Multivariate Self-Exciting Threshold Autoregressive Models with eXogenous Input

{PRELIMINARY VERSION– please do not quote}

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Abstract

This study defines a multivariate Self–Exciting Threshold Autoregressive with eXogenous input (MSETARX) models and present an estimation procedure for the parameters. The conditions for stationarity of the nonlinear MSETARX models is provided. In particular, the efficiency of an adaptive parameter estimation algorithm and LSE (least squares estimate) algorithm for this class of models is then provided via simulations.

Keywords: Multivariate Threshold, Nonlinear Time Series, MSETAR models, eXogenous input

JEL: C14, C22

1. Introduction

Recently there has been considerable interest in nonlinear time series analysis (Priestley (1988); Tong (1990); Brock et al. (1991); Terasvirta and Granger (1993); Terasvirta et al. (1994); Hansen (2011); Addo et al. (2014), and references therein), due primarily to the various limitations encountered with linear time series models in real applications. Many nonlinear time series models have been introduced in the literature and illustrated to be useful in some applications (Granger and Andersen (1978); Priestley (1988); Subba and Gabr (1984); Haggan and Ozaki (1981); Tong (1983, 1990)). For instance, Tong (1978, 1990) proposed the threshold autoregressive (TAR) model and showed its usefulness in describing the asymmetric limit cycle of the annual sunspot number. Let \( (\Omega, \mathcal{F}, P) \) be a probability space, \( \mathbb{R} = \bigcup_{j=1}^{l} \mathbb{R}_j \), \( \mathbb{R}_j = (r_{j-1}, r_j], \infty = r_0 < r_1 < \cdots < r_{l} = \infty \) a disjunctive decomposition of the real axis. Let \( d, p_1, \cdots, p_l \in \mathbb{Z}^+ \). Any solution of \( (y_t) \), of

\[
y_t + \sum_{j=1}^{l} y_{t,d}^{(j)} \left( a_{0}^{(j)} + \sum_{i=1}^{p_{j}} a_{i}^{(j)} y_{t-i} \right) = \sum_{j=1}^{l} y_{t,d}^{(j)} \mathcal{E}_{t}^{(j)}
\]

where

\[
y_{t,d}^{(j)} = \begin{cases} 
1; & y_{t-d} \in R_j \\
0; & y_{t-d} \notin R_j,
\end{cases}
\]

Preprint submitted to Elsevier July 30, 2014
is a univariate Self-Exciting Threshold Autoregressive process denoted by SETAR \((l, p_1, \cdots, p_l)\) with delay \(d\) (see Tong (1983, 1990) and the references therein). The process \((y_t)\) is assumed to be ergodic and its stationary distribution has a finite second moment. The process \((\varepsilon_t)_{i,j}\) in model equation (1) for each regime \(j\) is assumed to be a martingale difference sequence with respect to an increasing sequence of \(\sigma\)-field, denoted as \(\mathcal{F}_t\), i.e., \(E[\varepsilon_t^{(j)}|\mathcal{F}_{t-1}] = 0\). In this setting, the conditional variance of the process \((\varepsilon_t)_{i,j}\) can be a constant, \(E[(\varepsilon_t^{(j)})^2|\mathcal{F}_{t-1}] = \sigma^2\) or allowed for possibly asymmetric autoregressive conditional heteroscedasticity. The model equation (1) is nonlinear in time when the number of regimes \(l > 1\) and is a piecewise linear model in the threshold space \(y_{-\delta}\). Thus SETAR model (1) adopts a piecewise linear setting in such a fashion that regime switches are triggered by an observed variable crossing an unknown threshold. For a review on the asymptotic theory and inference for the SETAR model (1), see Tong (1990); Chan (1993); Qian (1998); Hansen (1997, 1999, 2000). Despite the simplicity of SETAR models, they have been shown to be able to capture economically interesting asymmetries, regime changes (such as periods of low/high stock market valuations, recessions/expansions, periods of low/high interest rates, etc), and empirically observed nonlinear dynamics relevant to economic data. For instance, Pfann et al. (1996) used a single-threshold SETAR model in describing the dynamic behaviour of the three-month US T-bill interest rate.

In analysing multivariate relationships between economic variables, the linear Vector Autoregression (VAR) models have gain popularity for empirical macroeconomic modelling, policy analysis and forecasting. However, the inability of these linear models to capture non-linear dynamics such as regime switching and asymmetric responses to shocks, has gained attention in macroeconomic research. For example, a significant number of empirical studies document asymmetries in the \(e\) effects of monetary policy on output growth (Rothman et al. (1999) and references therein). In this respect, the interest in nonlinear ARX time series and regression models has been increasing in econometrics as in other disciplines (Terasvirta and Granger (1993); Chen and Tsay (1993); Hubrich and Terasvirta (2013) and references therein). In this work, we consider the introduction of an exogenous input \((f_t)\) as an extension of the Multivariate SETAR model formulation and has a structural form of a nonlinear bivariate ARX model (Masry and Tjostheim (1997)). Unlike the multivariate threshold model proposed in Tsay (1998), we allow the possibility of the threshold variable to also be a multivariate process. In this case, the regime of the whole system is not necessarily determined by a single stationary subprocess. In otherwords, there exists thresholds for all subprocess of the multivariate process.

