Unified Formula for the Stationary Josephson Current in Planar Graphene Junctions

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The stationary Josephson current is theoretically studied in a ballistic graphene junction consisting of a monolayer graphene sheet on top of which a pair of superconducting electrodes is deposited. To characterize such a planar junction, we employ two parameters: coupling strength between the graphene sheet and the superconducting electrodes, and a potential drop induced in the graphene sheet by direct contact with the electrodes. We derive a general formula for the Josephson current by taking account of these parameters in addition to other basic parameters, such as temperature and chemical potential. The resulting formula applies for a wide range of the parameters and reproduces previously reported results in certain limits.

KEYWORDS: Josephson current, graphene junction, ballistic regime

1. Introduction

During more than a decade, the Josephson effect\(^1\) in a superconductor-graphene-superconductor (SGS) junction has attracted considerable theoretical\(^2\)–\(^12\) and experimental\(^13\)–\(^30\) attention. Most studies attempted to observe how the stationary Josephson current is affected by the unique band structure of a graphene sheet\(^,\)\(^31\),\(^32\) in which the conduction and valence bands touch conically at \(K\) and \(K\) points in the Brillouin zone (the Dirac points). In early experiments, such an attempt was not easy to succeed as a graphene sheet used to fabricate an SGS junction is not sufficiently clean so that electron motion cannot be ballistic in it. However, the encapsulation technique of a graphene sheet enables us to fabricate a nearly ideal SGS junction\(^13\),\(^24\)–\(^30\) in which the electron motion is ballistic. In such an SGS junction, the unique band structure of a graphene sheet manifest itself in various features of the Josephson current. To elucidate such features, a general theoretical description of the Josephson current is highly desirable.

We here briefly review a theoretical study of Titov and Beenakker\(^3\) that serves as a starting point of the theoretical approach to the Josephson effect in an SGS junction. The SGS junction considered in Ref. 3 is depicted in Fig. 1, where two superconductors \(S_1 (L/2 \leq x)\) and \(S_2 (x < -L/2)\) of width \(W\) are placed with separation \(L\) on top of a clean monolayer graphene sheet with the condition of \(L \ll W\). Reference 3 assumes that electron states in the graphene sheet are described by a massless Dirac equation, and that the carrier doping in the covered region of \(L/2 \leq |x|\) is described by an effective potential of a negative constant \(-U\).\(^33\) Reference 3 also assumes that the superconducting proximity effect on the graphene sheet is described by an energy-independent effective pair potential \(\Delta_{\text{eff}}\), which is constant in the covered region \((L/2 \leq |x|)\) and vanishes in the uncovered region \(|x| \leq L/2\). The important parameters characterizing the Josephson current in this model are \(L, U, \Delta_{\text{eff}}\) in addition to temperature \(T\) and chemical potential \(\mu\). By taking the limit of \(U \to \infty\), the authors of Ref. 3 derived a formula for the Josephson current at \(T = 0\) in the short junction limit of \(L \ll \xi\), where \(\xi\) is the superconducting coherence length. The formula, given in Eq. (19) of Ref. 3, applies for \(0 \leq \mu \leq 3\) under the condition of \(T = 0, L \ll \xi,\) and \(U \to \infty\).

The assumption of the energy-independent effective pair potential was examined in Refs. 9 and 10 to improve the description of the superconducting proximity effect. In Refs. 9 and 10, the proximity effect is described by treating the coupling between the graphene sheet and the superconducting electrodes in terms of a tunneling Hamiltonian.\(^35\),\(^36\) Instead of using \(\Delta_{\text{eff}}\), this approach adopts a parameter \(\Gamma\) that controls the strength of the tunnel coupling, enabling us to take account of energy dependence of the effective pair potential. It is shown that the resulting formula, given in Eq. (53) of Ref. 10, unifies the various behaviors of the Josephson critical current \(I_c\) as a function of \(T\) observed in a set of samples.\(^28\) Particularly, it succeeds to describe unusual \(T\) dependence of \(I_c\) in an SGS junction with relatively weak coupling strength. A drawback of this formula is that its application is restricted to the case of \(\mu\) being sufficiently away from the Dirac point. This is ascribed to a quasiclassical approximation used in its derivation.

