ON A UNIFORM BOUND FOR EXPONENTIAL SUMS MODULO $p^m$ FOR DELIGNE POLYNOMIALS

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Abstract. Let $f$ be a polynomial of degree $d > 1$ in $n$ variables over $\mathbb{Z}$. Let $f_d$ be the homogeneous part of degree $d$ of $f$ and $s$ be the dimension of the critical locus of $f_d$. In this paper, we prove Igusa’s conjecture for exponential sums with the exponent $(n - s)/(2(d - 1))$. This implies a weak solution for a recent conjecture raised by Cluckers and the author (2020) about an analogue of the results of Deligne (1974) and Katz (1999) for exponential sums over finite fields in the finite ring setting. Moreover, this also improves the result of Cluckers, Mustaţă and the author (2019) in case $n - s > 2(d - 1)$. In particular, this result improves the conditions $n - s > 2^d(d - 1)$ of Birch (1962) and $n - s > (d - 1)2^{d-2}$ of Browning-Prendiville (2017) on the validity of the estimation for the major arcs to $(n - s) > 4(d - 1)$. Therefore this result may have further applications on subjects related to the Hardy-Littlewood circle method such as the Hasse principle or distribution of rational points in algebraic varieties. On the other hand, we also improve the recent work of Cluckers, Kollár and Mustaţă (2019) on the strong monodromy conjecture in the range $(-\text{lct}(f) + J_f, 0]$ in case of bad reduction and bad Schwartz-Bruhat function. Namely, in the range $(-\text{lct}(f) + J_f, 0]$, the real part of any pole of the Igusa local zeta functions associated with $f$ and any Schwartz-Bruhat function over any $p$-adic field is a root of the Bernstein-Sato polynomial of $f$.

1. Introduction

Let $f(x)$ be a polynomial in $n$ variables over $\mathbb{Z}$ and of degree $d > 1$. Let $s = s(f)$ be the dimension of the critical locus of $f_d : \mathbb{C}^n \to \mathbb{C}$, where $f_d$ is the degree $d$ homogeneous part of $f$. When $f_d$ is smooth, i.e., $s = 0$, we say that $f$ is a Deligne polynomial. This name follows by the famous theorem of Deligne in [10] for exponential sums over finite fields. Let us recall this theorem.

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Theorem 1.1. Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a Deligne polynomial. Let $p$ be a large enough prime and $k$ be a finite field of characteristic $p$. Then
\[ ||k|| - n \sum_{x \in k^n} \psi(\text{Tr}_{k/\mathbb{F}_p}(f(x))) \leq (d - 1)^n |k|^{-n/2}, \]
where $\text{Tr}_{k/\mathbb{F}_p} : k \to \mathbb{F}_p$ is the trace function and $\psi : \mathbb{F}_p \to \mathbb{C}^\times$ is any non-trivial character of $\mathbb{F}_p$.

Remark 1.2. In the original statement, Deligne worked with a fixed prime $p$ and a Deligne polynomial $f \in \mathbb{F}_p[x_1, \ldots, x_n]$ of degree $d$, i.e., $(d, p) = 1$ and the critical locus of $f_d : \mathbb{F}_p^n \to \mathbb{F}_p$ has dimension 0. Theorem 1.1 follows by the fact that if $f \in \mathbb{Z}[x_1, \ldots, x_n]$ is a Deligne polynomial then $f \mod p$ is a Deligne polynomial if $p$ is large enough.

In [18], Katz extended Deligne’s theorem to more general setting where we allows $x$ running in the set of $k$-points of any $k$-scheme $Z$ with good enough geometric condition and $f$ is a regular function on $Z$. Katz’s result depends on the complexity of $Z$ and the singularities of $f$ at infinity. Where the complexity of $Z$ depends on a given embedding $Z \hookrightarrow \mathbb{P}^N$, the number $R$ and the degrees $D_1, \ldots, D_R$ of equations defining $Z$ in $\mathbb{P}^N$. However, if we only work on the setting of Theorem 1.1 but allow $s > 0$ then we can get a simple bound of the exponential sum by using Theorem 1.1 and induction on $s$ in a suitable manner.

Proposition 1.3. [9, Section 3.5] Let $f \in \mathbb{Z}[x_1, \ldots, x_n]$ be a polynomial of degree $d > 1$ and $s$ be given as above. Let $p$ be a large enough prime and $k$ be a finite field of characteristic $p$ then
\[ ||k|| - n \sum_{x \in k^n} \psi(\text{Tr}_{k/\mathbb{F}_p}(f(x))) \leq (d - 1)^n |k|^{-(n-s)/2}. \]

A natural question that one may ask is whether an analogue of Proposition 1.3 exists for exponential sums over finite rings $\mathbb{Z}/N\mathbb{Z}$, i.e., the exponential sums
\[ E_f(\psi) = \frac{1}{N^n} \sum_{x \in (\mathbb{Z}/N\mathbb{Z})^n} \psi(f(x)), \]
where $N \neq 0$ and $\psi$ is any group monomorphism from $\mathbb{Z}/N\mathbb{Z}$ to $\mathbb{C}^\times$. More general, we also ask this question for exponential sums
\[ E_f(\psi) = \frac{1}{|\mathcal{O}_K/I|^n} \sum_{x \in (\mathcal{O}_K/I)^n} \psi(f(x)), \]
where $\mathcal{O}_K$ is the ring of integers in a number field $K$, $I$ is any non-zero ideal of $\mathcal{O}_K$ and $\psi$ is any group monomorphism from $\mathcal{O}_K/I$ to $\mathbb{C}^\times$. This question lies in the context of Igusa’s conjecture on exponential sums and its relation with the Hasse principle as seen later. Inspired by the strong monodromy conjecture (see Section 2 for a formulation), Igusa’s conjecture on exponential sums (see for instance [17, 8]) and the recent work on the
minimal exponent of Mustaţă and Popa in [23], Cluckers and the author proposed in [9] a conjecture as follows:

**Conjecture 1.** Given $f$, $n$, $s$, and $d$ as above. For each $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that

$$\left| E_f(\psi) \right| \leq C_\varepsilon N^{-\frac{m}{2d} + \varepsilon}.$$  

for all integers $N \geq 1$, all group monomorphisms $\psi : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}^\times$.

By the Chinese Remainder Theorem, if one writes $N = \prod_{1 \leq i \leq r} p_i^{m_i}$ for distinct prime numbers $p_i$ and integers $m_i > 0$, then for each group monomorphism $\psi : \mathbb{Z}/N\mathbb{Z}$, there is a group monomorphism $\psi_i : \mathbb{Z}/p_i^{m_i} \to \mathbb{C}^\times$ for each $1 \leq i \leq r$ such that

$$E_f(\psi) = \prod_{1 \leq i \leq r} E_f(\psi_i).$$

Because of Formula (2) and Proposition 1.3, to prove Conjecture 1, it is sufficient to show that for each $\varepsilon > 0$, there is a constant $C_\varepsilon$ such that for all primes $p$, all $m > 0$ and all group monomorphisms $\psi : \mathbb{Z}/p^m \to \mathbb{C}^\times$ one has

$$\left| E_f(\psi) \right| \leq C_\varepsilon p^{m\left(-\frac{m}{2d} + \varepsilon\right)},$$

in addition, when $p$ is large enough and $m > 1$ one also needs

$$\left| E_f(\psi) \right| \leq p^{m\left(-\frac{m}{2d} + \varepsilon\right)}.$$  

Inequality (4) for large $p$ and $m > 1$ relates to Igusa’s conjecture on exponential sums. Let us explain this in more detail. Let $p$ be a prime and $m > 0$ be an integer. Let $\psi : \mathbb{Z}/p^m \mathbb{Z} \to \mathbb{C}^\times$ be a group monomorphism. There exists an additive character $\tilde{\psi}$ of $\mathbb{Q}_p$ of conductor $m$, i.e., $\psi|_{p^m \mathbb{Z}_p} = 1$ and $\tilde{\psi}|_{p^m-1 \mathbb{Z}_p} \neq 1$, such that

$$E_f(\psi) = E_f(\tilde{\psi}) := \int_{\mathbb{Z}_p^m} \tilde{\psi}(f(x))|dx|,$$

where $|dx|$ is the Haar measure on $\mathbb{Q}_p^m$ such that the volume of $\mathbb{Z}_p^m$ is 1. The integral in (5) helps us to generalize the notion of exponential sums to all non-Archimedean local fields. We refer to Section 2 for all details; here we mention the main ideas. Let $K$ be a number field and $\mathcal{O}_K$ be its ring of integers. Let $Z$ be a subscheme of $\mathbb{A}^n_{\mathcal{O}_K}$. Let $L$ be a non-Archimedean local field over $\mathcal{O}_K$, i.e., $L$ is endowed with a structure of $\mathcal{O}_K$-algebra. We associate to $Z$ the Schwartz-Bruhat function $\Phi_{Z,L} := 1_{\{x \in \mathcal{O}_L^n | \xi \in \mathcal{M}_L(x)\}}$ on $L^n$, where $\mathcal{O}_L$ is the ring of integers in $L$, $k_L$ is the residue field of $L$ and $\overline{\pi}$ is the reduction of $x$ modulo the maximal ideal $\mathcal{M}_L$ of $\mathcal{O}_L$. Let $f \in \mathcal{O}_K[x_1, \ldots, x_n]$ and $\psi$ be a non-trivial additive character of $L$. We associate to $f, \psi, Z$ the exponential sum

$$E_f(Z, \psi) = \int_{\mathcal{O}_L^n} \Phi_{Z,L}\psi(f(x))|dx|.$$
When $Z = \mathbb{A}^n_{\mathbb{Q}}$, we write $E_f(\psi)$ instead of $E_f(\mathbb{A}^n_{\mathbb{Q}}, \psi)$. Now we recall a simple form of Igusa’s conjecture on exponential sums.

**Conjecture 2.** Let $f \in \mathcal{O}_K[x_1, ..., x_n]$ be a non constant polynomial. Let $Z$ be a subscheme of $\mathbb{A}^n_{\mathbb{Q}}$. Let $\sigma$ be a positive real number. Suppose that for each non-Archimedean local field $L$ over $\mathcal{O}_K$ of large enough residue field characteristic, there is a constant $c_L$ such that

$$|E_f(Z, \psi)| \leq c_L |k_L|^{-m_{\psi}\sigma}$$

for all additive characters $\psi$ of $L$ of conductor $m_{\psi} > 1$ (see Section 2 for the definition of $m_{\psi}$). Then in the inequality (6) one can take $c_L$ independent of $L$ provided that $k_L$ is of large enough characteristic.

**Remark 1.4.** The supremum taken over all $\sigma$ satisfying the hypothesis of Conjecture 2 is the motivic oscillation index $\text{moi}_{K, Z}(f)$ of $f$ at $Z$ over $K$ (see [8, 24]). This index relates to the largest non-trivial pole of the Igusa local zeta functions (see Section 2 for the definition of these functions) associated with $f, Z, L$ when the local field $L$ is of large enough residue field characteristic. If $f - a$ has non-rational singularities at $Z(\mathbb{C})$ for some critical value $a$ of $f$ then Cluckers, Mustaţă and the author proved Conjecture 2 for any $\sigma < \text{moi}_{K, Z}(f) = \min_{a \in Z(\mathbb{C})} \text{lct}_a((f - f(a))$ in [8], where $\text{lct}_a(g)$ is the log-canonical threshold of $g$ at $a$ (see Section 2 for the definition of the log-canonical threshold). Recently, Cluckers, Kollár and Mustaţă showed in [6] that $\min_{a \in Z(\mathbb{C})} \text{lct}_a((f - f(a)) + J_f^2)$ is a lower bound of $\text{moi}_{K, Z}(f)$, where $J_f$ is the Jacobian ideal of $f$ and $\text{lct}_a((f - f(a)) + J_f^2)$ is the log-canonical threshold of the ideal $(f - f(a)) + J_f^2$ at $a$.

Within (4), we expect that $\sigma = (n - s)/d$ satisfies the hypothesis of Conjecture 2 for $Z = \mathbb{A}^n_2$. In other word, we should have $\text{moi}_{\mathbb{Q}, \mathbb{A}^n_2}(f) \geq (n - s)/d$. However, this seems to be a very hard problem at this moment.

**Remark 1.5.** Suppose one can show inequality (4) for $p > M$ and $m > 1$. To prove Conjecture 1, it remains to prove (3) for $p \leq M$. This question leads us to the study of non-trivial poles of the Igusa local zeta functions over $p$-adic fields where we do not know the existence of a log-resolution of $(f)$ with good reduction (see section 2 for the definitions). Up to present, there is no effective tools to deal with this question. Indeed, the main strategy in the works of Igusa is to find a log-resolution of $(f)$ and compute candidates of poles of the Igusa local zeta functions based on the numerical data associated with this log-resolution. More precisely, let $(\nu_i, N_i)_{i \in \mathcal{T}}$ be this numerical data, we need that $\min_{i \in \mathcal{T}, (\nu_i, N_i) \neq (1, 1)} \nu_i / N_i$ is at least $(n - s)/d$ to derive (3) for small $p$. However, it is not known how to construct such a log-resolution with this property in the general case.

