A LOWER BOUND ON THE SATURATION NUMBER, AND GRAPHS FOR WHICH IT IS SHARP

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Abstract. Let $H$ be a fixed graph. We say that a graph $G$ is $H$-saturated if it has no subgraph isomorphic to $H$, but the addition of any edge to $G$ results in an $H$-subgraph. The saturation number $\text{sat}(H, n)$ is the minimum number of edges in an $H$-saturated graph on $n$ vertices. Kázsonyi and Tuza, in 1986, gave a general upper bound on the saturation number of a graph $H$, but a nontrivial lower bound has remained elusive. In this paper we give a general lower bound on $\text{sat}(H, n)$ and prove that it is asymptotically sharp (up to an additive constant) on a large class of graphs. This class includes all threshold graphs and many graphs for which the saturation number was previously determined exactly. Our work thus gives an asymptotic common generalization of several earlier results. The class also includes disjoint unions of cliques, allowing us to address an open problem of Faudree, Ferrara, Gould, and Jacobson.

1. Introduction

Given a fixed forbidden graph $H$, what is the minimum number of edges that any graph $G$ on $n$ vertices can have such that $G$ contains no copy of $H$, but the addition of any single edge to $G$ results in a copy of $H$? This question is a variation of the well-known forbidden subgraph problem in extremal graph theory, which asks for the maximum number of edges in an $H$-free graph on $n$ vertices. Asking for the minimum number of edges instead (and tailoring the definition so that this is a sensible question) yields the notion of the saturation number of a graph $H$, first defined by Erdős, Hajnal, and Moon [4], albeit with slightly different terminology.

Definition 1. Let $H$ be a graph. For any graph $G$, we say that $G$ is $H$-free if it contains no subgraph isomorphic to $H$. We say that $G$ is $H$-saturated if it is $H$-free and for any $xy \in E(G)$, the graph $G + xy$ contains a subgraph isomorphic to $H$. For $n \geq |V(H)|$, let $\text{Sat}(H, n)$ denote the set of all $H$-saturated graphs on $n$ vertices, and let the saturation number of $H$ be

$$\text{sat}(H, n) = \min_{G \in \text{Sat}(H, n)} |E(G)|.$$  

In the event that $\text{Sat}(H, n) = \emptyset$, we adopt the convention that $\text{sat}(H, n) = \infty$. Note that this will only happen if $H$ has no edges.

In their paper introducing the concept, Erdős, Hajnal, and Moon [4] determined $\text{sat}(H, n)$ in the case where $H$ is a complete graph. Since then, the saturation numbers have been determined for various classes of graphs. A nice survey on these results and more was written by Faudree, Faudree, and Schmitt [6].

The best known general upper bound on $\text{sat}(H, n)$ was given by Kázsonyi and Tuza [9] and later slightly improved by Faudree and Gould [8]. However, as mentioned in [6] and in [7], there is no known nontrivial general lower bound on this
function. In this paper we give such a bound, and determine a class of graphs for which this bound is asymptotically sharp: for such graphs, we can prove that $\text{sat}(H, n) = \alpha_H n + O(1)$, where $\alpha_H$ and the $O(1)$ term depend on only $H$. (We remark that it is not known, in general, that the limit $\lim_{n \to \infty} \frac{\text{sat}(H, n)}{n}$ even exists, even though it is known [9] that $\text{sat}(H, n)$ is always bounded by a linear function of $n$; the existence of this limit was stated as a conjecture by Tuza [11].)

This class of graphs includes all threshold graphs as well as some non-threshold graphs. In particular, many previously-studied classes of graphs fall into this class, including cliques [4], stars [9], generalized books [1], disjoint unions of cliques [7], generalized friendship graphs [7], and several of the “nearly complete” graphs of [8]. Our result can be considered as an asymptotic common generalization of these previous results: at the cost of no longer determining the exact saturation number as in the previous results, we obtain a simple unified proof that gives the saturation number up to an additive constant number of edges.

