Abstract. Coxeter decompositions of hyperbolic simplices were studied in math.MG/0212010 and math.MG/0210067. In this paper we use the methods of these works to classify Coxeter decompositions of bounded convex pyramids and triangular prisms in the hyperbolic space \( \mathbb{H}^3 \).

Introduction

Let \( P \) be a convex polyhedron in the hyperbolic space \( \mathbb{H}^3 \).

**Definition 1.** A polyhedron \( P \) is called a **Coxeter polyhedron** if all dihedral angles of \( P \) are integer parts of \( \pi \).

**Definition 2.** A polyhedron \( P \) admits a **Coxeter decomposition** if \( P \) can be tiled by finitely many Coxeter polyhedra such that any two tiles having a common face are symmetric with respect to this face.

In this paper we classify Coxeter decompositions of bounded convex pyramids and triangular prisms in the hyperbolic space \( \mathbb{H}^3 \).

Basic definitions

If \( P \) admits a Coxeter decomposition we also say that \( P \) is a **quasi-Coxeter** polyhedron. The tiles in Definition 2 are called **fundamental polyhedra** and denoted by \( F \). Clearly, any two fundamental polyhedra are congruent to each other. A plane \( \alpha \) containing a face of a fundamental polyhedron is called a **mirror** if \( \alpha \) contains no face of \( P \).

In this paper any polyhedron is either a quasi-Coxeter polyhedron or a polyhedron bounded by the mirrors of some Coxeter decomposition. By the ”decomposition” we always mean a Coxeter decomposition.

**Definition 3.** Given a Coxeter decomposition of a polyhedron \( P \), a dihedral angle of \( P \) formed up by facets \( \alpha \) and \( \beta \) is called **fundamental** if no mirror contains \( \alpha \cap \beta \).

In this case the edge \( \alpha \cap \beta \) of \( P \) is called **fundamental** too.

A vertex \( A \) is called **fundamental** if no mirror contains \( A \).
1 Fundamental polyhedron

Let $P$ be a bounded quasi-Coxeter polyhedron. From now on $P$ is either a bounded pyramid or a bounded triangular prism in $H^3$.

**Notation.**
Denote by $\alpha \cap \beta$ the intersection of the sets $\alpha$ and $\beta$ in the inner part of $H^3$.
Let $\alpha$ be a face of a polyhedron, denote by $\overline{\alpha}$ a plane containing $\alpha$.

**Lemma 1.** Let $F$ be a Coxeter polyhedron.
Then either $F$ is a tetrahedron or $F$ has two faces $\alpha$ and $\beta$ such that $\alpha \cap \beta = \emptyset$.

**Proof.** For any polyhedron without obtuse angles we have $\alpha \cap \beta = \emptyset \Rightarrow \overline{\alpha} \cap \overline{\beta} = \emptyset$ (see [2]). Any Coxeter polyhedron has no obtuse angles. Thus, it is enough to prove that $F$ has faces $\alpha$ and $\beta$ such that $\alpha \cap \beta = \emptyset$.

Suppose that $\alpha \cap \beta \neq \emptyset$ for any faces $\alpha$ and $\beta$ of $F$. Let $A$ be an arbitrary vertex of $F$. Let $\alpha_1, \ldots, \alpha_k$ be all the faces of $F$ containing $A$. Let $\beta$ be any face of $F$ such that $A \notin \beta$. By the assumption, $\alpha_i \cap \beta \neq \emptyset \quad \forall i = 1, \ldots, k$. Therefore, $F$ has no faces except $\alpha_i$ and $\beta$, i.e. $F$ is a pyramid.

Suppose that $A$ is not an ideal vertex of $F$. Then $A$ is incident exactly to three faces of $F$ and $F$ is a tetrahedron. Suppose that $A$ is an ideal vertex of $F$ and the pyramid $F$ is not a tetrahedron. Consider two faces (say, $\alpha_1$ and $\alpha_3$) having no common edges. Since $A \notin H^3$, we have $\alpha_1 \cap \alpha_3 = \emptyset$.

**Lemma 2.** If $P$ is a bounded pyramid, then $F$ is a tetrahedron.

**Proof.** Suppose that $F$ is not a tetrahedron. By Lemma [1], $F$ has two faces $\alpha$ and $\beta$ such that $\overline{\alpha} \cap \overline{\beta} = \emptyset$. Let $F_0$ be a fundamental polyhedron in $P$; let $\alpha_0$ and $\beta_0$ be its disjoint faces. Consider a sequence of fundamental polyhedra $F_i \in P$, $i \in \mathbb{Z}$, such that $\alpha_i = \alpha_{i+1}$, if $i$ is odd, and $\beta_i = \beta_{i+1}$, if $i$ is even (see Fig. 1).

The sequence is finite, since $P$ contains finitely many fundamental polyhedra. The polyhedra $F_i$ cannot make a cycle, since $\alpha_i \cap \beta_i = \emptyset$. Let $F_k$ and $F_s$ be the endpoints of the sequence. Then $\alpha_k$ or $\beta_k$ belongs to some face $\gamma$ of $P$. Similarly, $\alpha_s$ or $\beta_s$ belongs to some face $\delta$ of $P$. Obviously, $\overline{\alpha_i} \cap \overline{\beta_j} = \emptyset$ for any $i, j$. But $\gamma$ intersects $\delta$, since $P$ is a bounded pyramid. The contradiction shows that $F$ is a tetrahedron.

**Remark.** The idea of the proof of Lemma 2 belongs to O. V. Schwarzman.

Let $\alpha$ be a mirror in a Coxeter decomposition of the triangular prism $P$. Then $\alpha$ intersects $P$ as shown in one of Fig. 2.1–2.16.
Definition 4. We say that a mirror is \textbf{pentagonal} if it intersects the prism as shown in Fig. 2.16.

