Angular momentum operator commutator against position and Hamiltonian of a free particle

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Abstract. Observables in Quantum mechanics are represented by operators. Its measurements cannot be determined simultaneously and the results are only a possibility. This study uses the rules of the commutator in determining the relation between angular momentum to the position and free particle Hamiltonian. The commutator will be zero if the operator that is connected can be determined simultaneously, and it will produce value if cannot be determined simultaneously. This study focused on the commutator of the angular momentum operator on the position and Hamiltonian of free particles in Cartesian coordinates. The result of the commutator of angular momentum operator to the position was zero (commut) if there wasn’t a component of the angular momentum that is equal to the position made by the commutation pair. While the results of the commutator angular momentum operator towards the free particle Hamiltonian indicated that angular momentum is the constant of motion.

1. Introduction
Quantum mechanics is a microscopic discussion that was born from Werner Karl Heisenberg's idea of the development of matrix mechanics and Erwin Schrodinger's approach to wave mechanics with the Schrodinger equation. Heisenberg explains the nature of a particle through position (x) and momentum (p). The hypothesis explains that it is impossible to know the exact position and momentum of a particle simultaneously [1]. Observables in physics can be determined directly and indirectly. Measurements in quantum mechanics cannot be determined simultaneously and the results are uncertain but only in the form of possibilities using some mathematical equations namely the Schrodinger equation and the operator method.

Schrodinger's equation is a second-order differential equation that must be solved in the case of nonrelativistic Quantum Physics. The solution to the Schrodinger equation is a wave function that provides information about the behavior of waves from particles. The conditions that must exist in Schrodinger's equations include not violating the law of conservation of energy, obeying the de Broglie hypothesis, must be linear and single value. Schrodinger's equation is divided into two, which is time dependent and free of time. The Schrodinger equation can be used to solve several cases in quantum mechanics, such as free particles, particles in potential boxes, potential wells, and potential ladders.

Some postulates underlying the formulation of Quantum mechanics include state representation, dynamic variable representation, system evolution, and motion constants. The operator method is a product of the postulate dynamic variable representation. The operator is a symbol of a mathematical command to do certain mathematical operations on the state behind it. The state in question can be a number, parameter, function and so on. Operators can also be defined as mathematical instructions that are imposed on a function and can be obtained simply from the derivative of an equation. Dynamic variables such as position, linear momentum, angular momentum, and energy can be determined with definite results in classical mechanics and in the form of probability in Quantum mechanics. Dynamic variables are represented by linear operators that will work on the functions of the system and convert them to other wave functions [1]. Operators are often written using capital letters and sometimes with inserts. \( \hat{x} \) represents the position operator and \( \hat{p} = -i\hbar \nabla \) represents the momentum operator [2]. Position operators and momentum operators can be developed into other dynamic variable operators.
Free particles are particles that do not have a working force ($\Sigma F = 0$) either between their particles or between particles and their environment so that their potential value is also zero [3]. One of the operators that is often used in Quantum mechanics is the energy operator. The energy operator is the same as the Hamiltonian operator which is the solution of the Hamiltonian equation. The Hamiltonian operator for a free particle system consists only of kinetic energy because its potential is zero.

There is a commutation relation of some operators in Quantum mechanics, namely commutators. The commutator can be zero or commutable if the observable (dynamic variable) that is related can be determined simultaneously. Commutators are not commutable because their dynamic variables cannot be determined simultaneously so they are not zero. The commutator can give meaning when it comes to a function. The function can be a wave function obtained from the Schrödinger equation which is the main foundation in Quantum mechanics.

This paper aims to determine the commutation relation of angular momentum with the position and free particle Hamiltonian. The focus of the study is on the operators in the Cartesian coordinates so that they can be transformed into other coordinates. The wave equation constant for free particles is difficult to determine because normalization integrals cannot be calculated from $-\infty$ to $\infty$. Therefore the wave function for the free particle system is not normalized so that the Hamiltonian operator is used.

