GOODWILLIE CALCULUS VIA ADJUNCTION AND LS COCATEGORY

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Abstract. In this paper, we show that for reduced homotopy endofunctors of spaces, $F$, and for all $n \geq 1$ there are adjoint functors $R_n, L_n$ with $T_n F \simeq R_n F L_n$, where $P_n F$ is the $n$-excisive approximation to $F$, constructed by taking the homotopy colimit over iterations of $T_n F$. This then endows $T_n$ of the identity with the structure of a monad and the $T_n F$'s are the functor version of bimodules over that monad. It follows that each $T_n F$ (and $P_n F$) takes values in spaces of symmetric Lusternik-Schnirelmann category $n$, as defined by Hopkins. This also recovers recent results of Chorny-Scherer relating vanishing Whitehead products and values of $n$-excisive functors. The spaces $T_n F(X)$ are in fact classically nilpotent (in the sense of Bernstein-Ganea) but not nilpotent in the sense of Biedermann and Dwyer. We extend the original constructions of dual calculus to our setting, establishing the $n$-co-excisive approximation for a functor, and dualize our constructions to obtain analogous results concerning constructions $T^n$, $P^n$, and LS category.

1. Introduction

The Lusternik-Schnirelmann category of a space $X$, denoted $\text{LScat}(X)$, is (one less than) the minimal number of open sets needed to cover $X$ which are contractible in $X$\footnote{Typically, LScat is renormalized by subtracting 1, so that contractible spaces have LScat 0.}. This was originally defined for manifolds, and is a lower bound for the number of critical points of a function on $X$ \cite{LS34}. The definition was broadened to arbitrary spaces, and later definitions include an inductive version by Ganea, ind LS cat, and symmetric version, symm LS cat, by Hopkins \cite{BG61, Hop84b}. Ganea showed that inductive category agrees with the original definition in suitable cases ($X$ path-connected, paracompact and locally contractible, e.g.) and since then all definitions of category have been shown to be equivalent (see \cite{FHT01} \S V Ch. 27 for a recent account), which is not true of their duals.

Inductive and symmetric category were both defined in a way that admit natural duals, the cocategory of a space. To understand their importance, we state the following inequalities, where “Whitehead length” is the maximum length of non-trivial Whitehead products (minus 1) in $\pi_n X$ and $\text{nil}(\Omega X)$ is the associated Berstein-Ganea nilpotence, that is, the number $n$ such that the homotopy commutators in $\Omega X$ of length strictly greater than $n$ vanish \cite{BG61, Hop84b, Gan60}:

\begin{equation}
\text{Whitehead length}(X) \leq \text{nil}(\Omega X) \leq \text{ind LS cocat}(X) \leq \text{sym LS cocat}(X).
\end{equation}

We use constructions in Goodwillie’s calculus of homotopy functors to reformulate Hopkins’s definition of symmetric LS cocat. We say that $F$ is a homotopy functor if it preserves weak equivalences. The analog to being polynomial of degree $n$ in this
setting is called $n$-excisive; homology theories (regarded as taking values in spaces) are 1-excisive functors. A homotopy functor $F$ can be approximated by a tower of $n$-excisive functors $P_n F$. Each $P_n F$ is defined as the homotopy colimit over a directed system of finite homotopy limit constructions, $T^k_n F$, that is, $P_n F(X) := \text{hocolim}(T_n F(X) \to T^{k+1}_n F(X) \to \cdots)$. In particular, we give the definition $T_n F(X) := \text{holim}_{U \in \mathcal{P}_{0}([n])} F(U \ast X)$, where $\mathcal{P}([n])$ is the power set on $[n] = \{0, \ldots, n\}$ and $\mathcal{P}_{0}([n]) = \mathcal{P}([n]) - \emptyset$.

We give here an explanation of Hopkins’ definition of symmetric LS cocategory [Hop84a, Section 3, p221-222] and how to translate from his language to ours. He lets $C_n$ be what we call $P_{0}([n])$. He defines, for a given space $X$, a functor $F^n$, as the homotopy inverse limit of a (punctured) cube. For $A \in \mathcal{P}_{0}([n]) =: C_n$, the $A$-indexed position of this $(n+1)$-cube is the homotopy colimit of $X$ mapping to $|A|$ different copies of a point, which we will explain shortly. He denotes this by $F^n_A$. Regarding $A$ as a finite ordered set, we can view $F^n_A$ as the homotopy pushout of the following:

\[
\begin{align*}
X & \quad \xleftarrow{\sim} \quad {0} \quad \xrightarrow{\sim} \quad {1} \quad \cdots \quad |A| - 1
\end{align*}
\]

That is, the $A$-indexed position of this $(n+1)$-cube is $A \ast X$.

Then $F^n := \text{holim}_{A \in \mathcal{P}_{0}([n])} F^n A \sim \text{holim}_{U \in \mathcal{P}_{0}([n])} U \ast X$. That is, we have shown that his $F^n$’s exactly the $T_n \mathcal{I}(X)$’s. He constructs a tower of these $F^n$’s:

\[
(\cdots \to \text{holim} F^n \to \text{holim} F^{n-1} \to \cdots \to \text{holim} F^1)
\]

which is therefore our $T_n \mathcal{I}$ tower,

\[
(\cdots \to T_n \mathcal{I}(X) \to T_{n-1} \mathcal{I}(X) \to \cdots \to T_1 \mathcal{I}(X)).
\]

In this language, denoting by $\mathcal{I}$ the identity functor, we may restate Hopkins’s definition as

**Definition 1.1.** For a space $X$, symmetric LS cocat$(X) \leq n$ iff $X$ is a retract of $T_n \mathcal{I}(X)$.

We prove the following:

**Theorem 1.2.** Let $F$ be a reduced homotopy endofunctor of topological spaces. For all $n \geq 1$, there exist (Quillen) adjoint functors $R_n, L_n$ such that $T_n F$ is homotopic to $R_n F L_n$. For $F$ which takes a point to a point (strongly, not just up to homotopy), we have that $T_n F = R_n F L_n$. Such functors include those which are basepoint-preserving homotopy functors of based, connected topological spaces. In particular, $T_n \mathcal{I} = R_n L_n$, i.e. has the structure of a monad for all $n$.

Which has as a consequence
Corollary 1.3. $T_n F$ are left and right $T_n I$-functors, as are the $P_n F$.

We leave the rigorous definition of being left or right $M$-functors (for $M$ a monad) to the background section. It is exactly the functor analog of being a left or right $M$-module. To avoid confusion with bi-functors, we choose to use the term “left and right $M$-functor” instead of “bi-$M$-functor” to denote having the structure of both a left and right $M$-functor.

There are several corollaries of Cor 1.3. We will refer the proof and discussion of them to section 4.2 but list them now.

Corollary 1.4. $T_n F$ naturally takes values in spaces of symmetric LS cocat $\leq n$, as do the $P_n F$.

Corollary 1.5. The Whitehead products of length $\geq n + 1$ vanish for $T_n F(X)$ and $P_n F(X)$ for any space $X$.

Corollary 1.6. For $F$ such that $T_n F$ is not $n$-excisive, $T_n F$ takes values in spaces which are classically nilpotent (in the sense of Berstein-Ganea) but not nilpotent in the sense of Biedermann and Dwyer (that is, not homotopy nilpotent groups once $\Omega$ is applied).

In answer to the question of an anonymous reviewer about the original hypotheses of Corollary 1.6, we also establish the following result. The proof is compact enough that we have included it in this paper as well, in Section 4 after the proof of Corollary 1.6.

For a thorough definition of analyticity, see Section 2.

Proposition 1.7. If $F$ is a $p$-analytic homotopy endofunctor of spaces for some $p$ and also for some $n$, $T_n F$ is $n$-excisive, then $F(X)$ is equivalent to $P_n F(X)$ for all $X$ of connectivity $\geq p$.

We point out that the constructions used in the definition of inductive LS category were proven by Deligiannis to have the structure of comonads [Del00]. Our proof is necessarily significantly different than a dualization of this result, as we lack an inductive definition of the $T_n$'s.

We also construct duals of the adjoints in the statement of Theorem 1.2, which we will call $R_n^a$ and $L_n^a$. These give rise to an alternate formulation of the dual Taylor tower, producing $n$-co-excisive approximations rather than co-degree-$n$ as in [McC01].

Dual to the normal case, we would then like to construct the $n$-co-excisive approximation to a functor, $P_n^a F$, as the homotopy limit over iterations of $T^n F = L^n F R^n$. We establish in the Appendix the counterpart of [Goo03] Lemma 1.9, which shows that the map $t^n : T^n F \to F$ factors through some co-cartesian cube. In [Goo03], the original Lemma was then combined with commutativity of finite pullbacks with filtered colimits to conclude that $\text{hocolim}(T_n F \to T_2^n F \to \cdots)$ produced a homotopy limit $(n + 1)$-cube—a cube whose initial space is equivalent to the homotopy limit of the rest of the cube—from a strongly cocartesian $(n + 1)$-cube.

The dual situation does not always happen. That is, we cannot always commute finite pushouts with (co)filtered homotopy limits of spaces. Since our current aim is not a complete re-write of the dual calculus theory to endofunctors of spaces, we choose to resolve the issue of commuting finite pushouts with (co)filtered homotopy limits by restricting
to functors landing in spectra if we need to consider $P^n F$. Then these approximations $P^n F$ do take strongly cartesian cubes to cocartesian ones.

We give here an explanation of Hopkins’ definition of symmetric LScategory\cite[Section 3, p221]{Hop84a}. He defines, for a given space $X$, a contravariant functor $F_n$, as the homotopy colimit of a (co-punctured) cube. Let $\mathcal{P}^1([n]) = \mathcal{P}([n]) \setminus [n]$. For $A \in \mathcal{P}^1([n]) = C_n^\op$, the $A$-indexed position of this $(n + 1)$-cube is the homotopy limit of $|A|$ different copies of a point mapping to $X$.

