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An asymptotic solution of the integral equation for the second moment function in geometric processes

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Abstract
In this study, we derive an asymptotic solution of the integral equation satisfied by the second moment function $M_2(t, a)$. We first find the Laplace transform $(M_2)_L(s, a)$ and then obtain $M_2(t, a)$ asymptotically by inversion. Further, we have derived the asymptotic expressions of $M_2(t, a)$ for some special lifetime distributions such as exponential, gamma, Weibull, lognormal and truncated normal. Finally, the asymptotic solution is compared with the numerical solution to evaluate its performance.

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1. Introduction

Geometric process (GP) is first introduced by [1] to facilitate the modelling of the deteriorating systems in which the consecutive working times of the system are stochastically decreasing and the successive repair times after failure are stochastically increasing. This process is defined in the following way.

Definition 1.1. Let \( N(t), t \geq 0 \) be a counting process and \( X_k \) be the interarrival time between \((k-1)\)th and \(k\)th event of this process for \( k = 1, 2, \ldots \). The counting process \( N(t), t \geq 0 \) is said to be a GP with the ratio parameter \( a \) if there exists a real number \( a > 0 \) such that \( a^{k-1}X_k, k = 1, 2, \ldots \) are independent and identically distributed random variables with a cumulative distribution function (cdf) \( F \).

The GP is also called quasi-renewal process with ratio parameter \( a = 1/a \) by [2].

Let \( N(t), t \geq 0 \) be a GP with the ratio parameter \( a \) and \( F_x \) be the distribution function of \( X_k, k = 1, 2, \ldots \). Then, it is obvious that \( F_x(x) = F(a^{k-1}x) \) for \( k = 1, 2, \ldots \). If \( a < 1 \), then \( \{X_k, k = 1, 2, \ldots \} \) is stochastically increasing. If \( a > 1 \), then \( \{X_k, k = 1, 2, \ldots \} \) is stochastically decreasing. When \( a = 1 \), the GP reduces to a renewal process (RP).

Some important characteristics of a GP are the mean value function \( M(t) \) and the second moment function \( M_2(t) \). These are defined as follows.

The mean value function of a GP, which is also called the geometric function, is given by

\[
M(t, a) = E(N(t)) = \sum_{k=1}^{\infty} F_1 * F_2 * \cdots * F_k(t), \quad t \geq 0,
\]

where \( * \) denotes Stieltjes convolution.
It is well known that the geometric function $M(t)$ satisfies the following integral equation, see [3].

$$M(t, a) = F(t) + \int_0^t M(a(t - x), a) dF(x), \quad t \geq 0. \quad (1.1)$$

If $a \leq 1$, the geometric function $M(t, a)$ is finite for all $t \geq 0$. Furthermore, if $F$ is continuous, then the integral equation (1.1) has a unique solution although $M(t, a)$ cannot be obtained in an analytical form. If $a > 1$, $M(t, a)$ is infinite for all $t > 0$ ([4] and [3]).

The second moment function of a GP is given by

$$M_2(t, a) = E(N^2(t)) = 2 \sum_{k=1}^{\infty} k(F_1 \ast F_2 \ast \cdots \ast F_k(t)) - \sum_{k=1}^{\infty} F_1 \ast F_2 \ast \cdots \ast F_k(t), \quad t \geq 0. \quad (1.2)$$

[5] shows that the second moment function of a GP satisfies the following integral equation.

$$M_2(t, a) = 2M(t, a) - F(t) + \int_0^t M_2(a(t - x), a) dF(x), \quad t \geq 0. \quad (1.3)$$

For $a \leq 1$, it is easily seen that a GP with ratio $a$ has finite moments of all orders. Hence for $a \leq 1$, the function $M_2(t, a)$ is finite for all $t \geq 0$. If $a > 1$ and $F(t) > 0$ for all $t > 0$, $M_2(t, a)$ is infinite for all $t > 0$ ([4] and [3]).

GP is widely used as a stochastic monotone model in many practical applications since its introduction. A considerable amount of the research on the GP has been published. For example, the GP has been applied in software reliability [6], maintenance [7], warranty cost analysis [8] and [9], modelling of an epidemic disease [10]. Numerical examples also show that the GP is of importance. In [3], model performance of the GP is validated by 14 real-world datasets according to criteria of the mean square error (MSE) and maximum percentage error. After validating the model performance of the GP, some characteristics such as mean value function $M(t)$ and the second moment function $M_2(t)$ are needed in most applications. Following applications of the GP can be given in order to illustrate how these functions are actually used.

[8] carries out the warranty cost analysis by using the GP model to compute the expected warranty cost and the variability of the warranty cost. In this model, $N(w)$ denotes the number of the product’s failure in $(0, w]$ where $w$ is the length of the warranty period. They conduct the analysis depending on the cost $C(w) = cN(w)$ where $c$ is a constant. Then, the expected warranty cost and the variability of the warranty cost are $E(C(w)) = cM_2(w)$ and $\text{Var}(C(w)) = c^2(M_2(w) + M^2(w))$. respectively. See [8] for the details.

[6] proposes a GP model to investigate software reliability and testing costs. In this model, $N(t)$ denotes the number of software faults in $(0, t]$ and the cost of fixing software fault $i$ is given by $C_i = c_0 + (i - 1)W$, $i = 1, 2, \ldots$ where $c_0$ is a constant and $W$ is an incremental cost variable with mean $c_0$. Then, the expected total debugging cost in $(0, t]$ is $C_1(t) = \frac{2c_0 - 3c_0}{2} M(t) + \frac{W}{2} M_2(t)$. In addition to above cost model, [6] also considers the cost of testing per unit time and assumes that it is a random variable $V$ with mean $c_V$. Then, the total expected testing and debugging cost in $(0, t]$ is given as $C_2(t) = cv + C_1(t)$. See [6] for the details.