In this paper we will prove a weaker version of Conjecture 1 for $(n - s)/(2(d - 1))$ instead of $(n - s)/d$. First of all, we will show that Conjecture 2 holds for $\sigma \leq (n - s)/(2(d - 1))$ as follows.
Theorem A. Suppose that $Z$ is a subscheme of $\mathbb{A}^n_{\mathbb{Q}}$ whose complexity bounded by $(R,D)$, i.e., $Z_K$ can be defined by $R$ polynomials in $n$ variables of degree at most $D$. There exist a constant $M$ depending only on $f$, $Z$ and a constant $C$ depending only on $n,d,R,D$ such that for all non-Archimedean local fields $L$ over $\mathcal{O}_K$ of residue field characteristic at least $M$ and all additive characters $\psi$ of $L$ of conductor $m_{\psi} \geq 2$ we have

$$|E_f(Z,\psi)| \leq C|k_L|^\frac{m_{\psi}(n-s)}{2(d-1)}.$$  

The main idea for the proof of Theorem A comes from the recent work of Veys and the author in [24], where we constructed a way to view Conjecture 2 as a conjecture for a family of exponential sums over finite fields. Then we can use a consequence of Katz’s result in [5] for exponential sums associated with weighted homogeneous polynomials over finite fields to give a uniform version of Theorem A when $Z$ is any closed point of $\mathbb{A}^n_{\mathbb{Q}}$. Combining this with an idea from [4] about constructible motivic functions helps us to prove Theorem A for any scheme $Z$.

To show (3) for $(n-s)/(2(d-1))$ instead of $(n-s)/d$, we will show first that $(n-s)/(2(d-1)) < \text{lct}_a((f-f(a)) + J_{f,a}^2)$ for all $a$. After that, we extend a result of Cluckers, Kollár and Mustaţă in [6] to the case of all $p$-adic fields and all Schwartz-Bruhat functions. Namely, we prove that:

Theorem B. Let $L$ be a $p$-adic field. Suppose that $f$ is a non-constant polynomial in $L[x_1,\ldots,x_n]$. For each Schwartz-Bruhat function $\Phi$ on $L^n$ and each $0 < \sigma < \sigma_0 := \min_{\nu \in \text{Supp}(\Phi)} \text{lct}_a((f-f(a)) + J_{f,a}^2)$, there is a constant $c(\sigma, \Phi) > 0$ such that for all additive characters $\psi$ of $L$ we have

$$|E_f(\Phi,\psi)| := \left| \int_{L^n} \Phi(x)\psi(f(x))dx \right| \leq c(\sigma, \Phi)|k_L|^{-m_{\psi}\sigma}. \quad (7)$$

As mentioned above, Igusa’s idea on the numerical data of log-resolution is hard to use. In fact, we do not know a way to construct a log resolution of $(f)$ with the numerical data $(\nu_i, N_i) \in T$ such that $\min_{\nu_i \in T, (\nu_i, N_i) \neq (1,1)} \nabla_{\nu_i}/N_i \geq \text{lct}_a((f) + J_{f,a}^2)$ for all singular points $a$ of $f$. In this paper, we will try to prove (7) directly, as in [6] but with a more advanced technique. More precisely, we need to control well twisted Igusa local zeta functions associated with higher ramification characters. As in [6], Theorem B also implies some consequence in Section 4 related to the strong monodromy conjecture in the ranges $(-\text{lct}(f, J(f)), 0]$ and $[-(n-s)/(2(d-1)), 0]$. This consequence extends the result in [6] to the context of all $p$-adic fields and all Schwartz-Bruhat functions.

By using Theorem A and Theorem B, we will show in Section 4 that:

Theorem C. Let $K, \mathcal{O}_K$ be as above. Suppose that $f$ is a non-constant polynomial in $\mathcal{O}_K[x_1,\ldots,x_n]$ of degree $d > 1$. Let $s = s(f)$ be as above. For each finite extension $K'$ of $K$ and $\varepsilon > 0$, there is a constant $c_{K',\varepsilon}$ such that for all non-zero ideals $I$ of $\mathcal{O}_{K'}$ and all group monomorphisms $\psi : \mathcal{O}_{K'}/I \rightarrow \mathbb{C}^\times$
we have

\[ |E_f(\psi)| \leq C_{K', \varepsilon}|O_{K'}/I|^{-\frac{2}{d(d-1)}+\varepsilon} \]

As mentioned in [17, 9], Conjecture 1 will help to obtain a simple condition for the validity of a certain adèlic Poisson summation formula and the estimation for the major arcs towards the smooth Hasse principle of \( f \). More precisely, we expect that this condition is \( n - s > 2d \). Furthermore, we may hope that there is an improvement of Birch’s result in [1] for the smooth Hasse principle related to this condition in the future.

Note that the current condition for the validity of the estimation for the major arcs relates to Birch’s version in [1] where he claimed that \( n - s > 2d(d-1) \) is a sufficient condition. In [2], Browning and Prendiville improved Birch’s condition to \( n - s > \frac{3}{2}2d(d-1) \). In fact, using Theorem C and the result of Igusa on zeta functions over Archimedean local fields in [17], we can improve Birch’s condition to \( n - s > 4(d-1) \) (see Section 5). Thus although Conjecture 1 has not been proved yet, the result of this paper is still good enough for further applications.

In fact, even if we can improve the estimation for the major arcs, it remains a huge challenge to overcome the estimation for the minor arcs to obtain a new form of the smooth Hasse principle (see Section 5). However, with the result of this paper, any improvement of the estimation for the minor arcs in the future may automatically imply a new form of the smooth Hasse principle. On the other hand, the estimation for the major arcs appears in many researches in number theory related to distribution of rational points in algebraic varieties under the Hardy-Littlewood circle method. Therefore, the result of this paper may streamline much future research related to this topic.

2. Notation and preliminary results

Let \( R \) be a commutative ring with unit \( 1 \neq 0 \) and \( f \in R[x_1, \ldots, x_n] \). We denote by \( Z(f) \) the \( R \)-scheme associated with the principal ideal \( (f) \). We also denote by \( C_f \) the \( R \)-scheme of critical points of \( f \), i.e. the closed subscheme of \( \mathbb{A}_R^n \) associated with the Jacobian ideal \( J_f \) generated by polynomials \( \frac{\partial f}{\partial x_i}, 1 \leq i \leq n \). If \( f \in R[x_1, \ldots, x_n] \) is a polynomial, we say that \( f \) is weighted homogeneous if there exists a tuple \( (r_1, \ldots, r_n, r) \in \mathbb{Z}_{>0}^{n+1} \) such that

\[ f(\lambda^{r_1}x_1, \ldots, \lambda^{r_n}x_n) = \lambda^r f(x_1, \ldots, x_n) \]

for all \( \lambda \in R' \), all \( R \)-algebras \( R' \). If \( R \) is a field and \( f \) is weighted homogeneous, then \( C_f \) is equal to the singular locus \( \text{Sing}(f) \) of the \( R \)-scheme \( Z(f) \).

Let \( F \) be a field, and, fix an algebraic closure \( \overline{F} \) of \( F \). Let \( K \) be a number field, and, denote by \( O_K \) the ring of integers in \( K \) and \( O_K^{\text{alg}} \) the integral closure of \( O_K \) in \( \overline{K} \). In this paper, a local field over \( O_K \) is a finite extension \( L \) of \( \mathbb{Q}_p \) or \( \mathbb{F}_p((t)) \) for some prime \( p \) such that moreover \( L \) is equipped with a structure of \( O_K \)-algebra. For each integer \( M \), we denote by \( \mathcal{L}_{K,M} \) the set
of all local fields over $\mathcal{O}_K$ of characteristic zero and of residue field characteristic at least $M$. We denote by $\mathcal{L}_{K,M}'$ the set of all local fields over $\mathcal{O}_K$ of characteristic at least $M$ and we put $\mathcal{L}_{K,M} = \mathcal{L}_{K,M} \cup \mathcal{L}_{K,M}'$.

Let $L$ be a local field over $\mathcal{O}_K$. We denote by $\mathcal{O}_L$ and $\mathcal{M}_L$ the valuation ring of $L$ and its maximal ideal, respectively, and by $k_L = \mathcal{O}_L / \mathcal{M}_L$ the residue field of $L$, of characteristic $p_L$ and cardinality $q_L$. For $x \in L$, we denote by $\text{ord}_L(x) \in \mathbb{Z} \cup \{+\infty\}$ its valuation and by $|x|_L = q_L^{-\text{ord}_L(x)}$ its absolute value. We fix a uniformizing parameter $\varpi_L$ for $\mathcal{O}_L$, and denote by $ac : L \to \mathcal{O}_L^\times$ the map given by $ac(x) = x\varpi_L^{-\text{ord}_L(x)}$ if $x \neq 0$ and $ac(0) = 0$. For $x \in \mathcal{O}_L$, we write $\overline{x}$ for the image of $x$ by the residue map $\mathcal{O}_L \to \mathcal{O}_L / \mathcal{M}_L = k_L$, and we put $\overline{ac(x)} = ac(\overline{x})$ if $x \in L$.

For each positive integer $n$, we endow $L^n$ with the Haar measure $|dx|$ which is normalized such that the volume of $\mathcal{O}_L^n$ is 1.

A multiplicative character $\chi$ of $\mathcal{O}_L^\times$ is a continuous homomorphism $\chi : \mathcal{O}_L^\times \to \mathbb{C}^\times$ with finite image. We also put $\chi(0) = 0$. If $\chi$ is a multiplicative character, the conductor $c(\chi)$ of $\chi$ is the smallest integer $c \geq 1$ such that $\chi$ is trivial on $1 + \varpi_L^c \mathcal{O}_L$. When $c(\chi) = 1$ then $\chi$ induces a multiplicative character $\overline{\chi} : k_L^\times \to \mathbb{C}^\times$. An additive character of $L$ is a continuous homomorphism $\psi : L \to \mathbb{C}^\times$. If $\psi$ is a non-trivial additive character, the conductor $m_\psi$ of $\psi$ is the integer $m$ such that $\psi$ is trivial on $\varpi_L^m \mathcal{O}_L$, but non-trivial on $\varpi_L^{m-1} \mathcal{O}_L$. If $\psi$ is the trivial additive character, we put $m_\psi = -\infty$. If we fix an additive character $\psi$ of conductor 0, then any non-trivial additive character of $L$ is of the form $\psi_\zeta : L \to \mathbb{C}^\times : x \mapsto \psi(\zeta x)$ for some non-zero element $\zeta$ of $L$. In particular, we have that $m_{\psi_\zeta} = -\text{ord}_L(\zeta)$.

Let $\Phi$ be a Schwartz-Bruhat function on $L^n$, i.e., a locally constant function with compact support. Let $f \in L[x_1, ..., x_n]$ be a non-constant polynomial. The exponential sum associated with $f$, $\Phi$ and an additive character $\psi$ is

$$E_f(\Phi, \psi) = \int_{L^n} \Phi(x)\psi(f(x))|dx|.$$ 

The Igusa local zeta function associated with $f$, $\Phi$ and a multiplicative character $\chi$ is

$$Z_\chi(f, L, \Phi, s) = \int_{L^n} \Phi(x)\chi(ac(f(x)))|f(x)|^s|dx|$$

where $s \in \mathbb{C}$ with $\Re(s) > 0$. It is well known that $Z_\chi(f, L, \Phi, s)$ is holomorphic in this region, that it has a meromorphic continuation to $\mathbb{C}$, and that it is a rational function in $q_L^{-s}$. The strong monodromy conjecture predicts a deep relation between poles of Igusa local zeta functions and roots of the Bernstein-Sato polynomial of $f$.

**Conjecture 3** (Strong monodromy conjecture). Let $f \in \mathcal{O}_K[x_1, ..., x_n]$ be a non-constant polynomial. Let $L$ be a local field over $\mathcal{O}_K$. If $s$ is a pole of $Z_\chi(f, L, \Phi, s)$ then $\Re(s)$ is a root of the Bernstein-Sato polynomial $b_f(s)$ of $f$ provided that $k_L$ is of large enough characteristic.
Remark 2.1. In Conjecture 3, we can allow all non-Archimedean local fields instead of non-Archimedean local fields of large enough residue field characteristic. More general, we can state Conjecture 3 for any non-constant polynomial \( f \) whose coefficients are in a \( p \)-adic field \( L \), and any Schwartz-Bruhat function \( \Phi \) on \( L^n \).