Regarding $A$ as a finite ordered set, we can view $F_n A$ as the homotopy pullback of the following:

\[
\begin{array}{cccc}
{} & \{0\} & \{1\} & \cdots & \{\vert A\vert - 1\} \\
\downarrow & & & & \\
X & & & & \\
\end{array}
\]

We replace these maps by fibrations (since we are taking a homotopy limit), which replaces a finite set $S$ by $X^{\Delta^S}$, using the convention that $\Delta^S = \Delta^{|S| - 1}$, the simplex with vertices induced by $S$, which is of dimension $\vert S\vert - 1$. We are then pulling back over the following diagram:

\[
\begin{array}{cccc}
X^{\Delta^0} & X^{\Delta^1} & \cdots & X^{\Delta^{|A| - 1}} \\
\downarrow & & & & \\
X & & & & \\
\end{array}
\]

That is, the $A$-indexed position of this $(n + 1)$ cube is homotopic to $\prod_{|A|} \Omega X$.

Then $F_n := \text{holim}_{A \in \mathcal{P}^1([n])} F_n A$. He constructs a directed system of these $F_n$’s, with the maps cofibrations:

\[
\text{hocolim } F_1 \to \text{hocolim } F_2 \to \cdots \to \text{hocolim } F_n \to \cdots.
\]

Hopkins states \cite[p.91]{Hop84b} that this sequence of $F_n$’s can be identified with the Milnor filtration of $X$ regarded as the classifying space of its loop space.

We dualize our adjoint functors to obtain another adjoint pair $R^n := R_n^\op$ and $L^n := L_n^\op$, such that $L^n R^n$ is equivalent to the $F_n$’s.

Since the dual calculus was defined before only in a (co)triple way, there was, before this paper, no $T^n F$ which one iterates to produce $P^n F$, the $n$-co-excisive approximation to a functor. Instead, we define $T^n F$ as the logical thing:

**Definition 1.8.** Given our adjoint pair $R^n := R_n^\op$ and $L^n := L_n^\op$, for all $n \geq 1$, we define $T^n F := L^n R^n$; $T^n \mathbb{I}$ is then the comonad $L^n R^n$. There is a natural map $t^n F : T^n F \to F$ which is the map from a (ho)colimit of a co-punctured diagram to its final entry.

In this language, we may re-state another of Hopkins’s definitions as

**Definition 1.9.** For a space $X$, symmetric LS cat$(X) \leq n$ iff the natural map $T^n \mathbb{I}(X) \to X$ has a section (up to homotopy).

\footnote{For the original usage, defined using cotriple/triple calculus, see \cite{McC01, BM04, Kuh04}.
}
The dualized theorem is then

**Theorem 1.10.** With our definitions as in \textit{Ls}, the functor given by
\[ P^n F := \text{holim}(\cdots \rightarrow (T^n)^2 F \rightarrow T^n F) \]
is \( n \)-co-excisive, with a map \( p^n F : P^n F \rightarrow F \) induced by the maps \( t^n F : (T^n)^k F \rightarrow F \). In the homotopy category, \( p^n F \) is the universal map to \( F \) from an \( n \)-co-excisive functor.

Which has as a consequence

**Corollary 1.11.** \( T^n F \) are left and right \( T^n \mathbb{I} \)-functors, as are the \( P^n F \).

As before, this extra structures implies that

**Corollary 1.12.** \( T^n F \) naturally takes values in spaces of symmetric LS cat \( \leq n \), as do the higher iterates \( (T^n)^k F \).

Due to the following inequality ([BG61])
\[ \text{cup-length } (X) \leq \text{LS cat}(X) \]
we conclude that

**Corollary 1.13.** The cup products of length \( \geq n + 1 \) vanish for \( T^n F(X) \) and the higher iterates \( (T^n)^k F \).

### 1.1. **Some remarks.**

It is worth noting that in this paper, we restrict ourselves to calculus “over a point”. Goodwillie has defined the theory more arbitrarily, for spaces over an arbitrary fixed space, and recent work has extended another model of the calculus to functors of spaces with maps factoring a fixed map (e.g. for a map \( f : A \rightarrow B \), a factorization is then a space \( X \) with maps \( \alpha, \beta \) such that \( \beta \circ \alpha = f \)), see ([BJME14]). It is possible that these more general forms of calculus then give altered versions of LS cocategory. That is, we expect that \( T_n \mathbb{I} \) for functors over \( Y \) classifies a sort of relative or fiberwise LS cocategory.

In analogy to the result in the “normal” case (see [AK98, Eld11], etc), we would expect that the homotopy colimit of the (first) partial approximation dual calculus tower
\[ \text{hocolim}(T^1 \mathbb{I}(X) \rightarrow T^2 \mathbb{I}(X) \rightarrow \cdots) \]
is the conilpotent analogue of the “\( Z \)-nilpotent completion of \( X \)”, for \( X \) 0-connected. However, any 0-connected space is recoverable by its loop space, which is reflected by the fact that the homotopy colimit of the partial approximation dual calculus tower for \( X \) 0-connected is just \( X \) again (which follows as a consequence of see [Hop84b, Theorem 3.2.1]).

We would like to additionally point out that some caution should be made in statements about the dual tower. The version of \( P^n F \) for any functor from spaces to spectra and any \( n \geq 0 \) found in [McC01] will be contractible on any space with a finite Postnikov tower. For example, it will vanish on \( S^1 \) (though not necessarily on \( S^2 \)).

Assuming a good definition of co-analyticity of a functor, one would expect a dual to [Eld13 Cor 1.4], which would give an equivalence between \( \text{hocolim}_\Delta F(\text{Hom}(\text{sk}_j \Delta^*, X)) \) and \( P^{\infty} F(X) \) for \( j \) bounded (below) by the co-analyticity of \( F \), with bounds being improved when one inputs spaces with lower connectivity. This would then lend support
to viewing spectra as a more natural place for the constructions, at least when we would like to consider $P^nF$ or $P^\infty F$.

For functors landing in spectra, the dual tower for spectra is a reasonable object to study (as we then may define negative homotopy groups), although the identity is linear, and all spectra would thus be considered of LS category 1.

As a consequence of this formally dual tower, something like the following should be true:

**Conjecture 1.14.** There is, for each $n$, a theory $P^n$, in the sense of Lawvere, with objects the free homotopy conilpotent groups of class $n$, products of $\Sigma P^nI$ applied to a product of 1-spheres. Then there should be a weak equivalence of categories between values taken on by functors of the form $\Sigma F$, where $F$ is n-co-excisive, and homotopy conilpotent groups of class $n$.

## 2. Background

This section contains a variety of information useful for non-experts. We first introduce terminology about cubical diagrams in section 2.1 and descriptions of our models for $\text{ho(co)lim}$ of (co)punctured cubes in section 2.2. These are necessary for the following constructions of Goodwillie calculus in section 2.3. We introduce the definition of a monad $M$, as well as left/right $M$-modules and the functor analog, left/right $M$-functors in section 2.4. We leave a discussion of the dual calculus for the appendix.

### 2.1. Cubes and cubical diagrams

We take $\Delta$ to be the category of finite ordered sets and monotone maps, with elements $[n] = \{0, 1, \ldots n\}$. If $S$ is a finite set, we denote by $\Delta^S$ the topological simplicial complex $\Delta^{[S-1]}$, so that $\Delta^n = \Delta^S$. We denote by $\mathcal{P}(S)$ the power set of the set $S$, which we will freely use to also mean the corresponding category with morphisms given by inclusion and objects the subsets of $S$. We can also use $\mathcal{P}(S)$ to mean its diagrammatic representation; the following is a diagrammatic representation of the category $\mathcal{P}([1])$.

\[
\begin{array}{ccc}
\emptyset & 
\rightarrow & \{0\} \\
\downarrow & & \downarrow \\
\{1\} & 
\rightarrow & \{0, 1\}
\end{array}
\]

We will denote by $\mathcal{P}_0(S)$ the subcategory without the empty set and $\mathcal{P}^1(S)$ the subcategory of $\mathcal{P}(S)$ with $S$ removed. An $(n+1)$-cube of spaces is then a functor from $\mathcal{P}([n])$ to spaces, with sub-diagrams given by restricting to $\mathcal{P}_0([n])$ or $\mathcal{P}^1([n])$, the punctured or co-punctured $(n+1)$-cube, respectively. For $X$ a $\mathcal{P}^1([n])$-diagram, rather than index $X$ by the subsets $S \in \mathcal{P}^1([n])$, it is customary to consider instead $X([n]-U)$, where $U \in \mathcal{P}_0([n])$.

We will commonly refer to a homotopy pullback square as *cartesian* and a homotopy pushout square as *cocartesian*. An $n$-cube $X$ is *cartesian* if its initial point, $X(\emptyset)$, is equivalent (along the natural map) to the homotopy limit of the rest of the diagram, i.e. if $X(\emptyset) \xrightarrow{\sim} \text{holim}_{U \in \mathcal{P}_0([n])} X(U)$ and *cocartesian* if $X([n]) \xleftarrow{\sim} \text{hocolim}_{U \in \mathcal{P}_0([n])} X([n]-U)$. The terms *strongly cocartesian* and *strongly cartesian* imply that every sub-2-face (i.e. every sub-square) is *cocartesian* (or, respectively, *cartesian*).
2.2. **Ho(co)lim for n-cubes.** For a punctured cube of spaces, \( X \), for the homotopy limit we use \( \text{Hom}_{\text{Top}P_0^{(n)}}(\Delta^S|_{S \in P_0^{(n)}}, X) \), the standard model. For \( X \) a punctured square, an element of this Hom-space is a map from the left diagram to the right diagram, where \( \Delta^i \) are topological simplices (the realizations of \( \Delta^i \), by common abuse of notation):

\[
\begin{array}{ccc}
\Delta^0 & \xrightarrow{d^i} & \Delta^1 \\
\downarrow & & \downarrow \\
X(0) & \xrightarrow{f} & X(\{0,1\})
\end{array}
\]

That is, a tuple \((x_0, x_1, \gamma) \in X(0) \times X(1) \times X(\{0,1\})\) such that the path \( \gamma \) in \( X(\{0,1\}) \) has \( \gamma(0) = f(x_0) \) and \( \gamma(1) = g(x_1) \).

