Masataka CHIDA

Selmer groups and central values of $L$-functions for modular forms
Tome 67, n° 3 (2017), p. 1231-1276.

<http://aif.cedram.org/item?id=AIF_2017__67_3_1231_0>

© Association des Annales de l’institut Fourier, 2017,
Certains droits réservés.

Cet article est mis à disposition selon les termes de la licence
CREATIVE COMMONS attribution – pas de modification 3.0 FRANCE.
http://creativecommons.org/licenses/by-nd/3.0/fr/

L’accès aux articles de la revue « Annales de l’institut Fourier »
(http://aif.cedram.org/), implique l’accord avec les conditions générales
d’utilisation (http://aif.cedram.org/legal/).

cedram

Article mis en ligne dans le cadre du
Centre de diffusion des revues académiques de mathématiques
http://www.cedram.org/
SELMER GROUPS AND CENTRAL VALUES OF 
L-FUNCTIONS FOR MODULAR FORMS

by Masataka CHIDA

Abstract. — In this article, we construct an Euler system using CM cycles on Kuga–Sato varieties over Shimura curves and show a relation with the central values of Rankin–Selberg \(L\)-functions for elliptic modular forms and ring class characters of an imaginary quadratic field. As an application, we prove that the non-vanishing of the central values of Rankin–Selberg \(L\)-functions implies the finiteness of Selmer groups associated to the corresponding Galois representation of modular forms under some assumptions.

Résumé. — Dans cet article, nous construisons un système d’Euler en utilisant les cycles CM sur les variétés de Kuga–Sato au-dessus de courbes de Shimura, et montrons une relation avec les valeurs centrales de fonctions \(L\) de Rankin–Selberg associées aux formes modulaires de poids 2 et aux caractères de classes d’un corps quadratique imaginaire. Comme application, nous prouvons que la non-annulation des valeurs centrales de fonctions \(L\) de Rankin–Selberg implique la finitude des groupes de Selmer associés à la représentation galoisienne de la forme modulaire sous certaines hypothèses.

1. Introduction

Let \(\ell\) be a prime and fix an embedding \(\iota_\ell : \overline{\mathbb{Q}} \to \mathbb{C}_\ell\), where \(\mathbb{C}_\ell = \hat{\mathbb{Q}}_\ell\). Let \(N\) be a positive integer and \(k\) an even positive integer. Let
\[f = \sum_{n=1}^{\infty} a_n(f) e^{2\pi i n z} \in S_k(\Gamma_0(N))^{\text{new}}\]
be a normalized cuspidal eigenform. Denote by \(E = \mathbb{Q}_\ell(\{a_n(f)\}_n)\) the Hecke field of \(f\) over \(\mathbb{Q}_\ell\) and fix a uniformizer \(\lambda\) of the ring of integers \(\mathcal{O}\) of \(E\). Denote the residue field of \(E\) by \(F\). Let
\[\rho_f : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E)\]

Keywords: Modular forms, Selmer groups, Bloch–Kato conjecture.
Math. classification: 11F67, 11R23.
be the Galois representation associated to \( f \). We put \( \rho_f^* = \rho_f \otimes E(\frac{2-k}{2}) \) and denote by \( V_f \) the representation space of \( \rho_f^* \). Fix a Gal(\( \mathbb{Q}/\mathbb{Q} \))-stable \( \mathcal{O} \)-lattice \( T_f \) and set \( A_f = V_f/T_f \). Let \( L \) be an abelian extension of \( \mathbb{Q} \) and \( \chi \) a character of the Galois group Gal(\( L/\mathbb{Q} \)). By the Bloch–Kato conjecture [5], it is expected that the central value of the \( L \)-function of \( f \) twisted by the character \( \chi \) is related to the order of the \( \chi \)-part of the Selmer group \( \text{Sel}(L, A_f) \) (see [22, §14] for the definition of the \( \chi \)-part). Kato [22] proved that the non-vanishing of the central value \( L(f, \chi, k/2) \) implies the finiteness of the \( \chi \)-part of the Selmer group \( \text{Sel}(L, A_f) \). Moreover, Kato showed a result on the upper bound of the size of the Selmer group in terms of the special values of \( L \)-functions using the Euler system of Beilinson–Kato elements in \( K_2 \) of modular curves. For an elliptic curve over \( \mathbb{Q} \) and an imaginary quadratic filed \( K \), similar results in the anticyclotomic setting are considered by Bertolini–Darmon [2] and Longo–Vigni [26] using the Euler system constructed from CM points on Shimura curves. These results were generalized to modular abelian varieties over totally real fields by Longo [25] and Nekovář [30]. In this paper, we will consider the generalization of these results for the central values of \( L \)-function associated to higher weight modular forms twisted by ring class characters over an imaginary quadratic field \( K \).

We fix an imaginary quadratic field \( K \) of discriminant \( D_K < 0 \) satisfying \( (N, D_K) = 1 \) and denote the integer ring of \( K \) by \( \mathcal{O}_K \). Then \( K \) determines a factorization \( N = N^+N^- \), where \( N^+ \) is divisible only by primes which splits in \( K \) and \( N^- \) is divisible only by primes which are inert in \( K \). Assume that

\((ST) \quad N^- \) is a square-free product of an odd number of inert primes.

Let \( k \geq 4 \) be an even integer.\(^{(1)}\) Fix an integer \( m \) such that \( (ND_K, m) = 1 \) and let \( K_m \) be the ring class field of \( K \) of conductor \( m \). Let \( \chi \) be a character of the Galois group \( \mathcal{G}_m = \text{Gal}(K_m/K) \). Then we can define the Rankin–Selberg \( L \)-function \( L(f/K, \chi, s) \) associated to \( f \) and \( \chi \). We define a complex number \( \Omega_{f,N^-} \) by

\[
\Omega_{f,N^-} = \frac{4^{k-1}\pi^k||f||_{\Gamma_0(N)}^2}{\xi_f(N^+, N^-)},
\]

where \( ||f||_{\Gamma_0(N)} \) is the Petersson norm of \( f \) and \( \xi_f(N^+, N^-) \) is the congruence number of \( f \) among cusp forms in \( S_k(\Gamma_0(N)) \). Then it is known that the value \( L(f/K, \chi, k/2)_{\Omega_{f,N^-}} \) belongs to \( E(\chi) \). Then

\(^{(1)}\) Although our proof also works for \( k = 2 \), similar results for the weight 2 case are already known by [2], [26], etc. Therefore we will focus on the higher weight case.
Bloch–Kato conjecture predicts a relation between the value $\frac{L(f/K,\chi,k/2)}{\Omega_{f,N^-}}$ and the size of the $\chi$-part of Selmer group $\text{Sel}(K_m,A_f)$.

We consider the following condition.

**HYPOTHESIS (CR$^+$).**

1. $\ell > k + 1$ and $\#(\mathbb{F}^\times)^{k-1} > 5$,
2. The restriction of the residual Galois representation $\overline{\rho}_f$ of $\rho_f$ to the absolute Galois group of $\mathbb{Q}(\sqrt{(-1)^{\frac{\ell-1}{2}}\ell})$ is absolutely irreducible,
3. $\overline{\rho}_f$ is ramified at $q$ if either (i) $q \mid N^-$ and $q^2 \equiv 1 \pmod{\ell}$ or (ii) $q \mid |N^+|$ and $q \equiv 1 \pmod{\ell}$,
4. $\overline{\rho}_f$ restricted to the inertia group of $\mathbb{Q}_q$ is irreducible if $q^2 \mid N$ and $q \equiv -1 \pmod{\ell}$.

Our main result is the following theorem.

**THEOREM 1.1.** — Let $\chi$ be a ring class character of conductor $m$. Suppose that $f$ is a cuspidal newform. Assume the following conditions:

1. $\ell$ does not divide $ND_K[K_m : K]$,
2. the residual Galois representation $\overline{\rho}_f$ satisfies the condition (CR$^+$).

If $c = \text{ord}_\lambda \left(\frac{L(f/K,\chi,k/2)}{\Omega_{f,N^-}}\right)$ is finite, then we have $\lambda^\frac{k}{2} \cdot \text{Sel}(K_m,A_f)\chi = 0$. In particular, if $L(f/K,\chi,k/2)$ is non-zero, then the $\chi$-part of the Selmer group $\text{Sel}(K_m,A_f)$ is finite.

**Remark 1.2.**

1. The assumption (ST) implies that $f$ is not a CM form. Hence the residual Galois representation $\overline{\rho}_f = \overline{\rho}_{f,\lambda}$ satisfies the condition (CR$^+$) for all but finitely many $\lambda$.
2. Let $\Omega_f^{\text{can}}$ be Hida’s canonical period defined by

$$\Omega_f^{\text{can}} = \frac{4^{k-1}\pi^k||f||_\Gamma_0(N)}{\eta_f(N)}$$

where $\eta_f(N)$ is the congruence number of $f$ among cusp forms in $S_k(\Gamma_0(N))$. Under the hypothesis (CR$^+$), one can show that

$$\Omega_{f,N^-} = u \cdot \Omega_f^{\text{can}}$$

for some $u \in \mathcal{O}^\times$, if we further assume that $\overline{\rho}_f$ is ramified at all primes dividing $N^-$. A similar result is given as a corollary of the anticyclotomic Iwasawa main conjecture studied in [9] under the ordinary condition. In this paper, we remove the ordinary condition. Also we should mention that Kings–Loeffler–Zerbes [23, 24] proved that the non-vanishing of $L$-values implies
the finiteness of Selmer groups in the case of Rankin–Selberg product of two modular forms in many cases, and their result contains the case of “modular forms twisted by ray class characters” (not necessarily ring class characters). However their result does not give a precise quantitative bound of the size of Selmer groups. Moreover we remark that our methods are amenable to being generalized to Hilbert modular forms. This is a big difference with Kato’s Euler system used in [22] or the Euler system of Rankin–Eisenstein classes used in [23, 24].

To prove our main theorem, we develop an analogue of the method of Bertolini–Darmon [2] on the Euler system obtained from CM points on Shimura curves. In [9], we used an Euler system obtained from CM points on Shimura curves and congruences between modular forms of higher weight and modular forms of weight two in the ordinary case. However, in the non-ordinary case it seems difficult to use such congruences. Therefore we choose to use CM cycles on Kuga–Sato varieties over Shimura curves instead of CM points. For the construction of our Euler system, we also use a level raising result (Theorem 6.3) for higher weight modular forms; the assumption (CR+) is necessary to obtain this level raising result. More precisely, under the assumption (CR+) we have a freeness result (Proposition 6.1) of the space of definite quaternionic modular forms as Hecke modules that is used as an important step in the proof of the level raising result. The freeness result is a generalization of [8, Proposition 6.8] to the “low weight crystalline case” which is closely related to “$R = T$” theorems and our case was considered by Taylor [36]. Then one can construct an Euler system using CM cycles and a level raising argument.

Moreover we show a relation between the Euler system and central values of Rankin–Selberg $L$-functions (Theorem 8.4), the so-called “first explicit reciprocity law” by Bertolini–Darmon. In the case of weight 2, the explicit reciprocity law is proved by Kummer theory and the theory of $\ell$-adic uniformization of Shimura curves. To show the explicit reciprocity law in the higher weight case, it is necessary to compute the image of CM cycles under the $\ell$-adic Abel–Jacobi map which is defined by Hochschild–Serre spectral sequence. Since it is difficult to compute the image of CM cycles directly, we give a different description of the image of CM cycles using the theory of vanishing cycles and the theory of $\ell$-adic uniformization of Shimura curves. This is a main ingredient of our proof.

This article is organized as follows. First, we review the theory of modular forms on quaternion algebras and the special value formula of Waldspurger in §2. Moreover, we recall basic facts on Galois cohomology and Selmer
groups in §3. In §4, we review the theory of vanishing cycles which is used in §5 and §6. In §5, we prepare some fundamental results on the cohomology of Shimura curves. In §6, we show a level raising result for higher weight modular forms and prove a key result to compute the image of CM cycles under the \( \ell \)-adic Abel–Jacobi map. In §7 and §8, we construct special cohomology classes using CM cycles on Kuga–Sato varieties and give a proof of the main theorem.

Acknowledgments. — The author expresses sincere gratitude to Jan Nekovář for helpful discussions, comments and encouragement. The author also thanks Yoichi Mieda greatly for his advice on the description of the \( \ell \)-adic Abel–Jacobi map using the monodromy pairing. He thanks Naoki Imai who answered for some questions kindly. He also thanks Ming-Lun Hsieh for helpful discussions on the special value formula for Rankin–Selberg \( L \)-functions and the freeness result for Hecke modules. The author thanks the referee for helpful comments and suggestions. The author is partly supported by JSPS KAKENHI Grant Number 23740015 and the research grant of Hakubi project of Kyoto University.

2. Theta elements and the special value formula

In this section, we recall the construction of the theta element and the relation with central values of anticyclotomic \( L \)-functions for modular forms following [8, §§2, 3 and 4].

Fix an embedding \( \iota_{\infty} : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C} \) and an isomorphism \( \iota : \mathbb{C} \cong \mathbb{C}_p \) for each rational prime \( p \), where \( \mathbb{C}_p \) is the \( p \)-adic completion of an algebraic closure of \( \mathbb{Q}_p \). Let \( \widehat{\mathbb{Z}} := \lim \leftarrow \mathbb{Z}/m\mathbb{Z} \) be the finite completion of \( \mathbb{Z} \). For a \( \mathbb{Z} \)-algebra \( A \), we denote \( A \otimes \mathbb{Z} \widehat{\mathbb{Z}} \) by \( \widehat{A} \).

Let \( K \) be an imaginary quadratic field of discriminant \( -D_K < 0 \) and let \( \delta = \sqrt{-D_K} \). Denote by \( z \mapsto \bar{z} \) the complex conjugate on \( K \). Define \( \theta \) by

\[
\theta = \frac{D' + \delta}{2}, \quad D' = \begin{cases} D_K & \text{if } 2 \nmid D_K, \\ D_K/2 & \text{if } 2 \mid D_K. \end{cases}
\]

Fix positive integers \( N^+ \) that are only divisible by prime split in \( K \) and \( N^- \) that are only divisible by primes inert in \( K \). We assume that \( N^- \) is the square-free product of an odd number of primes. Let \( B \) be the definite quaternion over \( \mathbb{Q} \) which is ramified at the prime factors of \( N^- \) and the archimedean place. We can regard \( K \) as a subalgebra of \( B \) (see [8, §2.2]). Write \( T \) and \( N \) for the reduced trace and norm of \( B \) respectively. Let
$G = B^\times$ be the multiplicative algebraic group of $B$ over $\mathbb{Q}$ and let $Z = \mathbb{Q}^\times$ be the center of $G$. Let $\ell \nmid N^-$ be a rational prime. Let $m$ be a positive integer such that $(m, N^+ N^- \ell) = 1$. We choose a basis of $B = K \oplus K \cdot J$ over $K$ such that 

- $J^2 = \beta \in \mathbb{Q}^\times$ with $\beta < 0$ and $Jt = tJ$ for all $t \in K$,
- $\beta \in (\mathbb{Z}_q^\times)^2$ for all $q \mid N^+$ and $\beta \in \mathbb{Z}_q^\times$ for $q \mid D_K$.

Fix a square root $\sqrt{\beta} \in \overline{\mathbb{Q}}$ of $\beta$. We fix an isomorphism $i^{(N^-)} = \prod_{q \mid N^-} i_q : \hat{B}^{(N^-)} \cong M_2(\hat{A}_f^{(N^-)})$ as follows. For each finite place $q \mid m \ell N^+$, the isomorphism $i_q : B_q \cong M_2(\mathbb{Q}_q)$ is defined by

$$i_q(\theta) = \begin{pmatrix} T(\theta) & -N(\theta) \\ 1 & 0 \end{pmatrix}, \quad i_q(J) = \sqrt{\beta} \cdot \begin{pmatrix} -1 & T(\theta) \\ 0 & 1 \end{pmatrix} \quad (\sqrt{\beta} \in \mathbb{Z}_q^\times).$$

For each finite place $q \nmid N^+ N^- \ell m$, choose an isomorphism $i_q : B_q := B \otimes_\mathbb{Q} \mathbb{Q}_q \cong M_2(\mathbb{Q}_q)$ such that $i_q(\mathcal{O}_K \otimes \mathbb{Z}_q) \subset M_2(\mathbb{Z}_q)$. From now on, we shall identify $B_q$ and $G(\mathbb{Q}_q)$ with $M_2(\mathbb{Q}_q)$ and $\text{GL}_2(\mathbb{Q}_q)$ via $i_q$ for $q \nmid N^-$. Finally, we define

$$i_K : B \hookrightarrow M_2(K), \quad a + bJ \mapsto i_K(a + bJ) := \begin{pmatrix} a & b \beta \\ b & a \end{pmatrix} \quad (a, b \in K)$$

and let $i_C : B \to M_2(\mathbb{C})$ be the composition $i_C = i_\infty \circ i_K$.

Fix a decomposition $N^+ \mathcal{O}_K = \mathfrak{m}^+ \mathfrak{m}^+$ once and for all. For each finite place $q$, we define $\varsigma_q \in G(\mathbb{Q}_q)$ as follows:

$$\varsigma_q = \begin{cases} 1 & \text{if } q \nmid N^+ m, \\
\delta^{-1} \begin{pmatrix} \theta & \bar{\theta} \\ 1 & 1 \end{pmatrix} & \text{if } q = q\bar{q} \text{ is split with } q \mid \mathfrak{m}^+, \\
\begin{pmatrix} q^n & 0 \\ 0 & 1 \end{pmatrix} & \text{if } q \mid m \text{ and } q \text{ is inert in } K \quad (n = \text{ord}_q(m)), \\
\begin{pmatrix} 1 & q^{-n} \\ 0 & 1 \end{pmatrix} & \text{if } q \mid m \text{ and } q \text{ splits in } K \quad (n = \text{ord}_q(m)). \end{cases}$$

Define $x_m : \mathbb{A}_K^\times \to G(\hat{\mathbb{A}})$ by

$$x_m(a) := a \cdot \varsigma \quad (\varsigma := \prod_q \varsigma_q).$$

This collection $\{x_m(a)\}_{a \in \mathbb{A}_K^\times}$ of points is called Gross points of conductor $m$ associated to $K$.
Let $O_{K,m} = \mathbb{Z} + mO_K$ be the order of $K$ of conductor $m$. For each positive integer $M$ prime to $N^-$, we denote by $R_M$ the Eichler order of level $M$ with respect to the isomorphisms $ \{ i_q : B_q \cong M_2(\mathbb{Q}_q) \}_{q|N^-}$. Then one can see that the inclusion map $K \rightarrow B$ is an optimal embedding of $O_{K,m}$ into the Eichler order $B \cap \widehat{R}_M(s)^{-1}$ (i.e. $(B \cap \widehat{R}_M(s)^{-1}) \cap K = O_{K,m}$) if $\text{ord}_q(M) \leq \text{ord}_q(m)$ for all primes $q|m$.