We recall the following proposition from [13], relating exponential sums to Igusa local zeta functions.

**Proposition 2.2** ([13], Proposition 1.4.4). Let \( L \) be a local field over \( \mathbb{Z} \) and \( \Phi \) be a Schwartz-Bruhat function on \( L^n \). Let \( f \in L[x_1, ..., x_n] \) be a non-constant polynomial and \( \psi \) be an additive character of \( L \) of positive conductor. We write \( m \) for \( m_\psi \) and \( t \) for \( q_L^{-s} \). There is \( u \in \mathcal{O}_L^\times \) such that

\[
E_f(\Phi, \psi) = Z_1(f, L, \Phi, 0) + \text{Coeff}_{m-1} \left( \frac{(t-q_L)Z_1(f, L, \Phi, s)}{(q_L-1)(1-t)} \right)
+ \sum_{\chi \neq 1} g_{\chi^{-1}, \psi}(u) \text{Coeff}_{m-c(\chi)}(Z_{\chi}(f, L, \Phi, s)),
\]

where 1 stands for the trivial character on \( \mathcal{O}_L^\times \), the summation index \( \chi \) runs over all non-trivial multiplicative characters and

\[
g_{\chi, \psi} = \frac{q_L^{1-c(\chi)}}{q_L-1} \sum_{v \in (\mathcal{O}_L / M_{\chi}^{(\chi)}))} \chi(v) \psi(w_L^{m-c(\chi)} v).
\]

For \( L \) a local field over \( \mathbb{Z} \) and \( Z \) a subset of \( k_L^n \), we set \( \Phi_{Z,L} := 1_{\{x \in \mathcal{O}_L^n, x \in Z\}} \). For \( Z \) a subscheme of \( \mathcal{A}_0^0 \), we associate to \( Z \) a family of Schwartz-Bruhat functions \( (\Phi_{Z,L})_L \) where \( L \) runs over the set of local fields over \( \mathcal{O}_K \) and \( \Phi_{Z,L} := \Phi_{Z(k_L),L} \). To simplify notation, if \( Z \) is a subscheme of \( \mathcal{A}_0^0 \) or \( Z \subset k_L^n \) we will write \( Z_{\chi}(f, L, Z, s) \) and \( E_f(Z, \psi) \) instead of \( Z_{\chi}(f, L, \Phi_{Z,L}, s) \) and \( E_f(\Phi_{Z,L}, \psi) \) respectively. In addition, if \( Z = \{P\} \) for a point \( P \in k_L^n \) we also write \( Z_{\chi}(f, L, P, s) \) and \( E_f(P, \psi) \) instead of \( Z_{\chi}(f, L, \{P\}, s) \) and \( E_f(\{P\}, \psi) \) respectively.

If \( Z \) is a subscheme of \( \mathcal{A}_0^0 \) and \( \Phi = \Phi_{Z,L} \), we can simplify the formula of Proposition 2.2 by the following properties.

**Proposition 2.3.** [12, Theorem 2.1] With the above notation, suppose that \( f((Z \cap C_f)(\mathbb{C})) = 0 \) then there exist an integer \( M \) depending only on \( f \) and a finite set \( \mathcal{N} \) of positive integers such that for all local fields \( L \in \hat{\mathcal{L}}_{K,M} \) we have

\[
Z_{\chi}(f, L, Z, s) = 0
\]

provided that at least one of the following conditions holds

i. \( \chi^N \neq 1 \) for all \( N \in \mathcal{N} \),

ii. \( c(\chi) > 1 \),

and

\[
Z_{\chi}(f, L, Z, s) = Z_{\chi}(f, L', Z, s)
\]
if $k_L = k_{L'}$, $c(\chi) = c(\chi') = 1$ and $\mathfrak{p} = \mathfrak{p}'$. Moreover we have

$$E_f(Z, \psi) = 0$$

for all additive characters $\psi : L \to \mathbb{C}^\times$ of conductor at least 2 if $p_L > M$ and

$$(Z \cap C_f)(\mathbb{C}) = \emptyset$$

Now we recall notation and results about log-resolutions of ideals. Let $F$ be a field and $\mathfrak{a}$ be a non-zero ideal of $F[x_1, \ldots, x_n]$. A log-resolution of $\mathfrak{a}$ is a projective, bi-rational $F$-morphism $\pi : X' \to \mathbb{A}_F^n$ such that $X'$ is smooth, $\mathfrak{a}O_{X'}$ is the ideal of a Cartier divisor $D$ such that $D + K_{X'/X}$ is a divisor with simple normal crossings. This means that at every point $Q \in X'$ there are local coordinates $y_1, \ldots, y_n$ such that $D + K_{X'/X}$ is defined by $(y_1^{d_1}, \ldots, y_n^{d_n})$ for some $d_1, \ldots, d_n \in \mathbb{Z}_{\geq 0}$. We also require that $\pi$ is an isomorphism over the complement of the zero locus $Z(\mathfrak{a})$ of $\mathfrak{a}$. The existence of log-resolution when $F$ has characteristic zero follows by the work of Hironaka (see [16, page 142, Main Theorem II]). Now, we suppose that $F$ has characteristic zero. Let $\pi$ be a log-resolution of an ideal $\mathfrak{a}$ as above. We can write $D = \sum_{i \in T} N_i E_i$ and $K_{X'/X} = \sum_{i \in T} (\nu_i - 1)E_i$, where $\sum_{i \in T} E_i$ is a divisor of $X'$ with simple normal crossings and $\nu_i, N_i$ are positive integers for all $i \in T$. The set $(\nu_i, N_i)_{i \in T}$ will be called the numerical data associated with $\pi$. Let $P$ be a point in $Z(\mathfrak{a})(\mathbb{F})$. The log-canonical threshold $lct_P(\mathfrak{a})$ of $\mathfrak{a}$ at $P$ is given by the following formula:

$$lct_P(\mathfrak{a}) = \min_{i \in T, P \in \pi(E_i)(\mathbb{F})} \frac{\nu_i}{N_i}$$

If $P \notin Z(\mathfrak{a})(\mathbb{F})$, we put $lct_P(\mathfrak{a}) = +\infty$. If $Z$ is a subscheme of $\mathbb{A}_F^n$ then the log-canonical threshold of $\mathfrak{a}$ at $Z$ will be the minimum in the set $\{lct_P(\mathfrak{a}) | P \in Z(\mathbb{F})\}$. To simplify notation, we will write $lct(\mathfrak{a})$ instead of $lct_Z(\mathfrak{a})$.

If $F \hookrightarrow F'$ is an injection for some field $F'$ then any log-resolution $\pi$ of $\mathfrak{a}$ induces a log-resolution $\pi_{F'}$ of $\mathfrak{a} \otimes_F F'$. Suppose that $F = K$ is a number field and that $F' = L$ is a local field over $O_K$ of characteristic zero. For $Z$ a closed subscheme of $X'_L = X' \otimes_K L$, we denote the reduction modulo $\mathcal{M}_L$ of $Z$ by $\overline{Z}$ (see [11]). We say that the log-resolution $(X', \pi)$ of $\mathfrak{a}$ has good reduction modulo $\mathcal{M}_L$ if $\overline{X'_L}$ and all $\mathcal{E}_{iL}$ are smooth, $\cup_{i \in T} \overline{\mathcal{E}}_{iL}$ has only normal crossings, and the schemes $\mathcal{E}_{iL}$ and $\mathcal{E}_{jL}$ have no common component whenever $i \neq j$. If in addition, $N_i \notin \mathcal{M}_L$ for all $i \in T$, we say that the log-resolution $(X', \pi)$ of $\mathfrak{a}$ has tame good reduction modulo $\mathcal{M}_L$. There exists an integer $M$ depending only on $\pi$ such that if $L \in \mathcal{L}_{K,M}$ then the log-resolution $(X', \pi)$ has good reduction mod $\mathcal{M}_L$ (see for instance [11, Theorem 2.4]).

Remark 2.4. Let a log-resolution $h$ for the principal ideal $(f)$ be given, the claims of Proposition 2.3 hold for a local field $L$ over $O_K$ if there is $\tilde{L} \in \mathcal{L}_{K,1}$ such that $k_L \simeq k_{\tilde{L}}$ and $h$ has tame good reduction modulo $\mathcal{M}_{\tilde{L}}$. Thus the integer $M$ in Proposition 2.3 can be taken as the largest prime $p$ such that the log-resolution $h$ does not have tame reduction modulo $pK_p$ for some
non-Archimedean place $p$ of $\mathcal{O}_K$ over $p$, where $K_p$ is the completion of $K$ at $p$.

Let $k$ be a field over $\mathcal{O}_K$, i.e. $k$ is equipped with a structure of $\mathcal{O}_K$-algebra $\varphi : \mathcal{O}_K \to k$. If $Z$ is a subscheme of $\mathbb{A}^n_{\mathcal{O}_K}$, we write $Z_k$ instead of $Z \otimes_{\text{Spec}(\mathcal{O}_K)} \text{Spec}(k)$, as usual. Let $f \in \mathcal{O}_K[x_1, \ldots, x_n]$. From now on, we also write $f$ for the polynomial $\varphi(f) \in k[x_1, \ldots, x_n]$. For each $x = (x_1, \ldots, x_n) \in k[[t]]^n$ one can write
\[ x_j = \sum_{i \geq 0} x_{ij} t^i. \]

For $m$ a non-negative integer and $x \in k[[t]]^n$, we set
\[ x^{(m)} := (x_{ij})_{0 \leq i \leq m, 1 \leq j \leq n} \in k^{(m+1)n}. \]

Using this notation one has
\[ f(x) = \sum_{i \geq 0} f_i(x^{(i)}) t^i, \]

where $f_i$ is a polynomial in $k[x^{(i)}]$ for all $i \geq 0$. If $Z$ is a subscheme of $\mathbb{A}^n_k$, we identify $Z(k) + (tk[[t]]/(t^{m+1}))^n \subset (k[[t]]/(t^{m+1}))^n$ with the set of $k$-points of the $k$-scheme $Z^{(m)} := \left( Z \otimes_{\text{Spec}(k)} \mathbb{A}^n_k \right)$. Then we can view $f_m$ as a regular function on $Z^{(m)}$. If $Z = \{ P \}$ for a point $P \in k^n$, we write $P^{(m)}$ instead of $\{ P \}^{(m)}$.

In this paper, we will need some important results from [24].

**Proposition 2.5.** [24, Proposition 4.3 and 5.4] Let $Z$ be a subscheme of $\mathbb{A}^n_{\mathcal{O}_K}$.

With the notation as above, if $L, L' \in \mathbb{L}_{K,m+1}$ for some $m \geq 2$ and $\psi$ is any additive character of $L$ of conductor $m$ then there exist elements $a \in k_L^\times$ and $b \in k_L$ depending only on $\psi$ and satisfying
\[
E_f(Z, \psi) = q_L^{-mn} \sum_{x^{(m-1)} \in Z^{(m-1)}(k_L)} \exp_{L/F_{PL}}(\text{Tr}_{k_L/F_{PL}}(bf_0(x^{(0)}) + af_{m-1}(x^{(m-1)})))
\]

where $\exp_{L/F_{PL}} : F_{PL} \to \mathbb{C}^\times$, $z \mapsto \exp\left( \frac{2\pi i z}{2}\right)$ is the standard additive character of $F_{PL}$.

Moreover, suppose that $f(Z(\mathbb{C})) = 0$ and $M$ is as in Remark 2.4. Let local fields $L, L' \in \mathbb{L}_{K,M}$ such that $k_L \simeq k_{L'}$. If $\psi$ is an additive character of $L$ of conductor $m \psi$ and $2$ then there exists an additive character $\psi'$ of $L'$ of conductor $m \psi' = m \psi$ such that
\[ E_f(Z, \psi) = E_f(Z, \psi'). \]

**Remark 2.6.** In the proof of [24, Proposition 5.4], it is not clear that one can take $M$ as in Remark 2.4. However, we can use Proposition 2.2 and 2.3 to see this fact.

