Impact of non-local electrodynamics on flux noise and inductance of superconducting wires

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We present exact numerical calculations of supercurrent density, inductance, and impurity-induced flux noise of cylindrical superconducting wires in the non-local Pippard regime, which occurs when the Pippard coherence length is larger than the London penetration depth. In this regime the supercurrent density displays a peak away from the surface of the superconductor, signalling a breakdown of the usual approximation of local London electrodynamics with a renormalized penetration depth. Our calculations show that the internal inductance and the bulk flux noise power increases with increasing non-locality. In contrast, the kinetic inductance is reduced and the surface flux noise remains the same. As a result, impurity spins in the bulk may dominate the flux noise in superconducting qubits in the Pippard regime, such as the ones using aluminum superconductors with large electron mean free path.

I. INTRODUCTION

In the past few years there has been astounding progress on the design of superconducting circuits for large scale quantum computing [1]. Specific purpose circuits for quantum annealing containing over two thousand qubits have been developed and are currently being benchmarked against classical algorithms [2], and universal circuits containing nine qubits with over one thousand quantum bits were successfully demonstrated [3].

While designing these circuits required careful control of circuit parameters such as self and mutual inductances, Josephson critical currents, capacitances, etc, most of the design was done without numerical prediction of circuit parameters. This happened because the tools for numerical modeling of superconducting circuits are much more scarce [4], in contrast e.g. to the tools available to the semiconductor industry.

A key difference between superconductors and normal conductors is that in the former electrical current is carried by Cooper pairs, which are objects that manifest quantum behavior. As a consequence the charge of a Cooper pair can not be localized within less than a length scale \( \xi \), the uncertainty in the distance between the two electrons forming the Cooper pair, which is also known as Pippard coherence length. The impact of \( \xi \) is that it makes the response of a superconductor to electromagnetic fields inherently non local, in the sense that a field applied at a certain point \( \mathbf{r} \) will affect Cooper pairs within a radius \( \xi \) of \( \mathbf{r} \). If we describe magnetic fields using a vector potential \( \mathbf{A}(\mathbf{r}) \) in the London gauge (defined by \( \nabla \cdot \mathbf{A} = 0 \) with \( \mathbf{A} \cdot \hat{n} = 0 \), where \( \hat{n} \) is the unit vector perpendicular to the surface of the superconductor), we get the following Pippard relation between current density and vector potential [5],

\[
J(\mathbf{r}) = -\frac{1}{4\pi\mu_0\lambda_L(T)^2} \int_{\text{SC}} \frac{R[\mathbf{R} \cdot \mathbf{A}(\mathbf{r}')]}{R^2} e^{-\frac{2}{\xi} d^2} d^2r',
\]

where the integral is taken over the superconducting wire, with \( \mathbf{R} = \mathbf{r} - \mathbf{r}' \), and \( \lambda_L(T) \) the bare London penetration depth. Here \( \xi_0 \) is the Pippard coherence length for pure materials; \( \xi \) is impacted by the presence of impurities according to \( \frac{1}{\xi} = \frac{1}{\xi_0} + \frac{1}{l} \), where \( l \) is the mean free path for electrons in the normal state (\( l < \infty \) due to effects such as electron-impurity scattering).

We remark that while Eq. (1) was originally proposed as a phenomenological relation, it is actually quite close to the microscopic result obtained using the Bardeen-Cooper-Schrieffer (BCS) theory. Moreover, the integral over \( \mathbf{r}' \) in Eq. (1) has a sharp cut off at the surface of the wire. This prescription is equivalent to assuming diffusive scattering of electrons at the surface, i.e. the surface is assumed to be rough so that electrons coming from the surface have no memory of any previous exposure to the vector potential.

