Gravitational lensing in 4-D Einstein-Gauss-Bonnet gravity in the presence of plasma

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Abstract

In this paper we have assumed a weak-field regime to explore the gravitational lensed photons in a 4 dimensional Einstein-Gauss-Bonnet gravity, which is very much in the limelight these days. The investigation is conducted in three distinct paradigms: uniform plasma, singular isothermal sphere and a non-singular isothermal sphere. The lensing angle associated with the distribution factor of the medium is individually derived for each case and further utilized to study the magnification of the image source brightness, selectively for the uniform plasma and singular isothermal sphere. The attained results are brought forth in contrast with the standard Schwarzschild geometry.

Keywords: Einstein Gauss-Bonnet gravity, gravitational lensing, weak-field, plasma medium

1. Introduction

As a matter of fact, the gravitational field of galaxy clusters and black holes is generally known to possess the ability to lens a nearby passing light emitted from a source. The process takes place when the gravitational perturbations of the dark matter as well as the black holes distort and magnify the incoming photons in its domain. This particular phenomenon has not only helped the astronomers to discover the existence of invisible black holes but also perceived the geometrical structure of the Universe. Refs [1–4] render specific details regarding the fundamental tools required to study the lensing mechanism in the black hole vicinity.

Virbhadra and Ellis rekindled the lensing theory by considering a strong-field regime for the classical Schwarzschild geometry and naked singularities [5, 6]. Bozza et al. based on the Virbhadra-Ellis lens model developed an analytical approach to highlight the process [7], afterwards Bozza independently extended the investigation for the Schwarzschild, Reissner-Nordström and Janis-Newman-Winicour spacetimes in the following paper [8]. The spinning Kerr black hole is explored by the authors in [9–12] to reveal its lensing properties. Some of the noticeable articles [13–21] manifest an elaborate analysis in regard to the above referred field by considering the Reissner-Nordström, Braneworld, Kerr-Taub-NUT, Born-infeld and Lee–Wick gravities.

The latter analysis has equally remained the subject of interest for a weak-field scenario. Bisnovatyi-Kogan and Tsupko with a great finesse laid the groundwork for a weak-field lensing in the Schwarzschild plasma [22] and thereupon continued their research for a variety of plasma mediums [23–26]. A rotating massive and compact objects, non-Schwarzschild geometry, Bardeen, Hayward, braneworld, boosted Kerr and Kerr-Newman gravities are addressed accordingly in [27–36].

In this paper our prime focus is to re-establish the process of weak-field gravitational lensing by taking into account the articles [22, 27, 32], in a plasma background of 4D Einstein-Gauss-Bonnet gravity (4D-EBG) [37], where D refers to the dimension of the space-time. Boulware et al. acquired a spherically symmetric static solution of EGB gravity in higher dimensions [38] by coupling the Einstein field action with the Gauss-Bonnet combination [39]. A distinctive analysis has been carried out in [40] to identify the effects of thermodynamics in higher dimensional EGB Anti–de Sitter space-time. As of late, Glavan and Lin devised a modified gravity of Gauss-Bonnet theory in four dimensions by ignoring the Lovelock’s theorem [41], which defines the theory of gravity in 4D only if the cosmological constant is provided, but rather made it possible by reformulating the Gauss-Bonnet coupling constant α → α/D − 4 confined to the limit D → 4 [37]. Howbeit, Metin et al. passed contradictory remarks against this method in [42] by revealing the lack of continuity of the EGB theory at D = 4. Shortly after that, similar comments regarding the rescaled term 1/D − 4 were addressed by Arrechea et al. in the letter [43] and observed some ill-defined terms affecting the metric perturbations. Although, the authors have now constructed positive arguments in response to the linear perturbations in the 4D-EBG background [44], hence, we shall employ linear weak-field approximations to study the lensing phenomenon in 4D-EBG gravity.

