Ratios of $B$ and $D$ Meson Decay Constants in Improved Mock Meson Model

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Abstract

We calculate the ratio $f_{B_s}/f_{B_d}$ by following the Oakes’ method which is based on chiral symmetry breaking, but we improve his calculation by performing the calculation of the factor $<0|\bar{b}\gamma_5 s|B_s>/ <0|\bar{b}\gamma_5 d|B_d>$ in the mock meson model. In this calculation we improved the mock meson model by using the value of the parameter $\beta$ which is obtained by the variational method in the relativistic quark model. We also calculate $f_{D_s}/f_{D_d}$, and then $(f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d})$. In this method we also obtain the ratio $f_{B_s}/f_{D_s}$ and $f_{B_d}/f_{D_d}$ which are important for the knowledge of CP violation and $B$-$\bar{B}$ mixing.

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Oakes [1] calculated the ratio \( f_{Bs} / f_{Bd} \) based on the usual assumption that chiral symmetry is broken by quark mass terms \( H_1(x) = \sum m_i \bar{q}_i(x)q_i(x) \). Using the local relation \( \partial^\mu A_\mu(x) = -i[Q_5, H_1(x)] \), he obtained

\[
\frac{f_{Bs}}{f_{Bd}} = \left( \frac{M_{Bd}}{M_{Bs}} \right)^2 \frac{m_b + m_s}{m_b + m_d} \frac{\langle 0|\bar{b}\gamma_5 s|B_s \rangle}{\langle 0|\bar{b}\gamma_5 d|B_d \rangle},
\]

where the decay constants are defined by

\[
\langle 0|\bar{b}\gamma_\mu \gamma_5 s|B_s(K) \rangle = i f_{Bs} K^\mu, \quad \langle 0|\bar{b}\gamma_\mu \gamma_5 d|B_d(K) \rangle = i f_{Bd} K^\mu.
\]

Through the symmetry consideration, he took the value of \( \langle 0|\bar{b}\gamma_5 s|B_s \rangle / \langle 0|\bar{b}\gamma_5 d|B_d \rangle \) as 1, and then obtained the results

\[
f_{Bs} / f_{Bd} = 0.989, \quad f_{Ds} / f_{Dd} = 0.985, \quad (f_{Bs} / f_{Bd}) / (f_{Ds} / f_{Dd}) = 1.004.
\]

Because of the light quark SU(3) flavor symmetry the factor \( \langle 0|\bar{b}\gamma_5 s|B_s \rangle / \langle 0|\bar{b}\gamma_5 d|B_d \rangle \) is almost 1. However, since we are interested in the ratios in (3) which are by themselves very close to 1, the deviation of the above factor from 1 is still important even though it is very small. Therefore it would be nice if we could calculate the small correction of the symmetry consideration. In this Letter we calculate this factor in the mock meson model of Capstick, Godfrey, and Isgur [2, 3, 4], and improve the Oakes’ results in (3). We also apply this approach to the calculations of \( f_{Bs} / f_{Ds} \) and \( f_{Bs} / f_{Dd} \), where the calculations of the above factor are essential since we can not apply the light quark flavor symmetry in these cases. The knowledge of these ratios are important since they allow us to get the values of \( f_{Bs} \) and \( f_{Bd} \) from the experimentally obtained value of \( f_{Ds} \). The information of \( f_{Bs} \) and \( f_{Bd} \) is very important, since it is crucial for the magnitude of CP violation and the size of \( B-\bar{B} \) mixing.

The mock meson state is represented by [2, 3, 4]

\[
|M(K)\rangle = \int d^3 p \Phi(p) \chi_{ss} \phi_{cc} |q\left(\frac{m_q}{m} K + p, s\right)\bar{Q}\chi\left(\frac{m_Q}{m} K - p, s\right)\rangle,
\]
where $K$ is the mock meson momentum, $m \equiv m_q + m_Q$, and $\Phi(p)$, $\chi_{ss}$, and $\phi_{cc}$ are momentum, spin, and color wave functions respectively. We take the momentum wave function $\Phi(p)$ as a Gaussian wave function
\begin{equation}
\Phi(p) = \frac{1}{(\sqrt{2\pi})^{3/2}} e^{-p^2/2\beta^2}.
\end{equation}

We note that we normalized the mock meson state $|M(K)\rangle$ according to $<M(K')|M(K)> = \delta^3(K' - K)$ [3], and we consider only pseudoscalar mesons in the present work. In this normalization of the mock meson state, (2) becomes [5]
\begin{equation}
<0|\bar{b}\gamma^\mu \gamma_5 s|B_s(K)> = \frac{if_{B_s}K^\mu}{\sqrt{2E_{B_s}}}, \quad <0|\bar{b}\gamma^\mu \gamma_5 d|B_d(K)> = \frac{if_{B_d}K^\mu}{\sqrt{2E_{B_d}}}.
\end{equation}

By applying the local relation $\partial\mu A_\mu(x) = -i [Q_5, H_1(x)]$ to (6), we get
\begin{equation}
f_{B_s} = \frac{\sqrt{2E_{B_s}}(m_b + m_s)}{(M_{B_s})^2} <0|\bar{b}\gamma_5 s|B_s(K)>,
\end{equation}
and a similar expression for $f_{B_d}$.