2. Methods
There are two major operators in quantum mechanics, namely the momentum operator $\hat{p}$ and energy or Hamiltonian operator $\hat{H}$. The linear momentum operator is expressed as:

$$\hat{p} = -i\hbar \nabla$$

and each linear momentum component can be written:

$$\hat{p}_x = -i\hbar \frac{\partial}{\partial x}$$

$$\hat{p}_y = -i\hbar \frac{\partial}{\partial y}$$

$$\hat{p}_z = -i\hbar \frac{\partial}{\partial z}$$

The linear momentum operator and position operator are used to define the angular momentum operator [4]. In other words, angular momentum operators are formed from linear momentum operators and position operators. Classical physics defines angular momentum as:

$$L = \vec{r} \times \vec{p}$$

where $\vec{r}$ is a displacement vector and $\vec{p}$ is linear momentum. The component of angular momentum in the Cartesian coordinates can be obtained from the calculation cross product explicitly:

$$L = \vec{r} \times \vec{p} = \begin{vmatrix} i & j & k \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)i + (zp_x - xp_z)j + (xp_y - yp_x)k$$

Then obtained:

$$L_x = yp_z - zp_y$$

$$L_y = zp_x - xp_z$$

$$L_z = xp_y - yp_x$$

using the coordinate representation of the momentum operator, the components of the angular momentum operator in quantum mechanics can be written using ordinary substitution for each component of linear momentum [2].

$$\hat{L}_x = y \left( -i\hbar \frac{\partial}{\partial z} \right) - z \left( -i\hbar \frac{\partial}{\partial y} \right) = -i\hbar \left( y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y} \right)$$

$$\hat{L}_y = z \left( -i\hbar \frac{\partial}{\partial x} \right) - x \left( -i\hbar \frac{\partial}{\partial z} \right) = -i\hbar \left( z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right)$$

$$\hat{L}_z = x \left( -i\hbar \frac{\partial}{\partial y} \right) - y \left( -i\hbar \frac{\partial}{\partial x} \right) = -i\hbar \left( x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right)$$
After obtaining the total angular momentum operator and the angular momentum of each component, the quadaric angular momentum and the total angular momentum squared can be determined. Squares defined total angular momentum:

\[ L^2 = \mathbf{\mathbf{L}} \cdot \mathbf{L} \]  
\[ L^2 = (L_x \mathbf{\mathbf{i}} + L_y \mathbf{\mathbf{j}} + L_z \mathbf{\mathbf{k}}). (L_x \mathbf{\mathbf{i}} + L_y \mathbf{\mathbf{j}} + L_z \mathbf{\mathbf{k}}) = L_x^2 + L_y^2 + L_z^2 \]  
(11)

Then the total angular momentum operator can be written as follows:

\[ \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \]  
(12)

So that the squared angular momentum of each component can be obtained by following the definition of equation (11):

\[ \hat{L}_x^2 = (yp_x - zp_y)(yp_x - zp_y) = yp_xyp_x - zp_xzp_x - yzp_xzp_y + zp_yzp_y \]
\[ = y(-i\hbar \frac{d}{dz})y(-i\hbar \frac{d}{dy})z(-i\hbar \frac{d}{dy}) - y(-i\hbar \frac{d}{dz})z(-i\hbar \frac{d}{dy}) + z(-i\hbar \frac{d}{dy})z(-i\hbar \frac{d}{dy}) = 0 + i\hbar z \left( -i\hbar \frac{d}{dx} \right) + i\hbar y \left( -i\hbar \frac{d}{dz} \right) = \hbar^2 \left( z \frac{d}{dx} + y \frac{d}{dz} \right) \]  
(13)

\[ \hat{L}_y^2 = z \left( -i\hbar \frac{d}{dx} \right) \left( -i\hbar \frac{d}{dz} \right) - x \left( -i\hbar \frac{d}{dx} \right) z \left( -i\hbar \frac{d}{dz} \right) - x \left( -i\hbar \frac{d}{dx} \right) \left( -i\hbar \frac{d}{dz} \right) = 0 + i\hbar x \left( -i\hbar \frac{d}{dy} \right) + i\hbar y \left( -i\hbar \frac{d}{dx} \right) = \hbar^2 \left( x \frac{d}{dx} + z \frac{d}{dy} \right) \]  
(14)

\[ \hat{L}_z^2 = x \left( -i\hbar \frac{d}{dy} \right) y \left( -i\hbar \frac{d}{dx} \right) - y \left( -i\hbar \frac{d}{dy} \right) x \left( -i\hbar \frac{d}{dx} \right) - y \left( -i\hbar \frac{d}{dy} \right) \left( -i\hbar \frac{d}{dx} \right) = 0 + i\hbar y \left( -i\hbar \frac{d}{dy} \right) + i\hbar x \left( -i\hbar \frac{d}{dy} \right) = \hbar^2 \left( y \frac{d}{dy} + x \frac{d}{dx} \right) \]  
(15)

