A BICATEGORY OF REDUCED ORBIFOLDS FROM THE POINT OF VIEW OF DIFFERENTIAL GEOMETRY - I

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Abstract. We describe a bicategory \((\text{Red Orb})\) of reduced orbifolds in the framework of classical differential geometry (i.e. without any explicit reference to notions of Lie groupoids or differentiable stacks, but only using orbifold atlases, local lifts and changes of charts). In order to construct such a bicategory, we first define a 2-category \((\text{Red Atl})\) whose objects are reduced orbifold atlases (on any paracompact, second countable, Hausdorff topological space). The definition of morphisms is obtained as a slight modification of a definition by A. Pohl, while the definitions of 2-morphisms and compositions of them is new in this setup. Using the bicalculus of fractions described by D. Pronk, we are able to construct the bicategory \((\text{Red Orb})\) from the 2-category \((\text{Red Atl})\). We prove that \((\text{Red Orb})\) is equivalent to the bicategory of reduced orbifolds described in terms of proper, effective, étale Lie groupoids by D. Pronk and I. Moerdijk and to the 2-category of reduced orbifolds described by several authors in the past in terms of a suitable class of differentiable Deligne-Mumford stacks.

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Introduction

A well-known issue in mathematics is that of modeling geometric objects where points have non-trivial groups of automorphisms. In topology and differential geometry the standard approach to these objects (when each point has a finite group of automorphisms) is through orbifolds. This concept was formalized for the first time by Ikikro Satake in 1956 in [Sa] with some different hypotheses than the current ones, although the informal idea dates back at least to Henri Poincaré (for example, see [Poi]). Currently there are at least 3 main approaches to orbifolds:

1. via orbifold atlases and “good maps” between them, as described in [CR],
2. via a class of Lie groupoids, namely proper, étale groupoids (see for example [PR], [M] and [MM]),
3. via a class of $C^\infty$-Deligne-Mumford stacks (see for example [J1] and [J2]).

On the one hand, the approach in (1) gives rise to a 1-category. On the other hand, the approach in (2) gives rise to a bicategory (i.e. almost a 2-category, except that compositions of 1-morphisms is associative only up to canonical 2-morphisms) and the approach in (3) gives rise to a 2-category. It was proved in [Pr] that (2) and (3) are equivalent bicategories. Since (2) and (3) are compatible approaches, then one might argue that:

(i) there should also exist a non-trivial structure of 2-category or bicategory having as objects orbifold atlases or equivalence classes of them (i.e. orbifold structures);
(ii) the structure of (i) should be compatible with the approaches of (2) and (3) and it should replace the approach of (1) (since (1) gives rise only to a category instead of a 2-category or bicategory).

In the present paper we manage to prove both (i) and (ii) for the family of all reduced orbifolds, i.e. orbifolds that are locally modeled on open connected sets of some $\mathbb{R}^n$, modulo finite groups acting smoothly and effectively on them. In order to do that, we proceed as follows.

- We describe a 2-category $(\text{Red Atl})$ whose objects are reduced orbifold atlases on any paracompact, second countable, Hausdorff topological space. The definition of morphisms is obtained as a slight modification of an analogous definition given by Anke Pohl in [P], while the notion of 2-morphisms (and compositions of them) is new in this setup (see Definitions 1.11 and 1.21). Such notions are useful for differential geometers mainly because they don’t require any previous knowledge of Lie groupoids and/or differentiable stacks. In Proposition 3.4 we will prove that $(\text{Red Atl})$ is a 2-category, but it is still not the structure that we want to get in (i); indeed in $(\text{Red Atl})$ different orbifold atlases that represent the same orbifold structure are not related by an isomorphism neither by an internal equivalence.
- We recall briefly the definition of the 2-category $(\text{PE}g\text{pd})$, whose objects are proper, étale groupoids and we describe in Theorem 4.18 a 2-functor $\mathcal{F}^{\text{red}} : (\text{Red Atl}) \to (\text{PE}g\text{pd})$.
- In [Pr] Dorette Pronk proved that the set $\mathcal{W}_{\text{PE}g\text{pd}}$ of all Morita equivalences (also known as weak equivalences or essential equivalences) in $(\text{PE}g\text{pd})$ admits a right bicalculus of fractions. Roughly speaking, this amounts to saying that it is possible to construct a bicategory $(\text{PE}g\text{pd})\left[\mathcal{W}^{-1}_{\text{PE}g\text{pd}}\right]$ and a pseudofunctor

$$L\mathcal{W}_{\text{PE}g\text{pd}} : (\text{PE}g\text{pd}) \to (\text{PE}g\text{pd})\left[\mathcal{W}^{-1}_{\text{PE}g\text{pd}}\right]$$
that sends each weak equivalence to an internal equivalence and that is universal
with respect to this property (see Proposition 5.9). The bicategory obtained in
this way is the bicategory that we mentioned in (b) above if we restrict to the
case of reduced orbifolds.

• In \((\text{Red Atl})\) we consider a class \(W_{\text{Red Atl}}\) of morphisms (that we call “refine-
ments” of reduced orbifold atlases, see Definition 6.1) and we prove that such a
class admits a right bicalculus of fractions. Therefore, we are able to construct
a bicategory \((\text{Red Orb})\) and a pseudofunctor

\[
\mathcal{U}_{W_{\text{Red Atl}}} : (\text{Red Atl}) \to (\text{Red Orb}) := (\text{Red Atl}) \left[ W_{\text{Red Atl}}^{-1} \right]
\]

that sends each refinement to an internal equivalence and that is universal with
respect to this property (see Proposition 7.1). Objects in this new bicategory
are again reduced orbifold atlases; a morphism from an atlas \(\mathcal{X}\) to an atlas \(\mathcal{Y}\)
is a triple consisting of a reduced orbifold atlas \(\mathcal{X}'\), a refinement \(\mathcal{X}' \to \mathcal{X}\) and a
morphism \(\mathcal{X}' \to \mathcal{Y}\). In other terms, a morphism from \(\mathcal{X}\) to \(\mathcal{Y}\) is given firstly by
replacing \(\mathcal{X}\) with a “refined” atlas \(\mathcal{X}'\) (keeping track of the refinement), then by
considering a morphism from \(\mathcal{X}'\) to \(\mathcal{Y}\). We refer to Lemma 7.8 for the description
of 2-morphisms in this bicategory.

• Using the axiom of choice and the results about bicategories of fractions that
we proved in our previous papers [T3] and [T4], we are able to prove that:

**Theorem A** (Proposition 8.2 and Theorem 8.4). There is an equivalence of
bicategories \(\mathcal{G}^{\text{red}}\) making the following diagram commute:

\[
\begin{array}{ccc}
(\text{Red Atl}) & \xrightarrow{\mathcal{F}^{\text{red}}} & (\mathcal{PE}\mathcal{E}\mathcal{\acute{E}}\mathcal{\acute{E}}\mathcal{\acute{Y}}\mathcal{pd}) \\
\mathcal{U}_{W_{\text{Red Atl}}} & \downarrow & \\
(\text{Red Orb}) & \xrightarrow{\mathcal{G}^{\text{red}}} & (\mathcal{PE}\mathcal{E}\mathcal{\acute{E}}\mathcal{\acute{E}}\mathcal{\acute{Y}}\mathcal{pd}) \left[ W_{\mathcal{PE}\mathcal{E}\mathcal{\acute{E}}\mathcal{\acute{Y}}\mathcal{pd}}^{-1} \right].
\end{array}
\]

This proves that the approach described in \((\text{Red Orb})\) is compatible with the
approach (2) to reduced orbifolds in terms of proper, effective, étale Lie groupoids.
Since (2) and (3) are equivalent approaches by [Pr], this implies at once that:

**Theorem B** (Theorem 9.1). \((\text{Red Orb})\) is equivalent to the 2-category \((\text{Orb}^{\text{eff}})\)
of effective orbifolds described as a full 2-subcategory of the 2-category of \(C^\infty\-
Deligne-Mumford stacks.

Even if we will not use explicitly the language of stacks in all this paper, we think
that it is important to remark the following 2 facts:

- in the language of (differentiable) stacks, the notion of objects is long and com-
  plicated to be stated precisely: it requires the notion of pseudofunctor (or the
  notion of category fibered in groupoids), Grothendieck topology and descent
  conditions. Having described that, morphisms (and 2-morphisms) are almost
  straightforward to define and the resulting structure is that of a 2-category;

- in the language used in the present paper (that is mostly intended to be used
  by differential geometers), objects are very easy to describe: they are simply
  reduced orbifold atlases; as we mentioned above, also morphism are easy to
describe. On the contrary, the definitions of 2-morphisms between such objects
will be a bit longer (see Lemma 7.8) and the resulting structure will be that of a
bicategory (hence composition of morphisms is associative only up to canonical 2-morphisms).

To summarize, this paper provides a suitable bicategory of reduced orbifolds, that is equivalent to the already known bicategories of reduced orbifolds that are standard in the literature. Its main advantage is that its objects are reduced atlases, so it gives a description that is closer to classical differential geometry than the descriptions (2) and (3) given in terms of Lie groupoids or differentiable stacks.

In the literature there are already other attempts to define morphisms of (reduced) orbifolds in terms that are useful for differential geometers:

(a) the “smooth maps” defined for example in [ALR, Definition 1.3];
(b) the “good maps” described by Weimin Chen and Yongbin Ruan in [CR];
(c) the orbifold maps described by Anke Pohl (only for the reduced case) in [Po].

The maps in (a) were the first ones to be defined, but it turned out that they were not good enough: in general one could not pullback orbifold vector bundles along such maps (and in the case when this was possible, the pullback was not unique up to isomorphism). This led Chen and Ruan to introduce the concept of good maps. Such maps proved to be good enough in order to define pullbacks of orbifold vector bundles (and fiber products under some assumptions), and they are currently frequently used in mathematical physics and differential geometry when dealing with orbifolds. However, they are bad-behaved for the following 2 reasons:

- not all smooth maps of manifolds are good maps;
- fiber products (when they exist) do not have the universal property of fiber products in a category; in particular, pullbacks of orbifold vector bundles do not have the universal property.

The first problem is just a technical mistake in the definition of good maps, and it can be corrected without much trouble by simply relaxing a bit the technical assumptions on good maps. However, the second problem is much worse and it cannot be corrected easily. We will exhibit examples of both bad behaviors in the next paper [T6].

The definition given in (c) solves the first problem (but not the second one). However, composition of morphisms is not well-defined in [Po] (we will show also this fact in [T6]).

Both in case (b) and in case (c), the bad problems quickly mentioned above are a consequence of completely ignoring the fact that orbifolds have much more structure than that of a usual 1-category. Actually, Theorem B proves that $(\text{Red Orb})$ has a non-trivial structure of bicategory because it is equivalent to $(\text{Orb}^{\text{eff}})$ (that has a non-trivial structure of 2-category). In [T6] we will prove that the bad behaviors of (b) and (c) are given by the following reasons.

- The category (b) of reduced orbifolds with good maps is equivalent to the homotopy category of $(\text{Red Orb})$, i.e. the 1-category obtained by identifying any pair of 1-morphisms of $(\text{Red Orb})$ whenever there is an invertible 2-morphism between them. Now the problem is the following: given any weak fiber product in a non-trivial bicategory $\mathscr{B}$, the corresponding commutative square in the homotopy category $\text{Ho}(\mathscr{B})$ not necessarily have the universal property of fiber products in a 1-category. This leads to all the problems mentioned above for fiber products of good maps in (b). On the contrary, we will show in [T6] that weak pullbacks of vector bundles in $(\text{Red Orb})$ have the universal property of weak fiber products in that bicategory.
The definition of maps according to Pohl is obtained in 2 steps. First of all, one gets a notion of “representative” of map, that corresponds to the notion of 1-morphism in \((P\mathcal{O}\mathcal{r}b_{pd})\); the 1-category \(\mathcal{C}\) obtained in this way has objects given by reduced orbifold atlases. Since Pohl wants to identify orbifold atlases that give the same orbifold structure, she has to construct a new 1-category \(\tilde{\mathcal{C}}\), where morphisms are equivalence classes of the “representatives” mentioned above. The way used by Pohl for identifying morphisms takes into account a certain class of commutative diagrams of \((P\mathcal{O}\mathcal{r}b_{pd})\), without considering the existence of 2-commutative diagrams. Because of that, one gets that composition of morphisms in \(\mathcal{C}\) is not well-defined. The problem can be solved by quotienting the set of morphisms in \(\mathcal{C}\) by a bigger equivalence relation (taking into account the role played by 2-morphisms). In this way one would get a 1-category \(\tilde{\mathcal{C}}\), that is equivalent to the homotopy category of \((\text{Red Orb})\). As such, \(\tilde{\mathcal{C}}\) would solve the bad-behaved definition of composition given by Pohl, but it would still carry the problems about fiber products mentioned before for (b).

To summarize, both the category constructed by Chen-Ruan (in the reduced case) and the one defined by Pohl have some serious drawbacks, mainly induced by ignoring the crucial role played by 2-morphisms. On the contrary, the bicategory \((\text{Red Orb})\) constructed in the present paper solves such problems (and it is equivalent to the standard approach to reduced orbifolds because of Theorems A and B).

Apart from that, only one important problem remains open: we have described a bicategory structure that solves problems (i) and (ii) by restricting to the case of reduced orbifolds. Is it possible to give an analogous description of a bicategory \((\text{Orb})\) also in the more general case of (possibly) non-reduced orbifolds? Since the bicategories of (2) and (3) are defined (and equivalent) also in this more general setup, in principle this should be possible, but it seems that this will require much more work.

1. Reduced orbifold atlases

Let us review some basic definitions about reduced orbifolds.

**Definition 1.1.** \([\text{MP} \ \S \ 1]\) Let \(X\) be a paracompact, second countable, Hausdorff topological space and let \(X' \subseteq X\) be open and non-empty. Then a reduced orbifold chart (also known as reduced uniformizing system) of dimension \(n\) for \(X'\) is the datum of:

- a connected open subset \(\tilde{X}\) of \(\mathbb{R}^n\);
- a finite group \(G\) of smooth automorphisms of \(\tilde{X}\);
- a continuous, surjective and \(G\)-invariant map \(\pi: \tilde{X} \to X'\), which induces an homeomorphism between \(\tilde{X}/G\) and \(X'\), where we give to \(\tilde{X}/G\) the quotient topology (in particular, \(\pi\) is an open map).

For every point \(\tilde{x} \in \tilde{X}\), we denote by \(\text{Stab}(G, \tilde{x})\) the stabilizer of \(\tilde{x}\) in \(G\).

**Remark 1.2.** We will always assume that \(G\) acts effectively; the orbifolds that have this property are usually called reduced or effective. Some of the current literature on orbifolds assumes that \(\tilde{X}\) is only a connected smooth manifold of dimension \(n\) instead of an open connected subset of \(\mathbb{R}^n\). This makes a difference for the definition of charts, but the arising notion of orbifold is not affected by that. To be more precise, to any orbifold atlas (see below) where the \(\tilde{X}\)'s are connected smooth manifolds of dimension \(n\), one can associate easily another orbifold atlas where the \(\tilde{X}\)'s are open connected subsets of \(\mathbb{R}^n\) and the 2 orbifold atlases give rise to the same orbifold structure (see below).
The following definition is a special case of [MP] § 2.1.

**Definition 1.3.** Let us fix any pair of reduced charts \((\tilde{X}_1, G_1, \pi_1)\) and \((\tilde{X}_2, G_2, \pi_2)\) for subsets \(X_1, X_2\) of \(X\). Then a *change of charts* from \((\tilde{X}_1, G_1, \pi_1)\) to \((\tilde{X}_2, G_2, \pi_2)\) is any diffeomorphism \(\lambda : \tilde{Y}_1 \rightarrow \tilde{Y}_2\) such that:

- \(\tilde{Y}_1\) is any connected component of \(\pi_1^{-1}(Y)\) for some open non-empty subset \(Y\) of \(X_1\) (since the action of \(G_1\) on \(\tilde{X}_1\) permutes such connected components, then \(\pi_1(\tilde{Y}_1) = Y\));
- \(\tilde{Y}_2\) is an open subset of \(\tilde{X}_2\);
- \(\pi_2 \circ \lambda = \pi_1|_{\tilde{Y}_1}\).

Using [MP] Lemma A.2 and the fact that \(\lambda\) is a diffeomorphism, it turns out that \(Y\) is contained also in \(X_2\) and that \(\tilde{Y}_2\) is a connected component of \(\pi_2^{-1}(Y)\). So the inverse of any change of charts is again a change of charts. If \(\lambda\) is any change of charts, we denote by \(\text{dom}\lambda\) its domain and by \(\text{cod}\lambda\) its codomain. If \(\tilde{x} \in \text{dom}\lambda\), then \(\text{germ}_{\tilde{x}}\lambda\) denotes the germ of \(\lambda\) at \(\tilde{x}\). An *embedding* is any change of charts \(\lambda\) as before, such that \(\text{dom}\lambda = \tilde{X}_1\). 2 charts as before are called *compatible* if for each pair \(\tilde{x}_1 \in \tilde{X}_1, \tilde{x}_2 \in \tilde{X}_2\) with \(\pi_1(\tilde{x}_1) = \pi_2(\tilde{x}_2)\), there exists a change of charts \(\lambda\) from \(\tilde{X}_1, G_1, \pi_1\) to \(\tilde{X}_2, G_2, \pi_2\), with \(\tilde{x}_1 \in \text{dom}\lambda\). Up to composing \(\lambda\) with an element of \(G_2\), this is the same as requiring that there exists a change of charts \(\lambda\) such that \(\tilde{x}_1 \in \text{dom}\lambda\) and \(\lambda(\tilde{x}_1) = \tilde{x}_2\).

**Remark 1.4.** Let us suppose that we have any change of charts \(\lambda : \tilde{Y}_1 \rightarrow \tilde{Y}_2\) from \((\tilde{X}_1, G_1, \pi_1)\) to \((\tilde{X}_2, G_2, \pi_2)\). Then let us fix any point \(g_1 \in G_1\) and let us suppose that \(g_1(\tilde{Y}_1) \cap \tilde{Y}_1 \neq \emptyset\); by the hypothesis on \(\tilde{Y}_1\), we conclude that necessarily \(g_1(\tilde{Y}_1) = \tilde{Y}_1\). Therefore we can consider the subgroup of \(G_1\):

\[G_1(\tilde{Y}_1) := \{g_1 \in G_1 \text{ s.t. } g_1(\tilde{Y}_1) \cap \tilde{Y}_1 \neq \emptyset\} = \{g_1 \in G_1 \text{ s.t. } g_1(\tilde{Y}_1) = \tilde{Y}_1\}.\]

By [MM] Lemma 2.10, we have that the group \(G_1(\tilde{Y}_1)\) acts effectively on \(\tilde{Y}_1\), so the triple \((\tilde{Y}_1, G_1(\tilde{Y}_1), \pi_1|_{\tilde{Y}_1})\) is a reduced orbifold chart; moreover, \(\lambda\) can be considered as an embedding from \((\tilde{Y}_1, G_1(\tilde{Y}_1), \pi_1|_{\tilde{Y}_1})\) to \((\tilde{X}_2, G_2, \pi_2)\).

Using Remark 1.4 the following definition is equivalent to [MP] § 1.

**Definition 1.5.** Let \(X\) be a paracompact, second countable, Hausdorff topological space; a *reduced orbifold atlas of dimension* \(n\) on \(X\) is any family \(\mathcal{X} = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}\) of reduced orbifolds charts of dimension \(n\), such that:

(i) the family \(\{X_i := \pi_i(\tilde{X}_i)\}_{i \in I}\) is an open cover of \(X\);
(ii) every pair of charts in \(\mathcal{X}\) is compatible.

Given any orbifold atlas \(\mathcal{X}\) as before and any pair \((i, i') \in I \times I\), we denote by \(\text{Ch}(\mathcal{X}, i, i')\) the set of all changes of charts \(\lambda\) from \((\tilde{X}_i, G_i, \pi_i)\) to \((\tilde{X}_{i'}, G_{i'}, \pi_{i'})\) and we set \(\text{Ch}(\mathcal{X}) := \bigcup_{(i, i') \in I \times I} \text{Ch}(\mathcal{X}, i, i')\).

**Definition 1.6.** [MP] § 1 Let \(\mathcal{X}\) and \(\mathcal{X}'\) be reduced orbifold atlases for the same topological space \(X\). We say that they are *equivalent* if their union is again an orbifold atlas for \(X\), i.e. if and only if any chart of \(\mathcal{X}\) is compatible with any chart of \(\mathcal{X}'\); a *reduced orbifold structure* of dimension \(n\) on \(X\) is any equivalence class with respect to compatibility of atlases. A *reduced orbifold* of dimension \(n\) is any pair \((X, [\mathcal{X}])\) consisting of a paracompact, second countable, Hausdorff topological space \(X\) and a reduced orbifold structure \(\mathcal{X}\) on \(X\). Any atlas in \([\mathcal{X}]\) is called a reduced orbifold atlas for \((X, [\mathcal{X}])\). The notion of being compatible gives also rise to a partial order on the set of reduced orbifold atlases for \(X\); it turns out that given any reduced orbifold atlas there is exactly one maximal atlas associated to it.
with respect of this definition, so a reduced orbifold structure can be equivalently
defined as a maximal reduced orbifold atlas.

**Definition 1.7.** [CR § 4.1] Let \( f : X \to Y \) be any continuous map between
topological spaces and let \( X' \subseteq X \) and \( Y' \subseteq Y \) be open subsets such that \( f(X') \subseteq Y' \). Let us suppose that there are reduced orbifold charts \((\tilde{X}, G, \pi)\) for \( X' \) and
\((\tilde{Y}, H, \chi)\) for \( Y' \). Then a local lift of \( f \) with respect to these 2 charts is any smooth
map \( \tilde{f} : \tilde{X} \to \tilde{Y} \), such that \( \chi \circ \tilde{f} = f \circ \pi \).

**Definition 1.8.** Let us fix any reduced orbifold atlas \( \mathcal{A} \) as before and let \( P \) be any
subset of \( Ch(\mathcal{A}) \). We say that \( P \) is a good subset of \( Ch(\mathcal{A}) \) if the following property holds:

(\( \text{GS} \)) for each \( \lambda \in Ch(\mathcal{A}) \) and for each \( \tilde{\lambda} \in \text{dom } \lambda \) there exists \( \hat{\lambda} \in P \) such that
\( \tilde{\lambda} \in \text{dom } \hat{\lambda} \) and \( \text{germ}_{\tilde{\lambda}} \lambda = \text{germ}_{\hat{\lambda}} \hat{\lambda} \).

Since \( P \) is a subset of \( Ch(\mathcal{A}) \), for each \((i, i') \in I \times I\) we write for simplicity \( P(i, i') := P \cap Ch(\mathcal{A}, i, i') \) and \( P(i, -) := \bigsqcup _{i' \in I} P(i, i') \).

**Remark 1.9.** In the notations of [Po], \( \text{GS} \) is the condition that \( P \) generates the pseudogroup \( Ch(\mathcal{A}) \) inside the larger pseudogroup \( \Psi(\mathcal{A}) \) defined and used in [Po]; such a pseudogroup is obtained by taking into account all changes of charts of \( \mathcal{A} \) with
a more general definition than the one used in the present paper. In [Po] there are other 2 technical conditions (axioms of “quasi-pseudogroup”), but
they are implied by \( \text{GS} \) in our case, so we can omit them. Under this remark, our
definition of morphism of orbifold atlases \( \mathcal{A} \to \mathcal{Y} \) (Definition 1.10 and 1.11 below) is equivalent to the definition of “orbifold map with domain atlas \( \mathcal{A} \) and range atlas \( \mathcal{Y} \)”
stated in [Po] Definitions 4.4 and 4.10. To be more precise, our “representatives of
morphisms” (Definition 1.11 below) are a subset of the representatives given in [Po]
Definitions 4.4, but the sets of equivalence classes described in Definition 1.11 and
in [Po] Definition 4.10 will be the same.

**Definition 1.10.** Let us fix any pair of reduced orbifold atlases \( \mathcal{A} = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I} \) and
\( \mathcal{Y} = \{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J} \) for \( X \) and \( Y \) respectively. Then a representative of a morphism from \( \mathcal{A} \) to \( \mathcal{Y} \) is any tuple \( \tilde{f} := (f, \pi, \{f_i\}_{i \in I}, P_f, \nu_f) \) that satisfies the
following conditions:

\begin{itemize}
  \item[(M1)] \( f : X \to Y \) is any continuous map;
  \item[(M2)] \( \pi : I \to J \) is any set map such that \( f(\pi(i), \tilde{X}_i) \subseteq \chi_j(\tilde{Y}_j) \) for each \( i \in I \);
  \item[(M3)] for each \( i \in I \), the map \( \tilde{f}_i \) is a local lift of \( f \) with respect to the orbifold
charts \((\tilde{U}_i, G_i, \pi_i) \in \mathcal{A} \) and \((\tilde{V}_j, H_j, \chi_j) \in \mathcal{Y} \);
  \item[(M4)] \( P_f \) is any good subset of \( Ch(\mathcal{A}) \);
  \item[(M5)] \( \nu_f : P_f \to Ch(\mathcal{Y}) \) is any set map that assigns to each \( \lambda \in P_f(i, i') \) a change
of charts \( \nu_f(\lambda) \in Ch(\mathcal{Y}, \pi(i), \pi(i')) \), such that:
    \begin{itemize}
      \item[(a)] \( \text{dom } \nu_f(\lambda) \) is an open set containing \( \tilde{f}_i(\text{dom } \lambda) \),
      \item[(b)] \( \text{cod } \nu_f(\lambda) \) is an open set containing \( \tilde{f}_i(\text{cod } \lambda) \),
      \item[(c)] \( \tilde{f}_i \circ \lambda = \nu_f(\lambda) \circ \tilde{f}_i(\text{dom } \lambda) \),
      \item[(d)] for all \( i \in I \), for all \( \lambda, \lambda' \in P_f(i, -) \) and for all \( \tilde{x}_i \in \text{dom } \lambda \cap \text{dom } \lambda' \) with
\( \text{germ}_{\tilde{x}_i} \lambda = \text{germ}_{\tilde{x}_i} \lambda' \), we have
\[ \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \nu_f(\lambda) = \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \nu_f(\lambda') \],
      \item[(e)] for all \( (i, i', i'') \in I^3 \), for all \( \lambda_1 \in P_f(i, i') \), for all \( \lambda_2 \in P_f(i', i'') \) and for all \( \tilde{x}_i \in \lambda_1^{-1}(\text{cod } \lambda_1 \cap \text{dom } \lambda_2) \), we have
\end{itemize}
\end{itemize}
Definition 1.11. Given 2 representatives of morphisms from $X = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}$ to $Y = \{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ as follows:

\[ \tilde{f} = \left( f, \tilde{f}, \left\{ \tilde{f}_i \right\}_{i \in I}, \{P_f, \nu_f\} \right) \quad \text{and} \quad \tilde{f}' = \left( f', \tilde{f}', \left\{ \tilde{f}'_i \right\}_{i \in I}, \{P_{f'}, \nu_{f'}\} \right), \]

we say that $\tilde{f}$ is equivalent to $\tilde{f}'$ if and only if $f = f'$, $\tilde{f} = \tilde{f}'$, $\tilde{f}_i = \tilde{f}'_i$ for all $i \in I$, and

\[ \text{germ}_{\tilde{f}_{i}(\tilde{\xi}_i)} \nu_f(\lambda) = \text{germ}_{\tilde{f}'_{i}(\tilde{\xi}_i)} \nu_{f'}(\lambda') \tag{1.1} \]

for all $i \in I$, for all $\lambda \in P_f(i, -)$, $\lambda' \in P_{f'}(i, -)$ and for all $\tilde{\xi}_i \in \text{dom} \lambda \cap \text{dom} \lambda'$ with $\text{germ}_{\tilde{\xi}_i} \lambda = \text{germ}_{\tilde{\xi}_i} \lambda'$. This defines an equivalence relation (it is reflexive by (GS)). The equivalence class of $\tilde{f}$ will be denoted by

\[ [\tilde{f}] = \left( f, \tilde{f}, \left\{ \tilde{f}_i \right\}_{i \in I}, \{P_f, \nu_f\} \right) : X \rightarrow Y \tag{1.2} \]

and it is called a morphism of reduced orbifold atlases from $X$ to $Y$ over the continuous map $\tilde{f} : X \rightarrow Y$.

Lemma 1.12.

(i) given any reduced orbifold chart $(\tilde{X}, G, \pi)$ on any topological space and any change of charts $\lambda$ from $(\tilde{X}, G, \pi)$ to itself, there is a unique $g \in G$ such that $\lambda = g|_{\text{dom} \lambda}$;

(ii) let us fix any pair of reduced orbifold atlases $X := \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}$ and $Y := \{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ and any morphism as in (1.2); for each $i \in I$ there is a group homomorphism $\tilde{f}_i : G_i \rightarrow H_{f(i)}$, such that for each $\lambda \in P_f(i, i)$ we have $\nu_f(\lambda) = \tilde{f}_i(g_i)|_{\text{dom} \nu_f(\lambda)}$, where $g_i$ is the unique element of $G_i$ as in (i);

(iii) for each $i \in I$ and for each $g_i \in G_i$ we have $\tilde{f}_i \circ g_i = \tilde{f}_i(g_i) \circ \tilde{f}_i$. 

Proof. Since $\text{dom} \lambda$ is connected, claim (i) is a straightforward consequence of (MP) Proposition A.1. Now let us fix any representative $(P_f, \nu_f)$ for $[P_f, \nu_f]$, any $g_i \in G_i$ and any point $\tilde{x}_i \in \tilde{X}_i$; since $P_f$ satisfies condition (MP), then there exists a (in general non-unique) $\lambda \in P_f(i, i)$ such that $\tilde{x}_i \in \text{dom} \lambda$ and $\lambda = g_i|_{\text{dom} \lambda}$ (a priori, the second condition holds only in a neighborhood $\tilde{X}'$ of $\tilde{x}_i$ in $\text{dom} \lambda$; by (i) we conclude that the same relation holds everywhere on $\text{dom} \lambda$). By (ii) applied on $\tilde{Y}$, we get that $\nu_f(\lambda)$ is the restriction of a unique object $\tilde{f}_i(g_i, \tilde{x}_i, \lambda)$ in $H_{f(i)}$.