Definition 5. We say that a triangular quasi-Coxeter prism $P$ is \textbf{minimal} if any prism $P'$ inside $P$ is fundamental.

Lemma 3. Let $P$ be a minimal prism and $F$ be a fundamental polyhedron. Suppose that $F$ is neither a tetrahedron nor a triangular prism. Then any mirror is pentagonal and any dihedral angle of $P$ is fundamental.
Proof. Suppose that the decomposition contains a tetrahedron or a pyramid. Then by Lemma 2 $F$ is a tetrahedron. Hence, $P$ contains neither pyramid nor tetrahedron. Since $P$ is minimal, $P$ contains no smaller triangular prisms. Therefore, any mirror is pentagonal (see Fig. 2). Clearly, all dihedral angles of $P$ are fundamental.

Lemma 4. Let $P$ be a triangular prism. Then $F$ is either a tetrahedron or a triangular prism.

Proof. It is sufficient to prove the lemma for a minimal prism $P$. Suppose that $F$ is neither a tetrahedron nor a triangular prism. Then by Lemma 3 any mirror is pentagonal and any dihedral angle of $P$ is fundamental.

Let $ABCDE$ be any pentagonal mirror (see Fig. 3). This mirror separates the points $M_1$, $N_1$, $N_3$ from the points $M_2$, $M_3$, $N_2$.

![Figure 3: A pentagonal mirror.](image)

We say that a polyhedron $W$ is good if it contains the points $M_2$, $M_3$, $N_2$ and if it can be cut out of $P$ by a single pentagonal mirror. Let $W$ be a minimal good polyhedron (i.e. no polyhedron inside $W$ is good). The minimal polyhedron exists, since the decomposition contains finitely many mirrors. Suppose that $W$ has vertices $M_2M_3N_2ABCDE$ (see Fig. 4).

Suppose that $W$ has a non-fundamental dihedral angle $\alpha$. By Lemma $\alpha$ is an angle formed up by $ABCDE$ and some face of $P$. Let $\Pi$ be a mirror decomposing the angle $\alpha$. Clearly, $\Pi$ does not separate the points $M_1$, $N_1$ and $N_3$ one from another. Since $\Pi$ is pentagonal, it separates all the points above from the points $M_2$, $M_3$, $N_2$. Therefore, $\Pi$ cuts out of $P$ a good polyhedron contained in $W$. This contradicts to the minimal property of $W$.

Thus, any dihedral angle of $W$ is fundamental, and $W$ is a Coxeter polyhedron. The prolongations of two disjoint faces cannot intersect each other (see Fig. 2). This property is broken in the polyhedron $W$ (consider the faces $AEN_2$ and $CDM_3$ which...
prolongations have a common line $N_1N_3$). The contradiction shows that $F$ is either a tetrahedron or a triangular prism.

\hfill \Box

2 Decompositions of pyramids.

This section describes the way to classify all Coxeter decompositions of bounded hyperbolic pyramids.

Suppose that a pyramid $P$ has only five vertices $OA_1A_2A_3A_4$ (where $A_1A_2A_3A_4$ is a base of $P$ and $O$ is an apex). By Lemma 2 $F$ is a tetrahedron. Suppose that any edge $OA_i$ is fundamental. Consider a small sphere $s$ centered in $O$. The intersection $s \cap P$ is a spherical quadrilateral $q$. Any angle of $q$ is fundamental. This contradicts to the fact that the sum of the angles of a spherical quadrilateral should be greater than $2\pi$.

Hence, we can assume that there is a mirror $m$ through $OA_1$. This mirror decomposes the pyramid into two smaller pyramids. One of the smaller pyramids is a tetrahedron and another is either a tetrahedron or a small quadrilateral pyramid. Thus, any minimal quadrilateral pyramid consists of two tetrahedra (not necessary fundamental). The Coxeter decompositions of hyperbolic tetrahedra are classified in [3]. Therefore, we can find all decompositions of minimal quadrilateral pyramids. Using the decompositions of minimal pyramids we can find the decompositions of greater pyramids too. So, using this algorithm, it is possible to find all decompositions of quadrilateral pyramids.

Analogously, it is possible to classify the decompositions of pyramids with bigger number of vertices. Any pyramid $OA_1...A_n$ has a decomposed edge $OA_i$. So, the pyramid is decomposed into two smaller pyramids which possible decompositions we should enumerate at the previous steps.

This procedure realized by a computer leads to a large list of quadrilateral quasi-Coxeter pyramids, several pentagonal quasi-Coxeter pyramids, and exactly one hexagonal quasi-Coxeter pyramid. See Table 2 for the result.

3 Decompositions of prisms into prisms

In this section both $P$ and $F$ are triangular prisms. For any prism we say that a triangular face is a base and a quadrilateral face is a side.

Lemma 5. No base of $F$ belongs to a side of $P$.

Proof. Suppose that a side $s$ of $P$ contains a base of a fundamental prism $F_1$. Consider a sequence of fundamental prisms $F_1, F_2,...,F_n,...$, such that $F_i \in P$, and $F_i$
and \( F_{i+1} \) have a common base. This chain is finite, since \( P \) contains finitely many fundamental prisms. The prolongations of the bases of \( F_1 \) have no common points, since \( F_1 \) is a Coxeter prism. So, the prolongations of the bases of two different prisms \( F_i \) have no common points. Therefore, the prisms \( F_1, F_2, \ldots, F_n \), cannot make a cycle and the sequence has an endpoint \( F_n \). One of the bases of \( F_n \) belongs to some face \( b \) of \( P \) (otherwise the sequence has a prolongation). The face \( s \) cannot intersect the face \( b \), since these faces are prolongations of the bases of \( F_i \). This contradicts to the fact that the side \( s \) intersects each face of \( P \).

\[
\square
\]

