Free Energy of a Dilute Bose Gas:
Lower Bound

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Abstract

A lower bound is derived on the free energy (per unit volume) of a 
homogeneous Bose gas at density \( \rho \) and temperature \( T \). In the dilute 
regime, i.e., when \( a^3 \rho \ll 1 \), where \( a \) denotes the scattering length of the 
pair-interaction potential, our bound differs to leading order from the 
expression for non-interacting particles by the term \( 4\pi a \left( 2\rho^2 - [\rho - \rho_c]^2 \right) \). 
Here, \( \rho_c(T) \) denotes the critical density for Bose-Einstein condensation 
(for the non-interacting gas), and \( [\cdot]_+ = \max\{\cdot, 0\} \) denotes the positive part. Our bound is uniform in the temperature up to temperatures 
of the order of the critical temperature, i.e., \( T \sim \rho^{2/3} \) or smaller. One 
of the key ingredients in the proof is the use of coherent states to 
extend the method introduced in [17] for estimating correlations to 
temperatures below the critical one.

1 Introduction and Main Result

The advance of experimental techniques for studying ultra-cold atomic gases 
has triggered numerous investigations on the properties of dilute quantum 
gases. From a mathematical point of view, several rigorous results have 
been obtained over the last few years. (See [8] for an overview.) The first of 
these, which has inspired much of the later work, was a study of the ground

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state energy of a Bose gas with repulsive interaction at low density $\varrho$. Per unit volume, it is given by

$$e_0(\varrho) = 4\pi a\varrho^2 + o(\varrho^2) \quad \text{for } a^3\varrho \ll 1 \quad (1.1)$$

in three spatial dimensions. Here, $a > 0$ denotes the scattering length of the interparticle interaction, and units are chosen such that $\hbar = 2m = 1$, with $m$ the mass of the particles. A lower bound on $e_0(\varrho)$ of the correct form (1.1) was proved by Lieb and Yngvason in [11]. Much earlier, Dyson [2] had already proved an upper bound of the desired form, at least in the special case of hard-sphere particles. An extension of his calculation to arbitrary repulsive interaction potentials was given in [9].

The methods introduced in [11] have been extended to treat the case of fermions as well, for the study of both the ground state energy [7] and the free energy at positive temperature [16]. We are concerned here with the extension of (1.1) to positive temperature, at least as far as a lower bound is concerned. That is, our goal is to derive a lower bound on the free energy of a dilute Bose gas at density $\varrho$ and temperature $T$. Much of the complication in such an estimate is caused by the existence of a Bose-Einstein condensate for temperatures below some critical temperature. Although the existence of a condensate for interacting Bose gases has so far eluded a mathematical proof, its presence can easily be shown in the case of non-interacting particles. A short review of the Bose gas without interaction among the particles is given in Subsection 1.2 below.

One of the main ingredients in our estimate is a method to quantify correlations present in the state of the interacting system. This method has been introduced in [17]; it does not immediately apply below the critical temperature for Bose-Einstein condensation, however. We have been able to overcome this difficulty with the aid of coherent states.

### 1.1 Definition of the Model

We consider a system of $N$ bosons, confined to a three-dimensional flat torus of side lengths $L$, which we denote by $\Lambda$. The one-particle state space is thus $L^2(\Lambda, dx)$, and the Hilbert space for the system is the symmetric $N$-fold tensor product $\mathcal{H}_N = L^2_{\text{sym}}(\Lambda^N, d^N x)$, i.e., the space of square integrable functions of $N$ variables that are invariant under exchange of any pair of variables. The Hamiltonian is given as

$$H_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} v(d(x_i, x_j)). \quad (1.2)$$
Here, $\Delta$ denotes the Laplacian on $\Lambda$, and $d(x,y)$ denotes the distance between points $x$ and $y$ on the torus $\Lambda$. The particle interaction potential $v: \mathbb{R}_+ \mapsto \mathbb{R}_+ \cup \{\infty\}$ is assumed to be a non-negative and measurable function. It is allowed to take the value $+\infty$ on a set of positive measure, corresponding to hard sphere particles. In this case, the domain of the Hamiltonian has to be suitably restricted to functions that vanish on the set where the interaction potential is infinite. We assume that $v$ has a finite range $R_0$, i.e., $v(r) = 0$ for $r > R_0$. In particular, it has a finite scattering length, which we denote by $a$. We will recall the definition of $a$ in Subsection 1.3 below.

We note that in a concrete realization of $\Lambda$ as the set $[0, L]^3 \subset \mathbb{R}^3$, $\Delta$ is the Laplacian on $[0, L]^3$ with periodic boundary conditions. Moreover, the distance $d(x,y)$ is given as $d(x,y) = \min_{k \in \mathbb{Z}^3} |x - y - kL|$. Note also that $v(d(x,y)) = \sum_{k \in \mathbb{Z}^3} v(|x - y - kL|)$ if $L > 2R_0$.

The free energy (per unit volume) of the system at inverse temperature $\beta = 1/T > 0$ and density $\rho > 0$ is given by

$$f(\beta, \rho) = -\frac{1}{\beta} \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \ln \text{Tr}_{\mathcal{H}_N} \exp(-\beta H_N),$$

(1.3)

where $\lim$ stands for the usual thermodynamic limit $L \to \infty$, $N \to \infty$ with $\rho = N/|\Lambda|$ fixed. Here, we denote the volume of $\Lambda$ by $|\Lambda| = L^3$. Existence of the thermodynamic limit in (1.3) can be shown by standard methods, see, e.g., [14, 15].

