The minimal Morse components of translations on flag manifolds are normally hyperbolic

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Abstract

We prove that each minimal Morse components of a flow $g^t$ of translations of $G$ into its flag manifolds is normally hyperbolic, where $G$ is a connected semi-simple real Lie group. Previously, this was only known for flows $g^t$ with no unipotent component and it seemed unknown whether this was true even for general translations on the projective space. We also give a brief survey of some previous results.

1 Introduction

The subject matter of this paper is the dynamics of a flow $g^t$ of translations of $G$ into its flag manifolds $F_\Theta$. Here $G$ is a connected real Lie group with semi-simple Lie algebra $\mathfrak{g}$. The flow $g^t$ is either given by the iteration of some $g \in G$, for $t \in \mathbb{Z}$, or by $\exp(tX)$, for $t \in \mathbb{R}$, where $X \in \mathfrak{g}$. The flag manifold is a homogeneous space of $G$ and $g^t$ acts on it by left translations. Throughout the article we will assume that $G$ is a linear group, since $G$ acts on its flag manifolds by the adjoint action.

This flow has many remarkable geometric, algebraic and numerical properties and was studied by many authors in both the semi-simple context and

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also in the classical context where \( G = \text{Gl}(n, \mathbb{R}) \) and \( \mathbb{F}_\Theta \) the classical manifolds of nested subspaces of \( \mathbb{R}^n \), such as the projective space and the grassmanians. First we give a brief survey of some of these results. Duistermaat et al. (1983, [6]) consider a continuous-time flow generated by a hyperbolic element \( H \in \mathfrak{g} \) acting on the flag manifolds of \( \mathfrak{g} \), describe its fixed point set and their stable manifolds and prove that it is a Morse-Bott flow. In particular, they show that each connected component of its fixed points and its stable manifold are given, respectively, by the following orbits \( G_H w b_\Theta \) and \( P_H^{-} w b_\Theta \), where \( G_H \) is the centralizer of \( H \) in \( G \), \( P_H^{-} \) is the opposed parabolic subgroup associated to \( H \), \( w \) is some element of the Weyl group of \( G \) and \( b_\Theta \) is a suitable base point in \( \mathbb{F}_\Theta \). Hermann (1983, [10]) consider a continuous-time flow generated by an element which is the sum of two commuting elements of \( \mathfrak{g} \), one of which is hyperbolic and the other elliptic. Batterson (1977, [4]) considers the discrete-time flow generated by an element \( g \in \text{Gl}(n) \) on Grassmanians, for real and complex matrices, characterizes the structurally stable ones and describe their conjugacy classes. Ammar and Martin (1986, [1]) extends the results of Batterson to the classical flag manifolds and considers some of its numerical aspects, following Hermann (1979, [11]). Shub and Vasquez (1987, [18]) gives an elementary approach to the relation between the QR algorithm of numerical linear algebra and the discrete-time flow generated by an element \( g \in \text{Gl}(n) \) on the maximal flag manifold. Ayala et al (2003, [3]) consider continuous-time flow generated by a matrix \( X \in \mathfrak{gl}(n) \) on classical flag manifolds in order to characterize dynamically some spectral properties of matrices. Recently, flows in flag bundles and related techniques of Lie theory appear in the analysis of the flows considered in [2, 8].

Now we give a brief survey of some of our results about these subjects. We start our investigations in a broader context, considering flows on flag bundles. In fact, we consider a flow on flag bundle arising from a flow on the corresponding principal bundle which commutes with the action of the structural group \( G \), under the assumption that the induced flow on the base is chain transitive. When the base of the bundle is a point, this gives precisely a flow of translations. In (2007, [13]) we describe its chain recurrent set, showing that each chain transitive component is an associated subbundle whose typical fiber is a connected component of the fixed points a flow generated by some hyperbolic element \( H \) of \( \mathfrak{g} \). Thus it follows that these are the minimal Morse components of the flow on the flag bundle. In (2009, [14]), we describe these constructions through an invariant \( G_H \)-reduction of the principal bundle, which allows us to describe the stable sets of the mini-
mal Morse components and to prove that, with some additional assumptions, they are normally hyperbolic, so we could compute their Conley index. This construction requires a $G_H$-equivariant linearization of the flow generated by $H$ around a connected component of the its fixed points which, unfortunately, we were only able to construct in some situations. We also show that each minimal Morse component is in fact a flag subbundle, whose fiber is a flag manifold of the semi-simple component of $G_H$ (see also [17]). In (2010, [7]) we shift our focus to the more concrete situation of a flow of translations $g^t$ on a flag manifold of $G$. Our main tool is the Jordan decomposition of $g^t$ which provides us the following commutative decomposition (see Lemma 3.1 of [7])

