Network Models of Quantum Percolation
and
Their Field-Theory Representations

Dung-Hai Lee *

*IBM Research Division, T. J. Watson Research Center, Yorktown Heights, NY 10598

Abstract

We obtain the field-theory representations of several network models that are relevant to 2D transport in high magnetic fields. Among them, the simplest one, which is relevant to the plateau transition in the quantum Hall effect, is equivalent to a particular representation of an antiferromagnetic SU(2N) (N \to 0) spin chain. Since the later can be mapped onto a $\theta \neq 0$, $U(2N)/U(N) \times U(N)$ sigma model, and since recent numerical analyses of the corresponding network give a delocalization transition with $\nu \approx 2.3$, we conclude that the same exponent is applicable to the sigma model.

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In strong magnetic fields, a two-dimensional electron gas exhibits the quantum Hall effect\[1\] (in this paper we will restrict ourselves to the integer effect) over a wide range of sample disorder.[2] The hallmark of the quantum Hall effect is that in the neighborhood of a series of 'magic' filling factors \(f = S_{xy}\), \(\sigma_{xy}\) exhibits quantized plateaus at values \(S_{xy}e^2/h\) while \(\sigma_{xx} \to 0\).[1] Between these filling factors, \(\sigma_{xy}\) interpolates between the plateaus, and \(\sigma_{xx}\) peaks in the middle of the transitions. By analyzing the temperature dependence of the widths of the transitions, Wei et al.[3] have shown convincingly that, if extrapolated to zero temperature, the plateau transition is a continuous phase transition. The single diverging length scale at this transition is the electron localization length \(\xi \propto |B - B_c|^{-\nu}\). Although Wei et al.’s results do not give \(\nu\) directly, a recent experiment using narrow Hall bars suggests \(\nu \approx 2.3\).[4] Moreover, considerable numerical work \[5\] has been done for the plateau transition of non-interacting electrons. The result for \(\nu\) is consistent with the experimental findings.[6]

It is known that the effective theory for zero-temperature, long-wavelength electronic transport in disordered media is a non-linear sigma model with symplectic symmetry.[7] In 2D this model has a finite localization length (hence gives an insulating state) for arbitrarily weak disorders. This result holds when a weak magnetic field is applied, where the symmetry of the sigma model is reduced to unitary. These results are in apparent contradiction with the quantum Hall effect. In trying to reconcile them, Levine, Libby and Pruisken[8] proposed that the proper field theory for electronic transport in strong magnetic fields is a \(N \to 0\), \(U(2N)/U(N) \times U(N)\) sigma model with a topological term. Subsequently, starting from a microscopic basis, Pruisken[9] derived the previously postulated sigma model

\[
\mathcal{L} = \frac{1}{8}\sigma_{xx}Tr[(\partial_\mu Q)^2] - \frac{1}{8}\sigma_{xy}\epsilon_{\mu\nu}Tr[Q\partial_\mu Q\partial_\nu Q],
\]

where \(\mu = x, y\), \(\epsilon_{xy} = -\epsilon_{yx} = 1\), and \(\sigma_{\mu\nu}\) are dimensionless conductivities in units of \(e^2/h\). In Eq.(1) \(Q = u^+\Lambda u\), where \(u\) is a \(2N\times2N\) unitary matrix, and \(\Lambda\) is a diagonal matrix with \(\Lambda^a_a = 1\).
for $a \leq N$ and $\Lambda_a^a = -1$ for $N < a \leq 2N$. Pruisken et al. argued that with the addition of the second (topological) term, Eq.(1) possesses stable fixed points at $(\sigma_{xx}, \sigma_{xy}) = (0, n)$ (corresponding to the quantized plateaus) and critical points at $(constant, n + \frac{1}{2})$ (corresponding to the transition points between the plateaus). Using the sigma model as the paradigm, Pruisken conjectured the scaling behavior of the plateau transition. This conjecture was later verified experimentally. Although Pruisken’s arguments are compelling, serious calculations of the critical properties are absent. It is still an open question as to whether Eq.(1) can produce the experimentally or numerically observed exponent.