This has a natural left adjoint, which takes a space \( X \) and sends it to the punctured cubical diagram \( S \mapsto X \times \Delta^S \) (see Example 8.13 of [Dug08]).

2.3. **Goodwillie Calculus.** Not much background in Goodwillie calculus is needed to understand our results. Information regarding the dual calculus may be found in the appendix. In this paper, we restrict ourselves to calculus “over a point”. Goodwillie has defined the theory more arbitrarily, for spaces over an arbitrary fixed space, and recent work has extended another model of the calculus to functors of spaces with maps factoring a fixed map (e.g. for a map \( f : A \to B \), a factorization is then a space \( X \) with maps \( \alpha, \beta \) such that \( \beta \circ \alpha = f \)). We will tend to assume our domain and codomain to be topological spaces over a point (outside of the dual calculus setting, where to talk about \( P^n \), we need to work stably, and use endofunctors of (Bousfield-Friedlander) spectra).

2.3.1. **Definitions and constructions.** In [Goo90, Goo91], Goodwillie establishes the following definition, in analogy with a function being polynomial of degree 1 or \( n \):

**Definition 2.1.** A functor \( F \) is excisive (i.e. 1-excusive) if it takes cocartesian squares to cartesian squares and \( n \)-excisive if it takes strongly cocartesian \((n+1)\)-cubes to cartesian ones.

Homology theories (viewed as functors \( X \mapsto \Omega^\infty(\Sigma^\infty X \wedge E) \) for some spectrum \( E \)) are then nice excisive functors. In particular, a functor \( F \) is excisive, reduced, and preserves filtered colimits if and only if it is a reduced homology theory in this sense.

We will now give the constructions necessary to produce the \( n \)-excisive approximations to a functor \( F \), \( P_n F \), which are assembled from finite limit constructions, \( T_n F \). We let \( * \) denote the topological join (over a point).

We recall the following definition:

\[ T_n F(X) := \text{holim}_{U \in P_0^{(n)}} F(U * X). \]

We have a natural transformation \( F(X) \to T_n F(X) \), given by the natural map

\[ F(X) = F(\emptyset * X) \to \text{holim}_{U \in P_0^{(n)}} (U \mapsto F(U * X)). \]
That is, the map from the initial object of the square, $F(X)$, to the homotopy pullback of the rest, $T_n F(X)$. We can take $T_n$ of $T_n F$, and also have the same natural transformation from initial to homotopy pullback, now $T_n F(X) \rightarrow T_n(T_n F(X)) =: T_n^2 F(X)$. For $n = 1$, see Figure 1.

The degree $n$ polynomial approximation to $F$, $P_n F$, is constructed as the homotopy colimit, $P_n F(X) := \text{hocolim}(T_n F(X) \rightarrow T_n^2 F(X) \rightarrow \cdots)$.

It is not immediately obvious that this is in fact $n$-excisive and universal (up to homotopy). We refer the reader to [Goo90, Goo03] for the details, especially Lemma 1.9 of [Goo03] with alternate proof provided by Charles Rezk [Rez08].

\[ T_n^2 F(X) := \text{holim} \begin{pmatrix} T_1 F([0] \ast X) \rightarrow T_1 F([1] \ast X) \rightarrow T_1 F([0,1] \ast X) \end{pmatrix} \]

\[ \simeq \text{holim} \begin{pmatrix} F([1] \ast [0] \ast X) \rightarrow F([1] \ast [0,1] \ast X) \rightarrow F([0,1] \ast [0,1] \ast X) \end{pmatrix} \]

\[ \overset{\text {Figure 1. } T_1^2 F(X)}{\longrightarrow} \]

The collection of polynomial approximations to a functor $F$, $\{P_n F\}_{n \geq 0}$, comes with natural fibrations $P_n F(X) \rightarrow P_{n-1} F(X)$ for all $n \geq 1$.

With these maps we form a tower, the Goodwillie (Taylor) tower of $F(X)$:

$\cdots \rightarrow P_n F(X) \rightarrow P_{n-1} F(X) \rightarrow \cdots \rightarrow P_1 F(X) \rightarrow P_0 F(X)$.

Since we are restricting ourselves in this work to calculus over a point, $P_0 F(X) = F(*)$; in general, $P_0 F(X) = F(Y)$ for whatever space $Y$ we were working over.

We denote by $P_\infty F(X)$ the homotopy inverse limit of this tower.

2.3.2. Analyticity and convergence. Heuristically, we say that a functor $F$ is $\rho$-analytic if its failure to be $n$-excisive for all $n$ is bounded with a bound depending on $\rho$; $\rho$-analytic implies $(\rho + 1)$ analytic, which is a weaker condition. This gives rise to a notion of “radius of convergence of a functor”.

More precisely, $F$ is a $\rho$-analytic functor when there exists a $q$ such that for all $n$, $F$ takes a strongly co-Cartesian $(n+1)$ cube $X$ (with connectivities of the maps $X(\emptyset) \rightarrow X(s) k_s > \rho$) to a cube which is $(np - q + \sum k_s)$-cartesian. That is, the map $F(X(\emptyset))$ to the homotopy limit of the rest of the cube is $(np - q + \sum k_s)$-connected (see definition 4.1 and 4.2 of [Goo91]). This is the bound on the failure of the target cube to be cartesian, i.e. the bound on the failure of $F$ to be $n$-excisive for all $n$.

**Proposition 2.2.** [Goo03 Theorem 1.13] If $F$ is at least $\rho$-analytic and $X$ is $k$-connected for $k$ at least $\rho$ (i.e. if $X$ is in $F$’s “radius of convergence”), then $F(X) \simeq P_\infty(X)$. 
As towers give rise to spectral sequences, so does the Goodwillie tower, and this property of an analytic functor can be read as a statement about convergence of the spectral sequence associated to the Goodwillie tower of $F$.

Some of the earliest and most powerful results of Goodwillie calculus relate to analyticity and other properties which follow. Examples of 1-analytic functors include $\mathbb{I}_{\text{Top}}$, Waldhausen’s algebraic $K$-theory functor, and $TC$, the topological cyclic homology of a space. For a $\rho$-connected CW complex $K$, the functor $X \mapsto \Omega^\infty \Sigma^\infty \text{Map}(K, X)$ is $\rho$-analytic.

2.4. Monads $M$ and left/right $M$-Functors. We would like to point out that monads are also sometimes called “triples”, especially in the more algebraic literature, and in some of the Goodwillie calculus constructions such as those of [JM04, BEJM14, BJME14].

We first recall relevant definitions of a monad $M$ and $M$-Functor, which is the functor extension of the notion of a module over the monad $M$:

**Definition 2.3.** [ML98, p.133] A monad $M = \langle M, \eta, \mu \rangle$ in a category $C$ consists of a functor $M : C \to C$ and two natural transformations

\[
\eta : \text{Id}_C \to M \\
\mu : M^2 \to M
\]

which make the following commute

\[
\begin{array}{ccc}
M^3 & \xrightarrow{\mu \mu} & M^2 \\
\downarrow \mu M & & \downarrow \mu \\
M^2 & \xrightarrow{\mu} & M
\end{array}
\quad
\begin{array}{ccc}
\text{Id}_C \circ M & \xrightarrow{\eta M} & M^2 \\
\downarrow \mu & & \downarrow \mu \\
M & \xrightarrow{\mu} & M
\end{array}
\]

**Definition 2.4.** [ML98, p.136] If $M = \langle M, \eta, \mu \rangle$ is a monad in a category $C$, we have notions of left and right $M$-module as follows. A left $M$-module (Referred to in [ML98, p.136] as an $M$-algebra) $\langle x, h \rangle$ is a pair consisting of an object $x \in C$ and an arrow $h : Mx \to x$ of $C$ which makes both of the following diagrams commute (assoc law, unit law):

\[
\begin{array}{ccc}
M^2 x & \xrightarrow{M h} & M x \\
\downarrow \mu_x & & \downarrow h \\
M x & \xrightarrow{h} & x
\end{array}
\quad
\begin{array}{ccc}
x & \xrightarrow{\eta x} & M x \\
\downarrow 1 & & \downarrow h \\
x & \xrightarrow{h} & x
\end{array}
\]

A right $M$-module $\langle x', h' \rangle$ is a pair consisting of an object $x' \in C$ and an arrow $h' : xM \to x$ of $C$ (here $xM$ means $x$ with a right $M$-action) which makes both of the following diagrams commute (co-assoc law, co-unit law):

\[
\begin{array}{ccc}
x' M^2 & \xrightarrow{h' M} & x' M \\
\downarrow \mu_{x'} & & \downarrow h' \\
x' M & \xrightarrow{h'} & x'
\end{array}
\quad
\begin{array}{ccc}
x' & \xrightarrow{\eta_{x'}} & x' M \\
\downarrow 1 & & \downarrow h' \\
x' & \xrightarrow{h'} & x'
\end{array}
\]

The following is a slight modification of Definition 9.4 from [May72]. There it is called an $M$-functor. We re-name it so as to be able to talk about functors with both right and left $M$ "actions". This is the functor-level analog of being an $M$-module.
Definition 2.5. Let \((M, \mu, \eta)\) be a monad in \(C\). A right \(M\)-functor \((G, \lambda)\) in a category \(D\) is a functor \(G : C \to D\) together with a natural transformation of functors \(\lambda : GM \to G\) such that the following diagrams are commutative

\[
\begin{align*}
GM^2 \xrightarrow{G\mu} GM & \quad \text{and} \quad G \xrightarrow{G\eta} GM \\
\lambda & \downarrow \quad \lambda \\
GM \xrightarrow{\lambda} G & \quad \quad G \xrightarrow{\lambda} G
\end{align*}
\]

A left \(M\)-functor \((G, \lambda')\) in a category \(D\) is a functor \(G : C \to D\) together with a natural transformation of functors \(\lambda' : MG \to G\) such that the following diagrams are commutative

\[
\begin{align*}
M^2G \xrightarrow{\mu G} MG & \quad \text{and} \quad G \xrightarrow{\eta G} MG \\
\lambda' & \downarrow \quad \lambda' \\
MG \xrightarrow{\lambda'} G & \quad \quad G \xrightarrow{\lambda'} G
\end{align*}
\]

Note that for an adjoint pair of functors \(L : A \to B, R : B \to A\), with unit and counit \(\eta : Id_A \to RL, \epsilon : LR \to Id_B\) we have a natural monad \(M := RL\) with multiplication \(\mu := R\epsilon L : RLRL \to RL\) and unit given by the unit of the adjunction.