We are interested in a bound on $f$ in the case of a dilute gas, meaning that $a^3 \rho$ is small. The dimensionless parameter $\beta \rho^{2/3}$ is of order one (or larger), however. Note that sometimes in the literature the case of small $\rho$, but fixed $a$ and $\beta$, is understood with the term “dilute”. This corresponds to a high-temperature (classical) limit and is not what we want to study here.

### 1.2 Ideal Bose Gas

In the case of vanishing interaction potential ($v \equiv 0$), the free energy can be evaluated explicitly. It is given as

$$f_0(\beta, \rho) = \sup_{\mu \leq 0} \left\{ \mu \rho + \frac{1}{(2\pi)^3 \beta} \int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta (p^2 - \mu)} \right) \right\},$$

(1.4)
The supremum is uniquely attained at \( \mu = \mu_0(\beta, \varrho) = d/d\varrho f(\beta, \varrho) \leq 0 \). If \( \varrho \) is bigger than the critical density

\[
\varrho_c(\beta) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp \frac{1}{e^{\beta p^2} - 1} = (4\pi \beta)^{-3/2} \sum_{\ell \geq 1} \ell^{-3/2},
\]

the supremum is attained at \( \mu_0 = 0 \), whereas for \( \varrho < \varrho_c(\beta) \), it is attained at some \( \mu_0 = \mu_0(\beta, \varrho) < 0 \). In particular,

\[
\frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp \frac{1}{e^{\beta (p^2 - \mu_0)} - 1} = \min\{\varrho, \varrho_c(\beta)\}.
\]

Note also that the scaling relation \( f_0(\beta, \varrho) = \varrho^{5/3} f_0(\beta \varrho^{2/3}, 1) \) holds for an ideal Bose gas. In particular, the dimensionless quantity \( \beta \varrho^{2/3} \) is the only relevant parameter.

### 1.3 Scattering Length

The scattering length of a potential \( v \) can be defined as follows (see Appendix A in [12], or Appendix C in [8]): For \( R \geq R_0 \),

\[
\frac{4\pi a}{1 - a/R} = \inf \left\{ \int_{|x| \leq R} dx \left( |\nabla \phi(|x|)|^2 + \frac{1}{2} v(|x|) \phi(|x|)^2 \right) : \phi : [0, R] \mapsto \mathbb{R}_+ , \phi(R) = 1 \right\}.
\]

For this definition to make sense, \( v \) need not necessarily be positive, one only has to assume that \( -\Delta + \frac{1}{2} v \) (as an operator on \( L^2(\mathbb{R}^3) \)) does not have any negative spectrum. We will restrict our attention to non-negative \( v \), however. The infimum in (1.7) is attained uniquely. Moreover, the minimizer has a trivial dependence on \( R \): for some function \( \phi_v \) (independent of \( R \)) it can be written as \( \phi_v(|x|)/\phi_v(R) \). Note that \( a \) is independent of \( R \), and also that \( \phi_v(|x|) = 1 - a/|x| \) for \( |x| \geq R_0 \).

### 1.4 Main Theorem

Our main result is the following lower bound on the free energy, defined in (1.3). It gives a bound on the leading order correction, compared with a non-interacting gas, in the case of small \( a^2 \varrho \) and fixed \( \beta \varrho^{2/3} \).
THEOREM 1 (Lower bound on free energy of dilute Bose gas).

There is a function $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$, uniformly bounded on bounded subsets of $\mathbb{R}_+$, and an $\alpha > 0$ such that

$$f(\beta, \rho) \geq f_0(\beta, \rho) + 4\pi a \left(2\rho^2 - [\rho - \rho_c(\beta)]^2\right) \left(1 - o(1)\right),$$  \hspace{1cm} (1.8)

with

$$o(1) \leq C \left((\beta \rho^{2/3})^{-1}\right) (a \rho^{1/3})^\alpha.$$ \hspace{1cm} (1.9)

Here, $[\cdot]_+ = \max\{\cdot, 0\}$ denotes the positive part. In the case of non-interacting particles, the expression $[\rho - \rho_c(\beta)]_+$ is just the condensate density.

Remarks.

1. Since $C(t)$ is uniformly bounded for bounded $t$, our estimate is uniform in the parameter $(\beta \rho^{2/3})^{-1}$ as long as it stays bounded. I.e., our result is uniform in the temperature for temperatures not much greater than the critical temperature (for the non-interacting gas). In particular, we recover the result (1.1) in the zero temperature limit. The error term is worse, however; in [11], it was shown that the exponent $\alpha$ can be taken to be $\alpha = 3/17$ at $T = 0$, whereas our proof shows that $\alpha$ can be chosen slightly larger than 0.00087 (independent of $T$). This value has no physical significance, however, it merely reflects the multitude of estimates needed to arrive at our result.

2. The error term, $o(1)$, in our lower bound depends on the interaction potential $v$, besides its scattering length $a$, only through its range $R_0$. This dependence could in principle be displayed explicitly. By cutting off the potential in a suitable way, one can then extend the result to infinite range potentials (with finite scattering length). See Appendix B in [9] for details.