The rest of the paper is organized as follows. In Section 2 we state and prove our general lower bound on the saturation number, and prove an upper bound on the saturation number of the graph $H'$ obtained from a graph $H$ by adding a dominating vertex. In Section 3 we define the sat-sharp graphs to be the graphs whose saturation numbers are asymptotically equal to the lower bound of Section 2 and prove that this class of graphs is closed under adding isolated vertices and dominating vertices. In Section 4 we discuss threshold graphs, which are contained within the class of sat-sharp graphs and encompass several graphs whose saturation numbers were previously determined. Finally, in Section 5 we prove that any graph consisting of a disjoint union of cliques is sat-sharp, and discuss the implications of this.

## 2. A WEIGHT FUNCTION AND SOME GENERAL BOUNDS

In this section, we will define a weight function for a general graph $H$, and prove that it gives a lower bound on the saturation number $\text{sat}(H, n)$. We will also prove a general bound relating the saturation number of $H$ to the saturation number of the graph $H'$ obtained from $H$ by adding a dominating vertex.

**Definition 2.** For a vertex $x$ in a graph $G$, let $N_G(x)$ and $N_G[x]$ denote the open and closed neighborhoods of $x$ respectively:

- $N_G(x) = \{ y \in V(G) : xy \in E(G) \}$,
- $N_G[x] = N_G(x) \cup \{x\}$.

Let $d_G(x) = |N_G(x)|$ denote the degree of $x$, and for a vertex set $S$, let $d_{G,S}(x)$ denote the number of neighbors of $x$ in the set $S$:

$$d_{G,S}(x) = |N(x) \cap S|.$$  

When the graph $G$ is clear from context, we omit it from our notation and simply write $N(v)$, $d(v)$, or $d_S(v)$ as appropriate.

**Definition 3.** Let $uv$ be an edge in a graph with $d(u) \leq d(v)$. Define the weight $\text{wt}(uv)$ of the edge $uv$ by

$$\text{wt}(uv) = 2|N(u) \cap N(v)| + |N(v) - N(u)|.$$
Define the weight of the graph $H$ by

$$\text{wt}(H) = \min_{uv \in E(H)} \text{wt}(uv).$$

If $E(H) = \emptyset$, we define $\text{wt}(H) = \infty$.

**Lemma 4.** For every graph $H$, there exists a constant $c'_H$ such that

$$\text{sat}(H, n) \geq \frac{\text{wt}(H) - 1}{2} n - c'_H.$$

**Proof.** First observe $\text{wt}(H) \geq 1$ for all $H$ and that the claim is trivial when $\text{wt}(H) = 1$, so (as $\text{wt}(H)$ is an integer) we may assume that $\text{wt}(H) \geq 2$. Let $G$ be an $H$-saturated graph, let $x^*$ be a vertex of minimum degree in $G$, and let $B = N_G(x^*)$, so that $|B| = d_G(x^*)$. Observe that if $d_G(x^*) \geq \text{wt}(H) - 1$, then the degree-sum formula immediately gives $|E(G)| \geq \frac{\text{wt}(H) - 1}{2} n$, so we may assume that $d_G(x^*) < \text{wt}(H) - 1$. As both of these quantities are integers, we have $d_G(x^*) \leq \text{wt}(H) - 2$.

Consider any vertex $y \in V(G) - N[x^*]$. By hypothesis, the graph $G + x^*y$ contains a copy of $H$. Let $\phi : V(H) \to V(G + x^*y)$ be an embedding of $H$ into $G + x^*y$. Since $G$ is $H$-free, the new edge $x^*y$ must be the image of some edge $uv \in E(H)$. We may take our notation so that $d_H(u) \leq d_H(v)$.

We first claim that $d_G(y) \geq a + b - 1$; this requires considering two cases, depending on whether $y = \phi(u)$ or $y = \phi(v)$. If $y = \phi(u)$, then

$$d_G(y) \geq \delta(G) = d_G(x^*) \geq d_H(v) - 1 = a + b - 1.$$  

Similarly, if $y = \phi(v)$, then $d_G(y) \geq d_H(v) - 1 \geq a + b - 1$. This establishes the claim.