A short overview of the paper is as follows. In Section 2 we define the multivariate SETAR process with exogenous input denoted MSETARX model as an extension of the multivariate SETAR model. In Section 3 we find conditions for stationarity of the MSETARX models, whereas Section 4 is used to present the LSE (least squares estimate) algorithm and an adaptive parameter estimation algorithm (Arnold and Gunther (2001); Leistritz et al. (2006)) based on the stochastic gradient principles for linear systems shown to be suitable for nonlinear systems. The performance of the proposed algorithms for estimating the parameters of Multivariate SETARX models is evaluated via simulations in Section 4.5. In Section 5 the modeling procedure for the MSETARX models and problems of estimation are briefly considered.

2. Multivariate SETARX models

Consider a \(D\)-dimensional time series \(y_t = (y_{1t}, \cdots, y_{Dt})^T\) such that \(L_1, \cdots, L_D \in \mathbb{Z}^+\), for each \(1 \leq i \leq D\), \((R_j^i)_{j=1,2,\cdots,l_i}\) a disjuction decomposition of the real axis: \(R = \bigcup_{j=1}^{l_i} R_j^i; i \in \{1, \cdots, D\}.\)
Let \( L = \max(L_1, L_2, \ldots, L_D) \) and \( R_j = \Phi: j = L_i + 1, \ldots, L_i \). Then any solution \((y_t)\) of
\[
y_t + \sum_{J \in [1, \ldots, L_i]} y_{i,J}(f_0 + \sum_{i=1}^{p_J} A_{i,J} y_{t-i}) = \sum_{J \in [1, \ldots, L_i]} y_{i,J} \alpha_{i,J}(J)
\]
is called a multivariate SETAR process denoted MSETAR \((L, p_J; J \in [1, \ldots, L])\), where \( y_{i,J} : \{1, \ldots, L\} \mapsto [0, 1) \) is the indicator variable defined by the following relation:
\[
y_{i,J} = \begin{cases} 1 & \text{if } d_i \in \{R_j: j \in (1, \ldots, L_i)\} \\
0 & \text{otherwise}
\end{cases}
\]
and \( \{\alpha_{i,J}, f_i\} \) be a sequence of martingale differentials with respect to an increasing sequence of \( \sigma \)-field \( \{F_t\} \) such that
\[
\sup_{t \geq 0} \mathbb{E}[||\alpha_{i,J}||^2 | F_t] = 0 \quad \text{a.s.} \quad \sup_{t \geq 0} \mathbb{E}[||\alpha_{i,J}||^2 | F_t] = \sigma^2 < \infty \quad \text{a.s.} \quad \sup_{t \geq 0} \mathbb{E}[||\alpha_{i,J}||^2 | F_t] < +\infty \quad \text{a.s.}
\]
for some \( a > 2 \) and \( ||\cdot|| \) be a matrix norm.

Now consider a \( D \)-dimensional time series \((y_t) = (y_{1,t}, \ldots, y_{D,t})^T \) and a \( k \)-dimensional inputs \((f_t) = (f_{1,t}, \ldots, f_{k,t})^T \) such that \( L_1, \ldots, L_D \in \mathbb{Z}^+ \), for each \( 1 \leq i \leq D \), \((R_j)_{j=1,2,\ldots} \) a disjunction decomposition of the real axis: \( R = \bigcup_{j=1}^{J} R_j \), \( i \in \{1, \ldots, D\} \). Let \( L = \max(L_1, L_2, \ldots, L_D) \) be the maximum of the number of regimes for each subprocess of \((y_t)\) and \( R_j = \Phi: j = L_i + 1, \ldots, L_i \). Then any solution \((y_t)\) of
\[
\begin{align*}
y_t + \sum_{J \in [1, \ldots, L_i]} y_{i,J} (\alpha_{i,J} + \sum_{i=1}^{p_J} A_{i,J} y_{t-i} + \Lambda(J) f_t) &= \sum_{J \in [1, \ldots, L_i]} y_{i,J} \beta_{i,J}(J) \\
f_t &= \sum_{i=1}^{q} \Xi_{iJ} f_{t+i} + \eta_t
\end{align*}
\]
is called a multivariate SETAR process with exogenous input denoted MSETARX \((L, p_J, q; J \in [1, \ldots, L])\). The variables \((y_t)\) and \((f_t)\) in model \((4)\) are endogenous and exogenous, respectively, and the econometrics significance of estimating the relationship between \((y_t)\) and \((f_t)\) is well known. The model equation \((4)\) can be rewritten as
\[
y_t + \sum_{J \in [1, \ldots, L_i]} y_{i,J} (\alpha_{i,J} + \sum_{i=1}^{p_J} A_{i,J} y_{t-i} + \Lambda(J) \sum_{r=1}^{q} \Xi_{i,J} f_{t+r}) = \sum_{J \in [1, \ldots, L_i]} y_{i,J} \omega_{i,J}(J)
\]
where \( \omega_{i,J} = \alpha_{i,J} - \Lambda(J) \eta_t \), \( \alpha_{i,J} \) and \( \omega_{i,J} \) are \( D \times 1 \) vectors, \( A_{i,J} \) are \( D \times D \) coefficient matrices, \( \Lambda(J) \) are \( D \times k \) coefficient matrices, \( \Xi_{i,J} \) are \( k \times k \) coefficient matrices, and \((f_t)\) is \( k \times 1 \) vector. When \( \Lambda(J) = 0 \) for all \( J \in [1, \ldots, L] \), \((5)\) becomes a MSETAR model \((3)\).