This freshly proposed theory of 4D-EBG has up till now been studied for various astrophysical observations to uncover the in-
tricate features of the space-time gravity. By considering the indicated gravity the gravitational collapse of a homogenous dust sphere is tacitly presented in [45]. The superradiance and stability along with the quasinormal modes of a charged 4D-EGB black hole are worked out in [46] and the EGB model endowed with a scalar field [47] is used to explore its stability by utilizing the odd parity perturbations method. In Refs [48, 49] the authors investigated the dynamics of spinning test particles, the innermost stable circular orbits and shadow cast in surroundings of a static 4D-EGB gravity; whereas a cognitive analysis for a rotating counterpart has been reviewed in [50]. The validity of strong cosmic censorship for charged de Sitter black hole is studied in in the form of a static 4D-EGB gravity; whereas a cognitive analysis for a rotating counterpart has been reviewed in [50]. The validity of strong cosmic censorship for charged de Sitter black hole is studied in the context of scalar and electromagnetic perturbation [51]. The quasinormal modes of the Dirac field and the perturbative as well as the non-perturbative modes are, respectively, inspected in the [52, 53]. Various aspects of black holes such as gravitational lensing, optical properties, thermodynamics and particle acceleration are profoundly examined in [54–58]. One may get relevant information concerning the 4D-EGB gravity from the research performed in the Refs [59–76].

The paper is organised as follows. In Sec. (2) the deflection angle of the massless particles is precisely studied in the light of a weak-field approximation by considering the EGB gravity embedded in three different mediums, i.e. uniform plasma, singular isothermal sphere and a non-singular isothermal sphere. Sec. (3) provides a detailed logical reasoning about the apparent magnification of the source star. Finally, in Sec. (4) we have summarized our main results.

2. Weak-field lensing in the presence of plasma

The action of D-dimensional Einstein-Gauss-Bonnet theory with a redefined coupling constant $\alpha \to \frac{\alpha}{d-2}$ is expressed by the relation

$$S = \frac{1}{16\pi} \int d^D x \sqrt{-g} (R + \frac{\alpha}{d-2} G),$$

where $\alpha$ is a dimensionless Gauss–Bonnet (GB) coupling parameter and $G$ is the Gauss–Bonnet invariant defined by the expression

$$G = R^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} - 4 R^{\alpha\beta} R_{\alpha\beta} + R^2,$$

$R$ is the Ricci scalar, $R_{\alpha\beta}$ and $R_{\mu\nu\rho\sigma}$ denote, respectively, the Ricci and Riemann tensors. The action $S$ in a 4 dimensional analysis yields the line-element of a non rotating 4D-EGB gravity in the form

$$ds^2 = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

where

$$f(r) = 1 + \frac{\rho_0}{2\alpha} \left(1 - \sqrt{1 + \frac{4\alpha R_0}{r^2}}\right),$$

here $R_0 = 2M$, the value of GB parameter $\alpha/M^2$ lies in the range [-8,1] [72]. Note that, $\alpha > 1$ corresponds to naked singularities and $\alpha < -8$ leads to complex-valued metric in the outer region of the event horizon[49]. Moreover, Schwarzschild metric is recovered when $\alpha \to 0$. We shall take the series expansion of $f(r)$ up to order $O(R_0^2)$ for a more exhaustive evaluation,

$$f(r) = 1 - \frac{R_0}{r} + \frac{\alpha R_0^2}{r^2}. \quad (5)$$

In this section our main concern is to unravel the effects of gravitational lensing in the background of 4D-EGB gravity surrounded by a plasma considering a weak-field approximation defined as follows,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta},$$

where $\eta_{\alpha\beta}$ and $h_{\alpha\beta}$ connote the Minkowski metric and perturbation metric, respectively.

$$\eta_{\alpha\beta} = \text{diag}(-1,1,1,1), \quad h_{\alpha\beta} \ll 1, \quad h_{\alpha\beta} \to 0 \quad \text{under} \quad \chi' \to \infty, \quad$$

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}, \quad h_{\alpha\beta} = h_{\alpha\beta}. \quad (7)$$

Before proceeding any further we shall briefly recall the layout presented in [22, 27, 32] to elicit the general expression of the angle of deflection. The correlation between the phase velocity $v$ and the 4-momentum $p^\alpha$ considering a static case is expressed as [1].