Let $k \geq 2$ be an even integer. For a ring $A$, we denote by $L_k(A) = \text{Sym}^{k-2}(A^2)$ the set of homogeneous polynomials in two variables of degree $k-2$ with coefficients in $A$. We write

$$L_k(A) = \bigoplus_{-\frac{k}{2} < r < \frac{k}{2}} A \cdot v_r \quad (v_r := X^{\frac{k-2}{2} - r}Y^{\frac{k-2}{2} + r}).$$

Also we let $\rho_k : \text{GL}_2(A) \rightarrow \text{Aut}_A L_k(A)$ be the unitary representation defined by

$$\rho_k(g)P(X,Y) = \det(g)^{-\frac{k-2}{2}} \cdot P((X,Y)g) \quad (P(X,Y) \in L_k(A)).$$

If $A$ is a $\mathbb{Z}_(\ell)$-algebra with $\ell > k - 2$, we define a perfect pairing

$$\langle , \rangle_k : L_k(A) \times L_k(A) \rightarrow A$$

by

$$\langle \sum_i a_i v_i, \sum_j b_j v_j \rangle_k = \sum_{-\frac{k}{2} < r < \frac{k}{2}} a_r b_{-r} \cdot (-1)^{\frac{k-2}{2} + r} \frac{\Gamma(k/2 + r)\Gamma(k/2 - r)}{\Gamma(k - 1)}.$$

For $P, P' \in L_k(A)$, this pairing satisfies

$$\langle \rho_k(g)P, \rho_k(g)P' \rangle_k = \langle P, P' \rangle_k.$$

Via the embedding $i_C$, we obtain a representation

$$\rho_{k,\infty} : G(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times \xrightarrow{i_C} \text{GL}_2(\mathbb{C}) \rightarrow \text{Aut}_C L_k(\mathbb{C}).$$

Then $C \cdot v_r$ is the eigenspace on which $\rho_{k,\infty}(t)$ acts by $(t/t)^r$ for $t \in (K \otimes_{\mathbb{Q}} \mathbb{C})^\times$. If $A$ is a $K$-algebra and $U \subset G(\mathbb{A}_f)$ is an open compact subgroup, we denote by $S_k^B(U, A)$ the space of modular forms of weight $k$ defined over $A$, consisting of functions $f : G(\mathbb{A}_f) \rightarrow L_k(A)$ such that

$$f(\alpha gu) = \rho_{k,\infty}(\alpha) f(g) \text{ for all } \alpha \in G(\mathbb{Q}) \text{ and } u \in U.$$

Set $S_k^B(A) := \lim_{\rightarrow_U} S_k^B(U, A)$. Let $\mathcal{A}(G)$ be the space of automorphic forms on $G(\mathbb{A})$. For $\mathbf{v} \in L_k(\mathbb{C})$ and $f \in S_k^B(\mathbb{C})$, we define a function $\Psi(\mathbf{v} \otimes f) : G(\mathbb{Q}) \backslash G(\mathbb{A}) \rightarrow \mathbb{C}$ by

$$\Psi(\mathbf{v} \otimes f)(g) := \langle \rho_{k,\infty}(g_\infty)\mathbf{v}, f(g_f) \rangle.$$

TOME 67 (2017), FASCICULE 3
Then the map \( \mathbf{v} \otimes f \mapsto \Psi(\mathbf{v} \otimes f) \) gives rise to a \( G(\mathbb{A}) \)-equivariant morphism \( L_k(\mathbb{C}) \otimes \mathcal{S}_B^B(\mathbb{C}) \to \mathcal{A}(G) \). Let \( \omega \) be a unitary Hecke character of \( \mathbb{Q} \). We write \( \mathcal{S}^B_k(U, \omega, \mathbb{C}) = \{ f \in \mathcal{S}^B_k(U, \mathbb{C}) \mid f(zg) = \omega(z)f(g) \text{ for all } z \in \mathbb{Z}(\mathbb{A}) \} \). Let \( \mathcal{A}^B_k(U, \omega, \mathbb{C}) \) be the space of automorphic forms on \( G(\mathbb{A}) \) of weight \( k \) and central character \( \omega \), consisting of functions \( \Psi(f \otimes \mathbf{v}) : G(\mathbb{A}) \to \mathbb{C} \) for \( f \in \mathcal{S}^B_k(U, \omega, \mathbb{C}) \) and \( \mathbf{v} \in L_k(\mathbb{C}) \). For each positive integer \( M \), we put

\[
\mathcal{S}^B_k(M, \mathbb{C}) = \mathcal{S}^B_k(\widehat{\mathbb{R}}_M, 1, \mathbb{C}), \\
\mathcal{A}^B_k(M, \mathbb{C}) = \mathcal{A}^B_k(\widehat{\mathbb{R}}_M, 1, \mathbb{C}),
\]

where \( 1 \) is the trivial character.

Let \( \pi \) be a unitary cuspidal automorphic representation on \( \text{GL}_2(\mathbb{A}) \) with trivial central character and \( \pi' \) the unitary irreducible cuspidal automorphic representation on \( G(\mathbb{A}) \) with trivial central character attached to \( \pi \) via the Jacquet–Langlands correspondence [18]. Let \( \pi'_{\text{fin}} \) denote the finite constituent of \( \pi' \). Let \( R := R_{N+} \) be an Eichler order of level \( N^+ \). The multiplicity one theorem together with our assumptions implies that \( \pi'_{\text{fin}} \) can be realized as a unique \( G(\mathbb{A}_f) \)-submodule \( \mathcal{S}^B_k(\pi'_{\text{fin}}) \) of \( \mathcal{S}^B_k(\mathbb{C}) \) and \( \mathcal{S}^B_k(N^+, \mathbb{C})[\pi'_{\text{fin}}] := \mathcal{S}^B_k(\pi'_{\text{fin}}) \cap \mathcal{S}^B_k(N^+, \mathbb{C}) \) is one dimensional. We fix a nonzero newform \( f_{\pi'} \in \mathcal{S}^B_k(N^+, \mathbb{C})[\pi'_{\text{fin}}] \). Define the automorphic form \( \varphi_{\pi'} \in \mathcal{A}^B_k(N^+, \mathbb{C}) \) by

\[
\varphi_{\pi'} := \Psi(\mathbf{v}_0^* \otimes f_{\pi'}) \quad (\mathbf{v}_0^* = D_K^{\frac{k+2}{2}} \cdot \mathbf{v}_0).
\]

Define the local Atkin–Lehner element \( \tau^N_q \in G(\mathbb{Q}_q) \) by \( \tau^N_q = J \) for \( q \nmid \infty N^- \), \( \tau^N_q = 1 \) for finite place \( q \nmid N \) and \( \tau^N_q = \begin{pmatrix} 0 & 1 \\ -N^+ & 0 \end{pmatrix} \) if \( q | N^+ \).

Let \( \tau^{N^+}_q := \prod_q \tau^N_q \in G(\mathbb{A}) \). Let \( \text{Cl}(R) \) be a set of representatives of \( B^\times / \widehat{B}^\times \mathbb{Q}^\times \) in \( \widehat{B}^\times = G(\mathbb{A}_f) \). Define the inner product of \( f_{\pi'} \) with itself by

\[
\langle f_{\pi'}, f_{\pi'} \rangle_R := \sum_{g \in \text{Cl}(R)} \frac{1}{\# \Gamma_g} \cdot \langle f_{\pi'}(g), f_{\pi'}(g^{-1} \tau^{N^+}) \rangle_k,
\]

where \( \Gamma_g := (B^\times \cap g \widehat{B}^\times \mathbb{Q}^\times )/\mathbb{Q}^\times \).

Let \( \ell \nmid N^- \) be a rational prime. Let \( \overline{\lambda} \) and \( I \) be the primes of \( \overline{\mathbb{Q}} \) and \( K \) induced by \( \ell \overline{\lambda} \) respectively. We recall the description of \( \ell \)-adic modular forms on \( B^\times \). Let \( A \) be a \( \mathcal{O}_{K_l} \)-algebra. For an open compact subgroup \( U \subset \widehat{B}^\times \), we define the space of \( \ell \)-adic modular forms of weight \( k \) and level \( U \) by

\[
\mathcal{S}^B_k(U, A) := \{ \hat{f} : \hat{B} \to L_k(A) \mid \hat{f}(\alpha g u) = \rho_k(u\ell^{-1})\hat{f}(g), \alpha \in B^\times, u \in U\mathbb{Q} \}.
\]
Also we write $S^B_k(N^+, A) := S^B_k(\hat{\mathcal{R}}^\times, A)$. We let $i_{K_i} : B \hookrightarrow M_2(K_i)$ be the composition $i_{K_i} := i_\ell \circ i_K$. Define $\rho_{k,\ell} : B^\times \rightarrow \text{Aut}L_k(\mathbb{C}_\ell)$ by
$$\rho_{k,\ell}(g) := \rho_k(i_{K_i}(g)).$$
By definition, $\rho_{k,\ell}$ is compatible with $\rho_{k,\infty}$ in the sense that $\rho_{k,\ell}(g) = \rho_{k,\infty}(g)$ for every $g \in B^\times$, and one can check that
$$\rho_{k,\ell}(g) = \rho_k(\gamma_i u_\ell(g)\gamma_i^{-1}),$$
where $\gamma_i := \begin{pmatrix} \sqrt{\beta} & \theta \\ -1 & \theta^{-1} \end{pmatrix} \in \text{GL}_2(K_i)$.
Here $i_\ell : B_\ell \simeq M_2(\mathbb{Q}_\ell)$ is the fixed isomorphism. If $\ell$ is invertible in $A$, there is an isomorphism:
$$S^B_k(N^+, A) \cong S^B_k(N^+, A), \ f \mapsto \hat{f}(g) := \rho_k(\gamma_1^{-1})\rho_{k,\ell}(g_\ell)f(g).$$
Let $\mathbb{Q}(f)$ be the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients of the newform $f = \text{fin} \in \mathbb{S}^B_k(\Gamma_0(N))$. Let $\mathcal{O} \subset \mathbb{C}_\ell$ be the completion of the ring of integers of $\mathbb{Q}(f)$ with respect to $\lambda' = \lambda \cap \mathbb{Q}(f)$. Fix a uniformizer $\lambda$ in $\mathcal{O}$. The $\mathcal{O}$-module $S^B_k(N^+, \mathcal{O})[\pi_{\text{fin}}'] := S^B_k(N^+, \mathcal{O}) \cap S^B_k(N^+, \mathcal{C}_\ell)[\pi_{\text{fin}}']$ has rank one. We say $f_{\pi'} \in S^B_k(N^+, \mathbb{C})[\pi_{\text{fin}}']$ is $\lambda$-adically normalized if $\hat{f}_{\pi'}$ is a generator of $S^B_k(N^+, \mathcal{O})[\pi_{\text{fin}}']$ over $\mathcal{O}$. This is equivalent to the following condition:
$$\hat{f}_{\pi'}(g_0) \not\equiv 0 \pmod{\lambda} \quad \text{for some } g_0 \in \mathbb{A}_f.$$
Now we define the theta elements. For a positive integer $m$, let $\mathcal{G}_m = K^x \backslash \hat{K}^x / \hat{\mathcal{O}}^x_{K,m}$ be the Picard group of the order $\mathcal{O}_{K,m}$. We identify $\mathcal{G}_m$ with the Galois group of the ring class field $K_m$ of conductor $m$ over $K$ via geometrically normalized reciprocity law.

Denote by $[\cdot]_m : \hat{K}^x \rightarrow \mathcal{G}_m, a \mapsto [a]_m$ the natural projection map. We consider the automorphic form $\varphi_{\pi'} = \Psi(v_0^* \otimes f_{\pi'})$. It is easy to see that the function
$$\hat{\varphi}_{\pi'} : \hat{K}^x \rightarrow \mathbb{C}, \ a \mapsto \hat{\varphi}_{\pi'}(a) := \varphi_{\pi'}(x_m(a))$$

factors through $\mathcal{G}_m$, so we can extend $\hat{\varphi}_{\pi'}$ linearly to be a function $\hat{\varphi}_{\pi'} : \mathbb{C}[\mathcal{G}_m] \rightarrow \mathbb{C}$. Let $P_m := [1]_m \in \mathcal{G}_m$ be the distinguished Gross point of conductor $m$. We put
$$\hat{\varphi}_{\pi'}(\sigma(P_m)) = \varphi_{\pi'}(x_m(a)) \quad \text{if } \sigma = [a]_m \in \mathcal{G}_m.$$  
We define the theta element $\Theta(f_{\pi'}) \in \mathbb{C}[\mathcal{G}_m]$ by
$$\Theta(f_{\pi'}) := \sum_{\sigma \in \mathcal{G}_m} \hat{\varphi}_{\pi'}(\sigma(P_m)) \cdot \sigma.$$  
Then we have the following special value formula.
Proposition 2.1. — Let \( \chi \) be a character of \( \mathcal{G}_m \). Then we have
\[
\chi(\Theta(f_{\pi'})^2) = \Gamma(k/2)^2 \cdot \frac{L(f_{\pi}/K, \chi, k/2)}{\Omega_{\pi,N^-}} \cdot (-1)^{\frac{k}{2}} \cdot m \cdot D_K^{k-1} \cdot \frac{\#O_K/2}{2} \times \sqrt{-D_K}^{-1} \cdot \chi(\mathfrak{N}^+) ,
\]
where
\[
\Omega_{\pi,N^-} = \frac{4^{k-1} \pi^k ||f_{\pi}|| \Gamma_0(N)}{(f_{\pi'}, f_{\pi'})_R}
\]
is the \( \lambda \)-adically normalized period for \( f \).

Proof. — This formula is a special case of Hung’s result [15, Proposition 5.3]. Also see [8, Proposition 4.3] for the case where \( \chi \) is an unramified character. \( \square \)

3. Selmer groups for modular forms

3.1. Definition of Selmer groups

First we recall the definition of Selmer groups following Bloch–Kato [5]. Let \( f \) be a cuspidal Hecke eigenform of weight \( k \) with respect to \( \Gamma_0(N) \). Let \( \mathbb{Q}(f) \) denote the Hecke field generated by the eigenvalues \( \{a_q(f)\} \) of the Hecke operators \( \{T_q\} \). Let \( \lambda \) be the prime of \( \mathbb{Q}(f) \) above the prime \( \ell \) induced by the fixed embedding \( \iota_\ell \). Denote \( E = \mathbb{Q}(f)_\lambda \). Also we denote the integer ring of \( E \) by \( \mathcal{O} \) and the uniformizer by \( \lambda \) and write \( \mathcal{O}_n = \mathcal{O}/\lambda^n \mathcal{O} \). Then there exist a 2-dimensional Galois representation
\[
\rho_f = \rho_{f,\lambda} : G_\mathbb{Q} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \text{GL}_2(E)
\]
such that \( \det(1 - \rho_f(\text{Frob}_q) \cdot X) = 1 - a_q(f)X + q^{k-1}X^2 \) for any prime \( q \) satisfying \( q \nmid \ell N \). Let \( V_f \) be the representation space of \( \rho_f \otimes \varepsilon_\ell^{2-k} \), where \( \varepsilon_\ell : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_\ell^\times \) is the \( \ell \)-adic cyclotomic character. We choose a \( G_\mathbb{Q} \)-stable \( \mathcal{O} \)-lattice \( T_f \) in \( V_f \), and denote \( A_f = V_f/T_f \). Then there is an exact sequence \( 0 \to T_f \xrightarrow{i} V_f \xrightarrow{\rho_f} A_f \to 0 \).

For a finite extension \( F/\mathbb{Q}_p \), Bloch–Kato [5] defined the finite part of Galois cohomology groups by
\[
H^1_f(F, V_f) := \begin{cases}
\text{Ker} \left[ H^1(F, V_f) \to H^1(F^\text{ur}, V_f) \right] & \ell \neq p, \\
\text{Ker} \left[ H^1(F, V_f) \to H^1(F, V_f \otimes_{\mathbb{Q}_p} \text{B}_{\text{cris}}) \right] & \ell = p,
\end{cases}
\]
where $B_{\text{cris}}$ is the $p$-adic period ring defined by Fontaine and $F^{\text{ur}}$ is the maximal unramified extension of $F$. Also we denote

$$H^1_f(F, T_f) = i^{-1}(H^1_f(F, V_f))$$

and

$$H^1_f(F, A_f) = \text{Im} \left[ H^1_f(F, V_f) \hookrightarrow H^1(F, V_f) \xrightarrow{pr} H^1(F, A_f) \right].$$

For a number field $F$, we define the $\lambda$-part of the Selmer group of $f$ by

$$\text{Sel}(F, A_f) = \text{Ker} \left[ H^1(F, A_f) \rightarrow \prod_v H^1(F_v, A_f) \right].$$

We also define

$$H^1_f(F, V_f) = \text{Ker} \left[ H^1(F, V_f) \rightarrow \prod_v H^1(F_v, V_f) \right].$$

Moreover we set $A_{f,n} = A_f[\lambda^n] = \text{Ker}[A_f \xrightarrow{\lambda^n} A_f]$ and $T_{f,n} = T_f/\lambda^n T_f$. Then there exists a Galois-equivariant bilinear pairing $T_f \times T_f \rightarrow O(1)$ such that the induced pairings on $T_{f,n} \cong A_{f,n}$ are non-degenerate for all $n$. For details, see Nekovář [27, Proposition 3.1].

**Proposition 3.1.** — The pairing above induces the local Tate pairing

$$\langle , \rangle_v : H^1(F_v, T_f) \times H^1(F_v, A_f) \rightarrow H^2(F_v, E/O(1)) \cong E/O,$$

$$\langle , \rangle_v : H^1(F_v, T_{f,n}) \times H^1(F_v, A_{f,n}) \rightarrow H^2(F_v, O_n(1)) \cong O_n,$$

for each place $v$ of $F$. The local Tate pairing is perfect and satisfies the following properties.

1. The pairing $\langle , \rangle_v$ makes $H^1_f(F_v, T_f)$ and $H^1_f(F_v, A_f)$ into exact annihilators of each other at any place $v$.
2. If $x$ and $y$ belong to $H^1(F, A_{f,n})$, then

$$\sum_v \langle x, y \rangle_v = 0,$$

where the sum is over all places $v$ of $F$ but is a finite sum.

**Proof.** — See Besser [4, Proposition 2.2].

**Definition 3.2.** — For each place $v$, we define $H^1_f(F_v, A_{f,n})$ to be the preimage of $H^1_f(F_v, A_f)$ in $H^1(F_v, A_{f,n})$. Then we let

$$\text{Sel}(F, A_{f,n}) = \text{Ker} \left[ H^1(F, A_{f,n}) \rightarrow \prod_v H^1(F_v, A_{f,n}) \right].$$
Also we define $H^1_f(F_v, T_{f,n})$ to be the image of $H^1_f(F_v, T_{f})$ in $H^1(F_v, T_{f,n})$. Moreover we define the singular part of local cohomology group $H^1_{\text{sing}}(F_v, T_{f,n})$ to be the quotient

$$H^1_{\text{sing}}(F_v, T_{f,n}) = \frac{H^1(F_v, T_{f,n})}{H^1_f(F_v, T_{f,n})}.$$  

If $v$ does not divide $N$, then we have

$$H^1_{\text{sing}}(F_v, T_{f,n}) = H^1(F_{\text{ur}}, T_{f,n})^{G_{F_v}}.$$  

By Proposition 3.1, $H^1_f(F_v, A_{f,n})$ and $H^1_{\text{sing}}(F_v, T_{f,n})$ are the Pontryagin dual of each other.

