1.2a)-(1.2b), is a unital \(^\ast\)-Hopf algebra, one can define on this noncommutative space–time the bicovariant differential calculus \[47, 48, 26\]. Following the results obtained in \[24\] Podleś investigated differential calculi on more general deformations of the Minkowski space algebra by considering the generators \( y_\mu \) satisfying the relations \[29\]

\[ (R - 1)^{\mu\nu\rho\tau}(y_\rho y_\tau - Z^{\mu\nu\rho} y_\rho + T^{\mu\nu\rho}) = 0 . \quad (3.1) \]

The algebra (1.2a)-(1.2b) is obtained by putting \( R = \tau \) i.e. \( R^{\mu\nu\rho\tau} = \delta^{\mu\tau}\delta^{\nu\rho}, T^{\mu\nu\rho\tau} = 0 \) and

\[ Z^{\mu\nu\rho} = \frac{i}{\kappa} (\delta^\mu_0 g^\nu_\rho - \delta^\nu_0 g^\mu_\rho) . \quad (3.2) \]

Podleś in \[29\] looked for the condition restricting the algebra (3.1) which implies the existence of four-dimensional covariant differential calculus.\(^4\) It appears that one can relate the problem of existence of 4-dimensional covariant calculus with the vanishing of a certain constant four-tensor \( F \). Using the bicovariance (with respect to the coproduct on the Minkowski space), covariance (with respect to the Poincaré group coaction) and the algebra commutation relations (1.1a) we determine that in our case the Podleś condition reads:

\[ F^\mu_\rho_\nu\sigma = (\frac{i}{\kappa})^2 g_{00}(g^{\nu_\rho \delta^\mu_\sigma} - g^{\mu_\rho \delta^\nu_\sigma}) = 0 . \quad (3.3) \]

We see that the condition \( g_{00} = 0 \) implies the classical dimension \( D = 4 \) of differential calculus; following \[24, 28\] one obtains that for \( g_{00} \neq 0 \) we get \( D = 5 \).

\(^4\) In \[29\] there are considered only covariant differential calculi. It appears however that for the case of \( \kappa \)-deformed Minkowski space (1.2a)-(1.2b) the Podleś differential calculi are bicovariant for any metric \( g_{\mu\nu} \).
Using the general construction of bicovariant \( \ast \)-calculi by Woronowicz [47] two cases \( g_{00} \neq 0 \) and \( g_{00} = 0 \) should be considered separately. We recall that the bicovariant noncommutative \( \ast \)-differential calculus on \( \kappa \)-Minkowski space \( \mathcal{M}_\kappa \) is obtained if we choose in the algebra of functions on \( \mathcal{M}_\kappa \) the ideal \( R \) which satisfies the properties

i) \( R \subset \ker \varepsilon \),

ii) \( R \) is ad-invariant under the action of \( \mathcal{U}_\kappa (\mathcal{P}) \)

iii) \( a \in R \Rightarrow S(a)^* \in R \).

We do not intend to elaborate here on the Woronowicz theory; we would like however to provide here short more intuitive description. To make things more clear let us appeal to the classical case. In order to define covariant calculus on Lie group it is sufficient to consider (co-)tangent space at one point, say the group unit. To construct vectors one can take the Taylor expansion of any function around \( e \). The derivatives are then given by linear terms in such an expansion. The value of \( f \) at \( e \) is irrelevant so we can assume \( f(e) = 0 \) this is the origin of \( \ker \varepsilon \) in the above definitions. Also, higher order terms are irrelevant; this can be taken into account by considering the ideal which consists of functions with their Taylor expansion starting from quadratic or higher order terms (this is counterpart of \( R \) above) and dividing out by it. The Woronowicz construction is a straightforward generalization of ”classical” procedure. The differential calculi are described by the nontrivial generators which span \( \Delta = \frac{\ker \varepsilon}{R} \).

It appears that in case of \( \kappa \)-Minkowski space one can introduce the following basis in \( R \) satisfying the properties i) – iii)

\[
X^{\mu \nu} = y^\mu y^\nu + \frac{i}{\kappa} (g^{\mu \nu} y_0 - \delta^\mu_0 y^\nu).
\]

(3.4)

In addition, for \( g_{00} \neq 0 \) \( R = \ker \varepsilon \) and in order to obtain nontrivial \( D \) one has to reduce the basis (3.5) by substracting the trace

\[
\tilde{X}^{\mu \nu} = X^{\mu \nu} - \frac{1}{4} g^{\mu \nu} X^\rho_\rho.
\]

(3.5)

We obtain two distinct cases:

a) \( g_{00} \neq 0 \)

In such a case using the kernel \( R \) with basis (3.5) one gets \( D \) span by five generators \((y_0, y_i, \varphi = y^\mu y_\mu + \frac{2i}{\kappa} y_0) \). Using general techniques presented in [17], one obtains the following five-dimensional basis of bi-invariant forms

\[
\omega^\mu = dy^\mu, \quad \Omega = d\varphi - 2y_\alpha dy^\alpha.
\]

(3.6)

The commutation relations between the one-forms and the generators \( y^\mu \) of the algebra of functions \( f(y^\mu) \) an \( \kappa \)-Minkowski space are the following

\[
[dy^\mu, y^\nu] = \frac{i}{\kappa} g^{0 \mu} dy^\nu - \frac{i}{\kappa} g^{\mu \nu} dy^0 + \frac{1}{4} g^{\mu \nu} \Omega,
\]

(3.7a)

\[
[\Omega, y^\mu] = -\frac{4}{\kappa^2} g_{00} dy^\mu.
\]

(3.7b)
We see from the relation (3.7b) that the differential $dy^\mu$ is a coboundary, i.e.

