Limit Distribution Theory for Smooth Wasserstein Distance with Applications to Generative Modeling

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Abstract

The 1-Wasserstein distance ($W_1$) is a popular proximity measure between probability distributions. Its metric structure, robustness to support mismatch, and rich geometric structure fueled its wide adoption for machine learning tasks. Such tasks inherently rely on approximating distributions from data. This surfaces a central issue — empirical approximation under Wasserstein distances suffers from the curse of dimensionality, converging at rate $n^{-1/d}$ where $n$ is the sample size and $d$ is the data dimension; this rate drastically deteriorates in high dimensions. To circumvent this impasse, we adopt the framework of Gaussian smoothed Wasserstein distance $W_1^{(\sigma)}$, where both probability measures are convolved with an isotropic Gaussian distribution of parameter $\sigma > 0$. In remarkable contrast to the original Wasserstein distance $W_1$, the empirical convergence rate under $W_1^{(\sigma)}$ is $n^{-1/2}$ in all dimensions. Inspired by this fact, the present paper conducts an in-depth study of the statistical properties of the smooth Wasserstein distance $W_1^{(\sigma)}$. We derive the limit distribution of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$ for all $d$, where $P_n$ is the empirical measure of $n$ independent observations from $P$. In arbitrary dimension, the limit is characterized as the supremum of a tight Gaussian process indexed by 1-Lipschitz functions convolved with a Gaussian density. Building on this result we derive concentration inequalities, bootstrap consistency, and explore generative modeling with $W_1^{(\sigma)}$ under the minimum distance estimation framework. For the latter, we derive measurability, almost sure convergence, and limit distributions for optimal generative models and their corresponding smooth Wasserstein error. These results promote the smooth Wasserstein distance $W_1^{(\sigma)}$ as a powerful tool for learning and statistical inference in high dimensions.

Keywords: Empirical process, Gaussian kernel, Gaussian process, limit distribution, minimum Wasserstein estimator, Wasserstein distance.

1. Introduction

The Wasserstein distance has seen a surge of applications in machine learning (ML). This includes data clustering (Ho et al., 2017), domain adaptation (Courty et al., 2016, 2014), generative modeling (Arjovsky et al., 2017; Gulrajani et al., 2017; Tolstikhin et al., 2018; Adler and Lunz, 2018), image recognition (Rubner et al., 2000; Sandler and Lindenbaum, 2011; Li et al., 2013), and word and document embedding (Alvarez-Melis and Jaakkola, 2018; Yurochkin et al., 2019; Grave et al., 2019). The popularity of the Wasserstein distance stems from its many beneficial attributes, such as metric structure, robustness to support mismatch, compatibility to gradient-based optimization via dual representation, and rich geometric properties (e.g., it defines a constant-speed geodesic on the space of probability measures). These attributes are especially useful for generative modeling (Arjovsky et al., 2017; Gulrajani et al., 2017), which is the chief application we consider based on the theory developed herein.

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The Wasserstein distance is a special case of the optimal transport (OT) problem, which dates back to Monge in the late 18th century (Monge, 1781). However, it was only until Kantorovich’s seminal work (Kantorovich, 1942) that a well-posed mathematical framework was devised. The Kantorovich OT problem between Borel probability measures $P$ and $Q$ on $\mathbb{R}^d$ with cost $c(x, y)$ is defined by

$$\inf_{\pi \in \Pi(P, Q)} \int c(x, y) \, d\pi(x, y), \quad (1)$$

where $\Pi(P, Q)$ is the set of couplings (or transport plans) between $P$ and $Q$. The 1-Wasserstein distance ($W_1$) (Villani, 2008, Chapter 6), on which we focus here, is the value (1) for the Euclidean distance cost $c(x, y) = ||x - y||$. While the Wasserstein distance $W_1$ is a popular distance measure between probability distributions, the ability to approximate it from samples (as typically required in data science) drastically deteriorates in high dimensions. This fact is called ‘curse of dimensionality’ for the empirical approximation.

Empirical approximation refers to the problem of approximating in $W_1$ a probability measure $P$ with its empirical version $P_n := n^{-1} \sum_{i=1}^n \delta_{X_i}$ based on $n$ independently and identically distributed (i.i.d.) observations $X_1, \ldots, X_n$ from $P$. The convergence rate of the expected distance is

$$\mathbb{E}[W_1(P_n, P)] \lesssim \begin{cases} n^{-1/2} & \text{if } d = 1 \\ n^{-1/2} \log n & \text{if } d = 2, \\ n^{-1/d} & \text{if } d \geq 3 \end{cases}, \quad (2)$$

up to constants independent of $n$ for any $P$ with sufficiently many moments (Fournier and Guillin, 2015, Theorem 1). These rates are known to be sharp. Arguably, the $n^{-1/d}$ rate is slow even for moderately large $d$ (say, $d = 5$), which renders $W_1$ as all but impossible to estimate from the sample in high dimensions.

To alleviate this impasse, Goldfeld et al. (2019) recently proposed the Gaussian-smoothed OT framework, where $W_1$ is measured between distributions after they are convolved with an isotropic Gaussian distribution. Specifically, for any $\sigma > 0$, Goldfeld et al. (2019) proposed the Gaussian-smoothed Wasserstein distance

$$W_1^{(\sigma)}(P, Q) := W_1(P \ast \mathcal{N}_\sigma, Q \ast \mathcal{N}_\sigma),$$

where $\mathcal{N}_\sigma = \mathcal{N}(0, \sigma^2 I_d)$ is the $d$-dimensional isotropic Gaussian measure with parameter $\sigma$, and $P \ast \mathcal{N}_\sigma$ denotes the convolution between $P$ and $\mathcal{N}_\sigma$. Goldfeld and Greenewald (2020) showed that $W_1^{(\sigma)}$ inherits the metric structure of $W_1$ and induces the same topology on the space of probability measures as $W_1$. In addition, Goldfeld et al. (2019) established that it alleviates the curse of dimensionality, converging as $\mathbb{E}[W_1^{(\sigma)}(P_n, P)] = O(n^{-1/2})$ in all dimensions under a sub-Gaussian condition on $P$. Motivated by this rate result, the present paper studies the limit distribution of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$, and develops limit theory for the minimum distance estimation under $W_1^{(\sigma)}$. Our main contributions are delineated next.

1.1. Limit distribution theory for smooth Wasserstein distance

Given the rate result (2), one natural question would be whether, when properly scaled, $W_1(P_n, P)$ has a limit distribution. A proof of existence and characterization of the limit is available, however, only in the one-dimensional case (del Barrio et al., 1999). Indeed, when $d = 1$, we have
Limit Distribution Theory for Smooth $W_1$

$W_1(P_n, P) = \|F_n - F\|_{L^1(\mathbb{R})}$, where $F_n$ and $F$ are the distribution functions of $P_n$ and $P$, respectively, and $\sqrt{n}(F_n - F)$ satisfies the central limit theorem (CLT) in $L^1(\mathbb{R})$ under mild moment conditions. To the best our knowledge, it is unknown whether $n^{1/d}W_1(P_n, P)$ has a nondegenerate limit in general.

This state of affairs changes when adopting $W_1^{(\sigma)}$ as the figure of merit. In a sharp contrast to the original $W_1$ case, our main result establishes the convergence in distribution of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$ to the supremum of a tight Gaussian process for all $d$. Our analysis relies on the Kantorovich-Rubinstein (KR) duality for $W_1$ (cf. Villani, 2008), which implies that

$$W_1(P_n, P) = \sup_{f \in \text{Lip}_1(\mathbb{R})} \frac{1}{n} \sum_{i=1}^n (f(X_i) - Pf),$$

where $\text{Lip}_1(\mathbb{R})$ denotes the set of Lipschitz continuous functions on $\mathbb{R}$ with Lipschitz constant bounded by 1, and $Pf = \int f \, dP$. Thus, the empirical smooth Wasserstein distance $W_1^{(\sigma)}(P_n, P)$ can be seen as the supremum of the empirical process indexed by the function class comprising 1-Lipschitz functions convoluted with a Gaussian density. Leveraging the result of van der Vaart (1996), we show that this indexing function class is Donsker (i.e., satisfies the uniform central limit theorem) under a polynomial moment condition on $P$. By the continuous mapping theorem, the Donsker result implies that $\sqrt{n}W_1^{(\sigma)}(P_n, P)$ converges in distribution to the supremum of a tight Gaussian process. We also show that the nonparametric bootstrap is able to consistently estimate the distributional limit of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$.

Deriving distributional limits for empirical Wasserstein distances has been an active research topic in the recent statistics and probability theory literature. Sommerfeld and Munk (2018) and Tameling et al. (2019) consider the empirical $p$-Wasserstein distances $\sqrt{n}W_p(P_n, P)$ when the target measure $P$ is supported on finite or countable spaces, respectively. Using the directional functional delta method (Dumbgen, 1993), they derive limit distributions for $1 \leq p < \infty$. The recent paper by del Barrio and Loubes (2019) shows asymptotic normality of $\sqrt{n}(W_2(P_n, Q) - \mathbb{E}[W_2(P_n, Q)])$, in arbitrary dimension, but under the assumption that $Q \neq P$. The limit distribution for the empirical 2-Wasserstein distance with $Q = P$ is known only when $d = 1$ (del Barrio et al., 2005). The key observation therein is that when $d = 1$, the empirical 2-Wasserstein distance coincides with the $L^2$ distance between the empirical and population quantile functions, resulting in a non-Gaussian limit distribution. Importantly, none of the techniques employed in these works are applicable in our case, which therefore requires a different analysis as described above.

1.2. Generative modeling with smooth Wasserstein distance

As an application of the limit distribution result we consider generative modeling. Assume that $X_1, \ldots, X_n$ are i.i.d. random variables from a Borel probability measure $P$ on $\mathbb{R}^d$. The goal is to use this sample to learn a generative model $Q_\theta$, $\theta \in \Theta$, that approximates $P$ under a statistical divergence $\delta$:

$$\hat{\theta}_n \in \arg\min_{\theta \in \Theta} \delta(P_n, Q_\theta).$$

1. The reader is referred to, e.g., Ledoux and Talagrand (1991); van der Vaart and Wellner (1996); Giné and Nickl (2016) as useful references on modern empirical process theory.
2. Recall that $\delta$ is a statistical divergence if $\delta(P, Q) = 0 \iff P = Q$. 


We consider here the case where $\delta = W_1^{(\sigma)}$. This framework can be viewed as approximating a kernel-density estimator of $P$ by a smoothed parametric model $Q_\theta + \mathcal{N}_\sigma$. The KR duality relates the optimization problem $\inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta)$ to the minimax formulation of generative adversarial networks (GANs) (Goodfellow et al., 2014; Arjovsky et al., 2017). This correspondence, which remains valid for $W_1^{(\sigma)}$, is central to the practical success of Wasserstein GANs (WGANs).

The setup in (4) with the original Wasserstein distance $W_1$ is called minimum Wasserstein estimation (MWE) (Bassetti et al., 2006; Bernton et al., 2019); see also Belili et al. (1999); Bassetti and Regazzini (2006). We thus refer to the $\delta = W_1^{(\sigma)}$ case as smooth-MWE (S-MWE). As for the original Wasserstein distance $W_1$, the limit distribution of the MWE is known only when $d = 1$ (Bernton et al., 2019). Remarkably, the characterized limit of the empirical smooth Wasserstein distance allows us to provide a thorough asymptotic analysis of the S-MWE in all dimensions.