Quite independently, in an insightful paper, Chalker and Coddington proposed a network model to describe the ‘quantum percolation’ of the semi-classical (equal-potential) orbitals of non-interacting electrons in a strong magnetic field and a smooth random potential. Both that work and a recent extension by Lee, Wang and Kivelson show that the network model has a plateau transition with $\nu = 2.4 \pm 0.2$. In brief, the network model consists of a checker-board of low-potential- energy plaquettes (marked by ‘1’ in Fig.1a) that are occupied by electrons. At the boundary of each plaquette, there is an edge state at the Fermi energy, in which an electron undergoes $\vec{E} \times \vec{B}$ drift in the direction indicated by the arrows. In this model, quantum tunneling takes place at the centers of the open squares only. The effect of changing the Fermi energy is reflected in the modification of the tunneling matrix elements. The fact that, in reality, the semi-classical orbitals have different spatial extent is reflected in the random Aharonov-Bohm phases the electrons accumulate when reaching the tunneling points. In numerical calculations, the localization length $\xi$ is determined in cylindrical systems. Using finite-size scaling the thermodynamic $\xi$ was deduced, from which the authors of Ref.[13] concluded that $\nu = 2.4 \pm 0.2$. Since to calculate $\nu$ (either numerical or analytically) using Eq.(1) is notoriously difficult, one purpose of the
present work is to establish a connection between the network model and the sigma model, so that the exponent obtained for the former can be used for the latter.

Due to a recent experiment on the spin-unresolved plateau transition\[14\], questions concerning the nature of plateau transitions in the presence of a strong inter-Landau level mixing have been raised. In this paper we generalize the network model,\[15\] and obtain the corresponding field-theory representation to describe this situation as well. Finally, to demonstrate the importance of the random phase on the critical properties of the delocalization transition, we consider a network model in its absence. We show that the transition in this model belongs to a different universality class.

A representation of the network model is to view it as a one dimensional (\(\hat{x}\)) stacking of alternating upper-right (\(\hat{y}\)) and lower-left (\(-\hat{y}\)) moving edge states shown in Fig.1b. Electrons that tunnel at the centers of the open squares can be localized or extended depending on the assignment of the (random) tunneling matrix elements. To capture the linear dispersion and the chiral nature of the edge states\[16\], we use the following Hamiltonian to model the eigenstates of the network near the Fermi energy (\(E = 0\)):

\[
\hat{H} = \sum_x (-1)^{x/a} \int dy v \psi^+(x, y) \frac{\partial}{\partial y} \psi(x, y) - \sum_x \sum_y [t'(x, y) e^{i\phi(x, y)} \psi^+(x + a, y) \psi(x, y) + h.c]. \tag{2}
\]

Here \(\psi\) is the electron annihilation operator, the \(\phi\)'s are the random Aharonov-Bohm phases for electrons at the Fermi energy, and the \(t'\)'s are random (positive) tunneling matrix elements. In the second term, the \(y\)-sum is extended over the discrete set of \(y\) coordinates at which tunneling takes place. Here, without changing the long-distance physics, we have assumed that the value of the drift velocity \(v\) is uniform. Since the effects of changing \(E\) can be absorbed into a dependence of \(\phi\) and \(t'\) on \(E\), we can concentrate on \(E = 0\) without losing generality. In order to study the critical behavior of the conductivities, we consider
the following transport action:

$$S = \sum_x \int dy \eta S_p \bar{\psi}_p(x,y) \psi_p(x,y) - H$$

$$\rightarrow \sum_x \int dy [i\pi \rho \eta S_p \bar{\psi}_p(x,y) \psi_p(x,y) - (-1)^{x/a} \bar{\psi}_p(x,y) \frac{\partial_y}{i} \psi_p(x,y)]$$

$$+ \sum_x \sum_y [t(x,y)e^{i\phi(x,y)} \bar{\psi}_p(x+a,y) \psi_p(x,y) + h.c.]. \quad (3)$$

In the above, repeated indices $p = \pm$ are summed over, $\psi_p$ and $\bar{\psi}_p$ are Grassmann fields, $S_p \equiv \text{sign}(p)$, $\rho \equiv 1/(\pi v)$ is the linear density of states, $\eta$ is a positive infinitesimal, and $t \equiv \pi \rho t'$. In going from the first to the second lines in Eq.(3) we have rescaled the fermion fields so as to absorb the edge velocity $v$.

