Higher order heavy quark Green’s functions in Coulomb gauge

C. Popovici,1,2 P. Watson,1 and H. Reinhardt1
1Institut für Theoretische Physik, Universität Tübingen, Auf der Morgenstelle 14, D-72076 Tübingen, Germany
2Centro de Física Computacional, Departamento de Física, Universidade de Coimbra, 3004-516 Coimbra, Portugal

The Dyson-Schwinger equation for the 4-point quark Green’s functions is studied. In the limit of the heavy quark mass and with the truncation to include only the dressed two point functions for the Yang-Mills sector, we provide an exact solution for the 4-point quark Green’s functions, in both quark-antiquark and diquark channels, and show that the corresponding poles relate to the bound state energy of the heavy quark systems. Moreover, a natural separation between physical and unphysical poles in the Green’s functions emerges.

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I. INTRODUCTION

Since its foundation more than 30 years ago, one of the main goals of Quantum Chromodynamics [QCD] is to understand the structure of hadrons via the interactions of its elementary degrees of freedom, the quarks and gluons. In order to approach this fundamental question, one of the most appropriate starting points is to consider Coulomb gauge. In this gauge, there is a natural connection to the would-be physical degrees of freedom [1], and an appealing picture of confinement exists: the so-called Gribov-Zwanziger scenario of confinement [1–3], whereby the temporal component of the gluon propagator provides for a long range confining force in the infrared, while the transversal spatial component is infrared suppressed (and therefore does not appear as an asymptotic state). In the last years, important progress has been made in investigating the Yang-Mills sector of the theory in Coulomb gauge: the variational method in the Hamiltonian approach [4–7], the Lagrange-based (Dyson-Schwinger) functional formalism [8, 9] and also lattice calculations [10, 11] (see also [12–15]). On the lattice, the temporal gluon propagator appears to be largely energy independent and behaves like \(1/\vec{q}^4\) in the infrared, whereas the static spatial gluon propagator is found to be vanishing in the infrared, in agreement with the Gribov formula [10]. In the framework of the functional formalism it has been shown that the total color charge of the system is conserved and vanishing, and the well-known energy divergence problem in Coulomb gauge disappears [16]. Moreover, within this approach, the Slavnov-Taylor identities [17] have been derived and perturbative results have been provided [9, 18]. In the quark sector, perturbative results have been presented [19], and in the heavy mass limit the confining potential has been studied, both for two and three quark systems [20, 21].

In general, the underlying equation for the description of meson bound states is the two-body homogeneous Bethe-Salpeter equation. In the rainbow-ladder approximation, this equation has been successfully used to describe the properties of light mesons (see, for example [22, 23] and for a recent review [24]), where the driving mechanism is the chiral symmetry breaking. Beyond this approximation, models with dressed vertex contributions [25–30] and unquenching effects [31–33] have been considered, and more sophisticated numerical methods to solve both the homogeneous and the inhomogeneous Bethe-Salpeter equation have been recently developed [34]. In spite of this success, an exact derivation of the meson or diquark bound state energies via Green’s functions techniques (i.e., using the Dyson-Schwinger equations as opposed to the Bethe-Salpeter equation) has not been yet reported. The difficulty stems from the fact that the (irreducible) interaction kernel contains higher order vertex functions which in general can not be calculated exactly. The connection between the singularities of the Green’s functions and the bound states dates back to the original work of Gell-Mann and Low [35], who gave a rigorous proof of the two-particle Bethe-Salpeter equation. It is, moreover, one of the longstanding problems in relativistic quantum field theory, and in particular phenomenological studies of meson and baryon states, to identify the physical solutions of the poles of the Green’s functions and to separate them from possible unphysical ones. For an analysis of the appearance of unphysical poles in QED bound states, see for example Ref. [36] and references therein.

Based on the heavy mass expansion underlying the Heavy Quark Effective Theory [HQET] [37, 38] and with the truncation of the Yang-Mills sector to include only dressed two-point functions, we study nonperturbatively quark-antiquark and diquark states using Green’s functions techniques. By means of functional methods, we explicitly derive the Dyson-Schwinger equation for the quark 4-point Green’s functions and give an exact, analytical solution. This will enable us to verify that bound states are related to the occurrence of physical poles in the Green’s functions and hence we will be able to provide a direct connection between the homogeneous Bethe-Salpeter equation (considered previously in Ref. [20]) and the singularities of the Green’s function, at least within the scheme considered here.
Moreover, within the employed approximation we will show that the physical and unphysical poles naturally separate.

This paper is organized as follows. In Sec. II we review the functional formalism and derive the equations of motion, along with the relevant higher order functional derivatives. We present the explicit derivation of the Dyson-Schwinger equations for the 4-point Green’s functions for quark-antiquark systems. In Sec. III we give a brief survey of the results obtained for heavy quark systems. After making an expansion of the generating functional of Coulomb gauge QCD in powers of $1/m$, where $m$ is the quark mass, and keeping only the leading term, we present the main steps in the derivation of the (anti)quark gap equation, and the corresponding temporal quark-gluon vertices. In Sec. IV we consider both the 1-particle irreducible and amputated connected 4-point quark-antiquark Green’s functions in the heavy mass limit. In this approximation, analytical solutions for the Green’s functions are provided and the singularity structure, in particular the separation of the physical and unphysical states, is discussed. Moreover, the correspondence with the bound state solution emerging from the homogeneous Bethe-Salpeter equation is emphasized. Sec. V is devoted to diquark systems. Again, the separation of the poles and the connection to the Bethe-Salpeter equation (including flavor structure) is discussed. A summary and some concluding remarks are given in Sec. VI. In the Appendix, the explicit derivation of the Slavnov-Taylor identity for the quark-2 gluon vertex is presented.

II. DYSON-SCHWINGER EQUATIONS FOR 4-POINT FUNCTIONS

We begin by briefly reviewing the functional formalism used to derive the basic equations that will lead to the Dyson-Schwinger equations and Slavnov-Taylor identities. Following the notations and conventions from [19, 20], we work in Minkowski space, with the metric $g_{\mu \nu} = \text{diag}(1, -1)$. Roman letters $(i, j . . .)$ denote spatial indices, greek letters $(\mu, \nu . . .)$ denote Lorentz indices and superscripts $(a, b, c . . .)$ stand for color indices in the adjoint representation. Unless otherwise specified, the Dirac flavor, spinor and (fundamental) color indices are labelled with a common index $(\alpha, \beta . . .)$. Also, configuration space coordinates may be denoted with subscript $(x, y, z . . .)$ when no confusion arises. The Dirac $\gamma$ matrices satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

Let us consider the explicit quark contribution to the full QCD generating functional [19]:

$$Z[\overline{\chi}, \chi] = \int \mathcal{D}\Phi \exp \left\{ i \int d^4x \overline{\chi}_\alpha(x) \left[ i\gamma^0 D_0 + i\gamma^i \vec{D} - m \right]_{\alpha\beta} q_\beta(x) \right\} \times \exp \left\{ i \int d^4x \left[ \overline{\chi}_\alpha(x) q_\alpha(x) + \overline{q}_\alpha(x) \chi_\alpha(x) \right] + i\mathcal{S}_{YM} \right\}. \tag{2.1}$$

In the above, $\mathcal{S}_{YM}$ represents the Yang-Mills part of the action and $\mathcal{D}\Phi$ generically denotes the functional integration measure over all fields. $q_\alpha$ denotes the full quark field, $\overline{q}_\alpha$ is the conjugate (or antiquark) field, and $\overline{\chi}_\alpha$, $\chi_\alpha$ are the corresponding sources. The temporal and spatial components of the covariant derivative (in the fundamental color representation) are given by

$$D_0 = \partial_0 - igT^a \sigma^a(x),$$
$$\vec{D} = \vec{\nabla} + igT^a \vec{A}^a(x), \tag{2.2}$$

where $\vec{A}$ and $\sigma$ refer to the spatial and temporal components of the gluon field, respectively, and the source terms for $\vec{A}$ and $\sigma$ (with sources $\vec{J}$ and $\rho$, respectively) are implicitly included in $\mathcal{S}_{YM}$. The structure constants of the $SU(N_c)$ group are denoted with $f^{abc}$, and the Hermitian generators $T^a$ satisfy $[T^a, T^b] = i f^{abc} T^c$ and are normalized via $\text{Tr}(T^a T^b) = \delta^{ab}/2$. Also note the Casimir factor associated with the quark gap equation:

$$C_F = \frac{N_c^2 - 1}{2N_c}. \tag{2.3}$$

The field equation of motion is derived from the generating functional, Eq. (2.1), and the observation that the integral of a total derivative vanishes, up to possible boundary terms. We assume that these terms do not contribute, although this is not obvious due to the Gribov problem [2] (see [8] and references therein), and obtain:

$$\int \mathcal{D}\Phi \frac{\delta}{\delta \Phi(y)} \exp \left\{ i \int d^4x \overline{\chi}_\alpha(x) \left[ i\gamma^0 D_0 + i\gamma^i \vec{D} - m \right]_{\alpha\beta} q_\beta(x) + i \int d^4x \left[ \overline{\chi}_\alpha(x) q_\alpha(x) + \overline{q}_\alpha(x) \chi_\alpha(x) \right] + i\mathcal{S}_{YM} \right\} = 0. \tag{2.4}$$

So far, the generating functional $Z[J]$ generates both connected and disconnected Green’s functions. The generating functional of connected Green’s functions is $W[J]$, where $W = \ln Z$. We introduce a bracket notation for the functional
derivatives of $W$, such that for a generic source denoted by $J_\alpha$ (in this case the index $\alpha$ refers to all the attributes of both the quark and gluon fields, including the type):

$$\frac{\delta W}{\delta t J_\alpha} = <\iota J_\alpha>.$$ (2.5)

The classical fields are (we use the same notation for the classical and quantum fields, as is standard):

$$\Phi_\alpha(x) = \frac{1}{Z} \int \mathcal{D}\Phi \Phi_\alpha(x) \exp \{iS\} = \frac{1}{Z} \frac{\delta Z}{\delta t J_\alpha(x)}.$$ (2.6)

The generating functional of the proper (one particle irreducible) Green’s function is the effective action $\Gamma$ (function of the classical fields) and is defined via the Legendre transform of $W[J]$:

$$\Gamma[\Phi] = W[J] - iJ_\alpha \Phi_\alpha.$$ (2.7)

In the above, we have used the common convention that the index $\alpha$ is either summed or integrated over as appropriate. This gives

$$<\iota J_\alpha> = \frac{\delta W}{\delta t J_\alpha} = \Phi_\alpha \quad \text{and} \quad <\iota \Phi_\alpha> = \frac{\delta \Gamma}{\delta \iota \Phi_\alpha} = -J_\alpha,$$ (2.8)

where the same bracket notation for derivatives of $\Gamma$ with respect to the classical fields is used – there is no confusion between the two sets of brackets since the derivatives with respect to the sources and fields are never mixed.

We now present the quark field equation of motion in terms of proper Green’s functions, arising from Eq. (2.4) (and from which the gap equation and the 4-point quark Green’s functions will be derived):

$$<\iota \bar{q}_\alpha(x)> = \iota \left[ v^\gamma_0 \partial_0 x + v^\gamma \cdot \nabla_x - m \right]_{\alpha\beta} q_\beta(x) + \left[ gT^{c\gamma} \sigma^c(x)q_\beta(x) + <\iota \bar{q}_\gamma(x)\iota \bar{q}_\beta(x)> \right].$$ (2.9)

We retain for the moment the spatial part in the above equation, although later on this will be truncated out within the employed approximation scheme.