$$g^t = e^t h^t u^t,$$

where $h^t = \exp(tH)$, with $H \in \mathfrak{g}$ hyperbolic, $u^t = \exp(tN)$, with $N \in \mathfrak{g}$ nilpotent, and $e^t, u^t \in G_H$. Furthermore, we prove that $H$ can be taken in the closure of a Weyl chamber $\mathfrak{a}^+$ and that $e^t$ can be taken in the maximal compact subgroup $K$ of the Iwasawa decomposition with respect to this chamber. In this context, we show that each minimal Morse component and its stable manifold are given, respectively, by the connected component of the fixed points of the hyperbolic component $h^t$ and its stable manifold. This puts the previous results about flows on flag bundles in a more concrete context and generalizes the results of Duistermaat et al. [6]. We were also able to show that the recurrent set is given by the fixed point of the unipotent component $u^t$ inside the minimal Morse components.

There are still some open problems. It remains to describe the geometry of the recurrent set that is, the geometry of the fixed points of the unipotent component $u^t$ inside the Morse components. There are some results that describe the geometry of fixed points of unipotent flows on classical flag manifolds [9]. We can apply these results to general flows of translations on classical flag manifolds, since we have shown that each minimal Morse component is a flag manifold of the semi-simple component of $G_H$, which in this case is a product of classical flag manifolds. It also remains to describe the geometry of the minimal sets and to investigate whether the minimal Morse components are normally hyperbolic.

In this article we deal with this last problem, showing that each minimal Morse components of $g^t$ in $F_{\Theta}$ is indeed normally hyperbolic, generalizing the results of [6], since this already happens when the flow is Morse-Bott. Previously, this was only known for flows $g^t$ with no unipotent component (see
Proposition 5.7 of [7]) and it seemed unknown whether this was true even for general translations on the projective space. As a consequence of the normal hyperbolicity, we get a linearization of $g^t$ around each minimal Morse set. It would be nice if this linearization could be made $G_H$-equivariant so that it could be used to provide the linearization of the flows on flag bundles but we still do not know whether this can be done.

We describe briefly the structure of the article. In order to shorten the exposition, we refer to Section 2 of [14] for some standard results and notations about Morse decompositions and semi-simple Lie theory (see also [5, 12]). In Section 2, we prove our main result by constructing a $K$-invariant metric on the flag manifold in a suitable manner, where $K$ is a maximal compact subgroup of $G$. This construction is related to reductive homogenous spaces of $K$, but, as a model for the tangent space, instead of a subspace of the Lie algebra of $K$, we use a subspace of the Lie algebra of $G$ which is more appropriate to study of the $G$-action. In Section 3, we give applications to linearization, linear flows with periodic coefficients, and we present some examples.

2 Main result

We start this section recalling the construction of a $G_H$-invariant vector subbundle of the tangent bundle $T\mathbb{F}_\Theta$ of $\mathbb{F}_\Theta$ restricted to $\text{fix}_\Theta(H, w)$ which complements the tangent bundle $T\text{fix}_\Theta(H, w)$ of $\text{fix}_\Theta(H, w)$ (see [14]). Next we introduce a suitable metric on $\mathbb{F}_\Theta$ for which this subbundle is the normal bundle of $\text{fix}_\Theta(H, w)$. By using this metric, we show that $\text{fix}_\Theta(H, w)$ is normally hyperbolic for $g^t$.

Given $X \in \mathfrak{g}$ and $x \in \mathbb{F}_\Theta$, denote by $X \cdot x$ the corresponding induced tangent vector at $x$. Recall that $g \in G$ acts on the tangent bundle $T\mathbb{F}_\Theta$ by its differential and that

$$g(X \cdot x) = gX \cdot gx,$$

where $G$ acts on $\mathfrak{g}$ by its adjoint action. Given a subset $I \subset \mathfrak{g}$ and $x \in \mathbb{F}_\Theta$, we denote by

$$I \cdot x = \{X \cdot x \in T_x\mathbb{F}_\Theta : X \in I\}$$

the set of corresponding induced tangent vectors at $x$.