By redefining $\psi_p \to \psi_p(i\psi_p)$ and $\bar{\psi}_p \to -i\bar{\psi}_p(\bar{\psi}_p)$ for even (odd) $x/a$'s, the action becomes

$$S = \sum_x \int dy [(-1)^{x/a} \pi \rho \eta S_p \bar{\psi}_p(x,y) \psi_p(x,y) + \bar{\psi}_p(x,y) \frac{\partial_y}{i} \psi_p(x,y)]$$

$$+ \sum_x \sum_y [t(x,y)e^{i\phi(x,y)} \bar{\psi}_p(x+a,y) \psi_p(x,y) + h.c.]. \quad (4)$$

Next, we replicate the action and integrate out the random $\phi$'s and $t$'s to obtain

$$S = \sum_x \int dy [(-1)^{x/a} \pi \rho \eta S_p \bar{\psi}_{p\alpha}(x,y) \psi_{p\alpha}(x,y) + \bar{\psi}_{p\alpha}(x,y) \frac{\partial_y}{i} \psi_{p\alpha}(x,y)]$$

$$+ \sum_x \int dy F_x [\bar{\psi}_{p\alpha}(x+a,y) \psi_{p'\beta}(x+a,y) \bar{\psi}_{p'\beta}(x,y) \psi_{p\alpha}(x,y)]. \quad (5)$$

We note that because $\phi$ is random, after integrating it out only the charge neutral combinations of $\psi$'s and $\psi^+$'s are present in Eq.(5). Moreover, the symmetry of the replicated action guarantees that only the SU(2N) invariant combinations appear at $\eta = 0$. As usual, the repeated replica indices $\alpha, \beta = 1...N$ are summed over, and $F_x(Z) = \frac{1}{a} < t^2(x) > Z + ...$. We note that $F$ has an explicit $x$-dependence. It reflects the fact that the electrons tunnel across the occupied/unoccupied regions for the even-odd/odd-even columns. As a result, the corresponding $< t^2 >$ are different.
If we view $y$ as the imaginary time $\tau$, Eq.(5) is the coherent-state path-integral action of a 1D quantum field theory described by the following Hamiltonian

$$\hat{H} = \sum_x \left[ (-1)^{x/a} \pi \rho \eta S_p \psi_{\alpha}^+(x) \psi_{\alpha}(x) \right] + \sum_x F_x \left[ \psi_{\alpha}^+(x + a) \psi_{\alpha'}(x + a) \psi_{\alpha}(x) \right]. \quad (6)$$

The relation between $F$ in Eq.(6) and $\tilde{F}$ in Eq.(5) is that upon normal ordering $F \rightarrow \tilde{F}$. The relation to the SU(2N) spin chain becomes explicit after we rewrite Eq.(6) in terms of the SU(2N) generators $\hat{S}_a \equiv \psi_a^+ \psi_b - \delta_{ab} \frac{1}{2N} \sum_c \psi_c^+ \psi_c$, where $a \equiv (p, \alpha)$ and takes on $2N$ values. The final spin Hamiltonian is

$$\hat{H} = \sum_x \left[ (-1)^{x/a} \pi \rho \eta \Lambda \hat{S}(x) \right] + \sum_x F_x \left[ Tr[\hat{S}(x + a) \hat{S}(x)] \right], \quad (7)$$

where $\Lambda$ is the diagonal c-number matrix defined earlier, and $Tr[\Lambda \hat{S}] \equiv \sum_{a,b} \Lambda^b \hat{S}^a$, $Tr[\hat{S} \hat{S}'] \equiv \sum_{a,b} \hat{S}_a^b \hat{S}_b^a$. Since $\hat{H}$ commutes with the site occupation number $n(x) = \sum_a \psi_a^+(x) \psi_a(x)$, Hilbert spaces corresponding to different $\{n(x)\}$ decouple. The ground state of Eq.(7) lies in the Hilbert space where $n(x) = N$ for all $x$. In this Hilbert space, a particular representation for SU(2N) is realized. This representation is characterized by a Young tableau with a single column of length $N$.