Then in the rest frames of the mesons we have
\begin{equation}
f_{B_s} \frac{f_{B_s}}{f_{B_d}} = \left(\frac{M_{B_d}}{M_{B_s}}\right)^{3/2} \left(\frac{m_b + m_s}{m_b + m_d}\right) \frac{<0|\bar{b}\gamma_5 s|B_s(0)>}{<0|\bar{b}\gamma_5 d|B_d(0)>},
\end{equation}
where
\begin{equation}
<0|Q \gamma_5 q|P_{qQ}(0)> = g(\beta) = 2\sqrt{3} \int d^3 p \Phi(p) \left(\frac{E_q + m_q E_Q + m_Q}{2E_q 2E_Q}\right)^{1/2} \times \left(1 + \frac{p^2}{(E_q + m_q)(E_Q + m_Q)}\right).
\end{equation}

We note that the power of $M_{B_d}/M_{B_s}$ in (8) is 3/2 instead of 2 in (4), since we use the normalization $<M(K')|M(K)> = \delta^3(K' - K)$, whereas Oakes used the more common normalization $<M(K')|M(K)> = 2E \delta^3(K' - K)$. We also note that the sign inside the last parenthesis in (9) is plus, whereas that in Eq. (3) of Ref. [3] is minus. This difference originates from the fact that we consider the pseudoscalar quantity of the left hand side of (9), whereas Ref. [3] considered the zeroth component of the four vector $<0|\bar{q}\gamma^\mu(1 - \gamma_5)|M(K)>$. In order to
calculate the ratio of the decay constants from (8), we should calculate the matrix
element $<0|\bar{Q}\gamma_5q|Pq\bar{Q}(0)>$. This matrix element is a function of $\beta$, which we call
$g(\beta)$. We performed numerical calculations for the $g(\beta)$ of $B_s$, $B_d$, $D_s$, and $D_d$
respectively, and present the results in Fig. 1.

Then the problem is what values we should use for $\beta$. Capstick and Godfrey
used the values of $\beta$ obtained from the effective harmonic oscillator potential [3, 4]. Here, we improve this treatment by applying the variational meth
od to the relativistic hamiltonian [6, 7]

$$H = \sqrt{p^2 + m_q^2} + \sqrt{p^2 + m_q^2} + V(r),$$

where $r$ and $p$ are the relative coordinate and its conjugate momentum. The
hamiltonian in (10) represents the energy of the meson in the meson rest frame, since in this reference frame the momentum of each quark is the same in magnitude
as that of the conjugate momentum of the relative coordinate. In Ref. [7] we
obtained the values of $\beta$ which minimize the expectation value of the hamiltonian
in (10). We considered six different potentials for $V(r)$ in (10), and the averages
of the minimizing $\beta$ values, $\bar{\beta}$, obtained from six potentials are given by [7]

$$\bar{\beta}_{B_s} = 0.579 \text{ GeV}, \quad \bar{\beta}_{B_d} = 0.558 \text{ GeV}, \quad \bar{\beta}_{D_s} = 0.492 \text{ GeV}, \quad \bar{\beta}_{D_d} = 0.475 \text{ GeV}.$$  \hspace{1cm} (11)

In the above calculations we used the current quark masses of the light quarks as
$m_d=9.9$ MeV and $m_s=199$ MeV, which were given by Dominguez and Rafael [8].

By using the values in (11) for $\bar{\beta}$, we can obtain from the $g(\beta)$ in Fig. 1 the
values of the matrix elements $g(\bar{\beta}) \equiv <0|\bar{Q}\gamma_5q|Pq\bar{Q}(0)>$ in (3), which are given by

$$g_{B_s}(0.579) = 8.657, \quad g_{B_d}(0.558) = 7.494, \quad g_{D_s}(0.492) = 7.418, \quad g_{D_d}(0.475) = 6.533.$$  \hspace{1cm} (12)

Then, using the meson [3] and the current quark [8] masses given by

$$M_{B_s} = 5.375 \text{ GeV}, \quad M_{B_d} = 5.279 \text{ GeV}, \quad M_{D_s} = 1.969 \text{ GeV}, \quad M_{D_d} = 1.869 \text{ GeV},$$

$$m_b = 4.397 \text{ GeV}, \quad m_c = 1.306 \text{ GeV}, \quad m_s = 0.199 \text{ GeV}, \quad m_d = 0.0099 \text{ GeV},$$  \hspace{1cm} (13)
we obtain the following ratios of the decay constants from (8):

\[
\frac{f_{B_s}}{f_{B_d}} = 1.173, \quad \frac{f_{D_s}}{f_{D_d}} = 1.201, \quad \frac{f_{D_s}}{f_{B_s}} = 1.266, \quad \frac{f_{D_d}}{f_{B_d}} = 1.236. \tag{14}
\]

