Abstract

We consider sparse random intersection graphs with the property that the clustering coefficient does not vanish as the number of nodes tends to infinity. We find explicit asymptotic expressions for the correlation coefficient of degrees of adjacent nodes (called the assortativity coefficient), the expected number of common neighbours of adjacent nodes, and the expected degree of a neighbour of a node of a given degree \( k \). These expressions are written in terms of the asymptotic degree distribution and, alternatively, in terms of the parameters defining the underlying random graph model.

key words: assortativity, clustering, power law, random graph, random intersection graph

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1 Introduction

Assortativity and clustering coefficients are commonly used characteristics describing statistical dependency of adjacency relations in real networks ([18], [2], [20]). The assortativity coefficient of a simple graph is the Pearson correlation coefficient between degrees of the endpoints of a randomly chosen edge. The clustering coefficient is the conditional probability that three randomly chosen vertices make up a triangle, given that the first two are neighbours of the third one.

It is known that many real networks have non-negligible assortativity and clustering coefficients, and a social network typically has a positive assortativity coefficient ([18], [21]). Furthermore, Newman et al. [21] remark that the clustering property (the property that the clustering coefficient attains a non-negligible value) of some social networks could be explained by the presence of a bipartite graph structure. For example, in the actor network two actors are adjacent whenever they have acted in the same film. Similarly, in the collaboration network authors are declared adjacent whenever they have coauthored a paper. These networks exploit the underlying bipartite graph structure: actors are linked to films, and authors to papers. Such networks are sometimes called affiliation networks.

In this paper we study assortativity coefficient and its relation to the clustering coefficient in a theoretical model of an affiliation network, the so called random intersection graph. In a random intersection graph nodes are prescribed attributes and two nodes are declared adjacent whenever they share a certain number of attributes ([11], [15], see also [1], [13]). An attractive property of random intersection graphs is that they include power law degree distributions and have tunable clustering coefficient see [5], [6], [8], [12]. In the present paper we show that the assortativity coefficient of a random intersection graph is non-negative. It is positive in the case where the vertex degree distribution has a finite third moment and the clustering coefficient is positive. In this case we show explicit asymptotic expressions for the assortativity coefficient.
in terms of moments of the degree distribution as well as in terms of the parameters defining the random graph. Furthermore, we evaluate the average degree of a neighbour of a vertex of degree $k$, $k = 1, 2, \ldots$, (called neighbour connectivity, see [16], [23]), and express it in terms of a related clustering characteristic, see [3] below.

Let us rigorously define the network characteristics studied in this paper. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a finite graph on the vertex set $\mathcal{V}$ and with the edge set $\mathcal{E}$. The number of neighbours of a vertex $v$ is denoted $d(v)$. The number of common neighbours of vertices $v_i$ and $v_j$ is denoted $d(v_i, v_j)$. We are interested in the correlation between degrees $d(v_i)$ and $d(v_j)$ and the average value of $d(v_i, v_j)$ for adjacent pairs $v_i \sim v_j$ (here and below ‘$\sim$’ denotes the adjacency relation of $\mathcal{G}$).

We are also interested in the average values of $d(v_i)$ and $d(v_i, v_j)$ under the additional condition that the vertex $v_j$ has degree $d(v_j) = k$.

In order to rigorously define the averaging operation we introduce the random pair of vertices $(v_1^*, v_2^*)$ drawn uniformly at random from the set of ordered pairs of distinct vertices. By $E f(v_1^*, v_2^*) = \frac{1}{N(N-1)} \sum_{i \neq j} f(v_i, v_j)$ we denote the average value of measurements $f(v_i, v_j)$ evaluated at each ordered pair $(v_i, v_j)$, $i \neq j$. Here $N = |\mathcal{V}|$ denotes the total number of vertices. By $E^* f(v_1^*, v_2^*) = p_{v_1}^{-1} E \left( f(v_1^*, v_2^*) \mathbb{I}_{v_1 \sim v_2} \right)$ we denote the average value over ordered pairs of adjacent vertices. Here $p_{v_1} = P(v_1^* \sim v_2^*)$ denotes the edge probability and $\mathbb{I}_{v_1 \sim v_2} = 1$, for $v_i \sim v_j$, and 0 otherwise. Furthermore, $E^{*k} f(v_1^*, v_2^*) = p_{v_1}^{-1} E \left( f(v_1^*, v_2^*) \mathbb{I}_{v_1 \sim v_2} \mathbb{I}_{d(v_2) = k} \right)$, denotes the average value over ordered pairs of adjacent vertices, where the second vertex is of degree $k$.

Here $p_{v_1} = P(v_1^* \sim v_2^*, d(v_2^*) = k)$.

The average values of $d(v_i)$ and $d(v_i, v_j)$ on adjacent pairs $v_i \sim v_j$ are now defined as follows

\[
g(\mathcal{G}) = E^* d(v_1^*)d(v_2^*), \quad h(\mathcal{G}) = E^* d(v_1^*, v_2^*), \quad h_k(\mathcal{G}) = E^{*k} d(v_1^*, v_2^*).\]

We also define the average values

\[
b(\mathcal{G}) = E^* d(v_1^*), \quad b'(\mathcal{G}) = E^* d^2(v_1^*), \quad b_k(\mathcal{G}) = E^{*k} d(v_1^*)\]

and the correlation coefficient

\[
r(\mathcal{G}) = \frac{g(\mathcal{G}) - b'^2(\mathcal{G})}{b'(\mathcal{G}) - b'^2(\mathcal{G})}
\]

called the assortativity coefficient of $\mathcal{G}$, see [18], [19].

In the present paper we assume that our graph is an instance of a random graph. We consider two random intersection graph models: active intersection graph and passive intersection graph introduced in [10] (we refer to Sections 2 and 3 below for a detailed description). Let $G$ denote an instance of a random intersection graph on $N$ vertices. Here and below the number of vertices is non random. An argument bearing on the law of large numbers suggests that, for large $N$, we may approximate the characteristics $b(G)$, $b_k(G)$, $h(G)$ and $h_k(G)$ defined for a given instance $G$, by the corresponding conditional expectations

\[
b = E^* d(v_1^*), \quad b_k = E^{*k} d(v_1^*), \quad h = E^* d(v_1^*, v_2^*), \quad h_k = E^{*k} d(v_1^*, v_2^*),
\]

(1)

where now the expected values are taken with respect to the random instance $G$ and the random pair $(v_1^*, v_2^*)$. We assume that $(v_1^*, v_2^*)$ is independent of $G$. Similarly, we may approximate $r(G)$ by $r = \frac{g - b'^2}{b' - b'^2}$, where $b' = E^* d(v_1^*)$ and $g = E^* d^2(v_1^*)$.

The main results of this paper are explicit asymptotic expressions as $N \to +\infty$ for the correlation coefficient $r$, the neighbour connectivity $b_k$, and expected number of common neighbours $h_k$. 

2
defined in [12]. As a corollary we obtain that the random intersection graphs have tunable assortativity coefficient \( r \geq 0 \). Another interesting property is expressed by the identity

\[ b_k - h_k = b - h + o(1) \quad \text{as} \quad N \to +\infty \]  

(2)

saying that the average value of the difference \( d(v_i) - d(v_i, v_j) \) of adjacent vertices \( v_i \sim v_j \) is not sensitive to the conditioning on the neighbour degree \( d(v_j) = k \). That is, a neighbour \( v_j \) of \( v_i \) may affect the average degree \( d(v_i) \) only by increasing/decreasing the average number of common neighbours \( d(v_i, v_j) \). It is relevant to mention that \( h_k = (k-1)\alpha[k] \), where \( \alpha[k] = P(v_1^* \sim v_2^*|v_1^* \sim v_3^*, v_2^* \sim v_3^*, d(v_3^*) = k) \) measures the probability of an edge between two neighbours of a vertex of degree \( k \). In particular, we have

\[ b_k = (k-1)\alpha[k] + b - h + o(1) \quad \text{as} \quad N \to +\infty. \]  

(3)

The remaining part of the paper is organized as follows. In Section 2 we introduce the active random graph and present results for this model. The passive model is considered in Section 3. Section 4 contains proofs.

2 Active intersection graph

Let \( s > 0 \). Vertices \( v_1, \ldots, v_n \) of an active intersection graph are represented by subsets \( D_1, \ldots, D_n \) of a given ground set \( W = \{w_1, \ldots, w_m\} \). Elements of \( W \) are called attributes or keys. Vertices \( v_i \) and \( v_j \) are declared adjacent if they share at least \( s \) common attributes, i.e., we have \( |D_i \cap D_j| \geq s \).

In the active random intersection graph \( G_s(n, m, P) \) every vertex \( v_i \in V = \{v_1, \ldots, v_n\} \) selects its attribute set \( D_i \) independently at random (\[11\]) and all attributes have equal chances to belong to \( D_i \), for each \( i = 1, \ldots, n \). We assume, in addition, that independent random sets \( D_1, \ldots, D_n \) have the same probability distribution. Then, we have

\[ P(D_i = A) = \binom{m}{\binom{A}{s}}^{-1} P(|A|), \]  

(4)

for each \( A \subset W \), where \( P \) is the common probability distribution of the sizes \( X_i = |D_i|, 1 \leq i \leq n \) of selected sets. We remark that \( X_i, 1 \leq i \leq n \) are independent random variables.

We are interested in the asymptotics of the assortativity coefficient \( r \) and moments (11) in the case where \( G_s(n, m, P) \) is sparse and \( n, m \) are large. We address this question by considering a sequence of random graphs \( \{G_s(n, m, P)\}_n \), where the integer \( s \) is fixed and where \( m = m_n \) and \( P = P_n \) depend on \( n \). We remark that subsets of \( W \) of size \( s \) plays a special role, we call them joints: two vertices are adjacent if their attribute sets share at least one joint. Our conditions on \( P \) are formulated in terms of the number of joints \( X_i \) available to the typical vertex \( v_i \). We denote \( a_k = E(X_i^k) \). It is convenient to assume that as \( n \to \infty \) the rescaled number of joints \( Z_1 = (m)^{-1/2} n^{1/2} X_i^k \) converges in distribution. We also introduce the \( k \)-th moment condition

(i) \( Z_1 \) converges in distribution to some random variable \( Z \);

(ii) \( \theta < EZ^k < \infty \) and \( \lim_{n \to \infty} EZ^k_i = EZ^k \).

We remark that the distribution of \( Z \), denoted \( P_Z \), determines the asymptotic degree distribution of the sequence \( \{G_s(n, m, P)\}_n \) (see \[9, 6, 8, 25\]). We have, under conditions (i), (ii-1) that

\[ \lim_{m \to \infty} P_\theta(d(v_1) = k) = p_k, \quad p_k = (k!)^{-1} E(\left(z_1 Z\right)^{k}e^{-z_1 Z}), \quad k = 0, 1, \ldots \]  

(5)
Here we denote $z_k = E Z^k$. Let $d_s$ be a random variable with the probability distribution $P(d_s = k) = p_k$, $k = 0, 1, \ldots$. We call $d_s$ the asymptotic degree. It follows from (8) that the asymptotic degree distribution is a Poisson mixture, i.e., the Poisson distribution with a random (intensity) parameter $z_1Z$. For example, in the case where $P_Z$ is degenerate, i.e., $P(Z = z_1) = 1$, we obtain the Poisson asymptotic degree distribution. Furthermore, the asymptotic degree has a power law when $P_Z$ does. We denote

$$\delta_i = E d_i^s, \quad \overline{\delta}_i = E(d_i^s),$$

where $(x)_i = x(x - 1) \cdots (x - i + 1)$.

Another important characteristic of the sequence $\{G_s(n, m, P)\}_n$ is the asymptotic ratio $\beta = \lim_{m \to \infty} (m/n)^{1/2}$ if $\delta_i = E d_i^s$ then $\{G_s(n, m, P)\}_n$ satisfies the first order asymptotics of the clustering coefficient $\alpha = P(v_1 \sim v_2|v_1 \sim v_3, v_2 \sim v_3)$, see [6], [8]. Under conditions (i), (ii-2), and (7) we have

$$\alpha = \frac{a_1}{a_2} + o(1) = \frac{1}{\beta \sqrt{\delta_2 - \delta_1}} + o(1).$$

Furthermore, we have $\alpha = o(1)$ in the case where $(m/n)^{1/2} \to +\infty$. We remark that $\alpha = o(1)$ also in the case where the second moment condition (ii-2) fails and we have $E Z^2 = +\infty$, see [6]. To summarize, the clustering coefficient $\alpha$ does not vanish as $n, m \to \infty$ whenever the asymptotic degree distribution (equivalently $P_Z$) has finite second moment and $0 < \beta < \infty$.

Our next result shows a first order asymptotics of the neighbour connectivity $b_k$ and the expected number of common neighbours $h_k$.

