Stochastic description of geometric phase for polarized waves in random media

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Abstract

We present a stochastic description of multiple scattering of polarized waves in the regime of forward scattering. In this regime, if the source is polarized, polarization survives along a few transport mean free paths, making it possible to measure an outgoing polarization distribution. We consider thin scattering media illuminated by a polarized source and compute the probability distribution function of the polarization on the exit surface. We solve the direct problem using compound Poisson processes on the rotation group \( SO(3) \) and non-commutative harmonic analysis. We obtain an exact expression for the polarization distribution which generalizes previous works and design an algorithm solving the inverse problem of estimating the scattering properties of the medium from the measured polarization distribution. This technique applies to thin disordered layers, spatially fluctuating media and multiple scattering systems and is based on the polarization but not on the signal amplitude. We suggest that it can be used as a non-invasive testing method.

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(Some figures may appear in colour only in the online journal)

1. Introduction

Soon after its discovery in adiabatic quantum systems by Berry \cite{berry1981}, experimental evidence of a geometric phase for polarized light travelling along a bended optical fibre was shown by Tomita and Chiao \cite{tomita1983}. It was rapidly realized that any classical transverse polarized wave could exhibit a geometric phase \cite{berry1985} when travelling along a curved path in a three-dimensional space. This, however, is only possible if the wave remains polarized, and if the direction of propagation evolves smoothly with time. Note that this phase is not to be confused with the Pancharatnam phase \cite{pancharatnam1956}.
Figure 1. (Left) Description of the experimental setup. A source (S) emits a linearly polarized plane wave. The polarization is represented by the vertical vector. The wave is scattered by spherical inclusions of diameter $a$. The typical distance between these inclusions is $\ell$. The memory of the initial (horizontal) direction of propagation is $\ell^\star$. (Right) The pdf $p(\beta)$ of the direction of the polarization for the outgoing wave at different positions of the exit surface. The geometric phase $\beta$ is the angle between the incoming polarization and the outgoing polarization. The distribution is more peaked for positions closer to the source, because the average paths are straighter.

If one emits a plane monochromatic wave into a scattering medium, it will, on average, travel freely along a distance $\ell$, called the mean free path, before encountering a scatterer and then change its direction and travel freely to the next scatterer. The multiple scattering events will result in a spreading of the distribution of the wave vector with time [27]. Depending on the strength and density of the scatterers, this spreading will happen at different rates. This rate is characterized by the travel distance along which the memory of the direction of propagation is conserved. We call this distance the transport mean free path $\ell^\star$; it is always larger than the mean free path. After travelling the distance $\ell^\star$, the wave vectors of all the scattered waves are uniformly distributed.

Beyond typically one transport mean free path, the wave energy propagation in the multiple scattering regime is diffused: it follows the diffusion equation with an effective diffusion constant $D = c/\ell^\star$ [21] ($c$ is the velocity of the wave). Several multiple scattering regimes exist, depending on the relations between the wavelength $\lambda$, the diameter $a$ of the scatterers, the mean free path $\ell$, the transport mean free path $\ell^\star$ and the depth of the medium $L$ [9]. In this paper, we will investigate the regime where

$$\lambda < a \ll \ell \ll \ell^\star \lesssim L.$$  \hspace{1cm} (1)

The condition $\ell \ll \ell^\star$ states that the wave is preferentially scattered to a direction close to the incoming one. This is the so-called forward scattering regime. For spherical scatterers, this regime appears under the condition $\lambda < a$ [27, 16]. An illustration of these length scales is shown in figure 1 (left).

The transport mean free path $\ell^\star$ is the typical length scale of correlation between waves in the medium, while the mean free path is the length scale of the disorder. Measuring either the mean free path or the transport mean free path is therefore an important issue in experiments because it is a resolution scale for most of the physical quantities measurable using waves. Several techniques estimating the transport mean free path exist, using the outgoing light in various directions. The simplest of them uses the intensity transmission ratio [7] to measure $\ell^\star$ in the diffusion approximation. Diffusion models are also useful in anisotropic media [10] to
describe intensity leaks in directions orthogonal to the incoming source [14]. In a backscattering setup, the opening angle of the weak localization cone, in which an intensity enhancement due to interferences is observed, is related to the transport mean free path in the medium [26]. All these experiments rely on intensity measurements. An amplitude-independent method has been proposed to measure the mean free path \( \ell \) from the phase statistics of a Gaussian field [1].

