TELEPORTATION OF ATOMIC STATES VIA CAVITY QED FOR A CAVITY PREPARED IN A SUPERPOSITION OF ZERO AND ONE FOCK STATES

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Abstract
In this article we discuss two schemes of teleportation of atomic states. In the first scheme we consider atoms in a three-level cascade configuration and in the second scheme we consider atoms in a three-level lambda configuration. The experimental realization proposed makes use of cavity Quantum Electrodynamics involving the interaction of Rydberg atoms with a micromaser cavity prepared in a state $|\psi\rangle_C = (|0\rangle + |1\rangle)/\sqrt{2}$.

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1 INTRODUCTION

Quantum information and quantum computation are important and active fields of research [1, 2, 3]. Teleportation, proposed by Bennett et al [4], has important applications in quantum information and quantum computation [1]. There has been a lot of theoretical proposals of schemes of teleportation. In Ref. [5] it is proposed a teleportation scheme based on cavity QED for the teleportation of an atomic state where the cavities are prepared in a entangled state of zero and one Fock states. In Ref. [6] it is proposed a scheme to teleport an atomic state for atoms in a cascade configuration making use of cavities prepared in a coherent state. In Ref. [7] it is proposed a scheme to teleport an atomic state for atoms in a lambda configuration making use of cavities prepared in a coherent state. In Ref. [8] it is presented a scheme of teleportation where a superposition of zero and one Fock states is teleported via cavity QED using atoms in a lambda configuration. An interesting proposition of generating EPR states and realization of teleportation using a dispersive atom-field interaction is presented in [9]. Teleportation has already been realized experimentally [10, 11]. The superposition principle together with entanglement and its consequence non-locality are the main
ingredients in teleportation. The goal in teleportation is to reproduce an unknown quantum state of a given system in another system far apart of the original system. That is if Alice has a system prepared in an unknown state, by teleportation, she is able to transfer this state to Bob’s system. In order to do this, Alice and Bob share a Bell state (or EPR state) in which half of the Bell pair is with Alice and the other half is with Bob. Then Alice and Bob perform a certain prescription and communicate classically with each other. In the end of the process Bob gets a state identical to the state of the original state in which Alice system was prepared and the state of Alice´s system is destroyed since according to the no-cloning theorem it is not possible to clone a quantum state.

In this article we consider Rydberg atoms interacting with a superconducting cavity prepared in a superposition of a zero and a one Fock states. We develop two schemes of teleportation. In the first scheme we consider atoms in a three-level cascade configuration and in the second scheme we consider atoms in a three-level lambda configuration.

2 TELEPORTATION

2.1 ATOMS IN A CASCADE CONFIGURATION

First let us present a scheme to prepare a Bell states. We start assuming that we have a cavity prepared in the state

\[ |\psi\rangle_C = \frac{(|0\rangle + |1\rangle)}{\sqrt{2}}. \tag{2.1} \]

In order to prepare this state, we send a two-level atom \( A_0 \), with \(|f_0\rangle\) and \(|e_0\rangle\) being the lower and upper level respectively, through a Ramsey cavity \( R_0 \) in the lower state \(|f_0\rangle\) where the atomic states are rotated according to

\[ R_0 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{2.2} \]

that is,

\[ |f_0\rangle \rightarrow \frac{1}{\sqrt{2}}(i|e_0\rangle + |f_0\rangle), \tag{2.3} \]

and through \( C \), for \( A_0 \) resonant with the cavity. Under the Jaynes-Cummings dynamics (see for \( \Delta = 0 \) and \( gt = \pi/2 \)) we know that the state \(|f_0\rangle|0\rangle\) does not evolve, however, the state \(|e_0\rangle|0\rangle\) evolves to \(-i|f_0\rangle|1\rangle\). Then, for the cavity initially in the vacuum state \(|0\rangle\), we have

\[ \frac{|(f_0) + i|e_0\rangle)}{\sqrt{2}}|0\rangle \rightarrow |f_0\rangle\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) = |f_0\rangle|\psi\rangle_C. \tag{2.4} \]

Now let us consider a three-level cascade atom \( A_k \) with \(|e_k\rangle\), \(|f_k\rangle\) and \(|g_k\rangle\) being the upper, intermediate and lower atomic state (see Fig. 1). We assume that the transition \(|f_k\rangle \leftrightarrow |e_k\rangle\) is far enough from resonance with the cavity central frequency such that only virtual transitions occur between these states (only these states interact with field in cavity \( C \)). In addition we assume that the transition \(|e_k\rangle \leftrightarrow |g_k\rangle\) is highly detuned from the cavity frequency so that there will be no coupling with the cavity field. Here we are going to consider the effect of the atom-field interaction taking into account only levels \(|f_k\rangle\) and \(|g_k\rangle\). We do not consider level \(|e_k\rangle\) since it will not play any
role in our scheme. Therefore, we have effectively a two-level system involving states \( |f_k\rangle \) and \( |g_k\rangle \). Considering levels \( |f_k\rangle \) and \( |g_k\rangle \), we can write an effective time evolution operator (see (A.102))

\[
U_k(t) = e^{ia^\dagger a} |f_k\rangle\langle f_k| + |g_k\rangle\langle g_k|,
\]

(2.5)

where the second term above was put by hand just in order to take into account the effect of level \( |g_k\rangle \). In (2.5) \( a (a^\dagger) \) is the annihilation (creation) operator for the field in cavity \( C \), \( \varphi = g^2 \tau / \Delta \), \( g \) is the coupling constant, \( \Delta = \omega_e - \omega_f - \omega \) is the detuning where \( \omega_e \) and \( \omega_f \) are the frequencies of the upper and intermediate levels respectively and \( \omega \) is the cavity field frequency and \( \tau \) is the atom-field interaction time. Let us take \( \varphi = \pi \). Now, let us assume that we let atom \( A1 \) to interact with cavity \( C \) prepared in the state (2.1). Let us assume that atom \( A1 \) is prepared in a Ramsey cavity \( R1 \) in a coherent superposition according to the rotation matrix

\[
R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix},
\]

(2.6)

and we have

\[
|\psi\rangle_{A1} = \frac{1}{\sqrt{2}}(|f_1\rangle + |g_1\rangle).
\]

(2.7)

Taking into account (2.5), after atom \( A1 \) has passed through the cavity prepared in state (2.1), we get

\[
|\psi\rangle_{A1-C} = \frac{1}{2}(|f_1\rangle + |g_1\rangle)|0\rangle + (- |f_1\rangle + |g_1\rangle)|1\rangle,
\]

(2.8)

Now, if atom \( A1 \) enters a second Ramsey cavity \( R2 \) where the atomic states are rotated according to the rotation matrix (2.6), we have

\[
\frac{1}{\sqrt{2}}(|f_1\rangle + |g_1\rangle) \rightarrow |f_1\rangle,
\]

\[
\frac{1}{\sqrt{2}}(- |f_1\rangle + |g_1\rangle) \rightarrow |g_1\rangle.
\]

(2.9)

and, therefore,

\[
|\psi\rangle_{A1-C} = \frac{1}{2}(|f_1\rangle|0\rangle + |g_1\rangle|1\rangle),
\]

(2.10)

Now, let us prepare a two-level atom \( A2 \) in the Ramsey cavity \( R3 \). If atom \( A2 \) is initially in the state \( |g_2\rangle \), according to the rotation matrix (2.6), we have

\[
|\psi\rangle_{A2} = \frac{1}{\sqrt{2}}(|f_2\rangle + |g_2\rangle),
\]

(2.11)

and let us send this atom through cavity \( C \), assuming that for atom \( A2 \), as above for atom \( A1 \), the transition \( |f_2\rangle \leftrightarrow |g_2\rangle \) is highly detuned from the cavity central frequency. Taking into account (2.5), after the atom has passed through the cavity we get

\[
|\psi\rangle_{A1-A2-C} = \frac{1}{2}(|f_1\rangle(|f_2\rangle + |g_2\rangle)|0\rangle + |g_1\rangle(- |f_2\rangle + |g_2\rangle)|1\rangle),
\]

(2.12)

Then, atom \( A2 \) enters a Ramsey cavity \( R4 \) where the atomic states are rotated according to the rotation matrix (2.6), that is,

\[
\frac{1}{\sqrt{2}}(|f_2\rangle + |g_2\rangle) \rightarrow |f_2\rangle,
\]

\[
\frac{1}{\sqrt{2}}(- |f_2\rangle + |g_2\rangle) \rightarrow |g_2\rangle.
\]

