Breakdown of fiber bundles with stochastic load-redistribution

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Abstract

We study fracture processes within a stochastic fiber-bundle model where it is assumed that after the failure of a fiber, each intact fiber obtains a random fraction of the failing load. Within a Markov approximation, the breakdown properties of this model can be reduced to the solution of an integral equation. As examples we consider two different versions of this model that both can interpolate between global and local load redistribution. For the strength thresholds of the individual fibers, we consider a Weibull distribution and a uniform distribution, both truncated below a given initial stress. The breakdown behavior of our models is compared with corresponding results of other fiber-bundle models.

Keywords: fracture mechanics, fiber-bundle model, statistical physics, branching process

1. Introduction

Fracture processes in heterogeneous materials are an important technological problem that has attracted the interest of the scientific community since a long time [1–4]. Due to the complex interaction between failures and the subsequent redistribution of local stresses, the development of adequate statistical models for fracture propagation is an extremely hard and challenging undertaking. The probably most important class of approaches to the study of fracture processes is that of fiber-bundle models (FBM’s) [5–15]. Despite their simplicity, FBM’s are able to describe the main processes that can lead to a propagation of fractures and eventually to a complete breakdown of real heterogeneous materials.

Fiber-bundle models refer to a bundle of $N$ parallel fibers that are clamped at both ends and stretched by a common force $F$. The fibers have a stochastic distribution of individual strength thresholds, and the different versions of FBM’s that have been considered can be distinguished by their assumptions with respect to the stress redistribution after the failure of one of the fibers. The usual experimental setup considered and analyzed in the FBM literature can be described as follows: The force $F$ is gradually increased from zero until the weakest fiber breaks, and the transfer of its stress to the surviving fibers may then induce an avalanche of subsequent failures. If the fiber bundle reaches an equilibrium with no further failures, the force $F$ is increased again until the next fiber breaks, and this procedure is repeated up to the complete breakdown of the entire bundle. The main quantities of interest in connection with this procedure are the distribution of avalanche sizes and the ultimate strength of the fiber bundle, defined as the maximum stress $F/N$ the system can support before it breaks down completely. An alternative but equivalent procedure is to apply a finite force $F$ to the system, so that immediately all fibers with a strength threshold smaller than $F/N$ fail. The ultimate strength of the fiber bundle is then determined by the maximum $F/N$ value that does not lead to a failure of the entire system.

The oldest and most well-known FBM is that where the stress of a failing fiber is distributed equally between the surviving fibers (global load sharing, GLS) [5]. For this model, the strength of the fiber bundle as well as the form of the avalanche-size distribution can be determined analytically [6, 7].

Local load sharing (LLS) fiber-bundle models, on the other hand, are much more difficult to analyze [6, 8, 9]. In these models, the stress of a failing fiber is only transferred locally, typically to the surviving nearest neighbors. LLS models have been studied mainly via Monte-Carlo simulations and analytical results have only been obtained for one-dimensional models with essentially nearest-neighbor load transfer.

In most studies, the strength thresholds of the individual fibers are assumed to be distributed according to a Weibull distribution (typically with a Weibull index $k = 2$). For simplicity, however, uniform strength-threshold distributions (sometimes with a finite lower cutoff) have also been considered [6, 7, 11].

The two idealized extremes of global load sharing and of load transfer to nearest neighbors only are not adequate assumptions for most real systems. Attempts have therefore been made to interpolate between GLS and LLS behavior [10–13]. Hidalgo et al. [12], e.g., assume that the stress transfer after a failure decays as $1/r^\gamma$, where $r$ is the distance from the broken fiber, and they study the failure-propagation behavior of such a system as a function of $\gamma$. A similar model (with $\gamma = 3$) has been studied by Curtin [14].

In Ref. [16], we have introduced a class of failure-propagation models that can represent, in a stochastic sense, the main characteristics of realistic load-redistribution mechanisms, but are still amenable to an analytical treatment. We have applied our approach to an illustrative prototype exam-
ample for cascading failure propagation in large infrastructure networks, e.g., power grids. In particular, we have analyzed the probability of a complete system breakdown after an initial failure, and we have found that the model exhibits interesting critical dependencies on parameters that characterize the failure tolerance of the individual elements and the range of load redistribution after a failure.

In this paper, we apply our stochastic approach to the problem of fracture propagation in fiber bundles and analyze two models that interpolate between global and local stress redistribution. The first is a stochastic version of the \(r^{-1}\)-model of Hidalgo et al. [12], and the second a fiber-bundle equivalent to our prototype example of Ref. [16]. We consider an experimental setup where initially all fibers carry the same stress \(\sigma_0\) and where the individual strength thresholds \(\sigma_{th}\) are randomly distributed according to a probability density \(p_{th}(\sigma_{th})\) that is zero for \(\sigma_{th} < \sigma_0\). We then examine the consequences of the breaking of a single fiber and concentrate on the calculation of the following quantities:

(i) the no-cascade probability \(P_{nc}(\sigma_0)\), i.e., the probability that an initial failure does not induce any further failures;
(ii) the breakdown probability \(P_b(\sigma_0)\), i.e., the probability that an initial failure leads to a breakdown of the entire fiber bundle; and
(iii) the critical stress \(\sigma_0^c\), defined as the largest \(\sigma_0^c\) such that \(P_b(\sigma_0) = 0\) for all \(\sigma_0 \leq \sigma_0^c\).

We note that, in contrast to most of the other fiber-bundle models, our stochastic models neglect any spatial correlations in the load transfer after a failure. The only other model, as far as we are aware of, that also uses a stochastic (rather than a spatially correlated) stress redistribution is that of Dalton et al. [15], where it is assumed that the load of a failing fiber is transferred to a fixed number \((n = 1, 2, \ldots)\) of randomly chosen surviving fibers.

In addition, we use strength-threshold distributions that are truncated below the initially applied stress \(\sigma_0\)—in contrast to most models studied in the fiber-bundle literature. There exist, however, a number of investigations that also consider truncated threshold distributions [7, 11, 4], so that a direct comparison with our results can be made.

