ON MULTIPARTITE HAJNAL-SZEMERÉDI THEOREMS

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Abstract. Let $G$ be a $k$-partite graph with $n$ vertices in parts such that each vertex is adjacent to at least $\delta^*(G)$ vertices in each of the other parts. Magyar and Martin [20] proved that for $k = 3$, if $\delta^*(G) \geq \frac{2}{3}n + 1$ and $n$ is sufficiently large, then $G$ contains a $K_3$-factor (a spanning subgraph consisting of $n$ vertex-disjoint copies of $K_3$). Martin and Szemerédi [21] proved that $G$ contains a $K_4$-factor when $\delta^*(G) \geq \frac{3}{4}n$ and $n$ is sufficiently large. Both results were proved using the Regularity Lemma. In this paper we give a proof of these two results by the absorbing method. Our absorbing lemma actually works for all $k \geq 3$ and may be utilized to prove a general and tight multipartite Hajnal-Szemerédi theorem.

1. Introduction

Let $H$ be a graph on $h$ vertices, and let $G$ be a graph on $n$ vertices. Packing (or tiling) problems in extremal graph theory are investigations of conditions under which $G$ must contain many vertex-disjoint copies of $H$ (as subgraphs), where minimum degree conditions are studied the most. An $H$-matching of $G$ is a subgraph of $G$ which consists of vertex-disjoint copies of $H$. A perfect $H$-matching, or $H$-factor, of $G$ is an $H$-matching consisting of $\lfloor n/h \rfloor$ copies of $H$. Let $K_k$ denote the complete graph on $k$ vertices. The celebrated theorem of Hajnal and Szemerédi [6] says that every $n$-vertex graph $G$ with $\delta(G) \geq (k - 1)n/k$ contains a $K_k$-factor (see [11] for another proof).

Using the Regularity Lemma of Szemerédi [25], researchers have generalized this theorem for packing arbitrary $H$ [1, 15, 24, 16]. Results and methods for packing problems can be found in the survey of Kühn and Osthus [17].

In this paper we consider multipartite packing, which restricts $G$ to be a $k$-partite graph for $k \geq 2$. A $k$-partite graph is called balanced if its partition sets have the same size. Given a $k$-partite graph $G$, it is natural to consider the minimum partite degree $\delta^*(G)$, the minimum degree from a vertex in one partition set to any other partition set. When $k = 2$, $\delta^*(G)$ is simply $\delta(G)$. In most of the rest of this paper, the minimum degree condition stands for the minimum partite degree for short.

Let $G_k(n)$ denote the family of balanced $k$-partite graphs with $n$ vertices in each of its partition sets. It is easy to see (e.g., using the König-Hall Theorem) that every bipartite graph $G \in G_k(n)$ with $\delta^*(G) \geq n/2$ contains a 1-factor. Fischer [14] conjectured that if $G \in G_k(n)$ satisfies

$$\delta^*(G) \geq \frac{k - 1}{k}n,$$

then $G$ contains a $K_k$-factor and proved the existence of an almost $K_k$-factor for $k = 3, 4$. Magyar and Martin [20] noticed that the condition (1) is not sufficient for odd $k$ and instead proved the following theorem for $k = 3$. (They actually showed that when $n$ is divisible by 3, there is only one graph in $G_3(n)$, denoted by $\Gamma_3(n/3)$, that satisfies (1) but fails to contain a $K_3$-factor, and adding any new edge to $\Gamma_3(n/3)$ results in a $K_3$-factor.)

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Theorem 1 (20). There exists an integer $n_0$ such that if $n \geq n_0$ and $G \in \mathcal{G}_k(n)$ satisfies $\delta^*(G) \geq 2n/3 + 1$, then $G$ contains a $K_3$-factor.

On the other hand, Martin and Szemerédi 21 proved the original conjecture holds for $k = 4$.

Theorem 2 (21). There exists an integer $n_0$ such that if $n \geq n_0$ and $G \in \mathcal{G}_k(n)$ satisfies $\delta^*(G) \geq 3n/4$, then $G$ contains a $K_4$-factor.

Recently Keevash and Mycroft 9 and independently Lo and Markström 14 proved that Fischer’s conjecture is asymptotically true, namely, $\delta^*(G) \geq \frac{k-1}{k}n + o(n)$ guarantees a $K_k$-factor for all $k \geq 3$. Very recently, Keevash and Mycroft. 10 improved this to an exact result.

In this paper we give a new proof of Theorems 1 and 2 by the absorbing method. Our approach is similar to that of 19 (in contrast, a geometric approach was employed in 9). However, in order to prove exact results by the absorbing lemma, one needs only assume $\delta^*(G) \geq (1 - 1/k)n$, instead of $\delta^*(G) \geq (1 - 1/k + \alpha)n$ for some $\alpha > 0$ as in 19. In fact, our absorbing lemma uses an even weaker assumption $\delta^*(G) \geq (1 - 1/k - \alpha)n$ and has a more complicated absorbing structure.

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi 23, has been shown to be effective handling extremal problems in graphs and hypergraphs. One example is the re-proof of Posa’s conjecture by Levitt, Sárközy, and Szemerédi 18, while the original proof of Komlós, Sárközy, and Szemerédi 13 used the Regularity Lemma. Our paper is another example of replacing the regularity method with the absorbing method. Compared with the threshold $n_0$ in Theorems 1 and 2 derived from the Regularity Lemma, the value of our $n_0$ is much smaller.

Before presenting our proof, let us first recall the approach used in 20, 21. Given a $k$-partite graph $G \in \mathcal{G}_k(n)$ with parts $V_1, \ldots, V_k$, the authors said that $G$ is $\Delta$-extremal if each $V_i$ contains a subset $A_i$ of size $\lfloor n/k \rfloor$ such that the density $d(A_i, A_j) \leq \Delta$ for all $i \neq j$. Using standard but involved graph theoretic arguments, they solved the extremal case for $k = 3, 4$ 20, Theorem 3.1], 21, Theorem 2.1].

Theorem 3. Let $k = 3, 4$. There exists $\Delta$ and $n_0$ such that the following holds. Let $n \geq n_0$ and $G \in \mathcal{G}_k(n)$ be a $k$-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and (1) when $k = 4$. If $G$ is $\Delta$-extremal, then $G$ contains a $K_k$-factor.

To handle the non-extremal case, they proved the following lemma (20, Lemma 2.2] and 21, Lemma 2.2).

Lemma 4 (Almost Covering Lemma). Let $k = 3, 4$. Given $\Delta > 0$, there exists $\alpha > 0$ such that for every graph $G \in \mathcal{G}_k(n)$ with $\delta^*(G) \geq (1 - 1/k)n - \alpha$ on $G$ contains an almost $K_k$-factor that leaves at most $C(k)$ vertices uncovered or $G$ is $\Delta$-extremal.

To improve the almost $K_k$-factor obtained from Lemma 4, they used the Regularity Lemma and Blow-up Lemma 11. Here is where we need our absorbing lemma whose proof is given in Section 2. Our lemma actually gives a more detailed structure than what is needed for the extremal case when $G$ does not satisfy the absorbing property.

We need some definitions. Given positive integers $k$ and $r$, let $\Theta_{k \times r}$ denote the graph with vertices $a_{ij}, i = 1, \ldots, k, j = 1, \ldots, r$, and $a_{ij}$ is adjacent to $a_{i'j'}$ if and only if $i \neq i'$ and $j \neq j'$. In addition, given a positive integer $t$, the graph $\Theta_{k \times r}(t)$ denotes the blow-up of $\Theta_{k \times r}$, obtained by replacing vertices $a_{ij}$ with sets $A_{ij}$ of size $t$, and edges $a_{ij}a_{i'j'}$ with complete bipartite graphs between $A_{ij}$ and $A_{i'j'}$. Given $\epsilon, \Delta > 0$ and $t \geq 1$ (not necessarily an integer), we say that a $k$-partite graph $G$ is $(\epsilon, \Delta)$-approximate to $\Theta_{k \times r}(t)$ if each of its partition sets $V_i$ can be partitioned into $\bigcup_{j=1}^{t} V_{ij}$ such that $|V_{ij}| - t \leq ct$ for all $i, j$ and $d(V_{ij}, V_{i'j}) \leq \Delta$ whenever $i \neq i'$ 21.