**Lemma 6.** No side of \( F \) belongs to a base of \( P \).

**Proof.** Denote by \( b \) and \( s \) the bases of \( P \). Suppose that a side \( a_0 \) of a fundamental prism \( F_0 \) belongs to \( b \). Consider a sequence of fundamental prisms \( F_{-k}, \ldots, F_0, \ldots, F_l, \) \( F_i \in P, \) such that \( F_i \) and \( F_{i+1} \) have a common base. The sequence is finite, and the prisms \( F_i \) cannot make a cycle. Let \( a_{-k} \) and \( b_l \) be the bases of the endpoints \( F_{-k} \) and \( F_l \). By Lemma 5 the bases \( a_{-k} \) and \( b_l \) belong to two different bases of \( P \). If \( a_{-k} \) belongs to the base \( s \) then the prolongations of \( a_{-k} \) and \( a_0 \) have a common point (see Fig. 4). This is impossible, since the prolongations of the bases have no common points.

![Figure 4: No side of \( F \) belongs to a base of \( P \).](image)

**Lemma 7.** Let \( b \) be a base of \( P \), \( b_1 \) and \( b_2 \) be the bases of \( F \). Suppose that \( b \) is non-fundamental. Then the tiling of \( b \) by the faces of \( F \) is a Coxeter decomposition. A fundamental polyhedron of this decomposition is a triangle \( b_1 \) or \( b_2 \).

**Proof.** By Lemma 6 any tile is a triangle. The adjacent tiles of the tiling are the faces of \( F \) having a common edge. Thus, if one of the tiles is a base \( b_1 \) then the other tiles are the copies of this base. Clearly, this tiling is a Coxeter decomposition.

\[
\square
\]
By Lemma 5 any side $s$ of $P$ is tiled by the sides of $F$. This tiling is not necessary a Coxeter decomposition, but Lemma 8 and 9 show that this tiling is very simple.

**Lemma 8.** Let $s$ be a side of $P$. Then the tiling of $s$ is a "lattice" (see Fig. 5a), i.e. the tiling has the following properties:

1. any vertex of the face $s$ belongs to a unique tile;
2. any point of any edge of the quadrilateral $s$ belongs to either one or two tiles.
3. any point in the inner part of $s$ belongs to one, two or four tiles.

**Proof.** We say that an edge $AB$ of a tile is **horizontal** if $AB$ belongs to a base of a fundamental prism; otherwise we say that $AB$ is **vertical**. Let $ABCD$ be any tile which vertical edges are $AB$ and $CD$. Consider a sequence of tiles such that this sequence contains $ABCD$, and any two neighboring tiles have a common horizontal edge. We say that this sequence is **vertical**. By the same way construct a **horizontal** sequence where the neighboring tiles have a common vertical edge. Evidently, the vertical sequence ends at two different horizontal edges of $s$ and the horizontal sequence ends at two different vertical edges. Let us show that this condition is broken if the lemma is false.

1. Let $A$ be a vertex of the face $s$. Suppose that $A$ belongs to more then one tile (see Fig. 5b). Consider a vertical sequence through the tile $ABCD$. No pair of tiles from this sequence can be separated from each other by the line $AB$. Therefore, this sequence cannot reach the horizontal edge $AE$, and (1) is proved.

2. Let $A$ be an inner point of a horizontal edge $MN$. Suppose that $A$ belongs to more then two tiles. Then one of these tiles has no edges in $MN$ (see Fig. 5c). At least two lines through $A$ are vertical edges of some tiles. Consider a vertical sequence through the tile $ABCD$. It is evident that the angle $\angle BAE$ contains any tile from this sequence. Thus, the sequence cannot reach the horizontal edge $MN$.

The same reasoning works for the points on the vertical edges, and (2) is proved.
3). Let $A$ be an inner point of the tile $s$. Clearly, $A$ cannot be incident to exactly three tiles. Suppose that $A$ is a vertex of five or more tiles.

Let $a_1$ be any tile and $a_1, ..., a_k$ be an upper part of the vertical sequence containing $a_1$ (i.e. $a_k$ is the upper endpoint of the sequence). We say that $a_1$ is a tile of the level $k$. We say that a point $A$ is a point of the level $k$, if $A$ belongs to a tile of the level $k$. If $A$ belongs to several tiles, we define the level of $A$ as a maximum.

The part 2) of this proof establishes the lemma for the points of level 0. The same reasoning shows that if the lemma is true for the tiles of the level $k$ then it is true for the tiles of the level $k + 1$. Therefore, the lemma is true for a tile of any level.

Lemma 9. Any side of $P$ is tiled as shown in Fig. 6.

![Figure 6: Tilings of sides.](image)

Proof. Let $s$ be a side of $P$ and $b_1$, $b_2$, $b_3$ be the sides of $F$. Since $F$ is a Coxeter polyhedron, no angle of $b_i$ is obtuse. Therefore, any horizontal line is perpendicular to any vertical line (where the word "line" means an intersection of $s$ with some mirror). By the same reason the lines are perpendicular to the edges of $s$. At most one line may be perpendicular to the pair of opposite edges of a quadrilateral in $H^2$. Thus, there is at most one vertical line and at most one horizontal line.

Lemma 10. The bases of $P$ are tiled as shown in Fig. 3.

Proof. Let $s$ be a base of $P$. By Lemma 7, $s$ is either fundamental or a quasi-Coxeter. The Coxeter decompositions of triangles are classified in [5]. By Lemma 8, no edge of the triangle $s$ may be decomposed into more than two parts. The decompositions satisfying this condition are listed in Fig. 3.

Theorem 1. Let $P$ be a quasi-Coxeter triangular prism such that $F$ is a triangular prism. Then the decomposition is one of the decompositions listed in Table 4.
**Proof.** It follows from Lemma 8 that two bases of $P$ are decomposed in the same way. By Lemma 10 there are five possibilities for decompositions of bases. Consider two cases.