In Sect. 2, we introduce our stochastic load-redistribution model and describe its application to fracture processes. Subsequently, in Sect. 3, we describe a Markov approximation of the model which leads to a description in terms of generalized branching processes. In Sects. 4 and 5, we analyze the two different model variants mentioned above, and final conclusions are given in Sect. 6.

2. Stochastic load-redistribution model

2.1. Load-redistribution rule and cascade model

We shall consider a bundle consisting of \(N\) fibers subjected to an external force \(F\). We assume that the initial stress \(\sigma_0 = F/N\) of all fibers is equal and that the strength thresholds \(\sigma_{th}\) of the individual fibers are randomly distributed according to a probability density \(p_{th}(\sigma_{th})\). For our setup of fracture-propagation experiments, we assume that we start from a finite stress \(\sigma_0 > 0\) and that all fibers with thresholds smaller than \(\sigma_0\) have been removed [4]. Thus, the threshold distribution has to fulfill \(p_{th}(\sigma_{th}) = 0\) for all \(\sigma_{th} < \sigma_0\). On the other hand, we are interested in a situation where already an infinitesimal increase of the external force leads—with probability one in the limit \(N \to \infty\)—to the breaking of exactly one fiber, and we thus require \(p_{th}(\sigma_{th} = \sigma_0) > 0\).

When a fiber breaks, its stress has to be taken over by the remaining intact fibers of the bundle. In our stochastic load-redistribution model [16], we assume that this process can be described by a rule of the form

\[
\sigma \to \sigma' = \sigma + \sigma_1 \Delta.
\]  

Here, \(\sigma (\sigma')\) is the stress of an intact fiber before (after) the failure of a fiber with stress \(\sigma_1\), and the load-redistribution factor \(\Delta\) is a random number drawn independently from the same distribution \(p_{\Delta}(\Delta)\) for each of the intact fibers. Note that for the initial failure, both \(\sigma\) and \(\sigma_1\) are given by the initial stress \(\sigma_0\). In the special case \(\Delta = 1/(N - 1)\) of a uniform load-redistribution, the form (1) reduces exactly to a GLS rule. For a general non-uniform load redistribution, we require that the failed stress will be shared on average by the remaining intact elements, i.e., the mean of \(\Delta\) has to fulfill the condition

\[
\langle \Delta \rangle = \frac{1}{N-1}.
\]

Due to the stress increment a fiber has obtained after a failure event, its stress itself might be above its critical threshold. In general, the initially failing fiber might thus induce the failure of \(N^{(1)}_1 \geq 1\) other fibers, thereby starting a failure cascade. We then assume that all overloaded fibers fail simultaneously and that their stress is again redistributed according to the rule (1), where now both the pre-failure stresses \(\sigma_1\) of each of the \(N - N^{(1)}_1\) intact fibers and the stresses \(\sigma_1\) of each of the \(N^{(1)}_f\) failing fibers will, in general, be different random variables. If this process leads to the further overloading of \(N^{(2)}_g \geq 1\) fibers, it continues to a next cascade stage, and so on. Eventually, either the bundle stabilizes again, i.e., all fibers are stressed below their respective strength thresholds, or it breaks down completely, i.e., all \(N\) fibers fail.

Note that, in general, during the failure cascade, the load redistribution and hence the \(\Delta\)-distribution will be modified. Whereas the details of such a modification, which can depend on topological changes, are very difficult to model, one at least has to take into account one dominant effect: As the number of fibers \(N^{(s)}_{th}\) that are still intact at cascade stage \(s\) decreases, the mean \(\langle \Delta \rangle\) has to increase in accordance with Eq. (2) with \(N\) replaced by \(N^{(s)}_{th}\). Below, we will discuss how to fulfill this requirement for the chosen forms of load redistribution.

2.2. Truncated strength-threshold distributions

For our setup of fracture-propagation experiments, we have to truncate the distribution of strength thresholds below \(\sigma_{th} = \sigma_0\). In the literature on fiber-bundle models, the strength thresholds \(\sigma_{th}\) of the individual fibers are usually assumed to be...
distributed according to a Weibull distribution with density
\( k \sigma_{th}^{k-1} \exp(-\sigma_{th}^{k}) \), where mostly the Weibull index \( k = 2 \) is used. Truncation then leads to a distribution of the form
\[
p_{th}(\sigma_{th}) = \begin{cases} k \sigma_{th}^{k-1} \exp(-\sigma_{th}^{k}) & \text{if } \sigma_{th} \leq \sigma_{th} \\ 0 & \text{else.} \end{cases}
\] (3)

In addition, we shall also consider uniform distributions that are truncated below the initial stress \( \sigma_{0} \):
\[
p_{th}(\sigma_{th}) = \begin{cases} \frac{1}{1 - \sigma_{0}} & \text{if } \sigma_{0} \leq \sigma_{th} \leq 1 \\ 0 & \text{else.} \end{cases}
\] (4)

In Sects. 4 and 5, we summarize and discuss the corresponding results for \( P_{nc}(\sigma_{0}) \), \( P_{b}(\sigma_{0}) \) and \( \sigma_{0}^{0} \) for two different load-redistribution models and for the two threshold distributions (3) and (4).

3. Generalized-branching-process approximation

The dynamics of the stochastic cascade model described in the previous section and the quantities \( P_{nc}(\sigma_{0}) \), \( P_{b}(\sigma_{0}) \) and \( \sigma_{0}^{0} \) can only be obtained exactly by means of Monte-Carlo simulations. In the limit of large system sizes \( N \to \infty \), however, we can achieve an approximate description of the cascade dynamics by noting the following points:

(i) The failure of a fiber leaves the stress in the majority of the intact fibers nearly unaffected, i.e., maximally leads to changes of the order of \( 1/N \). Thus, along the failure cascade, the stress of the intact fibers is approximately given by the initial stress \( \sigma_{0} \).

(ii) The remaining number of fibers always stays infinitely large and thus the number of further failures induced by a failing fiber is distributed according to a Poisson distribution.

(iii) The interaction between different failures can be neglected, i.e., in the case of several induced failures, the failure cascades resulting from each of these failures can be treated as being independent.