For each prime $q$ and $G_{Q_q}$-module $M$, we denote

$$H^1_f(F_q, M) = \bigoplus_{v|q} H^1_f(F_v, M)$$

and

$$H^1_{\text{sing}}(F_q, M) = \bigoplus_{v|q} H^1_{\text{sing}}(F_v, M).$$

**Lemma 3.3.** Let $q$ be a prime which splits in $K$. Then

$$H^1_{\text{sing}}(K_{m,q}, T_{f,n}) = 0$$

for sufficiently large $m$.

**Proof.** Proceed as in the proof of [2, Lemma 2.4].

### 3.2. The Euler system argument

Here we give a generalization of the Euler system argument introduced by Bertolini–Darmon [2] to the case of higher weight modular forms.

**Definition 3.4.** A prime $p$ is said to be $n$-admissible if

1. $p$ does not divide $N\ell [K_m : K]$.
2. $p$ is inert in $K$.
3. $\lambda$ does not divide $p^2 - 1$.
4. $\lambda^n$ divides $p^k + p^{\frac{k-2}{2}} - \varepsilon \cdot a_p(f)$, where $\varepsilon = \pm 1$.

**Lemma 3.5.** Let $p$ be an $n$-admissible prime. Then $H^1_f(K_{m,p}, A_{f,n})$ and $H^1_{\text{sing}}(K_{m,p}, T_{f,n})$ are both isomorphic to $O_n[G_m]$. In particular, the $\chi$-parts of these groups are both isomorphic to $O_n$.

**Proof.** This is a direct generalization of [2, Lemma 2.6].
Define the map $\partial_p$ to be the composition
\[ H^1(K_m, A_{f,n}) \to H^1(K_{m,p}, A_{f,n}) \to H^1_{\text{sing}}(K_{m,p}, A_{f,n}). \]
If $\partial_p(\kappa) = 0$ for $\kappa \in H^1(K_{m,p}, A_{f,n})$ (resp. $H^1(K_{m,p}, T_{f,n})$), let $v_p(\kappa) \in H^1_f(K_{m,p}, A_{f,n})$ (resp. $H^1_f(K_{m,p}, T_{f,n})$) denote the natural image of $\kappa$ under $\partial_p$.

**Theorem 3.6** ([9, Theorem 6.3]). — Let $s \in H^1(K_m, A_{f,n})$ be a non-zero element. Then there exist infinitely many $n$-admissible primes $p$ such that $\partial_p(s) = 0$ and $v_p(s) \neq 0$.

**Definition 3.7.** — For a prime $p$, we define the compactified Selmer group $H^1_p(K_m, T_{f,n})$ to be
\[ H^1_p(K_m, T_{f,n}) = \text{Ker} \left[ H^1(K_m, T_{f,n}) \to \prod_{v|p} H^1(K_{m,v}, T_{f,n}) \right]. \]
Recall that $\lambda$ is the fixed uniformizer of $O$.

**Theorem 3.8.** — Let $t$ be a positive integer. Suppose that for all but finitely many $n$-admissible primes $p$ there exists an element $\kappa_p \in H^1_p(K_m, T_{f,n+t})^\chi$ such that $\lambda^{t-1} \partial_p(\kappa_p) \neq 0$. Then $\lambda^n \cdot \text{Sel}(K_m, A_{f,n+t})^\chi = 0$

**Proof.** — Assume that there exists an element $s$ in $\text{Sel}(K_m, A_{f,n+t})^\chi$ satisfying $\lambda^n s \neq 0$. By Theorem 3.6 and the assumption, we can take an $n+t$-admissible prime $p$ satisfying the following properties simultaneously:

1. $v_p(\lambda^n s) \neq 0$ and $\partial_p(\lambda^n s) = 0$.
2. there exists an element $\kappa_p \in H^1_p(K_m, T_{f,n+t})^\chi$ such that $\lambda^{t-1} \partial_p(\kappa_p) \neq 0$.

By the properties of the local Tate pairing, we have
\[ \sum_q \langle \lambda^{t-1} \partial_q(\kappa_p), v_q(\lambda^n s) \rangle_q = 0. \]
Since $H^1_f(K_{m,q}, A_{f,n})^\chi$ and $H^1_f(K_{m,q}, T_{f,n})^\chi$ are annihilators of each other, we have
\[ \langle \lambda^{t-1} \partial_q(\kappa_p), v_q(\lambda^n s) \rangle_q = 0 \text{ for } q \neq p. \]
Therefore $\langle \lambda^{t-1} \partial_p(\kappa_p), \text{res}_p(\lambda^n s) \rangle_q = 0$ by Proposition 3.1 (2). Since the local Tate pairing is perfect, the assumption $\lambda^{t-1} \partial_p(\kappa_p) \neq 0$ implies $v_p(\lambda^n s) = 0$. This gives a contradiction. □
4. Review of vanishing cycles

In §5 and §6 we will use the theory of vanishing cycles in several important steps. Therefore, in this section we briefly recall the theory of vanishing cycles following the exposition in Rajaei [33].

4.1. Vanishing cycles

Let $R$ be a characteristic 0 henselian discrete valuation ring with residue field $k$ of characteristic $p$. Fix a uniformizer $\varpi$ in $R$. Denote the fraction field of $R$ by $K$ and the maximal unramified extension of $K$ by $K^{ur}$. Let $X \to S = \text{Spec } R$ be a proper and generically smooth curve and $F$ a constructible torsion sheaf on $X$ whose torsion is prime to $p$. Let $i : X_k \to X$, $j : X_K \to X$, $\overline{i} : X_{\overline{K}} \to X_{\overline{O}_{K^{ur}}}$ and $\overline{j} : X_{\overline{K}} \to X_{\overline{O}_{K^{ur}}}$ be the canonical maps. By the proper base change theorem and the Leray spectral sequence for $j$, we have

$$R\Gamma(X_K, j^* F) = R\Gamma(X_{O_{K^{ur}}}, R\overline{j}_* \overline{i}^* F) \xrightarrow{\cong} R\Gamma(X_{\overline{K}}, i^* R\overline{j}_* \overline{i}^* F).$$

Then the adjunction morphism gives $\phi : \overline{i}_* F \to \overline{i}^* R\overline{j}_* \overline{i}^* F$. We define the vanishing cycles by

$$R\Phi F := \text{Cone}(\phi),$$

and the nearby cycles by

$$R\Psi F := \overline{i}^* R\overline{j}_* \overline{i}^* F.$$

Then we have a distinguished triangle

$$\to i^* F \to R\Psi F \to R\Phi F \xrightarrow{\delta} .$$

For $i > 0$, we have $R^i \Phi F = R^i \Psi F$. Let $\Sigma$ be the set of singular points of $X_K$. Assume that a neighbourhood of each singular point $x$ is (locally) isomorphic to the subscheme of $\mathbb{A}^2_S = S[t_1, t_2]$ with equation $t_1t_2 = a_x$ (denote $e_x := v(a_x) > 0$). When the special fiber $X_{\overline{K}}$ is reduced, Deligne [11] proved the sheaves $R^i \Phi F$ vanish for $i \neq 1$ and $R^1 \Phi F$ is supported at $\Sigma$, and the specialization map $H^1(X_K, F) \to H^1(X_{\overline{K}}, F)$ is injective. Now we have the specialization sequence

$$0 \to H^1(X_K, i^* F)(1) \to H^1(X_{\overline{K}}, F)(1) \xrightarrow{\beta} \bigoplus_{x \in \Sigma}(R^1 \Phi F)_x(1) \to H^2(X_K, i^* F)(1) \xrightarrow{\text{sp}(1)} H^2(X_{\overline{K}}, F)(1) \to 0.$$
Then we define the character group for the sheaf \( F \) on \( X \) by

\[
\mathcal{X}(F) = \ker \left[ \bigoplus_{x \in \Sigma} R^1 \Phi \mathcal{F}_x(1) \to \ker(\text{sp}(1)) \right],
\]

so that there is a short exact sequence

\[
0 \to H^1(X_\overline{k}, F)(1) \to H^1(X_\overline{K}, F)(1) \to \mathcal{X}(F) \to 0. \tag{4.1}
\]

For \( x \in \Sigma \), let \( (X_\overline{k})_x \) be the henselization of \( X_\overline{k} \) at \( x \) and \( B_x \) the set of two branches of \( X_\overline{k} \) at \( x \) (i.e. the irreducible components of \((X_\overline{k})_x\)). For \( x \in \Sigma \), we define the modules \( \mathbb{Z}(x) \) and \( \mathbb{Z}'(x) \) by

\[
\mathbb{Z}(x) := \text{Coker} \left[ \mathbb{Z} \xrightarrow{\text{diag}} \mathbb{Z}^{B_x} \right]
\]

and

\[
\mathbb{Z}'(x) := \ker \left[ \mathbb{Z}^{B_x} \xrightarrow{\sum} \mathbb{Z} \right].
\]

Choose an ordering for \( B_x \) for each \( x \in \Sigma \) and define a basis of \( \mathbb{Z}'(x) \) by \( \delta'_x := (1, -1) \). Denote the dual basis by \( \delta_x \in \mathbb{Z}(x) \). We set \( \Lambda = \mathbb{Z}_\ell \). For \( x \in \Sigma \), one has \( H^i_x(X_\overline{k}, R\Psi \Lambda) = 0 \) for \( i \neq 1, 2 \) and the trace map gives an isomorphism \( H^2_x(X_\overline{k}, R\Psi \Lambda) \xrightarrow{\sim} \Lambda(-1) \) and \( H^1_x(X_\overline{k}, R\Psi \Lambda) \xrightarrow{\sim} \mathbb{Z}(x) \otimes \Lambda \). Moreover we have \( R^1 \Phi \Lambda_x \xrightarrow{\sim} \mathbb{Z}'(x) \otimes \Lambda \). Therefore we have the perfect pairing

\[
(R^1 \Phi \Lambda)_x \times H^1_x(X_\overline{k}, R\Psi \Lambda) \to H^2_x(X_\overline{k}, R\Psi \Lambda) \xrightarrow{\sim} \Lambda(-1).
\]

This pairing gives the cospecialization map

\[
0 \to H^0(\tilde{X}_\overline{k}, R\Psi \Lambda) \to H^0(\tilde{X}_\overline{k}, \iota^* \Lambda) \to \bigoplus_{x \in \Sigma} H^1_x(X_\overline{k}, R\Psi \Lambda) \xrightarrow{\beta'} H^1(X_\overline{k}, \Lambda) \to H^1(X_\overline{k}, \iota^* \Lambda) \to 0,
\]

where \( \tilde{X}_\overline{k} \to X_\overline{k} \) is the normalization map.

### 4.2. Monodromy pairing

Let \( \ell \) be a prime different from \( p \) and let \( I \) be the inertia subgroup of \( \text{Gal}(\overline{K}/K) \). We consider the map \( t_\ell : I \to \mathbb{Z}_\ell(1) \) which is defined by \( \sigma \mapsto \sigma(\varpi^{1/\ell})/\varpi^{1/\ell} \), where \( \varpi \) is the uniformizer of \( R \). For \( \sigma \in I \) and \( x \in \Sigma \), we define the variation map

\[
\text{Var}(\sigma)_x : (R^1 \Phi \Lambda)_x \to H^1_x(X_\overline{k}, R\Psi \Lambda)
\]
by $a \mapsto -e_x t_\ell(\sigma)(a\delta_x)\delta_x$, and define the monodromy logarithm
$$N_x : (R^1\Phi\Lambda)_x(1) \to H^1_x(X_{\overline{K}}, R\Psi\Lambda)$$
by $N_x(t_\ell(\sigma)a) = \text{Var}(\sigma)(a)$ for $a \in (R^1\Phi\Lambda)_x$ and $\sigma \in I$. Then we have a commutative diagram
\[
\begin{array}{c}
(R^1\Phi\Lambda)_x(1) \xrightarrow{\sim} \mathbb{Z}'(x) \otimes \Lambda \\
\downarrow N_x \quad \quad \downarrow \phi_x \\
H^1_x(X_{\overline{K}}, R\Psi\Lambda) \xrightarrow{\sim} \mathbb{Z}(x) \otimes \Lambda,
\end{array}
\]
where the right vertical map $\phi_x$ is given by $\delta'_x \mapsto -e_x \delta_x$. Moreover we define the monodromy operator $N$ by the following diagram:
\[
\begin{array}{c}
H^1(X_{\overline{K}}, \Lambda)(1) = H^1(X_{\overline{K}}, R\Psi\Lambda)(1) \xrightarrow{\beta} \bigoplus_{x \in \Sigma} (R^1\Phi\mathcal{F})_x(1) \\
\downarrow N \\
H^1(X_{\overline{K}}, \Lambda) = H^1(X_{\overline{K}}, R\Psi\Lambda) \xleftarrow{\beta'} \bigoplus_{x \in \Sigma} H^1_x(X_{\overline{K}}, R\Psi\Lambda).
\end{array}
\]
Then we have an explicit description of the map $N$.

**Theorem 4.1** (Picard–Lefschetz formula [10]). — With notation as above, we have the following formula:
$$N(t_\ell(\sigma)a) = (\sigma - 1)a \quad \text{for} \quad a \in H^1(X_{\overline{K}}, \Lambda) \quad \text{and} \quad \sigma \in I.$$

Let $B$ be the set of irreducible components of $X_{\overline{K}}$. Define the modules $\mathbb{X}$ and $\hat{\mathbb{X}}$ by the exact sequences
\[
0 \rightarrow \mathbb{X} \rightarrow \bigoplus_{x \in \Sigma} \mathbb{Z}'(x) \rightarrow \mathbb{Z}^B \rightarrow \mathbb{Z} \rightarrow 0
\]
and
\[
0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}^B \rightarrow \bigoplus_{x \in \Sigma} \mathbb{Z}(x) \rightarrow \hat{\mathbb{X}} \rightarrow 0.
\]
Then the monodromy pairing
$$u : \mathbb{X} \otimes \mathbb{X} \rightarrow \mathbb{Z}$$
is given by the diagram
\[
\begin{array}{cccc}
\mathbb{X} & \xrightarrow{u_*} & \bigoplus_{x \in \Sigma} \mathbb{Z}'(x) \\
\downarrow u_* & & \downarrow \bigoplus_{x \in \Sigma} \phi_x \\
\hat{\mathbb{X}} & \xrightarrow{} & \bigoplus_{x \in \Sigma} \mathbb{Z}(x).
\end{array}
\]
Also we have
\[ \mathbb{X} \otimes \Lambda = \text{Im} \left[ H^1(\tilde{X}, \Lambda) \to \bigoplus_{x \in \Sigma} (R^1 \Phi \Lambda)_x(1) \right] \]
and
\[ \hat{\mathbb{X}} \otimes \Lambda = \text{Coker} \left[ H^0(\tilde{X}, \Lambda) \to \bigoplus_{x \in \Sigma} H^1_x(\tilde{X}, R \Psi \Lambda) \right] . \]

Therefore we obtain the diagram
\[
\begin{array}{c}
H^1(\tilde{X}, \Lambda)(1) \rightarrow \mathbb{X} \otimes \Lambda \rightarrow \bigoplus_{x \in \Sigma} (R^1 \Phi \Lambda)_x(1) \\
\downarrow N \quad \quad \downarrow \ u_* \otimes \Lambda \quad \quad \quad \downarrow \bigoplus N_x \\
H^1(\tilde{X}, \Lambda) \leftarrow \hat{\mathbb{X}} \otimes \Lambda \leftarrow \bigoplus_{x \in \Sigma} H^1_x(\tilde{X}, R \Psi \Lambda)
\end{array}
\]

Note that the cokernel of \( u_* \) is the group of connected components.

Let \( \mathcal{F} \) be a locally constant \( \mathbb{Z}_\ell \)-sheaf on \( X \). The cospecialization exact sequence is

\[
0 \rightarrow H^0(\tilde{X}, R \Psi \mathcal{F}) \rightarrow H^0(\tilde{X}, i^* \mathcal{F}) \rightarrow \bigoplus_{x \in \Sigma} H^1_x(\tilde{X}, R \Psi(\mathcal{F})) \xrightarrow{\beta'} H^1(\tilde{X}, \mathcal{F}) \rightarrow H^1(\tilde{X}, i^* \mathcal{F}) \rightarrow 0.
\]

Now we define the cocharacter group by
\[ \hat{\mathbb{X}}(\mathcal{F}) := \text{Im}(\beta') \].

Then we have a canonical isomorphism \((R^1 \Phi \mathcal{F})_x \simeq (R^1 \Phi \Lambda)_x \otimes \mathcal{F}_x\) and a natural map \( H^1_x(\tilde{X}, R \Psi \Lambda) \otimes \mathcal{F}_x \rightarrow H^1_x(\tilde{X}, R \Psi(\mathcal{F}))\). These maps give a generalization of the monodromy pairing
\[ \lambda : \mathbb{X}(\mathcal{F}) \rightarrow \hat{\mathbb{X}}(\mathcal{F}) \]
by composition of the maps
\[
\begin{array}{c}
H^1(\tilde{X}, \mathcal{F})(1) \rightarrow \mathbb{X}(\mathcal{F}) \rightarrow \bigoplus_{x \in \Sigma} (R^1 \Phi \mathcal{F})_x(1) \\
\downarrow N \quad \quad \downarrow \lambda \quad \quad \quad \downarrow \bigoplus N_x \otimes 1 \\
H^1(\tilde{X}, \mathcal{F}) \leftarrow \hat{\mathbb{X}}(\mathcal{F}) \leftarrow \bigoplus_{x \in \Sigma} H^1_x(\tilde{X}, R \Psi \mathcal{F})
\end{array}
\]

Then the monodromy operator \( N \) is described by the Picard–Lefschetz formula:
\[ N(t_\ell(\sigma)a) = (\sigma - 1)a \quad \text{for } a \in H^1(\tilde{X}, \mathcal{F}) \text{ and } \sigma \in I. \]
We define the component group by
\[ \Phi(\mathcal{F}) := \text{Coker} \left[ \lambda : \mathcal{X}(\mathcal{F}) \to \hat{\mathcal{X}}(\mathcal{F}) \right]. \]

5. Cohomology of Shimura curves

Let \( M \) be a positive integer and \( M = M^+M^- \) an integer decomposition of \( M \) such that \( M^- > 1 \) is a square-free product of an even number of primes and \( (M^+, M^-) = 1 \). Let \( B' \) be the indefinite quaternion algebra over \( \mathbb{Q} \) with discriminant \( M^- \). Fix a prime \( p \) dividing \( M^- \). Let \( B \) be the definite quaternion algebra over \( \mathbb{Q} \) with discriminant \( M^-/p \). We fix a \( \mathbb{Q} \)-embedding \( t' : K \to B' \) and an isomorphism \( \varphi_{B,B'} : \hat{B}(p) \cong \hat{B'}(p) \). Also we fix an Eichler order \( R_{M^+} \) of level \( M^+ \) in \( B \).

5.1. Moduli interpretation of Shimura curves

Fix a maximal order \( \mathcal{O}_{B'} \) of \( B' \).