The purpose of this paper is to give a general formula for the stationary Josephson current through a monolayer graphene sheet that can apply for a wide range of parameters. To do so, we adopt the model used in Ref. 10 and derive a general formula for the Josephson current without relying on a quasiclassical approximation. The resulting formula applies for arbitrary \(T, \mu, L, U,\) and \(\Gamma\),\(^37\) and reproduces the formulas of Refs. 3 and 10 in certain limits. The paper is organized as follows. In Sect. 2, we describe the model for the SGS junction and introduce a thermal Green’s function. In Sect. 3, we construct the thermal Green’s function and then derive a general formula for the Josephson current. In Sect. 4, we show that the resulting formula reproduces the results of Refs 3 and 10 in certain limits. In Sect. 5, the behavior of the Josephson critical current is numerically studied in...
a short junction limit. Section 6 is devoted to summary. We set $k_B = h = 1$ throughout the paper.

2. Model and Thermal Green’s Function

We consider an SGS junction of monolayer graphene as depicted in Fig. 1. We adopt a model of Ref. 10 and then introduce a thermal Green’s function that is convenient for subsequent analysis of the Josephson current.

In Fig. 1, two superconductors $S_1$ and $S_2$ of width $W$ are placed with separation $L$ on top of a clean monolayer graphene sheet, where $S_1$ and $S_2$ respectively occupy the regions of $L/2 \leq x$ and of $x \leq -L/2$. We assume that the pair potential is given by

$$\Delta(x) = \begin{cases} \Delta e^{i\varphi/2} & (L/2 < x) \\ 0 & (|x| < L/2) \\ \Delta e^{-i\varphi/2} & (x < -L/2), \end{cases} \quad (1)$$

where $\varphi$ serves as the phase difference between the two superconducting electrodes.

Let us assume that the coupling of the graphene sheet and the superconductors is described by a tunneling Hamiltonian. The resulting proximity effect on the graphene sheet is described by a self-energy [35,36] [see Eq. (7)]. The coupling with the superconductors also induces carrier doping in the graphene sheet; the carrier density in the covered region of $L/2 < |x|$ becomes higher than that in the uncovered region of $|x| < L/2$. We describe this by adding the effective potential of a negative constant $-U$ only in the covered region, resulting in the renormalization of chemical potential $\mu$:

$$\tilde{\mu} = \begin{cases} \mu & (|x| < L/2) \\ \mu + U & (L/2 < |x|). \end{cases} \quad (2)$$

Let us turn to the electron states in the graphene sheet. Low energy states appear in the two valleys located at the $K_+$ and $K_-$ points in the Brillouin zone, where the wave vector corresponding to the $K_\pm$ point is given by $K_\pm = \pm(2\pi/a)(2/3,0)$ with $a$ being the lattice constant of the graphene sheet. Within the effective mass approximation, the low energy states in the $K_\pm$ valley are described by the effective Hamiltonian $H_\pm$ defined by [38–40]

$$H_\pm = \begin{pmatrix} -\tilde{\mu} & \gamma k_\pm \\ \gamma k_\pm & -\tilde{\mu} \end{pmatrix}, \quad (3)$$

where $k_{\pm} = k_x \pm ik_y$ with $k_x = -i\partial_x$ and $k_y = -i\partial_y$.

The $2 \times 2$ form of $H_\pm$ reflects the fact that the unit cell of a hexagonal lattice contains A and B sites, and $\gamma$ is given by $\gamma = (\sqrt{3}/2)\gamma_0 a$, where $\gamma_0$ represents the nearest-neighbor transfer integral [38–40].

In the presence of superconducting proximity effect, we need to treat electron and hole states taking their coupling into account. The simplest way for doing so is to employ a Bogoliubov–de Gennes equation:

$$H_{\text{BdG}} \begin{pmatrix} \Psi_e \\ \Psi_h \end{pmatrix} = \epsilon \begin{pmatrix} \Psi_e \\ \Psi_h \end{pmatrix}, \quad (4)$$

where $\Psi_e$ and $\Psi_h$ are respectively the electron and hole wavefunctions, and the $4 \times 4$ Hamiltonian for the $K_+$ valley is given by [41]

$$H_{\text{BdG}} = \begin{pmatrix} H_+ \\ H_{\text{eff}}(x)^* \sigma_0 \\ \Delta_{\text{eff}}(x) \sigma_0 \\ -H_+ \end{pmatrix} \quad (5)$$

with $\sigma_0 = \text{diag}(1,1)$. Here, $\Delta_{\text{eff}}(x)$ is the effective pair potential, which is usually assumed to be an energy independent constant in the covered region. This widely accepted assumption for $\Delta_{\text{eff}}$ is justified only when the coupling between the graphene sheet and the superconducting electrodes is sufficiently strong [10,36]. To cope with arbitrary coupling strength, we employ the tunneling Hamiltonian model proposed by McMillan [35] instead of assuming the energy independent pair potential. The approach of McMillan is reformulated in Ref. 36 in the form specific to a hybrid graphene system.

