A differential operator for integrating one-loop scattering equations

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\textbf{ABSTRACT:} We propose a differential operator for computing the residues associated with a class of meromorphic $n$-forms that frequently appear in the Cachazo-He-Yuan form of the scattering amplitudes. This differential operator is conjectured to be uniquely determined by the local duality theorem and the intersection number of the divisors in the $n$-form. We use the operator to evaluate the tree-level amplitude of $\phi^3$ theory and the one-loop integrand of Yang-Mills theory from their CHY forms. The method can reduce the complexity of the calculation. In addition, the expression for the 1-loop four-point Yang-Mills integrand obtained in our approach has a clear correspondence with the Q-cut results.

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Cachazo, He and Yuan discovered a succinct form of writing the scattering amplitudes of various (quantum) field theories [1–3]. Not only does it make many duality properties manifest, e.g. the KLT relation [4] between gauge theory and gravity [5] as well as the KK relation [6] and the BCJ relation [7], but it is also a great platform for constructing new theories, with or without Lagrangian descriptions [8]. A flurry of activities ensued.

In four dimensions, tree-level scattering equations can be viewed as the constraints of the Roiban-Spradlin-Volovich (RSV) formula in $\mathcal{N} = 4$ super Yang-Mills [9, 10]. The $\mathcal{N} = 8$ supergravity tree amplitudes, proposed and derived from twistor string theory [11, 12], can also be included in this framework [13–15]. In $d$ dimensions, the polynomial form of the scattering equations is first obtained in [16]. The algebraic varieties corresponding to these homogeneous polynomials are studied by [17].

A few solutions of the tree-level scattering equations in four dimensions in some special kinematic limits are obtained: tree level scattering amplitudes for $n$ gluons and $n$ gravitons are computed in [18], the tree-level scattering equations are solved up to six-points [19], while a general relation between solutions of the tree-level scattering equations is proposed, and checked at special kinematic limits up to six points [20].

MHV tree-level amplitudes of gravity and gluons are obtained from the CHY amplitudes in [21]. A five-point scattering amplitude in Yang-Mills theory is obtained in [22] by exploiting the Vieta formula (relating the sum of the solutions of a polynomial equation to the coefficients of the polynomial). The elimination theory is applied to reduce the polynomials and obtained the residues of the scattering equations in [23–25].

Methods based on algebraic geometry are exploited to improve the efficiency of computations. Companion matrix method [26] and Bezoutian matrix method [27] are used to evaluate the CHY expressions, without solving the scattering equations, and checked against the 5-point amplitude in $\phi^3$ theory analytically, as well as higher point amplitudes numerically. In [28] polynomial reduction techniques are used to cast the scattering equations into the standard basis, called the H-basis. In [29] another prescription for such evaluation is proposed, using the polynomial reduction techniques,
and explicit results are given analytically up to 3-points at one loop in $\phi^3$ theory. These approaches do not depend on the particular theory and the method we are going to propose also belongs to this category.

Building blocks method was proposed by [30] in which higher-point CHY-integrands are reduced a la BCFW to basic building blocks. It was then further developed in [31] to a systematic $\Lambda$-algorithm, and then used in [32] to propose and compute a 1-loop CHY amplitude for the $n$-gon. Integration rules for higher-point amplitudes are derived in [33–38] to facilitate practical computations. High order poles were discussed in [39, 40].

In [41] one loop scattering amplitudes are obtained from tree-level ones in one higher spatial dimension. A universal all-loop CHY form was constructed from tree-level CHY forms in $\phi^3$ theory in [42] and checked against Q-cut results.

These methods have various degrees of success in constructing higher-loop amplitudes in scalar theories as successful methods have been developed to subtract the forward singularities arisen in gluing tree-level amplitudes to form one-loop amplitudes. These methods can hence be generalized in a straightforward way to the computations of scattering amplitudes in gauge theory and gravity at tree-level; but so far no general method is in sight for removing the forward singularities introduced when gluing tree amplitudes to form loop amplitudes in gauge or gravity theories.

On the other hand the scattering equations at loop levels are derived from ambitwistor string theory [43–45]. The CHY expressions are then extended to one and two loops for the bi-adjoint scalar, gauge theory, and gravity in [46–48].

In this paper, we propose a new method, also based on algebraic geometry, to evaluate CHY forms. A differential operator, constructed in a systematic procedure, is conjectured to capture the residue around the contour associated with the scattering equations. A one-to-one correspondence to the Q-cut results for the 1-loop four-point CHY expression in SYM is made manifest by our algorithm. Compared with other algebraic geometry based techniques, we avoids the task of finding the Gröbner basis [49] of the scattering equations. The construction of this operator demands information mostly from the integration contour; and therefore the complexity of the theory-specific factors in CHY forms has little impact on the procedure. The residues at the phantom poles resulted from factorization in the polynomial form of scattering equations naturally vanish in our method.
2 A differential operator for multivariable residues

The CHY-form provides a beautiful and compact expression for scattering amplitudes. Schematically, an $n$-particle scattering process reads

$$\int \frac{d\sigma_1 \cdots d\sigma_n}{\text{vol(Residual Symmetry)}} \delta(\text{Scattering Equations}) I, \tag{2.1}$$

with $\sigma_i$’s being complex variables which can, in turn, be related to the worldsheet coordinates of the vertex operators in string theory. The integrand $I$ depends on the underlying theory. These scattering equations are typically a set of rational equations in $\sigma_i$’s, whose coefficients encoding the dynamic information of the scattering process.

In the language of complex analysis, integrals with delta-functions are equivalent to residues around the contours defined by these delta-function constraints. The CHY-form (2.1) can be casted into a residue associated with a meromorphic form as follows,

$$\oint \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-m}}{h_1 \cdots h_{n-m}} T', \tag{2.2}$$

where $m$ is the number of the residual symmetry generators and $h_i$’s are polynomials originated from the scattering equations. These polynomials are, roughly speaking, the numerators in the polynomial scattering equations.

In this section, we introduce a differential operator for computing such residues.

**Conjecture 2.1.** Given a polynomial ideal $(f_1, f_2 \cdots f_k)$ in variables $z_1, z_2 \cdots z_k$, the polynomials are homogeneous and their degrees are $d_1, d_2, \cdots, d_k$ respectively. If the solution to the corresponding algebraic equations is an isolated point $p$, the residue associated with a meromorphic form has a differential interpretation. Namely, for a holomorphic function $R(z_i)$ in the neighbourhood of the point $p$,

$$\text{Res}_{(f_1, \ldots, f_k)}[R] \equiv \oint \frac{dz_1 \wedge \cdots \wedge dz_k}{f_1 \cdots f_k} R = D[R], \quad D = \sum_{\{s_i\}} a_{s_1s_2\cdots s_k} \partial^{s_1}_{r_1} \partial^{s_2}_{r_2} \cdots \partial^{s_k}_{r_k}, \tag{2.3}$$

where the coefficients $a_{s_1s_2\cdots s_k}$ are constants independent of $z_i$ and $\partial^{s_i}_{r_i} = \frac{\partial^{s_i}}{\partial z_{r_i}}$, $i \in [1, k]$. The summation is taken over all the solutions to the Frobenius equation:

$$\sum_{i=1}^{k} s_i = \sum_{h=1}^{k} d_h - k. \tag{2.4}$$
Furthermore, the differential operator $\mathbb{D}$ is uniquely determined by requiring the residue to satisfy the local duality theorem [50, 51] and to give the correct intersection number of the divisors $D_i = (f_i)$.  

We shall exploit this conjecture to evaluate CHY scattering equation (2.2). Typically, the polynomials $h_i$ in (2.2) are not homogeneous, and therefore include extra poles that are not solutions to the original scattering equations. We call these “spurious poles,” and the locations of them are easy to determine. In Section 4, we shall introduce a “homogenization” procedure to case these polynomials and the resulting integrals to meet the conditions of the conjecture.

The differential operator $\mathbb{D}$ computes the sum of residues around all the solutions of $h_1 = \cdots = h_{n-m} = 0$, including the spurious ones. Therefore it is crucial to demonstrate how to remove the contributions from the spurious poles. A calculation of CHY amplitudes can sometimes involve computing the residues at infinity. This is achieved using our conjecture, as demonstrated in Section 4. By computing residues at the finite poles, the spurious poles and the poles at infinity, scattering amplitude can be conveniently evaluated from the CHY forms.

3 A Warm-up example

In this section we study the 5-point tree-level amplitude in the massless $\phi^3$ theory and use it as a toy model to illustrate the evaluation of the multi-variable contour integrals that often appear in CHY-form by the proposed operator $\mathbb{D}$. We denote this amplitude as $A_{\phi^3}$ and its explicit expression is given in [16, 26, 27],

$$A_{\phi^3} = \oint_{h_1=h_2=0} \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (\sigma_1 - \sigma_2)} + \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (1 - \sigma_1) \sigma_2}.$$  

We remark on the motivation for this conjecture. Let us consider the integral,

$$\oint_{f_1=\cdots=f_k=0} \frac{g(z_1, \cdots, z_k) \, dz_1 \wedge \cdots \wedge dz_k}{f_1 \cdots f_k},$$

where $f_i$’s are homogeneous polynomials in $z_i$’s and the numerator $g$ is a monomial in $z_i$ of degree $M$. The residue is non-vanishing if and only if $M = \sum_{i=1}^k d_{f_i}$ [27]. Therefore, the differential operator that computes the residue can involve only derivatives of degree $M$. This observation generalizes naturally to the case in which the numerator is a polynomial.
where \( h_1, h_2 \) being polynomials in \( \sigma_1 \) and \( \sigma_2 \), are roots of the tree-level scattering equations. Their solutions take the following forms,

\[
\begin{align*}
  h_1 &= k_{13} \sigma_1 + k_{14} \sigma_2 + k_{12}, \\
  h_2 &= k_{45} \sigma_1 + k_{25} \sigma_1 \sigma_2 + k_{35} \sigma_2,
\end{align*}
\]

(3.2) \hspace{1cm} (3.3)

where \( k_{ij} = \frac{1}{2}(k_i + k_j)^2 \).

The two terms in (3.1) can be integrated separately. In the first term we denote the remaining factor in the denominator as \( h_0 = (\sigma_1 - \sigma_2) \). These three polynomials \( h_1, h_2, h_0 \) can be made homogeneous:

\[
\begin{align*}
  \tilde{h}_0 &= (\sigma_1 - \sigma_2), \\
  \tilde{h}_1 &= k_{13} \sigma_1 + k_{14} \sigma_2 + k_{12} \sigma_0, \\
  \tilde{h}_2 &= k_{45} \sigma_0 \sigma_1 + k_{25} \sigma_1 \sigma_2 + k_{35} \sigma_0 \sigma_2,
\end{align*}
\]

by introducing an extra variable \( \sigma_0 \) which will be later integrated out. Hence the first contour integral becomes,

\[
\oint \frac{d\sigma_1 \wedge d\sigma_2}{h_1 h_2 (\sigma_1 - \sigma_2)} = \oint \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{h_1 h_2 (\sigma_0 - 1)} \frac{1}{h_0} = -\oint \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{h_1 h_2 h_0} \frac{1}{(\sigma_0 - 1)},
\]

where in the second equality sign we have used the global residue theorem and the intersecting divisors are all generated by homogeneous polynomials. This contour integral is immediately computed by our conjecture. The corresponding differential operator takes the form,

\[
\mathbb{D} = a_{100} \frac{\partial}{\partial \sigma_1} + a_{010} \frac{\partial}{\partial \sigma_2} + a_{001} \frac{\partial}{\partial \sigma_0}.
\]

(3.5)

The computation of the integral is now translated to finding the values of the coefficients \( a_{100}, a_{010} \) and \( a_{001} \). The local duality theorem requires,

\[
\oint \frac{\tilde{h}_1 d\sigma_1 \wedge \tilde{h}_2 \wedge d\sigma_0}{h_1 h_2 h_0} = 0, \hspace{1cm} \oint \frac{\tilde{h}_0 d\sigma_1 \wedge \tilde{h}_2 \wedge d\sigma_0}{h_1 h_2 h_0} = 0.
\]

(3.6)

The intersection number of the divisors here is 2 and this yields,

\[
\oint \frac{d\tilde{h}_1 \wedge d\tilde{h}_2 \wedge d\tilde{h}_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = \oint \frac{\mathcal{J} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_0}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_0} = 2,
\]