**Proposition 2.7.** [24, Corollary 5.5] Let $(Z_i)_{i \in J}$ be a family of subschemes of $\mathbb{A}^n_{\mathcal{O}_K}$ and $(\sigma_i)_{i \in J}$ be a family of positive real numbers. With the notation
as above, suppose that for each \( m > 1 \), there exist an integer \( M_m \) and a positive constant \( C_m \) such that for all \( i \in J \), all local fields \( L \in \mathcal{L}'_{K,M,m} \) and all additive characters \( \psi \) of \( L \) of conductor \( m \) we have

\[
|E_f(Z_i, \psi)| \leq C_m |k|^{-m\sigma_i}.
\]

Then there exist an integer \( M \) and a positive constant \( C \) such that for all \( i \in J \), all local fields \( L \in \mathcal{L}_{K,M} \) and all additive characters \( \psi \) of \( L \) of conductor \( m_{\psi} > 1 \), we have

\[
|E_f(Z_i, \psi)| \leq Cm_{\psi}^{n-1} q_L^{-m_{\psi}\sigma_i}.
\]

Remark 2.8. As in the proof of [3, Claim 3.2.7], the constant \( M \) in Proposition 2.7 depends only on \( f \) and the set \( \{ M_i | i \in I \} \) for a finite set \( I \subset J \) depending only on \( f \). In particular, we can take \( I \) depending only on the numerical data associated with a log-resolution of \( f \). By using log-resolutions of families as in [22, Property 1.23], there exists a finite set \( A \) of numerical data such that if \( f \) is a polynomial of degree \( d \) in \( n \) variables over \( \mathbb{C} \), we can find a log-resolution of \( f \) whose numerical data belongs to \( A \). Therefore we can take \( I \) depending only on \( n, d \).

On the other hand, again by [3, Claim 3.2.7], we can take \( C \), depending only on the set \( \{ C_i | i \in I \} \).

Remark 2.9. Proposition 2.7 still holds for any family of subschemes of \( \mathbb{A}^n_{\mathcal{O}_K} \) if we use the convention that \( E_f(Z, \psi) = 0 \) for all additive characters \( \psi \) of \( L \) whenever \( Z \) is not defined over \( \mathcal{O}_L \).

3. Uniform bound for exponential sums modulo \( p^m \) when \( p \) is large enough

At the beginning of this section, we study the exponential sums at points \( E_f(P, \psi) \) where \( f \in \mathcal{O}_K[x_1, ..., x_n] \) is a non-constant polynomial, \( \psi \) is any additive character of conductor \( m_{\psi} > 1 \) on a local field \( L \) over \( \mathcal{O}_K \) of large enough residue field characteristic and \( P \in k_L^n \). After that, we will glue information on the exponential sums at points to obtain bound for the exponential sums associated with arbitrary subscheme \( Z \) of \( \mathbb{A}^n_{\mathcal{O}_K} \).

Let \( f \) be a polynomial in \( \mathcal{O}_K[x_1, ..., x_n] \). Let \( k \) be a field over \( \mathcal{O}_K \). Using the notation in Section 2, for each \( P = (a_1, ..., a_n) \in k^n \) and \( m \geq 1 \) we denote

\[
f_{P,m}(\tilde{x}(m)) = f_m(x_{P}^{(m)})
\]

where \( \tilde{x}(m) = (x_{ij})_{1 \leq i \leq m, 1 \leq j \leq n} \) and \( x_{P}^{(m)} = (x_{ij})_{0 \leq i \leq m, 1 \leq j \leq n} \) with \( x_{0j} = a_j \) for all \( 1 \leq j \leq n \). We use the convention that \( f_{P,m} = 0 \) if \( m < 0 \) and \( f_{P,0} = f(P) \). Moreover, we identify \( P^{(m)}(k) \) with \( k^{nm} \). From now on, we suppose that the degree \( d \) of \( f \) is at least 2.

Lemma 3.1. Let \( k \) be a field over \( \mathcal{O}_K \), \( P \in k^n \) be a critical point of \( f \) and \( m \geq 1 \). Then \( f_{P,m} \) is a weighted homogeneous polynomial and its critical
locus is
\[ \text{Sing}(f_{P,m}) = \cap_{1 \leq j \leq n, 0 \leq i \leq m-1} Z((F_j)_P, i), \]
where \( F_j = \frac{\partial f}{\partial x_j} \) for all \( 1 \leq j \leq n \).

**Proof.** It is sufficient to work with the case that \( P = (0, \ldots, 0) \) and \( f(P) = F_1(P) = \ldots = F_n(P) = 0 \). It is clear that \( f_{P,m} \) is weighted homogeneous for all \( m \geq 1 \). If \( x = (x_1(t), \ldots, x_n(t)) \in \mathbb{R}^n \) with
\[ x_j(t) = x_{1j}t + x_{2j}t^2 + \ldots \]
for \( 1 \leq j \leq n \) then we consider the formal function
\[ F(t, \tilde{x}^{(\infty)}) := f(x) = \sum_{i \geq 1} f_{P,i}(\tilde{x}^{(i)})t^i \]
where \( \tilde{x}^{(\infty)} = (x_{ij})_{1 \leq i, 1 \leq j \leq n} \). If \( 1 \leq \ell \) and \( 1 \leq j \leq n \) then
\[ \frac{\partial}{\partial x_{ij}} (F(t, \tilde{x}^{(\infty)})) = \frac{\partial}{\partial x_{ij}} \left( \sum_{i \geq 1} f_{P,i}(\tilde{x}^{(i)})t^i \right) = \sum_{i \geq 1} \frac{\partial f_{P,i}}{\partial x_{ij}} (\tilde{x}^{(i)})t^i \]
and
\[ \frac{\partial}{\partial x_{ij}} (F(t, \tilde{x}^{(\infty)})) = \frac{\partial}{\partial x_{ij}} (f(x_1(t), \ldots, x_n(t))) \\
= \frac{\partial f}{\partial x_j}(x_1(t), \ldots, x_n(t)) \frac{\partial}{\partial x_{ij}} (x_j(t)) \\
= F_j(x_1(t), \ldots, x_n(t))t^\ell \\
= \sum_{i \geq 1} (F_j)_P,i(\tilde{x}^{(i)})t^{i+\ell} \]
Hence,
\[ (9) \quad \frac{\partial f_{P,i}}{\partial x_{ij}} (\tilde{x}^{(i)}) = (F_j)_P,i-\ell(\tilde{x}^{(i-\ell)}) \]
for all \( \ell, j \) with the convention that \( (F_j)_P,i = 0 \) if \( i \leq 0 \). Thus the lemma follows from (9).

**Definition 3.2.** Let \( J_f \) be the Jacobian ideal of \( f \in \mathcal{O}_K[x_1, \ldots, x_n] \) as in Section 2. For each local field \( L \) over \( \mathcal{O}_K \) of characteristic zero we also write \( J_f \) for \( J_f \otimes \mathcal{O}_K L \). For each \( P \in k_L^n \) we set
\[ \sigma_P(L, J_f) = \min_{Q \in \mathcal{O}_L^n, \mathcal{Q} = P} \text{lct}_Q(J_f), \]
where \( \text{lct}_Q(J_f) \) is the log-canonical threshold of \( J_f \) at \( Q \).

**Remark 3.3.** We have \( \sigma_P(L, J_f) = +\infty \) if there is no critical point of \( f \) in the set \( \{ x \in \mathcal{O}_L^n \mid \mathcal{Q} = P \} \). Let \( h : X' \to k_K^n \) be a log-resolution of \( J_f \otimes \mathcal{O}_K K \). There is an integer \( M \) such that \( h \) has good reduction modulo \( \mathcal{M}_L \) provided that \( p_L > M \). Thus if \( p_L > M \) and \( P \in k_L^n \) is a critical point of \( f \) then there exists \( Q \) in \( \mathcal{O}_L^n \) such that \( \mathcal{Q} = P \) and \( Q \) is a critical point of \( f \).
Lemma 3.4. There exists an integer $M$ depending only on $f$ such that for all $m \geq 2$, all local fields $L \in \mathcal{L}_{K,M}$ and all critical points $P \in k^n_L$ of $f$ we have
\[
\dim(\operatorname{Sing}(f_{P,m-1})) \leq mn - (m-1)\sigma_P(L,J_f).
\]

Proof. Let $h, M$ be as in Remark 3.3. Suppose that $\operatorname{Div}(h^{-1}(J_f)) = \sum_{i \in T} N_i E_i$ and $K_{X'/X} = \sum_{i \in T} (\nu_i - 1)E_i$. Fix a local field $L \in \mathcal{L}_{K,M}$. By our assumption, $h$ induces a log-resolution $h_L$ for $J_f \otimes_{O_K} L$ and $h_L$ induces a log-resolution $h_{k_L}$ for $J_f \otimes_{O_K} k_L$. We can use Lemma 3.1 and the description of the log-canonical thresholds of $J_f$ in [15] to see that
\[
\dim(\operatorname{Sing}(f_{P,m-1|k_L})) \leq mn - (m-1)\min_{P \in h_{k_L}(\mathcal{F}_i)} \frac{\nu_i}{N_i}.
\]

By the definition of good reduction, one has
\[
\min_{P \in h_{k_L}(\mathcal{F}_i)} \frac{\nu_i}{N_i} = \sigma_P(L,J_f).
\]

Thus our lemma follows by combining (11) and (12). □

Proposition 3.5. For each $m \geq 2$, there exist a constant $M_m$ depending only on $m, f$ and a constant $B_m$ depending only on $m, n, d$ such that for all finite fields $k$ over $O_K$ of characteristic at least $M_m$ with structure homomorphism $\varphi : O_K \rightarrow k$, all $\alpha \in k^\times$ and all critical points $P \in k^n$ of $f$ we have
\[
||k||^{-mn} \sum_{\tilde{x}^{(m-1)} \in k^{(m-1)n}} \exp_k(a f_{P,m-1}(\tilde{x}^{(m-1)})) \leq B_m ||k||^{-m \sigma_P(L,J_f) + \sigma_P(L,J_f) - n}
\]
if $L$ is a local field over $O_K$ of characteristic zero such that $k_L = k$ and the structure homomorphism $\varphi : O_K \rightarrow k$ is the reduction modulo $M_L$.

Proof. The proposition follows by combining (10) with the discussion in [5, Theorem 7.4] and [18]. □

Remark 3.6. With a log-resolution $h$ as in Remark 3.3, the proof of Lemma 3.4 and the discussion in [18] imply that Proposition 3.5 holds if $h$ has good reduction modulo $M_L$ and $p_L > m$. In particular, with $M$ as in Remark 3.3, we can take $M_m = \max\{m, M\}$.

In Definition 3.2, if $Q \in L^n$ is a critical point of $f$ then $\text{lct}_Q(J_f) \leq n$ by [22, Property 1.14]. Thus we can combine Proposition 3.5, Definition 3.2 and Remark 3.3 to deduce that:

Proposition 3.7. For each $m \geq 2$, there exist a constant $M_m$ depending only on $m, f$ and a constant $B_m$ depending only on $m, n, d$ such that for all finite fields $k$ over $O_K$ of characteristic at least $M_m$ with structure homomorphism $\varphi : O_K \rightarrow k$, all $\alpha \in k^\times$ and all critical points $P \in k^n$ of $f$ we have
\[
||k||^{-mn} \sum_{\tilde{x}^{(m-1)} \in k^{(m-1)n}} \exp_k(a f_{P,m-1}(\tilde{x}^{(m-1)})) \leq B_m ||k||^{-m \sigma_P(L,J_f) + \sigma_P(L,J_f) - n}
\]
if $L$ is a local field over $\mathcal{O}_K$ of characteristic zero such that $k_L = k$ and the structure homomorphism $\varphi : \mathcal{O}_K \rightarrow k$ is the reduction modulo $M_L$.

Because of Propositions 2.5, 2.7, 3.7 and Remarks 2.8, 2.9, we deduce that:

**Proposition 3.8.** There exist a constant $M$ depending only on $f$ and a constant $C$ depending only on $n, d$ such that for all local fields $L \in \mathcal{L}_{K,M}$, all critical points $P \in k^n_L$ of $f$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$ we have

$$|E_f(P, \psi)| \leq C m_\psi^{n-1} q_L^{-\frac{-m_\psi s_P(L, J_f)}{2}},$$

where $\tilde{L}$ is a local field over $\mathcal{O}_K$ of characteristic zero with $k_L \simeq k_L$.

