Modular geometry of the symplectic group attached to a \( q \)-level system and to multiple \( q \)-dit mixtures

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Abstract. We study the commutation relations within the Pauli groups built on all decompositions of a given Hilbert space dimension \( q \), containing a square, into its factors. Illustrative low dimensional examples are the quartit (\( q = 4 \)) and two-qubit (\( q = 2^2 \)) systems, the octit (\( q = 8 \)), qubit/quartit (\( q = 2 \times 4 \)) and three-qubit (\( q = 2^3 \)) systems, and so on. In the single qudit case, e.g. \( q = 4, 8, 12, \ldots \), one defines a bijection between the \( \sigma(q) \) maximal commuting sets (with \( \sigma(q) \) the sum of divisors of \( q \)) of Pauli observables and the maximal submodules of the modular ring \( \mathbb{Z}_{q^2} \), that arrange into the projective line \( \mathbb{P}_1(\mathbb{Z}_q) \) and an independent set of size \( \sigma(q) - \psi(q) \) (with \( \psi(q) \) the Dedekind psi function). In the multiple qudit case, e.g. \( q = 2^2, 3^2, \ldots \), the Pauli graphs rely on symplectic polar spaces such as the generalized quadrangles \( GQ(2,2) \) (if \( q = 2^2 \)) and \( GQ(3,3) \) (if \( q = 3^2 \)). More precisely, in dimension \( p^n \) (\( p \) a prime) of the Hilbert space, the observables of the Pauli group (modulo the center) are seen as the elements of the \( 2n \)-dimensional vector space over the field \( \mathbb{F}_p \). In this space, one makes use of the commutator to define a symplectic polar space \( W_{2n-1}(p) \) of cardinality \( \sigma(p^{2n-1}) \), that encodes the maximal commuting sets of the Pauli group by its totally isotropic subspaces. Building blocks of \( W_{2n-1}(p) \) are punctured polar spaces (i.e. a observable and all maximum cliques passing to it are removed) of size given by the Dedekind psi function \( \psi(p^{2n-1}) \). For multiple qudit mixtures (e.g. qubit/quartit, qubit/octit and so on), one finds multiple copies of polar spaces, punctuation polar spaces, hypercube geometries and other intricate structures. Such structures play a role in the science of quantum information.

1. Introduction

The \( q \)-level quantum systems (also denoted \( q \)-dits, or qudits), and tensor products of them, possibly with a different number of levels in each factor, are basic constituents of quantum information processing. Multiple qubits, that are tensor products of two-qubit systems are routinely employed in quantum algorithms, but multiple copies of \( q \)-dits (with \( q > 2 \)) may turn to be more interesting in terms of self error-correction, and in relation to multipartite communication, as on the quantum Internet. The most general system would be a mixture of multiple qudits corresponding to the factors of a integer factorization of the Hilbert space dimension as \( q = \prod_i q_i^{p_i} \). Let us point out that, for a given dimension \( q \), there exists several such factorizations, leading to distinct quantum systems. In the lowest dimensional case involving a square, one has either \( q = 4 \) or \( q = 2^2 \), corresponding to the single quartit and two-qubit systems, respectively. It may be convenient to use a 4-level system (like the states of a nuclear spin \( \frac{3}{2} \)) to physically implement the two-qubit CNOT gate [1], and in some respect both systems display similar symmetries (like in the Bloch sphere representation) [2], but in general they have...
distinctive features (like in the Pauli group of observables and in the structure of the maximal commuting sets).

In this paper, we focus on the commutation relations of observables attached to a selected decomposition of the Hilbert space dimension $q$. The observables in a factor are defined from the action on a vector $|s\rangle$ of the $q_i$-dimensional Hilbert space of the $q_i$-dit Pauli group generated by two unitary $X$ (shift) and clock $Z$ operators via $X|s\rangle = |s + 1\rangle$ and $Z|s\rangle = \omega^s|s\rangle$, with $\omega$ a primitive $q_i$-th root of unity. Then the observables in dimension $q$ are obtained by taking tensor products over the $q_i$-dimensional observable of each factor. A Pauli graph is constructed by taking the observables as vertices and a edge joining two commuting observables. Maximal sets of mutually commuting observables, i.e., maximum cliques of the Pauli graph, are used to define a point/line incidence geometry with observables as points and maximum cliques as lines.

In recent papers, multiple qubits [3, 4], single qudits [5, 6, 7] and a few examples of qudit mixtures [9] were already explored. Further work was published to clarify this earlier work dealing with symplectic polar spaces of multiple qudits [10, 11, 12] and, in what concerns multiple qubits, its link to units in Clifford algebras [13] and to Lie algebras [14]. Prior to the advent of quantum information science, the incidence properties of the $q$-dimensional geometry and the relations to Clifford algebras were published in [15, 16]. The link of mutual unbiasedness to the general theory of angular momentum is explored in [17], and its link to Feynman’s path integral may be found in [18].