We claim that $\tilde{f}_i(g_i, \tilde{x}_i, \lambda)$ does not depend on $\tilde{x}_i$ or $\lambda$ (but only on $g_i$ and on $(P_f, \nu_f)$). So let us fix another point $\tilde{x}'_i \in \tilde{X}_i$ and another $\lambda' \in P_f$ such that $\tilde{x}'_i \in \text{dom} \lambda'$ and $\lambda' = g_i|_{\text{dom} \lambda'}$. Since $\tilde{X}_i$ is connected by definition of reduced orbifold chart, then there exists a continuous path $\gamma : [0, 1] \rightarrow \tilde{X}_i$, which joins $\tilde{x}_i$ and $\tilde{x}'_i$. For any $t \in [0, 1]$, we choose $\lambda_t \in P_f$ such that $\gamma(t) \in \text{dom} \lambda_t$ and
\( \lambda_t = g_t|_{\text{dom } \lambda_t} \). By compactness, we can cover \( \gamma([0, 1]) \) by a finite number of open sets \( \{ \text{dom } \lambda_t \}_{1 \leq t \leq r} \) and we can also assume that \( \text{dom } \lambda_t \) intersects \( \text{dom } \lambda_t' \) for each \( t = 1, \ldots, r - 1 \). For each \( t \), using (i) we get that \( \nu_t(\lambda_t) \) is the restriction of a unique object \( f(g_t, \gamma(t^t), \lambda_t) \) in \( H^t_\gamma(i) \). This proves that for each \( t = 1, \ldots, r - 1 \) we have \( \hat{f}_t(g_t, \gamma(t^t), \lambda_t) = f(g_t, \gamma(t^t+1), \lambda_{t+1}) \), which implies that \( \hat{f}_t(g_t, \bar{x}_t, \lambda) = \hat{f}_t(g_t, \bar{x}_t', \lambda') \). Therefore, we have proved that \( \hat{f} \) is a well-defined set map from \( G_t \) to \( H^t_\gamma(i) \) which depends only on \( (P_f, \nu_f) \). The fact that \( \hat{f}_t \) is a group homomorphism is a simple consequence of conditions (M5e) and (M5f). Using (1.1), \( \hat{f}_t \) does not depend on the representative \( (P_f, \nu_f) \) chosen for \( [P_f, \nu_f] \). Claim (iii) is a direct consequence of (M5a) and (ii). \( \square \)

**Definition 1.13.** Let us fix any reduced orbifold atlas \( \mathcal{X} := \{ (\bar{X}_i, G_i, \pi_i) \}_{i \in I} \) on any topological space \( X \) and any set \( \{ (\bar{X}_{i'}, G_{i'}, \pi_{i'}) \}_{i' \in I'} \) of reduced orbifold charts on \( X \), that are compatible with the charts of \( \mathcal{X} \) (with the set \( I' \) disjoint from \( I \)). Then the family

\[
\mathcal{X}' := \{ (\bar{X}_i, G_i, \pi_i) \}_{i \in I} \coprod \{ (\bar{X}_{i'}, G_{i'}, \pi_{i'}) \}_{i' \in I'}
\]

is again a reduced orbifold atlas for \( X \). Moreover, there is an obvious inclusion \( \nu_{\mathcal{X}, \mathcal{X}'} : \text{Ch}(\mathcal{X}) \to \text{Ch}(\mathcal{X}') \). Therefore we can consider the following morphism:

\[
i_{\mathcal{X}, \mathcal{X}'} := \left( \text{id}_X, I \hookrightarrow (I \coprod I') \right), \left\{ \text{id}_{\bar{X}_i} \right\}_{i \in I}, \{ \text{Ch}(\mathcal{X}), \nu_{\mathcal{X}, \mathcal{X}'} \} : \mathcal{X} \hookrightarrow \mathcal{X}'.
\]

We call any such morphism an inclusion of reduced orbifold atlases. In particular, for every reduced orbifold atlas \( \mathcal{X} \) as before, we denote by

\[
i_{\mathcal{X}} : \mathcal{X} \hookrightarrow \mathcal{X}^\text{max}
\]
the inclusion \( i_{\mathcal{X}, \mathcal{X}^\text{max}} \) of \( \mathcal{X} \) into the maximal atlas \( \mathcal{X}^\text{max} \) associated to it.

Now we need to define the composition of morphisms of reduced orbifold atlases. In order to do that, we follow [Po, Construction 5.9], with the only differences due to Remark 1.9.

**Construction 1.14.** Let us fix any triple of orbifold atlases

\[
\mathcal{X} = \left\{ (\bar{X}_i, G_i, \pi_i) \right\}_{i \in I}, \quad \mathcal{Y} := \left\{ (\bar{Y}_j, H_j, \chi_j) \right\}_{j \in J}, \quad \mathcal{Z} = \left\{ (\bar{Z}_l, K_l, \eta_l) \right\}_{l \in L}
\]

for 3 topological spaces \( X, Y \) and \( Z \) respectively. Let us also fix 2 morphisms

\[
[f] = \left( f, \bar{f}, \left\{ \hat{f}_i \right\}_{i \in I}, [P_f, \nu_f] \right) : \mathcal{X} \to \mathcal{Y},
\]

\[
[g] = \left( g, \bar{g}, \left\{ \hat{g}_j \right\}_{j \in J}, [P_g, \nu_g] \right) : \mathcal{Y} \to \mathcal{Z}.
\]

Then we define a composition

\[
[g] \circ [f] := \left( g \circ f, \bar{g} \circ \bar{f}, \left\{ \hat{g}_j \circ \hat{f}_i \right\}_{i \in I}, [P_g \circ f, \nu_g \circ f] \right) : \mathcal{X} \to \mathcal{Z}.
\]

Here we construct the class \([P_g \circ f, \nu_g \circ f]\) as follows: first of all, let us fix representatives \( (P_f, \nu_f) \) for \([P_f, \nu_f]\) and \((P_g, \nu_g)\) for \([P_g, \nu_g]\). Then let us fix any \( i \in I \), any \( \lambda \in P_f(i, -) \) and any point \( \bar{x}_i \in \text{dom } \lambda \). Since \( P_g \) is a good subset of \( \text{Ch}(\mathcal{Y}) \), then by condition (GS) there are a (non-unique) \( \omega_{\bar{x}_i} \) and \( \nu_f(\lambda) \in P_f(\bar{x}_i, -) \) and an open set
\[
\tilde{Y}_{f_i}(\tau_i) \subseteq \text{dom } \nu_f(\lambda) \cap \text{dom } \omega_{f_i}(\tau_i, \nu_f(\lambda)) \subseteq \tilde{Y}_{f_i}(\lambda),
\]
such that \( \tilde{f_i}(\tau_i) \in \tilde{Y}_{f_i}(\tau_i, \nu_f(\lambda)) \) and
\[
\left( \nu_f(\lambda) \right) \bigg|_{\tilde{Y}_{f_i}(\tau_i, \nu_f(\lambda))} = \left( \omega_{f_i}(\tau_i, \nu_f(\lambda)) \right) \bigg|_{\tilde{Y}_{f_i}(\tau_i, \nu_f(\lambda))}.
\]
For each pair \((\lambda, \tau_i)\) as before, there exists a (non-unique) open connected subset \(\tilde{X}_{\tau_i, \lambda} \subseteq \tilde{f_i}^{-1}\left( \tilde{Y}_{f_i}(\tau_i, \nu_f(\lambda)) \right) \cap \text{dom } \lambda\), such that:

- \(\tau_i \in \tilde{X}_{\tau_i, \lambda}\);
- \(\tilde{X}_{\tau_i, \lambda}\) is invariant under the action of \(\text{Stab}(G_i, \tau_i)\);
- for each \(g_i \in G_i \setminus \text{Stab}(G_i, \tau_i)\) we have \(g_i(\tilde{X}_{\tau_i, \lambda}) \cap \tilde{X}_{\tau_i, \lambda} = \emptyset\).

(in this way, \(\lambda\) is still a change of charts if restricted to \(\tilde{X}_{\tau_i, \lambda}\)). Then for each \(i \in I\) and for each \(\lambda \in P_f(i, -)\) we choose any set of points \(\{\tau_i\}_{\lambda \in E(\lambda)} \subseteq \tilde{X}_i\) such that the set \(\{\tilde{X}_{\tau_i, \lambda}\}_{\lambda \in E(\lambda)}\) is a covering for \(\text{dom } \lambda\) and such that if \(e \neq e'\), then \(\tilde{X}_{\tau_i, \lambda} \neq \tilde{X}_{\tau_i, \lambda'}\). Then we consider the set:
\[
P_{gof} := \left\{ \lambda | \tilde{X}_{\tau_i, \lambda} \quad \forall i \in I, \forall \lambda \in P_f(i, -), \forall e \in E(\lambda) \right\}.
\]
In general, given any element \(\lambda' \in P_{gof}\), there can be more than one \(\lambda \in P_f\), such that \(\lambda | \tilde{X}_{\tau_i, \lambda} = \lambda'\) (for some \(e \in E(\lambda)\)); therefore for any such \(\lambda' \in P_f\), using the axiom of choice we make an arbitrary choice of \((\lambda, e)\) with that property. For that choice, we fix also a choice of \(\tilde{Y}_{f_i}(\tau_i, \nu_f(\lambda))\) and of \(\omega_{f_i}(\tau_i, \nu_f(\lambda))\) as before.

Since \(P_f\) is a good subset of \(Ch(\mathcal{X})\), then also \(P_{gof}\) is a good subset of \(Ch(\mathcal{X})\). Then for each \(\lambda | \tilde{X}_{\tau_i, \lambda} \in P_{gof}\) we set
\[
\nu_{gof}^\text{ind}(\lambda | \tilde{X}_{\tau_i, \lambda}) := \omega_{f_i}(\tau_i, \nu_f(\lambda)).
\]
So we have defined a set map \(\nu_f\) from \(P_{gof}\) to \(Ch(\mathcal{Y})\); a direct computation proves that \((P_{gof}, \nu_{gof}^\text{ind}) \in [P_f, \nu_f]\). Then we simply define
\[
\nu_{gof}(\lambda | \tilde{X}_{\tau_i, \lambda}) := \nu_g(\omega_{f_i}(\tau_i, \nu_f(\lambda))) = \nu_g \circ \nu_{f_i}^\text{ind}(\lambda | \tilde{X}_{\tau_i, \lambda})
\]
for every \(\lambda | \tilde{X}_{\tau_i, \lambda} \in P_{gof}\) and it is easy to verify that \(\nu_{gof}\) satisfies properties \((\text{M5a}) - (\text{M5d})\). The construction of \(P_{gof}\) and \(\nu_{gof}\) depends on some choices, but it can be proved that the equivalence class \([P_{gof}, \nu_{gof}]\) does not depend on such choices. In this way we have defined a notion of composition of morphisms of reduced orbifold atlases.

**Lemma 1.15.** The composition of morphisms of reduced orbifold atlases is associative.

The proof is obvious for what concerns the composition of maps of the form \(f_i, \tilde{f}_i\) and \(\tilde{f}_i\); the proof of the associativity on the pairs of the form \([P_f, \nu_f]\) is straightforward, so we omit it.

**Remark 1.16.** Since reduced orbifold structures are given by equivalence classes of reduced orbifold atlases (see Definition \ref{definition}), then one would like to define compositions also for every pair of morphisms \([f] : \mathcal{X} \to \mathcal{Y}\) and \([g] : \mathcal{Y} \to \mathcal{Z}\), whenever \([\mathcal{Y}] = [\mathcal{Y'}]\). We will provide such a definition in the paper \ref{paper} (at the cost of quotienting out by an equivalence relation induced by the 2-morphisms that we are going to define below). As we mentioned in the Introduction, currently in the
literature there are also other 2 definitions of morphisms between orbifolds. One is given by “good maps” defined by Weimin Chen and Yongbin Ruan in [CR] (such a definition is given also for the (possibly) non-reduced case); the other one is given by Anke Pohl in [Po]. We will describe in [T6] the relations between our definition of morphism and those of [CR] and [Po].

Definition 1.17. Let us fix any pair of paracompact, second countable, Hausdorff topological spaces $X$ and $Y$, any open embedding $f : X \hookrightarrow Y$ and any reduced orbifold atlas $\mathcal{X} = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}$ on $X$. Then we set

$$f_* (\mathcal{X}) := \left\{ \left( \tilde{X}_i, G_i, f \circ \pi_i \right) \right\}_{i \in I}.$$  \hspace{1cm} (1.4)

Since $f$ is an open embedding, then $f_* (\mathcal{X})$ is a family of compatible reduced orbifold charts over $Y$, hence $f_* (\mathcal{X})$ is a reduced orbifold atlas over $f (X)$ (with the topology given by the fact that $f(X)$ is open in $Y$). Moreover, $f$ induces a morphism $[f^\text{ind}] : \mathcal{X} \rightarrow f_* (\mathcal{X})$: on the topological level it is simply given by $f$, while the rest of the structure is trivial: $\mathcal{T}$ is the identity of $I$, each $f_i$ is an identity, $P_f$ is the entire set $\mathcal{Ch}(\mathcal{X})$ and $\nu_f$ associates to any change of chart in $\mathcal{X}$ the same change of charts in $f_* (\mathcal{X})$. In particular, if $f$ is an homeomorphism, then $f_* (\mathcal{X})$ is a reduced orbifold atlas on $Y$ and $[\hat{f}^\text{ind}]$ is an isomorphism (with respect to the definition of composition given above).

As we said in the introduction, our first aim is to construct a 2-category (Red Atl) of reduced orbifold atlases. Roughly speaking, a 2-category is the datum of objects, morphisms and “morphism between morphisms” (known as 2-morphisms or, sometimes, natural transformations), together with identities and compositions of morphisms and 2-morphisms (for more details we refer e.g. to [Lei]). First of all, we have to define a notion of 2-morphism in this setup.

Definition 1.18. Let us fix any pair of reduced orbifold atlases $\mathcal{X} = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}$ and $\mathcal{Y} :\{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ over $X$ and $Y$ respectively. Moreover, let us fix 2 morphisms from $\mathcal{X}$ to $\mathcal{Y}$ over the same continuous function $f : X \rightarrow Y$:

$$[\hat{f}^m] := \left( f, \mathcal{T}, \left\{ \tilde{f}_i^m \right\}_{i \in I}, \left\{ P_f^m, \nu_{f^m} \right\} \right) \quad \text{for } m = 1, 2.$$  \hspace{1cm} (1.5)

Then a representative of a 2-morphism from $[\hat{f}^1]$ to $[\hat{f}^2]$ is any set of data:

$$\delta := \left\{ \left( \tilde{X}_i^a, \delta_i^a \right) \right\}_{i \in I, a \in A(i)},$$  \hspace{1cm}

such that:

(2Ma) for all $i \in I$ the set $\{ \tilde{X}_i^a \}_{a \in A(i)}$ is an open covering of $\tilde{X}_i$;

(2Mb) for all $i \in I$ and for all $a \in A(i)$, $\delta_i^a$ is a change of charts in $\mathcal{Y}$ with

$$\tilde{f}_i^1 \left( \tilde{X}_i^a \right) \subseteq \text{dom} \delta_i^a \subseteq \tilde{Y}_i^{\text{\textsuperscript{T}}(i)}, \quad \tilde{f}_i^2 \left( \tilde{X}_i^a \right) \subseteq \text{cod} \delta_i^a \subseteq \tilde{Y}_i^{\text{\textsuperscript{T}}(i)};$$

(2Mc) for all $i \in I$, for all $a \in A(i)$ and for all $\tilde{x}_i \in \tilde{X}_i^a$ we have

$$\tilde{f}_i^2 (\tilde{x}_i) = \delta_i^a \circ \tilde{f}_i^1 (\tilde{x}_i);$$

(2Md) for all $i \in I$, for all $a, a' \in A(i)$ and for all $\tilde{x}_i \in \tilde{X}_i^a \cap \tilde{X}_i^{a'}$ we have

$$\text{germ} \tilde{f}_i^1 (\tilde{x}_i) \delta_i^a = \text{germ} \tilde{f}_i^1 (\tilde{x}_i) \delta_i^{a'};$$
Let us suppose that there exist data as in (1.6) that satisfy condition (1.7). Then by Definition 1.11 we conclude that
\[ \tilde{x}_i \in \text{dom } \lambda^m, \quad \text{germ}_{\tilde{x}_i} \lambda^m = \text{germ}_{\tilde{x}_i} \lambda \quad \text{for } m = 1, 2 \] (1.7)
and
\[ \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \nu_{f^m}(\lambda^2) \cdot \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \delta_i^a = \text{germ}_{\tilde{f}_i(\lambda(\tilde{x}_i))} \delta_i^a \cdot \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \nu_{f^1}(\lambda^1). \] (1.8)

**Remark 1.19.** Given any \( i \in I \) and any pair \( a, a' \in A(i) \), by (2Me) we get that \( \delta_i^a \) coincides with \( \delta_i^{a'} \) on the set \( \tilde{f}_i^1(\tilde{X}_i^a \cap \tilde{X}_i^{a'}) \), that in general is not an open set; actually, by (2Md) \( \delta_i^a \) and \( \delta_i^{a'} \) coincide on some open set containing \( \tilde{f}_i^1(\tilde{X}_i^a \cap \tilde{X}_i^{a'}) \).

We remark also that both the left hand side and the right hand side of (1.8) are well-defined. Indeed,
\[ \delta_i^a \circ \tilde{f}_i^1(\tilde{x}_i) = \tilde{f}_i^2(\tilde{x}_i) \]
by (1.5) and
\[ \nu_{f^1}(\lambda^1) \left( \tilde{f}_i^1(\tilde{x}_i) \right) = \tilde{f}_i^1 \left( \lambda^1(\tilde{x}_i) \right) = \tilde{f}_i^1 \left( \lambda(\tilde{x}_i) \right) \]
by (M5c) and (1.7).

**Remark 1.20.** Let us suppose that there exist data as in (1.6) that satisfy conditions (1.7) and (1.8). Let us suppose that \( (P_{f^m}, \nu_{f^m}, \lambda^m) \) for \( m = 1, 2 \) is another set of data as (1.6) that satisfies condition (1.7). Then by Definition 1.11 we conclude that
\[ \text{germ}_{\tilde{f}_i^m(\tilde{x}_i)} \nu_{f^m}(\lambda^m) = \text{germ}_{\tilde{f}_i^m(\tilde{x}_i)} \nu_{f^1}(\lambda^m) \quad \text{for } m = 1, 2, \]
so (1.8) is verified also by the new set of data. Therefore, (2Md) is equivalent to:

(2Mo′) for all \( (i, i') \in I \times I \), for all \( (a, a') \in A(i) \times A(i') \), for all \( \lambda \in \text{Ch}(\mathcal{X}, i, i') \), for all \( \tilde{x}_i \in \text{dom } \lambda \cap \tilde{X}_i^a \) such that \( \lambda(\tilde{x}_i) \in \tilde{X}_i^{a'} \) and for all data (1.6) that satisfy (1.7), we have that (1.8) holds.

**Definition 1.21.** Let us fix any pair of reduced orbifold atlases \( \mathcal{X}, \mathcal{Y} \) and any pair of morphisms \( [\tilde{f}^1], [\tilde{f}^2] \) from \( \mathcal{X} \) to \( \mathcal{Y} \) over the same continuous map (as in Definition 1.18). Moreover, let us fix any pair of representatives of 2-morphisms from \( [\tilde{f}^1] \) to \( [\tilde{f}^2] \):
\[ \delta := \left\{ \left( \tilde{X}_i^a, \delta_i^a \right) \right\}_{\tilde{x}_i \in I, a \in A(i)} \quad \text{and} \quad \overline{\delta} := \left\{ \left( \tilde{X}_i^{\overline{a}}, \overline{\delta}_i^{\overline{a}} \right) \right\}_{\tilde{x}_i \in I, a \in A(i)}. \]

Then we say that \( \delta \) is equivalent to \( \overline{\delta} \) if and only if for all \( i \in I \), for all pairs \( (a, \overline{a}) \in A(i) \times \overline{A}(i) \) and for all \( \tilde{x}_i \in \tilde{X}_i^a \cap \tilde{X}_i^{\overline{a}} \) (if non-empty) we have
\[ \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \delta_i^a = \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \overline{\delta}_i^{\overline{a}}. \]
This definition gives rise to an equivalence relation (it is reflexive by (2Md)). We denote by \([\delta] : [\tilde{f}^1] \Rightarrow [\tilde{f}^2]\) the class of any \( \delta \) as before and we say that \([\delta]\) is a 2-morphism from \( [\tilde{f}^1] \) to \( [\tilde{f}^2] \).
2. Vertical and horizontal compositions of 2-morphisms

Construction 2.1. Let us fix 2 reduced orbifold atlases $\mathcal{X} = \{(\tilde{X}_i,G_i,\pi_i)\}_{i \in I}$, $\mathcal{Y} = \{(\tilde{Y}_j,H_j,\chi_j)\}_{j \in J}$ for $X$ and $Y$ respectively, any continuous map $f : X \to Y$ and any triple of morphisms from $\mathcal{X}$ to $\mathcal{Y}$ over $f$:

$$[\tilde{f}^m] := \left( f, \overline{f}^m, \left\{ \overline{f}^m_i \right\}_{i \in I}, [P_{f^m}, \nu_{f^m}] \right) \quad \text{for} \quad m = 1, 2, 3.$$

In addition, let us fix any 2-morphism $[\delta] : [\tilde{f}^1] \Rightarrow [\tilde{f}^2]$ and any 2-morphism $[\sigma] : [\tilde{f}^2] \Rightarrow [\tilde{f}^3]$. We want to define a vertical composition $[\sigma] \circ [\delta] : [\tilde{f}^1] \Rightarrow [\tilde{f}^3]$; in order to do that, let us fix any representative

$$\sigma = \left\{ \left( \tilde{X}_i^a, \sigma_i^a \right) \right\}_{i \in I, a \in B(i)}$$

for $[\sigma]$. As we did in Construction 1.14, we can always choose a (non-unique) representative

$$\delta = \left\{ \left( \tilde{X}_i^a, \delta_i^a \right) \right\}_{i \in I, a \in A(i)}$$

for $[\delta]$, such that for each $i \in I$ and for each $(a, b) \in A(i) \times B(i)$, the set map $\delta_i^a$ restricted to the set

$$\tilde{Y}_{i}^{a,b} := (\delta_i^a)^{-1} \left( \text{cod} \delta_i^a \cap \text{dom} \sigma_i^b \right)$$

(2.1)

(if non-empty) is again a change of charts, so that also $\theta_i^{a,b} := \sigma_i^b \circ \delta_i^a|_{\tilde{Y}_{i}^{a,b}}$ is a change of charts of $\mathcal{Y}$. Then for each $i \in I$ and for each $(a, b) \in A(i) \times B(i)$ we set $\tilde{X}_{i}^{a,b} := \tilde{X}_i^a \cap \tilde{X}_i^b$; if $\tilde{X}_{i}^{a,b}$ is non-empty, then also $\tilde{Y}_{i}^{a,b}$ is non-empty. Then we define:

$$\theta := \left\{ \left( \tilde{X}_{i}^{a,b}, \theta_i^{a,b} \right) \right\}_{i \in I, (a,b) \in A(i) \times B(i) \text{ s.t. } \tilde{X}_{i}^{a,b} \neq \emptyset}.$$

A straightforward proof shows that:

Lemma 2.2. The collection $\theta$ so defined is a representative of a 2-morphism from $[\tilde{f}^1]$ to $[\tilde{f}^3]$. Moreover, the class of $\theta$ does not depend on the choices of representatives $\delta$ for $[\delta]$ and $\sigma$ for $[\sigma]$.

Therefore, it makes sense to give the following definition.

Definition 2.3. Given any pair $[\delta], [\sigma]$ as before, we define their vertical composition as:

$$[\sigma] \circ [\delta] := [\theta] : [\tilde{f}^1] \Rightarrow [\tilde{f}^3].$$

Construction 2.4. Let us fix any triple of reduced orbifold atlases

$$\mathcal{X} = \left\{ \left( \tilde{X}_i, G_i, \pi_i \right) \right\}_{i \in I}, \quad \mathcal{Y} = \left\{ \left( \tilde{Y}_j, H_j, \chi_j \right) \right\}_{j \in J}, \quad \mathcal{Z} = \left\{ \left( \tilde{Z}_l, K_l, \eta_l \right) \right\}_{l \in L}$$

for $X$, $Y$ and $Z$ respectively. Let us also fix any set of morphisms

$$[\tilde{f}^m] := \left( f, \overline{f}^m, \left\{ \overline{f}^m_i \right\}_{i \in I}, [P_{f^m}, \nu_{f^m}] \right) : \mathcal{X} \to \mathcal{Y} \quad \text{for} \quad m = 1, 2,$$

$$[\tilde{g}^m] := \left( g, \overline{g}^m, \left\{ \overline{g}^m_j \right\}_{j \in J}, [P_{g^m}, \nu_{g^m}] \right) : \mathcal{Y} \to \mathcal{Z} \quad \text{for} \quad m = 1, 2.$$
Moreover, let us suppose that we have fixed any 2-morphism \([\bar{\delta}] : [\bar{f}^1] \Rightarrow [\bar{f}^2]\) and any 2-morphism \([\xi] : [\bar{g}^1] \Rightarrow [\bar{g}^2]\). Our aim is to define an horizontal composition \([\xi] * [\bar{\delta}] : [\bar{g}^1] \circ [\bar{f}^1] \Rightarrow [\bar{g}^2] \circ [\bar{f}^2]\). In order to do that, we fix any representative \((P_{g^1}, \nu_{g^1})\) for \([P_{g^1}, \nu_{g^1}]\) and any representative

\[ \delta := \left\{ (\tilde{X}^i_a, \tilde{\delta}^i_a) \right\}_{i \in I, \pi \in A(i)} \]

for \([\bar{\delta}]\). For any \(i \in I\) and any \(\pi \in A(i)\) we have that \(\delta^i_\pi \in Ch(\mathcal{Y})\). Since \(P_{g^1}\) satisfies condition \((CS)\), then as in the previous constructions we can use \(\delta\) in order to get another representative

\[ \delta := \left\{ (\tilde{X}^i_a, \delta^i_a) \right\}_{i \in I, \pi \in A(i)} \]

for \([\bar{\delta}]\), such that for each \(i \in I\) and \(a \in A(i)\), the change of charts \(\delta^i_a\) is the restriction of a change of charts \(\delta^i_\pi \in P_{g^1}\) (in general, such a change of charts is not unique, we fix any arbitrary choice of \(\delta^i_a\)’s with this property). We choose also any representative \(\xi := \{(\hat{Y}^j_\xi, \xi^j_\xi)\}_{\xi \in I, \xi \in C(\mathcal{Z}^2(i))}\) for \([\xi]\). Let us fix any \(i \in I, a \in A(i), c \in C(\mathcal{Z}^2(i))\) and any point \(\pi_i \in \tilde{X}_i\) such that the point \(\pi_i := \tilde{\gamma}_{\mathcal{Z}^1(i)} \circ \tilde{f}^1_i(\pi_i)\) belongs to the set

\[ \tilde{Z}_{\pi_i} := \nu^1_\pi(\delta^i_\pi)^{-1} \right\} \left( \text{dom} \nu^i_\pi(\delta^i_\pi) \right). \]

Then there exists a (non-unique) open connected subset \(\tilde{Z}_{\pi_i} \subseteq \tilde{Z}_{\pi_i}^a\), such that:

- \(\tilde{Z}_{\pi_i} \subseteq \tilde{Z}_{\pi_i}^a\);
- \(\tilde{Z}_{\pi_i}^a\) is invariant under the action of \(\text{Stab}(H_{\mathcal{Z}^1(i)}, \pi_i)\);
- for all \(h \in H_{\mathcal{Z}^1(i)} \setminus \text{Stab}(H_{\mathcal{Z}^1(i)}, \pi_i)\) we have \(h(\tilde{Z}_{\pi_i}^a) \cap \tilde{Z}_{\pi_i}^a = \emptyset\)

(in this way, \(\nu^1_\pi(\delta^i_\pi)\) is again a change of charts if restricted to \(\tilde{Z}_{\pi_i}^a\)). We define also

\[ \tilde{X}_{\pi_i}^a := \left( \tilde{X}_{\pi_i} \cap (\tilde{g}^1_{\mathcal{Z}^1(i)} \circ \tilde{f}^1_i)^{-1} \right) \left( \tilde{Z}_{\pi_i}^a \right), \]

\[ \tilde{X}_{\pi_i}^a \cap \tilde{Z}_{\pi_i}^a \cap (\tilde{g}^1_{\mathcal{Z}^1(i)} \circ \tilde{f}^1_i)^{-1} \left( \tilde{Z}_{\pi_i}^a \right). \]

For each \((i, a, c)\) as before, we choose any set of points \(\{\pi_i^e\}_{e \in E(i, a, c)}\) such that the corresponding sets of the form \(\tilde{X}_{\pi_i}^a\), \(\tilde{Z}_{\pi_i}^a\) are a covering for \(\tilde{X}_{\pi_i}^a\) (if \(\tilde{X}_{\pi_i}^a\) is empty, we set \(E(i, a, c) := \emptyset\)). For simplicity, we rename each \(\tilde{X}_{\pi_i}^a\) as \(\tilde{X}_{\pi_i}^a\) and analogously for the sets of the form \(\tilde{Z}_{\pi_i}^a\). Then for each \(i \in I, (a, c) \in A(i) \times C(\mathcal{Z}^2(i))\) and \(e \in E(i, a, c)\) we define

\[ \gamma^a_{i, e} := \xi^a_{\mathcal{Z}^1(i)} \circ \nu^1_\pi(\delta^i_\pi) \mid \tilde{Z}_{\pi_i}^a,\}

and we set

\[ \gamma := \left\{ (\tilde{X}_{\pi_i}^a, \gamma^a_{i, e}) \right\}_{i \in I, (a, c) \in A(i) \times C(\mathcal{Z}^2(i)), e \in E(i, a, c)}. \]

A direct check proves that:

**Lemma 2.5.** The collection \(\gamma\) so defined is a representative of a 2-morphism from \([\bar{g}^1] \circ [\bar{f}^1]\) to \([\bar{g}^2] \circ [\bar{f}^2]\). Moreover, the class of \(\gamma\) does not depend on the representatives \((P_{g^1}, \nu_{g^1}), \delta\) and \(\xi\) chosen for \([P_{g^1}, \nu_{g^1}], [\bar{\delta}]\) and \([\bar{\xi}]\) respectively.
So it makes sense to give the following definition.