Specifically, we establish almost sure convergence of the infimum and argmin of $W_1^{(\sigma)}(P_n, Q_\theta)$ towards those of $W_1^{(\sigma)}(P, Q_\theta)$, and characterize the limit distributions of $\sqrt{n}\inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta)$ and $\sqrt{n}(\hat{\theta}_n - \theta^*)$ when $P = Q_{\theta^*}$ for some $\theta^* \in \Theta$. Building on the argument from Bernton et al. (2019, Section B.2.2), the result naturally extends to the ‘misspecified’ setup where $P$ does not belong to the parametric family. The S-MWE limit distribution proofs are based on the method of Pollard (1980) for minimum distance estimation analysis over normed spaces. We view the Gaussian-smoothed empirical process as a random element with values in the space of bounded functionals on the 1-Lipschitz functions, equipped with the uniform norm. Observing that the dual form of $W_1^{(\sigma)}(P_n, P)$ (cf. Equation (3)) coincides with that uniform norm, the weak convergence of $\sqrt{n} W_1^{(\sigma)}(P_n, P)$ is leveraged to control $\sqrt{n}\inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta)$ and $\sqrt{n}(\hat{\theta}_n - \theta^*)$. This highlights the benefit of $W_1^{(\sigma)}$ compared to classic $W_1$ for high-dimensional analysis and applications.

1.3. Notation

Let $\| \cdot \|$ denote the Euclidean norm, and $x \cdot y$, for $x, y \in \mathbb{R}^d$, designate the inner product of $x$ and $y$. For any probability measure $Q$ on a measurable space $(S, \mathcal{S})$ and any measurable real function $f$ on $S$, we use the notation $Q f := \int_S f \, dQ$ whenever the integral exists. We write $a \lesssim b$ when $a \leq C b$ for a constant $C$, that depends only on $x$ ($a \lesssim b$ means $a \leq C b$ for an absolute constant $C$).

In the present paper, we denote by $(\Omega, \mathcal{A}, \mathbb{P})$ the underlying probability space on which all random variables are defined. The class of Borel probability measures on $\mathbb{R}^d$ is denoted by $\mathcal{P}(\mathbb{R}^d)$. The subset of measures with finite first moment is denoted by $\mathcal{P}_1(\mathbb{R}^d)$, i.e., $P \in \mathcal{P}_1(\mathbb{R}^d)$ whenever $\int \|x\| \, dP(x) < \infty$. The convolution of $P, Q \in \mathcal{P}(\mathbb{R}^d)$ is denoted by $(P * Q)(A) := \int \int I_A(x + y) \, dP(x) \, dQ(y)$, where $I_A$ is the indicator of $A$. The convolution of measurable functions $f, g$ on $\mathbb{R}^d$ is denoted as $f * g(x) = \int f(x - y) g(y) \, dy$. We use the shorthand $\mathcal{N}_\sigma := \mathcal{N}(0, \sigma^2 I_d)$ for the isotropic Gaussian measure of parameter $\sigma$, and $\varphi_\sigma$ for its density function $\varphi_\sigma(x) = (2\pi\sigma^2)^{-d/2} e^{-\|x\|^2/(2\sigma^2)}$, where $x \in \mathbb{R}^d$.

For any non-empty set $\mathcal{T}$, let $\ell^\infty(\mathcal{T})$ denote the space of all bounded functions $f : \mathcal{T} \to \mathbb{R}$, equipped with the uniform norm $\|f\|_{\ell^\infty(\mathcal{T})} := \sup_{t \in \mathcal{T}} |f(t)|$. We use $\text{Lip}_1(\mathbb{R}^d)$ for the set of Lipschitz continuous functions on $\mathbb{R}^d$ with Lipschitz constant bounded by 1, i.e., $\text{Lip}_1(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R} : |f(x) - f(y)| \leq \|x - y\| \forall x, y \in \mathbb{R}^d\}$. When $d$ is clear from the context we use the shorthand $\text{Lip}_1$ for $\text{Lip}_1(\mathbb{R}^d)$. A stochastic process $G := (G(t))_{t \in \mathcal{T}}$ indexed by $\mathcal{T}$ is Gaussian if $(G(t_i))_{i=1}^k$.

3. The proof of Bernton et al. (2019) for $d = 1$ relies on the limit distribution result from del Barrio et al. (1999).
are jointly Gaussian for any finite collection \( \{t_i\}_{i=1}^k \subset T \). If \( G \) is sample bounded, we view it as a mapping from the sample space into \( \ell^\infty(T) \).

2. Background and preliminaries

Recall that the 1-Wasserstein distance \( W_1(P, Q) \) for \( P, Q \in \mathcal{P}_1(\mathbb{R}^d) \) is defined by

\[
W_1(P, Q) := \inf_{\pi \in \Pi(P, Q)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \|x - y\| \, d\pi(x, y),
\]

where \( \Pi(P, Q) \) is the set of all couplings of \( P \) and \( Q \). The KR duality yields that

\[
W_1(P, Q) = \sup_{f \in \text{Lip}_1} \int_{\mathbb{R}^d} f(P - Q).
\]

See Dudley (2002, Theorem 11.8.2) and Villani (2008, Remark 6.5).

2.1. Empirical approximation and limit distribution

For given \( P \in \mathcal{P}_1(\mathbb{R}^d) \), let \( X_1, \ldots, X_n \sim P \) be i.i.d. Let \( P_n = n^{-1} \sum_{i=1}^n \delta_{X_i} \) be the empirical distribution of \( X_1, \ldots, X_n \), where \( \delta_x \) is the Dirac measure at \( x \). Convergence in \( W_1 \) is equivalent to weak convergence plus convergence of the first moment (Villani, 2008, Corollary 6.18). It is therefore not hard to see from the Varadharajan theorem (Dudley, 2002, Theorem 11.4.1) and the strong law of large numbers that \( W_1(P_n, P) \to 0 \) as \( n \to \infty \) almost surely (a.s.) without any additional assumptions. The convergence rate of the empirical Wasserstein distance received much attention in the literature; see, e.g., Dudley (1969); Bolley et al. (2007); Boissard (2011); Dereich et al. (2013); Boissard and Guic (2014); Fournier and Guillin (2015); Weed and Bach (2019); Lei (2020). Sharp rates for \( W_1(P_n, P) \) are known in all dimensions\(^5\); see equation (2) extracted from Fournier and Guillin (2015). In contrast, limiting distribution results for scaled \( W_1(P_n, P) \) are known only for \( d = 1 \).

Indeed, Theorem 2 in Giné and Zinn (1986) yields that, when \( d = 1 \), the class \( \text{Lip}_1(\mathbb{R}) \) is \( P \)-Donsker if and only if \( \sum_j P([-j, j])^{1/2} < \infty \), which is satisfied if \( P \) has finite \( 2 + \epsilon \) moment for some \( \epsilon > 0 \). Under this condition, from the KR duality (cf. equation (3)), we have

\[
\sqrt{n}W_1(P_n, P) \overset{d}{\to} \sup_{f \in \text{Lip}_1(\mathbb{R})} G_P(f) \tag{5}
\]

for some tight Gaussian process \( G_P \) in \( \ell^\infty(\text{Lip}_1(\mathbb{R})) \). This also shows that the \( n^{-1/2} \) rate is sharp for \( W_1(P_n, P) \) when \( d = 1 \). In addition, by del Barrio et al. (1999), \( \sup_{f \in \text{Lip}_1(\mathbb{R})} G_P(f) \overset{d}{=} \int_0^1 B(F(t)) \, dt \), where \( (B(t))_{t \in [0, 1]} \) is a Brownian bridge and \( F(t) = P((-\infty, t]) \) denotes the cumulative distribution function of \( P \). Indeed, in the \( d = 1 \) case, we have \( W_1(P_n, P) = \|F_n - F\|_{L^1(\mathbb{R})} \), where \( F_n(t) = P_n((-\infty, t]) \) is the empirical distribution function. Thus, weak convergence in (5) also follows from the fact that \( \sqrt{n}(F_n - F) \) satisfies the CLT in \( L^1(\mathbb{R}) \) (see del Barrio et al. (1999) for details). del Barrio et al. (1999) also show that the condition \( \sum_j P([-j, j])^{1/2} < \infty \) is necessary for \( \sqrt{n}W_1(P_n, P) \) to be stochastically bounded.

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\(^4\) Those references also contain results on the more general Wasserstein distance and non-Euclidean spaces.

\(^5\) Except \( d = 2 \), where a log factor is possibly missing.
The arguments in those papers, however, do not carry over to general $d$. For $d \geq 2$, in general, the function class Lip$_1(\mathbb{R}^d)$ is no longer Donsker; if it was, then $\mathbb{E}[W_1(P_n, P)]$ would be of order $O(n^{-1/2})$, contradicting existing results on lower bounds on the rate of convergence of $W_1(P_n, P)$. See the discussion after Theorem 1 in Fournier and Guillin (2015).

2.2. Smooth Wasserstein distance

We are interested in the case of $d \geq 2$, and consider instead the smooth $W_1$ distance as in Goldfeld et al. (2019); Goldfeld and Greenewald (2020): $W_1^{(\sigma)}(P, Q) := W_1(P * \mathcal{N}_\sigma, Q * \mathcal{N}_\sigma)$. Goldfeld et al. (2019) show that $W_1^{(\sigma)}(P_n, P) = O_P(n^{-1/2})$ for all $d$ and any sub-Gaussian $P$.

Our first goal is to characterize the limit distribution and derive concentration inequalities for $\sqrt{n}W_1^{(\sigma)}(P_n, P)$, while relaxing the sub-Gaussian assumption on $P$. Corollary 1 of Goldfeld and Greenewald (2020) establishes Gaussian concentration of $W_1^{(\sigma)}(P_n, P)$ when $P$ has bounded support, but we generalize and strengthen this result herein. To simplify discussions, henceforth we assume $0 < \sigma \leq 1$.

3. Limit distribution theory for smooth Wasserstein distance

We first derive a limit distribution of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$. Let Lip$_{1,0} := \{f \in$ Lip$_1 : f(0) = 0\}$. Assume that $P\|x\|^2 < \infty$. Let $G^{(\sigma)}_P = (g^{(\sigma)}_P(f))_{f \in \text{Lip}_{1,0}}$ be a centered Gaussian process with covariance function

$$\mathbb{E} \left[ G^{(\sigma)}_P(f)G^{(\sigma)}_P(g) \right] = \text{Cov}_P(f * \varphi_\sigma, g * \varphi_\sigma), \quad \forall f, g \in \text{Lip}_{1,0}.$$  

It is not difficult to see that $|f * \varphi_\sigma(x)| \leq \|x\| + \sigma \sqrt{d}$ (cf. Section A.1), so that $P|f * \varphi_\sigma|^2 < \infty$ for all $f \in \text{Lip}_{1,0}$ (which ensures the covariance function above to be well-defined). A Gaussian process $G = (G(t))_{t \in \mathcal{T}}$ is tight in $\ell^\infty(\mathcal{T})$ if and only if $\mathcal{T}$ is totally bounded for the pseudo-metric $d_G(s, t) = \sqrt{\mathbb{E} \left[ |G(s) - G(t)|^2 \right]}$ and $G$ has sample paths a.s. uniformly $d_G$-continuous (van der Vaart and Wellner, 1996, Section 1.5). A version of a stochastic process is another stochastic process with the same finite dimensional distributions.

