Instanton-induced scalar potential
for the universal hypermultiplet

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Abstract

We calculate the scalar potential in the gauged N=2 supergravity with a single hypermultiplet, whose generic quaternionic moduli space metric has an abelian isometry. This isometry is gauged by the use of a graviphoton gauge field. The hypermultiplet metric and the scalar potential are both governed by the single real potential that is a solution to the 3d (integrable) continuous Toda equation. An explicit solution, controlled by the Eisenstein series $E_{3/2}$, is found in the case of the D-instanton-corrected universal hypermultiplet moduli space metric having an $U(1) \times U(1)$ isometry, with one of the isometries being gauged.
1 Introduction

The *Universal Hypermultiplet* (UH) sector of the *Calabi-Yau* (CY) compactified type-IIA superstrings/M-theory is a good place to study non-perturbative quantum corrections within the effective N=2 supergravity in four or five spacetime dimensions [1]. The UH contains a dilaton $\phi$, an axion $D$ and a RR-type complex scalar $C$ as the bosonic field components, while the UH is present in *any* CY compactification of type-IIA superstrings. The classical UH moduli space is given by a symmetric (homogeneous) quaternionic space $SU(2,1)/U(2)$ [2], while its metric and isometries are not protected against quantum corrections on the type-IIA side. The perturbative (type-IIA superstring loop) corrections to the UH metric are known to be limited to the one-loop order, being proportional to the Euler characteristics of CY [3]. The origin of the non-perturbative corrections is also well understood [1]: they appear due to the so-called D-instantons and five-brane instantons [4]. The former are the Euclidean D2-branes wrapped about the supersymmetric 3-cycles of CY, whereas the latter are the Euclidean BPS five-branes wrapped about the entire CY space [1]. The relevant instanton solutions saturating the BPS bound, as well as the corresponding instanton actions, were calculated in ref. [5].

The quantum UH moduli space metric is highly constrained by unbroken symmetries. This metric must be quaternionic because of unbroken N=2 local supersymmetry in four or five uncompactified spacetime dimensions [6]. As regards a single hypermultiplet (like UH) with the *four*-dimensional moduli space, the quaternionic condition amounts to the *Einstein-Weyl* equations (see sect. 3 for details). The quantized brane charges (or the flux quantization condition of the antisymmetric tensor field in M-theory) imply discrete identifications for the UH scalars, which break most of the continuous classical symmetries of $SU(2,1)$. Nevertheless, the $U(1)$ rotations of the RR-scalar, $C \rightarrow e^{i\alpha}C$, survive after taking into account the instanton corrections. The extra abelian symmetry associated with constant shifts of the axion, $D \rightarrow D + \delta$, also survives when merely D-instantons are taken into account and the five-brane instantons are suppressed [7]. These observations are consistent with the known instanton actions [5]. In particular, the exact UH metric is governed by the single pre-potential that is a solution to the three-dimensional (integrable) Toda equation [7]. The D-instanton corrected quantum moduli space metric of the UH is supposed to be $SL(2,\mathbb{Z})$-duality invariant. Its explicit form was found in ref. [7], in terms of the $E_{3/2}$ Eisenstein series, in agreement with the supersymmetric completion of the $R^4$-terms in ten-dimensional superstrings [8]. Some explicit results about the five-brane instanton corrected UM moduli space metric were obtained in ref. [9], in
terms of the particular exact solution to the Painlevé VI (integrable) equation [10].

An addition of non-trivial fluxes of the NS-NS and R-R three-forms in ten dimensions amounts to gauging some Peccei-Quinn-type isometries of the UH moduli space in the effective N=2 supergravity [11]. As a result of the gauging, the UH gets the non-trivial scalar potential whose critical points determine the vacua of the theory [11]. An explicit gauging of all abelian isometries of the classical UH moduli space metric was performed in ref. [12]. As regards the quantum UH metric, one can merely gauge the single abelian $U(1)$ isometry that survives after adding quantum instanton corrections. Gauging the abelian isometries of the classical UH moduli space metric gives rise to the scalar potential with the unphysical run-away behaviour, or no critical points in the weak-coupling region where perturbation theory applies [12]. Since the classical UH scalar potential is not protected against quantum corrections, it is more physically reasonable to examine the instanton-corrected UH scalar potential.

Another important motivation to study gauging of an abelian isometry of the instanton-corrected hypermultiplet metric is its relevance to the brane-world scenario [13] in the effective five-dimensional N=2 gauged supergravity. The brane world scenario (with gravity trapped near a domain wall) needs a scalar potential with at least two IR critical points, in order to achieve an exponential suppression on both sides of the wall. In the context of the gauged N=2 supergravity, this can only be achieved by gauging an isometry of a non-homogeneous hypermultiplet moduli space [14]. Some explicit examples of such construction were given in refs. [15, 16]. Unfortunately, the hypermultiplet metrics used in refs. [15, 16] were chosen ad hoc, they have unphysical regions, and they were not derived from some underlying theory (like superstrings or M-theory). The instanton-corrected hypermultiplet moduli space is not homogeneous, while it also gives the natural physical input towards a possible brane world scenario in the CY compactified type-IIA superstrings or M-theory.