$$\frac{v^2}{c^2} = n^2 = 1 + \frac{p_\alpha p^\alpha}{(p^\alpha \sqrt{-g_{\alpha\beta}})^2}. \quad (8)$$

By taking into account a dispersive medium, Synge [1] remodelled the Fermat’s least action principle to describe the photon trajectories. Henceforth, the variational principle $\delta(\int p_\alpha dx^\alpha) = 0$ was applied with the condition,

$$H(x', p_\alpha) = \frac{1}{2} [p_\beta p_\beta - (n^2 - 1)(p_0 \sqrt{-g_{\alpha\beta}})]^2. \quad (9)$$

The above expression governs the equations of motion by the following system of differential equations $\frac{dx}{dt} = \frac{\partial H}{\partial p_\alpha}, \frac{dp_\alpha}{dt} = -\frac{\partial H}{\partial x}$, here $\lambda$ is the affine parameter. It is of utmost importance to define the refractive index $n$ properly in order to study the plasma effects clearly in the black hole vicinity, therefore by the implication of [22] we define it as

$$n^2 = 1 - \frac{\omega_e^2}{[\omega(x')]^2}, \quad \omega_e^2 = \frac{4\pi e^2 N(x')}{m} = K_e N(x'). \quad (10)$$

Here, $\omega_e$ is the plasma electron frequency and $\omega(x')$ is a space coordinate function termed as the photon frequency. The notations $e$, $m$ and $N(x')$ denote the charge, mass and electron concentration, correspondingly. The validity of the inequality, $\omega^2 > \omega_e^2$ is central to the propagation of light through the plasma medium. While considering a non-rotating gravitational field with a static medium, the photon energy reads [1]
After applying the null approximation \( g_{00} \) takes the form
\[
H(x', p_\alpha) = \frac{1}{2} [\omega^2 p_\alpha p_\beta + \frac{\omega^2 h^2}{c^2}],
\]
where \( h \) is the Planck’s constant. Generally, for any arbitrary medium, photons in a flat space-time move along a straight path, while on the other hand, bent trajectories are followed in a curved space-time. Thus, we assume the motion specifically along the z-axis and take in the null approximations [22] to avoid any small deviations from the straight path. In this case the components of the 4-momentum are as below
\[
p^a = \left( \frac{\hbar \omega}{c}, 0, 0, \frac{n \hbar \omega}{c} \right), \quad p_0 = \left( -\frac{\hbar \omega}{c}, 0, 0, \frac{n \hbar \omega}{c} \right).
\]

Note that, in the forthcoming discussion we shall utilize special notations at infinity: \( \omega(\infty) = \omega, \omega_\alpha(\infty) = \omega_0 \) and \( n(\infty) = \sqrt{1 - \frac{\omega_\alpha^2}{\omega^2}} = n \). Since we are considering a diagonal metric therefore the components of the metric tensor \( g_{\alpha\beta} \) dissolve for all \( \alpha \neq \beta \). Hence, after using (12) we get the following set of equations
\[
\frac{dx_i}{d\lambda} = g^{i\lambda} p_j, \quad \frac{dp_i}{d\lambda} = -\frac{1}{2} g^{i\lambda} p_j p_\lambda - \frac{1}{2} g^{0\lambda} p_0^2 - \frac{1}{2} \frac{\hbar^2}{c^2} K_\alpha N, i.
\]
After applying the null approximation (14) reduces to
\[
\frac{dz}{d\lambda} = \frac{n \hbar \omega}{c}.
\]
The photon momentum in terms of 3-dimensional standard unit vector \( u_i = (0, 0, 1) \) can be expressed as
\[
p_i = \frac{n \hbar \omega}{c} (0, 0, 1) = \frac{n \hbar \omega}{c} u_i.
\]
By substituting the above equation in (14) we get,
\[
\frac{d}{d\lambda} \left( \frac{n \hbar \omega}{c} u_i \right) = \frac{1}{2} g^{i\lambda} p_j p_\lambda - \frac{1}{2} g^{0\lambda} p_0^2 - \frac{1}{2} \frac{\hbar^2}{c^2} K_\alpha N, i.
\]
Owing to the preceding assumption made i.e. the motion takes place only along z axis, we are confined to consider only those components of the unit vector which are perpendicular to the initial direction of propagation (see [22]). Finally, complying with the null approximation in addition to a weak gravitational field (17) appears as
\[
\frac{du_i}{dz} = \frac{1}{2} \left( h_{33,i} + \frac{1}{n^2} h_{00,i} \right), \quad i = 1, 2.
\]

