N=4 superconformal n-particle mechanics via superspace

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Abstract

We revisit the (untwisted) superfield approach to one-dimensional multi-particle systems with $N=4$ superconformal invariance. The requirement of a standard (flat) bosonic kinetic energy implies the existence of inertial (super-)coordinates, which is nontrivial beyond three particles. We formulate the corresponding integrability conditions, whose solution directly yields the superpotential, the two prepotentials and the bosonic potential. The structure equations for the two prepotentials, including the WDVV equation, follow automatically. The general solution for translation-invariant three-particle models is presented and illustrated with examples. For the four-particle case, we take advantage of known WDVV solutions to construct a $D_3$ and a $B_3$ model, thus overcoming a previously-found barrier regarding the bosonic potential. The general solution and classification remain a challenge.
1 Introduction and Summary

Although conformal multi-particle quantum mechanics (in one space dimension) is a subject with a long and rich history, its $\mathcal{N}=4$ superconformal extension has been achieved only recently [1, 2, 3, 4, 5, 6, 7, 8]. Enlarging the conformal algebra $su(1,1)$ to $su(1,1|2)$ (with central charge) imposes severe constraints on the particle interactions, which are not easily solved. Firstly, there is a nonzero prepotential $F$ which must obey a quadratic homogeneous differential equation of third order known as the Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equation [9, 10]. The general solution to the WDVV equation is unknown, but various classes of solutions, based on (deformed) Coxeter root systems, have been found [11, 12, 13, 14, 6].

Secondly, a second prepotential $U$ is subject to a linear homogeneous differential equation of second order, in a given $F$ background. With the known $F$ solutions, a nonzero $U$ (needed for nonzero central charge) has been constructed only for up to three particles, where the WDVV equation on $F$ is still empty. The general $U$ solution is known only for the highly symmetric cases based on the dihedral root systems $I_2(p)$, where it depends on three parameters [6, 7]. The bosonic potential $V_B$ of the multi-particle system is readily computed from $U$ and $F$. However, beyond three particles, where the WDVV equation is effective, the standard logarithmic ansatz for $U$ is insufficient and must be enriched by a suitably chosen homogeneous function of degree zero.

Since supersymmetry relates the two prepotentials $F$ and $U$ to a superpotential $G$, a superspace approach to these systems should simplify the analysis. This is what we present in the current paper. We profit from the fact that $\mathcal{N}=4$, $d=1$ superspace is well developed [15, 16, 17, 18, 19]. Our goal is to construct an $\mathcal{N}=4$ superconformally invariant one-dimensional system for $n$ bosonic and $4n$ fermionic physical components. Hence we need $n$ copies of an $\mathcal{N}=4$ superfield of type $(1,4,3)$, containing 1 physical bosonic, 4 fermionic and 3 auxiliary bosonic degrees of freedom. Such a supermultiplet is known for a long time [15], and its general action in superfields as well as in components was constructed in [1]. Everything in this action depends on a single bosonic function $G$ of the $n$ superfields, namely the superpotential. In this paper we investigate the situation where two additional properties hold:

- the system is superconformally invariant,
- the bosonic kinetic term is of standard (flat) form in suitable coordinates.

The first condition can rather easily be satisfied. Contrary to naive expectations, however, the second property implies rather intricate constraints on the superpotential (see equation (2.7) below), which are not solvable in general. To overcome these problems (at least partially) and to find explicit examples of $\mathcal{N}=4$ $n$-particle systems with the desired properties, we developed an approach whose main features are summarized as follows.

We start from the most general $\mathcal{N}=4$ supersymmetric action for $n$ untwisted $(1,4,3)$ superfields $u^A(t)$. Imposing our two properties turns out to be equivalent to the existence of `inertial super-coordinates' $y^i(t)$ (the $n$ particle locations) together with integrability and homogeneity conditions on our superfields $u^A$ as functions of the $y^i$. From these conditions we derive the existence and all properties of the two prepotentials $F(y)$ and $U(y)$, including the WDVV equation! What is more, an explicit construction for $U$, the third derivatives of $F$ and the superpotential $G$ is found in terms of the quadratic homogeneous functions $u^A(y)$. Since the homogeneity requirement is easy to fulfil, the only nontrivial task is to solve the integrability condition on $u^A(y)$ (equation (4.17) or (4.33) below).

To obtain explicit solutions, the low-dimensional cases of $n = 2, 3, 4$ are investigated in detail. Here, we must distinguish between translation non-invariant irreducible systems and translation invariant reducible systems of $n$ particles. Any latter one may be constructed from a former one (with $n−1$ 'particles') by embedding the former into one dimension higher and orthogonally to the center-of-mass coordinate to be added. For $n=2$, we reduce the problem to solving certain ordinary differential equations, which is then done for the dihedral systems and for three examples including the $A_2$ Calogero case. Their lift to translation-invariant three-particle models is presented very explicitly. For $n=3$ we encounter a system of partial differential equations, which we cannot solve in general. For a known prepotential $F$, however, the problem simplifies somewhat (equation (4.34) below), and we manage to find the explicit form of $u^A(y)$ for two models based on the $D_3 \simeq A_2$ and $B_3$ root systems. The ensuing prepotentials $U$ and bosonic potentials $V_B$ are new in the literature and, for the first time, overcome the $n=3$ barrier of [6].

Finally, we outline how to construct the corresponding translation invariant $n=4$ models.

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1 more precisely, translation-invariant irreducible systems of two coupled relative coordinates plus the decoupled center-of-mass coordinate
2 $\mathcal{N}=4$ supersymmetric $n$-particle systems

In $\mathcal{N}=4$ superspace one may define two sets of $\mathcal{N}=4$ superfields with one physical boson among the components, namely $u^A$ and $\phi^A$, restricted by the constraints

$$[D^a, \overline{D}_a] u^A = 0 \quad (a) \quad \text{and} \quad (D^a \overline{D}^b + D^b \overline{D}^a) \phi^A = 0 \quad (b). \quad (2.1)$$

These constraints define standard and twisted $\mathcal{N}=4$ supermultiplets, respectively [15, 16]. From (2.1a) it immediately follows that

$$\partial_t D^2 u^A = \partial_{\bar{t}} \overline{D}^2 u^A = 0 , \quad (2.2)$$

where $D^2 = D^a D_a$ and $\overline{D}^2 = \overline{D}_a \overline{D}^a$. Clearly, these equations result in the conditions

$$D^2 u^A = i m^A \quad \text{and} \quad \overline{D}^2 u^A = -i m^A , \quad (2.3)$$

where $m^A$ is an arbitrary set of constants.

Considering only standard (nontwisted) superfields for the time being, the most general $\mathcal{N}=4$ supersymmetric action reads

$$S = - \int dt \, d^2 \theta \, d^2 \bar{\theta} \, G(u^A) \quad \text{with} \quad A = 1, \ldots, n , \quad (2.4)$$

where $G(u^A)$ is an arbitrary function of a set of superfields $u^A$ subject to (2.3).

The bosonic part of the action (2.4) has the very simple form (we use the same notation for superfields and their leading components)

$$S_B = \frac{1}{2} \int dt \left[ G_{AB} \partial_t u^A \partial_t u^B - G_{AB} m^A m^B \right] \quad (2.5)$$

with the evident notation

$$G_A = \frac{\partial G}{\partial u^A} , \quad G_{AB} = \frac{\partial^2 G}{\partial u^A \partial u^B} , \quad G_{ABC} = \frac{\partial^3 G}{\partial u^A \partial u^B \partial u^C} , \quad \text{etc.} \quad (2.6)$$

This action has firstly been analyzed in [1].

We are interested in the subset of actions (2.4) which features superconformal invariance and a flat kinetic term for the bosonic variables. The second requirement means that the Riemann tensor for the metric $G_{AB}$ has to vanish. One may check that this condition results in the equations

$$G_{ABX} G^{XY} G_{YCD} - G_{ACX} G^{XY} G_{YBD} = 0 \quad \text{with} \quad G^{XY} G_{YZ} = \delta^X_X . \quad (2.7)$$

It is not clear how to find the solutions to this equation in full generality.