Now consider $n^+_H$ the sum of the positive/negative eigenspaces of $\text{ad}(H)$.
in \( g \). Define the subsets of \( T^F_\Theta \) given by
\[
V^\pm_\Theta(H, w) = \bigcup \{ n_H^\pm \cdot x : x \in \text{fix}_\Theta(H, w) \},
\]
in such a way that the fiber above \( x \in \text{fix}_\Theta(H, w) \) is given by
\[
V^\pm_\Theta(H, w)_x = n_H^\pm \cdot x.
\]
We have that \( G_H \) normalizes \( n_H^\pm \), since it is a sum of eigenspaces of \( \text{ad}(H) \), and thus \( V^\pm_\Theta(H, w) \) are \( G_H \)-invariant. We denote the Whitney sum of the these subsets by
\[
V_\Theta(H, w) = V^+_\Theta(H, w) \oplus V^-_\Theta(H, w). \tag{1}
\]
The subsets of \( T^F_\Theta \) defined above are in fact subbundles. In order to provide a more detailed description of them, we define the family of subspaces \( \mathfrak{f}^\pm_x \subset n_H^\pm \) given by
\[
\mathfrak{f}^\pm_x = k(n_H^\pm \cap w n^-_\Theta)
\]
for \( x = k w b_\Theta \), where \( k \in K_H \). We have the following result (See Proposition 3.1 of [14]).

**Proposition 2.1** Each \( \mathfrak{f}^\pm_x \) is well defined and \( \mathfrak{f}^\pm_{kx} = k \mathfrak{f}^\pm_x \), for every \( k \in K_H \). Furthermore, the map
\[
X \in \mathfrak{f}^\pm_x \mapsto X \cdot x \in n_H^\pm \cdot x
\]
is a linear isomorphism, for each \( x \in \text{fix}_\Theta(H, w) \). In particular, the fiber of \( V^\pm_\Theta(H, w) \) above \( x \in \text{fix}_\Theta(H, w) \) is given by
\[
V^\pm_\Theta(H, w)_x = \mathfrak{f}^\pm_x \cdot x.
\]

The next result shows that the subbundle \( V_\Theta(H, w) \) complements \( T_{\text{fix}_\Theta}(H, w) \) inside \( T^F_\Theta \) (See Proposition 3.3 of [14]).

**Proposition 2.2** We have that
\[
T^F_\Theta|_{\text{fix}_\Theta(H, w)} = T_{\text{fix}_\Theta}(H, w) \oplus V_\Theta(H, w).
\]
Now we introduce a suitable Riemannian metric in $F_{\Theta}$. We have that $n_\Theta$ complements the isotropy subalgebra $p_{\Theta}$ at $b_{\Theta}$ and that $K_{\Theta}$ is the the isotropy of $b_{\Theta}$ by $K$. Consider the restriction to $n_{\Theta}$ of the Cartan inner product $\langle \cdot, \cdot \rangle$. We have that $n_{\Theta}$ is a $K_{\Theta}$-invariant subspace, so that $\langle \cdot, \cdot \rangle$ is a $K_{\Theta}$-invariant inner product in $n_{\Theta}$. By a standard construction, this allows us to define a $K$-invariant Riemannian metric in $F_{\Theta}$ such that

$$\langle Y \cdot b_{\Theta}, Z \cdot b_{\Theta} \rangle_{b_{\Theta}} = \langle Y, Z \rangle,$$

for $Y, Z \in n_{\Theta}$. With this metric, we have that $V_{\Theta}(H, w)$ is the normal bundle of $\text{fix}_\Theta(H, w)$ in $F_{\Theta}$. This metric also has the following properties.

**Lemma 2.3** Writing $v \in V^\pm_{\Theta}(H, w)$ as

$$v = Y \cdot x, \quad Y \in n_{H}^\pm, \quad x \in \text{fix}_\Theta(H, w),$$

we have that $|v| \leq |Y|$, where the equality holds whenever $Y \in I^\pm_x$.

**Proof:** Let $x = kwb_{\Theta}$, where $k \in K_H$. First let us show that equality holds when $Y \in I^\pm_x$. First, we take $m \in M_{\ast}$ a representative of $w$ such that $\text{Ad}(m)|_{a} = w$. Thus, by the definition of $I^\pm_x$, there exists $Z \in I^\pm_{b_{\Theta}}$ such that $Y = kmZ$. It follows that $v = km(Z \cdot b_{\Theta})$ and that $|Y| = |Z|$, where we use that the Cartan inner-product is $K$-invariant. By the $K$-invariance of the Riemannian metric, it follows that $|v| = |Z| = |Y|$.