The following theorem characterizes the limit distribution of $\sqrt{n}W_1^{(\sigma)}(P_n, P)$ in all dimensions.

**Theorem 1 (Limit distribution for $W_1^{(\sigma)}$)** Assume that $P\|x\|^2 < \infty$. Let $\mathbb{R}^d = \bigcup_{j=1}^{\infty} I_j$ be a partition of $\mathbb{R}^d$ into bounded convex sets with nonempty interior such that $K := \sup_j \text{diam}(I_j) < \infty$. If

$$\sum_{j=1}^{\infty} M_j P(I_j)^{1/2} < \infty \quad \text{with} \quad M_j := \sup_{I_j} \|x\|,$$

then there exists a version of $G^{(\sigma)}_P$ that is tight in $\ell^\infty(\text{Lip}_{1,0})$, and denoting the tight version by the same symbol $G^{(\sigma)}_P$, we have $\sqrt{n}W_1^{(\sigma)}(P_n, P) \overset{d}{\to} \sup_{f \in \text{Lip}_{1,0}} G^{(\sigma)}_P(f) =: L^{(\sigma)}_P$. In addition, we have

$$\sqrt{n} \mathbb{E} \left[ W_1^{(\sigma)}(P_n, P) \right] \lesssim_{d,K} \sigma^{-[d/2]} \sum_{j=1}^{\infty} M_j P(I_j)^{1/2}.$$  

(7)
The theorem is proved in Appendix A.1. The key idea is to use the KR duality and translate the Gaussian convolution in the measure space to the convolution of Lipschitz functions with a Gaussian density. We then show that this class of Gaussian-smoothed Lipschitz functions is $P$-Donsker by bounding the metric entropy of the function class restricted to each $I_j$. The proof of the theorem thus substantially relies on empirical process theory, and significantly differs from the techniques used in Goldfeld et al. (2019).

**Remark 2 (Discussion on Condition (6))** Let $\{I_j\}$ consist of cubes with side length 1 and integral lattice points as vertices. Then, it is not difficult to see that

$$
\sum_{j=1}^{\infty} M_j P(I_j)^{1/2} \leq \sum_{k=1}^{\infty} k^d P(\|x\|_\infty > k)^{1/2} \lesssim \int_1^{\infty} t^d P(\|x\|_\infty > t)^{1/2} \, dt.
$$

By Markov’s inequality, the last term is finite if there exists an $\epsilon > 0$ such that $P|x_j|^{2(d+1)+\epsilon} < \infty$ for all $j$. Alternatively, we can state a sufficient condition for (6) by using the Lorentz norm. The Lorentz (quasi-)norm $\|\xi\|_{p,q}$ for $0 < p,q < \infty$ and a real-valued random variable $\xi$ is defined by $\|\xi\|_{p,q} := \left(q \int_0^{\infty} (t^p P(\xi > t))^{q/p} \, dt \right)^{1/q}$. See Ledoux and Talagrand (1991, p. 279). Thus, Condition (6) holds if $\|X\|_{2(d+1),d+1} < \infty$ for all $j$, where $X \sim P$.

**Remark 3 (Relaxed condition for expected value convergence)** Proposition 1 in Goldfeld et al. (2019) shows that $E[\mathcal{W}_{1}^{(\sigma)}(P_n,P)] = O(n^{-1/2})$ under a sub-Gaussian condition on $P$. Theorem 1 substantially relaxes this moment condition, in addition to deriving a limit distribution.

The distribution of the limit variable $L^{(\sigma)}_P$ is in general intractable, but we can deduce the following (see Section A.2 for the proof).

**Lemma 4 (Distribution of $L^{(\sigma)}_P$)** Assume the conditions of Theorem 1 and that $P$ is not a point mass. Then the distribution of $L^{(\sigma)}_P$ is absolutely continuous with respect to (w.r.t.) Lebesgue measure and its density is positive and continuous on $(0,\infty)$ except for at most countably many points.

The proof of Theorem 1, combined with Lemma 4, implies that we can estimate the distribution of $L^{(\sigma)}_P$ by the bootstrap (van der Vaart and Wellner, 1996, Chapter 3.6). Let $X^B_1, \ldots, X^B_n$ denote an i.i.d. sample from $P_n$ conditionally on $X_1, \ldots, X_n$, and let $P^B = n^{-1} \sum_{i=1}^{n} X^B_i$ denote the empirical distribution of the bootstrap sample. Let $\mathbb{P}^B$ denote the probability measure induced by the bootstrap (i.e., the conditional probability given $X_1, X_2, \ldots$).

**Corollary 5 (Bootstrap consistency)** Assume the conditions of Theorem 1 and that $P$ is not a point mass. Then, we have $\sup_{t \geq 0} |\mathbb{P}^B(\sqrt{n} \mathcal{W}_1^{(\sigma)}(P^B_n, P_n) \leq t) - \mathbb{P}(L^{(\sigma)}_P \leq t)| \to 0$ a.s.

This corollary, together with continuity of the distribution function of $L^{(\sigma)}_P$, implies that for $\tilde{\alpha} := \inf \{ t \geq 0 : \mathbb{P}(\sqrt{n} \mathcal{W}_1^{(\sigma)}(P^B_n, P_n) \leq t) \geq 1 - \alpha \}$ (which can be computed numerically), we have $\mathbb{P}(\sqrt{n} \mathcal{W}_1^{(\sigma)}(P^B_n, P) > \tilde{\alpha}) = \alpha + o(1)$.

Next, we consider a quantitative concentration inequality for $\mathcal{W}_1^{(\sigma)}(P_n,P)$. For $\alpha > 0$, let $\|\xi\|_{\psi_\alpha} := \inf \{ C > 0 : E[e^{\|\xi\|_{\psi_\alpha}}] \leq 2 \}$ be the Orlitz $\psi_\alpha$-norm for a real-valued random variable $\xi$ (if $\alpha \in (0,1)$, then $\|\cdot\|_{\psi_\alpha}$ is a quasi-norm). In Section A.4 we prove the following.
Corollary 6 (Concentration inequality) Assume $\mathbb{E}[W_1^{(\sigma)}(P_n, P)] < \infty$. The following hold:

(i) If $P$ is compactly supported with support $X$, then

$$
\mathbb{P} \left( W_1^{(\sigma)}(P_n, P) \geq \mathbb{E}[W_1^{(\sigma)}(P_n, P)] + t \right) \leq e^{-\frac{nt^2}{C \text{diam}(X)^2}}, \quad \forall t > 0.
$$

(ii) If $\|X\|_{\psi_\alpha} < \infty$ for some $\alpha \in (0, 1]$, where $X \sim P$, then for any $\eta > 0$, there exists a constant $C = C_{\eta, \alpha}$ depending only on $\eta, \alpha$ such that

$$
\mathbb{P} \left( W_1^{(\sigma)}(P_n, P) \geq (1 + \eta)\mathbb{E}[W_1^{(\sigma)}(P_n, P)] + t \right) \leq \exp \left( -\frac{nt^2}{C(\|x\|^2 + \sigma^2 d)} \right) + 3 \exp \left( -\frac{nt}{C \left( \max_{1 \leq i \leq n} \|X_i\|_{\psi_\alpha} + \sigma \sqrt{d} \right)^{\alpha} } \right), \quad \forall t > 0.
$$

(iii) If $\|X\|_q < \infty$ for some $q \in [1, \infty)$, then for any $\eta > 0$, there exists a constant $C = C_{\eta, q}$ depending only on $\eta, q$ such that

$$
\mathbb{P} \left( W_1^{(\sigma)}(P_n, P) \geq (1 + \eta)\mathbb{E}[W_1^{(\sigma)}(P_n, P)] + t \right) \leq \exp \left( -\frac{nt^2}{C(\|x\|^2 + \sigma^2 d)} \right) + \frac{C(\mathbb{E}[\max_{1 \leq i \leq n} \|X_i\|^q] + \sigma^q d^{q/2})}{nt^{q/2}}, \quad \forall t > 0.
$$

We next apply the results of this section to generative modeling via S-MWE.

4. Generative Modeling

Consider the unsupervised learning task of generative modeling. Let $X_1, \ldots, X_n$ be an i.i.d. sample from $P \in \mathcal{P}_1(\mathbb{R}^d)$. The goal is to use this sample to learn a generative model $Q_\theta$, where $\theta \in \Theta \subset \mathbb{R}^{d_\theta}$, that approximates $P$ under a statistical divergence $\delta$. Namely, an optimal (w.r.t. $\delta$) generative model is given by a parameter satisfying:

$$
\theta_\star \in \arg\min_{\theta \in \Theta} \delta(P, Q_\theta).
$$

Henceforth, we focus on the 1-Wasserstein distance and its smooth version in the role of $\delta$.

4.1. Generative modeling with Wasserstein distance

The 1-Wasserstein distance has received much attention in generative modeling literature (Arjovsky et al., 2017; Gulrajani et al., 2017; Tolstikhin et al., 2018; Adler and Lunz, 2018). This popularity stems from several beneficial properties that $W_1$ possesses. First, it is robust to support mismatch, i.e., $W_1(P, Q)$ is a meaningful distance measure between any $P, Q \in \mathcal{P}_1(\mathbb{R}^d)$ even when $\text{supp}(P) \cap \text{supp}(Q) = \emptyset$. This stands in contrast to other statistical divergences, e.g., $f$-divergences (Csiszár and Shields, 2004), which become vacuous when the supports are not aligned. Such robustness is crucial in
practice since supp(Q_\theta) oftentimes deviates from supp(P), especially in the first iterations of optimization. Robustness avoids unwanted pathologies during optimization.

A second reason central to the popularity of W_1 is its compatibility to generative adversarial networks (GANs) (Goodfellow et al., 2014). A GAN is an implementation of the minimax objective

$$\inf_{\theta \in \Theta} \sup_{f \in \mathcal{F}} \mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{Y \sim Q_\theta}[f(Y)],$$

(9)

where f is a discriminator function that aims to maximally differentiate inputs X \sim P from the true data distribution from (synthetic) inputs Y \sim Q_\theta generated by the model.\(^6\) The generator Q_\theta and the discriminator f are iteratively optimized until convergence, yielding a Q_\theta that is indistinguishable from P even by an optimal discriminator. Interestingly, if \delta = W_1, then the principled form (8) and the implementable minimax framework (9) coincide. Indeed, the KR duality implies that

$$\inf_{\theta \in \Theta} W_1(P, Q_\theta) = \inf_{\theta \in \Theta} \sup_{f \in \text{Lip}_1(\mathbb{R}^d)} \mathbb{E}_{X \sim P}[f(X)] - \mathbb{E}_{Y \sim Q_\theta}[f(Y)],$$

(10)

where \mathcal{F} = \text{Lip}_1(\mathbb{R}^d) is the discriminator class. GANs with 1-Lipschitz discriminators, termed WGANs (Arjovsky et al., 2017), attain state-of-the-art performance and are widely used in practice.