**Definition 2.6.** Given any pair $[\delta], [\xi]$ as before, we define their *horizontal composition* as:

$$[\xi] \ast [\delta] := [\gamma] : [\hat{g}^1] \circ [\hat{f}^1] \Longrightarrow [\hat{g}^2] \circ [\hat{f}^2].$$

3. The 2-category $(\text{Red Atl})$

**Definition 3.1.** Given any reduced orbifold atlas $X = \{ (\tilde{X}_i, G_i, \pi_i) \}_{i \in I}$ on a topological space $X$, we define the identity of $X$ as the morphism $\text{id}_X := \left( \text{id}_{\tilde{X}_i}, \text{id}_{G_i}, \{ \text{id}_{\tilde{X}_i} \}_{i \in I}, \text{Ch}(X), \nu_{\text{id}} \right) : X \rightarrow X$ where $\nu_{\text{id}}$ is the identity on $\text{Ch}(X)$ (this is a special case of Definition 1.13). Given any pair of reduced orbifold atlases $X$ and $Y$ and any morphism $[\hat{f}] = (f, \hat{f}_i, \{ P_f, \nu_f \})$ from $X$ to $Y$, we define the 2-identity $i_{[\hat{f}]}$ as the class of $\{ (\tilde{X}_i, \text{id}_{\tilde{X}_i}) \}_{i \in I}$.

Moreover, for each reduced orbifold atlas $X$, we set $i_X := i_{\text{id}_X}$.

A direct check proves that:

**Lemma 3.2.**

(a) The morphisms and 2-morphisms of the form $\text{id}_-$ and $i_-$ are the identities with respect to $\circ$ and $\circ \circ$ respectively. Moreover, any 2-morphism is invertible with respect to $\circ$.

(b) Let us fix any pair of morphisms of reduced orbifolds $[\hat{f}^1], [\hat{f}^2] : X \rightarrow Y$ and any 2-morphism $[\delta] : \iota_Y \circ [\hat{f}^1] \Rightarrow \iota_Y \circ [\hat{f}^2]$ (where $\iota_Y$ is the inclusion $Y \hookrightarrow Y^{\max}$, see Definition 1.13). Then there is a unique 2-morphism $[\delta'] : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ such that $[\delta] = i_{\iota_Y} \ast [\delta']$.

The following proof is long but completely straightforward, so we omit it.

** Lemma 3.3.** Given any diagram as follows

![Diagram](image)

we have

$$\left( [\eta] \circ [\xi] \right) \ast \left( [\sigma] \circ [\delta] \right) = \left( [\eta] \ast [\sigma] \right) \circ \left( [\xi] \ast [\delta] \right).$$

**Proposition 3.4.** The definitions of reduced orbifold atlases, morphisms and 2-morphisms, compositions $\circ, \circ \circ, \ast$ and identities give rise to a 2-category, that we denote by $(\text{Red Atl})$.

**Proof.** In order to construct a 2-category, we define some data as follows.

1. The class of objects is the set of all the reduced orbifold atlases $\mathcal{X}$ for any paracompact, second countable, Hausdorff topological space $X$.!
2. If $\mathcal{X}$ and $\mathcal{Y}$ are reduced atlases for $X$ and $Y$ respectively, we define a small category $(\text{Red.Atl})(\mathcal{X}, \mathcal{Y})$ as follows: the space of objects is the set of all morphisms $[f]: \mathcal{X} \to \mathcal{Y}$ over any continuous map $f: X \to Y$; for any pair of morphisms $[f]$ and $[g]$ over $f$ and $g$ respectively, using Definition 1.18 and 1.21 we set:

$$\left( (\text{Red.Atl})(\mathcal{X}, \mathcal{Y}) \right)([f], [g]) := \begin{cases} \text{all 2-morphisms } [f] \Rightarrow [g] & \text{if } f = g, \\ \emptyset & \text{otherwise.} \end{cases}$$

The composition in any such category is the vertical composition $\circ$, that is clearly associative; the identity over any object $[f]$ is given by $i_{[f]}$. By Lemma 3.2(a) we get that actually any such category is an internal groupoid in $(\text{Sets})$, i.e. a category where all the morphisms are invertible.

3. For every triple of reduced atlases $\mathcal{X}, \mathcal{Y}, \mathcal{Z}$, we define a functor “composition”

$$(\text{Red.Atl})(\mathcal{X}, \mathcal{Y}) \times (\text{Red.Atl})(\mathcal{Y}, \mathcal{Z}) \to (\text{Red.Atl})(\mathcal{X}, \mathcal{Z})$$

as $\circ$ on any pair of morphisms and as $*$ on any pair of 2-morphisms. We want to prove that this gives rise to a functor. It is easy to see that identities are preserved, so one needs only to prove that compositions are preserved, i.e. that the interchange law (see [Bo, Proposition 1.3.5]) is satisfied. This is exactly the statement of Lemma 3.3.

All the other necessary proofs that $(\text{Red.Atl})$ is a 2-category are trivial, so we omit them. \hfill $\Box$

4. From reduced orbifold atlases to proper, effective, étale groupoids

The aim of this section is to define a 2-functor $\mathcal{F}^{\text{red}}$ from $(\text{Red.Atl})$ to the 2-category of proper, effective, étale Lie groupoids. We recall briefly the necessary definitions and notations.

Definition 4.1. [Ler] Definition 2.11] A Lie groupoid is the datum of 2 smooth (Hausdorff, paracompact) manifolds $\mathcal{X}_0, \mathcal{X}_1$ and five smooth maps:

- $s, t : \mathcal{X}_1 \rightrightarrows \mathcal{X}_0$, such that both $s$ and $t$ are submersions (so that the fiber products of the form $\mathcal{X}_1 \times_s \cdots \times_s \mathcal{X}_1$ (for finitely many terms) are manifolds); these 2 maps are usually called source and target of the Lie groupoid;
- $m : \mathcal{X}_1 \times_s \mathcal{X}_1 \to \mathcal{X}_1$, called multiplication;
- $\iota : \mathcal{X}_1 \to \mathcal{X}_1$, known as inverse of the Lie groupoid;
- $e : \mathcal{X}_0 \to \mathcal{X}_1$, called identity;

which satisfy the following axioms:

1. $s \circ e = 1_{\mathcal{X}_0} = t \circ e$;
2. if we denote by $p_{1}$ and $p_{2}$ the 2 projections from $\mathcal{X}_1 \times_s \mathcal{X}_1$ to $\mathcal{X}_1$, then we have $s \circ m = s \circ p_{1}$ and $t \circ m = t \circ p_{2}$;
3. the 2 morphisms $m \circ (1_{\mathcal{X}_1} \times m)$ and $m \circ (m \times 1_{\mathcal{X}_1})$ from $\mathcal{X}_1 \times_s \mathcal{X}_1 \times_s \mathcal{X}_1$ to $\mathcal{X}_1$ are equal;
4. the 2 morphisms $m \circ (e \circ s, 1_{\mathcal{X}_1})$ and $m \circ (1_{\mathcal{X}_1}, e \circ t)$ from $\mathcal{X}_1$ to $\mathcal{X}_1$ are both equal to the identity of $\mathcal{X}_1$;
5. $\iota \circ \iota = 1_{\mathcal{X}_1}, s \circ \iota = t$ (and therefore $t \circ \iota = s$); moreover, we require that $m \circ (1_{\mathcal{X}_1}, \iota) = e \circ s$ and $m \circ (\iota, 1_{\mathcal{X}_1}) = e \circ t$.

In other terms, a Lie groupoid is an internal groupoid in the category of smooth manifolds, such that $s$ and $t$ are submersions. For simplicity, we will denote any Lie groupoid as before by $(\mathcal{X}_1 \rightrightarrows \mathcal{X}_0)$ or $\mathcal{X}_$. In the literature one can also find the
notations \((\mathcal{X}_0, \mathcal{X}_1), (U, R, s, t, m, e, i)\) and \(R_s \times_t R \xrightarrow{m} R \xrightarrow{i} R_1 \xrightarrow{\psi_1} U \xrightarrow{\psi_0} R\) (where \(U\) is the set \(\mathcal{X}_0\) and \(R\) is the set \(\mathcal{X}_1\) in our notations).

In the following pages, even if we will deal with several Lie groupoids, we will denote by \(s\) the source morphism of any such object, and analogously for the morphisms \(t, m, e\) and \(i\). This will not create any problem, since it will be always clear from the context what is the Lie groupoid we are working with.

**Definition 4.2.** \([\mathbf{M}]\) § 2.1] Given 2 Lie groupoids \(\mathcal{X}_\bullet\) and \(\mathcal{Y}_\bullet\), a morphism between them is any pair \(\psi_\bullet = (\psi_0, \psi_1)\), where \(\psi_0 : \mathcal{X}_0 \to \mathcal{Y}_0\) and \(\psi_1 : \mathcal{X}_1 \to \mathcal{Y}_1\) are smooth maps, which together commute with all structure morphisms of the 2 Lie groupoids.

In other words, we require that \(s \circ \psi_1 = \psi_0 \circ s, t \circ \psi_1 = \psi_0 \circ t, \psi_0 \circ e = e \circ \psi_0, \psi_1 \circ m = m \circ (\psi_1 \times \psi_1)\) and \(\psi_1 \circ i = i \circ \psi_1\).

**Definition 4.3.** \([\mathbf{PS}]\) Definition 2.3] Let us suppose that we have fixed 2 morphisms of Lie groupoids \(\psi_m : \mathcal{X}_\bullet \to \mathcal{Y}_\bullet\) for \(m = 1, 2\). Then a natural transformation (also known as 2-morphism) \(\alpha : \psi_1 \Rightarrow \psi_2\) is the datum of any smooth map \(\alpha : \mathcal{X}_0 \to \mathcal{Y}_1\), such that the following conditions hold:

\[
\begin{align*}
\text{(NT1)} & \quad s \circ \alpha = \psi_1^0 \quad \text{and} \quad t \circ \alpha = \psi_2^0; \\
\text{(NT2)} & \quad m \circ (\alpha \circ s, \psi_1^1) = m \circ (\psi_2^1, \alpha \circ t).
\end{align*}
\]

There are well-known notions of identities, compositions of morphisms, vertical and horizontal compositions of natural transformations, obtained in analogy with the corresponding notions in the 2-category of small categories. In particular, we have:

**Proposition 4.4.** \([\mathbf{PS}]\) § 2.1] The data of Lie groupoids, morphisms, and natural transformations between them (together with compositions and identities) form a 2-category, known as \(\mathbf{LieGpd}\).

**Definition 4.5.** \([\mathbf{M}]\) § 1.2 and § 1.5] A Lie groupoid \(\mathcal{X}_\bullet\) is called proper if the map \((s, t) : \mathcal{X}_1 \to \mathcal{X}_0 \times \mathcal{X}_0\) is proper; it is called étale if the maps \(s\) and \(t\) are both étale (i.e. local diffeomorphisms). Since each étale map is a submersion, in general we will simply write “étale groupoid” instead of “étale Lie groupoid”.

**Remark 4.6.** Given any Lie groupoid \((\mathcal{X}_1 \xrightarrow{\psi_1} \mathcal{X}_0)\), we can define an equivalence relation \(\sim_\mathcal{X}\) on \(\mathcal{X}_0\) by saying that \(x_0 \sim_\mathcal{X} x'_0\) if and only if there is \(x_1 \in \mathcal{X}_1\) such that \(s(x_1) = x_0\) and \(t(x_1) = x'_0\). We give to the set \(|\mathcal{X}_0| := \mathcal{X}_0/\sim_\mathcal{X}\) the quotient topology and we call it the underlying topological space of \(\mathcal{X}_\bullet\); we denote by \(pr_{\mathcal{X}_\bullet} : \mathcal{X}_0 \to |\mathcal{X}_\bullet|\) the quotient map. Given another Lie groupoid \((\mathcal{Y}_1 \xrightarrow{\psi_1} \mathcal{Y}_0)\) and any morphism \((\psi_0, \psi_1) : (\mathcal{X}_1 \xrightarrow{\psi_1} \mathcal{X}_0) \to (\mathcal{Y}_1 \xrightarrow{\psi_1} \mathcal{Y}_0)\), there is a unique set map \(|\psi_\bullet| : |\mathcal{X}_\bullet| \to |\mathcal{Y}_\bullet|\) (called the underlying set map of \(\psi_\bullet\)), making the following diagram commute

\[
\begin{array}{ccc}
\mathcal{X}_0 & \xrightarrow{\psi_0} & \mathcal{Y}_0 \\
pr_{\mathcal{X}_\bullet} \downarrow & & \downarrow pr_{\mathcal{Y}_\bullet} \\
|\mathcal{X}_\bullet| & \xrightarrow{|\psi_\bullet|} & |\mathcal{Y}_\bullet|.
\end{array}
\]

Such a map is defined by \(|\psi_\bullet|(pr_{\mathcal{X}_\bullet}(x_0)) := pr_{\mathcal{Y}_\bullet} \circ \psi_0(x_0)\) for all \(x_0 \in \mathcal{X}_0\). Then \(|\psi_\bullet|\) is well-defined by definition of \(\sim_\mathcal{X}\) and \(\sim_\mathcal{Y}\). Since \(|\psi_\bullet|\) is the unique map making (4.1) commute, then given any pair of composable morphisms \(\psi_\bullet\) and \(\xi_\bullet\), we have \(|\xi_\bullet| \circ |\psi_\bullet| = |\xi_\bullet| \circ |\psi_\bullet|\).
If we assume that $\mathcal{X}$ and $\mathcal{Y}$ are proper and étale, then $\text{pr}_{\mathcal{X}}$ and $\text{pr}_{\mathcal{Y}}$ are open maps as a consequence of [Ler, Proposition 2.23] and the induced map $|\psi|_{\mathcal{X}}$ is continuous. Indeed, for any open set $A' \subseteq |\mathcal{Y}|$, we have that $\text{pr}_{\mathcal{X}}^{-1}(|\psi|_{\mathcal{X}}^{-1}(A')) = \psi^{-1}_0((\text{pr}_{\mathcal{Y}})^{-1}(A'))$ is open in $|\mathcal{X}|$; since $\text{pr}_{\mathcal{Y}}$ is surjective and open, then $|\psi|_{\mathcal{X}}^{-1}(A')$ is equal to the open set $\text{pr}_{\mathcal{X}}^{-1}(\psi^{-1}_0((\text{pr}_{\mathcal{Y}})^{-1}(A')))$ of $|\mathcal{X}|$.

**Remark 4.7.** Let $\mathcal{X}$ be a proper étale groupoid and let us fix any pair of points $x_0, x'_0 \in \mathcal{X}_0$. Since both $s$ and $t$ are étale, for every point $x_1$ in $\mathcal{X}_1$ such that $s(x_1) = x_0$ and $t(x_1) = x'_0$, we can find a sufficiently small open neighborhood $W_{x_1}$ of $x_1$ where both $s$ and $t$ are invertible. Then we can define a set map

$$t \circ (s|_{W_{x_1}})^{-1} : s(W_{x_1}) \rightarrow t(W_{x_1}).$$

Such a map is actually a diffeomorphism from an open neighborhood of $x_0$ to an open neighborhood of $x'_0$; moreover, it is easy to see that it commutes with the projection $\text{pr}_{\mathcal{X}}$. So for each pair of points $x_0, x'_0$ as above we can define a set map:

$$\kappa_{\mathcal{X}}(x_0, x'_0, -) : \{ x_1 \in \mathcal{X}_1 \st s(x_1) = x_0 \text{ and } t(x_1) = x'_0 \} \rightarrow \{ \text{germ}_{x_0} f \not\in \text{diff} \text{morphisms } f \text{ around } x_0 \text{ s. t. } f(x_0) = x'_0 \text{ and } \text{pr}_{\mathcal{X}} \circ f = \text{pr}_{\mathcal{X}} \}$$

by setting

$$\kappa_{\mathcal{X}}(x_0, x'_0, x_1) := \text{germ}_{x_0} (t \circ (s|_{W_{x_1}})^{-1}) = \text{germ}_{x_1} \cdot (\text{germ}_{x_1} s)^{-1}. \quad (4.3)$$

We claim that $\kappa_{\mathcal{X}}(x_0, x'_0, -)$ is surjective. For that, we have to consider 2 cases separately: if $\text{pr}_{\mathcal{X}}(x_0) \neq \text{pr}_{\mathcal{X}}(x'_0)$, then both the first and the second set in (4.2) are empty, so $\kappa_{\mathcal{X}}(x_0, x'_0, -)$ is a bijection. If $\text{pr}_{\mathcal{X}}(x_0) = \text{pr}_{\mathcal{X}}(x'_0)$, this means that there is a (in general non-unique) point $x_1 \in \mathcal{X}_1$, such that $s(x_1) = x_0$ and $t(x_1) = x'_0$. Let us fix any diffeomorphism $f$ as in the second line of (4.2). Let us denote by $W_{x_1}$ any open neighborhood of $x_1$ such that both $s$ and $t$ are invertible if restricted to such a set. Then the function

$$g := s \circ (t|_{W_{x_1}})^{-1} \circ f|_{f^{-1} \circ s(W_{x_1})}$$

is a diffeomorphism around $x_0$, it fixes $x_0$ and it commutes with $\text{pr}_{\mathcal{X}}$. As a simple consequence of [M, Theorem 2.3], there is a (in general non-unique) point $\tilde{x}_1$ in $\mathcal{X}_1$, such that $s(\tilde{x}_1) = x_0 = t(\tilde{x}_1)$ and $\kappa_{\mathcal{X}}(x_0, x'_0, \tilde{x}_1) = \text{germ}_{x_0} g$. This implies that

$$\text{germ}_{x_0} f = \kappa_{\mathcal{X}}(x_0, x'_0, x_1) \cdot \kappa_{\mathcal{X}}(x_0, x'_0, \tilde{x}_1) = \kappa_{\mathcal{X}}(x_0, x'_0, m(\tilde{x}_1, x_1)),$$

so we have proved that $\kappa_{\mathcal{X}}(x_0, x'_0, -)$ is surjective.

**Definition 4.8.** [Mi, example 1.5] Let us fix any proper, étale groupoid $\mathcal{X}$. We say that $\mathcal{X}$ is effective (or reduced) if $\kappa_{\mathcal{X}}(x_0, x_0, -)$ is injective for all $x_0 \in \mathcal{X}_0$.

**Lemma 4.9.** Let us fix any proper, effective, étale groupoid $\mathcal{X}$. Then the set map $\kappa_{\mathcal{X}}(x_0, x'_0, -)$ is a bijection for every pair of points $x_0, x'_0$ in $\mathcal{X}_0$.

**Proof.** Let us fix any pair of points $x_1, \tilde{x}_1 \in \mathcal{X}_1$ such that $s(x_1) = x_0 = s(\tilde{x}_1)$, $t(x_1) = x'_0 = t(\tilde{x}_1)$ and $\kappa_{\mathcal{X}}(x_0, x'_0, x_1) = \kappa_{\mathcal{X}}(x_0, x'_0, \tilde{x}_1)$. Then:

$$\kappa_{\mathcal{X}}(x_0, x'_0, m(x_1, i(\tilde{x}_1))) = (\kappa_{\mathcal{X}}(x_0, x'_0, \tilde{x}_1))^{-1} \cdot \kappa_{\mathcal{X}}(x_0, x'_0, x_1) = \text{germ}_{x_0} \cdot \text{id} = \kappa_{\mathcal{X}}(x_0, x_0, e(x_0));$$

since $\mathcal{X}$ is effective, this implies that $m(x_1, i(\tilde{x}_1)) = e(x_0)$, i.e. that $x_1 = \tilde{x}_1$, so we have proved that $\kappa_{\mathcal{X}}(x_0, x'_0, -)$ is injective; we have already said above that it is surjective (even without the hypothesis of effectiveness), so we conclude. \(\square\)
Definition 4.10. We define the 2-categories \((\mathcal{E} \mathfrak{Gpd})\), \((\mathcal{P}\mathcal{E} \mathfrak{Gpd})\) and \((\mathcal{P}\mathcal{E}\mathcal{E} \mathfrak{Gpd})\) as the full 2-subcategories of \((\operatorname{Lie} \mathfrak{Gpd})\) obtained by restricting the objects to étale groupoids, respectively to proper, étale Lie groupoids, respectively to proper, effective, étale groupoids (morphisms and 2-morphisms are simply restricted according to that).

Construction 4.11. (adapted from [Po, Construction 2.4] and from [Pr2 § 4.4]) Let us fix any reduced orbifold atlas \(X = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}\) of dimension \(n\). Then we define \(\mathcal{F}_0^{\text{red}}(X) := (\mathcal{I}_1 \xrightarrow{\iota} \mathcal{I}_0)\) as the following Lie groupoid.

- The manifold \(\mathcal{I}_0\) is defined as \(\bigcup_{i \in I} \tilde{X}_i\), with the natural smooth structure given by the fact that each \(\tilde{X}_i\) is an open subset of \(\mathbb{R}^n\).
- As a set, we define
  \[\mathcal{I}_1 := \{\text{germ}_{\tilde{X}_i} \lambda \mid \forall i \in I, \forall \lambda \in C(\mathcal{X}, i, -), \forall \tilde{x}_i \in \text{dom} \lambda\}.\]
  For each \(i \in I\) and for each \(\lambda \in C(\mathcal{X}, i, -)\) we set
  \[\mathcal{I}_1(\lambda) := \{\text{germ}_{\tilde{X}_i} \lambda \mid \forall \tilde{x}_i \in \text{dom} \lambda\} \subseteq \mathcal{I}_1.\]

Then the topological and differentiable structure on \(\mathcal{I}_1\) are given by the germ topology and by the germ differentiable structure. In other terms, we choose as charts for \(\mathcal{I}_1\) all the bijections of the form:

\[
\tau_{\lambda} : \begin{array}{ccc}
\mathcal{I}_1(\lambda) & \to & \text{dom} \lambda \subseteq \tilde{X}_i \subseteq \mathbb{R}^n \\
\text{germ}_{\tilde{X}_i} \lambda & \mapsto & \tilde{x}_i
\end{array}
\]

(4.4) for each \(i \in I\) and for each \(\lambda \in C(\mathcal{X}, i, -)\) (one needs only to show that any pair of charts \(\tau_{\lambda_1}, \tau_{\lambda_2}\) are compatible on their common domain, but this is easy). Therefore, by construction any morphism of the form \(\tau_{\lambda}\) is a diffeomorphism from \(\mathcal{I}_1(\lambda)\) to \(\text{dom} \lambda\).

- The structure maps are defined as follows:

\[
\begin{align*}
  s(\text{germ}_{\tilde{X}_i} \lambda) & := \tilde{x}_i, \\
  t(\text{germ}_{\tilde{X}_i} \lambda) & := \lambda(\tilde{x}_i), \\
  m(\text{germ}_{\tilde{X}_i} \lambda, \text{germ}_{\lambda(\tilde{x}_i)} \lambda') & := \text{germ}_{\lambda(\tilde{x}_i)} \lambda' \cdot \text{germ}_{\tilde{X}_i} \lambda, \\
  i(\text{germ}_{\tilde{X}_i} \lambda) & := \text{germ}_{\tilde{X}_i} \lambda^{-1} \cdot \text{id}_{\tilde{X}_i}, \\
  e(\tilde{x}_i) & := \text{germ}_{\tilde{X}_i} \text{id}_{\tilde{X}_i}.
\end{align*}
\]

A direct check proves that \(s\) and \(t\) are both étale, that \(m, e, i\) are smooth and that axioms \([\text{LG1}] - [\text{LG5}]\) are satisfied, so \(\mathcal{F}_0^{\text{red}}(X)\) is an étale groupoid.

Lemma 4.12. For every reduced orbifold atlas \(X\), the étale groupoid \(\mathcal{F}_0^{\text{red}}(X)\) is proper and effective, i.e. it belongs to \((\mathcal{P}\mathcal{E}\mathcal{E} \mathfrak{Gpd})\).

Proof. For each \(x_0 \in \mathcal{I}_0\) we have to prove that the map \(\kappa_{\mathcal{X}}(x_0, x_0, -)\) described in Remark [4.7] is injective. By definition of \(\mathcal{I}_0\), each such \(x_0\) is equal to \(\tilde{x}_i \in \tilde{X}_i\) for some \(i \in I\). Moreover, any \(x_1 \in \mathcal{I}_1\) such that \(s(x_1) = x_0 = t(x_1)\) is necessarily equal to \(\text{germ}_{\tilde{X}_i} \lambda\) for some \(\lambda \in C(\mathcal{X}, i, i)\) with \(\tilde{x}_i \in \text{dom} \lambda\) and \(\lambda(\tilde{x}_i) = \tilde{x}_i\). For any such \(\lambda\), we have

\[
\kappa_{\mathcal{X}}(\tilde{x}_i, \tilde{x}_i, \text{germ}_{\tilde{X}_i} \lambda) = \text{germ}_{\tilde{X}_i} \left( t \circ s \mid_{\mathcal{X}_i(\lambda)} \right)^{-1} = \text{germ}_{\tilde{X}_i} \lambda,
\]

so \(\kappa_{\mathcal{X}}(\tilde{x}_i, \tilde{x}_i, -)\) is injective, hence \(\mathcal{F}_0^{\text{red}}(X)\) is effective. We need only to prove that \((s, t) : \mathcal{I}_1 \to \mathcal{I}_0 \times \mathcal{I}_0\) is proper. Let us fix any compact set \(K \subseteq \mathcal{I}_0 \times \mathcal{I}_0\) and let \(\{q^n\}_{m \in \mathbb{N}}\) be any sequence in \((s, t)^{-1}(K) \subseteq \mathcal{I}_1\). Up to passing to a subsequence we can always assume that the sequence \((s, t)(q^m) \in K\) converges to some point \((\tilde{x}_i, \tilde{x}_i) \in \tilde{X}_i \times \tilde{X}_i\). So there is \(m_1\) such that for each \(m > m_1\) we have...
(s, t)(q^m) ∈ X \times \tilde{X}_\nu$, hence we can write $(s, t)(q^m) =: (\tilde{x}^m_i, \tilde{x}^m_\nu)$. By definition of \( X \times \tilde{X}_\nu \), we have associated to each object of \( \mathcal{X}_1 \), for \( m > m_1 \) there is a (non-unique) change of charts \( \lambda^m \) from \( (X_i, \tilde{X}_i, \pi_i) \) to \( (\tilde{X}_\nu, G_\nu, \pi_\nu) \), such that

\[
\tilde{x}^m_i \in \text{dom} \lambda^m, \quad q^m = \text{germ}_{\tilde{x}^m_i} \lambda^m, \quad \lambda^m(\tilde{x}^m_i) = x^m_\nu.
\]

From this, we get

\[
\pi_i(\tilde{x}^m_i) = \pi_\nu \circ \lambda^m(\tilde{x}^m_i) = \pi_\nu(x^m_\nu) \quad \forall m > m_1.
\]

By considering the limit for \( m \to \infty \), we get that \( \pi_i(\tilde{x}_i) = \pi_\nu(x_\nu) \). Since \( \mathcal{X} \) is a reduced orbifold atlas, then there exists a change of charts \( \lambda \) from \( X_i, G_i, \pi_i \) to \( (\tilde{X}_\nu, G_\nu, \pi_\nu) \), such that \( \tilde{x}_i \in \text{dom} \lambda \). So there is \( m_2 \geq m_1 \), such that for all \( m > m_2 \) we have \( \tilde{x}^m_i \in \text{dom} \lambda \). Then for each such \( m \) there exists a chart \( (X^m, G^m, \pi_i | \tilde{x}^m_i) \) such that \( \tilde{x}^m_i \subset X^m \cap \text{dom} \lambda \). For each \( m > m_2 \) we have that both \( \lambda^m \) and \( \lambda^m \) (suitably restricted) can be considered as embeddings from \( (X^m, G^m, \pi_i | \tilde{x}^m_i) \) to \( (\tilde{X}_\nu, G_\nu, \pi_\nu) \). So by [MP] Lemma A.1 there exists a unique \( y^m \in G_\nu \), such that \( \lambda^m | \tilde{x}^m_i = g^m \circ \lambda | \tilde{x}^m_i \). Since \( G_\nu \) is a finite set, then after passing to a subsequence we can assume that \( g^m \) is the same for all \( m > m_2 \); we denote such a map by \( g \). Then by definition of the differentiable structure on \( \mathcal{X}_1 \) we have

\[
\lim_{m \to +\infty} q^m = \lim_{m \to +\infty} \text{germ}_{\tilde{x}^m_i} \lambda^m = \lim_{m \to +\infty} \text{germ}_{\tilde{x}^m_i} g \circ \lambda = \text{germ}_{\tilde{x}_i} g \circ \lambda.
\]

So this proves that \( (s, t)^{-1}(K) \) is compact, so \( (s, t) \) is proper. \( \square \)

Until now we have associated to each object of \( (\text{Red.Atl}) \) an object of \( (\mathcal{PE}\mathcal{E}\mathcal{G}\mathcal{pd}) \); we want to do the same for morphisms and 2-morphisms.