3. For $\rho \leq \rho_c(\beta)$ (i.e., above the critical temperature), the leading order correction term is given by $8\pi a \rho^2$, compared with $4\pi a \rho^2$ at zero temperature. The additional factor 2 is an exchange effect; heuristically speaking, it is a result of the symmetrization of the wave functions. This symmetrization only plays a role if the particles are in different one-particle states, which they are essentially always above the critical temperature. Below the critical temperature, however, a macroscopic number of particles occupies the zero-momentum state; there is no exchange effect among these particles, which explains the subtraction of the square of the condensate density in (1.8).
4. We note that \( f_0(\beta, \varrho) \) has a discontinuous third derivative with respect to \( \varrho \) at \( \varrho = \varrho_c(\beta) \) or, equivalently, a discontinuous third derivative with respect to \( T = 1/\beta \) at the critical temperature. Since the specific heat \( c_V(\beta, \varrho) \) can be expressed in terms of the free energy as \( c_V(\beta, \varrho) = -T d^2/(dT)^2 f(\beta, \varrho) \), it has a discontinuous derivative (with respect to \( T \)) at the critical temperature. The first order correction term in (1.8) has a discontinuous second derivative at this value. Considering only this term and neglecting higher order corrections, this would mean that the specific heat is actually discontinuous at the critical temperature.

5. Our method applies also to particles with internal degrees of freedom, e.g., to particles with nonzero spin. For simplicity, we treat only the case of spinless particles here.

6. Although we provide only a lower bound in this paper, one can expect that the second term in (1.8) gives the correct leading order correction to the free energy (see, e.g., [4, Chapter 12.4]). To prove this, one has to derive an appropriate upper bound on \( f(\beta, \varrho) \), which has not yet been achieved, however. We note that a naive upper bound using first order perturbation theory yields (1.8) with \( 4\pi a \) replaced by \( \frac{1}{2} \int dx v(|x|) \), which need not be finite, however (and is always strictly greater than \( 4\pi a \)).

The remainder of this paper is devoted to the proof of Theorem 1. The proof is quite lengthy and is split into several subsections. To guide the reader, we start every subsection with a short summary of what will be accomplished.

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2 Proof of Theorem 1

In the following, we find it convenient to think of \( \Lambda \) as the set \([0, L]^3\) embedded in \( \mathbb{R}^3 \). We will also assume \( L \) to be large. In particular, \( L > 2R_0 \), but \( L \) will also be assumed to be large compared with several other parameters (with are independent of \( L \)) appearing below. This is justified since we are only interested in quantities in the thermodynamic limit \( L \to \infty \).

In many places in our proof, the Heaviside step function \( \theta \) will appear. We point out that we use the convention that \( \theta \) equals 1 at the origin, i.e., \( \theta(t) = 0 \) for \( t < 0 \), and \( \theta(t) = 1 \) for \( t \geq 0 \).
2.1 Reduction to Integrable Potentials

Recall that we do not want to restrict our attention to interaction potentials that are integrable. For the Fock space treatment in the next subsection, it will be necessary that $v$ has finite Fourier coefficients, however. As a first step, we will therefore replace the interaction potential $v$ by a smaller potential $\tilde{v}$ whose integral is bounded by some number $8\pi \varphi$. The scattering length of the new potential will be smaller than $a$, however. In the following lemma, we show that as long as $\varphi$ is much greater than $a$, the change in the scattering length remains small.

**Lemma 1.** Let $v : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \cup \{\infty\}$ have finite scattering length $a$. For any $\varepsilon > 0$, there exists a $\tilde{v}$, with $0 \leq \tilde{v}(r) \leq v(r)$ for all $r$, such that $\int_0^\infty dr r^2 \tilde{v}(r) \leq 2\varphi$, and such that the scattering length of $\tilde{v}$, denoted by $\tilde{a}$, satisfies

$$
\tilde{a} \geq a \left( 1 - \sqrt{a/\varphi} \right) (1 - \varepsilon).
$$

**Proof.** Without loss of generality, we may assume that $\varphi > a$. Let $R = \inf\{s : \int_s^\infty dr r^2 v(r) < \infty\}$. We note that $R$ is finite; in fact $R \leq a$. This follows from the fact that $2a \geq \int_0^\infty dr r^2 v(r) |\phi_v(r)|^2$, where $\phi_v$ denotes the minimizer of (1.7) (for $R = \infty$), as introduced in Subsection 1.3. Since it satisfies $\phi_v(r) \geq 1 - a/r$ (see Appendix B in [12]), $\int_s^\infty dr r^2 v(r)$ is finite for $s > a$.

Assume first that $\int_R^\infty dr r^2 v(r) \geq 2\varphi$. The function $s \mapsto \int_s^\infty dr r^2 v(r)$ is continuous for $s > R$. We can thus choose $s \geq R$ such that $\int_s^\infty dr r^2 v(r) = 2\varphi$, and $\tilde{v}(r) = v(r)\theta(r-s)$.

To obtain an upper bound on $a$, we can use a trial function $\phi(r) = (\phi_\tilde{v}(r) - \phi_\tilde{v}(s)r/s)\theta(r-s)$ in the variational principle (1.7). We note that $\phi$ is a non-negative function, since $\phi_\tilde{v}(r)$ is monotone increasing in $r$ [12]. By partial integration, using the variational equation $-\Delta \phi_\tilde{v}(|x|) + 4\pi v(|x|)\phi_\tilde{v}(|x|) = 0$, we have

$$
4\pi a \leq \int_{\mathbb{R}^3} dx \left( |\nabla \phi(|x|)|^2 + \frac{1}{2} v(|x|)|\phi(|x|)|^2 \right) \tag{2.1.2}
$$

$$
= 4\pi (\tilde{a} + s\phi_\tilde{v}(s)) + 2\pi \phi_\tilde{v}(s) \int_s^\infty dr r^2 v(r) \frac{s}{r} \left( \phi_\tilde{v}(s) \frac{s}{r} - \phi_\tilde{v}(r) \right).
$$