Let \( S \) be a \( \mathbb{Z}[1/M] \)-scheme. A triple \( (A, \iota, C) \) is called an abelian surface with quaternionic multiplication with level \( M^+ \)-structure over \( S \) if

1. \( A \) is an abelian scheme over \( S \) of relative dimension 2,
2. \( \iota : \mathcal{O}_{B'} \to \text{End}_S(A) \) is an inclusion defining an action of \( \mathcal{O}_{B'} \) on \( A \),
3. \( C \) is a subgroup scheme of \( A \) of order \( (M^+)^2 \) which is stable and locally cyclic under the action of \( \mathcal{O}_{B'} \).

We denote by \( \mathcal{F}_{M^+,M^-} \) the functor from the category of schemes over \( \mathbb{Z}[1/M] \) to the category of sets which associates to a scheme \( S \) the set of isomorphism classes of abelian surfaces with quaternionic multiplication with level \( M^+ \)-structure over \( S \). If \( M^- \) is strictly greater than 1, the functor \( \mathcal{F}_{M^+,M^-} \) is coarsely representable by a scheme \( X_{M^+,M^-} \) over \( \mathbb{Z}[1/M] \) with smooth fibers. The scheme \( X_{M^+,M^-} \) is a smooth projective geometrically connected curve over \( \mathbb{Z}[1/M] \).

Let \( d \geq 1 \) be an integer relatively prime to \( M \) and \( S \) a \( \mathbb{Z}[1/Md] \)-scheme. A quadruple \( (A, \iota, C, \nu) \) is called an abelian surface with quaternionic multiplication by \( \mathcal{O}_{B'} \) with level \( M^+ \)-structure and full level \( d \)-structure if \( (A, \iota, C) \) is a triple as above and

\[ \nu : (\mathcal{O}_{B'}/d\mathcal{O}_{B'})_S \to A[d] \]

is an \( \mathcal{O}_{B'} \)-equivariant isomorphism from the constant group scheme \( (\mathcal{O}_{B'}/d\mathcal{O}_{B'})_S \) to the group scheme of \( d \)-division points of \( A \).
If \( d \geq 4 \), we have a fine moduli scheme representing the functor \( F_{M^+,M^-,d} \) from the category of schemes over \( \mathbb{Z}[1/Md] \) to the category of sets which associates to a scheme \( S \) the set of isomorphism classes of abelian surfaces with quaternionic multiplication with level \( M^+ \)-structure over \( S \) and full level \( d \)-structure. We denote it by \( X_{M^+,M^-,d} \). Then the Shimura curve \( X_{M^+,M^-,d} \) is a smooth projective curve over \( \mathbb{Z}[1/Md] \). We have a natural Galois covering

\[
\psi : X_{M^+,M^-,d} \rightarrow X_{M^+,M^-}
\]

with Galois group \( G_d \) isomorphic to \( G'_d/\{\pm 1\} \), where

\[
G'_d := (\mathcal{O}_{B'/d\mathcal{O}_{B'}}) \times \simeq (R'/dR') \times
\]

obtained by forgetting the level \( d \)-structure. We set

\[
U_d = \left\{ g = (g_v)_v \in \hat{R}_{M^+}^\times \mid g_v \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod d \text{ if } v \mid d \right\}.
\]

The complex uniformization of the Shimura curve \( X = X_{M^+,M^-,d} \) is given by

\[
X(\mathbb{C}) = B'^\times \backslash (\mathbb{C} \smallsetminus \mathbb{R}) \times \hat{B}_d^\times /U'_d,
\]

where \( U'_d = \varphi_{B',B'}(U_{d}(p))\mathcal{O}_{B'_p} \). For \( z' \in \mathbb{C} \) and \( b' \in \hat{B}'^\times \), we will denote by \([z',b']_C\) the point on \( X(\mathbb{C}) \) represented by \((z',b')\).

For a prime \( q \) dividing \( M^+ \), we can consider a model of \( X = X_{M^+,M^-,d} \) over \( \mathbb{Z}_q \) using a variant of the moduli functor \( F_{M^+,M^-,d} \). The resulting canonical model \( X_{\mathbb{Z}_q} \) is a nodal model, that is,

1. \( X_{\mathbb{Z}_q} \) is proper and flat over \( \mathbb{Z}_q \), and its generic fiber is \( X \),
2. the irreducible components of the special fiber \( X_{F_q} \) are smooth, and
3. the only singularities of \( X_{F_q} \) are ordinary double points.

For a prime \( p \) which divides \( M^- \), one may define a model \( X_{\mathbb{Z}_p} \) of \( X \) over \( \mathbb{Z}_p \) via moduli scheme. The model \( X_{\mathbb{Z}_p} \) is a nodal model. Moreover, the irreducible components of \( X_{F_p} \) are rational curves.

### 5.2. \( p \)-adic uniformization of Shimura curves

Let \( \mathcal{H}_p \) be the Drinfeld’s \( p \)-adic upper half plane. Then \( \mathbb{C}_p \)-valued points of \( \mathcal{H}_p \) are given by \( \mathcal{H}_p(\mathbb{C}_p) = \mathbb{P}^1(\mathbb{C}_p) \smallsetminus \mathbb{P}^1(\mathbb{Q}_p) \). Let \( \hat{\mathcal{H}}_p \) be a formal model of \( \mathcal{H}_p \) and there is a natural action of \( B^\times \) on \( \hat{\mathcal{H}}_p \) via \( \iota_p \). Fix a nodal model \( X_{\mathbb{Z}_p} \). Write \( \hat{X}_{\mathbb{Z}_p} \) for the formal completion of \( X_{\mathbb{Z}_p} \) along its special fiber. Then \( \hat{X}_{\mathbb{Z}_p} \) is canonically identified with

\[
B^\times \hat{\mathcal{H}}_p \hat{\otimes}_{\mathbb{Z}_p} \hat{\mathbb{Z}}_{\mathbb{Z}_p}^{\text{ur}} \times \hat{B}^{(p)} \times /U'_d^{(p)},
\]

\( \text{TOME 67 (2017), FASCICULE 3} \)
where the action of \( b \in B^\times \) on \( \hat{Z}_p^{ur} \) is given by \( \text{Frob}_p^\text{ord},N(b) \) ([7, Theorem 5.2]). Let \( X^{an} \) be the rigid analytification of \( X_{Z_p} \otimes \mathbb{Q}_p \), then \( X^{an} \) is identified with

\[
B^\times \backslash \mathcal{H}_p \widehat{\otimes} \mathbb{Q}_p \hat{\mathcal{O}}_p^{ur} \times \hat{B}^{(p)} \times U_d^{(p)},
\]

and \( X_{Z_p}(\mathbb{C}_p) \) is identified with

\[
B^\times \backslash \mathcal{H}_p(\mathbb{C}_p) \times \hat{B}^{(p)} \times U_d^{(p)}.
\]

### 5.3. Bad reduction of Shimura curves

Fix a quadratic unramified extension \( \mathbb{Q}_{p^2} \) of \( \mathbb{Q}_p \) which is unique up to isomorphism. We denote the ring of integer of \( \mathbb{Q}_{p^2} \) by \( \mathbb{Z}_{p^2} \) and the residue field by \( \mathbb{F}_{p^2} \). For \( p \nmid M^- \), the dual graph \( \mathcal{G}_p(X) \) of the special fiber of \( X_{\mathbb{F}_{p^2}} \) is defined to be the finite graph determined by the following properties.

1. The set of vertices \( V(\mathcal{G}_p) \) is the set of irreducible components of special fiber \( X_{\mathbb{F}_{p^2}} \).
2. The set of edges \( E(\mathcal{G}_p) \) is the set of singular points of \( X_{\mathbb{F}_{p^2}} \).
3. Two vertices \( v \) and \( v' \) are joined by an edge if \( v \) and \( v' \) intersect at the singular point \( e \).

Then the dual graph \( \mathcal{G}_p(X) \) is identified with \( \mathcal{T}_p/\Gamma \), where \( \mathcal{T}_p = (\mathcal{E}_p(\mathcal{T}_p), \mathcal{V}_p(\mathcal{T}_p)) \) is the Bruhat–Tits tree for PGL\(_2(\mathbb{Q}_p) \), and the \( p \)-adic uniformization of \( \hat{X}_{Z_p} \) induces the following identifications:

1. The set \( E(\mathcal{G}_p) \) is identified with the double coset space \( B^\times \backslash \hat{B}^\times / U_d^{(p)} \), where
   \[
   U_d^{(p)} = \left\{ g = (g_v)_v \in U_d \mid g_p \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod p \right\}.
   \]
2. The set \( V(\mathcal{G}_p) \) is identified with \( (B^\times \backslash \hat{B}^\times / U_d) \times \mathbb{Z}/2\mathbb{Z} \).

### 5.4. CM points on Shimura curves

Let \( z' \) be a point in \( \mathbb{C} \setminus \mathbb{R} \) fixed by \( \iota_\infty(K^\times) \subset \text{GL}_2(\mathbb{R}) \). We define the set of CM points unramified at \( p \) on the Shimura curve \( X \) by

\[
\text{CM}^{p-\text{ur}}_K(X) = \left\{ [z', b']_\mathbb{C} \mid b' \in \hat{B}^\times, b'_p = 1 \right\} \subset X(K^{ab}).
\]

Let \( \text{rec}_K : \hat{K}^\times \rightarrow \text{Gal}(K^{ab}/K) \) be the geometrically normalized reciprocity map. Then by Shimura’s reciprocity law we have

\[
\text{rec}_K(a)[z', b']_\mathbb{C} = [z', \ell(a)b']_\mathbb{C}.
\]

Hence one has \( \iota_p : \text{CM}^{p-\text{ur}}_K(X) \hookrightarrow X(K_p) \).
5.5. Ribet’s exact sequence for higher weight modular forms

Let \( k \) be a positive even integer. Let \( \mathcal{F}_k \) be the lisse \( \ell \)-adic sheaf on the Shimura curve \( X = X_{M+,M-,d} \) which is defined in Diamond-Taylor [13, §3]. We will use the sheaf \( \mathcal{F} = \mathcal{F}_k(\frac{k-2}{2}) \otimes \mathcal{O} \).

Denote the character group and the cocharacter group associated to the integral model \( X_{\mathbb{Z}} \) of the Shimura curve \( X_{M+,M-,d} \) and the sheaf \( \mathcal{F} \) by \( \Phi_p(M^+,M^-,d) \) and \( \widetilde{\Phi}_p(M^+,M^-,d) \). Also we denote by \( \Sigma_p = \Sigma_p(M^+,M^-,d) \) be the set of singular points of the special fiber of \( X_{\mathbb{Z}} \) at \( p \).

We fix a prime \( q \) dividing \( M^- \) such that \( q \neq p \). Let \( \mathcal{T} \) be the Hecke algebra acting on the character group \( \Phi_p(M^+,M^-,d) \). Let \( \mathcal{T}' \) be the Hecke algebra acting on \( \Phi_q(M^{+pq},M^{-/pq},d) \). Let \( \mathcal{T}'' \) be the Hecke algebra acting on \( \Phi_q(M^+,M^-/pq,d)^2 \). Let \( \mathcal{T}_v \) be the polynomial ring with \( \mathbb{Z} \)-coefficient generated by indeterminates \( \tilde{T}_v \) for \( v \nmid Md \) and \( \tilde{U}_v \) for \( v|Md \).

**Proposition 5.1.** — Let \( \mathfrak{m} \) be a non-Eisenstein maximal ideal.

1. (Ribet’s exact sequence) There is a Hecke equivariant exact sequences
   
   \[
   0 \to \widetilde{\Phi}_q(M^+,M^-/pq,d)^2 \to \widetilde{\Phi}_q(M^{+pq},M^-/pq,d)_m \to \widetilde{\Phi}_p(M^+,M^-,d)_m \to 0
   \]
   
   and
   
   \[
   0 \to \Phi_p(M^+,M^-,d)_m \to \Phi_q(M^{+pq},M^-/pq,d)_m \to \Phi_q(M^+,M^-/pq,d)^2_m \to 0.
   \]

2. The action of \( U'_p \in \mathcal{T}' \) on \( \Phi_q(M^+,M^-/pq,d)^2 \) is given by \((x,y) \mapsto (T_p^kx - p^{k-\frac{4}{k}}y, p^{k-1}x)\).

**Proof.** — These results are explained in Rajaei [33, §3.2]. \( \square \)

The Hecke algebra \( \mathcal{T}' \) is isomorphic to the Hecke algebra acting on \( S_k^B(U_d,\mathcal{O}) \), the space of quaternionic modular forms on \( B \) of level \( U_d \). The Hecke algebra \( \mathcal{T} \) is isomorphic to the Hecke algebra acting on \( S_k^{B'}(U'_d,\mathcal{O}) \). Also the Hecke algebra \( \mathcal{T}'' \) is isomorphic to the Hecke algebra acting on the space of quaternionic modular forms on \( B' \) of level \( U'_d \) which are old at \( p \).

**Lemma 5.2.** — There is a canonical map

\[
\omega_p : \text{Ker}[\text{sp}(1)] \to \Phi_p(M^+,M^-,d),
\]

where \( \text{sp}(1) : H^2(X_{\mathbb{Z}} \otimes \overline{\mathbb{F}_{p^2}},\mathcal{F})(1) \to H^2(X_{\mathbb{Z}} \otimes \overline{\mathbb{Q}_{p^2}},\mathcal{F})(1) \) is the specialization map.
Proof. — For $c \in \text{Ker}(\text{sp}(1))$, let $\tilde{c}$ be a lift of $c$ by the map
\[
\bigoplus_{x \in \Sigma_p} (R^1\Phi_F)_x(1) \to H^2(X_{Z_p} \otimes \mathbb{F}_p^2, \mathcal{F})(1).
\]
Then the monodromy pairing induces the map
\[
\bigoplus_{x \in \Sigma_p} (R^1\Phi_F)_x(1) \to \bigoplus_{x \in \Sigma_p} H^1_x(X_{Z_p} \otimes \mathbb{F}_p^2, R\Psi_F).
\]
Also we have a natural surjective map
\[
H^1_x(X_{Z_p} \otimes \mathbb{F}_p^2, R\Psi_F) \to \bigoplus_{x \in \Sigma_p} H^1_x(X_{Z_p} \otimes \mathbb{F}_p^2, R\Psi_F).
\]
Moreover one can see that the image of $\tilde{c}$ in the component group via the composition of the two maps above does not depend on the choice of lift of $c$. Then we define $\omega_p(c)$ by the natural image of $\tilde{c}$. □

Proposition 5.3. — Let $m$ be a non-Eisenstein maximal ideal. Then the map $\omega_p$ induces a $\tilde{T}$-equivariant isomorphism
\[
\varpi_p : \mathcal{X}_q(M^+q, M^-/pq, d)_m \times \mathcal{X}_q(M^+q, M^-/pq, d)_m/((U'_p)^2 - p^{k-2}) \to \Phi_p(M^+, M^-, d)_m.
\]

Proof. — Write $\mathcal{X}_p$ for $X_p(M^+, M^-, d)_m$, $\mathcal{X}_q'$ for $X_q(M^+pq, M^-/pq, d)_m$ and $\mathcal{X}_q''$ for $X_q(M^+q, M^-/pq, d)_m$. Let
\[
\lambda''_q : \mathcal{X}_q'' \times \mathcal{X}_q'' \to \mathcal{X}_q'' \times \mathcal{X}_q''
\]
and
\[
\lambda'_q : \mathcal{X}_q' \to \mathcal{X}_q''
\]
be the monodromy pairings, hence the cokernels are $\Phi'_q \times \Phi''_q$ and $\Phi_q$, where $\Phi''_q = \Phi_q(M^+q, M^-/pq, d)_m$ and $\Phi'_q = \Phi_q(M^+pq, M^-/pq, d)_m$. Let
\[
i : \mathcal{X}_p \to \mathcal{X}_q'
\]
be the map as in the second exact sequence of Proposition 5.1(1) and
\[
(5.1) \quad \delta''_q : \mathcal{X}_q'' \times \mathcal{X}_q'' \to \mathcal{X}_q''/\lambda'_q(i(\mathcal{X}_p))
\]
the map obtained by the first exact sequence of Proposition 5.1(1). Then the cokernel of $\delta''_q$ is $\Phi_p = \Phi_p(M^+, M^-, d)_m$. Let
\[
j_0 : \mathcal{X}_q'' \times \mathcal{X}_q'' \to \hat{\mathcal{X}}_q''/\lambda'_q(i(\mathcal{X}_p)).
\]
be the composition of the map $\lambda''_q$ with $\xi : \hat{\mathcal{X}}_q'' \times \hat{\mathcal{X}}_q'' \to \hat{\mathcal{X}}_q''$ as in the first exact sequence of Proposition 5.1(1). Moreover we define the map
\[
\sigma : \mathcal{X}_q'' \times \mathcal{X}_q'' \to \mathcal{X}_q'' \times \mathcal{X}_q''
\]
by
\[
(x, y) \mapsto ((p + 1)x + T_p y, p \frac{k-2}{2} T_p x + (p + 1)y).
\]
One obtains a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & X''_q & \times & X''_q & \longrightarrow & \Phi''_q & \times & \Phi''_q & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X''_q & \times & X''_q & \longrightarrow & \Phi'_q & \longrightarrow & 0
\end{array}
\]

In fact \(\Phi'_q = \Phi''_q = 0\). A direct calculation shows that the composition of the morphism

\[(x, y) \mapsto (-p^{k-2} x, T_p x - p^{k-2} y)\]

of \(X''_q \times X''_q\) with \(\sigma\) gives the action of \((U'_p)^2 - p^{k-2}\). By the snake lemma, we have the isomorphism

\[X''_q \times X''_q / ((U'_p)^2 - p^{k-2}) \rightarrow \Phi_p(M^+, M^-, d)_m.\]

5.6. Integral Hodge theory following Jordan–Livné

In this section, we give a different description of the component groups following Jordan–Livné [21]. Let \(X_{Z_p}\) be the integral model of the Shimura curve \(X_{M^+,M^-}\) over \(\mathbb{Z}_p\) discussed in the beginning of §5.1. Let \(X_s\) be the special fiber of \(X_{Z_p}^{ur} = X_{Z_p} \otimes \mathbb{Z}_p^{ur}\) and \(X_\eta\) the generic fiber. Define

\[C^0(G_p, \mathcal{F}) := \bigoplus_{y \in I} \mathcal{F}_y (\cong H^2(X_s, \mathcal{F}(1))\]

and

\[C^1(G_p, \mathcal{F}) := \bigoplus_{x \in \Sigma_p} \mathcal{F}_x \left( \cong \bigoplus_{x \in \Sigma_p} (R^1\Phi^\mathcal{F})_x(1) \right),\]

where \(\mathcal{G}_p = \mathcal{G}_p(X)\) is the dual graph of the special fiber of \(X_{Z_p}\) and \(I\) is the set of irreducible components of \(X_s\). We fix an orientation of \(\mathcal{G}_p\), that is, a pair of maps \(s, t : \mathcal{E}(\mathcal{G}_p) \rightarrow \mathcal{V}(\mathcal{G}_p)\) such that \(s(e)\) and \(t(e)\) are the end of the edge \(e\). Consider the map

\[d : C^0(G_p, \mathcal{F}) \rightarrow C^1(G_p, \mathcal{F})\]

defined by \((y \mapsto f_y) \mapsto (x \mapsto f_{t(x)} - f_{s(x)})\), where \(f_y \in \mathcal{F}_y\) and \(f_{s(x)}, f_{t(x)} \in \mathcal{F}_x = H^0_x(s(x), r^*\mathcal{F})(1) = H^0_x(t(x), r^*\mathcal{F})(1)\) (where \(r^* : \tilde{X}_s \rightarrow X_s\) is the normalization map). Note that \(r^*\mathcal{F}\) is a constant sheaf on \(t(x) \cup s(x)\). Then we define the cohomology \(H^i(G_p, \mathcal{F})\) by the exact sequence

\[0 \rightarrow H^0(G_p, \mathcal{F}) \rightarrow C^0(G_p, \mathcal{F}) \xrightarrow{d} C^1(G_p, \mathcal{F}) \rightarrow H^1(G_p, \mathcal{F}) \rightarrow 0.\]
On the other hand, we consider the map
\[ \delta : C^1(G_p, \mathcal{F}) \to C^0(G_p, \mathcal{F}) \]
defined by \( (x \mapsto f_x) \mapsto (y \mapsto \sum_{t(x) = y} f_x). \) The Laplacian \( \Box = \Box_i : C^i(G_p, \mathcal{F}) \to C^i(G_p, \mathcal{F}) \) is defined by \( \Box_i = d\delta + \delta d. \) Hence we have \( \Box_0 = \delta d \) and \( \Box_1 = d\delta. \) A cochain \( c \) is called harmonic if \( \Box_i c = 0. \) Let \( \mathbb{H}^i \) be the \( \mathcal{O} \)-module of all harmonic cochains.