At this stage, it is useful to introduce some notation for multiple functional derivatives with respect to fields and sources, which will be later on used to derive the Dyson-Schwinger equations. Consider the following partial differentiation expressions arising from the Legendre transform, Eq. (2.7) (we omit the configuration space arguments for clarity):

$$\frac{\delta}{\delta \iota \Phi_\beta} <X(J)> = -iS[\gamma] <\Phi_\beta \iota \Phi_\gamma> <\iota J_\gamma X(J)>,$$ (2.10a)

$$\frac{\delta}{\delta \iota J_\beta} <Y(\Phi)> = iS[\gamma] <\iota J_\beta \iota J_\gamma> <\iota \Phi_\gamma Y(\Phi)>,$$ (2.10b)

where $S[\gamma] = \pm 1$ accounts for the fact that the fields may be Grassmann-valued, i.e. a minus sign appears when the index $\gamma$ refers to the following contribution to the sum (over “type” $\gamma$)

$$<\ldots \iota q_\gamma \ldots > <\iota \bar{q}_\gamma \ldots > <\iota q_\gamma \ldots > <\iota q_\gamma \ldots >.$$ (2.11)

Combining Eq. (2.10a) and Eq. (2.10b) recursively, one easily finds the following useful relationship:

$$\frac{\delta}{\delta \iota \Phi_\beta} <\iota J_\beta \iota J_\alpha> = -\eta_{\beta\delta} S[\gamma, \kappa] <\iota J_\beta \iota J_\kappa> <\iota \Phi_\kappa \Phi_\beta \iota \Phi_\gamma> <\iota J_\gamma \iota J_\alpha>,$$ (2.12)

where the factor $\eta_{\beta\delta} = -1$ if fields/sources of the type $\delta, \beta$ anticommute. Of course, this type of relationship is standard, but the notation allows one to follow the various multiple derivatives and keep track of the signs efficiently.

The quark gap equation is derived by taking the functional derivative of the quark field equation of motion (in configuration space), Eq. (2.9), with respect to $\iota q_\gamma(z)$, and omitting the terms which will eventually vanish when the sources are set to zero:

$$<\iota \bar{q}_\alpha(x) q_\gamma(z)> = \iota \left[ v^\gamma_0 \partial_0 x + v^\gamma \cdot \nabla_x - m \right]_{\alpha\gamma} \delta(x - z)$$

$$- \int d^4y \delta(x - y) \left[ \Gamma^{(0)\alpha}_{\bar{q}q\alpha\beta}(y) \right]_{\gamma\alpha\beta} \frac{\delta}{\delta \iota q_\gamma(z)} <\iota q^3(y)\iota \bar{q}_\beta(x)> + \Gamma^{(0)\alpha}_{\bar{q}q\alpha\beta}(y) \frac{\delta}{\delta \iota q_\gamma(z)} <\iota J_3^\alpha(y)\iota \bar{q}_\beta(x)>.$$ (2.13)
where the (configuration space) tree-level quark-gluon vertices, obtained from the quark equation of motion Eq. (2.9), are given by

\[ \Gamma^{(0)\alpha}_{\bar{q}q\alpha\bar{\beta}} = \left[ gT^{\alpha\lambda}_{\bar{q}q\lambda} \right]_{\alpha\bar{\beta}}, \quad (2.14) \]

\[ \Gamma^{(0)\alpha}_{\bar{q}qA\alpha\bar{\beta}} = -\left[ gT^{\alpha\lambda}_{\bar{q}qA\lambda} \right]_{\alpha\bar{\beta}}, \quad (2.15) \]

(omitting the trivial \(\delta\)-function configuration space dependence). We now use the formula Eq. (2.12) to calculate the functional derivatives appearing in the bracket and obtain (for simplicity, we write the spatial arguments of the external fields/sources as subscripts and omit those for the internal fields/sources that are implicitly summed and integrated over):

\[ \frac{\delta}{\delta \bar{q}_{\gamma z}} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle = -S[\lambda, \kappa] \langle \bar{q} q^{\alpha}_y \bar{J}_\lambda \rangle \langle \bar{\Phi}_{\lambda \mu q_{\gamma z} \tau} \rangle \langle \tau J_\kappa \bar{\chi} \beta_x \rangle, \quad (2.16) \]

\[ \frac{\delta}{\delta \bar{q}_{\gamma z}} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle = -S[\lambda, \kappa] \langle \bar{q} q^{\alpha}_y \bar{J}_\lambda \rangle \langle \bar{\Phi}_{\lambda \mu q_{\gamma z} \tau} \rangle \langle \tau J_\kappa \bar{\chi} \beta_x \rangle. \quad (2.17) \]

When sources are set to zero, the brackets on the right-hand side correspond to connected and proper Green’s functions. We can then identify the field/source type \(\lambda\) with the gluon (temporal for the upper expression, spatial for the lower) and \(\kappa\) with the antiquark, and notice that in both cases \(S[\lambda, \kappa] = +1\). Introducing our conventions and notations for the Fourier transform, we have for a general two-point function (connected or proper) which obeys translational invariance:

\[ \langle \tau J_{\alpha y} \tau J_{\beta x} \rangle = \int d k \ W_{\alpha\beta}(k) e^{-ik \cdot (y - x)}, \quad (2.18) \]

\[ \langle \tau \Phi_{\alpha y} \tau \Phi_{\beta x} \rangle = \int d k \Gamma_{\alpha\beta}(k) e^{-ik \cdot (y - x)}, \quad (2.19) \]

and for the three-point (proper vertex) function

\[ \langle \tau \Phi_{\alpha y} \tau \Phi_{\beta x} \tau \Phi_{\gamma z} \rangle = \int d k_1 \ d k_2 \ d k_3 \Gamma_{\alpha\beta\gamma}(k_1, k_2, k_3) e^{-ik_1 \cdot (y - x) - ik_2 \cdot x - ik_3 \cdot z} (2\pi)^4 \delta(k_1 + k_2 + k_3), \quad (2.20) \]

(similarly for higher \(n\)-point functions) where \(d k = d^4 k/(2\pi)^4\). The propagator \(W_{\alpha\beta}(k)\) and proper (1PI) two-point function \(\Gamma_{\alpha\beta}(k)\) are related via the Legendre transform. For the quark propagator, we have the standard relation

\[ W_{\bar{q}q\alpha\beta}(k) \Gamma_{\bar{q}q\gamma\beta}(k) = \delta_{\alpha\beta}. \quad (2.21) \]

We now insert the expressions Eqs. (2.16), (2.17) into Eq. (2.13). After Fourier transforming to momentum space and with the definitions Eqs. (2.18, 2.19, 2.20) we obtain the quark Dyson-Schwinger (or gap) equation:

\[ \Gamma_{\bar{q}q\alpha\gamma}(k) = \Gamma^{(0)\alpha}_{\bar{q}q\alpha\gamma}(k) + \int d \omega \left\{ \Gamma^{(0)\alpha}_{\bar{q}q\alpha\gamma}(k, -\omega) W_{\bar{q}q\beta\gamma}(\omega) \Gamma^{b}_{\bar{q}q\gamma\beta}(\omega, -k - \omega) W_{\sigma\gamma}(k - \omega) \right. \]

\[ + \left( \Gamma^{(0)\alpha}_{\bar{q}q\alpha\gamma}(k, -\omega) W_{\bar{q}q\beta\gamma}(\omega) \Gamma^{b}_{\bar{q}q\gamma\beta}(\omega, -k - \omega) W_{\sigma\gamma}(k - \omega) \right) \}. \quad (2.22) \]

Let us now derive the Dyson-Schwinger equation for the one-particle irreducible (1PI) 4-point function. As an illustration of the functional differentiation techniques, we present the explicit derivation of the first term in this expression and notice that the rest of the terms follow from an identical calculation (as explained below, the \(\bar{A}\)-terms will be eliminated within our truncation scheme and hence in order to keep things simple, in the following derivations we will drop them completely). The goal is to derive the 1PI four-point function \(\langle \bar{q} q_{\alpha y} \bar{q}_{\gamma z} \bar{q}_{\tau} \bar{q}_{\rho} \rangle\), and hence on the right hand side we will need, for example, a term of the type

\[ \frac{\delta^2}{\delta \bar{q}_{\tau r} \delta \bar{q}_{\gamma z} \delta \bar{q}_{\rho \gamma}} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle. \]

Functionally differentiating Eq. (2.16) with respect to \(\bar{q}_{\tau r}\) and using the product rule, we obtain:

\[ \frac{\delta^2}{\delta \bar{q}_{\tau r} \delta \bar{q}_{\gamma z} \delta \bar{q}_{\rho \gamma}} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle = -S[\lambda, \kappa] \left\{ \frac{\delta}{\delta \bar{q}_{\tau r}} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle \right. \]

\[ + \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle \langle \bar{q} q^{\alpha}_y \bar{q}_{\gamma z} \tau \Phi_{\gamma} \rangle \langle \tau J_\kappa \bar{\chi} \beta_x \rangle \]

\[ + \eta_{\tau r} \eta_{\rho \gamma} \langle \bar{q} q^{\alpha}_y \bar{\chi} \beta_x \rangle \langle \bar{q} q^{\alpha}_y \bar{q}_{\gamma z} \tau \Phi_{\gamma} \rangle \left[ \frac{\delta}{\delta \bar{q}_{\tau r}} \langle \tau J_\kappa \bar{\chi} \beta_x \rangle \right]. \quad (2.23) \]
For clarity of presentation, we only retain the first term in the product (and denote the rest with dots). Again, we make use of the formula Eq. (2.12) and obtain:

\[ \frac{\delta^2}{\delta q_{w}\delta q_{z}} \langle \Phi_\mu \rangle = S[\lambda, \kappa, \mu, \nu] \langle \Phi_\mu \rangle \]

A last functional derivative with respect to the quark field \( q_{\mu} \) gives

\[ \frac{\delta^3}{\delta q_{\mu}\delta q_{w}\delta q_{z}} \langle \Phi_\mu \rangle = S[\lambda, \kappa, \mu, \nu] \]

Again, we take only the first term from the above sum and as before we use the formula Eq. (2.12). We obtain:

\[ \frac{\delta^3}{\delta q_{\mu}\delta q_{w}\delta q_{z}} \langle \Phi_\mu \rangle = -S[\lambda, \kappa, \mu, \nu, \varepsilon, \delta] \langle \Phi_\mu \rangle \]

As for the gap equation, when sources/fields are set to zero, we can identify the possible contributing field types denoted by internal indices and determine \( S[\lambda, \kappa, \mu, \nu, \varepsilon, \delta] \). Straightening out the ordering of the external quark lines, the full Dyson-Schwinger equation for the proper (1PI) four-quark Green’s function in configuration space reads

\[ \langle \Phi_{\alpha} \rangle = [g_{A}T^{a}]_{\alpha\beta} \int dy \delta(x - y) \]

where the dots represent the \( \bar{A} \) vertex terms which are not considered here and we have already replaced the tree-level temporal quark-gluon vertex with its expression Eq. (2.14). The expression Eq. (2.27) is diagrammatically represented in Fig. 1 and the terms have been reordered such that the first term corresponds to the diagram (a) of Fig. 1, the second term corresponds to diagram (b) and so on. The term presented explicitly above corresponds to diagram (i). Also, notice the minus sign in the term corresponding to diagram (h) — this is the origin of the minus sign arising in the kernel of the homogeneous Bethe-Salpeter equation considered in Ref. [20].