We introduce the $4 \times 4$ thermal Green’s function $G(r, r'; \omega)$ with $\omega = (2n+1)\pi T$, which obeys

$$i\omega \tau^0 - H - \Sigma \downarrow \uparrow G(r, r'; \omega) = \gamma^0 \delta(r - r'), \quad (6)$$

where $H = \text{diag}(H_+, -H_+)$, and $\gamma^0 = \text{diag}(1,1,1,1)$. The self-energy $\Sigma$, representing the proximity effect mediated by quasiparticle tunneling, is given by [9,36]

$$\Sigma = \frac{-i\theta(|x| - L/2)}{\sqrt{\Delta^2 + \omega^2}} \begin{pmatrix} i\omega & \Delta(x) \\ \Delta(x)^* & i\omega \end{pmatrix} \otimes \sigma_0, \quad (7)$$

where $\Gamma$ represents the strength of the tunnel coupling and $\theta(x)$ is the Heaviside step function. The off-diagonal elements are regarded as an energy dependent effective pair potential, while the diagonal elements describe renormalization of a quasiparticle energy. Here and hereafter, we restrict our consideration to quasiparticle states in the $K_+$ valley as those in the $K_-$ valley equivalently contribute to the Josephson current. A brief comment on $G(x, x'; q, \omega)$ is given in Appendix A.

3. Formulation

We derive a general formula for the Josephson current by using an analytical expression of the thermal Green’s function on the basis of the argument originally given by Ishii [42,43] and later developed by Furusaki and Tsukada [44–46].

Hereafter, we restrict our attention to the regime of electron doping: $0 \leq \mu \leq \mu + U$. Assuming that our system is translationally invariant in the $y$ direction, we perform the Fourier transformation:

$$G(x, x'; q, \omega) = \int d(y - y') e^{-i(qy - q'y')} G(r, r'; \omega), \quad (8)$$
which we explicitly express as
\[ G(x, x'; q, \omega) = \begin{pmatrix} g(x, x'; q, \omega) & f'(x, x'; q, \omega) \\ f^*(x, x'; q, \omega) & g^*(x, x'; q, \omega) \end{pmatrix}, \]
(9)

