Fractionalization in Three-Components Fermionic Atomic Gases in a One-Dimensional Optical Lattice

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We study a three-components fermionic gas loaded in a one-dimensional optical trap at half-filling. We find that the system is fully gapped and may order into 8 possible phases: four $2k_F$ atomic density wave and spin-Peierls phases with all possible relative $\pi$ phases shifts between the three species. We find that trionic excitations are unstable toward the decay into pairs of kinks carrying a fractional number, $Q = 3/2$, of atoms. These sesquions eventually condense upon small doping and are described by a Luttinger liquid. We finally discuss the phase diagram of a three component mixture made of three hyperfine level of $^6\text{Li}$ as a function of magnetic field.

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A problem analogue to quark confinement in particle physics has been recently addressed in systems with multi-components fermionic atoms loaded in an optical lattice. These studies strongly support the formation of “baryonic” states made of bound states of $n > 2$ atoms. In one dimension, for example, trionic states, made of bound state of three atoms, were predicted to be stable at generic densities and sufficiently low temperatures, typically of the order $\sim 30 - 100\text{ nK}$. This result opens the exciting possibility to probe in a new context and in future experiments a simplified version of the “quark” confinement phenomenon in quantum chromodynamics. In all these previous studies the attention has been drawn on the formation of baryonic (or molecular) states that contains an integer number of atoms. Here we shall focus on the intriguing situation where the low-energy elementary excitations carry a fractional number of atoms. Although it may appears counter-intuitive, fractionalization of quantum numbers is a well established phenomenon in condensed matter physics. Celebrated examples are fractionally charged excitations in the quantum Hall state and in quasi-one-dimensional polymers. In this work we shall present strong arguments that fractionalization can also occur in ultra-cold atomic physics. We shall give evidences that at densities close to half-filling a three-components fermionic mixture loaded in a one-dimensional optical trap may support low-energy excitations carrying a fractional number, $Q = 3/2$, of atoms, the sesquions.

When loaded in a one-dimensional optical lattice of wavelength $\lambda$, a three-components mixture is well described, away from resonance, by a Hubbard-type hamiltonian of the form:

$$\mathcal{H} = -t \sum_{\langle i,a \rangle} \left[ \hat{c}_{i,a}^\dagger \hat{c}_{i+1,a} + \text{H.C.} \right] + \sum_{i,a} U_{ab} \rho_{i,a} \rho_{i,b} \eqno(1)$$

where $\hat{c}_{i,a}^\dagger$ is the creation operator for a fermionic atom of species $a = (1,2,3)$, at site $i$, and $\rho_{i,a} = \hat{c}_{i,a}^\dagger \hat{c}_{i,a}$ is the local density of the atomic species $a$. The parameters $t$ and the couplings $U_{ab}$ can be expressed in term of the recoil energy, the laser intensity and wavelength as well as the $s$-wave scattering lengths $s_{ab}(B)$ between the species. For generic external magnetic fields $B$, the $s_{ab}$ are in general different and so are the couplings $U_{ab}$. Thus the physical symmetry group of (1) is $U(1)^3$ corresponding to the conservation of the number of atoms of each species. Such a small symmetry, which is an essential feature of atomic mixtures, make the elucidation of the physics associated with (1) a difficult task. However, as we shall see, much can be said in the weak coupling, low-energy, limit. The physics described by (1) strongly depends on the density of atoms $\bar{\rho} = \rho_1 = \rho_2 = \rho_3$. Away from half-filling, i.e. when $\bar{\rho} \neq 1/2$, it was shown in Ref. 11 that for generic couplings the dominant fluctuations consists into massless $2k_F$ Atomic Density Waves (ADW) and massless trionic excitations carrying total atomic number $Q = 3$. At half filling, when $\bar{\rho} = 1/2$, the physics is, as we shall see, radically different.

