On a Growing Transverse Mode as a Post-Newtonian Effect in the Large-Scale Structure Formation

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We point out the existence of a new type of growing transverse mode in the gravitational instability. This appears as a post-Newtonian effect to Newtonian dynamics. We demonstrate this existence by formulating the Lagrangian perturbation theory in the framework of the cosmological post-Newtonian approximation in general relativity. Such post-Newtonian order effects might produce characteristic appearances of large-scale structure formation, for example, through the observation of anisotropy of the cosmic microwave background radiation (CMB).

§1. Introduction

The investigation of nonlinear, large-scale structure formation of the universe has been a major research subject in cosmology. As long as the scales of the density fluctuations are much smaller than the horizon scale, the Newtonian approximation is sufficiently accurate to describe their evolution up to the non-linear regime, where the density contrast becomes larger than unity. On the other hand, the region in which the large-scale structure is observed is steadily growing. For example, SDSS (Sloan Digital Sky Survey) will cover over a region of several hundred megaparsecs. It is not clear if the application of Newtonian theory is appropriate for such a wide region of spacetime. In fact fluctuations relevant for such large-scale structures are not always much smaller than horizon scales in the past. For example, the horizon scale at the decoupling time is on the order of \(c t_{\text{dec}}(1+z_{\text{dec}}) \sim 80h^{-1}\text{Mpc}\) in the present physical length. This suggests that we must employ a relativistic description for the evolution of fluctuations larger than or equivalent to such scales.

Thus to understand the evolution of the large-scale structure of the universe, it is important and necessary to have some formalism to clearly evaluate the effect of general relativistic corrections to the Newtonian dynamics. Naturally, we also require the formalism to agree with the gauge invariant linear theory developed by Bardeen, and Kodama and Sasaki in the linear regime.

We develop such a formalism based on the Lagrangian perturbation theory in the post-Newtonian (PN) framework in which it has been shown that the formalism can be applied even for perturbations larger than the present horizon scale \(\sim 2000h^{-1}\text{Mpc}\). The Newtonian Lagrangian picture has also been developed by Buchert, where it is expected that the approach gives a good approximation up to a certain stage of the non-linear regime. There have also been some
studies based on relativistic Lagrangian perturbation theory. However, because of the gauge condition adopted in these studies, namely, the synchronous comoving coordinates, it is not easy to have contact with the Newtonian Lagrangian approach fully developed and used for the numerical simulation. Thus we shall study the Lagrangian perturbation in coordinates where comparison with the Newtonian case is most easily carried out.

§2. Cosmological post-Newtonian approximation

As stated above, we wish to develop the Lagrangian perturbation theory formulated in a coordinate system which has a straightforward Newtonian limit. For this purpose it is convenient to use the (3+1) formalism motivated by the following considerations. Let us first assume that there exists a congruence of time-like worldlines from which the spacetime looks isotropic. We shall call the family of the worldlines basic observers, who see no dipole component of the cosmic microwave background radiation (CMB). We can regard that any one of these observers at the spacetime point $x$ moves with 4-velocity $n^\mu(x)$, without loss of generality. The tangent vector $n^\mu$ is normalized as $n^\mu n_\mu = -1$. These observers are used to foliate the spacetime by their simultaneous surfaces: $t = \text{const}$. These considerations imply that matter, such as a galaxy, moves with a velocity in our coordinates according to the geodesic equation. Hence we naturally expect that the dynamics of nonrelativistic matter is described by the cosmological Newtonian picture at the lowest order. Fortunately, there have been some studies on the post-Newtonian approximation applied to cosmology in the (3+1) formalism. We shall follow these approaches to work in the following coordinates:

$$\begin{align*}
    ds^2 &= -(\alpha^2 - \beta_i \beta^i)(cdt)^2 + 2\beta_i (cdt)dx^i + a^2(t)[(1 - 2\psi)\tilde{\gamma}^{(B)}_{ij} + h_{ij}]dx^i dx^j, \\
    \tilde{\gamma}^{(B)}_{ij} h_{ij} &= 0.
\end{align*}$$

(1)

where $\tilde{\gamma}^{(B)}_{ij} h_{ij} = 0$. We employ gauge conditions termed the “longitudinal gauge” or “Newtonian gauge” with regard to the scalar modes of the metric perturbations:

$$\begin{align*}
    \tilde{\gamma}^{(B)}_{ij} \tilde{D}^{(B)}_j \beta_i &= 0, \\
    \tilde{\gamma}^{(B)}_{jk} \tilde{D}^{(B)}_k h_{ij} &= 0.
\end{align*}$$

(2)

Here $\tilde{D}^{(B)}_i$ denotes the covariant derivative with respect to the background metric $\tilde{\gamma}^{(B)}_{ij}$, for which we take the spatial metric of the Friedmann-Robertson-Walker (FRW) geometry as $\tilde{\gamma}^{(B)}_{ij} = \delta_{ij}/(1 + K r^2/4)^2$. The quantity $K$ is the curvature parameter of FRW models and $r^2 = x_1^2 + x_2^2 + x_3^2$. We treat the metric perturbations as small quantities, so we are able to regard tensorial quantities in our equations as tensors with respect to the background spatial metric $\tilde{\gamma}^{(B)}_{ij}$. The above condition (2) guarantees that $h_{ij}$ and $\beta_i$ contain only the tensor mode, which represents the freedom of the gravitational wave in the PN order, and the vector mode, respectively.