1). Suppose that the bases are fundamental. Then the sides are either fundamental or decomposed like in Fig. 6(c). Obviously all the sides of $P$ are decomposed in the same way. This leads to the decomposition shown in the upper part of Table 1 (since any mirror intersects a boundary of $P$, there cannot be another decomposition).

2). Suppose that the bases are decomposed. Each decomposition of the base corresponds to two possible decompositions of the sides (with the horizontal line or without). Thus, each decomposition of the base corresponds to two decompositions of $P$. Some of the dihedral angles of $F$ are uniquely determined by the combinatorial structure of the decomposition. We need only to check that the rest dihedral angles of $F$ can be prescribed in such a way that the dihedral angles satisfies the condition of Andreev’s theorem (see [1]). If the bases are decomposed as in Fig. 3(a) or 3(c), then it is possible to satisfy the Andreev’s theorem. Otherwise, it is impossible.

4 \ Decompositions of prisms into tetrahedra

In this section $P$ is a triangular prism and $F$ is a tetrahedron.

Any face of $P$ is obviously tiled by triangles, but in most cases this tiling is not a Coxeter decomposition. At first, we derive some information about tilings of bases.

**Definition 6.** Let $ABCD$ be a fundamental tetrahedron. An edge $AB$ is called $k$-edge if the dihedral angle formed by $ABC$ and $ABD$ equals $\frac{\pi}{k}$.

**Lemma 11.** Let $p$ be a base of $P$. If any flat angle of $p$ is fundamental then the tiling of $p$ consists of a unique triangle. If $p$ has a flat angle decomposed into three parts then $p$ is bounded by 2-edge, 3-edge and 5-edge.

**Proof.** Let $f$ be a face of $F$ and $\bar{f}$ be a plane containing $f$. Obviously, $\bar{f}$ is tiled by triangles congruent to the faces of $F$. To find this tiling consider a face $f$ and the
adjacent to $f$ triangles $f_1$, $f_2$ and $f_3$ at the plane $\bar{f}$. These triangles $f_i$ are congruent either to $f$ or to the other faces of $F$ (we obtain the same face if the dihedral angle is equal to $\frac{\pi}{k}$, where $k$ is even, otherwise we obtain another face). Adding face by face we can prolong the tiling. The only problem is how many triangles are incident to a fixed vertex.

To solve this problem suppose that $A$ is a vertex of $F$ incident to $k$-edge, $l$-edge and $m$-edge. Suppose that $\bar{f}$ contains $A$. Then a decomposition of a small three-dimensional spherical neighborhood of $A$ is similar to a Coxeter decomposition of a sphere with fundamental triangle $(\frac{\pi}{k}, \frac{\pi}{l}, \frac{\pi}{m})$. An intersection of the neighborhood with $\bar{f}$ corresponds to a spherical line in the decomposition of the sphere. So, to find how many rays starting from $A$ belong to $\bar{f}$, it is sufficient to count a number of vertices incident to a corresponding spherical line.

Thus, for any face of any Coxeter tetrahedron we can find a tiling of the corresponding plane. In these tilings any triangle with fundamental angles is a face of $F$. This proves the first part of the lemma. A triangle with an angle decomposed into three parts was found in one of these tilings only. This triangle is bounded by 2-edge, 3-edge and 5-edge and tiled as shown in Fig. 8.

![Figure 8: The edge labeled by $k$ is a $k$-edge ($k = 2, 3, 5$).](image)

**Lemma 12.** Let $A$ be a non-fundamental vertex of $P$. Then there exists a non-fundamental edge incident to $A$.

**Proof.** Suppose that any edge incident to $A$ is fundamental (i.e. three dihedral angles incident to $A$ are fundamental). Consider a small sphere $s$ centered in $A$. The decomposition of $P$ restricted to $s$ is a Coxeter decomposition of a spherical triangle $p = s \cap P$. Evidently, any angle of $p$ is fundamental. It is easy to check that in this condition $p$ is decomposed as shown in Fig. 8(a). Any angle of $p$ is a right angle.

Thus the neighborhood of $A$ is decomposed as shown in Fig. 8(b). Any edge of $P$ incident to $A$ is a 2-edge. Consider a base $ABC$ of the prism $P$. The angle $A$ of this triangle is decomposed into three parts by the lines $AN$ and $AM$. It follows from Lemma 11 that the sides of $ABC$ should be 2-edge, 3-edge and 5-edge. This contradicts to the fact that $AB$ and $AC$ are 2-edges.
Figure 9: If $A$ is a non-fundamental vertex but the edges ended in $A$ are fundamental, then $AB$ and $AC$ are 2-edges and the tiling of $\triangle ABC$ is impossible.

Lemma 13. Let $P$ be a triangular prism admitting a Coxeter decomposition into tetrahedra. Then $P$ has a non-fundamental dihedral angle.

Proof. Suppose that any dihedral angle of $P$ is fundamental. Then by Lemma 12 any vertex of $P$ is fundamental. By Lemma 11 any base of $P$ is congruent to a single face of $F$. Consider a fundamental tetrahedron $F_0$ containing a base $\alpha$. Since any dihedral angle of $P$ is fundamental, the sides of $P$ are the faces of $F_0$. This is impossible.

How to find all decompositions of prisms into tetrahedra

We have already proved that any prism with tetrahedral fundamental polyhedron has a non-fundamental dihedral angle. It follows from Fig. 3(1)–(4) that any prism consists of the tetrahedra, the smaller prisms and the quadrilateral pyramid. If we know the decompositions of the smaller parts we can find the decomposition of the whole prism.

Recall that a prism $P$ is called minimal if $P$ is non-fundamental and any prism inside $P$ is fundamental.