[9] constructs the cost analysis of the repair-limit risk-free policy by using the GP. In this model, $N_1(w)$ and $N_2(w)$ denote the number of repairs and the number of replacement under the warranty in $(0, w]$, respectively, where $w$ is the length of the warranty period. They carry out the analysis depending on the cost $C(w) = c_1N_1(w) + c_2N_2(w)$ where $c_1$ and $c_2$ are the repair cost per failure and the replacement cost per unit, respectively. They investigate the measures $E(C(w))$ and $\text{Var}(C(w))$ which include the functions $M(t)$ and $M_2(t)$. See [9] for the details.

The analytical forms of the geometric function $M(t, a)$ and the second moment function $M_2(t, a)$ do not exist. In the literature, many studies on their computations have been done. Some of the known studies on the geometric function are as follows. When the distribution of the first interarrival time has an exponential distribution, upper and lower bounds for the geometric function $M(t, a)$ are obtained by [4], and [11] obtains power series expansion for $M(t, a)$ by using the integral equation (1.1). [12] proposes a numerical solution to the integral equation given in (1.1) and applies this method to four common lifetime distributions, namely, exponential, gamma, Weibull and lognormal. These proposed methods for the geometric function have the considerable amount of computation needed to obtain the function with a desired accuracy. In practice, an asymptotic formula for $M(t, a)$ should be useful for applications requiring the geometric function $M(t, a)$. [3] gives an asymptotic expression for the geometric function by using its Laplace transform in the following theorem.

Let $(N(t), t \geq 0)$ be a GP with the ratio parameter $a$. Assume that the first interarrival time $X_1$ has a distribution function $F$ and probability density function (pdf) $f$ with $E(X_i) = \lambda$, $\text{Var}(X_i) = \sigma^2$ and $E\left(X_i^k\right) = \mu_k$, $k = 1, 2, \ldots$ .

**Theorem 1.1 ([3])**. Let $a \leq 1$. If $\mu_4 < \infty$ then

$$M(t, a) = \frac{t}{\lambda} + \frac{\sigma^2 - \lambda^2}{2\lambda^2} + (a - 1) \left( \frac{t^2}{2\lambda^2} + \frac{(\sigma^2 - \lambda^2)^2}{2\lambda^4} \right) + (a - 1)^2 \left( \frac{t^3}{3\lambda^3} + \frac{3(\sigma^2 - \lambda^2)^2}{4\lambda^4} \right) + \frac{9(\lambda^2 + \sigma^2)^2 - 12\lambda^2\sigma^2 - 4\lambda\mu_3}{12\lambda^5} t + \frac{(9\lambda^2 + 15\sigma^2)(\lambda^2 + \sigma^2)^2 - 4\lambda\mu_3 (3\lambda^2 + 4\sigma^2) + 3\lambda^2\mu_4}{24\lambda^6} + o(1).$$
To the best of our knowledge, some studies on the second moment function $M_2(t, a)$ are as follows. [13] proposes a numerical approximation and Monte Carlo estimation for the second moment function depending on the convolutions of the distribution functions. [5] adapts Tang and Lam’s method to compute the integral equation (1.3) and applies this method to exponential, gamma, Weibull and lognormal distributions. [5] also derives a power series expansion for the second moment function by the help of the integral equation (1.3) under the assumption of the first interarrival time $X_1$ having an exponential distribution. They give some computational procedures for the variance function after the calculation of $M_2(t, a)$.

The aim of this study is to derive an asymptotic solution of the integral equation (1.3). For this purpose, we first find the Laplace transform $(M_2)_L(s, a)$ and then obtain $M_2(t, a)$ asymptotically by inversion. By the obtained asymptotic solution, the second moment function $M_2(t, a)$ is calculated rapidly for an arbitrary distribution $F$. This approximation also works well for even small $t$ values. For some special lifetime distributions such as exponential, gamma, Weibull, lognormal and truncated normal, we have given the asymptotic expressions of $M_2(t, a)$. Then, the asymptotic solution is compared with the numerical solution proposed by [5] to evaluate its performance.

2. An asymptotic solution to the integral equation (1.3)

In this section, we give an asymptotic solution of the integral equation (1.3) for $M_2(t, a)$ by using its Laplace transform. Let $(N(t), t \geq 0)$ be a GP with the ratio parameter $a$. Assume that the first interarrival time $X_1$ has a distribution function $F$ and pdf $f$ with $E(X_1) = \lambda$, $\text{Var}(X_1) = \sigma^2$ and $E(X_1^k) = \mu_k, k = 1, 2, \ldots$

**Theorem 2.1.** Let $a \leq 1$. If $\mu_5 < \infty$, then the asymptotic solution of (1.3) is obtained as

$$
M_2(t, a) = \frac{t^2}{\lambda^2} + \frac{(2\sigma^2 - \lambda^2)t}{\lambda^3} + \frac{6\lambda^4 + 9\lambda^2 \sigma^2 + 9\lambda^4 - 4\lambda \mu_3}{6\lambda^4} + (a - 1) \left( \frac{t^3}{\lambda^4} + \frac{3(2\sigma^2 - \lambda^2)t^2}{2\lambda^4} + \frac{18\lambda^4 + 3\lambda^2 \sigma^2 + 6\lambda^4 - 2\lambda \mu_3}{2\lambda^5} \right) t + (a - 1)^2 \left( \frac{t^4}{\lambda^4} - \frac{2(\lambda + 17\lambda^2 - 29\sigma^2)t^3}{6\lambda^5} + \frac{6\lambda^3 + 23\lambda^4 - 6\lambda^2 \sigma^2 + 9\lambda^2 \sigma^2 + 40\lambda^4 - 12\lambda \mu_3}{4\lambda^6} \right) t^2 + \left( 12\lambda^6 - 30\lambda^5 + 12\lambda^3 \sigma^2 + 159\lambda^4 \sigma^2 - 30\lambda^2 \sigma^4 + 222\lambda^2 \sigma^4 + 135\lambda^6 + 8\lambda^2 \mu_3 - 38\lambda^3 \mu_3 - 110\lambda \mu_3 \mu_5 + 30\lambda^6 + 675\lambda^2 \mu_3 + 210\lambda^8 - 4\lambda^4 \mu_3 - 44\lambda^5 \mu_3 + 20\lambda^4 \sigma^2 \mu_3 - 424\lambda^3 \sigma^2 \mu_3 - 300\lambda \mu_4^2 - 2\lambda^3 \mu_4 + 37\lambda^4 \mu_4 + 70\lambda^2 \sigma^2 \mu_4 - 8\lambda^3 \mu_5 \right) + o(1).
$$

**Proof.** The proof is completed in the following way. The Laplace transform of the second moment function $(M_2)_L(s, a)$ is obtained by taking the Laplace transform of both sides of the integral equation (1.3). Then, we expand $(M_2)_L(s, a)$ into a Taylor series with respect to $a$ about $a = 1$. At the end, an asymptotic expression for $M_2(t, a)$ will be found by taking the inverse Laplace transform of $(M_2)_L(s, a)$.