**Theorem 1.** Let $s > 0$ be an integer. Let $m, n \to \infty$. Assume that (i) and (7) are satisfied. In the case where (ii-3) holds we have

$$r = \frac{a_1}{\beta^{-1}(a_1 a_3 - a_2^2) + a_2} + o(1)$$

$$= \frac{1}{\sqrt{\beta \delta_3 \delta_1} - \delta_2 + \sqrt{\delta_2 \delta_1}} + o(1).$$

In the case where (ii-2) holds and $E Z^3 = \infty$ we have $r = o(1)$.

We note that the inequality $a_1 a_3 \geq a_2^2$, which follows from Hölder’s inequality, implies that the ratio in the right hand side of (9) is positive.

**Remark 1.** In the case where (i), (ii-2) hold and $(m/n)^{1/2} \to +\infty$ we have $r = o(1)$.

Our next result shows a first order asymptotics of the neighbour connectivity $b_k$ and the expected number of common neighbours $h_k$.

**Theorem 2.** Let $s \geq 1$ and $k \geq 0$ be integers. Let $m, n \to \infty$. Assume that (i), (ii-2) and (7) hold. We have

$$b = 1 + \beta^{-1}a_2 + o(1), \quad h = \beta^{-1}a_1 + o(1)$$

and

$$h_{k+1} = \frac{a_1}{\beta} \frac{k}{k + 1} \frac{p_k}{p_{k+1}} + o(1),$$

$$b_{k+1} = 1 + \beta^{-1}(a_2 - a_1) + h_{k+1} + o(1).$$

Here $a_1 = (\beta \delta_1)^{1/2} + o(1)$ and $a_2 = \beta \delta_2 / \delta_1 + o(1)$.
We remark that the distribution of the random graph $G_s(n, m, P)$ is invariant under permutation of its vertices (we refer to this property as the symmetry property in what follows). Therefore, we have $b = \mathbb{E}(d(v_1)|v_1 \sim v_2)$ and $b_{k+1} = \mathbb{E}(d(v)|v_1 \sim v_2, d(v) = k + 1)$. In particular, the increment $b_{k+1} - b$ shows how the degree of $v_2$ affects the average degree of its neighbour $v_1$. By (11), (12), and (13), we have $b_{k+1} - b = \frac{p}{2} \left( \frac{k}{k+1} \frac{p}{p_{k+1}} - 1 \right) + o(1)$. In Examples 1 and 2 below we evaluate this quantity for a power law asymptotic degree distribution and the Poisson asymptotic degree distribution.

**Example 1.** Assume that the asymptotic degree distribution has a power law, i.e., for some $c > 0$ and $\gamma > 3$ we have $p_k = (c + o(1))k^{-\gamma}$ as $k \to +\infty$. Then

$$\frac{k}{k+1} \frac{p}{p_{k+1}} - 1 = \frac{\gamma - 1}{k} + o(k^{-1}).$$

Hence, for large $k$, we obtain as $n, m \to +\infty$ that $b_{k+1} - b \approx k^{-1}(\gamma - 1)(\delta_1/\beta)^{1/2}$.

**Example 2.** Assume that the asymptotic degree distribution is Poisson with mean $\lambda > 0$, i.e., $p_k = e^{-\lambda} / k!$. Then

$$\frac{k}{k+1} \frac{p}{p_{k+1}} - 1 = \frac{\lambda}{\lambda} - 1$$

and, for large $k$, we obtain as $n, m \to +\infty$ that

$$b_{k+1} - b \approx (\lambda/\beta)^{-1/2}k.$$ (14)

Our interpretation of (14) is as follows. We assume, for simplicity, that $s = 1$. We say that an attribute $w \in W$ realises the link $v_i \sim v_j$, whenever $w \in D_i \cap D_j$. We note that in a sparse intersection graph $G_1(n, m, P)$ each link is realised by a single attribute with a high probability.

We also remark that in the case of the Poisson asymptotic degree distribution, the sizes of the random sets, defining intersection graph, are strongly concentrated about their mean value $a_1$. Now, by the symmetry property, every element of the attribute set $D_2$ of vertex $v_2$ realises about $k/|D_2| \approx k/a_1$ links to some neighbours of $v_2$ other than $v_1$. In particular, the attribute responsible for the link $v_1 \sim v_2$ attracts to $v_1$ some $k/a_1$ neighbours of $v_2$. Hence, $b_{k+1} - b \approx a_1^{-1}k \approx (\beta/\lambda)^{-1/2}k$.

Finally, we remark that (11), (12), and (13) imply (2).

### 3 Passive intersection graph

A collection $D_1, \ldots, D_n$ of subsets of a finite set $W = \{w_1, \ldots, w_m\}$ defines the passive adjacency relation between elements of $W$: $w_i$ and $w_j$ are declared adjacent if $w_i, w_j \in D_k$ for some $D_k$. In this way we obtain a graph on the vertex set $W$, which we call the passive intersection graph, see [11]. We assume that $D_1, D_2, \ldots, D_n$ are independent random subsets of $W$ having the same probability distribution. In particular, their sizes $X_i = |D_i|$, $1 \leq i \leq n$ are independent random variables with the common distribution $P$. The **passive random intersection graph** defined by the collection $D_1, \ldots, D_n$ is denoted $G^*_1(n, m, P)$.

We shall consider a sequence of passive graphs $\{G^*_1(n, m, P)\}_n$, where $P = P_n$ and $m = m_n$ depend on $n = 1, 2, \ldots$. We remark that, in the case where $\beta_n = mn^{-1}$ is bounded and it is bounded away from zero as $n, m \to +\infty$, the vertex degree distribution can be approximated by a compound Poisson distribution (13). More precisely, assuming that $\beta_n \to \beta \in (0, +\infty)$; (iii) $X_1$ converges in distribution to a random variable $Z$;
In the case where $\beta > 0$ we put $\text{P}(Z_1 = 0) = 1$. The random variable $\Lambda$ is independent of the sequence $Z_1, Z_2, \ldots$ and has Poisson distribution with mean $E\Lambda = \beta^{-1}EZ$. We note that the asymptotic degree $d_{ss}$ has a power law whenever $Z$ has a power law. Furthermore, we have $Ed_{ss} < \infty \iff E\Lambda^{i+1} < \infty, i = 1, 2, \ldots$.

In Theorems 3, 4 below we express the moments $b_i = \delta_i$.

Remark 2. We note that $y_* := 2y_{24} + 2y_{23} - y_3^2$ is always non-negative. Hence, for large $n, m$ we have $r \geq 0$. To show that $y_* \geq 0$ we combine the identity $2y_* = \text{E}(X_1, X_2)$, where

$$y(i, j) = y'(i, j) + y'(j, i), \quad y'(i, j) = (i)_{2}((j)_{4} + (i)_{3})_{3} - (i)_{3}((j)_{3})_{3},$$

with the simple inequality

$$y(i, j) = (i)_{2}((j)_{4} - 2^2 + (j - 2^2 - 2(i - 2)(j - 2)) \geq 0.$$
Theorem 4. Let $m, n \to \infty$. Assume that $eta_n \to \beta \in (0, +\infty)$ and (iii), (v) hold. Then

\begin{align}
    b &= 1 + \beta_n^{-1}y_2 + y_2^{-1}y_3 + O(n^{-1}) = \delta_{x2} \delta_{x1}^{-1} + o(1), \\
    h &= y_2^{-1}y_3 + O(n^{-1}) = \delta_{x2} \delta_{x1}^{-1} - 1 - \delta_{x1} + o(1).
\end{align}

Assuming, in addition, that $P(d_{**} = k) > 0$, where $k > 0$ is an integer, we have

\begin{align}
    h_k &= k^{-1}E(d_{2*}|d_{**} = k) + o(1), \\
    b_k &= 1 + \beta^{-1}y_2 + h_k + o(1) = 1 + \delta_{x1} + h_k + o(1).\end{align}

Here $d_{2*} = \sum_{1 \leq i \leq \Lambda(\tilde{Z}_1)}$.

We remark that (18), (19), (20), (21) imply (2).

4 Proofs

Proofs for active and passive graphs are given in Section 4.1 and Section 4.2 respectively. We note that the probability distributions of $G_s(n, m, P)$ and $G_1^s(n, m, P)$ are invariant under permutations of the vertex sets. Therefore, for either of these models we have

\begin{align}
    b &= E_{12}d(\omega_1), \\
    b_k &= E_{12}(d(\omega_2)|d(\omega_1) = k), \\
    h &= E_{12}d(\omega_1, \omega_2), \\
    h_k &= E_{12}(d(\omega_1, \omega_2)|d(\omega_1) = k).
\end{align}

Here $\omega_1 \neq \omega_2$ are arbitrary fixed vertices and $E_{12}$ denotes the conditional expectation given the event $\omega_1 \sim \omega_2$. In the proof $P$ and $\tilde{E}$ (respectively, $P_*$ and $\tilde{E}_*$) denote the conditional probability and expectation given $X_1, \ldots, X_n$ (respectively, $D_1, D_2, X_1, \ldots, X_n$). Limits are taken as $n$ and $m = m_n$ tend to infinity. We use the shorthand notation $f_k(\lambda) = e^{-\lambda \beta^k}/k!$ for the Poisson probability.

4.1 Active graph

Before the proof we introduce some more notation. Then we state and prove auxiliary lemmas. Afterwards we prove Theorem 1 Remark 1 and Theorem 2.

The conditional expectation given $D_1, D_2$ is denoted $E_*$. The conditional expectation given the event $v_1 \sim v_2$ is denoted $E_{12}$. We denote

\begin{align}
    Y_i &= \binom{X}{i}, \\
    d_i &= d(v_i), \\
    d_i' &= d_i - 1, \\
    d_{ij} &= d(v_i, v_j), \\
    \mathbb{I}_i &= \mathbb{I}_{\{X_i < m^{v_i}\}}, \\
    \mathbb{II}_i &= 1 - \mathbb{I}_i, \\
    \delta_{ij} &= 1 - \mathbb{I}_i - \mathbb{I}_j - (m^{1/2} - 1)^{-1}
\end{align}

and introduce events

\begin{align}
    \mathcal{E}_{ij} &= \{|D_i \cap D_j| = s\}, \\
    \mathcal{E}_{ij}' &= \{|D_i \cap D_j| \geq s + 1\}, \\
    \mathcal{E}_{ij}'' &= \{|D_i \cap D_j| \geq s\}.
\end{align}

Observe that $\mathcal{E}_{ij}$ is the event that $v_i$ and $v_j$ are adjacent in $G_s(n, m, P)$. We denote

\begin{align}
    p_e &= P(\mathcal{E}_{ij}), \\
    a_i &= EY_i, \\
    x_i &= EX_i, \\
    z_i &= E\tilde{Z}_i, \\
    \tilde{m} &= \binom{m}{s}, \\
    \beta_n &= \frac{\tilde{m}}{n}.
\end{align}

We remark that the distributions of $X_i = X_{ni}, Y_i = Y_{ni}$ and $Z_i = Z_{ni} = (n/\tilde{m})^{1/2}Y_{ni}$ depend on $n$.

The following inequality is referred to as LeCam’s lemma, see e.g., [26].
Lemma 1. Let $S = \mathbb{I}_1 + \mathbb{I}_2 + \cdots + \mathbb{I}_n$ be the sum of independent random indicators with probabilities $P(\mathbb{I}_i = 1) = p_i$. Let $\Lambda$ be Poisson random variable with mean $p_1 + \cdots + p_n$. The total variation distance between the distributions $P_S$ and $P_\Lambda$ of $S$ and $\Lambda$

$$
\sup_{A \in \{0,1,2,\ldots\}} |P(S \in A) - P(\Lambda \in A)| \leq 2 \sum_i p_i^2.
$$

(24)

Lemma 2. (15) Given integers $1 \leq s \leq k_1 \leq k_2 \leq m$, let $D_1, D_2$ be independent random subsets of the set $W = \{1, \ldots, m\}$ such that $D_1$ (respectively $D_2$) is uniformly distributed in the class of subsets of $W$ of size $k_1$ (respectively $k_2$). The probabilities $p' := P(|D_1 \cap D_2| = s)$ and $p'' := P(|D_1 \cap D_2| \geq s)$ satisfy

$$
\left(1 - \frac{s}{\binom{m}{k_1} - \binom{k_1 - s}{k_1} - \binom{m - s}{k_1}}\right) p_{k_1,k_2,s}^* \leq p' \leq p'' \leq p_{k_1,k_2,s}^*.
$$

(25)

Here we denote $p_{k_1,k_2,s}^* = \binom{k_1}{s} \binom{k_2}{m-s}^{-1}$.