**Construction 4.13.** (adapted from [Po] Proposition 4.7) Let us fix any pair of reduced orbifold atlases \( \mathcal{X} := \{(X_i, G_i, \pi_i)\}_{i \in I} \) and \( \mathcal{Y} := \{(Y_j, H_j, \chi_j)\}_{j \in J} \) for \( X \) and \( Y \) respectively and any morphism \( [\tilde{f}] : \mathcal{X} \to \mathcal{Y} \) with representative given by

\[
\tilde{f} := ([f], \overline{\{f_i\}_{i \in I}}, P_f, \nu_f).
\]

We set

\[
\mathcal{X}_0^\text{red}(\mathcal{X}) := \left( \mathcal{X}_0 \to \mathcal{X}_1 \right) \quad \text{and} \quad \mathcal{X}_0^\text{red}(\mathcal{Y}) := \left( \mathcal{Y}_0 \to \mathcal{Y}_1 \right),
\]

where

\[
\mathcal{X}_0 := \prod_{i \in I} \tilde{X}_i, \quad \mathcal{X}_1 := \left\{ \text{germ}_{\tilde{x}_i} \lambda \, \forall i \in I, \, \forall \lambda \in \text{Ch}(\mathcal{X}_i, \nu_i, -), \, \forall \tilde{x}_i \in \text{dom} \lambda \right\},
\]

\[
\mathcal{Y}_0 := \prod_{j \in J} \tilde{Y}_j, \quad \mathcal{Y}_1 := \left\{ \text{germ}_{\tilde{y}_j} \omega \, \forall j \in J, \, \forall \omega \in \text{Ch}(\mathcal{Y}_j, \chi_j, -), \, \forall \tilde{y}_j \in \text{dom} \omega \right\}.
\]

Then we define a set map \( \psi_0 : \mathcal{X}_0 \to \mathcal{Y}_0 \) as

\[
\psi_0|_{\tilde{x}_i} := \tilde{f}_i : \tilde{X}_i \to \tilde{Y}_j(\tilde{x}_i) \subseteq \mathcal{Y}_0
\]

for all \( i \in I \). Now let \( x_i \) be any point in \( \tilde{X}_i \) and let \( \tilde{x}_i := s(x_i) \in \tilde{X}_i \) for some \( i \in I \). Since \( P_f \) is a good subset of \( \text{Ch}(\mathcal{X}_i) \), then there is a (non-unique) \( \lambda \in P_f(i, -) \) such that \( x_i = \text{germ}_{\tilde{x}_i} \lambda \). We set

\[
\psi_1(x_i) := \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \nu_f(\lambda) \in \mathcal{Y}_1.
\]
If \( \lambda' \) is another element of \( P_f(i, -) \) such that \( x_1 = \text{germ}_{x_1} \lambda' \), then property \( \text{(M5d)} \) for \( \hat{f} \) proves that \( \text{germ}_{\hat{f}(\tilde{x}_1)} \psi_f(\lambda') = \text{germ}_{\hat{f}(\tilde{x}_1)} \nu_f(\lambda') \), so \( \psi_1 \) is a well-defined set map from \( \mathcal{X} \) to \( \mathcal{Y} \). A direct check proves that both \( \psi_0 \) and \( \psi_1 \) are smooth and that the pair \( (\psi_0, \psi_1) \) satisfies Definition \( \text{NT}1 \) so it is a morphism of Lie groupoids from \( \mathcal{F}_0^{\text{red}}(\mathcal{X}) \) to \( \mathcal{F}_0^{\text{red}}(\mathcal{Y}) \). Now let us suppose that

\[
\hat{f}' := \left( f, \mathcal{T}, \{ \hat{f}'_i \}_{i \in I}, P_{f}', \nu_f' \right)
\]

is another representative for \( [\hat{f}] \). Then by Definition \( \text{NT1} \) we get that the morphism from \( \mathcal{F}_0^{\text{red}}(\mathcal{X}) \) to \( \mathcal{F}_0^{\text{red}}(\mathcal{Y}) \) associated to \( \hat{f}' \) coincides with \( (\psi_0, \psi_1) \). Therefore it makes sense to set

\[
\mathcal{F}_1^{\text{red}}([\hat{f}]) := (\psi_0, \psi_1) : \mathcal{F}_0^{\text{red}}(\mathcal{X}) \longrightarrow \mathcal{F}_0^{\text{red}}(\mathcal{Y}).
\]

**Construction 4.14.** Now let us fix any pair of atlases \( \mathcal{X} \) and \( \mathcal{Y} \) for \( X \) and \( Y \) respectively and any pair of morphisms \( [\hat{f}^1], [\hat{f}^2] : \mathcal{X} \to \mathcal{Y} \) over a continuous function \( f : X \to Y \), with representatives

\[
\hat{f}^m := \left( f, \mathcal{T}^m, \{ \hat{f}^m_i \}_{i \in I}, P_{f}^m, \nu_f^m \right) \quad \text{for } m = 1, 2.
\]

Let us also fix any 2-morphism \( [\delta] : [\hat{f}^1] \Rightarrow [\hat{f}^2] \) and any representative

\[
\delta := \left\{ \left( \tilde{X}_i^a, \delta_i^a \right) \right\}_{i \in I, a \in A(i)}
\]

for it. Let us set

\[
\mathcal{F}_0^{\text{red}}(\mathcal{X}) =: \left( \mathcal{X}_1 \overset{s_1}{\rightarrow} \mathcal{X}_0 \right), \quad \mathcal{F}_0^{\text{red}}(\mathcal{Y}) =: \left( \mathcal{Y}_1 \overset{s_1}{\rightarrow} \mathcal{Y}_0 \right),
\]

\[
\mathcal{F}_1^{\text{red}}([\hat{f}^m]) =: (\psi_0^m, \psi_1^m) \quad \text{for } m = 1, 2.
\]

Then let us define a set map \( \hat{\delta} : \mathcal{X}_0 = \bigsqcup_{i \in I} \tilde{X}_i \to \mathcal{Y}_1 \) as

\[
\hat{\delta}(\tilde{x}_i) := \text{germ}_{\hat{\delta}_i(\tilde{x}_i)} \delta_i^a
\]

for every \( i \in I \), for every \( a \in A \) and for every \( \tilde{x}_i \in \tilde{X}_i^a \); this is well-defined by property \( \text{(2Ma)} \) for \( \delta \). We claim that \( \hat{\delta} \) is a natural transformation from \( (\psi_0, \psi_1) \) to \( (\psi_0^m, \psi_1^m) \). Clearly \( \hat{\delta} \) is smooth, indeed on each open subset of \( \mathcal{X}_0 \) of the form \( \tilde{X}_i^a \) we have that \( \hat{\delta} \) coincides with the composition of \( \hat{f}_i^1 \) (that is smooth by definition of local lift) and of the inverse of the chart \( \tau_{\tilde{x}_i} \) for \( \mathcal{Y}_1 \) (see \( \text{(4.4)} \)). Moreover, let us fix any \( i \in I \), any \( a \in A(i) \) and any \( \tilde{x}_i \in \tilde{X}_i^a \). Then

\[
s \circ \hat{\delta}(\tilde{x}_i) = s \left( \text{germ}_{\hat{\delta}_i(\tilde{x}_i)} \delta_i^a \right) = \hat{f}_i^1(\tilde{x}_i) = \psi_0(\tilde{x}_i),
\]

\[
t \circ \hat{\delta}(\tilde{x}_i) = t \left( \text{germ}_{\hat{\delta}_i(\tilde{x}_i)} \delta_i^a \right) = \delta_i^a \circ \hat{f}_i^1(\tilde{x}_i) \overset{\text{2Ma}}{=} \hat{f}_i^2(\tilde{x}_i) = \psi_1(\tilde{x}_i),
\]

so \( \hat{\delta} \) satisfies axiom \( \text{NT}1 \). Now let us fix any point \( x_1 \in \mathcal{X}_1 \) and let us set \( \tilde{x}_i := s(x_1) \), \( \pi_i := t(x_1) \) for a unique pair \( (i, i') \in I \times I \). Since both \( P_{f_1} \) and \( P_{f_2} \) are good subsets of \( \text{Ch}(\mathcal{X}) \), then for each \( m = 1, 2 \) there exists \( \lambda^m \in P_{f}^m(i, i') \) such that \text{germ}_{x_1} \lambda^m = x_1 \). By property \( \text{2Ma} \) there exist \( a \in A(i) \) and \( a' \in A(i') \) such that \( \tilde{x}_i \in \tilde{X}_i^a \) and \( \pi_i \in \tilde{X}_{i'}^{a'} \). Then:

\[
(m \circ (\hat{\delta} \circ s, \psi_1^2))(x_1) = m \left( \hat{\delta}(\tilde{x}_i), \psi_1^2 \left( \text{germ}_{x_1} \lambda^2 \right) \right) \overset{\text{4.5a}}{=}
\]
m \left( \text{germ}_{f_i(z)} \delta_i^a, \text{germ}_{f_i(z)} \nu_i^2(\lambda^2) \right) = \text{germ}_{f_i(z)} \nu_i^2(\lambda^2) \cdot \text{germ}_{f_i(z)} \delta_i^a \quad (\text{LT})

\text{germ}_{f_i(\lambda(z))} \delta_i^a \cdot \text{germ}_{f_i(z)} \nu_i^1(\lambda^1) = m \left( \psi_1^i \left( \text{germ}_{x_i} \lambda^1 \right), \tilde{\delta} \left( \lambda^1(\tilde{x}_i) \right) \right) =

= \left( m \circ \left( \psi_1^i, \tilde{\delta} \circ \tau \right) \right) (x_1).

So \tilde{\delta} satisfies also axiom (NT2); therefore \tilde{\delta} is a natural transformation from \( F^1 \) to \( F^2 \). By Definition 1.21 we get that \( \tilde{\delta} \) depends only on \( \delta \) and not on the representative \( \delta \) chosen for that class. So it makes sense to set:

\[ F^2_2(\delta) := \tilde{\delta} : F^2_2(\tilde{f}) = \Rightarrow F^2_2(\tilde{f}^2). \]

A direct check proves that:

**Lemma 4.15.** For every pair of composable morphisms \([\tilde{f}] : \mathcal{X} \to \mathcal{Y} \), \([\tilde{g}] : \mathcal{Y} \to \mathcal{Z} \) we have \( F^1_2([\tilde{g}] \circ [\tilde{f}]) = F^1_2([\tilde{g}]) \circ F^1_2([\tilde{f}]) \). For every reduced orbifold atlas \( \mathcal{X} \) we have \( F^1_2(\text{id}_X) = \text{id}_F^1 \) for every morphism \([\tilde{f}] \) between reduced orbifold atlases we have \( F^1_2(i_{[\tilde{f}]}) = \tilde{f} \).

**Lemma 4.16.** Let us fix any diagram in \((\text{Red Atl})\) as follows:

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\tilde{f}} & \mathcal{Y} \\
\downarrow \delta & \iff & \downarrow \sigma \\
\mathcal{X} & \xrightarrow{\tilde{f}} & \mathcal{Y} \\
\end{array}
\]

Then \( F^2_2(\sigma) \circ [\delta] = F^2_2([\sigma]) \circ F^2_2([\delta]) \).

**Proof.** Let us set representatives

\[ \tilde{f}^m := \left( f, f^m, \left\{ \tilde{f}_i^m \right\}, \right) \quad \text{for } m = 1, 2, 3 \]

for each \( \tilde{f}^m \) and representatives

\[ \delta = \left\{ \left( X_i^a, \delta_i^a \right) \right\}_{i \in I, a \in A(i)} \quad \text{and} \quad \sigma = \left\{ \left( X_i^b, \sigma_i^b \right) \right\}_{i \in I, b \in B(i)} \]

for \( \delta \) and \( \sigma \) respectively, as in Construction 2.1. Then let us set \( F^2_2([\delta]) := \tilde{\delta} \) and \( F^2_2([\sigma]) := \tilde{\sigma} \); let us fix any \( i \in I \), any \( (a,b) \in A(i) \times B(i) \) and any \( \tilde{x}_i \in X_{i,a}^a := \tilde{X}_{i,a}^a \cap \tilde{X}_{i,b}^b \). Then

\[ \left( F^2_2([\sigma]) \circ [\delta] \right)(\tilde{x}_i) = \left( \tilde{\sigma} \circ \tilde{\delta} \right)(\tilde{x}_i) \overset{(\ast)}{=} m \left( \tilde{\delta}(\tilde{x}_i), \tilde{\sigma}(\tilde{x}_i) \right) =
\]

\[ = m \left( \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \delta_i^a, \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \sigma_i^b \right) = \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \sigma_i^b \cdot \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \delta_i^a =
\]

\[ = \text{germ}_{\tilde{f}_i(\tilde{x}_i)} \sigma_i^b \circ \delta_i^a \circ \lambda_i^{a,b} = F^2_2 \left[ [\sigma] \circ [\delta] \right](\tilde{x}_i), \]

where \( \lambda_i^{a,b} \) is defined as in (2.1) and \( (\ast) \) is the definition of vertical composition of 2-morphisms in \((\text{Lie Gpd})\), hence also in \((\text{PEE Gpd})\). \( \square \)
Lemma 4.17. Let us fix any diagram in \((\text{RedAtl})\) as follows:

\[
\begin{array}{c}
\chi \xrightarrow{f} Y \xleftarrow{g} \zeta
\end{array}
\]

Then \(\mathcal{F}_{2}^{\text{red}}([\xi] \star [\delta]) = \mathcal{F}_{2}^{\text{red}}([\xi]) \star \mathcal{F}_{2}^{\text{red}}([\delta])\).

Proof. Let us set representatives

\[
\hat{f}^{m} := \left( f, \mathcal{T}^{m}, \{ \hat{f}^{m} \}_{i \in I}, P_{f^{m}}, \nu_{f^{m}} \right) \quad \text{for} \quad m = 1, 2,
\]

\[
\hat{g}^{m} := \left( g, \mathcal{G}^{m}, \{ \hat{g}^{m} \}_{j \in J}, P_{g^{m}}, \nu_{g^{m}} \right) \quad \text{for} \quad m = 1, 2
\]

for \([\hat{f}^{m}]\) and \([\hat{g}^{m}]\) respectively and representatives

\[
\delta = \left\{ \left( \hat{X}^{a,c,e}_{i}, \hat{\delta}^{a,c,e}_{i} \right) \right\}_{i \in I, (a,e) \in A(i)}, \quad \xi = \left\{ \left( \hat{Y}^{c}_{j}, \hat{\xi}^{c}_{j} \right) \right\}_{j \in J, c \in C(j)}
\]

for \([\delta]\) and \([\xi]\) respectively, as in Construction 2.4 let us denote by

\[
\gamma = \left\{ \left( \hat{X}^{a,c,e}_{i}, \hat{\gamma}^{a,c,e}_{i} \right) \right\}_{i \in I, (a,e) \in A(i) \times C(\mathcal{T}^{2}(i)), e \in E(i,a,c)}
\]

a representative for \([\xi] \star [\delta]\), obtained as in the already mentioned construction. Then let us set:

\[
\mathcal{F}^{\text{red}}_{0}(\chi) := \left( \mathcal{X}^{e}_{1} \xrightarrow{\phi^{e}_{1}} \mathcal{X}^{e}_{0} \right), \quad \mathcal{F}^{\text{red}}_{0}(\chi) := \left( \mathcal{Y}^{e}_{1} \xrightarrow{\phi^{e}_{1}} \mathcal{Y}^{e}_{0} \right), \quad \mathcal{F}^{\text{red}}_{0}(\chi) := \left( \mathcal{Z}^{e}_{1} \xrightarrow{\phi^{e}_{1}} \mathcal{Z}^{e}_{0} \right),
\]

\[
\mathcal{F}^{\text{red}}_{1}([\hat{f}^{m}]) := \left( \psi^{m}_{0}, \psi^{m}_{1} \right), \quad \mathcal{F}^{\text{red}}_{1}([\hat{g}^{m}]) := \left( \phi^{m}_{0}, \phi^{m}_{1} \right) \quad \text{for} \quad m = 1, 2,
\]

\[
\mathcal{F}^{\text{red}}_{2}([\delta]) := \hat{\gamma}_{i}, \quad \mathcal{F}^{\text{red}}_{2}([\xi]) := \hat{\xi}_{j}.
\]

Let us fix any \(i \in I\), any \((a,c) \in A(i) \times C(\mathcal{T}^{2}(i))\), any \(e \in E(i,a,c)\) and any point \(\hat{x}_{i} \in \hat{X}^{a,c,e}_{i}\). Then we have

\[
\left( \mathcal{F}^{\text{red}}_{2}([\xi] \star \mathcal{F}^{\text{red}}_{2}([\delta])) \left( \hat{x}_{i} \right) \right)^{(\ast)} = m \circ \left( \phi^{e}_{1} \circ \phi^{a,c,e}_{1}(\hat{x}_{i}), \xi \circ \psi^{a,c,e}_{1}(\hat{x}_{i}) \right) = \mathcal{F}^{\text{red}}_{2} \left( \left[ \xi \right] \star \left[ \delta \right] \right)(\hat{x}_{i}),
\]

where \(\hat{\delta}^{a,c,e}_{i}\) and \(\hat{\xi}^{a,c,e}_{i}\) are defined as in Construction 2.3 and \((\ast)\) is the definition of horizontal composition in \((\text{LieGpd})\).

\(\square\)

Lemmas 4.12, 4.15, 4.16 and 4.17 prove that:

Theorem 4.18. The data \(\mathcal{F}^{\text{red}} := (\mathcal{F}^{\text{red}}_{0}, \mathcal{F}^{\text{red}}_{1}, \mathcal{F}^{\text{red}}_{2})\) define a 2-functor from \((\text{RedAtl})\) to \((\text{PEÊGpd})\).

We state some properties of \(\mathcal{F}^{\text{red}}\) that we are going to use soon.
Lemma 4.19. (adapted from \[Po\] Proposition 4.9) Let us fix any pair of reduced orbifold atlases \(X, Y\) for \(X\) and \(Y\) respectively. Let us set \(F_0^{\text{red}}(X) := (\tilde{X}_i \overset{\sim}{\to} \tilde{X}_i)\) and \(F_0^{\text{red}}(Y) := (\tilde{Y}_j \overset{\sim}{\to} \tilde{Y}_j)\). Let us also fix any morphism \((\psi_0, \psi_1) : (\tilde{X}_i \overset{\sim}{\to} \tilde{X}_i) \to (\tilde{Y}_j \overset{\sim}{\to} \tilde{Y}_j)\). Then there is a unique morphism \([\tilde{f}] : X \to Y\) in \((\text{Red, Atl})\), such that \(F_1^{\text{red}}([\tilde{f}]) = (\psi_0, \psi_1)\).

Proof. Let us suppose that \(X = \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}\) and \(Y = \{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J}\). Since each \(\tilde{X}_i\) is connected by definition of orbifold atlas, then the morphism \(\psi_0 : \tilde{X}_i \to \tilde{Y}_j\) induces a set map \(f : I \to J\) such that \(\psi_0(\tilde{X}_i) \subseteq \tilde{Y}_{f(i)}\) for every \(i \in I\). For each \(i \in I\) we set \(\tilde{f}_i := \psi_0|_{\tilde{X}_i} : \tilde{X}_i \to \tilde{Y}_{f(i)}\).

Now let us consider the continuous and surjective maps
\[
\pi : \tilde{X}_i = \prod_{i \in I} \tilde{X}_i \to X \quad \text{and} \quad \chi : \tilde{Y}_j = \prod_{j \in J} \tilde{Y}_j \to Y
\]
defined as \(\pi|_{\tilde{X}_i} := \pi_i\) for each \(i \in I\) and \(\chi|_{\tilde{Y}_j} := \chi_j\) for each \(j \in J\). If we consider the equivalence relation \(\sim_X\) on \(\tilde{X}_0\) defined in Remark 4.10 and the induced quotient \(\pi^{\sim}_X : \tilde{X}_0 \to |\tilde{X}_0| = \tilde{X}_0/\sim_X\), then we can construct easily an homeomorphism \(\varphi_X : |\tilde{X}_0| \sim X\), such that \(\varphi_X \circ \pi^{\sim}_X = \pi\). Analogously, there is an homeomorphism \(\varphi_Y : |\tilde{Y}_0| \sim Y\) such that \(\varphi_Y \circ \pi^{\sim}_Y = \chi\). Then we define a continuous map \(f : X \to Y\) as \(f := \varphi_Y \circ \varphi_X^{-1} \circ \pi = \varphi_Y \circ \varphi_X^{-1} \circ \varphi_X \circ \chi \circ \psi_0\), hence
\[
f \circ \pi = \varphi_Y \circ \varphi_X^{-1} \circ \pi = \varphi_Y \circ \varphi_X^{-1} \circ \varphi_X \circ \chi \circ \psi_0 = \varphi_Y \circ \varphi_X^{-1} \circ \chi \circ \psi_0,
\]
so the collection \(\{\tilde{f}_i\}_{i \in I}\) satisfies condition (4.7) (see Definition 4.2). Now let us fix any \(i \in I\), any \(\tilde{\tau}_i \in \tilde{X}_i\), and any \(\lambda \in \text{Ch}(X, i, -)\) with \(\tilde{\tau}_i \in \text{dom} \lambda\). Using the construction of \(\mathcal{U}_i\) and the fact that \((\psi_0, \psi_1)\) commutes with \(s\) (see Definition 4.2), there is a (non-unique) \(\omega \in \text{Ch}(Y, f(i), -)\) such that
\[
\psi_1(\text{germ}_{\tilde{\tau}_i, \lambda}) = \text{germ}_{\psi_0(\tilde{\tau}_i), \omega} = \text{germ}_{\tilde{f}_i(\pi_i), \omega}.
\]
Now \(\psi_1\) is continuous, so the set \(A := \psi_1^{-1}(|\tilde{\tau}_i|) \cap \tilde{X}_i\) is an open set (for the notations used here, see (4.4)); \(A\) is non-empty since it contains \(\text{germ}_{\tilde{\tau}_i, \lambda}\). For each point \(\tilde{x}_i \in s(A) \subseteq \tilde{X}_0\), using the definition of \(A\) (and the fact that \(s\) commutes with \((\psi_0, \psi_1)\), we have
\[
\psi_1(\text{germ}_{\tilde{x}_i, \lambda}) = \text{germ}_{\psi_0(\tilde{x}_i), \omega} = \text{germ}_{\tilde{f}_i(\pi_i), \omega}
\]
(see (4.4)) holds not only for \(\tilde{\tau}_i\), but also for every \(\tilde{x}_i\) in an open neighborhood of \(\tilde{\tau}_i\). Now let us choose any open connected subset \(\tilde{X}_{\lambda, \pi_i, \omega} \subseteq s(A) \subseteq \tilde{X}_0\), such that:

- \(\pi_i \in \tilde{X}_{\lambda, \pi_i, \omega}\);
- \(\tilde{X}_{\lambda, \pi_i, \omega}\) is invariant under the action of \(\text{Stab}(G_i, \pi_i)\);
- for all \(g \in G_i \setminus \text{Stab}(G_i, \pi_i)\) we have \(g(\tilde{X}_{\lambda, \pi_i, \omega}) \cap \tilde{X}_{\lambda, \pi_i, \omega} = \emptyset\)

(in this way, \(\lambda\) is again a change of charts if restricted to \(\tilde{X}_{\lambda, \pi_i, \omega}\)). Then let us set
\[
P_f := \{\lambda|_{\tilde{X}_{\lambda, \pi_i, \omega}} \mid \forall i \in I, \forall \lambda \in \text{Ch}(X, i, -), \forall \pi_i \in \text{dom} \lambda\};
\]
if we have 2 (or more) collections $(\lambda, \mathcal{X}_i, \omega)$ and $(\lambda', \mathcal{X}_i', \omega')$ such that

$$\lambda|_{\mathcal{X}_i, \omega} = \lambda'|_{\mathcal{X}_i', \omega'}, \quad (4.8)$$

then we simply make an arbitrary choice of a triple $(\lambda, \mathcal{X}_i, \omega)$ associated to the morphism $\hat{f}$ in $P_f$. Then for each $\lambda|_{\mathcal{X}_i, \omega} \in P_f$ we set $\nu_f(|\lambda|_{\mathcal{X}_i, \omega}) := \omega$ and it is easy to see that the collection

$$\hat{f} := \left( f, \mathcal{T}, \{ \hat{f}_i \}_{i \in I}, P_f, \nu_f \right)$$

is a representative of a morphism from $\mathcal{X}$ to $\mathcal{Y}$. The collection $\hat{f}$ depends on some choices, but the class $[\hat{f}]$ depends only on $(\psi_0, \psi_1)$. A direct check using (4.7) proves that $F^{\text{red}}_1([\hat{f}]) = (\psi_0, \psi_1)$; moreover it is easy to see that if $[\hat{f}^1], [\hat{f}^2] : \mathcal{X} \to \mathcal{Y}$ are such that $F^{\text{red}}_1([\hat{f}^1]) = F^{\text{red}}_1([\hat{f}^2])$, then $[\hat{f}^1] = [\hat{f}^2]$. This suffices to complete the proof. \hfill $\square$

Lemma 4.19 proves that $[\hat{f}]$ as above is unique. So the previous proof implies that:

**Corollary 4.20.** Let us fix any reduced orbifold atlas $\mathcal{X} = \{ (\tilde{X}_i, G_i, \pi_i) \}_{i \in I}$ on a topological space $X$ and let us set $\mathcal{X}_* := F_0^{\text{red}}(\mathcal{X})$. Then there is a canonical homeomorphism $\varphi_\mathcal{X} : |\mathcal{X}_*| \xrightarrow{\simeq} X$ (sending each point $[\tilde{x}_i] \in |\mathcal{X}_*|$ to $\pi_i(\tilde{x}_i)$ for each point $\tilde{x}_i \in \tilde{X}_i$, and for each $i \in I$). Here “canonical ” means that given any other reduced orbifold atlas $\mathcal{Y}$ on a topological space $Y$ and any morphism of reduced orbifold atlases $\hat{f} : \mathcal{X} \to \mathcal{Y}$ over a continuous map $f : X \to Y$, we have

$$f \circ \varphi_\mathcal{X} = \varphi_\mathcal{Y} \circ F^{\text{red}}_1([\hat{f}]),$$

where $F^{\text{red}}_1([\hat{f}])$ is the continuous map from $|\mathcal{X}_*|$ to $|\mathcal{Y}_*|$ associated to $F^{\text{red}}_1([\hat{f}])$ by Remark 4.6.

**Lemma 4.21.** Let us fix any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$ for 2 topological spaces $X$ and $Y$ respectively, and any pair of morphisms $\hat{f}_m : \mathcal{X} \to \mathcal{Y}$ for $m = 1, 2$ with representatives

$$\hat{f}_m := \left( f, \mathcal{T}, \{ \hat{f}_m \}_{i \in I}, P_f, \nu_f \right)$$

for $m = 1, 2$.

Let us set

$$F_0^{\text{red}}(\mathcal{X}) := \mathcal{X}_*, \quad F_0^{\text{red}}(\mathcal{Y}) := \mathcal{Y}_*, \quad F_1^{\text{red}}([\hat{f}_m]) := \varphi_*^{\omega_m} \quad \text{for } m = 1, 2.$$

Let us also fix any natural transformation $\alpha : \varphi^1 \Rightarrow \varphi^2$. Then there exists a unique 2-morphism $\delta : [\hat{f}^1] \Rightarrow [\hat{f}^2]$ such that $F_2^{\text{red}}(\delta) = \alpha$.

**Proof.** By Definition 4.3 $\alpha$ is a smooth map from $\mathcal{X}_0$ to $\mathcal{Y}_0$ such that $s \circ \alpha = \psi_0^1$, so for each $i \in I$ and for each $\mathcal{X}_i \in \tilde{X}_i \subseteq \mathcal{X}_0$, we can choose a change of charts $\delta^i_\mathcal{X}$ of $\mathcal{X}_i$, such that

$$\alpha(\mathcal{X}_i) = \text{germ}_{\psi^1_0(\mathcal{X}_i)} \delta^i_\mathcal{X} = \text{germ}_{\psi^2_0(\mathcal{X}_i)} \delta^i_\mathcal{X}.$$

(4.9)

For each $\mathcal{X}_i \in \tilde{X}_i$, we consider the set

$$\mathcal{Y}_i(\delta^i_\mathcal{X}) := \left\{ \text{germ}_{\psi^1_0(\mathcal{X}_i)} \delta^i_\mathcal{X} : \forall \eta^i_\mathcal{Y} \in \text{dom} \delta^i_\mathcal{X} \right\} \subseteq \mathcal{Y}_i.$$

By Construction 4.11 (for $\mathcal{Y}_i$ instead of $\mathcal{X}_i$), $\mathcal{Y}_i(\delta^i_\mathcal{X})$ is open in $\mathcal{Y}_i$; moreover the map $\tau : \mathcal{Y}_i(\delta^i_\mathcal{X}) \to \text{dom} \delta^i_\mathcal{X}$, defined by
is a diffeomorphism to an open connected subset of some $\mathbb{R}^n$ (here $n$ is the dimension of the atlas $\mathcal{F}$). Then we define an open set:

$\tilde{X}^\alpha_i := \alpha^{-1}(\mathfrak{F}(\delta^\alpha_i)) \cap \tilde{X}_i$.

For each $i \in I$ we choose any collection $\{\mathfrak{F}_i\}_{a \in A(i)}$ such that the family $\{\tilde{X}^\alpha_i\}_{a \in A(i)}$ is an open covering of $\tilde{X}_i$. For simplicity of notations, we set $\delta^\alpha_i := \delta^\alpha_i$ and $\tilde{X}^\alpha_i := \tilde{X}^\alpha_i$. We claim that the collection $\delta := \{(\tilde{X}^\alpha_i, \delta^\alpha_i)\}_{i \in I, a \in A(i)}$ is a representative of a 2-morphism from $[\tilde{f}^1]$ to $[\tilde{f}^2]$.

