Adiabatic transfer of amplitude using STIRAP-like protocols generalizes to many bipartite graphs

Koen Groenland,1,2,∗ Carla Groenland,3 and Reinier Kramer4
1QuSoft, CWI, Science Park 123, 1098 XG Amsterdam, the Netherlands
2Institute of Physics, University of Amsterdam, Science Park 904, 1098 XH Amsterdam, the Netherlands
3Mathematical Institute, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter (550), Woodstock Road, Oxford OX2 6GG, United Kingdom
4Korteweg-de Vries Institute, University of Amsterdam, Science Park 107, P.O. Box 94248, 1090 GE Amsterdam, the Netherlands

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Adiabatic passage techniques, used to drive a system from one quantum state into another, find widespread application in physics and chemistry. We focus on the techniques called STImulated Raman Adiabatic Passage (STIRAP) and Coherent Tunnelling by Adiabatic Passage (CTAP), which employ a unique zero-energy eigenstate to transfer amplitude between ends of a linear chain. We find that many more general physical systems can use the same protocol, namely those with a (semi-)bipartite interaction graph which allows a perfect matching both when the sender is removed and when the receiver is removed. Many of the favorable stability properties of STIRAP/CTAP are inherited. We numerically test transfer between the leaves of a tree, and find surprisingly accurate transfer, especially when straddling is used. Our results open up new possibilities for coherent control and quantum state transfer in more general systems, and show that conventional STIRAP/CTAP is resilient to a large class of perturbations.

CONTENTS

I. Introduction 1

II. Conventional STIRAP 2

III. Generalizing STIRAP 2

IV. Applications 4

V. Simulations 4

VI. Conclusion 5

VII. Acknowledgments 5

References 5

i. Proofs of claims made in the main text 7
1. Proof of equivalence between 2a and 2b in Thm. 1 7
2. Dangling vertices do not affect the the nullity 8
3. Graphs with certain matchings make our protocol work almost surely 8

ii. Robustness guarantees 9

iii. On the eigenvalue gap in generalized STIRAP 10
1. Interlaced eigenvalues 10
2. Numerical scaling 10

∗ Corresponding author, K.L.Groenland@cwi.nl

I. INTRODUCTION

Originally conceived as a scheme to transfer amplitude between the low-lying states of a Λ system, STImulated Raman Adiabatic Passage (STIRAP) [1] has proven to provide practical and high fidelity state manipulation due to the inherent stability to experimental imperfections [2]. It is now widely adopted in fields where accurate control of quantum states is vital, such as high precision measurement [3, 4], studies of atoms and molecules [5–9], and quantum information processing [10–14]. A closely related protocol, Coherent Tunnelling by Adiabatic Passage (CTAP) [15], is mathematically equivalent, but was proposed in the context of spatial transport of particles, such as electrons through nearby quantum dots or ultracold atoms through optical wells [16, 17].

With the advent of quantum information processing, accurate control in increasingly large systems has become an important scientific challenge [18]. The majority of work on CTAP and STIRAP focuses on either a small number of sites (typically 3) [2, 16], or extensions to larger systems that form a linear chain [15, 19]. Some notable exceptions are Refs. [20–24] which address different geometries, such as two- and even higher dimensional grids or branching trees. These protocols typically work through a clever mapping of such systems to the well-understood case of the 1D chain.

We present a completely different approach to find more general configurations that allow transfer protocols, by describing a system’s interactions in the language of graphs: the vertices represent basis states and edges represent interactions. We look at bipartite interactions, where the basis states can be separated into two sets, such that each state interacts only with states outside its own set. If the two sets differ in size by one, then
amplitude transfer between states in the bigger set may be possible. We can guarantee successful transfer when certain graph properties are satisfied, as made precise in Thm. 1.

Interestingly, our treatment naturally provides a means to transfer amplitude to one out of multiple potential receivers, generalizing Ref. [22]. We find that the final state need not yet be known when starting the protocol, which could be an advantage in quantum information processing.

Our results advance the fields of STIRAP and CTAP in two ways. Firstly, they open the way to practical adiabatic passage in more general systems. Secondly, they shed light on possible perturbations in conventional STIRAP and their effect: we find that many perturbations, as long as they satisfy our assumptions, do not cause a qualitatively different effect on the state’s evolution during the protocol.

Our work is closely related the field of perfect state transfer (PST), which addresses the same goal of transfer between two states $|a\rangle,|b\rangle$ in general graphs [25]. However, PST is concerned with quenches such that $|b|e^{-iHt}|a\rangle = 1$ for a time-independent Hamiltonian $H$. Therefore, PST is typically faster than adiabatic transfer, but it puts stringent constraints on the precise interaction strengths.