In this paper, we focus on quantum systems of Pauli observables defined over the Hilbert space of dimension $q$ containing a square. In the single qudit case, studied in Sec. 2, the maximal mutually commuting sets of observables in the Hilbert space of dimension $q$ are mapped bijectively to the maximal submodules over the ring $\mathbb{Z}_q$ [5, 6]. If $q$ contains a square, there are $\psi(q) = q\prod_{p|q}(1 + \frac{1}{p})$ points on the projective line $P_1(\mathbb{Z}_q)$ (in the Dedekind function $\psi(q)$), the product is taken over all primes $p$ dividing $q$ and the remaining $\sigma(q) - \psi(q) \neq 0$ independent points (with $\sigma(q)$ the sum of divisors function) is playing the role of a reference frame and possess their own modular substructure. The number theoretical properties of the modular ring $\mathbb{Z}_q$ are used to count the cardinality of the symplectic group $\text{Sp}(2, \mathbb{Z}_q)$ [7, 19, 20]. In Sec. 3, we remind the established results concerning the point/line geometries attached to multiple qudit systems in dimension $p^n$, that symplectic polar spaces $W_{2n-1}(p)$ of order $p$ and rank $n$ govern the commutation structure of the observables. Here, the number theoretical functions $\sigma(p^{2n-1})$ and $\psi(p^{2n-1})$ are found to count the number of observables in the symplectic polar space and in the punctured polar space, respectively. In Sec. 4, we study composite systems when at least one of the factors $q_i$ of the Hilbert space dimension is a square. It is shown, that the non-modularity leads to a natural splitting of the Pauli graph/geometry into several copies of basic structures such as polar spaces, punctured polar spaces and related hyperdimensional structures.

A few properties of the structures we have checked are in table 1. Details are given in the subsequent sections.

Most calculations are performed on Magma [21]. High dimensional computations have been made possible thanks to the supercomputer facilities of the Mésocentre de calcul at University of Franche-Comté.

2. Pauli graph/geometry of a single qudit
A single qudit is defined by a Weyl pair $(X, Z)$ of shift and clock cyclic operators satisfying

$$ZX - \omega XZ = 0,$$ (1)
| $q$ | name [Ref.] | # cliques | geometry | spectrum | aut. group |
|-----|-------------|-----------|----------|----------|------------|
| $4$ | quartit [6, 2] | $6 + 1$ | $\mathcal{P}(\mathbb{Z}_4)^3$ | $\{4^4, 0^{4-q}, -2^4\}$ | $G_{48} = \mathbb{Z}_2 \times S_4$ |
| $2^2$ | 2-qubit [3, 4] | $15$ | $GQ(2,2)$ | $\{6^4, 1^{q-3}, -3^2\}$ | $S_6$ |
| $8$ | octit [6, 2] | $12 + 3$ | $\mathcal{P}(\mathbb{Z}_8)^3$ | $\{8^4, 0^{4-q}, -4^4\}$ | $2^2 \times (\mathbb{Z}_4 \times G_{48})$ |
| $2.2^2$ | qubit/quartit [9] | $36 + 3$ | $3 \times GQ(2,2)$ | $\{5^4, 1^6, -1^2, -3^3\}$ | $G_{48} \times S_3$ |
| $2^3$ | 3-qubit [3, 4] | $135$ | $W_5(2)$ | $\{30^4, 3^{35}, 9^2, 5^{27}\}$ | $\text{Sp}(6,2)$ |
| $9$ | 9-dit [6] | $12 + 1$ | $\mathcal{P}(\mathbb{Z}_9)^3$ | $\{9^4, 0^{4-q}, -3^4\}$ | $G_{648} \times G_{48}$ |
| $3^2$ | 2-qutrit [3] | $40$ | $GQ(3,3)$ | $\{25^{10}, 5^{24}, -1^{40}, -7^{15}\}$ | $2^{10} \times W(E_6)$ |
| $12$ | 12-dit [6] | $24 + 4$ | $\mathcal{P}(\mathbb{Z}_{12})^3$ | $\{12^4, 2^8, 0^{4q+4}, -4^4, -6^4\}$ | $Z_{12}^4 \times G_{144}$ |
| $2^3.3$ | 2-qubit/quartit | $24 + 4$ | as above | as above | $S_6^3 \times S_4$ |
| $16$ | 16-dit [6] | $24 + 7$ | $\mathcal{P}(\mathbb{Z}_{16})^3$ | $\{16^4, 0^{4q+4}, -8^4\}$ | $A_2^2 \times G_{48}$ |
| $2^4$ | qubit/octit | $72 + 15$ | $6 \times GQ(2,2)$ | $\{5^4, 1^6, -1^2, -3^3\}$ | $G_{48} \times S_6$ |
| $4^2$ | 2-quartit | $120 + 31$ | 15-cube | $\{3^4, 3^4, -1^2, -3^2\}$ | $G_{48} \times S_{15}$ |
| $2^4.4^2$ | 2-qutrit/quartit | $300 + 31$ | $3 \times W_5(2)$ | $\{13^4, 5^{25}, 3^9, -1^{70}, -5^8, -7^{10}\}$ | $(Z_2^4 \times S_6)^3 \times S_3$ |
| $2^4$ | 4-qubit [3, 4] | $2395$ | $W_7(2)$ | $\{120^4, 7^{35}, -3^{19}\}$ | $\text{Sp}(8,2)$ |
| $18$ | 18-dit [6] | $36 + 3$ | $\mathcal{P}(\mathbb{Z}_{18})^3$ | $\{18^4, 3^6, 0^{6q+4}, -6^4, -9^4\}$ | $Z_3^4 \times (Z_2^4 \times G_{144})^2$ |
| $2^3.2^9$ | qubit/9-dit | $36 + 3$ | as above | as above | as above |
| $2^4.3^3$ | 2-qutrit/quartit [9] | $120$ | $3 \times GQ(3,3)$ | $\{12^4, 2^8, -4^{15}\}$ | $W(E_6)^3 \times G_{48}$ |
| $24$ | 24-dit [6] | $48 + 12$ | $\mathcal{P}(\mathbb{Z}_{24})^3$ | $\{24^4, 4^8, 0^{8q+4}, -8^4, -12^4\}$ | $G_{24^3} \times (Z_2^4 \times G_{144})^2$ |
| $2^3.3^3$ | 2-qubit/3-dit/4-dit | $144 + 12$ | see Sec. 4 | see Sec. 4 | |
| $2^4.3$ | 3-qubit/quartit | $540$ | $4 \times W_5(2)$ | $\{56^{1}, 14^{15}, 2^{15}, -4^{25}\}$ | $\text{Sp}(6,2)^4 \times S_4$ |