Note that we need to treat only \( g(x, x'; q, \omega) \) and \( f'(x, x'; q, \omega) \). Let us consider them in the uncovered region of \( |x| < L/2 \). It is convenient to define the wave numbers in the \( x \)-direction as
\[ k_x = \frac{\text{sgn}_\omega}{\gamma} \sqrt{\left( \frac{\mu + i\omega}{\gamma} \right)^2 - q^2}, \]
(10)
\[ k_h = \frac{\text{sgn}_\omega}{\gamma} \sqrt{\left( \frac{\mu - i\omega}{\gamma} \right)^2 - q^2}, \]
(11)
where \( \text{Im}\{k_x\} > 0 \) and \( \text{Im}\{k_h\} < 0 \), and \( \text{sgn}_\omega \) represents the sign of \( \omega \). It is also convenient to introduce
\[ e^{\pm i\phi_x} = \frac{\gamma(k_x \pm iq)}{\mu \\ 0}, \]  
\[ e^{\pm i\phi_h} = \frac{\gamma(k_h \pm iq)}{\mu - iq}. \]
(12)
(13)
This is equivalent to define
\[ \cos \phi_x = \frac{\gamma k_x}{\mu + iq}, \quad \sin \phi_x = \frac{\gamma q}{\mu + iq}, \]
(14)
\[ \cos \phi_h = \frac{\gamma k_h}{\mu - iq}, \quad \sin \phi_h = \frac{\gamma q}{\mu - iq}. \]
(15)
A general solution of \( g(x, x'; q, \omega) \) is written as
\[ g(x, x'; q, \omega) = \left[ -i v_e \theta(x - x') + c_{++} \right] e^{ik_x(x-x')}\Lambda_x^{++} + \left[ i v_e \theta(x' - x) + c_{--} \right] e^{-ik_x(x-x')}\Lambda_x^{--} + c_{+-} e^{ik_x(x+x')}\Lambda_x^{+-} + c_{-+} e^{-ik_x(x+x')}\Lambda_x^{-+}, \]
(16)
where \( v_e = \gamma \cos \phi_x \) and
\[ \Lambda_x^{++} = \frac{1}{2} \begin{pmatrix} 1 & e^{i\phi_x} \\ e^{-i\phi_x} & 1 \end{pmatrix}, \]
(17)
\[ \Lambda_x^{--} = \frac{1}{2} \begin{pmatrix} -1 & -e^{i\phi_x} \\ e^{-i\phi_x} & 1 \end{pmatrix}, \]
(18)
\[ \Lambda_x^{+-} = \frac{1}{2} \begin{pmatrix} e^{-i\phi_x} & -1 \\ 1 & e^{i\phi_x} \end{pmatrix}, \]
(19)
\[ \Lambda_x^{-+} = \frac{1}{2} \begin{pmatrix} e^{i\phi_x} & -1 \\ 1 & e^{-i\phi_x} \end{pmatrix}. \]
(20)
A general solution of \( f'(x, x'; q, \omega) \) is written as
\[ f'(x, x'; q, \omega) = d_{++} e^{i(k_x x - k_h x')} \Lambda_h^{++} + d_{--} e^{-i(k_x x - k_h x')} \Lambda_h^{--} + d_{+-} e^{i(k_x x + k_h x')} \Lambda_h^{+-} + d_{-+} e^{-i(k_x x + k_h x')} \Lambda_h^{-+}, \]
(21)
where
\[ \Lambda_h^{++} = \frac{1}{2} \begin{pmatrix} e^{\frac{\gamma}{2}(\phi_h - \phi_x)} & -e^{\frac{\gamma}{2}(\phi_h + \phi_x)} \\ e^{-\frac{\gamma}{2}(\phi_h + \phi_x)} & e^{-\frac{\gamma}{2}(\phi_h - \phi_x)} \end{pmatrix}, \]
(22)
\[ \Lambda_h^{--} = \frac{1}{2} \begin{pmatrix} e^{-\frac{\gamma}{2}(\phi_h + \phi_x)} & -e^{-\frac{\gamma}{2}(\phi_h - \phi_x)} \\ e^{\frac{\gamma}{2}(\phi_h - \phi_x)} & e^{\frac{\gamma}{2}(\phi_h + \phi_x)} \end{pmatrix}, \]
(23)
\[ \Lambda_h^{+-} = \frac{1}{2} \begin{pmatrix} e^{-\frac{\gamma}{2}(\phi_h + \phi_x)} & e^{\frac{\gamma}{2}(\phi_h - \phi_x)} \\ -e^{-\frac{\gamma}{2}(\phi_h - \phi_x)} & -e^{\frac{\gamma}{2}(\phi_h + \phi_x)} \end{pmatrix}, \]
(24)
\[ \Lambda_h^{-+} = \frac{1}{2} \begin{pmatrix} e^{\frac{\gamma}{2}(\phi_h - \phi_x)} & -e^{\frac{\gamma}{2}(\phi_h + \phi_x)} \\ -e^{-\frac{\gamma}{2}(\phi_h + \phi_x)} & e^{-\frac{\gamma}{2}(\phi_h - \phi_x)} \end{pmatrix}. \]
(25)
The Josephson current is formally expressed as
\[ I(\varphi) = 4eW \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} T \sum_{\omega} \text{tr} \left\{ j_x g(x; q, \omega) \right\}, \]
(26)
where the factor 4 comes from the spin and valley degeneracies, the current operator \( j_x \) is defined by
\[ j_x = e\gamma \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \]
(27)
and \( g(x; q, \omega) \equiv \frac{1}{2} \left[ \Lambda(x, x; q, \omega) + g(x, x + 0; q, \omega) \right] \). Substituting Eq. (16) into Eq. (26), we obtain
\[ I(\varphi) = 4eW \int_{-\infty}^{+\infty} \frac{dq}{2\pi} T \sum_{\omega} e^{i\varphi(\omega)} c_{++}(\omega) - c_{--}(\omega). \]
(28)
The unknown coefficients \( c_{++} \) and \( c_{--} \) are determined by a boundary condition at \( x = \pm L/2 \) for \( g(x, x'; q, \omega) \) and \( f'(x, x'; q, \omega) \), which we briefly describe below. By solving the Bogoliubov–de Gennes equation in the covered region of \( L/2 \leq |x| \) (see Appendix B), we find a relation between the electron wavefunction \( \Psi_e \) and the hole wavefunction \( \Psi_h \), which is expressed by using
\[ \tilde{\omega} = \left( 1 + \frac{\Gamma}{\sqrt{\omega^2 + \Delta^2}} \right) \omega, \]
(29)
\[ \tilde{\Delta} = \frac{\Gamma}{\sqrt{\omega^2 + \Delta^2}}, \]
(30)
\[ \Omega = \text{sgn}_\omega \sqrt{\omega^2 + \Delta^2}, \]
(31)
and \( \chi \) defined by
\[ e^{\pm i\chi} = \frac{\gamma(\varphi \pm iq)}{\mu + U}. \]
(32)
with
\[ p = \text{sgn}_\omega \sqrt{\left( \frac{\mu + U}{\gamma} \right)^2 - q^2}, \]
(33)
where \( U \) is assumed to be the largest energy scale in our model. Let \( \Psi_{e}^+ (\Psi_h^+) \) and \( \Psi_{e}^- (\Psi_h^-) \) be respectively the right-going and left-going components of \( \Psi_e (\Psi_h) \). At \( x = \pm L/2 \), they satisfy
\[ \Psi_{e}^\pm = B(\pm L/2) \Psi_{h}^\pm, \]
(34)
with
\[ B(\pm L/2) = \frac{e^{\pm i\varphi/2}}{\Delta \cos \chi} \left( \tilde{\omega} \cos \chi \mp i\Omega \sin \chi \right) \left( \pm \Omega \mp \tilde{\omega} \cos \chi \mp i\Omega \sin \chi \right), \]
(35)
The derivation of Eq. (34) is outlined in Appendix B. Equation (34), serving as the boundary condition, gives
a set of coupled equations:

\[
\begin{align*}
\left(-\frac{i}{v_c} + c_{++}\right) e^{i k_c \frac{L}{2}} \Lambda_+ + c_{--} e^{-i k_c \frac{L}{2}} \Lambda_- \\
= B(L/2) \left( d_{++} e^{i k_c \frac{L}{2}} \Lambda_+ + d_{--} e^{-i k_c \frac{L}{2}} \Lambda_- \right), \\
\end{align*}
\]

(36)

\[
\begin{align*}
c_{--} e^{-i k_c \frac{L}{2}} \Lambda_- + c_{++} e^{i k_c \frac{L}{2}} \Lambda_+ \\
= B(L/2) \left( d_{--} e^{-i k_c \frac{L}{2}} \Lambda_- + d_{++} e^{i k_c \frac{L}{2}} \Lambda_+ \right), \\
\end{align*}
\]

(37)

Solving these equations, we obtain

\[
c_{++}(\phi) = c_{--}(-\phi) = -ie^{-i \frac{\phi}{2v_c}} \xi 
\]

(40)

with

\[
\xi = e^{i(k_c-k_h) \frac{L}{2}}
\]

\[
\times \left[ \hat{\omega} \cos \chi \cos \left( \frac{\phi_c + \phi_h}{2} \right) \\
- \Omega \cos \left( \frac{\phi_c - \phi_h}{2} \right) - \sin \chi \sin \left( \frac{\phi_c + \phi_h}{2} \right) \right] \\
\times \left[ -i\hat{\omega} \cos \chi \cos \left( \frac{\phi_c + \phi_h}{2} \right) \sin \left( k_c - k_h \right) \frac{L}{2} + \frac{\phi}{2} \right] \\
+ \Omega \cos \left( \frac{\phi_c - \phi_h}{2} \right) - \sin \chi \sin \left( \frac{\phi_c + \phi_h}{2} \right) \right] \\
\times \cos \left( k_c + k_h \right) \frac{L}{2} + \frac{\phi}{2} \right], \\
\end{align*}
\]

(41)

\[
\Xi = \frac{1}{2} \left[ \hat{\omega}^2 \cos^2 \chi \cos^2 \left( \frac{\phi_c + \phi_h}{2} \right) \\
+ \Omega^2 \left( \cos \left( \frac{\phi_c - \phi_h}{2} \right) - \sin \chi \sin \left( \frac{\phi_c + \phi_h}{2} \right) \right)^2 \right] \\
\times \cos \left( (k_c - k_h)L \right) \\
- i\hat{\omega} \cos \chi \cos \left( \frac{\phi_c + \phi_h}{2} \right) \\
\times \left( \cos \left( \frac{\phi_c - \phi_h}{2} \right) - \sin \chi \sin \left( \frac{\phi_c + \phi_h}{2} \right) \right) \right] \\
\times \cos \left( (k_c - k_h)L \right) \\
+ \frac{1}{2} \left[ \hat{\omega}^2 \cos^2 \chi \cos \phi_c \cos \phi_h \cos \phi \right. \\
\]
and \(\text{sgn}_x\) represents the sign of \(q\). Hence, \(\Xi\) in Eq. (43) is reduced to \(\Xi_{SJL}\) for \(\gamma q < \mu\) and \(-\Xi_{SJL}\) for \(\mu < \gamma q\), where