II. CONVENTIONAL STIRAP

The conventional protocol (Fig. 1) deals with a three-dimensional quantum system, consisting of eigenstates \( \{|j\rangle\}_{j=1}^{3} \) of some background Hamiltonian. To transfer amplitude from \(|1\rangle\) to \(|3\rangle\), a sequence of two laser pulses is applied: the Stokes pulse coupling \(|2\rangle \leftrightarrow |3\rangle\), and the Pump pulse coupling \(|1\rangle \leftrightarrow |2\rangle\). Throughout this work, we consider only the interaction picture and assume the rotating wave approximation to hold. The system’s Hamiltonian then becomes

\[
H(t) = \begin{pmatrix}
0 & \Omega_P(t) & 0 \\
\Omega_P(t) & \varepsilon & \Omega_S(t) \\
0 & \Omega_S(t) & 0
\end{pmatrix}.
\]

Here, \( \Omega_{S/P} \) denotes the Rabi frequency (amplitude) of the Stokes and Pump lasers, respectively, and \( \varepsilon \) absorbs the off-resonances, assuming both are equal in size. One can check that one instantaneous eigenstate of \( H \) is the zero energy ‘dark state’ \(|z\rangle\) given by

\[
|z(t)\rangle = \frac{1}{\mathcal{N}} \begin{pmatrix}
\Omega_P^{-1}(t) \\
\varepsilon \\
-\Omega_S^{-1}(t)
\end{pmatrix},
\]

where \( \mathcal{N} \) denotes the normalization. Assuming that the relevant energy gaps are large compared to the time derivative of the Hamiltonian \( \frac{d}{dt}H \), the adiabatic theorem states that a system remains in its instantaneous eigenstates [26]. The dark state \(|z\rangle\) has precisely the property that it transitions from \(|1\rangle\) to \(|3\rangle\) as \( \Omega_S \) is gradually diminished while \( \Omega_P \) is increased. Note the counterintuitive order of the pulses, as indicated in Fig. 1. A key property of STIRAP is that the excited state \(|2\rangle\) is never populated during this process, hence the protocol is independent of decoherence due to emission from this state. Thanks to this, and the inherent stability of adiabatic methods [27], the protocol is relatively stable to experimental imperfections, and is broadly adopted in practice [2].

III. GENERALIZING STIRAP

We observe that a key property of STIRAP/CTAP is the existence of a unique zero-energy eigenstate at all times, and that this state is localizable by lowering couplings incident to a particular site. This leads us to our main question: which other physical configurations permit precisely one zero eigenvector, even when uncoupling a certain site?

We capture the more general configurations in the language of weighted graphs \( G = (V,E,w) \). Here, the collection of vertices \( V = \{|j\rangle\}_{j=1}^{\dim(\mathcal{H})} \) corresponds to a set of basis states of Hilbert space \( \mathcal{H} \). Two vertices \( u,v \in V \) are connected by an edge \((u,v) \in E\) if and only if an interaction that couples states \(|u\rangle\) and \(|v\rangle\) can be applied. The weights \( w : E \to \mathbb{C} \) assign a complex amplitude to each of the interactions. Weights evaluated on non-existent edges are zero: \( w_{uv} = 0 \) for all \((u,v) \notin E\). The adjacency matrix \( A_G \) of a graph is then defined as the matrix of weights, with matrix elements \((A_G)_{uv} = w_{uv}\).
As long as we impose hermiticity through \( w_{uv} = w_{vu}^* \), the control Hamiltonian \( H_G \) can be derived from a given graph \( G \), through

\[
H_G(t) = \sum_{u,v \in V} f_{uv}(t) w_{uv} |u⟩⟨v| , \quad f_{uv}(t) = f_{vu}^*(t). 
\]

In this definition of the control Hamiltonian, we assume arbitrary time-dependent control over each allowed interaction, by tuning the controls \( f_{uv}(t) \). We separate the interaction strengths into \( f \) and \( w \) because a clear distinction between graph properties and control fields is needed later.

The graph \( G \) from which \( H_G \) is derived will be called the interaction graph, which restricts the allowed interactions in \( H \). Note that only systems whose interactions have a certain sense of resonance can be represented this way.

Thanks to the mapping to graphs, we can use various notions from graph theory. We denote with \( G - v \) the graph \( G \) in which the vertex \( v \) and all the edges incident to \( v \) are removed. A bipartite graph has a vertex set \( V \) which can be separated into two independent subsets \( V_1, V_2 \) such that each edge \((u, v) \in E \) must run between \( V_1 \) and \( V_2 \) (that is, \( u \in V_1 \) and \( v \in V_2 \) or vice-versa). A \textit{semi-bipartite graph} \cite{28, 29} is a bipartite graph in which edges within \( V_2 \) are allowed (including self-loops), but edges within \( V_1 \) are still prohibited. For example, the graph in Fig. 1 is semi-bipartite with \( V_1 = \{1, 3\} \), but not bipartite unless \( \varepsilon = 0 \).

We are now ready to state our main result.

**Theorem 1.** Let \( G = (V, E, w) \) be a connected, weighted, semi-bipartite graph with parts \( V_1 \) and \( V_2 \). Let \( P = \{p_j\}_{j=1}^2 \subseteq V_1 \). Then, for any \( a, b \in P \), there exists a control Hamiltonian \( H_G(t) \) which adiabatically transfers amplitude from \( a \) to \( b \), as long as the following holds:

1. \(|V_1| = |V_2| + 1\);

2. Either of the following:
   1a. For all \( p_j \), \( \det(A_{G-p_j}) \neq 0 \);
   1b. \( A_G \) has a unique zero eigenvector, which has nonzero amplitude on each \( p_j \).

Note that \( \det(A) \neq 0 \) implies that \( A \) does not have a zero eigenvalue. In the Supplemental Material, we show that requirements 2a and 2b are equivalent whenever 1 holds \cite{30}.