Table 1. The main properties of the studied Pauli graphs. The first and second column gives the selected decomposition of $q$ and the name of the corresponding Pauli system, respectively. Third column represents the number of maximal sets of mutually commuting observables of size $q - 1$ (i.e. the number of maximum cliques in the corresponding Pauli graph) and how it splits into two numbers of geometrical significance explained in the paper. The fourth column provides a geometry that may be identified. The fifth column provides the spectrum of the Pauli graph, that of its dual geometry or that of an important subgraph, depending on context (see the corresponding section for details). The automorphism group of the selected geometry is given in the last column. The notation $S_n$, $A_n$ and $D_n$ is for the symmetric, alternating and dihedral group, respectively. Symbols $\times$, $\rtimes$ and $.$ are for the direct, semidirect and not semidirect products of groups, respectively.

The notation $W_{2n-1}(p)$ is for the symplectic polar space of order $p$ and rank $n$ [3, 4]. The polar space $W_3(2)$ is the generalized (self-dual) quadrangle of order two $GQ(2,2)$, also called the doily. The notation $W_{2n-1}(p)^{-}$ means the polar space $W_{2n-1}(p)$ minus a perp-set (i.e. a point and the maximum cliques passing through it). Whenever multiple polar spaces are featured in the table, it means that we are dealing with the mutual incidence of cliques at multiple points (see Sec. 4 for details).

† The incidence geometry is associated to the maximum cliques of the Pauli graph and the spectrum is that of all cliques (see Sec. 2 for details).

where $\omega = \exp \frac{2\pi i}{q}$ is a primitive $q$-th root of unity and $0$ is the null $q$-dimensional matrix. In the standard computational basis $\{|s\}, s \in \mathbb{Z}_q\}$, the explicit form of the pair is as follows

$$X = \begin{pmatrix}
0 & 0 & \ldots & 0 & 1 \\
1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & \ldots & 0 & 0 \\
0 & 0 & \ldots & 1 & 0 \\
\end{pmatrix}, \quad Z = \text{diag}(1, \omega, \omega^2, \ldots, \omega^{q-1}).$$

(2)
The Weyl pair generates the single qudit Pauli group $\mathcal{P}_q = (X, Z)$, of order $q^3$, where each element may be written in a unique way as $\omega^a X^b Z^c$, with $a, b, c \in \mathbb{Z}_q$.

It will be shown in this section that the study of commutation relations in an arbitrary single qudit system may be based on the study of symplectic modules over the modular ring $\mathbb{Z}_q^2$, and conversely that the elegant number theoretical relations underlying the isotropic lines of $\mathbb{Z}_q^2$ have their counterpart in the maximal commuting sets of a qudit system. Our results may be found in various disguises in several publications where the proofs are given [6, 7, 19, 20].

Let us start with the Weyl pair property (1) and write the group theoretical commutator as

$$[X, Z] = XZX^{-1}Z^{-1} = \omega^{-1}I_q$$

(where $I_q$ is the $q$-dimensional identity matrix), so that one gets the expression

$$[\omega^a X^b Z^c, \omega^{a'} X^{b'} Z^{c'}] = \omega^{cb' - c'b} I_q,$$

meaning that two elements of $\mathcal{P}_q$ commute if only if the determinant $\Delta = \text{det} \begin{pmatrix} b' & b \\ c' & c \end{pmatrix}$ vanishes.

Two vectors such that their symplectic inner product $[(b', c'), (b, c)] = \Delta = b'c - bc'$ vanishes are called perpendicular. Thus, from (3), one can transfer the study of commutation relations within the group $\mathcal{P}_q$ to the study of perpendicularity of vectors in the ring $\mathbb{Z}_q^2$ [6].

From (3), one gets the important result that the set $\mathcal{P}'_q$ of commutators (also called the derived subgroup) and the center $Z(\mathcal{P}_q)$ of the Pauli group $\mathcal{P}_q$ are identical, and one is led to the isomorphism

$$(\mathcal{P}_q / Z(\mathcal{P}_q), \times) \cong (\mathbb{Z}_q^2, +),$$

i.e. multiplication of observables taken in the central quotient $\mathcal{P}_q / Z(\mathcal{P}_q)$ transfers to the algebra of vectors in the $\mathbb{Z}_q$ module $\mathbb{Z}_q^2$ endowed with the symplectic inner product $\cdot, \cdot$.

### Isotropic lines of the lattice $\mathbb{Z}_q^2$

Let us now define an isotropic line as a set of $q$ points on the lattice $\mathbb{Z}_q^2$ such that the symplectic product of any two of them is 0(mod $q$). From (4), to such an isotropic line corresponds a maximal commuting set in $\mathcal{P}_q / Z(\mathcal{P}_q)$.