Now, observe that regardless of whether $y = \phi(u)$ or $y = \phi(v)$, we have

$$\phi(N_H(u) \cap N_H(v)) \subseteq N_G(x^*) = B.$$  

So in $G$, the vertex $y$ has $b$ guaranteed neighbors in $B$, together with at least $a - 1$ additional edges which may go to $B$ or may go to $\overline{B} - x^*$, where $\overline{B} = V(G) - B$. Therefore,

$$2d_{G,B}(y) + d_{G,B}(y) \geq 2b + a - 1 = \text{wt}(uv) - 1 \geq \text{wt}(H) - 1$$

for all $y \in \overline{B} - x^*$. 

Now, note that $\sum_{x \in B} d_G(x) \geq \sum_{y \in \overline{B}} d_G(y)$. So it follows that
\[
|E(G)| = \frac{1}{2} \left( \sum_{x \in B} d_G(x) + \sum_{y \in B} d_G(y) \right) \\
\geq \frac{1}{2} \left( \sum_{y \in B} d_G(y) + \sum_{y \in \overline{B}} d_G(y) \right) \\
= \frac{1}{2} \sum_{y \in B} (2d_G(y) + d_G, \overline{B}(y)) \\
\geq \frac{2d_G(x^*) + (\text{wt}(H) - 1) |\overline{B} - x^*|}{2} \\
= \frac{2d_G(x^*) + (\text{wt}(H) - 1)(n - 1 - d_G(x^*))}{2} \\
= \frac{\text{wt}(H) - 1}{2}n - c_H',
\]
where
\[
c_H' = \frac{d_G(x^*)(\text{wt}(H) - 3) + (\text{wt}(H) - 1)}{2}.
\]
Since we assume $0 \leq d_G(x^*) \leq \text{wt}(H) - 2$, the value $c_H'$, considered as a formal function of the quantity $d_G(x^*)$, is maximized at $d_G(x^*) = \text{wt}(H) - 2$ whenever $\text{wt}(H) \geq 2$ (the case $\text{wt}(H) = 2$, which would imply that this formal function has a negative derivative in $d_G(x^*)$, implies that $d_G(x^*) = 0$). Therefore,
\[
\text{sat}(H, n) \geq \frac{\text{wt}(H) - 1}{2}n - \frac{\text{wt}(H)^2 - 4 \text{wt}(H) + 5}{2}
\]
for $\text{wt}(H) \geq 2$. \hfill \Box

A central goal of this paper is to explore the effect on the saturation number of the operation of adding a dominating vertex to $H$, as shown in Figure 1. It turns out that this gives a general upper bound on the saturation number of the new graph in terms of the saturation number of $H$; we wish to know when this upper bound is sharp. We believe that this upper bound is in the same general spirit as Lemma 9 of Kászonyi–Tuza [9].

**Lemma 5.** If $H'$ is obtained from $H$ by adding a dominating vertex $v^*$, then for all $n \geq |V(H')|$, we have $\text{sat}(H', n) \leq (n - 1) + \text{sat}(H, n - 1)$. 

![Figure 1. Forming $H'$ by adding a dominating vertex $v^*$ to the graph $H$.](image)

\[
\text{sat}(H, n) \geq \frac{\text{wt}(H) - 1}{2}n - \frac{\text{wt}(H)^2 - 4 \text{wt}(H) + 5}{2}
\]
Proof: It suffices to produce an $H'$-saturated graph with at most the indicated number of edges. Let $G$ be a minimum $H$-saturated graph on $n - 1$ vertices, and let $G'$ be obtained from $G$ by adding a new dominating vertex $x^*$. It is clear that $|E(G')| = (n - 1) + \text{sat}(H, n - 1)$; we show that $G'$ is $H'$-saturated.

First we argue that $G'$ is $H'$-free. Suppose to the contrary that $\phi : V(H') \to V(G')$ is an embedding of $H'$ into $G'$. If $x^* \notin \phi(V(H'))$, then $\phi$ is an embedding of $H'$ into $V(G')$. Hence $G'$ has a copy of $H'$ and thus a copy of $H$, contradicting the $H$-saturation of $G$. If $\phi(v^*) = x^*$, then the restriction of $\phi$ to $V(H)$ is an embedding of $H$ into $G$, again a contradiction.