Proposition 2.6. For an adjoint pair of functors \(L : A \to B, R : B \to A\) and an endofunctor \(F : B \to B\), we have that \(RF L\) has the structure of a left and right \(RL\)-functor, and dually \(LF R\) a left and right \(LR\)-functor.

We will illustrate this for the bimodule over the monad case and the dual proof follows by dualizing our arguments and flipping the arrows in our diagrams. The structure map \(\lambda := R\epsilon\) and costructure map \(\lambda' := R\epsilon\).

Right \(RL\)-functor structure:

\[
\begin{align*}
RFL \circ RL \circ RL & \xrightarrow{\lambda} RFL \circ RL & RFL & \xrightarrow{(RFL)\eta} RFL \circ RL \\
(RFL)\mu & \downarrow \quad \lambda \\
RFL \circ RL & \xrightarrow{\lambda} RFL & RFL & \xrightarrow{\lambda} RFL
\end{align*}
\]

Left \(RL\)-functor structure:

\[
\begin{align*}
RL \circ RL \circ RFL & \xrightarrow{\lambda'} RL \circ RFL & RL \circ RFL & \xrightarrow{\eta(RFL)} RL \circ RFL \\
\mu(RFL) & \downarrow \quad \lambda' \\
RL \circ RFL & \xrightarrow{\lambda'} RFL & RFL & \xrightarrow{\lambda'} RFL
\end{align*}
\]

2.5. Quillen Functors. We review the definitions of Quillen functor and adjunction and include a list of useful properties.

Definition 2.7. [DHKS05 14.1.] Given two model categories \(M\) and \(N\), a Quillen adjunction is an adjunction

\[
f : M \rightleftharpoons N : f'
\]

of which
(i) the left adjoint, $f$, is a left Quillen functor; a functor which preserves cofibrations and trivial cofibrations, and

(ii) the right adjoint, $f'$, is a right Quillen functor; a functor which preserves fibrations and trivial fibrations.

We also provide a list of properties which these functors satisfy.

**Proposition 2.8.** [DHKS05, 14.2.(ii)-(iii)] Quillen functors satisfy the following properties:

(i) Every right adjoint of a left Quillen functor is a right Quillen functor and every left adjoint of a right Quillen functor is a left Quillen functor.

(ii) The opposite of a left Quillen functor is a right Quillen functor and the opposite of a right Quillen functor is a left Quillen functor.

We are considering adjoint pairs $(L, R)$ as well as their opposites $(L^{op}, R^{op})$. This proposition lets us reduce the work of showing both pairs are Quillen adjunctions to just showing it for one of the four functors.

3. **Proofs of main results**

We will prove in this section the following theorem and its consequences:

**Theorem 1.2** Let $F$ be a reduced homotopy endofunctor of topological spaces. For all $n \geq 1$, there exist adjoint functors $R_n, L_n$ such that $T_n F$ is weakly equivalent to $R_n F L_n$. For $F$ which takes a point to a point (strongly, not just up to homotopy), we have that $T_n F = R_n F L_n$. Such functors include those which are basepoint-preserving homotopy functors of based, connected topological spaces. In particular, $T_n I = R_n L_n$, i.e. has the structure of a monad for all $n$.

Here, we use the term **reduced** as Goodwillie did in [Goo03], to mean that $F$ takes contractible spaces to contractible spaces.

One key for this proof is to determine the correct categories to be working between.

3.1. We will first provide the proof in the case $n = 1$ to motivate the general proof.

Recall the definition of $T_1 F(X) := \lim_{U \in P_0([n])} F(U * X)$. For $n = 1$, this gives us the following homotopy pullback square:

$$
\begin{array}{ccc}
T_1 F(X) & \longrightarrow & F(CX) \\
\downarrow & & \downarrow \\
F(CX) & \longrightarrow & F(\Sigma X).
\end{array}
$$

For reduced functors from based spaces to based spaces, $T_1 F(X) \simeq \Omega F(\Sigma X)$, and $\Sigma, \Omega$ are adjoints between those categories. If we relax to reduced functors of unbased spaces, we have to be slightly more careful to get our adjunctions.

There is a (clear) equivalence of categories between spaces and diagrams of the form

$$
* \leftarrow X \rightarrow *
$$

GOODWILLIE CALCULUS VIA ADJUNCTION AND LS COCATEGORY

[11]

[DHKS05]
where * is a point and $X$ is a space. However, the category of dual diagrams,

$$* \to Y \leftarrow *$$

for $Y$ a space, is equivalent to the category of spaces with two basepoints, which we denote by $\text{Top}_{*1,*2}$. We can see that we have an adjunction

$$\text{Unreduced Suspension} : \text{Top} \xrightarrow{\sim} \text{Top}_{*1,*2} : \text{Paths (between } *1 \text{ and } *2)$$

such that for $F$ reduced, $T_1 F(X)$ is equivalent to $X \mapsto SX$ followed by $F$ (which remains in $\text{Top}_{*1,*2}$ because $F$ is reduced) and then by taking paths.

The general case will not involve spaces with a multitude of basepoints, but cubical diagrams $X$ which are similarly “reduced”, i.e. $X(S)$ is a point whenever $|S| = 1$.

3.2. Proof of the general case, arbitrary $n$. We will be working with the categories $\text{Top}$, $\text{Fun}(\mathcal{P}_0([n]), \text{Top}) = \text{Top} \mathcal{P}_0([n])$ and $\tilde{\text{Fun}}(\mathcal{P}_0([n]), \text{Top})$, where the latter is the subcategory of reduced punctured cubical diagrams of spaces. That is, each diagram $X \in \tilde{\text{Fun}}(\mathcal{P}_0([n]), \text{Top})$ has the property that $X(S)$ is a point, for $|S| = 1$.

There are adjunctions between these categories as follows:

$$\text{holim} \xleftarrow{\sim} \text{red} \xrightarrow{\sim} \tilde{\text{Fun}}(\mathcal{P}_0([n]), \text{Top}).$$

We let $L_n := \text{red} \circ (S \mapsto - \times \Delta^S)$ and $R_n := \text{holim} \circ \text{inc}$. We will elaborate on these.

3.3. Holim and $- \times \Delta^S$. The standard model which we use for the homotopy limit of a punctured cube $X$ is $\text{hom}_{\text{Top} \mathcal{P}_0([n])}(\Delta^S|_{S \in \mathcal{P}_0([n])}, X)$. It has a natural left adjoint, which takes a space $X$ and sends it to the punctured cubical diagram $S \mapsto X \times \Delta^S, S \in \mathcal{P}_0([n])$.

$$\text{Hom}_{\text{Top} \mathcal{P}_0([n])}(Y, \text{Hom}_{\text{Top} \mathcal{P}_0([n])}(\Delta^S|_{S \in \mathcal{P}_0([n])}, X)) \cong \text{Hom}_{\text{Top} \mathcal{P}_0([n])}(Y \times (\Delta^S|_{S \in \mathcal{P}_0([n])}), X).$$

where $Y \times (\Delta^S|_{S \in \mathcal{P}_0([n])})$ is precisely $S \mapsto Y \times \Delta^S, S \in \mathcal{P}_0([n])$. This adjunction is established in [BK72, CH XI, §3]

This arises from the levelwise/pointwise adjunctions, which for each $S$ are of the form $\text{Hom}_{\text{Top} \mathcal{P}_0([n])}(Y, \text{Hom}_{\text{Top} \Delta^S, X(S))} \cong \text{Hom}_{\text{Top} \mathcal{P}_0([n])}(Y \times \Delta^S, X(S))$.

3.4. Reduction and inclusion. There is also a natural left adjoint to the inclusion of reduced punctured cubical diagrams into punctured cubical diagrams. This takes a diagram $Y$ to a diagram $\text{red}(Y)$ such that

$$\text{red}(Y)(S) := \text{colim}(Y(S) \leftarrow \coprod_{j \in S} Y(\{j\}) \to S).$$

Note: it is not alarming that this is not a priori a homotopy colimit, because we want a diagram that is honestly reduced, with $\text{red}(Y)(S)$ to be a point, not just contractible, for $|S| = 1$. 
To establish this adjunction, it suffices to show that any map from $Y \in \text{Top}^{\text{P}_0(n)}$ to $X \in \text{Fun}(\text{P}_0([n]), \text{Top})$ must factor through $\text{red}(Y) := S \mapsto \text{red}(Y)(S)$.

Consider $n = 2$, a map of punctured squares.

This map of diagrams is the same as having a square of the following form:

The maps to the final space must factor through the colim of the rest, which is exactly $\text{red}(Y)(\{0,1\})$.