The representation in equation \((5)\) shows that the MSETARX \((L, p_J, q; J \in [1, \ldots, L])\) model \((5)\) has approximately the same structure as the MSETAR \((L, p_J; J \in [1, \ldots, L])\) model \((3)\) with exogenous variables or factors \((f_t)\). For simplicity, we assume the exogenous inputs enter the model in a linear autoregressive fashion. It is worth pointing out that the dynamics of process \((f_t)\), could be captured by suitable linear/nonlinear model, principal components, and among other model specifications. Unlike the multivariate threshold model in [Tsay, 1998], the threshold space is of dimension equal to the dimension of the multivariate process. Thus there exists thresholds for all subprocess of the multivariate process \((3)\). In this case, the regime of the whole system is not necessarily determined by a single stationary subprocess, as in [Tsay, 1998].
Assumption 1. Let \( \{\xi_t^{(j)}, F_t\} \) and \( \{\eta_t, F_t\} \); \( \forall J \in \{1, \ldots, L\}^0 \) be two independent sequence of martingale difference with respect to an increasing sequence of \( \sigma \)-field \( \{F_t\} \) such that

\[
\sup_{t \geq 0} E[||\xi_t^{(j)}||^2|F_t] = \tilde{T}_x < \infty \quad \text{a.s} \quad \text{and} \quad \sup_{t \geq 0} E[||\eta_t||^2|F_t] = \tilde{T}_\eta < \infty \quad \text{a.s}
\]

This ensures that \( \{\omega_t^{(j)}, F_t\} \) is a sequence of martingale difference with respect to an increasing sequence of \( \sigma \)-field \( \{F_t\} \) where

\[
\sup_{t \geq 0} E[||\omega_t^{(j)}||^2|F_t] = \tilde{T}_\omega < \infty \quad \text{a.s}
\]

Simple orthogonality assumptions on the errors \( \omega_t^{(j)} \) are insufficient to identify nonlinear models (Caner and Hansen (2004)) and as such it is important that Assumption 1 holds.

Let \( p = \max\{p|J| \in \{1, \ldots, L\}^0\} \) and \( q \) be the model orders for model (5). Now, suppose that \( \omega_t^{(j)} \), and \( p \) be regime independent. We can rewrite model equation (5) as

\[
y_t = \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} (\tilde{\Theta}_1^{(j)})^T \Phi_{1,J-1} + \Lambda^{(j)} f_t + \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} e_t^{(j)}
\]

where \( (\tilde{\Theta}_1^{(j)})^T = -[A_1^{(j)}, \ldots, A_p^{(j)}], \Lambda^{(j)} = \Lambda^{(j)} \Xi_1, \Lambda^{(j)} \Xi_2, \ldots, \Lambda^{(j)} \Xi_q] \).

\[
\Phi^T = [y_t, y_{t-1}, \ldots, y_{t-p+1}, f_t, f_{t-1}, \ldots, f_{t-q+1}] \quad \text{and the notation} \quad \zeta^T \quad \text{denotes the transpose of} \quad \zeta.
\]

We remark that the MSETARX model with the representation (6) permits us to make use of the Arnold and Gunther (2001) proposed adaptive parameter estimation algorithm for the MSETAR model (3).