**Effective Low Energy Hamiltonian.** The low energy effective theory associated with the Hubbard Hamiltonian (1) can be derived, as usual, from the linearization at the two Fermi points $\pm k_F$ of the dispersion relation of free three-component fermions:

$$c_{i,a} \sim \Psi_{aR} e^{ik_F x} + \Psi_{aL} e^{-ik_F x} \quad a = (1,2,3), \quad \eqno(2)$$

where $x = i a_0$, $a_0 = \lambda/2$ is the lattice spacing, $\lambda$ the laser wavelength, and $k_F = \pi \rho/\!\!\!\rho_0$ is the Fermi wave-vector. Finally $\bar{\rho} = 1/2$ is the density per species. In the weak coupling limit $|U_{ab}|/t \ll 1$ the effective hamiltonian associated with (1) is found to be:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 \quad \eqno(3)$$

with

$$\mathcal{H}_0 = -i v_F \sum_a (\Psi_{aR}^\dagger \partial_x \Psi_{aR} - \Psi_{aL}^\dagger \partial_x \Psi_{aL}) \quad \eqno(4)$$

and

$$\mathcal{H}_1 = \sum_{a < b} \left( \mu_{ab} h_{aR} h_{bL} + \lambda_{ab}^- J_{aR}^I J_{ab}^- L + \lambda_{ab}^+ J_{aR}^I J_{ab}^+ L \right) + (R \leftrightarrow L) \quad \eqno(5)$$
rents they generate the 15 couplings \( \lambda \) an general hamiltonian for the three species problem with later symmetry down to \( U \). The one-loop RG equations associated with (3) are given there of the lattice hamiltonian (1) depends on the asymptotic behavior of some positive coupling. As (8) is invariant under the simultaneous change \( \epsilon_a \rightarrow -\epsilon_a \), there are 8 independent fixed points, with \((\epsilon_1, \epsilon_2, \epsilon_3) = (+++), (-++), (+-) \) and \((+-+)\), that describe phases, labelled \( A \) and \( B \), with qualitatively different physical properties. The phases \( A \) and \( B \) are generalized 2\( k_F \), Spin-Peierls (SP) and Atomic Density Wave (ADW) phases. The \( \epsilon_a \) account for all possible relative \( \pi \)-phase shifts between the species \( a \). A pictorial representation of the ground states is presented in Fig.1. The corresponding lattice order parameters can be readily obtained from the structure of the interacting part of (8) and are given by:

\[
C^{+}_{\epsilon_1 \epsilon_2 \epsilon_3} = \sum_a \left( \frac{(-1)^a}{2} \epsilon_a (c_{ai} c_{ai+1} + \text{h.c}) \right) \\
C^{-}_{\epsilon_1 \epsilon_2 \epsilon_3} = \sum_a \left( \frac{(-1)^a}{2} \epsilon_a (c_{ai} c_{ai+1} + \text{h.c}) \right)
\]

In each phase the ground state is doubly degenerated and when \( C^{+}_{\epsilon_1 \epsilon_2 \epsilon_3} \neq 0 \) there is spontaneous symmetry breaking of translational invariance by one lattice site. As a consequence we expect kinks (or solitonic) excitations that interpolate between the two ground states to be present in the spectrum. As we shall see these have fractional quantum numbers.

\textbf{Spectrum and Fractionalization.} Remarkably enough, the different Hamiltonians (8) can be brought to the same form by mean of duality transformations \((10)\):

\[
H^{\pm}_{\epsilon_1 \epsilon_2 \epsilon_3}(G, \Psi) = \mathcal{H}^{\pm}(G, \omega^{\pm}_{\epsilon_1 \epsilon_2 \epsilon_3}(\Psi)),
\]



\[
\omega^{\pm}_{\epsilon_1 \epsilon_2 \epsilon_3}(\Psi_{aR}) = e^{\frac{1}{2}(1+\epsilon_{ai})} \epsilon_a \Psi_{aR}.
\]
We therefore find that the elucidation of the low-energy physics described by the fixed points hamiltonians $\mathcal{H}_A^\pm$ stem from the knowledge of those of $H_{++}$. The latter hamiltonian has an enlarged $SO(6)$ symmetry generated by the $\mathcal{J}^A = \int dx (\mathcal{J}_A^2 + \mathcal{J}_B^2)$. The DSE is only approximate and we expect residual symmetry breaking operators to survive even in the low-energy limit. As the $SO(6)$ GN model is integrable so that its spectrum is exactly known and by duality the one in the other phases $A^\pm_1$, these particles are described by the same quantum numbers since the duality transformations do not affect the 3 conserved charges (or Cartan generators):

$$ q_a = \int dx (\Psi^a_{aR} \Psi^a_{aL} + \Psi^a_{aL} \Psi^a_{aR}), \ a = (1, 2, 3). $$