What remains is to explain how this extends to higher dimensional cubical diagrams.

There is a natural map from $Y$ to $\text{red}(Y)$. One way to see this is to view $\text{red}(Y)$, the reduction, as a the cofiber of a bunch of maps at once into $Y(S)$, and it is clear that for $Y(j) \to Y(S)$ ($j \in S$) that $Y(S)$ maps to the cofiber of this map.

Given that, and given a map $f : Y \to X$ for $X$ reduced, we have

That is, it must factor through the reduced cube. □

3.5. **Composed adjunctions and the topological join.** The composition of the two left adjoints sends a space $X$ to the diagram

$$S \mapsto \text{colim}(X \times \Delta^S \leftarrow X \times S \rightarrow S)$$
The common model for the join of two spaces, $X$ and $Y$, is the following, where $C$ is the cone:

$$\text{colim}(X \times CY \leftarrow X \times Y \rightarrow Y)$$

Since the lefthand map is a cofibration, this colim is also a homotopy colim.

For a set $S$ considered as a discrete space, $\Delta^S$ is homotopic to $CS$, and the inclusion $CS \rightarrow \Delta^S$ is a cofibration. That is, we have a map of diagrams with all arrows homotopy equivalences and at least one is also a cofibration, so they have homotopic colimits. This suffices because, thanks to [DI04, Appendix], we do not need cofibrancy on the objects as well.

As a result, we have that the composition of our two left adjoints above is not just a colim but a hocolim, and its pushout it is homotopic to $X \ast S$, so we may take it as a model for the join. That is, $L_n(X) = S \mapsto X \ast S$.

3.6. Relating this adjunction to the Goodwillie Calculus. Recall that $T_n F(X)$ is formed by applying $F$ to the diagram $S \mapsto X \ast S$ (for $S \in \mathcal{P}_0([n])$), i.e. $F \circ L_n$, and following this with the appropriate homotopy limit functor.

For a very general homotopy functor $F$, it will not be true that $F$ of a reduced punctured cube will again be a reduced punctured cube. The condition necessary is that $F$ of a point is a point – on the nose, not up to homotopy.

However, there is a large class of functors for which this is true. Namely, basepoint-preserving homotopy functors of based, connected spaces. As we will want to compare our results with those of Biedermann and Dwyer, it is important to note that they restrict to functors which are of this type, specifically, with spaces replaced by based, connected simplicial sets.

Given such an $F$, the holim of punctured cubes is the same as $\text{holim} \circ \text{inc} = R_n$, and $T_n F = R_n F L_n$.

That is, we have established an adjunction between $\text{Top}$ and $\tilde{\text{Fun}}(\mathcal{P}_0([n]), \text{Top})$ for each $n$ such that $T_n F(X) = R_n F(L_n X)$.

3.6.1. Functors which are reduced, not based. In terms of other applications the author has in mind, it would be best if we could restrict ourselves not just to functors which are based. What can we then do to rectify the situation if $F$ is almost good enough, if $F$, up to homotopy, takes a point to a point? That is, if $X$ is contractible, that $F(X)$ is also contractible, i.e. reduced functors.

For such an $F$, consider $T_n \tilde{F} := R_n \circ \text{red} \circ F \circ L_n$.

If all the spaces involved are connected (if $F$ lands in connected spaces), then we can contract $F(S)$, $|S| = 1$ to a point for all $S$ and not disturb the homotopy type of the homotopy limit. That is, for $F$ taking values in connected spaces (and reduced), $T_n F \overset{\sim}{\rightarrow} T_n \tilde{F}$ is a homotopy equivalence.
It is important to note, that if $F(\text{point})$ is a point, then $T_n F \xrightarrow{\sim} T_n \tilde{F}$ is an isomorphism.

### 3.7. Quillen adjunction.

By definition 2.7, to establish that our adjunctions are Quillen pairs, since we already have that they are adjunctions, we just need to check that either $L_n$ preserves cofibrations and trivial cofibrations or $R_n$ preserves fibrations and trivial fibrations.

Top has the usual model category structure with (Serre) fibrations and cofibrations and with weak equivalences as weak homotopy equivalences. Both diagram categories will be taken with the levelwise model structure induced by this model structure in Top. We have that

$$X \mapsto (S \mapsto X * S)$$

preserves cofibrations and trivial cofibrations, as follows. Given a (trivial) cofibration $X \xrightarrow{\sim} Y$, consider

where the cube is cocartesian (because the top and bottom squares are cocartesian) and the three vertical maps $X \xrightarrow{\sim} Y$, $X \times \Delta^s \xrightarrow{\sim} Y \times \Delta^s$, and $S \xrightarrow{\sim} S$ are all (trivial) cofibrations.

As cofibrations in Top are stable under cobase change, the map $X * S \to Y * S$ is also a cofibration.

In the case of considering a trivial cofibration, homotopy invariance of homotopy colimits (i.e. topological join) gives us that the map $X * S \to Y * S$ is a weak equivalence, i.e. it is also a trivial cofibration. This will hold for all $S$ and (trivial) cofibrations are defined levelwise, so we have shown that this is not just a left adjoint but also a left Quillen functor.

**Remark 3.1.** We would like to point out that if we start with fibrations of cosimplicial spaces and consider the induced cubical diagrams, these will still be fibrations in the levelwise structure as Reedy fibrations are also levelwise fibrations. These diagrams are obtained by precomposing with the functor $\sigma_n : \mathcal{C}_0([n]) \to \Delta_{\leq n}$ which sends $S$ to
and inclusions to the induced coface maps. So, fibrations of cosimplicial things will go to fibrations in Top when following $c_n$ by this Quillen adjunction.

Using Prop. 2.8(iii), we can conclude that the duals will also be Quillen adjoints since our model for hocolim is holim.$^{39}$

4. LS cocategory and related corollaries

The purpose of this section is to provide further discussion of the relationship between LS cocategory and the constructions of Goodwillie calculus, including proofs and more details around the corollaries of Theorem 1.2 and also of Proposition 1.7.

4.1. Goodwillie Calculus and nilpotence. There is a beautiful paper by Arone and Kankaanrinta [AK98] which is likely the first link in the literature between Goodwillie calculus and nilpotence. The identity functor of spaces is a priori 1-analytic [Goo91], which implies that $P_\infty(I)(X) \sim I(X)$ for $X$ at least 1-connected, and this work of Arone and Kankaanrinta extends the equivalence to include $X$ nilpotent by showing that the $n$th degree polynomial approximations of the identity functor are equivalent to those of $Z_\infty X$, the $Z$-nilpotent completion of $X$. Additionally, for $X$ a 0-connected space, $P_\infty(id(X)) \simeq Z_\infty X$.

This connection was then strengthened via work of Biedermann and Dwyer [BD10]. They defined a theory (in the sense of Lawvere) for each $n$ whose algebras they call “homotopy nilpotent groups” of class $n$. These theories, $\Omega P_n$, are built from $\Omega P_n$ and allow one to now see that classical nilpotence is roughly a $\pi_0$-level (or really $\pi_1$-level, due to the $\Omega$-shift) version of this kind of homotopy nilpotence; $\pi_0\Omega P_n = Nil_n$, the theory governing classical nilpotence. In this paper they also show one half of an equivalence of categories (the other half is [BD]), between values of functors $\Omega F$ for $F$ $n$-excisive and $\Omega P_n$-algebras.

More work in this direction has also been done recently by Chorny and Scherer [CS12], who showed through a direct construction that the iterated Whitehead products of length $(n + 1)$ vanish in any value of an $n$-excisive functor. The author has become aware that independently, Christina Costoya and Antonio Viruel have shown that the $P_n F$’s take values in spaces with inductive cocategory $n$.

4.2. Results and proofs. This section contains the statements and proofs of the corollaries of Theorem 1.2 and Corollary 1.3 which relate to the relationship with LS cocat.

**Corollary 1.3** $T_n F$ are left and right $T_n I$-functors, as are the $P_n F$.

*Proof.* This corollary follows immediately from combining Proposition 2.6 with Theorem 1.2. $\square$

Cor 1.3 establishes structure that gives us maps which express $T_n F(X)$ as a retract of $T_n I(T_n F(X))$, combining with Theorem 1.2 so, we also have

**Corollary 1.4** $T_n F$ naturally takes values in spaces of symmetric LS cocat $\leq n$, as do the $P_n F$. 


Proof. Once we have this for each $T_n F$ and $T_n^b F$, these homotopy retract maps clearly induce the same structure on the homotopy colimit, $P_n F := \hocolim_k T_n^b F$.

This follows immediately from the retract structure established in Corollary 1.3. There is always a map, for $X$ a space, $X = I(X) \to T_n I(X)$. So also for a space $T_n F(X)$, we have a map $T_n F(X) \to T_n I \circ T_n F(X)$. Writing in the adjunctions, we have the following:

\[
\begin{array}{ccc}
T_n F X & \longrightarrow & T_n I \circ T_n F(X) \\
\| & \| & \| \\
R_n F L_n & \longrightarrow & R_n L_n R_n F L_n X
\end{array}
\]

The counit of the adjunction, $\epsilon : L_n R_n \to \|$, gives us our map

\[
R_n L_n R_n F L_n X \to R_n F L_n
\]

Recall that the counit and the unit of the adjunction give that the following composition

\[
\begin{array}{ccc}
R_n & \longrightarrow & R_n L_n R_n F L_n X \\
\eta R_n & \longrightarrow & R_n L_n R_n F L_n X \\
\epsilon & \longrightarrow & R_n
\end{array}
\]

is the identity.