(2.13)
and we get
\[ |\psi\rangle_{A1-A2-C} = \frac{1}{\sqrt{2}}( |f_1\rangle |f_2\rangle |g_1\rangle |g_2\rangle |0\rangle + |g_1\rangle |g_2\rangle |1\rangle), \] (2.14)

In order to disentangle the atomic states of the cavity field state we now send a two-level atom \(A3\), resonant with the cavity, with \(|f_3\rangle\) and \(|e_3\rangle\) being the lower and upper level respectively, through \(C\). If \(A3\) is sent in the lower state \(|f_3\rangle\), under the Jaynes-Cummings dynamics (see (A.99), for \(\Delta = 0\) and \(gt = \pi/2\)) we know that the state \(|f_3\rangle |0\rangle\) does not evolve, however, the state \(|f_3\rangle |1\rangle\) evolves to \(-i|e_3\rangle |0\rangle\). Then we get
\[ |\psi\rangle_{A1-A2-C} = \frac{1}{\sqrt{2}}(|f_1\rangle |f_2\rangle |f_3\rangle - i |g_1\rangle |g_2\rangle |e_3\rangle) |0\rangle, \] (2.15)

Now we let atom \(A3\) to enter a Ramsey cavity \(R5\) where the atomic states are rotated according to the rotation matrix
\[ R = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \] (2.16)

that is,
\[ |e_3\rangle \rightarrow \frac{1}{\sqrt{2}}(|e_3\rangle + i |f_3\rangle), \]
\[ |f_3\rangle \rightarrow \frac{1}{\sqrt{2}}(i |e_3\rangle + |f_3\rangle), \] (2.17)

and we get
\[ |\psi\rangle_{A1-A2} = \frac{1}{2}(|f_1\rangle |f_2\rangle (i |e_3\rangle + |f_3\rangle) - i |g_1\rangle |g_2\rangle (|e_3\rangle + i |f_3\rangle)), \] (2.18)

and if we detect atom \(A3\) in state \(|f_3\rangle\) finally we get the Bell state
\[ |\Phi^+\rangle_{A1-A2} = \frac{1}{\sqrt{2}}(|f_1\rangle |g_2\rangle + |g_1\rangle |f_2\rangle), \] (2.19)

which is an entangled state of atoms \(A1\) and \(A2\). If we detect atom \(A3\) in state \(|e_3\rangle\) we get
\[ |\Phi^-\rangle_{A1-A2} = \frac{1}{\sqrt{2}}(|f_1\rangle |g_2\rangle - |g_1\rangle |f_2\rangle), \] (2.20)

Now, if we apply an extra rotation on the states of atom \(A2\) in (2.19) in a Ramsey cavity \(R\), according to the rotation matrix
\[ R = |f_2\rangle <g_2| + |g_2\rangle <f_2|, \] (2.21)

we get
\[ |\Psi^+\rangle_{A1-A2} = \frac{1}{\sqrt{2}}(|f_1\rangle |g_2\rangle + |g_1\rangle |f_2\rangle), \] (2.22)

and applying (2.21) to (2.20) we get
\[ |\Psi^-\rangle_{A1-A2} = \frac{1}{\sqrt{2}}(|f_1\rangle |g_2\rangle - |g_1\rangle |f_2\rangle). \] (2.23)

The states (2.19), (2.20), (2.22) and (2.23) form a Bell basis which are a complete orthonormal basis for the system \(A1-A2\).
Now let us see how to distinguish the four states which form the Bell basis. Let us assume we have a cavity $C$ prepared in the state (2.1). Notice that if we send atoms $A_1$ and $A_2$ through $C$ in the state (2.19) or (2.20) we have
\[
| \Phi^\pm \rangle_{A_1-A_2} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \rightarrow | \Phi^\pm \rangle_{A_1-A_2} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}},
\] (2.24)
and if we send atoms $A_1$ and $A_2$ through $C$ in the state (2.22) or (2.23) we have
\[
| \Psi^\pm \rangle_{A_1-A_2} \frac{(|0\rangle + |1\rangle)}{\sqrt{2}} \rightarrow | \Psi^\pm \rangle_{A_1-A_2} \frac{(|0\rangle - |1\rangle)}{\sqrt{2}}.
\] (2.25)
Now if we send an atom $A_5$ through $C$ in the state $|f_5\rangle$, resonant with the cavity, with $|f_5\rangle$ and $|e_5\rangle$ being the lower and upper level respectively, for $gt = \pi/2$, we have
\[
|f_5\rangle(|0\rangle + |1\rangle) \rightarrow (|f_5\rangle - i|e_5\rangle)|0\rangle,
\] (2.26)
\[
|f_5\rangle(|0\rangle - |1\rangle) \rightarrow (|f_5\rangle + i|e_5\rangle)|0\rangle.
\] Now we send atom $A_5$ through a Ramsey cavity $R$, where the states are rotated according to the rotation matrix (2.16), that is, we have
\[
\frac{1}{\sqrt{2}}(i \mid e_5\rangle + |f_5\rangle) \rightarrow i \mid e_5\rangle,
\]
\[
\frac{1}{\sqrt{2}}(-i \mid e_5\rangle + |f_5\rangle) \rightarrow |f_5\rangle,
\] (2.27)
Therefore, the detection of $|f_5\rangle$ corresponds to the detection of $| \Phi^\pm \rangle_{A_1-A_2}$ and of $|e_5\rangle$ corresponds to the detection of $| \Psi^\pm \rangle_{A_1-A_2}$. Now we have to distinguish $(| \Psi^+ \rangle_{A_1-A_2}, | \Phi^\pm \rangle_{A_1-A_2})$ from $(| \Psi^- \rangle_{A_1-A_2}, | \Phi^- \rangle_{A_1-A_2})$. In order to do this we notice that, defining
\[
\Sigma_x = \sigma_x^1 \sigma_x^2,
\] (2.28)
where
\[
\sigma_x^k = | f_k \rangle \langle g_k | + | g_k \rangle \langle f_k |
\] (2.29)
we have
\[
\Sigma_x \mid \Psi^\pm \rangle_{A_1-A_2} = \pm \mid \Psi^\pm \rangle_{A_1-A_2},
\]
\[
\Sigma_x \mid \Phi^\pm \rangle_{A_1-A_2} = \pm \mid \Phi^\pm \rangle_{A_1-A_2}.
\] (2.30)
Therefore, we can distinguish between $(| \Psi^+ \rangle_{A_1-A_2}, | \Phi^\pm \rangle_{A_1-A_2})$ and $(| \Psi^- \rangle_{A_1-A_2}, | \Phi^- \rangle_{A_1-A_2})$ performing measurements of $\Sigma_x = \sigma_x^1 \sigma_x^2$. In order to do so we proceed as follows. We make use of
\[
K_k = \frac{1}{\sqrt{2}} \left[ \begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right],
\] (2.31)
or
\[
K_k = \frac{1}{\sqrt{2}}(| f_k \rangle \langle f_k | - | f_k \rangle \langle g_k | + | g_k \rangle \langle f_k | + | g_k \rangle \langle g_k |),
\] (2.32)
to gradually unravel the Bell states. The eigenvectors of the operators $\sigma_x^k$ are
\[
| \psi_{\pm}^k \rangle = \frac{1}{\sqrt{2}}(| f_k \rangle \pm | g_k \rangle),
\] (2.33)
and we can rewrite the Bell states as

\[ | \Phi^\pm \rangle_{A1-A2} = \frac{1}{2}[|\psi^1_x, +\rangle(|f_2\rangle \pm |g_2\rangle) + |\psi^1_x, -\rangle(|f_2\rangle \mp |g_2\rangle)], \]

\[ | \Psi^\pm \rangle_{A1-A2} = \frac{1}{2}[|\psi^1_x, +\rangle(|g_2\rangle \pm |f_2\rangle) + |\psi^1_x, -\rangle(|g_2\rangle \pm |f_2\rangle)]. \] (2.34)

Let us take for instance (2.19)

\[ | \Phi^+ \rangle_{A1-A2} = \frac{1}{\sqrt{2}}(|f_1\rangle |f_2\rangle + |g_1\rangle |g_2\rangle). \] (2.35)

Applying \( K_1 \) to this state we have

\[ K_1 | \Phi^+ \rangle_{A1-A2} = \frac{1}{2}[|f_1\rangle(|f_2\rangle - |g_2\rangle) + |g_1\rangle(|f_2\rangle + |g_2\rangle)]. \] (2.36)

Now, we compare (2.36) and (2.34). We see that the rotation by \( K_1 \) followed by the detection of \( |g_1\rangle \) corresponds to the detection of the state \( |\psi^1_x, +\rangle \) whose eigenvalue of \( \sigma^1_x \) is +1. After we detect \( |g_1\rangle \), we get

\[ | \psi \rangle_{A2} = \frac{1}{\sqrt{2}}(|f_2\rangle + |g_2\rangle), \] (2.37)

that is, we have got

\[ | \psi \rangle_{A2} = |\psi^2_x, +\rangle. \] (2.38)

If we apply (2.32) for \( k = 2 \) to the state (2.38) we get

\[ K_2 | \psi \rangle_{A2} = |g_2\rangle. \] (2.39)

We see that the rotation by \( K_2 \) followed by the detection of \( |g_2\rangle \) corresponds to the detection of the state \( |\psi^2_x, +\rangle \) whose eigenvalue of \( \sigma^2_x \) is +1. The same applies to (2.22).