3 Imposing $\mathcal{N}=4$ superconformal symmetry

In one dimension the most general superconformal group is $D(2,1;\alpha)$ [20]. Here we restrict our consideration to the special case of $\alpha = -1$, which corresponds to $SU(1,1|2)$ symmetry. The main reason for this is our wish to retain the potential term in (2.5) with nonzero parameters $m^A$. The presence of these constants in the defining superfield constraints (2.3) fixes the scale weight of our superfields $u^A$ under dilatation (the auxiliary components of $u^A$ must have zero weight). This reduces the full superconformal group $D(2,1;\alpha)$ to $SU(1,1|2)$.

The superconformal group $SU(1,1|2)$ has natural realization in $\mathcal{N}=4, d=1$ superspace [15] via

$$\delta t = E - \frac{i}{2} \theta^a D_a E - \frac{i}{2} \partial_a \overline{D}^a E , \quad \delta \theta^a = -\frac{i}{2} \overline{D}^a E , \quad \delta \bar{\theta}_a = -\frac{i}{2} D_a E , \quad (3.1)$$

where the superfunction $E(t, \theta, \bar{\theta})$ collects all $SU(1,1|2)$ parameters:

$$E = f - 2i(\varepsilon \bar{\theta} - \bar{\theta} \varepsilon) + \theta^a \overline{\theta}^b B_{(ab)} + 2(\partial_t \varepsilon \bar{\theta} + \theta \partial_t \bar{\theta} + \theta \bar{\theta} \varepsilon \bar{\theta}) + \frac{1}{2} (\theta \bar{\theta})^2 \partial_t^2 f \quad (3.2)$$
\[ f = a + bt + ct^2 \quad \text{and} \quad \varepsilon^a = \epsilon^a + t\eta^a. \]

The bosonic parameters \(a, b, c\) and \(B_{(ab)}\) correspond to translations, dilatations, conformal boosts and rigid SU(2) rotations, while the fermionic parameters \(\varepsilon^a\) and \(\eta^a\) correspond to Poincaré and conformal supersymmetries, respectively.

It is important that by definition the function \(E\) obeys the conditions

\[ D^2E = \overline{D}^2E = \left[D^a, \overline{D}_a\right]E = 0 \quad \text{and} \quad \partial_t \overline{D}_E = \partial_t D^a E = \partial_t (a \overline{D}^b E) = 0. \]

Keeping in mind the transformation properties of the covariant spinor derivatives \(D^a\) and \(\overline{D}_a\),

\[ \delta D^a = -\frac{i}{2} (D^a \overline{D}_b) E^b \quad \text{and} \quad \delta \overline{D}_a = -\frac{i}{2} (D_a D^b E) \overline{D}_b, \]

one may check that the constraints (2.1) are invariant under the \(\mathcal{N}=4\) superconformal group if the superfields \(u^A\) and \(\phi^A\) transform like

\[ \delta u^A = \partial_t E u^A \quad \text{and} \quad \delta \phi^A = 0, \]

respectively. Thus, the superfields \(\phi^A\) are superconformal scalars while the \(u^A\) are vectors.

It is our goal to construct superconformally invariant actions for a set of \(n\) supermultiplets \(u^A\) with \(A = 1, \ldots, n\). The variation of the general action (2.4) under the superconformal transformations (3.1) and (3.6) takes the form

\[ \delta S = \int dt d^2\theta d^2\bar{\theta} \partial_t E \left(-G + u^A G_A\right), \]

which is nullified by the condition

\[ u^A G_A - G = a_A u^A, \]

where \(a_A\) is an arbitrary set of constants. The right-hand side disappears after integration over superspace due to the constraints (3.4) and (2.1).

### 4 Inertial coordinates

Since we require the metric \(G_{AB}(u)\) to be flat, there must exist inertial coordinates \(y^i\), in which the flatness (2.7) becomes trivial because the bosonic action takes the form

\[ S_B = \int dt \left[\frac{1}{2} \delta y^i \partial_i y^j \partial_t y^j - V_B(y)\right]. \]

The price for this is a more complicated bosonic potential \(V_B\). Therefore, we can construct models of the required type by finding the transformation \(u^A = u^A(y)\) to inertial coordinates, with Jacobian

\[ u^A_i = \frac{\partial u^A}{\partial y^i}(y) \quad \text{and inverse} \quad \frac{\partial y^i}{\partial u^A}(u(y)) = (u^{-1})^i_A. \]

After transforming to the \(y\)-frame, the superconformal transformations (3.6) become nonlinear,

\[ \delta y^i = (u^{-1})^i_A u^A \partial_t E. \]

However, the action (4.1) is invariant only when the transformation law is \(\delta y^i = \frac{1}{2} y^i \partial_t E\). This demand restricts the variable transformation by

\[ (u^{-1})^i_A u^A = \frac{1}{2} y^i \quad \rightarrow \quad y^i u^A_i = 2 u^A. \]

Hence, superconformal invariance requires \(u^A\) to be a homogeneous quadratic function of the \(y^i\).

A rigid SU(2) rotation brings the constraints (2.1) and (2.3) into the equivalent form

\[ D^2 u^A = 0, \quad \overline{D}^2 u^A = 0, \quad \left[D^a, \overline{D}_a\right] u^A = 2 m^A. \]
In the new coordinates, they become

\[ u^A_i D^2 y^i + (\partial_i u^A_j) D^a y^i D_a y^j = 0, \]
\[ u^A_i \bar{D}^2 y^i + (\partial_i u^A_j) \bar{D}_a y^i \bar{D}^a y^j = 0, \]
\[ u^A_i [D^a, \bar{D}_a] y^i + 2(\partial_i u^A_j) D^a y^i \bar{D}^a y^j = 2 m^A, \]

which we rewrite as [2]

\[ D^2 y^i - f^i_{kj} D^a y^k D_a y^j = 0, \]
\[ \bar{D}^2 y^i - f^i_{kj} \bar{D}_a y^k \bar{D}^a y^j = 0, \]
\[ [D^a, \bar{D}_a] y^i - 2 f^i_{kj} D^a y^k \bar{D}_a y^j + 2 U^i = 0, \]

after introducing a flat connection and a covariantly constant vector via

\[ f^i_{kj} = - (u^{-1} \partial_k u^i)_j = - (u^{-1})^i_A u^A_{kj} \quad \text{and} \quad U^i = - (u^{-1})^i_A m^A, \]

in obvious notation. By construction, the integrability conditions of the system (4.7),

\[ \partial_k f^i_{mj} - f^i_{kj} \partial_m f^j_{mn} = 0, \]
\[ \partial_j U^i - f^i_{jk} U^k = 0, \]

are automatically satisfied. No restriction (besides invertibility) on the matrix \((u^{-1}_A)\) appears.

Let us come back to the superfield action (2.4) and consider the superpotential as a function of the inertial coordinates, writing again \(G(y)\) in place of \(G(u(y))\) in a slight abuse of notation, so that \(G_i, G_{ij}\) etc. denote its derivatives with respect to \(y\). After integration over the \(\theta_s\) in (2.4) and using the constraints (4.7) we arrive at

\[ S_B = -\frac{1}{2} \int dt \left[ \left( G_{ij} + G_k f^k_{ij} \right) \partial_i y^i \partial_j y^j - \partial_k \left( G_i \left( u^{-1} \right)^i_A \right) \left( u^{-1} \right)_B^k m^A m^B \right], \]

which may also be obtained by directly subjecting (2.5) to the change of variables. Comparing with the defining property (4.1), we read off that

\[ G_{ij} + G_k f^k_{ij} = -\delta_{ij}, \]

which simplifies the potential term to\(^2\)

\[ V_B = \frac{1}{2} \delta_{ij} \left( u^{-1} \right)^i_A \left( u^{-1} \right)_B^j m^A m^B = \frac{1}{2} \delta_{ij} U^i U^j. \]

Differentiating the condition (4.12) with respect to \(y^m\) we get

\[ 0 = G_{ijm} + G_m f^k_{ij} + G_k \partial_m f^k_{ij} = G_{ijm} - \delta_{mk} f^{k}_{ij} + G_k \left( \partial_m f^k_{ij} - f^{k}_{im} f^l_{lij} \right). \]

In view of (4.9), antisymmetrizing in \(i\) and \(m\) yields\(^3\)

\[ \delta_{klm} f^k_{ij} = 0 \quad \rightarrow \quad \delta_{mk} f^k_{ij} \equiv f_{mij} = f_{mij} = f_{ijm}, \]

so that our flat connection is symmetric in all three indices.