Lemma 3. Let $s > 0$ be an integer. Let $m, n \to \infty$. Assume that conditions (i) and (ii-3) hold. Denote $X_{n1} = m^{-1/2}n^{1/(2s)}X_n^1 \mathbb{1}(X_n \geq s)$. We have

$$
\lim_{A \to +\infty} \sup_n \mathbb{E}Z_{n1}^s \mathbb{1}(X_n \geq A) = 0,
$$

(26)

$$
\sup_n \mathbb{E}\tilde{X}_{n1}^s < \infty, \quad \lim_{A \to +\infty} \sup_n \mathbb{E}X_{n1}^s \mathbb{1}(X_n \geq A) = 0.
$$

(27)

For any $0 \leq u \leq 3$ and any sequence $A_n \to +\infty$ as $n \to \infty$ we have

$$
\mathbb{E}Z_{n1}^s \mathbb{1}(X_n \geq A_n) = o(1), \quad \mathbb{E}\tilde{X}_{n1}^s \mathbb{1}(X_n \geq A_n) = o(1).
$$

(28)

Proof of Lemma 3. The uniform integrability property (26) of the sequence $\{Z_{n1}^s\}$ is a simple consequence of (i) and (ii-3), see, e.g., Remark 1 in [5]. The first and second identity of (27) follows from (ii-3) and (26) respectively. Finally, (28) follows from (26) and (27).

Lemma 4. In $G_s(n,m,P)$ the probabilities of events $E_{ij} = \{v_i \sim v_j\}$, $E_{12}^s$, $E_{12}''$, see (23), and $B_i = \{|D_1 \cap (D_1 \cup D_2)| \geq s+1\}$ satisfy the inequalities

$$
Y_1Y_2m^{-1} \delta_{12} \leq \tilde{P}(E_{12}^s) \leq \tilde{P}(E_{12}^s) \leq Y_1Y_2\tilde{m}^{-1},
$$

(29)

$$
Y_1Y_2m^{-1} \delta_{ij} \leq \tilde{P}(E_{ij}) \leq Y_1Y_2\tilde{m}^{-1}, \quad \text{for} \quad \{i,j\} \neq \{1,2\},
$$

(30)

$$
\tilde{P}(E_{12}'' \leq Y_1Y_2X_1X_2(\tilde{m}m)^{-1}),
$$

(31)

$$
\tilde{P}(E_{ij}) \leq 2^s((s+1)!\tilde{m}m)^{-1} Y_1X_1(X_1^{s+1} + X_2^{s+1}).
$$

(32)

We recall that $Y_1$ and $\delta_{ij}$ are defined in (23).

Proof of Lemma 4. The right hand side of (29), (30) and inequality (31) are immediate consequences of (25). In order to show the left hand side inequality of (29) and (30) we apply the left hand side inequality of (25). We only prove (29). We have, see (23),

$$
\tilde{P}(E_{12}^s) = \mathbb{E}e_{12} Y_1Y_2m^{-1} \delta_{12} \geq \mathbb{E}e_{12} \geq m^{-1} Y_1Y_2\tilde{m}^{-1} \geq Y_1Y_2\tilde{m}^{-1},
$$

(33)

In order to show (32) we apply the right-hand side inequality of (25) and write

$$
\tilde{P}(E_{ij}) \leq \frac{(|D_1 \cup D_2|)(D_1)}{s+1} \left(\frac{X_1}{s+1}\right)^{-1} \leq \left(\frac{X_1}{s+1}\right)^{-1} \frac{X_1}{s+1} \left(\frac{X_1}{s+1}\right)^{-1}.
$$

(34)

Invoking the inequalities $\left(\frac{X_1}{s+1}\right)^{-1} \leq \frac{X_1}{s+1}$ and

$$
(X_1 + X_2)(s+1) \leq (X_1 + X_2)^{s+1} \leq 2^s(X_1^{s+1} + X_2^{s+1})
$$

we obtain (32).
Lemma 5. Assume that conditions of Theorem \[2\] are satisfied. Let \( k \geq 0 \) be an integer. For \( d_1 = \sum_{4 \leq t \leq n} I_{\xi_1t} \) and \( \Delta = P_*(d_1^* - k) - f_k(\beta^{-1} a_1 Y_1) \) we have

\[
E_*|\Delta| \leq R_1^* + R_2^* + R_3^* + R_4^*,
\]

where \( R_1^* = n\bar{m}^{-1} E_* Y_1 Y_4 |1 - \delta_{14}| \) and

\[
R_2^* = n^{1/2} \bar{m}^{-1} a_2^{1/2} Y_1, \quad R_3^* = a_1 Y_1 (n - 3) \bar{m}^{-1} - \beta^{-1}, \quad R_4^* = 2n\bar{m}^{-2} a_2 Y_1^2.
\]

Proof of Lemma 5. We denote \( \tilde{S} = \tilde{E}_* d_1^* = \sum_{4 \leq t \leq n} \tilde{P}_* (\xi_{1t}) \) and \( \tilde{S}_1 = \bar{m}^{-1} \sum_{4 \leq t \leq n} Y_t \) and write

\[
\Delta = \Delta_1 + \Delta_2, \quad \Delta_1 = \tilde{P}_*(d_1^* = k) - f_k(\tilde{S}), \quad \Delta_2 = f_k(\tilde{S}) - f_k(\beta^{-1} a_1 Y_1).
\]

We have, by Lemma 4, \( |\Delta_1| \leq 2 \sum_{4 \leq t \leq n} \tilde{P}_* (\xi_{1t}) \). Invoking (30) we obtain \( E_*|\Delta_1| \leq R_1^* \). Next, we apply the mean value theorem \( |f_k(\lambda') - f_k(\lambda'')| \leq |\lambda' - \lambda''| \) and write

\[
|\Delta_2| \leq |\tilde{S} - \beta^{-1} a_1 Y_1| \leq r_1^* + r_2^* + R_3^*,
\]

where \( r_1^* = |\tilde{S} - Y_1 \tilde{S}_1| \) and \( r_2^* = Y_1|\tilde{S}_1 - (n - 3) \bar{m}^{-1} a_1| \). Note that by (30),

\[
r_1^* \leq \sum_{4 \leq t \leq n} (\tilde{P}_* (\xi_{1t}) \bar{m}^{-1} Y_t) \leq \sum_{4 \leq t \leq n} \bar{m}^{-1} Y_t|1 - \delta_{1t}|
\]

and, by symmetry, \( E_* r_2^* \leq R_1^* \). Finally, we have

\[
E_* r_2^* = Y_1 E_* (\tilde{S}_1 - E_* \tilde{S}_1) \leq Y_1 \left( E_* (\tilde{S}_1 - E_* \tilde{S}_1)^2 \right)^{1/2} \leq R_2^*.
\]

Lemma 6. Let \( m, n \to \infty \). Assume (i), (ii-3) and (7) hold. Then

\[
E_{12} d_1^* d_2^* = n\bar{m}^{-1} a_1 + n^2 \bar{m}^{-2} a_2^2 + o(1),
\]

\[
E_{12} d_1^* = n\bar{m}^{-1} a_2 + o(1),
\]

\[
E_{12} d_1^* = n\bar{m}^{-1} a_2 + o(1),
\]

\[
E_{12} d_1^* = n\bar{m}^{-1} a_2 + o(1),
\]

Proof of Lemma 6. Proof of (37). In order to prove (37) we write

\[
E_{12} d_1^* d_2^* = p_e^{-1} E \kappa, \quad \kappa := I_{\xi_{12}} d_1^* d_2^*, \quad p_e := P(\xi_{12})
\]

and invoke the identities

\[
E \kappa = n\bar{m}^{-2} a_1^3 + n^2 \bar{m}^{-3} a_1^2 a_2^2 + o(\bar{m}^{-1}),
\]

\[
p_e = \bar{m}^{-1} a_1^2 (1 + o(1)).
\]

Proof of Lemma 6. Proof of (37). In order to prove (37) we write

\[
E_{12} d_1^* d_2^* = p_e^{-1} E \kappa, \quad \kappa := I_{\xi_{12}} d_1^* d_2^*, \quad p_e := P(\xi_{12})
\]

and invoke the identities

\[
E \kappa = n\bar{m}^{-2} a_1^3 + n^2 \bar{m}^{-3} a_1^2 a_2^2 + o(\bar{m}^{-1}),
\]

\[
p_e = \bar{m}^{-1} a_1^2 (1 + o(1)).
\]

Note that (33) follows from (30) and (25). Let us prove (42). To this aim we write

\[
E \kappa = E \left( I_{\xi_{12}} \tilde{E}_* (d_1^* d_2^*) \right) = E (\tilde{\kappa}_1 + \tilde{\kappa}_2),
\]

where \( \tilde{\kappa}_1 = I_{\xi_{12}} \tilde{E}_* d_1^* d_2^* \) and \( \tilde{\kappa}_2 = I_{\xi_{12}} \tilde{E}_* d_2^* d_1^* \), and show that

\[
E \tilde{\kappa}_1 = n\bar{m}^{-2} a_1^3 + n^2 \bar{m}^{-3} a_1^2 a_2^2 + o(\bar{m}^{-1}), \quad E \tilde{\kappa}_2 = o(\bar{m}^{-1}).
\]
Let us prove (44). Assuming that $\mathcal{E}_{12}$ holds we can write $d_i' = \sum_{t=3}^n \tilde{\mathbb{E}}(\mathcal{E}_t \cap \mathcal{E}_2)$, $i = 1, 2$, and
\begin{equation}
\tilde{\mathbb{E}}_t d_1' d_2' = S_1 + S_2, \quad S_1 = \sum_{3 \leq t \leq n} \tilde{\mathbb{P}}_*(\mathcal{E}_t \cap \mathcal{E}_2), \quad S_2 = 2 \sum_{3 \leq t < u \leq n} \tilde{\mathbb{P}}_*(\mathcal{E}_t \cap \mathcal{E}_u). \tag{45}
\end{equation}
To show the first identity of (44) we write $\mathbb{E} \hat{x}_t' = \mathbb{E} \mathbbm{1}_{\mathcal{E}_{12}} S_1 + \mathbb{E} \mathbbm{1}_{\mathcal{E}_{12}} S_2 =: I_1 + I_2$ and evaluate
\begin{equation}
I_1 = n\tilde{m}^{-2}a_1^3 + o(n\tilde{m}^{-2}), \quad I_2 = n^2\tilde{m}^{-3}a_1^2a_2^2 + o(n^2\tilde{m}^{-3}). \tag{46}
\end{equation}
We first evaluate $I_1$. Given $t \geq 3$, consider events
\begin{equation}
A_t = \{ |(D_1 \cap D_2) \cap D_t| = s \} \quad \text{and} \quad B_t = \{ |D_t \cap (D_1 \cup D_2)| \geq s + 1 \}. \tag{47}
\end{equation}
Assuming that $\mathcal{E}_{12}'$ holds we have that $A_t$ implies $\mathcal{E}_{12} \cap \mathcal{E}_{2t}$ and $\mathcal{E}_{1t} \cap \mathcal{E}_{2t}$ implies $A_t \cup B_t$. Hence, $\tilde{\mathbb{P}}_*(A_t) \leq \tilde{\mathbb{P}}_*(\mathcal{E}_t \cap \mathcal{E}_{2t}) \leq \tilde{\mathbb{P}}_*(A_t \cup B_t)$. Now, we invoke the identity $\tilde{\mathbb{P}}_*(A_t) = \tilde{m}^{-1}Y_t$ and write
\begin{equation}
\mathbbm{1}_{\mathcal{E}_{12}'} \tilde{m}^{-1}Y_t = \mathbbm{1}_{\mathcal{E}_{12}'} \tilde{\mathbb{P}}_*(A_t) \leq \mathbbm{1}_{\mathcal{E}_{12}'} \tilde{\mathbb{P}}_*(\mathcal{E}_t \cap \mathcal{E}_{2t}) \leq \mathbbm{1}_{\mathcal{E}_{12}'} \left( \tilde{\mathbb{P}}_*(A_t) + \tilde{\mathbb{P}}_*(B_t) \right). \tag{48}
\end{equation}
From (48) and (32) we obtain, by the symmetry property,
\begin{equation}
\frac{n-2}{\tilde{m}}\mathbb{P}(\mathcal{E}_{12}') \mathbb{E}Y_3 \leq I_1 \leq \frac{n-2}{\tilde{m}}\mathbb{P}(\mathcal{E}_{12}') \mathbb{E}Y_3 + \frac{n-2}{\tilde{m}m} \mathbb{E}\tilde{\mathbb{P}}(\mathcal{E}_{12}') R_1, \tag{49}
\end{equation}
where $R_1 = Y_3X(X_1 + X_2)$. Next, we evaluate $\tilde{\mathbb{P}}(\mathcal{E}_{12}')$ and $\mathbb{P}(\mathcal{E}_{12}') = \mathbb{E}\tilde{\mathbb{P}}(\mathcal{E}_{12}')$ using (29):
\begin{equation}
\tilde{m}\mathbb{P}(\mathcal{E}_{12}') \mathbb{E}Y_3 = a_1^3 + o(1), \quad \tilde{m}\mathbb{E}\tilde{\mathbb{P}}(\mathcal{E}_{12}') R_1 = O(1). \tag{50}
\end{equation}
Combining these relations with (49) we obtain the first relation of (44).
Let us evaluate $I_2$. We write
\begin{equation}
\tilde{\mathbb{E}}_{\mathcal{E}_{12}'} \tilde{\mathbb{P}}_*(\mathcal{E}_t \cap \mathcal{E}_{2u}) = \tilde{\mathbb{E}}_{\mathcal{E}_{12}'} \tilde{\mathbb{P}}_*(\mathcal{E}_t) \tilde{\mathbb{P}}_*(\mathcal{E}_{2u}) = \tilde{\mathbb{P}}(\mathcal{E}_{12}') \tilde{\mathbb{P}}(\mathcal{E}_{1t}) \tilde{\mathbb{P}}(\mathcal{E}_{2u}) \tag{51}
\end{equation}
and apply (29) to each probability in the right-hand side. We obtain
\begin{equation}
\tilde{m}^{-3}(Y_1^2Y_2^2Y_3 - R_{tu}) \leq \tilde{\mathbb{P}}(\mathcal{E}_{12}') \tilde{\mathbb{P}}(\mathcal{E}_t) \tilde{\mathbb{P}}(\mathcal{E}_{2u}) \leq \tilde{m}^{-3}Y_1^2Y_2^2Y_3, \tag{52}
\end{equation}
where $R_{tu} = Y_3Y_2Y_3(1 - \delta_1\delta_2\delta_{2u})$ satisfies $\mathbb{E}R_{tu} = o(1)$, see (28). Now, by the symmetry property, we obtain from (52) the second relation of (44)
\begin{equation}
I_2 = (n-2)\mathbb{E}\tilde{\mathbb{P}}(\mathcal{E}_{12}') \tilde{\mathbb{P}}(\mathcal{E}_{1t}) \tilde{\mathbb{P}}(\mathcal{E}_{2u}) = n^2\tilde{m}^{-3}a_1^2a_2^2 + o(n^2\tilde{m}^{-3}). \tag{53}
\end{equation}
To prove the second bound of (44) we write, see (15), $\tilde{x}_2 = \mathbbm{1}_{\mathcal{E}_{12}'} (S_1 + S_2)$ and show that
\begin{equation}
I_3 := \mathbb{E}\mathbbm{1}_{\mathcal{E}_{12}'} S_1 \leq x_{s+1}x_{s+1}x_{s}/(\tilde{m}^2m), \quad I_4 := \mathbb{E}\mathbbm{1}_{\mathcal{E}_{12}'} S_2 \leq x_{s+1}^2x_{s}^2n^2/(\tilde{m}^3m). \tag{54}
\end{equation}
Here $x_{s+1}, x_{s+1}, x_{s} = O(1)$, by (27). Let us prove (54). We have, see (29),
\begin{equation}
S_1 \leq \sum_{3 \leq t \leq n} \tilde{\mathbb{P}}_*(\mathcal{E}_t) \leq \sum_{3 \leq t \leq n} Y_t \tilde{m}^{-1}. \tag{55}
\end{equation}
Furthermore, by the symmetry property and (31), we obtain
\begin{equation}
I_3 = \mathbb{E}(\mathbbm{1}_{\mathcal{E}_{12}'} S_1) = \mathbb{E}(\tilde{\mathbb{P}}'''(\mathcal{E}_{12}') S_1) \leq (n-2)(\tilde{m}^2m)^{-1} \mathbb{E}Y_1^2Y_2^2Y_3X_1X_2.
\end{equation}
Since the expected value in the right hand side does not exceed \(x_{2s+1}x_{s+1}x_s\), we obtain the first bound of (52). In order to prove the second bound we write, cf. (50),
\[
\mathbb{E}\xi_{12}^{\nu} \hat{P}_{s}(\mathcal{E}_{11} \cap \mathcal{E}_{2n}) = \hat{P}(\mathcal{E}_{11}^{\nu}) \hat{P}(\mathcal{E}_{2n}) \leq \bar{m}^{-3}m^{-1}Y_1^2Y_2^2Y_1Y_2X_1X_2.
\]
In the last step we used (29), (30). Now (43), (55) and (56) imply (39).