The explicit forms of Einstein equations in terms of the above variables can be found in previous works. In the following, we only consider the Einstein-de Sitter background universe ($K = \Lambda = 0$) and use Cartesian coordinates for the sake
of simplicity:

\[ H(t)^2 = \left( \frac{\dot{a}(t)}{a(t)} \right)^2 = \frac{8\pi G \rho_0(t)}{3}, \]  

(3)

where \( H(t) \) and \( \rho_0(t) \) are the Hubble parameter and the homogeneous density of the background FRW universe, respectively. For simplicity, we assume that the above equation is derived by the usual averaging method \[13\] and we may regard \( \rho \) as an average value over the volume as large as the horizon scale \( (ct)^3 \):

\[ \langle \rho \rangle (ct)^3 \equiv \rho_0(t). \]

The metric required for the usual Eulerian Newtonian picture in the perturbed FRW universe is known to take the following form:

\[ ds^2 = -\left(1 + 2\frac{\Phi_N}{c^2}\right)c^2 dt^2 + a^2(t) \left(1 - 2\frac{\Phi_N}{c^2}\right) \delta_{ij} dx^i dx^j. \]  

(4)

The quantity \( \Phi_N \) is a cosmological Newton-like gravitational potential related to the matter density fluctuation field \( \delta_N \) via the Poisson equation

\[ \Delta x \Phi_N = 4\pi G a^2 \rho_0 \delta_N, \]  

(5)

where \( \delta_N(x, t) \equiv (\rho_N(x, t) - \rho_0(t))/\rho_0(t) \), and \( \Delta x \) is the Laplacian. The above metric is usually used to give an accurate description of the trajectories of nonrelativistic fluid elements on scales much smaller than the Hubble distance \( cH^{-1} \).

We expand the basic equations in powers of \( c^{-1} \) to obtain the post-Newtonian approximation with the condition that the lowest order metric takes the above Newtonian form \[4\]. Thus the PN terms of all metric variables used in this paper may be expanded as follows: \[4\]

\[ \alpha = 1 + \frac{\phi}{c^2} + \frac{\alpha^{(4)}}{c^4} + O(c^{-6}), \]
\[ \psi = \frac{\psi^{(2)}}{c^4} + \frac{\psi^{(4)}}{c^4} + O(c^{-6}), \]
\[ \beta^i = \frac{\beta^{(3)i}}{c^4} + O(c^{-5}), \]
\[ h_{ij} = \frac{h_{ij}^{(4)}}{c^4} + O(c^{-5}), \]  

(6)

where \( \phi \) is the Newtonian gravitational potential as stated below. It should be noted that we consider only the PN expansion of \( \beta^i \), not of \( \beta \). If we adopt a dust model \( T_{\mu\nu} = \rho c^2 u_\mu u_\nu \) for the energy-momentum tensor of matter, the metric up to order \( c^{-2} \) agrees with \[4\] by substituting the above expressions into Einstein equations. From the other substitutions, we can derive the relations between the metric perturbations and the matter variables in each order of \( c^{-n} \). We shall here present only relevant equations in our calculations. The lowest order field equation gives us

\[ \Delta x \phi = \Delta x \psi^{(2)} = 4\pi G a^2 (\rho - \rho_b). \]  

(7)

The field equations at the next order, that is, at the first PN order take the following forms for \( \alpha^{(4)} \) and \( \beta^{(3)i} \):

\[ \Delta x \alpha^{(4)} = \phi_{,k} \phi_{,k} + 4\pi G a^2 \left(2pa^2 v^2 - \rho \phi + 3\rho_b \phi \right) - 3a^2 \left(\frac{\partial^2 \phi}{\partial t^2} + 3\frac{\dot{a} \partial \phi}{a \partial t}\right), \]  

(8)
\[ \Delta x \beta^{(3)i} = 16\pi Ga^2 \rho v^i + 4 \left( \frac{\partial \phi}{\partial t} + \frac{\dot{a}}{a} \phi, i \right), \]  

(9)

where we have used the Newtonian order equations (7) and the gauge conditions (2), and \( v^i \) is defined as \( v^i/c = u^i/u^0 \). We note that \( v^i \) represents a peculiar velocity of a fluid element in the comoving frame, so the physical peculiar velocity is \( av^i \).