Since $sF_L(s) = f_L(s),

$$(M_2)_L(s, a) = \int_0^\infty e^{-st}M_2(t, a)\,dt = \int_0^\infty e^{-st} \left( 2M(t, a) - F(t) + \int_0^t M_2(a(t - x), a)f(x)\,dx \right)\,dt = 2M_L(s, a) - F_L(s) + \int_0^\infty \int_0^t e^{-st}M_2(a(t - x), a)f(x)\,dx\,dt = 2M_L(s, a) - \frac{f_L(s)}{s} + \int_0^\infty \int_0^t I(x < t)e^{-st}M_2(a(t - x), a)f(x)\,dx\,dt = 2M_L(s, a) - \frac{f_L(s)}{s} + \int_0^\infty f(x)\int_x^\infty e^{-st}M_2(a(t - x), a)\,dt\,dx.$$  

By substituting $a(t - x) = y$, we obtain

$$
(M_2)_L(s, a) = 2M_L(s, a) - \frac{f_L(s)}{s} + \frac{1}{a} \int_0^\infty f(x)\int_0^\infty e^{-s\left(\frac{x}{a} + y\right)}M_2(y, a)\,dy\,dx = 2M_L(s, a) - \frac{f_L(s)}{s} + \frac{1}{a} (M_2)_L \left( \frac{s}{a}, a \right) f_L(s).
$$

(2.2)
\((M_2)_L(s, a)\) can be expanded into a Taylor series with respect to \(a\) about \(a = 1\) as

\[
(M_2)_L(s, a) = (M_2)_L(s, 1) + \frac{\partial (M_2)_L(s, a)}{\partial a} \bigg|_{a=1} (a - 1) + \frac{\partial^2 (M_2)_L(s, a)}{\partial a^2} \bigg|_{a=1} \frac{(a - 1)^2}{2} + o\left((a - 1)^2\right)
\]

(2.3)

The first three terms in (2.3) can be expressed based on the function \(f_t\) and its derivatives as follows.

\(M(t, 1)\) and \(M(t, 1)\) are the mean value and second moment functions of the renewal process \(\{N(t), t \geq 0\}\) with the interarrival time distribution function \(F\). It is easily seen that

\[
M_L(s, 1) = \frac{f_t(s)}{s (1 - f_t(s))}.
\]

Then, the Laplace transform of \(M_2(t, 1)\) is found as

\[
(M_2)_L(s, 1) = M_L(s, 1) \left(1 + \frac{f_t(s)}{1 - f_t(s)}\right)
= \frac{f_t(s) (1 + f_t(s))}{s (1 - f_t(s))^2},
\]

and so this function has been written as a function of \(f_t\).

It is easily obtained that

\[
\frac{\partial (M_2)_L(s, 1)}{\partial s} = -\frac{f_t(s)^3 + sf_t'(s) + f_t(s) (3sf_t''(s) - 1)}{s^2 (f_t(s) - 1)^3}
\]

and

\[
\frac{\partial^2 (M_2)_L(s, 1)}{\partial s^2} = \frac{2f_t(s)^4 - 2f_t(s)^3 + f_t(s)^2 (6sf_t'(s) - 3s^2f_t''(s) - 2) + 2f_t(s) (1 - 2sf_t'(s) + 3s^2f_t'(s)^2 + s^2f_t''(s))}{s^3(f_t(s) - 1)^3}
\]

Further, [3] gives

\[
\frac{\partial M_L(s, 1)}{\partial s} = \frac{sf_t'(s) - f_t(s) (1 - f_t(s))}{s^2 (1 - f_t(s))^2},
\]

\[
\frac{\partial M_L(s, a)}{\partial a} \bigg|_{a=1} = \frac{f_t(s) f_t'(s)}{(1 - f_t(s))^3},
\]

\[
\frac{\partial^2 M_L(s, 1)}{\partial s^2} = \frac{2f_t(s) (1 - f_t(s))^2 + (s^2f_t''(s) - 2sf_t'(s))(1 - f_t(s)) + 2s^2(f_t'(s))^2}{s^3 (1 - f_t(s))^3},
\]

\[
\frac{\partial^2 M_L(s, a)}{\partial a^2} \bigg|_{a=1} = \frac{f_t(s) f_t'(s)}{1 - f_t(s)} \left((s^2f_t''(s) + 2sf_t'(s)) (1 - f_t(s))^2 + 2 s (f_t(s) f_t''(s) + f_t'(s))^2 \right) + f_t(s) f_t'(s) \left(1 - f_t(s) + 2s (f_t'(s))^2 (1 + f_t(s))\right).
\]

By substituting \(u = s/a\) and differentiating both sides of (2.2) with respect to \(a\), we see that

\[
\frac{\partial (M_2)_L(s, a)}{\partial a} = 2 \frac{\partial M_L(s, a)}{\partial a} + f_t(s) \left(-\frac{1}{a^2} (M_2)_L(u, a) - \frac{s}{a^2} \frac{\partial (M_2)_L(u, a)}{\partial u} + \frac{1}{a} \frac{\partial (M_2)_L(u, a)}{\partial a}\right).
\]

(2.5)

Since

\[
\frac{\partial M_L(s, a)}{\partial a} = f_t(s) \left(-\frac{1}{a^2} M_L(u, a) - \frac{s}{a^2} \frac{\partial M_L(u, a)}{\partial u} + \frac{1}{a} \frac{\partial M_L(u, a)}{\partial a}\right)
\]

by [3], taking \(a = 1\) in (2.5) we find

\[
\frac{\partial (M_2)_L(s, a)}{\partial a} \bigg|_{a=1} = \frac{f_t(s)}{1 - f_t(s)} \left(-2M_L(s, 1) + 2 \frac{\partial M_L(s, a)}{\partial a} \bigg|_{a=1} = \frac{1}{a} - (M_2)_L(s, 1) - s \frac{\partial (M_2)_L(s, 1)}{\partial s}\right).
\]
Then,
\[
\frac{\partial (M_2)_l (s, a)}{\partial a} \bigg|_{a=1} = - \frac{3f_l (s) (1 + f_l (s)) f_l' (s)}{(f_l (s) - 1)^4}.
\] (2.6)

and so this function has been expressed as a function of \( f_l \) and \( f_l' \).