$$dy^\mu = -\frac{\kappa^2}{4g_{00}} [\Omega, y^\mu] \quad (3.8)$$

and the classical limit $\kappa \to \infty$ is singular. The exterior products of one-forms remains classical

$$dy^\mu \wedge dy^\nu = -dy^\nu \wedge dy^\mu,$$
$$\Omega \wedge dy^\mu = -dy^\mu \wedge \Omega \quad (3.9)$$

and the Cartan-Maurer equation for $\Omega$ takes the form:

$$d\Omega = -2dy^\mu \wedge dy^\mu. \tag{3.10}$$

b) $g_{00} = 0$

In such a case the kernel $R$ can be chosen with basis (3.4) and one obtains $\mathcal{D}$ span by four generators $(y_i, y_0)$. One gets the fourdimensional differential calculus and the basic one-forms are the differentials $dy^\mu$ satisfying the relation

$$[y^\mu, dy^\nu] = \frac{i}{\kappa} (g^{\mu\nu} g_{0i} dy^i - \delta^\nu_0 dy^\mu). \quad (3.11)$$

Because the one-form $\Omega$ commutes with the one-forms $dy^\mu$ (see (3.8)), the deformed calculus similarly like in classical case is not a coboundary one.

The relation (3.11) is the only one which is deformed — other relations of the differential calculus remain classical.

It should be added that the left action of $\kappa$-Poincaré group $\mathcal{P}_\kappa$ on $\kappa$-Minkowski space

$$\rho_L(y^\mu) = \Lambda^\mu_\nu \otimes y^\nu + a^\mu \otimes I \tag{3.12}$$

describe the homomorphism and introduce the covariant action on the one-forms

$$\tilde{\rho}_L(\omega^\mu) = \Lambda^\mu_\nu \otimes \omega^\nu,$$
$$\tilde{\rho}_L(\Omega^\mu) = 1 \otimes \omega^\mu. \quad (3.13)$$

The relations (3.7a)-(3.7b) (as well as (3.11)) are covariant under the action of $\mathcal{P}_\kappa$ given by (3.12)-(3.13). One can introduce the right action of the $\kappa$-Poincaré group which is a homomorphism

$$\rho_R(y^\mu) = y^\nu \otimes \tilde{\Lambda}^\mu_\nu - 1 \otimes \tilde{a}^\nu \tilde{\Lambda}_\nu^\mu, \tag{3.14}$$

if $(\tilde{\Lambda}^\mu_\nu, \tilde{a}^\nu)$ are the generators of the quantum Poincaré group $\mathcal{P}^{(g_{\mu\nu})}_{-\kappa}$, satisfying the relations of $\mathcal{P}_\kappa^{(g_{\mu\nu})}$ with changed sign of $\kappa$. 

10
4 \( \kappa \)-deformed Vector Fields and Differential Realizations of \( \kappa \)-deformed Poincaré Algebra on \( \kappa \)-Minkowski Space \( \mathcal{M}_\kappa \)

Let us introduce on \( \mathcal{M}_\kappa \) a polynomial function \( f(y) \) of four variables \( y_\mu \), which formally can be extended to an analytic function. The product of generators \( (y_i, y_0) \) will be called normally ordered \[12\] if all generators \( y_0 \) stay to the left\(^5\). In such a way one can uniquely relate with any analytic function \( f(y) \) on \( \mathcal{M}_\kappa \) other function : \( f(y) : \).

a) \( \kappa \)-deformed vector fields

In the general case the differential of any function \( f \) is described by five partial derivatives. If we choose the left derivatives, we obtain

\[
    df = \partial_\mu f dy^\mu + \partial_\Omega f \cdot \Omega = X_\mu : f : dy^\mu + X_\Omega : f : \Omega , \tag{4.1}
\]

In order to obtain the explicit form of the vector fields \( \chi_\mu, \chi_\Omega \) one can follow the straightforward although rather tedious strategy. We take the differential of any normally ordered \( f \) and then use the commutation rules (1.1a) and (3.8) to get again normally ordered expression and differentials standing to the right. The results reads

\[
    X_0 : f : = -i : [\kappa(e^{\frac{i}{2}a_0} - 1) - \frac{\eta_{00} M^2}{2\kappa}(\frac{1}{i} \partial_\mu)] f(y) :,
\]
\[
    X_i : f : = [e^{\frac{i}{2}a_0} \partial_i + ig_{i0} \tau^2(\frac{1}{i} \partial_\mu)] f(y) :, \tag{4.2}
\]
\[
    X_\Omega : f : = -\frac{1}{8} \tau^2(\frac{1}{i} \partial_\mu) f(y) :,
\]

where \( \tau^2 \) is given by the formula (2.12).

It is interesting to observe that the formulae (4.2) can be written in the following way

\[
    X_\mu : f := P_\mu(\frac{1}{i} \partial_\mu) f(y) :, \tag{4.3}
\]

where the relations \( P_\mu(\frac{1}{i} \partial_\mu) \) are obtained from (2.23a)-(2.23b) (see also (2.24a)-(2.24b)) by substituting \( P_\mu = \frac{1}{i} \partial_\mu \). Indeed, in [12] it has been shown firstly for \( g_{\mu\nu} \equiv \eta_{\mu\nu} \) that

\[
    P_\mu : f(y) := \frac{1}{i} : \frac{\partial}{\partial y^\mu} f(y) :. \tag{4.4}
\]

The relation (4.4) remains valid for any choice of the metric \( g_{\mu\nu} \). Note that (4.3) and (4.4) provide the relation between Woronowicz [47] and duality-inspired bases in the ”Lie-algebra” of \( \kappa \)-Poincaré group, given in [25, 48].

