Collective fluctuations in networks of noisy components

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\textbf{Abstract.} Collective dynamics result from interactions among noisy dynamical components. Examples include heartbeats, circadian rhythms and various pattern formations. Because of noise in each component, collective dynamics inevitably involve fluctuations, which may crucially affect the functioning of the system. However, the relation between the fluctuations in isolated individual components and those in collective dynamics is not clear. Here, we study a linear dynamical system of networked components subjected to independent Gaussian noise and analytically show that the connectivity of networks determines the intensity of fluctuations in the collective dynamics. Remarkably, in general directed networks including scale-free networks, the fluctuations decrease more slowly with system size than the standard law stated by the central limit theorem. They even remain finite for a large system size when global directionality of the network exists. Moreover, such non-trivial behavior appears even in undirected networks when nonlinear dynamical systems are considered. We demonstrate it with a coupled oscillator system.

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1. Introduction

Understanding fluctuations in dynamically ordered states and physical objects, which consist of networks of interacting components, is an important issue in many disciplines ranging from biology to engineering. When each constituent component of a system is noisy due to, e.g., thermal fluctuations, it generally occurs that the entire system collectively fluctuates in time. Such collective fluctuations may be advantageous or disadvantageous for the functioning of the system depending on the situation. For example, a reduction in noise is likely to improve information processing in retinal neural networks [1]–[4]. The precision of biological circadian clocks [5]–[8] may be improved by a reduction in collective fluctuations (i.e. fluctuations in collective activities). On the other hand, maintaining a certain amount of fluctuations in an ordered state is advantageous for stochastic resonance [9] and Brownian motors [10].

Despite the relevance of collective fluctuations in a variety of systems, theoretical frameworks that formulate collective fluctuations are missing. The central limit theorem states that, if the dynamical order is simply the averaged activity of noisy components, the standard deviation of the collective fluctuation would decrease with the number $N$ of noisy components as $N^{-1/2}$. However, scaling is unclear in systems of interacting components. Clarifying the property of collective fluctuations in such systems will give us insights into the mechanisms and design principles underlying the regulation of noise in, for example, living organisms and chemical reactions, and also into possible ways to control fluctuations in collective dynamics.

In this study, we analyze an ensemble of components subjected to independent Gaussian noise that interact on general networks, including complex networks and regular lattices. We first consider a linear dynamical system, which can be regarded as a linearization of various systems, such as networks of periodic or chaotic oscillators [11, 12], the overdamped limit of elastic networks [13], a consensus problem treated in control theory [14]. We show that collective fluctuations are determined by the connectivity of networks. It turns out that the scaling $N^{-1/2}$ is the tight lower bound, which is obtained for undirected networks. General
directed networks yield a slower or non-vanishing decay of collective fluctuations with an increase in $N$. We then argue that such non-trivial behavior appears even in undirected networks when nonlinear systems are considered. In particular, we show that linearization of coupled nonlinear oscillator systems on undirected media yields linear dynamics on asymmetric networks, such that the slow decay of the collective fluctuation is relevant.

2. Model and analysis

Consider a network of $N$ components obeying

$$\dot{x}_i = \sum_{j=1}^{N} w_{ij}(x_j - x_i) + \sqrt{D_i} \xi_i(t) \quad (1 \leq i \leq N),$$  \hspace{1cm} (1)

where $x_i$ is the state (or the position) of the $i$th component, $\sqrt{D_i}$ is the intensity of noise, $\xi_i$ is the independent Gaussian (generally colored) noise, and $w_{ij}$ is the intensity of coupling and can be also regarded as originating from the Jacobian matrix of underlying nonlinear dynamical systems such as the coupled oscillator systems that we consider later. We allow negative weights and asymmetric coupling; $w_{ij}$ can be negative or different from $w_{ji}$. Equation (1) is a multivariate Ornstein–Uhlenbeck process [15, 16].

For convenience, we represent equation (1) as

$$\dot{x} = -L x + p,$$  \hspace{1cm} (2)

where $x \equiv (x_1 \ldots x_N)^\top$ ($\top$ denotes the transpose), $p \equiv (\sqrt{D_1} \xi_1 \ldots \sqrt{D_N} \xi_N)^\top$, and $L = (L_{ij})$ is the asymmetric Laplacian defined by $L_{ij} = \delta_{ij} \sum_{i' \neq i} w_{ii'} - (1 - \delta_{ij}) w_{ij}$ [17, 18]. $L$ always has a zero eigenvalue with the right eigenvector $u \equiv (1 \ldots 1)^\top$, i.e. $Lu = 0$. This eigenvector is associated with a global translational shift in state $x$ and corresponds to the fact that such a shift keeps equation (1) invariant. We assume the stability of the ordered state represented by $x_1 = \cdots = x_N$ in the absence of the noise (i.e. $D_i = 0$ for all $i$); the system relaxes to the ordered state from any initial condition. This is equivalent to assuming that the real parts of all the eigenvalues of $L$ are positive except for one zero eigenvalue, i.e. $0 \equiv \lambda_1 \leq \Re \lambda_2 \leq \cdots \leq \Re \lambda_N$. This is a non-trivial condition for general networks with negative weights. However, for networks with only non-negative weights, i.e. $w_{ij} \geq 0$ (1 $\leq i, j \leq N$), this property holds true when the network is strongly connected or all the nodes are reachable by a directed path from a single node [17, 19, 20].