When \( \mathbf{A}(\mathbf{r}) \) is approximately constant over the range \( \xi \), the integrand can be approximated by a delta function and the Pippard relation reduces to the London relation,

\[
J(\mathbf{r}) = -\frac{1}{\mu_0\lambda_L(T)^2} \mathbf{A}(\mathbf{r}),
\]

with \( \lambda_L(T) = \lambda_L(0)\sqrt{\xi_0/\xi} \) the London penetration depth for impure superconductors. This relation is “local” in the sense that the current density at any given point \( \mathbf{r} \) within the superconductor responds only to the vector potential at that same point \( \mathbf{r} \).

All modeling of superconducting devices to date was done using the local London relation Eq. (2) [4, 6–8]. This can only be justified when \( \xi \) is smaller than the characteristic length scale of \( \mathbf{A}(\mathbf{r}) \), which is set either by the smallest linear length scale of the wire, or by \( \lambda_L'(T) \), whichever is smaller.

Table I shows the zero-temperature bare penetration depth \( \lambda_L(0) \) and Pippard coherence length \( \xi_0 \) for some common superconducting materials.

We see from Table I that the Pippard coherence length is much larger than the penetration depth for most common superconductors. In the regime of large wire ge-
TABLE I. Zero-temperature “bare” London penetration depth $\lambda_L(0)$ and intrinsic Pippard coherence length $\xi_0$ for pure superconductors [9].

| Superconductor | $\lambda_L(0)$ (nm) | $\xi_0$ (nm) |
|---------------|---------------------|---------------|
| Al            | 16                  | 1600          |
| In            | 19                  | 490           |
| Nb            | 39                  | 38            |
| Pb            | 37                  | 83            |
| Sn            | 35                  | 250           |

ometry with $\xi \gg \lambda'_L$ this effect can be taken into account through a simple renormalization of the penetration depth, implying that local electrodynamics is still valid with a simple substitution of $\lambda_R$ for $\lambda'_L$ in Eq. (2) (See e.g. Section 3.11 of [10]):

$$\lambda_R \approx (\lambda'_L \xi)^{3/4}. \quad (3)$$

Here we perform exact numerical calculations of current density and vector potential for superconducting wires in the non-local regime. We show that local electrodynamics generally breaks down, impacting circuit parameters such as inductance and flux noise.

The article is organized as follows. Section II describes analytical solutions for the current density and vector potential inside a cylindrical wire using local electrodynamics. Section III presents our numerical method for exact solution of the self-consistent relations arising in non-local electrodynamics; Section IV describes our explicit numerical results, their comparison to the local case, and to simple approximations based on $\lambda_R$. Section V applies these results to calculations of internal and kinetic inductance, and Section VI to calculation of flux noise due to impurities at the surface and the bulk of the wires. Finally, Section VII presents our conclusions.

II. LOCAL ELECTRODYNAMICS: EXACT ANALYTICAL SOLUTION

![FIG. 1. Infinite wire with cylindrical geometry. The current is assumed to flow along $+\hat{z}$.

A well known challenge in the computation of superconducting current density and internal magnetic field is the presence of singularities at the wire’s edge. Even for the simpler local case, these singularities require the use of uncontrolled approximations, such as the ones made for thin films and strip lines [6–9, 11–13]. Here we avoid edge singularities by focusing on a much simpler wire geometry, the infinitely long, straight, and cylindrical superconducting wire shown in Fig. 1. As we show here, the choice of an edgeless geometry greatly simplifies the calculations and makes the non-local case exactly solvable, without the use of approximations that are hard to justify. Nevertheless, it should be noted that the cylindrical wire provides a realistic model of a coaxial cable with return current flowing on an external cylinder concentric with the wire; note that in this case the return current produces zero magnetic field inside the wire, and does not affect its SC current density.