Now let us show that the inequality hold when $Y \in n_{H}^\pm$. By Proposition 2.1 we have that $v = Z \cdot x$, for some $Z \in I^\pm_x = kwI^\pm_{b_{\Theta}}$. By the first part of the proof, we have that $|v| = |Z|$. Since $v = Y \cdot x = Z \cdot x$, it follows that $(Y - Z) \cdot x = 0$, which shows that $Y - Z$ belongs to $kw_p_{\Theta}$, which is the kernel of the map $X \in g \mapsto X \cdot x$. Since $kw_{b_{\Theta}}$ is orthogonal to $kw_{p_{\Theta}}$, it follows that

$$|Y|^2 = |Z|^2 + |Y - Z|^2 \geq |Z|^2 = |v|^2.$$

Next we show that each Morse component of the flow of translations $g^t$ on $F_{\Theta}$ is normally hyperbolic. First we need the following lemma.

**Lemma 2.4** We have that

$$|h^t Y| \leq e^{-\mu t}|Y|, \quad \text{for} \quad Y \in n_H, \quad t \geq 0.$$
and

\[ |h^t Y| \leq e^{\mu t} |Y|, \quad \text{for} \quad Y \in \mathfrak{n}_H, \quad t \leq 0 \]

where

\[ \mu = \min\{\alpha(H) : \alpha(H) > 0, \alpha \in \Pi\}. \]

**Proof:** For \( Y \in \mathfrak{n}^+_H \), we have that \( h^t Y = e^{t \text{ad}(H)} Y \), where \( e^{t \text{ad}(H)} \) is \( \langle \cdot, \cdot \rangle \)-symmetric with eigenvalues in \( \mathfrak{n}^+_H \) given by

\[ \{e^{\pm\alpha(H)t} : \alpha(H) > 0, \alpha \in \Pi\}, \]

since \( \text{ad}(H) \) is \( \langle \cdot, \cdot \rangle \)-symmetric with eigenvalues in \( \mathfrak{n}^+_H \) given by

\[ \{\pm\alpha(H) : \alpha(H) > 0, \alpha \in \Pi\}. \]

Writing \( Y \) as the orthogonal sum of eigenvectors \( Y = \sum_{\alpha} Y_{\alpha} \), we have that

\[ |h^t Y| = |\sum_{\alpha} e^{\pm\alpha(H)t} Y_{\alpha}| \leq \sum_{\alpha} e^{\pm\alpha(H)t} |Y_{\alpha}|. \]

For \( t > 0 \) and \( Y \in \mathfrak{n}_H^+ \), we have that

\[ |h^t Y| \leq e^{-\mu t} \sum_{\alpha} |Y_{\alpha}| = e^{-\mu t} |Y|. \]

since \( e^{-\alpha(H)t} < e^{-\mu t} \), for all \( \alpha \in \Pi \) with \( \alpha(H) > 0 \). For \( t < 0 \) and \( Y \in \mathfrak{n}_H \), we have that

\[ |h^t Y| \leq e^{\mu t} \sum_{\alpha} |Y_{\alpha}| = e^{\mu t} |Y|. \]

since \( e^{\alpha(H)t} < e^{\mu t} \), for all \( \alpha \in \Pi \) with \( \alpha(H) > 0 \).

Therefore we prove the main result of the article.

**Theorem 2.5** The bundles \( V^\pm_G(H, w) \) are \( g^t \)-invariant and there exist positive numbers \( c \) and \( \lambda \) such that

\[ |g^t v| \leq c e^{-\lambda t} |v|, \quad \text{for} \quad v \in V^-_G(H, w), \quad t \geq 0 \]

and

\[ |g^t v| \leq c e^{\lambda t} |v|, \quad \text{for} \quad v \in V^+_G(H, w), \quad t \leq 0. \]

Therefore \( \text{fix}_G(H, w) \) is normally hyperbolic for \( g^t \).
Proof: We recall some properties of the Jordan decomposition of $g^t$. By Lemma 3.1 of \cite{7}, we have the following commutative decomposition

$$g^t = e^t h^t u^t,$$

where $h^t = \exp(tH)$, with $H \in \mathfrak{g}$ hyperbolic, $u^t = \exp(tN)$, with $N \in \mathfrak{g}$ nilpotent, and $e^t, u^t \in G_H$, the centralizer of $H$ in $G$. Besides that, we can assume that $H \in \mathfrak{cl}^{+}$ and that $e^t \in K_H$, the centralizer of $H$ in $K$.