(3.7)
where $J = \det(\frac{\partial h_{\mu}}{\partial \sigma_{j}})$ is the Jacobian of integral parameter transformation. In the language of the differential operator, conditions (3.6) and (3.7) read,

$$\mathbb{D} h_{1} = 0, \quad \mathbb{D} h_{0} = 0, \quad \mathbb{D} J = 2. \quad (3.8)$$

Solving for $a$’s the constraints above, we obtain,

$$a_{100} = -\frac{2k_{12}}{G_{1}}, \quad a_{010} = -\frac{2k_{12}}{G_{1}}, \quad a_{001} = -\frac{2(k_{13} + k_{14})}{G_{1}}.$$

where

$$G_{1} = -2k_{12}^{2}k_{134} + 2k_{13}k_{12}k_{123} + 2k_{14}k_{12}k_{123} + 2k_{13}k_{12}k_{124} + 2k_{14}k_{12}k_{124}.$$

Thus the action of the differential operator gives,

$$\int_{h_{0}=h_{2}=h_{0}=0} \frac{d\sigma_{1} \wedge d\sigma_{2} \wedge d\sigma_{0}}{h_{1}h_{2}h_{0}} \frac{1}{(\sigma_{0} - 1)} = \mathbb{D} \left( \frac{1}{(\sigma_{0} - 1)} \right) = -\frac{a_{001}}{(\sigma_{0} - 1)^{2}} \bigg|_{\sigma_{0} \rightarrow 0} = \frac{2(k_{13} + k_{14})}{G_{1}}. \quad (3.9)$$

Similarly, the second term in (3.1) can also be related to another integral in which the intersecting divisors are originated from homogeneous polynomials only. The residue of the latter is then represented by a second order differential operator $\mathbb{D}$,

$$\mathbb{D} = a_{002} \frac{\partial^{2}}{\partial \sigma_{0}^{2}} + a_{011} \frac{\partial}{\partial \sigma_{0}} \frac{\partial}{\partial \sigma_{2}} + a_{020} \frac{\partial^{2}}{\partial \sigma_{2}^{2}} + a_{101} \frac{\partial}{\partial \sigma_{1}} \frac{\partial}{\partial \sigma_{0}} + a_{110} \frac{\partial}{\partial \sigma_{1}} \frac{\partial}{\partial \sigma_{2}} + a_{200} \frac{\partial^{2}}{\partial \sigma_{1}^{2}}.$$

The residue computed by this operator is,

$$\mathbb{D} \left( \frac{1}{(\sigma_{0} - 1)} \right) = -\frac{32k_{13}k_{14}(k_{13}(k_{124} + k_{134}) - k_{14}k_{123})}{G_{2}},$$

$$G_{2} = 32k_{12}k_{13}k_{14}k_{123}(-k_{14}k_{123} + k_{12}k_{124} + k_{13}k_{124} + k_{12}k_{134} + k_{13}k_{134}).$$

Putting together the two terms, we obtain,

$$A_{\phi^{3}} = -\frac{2(k_{13} + k_{14})}{G_{1}} + \frac{32k_{13}k_{14}(k_{13}(k_{124} + k_{134}) - k_{14}k_{123})}{G_{2}}. \quad (3.10)$$

The expression is identical with those in [16, 26, 27].

## 4 Direct evaluation of one-loop CHY-form

In this section we provide a systematic algorithm that exploits the conjecture (2.1) in the calculation of CHY forms.
Before discussing the technical details, let us first sketch out the steps in words. The polynomial form of scattering equations is our starting point. The general transformations from the standard scattering equations to the polynomial ones are given in [16]. These transformations introduce a Jacobian into the CHY expression that is simply the Vandermonde determinant. The polynomial equations are not entirely equivalent to the original equations, rather, they bring in extra solutions that do not satisfy the original.

As explained in Section 2, the evaluation of a CHY-form boils down to computing the residue (2.2). Since the polynomial equations have extra solutions, (2.2) can be obtained by computing the sum of the residues at all the poles of said polynomials first and then removing the contribution from the extra poles. To calculate these residues directly can be quite demanding and this is where Conjecture 2.1 comes in handy.

Depending on the specific expression of the CHY-form, we may encounter two kinds of meromorphic forms: one that is regular at infinity and one that has non-vanishing residues at infinity. To compute the total residue in the former category, we adopt a straightforward procedure of homogenizing the polynomials such that all the poles are condensed at one single isolated point. Hence the total residue can be immediately determined by our conjecture. The latter category can be attacked in a similar fashion, after we embed the complex manifold on which the corresponding meromorphic form lives into a compact one and make a point at infinity well-defined. This process will be discussed in detail.

As for the residues at the phantom poles, we observe that these poles are trivial to locate, however, for reasons that will become clear later, the aforementioned homogenization does not work for this case. A modified conjecture will be given for computing such residues.

### 4.1 Polynomial scattering equations

An $n$-point scattering amplitude in $D$ dimensions takes the form,

$$
\mathcal{A}_n^{l=1} = \int \frac{d^D \ell}{\ell^2} T_n^{l=1} \text{(kinematics)},
$$

(4.1)

where $\ell$ denotes the loop momentum and the exact expression for the integrand $T_n^{l=1}$ depends on the underlying quantum field theory. As shown in [46–48], in a variety of
theories, the integrand has a CHY representation that schematically reads,
\[
\mathcal{T}_{n}^{l=1} = \oint_{f_{1}=\cdots=f_{n-1}=0} d\sigma_{1} \wedge \cdots d\sigma_{n-1} \frac{\mathcal{N}(\sigma_{i})}{\mathcal{D}(\sigma_{i})},
\]
where \(\mathcal{N}(\sigma_{i})\) and \(\mathcal{D}(\sigma_{i})\) are polynomials in \(\sigma_{i}\)'s and encode the kinematic information of the scattering amplitude. The rational function \(f_{i}\) denotes the \(i\)-th one-loop scattering equation, originally derived in the context of the high-energy limit of string theory in [52], and later re-discovered in ambitwistor string in [43, 44],
\[
f_{i} = \frac{\ell \cdot k_{i}}{\sigma_{i}} + \sum_{j \neq i} k_{i} \cdot k_{j} = 0.
\]
While these equations are difficult to solve analytically, Dolan et al introduced in [16] the following transformations to make them more friendly,
\[
g_{m} = \sum_{i=1}^{n} \sigma_{i}^{m+1} f_{i}, \quad m \in \{-1, 0, \cdots\}. \tag{4.4}
\]
By counting the degrees of freedom, obviously only \(n - 1\) equations are independent. We choose the \((n - 1)\) ones with the lowest degrees, that is, \(g_{i} = 0 \quad (i = 1, \cdots, n - 1)\).\(^2\)
The transformations from \(\{f_{1}, \cdots, f_{n-1}\}\) to \(\{g_{1}, \cdots, g_{n-1}\}\) bring into the integrand the following Jacobian that can be easily computed [16],
\[
\mathcal{J} = \det \begin{pmatrix} \sigma_{1}(\sigma_{1} - \sigma_{n}) & \sigma_{2}(\sigma_{2} - \sigma_{n}) & \cdots & \sigma_{n-1}(\sigma_{n-1} - \sigma_{n}) \\ \sigma_{1}(\sigma_{1}^{2} - \sigma_{n}^{2}) & \sigma_{2}(\sigma_{2}^{2} - \sigma_{n}^{2}) & \cdots & \sigma_{n-1}(\sigma_{n-1}^{2} - \sigma_{n}^{2}) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1}(\sigma_{1}^{n-1} - \sigma_{n}^{n-1}) & \sigma_{2}(\sigma_{2}^{n-1} - \sigma_{n}^{n-1}) & \cdots & \sigma_{n-1}(\sigma_{n-1}^{n-1} - \sigma_{n}^{n-1}) \end{pmatrix}
= \sigma_{1} \sigma_{2} \cdots \sigma_{n-1} \prod_{1<i<j<n} (\sigma_{i} - \sigma_{j}). \tag{4.5}
\]
The explicit expressions for these new equations read,
\[
g_{m} = \sum_{i=1}^{n} p \cdot k_{i} \sigma_{i}^{m} + \sum_{i<j} k_{i} \cdot k_{j} \left( \sum_{r=1}^{m-1} \sigma_{i}^{r} \sigma_{j}^{m-r} \right) = 0, \quad m = 1, 2, \cdots, n - 1. \tag{4.6}
\]
These polynomial equations already provide us a much better platform than \(f_{i}\) since each of them is of degree \(m\) while all the \(f_{i}\)'s give rise to equations of degree \(n - 2\) (with \(^2\) \(g_{-1}\) is still a rational function and \(g_{0} = 0\) is satisfied trivially by the conservation of momentum.
the choice of gauge $\sigma_n = 1$). We can further simplify the equations by applying a few more linear transformations given as follows,

\[ h_1 = g_1, \]
\[ h_2 = g_2 - g_1 \left( \sum_i \sigma_i \right), \]
\[ h_3 = g_3 - g_2 \left( \sum_i \sigma_i \right) + \frac{g_1}{2} \left( \sum_{i \neq j} \sigma_i \sigma_j \right), \]
\[ \cdots \]
\[ h_{n-1} = g_{n-1} - g_{n-2} \left( \sum_i \sigma_i \right) + \frac{g_{n-3}}{2} \left( \sum_{i_1 \neq i_2} \sigma_{i_1} \sigma_{i_2} \right) + \ldots \]
\[ + \frac{g_{n-1-m}}{(-)^{m+1}} \left( \sum_{i_1 \neq \cdots \neq i_m} \prod_{r=1}^{m} \sigma_{i_r} \right) + \ldots + \frac{(-)^{n-m}g_1}{(n-2)!} \left( \sum_{i_1 \neq \cdots \neq i_{n-2}} \prod_{r=1}^{n-2} \sigma_{i_r} \right). \]  

(4.7)

The Jacobian for the additional transformations above is simply 1. Explicitly, we write down the $h_i$'s,

\[ h_1 = \sum_{i=1}^{n} l_i \sigma_i \]
\[ h_m = (-)^{m-1} \sum_{i_1 < i_2 \ldots < i_m}^{n} \sigma_{i_1 \ldots i_m} l_{i_1 \ldots i_m} \]  

(4.8)

where $m \in \{2, \cdots, n-1\}$ and we have used the following notations,

\[ \sigma_{i_1 \ldots i_m} \equiv \prod_{r=1}^{m} \sigma_r, \quad l_i \equiv l \cdot k_i, \quad l_{i_1 \ldots i_m} \equiv \left( l \cdot k_{i_1 \ldots i_m} - \frac{1}{2} k_{i_1 \ldots i_m}^2 \right), \quad k_{i_1 \ldots i_m} \equiv \sum_{r=1}^{m} k_{i_r}. \]

As mentioned before, the polynomial equations $h_i = 0 (i = 1, \cdots n-1)$ have more solutions than the original scattering equations. These extra solutions locate at $\sigma_1 = \sigma_2 = \cdots = \sigma_{n-1} = 1$ in the gauge $\sigma_n = 1$. 

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Now the one-loop CHY-form (4.2) can be rewritten as the following

\[
I_n^{l=1} = \oint_{h_1=\cdots=h_{n-1}=0} \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N'(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)} - \oint_{\sigma_1=\cdots=\sigma_{n-1}=0} \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N'(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)}
\]

\[
= \oint_{h_1=\cdots=h_{n-1}=0} \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N_{\text{reg}}(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)} + \oint_{h_1=\cdots=h_{n-1}=0} \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N_{\infty}(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)}
\]

\[
- \oint_{\sigma_1=\cdots=\sigma_{n-1}=1} \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N'(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)},
\]

(4.9)

where \(N'(\sigma_i) = J(\sigma_i)N(\sigma_i) = N_{\text{reg}} + N_{\infty}\). In the second equal sign we have separated the numerator \(N'\) into two parts, with

\[
\deg(N_{\text{reg}}) < \deg(h_1) + \cdots + \deg(h_{n-1}) + \deg(D) - (n - 1),
\]

(4.10)

\[
\deg(N_{\infty}) \geq \deg(h_1) + \cdots + \deg(h_{n-1}) + \deg(D) - (n - 1).
\]

(4.11)

The integrand containing \(N_{\text{reg}}\) has no residues at infinity while the one containing \(N_{\infty}\) does. In the rest of this section, we calculate these three terms in (4.9), using our conjecture.