Simpler formalism is obtained for \( \eta_{00} = 0 \), when \( dx^\mu \) describes the basic one-forms. In such a case the formula (4.2) can be shortened, and one obtains

\[
    df = \partial_\mu f dy^\mu = X_\mu : f : dy^\mu . \tag{4.5}
\]

\(^5\)Firstly such ordering was proposed in [50] and applied to \( \kappa \)-Poincaré in [17]; see also [36, 46].
We shall consider further the case \( g_{00} = 0 \) and the choice of the parameters in the inverse deformation map occurring in \[2.21\]. Because we define classical fields in normally ordered form, we shall write down also the formulae for multiplying the normally ordered functions by the coordinate \( y^\mu \) from the left as well as from the right. One gets

\[
y^\mu_L f(y) \equiv y^\mu f(y) =: [y^0 \delta_0^\nu (1 - e^{-\frac{i}{\kappa} \partial_0}) + y^\mu e^{-\frac{i}{\kappa} \partial_0}] f(y) :,
\]

\[
y^\mu_R f(y) \equiv f(y) y^\mu =: (y^\mu - \frac{i}{\kappa} \delta_0^\mu y^k \partial_k) f(y) :.
\]

The commutation relations between the coordinates \( y^\mu_L \) and \( y^\mu_R \) and left partial derivatives \( \partial_\mu \) are described by two sets of formulas. Simpler relations are obtained for the multiplicative operators \( y^\mu_R \), acting on the right

\[
[\partial_\mu, y^\nu_R] = \delta_\mu^\nu + \frac{i}{\kappa} (\partial_0 \delta_\mu^\nu - g_{0\mu} g^{\mu\rho} \partial_\rho) \tag{4.7}
\]

For completeness we write also the other relations

\[
[\partial_0, y^0_L] = 1 + \frac{i}{\kappa} \partial_0 ,
[\partial_0, y^j_L] = 0 ,
[\partial_i, y^0_L] = \frac{i}{\kappa} (\partial_i - g_0 (1 + \frac{i}{\kappa} \partial_0)^{-1} (g^{0\mu} \partial_\mu + \frac{2}{\kappa} g^{\mu\rho} \partial_\mu \partial_\rho)) ,
[\partial_i, y^j_L] = \delta^j_i - \frac{i}{\kappa} g_{0i} (1 + \frac{i}{\kappa} \partial_0)^{-1} g^{\mu\rho} \partial_\mu . \tag{4.8}
\]

Let us observe that

i) The relations \[4.8\] contain nonlocal operators \((1 + \frac{i}{\kappa} \chi_0)^{-1}\), which are however well defined.

ii) One can also introduce the right partial derivatives, replacing \[4.1\] with the following formulae:

\[
df = dy^\mu \tilde{\partial}_\mu f = dy^\mu \tilde{\chi}_\mu : f : . \tag{4.9}
\]

Both partial derivatives \( \partial_\mu \), \( \tilde{\partial}_\mu \) are related by the \(*\)-operation (Hermitean conjugation) in the Hopf algebra \[1.2a\]-\[1.2b\]

\[
\partial_\mu f = (\tilde{\partial}_\mu f^*)^* . \tag{4.10}
\]

The vector fields \( \tilde{\chi}_\mu \) satisfy simpler relations with the coordinates \( y^\nu_L \).

**b) Differential realizations of \( \kappa \)-Poincaré algebra \( \mathcal{U}_\kappa(\mathcal{P}(g_{\mu\nu})) \) on \( \kappa \)-deformed Minkowski space**

The differential realizations of \( \kappa \)-Poincaré algebra have been firstly given for standard choice of the metric \((g_{\mu\nu} = \eta_{\mu\nu})\) on commuting fourmomentum space \[11, 12, 34, 35\]. In spinless case this realization can be extended to the case of arbitrary metric \( g_{\mu\nu} \) (see the formulae \[2.2a\]-\[2.2d\]) in the following way:

\[
\mathcal{P}_\mu \tilde{f}(p) = p_\mu \tilde{f}(p) , \tag{4.11}
\]
\[ \mathcal{M}^{ij} \tilde{f}(p) = i\{\kappa(g^{0i} \frac{\partial}{\partial p_j} - g^{0j} \frac{\partial}{\partial p_i})(1 - e^{-i \frac{\phi}{\alpha}}) - (g^{is}p_s \frac{\partial}{\partial p_j} - g^{is}p_s \frac{\partial}{\partial p_i})\} \tilde{f}(p), \]

\[ \mathcal{M}^{00} \tilde{f}(p) = i\{[\kappa g^{00}(1 - e^{-i \frac{\phi}{\alpha}}) + g^{ik}p_k] \frac{\partial}{\partial p_0} - \frac{i\kappa}{2} g^{00}(1 - e^{-2i \frac{\phi}{\alpha}}) + g^{0s}p_s e^{-i \frac{\phi}{\alpha}} \} \frac{\partial}{\partial p_i} + g^{0i}(e^{-i \frac{\phi}{\alpha}} - 1)p_k \frac{\partial}{\partial p_s} + \frac{1}{2\kappa} g^{rs}p_rp_s \frac{\partial}{\partial p_i} \frac{\partial}{\partial p_k} \} \tilde{f}(p). \] (4.13)