1Here we follow the definition of $(\epsilon, \Delta)$-approximation in 20, 21. It seems natural to require that $d(V_{ij}, V_{i'j}) \geq 1 - \Delta$ whenever $i \neq i'$ and $j \neq j'$ as well. However, this follows from $d(V_{ij}, V_{i'j}) \leq \Delta (i \neq i')$ when $\delta^*(G) \geq (1 - 1/r)rt$.\]
Lemma 5 (Absorbing Lemma). Given $k \geq 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_1 > 0$ such that the following holds. Let $n \geq n_1$ and $G \in \mathcal{G}_k(n)$ be a $k$-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha n$. Then one of the following cases holds.

1. $G$ contains a $K_k$-matching $M$ of size $|M| \leq 2(k - 1)\alpha^{k-2}n$ in $G$ such that for every $W \subset V \setminus V(M)$ with $|W \cap V_i| = \cdots = |W \cap V_k| \leq \alpha^{k-6}n/4$, there exists a $K_k$-matching covering exactly the vertices in $V(M) \cup W$.

2. We may remove some edges from $G$ so that the resulting graph $G'$ satisfies $\delta^*(G') \geq (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k \times k}(\alpha n)$.

The $K_k$-matching $M$ in Lemma 5 has the so-called absorbing property: it can absorb any balanced set with a much smaller size.

Proof of Theorems 7 and 3. Let $k = 3, 4$. Let $\alpha \ll \Delta$, where $\Delta$ is given by Theorem 3 and $\alpha$ satisfies both Lemmas 4 and 5. Suppose that $n$ is sufficiently large. Let $G \in \mathcal{G}_k(n)$ be a $k$-partite graph satisfying $\delta^*(G) \geq (2/3)n + 1$ when $k = 3$ and 11 when $k = 4$. By Lemma 5, either $G$ contains a subgraph which is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k \times k}(\alpha n)$ or $G$ contains an absorbing $K_k$-matching $M$. In the former case, for $i = 1, \ldots, k$, we add or remove at most $\frac{\Delta n}{6k}$ vertices from $V_i$ to obtain a set $A_i \subset V_i$ of size $|n/k|$. For $i \neq i'$, we have

$$e(A_i, A_{i'}) \leq e(V_i, V_{i'}) + \frac{\Delta n}{6k}(|A_i| + |A_{i'}|).$$

which implies that $d(A_i, A_{i'}) \leq \Delta$. Thus $G$ is $\Delta$-extremal. By Theorem 3, $G$ contains a $K_k$-factor. In the latter case, $G$ contains a $K_k$-matching $M$ of size $|M| \leq 2(k - 1)\alpha^{k-2}n$ such that for every $W \subset V \setminus V(M)$ with $|W \cap V_i| = \cdots = |W \cap V_k| \leq \alpha^{k-6}n/4$, there exists a $K_k$-matching on $V(M) \cup W$. Let $G' = G \setminus V(M)$ be the induced subgraph of $G$ on $V(G) \setminus V(M)$, and $n' = |V(G')|$. Clearly $G'$ is balanced. As $\alpha \ll 1$, we have

$$\delta^*(G') \geq \delta^*(G') - |M| \geq \left(1 - \frac{1}{k}\right)n - 2(k - 1)\alpha^{k-2}n \geq \left(1 - \frac{1}{k} - \alpha\right)n'.$$

By Lemma 4, $G'$ contains a $K_k$-matching $M'$ such that $|V(G') \setminus V(M')| \leq C$. Let $W = V(G') \setminus V(M')$. Clearly $|W \cap V_i| = \cdots = |W \cap V_k|$. Since $C/k \leq \alpha^{k-6}n/4$ for sufficiently large $n$, by the absorbing property of $M$, there is a $K_k$-matching $M''$ on $V(M) \cup W$. This gives the desired $K_k$-factor $M' \cup M''$ of $G$.

Remarks.

- Since our Lemma 5 works for all $k \geq 3$, it has the potential of proving a general multipartite Hajnal-Szemerédi theorem. To do it, one only needs to prove Theorem 3 and Lemma 4 for $k \geq 5$.

- Since our Lemma 5 gives a detailed structure of $G$ when $G$ does not have desired absorbing $K_k$-matching, it has the potential of simplifying the proof of the extremal case. Indeed, if one can refine Lemma 4 such that it concludes that $G$ either contains an almost $K_k$-factor or it is approximate to $\Theta_{k \times k}(\alpha n)$ and other extremal graphs, then in Theorem 3 we may assume that $G$ is actually approximate to these extremal graphs.

- Using the Regularity Lemma, researchers have obtained results on packing arbitrary graphs in $k$-partite graphs, see [26, 8, 3, 2] for $k = 2$ and [22] for $k = 3$. With the help of the recent
result of Keevash–Mycroft \cite{9} and Lo-Markström \cite{19}, it seems not very difficult to extend these results to the $k \geq 4$ case (though exact results may be much harder). However, it seems difficult to replace the regularity method by the absorbing method for these problems.

2. Proof of the Absorbing Lemma

In this section we prove the Absorbing Lemma (Lemma 5). We first introduce the concepts of reachability.

Definition 6. In a graph $G$, a vertex $x$ is reachable from another vertex $y$ by a set $S \subseteq V(G)$ if both $G[x \cup S]$ and $G[y \cup S]$ contain $K_k$-factors. In this case, we say $S$ connects $x$ and $y$.

The following lemma plays a key role in constructing absorbing structures. We postpone its proof to the end of the section.

Lemma 7 (Reachability Lemma). Given $k \geq 3$ and $\Delta > 0$, there exists $\alpha = \alpha(k, \Delta) > 0$ and an integer $n_2 > 0$ such that the following holds. Let $n \geq n_2$ and $G \in \mathcal{G}_k(n)$ be a $k$-partite graph on $V_1 \cup \cdots \cup V_k$ such that $\delta^*(G) \geq (1 - 1/k)n - \alpha n$. Then one of the following cases holds.

1. For any $x$ and $y$ in $V_i$, $i \in [k]$, $x$ is reachable from $y$ by either at least $\alpha^3 n^{k-1} (k-1)$-sets or at least $\alpha^3 n^{2k-1} (2k-1)$-sets in $G$.

2. We may remove some edges from $G$ so that the resulting graph $G'$ satisfies $\delta^*(G') \geq (1 - 1/k)n - \alpha n$ and is $(\Delta/6, \Delta/2)$-approximate to $\Theta_{k \times k}(\frac{n}{k})$.

With the aid of Lemma 7, the proof of Lemma 5 becomes standard counting and probabilistic arguments, as shown in \cite{7}.

Proof of Lemma 5. We assume that $G$ does not satisfy the second property stated in the lemma.

Given a crossing $k$-tuple $T = (v_1, \cdots, v_k)$, with $v_i \in V_i$, for $i = 1, \cdots, k$, we call a set $A$ an absorbing set for $T$ if both $G[A]$ and $G[A \cup T]$ contain $K_k$-factors. Let $\mathcal{L}(T)$ denote the family of all $2k(k-1)$-sets that absorb $T$ (the reason why our absorbing sets are of size $2k(k-1)$ can be seen from the proof of Claim 8 below).

