Cohomology of GKM-sheaves

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Abstract

Let $T$ be a compact torus and $X$ be a a finite $T$-CW complex (e.g. a compact $T$-manifold). In [2], the second author introduced a functor which assigns to $X$ a so called GKM-sheaf $\mathcal{F}_X$ whose ring of global sections $H^0(\mathcal{F}_X)$ is isomorphic to the equivariant cohomology $H^*_T(X)$ whenever $X$ is equivariantly formal (meaning that $H^*_T(X)$ is a free module over $H^*(BT)$). In the current paper we prove more generally that $H^0(\mathcal{F}_X) \cong H^*_T(X)$ if and only if $H^*_T(X)$ is reflexive, and find a geometric interpretation of the higher cohomology $H^n(\mathcal{F}_X)$ for $n \geq 1$.

1 Introduction

Let $T = (S^1)^r$ be a compact torus Lie group and let $X$ be a finite $T$-CW complex (such as a compact, smooth $T$-manifold). GKM theory provides techniques for computing the equivariant cohomology ring $H^*_T(X) := H^*(ET \times_T X; \mathbb{C})$. For a large class of $T$-manifolds, now called GKM-manifolds, Goresky, Kottwitz and MacPherson [5] showed that the cohomology ring $H^*_T(X)$ can be encoded combinatorially in a finite graph (the GKM-graph or moment graph) with edges labelled by non-trivial weights $\alpha \in \Lambda := Hom(T, S^1)$. GKM-theory has since developed in several directions: combinatorially by Guillemin and Zara [7–9], to a broader range of spaces by Guillemin and Holm [6], and to equivariant intersection cohomology by Braden and MacPherson [3] who introduced the notion of $\Gamma$-sheaves on a GKM-graph.

In [2] the second author introduced GKM-sheaves which provide a unified framework for the above constructions. Given a finite $T$-CW complex $X$, we associate a sheaf $\mathcal{F}_X$ whose ring of global sections $H^0(\mathcal{F}_X)$ is isomorphic to $H^*_T(X)$ whenever $H^*_T(X)$ is a free module over the cohomology of a point. In the current paper we improve this result by proving that $H^0(\mathcal{F}_X) \cong H^*_T(X)$ if and only if $H^*_T(X)$ is a reflexive module (equivalently a 2-syzygy). Furthermore we show that $H^n(\mathcal{F}_X) = 0$ for $n \geq 2$ and that

**Theorem 1.1.** If $X$ is a finite $T$-CW complex and $H^*_T(X)$ is reflexive, then there is a natural exact sequence

$$0 \to H^0(\mathcal{F}_X) \to H^*_T(X_0) \to H^*_T(X_1, X_0) \to H^1(\mathcal{F}_X) \to 0$$

where $X_0 = X^T$ is the fixed point set, $X_1$ is the union of all orbits of dimension one or less, and $\delta$ is the coboundary map in the long exact sequence of the pair $(X_1, X_0)$. 

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Outline: In §2, §3, and §4 we review background material on equivariant cohomology, GKM sheaves, and sheaf cohomology respectively. In §5 we study the cohomology of GKM-sheaves and prove that $H^n(F) = 0$ for $n \geq 0$ and produce chain complexes to calculate $H^1(F)$. In §6 we study the cohomology of the GKM-sheaf $F_X$ associated to $T$-space and interpret it geometrically.

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2 Equivariant Cohomology

Let $T \cong (\mathbb{S}^1)^r$ be a compact torus Lie group. The universal $T$-bundle, $ET \to BT$ is a principal bundle for which $ET$ is contractible and whose base $BT$ is homotopy equivalent to the $r$-fold product space $\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty$. Given a $T$-space $X$, the Borel homotopy quotient $X_{hT} := ET \times_T X$ is the total space of the associated fibre bundle

$$X \mapsto ET \times_T X \mapsto BT.$$  \hfill (1)

The equivariant cohomology of $X$ is the singular cohomology of the homotopy quotient

$$H^*_T(X) := H^*(X_{hT}; \mathbb{C}).$$

We use complex coefficients throughout. More generally, if $Y \subseteq X$ is a $T$-invariant subspace then

$$H^*_T(X, Y) = H^*(X_{hT}, Y_{hT}).$$

Given a $T$-space $X$, the constant map to a point $X \to pt$ is equivariant. The induced morphism $H^*(pt) \to H^*_T(X)$ makes $H^*_T(X)$ an algebra over $H^*(BT) = H^*(BT)$. By the Kunneth formula

$$H^*(BT) = H^*(\mathbb{C}P^\infty \times \ldots \times \mathbb{C}P^\infty) \cong \mathbb{C}[x_1, \ldots, x_r]$$