Under these assumptions, the cascade dynamics becomes Markovian [18] if we choose the point process of the stresses of the failed fibers as underlying state space. This point process on the semi-infinite interval \([\sigma_{0}, \infty)\) is independent [17], and the failure dynamics can thus be described by a generalized branching process [19] with characteristic functional
\[
G[u; \sigma_{t}] = \exp\left\{ \mu(\sigma_{t}) \left[ \int \mathrm{d}\sigma_{t}' \ P(\sigma_{t}'|\sigma_{t} > \sigma_{th}; \sigma_{t}) \ e^{-u(\sigma_{t}') - 1} \right] \right\}
\] (5)

for the point process induced by a single failure with stress \( \sigma_{t} \). Here, \( P(\sigma_{t}'|\sigma_{t} > \sigma_{th}; \sigma_{t}) \) denotes the conditional probability density that the induced failure resulting from the breaking of a fiber with stress \( \sigma_{t} \) occurs with a stress \( \sigma_{t}' \). For given distributions of the load-redistribution factors \( \Lambda \) and the critical thresholds \( \sigma_{th} \), this quantity can be readily calculated from Eq. (1). This also holds true for the mean number of failures,
\[
\mu(\sigma_{t}) = (N - 1) P(\sigma_{t}' > \sigma_{th}|\sigma_{t}).
\] (6)

induced by the breaking of a fiber with stress \( \sigma_{t} \). We remark that in order for this quantity to be finite in the limit \( N \to \infty \), the probability \( P(\sigma_{t}' > \sigma_{th}|\sigma_{t}) \) for the induced failure of a given intact fiber has to vanish as \( 1/N \). This is in accordance with the requirement (2) for the mean of the load-redistribution factors \( \Lambda \). Finally, in Eq. (5), \( u \) denotes an arbitrary non-negative test function on the interval \([\sigma_{0}, \infty)\).

For later use, we note that from the mean number of induced failures, one directly obtains the no-cascade probability, which is given by
\[
P_{nc}(\sigma_{0}) = [1 - P(\sigma_{t}' > \sigma_{th}|\sigma_{t})]^{N-1}.
\] (7)

In the limit \( N \to \infty \), this relation becomes
\[
P_{nc}(\sigma_{0}) = \exp\{-\mu(\sigma_{0})\}.
\] (8)

In principle, the properties of the later cascade stages and thus the full cascade dynamics can be obtained in a recursive way from the functional (5) (see Ref. [19]). Here, we are only interested in the question whether an initial failure leads to the breakdown of the entire fiber bundle. It can be shown [19] that this question can be answered by solving the integral equation
\[
1 - P_{b}(\sigma_{t}) = \exp\left\{ -\mu(\sigma_{t}) \int_{\sigma_{t}}^{\infty} \mathrm{d}\sigma_{t}' P_b(\sigma_{t}') | \sigma_{t}' > \sigma_{th}; \sigma_{t} \right\} \times P_{b}(\sigma_{t}')
\] (9)

for the probability \( P_{b}(\sigma_{t}) \) that an initial failure with stress \( \sigma_{t} \) leads to the breakdown of the entire bundle. Using Eq. (6) and the load-redistribution rule (1), together with the fact that, within the approximation considered, the pre-failure stress \( \sigma \) is just given by the initial stress \( \sigma_{0} \), we can rewrite the integral equation (9) in the form
\[
1 - P_{b}(\sigma_{t}) = \exp\left\{ -\frac{1}{\sigma_{t}} \int_{\sigma_{t}}^{\infty} \mathrm{d}\sigma_{t}' F_{\sigma_{t}}(\sigma_{t}') \hat{P}_{\Lambda}\left(\frac{\sigma_{t}' - \sigma_{0}}{\sigma_{t}}\right) \right\} \times P_{b}(\sigma_{t}')
\] (10)

where \( \hat{P}_{\Lambda}(\Delta) = \lim_{N \to \infty} N P_{\Lambda}(\Delta) \). Here, we have also used the relation
\[
p(\sigma_{t}'|\sigma_{t} > \sigma_{th}; \sigma_{t}) P(\sigma_{t}' > \sigma_{th}|\sigma_{t}) = p(\sigma_{t}'|\sigma_{t} > \sigma_{th}|\sigma_{t})
\] (11)

and introduced the cumulative distribution function \( F_{\sigma_{t}}(\sigma_{t}) = \int_{\sigma_{t}}^{\infty} \mathrm{d}\sigma_{t}' p_{\sigma_{t}}(\sigma_{t}') \) corresponding to the threshold distribution \( P_{\sigma_{t}}(\sigma_{t}) \). Under quite general assumptions [19], the unique solution of the integral equations (9) or (10) can be found by means of an iterative procedure starting from an arbitrary initial guess for \( P_{b}(\sigma_{t}) \). The so obtained probability function can then be evaluated at the initial stress \( \sigma_{0} \) to obtain the probability \( P_{b}(\sigma_{0}) \) for the breakdown of a fiber bundle in the setup described in Sect. 1.

We finally remark that the interpretation of Eq. (9) becomes clear if one writes the exponential on the right-hand side in the form
\[
1 - P_{b}(\sigma_{t}) = \lim_{N \to \infty} \left[ 1 - \int \mathrm{d}\sigma_{t}' p(\sigma_{t}'|\sigma_{t} > \sigma_{th}|\sigma_{t}) P_{b}(\sigma_{t}') \right]^{N},
\] (12)
where we have again used Eqs. (6) and (11). Hence, the probability that no breakdown occurs after the failure of a fiber with stress \( \sigma_f \) is equal—in the limit \( N \to \infty \)—to the probability that none of the induced failures with stress \( \sigma'_f \) leads to a breakdown.

4. Stochastic model for range-dependent load redistribution (\( \gamma \)-model)

Hidalgo et al. [12] proposed a FBM where the stress transfer after a failure decays with the distance \( r \) between the failing fiber and the one affected by the failure as a power law \( Z/r^\gamma \). Here, \( Z \) is a normalization factor which ensures that the total load is conserved. They furthermore assumed that all fibers are arranged on a two-dimensional square lattice. By varying the exponent \( \gamma > 0 \), they were then able to study the transition between a GLS rule (for \( \gamma \to 0 \)) and a LLS rule (for \( \gamma \to \infty \)). Note that due to the infinite range of the power-law transfer function, the latter situation of a strictly local load-transfer can only be achieved in an approximate sense. For the case of Weibull-distributed strength thresholds, a Monte-Carlo analysis of this range-dependent load-transfer model showed that for an exponent \( \gamma \lesssim 2 \), the model behaves essentially as a FBM with GLS rule and, in particular, a finite critical stress value was observed. For larger \( \gamma \), there is a transition to the LLS case with a critical stress that vanishes in the large system-size limit. In Ref. [11], the same model has been analyzed for uniform threshold distributions with a lower cutoff \( \sigma_L \), and in this case, the critical stress remains finite for all values of \( \gamma \) (if \( \sigma_L > 0 \)).