In order to prove that, let us fix any $i \in I$, any $a \in A(i)$ and any $\tilde{x}_i \in \tilde{X}^\alpha_i$. By definition of $\tilde{X}^\alpha_i$ and using condition (NT1) (see Definition 4.3), we have:

$$\alpha(\tilde{x}_i) = \text{germ}_{\psi^\alpha_i(\tilde{x}_i)} \delta^\alpha_i = \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^\alpha_i$$

(in other terms, (1.10) holds not only for the point $\mathfrak{F}_i$, but also for any $\tilde{x}_i$ in an open neighborhood of $\mathfrak{F}_i$). Again by (NT1) we have $t \circ \alpha = \psi^\alpha_i$; so for each $\tilde{x}_i \in \tilde{X}^\alpha_i$ we have

$$\tilde{f}^2_\lambda(\tilde{x}_i) = \psi^\alpha_i(\tilde{x}_i) = t \circ \alpha(\tilde{x}_i) = t \left( \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^\alpha_i \right) = \delta^\alpha_i \circ \tilde{f}^1_\delta(\tilde{x}_i),$$

so in particular

$$\tilde{f}^2_\lambda(\tilde{X}^\alpha_i) \subseteq \text{dom} \delta^\alpha_i \quad \text{and} \quad \tilde{f}^2_\lambda(\tilde{X}^\alpha_i) \subseteq \text{cod} \delta^\alpha_i;$$

therefore properties $\text{(2M1)}, \text{(2M2)}$ and $\text{(2M3)}$ (see Definition 1.18) are verified for $\delta$. If $a$ and $a'$ are indices in $A(i)$ and $\tilde{x}_i \in \tilde{X}^\alpha_i \cap \tilde{X}^\alpha_i$, then by (4.10) we have

$$\text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^\alpha_i = \alpha(\tilde{x}_i) = \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^{a'}_i,$$

so $\text{(2M1)}$ holds. Now let us fix any $(i, i') \in I \times I$, any $(a, a') \in A(i) \times A(i')$, any $\lambda \in Ch(\tilde{X}_i, \lambda)$ and any $\tilde{x}_i \in \tilde{X}_i$ such that $\lambda(\tilde{x}_i) \in \tilde{X}^\alpha_i$. Since $P_{f_1}$ and $P_{f_2}$ are both good subsets of $Ch(X)$, then for each $m = 1, 2$ there exists $\lambda^m \in P_{f_m}(i, i')$ such that $\tilde{x}_i \in \tilde{X}^\alpha_i$ and $\text{germ}_{\tilde{x}_i} \lambda^m = \text{germ}_{\tilde{x}_i} \lambda$. We recall (see Construction 4.13) that

$$\psi^m_i \left( \text{germ}_{\tilde{x}_i} \lambda \right) = \text{germ}_{\tilde{f}^m_\delta(\tilde{x}_i)} \nu_f(\lambda^m) \quad \text{for} \quad m = 1, 2.$$

Therefore:

$$\text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \nu_f(\lambda^2) \cdot \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^\alpha_i = m \left( \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \delta^\alpha_i, \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \nu_f(\lambda^2) \right)_{\text{adj}},$$

$$= \left( m \circ (\alpha \circ s, \psi^m_i) \right) \left( \text{germ}_{\tilde{x}_i} \lambda^2 \right) \left( m \circ (\psi^m_i, \alpha \circ t) \right) \left( \text{germ}_{\tilde{x}_i} \lambda^1 \right) =$$

$$= \left( \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \nu_f(\lambda^2), \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \nu_f(\lambda^1) \right)_{\text{adj}}$$

$$= \text{germ}_{\tilde{f}^1_\delta(\lambda(\tilde{x}_i))} \delta^{a'}_i \cdot \text{germ}_{\tilde{f}^1_\delta(\tilde{x}_i)} \nu_f(\lambda).$$

So also property $\text{(2M2)}$ holds. Therefore $\delta$ is a representative of a 2-morphism from $[\tilde{f}^1]$ to $[\tilde{f}^2]$. Different choices of the sets $\{\mathfrak{F}_i\}_a$ and $\{\delta^\alpha_i\}_a$ give rise to different $\delta'$s, but their equivalence class $[\delta]$ is the same. Now by (1.10) we get that $\tilde{f}^2_\lambda(\delta^\alpha_i) = \alpha$; moreover, a direct computation proves that if $[\delta^1]$ and $[\delta^2]$ are such that $\tilde{f}^2_\lambda(\delta^1) = \tilde{f}^2_\lambda(\delta^2)$, then $[\delta^1] = [\delta^2]$. This suffices to conclude. □
Theorem 5.2. We recall the following fundamental result:

\[ \mathcal{F}_0^{\text{red}}(X, Y) : (\text{Red}_\text{Atl})(X, Y) \longrightarrow (\mathcal{PE}_\text{Ed})(\mathcal{F}_0^{\text{red}}(X), \mathcal{F}_0^{\text{red}}(Y)) \]

is a bijection on objects and morphisms (i.e. on 1-morphisms and 2-morphisms of (\text{Red}_\text{Atl}) and of (\mathcal{PE}_\text{Ed})). \( \mathcal{F}_0^{\text{red}} \) is not injective; indeed, given any homeomorphism \( f : X \to Y \) and any reduced orbifold atlas \( X \) on \( X \), by Definition 1.17 we get that \( \mathcal{F}_0^{\text{red}}(X) = \mathcal{F}_0^{\text{red}}(f(X)) \). It is not difficult to see that actually this is the only point where \( \mathcal{F}_0^{\text{red}} \) fails to be injective. We will see in Lemma 5.11 below that \( \mathcal{F}_0^{\text{red}} \) is surjective only up to “Morita equivalences”.

5. Morita equivalences between étale groupoids

As we mentioned in the Introduction, the bicategory of reduced orbifold atlases described in terms of proper, effective and étale groupoids is not \((\mathcal{PE}_\text{Ed})\), but a bicategory obtained from \((\mathcal{PE}_\text{Ed})\) by selecting a suitable class of morphisms (Morita equivalences, see below) and by “turning” them into internal equivalences. We briefly recall the axioms that are needed for that construction. We do that firstly because we will need the explicit description of the bicategory obtained from \((\mathcal{PE}_\text{Ed})\) by applying such a procedure; secondly, because later we will need to perform an analogous construction on \((\text{Red}_\text{Atl})\).

Definition 5.1. (\cite{Pr} § 2.1) in the special case of 2-categories instead of bicategories) Let us fix any 2-category \( \mathcal{C} \) and any class \( W \) of morphisms in \( \mathcal{C} \). We recall that the pair \((\mathcal{C}, W)\) is said to admit a right bicalculus of fractions if the following conditions are satisfied:

(BF1) for every object \( A \) of \( \mathcal{C} \), the 1-identity \( \text{id}_A \) belongs to \( W \);

(BF2) \( W \) is closed under compositions;

(BF3) for every morphism \( w : A \to B \) in \( W \) and for every morphism \( f : C \to B \), there are an object \( D, \) a morphism \( w' : D \to C \) in \( W \), a morphism \( f' : D \to A \) and an invertible 2-morphism \( \alpha : f \circ w' \Rightarrow w \circ f' \);

(BF4) (a) given any morphism \( w : B \to A \) in \( W \), any pair of morphisms \( f_1, f_2 : C \to B \) and any \( \alpha : w \circ f_1 \Rightarrow w \circ f_2 \), there are an object \( D, \) a morphism \( v : D \to C \) in \( W \) and a 2-morphism \( \beta : f_1 \circ v \Rightarrow f_2 \circ v \), such that \( \alpha \ast i_v = i_w \ast \beta \);

(b) if \( \alpha \) in (a) is invertible, then so is \( \beta \);

(c) if \( (D', v', D' \to C, \beta', f_1' \circ v' \Rightarrow f_2' \circ v') \) is another triple with the same properties of \( (D, v, \beta) \) in (a), then there are an object \( E, \) a pair of morphisms \( u : E \to D, \) \( u' : E \to D' \) and an invertible 2-morphism \( \zeta : v \circ u \Rightarrow v' \circ u' \), such that \( v \circ u \) belongs to \( W \) and

\[ \left( \beta' \ast i_{u'} \right) \circ \left( i_{f_1} \ast \zeta \right) = \left( i_{f_2} \ast \zeta \right) \circ \left( \beta \ast i_u \right) ; \]

(BF5) if \( w : A \to B \) is a morphism in \( W, \) \( v : A \to B \) is any morphism and if there is an invertible 2-morphism \( \alpha : v \Rightarrow w \), then also \( v \) belongs to \( W \).

We recall the following fundamental result:

Theorem 5.2. (\cite{Pr} Theorem 21) Given any 2-category or bicategory \( \mathcal{C} \) and any class \( W \) as before, there are a bicategory \( \mathcal{C} \left[ W^{-1} \right] \) (called right bicategory of fractions) and a pseudofunctor \( \mathcal{U}_W : \mathcal{C} \to \mathcal{C} \left[ W^{-1} \right] \) that sends each element of \( W \) to an internal equivalence and that is universal with respect to such a property.

Remark 5.3. In the notations of \cite{Pr}, \( \mathcal{U}_W \) is called bifunctor, but this notation is no more in use. For the precise meaning of “universal” we refer directly to \cite{Pr} § 3.2. In particular, the bicategory \( \mathcal{C} \left[ W^{-1} \right] \) is unique up equivalences of bicategories.
In [Pr] the theorem above is stated with (BF1) replaced by the slightly stronger hypothesis (BF1)′: all the equivalences of \( \mathcal{C} \) are in \( \mathbf{W} \). By looking carefully at the proofs in [Pr], it is easy to see that the only part of axiom (BF1)′ that is really used in all the computations is (BF1), so we are allowed to state the theorem of [Pr] under such less restrictive hypothesis.

We refer to [Pr, § 2.2, 2.3, 2.4] and to our paper [T2] for more details on the construction of bicategories of fractions and to [Lei, § 1.5] for a general overview on bicategories and pseudofunctors. Note that even if \( \mathcal{C} \) is a 2-category, in general \( \mathcal{C}[\mathbf{W}^{-1}] \) is only a bicategory (with trivial unitors but possibly non-trivial associators). In other terms, in \( \mathcal{C}[\mathbf{W}^{-1}] \) in general the composition of morphisms and the horizontal compositions of 2-morphisms are associative only up to canonical invertible 2-morphisms.

We recall also the following definition, that will be very useful soon.

**Definition 5.4.** [T3, Definition 2.1] Let us fix any bicategory \( \mathcal{C} \) and any class \( \mathbf{W} \) of morphisms in it (not necessarily satisfying conditions (BF)). Then we define a class \( \mathbf{W}_{\text{sat}} \) as the class of all morphisms \( f : B \to A \) in \( \mathcal{C} \), such that there is a pair of objects \( C,D \) and a pair of morphisms \( g : C \to B \), \( h : D \to C \), such that both \( f \circ g \) and \( g \circ h \) belong to \( \mathbf{W} \). We call \( \mathbf{W}_{\text{sat}} \) the (right) saturation of \( \mathbf{W} \); we say that \( \mathbf{W} \) is (right) saturated if it coincides with its saturation.

We recall (see [M, § 2.4]) that a morphism \( (\psi_0, \psi_1) : (\mathcal{X}_1 \stackrel{s}{\leftrightarrow} \mathcal{X}_0) \to (\mathcal{Y}_1 \stackrel{s}{\leftrightarrow} \mathcal{Y}_0) \) between Lie groupoids is a Morita equivalence (also known as weak equivalence or essential equivalence) if and only if the following 2 conditions hold:

1. **(ME1)** the smooth map \( t \circ \pi^1 : \mathcal{B}_s \times_{\psi_0} \mathcal{X}_0 \to \mathcal{B}_0 \) is a surjective submersion (here \( \pi^1 \) is the projection \( \mathcal{B}_s \times_{\psi} \mathcal{X}_0 \to \mathcal{B}_s \) and the fiber product is a manifold since \( s \) is a submersion);
2. **(ME2)** the following square is cartesian (it is commutative by Definition 4.2):
   \[
   \begin{array}{ccc}
   \mathcal{X}_1 & \xrightarrow{\psi_1} & \mathcal{Y}_1 \\
   (s,t) \downarrow & & \downarrow (s,t) \\
   \mathcal{X}_0 \times \mathcal{X}_0 & \xrightarrow{(\psi_0 \times \psi_0)} & \mathcal{Y}_0 \times \mathcal{Y}_0.
   \end{array}
   \]

Any 2 Lie groupoids \( \mathcal{X}_\ast \) and \( \mathcal{Y}_\ast \) are said to be *Morita equivalent* (or *weakly equivalent* or *essentially equivalent*) if and only if there are a Lie groupoid \( \mathcal{Z}_\ast \) and 2 Morita equivalences as follows:

\[
\begin{array}{ccc}
\mathcal{X}_\ast & \xrightarrow{\psi_1^2} & \mathcal{Z}_\ast \\
\psi_1^{-1} & \xrightarrow{\psi_2} & \mathcal{Y}_\ast.
\end{array}
\]

This is actually an equivalence relation, see for example [MM, Chapter 5]. Given any Morita equivalence as above, the induced set map \( |\psi_\ast| \) is surjective by (ME1) and it is injective as a consequence of (ME2). If both \( \mathcal{X}_\ast \) and \( \mathcal{Y}_\ast \) are étale, then by [MM, Exercise 5.16(4)] the map \( \psi_0 \) is étale, hence open. If in addition \( \mathcal{X}_\ast \) and \( \mathcal{Y}_\ast \) are also proper, then \( \text{pr}_{\mathcal{Y}_\ast} : \mathcal{Y}_0 \to |\mathcal{Y}_\ast| \) is also open and \( |\psi_\ast|^{-1} \) is continuous (see Remark 4.6). Since the projection \( \text{pr}_{\mathcal{X}_\ast} : \mathcal{X}_0 \to |\mathcal{X}_\ast| \) is continuous and surjective, then diagram (4.1) proves that the induced map \( |\psi_\ast| \) is open, i.e. that \( |\psi_\ast|^{-1} \) is continuous, so we have:
Lemma 5.5. Let us fix any Morita equivalence $\psi : \mathcal{X} \to \mathcal{Y}$ between proper, étale groupoids. Then the induced continuous map $[\psi] : [\mathcal{X}] \to [\mathcal{Y}]$ is an homeomorphism.

Lemma 5.6. Let us fix any pair of proper, effective, étale groupoids $\mathcal{X}, \mathcal{Y}$ and any pair of Morita equivalences $\psi_0, \psi_1 : \mathcal{X} \to \mathcal{Y}$. Then the following facts are equivalent:

(a) the topological maps $[\psi_0]$ and $[\psi_1]$ (see Remark 4.3) coincide;
(b) there exists a natural transformation $\alpha : \psi_1 \Rightarrow \psi_2$ in $(\mathcal{P}E \mathcal{Gpd})$;
(c) there exists a unique natural transformation $\alpha : \psi_1 \Rightarrow \psi_2$ in $(\mathcal{P}E \mathcal{Gpd})$.

Proof. Let us assume (a) and let us prove (b). Using [MM Exercise 5.16(4)] we get that both $\psi_0$ and $\psi_0'$ are étale maps, so for every point $x_0$ in $\mathcal{X}_0$ there is an open neighborhood $W_{x_0}$ of $x_0$, such that both $\psi_1^0$ and $\psi_0^0$ are diffeomorphisms if restricted to $W_{x_0}$. Then the map $f_{x_0}$ defined by

$$f_{x_0} := \psi_1^0 \circ (\psi_0^1|_{W_{x_0}})^{-1} : \psi_1^0(W_{x_0}) \to \psi_0^0(W_{x_0})$$

is a diffeomorphism from an open neighborhood of $\psi_1^0(x_0)$ to an open neighborhood of $\psi_0^0(x_0)$. Moreover, since the topological maps $[\psi_1^0]$ and $[\psi_0^0]$ (both defined from $|\mathcal{X}_0|$ to $|\mathcal{Y}_0|$) coincide, then we get easily that $f_{x_0}$ commutes with the projection $pr_\mathcal{Y} : \mathcal{Y}_0 \to |\mathcal{Y}_0|$ (see Remark 4.3). By Lemma 4.9 (applied to $\mathcal{Y}_0$), the set map $\kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), -)$ is a bijection. Therefore, it makes sense to define

$$\alpha(x_0) := \kappa_{\mathcal{Y}_0}(\psi_1^0(x_0), \psi_0^0(x_0), -)^{-1}(\text{germ}_{\psi_1^0(x_0)} f_{x_0}) \quad \forall x_0 \in \mathcal{X}_0.$$ 

So we have defined a set map $\alpha : \mathcal{X}_0 \to \mathcal{Y}_0$. Given any point $x_0 \in \mathcal{X}_0$, by definition of $\kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), -)$ (see 4.12) we have

$$s \circ \alpha(x_0) = \psi_0^1(x_0), \quad t \circ \alpha(x_0) = \psi_0^2(x_0).$$

Since both $s$ and $\psi_0^1$ are étale, the first identity implies that $\alpha$ is an étale map; moreover (5.2) proves that condition (NT1) holds for $\alpha$ (see Definition 4.3). We want to prove also that condition (NT2) is satisfied. In order to prove that, let us fix any point $x_1 \in \mathcal{X}_1$ and let us set $x_0 := s(x_1)$ and $x_0' := t(x_1)$. Then we have:

$$\text{germ}_{\psi_0^1(x_0)} f_{x_0} = \text{germ}_{x_0} \psi_0^2 \cdot (\text{germ}_{x_0} \psi_1^0)^{-1} =$$

$$= \text{germ}_{x_0} \psi_0^2 \cdot \text{germ}_{x_1} s \cdot (\text{germ}_{x_1} t)^{-1} \cdot \text{germ}_{x_1} t \cdot (\text{germ}_{x_1} s)^{-1} \cdot (\text{germ}_{x_0} \psi_1^0)^{-1} \quad \text{(a)}$$

$$= \text{germ}_{\psi_0^1(x_1)} s \cdot \text{germ}_{x_0} \psi_1^0 \cdot (\text{germ}_{x_1} t)^{-1},$$

$$= \text{germ}_{x_1} t \cdot (\text{germ}_{x_1} \psi_1^0)^{-1} \cdot (\text{germ}_{\psi_1^0(x_1)} s)^{-1} \quad \text{(a)}$$

$$= \text{germ}_{\psi_0^1(x_1)} s \cdot (\text{germ}_{\psi_0^1(x_1)} t)^{-1} \cdot \text{germ}_{x_0} \psi_0^2,$$

$$= (\text{germ}_{\psi_0^1(x_1)} t \cdot (\text{germ}_{\psi_0^1(x_1)} s)^{-1} =$$

$$= (\kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), \psi_0^1(x_1))^{-1} \cdot \text{germ}_{\psi_0^1(x_1)} f_{x_0} \cdot \kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), \psi_1^0(x_1)).$$

where all the identities denoted by (a) are a consequence of Definition 4.2 for $\psi_0^0$ and $\psi_1^0$. This implies that:

$$\kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), \psi_1^0(x_1)) \cdot \text{germ}_{\psi_0^1(x_0)} f_{x_0} =$$

$$= \text{germ}_{\psi_0^1(x_0)} f_{x_0} \cdot \kappa_{\mathcal{Y}_0}(\psi_0^0(x_0), \psi_0^0(x_0), \psi_1^0(x_1)).$$

(5.3)
Using the definition of $\kappa_Y(\cdot, \cdot, \cdot)$ (see Remark 4.7 with $\mathcal{Y}_\bullet$ replaced by $\mathcal{Y}_\bullet$), it is easy to prove that given any pair of objects $y_1, y_1 \in Y_1$ with $t(y_1) = s(y_1)$, we have

$$\kappa_Y(s(y_1), t(y_1), m(y_1, y_1)) = \kappa_Y(s(y_1), t(y_1), y_1) \cdot \kappa_Y(s(y_1), t(y_1), y_1).$$

We apply this fact to (5.3) together with the definition of $\alpha, \beta$.

In other terms, we have $m(\alpha(x_0), \psi^1(x_1)) = m(\psi^1(x_1), \alpha(x_0))$. In other terms, we have $m(\alpha(x_0), \psi^1(x_1)) = m(\psi^1(x_1), \alpha(x_0))$ for every point $x_1$ in $\mathcal{Y}_1$. This proves that condition (NT2) holds for $\alpha$, so we have constructed a natural transformation $\alpha : \psi^1 \Rightarrow \psi^2$ as required in (b).

Now let us assume (b) and let us prove that (c) holds. So let us suppose that there is a pair of natural transformations $\alpha, \beta : \psi^1 \Rightarrow \psi^2$. We denote by $\mathcal{Y}_0^{\text{reg}}$ the subset of $\mathcal{Y}_0$ consisting of regular points, namely those points $x_0$ such that $(s, t)^{-1}(x_0, x_0) = \{ e(x_0) \} \subset \mathcal{Y}_1$. Since $\mathcal{Y}_\bullet$ is effective by hypothesis, then $\mathcal{Y}_0^{\text{reg}}$ is open and dense in $\mathcal{Y}_0$. Let us fix any point $x_0$ in $\mathcal{Y}_0^{\text{reg}}$; since $\psi^1$ is a Morita equivalence, then by (ME2) (see Definition 5.4) we get that $\psi^1(x_0)$ belongs to $\mathcal{Y}_0^{\text{reg}}$. Moreover, by (NT1) (see Definition 4.3) we have

$$s \circ m(\alpha(x_0), i \circ \beta(x_0)) = s \circ \alpha(x_0) = \psi^1(x_0)$$

and

$$t \circ m(\alpha(x_0), i \circ \beta(x_0)) = t \circ i \circ \beta(x_0) = s \circ \beta(x_0) = \psi^2(x_0),$$

hence

$$m(\alpha(x_0), i \circ \beta(x_0)) \in (s, t)^{-1}\{ \psi^1(x_0), \psi^2(x_0) \} = \{ e \circ \psi^1(x_0) \},$$

so $\alpha(x_0) = \beta(x_0)$ for each $x_0 \in \mathcal{Y}_0^{\text{reg}}$. Now let us denote by $n$ the dimension of the étale groupoid $\mathcal{Y}_\bullet$ (i.e. the dimension of $\mathcal{Y}_0$, equivalently of $\mathcal{Y}_1$). If $n = 0$, since $\mathcal{Y}_\bullet$ is effective we have that $\mathcal{Y}_0 = \mathcal{Y}_0^{\text{reg}}$, so $\alpha = \beta$. If $n \geq 1$, then each connected component of $\mathcal{Y}_0$ contains a dense subset where $\alpha$ and $\beta$ coincide. Since such functions are both continuous (by Definition 4.3), with target in the Hausdorff space $\mathcal{Y}_1$, then we conclude that $\alpha$ and $\beta$ coincide everywhere, so (c) holds.

Lastly, let us assume that (c) holds and let us prove (a). Given any point $x_0 \in \mathcal{Y}_0$, using condition (NT1) and Remark 4.6 we get that

$$|\psi^1|(pr_{\mathcal{Y}_\bullet}(x_0)) = pr_{\mathcal{Y}_\bullet} \circ \psi^1(x_0) = pr_{\mathcal{Y}_\bullet} \circ s \circ \alpha(x_0) =$$

$$= pr_{\mathcal{Y}_\bullet} \circ t \circ \alpha(x_0) = pr_{\mathcal{Y}_\bullet} \circ \psi^1(x_0) = |\psi^2|(pr_{\mathcal{Y}_\bullet}(x_0)),$$

so (a) holds. \qed

Note that the previous Lemma is false if we remove the hypothesis of effectiveness.

We denote by $\mathcal{W}^{\text{gpd}}$ the set of all Morita equivalences in $(\mathcal{E} \mathcal{G} \mathcal{P} \mathcal{D})$, i.e. the set of all Morita equivalences between étale groupoids. Then we have the following useful result (the second part of which was also proved in [PS, Lemma 8.1]).
Lemma 5.7. [13] Corollary 4.2(b) and (c) and Proposition 2.11(ii)] The class \( W_{\mathcal{E}\mathcal{G}_{pd}} \) is right saturated. In particular, it satisfies the “2-of-3-property”, i.e. given any pair of morphisms \( \phi : \mathcal{X} \to \mathcal{Y} \) and \( \psi : \mathcal{Y} \to \mathcal{Z} \) between étale groupoids, if any 2 out of \( \{ \phi, \psi, \psi \circ \phi \} \) are Morita equivalences, then so is the third one.

Moreover we have:

Proposition 5.8. [Pr § 4.1] The set \( W_{\mathcal{E}\mathcal{G}_{pd}} \) admits a right bic calculus of fractions, so there are a bicategory \((\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{E}\mathcal{G}_{pd}}]\) and a pseudofunctor

\[
\mathcal{U}_{W_{\mathcal{E}\mathcal{G}_{pd}}}(\mathcal{E}\mathcal{G}_{pd}) \to (\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{E}\mathcal{G}_{pd}}]
\]

that sends each Morita equivalence between étale groupoids to an internal equivalence and that is universal with respect to such a property (see Remark [13]).

We denote by \( W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}} \) respectively by \( W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}} \), the set of all Morita equivalences between proper and étale groupoids, respectively between proper, effective and étale groupoids. Then we have the following standard result:

Proposition 5.9. The sets \( W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}} \) and \( W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}} \) admit a right bic calculus of fractions (in \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})\) and \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})\) respectively), so there are a pair of bicategories \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}]\) and \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}]\) and pseudofunctors

\[
\mathcal{U}_{W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}}(\mathcal{P}\mathcal{E}\mathcal{G}_{pd}) \to (\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}]
\]

(5.4)

that send each Morita equivalence between proper, (effective) étale groupoids to an internal equivalence and that are universal with respect to such a property (see Remark [13]). Moreover, both \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}]\) and \((\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}]\)

are full bi-subcategories of \((\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{E}\mathcal{G}_{pd}}]\) and we have a commutative diagram as follows:

\[
\begin{array}{ccc}
(\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}] & \to & (\mathcal{P}\mathcal{E}\mathcal{G}_{pd})[W^{-1}_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}] \\
\mathcal{U}_{W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}} & \searrow & \mathcal{U}_{W_{\mathcal{P}\mathcal{E}\mathcal{G}_{pd}}} \\
(\mathcal{P}\mathcal{E}\mathcal{G}_{pd}) & \hookrightarrow & (\mathcal{P}\mathcal{E}\mathcal{G}_{pd}) \\
\end{array}
\]

(5.4)

where each map without a name denotes an embedding as full 2-subcategory or as full bi-subcategory.

Proof. By [MM] Proposition 5.26, if \( \mathcal{X} \) and \( \mathcal{Y} \) are Morita equivalent Lie groupoids, then the first Lie groupoid is proper if and only if the second one is so. Moreover, by [MM] Example 5.21(2) if \( \mathcal{X} \) and \( \mathcal{Y} \) are both étale and they are Morita equivalent, then the first Lie groupoid is effective if and only if the second one is so (note that being effective is not preserved by Morita equivalences if we remove the étale condition).
Now axioms (BF1), (BF2) and (BF3) are easily verified for the set $\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}$. Let us consider (BF3), so let us fix any triple of proper, effective, étale groupoids and any pair of morphisms as follows

$$\mathcal{X} \xrightarrow{\psi} \mathcal{Y} \xrightarrow{\phi} \mathcal{Z},$$

with $\psi$ Morita equivalence. By Proposition 5.8 we get that (BF3) holds in $(\mathcal{E}\mathcal{G}\text{pd})$; therefore there are an étale groupoid $\mathcal{U}$, a Morita equivalence $\psi'$ and a morphism $\phi'$ as follows

$$\mathcal{X} \xrightarrow{\phi'} \mathcal{U} \xrightarrow{\psi'} \mathcal{Z},$$

and a natural transformation $\alpha : \psi' \circ \phi' \Rightarrow \phi' \circ \psi'$. Now $\mathcal{X}$ and $\mathcal{Y}$ are étale groupoids that are weakly equivalent and the first one is proper and effective; so also the second one is proper and effective. Therefore axiom (BF3) holds for the set $\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}$. An analogous proof shows that also (BF4) holds for $\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}$.

The proofs for the set $\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}$ are analogous. Therefore there are bicategories and pseudofunctors as in the claim. The last part of the claim is straightforward by looking at the explicit construction of the bicategories of fractions in [PT, § 2.2 and 2.3] and using the remarks at the beginning of this proof. □

The bicategory $(\mathcal{P}\mathcal{E}\mathcal{G}\text{pd})[\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}^{-1}]$ is usually called in the literature the bicategory of orbifolds (from the point of view of Lie groupoids); its bi-subcategory $(\mathcal{P}\mathcal{E}\mathcal{G}\text{pd})[\mathcal{W}_{PE\mathcal{E}\mathcal{G}\text{pd}}^{-1}]$ is usually called the bicategory of effective (or reduced) orbifolds. We refer to Description 7.9 below for an explicit description of the last bicategory mentioned above.

### 6. Weak equivalences, unit weak equivalences and refinements in $(\text{Red}\mathcal{A}\mathcal{t}l)$

In this section we introduce the notions of weak equivalences, unit weak equivalences and refinements in $(\text{Red}\mathcal{A}\mathcal{t}l)$. Using the 2-functor $\mathcal{F}^{\text{red}}$, the definition of weak equivalences will match with the notion of Morita equivalences between proper, effective, étale groupoids (see Proposition 6.9 below).

**Definition 6.1.** Let us fix any pair of reduced orbifold atlases $\mathcal{X} = \{\tilde{X}_i, G_i, \pi_i\}_{i \in I}$ on $X$ and $\mathcal{Y} = \{\tilde{Y}_i, G_i, \pi_i\}_{i \in I}$ on $Y$ and any morphism

$$[\tilde{w}] := \left( w, \tilde{w}, \{\tilde{w}_i\}_{i \in I}, [P_w, \nu_w] \right) : \mathcal{X} \rightarrow \mathcal{Y}. \quad (6.1)$$

Then we say that $[\tilde{w}]$ is a refinement if and only if the following 2 conditions hold:

- (REF1) $X = Y$ and the continuous map $w : X \rightarrow X$ is equal to $\text{id}_X$;
- (REF2) for each $i \in I$ the smooth map $\tilde{w}_i$ is an open embedding; assuming (REF1), this implies that $[\mathcal{X}] = [\mathcal{Y}]$.

We say that $[\tilde{w}]$ is a unit weak equivalence of reduced orbifold atlases if and only if it satisfies condition (REF1) and

- (UWE) for each $i \in I$ the chart $(\tilde{X}_i, G_i, \pi_i)$ on $X$ is compatible with the atlas $\mathcal{Y}$; assuming (REF1), this is equivalent to imposing that $[\mathcal{X}] = [\mathcal{Y}]$.

We say that $[\tilde{w}]$ is a weak equivalence of reduced orbifold atlases if and only if it satisfies the following conditions:

- (WE1) the continuous map $w : X \rightarrow Y$ is an homeomorphism;
(WE2) for each $i \in I$ the chart $(\tilde{X}_i, G_i, w \circ \pi_i)$ on $Y$ is compatible with the atlas $\mathcal{Y}$; assuming (WE1), this is equivalent to imposing that $[w_*(\mathcal{X})] = [\mathcal{Y}]$ (see Definition 1.17).

We say that $[\tilde{w}]$ is an open embedding of reduced orbifold atlases if and only if it satisfies the following conditions:

(OE1) the continuous map $w : X \to Y$ is a topological open embedding;

(OE2) for each $i \in I$ the chart $(\tilde{X}_i, G_i, w \circ \pi_i)$ on $Y$ is compatible with the atlas $\mathcal{Y}$; assuming (OE1), this is equivalent to imposing that $[w_*(\mathcal{X})] = [\mathcal{Y}]_{w(X)}$, where $\mathcal{Y}_{w(X)}$ is the reduced orbifold atlas induced by $\mathcal{Y}$ on the open set $w(X) \subseteq Y$.

So we have the following chain of inclusions:

$\{\text{refinements}\} \subset \{\text{unit weak equivalences}\} \subset \{\text{weak equivalences}\} \subset \{\text{open embeddings}\}$.

**Lemma 6.2.** Let us fix any pair of reduced orbifold atlases

\[ \mathcal{X} = \left\{ \left( \tilde{X}_i, G_i, \pi_i \right) \right\}_{i \in I}, \quad \mathcal{Y} = \left\{ \left( \tilde{Y}_j, H_j, \chi_j \right) \right\}_{j \in J} \quad (6.2) \]

and any open embedding $[\tilde{w}]$ as in (19). Then for each $i \in I$ the smooth map $\tilde{w}_i : \tilde{X}_i \to \tilde{Y}_{\pi(i)}$ is étale (i.e. a local diffeomorphism).