\[
\Xi_{SJL} = \omega^2 \left( \cos^2 \chi \cos^2 \phi + (\sin \chi - \sin \phi)^2 \sin^2 kL \right)
+ \Delta^2 \left( \cos^2 \chi \cos^2 \phi + (\sin \chi - \sin \phi)^2 \sin^2 kL \right)
- \cos^2 \chi \cos^2 \phi \sin^2 \frac{\omega}{2}.
\]  

(46)

We obtain the expression of the Josephson current in the short junction limit:

\[
I_{SJL}(\phi) = \frac{eW}{\pi} \int_{-\infty}^{+\infty} dq T \sum_{\omega} \frac{\tau(q) \Delta^2 \sin \varphi}{\omega^2 + \Delta^2 \left[ 1 - \tau(q) \sin^2 \frac{\omega}{2} \right]},
\]

(47)

where

\[
\tau(q) = \frac{\cos^2 \chi \cos^2 \phi}{\cos^2 \chi \cos^2 \phi + (\sin \chi - \sin \phi)^2 \sin^2 kL}.
\]

(48)

Let us restrict our consideration to the strong coupling limit of \(\Gamma \to \infty\), where \(\omega / \Delta\) can be replaced with \(\phi / \Delta\).

After performing the summation over \(\omega\), we find

\[
I_{SJL}(\varphi) = \frac{eW}{2\pi} \int_{-\infty}^{+\infty} dq T \frac{\tau(q) \sin \varphi}{\sqrt{1 - \tau(q) \sin^2 \frac{\omega}{2}}} \times \tanh \left( \frac{\Delta}{2T} \sqrt{1 - \tau(q) \sin^2 \frac{\omega}{2}} \right).
\]

(49)

At \(T = 0\), this expression is reduced to Eq. (19) of Ref. 3 in the limit of \(U \to \infty\), where \(\cos^2 \chi = 1\) and \(\sin \chi = 0\).

Equation (49) should be regarded as an extension of the result of Kulik and Omelyanchuk.\(^{48}\)

4.2 High carrier-density limit

Let us next consider the high carrier-density limit of \(\gamma / L, \Delta_0 \ll \mu\). In this limit, we can approximate that

\[
k_e = k + \frac{\mu}{\gamma^2 k} i \omega,
\]

(50)

\[
k_h = k - \frac{\mu}{\gamma^2 k} i \omega,
\]

(51)

and \(\phi_e = \phi_h = \phi\). Hence, \(\Xi\) in Eq. (43) is reduced to

\[
\Xi_{HCL} = \frac{1}{2} \left[ \omega^2 \cos^2 \chi \cos^2 \phi + \Omega^2 (1 - \sin \chi \sin \phi)^2 \right] \times \cosh \left( \frac{2\omega L}{v_x} \right)
+ \tilde{\omega} \Omega \cos \chi \cos \phi (1 - \sin \chi \sin \phi) \sinh \left( \frac{2\omega L}{v_x} \right)
- \frac{1}{2} \Omega^2 (\sin \chi - \sin \phi)^2 \cos 2kL
+ \frac{1}{2} \Delta^2 \cos^2 \chi \cos^2 \phi \cos \varphi,
\]

(52)

where \(v_x = \gamma \cos \phi\). We obtain the expression of the Josephson current in the high carrier-density limit:

\[
I_{HCL}(\varphi) = \frac{eW}{\pi} \int_{-\infty}^{+\infty} dq T \sum_{\omega} \frac{\Delta^2 \cos^2 \chi \cos^2 \phi}{\Xi_{HCL}} \sin \varphi.
\]

(53)

This expression is equivalent to Eq. (53) of Ref. 10, derived by using a quasiclassical Green’s function approach.

5. Numerical Result

We focus on the short junction limit of \(L \ll \xi\) with heavy doping in the covered region (i.e., \(\gamma / L, \Delta_0 \ll U\)), which is particularly important in actual experiments.