In the following, we study a stochastic version of this model, which we shall call “\( \gamma \)-model”. It is based on the assumption that the position of the fibers is uniformly distributed within the two-dimensional cross-section of the bundle. Upon failure of a fiber we then randomly pick the affected fibers from this uniform distribution and calculate the load-transfer factor \( \Delta \) according to the (random) distance. We will now first derive the corresponding \( \Delta \)-distribution, then analyze the properties of the resulting model, and finally compare its results to the ones obtained in Ref. [11].

4.1. Distribution of load-redistribution factors

For reasons of simplicity, we assume that the broken fiber is in the center of a hollow cylinder with inner (outer) radius \( r_{\text{min}} \) (\( r_{\text{max}} \)) containing \( N-1 \) intact fibers uniformly distributed with area density \( \varrho \) in the cross-sectional area of size \( A = \pi r_{\text{max}}^2 - \pi r_{\text{min}}^2 \). Note that in contrast to Refs. [11, 12], we consider a uniform distribution of the fiber positions and thus have to introduce a lower cutoff \( \Delta \) for the distance \( r \) to prevent a divergence at small distances.

For this uniform spatial distribution and the given distance dependence of the load transfer, we can then readily derive the probability distribution for the load-redistribution factors \( \Delta \) appearing in Eq. (1):

\[
p_\Delta(\Delta) := \frac{2\pi}{A} \int_{r_{\text{min}}}^{r_{\text{max}}} \text{d}r \frac{\Delta}{(\Delta - Z/r^\gamma)}
\]

\[
to \frac{2\pi Z^2}{A\gamma} \Delta - \frac{2\pi Z^2}{A\gamma} \Delta
\]

\[
\text{otherwise.}
\]

Here, the lower and upper cutoffs for \( \Delta \) are given by \( \Delta_{\text{min}} = Z/r_{\text{max}}^\gamma \) and \( \Delta_{\text{max}} = Z/r_{\text{min}}^\gamma \), respectively. In the stochastic model, we fix the constant \( Z \) by imposing Eq. (2), which states that, on average, the load is redistributed to the remaining elements. This yields the normalization constant

\[
Z = \frac{2 - \gamma}{2(N-1)} \frac{r_{\text{max}}^2 - r_{\text{min}}^2}{r_{\text{max}}^{2\gamma} - r_{\text{min}}^{2\gamma}}
\]

for \( \gamma \neq 2 \); the special case \( \gamma = 2 \) can be readily treated by considering the limit \( \gamma \to 2 \).

We can write Eqs. (14) and (15) in a more convenient form by introducing a dimensionless length \( L = r_{\text{max}}/r_{\text{min}} \). This first allows us to express the number of intact fibers as

\[
N - 1 = s (L^2 - 1) ,
\]

where \( s = \pi r_{\text{min}}^2 \varrho \approx 1 \) is the average number of fibers in the vicinity of the failing fiber. From Eq. (16), we obtain, e.g., \( s = \pi/4 \) if we assume that the model describes the continuous approximation of fibers located on a quadratic lattice (with lattice constant \( r_{\text{min}} \)) consisting of \( N \sim \pi (L/2)^2 \) sites inside a circle of the given radius. Furthermore, we can write the \( \Delta \)-cutoffs in the form

\[
\Delta_{\text{min}} = \frac{1}{2s} \frac{1}{L^2} \frac{2 - \gamma}{1 - L^2 - 1} \frac{1}{r_{\text{max}}^\gamma} \gamma
\]

\[
\Delta_{\text{max}} = \frac{1}{2s} \frac{1}{L^2 - 1} \frac{2 - \gamma}{L^2 - 1} = L^2 \Delta_{\text{min}} ,
\]

where again the case \( \gamma = 2 \) has to be treated as limit. The probability distribution is then given by the power-law form

\[
p_\Delta(\Delta) = \frac{2 - \gamma}{L^2 - 1} \frac{1}{\Delta_{\text{max}}} \left( \frac{\Delta}{\Delta_{\text{max}}} \right)^{-\frac{2\gamma}{\gamma - 2}}
\]

for \( \Delta_{\text{min}} \leq \Delta \leq \Delta_{\text{max}} \).

For later use, we note that in the limit of large system sizes, the lower \( \Delta \)-cutoff always scales to zero: \( \Delta_{\text{min}} \to 0 \) for \( L \to \infty \). The behavior of the upper cutoff, however, strongly depends on the exponent \( \gamma \): For \( \gamma \leq 2 \), \( \Delta_{\text{max}} \) also vanishes in the limit \( L \to \infty \), whereas for \( \gamma > 2 \), it converges to a finite value: \( \Delta_{\text{max}} \to (\gamma - 2)/2s \) for \( L \to \infty \).

4.2. Mean number of directly induced fiber failures

We now calculate the mean number \( \mu(\sigma_f) \) of fiber failures resulting directly from the breaking of a fiber with stress \( \sigma_f \). From rule (1) and Eq. (6), we obtain

\[
\mu(\sigma_f) = (N - 1) P(\sigma_f > \sigma_0)\Delta > \sigma_0)
\]

\[
 = (N - 1) \int \text{d}\Delta p_\Delta(\Delta) F_\sigma(\sigma_0 + \sigma_1 \Delta) .
\]
With Eqs. (16)–(19), we can write this expression in the form

\[ \mu(\sigma_l) = \frac{2s}{\gamma} \int_L^\infty dx x^{-\frac{\gamma+1}{2}} F_{\text{th}}(\sigma_0 + \sigma_l \Delta_{\text{max}} x) \]. \hspace{1cm} (21)

For the evaluation of this integral in the limit \( L \to \infty \), where the integrand becomes singular, it is useful to consider the cases \( \gamma \leq 2 \) and \( \gamma > 2 \) separately.