The deflection angle is basically defined by \( \dot{a} = u_{+\infty} - u_{-\infty} \), thus, (18) leads us to a general expression for it as below
\[
\dot{a}_i = \frac{1}{2} \int_{-\infty}^{\infty} \left( h_{33,i} + \frac{1}{n^2} h_{00,i} - \frac{K_\alpha}{\omega^2 - \omega_\alpha^2} N, i \right) dz.
\]

The \( \pm \) signs of \( \dot{a}_i \) determines the deflection towards and away from the central object, respectively. At large \( r \) we have \( \frac{b^2}{r^2} >> (\frac{\hbar}{c})^2 \), thereby the black hole metric is approximated to
\[
ds^2 = ds_0^2 + \frac{\left( R_s}{r} - \frac{\alpha R_s^2}{r^3} \right) dr^2 + \left( R_s}{r} - \frac{\alpha R_s^2}{r^3} \right) d\phi^2
\]
where \( ds_0^2 = -dt^2 + dx^2 + dy^2 \). In the Cartesian coordinates the components \( h_{\alpha\beta} \) can be written as
\[
h_{00} = \left( R_s}{r} - \frac{\alpha R_s^2}{r^3} \right), \quad h_{ik} = \left( R_s}{r} - \frac{\alpha R_s^2}{r^3} \right) n_i n_k, \quad h_{33} = \left( R_s}{r} - \frac{\alpha R_s^2}{r^3} \right) \cos^2 x,
\]
where \( \cos x = z/\sqrt{b^2 + z^2} \) and \( r = \sqrt{b^2 + z^2} \), \( b \) is the impact parameter signifying the closest approach of the photons to the black hole. Using the above mentioned expressions in the formula (19) one can compute the light deflection angle with respect to \( b \) for a black hole surrounded by plasma.

![Figure 1: Schematic representation of the gravitational lensing system typically based on the black hole lens, source star and the image formed.](image-url)
\[
\hat{\alpha}_b = \int_{\infty}^{\infty} \frac{b}{2r} \left( \frac{R_s}{r} \frac{\alpha R_s^2}{r^4} \frac{\cos^2 \chi}{r} \right) \partial_r \left( \frac{R_s}{r} \frac{\alpha R_s^2}{r^4} \frac{\omega^2}{\omega_r^2} \right) dZ - \frac{K_s}{\omega_r^2 - \omega_r^2} \partial_r N dZ. \quad (22)
\]

In the light of foregoing discussion, we can easily examine the impact of different plasma mediums on the photon deflection angle as follows:

2.1 Uniform Plasma

First of all, we consider the photon geodesics when plasma is uniformly distributed in the black hole surroundings. In homogenous plasma medium the refractive index \(n\) particulary counts on the photon frequency with \(\omega_0\) as a constant quantity which ultimately leads to the approximation \(1 - n \ll \frac{\omega_0}{\omega}\) [33]. Subsequently, the corresponding constraints annihilate the term \(\partial_r N\) and the angle of deflection takes the form

\[
\hat{\alpha}_{\text{uni}} = \left( \frac{R_s}{b} - \frac{3\pi \alpha R_s^2}{16b^4} \right) + \left( \frac{R_s}{b} - \frac{3\pi \alpha R_s^2}{4b^4} \right) \frac{1}{1 - \frac{\omega_0^2}{\omega^2}}. \quad (23)
\]