We assume the current density is parallel to the $z$-axis, and by symmetry it only depends on the cylindrical coordinate $\rho$, $J(r) = J(\rho)\hat{z}$. In the London gauge the vector potential assumes a similar form, $A(r) = A(\rho)\hat{z}$. Combining the Maxwell equation $\nabla \times B = \mu_0 J$ with Eq. (2) we get the usual Poisson equations for the local case,

$$\nabla^2 A(\rho) = \frac{1}{\lambda_L^2} A(\rho), \quad \nabla^2 J(\rho) = \frac{1}{\lambda_L^2} J(\rho). \quad (4)$$

The exact solutions normalized by total current $I$ are given by the modified Bessel functions of the first kind $I_n(x)$,

$$A(\rho) = -\frac{\lambda_L \mu_0 I}{2\pi R} \frac{I_0(\rho/\lambda_L)}{I_1(R/\lambda_L)}, \quad 0 \leq \rho \leq R, \quad (5)$$

and

$$J(\rho) = \frac{I}{2\pi R \lambda_L^2} \frac{I_0(\rho/\lambda_L)}{I_1(R/\lambda_L)}, \quad 0 \leq \rho \leq R. \quad (6)$$

The internal magnetic field $B = B_\theta(\rho)\hat{\theta}$ is found by taking the curl of vector potential, or just using Ampere’s Law,

$$B_\theta(\rho) = \frac{\mu_0 I}{2\pi R} \frac{I_1(\rho/\lambda_L)}{I_1(R/\lambda_L)}, \quad 0 \leq \rho \leq R. \quad (7)$$

These equations will be the baseline for comparison with the non-local current densities and vector potentials calculated in the next section.

III. EXACT NUMERICAL METHOD TO TREAT NON-LOCAL ELECTRODYNAMICS

In the case where the spatial variation of the field is smaller than the effective size of the Cooper pairs (defined by the parameter $\xi$), non-local effects must be taken into account. Since all the fields are independent of $z$ (the wire extends infinitely in the $z$ direction), we can evaluate the integral over $\rho^2$ in Eq. (1), leading to an effective two-
The second simplifying result is that the surface term in Eq. (13) can be shown to be exactly equal to $A_0$, the value of the vector potential at the surface:

$$
R \int_0^{2\pi} d\theta' \left. \frac{\partial G}{\partial \rho'} A(\rho') \right|_{\rho' = R} = A_0 \int_0^{2\pi} \frac{1}{2\pi} \left[ \frac{1 - \rho^2}{\rho^2 + 1 - 2\rho \cos(\theta - \theta')} \right] d\theta'.
$$

The last identity can be checked with Mathematica. Altogether the equation for $A(\rho)$ becomes,

$$
A(\rho) = A_0 - \mu_0 \int_0^R d\rho' \rho' \int_0^{2\pi} d\theta' G(\rho, \rho', \theta') J(\rho').
$$

We can now replace $J(\rho)$ with the 2d Pippard relation defined in Eq. (8):

$$
A(\rho) = A_0 + \int_0^R d\rho' \rho' \int_0^R d\rho'' \rho'' K(\rho, \rho', \rho'') A(\rho''),
$$

where the kernel $K(\rho, \rho', \rho'')$ intertwines the Green’s function with the Pippard relation,

$$
K(\rho, \rho', \rho'') = \frac{1}{4\pi L} \int_0^R d\rho' d\rho'' \left[ \int_0^{2\pi} d\theta' G(\rho, \rho', \theta') \right] \times \int_0^R d\rho'' \rho'' K(\rho', \rho'').
$$

Equation (17) is a Fredholm integral equation, which can be solved through a variety of numerical methods.

The numerical solutions presented in the following section were obtained with FIE, the Fredholm integral equation solver developed by Atkinson et al. \cite{Atkinson}. It uses the Nyström method, which discretizes into a mesh $\{\rho_i\}$ so that the integral part can be converted into a matrix times the vector $\{A(\rho_i)\}$. Equation (17) then becomes an inhomogeneous linear system of equations, with the
set of $A(\rho)$’s as the unknowns, and inhomogeneity equal to $A_0$. The system can be solved numerically with matrix inversion, and each $A(\rho_i)$ becomes proportional to $A_0$.