\[ I_4 = \mathbb{E}(\mathbb{E}_{12}^{\nu}s_2) \leq (n-2)\bar{m}^{-3}m^{-1}Y_1^2Y_2^2Y_3Y_4X_1X_2 \leq n^2\bar{m}^{-3}m^{-1}x_{2s+1}^2. \]

Proof of (38). We write, by the symmetry property,
\[
E_{12}d_1 = p_e^{-1}E \sum_{3 \leq t \leq n} \mathbb{I}_{\xi_{11}} \mathbb{I}_{\xi_{12}} = (n-2) p_e^{-1} E \mathbb{I}_{\xi_{13}} \mathbb{I}_{\xi_{12}}
\]
and evaluate using (29), (30)
\[
E \mathbb{I}_{\xi_{12}} \mathbb{I}_{\xi_{13}} = E \hat{P}(\mathcal{E}_{12}) \mathbb{P}(\mathcal{E}_{13}) = \bar{m}^{-2}EY_1^2Y_2Y_3 + o(\bar{m}^{-2}) = \bar{m}^{-2}a_1^2a_2 + o(\bar{m}^{-2}).
\]
Invoking this relation and (43) in (54) we obtain (38).

Proof of (39). Assuming that the event \(\xi_{12}\) holds we write
\[
(d_1')^2 = (\sum_{3 \leq t \leq n} \mathbb{I}_{\xi_{11}})^2 = d_1^2 + \sum_{3 \leq t < u \leq n} \mathbb{I}_{\xi_{11}} \mathbb{I}_{\xi_{1u}}
\]
and evaluate the expected value
\[
E_{12}(d_1')^2 = E_{12}d_1^2 + p_e^{-1}(n-2)_2 \nu^*.
\]
Here \(\nu^* = E \mathbb{I}_{\xi_{12}} \mathbb{I}_{\xi_{13}} \mathbb{I}_{\xi_{14}}\). We have
\[
\nu^* = E \hat{P}(\mathcal{E}_{12}) \mathbb{P}(\mathcal{E}_{13}) = \bar{m}^{-3}EY_1^3Y_2Y_3 + o(\bar{m}^{-3}).
\]
In the last step we used (29), (30). Now (43), (55) and (56) imply (39).

Proof of (40). We note that \(d_{12} = \sum_{3 \leq t \leq n} \mathbb{I}_{\xi_{11}} \mathbb{I}_{\xi_{2u}}\) and \(E \mathbb{I}_{\xi_{12}} d_{12} = E \mathbb{I}_{\xi_{12}} s_1\), see (43). Next, we write
\[
E_{12}d_{12} = p_e^{-1} E \mathbb{I}_{\xi_{12}} s_1 = p_e^{-1} (I_1 + I_3).
\]
and evaluate the quantity in the right hand side using (43) and (46), (52).

Proof of Theorem 1. It is convenient to write \(r\) in the form
\[
r = \eta/\xi, \quad \text{where} \quad \eta = E_{12}d_1'd_2' - (E_{12}d_1')^2, \quad \xi = E_{12}(d_1')^2 - (E_{12}d_1')^2. \tag{57}
\]
In the case where (ii-3) holds we obtain (9) from (37), (38), (39) and (57). Then we derive (10) from (9) using the identities
\[
a_i = \beta^{i/2}z_i + o(1), \quad \overline{d}_i = z_i \overline{z}_i^i, \quad i = 1, 2, 3. \tag{58}
\]
Now we consider the case where (ii-2) holds and \(EZ^3 = \infty\). It suffices to show that
\[
\eta = O(1) \quad \text{and} \quad \lim \inf \xi = +\infty. \tag{59}
\]

11
Before the proof of (59) we remark that (43) holds under condition (ii-2). In order to prove the first bound of (59) we show that $E_{12}d_1'd_2' = O(1)$ and $E_{12}d_1' = O(1)$. To show the first bound we write $E_{12}d_1'd_2' = p_e^{-1}E\|\xi_{12}\|d_1'd_2'$ and evaluate

$$E\|\xi_{12}\|d_1'd_2' = \sum_{3 \leq t \leq n} \|\xi_{12}\|\xi_{2u} + \sum_{3 \leq t, u \leq n, t \neq u} \|\xi_{12}\|\xi_{2u} = (n - 2)\varepsilon_1^* + (n - 2)\varepsilon_2^*,$$

where

$$\varepsilon_1^* = E\|\xi_{12}\|\xi_{23} \leq E\|\xi_{12}\|\xi_{13} \leq \tilde{m}^{-2}a_2a_1^2 = O(n^{-2});$$

$$\varepsilon_2^* = E\|\xi_{12}\|\xi_{11} \xi_{23} \leq \tilde{m}^{-3}a_2^2a_1^2 = O(n^{-3}).$$

In the last step we used (29) and (30). We note that (43), (60) and (61), (62) imply $E_{12}d_1'd_2' = O(1)$. Similarly, the bound $E_{12}d_1' = O(1)$ follows from (43) and the simple bound, cf. (54),

$$E_{12}d_1' = p_e^{-1}(n - 2)E\|\xi_{12}\|\xi_{13} \leq p_e^{-1}n\tilde{m}^{-2}a_2a_1^2. \tag{63}$$

In order to prove the second relation of (59) we show that $\liminf E_{12}(d_1')^2 = +\infty$. In view of (43) and (55) it suffices to show that $\liminf n^3\varepsilon^* = +\infty$. It follows from the left-hand side inequality of (24) that

$$n^3\varepsilon^* \geq n^3E_112_3\|\xi_{13}\|\xi_{13} \geq E_112_3\|\xi_{13}\|\xi_{14} \geq E_112_3\|\xi_{14}\|_{Z_2}Z_2Z_4(1 - O(m^{-1/2}))^3, \tag{64}$$

where, by the independence of $Z_1, \ldots, Z_4$, we have $E_112_3\|\xi_{14}\|_{Z_2}Z_2Z_4 = (E_1Z_2^3)(E_2Z_2)^3$. Finally, (i) combined with (ii-2) imply $E_1Z_2 = z_1 + o(1)$, and (i) combined with $EZ^3 = \infty$ imply $\liminf E_1Z_1^3 = +\infty$. \hfill \Box

**Proof of Remark 1.** Before the proof we introduce some notation and collect auxiliary inequalities. We denote

$$h = h_n = m^{1/2}n^{-1/48}, \quad \tilde{h} = \tilde{h}_n = (\frac{h}{n})^{1/2} \beta^{-1/2}$$

and observe that, under the assumption of Remark 1, $\beta, h_n, \tilde{h}_n \rightarrow +\infty$ and $h_n = o(m^{1/2})$. We further denote

$$I_{ih} = I_{(X_i < h)} \quad \tilde{I}_{ih} = 1 - I_{ih}, \quad \delta_{ijh} = 1 - \tilde{I}_{ih} - \tilde{I}_{jh} - \varepsilon_h,$$

where $\varepsilon_h = h^2(m - h)^{-1}$, and remark that $I_{ih} = I_{(Z_i < \tilde{h})}$ and $\varepsilon_h = o(1)$. We observe that conditions (i), (ii-k) imply, for any given $u \in \{0, k\}$, that

$$EZ_1^u = z_u + o(1), \quad EZ_1^u \tilde{I}_{ih} = z_u + o(1), \quad EZ_1^u \tilde{I}_{1h} = o(1). \tag{65}$$

Now from (26) we derive the inequalities

$$EZ_1Z_2\delta_{12h} \leq EZ_1Z_2\tilde{I}_{1h}\tilde{I}_{2h}(1 - \varepsilon_h) \leq nE\|\xi_{12}\|\tilde{I}_{1h}\tilde{I}_{2h} \leq nE\|\xi_{12}\| \leq EZ_1Z_2. \tag{66}$$

Then invoking in (66) relations $EZ_1 = z_1 + o(1)$ and $EZ_1Z_2\delta_{12h} = z_1^2 + o(1)$, which follow from (65) for $u = 1$, we obtain the relation

$$np_e = nE\|\xi_{12}\| = z_1^2 + o(1). \tag{67}$$
Similarly, under conditions (i), (ii-2), we obtain the relations
\[ n^2 E_{\xi_{12}} I_{\xi_{13}} = z_2^2 + o(1), \] (68)
\[ n^3 E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{24}} = z_2^2 z_3^2 + o(1), \] (69)
and, under conditions (i), (ii-3), we obtain
\[ n^3 E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{4}} = z_3^2 + o(1). \] (70)

Let us prove the bound \( r = o(1) \) in the case where (i), (ii-2) hold and \( E Z^3 = +\infty \). In order to prove \( r = o(1) \) we show \( (69) \). Proceeding as in \( (60), (61), (62), (63) \) and using \( (67) \) we show the bounds \( E_{12} d_1^2 = O(1) \) and \( E_{12} d_2 = O(1) \), which imply the first bound of \( (69) \). Next we show the second relation of \( (69) \). In view of \( (55) \) and \( (67) \) it suffices to prove that \( \limsup n^3 \kappa^* = +\infty \). In the proof we proceed similarly as in \( (64) \) above, but now we use the product \( I_{1h} I_{2h} I_{3h} I_{4h} \) instead of \( I_{1} I_{2} I_{3} I_{4} \). We obtain
\[ n^3 \kappa^* \geq \left( E_{1h} Z_{1}^3 \right) \left( E_{2h} Z_{2} \right)^3 (1 - \varepsilon_h)^3. \]
Here \( E_{2h} Z_{2} = \varepsilon_h + 1 \), see \( (65) \). Furthermore, under conditions (i) and \( E Z^3 = +\infty \) we have \( E_{1h} Z_{1}^3 \rightarrow +\infty \). Hence, \( n^3 \kappa^* \rightarrow +\infty \).