From the basic duality relation [12, 34]

\[ \langle \tilde{f}(\mathcal{P}), : f(y) : \rangle = \tilde{f}\left(\frac{1}{i} \frac{\partial}{\partial y}\right)f(x)|_{x=0} \] (4.14)

one can derive the formula [4.14]

\[ \langle \mathcal{P}_\mu \tilde{f}(\mathcal{P}), : f(y) : \rangle = \left. \left(\frac{1}{i} \frac{\partial}{\partial y}\right)\tilde{f}(\frac{1}{i} \frac{\partial}{\partial y})f(y)\right|_{y=0} = \langle \tilde{f}(\mathcal{P}), : \frac{1}{i} \frac{\partial}{\partial y}f(y) : \rangle \] (4.15)

as well as the relation valid for any \( f(\mathcal{P}) \) which can be expanded in power series

\[ \langle \frac{\partial \tilde{f}(\mathcal{P})}{\partial y^\nu}, : f(y) : \rangle = \left. \left[ \frac{\partial \tilde{f}(\mathcal{P})}{\partial y^\nu} \right|_{y=0} \frac{1}{i} \frac{\partial}{\partial y^\nu} \right] f(y)|_{y=0} = \left. \tilde{f}(\frac{1}{i} \frac{\partial}{\partial y})iy_\mu f(y)\right|_{y=0} = \langle \tilde{f}(\mathcal{P}), : iy_\mu f(y) : \rangle. \] (4.16)

From the relation (4.16) follows that the differential realization (4.11)-(4.13) on commuting fourmomentum space one can express by making the replacements

\[ P_\mu \leftrightarrow \frac{1}{i} \frac{\partial}{\partial y^\mu}, \quad \frac{1}{i} \frac{\partial}{\partial p^\mu} \leftrightarrow y^\mu \] (4.17)

as the differential realizations on the normally ordered functions on noncommutative Minkowski space. Denoting

\[ \mathcal{M}^{\mu\nu} : f(y) := \tilde{\mathcal{M}}^{\mu\nu}f(y) : \] (4.18)

one obtains the formula (4.5) as well as

\[ M^{ij} : f(y) := : \kappa(1 - e^{-i \frac{\phi}{2\alpha}})(g^{0i}y^j - g^{0j}y^i)i(g^{is}y^j - g^{is}y^i) \frac{\partial}{\partial y^s}f(y) :, (4.19a) \]

\[ M^{0i} : f(y) := : -iy^0(ikg^{00}(1 - e^{-i \frac{\phi}{2\alpha}}) - g^{ik} \frac{\partial}{\partial x^k} \kappa \frac{1}{2} g^{00}y^i(1 - e^{-2i \frac{\phi}{\alpha}}) \right) \]

\[ -ix^ig^{0k} \frac{\partial}{\partial y^k}e^{-i \frac{\phi}{\alpha}} + iy^0(e^{-i \frac{\phi}{\alpha}} - 1)y^k \frac{\partial}{\partial y^k} \]
\[ -\frac{1}{2\kappa} y^i g^{rs} \frac{\partial^2}{\partial y^r \partial y^s} + \frac{1}{\kappa} g^{is} y^r \frac{\partial^2}{\partial y^r \partial y^s} f(y) : . \]  

(4.19b)

The relations between the functions : \( \varphi(x) : \) and \( \tilde{f}(p) \) carrying respectively the realizations \( (4.19a)-(4.19b) \) and \( (4.12)-(4.13) \) can be derived from the following normally ordered Fourier transform

\[ : f(y) := \int d^4 p \tilde{f}(p) : e^{-ip_\mu y^\mu} :, \]  

(4.20)

where \( y^\mu \) satisfies the relations \( (1.2a)-(1.2b) \).

The realization \( (4.12)-(4.13) \) can be simplified if we introduce the nonlinear Fourier transform

\[ : f(x) := \int d^4 q \tilde{F}(q) : e^{-iP_\mu(q)y^\mu} :, \]  

(4.21)

where \( P_\mu(q) \) is given by the relations \( (2.13a)-(2.13b) \). Because it can be checked that the nonlinearities in \( (4.12)-(4.13) \) are described as follows:

\[ \mathcal{M}^{\mu\nu}(P, \frac{\partial}{\partial P}) = q^{\mu}(P) \frac{\partial P_\rho}{\partial q_\nu(P)} \frac{\partial}{\partial P_\rho} - q^{\nu}(P) \frac{\partial P_\rho}{\partial q_\mu(P)} \frac{\partial}{\partial P_\rho}, \]  

(4.22)

provided \( q^{\mu}(P) \) describes the inverse deformation map (see \( (2.23a)-(2.23b) \)), one obtains

\[ \mathcal{M}^{\mu\nu} : f(x) := \int d^4 q \tilde{M}^{\mu\nu}(q) \tilde{F}(q) : e^{-iP_\mu(q)y^\mu} :, \]  

(4.23)

where we get classical Loentz algebra realization

\[ \tilde{M}^{\mu\nu} = \frac{1}{i} (q^{\mu} \frac{\partial}{\partial q_\nu} - q^{\nu} \frac{\partial}{\partial q_\mu}). \]  

(4.24)

We see therefore that the nonlinear Fourier transform \( (4.21) \) relates the \( \kappa \)-covariant functions on noncommutative \( \kappa \)-Minkowski space with the functions on commutative classical fourmomentum space transforming under classical relativistic symmetries.

5 \( \kappa \)-deformed Klein-Gordon Fields on \( \kappa \)-deformed Noncommutative Minkowski Space

a) \( \kappa \)-Deformed Minkowski Space and Fifth Dimension

Let us consider the scalar field \( \Phi(y) \) on the noncommutative \( \kappa \)-Minkowski space \( (1.3) \) as the following normally ordered Fourier transform

\[ \Phi(y) = \frac{1}{(2\pi)^4} \int d^4 p \tilde{\Phi}(p) : e^{ipy} :, \]  

(5.1)

where we recall that \( py \overset{df}{=} p_\mu g^{\mu\nu} y_\nu = p_\mu y^\mu \) and

\[ : e^{ipy} := e^{p_{0} x^{0}} e^{ip_{i} x^{i}}. \]  