### 4.2 Residues at finite poles

In this subsection, we demonstrate how to utilize Conjecture 2.1 in computing the first term in the second equal sign of (4.9). This term is equal to the sum of residues associated with the \(n\)-form regular at infinity:

\[
\Omega = \frac{d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} N_{\text{reg}}(\sigma_i)}{h_1 \cdots h_{n-1} D(\sigma_i)}.
\]

(4.12)

This residue can not be evaluated by the conjecture yet, but will be after the so-called “homogenization” procedure.

Consider one of the polynomials \(h_i\) that is of degree \(d\) and reads,

\[
h_i = \sum_{\{s\}} \alpha_{s_1 \cdots s_{n-1}} \sigma_1^{s_1} \cdots \sigma_{n-1}^{s_{n-1}},
\]

(4.13)

where \(\alpha\)'s are constants of which the explicit expressions are not important and we have \(0 \leq s_i \leq d\) for \(i = 1, \cdots n - 1\) and \(s_1 + \cdots + s_{n-1} \leq d\) for every term in the summation.
We introduce an additional variable $\sigma_0$ and define the homogenization of $h_i$ as the following,

$$\tilde{h}_i = \sum_{\{s\}} \alpha_{s_1 \ldots s_{n-1}} \sigma_0^{s_0} \cdot \cdots \cdot \sigma_1^{s_1} \cdot \cdots \cdot \sigma_{n-1}^{d-s_1-\cdots-s_{n-1}}.$$ \hspace{1cm} (4.14)

When $\sigma_0 = 1$ we recover the original polynomial $\tilde{h}_i = h_i$. We homogenize all the polynomials $h_i$, $(i = 1, \ldots, n - 1)$ as well as $h_0 = D(\sigma_i)$. We construct a new meromorphic form from these homogenized polynomials as follows,

$$\tilde{\Omega} = \frac{N_{\text{reg}}(\sigma_i)}{\tilde{h}_1 \cdots \tilde{h}_{n-1} \tilde{h}_0 (\sigma_0 - 1)} d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} \wedge d\sigma_0.$$ \hspace{1cm} (4.15)

Since the function in $\tilde{\Omega}$ is regular at infinity, the global residue theorem leads to,

$$0 = \sum_p \text{Res}_{\{D_1, \ldots, D_{n-1}, \sigma_0 - 1\}, p} \tilde{\Omega} + \text{Res}_{\{D_1, \ldots, D_{n-1}, D_0\}, p'} \tilde{\Omega},$$ \hspace{1cm} (4.16)

where the divisors are $D_i = (\tilde{h}_i)$, $(i = 0, 1, \ldots, n - 1)$. The first term above recovers the original residues associated with $\Omega$ while the second term contains divisors generated solely by homogeneous polynomials. These divisors intersect at only an isolated point that is the origin. Conjecture 2.1 now applies to the second term straightaway.

According to our conjecture, the following differential operator $\mathbb{D}$ fully characterizes the relevant local information of the residue

$$\mathbb{D} = \sum_{\{s\}} a_{s_0 \ldots s_{n-1}} \partial_0^{s_0} \cdot \cdots \cdot \partial_{n-1}^{s_{n-1}},$$ \hspace{1cm} (4.17)

where the summation is taken over all the non-negative solutions to the Frobenius equation $s_0 + \cdots + s_{n-1} = \text{ord}(\mathbb{D})$ and

$$\text{ord}(\mathbb{D}) = \deg(\tilde{h}_1) + \cdots + \deg(\tilde{h}_{n-1}) + \deg(\tilde{h}_n) - n.$$ \hspace{1cm} (4.18)

The coefficients $a_{s_0 \ldots s_{n-1}}$ in the operator $\mathbb{D}$ are uniquely fixed by the local duality theorem and the intersection number of the divisors $D_i$'s.

The local duality theorem yields,

$$0 = \oint_{\tilde{h}_0 = \tilde{h}_1 = \cdots = \tilde{h}_{n-1} = 0} \frac{P Q \, d\sigma_0 \wedge d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1}}{\tilde{h}_0 \cdots \tilde{h}_{n-1}},$$ \hspace{1cm} (4.19)

where $Q$ is a polynomial of degree $d_Q$ in the ideal $\langle \tilde{h}_0, \tilde{h}_1, \cdots, \tilde{h}_{n-1} \rangle$ and $P$ is a holomorphic function in the neighbourhood of the intersecting point. To extract enough
constraints from (4.19), we need at most have \( Q \) run over the homogeneous polynomials \( \tilde{h}_i \) and \( P \) all the monomials of degree \( M - d_Q \) for each \( Q \). The residue of \( PQ \) vanishes in each case. Namely the constraints read,

\[
\mathbb{D} \left( \tilde{h}_j \prod_{i=0}^{n-1} \sigma_i^{r_i} \right) = 0, \quad j = 0, \cdots, n - 1,
\]

(4.20)

for all possible solutions to the equation \( \sum_{i=0}^{n-1} r_i = M - \deg(\tilde{h}_j) \) where all \( r_i \)'s are non-negative integers. These constraints are not completely independent, in fact they have a large redundancy. By probing a large number of non-trivial examples, we observed that these equations are enough to fix the coefficients \( a_{s_0 \cdots s_{n-1}} \) up to a global scalar factor.  

We only need one more inhomogeneous equation to determine this factor and the intersection number defined as follows is a natural choice,

\[
\prod_{i=0}^{n-1} \deg(\tilde{h}_i) = \oint_{\tilde{h}_0 = \cdots = \tilde{h}_{n-1} = 0} \frac{d\tilde{h}_0 \wedge \cdots \wedge d\tilde{h}_{n-1}}{\tilde{h}_0 \cdots \tilde{h}_{n-1}}.
\]

(4.21)

This translates to the inhomogeneous constraint,

\[
\mathbb{D} \left[ \det(\partial_\tilde{h}_j) \right] = \prod_{i=0}^{n-1} \deg(\tilde{h}_i).
\]

(4.22)

Combining the independent equations from (4.20) and (4.22), we have just enough constraints to determine the coefficients \( a_{s_0 \cdots s_{n-1}} \). Therefore we have obtained the value of the residue at the origin, i.e

\[
\text{Res}_{\{D_1, \cdots, D_{n-1}, D_n\}, p'} \tilde{\Omega} = \mathbb{D} \left[ \frac{\mathcal{N}_{\text{reg}}(\sigma_i)}{\sigma_0 - 1} \right].
\]

(4.23)

4.3 Residues at infinity

Now we move on to the second term in (4.9). Due to the degree of \( \mathcal{N}_\infty \), we need to deal with the poles at infinity.

\[\text{\footnote{We have not yet found a simple way to show that the rank of the homogeneous constraints is exactly (the number of } a_{s_0 \cdots s_{n-1}} - 1. \text{ We have observed this property in all the examples considered and believe this is true in general. In principle, if the homogeneous constraints are constructed with the generators of } \mathcal{O}/I \text{ (} \mathcal{O} \text{ is the polynomial ring of the ideal } I = \langle \tilde{h}_0, \cdots, \tilde{h}_{n-1} \rangle) \text{, these constraints are of less redundancy. This property may help in determining the rank of the homogenous constraints.}}\]
According to the global residue theorem, the residues of a meromorphic $n$-form on a compact $n$-dimensional complex manifold sum up to zero. This gives an equation relating residues at different poles (of the form) on the manifold, and thus provides an alternative method to compute residues: if we are interested in the sum of residues at certain poles, we can compute that of all the other poles on the manifold instead.

Since we are dealing with forms on $\mathbb{C}^n$ which is non-compact, to use the method above, we embed $\mathbb{C}^n$ into a compact $n$-dim manifold and use the global residue theorem there. A natural choice for the compact manifold is $\mathbb{C}\mathbb{P}^n$, where $\mathbb{C}^n$ can be identified with one of the standard coordinate patches, say $U_0 = \{[z_i]_{i=0,...,n} \in \mathbb{C}\mathbb{P}^n|z_0 \neq 0\}$, of $\mathbb{C}\mathbb{P}^n$. Then the point(s) at infinity are simply those in the complement of $U_0$, i.e $\{[z_i]_{i=0,...,n} \in \mathbb{C}\mathbb{P}^n|z_0 = 0\}$. In order to use the global residue theorem, we should not only extend the manifold from $\mathbb{C}^n$ to $\mathbb{C}\mathbb{P}^n$, but also extend the original differential form to the whole $\mathbb{C}\mathbb{P}^n$. Namely, we now regard the original differential form as a local expression on the coordinate patch $U_0$, and extend\textsuperscript{4} it naturally to $\mathbb{C}\mathbb{P}^n$ by the homogenous coordinates. Depending on the original differential form, the extended form may develop poles at infinity, and in that case the global residue theorem simply says that the sum of residues of finite poles and that of poles at infinity is zero. Note that now those points at infinity is in fact no different from those in $\mathbb{C}^n$, they are at infinity only w.r.t the patch $U_0$. In summary, to compute the residue of a pole at infinity we need go to a patch covering that point and then compute as usual.

However, the discussion above is usually not convenient in practice. Here we introduce another method. Suppose we want to compute the sum of residues of all finite poles of the differential form,
\[
\Omega = \frac{1}{h_1...h_{n-1}} \frac{N_\infty}{D} d\sigma_1 \wedge ... \wedge d\sigma_{n-1}. \tag{4.24}
\]

Then we can consider the global residue theorem for the form
\[
\Gamma = \frac{N_\infty}{h_1...h_{n-1}h_0} d\sigma_1 \wedge ... \wedge d\sigma_{n-1} \wedge d\sigma_0 , \tag{4.25}
\]

where $h_0 := \tilde{D}\sigma_0^m (\sigma_0 - 1)$, $\tilde{h}_i$ and $\tilde{D}$ mean the homogenized version of $h_i$ and $D$ as in (4.14) and $m$ is a positive integer such that the following equation is valid
\[
\deg(N_\infty) = \deg(\tilde{h}_1) + \deg(\tilde{h}_2) + ... + \deg(\tilde{h}_{n-1}) + \deg(h_0) - (n + 1). \tag{4.26}
\]
\textsuperscript{4}which will become clear in later example.
With these choices, the form $\Gamma$ has no pole at infinity and the sum of all finite residues simply vanishes. Now the set of poles of $\Gamma$ consists of two parts

$$S_0 = \{ \tilde{h}_1 = \ldots = \tilde{h}_{n-1} = 0, \tilde{D}\sigma_0^m = 0 \}$$  \hspace{1cm} (4.27)
$$S_1 = \{ \tilde{h}_1 = \ldots = \tilde{h}_{n-1} = 0, \sigma_0 = 1 \}.$$  \hspace{1cm} (4.28)

The global residue theorem thus reads

$$\sum_{p \in S_0} \text{Res}_{\{ (\tilde{h}_1, \ldots, (\tilde{h}_{n-1}), (\sigma_0^m)) \}, p} \Gamma + \sum_{p' \in S_1} \text{Res}_{\{ (\tilde{h}_1, \ldots, (\tilde{h}_{n-1}), (\sigma_0-1)) \}, p'} \Gamma = 0$$  \hspace{1cm} (4.29)

The second term actually equals to the sum of residues of all the finite poles of $\Omega$, which can been seen by writing it in terms of contour integral and then integrate out $\sigma_0$:

$$\sum_{p' \in S_1} \text{Res}_{\{ (\tilde{h}_1, \ldots, (\tilde{h}_{n-1}), (\sigma_0-1)) \}, p'} \Gamma = \oint \frac{N_{\infty}}{\tilde{h}_1 \ldots \tilde{h}_{n-1} \cdot h_0} d\sigma_1 \wedge \ldots \wedge d\sigma_{n-1} \wedge d\sigma_0$$

$$= \oint \frac{1}{\tilde{h}_1 \ldots \tilde{h}_{n-1}} \frac{N_{\infty}}{D} d\sigma_1 \wedge \ldots \wedge d\sigma_{n-1}$$

$$= \sum_{\text{finite poles}} \text{Res}_{\{ (\tilde{h}_1, \ldots, (\tilde{h}_{n-1}), \sigma_0-1) \}, p} \Omega.$$  \hspace{1cm} (4.30)

The first term, on the other hand, is of the form meeting conditions of Conjecture 2.1 and therefore can be computed accordingly. Thus the problem of finding the sum of residues of finite poles of $\Omega$ has been turned into that for $\Gamma$ which can be done using Conjecture 2.1.