By differentiating the both sides of Eq. (2.5) with respect to \( a \), it is found that
\[
\frac{\partial^2 (M_2)_l (s, a)}{\partial a^2} = \frac{f_l (s)}{1 - f_l (s)} \left( 4M_l (s, 1) + 8s \frac{\partial M_l (s, a)}{\partial s} - 4 \frac{\partial M_l (s, a)}{\partial a} \bigg|_{a=1} + 2s^2 \frac{\partial^2 M_l (s, 1)}{\partial s^2} - 4s \frac{\partial^2 M_l (s, a)}{\partial s \partial a} \bigg|_{a=1} \right.
\]
\[
+ \frac{\partial^2 M_l (s, a)}{\partial a^2} \bigg|_{a=1} + 2 \left( M_2 (s) (s + 1) + 4s \frac{\partial (M_2)_l (s, a)}{\partial s} \bigg|_{a=1} - 2 \frac{\partial (M_2)_l (s, a)}{\partial a} \bigg|_{a=1} + s^2 \frac{\partial^2 (M_2)_l (s, a)}{\partial s^2} \right)
\]
\[
- 2s \frac{\partial^2 (M_2)_l (s, a)}{\partial s \partial a} \bigg|_{a=1} \right).
\]

Hence, we have
\[
\frac{\partial^2 (M_2)_l (s, a)}{\partial a^2} \bigg|_{a=1} = \frac{f_l (s)}{1 - f_l (s)} \left( 4M_l (s, 1) + 8s \frac{\partial M_l (s, a)}{\partial s} - 4 \frac{\partial M_l (s, a)}{\partial a} \bigg|_{a=1} + 2s^2 \frac{\partial^2 M_l (s, 1)}{\partial s^2} - 4s \frac{\partial^2 M_l (s, a)}{\partial s \partial a} \bigg|_{a=1} \right.
\]
\[
+ \frac{\partial^2 M_l (s, a)}{\partial a^2} \bigg|_{a=1} + 2 \left( M_2 (s) (s + 1) + 4s \frac{\partial (M_2)_l (s, a)}{\partial s} \bigg|_{a=1} - 2 \frac{\partial (M_2)_l (s, a)}{\partial a} \bigg|_{a=1} + s^2 \frac{\partial^2 (M_2)_l (s, a)}{\partial s^2} \right)
\]
\[
- 2s \frac{\partial^2 (M_2)_l (s, a)}{\partial s \partial a} \bigg|_{a=1} \right).
\]

The partial derivative of (2.6) with respect to \( s \) can be given as
\[
\frac{\partial^2 (M_2)_l (s, a)}{\partial a \partial s} \bigg|_{a=1} = \frac{3 (f_l' (s))^2 (2f_l (s)^2 + 5f_l (s) + 1) - f_l (s) f_l'' (s) (f_l (s) - 1))}{(f_l (s) - 1)^5}.
\]

Then, we obtain the third term in (2.3) as a function of \( f_l, f_l' \) and \( f_l'' \) by
\[
\frac{\partial^2 (M_2)_l (s, a)}{\partial a^2} \bigg|_{a=1} = - \frac{f_l (s)}{(f_l (s) - 1)^6} \left( 2 (f_l (s) - 1) f_l' (s) (2 (s - 1) f_l (s)^2 + 5f_l (s) + 3)
\]
\[
+ 2sf_l' (s)^2 \left( f_l (s)^2 - 15f_l (s) - 10 \right) - 3sf_l'' (s) + 3sf_l (s)^3 f_l'' (s) + f_l (s)^2 (7sf_l'' (s)
\]
\[
- 8sf_l' (s)^2) - f_l (s) (4sf_l' (s)^2 + 7sf_l'' (s)) \right).
\] (2.7)

It can be written that as \( s \to 0 \)
\[
f_l (s) = 1 - \lambda s + \frac{1}{2} (\lambda^2 + \sigma^2) s^2 - \frac{1}{6} \mu s^3 + \frac{1}{24} \mu s^4 - \frac{1}{120} \mu s^5 + o(s^5).
\]

Hence, we obtain
\[
f_l' (s) = - \lambda + (\lambda^2 + \sigma^2) s - \frac{1}{2} \mu s^2 + \frac{1}{6} \mu s^3 - \frac{1}{24} \mu s^4 + o(s^4)
\]
and
\[
f_l'' (s) = (\lambda^2 + \sigma^2) - \mu s + \frac{1}{2} \mu s^2 - \frac{1}{6} \mu s^3 + o(s^3).
\]