We are concerned with collective fluctuations in dynamics given by equation (1). To quantify their intensity, we decompose $x$ as

$$x(t) = y(t) u + \rho(t),$$  \hspace{1cm} (3)

where $y(t)$ describes the one-dimensional component along $u$ and $\rho(t)$ is the $(N-1)$-dimensional remainder mode. Note that $y(t) = v x(t)$, where the row vector $v \equiv (v_1 \ldots v_N)$ is the left eigenvector of $L$ corresponding to the zero eigenvalue, i.e. $v L = 0$, and is normalized as $v u = 1$, i.e. $\sum_{i=1}^{N} v_i = 1$ (see appendix A for detailed descriptions). We call $y(t) u$ the collective mode. In the absence of noise, the dynamical equation for $y(t)$ is given by

$$\dot{y} = v \dot{x} = -v L x = 0.$$  \hspace{1cm} (4)

Therefore, $y(t)$ is a conserved quantity of the dynamics. The remainder mode $\rho(t)$ is associated with relative motions among the components. Because of the stability assumption, $\rho(t)$
asymptotically vanishes with characteristic time \((\text{Re} \lambda_2)^{-1}\). Therefore, all the values of \(x_i\) \((1 \leq i \leq N)\) eventually go to the same value \(y\) that is determined by the initial condition, i.e. \(y = vx(0)\).

In the presence of noise, we obtain

\[
\dot{y} = v\dot{x} = v(-Lx + p) = \nu \bar{p} = \sum_{i=1}^{N} v_i \sqrt{D_i} \xi_i(t).
\]

Because \(\xi_i\) is the independent Gaussian noise, this equation reduces to

\[
\dot{y}(t) = \left[ \sum_{i=1}^{N} v_i^2 D_i \right] \xi(t) \equiv \sigma \xi(t),
\]

where \(\xi(t)\) is the Gaussian noise having the same statistical property as that of each \(\xi_i(t)\). Thus, \(\nu(t)\) performs the Brownian motion with effective noise strength \(\sigma\) and is unbounded. The remainder mode \(\rho(t)\) fluctuates around zero because of its decaying nature. Therefore, the long-time behavior of \(x_i(t) = y(t) + \rho_i(t)\) is approximately described by a single variable \(y(t)\) for any \(i\). We denote as \(\sigma\), which depends on the structure of the network, the intensity of collective fluctuations. \(\sigma\) can be calculated for a given network.

In practice, the average activity of the population, \(\bar{x} \equiv \sum_{i=1}^{N} x_i / N\), but not the activity at individual nodes, may be observed. Because \(\bar{x} = y + \sum_{i=1}^{N} \rho_i / N\) and \(\sum_{i=1}^{N} \rho_i / N\) can be neglected in the long run, \(\sigma\) also characterizes the fluctuations of \(\bar{x}\).

3. Collective fluctuations in various networks

3.1. General properties

We assume for simplicity that \(D_i = 1\) \((1 \leq i \leq N)\) so that \(\sigma = \sqrt{\sum_{i=1}^{N} v_i^2}\). It is straightforward to extend the following results to the case of heterogeneous \(D_i\). The vector \(\nu\) is uniform, i.e. \(v_i = 1/N\) \((1 \leq i \leq N)\) if and only if \(k_i^{\text{in}} = k_i^{\text{out}}\) \((1 \leq i \leq N)\), where \(k_i^{\text{in}} \equiv \sum_{j=1}^{N} w_{ij}\) and \(k_i^{\text{out}} \equiv \sum_{j=1}^{N} w_{ji}\) are indegree and outdegree, respectively [18]. Undirected networks satisfy this condition. In this case, we obtain \(\sigma = N^{-1/2}\), which agrees with the central limit theorem. The normalization condition \(\sum_{i=1}^{N} v_i = 1\) guarantees that \(\sigma \geq N^{-1/2}\) for any \(\nu\). Therefore, undirected networks are the best for reducing collective fluctuations. In the case of directed or asymmetrically weighted networks, \(v_i\) is generally heterogeneous, and \(\sigma > N^{-1/2}\). We will show later that this is also the case for nonlinear systems on undirected networks. When the weight \(w_{ij}\) is non-negative for any \(i\) and \(j\), the Perron–Frobenius theorem guarantees that \(v_i\) is non-negative for all \(i\) [21]. In this case, we obtain

\[
\frac{1}{\sqrt{N}} \leq \sigma \leq 1.
\]