4.2. Relation to smooth Wasserstein distance

Since we only have access to the sample X_1, \ldots, X_n from the data distribution, we have to replace P in (10) with an estimator \hat{P}_n:

$$\inf_{\theta \in \Theta} W_1(\hat{P}_n, Q_\theta).$$

(11)

We next relate the optimization problem (11) to the smooth Wasserstein distance W_1(\sigma). The model class is chosen by the system designer, so we set it to Gaussian smoothed distributions Q_\theta, i.e., Q_\theta * N_\sigma, for \theta \in \Theta. Next, we approximate P by a Gaussian mixture P_n * N_\sigma.\(^7\) Here, \sigma is a hyperparameter, which is fixed while optimizing over \Theta. With these choices, the optimization problem (11) becomes \inf_{\theta \in \Theta} W_1(\sigma)(P_n, Q_\theta), which is the object of interest in this section.

Goldfeld and Grenewald (2020) showed that W_1(\sigma) shares many of the properties of classic W_1: (i) W_1(\sigma) is a metric on P_1(\mathbb{R}^d) that induces the same topology as W_1; (ii) it is robust to support mismatch; (iii) it is a continuous and monotonically nonincreasing function of \sigma \in [0, \infty); and (iv) it fits the GAN framework just as W_1 does, since W_1(\sigma) is simply W_1 between perturbed distributions.

In what follows, we study almost sure convergence of the infimum and argmin solutions of W_1(\sigma)(P_n, Q_\theta) towards those of W_1(\sigma)(P, Q_\theta), and characterize their limit distributions in all dimensions. Similar questions were addressed for classic W_1 and the sliced Wasserstein distance in Bernton et al. (2019) and Nadjaï et al. (2019), respectively. The former developed limit distribution results only for d = 1 (since a limit distribution for W_1(P_n, P) is known only in the one-dimensional case). For Nadjaï et al. (2019), results are given for arbitrary d. However, recalling that the sliced Wasserstein distance is an average of one-dimensional Wasserstein distances (via projections of d-dimensional measures), their results rely heavily on the limit distribution result of \sqrt{n}W_1(P_n, P) in d = 1. Remarkably, while W_1(\sigma) involves no dimensionality reduction of the high-dimensional distribution, the Gaussian smoothing allows us to develop a through asymptotic analysis of the S-MWE for arbitrary d.

\(^6\) In practice, the class \mathcal{F} is parametrized by a deep neural network (DNN) f_\phi, with parameters \phi \in \Phi.

\(^7\) If P has a density, then P_n * N_\sigma amounts to a kernel density estimator with a Gaussian kernel of width \sigma.
4.3. Measurability and consistency results

Let $P \in \mathcal{P}_1(\mathbb{R}^d)$ and assume that $\{Q_\theta: \theta \in \Theta\} \subset \mathcal{P}_1(\mathbb{R}^d)$. In addition, we henceforth will assume (without further mentioning) that the parameter space $\Theta \subset \mathbb{R}^{d_0}$ is compact with nonempty interior. This is a standard assumption in asymptotic statistics, but we note that boundedness of $\Theta$ can be replaced by weaker assumptions with some adjustments to the proofs of Theorems 8 and 9 below.

We first consider measurability of argmin solutions. Recall the definition of the weak topology on $\mathcal{P}(\mathbb{R}^d)$.

**Definition 7 (Weak topology on $\mathcal{P}(\mathbb{R}^d)$)** The weak topology on $\mathcal{P}(\mathbb{R}^d)$ is induced by integration against the set $C_b(\mathbb{R}^d)$ of bounded and continuous functions. We say that $(\mu_k)_{k \in \mathbb{N}}$ converges weakly to $\mu$, if $\mu_k \to \mu$, if $\mu_k(f) \to \mu(f)$ for all $f \in C_b(\mathbb{R}^d)$.

The following theorem is proved in Section B.2. The proof relies on Corollary 1 in Brown and Purves (1973), which provides a sufficient condition for the desired measurability.

**Theorem 8 (S-MWE measurability)** Assume that the map $\theta \mapsto Q_\theta$ is continuous relative to the weak topology, i.e., $Q_\theta \to Q_\bar{\theta}$ whenever $\theta \to \bar{\theta}$ in $\Theta$. Then, for every $n \in \mathbb{N}$, there exists a measurable function $\omega \mapsto \bar{\theta}_n(\omega)$ such that $\bar{\theta}_n(\omega) \in \text{argmin}_{\theta \in \Theta} W_1^\sigma(P_n(\omega), Q_\theta)$ for every $\omega \in \Omega$ (this also implies that $\text{argmin}_{\theta \in \Theta} W_1^\sigma(P_n(\omega), Q_\theta)$ is nonempty).

Next, we consider almost sure convergence of the infimum and argmin solutions of $W_1^\sigma(P_n, Q_\theta)$ to those of $W_1^\sigma(P, Q_\theta)$.

**Theorem 9 (S-MWE consistency)** Assume that the map $\theta \mapsto Q_\theta$ is continuous relative to the weak topology. Then, we have $\inf_{\theta \in \Theta} W_1^\sigma(P_n, Q_\theta) \to \inf_{\theta \in \Theta} W_1^\sigma(P, Q_\theta)$ a.s. In addition, there exists an event with probability one on which the following holds: for any sequence $\{\bar{\theta}_n\}_{n \in \mathbb{N}}$ of measurable estimators such that $W_1^\sigma(P_n, Q_{\bar{\theta}_n}) \leq \inf_{\theta \in \Theta} W_1^\sigma(P_n, Q_\theta) + o(1)$, the set of cluster points of $\{\bar{\theta}_n\}_{n \in \mathbb{N}}$ is included in $\text{argmin}_{\theta \in \Theta} W_1^\sigma(P, Q_\theta)$. In particular, if $\text{argmin}_{\theta \in \Theta} W_1^\sigma(P, Q_\theta)$ is unique, i.e., $\text{argmin}_{\theta \in \Theta} W_1^\sigma(P, Q_\theta) = \{\theta^*\}$, then $\bar{\theta}_n \to \theta^*$ a.s.

The proof relies on Theorem 7.33 in Rockafellar and Wets (2009). To apply the theorem, we verify epi-convergence of the extended version of the map $\theta \mapsto W_1^\sigma(P_n, Q_\theta)$ toward that of $\theta \mapsto W_1^\sigma(P, Q_\theta)$. See Section B.3 for details.

4.4. Limit distribution results

We study limit distributions in the S-MWE framework. Results are presented for the ‘well-specified’ setting, i.e., when $P = Q_\theta$, for some $\theta^*$ in the interior of $\Theta \subset \mathbb{R}^{d_0}$. We note that extensions to the ‘misspecified’ setting is straightforward by following the lines of Bernton et al. (2019, Theorem B.8). Our derivation leverages the method from Pollard (1980) for analysis of minimum distance estimation over normed spaces. To make the connection, we need some definitions.

Recall that $\text{Lip}_{1,0} := \{f \in \text{Lip}_1: f(0) = 0\}$ and for any $G \in \mathcal{C}^\infty(\text{Lip}_{1,0})$, define $\|G\|_{\text{Lip}_{1,0}} := \sup_{f \in \text{Lip}_{1,0}} |G(f)|$. With any $Q \in \mathcal{P}_1(\mathbb{R}^d)$, associate the function $Q^\sigma: \text{Lip}_{1,0} \to \mathbb{R}$ by $Q^\sigma(f) := Q(f \ast \varphi_\sigma) = (Q \ast N_\sigma)(f)$. Note that $\|Q^\sigma\|_{\text{Lip}_{1,0}} := \sup_{f \in \text{Lip}_{1,0}} |Q^\sigma(f)|$ is finite as $Q \in \mathcal{P}_1(\mathbb{R}^d)$.
and $|f \ast \varphi_\sigma(x)| \leq \|x\| + \sigma \sqrt{d}$ for any $f \in \text{Lip}_1$. Thus, $Q_\sigma \in \ell^\infty(\text{Lip}_1)$ for any $Q \in \mathcal{P}_1(\mathbb{R}^d)$. Finally, observe that $W_1^{(\sigma)}(P, Q) = \|P^{(\sigma)} - Q^{(\sigma)}\|_{\text{Lip}_1}$ for any $P, Q \in \mathcal{P}_1(\mathbb{R}^d)$ (cf. Section A.1).

**Theorem 10 (S-MWE infimum limit distribution)** Let $P$ satisfy the conditions of Theorem 1. In addition, suppose that (i) the map $\theta \mapsto Q_\theta$ is continuous relative to the weak topology; (ii) $P \neq Q_\theta$ for any $\theta \neq \theta^*$; (iii) there exists a vector-valued functional $D^{(\sigma)} \in (\ell^\infty(\text{Lip}_1))^d_0$ such that $\|Q_\sigma - Q_\theta - \langle \theta - \theta^*, D^{(\sigma)} \rangle\|_{\text{Lip}_1} = o(\|\theta - \theta^*\|)$ as $\theta \to \theta^*$, where $\langle t, D^{(\sigma)} \rangle := \sum_{i=1}^d t_i D_i^{(\sigma)}$ for $t \in \mathbb{R}^d_0$; (iv) the derivative $D^{(\sigma)}$ is nonsingular in the sense that $\langle t, D^{(\sigma)} \rangle \neq 0$, i.e., $(t, D^{(\sigma)}) \in \ell^\infty(\text{Lip}_1)$ is not the zero functional for all $0 \neq t \in \mathbb{R}^d$. Then, we have

$$\sqrt{n} \inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta) \to \inf_{t \in \mathbb{R}^d_0} \left\| G_P^{(\sigma)} - \langle t, D^{(\sigma)} \rangle \right\|_{\text{Lip}_1},$$

where $G_P^{(\sigma)}$ is the Gaussian process from Theorem 1.

Theorem 10 is proved in Section B.4 via an adaptation of the argument from Pollard (1980, Theorem 4.2).

**Remark 11 (Norm differentiability)** Condition (iii) in Theorem 10 is called ‘norm differentiability’ in Pollard (1980). In these terms, the theorem assumes that the map $\theta \mapsto Q_\theta \in \ell^\infty(\text{Lip}_1)$, is norm differentiable around $\theta^*$ with derivative $D^{(\sigma)}$. This allows approximating the map $\theta \mapsto Q_\theta^{(\sigma)}$ by the affine function $Q_\theta^{(\sigma)} + \langle \theta - \theta^*, D^{(\sigma)} \rangle$ near $\theta^*$. Together with the result of Theorem 1 and the right reparameterization, norm differentiability is key for establishing the theorem.