### 4.4 Residues at spurious poles

Now we are left with the last residue at the pole $\sigma_1 = \ldots \sigma_{n-1} = 1$. We call this pole “spurious" since it is not present in the solution to the original scattering equations. Notice that here we are interested in the residue at a particular pole, and thus the previous ansatz (2.3) does not apply because it computes the sum of residues of all finite poles. To deal with the current case, first we need to consider another homogenized version of the polynomial scattering equations $\{ \hat{h}_i \}$. From $\{ \hat{h}_i \}$ we then construct a differential operator $\hat{D}$.\(^5\) This operator is conjectured to give the residue at the spurious pole.

\(^5\)This may sound the same with the process dealing with the finite poles, i.e $\{ h_i \} \rightarrow \{ \hat{h}_i \} \rightarrow \hat{D}$, but actually both $\{ \hat{h}_i \}$ and $\hat{D}$ are constructed in ways different from the previous versions, as explained in detail in the following.
As we have mentioned above, the residue at a given pole depends only on the local information of that pole. Hence a natural step to take is to parallel transport the coordinate system by \( \sigma_i \to \sigma_i + 1 \) such that the pole becomes the origin. This is convenient for later discussion in this part.

Here we construct \( \{ \hat{h}_i \} \) from \( \{ h_i \} \). In the neighbourhood of the pole, now at the origin, each polynomial \( h_i \) can be separated into its leading order term \( L(h_i) \) and higher order terms \( H(h_i) \). Let \( d_{L_i} \) be the degree of the leading order term \( L(h_i) \). For each term \( m \) in \( H(h_i) \) of degree \( d_m \), we construct \( \hat{m} \) by substituting some of the \( \sigma_i \) factors with constants (w.r.t. \( \sigma \)'s) \( t_i \)'s such that \( \hat{m} \) has a power in \( \sigma \)'s the same with \( L(h_i) \).\(^6\)

In this way all higher order terms are degraded to the degree of the leading term, with which we then define the homogenous polynomials \( \{ \hat{h}_i \} \)

\[
H(h_i) = \sum_a m_a(\sigma_i) \to \hat{H}(h_i) = \sum_a \hat{m}_a(\sigma_i, t_i), \quad \hat{h}_i = L(h_i) + \hat{H}(h_i). \quad (4.31)
\]

Now we proceed to define the operator \( \hat{D} \). Those hatted polynomials \( \hat{h}_i \)'s are homogeneous, so exactly as in Section (2) we could define the differential operator \( D \) corresponding to the integration with \( \hat{h}_i \)'s. \( D \) is a differential operator with coefficients \( a_{s_0 \cdots s_{n-1}} \) being functions of \( t_i \)'s. Namely,

\[
\oint_{\hat{h}_1 = \cdots = \hat{h}_{n-1} = 0} d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} \quad \iff \quad D = \sum_{\{ s_i \}} a_{s_1 \cdots s_{n-1}}(t_j) \partial_{r_1}^{s_1} \partial_{r_2}^{s_2} \cdots \partial_{r_{n-1}}^{s_{n-1}}. \quad (4.32)
\]

From \( D \) we now define the differential operator \( \hat{D} \) that acts on an arbitrary function \( F \) in the following way,

\[
\hat{D} F = \sum_{\{ s_0 \cdots s_{n-1} \}} \partial_{r_1}^{s_1} \partial_{r_2}^{s_2} \cdots \partial_{r_{n-1}}^{s_{n-1}} (a_{s_1 \cdots s_{n-1}}(\sigma_j) F). \quad (4.33)
\]

Based on large amount of numerical test, we propose the following conjecture to compute the residue at a particular pole:

**Conjecture 4.2.** If the solution of \( \langle L(h_0) = 0, L(h_1) = 0, \cdots, L(h_{n-1}) = 0 \rangle \) is an isolated point \( p \), then the residue at this pole defined by

\[
\text{Res}_p[\mathcal{R}(\sigma_i)] := \oint_{h_1 = \cdots = h_{n-1} = 0} d\sigma_1 \wedge \cdots \wedge d\sigma_{n-1} \mathcal{R}(\sigma_i) / h_1 \cdots h_{n-1} \quad (4.34)
\]

\(^6\)For instance, if \( m = \sigma_1 \sigma_2 \sigma_3^2 \) and \( d_{L_i} = 2 \), the replacement can be \( \hat{m} = t_1 t_2 \sigma_3^2 \). Or replace \( \sigma_2 \cdot \sigma_2 \) with \( t_1 \sigma_2 \) if \( d_{L_i} = 1 \). Note that the particular choice of the replaced variables does not affect the final result. We have yet not found a direct proof for this property, but have checked it against many numerical or analytical examples.
can be obtained by $\hat{\mathcal{D}}$ in the following way

$$\text{Res}_p[\mathcal{R}(\sigma_i)] = \hat{\mathcal{D}}[\mathcal{R}(\sigma_i)]$$

(4.35)

where $\mathcal{R}(\sigma_i)$ is a holomorphic function in the neighborhood of the origin.

Besides a large number of numerical checks, this conjecture has also passed the analytical check for the one-loop super Yang-Mills integrand with four external particles.  

7 It is possible to get some intuition about this conjecture by considering a simpler scenario. Let $f_i$ be a set of inhomogeneous polynomials and $L(f_i)$ their respective leading order terms. The polynomials $\hat{f}_i$ are defined the same way as in (4.31). Suppose there exists a transformation matrix $M$ such that,

\[
\begin{pmatrix}
\vdots \\
L(f_i) \\
\vdots \\
\end{pmatrix} = M(\sigma_i) \begin{pmatrix}
\vdots \\
f_i \\
\vdots \\
\end{pmatrix}, \quad \begin{pmatrix}
\vdots \\
L(f_i) \\
\vdots \\
\end{pmatrix} = M(t_i) \begin{pmatrix}
\vdots \\
\hat{f}_i \\
\vdots \\
\end{pmatrix}.
\]

Then we have, for a rational function $\mathcal{F}$

\[
\oint \frac{d\sigma_1 \cdots d\sigma_i \cdots}{f_1 \cdots f_i \cdots} \oint \frac{d\sigma_1 \cdots d\sigma_i \cdots}{L(f_1) \cdots L(f_i) \cdots} \oint \frac{d\sigma_1 \cdots d\sigma_i \cdots}{L(f_1) \cdots L(f_i) \cdots} = \oint \frac{d\max(\sigma_i) d\sigma_1 \cdots d\sigma_i \cdots}{L(f_1) \cdots L(f_i) \cdots} \oint \frac{d\max(t_i) d\sigma_1 \cdots d\sigma_i \cdots}{L(f_1) \cdots L(f_i) \cdots}.
\]

Let $\mathcal{D}$, $\mathcal{D}_L$ and $\hat{\mathcal{D}}$ denote the differential operators corresponding to the integrations with $f_i$, $L(f_i)$ and $\hat{f}_i$ in the denominator respectively. It can be verified that,

\[
\mathcal{D}\mathcal{F} = \mathcal{D}_L(\det M(\sigma_i)\mathcal{F}) = \mathcal{D}_L \left( \lim_{t_i \to \sigma_i} \det M(t_i) \mathcal{F} \right) = \hat{\mathcal{D}} \left( \lim_{t_i \to \sigma_i} \det M(t_i) \mathcal{F} \right).
\]

Of course it is not obvious whether such a transformation exists and we can not prove our conjecture in general.
Here we are interested in mainly the mathematical properties of the CHY-form of this one-loop integrand and its explicit expression is given in \[46, 47\],

\[
\mathcal{I}_4^{i=1} = \oint_{f_1=\cdots=f_3=0} \frac{d\sigma_1 \cdots d\sigma_3}{f_1 \cdots f_3} \text{Pf}(M_4) PT_4 \prod_{i=1}^{3} \frac{1}{\sigma_i} .
\]  

(5.1)

The well-known Parke-Taylor factor reads

\[
PT_4 = \sum_{\gamma} \frac{1}{\sigma_{\gamma(1)}(\sigma_{\gamma(1)} - \sigma_{\gamma(2)})(\sigma_{\gamma(2)} - \sigma_{\gamma(3)})(\sigma_{\gamma(3)} - \sigma_{\gamma(4)})},
\]

(5.2)

where we sum over all the $S_4$ permutations of the indices. The Pfaffian in this case is simply a constant. The polynomial scattering equations are easy to construct as in Section 4.1. Substituting the Vandermonde determinant and the Parke-Taylor factor, we arrive at the simple contour integral form of the 4-point integrand,

\[
\mathcal{I}_4^{i=1} = \oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4 - \oint_{\sigma_1=\sigma_2=\sigma_3=1} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_4 ,
\]

(5.3)

where the polynomials $h_i$'s are given in (4.7) and we have chosen the gauge $\sigma_4 = 1$. The function $\mathcal{R}_4$ reads,

\[
\mathcal{R}_4 = \frac{(\sigma_2 - 1)(\sigma_1 - \sigma_3)[\sigma_1^2 \sigma_3 + \sigma_2 \sigma_3 + \sigma_1 \sigma_2 (\sigma_2 + \sigma_3 (\sigma_3 - 4))] \sigma_1 \sigma_2 \sigma_3}{\sigma_1 \sigma_2 \sigma_3} .
\]

(5.4)

We consider the integral around the origin first. The integrand is a sum of meromorphic functions and the terms can be put into three categories, depending on their singularities: (1) functions that have only poles originating from the scattering equations, i.e poles of $h_i$'s; (2) functions that have poles originating from $h_i$'s and other factors, such as $\sigma_i$'s, but are regular at infinity; (3) functions that have poles at infinity. The residues of those in the first category are obviously zero and we drop these terms from now on. There are only four surviving terms and we denote them as $\mathcal{R}_4 = \sum_{i=1}^{4} \mathcal{R}_4^{(i)}$. The first three terms read,

\[
\mathcal{R}_4^{(1)} = \frac{(\sigma_2 - 1)\sigma_3}{\sigma_1} , \quad \mathcal{R}_4^{(2)} = \frac{(\sigma_1 - \sigma_3)\sigma_1}{\sigma_2} , \quad \mathcal{R}_4^{(3)} = \frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{\sigma_3} .
\]

(5.5)

These terms do not have non-zero residues at infinity. The residue corresponding to the last one, however, is non-vanishing at infinity and needs to be taken care of differently,

\[
\mathcal{R}_4^{(4)} = \sigma_2 \sigma_3 (\sigma_3 - \sigma_1) .
\]

(5.6)
The meromorphic forms corresponding to \( R^{(i)} \) are

\[
\Gamma_1 = \frac{(\sigma_2 - 1)\sigma_3}{h_1 h_2 h_3 \cdot \sigma_1} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \tag{5.7}
\]

\[
\Gamma_2 = \frac{(\sigma_1 - \sigma_3)\sigma_1}{h_1 h_2 h_3 \cdot \sigma_2} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \tag{5.8}
\]

\[
\Gamma_3 = \frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{h_1 h_2 h_3 \cdot \sigma_3} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \tag{5.9}
\]

\[
\Gamma_4 = \frac{\sigma_2 \sigma_3 (\sigma_3 - 1)}{h_1 h_2 h_3} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3, \tag{5.10}
\]

### 5.1 Computing the residues at finite poles

The residues associated with \( \Gamma_1, \Gamma_2, \) and \( \Gamma_3 \) can all be obtained the same way, following our conjecture, and here we just take \( \Gamma_1 \) as an example. First, we homogenize the factors \( h_1, h_2, h_3 \) with an auxiliary variable \( \sigma_0 \) and the residue associated with \( \Gamma_1 \) becomes,

\[
\oint_{h_1 = h_2 = h_3 = 0} \Gamma_1 = \oint_{\tilde{h}_1 = \tilde{h}_2 = \tilde{h}_3 = \sigma_0 - 1} \tilde{\Gamma}_1 \tag{5.11}
\]

where

\[
\tilde{\Gamma}_1 = \frac{(\sigma_2 - 1)\sigma_3}{h_1 h_2 h_3 \cdot \sigma_1 (\sigma_0 - 1)} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0. \tag{5.12}
\]