A flat connection as defined in (4.8) can be totally symmetric (after lowering all indices) if and only if the inverse Jacobian is integrable,

\[ \frac{\partial y^i}{\partial u^A}(u(y)) \equiv \left( u^{-1} \right)^i_A =: w_{A,i} = \partial_i w_A = \frac{\partial w_A}{\partial y^i}(y), \]

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\(^2\)This is the classical bosonic potential. After quantization, it picks up an additional contribution of \(\frac{1}{8} \hbar^2 f_{ijk} f_{ijk}\).

\(^3\)Since the inertial metric is euclidean, we may freely raise and lower inertial indices.
which establishes the existence of a set of superfields $w_A$ ‘dual’ to $u^A$. The $w_A$ can be shifted by integration constants. It is instructive to rewrite formulae by replacing $u^{-1}$ by $w$. Beginning with

$$w_{A,i} u^B_i = \delta_A^B \quad \longleftrightarrow \quad w_{A,i} u^A_i = \delta_{ij}$$

(4.17)

and the superconformality condition (4.4),

$$y^i u^A_i = 2u^A \quad \longleftrightarrow \quad w_{A,i} u^A_i = \frac{1}{2} y^i ,$$

(4.18)

we introduce the notation $p_A q^A = p \cdot q$ and find by repeated differentiation and contraction with $y^k$ that

$$y^k u^A_k = 2u^A \quad \longrightarrow \quad y^k u^A_{ki} = u^A_i \quad \text{and} \quad y^k w_{A,ki} = -w_{A,i} \quad \longrightarrow \quad y^k w_{A,k} = c_A$$

(4.19)

$$3 w_{ij} \cdot u = -\frac{3}{2} \delta_{ij} = w \cdot u - \partial_i \partial_j (w \cdot u) ,$$

$$f_{ijk} = -w_{ijk} \cdot u = w_{ij} \cdot u_k = w_{i} \cdot u_{jk} = w_{ij} \cdot u_k - \delta_i \delta_j \delta_k (w \cdot u) ,$$

(4.20)

with some constants $c_A$. In the last line, indices may be permuted freely. Playing a bit more, one finds

$$w_{ij} \cdot u_k = w_{ij} \cdot u_m w_m \cdot u_{kl} = w_{ij} \cdot u_{kl} w_{m} \cdot u_{m} = w_{kl} \cdot u_{ij} ,$$

(4.22)

$$2 \partial_i f_{ijk} = \partial_i (w_{jk} \cdot u_i - w_i \cdot u_{jk}) = w_{jkl} \cdot u_i - w_i \cdot u_{jkl} ,$$

(4.23)

proving that

$$\partial_i f_{kl} = 0 \quad \longrightarrow \quad f_{ijk} = \partial_i \partial_j \partial_k F \quad \text{and} \quad f^{[k}_{i m} f^l_{m]} = 0 .$$

(4.24)

Hence, there exists a prepotential $F$ obeying the WDVV equation.

The homogeneity relations (4.19) imply that there exists a ‘radial coordinate’,

$$c \cdot u_k = y^k \quad \longrightarrow \quad c \cdot u = \frac{1}{2} y^k y^k =: \frac{1}{2} R^2 .$$

(4.25)

In view of this, it is reasonable to choose

$$u^i = R^2 \quad \text{and} \quad w_1 = \frac{1}{2} \ln R + \sqrt{1} \quad \longrightarrow \quad c_1 = \frac{1}{2} \quad \text{and} \quad c_{A>1} = 0 .$$

(4.26)

Furthermore, contractions of $f_{ijk}$ simplify,

$$w_{A,i} f_{ijk} = w_{A,jk} , \quad u^A_i f_{ijk} = -u^A_{jk} \quad \text{and} \quad y^i f_{ijk} = -\delta_{jk} ,$$

(4.27)

and the vector $U_i = \delta_{ij} U^j$ obeys

$$U_i = -w_{A,i} m^A = \partial_i U \quad \longrightarrow \quad U = -w_A m^A \quad \text{and} \quad y^i U_i = -c_A m^A =: -C .$$

(4.28)

Thus, all the ‘structure equations’ of [5] are fulfilled precisely by

$$\partial_i \partial_j \partial_k F = f_{ijk} \quad \text{and} \quad \partial_i U = U_i$$

(4.29)

and the central charge $C$.

With the help of the ‘dual superfields’ $w_A$, one can give a simple expression for the superpotential $G(y)$, namely

$$G = -u^A w_A = -\frac{1}{2} R^2 \ln R - R^2 \sqrt{1} - u^{A>1} w_{A>1} .$$

(4.30)

Employing the relations above, it is readily verified that this function indeed obeys (4.12) and thus leads to the bosonic action (4.1). In the inertial coordinates, the superconformality condition (3.8) acquires the form

$$y^i G_i - 2G - 2a_A u^A = 0 .$$

(4.31)

The superpotential $G$ given by (4.30) does satisfy this constraint, provided the constants $c_A$ and $a_A$ are related as $c_A = -2 a_A$. In view of (4.25), this yields the homogeneity relation

$$y^i G_i - 2G + \frac{1}{2} y_i y^i = 0 .$$

(4.32)

Incidentally, the prepotential $F$ defined in (4.29) respects just the same homogeneity relation, as is found by twice integrating the last equation in (4.27).
So, for the construction of \( N=4 \) superconformal mechanics models, in principle one needs to solve only two equations, namely \( (4.17) \) and \( (4.18) \). All other relations and conditions follow from these! The homogeneity condition \( (4.18) \) is easy to satisfy: the \( u^A \) must be homogeneous of degree two as functions of \( y \). Nontrivial, however, is the integrability condition \( (4.17) \). Its derivative may be recast in a different form:

\[
0 = w_{ik} u_j + w_i u_{jk} = -w_{jk} u_i + w_i u_{jk} \quad \rightarrow \quad u_i^A u_j^B = 0 ,
\]

(4.33)
after contracting with two Jacobians. This equation looks deceptively simple. Contracting it with \( w_{A,k} w_{B,l} \) we reproduce the total symmetry of \( f_{ijk} = w_i u_{jk} \) and thus the integrability \( w_{A,i} = \partial_i w_A \). If \( F \) is known otherwise, e.g. from solving the WDVV equation, it is easier to reconstruct \( u^A \) or \( w_A \) from \( (4.27) \).

\[
u_{A}^i + f_{ijk} u_{A}^k = 0 \quad \text{and} \quad w_{A,ij} - f_{ijk} w_{A,k} = 0 .
\]

(4.34)
With \( f_{ijk} \) being totally symmetric, any one of these equations is equivalent to \( (4.33) \). Their advantage is the linearity, which allows superpositions, as long as we respect \( (4.25) \).

It is also worthwhile to consider \( w_A \) as a function of the \( u^A \), i.e.

\[
w_A = w_A (y(u)) \quad \rightarrow \quad \partial B = w_{B,i} \partial_i \quad \text{and} \quad \partial_i = u_i^A \partial A .
\]

(4.35)
Then,

\[
w_{AB} = \partial_B w_A = w_{B,i} w_{A,i} = w_{BA},
\]

(4.36) and the bosonic potential \( (4.13) \) can be rewritten as

\[
V_B = \frac{1}{2} m \cdot w_i m \cdot w_i = \frac{1}{2} m^A m^B w_{AB} .
\]

(4.37) Furthermore, we can directly reconfirm \( (3.8) \) and discover that

\[
G_{AB} = -w_{AB} \quad \text{and} \quad G_A = -w_A - \frac{1}{2} c_A \quad \rightarrow \quad w_A = -\partial_A (G + \frac{1}{2} c u) = -\partial_A (G + \frac{1}{2} u^1) .
\]

(4.38)
We note that, since \( w_A \) is only determined up to a constant, a linear function of \( u^A \) may be added to \( G \), e.g. to achieve \( w_A = -\partial_A G \). As expected, the superpotential \( G(y) \) determines both \( U \) and \( f_{ijk} \).

\[
U = m^A \partial A G \quad \text{and} \quad G_{ij} + G_k f^k_{ij} = -\delta_{ij} ,
\]

(4.39)
albeit rather indirectly.