Our paper is organized as follows. In sect. 2 we review the relevant facts about gauging an abelian isometry of a hypermultiplet metric in N=2 supergravity. Our discussion is limited to the scalar potential of a single hypermultiplet. In sect. 3 we discuss the relation between the Einstein-Weyl spaces with an $U(1)$ abelian isometry and integrable systems. An explicit solution to the D-instanton-induced UH scalar potential is given in sect. 4. The critical points of the scalar potentials are discussed in sect. 5. Sect. 6 is our conclusion. A brief review of the Einstein-Weyl geometry and a summary of our notation are given in Appendix.

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1There is no difference in treating hypermultiplets in N=2 supergravities in four and five spacetime dimensions.
2  Hypermultiplet scalar potential in the gauged N=2 supergravity

Our purpose in this section is to provide minimum information needed to calculate a scalar potential in the gauged N=2 supergravity with a single charged hypermultiplet whose quaternionic metric has an abelian isometry. This will serve as the pre-requisite for the subsequent sections.

The detailed structure of a generic gauged N=2 supergravity theory with a hypermultiplet matter in four or five spacetime dimensions is well known (see, e.g. ref. [17] for a recent account). In our case, all the relevant formulæ can be extracted from the most recent paper [15] that we are going to follow in this section.

The field contents of an N=2 supergravity multiplet is given by a graviton $e^{a}_{\mu}$, two Majorana gravitinos $\psi^{a}_{\mu i}$, $i = 1, 2$, and a graviphoton (an abelian vector gauge field) $A_{\mu}$. The field contents of a hypermultiplet is given by four real hyperscalars $q^{X}$ and a Dirac hyperino $\eta_{\alpha}$ (for definiteness, we refer to four spacetime dimensions, $\mu = 0, 1, 2, 3$, etc.). In the case of the UH, the scalars $q^{X}$ represent a dilaton, an axion and a complex RR-type scalar in an arbitrary (non-linear sigma-model) parametrization.

The relevant bosonic part of the hypermultiplet low-energy effective action in N=2 supergravity is given by

$$e^{-1}L = -\frac{1}{2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}g_{XY}D_{\mu}q^{X}D^{\mu}q^{Y} - g^{2}V ,$$

where we have added the standard kinetic terms for the graviton (Einstein-Hilbert) and the graviphoton (Maxwell). The kinetic terms of the hypermultiplet in eq. (2.1) are given by the gauged Non-Linear Sigma-Model (NLSM) with the four-dimensional quaternionic metric $g_{XY}(q)$, and the gauge-covariant derivatives

$$D_{\mu}q^{X} = \partial_{\mu}q^{X} + gA_{\mu}k^{X}(q) ,$$

in terms of the Killing vector $k^{X}(q)$ of the gauged isometry of the hypermultiplet moduli space parameterized by $q^{X}$, and the gauge coupling constant $g$. The scalar potential $V(q)$ in eq. (2.1) is given by [11]

$$V = -4P^{k}P^{k} + \frac{3}{4}g_{XY}k^{X}k^{Y} ,$$

$^{2}$The 4-dimensional (curved) spacetime should not be confused with the 4-dimensional (curved) hypermultiplet moduli space. We also distinguish between the hypermultiplet moduli space and its tangent space, as well as between the corresponding indices — see Appendix.
where a triplet of the Killing pre-potentials $P^k$ has been introduced,

$$P^k = -\frac{1}{4} D_X k_Y J^{XYk} = -\frac{1}{4} \partial_X k_Y J^{XYk}, \quad k = 1, 2, 3,$$  

(2.4)

in terms of the complex structures $J^{XYk} = -J^{YXk}$ of the quaternionic metric $g_{XY}$. However, eq. (2.3) is not convenient for our purposes, since it requires a calculation of the quaternionic structure $J^{XYk}$ that is not really needed.

In fact, the structure of the scalar potential $V$ in eq. (2.1) is dictated by another scalar function known as the superpotential $W$ that can be read off from the gravitino supersymmetry transformation law [17],

$$\delta \psi_{\mu i} = D_\mu \varepsilon_i + \frac{ig}{\sqrt{6}} \gamma_\mu P_{ij} \varepsilon^j + \ldots, \quad i, j = 1, 2,$$  

(2.5)

where merely the bosonic terms have been written down on the right-hand-side. The superpotential $W$ is defined in terms of the Killing pre-potentials [17],

$$W^2 = \frac{1}{3} P_{ij} P^{ij} = \frac{2}{3} P^k P^k,$$  

(2.6)

whereas the scalar potential $V$ is related to the superpotential $W$ as follows [15]:

$$V = -6W^2 + \frac{9}{2} g^{XY} \partial_X W \partial_Y W.$$  

(2.7)

In the special case of a single hypermultiplet, the quaternionic identity $J^{k}_{XY} J^{k}_{ZW} = -\varepsilon_{XYZW} + (\delta_{XZ} \delta_{YW} - \delta_{XW} \delta_{YZ})$ allows one to rewrite eq. (2.6) into a simpler form [15]

$$W^2 = \frac{1}{3} dK \wedge * dK - \frac{1}{6} dK \wedge dK,$$  

(2.8)

in terms of the Killing one-form $K = k_X dq^X$ and the Hodge star operation ($*$) alone.