It is known that in the forward scattering regime, the polarization is parallel transported during the propagation, which may result in the existence of a geometric phase [15]. In the eikonal approximation, each ray possesses a geometric phase depending on its geometry. At a given point, the observed phase is not uniquely defined, but has a distribution related to the distribution of paths from the source. This distribution was recently shown to depend on the outgoing direction of the ray [22]. In figure 1 (right), examples of outgoing polarization probability density functions are shown, depending on the position of the exit wave.

The distribution of geometric phase from multiple scattering of a polarized incident wave was addressed ten years ago [12] without any condition on the final direction. The calculation was based on the Brownian motion at the surface of a sphere and followed an approach developed earlier by Antoine et al [2]. However, multiple scattering is rigorously a Brownian motion on a sphere only in the limit of very weak and dense scatterers. A more general model would be a random walk with finite steps.

In media with a thickness \( L \) of the order of \( \ell^* \), the outgoing wave vectors are not evenly distributed. The joint distribution of direction and polarization state, related to path statistics, contains information concerning the scattering events that are not yet randomized. The number of scattering events is of the order of \( L/\ell \) and in the case where this number is not very large \( (L/\ell \simeq 10) \), the usual approximations are not suited. For this reason, attention was recently brought to stochastic processes for the description of a scattering system where the number of scattering events is small.

Stochastic models have been introduced 15 years ago in multiple scattering [17] to solve direct problems. Recently, Le Bihan and Margerin [13] showed that a stochastic model for the direction of propagation can be used to solve inverse problems. They computed the mean free path in a medium from the distribution of outgoing directions. We extend this model to take polarization into account and estimate the transport mean free path \( \ell^* \) from the geometric phase distribution.

2. Scattering of polarized waves

We consider scattering of polarized waves and more precisely the transport of polarization during scattering. Adopting a geometrical point of view, polarization can be seen as a particular direction of the electric field. This particular direction evolves along a trajectory. The geometric phase appears in comparing the polarizations of different trajectories. Therefore, without loss of generality, we consider linearly polarized waves for which the particular direction is the polarization itself. This easily extends to elliptical polarized waves, for which the great axis orientation takes the role of the linear polarization. For circular waves, as they have a circular symmetry, the definition of the geometric phase must be done using interferences.

Polarization of transverse waves lies in the plane orthogonal to the direction of propagation. We describe linearly polarized plane waves with two vectors: the direction of propagation \( k \) and the direction of polarization \( p \), with \( p \perp k \). We consider only media with isotropic uniform absorption; the wave amplitude is therefore completely determined for all paths and does not play any relevant role in our geometric description of the parallel transport.
The direction $k$ lies on the unit sphere $S^2 \subset \mathbb{R}^3$. The direction of polarization $p$ lies in the tangent plane $T_k S^2$. The frame $F$ is fully determined by $k$ and $p$ and contains all information concerning the polarization and direction of propagation. (The third vector of $F$ is indeed $k \times p$.)

We follow the wave state along a ray using the frame $F$. Between scattering events, $F$ remains constant. The changes of $F$ occur at scattering events. As $F$ is a frame, any change, or jump, of $F$ corresponds to a rotation matrix. The scattering angle $\theta$ is random, following a probability distribution function (pdf) called phase function $\Phi(\theta)$ that depends on $a/\lambda$ and on the nature of the scatterer. The average value of $\langle \cos \theta \rangle = \int \Phi(\theta) \cos \theta \sin \theta d\theta / 2$ is called the scattering anisotropy and is noted $g$. Note that we have the relation $\ell^* = \ell / (1 - g)$. In the forward scattering regime, the main scattering angle $\theta$ is small and $g$ is close to 1; therefore, $\ell^* \gg \ell$ as mentioned in the introduction. We use the ZYZ convention for the Euler angles describing the rotations of $SO(3)$ and note the angles $(\psi, \theta, \phi) \in [-\pi, \pi] \times [0, \pi] \times [-\pi, \pi]$.