Hence we can assume that $\phi(v^*) \neq x^*$ and there is some vertex $w^* \in V(H)$ with $\phi(w^*) = x^*$. Construct a new embedding $\phi_0 : V(H) \to V(G)$ by letting $\phi_0(w^*) = \phi(v^*)$ and taking $\phi_0(w) = \phi(w)$ for all $w \neq w^*$. Since $\phi(v^*)$ dominates the image of $\phi$ in $G$ (as $x^*$ was a dominating vertex of $H$), we see that $\phi_0$ is still a valid embedding. Hence we have again obtained a copy of $H$ in $G$, a contradiction. We conclude that $G'$ is $H'$-free.

Finally we argue that adding any missing edge to $G'$ produces a new copy of $H'$. Since $x^*$ is dominating, any missing edge in $G'$ is an edge of the form $yz$ where $y, z \in V(G)$. Now $G + yz$ contains a copy of $H$, since $G$ is $H$-saturated; adding the dominating vertex $x^*$ to this copy of $H$ gives a copy of $H'$ in $G' + yz$. □

3. Sat-sharp graphs

For a graph $H$, let $\text{satlim}(H) = \lim_{n \to \infty} \frac{\text{sat}(H, n)}{n}$, provided that this limit exists. Say a graph $H$ is sat-sharp if $\text{satlim}(H) = \frac{\text{wit}(H) - 1}{2}$. Moreover, say that a graph $H$ is strongly sat-sharp if $\text{sat}(H, n) = \frac{\text{wit}(H) - 1}{2}n + O(1)$. Note that any strongly sat-sharp graph is also sat-sharp. Also, note that by adopting the convention that $w(H) = \infty$ when $E(H) = \emptyset$, we can conclude that any graph with no edges is strongly sat-sharp since $\text{sat}(H, n) = \infty$ for all $n \geq |V(H)|$.

In this section we will show that the classes of sat-sharp graphs and strongly sat-sharp graphs are each closed under adding isolated and dominating vertices.

To express these results concisely, we write statements like “if $H$ is (strongly) sat-sharp, then $H'$ is (strongly) sat-sharp” as shorthand for the pair of statements “if $H$ is sat-sharp, then $H'$ is sat-sharp; if $H$ is strongly sat-sharp, then $H'$ is strongly sat-sharp”.

As $K_1$ is strongly sat-sharp, these closure results immediately imply that all threshold graphs are strongly sat-sharp (as we will discuss in Section 4). They also imply that any graph $H$ which can be proven to be (strongly) sat-sharp gives rise to many (strongly) sat-sharp graphs derived from $H$ by these operations. In particular, we will prove in Section 5 that a disjoint union of cliques is strongly sat-sharp, although it is not in general a threshold graph; this implies that any graph obtained from a disjoint union of cliques via these operations is also strongly sat-sharp.

Lemma 6. If $H$ is a (strongly) sat-sharp graph, and $H'$ is obtained from $H$ by adding isolated vertices, then $H'$ is (strongly) sat-sharp, and $\text{satlim}(H') = \text{satlim}(H)$.

Proof. For all $n \geq |V(H')|$, a graph $G$ is $H'$-saturated if and only if it is $H$-saturated, hence $\text{sat}(H', n) = \text{sat}(H, n)$ for all sufficiently large $n$. □
To handle the operation of adding a dominating vertex, we prove the following two lemmas, which taken together show that the class of (strongly) sat-sharp graphs is closed under the operation of adding a dominating vertex.

**Lemma 7.** Let $H$ be a $k$-vertex (strongly) sat-sharp graph, and let $H'$ be obtained from $H$ by adding a dominating vertex $v^*$. If $H$ has no isolated vertices, or if $\text{wt}(H) \leq k-2$, then $\text{wt}(H') = 2 + \text{wt}(H)$ and $\text{satlim}(H') = 1 + \text{satlim}(H)$. Moreover, $H'$ is also (strongly) sat-sharp.