Thus the only NLSM reparametrization-invariant input needed to calculate the scalar potential $V(q)$ is given by a quaternionic metric $g_{XY}$ and its Killing vector $K$, by using eqs. (2.7) and (2.8). In physical applications we should choose a particular parametrization of the hypermultiplet moduli space, in which the abelian isometry of the metric is manifest (see sect. 3).

It is worth noticing that the scalar potential $V(q)$ is obviously dependent upon the chosen NLSM parametrization of the hypermultiplet scalars, whereas the critical points (vacua) of the scalar potential are parametrization-independent. Under a reparametrization $q = q(\tilde{q})$ we have

$$\frac{\partial V}{\partial \tilde{q}} = \frac{\partial q}{\partial \tilde{q}} \frac{\partial V}{\partial q},$$  

(2.9)

so that $\partial V/\partial q = 0$ is equivalent to $\partial V/\partial \tilde{q} = 0$ because of $\det(\partial q/\partial \tilde{q}) \neq 0$. 

5
3 Einstein-Weyl metrics with an abelian isometry

An N=2 locally supersymmetric NLSM with any number of hypermultiplets has a quaternionic metric \([6]\). In the case of a single hypermultiplet, N=2 local supersymmetry amounts to the Einstein-Weyl conditions (with a negative scalar curvature) on the NLSM metric \(g\), \[ W^{-}_i = 0 \, , \quad (Ric)_{ab} = \Lambda \delta_{ab} \, , \quad \Lambda < 0 \, . \] (3.1)

Given an isometry of the metric \(g\) with the associated Killing 1-form \(K = K_X dq^X\) and \(D_X K_Y + D_Y K_X = 0\), one can decompose the 2-form \(dK = \partial_X K_Y dq^X \wedge dq^Y\) with respect to the basis \((A.8)\),

\[ dK = (dK^+_i)\Xi^+_i + (dK^-_i)\Xi^-_i \, , \] (3.2)

and extract the quaternionic structure (i.e. three complex structures \(J^k\) obeying the quaternionic algebra \(J^i J^j = -\delta^{ij} + \varepsilon^{ijk} J^k\)) in terms of the Killing form \([18, 19, 20]\),

\[ J = \frac{dK^-}{\sqrt{\sum_i (dK^-_i)^2}} \Xi^-_i \, . \] (3.3)

An important theorem due to Przanowski \([18]\) and Tod \([19]\) claims that any Einstein-Weyl metric can be locally written down in adapted coordinates (with a Killing vector \(\partial_t\)) as follows:

\[ g = \frac{1}{w^2} \left\{ \frac{1}{P} (dt + \Theta)^2 + P \left[ e^u (d\mu^2 + d\nu^2) + dw^2 \right] \right\} \, . \] (3.4)

in terms of real local coordinates \((t, w, \mu, \nu)\), 1-form \(\Theta = \Theta_1 dw + \Theta_2 d\mu + \Theta_3 d\nu\), and two potentials \(P = P(w, \mu, \nu)\) and \(u = u(w, \mu, \nu)\). Imposing the Einstein-Weyl conditions (3.1) on the metric (3.4) yields \([18, 19]\)

\[ P = \frac{3}{2\Lambda} (w \partial_w u - 2) \, , \] (3.5)

\[ (\partial_\mu^2 + \partial_\nu^2) u + \partial_w^2 (e^u) = 0 \, , \] (3.6)

and

\[ -d\Theta = (\partial_\nu P) d\mu \wedge dw + (\partial_\mu P) d\nu \wedge dw + \partial_w (Pe^u) d\nu \wedge d\mu \, . \] (3.7)

As is clear from eqs. (3.5), (3.6) and (3.7), the metric \(g\) is controlled by the single prepotential \(u\) obeying the non-linear (integrable) three-dimensional (continuous) Toda equation (3.6).

\(^3\)Our notation is given in Appendix.
In accordance to eq. (3.4), let’s choose the vierbein as

\[ e_0 = \frac{(dt + \Theta)}{w \sqrt{P}} , \quad e_2 = \sqrt{P} \exp(\frac{1}{2} w^\mu \frac{d\mu}{w}) , \]
\[ e_1 = \sqrt{P} \frac{dw}{w} , \quad e_3 = \sqrt{P} \exp(\frac{1}{2} w^\nu \frac{d\nu}{w}) . \]  

(3.8)

The Killing vector \( K^X = (1, 0, 0, 0) \) yields the Killing 1-form

\[ K = \frac{1}{w^2 P} (dt + \Theta) = \frac{1}{w \sqrt{P}} e_0 . \]  

(3.9)

The square of the Killing vector is given by

\[ K^2 = g_{XY} K^X K^Y = g_{tt} = \frac{1}{w^2 P} . \]  

(3.10)

The coordinate \( w \) in terms of the Killing vector \( K \) reads [20]

\[ w = -\Lambda/3 \frac{1}{\sqrt{\sum_i (dK_i^-)^2}} . \]  