We can apply this to $F L_n X$ to realize it as our maps

\[
R_n F L_n X \to R_n L_n R_n F L_n X \to R_n F L_n
\]

□

Theorem 2.1 of [CS12] states that the Whitehead products of length $\geq n + 1$ vanish in $P_n F(X)$ for any space $X$. Corollary 1.5 recovers and extend this result to the $T_n$ as well:

**Corollary 1.5** The Whitehead products of length $\geq n + 1$ vanish for $T_n F(X)$ and $P_n F(X)$ for any space $X$.

Proof. Combining Corollary 1.4 with the following chain of inequalities, we conclude our result:

Whitehead length($X$) \leq \text{nil}(\Omega X) \leq \text{ind LScocat}(X) \leq \text{sym LScocat}(X).

□

We also have the following interesting difference between the $T_n$’s and $P_n$’s, which we thank Boris Chorny and Georg Biedermann for pointing out.

We first mention that Biedermann and Dwyer, in [BD10] Def 5.2, 5.4, define $X$ to be a homotopy nilpotent group of class $n$ if it is (the underlying space of) a homotopy algebra over a certain theory (in the sense of Lawvere), as spaces which admit $k$–ary operations parametrized by $\Omega P_n \| (\bigvee_k S^1)$.

**Corollary 1.6** For $F$ such that $T_n F$ is not $n$-excisive, $T_n F$ takes values in spaces which are classically nilpotent (in the sense of Berstein-Ganea) but not nilpotent in the sense of Biedermann and Dwyer (that is, not homotopy nilpotent groups once $\Omega$ is applied).
Proof. Classical nilpotence follows from the inequalities of [1] and the fact that $T_n F$ take values in spaces of symm LSccat $\leq n$.

There is an equivalence of categories $[BD] \{\text{values of functors } \Omega F, \ F n \text{ - excisive}\} \sim \{\text{Homotopy Nilpotent Groups of class } \leq n\}$.

Thus, under our hypotheses, $\Omega T_n F$ will not be a homotopy nilpotent group (using the above equivalence of categories) in the sense of Biedermann and Dwyer. □

**Proposition 1.7** If $F$ is a $p$-analytic homotopy endofunctor of spaces for some $p$ and also for some $n$, $T_n F$ is $n$-excisive, then $F(X)$ is equivalent to $P_n F(X)$ for all $X$ of connectivity $\geq p$.

**Proof.** Proposition 2.2 stated that if $F$ is $p$-analytic, then for $X$ at least $p$-connected, $P_\infty F(X) \simeq F(X)$.

Corollary 1.4 of [Eld13] states that for $F$ a $p$-analytic homotopy endofunctor of spaces and some space $X$,

$$P_\infty F(X) \simeq \text{holim}(\cdots \rightarrow T_n F(X) \rightarrow \cdots \rightarrow T_k F(X) \rightarrow T_k^1 F(X))$$

i.e. of the kth “row” of these things you use to build the $P_n F$'s, for all $k \geq \max(p - 1 - \text{conn}(X) - 1, 0)$.

We have assumed that $T_n F$ is $n$-excisive, which we implies that $T_n F$ is equivalent to $P_n F$ [1]. $T_n F$ $n$-excisive means that $T_n(T_n F) \simeq T_n F$, in particular, $T_n^k F$ is also equivalent to $T_n F$ and $n$-excisive. Similar argument means that excisiveness holds for the higher $T_{n+1} F$ and $T_{n+1}^k F$. So the holim of each row will be $P_n F$; in particular, we know the holim of the rows at level $k \geq p$ (i.e. greater than or equal to the analyticity of $F$) will have this property, therefore $P_\infty F \simeq P_n F$.

And on its radius of convergence, $F$ is equivalent to $P_\infty F$, which we just said was equivalent to $P_n F$. So, on its radius of convergence, $F$ is equivalent to $P_n F$. □

5. Dual adjunction

Historically, dual calculus has only been defined rigorously in the additive calculus, of [cite Johnson-McCarthy]. When we wish to speak of a dual calculus for excisive functors, we have two choices. Either we define this rigorously, then make our theorems about their structure, or we define the constructions using our adjunctions and then prove that they give rise to the (homtopy-universal) $n$-co-excisive approximations to a functor. We choose the latter route.

We dualize our adjoint functors to obtain another adjoint pair $R^n := R^n_{op}$ and $L^n := L^n_{op}$, such that $L^n R^n$ is equivalent to the $F_n$'s.

Since the dual calculus was defined before only in a (co)triple way, there was, before this paper, no $T^n F$ which one iterates to produce $P^n F$, the $n$-co-excisive approximation to a functor. Instead, we define $T^n F$ as the logical thing:

**Definition 1.8** Given our adjoint pair $R^n := R^n_{op}$ and $L^n := L^n_{op}$, for all $n \geq 1$, we define $T^n F := L^n F R^n$; $T^n_\Box$ is then the comonad $L^n R^n$. There is a natural map

---

3This is established by Goodwillie on the bottom of p.661 of [Goo03].
Remark 5.1. It is important to point out that for these dual results, we do not need to assume that our spaces are based. We address what happens when based and unbased.

5.1. Establishing the dual adjunction. We will be working with the categories Top, $\text{Fun}(\mathcal{P}^1([n]), \text{Top}) = \text{Top}^{\mathcal{P}^1([n])}$ and $\text{Fun}(\mathcal{P}^1([n]), \text{Top})$, where the latter is the subcategory of “co-reduced” co-punctured cubical diagrams of spaces. That is, each diagram $X \in \text{Fun}(\mathcal{P}^1([n]), \text{Top})$ has the property that $X([n] - S)$ is contractible, for $|S| = 1$.

There are adjunctions between these categories as follows:

$$\text{Fun}(\mathcal{P}^1([n]), \text{Top}) \xrightarrow{\text{inc}} \text{Fun}(\mathcal{P}^1([n]), \text{Top}) \xleftarrow{\text{cor}} \text{Top} \xrightarrow{\text{hocolim}} \text{Top}.$$ 

We let $R^n$ be the composition of the two leftward arrows and $L^n$ be the composition of the two rightward arrows, $\text{hocolim} \circ \text{inc}$. We will elaborate on these.

The first step is to dualize the process of taking a space and producing a diagram $S \mapsto \text{colim}(X \times \Delta^S \leftarrow X \times S \rightarrow S)$, by dualizing the processes we composed to get this diagram. We still would like to start with a space and produce a diagram which is a sort of “co-reduced” cube.

The previous first step was to consider the natural model for a homotopy limit and its adjoint. In a similar way, we can consider the natural model for a homotopy colimit, following Dugger [Dug08, Section 8.10].

Given two diagrams, $X, Y$ where $X : I \rightarrow \text{Top}$ and $Y : I^{\text{op}} \rightarrow \text{Top}$, the tensor product of diagrams $X \otimes Y$ is defined as the coend $X \otimes Y = \text{coeq}\left(\bigoplus_{i \rightarrow j} X_i \times Y_j \longrightarrow \prod_i X_i \times Y_i\right)$.

Example 8.12 of [Dug08] gives as the model for $\text{hocolim}_I X$ (for $X$ an $I$-diagram) as $X \otimes B(- \downarrow I)^{\text{op}}$. Where $B(- \downarrow I)^{\text{op}} : I^{\text{op}} \rightarrow \text{Top}$ is the functor $i \rightarrow B(i \downarrow I)^{\text{op}}$.

For any copunctured diagram $X : \mathcal{P}^1([n]) \rightarrow \text{Top}$, we form its homotopy colimit by tensoring with the diagram $\Delta^S$ (which we will denote by $\Delta^{\leq n} \circ c_n$; this is our $B(- \downarrow I)^{\text{op}}$ see in particular Example 8.13 of [Dug08].).

Thanks to the fact that this is an actual tensor product, we get an adjunction very similar to the homotopy limit case:

$$(\Delta^{\leq n} \circ c_n) \otimes (-) : \text{Top}(\mathcal{P}^1([n])) \longrightarrow \text{Top} : \text{Hom}(\Delta^{\leq n} \circ c_n, -)$$

That is, the right adjoint sends $[n] - S$ to $X^{\Delta^S}$ (the dual of sending $S$ to $X \times \Delta^S$). This adjunction relies on observations that one can find nice exposition on (including proofs) in Dan Dugger’s primer on homotopy colimits [Dug08].

We next need the universal co-reduced diagram. We are using the convention now of indexing our diagrams in $\mathcal{P}^1([n])$ by sets in $\mathcal{P}_0([n])$ by considering where we send...
\([n] - S\) for varying \(S\) in \(\mathcal{D}_0([n])\). Dualizing the construction for reduced diagram yields

\[
[n] - S \mapsto \lim \left( \prod_{S^*} \mathcal{X}([n] - S) \longrightarrow \prod_{s \in S} \mathcal{X}([n] - s) \right).
\]

Now, about the maps. The map into the product is the map induced by each map from the original diagram between \(X([n] - S)\) and \(X([n] - s)\) for all \(s \in S\). The map from \(\ast = \prod_{s \in S} \ast\) is the choice of a point in each copy of \(\mathcal{X}([n] - s)\). If each space is based, this is the base point. If not, we need to start with connected spaces and have made a choice of base point. This map is the opposite of collapsing each space in \(\prod_{s \in S} \mathcal{X}(s)\) to the point indexing it. Pre-composing with the right adjoint to hocolim gives us

\[
X \mapsto \left( [n] - S \mapsto \lim \left( X^{\Delta S} \longrightarrow \prod_{s \in S} X \right) \right)
\]

using that \(X^{\Delta 0} \cong X\), so the final element is unchanged, which is our \(R^n\). Note that this limit is actually a homotopy limit because the bottom map is a fibrant replacement of the diagonal. Then \(L^n\) is inclusion followed by hocolim.

If we allowed the final element, i.e. \(S = [n]\), this clearly yields \(X\). Then, for every singleton \(s \in S\), we get \(P_s X\), i.e. paths in \(X\) based at whatever point was chosen by the map of \(s\) into \(X\). If \(X\) is based already, these are all copies of the “normal” based path space, \(P_s X\). Then we have for \(S = \{i, j\}\), loop spaces.