Requirement 2 is not very intuitive, but is virtually implied by a perfect matching. A graph contains a perfect matching whenever one can ‘marry’ pairs of vertices connected by an edge, such that all vertices are paired with precisely one partner. In the Supplemental Material, we prove that if \( G - p_j \) allows such a perfect matching (for all \( p_j \in P \)), then upon choosing \( w_{uv} \) at random, requirement 2 is satisfied with probability 1. Moreover, a ‘dangling vertex’ which has just a single neighbor, can be added or removed from the graph together with its unique neighbor, and doing so will not change the number of zero eigenvectors of the graph \cite{30}. This allows straightforward generation of viable graphs with long-distance connectivity out of well understood basis graphs.

To prove Thm. 1, let us first show that requirement 1 implies the existence of a zero-energy eigenstate. For now, set \( f_{uv} = 1 \), such that \( H_G = A_G \). Let \( V \) denote the vector space spanned by the states corresponding to the vertices in \( V \). Likewise, we use \( V_1, V_2 \) to denote the subspaces corresponding to subsets \( V_1, V_2 \). We order the basis of \( V \) by first stating the elements of \( V_1 \) and then the elements of \( V_2 \). In this basis, the interaction graph has the form

\[
A_G = \begin{pmatrix} 0 & B^T \\ B & C \end{pmatrix},
\]

where \( B \) is a matrix of size \(|V_1| \times |V_2|\) and \( C \) has size \(|V_2| \times |V_2|\).

Thanks to \(|V_1| = |V_2| + 1\), there must exist a zero-energy eigenvector \(|z⟩ = (\bar{z}, 0) \in V_1 \) whose nonzero amplitudes \( \bar{z} \) are only located on sites in \( V_1 \). This holds because in the eigenvalue equation,

\[
\begin{pmatrix} 0 & B^T \\ B & C \end{pmatrix} \begin{pmatrix} \bar{z} \\ 0 \end{pmatrix} = 0,
\]

the system of equations \( B^T \bar{z} = 0 \) has \(|V_1| \) variables and \(|V_2| \) constraints, hence it must always have at least one non-trivial solution.

Importantly, \(|z⟩\) has many favorable stability properties. Its eigenvalue is \textit{exactly} 0, independent of changes to \( w_{uv} \), as long as the graph remains semi-bipartite. This makes the state’s dynamical phase easy to track. Moreover, it has exactly 0 amplitude on \( V_2 \), which makes it insensitive to any decoherence on sites in \( V_2 \). The state \(|z⟩\) generalizes the ‘dark state’ of conventional STIRAP and CTAP, inheriting important features that make these protocols attractive for practical purposes.

In order for the adiabatic theorem to apply, we should guarantee that \textit{no other} states with zero energy exist. We will give a simple choice of time-dependent controls \( f_{uv}(t) \) that, given assumptions 1 and 2, satisfy

1. For \( 0 \leq t \leq T \), the state \(|z(t)⟩\) is the unique zero-energy eigenstate,

2. \( H_G(0)|a⟩ = 0 \), such that at time \( t = 0 \) the unique zero eigenstate is precisely the state \(|z(0)⟩ = |a⟩\),

3. \( H_G(T)|b⟩ = 0 \), such that at time \( t = T \) the unique zero eigenstate is precisely the state \(|z(T)⟩ = |b⟩\).

A solution is as follows. We will assume that, for each vertex \( v \in V_1 \), the incident couplings are changed proportionally. Furthermore, we take all couplings within \( V_2 \) to be equal to one. Therefore, we set

\[
\begin{align*}
f_{uv}(t) &= f_o(t) \quad \forall u \in V_2, v \in V_1; \\
f_{uv}(t) &= 1 \quad \forall u, v \in V_2.
\end{align*}
\]
We will henceforth call such a setting ‘commensurate couplings’. This means that we can write

$$H_G(t) = F(t)A_G F^*(t),$$

where $F(t) = \text{diag}(f_1(t), \ldots, f_{|V_i|}(t), 1, \ldots, 1)$.

We then set

$$f_a(0) = 0;$$
$$f_b(T) = 0;$$
$$f_v(t) \neq 0 \text{ for all } v \notin P;$$

No two $f_v(t)$ may be zero simultaneously.

As long as all the $f_a$ are non-zero, there is still a unique zero eigenvector of $H_G(t)$,

$$|z(t)| \propto F(t)^{-1}|z\rangle,$$

as can be seen from Eq. 1. Because $F$ is diagonal, $|z(t)|$ is still located on $V_1$. It is unique, because given any zero eigenvector $|w\rangle$ of $H_G(t)$, $F(t)|w\rangle$ is an eigenvector of $A_G$, hence must be equal, up to scaling, to $|z\rangle$.

Special care has to be taken when reducing weights to zero. When reducing $f_a$ ($p \in P$) towards zero, assumption 2a guarantees that no zero eigenvectors occur on $G - p$, hence $|p\rangle$ must then be the unique zero eigenstate. Note that $2b$ also implies $2a$ [30].

This shows that any controls $f_v$ satisfying Eq. 2 indeed pertain a unique zero-energy eigenstate, and provide the correct initial and final state at times $t = 0$ and $t = T$.