The rotation matrix $P$ acting on the frame $F$ at a scattering event is fully determined by the incoming direction $k$ and the outgoing direction $k'$ and the requirement that the vector $p$ is parallel transported [15]. On the unit sphere, jumps of the stochastic process are represented by geodesics (arcs of great circles). Moreover, parallel transport in the direct space implies parallel transport in the phase space from $T_k S^2$ to $T_{k'} S^2$ [11] and therefore the angle between $p$ and the geodesic remains constant. As a consequence, the Euler angles $\psi$ and $\phi$ of the rotation matrix $P$ are related by (see figure 2)

$$\psi = -\phi.$$ (2)

Parallel transport of polarization through a scattering event may therefore be described by a rotation matrix $P(\theta, \psi)$, $\ell$ is the length of the geodesic on the unit sphere and $\theta$ is the angle
between this geodesic and the polarization vector \( p \). The relation between the incoming and outgoing frames is

\[
F' = F P(\theta, \psi).
\]  

(3)

Note that the rotation matrix \( P \) acts on the right of \( F \). An expression in terms of left action of rotation matrices can be obtained, but would lead to a random process over \( SO(3) \) with dependent increments. The independence of increments will be useful for the repeated action (3) on \( F \) that we present in the next section.

### 3. Multiple scattering process

We consider a plane wave with linear polarization, corresponding to the frame \( F_0 \) as described in the previous section. The scattering events are described as random rotations acting on the right of \( F_0 \). The frame \( F_0 \) changes according to equation (3) for each scattering event, so that it becomes the frame \( F_n \) after \( n \) scattering events:

\[
F_n = F_0 P(\theta_1, \psi_1) P(\theta_2, \psi_2) \cdots P(\theta_n, \psi_n).
\]

(4)

The scattering events are independent and the scatterers are identical; the random rotations \( P(\theta_m, \psi_m) \) are therefore independent and identically distributed.

From the source to the observer, a wave may encounter a variable number of scatterers. The number of scattering events at a time \( t \) can be taken as a Poisson process \( N(t) \) [13]. The Poisson parameter \( \eta \) is the average rate at which a wave encounters a scatterer. From the definition of the mean free path \( \ell \), we have \( \eta = c/\ell \); \( \eta \) is the inverse of the mean free time. From these considerations, we obtain the process \( F_t \) describing the distribution of polarization after a time \( t \) as

\[
F_t = F_0 \prod_{k=1}^{N(t)} P(\theta_k, \psi_k),
\]

(5)

where \( \prod \) denotes the right-sided product on \( SO(3) \). This equation expresses the stochastic process \( F_t \) as a compound Poisson process (CPP) on \( SO(3) \) with parallel transport. We denote by \( p_\nu \) the distribution of the process \( F_t \) at a given time \( t \), where \( \nu = \eta t \) is the average number of scattering events. A Poisson process is a pure jump process with independent increments and we have

\[
p_\nu = \sum_{n=0}^{\infty} e^{-\nu} \frac{\nu^n}{n!} p_{F^n},
\]

(6)

where \( e^{-\nu} \frac{\nu^n}{n!} \) is the probability to have exactly \( n \) scattering events in the time interval \([0, t]\). Thanks to equation (4) and the independence of the parallel transport rotations, \( p_{F^n} \) is given by [4]

\[
p_{F^n} = p_{F_0} * p_{F_0} * \cdots * p_{F_0} = \Phi^{*n}.
\]

(7)

Here, the symbol * is the convolution product on \( SO(3) \) and \( \Phi^{*n} \) denotes the result of the convolution of \( n \) identical functions \( \Phi \). We have used the initial condition \( p_{F_0} = \delta(I_3 - F_0) \). In the case where the source is a distribution of directions of propagation and polarization, the distribution \( p_{F_0} \) is the convolution of the initial distribution \( p_{F_0} \) with \( \Phi^{*n} \).
4. Harmonic analysis

We use a harmonic analysis on $SO(3)$ to transform expression (7). The harmonic analysis on $SO(3)$ is similar to Fourier series. We briefly introduce the harmonic analysis on $SO(3)$ and use its properties to obtain the main result of this work.