\section{Two-dimensional systems}

In the simplest situation of \( n=2 \), all equations can be solved in principle. Indeed, the integrability condition \( (4.33) \) then merely implies that \( u^1 \) and \( u^2 \) are homogeneous quadratic functions of \( y^1 \) and \( y^2 \). So – moving the inertial index down for notational simplicity – let us take

\[
u^1 = y^1 + y^2 = R^2 \quad \text{and} \quad u^2 = R^2 h(\varphi) \quad \text{with} \quad \tan \varphi = \frac{w_2}{y_1} .
\]

(5.1)
With

\[
\begin{pmatrix} u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 2y_1 \\ 2y_2 \\ 2y_1 h - y_2 h' \end{pmatrix} \begin{pmatrix} 2y_2 \\ 2y_2 h + y_1 h' \end{pmatrix}
\]

(5.2)
the condition \( (4.33) \) is identically satisfied. Inversion of this matrix produces

\[
w = \begin{pmatrix} \frac{y_1}{2R^2} & \frac{y_1 h}{2R^2} & -\frac{y_2}{2R^2} \\ \frac{y_2}{2R^2} & \frac{y_2 h}{2R^2} & \frac{y_1}{2R^2} \\ \frac{y_1 h'}{2R^2} & \frac{y_2 h'}{2R^2} & \frac{y_1}{2R^2} \end{pmatrix} = \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{1,2} & w_{2,2} \end{pmatrix},
\]

(5.3)
and we read off that

\[
w_1 = \frac{1}{2} \ln R + \bar{w}_1(\varphi) \quad \text{and} \quad w_2 = w_2(\varphi) \quad \text{with} \quad \bar{w}_1 = -\frac{h}{R} \quad \text{and} \quad w^2_2 = \frac{1}{R^2} .
\]

(5.4)
This yields the superpotential

\[
G = -\frac{1}{2} R^2 \ln R - R^2 \bar{g}(\varphi) \quad \text{with} \quad \bar{g} = \bar{w}_1 + h w_2 \quad \rightarrow \quad \bar{g}' = h w_2 = \frac{w_2}{w_1} .
\]

(5.5)
Let us make a matching ansatz for the WDVV prepotential,
\[ F = -\frac{1}{2}R^2 \ln R - R^2 \tilde{f}(\varphi). \] (5.6)

Then, from \( \partial_i \partial_j \partial_k F = -w_i u_{jk} \) we learn that
\[ \tilde{f}''' + 4\tilde{f} = \frac{h''}{h'} = \frac{w_i' w_j'}{w_i w_j} \quad \longrightarrow \quad h' \propto e^{\tilde{f}''' + 4\tilde{f}}, \] (5.7)
and the bosonic potential reads
\[ V_B = (\frac{w_i' h}{w_i h'})^2 + (\frac{m^2 h}{w_i h'})^2 \quad \text{with} \quad C = \frac{1}{2}m^4. \] (5.8)

### 5.1 Dihedral systems

A highly symmetric class of models is based on the dihedral root systems \( I_2(p) \) for \( p \in \mathbb{N} \),
\[ \alpha \cdot y = \cos(k\pi/p) y_1 + \sin(k\pi/p) y_2 \quad \text{for} \quad k = 0, 1, \ldots, p-1. \] (5.9)

The WDVV prepotential for these systems was found to be [6]
\[ F = -\frac{1}{2}f_R R^2 \ln R - \frac{1}{2}f_p \sum_{\alpha \in I_2(p)} (\alpha \cdot y)^2 \ln |\alpha \cdot y| \quad \text{with} \quad f_R + \frac{1}{2} f_p = 1, \] (5.10)
which corresponds to
\[ \tilde{f}(\varphi) = \frac{1}{2}f_p \sum_{k=0}^{p-1} \cos^2(\varphi - \frac{kn}{p}) \ln |\cos(\varphi - \frac{kn}{p})|. \] (5.11)

Differentiating, we obtain (modulo irrelevant integration constants)
\[ \ln h' = \tilde{f}''' + 4\tilde{f} = \frac{1}{2}f_p \sum_{k=0}^{p-1} \ln |\cos(\varphi - \frac{kn}{p})| \quad \longrightarrow \quad h' \propto \left[ \sin(p\varphi + p\frac{\varphi}{2}) \right]^{f_p} \] (5.12)
and thus
\[ h(\varphi) = h_0 \cos(p\varphi + p\frac{\varphi}{2}) \quad \text{with} \quad p \in \mathbb{N}. \] (5.13)

This result simplifies for
- \( f_p = +1 : \quad h(\varphi) = h_0 \cos(p\varphi + p\frac{\varphi}{2}) \)
- \( f_p = -1 : \quad h(\varphi) = h_0 \ln \tan(p\varphi + p\frac{\varphi}{2}) \)
- \( f_p = 0 : \quad h(\varphi) = h_0 \varphi \)

for which \( w_A \) and \( G \) are readily computed. With impunity the roots may be rotated by a common angle \( \delta \), which corresponds to \( \varphi \rightarrow \varphi - \delta \) in all equations. We provide three simple examples.

### 5.2 First example

An easy choice is
\[ h(\varphi) = \sin 2\varphi \quad \leftrightarrow \quad u^2 = 2y_1y_2 \quad \text{so that} \quad u^1 \pm u^2 = (y_1 \pm y_2)^2, \] (5.14)
which leads to
\[ u^{-1} = \left( \begin{array}{cc} \frac{y_1}{2(y_1^2 - y_2^2)} & -\frac{y_2}{2(y_1^2 - y_2^2)} \\ -\frac{y_2}{2(y_1^2 - y_2^2)} & \frac{y_1}{2(y_1^2 - y_2^2)} \end{array} \right) = \left( \begin{array}{cc} w_{1,1} & w_{2,1} \\ w_{1,2} & w_{2,2} \end{array} \right). \] (5.15)
This integrates to
\[ w_A = \frac{1}{8} \ln |y_1 + y_2| \pm \frac{i}{8} \ln |y_1 - y_2| = \frac{1}{8} \ln |u^1 + u^2| \pm \frac{i}{8} \ln |u^1 - u^2| \] (5.16)
with the upper (lower) sign corresponding to \( A=1 \) (\( A=2 \)), and further yields the superpotential
\[ G = -\frac{1}{4}(u^1 + u^2)^2 \ln |u^1 + u^2| - \frac{1}{4}(u^1 - u^2)^2 \ln |u^1 - u^2| \]
\[ = -\frac{1}{4}(y_1 + y_2)^2 \ln |y_1 + y_2| - \frac{1}{4}(y_1 - y_2)^2 \ln |y_1 - y_2| = F \] (5.17)
as well as \( (c_1, c_2) = (1, 0) \). It is obvious that \( G = -u^4 w_A \). Depending on the value of \( (m^1, m^2) \), the bosonic potential is a linear combination of \( (y_1 + y_2)^{-2} \) and \( (y_1 - y_2)^{-2} \). We recognize the roots of \( D_2 \) here. A rotation by \( \delta = \frac{\pi}{4} \) produces the (decoupled) \( I_2(2) = A_1 \oplus A_1 \) system with \( h = \cos 2\varphi \) as well as \( f_p = 1 \) and \( f_R = 0 \). The decoupling of the center of mass \( u^1 + u^2 = (y_1 + y_2)^2 \) renders this example a bit trivial.
5.3 Second example