The last term is negative and can be dropped for an upper bound. To obtain an upper bound on $s\phi_\tilde{v}(s)$, we note that $\phi_\tilde{v}(s) \geq 1 - \tilde{a}/s$, and hence

$$
s\phi_\tilde{v}(s) \leq \frac{\tilde{a}}{1/\phi_\tilde{v}(s) - 1}. \tag{2.1.3}
$$
For an upper bound on \( \phi_\bar{v}(s) \), we use again the monotonicity of \( \phi_\bar{v}(r) \), which allows us to estimate
\[
a \geq \bar{a} \geq \frac{1}{2} \int_s^\infty dr \, r^2 \phi_\bar{v}(r)^2 \geq \phi_\bar{v}(s)^2 \phi.
\] (2.1.4)
This yields \( \phi_\bar{v}(s) \leq \sqrt{a/\phi} \).

Altogether, we have thus shown that
\[
a \leq \bar{a} + s \phi_\bar{v}(s) \leq \bar{a} \left( 1 + \frac{1}{\sqrt{\phi/a - 1}} \right).
\] (2.1.5)
This proves (2.1.1) (with \( \varepsilon = 0 \)) under the assumption that \( \int_R^\infty dr \, r^2 v(r) \geq 2 \phi \).

Consider now the case when \( \int_R^\infty dr \, r^2 v(r) = 2 \phi - T \) for some \( T > 0 \). If \( R = 0 \), we can take \( \bar{v} = v \), and there is nothing to prove. Hence we can assume that \( R > 0 \). By definition, we have that \( \int_{R(1-\varepsilon)}^R dr \, r^2 v(r) = \infty \) for any \( \varepsilon > 0 \). Hence there exists a \( \tau \) (depending on \( T \) and \( \varepsilon \)) such that \( \int_{R(1-\varepsilon)}^R dr \, \min\{v(r), \tau\} = T \). We can then take
\[
\bar{v}(r) = \begin{cases} v(r) & \text{for } r \geq R \\ \min\{v(r), \tau\} & \text{for } (1-\varepsilon)R \leq r < R \\ 0 & \text{otherwise}. \end{cases}
\] (2.1.6)
Applying the same argument as in (2.1.2), with \( s = R \), we have \( a \leq \bar{a} + R \phi_\bar{v}(R) \). Now \( R \leq a \), and \( \phi_\bar{v}(R(1-\varepsilon)) \leq \sqrt{a/\phi} \) using the same argument as in (2.1.4), noting that \( \bar{v}(r) = 0 \) for \( r \leq R(1-\varepsilon) \). Moreover, since \( |\nabla \phi_\bar{v}(|x|)| \leq \bar{a}/|x|^2 \), as shown in [9, Eq. (3.33)], \( |\phi_\bar{v}(R(1-\varepsilon)) - \phi_\bar{v}(R)| \leq \varepsilon \bar{a} R^{-1}/(1-\varepsilon) \), and thus
\[
a \leq \bar{a} \frac{1}{1-\varepsilon} + a \sqrt{\frac{a}{\phi}}.
\] (2.1.7)
This finishes the proof of the lemma.

As an example, consider the case of a pure hard sphere interaction, i.e., \( v(r) = \infty \) for \( r \leq a \), and \( v(r) = 0 \) for \( r > a \). In this case, we can choose \( \bar{v}(r) = 6 \varphi a^{-3} \theta(a - r) \). The scattering length of \( \bar{v} \) is given by \( \bar{\alpha} = a(1 - \sqrt{a/(6 \varphi)}) \tan \sqrt{6 \varphi/a} \) in this case. Note that \( \tanh t \leq 1 \) for all \( t \). In particular, \( \bar{a} \geq a(1 - \sqrt{a/(6 \varphi)}) \).

For a lower bound, we can simply replace \( v \) by \( \bar{v} \), i.e., we have \( H_N \geq \bar{H}_N \), with
\[
\bar{H}_N = \sum_{i=1}^{N} -\Delta_i + \sum_{1 \leq i < j \leq N} \bar{v}(d(x_i, x_j)).
\] (2.1.8)
If we choose $\varepsilon \leq \sqrt{a/\varphi}$, the error in the scattering length $\tilde{a}$ is of the order $\sqrt{a/\varphi}$. We will choose $\varphi \gg a$ below.

2.2 Fock Space

In the following, it will be convenient to give up the restriction on the particle number and work in Fock space instead. This has the advantage that the problem of condensation can be dealt with with the aid of coherent states, which will be introduced in the next subsection. Our treatment differs slightly from the usual grand canonical ensemble since we do not simply introduce a chemical potential as Lagrange multiplier to control the number of particles, but we add a quadratic expression in $N$ to the Hamiltonian. This gives a better control on the particle number.