Once $A(\rho)$ is obtained, $J(\rho)$ can be calculated by numerical integration of the Pippard relation Eq. (8), and the total current $I$ becomes a constant times $A_0$. Hence, $A_0$ is the constant of integration that sets the total current $I$ flowing through the wire.

IV. VECTOR POTENTIAL AND CURRENT DENSITY IN THE NON-LOCAL REGIME

We now show exact numerical calculations of vector potential and current density in the cylindrical wire, using the method described in Sec. III. In all cases we assumed penetration depth $\lambda'_L = 70$ nm ($\lambda'_L$ is assumed fixed, i.e. independent of $\xi$), with varying values of wire radius $R$ and Pippard coherence length $\xi$. Section IV B compares our exact solutions to a much simpler numerical approximation based on the renormalized penetration depth of Eq. (3).

A. Exact numerical results

Figure 2 compares local and non-local ($\xi = 200$ nm) calculations of $A(\rho)$ and $J(\rho)$. The quantities were normalized by $A_{\text{norm}} = 2\pi \int_0^R d\rho A(\rho)$ and $J_{\text{norm}} = 2\pi \int_0^R d\rho \rho J(\rho) = I/(\pi R^2)$, allowing the local solution for $A$ and $J$ to be represented by the same curve. In the non-local case, $J(\rho)$ is no longer directly proportional to $A(\rho)$, signaling the break down of the London relation Eq. (2).

Figure 3 shows calculations of $A$ and $J$ for $R = 1000$ nm and different values of $\xi$. Note the convergence to the local solution as $\xi \to 0$. As $\xi$ increases, the distribution of $A(\rho)$ and $J(\rho)$ is flattened.

These results show that non-local electrodynamics leads to qualitatively different behaviour for the current density. Note that in the local regime, $J(\rho)$ is always peaked at the edge of the wire, and decreases exponentially towards the interior. In contrast, the non-local regime shows that $J(\rho)$ is peaked off the edge of the wire, without a simple exponential fall off towards the interior. Under London electrodynamics, it was assumed that the current in superconductors always flowed along the boundary; however, as all superconductors have a degree of non-locality, we can see that this does hold true, and a significant portion of the current can flow away from the surface.

FIG. 3. (Color online) (a) Vector potential and (b) current density for different values of the non-local parameter $\xi$ for a superconducting cylinder with radius $R = 1000$ nm.

FIG. 4. Compares the vector potential calculated using the local approximation with $\lambda R = (\lambda'_L^2 \xi)^{1/3}$ to the exact solution of the Fredholm integral equation, for $\xi = 500$ nm.
\[ L_k = \frac{\mu_0^2}{T^2} \int_{SC} J^2(r) d^3r. \]  

In contrast, internal inductance depends on the magnetic field inside the wire,

\[ L_{\text{int}} = \frac{1}{\mu_0 T^2} \int_{SC} B^2(r) d^3r. \]

In the local case the inductances can be calculated by plugging Eq. (6) for the current density and Eq. (7) for the magnetic field. After performing the integrals the local inductances for a wire of length \( l \) become

\[ L_k^{\text{local}} = \frac{\mu_0 l}{2\pi R^2 I_1^2(R/\lambda_L')} \int_0^R d\rho \rho I_1^2(\rho/\lambda_L'), \]  

\[ L_{\text{int}}^{\text{local}} = \frac{\mu_0 l}{2\pi R^2 I_1^2(R/\lambda_L')} \int_0^R d\rho \rho I_1^2(\rho/\lambda_L'). \]