**Proof of Lemma 7.** Let $\varepsilon(H,n) = \text{sat}(H,n) - \text{satlim}(H)n$, so that $\varepsilon(H,n) = o(n)$ when $H$ is sat-sharp and $\varepsilon(H,n) = O(1)$ when $H$ is strongly sat-sharp.

By Lemma 6 we have

$$\text{sat}(H',n) \leq (n-1) + \text{sat}(H,n-1) = (\text{satlim}(H) + 1)n + \varepsilon(H,n-1) - 1.$$  

If we can prove that $\text{wt}(H') \geq \text{wt}(H) + 2$, then Lemma 4 will give

$$\text{sat}(H',n) \geq \frac{\text{wt}(H') - 1}{2} n - c'_H = (\text{satlim}(H) + 1)n - c'_H.$$  

In particular, this implies that $\text{satlim}(H') = \frac{\text{wt}(H') - 1}{2}$ and that $|\varepsilon(H',n)| \leq |\varepsilon(H,n)| + |c'_H| + 1$, so if $H$ is (strongly) sat-sharp, then $H'$ is (strongly) sat-sharp.

An edge $e \in E(H)$ can be viewed (and its weight computed) either as an edge of $H$ or as an edge of $H'$. We will use $\text{wt}(e)$ and $\text{wt}'(e)$ to refer to the weight of such an edge computed in $H$ or $H'$, respectively. Observe that if $uv \in E(H)$, then when we pass to $H'$, we add $v^*$ as a new element of $N(u) \cap N(w)$ and change nothing else about the sets $N(u) \cap N(w)$ or $N(w) - N(u)$. Hence, $\text{wt}'(e) = \text{wt}(e) + 2$ for all $e \in E(H')$.

The only remaining edges of $H'$ are edges of the form $v^*u$ for $u \in V(H)$. We claim that all such edges have weight at most $2 + \text{wt}(H)$. If $u$ is isolated in $H$, then we have

$$\text{wt}'(uv^*) = 2 |N(u) \cap N(v^*)| + |N(v^*) - N(u)| = 0 + k = k \geq 2 + \text{wt}(H),$$

where the last inequality follows from the assumption that $\text{wt}(H) \leq k-2$ (since we assumed that $u$ is isolated and that $H$ either obeys this weight inequality or is isolate-free).

On the other hand, if $u$ is not isolated in $H$, let $ut$ be another edge incident to $u$. Observe that

$$\text{wt}'(v^*u) = 2 |N(u) \cap N(v^*)| + |N(v^*) - N(u)|$$

$$= 2d_H(u) + (k - d_H(u))$$

$$= d_H(u) + k$$

Note that Lemma 8 does not actually require the graph $H$ to be sat-sharp, although that is the main case we are concerned with. In the case where $H$ has no edges and so $\text{wt}(H) = \infty$, the hypothesis of Lemma 8 applies, yielding $\text{satlim}(K_{1,k}) = \frac{k-1}{2}$; this is an asymptotic version of the exact result of Kászonyi and Tuza [9].

**Lemma 8.** Let $H$ be a $k$-vertex graph with an isolated vertex $u$, and let $H'$ be obtained from $H$ by adding a dominating vertex $v^*$. If $\text{wt}(H) > k - 2$, then $H'$ is strongly sat-sharp, with $\text{wt}(H') = k$ and $\text{satlim}(H') = \frac{k-1}{2}$.

Proof of Lemma 8. Let $\varepsilon(H,n) = \text{sat}(H,n) - \text{satlim}(H)n$, so that $\varepsilon(H,n) = o(n)$ when $H$ is sat-sharp and $\varepsilon(H,n) = O(1)$ when $H$ is strongly sat-sharp. By Lemma 5 we have

$$\text{sat}(H',n) \leq (n-1) + \text{sat}(H,n-1) = (\text{satlim}(H) + 1)n + \varepsilon(H,n-1) - 1.$$  

If we can prove that $\text{wt}(H') \geq \text{wt}(H) + 2$, then Lemma 4 will give

$$\text{sat}(H',n) \geq \frac{\text{wt}(H') - 1}{2} n - c'_H = (\text{satlim}(H) + 1)n - c'_H.$$  