The global residue theorem yields,

\[
\oint_{\tilde{h}_1 = \tilde{h}_2 = \tilde{h}_3 = \sigma_0 - 1} \tilde{\Gamma}_1 = -\text{Res}_{\{D_1, D_2, D_3, (\sigma_0)\}, p} \tilde{\Gamma}_1. \tag{5.13}
\]

where \( D_i = (\tilde{h}_i) \ (i = 1, 2, 3) \). All the intersecting divisors on the right-hand-side are generated by homogeneous polynomials whose common zero is assumed to be a single point, which thus must be the origin. The right-hand-side is corresponding to a third-order differential operator, as conjectured in 2.1,

\[
\mathbb{D} = \sum_{0 \leq s_i \leq 3, \ s_0 + s_1 + s_2 + s_3 = 3} a_{s_0 s_1 s_2 s_3} \frac{\partial^{s_0}}{\partial \sigma_0^{s_0}} \frac{\partial^{s_1}}{\partial \sigma_1^{s_1}} \frac{\partial^{s_2}}{\partial \sigma_2^{s_2}} \frac{\partial^{s_3}}{\partial \sigma_3^{s_3}}. \tag{5.14}
\]

There are 20 coefficients \( a_{s_0 s_1 s_2 s_3} \) to be determined. The local duality theorem yields,

\[
\mathbb{D} \left( \sigma_1 \sigma_3 \hat{h}_0 \right) = \mathbb{D} \left( \sigma_1 \sigma_3 \hat{h}_1 \right) = \mathbb{D} \left( \sigma_1 \sigma_2 \hat{h}_2 \right) = \mathbb{D} \left( \sigma_1 \sigma_2 \hat{h}_3 \right) = 0 \quad 0 \leq i, j \leq 3. \tag{5.15}
\]
These constraints have a huge redundancy and if one carefully computes the rank of the constraint matrix, it is in fact 19. The intersection number of the divisors in this case is 6 and this demands,
\[ \mathbb{D} \left( \det \left[ \frac{\partial \tilde{h}_i}{\partial \sigma_j} \right] \right) = 6. \quad (5.16) \]

Now we have 20 conditions that fix the coefficients completely. These constraints are simple and linear conditions and solving them possesses no difficulty at all. Substituting the solution into \( \mathbb{D} \) the residue associated with \( \Gamma_1 \) is given by
\[ - \text{Res}_{D_1, D_2, D_3, \{\sigma_0\}}, p \tilde{\Gamma}_1 = -\mathbb{D} \left( \frac{(\sigma_2 - 1)\sigma_3}{\sigma_0 - 1} \right) = -\mathbb{D} \left( \frac{-\sigma_3}{\sigma_0 - 1} \right) \]
\[ = \frac{1}{\ell \cdot k_1 \ell \cdot k_4 (k_{12} + \ell \cdot k_1 + \ell \cdot k_2)}. \quad (5.17) \]

Likewise, the terms \( R_4^{(2)} \) and \( R_4^{(3)} \) give rise to the following residues respectively,
\[ -\mathbb{D} \left[ \frac{\sigma_1 (\sigma_1 - \sigma_3)}{\sigma_0 - 1} \right] = -\mathbb{D} \left[ \frac{\sigma_1^2}{\sigma_0 - 1} \right] = \frac{1}{\ell \cdot k_1 \ell \cdot k_2 \ell \cdot k_3 (k_{12} + \ell \cdot k_1 + \ell \cdot k_2 + \ell \cdot k_3)}, \quad (5.18) \]
\[ -\mathbb{D} \left[ \frac{\sigma_1 \sigma_2 (\sigma_2 - 1)}{\sigma_0 - 1} \right] = -\mathbb{D} \left[ \frac{\sigma_1 \sigma_2^2}{\sigma_0 - 1} \right] = \frac{1}{\ell \cdot k_2 \ell \cdot k_3 (k_{34} + \ell \cdot k_3 + \ell \cdot k_4)}. \quad (5.19) \]

### 5.2 Computing the residues at infinity

Now we discuss the term \( R_4^{(4)} \) which has a non-zero residue at infinity. The standard method to obtain the residue at infinity is discussed in Appendix B. Here we obtain the residue also by our ansatz. For that, we consider the form
\[ \tilde{\Gamma}_4 = \frac{(\sigma_3 - \sigma_1)\sigma_2 \sigma_3}{\tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \sigma_0 (\sigma_0 - 1)} d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0. \quad (5.20) \]

According to the global residue theorem, we know that its residue at infinity is zero, i.e residues of all finite poles sum up to zero ([50, 51]). We choose the four divisors (for this 4-dim space) by
\[ D_1 = (\tilde{h}_1), \ D_2 = (\tilde{h}_2), \ D_3 = (\tilde{h}_3), \ D_4 = (\sigma_0 (\sigma_0 - 1)). \quad (5.21) \]

The global residue theorem leads to
\[ 0 = \sum_p \text{Res}_{D_1, D_2, D_3, \{\sigma_0 (\sigma_0 - 1)\}}, p \tilde{\Gamma}_4 \]
\[ = \sum_p \text{Res}_{D_1, D_2, D_3, \{\sigma_0\}}, p \tilde{\Gamma}_4 + \sum_{p'} \text{Res}_{D_1, D_2, D_3, \{\sigma_0 - 1\}}, p' \tilde{\Gamma}_4. \quad (5.22) \]
The second term, if integrated over $\sigma_0$ first, simply returns to the original integral

$$\oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1h_2h_3} R_4^{(4)}$$

Now according the anstaz 2.3, the first term is explicitly

$$D_3 \left[ \frac{\sigma_2\sigma_3(\sigma_3 - \sigma_1)}{\sigma_0 - 1} \right] = D_3 \left[ \frac{\sigma_2\sigma_3^2}{\sigma_0 - 1} \right] = -\frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (-l \cdot k_4)}. \quad (5.23)$$

Thus from (5.22) we immediately get

$$\oint_{h_1=h_2=h_3=0} \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1h_2h_3} R_4^{(4)} = \frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (-l \cdot k_4)}.$$

### 5.3 Computing the residues at spurious poles

We are now left with the spurious pole at $\sigma_1 = \cdots = \sigma_{n-1} = 1$. The parameter transformation $\sigma_i \to \sigma_i + 1$ shifts the pole to the origin. The polynomial scattering equations are directly read off from (4.8). Unlike the first integral in (5.3), now we have to take all the terms in $R_4$ into consideration. That is, we are computing the residue at the origin associate with the form,

$$\Gamma_{\text{spurious}} = \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1(\sigma_i + 1)h_2(\sigma_i + 1)h_3(\sigma_i + 1)} R_4(\sigma_i + 1). \quad (5.24)$$

Following the discussion in Section 4.4, we homogenize the shifted polynomials $h(\sigma_i + 1)$ as follows,

$$\hat{h}_1 = l_1\sigma_1 + l_2\sigma_2 + l_3\sigma_3,$$
$$\hat{h}_2 = -\hat{l}_{12}\sigma_1\sigma_2 - \hat{l}_{13}\sigma_1\sigma_3 - \hat{l}_{23}\sigma_2\sigma_3,$$
$$\hat{h}_3 = -\hat{l}_4 t_1\sigma_2\sigma_3 + \hat{k}_{12}\sigma_1\sigma_2 + \hat{k}_{13}\sigma_1\sigma_3 + \hat{k}_{23}\sigma_2\sigma_3. \quad (5.25)$$

At the moment the quantity $t_1$ in $\hat{h}_3$ is regarded as a number and the degrees of the polynomials are $\text{deg}(\hat{h}_1) = 1$, $\text{deg}(\hat{h}_2) = 2$ and $\text{deg}(\hat{h}_3) = 2$. The intersection number of the divisors $\hat{D}_i = (\hat{h}_i), \ (i = 1, 2, 3)$ is 4. Now we consider the following residue,

$$\text{Res}_{\{\hat{D}_1, \hat{D}_2, \hat{D}_3\}} \hat{\Gamma}_{\text{spurious}}, \quad \text{with} \quad \hat{\Gamma}_{\text{spurious}} = \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{\hat{h}_1\hat{h}_2\hat{h}_3} R_4(\sigma_i + 1). \quad (5.26)$$
To obtain this residue using Conjecture 4.2, a second-order differential operator is to be worked out,

$$
\mathbb{D} = a_{002} \frac{\partial^2}{\partial \sigma_3^2} + a_{011} \frac{\partial^2}{\partial \sigma_2 \partial \sigma_3} + a_{020} \frac{\partial^2}{\partial \sigma_2^2} + a_{101} \frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} + a_{110} \frac{\partial^2}{\partial \sigma_1 \partial \sigma_2} + a_{200} \frac{\partial^2}{\partial \sigma_1^2} (5.27)
$$

The local residue theorem and the intersection number conditions demand,

$$
\mathbb{D}(\sigma_1 \hat{h}_1) = 0, \quad \mathbb{D}(\sigma_2 \hat{h}_1) = 0, \quad \mathbb{D}(\sigma_3 \hat{h}_1) = 0, \quad \mathbb{D}\hat{h}_2 = 0, \quad \mathbb{D}\hat{h}_3 = 0, \quad \mathbb{D}\hat{f} = 4. \quad (5.28)
$$

where $\hat{f} \equiv \text{det}(\partial_i \hat{h}_j)$. These constraints are easily solved. Note that the condition matrix here is invertible as $t_1 \to 0$. Substituting $t_1 = \sigma_1$ back in the expressions for the coefficients $a_{ijk}'s$, the ansatz (4.35) for the inhomogenous case leads to,

$$
\hat{\mathbb{D}} (\mathcal{I}_4(\sigma_i + 1)) \\
\equiv \frac{\partial^2}{\partial \sigma_3^2} [a_{002} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_2 \partial \sigma_3} [a_{011} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_2^2} [a_{020} \mathcal{R}_4(\sigma_i + 1)] \\
+ \frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} [a_{101} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_1 \partial \sigma_2} [a_{110} \mathcal{R}_4(\sigma_i + 1)] + \frac{\partial^2}{\partial \sigma_1^2} [a_{200} \mathcal{R}_4(\sigma_i + 1)] \\
= \text{Res}_{\{h_1(\sigma_1+1), h_2(\sigma_1+1), h_3(\sigma_1+1)\}, p} \Gamma_{\text{spurious}} (5.29)
$$

Fortunately, the explicit expressions for the coefficients $a_{ijk}$ are not necessary for computing this residue. For instance, consider the term

$$
\frac{\partial^2}{\partial \sigma_1 \partial \sigma_3} a_{101} \mathcal{R}_4(\sigma_i + 1)|_{\sigma_i = 0} = \left[ \frac{1}{\sigma_2 + 1} + \frac{1}{(\sigma_3 + 1)^2} + \frac{1}{\sigma_1 + 1} + 3 \right] \frac{\partial a_{101}(\sigma_1)}{\partial \sigma_1} \bigg|_{\sigma_i = 0} (5.30)
$$

This vanishes since $\frac{\partial a_{101}(\sigma_1)}{\partial \sigma_1}$ is holomorphic in the neighbourhood of the origin and the factor multiplying this derivative is zero at the origin. This is true for all the terms in the action of $\hat{\mathbb{D}}$ when the condition matrix is invertible. Hence the residue at the spurious pole is vanishing.

### 5.4 Summary of the four-point integrand

Now we conclude this section by summarizing the CHY-form for the 4-point 1-loop SYM integrand evaluated by our ansatz. Explicitly, the final result reads,

$$
\mathcal{I}_4^{t=1} = \oint_{h_1=h_2=h_3=0} (\Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4) \\
= \frac{1}{\ell \cdot k_1 \ell \cdot k_4 (\ell \cdot k_1 + \ell \cdot k_2 + k_{12})} + \frac{1}{\ell \cdot k_1 \ell \cdot k_2 (\ell \cdot k_2 + \ell \cdot k_3 + k_{23})} \\
- \frac{1}{\ell \cdot k_2 \ell \cdot k_3 (\ell \cdot k_1 + \ell \cdot k_2 - k_{12})} - \frac{1}{\ell \cdot k_3 \ell \cdot k_4 (\ell \cdot k_2 + \ell \cdot k_3 - k_{23})}. (5.31)
$$
The four terms in this expression have a one-to-one correspondence with the four forward-limit channels in the Q-cut representation of the same amplitude \([59, 60]\). This is a consequence of the string origin of the CHY representations. The singular behaviour of the CHY-form is inherited from the worldsheet factorization structure \([43, 44, 48]\), which is naturally related to the forward limit.