The next step is to derive the connection between the amputated connected 4-point quark-antiquark Green’s function (used conventionally in the Bethe-Salpeter equation), and the 1PI Green’s function. Starting with the identity

\[ \langle u_{\beta} \bar{u}_{\gamma} \rangle = \delta_{\beta\gamma}, \]
we take two subsequent functional derivatives with respect to generic sources, \( iJ_\delta \) and \( iJ_\lambda \). Using Eq. (2.10b) recursively, choosing \( \delta, \lambda \) as needed, we eventually obtain the standard expression (in terms of our notation and conventions) for the 4-point quark-antiquark connected Green’s function, written in terms of 1PI Green’s functions (and omitting the \( \vec{A} \)-vertex contributions, which will not play a role here):

\[
\langle \chi_\alpha \chi_\gamma \chi_\lambda \chi_\eta \rangle = \langle \chi_\alpha \chi_\gamma \rangle \langle \chi_\lambda \chi_\eta \rangle - \Big[ \langle \chi_\alpha \chi_\gamma \rangle \langle \sigma_a \gamma \chi_\lambda \chi_\eta \rangle - \langle \chi_\lambda \chi_\eta \rangle \langle \sigma_a \gamma \chi_\alpha \chi_\gamma \rangle \Big] \langle \rho^a_{\alpha} \rho^a_{\gamma} \rangle + \Big[ \langle \chi_\alpha \chi_\gamma \rangle \langle \sigma_a \gamma \chi_\lambda \chi_\eta \rangle \langle \rho^a_{\alpha} \rho^a_{\gamma} \rangle - \langle \chi_\lambda \chi_\eta \rangle \langle \sigma_a \gamma \chi_\alpha \chi_\gamma \rangle \langle \rho^a_{\alpha} \rho^a_{\gamma} \rangle \Big] .
\] (2.29)

The relation Eq. (2.29) can be further simplified by introducing the fully amputated Green’s function, i.e. dividing by the quark propagators (cut the quark legs), as shown in Fig. 2. Before we consider the explicit form of Eqs. (2.27, 2.29), it is necessary to briefly recall the heavy mass expansion and the truncation scheme that will be applied in this study. This is the topic of the next section.

III. HEAVY QUARK MASS EXPANSION

Consider again the quark contribution to the full QCD generating functional, Eq. (2.1). In the following, we briefly describe the derivation of the quark (and antiquark) propagator, in the heavy mass limit and under the truncation of the Yang-Mills sector to include only the dressed two-point functions. The presentation follows the original work of Ref. [20].
The full quark field is decomposed according to the heavy quark transformation
\[ q_\alpha(x) = e^{-imx^0} [h(x) + H(x)]_\alpha, \quad h_\alpha(x) = e^{imx_0} [P_+ q(x)]_\alpha, \quad H_\alpha(x) = e^{imx_0} [P_- q(x)]_\alpha \] (3.1)
(similarly for the antiquark field), where the two components \( h \) and \( H \) are introduced with the help of the spinor projectors
\[ P_\pm = \frac{1}{2} (1 \pm \gamma^0). \] (3.2)
This corresponds to a particular case of the heavy quark transform underlying HQET [37], but in the functional approach considered here it is simply an arbitrary decomposition that will turn out to be very useful in Coulomb gauge. As will shortly be discussed, this will lead to the suppression of the spatial gluon propagator and spatial quark-gluon vertex at leading order in the mass expansion.

The quark fields, decomposed according to Eq. (3.1), are now inserted into the generating functional Eq. (2.1). Further, the \( H \)-fields are integrated out, and an expansion in the heavy quark mass is performed (throughout this work, we will follow the usual terminology and use the expression “mass expansion”, instead of “expansion in the inverse mass”). At leading order, the generating functional reads:
\[ Z[\chi, \chi] = \int \mathcal{D} \Phi \exp \left\{ i \int d^4x \bar{\chi}_\alpha(x) [i \partial_0 + g T^a \sigma^a(x)]_{\alpha \beta}(x) h_\beta(x) \right\} \times \exp \left\{ i \int d^4x \left[ e^{-imx_0} \bar{h}_\alpha(x) h_\alpha(x) + e^{imx_0} \bar{h}_\alpha(x) \chi_\alpha(x) \right] + i \mathcal{S}_M \right\} + \mathcal{O}(1/m), \] (3.3)
where the temporal component of the covariant derivative \( D_0 \) has been written explicitly. Notice that, as a consequence of the decomposition Eq. (3.1), the spin degrees of freedom have decoupled from the system. Importantly, in the above we have kept the full quark and antiquark sources (rather than the ones corresponding to the large components of the quark field \( h \), as is usually done in HQET). This means that the methods of functional formalism are preserved at leading order in the mass expansion, and in turn this implies that we are allowed to use the full Dyson-Schwinger equations of QCD, i.e. the gap and 4-point Dyson-Schwinger equations of the last section, replacing, however, the kernels, propagators and vertices by their leading order in the mass expansion.

The gap equation Eq. (2.22) is solved together with the Slavnov-Taylor identity. This is derived from the invariance of the QCD action under a time-dependent Gauss-BRST transform [20], and in Coulomb gauge it has the following expression:
\[ k_3^0 \Gamma_{q\sigma\alpha\beta}^\dag(k_1, k_2, k_3) = i k_3^2 \Gamma_{q\sigma\alpha\beta i}^\dag(k_1, k_2, k_3) \Gamma_{\delta\rho\sigma\beta}^{\dag(-k_3)} \] 
\[ + \Gamma_{q\sigma\alpha\beta}(k_1) \left( \Gamma_{q\sigma\alpha\beta i}^\dag(k_1 + q_0, k_3 - q_0; k_2) + ig T^d \right)_{\delta\beta} \] 
\[ + \left[ \Gamma_{q\sigma\alpha\beta}^\dag(k_1 + q_0, k_3 - q_0; k_1) - ig T^d \right]_{\alpha\delta} \Gamma_{q\sigma\beta}(-k_2) \] (3.4)
where \( k_1 + k_2 + k_3 = 0, q_0 \) is an arbitrary energy injection scale (arising from the noncovariance of Coulomb gauge [17]), \( \Gamma_{\gamma \sigma} \) is the ghost proper two-point function, \( \tilde{\Gamma}_{q\sigma} \) and \( \tilde{\Gamma}_{q\sigma \gamma} \) are ghost-quark kernels associated with the Gauss-BRST transform. Since in the generating functional Eq. (3.3) the tree-level spatial quark gluon vertex \( \Gamma_{q\sigma A}^{(0)} \) appears at \( O(1/m) \) in the mass expansion (which is here neglected), and furthermore, under our truncation the pure Yang-Mills vertices are also neglected, it follows that the Dyson-Schwinger equation for the nonperturbative spatial quark-gluon vertex provides the result that \( \Gamma_{q\sigma A}^{(0)} \sim O(1/m) \) (see [20] for a complete discussion and justification of this truncation). Moreover, the ghost-quark kernels involve pure Yang-Mills vertices and can be also neglected. Thus, under our truncation scheme, the Slavnov-Taylor identity reduces to
\[ k_3^0 \Gamma_{q\sigma\alpha\beta}^\dag(k_1, k_2, k_3) = \Gamma_{q\sigma\alpha\beta}(k_1) \left( ig T^d \right)_{\delta\beta} - [ig T^d]_{\alpha\delta} \Gamma_{q\sigma\beta}(-k_2) + \mathcal{O}(1/m). \] (3.5)
We then insert this into Eq. (2.22), together with the tree-level quark proper two-point function
\[ \Gamma_{q\sigma\alpha\beta}^{(0)}(k) = i \delta_{\alpha\beta} [k_0 - m] + \mathcal{O}(1/m) \] (3.6)
and the tree level quark gluon vertex
\[ \Gamma_{q\sigma\alpha\beta}^{(0)}(k_1, k_2, k_3) = [g T^a]_{\alpha\beta} + \mathcal{O}(1/m) \] (3.7)
that follow from the generating functional Eq. (3.3). The nonperturbative temporal gluon propagator reads [9]:

$$ W_{\sigma\sigma}^{ab}(k) = \delta^{ab} W_{\sigma\sigma}(\mathbf{k}) = \delta^{ab} \frac{1}{k^2} D_{\sigma\sigma}(\mathbf{k}^2). \quad (3.8) $$

Lattice results [11] motivate that the dressing function $D_{\sigma\sigma}$ is largely independent of energy and moreover, $D_{\sigma\sigma}$ is infrared divergent and likely to behave as $1/k^2$ for vanishing $\mathbf{k}^2$ (we will only use the explicit form of $D_{\sigma\sigma}$ in the last step of the calculation). Putting all this together, we find the following solution to Eq. (2.22) for the heavy quark propagator:

$$ W_{\bar{q}q_{\alpha\beta}}(k) = \frac{-\delta^{\alpha\beta}}{|k_0 - m - \mathcal{I}_r + i\varepsilon|} + \mathcal{O}(1/m), \quad (3.9) $$

with the constant $[d\tilde{\omega} = d^3\tilde{\omega}/(2\pi)^3]$

$$ \mathcal{I}_r = \frac{1}{2} g^2 C_F \int d\bar{\mathbf{\omega}} D_{\sigma\sigma}(\mathbf{\bar{\omega}}) + \mathcal{O}(1/m). \quad (3.10) $$

The (implicit) regularization of $\mathcal{I}_r$ is signaled by the symbol “$r$”. When solving Eq. (2.22), the ordering of the integration is set such that the temporal integral is performed first, under the condition that the spatial integral is regularized and finite. With the solution Eq. (3.9), we return to the Slavnov-Taylor identity, Eq. (3.5), and find that nonperturbatively the temporal quark-gluon vertex remains bare:

$$ \Gamma^0_{\bar{q}q_{\sigma\alpha\beta}}(k_1, k_2, k_3) = [gT^a]_{\alpha\beta} + \mathcal{O}(1/m). \quad (3.11) $$

Let us now briefly review the properties of the propagator Eq. (3.9) (see also [20]). As opposed to the conventional QCD quark propagator, which possesses a pair of simple poles, in this case we only have a single pole in the complex $k_0$-plane, and the explicit Feynman prescription becomes important. It then follows that the closed quark loops (constructed from virtual quark-antiquark pairs) vanish due to the energy integration, and this implies that the theory is quenched in the heavy mass limit:

$$ \int \frac{dk_0}{|k_0 - m - \mathcal{I}_r + i\varepsilon| |k_0 + p_0 - m - \mathcal{I}_r + i\varepsilon|} = 0. \quad (3.12) $$

Notice also that the propagator Eq. (3.9) is diagonal in the outer product of the fundamental color, flavor and spinor spaces. This corresponds to the decoupling of the spin from the heavy quark system, due to the decomposition Eq. (3.1). Indeed, $W_{\bar{q}q}^{(0)}$ differs from the heavy quark tree-level propagator [37] only through the mass term, and this is due to the fact that we retain the sources of the full quark fields, while in HQET one uses the sources for the large $h$-fields directly. Finally, we also stress that the position of the pole has no physical meaning (the quark can never be on-shell). Since the poles in the quark propagator are situated at infinity (this becomes apparent after the regularization in $\mathcal{I}_r$ is removed assuming, as indicated by the lattice [11], that the temporal gluon propagator is infrared enhanced), one requires infinite energy to create a single quark from the vacuum or, if a hadronic system is considered, only the relative energy (derived from the Bethe-Salpeter equation) is important. Indeed, it is precisely the exact cancellation of the divergent constants that lead to the separation of physical and unphysical poles.