Fig. (2) illustrates the plots of the photon deflection angle \(\hat{\alpha}_{\text{uni}}\) as a function of the impact parameter \(b\) for various coupling constant \(\alpha\) (upper panel) and plasma parameters \(\frac{\omega_0^2}{\omega^2}\) (lower panel). An increase is examined in the deflection angle for smaller values of the impact parameter \(b\), which means that a massless particle passing too close to the black hole surroundings basically enhances its deviating tendency. Fig. (3) is a visualisation of the deflection angle distinctly with respect to \(\frac{\omega_0^2}{\omega^2}\) and \(\alpha\). The deflection angle is maximum due to high plasma distribution (upper panel) and is seen to be strictly decreasing against an increasing coupling parameter \(\alpha\) (lower panel), for instance, taking \(\alpha = 0\), the Schwarzschild gravity ensures the highest degree of deviation \(\hat{\alpha}_{\text{uni}}\). We deduce that, as expected, the existence of plasma in the black hole vicinity, contrarioise to the vacuum case \(\frac{\omega_0^2}{\omega^2} = 0\), contributes to the photon motion.

2.2 Singular Isothermal sphere

A singular isothermal sphere (SIS) is the most favourable model to comprehend the peculiar features of gravitational lensed photons. It was primarily introduced in [77, 78] to explore the len's property of the galaxies and clusters. Generally, SIS is a spherical gas cloud with a singularity located at its center where the density tends to infinity. The density distribution of a SIS is given by

\[
\rho(r) = \frac{\sigma_v^2}{2\pi r^2}, \quad (24)
\]

where \(\sigma_v^2\) refers to a one-dimensional velocity dispersion. The plasma concentration admits the following analytic expression

\[
N(r) = \frac{\rho(r)}{km\rho}, \quad (25)
\]
here $m_p$ is the proton mass and $\kappa$ is a dimensionless constant coefficient generally associated to the dark matter universe [22]. Utilizing (10,24,25) the plasma frequency takes the form

$$\omega_p^2 = K_e N(r) = \frac{K_e \sigma_v^2}{2\pi m_p r_c^2}.$$  (26)

We reckon with the above mentioned properties of the SIS and compute the angle of deflection $\hat{\alpha}_{SIS}$ as below [22].

$$\hat{\alpha}_{SIS} = \left( \frac{2R_s}{b} - \frac{15\pi \alpha R_s^2}{16b^4} \right) + \frac{R_s}{b^2} \omega_p^2 \left( \frac{1}{2} - \frac{3}{8b^2} + \frac{5\alpha R_s^2}{8b^4} \right).$$  (27)

These calculations brings up a supplementary plasma constant $\omega_p^2$ which has the following analytic expression.

$$\omega_p^2 = \frac{\sigma_v^2 K_e}{2\pi m_p r_c^2}.$$  (28)

In order to assimilate the influence of SIS on the photon trajectory we plotted the deflection angle $\hat{\alpha}_{SIS}$ as a function of the impact parameter $b$, see Fig. (4), interestingly, we see that the uniform plasma and SIS medium share common features regarding the parameter $b$. Note that, the quantity $\frac{\omega}{c}$ identifies the distribution of SIS in the black hole vicinity, thus we detect the photon sensitivity to the specified parameter along with the coupling constant parameter $\alpha$, by means of a graphical analysis in Fig. (5). We examined that $\hat{\alpha}_{SIS}$ increases when $\frac{\omega}{c}$ increases (upper panel) and, conversely, $\hat{\alpha}_{SIS}$ decreases when $\alpha$ increases (lower panel). Hence, the presence of SIS in the black hole surroundings to some extent affects the intervening massless particles.