**Proof.** Let us fix any $i \in I$ and any $\tilde{x}_i \in \tilde{X}_i$. By definition of morphism in (Red Atl), we have

\[ w \circ \pi_i = \chi_{\pi(i)} \circ \tilde{w}_i, \quad (6.3) \]

so $w \circ \pi_i(\tilde{x}_i)$ belongs to $\chi_{\pi(i)}(\tilde{Y}_{\pi(i)})$. By (OE2), the chart $(\tilde{X}_i, G_i, w \circ \pi_i)$ is compatible with the atlas $\mathcal{Y}$, so in particular it is compatible with $(\tilde{Y}_{\pi(i)}, H_{\pi(i)}, \chi_{\pi(i)})$. Therefore there exists a change of charts $\lambda$ from $(\tilde{X}_i, G_i, w \circ \pi_i)$ to $(\tilde{Y}_{\pi(i)}, H_{\pi(i)}, \chi_{\pi(i)})$, such that $\tilde{x}_i \in \text{dom } \lambda$. By Definition 1.3 we have

\[ \chi_{\pi(i)} \circ \lambda = w \circ \pi_i|_{\text{dom } \lambda}. \quad (6.4) \]

Then let us consider the map

\[ \overline{\lambda} := \tilde{w}_i \circ \lambda^{-1} : \text{cod } \lambda \longrightarrow \tilde{Y}_{\pi(i)}. \quad (6.5) \]

For each $\overline{y} \in \text{cod } \lambda$ we have

\[ \chi_{\pi(i)} \circ \overline{\lambda}(\overline{y}) = \chi_{\pi(i)} \circ \tilde{w}_i \circ \lambda^{-1}(\overline{y}) \quad \text{(6.6)} \]

\[ = w \circ \pi_i \circ \lambda^{-1}(\overline{y}) = \chi_{\pi(i)}(\overline{y}). \quad \text{(6.7)} \]

So for each $\overline{y}$ as before, there exists a (in general non-unique) $h \in H_{\pi(i)}$ such that $\overline{\lambda}(\overline{y}) = h(\overline{y})$. Since $\text{cod } \lambda$ is connected, then by [MM] Lemma 2.11 there is a unique $h \in H_{\pi(i)}$ such that $\overline{\lambda} = h|_{\text{cod } \lambda}$. Therefore,

\[ \tilde{w}_i|_{\text{dom } \lambda} \circ \lambda = \overline{\lambda} \circ \lambda. \quad \text{(6.8)} \]

So we have proved that for each $i \in I$ the map $\tilde{w}_i$ coincides locally with a diffeomorphism. \hfill $\square$

**Remark 6.3.** The previous lemma shows that the morphisms called “lift of the identity” in [PG] Definition 5.8] coincide with the unit weak equivalences defined before.

**Lemma 6.4.** Let us fix the following data:
(a) a pair of reduced orbifold atlases $\mathcal{X} := \{(\tilde{X}_i, G_i, \pi_i)\}_{i \in I}$ over $X$ and $\mathcal{Y} := \{(\tilde{Y}_j, H_j, \chi_j)\}_{j \in J}$ over $Y$;
(b) a topological open embedding $w : X \hookrightarrow Y$;
(c) a set map $\mathbb{W} : I \to J$;
(d) for each $i \in I$ an étale map $\tilde{w}_i : \tilde{X}_i \to \tilde{Y}(i)$ such that $\chi_{\tilde{w}_i} \circ \tilde{w}_i = w \circ \pi_i$.

Then there is a unique class $[P_w, \nu_w]$, such that the collection $[\tilde{w}] := (w, \mathbb{W}, \{\tilde{w}_i\}_{i \in I}, [P_w, \nu_w])$ is a morphism of reduced orbifold atlases. Moreover, in this case $[\tilde{w}]$ is actually an open embedding of reduced orbifold atlases.

**Remark 6.5.** Combining this with Lemma 6.2, this means that each open embedding $[\tilde{w}]$ is completely determined by an underlying topological open embedding and by a collection of étale local liftings. In particular, each refinement is completely determined by a collection of open embeddings from each chart of $\mathcal{X}$ to some charts of $\mathcal{Y}$, commuting with the projections.

**Proof of Lemma 6.4.** Let us fix any pair $i, i' \in I$, any $\lambda \in \text{Ch}(\mathcal{X}, i, i')$ and any point $\tilde{x}_i \in \text{dom} \lambda$. Since $\tilde{w}_i$ and $\tilde{w}_{i'}$ are both étale, then there are open neighborhoods $\tilde{X}_i(\tilde{x}_i)$ of $\tilde{x}_i$ in $\text{dom} \lambda$, respectively $\tilde{X}_{i'}(\lambda(\tilde{x}_i))$ of $\lambda(\tilde{x}_i)$ in $\text{cod} \lambda$, where $\tilde{w}_i$, respectively $\tilde{w}_{i'}$, is invertible. Up to restricting $\tilde{X}_i(\tilde{x}_i)$ to $\lambda^{-1}(\tilde{X}_{i'}(\lambda(\tilde{x}_i)))$, we can assume that $\lambda(\tilde{X}_i(\tilde{x}_i)) = \tilde{X}_{i'}(\lambda(\tilde{x}_i))$. So it makes sense to consider the diffeomorphism

$$\nu_w(\lambda, \tilde{x}_i) := \tilde{w}_{i'} \circ \lambda \circ \tilde{w}_i^{-1} : \tilde{w}_i(\tilde{X}_i(\tilde{x}_i)) \to \tilde{w}_{i'}(\tilde{X}_{i'}(\lambda(\tilde{x}_i))).$$

By (d), this is a change of charts of $\mathcal{Y}$. Now for each $\lambda \in \text{Ch}(\mathcal{X}, i, i')$ we choose a collection of points $\{\tilde{x}_{i,t}\}_{t \in T(\lambda)} \subset \text{dom} \lambda$, such that the family $\{\tilde{X}_i(\tilde{x}_{i,t})\}_{t \in T(\lambda)}$ covers $\text{dom} \lambda$ (and such that $\tilde{X}_i(\tilde{x}_{i,t}) \neq \tilde{X}_i(\tilde{x}_{i',t})$ for each $t \neq t'$). Then we define

$$P_w := \left\{ \lambda|_{\tilde{X}_i(\tilde{x}_{i,t})} \quad \forall i \in I, \; \forall \lambda \in \text{Ch}(\mathcal{X}, i, -), \; \forall t \in T(\lambda) \right\};$$

for each $\lambda|_{\tilde{X}_i(\tilde{x}_{i,t})}$ in such a set, we define $\nu_w(\lambda|_{\tilde{X}_i(\tilde{x}_{i,t})}) := \nu_w(\lambda, \tilde{x}_{i,t})$. Then it is easy to see that the class $[P_w, \nu_w]$ is such that $(w, \mathbb{W}, \{\tilde{w}_i\}_{i \in I}, [P_w, \nu_w])$ is a morphism of reduced orbifold atlases. The class $[P_w, \nu_w]$ is unique is a direct consequence of [M56] (see Definition 1.10): given any $\lambda$ in $P_w(i, i')$, since $\tilde{w}_i$ is a diffeomorphism if restricted enough in source and target, then for each $\tilde{x}_i \in \text{dom} \lambda$ the value of $\nu_w(\lambda)$ around $\tilde{w}_i(\tilde{x}_i)$ is completely determined by [M56]. In other terms, the class $[P_w, \nu_w]$ is uniquely determined.

**Lemma 6.6.** Let us fix the following data:

- a finite number of reduced orbifold atlases $\mathcal{X}^1, \ldots, \mathcal{X}^r$ over a topological space $X$, all belonging to the same orbifold structure $[\mathcal{X}]$;
- a reduced orbifold atlas $\mathcal{X}'$ over a topological space $X'$;
- an homeomorphism $w : X' \to X$;
- a collection of weak equivalences of reduced orbifold atlases $[\tilde{w}^m] : \mathcal{X}' \to \mathcal{X}^m$ for $m = 1, \ldots, r$, all defined over $w$.

Then there are a reduced orbifold atlas $\overline{\mathcal{X}}$ over $X$ and a weak equivalence $[\overline{v}] : \overline{\mathcal{X}} \to \mathcal{X}'$, such that:

- $[\overline{v}]$ is defined over the homeomorphism $w^{-1} : X \to X'$;
- each local lift of $[\overline{v}]$ is an open embedding;
- for each $m = 1, \ldots, r$, $[\tilde{w}^m] \circ [\overline{v}]$ is a refinement.

In particular, if $\mathcal{X}' = X$ and $w = \text{id}_X$, then also $[\overline{v}]$ is a refinement.

**Proof.** Let us suppose that $\mathcal{X}' = \{(\tilde{X}_i', G_i', \pi_i')\}_{i \in I}$ and that

$$[\tilde{w}^m] := \left( w, \mathbb{W}^m, \{\tilde{w}_i^m\}_{i \in I}, [P_{w^m}, \nu_{w^m}] \right) : \mathcal{X}' \to \mathcal{X}^m \quad \text{for} \; m = 1, \ldots, r.$$
By Lemma 6.2 for each $i \in I$ and for each $m = 1, \ldots, r$ the smooth map $\tilde{w}_i^m$ is a local diffeomorphism, so for each $i \in I$ there exist a (non-unique) open covering $\{ \tilde{X}_{i,a} \}_{a \in A(i)}$ of $\tilde{X}_i$ such that for each $m = 1, \ldots, r$, the map $\tilde{w}_i^m$ is an open embedding if restricted to any $\tilde{X}_{i,a}$. For each $i \in I$ and for each $a \in A(i)$ there exists a (non-unique) open covering $\{ \tilde{X}_{i,a,b} \}_{b \in B(i,a)}$ of $\tilde{X}_{i,a}$, such that each $\tilde{X}_{i,a,b}$ is the domain of a chart $(\tilde{X}_{i,a,b}, G_{i,a,b}, \nu_{i,a,b})$ compatible with $\chi'$. Then we define a reduced orbifold atlas on $\tilde{X}$ as follows:

$$\mathcal{X} := \left\{ \left( \tilde{X}_{i,a,b}, G_{i,a,b}, \nu_{i,a,b} \right) \right\}_{i \in I, a \in A(i), b \in B(i,a)}.$$  

Now we consider the set map $\nabla$ sending each triple $(i, a, b)$ to $i$; for each such triple we define $\tilde{v}_{i,a,b}$ as the inclusion of $\tilde{X}_{i,a,b}$ in $\tilde{X}_i$. Now let us fix any change of charts

$$\lambda \in C_h (\mathcal{X}, (i, a, b), (i', a', b')).$$  

By definition of change of charts, we have $w \circ \pi_{i'} \circ \lambda = w \circ \pi_i$; since $w$ is an homeomorphism, this implies that $\pi_{i'} \circ \lambda = \pi_i$, so $\lambda$ can be considered as a change of charts in $C_h (\chi', i, i')$; we denote by $\nu_i (\lambda)$ such a change of charts. Then we get easily that the collection

$$[v] \ := \ w^{-1}, \vec{\nabla}, \{ \tilde{v}_{i,a,b} \}_{i,a,b}, [C_h (\mathcal{X}), \nu_v] : \mathcal{X} \to \chi'$$  

is a morphism of reduced orbifold atlases. Now for each triple $(i, a, b)$ and for each $m = 1, \ldots, r$ the morphism $\tilde{w}_i^m \circ \tilde{v}_{i,a,b}$ is an open embedding because by construction $\tilde{w}_i^m$ is an open embedding if restricted to $\tilde{X}_{i,a,b} \subseteq \tilde{X}_i$. Moreover, the morphism $[\tilde{w}_i^m] \circ [v]$ is defined over $w \circ w^{-1} = 1_{\tilde{X}}$. Therefore, $[\tilde{w}_i^m] \circ [v]$ is a refinement for each $m = 1, \ldots, r$. □

The following lemmas are on the same line of [Po, Propositions 5.3 and 6.2]; the significant difference is given by the fact that we consider all the weak equivalences instead of restricting only the unit weak equivalences considered in [Po].

**Lemma 6.7.** If $[\tilde{w}] : \mathcal{X} \to \mathcal{Y}$ is a weak equivalence of reduced orbifold atlases, then $\mathcal{F}_{\text{red}} (\mathcal{X})$ is a Morita equivalence of proper, effective, étale groupoids.

**Proof.** Let us use the notations of (6.1) and (6.2) and let us set:

$$\mathcal{F}_{\text{red}} (\mathcal{X}) := \left( \mathcal{X}_1 \overset{\pi^1}{\to} \mathcal{F}_0 \right), \quad \mathcal{F}_{\text{red}} (\mathcal{Y}) := \left( \mathcal{Y}_1 \overset{\pi^1}{\to} \mathcal{F}_0 \right), \quad \mathcal{F}_{\text{red}} ([\tilde{w}]) := (\psi_0, \psi_1).$$

(6.6)

By Lemma 6.2 each $\tilde{w}_i$ is étale; therefore also $\psi_0 = \bigsqcup_{i \in I} \tilde{w}_i$ is étale. So also the induced projection (see property (WE1))

$$\pi^1 : \mathcal{F}_1 \times_{\psi_0} \mathcal{F}_0 \to \mathcal{Y}_1$$

is étale. Since $t$ is also étale, we conclude that $t \circ \pi^1$ is étale, so in particular it is a submersion. Let us prove that it is also surjective. Let us fix any point $\tilde{y} \in \tilde{Y}_i \subseteq \mathcal{Y}_0$; since $w$ is an homeomorphism by (WE1), then it makes sense to define $x := w^{-1}(\chi_j (\tilde{y}))$. Let us choose any chart $(\tilde{X}_i, G_i, \pi_i)$ in $\mathcal{X}$ and any $\tilde{x}_i \in \tilde{X}_i$, such that $\pi_i (\tilde{x}_i) = x$. Then by definition of morphism in $(\text{Red}, \text{Atl})$ we have:

$$\chi_{(i)} \circ \tilde{w}_i (\tilde{x}_i) = w \circ \pi_i (\tilde{x}_i) = w (x) = \chi_j (\tilde{y}).$$

Since $\mathcal{Y}$ is a reduced orbifold atlas, there exists a change of charts $\omega$ from $(\tilde{Y}_m (i), H_{m (i)}, \chi_{(i)} (i))$ to $(\tilde{Y}_j, H_j, \chi_j)$, such that $\tilde{w}_i (\tilde{x}_i) \in \text{dom} \omega$ and $\omega (\tilde{w}_i (\tilde{x}_i)) = \tilde{y}$. Then we set $p := \text{germ}_{\tilde{w}_i (\tilde{x}_i)} \omega (\tilde{x}_i)$ and we get that $p$ belongs to the fiber product $\mathcal{Y}_1 \times_{\psi_0} \mathcal{F}_0$;
moreover $t \circ \pi^1(p) = \tilde{y}_j$. So we have proved that $t \circ \pi^1$ is surjective, so \( \text{(ME1)} \) holds.

In order to prove that $(\psi_0, \psi_1)$ is a Morita equivalence, we have also to prove \( \text{(ME2)} \), i.e., we have to show that the commutative square \( \text{(5.1)} \) has the universal property of fiber products. So let us fix any smooth manifold $M$ together with any pair of smooth maps $a = (a_1, a_2) : M \to X_0 \times X_0$ and $b : M \to Y_1$, such that $(s, t) \circ b = (\psi_0 \times \psi_0) \circ (a_1, a_2)$. We have to prove that there is a unique smooth map $c : M \to Y_1$, making the following diagram commute:

\[
\begin{array}{ccc}
M & \xrightarrow{c} & Y_1 \\
\downarrow & & \downarrow \\
X_0 \times X_0 & \xrightarrow{(s, t) \circ b} & Y_0 \times Y_0.
\end{array}
\]

We have already said that $\psi_0$ is étale; moreover by definition of étale groupoid we have $\psi_0 \circ s = s \circ \psi_1$ and the 2-morphisms $s$ in the previous identity are étale; hence $\psi_1$ is étale. So if we fix any point $x_1 \in Y_1$ and we set $x_0 := s(x_1)$ and $x'_0 := t(x_1)$, we have:

\[
\begin{align*}
\kappa_{X}(x_0, x'_0, x_1) & \overset{(4.3)}{=} \text{germ}_{x_1} t \cdot (\text{germ}_{s, x_1} s)^{-1} = \\
& = \text{germ}_{x_1} t \cdot (\text{germ}_{x_1} \psi_1)^{-1} \cdot \text{germ}_{x_1} \psi_1 \cdot (\text{germ}_{s, x_1} s)^{-1} = \\
& = (\text{germ}_{x_0} \psi_0)^{-1} \cdot \text{germ}_{x_0} \psi_1 \cdot (\text{germ}_{x_1(x_1)} s)^{-1} \cdot \text{germ}_{x_0} \psi_0.
\end{align*}
\]

By Lemma \( \text{(4.9)} \) the set map $\kappa_{X}(x_0, x'_0, -)$ is a bijection, so:

\[
x_1 = \kappa_{X}(x_0, x'_0, -)^{-1} \left((\text{germ}_{x_0} \psi_0)^{-1} \cdot \kappa_{Y}(\psi_0(x_0), \psi_0(x'_0), \psi_1(x_1)) \cdot \text{germ}_{x_0} \psi_0\right).
\]

So for every point $m \in M$, if $c(m)$ exists making \( \text{(6.7)} \) commute, then it is necessarily equal to

\[
c(m) = \kappa_{X}(a_1(m), a_2(m), -)^{-1}(g(m)),
\]

where $g(m)$ is the germ defined as follows:

\[
g(m) := (\text{germ}_{a_2(m)} \psi_0)^{-1} \cdot \kappa_{Y}(\psi_0 \circ a_1(m), \psi_0 \circ a_2(m), b(m)) \cdot \text{germ}_{a_1(m)} \psi_0.
\]

This proves the uniqueness of a morphism $c$ as in \( \text{(5.7)} \). Moreover, using the definition of $\kappa_{X}(\cdot, \cdot, \cdot)$, the previous lines actually give rise to a well-defined set map $c : M \to Y_1$, making \( \text{(6.7)} \) commute. In particular, we have $s \circ c = a_1$; since $a_1$ is smooth by hypothesis and $s$ is étale, this implies that $c$ is smooth, so we have proved that property \( \text{(ME2)} \) holds for $(\psi_0, \psi_1)$. \( \square \)

**Lemma 6.8.** Let $X$ and $Y$ be reduced orbifold atlases (for $X$ and $Y$ respectively). Let $(\psi_0, \psi_1) : F_0^{\text{red}}(X) \to F_0^{\text{red}}(Y)$ be a Morita equivalence and let $[\hat{w}] : X \to Y$ be the unique morphism such that $F_1^{\text{red}}([\hat{w}]) = (\psi_0, \psi_1)$ (see Lemma \( \text{(4.19)} \)). Then $[\hat{w}]$ is a weak equivalence of reduced orbifold atlases.
Proof. Let us use the notations of (6.2) and (6.16) and let us denote by

\[ \tilde{\omega} = \left( w, \overline{w}, \{ \tilde{w}_i \}_{i \in I}, \{ F_w, \nu_w \} \right) : \mathcal{X} \to \mathcal{Y} \]

the unique morphism obtained from \( \psi \) using Lemma 5.19. We recall that in the proof of such lemma, we defined the continuous map \( w : X \to Y \) (denoted by \( f \) in the mentioned lemma), as \( w := \varphi_Y \circ \psi \circ \varphi_X^{-1} \), where both \( \varphi_X \) and \( \varphi_Y \) are homeomorphisms. Since \( \psi \) is a Morita equivalence of étale groupoids, then by Lemma 5.5 we have that also \( |\psi_*| \) is an homeomorphism, so property (WE1) is satisfied.

In order to prove that \( \tilde{\omega} \) is a weak equivalence, we need also to prove that for each \( i \in I \) the chart \( (\tilde{X}_i, G_i, w \circ \pi_i) \) on \( Y \) is compatible with the atlas \( \mathcal{Y} \). So let us fix any index \( j \in J \) and any pair of points \( (\tilde{x}_i, \tilde{y}_j) \in \tilde{X}_i \times \tilde{Y}_j \) such that \( w \circ \pi_i(\tilde{x}_i) = \chi_j(\tilde{y}_j) \).

Then we have

\[ \chi_{\pi(i)} \circ \tilde{w}_j(\tilde{x}_i) = w \circ \pi_i(\tilde{x}_i) = \chi_j(\tilde{y}_j). \]

Since \( \mathcal{Y} \) is a reduced orbifold atlas, then there exists a change of charts \( \omega \) from \( (\tilde{Y}_j, H_j, \chi_j) \) to \( (\tilde{Y}, H, \chi) \), such that \( \tilde{w}_i(\tilde{x}_i) \in \text{dom } \omega \). Since \( \omega \) is a change of charts, then \( \chi_j \circ \omega = \chi_{\pi(i)} \). Moreover, since \( \psi_0 \) is étale (see (MM) Exercise 5.16(4))), then the map \( \tilde{w}_i = \psi_0 \circ \tilde{\omega} \) is locally a diffeomorphism. Therefore there exists an open neighborhood \( \tilde{X} \) of \( \tilde{x}_i \), contained in \( \tilde{w}_i^{-1}(\text{dom } \omega) \), such that \( \tilde{w}_i \) is an embedding if restricted to \( \tilde{X} \). Then \( \omega := \omega \circ \tilde{w}_i |_{\tilde{X}} \) is a smooth embedding from \( \tilde{X} \subseteq \tilde{\tilde{X}} \) to \( \tilde{Y}_j \). Up to restricting \( \tilde{X} \) to a smaller neighborhood of \( \tilde{x}_i \), we can always assume that \( \tilde{X} \) is the domain of a change of charts of \( \mathcal{X} \). Moreover, we have

\[ \chi_j \circ \tilde{\omega} = \chi_j \circ \omega \circ \tilde{w}_i |_{\tilde{X}} = \chi_{\pi(i)} \circ \tilde{w}_i |_{\tilde{X}} = w \circ \pi_i |_{\tilde{X}}, \]

so \( \tilde{\omega} \) is a change of charts from \( (\tilde{X}_i, G_i, w \circ \pi_i) \) to \( (\tilde{Y}_j, H_j, \chi_j) \) with \( \tilde{x}_i \in \text{dom } \tilde{\omega} \). So we have proved that the chart \( (\tilde{X}_i, G_i, w \circ \pi_i) \) is compatible with \( \mathcal{Y} \) for every \( i \in I \), i.e. condition (WE2) holds. \( \square \)

By combining Lemmas 6.7 and 6.8 we get:

**Proposition 6.9.** Given any 2 reduced orbifold atlases \( \mathcal{X}, \mathcal{Y} \), the bijection

\[ \{ \text{morphisms } [f] : \mathcal{X} \to \mathcal{Y} \text{ in } (\text{Red Atl}) \} \to \{ \text{morphisms } \phi_* : \mathcal{F}_\text{red}^0(\mathcal{X}) \to \mathcal{F}_\text{red}^0(\mathcal{Y}) \text{ in } (\mathcal{PE\tilde{C}} \mathcal{Gpd}) \} \]

of Lemma 5.19 induces a bijection between weak equivalences of reduced orbifold atlases and Morita equivalences of proper, effective, étale groupoids.

Then we are able to compute the right saturation (see Definition 5.4) of the class \( W_{\text{Red Atl}} \) of all refinements.

**Lemma 6.10.** The right saturation \( W_{\text{Red Atl sat}} \) is the class of all weak equivalences of reduced orbifold atlases.

**Proof.** Let us fix any morphism \( [f] : \mathcal{Y} \to \mathcal{X} \) in \( W_{\text{Red Atl sat}} \). By Definition 5.3 this implies that there are a pair of reduced orbifold atlases \( \mathcal{U}, \mathcal{Z} \) and a pair of morphisms \( [h] : \mathcal{U} \to \mathcal{Z} \) and \( [g] : \mathcal{Z} \to \mathcal{Y} \), such that both \( [f] \circ [g] \) and \( [g] \circ [h] \) are refinements (hence weak equivalences of reduced orbifold atlases). So by Proposition 6.9 we have that both \( \mathcal{F}_2^{\text{red}}([f]) \circ \mathcal{F}_2^{\text{red}}([g]) \) and \( \mathcal{F}_2^{\text{red}}([g]) \circ \mathcal{F}_2^{\text{red}}([h]) \) are Morita equivalences of étale groupoids. In other terms, the morphism \( \mathcal{F}_2^{\text{red}}([f]) \) belongs to the right saturation of the class \( W_{\mathcal{Gpd}} \) of Morita equivalences between étale groupoids. So
by Lemma [6.7] we conclude that actually \( F^\text{red}(\hat{f}) \) is a Morita equivalence. Again by Proposition [6.9] this implies that \( \hat{f} \) is a weak equivalence of reduced orbifold atlases. So we have proved that \( W_{\text{Red,Atl,sat}} \) is contained in the set of all weak equivalences of reduced orbifold atlases. Conversely, let us suppose that \( \hat{f} : Y \to X \) is a weak equivalence. Then by Lemma [6.6] there are a reduced orbifold atlas \( Z \) and a weak equivalence \([\hat{g}] : Z \to Y\) such that \( \hat{f} \circ [\hat{g}] \) is a refinement. Applying Lemma [6.6] a second time on \([\hat{g}]\), there are a reduced orbifold atlas \( U \) and a weak equivalence \([\hat{h}] : U \to Z\) such that \([\hat{g}] \circ [\hat{h}]\) is a refinement. Therefore, the morphism \( \hat{f} \) belongs to the right saturation \( W_{\text{Red,Atl,sat}} \). This suffices to conclude.

Lemma 6.11. Let us fix any proper, effective, étale groupoid \((\mathcal{X}', \to, \mathcal{X}_0)\). Then there are a reduced orbifold atlas \( X \) and a Morita equivalence \((\psi_0, \psi_1) : F^\text{red}(X) \to (\mathcal{X}', \to, \mathcal{X}_0)\).

Proof. Given \((\mathcal{X}', \to, \mathcal{X}_0)\), the reduced orbifold atlas \( X \) is obtained as in the last part of the proof of Theorem 4.1 in [MP]. In [T1] Lemmas 4.7, 4.8 and 4.9 were done in the category of complex manifolds, but they can be easily adapted to the case of smooth manifolds, so we omit the details.

Lemma 6.12. Let us fix any pair of reduced orbifold atlases \( X, Y \) and any pair of open embeddings \([\hat{w}^m] : X \to Y \) for \( m = 1, 2 \). Then the following facts are equivalent:

(a) the underlying topological maps \( w^1 \) and \( w^2 \) coincide;
(b) there exists a 2-morphism \([\hat{\alpha}] : [\hat{w}^1] \Rightarrow [\hat{w}^2] \) in \((\text{Red,Atl})\);
(c) there exists a unique 2-morphism \([\hat{\alpha}] : [\hat{w}^1] \Rightarrow [\hat{w}^2] \) in \((\text{Red,Atl})\).

In particular, (c) holds if we consider any pair of refinements (or more generally, any pair of unit weak equivalences).

Proof. Clearly (b) implies (a) by Definitions [1.18] and [1.21] of 2-morphism of reduced orbifold atlases, so let us prove that (a) implies (b). Let

\[
X := \left\{ (\hat{X}_i, G_i, \pi_i) \right\}_{i \in I}, \quad Y := \left\{ (\hat{Y}_j, H_j, \chi_j) \right\}_{j \in J}
\]

and let

\[
\hat{w}^m := (w^m, \omega^m, \{\hat{w}^m_i\}_{i \in I}, P^m_i, \nu^m_i)
\]

be representatives for \([\hat{w}^1]\) and \([\hat{w}^2]\) respectively, with \( w^1 = w^2 \). By Lemma 6.2 for each \( i \in I \) both \( \hat{w}^1_i \) and \( \hat{w}^2_i \) are étale. Therefore for each \( i \in I \) there exists an open covering \( \{\hat{X}^a_i\}_{a \in A(i)} \) of \( \hat{X}_i \) such that both \( \hat{w}^1_i \) and \( \hat{w}^2_i \) are diffeomorphisms if restricted to any \( \hat{X}^a_i \). Up to replacing \( \{\hat{X}^a_i\}_{a \in A(i)} \) by a finer covering, we can assume that each \( \hat{X}^a_i \) is the domain of a change of charts of \( X \). Since both \([\hat{w}^1]\) and \([\hat{w}^2]\) are morphisms in \((\text{Red,Atl})\), then for each \( i \in I \) we have

\[
\chi^a_{\omega^m(i)} \circ \hat{w}^1_i = w^1 \circ \pi_i \quad \text{and} \quad \chi^a_{\omega^m(i)} \circ \hat{w}^2_i = w^2 \circ \pi_i = w^1 \circ \pi_i. \quad (6.8)
\]

Then for each \( i \in I \) and for each \( a \in A(i) \) we set:

\[
\delta^a_i := \hat{w}^2_i \circ (\hat{w}^1_i |_{\hat{X}^a_i})^{-1}.
\]

By construction, each \( \delta^a_i \) is a diffeomorphism; moreover, by (6.8) each \( \delta^a_i \) is a change of charts in \( \mathcal{C}(\hat{Y}, \omega^m(i), \hat{w}^m(i)) \). Then it is easy to see that the family \( \delta := \{(\hat{X}^a_i, \delta^a_i)\}_{i \in I, a \in A(i)} \) satisfies properties (2Main) – (2Main), so \( [\hat{\alpha}] \) is a 2-morphism.
from $[\hat{w}]^1$ to $[\hat{w}]^2$. Therefore (a) implies (b).

Since (c) implies (b), we need only to prove the opposite implication. Let us suppose that there exists another 2-morphism $[\overline{\delta}] : [\hat{w}]^1 \Rightarrow [\hat{w}]^2$ with representative

$$
\overline{\delta} = \left\{ \left( \hat{X}_i^\delta, \delta_i^\delta \right) \right\}_{\delta \in \mathcal{E} \in \mathcal{A}}.
$$

Let us fix any $i \in I$ and any pair $(\alpha, \overline{\alpha}) \in \mathcal{A}(i) \times \overline{\mathcal{A}}(i)$ such that $\hat{X}_i^\alpha \cap \hat{X}_i^\overline{\alpha} \neq \emptyset$. Then by property (2Mc) for $\delta$ and $\overline{\delta}$, we get that $\delta_i^\alpha$ coincides with $\delta_i^\overline{\alpha}$ on the set $\hat{w}_i^1(\hat{X}_i^\alpha \cap \hat{X}_i^\overline{\alpha})$; such a set is open because $\hat{w}_i^1$ is étale by Lemma 6.2. Therefore, for each $\hat{x}_i \in \hat{X}_i^\alpha \cap \hat{X}_i^\overline{\alpha}$ we have

$$
geq_{\hat{w}_i^1(\hat{x}_i)} \delta_i^\alpha = \delta_i^\overline{\alpha},$$

so $[\delta] = [\overline{\delta}]$ by Definition 1.21. \qed

7. The bicategories $(\text{Red Orb})$ and $(\mathcal{P} \mathcal{E} \mathcal{E} \mathcal{P} \mathcal{d})$ $\left[ W_{\mathcal{P} \mathcal{E} \mathcal{E} \mathcal{P} \mathcal{d}}^{-1} \right]

In this section we will prove that the pair $((\text{Red Atl}), W_{\text{Red Atl}})$ admits a right bicalculus of fractions and we will give a simple description of the associated bicategory of fractions $(\text{Red Orb})$. We will also recall briefly the description of the bicategory $(\mathcal{P} \mathcal{E} \mathcal{E} \mathcal{P} \mathcal{d})$ $\left[ W_{\mathcal{P} \mathcal{E} \mathcal{E} \mathcal{P} \mathcal{d}}^{-1} \right]$. In the next section we will prove that such 2 bicategories are equivalent.