**Corollary 3.9.** Let $f$ be a polynomial in $n$ variables over $\mathcal{O}_K$ and of degree $d > 1$. Let $s$ be as in Section 1. There exist a constant $M$ depending only on $f$ and a constant $C$ depending only on $n, d$ such that for all local fields $L \in \mathcal{L}_{K,M}$, all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$ and all critical points $P \in k^n_L$ of $f$ we have

$$|E_f(P, \psi)| \leq C m_\psi^{n-1} q_L^{-\frac{-m_\psi (n-s)}{2(d-1)}}.$$

**Proof.** As in Section 1, we denote by $f_d$ the homogeneous part of degree $d$ of $f$. Let $L$ be a local field over $\mathcal{O}_K$ of characteristic 0. Because of Proposition 3.8, Definition 3.2 and Remark 3.3, it remains to show that

$$\text{lct}_Q(J_f) \geq \frac{n-s}{d-1} \tag{13}$$

if $Q \in \mathcal{O}_L^n$ is a critical point of $f$. First of all, we can suppose that $Q = 0$ by a suitable changing of coordinates. We can take a hyperplane $H$ passing through the origin $0$ such that $f_d|H \neq 0$ and $\dim(\text{Sing}(f_d|H)) = s - 1$. Because of [22, Property 1.17] we have $\text{lct}_0(J_f) \geq \text{lct}_0(J_f|H)$. It is easy to see that $J_f|H \subset J_f|H$ thus we can use [22, Property 1.12] to deduce that $\text{lct}_0(J_f) \geq \text{lct}_0(J_f|H) \geq \text{lct}_0(J_f|H)$. Therefore we can reduce to the case $s = 0$ by induction. Let us check (13) when $s = 0$. Since log canonical threshold is stable under changing of coordinates, for each $a \neq 0$ we have $\text{lct}_0(a^{d-1} J_f(x/a)) = \text{lct}_0(J_f(x/a)) = \text{lct}_0(J_f(x))$. Because of the semi-continuity of the log canonical threshold in [22, Property 1.24], we have $\text{lct}_0(a^{d-1} J_f(x/a)) \geq \text{lct}_0(J_{f_d})$. Note that $\dim(\text{Sing}(f_d)) = 0$ and $f_d$ is homogeneous. Thus for each $x \in tL[[t]]^n$, $\text{ord}_t(J_{f_d}(x)) := \min_{g \in J_{f_d}} \text{ord}_t(g(x)) > m$ if $\text{ord}_t(x) > \left[\frac{m}{d-1}\right]$. Where, as usual, $\left[\frac{m}{d-1}\right]$ is the integer part of $\frac{m}{d-1}$. Therefore the description of the log-canonical threshold in [15] implies $\text{lct}_0(J_{f_d}) = \frac{n}{d-1}$, as desired.

**Corollary 3.10.** There exist a constant $M$ depending only on $f$ and a constant $C$ depending only on $n, d$ such that for all local fields $L \in \mathcal{L}_{K,M}$, all
additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$ and all critical points $P \in k^n_L$ of $f$ we have

$$|E_f(P, \psi)| \leq C m_\psi^{n-1} p^{-m_\psi \tilde{\sigma}_{P,L}},$$

with $\tilde{L}$ as in Proposition 3.8 and with

$$\tilde{\sigma}_{P,\tilde{L}} = \min_{Q \in \mathcal{C}_f(\tilde{L}), P = \frac{Q}{d}} \frac{n - s_Q}{2(d - 1)}.$$  

Where if $Q \in C_f(\tilde{L})$ then $d_Q > 1$ and $s_Q$ are natural numbers such that one can write $f(x + Q) = f(Q) + f_{d_Q}(x) + \ldots + f_d(x)$ with $f_{d_Q} \neq 0$, $f_i$ is homogeneous for all $i$ and $s_Q = \dim(\text{Sing}(f_{d_Q}))$.

**Proof.** As in the proof of Corollary 3.9, it suffices to show that $\text{lct}_0(J_f) = \frac{n}{d_0 - 1}$ if $0$ is a critical point of $f$ and $s_0 = 0$. Since $\dim(\text{Sing}(f_{d_Q})) = 0$, if $x \in t\tilde{L}[t]^n$ then $\text{ord}_L(J_f(x)) > m$ iff $\text{ord}_L(x) > \lceil \frac{m}{d_0 - 1} \rceil$. The same argument as in Corollary 3.9 yields

$$\text{lct}_0(J_f) = \frac{n}{d_0 - 1}.$$

We need the following lemma to go from exponential sums at points to exponential sums associated with schemes.

**Lemma 3.11.** Let $K$ be a number field. Let $f$ be a non-constant polynomial in $\mathcal{O}_K[x_1, \ldots, x_n]$. Let $H_1$ and $H_2$ be linear subspaces of $\mathbb{A}^n$, of dimension $n - r$, resp. dimension $r$, defined over $\mathcal{O}_K$, and so that $H_1(\mathbb{C}) + H_2(\mathbb{C}) = \mathbb{C}^n$. Then there are $M > 0$ and a nonzero polynomial $g$ in $r$ variables defined over $\mathcal{O}_K$ such that for any $b \in H_2(\mathcal{O}_K^{\text{alg}})$, and $Q \in b + H_1(\mathcal{O}_K^{\text{alg}})$, any local field $L$ over $\mathcal{O}_K[b,Q]$ with residue field characteristic at least $M$, and any multiplicative character $\chi$ on $\mathcal{O}_L^\times$ with $c(\chi) = 1$, if one has $\text{ord}_L(g(b)) = 0$, then

$$Z_\chi(f, L, Q, s) = q_{L,\mathbb{C}}^{-s} Z_\chi(f_{|b + H_1}, L, Q, s).$$

Furthermore, the degree of $g$ can be bounded in terms of $d = \deg(f)$.

**Proof.** This is a variant of [4, Lemma 4.1], with similar proof based on Denef-Pas cell decomposition [25] and the study of parameter integrals of [7], which works uniformly for big enough $p_L$ since $c(\chi) = 1$. To see our claim on the degree of $g$, we can proceed similarly as in [4, Lemma 4.1] but with more parameters come from coefficients defining $f, H_1, H_2$. \hfill $\square$

**Proposition 3.12.** Let $Z$ be a subscheme of $\mathbb{A}_K^n$ whose complexity is bounded by $(R, D)$. Moreover, suppose that $f(Z(\mathbb{C})) = 0$. There exist a constant $M$ depending only on $Z, f$ and a constant $C$ depending only on $n, d, R, D$ such that for all local fields $L \in \mathcal{L}_{K,M}$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$, we have

$$|E_f(Z, \psi)| \leq C m_\psi^{n-1} p^{-m_\psi (n-s)}.$$
Proof. We follow a strategy from [4], and use induction on the dimension of the intersection of $Z$ with the singular locus of $f$. If the intersection of $Z$ with the singular locus of $f$ has dimension zero, there is nothing to prove by Proposition 2.3 and Corollary 3.9. Now let $Z$ be general with $f(Z(\mathbb{C})) = 0$. We may suppose that $Z$ is contained in the singular locus of $f$, again since the exponential sum vanishes away from the singular locus as in Proposition 2.3. Thus we can suppose that the dimension of $Z$ is $r \leq \dim(\text{Sing}(f))$.

Note the easy fact that $\dim(\text{Sing}(f)) \leq \dim(\text{Sing}(f_d)) = s$ and thus $r \leq s$. There exist a closed sub-variety $Z''$ defined over $\mathcal{O}_K$ of $Z$ of codimension at least 1, and a linear subspace $H_1$ defined over $\mathcal{O}_K$ of dimension $n - r$, such that $f_d|_{H_1} \neq 0$, $\dim(\text{Sing}(f_d|_{H_1})) = s - r$ and the set of $\mathbb{C}$-rational points on $(Z \setminus Z'') \cap (a + H_1)$ is finite for any $a \in Z(\mathbb{C}) \setminus Z''(\mathbb{C})$, and such that there is a linear subspace $H_2$ also defined over $\mathcal{O}_K$ such that $H_1(\mathbb{C}) + H_2(\mathbb{C}) = \mathbb{C}^n$, the sum being direct.

By Lemma 3.11, there is a closed subscheme $Z'$ of $Z$, also defined over $\mathcal{O}_K$, such that the relation (15) between $\mathcal{Z}(f, L, Q, s)$ and $\mathcal{Z}(f|_{Q + H_1}, L, Q, s)$ holds for any $Q$ as soon as $\overline{Q} \in Z(k_L) \setminus Z'(k_L)$ and $p_L > M_0$. Let $g$ be as in Lemma 3.11. Because of the conditions that $(Z \setminus Z'') \cap (a + H_1)$ is finite for any $a \in Z(\mathbb{C}) \setminus Z''(\mathbb{C})$, we have $\dim(Z') \leq \max(\dim(Z''), \dim(Z(g))) = r - 1 < \dim(Z)$. Put $Z_1 = Z \setminus (Z' \cup Z'')$. If $p_L > M_0$ then by (15) we have

$$\mathcal{Z}(f, L, Z_1, s) = \mathcal{Z}(f, L, Z, s) - \mathcal{Z}(f, L, Z' \cup Z'', s) = q_L^{-\dim(Z_1)} \sum_{\overline{Q} \in Z_1(k_L)} \mathcal{Z}(f|_{Q + H_1}, L, Q, s).$$

We can shrink $Z_1$ such that $\dim(Z \setminus Z_1) < \dim(Z)$. $Z_1$ is defined over $\mathcal{O}_K$ and there exists a common log-resolution $h_1$ defined over $K$ of singularities of family $(f|_{Q + H_1})_{Q \in Z_1}$. By the definition of good reduction, there exist regular functions $g_1, ..., g_e$ on $Z_1$ defined over $K$ such that if $L$ is a local field over $\mathcal{O}_K$ of characteristic zero and $h_1, g_1, ..., g_e$ are defined over $\mathcal{O}_L$ then the specialization $h_{1Q}$ of $h_1$ at $Q \in Z_1(L) \cap \mathcal{O}_L^e$ has good reduction modulo $\mathcal{M}_L$ iff $\overline{h_1(Q)} \neq 0$ for all $1 \leq i \leq e$. For each point $Q \in Z_1(\overline{K})$ and $L \in \mathcal{L}_{K,1}$ such that $Q \in \mathcal{O}_L^e$ then $h_{1Q}$ will have good reduction modulo $\mathcal{M}_L$ if $p_L$ is large enough. Thus we see that $g_1, ..., g_e$ must be invertible functions on $Z_1(\mathbb{C})$. There exists an integer $M_1 > M_0$ such that $h_1, g_1, ..., g_e$ are defined over $\mathcal{O}_L$ if $L \in \mathcal{L}_{K,M_1}$. By logical compactness (see for instance [20, Corollary 2.2.10]), we can enlarge $M_1$ such that $\overline{g_1}, ..., \overline{g_e}$ are invertible functions on $Z_1(\mathbb{F}_p)$ if $p > M_1$. Therefore, if $L \in \mathcal{L}_{K,M_1}$ and $Q \in Z_1(L) \cap \mathcal{O}_L^e$, then $h_{1Q}$ has good reduction modulo $\mathcal{M}_L$ if $\overline{Q} \in Z_1(k_L)$. By enlarging $M_1$ if needed, $h_{1Q}$ has tame good reduction modulo $\mathcal{M}_L$ if $L \in \mathcal{L}_{K,M_1}$ and $\overline{Q} \in Z_1(k_L)$.

Thus Propositions 2.2, 2.3, 2.5 and Remark 2.4 yield

$$E_f(Z_1, \psi) = q_L^{-\dim(Z_1)} \sum_{\overline{Q} \in Z_1(k_L)} E_{f|_{Q + H_1}}(Q, \psi).$$
for all local fields \( L \in \tilde{\mathcal{L}}_{K,M_1} \) and all additive characters \( \psi \) of \( L \) of conductor \( m_\psi \geq 2 \).

By shrinking \( Z_1 \) again, we may suppose that there exists a common log-resolution \( h_2 \) of singularities of family \((J_{f|Q+H_1})_{Q \in Z_1}\). By the same reason as above, we also have an integer \( M_2 > M_1 \) such that if \( L \in \mathcal{L}_{K,M_2} \), \( Q \in Z_1(L) \cap \mathcal{O}_L^n \) and \( \mathcal{Q} \in Z_1(k_L) \) then the specialization \( h_2Q \) of \( h_2 \) at \( Q \) will have good reduction modulo \( \mathcal{M}_L \). Thus the discussion in Remark 3.6 implies a uniform version of Proposition 3.7 for the family of polynomials \((f|Q+H_1)_{Q \in Z_1(k_L)}\) if \( p_L > M_2 \). By combining this uniform version with Proposition 2.5 and the proof of Corollary 3.9, we deduce that for each \( m \geq 2 \) there exist a constant \( M_m \) depending only on \( f, Z, m \) and a constant \( B_m \) such that for all local fields \( L \in \tilde{\mathcal{L}}_{K,M_m} \), all additive characters \( \psi \) of \( L \) of conductor \( m \) and all \( Q \) with \( \mathcal{Q} \in Z_1(k_L) \), we have

\[
|E_{f|Q+H_1}(Q, \psi)| \leq B_m q_L^{-\frac{m(n-s)}{2(d-1)}}.
\]

Thus for all local fields \( L \in \tilde{\mathcal{L}}_{K,M_m} \) and all additive characters \( \psi \) of \( L \) of conductor \( m \geq 2 \), we have

\[
|E_f(Z_1, \psi)| = q_L^{-\dim(Z_1)} \sum_{Q \in Z_1(k_L)} E_{f|Q+H_1}(Q, \psi)
\leq |Z_1(k_L)| q_L^{-\dim(Z_1)} B_m q_L^{-\frac{m(n-s)}{2(d-1)}}
\leq c_Z B_m q_L^{-\frac{m(n-s)}{2(d-1)}}.
\]

for a constant \( c_Z \), by the Lang-Weil estimate. Note that \( c_Z \) depends only on the complexity of \( Z \) (see for instance [19]). Because of Proposition 2.7 and Remark 2.8, there exist a constant \( M \) depending only on \( Z, f \) and a constant \( C \) depending only on \( n, d \) together with the complexity of \( Z \) such that for all local fields \( L \in \tilde{\mathcal{L}}_{K,M} \) and all additive characters \( \psi \) of \( L \) of conductor \( m_\psi \geq 2 \), we have

\[
|E_f(Z_1, \psi)| \leq C m_\psi^{n-1} q_L^{-\frac{m_\psi(n-s)}{2(d-1)}}.
\]

This reduces the problem to \( Z' \cup Z'' \) instead of \( Z \). Since Lemma 3.11 and our argument on log-resolutions of families, the complexity of \( Z' \cup Z'' \) can be bounded in terms of the complexity of \( f, Z \). Hence, our claim follows by induction on the dimension. \( \square \)

In fact, we can remove the term \( m_\psi^{n-1} \) in the conclusion of Proposition 3.12.