The Josephson critical current \(I_c\) defined by

\[
I_c = \max_{\varphi} \{ I(\varphi) \}
\]

(54)

is numerically calculated as a function of \(T\) in the low carrier-density case of \(\mu / \Delta_0 = 1.0\) and the high carrier-density case of \(\mu / \Delta_0 = 200.0\). The critical current is also calculated as a function of \(\mu\). In every case, we set \(\Gamma / \Delta_0 = 1, 20,\) and \(200.0\). The following parameters are employed: \(L = 200\) nm, \(W = 4\) \(\mu\)m, \(\gamma_0 = 2.8\) eV, \(a = 0.246\) nm, and \(\Delta_0 = 120\) meV. The coherence length is estimated as \(\xi = 2.4\) \(\mu\)m, which is much longer than \(L\). The behavior of \(I_c\) in the short junction limit is fully described by Eq. (47). The amplitude of the pair potential is determined by the gap equation

\[
1 = \lambda_{\text{int}} \int_0^{\epsilon_D} \frac{d \tan \left( \sqrt{\epsilon^2 + \Delta_0^2} / 2T \right) / \sqrt{\epsilon^2 + \Delta_0^2}}{\sqrt{\epsilon^2 + \Delta_0^2}},
\]

(55)

where \(\lambda_{\text{int}}\) is the dimensionless interaction constant, and the Debye energy is chosen as \(\epsilon_D / \Delta_0 = 200.0\).

Figure 2 shows \(I_c\) in the high carrier-density case of \(\mu / \Delta_0 = 200.0\) normalized by

\[
I_0 = e \Delta_0 \frac{W}{\pi \gamma}
\]

(56)

as a function of \(T / T_c\) with \(\Gamma / \Delta_0 = 1, 20,\) and \(200.0\). The \(I_c\) curve is convex upward for \(\Gamma / \Delta_0 = 200.0\), whereas it becomes convex downward for \(\Gamma / \Delta_0 = 1.0\). Figure 3 shows \(I_c\) in the low carrier-density case of \(\mu / \Delta_0 = 1.0\) normalized by

\[
I_0 = e \Delta_0 \frac{W}{\pi L}
\]

(57)

as a function of \(T / T_c\) with \(\Gamma / \Delta_0 = 1, 20,\) and \(200.0\). The \(I_c\) curve also shows a crossover from convex upward to convex downward with decreasing \(\Gamma / \Delta_0\).

As noted in the previous section, Eq. (47) reproduces
Eq. (19) of Ref. 3 at $T = 0$ if $\Gamma$ and $U$ are sufficiently large. Thus, the resulting $I_c$ in the strong coupling case of $\Gamma/\Delta_0 = 2000$ is expected to reproduce the corresponding results of Ref. 3. Indeed, for $\Gamma/\Delta_0 = 2000$, $I_c/I_0$ in the case of $\mu/\Delta_0 = 200$ is 1.228 at $T = 0$, which is quantitatively consistent with Eq. (22) of Ref. 3. Similarly, $I_c/I_0$ in the case of $\mu/\Delta_0 = 1$ is 1.321 at $T = 0$, which is also quantitatively consistent with Eq. (21) of Ref. 3.

Figure 4 shows $I_c$ as a function of $\mu$ for $\Gamma/\Delta_0 = 1$, 20, and 2000 at $T/T_c = 0.01$, where $I_c$ is normalized by $I_0 = e\Delta_0 W^2/\pi L$.

6. Summary

Adopting a simple model of superconductor-graphene-superconductor junctions, we derive a general formula for the stationary Josephson current. The resulting formula contains $T$, $\mu$, $L$, $U$, and $\Gamma$ as important parameters and is applicable for arbitrary values of these parameters, where $T$ is temperature, $\mu$ is chemical potential, $L$ is the separation between two superconducting electrodes, $U$ controls the carrier doping in the graphene sheet, and $\Gamma$ represents the tunneling strength between the graphene sheet and the superconducting electrodes. We show that it reproduces the formula of Ref. 3 in the limit of $L \ll \xi$, $U \to \infty$, and $\Gamma \to \infty$ at $T = 0$. We also show that it is reduced to the formula of Ref. 10 in the limit of $\gamma/L, \Delta_0 \ll \mu$, where $\gamma$ is the velocity of an electron in a graphene sheet and $\Delta_0$ is the pair potential at $T = 0$.