As discussed in Ref. [20] (and visible in the single pole structure of the quark propagator), the mass expansion leads to the separation of the quark and antiquark states. Whilst virtual $q\bar{q}$-loops cancel due to the energy integration, Eq. (3.12), physical bound states for $q\bar{q}$ pairs should still exist. This means that we have to consider the gap equation for the antiquark propagator (and its Feynman prescription) separately. The gap equation for the antiquark propagator is derived from the full QCD gap equation Eq. (2.22), by reversing the ordering of the quark and antiquark functional derivatives that lead to the quark Green’s functions:

$$ -\Gamma_{\bar{q}q_{\alpha\sigma}}(-k) = -\Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)}(-k) - \int d\omega \left\{ \Gamma_{\bar{q}q_{\alpha\beta}}^{(0)}(-k, \omega, k - \omega) W_{\bar{q}q_{\gamma\beta}}(-\omega) \Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)a}(-\omega, k, \omega - k) W_{\sigma\sigma}^{ab}(k - \omega) \right. \\
+ \Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)}(-k, \omega, k - \omega) W_{\bar{q}q_{\gamma\beta}}(-\omega) \Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)a}(-\omega, k, \omega - k) W_{\bar{A}A_{ij}}^{ab}(k - \omega) \left. \right\}. \quad (3.13) $$

Again, we apply our truncation scheme and the above equation reduces to:

$$ \Gamma_{\bar{q}q_{\alpha\sigma}}(-k) = \Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)}(-k) + \int d\omega \Gamma_{\bar{q}q_{\alpha\beta}}^{(0)}(-k, \omega, k - \omega) W_{\bar{q}q_{\gamma\beta}}(-\omega) \Gamma_{\bar{q}q_{\alpha\sigma}}^{(0)a}(-\omega, k, \omega - k) W_{\sigma\sigma}^{ab}(k - \omega) + \mathcal{O}(1/m). \quad (3.14) $$
FIG. 3: Crossed ladder diagram that contributes to the 1PI 4-point Green’s function. The upper line denotes the quark propagator, the lower one, the antiquark propagator and springs denote the temporal gluon propagator.

In similar manner, we derive the Slavnov-Taylor identity for the antiquark-gluon vertex:

\[
-\kappa_0^\dagger \Gamma_\sigma^{q\bar{q} \alpha}(k_2, k_1, k_3) = + \Gamma_\sigma^{q\bar{q} \alpha}(k_2) \left[ igT^a \right]_\delta^\dagger T^\dagger \Gamma_\sigma^{q\bar{q} \alpha}(-k_1) + \mathcal{O}(1/m). \tag{3.15}
\]

The following antiquark propagator is obtained, as a solution to Eq. (3.14):

\[
W_{\bar{q}q \alpha \beta}(k) = \frac{-i \delta_{\alpha \beta}}{k_0 + m - i \varepsilon} + \mathcal{O}(1/m). \tag{3.16}
\]

The corresponding temporal antiquark-gluon vertex is given by:

\[
\Gamma_{\bar{q}\sigma \alpha \beta}^a(k_1, k_2, k_3) = - \left[gT^a\right]_\delta^\alpha + \mathcal{O}(1/m). \tag{3.17}
\]

In the above, notice the Feynman prescription of the propagator, as well as the sign of the loop correction. This will have the consequence that the Bethe-Salpeter equation for the quark-antiquark states can be interpreted as a bound state equation. There, the quark and the antiquark do not create a virtual quark-antiquark pair (as in the case of the closed quark loops), but a system composed of two separate unphysical particles (i.e., they are not connected by a single primitive vertex). Moreover, in the Bethe-Salpeter equation the so-called crossed box contributions (i.e., nonplanar diagrams that contain any combinations of nontrivial interactions allowed within our truncation scheme) cancel – in fact, this type of cancellation will also appear in the 4-point Green’s function. This is due to the temporal integration performed over multiple propagators with the same relative sign for the Feynman prescription (similar to Eq. (3.12), but in this case the terms originate from internal quark or antiquark propagators). As a consequence, the Bethe-Salpeter kernel reduces to the ladder truncation [20].

IV. SOLUTION IN THE HEAVY MASS LIMIT

Having reviewed the properties of the various vertices and propagators under truncation and in the heavy mass limit, we can now return to the formula for the 4-point Green’s function Eq. (2.27) (combined with Eq. (2.29)), and the corresponding diagrammatic representations Fig. 1, Fig. 2, and apply our truncation scheme in the heavy quark limit, at leading order in the mass expansion.

A. One particle irreducible Green’s function

For the quark-antiquark system, we consider the flavor non-singlet Green’s function in the s-channel, where the quark and the antiquark are regarded as two distinct flavors (but with equal masses). Hence, the diagrams (a), (c) and (i) of Fig. 1 are excluded. The diagram (b) (crossed ladder type exchange diagram) explicitly reads in momentum space and without the minus sign prefactor (see also Fig. 3):

\[
\int d\omega \left[ \Gamma_{\bar{q}\bar{q} \sigma \alpha \beta}^a(p_1, -p_1 - \omega, \omega) W_{\bar{q}q \sigma \alpha \beta}(p_1 + \omega, p_4, -p_1 - p_4 - \omega) \right] \times \left[ \Gamma_{\bar{q}q \mu \nu}^c(p_3, \omega - p_3, -\omega) W_{\bar{q}q \mu \nu}(p_3 - \omega, p_2, p_1 + p_4 + \omega) \right] W_{\sigma \lambda \gamma}^d(-\omega) W_{\sigma \lambda \gamma}^d(p_1 + p_4 + \omega) \tag{4.1}
\]
The above expression refers to the full QCD case (but omitting the spatial $W_{AA}$ contributions). The lower line corresponds to the antiquark, so we first rewrite it in terms of the (full QCD) antiquark propagators and vertices, using $W_{qq\lambda}(p_3 - \omega)$ as $-W_T^{q\lambda}(\omega - p_3)$ and $\Gamma_{qq\sigma\tau \mu} = -\Gamma_{\overline{q}q\sigma\tau \mu}$. Then we apply the mass expansion by inserting the appropriate propagators Eqs. (3.9, 3.16) and vertices Eqs. (3.11, 3.17). Noticing the energy independence of the temporal gluon propagator, one sees then immediately that the energy integral vanishes, due to the fact that both quark and antiquark propagators have the same Feynman prescription relative to the appropriate propagators Eqs. (3.9, 3.16) and vertices Eqs. (3.11, 3.17). Noticing the energy independence of the crossed ladder contribution to the kernel of the homogeneous Bethe-Salpeter equation in Ref. [20].

Turning to the diagram (d), this contribution involves the $\Gamma_{q\overline{q}}$ vertex. In the Appendix it is shown that under the scheme considered here, the Slavnov-Taylor identity furnishes the result that the nonperturbative quark-2 gluon vertex (and hence the diagram (d)) vanishes. Again, this result is analogous to the discussion of the kernel in the homogeneous Bethe-Salpeter equation from [20].

Let us for the moment discard the diagrams (f) and (g), which include the 1PI 4-point quark Green’s function, and the diagram (e), containing a 4 quark-glue vertex. Then we are only left with the diagram (h) and the rainbow-ladder term (j). This simplification enables us to derive a solution for the corresponding (truncated) equation for the 1PI 4-point quark Green’s function. With this result at hand, we will then return to the diagrams (f), (g) and (e), and explicitly show that they cancel (and hence our assumption is justified). With these observations, Eq. (2.27) reduces to:

\[ \int d\omega_0 W_{qq}(p_1 + \omega)W_{q\bar{q}}(\omega - p_3) \sim \int d\omega_0 \left\{ \frac{1}{\omega_0 + p_{10}^\mu - m - I_\epsilon + \epsilon c} - \frac{1}{\omega_0 - p_{30}^\mu + m - I_\epsilon + \epsilon c} \right\} = 0. \] (4.2)

That this crossed ladder diagram vanishes within the scheme considered here is exactly analogous to the absence of the crossed ladder contribution to the kernel of the homogeneous Bethe-Salpeter equation in Ref. [20].

It is convenient to express the resulting equation in momentum space. We define:

\[ <\overrightarrow{\sigma}_{\alpha x} q_7 z q_{\tau w} q_{\mu t}> = \int d k_1 d k_2 d k_3 d k_4 e^{-i k_1 \cdot z - i k_2 \cdot w - i k_3 \cdot x - i k_4 \cdot z} \Gamma^{(4)}_{\overline{\alpha} \gamma \gamma \tau \eta}(k_1, k_2, k_3, k_4)(2\pi)^4 \delta(k_1 + k_2 + k_3 + k_4) \] (4.4)

and arrive at the following Dyson-Schwinger equation for the 1PI 4-point quark Green’s function in the $s$-channel (shown diagrammatically in Fig. 4):

\[ \Gamma^{(4)}_{\alpha \gamma \gamma \tau \eta}(p_1, p_2, p_3, p_4) = - \int d\omega \left[ \Gamma^{(0)a}_{q\overline{q}\alpha \beta}(p_1, -p_1 - \omega, \omega)W_{q\overline{q}\beta}(p_1 + \omega)\Gamma^{(0)\beta \tau \eta}_{q\overline{q} \sigma \tau \mu}(p_1 + \omega, p_4, -p_1 - p_4 - \omega) \right] \]
\[ \times \left[ \Gamma^{(0)\beta \tau \eta}_{q\overline{q} \sigma \tau \mu}(p_3, p_2 - \omega, p_1 + p_4 + \omega)W_{q\overline{q}\mu \epsilon}(\omega - p_2)\Gamma^{(0)\mu \epsilon \tau \eta}_{q\overline{q} \sigma \tau \mu}(\omega - p_2, P_1, p_4) \right] W^{a\lambda}(\omega)W^{\mu \sigma}(p_1 + p_4 + \omega) \]
\[ - \int d\omega \left[ \Gamma^{(0)a}_{q\overline{q}\alpha \beta}(p_1, -p_1 - \omega, \omega)W_{q\overline{q}\beta\mu}(p_1 + \omega)\Gamma^{(4)}_{\nu \mu \tau \eta}(p_1 + \omega, p_2 - \omega, p_3, p_4) \right] W_{q\overline{q}\mu \epsilon}(\omega - p_2)\Gamma^{(0)\beta \tau \eta}_{q\overline{q} \sigma \tau \mu}(\omega - p_2, P_1, p_4) \]
\[ \times \Gamma^{(0)\beta \tau \eta}_{q\overline{q} \sigma \tau \mu}(-\omega). \] (4.5)

In order to proceed, we make the following assumption for the energy and momentum dependence of the function $\Gamma^{(4)}$:

\[ \Gamma^{(4)}(p_1, p_2, p_3, p_4) = \Gamma^{(4)}(P_0; \bar{p}_1 + \bar{p}_4), \] (4.6)
with $P_0 = p_1^0 + p_2^0$. This implies that in the above equation the 4-point function $\Gamma_{\nu\mu\tau\eta}^{(4)}(p_1 + \omega, p_2 - \omega, p_3, p_4)$ does not depend on the integration variable $\omega$, and hence we can separate the energy and three-momentum integrals.