2.3. Non-Singular Isothermal gas sphere

Now we further proceed to study the motion of photons considering a non-singular isothermal sphere (NSIS) which is a more reasonable and physical setup for the analysis. Unlike the SIS, in this lens model the singularity is bounded by a finite core at the origin of the gas cloud whereby the density distribution is defined as [79, 80]

$$\rho(r) = \rho_0 \frac{\sigma_v^2}{2\pi (r^2 + r_c^2)} = \rho_0 \left( \frac{\sigma_v^2}{2\pi r_c^2} \right).$$  (29)

Here the core radius is represented by $r_c$. The plasma concentration for NSIS using (25) becomes

$$N(r) = \frac{\sigma_v^2}{2\pi m_p (r^2 + r_c^2)}. \hspace{1cm} \text{(30)}$$

We compute the plasma frequency from (10,29,30) as follows

$$\omega_p^2 = \frac{K_e \sigma_v^2}{2\pi m_p (r^2 + r_c^2)}. \hspace{1cm} \text{(31)}$$

![Figure 4: Plot of the deflection angle $\hat{\alpha}_{SIS}$ as a function of the impact parameter $b$ for $\frac{\omega}{c} = 0.5$ (upper panel) and $\alpha = 0.5$ (lower panel).](image1)

![Figure 5: Plot of the deflection angle $\hat{\alpha}_{SIS}$ as a function of $\frac{\omega}{c}$ (upper panel) and $\alpha$ (lower panel) for a fixed impact parameter $b = 3$.](image2)
The angle of deflection obtained by the deviation of photons in NSIS gravitational lens setup in accordance with the properties in the latter discussion is as below

\[
\hat{\alpha}_{\text{NSIS}} = \left(\frac{2R_s}{b} - \frac{15\pi aR_s^2}{16b^4}\right) + \frac{R_s^2\omega^2_s}{c^2} \left(\frac{R_s}{b\pi r_c^2} + \frac{b}{2(\sqrt{b^2 + r_c^2})^3}\right) - \frac{bR_s\tanh^{-1}\left(\frac{\sqrt{b^2 + r_c^2}}{\sqrt{b^2 + r_c^2}}\right)}{\pi r_c^2(\sqrt{b^2 + r_c^2})} \left(\frac{2}{r_c^2} + \frac{3}{4b^2} - \frac{1}{b^2r_c^2}\right) - \frac{2b}{r_c^2(\sqrt{b^2 + r_c^2})}. \tag{32}
\]

As previously executed we employ the same graphical interpretation to unfold the properties of NSIS concerning the photon motion. Here, the distribution of NSIS is associated with the parameter \(\omega^2\). It is evidently revealed from Fig. (6,7) that the behaviour of the impact parameter \(b\), coupling constant \(\alpha\) and the parameter \(\omega^2\) cannot be distinguished from a specific point of view when brought in comparison with the uniform plasma and SIS case. Nevertheless, one can at least figure out the distribution which has the most pronounced effect on the deflection angle. Fig. (8) is a visual juxtaposition of the \(\hat{\alpha}_{\text{uni}}, \hat{\alpha}_{\text{SIS}}\) and \(\hat{\alpha}_{\text{NSIS}}\) as a function of the impact parameter and the coupling constant. It is quite obvious that the deflection is maximum when the black hole is surrounded by a uniform plasma medium. The final result can therefore be encapsulated in a mathematical expression as, \(\hat{\alpha}_{\text{uni}} > \hat{\alpha}_{\text{SIS}} > \hat{\alpha}_{\text{NSIS}}\).
3. Lens equation and magnification in the presence of plasma

We now focus on the magnification of the image source brightness in the presence of plasma using the angle of deflection $\theta$ discussed in our previous sections, particularly for the uniform plasma and SIS case. Fig. (1) is a formal visualization of the gravitational lensing system where the black hole function as a lens to the distant source (star) and generates an image. In the diagram $D_\alpha$, $D_\beta$, and $D_0$ represent the distances from the source to the observer, from the lens to the observer, and from the source to the lens, respectively. The angular position of the source and image are correspondingly specified by $\beta$ and $\theta$. The relation which finds the angular position of the image, usually termed as the gravitational lens equation is defined as follows [27, 32]

$$\theta D_\alpha = \beta D_\alpha + \tilde{\alpha} D_0. \quad (33)$$

Generally, in this setup the impact parameter $b$ admits the expression $b = D_\beta \theta$ which on substitution in (33) along with a newly introduced quantity $\xi(\theta) = \{\tilde{\alpha} b\} \theta$ yields the following equation

$$\beta = \theta - \frac{D_0}{D_\alpha} \frac{\xi(\theta)}{\tilde{\alpha}} D_\beta \theta. \quad (34)$$