Similarly, the material equations up to the first PN order become

\[ \frac{\partial}{\partial t} \left[ \rho a^3 \left\{ 1 + \frac{1}{c^2} \left( \frac{1}{2} a^2 v^2 - 3\phi \right) \right\} \right] + \frac{\partial}{\partial x^j} \left[ \rho a^3 v^i \left\{ 1 + \frac{1}{c^2} \left( \frac{1}{2} a^2 v^2 - 3\phi \right) \right\} \right] + O(c^{-4}) = 0, \]  

(10)

\[ \frac{\partial v^i}{\partial t} + v^j \frac{\partial v^i}{\partial x^j} + 2 \frac{\dot{a}}{a} v^i + \frac{1}{c^2} v^i \left[ \frac{\partial}{\partial t} \left( \frac{1}{2} a^2 v^2 - 3\phi \right) + v^j \frac{\partial}{\partial x^j} \left( \frac{1}{2} a^2 v^2 - 3\phi \right) \right] = - \frac{1}{a^2} \phi, i - \frac{1}{c^2} \frac{1}{a^2} \alpha^{(4)i}, i + 3\phi, i + a^2 v^2 \phi, i + \frac{\partial}{\partial t} \left( a^2 \beta^{(3)i} \right) + a^2 v^j \left( \beta^{(3)i}, j - \beta^{(3)j}, i \right) \right] + O(c^{-4}). \]  

(11)

The lowest order in these two equations reduces to the Newtonian Eulerian equations of hydrodynamics for a pressureless fluid, and the terms of order \( c^{-2} \) provide the first PN corrections. Equations (10) and (11) are our basic equations for the Lagrangian approach to the trajectory field of matter fluid elements.

§3. PN Lagrangian perturbation approach to large-scale structure formation in the perturbed FRW universe

3.1. Basic equations for the transverse part of the trajectory field of the fluid elements

We shall rewrite the above set of post-Newtonian equations in the Lagrangian coordinates and then solve them perturbatively by extending the formalism in the Newtonian theory developed by Buchert.\[10\] Namely we concentrate on the integral curves \( x = f(X, t) \) of the velocity field \( v(x, t) \):

\[ \frac{df}{dt} \left( = \dot{f} \right) = \frac{\partial f}{\partial t} \bigg|_X = v(f, t), \quad f(X, t_I) \equiv X, \]  

(12)

where \( X \) denotes the Lagrangian coordinates which label each fluid element, \( x \) is the position of such an element in Eulerian space at time \( t \), and \( t_I \) is the initial time when the Lagrangian coordinates are defined. It should be noted that we introduce the Lagrangian coordinates on the comoving coordinate because we have already derived basic equations by using the FRW metric defined with the comoving coordinates. As long as the mapping \( f \) is invertible, we can give the inverse of the deformation tensor \( f_{ij} \), which is written in terms of variables \( (X, t) \):

\[ \frac{\partial X^i}{\partial x^j} \bigg|_{t_I} \equiv h_{i,j}(X, t) = \frac{1}{2} \epsilon_{iab} \epsilon_{jcd} f_{c[a} f_{d]b}, \]  

(13)
where $J$ is the determinant of the deformation tensor $f_{ij}$, and $\epsilon_{ijk}$ is a totally antisymmetric quantity with $\epsilon_{123} = +1$. The comma and the vertical slash in the subscript denote partial differentiation with respect to the Eulerian coordinates and the Lagrangian coordinates, respectively.

The continuity equation (10) then may be rewritten as
\[
\frac{d}{dt} \left[ \rho a^3 \left( 1 + \frac{1}{c^2} A(\mathbf{X}, t) \right) J(\mathbf{X}, t) \right] + O(c^{-4}) = 0,
\]
where
\[
A(\mathbf{X}, t) \equiv \frac{1}{2} a^2(t) f^2 - 3\phi(\mathbf{X}, t).
\]

Thus the density field is integrated exactly up to the first PN order in the Lagrangian picture as in the Newtonian case:
\[
\rho(\mathbf{X}, t) a^3 J(\mathbf{X}, t) = \left( 1 + \frac{1}{c^2} A(\mathbf{X}, t) \right)^{-1} \hat{\rho} (\mathbf{X}) \hat{a}^3 \left( 1 + \frac{1}{c^2} \hat{A} (\mathbf{X}) \right) + O(c^{-4}),
\]
where the quantities with $\circ$, such as $\hat{a}$, are the quantities at the initial time $t_I$, $\Delta_X \hat{\phi} = 4\pi G \hat{a}^2 \left( \hat{\rho} - \hat{\rho}_b \right)$ with $\Delta_X$ representing the Laplacian with respect to the Lagrangian coordinate, and $\hat{J} = 1$.