Then the quadratic mometum operator can be rewritten neatly as follows:

\[ \hat{L}_x^2 = \hbar^2 \left( z \frac{d}{dx} + y \frac{d}{dy} \right) \]
\[ \hat{L}_y^2 = \hbar^2 \left( x \frac{d}{dx} + z \frac{d}{dy} \right) \]
\[ \hat{L}_z^2 = \hbar^2 \left( y \frac{d}{dy} + x \frac{d}{dx} \right) \]

After determining the angular momentum operator both the total angular momentum and each component are first order and second order (squared), then determine the Hamiltonian operator in free particles. The Hamiltonian operator is an energy operator that has a constant value and is one of the main operators in the mathematical formulation of quantum mechanics. The form of the Hamiltonian operator in the Schrodinger equation can be written simply as:

\[ \hat{H}\psi = E\psi \]  
(17)

where \( \hat{H} \) is the Hamiltonian operator which can be written as follows:

\[ \hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V \]  
(18)

Hamilton's function is an alternative form of Newton's equations of motion using the total energy function. In the classical system, Hamilton's function is only a total energy system expressed in particle coordinates and conjugated momentum [5]. The kinetic energy of a particle with mass \( m \) can be written as:
The Hamiltonian can be written as follows:

$$H = T + V = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + V(x,y,z)$$  \hspace{1cm} (20)

Thus, \( H = E \).

Overall the Hamilton function \( H \) can be changed to the Hamiltonian operator \( \hat{H} \) by applying a few simple patterns as follows:

a. Using the classical equation \( H \) for all particles and each component of linear momentum in Cartesian coordinates.

b. The linear momentum component can be replaced by a linear momentum operator \( \hat{p} = -i\hbar \frac{\partial}{\partial r} \)

c. The position coordinates without being changed.

If the three patterns are applied, the form of Hamilton's function will be the Schrödinger equation which is also a Hamiltonian operator [5].

Because the potential value of the free particle system is zero, the Hamiltonian operator in the free particle system consists only of kinetic energy [6] and can be written as follows:

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2$$  \hspace{1cm} (21)

where \( \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \).

After obtaining the value of the angular momentum operator and the Hamiltonian operator, we can find the relationship between angular momentum and position with the Hamiltonian by using the commutator rule. Suppose that \( \hat{A} \) and \( \hat{B} \) are operators, then the commutators of \( \hat{A} \) and \( \hat{B} \) can be written as:

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$  \hspace{1cm} (22)

If \( [\hat{A}, \hat{B}] = 0 \), then operators \( \hat{A} \) and \( \hat{B} \) said to be commutable. Generally in quantum mechanics \( \hat{A}\hat{B} \neq \hat{B}\hat{A} \) so that \( [\hat{A}, \hat{B}] \neq 0 \). The properties of the commutator [4] include:

a. \( [\hat{A}, \hat{B}] = -[\hat{B}, \hat{A}] \)  \hspace{1cm} (23)

b. \( [\hat{A} + \hat{B}, \hat{C}] = [\hat{A}, \hat{C}] + [\hat{B}, \hat{C}] \)  \hspace{1cm} (24)

c. \( [\hat{A}, \hat{B}\hat{C}] = [\hat{A}, \hat{B}]\hat{C} + \hat{B}[\hat{A}, \hat{C}] \)  \hspace{1cm} (25)

da. If \( \hat{x} \) is the position operator and \( \hat{p} \) is the momentum operator, then \( [\hat{x}, \hat{p}] = i\hbar \). 