The only couplings that actually require time-dependent control are those directly connected to sender and receiver; controlling any of the other couplings is optional. Moreover, the condition $\det(A_{G - P_j}) \neq 0$ guarantees that we can drop the requirement of commensurate couplings for a single $p_j$ as long as all other sites remain commensurately coupled: the rank of $A_G$ must be at least that of $A_{G - P_j}$, which shows that for any couplings between $p_j$ and the rest of the graph, there can be at most one zero-energy state. Bringing all these couplings to zero must move this state to $p_j$. This freedom gives the protocol a convenient stability to imperfect controls when bringing the weights adjacent to $p_j$ down to zero.

By the adiabatic theorem, a sufficiently high total time $T$ allows state transfer at arbitrary accuracy. The time scale is determined by the gap in the spectrum around the zero eigenvalue, as opposed to the well-studied gap between the lowest and second lowest energy [31]. To our best knowledge, little is known about the gap around zero, and characterizing its scaling is an interesting open problem. Interestingly, we numerically find that the gain of most grids scales as $1/|V|$, but also find exceptions that decay faster than a polynomial in the graph size $|V|$ [30].

**IV. APPLICATIONS**

The mathematical framework we consider applies to various realistic cases, such as discrete electronic levels coupled by laser fields [2], electrons hopping through quantum dots [15, 32], cold atoms or atomic condensates in optical potentials [17, 33, 34], or XX spin models in a single-excitation sector [35].

Our results show that CTAP and STIRAP work on many more systems than known previously. Moreover, decoherence mechanisms that can be turned into graphs which satisfy our assumptions should not have disrupting effects on the protocols. The most interesting application, however, might be in quantum information processing. Note that our protocols can transmit quantum information, for example when the transported state represents the position of a quantum particle with internal degrees of freedom, or when a superposition between a shared vacuum and an excitation on a graph may be made.

Ref. [36] observes that individual quantum processors based on quantum dots are limited in size, raising the need for communication between nearby processors. Our results readily generalize the CTAP protocol [15] to transfer electrons through a network of quantum dots, and the possibility to use more general graphs may be of great benefit for larger quantum computer architectures.

Another new application is in a delayed transfer scheme, previously addressed in Ref. [37]. The sender $a$ can initialize the system into the dark state $|z\rangle$ and leave it at that, such that any party in $P$ can retrieve the quantum state, at any time they like. This opens the possibility to share unclonable quantum information among many parties without yet knowing which party is required to obtain the information.

**V. SIMULATIONS**

As an example of the new possibilities in state transfer, we construct regular binary trees of depth $k$. To guarantee that requirement 1 is always fulfilled, we put another vertex on each edge (the white vertices), leading to graphs as shown in Fig. 2. The possible communicating parties $P$ are chosen to be the leaves (endpoints) of the tree, allowing $|P| = 2^k$ parties to be connected. We choose $a$ and $b$ at maximum distance from each other.

We define the transfer error as $\mathcal{E} = 1 - |\langle b | U_T | a \rangle|$, where $U_T$ denotes the unitary time-evolution operator as found by numerically solving Schrödinger’s Equation, and $T$ is the total protocol’s time. We choose simple time-dependent couplings $f_a = Jt/T$ and $f_b = J(1 - t/T)$, while all other controls remain $f_v = 1$. Moreover, we define $T^*$ as the lowest time for which $\mathcal{E} < 0.05$, setting a bar for transfer with 95% fidelity.

Owing to the exponentially large size $|V|$ of the graphs, the time required rapidly increases with $k$ (Fig. 2). Interestingly, we find that the technique of straddling [15, 19], in which all controls $f_v$ except for $f_a$ and $f_b$ are multiplied by a factor $s$, flattens the scaling down to roughly $T^* \approx 10\sqrt{k}$, up to a certain $k$ where the steep increase is observed again. Although Ref. [38] already predicted a favorable scaling $T^* \propto \sqrt{n}$ for linear chains of length
FIG. 2. Simulation results on tree graphs. A tree of depth \( k = 2 \) is shown on the top-left, with receivers \( a \) and \( b \) maximally separated. The top-right shows the ideal state evolution over time, and the energy levels during the protocol. The times \( T^* \) required for constant fidelity increase steeply with the exponential size \( |V| \) of the graph (bottom), except when sufficiently strong straddling is applied, leading to \( T^* \propto k^{0.5} \) (dashed line).

\( n \) in the strong straddling limit, it is surprising that in this case as similar better-than-linear scaling is found for much larger graphs, whose gap decays exponentially in \( k \).

There are various reasons to believe that the strong straddling scaling cannot remain valid for increasingly large systems, for example due to Lieb-Robinson bounds [39]. Still, with a modest straddling factor \( s = 10 \), transfer at favorable scaling \( T^* \propto \sqrt{\log(|P|)} \) is observed for graphs of up to 1000 sites, showing that near-term experiments can benefit from this favorable scaling.

VI. CONCLUSION

To summarize, we extend the set of graphs in which STIRAP-like protocols are known to work. The sufficient requirements are made precise in assumptions 1 and 2.

We inherit the most important properties of the conventional protocols: the adiabatic controls do not require precise amplitudes or timings, the system’s energy is exactly zero at all times, and the fidelity is largely insensitive to decay on sites in \( V_2 \). Various extensions, such as straddling and multi-party transfer, can be readily incorporated. In the studied example of tree-shaped graphs, we find that with mild straddling the fidelities are much better than naively expected.