**Proposition 3.13.** Let \( Z \) be a subscheme of \( \mathbb{A}_K^n \) whose complexity is bounded by \((R,D)\). Suppose that \( f(Z(\mathbb{C})) = 0 \). There exist a constant \( M_3 \) depending only on \( f, Z \) and a constant \( C_3 \) depending only on \( n, d, R, D \) such that for
all local fields $L \in \tilde{L}_{K, M_1}$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$, we have
\[
|E_f(Z, \psi)| \leq C_3 q_L^{\frac{-m_\psi(n-s)}{2(d-1)}}.
\]

Proof. We use the proof of [6, Proposition 5.2] and repeat the proof of Lemma 3.4 for the ideal $(f) + J_f^2$ to see that there exist a constant $M_1$ depending only on $f$ and a constant $C_1$ depending only on $n, d, R, D$ such that for all local fields $L \in \tilde{L}_{K, M_1}$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 2$, we have
\[
|E_f(Z, \psi)| \leq C_1 m_\psi^{n-1} q_L^{-(m-1)\sigma_Z(\tilde{L}, (f) + J_f^2)},
\]
where $\tilde{L}$ is a local field over $O_K$ of characteristic 0 with $k_{\tilde{L}} \simeq k_L$ and
\[
\sigma_Z(\tilde{L}, (f) + J_f^2) = \min_{Q \in O_L^J, Q \in Z(k_L)} \text{lct}_Q((f) + J_f^2).
\]
If $d = 2$, we can suppose that $f(0) = 0$. By changing of coordinates, we can suppose that $f = x_1^2 + \ldots + x_n^2$. As in the proof of Proposition 3.12, we can suppose that $Z \subset \text{Sing}(f)$ thus we can suppose that $Z = \{0\} \times Z_1$ for some subscheme $Z_1$ of $\mathbb{A}^n_{K, \psi}$. The integral form of $E_f(Z, \psi)$ yields our claim.

Let us suppose that $d > 2$, $f(0) = 0$ and 0 is a critical point of $f$. We will show that
\[
\sigma_Z(\tilde{L}, (f) + J_f^2) > \frac{n - s}{2(d-1)}.
\]
Because of the definition of $\sigma_Z(\tilde{L}, (f) + J_f^2)$, it is sufficient to show that
\[
\text{lct}_0(f, J_f^2) > \frac{n - s}{2(d-1)}.
\]
As in the proof of Corollary 3.9, we have
\[
\text{lct}_0(f, J_f^2) \geq \text{lct}_0(f_d, J_{f_d}^2),
\]
moreover, we only need to prove that
\[
\text{lct}_0(f_d, J_{f_d}^2) > \frac{n}{2(d-1)}
\]
if $\dim(\text{Sing}(f_d)) = 0$. In this case, suppose that $\text{lct}_0(f_d, J_{f_d}^2) = \text{lct}_0(J_{f_d}^2) = n/(2(d-1))$. Since $\dim(\text{Sing}(f_d)) = 0$ we remark again that if $x \in t\tilde{L}[[t]]^n$ then $\text{ord}_t(J_{f_d}^2(x)) > m$ iff $\text{ord}_t(x) > m/(2(d-1))$. Therefore if $d > 2$ then for all large enough $m$ we have
\[
\dim(\{x \in (t\tilde{L}[[t]]/(t^{m+1}))^n| \text{ord}_t(f_d, J_{f_d}^2)(x) > m\}) < \dim(\{x \in (t\tilde{L}[[t]]/(t^{m+1}))^n| \text{ord}_t(J_{f_d}^2(x)) > m\}).
\]
We can use [21, Corollary 3.4] to get a contradiction. Thus (17) holds.

Let $m_0$ be the smallest integer such that
\[
m_0(\sigma_Z(\tilde{L}, (f) + J_f^2) - \frac{n - s}{2(d-1)}) > \sigma_Z(\tilde{L}, (f) + J_f^2).
\]
Inequality (16) and the definition of $m_0$ imply that there exists an integer $M_2$ depending only on $C_1, f$ such that for all local fields $L \in \tilde{L}_{K,M_2}$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq m_0$, we have

$$|E_f(Z, \psi)| \leq q_L^{-m_\psi(n-s)}2^{(d-1)}$$

provided that the complexity of $Z$ is bounded by $(R, D)$.

Let us fix a scheme $Z$ whose complexity bounded by $(R, D)$ and let $M, C$ be as in Proposition 3.12 with respect to $Z$. We can finish our proof by using (19) and putting $M_3 = \max\{M, M_2\}$, $C_3 = Cm_0^{n-1}$.

**Proof of Theorem A.** The assertion follows by combining Proposition 3.13 and [24, Proof of Proposition 5.4(2)].

**Corollary 3.14.** There exist a constant $M$ depending only on $f$ and a constant $C$ depending only on $n, d$ such that for all local fields $L \in \tilde{L}_{K,M}$ and all additive characters $\psi$ of $L$ of conductor $m_\psi \geq 1$ we have

$$|E_f(\psi)| \leq Cq_L^{-m_\psi(n-s)}.$$  

**Proof.** Because of Theorem A and Proposition 1.3, it remains to show that $C$ depends only on $n, d$. In the proof of Proposition 3.13 we see that $C$ depends only on $n, d, m_0$. Therefore $C$ depends only on $n, d$ and the set $\{\text{lct}_P((f) + J^2_f)|P \in A^n_C\}$. We can use the argument in [22, Property 1.23] to show that the set $\{\text{lct}_P((g) + J^2_g)|P \in A^n_C, g \in C[x_1, ..., x_n] \setminus \mathbb{C}, \deg(g) \leq d\}$ is finite. Therefore $C$ depends only on $n, d$.

**Remark 3.15.** In Corollary 3.14, we can not take $M$ depending only on $n, d$. Indeed, let us look at the following example.

Let $d > 1$. For each prime number $p$ and each integer $m \geq 2$, we set

$$f_{p,m}(x_1, ..., x_n) = x_1^d + p^m(x_2^d + ... + x_n^d).$$

Let $2 \leq r \leq d$ and $\ell \in \mathbb{N}$. Then

$$E_{f_{p,m}}(\psi) = p^{-m_0}p^{r/d}$$

for all additive characters $Q_p$ of conductor $m_\psi = m = d\ell + r$. We remark that $s(f_{p,m}) = 0$ for all $p, m$. Suppose that $n \geq 2$. It is impossible to have a constant $C$ such that

$$p^{-m_0}p^{r/d} \leq Cp_\ell^{2(\frac{m_0}{d})}$$

for all large enough primes $p$ and all positive integers $m$ of the form $d\ell + r$.

4. **Exponential sums modulo $p^m$ for small $p$ and the strong monodromy conjecture**

To begin this section, let us try to show a weaker version of Theorem B for $\text{lct}(J^2_f)$ instead of $\text{lct}(f, J^2_f)$. 
Proposition 4.1. Let $L$ be a $p$-adic field. Suppose that $f$ is a non-constant polynomial in $L[x_1, ..., x_n]$. For each Schwartz-Bruhat function $\Phi$ on $L^n$ and each $0 < \sigma < \frac{\text{lct}(J_f)}{2}$ there exists a constant $c(\sigma, \Phi) > 0$ such that for all additive characters $\psi$ of $L$ we have

$$|E_f(\Phi, \psi)| \leq c(\sigma, \Phi)q_{L}^{-m_\psi \sigma},$$

where $\text{lct}(J_f) = \min_{x \in \text{Supp}(\Phi)} \text{lct}(J_f)$.

Proof. Let $\varpi_L$ be a uniformizing parameter of $\mathcal{O}_L$. It is clear that

$$|E_f(\Phi, \psi)| \leq \int_{L^n} |\Phi(x)||dx| \leq C$$

for a constant $C$ depending only on $\Phi$ and all additive characters $\psi$ of $L$. Therefore it is sufficient to deal with additive characters $\psi$ of large enough conductor. On the other hand, we can suppose that $\Phi = 1_{a + \varpi_L^r \mathcal{O}_L^n}$ for some $a \in L^n$ and $e \in \mathbb{Z}$. By changing of coordinates we can suppose that $a = 0$ and $e = 0$. Multiplying $f$ by $\varpi_L^e$ if needed, we can suppose that $f \in \mathcal{O}_L[x_1, ..., x_n]$.

If $f$ has no critical point in $\mathcal{O}_L$, then $E_f(\Phi, \psi) = 0$ if $m_\psi$ is large enough. In this case we also have $\text{lct}(J_f) = +\infty$, so the claim is trivial.

Suppose that $f$ has a critical point in $\mathcal{O}_L^n$. For each $m \geq 0$, we set

$$N_m = |\{x \mod \varpi_L^n \mathcal{O}_L | x \in \mathcal{O}_L^n, \text{ord}_L(J_f^2(x)) \geq m \}|,$$

where $J_f$ is the Jacobian ideal of $f$ over $\mathcal{O}_L$ as mentioned in Section 2 and $\text{ord}_L(J_f^2(x)) = \min_{g \in J_f^2} \text{ord}_L(g(x))$. We also consider the Igusa local zeta function associated with $J_f^2$ given by

$$Z_1(J_f^2, L, \Phi, s) = \int_{\mathcal{O}_L^n} q_{L}^{-\text{ord}_L(J_f^2(x))s} |dx|$$

By repeating the argument of Igusa in [17] for a log-resolution of $J_f^2$, it follows that $Z_1(J_f^2, L, \Phi, s)$ is a rational function in $T = q_{L}^{-s}$ such that the real part of any pole is at most $-\text{lct}_L(J_f^2) > -\infty$ and the multiplicity of any pole is at most $n$. By the same argument as in [6, Proposition 5.2] we have

$$\int_{\mathcal{O}_L^n} \psi(f(x))|dx| = \int_{\{x \in \mathcal{O}_L^n | \text{ord}_L(J_f^2(x)) \geq m_\psi - 1\}} \psi(f(x))|dx|$$

whenever $m_\psi \geq 1$. Equality (20) implies that

$$|E_f(\Phi, \psi)| \leq N_{m_\psi - 1} q_{L}^{-n(m_\psi - 1)}$$

if $m_\psi \geq 1$. In addition, by a simple calculation we have

$$N_{m_\psi - 1} q_{L}^{-n(m_\psi - 1)} = \sum_{j \geq m_\psi - 1} \text{Coeff}_T, Z_1(J_f^2, L, \Phi, s)$$
where $T = q_L^{-s}$ as above. Since the real part of any pole of $Z_1(J_f^2, L, \Phi, s)$ is
at most $-\lct{\Phi}(J_f^2)$, there exists a constant $C > 0$ such that
\[
|\text{Coeff}_{C_j}Z_1(J_f^2, L, \Phi, s)| \leq C j^{n-1} q_L^{-j \lct{\Phi}(J_f^2)}
\]
for all $j \geq 1$. For each $0 < \varepsilon < \lct{\Phi}(J_f^2)$, there exists $C_\varepsilon > 0$ such that $C j^{n-1} < C_\varepsilon q_L^j$ for all $j \geq 1$. Thus we have
\[
|E_f(\Phi, \psi)| \leq \sum_{j \geq m_\psi - 1} C_\varepsilon q_L^{-j (\lct{\Phi}(J_f^2) - \varepsilon)}
\]
\[
= C_\varepsilon q_L^{-(m_\psi - 1)(\lct{\Phi}(J_f^2) - \varepsilon)} (1 - q_L^{-\lct{\Phi}(J_f^2) + \varepsilon})^{-1}
\]
whenever $m_\psi \geq 2$. Note that $\lct{\Phi}(J_f^2) = \lct{\Phi}(J_f)/2$ by [22, Property 1.13]. Therefore, to prove the assertion, it suffices to set
\[
e(\lct{\Phi}(J_f)/2 - \varepsilon, \Phi) = C_\varepsilon q_L^{\lct{\Phi}(J_f^2) - \varepsilon} (1 - q_L^{\lct{\Phi}(J_f^2) + \varepsilon})^{-1}
\]
for each $0 < \varepsilon < \lct{\Phi}(J_f)/2$. \hfill \Box

**Proof of Theorem B.** Let us fix a uniformizing parameter $\infty_L$ of $\mathcal{O}_L$. We can suppose that $C_f \cap \text{Supp}(\Phi) \subset f^{-1}(0)$ by using the proof of [14, Proposition 2.7]. As in the proof of Proposition 4.1, we can suppose that $\Phi = 1_{\mathcal{O}_L^n}$, $f \in \mathcal{O}_L[x_1, \ldots, x_n]$ and $f$ has a critical point in $\mathcal{O}_L^n$. Moreover, we only need to prove the assertion for additive characters of large enough conductor.