Summarizing, we have two possible sequences of atomic state rotations through \( K_k \) and detections of \( |f_k\rangle \) or \( |g_k\rangle \) and the corresponding states \( |\psi^k_x, \pm\rangle \) where \( k = 1 \) and \( 2 \) which corresponds to the measurement of the eigenvalue +1 of the operator \( \Sigma_x \) given by (2.30) and the detection of (2.19) or (2.22) corresponds to

\[ (K_1, |g_1\rangle)(K_2, |g_2\rangle) \leftrightarrow |\psi^1_x, +\rangle |\psi^2_x, +\rangle, \]

\[ (K_1, |f_1\rangle)(K_2, |f_2\rangle) \leftrightarrow |\psi^1_x, -\rangle |\psi^2_x, -\rangle. \] (2.40)

Considering (2.20) and (2.23) we have

\[ (K_1, |g_1\rangle)(K_2, |f_2\rangle) \leftrightarrow |\psi^1_x, +\rangle |\psi^2_x, -\rangle, \]

\[ (K_1, |f_1\rangle)(K_2, |g_2\rangle) \leftrightarrow |\psi^1_x, -\rangle |\psi^2_x, +\rangle. \] (2.41)

which corresponds to the measurement of the eigenvalue −1 of the operator \( \Sigma_x \) given by (2.30).

Let us now assume that Alice keeps with her the half of the Bell state (2.19) consisting of atom \( A2 \) and Bob keeps with him the other half of this Bell state, that is, atom \( A1 \). Then, they separate and let us assume that they are far apart from each other. Later on, Alice decides to teleport the state of an atom \( A4 \) prepared in an unknown state

\[ | \psi \rangle_{A4} = \zeta |f_4\rangle + \xi |g_4\rangle \] (2.42)
Now let us write the state formed by the direct product of the Bell state and the unknown state $|\Phi^\pm\rangle_{A1-A2} |\psi\rangle_{A4}$, that is,

$$
|\psi\rangle_{A1-A2-A4} = \frac{1}{\sqrt{2}}\left[\xi(|f_1\rangle |f_2\rangle |f_4\rangle + |g_1\rangle |g_2\rangle |f_4\rangle) + \xi(|f_1\rangle |f_2\rangle |g_4\rangle + |g_1\rangle |g_2\rangle |g_4\rangle)\right].
$$

(2.43)

First Alice prepares a cavity $C$ in the state (2.1). Taking into account (2.5) with $\varphi = \pi$, after atoms $A2$ and $A4$ fly through the cavity we have

$$
|\psi\rangle_{A1-A2-A4-C} = \frac{1}{2}\left[\xi(|f_1\rangle |f_2\rangle |f_4\rangle(|0\rangle + |1\rangle) + \xi(|f_1\rangle |f_2\rangle |g_4\rangle(|0\rangle - |1\rangle) + \xi(|f_1\rangle |f_2\rangle |g_4\rangle(|0\rangle + |1\rangle)\right].
$$

(2.44)

Now, making use of the Bell basis involving atom $A1$ and $A2$ we can rewrite (2.43) as

$$
|\psi\rangle_{A1-A2-A4-C} =
\frac{1}{2\sqrt{2}}[\left|\Phi^+\right\rangle_{A2-A4}(\zeta |f_1\rangle + \xi |g_1\rangle)(|0\rangle + |1\rangle) + \\
\left|\Phi^-\right\rangle_{A2-A4}(\zeta |f_1\rangle - \xi |g_1\rangle)(|0\rangle - |1\rangle) + \\
\left|\Psi^+\right\rangle_{A2-A4}(\zeta |g_1\rangle + \xi |f_1\rangle)(|0\rangle - |1\rangle) + \\
\left|\Psi^-\right\rangle_{A2-A4}(\zeta |g_1\rangle - \xi |f_1\rangle)(|0\rangle - |1\rangle)].
$$

(2.45)

Now Alice sends an atom $A5$ through $C$ in the state $|f_5\rangle$, resonant with the cavity, with $|f_5\rangle$ and $|e_5\rangle$ being the lower and upper level respectively. For $gt = \pi/2$, we have

$$
|f_5\rangle(|0\rangle + |1\rangle) \rightarrow (|f_5\rangle - i|e_5\rangle)|0\rangle,
$$

(2.46)

$$
|f_5\rangle(|0\rangle - |1\rangle) \rightarrow (|f_5\rangle + i|e_5\rangle)|0\rangle,
$$

and

$$
|\psi\rangle_{A1-A2-A4-A5-C} =
\frac{1}{2\sqrt{2}}[\left|\Phi^+\right\rangle_{A2-A4}(\zeta |f_1\rangle + \xi |g_1\rangle)(|f_5\rangle - i|e_5\rangle) + \\
\left|\Phi^-\right\rangle_{A2-A4}(\zeta |f_1\rangle - \xi |g_1\rangle)(|f_5\rangle - i|e_5\rangle) + \\
\left|\Psi^+\right\rangle_{A2-A4}(\zeta |g_1\rangle + \xi |f_1\rangle)(|f_5\rangle + i|e_5\rangle) + \\
\left|\Psi^-\right\rangle_{A2-A4}(\zeta |g_1\rangle - \xi |f_1\rangle)(|f_5\rangle + i|e_5\rangle)|0\rangle.
$$

(2.47)

Now, Alice sends atom $A5$ through a Ramsey cavity $R6$, where the states are rotated according to the rotation matrix (2.16), that is,

$$
\frac{1}{\sqrt{2}} (|f_5\rangle + |e_5\rangle) \rightarrow i |e_5\rangle,
$$

$$
\frac{1}{\sqrt{2}} (-i |e_5\rangle + |f_5\rangle) \rightarrow |f_5\rangle.
$$

(2.48)

and we have

$$
|\psi\rangle_{A1-A2-A4-A5-C} =
\frac{1}{2}[\left|\Phi^+\right\rangle_{A2-A4}(\zeta |f_1\rangle + \xi |g_1\rangle)|f_5\rangle + \\
\left|\Phi^-\right\rangle_{A2-A4}(\zeta |f_1\rangle - \xi |g_1\rangle)|f_5\rangle + \\
i |\Psi^+\rangle_{A2-A4}(\zeta |g_1\rangle + \xi |f_1\rangle)|e_5\rangle + \\
i |\Psi^-\rangle_{A2-A4}(\zeta |g_1\rangle - \xi |f_1\rangle)|e_5\rangle)|0\rangle.
$$