**Remark 12 (Primitive regularity conditions for norm differentiability)** Suppose that $\{Q_\theta\}_{\theta \in \Theta}$ is dominated by a common Borel measure $\nu$ on $\mathbb{R}^d$, and let $q_\theta$ denote the density of $Q_\theta$ with respect to $\nu$: $dQ_\theta = q_\theta d\nu$. Then, $Q_\theta \ast N_p$ has Lebesgue density $x \mapsto \varphi_\sigma(x - t)q_\theta(x) d\nu(t)$. Suppose in addition that $q_\theta$ admits a Taylor expansion of the form $q_\theta(x) = q_\theta(x) + \hat{q}_\theta(x) \cdot (\theta - \theta^*) + r_\theta(x) \cdot (\theta - \theta^*)$ with $r_\theta(x) = o(\|\theta - \theta^*\|)$. Then, it is not difficult to see that Condition (iii) holds with $D^{(\sigma)}(f) = \int f(x) \int \varphi_\sigma(x - t)q_\theta(t) d\nu(t) dx = \int (f \ast \varphi_\sigma)(t) \hat{q}_\theta(t) d\nu(t)$, for $f \in \text{Lip}_1$, provided that $\int (1 + |t|)\|\hat{q}_\theta(t)\| d\nu(t) < \infty$ and $\int (1 + |t|)\|\hat{\sigma}_\theta(t)\| d\nu(t) = o(\|\theta - \theta^*\|)$ (use the fact that $|f(t)| \leq \|t\|$, for any $f \in \text{Lip}_1$).

We next consider convergence in distribution of solutions. Optimally, one would be interested in the limit distribution of $\sqrt{n}(\hat{\theta}_n - \theta^*)$ with $\hat{\theta}_n \in \arg\min_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta)$. However, a limit is guaranteed to exist only when the (convex) function $t \mapsto \|G_P^{(\sigma)} - \langle t, D^{(\sigma)} \rangle\|_{\text{Lip}_1}$ has a unique minimum a.s. (see Corollary 14 below for details). To avoid making such a stringent assumption, instead of $\sqrt{n}(\hat{\theta}_n - \theta^*)$, we consider the set of approximate minimizers $\hat{\Theta}_n := \{\theta \in \Theta : W_1^{(\sigma)}(P_n, Q_\theta) \leq \inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, Q_\theta^\prime) + \lambda_n/\sqrt{n}\}$, where $\{\lambda_n\}_{n \in \mathbb{N}}$ is an arbitrary $o_P(1)$ sequence.

We will show that $\hat{\Theta}_n \subset \theta^* + n^{-1/2}K_n$ for some (random) sequence of compact convex sets $\{K_n\}_{n \in \mathbb{N}}$ with inner probability approaching one. Resorting to inner probability seems inevitable since the event $\{\hat{\Theta}_n \subset \theta^* + n^{-1/2}K_n\}$ need not be measurable in general (see Pollard, 1980, Section 7). To define such sequence $\{K_n\}_{n \in \mathbb{N}}$, for any $L \in \ell^\infty(\text{Lip}_1)$ and $\beta \geq 0$, let

$$K(L, \beta) := \left\{ t \in \mathbb{R}^d_0 : \|L - \langle t, D^{(\sigma)} \rangle\|_{\text{Lip}_1} \leq \inf_{t' \in \mathbb{R}^d} \|L - \langle t', D^{(\sigma)} \rangle\|_{\text{Lip}_1} + \beta \right\}.$$
Lemma 7.1 of Pollard (1980) ensures that for any $\beta \geq 0$, $L \mapsto K(L, \beta)$ is a measurable map from $\ell^\infty(\text{Lip}_{1,0})$ into $\mathbb{R}$—the class of all compact, convex, and nonempty subsets of $\mathbb{R}^d_0$—endowed with the Hausdorff topology. That is, the topology induced by the metric $d_H(K_1, K_2) := \inf \{ \delta > 0 : K^\delta_1 \supseteq K_2, K^\delta_2 \supseteq K_1 \}$, where $K^\delta := \bigcup_{x \in K} \{ y \in \mathbb{R}^d_0 : \| y - x \| \leq \delta \}$ is the $\delta$-blowup of $K$.

**Theorem 13 (S-MWE solution limit distribution)** Under the conditions of Theorem 10, there exists a sequence of nonnegative reals $\beta_n \downarrow 0$ such that (i) $\mathbb{P}_* \left( \hat{\Theta}_n \in \theta^* + n^{-1/2}K(\mathbb{G}^{(\sigma)}_n, \beta_n) \right) \to 1$, where $\mathbb{G}^{(\sigma)}_n := \sqrt{n}(P^{(\sigma)}_n - P^{(\sigma)})$ is the (smooth) empirical process and $\mathbb{P}_*$ denotes inner probability; and (ii) $K(\mathbb{G}^{(\sigma)}_n, \beta_n) \Rightarrow K(G^{(\sigma)}_P, 0)$ as $\mathbb{R}$-valued random variables.

Given Theorem 10, the proof of Theorem 13 follows by a verbatim repetition of the argument from Pollard (1980, Section 7.2). The details are therefore omitted. If $\arg\min_{t \in \mathbb{R}^d_0} \| G^{(\sigma)}_P - \langle t, D^{(\sigma)} \rangle \|_{\text{Lip}_{1,0}}$ is unique a.s. (a nontrivial assumption), then Theorem 13 simplifies as follows:

**Corollary 14 (S-MWE solution limit distribution under simplified setting)** Assume the conditions of Theorem 10. Let $\{ \hat{\theta}_n \}_{n \in \mathbb{N}}$ be a sequence measurable estimators such that $W_1^{(\sigma)}(P_n, Q_{\hat{\theta}_n}) \leq \inf_{\theta \in \Theta} W_1^{(\sigma)}(P_n, P_\theta) + o_P(n^{-1/2})$. Then, provided that $\arg\min_{t \in \mathbb{R}^d_0} \| G^{(\sigma)}_P - \langle t, D^{(\sigma)} \rangle \|_{\text{Lip}_{1,0}}$ is unique a.s., we have $\sqrt{n}(\hat{\theta}_n - \theta^*) \xrightarrow{d} \arg\min_{t \in \mathbb{R}^d_0} \| G^{(\sigma)}_P - \langle t, D^{(\sigma)} \rangle \|_{\text{Lip}_{1,0}}$.

This corollary follows from the proof of Theorem 10 combined with the argument given at the end of p. 63 in Pollard (1980) (plus some modifications), or the result of Theorem 13 combined with the argument given at the end of p. 67 in Pollard (1980). We provide a separate and direct proof of the corollary in Section B.5 using the ‘convexity’ argument for the reader’s convenience.

5. Summary and concluding remarks

This paper studied the statistical properties of the smoothed Wasserstein distance $W_1^{(\sigma)}$. Specifically, we have shown that, in arbitrary dimension, $\sqrt{n}W_1^{(\sigma)}(P_n, P)$ converges in distribution under a polynomial moment condition on $P$, and the limit is characterized as the supremum of a tight Gaussian process. This result for $W_1^{(\sigma)}$ contrasts the classic $W_1$ case, where a limit distribution is known only in $d = 1$. We have also established the bootstrap consistency and concentration inequalities for $W_1^{(\sigma)}(P_n, P)$. As an application, we have developed limit distribution results for minimum distance estimation with the smoothed Wasserstein distance in arbitrary dimension.

These strong statistical guarantees highlight the virtue of $W_1^{(\sigma)}$ for high-dimensional generative modeling and inference tasks. Future research trajectories include analysis for $\sigma \to 0$ with a sufficiently slow rate, as a proxy for classic $W_1$, and smooth Wasserstein distances of order $p \neq 1$. Additional statistical questions under $W_1^{(\sigma)}$, such as hypothesis testing, will also be considered.
Appendix A. Proofs for Section 2

A.1. Proof of Theorem 1

Recall that $\varphi_\sigma$ is the density function of $N(0, \sigma^2 I_d)$, i.e., $\varphi_\sigma(x) = (2\pi\sigma^2)^{-d/2} e^{-\|x\|^2/(2\sigma^2)}$ for $x \in \mathbb{R}^d$. Noting that the measure $P_n \ast N_\sigma$ has density $x \mapsto \frac{1}{n} \sum_{i=1}^n \varphi_\sigma(x - X_i) = \frac{1}{n} \sum_{i=1}^n \varphi_\sigma(x_i - x)$, we arrive at the expression

$$W^{(\sigma)}_1(P_n, P) = \sup_{f \in \text{Lip}_1} \left[ \frac{1}{n} \sum_{i=1}^n f \ast \varphi_\sigma(X_i) - Pf \ast \varphi_\sigma \right].$$  \hfill (12)

The RHS of (12) does not change even if we replace $f$ by $f - f(x^*)$ for any fixed point $x^*$ (as $\int_{\mathbb{R}^d} \varphi_\sigma(x^* - y) dy = 1$). Thus, the problem boils down to showing that the function class

$$\tilde{F} := \tilde{F}_{\sigma,d} := \{ f \ast \varphi_\sigma : f \in \text{Lip}_{1,0} \}$$

with $\text{Lip}_{1,0} := \{ f \in \text{Lip}_1 : f(0) = 0 \}$ is $P$-Donsker. Pick any $f \in \text{Lip}_{1,0}$, and consider $f_\sigma(x) := f \ast \varphi_\sigma(x) = \int f(y) \varphi_\sigma(x - y) dy$.

We see that, since $|f(y)| \leq |f(0)| + \|x\| = \|x\|$, $|f_\sigma(x)| \leq \int \|y\| \varphi_\sigma(x - y) dy \leq \int (\|x\| + \|x - y\|) \varphi_\sigma(x - y) dy$

$$\leq \|x\| + \int \|y\| \varphi_\sigma(y) dy \leq \|x\| + \left( \int_{\mathbb{R}^d} \|y\|^2 \varphi_\sigma(y) dy \right)^{1/2}$$

$$= \|x\| + \sigma \sqrt{d}.$$

In general, for a vector $k = (k_1, \ldots, k_d)$ of $d$ nonnegative integers, define the differential operator

$$D^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \cdots \partial x_d^{k_d}},$$

with $|k| = \sum_{i=1}^d k_i$. We next give a uniform bound on the derivatives of $f_\sigma$, for any $f \in \text{Lip}_1$.

Lemma 15 (Uniform bound on derivatives) For any $f \in \text{Lip}_1$ and any nonzero multiindex $k = (k_1, \ldots, k_d)$, we have

$$|D^k f_\sigma(x)| \leq \sigma^{-|k|+1} \sqrt{(|k| - 1)!}, \quad \forall x \in \mathbb{R}^d.$$

Proof Let $H_m(z)$ denote the Hermite polynomial of degree $m$ defined by

$$H_m(z) = (-1)^m e^{z^2/2} \left[ \frac{d^m}{dz^m} e^{-z^2/2} \right], \quad m = 0, 1, \ldots.$$
Note that for $Z \sim \mathcal{N}(0, 1)$, $\mathbb{E}[H_m(Z)^2] = m!$.

A straightforward computation shows that

$$D_x^k \varphi_\sigma(x - y) = \varphi_\sigma(x - y) \left[ \prod_{j=1}^d (-1)^{k_j} \sigma^{-k_j} H_{k_j}((x_j - y_j)/\sigma) \right]$$

for any multiindex $k = (k_1, \ldots, k_d)$, where $D_x$ means that the differential operator is applied to $x$.