Figures 6 and 7 compare these local results to explicit numerical calculations of the inductances in the non-local regime. From these plots it is clear that kinetic inductance decreases with increasing Pippard coherence length \( \xi \); in contrast, internal inductance shows the opposite behaviour.
The performance of Superconducting Quantum Interference Devices (SQUIDs) and other superconducting circuits is limited by the presence of intrinsic flux noise, whose origin is believed to be due to the time dependent fluctuation of spin impurities located within the surfaces and interfaces forming the device [16–22]. Consider a set of localized impurities labeled by $i = 1, 2, \ldots, N$. Each impurity is located at position $\mathbf{R}_i$, and its magnetic moment is described by the dimensionless spin operator $\mathbf{s}_i$. The flux sensed by a circuit due to the presence of impurities can be written as [13]

$$\Phi = - \sum_i F_i \cdot \mathbf{s}_i,$$

with a flux vector $F_i$ pointing along the magnetic field produced by the device’s current density,

$$F_i = \frac{g \mu_B}{I} \mathbf{B}(\mathbf{R}_i).$$

Here $g \mu_B$ is the magnetic moment of the electronic impurity, with $\mu_B$ the Bohr magneton.

For a cylindrical wire we have $F_i = F_\theta(\rho_i) \hat{\theta}$, i.e. the flux vector only depends on the spin’s radial position $\rho_i$. Figure 10 shows $F_\theta$ as a function of $\rho_i$ for a wire of radius $R = 1000$ nm. In the local case $F_\theta$ is peaked at the surface, so the device is mostly sensitive to impurities located at the surfaces and interfaces. In contrast, for the non-local case the device becomes quite sensitive to impurities in the bulk.

We now turn to calculations of the total flux noise power (integrated over all frequencies) produced by the impurity spins, $\langle (\delta \Phi)^2 \rangle = \langle \Phi^2 \rangle - \langle \Phi \rangle^2$. At large temperatures (larger than any ordering temperature for the spins) the flux noise power due to spins at the surface of the wire is given by [13]

$$\langle (\delta \Phi)^2 \rangle = \frac{S(S + 1)}{3} \sigma_2 \int_{\text{surface}} d^2r |\mathbf{F}(r)|^2,$$

$$= \frac{S(S + 1)(g \mu_B \rho_0)^2 \sigma_2 l}{6\pi R}.$$  

Here $S$ is the spin quantum number of the impurity species, and $\sigma_2$ is the impurity areal density. In the second line we used Ampère’s law to compute the magnetic

non-local regime shows the opposite behaviour: Internal inductance is always larger than kinetic inductance, and this disparity increases as $\xi$ increases. The ratio of inductances is plotted in Fig. 8.

Finally, Fig. 9 shows the sum of kinetic and internal inductances; as $\xi$ increases the total inductance becomes significantly larger than in the local case.
field at the surface of the wire. This exact result shows that the surface flux noise is the same for the local and non-local regimes. Note how Eq. (24) scales proportional to \( \sigma_2/L \), a result that is quite similar to the \( \sigma_2/W \) scaling obtained in approximate calculations of flux noise in the local regime in thin film wires [13, 23] (\( W \) is the lateral width of the thin film).

The flux noise power in the bulk is given by

\[
\langle (\delta \Phi)^2 \rangle = \frac{S(S + 1)}{3} \sigma_3 (g \mu_B)^2 \int_{\text{sc}} d^3r |F(r)|^2
= \frac{S(S + 1)}{3} \sigma_3 (g \mu_B)^2 \mu_0 L_{\text{int}}. \tag{25}
\]

Note how the bulk noise power is directly proportional to the internal inductance \( L_{\text{int}} \), which is quite sensitive to non-local effects.

Figure 11 shows calculations of the flux noise power due to surface and bulk spins as a function of wire radius, for local and non-local cases with \( \xi = 100, 200, 500, 1000 \) nm. We assumed \( S = 1/2 \) impurities with surface density \( \sigma_2 = 5 \times 10^{12} \) cm\(^{-2} \) and bulk density \( \sigma_3 = 1 \times 10^{18} \) cm\(^{-3} \) (average distance between impurities \( \sim 1 \) \( \mu \)m). Here we see that the bulk flux noise power increases significantly as \( \xi \) increases. For the local case the surface noise power is larger than the bulk one; however for this choice of densities the bulk noise power becomes larger than the surface when \( \xi > 100 \) nm. At large \( \xi \) bulk power can be as much as four times larger than the surface power.