Now we prove the bound \( r = o(1) \) in the case where (i), (ii-3) hold. We shall show that
\[ \eta = o(1) \quad \text{and} \quad \liminf \xi > 0. \] (71)

Let us prove the second inequality of \( (71) \). Combining the first identity of \( (63) \) with \( (67) \) and \( (68) \) we obtain
\[ E_{12} d_1^2 = z_2 + o(1). \] (72)
Next, combining \( (55) \) with \( (67) \) and \( (70) \) we obtain
\[ E_{12} (d_1^2) = E_{12} d_1^2 + z_1 z_3 + o(1). \] (73)

It follows from \( (72), (73) \) and the inequality \( z_1 z_3 \geq z_2^2 \), which follows from Hölder’s inequality, that \( \xi = z_2 + z_1 z_3 = z_2^2 + o(1) \geq z_2 + o(1) \). We have proved the second inequality of \( (71) \).

Let us prove the first bound of \( (71) \). In view of \( (60) \) and \( (72) \) it suffices to show that
\[ p_e^{-1} n^2 \kappa^* = z_2^2 + o(1), \quad p_e^{-1} n \kappa^* = o(1). \] (74)

We note that the first relation of \( (71) \) follows from \( (67), (69) \). To prove the second bound of \( (71) \) we need to show that \( \kappa^* = o(n^{-2}) \). We split
\[ \kappa^* = E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{23}}^2 + E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{23}} \]
and estimate, using \( (30) \) and \( (31) \),
\[ E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{23}}^2 \leq E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{23}} \leq \tilde{m}^{-1} m^{-1} Y_{2}^2 X_{1} X_{2} Y_{3} = O(n^{-2-s^{-1}}). \]
In the last step we combined the inequality \( Y_{1}^u \leq X_{1}^u (X_{1} \geq s) \) and \( (27) \). Furthermore, using the right-hand side inequality of \( (35) \) we write
\[ E_{\xi_{12}} I_{\xi_{13}} I_{\xi_{23}} \leq m^{-1} Y_{3} + E_{\xi_{12}} \tilde{P}_s(B_3) \]
and estimate, by \( (29) \) and \( (32) \),
\[ E_{\xi_{12}} \tilde{m}^{-1} Y_{3} \leq \tilde{m}^{-1} E_{\xi_{12}} Y_{2} Y_{3} = O(n^{-2} s^{-1/2}), \]
\[ E_{\xi_{12}} \tilde{P}_s(B_3) \leq \tilde{m}^{-1} m^{-1} E_{\xi_{12}} Y_{2} Y_{3} X_{3} (X_{1}^{s+1} + X_{2}^{s+1}) = O(n^{-2-s^{-1}}). \]

\[ \square \]
Proof of (12), (13) follow from (22) and (33), (10). Before the proof of (12) and (13) we introduce some notation. Given two sequences of real numbers \( \{A_n\} \) and \( \{B_n\} \) we write \( A_n \simeq B_n \) (respectively \( A_n \simeq 0 \)) to denote the fact that \( A_n - B_n = o(n^{-2}) \) (respectively \( A_n = o(n^{-2}) \)). We denote \( p_* = P(v_1 \sim v_2, d'_1 = k) \) and introduce random variables, see (23), \( \|.\| = \|1_{I_{12}}\|_2, \|.\| = 1 - \|.\| \), and

\[
\tau_1 = 1_{I_{12}}, \quad \tau_2 = 1_{I_{12}^c}, \quad \tau_3 = 1_{I_{12} \cap I_{13} \cap I_{12}^c}, \quad \tau_4 = 1_{I_{12}^c \cap I_{13}^c}, \quad \tau_5 = 1_{I_{12}^c}.
\]

Here \( \tau = 1_{I_{23} \cap (d'_1 = k)} \) and \( \tau^* = 1_{I_{11}} 1_{I_{23}} 1_{(d'_1 = k - 1)} \), and \( d'_1 = \sum_{4 \leq t \leq n} I_{I_{11}} \). We remark that the identity \( 1_{I_{12}} = 1_{I_{12}^c} + 1_{I_{12}^c} \) in combination with \( 1 = 1_{I_{13}} + 1_{I_{13}} \) implies

\[
\tau_1 = \tau_2 + \tau_3 + \tau_4.
\]

Proof of (12), (13). In view of (22) we can write

\[
h_{k+1} = E_{12}(d_{12}|d'_{1} = k) = p_*^{-1}E_{12}1_{I_{12}} 1_{(d'_{1} = k)} d_{12}, \quad b_{k+1} - 1 = E_{12}(d'_{12}|d'_{1} = k) = p_*^{-1}E_{12}1_{I_{12}^c} 1_{(d'_{1} = k)} d'_{2}.
\]

Furthermore, by the symmetry property, we have

\[
E_{12}1_{I_{12}} 1_{(d'_{1} = k)} d_{12} = (n - 2)E_{12}1_{I_{12}} \tau^*, \quad E_{12}1_{I_{12}^c} 1_{(d'_{1} = k)} d'_{2} = (n - 2)E_{12} \tau_1.
\]

We note that (76), (77) combined with the identities \( 1_{I_{12}} \tau^* = \tau_4 + \tau_5 \) and (75) imply

\[
h_{k+1} = (n - 2)p_*^{-1}E(\tau_4 + \tau_5), \quad b_{k+1} - 1 = (n - 2)p_*^{-1}E(\tau_2 + \tau_3 + \tau_4),
\]

and observe that (12), (13) follow from (78) and the relations

\[
p_* = n^{-1}(k + 1)p_{k+1} + o(n^{-1}), \quad E\tau_3 = n^{-2}a^{-1}(k + 1)(a_2 - a_1)p_{k+1} + o(n^{-2}), \quad E\tau_4 = n^{-2}a^{-1}k a_1 p_{k} + o(n^{-2}), \quad E\tau_5 = o(n^{-2}), \quad i = 2, 5.
\]

It remains to prove (79), (80), (81), (82). In order to show (82) we combine the inequalities

\[
\tau_1 \leq 1_{I_{12}} 1_{I_{23}} = 1_{I_{12}} 1_{I_{23}} (1^* + 1^*) \leq 1_{I_{12}} 1_{I_{23}} 1^* + 1_{I_{12}^c} 1_{I_{23}^c} 1^*
\]

with the inequalities, which follow from (30) and (31),

\[
E I_{I_{12}^c} 1_{I_{23}} 1^* \leq E P(I_{12}^c) \hat{P}_*(I_{23}^c) 1^* \leq (m^2 m^{-1})^{-1} E Y_1 Y_2 Y_3 X_1 X_2 1^* = O(n^{-2}m^{-1/2})
\]

In the last step we used the bound \( E Y_1 Y_2 Y_3 1^* = o(1) \), which holds under conditions (i), (ii-2).

Proof of (81). We have

\[
E\tau_4 = E I_{I_{12}^c} \hat{P}_*(I_{23}^c \cap I_{13}^c) \hat{P}_*(d'_{1} = k - 1).
\]
We first replace in \((83)\) the probability \(\hat{P}_* (\mathcal{E}_3 \cap \mathcal{E}_{13})\) by \(\hat{P}_* (\mathcal{A}_3) = Y_3 / \bar{m}\) using \((17)\), \((18)\). Then we replace \(\hat{P}_* (d'_i = k - 1)\) by \(f_{k-1}(\beta^{-1} a_1 Y_1)\) using Lemma \(5\). Finally, we replace \(I_{\mathcal{E}_{12}}\) by \(\bar{m}^{-1} Y_1 Y_2\) using \((29)\). We obtain

\[
E_\tau_4 \simeq \bar{m}^{-1} E I_{\mathcal{E}_{12}} Y_3 \hat{P}_* (d'_i = k - 1) \quad (84)
\]

\[
\simeq \bar{m}^{-1} E I_{\mathcal{E}_{12}} Y_3 f_{k-1} (\beta^{-1} a_1 Y_1) \quad (85)
\]

\[
\simeq \bar{m}^{-2} E Y_1 Y_2 f_{k-1} (\beta^{-1} a_1 Y_1) \quad (86)
\]

\[
= n^{-2} \beta^{-2} \bar{a}_1^2 E Y_1 f_{k-1} (\beta^{-1} a_1 Y_1). \quad (87)
\]

Here \((84)\) follows from the bound \(E I_{\mathcal{E}_{12}} \hat{P}_* (B_3) = o(n^{-2})\). To show this bound we write

\[
I_{\mathcal{E}_{12}} \hat{P}_* (B_3) = I_{\mathcal{E}_{12}} \hat{P}_* (B_3) (\mathbb{I}^* + \mathbb{I}^t) \leq I_{\mathcal{E}_{12}} \hat{P}_* (B_3) \mathbb{I}^* + I_{\mathcal{E}_{12}} \hat{P}_* (B'_3) \mathbb{I}^t,
\]

where \(B'_3 = \{D_3 \cap (D_1 \cup D_2) | \geq s\}\), and estimate, see \((29)\), \((32)\), \((34)\),

\[
E I_{\mathcal{E}_{12}} \hat{P}_* (B'_3) \mathbb{I}^t \leq \bar{m}^{-2} m^{-1} E Y_1 Y_2 Y_3 (X_1^{s+1} + X_2^{s+1}) \mathbb{I}^t
\]

\[
\leq \bar{m}^{-2} m^{-3/4} E Y_1 Y_2 Y_3 (X_1^{s+1} + X_2^{s+1})
\]

\[
= O(n^{-2} m^{-3/4}),
\]

\[
E I_{\mathcal{E}_{12}} \hat{P}_* (B'_3) \mathbb{I}^t \leq \bar{m}^{-2} E Y_1 Y_2 Y_3 (X_1^{s+1} + X_2^{s+1}) \mathbb{I}^t
\]

\[
\leq o(n^{-2}).
\]

Furthermore, \((85)\) follows from the bounds \(E I_{\mathcal{E}_{12}} Y_3 R_j^* = o(n^{-1}), 1 \leq j \leq 4\), see \((33)\). We show these bound using \((29)\). For \(1 \leq j \leq 3\) the proof is obvious. For \(j = 4\) we need to show that \(E I_{\mathcal{E}_{12}} Y_1^2 Y_3 = o(1)\). For this purpose we write (using the inequality \(I_{\mathcal{I}_1} Y_1 \leq I_{\mathcal{I}_1} m^{s/4}\))

\[
I_{\mathcal{E}_{12}} Y_1^2 Y_3 = I_{\mathcal{E}_{12}} Y_1^2 Y_3 (I_{\mathcal{I}_1} + \mathbb{I}) \leq m^{s/4} I_{\mathcal{E}_{12}} Y_1 Y_3 I_{\mathcal{I}_1} + Y_1^2 Y_3 \mathbb{I}_1
\]

and note that the expected values of both summands in the right hand side tend to zero as \(n \to +\infty\). Finally, \((86)\) follows from \((29)\) and implies directly \((87)\).

Now we derive \((81)\) from \((87)\). We observe that

\[
k^{-1} \beta^{-1} a_1 E Y_1 f_{k-1} (\beta^{-1} a_1 Y_1) = E f_k (\beta^{-1} a_1 Y_1) \to E f_k (z_1 Z)
\]

(here we use the fact that the weak convergence of distributions \(i\) implies the convergence of expectations of smooth functions). Furthermore, by \((5)\), \(E f_k (z_1 Z) = p_k\). Hence, \((87)\) implies

\[
E_\tau_4 \simeq n^{-2} \beta^{-1} \bar{a}_1 E f_k (z_1 Z) = n^{-2} \beta^{-1} \bar{a}_1 p_k.
\]

**Proof of \((87)\).** Introduce the event \(\mathcal{C} = \{D_3 \cap (D_1 \setminus D_2) = \emptyset\}\), probability \(\tilde{P}_* (\mathcal{E}_3 \cap \mathcal{C} \cap \overline{\mathcal{E}_{13}})\), and random variable \(H = \bar{m}^{-1} (Y_2 - 1) Y_3\). We obtain \((80)\) in several steps. We show that

\[
E_\tau_3 \simeq E I_{\mathcal{E}_{12}} \mathbb{I}_1 (d'_i = k) \quad (88)
\]

\[
\simeq E I_{\mathcal{E}_{12}} H \mathbb{I}_1 (d'_i = k) \quad (89)
\]

\[
\simeq E I_{\mathcal{E}_{12}} H f_k (\beta^{-1} a_1 Y_1) \quad (90)
\]

\[
\simeq \bar{m}^{-1} E Y_1 Y_2 H f_k (\beta^{-1} a_1 Y_1) \quad (91)
\]

\[
\simeq \bar{m}^{-2} (a_2 - a_1) (k + 1) \beta p_{k+1}. \quad (92)
\]
We note that (88) is obtained by replacing \( \mathbb{I}_{E_{23}} \) by the product \( I_{E_{12}} \mathbb{I}_{C} \) in the formula defining \( \tau_3 \).