By using the above expressions in (2.4), (2.6) and (2.7), it follows that
\[
(M_2)_l (s, 1) = \frac{2}{\lambda^2 s^2} + \frac{2 \sigma^2 - \lambda^2}{\lambda^2 s^2} + 6 \lambda^2 + 9 \lambda^2 \sigma^2 + 9 \sigma^4 - 4 \lambda \mu s + O(1)
\] (2.8)

\[
\frac{\partial (M_2)_l (s, a)}{\partial a} \bigg|_{a=1} = \frac{6}{\lambda^3 s^4} + \frac{3 (2 \sigma^2 - \lambda^2)}{\lambda^4 s^4} + \frac{3 \lambda^4 + 3 \lambda^2 s^2 + 6 \lambda^2 s^4 - 2 \lambda \mu s + O(1)}{2 \lambda^5 s^2}
\] (2.9)

and
\[
\frac{\partial^2 (M_2)_l (s, a)}{\partial a^2} \bigg|_{a=1} = \frac{48}{\lambda^5 s^4} + \frac{58 \sigma^2 - 2 \lambda (2 + 17 \lambda)}{\lambda^3 s^4} + \frac{\lambda^2 (6 + 23 \lambda) + 3 \lambda^2 s^2 (3 \lambda - 2) + 40 \sigma^4 - 12 \lambda \mu s}{\lambda^5 s^3}
\]
\[
+ \frac{1}{s^2} \left( \frac{45 \sigma^6}{2 \lambda} - \frac{30 \sigma^4 + 110 \lambda \sigma^2 \mu s}{6 \lambda^6} + \frac{22 \sigma^4 + 8 \mu s + 15 \mu s}{6 \lambda^5} + \frac{12 \lambda^2 - 38 \mu s}{6 \lambda^4} + \frac{53 \sigma^2}{2 \lambda^3} \right)
\]
\[
- \frac{1}{s^2} \left( \frac{35 \sigma^8}{2 \lambda^8} - \frac{30 \sigma^6 + 300 \sigma^4 \mu s}{12 \lambda^7} + \frac{675 \sigma^6 + 20 \sigma^2 \mu s + 40 \mu s^2 + 70 \sigma^7 \mu s}{12 \lambda^6} \right).
\]
Thus, inverting (2.11) yields

$$- \frac{18\sigma^4 + 424\sigma^2 \mu_3 + 2\mu_4 + 8\mu_5}{12\lambda^5} + \frac{795\sigma^4 - 4\mu_3 + 37\mu_4}{12\lambda^4} - \frac{18\sigma^2 + 144\mu_3}{12\lambda^3} + \frac{135\sigma^2}{4\lambda^2}$$

$$+ \frac{3}{2\lambda} + \frac{25}{4} ) + 0(1).$$

Using Eqs. (2.8)–(2.10) in Eq. (2.3), we see that

$$(M_2)_t(s, a) = \frac{2}{s^3\lambda^2} + \frac{2\sigma^2 - \lambda^2}{s^3\lambda^3} + \frac{6\lambda^4 + 9\lambda^2 \sigma^2 + 9\sigma^4 - 4\lambda \mu_3}{6s\lambda^4} + (a - 1) \left( \frac{6}{s^2\lambda^3} + \frac{3(2\sigma^2 - \lambda^2)}{s^2\lambda^4} \right)$$

$$+ \frac{3\lambda^4 + 3\lambda^2 \sigma^2 + 6\sigma^4 - 2\lambda \mu_3}{2s^3\lambda^5} \right) + (a - 1)^2 \left( \frac{24}{s^2\lambda^4} + \frac{29\sigma^2 - \lambda (2 + 17\lambda)}{s^3\lambda^5} \right)

$$+ \frac{3}{2\lambda} + \frac{25}{4} ) + 0(1).$$

Thus, inverting (2.11) yields

$$M_2(t, a) = \frac{t^2}{\lambda^2} + \frac{(2\sigma^2 - \lambda^2)t}{\lambda^3} + \frac{6\lambda^4 + 9\lambda^2 \sigma^2 + 9\sigma^4 - 4\lambda \mu_3}{6s\lambda^4} + (a - 1) \left( \frac{t^3}{\lambda^3} + \frac{3 (2\sigma^2 - \lambda^2) t^2}{2\lambda^4} \right)$$

$$+ \left( \frac{3\lambda^4 + 3\lambda^2 \sigma^2 + 6\sigma^4 - 2\lambda \mu_3}{2s^3\lambda^5} \right) + (a - 1)^2 \left( \frac{t^4}{\lambda^4} - \frac{(2\lambda + 17\lambda^2 - 29\sigma^2) t^3}{6\lambda^5} \right)$$

$$+ \frac{6\lambda^3 + 23\lambda^4 - 6\lambda \sigma^2 + 9\lambda^2 \sigma^2 + 40\sigma^4 - 12\lambda \mu_3}{4s\lambda^8} \right) + \left( \frac{12\lambda^6 - 30\lambda^5 + 12\lambda^3 \sigma^2 + 159\lambda^4 \sigma^2}{12\lambda^7} \right)$$

$$+ \frac{30\lambda^4 + 222\lambda^2 \sigma^4 + 135\sigma^6 + 8\lambda^2 \mu_3 - 38\lambda^3 \mu_3 - 110\lambda \sigma^2 \mu_3 + 15\lambda^2 \mu_4}{12\lambda^8} \right)$$

$$+ \frac{1}{243\lambda^8} \left( 75\lambda^8 + 18\lambda^7 - 18\lambda^5 \sigma^2 + 405\lambda^6 \sigma^2 - 18\lambda^3 \sigma^4 + 795\lambda^4 \sigma^4 - 30\lambda \sigma^6 + 675\lambda^2 \sigma^6 \right)$$

$$+ 210\sigma^8 - 4\lambda^4 \mu_3 + 144\lambda^3 \mu_3 + 20\lambda^2 \sigma^2 \mu_3 - 424\lambda^3 \sigma^2 \mu_3 - 300\sigma^4 \mu_3 + 40\lambda^2 \mu_3 - 2\lambda^3 \mu_4$$

$$+ 37\lambda^4 \mu_4 + 70\lambda^2 \sigma^2 \mu_4 - 8\lambda^3 \mu_5 \right) + g(t) , \quad \text{as } t \to \infty.$$
A stochastic process \( \{X_k, k = 1, 2, \ldots \} \) is said to be a TGP if there exists real numbers \( a_i > 0, i = 1, 2, \ldots \) and integers \( \{1 = m_1 < m_2 < \cdots \} \) such that for each \( i = 1, 2, \ldots \), \( a_i^{-m_i} X_k, m_i \leq k < m_{i+1} \) forms a RP with a common cdf.

The monotonicity property of the TGP varies according to the value of ratio parameter \( a_i \). If \( a_i < 1 \), then \( \{X_k, m_i \leq k < m_{i+1}\} \) is stochastically increasing. If \( a_i > 1 \), then \( \{X_k, m_i \leq k < m_{i+1}\} \) is stochastically decreasing. When \( a_i = 1 \), \( \{X_k, m_i \leq k < m_{i+1}\} \) forms a RP.