where each class $x_i$ has degree two. More invariantly, there is a natural isomorphism between $H^*(BT)$ and the ring of complex valued polynomials functions on the Lie algebra $t$

$$H^*(BT) \cong \mathbb{C}[t].$$

The weight lattice

$$\Lambda := \{\alpha : T \to \mathbb{S}^1\} \hfill (2)$$

is the set of Lie group homomorphisms from $T$ to $\mathbb{S}^1 = U(1)$. It forms a group under multiplication and there is a natural injection

$$\Lambda \hookrightarrow H^2(BT)$$

which sends $\alpha \in \Lambda$ to the tangent map $d\alpha : t \to u(1) = i\mathbb{R}$ regarded as a homogeneous linear polynomial in $\mathbb{C}[t]$.

The Borel Localization Theorem is central to GKM theory. We require only the following basic version (see [4]).

**Theorem 2.1** (Localization Theorem). Let $X$ be a finite $T$-CW complex with fixed point set $i : X^T \hookrightarrow X$. Then the kernel and cokernel of $i^* : H^*_T(X) \to H^*_T(X^T)$ are both torsion $H^*(BT)$-modules. In particular if $H^*_T(X)$ is torsion free then $i^*$ is injective.
2.1 Atiyah-Bredon Sequence

Let \( R = H^*(BT) \cong \mathbb{C}[x_1, \ldots, x_r] \). A finitely generated \( R \)-module \( M \) is said to be a \( j \)-th syzygy if there exists an exact sequence

\[
0 \to M \to F^1 \to F^2 \to \cdots \to F^j
\]

where the \( \{F^i\}_{i \in \{1, \ldots, j\}} \) are finitely generated free \( R \)-modules. According to ([1] Prop. 2.3):

- \( M \) is an \( r \)-syzygy if and only if \( M \) is free,
- \( M \) is a 1-syzygy if and only if \( M \) is torsion free,
- \( M \) is a 2-syzygy if and only if \( M \) is reflexive, meaning the natural map
  \[
  M \to Hom_R(Hom_R(M, R), R)
  \]
  is an isomorphism.

Given a \( T \)-space \( X \), define \( X_i \) to be the union of all orbits of dimension less than or equal to \( i \),

\[
X_i := \{ x \in X \mid \dim(T \cdot x) \leq i \}.
\]

We call \( X_i \) the \( i \)-skeleton of \( X \). In particular, \( X_{-1} = \phi \), \( X_0 = X^T \), and \( X_r = X \).

The following is due to Allay-Franz-Puppe ([1] Theorem 5.7).

**Theorem 2.2.** Let \( j \geq 0 \) and let \( T \) be a torus of rank \( r \), and \( X \) be a finite \( T \)-CW complex. Consider the sequence

\[
0 \to H^*_T(X) \to H^*_T(X_0) \xrightarrow{\delta} H^*_T(X_1, X_0) \xrightarrow{\delta_1} \cdots \xrightarrow{\delta_r} H^*_T(X_r, X_{r-1}) \to 0,
\]

where \( \delta_i \) is the boundary map of the triple \((X_{i-1}, X_i, X_{i+1})\). Then (3) is exact for all the positions \( i \leq j - 2 \) if and only if \( H^*_T(X) \) is \( j \)-th syzygy. In particular, the sequence

\[
0 \to H^*_T(X) \to H^*_T(X_0) \xrightarrow{\delta} H^*_T(X_1, X_0)
\]

is exact if and only if \( H^*_T(X) \) is a 2-syzygy if and only if \( H^*_T(X) \) is reflexive.

The sequences (3) and (4) are known as the Atiyah-Bredon sequence and the Chang-Skjelbred sequence respectively. Observe that if (4) is exact, then

\[
H^*_T(X) \cong \ker(\delta).
\]

GKM theory is concerned with calculating \( \ker(\delta) \).
3 GKM-Sheaves

Recall the weight lattice $\Lambda := \text{Hom}(T, S^1)$ from (2). Declare two weights $\alpha, \beta$ to be projectively equivalent if $\alpha^n = \beta^m$ for some $m, n \in \mathbb{Z}$. The set of projective weights $\mathbb{P}(\Lambda)$ is the set of non-zero weights in $\Lambda$ modulo projective equivalence. The elements of $\mathbb{P}(\Lambda)$ are in one to one correspondence with the codimension one subtori of $T$ by the rule

$$\alpha \in \mathbb{P}(\Lambda) \leftrightarrow \ker_0(\tilde{\alpha}) \leq T$$

where $\tilde{\alpha} \in \Lambda$ is a representative of $\alpha$, and $\ker_0(\tilde{\alpha})$ is the identity component of the kernel of $\tilde{\alpha} : T \rightarrow S^1$. We denote $\ker(\alpha) = \ker_0(\tilde{\alpha})$.