Let $v \in V^\pm_\Theta(H, w)$. Writing $v = Y \cdot x$, where $Y \in \mathfrak{t}^\pm$ and $x \in \text{fix}_\Theta(H, w)$, it follows, from Lemma 2.3, that $|v| = |Y|$. Using the same lemma and that $g^t Y \in \mathfrak{n}^+_H$, it follows that

$$|g^t v| = |g^t Y \cdot g^t x| \leq |g^t Y|.$$ 

Now it is enough to show that the inequalities hold for $g^t$ restricted to $\mathfrak{n}^+_H$.

This follows from standard linear algebra and we will sketch the argument here for the readers’ convenience. We will consider only the case in which $Y \in \mathfrak{n}^-_H$, the other case being completely analogous. By Lemma 2.4 there exists $\mu > 0$ such that $|h^t Z| \leq e^{-\mu t}|Z|$, for $t \geq 0$ and $Z \in \mathfrak{n}^-_H$. Since we can assume that $e^t \in K_H$, it follows that

$$|g^t Y| = |h^t u^t Y| \leq e^{-\mu t}|u^t Y|.$$ 

Since $u^t = \exp(tN)$, for some nilpotent $N \in \mathfrak{g}$, we have that $u^t Y = e^{t \text{ad}(N)} Y$. By the triangle inequality, we have that

$$|e^{t \text{ad}(N)} Y| \leq \sum_{k \geq 0} \frac{t^k}{k!} ||\text{ad}(N)^k||Y| = p(t)|Y|,$$

where $|| \cdot ||$ is the operator norm induced by the norm $| \cdot |$ in $\mathfrak{n}^-_H$ and $p(t)$ is a polynomial, since $\text{ad}(N)$ is nilpotent. Thus, we have that

$$|g^t Y| \leq e^{-\mu t} p(t)|Y|,$$

so that there exists positive numbers $c$ and $\lambda$ such that

$$|g^t Y| \leq c e^{-\lambda t}|Y|, \quad t \geq 0.$$ 

It follows that $V^\pm_\Theta(H, w)$ are the unstable/stable bundle of $g^t$. 

\end{proof}
3 Applications and examples

Using Theorem 2.5 and the main result of [16], we obtain the following linearization result.

**Corollary 3.1** There exists a differentiable map $\psi : V_\Theta(H, w) \to F_\Theta$ which takes the null section $V_\Theta(H, w)_0$ to $\text{fix}_\Theta(H, w)$ and such that:

(i) Its restriction to some neighborhood of $V_\Theta(H, w)_0$ in $V_\Theta(H, w)$ is a $g^t$-equivariant diffeomorphism onto some neighborhood of $\text{fix}_\Theta(H, w)$ in $F_\Theta$.

(ii) Its restrictions to $V_\Theta^\pm(H, w)$ are $g^t$-equivariant diffeomorphisms, respectively, onto the submanifolds $N^\pm_H\text{fix}_\Theta(H, w)$.

**Proof:** It is enough to note that the action of $g^t$ on $V_\Theta(H, w)$ is given by the restriction of the differential of the action of $g^t$ on $F_\Theta$ and also that the equivariance property is equivalent to the conjugation property of [16]. $\square$

Next, we give two examples of flows with non-trivial unipotent part where this linearization can be visualized.

**Example 3.2** Let $X = H + N$ in $\mathfrak{sl}(3, \mathbb{R})$, where the Jordan components are given by

$$H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and consider $g^t = \exp(tX)$ acting on the projective plane, which is a minimal flag manifold of $G = \text{Sl}(3, \mathbb{R})$. Its attractor (repeller) is the projectivization of the eigenspace of $H$ associated to the largest (smallest) eigenvalue. Figure 1 shows the phase portrait of the flow on the projective plane and its linearization around the repeller, which is a linear flow on the Möbius strip over the unipotent flow on the projective line.