Note that we can adapt the argument in the proof of Proposition 4.1 whenever we have a positive integer $r$ independent of additive characters such that if $m_\psi$ is large enough then
\[
\int_{\mathcal{O}_L^n} \psi_{(x)}(f(x)) |dx| = \int_{\{x \in \mathcal{O}_L^n | \text{ord}_L(f(x)) \geq m_\psi - r\}} \psi(f(x)) |dx|.
\]
In more detail, using (21), we can use the same discussion as in [6, Proposition 5.2] to show that
\[
\int_{\mathcal{O}_L^n} \psi(f(x)) |dx| = \int_{\{x \in \mathcal{O}_L^n | \text{ord}_L(J_f^2(x)) \geq m_\psi - 1, \text{ord}_L(f(x)) \geq m_\psi - r\}} \psi(f(x)) |dx|
\]
for all additive characters $\psi$ of large enough conductor. Therefore, we can bound $|E_f(\Phi, \psi)|$ by the volume of the set
\[
\{x \in \mathcal{O}_L^n | \text{ord}_L(f(x)) \geq m_\psi - r, \text{ord}_L(J_f^2) \geq m_\psi - r\}.
\]
Thus we can proceed as in the proof of Proposition 4.1 for ideal $(f) + J_f^2$ instead of $J_f^2$ to obtain our claim.

Let us prove (21). We fix a log-resolution $h : X' \to \mathbb{A}_L^n$ of $(f)$ associated with divisors $(E_i)_{i \in T}$ and numerical data $(\nu_i, N_i)_{i \in T}$. We put $r_0 = \max_{i \in T} \text{ord}_L(N_i) + 1$. Note that $h$ is projective thus $h^{-1}(\mathcal{O}_L^n)$ is compact. For each point $P \in h^{-1}(\mathcal{O}_L^n)$, we modify local coordinates at $P$ as in [17, Section 3, Chapter 3]. More precisely, if $P \in h^{-1}(\mathcal{O}_L^n)$, we have an open
compact neighbourhood $U_P$ of $P$ and local coordinates $y_1, ..., y_n$ such that the following conditions hold:

i. $\text{ord}_L(y_i(Q)) \geq 0$ for all $Q \in U$ and $1 \leq i \leq n$,

ii. $f \circ h_U = \eta \prod_{i=1}^{n} y_i^{N_i}$,

iii. $h^*(dx_1 \wedge ... \wedge dx_n)|_U = \vartheta y_1^{\nu_1-1}...y_n^{\nu_n-1}dy_1 \wedge ... \wedge dy_n$,

iv. $|\eta|, |\vartheta|$ are non-zero constant functions on $U$,

v. $\eta(y_1,...,y_n) = \vartheta(P)(1+\sum_{I \in \mathbb{N}^n, |I| \geq 1} \alpha_I y^I)$ where $\alpha_I \in \mathcal{O}_L$ for all $I$,

vi. $\vartheta(y_1,...,y_n) = \vartheta(P)(1+\sum_{I \in \mathbb{N}^n, |I| \geq 1} \beta_I y^I)$ where $\beta_I \in \mathcal{O}_L$ for all $I$.

Since $h^{-1}(\mathcal{O}_L^n)$ is compact, we can select a finite set $\mathcal{I}$ of points in $h^{-1}(\mathcal{O}_L^n)$ such that $h^{-1}(\mathcal{O}_L) \subset \bigcup_{P \in \mathcal{I}} U_P$. For distinct points $P, P'$ in $\mathcal{I}$ then $U_P \setminus U_{P'}$ is still a compact open set. Therefore, replacing pair $(U_P, U_{P'})$ by pair $(U_P \setminus U_{P'}, U_{P'})$ if needed, we can suppose that $\bigcup_{P \in \mathcal{I}} U_P$ is a disjoint union. Replacing $f$ by $\varphi_L f$ if necessary, we also can suppose that $\eta(P) \in \mathcal{O}_L$ for all $P \in \mathcal{I}$. We put $r_1 = \max_{P \in \mathcal{I}} \text{ord}_L(\eta(P))$.

For each $P \in \mathcal{I}$, let the local coordinates $y_1, ..., y_n$ on $U_P$ as above. We set $g = (y_1, ..., y_n)$. Since $U_P$ is a compact open set, $g(U_P)$ and $U_P \cap h^{-1}(\mathcal{O}_L^n)$ are also compact open sets. Thus there exists a positive integer $r_P$ such that if $h(Q) \in \mathcal{O}_L^n$ and $Q, Q' \in U_P$ then $g(Q)(1 + \varphi_L^{r_P} \mathcal{O}_L)^n \subset g(U_P)$ and $h(Q') \in \mathcal{O}_L^n$ whenever $g(Q') \in g(Q)(1 + \varphi_L^{r_P} \mathcal{O}_L)^n$. We put $r_2 = \max_{P \in \mathcal{I}} r_P$ and $r = r_0 + r_2$.

Let $\chi$ be a multiplicative character of $\mathcal{O}_L^\times$ of conductor $c(\chi) > r$. Suppose that $Q \in h^{-1}(\mathcal{O}_L^n)$ and $f(h(Q)) \neq 0$. Let $P \in \mathcal{I}$ such that $Q \in U_P$. Let the local coordinates $y_1, ..., y_n$ and functions $\eta, \vartheta$ on $U_P$ as above. We still set $g = (y_1, ..., y_n)$. Because of the definition of $r_2$, we have

$$\int_{h^{-1}(g(Q)(1+\varphi_L^{r_2} \mathcal{O}_L)^n))} \chi(ac(f(x)))|f(x)|^s|dx_1 \wedge ... \wedge dx_n|$$

$$= \int_{(1+\varphi_L^{r_2} \mathcal{O}_L)^n} \chi(ac(\eta(y_1(Q)z_1, ..., y_n(Q)z_n) \prod_{1 \leq i \leq n} (y_i(Q)z_i)N_i))|f(h(Q))|^s \times$$

$$|\vartheta(P)| \prod_{1 \leq i \leq n} |y_i(Q)|^\nu_i|dz_1 \wedge ... \wedge dz_n|$$

$$= \chi(ac(\prod_{1 \leq i \leq n} y_i(Q)N_i))|f(h(Q))|^s |\vartheta(P)| \prod_{1 \leq i \leq n} |y_i(Q)|^\nu_i \times q_L^{-r2^n} \times$$

$$\int_{\mathcal{O}_L^n} \chi(ac(\eta(y_1(Q)(1+\varphi_L^{r_2} t_1), ..., y_n(Q)(1+\varphi_L^{r_2} t_n)) \prod_{1 \leq i \leq n} (1 + \varphi_L^{r_2} t_i)N_i))|dt_1 \wedge ... \wedge dt_n|$$

We recall that

$$\eta(y_1, ..., y_n) = \eta(P)(1 + \sum_{|I| \geq 1} a_I y^I)$$

and $a_I \in \mathcal{O}_L$ for all $I$. Suppose that $\text{ord}_L(y_i(Q)) \geq r_0$ and $N_i > 0$ for some $1 \leq i \leq n$. Without loss of generality, we can suppose that $i = 1$. Fix a
tuple \((t_1, t_2, \ldots, t_n) \in \mathcal{O}_L^n\). We have

\[
\begin{align*}
\text{ord}_L \eta(y_1(Q)(1 + \omega_L^{r_1}t_1), \ldots, y_n(Q)(1 + \omega_L^{r_2}t_n)) \\
- \eta(y_1(Q)(1 + \omega_L^{r_1}(t_1 + \omega_L^{c(\chi)}-r_2-r_0u)), \ldots, y_n(Q)(1 + \omega_L^{r_2}t_n))) \\
\geq \text{ord}_L(y_1(Q)) + c(\chi) - r_0 + \text{ord}_L(\eta(P)) \\
\geq c(\chi) + \text{ord}_L(\eta(P))
\end{align*}
\]

for all \(u \in \mathcal{O}_L\). Since \(|\eta|\) is constant in \(U_P\) we have

\[
\begin{align*}
\chi(ac(\eta(y_1(P)(1 + \omega_L^{r_1}t_1), \ldots, y_n(P)(1 + \omega_L^{r_2}t_n)))) \\
= \chi(ac(\eta(y_1(P)(1 + \omega_L^{r_1}(t_1 + \omega_L^{c(\chi)}-r_2-r_0u)), \ldots, y_n(P)(1 + \omega_L^{r_2}t_n))))
\end{align*}
\]

for all \(u \in \mathcal{O}_L\). It is easy to verify that

\[
\int_{\mathcal{O}_L} \chi(ac(1 + \omega_L^{r_1}t_1 + \omega_L^{c(\chi)}-r_2-r_0u) N_1) |du| = 0
\]

since \(\text{ord}_L(N_1) < r_0\). Therefore, the Fubini theorem yields

\[
\int_{h^{-1}(g(Q),(1+\omega_L^{r_1} \mathcal{O}_L)^n))} \chi(ac(f(x)))|f(x)|^s|dx_1 \wedge \ldots \wedge dx_n| = 0
\]

if \(c(\chi) > r, \text{ord}_L(y_1(P)) \geq r_0\) and \(N_i > 0\) for some \(1 \leq i \leq n\).

Let

\[
W = \{b \in \mathcal{O}_L^n | \text{ord}(f(b)) \leq r_1 + r_0 \sum_{i \in \mathcal{T}} N_i\}
\]

and \(V = \mathcal{O}_L^n \setminus W\). Since \(C_f \cap \mathcal{O}_L^n \subset f^{-1}(0)\), we see that \(W \cap C_f = \emptyset\). Moreover, \(W\) is open and compact thus \(E_f(1_W, \psi) = 0\) if \(m_\psi\) is large enough.

Let \(m\) be a large enough integer. We put \(V_m = \{b \in V | \text{ord}(f(b)) < m - r\}\). We will use Proposition 2.2 to compute \(E_f(1_{V_m}, \psi)\) if \(m_\psi = m\).

Let \(b \in V, P \in \mathcal{T}, Q \in h^{-1}(b) \cap U_P\) and the local coordinates \(y_1, \ldots, y_n\) on \(U_P\) as above. The definitions of \(V\) and \(r_1\) imply that there exists \(1 \leq i \leq n\) such that \(N_i > 0\) and \(\text{ord}_L(y_i(Q)) > r_0\). The above discussion together with the compactness of \(h^{-1}(V_m)\) and the definition of \(r_2\) imply that \(Z_{\chi}(f, L, 1_{V_m}, s) = 0\) if \(c(\chi) > r\). Therefore, in Proposition 2.2, it remains to deal with \(Z_{\chi}(f, L, 1_{V_m}, s)\) if \(c(\chi) \leq r\). Because of the definition of Igusa local zeta function, \(Z_{\chi}(f, L, 1_{V_m}, s)\) is a polynomial in \(q_L^{-s}\) of degree at most \(m - r - 1 < m - c(\chi)\) if \(c(\chi) \leq r\). Therefore Proposition 2.2 yields \(E_f(1_{V_m}, \psi) = 0\) if \(m_\psi = m\). Thus (21) is proved.

\[\square\]

**Proof of Theorem C.** When \(K' = \mathbb{Q}\), our theorem follows by combining Theorem B, Formula (2), Corollary 3.14 and the fact that

\[
\text{lct}_a ((f - f(a)) + J_2^7) > (n - s)/(2(d - 1))
\]

for all \(a\) as in the proof of Proposition 3.13.