For example, for \(X\) based and \([n] = \{0, 1\}\), \(R^1 X\) is \(P_1 X \leftarrow \Omega X \rightarrow P_s X\).

6. **LS Category and related corollaries**

The purpose of this section is to provide further discussion of the relationship between LS category and the constructions of Goodwillie calculus, including proofs and more details around the corollaries of Theorem 1.10.

6.1. **Versions and equivalence of the notions of category.** There are three major, equivalent notions of LS category for a space, those of Ganea, Whitehead and Hopkins. We have already mentioned a proof of this equivalence ([FHT01, §V Ch. 27]), but mention earlier work to emphasize what fails when trying to dualize.

Doeraene [Doe93, Theorem 3.11] was the first to show the equivalence of the Ganea and Whitehead definitions, which depended on showing that the objects of Whitehead’s definition (fat wedges) and the Ganea spaces (which we will define shortly) are nothing but particular types of joins. There is a homotopy pullback (where \(W_n\) is for \(n\)th Whitehead space)
and that this implies the equivalence of the two versions. The link between this homotopy pullback square above fact and Mather’s Cube theorem appears first in Doeraene’s thesis as an application of the “Join theorem” – that the join of homotopy pull backs is a homotopy pullback. See also [DEH13, Theorem 25]. The failure of dualizability of Mather’s Cube theorem is the obstruction to the equivalence of the definitions of LS cocategory.

Hopkins in [Hop84a] establishes the equivalence of Ganea’s and his notions. We reiterate a small part of the definitions and argument here.

6.1.1. Equivalence of Hopkins and Ganea definitions. Let $G_0 X = P X \to X$ and $G_{i+1} X$ is the mapping cone $G_i X \cup C F_i X$, with $F_i X \to G_i X$ the inclusion of the homotopy fiber of $G_i X \to X$. Then $F_i X$ is the join of $(i + 1)$ copies of $\Omega X$ and the fibration $G_i X \to X$ has the homotopy type of the $(i + 1)$-fold fiber join of the universal fibration over $X$. A space $X$ has inductive (Ganea) LS category $\leq n$ precisely when the map $G_n X \to X$ admits a section.

A homotopy covering of a space $X$ is a contravariant functor $F : \mathcal{P}(S) \to \text{Top}$ for some set $S$, with maps $p : \text{hocolim} F \to X$ and $s : X \to \text{hocolim} F$ such that $p \circ s$ is homotopy to the identity map of $X$. We say $X$ can be homotopy covered by a collection of maps $\{p_a : U_a \to X | a \in S\}$ if there is a homotopy covering $F : \mathcal{P}(S) \to \text{Top}$ together with homotopy equivalences $F\{a\} \simeq U_a$ such that of all $a$, the squares

\[ \begin{array}{ccc}
F\{a\} & \to & \text{hocolim} F \\
\downarrow & & \downarrow \\
U_a & \to & X
\end{array} \]

 commute up to homotopy. Then an equivalent formulation of Hopkins’ definition is [Hop84a, p.215] that $X$ is (symm) cat $\leq n$ precisely when $X$ can be homotopy covered by $(n + 1)$ points. Hopkins [Hop84a, p.217] explains that $X$ can be homotopy covered by $n$ copies of a fibration $p : E \to X$ if and only if the $n$-fold fiber join of $p$ admits a section. That is, taking the standard fibration $P X \to X$ (for $P X$ the contractible path space) for $p$, Hopkins’ definition is equivalent to Ganea’s.

6.2. Consequences and corollaries. Returning to our functor-calculus language and recalling that we defined $T^n F$ as $L^n R^n$ where $L^n, R^n$ are natural dualizations of $L_n, R_n$, we re-state Hopkins’s relevant definitions as

**Definition 6.9** For a space $X$, symmetric LS cat$(X) \leq n$ iff the natural map $T^n \| (X) \to X$ has a section (up to homotopy).
As we are defining $T^n$ this way, we did not strictly dualize Theorem 1.2. The related result (Theorem 1.10) – that this definition of $T^n$ leads to a $n$-co-excisive approximation – is left for the appendix.

However, given our above definition and Theorem 1.8, we have the following corollaries as consequences, which are the duals of those in Section 4.

Corollary [1.11] $T^nF$ are left and right $T^n1$-functors, as are the $P^nF$.

Proof. As with the analogous result [1.3], we point out that showing this for the $T^nF$ implies it for the $P^nF$ since a limit of spaces which have sections to the same space also has such sections.

This corollary follows immediately from combining the definition of $T^nF$ (Definition 1.8) with Proposition 2.6.

Given this corollary, as before, this extra structures implies that

Corollary [1.12] $T^nF$ naturally takes values in spaces of symmetric $LS\,cat \leq n$, as do the higher iterates $(T^n)^kF$.

This proof is the exact dual of the proof of Corollary 1.3, we leave it for the interested reader.

Corollary [1.13] The cup products of length $\geq n + 1$ vanish for $T^nF(X)$ and the higher iterates $(T^n)^kF$.

Proof. Similar to Whitehead length, we define cup-length to be the maximum length of non-trivial cup products (minus 1) in $H^*X$. Combining Corollary 1.12 with the following inequality ([BG61]),

\[
\text{cup-length } (X) \leq LS\,cat(X)
\]

we conclude our result.

Appendix A. Dual Goodwillie Calculus

In this appendix, we further develop the dual calculus theory, which includes proving Theorem 1.10, which states that $P^nF$, the holim of the iterated $T^nF$'s as we have defined them, is in fact the $n$-co-excisive approximation to a functor $F$ (when $F$ lands in Spectra).

We first point out that the original form of the dual calculus and results derived therefrom may be found in [McC01, Kuh04, BM04]. A dual tower has stages which naturally map to the functor. $K$-theory is natural to map out of, e.g. the trace map from $K$-theory to TC. McCarthy originally constructed the (algebraic/cotriple) Dual Calculus in the hopes that having an approximation to $K$ theory which mapped into it would be illuminating.

One may form the Eckmann-Hilton dual of Goodwillie’s calculus theory, switching cocartesian to cartesian everywhere. That is,
Definition A.1. For $F$ a homotopy endofunctor of spaces, $F$ is co-excisive if it takes homotopy pullbacks to homotopy pushouts and $n$-co-excisive if it takes strongly cartesian $(n + 1)$-cubes to cocartesian ones.

To prove Theorem 1.10, we will first establish the dual of a key lemma needed to show that the approximations do in fact take strongly cartesian cubes to cocartesian ones, and leave further development of this theory and its background to future work.

That is, we establish the counterpart of [Goo03, Lemma 1.9], which shows that the map $t^n : T^n F \to F$ factors through some co-cartesian cube. In [Goo03], Goodwillie combined the original Lemma with commutativity of finite pullbacks with filtered colimits to conclude that the construction producing $P_n F$ produces a homotopy limit cube from a strongly cocartesian $(n + 1)$-cube.

However, it is important for us that we cannot always commute finite pushouts with (co)filtered homotopy limits of spaces. We choose currently to resolve the issue of commuting finite pushouts with (co)filtered homotopy limits by restricting to functors landing in spectra if we need to consider $P^n F$. Then these approximations $P^n F$ do take strongly cartesian cubes to cocartesian ones, as we will show.

Recall that $T^n F$ is given by our dualization of the functors we use to decompose $T_n$. That is, we (in Definition 1.8) let $T^n F := L^n \circ F \circ R^n$ and $(T^n)^k F := (L^n)^k \circ F \circ (R^n)^k$.

Recall (from section 5) that $R^n (X)$ is

$$X \mapsto ([n] - S \mapsto \lim \left( X^{\Delta^S} \longrightarrow \prod_{s \in S} X \right))$$

and $L^n$ is the inclusion followed by hocolim.

The adjunctions are between these categories as follows:

$$R^n : \text{Fun}(\mathcal{S}^1 ([n]), \text{Top}) \xrightarrow{\text{inc}} \text{Fun}(\mathcal{S}^1 ([n]), \text{Top}) \xrightarrow{\text{hocolim}} \text{Top} : L^n$$

We then construct the $n$-co-excisive approximation,

$$P^n F(X) := \text{holim}(\cdots \to (T^n)^2 F(X) \to T^n F(X)).$$

To show that this is the $n$-co-excisive approximation to $F$, we first need to show that it takes strongly cartesian diagrams to cartesian ones (i.e. Theorem 1.10). We do this by dualizing the proof which was provided by Charles Rezk [Rez08] of Lemma 1.9 of [Goo03].

Proposition A.2. (Dual of [Goo03 Lemma 1.9]) Let $X$ be any strongly cartesian $(n+1)$-cube and $F$ be any homotopy functor. The map of cubes $t^n F(X) : T^n F(X) \to F(X)$ factors through some cocartesian cube.

which we use to show
Theorem 1.10. With our definitions as in 1.8 and for functors $F$ landing in Spectra, the functor given by

$$P^n F(X) := \operatorname{holim}(\cdots \to (T^n)^2 F(X) \to T^n F(X)),$$

is $n$-co-excisive. In the homotopy category, $p^n F : P^n F \to F$, induced by the map $t^n$ and its iterates, is the universal map to $F$ from an $n$-co-excisive functor.

A.1. Proof of Prop A.2, Dual of [Goo03, Lemma 1.9]. To prove Prop A.2, we first need some setup and a lemma.

Let $U \in [n] - S$ and

$$\mathcal{X}^U([n] - S) := \operatorname{holim}
\begin{pmatrix}
\prod_{u \in U} \mathcal{X}([n] - S - \{u\}) \\
\mathcal{X}([n] - S) \to \prod_{u \in U} \mathcal{X}([n] - S)
\end{pmatrix}$$

Lemma A.3. If $\mathcal{X}$ is already strongly cartesian, then $\mathcal{X}^U([n] - S) \simeq \mathcal{X}([n] - S - U)$.