Let us consider a TGP \( \{X_k, k = 1, 2, \ldots\} \) proposed in [10] to model the SARS data, where \( X_k \) is the number of infected people with SARS on the \( k \)th day. Then, a corresponding counting process \( \{N(t), t \geq 0\} \) can be defined as

\[
N(t) = \sum_{k=1}^{\lceil t \rceil} X_k, \quad t \geq 0,
\]

where \( \lceil t \rceil \) denotes the greatest integer which is not more than \( t \). \( N(t) \) denotes the total number of infected cases in \( t \) days. In principle, there exists an \( i^* \geq 2 \) such that \( 1 = m_1 < \cdots < m_{i^*} = \infty \) for the TGP. Then, \( N(t) \) can be written as

\[
N(t) = \begin{cases} 
\sum_{j=m_i}^{\lceil t \rceil-1} \sum_{k=m_i}^{j-1} X_k, & m_i \leq t < m_{i+1}, i = 0, 1, \ldots, i^* - 2 \\
\sum_{k=m_i}^{\lceil t \rceil-1} \sum_{j=k}^{\lceil t \rceil-1} X_k, & t \geq m_{i^* - 1}
\end{cases}
\]

where \( m_0 = 0 \) and \( X_0 = 0 \). Depending on the definition of \( N(t) \), this counting process has finite moments of all orders. As an example, we give the mean value function of this process as follows. Let \( E(X_{m}) = \mu_{j} \) for \( j = 1, 2, \ldots \). It is clear from the definition of the TGP that \( E(X_k) = a_j^{m_j-k} \mu_j \) for \( k = m_j, m_j + 1, \ldots, m_{j+1} - 1, j = 1, 2, \ldots \). Then \( E(N(t)) \) is obtained as

\[
E(N(t)) = \begin{cases} 
\sum_{j=m_i}^{\lceil t \rceil-1} a_j^{-1} \left( 1 - a_j^{-(m_j-1-m_i)} \right) \mu_j + \frac{a_j}{a_i - 1} \left( 1 - a_j^{-(\lceil t \rceil - m_i)} \right) \mu_i, & m_i \leq t < m_{i+1}, i = 0, 1, \ldots, i^* - 2 \\
\sum_{j=m_i}^{\lceil t \rceil-1} a_j^{-1} \left( 1 - a_j^{-(m_j-1-m_i)} \right) \mu_j + \frac{a_{i^*-1}}{a_{i^*-1} - 1} \left( 1 - a_{i^*-1}^{-(\lceil t \rceil - m_{i^*-1})} \right) \mu_{i^*-1}, & t \geq m_{i^*-1}
\end{cases}
\]

Doubly Geometric Process (DGP):

**Definition 3.2.** Let \( \{X_k, k = 1, 2, \ldots\} \) be the sequence of independent and the positive-valued random variables. The counting process \( \{N(t), t \geq 0\} \) is said to be a DGP with the parameter \( a \) if there exists a real number \( a > 0 \) such that \( a^{-k}X_k, k = 1, 2, \ldots \) are independent and identically distributed with a common cdf \( \Phi \) where \( h(k) > 0 \) is a function of \( k \) with \( h(1) = 1 \) for \( k = 1, 2, \ldots, [14] \).

[14] gives some properties of the DGP by choosing \( h(k) = (1 + \log(k))^b \) where \( b \) is a real number. The reason that [14] chooses \( h(k) = (1 + \log(k))^b \) is: [14] fits the DGP with different \( h(k) \), which are \( h(k) = b^{1+b \log(k)} \) and \( 1 + b \log(k) \), on ten real data sets and finds that the DGP with \( h(k) = (1 + \log(k))^b \) performs better than the other three \( h(k) \)'s. Let \( \{N(t), t \geq 0\} \) be a DGP with the parameters \( a \) and \( b \), and \( F_k \) be the distribution function of \( X_k, k = 1, 2, \ldots \). Then \( F_k(x) = \Phi \left( a^{-k} X_k^{1+b \log(k)} \right) \) for \( k = 1, 2, \ldots \).

[14] gives the monotonicity property of the DGP as follows: if \( 0 < a < 1, b < 0 \) and \( P(X_1 > 1) = 1 \) or if \( 0 < a < 1, 0 < b < 4.898226 \) and \( P(0 < X_1 < 1) = 1 \), then \( \{X_k, k = 1, 2, \ldots\} \) is stochastically increasing. If \( a > 1, b < 0 \) and \( P(X_1 > 1) = 1 \) or if \( a > 1, 0 < b < 4.898226 \) and \( P(X_1 > 1) = 1 \), then \( \{X_k, k = 1, 2, \ldots\} \) is stochastically decreasing. If \( (1 + \log(k + 1))^b \log((y - k \log(a)) + (1 + \log(k))^b (k - 1) \log(a) - \log(y)) \) varies between negative and positive values, then \( \{X_k, k = 1, 2, \ldots\} \) is not stochastically monotone over \( k \)'s, where \( y \) represents all possible values on \( X_k, k = 1, 2, \ldots \).