Moreover, with this Ansatz we will be allowed to Fourier transform the resulting spatial integral back to coordinate space, as shall be explained shortly below.

We identify the antiquark component of Eq. (4.5) (lower line of Fig. 4) and insert the expressions Eqs. (3.9, 3.16), for the quark and antiquark propagators, along with the vertices Eqs. (3.11, 3.17). We also note the Fierz identity for the generators

$$2 [T^a]_{\alpha\beta} [T^a]_{\delta\gamma} = \delta_{\alpha\gamma} \delta_{\beta\delta} - \frac{1}{N_c} \delta_{\alpha\beta} \delta_{\gamma\delta}. \quad (4.7)$$

After completing the energy integration and with the definition Eq. (3.8) for the temporal gluon propagator, Eq. (4.5) simplifies to:

$$[P_0 - 2\mathcal{L}_r + 2i\varepsilon] \Gamma_{\alpha\beta\gamma\eta}^{(4)}(P_0; \bar{p}_1 + \bar{p}_4) = \frac{g^4}{4} \left[ \left( \frac{N_c - 2}{N_c} \right) \delta_{\alpha\gamma} \delta_{\tau\eta} + \frac{1}{N_c} \delta_{\alpha\eta} \delta_{\tau\gamma} \right] \int d\vec{\omega} W_{\sigma\sigma} (\vec{\omega}) W_{\sigma\sigma} (\bar{p}_1 + \bar{p}_4 + \vec{\omega})$$

$$+ i \frac{g^2}{2} \left[ \delta_{\alpha\gamma} \delta_{\mu\nu} - \frac{1}{N_c} \delta_{\alpha\mu} \delta_{\gamma\nu} \right] \int d\vec{\omega} W_{\sigma\sigma} (\vec{\omega}) \Gamma_{\nu\mu\tau\eta}^{(4)}(P_0; \bar{p}_1 + \bar{p}_4 + \vec{\omega}). \quad (4.8)$$

Let us now make the following color decomposition for the function $\Gamma^{(4)}$:

$$\Gamma_{\alpha\beta\gamma\eta}^{(4)} = \delta_{\alpha\gamma} \delta_{\tau\eta} \Gamma_1^{(4)} + \delta_{\alpha\eta} \delta_{\tau\gamma} \Gamma_2^{(4)}. \quad (4.9)$$

where $\Gamma_1^{(4)}$ and $\Gamma_2^{(4)}$ are scalar functions. At this point, it is convenient to Fourier transform back to coordinate space. In general, since Eq. (4.5) might in principle contain momentum-dependent vertex functions, as well as mixing of energy and three-momentum variables, this transformation could not be carried out. However, in our case (Eq. (4.8)) momentum-dependent vertices are absent and moreover, with the Ansatz Eq. (4.6), the energy and three-momentum integrals have separated such that the spatial integral is performed only over two functions (spatial gluon propagator and the spatial component of the quark 4-point function, both functions of momentum squared). Hence the Fourier transform simplifies to:

$$\int d\vec{\omega} W_{\sigma\sigma} (\vec{\omega}) \Gamma^{(4)}(P_0; (\vec{q} + \vec{\omega})^2) = \int d\vec{x} e^{-i\vec{x} \cdot \vec{\omega}} W_{\sigma\sigma}(x) \Gamma^{(4)}(P_0; x), \quad (4.10)$$

with $x = |\vec{x}|$. Sorting out the color factors, it is straightforward to obtain for the components $\Gamma_1^{(4)}$, $\Gamma_2^{(4)}$:

$$\Gamma_1^{(4)}(P_0; x) = i \left( \frac{g^2}{2N_c} \right)^2 \frac{W_{\sigma\sigma}(x)^2 N_c [(P_0 - 2i\varepsilon) (N_c^2 - 2) + ig^2 C_F W_{\sigma\sigma}(x)]}{[P_0 - 2\mathcal{L}_r + i \frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon][P_0 - 2\mathcal{L}_r - ig^2 C_F W_{\sigma\sigma}(x) + 2i\varepsilon]} \right)^2$$

$$\Gamma_2^{(4)}(P_0; x) = i \left( \frac{g^2}{2N_c} \right)^2 \frac{W_{\sigma\sigma}(x)^2 N_c [(P_0 - 2i\varepsilon) (N_c^2 - 2) + ig^2 C_F W_{\sigma\sigma}(x)]}{[P_0 - 2\mathcal{L}_r + i \frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon][P_0 - 2\mathcal{L}_r - ig^2 C_F W_{\sigma\sigma}(x) + 2i\varepsilon]} \right)^2 \quad (4.11)$$

where $x$ is the separation associated with the momentum $\vec{p}_1 + \vec{p}_4$. Inserting the above results into the decomposition
Eq. (4.9), we find the final formula for the 1PI quark Green’s function:

\[
\Gamma^{(4)}_{\alpha\gamma\tau\eta}(P_0; x) = \frac{i \left( \frac{g^2}{2N_c} \right)^2 W_{\sigma\sigma}(x)^2}{P_0 - 2I_r + i \frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2\epsilon} \times \left\{ \delta_{\alpha\gamma} \delta_{\tau\eta} \frac{(P_0 - 2I_r) N_c (N_c^2 - 2) + ig^2 N_c C_F W_{\sigma\sigma}(x)}{P_0 - 2I_r - ig^2 C_F W_{\sigma\sigma}(x) + 2\epsilon} + \delta_{\alpha\eta} \delta_{\tau\gamma} \right\}.
\] (4.12)

Having derived the solution Eq. (4.12) for the 1PI Green’s function, we return to the diagrams (f), (g) and (e) and show that they do not contribute to the final result. To see this, we first consider the diagram (g) and study the energy integral (see also Fig. 5). Noticing that the energy dependence of the internal four-point function can be written as

\[
\Gamma^{(4)}(P_0 + \omega_0) \sim \frac{\omega_0^n}{\omega_0 + X + i\epsilon},
\] (4.13)

where \(X\) is a combination of constants, \(n = 1, 2\) and \(m = 0, 1\), the energy integral takes the form

\[
\int d\omega_0 \omega_0^m \prod_{i=1}^{2+n} \frac{1}{|\omega_0 + X_i + i\epsilon|} = 0.
\] (4.14)

Clearly, this integral is a generalization of Eq. (3.12) and this vanishes, just as for the loop corrections in the kernel of the Bethe-Salpeter equation from Ref. [20] and the diagram (b) from above. An identical calculation for the diagram (f), recalling that the lower line corresponds to an antiquark propagator, leads us to the fact that this integral is also vanishing.

Finally, we are now in the position to show that the diagram (e), containing the 4 quark-gluon vertex, is also vanishing. The argumentation is based on our previous findings, namely that the diagrams (f) and (g), containing the 1PI quark Green’s functions, are zero. Because we exclude the Yang-Mills vertices, the external gluon of the five-point (proper) vertex can only couple to the (upper) quark or (lower) antiquark line. Then the 4 quark-gluon vertex at lowest order contains the diagrams shown in Fig. 6. The crossed diagrams vanish, based on integrals of the type Eq. (3.12) or Eq. (4.14), and in fact this can be generalized to the case with an arbitrary number of external gluon legs. The remaining contributions can then be included in the diagram (e) and can be written as a combination of diagrams of the form shown in Fig. 7, where the boxes contain an arbitrary number of internal gluon lines (ladder resummation). On the other hand, as a result of the Dyson-Schwinger equation for the 1PI quark Green’s functions (as in Fig. 4), the 4-point functions contained in the diagrams (f) and (g) can also be written as a ladder resummation,
and with the explicit gluon lines, diagrams (f), (g) coincide precisely with the internal substructure of the two terms in the diagram Fig. 7. Hence, the perturbative series of diagram (e) has been reorganized such that although the function $\Gamma_{\bar{q}q\sigma\tau}$ itself does not vanish, this 5-point interaction vertex and the gluon line on top of it do indeed vanish at every order perturbatively. In turn, this implies that our original assumption is correct and the solution Eq. (4.12) is valid at every order in perturbation theory. As a side remark, we note that the nonvanishing of the 5-point function relates to the existence of three-quark bound states in the Faddeev equation, in the ladder approximation [21].

### B. Amputated Green’s function

In the following, we consider the Dyson-Schwinger equation for the fully amputated 4-point quark-antiquark Green’s function in the $s$-channel (which we denote by $G^{(4)}$) and, with the simplifications outlined in the previous Section, we will derive a solution to this equation. The motivation for studying the fully amputated Green’s function in addition to the proper function is the fact that the homogeneous Bethe-Salpeter equation is based on the amputated 4-point function. Since the reduction from Eq. (2.27) to Eq. (4.3) is valid for the 1PI Green’s function, this must also hold for the amputated Green’s function. We will explicitly verify that this is satisfied, and will analyze the position of the poles and show that they relate to the Bethe-Salpeter equation for physical states.

The appropriate Dyson-Schwinger equation for the fully amputated 4-point quark-antiquark Green’s function can thus be obtained from the formula Eq. (4.3), by replacing the 1PI Green’s function $\Gamma^{(4)}$ with the amputated Green’s function $G^{(4)}$ using the expression Eq. (2.29) and cutting the legs. This equation is the inhomogeneous ladder Bethe-Salpeter equation and writing it in terms of the usual variables reads, in momentum space (see also Fig. 8):

$$
G^{(4)}_{\alpha\gamma;\tau\eta}(p_+, p_-, k_+, k_-) = W_{\sigma\tau}(\vec{p} - \vec{k}) \left[ \Gamma^{(4)}_{\bar{q}q\sigma\tau} \right]_{\alpha\eta} \left[ \Gamma_{\bar{q}q\sigma\tau} \right]_{\gamma}\nonumber
\int dq \left[ \Gamma^{(4)}_{\bar{q}q\sigma\tau} \right]_{\alpha\eta} \left[ W_{\bar{q}q}(q^-) \right]_{\mu\gamma} W^{ab}_{\sigma\tau}(\vec{p} - \vec{q}) G^{(4)}_{\mu\nu;\eta\tau}(q_+, q_-; k_+, k_-) \nonumber
\right). (4.15)
$$

In the above, the momenta of the quarks are given by $p_+ = p + \epsilon P$, $p_- = p - (1 - \xi) P$ (similarly for $k$ and $g$), and $\xi$ is the momentum sharing fraction. $P$ indicates the dependence on the total four momentum, which will become important for the investigation of the bound-state contributions to the Green’s function. It comes as no surprise that the inhomogeneous Bethe-Salpeter equation is already in the ladder truncation, similar to Eq. (4.3).