The case \(\sigma = 1\) is realized by a feedforward network, in which a certain component \(i_0\) has no inward connection (i.e. \(k_{i_0}^{\text{in}} = 0\)). Then, \(v_{i_0} = 1\) and \(v_i = 0\) for \(i \neq i_0\), which yields \(\sigma = 1\) irrespective of \(N\); the collective fluctuations are not reduced at all with an increase in \(N\). When negative weights are allowed, some elements of \(\nu\) may assume negative values. Then, \(\sigma\) may be larger than 1, in which case collective fluctuations are larger than individual noise.
Figure 1. Schematic diagram of (a) the directed scale-free network, (b) the directed chain and (c) the directed two-dimensional lattice. The numbers in part (b) indicate the indices of the nodes, while those in part (c) indicate the layer index.

We note that $\sigma^{-2}$ is the so-called inverse participation ratio \cite{22}. $\sigma^{-2}$ can be interpreted as the effective number of components that participate in collective activities; the remaining components are slaved.

3.2. Directed scale-free networks

We demonstrate our theory by using some example networks. First, we consider directed scale-free networks, schematically shown in figure 1(a) in which $k_{in}^i$ and $k_{out}^i$ independently follow the distributions $p(k_{in}) \propto k_{in}^{-\gamma_{in}}$ and $p(k_{out}) \propto k_{out}^{-\gamma_{out}}$, respectively. By assuming that the values of $v_i$ of adjacent nodes are independent of each other, we obtain

$$\sum_{j=1}^{N} w_{ji} v_j \approx \sum_{j=1}^{N} w_{ji} \bar{v}_{i} = k_{out}^i \bar{v},$$

(8)

where

$$\bar{v} \equiv \frac{\sum_{i=1}^{N} v_i}{N} = \frac{1}{N},$$

(9)
Therefore,
\[ v_i = \frac{\sum_{j=1}^{N} w_{ji} v_j}{\sum_{j=1}^{N} w_{ij}} \approx \frac{k_i^{\text{out}}/k_i^{\text{in}}}{\sum_{j=1}^{N} (k_j^{\text{out}}/k_j^{\text{in}})}. \]  
(10)

This approximation is sufficiently accurate for uncorrelated networks [18]. For \( p(k^{\text{in}}) \propto k^{-\gamma^{\text{in}}} \) and \( p(k^{\text{out}}) \propto k^{-\gamma^{\text{out}}} \), we obtain
\[ v_i \approx \frac{k_i^{\text{out}}/k_i^{\text{in}}}{N \langle k^{\text{out}} \rangle \langle (k^{\text{in}})^{-1} \rangle} \]  
(11)

and
\[ \sigma \approx \frac{\langle (k^{\text{out}})^2 \rangle \langle (k^{\text{in}})^{-2} \rangle}{N \langle k^{\text{out}} \rangle^2 \langle (k^{\text{in}})^{-1} \rangle^2}, \]  
(12)

where \( \langle \cdot \rangle \) is the ensemble average. When \( \gamma^{\text{out}} < 2 \), a winner-take-all network is generated [23, 24], and there exists a node \( i \) such that \( k_i^{\text{out}} = O(N) \) and \( v_i = O(1) \). When \( \gamma^{\text{out}} \geq 2 \), the extremal criterion results in the maximum degree increasing with \( N \) as \( N^{1/(\gamma^{\text{out}}-1)} \) (\( \gamma^{\text{out}} \geq 2 \)) in many networks [24, 25]. Then, we obtain [26]

\[ \langle k^{\text{out}} \rangle \propto \begin{cases} N^{2-\gamma^{\text{out}}}, & (\gamma^{\text{out}} < 2), \\ \ln N, & (\gamma^{\text{out}} = 2), \\ O(1), & (\gamma^{\text{out}} > 2), \end{cases} \]  
(13)

\[ \langle (k^{\text{out}})^2 \rangle \propto \begin{cases} N^{-\gamma^{\text{out}}+3}, & (\gamma^{\text{out}} < 2), \\ N^{(-\gamma^{\text{out}}+3)/(\gamma^{\text{out}}-1)}, & (2 \leq \gamma^{\text{out}} < 3), \\ \ln N, & (\gamma^{\text{out}} = 3), \\ O(1), & (\gamma^{\text{out}} > 3), \end{cases} \]  
(14)

and
\[ \langle (k^{\text{in}})^{-1} \rangle, \langle (k^{\text{in}})^{-2} \rangle = O(1). \]  
(15)