(3.11)

By using the identities

\[ \Xi_i^+ \wedge \Xi_j^+ = -\Xi_i^- \wedge \Xi_j^- = 2 \delta_{ij} e_0 \wedge e_1 \wedge e_2 \wedge e_3 \]  

(3.12a)

and

\[ \Xi_i^+ \wedge \Xi_j^- = \Xi_i^- \wedge \Xi_j^+ = 0 , \]  

(3.12b)

and substituting the decomposition (3.2) into eq. (2.8) allows us to simplify the superpotential \( W \) to the form

\[ W^2 = (dK_i^-)^2 + \frac{1}{3} (dK_i^+)^2 = \frac{\Lambda^2}{9 w^2} + \frac{1}{3} (dK_i^+)^2 , \]  

(3.13)

where eq. (3.11) has been used.

It is straightforward to calculate the 2-form \( dK \) from eqs. (3.8) and (3.9). We find

\[ dK = \frac{1}{\sqrt{P}} \left( 1 + \frac{w}{2 P \partial_w P} \right) e_0 \wedge e_1 + \frac{w \partial_\mu P}{2 P \sqrt{e^\mu P}} e_0 \wedge e_2 + \frac{w \partial_\mu P}{2 P \sqrt{e^\mu P}} e_0 \wedge e_3 + \frac{w \partial_\nu P}{P \sqrt{e^\nu P}} e_3 \wedge e_1 + \frac{w}{\sqrt{P}} \left( \partial_w u + \frac{\partial_w P}{P} \right) e_2 \wedge e_3 . \]  

(3.14)
Equations (3.2) and (A.8) now imply
\[(dK_i^+)\Xi_i^+ = \frac{1}{2\sqrt{P}} \left( 1 + \frac{3w}{2P} \partial_w P + w \partial_w u \right) \Xi_i^+ \]
\[+ \frac{3w \partial_w P}{2P \sqrt{e^u P}} \Xi_i^+ + \frac{3w \partial_v P}{2P \sqrt{e^u P}} \Xi_3^+ , \]
and hence, we get
\[(dK_i^+)^2 = \frac{1}{4P} \left( 1 + w \partial_w u + \frac{3}{2P} w \partial_w P \right)^2 + \frac{9w^2 e^{-u}}{4P^3} \left[ (\partial_\mu P)^2 + (\partial_\nu P)^2 \right] . \]

We conclude that both the hypermultiplet metric (3.4) and the scalar potential (2.7) are dictated by a solution $u(w, \mu, \nu)$ of the Toda equation (3.6) via eqs. (3.5), (3.7), and eqs. (2.7), (3.13) and (3.16), respectively.

By substituting eq. (3.16) into eq. (3.13) and using eq. (3.5), we find the superpotential in the form
\[W^2 = \frac{\Lambda^2}{9w^2} + \frac{1}{12P} \left( 3 + \frac{2\Lambda}{3} P + \frac{3}{2P} w \partial_w P \right)^2 \]
\[+ \frac{3w^2 e^{-u}}{4P^3} \left[ (\partial_\mu P)^2 + (\partial_\nu P)^2 \right] . \]

Until this point no approximation was made, so that we actually discussed a derivation of exact solutions to the hypermultiplet moduli space metric and the scalar potential. Unfortunately, despite of the fact that the Toda equation (3.6) is known to be integrable (this equation appears in the large-$N$ limit of the standard (two-dimensional) Toda system for $SU(N)$ [21]), it is very hard to obtain its explicit solutions [22]. This is apparently the price to pay for getting the exact solution describing both five-brane and two-brane instanton corrections to the UH metric and its scalar potential [9].

It is instructive to see how this problem simplifies in the hyper-Kähler limit for the hypermultiplet metric, when $N=2$ supergravity decouples. This limit appears when $\Lambda \rightarrow 0$ above, since $\Lambda$ is proportional to the gravitational coupling constant [6]. In this limit the function $P$ becomes proportional to $\partial_w u$, whereas the non-linear Toda equation (3.6) becomes a linear equation on $P$ [23, 9],
\[(\partial_\mu^2 + \partial_\nu^2 + \partial_w^2) P = 0 . \]

The abelian isometry is tri-holomorphic in this limit, so that we obtain the standard Gibbons-Hawking Ansatz for a hyper-Kähler metric governed by a harmonic function $P(w, \mu, \nu)$ [24].
The scalar potential, originating from the gauging of the tri-holomorphic isometry in the hyper-Kähler limit is given by half of the Killing vector squared, or just $\frac{1}{2}P^{-1}$. For example, the (Gibbons-Hawking) multi-centre metrics are described by

$$P(\vec{X}) = \sum_{p=1}^{m} \frac{1}{|\vec{X} - \vec{X}_p|}, \quad \vec{X} = (w, \mu, \nu), \quad \vec{X}_p = \text{const.} \quad (3.19)$$

The critical points of the scalar potential are given by poles of $P$, i.e. they appear at $\vec{X} = \vec{X}_p$ in the case of eq. (3.19). Since the scalar potential vanishes at these points, N=2 supersymmetry remains unbroken. Our results are, therefore, consistent with a derivation of the hypermultiplet scalar potential by Scherk-Schwarz dimensional reduction from six dimensions in the hyper-Kähler limit [25].