The Fourier basis of functions $f \in L^2(SO(3), \mathbb{R})$ is made up of the Wigner $D$-matrices $(D^j)_{j \geq 0}$ [6, 4]. The Wigner $D$-matrices are $(2j+1) \times (2j+1)$ square matrices with coefficients

$$D^j_{m,n} (\psi, \theta, \varphi) = e^{-im\varphi} d^j_{m,n}(\theta) e^{-in\psi}. \quad (8)$$

The Fourier coefficients are matrices of the same size and are defined by $\tilde{f}^j_{m,n} = (2j + 1) \langle \Phi | D^j_{m,n} \rangle$, where $\langle \cdot \rangle$ represents the scalar product for function on $SO(3)$ [6].

Like in the Fourier theory for periodic functions, the transform of a convolution is a product (here, a matrix product):

$$\tilde{f}^j_1 \ast \tilde{f}^j_2 = \tilde{f}^j_1 \tilde{f}^j_2. \quad (9)$$

We deduce the Fourier coefficients of $p_\nu$ from equation (6)

$$\tilde{p}^j_\nu = \sum_{n=0}^{\infty} e^{-n\psi} \frac{\nu^n}{n!} (\tilde{\Phi}^j)^n = \exp[\nu(\tilde{\Phi}^j - I_{j+1})], \quad (10)$$

where exp denotes the matrix exponential. Using the notation $\Delta(x) = 2\pi \sum_k \delta(x - 2\pi k)$ to put the parallel transport constraint on the Fourier coefficients, we obtain

$$\tilde{\Phi}^j_{m,n} = (2j + 1) \frac{1}{8\pi^2} \int \int \Phi(\theta) \Delta(\psi + \psi) D^j_{m,n}(\psi, \theta, \varphi) \sin \theta d\theta d\varphi d\psi. \quad (11)$$

Using definition (8), we obtain the simpler expression

$$\tilde{\Phi}^j_{m,n} = (2j + 1) \frac{\delta_{m,n}}{2} \int_0^\pi \Phi(\theta) d^j_{m,n}(\theta) \sin \theta d\theta, \quad (12)$$

showing that the Fourier matrices $\tilde{\Phi}^j$ are diagonal for all $j$. Thanks to equation (10), it is clear that $\tilde{p}^j$ is also diagonal at all orders $j$.

Using the inverse Fourier formula [6], $p_\nu$ reads

$$p_\nu(\psi, \theta, \varphi) = \frac{1}{2\pi} \sum_{j=0}^{\infty} (2j + 1) \text{tr}(\tilde{p}^j_\nu D^j(\psi, \theta, \varphi)), \quad (13)$$

where tr denotes the trace and $\dagger$ denotes the Hermitian conjugation of elements from $\mathcal{M}_{2j+1}(\mathbb{C})$.

The variable $\beta = \psi + \psi$ represents the direction of polarization of the wave after its propagation through the medium, which is the geometric phase. Because $\tilde{p}^j_\nu$ is diagonal, only the functions $D^j_{m,n}$ appear in the expansion of expression (13). Referring to equation (8), we deduce that $p_\nu$ is precisely a function of $\psi + \psi$. The expansion of equation (13) yields the main result of this work:

$$p_\nu(\theta, \beta) = R_0(\theta, \nu) + 2 \sum_{m \geq 1} \cos(m\beta) R_m(\theta, \nu), \quad (14)$$

$$R_m(\theta, \nu) = \frac{1}{2\pi} \sum_{j \geq m} (2j + 1) e^{i\nu(\tilde{\Phi}^{j}_{m,n} - 1)} d^j_{m,n}(\theta). \quad (15)$$

Equation (14) is a general expression of the distribution of geometric phase in the multiple scattering regime for polarized waves. We have only assumed that the scatterers are spherical and identical. To obtain this result, we have only used the Fourier series on $SO(3)$, summed the
Poisson series (6) and used the Fourier inversion formula. As in [13], we obtain a semi-analytic expression of \( p_\nu \) that we use for statistical estimation in section 5.