The trajectory field $\mathbf{f}$ is determined by solving the PN equations of motion (11). As in the Newtonian case, we consider the rotation and the divergence of the equations of motion with respect to the Eulerian coordinates, respectively. In the Lagrangian picture, the former set becomes basically the evolution equation of the transverse part of the trajectory field, and the latter becomes the evolution equation of its longitudinal part. We shall here concentrate on the transverse part at the PN order. The longitudinal part will be discussed elsewhere in detail.

So by operating with rotation on Eq. (11) with respect to the Eulerian coordinate we obtain
\[
\epsilon_{ijk} \frac{\partial}{\partial x^j} \left[ \ddot{f}^k + \frac{\dot{\hat{a}}}{a} \dot{f}^k + \frac{1}{c^2} \dot{f}^k \hat{A} \right] = -\frac{J}{c^2} \frac{1}{a^2} a^2 \epsilon_{ijk} \frac{\partial}{\partial x^j} \left[ a^2 \dot{v}^2 \hat{\phi}_{,k} \right.
\]
\[
+ \frac{\partial}{\partial t} \left( a^2 \beta^{(3)k} \right) + a^2 \dot{v}^l \left( \beta^{(3)k}_{,l} - \beta^{(3)l}_{,k} \right) \Bigg] + O\left(c^{-4}\right),
\]

According to the usual procedure in the Lagrangian formalism, we can modify the above equation by using Eqs. (12) and (13) in the following form:
\[
\epsilon_{abc} \dot{f}_j | a \dot{f}_i | b \left[ \ddot{f}_j | c + \frac{\dot{\hat{a}}}{a} \dot{f}_j | c \right] = -\frac{J}{c^2} \frac{1}{a^2} \frac{\partial}{\partial t} \left( a^2 \epsilon_{ijk} \beta^{(3)k}_{,j} \right)
\]
\[
+ \epsilon_{ijk} \frac{\partial}{\partial x^j} \left\{ v^l \left( \beta^{(3)k}_{,l} - \beta^{(3)l}_{,k} \right) \right\} \Bigg] + \frac{2}{J} \epsilon_{abc} \dot{f}_i | b \dot{f}_i | c \hat{f}_j | c \hat{\phi}_{,a} \right]
\]
\[
- \frac{1}{a^2} \epsilon_{abc} \dot{f}_j | a \dot{f}_i | b \left( \ddot{f}_j | c \hat{A} + \dot{f}_j | c \hat{\dot{A}} \right) + O\left(c^{-4}\right),
\]

where we have not transformed the terms including the shift vector $\beta^{(3)i}$ in Eq. (18) into their forms in the Lagrangian picture, because the expression in the Eulerian
picture will make it easier to understand the expansion of the above equation in later discussion. We note that the right-hand side of Eq. (18) is of order \(c^{-2}\); namely, it has only quantities of first PN order.

The structure of the above equation allows us to solve for the displacement vector iteratively up to any desired order in \(c^{-n}\) in terms of the lower order displacement vectors and metric variables.

3.2. The solution of the transverse part at PN order

The solution of Eq. (3) in an Einstein-de Sitter background can be chosen as

\[ a(t) = \left( \frac{t}{t_0} \right)^{2/3}, \]

where \(t_0\) denotes the present time. We use this normalization for the scale factor since it makes the physical interpretation of the solution more transparent, partly because the length in the comoving frame is equal to the physical length at the present universe with this choice. Furthermore, we shall assume that the initial hypersurface exists at sufficiently early time so that the initial density contrast field is much smaller than unity. This allows us to adopt the order of the initial density contrast, say \(\lambda\), as a new perturbation parameter.

The parameter \(\lambda\) is assumed to be small and dimensionless. We formally split the initial density field accordingly:

\[ \dot{\rho}(X) = \dot{\rho}_b + \lambda \dot{\rho}_b \delta(X), \quad \langle \dot{\rho}(X) \rangle = \dot{\rho}_b, \]

where \(\delta\) denotes the initial density contrast field. This choice of the initial data is adequate, because the density field at some given time need not be perturbed in the Lagrangian framework.