Let \( \{N(t), t \geq 0\} \) be a DGP with the ratio parameter \( a \) and \( h(k) = (1 + \log(k))^b \). Suppose that the sequence of interarrival times \( \{X_k, k = 1, 2, \ldots\} \) follows this DGP. Denote \( S_n = \sum_{i=1}^{n} X_i \) with \( S_0 = 0 \). Now, consider a RP \( \{\tilde{N}(t), t \geq 0\} \) with the interarrival times \( \tilde{X}_k, k = 1, 2, \ldots \) having a common cdf \( \Phi \) which is the same as the cdf of \( X_1 \). Denote \( \tilde{S}_n = \sum_{i=1}^{n} \tilde{X}_i \) with \( \tilde{S}_0 = 0 \). If \( \tilde{X}_k, k = 1, 2, \ldots \) is stochastically decreasing, then \( \tilde{X}_k \leq f \tilde{X}_k, k = 1, 2, \ldots \). Thus, it can be shown that \( \tilde{N}(t) \leq f \tilde{N}(t) \) for each fixed \( t \) by using the fact that \( S_n \leq f S_n, n = 1, 2, \ldots \). If \( \tilde{X}_k, k = 1, 2, \ldots \) is stochastically increasing, then \( \tilde{X}_k \leq f \tilde{X}_k, k = 1, 2, \ldots \), and so, \( N(t) \leq f N(t) \) for each fixed \( t \) is well known that the RP \( \{N(t), t \geq 0\} \) has finite moments of all orders. Thus, the DGP \( \{N(t), t \geq 0\} \) possesses moments of \( k > 0 \) when \( \{X_k, k = 1, 2, \ldots\} \) is stochastically increasing. Then, we make following comments on the moment functions of the DGP \( \{N(t), t \geq 0\} \).

(i) If \( 0 < a < 1, b < 0 \) and \( P(X_1 > 1) = 1 \) then the DGP has finite moments of all orders.

(ii) If \( 0 < a < 1, 0 < b < 4.898226 \) and \( P(0 < X_1 < 1) = 1 \) then the DGP has finite moments of all orders.
In addition, by Proposition 5(ii) in [14],

(iii) If \( a > 1, b < 0 \) and \( P \left( 0 < X_1 < 1 \right) = 1 \) then the mean value function of the DGP is infinite and so the second moment function is also infinite.

However, for the other cases in terms of \( a, b \) and the distribution \( F \), it has not been determined yet whether the mean value and second moment functions of the DGP are finite.

Although the alternative processes provide more flexible models for wider application than the GP, they may create difficulties in mathematical derivations. For example, the integral equations (1.1) and (1.3) can be obtained from the fact that \( X_2 + X_3 + \ldots + X_k = aX_1 + aX_2 + \ldots + aX_{k-1} \) for the GP. However, such equality cannot be written for DGP. Therefore, it does not seem possible to obtain an integral equation for the DGP. Since the integral equations do not exist, Tang and Lam's method given in [12] or power series expansion given in [5] cannot be adapted to compute its mean value and second moment functions. In addition, the asymptotic expressions for the mean value and second moment functions of the DGP cannot be obtained by using the similar argument of this study since we use the Laplace transform of the integral equations (1.1) and (1.3). To the best of our knowledge, the only computational procedure for the mean value and second moment functions of the DGP is to obtain Monte Carlo estimation of these functions depending on the convolutions of the distribution functions proposed by [13].

4. Numerical examples

In this section, the performance of the proposed asymptotic solution is evaluated by comparing with the power series expansion and the numerical solution. To do this, we consider five numerical examples each with exponential distribution, gamma distribution, Weibull distribution, lognormal distribution and truncated normal distribution. In the first example, the approximate values are compared with the values computed by the power series expansion given in [5]. In other examples, the approximate values are compared with the values computed by the numerical method based on the trapezoidal integration rule with the step length \( h = 0.01 \) given in [5]. In all five examples, the ratio parameter \( a \) is chosen as \( a = 0.95, 0.975, 0.99 \) since it satisfies the condition \( 0.95 \leq a \leq 1.05 \) for many real data sets fitted by the GP [3]. The solutions are calculated up to time \( t = 10E \left( X_1 \right) \). After computation, the approximate and numerical solutions are plotted together in the same figure for comparison. The figures are presented for only \( a = 0.99 \). In these figures, we use solid and dash-dotted line for the asymptotic solution and numerical solution (power series expansion for only exponential distribution), respectively. For easy comparison, we also plot the 95% and 105% of the values of the numerical method as the lower and upper bounds of the second moment function \( M_2 \left( t, a \right) \) by using dotted lines.

**Example 1.** Consider a GP \( \{ N(t), t \geq 0 \} \) with ratio parameter \( a \leq 1 \) and assume that the first interarrival time \( X_1 \) has an exponential distribution \( \text{Exp} \left( \theta = 2 \right) \) with the pdf \( f \left( x \right) = \frac{1}{2}e^{-x/2}, x > 0 \). Then \( E \left( X_1 \right) = 2, \text{Var} \left( X_1 \right) = 4 \) and \( E \left( X_1^2 \right) = 2^{k} \Gamma \left( k + 1 \right), k = 1, 2, \ldots. \) From (2.1), an asymptotic solution is given by

\[
M_2 \left( t, a \right) = 0.008 + 0.5t + 0.24625t^2 - 0.00123t^3 + 0.0000625t^4 + o \left( 1 \right).
\]  

(4.1)

The approximate values of \( M_2 \left( t, a \right) \) calculated by (4.1) and the power series expansion with an absolute error not exceeding \( \epsilon = 10^{-6} \) are plotted in Fig. 1. The solutions are calculated up to time \( t = 10E \left( X_1 \right) = 20 \). Fig. 1 shows that the results obtained by the asymptotic solution and power series expansion are very close to each other. Asymptotic solution lies inside the lower and upper bounds. As expected, solutions are getting closer when \( t \) increases. In general, the absolute relative errors are not more than 1%. Note that for \( a = 0.975 \), similar comments can be made for the asymptotic solution. When \( a = 0.95 \), asymptotic solution lies inside the lower and upper bounds for the interval (1.5, 12] and it is very close to solution of the power series expansion. For the intervals (0, 1.5] and (12, 20], the deviation of the asymptotic solution is a bit larger. However, in general, the absolute relative errors are not more than 5%.

**Example 2.** Consider a GP \( \{ N(t), t \geq 0 \} \) with ratio parameter \( a \leq 1 \) and assume that the first interarrival time \( X_1 \) has a gamma distribution \( \Gamma \left( \alpha = 2, \beta = 1 \right) \) with the pdf \( f \left( x \right) = xe^{-x}, x > 0 \). Then \( E \left( X_1 \right) = 2, \text{Var} \left( X_1 \right) = 2 \) and \( E \left( X_1^2 \right) = \frac{\Gamma \left( k + 2 \right)}{\Gamma \left( k ight)} , k = 1, 2, \ldots. \) From (2.1), an asymptotic solution is given by

\[
M_2 \left( t, a \right) = 0.12651 - 0.000002t + 0.250002t^2 - 0.00126t^3 + 0.00000625t^4 + o \left( 1 \right).
\]  

(4.2)

The solutions are calculated up to time \( t = 10E \left( X_1 \right) = 20 \).