4 UH scalar potential induced by D-instantons

To get an explicit non-perturbative solution to the hypermultiplet scalar potential, we now consider the special case of the UH when the D-instanton contributions are included but the five-brane instantons are suppressed. The D-instanton corrections are of the order $e^{-1/g_{\text{string}}}$, whereas the five-brane instanton corrections are of the order $e^{-1/g_{\text{string}}^2}$ [4]. Hence, for sufficiently small string coupling $g_{\text{string}}$, we may hope that the D-instanton corrections dominate over the five-brane instanton corrections. In this case, there is another abelian isometry given by a shift of the axion, $D \rightarrow D + \delta$, which commutes with an $U(1)$ rotation of the RR-scalar, $C \rightarrow e^{i\alpha}C$, that is going to be gauged. As was pointed out in the second ref. [12], gauging a compact direction of the homogeneous hypermultiplet moduli space yields a fixed point, whereas gauging a non-compact direction yields a run-away solution.

Due to some recent advances in the mathematical literature [26], given two commuting and non-degenerate (i.e. hypersurface generating) abelian isometries (Killing vectors), one can completely solve the Einstein-Weyl equations (3.1) in adapted coordinates, where both isometries are manifest, in terms of a real potential depending upon two remaining coordinates and satisfying a linear equation.

The main result of ref. [26] is the theorem that any Einstein-Weyl metric (of non-vanishing scalar curvature) with two linearly independent Killing vectors can be written down in the from

$$g = \frac{F^2 - 4\rho^2(F^2_\rho + F^2_\eta)}{4F^2} \left(\frac{dp^2 + d\eta^2}{\rho^2}\right) + \frac{[(F - 2\rho F_\rho)\hat{\alpha} - 2\rho F_\eta \hat{\beta}]^2 + [2\rho F_\rho \hat{\alpha} - (F + 2\rho F_\rho)\hat{\beta}]^2}{F^2[F^2 - 4\rho^2(F^2_\rho + F^2_\eta)]}, \quad (4.1)$$
in some local coordinates \((\rho, \eta, \theta, t)\) inside an open region of the half-space \(\rho > 0\). Here \(\partial_\theta\) and \(\partial_t\) are two Killing vectors, while the one-forms \(\hat{\alpha}\) and \(\hat{\beta}\) are given by

\[
\hat{\alpha} = \sqrt{\rho} \, d\theta \quad \text{and} \quad \hat{\beta} = \frac{dt + \eta d\theta}{\sqrt{\rho}}.
\] (4.2)

The whole metric (4.1) is governed by a real function (= pre-potential) \(F(\rho, \eta)\) that is the eigenfunction of the Laplacian in the hyperbolic plane,

\[
\Delta_\mathcal{H} F \equiv \rho^2 \left( \partial^2_\rho + \partial^2_\eta \right) F = \frac{3}{4} F.
\] (4.3)

The Einstein-Weyl metric (4.1) has a negative scalar curvature provided that

\[
4\rho^2 (F^2_\rho + F^2_\eta) > F^2 > 0.
\] (4.4)

As was demonstrated in ref. [7], an unique (up to a normalization) \(SL(2, \mathbb{Z})\) duality-invariant solution to the master equation (4.3) is given by the Eisenstein series \(E_{3/2}(\rho, \eta)\). It has the Fourier expansion [27]

\[
4\pi \zeta(3) E_{3/2}(\rho, \eta) = 2\zeta(3) \rho^{3/2} + \frac{2\pi^2}{3} \rho^{-1/2} + 8\pi \rho^{1/2} \sum_{m \neq 0 \atop n \geq 1} \frac{|m|}{n} e^{2\pi imn} K_1(2\pi |mn| \rho),
\] (4.5)

where \(\zeta(3) = \sum_{m > 0} (1/m)^3\) and the modified Bessel function \(K_1(z)\) of the 3rd kind have been introduced. The asymptotic expansion of the hypermultiplet pre-potential in the perturbative region (large \(\rho\)) reads

\[
F(\rho, \eta) = 4\pi \zeta(3) E_{3/2}(\rho, \eta) = 2\zeta(3) \rho^{3/2} + \frac{2\pi^2}{3} \rho^{-1/2} + 4\pi^{3/2} \sum_{m,n \geq 1} \left( \frac{m}{n^3} \right)^{1/2} \times
\]

\[
\times \left[ e^{2\pi imn(\eta + i\rho)} + e^{-2\pi imn(\eta - i\rho)} \right] \left[ 1 + \sum_{k=1}^{\infty} \frac{\Gamma(k - 1/2)}{\Gamma(-k - 1/2)} \frac{1}{(4\pi mn\rho)^k} \right],
\] (4.6)

while it can be interpreted as a sum of the classical (tree level) term, the one-loop (perturbative) correction and the infinite D-instanton sum, respectively, in the apparent similarity to the known \(SL(2, \mathbb{Z})\) duality-invariant completion of the \(R^4\)-terms in the ten-dimensional type-IIB superstrings [8]. We expect that our result (4.6) can be reproduced from the ten-dimensional \(R^4\)-terms via CY compactification [7].