### 3. On the Stationarity of MSETARX model

In this section, we establish the conditions for the existence of a solution for the model equation (4). Let \( p = \max\{p|J| \in \{1, \ldots, L\}^0\} \) and \( q \) be the model orders for model (4). Now, suppose that \( p \) and \( q \) be regime independent and \( \theta_0^{(j)} = 0 \) for each \( J \). We rewrite model equation (4) in the form

\[
y_t = \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} (\tilde{\Theta}_1^{(j)})^T \Phi_{1,J-1} + \Lambda^{(j)} f_t + \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} e_t^{(j)}
\]

\[
f_t = (\tilde{\Theta}_2)^T \Phi_{2,J} + \eta_t
\]

where \( (\tilde{\Theta}_1^{(j)})^T = -[A_1^{(j)}, \ldots, A_p^{(j)}], \quad (\tilde{\Theta}_2)^T = -[\Xi_1, \Xi_2, \ldots, \Xi_q] \).

\[
\Phi^T = [y_t, y_{t-1}, \ldots, y_{t-p+1}^T, f_t, f_{t-1}, \ldots, f_{t-q+1}^T] \quad \text{and} \quad \Phi^T = [f_t, f_{t-1}, \ldots, f_{t-q+1}^T].
\]

The equation model (7)-(8) can be represented as a nonlinear ARX model (Masry and Tjøstheim (1997)) of the form :

\[
\begin{cases}
    y_t = g_1(y_{t-1}^T, \ldots, y_{t-p}^T) + g_2(f_t, \ldots, f_{t-q}^T) + \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} e_t^{(j)} \\
    f_t = g_3(f_{t-1}^T, \ldots, f_{t-q}^T) + \eta_t
\end{cases}
\]

with \( g_1(y_{t-1}, \ldots, y_{t-p}) = \sum_{J \in \{1, \ldots, L\}^0} y_{t,J}^{(j)} (\tilde{\Theta}_1^{(j)})^T \Phi_{1,J-1}, \quad g_2(f_t, \ldots, f_{t-q}) = \sum_{J \in \{1, \ldots, L\}^0} y_t^{(j)} \Lambda^{(j)} f_t, \quad \text{and} \quad g_3(f_{t-1}, \ldots, f_{t-q}^T) = (\tilde{\Theta}_2)^T \Phi_{2,J} \quad \text{the process} \quad \{f_t, y_t\} \quad \text{of the equation model (9)} \quad \text{is a Markov process.}
Lemma 1. Under Assumption 2, the result is known as in Lemma 3.1 in Masry and Tjøstheim (1997) and thus we are geometrically ergodic if \( A \) holds, the sup \( \sup_{\gamma \geq 1} E[\|\gamma\|^{1+\gamma}] < \infty \) for some \( \gamma > 0 \).

Remark 1. We denote \( \eta = (y_{t-1}, \ldots, y_{t-p}) \) and \( \beta = (f_t, \ldots, f_{T-q}) \). The multivariate SETARX model \( \mathbb{A} \) satisfies the following:

1. The functions \( g_1^{(j)}(\eta), g_2^{(j)}(\beta) \), and \( g_3(\beta) \) for each \( J \in \{1, \ldots, L\}^D \) are nonperiodic and bounded on compact sets, and \( g_2^{(j)}(\beta) = O(\|\beta\|^\gamma) \) as \( \|\beta\| \to 0 \) for some real \( \gamma \).
2. Assumption 2 holds, the sup \( \sup_{\gamma \geq 1} E[\|\gamma\|^{1+\gamma}] < \infty \) for some \( \gamma > 0 \).
3. There exist \( \alpha^{(j)} = [\alpha_1^{(j)}, \alpha_2^{(j)}, \ldots, \alpha_p^{(j)}] \) and \( \beta = [\beta_1, \beta_2, \ldots, \beta_{Q-1}] \), each of which may be the zero matrix, for each \( J \in \{1, \ldots, L\}^D \), where \( \alpha^{(j)} \) and \( \beta \) are matrices of dimension \( D \times D \) and \( \kappa \times \kappa \) respectively such that \( g_1^{(j)}(\eta) = \eta(\alpha^{(j)})^T + o(\|\eta\|) \) and \( g_3(\beta) = f(\beta)^T + o(\|\beta\|) \) as \( \|\eta\| \) and \( \|\beta\| \to \infty \). Then the \( D \times D \)-dimensional square matrix \( \mathbb{A} \) defined by 0 if \( \alpha^{(j)} = 0 \) and by

\[
\mathbb{A} = \begin{bmatrix}
O_D & O_D & \cdots & O_D & (\alpha_1^{(j)})^T \\
I_D & O_D & \cdots & O_D & (\alpha_2^{(j)})^T \\
O_D & I_D & \cdots & O_D & (\alpha_3^{(j)})^T \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O_D & O_D & \cdots & I_D & (\alpha_p^{(j)})^T
\end{bmatrix}
\]

otherwise, and the \( \kappa \times \kappa \) dimensional square matrix \( \mathbb{B} \) be defined by