Hence, we have

$$D^k f_\sigma(x) = \int f(y) \varphi_\sigma(x - y) \left[ \prod_{j=1}^d (-1)^{k_j} \sigma^{-k_j} H_{k_j}((x_j - y_j)/\sigma) \right] dy$$

$$= \int f(x - \sigma y) \varphi_1(y) \left[ \prod_{j=1}^d (-1)^{k_j} \sigma^{-k_j} H_{k_j}(y_j) \right] dy,$$

so that, by 1-Lipschitz continuity of $f$,

$$\left| D^k f_\sigma(x) - D^k f_\sigma(x') \right| \leq \|x - x\| \int \varphi_1(y) \left[ \prod_{j=1}^d \sigma^{-k_j} |H_{k_j}(y_j)| \right] dy.$$

Note that the integral on the RHS equals

$$\prod_{j=1}^d \sigma^{-k_j} \mathbb{E}[|H_{k_j}(Z)|] \leq \prod_{j=1}^d \sigma^{-k_j} \sqrt{\mathbb{E}[|H_{k_j}(Z)|^2]} = \prod_{j=1}^d \sigma^{-k_j} \sqrt{k_j!} \leq \sigma^{-|k|} \sqrt{|k|!},$$

where $Z \sim \mathcal{N}(0, 1)$. The conclusion of the lemma follows from induction on the size of $|k|$.

We will use the following technical result.

**Lemma 16 (Metric entropy bound for Hölder ball)** Let $\mathcal{X}$ be a bounded convex subset of $\mathbb{R}^d$ with nonempty interior. For given $N \in \mathbb{N}$ and $M > 0$, let $C_N^N(\mathcal{X})$ be the set of continuous real functions on $\mathcal{X}$ that are $N$-times differentiable on the interior of $\mathcal{X}$, and consider the Hölder ball with smoothness $N$ and radius $M$

$$C_M^N(\mathcal{X}) := \left\{ f \in C_N^N(\mathcal{X}) : \|f\|_{C_N^N(\mathcal{X})} \leq M \right\},$$

where $\|f\|_{C_N^N(\mathcal{X})} := \max_{0 \leq |k| \leq N} \sup_{x} |D^k f(x)|$ (the suprema are taken over the interior of $\mathcal{X}$). Then, the metric entropy of $C_M^N(\mathcal{X})$ (w.r.t. the uniform norm $\|\cdot\|_{\infty}$) can be bounded as

$$\log N \left( \epsilon M, C_M^N(\mathcal{X}), \|\cdot\|_{\infty} \right) \lesssim_{d, \text{diam}(\mathcal{X})} \epsilon^{-d/N}, \quad 0 < \epsilon \leq 1,$$

**Proof** [Lemma 16] See Theorem 2.7.1 in van der Vaart and Wellner (1996).
Theorem 1.1] The proof applies Theorem 1.1 in var der Vaart (1996) to the function class \( \hat{F} = \hat{F}_{\sigma,d} = \{ f * \varphi_\sigma : f \in \text{Lip}_1 \} \) to show that it is \( P \)-Donsker. We begin with noting that the function class \( \hat{F} \) has envelope \( \hat{F}(x) := \hat{F}_{\sigma,d}(x) := \| x \| + \sigma \sqrt{d} \). By assumption, \( P \hat{F}^2 < \infty \).

Next, for each \( j \), consider the restriction of \( \hat{F} \) to \( I_j \), denoted as \( \hat{F}_j = \{ f 1_{I_j} : f \in \hat{F} \} \). To invoke var der Vaart (1996, Theorem 1.1), we have to bound each \( \mathbb{E}[\| G_n \|_{\hat{F}_j}] \) where \( G_n := \sqrt{n}(P_n - P) \) and \( \| \cdot \|_{\hat{F}_j} = \sup_{f \in \hat{F}_j} | \cdot | \). In view of Lemma 15, \( \hat{F}_j \) can be regarded as a subset of \( C_M(I_j) \) with \( N = \lfloor d/2 \rfloor + 1 \) and \( M_j = (\sup_{I_j} \| x \| + \sigma \sqrt{d}) \sqrt{\sigma^{-|d/2|} \sqrt{d/2}} \). Thus, by Lemma 16, the \( L^2(Q) \)-metric entropy of \( \hat{F}_j \) for any probability measure \( Q \) on \( \mathbb{R}^d \) can be bounded as

\[
\log N(\epsilon M_j^1 Q(I_j)^{1/2}, \hat{F}_j, L^2(Q)) \lesssim_{d,K} \epsilon^{-d/(|d/2|+1)}.
\]

The square root of the RHS is integrable (w.r.t. \( \epsilon \)) around 0, so that by Theorem 2.14.1 in van der Vaart and Wellner (1996), we obtain

\[
\mathbb{E}[\| G_n \|_{\hat{F}_j}] \lesssim_{d,K} M_j^1 P(I_j)^{1/2} \lesssim_{d} \sigma^{-|d/2|} M_j P(I_j)^{1/2}
\]

with \( M_j = \sup_{I_j} \| x \| \). By assumption, the RHS is summable over \( j \).

By Theorem 1.1 in var der Vaart (1996) we conclude that \( \hat{F} \) is \( P \)-Donsker, which implies that there exists a tight version of \( P \)-Brownian bridge process \( G_P \) in \( \ell^\infty(\hat{F}) \) such that \( (G_n f)_{f \in \hat{F}} \) converges weakly in \( \ell^\infty(\hat{F}) \) to \( G_P \). Finally, the continuous mapping theorem yields that

\[
\sqrt{\mathbb{W}_1^2(P,n,P)} = \sup_{f \in \hat{F}} \mathbb{G}_n f \overset{d}{\to} \sup_{f \in \hat{F}} G_P(f) = \sup_{f \in \text{Lip}_1} G_P^0(f),
\]

where \( G_P^0(f) := G_P(f * \varphi_{\sigma}) \). By construction, the Gaussian process \( (G_P^0(f))_{f \in \text{Lip}_1} \) is tight in \( \ell^\infty(\text{Lip}_1) \). The moment bound (7) follows from summing up the moment bound for each \( \hat{F}_j \). This completes the proof.

A.2. Proof of Lemma 4

From the proof of Theorem 1 and the fact that \( \text{Lip}_1 \) is symmetric, we have \( L_P^1 = \| G_P \|_{\hat{F}} \) with \( \| \cdot \|_{\hat{F}} := \sup_{f \in \hat{F}} | \cdot | \). Since \( G_P \) is a tight Gaussian process in \( \ell^\infty(\hat{F}) \), \( \hat{F} \) is totally bounded for the pseudometric \( d_P(f,g) = \sqrt{\text{Var}_P(f-g)} \), and \( G_P \) is a Borel measurable map into the space of \( d_P \)-uniformly continuous functions \( C_u(\hat{F}) \) equipped with the uniform norm \( \| \cdot \|_{\hat{F}} \). Let \( F \) denote the distribution function of \( L_P^1 \), and define

\[
r_0 := \inf \{ r \geq 0 : F(r) > 0 \}.
\]

From (Davydov et al., 1998, Theorem 11.1), \( F \) is absolutely continuous on \((r_0, \infty)\), and there exists a countable set \( \Delta \subset (r_0, \infty) \) such that \( F' \) is positive and continuous on \((r_0, \infty) \setminus \Delta \). The theorem however does not exclude the possibility that \( F \) has a jump at \( r_0 \), and we will verify that (i) \( r_0 = 0 \) and (ii) \( F \) has no jump at \( r = 0 \), which lead to the conclusion. The former follows from p. 57 in Ledoux and Talagrand (1991). The latter is trivial since

\[
F(0) - F(0-) = \mathbb{P}(L_P^1 = 0) \leq \mathbb{P}(G_P(f) = 0),
\]

for any \( f \in \hat{F} \). Because \( G_P \) is Gaussian we have \( \mathbb{P}(G_P(f) = 0) = 0 \) unless \( f \) is constant \( P \)-a.s. \( \blacksquare \)
A.3. Proof of Corollary 5

From Theorem 3.6.2 in van der Vaart and Wellner (1996) applied to the function class $\tilde{F}$, together with the continuous mapping theorem, we see that conditionally on $X_1, X_2, \ldots$, 

$$\sqrt{n}W_1^{(\sigma)}(P_n, P_\sigma) = \sup_{f \in \mathcal{F}} \sqrt{n}(P_n - P_\sigma) f \xrightarrow{d} L_P^{(\sigma)}$$

for almost every realization of $X_1, X_2, \ldots$. The desired conclusion follows from the fact that the distribution function of $L_P^{(\sigma)}$ is continuous (cf. Lemma 4) and Polya’s theorem (cf. Lemma 2.11 in van der Vaart (1998)).

A.4. Proof of Corollary 6

Case (i) is Corollary 1 in Goldfeld and Greenewald (2020). Cases (ii) and (iii) follow from Theorems 4 and 2 in Adamczkak (2010) and Adamczkak (2008), respectively, applied to the function class $\tilde{F}$ using the envelope function $\tilde{F}(x) = \|x\| + \sigma \sqrt{d}$. We omit the details for brevity.

Appendix B. Proofs for Section 4

B.1. Preliminaries

The following technical lemmas will be needed.

Lemma 17 (Continuity of $W_1^{(\sigma)}$). The smooth Wasserstein distance $W_1^{(\sigma)}$ is lower semicontinuous (l.s.c.) relative to the weak convergence on $\mathcal{P}(\mathbb{R}^d)$ and continuous in $W_1$. Explicitly, (i) if $\mu_k \to \mu$ and $\nu_k \to \nu$, then 

$$\liminf_{k \to \infty} W_1^{(\sigma)}(\mu_k, \nu_k) \geq W_1^{(\sigma)}(\mu, \nu);$$

and (ii) if $W_1(\mu_k, \mu) \to 0$ and $W_1(\nu_k, \nu) \to 0$, then 

$$\lim_{k \to \infty} W_1^{(\sigma)}(\mu_k, \nu_k) = W_1^{(\sigma)}(\mu, \nu).$$

Proof Part (i). We first note that if $\mu_k \to \mu$, then $\mu_k * \mathcal{N}_\sigma \to \mu * \mathcal{N}_\sigma$. This follows from the facts that weak convergence is equivalent to pointwise convergence of characteristic functions, and the Gaussian measure has a nonvanishing characteristic function $E_X \sim \mathcal{N}_\sigma \left[ e^{it \cdot X} \right] = e^{-\sigma^2 \|t\|^2 / 2} \neq 0$ for all $t \in \mathbb{R}^d$. Now, if $\mu_k \to \mu$ and $\nu_k \to \nu$, then $\mu_k * \mathcal{N}_\sigma \to \mu * \mathcal{N}_\sigma$ and $\nu_k * \mathcal{N}_\sigma \to \nu * \mathcal{N}_\sigma$. From the lower semicontinuity of $W_1$ relative to the weak convergence (cf. Remark 6.10 in Villani (2008)), we conclude that $\liminf_{k \to \infty} W_1^{(\sigma)}(\mu_k, \nu_k) = \liminf_{k \to \infty} W_1(\mu_k * \mathcal{N}_\sigma, \nu_k * \mathcal{N}_\sigma) \geq W_1(\mu * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) = W_1^{(\sigma)}(\mu, \nu)$.

