Estimation the Shape Parameter for Power Function Distribution

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Abstract. In the paper estimate of the shape parameter for power function distribution was proposed. For different sample sizes (small, medium, and large). Using different methods, Maximum likelihood method, Moment method, Shrinkage methods, and Least square method. mean square error (MSE) was implemented as an indicator of performance and comparisons of performance have been carried out through data analysis and computer simulation between the estimation methods according to the applied indicator. It was observed from the results that the shrinkage method (constant weight factor (\(sh_2\)) estimates for the shape parameter are the best in performance for each case.

1. Introduction

The power function distribution an important distribution for it is used in daily life and most distributions such as Rayleigh distribution, gamma distribution, and Weibull distribution has a relationship with the power function distribution. It has an inverse relationship with Pareto distribution the negative moments for Pareto distribution are simply the moment for the power function distribution \[1\]|2]. The first to use the power function distribution. Malik H J in 1967, found a precise expression of the moment for power function distribution and he studied the exact moment for power function distribution \[7\]. The power function distribution is flexible distribution and used in many applications such as engineering, economics, natural sciences, and applied mathematics. The power function distribution is a special case for Pareto family distributions\[3\]|[4]. The problem of estimating unknown parameters in statistical distribution is one of the most important problems. [10]. The reliability function, which is a monotonically decreasing function of the lifetime, the study of reliability is an important task to develop future plans to develop performance and quality for the equipment and estimate the reliability system in stress-strength when follows power function for different shape parameter \[12\].

The aim of this study is to comparisons of performance the shape parameter (\(\alpha\)) based on power function distribution (PD) have been carried out through data analysis and computer simulation between among four estimation methods, Maximum likelihood method (MLE), Moment method(MOM), Shrinkage methods(SH) and Least square method(LS) according to the applied indicator. and obtain the best using (MSE) and find the best method.

2. Methodology
Let $x$ be random variable from power distribution with shape parameter \((\alpha)\) and scale parameter \((\gamma)\); then the probability density function \((pdf)\) is defined as following \([3]\)

$$f(x, \alpha, \gamma) = \alpha \gamma^\alpha x^{\alpha-1}, \quad 0 < x < \gamma^{-1}$$ \hspace{1cm} (1)

When \(\alpha\) and \(\gamma\) are often called the shape and scale parameters

While the cumulative distribution function \((c.d.f)\) for power function distribution

$$F(x, \alpha, \gamma) = x^\alpha \gamma^\alpha, \quad 0 < x < \gamma^{-1}$$ \hspace{1cm} (2)

And as a special case when \(\gamma = 1\) the \((p.d.f)\) for power function distribution will be

$$f(x, \alpha) = \alpha x^{\alpha-1}, \quad 0 < x < 1$$ \hspace{1cm} (3)

And the \((c.d.f)\) for power function distribution become

$$F(x, \alpha) = x^\alpha, \quad 0 < x < 1$$ \hspace{1cm} (4)

For estimation of shape parameter for power function distribution, we need to find the random variable of cumulative distribution function then

$$x = \left[ f(x) \right]_1^\gamma$$ \hspace{1cm} (5)

3. Estimation methods of the shape parameter of power function distribution

3.1 Maximum likelihood method (MLE) \([8]/[11]\)

The Maximum likelihood method is one of the most reliable and popular methods using to obtaining point estimation for the parameter in all distributions, The Maximum likelihood method has many excellent statistical properties, so the statisticians prefer the MLE for other methods, Let \(x_1, x_2, \ldots, x_n\) be a random sample of \(pow(\alpha, 1)\). The Maximum likelihood for \(\alpha\)

$$L = L(\alpha_i, x_i) = \prod_{i=1}^{n} f(x_i)$$

$$L = \alpha^n \prod_{i=1}^{n} x_i^{\alpha - 1}$$ \hspace{1cm} (6)

Taking the logarithm of the likelihood, yields

$$Ln(L) = n \ln \alpha + (\alpha - 1) \sum_{i=1}^{n} \ln x_i$$ \hspace{1cm} (7)

The partial derivation for the equation with respect to the shape parameter and equating the result to zero given as follows.

$$\hat{\alpha}_{mle} = \frac{-n \sum_{i=1}^{n} \ln x_i}{\sum_{i=1}^{n} x_i}$$ \hspace{1cm} (8)