In particular, this implies that $\text{satlim}(H') = \frac{\text{wt}(H') - 1}{2}$ and that $|\varepsilon(H',n)| \leq |\varepsilon(H,n)| + |c'_H| + 1$, so if $H$ is (strongly) sat-sharp, then $H'$ is (strongly) sat-sharp.
and that $\text{wt}(ut) \leq (d_H(u) - 1) + (k - 1)$ for any edge $ut \in E(H)$. It follows that $\text{wt}'(v^*u) \geq \text{wt}(ut) + 2 = \text{wt}'(ut)$. Hence, an edge of minimum weight in $H'$ is found among the edges of $H$, and the smallest such weight is $\text{wt}(H) + 2$. \hfill \Box

**Proof of Lemma** We again write $\text{wt}(e)$ to refer to the weight of an edge $e$ computed in $H$ and $\text{wt}'(e)$ to refer to the weight of an edge $e$ when computed in $H'$.

As previously discussed, we have $\text{wt}'(e) = \text{wt}(e) + 2$ for every edge $e \in E(H)$. On the other hand, considering the isolated vertex $u$, we see that $\text{wt}(uv^*) = k$, as $\vert N(u) \cap N(v^*) \vert = 0$ and $\vert N(v^*) - N(u) \vert = k$.

If $\text{wt}(H) + 2 > k$, then this implies $\text{wt}(H') = k$, with the only edges of minimum weight being those edges joining $v^*$ with an isolated vertex of $H$. This establishes the first claim of the lemma.

Lemma 4 now gives the lower bound

$$\text{sat}(H', n) \geq \frac{k - 1}{2} n - c'_{H'}.$$ 

We establish a matching upper bound by constructing an $H'$-saturated graph on $n$ vertices, for any $n \geq \vert V(H') \vert$.

Let any $n \geq \vert V(H') \vert$ be given, and write $n = qk + r$, where $0 \leq r < k$. Let $G$ be the $n$-vertex graph consisting of $q$ disjoint copies of $K_k$ and a single copy of $K_r$. Clearly

$$|E(G)| = \frac{n - r}{k} \binom{k}{2} + \binom{r}{2} \leq \frac{k - 1}{2} n.$$ 

So if we can argue that $G$ is $H'$-saturated, then we will have $\text{satlim}(H') = \frac{k - 1}{2}$, and we will have that $H'$ is strongly sat-sharp.

It is clear that $G$ is $H'$-free, since $H'$ is connected and has $k + 1$ vertices, while every component of $G$ has at most $k$ vertices. We claim that adding any edge to $G$ produces a subgraph isomorphic to $H$. Let $xy$ be a missing edge in $G$; we may assume that $y$ lies in a copy of $K_k$. Let $Q$ be the set of vertices of the copy of $K_k$ containing $y$.

Now observe that we can embed $H'$ into $G + xy$ by any injection $\phi : V(H') \rightarrow V(G)$ that satisfies:

- $\phi(u) = x$, and
- $\phi(v^*) = y$,
- $\phi(V(H) - \{u, v^*\}) = Q - y$,

and with $k - 1$ vertices in $Q - y$, there is enough room to complete the last part of the embedding. The key point is that there is no edge, in $H'$, from $u$ to any vertex of $H'$ except for $v^*$, and all vertices of $H'$ except for $u$ are being embedded into a clique of $G$, so any edges they require are present. Thus, $G$ is $H'$-saturated, which completes the proof. \hfill \Box

### 4. Threshold Graphs

A natural class of strongly sat-sharp graphs is the class of **threshold graphs**. A simple graph $G$ with vertex set $\{v_1, \ldots, v_n\}$ is a threshold graph if there exist weights $x_1, \ldots, x_n \in \mathbb{R}$ such that, for all $i \neq j$, we have $v_iv_j \in E(G)$ if and only if $x_i + x_j \geq 0$. Threshold graphs were first introduced by Chvátal and Hammer \cite{Chvatal1976, Hammer1976}, albeit with a slightly different definition than the one we give here.