It is not difficult to map the Calderbank-Petersen (=CP) Ansatz (4.1) into the more general Przanowski-Tod (=PT) Ansatz (3.4). In fact, this was already done in ref. [20]. We are going to pay a special attention to the PT coordinate \(w\) and the PT
potential $P$ in terms of the ‘active’ CP coordinates $(\rho, \eta)$ and the CP pre-potential $F(\rho, \eta)$. In terms of the related function $G = \sqrt{\rho} F$, (4.6)

the CP metric (4.1) can be rewritten to the form [20]

$$ g = \frac{1}{G^2} \left\{ \frac{1}{W} (dt + \Theta)^2 + W\gamma \right\} , $$

(4.7)

where we have used the notation [20]

$$ W = \frac{GG_{\rho}}{\rho(G_{\rho}^2 + G_{\eta}^2)} - 1 , \quad \Theta = \left( \frac{GG_{\eta}}{G_{\rho}^2 + G_{\eta}^2} - \eta \right) d\alpha , $$

(4.8a)

and

$$ \gamma = \rho^2 d\alpha^2 + (G_{\rho}^2 + G_{\eta}^2)(d\rho^2 + d\eta^2) . $$

(4.8b)

Let $K$ be the 1-form associated with the Killing vector $\partial_t$,

$$ K = \frac{dt + \Theta}{G^2 W} . $$

(4.9)

We can now explicitly compute the 2-form $dK$, as well as its SD and ASD parts, $dK^-$ and $dK^+$, like in the previous sect. 3. For example, one finds [20]

$$ dK^- = \frac{-1}{G \sqrt{G_{\rho}^2 + G_{\eta}^2}} (G_{\rho}\Xi^-_1 + G_{\eta}\Xi^-_2) . $$

(4.10)

This allows us to identify

$$ w = G \quad \text{and} \quad P = W . $$

(4.11)

More explicitly, we find

$$ w = \sqrt{\rho} F = \sqrt{\rho} E_{3/2}(\rho, \eta) , $$

(4.12)

and

$$ P = \frac{E_{3/2}^2}{2\rho^2} \left[ \left( \frac{E_{3/2}}{2\rho} + \frac{\partial E_{3/2}}{\partial \rho} \right)^2 + \left( \frac{\partial E_{3/2}}{\partial \eta} \right)^2 \right] - 1 . $$

(4.13)

Once the $P$-function is known, the Toda potential $u$ is easily obtained by integrating eq. (3.5). The final result for the D-instanton-corrected scalar potential (or the superpotential) of the UH, in terms of the Eisenstein series $E_{3/2}$, is not very illuminating, so that we do not write it down here. Instead, in the next sect. 5, we discuss its critical points.
5 Critical points of the scalar potential

According to our results in sect. 3, the critical points of the hypermultiplet scalar potential are given by *poles* of $P$ and $w$, if one assumes that those poles are isolated points. This is the case, as long as the scalar potential is controlled by a meromorphic function like the Eisenstein series. Those poles precisely correspond to the points where the gauged $U(1)$ Killing vector (3.9) vanishes, because of eq. (3.10):

$$K^2 = 0 \quad \text{is equivalent to} \quad w^2 P = \infty,$$

in agreement with the general results of refs. [12, 15, 17].

Generally speaking, eq. (5.1) defines a (null) surface in the hypermultiplet moduli space, either of real dimension zero or two, depending upon the rank of the two-form $dK$ on the surface [28]. If the rank is maximal, the null surface is just a point called *nut*. When the rank of $dK$ is two, the null two-dimensional surface is called a *bolt*.

A physical vacuum is supposed to allow a perturbative expansion around it, which amounts to analyticity of the Killing vector and a finite curvature at the critical point, in our situation. The good (physical) critical points are therefore described by the following (NLSM) reparametrization-invariant conditions [15]:

$$g_{XY} K^X K^Y \equiv K^2 = 0, \quad (D_X K_Y)(D^X K^Y) \equiv (D K)^2 \neq 0,$$

and

$$R_{XYZW} R^{XYZW} \neq \infty.$$  (5.3)

The critical points of the scalar potential in the particular $\text{N}=2$ gauged supergravity model of a hypermultiplet, based on the non-homogeneous Einstein-Weyl metric interpolating between two homogeneous quaternionic metrics of $SO(4,1)/SO(4)$ and $SU(2,1)/U(2)$, were analyzed in detail by Behrndt and Dall’Agata [15]. They found two good IR fixed points in their model [15]. Unfortunately, the hypermultiplet moduli space, used as an input in ref. [15], has a singularity, while it is not geodesically complete. Quantum instanton corrections to the classical hypermultiplet moduli space metric are expected to result in a regular and positive definite metric [1]. Further progress in this direction apparently requires an explicit knowledge of exact (non-separable) solutions to the 3d Toda equation (3.6).