In order to bound the error of this replacement we apply the inequality

\[
\mathbb{I}_{E_{23}} \mathbb{I}_{C} \leq \mathbb{I}_{E_{12}} \mathbb{I}_{C} + \mathbb{I}_{B_3},
\]

(93)

and invoke the bound \( \mathbb{E} \mathbb{I}_{E_{12}} \mathbb{I}_{E_{11}} \mathbb{I}_{B_3} \{ d_i' = k \} \leq \mathbb{E} \mathbb{I}_{E_{12}} \mathbb{P}_* (B_3) = o(n^{-2}) \), see the proof of (84) above.

We remark that the left hand side inequality of (93) is obvious. The right hand side inequality holds because the event \( E_{12} \) implies \( (E_{12} \cap C) \cup B_3 \).

In (89) we replace \( \tilde{p} \) by \( H \). To prove (89) we show that

\[
\mathbb{E} \mathbb{I}_{E_{12}} \tilde{p} \mathbb{I}_{\{ d_i' = k \}} \mathbb{I}_1 \simeq \mathbb{E} \mathbb{I}_{E_{12}} \tilde{p} \mathbb{I}_{\{ d_i' = k \}} \mathbb{I}_1 \simeq \mathbb{E} \mathbb{I}_{E_{12}} H \mathbb{I}_{\{ d_i' = k \}} \mathbb{I}_1 \simeq \mathbb{E} \mathbb{I}_{E_{12}} H \mathbb{I}_{\{ d_i' = k \}}.
\]

(94)

We remark that the first and third relations follow from the simple bounds, see (29), (30),

\[
\mathbb{E} \mathbb{I}_{E_{12}} \tilde{p} \mathbb{I}_{\{ d_i' = k \}} \mathbb{I}_1 \leq \mathbb{E} \mathbb{I}_{E_{12}} \mathbb{I}_{E_{23}} \mathbb{I}_1 \leq \tilde{m}^{-2} \mathbb{E} Y_1 Y_2 Y_3 \mathbb{I}_1 = o(n^{-2}),
\]

\[
\mathbb{E} \mathbb{I}_{E_{12}} H \mathbb{I}_{\{ d_i' = k \}} \mathbb{I}_1 \leq \tilde{m}^{-1} \mathbb{E} Y_1 Y_2 |H| \mathbb{I}_1 = o(n^{-2}).
\]

In order to show the second relation of (94) we split

\[
\tilde{p} = \mathbb{P}_* (E_{13} \cap E_{23} \cap C) \mathbb{P}_* (E_{23} \cap C) \mathbb{P}_* (C) =: \tilde{p}_1 \tilde{p}_2 \tilde{p}_3
\]

(95)

and observe that \( \tilde{p}_1 \) is the probability that the random subset \( D_3 \cap D_2 \) (of size \( s \)) of \( D_2 \) does not match the subset \( D_1 \cap D_3 \) (we note that \( |D_1 \cap D_2| = s \), since the event \( E_{12} \) holds). Hence, \( \tilde{p}_1 = 1 - Y_2^{-1} \). Furthermore, from (29) we obtain

\[
\tilde{p}_3 = 1 - \mathbb{P}_* (D_3 \cap (D_1 \setminus D_2) \neq \emptyset) \geq 1 - \mathbb{P}_* (D_3 \cap D_1 \neq \emptyset) \geq 1 - \tilde{m}^{-1} X_1 X_3.
\]

(96)

Finally, \( \tilde{p}_2 \) is the probability that the random subset \( D_3 \) of \( W \setminus (D_1 \setminus D_2) \) intersects with \( D_2 \) in exactly \( s \) elements. Taking into account that the event \( E_{12} \) holds we obtain (see (29), (33))

\[
\tilde{m}^{-1} Y_2 Y_3 \tilde{p}_2 \tilde{I}_3 (1 - m^{1/2}/(m' - X_1)) \leq \tilde{p}_2 \leq \tilde{m}^{-1} Y_2 Y_3.
\]

(97)

Here we denote \( \tilde{m}_1 \equiv (m'/s) \) and \( m' = |W \setminus (D_1 \setminus D_2)| = m - (X_1 - s) \). We remark that on the event \( \{ X_1 < m^{1/4} \} \) we have \( m' = m - O(m^{3/4}) \). Hence, for large \( m \), (97) implies

\[
\tilde{m}^{-1} Y_2 Y_3 \delta_{23} \tilde{I}_1 \leq \tilde{p}_2 \tilde{I}_1 \leq \tilde{m}^{-1} Y_2 Y_3 \tilde{I}_1 (1 + m^{-3/4} (s + o(1))).
\]

(98)

Now, collecting (96), (98), and the identity \( \tilde{p}_1 = 1 - Y_2^{-1} \) in (95) we obtain the inequalities

\[
\mathbb{I}_{E_{12}} \mathbb{I}_1 \delta_{23} H (1 - m^{-1} X_1 X_3) \leq \mathbb{I}_{E_{12}} \mathbb{I}_1 \tilde{p} \leq \mathbb{I}_{E_{12}} \mathbb{I}_1 H (1 + O(m^{-3/4}))
\]

(99)

that imply the second relation of (94).

In the proof of (90), (91), (92) we apply the same argument as in (85), (86), (87) above.

Proof of (77). We write

\[
p_* = \mathbb{E} \mathbb{I}_{E_{12}} \mathbb{P}_* (d_i' = k) = \mathbb{E} \mathbb{P} (E_{12}) \mathbb{P}_* (d_i' = k)
\]

and in the integrand of the right hand side we replace \( \mathbb{P}_* (d_i' = k) \) by \( f_k (\beta^{-1} a_1 Y_1) \) and \( \mathbb{P} (E_{12}) \) by \( \tilde{m}^{-1} Y_1 Y_2 \) using (35) and (29), respectively.
4.2 Passive graph

Before the proof we introduce some more notation. Then we present auxiliary lemmas. Afterwards we prove Theorems 3, 4.

By $E_{ij}$ we denote the conditional expectation given the event $\mathcal{E}_{ij} = \{w_i \sim w_j\}$. Furthermore, we denote

$$p_e = P(\mathcal{E}_{ij}), \quad D_{ij} = D_i \cap D_j, \quad X_{ij} = \lfloor D_{ij} \rfloor, \quad x_i = E(X_i), \quad y_i = E(X_i^1), \quad u_i = E(Z_i).$$

For $w \in W$, we denote $I_i(w) = I_{\{w \in D_i\}}$ and $I_i(w) = 1 - I_i(w)$, and introduce random variables

$$E(w) = \sum_{1 \leq i \leq n} l_i(w), \quad l_i(w) = I_i(w)(X_i - 1),$$

$$Q(w) = \sum_{1 \leq i < j \leq n} q_{ij}(w), \quad q_{ij}(w) = I_i(w)I_j(w)(X_i - 1),$$

$$S_1 = \sum_{1 \leq i \leq n} s_i, \quad S_2 = \sum_{1 \leq i < j \leq n} s_is_j, \quad s_i = I_i(w_1)I_i(w_2).$$

We say that two vertices $w_i, w_j \in W$ are linked by $D_k$ if $w_i, w_j \in D_k$. In particular, a set $D_k$ defines $(X_k^2)$ links between its elements. We note that $L_t = L(w_t)$ counts the number of links incident to $w_t$. Similarly, $Q_t = Q(w_t)$ counts the number of different parallel links incident to $w_t$ (a parallel link between $w$ and $w'$ is realized by a pair of sets $D_i, D_j$ such that $w', w'' \in D_i \cap D_j$). Furthermore, $S_1$ counts the number of links connecting $w_1$ and $w_2$ and $S_2$ counts the number of different pairs of links connecting $w_1$ and $w_2$. We denote the degree $d_t = d(w_t)$ and introduce event $\mathcal{L}_t = \{L_t = d_t\}$.

Lemma 7. The factorial moments $\bar{\delta}_{s_t} = E(d_{st})$ and $u_t = E(Z_t)$ satisfy the identities

$$\bar{\delta}_{s_t} = \beta^{-1}u_2, \quad \bar{\delta}_{s_2} = \beta^{-2}u_2^2 + \beta^{-1}u_3, \quad \bar{\delta}_{s_3} = \beta^{-3}u_2^3 + 3\beta^{-2}u_2u_3 + \beta^{-1}u_4. \quad (100)$$

Proof of Lemma 4. We only show the third identity of (100). The proof of the first and second identities is similar, but simpler. We color $z = z_1 + \cdots + z_r$ distinct balls using $r$ different colors so that $z_i$ balls receive $i$-th color. The number of triples of balls

$$(3)_3 = \sum_{i \leq r} (z_i)_3 + \sum_{i \leq r} (z_j)_2 \sum_{j \leq r \setminus \{1\}} z_j + \sum_{\{i, j, k\} \subset \{r\}} z_iz_jz_k. \quad (101)$$

Here the first sum counts triples of the same color, the second sum counts triples having two different colors, etc. We apply (101) to the random variable $(d_{st})$, where $d_{st} = \tilde{Z}_1 + \cdots + \tilde{Z}_\Lambda$. We obtain, by the symmetry property,

$$E(d_{st}) = E\Lambda E(\tilde{Z}_1^3) + E\Lambda E(\tilde{Z}_1^2)E\tilde{Z}_1 + E(\Lambda)^3(\bar{\delta})^3.$$

Now invoking the simple identities $E(\Lambda) = (E\Lambda)^t = (u_1\beta^{-1})^t$ and $E(\tilde{Z}) = u_{t+1}u_1^{-1}$ we obtain the third identity of (100).

Lemma 8. We have

$$EES_1 = n^{-1}\beta_n^{-2}y_2 + R'_1, \quad (102)$$

$$EL_1S_1 = n^{-1}\beta_n^{-2}(y_2 + y_3) + n^{-1}\beta_n^{-3}y_2^2 + R'_2, \quad (103)$$

$$EI_1L_1S_1 = n^{-1}\beta_n^{-2}(y_2 + 3y_3 + y_4) + 3n^{-1}\beta_n^{-3}y_2^2(y_2 + y_3) + n^{-1}\beta_n^{-4}y_2^3 + R'_3, \quad (104)$$

$$EI_1L_2S_1 = n^{-1}\beta_n^{-2}(y_4 + 3y_3 + y_2) + 2n^{-1}\beta_n^{-3}y_2^2(y_3 + y_2) + n^{-1}\beta_n^{-4}y_2^3 + R'_4. \quad (105)$$
where, for some absolute constant $c > 0$, we have $|R'_n| \leq cn^{-2}\beta_n^{-3}x_2$ and

| $R'_n$ | $\leq cn^{-2}(\beta_n^{-3} + \beta_n^{-4}x_4)$,  
| $R'_n$ | $\leq cn^{-2}\beta_n^{-3}(1 + \beta_n^{-1} + x_2 + \beta_n^{-2}x_2)x_4$,  
j = 3, 4.