For a more complicated case, consider

\[ h(\varphi) = \sqrt{\sin^4 \varphi + \cos^4 \varphi} \quad \implies \quad u^2 = \sqrt{y_1^4 + y_2^4} \quad \text{so that} \quad (u^1)^2 - (u^2)^2 = 2y_1^2 y_2^2. \]  

(5.18)

The matrix \( u^{-1} \) may easily be found to be

\[
\begin{pmatrix}
-\frac{y_2}{2y_1(y_1^2 - y_2^2)} & \frac{\sqrt{y_1^4 + y_2^4}}{2y_1(y_1^2 - y_2^2)} \\
\frac{y_2}{2y_1(y_1^2 - y_2^2)} & -\frac{\sqrt{y_1^4 + y_2^4}}{2y_1(y_1^2 - y_2^2)}
\end{pmatrix} = \begin{pmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{pmatrix}, \tag{5.19}
\]

which can be integrated to

\[
w_1 = \frac{1}{4} \ln y_1^2 + \frac{1}{4} \ln y_2^2 - \frac{1}{4} \ln |y_1^2 - y_2^2|, \\
w_2 = -\frac{1}{4} \ln y_1^2 - \frac{1}{4} \ln y_2^2 + \frac{1}{4} \ln \left(y_1^2 + \sqrt{y_1^4 + y_2^4}\right) + \frac{1}{4} \ln \left(y_2^2 + \sqrt{y_1^4 + y_2^4}\right) + \frac{1}{2\sqrt{2}} \ln \left(y_1^2 + y_2^2 + \sqrt{2} \sqrt{y_1^4 + y_2^4}\right). \tag{5.20}
\]

It is amusing to check that indeed

\[
(\partial_i w_A) u^A = \frac{1}{2} y_i, \quad (\partial_{ij} w_A) u^A = -\frac{1}{2} \delta_{ij} \quad \text{and} \quad G_A = -w_A - \frac{1}{2} c_A, \tag{5.21}
\]

as it must be by construction. The simplest form of the bosonic potential (4.13) occurs for the choice \((m_1, m_2) = (0, m)\), namely

\[
V_B \sim \frac{1}{y_1^2} + \frac{1}{y_2^2} + \frac{1}{(y_1 - y_2)^2} + \frac{1}{(y_1 + y_2)^2}. \tag{5.22}
\]

We recognize the roots of the \( I_2(4) = BC_2 \) system. This is not surprising, since \( h = \frac{1}{2} \sqrt{\cos 4\varphi + 3} \) is a simple deformation of the dihedral construction. This model is not translation invariant. In fact, the only two-dimensional model with this property is our first example above.

5.4 Third example

Finally, let us present the standard Calogero example based on the \( A_2 \) root system,

\[ h(\varphi) = \sin 3\varphi \quad \implies \quad u^2 = \frac{3y_1^2 y_2 - y_1^3}{\sqrt{y_1^4 + y_2^4}}. \]  

(5.23)

It leads to

\[
u^{-1} = \begin{pmatrix}
\frac{9}{y_1} & \frac{2y_1}{y_1^2 - 3y_2^2} + \frac{y_1}{6(y_1^2 + y_2^2)} & \frac{y_2 \sqrt{y_1^4 + y_2^4}}{3(y_1 + y_2)} \\
\frac{2y_1}{y_1^2 - 3y_2^2} + \frac{y_1}{6(y_1^2 + y_2^2)} & \frac{y_2}{y_1^2 - 3y_2^2} + \frac{y_2}{6(y_1^2 + y_2^2)} & \frac{3(y_1^2 - y_2^2)}{3(y_1^2 + y_2^2)} \\
\end{pmatrix} = \begin{pmatrix} w_{11} & w_{21} \\ w_{12} & w_{22} \end{pmatrix}, \tag{5.24}
\]

which produces

\[
w_1 = \frac{1}{9} \ln \left|y_1 (y_1 - \sqrt{3} y_2)(y_1 + \sqrt{3} y_2)\right| + \frac{1}{9} \ln R = \frac{1}{18} \ln |u^1 + u^2| + \frac{1}{18} \ln |u^1 - u^2| + \frac{5}{36} \ln |u^1|, \\
w_2 = \frac{1}{9} \ln \left|\frac{y_1(y_2 - \sqrt{3} y_1 + 2R)(y_1 + \sqrt{3} y_2)(y_1 + 2R)}{(y_1 + \sqrt{3} y_2)(y_1 - \sqrt{3} y_2)(y_2 + 2R)}\right| = \frac{1}{18} \ln |u^1 + u^2| - \frac{1}{18} \ln |u^1 - u^2| \tag{5.25}
\]

and the superpotential

\[ G = \frac{1}{18} \left(u^1 + u^2\right) \ln |u^1 + u^2| + \frac{1}{18} \left(u^1 - u^2\right) \ln |u^1 - u^2| + \frac{5}{36} u^1 \ln |u^1|. \tag{5.26}
\]

For the bosonic potential \( V_B \), please proceed to the following section.
6 Embedding into three dimensions

The generic two-dimensional system is irreducible and thus not translation invariant. To generate translation-invariant models, we may take the inertial \( y \) coordinates as relative coordinates in a three-particle system, whose absolute coordinates \((x^1, x^2, x^3)\) comprise the center-of-mass combination

\[
\begin{align*}
  u^0 &= (x^1 + x^2 + x^3)^2 \\
  u^1 &= u^1(x^\mu) \\
  u^2 &= u^2(x^\mu)
\end{align*}
\]

live in the ‘relative-motion plane’ orthogonal to the center-of-mass motion. Our notation reflects the \( 3 = 1+2 \) split of this reducible system. To find the relation between the 3d coordinates \( x^\mu \) and the 2d coordinates \( y^i \), we have to formulate the embedding map \([6]\),

\[
y^i = M_i^\mu x^\mu \quad \text{with} \quad (M_i^\mu) = \begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{\sqrt{6}} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{(6.2)}
\]

Here, the matrix \( M \) effects a partial isometry,

\[
M M^T = \mathbb{1}_2 \quad \text{and} \quad M^T M = P = \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix},
\]

where \( P \) is the projection onto the relative-motion plane. By a slight abuse of notation, we write \( u^A(y=Mx) = u^A(x) \) and embed \((5.1)\),

\[
u^1(x) = x^\top P x = \frac{1}{4} \{(x^{12})^2 + (x^{23})^2 + (x^{31})^2 \} =: \tilde{R}^2 ,
\]

\[
u^2(x) = \tilde{R}^2 h(\varphi) \quad \text{with} \quad \sin \varphi = \frac{x^1 + x^2 - x^3}{\sqrt{6} R} \quad \text{and} \quad \cos \varphi = \frac{x^1}{\sqrt{2} R} \quad \text{(6.4)}
\]

This will automatically take care of the integrability condition \((4.33)\). Permutations of the \( x^\mu \) are generated by the reflection \( \varphi \rightarrow \pi - \varphi \) and a \( \frac{2\pi}{3} \) rotation in the relative-motion plane \([6]\). Therefore, if we want to describe a system of three identical particles, the function \( h(\varphi) \) better be invariant under these actions, for instance by taking

\[
h(\varphi) = \tilde{h}(\varphi) \tilde{h}(\varphi+\frac{2\pi}{3}) \tilde{h}(\varphi-\frac{2\pi}{3}) \quad \text{with} \quad \tilde{h}(\pi-\varphi) = \tilde{h}(\varphi) \quad \text{(6.5)}
\]