(i) \( \gamma \leq 2 \): As mentioned above, we then have \( \Delta_{\text{max}} \to 0 \) for \( L \to \infty \) and thus consider the Taylor expansion

\[ F_{\text{th}}(\sigma_0 + \sigma_l \Delta_{\text{max}} x) = \sum_{l=1}^{\infty} \frac{(\sigma_l \Delta_{\text{max}} x)^l}{l!} F_{\text{th}}(\sigma_0) \], \hspace{1cm} (22)

where we have used that \( F_{\text{th}}(\sigma_0) = 0 \). Inserting this expansion into Eq. (21), we find with Eq. (18) that only the first order term \( l=1 \) leads to a non-vanishing contribution in the limit \( L \to \infty \). This yields

\[ \mu(\sigma_l) \to \sigma_l p_{\text{th}}(\sigma_0) \]. \hspace{1cm} (23)

Note that this result is independent of the exponent \( \gamma \).

(ii) \( \gamma > 2 \): In this case, we can insert the finite asymptotic value for \( \Delta_{\text{max}} = (\gamma - 2)/2s \) for \( L \to \infty \), into Eq. (21) and write the mean number of failures as the improper integral

\[ \mu(\sigma_l) = \frac{2s}{\gamma} \int_0^\infty dx x^{-\frac{\gamma}{2}} F_{\text{th}}(\sigma_0 + \sigma_l (\gamma - 2)/(2s) x) \]. \hspace{1cm} (24)

Here, the convergence of the integral at the lower boundary is guaranteed since, because the strength-threshold distribution is truncated below \( \sigma_0 \), we have \( F_{\text{th}}(\sigma_0 + \sigma_l (\gamma - 2)/(2s) x) = O(x) \).

4.3. Breakdown probability

For the evaluation of the breakdown probability, we have to solve the integral equation (10), which in the present case assumes the form

\[ 1 - P_b(\sigma_l) = \exp \left\{ - \frac{2s}{\gamma \sigma_l \Delta_{\text{max}}} \int_{\sigma_0 + \sigma_l \Delta_{\text{max}}}^{\sigma_0 + \sigma_l \Delta_{\text{max}}} d\sigma_l' \frac{\sigma_l' - \sigma_0}{\gamma \sigma_l' \Delta_{\text{max}}} \right\} \times F_{\text{th}}(\sigma_l') P_b(\sigma_l') \]. \hspace{1cm} (25)

Alternatively, we can write this equation as

\[ 1 - P_b(\sigma_l) = \exp \left\{ - \frac{2s}{\gamma} \int_{\infty}^\infty dx x^{-\frac{\gamma}{2}} F_{\text{th}}(\sigma_0 + \sigma_l \Delta_{\text{max}} x) \right\} \times P_b(\sigma_0 + \sigma_l \Delta_{\text{max}} \Delta_{\text{max}}) \]. \hspace{1cm} (26)

The integral on the right-hand side of this equation has to be evaluated in the limit of infinitely large systems (\( L \to \infty \)). Again, we treat the cases \( \gamma \leq 2 \) and \( \gamma > 2 \) separately.

(i) \( \gamma \leq 2 \): As above, in Eq. (22), we expand the distribution function \( F_{\text{th}}(\sigma_l) \) around \( \sigma_{\text{th}} = \sigma_0 \) and now furthermore assume that such an expansion is also valid for the breakdown probability,

\[ P_b(\sigma_0 + \sigma_l \Delta_{\text{max}} \Delta_{\text{max}}) = \sum_{m=0}^{\infty} \frac{(\sigma_l \Delta_{\text{max}})^m}{m!} p^{(m)}(\sigma_0) \]. \hspace{1cm} (27)

Inserting these expansions into the integral in Eq. (26), we find with Eq. (18) that the various terms behave as \( L^{2(1-\gamma)} \) for large \( L \). Thus, only the lowest order terms \( l = 1 \) and \( m = 0 \) survive in this limit. The integral equation (26) hence simplifies to the transcendental equation

\[ 1 - P_b(\sigma_l) = \exp \left[ - \sigma_l p_{\text{th}}(\sigma_0) P_b(\sigma_0) \right], \hspace{1cm} (28) \]

which, again, is \( \gamma \)-independent. In this regime, the critical stress \( \sigma^c_0 \) is given by the condition that the mean number of directly induced failures equals unity:

\[ \mu(\sigma^c_0) = \sigma^c_0 p_{\text{th}}(\sigma_0) = 1 \hspace{1cm} (29) \]

(ii) \( \gamma > 2 \): As for the evaluation of the mean number of induced failures, cf. Eq. (24), we use the asymptotic value of \( \Delta_{\text{max}} \) and replace the integral in the limit \( L \to \infty \) by an improper one [cf. also remark after Eq. (24)]. This leads to the integral equation

\[ 1 - P_b(\sigma_l) = \exp \left\{ - \frac{2s}{\gamma} \int_0^\infty dx x^{-\frac{\gamma}{2}} F_{\text{th}}(\sigma_0 + \sigma_l (\gamma - 2)/(2s) x) \right\} \times P_b(\sigma_0 + \sigma_l (\gamma - 2)/(2s) x) \hspace{1cm} (30) \]

which, in general, can only be solved numerically, e.g., by means of an iterative procedure. In particular, the critical stress \( \sigma^c_0 \) is not determined by a simple relation like Eq. (29) but has to be determined from the full solution of Eq. (30).

4.4. Results

4.4.1. Uniform distribution of strength thresholds

In the case of a uniform distribution (4) of the strength thresholds, the mean number of induced failures (21) can be evaluated explicitly. With \( \sigma_l = \sigma_0 \) we then obtain from Eq. (8) the no-cascade probability

\[ P_{nc}(\sigma_0) = \exp \left\{ - \frac{\sigma_0}{1 - \sigma_0} \right\} \hspace{1cm} (31a) \]

if \( \gamma < 2 \) or \( \sigma_0 \leq [1 + (\gamma - 2)/(2s)]^{-1} \), and

\[ P_{nc}(\sigma_0) = \exp \left\{ s - \frac{\gamma}{2} \left[ (\gamma - 2)/(2s) \right]^{\gamma-1} \left[ \frac{\sigma_0}{1 - \sigma_0} \right]^{2/\gamma} \right\} \hspace{1cm} (31b) \]

otherwise. For the calculation of the breakdown probability \( P_b(\sigma_0) \), the transcendental equation (28) (for \( \gamma \leq 2 \) or the integral equation (30) (for \( \gamma > 2 \)) have to be solved numerically.