Claim 8. For every crossing $k$-tuple $T$, we have $|\mathcal{L}(T)| > \alpha^{4k-3} n^{2k(k-1)}$.

Proof. Fix a crossing $k$-tuple $T$. First we try to find a copy of $K_k$ containing $v_1$ and avoiding $v_2, \ldots, v_k$. By the minimum degree condition, there are at least

$$\prod_{i=2}^{k} \left( n - 1 - (i - 1) \left( \frac{1}{k} + \alpha \right) n \right) \geq \prod_{i=2}^{k} \left( n - (i - 1) \frac{n}{k} - ((k - 1)\alpha n + 1) \right)$$

such copies of $K_k$. When $n \geq 3k^2$ and $\frac{1}{\alpha} \geq 3k^2$, we have $(k - 1)\alpha n + 1 \leq n/(3k)$ and thus the number above is at least

$$\prod_{i=2}^{k} \left( n - (i - 1) \frac{n}{k} - \frac{n}{3k} \right) \geq \left( \frac{n}{k} \right)^{k-1}, \text{ when } k \geq 3.$$

Fix such a copy of $K_k$ on $\{v_1, u_2, u_3, \ldots, u_k\}$. Consider $u_2$ and $v_2$. By Lemma 7 and the assumption that $G$ does not satisfy the second property of the lemma, we can find at least $\alpha^3 n^{k-1} (k-1)$-sets or $\alpha^3 n^{2k-1} (2k-1)$-sets to connect $u_2$ and $v_2$. If $S$ is a $(k-1)$-set that connects $u_2$ and $v_2$, then $S \cup K$ also connects $u_2$ and $v_2$ for any $k$-set $K$ such that $G[K] \cong K_k$ and $K \cap S = \emptyset$. There are at least

$$(n-2) \prod_{i=2}^{k} \left( n - 1 - (i - 1) \left( \frac{1}{k} + \alpha \right) n \right) \geq \frac{n}{2} \left( \frac{n}{k} \right)^{k-1}$$
copies of $K_k$ in $G$ avoiding $u_2, v_2$ and $S$. If there are at least $\alpha^3n^{k-1}$ $(k-1)$-sets that connect $u_2$ and $v_2$, then at least

$$\alpha^3n^{k-1} \cdot \frac{n}{2} \left( \frac{n}{k} \right)^{k-1} \frac{1}{2^{k-1}} \geq 2\alpha^3n^{2k-1}$$

$(2k-1)$-sets connect $u_2$ and $v_2$ because a $(2k-1)$-set can be counted at most $\binom{2k-1}{k-1}$ times. Since $2\alpha^4 < \alpha^3$, we can assume that there are always at least $2\alpha^4n^{2k-1} (2k-1)$-sets connecting $u_2$ and $v_2$. We inductively choose disjoint $(2k-1)$-sets that connects $v_i$ and $u_i$ for $i = 2, \ldots, k$. For each $i$, we must avoid $T, u_2, \ldots, u_k$, and $i - 2$ previously selected $(2k-1)$-sets. Hence there are at least $2\alpha^4n^{2k-1} - (2k-1)(i-1)n^{2k-2} > \alpha^4n^{2k-1}$ choices of such $(2k-1)$-sets for each $i \geq 2$. Putting all these together, and using the assumption that $\alpha$ is sufficiently small, we have

$$|\mathcal{L}(T)| \geq \left( \frac{n}{k} \right)^{k-1} \cdot (\alpha^4n^{2k-1})^{k-1} > \alpha^{4k-3}n^{2k(k-1)}.$$
in each of the other color classes. Then either $H$ contains at least $e^2t^k$ copies of $K_k$, or $H$ is $(16kAe^{1/2k^2-2}, 16kAe^{1/2k^2-2})$-approximate to $\Theta_{k \times (k-1)}(t)$.

Proof. First we derive an upper bound for $|V_i|$, $i \in [k]$. Suppose for example, that $|V_k| \geq (k-1)(1 + \epsilon)t + ct$. Then if we greedily construct copies of $K_k$ while choosing the last vertex from $V_k$, by the minimum degree condition and $\epsilon \ll 1$, there are at least

$$|V_i| \geq (k-1)(1-\epsilon)t \cdot (k-2)(1-\epsilon)t \cdots (1-(2k-3)\epsilon)t \cdot ct \geq (k-1 - \frac{1}{2})(k-2 - \frac{1}{2}) \cdots (1 - \frac{1}{2})c t^k \geq \frac{2}{t}k^k$$

copies of $K_k$ in $H$, so we are done. We thus assume that for all $i$,

$$|V_i| \leq (k-1)(1 + \epsilon)t + ct < (k-1)(1 + 2\epsilon)t.$$  (4)

Now we proceed by induction on $k$. The base case is $k = 2$. If $H$ has at least $e^2t^2$ edges, then we are done. Otherwise $e(H) < e^2t^2$. Using the lower bound for $|V_i|$, we obtain that

$$d(V_1, V_2) < \frac{e^2t^2}{|V_1| \cdot |V_2|} \leq \frac{e^2}{(1-\epsilon)^2} < \epsilon.$$

Hence $H$ is $(2\epsilon, \epsilon)$-approximate to $\Theta_{2 \times 1}(t)$. When $k = 2$, $16kAe^{1/2k^2-2} = 256\epsilon$, so we are done.

Now assume that $k \geq 3$ and the conclusion holds for $k-1$. Let $H$ be a $k$-partite graph satisfying the assumptions and assume that $H$ contains less than $e^2t^k$ copies of $K_k$.

For simplicity, write $N_i(v) = N(v) \cap V_i$ for any vertex $v$. Let $V_i' \subset V_i$ be the vertices which are in at least $ct^{k-1}$ copies of $K_k$ in $H$, and let $\bar{V}_i = V_i \setminus V_i'$. Note that $|V_i'| < ct$ otherwise we get at least $e^2t^k$ copies of $K_k$ in $H$. Fix $v_0 \in \bar{V}_1$. For $2 \leq i \leq k$, by the minimum degree condition and $k \geq 3$,

$$|N_i(v_0)| \geq (k-1)(1-\epsilon)t - (1 + \epsilon)t = (k-2) \left(1 - \frac{k}{k-2}\epsilon\right)t \geq (k-2)(1 - 3\epsilon)t.$$  

On the other hand, following the same arguments as we used for (3), we derive that

$$|N_i(v_0)| \leq (k-2)(1 + 2\epsilon t).$$  

(5)

The minimum degree condition implies that a vertex in $N(v_0)$ misses at most $(1 + \epsilon)t$ vertices in each $N_i(v_0)$. We now apply induction with $k-1$, $t$ and $3\epsilon$ on $H[N(v_0)]$. Because of the definition of $V_i'$, we conclude that $N(v_0)$ is $(e', \epsilon')$-approximate to $\Theta_{(k-1) \times (k-2)}(t)$, where

$$e' : = 16(k-1)^4(3\epsilon)^{1/2k^3-2}.$$  

This means that we can partition $N_i(v_0)$ into $A_{i1} \cup \cdots A_{i(k-2)}$ for $2 \leq i \leq k$ such that

$$\forall 2 \leq i \leq k, \ 1 \leq j \leq k-2, \ (1 - \epsilon')t \leq |A_{ij}| \leq (1 + \epsilon')t \quad \text{and} \quad (6)$$

$$\forall 2 \leq i < i' \leq k, \ 1 \leq j \leq k-2, \ d(A_{ij}, A_{i'j}) \leq \epsilon'. \quad (7)$$

Furthermore, let $A_{i(k-1)} := V_i \setminus N(v_0)$ for $i = 2, \cdots, k$. By (5) and the minimum degree condition, we get that

$$1 - (3k-5)\epsilon t \leq |A_{i(k-1)}| \leq (1 + \epsilon)t,$$  

(8)

for $i = 2, \cdots, k$.