The limit of Brownian motion, which is made up of isotropic infinitesimal steps of spherical length \( \delta \) occurring at a strong rate \( \eta \), corresponds to the physical situation where scatterers are very weak but have a large spatial density. The Brownian motion in \( SO(3) \) was first studied by Perrin [19, 20]. Perrin showed that the Brownian motion in \( SO(3) \) has eigenvalues equal to \( j(j + 1)D + m^2(D_\psi - D) \), where \( D_\psi \) is the diffusion constant of the angular degree of freedom \( \psi \) [20, equation (42)]. Since the value of \( \psi \) for infinitesimal steps is constrained by parallel transport, we have \( D_\psi = 0 \) and we deduce that the expression for a single step \( \Phi_{n,m} = \exp[-\frac{1}{2} (j(j + 1) - m^2) \delta^2] \). Taking the limit \( \delta \to 0 \) and \( \eta \to \infty \) such that \( \delta^2 \nu = \delta^2 \eta t = D t \) remains constant, formula (14) gives the distribution of the geometric phase for a wave propagating in a heterogeneous continuous medium as equation (16) where \( \nu(\Phi_{n,m} - 1) \) should be replaced by \( -\frac{1}{2}[j(j + 1) - m^2]Dt \). Within the Brownian motion limit, taking \( \theta = 0 \) in equation (16) gives the result demonstrated by Antoine et al [2], and integrating over \( \theta \) we find the result obtained by Krishna et al [12]. Equation (14) is therefore a generalization of these known formulas that can be applied to an arbitrary phase function, also in the case of a small number of scattering events and for an arbitrary deviation angle \( \theta \).

As illustrations, we display the distribution obtained from formula (14) in the following two cases: discrete Henyey–Greenstein scatterers with anisotropy \( g = 0.8 \) on a range of Poisson parameter up to \( \nu = 20 \) on the one hand (figures 3 and 4) and diffusive limit with \( D = 1 \text{ rad}^2 \text{ s}^{-1} \) on the other hand (figures 5 and 6). Note that any phase function \( \Phi \) could be used to obtain similar results for Mie scatterers, resonant scatterers, etc. The exit angle is \( \theta = 0 \) for all the figures. We can observe that the behaviour of the distributions in figures 4 and 6 is very different, especially at short times, the distribution \( p_\nu \) is sharply peaked for the Henyey–Greenstein scatterers but in the case of Brownian motion, it is smooth. At long times, depolarization is observed in both cases. The depolarized state is reached with significantly distinct behaviours. This is an illustration that depolarization is significantly dependent on the
Figure 4. Distribution of the geometric phase $\beta$ for the values of the Poisson parameter $\nu = \eta t$: $\nu = 20$, $\nu = 25$, $\nu = 30$ and $\nu = 35$, corresponding to the vertical lines of figure 3.

Figure 5. Evolution of the density $p_{D_{t}}(\theta = 0, \beta)$ with time for a Gaussian phase function. This corresponds to the rotational Brownian motion on the sphere studied by Perrin in [20]. The curves are iso-density levels from 0.05 to 0.45 by increments of 0.05. The vertical lines correspond to the slices displayed in figure 6.

scattering properties of the medium. It is therefore natural to investigate how these scattering properties can be estimated from observations through a signal analysis.

It may be surprising that the non-commutativity in $SO(3)$ is not reflected in the main result equation (14). This is related to the local description we have performed. The distribution equation (14) describes the distribution of the geometric phase centred on a global geometric phase which depends on the direction $k_{\text{out}}$ of the outgoing trajectory and the azimuth $\chi$ of the end of the trajectory taken in the initial frame $F_{0}$. This geometric phase can be seen
Figure 6. Density $p_{\theta=0}(\theta, \beta)$ for a Gaussian phase function with the values $\Delta t = 0.03$, $\Delta t = 0.05$, $\Delta t = 0.07$ and $\Delta t = 0.09$ corresponding to the vertical lines of figure 5.

as the average polarization of trajectories following these boundary conditions. Its value is well known and is equal to $(1 - \cos \theta)\chi$. It has been already observed in the backscattering configuration ($\theta = \pi$) [8, 23].