In Fig. 1 we show the no-cascade probability \( P_{nc}(\sigma_0) \) and the breakdown probability \( P_b(\sigma_0) \) as a function of the initial stress \( \sigma_0 \) for different values of the exponent \( \gamma \) in Eq. (19). The
The value of the critical stress on the exponent down of the fiber bundle vanishes exactly. The dependence of the critical stress, as is shown in Fig. 1(a), is that condition (2) stays fulfilled during the entire cascade process. This is done by replacing \( \gamma \) in Eqs. (17)–(19) by \( L_{\text{eff}} = \sqrt{N_{\text{in}}/s + 1} \), where \( N_{\text{in}} \) denotes the number of remaining intact fibers.

With increasing initial stress \( \sigma_0 \), we observe a gradual decrease of the no-cascade probability from one to zero [cf. Fig. 1(a)]. In contrast, the breakdown probability [Fig. 1(b)] exhibits a critical behavior: There is a \( \gamma \)-dependent critical stress \( \sigma_c^\gamma \) such that for \( \sigma_0 \leq \sigma_c^\gamma \), the probability of a breakdown of the fiber bundle vanishes exactly. The dependence of the critical stress on the exponent \( \gamma \) is illustrated in Fig. 2. The value \( \sigma_c^\gamma = 1/2 \) for \( \gamma \leq 2 \) can be readily determined from Eqs. (4) and (29) and exactly reproduces the value of the FBM with GLS rule [10]. For \( \gamma > 2 \), and hence smaller effective “range” of the stress redistribution, we first observe a transition to a regime, where the critical stress decreases with increasing \( \gamma \) down to a minimal value \( \sigma_c^\gamma \approx 0.26 \). For even larger \( \gamma \), the critical stress increases again. We remark, however, that for large \( \gamma \), it becomes numerically rather difficult to find the precise location of the critical transition because the onset of the regime with a finite breakdown probability becomes more and more flat [cf. Fig. 1(b)].

It is interesting to compare our results with the ones obtained for the variable-range load-redistribution model of Ref. [11], in particular, the behavior for the case of the failure stress being equal to the cutoff stress [cf. Fig. 1(a) of Ref. [11]]. In both models, we observe a critical value of \( \gamma \approx 2 \) above which a transition from a GLS regime to one with short-ranged stress transfer and smaller \( \sigma_c^\gamma \)-value takes place. Within our model, we can trace back this transition to a change in the load-redistribution distribution, in particular, the asymptotic value of \( \Delta_{\text{max}} \). Furthermore, the critical stress values of both models agree rather well up to \( \gamma \lesssim 7 \). For even larger exponents \( \gamma \), we find an increase of the critical stress, which cannot be observed in the more microscopic model of Ref. [11]. This discrepancy probably results from a breakdown of the continuum approximation upon which the distribution (13) is based. The deficiency of this approximation for large \( \gamma \) is also reflected in the fact that for \( \gamma > 2 + 2s \), the asymptotic value for \( \Delta_{\text{max}} \) becomes larger than unity, which means that a single fiber may receive a stress increment that is higher than the stress of the failing fiber. In order to prevent such a pathological behavior, a more sophisticated load-redistribution model has to be used.

We finally note that the results from the Markov approximation, i.e., the generalized branching process description, agree very well with the ones obtained by Monte-Carlo simulations of the failure process. Around the critical transition, some deviations for the breakdown probability can be observed, which, however, decrease with increasing system size \( N \) and, thus, represent finite-size effects [16].

4.4.2. Weibull distribution of strength thresholds

For the case of the truncated Weibull distribution (3) of strength thresholds, we have to evaluate for \( \gamma > 2 \) both the no-cascade probability and the breakdown probability numerically from Eqs.(24) and (30), respectively.

The results as a function of the initial stress \( \sigma_0 \) are depicted in Fig. 3, where we have chosen here and in the following a Weibull index of \( k = 2 \). Comparing with the case of a uniform distribution of the strength thresholds, we find qualitatively the same behavior. In particular, we identify a critical transition at a \( \gamma \)-dependent stress \( \sigma_c^\gamma \) (cf. Fig. 2), and again, for \( \gamma \leq 2 \), the
result \( \sigma_0^\gamma = (1/k)^{1/k} \) for the GLS case [5] is recovered exactly from Eq. (29) together with the distribution (3).

Figure 4 shows the dependence of the breakdown probability on the exponent \( \gamma \) for fixed initial stress \( \sigma_0 \). In accordance with Eq. (28), the breakdown probability is \( \gamma \)-independent for \( \gamma \leq 2 \) and assumes the GLS value. In the case \( \gamma > 2 \), we find for \( \sigma_0 \geq 1 \) a regime with a monotonic decrease of \( P_b \) towards zero as a function of \( \gamma \). For smaller \( \sigma_0 \), the breakdown probability assumes a maximum at a certain \( \gamma \)-value and then decreases again towards zero. Finally, for \( \sigma_0 \) smaller than the critical stress of the GLS model but larger than the minimal stress observed in Fig. 2, an increase of \( \gamma \) eventually leads to a destabilization of the system, i.e., a non-vanishing breakdown probability, above some critical \( \gamma \)-value.

5. Simple bimodal load-redistribution model (\( \Delta_0 \)-model)

In Ref. [16], we have introduced a simple prototype model that interpolates between the limiting cases of global load redistribution and the transfer of the failing load to a single other element.

The model, which we shall call “\( \Delta_0 \)-model”, is characterized by a bimodal distribution of the load-redistribution factors \( \Delta \),

\[
\Delta = \begin{cases} 
\Delta_0 & \text{with probability } p_0 \\
0 & \text{with probability } 1 - p_0.
\end{cases} \tag{32}
\]

i.e., after the failure of an element with stress \( \sigma_f \), the stress \( \sigma \) of a still intact element is increased to \( \sigma' = \sigma + \Delta_0 \sigma_f \) with probability \( p_0 \) and remains unchanged with probability \( 1 - p_0 \).