(2.49)
Then, if she detects the lower state $| f_5 \rangle$ she gets
\[
| \psi \rangle_{A1-A2-A4} = \frac{1}{N} [ | \Phi^+ \rangle_{A2-A4}(\zeta | f_1 \rangle + \xi | g_1 \rangle) + | \Phi^- \rangle_{A2-A4}(-\zeta | g_1 \rangle + \xi | f_1 \rangle)],
\] (2.50)
where $N$ is a normalization constant. Now Alice follows the above prescription in order to distinguish $| \Phi^+ \rangle_{A2-A4}$ from $| \Phi^- \rangle_{A2-A4}$. That is, she proceeds according to (2.40) and (2.41). If Alice gets $(K_2, | g_2 \rangle)(K_4, | g_4 \rangle)$ or $(K_2, | f_2 \rangle)(K_4, | f_1 \rangle)$ this corresponds to the detection of $| \Phi^+ \rangle_{A2-A4}$ and Bob gets
\[
| \psi \rangle_{A1} = \zeta | f_1 \rangle + \xi | g_1 \rangle,
\] (2.51)
and he has to do thing since he got the correct state (2.42). If Alice gets $(K_2, | f_2 \rangle)(K_4, | g_4 \rangle)$ or $(K_2, | g_2 \rangle)(K_4, | f_1 \rangle)$ this corresponds to the detection of $| \Phi^- \rangle_{A2-A4}$ and Bob gets
\[
| \psi \rangle_{A1} = \zeta | f_1 \rangle - \xi | g_1 \rangle,
\] (2.52)
and he has to apply a rotation in the Ramsey cavity $R7$
\[
R = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},
\] (2.53)
to get (2.42). If Alice detects the upper state $| e_5 \rangle$ she gets
\[
| \psi \rangle_{A1-A2-A4} = \frac{1}{N} [ | \Psi^+ \rangle_{A2-A4}(\zeta | g_1 \rangle + \xi | f_1 \rangle) + | \Psi^- \rangle_{A2-A4}(-\zeta | f_1 \rangle + \xi | g_1 \rangle)],
\] (2.54)
Now Alice follows the above prescription in order to distinguish $| \Psi^+ \rangle_{A2-A4}$ from $| \Psi^- \rangle_{A2-A4}$. That is, she proceeds according to (2.40) and (2.41). If Alice gets $(K_2, | g_2 \rangle)(K_4, | g_4 \rangle)$ or $(K_2, | f_2 \rangle)(K_4, | f_1 \rangle)$ this corresponds to the detection of $| \Psi^+ \rangle_{A2-A4}$ and Bob gets
\[
| \psi \rangle_{A1} = \zeta | g_1 \rangle + \xi | f_1 \rangle,
\] (2.55)
and he has to apply a rotation in the Ramsey cavity $R7$
\[
R = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},
\] (2.56)
to get (2.42). If Alice gets $(K_2, | f_2 \rangle)(K_4, | g_4 \rangle)$ or $(K_2, | g_2 \rangle)(K_4, | f_1 \rangle)$ this corresponds to the detection of $| \Psi^- \rangle_{A2-A4}$ and Bob gets
\[
| \psi \rangle_{A1} = -\zeta | g_1 \rangle + \xi | f_1 \rangle,
\] (2.56)
and he has to apply a rotation in the Ramsey cavity $R7$
\[
R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},
\] (2.57)
to get (2.42).

Notice that the original state (2.42) is destroyed in the end of the teleportation process (it evolves to $| f_4 \rangle$ or $| g_4 \rangle$) in accordance with the no-cloning theorem [11,12]. In Fig. 2 we present the scheme of the teleportation process we have discussed above.
2.2 ATOMS IN A LAMBDA CONFIGURATION

Consider a three-level lambda atom (see Fig. 3) interacting with an electromagnetic field inside a cavity $C$. The states of the atom $|a\rangle$, $|b\rangle$ and $|c\rangle$, are so that the $|a\rangle \equiv |c\rangle$ and $|a\rangle \equiv |b\rangle$ transitions are in the far off resonance interaction limit. The time evolution operator for the atom-field interaction is given by [17] (see Appendix)

\[
U(\tau) = -e^{i\varphi a^\dagger a}|a\rangle\langle a| + \frac{1}{2}(e^{i\varphi a^\dagger a} + 1)|b\rangle\langle b| + \frac{1}{2}(e^{i\varphi a^\dagger a} - 1)|c\rangle\langle c| + \frac{1}{2}(e^{i\varphi a^\dagger a} - 1)|c\rangle\langle b| + \frac{1}{2}(e^{i\varphi a^\dagger a} + 1)|b\rangle\langle c|,
\]

(2.58)

where $a$ ($a^\dagger$) is the annihilation (creation) operator for the field in cavity $C$, $\varphi = 2g^2\tau/\Delta$, $g$ is the coupling constant, $\Delta = \omega_a - \omega_b - \omega = \omega_a - \omega_c - \omega$ is the detuning where $\omega_a$, $\omega_b$ and $\omega_c$ are the frequency of the upper level and of the two degenerate lower levels respectively and $\omega$ is the cavity field frequency and $\tau$ is the atom-field interaction time. For $\varphi = \pi$, we get

\[
U(\tau) = -\exp\left(i\pi a^\dagger a\right)|a\rangle\langle a| + \Pi_+|b\rangle\langle b| + \Pi_-|c\rangle\langle c| + \Pi_+|c\rangle\langle b| + \Pi_+|c\rangle\langle b|,
\]

(2.59)

where

\[
\Pi_+ = \frac{1}{2}(e^{i\pi a^\dagger a} + 1),
\]

\[
\Pi_- = \frac{1}{2}(e^{i\pi a^\dagger a} - 1).
\]

(2.60)

Let us first show how we can get Bell states making use of three-level lambda atoms interacting with a cavity field prepared in state $|2.1\rangle$. Consider an atom $A1$ in the state $|\psi\rangle_{A1} = |b_1\rangle$ and a cavity $C$ prepared in the state $|2.1\rangle$. We now let atom $A1$ to fly through the cavity $C$. Taking into account (2.59) the state of the system $A1 - C$ evolves to

\[
|\psi\rangle_{A1-C} = \frac{1}{\sqrt{2}}(|b_1\rangle|0\rangle - |c_1\rangle|1\rangle).
\]

(2.61)

Consider now another three-level lambda atom $A2$ prepared initially in the state $|b_2\rangle$, which is going to pass through the cavity. After this second atom has passed through the cavity, the system evolves to

\[
|\psi\rangle_{A1-A2-C} = \frac{1}{\sqrt{2}}(|b_1\rangle|b_2\rangle|0\rangle + |c_1\rangle|c_2\rangle|1\rangle).
\]

(2.62)

In order to disentangle the atomic states of the cavity field state we now send a two-level atom $A3$, resonant with the cavity, with $|f_3\rangle$ and $|e_3\rangle$ being the lower and upper levels respectively, through $C$. If $A3$ is sent in the lower state $|f_3\rangle$, under the Jaynes-Cummings dynamics (see (A.90)), for $\Delta = 0$ and $gt = \pi/2$ we know that the state $|f_3\rangle|0\rangle$ does not evolve, however, the state $|f_3\rangle|1\rangle$ evolves to $-i|e_3\rangle|0\rangle$. Then we get

\[
|\psi(\tau)\rangle_{A1-A2-A3-C} = \frac{1}{\sqrt{2}}(|b_1\rangle|b_2\rangle|f_3\rangle + i|c_1\rangle|c_2\rangle|e_3\rangle)|0\rangle.
\]

(2.63)

Now we let atom $A3$ to enter a Ramsey cavity $R1$ where the atomic states are rotated according to the rotation matrix (2.10), that is,

\[
|e_3\rangle \rightarrow \frac{1}{\sqrt{2}}(|e_3\rangle + i|f_3\rangle),
\]

\[
|f_3\rangle \rightarrow \frac{1}{\sqrt{2}}(|f_3\rangle + i|e_3\rangle),
\]

(2.64)
and we get
\[ |\psi\rangle_{A_1-A_2-A_3} = \frac{1}{2}[(|b_1\rangle|b_2\rangle|f_3\rangle(i\ | c_3\rangle + | f_3\rangle) - i|c_1\rangle|c_2\rangle(| e_3\rangle + i\ | f_3\rangle)]. \] (2.65)

and if we detect atom A3 in state | f_3\rangle finally we get the the Bell state
\[ |\Phi^+\rangle_{A_1-A_2} = \frac{1}{\sqrt{2}}(|b_1\rangle|b_2\rangle + |c_1\rangle|c_2\rangle), \] (2.66)

which is an entangled state of atoms A1 and A2. If we detect atom A3 in state | e_3\rangle we get
\[ |\Phi^-\rangle_{A_1-A_2} = \frac{1}{\sqrt{2}}(|b_1\rangle|b_2\rangle - |c_1\rangle|c_2\rangle). \] (2.67)

Now, if we apply an extra rotation on the states of atom A2 in (2.66) in a Ramsey cavity \( R \), according to the rotation matrix
\[ R = |b_2\rangle\langle c_2| + |c_2\rangle\langle b_2|, \] (2.68)
we get
\[ |\Psi^+\rangle_{A_1-A_2} = \frac{1}{\sqrt{2}}(|b_1\rangle|c_2\rangle + |c_1\rangle|b_2\rangle), \] (2.69)

and applying (2.68) to (2.67) we get
\[ |\Psi^-\rangle_{A_1-A_2} = \frac{1}{\sqrt{2}}(|b_1\rangle|c_2\rangle - |c_1\rangle|b_2\rangle). \] (2.70)