The \( A_2 \) Calogero model arises from the simple choice \( \tilde{h}(\varphi) = -\sqrt{3} \sin \varphi \), which gives

\[
h(\varphi) = -4 \sin(\varphi) \sin(\varphi+\frac{2\pi}{3}) \sin(\varphi-\frac{2\pi}{3}) = \sin 3\varphi = \frac{\sqrt{3} (2x^1 - x^2 - x^3)(2x^2 - x^3 - x^1)(2x^3 - x^1 - x^2)}{[(x^{12})^2 + (x^{13})^2 + (x^{23})^2]^2} \quad \text{(6.6)}
\]

and thus

\[
u^2 = \frac{\sqrt{3}}{3} \left( \frac{x^{12} + x^{13}}{(x^{12})^2 + (x^{13})^2 + (x^{23})^2} \right)^{\frac{1}{2}} \quad \text{(6.7)}
\]

which also follows directly from \((5.23)\). For this choice we integrate the matrix \((u^{-1})_A^\mu = \partial_\mu w_A \) to get

\[
w_0 = \frac{1}{6} \ln \left| x^1 + x^2 + x^3 \right| = \frac{1}{16} \ln u^0 ,
\]

\[
w_1 = \frac{1}{6} \ln \left| x^{12} x^{13} x^{23} \right| + \frac{1}{6} \ln \tilde{R} = \frac{1}{18} \ln |u^1 + u^2| + \frac{1}{18} \ln |u^1 - u^2| + \frac{5}{36} \ln |u^1| ,
\]

\[
w_2 = \frac{1}{9} \ln \left| \frac{x^{21} (x^{23} + x^{21} + \sqrt{6}\tilde{R}) (x^{12} + x^{13} - \sqrt{6}\tilde{R})}{x^{23} x^{13} (x^{21} + x^{23} - \sqrt{6}\tilde{R})} \right| = \frac{1}{18} \ln |u^1 + u^2| - \frac{1}{18} \ln |u^1 - u^2| .
\]

As expected, \( w_1(u) \) and \( w_2(u) \) agree with the functions in \((5.25)\). Beyond the center-of-mass term, the superpotential then reproduces the result of \((5.26)\) (see also \([7]\)),

\[
G = \frac{1}{18} u^0 \ln |u^0| + \frac{1}{18} (u^1 + u^2) \ln |u^1 + u^2| + \frac{1}{18} (u^1 - u^2) \ln |u^1 - u^2| + \frac{5}{36} u^1 \ln |u^1| \quad \text{(6.9)}
\]

The possible potential terms are specified by a choice of the constants \( m^A \) in the basic constraints on the superfields \((2.3)\). They completely agree with the results of \([6]\) on the \( A_2 \) model. The coupling \( m^0 \) goes
with the center of mass. The general bosonic potential for \( m^1 \neq 0 \) and \( m^2 \neq 0 \) is not very illuminating, so we display two special cases:

\[
V_B |_{m^1=0} = \frac{1}{8x}(m^2)^2 \left( \frac{1}{(x^2)^2} + \frac{1}{(x^3)^2} + \frac{1}{(x^4)^2} \right) + \frac{1}{23}(m^0)^2 \left( \frac{1}{(x^1+x^2+x^3)^2} \right), \quad (6.10)
\]

\[
V_B |_{m^2=0} = \frac{1}{8x}(m^1)^2 \left( \frac{1}{(x^2)^2} + \frac{1}{(x^3)^2} + \frac{1}{(x^4)^2} \right) + \frac{5}{24}(m^1)^2 \left( \frac{1}{(x^2)^2} + \frac{1}{(x^3)^2} + \frac{1}{(x^4)^2} \right) + \frac{1}{23}(m^0)^2 \left( \frac{1}{(x^1+x^2+x^3)^2} \right).
\]

Shifting \( \varphi \) by a constant should produce an equivalent formulation of the Calogero model. For instance,

\[
h(\varphi) = \cos 3\varphi \quad \rightarrow \quad u^2 = \frac{\sqrt{6} x^{12} x^{23} x^{31}}{(x^{12})^2 + (x^{13})^2 + (x^{13})^2}. \quad (6.11)
\]

In this case, we find

\[
w_0 = \frac{1}{6} \ln |x^1+x^2+x^3| = \frac{1}{18} \ln u^0, \\
w_1 = \frac{1}{9} \ln \left| (x^{13}+x^{23})(x^{21}+x^{31})(x^{32}+x^{12}) \right| + \frac{1}{9} \ln \tilde{R} = \frac{1}{18} \ln |u^1+u^2| + \frac{1}{18} \ln |u^1-u^2| + \frac{1}{18} \ln |u^1|,
\]

\[
w_2 = \frac{1}{9} \ln \left| \frac{(x^{21}+x^{23})(x^{12}+x^{31})(x^{12}+x^{31})}{(x^{13}+x^{23})(x^{13}+x^{23})} \right| = \frac{1}{18} \ln |u^1+u^2| - \frac{1}{18} \ln |u^1-u^2|,
\]

and obtain bosonic potentials

\[
V_B |_{m^1=0} = \frac{1}{8x}(m^2)^2 \left( \frac{1}{(x^{12})^2 + (x^{23})^2 + \text{cyclic}} \right) + \frac{1}{23}(m^0)^2 \left( \frac{1}{(x^1+x^2+x^3)^2} \right), \quad (6.13)
\]

\[
V_B |_{m^2=0} = \frac{1}{8x}(m^1)^2 \left( \frac{1}{(x^{12})^2 + (x^{23})^2 + \text{cyclic}} \right) + \frac{5}{24}(m^1)^2 \left( \frac{1}{(x^1+x^2+x^3)^2} \right) + \frac{1}{23}(m^0)^2 \left( \frac{1}{(x^1+x^2+x^3)^2} \right).
\]

Other translation and permutation invariant models may be constructed by embedding the root systems of \( I_2(3q) \) into three dimensions [6]. The next higher case is \( q = 2 \), i.e. the \( G_2 \) model, which is also obtained by combining the cases (6.6) and (6.11). The freedom of rescaling the short roots versus the long ones gives us a more general solution,

\[
F = -\frac{1}{4} f_S (x^1-x^2)^2 \ln |x^1-x^2| - \frac{1}{12} f_L (x^1+x^2-2x^3)^2 \ln |x^1+x^2-2x^3| + \text{cyclic} \\
-\frac{1}{4} f_R^2 \ln R - \frac{1}{6} (x^1+x^2+x^3)^2 \ln |x^1+x^2+x^3| \quad \text{with} \quad \frac{3}{2} f_S + \frac{3}{2} f_L + f_R = 1. \quad (6.14)
\]

The corresponding \( u^A \) are determined by (6.4), with

\[
h(\varphi) = h_0 \cos 3\varphi + f_S \sin 3\varphi \quad (6.15)
\]

For \( f_L = 1 \) or \( f_S = 1 \), this simplifies to \( h = h_0 \cos 3\varphi \) or \( h = h_0 \sin 3\varphi \), respectively. The ‘radial term’ proportional to \( R^2 \ln R \) may be eliminated in \( F \) by taking \( f_R = 0 \) whence \( f_L + f_S = \frac{2}{3} \). Thus,

\[
(f_L, f_S) = (1, -\frac{4}{3}) \quad \rightarrow \quad h(\varphi) = h_0 \cos 2/3(3\varphi),
\]

\[
(f_L, f_S) = (-\frac{4}{3}, 1) \quad \rightarrow \quad h(\varphi) = h_0 \sin 2/3(3\varphi). \quad (6.16)
\]

These cases and the corresponding bosonic potential were already featured in [5].