Taking the prime power decomposition of the Hilbert space dimension as $q = \prod_i p_i^{n_i}$, it is shown in (18) of [7] that the number of isotropic lines of the lattice $\mathbb{Z}_q^2$ reads

$$\eta(q) = \prod_i p_i^{n_i+1} - 1 \equiv \sigma(q),$$

where $\sigma(q)$ denotes the sum of divisor function $^1$

It may be checked from table 1 (column 3), that the number of maximum cliques in the Pauli graph of $\mathcal{P}_q$ [i.e. the number of maximal commuting set in $\mathcal{P}_q / Z(\mathcal{P}_q)$] in the considered single qudit decompositions $q = 4, 8, 9, 12, 16$ and 18 are $\sigma(4) = 1 + 2 + 4 = 7$, $\sigma(8) = 1 + 2 + 4 + 8 = 15$, $\sigma(9) = 13$, $\sigma(12) = 27$, $\sigma(16) = 31$ and $\sigma(18) = 39$, respectively.

Another important quantity is the number $\eta(q; x)$ of isotropic lines through a given point $x = (b, c)$ of the lattice. Denoting by $t_i = v_{p_i}(x)$ the $p_i$-valuation $^2$ of $x$, it is shown in (36) of [7] that one obtains

$$\eta(q; x) = \prod_i p_i^{t_i+1} - 1 \equiv \sigma(\tilde{q}(x)),$$

$^1$ The identification of $\eta(q)$ to $\sigma(q)$ is not provided in [7]. However, it is easy to see that the factors in (5) are $p_i^{t_i+1} - 1 = 1 + p_i + p_i^2 + \cdots + p_i^{t_i} = \sigma(p_i^t)$ and, since $\sigma(q)$ is multiplicative, (5) immediately follows. Similarly, the identification of $\eta(q; x)$ to $\sigma(\tilde{q}(x))$ given in (6) is easy to establish.

$^2$ The $p$-adic valuation $v_p(x)$ of an integer number $x$ is the highest exponent $t$ such that the power of prime $p^t$ divides $x$. 

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where \( \tilde{q}(x) = \prod_i p_i^{l_i} \leq q \) is a local dimension defined at the selected point \( x \).

The projective line \( \mathbb{P}_1(\mathbb{Z}_q) \) and the symplectic group \( \text{Sp}(2, \mathbb{Z}_q) \)

As shown in [7], a isotropic line of \( \mathbb{Z}_q^2 \) corresponds to a Lagrangian submodule, i.e. a maximal module such that the perpendicular module \( M^\perp = M \). Let us now specialize to Lagrangian submodules that are free cyclic submodules

\[
\mathbb{Z}_q(b, c) = \{(ub, uc) | u \in \mathbb{Z}_q\},
\]

for which the application \( u \rightarrow (ub, uc) \) is injective. Not all Lagrangian submodules are free cyclic submodules. A point \( x = (b, c) \) such that \( \mathbb{Z}_q(b, c) \) is free is called an admissible point, and the set of admissible points is called the projective line

\[
\mathbb{P}_1(\mathbb{Z}_q) = \{ \mathbb{Z}_q(b, c) | (b, c) \text{ is admissible} \}.
\]

Following theorem 5 in [6], the number of points of the projective line is

\[
|\mathbb{P}_1(\mathbb{Z}_q)| = \prod_i (p_i^{s_i} + p_i^{s_i - 1}) \equiv \psi(q),
\]

where \( \psi(q) = q \prod_{p|q} (1 + \frac{1}{p}) \) and the product is taken over all primes \( p \) dividing \( q \).

3 Note that one has \( \psi(q) \leq \sigma(q) \), where the equality holds if \( q \) is square-free integer.

In the considered single qudit decompositions \( q = 4, 8, 9, 12, 16 \) and \( 18 \), that contain a square, one gets \( \psi(4) = 4(1 + \frac{1}{2}) = 6 \), \( \psi(8) = 8(1 + \frac{1}{2}) = 12 \), \( \psi(9) = 12 \), \( \psi(12) = 24 \), \( \psi(16) = 24 \), \( \psi(18) = 36 \), as it is also shown in table 1 (column 3).

Then, still using theorem 5 in [6], the number of points of the projective line containing a selected vector \( x = (b, c) \) of the lattice reads as

\[
|\mathbb{P}_1(\mathbb{Z}_q; x)| = \psi(\tilde{q}(x)),
\]

where \( \tilde{q}(x) \) is the local dimension introduced in (6).

As for the projective line \( \mathbb{P}_1(\mathbb{Z}_q) \), the symplectic group \( \text{Sp}(2, \mathbb{Z}_q) \) contains interesting number theoretical features.

We defined an admissible vector \( (b, c) \) as one leading to a point of the projective line \( \mathbb{P}_1(\mathbb{Z}_q) \). If \( q = p^s \), there are \( p^{2s} - p^{2(s-1)} \) admissible vectors and, for arbitrary dimensions \( q = \prod_i p_i^{s_i} \), the number of admissible vectors is

\[
q^2 \prod_i (1 - \frac{1}{p_i^2}) = \phi(q)\psi(q) = J_2(q),
\]

where \( \phi(q) = q \prod_{p|q} (1 - \frac{1}{p}) \) is the Euler totient function and \( J_2(q) \) is known as the Jordan totient function.