We assume that the vorticity of the Newtonian displacement vector is negligible, which seems to be reasonable. We consider the following ansatz for the transverse perturbations at the first PN order superposed on the solved Newtonian irrotational trajectory field:

\[ f^i(X, t) = \langle N \rangle f^i(X, t) + \frac{1}{c^2} \left( \lambda Q^T_z(t)^{(PN)} \xi_1^{(1)}(X) + \lambda^2 Q^T_{zz}(t)^{(PN)} \xi_2^{(2)}(X, t) \right), \]

where

\[ \langle N \rangle f^i(X, t) = X^i + \lambda q_z(t) \psi_{N|i}^{(1)}(X) + \lambda^2 q_{zz}(t) \psi_{N|i}^{(2)}, \]

with

\[ q_z = \left( \frac{3}{2} \right) \left[ \left( \frac{t}{t_I} \right)^{2/3} - 1 \right], \]

\[ q_{zz} = \left( \frac{3}{2} \right)^2 \left[ -\frac{3}{14} \left( \frac{t}{t_I} \right)^{4/3} + \frac{3}{5} \left( \frac{t}{t_I} \right)^{2/3} - \frac{1}{2} + \frac{4}{35} \left( \frac{t}{t_I} \right)^{-1} \right], \]
and

$$\Delta X\Psi^{(1)}_N = -\frac{2}{3} \delta g,$$

$$\Delta X\Psi^{(2)}_N = \Psi^{(1)}_{N|ii} \Psi^{(1)}_{N|jj} - \Psi^{(1)}_{N|ij} \Psi^{(1)}_{N|ij},$$  \hspace{1cm} (22d)

$$\Delta X\Psi^{(2)}_N = \Psi_{N|ii}^{(1)} \Psi_{N|jj}^{(1)} - \Psi_{N|ij}^{(1)} \Psi_{N|ij}^{(1)}.$$  \hspace{1cm} (22e)

The initial conditions are imposed on the time coefficients $Q^T$ by the definition of the Lagrangian coordinate (12): $Q^T(t_1) = Q^T_{zz}(t_1) = 0$. The quantity $\delta g$ is defined as $\delta = -\frac{2}{3} \delta g / c^2$ and denotes the initial gauge invariant density contrast field. \hspace{1cm} (20)

Before considering the PN rotational flow in our coordinates, we remark that we can express the Newtonian gravitational potential constrained by Eq. (7) in terms of the above Newtonian solution $(N)^f_i$ following the Lagrangian perturbation formalism:

$$\phi_N(X, t) = -\lambda \frac{1}{t^2(1+z)^2} \Psi^{(1)}_N - \lambda^2 \left( 2a\dot{q}_{zz} + a^2 \ddot{q}_{zz} \right) \Psi^{(2)}_N
+ q_z \left( 2a\dot{q}_z + a^2 \ddot{q}_z \right) \tilde{\phi}(X) + O(\lambda^3),$$  \hspace{1cm} (24)

where

$$\tilde{\phi}(X) \equiv -\int dY \frac{[\Psi^{(1)}_{N|ij}(Y)\Psi^{(1)}_N(Y)]_{ij}}{4\pi|X - Y|}. \hspace{1cm} (25)$$

In the above derivation, we have used the Green function $\frac{1}{4\pi|X - Y|}$ of the Laplacian motivated by the following considerations. As seen in Eq.(7), the Newton-like gravitational potential in the cosmological situation is generated by the density contrast $\delta$. Then we can safely express such quantities using the Green function of the Laplacian, because the integrals over the horizon scale converge to a definite value by assuming that the density contrast field obeys periodic boundary condition on the horizon scale. We have found that the cosmological PN treatment to the large-scale structure formation is valid even for perturbations larger than the present horizon scale. \hspace{1cm} (21)

We need Eq. (18) to solve the transverse part $Q^T\mathbf{\Xi}_i$ of the PN displacement vector. Naturally, the order $c^0$ in this equation produces Newtonian counterparts in the case that the trajectory field is introduced on the comoving coordinates. \hspace{1cm} (6, 7)

Then, if we assume $J_N > \tilde{J}_{PN}/c^2$, where $\tilde{J}_{PN}$ is a term of order $c^{-2}$ in expanding $J$, and take into account the order of $\lambda$ in each term on the right-hand side, Eq. (18) becomes

$$\epsilon_{abc} f_{j[a} \dot{f}_{b]} \left[ f_{j[c} + 2a f_{j]} \right] = \frac{J_N}{c^3} \epsilon^i_{jkl} \frac{\partial}{\partial t} \left( a^2 \epsilon^{ijk}(\beta^{(3)k})_j \right)
- \frac{J_N}{c^3} \epsilon^{ijk} \frac{\partial}{\partial x} \left[ (N)^f_i (\beta^{(3)k})_l - \beta^{(3)l}_i \right] + O(c^{-4}, \lambda^3), \hspace{1cm} (26)$$
where we have neglected terms like $\phi_Nv_N^2$ in our approximation because they are of order $\lambda^3$ at most, and $J_N \equiv \det(\xi^{(N)}f_{ij})$. This equation shows that the shift vector constrained by the gauge condition (2) generates the first PN transverse flow. Inserting the ansatz (21) into Eq. (26), we obtain the following differential equation for solving the transverse velocity potential of the first PN displacement vector at the order $c^{-2}$:

$$
\lambda \left( \dot{Q}_z^T + \frac{2}{a} \dot{Q}_z^T \right) \left( \nabla_X \times (PN)_{(P)} \right)_i + \lambda^2 \left( \dot{Q}_z^T + 2 \frac{\dot{a}}{a} Q_z \right) \left( \nabla_X \times (PN)_{(T)} \right)_i \\
= -\frac{J_N}{a^2} \frac{\partial}{\partial t} \left( a^2 \left( \nabla_X \times \beta(3) \right)_i \right) - J_N \epsilon_{ijk} \frac{\partial}{\partial x^j} \left[ \varepsilon_X(\beta(3)_{ij} - \beta(3)_{ik}) \right] \\
- \lambda^2 \left( \dot{q}_z + \frac{2}{a} \dot{q}_z \right) \epsilon_{ijk} N_{[ik]} T_{(P)} N_{l[k]} \left( \nabla_X \times (PN)_{(P)} \right)_i \\
- \lambda^2 q_z \left( \epsilon_{ijk} \Psi_{N[ijl]}^{(1)} + \epsilon_{jkl} Q_{N[i]}^{(1)} \right) \left( \dot{Q}_z^T + 2 \frac{\dot{a}}{a} Q_z \right) \left( PN \right)_{l[j]} + O(\lambda^3). \quad (27)
$$

The form of this equation shows that it can be solved if we express the shift vector $\beta(3)^i$ in terms of already-known Newtonian trajectory field $\xi^{(N)}f^i$.

Thus we first must solve the shift vector with respect to the Lagrangian coordinate. The equation (27) shows that we need only the explicit form of its transverse part, say $\beta_T^{(3)i}$, for the situation in which we are interested here. We note that since the gauge condition (2) implies that the transverse part of the shift vector is a gauge invariant quantity, no ambiguities caused by the gauge freedom remains. The perturbation quantity $\beta_T^{(3)i}$ is constrained by the Einstein equation (9):

$$
\Delta_z \beta_T^{(3)i} = 4 \left( \frac{\partial \phi_N}{\partial t} + \frac{\dot{a}}{a} \phi_N \right) + 16\pi G a^2 \rho v_N^i.
$$

The right-hand side of this equation is shown to be divergenceless through the continuity equation (10) at Newtonian order. We have already obtained an expression for the peculiar velocity field and the peculiar gravitational potential (12) and (24) in terms of the Lagrangian coordinate, respectively. Therefore, it is adequate to make use of Eq. (28) for our purpose. Then if we express Eq. (28) in terms of the independent variables $X$ and $t$, we can obtain the following equations in simple form:

$$
\Delta_X \beta_T^{(3)i} - \lambda q_z \psi_{N[i]}^{(1)} \beta_T^{(3)i} \left( \psi_{N}[j] \beta_T^{(3)i} \right)_k + 3\lambda q_z \psi_{N[i]}^{(1)} \beta_T^{(3)i} \beta_T^{(3)i} \left( \psi_{N}[k] \right)_j - 2\lambda q_z \psi_{N[i]}^{(1)} \beta_T^{(3)i} \left( \psi_{N}[j] \right)_k \\
= \lambda^2 \frac{4}{t_1^2(1 + z_1)^2} \left( \frac{t}{t_1} \right)^{-1/3} \left( -\psi_{N[i]}^{(1)} \psi_{N}[j] + \psi_{N[i]}^{(1)} \psi_{N}[j] \psi_{N}[j] \right) + O(\lambda^3), \quad (29)
$$

where we have used Eqs. (22), (24) and the equation of mass conservation, (16). We are now in a position to be able to solve for the shift vector iteratively. Accordingly, we can conclude that the shift vector $\beta_T^{(3)i}$ is of second order in $\lambda$ and obtain the following equation at order $\lambda^2$ of Eq.(29):

$$
\Delta_X \beta_T^{(3)i} = \lambda^2 \frac{4}{t_1^2(1 + z_1)^2} \left( \frac{t}{t_1} \right)^{-1/3} \left( \psi_{N[i]}^{(1)} \psi_{N}[j] + \psi_{N[i]}^{(1)} \psi_{N}[j] \psi_{N}[j] \right) \psi_{N[i]}^{(2)}. \quad (30)
$$
The consistency of our formulation can easily be checked if one confirms that the right-hand side of this equation also satisfies the divergenceless condition with respect to the Lagrangian coordinate by using Eq. (22e). If we again use the Green function of the Laplacian, we can express the solution of this equation as

$$ \beta_T^{(3)i}(X,t) = \lambda^2 \frac{4}{t_f^2(1 + z)^2} \left( \frac{t}{t_f} \right)^{-1/3} \beta_T^{(3)i}(X) + O(\lambda^3), \quad (31) $$

where

$$ \beta_T^{(3)i}(X) := \int d^3 Y \frac{\psi^{(1)}_{N[j]}(Y)\psi^{(1)}_{N[i]}(Y) - \psi^{(1)}_{N[i]}(Y)\psi^{(1)}_{N[j]}(Y) - \psi^{(2)}_{N[i]}(Y)}{4\pi|X - Y|}. \quad (32) $$