Let $\mu_0 \leq 0$ be the chemical potential of the ideal Bose gas, which is the quantity that maximizes the expression in (1.4). Let $F = \bigoplus_N H_N$ be the bosonic Fock space over $L^2(\Lambda)$. Let $a_p^\dagger$ and $a_p$ denote the usual creation and annihilation operators of plane waves in $\Lambda$ with wave functions $L^{-3/2}e^{-ipx}$. We define a Hamiltonian $H$ on Fock space as

$$H = T + V + K + \mu_0 N,$$  

(2.2.1)

with

$$T = \sum_p (p^2 - \mu_0) a_p^\dagger a_p, \quad V = \frac{1}{2|\Lambda|} \sum_{p,k,l} \hat{v}(p) a_{k+p}^\dagger a_{l-p}^\dagger a_k a_l,$$  

(2.2.2)

and

$$K = 4\pi \tilde{a} C \frac{|\Lambda|}{|\Lambda|} (N - N)^2.$$  

(2.2.3)

Here and in the following, all sums are over $p \in \frac{2\pi}{L} \mathbb{Z}^3$. The Fourier transform of $\tilde{v}$ is denoted by $\hat{v}$, i.e., $\hat{v}(p) = \int_{\Lambda} dx \tilde{v}(|x|) e^{-ipx}$. It is uniformly bounded; in fact, $|\hat{v}(p)| \leq \tilde{v}(0) \leq 8\pi \varphi$, where $\varphi$ was introduced in the previous subsection. The number operator $\sum_p a_p^\dagger a_p$ is denoted by $\mathbb{N}$, whereas $N$ is just a parameter. The parameter $C$ is positive and will be chosen later on.

The Hamiltonian $H$ commutes with the number operator $\mathbb{N}$, and can be thought of a direct sum of its restrictions to definite particle number. Note that the restriction to particle number $N$ is just $\tilde{H}_N$, i.e., $H = \tilde{H}_N$ on the sector of particle number $N$. This implies, in particular, that

$$\text{Tr}_{\mathbb{N}} \exp (-\beta \tilde{H}_N) \leq \text{Tr}_F \exp (-\beta H) .$$  

(2.2.4)

We will proceed deriving an upper bound on the latter expression.
2.3 Coherent States

To obtain an upper bound on the partition function $\text{Tr}_\mathcal{F} \exp(-\beta \mathcal{H})$, we use the method of coherent states [10]. Effectively, this replaces the operators $a_p^\dagger$ and $a_p$ by numbers. This can be viewed as a rigorous version of part of the Bogoliubov approximation, where one replaces the operators $a_0^\dagger$ and $a_0$ by numbers. Such a replacement is particularly useful if the zero-mode is “macroscopically occupied”, i.e., if $a_0^\dagger a_0 \sim |\Lambda|$. We will use this method not only for $p = 0$, however, but for a whole range of momenta $|p| < p_c$ for some $p_c \geq 0$. Although not macroscopic, their occupation will be large enough to require this separate treatment.

To be more precise, let us pick some $p_c \geq 0$ and write $F = F_< \otimes F_>$, where $F_<$ and $F_>$ denote the Fock spaces corresponding to the modes $|p| < p_c$ and $|p| \geq p_c$, respectively. Let $M$ denote the number of $p \in \frac{2\pi}{\mathcal{L}} \mathbb{Z}^3$ with $|p| < p_c$. As shown in [10], the Berezin-Lieb inequality [1, 5] implies that

$$\text{Tr}_\mathcal{F} \exp(-\beta \mathcal{H}) \leq \int_{C^M} d^Mz \text{Tr}_{F_>} \exp\left(-\beta \mathbb{H}^s(\vec{z})\right).$$

(2.3.1)

Here, $\vec{z}$ denotes the vector $(z_1, \ldots, z_M) \in \mathbb{C}^M$, $d^Mz = \prod_{i=1}^M dz_i$ and $dz = \pi^{-1} dx dy$ with $x = \Re(z)$, $y = \Im(z)$. Moreover, $\mathbb{H}^s(\vec{z})$ is the upper symbol of the operator $\mathcal{H}$. It is an operator on $F_>$, parametrized by $\vec{z}$, and can be written in the following way. Let $|\vec{z}\rangle \in F_<$ denote the coherent state

$$|\vec{z}\rangle = \exp\left(\sum_{|p| < p_c} z_p a_p^\dagger - z_p^* a_p\right) |0\rangle \equiv U(\vec{z})|0\rangle,$$

(2.3.2)

with $|0\rangle$ the vacuum in the Fock space $F_<$. Then the lower symbol of $\mathcal{H}$ is given by $\mathbb{H}_s(\vec{z}) = \langle \vec{z}| \mathcal{H}|\vec{z}\rangle$. Since $a_p|\vec{z}\rangle = z_p|\vec{z}\rangle$, the lower symbol is obtained from the expression (2.2.1) by simply replacing all the $a_p$ by $z_p$ and the $a_p^\dagger$ by $z_p^*$ for all $|p| < p_c$. The upper symbol can be obtained from the lower symbol by replacing $|z_p|^2$ by $|z_p|^2 - 1$, for instance, and similarly with other polynomials in $z_p$; see, e.g., [10] for details. We can then write $\mathbb{H}^s(\vec{z})$ in the following way. Denoting by $N_s(\vec{z}) = |\vec{z}|^2 + \sum_{|p| \geq p_c} a_p^\dagger a_p$ the lower symbol of the number operator, we have

$$\mathbb{H}^s(\vec{z}) = \mathbb{H}_s(\vec{z}) - \Delta \mathbb{H}(\vec{z}),$$

(2.3.3)
\[ \Delta \mathbb{H}(z) = \sum_{|p| < p_c} (p^2 - \mu_0) + \frac{1}{2|\Lambda|} \left[ \hat{v}(0) \left( 2MN_s(z) - M^2 \right) \right. \\
+ 2 \sum_{|l| < p_c, |k| \geq p_c} \hat{v}(l-k)a_k^\dagger a_k + \sum_{|l| < p_c, |k| < p_c} \hat{v}(l-k) \left( 2|z_k|^2 - 1 \right) \\
+ \left. \frac{4\pi \tilde{a}C}{|\Lambda|} \left[ 2|z|^2 + M(2N_s(z) - 2N - M) \right] \right]. \tag{2.3.4} \]

Here, we have used that \( \hat{v}(p) = \hat{v}(-p) \).