When \(K'\) is an arbitrary number field, we still have an analogous formula of (2) as in [9, Section 1.8], thus our claim follows by the same discussion as in the case \(K' = \mathbb{Q}\). \[\square\]
Corollary 4.2. Let \( L \) be a \( p \)-adic field and \( f \) be a non-constant polynomial in \( L[x_1, \ldots, x_n] \). The strong monodromy conjecture holds in the range \((-\lct((f) + J^2_f), 0]\). More precisely, if \( \Phi \) is a Schwartz-Bruhat function on \( L^n \), \( \chi \) is a multiplicative character of \( \mathcal{O}_L^\times \) and \( s \) is a pole of \( Z_{\chi}(f, L, \Phi, s) \) such that \( \Re(s) \in (-\lct(\Phi((f) + J^2_f), 0] \) then \( \Re(s) \) is a root of the Bernstein-Sato polynomial of \( f \).

Proof. The assertion follows by combining Theorem B and the work of Igusa in [17] (see also in [13, Corollary 1.4.5] or [14, Proposition 2.7]).

Corollary 4.3. The assertion of Corollary 4.2 holds for the range \([- (n - s)/(2(d - 1)), 0]\) instead of the range \((-\lct(\Phi((f) + J^2_f), 0]\).

Proof. The claim follows by Corollary 4.2 and the proof of Proposition 3.13.

5. Estimation for the Major arcs

We recall the formulation of the major arcs in [1] and [2]. Let \( f \) be a homogeneous polynomial of degree \( d > 1 \) in \( \mathbb{Z}[x_1, \ldots, x_n] \) and \( s = s(f) \) as in Section 1. We suppose that the system \( f(x) = 0, \grad f(x) = (\frac{\partial f}{\partial x_1}(x), \ldots, \frac{\partial f}{\partial x_n}(x)) \neq 0 \) has a solution over \( \mathbb{R} \) and \( \mathbb{Q}_p \) for every prime number \( p \). In other words, this means that \( f \) has a smooth solution over each completion of \( \mathbb{Q} \). We say that the smooth Hasse principle holds for \( f \) if our assumption guarantees the existence of a non-trivial solution of \( f \) over \( \mathbb{Q} \). We also can ask the smooth Hasse principle for non-homogeneous polynomials without requiring the non-triviality of solution.

To study the smooth Hasse principle of \( f \), we follow the Hardy-Littlewood circle method. Let \( \omega : \mathbb{R}^n \rightarrow [0, +\infty) \) be a suitable weight function then we try to give an asymptotic formula for the function

\[
N_\omega(f, B) = \sum_{x \in \mathbb{Z}^n, f(x) = 0} \omega(x/B)
\]

when \( B \rightarrow +\infty \). Let us use the identity

\[
(23) \quad N_\omega(f, B) = \int_T S(\alpha, B) d\alpha
\]

where \( T = \mathbb{R}/\mathbb{Z} \), and

\[
S(\alpha, B) = \sum_{x \in \mathbb{Z}^n} \omega(x/B) e^{2\pi i\alpha f(x)}
\]

if \( \omega \) has good enough analytic properties. The Hardy-Littlewood circle method suggests us to divide the torus \( T \) into a set of major arcs \( \mathcal{M} \) and minor arcs \( \mathcal{m} \). More precisely, for each \( \delta > 0 \) we define the major arcs \( \mathcal{M} = \mathcal{M}(\delta) \) to be the set

\[
\mathcal{M}(\delta) = \cup_{q \leq B^\delta} \cup_{0 \leq a \leq q, (a, q) = 1} \{ \alpha \in T | |\alpha - a/q| \leq B^{\delta-d} \}
\]
Note that the union is disjoint if $3\delta < d$ and $B$ is large enough. We define the minor arcs to be the complement $\mathfrak{M}(\delta) = \mathbb{T} \setminus \mathfrak{M}(\delta)$. To deal with the smooth Hasse principle, we want to obtain the following asymptotic formulas

(24) \[ \int_{\mathfrak{M}} S(\alpha, B) d\alpha \sim c_f B^{n-d} \]

and

(25) \[ \int_{m} S(\alpha, B) d\alpha = o(B^{n-d}). \]

Here the constant $c_f$ is a product of local densities which will be positive if $f$ has a smooth solution over each completion of $\mathbb{Q}$. Therefore, to prove (24), we need to find conditions on $f$ such that the product of local densities is convergent. Birch’s condition for the existence of (24) and (25) is $n - s > 2d(d-1)$. This condition is improved by Browning and Prendiville in [2] to $n - s > \frac{3}{2}d^2(d-1)$ as mentioned in Section 1. It seems to be very difficult to improve much on Browning-Prendiville’s condition for the validity of (25) by the current method related to Weyl bound and Diophantine approximation. However, we can reduce much on the condition for the validity of (24) as follows.

**Theorem D.** Let $f$ be a non-constant polynomial of degree $d > 1$ in $\mathbb{Z}[x_1, ..., x_n]$. We recall $f_d, s$ as in Section 1. Suppose that $n - s > 4(d-1)$. Let a weight function $\omega : \mathbb{R}^n \to [0, +\infty)$ given by $\omega(x) = w(\rho^{-1}||x-x_0||)$ where $\rho \in (0, 1)$ is very close to 1, $||.||$ is the Euclidean norm in $\mathbb{R}^n$

\[ ||y||^2 = \sum_{i=1}^{n} y_i^2 \]

and $w : \mathbb{R} \to \mathbb{R}_{\geq 0}$ is given by $x \mapsto 0$ if $|x| \geq 1$ and $x \mapsto e^{1-x^2}$ if $|x| < 1$. Let $\delta$ be small enough. Then we have

(26) \[ \int_{\mathfrak{M}(\delta)} S(\alpha, B) d\alpha \sim c_f B^{n-d} \]

for a positive constant $c_f$ when $B \to +\infty$ provided that $f$ has a smooth solution over each completion of $\mathbb{Q}$.

**Proof.** Let $\alpha \in \mathfrak{M}(\delta)$ then we have $\alpha = \frac{a}{q} + \vartheta$ with integers $a,q$ such that $1 \leq q \leq B^\delta$ and $(a,q) = 1$. We use again the argument of [2, Lemma 5.1] to obtain:

(27) \[ S(\alpha, B) = q^{-n}B^n S_q(a) I(\vartheta B^d) + O(B^{n-1+2\delta}) \]

where

\[ S_q(a) = \sum_{y \in (\mathbb{Z}/q\mathbb{Z})^n} e^{\frac{2\pi i f(y)}{q}} \]

and

\[ I(\gamma) = \int_{\mathbb{R}^n} \omega(x)e^{2\pi i \gamma f(x)} dx \]
if \( \gamma \in \mathbb{R} \). Since the measure of \( \mathfrak{M}(\delta) \) is \( O(B^{-d+3\delta}) \) we have
\[
\int_{\mathfrak{M}(\delta)} S(\alpha, B)d\alpha = B^{n-d} \mathfrak{S}(B^\delta) \mathfrak{J}(B^\delta) + O(B^{n-d-1+5\delta})
\]
where
\[
\mathfrak{S}(R) = \sum_{1 \leq q \leq R} q^{-n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^*} S_q(a)
\]
and
\[
\mathfrak{J}(R) = \int_{-R}^R I(\gamma)d\gamma.
\]
Let \( c_1, \ldots, c_r \) be critical values of \( f \) over \( \mathbb{R} \). Let \( \varepsilon \) be a positive constant such that \( |c_i - c_j| > 2\varepsilon \) for all \( i \neq j \). For each \( 1 \leq i \leq r \), there exists a bounded open subset \( V_i \) of \( \mathbb{R}^n \) such that \( V_i \cap V_j = \emptyset \) for all \( i \neq j \) and \( U_i = f^{-1}[c_i - \varepsilon, c_i + \varepsilon] \cap \text{Supp}(\omega_i) \subset V_i \). For each \( 1 \leq i \leq r \) there exists a smooth function \( u_i \) such that \( u_i|_{U_i} = 1 \), \( 0 \leq u_i \leq 1 \) and \( u_i|_{\mathbb{R}^n \setminus V_i} = 0 \). We also put \( u_r = 1 \). Let \( \omega_1 := \omega \) and \( \omega_{i+1} = \omega_i - \omega_i u_i \) for \( 1 \leq i \leq r - 1 \). Then \( (u_i \omega_i)_{1 \leq i \leq r} \) are positive smooth functions with compact support, moreover we have \( \omega = \sum_{1 \leq i \leq r} \omega_i u_i \) and \( \text{Supp}(u_i \omega_i) \cap C_f \subset f^{-1}(c_i) \) for all \( 1 \leq i \leq r \). For each \( i \), we use [17, Theorem 1.6] to have the asymptotic expansion of oscillatory integral associated with the Schwartz function \( \omega_i u_i \). Then there exists a constant \( \alpha > 0 \) such that for each small enough \( \varepsilon > 0 \), there is a constant \( C_\varepsilon > 0 \) such that
\[
|I(\gamma)| \leq C_\varepsilon |\gamma|^{-\alpha + \varepsilon}
\]
for large enough \( |\gamma| \). So the limit
\[
\mathfrak{J} = \lim_{R \to +\infty} \mathfrak{J}(R)
\]
exists if one can show that \( \alpha > 1 \). Because of the work of Igusa in [17], \(-\alpha\) will be the real part of a certain pole of the function
\[
(s + 1)\mathcal{Z}_1(\omega_i u_i, f - c_i, s) = \int_{\mathbb{R}^n} \omega_i u_i |f(x) - c_i|^s |dx|
\]
or the function
\[
\mathcal{Z}_{-1}(\omega_i u_i, f - c_i, s) = \int_{\mathbb{R}^n} \frac{f(x) - c_i}{|f(x) - c_i|^s} \omega_i u_i |f(x) - c_i|^s |dx|
\]
for some \( 1 \leq i \leq r \). By our assumption, we have \( n - s > 2(d - 1) \). Thus Theorem A and [8, Proposition 3.10] imply that \( f - a \) has at worst rational singularities for all critical values \( a \) of \( f \). Again by the proof of [8, Proposition 3.10], we can obtain a log-resolution of \( (f - a) \) satisfying the condition \( (P) \) in [17, Lemma 6.3, Chapter 4] for each critical value \( a \) of \( f \). Therefore \( \alpha > 1 \) by looking at [17, Theorem 3.1, Chapter 2]. With this constant \( \alpha > 1 \) and \( 0 < \varepsilon < \alpha - 1 \) we have
\[
|\mathfrak{J} - \mathfrak{J}(R)| \leq 2 \int_R^{+\infty} C_\varepsilon \gamma^{-\alpha + \varepsilon} d\gamma = \frac{2C_\varepsilon R^{1+\varepsilon-\alpha}}{\alpha - 1 - \varepsilon} = O_\varepsilon(R^{1+\varepsilon-\alpha})
\]
when $R$ is large enough.

On the other hand, because of Corollary 4, if $n - s > 4(d - 1)$ then for each $0 < \varepsilon < \frac{n-s}{2(d-1)} - 2$ there exists a constant $C'_\varepsilon > 0$ such that

$$|S_q(a)| \leq C'_\varepsilon q^{n - \frac{n-s}{2(d-1)} + \varepsilon}$$

for all $q \geq 1$. Therefore

$$|\sum_{q > R} q^{-n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} S_q(a)| \leq C'_\varepsilon \sum_{q > R} q^{1 - \frac{n-s}{2(d-1)} + \varepsilon} = O_{\varepsilon}(R^{2 - \frac{n-s}{2(d-1)} + \varepsilon})$$

Thus $\mathcal{S}(R)$ converges to the singular series

$$\mathcal{S} = \sum_{1 \leq q} q^{-n} \sum_{a \in (\mathbb{Z}/q\mathbb{Z})^\times} S_q(a)$$

when $R \to +\infty$. Moreover we have

$$(30) \quad |\mathcal{S} - \mathcal{S}(B^\delta)| = O_{\varepsilon}(B^{(2 - \frac{n-s}{2(d-1)} + \varepsilon)})$$

Let $\varepsilon$ and $\delta$ be small enough. Because of (29) and (30), we obtain

$$\int_{\mathfrak{M}(\delta)} S(\alpha, B) d\alpha = \mathcal{S} B^{n-d} + o(B^{n-d-\vartheta(\varepsilon, \delta)})$$

for large enough $B$, where $\vartheta(\varepsilon, \delta)$ is a positive constant depending only on $\varepsilon, \delta$. Now we can look at the argument in [1, Section 6, Section 7] to see that $\mathfrak{I} > 0$ whenever $f$ has a smooth solution over $\mathbb{R}$ and

$$\mathcal{S} = \prod_{p \text{ primes}} \mathcal{S}_p$$

is the convergent product of the local densities $\mathcal{S}_p$, where

$$\mathcal{S}_p = \sum_{r \geq 0} p^{-rn} \sum_{a \in (\mathbb{Z}/p^r\mathbb{Z})^\times} S_p^r (a)$$

is positive whenever $f$ has a smooth solution over $\mathbb{Q}_p$. \hfill \Box

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