Following the same line of reasoning than (6) and (10), one may also define a finer structure of admissibility from the number \( J_2(\tilde{q}(x)) \), with \( \tilde{q}(x) = \prod_i p_i^{l_i} \leq q \) is the local dimension. If \( q = p^s \), one has \( \tilde{q}(x) = q \) and the structure is simpler than in the composite case such as \( q = 12 \) and \( q = 18 \).

The symplectic group \( \text{Sp}(2, \mathbb{Z}_q) \) is built from all matrices

\[
\begin{pmatrix}
  b' & b \\
  c' & c
\end{pmatrix}
\]

such that \( (b, c) \) is an admissible vector and the symplectic inner product, i.e. the determinant \( \Delta = b'c - bc' = 1 \). The cardinality of such a group is \( |\text{Sp}(2, \mathbb{Z}_q)| = qJ_2(q) \) [19, 20].

As for the relation (5), the identification of \( |\mathbb{P}_1(\mathbb{Z}_q)| \) to the Dedekind psi function \( \psi(q) \) is not provided in [6]. The proof is easy to establish since \( \psi(q) \) is a multiplicative function.
In the previous subsections, we investigated the bijection between sets of operators of the Pauli group $\mathcal{P}_q$ and vectors defined over the modular ring $\mathbb{Z}_q$. More precisely, from (4), elements of the central quotient of the Pauli group $\mathcal{P}_q/\mathbb{Z}(\mathcal{P}_q)$ were mapped to vectors of the lattice $\mathbb{Z}_q^2$ and, from (5) the $\sigma(q)$ isotropic lines of $\mathbb{Z}_q^2$ were mapped to its maximal commuting sets.

One can see these bijections in a clearer way by defining the Pauli graph $G_q$ of the qudit system. The Pauli graph $G_q$ is constructed by taking the observables as vertices and a edge joining two commuting observables. A maximal set of mutually commuting observables corresponds to a maximum clique of $G_q$, and one further defines a point/line incidence geometry with observables as points and maximum cliques as lines. One characterizes this geometry by creating a dual graph $G_q^*$ such that the vertices are the cliques and a edge joins two non-intersecting cliques. The connected component of $G_q^*$ corresponds to the graph of the projective line $\mathbb{P}_1(\mathbb{Z}_q)$ (as defined in previous papers [3]-[12]).

In the subsequent sections, we shall also introduce the graph $G^{(k)}_q$, in which the vertices are the maximum cliques of the Pauli graph $G_q$ and a edge joins two maximum cliques intersecting at $k$ points.

The Pauli graph of a qudit

For the four-level system, there are $4^2 - 1$ observables/vertices in the Pauli graph $G_4$. The $\sigma(4) = 7$ maximum cliques

$$cl := \{(X^2, Z^2, Z^2X^2), (X, X^2, X^3), (X^2, Z^2X, Z^2X^3), (Z, Z^2, Z^3), (ZX, Z^3X^2, Z^3X^3), (ZX^2, Z^2, Z^3X^2), (ZX^3, Z^2X^2, Z^3X)\}$$

are mapped to the following isotropic lines of $\mathbb{Z}_4^2$

$$il := \{(0,2), (2,0), (2,2)\}, \{(0,1), (0,2), (0,3)\}, \{(0,2), (2,1), (2,3)\},$$
$$\{(1,0), (2,0), (3,0)\}, \{(1,1), (2,2), (3,3)\}, \{(1,2), (2,0), (3,2)\},$$
$$\{(1,3), (2,2), (3,1)\}.$$  

From the latter list, one easily observes that non-admissible vectors belong to the first line $\{(0,2), (2,0), (2,2)\}$, that corresponds to the maximum clique $(X^2, Z^2, Z^2X^2)$. The remaing vectors in $\mathbb{Z}_4^2$ generate free cyclic submodules of the form (7).

The sequence of degrees in $G_q^*$ is obtained as $(1,0,0,0,6)$, meaning that the first clique given in (12) (of degree 0) intersects all the remaing ones, and that cliques number 2 to 7 in (12) (of degrees 4) form the projective line $\mathbb{P}_1(\mathbb{Z}_4)$. Indeed, one has $|\mathbb{P}_1(\mathbb{Z}_4)| = \psi(4) = 6$. There are $J_2(4) = \phi(4)\psi(4) = 12$ admissible points.

The graph $G_4^*$ is strongly regular, with spectrum $\{41, 0^{3+1}, -2^2\}$ (in the notations of [3]); the notation $0^{3+1}$ in the spectrum means that $0^4$ belongs to the projective line subgraph and there exists an extra 0 eigenvalue in the spectrum of $G_4^*$. The automorphism group of $\mathbb{P}_1(\mathbb{Z}_4)$ is found to be the direct product $G_{48} = \mathbb{Z}_2 \times S_4$ (where $S_4$ is the four-letter symmetric group).