6 Outlooks

So far we have developed a differential operator for the residue with respect to a general meromorphic form and exploited it in the study of the four-point CHY expressions at one loop. An immediate follow-up direction is to probe the one-loop CHY-form for higher points. Starting from 5-points, the integrands of amplitudes grow more complicated, in particular, nontrivial Pfaffians enter in the integrand in SYM and SUGRA. Nevertheless, these factors are rational functions and our method is expected to apply to higher-point integrands comfortably. Having collected more analytical data for higher-point expressions, other ways may be discovered to determine the exact form of the differential operator, without solving the corresponding constraints by brute force.

Our method also serves as a useful tool to investigate the higher-loop CHY-forms in Yang-Mills and gravity theory, as well as to explore the non-planar regime of these theories where new symmetries are likely to be hiding. At two loops, the construction of the integral basis, involving the integration-by-parts (IBP) relations among loop integrands, remains an interesting open question. CHY-forms may be a new playground for such construction and the conjectures for residues hopefully help us in finding similar relations at the level of CHY expressions. Many aspects of Yang-Mills and gravity outside the large-N limit are still uncharted at the moment. Constructing CHY-like representations for non-planar amplitudes is certainly of importance while the string origin of such representations may make some symmetries and algebraic structures, which are otherwise hard to observe, manifest.

Furthermore, this method also finds natural applications in a variety of aspects of scattering amplitudes, such as the Grassmannian integral form and the generalized unitarity cut.
A Numerical Checks for the Ansatz

In this appendix we present numerical checks of the ansatz for the differential operator used in the computation of the residues. To check the ansatz, we consider different ideals of polynomials $\langle h_1, h_2, \ldots, h_n \rangle$ and obtain the positions of the intersection point by solving the corresponding algebraic equations numerically. For some given integrand $g(\sigma_i)$ we compute its residues at these intersection points, both directly and using our conjectures. In all the examples we have considered, the numerical results obtained from both methods match beautifully.

To compute the residue directly at a given intersection point, we have to consider the nature of this intersection point first. There are two types of isolated intersection points, the singular ones and the non-singular ones. For a non-singular isolated point $p \in (h_1) \cap (h_2) \cdots \cap (h_n)$, the residue is

$$\text{Res}_p g(\sigma_i) d\sigma_1 \wedge \cdots \wedge d\sigma_n = \frac{g}{\mathcal{J}} \bigg|_p,$$

(A.1)

where $\mathcal{J} = \frac{\partial f_i}{\partial \sigma_j} \bigg|_p$ is nonzero for the non-singular point. For a singular isolated intersection point, we need to perform a deformation first. To guarantee that no information of the singularity gets lost, we need a semiuniversal deformation and a deformation generated by the Tjurina algebra [61, 62] is a nice candidate. For a singular isolated point defined by a set of generators of the ideal $\langle h_1, h_2 \cdots, h_n \in \mathcal{O}_{\mathbb{C}^n, p} \rangle$, the Tjurina algebra is

$$T_j = \mathcal{O}_{\mathbb{C}^n, p} / \langle \vec{h}_1, \vec{h}_2, \ldots, \vec{h}_n, \partial_{\sigma_1} \vec{h}, \ldots, \partial_{\sigma_n} \vec{h} \rangle,$$

where $\vec{h}$ is the n-column $(h_1, h_2, \cdots, h_n)$ and $\vec{e}_i$ is the unit n-column with its $i$-th element set to be 1. In this case, the algebra has a finite number of generators $\vec{g}_1, \cdots, \vec{g}_\tau$, where $\tau$ denotes the total number of the generators known as the Tjurina number. The semiuniversal deformation reads,

$$\bar{F}(\sigma, t) = \vec{h}(\sigma) + \sum_{i=1}^{\tau} t_i \vec{g}_i.$$

In principle one is supposed to perform such a deformation. However, in practice, we only have to use a sub-space in the Tjurina algebra such that the singular point of degree $d$ is decoupled into $d$ separated intersection points $p^i$. This process is easy to implement numerically by simply choosing each parameter $t_i$ to be a very small number. After
deforming the singular point, we sum up the residue over all the separated intersection points and obtain the residue at the original point,

\[
\text{Res}_p \frac{g(\sigma_i) d\sigma_1 \cdots d\sigma_n}{h_1 \cdots h_n} = \sum_{i=1}^{d} \frac{g}{\mathcal{F}} \bigg|_{p^i}.
\]

(A.2)

In numerical computations, the high precision of the results is guaranteed as long as the deformation parameters \( t_i \) are sufficiently small.

### A.1 Homogeneous ideals

In this section we take 6 homogeneous ideals as our examples to test Conjecture 2.1. (We have tested our ansatz against a lot more examples and believe our method to be quite robust.) The coefficients in these ideals are randomly generated integers. Half of the ideals contain 3 variables and the degrees of the polynomials in them range from 4 to 6. The rest examples are ideals consisting of polynomials of degree 2 and the number of variables ranges from 4 to 6. The integrand is chosen to be \( g(\sigma_i) = \frac{\sigma_{i+1}}{\sum_{i=1}^{\sigma_i+1}} \) for all examples, where \( n \) denotes the number of the variables in the ideal.\(^8\) The ideals are given below and the residues computed using the two methods are listed in Table 1.

\[
\mathcal{I}_1 = \langle 7\sigma_1^4 + 7\sigma_2\sigma_1^3 + 9\sigma_3\sigma_1^3 + 2\sigma_2^2\sigma_1^2 + 18\sigma_3^2\sigma_1^2 + 11\sigma_2\sigma_3\sigma_1^2 + 17\sigma_2^2\sigma_1 + 18\sigma_3^3\sigma_1
+23\sigma_2^2\sigma_3\sigma_1 + 14\sigma_2\sigma_3\sigma_1 + 9\sigma_2^4 + 16\sigma_3^4 + 20\sigma_2\sigma_3^3 + 19\sigma_3^2\sigma_3^2 + 12\sigma_2^3\sigma_3,

\sigma_1^4 + 23\sigma_2\sigma_1^3 + 12\sigma_3\sigma_1^3 + 13\sigma_2^2\sigma_1^2 + 22\sigma_3^2\sigma_1^2 + 22\sigma_2\sigma_3\sigma_1^2 + 6\sigma_3^2\sigma_1 + 16\sigma_3^3\sigma_1
+20\sigma_2^2\sigma_3\sigma_1 + 16\sigma_2\sigma_3\sigma_1 + 18\sigma_2^4 + 19\sigma_3^4 + 3\sigma_2\sigma_3^3 + 11\sigma_2^2\sigma_3 + 9\sigma_2^3\sigma_3,

\sigma_1^4 + 19\sigma_2\sigma_1^3 + 20\sigma_3\sigma_1^3 + 12\sigma_2^2\sigma_1^2 + 19\sigma_3^2\sigma_1^2 + 22\sigma_2\sigma_3\sigma_1^2 + 12\sigma_3^2\sigma_1 + 20\sigma_3^3\sigma_1
+10\sigma_2^2\sigma_3\sigma_1 + 17\sigma_2^2\sigma_3\sigma_1 + 5\sigma_4^4 + 3\sigma_3^4 + 11\sigma_2\sigma_3^3 + 17\sigma_2^2\sigma_3^2 + 22\sigma_3^2\sigma_3 \rangle,
\]

\(^8\)This function \( g \) is chosen such that none of the derivatives \( \partial_i \) ever acts trivially on the integrand.
\[ I_3^5 = \langle 6\sigma_1^5 + 2\sigma_2\sigma_1^4 + 3\sigma_3\sigma_1^4 + 9\sigma_2^2\sigma_1^2 + 6\sigma_2\sigma_3\sigma_1^3 + 3\sigma_2^3\sigma_1^2 \]
\[ + 12\sigma_3^2\sigma_1^2 + 12\sigma_2\sigma_3^2\sigma_1^2 + 2\sigma_2^2\sigma_3\sigma_1^2 + 13\sigma_4\sigma_1 + 8\sigma_3\sigma_1 + 11\sigma_2\sigma_3\sigma_1 \]
\[ + 9\sigma_2\sigma_3\sigma_1 + 4\sigma_2\sigma_3\sigma_1 + 8\sigma_2^2 + 7\sigma_3^2 + 5\sigma_2\sigma_3^2 + 6\sigma_2^2\sigma_3^2 + 5\sigma_2\sigma_3^2 \]
\[ + 7\sigma_1^2 + 10\sigma_2\sigma_3^4 + 3\sigma_3\sigma_1^4 + \sigma_2^2\sigma_3^2 + 10\sigma_2\sigma_3\sigma_1^2 + 6\sigma_2\sigma_3\sigma_1^2 + 13\sigma_2\sigma_3^2 \]
\[ + 13\sigma_3^2\sigma_1^2 + 9\sigma_2\sigma_3^2\sigma_1^2 + 4\sigma_2\sigma_3\sigma_1^2 + 5\sigma_2 + \sigma_3\sigma_1 + 13\sigma_4\sigma_1 + 3\sigma_2\sigma_3\sigma_1 \]
\[ + 2\sigma_2\sigma_3\sigma_1 + 9\sigma_2\sigma_3\sigma_1 + 10\sigma_3^5 + 10\sigma_2\sigma_3\sigma_1^3 + 13\sigma_2\sigma_3\sigma_1 + 11\sigma_2\sigma_3\sigma_1 + 8\sigma_2\sigma_3 \]
\[ + 2\sigma_5^5 + 10\sigma_2\sigma_3^4 + 4\sigma_2\sigma_3^4 + 4\sigma_2\sigma_3\sigma_1^2 + 4\sigma_2\sigma_3\sigma_1^2 + 8\sigma_2\sigma_3^2 \]
\[ + 8\sigma_3^2\sigma_1^2 + 3\sigma_2\sigma_3^2\sigma_1^2 + 5\sigma_2\sigma_1 + 8\sigma_2\sigma_3^2\sigma_1 + 7\sigma_2\sigma_3^2\sigma_1 + 9\sigma_2\sigma_3^2 \]
\[ + 13\sigma_3\sigma_1 + 8\sigma_5^2 + 12\sigma_3^5 + 7\sigma_2\sigma_3^4 + 2\sigma_2^2\sigma_3^2 + 7\sigma_2\sigma_3^2 \rangle, \]

\[ I_3^6 = \langle 3\sigma_1^6 + 4\sigma_2\sigma_1^5 + 2\sigma_3\sigma_1^5 + 5\sigma_2^2\sigma_1^4 + 2\sigma_2^2\sigma_1^4 + 4\sigma_2\sigma_3\sigma_1^4 + \sigma_2^2\sigma_1^3 + \sigma_3^2\sigma_1^3 + 2\sigma_2\sigma_3^2 \]
\[ + \sigma_3^2\sigma_1^2 + 2\sigma_2\sigma_1^2 + 3\sigma_2\sigma_1^2 + 5\sigma_2\sigma_3\sigma_1^2 + \sigma_2^2\sigma_3^2 + \sigma_2^2\sigma_3^2 + \sigma_2^2 + 3\sigma_2\sigma_3^2 + 3\sigma_2^2 + \sigma_3^2 + 4\sigma_2\sigma_3^2 + 3\sigma_2^2 \]
\[ + 3\sigma_2^2\sigma_1 + 4\sigma_2\sigma_3\sigma_1 + 3\sigma_2^2 + 2\sigma_3^2 + 3\sigma_2^2 + 3\sigma_2^2 + 3\sigma_2^2 \]
\[ + 4\sigma_2\sigma_3\sigma_1^2 + 2\sigma_2\sigma_3\sigma_1^2 + 3\sigma_2^2\sigma_3^2 + \sigma_2^2 + 3\sigma_2\sigma_3^2 + 3\sigma_2^2 + \sigma_3^2 \]
\[ + 5\sigma_2^2 + 2\sigma_2^2 + \sigma_3^2 + 5\sigma_2^2 + 3\sigma_2^2 + 4\sigma_2\sigma_3^2 + 3\sigma_2^2 + 3\sigma_2^2 + 3\sigma_2^2 \]
\[ + 4\sigma_2\sigma_3\sigma_1^2 + 2\sigma_2\sigma_3\sigma_1^2 + 3\sigma_2^2\sigma_3^2 + \sigma_2^2 + 3\sigma_2\sigma_3^2 + 3\sigma_2^2 + \sigma_3^2 \]
\[ + 4\sigma_2^2 + 2\sigma_2^2 + \sigma_3^2 + 5\sigma_2^2 + 3\sigma_2^2 + 4\sigma_2\sigma_3^2 + 3\sigma_2^2 + 3\sigma_2^2 \]
\[ + 2\sigma_2^2 + 4\sigma_2\sigma_3^2 + 3\sigma_2^2 + 3\sigma_2^2 + 4\sigma_2\sigma_3^2 \rangle, \]