Analogous to the SU(2) quantum spin chain, which in the large-spin limit is equivalent to the U(2)/U(1)xU(1)=O(3) sigma model, one can show that at $\eta = 0$ the SU(2N) spin chain of Eq.(7) is equivalent to the following $U(2N)/U(N) \times U(N)$ sigma model in the large representation limit[18, 19]

$$\mathcal{L} = \frac{M}{16} (\sqrt{1 - R^2}) Tr(\partial_{\mu} Q)^2 + \frac{M}{16} (1 - R) \epsilon_{\mu\nu} Tr[Q \partial_{\mu} Q \partial_{\nu} Q], \quad (8)$$

where $\mu = \tau, x$, and $R \equiv [F'_x(Z_0) - F'_x(0)]/[F'_x(Z_0) + F'_x(0)]$ (here $Z_0 \equiv -NM^2/4 \text{ and } F'(Z) \equiv dF/dZ$). For the network model $M = 1$, and Eq.(8) is massless at $R = 0$. For $R \neq 0$ the system remains massive. Thus we identify the transition from $R < 0$ to $R > 0$ as the
plateau transition. At the critical point \((R = 0)\) the \(\hat{x}\)-translational symmetry is restored. If we assume that a) the mapping from Eq.(7) to Eq.(8) is valid down to \(M = 1\), and b) at the critical point the system is described by a translationally-invariant SU(2N) spin chain with \(M=1\), then by comparing Eq.(1) and Eq.(8) we conclude that \(\sigma_{xx} = \sigma_{xy} = 1/2\) at the critical point. In any case, \(Q\) remains zero. As a result, the \(\sigma_{\mu\nu}\) in Eq.(1) are dimensionless, and hence are generically universal at the transition.

Modifications can be made so that the network model represents other physically relevant situations. For example, to study the effects of neighboring Landau-level mixing on the plateau transition we introduce two edge states with the same circulation in each plaquette of Fig.1. The Hamiltonian including the inter-‘channel’ mixing is given by

\[
\hat{H} = \sum_x (-1)^{x/a} \int dy v_\sigma \psi_\sigma^+(x, y) \frac{\partial_y}{\partial} \psi_\sigma(x, y) - \sum_x \sum_y [t'_\sigma(x, y)e^{i\phi_\sigma(x,y)}\psi_\sigma^+(x+a, y)\psi_\sigma(x, y) + h.c]
- \sum_x \sum_y [t'_{12}(x, y)e^{i\phi_{12}(x,y)}\psi_{12}^+(x+a, y)\psi_{2}(x, y) + h.c].
\]

(9)

Here \(\sigma = 1, 2\) is the channel index and is implicitly summed over. Straightforward generalization of the steps between Eq.(2) and Eq.(7) now gives

\[
\hat{H} = \sum_x (-1)^{x/a} \pi \rho_1 \eta \text{Tr} [\Lambda \hat{S}_\sigma(x)] + \sum_x F_{x,\sigma} (\text{Tr} [\hat{S}_\sigma(x+a)\hat{S}_\sigma(x)]) + \sum_x F_{x,12} (\text{Tr} [\hat{S}_1(x)\hat{S}_2(x)])
\]

(10)

In the above \(F_{x,\sigma}(Z) = \frac{\pi^2 \rho_1^2}{a} < t'^2_\sigma(x) > Z + ...\), and are antiferromagnetic interactions. On the contrary, \(F_{x,12}(Z) = -\frac{\pi^2 \rho_1 \rho_2}{a} < t'^2_{12}(x) > Z + ...\), and is ferromagnetic. Therefore inter-Landau mixing causes a ferromagnetic coupling between two otherwise decoupled SU(2N) spin chains.

In the absence of the inter-chain coupling there are two transitions as we tune \(R_1\) and \(R_2\) keeping \(R_1 > R_2\). \((R_\sigma \equiv [F'_{x+a,\sigma}(Z_0) - F'_{x,\sigma}(Z_0)]/[F'_{x+a,\sigma}(Z_0) + F'_{x,\sigma}(Z_0)]\) All three
massive phases break the translational symmetry and correspond to the (even-odd, even-odd), (even-odd, odd-even), and (odd-even, odd-even) ‘spin-Peierls’ phases. In the presence of a strong ferromagnetic inter-chain coupling we expect vertical spin pairs to form the $M = 2$ representation of SU(2N). At long wave-length, the problem is equivalent to a single antiferromagnetic spin chain in the $M = 2$ representation. In this case there are also two transitions as we tune $R_1$ and $R_2$. Among the three massive phases, two break translation symmetry and they correspond to the (even-odd) and (odd-even) spin-Peierls phases. The phase which remains translationally invariant is the SU(2N) analog of the Haldane phase.\cite{22}

Thus the phase structure and the universality class of the phase transition are preserved in the presence of a strong inter-Landau level mixing. Although we have not done calculations for intermediate inter-chain couplings, based on the knowledge about the SU(2) spin chains\cite{23} we expect the same behavior as in the zero and strong coupling limits.