Threshold graphs admit many equivalent characterizations. For our purposes, the following characterization is the most useful one.
Theorem 9 (Chvátal–Hammer \cite{3}; see also \cite{10}). For a simple graph $G$, the following are equivalent:

1. $G$ is a threshold graph;
2. $G$ can be obtained from $K_1$ by iteratively adding a new vertex which is either an isolated vertex, or dominates all previous vertices;

In fact, \cite{10} gives several other equivalent characterizations of threshold graphs, but this is the one we will be interested in. The results of Section 3, together with this characterization, immediately imply that all threshold graphs are strongly sat-sharp. Furthermore, when a construction sequence for a threshold graph $G$ is given, one can use the lemmas from Section 3 to easily compute $\text{wt}(G)$ by iteratively computing the weight of each intermediate subgraph, keeping track of the previous subgraph’s weight and whether or not it had an isolated vertex.

As discussed in the introduction, several graphs whose saturation numbers were determined in previous work fall into the class of strongly sat-sharp graphs. In particular, complete graphs \cite{4}, stars \cite{9}, generalized books \cite{1}, stars plus an edge \cite{5}, and “nearly complete” graphs \cite{8} of the form $K_t - H$ for $H \in \{K_1, K_4 - K_1, K_4 - K_2\}$ are all threshold graphs. Thus, all of these graphs are strongly sat-sharp, and their saturation number is determined (up to a constant number of edges) by the results of Section 3.

As a non-example, we note that among the “nearly complete” graphs of \cite{8}, the graph $K_t - 2K_2$ is not a threshold graph, and in fact \cite{8} prove that $\text{sat}(K_t - 2K_2, n) = \left(\frac{t}{2} - \frac{3}{2}\right)n + O(1)$, whereas Lemma 4 only gives the lower bound $\text{sat}(K_t - 2K_2, n) \geq \left(\frac{t}{2} - \frac{7}{2}\right)n$.

Kászonyi and Tuza \cite{9} observed the “irregularity” that if $H$ is the graph obtained from $K_4$ by adding a pendant edge, then $\text{sat}(H, n) \leq \frac{5}{2}n$ while $\text{sat}(K_4, n) = 2n - 3$, so that $\text{sat}(H, n) < \text{sat}(K_4, n)$ for sufficiently large $n$ even though $K_4 \subseteq H$. Both $K_4$ and the graph $H$ are threshold graphs; in terms of our weight function, the irregularity can be seen to arise from the fact that all edges of $K_4$ have weight 5 while the pendant edge of $H$ has weight 4.

5. $H$-saturated construction when $H$ is the disjoint union of cliques

Faudree, Ferrara, Gould, and Jacobson \cite{7} determined the saturation numbers of generalized friendship graphs $F_{t,p,\ell}$, consisting of $t$ copies of $K_p$ which all intersect in a common $K_\ell$ but are otherwise pairwise disjoint. When $\ell = 0$, this includes the case of $tK_p$, consisting of $t$ disjoint copies of $K_p$. They also determined the saturation numbers of two disjoint cliques, $K_p \cup K_q$, when $p \neq q$, but left determining the saturation number of three or more disjoint cliques with general orders as an open problem. Here, we give a proof that all of these graphs are strongly sat-sharp, and determine their saturation numbers up to an additive constant for all sufficiently large $n$.

Proposition 10. Let $2 \leq p_1 \leq \cdots \leq p_m$ be positive integers. The graph $H = K_{p_1} \cup \cdots \cup K_{p_m}$ is strongly sat-sharp. In particular,

$$(p_1 - 2)n - c'_H \leq \text{sat}(H, n) \leq (p_1 - 2)n + c_H$$

for some constants $c_H, c'_H$ depending only on $H$ and for all $n \geq \sum_{i=1}^{m} p_i$.