\[ I_4^2 = \langle 4\sigma_1^2 + 4\sigma_2\sigma_1 + 4\sigma_3\sigma_1 + 3\sigma_4\sigma_1 + \sigma_2^2 + 5\sigma_3^2 + \sigma_4^2 + 2\sigma_2\sigma_2 + 2\sigma_4^2 + 3\sigma_3^2, \]
\[ \sigma_1^2 + 2\sigma_2\sigma_1 + 2\sigma_3\sigma_1 + 4\sigma_4\sigma_1 + 5\sigma_2^2 + \sigma_4^2 + 4\sigma_2\sigma_3 + 5\sigma_2\sigma_2 + 2\sigma_3^2 \]
\[ + 4\sigma_2\sigma_3^2 + 4\sigma_3^2 + 5\sigma_2^2 + 4\sigma_2\sigma_3 + 3\sigma_3^2 + 2\sigma_2^2 + 3\sigma_3^2 + 3\sigma_4^2 + 4\sigma_2\sigma_3 + 3\sigma_3^2 \]
\[ + 4\sigma_2^2 + \sigma_3^2 + 3\sigma_4^2 + 3\sigma_4^2 + 3\sigma_3^2 + 3\sigma_4^2 + 3\sigma_3^2 + 3\sigma_4^2 \]
\[ + 4\sigma_2^2 + 4\sigma_2\sigma_3^2 + 3\sigma_2^2 + 3\sigma_3^2 + 3\sigma_4^2 + 3\sigma_3^2 + 3\sigma_4^2 \rangle, \]

\[ I_5^2 = \langle 5\sigma_1^2 + 5\sigma_2\sigma_1 + 3\sigma_3\sigma_1 + \sigma_4\sigma_1 + 2\sigma_3\sigma_1 + \sigma_2^2 + 5\sigma_2^2 + 5\sigma_2\sigma_4 + 3\sigma_4 + 5\sigma_4 \]
\[ + \sigma_2^2 + 3\sigma_2\sigma_1 + 3\sigma_3\sigma_1 + 4\sigma_4\sigma_1 + 5\sigma_3\sigma_1 + \sigma_2^2 + 3\sigma_3^2 + 5\sigma_4 \]
\[ + \sigma_2^2 + 3\sigma_2\sigma_3 + 5\sigma_2\sigma_4 + 5\sigma_2\sigma_4 + 2\sigma_3\sigma_4 + 3\sigma_4, \]
\[ 5\sigma_1^2 + 3\sigma_3\sigma_1 + 3\sigma_4\sigma_1 + 4\sigma_3^2 + 2\sigma_4^2 + 2\sigma_2\sigma_3 + 3\sigma_3\sigma_4 + \sigma_4\sigma_4 + 3\sigma_3\sigma_4 + 2\sigma_4\sigma_5 \]
\[ + 3\sigma_2\sigma_1 + 4\sigma_3\sigma_1 + 4\sigma_2\sigma_5 + 5\sigma_3\sigma_5 + 4\sigma_4\sigma_5, \]
\[ \sigma_2^2 + 5\sigma_1\sigma_2 + 4\sigma_3\sigma_2 + 3\sigma_4\sigma_2 + 2\sigma_5\sigma_2 + 3\sigma_3^2 + 2\sigma_4^2 + 4\sigma_1\sigma_3 + 2\sigma_1\sigma_4 + 3\sigma_3\sigma_4 + 3\sigma_3\sigma_5 + 5\sigma_4\sigma_5 \rangle, \]
We have also checked Conjecture 4.2 against non-homogeneous ideals. This case, however, is much more time-consuming to work out and we only present four simple examples below that can be easily processed in a short period of time. The function $g(\sigma_i)$...
remains the same as in the homogeneous case and the residues are listed in Table 2.

\[ I_3^{2,3} = \langle 2\sigma_1^3 + \sigma_3 \sigma_1^2 + \sigma_1^2 + 2\sigma_2 \sigma_1 + \sigma_3 \sigma_1 + \sigma_2^2 + 2\sigma_2 \sigma_3 + 2\sigma_2^2 \sigma_3 + 2\sigma_2 \sigma_3, \]
\[ = 2\sigma_1^3 + 2\sigma_2 \sigma_1^2 + 2\sigma_3 \sigma_1^2 + \sigma_2^2 \sigma_1 + \sigma_3^2 \sigma_1 + 2\sigma_2^2 + \sigma_3^2 + \sigma_2 \sigma_3, \]
\[ = 2\sigma_3^3 + \sigma_2^2 \sigma_1 + 2\sigma_3^2 \sigma_1 + 2\sigma_2 \sigma_1 + \sigma_2 \sigma_3 \sigma_1 + \sigma_3^2 + \sigma_3^2 + \sigma_2^2 + \sigma_2^2 \sigma_3 + 2\sigma_2 \sigma_3 \rangle, \]
\[ I_3^{2,4} = \langle 2\sigma_1^4 + \sigma_1^3 + \sigma_3^2 \sigma_1 + 2\sigma_1^3 + 2\sigma_3^3, \sigma_1^4 + \sigma_1^3 + 2\sigma_2 \sigma_1 + 2\sigma_3^4 + \sigma_2^2 \sigma_3^2 + 2\sigma_3^2, \]
\[ = 2\sigma_1^4 + 2\sigma_2 \sigma_1^2 + 2\sigma_3 \sigma_1 + \sigma_2 \sigma_3^2 + \sigma_2^2 \sigma_3, \]
\[ I_3^{3,4} = \langle 3\sigma_1^3 + \sigma_2 \sigma_3 \sigma_1 + 3\sigma_2^2 \sigma_3 + 3\sigma_2 \sigma_3^2, \sigma_1^4 + 5\sigma_2 \sigma_1^2 + 5\sigma_3^2 + \sigma_2^2 \sigma_3^3 + \sigma_2 \sigma_3^3 + 3\sigma_2 \sigma_3^3 + 5\sigma_3^3 \rangle, \]
\[ I_3^{2,3} = \langle 2\sigma_1^2 + 5\sigma_2 \sigma_1 + 2\sigma_3 \sigma_1^2, 2\sigma_2^2 + 3\sigma_1^2 \sigma_2 + 3\sigma_1 \sigma_3, 3\sigma_1^2 + \sigma_1 \sigma_3 + 2\sigma_1 \sigma_2 \sigma_3 + 2\sigma_2 \sigma_3 \sigma_4 + 5\sigma_1 \sigma_4 \rangle. \]

## B Standard method for dealing with poles at infinity

Here we present a standard method of calculating the residue at infinity for the following differential form

\[ \Omega = \left( \frac{\sigma_2 \sigma_3^2}{h_1 h_2 h_3} - \frac{\sigma_1 \sigma_2 \sigma_3}{h_1 h_2 h_3} \right) d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \]  

(B.1)

where \( h_i \)'s are the scattering equations in the gauge \( \sigma_4 = 1 \)

\[ h_1 = l_4 + l_1 \sigma_1 + l_2 \sigma_2 + l_3 \sigma_3, \]  

(B.2)
\[ h_2 = -l_1 l_4 \sigma_1 - l_2 l_4 \sigma_2 - l_3 l_4 \sigma_3 - l_1 l_2 \sigma_1 \sigma_2 - l_2 l_3 \sigma_2 \sigma_3 - l_1 l_3 \sigma_1 \sigma_3, \]  

(B.3)
\[ h_3 = l_1 l_2 l_4 \sigma_1 \sigma_2 + l_2 l_3 l_4 \sigma_2 \sigma_3 + l_1 l_3 l_4 \sigma_1 \sigma_3 + l_1 l_2 l_3 \sigma_1 \sigma_2 \sigma_3. \]  

(B.4)
Let us first recall how to calculate the residue at infinity in the single variable case. In that case we have, after changing variable \( z \rightarrow 1/\xi \),

\[
\oint_{z=\infty} f(z)dz = \oint_{\xi=0} f\left(\frac{1}{\xi}\right) d\left(\frac{1}{\xi}\right).
\]  

(B.5)

In the language of differential geometry we are actually considering the complex plane as one of two standard patches \( U_0 \) and \( U_1 \) covering \( \mathbb{C}P^1 \), i.e \( U_0 = \{[z_0, z_1]|z_0 \neq 0\} \) and \( U_1 = \{[z_0, z_1]|z_1 \neq 0\} \), where \( z_{0,1} \) are homogenous coordinates for \( \mathbb{C}P^1 \). Then the point of infinity is just the single point of \( \mathbb{C}P^1 \) that is missing in \( U_0 \), i.e \( \infty = [0, 1] \in \mathbb{C}P^1 \). It is not on \( U_0 \) but on \( U_1 \), and the change of variable \( z \rightarrow 1/\xi \) is just the coordinate transformation when we go from patch \( U_0 \) to \( U_1 \) where we can calculate the residue.

Similarly we can define the residue at infinity for the multivariable case. But in this case there is actually a hypersurface, instead of a single point, located at infinity. This can be seen as follows. To be specific we discuss the calculation in the form of (B.1). Thus we are considering the form \( \Omega \) on \( \mathbb{C}^3 \). Firstly we need to embed \( \mathbb{C}^3 \) into a compact manifold to be able to use the global residue theorem. \( \mathbb{C}P^3 \) is a natural choice. The original \( \mathbb{C}^3 \) can be identified with one of the standard patches covering \( \mathbb{C}P^3 \), say \( U_0 = \{[z_0, z_1, z_2, z_3]|z_0 \neq 0\} \). In that sense, what is now at infinity is the hypersurface \( \{[z_0, z_1, z_2, z_3]|z_0 = 0\} \). Eq. (B.1) is now interpreted as the local expression on \( U_0 \) of a form on \( \mathbb{C}P^3 \), i.e in terms of the homogenous coordinates,

\[
\Omega = \frac{(z_3/z_0 - z_1/z_0)(z_2/z_0)(z_3/z_0)}{h_1 h_2 h_3} d\left(\frac{z_1}{z_0}\right) \land d\left(\frac{z_2}{z_0}\right) \land d\left(\frac{z_3}{z_0}\right).
\]

(B.6)

And \( h_i \)'s are expressed in terms of homogenous coordinates as well

\[
h_1 = (l_4 z_0 + l_1 z_1 + l_2 z_2 + l_3 z_3)z_0^{-1} =: \tilde{h}_1 z_0^{-1},
\]

(B.7)

\[
h_2 = (-l_14 z_0 z_1 - l_24 z_0 z_2 - l_34 z_0 z_3 - l_12 z_1 z_2 - l_23 z_2 z_3 - l_13 z_1 z_3)z_0^{-2} =: \tilde{h}_2 z_0^{-2},
\]

(B.8)

\[
h_3 = (l_124 z_0 z_1 z_2 + l_234 z_0 z_2 z_3 + l_134 z_0 z_1 z_3 + l_123 z_1 z_2 z_3)z_0^{-3} =: \tilde{h}_3 z_0^{-3}.
\]

(B.9)

Furthermore the \( \tilde{h}_i, \ i = 1, 2, 3 \) as defined above are homogenous functions of degrees 1, 2, 3 respectively. It follows that

\[
\Omega = \frac{(z_3 - z_1)z_2 z_3}{h_1 h_2 h_3 z_0} \times (\sigma_0 d\sigma_1 \land d\sigma_2 \land d\sigma_3 - \sigma_1 d\sigma_0 \land d\sigma_2 \land d\sigma_3 - \sigma_2 d\sigma_1 \land d\sigma_0 \land d\sigma_3 - \sigma_3 d\sigma_1 \land d\sigma_2 \land d\sigma_0),
\]

from which \( \Omega \) is singular on \( \tilde{h}_1 = 0, \tilde{h}_2 = 0, \tilde{h}_3 = 0 \) and \( z_0 = 0 \). To define residues on \( \mathbb{C}P^3 \) we need to regard the denominator \( \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 z_0 \) as the product of three divisors.
There are multiple ways to do so, each of which leads to an equation of global residue theorem. One choice, however, is particularly simple, i.e

\[ D_1 = \tilde{h}_1 z_0, \]
\[ D_2 = \tilde{h}_2, \]
\[ D_3 = \tilde{h}_3. \]  

(B.11)

With this choice the global residue theorem reads, with \( S \) denoting the set of common zeros of these divisors,

\[ 0 = \sum_{p \in S} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega = \sum_{p \in S \cap U_0} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega + \sum_{p \in S_\infty} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega \]  

(B.12)

The first term is what we originally want to calculate, i.e the sum of all the residues not at infinity. The second term contains contributions from points at infinity. \( S_\infty \) denotes the set of poles at infinity, and is easily seen to be

\[ S_\infty = \{ p_1 := [0, 1, 0, 0], p_2 := [0, 0, 1, 0], p_3 := [0, 0, 0, 1] \} \]  

(B.13)

whose elements are on \( U_1, U_2 \) and \( U_3 \) respectively. We now go on to each of these three patches to compute residues there.