Definition 7. We say that a minimal prism is a prism of level $0$. A non-fundamental prism $P$ is a prism of level $k + 1$ if $P$ contains a prism of level $k$ but any prism inside $P$ contains no prism of level $k$.

At first, we classify decompositions of minimal prisms. Evidently, a minimal prism contains no mirrors shown in Fig. 3(2), 3(4). Thus, it contains a mirror shown in Fig. 3(1). Let this mirror be $A_1B_2B_3$ in Fig. 3. Then $A_1$ is a non-fundamental
vertex, and by Lemma 12 there is a mirror containing an edge ended in $A_1$. This edge cannot be $A_1B_1$, since the prism is minimal. Hence, one of the edges $A_1A_2$ and $A_1A_3$ (say, $A_1A_2$) is non-fundamental. Therefore, the prism is decomposed into three tetrahedra. Decompositions of tetrahedra are classified in [3]. Thus, we can classify decompositions of minimal prisms.

![Figure 10: A minimal prism consists of three (possibly non-fundamental) tetrahedra.](image)

Suppose that we know the classification of decompositions for the prisms of the levels smaller than $k + 1$. Now we can show how to find a classification for the prisms of the level $k + 1$. Suppose that $P$ has a dihedral angle decomposed as shown in Fig. 2(2)–2(4). By the definition, any prismatic part is a prism of the level smaller than $k + 1$. Thus, we know the list of possible decompositions of prismatic parts. Decompositions of tetrahedral parts are known too. Therefore, we can classify all possible decompositions of $P$. Suppose now that $P$ has no mirrors shown in Fig. 2(2)–2(4). Then $P$ has a mirror shown in Fig. 3(1). Moreover, there are two mirrors of this type decomposing $P$ into three tetrahedra (we use Lemma 12 again). Combining the decompositions of these tetrahedra we can find all possible decompositions of $P$.

Thus, we have an algorithm leading to the classification of decompositions for the prisms which level is smaller than any fixed number. In fact, there is no prism of level eight; hence, there cannot be prism of greater level. Therefore, this algorithm classifies all decompositions of bounded triangular prisms. See Table 2 for the classification.
Tables

Table 1 contains all decompositions of bounded triangular prisms which fundamental polyhedron is a triangular prism. Table 2 contains the list of decompositions of convex bounded pyramids and triangular prisms which fundamental polyhedron is a tetrahedron. Here we describe the structure of Table 2.

• Horizontal lines separate polyhedra with different fundamental tetrahedra. For any fundamental tetrahedron the table contains three columns. The left column contains the list of decompositions of bounded tetrahedra (the tetrahedron number 0 is the fundamental one). The right column lists decompositions of bounded triangular prisms. The column at the middle contains bounded pyramids. At first we list quadrilateral pyramids which can be combined from two (possibly non-fundamental) tetrahedra. Then, after a dotted line, we list the rest quadrilateral pyramids. After a new dotted line we list pentagonal pyramids and at last we list hexagonal pyramids. (There is no Coxeter decomposition of heptagonal pyramid).

The polyhedra and their decompositions are represented in Table 2 as it is described below.

• A tetrahedron with dihedral angles $\frac{k\pi}{q}$ $i = 1,\ldots,6$ is represented by the following diagram: the nodes of the diagram correspond to the faces of the tetrahedron, two nodes are connected by a $q$-fold edge decomposed into $k$ parts if the corresponding faces form up a dihedral angle $\frac{k\pi}{q}$. The numerator $k_i$ is a number of parts in the dihedral angle (for example, a fraction $\frac{2}{4}$ denotes a right angle decomposed into two parts). The faces of the tetrahedron are numbered by the following way: the nodes of the diagram have the numbers 0,1,2,3 from the left to the right. (The numeration will be used below).

• A quadrilateral pyramid $OA_1A_2A_3A_4$ with the base $A_1A_2A_3A_4$ is represented by eight dihedral angles

\[ (A_1\widehat{A}_2, A_2\widehat{A}_3, A_3\widehat{A}_4, A_4\widehat{A}_1; \widehat{OA}_1, \widehat{OA}_2, \widehat{OA}_3, \widehat{OA}_4). \]

The dihedral angles are the rational fractions, where the numerator $q$ means that the corresponding dihedral angle is decomposed into $q$ parts (we always omit a multiple $\pi$). A triangular face $OA_iA_{i+1}$ has a number $i$.

• By the same way we represent pentagonal and hexagonal pyramids (we use ten and twelve angles correspondingly).
• A triangular prism \(A_1A_2A_3B_1B_2B_3\) (where \(A_1A_2A_3\) and \(B_1B_2B_3\) are the bases) is represented by nine dihedral angles
\[
(A_3B_3, A_1B_1, A_2B_2; A_1A_2, A_2A_3, A_3A_1; B_1B_2, B_2B_3, B_3B_1).
\]
The dihedral angles are the rational fractions, where the numerator is a number of parts in the dihedral angle and the multiple \(\pi\) is omitted. The faces are numbered in the following way: \(A_1A_2A_3\) has number 0, \(B_1B_2B_3\) has number 1, nd \(A_iA_{i+1}B_{i+1}B_i\) has number \(i + 1\).

• Any decomposition which is a superposition of some other decompositions is labeled by a star.

Remark. In this notation each pyramid or triangular prism can be written by several ways (for instance, the order of angles changes while we rotate the polyhedron). To unify the notation we choose a record which stands first in the alphabetical order.