Let $A_{ij}' = V_i \setminus A_{ij}$ denote the complement of $A_{ij}$. Let $\bar{e}(A, B) = |A||B| - e(A, B)$ denote the number of non-edges between two disjoint sets $A$ and $B$, and $d(A, B) = \bar{e}(A, B)/(|A||B|) = 1 - d(A, B)$. Given two disjoint sets $A$ and $B$ (with density close to one) and $\alpha > 0$, we call a vertex $a \in A$ is $\alpha$-typical to $B$ if $\deg_B(a) \geq (1 - \alpha)|B|$.

Claim 10. Let $2 \leq i \neq i' \leq k$, $1 \leq j \neq j' \leq k-1$.

1. $d(A_{ij}, A_{i'j'}) \geq 1 - 3\epsilon'$ and $d(A_{ij}, A_{i'j}) \geq 1 - 3\epsilon'$.  

(1)
(2) All but at most $\sqrt{3\epsilon'}$ vertices in $A_{ij}$ are $\sqrt{3\epsilon'}$-typical to $A_{ij'}$; at most $\sqrt{3\epsilon'}$ vertices in $A_{ij}$ are $\sqrt{3\epsilon'}$-typical to $A_{ij'}$.

Proof. (1). Since $A_{ij} = \bigcup_{j' \neq j} A_{ij'}$, the second assertion $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$ immediately follows from the first assertion $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$. Thus it suffices to show that $d(A_{ij}, A_{ij'}) \geq 1 - 3\epsilon'$, or equivalently that $d(A_{ij}, A_{ij'}) \leq 3\epsilon'$.

Assume $j \geq 2$. By (1), we have $e(A_{ij}, A_{ij'}) \leq \epsilon|A_{ij}||A_{ij'}|$. So if $s$ satisfy $\deg_{ij} v < (1 - \epsilon)|A_{ij}|$, we have $|\bar{A}v| \leq \epsilon|A_{ij}|$, which implies that $e((A_{ij}, A_{ij'}) \leq (\epsilon + 2\epsilon')t|A_{ij}|$ for any $j' \neq j$ and $1 \leq j' \leq k - 1$. By (6) and (8), we have $|A_{ij'}| \geq (1 - \epsilon')t$. Hence

$$d(A_{ij}, A_{ij'}) \leq (\epsilon + 2\epsilon')t = \epsilon + 2\epsilon'
$$

where the last inequality holds because $\epsilon < \epsilon'$.

(2) Given two disjoint sets $A$ and $B$, if $d(A, B) \leq \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|A|$ vertices $a \in A$ satisfy $\deg_B a < (1 - \sqrt{\alpha})|B|$. Hence Part (2) immediately follows from Part (1). \hfill \square

We need a lower bound for the number of copies of $K_k$ in a dense $k$-partite graph.

**Proposition 11.** Let $G$ be a $k$-partite graph with vertex class $V_1, \ldots, V_k$. Suppose for every two vertex classes, the pairwise density $d(V_i, V_j) \geq 1 - \alpha$ for some $\alpha \leq (k + 1)^{-4}$, then there are at least $\frac{1}{2} \prod_i |V_i|$ copies of $K_k$ in $G$.

Proof. Given two disjoint sets $V_i$ and $V_j$, if $d(V_i, V_j) \leq \alpha$ for some $\alpha > 0$, then at most $\sqrt{\alpha}|V_i|$ vertices $v \in V_i$ satisfy $\deg_{V_j}(v) < (1 - \sqrt{\alpha})|V_j|$. Thus, by choosing $\frac{1}{2} \prod \alpha_i V_i$ greedy and the assumption $\alpha < (k + 1)^{-4}$, there are at least

$$(1 - \sqrt{\alpha})|V_i|(1 - 2\sqrt{\alpha})|V_2| \cdots (1 - k\sqrt{\alpha})|V_k| > (1 - (1 + \cdots + k)\sqrt{\alpha}) \prod_i |V_i| > \frac{1}{2} \prod_i |V_i|$$

copies of $K_k$ in $G$. \hfill \square

Let $\epsilon'' = 2k\sqrt{\epsilon'}$. Now we want to study the structure of $\tilde{V}_1$.

**Claim 12.** Given $v \in \tilde{V}_1$ and $2 \leq i \leq k$, there exists $j \in [k - 1]$, such that $|N_{A_{ij}}(v)| < \epsilon''t$.

Proof. Suppose instead, that there exist $v \in \tilde{V}_1$ and some $2 \leq i_0 \leq k$, such that $|N_{A_{ij}}(v)| \geq \epsilon''t$ for all $j \in [k - 1]$. By the minimum degree condition, for each $2 \leq i \leq k$, there is at most one $j \in [k - 1]$ such that $|N_{A_{ij}}(v)| < t/3$. Therefore we can greedily choose $k - 2$ distinct $j_i$ for $i \neq i_0$, such that $|N_{A_{ij_i}}(v)| \geq t/3$. Let $j_{i_0}$ be the the (unique) unused index. Note that

$$\forall i \neq i_0, \quad |A_{ij_0} - A_{ij_i}| \leq \frac{(1 + \epsilon')t}{t/3} < 4, \quad \text{and} \quad |A_{ij_0} - A_{ij_0}| \leq \frac{(1 + \epsilon')t}{\epsilon''t} < \frac{2}{\epsilon''}$$

So for any $i \neq i'$, by Claim 10 and the definition of $\epsilon''$, we have

$$d(N_{A_{ij}}, (v), N_{A_{ij'}}(v)) \leq \frac{3\epsilon' |A_{ij}||A_{ij'}|}{|N_{A_{ij}}(v)||N_{A_{ij'}}(v)|} \leq 3\epsilon' \cdot 4 \cdot \frac{2}{\epsilon''} = \frac{6}{k^2} \epsilon''.$$  \hfill (9)
Since $\epsilon \ll \epsilon'' \ll 1$, by Proposition 11 there are at least
\[
\frac{1}{2} \prod_i N_{A_{ij}}(v) \geq \frac{1}{2} \cdot \epsilon'' t \left( \frac{1}{2} \right)^{k-2} = \frac{\epsilon''}{2 \cdot 3^{k-2}} k-1 > \epsilon k^{-1}
\]
copies of $K_{k-1}$ in $N(v)$, contradicting the assumption $v \in \tilde{V}_1$.

Note that if $\deg_{A_{ij}}(v) < \epsilon'' t$, at least $|A_{ij}| - \epsilon'' t$ vertices of $A_{ij}$ are not in $N(v)$. By the minimum degree condition, (6) and (8), it follows that
\[
|A_{ij} \setminus N(v)| \leq (1 + \epsilon) t - (|A_{ij}| - \epsilon'' t) \leq (1 + \epsilon) t - (1 - \epsilon') t + \epsilon'' t \leq 2 \epsilon'' t. \tag{10}
\]
Fix a vertex $v \in \tilde{V}_1$. Given $2 \leq i \leq k$, let $\ell_i$ denote the (unique) index such that $|N_{A_{ii}}(v)| < \epsilon'' t$ (the existence of $\ell_i$ follows from Claim 12).