5. Inverse problem

In this section, we take an advantage of the Fourier expansion and propose a statistical estimation of the mean free path $\ell$ obtained from the measured distribution of the geometric phase $\beta$. This is achieved by estimating the Poisson parameter $\nu = \eta t$. The estimate of $\nu$ is denoted by $\hat{\nu}$.

The inverse problem is solved using an expectation–minimization (EM) approach [5] with the parametric CPP. The EM algorithm is based on the maximization of the log likelihood of the a posteriori distribution given in equation (6). Suppose that we are given a sample of size $M$ of observations (measurements) $F_m$, $1 \leq m \leq M$. We assume that each observation $F_m$ follows independently the probability law (14); the joint pdf is simply the product $\prod_m p_{\nu}(F_m)$, with $p_{\nu}(F) = p_{\nu}(\theta, \psi + \varphi)$, where $\varphi$, $\theta$ and $\psi$ are the Euler angles of $F$. The value of $\nu$ that maximizes this log likelihood is

$$\hat{\nu} = \arg \max_{\nu} \left( \sum_{m=1}^{M} \log p_{\nu}(F_m) \right). \quad (16)$$

The EM algorithm is an iterative procedure that almost surely converges to a local minimum [5]. The minimization step (M-step) consists in updating the estimate $\hat{\nu}$, while the expectation step (E-step) consists in updating $P(n \mid F, \hat{\nu}) = p_{F_n}(F) / p_{\hat{\nu}}(F) \times e^{-\hat{\nu}^2} \hat{\nu}^n / n!$. We obtain a sequence of estimates with the relation

$$\hat{\nu}_{i+1} = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{p_{\nu_i}(F_m)} \sum_{n=0}^{\infty} n p_{F_n}(F_m) e^{-\hat{\nu}_i} \hat{\nu}_i^n / n!. \quad (17)$$

In practice, the sums over $n$ in equations (17) and (6) and over $j$ in equation (15) are truncated to an arbitrarily fixed value $N$. At most $N$ scattering events are described by the model;
therefore, $N$ should be taken large compared to $L/\ell$. This EM algorithm is able to estimate one parameter, the Poisson parameter $\nu$ (or equivalently the mean free path $\ell = ct/\nu$), which means that we have considered that the phase function $\Phi$ is known. It is nonetheless possible to estimate simultaneously any finite number of parameters of $p_\nu$, such as the anisotropy $g$ of the scatterers, but the iteration formulas are more sophisticated.

We illustrate the behaviour of the EM algorithm in figure 7 which shows the convergence of the estimator $\hat{\nu}$ for different initial values $\hat{\nu}_0$. Acceleration procedures could be developed in the case of a large dataset from the experimental setup. Note that no local minima were reached in the simulation, leading to convergence in all the cases. However, as can be seen for initial values below 10 in figure 7, underestimated initial values lead to a systematic bias in $\hat{\nu}$. This suggests that high values should be privileged when processing a dataset, as it does not penalize convergence and leads to a more accurate estimate.

6. Conclusions

The description of the depolarization of multiply scattered waves can be made, in the forward scattering regime, through a compound Poisson process (CPP). This stochastic model predicts the distribution of the geometric phase in all directions, for discrete or continuous scattering media, generalizing existing results (forward outgoing direction or a spatially fluctuating medium). Moreover, the CPP model allows a more detailed description of the phenomenon as it provides the behaviour of this distribution as a function of the output scattering angle $\theta$. The present approach allows us to design an iterative procedure to estimate properties of the scattering medium through the measurement of the outgoing polarization distribution. We have illustrated this point by presenting an expectation–minimization (EM) algorithm for the estimation of the Poisson parameter, which is directly linked to the transport mean free path $\ell^*$. An interesting feature of this technique is that it relies on polarization rather than on amplitude measurements. Ongoing work consists in validating the proposed model and estimation algorithm on the measurement of polarization distribution from a laboratory experiment.
The stochastic model of section 3 and the distribution of geometric phase found in section 4 constitute powerful tools for numerical approaches of the polarization in multiple scattering media. It can be used even for small scatterers or in any other situation where the scattering anisotropy vanishes.

The EM procedure can be used on experimental data to estimate the mean free path from polarization statistics of any kind of polarized waves in multiple scattering media, provided that the limit of uniformly distributed polarization is not reached and the source polarization is known. Such waves can be light, scattered neutrons or elastic waves in solids.