• For any polyhedron except fundamental tetrahedra we need some way to reconstruct the decomposition. For this aim we put some numbers under the diagram of the polyhedron. These seven numbers (denoted by \(t,k,l\) and \(m,n,p,q\)) show the following:
  \(t\) is a number of the polyhedron (for any fundamental tetrahedron we start a new numeration),
  \(k\) is a number of fundamental tetrahedra in the decomposition,
  \(l\) is a number of gluings used to obtain this decomposition (if the decomposition was obtained by a gluing of two polyhedra with \(l = l_1\) and \(l = l_2\) then \(l = 1 + \max\{l_1, l_2\}\)),
  \(m\) and \(n\) are the numbers of the polyhedra which should be glued together to obtain the decomposition.
  \(p\) and \(q\) are the numbers of the glued faces of the polyhedra \(m\) and \(n\) respectively,
  The numbers \(m\) and \(n\) are accompanied by the labels: we write ”tet”, ”pyr” or ”pri” to show that a corresponding polyhedron is a tetrahedron, a quadrilateral pyramid or a triangular prism. (Any tetrahedron admitting a Coxeter decomposition consists of two smaller tetrahedra, so we omit the label ”tet” in the left column.)
Table 1: Coxeter decompositions of prisms into prisms.

$k, m = 4$ or 5

$k, m = 4$ or 5
Table 2. Coxeter decompositions of pyramids and prisms
which fundamental polyhedron is a tetrahedron