We further require that the sum of the induced stress increments is, on average, equal to the stress of the failing element and that \( \Delta_0 \leq 1 \). It follows that

\[
p_0 \Delta_0 = \frac{1}{N-1}, \quad \frac{1}{N-1} \leq \Delta_0 \leq 1. \tag{33}
\]

\( \Delta_0 = 1/(N-1) \) then corresponds to the limiting case of global stress redistribution and \( \Delta_0 = 1 \) to the case where the failing load is transferred, on average, to a single other element.

The probability that after the failure of a fiber with stress \( \sigma_f \), a still intact fiber also fails can be written as

\[
p_0 P(\sigma_{th} < \sigma_0 + \Delta_0 \sigma_f) = \frac{1}{N-1} \Delta_0 P(\sigma_{th} < \sigma_0 + \Delta_0 \sigma_f) \tag{34}
\]

and the mean number of induced failures becomes

\[
\mu(\sigma_f) = \frac{1}{\Delta_0} P(\sigma_{th} < \sigma_0 + \Delta_0 \sigma_f). \tag{35}
\]

In these expressions, we have neglected that a fraction of the still intact fibers at later cascade stages may carry a stress larger than \( \sigma_0 \). It can be shown, however, that this is a finite-size effect, i.e., this fraction vanishes as \( N \to \infty \) [16].

The no-cascade probability then follows directly from Eq. (8) by using Eq. (35) with \( \sigma_0 = \sigma_0' \):

\[
P_{nc}(\sigma_0) = \exp \left[ -\frac{1}{\Delta_0} P(\sigma_{th} < \sigma_0(1 + \Delta_0)) \right]. \tag{36}
\]

To calculate the breakdown probability \( P_b(\sigma_0) \), we use Eq. (10), which reduces in the present case to the recursion relation

\[
P_b(\sigma_n) = 1 - \exp[-\mu(\sigma_n) P_b(\sigma_{n+1})], \tag{37}
\]

where

\[
\sigma_n = \sigma_0 \frac{1 - \Delta_{n+1}}{1 - \Delta_0}, \quad n = 0, 1, \ldots \tag{38}
\]
This recursion can be solved numerically to an arbitrary degree of accuracy by starting at a high enough value of \( n \), say \( n_\epsilon \), and setting \( P_b(n_\epsilon) = 1 \).

Finally, it can be shown that the critical stress \( \sigma_0^c \) is determined by

\[
\mu(\sigma_0) = 1, \quad \sigma_{n \to \infty} = \frac{\sigma_0}{1 - \Delta_0},
\]

i.e., by

\[
\frac{1}{\Delta_0} P_c(\sigma_c - \sigma_0 < \Delta_0 \sigma_0) = 1.
\]

5.1. Results

5.1.1. Uniform distribution of strength thresholds

For a uniform threshold distribution (4) we obtain

\[
P(\sigma_c < \sigma_0 + \Delta_0 \sigma_0) = \min \left( 1, \frac{\Delta_0 \sigma_0}{1 - \sigma_0} \right),
\]

and it follows that

\[
P_{nc}(\sigma_0) = \begin{cases} 
\exp\left( -\frac{\sigma_0}{1 - \sigma_0} \right) & \text{if } \sigma_0 \leq \frac{1}{1 + \Delta_0} \\
\exp\left( -\frac{1}{\Delta_0} \right) & \text{otherwise}
\end{cases}
\]

and

\[
\sigma_0^c = \frac{1 - \Delta_0}{2 - \Delta_0}.
\]

The breakdown probability \( P_b(\sigma_0) \) is determined by numerically solving the recursion defined in Eqs. (37) and (38) with

\[
\mu(\sigma_n) = \frac{1}{\Delta_0} \min \left( 1, \frac{\Delta_0 \sigma_n}{1 - \sigma_0} \right).
\]

5.1.2. Weibull distribution of strength thresholds

For the case of a truncated Weibull distribution of strength thresholds, Eq. (3), with Weibull index \( k = 2 \), we have

\[
P(\sigma_{ib} < \sigma_0 + \Delta_0 \sigma_0) = 1 - \exp \left[ -\Delta_0 \sigma_0 (2\sigma_0 + \Delta_0 \sigma_0) \right]
\]

and obtain

\[
P_{nc}(\sigma_0) = \exp \left( -\frac{1}{\Delta_0} \left[ 1 - \exp \left( -\Delta_0 (2 + \Delta_0 \sigma_0^2) \right) \right] \right)
\]

and

\[
\sigma_0^c = (1 - \Delta_0) \left[ -\ln(1 - \Delta_0) / \Delta_0 \left( 2 + \Delta_0 \right) \right]^{1/2}.
\]

The corresponding results for arbitrary Weibull indices can be readily obtained. \( P_b(\sigma_0) \) is again determined numerically by solving the recursion of Eqs. (37) and (38), with

\[
\mu(\sigma_n) = \frac{1}{\Delta_0} \min \left( 1, \frac{\Delta_0 \sigma_n (2\sigma_0 + \Delta_0 \sigma_n)}{1 - \sigma_0} \right).
\]

5.1.3. Discussion

The behavior of \( P_{nc} \) and \( P_b \) as a function of \( \sigma_0 \) and \( \Delta_0 \) is illustrated in Figs. 5 and 6 for a truncated Weibull distribution of strength thresholds, and we note that the results for uniformly distributed strength thresholds (see Sect. 5.1.1) show a qualitatively similar behavior. Figure 7 shows the dependence of \( \sigma_0^c \) on \( \Delta_0 \) for both uniformly and Weibull distributed strength thresholds.

In Fig. 5 and 6, our analytical results are compared with those obtained from Monte-Carlo simulations. To ensure the validity of condition (2), we have chosen to keep \( \Delta_0 \) fixed as the number \( N_{in} \) of intact fibers decreases, and to use the scaling \( p_0 = \Delta_0 / N_{in} \).