Therefore, we obtain
\[ \sigma \propto \begin{cases} 1, & (\gamma^{\text{out}} < 2), \\ 1/\ln N, & (\gamma^{\text{out}} = 2), \\ N^{-1+(\gamma^{\text{out}}-1)^{-1}}, & (2 \leq \gamma^{\text{out}} < 3), \\ N^{-1/2}(\ln N)^{1/2}, & (\gamma^{\text{out}} = 3), \\ N^{-1/2}, & (\gamma^{\text{out}} > 3). \end{cases} \]  
(16)

The fairly heterogeneous case \( \gamma^{\text{out}} < 2 \), in which the average outdegree diverges as \( N \to \infty \), effectively yields a feedforward network. The case \( \gamma^{\text{out}} \geq 3 \), where the second moment of the outdegree converges for \( N \to \infty \), reproduces the central limit theorem. The latter result is shared by the directed version of the conventional random graph. The case \( 2 \leq \gamma^{\text{out}} < 3 \) yields a non-trivial dependence of \( \sigma \) on \( N \). Figure 2(a), we compare the scaling exponent \( \beta \), where \( \sigma \propto N^{-\beta} \), obtained from the theory (solid line; equation (16)) and numerical simulations of the configuration model [23, 27] with the power-law degree distribution with minimum degree 3 (open circles). The fitting procedure is explained in figure 2(b). Equation (16) roughly explains numerically obtained values of \( \beta \).
3.3. Directed lattices

The second example is the directed one-dimensional chain of $N$ nodes depicted in figure 1(b). We set $w_{i+1,i} = 1$ ($1 \leq i \leq N - 1$), $w_{i-1,i} = \epsilon$ ($2 \leq i \leq N$) and $w_{j,i} = 0$ ($j \neq i - 1, i + 1$). For this network, by solving $vL = 0$, we analytically obtain

$$v_i = \frac{(1 - \epsilon)\epsilon^{-1}}{1 - \epsilon^N} \quad (1 \leq i \leq N)$$

(17)

and

$$\sigma = \sqrt{\frac{1 - \epsilon}{1 + \epsilon} \frac{1 + \epsilon^N}{1 - \epsilon^N}}.$$  

(18)

The values of $\sigma$ for various $\epsilon$ and $N$ are plotted by solid lines in figure 3(a). Interestingly, for $\epsilon \neq 1$, $\lim_{N \to \infty} \sigma = (1 - \epsilon)/(1 + \epsilon)$; $\sigma$ is non-vanishing. We have also analytically derived $\sigma$ for directed $d$-dimensional lattices (see appendix B). The results for the two-dimensional lattice depicted in figure 1(c) are plotted by solid lines in figure 3(b). To confirm our theory, we also carried out direct numerical simulations of equation (1) with Gaussian white noise for these directed lattices. The results indicated by circles in figure 3 indicate excellent agreement with our theory.

A similar result is obtained for the Cayley tree (see appendix C).
Figure 3. Collective fluctuations for (a) the directed one-dimensional chain and (b) the directed two-dimensional lattice for various $N$ and $\epsilon$. The solid lines and the circles represent the theoretical and the numerical results, respectively. In (b), $l_{\text{max}}$ is the maximum distance from the center of the lattice. For both networks, we set $D_i = 1$ $(1 \leq i \leq N)$ and simulate equation (1) with the initial condition $x_i(t = 0) = 0$ $(1 \leq i \leq N)$. We measured $\sigma$ as the standard deviation of $\bar{x}(t = 10\,200) - \bar{x}(t = 200)$ obtained by conducting 2000 trials, which is then normalized by $\sqrt{10\,000}$. We disregard the first 200 time units as transient.

4. Oscillator dynamics

As an application of our theory to nonlinear systems, we examine noisy and rhythmic components. As a general, tractable, yet realistic model, we consider a network of phase oscillators [11, 28, 29], whose dynamical equation is given by

$$\dot{\phi}_i = \omega_i + \sum_{j=1}^{N} A_{ij} f(\phi_j - \phi_i) + \sqrt{D_i} \xi_i(t) \quad (1 \leq i \leq N),$$

where $\phi_i \in [0, 2\pi)$ and $\omega_i$ are the phase and the intrinsic frequency of the $i$th oscillator, respectively, $A_{ij}$ is the intensity of coupling and $f(\cdot)$ is a $2\pi$-periodic function. We assume that, in the absence of noise, all the oscillators are in a fully phase-locked state, i.e. $\phi_i(t) = \Omega t + \psi_i$, where $\Omega$ and $\psi_i$ are the constants derived from $\dot{\phi}_i = \Omega$ $(1 \leq i \leq N)$. Under sufficiently weak noise, we can linearize equation (19) around the phase-locked state. Letting $x_i = \phi_i - (\psi_i + \Omega t)$, we obtain equation (1), where $w_{ij} = A_{ij} f'(\psi_j - \psi_i)$ is the effective weight. The validity of linearizing equation (19) for small noise intensity is tested by carrying out direct numerical simulations of equation (19) with $\omega_i = \omega$ $(1 \leq i \leq N)$ and $f(\phi) = \sin \phi$. The relationship $\sigma \approx \sqrt{\sum_{i=1}^{N} D_i v_i^2}$ is satisfied in the directed one- and two-dimensional lattices, as shown in figures 4(a) and (b), respectively.