These are nothing but the total number of atoms of a given species $a$ and the particles of the spectrum are labelled by the set of quantum numbers $(q_1, q_2, q_3)$. The kinks quantum numbers are fractional: $S_{1a} = \pm 1/2(1, 1, 1), S_{2a} = \pm 1/2(1, -1, -1), S_{3a} = \pm 1/2(-1, 1, -1)$. The fermions are bound states of two kinks and have the same quantum numbers as the original lattice fermions: $\xi_{3} = (\mp 1, 0, 0), (0, \pm 1, 0)$ and $(0, 0, \pm 1)$. There are no other stable particles. In particular there are no stable trions in contrast with what happens at incommensurate fillings. Trionic excitations, $T^1 = c_1 c_2 c_3$, have total atomic number $Q = 3$, where

$$ Q = q_1 + q_2 + q_3, $$

and are unstable toward the decay into elementary kinks or fermions. A trion has quantum numbers $(1, 1, 1)$ and the most energetically favorable process is $T^1 \rightarrow S_{1a}^0 S_{0a}^0$ so that one may think of the kink $S_{1a}^0$ as “half” a trion. As it has total atomic number $Q = 3/2$ one may call it a sesquion. The existence of these fractional kinks as the lowest energy excitations in generic three-components ultra-cold atomic systems is an unexpected and non-trivial finding and constitute one of the main results of the present work. It is therefore worth discussing the stability of the above excitations. Indeed, as noticed in both Refs. [16,17], the DSE is only approximate and we expect residual symmetry breaking operators to survive even in the low-energy limit. As the $SO(6)$ GN particle spectrum is described by the conserved quantum numbers associated with the $U(1)$ symmetry of the problem, small residual anisotropy will result only into a small splitting of the particle spectrum. For large enough anisotropy and/or strong couplings it may eventually happens that the above description of the spectrum breaks down. We expect however that the DSE description of the hamiltonian (11) (and hence the stability of the sesquions) holds in a relatively large portion of the phase diagram. Indeed the accuracy of the DSE description has been checked numerically in the different context of the $SU(4)$ Hubbard model at half-filling where the adiabatic continuity of the $SO(8)$ GN spectrum in this case has been explicitly observed for small enough interactions [18].

Doping. To model small doping we consider adding a chemical potential term $H_Q = -\mu Q$ (we consider hole doping with $\mu > 0$). As $Q$ is invariant under the duality transformation it is sufficient to consider doping the $SO(6)$ Gross-Neveu model. The chemical potential term breaks the $SO(6)$ symmetry but since $H_{++}^+ = 0$ doping does not spoil integrability and the following picture emerges. At non zero $\mu$, the particle spectrum is splitted according to the values of $Q$. A particle with mass $m$ and atomic number $Q$ will lower its energy to $m - \mu Q$. When its energy becomes negative the ground state start to fill with these particles. When $\mu > 2m S/3$ the first particle that start filling the ground state is the sesquion $S_{10}^0$ of mass $m_S$ and $Q = 3/2$. As $\mu$ is increased further other particles would like to enter the ground state like other members of the $K$ multiplets with $Q = 1/2$ or the fermions with $Q = 1$. However the increase of the chemical potential is counteracted by the repulsion felt by the kinks and the fermions to the sesquions. As a result for $\mu > 2m S/3$ the ground state is only filled by sesquions which become massless excitations. The effective theory describing these massless fractional $Q = 3/2$ excitations is a Luttinger liquid with a stiffness $K$. At these dopings the kinks $S_{1a}^{12,3}$ and the fermions $\xi_{3}$ remain massive, with renormalized masses. Both the renormalized masses and the stiffness $K$ could be in principle computed from the Bethe ansatz solution in a similar way as done in Ref. [19]. At large doping, i.e. when $\mu >> m_S$, the $4K_F$ term $\lambda^+\lambda^-$ decouples and one recovers the physics described above for the generic filling case with massless $Q = 3$ trionic excitations. We may therefore expect that below some critical value of the density $\tilde{\rho} < \tilde{\rho}$ sesquions get confined into trions. The elucidation of the nature as well as the location of such a confinement/deconfinement transition goes beyond the scope of this work and will studied elsewhere.