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Appendix A: Components of the Green’s function

The thermal Green’s function $G(r, r'; \omega)$ is described by the effective Hamiltonian $\tilde{H}$ defined by

$$\tilde{H} = \begin{pmatrix} -\tilde{\mu}(x) & \gamma k_- - \tilde{\Delta}(x) & 0 \\ \gamma k_+ - \tilde{\mu}(x) & \tilde{\Delta}(x) & 0 \\ 0 & 0 & -\tilde{\Delta}^*(x) - \gamma k_+ - \tilde{\mu}(x) \end{pmatrix} ,$$

(A-1)

which possesses the particle-hole symmetry $41)$

$$\Theta^{-1} \tilde{H} \Theta = -\tilde{H} ,$$

(A-2)

where

$$\Theta = \begin{pmatrix} 0 & -\vartheta & 0 \\ \vartheta & 0 & 0 \end{pmatrix}$$

(A-3)

with $\vartheta = -i\sigma_y K$. Here, $\sigma_y$ is the $y$ component of Pauli matrix and $K$ denotes complex conjugate operator.

Let us express the thermal Green’s function as

$$G(r, r'; \omega) = \begin{pmatrix} g(r, r'; \omega) & f'(r, r'; \omega) \\ f(r, r'; \omega) & g'(r, r'; \omega) \end{pmatrix}.$$  (A-4)

Using a spectral representation with the help of the particle-hole symmetry, we can represent $g'(r, r'; \omega)$ and $f'(r, r'; \omega)$, respectively, in terms of $g(r, r'; \omega)$ and $f(r, r'; \omega)$. Here, we present only the final results,

$$g'(r, r'; \omega) = -\vartheta^{-1} g(r, r'; \omega) \vartheta ,$$

(A-5)

$$f'(r, r'; \omega) = \vartheta^{-1} f(r, r'; \omega) \vartheta .$$

(A-6)

Appendix B: Derivation of the Boundary Condition

By solving the Bogoliubov–de Gennes equation in a Matsubara representation, we present wavefunctions in the covered region of $L/2 \leq |x|$. The boundary condition, given in Eq. (34), is straightforwardly obtained from the resulting wavefunctions. The Bogoliubov–de Gennes equation in the region of $L/2 \leq x$ is written as

$$\left( i\tilde{\omega} r^0 - \tilde{H} \right) \begin{pmatrix} \Psi_e \\ \Psi_h \end{pmatrix} = 0 ,$$

(B-1)

where $\tilde{H}$ is given in Eq. (A-1), and $\tilde{\mu}$ and $\tilde{\Delta}(x)$ in it should read as $\tilde{\mu} = \mu + U$ and $\tilde{\Delta}(x) = \Delta e^{i\varphi}$. Hereafter, we assume that $U$ is much larger than $\Delta_0$.

It is convenient to define $\kappa$ as

$$\kappa = \frac{\mu + U}{\gamma^2 p} \Omega .$$

(B-2)

By using the wave number $q$ in the transverse direction in addition to $\kappa$, $p$, and $\chi$ (the latter two are defined in the text), the right-going wave function $\Psi^+ = \gamma (\Psi^+_e, \Psi^+_h)$ and the left-going wavefunction $\Psi^- = \gamma (\Psi^-_e, \Psi^-_h)$ in the
region of $L/2 \leq x$ are expressed as
\[
\begin{align*}
(\Psi^+_{h}) &= e^{i px - \kappa x + i q y} \\
(\Psi^-_{h}) &= e^{-i px - \kappa x + i q y}
\end{align*}
\]
From these equations, we can easily derive the boundary condition \[i.e., Eq. (34)] at $x = -L/2$.

The Bogoliubov–de Gennes equation in the region of $x \leq -L/2$ is equivalent to Eq. (B–1) if we set $\Delta(x) = \Delta e^{-i \omega}$. The right-going wavefunction $\Psi^+ = \Psi^+ (\Psi^+_{h}, \Psi^+_{i})$ and the left-going wavefunction $\Psi^- = \Psi^- (\Psi^-_{h}, \Psi^-_{i})$ in the region of $x \leq -L/2$ are expressed as
\[
\begin{align*}
(\Psi^+_{h}) &= e^{i px + \kappa x + i q y} \\
(\Psi^-_{h}) &= e^{-i px + \kappa x + i q y}
\end{align*}
\]
From these equations, we can easily derive the boundary condition \[i.e., Eq. (34)] at $x = L/2$.

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