**Claim 13.** We have $\ell_2 = \ldots = \ell_k$.

**Proof.** Otherwise, say $\ell_2 \neq \ell_3$, then we set $j_2 = \ell_3$ and for $3 \leq i \leq k$, greedily choose distinct $j_k, j_{k-1}, \ldots, j_3 \in [k-1] \setminus \{\ell_i\}$ such that $j_i \neq \ell_i$ (this is possible as $j_3$ is chosen at last). Let us bound the number of copies of $K_{k-1}$ in $\bigcup_{i=2}^{k-1} N_{A_{ii}}(v)$. By 11 we get $|N_{A_{ij}}(v)| \geq |A_{ij}| - 2 \epsilon'' t \geq t/2$ for all $i$. As in 9, for any $i \neq i'$, we derive that $d(N_{A_{ij}}(v), N_{A_{i'j'}}(v)) \leq 3 \epsilon'' \cdot 4 \cdot 4 = 48 \epsilon''$. Since $\epsilon'' < 1$, by Proposition 11 we get at least $\frac{1}{2} (\frac{1}{2})^{k-1} > \epsilon k^{-1}$ copies of $K_{k-1}$ in $N(v)$, a contradiction. \qed

We define $A_{ij} := \{v \in \tilde{V}_1 : |N_{A_{ij}}(v)| < \epsilon'' t\}$ for $j \in [k-1]$. By Claims 12 and 13 this yields a partition of $\tilde{V}_1 = \bigcup_{j=1}^{k-1} A_{ij}$ such that
\[
d(A_{ij}, A_{ij}) \leq \frac{\epsilon'' t |A_{ij}|}{|A_{ij}|} \leq \frac{\epsilon'' t}{1 - \epsilon'} \leq (1 + 2 \epsilon') \epsilon'' \quad \text{for} \quad i \geq 2 \quad \text{and} \quad j \geq 1. \tag{11}
\]

By 6, 8 and 10, as $(3k - 5) \epsilon \leq \epsilon'$, we have
\[
d(A_{ij}, A_{ij'}) \leq \frac{|A_{ij}| \epsilon'' t}{|A_{ij}|} \leq \frac{2 \epsilon'' t}{1 - \epsilon'} \leq 3 \epsilon'' \quad \text{for} \quad i \geq 2 \quad \text{and} \quad j \neq j'. \tag{12}
\]
We claim $|A_{ij}| \leq (1 + \epsilon) t + (1 + 2 \epsilon') \epsilon'' |A_{ij}|$ for all $j$. Otherwise, by the minimum degree condition, we have $\deg_{A_{ij}}(v) > (1 + 2 \epsilon') \epsilon'' |A_{ij}|$ for all $v \in A_{ij}$, and consequently $d(A_{ij}, A_{ij}) > (1 + 2 \epsilon') \epsilon''$, contradicting 11. We thus conclude that
\[
|A_{ij}| \leq \frac{1 + \epsilon}{1 - (1 + 2 \epsilon') \epsilon''} \leq (1 + 2 \epsilon') \epsilon'' t. \tag{13}
\]
Since $|V_i| \leq \epsilon t$, we have $|\bigcup_{j=1}^{k-1} A_{ij}| = |V_i \setminus V_i| \geq |V_i| - \epsilon t$. Using 13, we now obtain a lower bound for $|A_{ij}|, j \in [k-1]$: \[
|A_{ij}| \geq (k - 1)(1 - \epsilon) t - (k - 2)(1 + 2 \epsilon'') t - \epsilon t \geq (1 - 2k \epsilon'') t. \tag{14}
\]
It remains to show that for $2 \leq i \neq i' \leq k$, $d(A_{ii(k-1)}, A_{i'(k-1)})$ is small. Write $N(v_1 \cdots v_m) = \bigcap_{i \leq m} N(v_i)$.

**Claim 14.** $d(A_{ii(k-1)}, A_{i'(k-1)}) \leq 6 \epsilon''$ for $2 \leq i, i' \leq k$.

**Proof.** Suppose to the contrary, that say $d(A_{ii(k-1)}, A_{i'(k-1)}) > 6 \epsilon''$. We first select $k - 2$ sets $A_{ij}$ with $1 \leq i \leq k - 2$ and $1 \leq j \leq k - 2$ such that no two of them are on the same row or column - there are $(k - 2)!$ choices. Fix one of them, say $A_{11}, A_{22}, \cdots, A_{(k-2)(k-2)}$. We construct copies of $K_{k-2}$ in $A_{11} \cup A_{22} \cup \cdots \cup A_{(k-2)(k-2)}$ as follows. Pick arbitrary $v_1 \in A_{11}$. For $2 \leq i \leq k - 2$, we select $v_i \in N_{A_{ii}}(v_1 \cdots v_{i-1})$ such that $v_i$ is $\sqrt{3 \epsilon'}$-typical to $A_{(k-1)(k-1)}, A_{(k-1)(k-1)}$ and all $A_{ij}, i < j \leq k - 2$. By Claim 10 and 10, there are at least $(1 - (k - 2) \sqrt{3 \epsilon'}) |A_{ii}| - 2 \epsilon' t \geq t/2$ choices for each $v_i$. After selecting $v_1, \ldots, v_{k-2}$, we select adjacent vertices $v_{k-1} \in A_{(k-1)(k-1)}$ and $v_k \in A_{k(k-1)}$ such
that \( v_{k-1}, v_k \in N(v_1 \cdots v_{k-2}) \). For \( j \in \{k-1, k\} \), we know that \( N(v_1) \) misses at most \( 2\epsilon''t \) vertices in \( A_{j(k-1)} \), and at most \((k-3)3^{\frac{3}{4}}e''|A_{j(k-1)}|\) vertices of \( A_{j(k-1)} \) are not in \( N(v_2 \cdots v_{k-2}) \). Since \( d(A_{j(k-1)}, A_{k1}) > 6\epsilon'' \) and \( \epsilon'' = 2k\sqrt{\epsilon'} \), there are at least
\[
6\epsilon''|A_{(k-1)(k-1)}||A_{k(k-1)}| - 2\epsilon''t(|A_{(k-1)(k-1)}| + |A_{k(k-1)}|) - 2(k-3)3^{\frac{3}{4}}e''|A_{(k-1)(k-1)}||A_{k(k-1)}|
\]
\[
\geq (6\epsilon'' - 4\epsilon'') - 4(k-3)3^{\frac{3}{4}}e''|A_{(k-1)(k-1)}||A_{k(k-1)}|
\]
\[
= 12\sqrt{\epsilon''}|A_{(k-1)(k-1)}||A_{k(k-1)}| \geq 6\sqrt{\epsilon''} t^2
\]
such pairs \( v_{k-1}, v_k \). Together with the choices of \( v_1, \cdots, v_{k-2} \), we obtain at least \((k-2)!\left(\frac{\epsilon}{4}\right)^{k-2} 6\sqrt{\epsilon''} t^2 > \epsilon t^k \) copies of \( K_k \), a contradiction. □

In summary, by (9), (8), (13) and (14), we have \((1 - 2\epsilon''\epsilon')t \leq |A_{ij}| \leq (1 + 2\epsilon''\epsilon')t\) for all \( i \) and \( j \). In order to make \( \bigcup_{j=1}^{k-1} A_{ij} \) a partition of \( V_1 \), we move the vertices of \( V_1' \) to \( A_{11} \). Since \( |V_1'| < \epsilon t \), we still have \(||A_{ij}| - t| \leq 2k\epsilon''\epsilon'\) after moving these vertices. On the other hand, by (7), (11), and Claim 14 we have \( d(A_{ij}, A_{ij'}) \leq 6\epsilon'' \leq 2k\epsilon''\) for \( i \neq i' \) and all \( j \) (we now have \( d(A_{11}, A_{11}) \leq 2\epsilon'' \) for all \( i \geq 2 \) because \( |A_{11}| \) becomes slightly larger). Therefore \( H \) is \((2k\epsilon'', 2k\epsilon'')\)-approximate to \( \Theta_{kX(k-1)}(t) \). By the definitions of \( \epsilon'' \) and \( \epsilon' \),
\[
2k\epsilon'' = 4k^2\sqrt{\epsilon'} = 4k^2\sqrt{16(k-1)^4(3e)^{1/2k-3}} \leq 16k^4\epsilon^1/k^{2k-2},
\]
where the last inequality is equivalent to \((k-1)^23^{1/2k-2} \leq 1 \) or \( 3^{1/2k-1} \leq \frac{k}{k-1} \), which holds because
\[
3 \leq 1 + \frac{2k-1}{k-1} \leq (1 + \frac{1}{k-1})^2k \quad \text{for} \quad k \geq 2.
\]
This completes the proof of Lemma 9 □

We are ready to prove Lemma 7.