Now, let $X = H + N$ in $\mathfrak{sl}(2, \mathbb{R}) \times \mathfrak{sl}(2, \mathbb{R})$, where the Jordan components are given by

$$H = \left( \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right), \quad N = \left( \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} , \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right),$$

and consider $g^t = \exp(tX)$ acting on the projective plane, which is a minimal flag manifold of $G = \text{Sl}(2, \mathbb{R})$. Its attractor (repeller) is the projectivization of the eigenspace of $H$ associated to the largest (smallest) eigenvalue.
and consider $g^t = \exp(tX)$ acting on the torus, which is the maximal flag manifold of $G = \text{Sl}(2, \mathbb{R}) \times \text{Sl}(2, \mathbb{R})$. Identifying the torus with $S^1 \times S^1$, where each $S^1$ is the projective line of $\mathbb{R}^2$, $g^t$ acts on the first component by the exponential of $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and acts on the second component by the exponential of $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then, its attractor (repeller) is the cartesian product of the attractor (repeller) in the first component by $S^1$. Figure 2 shows the phase portrait of the flow on the torus and its linearization around the repeller, which is a linear flow on the cylinder over the unipotent flow on the projective line.

Figure 2: Dynamics on the torus and its linearization around the repeller.

Now we compute the Conley indices of the minimal Morse components of $g^t$ in $\mathbb{F}_\Theta$ by using Theorem 2.5 and the standard fact that the Conley index of a normally hyperbolic invariant manifold is given by the Thom space of its unstable bundle (see, for example, Theorem 7.4 of [14]). We also use the Thom isomorphism and that each minimal Morse component of $g^t$ in $\mathbb{F}_\Theta$
is, in fact, a flag manifold of $\mathfrak{g}(H)$. More precisely, let $\mathfrak{g}(H)$ be the semi-simple Lie subalgebra of $\mathfrak{g}$ determined by the simple roots which annihilate $H$ and let $\mathfrak{a}(H) = \mathfrak{g}(H) \cap \mathfrak{a}$. Then $\text{fix}_\Theta(H,w)$ is diffeomorphic to the flag manifold $\mathcal{F}_\Theta(H,w)$ of $\mathfrak{g}(H)$ determined by the orthogonal projection of $wH_\Theta$ onto $\mathfrak{a}(H)$, where $H_\Theta \in \text{cl}^{+}$ is such that $\Theta = \{ \alpha \in \Sigma : \alpha(H_\Theta) = 0 \}$ (see Proposition 3.11 of [14]).

**Corollary 3.3** The Conley index of the Morse component $\text{fix}_\Theta(H,w)$ is the homotopy class of the Thom space of the vector bundle $V^+_\Theta(H,w) \to \text{fix}_\Theta(H,w)$. In particular, we have the following isomorphism in cohomology

$$CH^{*+n_w}(\text{fix}_\Theta(H,w)) \simeq H^*(\mathcal{F}_\Theta(H,w)),$$

where $CH$ denotes the cohomology of the Conley index and $n_w$ is the dimension of $V^+_\Theta(H,w)$ as a vector bundle.

In the previous result, the cohomology coefficients are taken in $\mathbb{Z}_2$ in the general case and in $\mathbb{Z}$ when $V^+_\Theta(H,w)$ is orientable. We remark that [15] gives closed formulas, in terms of roots and their multiplicities, to decide when $V^+_\Theta(H,w)$ is orientable.

Now we apply our results to linear differential equations with periodic coefficients in $\mathfrak{g}$. Recall that $g^t = \exp(tX)$ is the solution of the linear differential equation with constant coefficients $g'(t) = Xg(t)$, with $g(0) = I$, where $I$ is the identity of $G$. Now consider the following linear differential equation

$$g'(t) = X(t)g(t),$$

where $t \in \mathbb{R} \to X(t) \in \mathfrak{g}$ is a continuous periodic map. By a standard construction, we can associate to the solutions of this equation a continuous time flow $\phi^t$ on $S^1 \times \mathcal{F}_\Theta$ (see Section 6 of [7]). The following result generalizes Proposition 6.5 of [7] and can be proved exactly in the same way presented in that article by replacing the linearization result presented in Section 3 of [14] by the one given in Corollary 3.1.

**Corollary 3.4** Each minimal Morse component of $\phi^t$ is homeomorphic to $S^1 \times \mathcal{F}_\Theta(H,w)$ and is normally hyperbolic.

Thus, we also have results for the Conley index similar to Corollary 3.3 which generalizes Theorem 6.6 of [7].
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