By making a special choice of $\beta = 0$ one may obtain a uniquely aligned or more precisely a collinear framework of the observer, the lens and the source. In such context due to symmetry around the lens the image appears in the form of a ring, often called the Einstein ring with a radius $R_0 = D_\beta \theta_0$. Here, $\theta_0$ is referred as the Einstein angle, which for the Schwarzschild space-time becomes

$$\theta_0 = \sqrt{2R_\alpha \frac{D_0}{D_\alpha D_\beta}}. \quad (35)$$

The lensing process by the astrophysical objects is trackable only through the apparent brightness of the source which can be computed utilizing the formula

$$\mu_\pm = \frac{I_\pm}{I_0} = \sum_k \left( \frac{\theta_k}{\beta} \right) \left( \frac{d \theta_k}{d \beta} \right), \quad k = 1, 2, \ldots, j \quad (36)$$

where $k$ denotes the number of images formed each indicated by the angular position $\theta_k$, $I_\pm$ is the total brightness of all the images and $I_0$ is the unlensed brightness of the source. However, this straightforward relation cannot be efficiently employed for a computation of the total magnification, therefore we follow the notion in [2] by using (36) in terms of the positive- and negative-parity images. As a customary, the positive parity image lies on the same side of the lens as the source for $\beta > 0$, whereas the negative-parity image lies on the opposite side of the lens from the source for $\beta < 0$. Thus, the verified formulae for the magnification of the source given in [2] are as follows

$$\mu_+ = \frac{1}{4} \left( \frac{x}{\sqrt{x^2 + 4}} + \frac{\sqrt{x^2 + 4}}{x} + 2 \right), \quad (37)$$

$$\mu_- = \frac{1}{4} \left( \frac{x}{\sqrt{x^2 + 4}} + \frac{\sqrt{x^2 + 4}}{x} - 2 \right). \quad (38)$$

where $x = \frac{\beta}{\theta_0}$ is a dimensionless quantity and the superscript $pl$ refers to the plasma presence. After performing some calculations with the aid of (37,38) we get the total magnification of the image as below

$$\mu_\text{tot} = \mu_+ + \mu_- = \frac{x^2 + 2}{x \sqrt{x^2 + 4}} \quad (39)$$

Henceforward, we shall concentrate on the uniform plasma and SIS case to examine the magnification of the apparent brightness of the source.

3.1. Uniform Plasma

Keeping in view the discussion regarding the 4D-EGB gravity surrounded by a uniform plasma the angular position of the image is smoothly worked out from (23,34) as below,

$$\mu_\text{uni} = \theta_0 \left[ \frac{1}{2} \left( 1 - \frac{3\pi \alpha R_\alpha}{16 b^3} \right) + \left( 1 - \frac{3\pi \alpha R_\alpha}{4 b^3} \right) \frac{1}{1 - \frac{\beta^2}{\theta_0^2}} \right]. \quad (40)$$

The value of $x$ requisite for the magnification of the source in uniform plasma thus takes the form

$$x = \frac{x_0}{\sqrt{2 \left( \frac{1 - \frac{3\pi \alpha R_\alpha}{16 b^3}}{1 - \frac{3\pi \alpha R_\alpha}{4 b^3}} \right) \left( 1 - \frac{\beta^2}{\theta_0^2} \right)}}. \quad (41)$$