In the above expression of the integration, we have again assumed that the homogeneous solution of Eq. (30) is zero. It should be noted that we consider $\beta^{(3)i}$, not $\beta_i^{(3)}$, throughout this paper. If one needs the quantity $\beta_i^{(3)}$, one finds that it has a growing mode by the definition $\beta_i^{(3)} = a^2\beta_T^{(3)i}$. The above equation is the expression of the shift vector that we have sought in order to solve Eq. (27). Thus we are able to express all perturbation quantities of the metric in our coordinates in terms of the trajectory field. This must be the case because it is only a dynamical variable in the Lagrangian description.

By inserting Eq. (31) into Eq. (27), we can obtain the following equation at the lowest order of $\lambda$:

$$ \left( \ddot{Q}_z^T + 2\frac{\dot{a}}{a} \dot{Q}_z^T \right) \nabla_X \times (PN)\Xi^{(1)}(X) = 0. \quad (33) $$

As an irrotational case, the form of this equation allows us to seek solutions of the form

$$ \dot{Q}_z^T + 2\frac{\dot{a}}{a} \dot{Q}_z^T = 0, \quad \text{with} \quad (PN)\Xi^{(1)}(X) = \Pi(X), \quad \nabla_X \cdot \Pi = 0, \quad (34) $$

where $\Pi$ is an unknown function determined by the initial conditions. The solutions of Eq. (34) can be easily found to have only a decaying mode. The first order solution may be safely ignored, because the decaying mode plays no physically important role:

$$ Q_z^{T(PN)}\Xi_i^{(1)} \approx 0. \quad (35) $$

Similarly, by inserting Eqs. (31) and (35) into Eq. (27), we can obtain the following equation at second order in $\lambda$:

$$ \left( \ddot{Q}_{zz}^T + 2\frac{\dot{a}}{a} \dot{Q}_{zz}^T \right) \nabla_X \times (PN)\Xi^{(2)} = -\frac{4}{t_f^4(1 + z)^2} \left( \frac{t}{t_f} \right)^{-4/3} \nabla_X \times \beta_T^{(3)}. \quad (36) $$

The solution of this equation can be found by solving the linear ordinary differential equations

$$ \ddot{Q}_{zz}^T + 2\frac{\dot{a}}{a} \dot{Q}_{zz}^T = \frac{4}{t_f^2} \left( \frac{t}{t_f} \right)^{-4/3}, \quad (37) $$
with \( \Delta X^{(PN)} \Xi^{(2)}_i = -\frac{1}{(1+z_I)^2 t_I^2} \left( -\Psi^{(1)}_{N|jj} N_{|ii} + \Psi^{(1)}_{N|i} \Psi^{(1)}_{N|i} + \Psi^{(2)}_{N|i} \right) \). (38)

Since we expect the quantity \( Q_T^{(PN)} \Xi^{(2)}_i \) to be much smaller than the order \( \lambda \) quantities such as \( \delta_g \) at the initial time, we can safely adopt the additional initial condition \( \dot{Q}_T^{(PN)}(t_I) = 0 \) as a longitudinal perturbation case besides the initial condition \( Q_T^{(PN)}(t_I) = 0 \). Then we can find the solution of Eq. (37),

\[
Q_T^{(PN)}(t_I) = \left( \frac{3}{2} \right) \left[ 4 \left( \frac{t}{t_I} \right)^{2/3} - 12 + 8 \left( \frac{t}{t_I} \right)^{-1/3} \right].
\] (39)

§4. Discussion and summary

We have found that the transverse part of the first PN displacement vector has a growing mode even if the Newtonian trajectory field does not have a transverse mode. We emphasize that the PN solution with the growing mode is a particular solution of the differential equation (36) and is generated by the initial density fluctuation field through Eq. (22d). Actually, we have found that the longitudinal part of the PN displacement vector first appears of the same order of its magnitude. Thus the existence of the growing transverse part in the PN displacement vector is interesting because such a mode is present in neither Newtonian theory nor the gauge invariant linear theory.