**Definition 5.4.** — We set
\[ \Phi'_p(M^+, M^-, d) := H^1(G_p, \mathcal{F})/\mathbb{H}^1 \]
and
\[ \Phi''_p(M^+, M^-, d) := \delta C^1(G_p, \mathcal{F})/\Box_0 C^0(G_p, \mathcal{F}). \]

**Remark 5.5.** — The definition of \( \Phi'' \) is different from the notation used in Jordan–Livné [21]. The definition of \( \Phi' \) corresponds to the Grothendieck’s description of the component group and the definition of \( \Phi'' \) corresponds to the Raynaud’s description of the component group.

Recall that \( \omega_p \) is the map defined in Lemma 5.2.

**Lemma 5.6 ([21, Proposition 2.14]).** — There are canonical identifications
\[ \Phi'_p(M^+, M^-, d) \cong \Phi''_p(M^+, M^-, d) \cong \Phi_p(M^+, M^-, d). \]
In particular, the map \( \omega_p \) is surjective.

Now for each irreducible component \( Y \) we fix a non-singular point \( P_Y \) on \( Y. \) Let \( \tilde{x} \) be a closed point of \( X_\eta \) such that \( x = \tilde{x} \mod p \) is non-singular. We may assume that \( x = P_Y \) for some irreducible component \( Y. \) Denote
\[ H^2_{\tilde{x}}(X_\eta, \mathcal{F})(1)^0 := \text{Ker} \left[ H^2_{\tilde{x}}(X_\eta, \mathcal{F})(1) \to H^2(X_\eta, \mathcal{F})(1) \right]. \]

**Lemma 5.7.** — There exists a natural map
\[ H^2_{\tilde{x}}(X_\eta, \mathcal{F}) \to H^2_x(X_s, R\Psi \mathcal{F}). \]

**Proof.** — Let \( z \) be the \( \mathbb{Z}_p^{ur} \)-valued point of \( X \) determined by \( \tilde{x}. \) Let \( \tilde{j}' : \tilde{x} \to z, \tilde{i}' : x \to z \) and \( \tilde{i}_z : \tilde{x} \to X_\eta \) be canonical maps. Also define \( \tilde{i}_x \) and \( \tilde{i}_z \) similarly. Then by definition one has
\[ H^2_{\tilde{x}}(X_\eta, \mathcal{F}) = H^2(\tilde{x}, R\tilde{i}_z^\ast \tilde{j}'^\ast \mathcal{F}) \]
and this is isomorphic to \( H^2(x, \tilde{i}'^\ast R\tilde{j}_s^\ast R\tilde{i}_z^\ast \tilde{j}'^\ast \mathcal{F}). \) It is known that the last cohomology group is isomorphic to \( H^2(x, \tilde{i}'^\ast R\tilde{i}_z^\ast R\tilde{j}_s^\ast \tilde{j}'^\ast \mathcal{F}) \) (See Fu [14, Proposition 8.4.9]). Therefore using the adjunction morphism we have a natural
map

\[ H^2(x, i^* R \partial j R \partial j^* \mathcal{F}) \to H^2(x, R \partial j R \partial j^* \mathcal{F}) = H^2_x(\mathcal{F}). \]

We define the reduction map \( \text{red}_p : H^2_x(\mathcal{F})(1)^0 \to H^2_x(i^* \mathcal{F})(1) \) by the composition of the maps

\[
H^2_x(\mathcal{F})(1)^0 \longrightarrow H^2_x(i^* \mathcal{F})(1)
\]

\[
\cong \bigoplus_{Y \in I} H^2_Y(i^* \mathcal{F})(1)
\]

\[
\cong H^2_x(i^* \mathcal{F})(1),
\]

where the first map is obtained by the above lemma and the second map is the inverse of the specialization map

\[
\text{sp}(1)_x : H^2_x(i^* \mathcal{F})(1) \to H^2_x(i^* \mathcal{F})(1)
\]

(since \( x \) is a smooth point, \( \text{sp}(1)_x \) is an isomorphism). Then the image of the reduction map is contained in the kernel of the specialization map

\[
\text{sp}(1) : H^2(i^* \mathcal{F})(1) \to H^2(i^* \mathcal{F})(1).
\]

Using the identification of component groups, we define the map

\[
d_p : H^2_x(\mathcal{F})(1)^0 \to \Phi_p(M^+, M^-, d)
\]

by the composition of the maps

\[
H^2_x(\mathcal{F})(1)^0 \xrightarrow{\text{red}_p} \text{Ker}[\text{sp}(1)] \xrightarrow{\cong} \delta C^1(\mathcal{G}, \mathcal{F})
\]

\[
\to \Phi''_p(M^+, M^-, d) \cong \Phi_p(M^+, M^-, d).
\]

Combining these facts, we have the following proposition.

**Proposition 5.8.** — For \( c \in H^2_x(\mathcal{F})(1)^0 \), we have

\[ d_p(c) = \omega_p(\text{red}_p(c)). \]

**Proof.** — By the definition of the maps, it suffices to show that \( \omega_p \) can be identified with the map

\[
\kappa : \text{Ker}[\text{sp}(1)] \xrightarrow{\cong} \delta C^1(\mathcal{G}, \mathcal{F}) \to \Phi''_p(M^+, M^-, d) \cong \Phi_p(M^+, M^-, d).
\]

By Lemma 5.6, the map \( \omega_p \) can be identified with the natural quotient map. Therefore the map \( \omega_p \) coincides with \( \kappa \).
6. Level raising of modular forms

In this section, we prove a level raising result for modular forms on quaternion algebras.

6.1. A freeness result for the space of modular forms

Let $N$ be a positive integer and $N = N^+N^-$ a integer factorization of $N$, where $N^-$ is a square-free product of an odd number of primes. Let $\tilde{f} : B^x \backslash \tilde{B}^x \to L_{k-2}(\mathcal{O})$ be a $\lambda$-adically normalized $\ell$-adic modular form corresponding to $f$ via the Jacquet–Langlands correspondence. Let $q$ be a prime number dividing $N^-$ and $p$ a prime number which does not divide $N$. Let $B'$ be the indefinite quaternion algebra over $\mathbb{Q}$ with discriminant $pN^-$. Choose a positive integer $d$ such that $(d,Np) = 1$ and $\ell \nmid \#(\mathcal{O}_{B'}/d\mathcal{O}_{B'})^\times$. Write $T_{B_{k}}(N^+)$ (resp. $T_{B'}(N^+)$) for the Hecke algebra acting on the space of $\ell$-adic modular forms $S_{B_{k}}(N^+)$ (resp. $S_{B'}(N^+,\mathcal{O})$). Set $\mathbb{T} = \mathbb{T}_{B_{k}}(N^+)$ and $\mathbb{T}[p] = \mathbb{T}_{B'}(N^+)\mathcal{O}$. We denote by $T_v$ and $U_v$ (resp. $T_v$ and $U_v$) the Hecke operators in $\mathbb{T}$ (resp. $\mathbb{T}[p]$). The modular form $\tilde{f}$ yields a surjective homomorphism

$$\lambda_f : \mathbb{T} \to \mathcal{O}_n.$$ 

We write $\mathcal{I}_f$ for the kernel of $\lambda_f$, and $\mathfrak{m}$ for the unique maximal ideal of $\mathbb{T}$ containing $\mathcal{I}_f$.

**Proposition 6.1.** — Assume that the residual Galois representation $\bar{\rho}_f$ satisfies $\text{CR}^+$. Then $S_{B_{k}}(N^+,\mathcal{O})\mathfrak{m}$ is a cyclic $\mathbb{T}\mathfrak{m}$-module.

**Proof.** — Since this proposition follows from the same argument as in [8, Proof of Proposition 6.8] and [36, §2 and §3], we only give a sketch of the proof. Let $M^+$ be an integer such that $(M^+,N^-) = 1$ and let $M = M^+N^-$. Write $S(M) = S_{B_{k}}M^+,\mathcal{O}$. Let $\mathbb{T}(M)$ be the Hecke algebra generated over $\mathcal{O}$ by the Hecke operators $T_q$ for $q \nmid M$ and $U_q$ for $q | M$ in $\text{End}_{\mathcal{O}} S(M)$. Let $\lambda_{\pi^\prime} : \mathbb{T}(N) \to \mathcal{O}$ be the $\mathcal{O}$-algebra homomorphism induced by $\pi'$. We denote by $N(\bar{\rho}_f)$ the Artin conductor of $\bar{\rho}_f$. Let $N_1^-$ be the product of prime factors of $N^-$ but not dividing $N(\bar{\rho}_f)$. We set $N_0 = N(\bar{\rho}_f)N_1^-$. By level lowering and raising, there exists a modular lifting $\lambda_0 : \mathbb{T}(N_0) \to \mathcal{O}$ such that $\lambda_0(T_q) = \lambda_{\pi^\prime}(T_q) \mod \mathfrak{m}\mathcal{O}$ for all $q \nmid N$. We write

$$N = N_0 \prod_{q} q^{m_q}.$$
Let $\Sigma$ be a set of prime factors of $N/N_\emptyset$ and set $N_\Sigma = N_\emptyset \prod_{q \in \Sigma} q^{m_q}$. Let $m_\Sigma$ be the maximal ideal of $T(N_\Sigma)$ generated by $m_\emptyset, T_q - \lambda_\emptyset(T_q)$ for $q \nmid N_\Sigma$ and $U_q - \lambda_{p^r}(U_q)$ for $q \mid N_\Sigma$. Let $T_\Sigma = T(N_\Sigma)m_\Sigma$ be the localization at $m_\Sigma$. Similarly, we denote the localization of $S(N_\Sigma)$ at $m_\Sigma$ by $S_\Sigma$. By [8, Lemma 6.3], we have a surjection $T_\Sigma \twoheadrightarrow T_\emptyset$. Let $\lambda_\Sigma : T_\Sigma \twoheadrightarrow T_\emptyset \xrightarrow{\lambda_\emptyset} O$ be the composition and $I_{\lambda_\Sigma}$ the kernel of $\lambda_\Sigma$. Set

$$S_\Sigma[\lambda_\Sigma] = \{x \in S_\Sigma \mid I_{\lambda_\Sigma}x = 0\}$$

and

$$S_\Sigma[\lambda_\Sigma]^{\perp} = \{x \in S_\Sigma \otimes_O E \mid \langle x, y \rangle_{R_\Sigma} = 1 \text{ for all } y \in S_\Sigma[\lambda_\Sigma]\},$$

where $R_\Sigma = R_{N_\Sigma/N}$ is an Eichler order of level $N_\Sigma/N^-$. Then $S_\Sigma[\lambda_\Sigma]^{\perp} \supset S_\Sigma[\lambda_\Sigma]$. We define the congruence module of $\lambda_\Sigma$ by $C(N_\Sigma) = S_\Sigma[\lambda_\Sigma]^{\perp}/S_\Sigma[\lambda_\Sigma]$ and the congruence ideal of $\lambda_\Sigma$ by $\eta_\Sigma = \lambda_\Sigma(\Ann_{T_\Sigma}(I_{\lambda_\Sigma}))$.

Let $M_{F_{Q_\ell}, O, k}$ denote the abelian category whose objects are finite length $O$-modules $D$ together with a distinguished submodule $D^0$ and $\text{Frob}_{Q_\ell} \otimes 1$-semilinear maps $\varphi_{1-k} : D \to D$ and $\varphi_0 : D^0 \to D$ such that

- $\varphi_{1-k}|D^0 = \ell^{k-1}\varphi_0$,
- $\text{Im} \varphi_{1-k} + \text{Im} \varphi_0 = D$.

Then there is a fully faithful, $\mathbb{Z}_\ell$-length preserving, $O$-additive, contravariant functor $M$ from $M_{F_{Q_\ell}, O, k}$ to the category of continuous $O[\text{Gal}((\overline{Q_\ell}/Q_\ell))]$-modules with essential image closed under the formation of sub-objects.

Consider the functor $D_\Sigma$ from the category of local Noetherian complete $O$-algebra with the residue field $k_O = O/m_O$ to the category of sets which sends $A$ with the maximal ideal $m_A$ to the isomorphism class of deformations $\rho : \text{Gal}(\overline{Q}/Q) \to \text{GL}_2(A)$ of $\overline{\rho}_f$ satisfying

1. $\det \rho = \varepsilon_\ell$, where $\varepsilon_\ell : \text{Gal}(\overline{Q}/Q) \to \mathbb{Z}_\ell^\times$ is the $\ell$-adic cyclotomic character,
2. $\rho$ is minimally ramified outside $N^- \Sigma$,
3. for each finite length quotient $A/I$ of $A$ the $O[\text{Gal}((\overline{Q_\ell}/Q_\ell))]$-module $(A/I)^2$ is isomorphic to $M(D)$ for some object $D$ of $M_{F_{Q_\ell}, O, k}$,
4. for $q || N_\Sigma/N_\emptyset$, there exists a unramified character $\delta_q : \text{Gal}(\overline{Q_q}/Q_q) \to A^\times$ such that $\rho|_{\text{Gal}(\overline{Q_q}/Q_q)} \sim \begin{pmatrix} \delta_q^{-1}\varepsilon_\ell & * \\ 0 & \delta_q \end{pmatrix}$ and $\delta_q(\text{Frob}_q) \equiv 1 \mod m_A$, and
(5) if \( q \mid N_1^- \), then \( \rho \big|_{\text{Gal}(Q_q/Q_q)} \) satisfies
\[
\rho \big|_{\text{Gal}(Q_q/Q_q)} \sim \begin{pmatrix} \pm \varepsilon_\ell & * \\ 0 & \pm 1 \end{pmatrix}
\]
with \( * \in m_A \).

Under the assumption \( (\text{CR}^+) \), it is known that \( D_\Sigma \) is represented by the universal deformation
\[
\rho_{R_\Sigma} : \text{Gal}(\overline{Q}/Q) \to \text{GL}_2(R_\Sigma).
\]
Then the universality of \( R_\Sigma \) gives rise to surjections of \( \mathcal{O} \)-algebras
\[
R_\Sigma \twoheadrightarrow R_\emptyset \xrightarrow{\lambda_\emptyset} \mathcal{O}.
\]

By the Taylor–Wiles argument in [36, §2], we deduce that \( S(\mathcal{O}_m)_{n_0} \) is a free \( \mathcal{T}_\emptyset \)-module of rank one and
\[
\#(\varphi_\emptyset/\varphi_\emptyset^2) = \#C(\mathcal{O}_\emptyset) = \#(\mathcal{O}/\eta_\emptyset).
\]

Using the argument in [36, §3], we have
\[
\#(\varphi_{Q_2}/\varphi_{Q_2}^2) = \#C(\mathcal{O}_{Q_2}) = \#(\mathcal{O}/\eta_{Q_2}),
\]
where \( Q_2 \) is the set of prime factors \( q \mid N/N_\emptyset \) with \( m_q = 2 \). By [8, Lemma 6.4] and [8, Corollary 6.7], the above equality implies
\[
\#(\varphi_\Sigma/\varphi_\Sigma^2) \mid \#C(\mathcal{O}_\Sigma) \mid (\mathcal{O}/\eta_\Sigma).
\]

Then the proposition follows from [12, Theorem 2.4].

**Proposition 6.2.** — Let \( \psi_f : S^B_k(N^+,\mathcal{O}) \to \mathcal{O} \) be the map defined by
\( h \mapsto \psi_f(h) := \langle \hat{f}, h \rangle_R \), where \( R = R_{N^+} \). Then \( \psi_f \) induces an isomorphism
\[
\psi_f : S^B_k(N^+,\mathcal{O})/I_f \cong \mathcal{O}_n.
\]

**Proof.** — By Proposition 6.1, \( S^B_k(N^+,\mathcal{O})_m \) is a cyclic \( \mathcal{T}_m \)-module. Hence \( S^B_k(N^+,\mathcal{O})/I_f \) is generated by a modular form \( g \). Since \( \psi_f \) is surjective and the Hecke operators in \( \mathcal{T} \) are self-adjoint with respect to the pairing \( \langle \ , \ , \rangle_R \), we have that \( \psi_f(g) = \langle f, g \rangle_R \in \mathcal{O}_n^\ast \) and the annihilator of \( g \) in \( \mathcal{T} \) is \( I_f \). Therefore we have an isomorphism \( S^B_k(N^+,\mathcal{O})/I_f \cong \mathcal{T}/I_f \cong \mathcal{O}_n. \)
6.2. Level raising

Theorem 6.3. — Let $p$ be an $n$-admissible prime. Assume that the residual Galois representation $\overline{\rho}_f$ satisfies (CR$^+$). Then

(1) There exists a surjective homomorphism
\[ \lambda_f^{[p]} : T^{[p]} \to O_n \]
such that $\lambda_f^{[p]}(T_q) = \lambda_f(t_q)$ for all $q \nmid Np$, $\lambda_f^{[p]}(U_q) = \lambda_f(u_q)$ for all $q \mid N$, and $\lambda_f^{[p]}(U_p) = \varepsilon \cdot p^{k-2} - \varepsilon \cdot \lambda_f(t_p)$, where $\varepsilon = \pm 1$ is such that $\lambda^n$ divides $p^{k-2} + p^{k-2} - \varepsilon \cdot \lambda_f(t_p)$.