When we use the values of the ratios in (14) together with the experimental value of \( f_{D_s} \) given by \( f_{D_s} = 265 \pm 68 \text{ MeV} \) \cite{7, 9, 10, 11}, we obtain \( f_{D_d} = 221 \pm 57 \text{ MeV}, \) \( f_{B_s} = 209 \pm 54 \text{ MeV}, \) and \( f_{B_d} = 178 \pm 46 \text{ MeV}. \)

In order to compare our results in (14) with the Oakes’ original approach, we calculated the ratios using the formula (1) with our values of the meson and current quark masses in (13). Here, we took the last factor in (1) as 1, as Oakes did. The results are presented in the row 2 of Table 1. As we see in Table 1, our results of the ratios \( f_{B_s}/f_{B_d} \) and \( f_{D_s}/f_{D_d} \) have been enhanced significantly compared with the Oakes’ results, since we have taken care of the meson structure through (8) and (9). However, the double ratio \( (f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d}) \) remains the same.

For comparisons, we summarize the results of various different calculations in Table 1. Grinstein did the calculation using the heavy quark effective theory, and Dominguez using the QCD sum rules. The row 6 is the results of the original calculations of Capstick and Godfrey. They used the values of the parameter \( \beta \) given by the effective harmonic oscillator potential, whereas we used the values given by the variational method in the relativistic quark model. Another important difference between their and our calculations is that they used the formula (2) directly and had the ambiguity of which component of the four vector should be used to obtain \( f_P \) as they discussed \cite{4}, whereas we followed the Oakes’ approach based on chiral symmetry breaking and then arrived at the formula (8) which has no such ambiguity. In the row 7 we present the results which the authors of this Letter obtained in the relativistic quark model \cite{7}. The rows 8-11 are the results of the lattice calculations, where the third column was estimated from their announced values of the first and second columns, and the fourth and fifth columns were estimated from their results of \( f_{D_s}, f_{B_s}, f_{D_d}, \) and \( f_{B_d}. \)
Acknowledgements

This work was supported in part by the Basic Science Research Institute Program, Ministry of Education, Project No. BSRI-94-2414, and in part by Daeyang Foundation at Sejong University in 1994.
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|   |          | $f_{B_s}/f_{B_d}$ | $f_{D_s}/f_{D_d}$ | $(f_{B_s}/f_{B_d})/(f_{D_s}/f_{D_d})$ | $f_{D_s}/f_{B_s}$ | $f_{D_d}/f_{B_d}$ |
|---|----------|------------------|------------------|---------------------------------|-----------------|-----------------|
| 1 | This Work | 1.173            | 1.201            | 0.977                           | 1.266           | 1.236           |
| 2 | Oakes Modif. | 1.006            | 1.030            | 0.977                           | —              | —              |
| 3 | Oakes [1] | 0.989            | 0.985            | 1.004                           | —              | —              |
| 4 | Grinstein [12] | —                | —                | 0.967                           | —              | —              |
| 5 | Dominguez [13] | 1.22±.02        | 1.21±.06        | 1.01±.05                        | —              | —              |
| 6 | Cap. Godf. [6] | 1.35±.18      | 1.21±.13        | 1.12±.19                        | 1.38±.16       | 1.55±.20       |
| 7 | Hwang Kim [7] | 1.053            | 1.045            | 1.008                           | 1.251           | 1.261           |
| 8 | ELC [14] | 1.08±.06         | 1.08±.02        | 1.00±.06                        | 1.03±.22       | 1.02±.21       |
| 9 | UKQCD [15] | 1.22±.04        | 1.18±.02        | 1.03±.04                        | 1.09±.04+.42   | 1.16±.05+.46   |
| 10 | BLS [16] | 1.11±.02±.05     | 1.11±.02±.05    | 1.00±.03±.06                   | 1.11±.06±.27   | 1.11±.08±.30   |
| 11 | MILC [17] | 1.13(2)(9)(4)    | 1.09(1)(4)(4)    | 1.04(2)(9)(5)                   | 1.18(3)(17)(13) | 1.22(5)(17)(19) |

Table 1: The ratios of the decay constants from different calculations.
Fig. 1. \( g_{B_s}(\beta) \equiv <0|\bar{b}\gamma_5 s|B_s(0)> \), \( g_{B_d}(\beta) \equiv <0|\bar{b}\gamma_5 d|B_d(0)> \), \( g_{D_s}(\beta) \equiv <0|\bar{c}\gamma_5 s|D_s(0)> \), and \( g_{D_d}(\beta) \equiv <0|\bar{c}\gamma_5 d|D_d(0)> \), as functions of the parameter \( \beta \).