| tetrahedra | pyramids | triangular prisms |
|------------|----------|-------------------|
| 1          |          |                   |
| 0          | $\frac{1111\cdot2121}{334\cdot342\cdot432}$ | $\frac{111\cdot121\cdot121}{334\cdot233\cdot233}$ 1, 3, 3, tet0, pir1, 0, 1 |
|            | 1, 2, 2, tet0, tet0, 0, 0 | 2, 3, 3, tet0, pir1, 1, 3 |
|            | $\frac{1111\cdot1212}{3333\cdot2332}$ | $\frac{111\cdot121\cdot121}{334\cdot234\cdot423}$ 3, 4, 3, tet0, pri1, 1, 0 |
|            | 2, 2, 2, tet0, tet0, 1, 1 | 4, 5, 3, tet0, pri3, 1, 1 |
| 2          |          |                   |
| 0          | $\frac{1111\cdot2121}{335\cdot352\cdot532}$ | $\frac{111\cdot211\cdot121}{335\cdot355\cdot352}$ 1, 3, 3, tet0, pir1, 0, 1 |
|            | 1, 2, 2, tet0, tet0, 0, 0 | 2, 3, 3, tet0, pir1, 1, 3 |
|            | $\frac{1111\cdot1212}{3333\cdot2332}$ | $\frac{111\cdot121\cdot121}{335\cdot235\cdot532}$ 3, 4, 3, tet0, pri1, 1, 0 |
|            | 2, 2, 2, tet0, tet0, 1, 1 | 4, 5, 3, tet0, pri3, 1, 1 |
| 3          |          |                   |
| 0          | $\frac{1111\cdot2121}{3344\cdot4232}$ | $\frac{111\cdot112\cdot121}{344\cdot234\cdot423}$ 1, 3, 3, tet0, pir1, 0, 3 |
|            | 1, 2, 2, tet0, tet0, 0, 0 | |
| tetrahedra | pyramids | triangular prisms |
|------------|----------|-------------------|
| 4          | (1111, 2121) | (111, 112, 121) |
| 0          | (3355, 5232) | (345, 234, 253) |
|            | 1, 2, 2, tet0, tet0, 0, 0 | 1, 3, 3, tet0, pir1, 1, 3 |
| 5          | (1111, 2121) | (111, 112, 121) |
| 0          | (3355, 5232) | (355, 235, 253) |
|            | 1, 2, 2, tet0, tet0, 0, 0 | 1, 3, 3, tet0, pir1, 0, 3 |
| 6          | (1111, 3121) | (111, 112, 131) |
| 0          | (2255, 5232) | (255, 225, 353) |
|            | 1, 2, 2, tet0, tet0, 1, 1 | 2, 4, 4, tet0, pir2, 1, 3 |
|            | (1111, 4121) | (111, 112, 211) |
|            | (2255, 5333) | (255, 225, 333) |
|            | 3, 4, 3, tet1, tet1, 0, 0 | 4, 6, 4, tet1, pir2, 0, 1 |
| 1          | (1111, 1221) | (112, 112, 111) |
| 2, 1       | (2255, 2333) | (255, 225, 323) |
|            | 4, 5, 4, tet1, pir2, 0, 1 | 5, 7, 4, pir1, pir2, 3, 3 |
| 2          | (1111, 41221) | (111, 113, 211) |
| 4, 2       | (2255, 52332) | (255, 232, 333) |
|            | 1, 4, 2, tet0, pir2, 1, 1 | 6, 7, 4, tet1, pir3, 0, 0 |
| 2          | (1111, 2222) | (112, 111, 113) |
| 4, 2       | (5555, 5333) | (355, 232, 333) |
|            | 2, 10, 5, pir4, pir4, 1, 1 | 7, 9, 5, tet1, pir5, 2, 0 |
| 2          | (1111, 2111) | (111, 211, 211) |
| 4, 2       | (2255, 335, 353) | (255, 335, 353) |
|            | 8, 9, 6, tet1, pir6, 0, 0 |
| tetrahedra | pyramids | triangular prisms |
|-----------|---------|------------------|
| 7         | 0       |                  |
| 1 2,1 0, 0, 0 | (1111.2121; 2 4 4.5 2 3 2) | (111.112.121; 2 4 5.2 2 4.2 5 2) |
| 1 2,2 tet0, tet0, 1, 1 | 1, 2, 2, tet0, tet0, 1, 1 |
| 1 2,2 tet0, tet0, 2, 2 | 2, 2, 2, tet0, tet0, 2, 2 |
| 1 3, 3, tet1, tet0, 0, 1 | (1111.3121; 2 2 4 5.2 3 3 3) | (111.113.121; 2 3 5 2 2.4 3 3 5 2) |
| 1 3, 3, tet1, tet1, 0, 0 | 3, 3, 3, tet1, tet0, 0, 1 |
| 1 4, 3, tet1, tet1, 0, 0 | (1111.4121; 2 2 4 5.3 3 3 3) | (111.113.131; 2 2 3 3 3 2 4 3 3 5 2) |
| 1 4, 3, tet1, tet1, 0, 0 | 4, 3, 3, tet1, tet1, 0, 0 |
| 1 5, 3, 3, tet2, tet0, 0, 2 | (1111.3121; 2 2 4 5.4 2 3 3) | (2111.1222.211; 4 5 5 5.3 3 3 5 2 2) |
| 1 5, 3, 3, tet2, tet0, 0, 2 | 5, 3, 3, tet2, tet0, 0, 2 |
| 1 6, 4, tet3, tet3, 0, 0 | (1111.2222; 3 3 3 3.3 3 3 3) | (2111.1222.311; 4 5 5 5.5 3 3 3 5 3) |
| 1 6, 4, tet3, tet3, 0, 0 | 6, 4, tet3, tet3, 0, 0 |
| 1 7, 6, 4, tet4, tet2, 0, 2 | (1211.1212; 2 4 3 5 2 3 2 5) | (111.112.122; 3 3 5 2 3 5 3 3 4) |
| 1 7, 6, 4, tet4, tet2, 0, 2 | 7, 6, 4, tet4, tet2, 0, 2 |
| 1 8, 4, tet4, tet3, 0, 1 | (1211.1213; 3 4 3 5 3 3 2 5) | (111.112.221; 3 3 5 2 3 3 3 5 3) |
| 1 8, 4, tet4, tet3, 0, 1 | 8, 4, tet4, tet3, 0, 1 |
| 1 9, 8, 4, tet4, tet4, 1, 1 | (1111.2222; 5 5 5 5.3 3 3 3 3) | (111.112.131; 3 3 5 2 3 5 5 5 2) |
| 1 9, 8, 4, tet4, tet4, 1, 1 | 9, 8, 4, tet4, tet4, 1, 1 |
| 1 10, 24, 6, tet5, pir11, 0, 2 | (1111.221.221; 5 5 5 5 5 2 5 5 2) | (111.112.221; 5 5 5 5 2 5 5 2) |
| 1 10, 24, 6, tet5, pir11, 0, 2 | 10, 24, 6, tet5, pir11, 0, 2 |
| 1 11, 24, 6, tet5, pir12, 1, 1 | (111.112.221; 5 5 5 5 5 5 5 2) | 11, 24, 6, tet5, pir12, 1, 1 |
| tetrahedra | pyramids | triangular prisms |
|------------|----------|-------------------|
| (11111, 22121) | (11111, 22121) | (211, 211, 211) |
| 10*, 12, 5, tet5, tet4, 0, 2 | 12, 6, 3, pri1, pri1, 2, 2 | 12, 6, 3, pri2, pri2, 3, 3 |
| (11111, 12121) | (11111, 12121) | (211, 211, 211) |
| 11*, 16, 5, tet5, tet5, 0, 0 | 13, 6, 3, tet1, pri1, pri1, 2, 2 | 13, 6, 3, pri2, pri2, 3, 3 |
| (11111, 21421) | (11111, 21421) | (211, 211, 211) |
| 12*, 16, 5, tet5, tet5, 1, 1 | 14, 7, 3, pri1, pri1, 2, 2 | 14, 7, 3, pri2, pri2, 3, 3 |
| (11111, 21421) | (11111, 21421) | (211, 211, 211) |
| 13*, 16, 5, tet5, tet5, 3, 3 | 15, 8, 3, pri2, pri2, 2, 2 | 15, 8, 3, pri3, pri3, 2, 1 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 14, 5, 5, tet1, pri3, 0, 1 | 16, 5, 4, tet0, pri5, 2, 0 | 16, 5, 4, pri3, pri3, 2, 1 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 15, 4, 4, tet0, pri5, 2, 1 | 17, 6, 4, tet0, pri5, 2, 0 | 17, 6, 4, pri4, pri4, 2, 1 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 16, 5, 5, tet1, pri3, 0, 1 | 18, 7, 4, tet1, pri5, 0, 1 | 18, 7, 4, pri3, pri3, 2, 1 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 17, 6, 5, tet1, pri3, 0, 1 | 19, 9, 4, pri5, pri5, 3, 3 | 19, 9, 4, pri3, pri3, 3, 3 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 18, 7, 5, tet1, pri3, 0, 1 | 20*, 12, 5, tet2, pri6, 2, 0 | 20*, 12, 5, pri6, pri6, 2, 0 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 19, 9, 5, tet1, pri3, 0, 1 | 21*, 14, 5, tet3, pri6, 1, 0 | 21*, 14, 5, pri6, pri6, 1, 0 |
| (11111, 12221) | (11111, 12221) | (211, 211, 211) |
| 20*, 12, 5, tet2, pri6, 2, 0 | 22*, 16, 5, pri6, pri6, 1, 0 | 22*, 16, 5, pri7, pri7, 1, 0 |