(2) Let $I_f^{[p]} \subset T^{[p]}$ denote the kernel of the homomorphism $\lambda_f^{[p]}$ and $\Phi_p(N^+, N^-)$ the component group associated to the Shimura curve $X_{N^+, N^-}$ and the lisse sheaf $\mathcal{I}$. Then there is a group isomorphism
\[ \Phi_p(N^+, N^-)/I_f^{[p]} \cong S_B^B(N^+, O)/\mathcal{I}_{\mathcal{F}} \cong O_n. \]

Proof. — Let $B'$ be the indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $N^-/q$ and $R'_{N^+, q}$ an Eichler order of level $N^+q$. Denote the Shimura curve associated to $U'_d$ by $X_{U'_d}$. Also we write the character group for the Shimura curve $X_{U'_d}$ and the lisse $\ell$-adic sheaf $\mathcal{F}$ at $q$ by $X_q(U'_d)$. Let $\Sigma_q(U'_d)$ be the set of singular points on the special fiber of $X_{U'_d}$. Moreover since $\Sigma_q(U'_d)$ is identified with $B^\times \backslash \tilde{B}^\times /U_d$, we obtain the identification
\[ \bigoplus_{x \in \Sigma_q(U'_d)} (R^1\Phi_3)_x \cong \bigoplus_{x \in \Sigma_q(U'_d)} L_k(O) \cong S_k^B(U_d, O). \]

Taking $\tilde{R}_{N^+, d}$-invariant part, we obtain the Hecke-equivariant isomorphism
\[ \bigoplus_{x \in \Sigma_q} (R^1\Phi_3)_x \cong S_k^B(N^+, O), \]
where $\Sigma_q$ is the set of singular points on the special fiber of a model of $X_{N^+, N^-}/q$. By [33, Proposition 5], we have
\[ X_q(N^+, N^-/q)_m \cong \left( \bigoplus_{x \in \Sigma_q} (R^1\Phi_3)_x \right)_m. \]

Therefore by Proposition 6.1 one obtains the isomorphism
\[ X_q(N^+, N^-/q)^2 / \mathcal{I}_f \cong O_n^2. \]
We denote by $T'_v$ and $U'_v$ the Hecke operators in $T[\nu]$. There is an action of $T[\nu]$ on $X_\eta(N^+q, N^-/q)^2$ induced by $t_v$ for $v \nmid N\nu$ and $u_v$ for $v | N$ and the Hecke operator $U'_v$ acts via the formula

$$(x, y) \mapsto (T''_v x - p^{-\frac{k-4}{2}} y, p^{k-1} x).$$

Since $p$ is $n$-admissible, the action of $t_p$ modulo $I_f$ is given by $\varepsilon \cdot (p^{\frac{k}{2}} + p^{\frac{k-2}{2}})$. Then the determinant of $U'_v + \varepsilon \cdot p^{\frac{k-2}{2}}$ is invertible on $X_\eta(N^+q, N^-/q)^2 / I_f$. These facts yield an isomorphism

$$X_\eta(N^+q, N^-/q)^2 / (I_f, U'_v - \varepsilon \cdot p^{\frac{k-2}{2}}) \cong X_\eta(N^+q, N^-/q)^2 / (I_f, U'_v - p^{k-2}) \cong \mathcal{O}_n.$$ 

Thus, the action of $T[\nu]$ on $X_\eta(N^+q, N^-/q)^2 / (I_f, U'_v - \varepsilon \cdot p^{\frac{k-2}{2}})$ is given via a surjective homomorphism

$$\lambda'_f : T[\nu] \to \mathcal{O}_n.$$ 

Denote the kernel of $\lambda'_f$ by $I'_f$. Then Propositions 5.1, 5.3 and the residual irreducibility of $\mathfrak{m}$ imply the existence of an isomorphism

$$\Phi_p(N^+, N^-) / I'_f \cong X_\eta(N^+q, N^-/q)^2 / (I_f, (U'_v)^2 - p^{k-2}).$$

This shows that $\lambda'_f$ factors through $T$ which gives $\lambda'_f$ and $\Phi_p(N^+, N^-) / I'_f$ is isomorphic to $\mathcal{O}_n$. Let $\mathfrak{m}[\nu]$ be the maximal ideal of $T[\nu]$ containing $I'_f$. The embedding $S_k^B(N^+, \mathcal{O})_{\mathfrak{m}[\nu]} \to S_k^B(N^+, \mathcal{O})_{\mathfrak{m}[\nu]}$ given by $x \mapsto (x, 0)$ induces an isomorphism

$$S_k^B(N^+, \mathcal{O})_{\mathfrak{m}[\nu]} / (\varepsilon T_p - p^{\frac{k}{2}} - p^{\frac{k-2}{2}}) \cong S_k^B(N^+, \mathcal{O})_{\mathfrak{m}[\nu]} / (U'_v - \varepsilon p^{\frac{k-2}{2}}).$$

Therefore we have

$$\Phi_p(N^+, N^-) / I'_f \cong S_k^B(N^+, \mathcal{O}) / I_f \cong \mathcal{O}_n.$$

Write $X^\nu_d$ for the Shimura curve $X_{N^+, N^- d}$, $X_{d, p}$ for $X_p(N^+, N^- p, d)$, $\widehat{X}_{d, p}$ for $\widehat{X}_p(N^+, N^- p, d)$ and $\Phi_{d, p}$ for $\Phi_p(N^+, N^- p, d)$. Also write $X^\nu$ for the Shimura curve $X_{N^+, N^-}$, $X_p$ for $X_p(N^+, N^-)$, $\widehat{X}_p$ for $\widehat{X}_p(N^+, N^-)$ and $\Phi_p$ for $\Phi_p(N^+, N^-)$.

**Proposition 6.4.** — Let $p$ be an $n$-admissible prime. Under the assumption (CR), the Galois representations $H^1(X^\nu \otimes \mathbb{Q}, \mathcal{F})(1) / I'_f$ and $T_{f, n}$ are isomorphic.
Proof. — Let \( m_f^{[p]} \) be the maximal ideal containing \( T_f^{[p]} \). Then \( \mathbb{T}^{[p]}/m_f^{[p]} \) is isomorphic to \( \mathcal{O}_1 = \mathbb{F} \). First we will show that \( H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathcal{F})(1)/m_f^{[p]} \) is isomorphic to \( T_{f,1} \). By (4.1) and the fact \( H^1(X^{[p]}_\mu \otimes \overline{\mathbb{F}}_{p^2}, \mathcal{F}) \cong \hat{X}_p \) (see Rajaei [33, p. 52 (3.5)]), one obtains an exact sequence

\[
0 \to (\hat{X}_p/m_f^{[p]}) \otimes \mu_\lambda \to H^1(X^{[p]} \otimes \overline{\mathbb{Q}}_{p^2}, \mathcal{F})/m_f^{[p]} \otimes \mu_\lambda \to X_p/m_f^{[p]} \to 0,
\]

where \( \mu_\lambda = \mathbb{Z}_p(1) \otimes \mathcal{O}/\mathcal{O} \). Taking the Galois cohomology over \( \mathbb{Q}_{p^2} \), we have the exact sequence

\[
X_p/m_f^{[p]} \to H^1(\mathbb{Q}_{p^2}, (\hat{X}_p/m_f^{[p]}) \otimes \mu_\lambda) \to H^1(\mathbb{Q}_{p^2}, H^1(X^{[p]} \otimes \overline{\mathbb{Q}}_{p^2}, \mathcal{F})/m_f^{[p]} \otimes \mu_\lambda).
\]

Since \( p \) is an admissible prime, \( \lambda \) does not divide \( p^2 - 1 \), hence we have the identification

\[
H^1(\mathbb{Q}_{p^2}, \hat{X}_p/m_f^{[p]} \otimes \mu_\lambda) \cong \hat{X}_p/m_f^{[p]}.
\]

By the main theorem of [6] and the Eichler–Shimura relation, \( H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathcal{F})(1)/m_f^{[p]} \) is semisimple over \( \mathbb{F}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})] \), we have that

\[
H^1(X^{[p]} \otimes \overline{\mathbb{Q}}, \mathcal{F})(1)/m_f^{[p]} \cong (T_{f,1})^r
\]

for some \( r \geq 1 \). Therefore \( H^1(\mathbb{Q}_{p^2}, H^1(X^{[p]} \otimes \overline{\mathbb{Q}}_{p^2}, \mathcal{F})(1)/m_f^{[p]} \) is isomorphic to \( H^1(\mathbb{Q}_{p^2}, T_{f,1})^r \). By Lemma 3.5, the \( \mathbb{F} \)-vector space \( H^1(\mathbb{Q}_{p^2}, T_{f,1})^r \) is 2r-dimensional. We claim that

\[
\dim_{\mathbb{F}} X_p/m_f^{[p]} \geq r.
\]

To see this, assume that \( \dim_{\mathbb{F}} X_p/m_f^{[p]} \leq r - 1 \). Then we have \( \dim_{\mathbb{F}} \hat{X}_p/m_f^{[p]} \geq r + 1 \) by the exact sequence (6.1), which implies \( \dim_{\mathbb{F}} \Phi_p/m_f^{[p]} \geq 2 \) by the definition of the component group. This gives a contradiction.

By the Picard–Lefschetz formula, the monodromy operator \( N \) is described as \( N(a \otimes t_\ell(\sigma)) = \sigma(a) - a \) for all \( a \in H^1(X^{[p]} \otimes \overline{\mathbb{Q}}_{p^2}, \mathcal{F})(1)/m_f^{[p]} \) and \( \sigma \in I \). One notices that the monodromy operator \( N \) acts on each piece \( T_{f,1} \), thus \( N \) defines the map \( N : T_{f,1}(-1) \otimes \mu_\lambda \to T_{f,1}(-1) \).

**Lemma 6.5.** — The map \( N : T_{f,1}(-1) \otimes \mu_\lambda \to T_{f,1}(-1) \) is the zero map. Equivalently, the monodromy pairing is the zero map. In particular, \( \hat{X}_p/m_f^{[p]} \) is isomorphic to \( \Phi_p/m_f^{[p]} \).

**Proof.** — If \( N \) is non-trivial, we have the inequality

\[
\dim_{\mathbb{F}} \text{Im} \left[ N : H^1(\mathbb{Q}_{p^2}, T_{f,1}(-1))^r \to H^1(\mathbb{Q}_{p^2}, T_{f,1})^r \right] \geq r.
\]

The definition of the monodromy pairing implies

\[
\text{Im}(N) = \text{Im} \left[ \lambda_p : X_p/m_f^{[p]} \to \hat{X}_p/m_f^{[p]} \right].
\]
where $\lambda_p$ is the monodromy pairing and its cokernel is the component group $\Phi_p/m_f^{[p]}$. Since

$$\dim_\mathbb{F} \hat{X}_p/m_f^{[p]} - \dim_\mathbb{F} \text{Im}(N) = \dim_\mathbb{F} \Phi_p/m_f^{[p]} = 1,$$

we have the inequality $\dim_\mathbb{F} H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/m_f^{[p]} \geq r + 1$ by (6.2). Hence one obtains

$$\dim_\mathbb{F} H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/m_f^{[p]} \geq 2r + 1.$$ 

This gives a contradiction. □

Since $\lambda \nmid (p^2 - 1)$, we have the identifications

$$H^1(Q_{p^2}, X_p/m_f^{[p]}) = \text{Hom}_{\text{unr}}(\text{Gal}((\overline{Q}_{p^2}/Q_{p^2}), X_p/m_f^{[p]}))$$

$$= \text{Hom}((\mathcal{O}/\lambda \mathcal{O}, X_p/m_f^{[p]}).$$

Therefore we have the exact sequence

(6.3)

$$\overline{\Phi}_p/m_f^{[p]} \rightarrow H^1(Q_{p^2}, H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/m_f^{[p]}) \rightarrow H^1_{\text{unr}}(Q_{p^2}, X_p/m_f^{[p]}),$$

where $\overline{\Phi}_p/m_f^{[p]}$ is a quotient of $\Phi_p/m_f^{[p]}$. Recall that $H^1(Q_{p^2}, H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/m_f^{[p]}$) can be decomposed as the direct sum of two $r$-dimensional subspaces. Furthermore, one subspace is generated by unramified cohomology classes and the other by ramified cohomology classes. By Theorem 6.3, the group $\Phi_p/m_f^{[p]}$ is isomorphic to $\mathcal{O}/\lambda \mathcal{O}$. Hence by the exact sequence (6.3) we have $r = 1$ and $\overline{\Phi}_p/m_f^{[p]} \cong \Phi_p/m_f^{[p]}$, Therefore $H^1(X^{[p]} \otimes Q_{p^2}, \mathcal{F})(1)/m_f^{[p]}$ is isomorphic to $T_{f,1}$.

Next we show that $H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/m_f^{[p]}$ is isomorphic to $T_{f,n}$. There is a natural $\text{Gal}(\overline{Q}/Q)$-equivariant projection

$$H^1(X^{[p]} \otimes \overline{Q}, \mathcal{F})(1)/m_f^{[p]} \rightarrow H^1(X^{[p]} \otimes \overline{Q}, \mathcal{F})(1)/m_f^{[p]}.$$ 

By the exact sequence

$$0 \rightarrow \hat{X}_p(1)/T_f^{[p]} \rightarrow H^1(X^{[p]} \otimes \overline{Q}_{p^2}, \mathcal{F})(1)/T_f^{[p]} \rightarrow X_p/T_f^{[p]} \rightarrow 0$$

and the fact that the group $\Phi_p/T_f^{[p]}$ is isomorphic to $\mathcal{O}_n$, we can take an element $t_1$ in $H^1(X^{[p]} \otimes \overline{Q}, \mathcal{F})(1)/T_f^{[p]}$ which generates a subgroup $C$ isomorphic to $\mathcal{O}_n$. Hence we can choose $t_1, t_2 \in H^1(X^{[p]} \otimes \overline{Q}, \mathcal{F})(1)/T_f^{[p]}$ such that $H^1(X^{[p]} \otimes \overline{Q}, \mathcal{F})(1)/T_f^{[p]} \cong \mathcal{O}_n \cdot t_1 \bigoplus \mathcal{O}_r \cdot t_2$ with $r \leq n$. Since the restriction of the residual Galois representation $\overline{\rho}_f$ to the absolute Galois
group of $\mathbb{Q}(\sqrt{(-1)^{\frac{p-1}{2}}})$ is absolutely irreducible by (CR$^+$)(2), $\mathfrak{p}_f$ is also absolutely irreducible. Thus one has

$$\mathfrak{p}_f(\mathbb{F}[G_\mathbb{Q}]) = \text{End}_\mathbb{F}(T_{f,1}) = \text{End}_{\mathcal{O}}(H^1(X[p] \otimes \mathcal{O}, \mathcal{F})(1)/m_f[p]).$$

Therefore there exist $h \in \mathfrak{p}_f(\mathbb{F}[G_\mathbb{Q}])$ such that $ht_2 = at_1 + bt_2$ with $a \in \mathcal{O}^\times$, $b \in \mathcal{O}$. This implies $r = n$ and $H^1(X[p] \otimes \mathcal{O}, \mathcal{F})(1)/T_f[p]$ is isomorphic to $\mathcal{O}^2$. Hence, $H^1(X[p] \otimes \mathcal{O}, \mathcal{F})(1)/T_f[p]$ is isomorphic to $T_{f,n}$. □

Let $O_{K,m} = \mathbb{Z} + mO_K$ be the order of the imaginary quadratic field $K$ of conductor $m$. Let $K_m$ be the ring class field of $K$ of conductor $m$. Write $\Phi_{p,m}$ for $\bigoplus_{p|m} \Phi_p$, where the sum is taken over the primes $p$ of $K_m$ and $\Phi_p$ denotes the component group associated to the Shimura curve $X[p]$ and the lisse sheaf $\mathcal{F}$ at $p$. Since the prime $p$ is inert in $K$, it splits completely in $K_m$. Hence, the choice of a prime of $K_m$ above $p$ identifies $\Phi_{p,m}$ with $\Phi_p[G_m]$. Therefore, we have an isomorphism

$$\Phi_{p,m}/T_f[p] \cong \mathcal{O}[G_m].$$

For $X = X[p]$ or $X_d[p]$, let $X_\eta$ be the generic fiber of $X_{Z_p} \otimes \mathbb{Z}_{\nu}$ and $X_s$ the special fiber. For a $\mathbb{Q}_p^{ur}$-valued point $x$ on $X$, denote

$$H^0_x(X_\eta, \mathcal{F})(1)^0 := \text{Ker}[H^2_x(X_\eta, \mathcal{F})(1) \rightarrow H^2(X_\eta, \mathcal{F})(1) \rightarrow H^2(X_\eta, \mathcal{F})(1)].$$

Then we have a canonical map

$$H^0_x(X_\eta, \mathcal{F})(1)^0 \rightarrow H^2(X_\eta, \mathcal{F})(1)^0 := \text{Ker}[H^2_x(X_\eta, \mathcal{F})(1) \rightarrow H^2(X_\eta, \mathcal{F})(1)].$$

Let $I_{Q_p}$ be the inertia group and $I_{Q_p}^t$ the tame inertia. By the Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(I_{Q_p}, H^j(X_\eta, \mathcal{F})(1)) \Rightarrow H^{i+j}(X_\eta, \mathcal{F})(1)$$

we obtain a map $H^2(X_\eta, \mathcal{F})(1)^0 \rightarrow H^1(I_{Q_p}, H^1(X_\eta, \mathcal{F})(1)).$ Assume that $d \geq 4$. Since $X_{d,Z_p}^p$ is semistable, $R\Psi F$ is tame (Illusie [16, Theorem 1.2]). Therefore this map induces

$$\alpha : H^2(X_{d,\eta}, \mathcal{F})(1)^0 \rightarrow H^1(I_{Q_p}^t, H^1(X_{d,\eta}^p, R\Psi \mathcal{F})(1)) \cong H^1(I_{Q_p}^t, H^1(X_{d,\eta}^p, \mathcal{F})(1)).$$

On the other hand, we have a map

$$H^2_x(X_{d,\eta}^p, \mathcal{F})(1)^0 \rightarrow H^1(I_{Q_p}^t, H^1(X_{d,\eta}^p, \mathcal{F})(1))$$

by the composition

$$H^2_x(X_{d,\eta}^p, \mathcal{F})(1)^0 \rightarrow H^2_x(X_{d,\eta}^p, \mathcal{F})(1)^0 \rightarrow H^1(I_{Q_p}^t, H^1(X_{d,\eta}^p, \mathcal{F})(1)).$$
where the map $\beta$ is induced by the monodromy pairing
\[
H^0(I_{Q_p}^t, X_{d,p})(\cong X_{d,p}) \xrightarrow{\lambda} H^1(I_{Q_p}^t, H^1(X_{d,p}^{[p]}, i^*F)(1)) \cong H_{d,p}.
\]

\[\rightarrow H^1(I_{Q_p}^t, H^1(X_{d,\eta}^{[p]}, \mathcal{F})(1)).\]

**Theorem 6.6.**

(1) Let $x$ be a $\mathbb{Q}_p^{ur}$-valued point on $X^{[p]}$ such that $x \mod p$ is a nonsingular point. Then, there is a commutative diagram

\[
\begin{array}{cccc}
H^2_x(X^{[p]}_\eta, \mathcal{F})(1)^0 & \xrightarrow{\Phi_p} & H^2(X^{[p]}_\eta, \mathcal{F})(1)^0 \\
\downarrow & & \downarrow \alpha \\
\Phi_p & \xrightarrow{\beta} & H^1(I_{Q_p}^t, H^1(X^{[p]}_{\eta}, \mathcal{F})(1)).
\end{array}
\]

(2) The map $\beta$ induces an isomorphism

\[
\Phi_p/I_f^{[p]} \simeq H^1_{sing}(\mathbb{Q}_p^2, T_{f,n}).
\]