Definition 1. A GKM-hypergraph $\Gamma$ consists of:

1. A finite set of vertices $\mathcal{V} = \mathcal{V}_\Gamma$.
2. An equivalence relation $\sim_\alpha$ on $\mathcal{V}$ for each $\alpha \in \mathbb{P}(\Lambda)$.

Given a GKM-hypergraph $\Gamma$, the set of hyperedges is defined to be

$$\mathcal{E} = \mathcal{E}_\Gamma := \{(S, \alpha) \in \wp(\mathcal{V}) \times \mathbb{P}(\Lambda) \mid S \text{ is an equivalence class for } \sim_\alpha\}$$

where $\wp(\mathcal{V})$ is the power set of $\mathcal{V}$. We have projection maps

- $a : \mathcal{E} \rightarrow \mathbb{P}(\Lambda)$ the axial function, and
- $I : \mathcal{E} \rightarrow \wp(\mathcal{V})$ the incidence function.

In particular, each hyperedge $e \in \mathcal{E}$ has associated to it a projective weight $a(e)$ and a non-empty subset $I(e) \subseteq \mathcal{V}$. We say a vertex $v \in \mathcal{V}$ is incident to $e \in \mathcal{E}$ if $v \in I(e)$. Given $\alpha \in \mathbb{P}(\Lambda)$ denote by $\mathcal{E}_\alpha := \{e \in \mathcal{E} \mid a(e) = \alpha\}$.

Let $\text{Top}(\Gamma)$ to be the topological space with underlying set $\mathcal{V} \cup \mathcal{E}$ generated by basic open sets $U_v = \{v\}$ for $v \in \mathcal{V}$, and $U_e = \{e\} \cup I(e)$ for $e \in \mathcal{E}$. Observe that for each $x \in \text{Top}(\Gamma)$, the set $U_x$ is smallest open set containing $x$.

Definition 2. Let $R := H^*(BT) \cong \mathbb{C}[t]$. A GKM-sheaf $\mathcal{F}$ is a sheaf of finitely generated, $\mathbb{Z}$-graded $R$-modules over $\text{Top}(\Gamma)$, satisfying the following conditions.

1. $\mathcal{F}$ is locally free (i.e, for every basic open set $U_x$, the stalk $\mathcal{F}(U_x) = \mathcal{F}_x$ is a free $R$-module).

2. For every hyperedge $e \in \mathcal{E}_\Gamma$, the restriction map $\text{res}_e : \mathcal{F}(U_e) \rightarrow \mathcal{F}(I(e))$ is an isomorphism upon inverting $a(e)$:

$$\mathcal{F}(U_e) \otimes_R R[a(e)^{-1}] \cong \mathcal{F}(I(e)) \otimes_R R[a(e)^{-1}]$$

3. The restriction map $\text{res}_e : \mathcal{F}(U_e) \rightarrow \mathcal{F}(I(e))$ is an isomorphism for all but a finite number of $e \in \mathcal{E}_\Gamma$. 


The main motivating example is the GKM-hypergraph $\Gamma_X$ and GKM-sheaf $\mathcal{F}_X$ associated to a finite $T$-CW complex $X$. The vertex set of $\Gamma_X$ is the set $V_X := \pi_0(X^T)$ of path components of the $T$-fixed point set $X^T$. Define $v_1 \sim_\alpha v_2$ if and only if $v_1$ and $v_2$ lie in the same path component of the fixed point set $X^{\ker(\alpha)}$. The hyperedges $e \in E_\alpha := \{ e \in E \mid a(e) = \alpha \}$ therefore correspond to path components of $X^{\ker(\alpha)}$ that intersect $X^T$ non-trivially.

Define the GKM-sheaf $\mathcal{F}_X$ over $\Gamma_X$, as follows. The stalk at a vertex $v \in \pi_0(X^T)$ is

$$\mathcal{F}_X(U_v) := H^*_T(v),$$

and at a hyperedge $e \in \pi_0(X^{\ker(\alpha)})$ is

$$\mathcal{F}_X(U_e) = \mathcal{F}_X(e \cup I(e)) = H^*_T(e)/t,$$

where $t$ is the torsion submodule of $H^*_T(e)$. The sheaf restriction maps $\text{res}_e : \mathcal{F}_X(U_e) \to \mathcal{F}_X(I(e))$ are identified with the natural map $H^*_T(e)/t \to H^*_T(e^T)$ which is well defined because $e^T \subset e$ and $H^*_T(e^T)$ is torsion free. This data completely determines $\mathcal{F}_X$.

The following result (Proposition 2.7 in [2]), relates the degree zero sheaf cohomology of $\mathcal{F}_X$ with the equivariant cohomology of $X$.