The right hand side of equation Eq. (4.15) does not depend on the external energy $p_0$, implying that the 4-point function $G^{(4)}(p_+, p_-; k_+, k_-)$ has to be independent on the relative energies $p_0, k_0$, and we further assume that $G^{(4)}$ depends on the relative spatial momentum $\vec{p} - \vec{k}$. We then identify the antiquark line as before, replace the heavy quark and antiquark propagators and vertices with the expressions Eqs. (3.9, 3.16, 3.11, 3.17), and perform the energy integration. We arrive at the following expression:

$$
G^{(4)}_{\alpha\gamma;\tau\eta}(P_0; \vec{p} - \vec{k}) = g^2 T_{\alpha\gamma}^{\tau\gamma} W_{\sigma\tau}(\vec{p} - \vec{k}) + g^2 T_{\alpha\gamma}^{\mu\gamma} T^{\mu\gamma}_{\tau\eta} P_0 - 2E + 2\epsilon x \int d\bar{q} W_{\sigma\tau}(\vec{p} - \vec{q}) G^{(4)}_{\mu\nu;\eta\tau}(P_0; \vec{q} - \vec{k}). \nonumber
\right) (4.16)
$$

Having integrated out the energy, it is convenient to rewrite the above formula back into coordinate space. Using the definition Eq. (4.4) and noting that the Fourier transform is as before given via Eq. (4.10), the equation Eq. (4.16) simplifies to (the separation $x$ corresponds to the momentum $\vec{p} - \vec{k}$):

$$
G^{(4)}_{\alpha\gamma;\tau\eta}(P_0; x) = g^2 T_{\alpha\gamma}^{\tau\gamma} W_{\sigma\tau}(x) + g^2 T_{\alpha\gamma}^{\mu\gamma} T^{\mu\gamma}_{\tau\eta} P_0 - 2E + 2\epsilon x \int d\bar{q} W_{\sigma\tau}(x) G^{(4)}_{\mu\nu;\eta\tau}(P_0; x). \nonumber
\right) (4.17)
$$
Again, we make a color decomposition of the function $G^{(4)}$:

$$G^{(4)}_{\alpha\gamma,\tau\eta} = \delta_{\alpha\gamma} \delta_{\tau\eta} G^{(4)}_1 + \delta_{\alpha\eta} \delta_{\tau\gamma} G^{(4)}_2,$$  \hspace{1cm} (4.18)

($G^{(4)}_1$ and $G^{(4)}_2$ are scalar functions), use the Fierz identity, Eq. (4.7), to sort out the color factors, and obtain the following results for the components $G^{(4)}_1$, $G^{(4)}_2$:

$$G^{(4)}_1(P_0; x) = \frac{g^2}{2} \frac{(P_0 - 2\mathcal{I}_r)^2 W_{\sigma\sigma}(x)}{[P_0 - 2\mathcal{I}_r + \frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon]} \left( P_0 - 2\mathcal{I}_r + i\frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon \right),$$  \hspace{1cm} (4.19)

$$G^{(4)}_2(P_0; x) = -\left( \frac{g^2}{2N_c} \right) \frac{(P_0 - 2\mathcal{I}_r) W_{\sigma\sigma}(x)}{P_0 - 2\mathcal{I}_r + i\frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon}. \hspace{1cm} (4.20)$$

Replacing these results in the formula Eq. (4.18), we get the final result for the function $G^{(4)}$:

$$G^{(4)}_{\alpha\gamma,\tau\eta}(P_0; x) = \frac{g^2}{2} \frac{(P_0 - 2\mathcal{I}_r) W_{\sigma\sigma}(x)}{[P_0 - 2\mathcal{I}_r + \frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon]} \left[ \delta_{\alpha\gamma} \delta_{\tau\eta} \frac{(P_0 - 2\mathcal{I}_r)}{P_0 - 2\mathcal{I}_r + i\frac{g^2}{2N_c} W_{\sigma\sigma}(x) + 2i\varepsilon} - \delta_{\alpha\eta} \delta_{\tau\gamma} \frac{1}{N_c} \right]. \hspace{1cm} (4.21)$$

A direct calculation shows that our result for the amputated 4-point function is related to the result Eq. (4.12) for the 1PI Green’s function, via the formula Eq. (2.29) in the $s$-channel (or alternatively, Fig. 2):

$$\Gamma^{(4)}_{\alpha\gamma,\tau\eta}(p_+, p_-, k_+, k_-) = G^{(4)}_{\alpha\gamma,\tau\eta}(p_+, p_-, k_+, k_-) - W_{\sigma\sigma}(\bar{p} - \bar{k}) \left[ \Gamma^{(4)}_{\bar{q}\bar{q}\sigma}_{\alpha\eta} \Gamma^{(4)}_{\bar{q}\bar{q}\sigma}_{\tau\gamma} \right]. \hspace{1cm} (4.22)$$

A few comments regarding the structure of the above equation are in order. Notice that despite the truncation, the denominator structure of the 1PI and amputated Green’s functions is identical in both color channels. Moreover, in this approach the physical and nonphysical poles disentangle automatically. Using the form Eq. (3.8) for the temporal gluon propagator, the denominator factor of the color singlet channel can be rewritten in the form

$$P_0 - g^2 \int d\bar{\omega} \frac{D_{\sigma\sigma}(\bar{\omega})}{\bar{\omega}^2} C_F \left[ 1 - e^{i\bar{\omega} \cdot \bar{x}} \right] \hspace{1cm} (4.23)$$

and hence the bound state (infrared confining) energy $P_0^{res}(x) = \sigma|\bar{x}|$ emerges as the finite pole position of the resonant component (first term in the bracket of Eq. (4.21) or Eq. (4.12)), for arbitrary number of colors, assuming that $D_{\sigma\sigma}(\bar{\omega}^2) \sim 1/\bar{\omega}^2$ as indicated on the lattice [11]. This provides an explicit analytical dependence of the 4-point Green’s function on the $\bar{q}\bar{q}$ bound state energy, which results from the homogeneous Bethe-Salpeter equation from Ref. [20]. The overall denominator factor for Eq. (4.21) is part of the normalization and has the explicit form

$$P_0 - g^2 \int d\bar{\omega} \frac{D_{\sigma\sigma}(\bar{\omega})}{\bar{\omega}^2} \left[ C_F + \frac{1}{2N_c} e^{i\bar{\omega} \cdot \bar{x}} \right] \hspace{1cm} (4.24)$$

which (like the quark propagator) implies a pole position at infinity when the regularization is removed. This factor does not appear in the homogeneous Bethe-Salpeter equation. Notice that the result, Eq. (4.21), is independent of $\xi$ (the momentum sharing fraction). That the physical pole position should be independent of $\xi$ is clear (c.f. Refs. [20, 23]); that the rest is independent of $\xi$ is presumably a consequence of the truncation scheme considered here.

V. 4-POINT GREEN’S FUNCTIONS FOR DIQUARKS

Let us now consider the diquark 4-point Green’s function. This can be easily obtained from the equation Eq. (2.27) for quark-antiquark systems, by interchanging the quark legs and inserting the appropriate minus signs. We obtain
Further, the diagrams (b) and (h) are zero, since the corresponding quark-gluon vertex and thus the diagram (a) is zero. A similar type of integral arises in the diagram (j), and hence this term is also zero.

Assume that the integrals (f), (i), containing the diquark 4-point vertex, and (g), containing the 5-point function, are not giving a contribution. Further, the diagrams (b) and (h) are zero, since the corresponding quark-gluon vertex has been replaced with its expression Eq. (2.14). Also, notice that the diquark is antisymmetric between quark and antiquark.

As in the $\bar{q}q$ case, the dots represent the $\bar{A}$ vertex terms which are not considered here and the tree-level temporal quark-gluon vertex has been replaced with its expression Eq. (2.14). Also, notice that the diquark is antisymmetric under the exchange of two quark legs ($\alpha \leftrightarrow \tau$ or $\gamma \leftrightarrow \eta$). We must therefore explicitly keep track of which quark is which and thus introduce flavor structure with an arbitrary number of equal mass flavors (before, we had to distinguish between quark and antiquark).

As before, we analyze the diagrammatic representation from Fig. 9 and show that the same type of cancellations occur. Starting with the diagram (a) in momentum space, we notice that this is a crossed ladder type exchange diagram (see also Fig. 10):

\[
\langle \bar{q}_a q_x q_y q_{g\gamma} q_{g\eta} \rangle = \left[ g\gamma^\alpha T^\alpha \right] \int dy \delta (x-y)
\]

\[
\times \left\{ - \left[ \bar{q}_a \Gamma^{0}_{\beta\gamma} \gamma^\alpha q_x q_y q_{g\gamma} q_{g\eta} \right] + \left[ \bar{q}_a \Gamma^{0}_{\beta\gamma} \gamma^\alpha q_x q_y q_{g\gamma} q_{g\eta} \right] \right\} \]

As in the $\bar{q}q$ case, the dots represent the $\bar{A}$ vertex terms which are not considered here and the tree-level temporal quark-gluon vertex has been replaced with its expression Eq. (2.14). Also, notice that the diquark is antisymmetric under the exchange of two quark legs ($\alpha \leftrightarrow \tau$ or $\gamma \leftrightarrow \eta$). We must therefore explicitly keep track of which quark is which and thus introduce flavor structure with an arbitrary number of equal mass flavors (before, we had to distinguish between quark and antiquark).

As before, we analyze the diagrammatic representation from Fig. 9 and show that the same type of cancellations occur. Starting with the diagram (a) in momentum space, we notice that this is a crossed ladder type exchange diagram (see also Fig. 10):

\[
\int d\omega \left[ \bar{q}_a \Gamma^{0}_{q\gamma q\beta \gamma} (p_1, \omega - p_1, -\omega) W_{q\gamma q\beta \gamma} (p_1 - \omega, p_3, \omega - p_1, -p_3) \right]
\]

\[
\times \left[ \bar{q}_a \Gamma^{0}_{q\gamma q\beta \gamma} (p_2, p_4 + \omega, p_1 + p_3 - \omega) W_{q\gamma q\beta \gamma} (p_4 - \omega, p_4, \omega) \right] W^{ab}_{\sigma\sigma} (\omega) W^{dc}_{\sigma\sigma} (p_1 + p_3 - \omega). \]  

It has been already shown that the integral over the quark propagators (with the same Feynman prescription) vanishes and thus the diagram (a) is zero. A similar type of integral arises in the diagram (j), and hence this term is also not giving a contribution. Further, the diagrams (b) and (h) are zero, since the corresponding quark-2 gluon vertex vanishes according to the Slavnov-Taylor identity, Eq. (A.14). As in the case of the $\bar{q}q$ system, let us for the moment assume that the integrals (f), (i), containing the diquark 4-point vertex, and (g), containing the 5-point function
\[ \Gamma_{\tilde{q}qq\sigma} \] are also zero, and solve the Dyson-Schwinger equation with the remaining terms. Having derived the solution, we will then return to the diagrams (f), (i) and (g) and show that the y vanish. The Dyson-Schwinger equation for the momentum space:

\[ \int \frac{d^4 \mathbf{p}}{(2\pi)^4} \frac{1}{(2\omega - (p_2 + p_1 + m - i\varepsilon))^4} \left[ \Gamma_{\tilde{q}qq\sigma}(p_1, p_2) \right] = \frac{1}{(2\omega - (p_2 + m - i\varepsilon))^4} \]

\[ \Gamma_{\tilde{q}qq\sigma}(p_1, p_2) = \frac{1}{(2\omega - (p_2 + m - i\varepsilon))^4} \left[ \Gamma_{\tilde{q}qq\sigma}(p_1, p_2) \times \Gamma_{\tilde{q}qq\sigma,(p_1 + \omega, p_2 - \omega, p_3 + \omega)} \right] \]

Inserting the expressions for the quark propagator Eq. (3.9) and vertex Eq. (3.11), resolving the color factors using the Fierz identity, Eq. (4.7), and simplifying the energy integrals, we have (noting the flavor channels with the \( \delta_f \) factors):

\[ \Gamma_{\tilde{q}qq\sigma}(p_1, p_2) = \frac{1}{(2\omega - (p_2 + m - i\varepsilon))^4} \left[ \right] \]