\[
\mathbb{B} = \begin{bmatrix}
O_\kappa & O_\kappa & \cdots & O_\kappa & (\beta_1)^T \\
I_\kappa & O_\kappa & \cdots & O_\kappa & (\beta_2)^T \\
O_\kappa & I_\kappa & \cdots & O_\kappa & (\beta_3)^T \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
O_\kappa & O_\kappa & \cdots & I_\kappa & (\beta_q)^T
\end{bmatrix}
\]

satisfy \( \varrho(\mathbb{A}) < 1 \) and \( \varrho(\mathbb{B}) < 1 \), where \( \varrho \) denotes the spectral radius, \( O \) denotes the \( t \)-dimensional zero square matrix and \( I \) denotes the \( t \)-dimensional unit square matrix.

Lemma 1. Under Assumption 2 \( \{f, y_t\} \) of the multivariate SETARX model \( \mathbb{A} \) represented as a nonlinear ARX model \( \mathbb{B} \) is \( \alpha \)-mixing with mixing coefficient \( \alpha(k) \sim e^{-\beta k} \) for some \( \beta > 0 \).

Proof. The result is known as in Lemma 3.1 in Masry and Tjøstheim (1997) and thus we do not provide the proof since it is roughly same. We refer the interested reader to remarks after Assumption 3.3 and Lemma 3.1 in Masry and Tjøstheim (1997) and the references therein.

Remark 1. Lemma 1 provides sufficient conditions for the multivariate SETAR process \( \{f, y_t\} \) to be stationary (Masry and Tjøstheim (1997); Tjøstheim (1990); Pham (1986)). The proof of this Lemma as in Lemma 3.1 in Masry and Tjøstheim (1997) implies geometric ergodicity and stronger conclusion of absolute regularity with an exponentially decreasing rate (Tjøstheim (1990); Pham (1986); Tweedie (1975, 1988)).

Lemma 2. Let \( \alpha_0^{(j)} = 0 \) for each \( J \) in model \( \mathbb{A} \) and \( p = 1 \). Assume that there is a D-cycle of indexes \( j_1 \to j_2 \to j_3 \to \cdots \to j_D \to j_1 \) with the notation \( A_{j_1}^{(j)} \) corresponding to \( A_{j_1}^{(j)} \) (mod the D-cycle) so that \( A_{j_1}^{(j_1)} = A_{j_1}^{(j)} \). The process \( \{y_t\} \) of the multivariate SETAR model \( \mathbb{A} \) is geometrically ergodic if

\[
\varrho \left( \prod_{i=1}^{D} -A_{j_1}^{(j)} \right) < 1
\]
where \( g \) denotes the spectral radius and the product notation \( \prod_{i=m}^{n} A^{(j)} = A^{(j_n)} \cdots A^{(j_{m+1})} A^{(j_m)} \) is interpreted as the identity matrix if \( n = m - 1 \).

**Proof.** The result about geometric ergodicity follows from Theorem 4.5 and equation model (4.12) in Tjøstheim (1990) with \( A_i = -A^{(j)}_i \) and \( k = D \).

4. Estimation of model parameters

In this section, we assume that assumption 2 and Lemma 2 are satisfied. We also assume the model orders \( p, q, d \), and \( L \), of model (4)–(6) are known. Let model (4) be represented as a MSETAR \((L, p, q; J, J^0)\) model (5) with exogenous variables or factors as in model (5)–(6). We propose to use estimation procedures based on the standard LSE approach and the concept of self-tuning regulators used in the study of adaptive control of stochastic linear systems (see Kumar and Varaiya (1986)). Arnold and Gunther (2001) has shown that algorithms for estimation of parameters based on the stochastic gradient principles for linear systems are also suitable for nonlinear systems. Alternatively, following Cai and Masry (2000), one can use local linear fitting plus the projection method to estimate components \( g^{(j)}(\cdot) \) and \( g^{(j)}(\cdot) \) of model equation (6). The function \( g^{(i)}(\cdot) \) can then be estimated directly using a standard approach or by kernel-type estimation (Masry and Tjøstheim (1995)).