The nature of the commutator [7] in addition to the four properties above, as follows:

a. \( [\hat{A}, \alpha\hat{B} + \hat{C}] = \alpha[\hat{A}, \hat{B}] + [\hat{A}, \hat{C}] \)  \hspace{1cm} (26)

b. \( [\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}] + [\hat{A}, \hat{C}]\hat{B} \)  \hspace{1cm} (27)

The commutation relationship of the position and linear momentum operators for each component can be written as follows [2,8-10]:

$$[\hat{x}, \hat{p}_x] = [\hat{y}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = i\hbar \hspace{1cm} (28)$$

$$[\hat{x}, \hat{p}_y] = [\hat{x}, \hat{p}_z] = [\hat{y}, \hat{p}_x] = [\hat{y}, \hat{p}_z] = [\hat{z}, \hat{p}_x] = [\hat{z}, \hat{p}_y] = 0 \hspace{1cm} (29)$$

Whereas the commutation relationship of the angular momentum operator for each component can be written as follows:

$$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z \hspace{1cm} (30)$$

The commutation relation of the operator of the square angular momentum with each component of angular momentum can be written as follows [11]:

$$[\hat{L}^2, \hat{L}_x] = [\hat{L}^2, \hat{L}_y] = [\hat{L}^2, \hat{L}_z] = 0 \hspace{1cm} (31)$$

with
\[ L^2 = 2\hbar^2 r, \nabla = 2\hbar (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}) \]  

Equation (32) is the total squared angular momentum obtained from the sum of the squared angular momentum of each component in equations (14), (15), and (16) .

\[
L^2 = L_x^2 + L_y^2 + L_z^2 \\
= \hbar^2 \left( z \frac{d}{dz} + y \frac{d}{dy} + x \frac{d}{dx} \right) + 2\hbar^2 \left( x \frac{d}{dx} + y \frac{d}{dy} + z \frac{d}{dz} \right) \\
= 2\hbar^2 \left( x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \right)
\]

When the angular momentum operator is commuted with a position or other angular momentum, it will give a description of simultaneous determination. But this is not the case if it is commuted by the Hamiltonian. Observables in Quantum mechanics if commuted by the Hamiltonian will relate to representations of constant motion. The following are some observable explanations that have been commuted to the Hamiltonian.

By taking the Hamiltonian operator in the form of a differential with time, so

\[
\frac{\partial}{\partial t}(\hat{O}) = \frac{\partial}{\partial t} \int \psi^* \hat{O} \psi \, dt
\]

where \( \hat{O} \) is a linear operator.

Using the equation

\[
(\hat{H}^\star)\psi^* = -i\hbar \left( \frac{\partial}{\partial t} \right) \psi^*
\]

and pay attention to nature Hermitian of Hamiltonian operator, will span the hierarchy formula

\[
\frac{\partial(\hat{O})}{\partial t} = \frac{i}{\hbar} \left\{ \hat{H}, \hat{O} \right\} + \frac{\partial \hat{O}}{\partial t}
\]

by using the analogy of the rules of Poison in classical mechanics, the following equation

\[
\frac{d\hat{O}}{dt} = \left\{ \hat{O}, \hat{H} \right\} + \frac{\partial \hat{O}}{\partial t}
\]

If \( \hat{O} \) is not an explicit function of time \( t \), i.e. \( \frac{d\hat{O}}{dt} = 0 \), then applies the relationship

\[
\frac{\partial(\hat{O})}{\partial t} = \frac{i}{\hbar} \left\{ \hat{H}, \hat{O} \right\}
\]

If \( \hat{O} \) is constant over time, then both sides of the equation (37) is zero. So each constant physical quantity of time, the operator will be commuted with the Hamiltonian operator. In other words, any commutation of the Hamiltonian operator is a constant for time. For \( \hat{O} \) is position \( \mathbf{x} \) or momentum \( \mathbf{p} \) obtained by the relationship

\[
\frac{\partial(\mathbf{x})}{\partial t} = \frac{i}{\hbar} \left\{ \hat{H}, \mathbf{x} \right\} = \frac{i}{\hbar} \left\{ \left( \frac{\mathbf{p}^2}{2m} + V, \mathbf{x} \right) \right\}
\]

Vcommute to \( \mathbf{x} \), namely \([V, \mathbf{x}] = 0 \), while

\[
\left[ \mathbf{p}, \mathbf{x} \right] = \mathbf{p}[\mathbf{p}, \mathbf{x}] + [\mathbf{p}, \mathbf{x}]\mathbf{p} = \left( 2\hbar i \right) [\mathbf{p}]
\]

in order to obtain the equation

\[
\frac{\partial(\mathbf{x})}{\partial t} = \frac{\mathbf{p}}{m}
\]

Equation (40) as expected is an equation of motion in mechanics.