7 Cont.
| tetrahedra | pyramids | triangular prisms |
|------------|----------|-------------------|
|            |          | \((111,112,112)\) |
|            |          | \((\frac{3}{35},\frac{3}{35},\frac{3}{35})\) |
|            |          | 23\(^*\), 32, 6, tet5, pri10, 0, 1 |
|            |          | \((\frac{1}{112},\frac{1}{111},\frac{1}{221})\) |
|            |          | \((\frac{5}{5},\frac{5}{2},\frac{5}{2})\) |
|            |          | 24, 12, 6, pri12, pri12, 2, 2 |
|            |          | \((111,121,121)\) |
|            |          | \((\frac{2}{245},\frac{5}{2},\frac{2}{5})\) |
|            |          | 25, 6, 2, tet0, pri16, 1, 1 |
|            |          | \((111,121,121)\) |
|            |          | \((\frac{2}{245},\frac{5}{2},\frac{2}{5})\) |
|            |          | 26, 7, 3, tet1, pri16, 0, 1 |
|            |          | \((111,211,211)\) |
|            |          | \((\frac{1}{211},\frac{1}{211},\frac{1}{35})\) |
|            |          | 27, 8, 3, tet1, pri17, 0, 0 |
|            |          | \((111,131,131)\) |
|            |          | \((\frac{1}{245},\frac{5}{35},\frac{2}{35})\) |
|            |          | 28, 8, 3, tet1, pri17, 0, 1 |
|            |          | \((111,121,211)\) |
|            |          | \((\frac{1}{245},\frac{2}{35},\frac{2}{35})\) |
|            |          | 29, 9, 3, tet1, pri27, 0, 1 |
|            |          | \((112,111,111)\) |
|            |          | \((\frac{1}{445},\frac{2}{32},\frac{2}{32})\) |
|            |          | 30, 14, 1, pri27, pri27, 3, 3 |
|            |          | \((111,131,211)\) |
|            |          | \((\frac{1}{245},\frac{5}{35},\frac{2}{35})\) |
|            |          | 31, 10, 3, tet1, pri28, 0, 1 |
|            |          | \((112,111,111)\) |
|            |          | \((\frac{1}{445},\frac{2}{32},\frac{2}{32})\) |
|            |          | 32, 14, 1, pri26, pri29, 3, 3 |
| tetrahedra | pyramids | triangular prisms |
|-----------|----------|------------------|
|           |          |                  |
| 8         |          |                  |
| 0         |          |                  |
| 1 2,1     | 0,0,0    |                  |
| 2 3,2     | 1,0,0,1  |                  |
| 3 4,2     | 1,1,2,2  |                  |
| 4 5,3     | 2,1,0,0  |                  |
| 5 6,3     | 2,2,1,1  |                  |
| (1111,2121) |          | (1111,1221)     |
|           |           |                  |
| (1111,2131) |          | (1111,1121)     |
| (1111,2141) |          | (1111,1122)     |
| (1111,2222) |          | (1111,1221)     |
| (1111,1322) |          | (1111,1123)     |
| (1111,1212) |          | (1111,1123)     |
| (1111,1422) |          | (1111,1213)     |
| (1111,2323) |          | (1111,1124)     |
| (1112,1213) |          | (1111,1124)     |
| (1111,3242) |          | (1111,1125)     |
| (1111,1412) |          | (1111,1126)     |
| (1111,2222) |          | (1111,1127)     |
|           | 1,2,2,tet0,tet0,1,1 | 1,11,6,tet1,pir7,1,3 |
|           | 2,4,4,tet2,tet0,0,1 | 2,7,5,tet2,pir2,0,3 |
|           | 3,6,4,tet2,tet2,0,0 | 3,10,5,tet2,pir5,0,2 |
|           | 4,6,4,tet2,tet2,2,2 | 4,10,5,tet2,pir5,3,3 |
|           | 5,7,4,tet3,tet2,0,3 | 5,12,6,tet2,pir7,3,3 |
|           | 6,8,5,tet4,tet2,0,3 | 6,13,6,tet2,pir8,3,3 |
|           | 7,9,5,tet4,tet3,0,0 | 7,13,6,tet3,pir7,0,2 |
|           | 8,10,5,tet4,tet4,1,1 | 8,14,6,tet4,pir7,0,3 |
|           | 9,11,5,tet5,tet4,0,0 | 9,16,6,tet4,pir9,1,2 |
|           | 10,15,5,tet4,pir7,1,1 | 10,15,5,tet4,pir3,0,1 |
|           | 11,18,6,tet4,pir9,0,4 | 11,18,6,tet4,pir9,0,2 |
|           | 12,23,5,tet4,pir11,0,1 | 12,23,5,tet4,pir11,0,1 |

**Note:** The table and diagram provide a visual representation of the tetrahedra, pyramids, and triangular prisms, along with their respective counts and configurations.
| tetrahedra | pyramids | triangular prisms |
|------------|----------|-------------------|
|            |          |                   |
|            |          |                   |
|            |          |                   |
|            |          |                   |
|            |          |                   |
| 9          |          |                   |
| 1          | 2, 2    | 1, 0, 1, 1        |
| 2          | 2, 2    | 0, 2, 2           |
| 3          | 4, 2    | 2, 2, 0, 0        |
|            |          |                   |
|            |          |                   |
|            |          |                   |
|            |          |                   |
|            |          |                   |
| 1, 4, 3   | tet1, tet1, 1, 1 |
| 2, 3, 3   | tet2, tet0, 0, 0 |
| 3, 4, 3   | tet2, tet1, 0, 0 |
| 4, 4, 3   | tet2, tet2, 1, 1 |
| 5, 6, 4   | tet3, tet3, 0, 2 |
| 6, 8, 4   | tet3, tet3, 0, 0 |
| 7, 8, 4   | tet3, tet3, 1, 1 |
| 8, 8, 4   | tet3, tet3, 3, 3 |
| 1, 6, 3   | tet2, pir3, 0, 2 |
| 1, 5, 4   | tet0, pir4, 0, 1 |
| 2, 6, 4   | tet1, pir4, 0, 1 |
| 3, 8, 5   | tet1, pir5, 0, 1 |
| 4, 10, 5  | tet2, pir8, 2, 3 |
| 5, 12, 5  | tet3, pir6, 0, 2 |
| 6, 12, 5  | tet3, pir7, 1, 1 |
| 7, 6, 4   | tet0, pri1, 0, 0 |
| 8, 7, 4   | tet1, pri1, 0, 0 |
| 9, 8, 4   | tet1, pri2, 0, 0 |
| 10, 16, 5 | tet3, pri5, 0, 1 |
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