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References

[1] Anache-Ménier D, van Tiggelen B A and Margerin L 2009 Phase statistics of seismic coda waves Phys. Rev. Lett. 102 248501
[2] Antoine M, Comtet A, Desbois J and Stéphane O 1991 Magnetic fields and Brownian motion on the 2-sphere J. Phys. A: Math. Gen. 24 2581–6
[3] Berry M V 1984 Quantal phase factors accompanying adiabatic changes Proc. R. Soc. A 392 45–57
[4] Chirikjian G S and Kyatkin A B 2001 Engineering Applications of Noncommutative Harmonic Analysis (Boca Raton, FL: CRC Press)
[5] Dempster A P, Laird D B and Rubin D B 1977 Maximum likelihood from incomplete data via the EM algorithm J. R. Stat. Soc. B 39 1–38
[6] Dieudonné J 1980 Special Functions and Linear Representations of Lie Groups (Providence, RI: American Mathematical Society)
[7] Garcia N, Genack A Z and Lisyansky A A 1992 Measurement of the transport mean free path of diffusing photons Phys. Rev. B 46 14475–8
[8] Helmscher A H, Eick A A, Mourant J R, Shen D, Freyer J P and Bigio J J 1997 Diffusive backscattering Mueller matrices of highly scattering media Opt. Express 1 441–53
[9] Ishimaru A 1978 Wave Propagation and Scattering in Random Media vols 1 and 2 (New York: Academic)
[10] Johnson P M, Faez S and Lagendijk A 2008 Full characterization of anisotropic diffuse light Opt. Express 16 7435–46
[11] Jordan T F and Maps J 2010 Change of the plane of oscillation of a Foucault pendulum from simple pictures Am. J. Phys. 78 1188
[12] Krishna M M G, Samuel J and Sinha S 2000 Brownian motion on a sphere: distribution of solid angles J. Phys. A: Math. Gen. 33 5965–71
[13] Le Bihan N and Margerin L 2009 Nonparametric estimation of the heterogeneity of a random medium using compound Poisson process modeling of wave multiple scattering Phys. Rev. E 80 016601
[14] Leonetti M and López C 2011 Measurement of transport mean-free path of light in thin systems Opt. Lett. 36 2824–6
[15] Maggs A C and Rossetto V 2001 Writhing photons and Berry phases in polarized multiple scattering Phys. Rev. Lett. 87 253901
[16] Mishchenko M 1991 Multiple Scattering of Light by Particles: Radiative Transfer and Coherent Backscattering (Cambridge: Cambridge University Press)
[17] Ning X, Papiez L and Sandison G 1995 Compound-Poisson-process method for the multiple scattering of charged particles Phys. Rev. E 52 5621–33
[18] Panchratnam S 1956 Generalized theory of interference and its applications i: coherent pencils Proc. Indian Acad. Sci. A 44 247
[19] Perrin F 1928 Étude mathématique du mouvement Brownien de rotation Ann. Éc. Norm 45 1–5
[20] Perrin F 1936 Mouvement Brownien d’un ellipsoïde (II) J. Phys. Radium 1 1–11
[21] Puse D J, Weitz D A, Chaikin P M and Herbolsheimer E 1988 Diffusing-wave spectroscopy Phys. Rev. Lett. 60 1134–7
[22] Rossetto V 2009 A general framework for multiple scattering of polarized waves including anisotropies and Berry phase Phys. Rev. E 80 056605
[23] Rossetto V and Maggs A C 2002 Writhing geometry of stiff polymers and scattered light Eur. Phys. J. B 118 323–6
[24] Segert J 1987 Photon Berry’s phase as a classical topological effect Phys. Rev. A 36 10–15
[25] Tomita A and Chiao R Y 1986 Observation of Berry’s topological phase by use of an optical fiber Phys. Rev. Lett. 57 937–40
[26] Van Albada M P and Lagendijk A 1985 Observation of weak localization of light in a random medium Phys. Rev. Lett. 24 2692–5
[27] van de Hulst H C 1957 Light Scattering by Small Particles (New York: Dover)