Proof of Lemma 7. First assume that \( G \in G_3(n) \) is minimal, namely, \( G \) satisfies the minimum partite degree condition but removing any edge of \( G \) will destroy this condition. Note that this assumption is only needed by Claim 20.

Given \( 0 < \Delta \leq 1 \), let
\[
\alpha = \frac{1}{2k} \left( \frac{\Delta}{24k(k-1)^{\sqrt{2k}}} \right)^{2k^{-1}}.
\]
(15)

Without loss of generality, assume that \( x, y \in V_1 \) and \( y \) is not reachable by \( \alpha^3n^{k-1} (k-1)\)-sets or \( \alpha^3n^{2k-1} (2k-1)\)-sets from \( x \).

For \( 2 \leq i \leq k \), define
\[
A_{1i} = V_i \cap (N(x) \setminus N(y)), \quad A_{ik} = V_i \cap (N(y) \setminus N(x)),
\]
\[
B_i = V_i \cap (N(x) \cap N(y)), \quad A_{i0} = V_i \setminus (N(x) \cup N(y)).
\]

Let \( B = \bigcup_{i=2} B_i \). If there are at least \( \alpha^3n^{k-1} \) copies of \( K_{k-1} \) in \( B \), then \( x \) is reachable from \( y \) by at least \( \alpha^3n^{k-1} (k-1)\)-sets. We thus assume there are less than \( \alpha^3n^{k-1} \) copies of \( K_{k-1} \) in \( B \).

Clearly, for \( i \geq 2 \), \( A_{11}, A_{1k}, B_i \) and \( A_{i0} \) are pairwise disjoint. The following claim bounds the sizes of \( A_{ik}, B_i \) and \( A_{i0} \).

Claim 15. \( \begin{align*}
(1) \quad (1 - k^2\alpha)^{\frac{3}{2k}} &< |A_{1i}|, |A_{ik}| \leq (1 + k\alpha)^{\frac{3}{2k}}, \\
(2) \quad (k - 2 - 2k\alpha)^{\frac{3}{2k}} &\leq |B_i| < (k - 2 + k(k-1)\alpha)^{\frac{3}{2k}}, \\
(3) \quad |A_{i0}| < (k + 1)\alpha n.
\end{align*} \)
Proof. For \( v \in V \), and \( i \in [k] \), write \( N_i(v) := N(v) \cap V_i \). By the minimum degree condition, we have \( |A_{i1}|, |A_{ik}| \leq (1/k + \alpha)n \). Since \( N_i(x) = A_{i1} \cup B_i \), it follows that
\[
|B_i| \geq \left( \frac{k-1}{k} - \alpha \right)n - \left( \frac{1}{k} + \alpha \right)n. \tag{16}
\]

If some \( B_i \), say \( B_k \), has at least \( \left( \frac{k-2}{k} + (k-1)\alpha \right)n \) vertices, then there are at least \( \prod_{i=2}^{k-1} |B_i| - (i-2) \left( \frac{1}{k} + \alpha \right) \) copies of \( K_{k-1} \) in \( B \). By \( [16] \) and \( |B_k| \geq \left( \frac{k-2}{k} - (k-1)\alpha \right)n \), this is at least
\[
\alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-1}{k} - \alpha \right)n - (i-1) \left( \frac{1}{k} + \alpha \right)n
= \alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-i}{k} - i\alpha \right)n
\geq \alpha n \cdot \prod_{i=2}^{k-1} \left( \frac{k-i-\frac{1}{k}}{k} \right)n
\geq \alpha n \cdot \frac{1}{2} \left( \frac{n}{k} \right)^{k-2}
\geq \alpha^2 n^{k-1}
\]

because \( 2k\alpha \leq 1 \).

This is a contradiction.

We may thus assume that \( |B_i| < \left( \frac{k-2}{k} + (k-1)\alpha \right)n \) for \( 2 \leq i \leq k \), as required for Part (2). As \( N_i(x) = A_{i1} \cup B_i \), it follows that
\[
|A_{i1}| > \left( \frac{k-1}{k} - \alpha \right)n - \left( \frac{k-2}{k} + (k-1)\alpha \right)n = \left( \frac{1}{k} - k\alpha \right)n.
\]
The same holds for \( |A_{ik}| \) thus Part (1) follows. Finally
\[
|A_{i0}| = |V_i| - |N_i(x)| - |A_{ik}| < n - \left( \frac{k-1}{k} - \alpha \right)n - \left( \frac{1}{k} - k\alpha \right)n = (k + 1)\alpha n,
\]
as required for Part (3). \( \square \)

Let \( t = n/k \) and \( \epsilon = 2k\alpha \). By the minimum degree condition, every vertex \( u \in B \) is nonadjacent to at most \( (1 + k\alpha)n/k < (1 + \epsilon)t \) vertices in other color classes of \( B \). By Claim \([15]\) \( |B_i| \geq (k-2-2k\alpha)\frac{n}{k} = (1-\epsilon)t \geq (k-2)(1-\epsilon)t \). Thus \( G[B] \) is a \( (k-1) \)-partite graph that satisfies the assumptions of Lemma \([9]\). We assumed that \( B \) contains less than \( \alpha^3 n^{k-1} < \epsilon^2 t^{k-1} \) copies of \( K_{k-1} \), so by Lemma \([9]\) \( B \) is \( (\alpha', \alpha') \)-approximate to \( \Theta((k-1)\times(k-2))(\frac{n}{k}) \), where
\[
\alpha' := 16(k-1)^4 (2k\alpha)^{1/2k-3}.
\]
This means that we can partition \( B_i \), \( 2 \leq i \leq k \), into \( A_{i2} \cup \cdots A_{i(k-1)} \) such that \( (1-\alpha')\frac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\frac{n}{k} \) for \( 2 \leq j \leq k-1 \) and
\[
\forall 2 \leq i < i' \leq k, 2 \leq j \leq k-1, \ d(A_{ij}, A_{i'j}) \leq \alpha'. \tag{17}
\]

Together with Claim \([15]\) Part (1), we obtain that (using \( k^2\alpha \leq \alpha' \))
\[
\forall 2 \leq i \leq k, 1 \leq j \leq k, \ (1-\alpha')\frac{n}{k} \leq |A_{ij}| \leq (1+\alpha')\frac{n}{k}. \tag{18}
\]

Let \( A^c_{ij} := V_i \setminus A_{ij} \) denote the complement of \( A_{ij} \). The following claim is an analog of Claim \([16]\) and its proof is almost the same – after we replace \( (1+\epsilon)t \) with \( (1+k\alpha)n/k \) and \( \epsilon \) with \( \alpha' \) (and we use \( \alpha \ll \alpha' \)). We thus omit the proof.