**Proof.** For the first part, it is enough to show the commutativity of the following diagram:

\[
\begin{array}{cccc}
H^2(X^{[p]}_\eta, i^*\mathcal{F})(1)^0 & \xrightarrow{\text{sp}(1)} & H^2(X^{[p]}_\eta, \mathcal{F})(1)^0 \\
\downarrow \omega_p & & \downarrow \alpha \\
\Phi_p & \xrightarrow{\beta} & H^1(I_{Q_p}^t, H^1(X^{[p]}_{\eta}, \mathcal{F})(1)).
\end{array}
\]

Fix a topological generator $\sigma$ of the tame inertia $I_{Q_p}^t$. First we work with the Shimura curve $X^{[p]}_d$ instead of $X^{[p]}$. By [34, Lemma 1.6], we have a distinguished triangle

\[
\rightarrow i^* Rj_* \Lambda \rightarrow R\Psi \Lambda \xrightarrow{\sigma^{-1}} R\Psi \Lambda \xrightarrow{+1},
\]

where $\Lambda = \mathbb{Z}_\ell$. Since the action of $\sigma$ on $i^* \mathcal{F}$ is trivial and $\mathcal{F}$ is extended to the model of $X$ smoothly, we have an isomorphism $i^* \mathcal{F} \otimes R\Psi \Lambda \cong R\Psi \mathcal{F}$. Therefore one has a distinguished triangle

\[
\rightarrow i^* Rj_* \mathcal{F} \rightarrow R\Psi \mathcal{F} \xrightarrow{\sigma^{-1}} R\Psi \mathcal{F} \xrightarrow{+1}.
\]

Let $\gamma$ be the composition of morphisms

\[
R\Phi \mathcal{F} \xrightarrow{\bigoplus_{x \in \Sigma}} \bigoplus_{x \in \Sigma} R\Phi x \xrightarrow{\bigoplus_{x \in \Sigma} Var(\sigma) x} \bigoplus_{x \in \Sigma} i_{x *} i_{x}^! R\Psi x \xrightarrow{\text{adj}} R\Psi \mathcal{F}.
\]
Then we have the following commutative diagram:

\[
\begin{array}{cccccc}
\longrightarrow & i^* \mathcal{F} & \longrightarrow & R \Psi \mathcal{F} & \longrightarrow & R \Phi \mathcal{F} \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\longrightarrow & i^* R j_* \mathcal{F} & \longrightarrow & R \Psi \mathcal{F} & \longrightarrow & R \Phi \mathcal{F}
\end{array}
\]

Taking cohomology \( H^i(X_{d,s},-) \), one has the following commutative diagram:

\[
\begin{array}{cccccc}
H^1(X_{d,s}, R \Psi \mathcal{F}) & \longrightarrow & H^1(X_{d,s}, R \Phi \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, i^* \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, R \Psi \mathcal{F}) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^1(X_{d,s}, R \Psi \mathcal{F}) & \longrightarrow & H^1(X_{d,s}, R \Phi \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, i^* R j_* \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, R \Psi \mathcal{F}) \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
H^1(X_{d,s}, \mathcal{F}) & \longrightarrow & H^1(X_{d,s}, \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, \mathcal{F}) & \longrightarrow & H^2(X_{d,s}, \mathcal{F}),
\end{array}
\]

where \( \gamma' \) is the composition of morphisms

\[
H^1(X_{d,s}, R \Phi \mathcal{F}) = \bigoplus_{x \in \Sigma} (R \Phi \mathcal{F})_x \xrightarrow{\text{Var}(\sigma)} \bigoplus_{x \in \Sigma} H^1_x(X_{d,s}, R \Psi \mathcal{F}) \rightarrow H^1(X_{d,s}, R \Psi \mathcal{F}).
\]

Then one can see that \( \delta : H^1(X_{d,\eta}, \mathcal{F}) \rightarrow H^2(X_{d,\eta}, \mathcal{F}) \) factors through the coinvariant \( H^1(X_{d,\eta}, \mathcal{F})_{\sigma^{-1}} \cong H^1(I_{Q_p}, H^1(X_{d,\eta}, \mathcal{F})) \) and the map

\[
H^1(I_{Q_p}, H^1(X_{d,\eta}, \mathcal{F})) \rightarrow H^2(X_{d,\eta}, \mathcal{F})
\]

coincides with the inverse of the map obtained via the Hochschild–Serre spectral sequence. Applying the projector \( \epsilon_d \) defined by

\[
\epsilon_d := \frac{1}{\# G_d} \sum_{g \in G_d} g \in \mathbb{Q}[G_d],
\]

the first part of the theorem follows. Since \( T_{f,n} \) is unramified at \( p \) and \( \lambda \) does not divide \( p \), one has

\[
H^1(I_{Q_p}, H^1(X_{d,\eta}, \mathcal{F})(1)/I_f^{[p]}) \cong H^1(I_{Q_p}, H^1(X_{d,\eta}, \mathcal{F})(1)/I_f^{[p]}).
\]

Therefore the second part follows from the discussions in the proof of Proposition 6.4 and the discussion after Lemma 6.5. \( \square \)
7. Kuga–Sato varieties and CM-cycles

7.1. The $\ell$-adic Abel–Jacobi map

Here we recall some basic facts on $\ell$-adic Abel–Jacobi map following Jannsen ([19], [20]).

Let $Y$ be a proper smooth variety over a field $F$ of characteristic zero. For an integer $i \geq 0$, write $\text{CH}^i(Y)$ for the Chow group of algebraic cycles defined over $F$ of codimension $i$ on $Y$ modulo rational equivalence. Fix a rational prime $\ell$. Then one may define the cycle class map $\text{cl}_\ell : \text{CH}^i(Y) \to H^{2i}(Y_{\overline{F}}, \mathbb{Z}_\ell(i))^\text{G}_F$ and we denote by $\text{CH}^i(Y)_0$ its kernel. Note that this definition does not depend on the choice of the prime $\ell$ by the Lefschetz principle and the comparison theorem between étale cohomology and singular cohomology.

The cycle class map $\text{cl}_\ell$ factors through $H^{2i}(Y, \mathbb{Z}_\ell(i))$, then the Hochschild–Serre spectral sequence $H^i(F, H^j(Y_{\overline{F}}, \mathbb{Z}_\ell(k))) \Rightarrow H^{i+j}(Y, \mathbb{Z}_\ell(k))$ induces the $\ell$-adic Abel–Jacobi map $AJ_\ell : \text{CH}^i(Y)_0 \to H^1(F, H^{2i-1}(Y_{\overline{F}}, \mathbb{Z}_\ell(k)))$.

By Jannsen [19] we have the following geometric description of the $\ell$-adic Abel–Jacobi map. Let $Z$ be a homologically trivial cycle on $X$ defined over $F$ of codimension $i$ representing an element in $\text{CH}^i(Y)_0$. Then the image of $Z$ under the $\ell$-adic Abel–Jacobi map is represented by the pull-back of the extension of $G_F$-modules

$$0 \to H^{2i-1}(Y_{\overline{F}}, \mathbb{Z}_\ell(i)) \to H^{2i-1}(Y_{\overline{F}} \setminus \mathbb{Z}_{\overline{F}}(i), \mathbb{Z}_\ell(i)) \to \text{Ker} \left[ H^{2i}(Y_{\overline{F}}, \mathbb{Z}_\ell(i)) \to H^{2i}(Y_{\overline{F}}, \mathbb{Z}_\ell(i)) \right] \to 0$$

by the map $\mathbb{Z}_\ell \to H^{2i}|_{\mathbb{Z}_{\overline{F}}}(Y_{\overline{F}}, \mathbb{Z}_\ell(i))$ sending 1 to $b(Z)$, where $b(Z)$ is the cohomology class of $\mathbb{Z}_{\overline{F}}$.

7.2. Kuga–Sato varieties over Shimura curves

To construct global cohomology classes in $H^1(K_m, T_{f,n})$ we will use the image of algebraic cycles on Kuga–Sato varieties under the $\ell$-adic Abel–Jacobi map. We keep the assumptions and notations as in §5. Now we suppose that $d$ is a prime greater than 3 which splits in $K$ and is prime to
Let \( \pi : A_{d[r]} \to X_{d[r]} \) be the universal abelian surface over the Shimura curve \( X_{d[r]} \). Then we define the Kuga–Sato variety

\[
\pi_k : W_{k,d[r]} \to X_{d[r]}
\]

by the \( \frac{k-2}{2} \)-fold fiber product over \( X_{d[r]} \) of \( A_{d[r]} \) with itself.

Since the action of \( \mathcal{O}_{B'} \) on \( A_{d[r]} \) induces an action of \( B' \times \mathcal{O} \) on \( R^q \pi_{\ast} \mathcal{Q}_\ell \), one may define

\[
\mathcal{L}_2 := \bigcap_{b \in B'} \text{Ker} \left[ b - 1 : R^q \pi_{\ast} \mathcal{Q}_\ell \to R^q \pi_{\ast} \mathcal{Q}_\ell \right]
\]

following Iovita–Spiess [17]. For an integer \( m \geq 2 \), let

\[
\Delta_m : \text{Sym}^m \mathcal{L}_2 \to \text{Sym}^{m-2} \mathcal{L}_2(-2)
\]

be the Laplace operator symbolically given by

\[
\Delta_m(x_1 \cdots x_m) = \sum_{1 \leq i < j \leq m} (x_i, x_j)x_1 \cdots \hat{x}_i \cdots \hat{x}_j \cdots x_m,
\]

where \((\ ,\ )\) is the non-degenerate pairing

\[
(\ ,\) : \mathcal{L}_2 \times \mathcal{L}_2 \to R^q \pi_{\ast} \mathcal{Q}_\ell \otimes R^q \pi_{\ast} \mathcal{Q}_\ell \xrightarrow{\cup} R^q \pi_{\ast} \mathcal{Q}_\ell \xrightarrow{\text{Tr}} \mathcal{Q}_\ell(-2).
\]

Let \( \mathcal{L}_{k-2} \) denote the kernel of \( \Delta_{k-2} \).

Then there exists a projector \( \epsilon_k \) defined as in Scholl [35] (see also Iovita–Spiess [17, §10]) such that

\[
\epsilon_d \cdot \epsilon_k H^{k-1}(W_{k,d[r]} \otimes \mathcal{Q}_\ell \otimes \mathcal{Q}_\ell) \otimes \mathcal{Q}_\ell E \cong \epsilon_d H^{1}(X_{d[r]} \otimes \mathcal{Q}_\ell \otimes \mathcal{L}_{k-2}) \otimes \mathcal{Q}_\ell E \cong H^{1}(X_{d[r]} \otimes \mathcal{Q}_\ell, \mathcal{F})(1) \otimes \mathcal{O} E,
\]

where \( \epsilon_d \) is the projector defined by

\[
\epsilon_d = \frac{1}{\#G_d} \sum_{g \in G_d} g \in \mathcal{Q}[G_d].
\]

Note that

\[
\epsilon_k H^{k-1}(W_{k,d[r]} \otimes \mathcal{Q}_\ell \otimes \mathcal{Q}_\ell) \cong \epsilon_k H^{*}(W_{k,d[r]} \otimes \mathcal{Q}_\ell \otimes \mathcal{Q}_\ell).
\]

### 7.3. Description on CM points

By the moduli interpretation of the Shimura curve \( X_{d[r]} \), a point on \( X_{d[r]} \) is represented by a triple \((A_{d}, \iota, C)\). For \( m \geq 0 \), there exists a point \( P_m = (A_m, \iota_m, C_m) \) such that \( \text{End}(P_m) \) is isomorphic to \( \mathcal{O}_{K,m} \), where \( \text{End}(P_m) \) is the ring of endomorphisms of \( A_m \) which commutes with the action of \( \iota_m \) and respect the level structure \( C_m \) and \( \mathcal{O}_{K,m} \) is the order of \( K \) of conductor
The point $P_m$ is called a CM point of level $m$. By the theory of complex multiplication, such a point $P_m$ is defined over $K_m$, where $K_m$ is the ring class field of $K$ of conductor $m$.

Using the complex uniformization of the Shimura curve $X[p]$, the CM points of level $m$ are defined by

$$P_m(a) := \left[ (z', \varphi_{B,B'}(a^{(p)}\zeta T^N)) \right] \in X[p](\mathbb{C})$$

for each $a \in \mathbb{R}$. By Shimura’s reciprocity law, one has $P_m(a) \in \text{CM}_K X[p] - \left( X[p](K_m) \right)$

and $P_m(a)^\sigma = P_m(ab)$ for $\sigma = \text{rec}_K(b) \in \mathbb{G}_m$. Set $P_m = P_m(1)$.

### 7.4. Definition of CM cycles

Here we construct CM cycles following Besser [3], Nekovář [27] and Iovita–Spiess [17]. Let $X = X_{N+,N-}$ be the Shimura curve defined in §4 and let $P_m = (A_m, \iota_m, C_m)$ be a CM point of level $m$. Then $A_m$ is defined over the ring class field $K_m$. Write $\text{NS}(A_m)$ for the Néron–Severi group of $A_m$. There is a natural right $B'^\times$-action on $\text{NS}(A_m) \otimes \mathbb{Q}$ given by $L \cdot b = \text{Nrd}(b)^{-1} \iota_m(b)^*(L)$ for $b \in B'^\times$ and $L \in \text{NS}(A_m) \otimes \mathbb{Q}$. Note that our normalization is different from the action used in [17].

Since $\text{End}(P_m) \simeq O_{K,m}$ and $A_m$ has endomorphism by the maximal order $O_{B'}$, $A_m$ has endomorphisms by an order $O_{B'} \otimes O_{K,m}$ in $B' \otimes K \simeq M_2(K)$. Hence $A_m$ is isogenous to a product $E_m \times E_m$, where $E_m$ is an elliptic curve with complex multiplication by $O_{K,m}$. Write $\Gamma_m$ for the graph of $\sqrt{D_K}$. Then, define $Z_m$ to be the image of the divisor $[\Gamma_m] - [E_m \times 0] - m^2[D_K][0 \times E_m]$ in $\text{NS}(A_m)$. It lies in the free rank one $\mathbb{Z}$-module $\langle [E_m \times 0], [0 \times E_m], \Delta_{E_m} \rangle \perp \subset \text{NS}(A_m)$, where $\Delta_{E_m}$ is the diagonal.

**Proposition 7.1.** — Assume that $A$ has complex multiplication by $O_{K,m}$. Then there exists an element $y_m$ in $\text{NS}(A) \otimes \mathbb{Q}$ such that

1. $\iota_m(b)^*(y_m) = y_m$ for any $b \in B'^\times$,
2. The self-intersection number of $y_m$ is $2D_K$.

Moreover, $y_m$ is uniquely determined up to sign by these properties.

**Proof.** — This is a direct generalization of [17, Proposition 8.2]. In particular, $y_m = m^{-1}Z_m$ satisfies the properties. \qed
Remark 7.2. — Since we use a different normalization for the action of $B' \times$ on $NS(A_m)_{\mathbb{Q}}$ from [17], the formula (1) in Proposition 7.1 is different from the corresponding formula in [17, Proposition 8.2].

Let $t$ denote the number of prime divisors of $Np$, $h_m$ the class number of $K_m$. Then there are exactly $2^t h_m$ CM points of conductor $m$ (see Bertolini–Darmon [1] for details).

Let $\mathcal{W}$ be the Atkin–Lehner group of order $2^t$ generated by all the Atkin–Lehner involutions $W_q^{+}$ with $q \mid N^+$ and $W_q^{-}$ with $q \mid N^- p$. Write $\mathcal{G}_m$ for the Galois group $\text{Gal}(K_m/K)$. One can identify the Galois group $\mathcal{G}_m$ with $\text{Pic}(\mathcal{O}_{K,m})$ via the geometrically normalized reciprocity map $\text{rec}_K: \hat{K}^\times \to \text{Gal}(K^{ab}/K)$ which sends a prime ideal $\mathfrak{p}$ to the geometric Frobenius at $\mathfrak{p}$. The group $\text{Pic}(\mathcal{O}_{K,m}) \times \mathcal{W}$ acts simply transitively on the set of CM points.

Recall that $d \geq 4$ is an integer relatively prime to $Np$ and

$$\pi: \mathcal{A} = \mathcal{A}^{[p]}_d \to X^{[p]}_d$$

is the universal abelian surface and

$$\psi: X^{[p]}_d \to X^{[p]}$$

the natural morphism. Let $P_m = P_m(1)$ be the CM point of level $m$ defined as above and let $\tilde{P}_m$ be any point on $X^{[p]}_d$ such that $\psi(\tilde{P}_m) = P_m$. The fiber $A_{\tilde{P}_m} = \pi^{-1}(\tilde{P}_m)$ is an abelian surface with $\text{End}_{\mathcal{O}_m}(A_{\tilde{P}_m}) \simeq \mathcal{O}_{K,m}$. By Proposition 7.1, there exist an element $y_m \in NS(A_{\tilde{P}_m})_{\mathbb{Q}}$ satisfying

1. $\iota_m(b)^*(y_m) = y_m$ for any $b \in B' \times$,
2. The self-intersection number of $y_m$ is $2D_K$.

which is uniquely determined up to sign.

Let $Y_{\tilde{P}_m}$ be an element of $\epsilon_d CH^1(A_{\tilde{P}_m})_{\mathbb{Q}}$ representing $y_{\tilde{P}_m}$. One may choose the elements $Y_{\tilde{P}_m}$ in such a way that

$$g_*(Y_{\tilde{P}_m}) = Y_{g_*(\tilde{P}_m)}$$

for all $\tilde{P}_m \in \psi^{-1}(P_m)$ and $g \in G_d$,

where $g: A_{\tilde{P}_m} \to A_{\tilde{P}_m}$ is the automorphism induced by $g \in G_d$.

Let $j_{k,m}: A_{\tilde{P}_m}^{k-2} \hookrightarrow A_{\tilde{P}_m}^{k-2} = W_{k,d}$ be the inclusion of the fiber over $\tilde{P}_m$ into the Kuga–Sato variety. We define the element $Z_{\tilde{P}_m}$ of $\epsilon_d \cdot \epsilon_d CH^2(\mathcal{A} \otimes K_m)_{\mathbb{Q}}$ as the image of $Y_{\tilde{P}_m}$ under the composition

$$\epsilon_d\epsilon_d CH^1(A_{\tilde{P}_m})_{\mathbb{Q}} \xrightarrow{j_{4,m}} \epsilon_d\epsilon_d CH^2(\mathcal{A} \otimes K_m)_{\mathbb{Q}} \xrightarrow{\epsilon_d} \epsilon_d \cdot \epsilon_d CH^2(\mathcal{A} \otimes K_m)_{\mathbb{Q}}.$$

We require that the elements $Y_{\tilde{P}_m}$ are compatible with the action of $\mathcal{W} \times \text{Pic}(\mathcal{O}_{K,m})$ (see Iovita–Spiess [17, p. 366] for details). Then we define the
CM cycle $Z_{m^{k-2}}$ of level $m$ by setting

$$Z_{m^{k-2}} := \epsilon_d \cdot \epsilon_k(j_{k,m})_*(Y_{P_m^{k-2}})$$

$$\in \epsilon_d \cdot \epsilon_k CH^{k/2}(W^{[p]}_{k,d} \otimes K_m)_Q \subset CH^{k/2}(W^{[p]}_{k,d} \otimes K_m)_Q.$$

8. Construction of Euler systems and the explicit reciprocity law

8.1. Construction of special cohomology classes

Let $p$ be an $n$-admissible prime. Here we give a description of the image of CM cycles under the $\ell$-adic Abel–Jacobi map following Nekovář [28]. Write $Z_m$ for the CM cycle of level $m$. Let $\kappa_p^{[p]}(m)$ be the image of CM cycle $Z_{k-2}^2$ under the $\ell$-adic Abel–Jacobi map $\epsilon_k \circ AJ_{\ell,E} : CH^{k/2}(W^{[p]}_{k,d} \otimes K_m)_E \to H^1(K_m, H^1(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1))_E$.