Now, let us assume that Alice keeps with her the half of the Bell state (2.66) consisting of atom A2 and Bob keeps with him the other half of this Bell state, that is, atom A1. Then they separate and let us assume that they are far apart from each other. Later on, Alice decides to teleport the state of an atom A4 prepared in an unknown state
\[ |\psi\rangle_{A4} = \zeta |b_4\rangle + \xi |c_4\rangle \] (2.71)
to Bob. First Alice prepares a cavity \( C \) in the state (2.1). Let us write the state formed by the direct product of the Bell state and the unknown state |\Phi^+\rangle_{A1-A2} |\psi\rangle_{A4}, that is,
\[ |\psi\rangle_{A1-A2-A4} = \frac{1}{\sqrt{2}}[\zeta(|b_1\rangle|b_2\rangle|b_4\rangle + |c_1\rangle|c_2\rangle|b_4\rangle] + \xi(|b_1\rangle|b_2\rangle|c_4\rangle + |c_1\rangle|c_2\rangle|c_4\rangle)]. \] (2.72)

Taking into account (2.39), after Alice let atoms A2 and A4 to fly through C prepared in the state (2.1), she gets
\[ \frac{1}{2\sqrt{2}}[|\psi\rangle_{A1-A2-A4-C} = \]
\[ \frac{1}{2\sqrt{2}}[|\Phi^+\rangle_{A2-A4}(\zeta |b_1\rangle + \xi |c_1\rangle)(|0\rangle + |1\rangle) + \]
\[ |\Phi^-\rangle_{A2-A4}(\zeta |b_1\rangle - \xi |c_1\rangle)(|0\rangle - |1\rangle) + \]
\[ |\Psi^+\rangle_{A2-A4}(\zeta |c_1\rangle + \xi |b_1\rangle)(|0\rangle + |1\rangle) + \]
\[ |\Psi^-\rangle_{A2-A4}(-\zeta |c_1\rangle + \xi |b_1\rangle)(|0\rangle - |1\rangle)]. \] (2.73)

In order to disentangle the atomic states of the cavity field state Alice sends a two-level atom A5, resonant with the cavity, with |f_5\rangle and |e_5\rangle being the lower and upper levels respectively, through C.
If $A5$ is sent in the lower state $|f_5\rangle$, under the Jaynes-Cummings dynamics (see (A.99), for $\Delta = 0$ and $gt = \pi/2$) we know that the state $|f_5\rangle|0\rangle$ does not evolve, however, the state $|f_5\rangle|1\rangle$ evolves to $-i|e_5\rangle|0\rangle$ and therefore she gets

$$\begin{align*}
|\psi\rangle_{A1-A2-A4-A5-C} &= \frac{1}{2\sqrt{2}} \left[ |\Phi^+\rangle_{A2-A4}(\zeta | b_1) + \xi | c_1\rangle)(|f_5\rangle - i|e_5\rangle) + \\
&|\Phi^-\rangle_{A2-A4}(\zeta | b_1) - \xi | c_1\rangle)(|f_5\rangle + i|e_5\rangle) + \\
&|\Psi^+\rangle_{A2-A4}(\zeta | c_1) + \xi | b_1\rangle)(|f_5\rangle - i|e_5\rangle) + \\
&|\Psi^-\rangle_{A2-A4}(-\zeta | c_1) + \xi | b_1\rangle)(|f_5\rangle + i|e_5\rangle)|0\rangle.\end{align*}$$

(2.74)

Now, if atom $A5$ enters a Ramsey cavity $R2$ where the atomic states are rotated according to the rotation matrix (2.16), we have

$$\begin{align*}
\frac{1}{\sqrt{2}}(i | e_5\rangle + |f_5\rangle) &\rightarrow i | e_5\rangle, \\
\frac{1}{\sqrt{2}}(-i | e_5\rangle + |f_5\rangle) &\rightarrow |f_5\rangle.\end{align*}$$

(2.75)

and we get

$$\begin{align*}
|\psi\rangle_{A1-A2-A4-A5} &= \frac{1}{2} \left[ |\Phi^+\rangle_{A2-A4}(\zeta | b_1) + \xi | c_1\rangle)|f_5\rangle + \\
i |\Phi^-\rangle_{A2-A4}(\zeta | b_1) - \xi | c_1\rangle)|e_5\rangle + \\
i |\Psi^+\rangle_{A2-A4}(\zeta | c_1) + \xi | b_1\rangle)|f_5\rangle + \\
i |\Psi^-\rangle_{A2-A4}(-\zeta | c_1) + \xi | b_1\rangle)|e_5\rangle\right].\end{align*}$$

(2.76)

If Alice detects $|f_5\rangle$ she gets

$$|\psi\rangle_{A1-A2-A4} = \frac{1}{N}[|\Phi^+\rangle_{A2-A4}(\zeta | b_1) + \xi | c_1\rangle|f_5\rangle + |\Psi^+\rangle_{A2-A4}(\zeta | c_1) + \xi | b_1\rangle)].$$

(2.77)

where $N$ is a normalization constant. If Alice detects ($|b_2\rangle | b_4\rangle$) or ($|c_2\rangle | c_4\rangle$) Bob gets

$$|\psi\rangle_{A1} = \zeta | b_1\rangle + \xi | c_1\rangle,$$

(2.78)

and he has to do nothing since this is the correct state (2.71). If she detects ($|b_2\rangle | c_4\rangle$) or ($|c_2\rangle | b_4\rangle$) Bob gets

$$|\psi\rangle_{A1} = \zeta | c_1\rangle + \xi | b_1\rangle,$$

(2.79)

and he has to apply a rotation in the Ramsey cavity $R3$

$$R_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

(2.80)

to get (2.71). If Alice detects $|e_5\rangle$ she gets

$$|\psi\rangle_{A1-A2-A4} = \frac{i}{N}[|\Phi^-\rangle_{A2-A4}(\zeta | b_1) - \xi | c_1\rangle|f_5\rangle + |\Psi^-\rangle_{A2-A4}(-\zeta | c_1) + \xi | b_1\rangle)].$$

(2.81)
If Alice detects ($|b_2\rangle |b_4\rangle$) or ($|c_2\rangle |c_4\rangle$) Bob gets

$$|\psi\rangle_{A1} = \zeta |c_1\rangle - \xi |b_1\rangle,$$

(2.82)

and he has to apply a rotation in the Ramsey cavity $R3$

$$R = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

(2.83)

to get (2.71). If she detects ($|b_2\rangle |c_4\rangle$) or ($|c_2\rangle |b_4\rangle$) Bob gets

$$|\psi\rangle_{A1} = -\zeta |c_1\rangle + \xi |b_1\rangle,$$

(2.84)

and he has to apply a rotation in the Ramsey cavity $R3$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

(2.85)

to get (2.71). In Fig. 4 we present the scheme of the teleportation process we have discussed above.

### 3 Conclusion

Concluding, we have presented two schemes of realization of atomic state teleportation making use of cavity QED. In the schemes presented here we use atoms interacting with a superconducting cavity prepared in a state ($|0\rangle + |1\rangle)/\sqrt{2}$ which is a state relatively easy to be prepared and handled and for which decoherence is not so drastic. In the first scheme we make use of atoms in a cascade configuration and in the second scheme we make use of atoms in a lambda configuration. The advantage of using a cascade atomic configuration is that the atomic state detection process is simpler than in the lambda configuration where we have states which are degenerated. On the other hand, for the cascade configuration we have to perform more atomic state rotations using Ramsey cavities than in the case of the lambda configuration. Nice alternative schemes also making use of atoms interacting with electromagnetic cavities have also been proposed in Refs. [6, 7, 5].