7 Irreducible three-dimensional systems

For irreducible systems beyond two dimensions, it is much more difficult to solve the integrability condition (4.17) or (4.33). We again lower the inertial index and generalize (5.1) to

\[
u^1 = y_1^2 + y_2^2 + y_3^2 = R^2 \quad \text{and} \quad u^2 = R^2 h(\vartheta, \varphi), \quad u^3 = R^2 k(\vartheta, \varphi), \quad (7.1)
\]

where \( \vartheta \) and \( \varphi \) are the two polar angles (declination and ascension) of the two-sphere,

\[
y_1 = R \sin \vartheta \cos \varphi, \quad y_2 = R \sin \vartheta \sin \varphi, \quad y_3 = R \cos \vartheta. \quad (7.2)
\]
The matrix \( (u^A)^T \) is straightforwardly inverted. Equating it to \( (w_{A,i})^T \) we discover that
\[
\begin{align*}
w_1 &= \frac{1}{2} \ln R + \mathbf{w}_1(\theta, \varphi) \quad \text{and} \quad w_2 = w_2(\theta, \varphi), \quad w_3 = w_3(\theta, \varphi) \quad \text{with} \quad (7.3) \\
\partial_\theta \mathbf{w}_1 &= -\frac{h_k - h_k h_s}{R_k h_k - h_k h_s}, \quad \partial_\varphi w_2 = \frac{k_g}{R_k h_k - h_k h_s}, \quad \partial_\varphi w_3 = -\frac{h_g}{h_s h_k - h_k h_s}, \quad (7.4)
\end{align*}
\]
from which one learns that
\[
\partial_\theta \mathbf{w}_1 + h \partial_\theta w_2 + k \partial_\theta w_3 = 0 \quad \text{and} \quad \partial_\varphi \mathbf{w}_1 + h \partial_\varphi w_2 + k \partial_\varphi w_3 = 0. \quad (7.5)
\]
The corresponding superpotential reads
\[
G = -\frac{1}{2} R^2 \ln R - R^2 \tilde{g}(\theta, \varphi) \quad \text{with} \quad \tilde{g} = \mathbf{w}_1 + h w_2 + k w_3, \quad (7.6)
\]
leading to
\[
\tilde{g}_\theta = h \partial_\theta w_2 + k \partial_\theta w_3 \quad \text{and} \quad \tilde{g}_\varphi = h \partial_\varphi w_2 + k \partial_\varphi w_3. \quad (7.7)
\]
Similarly, the analogous ansatz for \( F \) can be related to these functions, and \( V_B \) may be expressed through them as well, with \( C = \frac{1}{2} m^2 \).

In contrast to the \( n=2 \) case, the above equations do not admit solutions for an arbitrary choice of \( h(\theta, \varphi) \) and \( k(\theta, \varphi) \). In fact, it seems quite nontrivial to find an admissible pair \( (h,k) \) at all. This is related to the appearance of the WDVV equations. For completeness, we also display the integrability condition (4.33) for our ansatz (7.1),
\[
(k_\varphi h_\varphi - h_\varphi k_\varphi) + (k_\varphi h_\theta - h_\varphi k_\theta) \sin^2 \theta = (k_\varphi h_\varphi - h_\varphi k_\varphi) \sin 2\theta, \quad (7.8)
\]

### 7.1 \( D_3 \) solution

Since some solutions for the prepotential \( F \), based on Coxeter root systems \([11, 12]\), are known, we might as well take advantage of them and employ (4.34) to identify the inertial coordinates and superpotential for such cases. Most important is the \( A_3 \) case, as it generalizes the four-particle Calogero model. We use the \( D_3 \) parametrization of the roots and allow for a ‘radial term’ in the WDVV prepotential
\[
F = -\frac{1}{2} f_L \sum_{i<j} (y_i - y_j)^2 \ln |y_i - y_j| - \frac{1}{2} f_R \sum_{i<j} (y_i + y_j)^2 \ln |y_i + y_j| - \frac{1}{2} f_R R^2 \ln R \quad (7.9)
\]
with \( i, j = 1, 2, 3 \) and the restriction \( 4f_L + f_R = 1 \). For the special value \( (f_L, f_R) = (-\frac{1}{4}, 2) \) we discovered the solution
\[
\begin{align*}
u^1 &= R^2, \quad u^2 = R^2 I(\frac{y_1 - y_2}{y_1 + y_2}), \quad u^3 = R^2 I(\frac{y_1 - y_3}{y_1 + y_3}) \quad \text{with} \quad \nu^1_i = y_i + \sqrt{y_i^2 - y_3^2}, \quad (7.10)
\end{align*}
\]
where \( I(x) = \int_0^x \frac{dt}{\sqrt{1-t}} \) denotes an incomplete elliptical integral of the first kind. The inverse Jacobian yields
\[
\begin{align*}
w_{2,1} &= \frac{r_1 r_2}{y_3 R^4} \sqrt{\frac{2(y_1^2 - y_2^2)}{y_1^2 - y_3^2}} \left( y_1 y_2 \sqrt{y_2^2 - y_3^2} - (y_2^2 + y_3^2) \sqrt{y_1^2 - y_3^2} \right), \\
w_{2,2} &= \frac{r_1 r_2}{y_3 R^4} \sqrt{\frac{2(y_2^2 - y_3^2)}{y_2^2 - y_1^2}} \left( y_1 y_2 \sqrt{y_1^2 - y_3^2} - (y_1^2 + y_3^2) \sqrt{y_2^2 - y_3^2} \right), \\
w_{2,3} &= \frac{r_1 r_2}{R^4} \sqrt{\frac{2(y_1^2 - y_2^2)}{y_1^2 - y_3^2}} \left( y_1 \sqrt{y_1^2 - y_3^2} + y_2 \sqrt{y_2^2 - y_3^2} \right), \\
w_{3,1} &= \frac{r_2}{r_3 R^4} \sqrt{2(y_1^2 r_2^2 - y_2^2 r_3^2)} \left( r_1^2 r_2^2 y_1 - 3r_1 r_2^2 y_1 y_2 - r_2^2 (y_2^2 + y_3^2) + 2y_2 (y_2^2 + y_3^2) \right), \\
w_{3,2} &= \frac{r_2}{r_3 R^4} \sqrt{2(y_1^2 r_2^2 - y_3^2 r_3^2)} \left( r_1^2 r_2^2 y_2 + r_1^2 (y_2^2 - 2y_2^2 + y_3) + r_2^2 y_1 y_2 - 2y_1 y_2^2 \right), \\
w_{3,3} &= \frac{r_2}{r_3 R^4} \sqrt{2(y_1^2 r_2^2 - y_3^2 r_3^2)} \left( -r_1^2 r_2^2 (y_1^2 + y_2^2) + r_1^2 y_2 (2y_1^2 + 2y_2^2 - y_3^2) + r_2^2 y_1 y_2^2 - 2y_1 y_2 y_3^2 \right)
\end{align*}
\]

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and \( w_{1,j} \) in terms of elliptic integrals, which gives us \( U \) and the bosonic potential

\[
V_B = -\frac{2}{(R^2)^3} \left( \left( (m^2)^2 + (m^3)^2 \right) \left[ y_1 (y_2^2 - y_3^2)^{\frac{3}{2}} + y_2 (y_1^2 - y_3^2)^{\frac{3}{2}} \right] + \sqrt{2} m^2 m_\lambda \frac{(y_1^2 - y_2^2)^{\frac{3}{2}}}{y_3} \sqrt{r_1^2 + r_2^2} \frac{(r_2^2 - 2 y_2)}{y_1 r_1^2 - y_2 r_2^2} \right) + m^4 \text{-terms}. \tag{7.12}
\]

It is regular except for \( R \to 0 \).

To pass to the \( A_3 \) parametrization \((z_1, z_2, z_3)\), one has to apply the orthogonal transformation

\[
y_i = O_{ij} \ z_j \quad \text{with} \quad (O_{ij}) = \frac{1}{\sqrt{6}} \begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 1 & 0 \\ -\sqrt{2} & 0 & \sqrt{2} \end{pmatrix}, \tag{7.13}
\]

so that the six positive roots become

\[
\frac{2\sqrt{3} z_1}{\sqrt{6}}, \quad \frac{\sqrt{3} z_1 + 3 z_2}{\sqrt{6}}, \quad \frac{-\sqrt{3} z_1 + 3 z_2}{\sqrt{6}}, \quad \frac{2 z_2 - 2 \sqrt{2} z_3}{\sqrt{6}}, \quad \frac{\sqrt{3} z_1 - z_2 - 2 \sqrt{2} z_3}{\sqrt{6}}, \quad \frac{-\sqrt{3} z_1 - z_2 - 2 \sqrt{2} z_3}{\sqrt{6}}. \tag{7.14}
\]