As our requirements are sufficient but not necessary, we would be interested to see further work explore other graphs with unique zero eigenstates, and give guarantees on gap sizes for specific graphs. Moreover, we look forward to seeing state-of-the-art experiments test our results in practice.

VII. ACKNOWLEDGMENTS

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Appendix i: Proofs of claims made in the main text

1. Proof of equivalence between 2a and 2b in Thm. 1

Let us recall Thm. 1:

**Theorem 1.** Let $G = (V, E, w)$ be a connected, weighted, semi-bipartite graph with parts $V_1$ and $V_2$. Let $P = \{p_j\}_{j=1}^k \subseteq V_1$. Then, for any $a, b \in P$, there exists a control Hamiltonian $H_G(t)$ which adiabatically transfers amplitude from $a$ to $b$, as long as the following holds:

1. $|V_1| = |V_2| + 1$ ;
2. Either of the following:
   2a For all $p_j$, $\det(A_G - p_j) \neq 0$; 
   2b $A_G$ has a unique zero eigenvector, which has nonzero amplitude on each $p_j$.

In this section, we will prove that assumptions 2a and 2b are equivalent under the assumption of 1. More precisely, we prove the following proposition.

**Proposition 2.** Let $G = (V, E, w)$ be a weighted, semi-bipartite graph with parts $V_1$ and $V_2$, such that $|V_1| = |V_2| + 1$, and let $p \in V_1$. Then the following are equivalent:

a. $\det(A_G - p) \neq 0$;

b. $A_G$ has a unique zero eigenvector, which has non-zero amplitude on $p$.

**Proof.** We start with the implication from $a$ to $b$. By the start of the proof of Thm. 1, we know that $A_G$ has at least one zero eigenvector, so the rank of $A_G$ can be at most $|V| - 1$. However, as $\det(A_G - p) \neq 0$, it must be of maximal rank, which is also $|V| - 1$. As the rank of a submatrix gives a lower bound on the rank of a matrix, this shows that $\text{rk} A_G \geq |V| - 1$. Therefore, there is a unique zero eigenvector. Let this eigenvector be $v$, let its component at $p$ be $v_p$, and its components away from $p$ be $\tilde{v}$ (so $\tilde{v}$ is a vector with $|V| - 1$ components). We can write $A_G$ as a block matrix

$$A_G = \begin{pmatrix} 0 & b_p \\ b_p^T & A_{G-p} \end{pmatrix},$$

where we wrote the component $p$ as the first component for simplicity. As $v$ is a zero eigenvector, we get

$$0 = A_Gv = \begin{pmatrix} 0 & b_p \\ b_p^T & A_{G-p} \end{pmatrix} \begin{pmatrix} v_p \\ \tilde{v} \end{pmatrix} = \begin{pmatrix} b_p^Tv_p + A_{G-p}\tilde{v} \\ b_p^T\tilde{v} \end{pmatrix} \begin{pmatrix} v_p \\ \tilde{v} \end{pmatrix}.$$ 

If now $v_p = 0$, then $\tilde{v} \neq 0$, as an eigenvector cannot be zero, but then $A_{G-p}\tilde{v} \neq 0$, as $\det(A_{G-p}) \neq 0$. This is a contradiction, so we must have $v_p \neq 0$.

Now we prove the implication from $b$ to $a$ by contrapositive. Hence we assume $\det(A_{G-p}) = 0$, and show that there exists a zero eigenvector of $A_G$ whose $p$-component is zero. Again, for notational simplicity, we write the component $p$ as the first component, so we have

$$A_G = \begin{pmatrix} 0 & \tilde{B} \\ B^T & C \end{pmatrix} = \begin{pmatrix} 0 & 0 & b_p \\ 0 & 0 & \tilde{B} \\ b_p^T & \tilde{B}^T & C \end{pmatrix}.$$ 

From this, we get

$$A_{G-p} = \begin{pmatrix} 0 & \tilde{B} \\ B^T & C \end{pmatrix},$$

where, crucially, the sizes of $\tilde{B}$ and $C$ are equal by the assumption $|V_1| = |V_2| + 1$. Hence,

$$\det(A_{G-p}) = \pm \det(\tilde{B}\tilde{B}^T) = \pm \det(\tilde{B})^2.$$
Now, by assumption $\det(A_{G_p}) = 0$, so $\det(\tilde{B}^T) = 0$. Therefore, there exists a zero eigenvector $u$ of $\tilde{B}^T$. If we define $v = (0, u, 0)$, then

$$A_Gv = \begin{pmatrix} 0 & 0 & b_p \\ 0 & 0 & \tilde{B} \\ b_p^T & \tilde{B}^T & C \end{pmatrix} \begin{pmatrix} 0 \\ u \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{B}^T u \end{pmatrix} = 0,$$

so we have constructed a zero eigenvector of $A_G$ with zero amplitude on $p$, giving a contradiction.

**Remark 3.** In fact, the implication from $a$ to $b$ goes through even in the case $G$ is not semi-bipartite; the proof does not use this assumption. However, for the other direction, it is essential.

2. **Dangling vertices do not affect the nullity**

For an $(n \times n)$-matrix $A$, let $\eta(A) = n - \text{rk}(A)$ denote the nullity of the matrix.