4.1. Standard LSE Algorithm for Parameter Estimation

Consider the MSETAR \((L, p, q; J, J^0)\) model in equation (6):

\[
y_t = \sum_{J \in [1, \cdots, L]} y_{J \in [1, \cdots, L]} (\hat{\Theta}^{(j)}_J)^T \hat{\Phi}_{t-1} + \epsilon_t
\]

where \((\hat{\Theta}^{(j)}_J)^T = [a_0^{(j)}, A_1^{(j)}, \cdots, A_p^{(j)}, A^{(j)}_{-1}, A^{(j)}_{-2}, \cdots, A^{(j)}_{-q}]\), \(\hat{\Phi}_t = [y_t, y_{t-1}, \cdots, y_{t-p+1}, \epsilon_t, \epsilon_{t-1}, \cdots, \epsilon_{t-q+1}]\) with the autoregressive orders \(p, q\), delay \(d\), and thresholds known. Then the LSE is \(\hat{\Theta}^{(j)}_J = \sum_{J \in [1, \cdots, L]} y_{J \in [1, \cdots, L]} (\hat{\Phi}_{t-1}) (\hat{\Phi}_{t-1})^{-1} \hat{\Phi}_{t-1} y_t\). Following Kumar and Varaiya (1986) presentation of the stochastic gradient algorithm for ARX systems, the true parameter \(\Theta^{(j)}_J\) can also be estimated by the LSE using the recursion:

\[
\hat{\Theta}^{(j)}_{k+1} = \hat{\Theta}^{(j)}_k + y_{k+1}^T R_k^{-1} \hat{\Phi}_k (y_{k+1} - \hat{\Phi}_k^T \hat{\Theta}^{(j)}_k)
\]

\[
R_k = \sum_{J \in [1, \cdots, L]} y_{J \in [1, \cdots, L]} \hat{\Phi}_i \hat{\Phi}_i^T
\]

4.2. Algorithm for Adaptive Parameter Estimation

Let \(0 < \alpha \leq 1\), \(0 < v^{(j)} \leq 1\), \(p^* = \max\{p, d, q\}\), and \(\tilde{\Theta}\) be the coefficients of the MSETAR \((L, p, q; J, J^0)\) model in equation (6).

\[
\tilde{\Theta}^{(j)}_k = 0; \quad k \leq p^*
\]

\[
\tilde{\Theta}^{(j)}_{k+1} = \tilde{\Theta}^{(j)}_k + y_{k+1}^T \frac{\alpha \hat{\Phi}_k}{3_k} (y_{k+1} - \hat{\Phi}_k^T \tilde{\Theta}^{(j)}_k); \quad k \geq p^*
\]
This algorithm corresponds to the adaptive parameter estimation algorithm proposed by Arnold and Gunther (2001), with the control sequence being \((s_k^{(j)})^{-1}\) instead of \((r_k^{(j)})^{-1}\). The simulation results presented by the authors showed that as the control sequence \((r_k^{(j)})^{-1}\) becomes large, a further progress towards the true coefficients is prevented or slowed down since this control sequence which weight the prediction error decrease too fast. The relaxed control sequence \((s_k^{(j)})^{-1}\) have similar properties as \((r_k^{(j)})^{-1}\) with the convergence speed decreased by the factors \(r_k^{(j)}\) and in particular, improves the estimation accuracy (Arnold and Gunther (2001)). This algorithm was applied in Leistritz et al. (2006) for the analysis of biomedical signals.

4.3. Simulations

In this section, we carry out a simulation exercise to study the performance of the parameter estimation algorithm presented in Section 4.2 on MSETARX models. In this respect, we consider two data generating process (DGP) according to the following:

1. Consider a simulated 50,000 points of a two-dimensional MSETARX process with six-regimes, delay \(d = 6\), \(\Lambda^{(j)} = 0\) for all \(J \in \{1, \ldots, L\}^D\) in equation (5), standard normal noise \(N(0, 1)\) added to all regimes and autoregressive order \(p = 3\) defined by:

\[
y_t = \begin{cases}
    a_0^{(1)} + a_1^{(1)} y_{t-1} + a_2^{(1)} y_{t-2} + a_3^{(1)} y_{t-3} + \omega_t; & \text{for } R_1 := [-\infty, -0.50) \times [-\infty, 0) \\
    a_0^{(2)} + a_1^{(2)} y_{t-1} + a_2^{(2)} y_{t-2} + a_3^{(2)} y_{t-3} + \omega_t; & \text{for } R_2 := [-\infty, -0.50) \times [0, \infty) \\
    a_0^{(3)} + a_1^{(3)} y_{t-1} + a_2^{(3)} y_{t-2} + a_3^{(3)} y_{t-3} + \omega_t; & \text{for } R_3 := [-0.50, 0.50) \times [-\infty, 0) \\
    a_0^{(4)} + a_1^{(4)} y_{t-1} + a_2^{(4)} y_{t-2} + a_3^{(4)} y_{t-3} + \omega_t; & \text{for } R_4 := [-0.50, 0.50) \times (0.00, \infty) \\
    a_0^{(5)} + a_1^{(5)} y_{t-1} + a_2^{(5)} y_{t-2} + a_3^{(5)} y_{t-3} + \omega_t; & \text{for } R_5 := [0.50, \infty) \times [-\infty, 0) \\
    a_0^{(6)} + a_1^{(6)} y_{t-1} + a_2^{(6)} y_{t-2} + a_3^{(6)} y_{t-3} + \omega_t; & \text{for } R_6 := [0.50, \infty) \times (0.00, \infty)
\end{cases}
\]