**Claim 16.** Let \( 2 \leq i \neq i' \leq k, 1 \leq j \neq j' \leq k \), and \( \{j, j'\} \neq \{1, k\} \).
\[\begin{align*}
(1) \ d(A_{ij}, A_{ij'}) & \geq 1 - 3\alpha' \quad \text{and} \quad d(A_{ij}, A^c_{ij'}) \geq 1 - 3\alpha'. \\
(2) \ \text{All but at most} \ \sqrt{3\alpha'} \ \text{vertices in} \ A_{ij} \ \text{are} \ \sqrt{3\alpha'}-\text{typical to} \ A_{ij'}; \ \text{at most} \ \sqrt{3\alpha'} \ \text{vertices in} \ A_{ij} \\
\quad \text{are} \ \sqrt{3\alpha'}-\text{typical to} \ A^c_{ij'}.
\end{align*}\] \( \square \)
Now let us study the structure of $V_1$. Let $\alpha'' = 2k\sqrt{\alpha^*}$. Recall that $N(vx) = N(x) \cap N(v)$. Let $V'_1$ be the set of the vertices $v \in V_1$ such that there are at least $\alpha n^{k-1}$ copies of $K_{k-1}$ in each of $N(xv)$ and $N(yv)$. We claim that $|V'_1| < 2\alpha n$. Otherwise, since a $(k-1)$-set intersects at most $(k-1)n^{k-2}$ other $(k-1)$-sets, there are at least

$$2\alpha n \cdot \alpha' n^{k-1} (\alpha n^{k-1} - (k-1)n^{k-2}) > \alpha^3 n^{2k-1}$$

copies of $(2k-1)$-sets connecting $x$ and $y$, a contradiction.

Let $V_1 := V_1 \setminus V'_1$. The following claim is an analog of Claim 12 for Lemma 9 and can be proved similarly. The only difference between their proofs is that here we find at least $\alpha n^{k-1}$ copies of $K_{k-1}$ in each of $N(xv)$ and $N(yv)$, which contradicts the definition of $V_1$.

Claim 17. Given $v \in V_1$ and $2 \leq i \leq k$, there exists $j \in [k]$ such that $|N_{A_{ij}}(v)| < \alpha'' t$. □

Fix an vertex $v \in V_1$. Claim 17 implies that for each $2 \leq i \leq k$, there exists $\ell_i$ such that $|N_{A_{i\ell_i}}(v)| < \alpha'' t$. Our next claim is an analog of Claim 13 for Lemma 9 and can be proved similarly.

Claim 18. We have $\ell_2 = \ell_3 = \cdots = \ell_k$. □

We now define $A_{ij} := \{v \in V_1 : |N_{A_{ij}}(v)| < \alpha'' t\}$ for $j \in [k]$. By Claims 17 and 18, this yields a partition of $V_1 = \bigcup_{j=1}^k A_{ij}$ such that

$$d(A_{ij}, A_{ij}) < \frac{\alpha'' t |A_{ij}|}{|A_{ij}|} \leq \frac{\alpha'' t}{(1-\alpha'') t} < (1+2\alpha')\alpha''$$

for $i \geq 2$ and $j \geq 1$. (19)

For $v \in A_{ij}$, we have $|N_{A_{ij}}(v)| < \alpha'' t$ for $i \geq 2$. By the minimum degree condition and (18),

$$|A_{ij} \setminus N(v)| \leq (\frac{1}{k} + \alpha)n - (|A_{ij}| - \alpha'' t) < 2\alpha'' t.$$ (20)

By (18) and (20), we derive that

$$\bar{d}(A_{ij}, A_{ij'}) < \frac{|A_{ij}| \cdot 2\alpha'' t}{|A_{ij}| \cdot |A_{ij'}|} \leq \frac{2\alpha'' t}{(1-\alpha'') t} < 3\alpha''$$

for $i \geq 2$ and $j \neq j'$. (21)

We claim that $|A_{ij}| \leq (1+\alpha)t + (1+2\alpha')\alpha'' |A_{ij}|$ for all $j$. Otherwise, by the minimum degree condition, we have $\deg_{A_{ij}}(v) > (1+2\alpha')\alpha'' |A_{ij}|$ for all $v \in A_{ij}$, and consequently $d(A_{ij}, A_{ij}) > (1+2\alpha')\alpha''$, contradicting (19). We thus conclude that

$$|A_{ij}| \leq \frac{1+\alpha}{1-2\alpha'} \alpha''t < (1+2\alpha'') \frac{n}{k}.$$ (22)

Since $|V'_1| \leq 2\alpha n$, we have $|\bigcup_{j=1}^k A_{ij}| = |V_1 \setminus V'_1| \geq |V_1| - 2\alpha n$. Using (22), we now obtain a lower bound for $|A_{ij}|$, $j \in [k]$.

$$|A_{ij}| \geq n - (k-1)(1+2\alpha'') \frac{n}{k} - 2\alpha n \geq (1-2k\alpha'') \frac{n}{k}.$$ (23)

It remains to show that $d(A_{i1}, A_{i'1})$ and $d(A_{ik}, A_{i'k})$, $2 \leq i, i' \leq k$, are small. First we show that if both densities are reasonably large then there are too many reachable $(2k-1)$-sets from $x$ to $y$. The proof resembles the one of Claim 13.

Claim 19. For $2 \leq i \neq i' \leq k$, either $d(A_{i1}, A_{i'1}) \leq 6\alpha''$ or $d(A_{ik}, A_{i'k}) \leq 6\alpha''$.

Proof. Suppose instead, that say $d(A_{i(k-1)1}, A_{i'k})$, $d(A_{i(k-1)k}, A_{i'k}) > 6\alpha''$. Fix a vertex $v_1$ in $A_{i1}$, for some $2 \leq j \leq k - 1$, say $v_1 \in A_{ij}$. We construct two vertex disjoint copies of $K_{k-1}$ in $N(xv_1)$ and $N(yv_1)$ as follows. We first select $k-3$ sets $A_{ij}$ with $2 \leq i \leq k-2$ and $3 \leq j \leq k-1$ such that no two of them are on the same row or column – there are $(k-3)!$ choices. Fix one of them, say $A_2, \cdots, A_{(k-2)(k-1)}$. For $2 \leq i \leq k-2$, we select $v_i \in N_{A_{i(i+1)}}(v_1 \cdots v_{i-1})$ that is $\sqrt{3\alpha''}$-typical to $A_{(k-1)1}$, $A_{k1}$ and $A_{ij(i+1)}$, $i \leq j \leq k-2$. By Claim 16 and (20), there are at least

$$(1-(k-2)\sqrt{3\alpha''})|A_{i(i+1)}| - (\alpha' + \alpha'') \frac{n}{k} \geq \frac{n}{2k}$$
such \(v_i\). After selecting \(v_2, \ldots, v_{k-2}\), we select two adjacent vertices \(v_{k-1} \in A_{(k-1)1}\) and \(v_k \in A_{k1}\) such that \(v_{k-1}\) and \(v_k\) are in \(N(v_1 \cdots v_{k-2})\). For \(j = k - 1, k\), we know that \(N(v_1)\) misses at most \((k\alpha + \alpha' + \alpha'')n/k\) vertices in \(A_j\) and at most \((k - 3)\sqrt{3}\alpha'|A_{j1}|\) vertices of \(A_{j1}\) are not in \(N(v_2 \cdots v_{k-2})\). Since \(d(A_{(k-1)1}, A_{k1}) > 6\alpha''\), there are at least

\[
6\alpha''|A_{(k-1)1}|A_{k1}| - (k\alpha + \alpha' + \alpha'')\frac{n}{k}(|A_{(k-1)1}| + |A_{k1}|) - 2(k - 3)\sqrt{3}\alpha'|A_{(k-1)1}|A_{k1}| \geq 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2
\]
such pairs \(v_{k-1}, v_k\). Hence \(N(xv_1)\) contains at least