**Proposition 3.1.** Let $X$ be a finite $T$-CW complex. The space of global sections $H^0(\mathcal{F}_X)$ fits into an exact sequence of graded $R$-modules

$$0 \to H^0(\mathcal{F}_X) \xrightarrow{\delta} H^*_T(X_0) \xrightarrow{\delta} H^*_{T+1}(X_1, X_0).$$

(5)

We obtain a generalization of the main result of [2], which was originally proven only when $H^*_T(X)$ is a free module.

**Corollary 3.2.** Let $X$ be a finite $T$-CW complex. If $H^*_T(X)$ is reflexive, then

$$H^*_T(X) \cong H^0(\mathcal{F}_X).$$

**Proof.** Combine Proposition 3.1 with the Chang-Skjelbred sequence [1] which holds if $H^*_T(X)$ is reflexive. 

For later use, we state the following lemmas from [2].

**Lemma 3.3.** If $X$ is a finite $T$-CW complex and $H \subset T$ is a codimension one subtorus, then $H^*_T(X^H)$ is the direct sum of a free and a torsion $R$-module. If $H^*_T(X)$ is torsion free, then $H^*_T(X^H)$ is free.

**Proof.** This is Lemma 2.6 in [2].

**Lemma 3.4.** Let $X'_1$ be the union of those components of $X_1$ which do not intersect with $X_0$. Then $H^*_T(X_1, X_0)$ decomposes into

$$H^*_T(X_1, X_0) \cong \bigoplus_{e \in E} H^*_T(e, e^T) \oplus H^*_T(X'_1).$$

(6)

**Proof.** This is Proposition 2.7 in [2].
4 Sheaf Cohomology using the Godement Resolution

We summarize material from Iversen [10]. Given a sheaf $\mathcal{F}$ over a topological space $Y$, define the sheaf $C^0\mathcal{F}$ which sends open sets $U \subseteq Y$ to be the product of stalks

$$C^0\mathcal{F}(U) = \prod_{y \in U} \mathcal{F}_y$$

and whose restriction morphisms are given by projection. There is a natural monomorphism of sheaves,

$$\mathcal{F} \rightarrow C^0\mathcal{F}$$

which sends $s \in \mathcal{F}(U)$ to the product of germs $(s_y)_{y \in U} \in C^0(\mathcal{F})(U)$. Construct sheaves $\mathcal{F}^n$ for all $n \geq 0$ inductively by setting $\mathcal{F}^0 := \mathcal{F}$ and setting $\mathcal{F}^n$ equal to the cokernel sheaf of the natural monomorphism $\mathcal{F}^{n-1} \rightarrow C^0\mathcal{F}^{n-1}$ for all $n \geq 1$. Denote $C^n\mathcal{F} = C^0\mathcal{F}^n$. By construction we get short exact sequences of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow C^0\mathcal{F} \xrightarrow{d_0} C^1\mathcal{F} \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n\mathcal{F} \xrightarrow{d_n} \cdots$$

for all $n \geq 0$. Let $d_n := f_{n+1} \circ g_n$ be the composition,

$$C^n\mathcal{F} \xrightarrow{g_n} \mathcal{F}^{n+1} \xrightarrow{f_{n+1}} C^{n+1}\mathcal{F}$$

Theorem 4.1. The sequence of sheaves

$$0 \rightarrow \mathcal{F} \rightarrow C^0\mathcal{F} \xrightarrow{d_0} C^1\mathcal{F} \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n\mathcal{F} \xrightarrow{d_n} \cdots$$

is exact. It is called the Godement resolution of $\mathcal{F}$.

Proof. See [10]. □

Given an open set $U \subseteq Y$, define the chain complex

$$0 \xrightarrow{d_{n-1}} C^0\mathcal{F}(U) \xrightarrow{d_0} C^1\mathcal{F}(U) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} C^n\mathcal{F}(U) \rightarrow \cdots$$

which satisfies $d_n \circ d_{n-1} = 0$ for all $n \geq 0$. Define the degree $n$ cohomology of $\mathcal{F}$ on $U$ by

$$H^n(U; \mathcal{F}) := \frac{\ker(d_n)}{\text{im}(d_{n-1})}.$$ 

We use shorthand $H^n(\mathcal{F}) := H^n(Y, \mathcal{F})$. The sequence

$$0 \rightarrow \mathcal{F}(U) \rightarrow C^0\mathcal{F}(U) \xrightarrow{d_0} C^1\mathcal{F}(U)$$

is exact, which implies $H^0(U, \mathcal{F}) = \mathcal{F}(U)$.