Since \( \tilde{v} \) is a non-negative function, \( |\hat{v}(p)| \leq \hat{v}(0) \leq 8\pi \varphi \) for all \( p \). Hence we obtain the bound

\[ \Delta \mathbb{H}(z) \leq M(p_c^2 - \mu_0) + \frac{16\pi \varphi}{|\Lambda|} MN_s(z) + \frac{8\pi \tilde{a}C}{|\Lambda|} \left[ |z|^2 + M(N_s(z) - N) \right]. \tag{2.3.5} \]

(Here, we have used again the positivity of \( \tilde{v} \).) Denoting by \( K_s(z) = \langle z | K | z \rangle \) the lower symbol of \( K \) (and, similarly, for \( T \) and \( V \) below), we have

\[ K_s(z) = \frac{4\pi \tilde{a}C}{|\Lambda|} \left( (N_s(z) - N)^2 + |z|^2 \right) \geq \frac{4\pi \tilde{a}C}{|\Lambda|} (N_s(z) - N)^2. \tag{2.3.6} \]

We can use part of \( K_s(z) \) to estimate \( -\Delta \mathbb{H} \) from below independently of \( z \). More precisely, we have

\[ \frac{1}{2} K_s(z) - \Delta \mathbb{H}(z) \]
\[ \geq -M(p_c^2 - \mu_0) - \frac{8\pi N}{|\Lambda|}(2\varphi M + \tilde{a}C) - 32\pi \tilde{a}C \frac{(M + 1)^2}{|\Lambda|} \left( 1 + \frac{2\varphi}{\tilde{a}C} \right)^2 \]
\[ \equiv -Z^{(1)}. \tag{2.3.7} \]

Note that \( M \sim p_c^2|\Lambda| \) in the thermodynamic limit. We will choose the parameters \( p_c, \varphi \) and \( C \) such that \( Z^{(1)} \ll |\Lambda| a^2 \) for small \( a^1/3 \).

With the definition

\[ F_s(\beta) \equiv -\frac{1}{\beta} \ln \text{Tr}_{F_s} \exp \left( -\beta(T_s(z) + V_s(z) + \frac{1}{2}K_s(z)) \right), \tag{2.3.8} \]

(2.3.1) and the estimates above imply that

\[ -\frac{1}{\beta} \ln \text{Tr}_{F} \exp(-\beta \mathbb{H}) \geq \mu_0 N - \frac{1}{\beta} \ln \int_{CM} d^Mz \exp(-\beta F_s(\beta)) - Z^{(1)}. \tag{2.3.9} \]
Hence it remains to derive a lower bound on $F_{\vec{z}}(\beta)$.

Let $\Gamma_{\vec{z}}$ denote the Gibbs state of $T_s(\vec{z})+V_s(\vec{z})+\frac{1}{2}K_s(\vec{z})$ on $\mathcal{F}_\geq$, for inverse temperature $\beta$. Let $\Pi_0 = |0\rangle\langle 0|$ denote the vacuum state in $\mathcal{F}_\leq$. Denoting by $\Upsilon_{\vec{z}}$ the state $\Upsilon_{\vec{z}} = U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma_{\vec{z}}$ on the full Fock space $\mathcal{F}$, we can write

$$F_{\vec{z}}(\beta) = \text{Tr}_{\mathcal{F}}[(T + V + \frac{1}{2}K)\Upsilon_{\vec{z}}] - \frac{1}{\beta}S(\Upsilon_{\vec{z}}). \quad (2.3.10)$$

Here, $S(\Gamma) = -\text{Tr}_{\mathcal{F}} \Gamma \ln \Gamma$ denotes the von-Neumann entropy.

### 2.4 Relative Entropy and A Priori Bounds

In the following, we want to derive a lower bound on $F_{\vec{z}}(\beta)$. Although we do not have an upper bound available, we can assume an appropriate upper bound without loss of generality; if the assumption is not satisfied, there is nothing to prove (as far as a lower bound is concerned). This upper bound can be formulated as a bound on the relative entropy between the state $\Upsilon_{\vec{z}} = U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma_{\vec{z}}$ defined above and a simple reference state (describing non-interacting particles). Together with a bound on the total number of particles, this estimate on the relative entropy contains all the information we need in order to prove the desired properties of the state $\Upsilon_{\vec{z}}$ that will allow us to derive a lower bound on (2.3.10).