Proof of Lemma 3. We only show (105). The proof of remaining identities is similar or simpler. We write, for $t = 1, 2$, $L_t = L(w_t) = l_1(w_t) + L'_t$ and denote $\tau_j = E\tau_j = (m)^{-1}(X_j)_2$. We have, by the symmetry property,

\[
\begin{align*}
    EL_1L_2S_1 &= nES_1L_1L_2, \\
    ES_1L_1L_2 &= ES_1l_1(w_1)l_1(w_2) + 2ES_1l_1(w_1)L'_2 + ES_1L'_1L'_2, \\
    ES_1L'_1L'_2 &= (n - 1)ES_1l_2(w_1)l_2(w_2) + (n - 1)ES_1l_2(w_1)l_3(w_2), \\
    ES_1l_1(w_1)L'_2 &= (n - 1)ES_1l_1(w_1)l_2(w_2).
\end{align*}
\]

A straightforward calculation shows that

\[
\begin{align*}
    \tilde{E}s_1l_1(w_1)l_1(w_2) &= (X_1 - 1)^2\tau_1 = (m)^{-1}((X_1)_4 + 3(X_1)_3 + (X_1)_2), \\
    \tilde{E}s_1l_1(w_1)l_2(w_2) &= m^{-1}(X_1 - 1)(X_2 - 1)X_2\tau_1 = m^{-1}(m)_2^{-1}((X_1)_3 + (X_1)_2)(X_2)_2, \\
    \tilde{E}s_1l_2(w_1)l_2(w_2) &= (X_2 - 1)^2\tau_2 = (m)^{-2}(X_1)_2((X_2)_4 + 3(X_2)_3 + (X_2)_2), \\
    \tilde{E}s_1l_2(w_1)l_3(w_2) &= m^{-2}(X_2)_2(X_3)_2\tau_1 = m^{-2}(m)_2^{-1}(X_1)_2(X_2)_2(X_3)_2.
\end{align*}
\]

Invoking these expressions in the identity $ES_1l_i(w_1)l_j(w_u) = E\tilde{E}s_1l_i(w_1)l_j(w_u)$ we obtain expressions for the moments $ES_1l_i(w_1)l_j(w_u)$. Substituting them in (106) we obtain (105). \[\square\]

Lemma 9. We have

\[
\begin{align*}
    ES_2 &\leq 0.5n^{-2}\beta_n^{-4}x_2^2, \\
    EL_1S_2 &\leq n^{-2}\beta_n^{-4}x_2x_3 + 0.5n^{-2}\beta_n^{-5}x_3^2, \\
    EQ_1S_1 &\leq n^{-2}\beta_n^{-4}x_2x_3 + 0.5n^{-2}\beta_n^{-5}x_3^2, \\
    ENQ_2S_1 = ENQ_2S_1 &\leq n^{-2}\beta_n^{-4}(2x_2x_4 + 1.5\beta_n^{-1}x_2^2x_3 + 0.5\beta_n^{-2}x_2^3) + n^{-3}\beta_n^{-6}x_2^2x_4, \\
    ENQ_1S_1 &\leq n^{-2}\beta_n^{-4}(x_3^2 + x_2x_4) + 2.5n^{-2}\beta_n^{-5}x_2x_3 + 0.5n^{-2}\beta_n^{-6}x_4^2, \\
    ENQ_1S_1 &\leq n^{-2}\beta_n^{-4}(x_3^2 + x_2x_4) + 2.5n^{-2}\beta_n^{-5}x_3^2x_3 + 0.5n^{-2}\beta_n^{-6}x_2^2x_3, \\
    ENQ_1S_2 &\leq n^{-2}\beta_n^{-4}(x_2x_4 + x_3^2 + x_2x_3 + 0.5\beta_n^{-2}x_2^3) + n^{-3}\beta_n^{-6}x_2x_4, \\
    EQ_1S_1(w_1)(X_1 - 1)_2 &\leq 4n^{-2}\beta_n^{-3}y_2(y_3 + y_4 + \beta^{-1}y_2y_3).
\end{align*}
\]

Proof of Lemma 3. We only prove (110). The proof of remaining inequalities is similar or simpler. In the proof we use the shorthand notation $l_i = l_i(w_1)$ and $q_i = q_i(w_2)$.

To prove (110) we write, by the symmetry property,

\[
\begin{align*}
    EQ_2L_1S_1 &= \binom{n}{2}EQ_2L_1S_1, \\
    EQ_2L_1S_1 &= 2EQ_2l_1S_1 + (n - 2)EQ_2l_3S_1, \\
    EQ_2l_1S_1 &= EQ_2l_1S_1 + EQ_2l_2S_2 + (n - 2)EQ_2l_1S_3, \\
    EQ_2l_3S_1 &= EQ_2l_3S_1 + EQ_2l_3S_2 + EQ_2l_3S_3 + (n - 3)EQ_2l_3S_4
\end{align*}
\]

and invoke the inequalities

\[
\begin{align*}
    EQ_2l_1S_j &\leq m^{-4}x_2x_4, \\
    EQ_2l_3S_j &\leq m^{-5}x_2^2x_3, \\
    EQ_2l_3S_j &\leq m^{-6}x_2^2x_4, \\
    EQ_2l_3S_3 &\leq m^{-5}x_2^2x_3, \\
    EQ_2l_3S_4 &\leq m^{-6}x_4^4, \\
    j &= 1, 2.
\end{align*}
\]

18
These inequalities follow from the identity $E_{q_{12}l} s_j = E\tilde{E}_{q_{12}l} s_j$ and the upper bounds for the conditional expectations $E_{q_{12}l} s_j$ constructed below.  
For $i = 1$ and $j = 1, 2$, we have 
\[ E_{q_{12}l} s_j \leq E_{q_{12}l} = (X - 1)\tilde{E}_{q_{12}l}(w_1) \leq m^{-4}X_1^2X_2^2. \] (115) 
In the first inequality we use $s_j \leq 1$. In the second inequality we use the inequality 
\[ \tilde{E}_{q_{12}l}(w_1) = \eta \xi \leq m^{-4}X_1^2X_2^2. \] (116) 
Here $\eta = \tilde{E}(X_1 - 1)\ll_1(w_1)\ll_1(w_2)\ll_2(w_2) = 1$ and $\xi = \tilde{P}(\ll_1(w_1)\ll_1(w_2)\ll_2(w_2) = 1)$. We note that given $X_1, X_2, D_1$, the random variable $\eta$ evaluates the expected number of elements of $D_1 \setminus \{w_2\}$ that belong to the random subset $D_2 \setminus \{w_2\}$ (of size $X_2 - 1$). Hence, we have $\eta = (m - 1)^{-1}(X_1 - 1)(X_2 - 1)$. Furthermore, the probability 
\[ \xi = \tilde{P}(w_1, w_2 \in D_1) \times \tilde{P}(w_2 \in D_2) = \frac{(X_1)}{m} \times \frac{X_2}{m}. \] 
Combining obtained expressions for $\eta$ and $\xi$ we easily obtain (116). For $i = 1$ and $j = 3$, we write, by the independence of $D_1, D_2$ and $D_3$, 
\[ E_{q_{12}l} s_3 = (E_{q_{12}l})(E_{s_3}) \leq m^{-6}X_1^2X_2^2X_3^2. \] In the last step we used $E_{s_3} = (X_3)2(m/2)$ and $E_{q_{12}l} \leq m^{-4}X_1^2X_2^2$, see (115). For $i = 3$ and $j = 1, 2$, we write $E_{q_{12}l} s_j = (E_{q_{12}l}(w_1))(E_{s_3})$, by the independence of $D_1, D_2$ and $D_3$. Invoking the inequalities 
\[ E_{l_3} = (X_3 - 1)\tilde{P}(w_1 \in D_3) \leq m^{-1}X_2^2, \quad E_{q_{12}l}(w_1) = \eta \xi \leq m^{-4}X_1^3X_2^2, \]  
we obtain $E_{q_{12}l} s_1 \leq m^{-5}X_1^3X_2^2X_3^2$. Similarly, $E_{q_{12}l} s_2 \leq m^{-5}X_1^3X_2^3X_3^2$. For $i, j = 3$, we split $E_{(q_{12}l)} s_3 = (E_{q_{12}l})(E_{s_3})$ and write $E_{q_{12}l} = \eta \xi_1$. Here 
\[ \eta_1 = \tilde{E}(X_1 - 1)\ll_1(w_2)\ll_2(w_2) = 1, \quad \xi_1 = \tilde{P}(\ll_1(w_2)\ll_2(w_2) = 1). \] Invoking the identities $\eta_1 = (m - 1)^{-1}(X_1 - 1)(X_2 - 1)$ and $\xi_1 = m^{-2}X_1X_2$ we obtain 
\[ E_{q_{12}l} = \eta \xi_1 \leq m^{-3}X_1^2X_2^2. \] (117) 
Combining (117) with the identities $E_{l_3} s_3 = (X_3 - 1)E_{s_3} = (X_3 - 1)(X_3)2(m/2)$, we obtain the inequality $E_{q_{12}l} s_3 \leq m^{-5}X_1^2X_2^2X_3^2$. For $i = 3$ and $j = 4$ we write by the independence of $D_1, D_2, D_3, D_4$, and (117) 
\[ E_{q_{12}l} s_4 = (E_{q_{12}l})(E_{l_3})(E_{s_4}) \leq (m^{-3}X_1^2X_2^2)(m^{-1}X_2^2)(m^{-2}X_2^3) = m^{-6}X_1^2X_2^2X_3^2X_4^2. \] 
\[ \square \]

Proof of Theorem 3. In order to show (115) we write 
\[ r = \frac{E_{12d_1d_2} - (E_{d_2d_1})^2}{E_{12d_1}^2 - (E_{d_2d_1})^2} = \frac{p_0 E_{12d_1d_2} - (E_{12d_1})^2}{p_0 E_{12d_1}^2 - (E_{12d_1})^2} \] (118) 

19
and invoke the expressions
\[ p_c = \mathbb{E}S_1 + O(n^{-2} \beta_n^{-4}), \quad (119) \]
\[ \mathbb{E}d_1^1 1_{E_{12}} = \mathbb{E}L_1 S_1 + O(n^{-2} \beta_n^{-4}(1 + \beta_n^{-1})), \]
\[ \mathbb{E}d_1^2 1_{E_{12}} = \mathbb{E}L_1^2 S_1 + O(n^{-2} \beta_n^{-4}(1 + \beta_n^{-2})), \]
\[ \mathbb{E}d_1 d_2 1_{E_{12}} = \mathbb{E}L_1 L_2 S_1 + O(n^{-2} \beta_n^{-4}(1 + \beta_n^{-2})). \]

Now the identities of Lemma 8 complete the proof of (15).

Let us prove (119). We first write, by the inclusion-exclusion,
\[ S_1 - S_2 \leq 1_{E_{12}} \leq S_1, \quad (120) \]
\[ L_t - Q_t \leq d_t \leq L_t. \quad (121) \]

Then we derive from (121) the inequalities
\[ 0 \leq L_1 L_2 - d_1 d_2 \leq L_1 Q_2 + L_2 Q_1 \quad \text{and} \quad 0 \leq L_1^2 - d_1^2 \leq 2L_1 Q_1, \quad (122) \]
which, in combination with (120) and (121), imply the inequalities
\[ 0 \leq L_1 S_1 - d_1 1_{E_{12}} \leq L_1 S_2 + Q_1 S_1, \quad (123) \]
\[ 0 \leq L_1^2 S_1 - d_1^2 1_{E_{12}} \leq L_1^2 S_2 + 2L_1 Q_1 S_1, \]
\[ 0 \leq L_1 L_2 S_1 - d_1 d_2 1_{E_{12}} \leq L_1 L_2 S_2 + L_1 Q_2 S_1 + L_2 Q_1 S_1. \]

Finally, invoking the upper bounds for the expected values of the quantities in the right hand sides of (123) shown in Lemma 9 we obtain (119).

Now we derive (109) from (117). Firstly, using the fact that (iii), (v) imply the convergence of moments \( \mathbb{E}(X_i) \rightarrow \mathbb{E}(Z)_i \) for \( i = 2, 3, 4 \), we replace the moments \( y_i \) by \( u_i = \mathbb{E}(Z)_i \) in (115).

Secondly, we replace \( u_i \) by their expressions via \( \delta_{s_i} \). For this purpose we solve for \( w_2, u_3, u_4 \) from (100) and invoke the identities
\[ \overline{\delta}_{s_1} = \delta_{s_1}, \quad \overline{\delta}_{s_2} = \delta_{s_2} - \delta_{s_1}, \quad \overline{\delta}_{s_3} = \delta_{s_3} - 3\delta_{s_2} + 2\delta_{s_1}. \quad (124) \]

For \( \beta_n \rightarrow +\infty \) relation (115) remains valid and it implies \( r = 1 + o(1) \). For \( \beta_n \rightarrow 0 \) the condition \( n \beta_n^3 \rightarrow +\infty \) on the rate of decay of \( \beta_n \) ensures that the remainder terms of (119) and Lemma 8 are negligibly small. In particular, we derive (115) using the same argument as above. Letting \( \beta_n \rightarrow 0 \) in (115) we obtain the bound \( r = o(1) \). \( \square \)

**Proof of Theorem 4.** Before the proof we introduce some notation. We denote
\[ H = \sum_{1 \leq i \leq n} 1_{i} (w_1 + (X_i - 1)_2), \quad p_{k \epsilon} = P(w_2 \sim w_1, d_1 = k). \]

Given \( w_j, w_j \in W \) we write \( d_{ij} = d(w_i, w_j) \). A common neighbour \( w \) of \( w_i \) and \( w_j \) is called black if \( \{w, w_i, w_j\} \subset D_r \) for some \( 1 \leq r \leq n \), otherwise it is called red. Let \( d_{ij}' \) and \( d_{ij}'' \) denote the numbers of black and red common neighbours, so that \( d_{ij}' + d_{ij}'' = d_{ij} \). Let \( w_s \) be a vertex drawn uniformly at random from the set \( W' = W \setminus \{w_1\} \). By \( E_{1s} \) denote the event \( \{w_1 \sim w_s\} \). We assume that \( w_s \) is independent of the collection of random sets \( D_1 \ldots, D_n \) defining the adjacency relation of our graph.

20
In the proof we use the identity, which follows from (102), (119),
\[ p_c = n^{-1} \beta_n^{-2} y_2 + O(n^{-2}). \]  
(125)

We also use the identities, which follow from (100) and (124)
\[ 1 + \beta^{-1} w_2 + u_2^{-1} u_3 = \delta_2 \delta_{d_1}^{-1}, \quad \beta^{-1} u_2 = \delta_{d_1}. \]  
(126)

We remark that (126) in combination with relations \( y_i \to u_i \) as \( n, m \to +\infty \), imply the right hand side relations of (13), (19) and (21).