The quartit system

For the four-level system, there are $4^2 - 1$ observables/vertices in the Pauli graph $G_q$. The $\sigma(4) = 7$ maximum cliques

$$cl := \{(X^2, Z^2, Z^2X^2), (X, X^2, X^3), (X^2, Z^2X, Z^2X^3), (Z, Z^2, Z^3),$$
$$ (ZX, Z^3X^2, Z^3X^3), (ZX^2, Z^2, Z^3X^2), (ZX^3, Z^2X^2, Z^3X)\}$$

are mapped to the following isotropic lines of $\mathbb{Z}_4^2$

$$il := \{(0,2), (2,0), (2,2)\}, \{(0,1), (0,2), (0,3)\}, \{(0,2), (2,1), (2,3)\},$$
$$\{(1,0), (2,0), (3,0)\}, \{(1,1), (2,2), (3,3)\}, \{(1,2), (2,0), (3,2)\},$$
$$\{(1,3), (2,2), (3,1)\}.$$  

The 12-dit system

The main results for all qudit systems with $4 \leq q \leq 18$, such that $q$ contains a square, are given in Table 1. We take the composite dimension $q = 2^2 \times 3$ as our second illustration. There are $12^2 - 1 = 143$ observables in the Pauli graph $G_{12}$. There are $\sigma(12) = 28$ maximum cliques in $G_{12}$, as expected. The sequence of degrees in the dual graph $G_{12}^*$ is found as $(4,0,\ldots,24)$, i.e. there are four cliques of degree 0 and the remaining $\psi(12) = 24$ ones have degree 12 (as also seen from the spectrum given in Table 1).

Owing to the composite character of the dimension, the structure of $G_{12}$ is more complex than in the quartit case, see Fig. 1 of [6] for a picture. All four independent cliques intersect at the three vectors $(0,6), (6,0), (6,6)$, corresponding to the three observables $X^6, Z^6, X^6Z^6$. The
remaining 24 cliques intersect at 0, 1, 2, 3 or 5 points. The automorphism group of \( \mathbb{P}_1(\mathbb{Z}_{12}) \) is found to be \( \mathbb{Z}_2^2 \times G_{144} \), with \( G_{144} = A_4 \times D_8 \).

Remarkably, the automorphism groups of \( \mathbb{P}_1(\mathbb{Z}_{18}) \) and \( \mathbb{P}_1(\mathbb{Z}_{24}) \) encompass that of \( \mathbb{P}_1(\mathbb{Z}_{12}) \), as shown in Table 1.

3. Pauli graph/geometry for multiple qudits
In this section, we specialize on multiple qudits \( q = p^n \), when the qudit is a p-dit (with \( p \) a prime number). The multiple qudit Pauli group \( \mathcal{P}_q \) is generated from the \( n \)-fold tensor product of Pauli operators \( X \) and \( Z \) [defined in (2) with \( \omega = \exp(\frac{2\pi i}{p}) \)]. One has \( |\mathcal{P}_q| = p^{2n+1} \) and the derived group \( \mathcal{P}_q' \) equals the center \( Z(\mathcal{P}_q) \) so that \( |\mathcal{P}_q'| = p \).

Following [4, 10], the observables of \( \mathcal{P}_q/Z(\mathcal{P}_q) \) are seen as the elements of the \( 2n \)-dimensional vector space \( V(2n, p) \) defined over the field \( \mathbb{F}_p \), and one makes use of the commutator

\[
[\ldots] : V(2n, p) \times V(2n, p) \to \mathcal{P}_q'
\]

(14)
to induce a non-singular alternating bilinear form on \( V(2n, p) \), and simultaneously a symplectic form on the projective space \( PG(2n-1, p) \) over \( \mathbb{F}_p \).

Doing this, the \( |V(2n, q)| = p^{2n} \) observables of \( \mathcal{P}_q/Z(\mathcal{P}_q) \) are mapped to the points of the symplectic polar space \( W_{2n-1}(p) \) of cardinality \(^4\)

\[
|W_{2n-1}(p)| = \frac{p^{2n} - 1}{p - 1} = \sigma(p^{2n-1}),
\]

(15)
and two elements of \( \mathcal{P}_q/Z(\mathcal{P}_q), \) commute iff the corresponding points of the polar space \( W_{2n-1}(p) \) are collinear.

A subspace of \( V(2n, p) \) is called totally isotropic if the symplectic form vanishes identically on it. The polar space \( W_{2n-1}(p) \) can be regarded as the space of totally isotropic subspaces of the \( (2n-1) \)-dimensional projective space \( PG(2n-1, p) \). Such totally isotropic subspaces, also called generators \( G \), have dimension \( p^n - 1 \) and their number is

\[
|\Sigma(W_{2n-1}(p))| = \prod_{i=1}^{n} (1 + p^i).
\]

(16)
Let us call a spread \( S \) of a vector space a set of generators partitioning its points. The size of a spread of \( V(2n, p) \) is \( |S| = p^n + 1 \) and one has \( |V(2n, p)| - 1 = |S| \times |G| = (p^n + 1) \times (p^n - 1) = p^{2n} - 1 \), as expected.

Going back to the Pauli observables, a generator \( G \) corresponds to a maximal commuting set and a spread \( S \) corresponds to a maximum (and complete) set of disjoint maximal commuting sets. Two generators in a spread are mutually disjoint and the corresponding maximal commuting sets are mutually unbiased [3, 23].

Let us define the punctured polar space \( W_{2n-1}(p)' \) as the polar space \( W_{2n-1}(p) \) minus a perp-set (i.e. a point \( u \) and all the totally isotropic spaces passing though it)\(^5\). Then, one gets

\[
|W_{2n-1}(p)'| = \sigma(p^{2n-1}) - \sigma(p^{2n-3}) = \psi(p^{2n-1}),
\]

(17)
where \( \sigma(p^{2n-3}) \) is the size of a perp-set and \( \psi(q) \) is the Dedekind psi function.