**Lemma 4.** Let $G$ be a graph with a vertex $v$ of degree 1. Let $u$ be the unique neighbor of $v$. Then

$$\eta(A_G) = \eta\left( A_{G - \{v,u\}} \right).$$

**Proof.** Let $\tilde{G}$ denote the graph $G - \{v, u\}$. Assuming for convenience that $v$ and $u$ are the first and second column of the adjacency matrix $A_G$ respectively, we can write

$$A_G = \begin{pmatrix} 0 & w_{uv} & 0 \\ w_{vu} & w_{uu} & b \\ 0 & b^T & A_{\tilde{G}} \end{pmatrix}.$$

We can write any vector $\alpha$ as $(\alpha_v, \alpha_u, \tilde{\alpha})$. Now $w_{uv} \neq 0$ and

$$0 = A_G \alpha = \begin{pmatrix} 0 & w_{uv} & 0 \\ w_{vu} & w_{uu} & b \\ 0 & b^T & A_{\tilde{G}} \end{pmatrix} \begin{pmatrix} \alpha_v \\ \alpha_u \\ \tilde{\alpha} \end{pmatrix}$$

imply that $\alpha_u = 0$, $\alpha_v = -\frac{1}{w_{vu}} b \cdot \tilde{\alpha}$, and $A_{\tilde{G}} \tilde{\alpha} = 0$. Hence, we get a linear isomorphism $\text{ker} A_G \to \text{ker} A_{\tilde{G}} : (\alpha_v, \alpha_u, \tilde{\alpha}) \mapsto \tilde{\alpha}$ with inverse $\tilde{\alpha} \mapsto (-\frac{1}{w_{vu}} b \cdot \tilde{\alpha}, 0, \tilde{\alpha})$. As the nullity is the dimension of the kernel, this shows $\eta(A_G) = \eta(A_{\tilde{G}})$.

3. **Graphs with certain matchings make our protocol work almost surely**

A perfect matching in a graph $G$ is a set of disjoint edges that covers all the vertices.

**Theorem 5.** Let $G = (V, E, w)$ be a weighted semi-bipartite graph with parts $V_1$ and $V_2$ where $|V_1| = |V_2| + 1$. Let $P = \{p_i\}_{i=1}^k \subseteq V_1$. Suppose that for all $i$ there exists a perfect matching in $G - p_i$. Then, if weights $w_{uv}$ are chosen uniformly at random from $[0, 1]$ for all $uv \in E$, we find $\det(A_{G - p_i}) \neq 0$ for all $p_i$ with probability 1.

Note that the theorem exactly gives us condition (2a) required by the protocol.

**Proof.** It suffices to prove that $\det(A_{G - p_i}) \neq 0$ with probability 1 for a fixed $i \in \{1, \ldots, k\}$; the claim of the theorem then follows since a countable intersection of events with probability 1 still has probability 1.

Let $p = p_i$ be given. We will first permute the rows and columns of the matrix $A_{G - p}$ to bring it in a convenient form; such a permutation only affects the determinant of the matrix by a sign, which is irrelevant to us.

By assumption, there is a perfect matching on the graph $G - p$. Since $|V_1 - p| = |V_2|$ and there are no edges within $V_1$, any perfect matching must use only edges between $V_1$ and $V_2$. Let $u_1, v_1, \ldots, u_k, v_k \in E \cap (V_1 \times V_2)$ denote the edges given in a perfect matching on $G - p$. Permute the rows and columns such that the rows are in the order $u_1, v_1, u_2, v_2, \ldots$ and the columns are in the order $v_1, u_1, v_2, u_2, \ldots$. We show with an inductive argument that for all $\ell \in \{1, \ldots, k\}$, the matrix $A_\ell$ on the first $2\ell$ rows and columns has non-zero determinant with probability 1. This proves the claim.
For \( \ell = 1 \), we consider

\[
\det \begin{pmatrix} w_{u_1 v_2} & 0 \\ w_{v_1 u_2} & w_{u_1 v_1} \end{pmatrix} = w_{u_1 v_1}^2,
\]

since \( w_{u_1 u_1} = 0 \) as \( u_1 \in V_1 \). As \( w_{v_1 v_1} \) is sampled uniformly at random from \([0,1]\), this is non-zero with probability 1. Now suppose we have shown the statement up to some \( \ell \). We find

\[
\det \begin{pmatrix} A_\ell & b_1 \\ d_1 & w_{u_{\ell+1} v_{\ell+1}} \\ d_2 & w_{v_{\ell+1} u_{\ell+1}} \\ & w_{u_{\ell+1} v_{\ell+1}} \end{pmatrix} = \det(A_\ell)w_{u_{\ell+1} v_{\ell+1}}^2 + bw_{u_{\ell+1} v_{\ell+1}} + c
\]

for some \( b \) and \( c \) which do not depend on \( w_{u_{\ell+1} v_{\ell+1}} \), and where we may assume that \( \det(A_\ell) \neq 0 \). Since the other entries do not depend on \( w_{u_{\ell+1} v_{\ell+1}} \) and this gets sampled independently of the other entries, we may view \( \det(A_\ell) \), \( b \) and \( c \) as constants. Since there are at most two possible values in \([0,1]\) which make a quadratic polynomial \( ax^2 + bx + c \) equal to zero (if \( a \neq 0 \), with probability 1 the expression will be non-zero. Continuing until \( \ell + 1 = k \), we conclude \( \det(A_{G-p}) \neq 0 \) with probability 1 as desired.