(5.2)
The $\kappa$-invariant wave operator is given by the formulae

$$g^{\mu\nu} \partial_\mu \partial_\nu \Phi(y) = \frac{1}{(2\pi)^4} \int d^4 p \tilde{\Phi}(p) : g^{\mu\nu} \chi_\mu \chi_\nu e^{i p y} : ,$$  \hspace{1cm} (5.3)

where the nonpolynomial vector fields are given by the formulae (4.1)-(4.2). For any $g_{00} \neq 0$ one gets using the relation (2.16)

$$g^{\mu\nu} \chi_\mu \chi_\nu e^{i p y} := A^2 (g_{00} + \frac{4 \kappa^2}{M^2} p_\mu) : e^{i p y} : ,$$  \hspace{1cm} (5.4)

where $M^2$ is given by (2.12). For the special choice (2.19), following the relation (2.20) one obtains

$$g^{\mu\nu} \partial_\mu \partial_\nu \Phi(y) + \frac{g_{00}^2}{4 \kappa^2} (M^2)^2 \Phi(y) = -M^2 \Phi(y) ,$$  \hspace{1cm} (5.5)

Introducing the fifth derivative $\partial_\Omega \equiv \partial_4$ (see (4.1)) and five-dimensional metric tensor $(A, B = 0, 1, 2, 3, 4)$

$$g^{AB} = \begin{pmatrix} g^{\mu\nu} & 0 \\ 0 & g^{44} \end{pmatrix} , \quad g^{44} = \frac{16 g_{00}^0}{\kappa^2}$$  \hspace{1cm} (5.6)

one can write the relation (5.5) the following five-dimensional Klein-Gordon equation (see also [36])

$$g^{AB} \partial_A \partial_B \Phi(y) = -M^2 \Phi(y) .$$  \hspace{1cm} (5.7)

We obtain therefore the result that if $g_{00} \neq 0$ the deformed mass Casimir is described by the five-dimensional noncommutative wave operator, what is linked with the five dimensions of differential calculus.

In principle one can generalize the formulae (5.1)-(5.7) for the Fourier transform with any ordering of the noncommutative coordinates $y_\mu$. In particular following Podleś [29] it is interesting to describe the action on the noncommutative d’Alambertian (see (5.3)) on nonordered exponentials $e^{i p y}$. In such a case the Fourier transform (5.1) is nonordered

$$\tilde{\Phi}(y) = \frac{1}{(2\pi)^4} \int d^4 \tilde{p} \tilde{\Phi}(\tilde{p}) e^{i \tilde{p} y} .$$  \hspace{1cm} (5.8)

In order to compare the deformed mass shell conditions satisfied by the Fourier transforms $\tilde{\Phi}(p)$ (see (5.11)) and $\tilde{\Phi}(\tilde{p})$ we observe that (see also [36])

$$e^{i \tilde{p}_0 y_0 - \tilde{p}_y} = e^{i p_0 y_0} e^{-i \tilde{p}_0} = : e^{i p y} :$$  \hspace{1cm} (5.9)

where, due to the relation (1.2a),

$$p_i = \frac{\kappa}{\tilde{p}_0} (1 - e^{-p_0 / \kappa}) \tilde{p}_i , \quad p_0 = \tilde{p}_0 ,$$  \hspace{1cm} (5.10a)

or inversely

$$\tilde{p}_i = -\frac{p_0}{2 \kappa \sin \frac{p_0}{\kappa}} (1 + e^{-p_0 / \kappa}) p_i , \quad \tilde{p}_0 = p_0 .$$  \hspace{1cm} (5.10b)
For the case of standard $\kappa$-deformation ($g^{\mu\nu} = \eta^{\mu\nu}$) one can show that

$$
\mathcal{M}^2(p_\mu) = (2\kappa \sinh \frac{p_0}{2\kappa})^2 - \vec{p}^2 e^{-\frac{2\kappa}{p_0}}
$$

$$
= \frac{4\kappa^2}{p_0^2} (\sinh \frac{p_0}{2\kappa})^2 (p_0^2 - \vec{p}^2).
$$

(5.11)

The rhs describes the mass-shell condition obtained by Podleś in [29] derived however under the assumption that the fourdimensional differential calculus on quantum Minkowski space does exist.

It should be mentioned that the use of the Fourier decomposition of $\kappa$-deformed free KG field on normally ordered exponentials has the advantage of reproducing $\kappa$-deformed fourmomentum composition law $p''_\mu = \Delta^{(2)}(p, p')$

$$
p''_0 = p_0 + p'_0, \\
p''_i = p_i e^{-\frac{2\kappa}{p_0}} + p'_i,
$$

(5.12)

which is described by the coproduct relations for $\kappa$-deformed Poincaré algebra (see (2.3a)). Indeed, one obtains

$$
: e^{-p_\mu y^\mu} := e^{-p''_\mu y^\mu}.
$$

(5.13)

The $n$-fold product of normally ordered exponentials leads to the formula

$$
: e^{-ip^{(1)}_\mu y^\mu} : \ldots : e^{-ip^{(n)}_\mu y^\mu} := e^{-i\Delta^{(n)}(p^{(1)}_\mu \ldots p^{(n)}_\mu)y^\mu} :,
$$

(5.14)

where

$$
\Delta^{(n)}_0(p^{(1)}_\mu \ldots p^{(n)}_\mu) = \sum_{k=1}^{n} p^{(k)}_0
$$

(5.15a)

$$
\Delta^{(n)}_i(p^{(1)}_\mu \ldots p^{(n)}_\mu) = \sum_{k=1}^{n} p^{(k)}_i \exp \frac{1}{\kappa} \sum_{l=k+1}^{n} p^{(l)}_0.
$$

(5.15b)

The formulae (5.14)-(5.15b) are important if we wish to construct the local $\kappa$-Poincaré -covariant vertices, by introducing the local polynomials of the field $\Phi(y)$.