\[
(k - 3)! \left(\frac{n}{2k}\right)^{k-3} 6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2 \geq \sqrt{\alpha'}\left(\frac{n}{k}\right)^{k-1} \geq \sqrt{\alpha''}n^{-k}
\]
copies of \(K_{k-1}\). Let \(C\) be such a copy of \(K_{k-1}\). Then we follow the same procedure and construct a copy of \(K_{k-1}\) on \(N(yv_1) \setminus C\). After fixing \(k - 3\) sets \(A_{ij}\) with \(2 \leq i \leq k - 2\) and \(3 \leq j \leq k - 1\) such that no two of them are on the same row or column, there are still at least \(\frac{n}{2k}\) such \(v_i\) for \(2 \leq i \leq k - 2\). Then, as \(d(A_{ik}, A_{i'k}) > 6\alpha''\), there are at least \(6\sqrt{\alpha'}\left(\frac{n}{k}\right)^2\) choices of \(v_{k-1} \in A_{(k-1)k}\) and \(v_k \in A_{k}\) such that \(v_{k-1}\) and \(v_k\) are in \(N(v_1 \cdots v_{k-2})\). This gives at least \(\sqrt{\alpha''}n^{-k}\) copies of \(K_{k-1}\) in \(N(yv_1)\). Then, since there are at least \(|V_i| - |A_{11}| - |A_{ik}| \geq \alpha n\) choices of \(v_i\), totally there are at least \(\alpha n(\sqrt{\alpha''}n^{-k})^2 = \alpha^2n^{2k-2}\) reachable \((2k-1)\)-sets from \(x\) to \(y\), a contradiction. \(\square\)

Next we show that if any of \(d(A_{11}, A_{i'1})\) or \(d(A_{ik}, A_{i'k})\), \(2 \leq i, i' \leq k\), is sufficiently large, then we can remove edges from \(G\) such that the resulting graph still satisfies the minimum degree condition, which contradicts the assumption that \(G\) is minimal.

**Claim 20.** For \(2 \leq i \neq i' \leq k\), \(d(A_{i1}, A_{i'1}), d(A_{ik}, A_{i'k}) \leq 6k\sqrt{\alpha''}\).

**Proof.** Without loss of generality, assume that \(d(A_{2k}, A_{3k}) > 6k\sqrt{\alpha''}\). By Claim 19, we have \(d(A_{21}, A_{31}) < 6\alpha''\). Combining this with 17, we have \(d(A_{2j}, A_{3j}) < 6\alpha''\) for all \(j \in [k - 1]\). Now fix \(j \in [k - 1]\). The number of non-edges between \(A_{2j}\) and \(A_{3j}\) satisfies \(\bar{e}(A_{2j}, A_{3j}) > (1 - 6\alpha'')|A_{2j}||A_{3j}|\).

By the minimum degree condition and 18,

\[
\bar{e}(A_{2k}, A_{3j}) < (1 + k\alpha)\frac{n}{k}|A_{3j}| - (1 - 6\alpha'')|A_{2j}||A_{3j}| \leq 7\alpha''\frac{n}{k}|A_{3j}|
\]

Using 18 again, we obtain that

\[
d(A_{2k}, A_{3j}) \geq 1 - \frac{7\alpha''\frac{n}{k}|A_{3j}|}{|A_{2k}||A_{3j}|} \geq 1 - 8\alpha''
\]

This implies that \(d(A_{2k}, A_{3k}^c) \geq 1 - 8\alpha''\). Similarly we derive that \(d(A_{3k}, A_{2k}^c) \geq 1 - 8\alpha''\). For \(i = 2, 3\), define \(A_{ik}^T\) as the set of the vertices in \(A_{ik}\) that are \(\sqrt{8\alpha''}\)-typical to \(A_{(5 - i)k}^c\). Thus \(|A_{ik} \setminus A_{ik}^T| \leq \sqrt{8\alpha''}|A_{ik}|\).

Let \(A_{ik}^T = \{v \in A_{ik}^T : \deg_{A_{(5 - i)k}}(v) \leq \sqrt{8\alpha''}|A_{ik}^c|\}\) and \(A_{ik}^T = A_{ik}^T \setminus A_{ik}^T\). For \(u \in A_{2k}^T\), we have

\[
\deg_{\bar{V}_3}(u) = \deg_{A_{5k}^c}(u) + \deg_{A_{3k}}(u) > (1 - \sqrt{8\alpha''})|A_{3k}^c| + \sqrt{8\alpha''}|A_{3k}^c| = |A_{3k}^c|
\]

Since \(|A_{3k}^c| \geq \deg_{\bar{V}_3}(x)\) and \(|A_{3k}^c|\) is an integer, we conclude that \(\deg_{\bar{V}_3}(u) \geq \deg_{\bar{V}_3}(x) + 1\). Similarly we can derive that \(\deg_{\bar{V}_3}(v) \geq \deg_{\bar{V}_3}(x) + 1\) for every \(v \in A_{3k}^T\). If there is an edge \(uv\) joining some \(u \in A_{2k}^T\) and some \(v \in A_{3k}^T\), then we can delete this edge and the resulting graph still satisfies the minimum degree condition. Therefore we may assume that there is no edge between \(A_{2k}^T\) and \(A_{3k}^T\).
Then
\[ e(A_{2k}, A_{3k}) \leq e(A_{2k} \setminus A'_{2k}, A_{3k}) + e(A_{2k}, A_{4k} \setminus A'_{4k}) + e(A'_{2k}, A'_{4k} + e(A'_{2k}, A'_{4k}) \]
\[ \leq 2\sqrt{8\alpha''}A_{2k}|A_{3k}| + |A'_2|\sqrt{8\alpha''}|A'_{3k}| + |A'_{4k}|\sqrt{8\alpha''}|A'_{2k}| \]
\[ \leq \sqrt{8\alpha''}(2|A_{2k}|A_{3k} + |A_{2k}|A'_{3k} + |A_{3k}|A'_{2k}) \]
\[ = \sqrt{8\alpha''}(|A_{2k}||V_3| + |A_{4k}||V_2|) \]
\[ \leq 3\sqrt{\alpha''} \cdot 2k|A_{2k}|A_{3k} \]  
by (18).
Therefore \( d(A_{2k}, A_{3k}) \leq 6k\sqrt{\alpha''}. \)

In summary, by (18), (22) and (23), we have \((1 - 2k\alpha'') \frac{n}{k} \leq |A_{ij}| \leq (1 + 2\alpha'') \frac{n}{k}\) for all \(i \neq j\). In order to make \(\bigcup_{j=1}^k A_{ij}\) a partition of \(V_i\), we move the vertices of \(V_i'\) to \(A_{11}\) and the vertices of \(A_{10}\) to \(A_{12}\) for \(2 \leq i \leq k\). Since \(|V'_i| < 2\alpha n\) and \(|A_{10}| \leq (k+1)\alpha n\), we have \(||A_{ij}| - \frac{n}{k}| \leq 2k\alpha'' \frac{n}{k}\) after moving these vertices. On the other hand, by (17), (19), and Claim 20 we have \(d(A_{ij}, A_{ij}) \leq 6k\sqrt{\alpha''}\) for \(i \neq i'\) and all \(j\). (In fact, for \(i \geq 2\), we now have \(d(A_{11}, A_{11}) \leq 2\alpha''\) as we added at most \(2\alpha n\) vertices to \(A_{11}\). For \(i > i' \geq 2\), we now have \(d(A_{12}, A_{12}) \leq \alpha''\) and \(d(A_{12}, A_{12}) \leq \alpha'\) as we moved at most \((k+1)\alpha n\) vertices to \(A_{12}\).) Therefore after deleting edges, \(G\) is \((2k\alpha'', 6k\sqrt{\alpha''})\)-approximate to \(\Theta_{k \times k}(n/k)\). By (15), and the definitions of \(\alpha''\) and \(\alpha'\), \(G\) is \((\Delta/6, \Delta/2)\)-approximate to \(\Theta_{k \times k}(n/k)\).

\[ \square \]

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