**Remark 6.** From the proof of the theorem, it follows that various of the assumptions in the theorem can be relaxed: the only requirement is that the weights are chosen from a continuous distribution (in the sense that any particular value has probability 0) and this requirement is only necessary for the edges involved in the matching.

In fact, it is possible to show that the adjacency matrix of \( G \) is equivalent to a matrix with non-zero entries on the diagonal if and only if there is a perfect matching. Limited generalisation is also possible to non-bipartite graphs.

**Appendix ii: Robustness guarantees**

A lower bound on the determinants \( \det(A_{G-p}) \) gives the robustness guarantee that our protocol will keep working even if the weights cannot be held exactly at the aimed value. More precisely, if \( |\det(A_{G-p})| > \epsilon \) for some \( \epsilon > 0 \), then by continuity of the determinant, this remains true even if the entries of \( A_{G-p} \) (that is, the weights on the edges) get permuted by at most some \( \delta \). Since the determinant is a polynomial, we may expect \( \delta \) to be of a similar scale to \( \epsilon \). This implies that the uniqueness of the zero eigenvector would be guaranteed even if the weights of the edges are slightly perturbed. Note that the weights on the edges adjacent to \( p_i \) do not affect the determinant at all.

The proof of Theorem 5 extends to give a weak lower bound on the determinant.

**Theorem 7.** Let \( G \) be a semi-bipartite graph on vertex classes \( V_1 \) and \( V_2 \) with a perfect matching \( u_1 v_1, \ldots, u_\ell v_\ell \). Suppose the weights on some edges of \( G \), including the \( u_i v_i \), are chosen independently and uniformly at random from \([0,1]\). Then with probability at least \( (\frac{1}{2})^\ell \), we have \( |\det(A_G)| > (\frac{1}{2})^{3\ell-1} \).

**Proof.** We may assume there are no edges within \( V_1 \). (This assumption can be left out but makes the analysis easier.) As in the proof of Theorem 5, we reorder the columns to \( u_1, v_1, u_2, v_2, \ldots \) and the rows to \( v_1, u_1, v_2, u_2, \ldots \) and prove the claim for all submatrices \( A_\ell \) spanned by the first \( 2\ell \) rows and columns for all \( \ell \in \{1, \ldots, k\} \).

The statement is true for \( \ell = 1 \): \( \det(A_1) = w_{u_1 v_1}^2 > 1 \) with probability at least \( \frac{1}{2} \). Suppose now that \( |\det(A_\ell)| \geq (\frac{1}{2})^{3\ell-1} \) with probability at least \( (\frac{1}{2})^{\ell} \) for some \( \ell \). Again, we find

\[
\det \begin{pmatrix} A_\ell & b_1 \\ d_1 & w_{u_{\ell+1} v_{\ell+1}} \\ d_2 & w_{v_{\ell+1} u_{\ell+1}} \\ & w_{u_{\ell+1} v_{\ell+1}} \end{pmatrix} = \det(A_\ell)w_{u_{\ell+1} v_{\ell+1}}^2 + bw_{u_{\ell+1} v_{\ell+1}} + c
\]

takes the form \( ax^2 + bx + c \), where \( a, b, c \) do not depend on \( x = w_{u_{\ell+1} v_{\ell+1}} \) and can hence be viewed as constants by the independence assumption, and where \( |a| \geq (\frac{1}{2})^{3\ell-1} \) with probability at least \( (\frac{1}{2})^{\ell} \).

We can rewrite \( ax^2 + bx + c = a(x+b')^2 + c' \) for possibly different values \( b', c' \). Then \( |a(x+b')^2 + c'| \leq (\frac{1}{2})^{3(\ell+1)-1} \) if and only if \( a(x+b')^2 \in (-c' - (\frac{1}{2})^{3(\ell+1)-1}, -c' + (\frac{1}{2})^{3(\ell+1)-1}) \). The probability of this happening is maximized when \( b' = -\frac{1}{2}, |c'| = (\frac{1}{2})^{3(\ell+1)-1} \) and the sign of \( c' \) and \( a \) are different; we may assume \( a > 0 \) as the other case goes analogous. In this case the interval is \( (0, 2(\frac{1}{2})^{3(\ell+1)-1}) = (0, (\frac{1}{2})^{3\ell+1}) \). We find \( a \geq (\frac{1}{2})^{3\ell-1} \) with probability at
least \((\frac{1}{2})^\ell\), in which case independently with probability \(\frac{1}{2}\) we have \(|x + b'| \geq \frac{1}{2}\). Hence with probability at least \((\frac{1}{2})^{\ell+1}\), we find

\[
a(x + b')^2 \geq \left(\frac{1}{2}\right)^{3\ell-1} \left(\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^{3\ell+1} \notin \left[0, \left(\frac{1}{2}\right)^{3\ell+1}\right].
\]

\(\square\)

We cannot hope to do much better than the result above. Consider the case in which \(A = \text{diag}(a_1, a_2, \ldots, a_k, a_k)\) is a diagonal matrix, such that \(\det(A) = a_1^2 \cdots a_k^2\) where the \(a_i\) get chosen independently and uniformly at random. Using the law of large numbers or the Central Limit Theorem and the fact that \(-\log(U(0,1)) \sim \text{Exp}(1)\), it follows that \(a_1 \cdots a_k\) is concentrated around \((\frac{1}{2})^k\). In fact, one can prove using Chernoff bounds \([40]\) that

\[
\mathbb{P} \left( a_1 \cdots a_k \geq \left(0.5^{2/3}\right)^k \right) \leq e^{-1/(144)k}.
\]

Hence without further assumptions, we cannot hope to improve the exponential decay in the lower bound of the theorem.