It is seen from Fig. 2 that the asymptotic solution lies inside the lower and upper bounds in the interval [1.5, 20]. The deviation of this solution in the interval (0, 1.5) is a bit larger. However, in general, the absolute relative errors are not more than 3%. Note that for \( a = 0.975 \), the results are similar for this approximation. When \( a = 0.95 \), although asymptotic solution lies inside the lower and upper bounds for the interval (0.5,15], the deviation of the asymptotic solution in the intervals (0,0.5] and (15,20] is a bit larger and generally, the absolute relative errors are not more than 5%.
Example 3. Consider a GP \( \{ N(t), t \geq 0 \} \) with ratio parameter \( a \leq 1 \) and assume that the first interarrival time \( X_1 \) has a Weibull distribution \( W(\alpha = 2, \beta = 1) \) with the pdf \( f(x) = 2xe^{-x^2}, x > 0 \). Then \( E(X_1) = \sqrt{\frac{\pi}{2}} \), \( \text{Var}(X_1) = 1 - \frac{\pi}{4} \) and \( E(X_1^k) = \Gamma\left(1 + \frac{k}{2}\right), k = 1, 2, \ldots \). From (2.1), an asymptotic solution is given by

\[
M_2(t, a) = 0.24905 - 0.51447t + 1.28223t^2 - 0.01464t^3 + 0.00016212t^4 + o(1).
\]

(4.3)

The solutions are calculated up to time \( t = 10 \) since \( 10 > 10E(X_1) = 5\sqrt{\pi} \).

We can see from Fig. 3 that the results obtained by the asymptotic and numerical solution are very close to each other. The approximate solution lies inside the bounds. The absolute relative errors are generally smaller than 3%. Note that similar results can be yielded for \( a = 0.975 \). When \( a = 0.95 \), the values lie inside the bounds for the interval \([0, 7]\). Out of this interval, i.e. in the interval \([7, 10]\), the absolute relative errors are more than 5% but generally less than 10%.

Example 4. Consider a GP \( \{ N(t), t \geq 0 \} \) with ratio parameter \( a \leq 1 \) and assume that the first interarrival time \( X_1 \) has a lognormal distribution \( \text{LN}(\mu = 0, \tau^2 = 1) \) with the pdf \( f(x) = \frac{1}{x\sqrt{2\pi}}e^{-\frac{1}{2}(\ln x)^2}, x > 0 \). Then \( E(X_1) = \sqrt{e}, \text{Var}(X_1) = e(e - 1) \)
and \( E(X_1^k) = e^{k^2/2}, k = 1, 2, \ldots \). From (2.1), an asymptotic solution is given by

\[
M_2(t, a) = -4.43168 + 1.5372t + 0.35363t^2 - 0.00211t^3 + 0.00001354t^4 + o(1) .
\]  

(4.4)

The solutions are calculated up to time \( t = 18 \) since \( 18 > 10E(X_1) = 10\sqrt{\pi} \).

From Fig. 4, the deviation of asymptotic solution in the interval \((0, 7)\) is a bit larger. In general, the absolute relative errors are not more than 10\%. Out of this interval, i.e. in the interval \([7, 18]\), asymptotic solution is very close to the solution of the numerical method. The absolute relative errors are not more than 3\%. Note that for \( a = 0.975 \) and 0.95, the deviations of the asymptotic solution from the numerical solution are larger. To obtain an accurate approximation, it is not sufficient that \( t \) is large enough. At the same time, \( a \) should be close to 1.

**Example 5.** Consider a GP \( \{N(t), t \geq 0\} \) with ratio parameter \( a \leq 1 \) and assume that the first interarrival time \( X_1 \) has a truncated normal distribution \( \text{TN}(\mu = 2, \tau^2 = 1) \) with the pdf \( f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}}, x > 0 \) and \( b = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-2)^2}{2}} \, dx \).

Then, the first five moments of this truncated normal distribution are \( E(X_1) = 2.0552, E(X_1^2) = 5.1105, E(X_1^3) = 14.3315, \)
Fig. 5. Truncated normal distribution TN(2, 1) and \( a = 0.99 \).

\[
E(X_1^4) = 43.9945 \text{ and } E(X_1^5) = 145.315. \text{ From (2.1), an asymptotic solution is given by}
\]

\[
M_2(t, a) = 0.28059 - 0.28384t + 0.23885t^2 - 0.00118t^3 + 0.00000561t^4 + o(1).
\] (4.5)

The solutions are calculated up to time \( t = 21 \) since \( 21 > 10E(X_1) = 20.552. \)

It is seen from Fig. 5 that the asymptotic solution lies inside the lower and upper bounds and it is very close to solution of the numerical method. In general, the absolute relative errors are not more than 1%. As expected, the solutions are getting closer when \( t \) increases. Note that for \( a = 0.975 \), the results are similar. When \( a = 0.95 \), the deviation of the asymptotic solution in the interval \((0, 2]\) is a bit larger. Out of this interval, the asymptotic solution is very close to the solution of the numerical method. In general, the absolute relative errors are not more than 5%.

5. Conclusions

In this study, the asymptotic solution of the integral equation given for the second moment function of a GP is obtained by using Laplace transform of the integral equation. Further, we have derived the asymptotic expressions of \( M_2(t, a) \) for some special lifetime distributions such as exponential, gamma, Weibull, lognormal and truncated normal. Then, we consider five numerical examples to evaluate the performance of the solution given. According to the numerical examples, the asymptotic solution is generally accurate on the interval \((0, 10E(X_1)] \) at least for the distributions considered here. It is getting closer to the numerical solution when \( t \) increases and \( a \) approaches 1. Also, it can be concluded that the ratio parameter \( a \) is more effective than \( t \). That is, the asymptotic solution gets more closer to the numerical solution as \( a \) approaches 1 for fixed \( t \).

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