In the particular case of the D-instanton-corrected UH moduli space, controlled by the Eisenstein series $E_{3/2}$, the situation is much simpler. The $E_{3/2}$ has polynomial growth ($\sim \rho^{3/2}$) at weak coupling $\rho = +\infty$ that corresponds to the classical vacuum,
while all the one-loop and D-instanton quantum corrections disappear in that limit. This is consistent with our equations in sect. 4. When \( \rho \to +\infty \), we have

\[
E_{3/2} \sim \rho^{3/2}, \quad G \sim \rho^2, \quad \frac{\partial E_{3/2}}{\partial \eta} \to 0, \quad \text{and} \quad \mathcal{W} \to \text{const.} \neq 0, \quad (5.4)
\]

so that \( K^2 = G^{-2}\mathcal{W}^{-1} \sim \rho^{-2} \to 0 \) indeed. Hence, the classical limit

\[
\rho = +\infty \quad (5.5)
\]
corresponds to a fixed point of the D-instanton-induced scalar potential.

As regards finite values of \( \rho \neq 0 \) (i.e. strong coupling), the Eisenstein series \( E_{3/2} \) has no singularities, while the function \( G = \sqrt{\rho}E_{3/2} \) is finite even at \( \rho = 0 \). Hence, any other critical points are only possible when \( P = \infty \), i.e.

\[
\frac{\partial E_{3/2}}{\partial \rho} + \frac{E_{3/2}}{2\rho} = \frac{\partial E_{3/2}}{\partial \eta} = 0 \quad, \quad (5.6)
\]

where we have used eq. (4.13). Equation (5.6) has a solution, \( \rho = 0 \) and \( \eta \in \mathbb{Z} \), where we have taken into account that the Eisenstein series is periodic in \( \eta \) with period 1. Indeed, by using the relation \( K_1(z) \approx z^{-1} \) for small values of \( z \to 0 \), it is not difficult to verify that for small values of \( \rho \to 0 \) (at strong coupling), we have

\[
4\pi \zeta(3)E_{3/2}(\rho, \eta) \approx \rho^{-1/2} \left[ \frac{2\pi^2}{3} + 4 \sum_{m \neq 0} \sigma_{-2}(m)e^{2\pi i m \eta} \right], \quad (5.7)
\]

and

\[
\frac{\partial G}{\partial \eta} = \sqrt{\rho} \frac{\partial E_{3/2}}{\partial \eta} = -4 \frac{\zeta(3)}{\zeta(3)} \sum_{m=1}^{+\infty} m\sigma_{-2}(m) \sin(2\pi m \eta) \quad, \quad (5.8)
\]

where we have introduced the standard divisor function \([27]\]

\[
\sigma_s(m) = \sum_{0<d|m} d^s \quad, \quad (5.9)
\]

It is now clear that the only solution to eq. (5.6) is given by

\[
\rho = 0 \quad \text{and} \quad \eta \in \mathbb{Z} \quad. \quad (5.10)
\]

We conclude that the D-instanton-induced scalar potential of the universal hypermultiplet has the classical fixed point (5.5) at weak coupling and the fixed points (5.10) at strong coupling. The latter exactly appear at the points where the D-instantons are located, so that they are truly generated by the D-instantons.
6 Conclusion

Our results may have natural applications to the brane-world scenarios [13] within the effective N=2 supergravity originating from the CY compactified type-II superstrings and M-theory (see e.g., refs. [14, 29, 30]). They may also be applied to a description of possible renormalization group flows in the holographic approach to extended supergravity [31]. The standard five-dimensional spacetime metric respecting the four-dimensional Poincaré invariance, is given by

\[ ds^2_{5d} = e^{2U(r)} dx_{4d}^2 + dr^2, \]  

where \( U(r) \) is the warp factor [13]. The domain wall solutions in the gauged five-dimensional N=2 supergravity normally preserve half of the original supersymmetries. In the case of a single hypermultiplet supporting the domain wall, the BPS (flow) equations are given by [29]

\[ \frac{dU}{d\tau} = \pm gW, \quad \frac{dq^X}{d\tau} = \mp 3gg^{XY} \partial_Y W, \]  

where \( W \) is the superpotential and \( \tau \) is the flow parameter. The equations of motion are automatically satisfied for the BPS solutions to eq. (6.2). It would be interesting to investigate the BPS solutions to eq. (6.2) in the case of the instanton-generated scalar potential. The BPS walls in some N=2 supersymmetric non-linear sigma-models with hyper-Kähler metrics (in the absence of N=2 supergravity) were investigated in great detail in ref. [32].

The D-instanton corrections are given by powers of \( e^{-1/g_{\text{string}}} \), whereas the five-brane instantons contribute by powers of \( e^{-1/g_{\text{string}}^2} \) [4]. Our results apply when the former dominate over the latter, i.e. when \( g_{\text{string}} \) is sufficiently small. The exact quantum moduli space metric of UH is still governed by the same Toda equation (3.6), however, its general solution is unknown (see, however, ref. [9] for some explicit results about the five-brane instanton corrections to the UH metric).

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Appendix: Einstein-Weyl geometry

Our definitions and notation about the Einstein-Weyl spaces coincide with those used in refs. [18, 19, 20]; see also ref. [33] for more about the quaternionic geometry, and ref. [34] for more about the NLSM with quaternionic geometry and extended supersymmetry. We follow ref. [20] here.