We note that, for any state $\Gamma$ of the form $\Gamma = U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma_{\geq}$ for some state $\Gamma_{\geq}$ on $\mathcal{F}_{\geq}$,

$$\text{Tr}_{\mathcal{F}}[T \Gamma] - \frac{1}{\beta}S(\Gamma) = \text{Tr}_{\mathcal{F}_{\geq}}[T_s(\vec{z})\Gamma_{\geq}] - \frac{1}{\beta}S(\Gamma_{\geq})$$

$$\geq -\frac{1}{\beta} \ln \text{Tr}_{\mathcal{F}_{\geq}} \exp \left(-\beta T_s(\vec{z})\right). \quad (2.4.1)$$

In fact, the difference between the right and left sides of (2.4.1) is given by $\beta^{-1}S(\Gamma, \Omega_{\vec{z}})$, where $S$ denotes the relative entropy. For two general states $\Gamma$ and $\Gamma'$ on Fock space, it is given by

$$S(\Gamma, \Gamma') = \text{Tr}_{\mathcal{F}}(\ln \Gamma - \ln \Gamma'). \quad (2.4.2)$$

Note that the relative entropy is a non-negative quantity. The state $\Omega_{\vec{z}}$ is given by $\Omega_{\vec{z}} = U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma^0$, where $\Gamma^0$ is the Gibbs state of $T_s(\vec{z})$ on $\mathcal{F}_{\geq}$ (which is independent of $\vec{z}$).

For $\Gamma = \Upsilon_{\vec{z}}$, we have

$$S(\Upsilon_{\vec{z}}, \Omega_{\vec{z}}) = \text{Tr}_{\mathcal{F}_{\geq}} \Upsilon_{\vec{z}}(\ln \Upsilon_{\vec{z}} - \ln \Gamma^0) = S(\Gamma_{\vec{z}}, \Gamma^0). \quad (2.4.3)$$
From these considerations, together with the positivity of $\mathbb{V}$, we conclude that (2.3.10) is bounded from below by

$$F_\mathcal{Z}(\beta) \geq \frac{1}{\beta} \ln \text{Tr}_{x^s} \exp \left( - \beta T_{x^s}(\mathcal{Z}) \right) + \frac{1}{2} \text{Tr}_\mathcal{F} \left[ K \mathcal{Y} \mathcal{Z} \right] + \frac{1}{\beta} S(\Gamma_{\mathcal{Z}}, \Gamma^0). \quad (2.4.4)$$

Hence we can distinguish the following two cases:

A) The following lower bound on $F_\mathcal{Z}(\beta)$ holds:

$$F_\mathcal{Z}(\beta) \geq -\frac{1}{\beta} \ln \text{Tr}_{x^s} \exp \left( - \beta T_{x^s}(\mathcal{Z}) \right) + 8\pi |\Lambda| \bar{a} q^2. \quad (2.4.5)$$

B) Inequality (2.4.5) is false, in which case

$$S(\Gamma_{\mathcal{Z}}, \Gamma^0) \leq 8\pi |\Lambda| \bar{a} q^2 \quad (2.4.6)$$

and

$$\text{Tr}_\mathcal{F} \left[ K \mathcal{Y} \mathcal{Z} \right] \leq 16\pi |\Lambda| \bar{a} q^2. \quad (2.4.7)$$

From now on, will consider case B, i.e., we will assume (2.4.6) and (2.4.7) to hold. The lower bound we will derive on $F_\mathcal{Z}(\beta)$ below will actually be worse than the bound (2.4.5) above; i.e., the bound in case B holds in any case, irrespective of whether the assumptions (2.4.6) and (2.4.7) actually hold.

Although the relative entropy does not define a metric, it measures the difference between two states in a certain sense. In particular, it dominates the trace norm [13, Thm. 1.15]:

$$S(\Gamma, \Gamma') \geq \frac{1}{2} ||\Gamma - \Gamma'||_1. \quad (2.4.8)$$

This inequality is a special case of the fact that the relative entropy decreases under completely positive trace-preserving (CPT) maps. In fact, inequality (2.4.8) can be obtained using monotonicity under the CPT map $\Gamma \mapsto \text{Tr}_\mathcal{F} [P \Gamma] \oplus \text{Tr}_\mathcal{F} [(1 - P) \Gamma]$, where $P$ is the projection onto the subspace where $\Gamma - \Gamma' \geq 0$.

Although we have the upper bound (2.4.6) on the relative entropy, inequality (2.4.8) is of no use for us since the relative entropy is of the order of the volume of the system, while the right side of (2.4.8) never exceeds 2.

To make use of (2.4.8), we must not look at the state on the full Fock space (over the whole volume) but rather on its restriction to a small subvolume. We do this in Subsection 2.8 below. Again, the monotonicity of the relative entropy will be used in an essential way.
We note that (2.4.7) implies the following simple upper bound on $|\vec{z}|^2$.

From (2.2.3) and (2.4.7),

$$|\vec{z}|^2 - N \leq \text{Tr}_F[(N - N)\Upsilon^2] \leq \left(\text{Tr}_F[(N - N)^2\Upsilon^2]\right)^{1/2} \leq \frac{2}{\sqrt{C}}|\Lambda|\varrho,$$  

and hence

$$\varrho \equiv \frac{|\vec{z}|^2}{|\Lambda|} \leq \varrho \left(1 + \frac{2}{\sqrt{C}}\right).$$  

We will choose $C \gg 1$ below.

### 2.5 Replacing Vacuum

In the following, we want to derive a lower bound on the expectation value of the interaction energy $V$ in the state $\Upsilon^\vec{z}$, i.e., on

$$\text{Tr}_F[V \Upsilon^\vec{z}] = \text{Tr}_F[V_s(\vec{z})\Gamma^\vec{z}].$$  

For reasons that will be explained later (see Subsection 2.13), we find it necessary to replace the vacuum $\Pi_0$ on $\mathcal{F}_<$ in the definition of the state $\Upsilon^\vec{z} = U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma^\vec{z}$ by a more general quasi-free state. In this subsection, we show that such a replacement can be accomplished without significant errors.