Furthermore, equation (35) can also be written

\[
\frac{\partial(\mathbf{p})}{\partial t} = \frac{i}{\hbar} \left\{ \hat{H}, \mathbf{p} \right\} = \frac{i}{\hbar} \left\{ \left( \frac{\mathbf{p}^2}{2m} + V, \mathbf{p} \right) \right\} = -\frac{i}{\hbar} \left\{ \left[ \mathbf{p}, V \right] \right\}
\]

from the following equation:

\[
\left[ \mathbf{p} \right] V \psi - V[\mathbf{p}] \psi = -\hbar \left( \frac{d}{dx} \right) (V\psi) + i\hbar V \left( \frac{d\psi}{dx} \right) = -\hbar \left( \frac{d}{dx} \right) \psi
\]

obtained the equation:

\[
\left[ \mathbf{p}, V \right] = -\hbar \left( \frac{dV}{dx} \right) \psi
\]
so that
\[ \frac{d \langle \psi \rangle}{dt} = -\frac{\partial V}{\partial x} = \langle F_x \rangle \] (44)

Equation (44) according to the equation of motion in mechanics [12].

After proving that commutable observables with the Hamiltonian are motion constants, this can be used as an indicator in interpreting the results of observables in this study, namely angular momentum which is commuted with the free particle Hamiltonian. The commutator of the angular momentum operator against the free particle Hamiltonian is obtained by using the existing commutator rules.

3. Result and Discussion

Operators in Quantum mechanics are very important in representing observable quantities. Angular momentum is one of quantum mechanical observables which also plays a role in determining the shape of a system wave function. The wave function itself determines the state of the system. Commutator can be mentioned as a catalyst in physics to determine relationships between observables with a shorter process. In this paper a commutator of angular momentum with respect to position and free particle Hamiltonian are not applied to any wave function.

| Commutator  | Result |
|-------------|--------|
| \([\hat{L}_x, x]\) | 0 |
| \([\hat{L}_y, x]\) | \(-i\hbar z\) |
| \([\hat{L}_z, x]\) | \(ihy\) |
| \([\hat{L}_x, y]\) | \(i\hbar z\) |
| \([\hat{L}_y, y]\) | 0 |
| \([\hat{L}_z, y]\) | \(-i\hbar x\) |
| \([\hat{L}_x, z]\) | \(-ih y\) |
| \([\hat{L}_y, z]\) | \(ih x\) |
| \([\hat{L}_z, z]\) | 0 |
| \([\hat{L}_x, \hat{L}_y]\) | \(2\hbar^2 x\) |
| \([\hat{L}_y, \hat{L}_z]\) | \(2\hbar^2 z\) |
| \([\hat{L}_z, \hat{L}_x]\) | \(2\hbar^2 y\) |

Based on Table 1 obtained results \([\hat{L}_x, x] = [\hat{L}_y, y] = [\hat{L}_z, z] = [\hat{L}_x, x] = [\hat{L}_y, y] = [\hat{L}_z, z] = [\hat{L}_x, x^2] = [\hat{L}_y, y^2] = [\hat{L}_z, z^2] = [\hat{L}_x, x^2] = [\hat{L}_y, y^2] = [\hat{L}_z, z^2] = 0\) indicates that the angular momentum operator against position that is made into pairs of commutations each of which is commutable, so they can be determined simultaneously. The commutator of angular momentum operator against position that was not zero meant that the angular momentum and position of the commutation pair could not be determined simultaneously. The positive and negative values in the results were due to the multiplication of cross product vectors which also paid attention to the rules of right hand angular momentum. The resulting value was \(i\hbar\) multiplied by the position of the product of the vector.
The results of the commutator of squared angular momentum operator against position which were not zero or non commutated resulted in all positive values. This was because in its commutation there was a dot product multiplication for its quadratic angular momentum so that it did not pay attention to the direction. The result of the commutator of squared angular momentum operator against the position was $\hbar^2$ multiplied by the position used as the commutation pair, except for $L^2$ is worth 2$h^2$ because it was the total angular momentum.