### A Time evolution operator

#### A.1 Two-level atoms

Let us consider a two-level atom interacting with a cavity field, where $|e\rangle$ and $|f\rangle$ are the upper and lower states respectively, with $\omega_e$ and $\omega_f$ being the two atomic frequencies associated to these two states and $\omega$ the cavity field frequency (see Fig. 1). The Jaynes-Cummings Hamiltonian, under the rotating-wave approximation, is given by

$$H = \hbar a^\dagger a + \hbar \omega_e |e\rangle\langle e| + \hbar \omega_f |f\rangle\langle f| + \hbar g [a|e\rangle\langle f| + a^\dagger |f\rangle\langle e|],$$

(A.86)

where $a^\dagger$ and $a$ are the creation and annihilation operators respectively for the cavity field, $g$ is the coupling constant. We write

$$H = H_0 + H_I,$$

(A.87)
where we have settled
\[ H_0 = \hbar a^\dagger a + \hbar \omega_e |e\rangle\langle e| + \hbar \omega_f |f\rangle\langle f|, \]
\[ H_I = \hbar g |e\rangle\langle f| + a^\dagger |f\rangle\langle e|. \]  
(A.88)

Let's define the interaction picture
\[ |\psi_I\rangle = e^{\frac{i\hbar t}{\hbar}} |\psi_S\rangle, \]  
(A.89)
where
\[ i\hbar \frac{d}{dt} |\psi_S\rangle = H |\psi_S\rangle. \]  
(A.90)

Then, we get
\[ i\hbar \frac{d}{dt} |\psi_I\rangle = V_I |\psi_I\rangle, \]  
(A.91)
where
\[ V_I = e^{\frac{i\hbar t}{\hbar}} H_I e^{-\frac{i\hbar t}{\hbar}} = \hbar \begin{bmatrix} 0 & g e^{i\Delta t} a \\ g e^{-i\Delta t} a^\dagger & 0 \end{bmatrix} \]  
(A.92)
and
\[ \Delta = (\omega_e - \omega_f) - \omega. \]  
(A.93)

Considering
\[ |\psi_I(t)\rangle = U_I(t) |\psi_I(0)\rangle = U_I(t) |\psi_S(0)\rangle, \]  
(A.94)
we have to solve the Schrödinger’s equation for the time evolution operator
\[ i\hbar \frac{dU_I}{dt} = V_I U_I, \]  
(A.95)
where
\[ U_I(t) = \begin{bmatrix} u_{ee}(t) & u_{ef}(t) \\ u_{fe}(t) & u_{ff}(t) \end{bmatrix}, \]  
(A.96)
and
\[ U_I(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \]  
(A.97)

That is,
\[ i\frac{d}{dt} u_{ee}(t) = g e^{i\Delta t} a u_{ef}(t), \]
\[ i\frac{d}{dt} u_{ef}(t) = g e^{i\Delta t} a u_{ff}(t), \]
\[ i\frac{d}{dt} u_{fe}(t) = g e^{-i\Delta t} a^\dagger u_{ee}(t), \]
\[ i\frac{d}{dt} u_{ff}(t) = g e^{-i\Delta t} a^\dagger u_{ef}(t), \]  
(A.98)

which can be solved easily using, for instance, Laplace transformation, and we get
\[ U_I(t) = \begin{bmatrix} e^{i\Delta t}(\cos \mu t - i \frac{\Delta}{2\mu} \sin \mu t) & -i g e^{i\Delta t} \frac{1}{\mu} (\sin \mu t) a \\ -i g a^\dagger e^{-i\Delta t} \frac{1}{\mu} (\sin \mu t) & e^{-i\Delta t}(\cos \nu t + i \frac{\Delta}{2\nu} \sin \nu t) \end{bmatrix}, \]  
(A.99)
where we have defined

\[ \mu = \sqrt{g^2 a a^\dagger + \frac{\Delta^2}{4}}, \]

\[ \nu = \sqrt{g^2 a^\dagger a + \frac{\Delta^2}{4}}. \tag{A.100} \]

In the large detuning limit \((\Delta \gg g)\) we have

\[ \mu = \sqrt{g^2 a a^\dagger + \frac{\Delta^2}{4}} \approx \frac{\Delta}{2} + \frac{g^2 a a^\dagger}{\Delta}, \]

\[ \nu = \sqrt{g^2 a^\dagger a + \frac{\Delta^2}{4}} \approx \frac{\Delta}{2} + \frac{g^2 a^\dagger a}{\Delta}. \tag{A.101} \]

and we get easily

\[ U_d(t) = e^{-i\varphi(a^\dagger a+1)} |e\rangle\langle e| + e^{i\varphi a^\dagger a} |f\rangle\langle f|, \]

where \(\varphi = g^2 t/\Delta.\)

In the case we have a resonant interaction of an atom with cavity field \((\Delta = 0 \text{ in } (A.99))\), if the field is a very intense field we can treat it classically. That is, in the time evolution operator \((A.99)\) we set \(\Delta = 0\), and we substitute the creation and annihilation field operators according to \(a \rightarrow \eta e^{i\theta}\) and \(a^\dagger \rightarrow \eta e^{-i\theta}\) where \(\eta\) and \(\theta\) are c-numbers. Then, we have a semiclassical approach of the atom-field interaction in which the field is treated classically and the atoms according to quantum mechanics. In this case \((A.99)\) becomes

\[ U_{I,SC}(t) = \begin{bmatrix} \cos(g\eta t) & -ie^{i\theta} \sin(g\eta t) \\ -ie^{-i\theta} \sin(g\eta t) & \cos(g\eta t) \end{bmatrix}. \tag{A.103} \]

Now we take \(\theta = \pi/2\). If we choose \(g\eta t = \pi/4\) we have the rotation matrix

\[ R_{\frac{\pi}{4},\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \tag{A.104} \]

and for \(g\eta t = -\pi/4\) we have

\[ R_{\frac{\pi}{4},-\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}. \tag{A.105} \]

For \(g\eta t = \pi/2\) we get

\[ R_{\frac{\pi}{2},\frac{\pi}{2}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{A.106} \]

and for \(g\eta t = -\pi/2\)

\[ R_{\frac{\pi}{2},-\frac{\pi}{2}} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{A.107} \]

Now if we take \(\theta = \pi\) and \(g\eta t = \pi/4\) we have the rotation matrix

\[ R_{\pi,\frac{\pi}{4}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}, \tag{A.108} \]

Choosing the proper values of \(\theta, g, \eta\) and \(t\) we can get the rotation matrix we need to perform the rotation of the atomic states we desire in a Ramsey cavity.
Just as a remark, consider (A.102) and assume we have an intense field, that is, we can use a semiclassical approach. In this case we set \( a \rightarrow \eta e^{\text{i} \theta} \) and \( a^\dagger \rightarrow \eta e^{-\text{i} \theta} \) where \( \eta \) and \( \theta \) are c-numbers and \( a^\dagger a \rightarrow \eta^2 \). Defining \( \varphi a^\dagger a \rightarrow \varphi \eta^2 = \beta/2 \), (A.102) reads

\[
U_{d,SC} = \begin{bmatrix} e^{-i \beta/2} & 0 \\ 0 & e^{i \beta/2} \end{bmatrix},
\]

and for \( \beta = \pi \) we have

\[
U_\pi = i \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Any arbitrary \( 2 \times 2 \) unitary matrix \( M \) may be decomposed as [2]

\[
M = e^{i\alpha} \begin{bmatrix} e^{-i \beta/2} & 0 \\ 0 & e^{i \beta/2} \end{bmatrix} \begin{bmatrix} \cos \frac{\gamma}{2} & -\sin \frac{\gamma}{2} \\ \sin \frac{\gamma}{2} & \cos \frac{\gamma}{2} \end{bmatrix} \begin{bmatrix} e^{-i \delta/2} & 0 \\ 0 & e^{i \delta/2} \end{bmatrix},
\]

where \( \alpha, \beta, \gamma \) and \( \delta \) are real parameters. Therefore, we can use (A.103) and (A.109) to get a rotation matrix we need.