b) $\kappa$-deformed Klein-Gordon field induced by the light cone $\kappa$-deformation

From the proportionality of $g^{44}$ to $g_{00}$ it follows that if $g_{00} = 0$ the equation (5.7) contains only the fourdimensional noncommutative d’Alembert operator, i.e. it takes the form

$$
(g^{\mu\nu} \partial_\mu \partial_\nu + m^2)\Phi(y) = 0
$$

(5.16)

in accordance with the dimension four of differential calculus and the relation (2.22), which takes the form (we recall that $g_{00} = 0$):

$$
:g^{\mu\nu} \chi_\mu \chi_\nu e^{ipy} := -\mathcal{M}^2(p_\mu) : e^{ipy} :,
$$

(5.17)
where for the choice of null-plane $\kappa$-deformation $g_{00} = \delta_{33}$ one gets $(r, s = 1, 2)$

$$
\mathcal{M}^2(p_\mu) = 4\kappa p_3 \frac{p_0}{2\kappa} \sinh \frac{p_0}{2\kappa} + g^{rs} p_r p_s e^\frac{p_0}{2\kappa}.
$$

(5.18)

In such a case the solution of free KG field (5.16) can be written as follows:

$$
\Phi(y) = \frac{1}{(2\pi)^4} \int d^4 p \delta(4\kappa p_3 e^\frac{p_0}{2\kappa} \sinh \frac{p_0}{2\kappa} + g^{rs} p_r p_s e^\frac{p_0}{2\kappa} - m^2) a(p) : e^{ipy} : = \frac{1}{(2\pi)^4} \int \frac{d^2p dp_0}{4\kappa \sinh \frac{p_0}{2\kappa}} e^{-\frac{p_0}{2\kappa}} a(p_1, p_2, p_0) : e^{i(p_1 r + \omega_3 y^0 + p_0 y^3)} : ,
$$

(5.19)

where

$$
\omega_3(p_1, p_2, p_0) = \frac{e^{-\frac{p_0}{2\kappa}}}{4\kappa \sinh \frac{p_0}{2\kappa}} (\vec{p}^2 e^\frac{p_0}{2\kappa} - m^2).
$$

(5.20)

The relation (5.12) describes the mass-shell condition for $\kappa$-deformed null-plane dynamics. If $m = 0$ one gets from (5.20)

$$
\omega_3^{m=0}(p_1, p_2, p_0) = \frac{\vec{p}^2}{4\kappa \sinh \frac{p_0}{2\kappa}} \rightarrow \frac{\vec{p}^2}{2p_0},
$$

(5.21)

in accordance with the light-cone quantization kinematics [45].

### 6 $\kappa$-Deformed Klein-Gordon Fields on Commutative Space–Time and $\kappa$-Deformed Pauli–Jordan Commutator Function

a) Noncommutative action and integration over $\kappa$-Minkowski space.

In order to describe the noncommutative action and derive the field equations (5.5) or (5.7) from action principle it is sufficient to define the integral of the ordered exponential (5.9) over $\kappa$-deformed Minkowski space. Following [36] we postulate

$$
\frac{1}{(2\pi)^4} \int \int d^4 y : e^{ipy} := \delta^4(p).
$$

(6.1)

From (6.1) and (5.4) follows that

$$
\int \int d^4 y \Phi(y) = \int d^4 p \delta^4(p) \tilde{\Phi}(p) = \tilde{\Phi}(0).
$$

(6.2)

Further using (5.17) one gets

$$
\int \int d^4 y \Phi_1(y) \Phi_2(y) = \int d^4 p_1 \int d^4 p_2 \Phi_1(p_1) \Phi_2(p_2) \delta(\Delta^2(p_1, p_2))

= \int d^4 p \Phi_1(\vec{p}^1 p_0) \Phi_2(-\vec{p}^1 e^\frac{p_0}{2\kappa}, -p_0),
$$

(6.3)

i.e. we obtain $\kappa$-deformed convolution formula.
The formula (6.1) is invariant under the Poincaré transformation of the noncommutative $\kappa$-Minkowski coordinates, described by the formulae (3.13) (left action of the $\kappa$-Poincaré group) or (3.14) (right action of the $\kappa$-Poincaré group). Choosing the formulae (3.14) one should show that
\[ \int \int d^4y e^{i(p_\kappa \cdot y^\kappa \otimes \Lambda_\kappa^\nu - p_\kappa \otimes a^\kappa \Lambda_\kappa^\nu)} = \delta^4(p) \otimes 1, \] (6.4)
where $(a_\mu, \Lambda_\nu^\kappa)$ describe the noncommutative parameters of $\kappa$-deformed Poincaré group [16, 17, 25]. Indeed, it can be shown after nontrivial calculations (see [1], Appendix) that the formula (6.4) is valid.

We propose the noncommutative KG action in the following form
\[ S = \frac{1}{2} \int \int d^4y \Phi^+(y)(\widehat{\Box} + m^2)\Phi(y), \] (6.5)
where $\widehat{\Box} = \widehat{\partial}_\mu \widehat{\partial}^\mu$ and from the formula
\[ \Phi^+(y) = \frac{1}{(2\pi)^4} \int d^4p \Phi^+(p) \cdot e^{ip\widehat{\mu}} : \] (6.6)
it follows that
\[ \Phi^+(\widehat{p}, p_0) = e^{-\frac{3p_0}{\kappa}} \Phi^*(-e^{-\kappa} \widehat{p}, -p_0). \] (6.7)

b) Nonlocal commutative $\kappa$-deformed Klein-Gordon theory and $\kappa$-deformed star product multiplication.