Part (ii). Recall that $W_1^{(\sigma)}$ generates the same topology as $W_1$, i.e., 

$$W_1^{(\sigma)}(\mu_k, \mu) \to 0 \iff W_1(\mu_k, \mu) \to 0.$$ 

See Theorem 2 in Goldfeld and Greenewald (2020). So if $\mu_k \to \mu$ and $\nu_k \to \nu$ in $W_1$, then 

$$W_1^{(\sigma)}(\mu_k, \mu) = W_1(\mu_k * \mathcal{N}_\sigma, \mu * \mathcal{N}_\sigma) \to 0$$

and 

$$W_1^{(\sigma)}(\nu_k, \nu) = W_1(\nu_k * \mathcal{N}_\sigma, \nu * \mathcal{N}_\sigma) \to 0.$$ 

Thus, by Corollary 6.9 in Villani (2008), we have 

$$W_1^{(\sigma)}(\mu_k, \nu_k) = W_1(\mu_k * \mathcal{N}_\sigma, \nu_k * \mathcal{N}_\sigma) \to W_1(\mu_k * \mathcal{N}_\sigma, \nu_k * \mathcal{N}_\sigma) = W_1^{(\sigma)}(\mu, \nu).$$


Lemma 18 (Weierstrass criterion for the existence of minimizers)  Let $\mathcal{X}$ be a compact metric space, and let $f : \mathcal{X} \to \mathbb{R} \cup \{\pm \infty\}$ be l.s.c. (i.e., $\liminf_{x \to \overline{x}} f(x) \geq f(\overline{x})$ for any $\overline{x} \in \mathcal{X}$). Then, $\operatorname{argmin}_{x \in \mathcal{X}} f(x)$ is nonempty.

**Proof** See, e.g., p. 3 of Santambrogio (2010). □

B.2. Proof of Theorem 8

By Lemma 18, compactness of $\Theta$, and lower semicontinuity of the map $\theta \mapsto W_1^{(\sigma)}(P_n(\omega), Q_\theta)$ (cf. Lemma 17), we see that $\operatorname{argmin}_{\theta \in \Theta} W_1^{(\sigma)}(P_n(\omega), Q_\theta)$ is nonempty.

To prove the existence of a measurable estimator, we will apply Corollary 1 in Brown and Purves (1973). Consider the empirical distribution as a function on $\mathcal{X}^N$ with $\mathcal{X} = \mathbb{R}^d$, i.e., $\mathcal{X}^N \ni x = (x_1, x_2, \ldots) \mapsto P_n(x) = n^{-1} \sum_{i=1}^n \delta_{x_i}$. Observe that $\mathcal{X}^N$ and $\mathbb{R}^{d_0}$ are both Polish, $\mathcal{D} := \mathcal{X}^N \times \Theta$ is a Borel subset of the product metric space $\mathcal{X}^N \times \mathbb{R}^{d_0}$, the map $\theta \mapsto W_1^{(\sigma)}(P_n(x), Q_\theta)$ is l.s.c. by Lemma 17, and the set $\mathcal{D}_x = \{ \theta \in \Theta : (x, \theta) \in \mathcal{D} \} \subset \mathbb{R}^{d_0}$ is $\sigma$-compact (as any subset in $\mathbb{R}^{d_0}$ is $\sigma$-compact). Thus, in view of Corollary 1 of Brown and Purves (1973), it suffices to verify that the map $(x, \theta) \mapsto W_1^{(\sigma)}(P_n(x), Q_\theta)$ is jointly measurable.

To this end, we use the following fact: for a real function $\mathcal{Y} \times \mathcal{Z} \ni (y, z) \mapsto f(y, z) \in \mathbb{R}$ defined on the product of a separable metric space $\mathcal{Y}$ (endowed with the Borel $\sigma$-field) and a measurable space $\mathcal{Z}$, if $f(y, z)$ is continuous in $y$ and measurable in $z$, then $f$ is jointly measurable; see e.g., Lemma 4.51 in Aliprantis and Border (2006). Equip $\mathcal{P}_1(\mathbb{R}^d)$ with the metric $W_1$ and the associated Borel $\sigma$-field; the metric space $(\mathcal{P}_1(\mathbb{R}^d), W_1)$ is separable (Villani, 2008, Theorem 6.16). Then, since the map $\mathcal{X}^N \ni x \mapsto P_n(x) \in \mathcal{P}_1(\mathbb{R}^d)$ is continuous (which is not difficult to verify), the map $\mathcal{X}^N \times \Theta \ni (x, \theta) \mapsto (P_n(x), \theta) \in \mathcal{P}_1(\mathbb{R}^d) \times \Theta$ is continuous and thus measurable. Second, by Lemma 17, the function $\mathcal{P}_1(\mathbb{R}^d) \times \Theta \ni (\mu, \theta) \mapsto W_1^{(\sigma)}(\mu, Q_\theta) \in [0, \infty)$ is continuous in $\mu$ and l.s.c. (and thus measurable) in $\theta$, from which we see that the map $(\mu, \theta) \mapsto W_1^{(\sigma)}(\mu, Q_\theta)$ is jointly measurable. Conclude that the map $(x, \theta) \mapsto W_1^{(\sigma)}(P_n(x), Q_\theta)$ is jointly measurable. □

B.3. Proof of Theorem 9

The proof relies on Theorem 7.33 in Rockafellar and Wets (2009), and is reminiscent of that of Theorem B.1 in Bernton et al. (2019); we present a simpler derivation under our assumption. 9 To apply Theorem 7.33 in Rockafellar and Wets (2009), we extend the map $\theta \mapsto W_1^{(\sigma)}(P_n, Q_\theta)$ to the entire Euclidean space $\mathbb{R}^{d_0}$ as

$$g_n(\theta) := \begin{cases} W_1^{(\sigma)}(P_n, Q_\theta) & \text{if } \theta \in \Theta \\ +\infty & \text{if } \theta \in \mathbb{R}^{d_0} \setminus \Theta \end{cases}.$$  

Likewise, define

$$g(\theta) := \begin{cases} W_1^{(\sigma)}(P, Q_\theta) & \text{if } \theta \in \Theta \\ +\infty & \text{if } \theta \in \mathbb{R}^{d_0} \setminus \Theta \end{cases}.$$  

9. Theorem B.1 in Bernton et al. (2019) applies Theorem 7.31 in Rockafellar and Wets (2009). To that end, one has to extend the maps $\theta \mapsto W_{\mu}(\mu_n, \mu_0)$ and $\theta \mapsto W_{\nu}(\mu_s, \mu_0)$ to the entire Euclidean space $\mathbb{R}^{d_0}$. The extension was not mentioned in the proof of (Bernton et al., 2019, Theorem B.1), although this missing step does not affect their final result.
The function $g_n$ is stochastic, $g_n(\theta) = g_n(\theta, \omega)$, but $g$ is non-stochastic. By construction, we see that $\arginf_{\theta \in \mathbb{R}^{d_0}} g_n(\theta) = \arginf_{\theta \in \Theta} \mathcal{W}_1^{(\sigma)}(P_n, Q_\theta)$ and $\arginf_{\theta \in \mathbb{R}^{d_0}} g(\theta) = \arginf_{\theta \in \Theta} \mathcal{W}_1^{(\sigma)}(P, Q_\theta)$.

In addition, by Lemma 17, continuity of the map $\theta \mapsto Q_\theta$ relative to the weak topology, and closedness of the parameter space $\Theta$, we see that both $g_n$ and $g$ are l.s.c. (on $\mathbb{R}^{d_0}$). The main step of the proof is to show a.s. epi-convergence of $g_n$ to $g$. Recall the definition of epi-convergence (in fact, this is an equivalent characterization; see Rockafellar and Wets (2009, Proposition 7.29)):

**Definition 19 (Epi-convergence)** For extended-real-valued functions $f_n, f$ on $\mathbb{R}^{d_0}$ with $f$ being l.s.c., we say that $f_n$ epi-converges to $f$ if the following two conditions hold:

(i) $\liminf_{n \to \infty} \inf_{\theta \in K} f_n(\theta) \geq \inf_{\theta \in K} f(\theta)$ for any compact set $K \subset \mathbb{R}^{d_0}$; and

(ii) $\limsup_{n \to \infty} \inf_{\theta \in U} f_n(\theta) \leq \inf_{\theta \in U} f(\theta)$ for any open set $U \subset \mathbb{R}^{d_0}$.

We also need the concept of level-boundedness.

**Definition 20 (Level-boundedness)** For an extended-real-valued function $f$ on $\mathbb{R}^{d_0}$, we say that $f$ is level-bounded if for any $\alpha \in \mathbb{R}$, the set $\{\theta \in \mathbb{R}^{d_0} : f(\theta) \leq \alpha\}$ is bounded (possibly empty).

We are now in position to prove Theorem 9.

**Proof** [Proof of Theorem 9] By boundedness of the parameter space $\Theta$, both $g_n$ and $g$ are level-bounded by construction as the (lower) level sets are included in $\Theta$. In addition, by assumption, both $g_n$ and $g$ are proper (an extended-real-valued function $f$ on $\mathbb{R}^{d_0}$ is proper if the set $\{\theta \in \mathbb{R}^{d_0} : f(\theta) < \infty\}$ is nonempty). In view of Theorem 7.33 in Rockafellar and Wets (2009), it remains to prove that $g_n$ epi-converges to $g$ a.s. To verify property (i) in the definition of epi-convergence, recall that $P_n \to P$ in $W_1$ (and hence in $W_1^{(\sigma)}$) a.s. Pick any $\omega \in \Omega$ such that $P_n(\omega) \to P$ in $W_1$. Pick any compact set $K \subset \mathbb{R}^{d_0}$. Since $g_n(\cdot, \omega)$ is l.s.c., by Lemma 18, there exists $\theta_n(\omega) \in K$ such that $g_n(\theta_n(\omega), \omega) = \inf_{\theta \in K} g_n(\theta, \omega)$. Up to extraction of subsequences, we may assume $\theta_n(\omega) \to \theta^*(\omega)$ for some $\theta^*(\omega) \in K$. If $\theta^*(\omega) \notin \Theta$, then by closedness of $\Theta$, $\theta_n(\omega) \notin \Theta$ for all sufficiently large $n$. Thus, we have

$$\liminf_{n \to \infty} \inf_{\theta \in K} g_n(\theta, \omega) = \liminf_{n \to \infty} g_n(\theta_n(\omega), \omega) = +\infty,$$

so that $\liminf_{n \to \infty} \inf_{\theta \in K} g_n(\theta, \omega) \geq \inf_{\theta \in K} g(\theta)$. Next, consider the case where $\theta^*(\omega) \in \Theta$. In this case, $\theta_n(\omega) \in \Theta$ for all $n$ (otherwise, $+\infty = g_n(\theta_n(\omega), \omega) > g_n(\theta^*(\omega), \omega)$, which contradicts the construction of $\theta_n(\omega)$). Thus, $g_n(\theta_n(\omega), \omega) = \mathcal{W}_1^{(\sigma)}(P_n(\omega), Q_{\theta_n(\omega)})$, so that

$$\liminf_{n \to \infty} \inf_{\theta \in K} g_n(\theta(\omega), \omega) = \liminf_{n \to \infty} \mathcal{W}_1^{(\sigma)}(P_n(\omega), Q_{\theta_n(\omega)})$$

$$\geq \mathcal{W}_1^{(\sigma)}(P, Q_{\theta^*(\omega)})$$

$$= \inf_{\theta \in K} g(\theta),$$

where (a) follows from Lemma 17.