The results of the commutator of angular momentum operator against squared position that were not zero or non commutated resulted in a positive and negative value. This was because the same thing when position in first order, position commuted to the angular momentum, ie the multiplication of the cross product vector which also pays attention to the right hand rule of angular momentum. The resulting value was $i\hbar$ multiplied by the position of the product of the vector multiplied by the derivative of the position used as the commutation pair.

The results of the commutator of the squared angular momentum operator against squared position which were not zero or non commutated resulted in all positive values. This was because the same thing when the one-order position commuted to the angular momentum of the square, that in its commutation there was a dot product multiplication for its quadratic angular momentum so that it didn't pay attention to the direction. The resulting value was 2$h^2$ multiplied by the position of the square used as the commutation pair, except for $L^2$ it was worth 4$h^2$ because it was a total squared angle momentum.

**Tabel 2.** The Results of Operator Commutator Angular Momentum Against the Hamiltonian of Free Particles

| Commutator  | Result            |
|-------------|-------------------|
| $[\hat{L}_x, \hat{H}]$ | 0                  |
| $[\hat{L}_y, \hat{H}]$ | 0                  |
| $[\hat{L}_z, \hat{H}]$ | 0                  |
| $[\hat{L}_x, \hat{H}]$ | $-2\hbar^2\hat{H}_{yz}$ |
| $[\hat{L}_y, \hat{H}]$ | $-2\hbar^2\hat{H}_{xz}$ |
| $[\hat{L}_z, \hat{H}]$ | $-2\hbar^2\hat{H}_{xy}$ |
| $[\hat{L}^2, \hat{H}]$ | $-4\hbar^2\hat{H}$ |

The Hamiltonian function which is used in this study is for free particles where the potential is zero so that in the Hamiltonian operator only consists of kinetic energy. Based on **Table 2**, the commutator of the angular momentum operator against the free particle Hamiltonian produces a zero value. This shows that angular momentum is a constant due to commutative of the Hamiltonian [12]. However, the commutator of the angular momentum operator against the Hamiltonian produces a value of $-2\hbar^2$ multiplied by Hamiltonian on the axis which is in the component of squared angular momentum used as commutation pair. While the commutator of squared total angular momentum operator against the Hamiltonian produces a value of $-4\hbar^2 \hat{H}$.So that the squared angular momentum and the squared total angular momentum are not motion constants.

**4. Conclusion**

Based on the results and discussion above, it can be concluded that the angular momentum operator commutator against position will be commut or zero if angular momentum component does not exist a position which is used as commutation pair. It shows that paired angular momentum and position can be determined simultaneously. The angular momentum operator commutator against Hamiltonian of free particle will be commut or zero if it is commutated by angular momentum $L_x, L_y, L_z$. It shows that all of three angular momentum $L_x, L_y, L_z$ are included in the constant motion.
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References

[1] Purwanto A 2006 *Fisika Kuantum* (Yogyakarta: Gava Media Press) p.37&109
[2] Mustamin M F 2017 *Dasar Matematis Mekanika Kuantum* (Makassar: Infty Press) p.20-26
[3] Krane K S 1992 *Fisika Modern (Indonesia Translated)* (Jakarta: Universitas Indonesia Press) p.182
[4] McMahon D 2005 *Quantum mechanics* (New York: McGraw-Hill Professional) p.259-260, 167
[5] Hayward D O 2002 *Quantum mechanics for Chemists* (Cambridge: Thomas Graham House) p.115-116
[6] Saller H 2017 *International Journal Theoretical Physics* 56: 168-220 p.177
[7] Transtum M KandHuele J S V 2005 *Journal of Mathematical Physics* 46 (25) p.3
[8] ZhangX L, Sun Y L, Wang Q, LongZ W, and Jing J 2016 *The European Physical Journal Plus* 131: 229 p.3
[9] Jing J, Zhang Q Y, Wang Q, LongZ W, and Dong S H 2019 *The European Physical Journal Plus* C 79: 1 p.3
[10] Feigen M and Hakobyan T 2015 *Journal of High Energy Physics* 107 p.3
[11] Sani R and Kadri M 2017 *Fisika Kuantum* (Jakarta: Bumi Aksara) p.192
[12] Soedojo P 2001 *Azas-Azas Ilmu Fisika Jilid 4 Fisika Modern* (Yogyakarta: Gajah Mada University Press) p.101-102