3.2 Moment method (MOM)
In this section, we estimate the shape parameter for power function distribution by using the simplest technique commonly is the moment method, the general idea for moment method to determine the population moment \( M_\alpha = E(x^\alpha) \) [9]. Let \( x^{\text{pow}(\alpha)} \)

\[
E(x) = \frac{\alpha}{\alpha + 1}
\]

(9)

We equate the first moment with \( \sum_{i=1}^{n} x_i / n \) as below

\[
\frac{\alpha}{\alpha + 1} = \frac{\sum_{i=1}^{n} x_i}{n}
\]

(10)

By solving equation (5) we get

\[
\hat{\alpha}_{\text{mom}} = \frac{\sum_{i=1}^{n} x_i}{n - \sum_{i=1}^{n} x_i}
\]

(11)

3.3 Least square method (LS) [4][5]

The least squares method can be applied in reliability engineering mathematics problem, we assume that there is a linear relationship between two variables.

\[
F(x_i) = x_i^\alpha
\]

\[
x_i = [F(x_i)]^{1/\alpha}
\]

(12)

Take logarithm for both sides

\[
\ln x_i = \frac{1}{\alpha} \ln F(x_i)
\]

(13)

\[
y = ax + b
\]

\[
x = \ln F(x_i)
\]

\[
a = \frac{\sum_{i=1}^{n} x_i y_i - \frac{\sum_{i=1}^{n} x_i y_i}{n}}{\sum_{i=1}^{n} x_i^2 - \left[\frac{\sum_{i=1}^{n} x_i}{n}\right]^2}
\]

(14)

\[
\hat{a}_{\text{ls}} = \frac{\sum_{i=1}^{n} [\ln F(x_i)]^2 - \frac{[\sum_{i=1}^{n} \ln F(x_i)]^2}{n}}{\sum_{i=1}^{n} \ln F(x_i) \ln x_i - \frac{[\sum_{i=1}^{n} \ln F(x_i)]^2}{n}}
\]

(15)

3.4 Shrinkage estimator (SH) [3]

In 1968, Thompson introduced the basic reason for use prior estimation. He suggested the problem of shrink in the usual estimator \( \hat{\alpha} \) of parameter \( \alpha \) toward prior information. The shrinkage estimation method is to use \( \alpha \) as the initial value prior information \( \alpha_0 \) from the past and usual estimator \( \hat{\alpha}_{\text{ub}} \) through consideration them by shrinkage- weight factor \( \Phi(\hat{\alpha}) \) where \( 0 < \Phi(\hat{\alpha}) < 1 \) as below

\[
\hat{\alpha}_{\text{sh}} = \Phi(\hat{\alpha}) \alpha_{ub} + [1 - \Phi(\hat{\alpha})] \alpha_0
\]

(16)

Now, to find \( \alpha_{ub} \) from \( \hat{\alpha}_{\text{mle}} \)

\[
\hat{\alpha}_{\text{mle}} = \frac{\sum_{i=1}^{n} x_i}{\sum_{i=1}^{n} \ln x_i}
\]

x \text{pow}(\alpha)

\[
z = \sum_{i=1}^{n} \ln x_i
\]

z \text{IG}(n, 1)
\[
\frac{n}{n-1} \hat{\beta}_{mle} \neq \hat{\beta}_{mle}
\]

\[
\alpha_{ub} = \frac{n}{n-1} \hat{\beta}_{mle}
\]

\[
\alpha_{ub} = \frac{-n}{(n-1)} \sum_{i=1}^{n} \ln x_i
\]

(17)

3.4.1 shrinkage- weight function \((sh_1)[3][6]\)

In this section, the shrinkage- weight factor will be assumed as a function of \(n\), \(\varnothing(\hat{\beta}) = \left|\frac{\sin n}{n}\right|\). Put in (10), will be

\[
\hat{\beta}_{sh1} = \left(1 - \left|\frac{\sin n}{n}\right|\right)\alpha_0
\]

(18)

3.4.2 constant- shrinkage weight factor \((sh_2)[3][6]\)