\(^4\) The proof of this statement is given in [10]. The identification of \( |W_{2n-1}(p)| \) to \( \sigma(p^{2n-1}) \) is new in this context. It is reminiscent of (5) and has still unnoticed consequences about the structure of the polar space, as explained in the sequel of the paper. For \( q \)-level systems (single qudits), \( \sigma(q) \) and \( \psi(q) \) refer to the number of isotropic lines and the number of points of the projective line, respectively (as in (5) and (9)). For multiple qudits, one has \( q = p^{n-1} \) and \( \sigma(q) \) and \( \psi(q) \) refer to the number of points of the symplectic polar space \( W_{2n-1}(p) \) and of punctured polar space \( W_{2n-1}(p)' \), respectively (as in (15) and (17)).

\(^5\) In the graph context the symbol ‘ means a puncture in the graph. It is not the same symbol as in the derived subgroup \( G' \) of the group \( G \).
The Pauli graph of a multiple qudit

The symmetries carried by multiple qudit systems may also be studied with Pauli graphs. We define the Pauli graph $G_{p^n}$ of a multiple $p^n$-dit, as we did for the single qudit case, by taking the observables as vertices and a edge joining two commuting observables. A dual graph $G_{p^n}^\star$ is such that the vertices are the maximum cliques and a edge joins two non-intersecting cliques. One denotes $G_{p^n}^{(k)}$ the corresponding graph attached to the punctured polar space. Finally, one denotes $G_{p^n}^{(0)}$ the graph whose vertices are the maximum cliques of the Pauli graph $G_{p^n}$ and whose edges join two maximum cliques intersecting at $k$ points.

Actual calculations have been performed for two- and three-qubits, and for two- and three-qutrits. Main results are in Table 2 (see details in the corresponding subsections). Denoting $c$ the ratio between the cardinalities of $\text{aut}(G_{p^n}^{(k)})$ and $\text{aut}(G_{p^n}^{(0)})$, one observes that $c$ identifies to the size $\sigma(p^{2n-1})$ of the polar space $W_{2n-1}(p)$, except for the case of the 3-qubit system where $c$ is twice the number of cliques of the Pauli graph $G_3$. Thus, the space $W_{2n-1}(p)$ may be seen as a building block of Pauli systems. One may remind that $W_{2n-1}(p)$ contracts to $W_{2n-1}(p)$, as the size $\sigma(p^{2n-1})$ to the size $\psi(p^{2n-1})$, that the ratio of cardinalities of their automorphism groups is the number $c$, and anticipate on the structural role of $W_{2n-1}(p)$ in qudit mixtures, shown in Table 1 and Sec. 4. For more details, see [22].

### Table 2.

Comparison of the automorphism group of the dual Pauli graph $G_{p^n}^\star$, and that of its building block $G_{p^n}^{(k)}$, defined by removing a perp-set in the symplectic polar space. The ratio of sizes of both groups turns out to be the number of observables $\sigma(p^{2n-1})$ of the space, except for the case of the 3-qubit system where it is twice the number 135 of cliques in the Pauli graph $G_3$.

| $q = p^n$ | name | $\text{aut}(G_{p^n}^{(k)})$ | $\text{aut}(G_{p^n}^{(0)})$ | $\text{aut}(G_{p^n}^{(0)})$ | $c = \frac{|\text{aut}(G_{p^n}^{(k)})|}{|\text{aut}(G_{p^n}^{(0)})|}$ |
|-----------|------|-----------------|-----------------|-----------------|----------------------------------|
| $2^2$     | qubit $S_4$ | $S_4$ | $S_4$ | $S_2$ | $3 \equiv \sigma(2)$ |
| $2^3$     | 2-qubit $S_6$ | $S_6$ | $G_{48} = \mathbb{Z}_2 \times S_4$ | $15 \equiv \sigma(2^3)$ |
| $3^3$     | 3-qubit $S_3 \times G_{48}$ | $O^+(8,2)$ | $\mathbb{Z}_2 \times A_8$ | $2 \times 135 \neq 63 = \sigma(3^3)$ |
| $3^3$     | 2-qutrit $\mathbb{Z}_2^{364}W(E_6)$ | $W(E_6)$ | $G_{648} \times \mathbb{Z}_2$ | $40 \equiv \sigma(3^3)$ |
| $3^3$     | 3-qutrit $\mathbb{Z}_2^{364}G$ | $G$ | $(E_{243} \times \mathbb{Z}_2)W(E_6)$ | $364 \equiv \sigma(3^3)$ |

4. Pauli graph/geometry of multiple qudit mixtures

As before, $G_q$ is the Pauli graph whose vertices are the observables and whose edges join two commuting observables. A dual graph of the Pauli graph is $G_q^\star$ whose vertices are the maximum cliques and whose edges join two non-intersecting cliques. In this section, we also introduces $G_q^{(k)}$, the graph whose vertices are the maximum cliques and whose edges join two maximum cliques intersecting at $k$ points.

First of all, as shown in Sec. 6 of [5], a qudit mixture in composite dimension $q = p_1 \times p_2 \times \cdots \times p_r$ ($p_1$ a prime number), identifies to a single $q$-dit. Since the ring $\mathbb{Z}_q$ is isomorphic to the direct product $\mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \times \mathbb{Z}_{p_r}$, the commutation relations arrange as the $\sigma(q) \equiv \psi(q)$ isotropic lines of the lattice $\mathbb{Z}_q^2$, that reproduce the projective line $\mathbb{P}_1(\mathbb{Z}_q) = \mathbb{P}_1(\mathbb{Z}_{p_1}) \times \mathbb{P}_1(\mathbb{Z}_{p_2}) \times \cdots \times \mathbb{P}_1(\mathbb{Z}_{p_r})$.