**Proposition 7.1.** The pair $((\text{Red Atl}), W_{\text{Red Atl}})$ admits a right bicalculus of fractions, so there are a bicategory $(\text{Red Orb}) := (\text{Red Atl}) \left[ W_{\text{Red Atl}}^{-1} \right]$ and a pseudofunctor

$$
\mathcal{U}_{W_{\text{Red Atl}}} : (\text{Red Atl}) \longrightarrow (\text{Red Orb})
$$

that sends every refinement of reduced orbifold atlases to an internal equivalence and that is universal with respect to such a property (see Remark 5.3).

**Proof.** Condition (11Mc) is obviously satisfied and (11Mc2) is an easy consequence of the definition of compositions (see Construction 1.11). Let us consider (11Mc3), so let us fix any triple of reduced orbifold atlases $X, Y, Z$, any refinement $[\hat{w}] : X \rightarrow Y$ and any morphism $[f] : Z \rightarrow Y$. Using Lemmas 6.7 and 4.12 we have that $F_{\text{red}}([\hat{w}])$ is a Morita equivalence between proper, effective and étale groupoids; moreover we have proved in Proposition 5.9 that the set $W_{\mathcal{P} \mathcal{E} \mathcal{E} \mathcal{P} \mathcal{d}}$ satisfies (BF3). Therefore there exist a proper, effective, étale Lie groupoid $\mathcal{U}$, a Morita equivalence $\psi$, a morphism $\phi$ and a natural transformation $\alpha$ as follows:

$$
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\psi} & \mathcal{U} \\
\alpha & \Rightarrow & \phi
\end{array}
$$

$F_{\text{red}}(Z) \xrightarrow{F_{\text{red}}([f])} F_{\text{red}}(Y) \xrightarrow{F_{\text{red}}([\hat{g}])} F_{\text{red}}(X)$.

By Lemma 6.11 there exist a reduced orbifold atlas $\mathcal{U}$ and a Morita equivalence $\psi_{\text{red}} : F_{\text{red}}(\mathcal{U}) \rightarrow \mathcal{U}$. Lemmas 4.19 and 4.21 prove that there are a pair of morphisms $[\hat{w}], [\hat{g}]$ and a 2-morphism $[\theta]$ in $(\text{Red Atl})$ as follows
such that the 2-functor $\mathcal{F}_{\text{red}}$ maps such a diagram to

\[
\begin{array}{ccc}
\mathcal{F}_{\text{red}}(X) & \xrightarrow{\phi' \circ \psi''} & \mathcal{F}_{\text{red}}(Y) \\
\mathcal{F}_{\text{red}}(Z) & \xrightarrow{\alpha \circ \psi'} & \mathcal{F}_{\text{red}}(U)
\end{array}
\]

Now $\psi' \circ \psi''$ is a Morita equivalence (see condition (BF2) for the class $\mathcal{W}_{\text{red}}^\text{fpd}$). So by Lemma 6.8 we have that $[\hat{v}]$ is a weak equivalence of reduced orbifold atlases. By Lemma 6.6 there is a reduced orbifold atlas $V$ and a weak equivalence $[\hat{w}] : V \to U$, such that $[\hat{v}] \circ [\hat{w}]$ is a refinement. Then we set

$[\hat{w}] := [\hat{v}] \circ [\hat{u}], \quad [\bar{f}] := [\hat{g}] \circ [\hat{u}], \quad [\hat{a}] := [\hat{\theta}] \circ i_{\hat{u}}$.

By Lemma 7.2 (a) we get that $[\hat{a}]$ is an invertible 2-morphism, so the data $(V, [\hat{w}], [\bar{f}], [\hat{a}])$ prove that (BF3) holds for $\mathcal{W}_{\text{red}, \text{Atl}}$.

The proof that (BF3) holds follows the same ideas described for (BF2), so we omit it. Lastly, let us prove condition (BF3), so let us fix any pair of reduced orbifold atlases

$\mathcal{X} := \{(\tilde{X}_i,G_i,\pi_i)\}_{i \in I}, \quad \mathcal{Y} := \{(\tilde{Y}_j,H_j,\chi_j)\}_{j \in J}$

over $X$ and $Y$ respectively, any pair of morphisms

$[\hat{w}] := \left( w_1, w_2, \left\{ \tilde{w}^m_{1} \right\}_{1 \in I}, [P_{w_1}, P_{w_2}] \right) : \mathcal{X} \to \mathcal{Y}, \quad m = 1, 2$

and any 2-morphism

$[\hat{a}] := \left[ \left\{ \left( \tilde{X}_i^a, \delta^a_i \right) \right\}_{i \in I, a \in A(i)} \right] : [\hat{w}] \Longrightarrow [\hat{w}]$

in (Red, Atl). Moreover, let us suppose that $[\hat{w}]$ is a refinement. This implies that $X = Y$, $w_2 = \text{id}_X$, and that every smooth map $\tilde{w}^a_i$ is an open embedding. Now let us fix any $i \in I$, any $a \in A(i)$ and any point $\tilde{x}_i$ in the open set $\tilde{X}^a_i$. By (2Mc) we have that

$\tilde{w}^1_i(\tilde{x}_i) = (\delta^a_i)^{-1} \circ \tilde{w}^2_i(\tilde{x}_i)$

so $\tilde{w}^1_i$ locally coincides with an open embedding, hence $\tilde{w}^1_i$ is an étale map. Again by (2Mc) we get

$\tilde{x}_i = (\tilde{w}^1_i)^{-1} \circ \delta^a_i \circ \tilde{w}^2_i(\tilde{x}_i)$.

If we fix any other index $a' \in A(i)$ and any other point $\tilde{x}'_i \in \tilde{X}^a_i$, then we have also

$\tilde{x}'_i = (\tilde{w}^1_i)^{-1} \circ \delta^a_i \circ \tilde{w}^2_i(\tilde{x}'_i)$. 

Therefore, if \( \tilde{w}_i^1(\tilde{x}_i) = \tilde{w}_i^1(\tilde{x}_i') \), then \( \tilde{x}_i = \tilde{x}_i' \), i.e. \( \tilde{w}_i^1 \) is injective. So we conclude that the smooth map \( \tilde{w}_i^1 \) is an open embedding for each \( i \in I \), i.e. condition (REF2) holds for \( \tilde{w}^1 \).

By Lemma 6.12, the topological map \( w^1 \) coincides with \( w^2 = \text{id}_X \), so (REF1) holds. Therefore, we have proved that (BF5) holds for \( W_{\text{Red Atl}} \).

\[ \square \]

Remark 7.2. In a similar way we can prove that also the pairs

(a) \( (\text{Red Atl}) \) together with the class of all unit weak equivalences,

(b) \( (\text{Red Atl}) \) together with the class of all weak equivalences

satisfy axioms (BF1) - (BF5) (actually, (b) satisfies also the stronger axiom (BF1)', see Remark 5.3), so a bicalculus of fractions exists also for (a) and (b), and the resulting bicategories of fractions are equivalent to \( (\text{Red Orb}) \) (this is an easy consequence of [T2, Proposition 2.10] and Lemma 6.10). We prefer to use the class \( W_{\text{Red Atl}} \) as refinements instead of (a) or (b) because this leads to a bicategory \( (\text{Red Orb}) \) that is easier to describe and that is more close to the geometric intuition (see below for details).

Lemma 7.3. The set \( W_{\text{Red Atl}} \) satisfies a “weak 2-out-of-3 property”, i.e. given any pair of morphisms \( [f] : X \to Y \) and \( [g] : Y \to Z \) in (\( \text{Red Atl} \)), we have:

(i) if \( [f] \) and \( [g] \) belong to \( W_{\text{Red Atl}} \), then so does \( [g] \circ [f] \);

(ii) if \( [g] \) and \( [g] \circ [f] \) belong to \( W_{\text{Red Atl}} \), then so does \( [f] \).

The sets of unit weak equivalences and of all weak equivalences satisfy both the (strong) “2-out-of-3” property.

Note that if both \( [f] \) and \( [g] \circ [f] \) belong to \( W_{\text{Red Atl}} \), then in general it is not true that \( [g] \) belong to \( W_{\text{Red Atl}} \) (in general one can only prove that \( [g] \) is a unit weak equivalence).

Proof. (i) is simply (BF2) for the class \( W_{\text{Red Atl}} \) (see Proposition 7.1). For (ii), let us suppose that \( [f] \) and \( [g] \) are as in (5.3). Since \( [g] \) is a refinement, then \( Y = Z \), \( g = \text{id}_Z \) and \( \tilde{g}_j \) is an open embedding for each \( j \in J \). Since \( [g] \circ [f] \) is a refinement, then \( X = Z \), \( g \circ f = \text{id}_Z \) and \( \tilde{g}_j \circ \tilde{f}_i \) is an open embedding for each \( i \in I \). From this we get that \( \tilde{f} = \text{id}_X \) and that \( \tilde{f}_i \) is an open embedding for each \( i \in I \), i.e. \( [f] \) is a refinement.

The last part of the statement can be easily verified directly. For the class of all weak equivalences, one can also prove it using Lemmas 5.7 and 6.10. \[ \square \]

Corollary 7.4. Let us fix any triple of reduced orbifold atlases \( X, Y, Z \) over the same topological space \( X \) and any pair of refinements

\[ X \xrightarrow{[\tilde{w}^1]} Y \xleftarrow{[\tilde{w}^2]} Z. \]

Then there exists data in (\( \text{Red Atl} \)) as in the following diagram

\[ \xymatrix{ X \ar[r]^{[\tilde{w}^1]} & Y \ar[l]_{[\tilde{w}^2]} \ar@{=>}[d]^{[\tilde{v}^1]} \ar@{=>}[l]^{[\tilde{v}^2]} \ar@{=>}[r]^{[\tilde{v}^3]} \ar@{=>}[u]^{[\tilde{d}]} & Z \ar[u]^{[\tilde{v}^3]} \ar[l]_{[\tilde{v}^2]} \ar[r]^{[\tilde{v}^1]} & Z. } \]

such that also \([\tilde{v}^1]\) and \([\tilde{v}^2]\) are refinements.
weak equivalences of bicategories. In order to describe explicitly one such bicategory, different choices will give equivalent bicategories of fractions where objects, rem 21\[1\], and of the pseudofunctor \(U\) are refinements, therefore by Corollary 7.4 there are a reduced orbifold atlas \(X\) and a pair of refinements (hence, of weak equivalences) as in (7.2), so (1) holds.

Now let us assume that (3) holds and let us prove (1). So let us assume that \(X^1\) and \(X^2\) are equivalent atlases. Therefore, for \(m = 1, 2\) we can consider the inclusion \(\iota_{X^m}\) of \(X^m\) in the common maximal atlas \(X^{\text{max}}\) (see Definition 1.13). Both maps are refinements, therefore by Corollary 7.5 there are a reduced orbifold atlas \(X\) and a pair of refinements (hence, of weak equivalences) as in (7.2), so (1) holds.

Remark 7.6. For the explicit description of the bicategory of fractions \((\text{Red Orb})\) and of the pseudofunctor \(U_{\text{W\text{Red Atl}}}\), we refer mainly to the original construction in [P7] or to our previous paper [T2], where we have explained how to simplify the construction of associators and compositions of 2-morphisms in any bicategory of fractions. As it is stated in [P7], bicategories of fractions are unique only up to weak equivalences of bicategories. In order to describe explicitly one such bicategory, one has to make some choices as in the following description. By [P7] Theorem 21, different choices will give equivalent bicategories of fractions where objects, 1-morphisms and 2-morphisms are the same, but compositions of 1-morphisms and 2-morphisms are (possibly) different.

Description 7.7. Following [P7] § 2.2, 2.3 and 2.4, the bicategory \((\text{Red Orb})\) and the pseudofunctor \(U_{\text{W\text{Red Atl}}}\) can be described as follows.

- The objects of \((\text{Red Orb})\) are exactly the objects of \((\text{Red Atl})\), i.e. all the reduced orbifold atlases according to Definition 1.5.
- Given any pair of reduced orbifold atlases \(X, Y\), the 1-morphisms in \((\text{Red Orb})\) from the first atlas to the second one consist of all the triples \((X', [\hat{w}], [\hat{f}])\) where \(X'\) is any reduced orbifold atlas, \(\hat{w}\) is any refinement (see Definition 6.1) and \([\hat{f}]\) is any morphism of reduced orbifold atlases (see Definition 1.11), as follows

\[
\begin{array}{ccc}
X & \xrightarrow{[\hat{w}]} & X' \\
\downarrow & & \downarrow \leftarrow [\hat{f}] \\
\uparrow \downarrow & & \downarrow \uparrow \\
Y & \xrightarrow{[\hat{w}]} & Y
\end{array}
\]

(in particular, using Corollary 7.5 we have that \(X'\) is equivalent to \(X\)). In other terms, a morphism in \((\text{Red Orb})\) from \(X\) to \(Y\) consists firstly in replacing \(X\) with a “refined” atlas \(X'\) (keeping track of the refinement), then by considering a usual morphism of \((\text{Red Atl})\) from \(X'\) to \(Y\).
Given any pair of objects \( X, Y \) and any pair of morphisms \((X^m, [\hat{w}^m], [\hat{f}^m]) : X \to Y\) for \( m = 1, 2 \), according to the construction of bicategories of fractions in [Pr, § 2.3], a 2-morphism from \((X^1, [\hat{w}^1], [\hat{f}^1])\) to \((X^2, [\hat{w}^2], [\hat{f}^2])\) is an equivalence class of data \((X^3, [\hat{v}^1], [\hat{v}^2], [\mu], [\delta])\) in \((\text{Red Atl})\) as follows:

\[
\begin{array}{ccc}
X^1 \downarrow [\mu] & \longrightarrow & [\delta] \\
[\hat{w}^1] & \searrow & [\hat{v}^1] \\
X & \downarrow & X^3 \\
[\hat{w}^2] & \swarrow & [\hat{v}^2] \\
X^2 & \longleftarrow & X^3 \\
\end{array}
\]  

(7.3)

such that \([\hat{w}^1] \circ [\hat{v}^1]\) is a refinement (in [Pr] it is also required that \([\mu]\) is invertible, but this property is automatically satisfied by Lemma 3.2). Any other set of data

\[
\begin{array}{ccc}
X^1 \downarrow [\mu'] & \longrightarrow & [\delta'] \\
[\hat{w}^1] & \searrow & [\hat{v}^1] \\
X & \downarrow & X^3 \\
[\hat{w}^2] & \swarrow & [\hat{v}^2] \\
X^2 & \longleftarrow & X^3 \\
\end{array}
\]  

(7.4)

(with \([\hat{w}^1] \circ [\hat{v}^1]\) refinement) represents the same 2-morphism in \((\text{Red Orb})\) if and only if there is a set of data \((X^4, [\hat{z}], [\hat{z}'], [\sigma^1], [\sigma^2])\) as follows:

\[
\begin{array}{ccc}
X^1 \downarrow [\sigma^1] & \longrightarrow & [\sigma^1] \\
[\hat{v}^1] & \searrow & [\hat{v}^1] \\
X^3 & \downarrow & X^3 \\
[\sigma^2] & \swarrow & [\sigma^2] \\
X^4 & \longleftarrow & X^4 \\
\end{array}
\]  

(7.4)

such that \([\hat{w}^1] \circ [\hat{v}^1] \circ [\hat{z}]\) is a refinement,

\[
\left( i_{[f^2]} \ast [\sigma^2] \right) \circ \left( [\delta] \ast i_{[\hat{z}]} \right) \circ \left( i_{[f^1]} \ast [\sigma^1] \right) = [\delta'] \ast i_{[\hat{z}']}
\]  

(7.5)

and

\[
\left( i_{[\hat{w}^2]} \ast [\sigma^2] \right) \circ \left( [\mu] \ast i_{[\hat{z}]} \right) \circ \left( i_{[\hat{w}^1]} \ast [\sigma^1] \right) = [\mu'] \ast i_{[\hat{z}']}
\]  

(7.6)

We denote by

\[
[\text{Red}, \text{Orb}] : (X^1, [\hat{w}^1], [\hat{f}^1]) \implies (X^2, [\hat{w}^2], [\hat{f}^2])
\]  

(7.7)

the class of any such data (we refer to Lemma 7.8 below for a slightly simplified description of 2-morphisms).
• For the composition of 1-morphisms in \( \text{Red Orb} \) we have to do a preliminary step as follows: for every pair of morphisms in \( \text{Red Atl} \)

\[
\mathcal{X}' \xrightarrow{[f]} \mathcal{Y} \xleftarrow{[v]} \mathcal{Y}'
\]

(7.8)

with \([v]\) refinement, using the axiom of choice we choose any reduced orbifold atlas \( \mathcal{X}'' \), any pair of morphisms \([v']\), \([f']\) in \( \text{Red Atl} \) with \([v']\) refinement and any 2-morphism \([\delta]\) in \( \text{Red Atl} \) as follows

\[
\xymatrix{ \mathcal{X}' 
 & \mathcal{X}'' \ar[ld]^{[\delta]} \ar[rd]_{[f']} \\
\mathcal{X} 
 & \mathcal{Y} \ar[l]_{[\delta]} \ar[r]^{[v]} \\
\mathcal{Y}' \ar[u]_{[v']} 
}
\]

(7.9)

Such a choice is always possible by [B13] (see Proposition 7.1) but in general it is not unique. By [P2] § 2.2 we only have to force such a choice in the following special cases:

(a) whenever (7.8) is such that \( \mathcal{Y} = \mathcal{X}' \) and \([f] = \text{id}_Y\), then we have to choose \( \mathcal{X}'' := \mathcal{Y}', [f'] := \text{id}_{Y'}, [v'] := [v] \) and \([\delta] := i_{[v]}\);

(b) whenever (7.8) is such that \( \mathcal{Y} = \mathcal{X}' \) and \([v] = \text{id}_Y\), then we have to choose \( \mathcal{X}'' := \mathcal{X}', [f'] := [f], [v'] := \text{id}_{X'} \) and \([\delta] := i_{[f]}\).

Having fixed any set of such choices, given any pair of morphisms in \( \text{Red Orb} \) as follows:

\[
\xymatrix{ \mathcal{X} 
 & \mathcal{X}' 
 & \mathcal{Y} 
 & \mathcal{Y}' 
 & \mathcal{Z} \\
& \text{[\hat{w}]} \ar[ru] \ar[lu] & \text{[\hat{v}]} \ar[ru] \ar[lu] & \text{[\hat{g}]} \ar[ru] \ar[lu] & \text{[\hat{z}]} \ar[ru] \ar[lu] \\
}
\]

(with both \([\hat{w}]\) and \([\hat{v}]\) refinements), we use the fixed choice (7.9) and we set

\[
(\mathcal{Y}', [v], [g]) \circ (\mathcal{X}', [\hat{w}], [\hat{f}]) := (\mathcal{X}'', [\hat{w}] \circ [v'], [\hat{g}] \circ [f']) : \mathcal{X} \longrightarrow \mathcal{Z}.
\]

In this way in general the composition of morphisms in \( \text{Red Orb} \) is associative only up to canonical 2-isomorphisms, so \( \text{Red Orb} \) is a bicategory but not a 2-category.

• We omit the construction of the vertical and horizontal compositions for 2-morphisms (for details we refer to the original constructions in any bicategory of fractions, as described in [P2] § 2.3, or to the simplified version given in our previous paper [T2]). A priori the construction of such compositions depends on some additional choices involving axiom [B13]; by [T2] Theorem 0.5] actually the choices of (7.10) completely determine all the structure of \( \text{Red Orb} \).

We only remark that since each 2-morphism is invertible in \( \text{Red Atl} \), then it is not difficult to prove that the same property holds in \( \text{Red Orb} \). In particular, the inverse of any 2-morphism as in (7.10) is given by \([X^t, [v^2], [v^1], [\mu]^{-1}, [\delta]^{-1}]\).

• The pseudofunctor \( \mathcal{U}_W_{\text{Red Atl}} \) sends each reduced orbifold atlas \( \mathcal{X} \) to the same object in \( \text{Red Orb} \). For every morphism \([f] : \mathcal{X} \rightarrow \mathcal{Y}\) we have \( \mathcal{U}_W_{\text{Red Atl}}(\{f\}) = (\mathcal{X}, \text{id}_X, \{f\}) \). For every pair of morphisms \([f^m] : \mathcal{X} \rightarrow \mathcal{Y}\) for \( m = 1,2 \) and for every 2-morphism \([\delta] : [f^1] \Rightarrow [f^2] \) in \( \text{Red Atl} \) we have

\[
\mathcal{U}_W_{\text{Red Atl}}(\{\delta\}) = \left( \mathcal{X}, \text{id}_X, \text{id}_X, [i_{\text{id}_X}, \{\delta\}] : \mathcal{X}, \text{id}_X, \{f^1\} \Rightarrow \mathcal{X}, \text{id}_X, \{f^2\} \right).
\]
As we mentioned above, we can simplify a bit the description of 2-morphisms in $(\text{Red Orb})$ as follows.

**Lemma 7.8.** Let us fix any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$ and any pair of morphisms $f^m := (\mathcal{X}^m, [\hat{w}^m], [f^m]) : \mathcal{X} \to \mathcal{Y}$ in $(\text{Red Orb})$ for $m = 1, 2$. Then any 2-morphism from $f^1$ to $f^2$ in $(\text{Red Orb})$ is completely determined by a set of data as follows:

(a) a reduced orbifold atlas $\mathcal{X}^3$,
(b) a pair of refinements $[\hat{v}^m] : \mathcal{X}^3 \to \mathcal{X}^m$ for $m = 1, 2$,
(c) a 2-morphism $[\delta]$ in $(\text{Red Atl})$ as follows:

\[
\begin{array}{ccc}
\mathcal{X}^1 & \xrightarrow{[f^1]} & \mathcal{Y} \\
\downarrow{[\hat{v}^1]} & & \\
\mathcal{X}^3 & \xrightarrow{[\delta]} & \mathcal{Y} \\
\downarrow{[\hat{v}^2]} & & \\
\mathcal{X}^2 & \xrightarrow{[f^2]} & \\
\end{array}
\]  

(7.10)

Moreover, any set of data as above determines a 2-morphism in $(\text{Red Orb})$; any other set of data

\[
\begin{array}{ccc}
\mathcal{X}^1 & \xrightarrow{[f^1]} & \mathcal{Y} \\
\downarrow{[\hat{v}'^1]} & & \\
\mathcal{X}^3 & \xrightarrow{[\delta']} & \mathcal{Y} \\
\downarrow{[\hat{v}'^2]} & & \\
\mathcal{X}^2 & \xrightarrow{[f^2]} & \\
\end{array}
\]  

(7.11)

(with $[\hat{v}'^1]$ and $[\hat{v}'^2]$ refinements) determines the same 2-morphism as (7.10) if and only if there are a reduced orbifold atlas $\mathcal{X}^4$ and a pair of refinements

\[
\begin{array}{ccc}
\mathcal{X}^3 & \xrightarrow{[\hat{z}]} & \mathcal{X}^4 \\
\downarrow{[\hat{v}]} & & \\
\mathcal{X}^3 & \xrightarrow{[\delta]} & \\
\end{array}
\]  

(7.12)

such that

\[
\begin{pmatrix} i_{[f^1]} * [\sigma^2] \circ (\delta) \circ i_{[\hat{w}]} \end{pmatrix} \circ \begin{pmatrix} i_{[f^1]} \circ [\sigma^1] \end{pmatrix} = [\delta'] \circ i_{[\hat{w}']},
\]  

(7.13)

where $[\sigma^1], [\sigma^2]$ are the unique 2-morphisms filling diagram (7.14) (existence and uniqueness are a consequence of Lemma 6.12). Therefore, from now on we will denote each 2-morphism in $(\text{Red Orb})$ from $f^1$ to $f^2$ as a class $[\mathcal{X}^3, [\hat{v}^1], [\hat{v}^2], [\delta]]$, where the class of equivalence is the one induced by saying that

\[
\begin{pmatrix} \mathcal{X}^3, [\hat{v}^1], [\hat{v}^2], [\delta] \end{pmatrix} \sim \begin{pmatrix} \mathcal{X}^3, [\hat{v}'^1], [\hat{v}'^2], [\delta'] \end{pmatrix}
\]  

if and only if there are data $(\mathcal{X}^4, [\hat{z}], [\hat{z}'])$ as above, such that (7.14) holds.

**Proof.** Let us fix any 2-morphism in $(\text{Red Orb})$ represented by a set of data as in (7.3). Since each $f^m$ is a morphism in $(\text{Red Orb})$, then $[\hat{w}^1]$ and $[\hat{w}^2]$ are refinements. Moreover also $[\hat{w}^1] \circ [\hat{v}^1]$ is a refinement, so by Lemma (7.3) we have
that \([v^1]\) is a refinement. Using (BF2) for \((\text{Red Atl})\) on \([\mu]^{-1}\), also \([\tilde{w}^2] \circ [v^2]\) is also a refinement, so again by Lemma 7.9 we conclude that \([v^2]\) is a refinement, so we have obtained a set of data as in (a) – (c). Conversely, let us fix any set of data as in (a) – (c). Then by (BF2) for \((\text{Red Atl})\) we get that \([w^m] \circ [v^m]\) is a refinement for each \(m = 1, 2\). So by Lemma 6.12 there is a unique 2-morphism

\[
[\mu] : [\tilde{w}^1] \circ [v^1] \rightarrow [\tilde{w}^2] \circ [v^2].
\]

This proves that each set of data (a) – (c) completely determines a set of data as in (7.14), hence a 2-morphism from \(f^1\) to \(f^2\) in \((\text{Red Orb})\).

Now let us fix any other set of data as in (7.11) and let us denote by

\[
[\mu'] : [\tilde{w}^1] \circ [v'^{1}] \rightarrow [\tilde{w}^2] \circ [v'^{2}]
\]

the unique 2-morphism in \((\text{Red Atl})\) determined by Lemma 6.12. By Description 7.9, the 2-morphisms

\[
\left[X^3, [v^1], [v^2], [\mu], [\delta]\right] \quad \text{and} \quad \left[X'^3, [v'^1], [v'^2], [\mu'], [\delta']\right]
\]

coincide if and only if there are data \((X^4, \tilde{z}^1, \tilde{z}^2, [\sigma^1], [\sigma^2])\) as in (7.13), such that \([\tilde{w}^1] \circ [v^1] \circ [\tilde{z}]\) is a refinement and (7.5) and (7.6) hold. Since \([\tilde{w}^1] \circ [v^1]\) is a refinement, by Lemma 7.9 we conclude that \([\tilde{z}]\) is a refinement. Using (BF2) for \((\text{Red Atl})\) on \([\sigma^1]\), we conclude that also \([v'^1] \circ [\tilde{z}']\) is a refinement. Again by Lemma 7.9 this implies that \([\tilde{z}']\) is a refinement.

Conversely, let us suppose that there are a reduced orbifold atlas \(X^4\) and a pair of refinements \([\tilde{z}]\) and \([\tilde{z}']\) as in (7.11) such that (7.6) holds. Then (7.6) is automatically satisfied: indeed it is an equality between 2-morphisms having sources and target given by refinements, and Lemma 6.12 applies. This suffices to conclude.

**Description 7.9.** A description analogous to Description 7.7 holds for the bicategory \((\text{PE \tilde{E} Gpd})\) \([W^{-1}_{\text{PE \tilde{E} Gpd}}]\) and for the pseudofunctor \(\mathcal{U}_{W^{-1}_{\text{PE \tilde{E} Gpd}}}\) defined in (5.1). The objects of that bicategory are proper, effective, étale Lie groupoids; given \(\mathcal{X}_*\) and \(\mathcal{Y}_*\), a morphism from the first object to the second one is given by any data as follows, with \(\psi_*\) Morita equivalence:

\[
\xymatrix{ \mathcal{X}_* \ar[r]^\psi_* & \mathcal{X}'_* \ar[r]^\psi_* & \mathcal{Y}_* }.\]

Given any pair of morphisms in \((\text{PE \tilde{E} Gpd})\) \([W^{-1}_{\text{PE \tilde{E} Gpd}}]\)

\[
\left(\mathcal{X}_*, \psi^m_{\mathcal{X}}, \phi^m_{\mathcal{Y}}\right) : \mathcal{X}_* \rightarrow \mathcal{Y}_* \quad \text{for } m = 1, 2,
\]

using [PS] Lemma 8.1 a 2-morphism from the first morphism to the second one is any equivalence class of data \((\mathcal{X}_*, \mathcal{Y}_*, \xi^1_{\mathcal{X}}, \xi^2_{\mathcal{Y}}, \mu, \delta)\) in \((\text{PE \tilde{E} Gpd})\) as follows

\[
\xymatrix{ \mathcal{X}^m_* \ar[r]^-\psi^m_{\mathcal{X}} \ar[d]^-\mu & \mathcal{X}'^m_* \ar[r]^-\mu_{\mathcal{X}} \ar[d]^-\delta & \mathcal{Y}^m_* \ar[r]^-\phi^m_{\mathcal{Y}} & \mathcal{Y}^m_* }.
\]

(7.14)
such that both $\xi^1$ and $\xi^2$ are Morita equivalences (a priori we should also impose that $\mu$ is invertible, but this is always verified since each natural transformation is invertible in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})$). The equivalence relation on the set of data of the form $(\mathcal{D}^3, \xi^1, \xi^2, \mu, \delta)$ is analogous to the one given in Description 7.7, so we omit it.

In the next pages we will also need the following simplified description of 2-morphisms in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})[W^{-1}_{\mathcal{P}E\mathcal{E}\mathcal{Gpd}}]$.