On \( U_1 \) we set \( z_1 = 1 \) and the form \( \Omega \) is

\[ U_1 : \quad \Omega|_{U_1} = -\frac{(z_3 - 1)z_2z_3}{z_0 \cdot (\tilde{h}_1\tilde{h}_2\tilde{h}_3)} dz_0 \wedge dz_2 \wedge dz_3. \]  

(B.14)

The divisor choice (B.11) becomes\(^9\)

\[ U_1 : \quad D_1 = z_0\tilde{h}_1(z_1 = 1), \quad D_2 = \tilde{h}_2(z_1 = 1), \quad D_3 = \tilde{h}_3(z_1 = 1). \]  

(B.15)

Thus

\[ \oint_{p_1} \Omega|_{U_1} = \oint_{z_0 = z_2 = z_3 = 0} \left[ -\frac{(z_3 - 1)z_2z_3}{z_0 \cdot (\tilde{h}_1\tilde{h}_2\tilde{h}_3)} dz_0 \wedge dz_2 \wedge dz_3 \right] = 0 \]  

(B.16)

and the residue at \( p_1 \) vanishes.

\(^9\)Since \( \tilde{h}_1(z_1 = 1, z_1 = z_2 = z_3 = 0) \neq 0 \) the divisors are actually \( \{z_0, \tilde{h}_2, \tilde{h}_2, \tilde{h}_3\} \big|_{z_0 = 1} \).
On \( U_2 \) we set \( z_2 = 1 \) in \( \Omega \) and choose the divisors in the same fashion, and the residue at \( p_2 \) also vanishes

\[
\oint_{p_2} \Omega_{|U_2} = \oint_{z_0 = z_1 = z_3 = 0} \left[ - \frac{(z_3 - z_1)z_3}{z_0 \cdot \left( \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right)_{|z_2 = 1}} dz_0 \wedge dz_2 \wedge dz_3 \right] = 0. \tag{B.17}
\]

The residue at \( p_3 \) is nontrivial. On \( U_3 \) we set \( z_3 = 1 \) and have

\[
\oint_{p_3} \Omega_{|U_3} = \oint_{z_0 = z_1 = z_3 = 0} \left[ - \frac{(1 - z_1)z_2}{z_0 \cdot \left( \tilde{h}_1 \tilde{h}_2 \tilde{h}_3 \right)_{|z_2 = 1}} dz_1 \wedge dz_2 \wedge dz_0 \right] = -(l_{123} \cdot l_{23} \cdot l_3)^{-1}. \tag{B.18}
\]

Combining (B.16), (B.17) and (B.18) we get

\[
\sum_{p \in S_{\infty}} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega = -(l_{123} \cdot l_{23} \cdot l_3)^{-1}. \tag{B.19}
\]

Using momentum conservation, this is equal to

\[
\frac{1}{l \cdot k_3 (-k_{23} + l \cdot k_2 + l \cdot k_3) (l \cdot k_4)}. \tag{B.20}
\]

By (B.12) we then immediately get the value of \( \sum_{p \in S \cap U_0} \text{Res}_{\{D_1, D_2, D_3\}, p} \Omega \), which agrees with (5.23) found using our ansatz.

### C Obtaining the residue by inspection

As shown in Section 5, we can evaluate the CHY-form conveniently using our ansatz. This method in fact applies to all kinds of meromorphic differential forms, and the intrinsic structure of the scattering equations has not been fully explored in calculations. From the discussion on residues at infinity in section B, it seems the particular form of scattering equations in fact greatly simplifies the evaluation process of the standard method. A natural question is then whether that could also help simplify the calculation using the ansatz method. As an example, in this section we discuss the following term from the 4-point one-loop super Yang-Mills amplitude

\[
\oint \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3}{h_1 h_2 h_3} \mathcal{R}_{4}^{(1)} = - \oint \frac{d\sigma_1 \wedge d\sigma_2 \wedge d\sigma_3 \wedge d\sigma_0}{h_0 h_1 h_2 h_3} \frac{\sigma_3 (\sigma_2 - 1)}{(\sigma_0 - 1)}. \tag{C.1}
\]
Many equations from the local duality theorem contain only one of the $a_{ijkl}$'s. Such equations simply lead to the vanishing of those coefficients. The surviving equations from the local duality theorem are

\[
\begin{pmatrix}
0 & -l_{34} & 0 & -l_{23} & 0 & 0 & 0 & 0 \\
0 & 0 & l_4 & 0 & l_2 & 0 & 0 & 0 \\
l_3 & 0 & l_2 & 0 & 0 & 0 & 0 & 0 \\
l_4 & l_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & l_4 & 0 & l_2 & 0 & 0 & 0 & 3l_3 \\
0 & 0 & 0 & l_3 & 0 & l_2 & 0 & 0 \\
-l_{34} & 0 & -l_{24} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -l_{24} & -l_{23} & 0 & 0 \\
\end{pmatrix}
\begin{pmatrix}
a_{0012} \\
a_{0021} \\
a_{0102} \\
a_{0120} \\
a_{a_{0201}} \\
a_{0a_{0210}} \\
a_{0a_{0300}} \\
a_{0a_{0303}} \\
\end{pmatrix} = 0. \tag{C.2}
\]

The equation for the intersection number is

\[
(-2l_1l_4l_{12}, 2l_1l_3l_{12}, 2l_1l_4l_{13}, -2l_1l_3l_{14}, -2l_1l_2l_{13}, 2l_1l_2l_{14})
\cdot (a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{a_{0201}}, a_{0a_{0210}})^T = 6. \tag{C.3}
\]

From these equations it is possible to read off $a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{a_{0201}}, a_{0a_{0210}}$ by inspection

\[
(a_{0012}, a_{0021}, a_{0102}, a_{0120}, a_{a_{0201}}, a_{0a_{0210}})^T = \frac{1}{2l_1l_4l_{34}}, \frac{1}{-2l_1l_3l_{34}}, \frac{1}{-2l_1l_4l_{24}}, \frac{1}{2l_1l_3l_{23}}, \frac{1}{2l_1l_2l_{24}}, \frac{1}{-2l_1l_2l_{23}}. \tag{C.4}
\]

Namely the value of each $a$ is simply the reciprocal of the coefficient in front of it. We will see that this pattern also appears in later calculations for $\mathcal{R}_4^{(2)}, \mathcal{R}_4^{(3)}$ and $\mathcal{R}_4^{(4)}$. Now this knowledge is already enough for us to evaluate (C.1) using our conjecture because

\[
\text{Res}_{\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3, \tilde{h}_0\}} \left( \frac{\sigma_3(\sigma_2 - 1)}{(\sigma_0 - 1)} \right) = \mathbb{D} \left( \frac{\sigma_3\sigma_2 - \sigma_3}{\sigma_0 - 1} \right) = \mathbb{D} \left( \frac{-\sigma_3}{\sigma_0 - 1} \right) \tag{C.5}
\]

which involves only $a_{0012}$ and therefore

\[
\text{Res}_{\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}} (\mathcal{R}_4^{(1)}) = 2a_{0012} = \frac{1}{l_1l_4l_{34}}.
\]

The other three terms involving $\mathcal{R}_4^{(2)}, \mathcal{R}_4^{(3)}, \mathcal{R}_4^{(4)}$ can be calculated similarly. In evaluating the residue for $\mathcal{R}_4^{(2)}$, the intersection number equation is

\[
2l_1l_2l_{13}a_{2010} - 2l_2l_3l_{13}a_{1020} - 2l_2l_4l_{34}a_{0012} + 2l_2l_3l_{34}a_{0021} + 2l_2l_4l_{14}a_{1002} - 2l_1l_2l_{14}a_{2001} = 6.
\]

\[\text{Res}_{\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}} (\mathcal{R}_4^{(2)}) \]

\[\text{Res}_{\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}} (\mathcal{R}_4^{(3)}) \]

\[\text{Res}_{\{\tilde{h}_1, \tilde{h}_2, \tilde{h}_3\}} (\mathcal{R}_4^{(4)}) \]
As mentioned above, the \( a \)'s are again reciprocals of their respective coefficients

\[
(a_{2010}, a_{1020}, a_{0012}, a_{0021}, a_{1002}, a_{2001}) = \left( \frac{1}{2l_1l_2l_3}, \frac{1}{2l_2l_3l_1}, \frac{1}{2l_2l_4l_{34}}, \frac{1}{2l_1l_4l_{34}}, \frac{1}{2l_1l_2l_{14}}, \frac{1}{2l_1l_2l_{14}} \right)
\] (C.6)

which at the same time solve other equations from local duality theorem. Thus

\[
\text{Res}\{h_1, h_2, h_3\}(\mathcal{R}^{(2)}_4) = -2a_{2001} = \frac{1}{l_1l_2l_{14}}.
\]

For \( \mathcal{R}^{(3)}_4 \), the intersection equation is

\[-2l_2l_3l_4a_{0,2,0,1} + 2l_4l_3l_4a_{0102} - 2l_1l_3l_2a_{2100} + 2l_2l_3l_2a_{1200} + 2l_1l_3l_1a_{2001} - 2l_3l_4l_{14}a_{1002} = 6,\]

from whose coefficients we find the solution to be

\[
(a_{0201}, a_{0102}, a_{2100}, a_{1200}, a_{2001}, a_{1002}) = \left( \frac{1}{-2l_2l_3l_{24}}, \frac{1}{2l_4l_3l_{24}}, \frac{1}{-2l_1l_3l_{12}}, \frac{1}{2l_2l_3l_{12}}, \frac{1}{2l_1l_3l_{14}}, \frac{1}{-2l_3l_4l_{14}} \right).
\] (C.7)

So we have

\[
\text{Res}\{h_1, h_2, h_3\}(\mathcal{R}^{(3)}_4) = 2a_{1200} = \frac{1}{-l_2l_3l_{12}}.
\]

For \( \mathcal{R}^{(4)}_4 \), the intersection equation is

\[2l_4l_3l_3a_{1020} - 2l_1l_4l_{13}a_{2010} - 2l_1l_4l_{12}a_{2100} - 2l_2l_4l_{12}a_{1200} + 2l_4l_3l_{23}a_{0120} + 2l_2l_4l_{23}a_{0210} = 6.\]

And the solution is

\[
(a_{1020}, a_{2010}, a_{2100}, a_{1200}, a_{0120}, a_{0210}) = \left( \frac{1}{2l_4l_3l_{13}}, \frac{1}{-2l_1l_4l_{13}}, \frac{1}{-2l_1l_4l_{12}}, \frac{1}{-2l_2l_4l_{12}}, \frac{1}{-2l_3l_4l_{23}}, \frac{1}{2l_2l_4l_{23}} \right).
\] (C.8)

Thus

\[
\text{Res}\{h_1, h_2, h_3\}(\mathcal{R}^{(4)}_4) = -2a_{0120} = \frac{1}{l_3l_4l_{23}}.
\]

We expect such ease of finding the ansatz solution to appear also for higher point one-loop amplitudes, and will discuss it in future projects.
Acknowledgments

GC and TW thank Y. Zhang, E. Y. Yuan, R.J. Huang, Y.H. Zhou, K.J. Larsen, D. A. Kosower and H. Johansson for useful discussion and kind suggestions. GC, EC and TW have been supported by NSF of China Grant under Contract 11405084, the Open Project Program of State Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, China (No.Y5KF171CJ1).

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