Regime 1 \(R_1 := [-\infty, -0.50) \times [-\infty, 0)\), \(\omega_1^{(1)} = \begin{pmatrix} -0.02 \\ 0.00 \end{pmatrix}, \omega_2^{(1)} = \begin{pmatrix} 0.53 \\ 0.00 \end{pmatrix}, \omega_3^{(1)} = \begin{pmatrix} 0.30 \\ 0.00 \end{pmatrix}\)

\[
\omega_3^{(1)} = \begin{pmatrix} 0.00 \\ 0.30 \end{pmatrix}, \omega_0^{(1)} = \begin{pmatrix} 0.74 \\ -0.20 \end{pmatrix}
\]

Regime 2 \(R_2 := [-\infty, -0.50) \times [0, \infty)\), \(\omega_1^{(2)} = \begin{pmatrix} -0.02 \\ 0.00 \end{pmatrix}, \omega_2^{(2)} = \begin{pmatrix} 0.53 \\ 0.00 \end{pmatrix}, \omega_3^{(2)} = \begin{pmatrix} 0.30 \\ 0.00 \end{pmatrix}\)

\[
\omega_3^{(2)} = \begin{pmatrix} 0.00 \\ 0.30 \end{pmatrix}, \omega_0^{(2)} = \begin{pmatrix} -0.75 \\ -0.20 \end{pmatrix}
\]

Regime 3 \(R_3 := [-0.50, 0.50) \times [-\infty, 0)\), \(\omega_1^{(3)} = \begin{pmatrix} -0.94 \\ 0.00 \end{pmatrix}, \omega_2^{(3)} = \begin{pmatrix} 0.85 \\ 0.00 \end{pmatrix}, \omega_3^{(3)} = \begin{pmatrix} 0.30 \\ 0.00 \end{pmatrix}\)

\[
\omega_3^{(3)} = \begin{pmatrix} 0.00 \\ 0.30 \end{pmatrix}, \omega_0^{(3)} = \begin{pmatrix} 1.15 \\ -0.20 \end{pmatrix}
\]
5. Concluding remarks

The recent financial crisis of 2007-2009 has lead to a need for regulators and policy makers to understand and track systemic linkages. As the events following the turmoil in financial markets unfolded, it became evident that modern financial systems exhibit a high degree of interdependence and nonlinearity making it difficult in predicting the consequences of such an intertwined system. In this study, we define a nonlinear multivariate SETARX model useful in modeling economic relationships and to capture non-linear dynamics such as regime switching and asymmetric responses to shocks. We then present an estimation procedure for the parameters.

In general, testing linearity is the first step of a proper modelling strategy of nonlinear models as it is possible that a linear model could adequately capture the relationship considered. Nonlinear models are usually not identified when the underlying process is linear (Terasvirta et al. (1994), Hubrich and Terasvirta (2013), Hansen (1999), Tsay (1998), Addo et al. (2014)). The proposed test statistic for detecting threshold nonlinearity in vectors time series and the procedure for building multivariate threshold models discussed in Tsay (1998) could be performed.
on each subprocess in the MSETARX model setting. In this case, Arnold and Gunther (2001) suggests a reasonable choice of the delay to be $d^* = \arg\max\{\sum_d c(i)(d) | d \in \{1, \ldots, d_{\max}\}\}$ where $c(i)(d)$ is the value of the test statistic (Tsay (1998)) for each subprocess $i$. One could apply the Wald test procedure used in Balke (2000), which is a generalisation of Hansen (1996) approach, to test linearity. Another possibility of testing linear VAR model against a MSETARX model would be to generalise the approach of Strikholm and Terasvirta (2006) to multivariate models.

After the parameter estimation of model (5), it is necessary to evaluate the model by appropriate misspecification tests before putting it into practice. The general purpose is to find out if the assumptions made in the estimation step appear satisfied (Tsay (1998); Strikholm and Terasvirta (2006); Hansen (1997, 2000)). For more details about modelling strategies and issues of vector threshold autoregressive models, we refer interested readers to Tsay (1998); Hansen (2011). This model could be very useful in studying huge data sets such as the analysis of high-frequency financial data.

Many problems remain open for the multivariate SETARX models. For example, establishing a testing procedure in determining the number of regimes and the specification of the threshold space will required a careful investigation.

Acknowledgement

This research is supported by the Erasmus Mundus Fellowship. We are grateful to Lutz Leistritz for his support.
Appendix A. Estimation Results

We provide below the estimation of parameters obtained via LSE algorithm in Section 4.3 on the first simulated process in Section 4.3. The regime time corresponds to the number of temporal samples, where the multivariate process stayed in each regime.