Now we prove the left hand side relations of (18), (19) and (21), and the relation (20).

In order to show (18) we write \( b = p_c^{-1} E d_{12} \mathbb{E}_{\xi_{12}} \) and invoke identities (119), (103) and (125).

Proof of (12). We write \( h = p_c^{-1} E d_{12} \mathbb{E}_{\xi_{12}} \) and evaluate
\[ E d_{12} \mathbb{E}_{\xi_{12}} = n^{-1} \beta_n^{-2} y_3 + O(n^{-2}). \]  
(127)

Combining (125) with (127) we obtain (19). Let us show (127). Using the identity
\[ d_{12} = d'_{12} + d''_{12} = d'_{12} \mathbb{E}_{\mathcal{L}_1} + d''_{12} \mathbb{E}_{\mathcal{L}_1} + d''_{12} \]  
(128)

we write
\[ E d_{12} \mathbb{E}_{\xi_{12}} = E d'_{12} \mathbb{E}_{\xi_{12}} \mathbb{E}_{\mathcal{L}_1} + R_1 + R_2, \]  
(129)

where \( R_1 = E d''_{12} \mathbb{E}_{\xi_{12}} \mathbb{E}_{\mathcal{L}_1} \) and \( R_2 = E \mathbb{E}_{\mathcal{L}_1} d''_{12} \mathbb{E}_{\xi_{12}} \). Next, we observe that \( E \mathbb{E}_{\mathcal{L}_1} d''_{12} \mathbb{E}_{\xi_{12}} = E \mathbb{E}_{\mathcal{L}_1} d''_{ij} \mathbb{E}_{\xi_{12}}, \) for \( 2 \leq j \leq n \), and write
\[ E \mathbb{E}_{\mathcal{L}_1} d''_{12} \mathbb{E}_{\xi_{12}} = E \mathbb{E}_{\mathcal{L}_1} d''_{1} \mathbb{E}_{\xi_{12}} = E \mathbb{E}_{\mathcal{L}_1} H (m - 1)^{-1}. \]  
(130)

We explain the second identity of (130). We observe that \( H (m - 1)^{-1} \) is the conditional expectation of \( d''_{1} \mathbb{E}_{\xi_{1}}, \) given \( D_1, \ldots, D_n \). Indeed, any pair of sets \( D_i, D_j \) containing \( w_1 \) intersects in the single point \( w_1 \), since the event \( \mathcal{L}_1 \) holds. Consequently, each \( D_i \) containing \( w_1 \) produces \( X_i - 2 \) black common neighbours provided that \( w_1 \) hits \( D_i \). Since the probability that \( w_1 \) hits \( D_i \) equals \( (X_i - 1)/(m - 1) \), the set \( D_i \) contributes (on average) \( (m - 1)^{-1} \mathbb{I}_i(w_1)(X_i - 2) \) black vertices to \( d''_{1} \).

Now, by the symmetry property, we write the right-hand side of (130) in the form
\[ \frac{n}{m - 1} E \mathbb{E}_{\mathcal{L}_1} \mathbb{I}_1(w_1)(X_1 - 1)^2 = \frac{n}{m - 1} E \mathbb{I}_1(w_1)(X_1 - 1)^2 - R_3 = \frac{n}{(m - 1)^2} y_3 - R_3, \]  
(131)

where, \( R_3 = \frac{n}{m - 1} E \mathbb{E}_{\mathcal{L}_1} \mathbb{I}_1(w_1)(X_1 - 1)^2 \). Finally, we observe that (127) follows from (129), (130), (131) and the bounds \( R_i = O(n^{-2}), i = 1, 2, 3 \), which are proved below.

In order to bound \( R_i, i = 1, 2, \) we use the inequalities
\[ d''_{12} \leq d_1 \leq L_1, \quad \mathbb{I}_{\xi_{12}} \leq S_1, \quad \mathbb{I}_{\mathcal{L}_1} = \mathbb{I}_{\{L_1 \neq d_1\}} = \mathbb{I}_{\{Q_1 \geq 1\}} \leq Q_1 \]  
(132)

and write \( R_2 \leq E Q_1 L_1 S_1 \) and \( R_3 \leq n(m - 1)^{-1} E Q_1 \mathbb{I}_1(w_1)(X_1 - 1)^2. \) Then we apply (111) and (114). In order to bound \( R_1 \) we observe, that the number of red common neighbours of \( w_1, w_2 \) produced by the pair of sets \( D_i, D_j \) is
\[ a_{ij} = \mathbb{I}_i(w_1) \mathbb{I}_j(w_2) \mathbb{I}_i(w_1) \mathbb{I}_i(w_2) + \mathbb{I}_j(w_1) \mathbb{I}_i(w_2) \mathbb{I}_i(w_1) \mathbb{I}_j(w_2) \]  
(133)

Hence, on the event \( w_1, w_2 \in D_i \) we have \( d''_{12} \leq \sum_{2 \leq i < j \leq n} a_{ij} \), since elements of \( D_1 \setminus \{w_1, w_2\} \) are black common neighbours of \( w_1, w_2 \). From this inequality and the inequality \( \mathbb{I}_{\xi_{12}} \leq S_1 \) we obtain

\[ R_1 \leq E d''_{12} S_1 = n E d''_{12} s_1 \leq n \left( \frac{n - 1}{2} \right) E s_1 a_{23}. \]  
(133)
Furthermore, invoking in (133) identities

\[ E(s_1 a_{23}) = E(\bar{E}(s_1 a_{23}) = E(\bar{E}s_1)\bar{E}a_{23}) \quad \text{and} \quad s_1 = (X_1)_{2}/(m)_{2} \]

and inequalities

\[ \bar{E}a_{23} = 2\bar{E}I_{2}(w_1)I_{3}(w_2)I_{2}(w_2)X_{23} \leq 2\frac{X_{2}X_{3}(X_{2} - 1)(X_{3} - 1)}{m - 2} \]

we obtain \( R_{1} = O(n^{-2}) \).

**Proof of (20).** In the proof we use the fact that the random vector \((H, L_{1})\) converges in distribution to \((d_{2}, d_{*})\) as \( n \to +\infty \). We recall that \( H \) is described after (130). The proof of this fact is similar to that of the convergence in distribution of \( L_{1} = \sum_{1 \leq i \leq n} I_{i}(w_{1})(X_{i} - 1) \) to the random variable \( d_{*} \), see Theorems 5 and 7 of [6]. We note that the convergence in distribution of \((H, L_{1})\) implies the convergence in distribution of \( H1_{L_{1}=k} \) to \( d_{2}1_{\{d_{*}=k\}} \). Furthermore, since under condition (v) the first moment \( E H \) is uniformly bounded as \( n \to +\infty \) and \( Ed_{2} < \infty \), we obtain the convergence of moments

\[ E H1_{L_{1}=k} \to Ed_{2}1_{\{d_{*}=k\}} \quad \text{as} \quad n \to \infty. \] (134)

In order to prove (20) we write

\[ h_{k} = E(d_{12}|w_1 \sim w_2, d_1 = k) = p_{k}^{-1}E d_{12}I_{\{d_{1}=k\}} \]

and show that

\[ \frac{p_{k}}{n} = km^{-1}P(d_{*} = k) + o(n^{-1}), \] (135)

\[ Ed_{12}I_{\{d_{1}=k\}} = m^{-1}EH1_{L_{1}=k} + o(n^{-1}). \] (136)

We remark that (134) in combination with (135) and (136) implies (20). Let us show (135). In view of the identities \( p_{k} = P(w_{*} \sim w_{1}, d_1 = k) \), \( 2 \leq i \leq n \), we can write

\[ p_{k} = P(w_{*} \sim w_{1}, d_1 = k) = P(w_{*} \sim w_1|d_1 = k)P(d_1 = k). \]

Now, from the simple identity \( P(w_{*} \sim w_{1}|d_1 = k) = k(m - 1)^{-1} \) and the approximation \( P(d_1 = k) = P(d_{*} = k) + o(1) \), see [6], we obtain (135).

Let us show (136). Using (128) we obtain, cf. (129),

\[ Ed_{12}I_{\{d_{1}=k\}} = Ed'_{12}I_{\{d_{1}=k\}} + O(n^{-2}). \] (137)

Furthermore, proceeding as in (130), we obtain

\[ Ed'_{12}I_{\{d_{1}=k\}} = Ed'_{12}I_{\{d_{1}=k\}} + (m - 1)^{-1}EH1_{\{d_{1}=k\}}. \] (138)

Next, we invoke identity \( EH1_{\{d_{1}=k\}} = EH1_{\{L_{1}=k\}} \) and approximate, cf. (131),

\[ (m - 1)^{-1}EH1_{\{L_{1}=k\}} = (m - 1)^{-1}EH1_{\{L_{1}=k\}} + O(n^{-2}). \] (139)

Combining (137), (138) and (139) we obtain (136).

**Proof of (27).** Let \( d_{12} \) denote the number of neighbours of \( w_1 \), which are not adjacent to \( w_2 \), and let \( \bar{h}_{k} = E(d_{12}|w_1 \sim w_2, d_2 = k) \). We obtain (21) from the identity

\[ \bar{h}_{k} = E(d_{1}|w_1 \sim w_2, d_2 = k) = 1 + h_{k} + \bar{h}_{k} \]

22
and the relation $h_k = \beta_n^{-1}y_2 + o(1)$. In order to prove this relation we write
\[
\bar{h}_k = \eta^{-1}_k \tau, \quad \text{where} \quad \tau = E\bar{d}_{12}1\{d_2 = k\},
\]
and combine (135) with the identity
\[
\tau = km^{-1}\beta_n^{-1}1\{d_2 = k\}P(d_{ss} = k) + o(n^{-1}). \tag{140}
\]

It remains to prove (140). In the proof we use the shorthand notation
\[
\eta_i = 1\{w_1\}(w_2)(X_i - 1), \quad \eta_i' = \eta_i1\{\{d_2 = k\}\}I_{L_1}, \quad \eta_i'' = \eta_i1\{\{d_2 = k\}\}.
\]

Let us prove (140). Using the identity $1 = \bar{\eta}_{L_1} + \bar{\eta}_{L_1}$ we write
\[
\tau = E\bar{d}_{12}1\{\{d_2 = k\}\}I_{L_1} + R_4, \quad R_4 = E\bar{d}_{12}1\{\{d_2 = k\}\}I_{L_1}.
\]

Next, assuming that the event $L_1$ holds, we invoke the identity $\bar{d}_{12} = \sum_{1 \leq i \leq n} \eta_i$ and obtain
\[
E\bar{d}_{12}1\{\{d_2 = k\}\}I_{L_1} = E \sum_{1 \leq i \leq n} \eta_i' = nE\eta_1'.
\]

In the last step we used the symmetry property. Furthermore, from the identity
\[
E\eta_1' = E\eta_1'' - R_5, \quad R_5 = E\eta_11\{\{d_2 = k\}\}I_{L_1},
\]
we obtain $\tau = nE\eta_1'' + R_4 - nR_5$. We note that inequalities $\bar{d}_{12} \leq d_1 \leq L_1$ and (132) imply
\[
R_4 \leq EL_1S_1Q_1, \quad R_5 \leq E\eta_1(w_1)(X_1 - 1)S_1Q_1 = n^{-1}EL_1S_1Q_1.
\]

Now, from (111) we obtain $R_4 = O(n^{-2})$ and $R_5 = O(n^{-3})$. Hence, we have $\tau = nE\eta_1'' + o(n^{-2})$.

Finally, invoking the relation
\[
E\eta_1'' = km^{-1}y_2P(d_{ss} = k) + o(n^{-2}), \tag{141}
\]
we obtain (140). To show (111) we write
\[
E\eta_1'' = \kappa E\eta_1, \quad \kappa = E\left(\eta_1\{\{d_2 = k\}\}I_{D_1}\right), \tag{142}
\]
and observe that on the event $w_2 \notin D_1$ the quantity $\kappa$ evaluates the probability of the event $\{w_1 \sim w_2, d_2 = k\}$ in the passive random intersection graph defined by the sets $D_2, \ldots, D_3$ (i.e., the random graph $G^*_1(n - 1, m, P)$). We then apply (135) to the graph $G^*_1(n - 1, m, P)$ and obtain $\kappa = km^{-1}P(d_{ss} = k) + o(n^{-1})$. Here the remainder term does not depend on $D_1$. Substitution of this identity in (142) gives
\[
E\eta_1'' = \left(km^{-1}P(d_{ss} = k) + o(n^{-1})\right)E\eta_1.
\]

The following identities complete the proof of (111)
\[
E\eta_1 = E\|\{w_1\}(X_1 - 1) - E\|\{w_1\}I_{\{w_2\}}(X_1 - 1)
\]
\[
= n^{-1}y_2 - (m)^{-1}(y_3 + y_2).
\]

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