The sextit system

The simplest non-trivial case is in dimension $q = 6 = 2 \times 3$. The projective line may be pictured by the dual Pauli graph $G_6^\star$ of spectrum $\{6^1, 1^6, -2^3, -3^2\}$. It represents the complement of a
3 × 4 grid, or in graph theoretical language the complement \( \hat{L}(K_{3,4}) \) of the line graph over the complete bipartite graph \( K_{3,4} \) (see Fig. 1 of [8]). One finds 24 maximum cliques of size 3 in \( G_6 \) corresponding to the same number of non-complete sets of mutually unbiased bases. The symmetry of this new configuration is the semi-direct product \( G_{144} = A_4 \rtimes D_6 \) of two groups of order twelve, namely the four-letter alternating group \( A_4 \) and the dihedral group \( D_6 \). Until now, it is not known whether sets of mutually unbiased bases of size larger than three can be built [23, 24].

A summary of the main results for mixtures where at least one factor in the prime number decomposition of \( q \) contains a square is in Table 1. See [22] for details.

**The qubit/quartit and qubit/octit systems**

Let us illustrate the lowest case of a mixture where a factor is not a prime: the qubit/quartit system living in the Hilbert space dimension \( q = 2 \times 4 \). As shown in Table 1, the maximum cliques studied from the dual Pauli graph \( G_{2 \times 4} \) split into two parts, that are a set of 3 independent (non-intersecting) cliques and a connected component of 36 cliques. The single qudit in dimension \( q = 18 \) has a similar splitting since \( \sigma(18) = 39 \) and \( \psi(18) = 36 \).

Let us denote \( G_{2 \times 4}^{\sigma(c)} \) the connected subgraph of the dual Pauli graph. Its spectrum is again that \( \{16^1, 0^3, -8^4, 4^3\} \) of a regular graph. Maximum cliques of the Pauli graph intersect each other at 0, 1 or 3 points and there is a subgeometry of the qubit/quartit system found by taking the incidence graph of maximum cliques intersecting at 3 points. The spectrum of this latter graph \( G_{2 \times 4}^{(7)} = 5^1, 1^6, -1^2, -3^3 \) corresponds to three copies of the punctured Pauli graph associated to \( GQ(2, 2)' \).

Similarly, one considers the qubit/octit system living in the Hilbert space dimension \( q = 2 \times 8 \). As shown in Table 1, the maximum cliques studied from the Pauli graph \( G_{2 \times 8} \) split into two parts, that are a set of 15 independent (non-intersecting) cliques and a connected component of 72 cliques. Here, there exists no single qudit with such a splitting. Let us denote \( G_{2 \times 8}^{\sigma(c)} \) the connected subgraph of the dual Pauli graph. Its spectrum is \( \{32^1, 8^4, 0^{63}, -16^3\} \). Maximum cliques of the Pauli graph intersect each other at 0, 1, 3 or 7 points and there is a subgeometry of the qubit/octit system found by taking the incidence graph \( G_{2 \times 8}^{(7)} \) of maximum cliques intersecting at 7 points. The spectrum of this latter graph corresponds to six copies of of \( GQ(2, 2)' \). The automorphism group of this latter configuration is found to be the semidirect product of groups \( G_{48}^6 \rtimes S_6 \).

5. Conclusion

It has been shown for the first time that number theoretical functions \( \sigma(q) \) and \( \psi(q) \) enter into the structure of commutation relations of Pauli graphs and geometries. For single q-dits (in section 2), \( \sigma(q) \) and \( \psi(q) \) refer to the number of maximal commuting sets and the cardinality of the projective line \( \mathbb{P}_1(\mathbb{Z}_q) \), respectively. For multiple qudits, with dimension \( p^n \), \( p \) a prime number, (in section 3) the parameter \( q = p^{2n-1} \) enters in the function \( \sigma(q) \) to count the size of the symplectic polar space \( W_{2n-1}(p) \) (that carries the multiple qudit system), and enters in the function \( \psi(q) \) to count the size of the basic constituent: the punctured polar space \( W_{2n-1}^1(p) \). For multiple qudit mixtures, spaces \( W_{2n-1}(p) \) and \( W_{2n-1}^1(p) \) are also found to arise as constituents of the commutation structure.

The structural role of symplectic groups \( Sp(2n,p) \) has been found, as expected. Other important symmetry groups are \( G_{48} = \mathbb{Z}_2 \times S_4 \), \( G_{144} = A_4 \times D_6 \) and \( W(E_6) \). The group \( G_{48} \) is first of all the automorphism group of the single qudit Pauli group \( \mathbb{P}_1 \) and is important in understanding the CPT symmetry [25]. In this paper, it arises as the symmetry group of the quartit, of the punctured generalized quadrangle \( GQ(2,2)' \) and as a normal subgroup of many systems of qudits (as shown in Table 1). The torus group \( G_{144} \) occurs in the symmetries of the
6-dit, 12-dit, 18-dit and 24-dit systems. The Weyl group $W(E_6)$ happens to be central in the symmetries of three-qubit and multiple qutrit systems. The understanding of symmetries in the Hilbert space is important for the applications in quantum information processing.

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