Definition 3. For a sheaf $\mathcal{F}$ on $Y$ and a closed subset $A \subseteq Y$ we define

$$\Gamma_A(\mathcal{F}) = \{ s \in \mathcal{F}(Y) | \text{supp}(s) \subseteq A \}$$

where $\text{supp}(s) = \{ y \in Y | s_y \neq 0 \}$. If $A = \emptyset$ we write $\Gamma_0(\mathcal{F}) = \Gamma(\mathcal{F}) = \mathcal{F}(Y)$. 

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Given the chain complex

\[
0 \xrightarrow{d_{n-1}} \Gamma_A(C^0\mathcal{F}) \xrightarrow{d_0} \Gamma_A(C^1\mathcal{F}) \xrightarrow{d_1} \ldots \xrightarrow{d_n} \Gamma_A(C^n\mathcal{F}) \to \ldots
\]

define the local cohomology \( H^n_A(Y, \mathcal{F}) := \ker(d_n) / \text{im}(d_{n-1}) \).

**Proposition 4.2.** Let \( A \) be closed in \( Y \). A sheaf \( \mathcal{F} \) on \( Y \) gives rise to a long exact sequence

\[
0 \to H^0_A(Y, \mathcal{F}) \to H^0(Y, \mathcal{F}) \to H^0(Y - A, \mathcal{F}) \xrightarrow{\delta} H^1_A(Y, \mathcal{F}) \to H^1(Y, \mathcal{F}) \to \ldots
\]

**Proof.** See Proposition 9.2 in [10]. \( \square \)

## 5 Cohomology of GKM-Sheaves

### 5.1 The Godement Chain Complex for GKM-Sheaves

**Proposition 5.1.** If \( \Gamma \) is a GKM-hypergraph and \( \mathcal{F} \) is a sheaf on \( \text{Top}(\Gamma) \), then \( C^n\mathcal{F} = 0 \), for all \( n \geq 2 \).

**Proof.** The basic open sets of \( \text{Top}(\Gamma) \) are

(i) \( U_v := \{v\} \) for vertices \( v \in \mathcal{V} \).

(ii) \( U_e := \{e\} \cup I(e) \) for hyperedges \( e \in \mathcal{E} \).

Given a sheaf \( \mathcal{F} \) on \( \text{Top}(\Gamma) \), for each vertex \( v \) we have

\[
(C^0\mathcal{F})_v := \prod_{x \in U_v} \mathcal{F}_x = \mathcal{F}_v.
\]

The induced morphism \( \mathcal{F}_v \to (C^0\mathcal{F})_v = \mathcal{F}_v \) is an isomorphism, so the cokernel \( C^1\mathcal{F}_v \) is zero. Similarly, for all \( n \geq 1 \),

\[
C^n\mathcal{F}_v = 0.
\]

For each hyperedge \( e \), \( \mathcal{F}^2_e \) is the cokernel of the product of restriction maps

\[
\mathcal{F}_e \to \prod_{x \in U_e} \mathcal{F}_x \cong \mathcal{F}_e \times \mathcal{F}_{v_1}^1 \times \ldots \times \mathcal{F}_{v_k}^1.
\]

This is an isomorphism because \( \mathcal{F}^1_{v_i} = 0 \), so the cokernel \( \mathcal{F}^2_e \cong 0 \). We conclude \( \mathcal{F}^2 = 0 \) since all of its stalks vanish and consequently \( C^n\mathcal{F} = 0 \) for all \( n \geq 2 \). \( \square \)

**Corollary 5.2.** If \( \mathcal{F} \) is a sheaf over \( \text{Top}(\Gamma) \), then \( H^n(\mathcal{F}) = 0 \) for \( n \geq 2 \) and we have a natural exact sequence

\[
0 \to H^0(\mathcal{F}) \to \Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) \to H^1(\mathcal{F}) \to 0.
\]
Proof. By Proposition 5.1, \( C^n(\mathcal{F}) = \mathcal{F}^n = 0 \) for all \( n \geq 2 \), the chain complex for \( H^n(\mathcal{F}) \) is

\[
0 \rightarrow \Gamma(C^0\mathcal{F}) \xrightarrow{\delta} \Gamma(C^1\mathcal{F}) \rightarrow 0 \rightarrow 0 \rightarrow \cdots.
\] (12)

We want a more concrete description of (12).

**Lemma 5.3.** The stalks of \( \mathcal{F}^1 \) are as follows: \( \mathcal{F}^1_v = 0 \) for all vertices \( v \), and

\[
\mathcal{F}^1_e \cong \prod_{i=1}^{k} \mathcal{F}_{v_i} = \mathcal{F}(I(e))
\]

for all hyperedges \( e \), where \( I(e) = \{v_1, v_2, \ldots, v_k\} \) is the set of vertices incident to \( e \).