### 7.2 \( B_3 \) solution

Surprisingly, a simpler solution arises for the \( B_3 \) root system, with the WDVV prepotential

\[
F = -\frac{1}{2} f_S \sum_i y_i^2 \ln |y_i| - \frac{1}{2} f_L \sum_{i<j} (y_i - y_j)^2 \ln |y_i - y_j| - \frac{1}{2} f_L \sum_{i<j} (y_i + y_j)^2 \ln |y_i + y_j| \tag{7.15}
\]

lacking a radial term. The weights are constrained by \( f_S + 4 f_L = 1 \). For \((f_S, f_L) = (5, -1)\) we found the inertial coordinates

\[
u^1 = \frac{y_1^6}{(y_1^2 - y_2^2)(y_1^2 - y_3^2)}, \quad \nu^2 = \frac{y_2^6}{(y_2^2 - y_3^2)(y_2^2 - y_1^2)}, \quad \nu^3 = \frac{y_3^6}{(y_3^2 - y_1^2)(y_3^2 - y_2^2)}, \tag{7.16}
\]

which yield the dual coordinates

\[
w_1 = \frac{1}{2} \ln |y_1| + \frac{y_1^2 + y_2^2 - y_1 y_2}{8 y_1^3}, \quad w_2 = \frac{1}{2} \ln |y_2| + \frac{y_2^2 + y_3^2 - y_2 y_3}{8 y_2^3}, \quad w_3 = \frac{1}{2} \ln |y_3| + \frac{y_3^2 + y_1^2 - y_3 y_1}{8 y_3^3}. \tag{7.17}
\]

Note that this solution is outside the ansatz \((7.1)\) and somewhat peculiar since \( w_A \) contains rational parts but features logarithms of the short roots only. Moreover, it is invariant under permutations of the \( y_i \) but of course not translation invariant. No coordinate is distinguished as radial, but we have \( u^1 + u^2 + u^3 = y_1^2 + y_2^2 + y_3^2 = R^2 \). Due to \( c_A = 1/2 \), the central charge becomes \( C = \frac{1}{2}(m^1 + m^2 + m^3) \). We read off the second prepotential

\[
U = -\frac{1}{2} m^1 \ln |y_1| - m^1 \left( \frac{y_2^3 y_3^2 - y_1^3 y_3^2}{8 y_1^3} \right) + \text{cyclic}, \tag{7.18}
\]

thus obtaining the specific homogeneous function needed to overcome the \( n=3 \) barrier of \([6]\). It displays the expected singular behavior \( U \sim |y_i|^{1-f_S} \) for \( y_i \to 0 \) and has couplings only for the short roots. The rational parts of \( w_A \) drop out in the superpotential

\[
G = \frac{y_1^6 \ln |y_1|}{2(y_1^2 - y_2^2)(y_1^2 - y_3^2)} + \frac{y_2^6 \ln |y_2|}{2(y_2^2 - y_3^2)(y_2^2 - y_1^2)} + \frac{y_3^6 \ln |y_3|}{2(y_3^2 - y_1^2)(y_3^2 - y_2^3)} \tag{7.19}
\]

\[
= -\frac{1}{2} u^1 \ln |y_1(u)| - \frac{1}{2} u^2 \ln |y_2(u)| - \frac{1}{2} u^3 \ln |y_3(u)|,
\]

but we could not invert \((7.16)\) to obtain \( y_i(u) \). Finally, one may compute the bosonic potential

\[
V_B = \frac{1}{2 m^1} \left[ m^1 \left( \frac{y_2^3 y_3^2 - y_1^3 y_3^2}{8 y_1^3} \right) y_1 + m^2 \left( \frac{y_2^3 y_3^2 - y_1^3 y_3^2}{8 y_2^3} \right) y_2 - m^3 \left( \frac{2 y_1^3 y_3^2 - 3 y_1^3 y_3^2 + 4 y_3^3}{y_3^2} \right) \right]^{\frac{2}{3}} + \text{cyclic}, \tag{7.20}
\]

which features poles (up to tenth order) for the short roots only. Deviating from the above special values of \((f_L, f_S)\) destroys the simplicity of this solution.
8 Embedding into four dimensions

We may try to produce a translation-invariant four-particle model by repeating the previous story one dimension higher. To this end, we employ the embedding

$$y^i = M^i_\mu x^\mu \quad \text{with} \quad (M^i_\mu) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & \frac{1}{\sqrt{12}} & -\frac{3}{\sqrt{12}} \end{pmatrix}.$$  \hspace{1cm} (8.1)

where the partial isometry $M$ maps onto the relative-motion space due to

$$MM^T = I_3 \quad \text{and} \quad M^T M = P = \frac{1}{3} \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}.$$  \hspace{1cm} (8.2)

For embedding our $D_3$ solution as an $A_3$ model, we must apply the map $M$ to the $z^i$ coordinates, i.e.

$$y^i = (OM)^i_\mu x^\mu \quad \text{with} \quad ((OM)^i_\mu) = \frac{1}{2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$  \hspace{1cm} (8.3)

Together with the center of mass $y^0 = \frac{1}{3}(x^1+x^2+x^3+x^4)$, this is the triality map relating $D_4$ vectors to spinors. The center-of-mass degree of freedom is decoupled,

$$u^0 = (x^1+x^2+x^3+x^4)^2 \quad \longrightarrow \quad w_0 = \frac{1}{8} \ln \left| x^1+x^2+x^3+x^4 \right| = \frac{1}{16} \ln u^0.$$  \hspace{1cm} (8.4)

For the relative motion, our ansatz (7.1) extends to

$$u^1 = \bar{R}^2, \quad u^2 = \bar{R}^2 \, h(\vartheta, \varphi), \quad u^3 = \bar{R}^2 \, k(\vartheta, \varphi)$$

with

$$\bar{R}^2 = \frac{1}{4} \sum_{i<j}(x^i j)^2 \quad \text{and} \quad (y^1, y^2, y^3) = \bar{R} \left( \sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta \right).$$  \hspace{1cm} (8.5)

Models of identical particles require invariance under permutations of the $x^i$ coordinates. The permutation group $S_4$ acts on the two-sphere $(\vartheta, \varphi)$ as the Weyl group of $A_3$, i.e. by permuting the corners of a regular tetrahedron by via $\frac{2\pi}{3}$ rotations and reflections. Therefore, a permutation-invariant solution requires $h$ and $k$ to be $S_4$ invariant functions. Such functions are generated by taking some function $h(\vartheta, \varphi)$ and forming a symmetric combination from its pullbacks $(h \circ \pi)(\vartheta, \varphi)$ along the $S_4$ orbit. The simplest option just averages (8.5) over its $S_4$ orbit. This is admissible due to the linearity of (4.34) (assuming a permutation symmetric $F$ is given), but may result in a degenerate solution. In this way, our $D_3$ solution (7.10), after embedding into four dimensions via $y=OMx$ and averaging over $S_4$ permutations, may yield a totally symmetric four-particle system after all, although we have not checked this.

Another four-particle model is created by subjecting our $B_3$ solution to the embedding (8.1). Clearly, the corresponding four-dimensional superpotential $G(x)$ and bosonic potential $V_B(x)$ are not invariant under permutations of the $x^i$. This is hardly surprising, since this system started out being only $S_3$ symmetric, and so an $S_4$ average of the above solution is not consistent with the WDVV solution (7.15).

In order to produce a genuine four-particle $N=4$ Calogero system, one has to find a solution which combines the features of our $D_3$ and $B_3$ systems above, namely for

$$(f_S, f_L, f_R) = (0, \frac{1}{4}, 0) \quad \longrightarrow \quad F = -\frac{1}{8} \sum_{i<j}(y^i - y^j)^2 \ln |y^i - y^j| - \frac{1}{8} \sum_{i<j}(y^i + y^j)^2 \ln |y^i + y^j|.$$  \hspace{1cm} (8.6)

We know [6] that $U$ (and therefore some $w_A$) behaves as $|\alpha \cdot y|^{1-f_\alpha+\alpha}$ when crossing the wall $\alpha \cdot y=0$ for any root $\alpha$, thus no logarithms should occur in

$$U(y) \sim (y^i + y^j)^{1/2} \quad \text{for} \quad y^i \to \pm y^j \quad \text{hence} \quad U(x) \sim (x^i - x^j)^{1/2} \quad \text{for} \quad x^i \to x^j.$$  \hspace{1cm} (8.7)

It remains a challenge to construct the superpotential $G$ and prepotential $U$ belonging to (8.6).
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