![FIG. 11. Calculated flux noise power due to the presence of spin impurities in the cylindrical wire, for the local and non-local cases with \( \xi = 100, 200, 500, 1000 \) nm. We separate flux noise into contributions from spins at the surface (we assumed spin density \( \sigma_2 = 5 \times 10^{12} \) cm\(^{-2} \)) and in the bulk (spin density \( \sigma_3 = 1 \times 10^{18} \) cm\(^{-3} \)).](image)

### VII. CONCLUSIONS

In summary, we presented a exact numerical method to calculate the space dependence of the vector potential \( A(r) \) and current density \( J(r) \) in non-local superconductors using exact integration of the Fredholm integral equations. We showed numerical results for a cylindrical wire, and applied them to calculations of inductance and flux noise per unit length in the infinite wire.

As expected, both \( A(r) \) and \( J(r) \) become more evenly distributed inside the wire as the Pippard coherence length \( \xi \) increases and non-local effects become important. The vector potential was shown to be well described by the local theory with a renormalized penetration depth \( \lambda_R = (\lambda^2 \xi^2)^{1/2} \). It is interesting to note that this choice of renormalized penetration depth \( \lambda_R \) is specific to the cylindrical surface with diffusive scattering and is different from the result obtained in a flat surface (For a flat surface, \( \lambda_R = 0.65(\lambda^2 \xi_0)^{1/2} \), see Appendix 3 in [10]).

However, in the presence of non-locality the London relation \( J(r) = -A(r)/(\mu_0 \lambda_R^2) \) ceases to hold, in that \( J(r) \) develops a peak away from the surface of the wire, see Fig. 2. We also find that \( J \) decreases slower than \( A \) towards the centre of the wire. Nevertheless, we find that a good approximation to calculate \( J \) is to evaluate the Pippard relation Eq. (1) explicitly using the local \( A \) with renormalized penetration depth. The comparison between this approximate scheme and the exact solution is shown in Fig. 5.

We showed that non-local electrodynamics has a large impact on device parameters. While for local superconductors kinetic inductance is always larger than internal inductance, the opposite happens for non-local superconductors. Increasing \( \xi \) greatly enhances the penetration of magnetic field inside the wire, making the internal inductance dominant over the kinetic inductance. While we did not show calculations of mutual inductance between two wires we can infer from our results that mutual inductance will also be enhanced as \( \xi \) increases.

We also considered the impact of non-locality on the flux noise produced by the time-dependent fluctuation of impurities with spin. The amount of noise caused by each impurity scales quadratically with the magnetic field at the impurity location, see Eqs. (22) and (23). This observation has led many authors to assume that spins at the surface are the only ones causing flux noise [13, 18, 19, 21, 23]. The fact that the magnetic field at the surface of a cylindrical wire only depends on the total current inside the wire implies that the flux noise due to impurities at the surface is independent of \( \xi \). This is in stark contrast to the flux noise due to impurities in the bulk. As \( \xi \) increases, more magnetic field penetrates inside the wire, making the impurities in the bulk more relevant. As a result, devices made of non-local superconductors can be more sensitive to spin impurities in the bulk rather than in the surface. It all depends on the relative sizes of the surface and bulk impurity densities, and of \( \xi \). For example, our Fig. 11 shows that when the impurities in the surface/bulk are separated by \( \sim 1 \) \( \mu \)m, the bulk contribution will become greater than the surface for \( \xi > 100 \) nm. The surface/bulk difference can be quite dramatic for small wires.

In conclusion, we demonstrated that non-local electro-
dynamics has a large effect in determining the inductance and flux noise properties of superconducting wires. We hope that our results will contribute to the modelling and design of superconducting devices with $\xi$ comparable to wire length scales, as is often the case for devices made of aluminum.

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