**Proof.** That \( \mathcal{F}^1_v = 0 \) was proven in (11). Given a hyperedge \( e \), recall \( U_e := \{e\} \cup I(e) \). The stalk \( \mathcal{F}^1_e \) is the cokernel of the map

\[
\mathcal{F}_e \xrightarrow{\varepsilon} \prod_{x \in U_e} \mathcal{F}_x = \mathcal{F}_e \times \mathcal{F}_{v_1} \times \cdots \times \mathcal{F}_{v_k}
\]

where \( \varepsilon(s_e) = (s_e, \text{res}_{(e,v_1)}(s_e), \ldots, \text{res}_{(e,v_k)}(s_e)) \), where \( \text{res}_{(e,v)} : \mathcal{F}_e \rightarrow \mathcal{F}_v \) is the sheaf restriction map associated to \( U_v \subseteq U_e \). Because the first coordinate function of \( \varepsilon \) is the identity map on \( \mathcal{F}_e \) we obtain the isomorphism

\[
\mathcal{F}^1_e := \text{coker}(\varepsilon) \cong \prod_{i=1}^{k} \mathcal{F}_{v_i}
\]

simply by projecting onto the remaining factors. \( \square \)

**Proposition 5.4.** Let \( \mathcal{F} \) be a sheaf over \( \text{Top}(\Gamma) \). Then there is a commutative diagram,

\[
\begin{array}{ccc}
\Gamma(C^0\mathcal{F}) & \xrightarrow{\delta} & \Gamma(C^1\mathcal{F}) \\
\cong & & \cong \\
\prod_{x \in V \cup E} \mathcal{F}_x & \xrightarrow{\tilde{\delta}} & \prod_{e \in E} \prod_{v \in I(e)} \mathcal{F}_v \\
\end{array}
\]

where the vertical maps isomorphisms, \( \delta \) is as above and \( \tilde{\delta} \) sends \( s = (s_x)_{x \in V \cup E} \) to \( \tilde{\delta}(s) \) with factors

\[
\tilde{\delta}(s)_{(e,v)} = \text{res}_{(e,v)}(s_e) - s_v.
\] (13)

In particular, \( H^0(\mathcal{F}) \cong \text{ker}(\tilde{\delta}) \) and \( H^1(\mathcal{F}) \cong \text{coker}(\tilde{\delta}) \).
Proof. The isomorphism $\Gamma(C^0F) \cong \prod_{x \in V \cup E} F_x$ is the defining identity (7). By Lemma 5.3 we have

$$\Gamma(C^1F) := \prod_{x \in V \cup E} F_x \cong \prod_{e \in \mathcal{E}} \prod_{i=1}^k F_{v_i}.$$\[10]

The formula for $\tilde{\delta}$ is obtained by chasing through definition [1].

**Proposition 5.5.** Let $F$ be a GKM-sheaf and let $\mathcal{E}^{nd} \subseteq \mathcal{E}$ be the finite set of hyperedges for which $\text{res}_e$ is not an isomorphism. Then there is an exact sequence

$$0 \to H^0(\mathcal{F}) \to \bigoplus_{x \in \mathcal{E}^{nd}} F_x \xrightarrow{\beta} \bigoplus_{e \in \mathcal{E}^{nd}} \bigoplus_{v \in I(e)} F_v \to H^1(\mathcal{F}) \to 0$$

(14)

where $\beta$ is the morphism of finitely generated free $R$-modules defined by

$$\beta(s)_{(e,v)} = \text{res}_{(e,v)}(s_e) - s_v.$$

Proof. Let $\mathcal{E}^d := \mathcal{E} \setminus \mathcal{E}^{nd}$. We have a commuting diagram of $R$-modules with exact rows

$$\begin{array}{cccccccc}
0 & \to & \prod_{x \in \mathcal{E}^d} F_x & \xrightarrow{\psi} & \prod_{x \in V \cup \mathcal{E}} F_x & \xrightarrow{\phi} & \bigoplus_{x \in \mathcal{E}^{nd}} F_x & \to & 0 \\
& & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\beta} & & \\
0 & \to & \prod_{e \in \mathcal{E}^d v \in I(e)} F_v & \xrightarrow{\psi'} & \prod_{e \in \mathcal{E} v \in I(e)} F_v & \xrightarrow{\phi'} & \bigoplus_{e \in \mathcal{E}^{nd} v \in I(e)} F_v & \to & 0.
\end{array}$$

Where $\phi$, $\phi'$ are projections, $\psi$, $\psi'$ are inclusions, and $\gamma$ is defined by commutativity. By the Snake Lemma there is an exact sequence

$$0 \to \ker \gamma \to \ker(\tilde{\delta}) \xrightarrow{\phi} \ker(\beta) \to \text{coker}(\gamma) \xrightarrow{\psi'} \text{coker}(\tilde{\delta}) \xrightarrow{\phi'} \text{coker}(\beta) \to 0.$$\[11]

It is clear by definition of $\mathcal{E}^d$ that $\gamma$ is an isomorphism. Thus,

$$0 \to 0 \to \ker(\tilde{\delta}) \xrightarrow{\phi} \ker(\beta) \to 0 \to \text{coker}(\tilde{\delta}) \xrightarrow{\phi'} \text{coker}(\beta) \to 0$$

is exact so $\phi$ and $\phi'$ are isomorphisms. Compare Proposition 5.4.