Notice the distinctive energy, momentum and flavor structure of the equation, a suitable Ansatz for the diquark function is

\[ \Gamma_{\alpha\tau\gamma\eta}(p_1, p_2, p_3, p_4) = \delta_{\alpha\tilde{q}qq\sigma} \Gamma_{\alpha\tau\gamma\eta}(p_1, p_2, p_3, p_4) + \delta_{\alpha\tilde{q}qq\sigma} \Gamma_{\alpha\tau\gamma\eta}(p_1, p_2, p_3, p_4) \]

where the superscripts c and p stand for the crossed and parallel configurations, respectively, \( P_0 = p_0^c + p_0^p \) as before,
Putting the components together, we find
\[
\Gamma_{\alpha\tau\gamma}(P_0; \tilde{p}_1 + \tilde{p}_3) = (-1) \frac{g^2}{2N_c} \frac{1}{P^0 - 2m - 2\Delta + 2\varepsilon} \left\{ \frac{g^2}{2N_c} \left[ (N^2_c + 1) \delta_{\alpha\tau} \delta_{\tau\gamma} - 2N_c \delta_{\alpha\eta} \delta_{\tau\gamma} \right] \int d\bar{\omega} W_{\sigma\sigma}(\bar{\omega}) W_{\sigma\sigma}(\tilde{p}_1 + \tilde{p}_3 + \bar{\omega}) + [N_c \delta_{\alpha\tau} \delta_{\tau\lambda} - \delta_{\alpha\lambda} \delta_{\tau\gamma}] \int d\bar{\omega} W_{\sigma\sigma}(\bar{\omega}) \Gamma^{(c)}_{\lambda\kappa\gamma}(P_0; \tilde{p}_1 + \tilde{p}_3 + \bar{\omega}) \right\}, \tag{5.6}
\]
\[
\Gamma^{(p)}_{\alpha\tau\gamma}(P_0; \tilde{p}_1 + \tilde{p}_4) = \frac{g^2}{2N_c} \frac{1}{P^0 - 2m - 2\Delta + 2\varepsilon} \left\{ \frac{g^2}{2N_c} \left[ (N^2_c + 1) \delta_{\alpha\eta} \delta_{\tau\gamma} - 2N_c \delta_{\alpha\gamma} \delta_{\tau\gamma} \right] \int d\bar{\omega} W_{\sigma\sigma}(\bar{\omega}) W_{\sigma\sigma}(\tilde{p}_1 + \tilde{p}_4 + \bar{\omega}) - [N_c \delta_{\alpha\tau} \delta_{\tau\lambda} - \delta_{\alpha\lambda} \delta_{\tau\gamma}] \int d\bar{\omega} W_{\sigma\sigma}(\bar{\omega}) \Gamma^{(p)}_{\lambda\kappa\gamma}(P_0; \tilde{p}_1 + \tilde{p}_4 + \bar{\omega}) \right\}. \tag{5.7}
\]

Fourier transforming to configuration space as in the previous section and with \(x, y\) representing the separations associated with \(\tilde{p}_1 + \tilde{p}_4, \tilde{p}_1 + \tilde{p}_3\), respectively, the equations read
\[
\Gamma^{(c)}_{\alpha\tau\gamma}(P_0; y) = (-1) \frac{g^2}{2N_c} \frac{1}{P^0 - 2m - 2\Delta + 2\varepsilon} \left\{ \frac{g^2}{2N_c} \left[ (N^2_c + 1) \delta_{\alpha\gamma} \delta_{\tau\eta} - 2N_c \delta_{\alpha\eta} \delta_{\tau\gamma} \right] W_{\sigma\sigma}(y)^2 + [N_c \delta_{\alpha\tau} \delta_{\tau\lambda} - \delta_{\alpha\lambda} \delta_{\tau\gamma}] W_{\sigma\sigma}(y) \Gamma^{(c)}_{\lambda\kappa\gamma}(P_0; y) \right\}; \tag{5.8}
\]
\[
\Gamma^{(p)}_{\alpha\tau\gamma}(P_0; x) = (+1) \frac{g^2}{2N_c} \frac{1}{P^0 - 2m - 2\Delta + 2\varepsilon} \left\{ \frac{g^2}{2N_c} \left[ (N^2_c + 1) \delta_{\alpha\eta} \delta_{\tau\gamma} - 2N_c \delta_{\alpha\gamma} \delta_{\tau\gamma} \right] W_{\sigma\sigma}(x)^2 - [N_c \delta_{\alpha\tau} \delta_{\tau\lambda} - \delta_{\alpha\lambda} \delta_{\tau\gamma}] W_{\sigma\sigma}(x) \Gamma^{(p)}_{\lambda\kappa\gamma}(P_0; x) \right\}. \tag{5.9}
\]

Further, we make the color decomposition:
\[
\Gamma^{(c,p)}_{\alpha\tau\gamma} = \delta_{\alpha\gamma} \delta_{\tau\eta} \Gamma^{(c,p;1)} + \delta_{\alpha\eta} \delta_{\tau\gamma} \Gamma^{(c,p;2)}, \tag{5.10}
\]
where \(\Gamma^{(c,p;1,2)}\) are now scalar dressing functions. Defining the functions
\[
f_{\pm}(P_0; x) = (-1) \left( \frac{g^2}{2N_c} \right)^2 \frac{(N_c + 1)^2 W_{\sigma\sigma}(x)^2}{P^0 - 2m - 2\Delta - \frac{1}{2N_c}(1 + N_c)W_{\sigma\sigma}(x) + 2\varepsilon} \tag{5.11}
\]
it is then straightforward to show that the solution to the above equations can be written
\[
2\Gamma^{(c)}(P_0; y) = f_+(P_0; y) + f_-(P_0; y), \tag{5.12a}
\]
\[
2\Gamma^{(c)}(P_0; y) = f_+(P_0; y) - f_-(P_0; y), \tag{5.12b}
\]
\[
2\Gamma^{(p)}(P_0; x) = f_-(P_0; x) - f_+(P_0; x) = -2\Gamma^{(c)}(P_0; x), \tag{5.12c}
\]
\[
2\Gamma^{(p)}(P_0; x) = -f_+(P_0; x) - f_-(P_0; x) = -2\Gamma^{(c)}(P_0; x). \tag{5.12d}
\]

Putting the components together, we find
\[
\Gamma_{\alpha\tau\gamma}(P_0; x, y) = \delta f_{\alpha\gamma} \delta f_{\tau\eta} \left[ \delta_{\alpha\gamma} \delta_{\tau\eta} \Gamma^{(c)}(P_0; y) + \delta_{\alpha\eta} \delta_{\tau\gamma} \Gamma^{(c)}(P_0; y) \right] + \delta f_{\alpha\gamma} \delta f_{\tau\gamma} \left[ \delta_{\alpha\gamma} \delta_{\tau\gamma} \Gamma^{(p)}(P_0; x) + \delta_{\alpha\eta} \delta_{\tau\gamma} \Gamma^{(p)}(P_0; x) \right] \nonumber
\]
\[
= \frac{1}{2} \left\{ (\delta_{\alpha\tau} \delta_{\tau\eta} + \delta_{\alpha\eta} \delta_{\tau\gamma}) \left[ \delta f_{\alpha\gamma} \delta f_{\tau\gamma} f_+(P_0; y) - \delta f_{\alpha\eta} \delta f_{\tau\gamma} f_+(P_0; x) \right] + (\delta_{\alpha\tau} \delta_{\tau\eta} - \delta_{\alpha\eta} \delta_{\tau\gamma}) \left[ \delta f_{\alpha\gamma} \delta f_{\tau\gamma} f_-(P_0; y) + \delta f_{\alpha\eta} \delta f_{\tau\gamma} f_-(P_0; x) \right] \right\}. \tag{5.13}
\]

As in the case of the \(\bar{q}q\) systems, with this solution we return to the diagrams \(f, i\) and \((g)\). Writing out the explicit form of the energy integrals we notice that their form is identical to the quark-antiquark case, since the \(\varepsilon\) prescription is similar, regardless of the internal quark (or antiquark) propagator. Thus, these diagrams are also vanishing.
Analyzing the pole structure of the above solution, Eq. (5.13), we notice that, as in the case of the \( \bar{q}q \) systems, we have two different pole conditions (for either separation \( x \) or \( y \)):

\[
P_{0 \text{res}}(x) - 2m - 2\mathcal{I}_r - \frac{g^2}{2N_c}(1 \pm N_c)W_{\sigma\sigma}(x) = 0
\]

which can be rewritten as

\[
P_{0 \text{res}}(x) = 2m + g^2 \int_\mathcal{I}_r \frac{d\tilde{\omega}^2}{\tilde{\omega}^2} D_{\sigma\sigma}(\tilde{\omega}) \left[ C_F - \frac{1 \pm N_c}{2N_c} e^{i\tilde{\omega}^2 x} \right].
\] (5.15)

With the plus sign (corresponding to the color antisymmetric term or \( f_- \) in Eq. (5.13) above), we find the condition for the finite pole position (i.e., finite integral even with infrared enhanced \( D_{\sigma\sigma} \)) \( P_{0 \text{res}}(x), N_c = 2, -1 \); but with the minus sign (\( f_+ \), color symmetric), we obtain \( N_c = -2, 1 \). Hence, the only physical solution, with \( N_c = 2 \), corresponds to a color antisymmetric and flavor symmetric configuration, in agreement with our findings from Ref. [20] that bound states exist only for color antisymmetric \( SU(2) \) baryons (in that case flavor symmetry was implicit). Indeed, the physical poles in the flavor channels are the same (both correspond to \( f_- \)), as demanded by the symmetry of the system. The homogeneous equation makes no reference to the flavor structure and can only have the flavor symmetric part. Also notice that whereas for the \( \bar{q}q \) system there was a common unphysical pole in the normalization of \( \Gamma^{(4)} \) or \( G^{(4)} \) and then a distinct physical pole for the color singlet channel, in the diquark case each channel has a separate pole.

VI. SUMMARY AND CONCLUSIONS

In this paper, the Dyson-Schwinger equations for the 1PI and amputated 4-point Green’s functions, for quark-antiquark and diquark systems, have been considered. At leading order in the heavy quark mass expansion and with the truncation to include only the (nonperturbative) gluon propagator and neglect the pure Yang-Mills vertices and higher order interactions, analytic solutions for the Green’s functions have been obtained. We have found that the physical and unphysical poles disentangle, in contrast to the conventional phenomenological calculations. Indeed, this is one of the main results of this work, since in contemporary studies the interpretation of the spurious states appearing in the Bethe-Salpeter equation is still under debate. These results may hopefully be of further use in the construction of phenomenological models of mesons and baryons.

The color singlet pole of the 4-point Green’s function for \( \bar{q}q \) systems leads directly to the linearly rising potential (confining energy for the bound state), for arbitrary number of colors, and in turn this can be related to the \( \bar{q}q \) bound state energy arising from the homogeneous Bethe-Salpeter equation (considered in Ref. [20]). The second pole, which is situated at infinity as the (infrared) regularization is removed, is nonphysical and is part of the normalization; it is similar to the poles appearing in the gap equation and in the Faddeev vertex for baryons [21]. This simply means that the only relevant quantity is the bound state energy, i.e. the pole of the resonant component. In the case of the diquarks, the only physical solution, for \( N_c = 2 \) colors, corresponding to a color antisymmetric and flavor symmetric configuration, again verifies our result from Ref. [20] that bound states only appear for \( SU(2) \) baryons and otherwise the system is not physically allowed. The second pole, corresponding to a flavor antisymmetric configuration, is not contained within the homogeneous Bethe-Salpeter equation (there, the flavor symmetry is implicit). We thus confirm that within the scheme studied here, the homogeneous Bethe-Salpeter equation furnishes exactly the right physical poles with no other solution and these are explicitly contained within the full Green’s functions.