Given a four-dimensional Riemannian manifold of Euclidean signature with local coordinates $q^X$, $X = 0, 1, 2, 3$, and a metric $g = g_{XY} dq^X dq^Y$, let’s introduce a local basis (or a vierbein) $e_a = e_{aX} dq^X$ so that

$$g_{XY} = \delta_{ab} e_{aX} e_{bY}, \quad \text{or} \quad g = \sum_a e_a^2. \quad (A.1)$$

The ‘time’ 0-direction is associated with an abelian isometry of the metric in the main text of the paper. We use capital Latin letters for curved 4-vector indices and early lower-case Latin letters for flat (tangent) 4-vector indices, whereas middle lower-case Latin indices denote ‘spatial’ components of flat (tangent) 4-vector indices, $a = (0, k)$, $k = 1, 2, 3$, in the hypermultiplet moduli space (NLSM).

The spin connection (1-form) $\omega_{ab} = \omega_{abX} dq^X$ is fixed by the vierbein postulate, which means the covariant constancy of the vierbein,

$$de_a + \omega_{ab} \wedge e_b = 0. \quad (A.2)$$

The spin-connection is antisymmetric, $\omega_{ab} = -\omega_{ba}$, while it can be decomposed into the Self-Dual (SD) and Anti-Self-Dual (ASD) parts,

$$\omega_{ab}^\pm = \omega_{ab} \pm \frac{1}{2} \varepsilon_{abcd} \omega_{cd}, \quad \text{or, equivalently,} \quad \omega_i^\pm = \omega_{0i} \pm \frac{1}{2} \varepsilon_{ijk} \omega_{jk}. \quad (A.3)$$

The curvature 2-form is defined by

$$R_{ab} = d\omega_{ab} + \omega_{ad} \wedge \omega_{db} = \frac{1}{2} R_{ab,cd} e_c \wedge e_d, \quad (A.4)$$

while its SD and ASD components are given by

$$R_i^\pm = R_{0i} \pm \frac{1}{2} \varepsilon_{ijk} R_{jk}. \quad (A.5)$$

The Ricci tensor and the scalar curvature are defined by

$$(\text{Ric})_{ab} = R_{ac,bc}, \quad \text{and} \quad R = (\text{Ric})_{aa}. \quad (A.6)$$

The Weyl curvature is given by the traceless part of the curvature, viz.

$$W_{ab,cd} = R_{ab,cd} + \frac{R}{6} [\delta_{ac} \delta_{bd} - \delta_{ad} \delta_{bc}]$$

$$\quad - \frac{1}{2} [\delta_{ac} (\text{Ric})_{bd} - \delta_{ad} (\text{Ric})_{bc} + \delta_{bd} (\text{Ric})_{ac} - \delta_{bc} (\text{Ric})_{ad}]. \quad (A.7)$$
The 2-forms $e_a \wedge e_b$ serve as a basis in the space of all 2-forms, while they can also be decomposed into their SD and ASD parts,

$$\Xi_i^\pm = e_0 \wedge e_i \pm \frac{1}{2} \varepsilon_{ijk} e_j \wedge e_k . \quad (A.8)$$

In particular, we have

$$R_i^+ = A_{ij} \Xi_j^+ + B_{ij} \Xi_j^- , \quad (A.9)$$

$$R_i^- = B_{ij}^T \Xi_j^+ + C_{ij} \Xi_j^- ,$$

where the symmetric $3 \times 3$ real matrices $A$ and $C$, and the non-symmetric $3 \times 3$ real matrix $B$ have been introduced.

Similarly, the Weyl 2-form

$$W_{ab} = \frac{1}{2} W_{ab,cd} e_c \wedge e_d \quad (A.10)$$

can be decomposed into its SD and ASD parts as

$$W_i^+ = W_{0i} + \frac{1}{2} \varepsilon_{ijk} W_{jk} = W_{ij}^+ \Xi_j^+ , \quad (A.11)$$

$$W_i^- = W_{0i} - \frac{1}{2} \varepsilon_{ijk} W_{jk} = W_{ij}^- \Xi_j^- .$$

The *Einstein condition* (with a real constant $\Lambda$),

$$(\text{Ric})_{ab} = \Lambda \delta_{ab} , \quad (A.12)$$

is equivalent to

$$B_{ij} = 0 \quad \text{and} \quad \text{tr} A = \text{tr} C = \Lambda . \quad (A.13)$$

The self-duality of the *Weyl* tensor,

$$W_i^- = 0 , \quad (A.14)$$

is equivalent to

$$C_{ij} \propto \delta_{ij} . \quad (A.15)$$

Hence, the SD Weyl and Einstein conditions together imply

$$C_{ij} = \frac{\Lambda}{3} \delta_{ij} \quad \text{and} \quad R_i^- = \frac{\Lambda}{3} \Xi_i^- . \quad (A.16)$$

The only remaining matrix $A$ is symmetric and has $\text{tr} A = \Lambda$, so that there are five independent curvature components for a generic Einstein-Weyl metric.
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