Proof. First, note that $\text{wt}(H) = 2(p_1 - 2) + 1$. So by Lemma 4,

$$\text{sat}(H, n) \geq (p_1 - 2)n - c'_H$$
A LOWER BOUND ON THE SATURATION NUMBER

\[ G' = \underbrace{\cdots} + \underbrace{\cdots} + \underbrace{\cdots} \]

\[ I \]

Figure 2. Construction of the saturated graph \( G \) for \( H = K_3 \cup K_5 \cup K_6 \).

for some constant \( c_H' \).

On the other hand, let \( G \) be the graph on \( n \) vertices defined as the join, \( G = K_{p_1-2} \cup G' \) where \( G' = K_t \cup I \), the disjoint union of a clique on \( t = 1 + \sum_{i=2}^{m} p_i \) vertices and a set \( I \) with \( n - t - p_1 + 2 \) isolated vertices. Figure 2 shows the graph \( G \) that is constructed for \( H = H_4 \cup K_5 \cup K_6 \).

We claim that \( G \) is \( H \)-free and \( H \)-saturated. To see that \( G \) is \( H \)-free, consider its maximal cliques. Let \( Q \) denote the subgraph of \( G \) induced by the vertices of the \( K_{p_1-2} \) and the \( K_t \). Then \( Q \) is a maximal clique with \( p_1 - 2 + t \) vertices. All other maximal cliques of \( G \) are formed from the \( K_{p_1-2} \) and one vertex from \( I \). Therefore, if we were to find a copy of \( H \) in \( G \), then each of the disjoint cliques of \( H \) must be found in \( Q \). But \( Q \) only has \( |V(H)| - 1 \) vertices so this cannot happen.

To see that \( G \) is \( H \)-saturated, consider the graph \( G + xy \) for some \( xy /\in E(G) \). Without loss of generality, either \( x, y \in I \) or \( x \in I \) and \( y \in K_t \). In either case, the vertices of \( K_{p_1-2} \cup \{x, y\} \) form a \( p_1 \)-clique in \( G + xy \), while at least \( t - 1 = p_2 + \cdots + p_m \) vertices of the \( K_t \) remain disjoint from this clique and can be used to embed the remaining cliques of \( H \). So \( G \) is \( H \)-saturated.

Since \( G \) has \( (p_1 - 2)(n + 1 - \sum_{i=1}^{m} p_i) + (p_1 + \cdots + p_m - 1) \) edges, it follows that

\[ \text{sat}(H, n) \leq (p_1 - 2)n + c_H \]

for some constant \( c_H \). Therefore, \( H \) is strongly sat-sharp.

An immediate corollary to this proposition and the results of Section 3 is the following result.

**Corollary 11.** Let \( \ell \) and \( 2 \leq p_1 \leq \cdots \leq p_m \) be positive integers. Let \( H' = K_{p_1} \cup \cdots \cup K_{p_m} \), and let \( H = K_\ell \cup H' \). Then \( H \) is sat-sharp. In particular,

\[ (p_1 + \ell - 2)n - c'_H \leq \text{sat}(H, n) \leq (p_1 + \ell - 2)n + c_H \]

for some constants \( c_H, c'_H \) depending only on \( H \) and for all \( n \geq \ell + \sum_{i=1}^{m} p_i \).

Note that this class of graphs includes all generalized friendship graphs \( F_{t,p,\ell} \) for \( p \geq t + 2 \). Since \( F_{t,p,\ell} \) for \( p = \ell + 1 \) is a threshold graph, we already know from the discussion in Section 4 that it is strongly sat-sharp.

While a disjoint union of cliques is not, in general, a threshold graph, each of its components is a threshold graph. Proposition 10 therefore suggests that perhaps a disjoint union of threshold graphs is always strongly sat-sharp. More boldly, the following conjecture appears to be plausible:
Conjecture 12. If $H_1$ and $H_2$ are (strongly) sat-sharp graphs, then their disjoint union $H_1 + H_2$ is (strongly) sat-sharp. That is, the class of (strongly) sat-sharp graphs is closed under taking disjoint unions.

Conjecture 12 together with the other closure properties from Section 3 would immediately imply Proposition 10. We have found ad-hoc constructions for some small disjoint unions of particular threshold graphs which suggest that Conjecture 12 might hold, but it has been difficult to extract a general construction.

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