By the construction of the cohomology class, we have the following lemma.

**Lemma 8.1.** — The global cohomology class $\kappa^{[p]}(m) := \epsilon_d \kappa_p^{[p]}(m)$ belongs to $H^1(K_m, H^1(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1))$.

Let $\widetilde{P}_m$ be a lift of the CM point $P_m$ of level $m$. Let

$$\text{cl}_\ell : CH^{k-2}(A_{P_m^{k-2}}) \to \epsilon_k H^{k-2}(A_{P_m^{k-2}} \otimes \overline{K_m}, \mathbb{Z}_\ell(k-2/2))^{G_K}$$

be the cycle class map. Then $\epsilon_d \cdot \epsilon_k H^{k-2}(A_{P_m^{k-2}} \otimes \overline{K_m}, \mathbb{Z}_\ell(k-2/2))^{G_K}$ is isomorphic to $H^2_{\widetilde{P}_m}(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1)^{G_K}$. By an argument similar to the one in the proof of [28, (2.4) Proposition (2)], one can show that the image of $Y = Y_{P_m^{k-2}}$ is represented by the pull-back of the extension

$$0 \to H^1(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1) \to H^1(X^{[p]}_d \otimes \overline{K_m} \setminus \widetilde{P}_m \otimes \overline{K_m}, \mathcal{F})(1)$$

$$\to H^2_{\widetilde{P}_m \otimes \overline{K_m}}(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1)^{G_K} \to 0$$

by the map $\mathcal{O} \to H^2_{\widetilde{P}_m \otimes \overline{K_m}}(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1)^{G_K}$ sending 1 to $\epsilon_k b(Y)$, where $b(Y)$ is the cohomology class of $Y_{\overline{K_m}}$. We will compute the image

$$\epsilon_d \cdot \epsilon_k b(Y) \in H^2_{\widetilde{P}_m \otimes \overline{K_m}}(X^{[p]}_d \otimes \overline{K_m}, \mathcal{F})(1).$$
Recall that there is an elliptic curve $E_m$ with complex multiplication by $O_{K,m}$ defined over $K_m$ such that $A_{\widetilde{P}_m}$ is isogenous to $E_m \times E_m$. Then the Künneth formula and antisymmetrization gives a projection

$$\text{pr}_k : H^{k-2}\left( A_{\widetilde{P}_m} \otimes K_m, \mathbb{Z}_\ell \left( \frac{k-2}{2} \right) \right) \rightarrow H^1(E_m \otimes K_m, \mathbb{Z}_\ell) \otimes H^{k-2}\left( \frac{k-2}{2} \right) \rightarrow (\text{Sym}^{k-2} H^1(E_m \otimes K_m, \mathbb{Z}_\ell)) \left( \frac{k-2}{2} \right).$$

One obtains that the element $\epsilon_{k_{\text{cl}}} Y$ belongs to the space $(\text{Sym}^{k-2} H^1(E_m \otimes K_m, \mathbb{Q}_\ell))(k/2 - 1)$.

There exists a $B'_\ell \simeq B'_{\ell} \simeq \text{GL}_2(\mathbb{Q}_\ell)$-equivariant isomorphism

$$(\text{Sym}^{k-2} H^1(E_m \otimes K_m, \mathbb{Q}_\ell)) \left( \frac{k-2}{2} \right) \simeq (\text{Sym}^{k-2} H^1(E_m \otimes K_{m,p}, \mathbb{Q}_\ell)) \left( \frac{k-2}{2} \right) \simeq (\text{Sym}^{k-2} H^1(E_m \otimes F_{p^2}, \mathbb{Q}_\ell)) \left( \frac{k-2}{2} \right) \simeq L_k(\mathbb{Q}_\ell)$$

which preserves the intersection pairing. Therefore, we have an identification

$$H^2_{\widetilde{P}_m \otimes \overline{K}_{p^2}}(X_{d}^{[p]} \otimes \overline{K}_{p^2}, \mathcal{F})(1) \cong H^2_{\widetilde{P}_m \otimes \overline{F}_{p^2}}(X_{d}^{[p]} \otimes \overline{F}_{p^2}, \mathcal{F})(1) \cong L_k(\mathcal{O}),$$

where $\widetilde{P}_m = \widetilde{P}_m \mod p$.

**Lemma 8.2.** — The image of $\epsilon_d \cdot \epsilon_{k_{\text{cl}}}(Y)$ in $L_k(\mathbb{Q}_\ell)$ is given by $\mathbf{v}_0^*$ up to sign.

**Proof.** — This follows from the fact that both elements satisfy the same properties:

1. $\epsilon_d \cdot \epsilon_{k_{\text{cl}}}(Y)$ and $\mathbf{v}_0^*$ are eigenvectors for the action of $K$ with eigenvalue 1,
2. $\langle \epsilon_d \cdot \epsilon_{k_{\text{cl}}}(Y), \epsilon_d \cdot \epsilon_{k_{\text{cl}}}(Y) \rangle = \langle \mathbf{v}_0^*, \mathbf{v}_0^* \rangle = D_K^{k-2}$.

These properties characterize an element in $L_k(\mathbb{Q}_\ell)$ up to sign. □
By Proposition 6.4, we have an isomorphism \( H^1(X^{[p]} \otimes \overline{K_m}, \mathcal{F}(1))/I_f^{[p]} \simeq T_{f,n} \) as \( \text{Gal}(\overline{K_m}/K_m) \)-modules, therefore \( \kappa^{[p]}(m) \) defines a cohomology class \( \kappa_{f,n}^{[p]}(m) \) in \( H^1(K_m, T_{f,n}) \).

8.2. The explicit reciprocity law

By Theorem 6.6 and the description of the \( \ell \)-adic Abel–Jacobi map considered in the previous section, we have a commutative diagram

\[
\begin{array}{cccc}
\text{CH}^{k/2}(W_{k,d}^{[p]} \otimes K_m) & \xrightarrow{\varepsilon_d \cdot \varepsilon_k \cdot \text{cl}_f} & H^1(K_m, H^1(X^{[p]} \otimes \overline{K_m}, \mathcal{F}(1)) & \xrightarrow{\text{res}} \\
\bigoplus_{p|d} H^2(X^{[p]} \otimes K_{m,p}, \mathcal{F}(1)) & \xrightarrow{\omega_p} & H^1_{\text{sing}}(K_{m,p}, T_{f,n}). & \\
\end{array}
\]

**Proposition 8.3.** — For sufficiently large \( m \), there exists a positive integer \( M \) such that \( \text{red}_{\lambda^n}(\kappa_{f,n+M}^{[p]}(m)) \in H^1_p(K_m, T_{f,n}) \), where \( p \) is an \( n + M \)-admissible prime.

**Proof.** — For \( v|dN^+ \), by Lemma 3.3 and [4, Corollary 5.2] we have

\( \text{red}_{\lambda^n}(\text{res}_v \kappa_{f,n+M}^{[p]}(m)) \in H^1_f(K_{m,v}, T_{f,n}) \)

for sufficiently large \( M \). For \( v|N^- \), since \( H^0(K_{m,v}, A_f) \) is finite,

\( H^2(K_{m,v}, T_f) = \text{Hom}(H^0(K_{m,v}, A_f), E/O) \)

is also finite. Hence for sufficiently large \( M, \lambda^M H^2(K_{m,v}, T_f) = 0 \). The commutative diagram

\[
\begin{array}{cccc}
0 & \longrightarrow & T_f & \xrightarrow{\times \lambda^{n+M}} T_f & \xrightarrow{\text{red}_{\lambda^{n+M}}} T_{f,n} & \longrightarrow & 0 \\
\times \lambda^M & \downarrow & = & \downarrow & \text{red}_{\lambda^n} & & \\
0 & \longrightarrow & T_f & \xrightarrow{\times \lambda^n} T_f & \xrightarrow{\text{red}_{\lambda^n}} T_{f,n} & \longrightarrow & 0
\end{array}
\]

gives rise to

\[
\begin{array}{cccc}
H^1(K_{m,v}, T_f) & \xrightarrow{\text{red}_{\lambda^{n+M}}} & H^1(K_{m,v}, T_{f,n}) & \longrightarrow & H^2(K_{m,v}, T_f)[\lambda^{n+M}] \\
= & \downarrow & \text{red}_{\lambda^n} & \downarrow & \times \lambda^M \\
H^1(K_{m,v}, T_f) & \xrightarrow{\text{red}_{\lambda^n}} & H^1(K_{m,v}, T_{f,n}) & \longrightarrow & H^2(K_{m,v}, T_f)[\lambda^n].
\end{array}
\]
Therefore by the definition of $H^1_f(K_{m,v}, T_f)$ and the fact that

$$H^1_f(K_{m,v}, V_f) = H^1(K_{m,v}, V_f)$$

(see Besser [4, Proposition 4.1 (2)]), we have

$$\text{red}_\lambda^n(\text{res}_v(\kappa_{f,n+M}^{[p]}(m))) \in H^1_f(K_{m,v}, T_f).$$

For $v \nmid \text{N}d\ell p$, the CM cycle $Z^{k-2}$ is unramified at $v$, hence the class $\text{red}_\lambda^n(\kappa_{f,n+M}^{[p]}(m))$ is also unramified at $v$. For the case $v|\ell$, the Galois representation $H^1(X[p]_{\overline{K}_m}, F(1))$ is crystalline, since the Kuga–Sato variety $W_{k,d}$ has good reduction at $v$. Hence by Nekovář [29, Theorem 3.1(1)] and Nizioł [32, Theorem 3.2] the image of the $\ell$-adic Abel–Jacobi map is contained in $H^1_f$ (also see Nekovář [29, Theorem 3.1(2)] and Nekovář–Nizioł [31, Theorem B] for the general case). Therefore one has

$$\text{red}_\lambda^n(\text{res}_v(\kappa_{f,n+M}^{[p]}(m))) \in H^1_f(K_{m,v}, H^1(X[p] \otimes K_{m,v}, F)(1)/I_f^{[p]}).$$

Since the prime $\ell$ is greater than $k-1$, one can use the Fontaine–Laffaille theory. Choose a Galois-stable lattice $T$ in a crystalline representation of $G_{K_{m,v}}$ such that $T/\lambda^n T \cong H^1(X[p] \otimes K_{m,v}, F)(1)/I_f^{[p]}$. Denote $T_1 = T$ and $T_2 = T_f$. Let $D_i$ be a strongly divisible $\mathcal{O}$-lattice in $D_{\text{cris}}(V_i) = D_{dR}(V_i)$ (the equality follows from the facts that $K_{m,v} = \mathbb{Q}_p$ is an unramified extension of $\mathbb{Q}_p$ and $V_i$ are crystalline) for $i = 1, 2$, where $V_i = T_i \otimes_{\mathcal{O}} E$. Define $D_i^{k/2} = D_i \cap D_i^{k/2}(V_i)$ and $\phi_{k/2} = \lambda^{-k/2}\phi$, where $\phi$ is the Frobenius morphism. By the Fontaine–Laffaille theory, we have isomorphisms $D_1/\lambda^n D_1 \cong D_2/\lambda^n D_2$ and $D_1^{k/2}/\lambda^n D_1^{k/2} \cong D_2^{k/2}/\lambda^n D_2^{k/2}$. Moreover by Bloch–Kato [5, Lemma 4.5 (c)], $h^1(D_i) = \text{Coker}[D_i^{k/2} \rightarrow D_i]$ is isomorphic to $H^1_f(K_{m,v}, T_i)$. From these facts, it is easy to see

$$H^1_f(K_{m,v}, H^1(X[p] \otimes K_{m,v}, F)(1)/I_f^{[p]}) \cong H^1_f(K_{m,v}, T_f)$$

for $v|\ell$. Therefore we have $\text{red}_\lambda^n(\text{res}_v(\kappa_{f,n+M}^{[p]}(m))) \in H^1_f(K_{m,v}, T_f)$ for $v|\ell$. This completes the proof.

The relation between the image of the CM cycle in $H^1_{\text{sing}}(K_{m,p}, T_f)$ and the theta element $\Theta(f_{\pi'})$ is given by the following theorem.

THEOREM 8.4. — There exists a constant $u \in \mathcal{O}_n^\times$ such that

$$\partial_p(\text{red}_\lambda^n(\kappa_{n+M}^{[p]}(m))) \equiv u \cdot \Theta(f_{\pi'}) \mod \lambda^n.$$
Proof. — By Theorem 6.3, Theorem 6.6 and Lemma 8.2, one has
\[ \partial_p(\kappa_{f,n+M}(m)) = \sum_{[a] \in G_m} \langle t_0^*, \hat{f}(x_m(a)\tau^{N^+}) \rangle_k \cdot [a]_m = \Theta(f_{\pi'}) \in \mathcal{O}_{n+M}[G_m] \]
up to \( \mathcal{O}_{n+M}^\times \). Therefore the natural image of \( \partial_p(\kappa_{f,n+M}(m)) \) in \( \mathcal{O}_n[G_m] \) satisfies the same property.

Now, our main result (Theorem 1.1) follows from Proposition 8.3, Theorem 3.8 and Theorem 8.4.

Remark 8.5. — Assume that \( \bar{\rho}_f \) is ramified at all primes dividing \( N^- \). Then we have
\[ \Omega_{\pi,N^-} = u \cdot \Omega_{f}^{can} \text{ for some } u \in \mathcal{O}^\times. \]
This fact follows from Proposition 6.1 and the argument in [8, Proof of Proposition 6.1].

BIBLIOGRAPHY

[1] M. Bertolini & H. R. Darmon, “Heegner points on Mumford–Tate curves”, Invent. Math. 126 (1996), no. 3, p. 413-456.
[2] ———, “Iwasawa’s main conjecture for elliptic curves over anticyclotomic \( \mathbb{Z}_p \)-extensions”, Ann. Math. 162 (2005), no. 1, p. 1-64.
[3] A. Besser, “CM cycles over Shimura curves”, J. Algebr. Geom. 4 (1995), no. 4, p. 659-691.
[4] ———, “On the finiteness of III for motives associated to modular forms”, Doc. Math., J. DMV 2 (1997), no. 1, p. 31-46.
[5] S. Bloch & K. Kato, “L-functions and Tamagawa numbers of motives”, in The Grothendieck Festschrift, Vol. 1, Progress in Mathematics, vol. 86, Birkhäuser, Boston, MA, 1990, p. 333-400.
[6] N. Boston, H. W. Lenstra & K. A. Ribet, “Quotients of group rings arising from two-dimensional representations”, C. R. Acad. Sci. Paris Sér. I Math. 312 (1991), no. 4, p. 323-328.
[7] J.-F. Boutot & H. Carayol, “Uniformisation \( p \)-adique des courbes de Shimura: les théorèmes de Cerednik et de Drinfeld”, Astérisque 196–197 (1991), p. 45-158.
[8] M. Chida & M.-L. Hsieh, “Special values of anticyclotomic \( L \)-functions for modular forms”, to appear in J. reine angw. Math., available at http://dx.doi.org/10.1515/crelle-2015-0072.
[9] ———, “On the anticyclotomic Iwasawa main conjecture for modular forms”, Compos. Math. 151 (2015), no. 5, p. 863-893.
[10] P. Deligne, “La formule de Picard–Lefschetz”, in SGA 7 II, Exposé XV, Lecture Notes in Mathematics, vol. 340, Springer, 1973, p. 165-196.
[11] ———, “Le formalisme des cycles evanescents”, in SGA 7 II, Exposé XIII, Lecture Notes in Mathematics, vol. 340, Springer, 1973, p. 82-115.
[12] F. Diamond, “The Taylor–Wiles construction and multiplicity one”, Invent. Math. 128 (1997), no. 2, p. 379-391.
[13] F. Diamond & R. Taylor, “Non-optimal levels of mod \( l \) modular representations”, Invent. Math. 115 (1994), no. 3, p. 435-462.
[14] L. Fu, Étale cohomology theory, Nankai Tracts in Mathematics, vol. 13, World Scientific Publishing Co. Pte. Ltd., 2011, ix+611 pages.

[15] P.-C. Hung, “On the non-vanishing mod ℓ of central L-values with anticyclotomic twists for Hilbert modular forms”, J. Number Theory 173 (2017), p. 170-209.

[16] L. Illusie, “On semistable reduction and the calculation of nearby cycles”, in Geometric aspects of Dwork’s theory, vol. I, II, Walter de Gruyter GmbH & Co. KG, Berlin, 2004, p. 785-803.

[17] A. Iovita & M. Spiess, “Derivatives of p-adic L-functions, Heegner cycles and monodromy modules attached to modular forms”, Invent. Math. 154 (2003), no. 2, p. 333-384.

[18] H. M. Jacquet & R. P. Langlands, Automorphic forms on GL(2), Lecture Notes in Mathematics, vol. 114, Springer, Berlin, 1970, vii+548 pages.

[19] U. Jannsen, “Continuous étale cohomology”, Math. Ann. 280 (1988), no. 2, p. 207-245.

[20] ———, Mixed motives and algebraic K-theory, second ed., Lecture Notes in Mathematics, vol. 1400, Springer, 1990, xiii+246 pages.

[21] B. W. Jordan & R. Livné, “Integral Hodge theory and congruences between modular forms”, Duke Math. J. 80 (1995), no. 2, p. 419-484.

[22] J. Nekovář, “p-adic Hodge theory and values of zeta functions of modular forms”, Astérisque 295 (2004), p. 117-290.

[23] ———, “p-adic Abel-Jacobi maps and p-adic heights”, in The Arithmetic and Geometry of Algebraic Cycles, (Banff, Canada, 1998), CRM Proc. and Lect. Notes 24, Amer. Math. Soc, Providence, R.I., 2000, p. 367-379.

[24] ———, “Level raising and anticyclotomic Selmer groups for Hilbert modular forms of weight two”, Can. J. Math. 64 (2012), no. 3, p. 588-668.

[25] W. Nizioł, “On the image of p-adic regulators”, Invent. Math. 127 (1997), no. 2, p. 375-400.

[26] ———, “The arithmetic of modular forms”, Doc. Math., J. DMV Extra Vol. (2006), p. 729-779, electronic.
Manuscrit reçu le 16 janvier 2016,
révisé le 15 septembre 2016,
accepté le 16 septembre 2016.

Masataka CHIDA
chida@math.tohoku.ac.jp
Mathematical Institute
Tohoku University
6-3, Aramaki Aza-Aoba, Aoba-ku
Sendai 980-8578 (Japan)