The $\kappa$-deformation implies the noncommutative $\kappa$-Minkowski space with noncommutative coordinates $y_\mu$ and commutative fourmomentum space (coordinates $p_\mu$). From the Fourier transform $\Phi(p)$ (see [5.1]) one can obtain also a standard relativistic field $\phi(x)$ on classical Minkowski space with coordinates $x_\mu$, by performing classical Fourier transform
\[ \phi(x) = \frac{1}{(2\pi)^4} \int d^4p \Phi(p) e^{ipx}, \] (6.8)
where for simplicity we employ in fourmomentum space the standard integration measure (the $\kappa$-invariant one is given by the formula (2.27)). In the limit $\kappa \to \infty$ the noncommutative Fourier transform (5.1) and classical one given by (6.8) coincide.

The multiplication of two field operators $\Phi_1(y), \Phi_2(y)$ is translated into homomorphic multiplication of their classical counterparts $\Phi_1(y), \Phi_2(y)$ if we postulate the following star multiplication of classical Fourier exponentials (see (5.12)–(5.13))
\[ e^{ipx} * e^{ip'x} = e^{i\Delta^{(2)}(p,p')x}. \] (6.9)
We obtain
\[ \Phi_1(y)\Phi_2(y) \leftrightarrow \phi_1(x) * \phi_2(x) = \frac{1}{(2\pi)^4} \int d^4p d^4p' \Phi_1(p)\Phi_2(p')e^{i\Delta^{(2)}(p,p')x} \] (6.10)
and one gets (compare with (6.3))
\[ \int \int d^4y \phi_1(y)\phi_2(y) = \int d^4x \phi_1(x) * \phi_2(x). \] (6.11)
Similarly using (6.7) one gets

$$
\int\int d^4y \Phi_1(y)^+\Phi_2(y) = \int d^4x [\exp -i{\frac{3}{\kappa}\phi_1(x)}] \ast \phi_2(x) \quad (6.12)
$$

$$
= \int d^3x dx_0\phi_1^*(\overrightarrow{x}, x_0 - \frac{3i}{\kappa}) \ast \phi_2(\overrightarrow{x}, x_0).
$$

We see that the local multiplication of the fields on noncommutative Minkowski space is replaced by nonlocal star-multiplication on classical Minkowski space, in accordance with the diagram depicted on Fig. 1 [1]:

**Figure 1:** Relation between noncommutative and commutative $\kappa$-deformed field theories.

c) $\kappa$-deformed scalar Green functions: commutator function and the propagator.

The $\kappa$-deformed commutator function (Pauli–Jordan function) is given by the formula (we assume $\kappa > 0$):

$$
\Delta^\kappa(x) = \frac{1}{(2\pi)^4} \int d^4p \varepsilon(p_0) \delta(M^2(p)(1 - \frac{M^2(p)}{4\kappa^2}) - m^2)e^{-ipx}
$$

$$
\equiv \Delta^+_\kappa(x) - \Delta^-_\kappa(x) \quad (6.13)
$$

where

$$
\Delta^\pm_\kappa(x) = \frac{1}{(2\pi)^4} \int d^4p \theta(\pm p_0) \delta(M^2(p)(1 - \frac{M^2(p)}{4\kappa^2}) - m^2)e^{-ipx}.
$$

(6.14)

We get

$$
\delta(M^2(p)(1 - \frac{M^2(p)}{4\kappa^2}) - m^2) = \delta(-\frac{1}{4\kappa^2}(M^2(p) - m^2_+)(M^2(p) - m^2_-))
$$
\[
\begin{align*}
\bar{\Delta}_\pm(x) & \equiv \frac{1}{(2\pi)^4} \sum_i \int \frac{d^3p}{\sqrt{|\bar{p}|^2 + m^2}} \left( -\frac{p_0}{\kappa} \mp \sqrt{1 + \frac{m^2}{\kappa^2}} \right) = 0 .
\end{align*}
\]

Further from \( M^2(p) - m^2 \pm 0 \) it follows that
\[
\begin{align*}
2\kappa^2 \left( e^{-\frac{p_0}{\kappa}} \pm \sqrt{1 + \frac{m^2}{\kappa^2}} \right) &= 0 .
\end{align*}
\]

For real \( p_0 \) the sgn " + " should be discarded. Then we obtain
\[
\begin{align*}
e^{-\frac{p_0}{\kappa}} &= \sqrt{1 + \frac{m^2}{\kappa^2}}
\end{align*}
\]

and from (6.19) it follows that
\[
\begin{align*}
\bar{p}^2 + m^2 &= 0 .
\end{align*}
\]

which is impossible. Because \( \frac{dM^2(p)}{dp_0} \neq 0 \) does not vanish, the only short distance \( (x^\mu \sim 0) \) singularities of \( \Delta^\pm_\kappa(x) \) can be generated by divergent integral over \( \bar{p} \). Let us observe however that the on-shell values of \( e^{-\frac{p_0}{\kappa}} \) behave as \( \frac{p_0}{|\bar{p}|} \) for \( \bar{p} \to \infty \). For \( \Delta^{(+)}_\kappa \) the condition \( p_0 > 0 \ (\kappa > 0) \) for large \( \bar{p} \) is not valid, the integration over \( |\bar{p}| \) is truncated, and therefore short distance singularities do not occur. For \( \Delta^{(-)}_\kappa \) the situation is different – we have two real solutions of \( \kappa \)-deformed mass-shell condition (for \( m_+ \) and \( m_- \)) with negative \( p_0 \)
\[
\begin{align*}
e^{-\frac{p_0}{\kappa}} &\sim \frac{\kappa}{|\bar{p}|}, \quad p_0 = -\kappa \ln\left( \frac{|\bar{p}|}{\kappa} \right) ,
\end{align*}
\]

where we sum up all solutions for \( p_0 \) satisfying the following \( \kappa \)-deformed mass-shell condition
\[
\begin{align*}
M^2(p)(1 - \frac{M^2(p)}{4\kappa^2}) - m^2 &= 0 .
\end{align*}
\]
then

\[
\left( \frac{dM^2(p)}{dp_0} \bigg|_{p_0=p_0^{(i)}} \sim 2|p| \right).
\]  