When there is some dispersion in $\psi_i$ in a phase-locked state, the relation $\sigma \approx N^{-1/2}$ may be violated even in undirected networks. This is because the effective weight is generally asymmetric (i.e. $w_{ij} \neq w_{ji}$) unless $f(\cdot)$ is an exact odd function. In reality, $f(\cdot)$ is usually not an odd function [11, 29--31]. As an example, we consider target patterns (i.e. concentric traveling waves), which naturally appear in spatially extended oscillator systems [11, 32]. We carry out direct numerical simulations of equation (19) on the two-dimensional undirected lattice with linear length $\sqrt{N} = 50$, $f(\phi) = \sin(\phi - \alpha) + \sin \alpha$ and $\alpha = \pi/4$. Such a function may be analytically derived from a general class of coupled oscillators [11], and it approximates

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Figure 4. Collective fluctuation for coupled phase oscillators in (a) the directed one-dimensional chain and (b) the directed two-dimensional lattice. The solid lines represent the theoretical results, and the circles represent the numerical results obtained by the direct numerical simulations of equation (19) with $f(\phi) = \sin \phi$. We set $D_i = 0.01$ ($1 \leqslant i \leqslant N$) and start with $x_i = 0$ ($1 \leqslant i \leqslant N$). We measure $\sigma$ as the standard deviation of $\bar{x}(t = 10000) - \bar{x}(t = 200)$ obtained by conducting 2000 trials, which is then normalized by $\sqrt{10000}$.

A variety of real systems [29, 31, 33]. We set $\omega_i = \omega_0 + \Delta \omega$ ($\Delta \omega \geqslant 0$) for $4 \times 4$ pacemaker oscillators in the center and $\omega_i = \omega_0$ for the other oscillators, where $\omega_0$ is arbitrary and is set to 1. A target pattern is formed when there is sufficient heterogeneity in the intrinsic frequency [11]. A region with high intrinsic frequency acts as a pacemaker. A snapshot for $\Delta \omega = 0.3$ is shown in figure 5(a). As observed, the radial phase gradient is approximately constant, which makes the effective network similar to the directed two-dimensional lattice depicted in figure 1(c). Therefore, as shown in figure 5(b), $v_i$ calculated numerically decreases almost exponentially with the distance from the center. We find that the dependence of $\sigma$ on $N$, shown in figure 5(c), is similar to that for directed lattices.

We emphasize that the network is undirected (i.e. $A_{ij} = A_{ji}$). We have also theoretically confirmed that our results are valid for the continuous oscillatory media under spatial block noise, which models chemical reaction–diffusion systems (see appendix D).

5. Conclusions

In summary, we have obtained the analytical relationship between collective fluctuations and the structure of networks. In undirected networks, the fluctuations decrease with the system size $N$ as $N^{-1/2}$; this result agrees with the central limit theorem. In general directed networks, the collective fluctuations decay more slowly. For example, in directed scale-free networks, we obtain $N^{-\beta}$ with $0 < \beta < 1/2$. In networks with global directionality, the fluctuations do not vanish for a large system size. We have also demonstrated that such non-trivial dependence appears even in undirected networks when nonlinear systems are considered. We have focused on systems of non-leaky components. The results for coupled leaky components will be reported elsewhere.

Our results are distinct from earlier results demonstrating the breach of the central limit theorem due to heavy-tailed noise [34] or the correlation between the noise in different elements [35, 36].

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Numerical results for the coupled oscillators on the two-dimensional undirected lattice. (a) Snapshot of \( \sin \phi_i \) and (b) eigenvector \( v \) (log scale) for \( \Delta \omega = 0.3 \), where \( r_1 \) and \( r_2 \) denote the spatial coordinates. (c) The dependence of \( \sigma \) on system size \( N \).

Finally, because our theory is based on a general linear model, it can be tested in a variety of experimental systems. An ideal experimental protocol is provided by photo-sensitive Belousov–Zhabotinsky reaction systems, in which the heterogeneity, noise intensity and system size can be precisely controlled by light stimuli [32]. Experiments with coupled oscillatory cells, such as cardiac cells and neurons under an appropriate condition, would also be interesting.