**Lemma 7.10.** Let us fix any 2 proper, effective, étale groupoids $\mathcal{X}$, $\mathcal{Y}$ and any pair of morphisms $\underline{g}^m := (\mathcal{X}^m, \psi^m, \phi^m) : \mathcal{X} \rightarrow \mathcal{Y}$ in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})[W^{-1}_{\mathcal{P}E\mathcal{E}\mathcal{Gpd}}]$ for $m = 1, 2$. Then any 2-morphism from $\underline{g}^1$ to $\underline{g}^2$ in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})[W^{-1}_{\mathcal{P}E\mathcal{E}\mathcal{Gpd}}]$ is completely determined by a set of data as follows:

- (a) a proper, effective and étale groupoid $\mathcal{D}^3$,
- (b) a pair of Morita equivalences $\xi^m : \mathcal{D}^m \rightarrow \mathcal{X}^m$ for $m = 1, 2$, such that $|\phi^1| \circ |\xi^1| = |\psi^2| \circ |\xi^2|$ (see Remark [1.8]),
- (c) a 2-morphism $\alpha$ in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})$ as follows:

\[\begin{array}{ccc}
\mathcal{D}^1 & \xrightarrow{\xi^1} & \mathcal{D}^3 \\
\downarrow \phi^1 & \downarrow \alpha & \downarrow \psi^3 \\
\mathcal{D}^2 & \xrightarrow{\xi^2} & \mathcal{D}^4 \\
\downarrow \phi^2 & & \downarrow \psi^4 \\
\mathcal{D}^0 & \xrightarrow{\xi^0} & \mathcal{D}^0
\end{array}\]  

(7.15)

Moreover, any set of data as above determines a 2-morphism in the bicategory $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})[W^{-1}_{\mathcal{P}E\mathcal{E}\mathcal{Gpd}}]$; any other set of data

\[\begin{array}{ccc}
\mathcal{D}^0 & \xrightarrow{\xi^0} & \mathcal{D}^3 \\
\downarrow \phi^0 & \downarrow \alpha' & \downarrow \psi^3 \\
\mathcal{D}^0 & \xrightarrow{\xi^0} & \mathcal{D}^0 \\
\downarrow \phi^0 & & \downarrow \psi^0 \\
\mathcal{D}^0 & \xrightarrow{\xi^0} & \mathcal{D}^0
\end{array}\]  

(7.16)

(with $\xi^0$ and $\xi^2$ Morita equivalences such that $|\psi^1| \circ |\xi^1| = |\psi^2| \circ |\xi^2|$) determines the same 2-morphism as (7.15) if and only if there are a proper, effective and étale groupoid $\mathcal{D}^3$ and a pair of Morita equivalences

\[\begin{array}{ccc}
\mathcal{D}^3 & \xrightarrow{\gamma^3} & \mathcal{D}^4 \\
\downarrow \gamma^1 & \downarrow \gamma^1 & \downarrow \gamma^1 \\
\mathcal{D}^3 & \xrightarrow{\gamma^3} & \mathcal{D}^3
\end{array}\]  

(7.17)

such that $|\xi^1| \circ |\gamma^1| = |\xi^0| \circ |\gamma^0|$ and

\[\left(i_{\phi^0} \ast \beta^0\right) \circ \left(\alpha \ast i_{\gamma^0}\right) \circ \left(i_{\phi^0} \ast \beta^1\right) = \alpha \ast i_{\gamma^1}.\]

where each $\beta^1$ is the unique natural transformation from $\xi^1 \circ \gamma^1$ to $\xi^2 \circ \gamma^0$ and $\beta^2$ is the unique natural transformation from $\xi^2 \circ \gamma^2$ to $\xi^3 \circ \gamma^1$. Therefore, from now on we will denote each 2-morphism in $(\mathcal{P}E\mathcal{E}\mathcal{Gpd})[W^{-1}_{\mathcal{P}E\mathcal{E}\mathcal{Gpd}}]$ from $\underline{g}^1$ to $\underline{g}^2$ as
the class $[\mathcal{F}^3, \xi^1, \xi^2, \alpha]$, where the class of equivalence is the one induced by saying that

$$
(\mathcal{F}^3, \xi^1, \xi^2, \alpha) \sim (\mathcal{F}'^3, \xi^1, \xi^2, \alpha')
$$

if and only if there are data $(\mathcal{F}'^3, \gamma, \gamma')$ as above, such that

as desired. For the existence and uniqueness of the $2$-morphism $\beta^2 : \xi^1 \circ \gamma' \Rightarrow \xi^2 \circ \gamma$, we have that

$$
|\psi^1_\ast| \circ (\xi^1 \circ \gamma) = |\psi^1_\ast| \circ |\xi^1| \circ |\gamma|;
$$

moreover, using the hypothesis we have

$$
|\psi^1_\ast| \circ (\xi^1 \circ \gamma') = |\psi^1_\ast| \circ |\xi^1| \circ |\gamma'|
$$

and

$$
|\psi^1_\ast| \circ |\xi^1| \circ |\gamma| = |\psi^1_\ast| \circ |\xi^2| \circ |\gamma'.
$$

Since $|\psi^1_\ast|$ is an homeomorphism (see Lemma 5.3), then the previous identities imply that $|\xi^1_\ast| \circ |\gamma| = |\xi^2_\ast| \circ |\gamma'|$; then the existence and uniqueness of $\beta^2$ is again a consequence of Lemma 5.6.

Proof. The proof of this result follows the same lines of the proof of Lemma 5.8. The only significant difference is that we use Lemma 5.6 instead of Lemma 6.12: this allows to prove the existence and uniqueness of a $2$-morphism $\beta^1 : \xi^1 \circ \gamma' \Rightarrow \xi^2 \circ \gamma$, as desired. For the existence and uniqueness of the $2$-morphism $\beta^2$, one proceeds as follows: using Lemma 5.6 and (b), we have that

$$
|\psi^1_\ast| \circ (\xi^1 \circ \gamma') = |\psi^1_\ast| \circ |\xi^1| \circ |\gamma'|
$$

Moreover, using the hypothesis we have

$$
|\psi^1_\ast| \circ (\xi^1 \circ \gamma') = |\psi^1_\ast| \circ |\xi^2| \circ |\gamma'.
$$

8. The pseudofunctor $\mathcal{G}^{red}$

Now we are almost ready to describe the pseudofunctor $\mathcal{G}^{red}$ mentioned in the Introduction. For that, we will only need the following result.

Theorem 8.1. [3] Theorem 0.3 and Remark 3.2 in the case when $\mathcal{A}$ and $\mathcal{B}$ are $2$-categories. Let us fix any pair of $2$-categories $\mathcal{A}$ and $\mathcal{B}$ and any pair of classes $W_{\mathcal{A}}$ and $W_{\mathcal{B}}$ of morphisms in $\mathcal{A}$ and $\mathcal{B}$ respectively, such that both $(\mathcal{A}, W_{\mathcal{A}})$ and $(\mathcal{B}, W_{\mathcal{B}})$ satisfy conditions [11]. Let us also fix any pseudofunctor $F : \mathcal{A} \to \mathcal{B}$ such that $F_1(W_{\mathcal{A}}) \subseteq W_{\mathcal{B}, \text{sat}}$ and let us assume the axiom of choice. Then there is a pseudofunctor

$$
\tilde{G} : \mathcal{A} \left[ W_{\mathcal{A}}^{-1} \right] \to \mathcal{B} \left[ W_{\mathcal{B}, \text{sat}}^{-1} \right]
$$

such that:

- $UW_{\mathcal{B}, \text{sat}} \circ F = \tilde{G} \circ UW_{\mathcal{A}}$;
- for each object $A_{\mathcal{A}}$, we have $\tilde{G}_0(A_{\mathcal{A}}) = F_0(A_{\mathcal{A}})$;
- for each morphism $(A'_{\mathcal{A}}, w_{\mathcal{A}}, f_{\mathcal{A}}) : A_{\mathcal{A}} \to B_{\mathcal{A}}$ in $\mathcal{A} \left[ W_{\mathcal{A}}^{-1} \right]$, we have

$$
\tilde{G}_1 \left( A'_{\mathcal{A}}, w_{\mathcal{A}}, f_{\mathcal{A}} \right) = \left( F_0(A'_{\mathcal{A}}), F_1(w_{\mathcal{A}}), F_1(f_{\mathcal{A}}) \right);
$$

- for each $2$-morphism

$$
\left[ A^3_{\mathcal{A}}, v^1_{\mathcal{A}}, v^2_{\mathcal{A}}, \mu_{\mathcal{A}}, \delta_{\mathcal{A}} \right] : \left( A^1_{\mathcal{A}}, w^1_{\mathcal{A}}, f^1_{\mathcal{A}} \right) \Rightarrow \left( A^2_{\mathcal{A}}, w^2_{\mathcal{A}}, f^2_{\mathcal{A}} \right)
$$

in $\mathcal{A} \left[ W_{\mathcal{A}}^{-1} \right]$, we have

$$
\tilde{G}_2 \left[ A^3_{\mathcal{A}}, v^1_{\mathcal{A}}, v^2_{\mathcal{A}}, \mu_{\mathcal{A}}, \delta_{\mathcal{A}} \right] = \left[ F_0(A^3_{\mathcal{A}}), F_1(v^1_{\mathcal{A}}), F_1(v^2_{\mathcal{A}}), F_2(\mu_{\mathcal{A}}), F_2(\delta_{\mathcal{A}}) \right].
$$

Then we have:
Proposition 8.2. If we assume the axiom of choice, there is a pseudofunctor

\[ G^{\text{red}} : (\text{Red Orb}) \rightarrow (\text{P E E Gpd}) \left[ W_{\text{P E E Gpd}}^{-1} \right] \]

such that:

1. for each reduced orbifold atlas \( \mathcal{X} \), \( G^{\text{red}}_0(\mathcal{X}) = F_0^{\text{red}}(\mathcal{X}) \);
2. for each morphism \((\mathcal{X}', [\hat{w}], [\hat{f}]) : \mathcal{X} \rightarrow \mathcal{Y}\) in \((\text{Red Orb})\), we have
   \[
   G^{\text{red}}_1 \left( \mathcal{X}', [\hat{w}], [\hat{f}] \right) = \left( F_0^{\text{red}}(\mathcal{X}), F_1^{\text{red}}([\hat{w}]), F_1^{\text{red}}([\hat{f}]) \right);
   \]
3. for each 2-morphism
   \[
   \left( \mathcal{X}', [\hat{v}^1], [\hat{v}^2], [\hat{\alpha}] \right) : \left( \mathcal{X}'^1, [\hat{w}^1], [\hat{f}^1] \right) \Rightarrow \left( \mathcal{X}'^2, [\hat{w}^2], [\hat{f}^2] \right)
   \]
   in \((\text{Red Orb})\), we have
   \[
   G^{\text{red}}_2 \left( \left[ \mathcal{X}', [\hat{v}^1], [\hat{v}^2], [\hat{\alpha}] \right] \right) = \left[ F_0^{\text{red}}(\mathcal{X}), F_1^{\text{red}}([\hat{v}^1]), F_1^{\text{red}}([\hat{v}^2]), F_2^{\text{red}}([\hat{\alpha}]) \right].
   \]
Moreover, we have \( U_{W_{\text{P E E Gpd}}} \circ F^{\text{red}} = G^{\text{red}} \circ U_{W_{\text{Red Atl}}} \).

Proof. Let us apply Theorem 5.1 with \( \mathcal{A} := (\text{Red Atl}) \), \( W_{\mathcal{A}} := W_{\text{Red Atl}} \) (i.e. all refinements of reduced orbifold atlases), \( \mathcal{B} := (\text{P E E Gpd}) \), \( W_{\mathcal{B}} := W_{\text{P E E Gpd}} \) (i.e. all Morita equivalences of proper, effective, étale groupoids) and \( F := F^{\text{red}} \). We recall that by Lemma 5.1 we have \( W_{\mathcal{B}, \text{sat}} = W_{\text{P E E Gpd}} \). Given any refinement \( \hat{w} \), by Proposition 6.2 we have that \( F_1^{\text{red}}([\hat{w}]) \) is a Morita equivalence, so we are in the hypothesis of Theorem 5.1. Then the claim follows at once using Lemmas 6.2 and 7.10 for the description of \( G^{\text{red}}_2 \).

In addition, we recall the following result. For the more general form of this statement, we refer to [14] Theorem 0.2. We state such a result here only in the special framework where:
- \( \mathcal{A} \) and \( \mathcal{B} \) are 2-categories and \( F \) is a 2-functor (also known as strict pseudofunctor), i.e. it preserves compositions and identities;
- \( U_{W_{\mathcal{A}}} \circ F = G \circ U_{W_{\mathcal{A}}} \) and the natural equivalence \( \kappa \) appearing in [14] Theorem 0.2 is the 2-identity of \( U_{W_{\mathcal{A}}} \circ F \).

Theorem 8.3. [14] Let us fix any pair of 2-categories \( \mathcal{A}, \mathcal{B} \) and any pair of classes of morphisms \( W_{\mathcal{A}}, W_{\mathcal{B}} \) such that both \((\mathcal{A}, W_{\mathcal{A}})\) and \((\mathcal{B}, W_{\mathcal{B}})\) satisfy conditions (B1). Moreover, let us fix any 2-functor \( F : \mathcal{A} \rightarrow \mathcal{B} \), such that \( F_1(W_{\mathcal{A}}) \subseteq W_{\mathcal{B}, \text{sat}} \).

In addition, let us suppose that there is a pseudofunctor \( G : \mathcal{A} \left( [W_{\mathcal{A}}]^{-1} \right) \rightarrow \mathcal{B} \left( [W_{\mathcal{B}}]^{-1} \right) \) such that \( U_{W_{\mathcal{A}}} \circ F = G \circ U_{W_{\mathcal{A}}} \), and let us assume the axiom of choice. Then \( G \) is an equivalence of bicategories if and only if \( F \) satisfies the following 5 conditions.

(A1) For any object \( A_{\mathcal{B}} \), there is a pair of objects \( A'_{\mathcal{A}} \) and \( A''_{\mathcal{A}} \) and a pair of morphisms \( w'_{\mathcal{A}} \) in \( W_{\mathcal{A}} \) and \( w''_{\mathcal{A}} \) in \( W_{\mathcal{A}, \text{sat}} \), as follows:

\[
\xymatrix{ F_0(A_{\mathcal{A}}) \ar@{<->}[r]^{w'_{\mathcal{A}}} & A'_{\mathcal{A}} \ar[r]^{w''_{\mathcal{A}}} & A''_{\mathcal{A}}. }
\]

(A2) Let us fix any triple of objects \( A^1_{\mathcal{A}}, A^2_{\mathcal{A}}, A^3_{\mathcal{A}} \) and any pair of morphisms \( w^1_{\mathcal{A}} \) in \( W_{\mathcal{A}} \) and \( w^2_{\mathcal{A}} \) in \( W_{\mathcal{A}, \text{sat}} \) as follows:

\[
\xymatrix{ F_0(A^1_{\mathcal{A}}) \ar[r]^{w^1_{\mathcal{A}}} & A^2_{\mathcal{A}} \ar[r]^{w^2_{\mathcal{A}}} & F_0(A^3_{\mathcal{A}}). }
\]

Then there is a 3-arrow \( A^3_{\mathcal{A}} \), a pair of morphisms \( w^1_{\mathcal{A}} \) in \( W_{\mathcal{A}} \) and \( w^2_{\mathcal{A}} \) in \( W_{\mathcal{A}, \text{sat}} \) as follows.
\begin{align*}
A_1^A & \xrightarrow{w_1^A} A_3^A \xrightarrow{w_2^A} A_2^A
\end{align*}

and a set of data \((A', z_1^B, z_2^B, \gamma_1^B, \gamma_2^B)\) as follows

\begin{align*}
\xymatrix{
A^B \ar[r]^{w_1^A} & A'\ar[d]_{\gamma_1^B} \\
F_0(A_1^B) & A'\ar[r]_{\gamma_2^B} & F_0(A_2^B)
}
\end{align*}

\[F_0(A_1^B) \xrightarrow{x_1^B} \cdots \xrightarrow{x_2^B} F_0(A_2^B)\]

such that \(z_1^B\) belongs to \(W_B\) and both \(\gamma_1^B\) and \(\gamma_2^B\) are invertible.

\((A3)\) Let us fix any pair of objects \(B_A, A_B\) and any morphism \(f_B : A_B \to F_0(B_A)\). Then there are an object \(A_A\), a morphism \(f_A : A_A \to B_A\) and data \((A'_A, v_1^A, v_2^A, \alpha_A)\) as follows

\begin{align*}
\xymatrix{
A'_B \ar[r]^{v_1^B} & A_B \ar[d]_{\alpha_B} \\
F_0(A_1^B) & F_0(A_B) \ar[r]_{f_B} & F_0(A_2^B)
}
\end{align*}

with \(v_1^A\) in \(W_B\), \(v_2^A\) in \(W_B, W\) and \(\alpha_B\) invertible.

\((A4)\) Let us fix any pair of objects \(A_A, B_A\), any pair of morphisms \(f_1^A, f_2^A : A_A \to B_A\) and any pair of \(2\)-morphisms \(\gamma_1^A, \gamma_2^A : f_1^A \Rightarrow f_2^A\). Moreover, let us fix any object \(A'_A\) and any morphism \(z_A : A'_A \to F_0(A_A)\) in \(W_B\). If \(F_2(\gamma_1^A)^* i_A = F_2(\gamma_2^A)^* i_A\), then there are an object \(A'_A\) and a morphism \(z_A : A'_A \to A_A\) in \(W_B\), such that \(\gamma_1^A = \gamma_2^A\). \(\alpha_A\) invertible.

\((A5)\) Let us fix any triple of objects \(A_A, B_A, A_B\), any pair of morphisms \(f_1^A, f_2^A : A_A \to B_A\), any morphism \(v_A : A_A \to F_0(A_A)\) in \(W_B\) and any \(2\)-morphism

\begin{align*}
\xymatrix{
A_A \ar[r]^{v_A} & F_0(A_A) \ar[d]_{\alpha_A} \\
F_0(A_1^B) & F_0(A_B) \ar[r]_{f_B} & F_0(A_2^B)
}
\end{align*}

Then there are a pair of objects \(A'_A, A'_B\), a triple of morphisms \(v_A : A'_A \to A_A\) in \(W_B\), \(z_A : A'_A \to F_0(A'_A)\) in \(W_B\) and \(z_A' : A'_A \to A'_B\), a \(2\)-morphism

\begin{align*}
\xymatrix{
A'_A \ar[r]^{v_A} & A_A \ar[d]_{\alpha_A} \\
A'_B & B_B \ar[l]_{f_B}
}
\end{align*}

and an invertible \(2\)-morphism

\begin{align*}
\text{and a set of data } (A'_A, z_1^B, z_2^B, \gamma_1^B, \gamma_2^B) \text{ as follows}
\end{align*}
such that \( \alpha_{\mathcal {A} \mathcal {B}} \ast i_{\mathcal {A} \mathcal {B}}' \) coincides with the following composition:

Then we have:

**Theorem 8.4.** The pseudofunctor \( G_{\text{red}} \) described in Proposition 8.2 (using the axiom of choice) is an equivalence of bicategories.

**Proof.** Let us verify condition (A1), so let us fix any \( \mathcal {X} \) in \( (\mathcal {PE} \mathcal {E} \mathcal {G} \mathcal {pd}) \); by Lemma 6.11 there are a reduced orbifold atlas \( \mathcal {X} \) and a Morita equivalence \( \psi \) : \( \mathcal {X} \rightarrow \mathcal {Y} \). Therefore, (A1) holds if we choose the following set of data:

Let us consider (A2), so let us fix any pair of reduced orbifold atlases \( \mathcal {X}^1, \mathcal {X}^2 \) and any \( \mathcal {X} \) in \( (\mathcal {PE} \mathcal {E} \mathcal {G} \mathcal {pd}) \), together with any pair of Morita equivalences as follows:

By Lemma 6.11 there are a reduced orbifold atlas \( \mathcal {Y} \) and a Morita equivalence \( \phi : \mathcal {Y} \rightarrow \mathcal {X}^1 \). By Proposition 6.9 there is a weak equivalence \( [\hat{v}] : \mathcal {Y} \rightarrow \mathcal {X}^1 \), such that \( \mathcal {F}^1_0([\hat{v}]) = \psi^1 \circ \phi \). Since \( [\hat{v}] \) a weak equivalence, then by Lemma 6.6 there are a reduced orbifold atlas \( \mathcal {X}^3 \) and a weak equivalence \( [\hat{u}] : \mathcal {X}^3 \rightarrow \mathcal {Y} \), such that the morphism

is a refinement. We set \( \xi := \mathcal {F}^1_1([\hat{u}]); \) this morphism is a Morita equivalence by Lemma 6.7 and we have

Again by Proposition 6.9 there is a unique weak equivalence \( [\hat{w}] : \mathcal {X}^3 \rightarrow \mathcal {X}^2 \), such that

By Lemma 6.10 we have that \([\hat{w}]\) belongs to the right saturation of \( \mathcal {W}_{\text{Red Atl}} \). Then (A2) is satisfied by the following set of data:
Let us consider (A3), so let us fix any pair of reduced orbifold atlases $\mathcal{X}$, any object $\mathcal{F}_0$ in $(\mathcal{PEE}\mathcal{Gpd})$ and any morphism $\phi : \mathcal{F}_0 \to \mathcal{F}_0^\text{red}(\mathcal{Y})$. By Lemma 6.11 there are a reduced orbifold atlas $\mathcal{X}$ and a Morita equivalence $\psi : F_0^\text{red}(\mathcal{X}) \to \mathcal{F}_0$. By Lemma 4.21 there is a unique weak equivalence $[\hat{f}] : \mathcal{X} \to \mathcal{Y}$, such that $F_0^\text{red}([\hat{f}]) = \phi \circ \psi$. Then (A3) is easily verified with $A'_0 := F_0^\text{red}(\mathcal{X})$, $v_0^1 := \psi$ and $v_0^2 := \text{id}_{F_0^\text{red}(\mathcal{X})}$.

Let us consider (A3), so let us fix any reduced orbifold atlas $\mathcal{X}$, any pair of morphisms $[\hat{f}], [\hat{f}'] : \mathcal{X} \to \mathcal{Y}$ and any pair of 2-morphisms $[\gamma], [\gamma'] : [\hat{f}] \Rightarrow [\hat{f}']$ in $(\text{RedAtl})$. Moreover, let us fix any object $\mathcal{F}_0$ in $(\mathcal{PEE}\mathcal{Gpd})$ and any Morita equivalence $\psi : \mathcal{F}_0 \to \mathcal{F}_0^\text{red}(\mathcal{X})$, such that

$$F_0^\text{red}([\gamma]) \ast v_0 = F_0^\text{red}([\gamma']) \ast v_0. \quad (8.1)$$

By Lemma 6.11 there are a reduced orbifold atlas $\mathcal{Z}$ and a Morita equivalence $\phi : F_0^\text{red}(\mathcal{Z}) \to \mathcal{F}_0$. By Proposition 6.9 there is a unique weak equivalence $[\hat{u}] : \mathcal{Z} \to \mathcal{X}$ such that $F_0^\text{red}([\hat{u}]) = \psi \circ \phi$. By Lemma 6.6 there are a reduced orbifold atlas $\mathcal{U}$ and a weak equivalence $[\hat{v}] : \mathcal{U} \to \mathcal{Z}$, such that $[\hat{z}] := [\hat{u}] \circ [\hat{v}]$ is a refinement. So:

$$F_0^\text{red}([\gamma] \ast [\hat{u}]) = F_0^\text{red}([\gamma']) \ast v_0 \ast [\hat{u}] \ast [\hat{v}] = F_0^\text{red}([\gamma'] \ast [\hat{u}] \ast [\hat{v}]). \quad (8.1)$$

By Lemma 4.21 this implies that $[\gamma] \ast [\hat{u}] = [\gamma'] \ast [\hat{v}]$, so (A3) holds.

Lastly, let us prove (A5), so let us fix any pair of reduced orbifold atlases $\mathcal{X}, \mathcal{Y}$, any object $\mathcal{F}_0$ in $(\mathcal{PEE}\mathcal{Gpd})$, any pair of morphisms $[\hat{f}], [\hat{f}'] : \mathcal{X} \to \mathcal{Y}$, any Morita equivalence $\psi : \mathcal{F}_0 \to \mathcal{F}_0^\text{red}(\mathcal{X})$ and any natural transformation $\alpha : F_0^\text{red}([\hat{f}']) \circ \psi \Rightarrow F_0^\text{red}([\hat{f}]) \circ \psi$. By Lemma 6.11 there are a reduced orbifold atlas $\mathcal{Z}$ and a Morita equivalence $\phi : F_0^\text{red}(\mathcal{Z}) \to \mathcal{F}_0$. By Proposition 6.9 there is a unique weak equivalence $[\hat{u}] : \mathcal{Z} \to \mathcal{X}$ such that $F_0^\text{red}([\hat{u}]) = \psi \circ \phi$. By Lemma 6.6 there are a reduced orbifold atlas $\mathcal{X}'$ and a weak equivalence $[\hat{v}] : \mathcal{X}' \to \mathcal{Z}$, such that $[\hat{z}] := [\hat{u}] \circ [\hat{v}]$ is a refinement. Then let us consider the 2-morphism

$$\alpha \ast v_0 \circ F_0^\text{red}([\hat{v}]) : F_0^\text{red}([\hat{f}']) \circ [\hat{u}] \circ [\hat{v}] \Rightarrow F_0^\text{red}([\hat{f}]) \circ [\hat{u}] \circ [\hat{v}]. \quad (8.2)$$

By Lemma 4.21 there is a unique 2-morphism

$$[\hat{v}] : [\hat{f}'] \circ [\hat{u}] \circ [\hat{v}] \Rightarrow [\hat{f}] \circ [\hat{u}] \circ [\hat{v}]$$

in $(\text{RedAtl})$, such that $F_0^\text{red}([\hat{v}])$ is equal to (8.2). Then (A5) is satisfied if we choose $A'_0 := \mathcal{X}'$, $A'_0 := F_0^\text{red}(\mathcal{X}')$, $v_0^1 := [\hat{u}] \circ [\hat{v}] : \mathcal{X}' \to \mathcal{X}$, $z_{a^0} := \text{id}_{F_0^\text{red}(\mathcal{X}')}$,

$$z_{a^0} := \phi \circ F_0^\text{red}([\hat{v}]) : F_0(\mathcal{X}') \to \mathcal{F}_0.$$
(this is a Morita equivalence because composition of Morita equivalences), $\alpha := [\delta]$ and if we define $\sigma$ as the 2-identity of

$$F_1^{\text{red}}([\hat{u}] \circ [\hat{v}]) = \psi \circ \phi \circ F_1^{\text{red}}([\hat{v}]).$$

□

Remark 8.5. As we said in the Introduction, $(\mathcal{P}E \mathcal{G}pd \mathcal{W}^{-1})$ is the bicategory of reduced differentiable orbifolds in the language of Lie groupoids; so the previous theorem proves that the bicategory $(\mathcal{R}ed \mathcal{O}rb)$ just defined is the first known bicategory of reduced orbifolds in the language of reduced orbifold atlases. Compared to $(\mathcal{P}E \mathcal{G}pd \mathcal{W}^{-1})$, the main advantage of $(\mathcal{R}ed \mathcal{O}rb)$ for differential geometers is the fact that all the definitions used for the construction of such a bicategory do not require any knowledge of Lie groupoids or differentiable stacks, but they use only the notion of reduced orbifold atlases, local lifts and changes of charts.

9. An equivalence between $(\mathcal{R}ed \mathcal{O}rb)$ and the 2-category of effective orbifolds described in terms of differentiable stacks

As we mentioned in the introduction, a very convenient way to define a 2-category of orbifolds is by exhibiting it as a full 2-subcategory of the 2-category of $C^\infty$-stacks (these are called “differentiable stacks” in several papers, see for example [Pr]). For the Grothendieck topology used for such stacks, we refer to [J2, Definition 8.1]. A $C^\infty$-stack is called an orbifold (see [J2, Definition 9.25]) if it is equivalent to the stack $[\mathcal{X} : \mathcal{G} \to \mathcal{X}]$ associated to a proper, étale groupoid $(\mathcal{X} : \mathcal{G} \to \mathcal{X})$. In particular (see again [J2, Definition 9.25]) every orbifold is a separated, locally finitely presented Deligne-Mumford $C^\infty$-stack. An orbifold $\mathcal{X}$ is called effective or reduced (see [J1, Definition 1.9.4]) if for every point $[x] \in \mathcal{X}$ there exists a linear effective action of $G := \text{Iso}_X([x])$ on some $\mathbb{R}^n$, a $G$-invariant open neighborhood $\tilde{X}$ of 0 in $\mathbb{R}^n$ and a 1-morphism $i : \tilde{X}/G \to \mathcal{X}$, which is an equivalence with an open neighborhood of $x$ in $\mathcal{X}$ with $i_{\text{top}}(0) = [x]$ (if $\mathcal{X}$ is not effective, we are in the same setup but the action of each $G$ is not required to be effective). Equivalently, an orbifold is effective if and only if it is associated to a proper, étale, effective groupoid.

According to [J2] we write $(\mathcal{O}rb)$ and $(\mathcal{O}rb^{\text{eff}})$ for the full 2-subcategories of orbifolds, respectively of effective orbifolds, in the 2-category of $C^\infty$-stacks (or, equivalently, in the 2-category of Deligne-Mumford $C^\infty$-stacks). We recall that by [Pr, Corollary 43] there is an equivalence of bicategories

$$\tilde{\mathcal{H}} : (\mathcal{E} \mathcal{G}pd \mathcal{W}^{-1}) \to (\mathcal{C}^\infty\text{-Stacks})$$

and that by [J2, Theorem 9.26] there is an equivalence of bicategories induced by $\tilde{\mathcal{H}}$:

$$\mathcal{H} : (\mathcal{P}E \mathcal{E} \mathcal{G}pd \mathcal{W}^{-1}) \to (\mathcal{O}rb).$$

Therefore we get easily that there is also an equivalence of bicategories induced by $\tilde{\mathcal{H}}$:

$$\mathcal{H}^{\text{red}} : (\mathcal{P}E \mathcal{E} \mathcal{G}pd \mathcal{W}^{-1}) \to (\mathcal{O}rb^{\text{eff}}).$$

By considering the composition:
we conclude by Theorem 8.4 that:

**Theorem 9.1.** Assuming the axiom of choice, there is an equivalence between the bicategory \((\text{Red} \, \text{Orb})\) and the 2-category \((\text{Orb}^{\text{eff}})\) of effective orbifolds described as a full 2-subcategory of the 2-category of \(C^\infty\)-Deligne-Mumford stacks.

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