To verify property (ii) in the definition of epi-convergence, pick any open set $U \subset \Theta$. It is enough to consider the case where $U \cap \Theta \neq \emptyset$. Let $\{\theta_n\}_{n=1}^{\infty} \subset U$ be a sequence with
\[ \lim_{n \to \infty} g(\theta'_n) = \inf_{\theta \in \mathcal{U}} g(\theta). \]

Since \(\inf_{\theta \in \mathcal{U}} g(\theta)\) is finite, we may assume that \(\theta'_n \in \mathcal{U} \cap \Theta\) for all \(n\). Thus, we have

\[
\limsup_{n \to \infty} \inf_{\theta \in \mathcal{U}} g_n(\theta, \omega) \leq \limsup_{n \to \infty} g_n(\theta'_n, \omega)
= \limsup_{n \to \infty} W_1^\sigma(P_n(\omega), Q_{\theta'_n})
\leq \lim_{n \to \infty} W_1^\sigma(P_n(\omega), P) + \liminf_{n \to \infty} W_1^\sigma(P, Q_{\theta'_n})
= \inf_{\theta \in \mathcal{U}} g(\theta).
\]

Conclude that \(g_n\) epi-converges to \(g\) a.s. This completes the proof. \(\blacksquare\)

**B.4. Proof of Theorem 10**

Recall that \(P = Q_{\theta^*}\). Condition (ii) implies that \(\arg\min_{\theta \in \Theta} W_1^\sigma(P, Q_\theta) = \{\theta^*\}\). Hence, by Theorem 9, for any neighborhood \(N\) of \(\theta^*\),

\[
\inf_{\theta \in \Theta} W_1^\sigma(P_n, Q_\theta) = \inf_{\theta \in N} W_1^\sigma(P_n, Q_\theta)
\]

with probability approaching one.

Define \(R^\sigma_\theta := Q_\theta^\sigma - P^\sigma - \langle \theta - \theta^*, D^\sigma \rangle \in \ell^\infty(\text{Lip}_1, 0)\), and choose \(N_1\) as a neighborhood of \(\theta^*\) such that

\[
\|\langle \theta - \theta^*, D^\sigma \rangle\|_{\text{Lip}_1, 0} - \|R^\sigma_\theta\|_{\text{Lip}_1, 0} \geq \frac{1}{2} C, \quad \forall \theta \in N_1,
\]

for some constant \(C > 0\). Such \(N_1\) exists since conditions (iii) and (iv) ensure the existence of an increasing function \(\eta(\delta) = o(1)\) (as \(\delta \to 0\)) and a constant \(C > 0\) such that \(\|R^\sigma(\theta)\|_{\text{Lip}_1, 0} \leq \|\theta - \theta^*\|\eta(\|\theta - \theta^*\|)\) and \(\|\langle t, D^\sigma \rangle\|_{\text{Lip}_1, 0} \geq C\|t\|\) for all \(t \in \mathbb{R}^d_0\).

For any \(\theta \in N_1\), the triangle inequality and (16) imply that

\[
W_1^\sigma(P_n, Q_\theta) \geq C\frac{1}{2} \|\theta - \theta^*\| - W_1^\sigma(P_n, P).
\]

For \(\xi_n := \frac{4\sqrt{n}}{C} W_1^\sigma(P_n, P)\), consider the (random) set \(N_2 := \{\theta \in \Theta : \sqrt{n}\|\theta - \theta^*\| \leq \xi_n\}\). Note that \(\xi_n\) is of order \(O_p(1)\) by Theorem 1. By the definition of \(\xi_n\), \(\inf_{\theta \in N_1} W_1^\sigma(P_n, Q_\theta)\) is unchanged if \(N_1\) is replaced with \(N_1 \cap N_2\); indeed, if \(\theta \in N_2^c\), then \(W_1^\sigma(P_n, Q_\theta) > \frac{C}{2}\sqrt{n} - W_1^\sigma(P_n, P) = W_1^\sigma(P_n, P)\), so that \(\inf_{\theta \in N_2^c} W_1^\sigma(P_n, Q_\theta) > W_1^\sigma(P_n, P) \geq \inf_{\theta \in N_1} W_1^\sigma(P_n, Q_\theta)\).
Reparametrizing \( t := \sqrt{n}(\theta - \theta^*) \) and setting \( T_n := \{ t \in \mathbb{R}^{d_0} : \|t\| \leq \xi_n, \theta^* + t/\sqrt{n} \in \Theta \} \), we have the following approximation

\[
\sup_{t \in T_n} \sqrt{n} \left\| \frac{P_n^{(\sigma)} - Q_{\theta^* + t/\sqrt{n}}}{{\text{Lip}}_{1,0}} \right\|_{\text{Lip}_{1,0}} = \sup_{t \in T_n} \sqrt{n} \left\| \frac{P_n^{(\sigma)} - P^{(\sigma)}}{=G_n^{(\sigma)}} - \left\langle t, D^{(\sigma)} \right\rangle \right\|_{\text{Lip}_{1,0}} \leq \sup_{t \in T_n} \sqrt{n} \left\| R_n^{(\sigma)}\frac{t}{\sqrt{n}} \right\|_{\text{Lip}_{1,0}} \leq \xi_n \eta(\xi_n/\sqrt{n}) = o_P(1). \tag{18}
\]

Observe that any minimizer \( t^* \in \mathbb{R}^{d_0} \) of the function \( h_n(t) := \left\| G_n^{(\sigma)} - \left\langle t, D^{(\sigma)} \right\rangle \right\|_{\text{Lip}_{1,0}} \) satisfies \( \|t^*\| \leq \xi_n \); indeed if \( \|t^*\| > \xi_n \), then \( h_n(t^*) \geq C \|t^*\| - \|G_n^{(\sigma)}\|_{\text{Lip}_{1,0}} = C \|t^*\| - \sqrt{n} W_1^{(\sigma)}(P_n, P) = 3\sqrt{n} W_1^{(\sigma)}(P_n, P) = 3h_n(0) \), which contradicts the assumption that \( t^* \) is a minimizer of \( h_n(t) \).

Since by assumption \( \theta^* \in \text{int}(\Theta) \), the set of minimizers of \( h_n \) lies inside \( T_n \). Conclude that

\[
\inf_{\theta \in \Theta} \sqrt{n} W_1^{(\sigma)}(P_n, Q_{\theta}) = \inf_{t \in \mathbb{R}^{d_0}} \left\| G_n^{(\sigma)} - \left\langle t, D^{(\sigma)} \right\rangle \right\|_{\text{Lip}_{1,0}} + o_P(1). \tag{19}
\]

Now, from the proof of Theorem 1 and the fact that the map \( G \Rightarrow (G(f * \varphi_0))_{f \in \text{Lip}_{1,0}} \) is continuous (indeed, isometric) from \( \ell^\infty(\tilde{F}) \) into \( \ell^\infty(\text{Lip}_{1,0}) \), we see that \( (G_n^{(\sigma)} f)_{f \in \text{Lip}_{1,0}} \to G_P^{(\sigma)} \) weakly in \( \ell^\infty(\text{Lip}_{1,0}) \).

Applying the continuous mapping theorem to \( L \Rightarrow \inf_{t \in \mathbb{R}^{d_0}} \| L - \left\langle t, D^{(\sigma)} \right\rangle \|_{\text{Lip}_{1,0}} \) and using the approximation (19), we obtain the conclusion of the theorem. \( \blacksquare \)

**B.5. Direct proof of Corollary 14**

The proof relies on the following result on weak convergence of argmin solutions of convex stochastic functions. The following lemma is a simple modification of Theorem 1 in Kato (2009). Similar techniques can be found in Pollard (1991) and Hjort and Pollard (1993).

**Lemma 21** Let \( H_n(t) \) and \( H(t) \) be convex stochastic functions on \( \mathbb{R}^{d_0} \). Suppose that (i) \( \arg \min_{t \in \mathbb{R}^{d_0}} H(t) \) is unique a.s., and (ii) for any finite set of points \( t_1, \ldots, t_k \in \mathbb{R}^{d_0} \), we have \( (H_n(t_1), \ldots, H_n(t_k)) \to_d (H(t_1), \ldots, H(t_k)) \). Then, for any sequence \( \{\hat{t}_n\}_{n \in \mathbb{N}} \) such that \( H_n(\hat{t}_n) \leq \inf_{t \in \mathbb{R}^{d_0}} H_n(t) + o_P(1) \), we have \( \hat{t}_n \to_d \arg \min_{t \in \mathbb{R}^{d_0}} H(t) \).

**Proof** [Proof of Corollary 14] By Theorem 9, \( \tilde{\theta}_n \to \theta^* \) in probability. From equation (17) and the definition of \( \tilde{\theta}_n \), we see that, with probability approaching one,

\[
\inf_{\theta \in \Theta} \sqrt{n} W_1^{(\sigma)}(P_n, Q_{\theta}) + o_P(1) \geq \sqrt{n} W_1^{(\sigma)}(P_n, Q_{\tilde{\theta}_n}) \geq \frac{C}{2} \sqrt{n} \| \hat{\theta}_n - \theta^* \| - \sqrt{n} W_1^{(\sigma)}(P_n, P), \tag{19}
\]

which implies that \( \sqrt{n} \| \hat{\theta}_n - \theta^* \| = o_P(1) \). Let \( H_n(t) := \left\| G_n^{(\sigma)} - \left\langle t, D^{(\sigma)} \right\rangle \right\|_{\text{Lip}_{1,0}} \) and \( H(t) := \left\| G_P^{(\sigma)} - \left\langle t, D^{(\sigma)} \right\rangle \right\|_{\text{Lip}_{1,0}} \). Both \( H_n(t) \) and \( H(t) \) are convex in \( t \). Then, from equation (18), for
\[ \hat{t}_n := \sqrt{n}(\hat{\theta}_n - \theta^*) = O_P(1), \] we have
\[ \sqrt{n}W_1^{(\sigma)}(P_n, Q_{\hat{\theta}_n}) = H_n(\hat{t}_n) + o_P(1). \]
Combining the result (19) and the definition of \( \hat{\theta}_n \), we see that \( H_n(\hat{t}_n) \leq \inf_{t \in \mathbb{R}^{d_0}} H_n(t) + o_P(1). \) Since \( G_n^{(\sigma)} \) converges weakly to \( G_P^{(\sigma)} \) in \( \ell^\infty(\text{Lip}_{1,0}) \), by the continuous mapping theorem, we have
\[ (H_n(t_1), \ldots, H_n(t_k)) \overset{d}{\to} (H(t_1), \ldots, H(t_k)) \]
for any finite number of points \( t_1, \ldots, t_k \in \mathbb{R}^{d_0}. \)
By assumption, \( \arg\min_{t \in \mathbb{R}^{d_0}} H(t) \) is unique a.s. Hence, by Lemma 21, we conclude that
\[ \hat{t}_n \overset{d}{\to} \arg\min_{t \in \mathbb{R}^{d_0}} H(t). \]

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