In this section, assumed \(\varnothing(\hat{\beta}) = \beta\) the constant- shrinkage weight factor and \(K=0.001, 1-K=0.999\) put in (10), will be

\[
\hat{\beta}_{sh2} = K\alpha_{ub} + (1 - K)\alpha_0
\]

(19)

3.4.3 Beta- shrinkage weight factor \((sh_3)[3][6]\)

In this section, \(\varnothing(\hat{\beta}) = \beta(1, n)\) the Beta- shrinkage weight factor, put in (10), will be

\[
\hat{\beta}_{sh3} = \beta(1, n)\alpha_{ub} + [1 - \beta(1, n)]\alpha_0
\]

(20)

4. Simulation study

Let \(x\) according to the uniform distribution on the interval \((0,1)\)

\[
F(x_i) = x_i^\alpha
\]

\(x_i = [F(x_i)]^{\frac{1}{\alpha}}, i = 1, 2, ..., n\)

Step 1. By equation (8) calculated \(\hat{\beta}_{mle}\)

Step 2. by equations (11) calculated \(\hat{\beta}_{mom}\).

Step 3. by equations (15) calculated \(\hat{\beta}_{LS}\).

Step 4. by equations (18),(19), and (20), calculated \(\hat{\beta}_{sh_i}\) for \(i = 1, 2, 3\)

Step 5. by \(MSE(\gamma) = \frac{\sum (\gamma - \gamma_i)^2}{n}\) calculated MSE for all method and cases

And by using random sample \(n=30, 50, 100\).

We obtain the result shown in the following tables

| Table 1: Estimation of the shape parameter of (PD) when alpha= 1 |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| n    | Mle     | Mom     | Sh1    | Sh2    | Sh3    | LS    |
| 30   | 1.028525 | 1.016048 | 0.934320 | 0.998005 | 0.933525 | 1.027646 |
| 50   | 1.019068 | 1.014803 | 0.989511 | 0.998001 | 0.960026 | 1.025030 |
| 100  | 1.007879 | 1.003996 | 0.989883 | 0.998002 | 0.980021 | 1.009319 |

| Table 2: Shown the MSE values when alpha= 1 |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| n    | mse_mle | mse_mom | mse_sh1 | mse_sh2 | mse_sh3 | mse_Ls | Best |
| 30   | 0.039919072 | 0.047077758 | 0.000039626 | 0.000000036 | 0.000004592 | 0.064047639 | mse_sh2 |
| 50   | 0.022481145 | 0.028972695 | 0.000000584 | 0.000000021 | 0.000008497 | 0.044030383 | mse_sh2 |
| 100  | 0.010430490 | 0.013606705 | 0.000000260 | 0.000000100 | 0.000001016 | 0.018322772 | mse_sh2 |

| Table 3: Table 1: Estimation of shape parameter of (PD) when alpha= 2 |
|----------------|----------------|----------------|----------------|----------------|----------------|----------------|
| n    | Mle     | Mom     | Sh1    | Sh2    | Sh3    | LS    |
| 30   | 2.058343 | 2.037616 | 1.868600 | 1.996010 | 1.867008 | 2.038601 |
Numerical result and conclusion

Consequently, for practical work and taking the mean square error as the indicator of preference between the different estimator methods, the following result obtained:

1. For the conventional methods and for different sample sizes the following results are obtained:
   - Small sample size (n=30)
     i. When alpha= 1, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     ii. When alpha= 2, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     iii. When alpha= 3, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
   - Medium sample size (n=50)
     i. When alpha= 1, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     ii. When alpha= 2, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     iii. When alpha= 3, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
   - Large sample size (n=100)
     i. When alpha= 1, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     ii. When alpha= 2, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.
     iii. When alpha= 3, Sh2 estimator method was given the best results, then his methods Sh1, Sh3, MLE, MOM, and LS respectively.

2. The result of the simulation (estimate the shape parameter) of power function distribution for studied methods for different sample sizes (n=30, 50, 100). From tables, it is clear that, the shrinkage (constant weight factor ($s_h^2$)) estimates for the $\alpha$ shape parameter, are the best in performance for each case. Then his first shrinkage way ($s_h^1$) comes after.
From the tables for each (n), we conclude that the suggested Shrinkage method has a good performance based on constant - weight factor ($sh_2$) has the minimum mean square error (MSE) in the majority of cases.

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