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Appendix A. Derivation of the collective mode

To derive the collective mode \( y(t)u \), we note that there exists a non-singular matrix \( P \) such that \( \tilde{L} \equiv P^{-1}LP \) is its Jordan canonical form [17, 21]. We assume that \( \tilde{L}_{11} = \lambda_1 = 0 \) and \( \tilde{L}_{ii} = \tilde{L}_{ji} = 0 \) (2 \( \leq i \leq N \)) without loss of generality. The submatrix \( (\tilde{L}_{ij}) \) (2 \( \leq i, j \leq N \)) corresponds to the \( N - 1 \) modes with the eigenvalues \( \lambda_2, \ldots, \lambda_N \). Because the first column of \( LP = P\tilde{L} \) is equal to \( (0 \ldots 0)^{\top} \), the first column of \( P \) is equal to the right eigenvector of \( L \) corresponding to \( \lambda_1 = 0 \), i.e. \( u = (1 \ldots 1)^{\top} \). Because the first row of \( P^{-1}L = \tilde{L}P^{-1} \) is equal to \( (0 \ldots 0) \), the first row of \( P^{-1} \) is equal to the left eigenvector of \( L \) corresponding to \( \lambda_1 = 0 \),

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Similarly, we obtain undirected networks, the central limit theorem is recovered, i.e. \( \lim_{\epsilon \to 0} \sigma = 1 \). In the case of undirected networks, the central limit theorem is recovered, i.e. \( \lim_{\epsilon \to 1} \sigma = N^{1/2} \). In the limit of infinite space, we obtain
\[
\lim_{\ell_{\text{max}} \to \infty} \sigma = \frac{(1 - \epsilon)(1 + \epsilon^2)}{(1 + \epsilon)^3}.
\]
For a general dimension $d$, layer 0 has a single root node, and layer $\ell$ ($1 \leq \ell \leq \ell_{\max}$) has
\[ N^{(d)}_{\ell} \equiv \sum_{d'=1}^{\ell} \frac{d!}{d'!(d-d')!} \frac{(\ell-1)!}{(d'-1)!(\ell-d')!} 2^{d'} \]
nodes. $d'$ is the number of coordinates among the $d$ coordinates to which non-zero values are assigned, and the factor $2^{d'}$ takes care of the fact that reversing the sign of any coordinate does not change the layer of the node. Similar to the case of the two-dimensional lattice, the value of $v_i$ for any node in layer $\ell$ in a $d$-dimensional lattice, denoted by $v^{(d)}_{\ell}$, is given by
\[ v^{(d)}_{\ell} = [T^{(d)}_{\ell_{\max}}(\epsilon)]^{-1} \epsilon^\ell \quad (0 \leq \ell \leq \ell_{\max}), \quad (B.5) \]
where
\[ T^{(d)}_{\ell_{\max}}(z) = 1 + \sum_{\ell=1}^{\ell_{\max}} N^{(d)}_{\ell} z^\ell. \quad (B.6) \]
From equation (B.5), we obtain
\[ \sigma = \sqrt{\frac{T^{(d)}_{\ell_{\max}}(\epsilon^2)}{T^{(d)}_{\ell_{\max}}(\epsilon)}}. \quad (B.7) \]
Note that $\lim_{\epsilon \to 0} \sigma = 1$ and $\lim_{\epsilon \to 1} \sigma = N^{-1/2}$. In the limit $\ell_{\max} \to \infty$, equation (B.6) becomes
\[
\lim_{\ell_{\max} \to \infty} T^{(d)}_{\ell_{\max}}(z) = 1 + \sum_{d'=1}^{\infty} \frac{d!}{d'!(d-d')!} 2^{d'} \sum_{\ell=d}^{\infty} \frac{(\ell-1)!}{(d'-1)!(\ell-d')!} z^\ell \\
= 1 + \sum_{d'=1}^{\infty} \frac{d!}{d'!(d-d')!} 2^{d'} \left( \frac{z}{1-z} \right)^{d'} \\
= \left( \frac{1+z}{1-z} \right)^d. \quad (B.8)
\]
Substituting equation (B.8) into equation (B.7) yields
\[ \lim_{\ell_{\max} \to \infty} \sigma = \left[ \frac{(1-\epsilon)(1+\epsilon^2)}{(1+\epsilon)^3} \right]^{d/2}. \quad (B.9) \]