Proof. Since strongly cartesian is a property of the sub-2-faces, we will show this for an arbitrary sub-2-face of $\mathcal{X}$. Let $U = \{u_1, u_2\}$. Strongly cartesian gives us that the following is a homotopy pullback square

\begin{center}
\begin{tikzcd}
\mathcal{X}([n] - S - \{u_1, u_2\}) = \operatorname{holim}
\begin{pmatrix}
\mathcal{X}([n] - S - \{u_1\}) \\
\mathcal{X}([n] - S - \{u_2\}) \to \mathcal{X}([n] - S)
\end{pmatrix}
\end{tikzcd}
\end{center}

**Figure 2.** Definition of $\mathcal{X}([n] - S - \{u_1, u_2\})$

where we take as model for the holim the space of maps from $\Delta < 1 \circ c_1$ into this diagram, which is the same model as we used previously in this paper for the holim of a punctured square.

We will show that $\mathcal{X}^{\{u_1, u_2\}}([n] - S) \simeq \mathcal{X}([n] - S - \{u_1, u_2\})$.

We have that

\begin{center}
\begin{tikzcd}
\mathcal{X}^{\{u_1, u_2\}}([n] - S) = \operatorname{holim}
\begin{pmatrix}
\mathcal{X}([n] - S - \{u_1\}) \times \mathcal{X}([n] - S - \{u_2\}) \\
\mathcal{X}([n] - S) \to \mathcal{X}([n] - S)_{(u_1)} \times \mathcal{X}([n] - S)_{(u_2)}
\end{pmatrix}
\end{tikzcd}
\end{center}

**Figure 3.** Definition of $\mathcal{X}^{\{u_1, u_2\}}([n] - S)$

where the horizontal map in Figure 3 is the diagonal and the vertical map is the inclusion of $\mathcal{X}([n] - S - \{u_i\})$ into the $\{u_i\}$-indexed copy of $\mathcal{X}([n] - S)$ (index denoted by subscript).
Let us examine a point in the homotopy pullback (defined in Figure 3), keeping in mind that a map into a product is determined by a map into each factor. An element of the homotopy pullback is the following data

\[ x \in \mathcal{X}([n] - S) \]
\[ (y, z) \in \mathcal{X}([n] - S - \{u_1\}) \times \mathcal{X}([n] - S - \{u_2\}) \]
\[ (y', z') = \text{img}(y, z) \text{ in } \mathcal{X}([n] - S)_{\{u_1\}} \times \mathcal{X}([n] - S)_{\{u_2\}} \]
\[ \gamma : I \to \mathcal{X}([n] - S)_{\{u_1\}} \times \mathcal{X}([n] - S)_{\{u_2\}} \]

where \( \gamma \) may be expressed as a path in each coordinate, \( \gamma = (\gamma_1, \gamma_2) \) such that

\[ \gamma_1 : I \to \mathcal{X}([n] - S)_{\{u_1\}} \quad \gamma_1(0) = y' \quad \gamma_1(1) = x \]
\[ \gamma_2 : I \to \mathcal{X}([n] - S)_{\{u_2\}} \quad \gamma_2(0) = x \quad \gamma_2(1) = z' \]

The point \( x \) was then effectively superfluous. Note that we now have \( \tilde{\gamma} = \gamma_1 \ast \gamma_2 \) between \( y' \) and \( z' \) in \( \mathcal{X}([n] - S) \), which gives the corresponding point in the homotopy limit given in Figure 2 \( \mathcal{X}([n] - S - \{u_1, u_2\}) \). There is a clear (up to homotopy) inverse to this process, and we have that the holims are the same. \( \square \)

**Proof of Lemma A.2.** Given Lemma A.3, we now point out

1. How the map \( t^nF \) factors through this cube:

   \[ \text{Rezk}[\text{Rez08}] \text{ observes that there is a natural map } \mathcal{X}_U(S) \to \mathcal{X}(S) \ast U = L_n(\mathcal{X}(S))(U) \text{ which induces the factorization} \]
   \[ t_nF(\mathcal{X}(S)) : F(\mathcal{X}(S)) \to \text{holim}_{U \in \mathcal{P}_0([n])} F(\mathcal{X}_U(S)) \to T_n F(\mathcal{X}(S)). \]
   The dual is a natural map \( R^n(\mathcal{X}([n] - S))(U) \to \mathcal{X}^U([n] - S) \), inducing a factorization
   \[ T^nF(\mathcal{X}([n] - S)) \to \text{hocollim}_{U \in \mathcal{P}_1([n])} F(\mathcal{X}^U([n] - S)) \to F(\mathcal{X}([n] - S)). \]

   We provide the map after recalling the two objects involved:

   \[ R^n(\mathcal{X}([n] - S))(U) = \text{holim} \left( \begin{array}{c}
   \mathcal{X}(n) \ast U \\
   \mathbb{U}
   \end{array} \right) \]

   and

   \[ \mathcal{X}^U([n] - S) = \text{holim} \left( \begin{array}{c}
   \prod_{u \in U} \mathcal{X}(n) - S - \{u\} \\
   \mathbb{U}
   \end{array} \right) \]
Comparing the diagrams, we note that \( X([n] - S) \to \prod_{u \in U} X([n] - S) \) is a fibrant replacement of the diagonal and factors naturally through \( X([n] - S) \) as a result.

The map \( * \to \prod_{u \in U} X([n] - S) \), as before, is the map to the basepoint. This factors through \( \prod_{u \in U} X([n] - S - \{u\}) \).

(2) Why this cube will be cocartesian:

We can consider \( X^U \) as two sub-cubes which differ by an element \( \{u\} \in U \), we have that the maps \( X([n] - U - \{u\}) \to X([n] - U) \) are isomorphisms; for nonempty \( U \), the cube is cocartesian.

\[ \square \]

A.2. Proof of Theorem 1.10. There are three parts of this. One, that \( P^n F \) is actually \( n \)-co-excisive. Then existence and uniqueness of a map which ‘co’ factors a map \( v : P \to F \) for \( P \) a random co-\( n \)-excisive functor.

A.2.1. Co-\( n \)-excisiveness. In [Goo03], the original Lemma [Goo03, Lemma 1.9] the counterpart of Prop A.2 was then combined with commutativity of finite pullbacks with filtered colimits to conclude that \( \text{hocolim}(T_n F \to T^2_n F \to \cdots) \) produced a homotopy limit cube from a strongly cocartesian \( (n + 1) \)-cube.

We cannot always commute finite pushouts with (co)filtered homotopy limits of spaces. Since our current aim is not a complete re-write of the dual calculus theory to endofunctors of spaces, we choose to resolve the issue of commuting finite pushouts with (co)filtered homotopy limits by restricting to functors landing in spectra.

Let \( X \) be a strongly cartesian \( (n + 1) \)-cube. By Prop A.2, each of the maps of

\[
\text{holim}(T^n F \to (T^n)^2 F \to \cdots)
\]

factors through some cocartesian cube. Then the functor \( P^n F \) is given by this sequential holim of cocartesian cubes. Since pushouts and pullbacks agree in spectra, we can commute finite pushouts with (co)filtered homotopy limits and conclude that the holim of cocartesian cubes is again cocartesian. That is, that \( P^n F \) is \( n \)-co-excisive.

A.2.2. Existence of a co-factorization. We follow the proof in [Goo03]. We first show uniqueness in a similar way. Let \( P \) be some \( n \)-co-excisive functor and \( P \to F \) a weak map (a zig zag of maps is a “weak map”; it is a map in the homotopy category). We then have a commutative square

\[
\begin{array}{ccc}
P^n F & \xrightarrow{p^n u} & P^n F \\
p^n P & \xrightarrow{u} & p^n F \\
P & \xrightarrow{u} & F 
\end{array}
\]

Due to \( n \)-co-excisiveness of \( P \), we get that \( p^n P \) is invertible as a weak map, giving us our (in the homotopy category) co-factorization of \( u \), re-writing the above square while taking into account this invertability:
A.2.3. Uniqueness of a cofactorization. We need to show that if $P$ is $n$-co-excisive, then a weak map $v : P \to P^nF$ is determined by the composition $p^nF \circ v$ (that is, comes from a weak map $P \to F$).

It suffices to show that in the following diagram of weak maps, those labeled with $\sim$ are in fact invertible

\[
\begin{array}{ccc}
P^n P & \xrightarrow{P^n v} & P^n P^n F \\
\downarrow & \sim & \downarrow \\
P^n P & \xrightarrow{p^n F} & P^n F
\end{array}
\]

for then, $v$ is determined by $P^n v$, which is determined by $P^n p^n F \circ P^n v = P^n (p^n F \circ v)$, which is clearly determined by $p^n F \circ v$.

Since $P$ and $P^n F$ are $n$-co-excisive, the vertical marked weak maps are invertible. For the remaining map $P^n (p^n F)$ to be an equivalence, it is sufficient for $P^n (t^n F)$ to be an equivalence, as $p^n$ is the map induced by taking the limit other the iterations of $t^n$). Then

\[
P^n F \xrightarrow{P^n (t^n F)} P^n T^n F := P^n (L^n F R^n)
\]

Using that we’re landing in spectra, since $L^n$ is a finite hocolim and in spectra, holims commute with finite hocolims, we can pull $P^n$ past $L^n$

\[
P^n (L^n F R^n) \simeq L^n P^n F R^n
\]

and as $P^n F$ is $n$-co-excisive, it takes the strongly cartesian cube that $R^n$ outputs to a cocartesian one. That is, the composition

\[
P^n F \xrightarrow{P^n (t^n F)} P^n T^n F := P^n (L^n F R^n) \simeq L^n P^n F R^n
\]

is an equivalence. That is, $P^n (t^n F)$ is an equivalence.

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