where $x_0 = \beta/\theta_0$. In the upper panel of Fig. (9) the total magnification is plotted against the plasma parameter $\frac{x_0}{\theta_0}$ for various coupling parameters $\alpha$. The magnification is observed to increase for increasing uniform plasma distribution. Hence, we restore the fact that the presence of uniform plasma typically increases the magnification and immensely reduces in case of the vacuum[22]. In the lower panel, for different values of $\frac{x_0}{\theta_0}$ the behaviour of $\mu_\text{uni}$ is shown by varying $\alpha$. Generally, for increasing $\alpha$, $\mu_\text{tot,uni}$ decreases and is maximum for the Schwarzschild back hole ($\alpha = 0$). For the purpose of additional details we hereby discuss the magnification ratio of uniform plasma to vacuum assuming the positive- and negative-parity images, see Fig.(10). The lower and upper curves depict the plots for $\mu_\text{uni}^+ / \mu_\text{uni}$ and $\mu_\text{uni}^- / \mu_\text{uni}$, respectively. It is observed that the two curves meet at a single point each time $x_0 \to 0$, for $\frac{x_0}{\theta_0} = 0.5$ and 0.9 the corresponding values of $x_0$ recorded are 1.21766 and 2.31176. On the whole, at large $x_0$ the ratio $\mu_\text{uni}^+ / \mu_\text{uni}^-$ tends to unity for each $\frac{x_0}{\theta_0}$ but exclusive for $\mu_\text{uni}^- / \mu_\text{uni}$ the magnification factor is greatly amplified for a larger plasma distribution.
3.2. Singular Isothermal sphere

We shall revisit the above analysis in a similar manner to discover the influence of SIS on magnification of the image source. Hence, the angular position of the image in this background is obtained from (27,34) as below,

$$
\theta_{\text{SIS}}^0 = \theta_0 \sqrt{\frac{1}{2} \left( 2 - \frac{15\pi\alpha R_s}{16b^3} + \frac{R_s\omega^2}{b\omega^2} \left( \frac{1}{2} - \frac{2R_s}{3b\pi} + \frac{5\alpha R_s^2}{8b^2} \right) \right)}. \quad (42)
$$

The value of \( x \) for the magnification of the source in the SIS surroundings is thus expressed in the form

$$
\lambda = \frac{x_0}{\sqrt{\frac{1}{2} \left( 2 - \frac{15\pi\alpha R_s}{16b^3} + \frac{R_s\omega^2}{b\omega^2} \left( \frac{1}{2} - \frac{2R_s}{3b\pi} + \frac{5\alpha R_s^2}{8b^2} \right) \right)}}. \quad (43)
$$

where \( x_0 = \frac{\beta}{\beta_0} \). The behaviour of total magnification of the image source dependence of the SIS distribution \( N_1 \) for distinct \( \alpha \) is displayed in the upper panel of Fig. (11). It is verified that the presence of SIS amplifies the luminosity of the source star. Alternatively, in the lower panel by increasing the \( \alpha \) parameter.
a decrease is observed for $\mu_{tot,SIS}^3$. In Fig. (12), the magnification ratio of SIS to vacuum operates analogous to the uniform plasma case when adjusted with the same parameters, but nonetheless, when the black hole is surrounded by a uniform plasma medium, a considerable magnification is perceived by a far-off observer (see Fig. (13)).

4. Conclusion

We discussed categorically the weak-field lensing phenomenon in the background of a 4 dimensional Einstein-Gauss-Bonnet gravity environment by a uniform plasma, singular isothermal sphere and non-singular isothermal sphere. The deflection angle for each case is incisively discussed to evince the most effective medium. Interestingly, the uniform plasma is investigated to deviate the incoming photons at a very large angle. Moreover, the SIS deflection angle is greater to a certain extent than that of the NSIS, even though the difference could be regarded as negligible. In general, the light lensed by a nearby black hole modifies the spectrum of the source star that results in the apparent magnification of the image source luminosity. This behaviour has remained the focus of our analysis by considering a plasma environment and we have successfully achieved the fact that the presence of a uniform plasma significantly enhances the magnification. Finally, all along the examination, the Schwarzschild black hole compared to the 4D-EBG black hole emerged as an effective gravity.

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Figure 13: Plots for $\mu_+^3/\mu_+$ (lower curve) and $\mu_-^3/\mu_-$ (upper curve) of the image magnifications in the presence of uniform plasma and SIS. The fixed parameters used are $R_s = 2$, $\beta = 3$, $\nu = 0.5$, $\omega = 0.5$, $\omega = 0.5$, and $x_0 = 0.055$. 

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