**Appendix C. Collective fluctuations in the Cayley tree**

Consider a Cayley tree with degree $k$ and a specific root node. We assume that the maximum distance from the root node is equal to $\ell_{\max}$. The edges descending from the root node and those approaching the root node are assigned weight 1 and $\epsilon$, respectively. The exact value of $v_i$ in layer $\ell$, denoted by $v_{\ell}$ without confusion, is obtained via
\[ [1 + (k-1) \epsilon] v_{\ell} = \epsilon v_{\ell-1} + (k-1) v_{\ell+1} \quad (\ell \geq 1). \quad (C.1) \]
By solving equation (C.1), we obtain
\[ v_{\ell} = \frac{1 - (\epsilon k)}{1 - (\epsilon k)^{\ell_{\max}+1}} \epsilon^\ell \quad (0 \leq \ell \leq \ell_{\max}). \quad (C.2) \]
From equation (C.2), we obtain
\[ \sigma = \frac{1 - (ek)}{1 - (ek)^{\ell_{\text{max}}+1}} \sqrt{\frac{1 - (e^2k)^{\ell_{\text{max}}+1}}{1 - (e^2k)}}. \]  
(C.3)

Note that \( \lim_{\epsilon \to 0} \sigma = 1 \) and \( \lim_{\epsilon \to 1} \sigma = \sqrt{(1 - k)/(1 - k^{\ell_{\text{max}}+1})} = N^{-1/2} \). The infinite-size limit exists only when \( \epsilon < 1 \) and it is equal to
\[ \lim_{\ell_{\text{max}} \to \infty} \sigma = \frac{1 - \epsilon k}{\sqrt{1 - e^2k}}. \]  
(C.4)

Appendix D. Target patterns in continuous media under spatial block noise

We show that our results for the coupled oscillator system in the \( d \)-dimensional lattice are also valid for that in the continuous Euclidean space. We assume that Gaussian spatial block noise is applied. This type of noise has been used in experiments [32].

We consider the \( d \)-dimensional nonlinear phase diffusion equation given by
\[ \partial_t \phi(r, t) = \omega + v \nabla^2 \phi + \mu (\nabla \phi)^2 + s(r), \]  
(D.1)
where \( r \in \mathbb{R}^d \) is the spatial coordinate, \( \omega > 0 \) is the intrinsic frequency, \( v > 0 \) is the diffusion constant and \( \mu > 0 \) is the coefficient of the nonlinear term [11]. The term \( s(r) \) represents the localized heterogeneity, which is positive near the origin and vanishing otherwise.

The synchronous solution corresponding to the target pattern is written as \( \phi(r, t) = \Omega t + \psi(r) \), where \( \Omega \) and \( \psi(r) \) satisfy
\[ \Omega = \omega + v \nabla^2 \psi + \mu (\nabla \psi)^2 + s(r). \]  
(D.2)
Let \( x(r, t) \) be a small deviation from the target pattern defined by \( x \equiv \phi - (\Omega t + \psi) \). Linearizing equation (D.1) using \( x(r, t) \), we obtain \( \partial_t x(r, t) = \mathcal{L} x \), where the linear operator \( \mathcal{L} \) is given by
\[ \mathcal{L} x = v \nabla^2 x + 2 \mu (\nabla \psi) \cdot (\nabla x). \]  
(D.3)

We define the inner product as
\[ [x_1(r), x_2(r)] = \int dr \ x_1(r) x_2(r). \]  
(D.4)
We define the adjoint operator \( \mathcal{L}^\dagger \) as \( [x_1, \mathcal{L} x_2] = [\mathcal{L}^\dagger x_1, x_2] \), that is,
\[ \mathcal{L}^\dagger x = v \nabla^2 x - 2 \mu \nabla \cdot (x \nabla \psi). \]  
(D.5)
Note that \( \mathcal{L} \) is self-adjoint when \( \nabla \psi = 0 \).

Because of the translational symmetry in equation (D.1) with respect to \( \phi \), \( \mathcal{L} \) has one zero eigenvalue. Let the right and left eigenfunctions of \( \mathcal{L} \) corresponding to the zero eigenvalue be \( u(r) \) and \( v(r) \), respectively, i.e. \( \mathcal{L} u = 0 \) and \( \mathcal{L}^\dagger v = 0 \). Trivially, \( u(r) = 1 \). The normalization condition \( [v(r), u(r)] = 1 \) then implies that \( \int dr \ v(r) = 1 \).

Now, we introduce the perturbation to equation (D.1) as follows:
\[ \partial_t \phi(r, t) = \omega + v \nabla^2 \phi + \mu (\nabla \phi)^2 + s(r) + \sqrt{D} \xi(r, t), \]  
(D.6)
where \( \xi(r, t) \) represents a weak perturbation to the target pattern. Similarly to equation (3), we decompose \( x \) into
\[ x(r, t) = y(t) u(r) + \rho(r, t), \]  
(D.7)
where \( y(t)u(r) \) is the collective mode. The dynamical equation for \( y \) is then obtained as

\[
\dot{y} = \sqrt{D} \int dr \, v(r) \xi(r, t). \tag{D.8}
\]