**Corollary 5.6.** If $\mathcal{F}$ is a GKM-sheaf, then $H^0(\mathcal{F})$ is reflexive.

Proof. We see from (11) we have an exact sequence $0 \to H^0(\mathcal{F}) \to F_0 \to F_1$ where $F_0$ and $F_1$ are finitely generated free modules so $H^0(\mathcal{F})$ is a 2-syzygy, hence reflexive.
5.2 Local Cohomology of a GKM-Sheaf

Let $\mathcal{F}$ be a GKM-sheaf over $\text{Top}(\Gamma) = V \cup E$. The set of vertices $V$ is an open set and the set of edges $E$ is a closed set, so we obtain a long exact sequence by Proposition 4.2

$$0 \to H^0(\text{Top}(\Gamma), \mathcal{F}) \to H^0(V, \mathcal{F}) \to H^1_{\mathcal{E}}(\text{Top}(\Gamma), \mathcal{F}) \to H^1(\text{Top}(\Gamma), \mathcal{F}) \to \cdots.$$ 

Since $V$ is discrete, $H^i(V, \mathcal{F}) = 0$ for all $i \geq 1$ and

$$0 \to H^0(\text{Top}(\Gamma), \mathcal{F}) \to H^0(V, \mathcal{F}) \to H^1_{\mathcal{E}}(\text{Top}(\Gamma), \mathcal{F}) \to H^1(\text{Top}(\Gamma), \mathcal{F}) \to 0 \quad (15)$$

is exact.

**Lemma 5.7.** The Godement chain complex for $H^*_E(\text{Top}(\Gamma), \mathcal{F})$ is given by

$$0 \to \prod_{e \in E} \mathcal{F}_e \xrightarrow{\prod_{e \in E} \text{res}_e} \prod_{e \in E} \prod_{v \in \tilde{I}(e)} \mathcal{F}_v \to 0. \quad (16)$$

**Proof.** Applying Definition 3 and Lemma 5.3 we have

$$\Gamma_E(C^0\mathcal{F}) = \{ s \in \Gamma(C^0\mathcal{F}) \mid s_\emptyset = 0, \forall v \in V \} = \prod_{e \in E} \mathcal{F}_e$$

$$\Gamma_E(C^1\mathcal{F}) = \{ s \in \Gamma(C^1\mathcal{F}) \mid s_\emptyset = 0, \forall v \in V \} = \prod_{e \in E} \prod_{v \in \tilde{I}(e)} \mathcal{F}_v,$$

and the boundary map $\Gamma_E(C^0\mathcal{F}) \to \Gamma_E(C^1\mathcal{F})$ is the natural one. □

**Proposition 5.8.** If $\Gamma$ is a GKM-hypergraph and $\mathcal{F}$ is a GKM-sheaf on $\text{Top}(\Gamma)$, then

$$H^0(V, \mathcal{F}) \cong \bigoplus_{v \in V} \mathcal{F}_v,$$

$$H^1_{\mathcal{E}}(\text{Top}(\Gamma), \mathcal{F}) \cong \bigoplus_{e \in E^{nd}} \text{coker}(\text{res}_e).$$

**Proof.** Since $V$ is discrete, $H^0(V) = \prod_{v \in V} \mathcal{F}(\{v\}) = \bigoplus_{v \in V} \mathcal{F}_v$. By Lemma 5.7 we get

$$H^1_{\mathcal{E}}(\text{Top}(\Gamma), \mathcal{F}) = \text{coker}\left(\prod_{e \in E} \text{res}_e\right) = \prod_{e \in E^{nd}} \text{coker}(\text{res}_e).$$

By Definition 2, $\text{coker}(\text{res}_e) \neq 0$ only for $e$ in a finite subset $E^{nd} \subseteq E$ so

$$\prod_{e \in E^{nd}} \text{coker}(\text{res}_e) \cong \bigoplus_{e \in E^{nd}} \text{coker}(\text{res}_e).$$

□
6 Geometric meaning of GKM-sheaf cohomology

Theorem 6.1. Let $X$ be a finite $T$-$CW$ complex. Then

$$H^*_T(X) \cong H^0(F_X)$$

if and only if $H^*_T(X)$ is reflexive.