We get

\[
\Delta^{(-)}_\kappa(x) = \frac{1}{(2\pi)^4 \sqrt{1 + \frac{m^2}{\kappa^2}}} \int_0^\infty \frac{d|p|}{|p|^2} e^{i\kappa \ln(\frac{|p|}{\kappa}) x^0} e^{-\kappa \ln(\frac{|p|}{\kappa}) x_0} \cdot
\]

\[
= \frac{2}{(2\pi)^3 \sqrt{1 + \frac{m^2}{\kappa^2}}} \int_0^\infty \frac{d|p|}{|p|^4} \sin(|p| x^0) (\frac{|p|}{\kappa})^{i\kappa x_0} \cdot
\]

\[
= \frac{-i}{(2\pi)^2 \sqrt{1 + \frac{m^2}{\kappa^2}}} \left( \int_0^\infty dp e^{ip |\bar{x}| (\frac{p}{\kappa})^{i\kappa x_0}} - \int_0^\infty dp e^{ip |\bar{x}| (\frac{p}{\kappa})^{i\kappa x_0}} \right) \cdot
\]

\[
= \frac{-i\kappa}{(2\pi)^2 |\bar{x}| \sqrt{1 + \frac{m^2}{\kappa^2}} (\kappa |\bar{x}|)^{1+i\kappa x_0}} \cdot
\]

\[
\]  

(6.25)

Using (6.13) we obtain finally that

\[
\Delta_\kappa(x) \sim \frac{-2\kappa \Gamma(1 + i\kappa x_0) \cosh(\frac{\pi}{2} \kappa x^0)}{(2\pi)^2 \sqrt{1 + \frac{m^2}{\kappa^2}} |\bar{x}| (\kappa |\bar{x}|)^{1+i\kappa x_0}}.
\]  

(6.26)

Further steps is to calculate the \(\kappa\)-deformed Feynman propagator and simple Feynman diagrams, e.g. self-energy diagram in \(\kappa\)-deformed \(\Phi^4\) theory. It has been already observed in \[\text{[1]}\] that at the \(\kappa\)-deformed Feynman vertices the fourmomentum is not conserved, because the energy-momentum conservation is replaced as follows

\[
\delta \left( \sum_{n=1}^{i=1} p_\mu^{(i)} \right) \rightarrow \delta \left( \Delta_\mu^{(n)}(p_\mu^{(1)}, \ldots, p_\mu^{(n)}) \right),
\]  

where \(\Delta_\mu^{(n)}\) is given by the formulae (5.15a-b). The renormalization of self-energy diagrams in \(\kappa\)-deformed \(\Phi^4\) theory, in particular the problem of \(\text{UV/IR}\) divergencies in such a framework, is now under consideration.

\[\text{[51]}\]

Recently the nonconservation of four-momenta for Lie-algebraic noncommutative space-times has been rediscovered in \[\text{[51]}\].
7 Discussion

In this paper we present the results related with so-called $\kappa$-deformations of relativistic symmetries which introduce the elementary length $\lambda_\kappa = \frac{\hbar}{\kappa c}$ as third fundamental constant besides $c$ and $\hbar$. The appearance of this third universal constant implies the existence of new domain of ultrashort distances $|x| \leq \lambda_\kappa$, where new noncommutative physics should be applied. Calling this domain $\kappa$-relativistic physics we obtain the following relation between the theories (see Table 1).

| $\hbar = 0$ | $\hbar \neq 0$ |
|------------|---------------|
| $c = \infty$, $\kappa = \infty$ | $c$ finite, $\kappa = \infty$ |
| nonrelativistic | relativistic |
| classical physics | classical physics |
| $\kappa$-relativistic |
| $\hbar = 0$ | $\hbar \neq 0$ |
| nonrelativistic | relativistic |
| quantum physics | quantum physics |
| $\kappa$-relativistic |

In this table fourth pair of possibilities - $\kappa$-nonrelativistic physics - is not included because very short distances imply large velocities of test particles probing the short-distance behaviour, so the nonrelativistic restriction of velocities is not reasonable for the distances $|x| < \lambda_\kappa$.

The need for third fundamental constant has been also advocated by astrophysical considerations, where without Hopf-algebraic framework the simplest deformations of the mass-shell condition

$$ p^2_0 = p^2 + m^2 \rightarrow p^2_0 = p^2 + m^2 + \alpha \frac{p^3}{\kappa}, \quad (7.1) $$

has been extensively studied (see for example [44, 52]). In (6.1) $\alpha$ is a dimensionless parameter and $\kappa$ can be identified with the Planck mass ($\sim 10^{19}$Gev in energy units).

At present the crucial question remains open how to incorporate full Hopf algebra structure of $\kappa$-deformed symmetry algebras into the phenomenological description of the ultra-short distance corrections implied by quantum gravity. One of the points to be understood is the physical meaning of nonabelian symmetry of the quantum coproduct rules, in particular the problem of physically plausible description of quantum-deformed multiparticle states and energy-momentum conservation in $\kappa$-deformed field theory. These problems are now under continuous considerations.

Acknowledgments

One of the authors (J.L.) would like to thank J. Kowalski-Glikman for his interest in the subject, discussions and inducing us to finish this paper, which was written in its preliminary version in 1997–98.

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\footnote{In fact the efforts to obtain the nontrivial deformation of nonrelativistic physics by considering the limit $c \to \infty$ in the framework of $\kappa$-deformed Poincaré symmetries were rather not successful.}
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