Buchert has shown that the Newtonian trajectory field has no growing transverse mode up to third order in \( \lambda \) when the initial velocity field is assumed to be irrotational. In his work, the solution of the transverse part \( q_T^{(N)}(N) \Xi^{(3)}_i \) at order \( \lambda^3 \) takes the form

\[
q_T^{(N)}(N) \Xi^{(3)}_i = \left( \frac{3}{2} \right)^3 \left[ \frac{1}{14} \left( \frac{t}{t_I} \right)^2 - \frac{3}{14} \left( \frac{t}{t_I} \right)^{4/3} + \frac{1}{10} \left( \frac{t}{t_I} \right)^{2/3} + \frac{1}{2} - \frac{4}{7} \left( \frac{t}{t_I} \right)^{-1/3} + \frac{4}{35} \left( \frac{t}{t_I} \right)^{-1} \right],
\] (40a)

with

\[
\Delta X^{(N)} \Xi^{(3)}_i = \Psi^{(1)}_{N|ijk} \Psi^{(2)}_{N|jk} - \Psi^{(1)}_{N|kjj} \Psi^{(2)}_{N|k} + \Psi^{(1)}_{N|k} \Psi^{(2)}_{N|k} - \Psi^{(1)}_{N|kj} \Psi^{(2)}_{N|k},
\] (40b)

He has pointed out that the existence of the above transverse part ensures the conservation of the vorticity flow along the fluid flow up to order \( \lambda^3 \).

To see the importance of the PN effect on Newtonian dynamics, let us make a simple order estimation. Consider the initial density fluctuation \( \delta_g(t_0) \) with the characteristic length \( l \) in the comoving frame. Note that due to \( a(t_0) \), we may regard the length \( l \) as the physical scale at the present time \( t_0 \). By using Eqs. (22d), (22c), (38), (39), and (40), the above transverse parts of the Newtonian displacement vector and the PN displacement vector may be estimated respectively roughly at some time as

\[
q_T^{(N)}(N) \Xi^{(3)}_i \sim a^3 \left( \delta_g(t_0) \right)^3.
\]
\[
\frac{1}{c^2} Q^{T (PN)}_{zzz} \Xi^{(2)}_i \sim a \left( \frac{1}{ct_0} \right)^2 \left( \delta g(t_0) \right)^2,
\]

where \( \delta g(t_0) \) is defined by \((1 + z_N) \delta g(t) \) representing the present value of the density contrast for the scale \( l \) extrapolated by the linear theory. This expression (41) suggests the following interpretations. As long as we consider the evolution of the transverse part with scale \( l \) much smaller than the present horizon scale \( ct_0 \), it is described well only by Newtonian theory, including the transverse part, because the Newtonian transverse part \( q_{zzz}^{(N)} \Xi^{(3)}_i \) remains much larger than the PN transverse part \( Q^{T (PN)}_{zzz} \Xi^{(2)}_i \) from the initial time. This may not be correct for fluctuations with larger scale. In fact, from Eq. (41) we can evaluate the redshift \( z_N \) when the magnitude of the Newtonian transverse part begins to become larger than that of the PN transverse part for the density fluctuation field with scale \( l \):

\[
1 + z_N \sim \frac{ct_0}{l} \left( \delta g(l) \right)^{1/2}.
\]

Noting that \( ct_0 \) is on the order of the present horizon scale and \( \sim 2000h^{-1}\) Mpc, it is found that this redshift is smaller for fluctuations with larger scale. This means that the PN transverse part might play an important role for the early evolution of large-scale structure. Since some fluctuations with large scale become localized smaller in comparison with the horizon scale, which grows larger, it is naturally described more accurately by Newtonian dynamics later. For example, SDSS is expected to survey a cubic region of several hundred megaparsecs length. Then the above estimate gives \( z_N \sim 4 \) for fluctuations with scale \( 50h^{-1}\) Mpc at the present time, if we estimate the magnitude of the present density contrast as \( \delta g(50) \) \( \sim 10^{-2} \), according to the power spectrum of the density contrast in the standard CDM model normalized by the COBE observation.

We can solve for the trajectory field, namely the integral curve of velocity field \( v \) in our coordinates, up to the 1st PN order, so it will be interesting to see how the relativistic effect on Newtonian dynamics appears as a characteristic pattern in the large-scale structure formation. It will be easy to investigate this problem expanding the Newtonian simulation based on the Lagrangian perturbation approach. Also, as stated above, the PN characteristic effect might appear on larger scales. In particular, the effect may induce secondary temperature anisotropies of CMB, as the Rees-Sciama effect, and potentially be observable. This reason in the anisotropy may be interpreted as follows. As stated in §2, we foliate the spacetime by simultaneous surfaces of the basic observers who see no dipole component of CMB using the \((3+1)\) formalism. Thus, it would be natural for the existence of the shift vector to influence the anisotropy of CMB. These works are now in progress.

Since the observable region of the large-scale structure increases steadily, we expect that our work will play some important role in the future.

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