To compare the results of this model (\( \Delta_0 \)-model) with those of the model analyzed in Section 4 (\( \gamma \)-model), we first make the following observations. The \( \gamma \)-model reproduces the GLS-limit if \( \gamma \leq 2 \) and (in a particular sense) approaches an LLS-limit as \( \gamma \to \infty \), while in the \( \Delta_0 \)-model, the GLS-limit is reproduced if \( \Delta_0 \to 0 \) and the LLS-limit (transfer of the failing load to a single surviving fiber) for \( \Delta_0 \to 1 \). Because of the different nature of the two \( \Delta \)-distributions, an exact relation between \( \gamma \) and \( \Delta_0 \) cannot be derived. A comparison of Figs. 2 and 7, however, suggests that for \( 2 \leq \gamma \leq 6 \), a rough correspondence between the \( \gamma \)- and the \( \Delta_0 \)-model is obtained if we set

\[
2 / \gamma = 1 - \Delta_0.
\]
of a failing fiber that is transferred to the surviving fibers is assumed to be a random variable, and we have considered two different distributions for the $\Delta$-values. The first ($\gamma$-model) refers to a stochastic version of the range-dependent load redistribution model of Hidalgo et al. [12], and the second ($\Delta_0$-model) to a model with a simple bimodal $\Delta$-distribution that can also interpolate between the two limiting cases of global and local load sharing. For the distribution of strength thresholds, we have also considered two different cases, a uniform and a Weibull distribution, both truncated below some finite stress $\sigma_0$.

While our models neglect any spatial correlations in the load redistribution after a failure, they have the advantage that they can be treated analytically, in contrast to most of the existing fiber bundle models that can only be analyzed via Monte-Carlo simulations.

In the limit of global load sharing ($\gamma < 2$ in the $\gamma$-model or $\Delta_0 \to 0$ in the $\Delta_0$-model), our models not only recover the known exact results for the critical stress $\sigma_{\text{crit}}^0$, but also give the exact behavior of the breakdown probability for $\sigma_0 > \sigma_{\text{crit}}^0$. In this GLS limit, the recursion relations for the determination of $P_b(\sigma_0)$ are reduced to simple transcendental equations

$$P_b(\sigma_0) = 1 - \exp\left[\frac{\sigma_0}{1 - \sigma_0} P_b(\sigma_0)\right]$$

for a truncated uniform strength-threshold distribution, and

$$P_b(\sigma_0) = 1 - \exp\left[-2\sigma_0^2 P_b(\sigma_0)\right]$$

for a truncated Weibull distribution with index $k = 2$. Eqs. (50) and (51) can easily be derived from the recursion of Eq. (37) by taking the limit $\Delta_0 \to 0$, or from the transcendental equation (28) for $\gamma \leq 2$.

With our stochastic models, we can also determine the critical stress $\sigma_{\text{crit}}^0$ and the behavior of $P_b(\sigma_0)$ for $\sigma_0 > \sigma_{\text{crit}}^0$ in the case of a more localized stress redistribution ($\gamma > 2$ in the $\gamma$-model or $\Delta_0 > 0$ in the $\Delta_0$-model). As already discussed in Sect. 4.4.1, our $\gamma$-model results for $\sigma_{\text{crit}}^0$ (for truncated uniform strength-threshold distributions) agree very well with the corresponding results of Raischel et al. [11] up to $\gamma \approx 6$ or 7. It is quite remarkable that a stochastic model that neglects any spatial correlations can so accurately reproduce the behavior of a microscopically more adequate model. In addition, our analytical solution allows us to trace back the onset of the transition between the GLS and LLS behavior at $\gamma = 2$ in the scaling of the upper cutoff $\Delta_{\text{max}}$ of the $\Delta$-distribution in the limit of infinite system sizes.

In the case of strength-threshold distributions that are not truncated, the usual procedure is to gradually increase the external force $F$ from zero up to the complete breakdown of the entire bundle. The critical strength of the fiber bundle is then defined as the maximum stress $F/N$ the system can support before it breaks down.

In global load sharing models, the surviving fibers always carry the same stress, so that the critical fiber-bundle strength can be written as

$$\sigma_{\text{bundle}}^0 = \sigma_c \left[1 - F_{\text{th}}(\sigma_c)\right].$$

(52)

6. Conclusions

In this paper, we have introduced and analyzed a new fiber-bundle model with stochastic load redistribution. The fraction $\Delta$
where $\sigma_c$ is the stress a surviving fiber carries at breakdown and $1 - F_{th}(\sigma_c)$ is the fraction of surviving fibers. This result is also recovered within our approach, with $\sigma_c = \sigma_c^0$.

For nearest-neighbor LLS models, however, $\sigma^c_{\text{bundle}}$ vanishes in the limit of large system sizes, $\sigma^c_{\text{bundle}} \propto 1/\sqrt{N}$. As our stochastic models neglect spatial correlations, they cannot describe such situations.

We can, however, compare the results of our models with that of Ref. [15], where also a stochastic load redistribution model is used. Here, $\sigma^c_{\text{bundle}}$ remains finite, even if the failing load is transferred only to a small, fixed number ($n = 1, 2, \ldots$) of randomly chosen surviving fibers. For $n = 2$ and for a uniform strength-threshold distribution, e.g., it is found that $\sigma^c_{\text{bundle}} \approx 0.2$ for large systems. This can be compared with the corresponding result of our $\Delta_0$-model. If we choose $\Delta_0 = 0.5$, so that the failing load is, on average, transferred to two surviving fibers, we obtain $\sigma^c_{\text{bundle}} = (1 - \Delta_0)/(2 - \Delta_0) = 1/3$ for a uniform distribution of strength thresholds, i.e.,

$$\sigma^c_{\text{bundle}} = \sigma^c_0(1 - \sigma^c_0) = 2/3 = 0.22.$$ (53)

The discrepancy between the two models can be attributed to our assumption that at breakdown all surviving fibers carry the same stress $\sigma^c_0$, whereas it is shown in Ref. [15] that at breakdown, the surviving fibers have a broad distribution of stresses, with a pronounced exponential tail.

It remains to be investigated whether our approach can be adapted to correctly analyze the behavior of fiber-bundle models with a strength-threshold distribution that is not truncated.

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