Three-Species Problem and Experiments. The phase diagram in the three-dimensional space $(U_{12}/t, U_{23}/t, U_{31}/t)$ is rich and complex, revealing the delicate balance between the different competing orders. We find that among the 8 possible phases only 5 are stabilized: the four ADW phases $A^\pm_1 c_2 c_3$ and the uniform SP phase $A^+_++$. We show in Fig.2, two projections of the phase diagram in the $(U_{23}/t, U_{31}/t)$ plane for typical value of $U_{12}/t = \pm 0.5$. Though it is
scattering lengths. Taking for example B | is an isotropic ray for sufficiently large anisotropies. In experiments once the phase is destabilized in favor of various ADW phases of the anisotropic Hubbard model for fixed values of the potential in (1) and ii) that in the frustrated regions, we observe that: i) in the unfrustrated region, a uniform SP phase A₁⁺⁻ occurs in the frustrated region (i.e. when (Πₐ<ₙ Uₐₙ) > 0) around the isotropic points |Uₐₙ| = U.

difficult to draw any general quantitative picture of the phase diagram we observe that: i) in the unfrustrated regions, where (Πₐ<ₙ Uₐₙ) < 0, the ADW phases that are stabilized are the one that minimize the density-density potential in (1) and ii) that in the frustrated regions, when (Πₐ<ₙ Uₐₙ) > 0, the kinetic energy term play an important role when all couplings are of the same order of magnitude. In particular in the vicinity of the isotropic points |Uₐₙ| = U a uniform SP phase A₁⁺⁻ is likely to be stabilized. Eventually the above SP phase is destabilized in favor of various ADW phases for sufficiently large anisotropies. In experiments once the optical lattice parameters and the density are fixed the only control parameter is the external magnetic field B. The phase diagram as a function of the magnetic field B is a line in the three-dimensional space (U₁₂(B)/t, U₂₃(B)/t, U₃₁(B)/t) which dependence on B essentially depends on the mixture through the s-wave scattering lengths. Taking for example a mixture made of a balanced population of three hyperfine states of ⁶Li atoms, |F, m_F⟩ = |1⟩ = |1/2, 1/2⟩, |2⟩ = |1/2, −1/2⟩, and |3⟩ = |3/2, −3/2⟩, with typical optical lattice parameter and a laser wavelength λ = 1µm, we find a weak coupling regime where a non trivial SP phase may be observed. At half-filling using our one loop RG equations the following phase diagram as a function of the magnetic field emerges. For small fields B < Bₙ₁, a uniform ADW phase A₁⁻⁺⁺ is stabilized while at larger fields B > Bₙ₂ an ADW phase A₁⁻⁻⁺, where the species labeled 1 is in phase opposition with species 2 and 3, shows off. This match the result of Ref. [11] for U = 1/2 where at these filings ADW phases of the same type with quasi-long range order were predicted. The essential difference with the above case is that when U = 1/2 an intermediate uniform SP phase A₁⁺⁻⁺ is locked in the region Bₙ₂ < B < Bₙ₁. Within the one loop accuracy we find Bₙ₂ ~ 560G and Bₙ₃ ~ 540G. This is an interesting result from the experimental point of view since in both A₁⁺⁻⁺ and A₁⁻⁻⁺ phases we expect that effect of the three-body losses will be considerably reduced. The knowledge of actual values of the binding energies of the sesquions and hence of the temperature scale below which these phases could be stabilized could be used for thermal pulses. A universality call for an alternative approach such like numerical calculation. [21] To summarize we have shown that in the vicinity of half-filling fractional excitations carrying Q = 3/2 atoms, the sesquions, are the relevant low energy excitations in a generic three-components Fermi mixture. These sesquions are likely to get confined into Q = 3 trionic excitations when one moves sufficiently far away from half-filling. We therefore expect that both the confined (trionic) and unconfined (sesquionic) phases could be probed in future experiments.

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