Let us assume that \( \xi(r, t) \) is the Gaussian spatial block noise characterized by

\[
\xi(r, t) = \xi_\ell(t), \quad r \in \mathbb{R}^d(\ell), \tag{D.9}
\]

\[
\langle \xi_\ell(t) \xi_{\ell'}(t') \rangle = \delta_{\ell,\ell'}C(|t-t'|), \tag{D.10}
\]

where \( \ell \) is the vector index for the block \( \mathbb{R}^d(\ell) \). Using equation (D.9), equation (D.8) is transformed into

\[
\dot{y} = \sqrt{D} \sum_\ell v_\ell \xi_\ell(t), \tag{D.11}
\]

where

\[
v_\ell = \int_{\mathbb{R}^d(\ell)} dr \, v(r). \tag{D.12}
\]

From equation (D.11), we find that the intensity of the collective fluctuation is given by

\[
\sigma = \sqrt{D \sum_\ell v_\ell^2}. \tag{D.13}
\]

Note that \( v_\ell \) satisfies the normalization condition as follows:

\[
\sum_\ell v_\ell = \sum_\ell \int_{\mathbb{R}^d(\ell)} dr \, v(r) = \int dr \, v(r) = 1. \tag{D.14}
\]

References

[1] Lamb T D and Simon E J 1976 J. Physiol. (London) 263 257
[2] Smith R G and Vardi N 1995 Vis. Neurosci. 12 851
[3] DeVries S H, Qi X, Smith R, Makous W and Sterling P 2002 Curr. Biol. 12 1900
[4] Bloomfield S A and Volgyi B 2004 Vis. Res. 44 3297
[5] Wilders R and Jongsma H J 1993 Biophys. J. 65 2601
[6] Enright J T 1980 Science 209 1542
[7] Garcia-Ojalvo J, Elowitz M B and Strogatz S H 2004 Proc. Natl Acad. Sci. USA 101 10955
[8] Herzog E D, Aton S J, Numano R, Sakaki Y and Tei H 2004 J. Biol. Rhythms 19 35
[9] Gammaitoni L, Hänggi P, Jung P and Marchesoni F 1998 Rev. Mod. Phys. 70 223
[10] Reimann P 2002 Phys. Rep. 361 57
[11] Kuramoto Y 1984 Chemical Oscillations, Waves, and Turbulence (New York: Springer)
[12] Pikovsky A, Rosenblum M and Kurths J 2001 Synchronization: A Universal Concept in Nonlinear Sciences (Cambridge: Cambridge University Press)
[13] Togashi Y and Mikhailov A 2007 Proc. Natl Acad. Sci. USA 104 8697
[14] Olfati-Saber R, Fax J A and Murray R M 2007 Proc. IEEE 95 215
[15] Risken H 1989 The Fokker–Planck Equation 2nd edn (Berlin: Springer)
[16] Van Kampen N G 2007 Stochastic Processes in Physics and Chemistry 3rd edn (Amsterdam: Elsevier)
[17] Arenas A, Díaz-Guilera A, Kurths J, Moreno Y and Zhou C 2008 Phys. Rep. 469 93
[18] Masuda N, Kawamura Y and Kori H 2009 New J. Phys. 3 113002
[19] Ermentrout G B 1992 SIAM J. Appl. Math. 52 1665

New Journal of Physics 12 (2010) 093007 (http://www.njp.org/)
[20] Agaev R P and Chebotarev P Y 2000 Autom. Remote Control 61 1424
[21] Horn R A and Johnson C R 1985 Matrix Analysis (Cambridge: Cambridge University Press)
[22] Derrida B and Flyvbjerg H 1987 J. Phys. A: Math. Gen. 20 5273
[23] Albert R and Barabási A-L 2002 Rev. Mod. Phys. 74 47
[24] Dorogovtsev S N, Goltsev A V and Mendes J F F 2008 Rev. Mod. Phys. 80 1275
[25] Newman M E J 2005 Contemp. Phys. 46 323
[26] Sood V, Antal T and Redner S 2008 Phys. Rev. E 77 041121
[27] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D-U 2006 Phys. Rep. 424 175
[28] Winfree A T 1967 J. Theor. Biol. 16 15
[29] Kiss I Z, Rusin C G, Kori H and Hudson J L 2007 Science 316 1886
[30] Brown E, Moehlis J and Holmes P 2004 Neural Comput. 16 673
[31] Galán R F, Ermentrout G B and Urban N N 2005 Phys. Rev. Lett. 94 158101
[32] Mikhailov A S and Showalter K 2006 Phys. Rep. 425 79
[33] Tsubo Y, Takada M, Reyes A D and Fukai T 2007 Eur. J. Neurosci. 25 3429
[34] Bouchaud J P and Georges A 1990 Phys. Rep. 195 127
[35] Kaneko K 1990 Phys. Rev. Lett. 65 1391
[36] Zohary E, Shadlen M N and Newsome W T 1994 Nature 370 140