### A.2 Three-level lambda atoms

We start with the Hamiltonian of a degenerate three-level lambda atom (see Fig. 3) interacting with a field cavity mode

\[
H = \hbar \omega a^\dagger a + \hbar \omega_a |a\rangle\langle a| + \hbar \omega_b |b\rangle\langle b| + \hbar \omega_c |c\rangle\langle c| + \hbar a (g_1 |a\rangle\langle b| + g_2 |a\rangle\langle c|) + \hbar a^\dagger (g_1^* |b\rangle\langle a| + g_2^* |c\rangle\langle a|),
\]

where \(|a\rangle, |b\rangle \) and \(|c\rangle \) are the upper and the two degenerated lower atomic levels respectively, \( a \) (\( a^\dagger \)) is the annihilation (creation) field operator and \( g_1 \) and \( g_2 \) are the coupling constants corresponding to the transitions \(|a\rangle \rightleftharpoons |c\rangle \) and \(|a\rangle \rightleftharpoons |b\rangle \), respectively. In the interaction picture,

\[
V = \hbar \left[ e^{i \Delta_1 t} g_1 a |a\rangle\langle b| + e^{-i \Delta_1 t} g_1^* a^\dagger |b\rangle\langle a| \right] + \hbar \left[ e^{i \Delta_2 t} g_2 a |a\rangle\langle c| + e^{-i \Delta_2 t} g_2^* a^\dagger |c\rangle\langle a| \right],
\]

where

\[
\Delta_1 = \omega_a - \omega_b - \omega, \\
\Delta_2 = \omega_a - \omega_c - \omega.
\]

The time evolution operator

\[
U(t) = \begin{bmatrix} u_{aa} & u_{ab} & u_{ac} \\ u_{ba} & u_{bb} & u_{bc} \\ u_{ca} & u_{cb} & u_{cc} \end{bmatrix},
\]

whose initial condition is given by

\[
U(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},
\]

should satisfy the Schrödinger equation of motion

\[
\frac{i \hbar}{m} \frac{dU}{dt} = VU = \begin{bmatrix} \zeta_1 u_{ba} + \zeta_2 u_{ca} & \zeta_1 u_{bb} + \zeta_2 u_{cb} & \zeta_1 u_{bc} + \zeta_2 u_{cc} \\ \zeta_1^t u_{aa} & \zeta_1^t u_{ab} & \zeta_1^t u_{ac} \\ \zeta_2^t u_{aa} & \zeta_2^t u_{ab} & \zeta_2^t u_{ac} \end{bmatrix},
\]

(1.17)
where
\begin{align*}
\zeta_1 &= \hbar e^{i\Delta_1 t} g_1 a, \\
\zeta_2 &= \hbar e^{i\Delta_2 t} g_2 a.
\end{align*}
(A.118)

Observe that this equation may be grouped in three sets of coupled differential equations. One for \(u_{aa}, u_{ba}\) and \(u_{ca}\), i.e.,
\begin{align*}
\hbar \frac{du_{aa}}{dt} &= \zeta_1 u_{ba} + \zeta_2 u_{ca}, \\
\hbar \frac{du_{ba}}{dt} &= \zeta_1^{\dagger} u_{aa}, \\
\hbar \frac{du_{ca}}{dt} &= \zeta_2^{\dagger} u_{aa},
\end{align*}
(A.119)

another involving only \(u_{ab}, u_{bb}\) and \(u_{cb}\),
\begin{align*}
\hbar \frac{du_{ab}}{dt} &= \zeta_1 u_{bb} + \zeta_2 u_{cb}, \\
\hbar \frac{du_{bb}}{dt} &= \zeta_1^{\dagger} u_{ab}, \\
\hbar \frac{du_{cb}}{dt} &= \zeta_2^{\dagger} u_{ab},
\end{align*}
(A.120)

and, finally, one involving \(u_{ac}, u_{bc}\) and \(u_{cc}\),
\begin{align*}
\hbar \frac{du_{ac}}{dt} &= \zeta_1 u_{bc} + \zeta_2 u_{cc}, \\
\hbar \frac{du_{bc}}{dt} &= \zeta_1^{\dagger} u_{ac}, \\
\hbar \frac{du_{cc}}{dt} &= \zeta_2^{\dagger} u_{ac}.
\end{align*}
(A.121)

Let us take \(\Delta_1 = \Delta_2 = \Delta = -i\eta\)
\begin{align*}
\zeta_1 &= \hbar e^{i\Delta t} g_1 a = \hbar \alpha_1 e^{\eta t}, \\
\zeta_2 &= \hbar e^{i\Delta_2 t} g_2 a = \hbar \alpha_2 e^{\eta t}.
\end{align*}
(A.122)

Notice that all the three systems of differential equations above are of the form
\begin{align*}
i dx \over dt &= \alpha_1 e^{\eta t} y + \alpha_2 e^{\eta t} z, \\
i e^{\eta t} dy \over dt &= \alpha_1^{\dagger} x, \\
i e^{\eta t} dz \over dt &= \alpha_2^{\dagger} x.
\end{align*}
(A.123)

We can solve the systems of differential equations using, for instance, Laplace transformation and we have for the degenerate case
\[
u_{aa}(t) = \frac{e^{i\Delta t}}{\sqrt{\mu}} \left[ \sqrt{\mu} \cos \sqrt{\mu} t - \frac{\Delta}{2} \sin \sqrt{\mu} t \right],
\]
where

\[ u_{ab}(t) = -i \frac{e^{i \frac{\Delta}{4}}}{\sqrt{\mu}} \sin \sqrt{\mu} \alpha_1, \]

\[ u_{ac}(t) = -i \frac{e^{i \frac{\Delta}{4}}}{\sqrt{\mu}} \sin \sqrt{\mu} \alpha_2, \]

\[ u_{ba}(t) = -i \alpha_1^+ e^{-i \frac{\Delta}{4}} \sin \sqrt{\mu} t, \]

\[ u_{bb}(t) = 1 + \alpha_1^+ \alpha_1 \frac{1}{\sqrt{\nu}} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2 \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu} t + \sqrt{\nu} \cos \sqrt{\nu} t \right) - \sqrt{\nu} \right], \]

\[ u_{bc}(t) = \alpha_1^+ \alpha_2 \frac{1}{\sqrt{\nu} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2} \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu} t + \sqrt{\nu} \cos \sqrt{\nu} t \right) - \sqrt{\nu} \right], \]

\[ u_{ca}(t) = -i \alpha_1^+ e^{i \frac{\Delta}{4}} \sin \sqrt{\mu} t, \]

\[ u_{cb}(t) = \alpha_2^+ \alpha_1 \frac{1}{\sqrt{\nu} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2} \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu} t + \sqrt{\nu} \cos \sqrt{\nu} t \right) - \sqrt{\nu} \right], \]

\[ u_{cc}(t) = 1 + \alpha_2^+ \alpha_2 \frac{1}{\sqrt{\nu} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2} \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu} t + \sqrt{\nu} \cos \sqrt{\nu} t \right) - \sqrt{\nu} \right], \quad (A.124) \]

where

\[ \mu = \frac{\Delta^2}{4} + \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2, \]

\[ \nu = \frac{\Delta^2}{4} + \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2. \quad (A.125) \]

It is easy to show that for the non-degenerate case, i.e.,

\[ \alpha_1 = g_1 a_1, \]

\[ \alpha_2 = g_2 a_2, \quad (A.126) \]

we obtain

\[ u_{aa}(t) = e^{i \frac{\Delta}{4}} \left[ \sqrt{\mu} \cos \sqrt{\mu} t - i \frac{\Delta}{2} \sin \sqrt{\mu} t \right], \]

\[ u_{ab}(t) = -i e^{i \frac{\Delta}{4}} \sin \sqrt{\mu} t \alpha_1, \]

\[ u_{ac}(t) = -i e^{i \frac{\Delta}{4}} \sin \sqrt{\mu} t \alpha_2, \]

\[ u_{ba}(t) = -i \alpha_1^+ e^{-i \frac{\Delta}{4}} \sin \sqrt{\mu} t, \]

\[ u_{bb}(t) = 1 + \alpha_1^+ \alpha_1 \frac{1}{\sqrt{\nu_1} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2} \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu_1} t + \sqrt{\nu_1} \cos \sqrt{\nu_1} t \right) - \sqrt{\nu_1} \right], \]

\[ u_{bc}(t) = \alpha_1^+ \alpha_2 \frac{1}{\sqrt{\nu_1} \alpha_1^+ \alpha_1 + \alpha_2^+ \alpha_2} \left[ e^{-i \frac{\Delta}{4}} \left( i \frac{\Delta}{2} \sin \sqrt{\nu_1} t + \sqrt{\nu_1} \cos \sqrt{\nu_1} t \right) - \sqrt{\nu_1} \right], \]

\[ u_{ca}(t) = -i \alpha_2^+ e^{-i \frac{\Delta}{4}} \sin \sqrt{\mu} t, \]
\[ u_{cb}(t) = \alpha^\dagger_2 \alpha_1 \frac{1}{\alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2} \left[ e^{-i \frac{\Delta}{2} t} \left( i \frac{\Delta}{2 \sqrt{\nu_1}} \sin \sqrt{\nu_1} t + \cos \sqrt{\nu_1} t \right) - 1 \right], \]
\[ u_{cc}(t) = 1 + \alpha^\dagger_2 \alpha_2 \frac{1}{\sqrt{\nu_2} \alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2} \left[ e^{-i \frac{\Delta}{2} t} \left( i \frac{\Delta}{2 \sqrt{\nu_2}} \sin \sqrt{\nu_2} t + \cos \sqrt{\nu_2} t \right) - \sqrt{\nu_2} \right], \tag{A.127} \]

where
\[ \mu = \frac{\Delta^2}{4} + \alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2, \]
\[ \nu_1 = \frac{\Delta^2}{4} + \alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2, \]
\[ \nu_2 = \frac{\Delta^2}{4} + \alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2. \tag{A.128} \]