Regime 1 \( R_1^t := [\infty, -0.50) \times [\infty, 0) \), \( \hat{\alpha}_1^{(1)} = \begin{pmatrix} -0.0278 & -0.0169 \\ 0.0027 & 0.2812 \end{pmatrix} \), \( \hat{\alpha}_2^{(1)} = \begin{pmatrix} 0.5275 & 0.0025 \\ 0.0013 & 0.3073 \end{pmatrix} \), \( \hat{\beta}_3^{(1)} = \begin{pmatrix} 0.0069 & 0.5419 \\ -0.0005 & 0.3046 \end{pmatrix} \), \( \hat{\beta}_0^{(1)} = \begin{pmatrix} 0.7399 \\ -0.2012 \end{pmatrix} \) (regime time: 10927).

Regime 2 \( R_2^t := [\infty, -0.50) \times [0, \infty) \), \( \hat{\alpha}_1^{(2)} = \begin{pmatrix} -0.0156 & 0.0009 \\ 0.0033 & 0.2935 \end{pmatrix} \), \( \hat{\alpha}_2^{(2)} = \begin{pmatrix} 0.5317 & -0.0051 \\ 0.0043 & 0.3102 \end{pmatrix} \), \( \hat{\beta}_3^{(2)} = \begin{pmatrix} -0.0012 & 0.5173 \\ -0.0021 & 0.2904 \end{pmatrix} \), \( \hat{\beta}_0^{(2)} = \begin{pmatrix} -0.7404 \\ -0.1951 \end{pmatrix} \) (regime time: 8770).

Regime 3 \( R_3^t := [-0.50, 0.50) \times [\infty, 0) \), \( \hat{\alpha}_1^{(3)} = \begin{pmatrix} -0.9417 & -0.0008 \\ 0.0143 & 0.2859 \end{pmatrix} \), \( \hat{\alpha}_2^{(3)} = \begin{pmatrix} 0.8602 & 0.0040 \\ 0.0211 & 0.3003 \end{pmatrix} \), \( \hat{\beta}_3^{(3)} = \begin{pmatrix} 0.0067 & 0.8483 \\ -0.0004 & 0.3014 \end{pmatrix} \), \( \hat{\beta}_0^{(3)} = \begin{pmatrix} 1.1337 \\ -0.2408 \end{pmatrix} \) (regime time: 3932).

Regime 4 \( R_4^t := [-0.50, 0.50) \times (0, \infty) \), \( \hat{\alpha}_1^{(4)} = \begin{pmatrix} -0.9302 & -0.0210 \\ -0.0023 & 0.3142 \end{pmatrix} \), \( \hat{\alpha}_2^{(4)} = \begin{pmatrix} 0.8631 & 0.0033 \\ -0.0135 & 0.3116 \end{pmatrix} \), \( \hat{\beta}_3^{(4)} = \begin{pmatrix} 0.0066 & 0.8497 \\ 0.0079 & 0.2789 \end{pmatrix} \), \( \hat{\beta}_0^{(4)} = \begin{pmatrix} 0.7101 \\ 0.1960 \end{pmatrix} \) (regime time: 3235).

Regime 5 \( R_5^t := [0.50, \infty) \times [\infty, 0.00) \), \( \hat{\alpha}_1^{(5)} = \begin{pmatrix} -1.1008 & -0.0056 \\ 0.0032 & 0.2923 \end{pmatrix} \), \( \hat{\alpha}_2^{(5)} = \begin{pmatrix} -0.2918 & -0.0012 \\ 0.0019 & 0.3106 \end{pmatrix} \), \( \hat{\beta}_3^{(5)} = \begin{pmatrix} 0.0066 & -0.2971 \\ 0.0029 & 0.2958 \end{pmatrix} \), \( \hat{\beta}_0^{(5)} = \begin{pmatrix} -0.7595 \\ 0.1927 \end{pmatrix} \) (regime time: 9697).

Regime 6 \( R_6^t := [0.50, \infty) \times (0.00, \infty) \), \( \hat{\alpha}_1^{(6)} = \begin{pmatrix} -1.0995 & 0.0087 \\ 0.0013 & 0.3147 \end{pmatrix} \), \( \hat{\alpha}_2^{(6)} = \begin{pmatrix} 0.3011 & -0.0150 \\ 0.0029 & 0.2904 \end{pmatrix} \), \( \hat{\beta}_3^{(6)} = \begin{pmatrix} -0.0004 & 0.3033 \\ 0.0026 & 0.2996 \end{pmatrix} \), \( \hat{\beta}_0^{(6)} = \begin{pmatrix} 1.1508 \\ 0.1942 \end{pmatrix} \) (regime time: 13433).

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