**Appendix iii: On the eigenvalue gap in generalized STIRAP**

The eigenvalue gap between the ground state and first excited state is an active field of research. However the gap between a zero eigenvalue and the nearest non-zero eigenvalue seems to have received significantly less interest. Below, we share our preliminary findings on the gap around eigenvalue 0.

1. **Interlaced eigenvalues**

When matrix entries get chosen from \([0,1]\), all eigenvalues satisfy \(|\lambda| \leq \max_d(G)\) for \(\max_d(G)\) the maximum degree of \(G\). Since the determinant is the product of the eigenvalues, this gives the lower bound \(|\lambda| \geq \frac{\det(A_G)}{\max_d(A_G)}\) for a graph \(G\) on \(n\) vertices. We can obtain a lower bound on the eigenvalue gap using the following well-known result, which follows from the fact that \(A_{G-p}\) is a principal submatrix of \(A_G\) \([41]\).

**Theorem 8** (Cauchy interlacing theorem). Let \(G\) be a graph with a vertex \(p\). Let \(\lambda_1 \leq \cdots \leq \lambda_{n+1}\) be the eigenvalues of \(A_G\) and \(\mu_1 \leq \cdots \leq \mu_n\) the eigenvalues of \(A_{G-p}\). Then

\[
\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \lambda_n \leq \mu_n \leq \lambda_{n+1}.
\]

In our set-up, one of the \(\lambda_i\) will be equal to 0, and the theorem shows that that the gap to the second absolutely smallest eigenvalue is at least \(\min_i |\mu_i|\), and hence the eigenvalue gap

\[
\Delta(G) \geq \max_{\mu \in V_1} \min_{\mu \text{ eigenvalue of } G-p} |\mu|.
\]

Along with the bounds on \(\det(A_{G-p})\) from the previous section, we can use this to obtain a lower bound on the eigenvalue gap of \(G\). This bound is very weak, and based on the experiments outlined below it is our expectation that vastly better bounds can be obtained.

2. **Numerical scaling**

Fig. 3 depicts the scaling of the energy gap around the zero energy state, as a function of the number of vertices \(|V|\), for various types of graphs. For most graphs, we considered the unweighted versions, setting \(w_{uv} = 1\) whenever the corresponding edge is present. Some graphs have the annotation ‘random’, which means that the graphs typically do not have a unique zero eigenvalue when all weights equal one; using our results we can ensure the existence of a unique zero eigenvalue by multiplying each weight \(w_{uv}\) with a random number chosen independently and uniformly from between 0 and 2. We took the average energy gap over 50 such perturbations.

These results show that the energy gap often decays roughly as \(\Delta E \propto |V|^{-1}\) or better, similarly to conventional STIRAP over a linear chain, with hexagonal grids being an exception.

The precise details of the graph generation is as follows.
We generate star graphs by connecting $k$ ‘arms’, linear chains of length $m$, to a single center vertex. Interestingly, the eigenvalue gaps do not change as the number of arms increases. We fix the number of arms to 3 and vary the chain lengths to make larger graphs.

The hexagonal grids consist of unit cells of size 2. We take $k^2$ copies of these unit cells and place them on a $k \times k$ square grid, which is connected as indicated in Fig. 3. To enforce an odd number of sites, we remove a single site in the top-right corner, leading to $2k^2 - 1$ sites in total. Interestingly, the hexagonal grids are the only graph configuration we considered whose gap decays superpolynomially (yet slower than an exponential). Randomly perturbing weights does not change this behavior.

The square grids are chosen to have $k$ by $k$ vertices, where $k$ is an odd number.

The bipartite graphs consist of two parts of size $m + 1$ and $m$, respectively. Each potential edge which can be laid to connect the two parts is added with probability $p = 0.81$. Because these graphs are also random, for each datapoint, we also averaged the gap size over 50 random instantiations of the edge set. The thickness of the line indicates the standard deviation.

Lastly, the subdivided binary trees are generated as in the main text: starting from a complete binary tree of certain depth, we create an additional vertex on each edge, which makes sure that $|V_1| = |V_2| + 1$. 

FIG. 3. Scaling of the eigenvalue gap $\Delta E$ between the unique zero eigenvalue and the closest other eigenvalue, on a log-log scale. These are calculated for various bipartite graphs of various sizes $|V|$. The annotation (random) indicates that the weights were randomly chosen in the interval $(0,2)$ to guarantee a unique zero eigenvector. The lower dashed line indicates $\Delta E = 1/|V|$, and the upper dashed line follows $\Delta E = 10/\sqrt{|V|}$. Interestingly, for most of the graphs we study, the gaps decay scales proportional to $1/|V|$ or better. Hexagonal grids are an exception, as these are found to decay superpolynomially.