Returning to the degenerate case, in the large detuning limit, we have
\[ \sqrt{\mu} \approx \frac{\Delta^2}{2} + \frac{\alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2}{\Delta} = \frac{\Delta}{2} + \frac{|g_1|^2 + |g_2|^2}{\Delta} a^\dagger a, \]
\[ \sqrt{\nu} \approx \frac{\Delta^2}{2} + \frac{\alpha^\dagger_1 \alpha_1 + \alpha^\dagger_2 \alpha_2}{\Delta} = \frac{\Delta}{2} + \frac{|g_1|^2 + |g_2|^2}{\Delta} a^\dagger a, \tag{A.129} \]

and we get
\[ u_{aa}(t) = \exp \left( i \frac{|g_1|^2 + |g_2|^2}{\Delta} t a^\dagger a \right), \]
\[ u_{ab}(t) = 0, \]
\[ u_{ac}(t) = 0, \]
\[ u_{ba}(t) = 0, \]
\[ u_{bb}(t) = 1 + \frac{|g_1|^2}{|g_1|^2 + |g_2|^2} \left[ \exp \left( i \frac{|g_1|^2 + |g_2|^2}{\Delta} t a^\dagger a \right) - 1 \right], \]
\[ u_{bc}(t) = \frac{g^*_2 g_1}{|g_1|^2 + |g_2|^2} \left[ \exp \left( i \frac{|g_1|^2 + |g_2|^2}{\Delta} t a^\dagger a \right) - 1 \right], \]
\[ u_{ca}(t) = 0, \]
\[ u_{cb}(t) = \frac{g_2^* g_1}{|g_1|^2 + |g_2|^2} \left[ \exp \left( i \frac{|g_1|^2 + |g_2|^2}{\Delta} t a^\dagger a \right) - 1 \right], \]
\[ u_{cc}(t) = 1 + \frac{|g_2|^2}{|g_1|^2 + |g_2|^2} \left[ \exp \left( i \frac{|g_1|^2 + |g_2|^2}{\Delta} t a^\dagger a \right) - 1 \right]. \tag{A.130} \]

If
\[ g_1 = g e^{i \varphi_1}, \]
\[ g_2 = g e^{i \varphi_2}, \tag{A.131} \]

we finally have
\[ u_{aa}(t) = \exp \left( i \frac{g^2 t}{\Delta} \right) \exp \left( i \frac{g^2 t}{\Delta} a^\dagger a \right), \]

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$$u_{ab}(t) = 0,$$
$$u_{ac}(t) = 0,$$
$$u_{ba}(t) = 0,$$
$$u_{bb}(t) = \frac{1}{2} \left[ \exp \left( i \frac{2g^2t}{\Delta} a^\dagger a \right) + 1 \right],$$
$$u_{bc}(t) = \frac{e^{i(\varphi_1 - \varphi_2)}}{2} \left[ \exp \left( i \frac{2g^2t}{\Delta} a^\dagger a \right) - 1 \right],$$
$$u_{ca}(t) = 0,$$
$$u_{cb}(t) = \frac{e^{-i(\varphi_1 - \varphi_2)}}{2} \left[ \exp \left( i \frac{2g^2t}{\Delta} a^\dagger a \right) - 1 \right],$$
$$u_{cc}(t) = \frac{1}{2} \left[ \exp \left( i \frac{2g^2t}{\Delta} a^\dagger a \right) + 1 \right],$$
(A.132)

which agrees with the result obtained in [17].

**Figure Captions**

**Fig. 1-** Energy states scheme of a three-level atom where $|e\rangle$ is the upper state with atomic frequency $\omega_e$, $|f\rangle$ is the intermediate state with atomic frequency $\omega_f$, $|g\rangle$ is the lower state with atomic frequency $\omega_g$ and $\omega$ is the cavity field frequency and $\Delta = (\omega_e - \omega_f) - \omega$ is the detuning. The transition $| f \rangle \equiv \omega_e$, $| g \rangle$ is far enough of resonance with the cavity central frequency such that only virtual transitions occur between these levels (only these states interact with field in cavity $C$). In addition we assume that the transition $| e \rangle \equiv \omega_c$, $| g \rangle$ is highly detuned from the cavity frequency so that there will be no coupling with the cavity field in $C$.

**Fig. 2-** Set-up for teleportation process. (a) Preparation of a Bell state: we send atom $A1$ through a Ramsey cavity $R1$, cavity $C$ prepared initially in a coherent state $(|0\rangle + |1\rangle)/\sqrt{2}$ and through a Ramsey cavity $R2$ and atom $A2$ through a Ramsey cavity $R3$, cavity $C$ and through a Ramsey cavity $R4$. Then we send a two-level atom $A3$ resonant with the cavity through $C$ in the lower state $|f_3\rangle$ and through Ramsey cavity $R5$ and detect the upper state $|e_3\rangle$ or $|f_3\rangle$ in $D3$. (b) Alice and Bob meet and generate a Bell state involving atoms $A1$ and $A2$. Then they separate and Alice keeps atom $A2$ with her and Bob keeps atom $A1$ with him. Later on Alice decides to teleport an unknown state prepared in atom $A4$ to Bob. Alice sends atoms $A2$ and $A4$ through a cavity $C$ prepared initially in a state $(|0\rangle + |1\rangle)/\sqrt{2}$. After atoms $A2$ and $A4$ have flown through $C$ she sends a two-level atom $A5$ resonant with the cavity through $C$ in the lower state $|f_5\rangle$ and through a Ramsey cavity $R6$ and then detects $|e_5\rangle$ or $|f_5\rangle$ in $D5$. Then she must perform a measurement of the remaining Bell states of the Bell basis. For this purpose she sends atom $A2$ through the Ramsey cavity $K_2$ and $A4$ through Ramsey cavity $K_4$. Then, she calls Bob and informs him the result of her atomic detections in detectors $D2$ and $D4$. Depending on the results of the Alice’s atomic detections and which atomic state of $A5$ she detected, Bob has or not to perform an extra rotation in the Ramsey cavity $R7$ on the states of his atom $A1$.

**Fig. 3-** Energy level scheme of the three-level lambda atom where $|a\rangle$ is the upper state with atomic frequency $\omega_a$, $|b\rangle$ and $|c\rangle$ are the lower states with atomic frequency $\omega_b$ and $\omega_c$, $\omega$ is the cavity field frequency and $\Delta = \omega_a - \omega_b - \omega = \omega_a - \omega_c - \omega$ is the detuning.
Fig. 4- Set-up for teleportation process. (a) Preparation of a Bell state: we send atoms A1 and A2 through a cavity C prepared initially in a coherent state \((|0⟩ + |1⟩)/\sqrt{2}\). Then we send a two-level atom A3 resonant with the cavity through C in the lower state \(|f_3⟩\) and through a Ramsey cavity R1 and detect the upper state \(|e_3⟩\) or \(|f_3⟩\) in D3. (b) Alice and Bob meet and generate an Bell state involving atoms A2 and A4. Then they separate and Alice keeps atom A2 with her and Bob keeps atom A1 with him. Later on Alice decides to teleport an unknown state prepared in atom A4 to Bob. She sends atoms A2 and A4 through a cavity C prepared initially in a state \((|0⟩ + |1⟩)/\sqrt{2}\). After atoms A2 and A4 have flown through C Alice sends a two-level atom A3 prepared initially in the lower state \(|f_3⟩\) and resonant with the cavity through C, through a Ramnsey cavity R2, and detects \(|e_3⟩\) or \(|f_3⟩\) in D3. Then she detects the states of atoms A2 and A4 in detectors D2 and D4 and calls Bob and inform him the result of her atomic detections. Depending on the results of Alice’s atomic detections Bob has or not to perform an extra rotation in the Ramsey cavity R3 on the states of his atom A4.

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Fig. 1 - E. S. Guerra

\[
\begin{align*}
\Delta & \quad |e\rangle \\
- & \quad - \quad - \quad - \\
\arrow & \quad |f\rangle \\
\quad & \quad |g\rangle
\end{align*}
\]
Fig. 2a - E. S. Guerra
Fig. 2b - E. S. Guerra
Fig. 3 - E. S. Guerra
Fig. 4a - E. S. Guerra
Fig. 4b - E. S. Guerra