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Appendix: Slavnov-Taylor identities

In this Appendix we consider the Slavnov-Taylor identity for the quark-2 gluon vertex appearing in the text. The derivation is similar to the Slavnov-Taylor identity for the quark-gluon vertex presented in Ref. [20]. We start with
the full QCD action in the standard, second order formalism:

\[
S_{QCD} = \int d^4x \left\{ \bar{\psi}_\alpha \left[ i\gamma^0 \partial_0 + g T^a \gamma^0 \sigma^a + i \gamma^\lambda \nabla - g T^a \gamma_\lambda \tilde{A}^a - m \right]_\alpha _\beta q_\beta - \frac{1}{4} F^a_{\mu \nu} F^{a \mu \nu} \right\} \quad (A.1)
\]

where the (antisymmetric) field strength tensor is defined in terms of the gauge potential \(A^a_\mu\):

\[
F^a_{\mu \nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \quad (A.2)
\]

The action is invariant under a local \(SU(N_c)\) gauge transform characterized by the parameter \(\theta^a_\alpha\):

\[
U_x = \exp \{-i \theta^a_\alpha T^a \} \quad (A.3)
\]

such that for infinitesimal \(\theta^a_\alpha\), the fields transform as (with the notation \(\sigma \equiv A_0\))

\[
\delta \sigma_a = -\frac{1}{g} \partial_0 \theta^a - f^{abc} \sigma^b \theta^c, \quad \delta \bar{A}^a = \frac{1}{g} \nabla \theta^a - f^{abc} \bar{A}^b \theta^c,
\]

\[
\delta q_\alpha = -i \theta^a [T^a]_\alpha _\beta q_\beta, \quad \delta \bar{q}_\alpha = i \bar{\theta}^a \bar{q}_\beta [T^a]_\beta _\alpha. \quad (A.4)
\]

The action Eq. (A.1) is invariant under gauge transformations and hence the functional integral \(Z\) is divergent by virtue of a zero mode. To surmount this problem we use the Faddeev-Popov technique and introduce a gauge-fixing term along with an associated ghost term. In Coulomb gauge (\(\nabla \cdot \tilde{A} = 0\)), the new term in the action reads:

\[
S_{FP} = \int d^4x \left[ -\lambda^a \nabla \cdot \tilde{A}^a - \bar{c}^a \nabla \cdot \tilde{D}^{ab} c^b \right], \quad (A.5)
\]

where the gauge fixing condition is implemented with the help of the Lagrange multiplier \(\lambda^a\), and \(\bar{c}^a\) and \(c^b\) are the Grassmann-valued ghost fields. The spatial covariant derivative (in the adjoint representation) is given by:

\[
\tilde{D}^{ab} = \delta^{ab} \nabla - g f^{aef} \bar{A}^f. \quad (A.6)
\]

The action is invariant under a Gauss-BRST transform [1] whereby the infinitesimal spacetime-dependent parameter \(\theta_x^a\) is factorized into two Grassmann-valued components: \(\theta_x^a = c^a_\lambda \delta \lambda_\lambda\). In Coulomb gauge, the gauge fixing term does not involve any time derivatives and hence one can define the infinitesimal variation \(\delta \lambda_\beta\) (not to be confused with the Lagrange multiplier \(\lambda^a\)) to be \(time-dependent\). The variations of the new fields read:

\[
\delta \bar{c}^a = \frac{1}{g} \lambda^a \delta \lambda_\lambda, \quad \delta c^a = -\frac{1}{2} f^{abc} c^b \delta \lambda_\lambda, \quad \delta \lambda^a = 0. \quad (A.7)
\]

The generating functional is given by

\[
Z[J] = \int D\Phi \exp \{ iS_{QCD} + iS_{FP} + iS_s \}, \quad (A.8)
\]

with the source term

\[
S_s = \int d^4x \left[ \rho^a \sigma^a + \tilde{f}^a \tilde{A}^a + \bar{c}^a \eta^a + \bar{\eta}^a c^a + \xi^a \lambda^a + \bar{\xi}^a \lambda^a + \bar{\eta}^a q^a \right]. \quad (A.9)
\]

The Slavnov-Taylor identities in Coulomb gauge are derived from the observation that the Gauss-BRST transform can be regarded as a change of integration variables under which the generating functional is invariant. Provided that the Jacobian factor is trivial [8], we are left with an equation where only the source term varies:

\[
0 = \left. \int D\Phi \frac{\delta}{\delta [\delta \lambda_\lambda]} \exp \{ iS_{QCD} + iS_{FP} + iS_s + i\delta S_s \} \right|_{\delta \lambda_\lambda = 0}
\]

\[
= \int D\Phi \exp \{ iS_{QCD} + iS_{FP} + iS_s \} \int d^4x \delta(t-x_0) \left\{ -\frac{1}{g} \left( \partial_\mu \rho_x^a \right) c^a_x + f^{abc} \rho_x^a c^b_x c^c_x - \frac{1}{g} f^{abc} \bar{c}_x^a \bar{c}_x^b \bar{c}_x^c \right\} \quad (A.10)
\]
The $\delta(t-x_0)$ constraint, appearing because of the time-dependent variation $\delta \lambda_i$, leads in principle to a non-trivial energy injection into the ghost lines of the Slavnov-Taylor identities (the ghost functions are, however, discarded in our truncation scheme). The above formula can be reexpressed by using the definitions Eq. (2.8) for the connected energy injection into the ghost lines of the Slavnov-Taylor identities (the ghost functions are, however, discarded in classical field

Note that functional derivatives involving the Lagrange multiplier result merely in a trivial identity such that the classical field $\lambda^a_i$ can be set to zero [17]. Further, one functional derivative with respect to $\nu^a_0$ must be taken, and it follows that the Slavnov-Taylor identities are functional derivatives of (as we will take further derivatives, all the fields and sources must be retained):

$$
0 = \int d^4 x \delta(t - x_0) \left\{ \frac{1}{g} \left( \partial_0^a <\sigma_0^a> + \frac{f_{abc}}{g} <\sigma_0^a> \right) e^c_i - f^{abc} <\sigma_0^a> \left[ <\eta_0^b \eta_0^c> + \sigma_0^c e^a_i \right] \right\} + \frac{1}{g} \left[ \frac{\nabla_0^i}{(-\nabla_0^2)} <\eta_0^a \eta_0^i> - 1 <\eta_0^a \eta_0^i> \right] + \frac{1}{g} \left[ - \frac{f^{abc}}{g} <\sigma_0^a> \left[ <\eta_0^b \eta_0^c> + \sigma_0^c e^a_i \right] \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right]. \tag{A.11}
$$

Note that functional derivatives involving the Lagrange multiplier result merely in a trivial identity such that the classical field $\lambda^a_i$ can be set to zero [17]. Further, one functional derivative with respect to $\nu^a_0$ must be taken, and it follows that the Slavnov-Taylor identities are functional derivatives of (as we will take further derivatives, all the fields and sources must be retained):

Starting with the above equation, the procedure is now to functionally differentiate with respect to the quark, antiquark and gluon fields. In principle, the resulting equation contains a large number of terms, however most of them simplify in the heavy mass limit and under truncation. To be specific, after taking the functional derivatives there are four categories of terms entering the equation — three of them are vanishing and one remains. Firstly, the terms multiplied by a spatial quark-gluon vertex do not contribute, since this vertex is suppressed at leading order in the mass expansion (as explained previously in Sec. III). Secondly, the terms containing a 4-point function $\Gamma_{q\bar{q}A\sigma}$ are also of $O(1/m)$, due to the fact that in the corresponding Dyson-Schwinger equation at least one vertex in each loop term must be at tree-level (just as for the $\Gamma_{q\bar{q}A}$ vertex), and we are at liberty to chose this to be $\Gamma_{q\bar{q}A}$ (suppressed by the mass expansion) or a pure Yang-Mills contribution (explicitly truncated out). Thirdly, the ghost kernels arising from the functional derivatives also vanish, since they necessarily involve Yang-Mills vertices. Hence, under truncation and in the heavy mass limit, only the terms that involve a temporal quark-gluon vertex will survive. Explicitly, the equation Eq. (A.12), from which the Slavnov-Taylor identity for the quark-2 gluon vertex is derived, reduces to:

$$
0 = \int d^4 x \delta(t - x_0) \delta(z - x) \left\{ - \frac{g}{4} \left( \partial_0^a <\sigma_0^d> + f^{abc} <\sigma_0^a> \right) \sigma_0^c (t_i x_j) \right\} + \frac{1}{g} \left[ \frac{\nabla_0^i}{(-\nabla_0^2)} <\eta_0^a \eta_0^i> - 1 <\eta_0^a \eta_0^i> \right] + \frac{1}{g} \left[ - \frac{f^{abc}}{g} <\sigma_0^a> \left[ <\eta_0^b \eta_0^c> + \sigma_0^c e^a_i \right] \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right]. \tag{A.13}
$$

We now functionally differentiate with respect to $\eta_{t,y}, \eta_{t,p}$ and $\sigma^a_w$ and arrive at the following expression, after setting the sources to zero:

$$
0 = \int d^4 x \delta(t - x_0) \delta(z - x) \left\{ - \frac{g}{4} \left( \partial_0^a <\sigma_0^d> + f^{abc} <\sigma_0^a> \right) \sigma_0^c (t_i x_j) \right\} + \frac{1}{g} \left[ \frac{\nabla_0^i}{(-\nabla_0^2)} <\eta_0^a \eta_0^i> - 1 <\eta_0^a \eta_0^i> \right] + \frac{1}{g} \left[ - \frac{f^{abc}}{g} <\sigma_0^a> \left[ <\eta_0^b \eta_0^c> + \sigma_0^c e^a_i \right] \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right] + \frac{1}{2} \frac{f^{abc}}{g} <\eta_0^a \eta_0^i> \left[ (\eta_0^b \eta_0^c) + e^a_i \right]. \tag{A.14}
$$

Identifying $<\eta_{t,y} \eta_{t,y} \sigma^a_w>$ with $\Gamma_{q\bar{q}A}$ and Fourier transforming, one obtains (the momentum dependence of the right hand side is trivial)

$$
\Gamma_{q\bar{q}A}^{de} \sim f_{de}^{a} \Gamma_{q\bar{q}A}^{a} + \Gamma_{q\bar{q}A}^{c} \Gamma_{q\bar{q}A}^{d} - i T^{d} T^{e} - i T^{d} T^{e} - i T^{d} T^{e} - i T^{d} T^{e} = 0. \tag{A.15}
$$

which under truncation (where $\Gamma_{q\bar{q}A} = \Gamma_{q\bar{q}A}^{(0)}$) gives

$$
\Gamma_{q\bar{q}A}^{de} \sim f_{de}^{a} \Gamma_{q\bar{q}A}^{a} + \Gamma_{q\bar{q}A}^{c} \Gamma_{q\bar{q}A}^{d} + f_{de}^{a} \Gamma_{q\bar{q}A}^{a} + \Gamma_{q\bar{q}A}^{c} \Gamma_{q\bar{q}A}^{d} = 0. \tag{A.16}
$$