In addition to the quantum spin chains, the network model contains another interesting field theory. In specific, if we set the random phases $\theta = 0$, hold $t(x+a, y) + t(x, y)$ fixed and let $t(x+a, y) - t(x, y)$ be random, the network model is equivalent to a 2N-component Gross-Neveu model\cite{24} in the limit $N \to 0$. Since the latter model has a different localization length exponent, we conclude that the random phase is essential in determining the universality class of the delocalization transition. To obtain the field theory we group each adjacent upper-right and lower-left moving edge states into a doublet. For each doublet we define $\psi_+$ and $\psi_-$ as the annihilation operators for the upper-right and lower-left moving electrons respectively. Moreover, we construct a Dirac spinor $\Psi$ such that $\Psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$. It is simple to show that in the continuum limit Eq.(2) reduces to

$$H = \sum_x \int dy \Psi^+(x, y) \left\{ \sigma_z p_y + [(t(x+1, y) - t(x, y))\sigma_x + a[t(x+a, y) + t(x, y)]\sigma_y p_x \right\} \Psi(x, y).$$

Here $p_\mu = \frac{\partial}{\partial x}$ ($\mu = x, y$), and $\sigma_\mu$ are the Pauli matrices. In obtaining the above we have
flipped the signs of $\Psi$ and $\Psi^+$ for every other doublets. After going to the spinor basis that diagonalize $\sigma_y$, we let $\sum_x \rightarrow \int \frac{dx}{2a}$, $\frac{t(x,y)+t(x+a,y)}{2}y \rightarrow y$ and $x/2a \rightarrow x$. As the result, we obtain

$$ H = \int dxdy \Psi^+(x,y) [\sigma_\mu p_\mu + m(x,y)\sigma_z] \Psi(x,y), \quad (12) $$

where $m(x,y) = 2[t(x+a,y) - t(x,y)]/[t(x+a,y) + t(x,y)]$. We recognize that Eq.(12) is the Hamiltonian for 2D Dirac electrons with random masses. To study the conductivities we study the following $E = 0$ transport action

$$ S = \int d^2x \bar{\Psi}_\mu [\gamma_\mu \partial_\mu + m - i\eta S_p \sigma_z] \Psi_p. \quad (13) $$

Here we have let $x_\mu \rightarrow \epsilon_{\mu\nu} x_\nu$, and redefined $\bar{\Psi}$ so that $\bar{\Psi} \rightarrow -\bar{\Psi} \sigma_z$. Finally, we replicate and integrating out $\delta m = m - \langle m \rangle$ to obtain:

$$ S = \int d^2x \bar{\Psi}_{p\alpha} [\gamma_\mu \partial_\mu + \langle m \rangle - m - i\eta S_p \sigma_z] \Psi_{p\alpha} - \frac{g}{2}(\bar{\Psi}_{p\alpha} \Psi_{p\alpha})(\bar{\Psi}_{p'\beta} \Psi_{p'\beta}), \quad (14) $$

where $g = \langle (\delta m)^2 \rangle$. In the limit $\eta \rightarrow 0$, Eq.(14) describes the 2N-component Gross-Neveu model. We have checked numerically that for small $\langle \delta m^2 \rangle$ the phase transition of the network model has $\nu = 1$, consistent with the result of the perturbative analysis of Eq.(14).

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[ * New address: Department of physics, University of California at Berkeley, Berkeley, CA 94720]
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**Figure 1.** a) The network model. Here the arrows indicate the direction of the edge velocity, and the open squares enclose the tunneling points. b) A representation of the network model as alternating $\hat{y}$ and $-\hat{y}$-moving fermions that tunnel at the centers of the squares.