Proof. Suppose $H^*_T(X) \cong H^0(F_X)$. By Corollary 5.6, we conclude that $H^*_T(X)$ is reflexive.

Conversely, suppose that $H^*_T(X)$ is reflexive. By Theorem 2.2, the Chang-Skjelbred sequence

$$0 \to H^*_T(X) \to H^*_T(X_0) \overset{\delta}{\to} H^{*+1}_T(X_1, X_0)$$

is exact, so $H^*_T(X) \cong \ker(\delta)$ and $\ker(\delta) \cong H^0(F_X)$ by Proposition 3.1.

Lemma 6.2. Let $X'_1 \subseteq X_1$ be the union of path components that do not intersect $X_0$. Suppose $H^*_T(X)$ is torsion free. Then $X'_1 = \emptyset$.

Proof. Observe that $X_1$ can be written as follows:

$$X_1 = \bigcup_{H \leq T} X^H$$

where the union is indexed by codimension one subtori $H$. Since $H^*_T(X)$ is torsion free, $H^*_T(X^H)$ is free by Lemma 3.3. By the Localization theorem 2.1, every path component of $X^H$ must intersect $X_0$ so $X^H \cap X'_1 = \emptyset$. We conclude $X'_1 = \emptyset$.

Lemma 6.3. Suppose $H^*_T(X)$ is torsion free. Then $H^0(\mathcal{V}, F_X) \cong H^*_T(X_0)$ and $H^1_\mathcal{E}(\text{Top}(\Gamma), F_X) \cong H^{*+1}_T(X_1, X_0)$.

Proof. The vertices $v \in \mathcal{V}$ correspond path components of $X_0$, so

$$H^0(\mathcal{V}, F) = \bigoplus_{v \in \mathcal{V}} F_v = \bigoplus_{v \in \mathcal{V}} H^*_T(v) = H^*_T(X_0).$$

The hyperedges $e \in \mathcal{E}$ for which $a(e) = \alpha$ correspond to path components of $X^{\ker(\alpha)}$ that intersec non-trivially with $X_0$. Combine Lemma 3.4 with Lemma 6.2 to get

$$H^*_T(X_1, X_0) \cong \bigoplus_{e \in \mathcal{E}} H^*_T(e, e^T). \quad (17)$$

Claim 6.1. If $H^*_T(X)$ is torsion free, then $F_X(U_e) \cong H^*_T(e)$.

Proof. Recall that by definition $F_X(U_e) := H^*_T(e)/t$ where $t$ is the torsion submodule so it is enough to show $H^*_T(e)$ is torsion free for all $e \in \mathcal{E}$. Since $H^*_T(X)$ is a submodule of finitely generated free $R$-module, it is torsion free. Apply Lemma 3.3.
The restriction morphism \( \text{res}_e : \mathcal{F}(U_e) \to \mathcal{F}(I(e)) \) is identical with the natural map \( H^*_T(e) \to H^*_T(e^T) \) which is injective by the Localization Theorem \ref{thm:localization}. The long exact sequence for the pair \((e, e^T)\) implies

\[
H^{*+1}_T(e, e^T) = \text{coker}(\text{res}_e).
\]

Applying Proposition \ref{prop:injective}, we have

\[
H^1_t(\text{Top}(\Gamma), \mathcal{F}) = \bigoplus_{e \in E} H^{*+1}_T(e, e^T).
\]

Combining with \((17)\), we conclude

\[
H^1_t(\text{Top}(\Gamma), \mathcal{F}) \cong H^{*+1}_T(X_1, X_0).
\]

\[\square\]

**Proof of Theorem \ref{thm:main}** Since \( H^*_T(X) \) is reflexive, Theorem \ref{thm:reflexive} implies that

\[
0 \to H^0(\mathcal{F}_X) \to H^*_T(X_0) \overset{\delta}{\to} H^{*+1}_T(X_1, X_0)
\]

is exact. From Lemma \ref{lem:exact} and \((15)\) we have an isomorphism of exact sequences.

\[
\begin{array}{cccccc}
0 & \to & H^*_T(X) & \to & H^*_T(X_0) & \overset{\delta}{\to} & H^{*+1}_T(X_1, X_0) & \to & \text{coker}(\delta) & \to & 0 \\
& & \cong & & \cong & & \cong & & \cong & & \\
0 & \to & H^0(\mathcal{F}_X) & \to & H^0(Y, \mathcal{F}_X) & \to & H^1_t(\text{Top}(\Gamma), \mathcal{F}_X) & \to & H^1(\mathcal{F}_X) & \to & 0 \\
\end{array}
\]

\[\square\]

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