Let $\Pi$ denote a (particle-number conserving) quasi free state on $\mathcal{F}_<$. It is completely determined by its one-particle density matrix, which we choose to be given as

$$\pi = \sum_{|p| < p_c} \pi_p|p\rangle\langle p|.$$  

Here, $|p\rangle \in L^2(\Lambda)$ denotes a plane wave of momentum $p$. We denote the trace of $\pi$ by $P = \sum_{|p| < p_c} \pi_p$. Let $\Upsilon^\vec{z}_\pi$ denote the state $\Upsilon^\vec{z}_\pi \equiv U(\vec{z})\Pi_0 U(\vec{z})^\dagger \otimes \Gamma^\vec{z}$ on $\mathcal{F}$. We want to derive an upper bound on the difference

$$\text{Tr}_F[V(\Upsilon^\vec{z}_\pi - \Upsilon^\vec{z})] = \text{Tr}_F[V(U(\vec{z}) (\Pi - \Pi_0) U(\vec{z})^\dagger \otimes \Gamma^\vec{z})].$$  

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A simple calculation yields

\begin{equation}
(2.5.3) = \frac{1}{2|\Lambda|} \tilde{\gamma}(0) \left( P^2 + 2P \text{Tr}_{\mathcal{F}} \left[ N_b \langle \bar{z} \rangle \Gamma_z \right] - 2\sum_{|k| < p_c} \pi_k |z_k|^2 \right) \\
+ \frac{1}{2|\Lambda|} \sum_{|k| < p_c, |l| < p_c} \tilde{\gamma}(k - l) \left[ \pi_k \pi_l + 2|z_k|^2 \pi_l \right] \\
+ \frac{1}{|\Lambda|} \sum_{|k| < p_c, |l| \geq p_c} \tilde{\gamma}(k - l) \pi_k \text{Tr}_{\mathcal{F}} \left[ a_l^\dagger a_l \Gamma_z \right] \\
\leq \frac{8\pi \varphi}{|\Lambda|} \left( P^2 + 2P \text{Tr}_{\mathcal{F}} \left[ N \Upsilon \bar{z} \right] \right) .
\end{equation}

Here we have used again that $|\tilde{\gamma}(k)| \leq \tilde{\gamma}(0) \leq 8\pi \varphi$. It follows easily from (2.4.7) (compare with (2.4.9)) that $\text{Tr}_{\mathcal{F}} \left[ N \Upsilon \bar{z} \right]$ satisfies $\leq N(1 + 2/\sqrt{C})$. Hence we obtain from (2.5.4) that

\begin{equation}
\text{Tr}_{\mathcal{F}} \left[ N \Upsilon \bar{z} \right] \geq \text{Tr}_{\mathcal{F}} \left[ N \Upsilon \bar{z} \right] - Z^{(2)},
\end{equation}

with

\begin{equation}
Z^{(2)} = \frac{8\pi \varphi P^2}{|\Lambda|} + \frac{16P\pi \varphi}{|\Lambda|} N \left( 1 + \frac{2}{\sqrt{C}} \right) .
\end{equation}

Recall that $C \gg 1$ and $\varphi \gg a$. Hence $Z^{(2)} \ll |\Lambda|a g^2$ as long as $\varphi P \ll aN$.

Note that the effect of the replacement of $\Upsilon \bar{z}$ by $\Upsilon \bar{z}_\pi$ on the kinetic energy is

\begin{equation}
\text{Tr}_{\mathcal{F}} \left[ T \Upsilon \bar{z} \right] = \text{Tr}_{\mathcal{F}} \left[ T \Upsilon \bar{z}_\pi \right] - \sum_{|p| < p_c} (p^2 - \mu_0) \pi_p .
\end{equation}

We have thus obtained the lower bound

\begin{equation}
F_{\bar{z}}(\beta) \geq \text{Tr}_{\mathcal{F}} \left[ (T + V) \Upsilon \bar{z}_\pi \right] - \frac{1}{\beta} S(\Upsilon \bar{z}) \\
- \sum_{|p| < p_c} (p^2 - \mu_0) \pi_p + \frac{1}{2} \text{Tr}_{\mathcal{F}} \left[ K \Upsilon \bar{z} \right] - Z^{(2)} .
\end{equation}

2.6 Dyson Lemma

Since the interaction potentials in $V$ are very short range and strong (compared with the average kinetic energy per particle), we cannot directly obtain information on the expectation value of $V$ in the state $\Upsilon \bar{z}_\pi$. In fact, we cannot even expect that it yields the desired correction to the free energy, since part of the interaction energy leading to the second term in (1.8) is actually kinetic energy! Hence we will first derive a lower bound on $V$ in terms of
“softer” and longer ranged potentials, with the aid of part of the kinetic energy. More precisely, we will use only the high momentum part of the kinetic energy for this task, since this is the relevant part contributing to the interaction energy. The appropriate lemma to achieve this was derived in [7]; part of the idea for such an estimate is already contained in the paper by Dyson [2]. For this reason, we refer to this estimate as “Dyson Lemma”.

Our goal is to derive an appropriate lower bound on the Hamiltonian $T + V$. Let $\chi : \mathbb{R}^3 \mapsto \mathbb{R}$ be a radial function, $0 \leq \chi(p) \leq 1$, and let

$$h(x) = \frac{1}{|\Lambda|} \sum_{p} (1 - \chi(p)) e^{-ipx}.$$ \hspace{1cm} (2.6.1)

We assume that $\chi(p) \to 1$ as $|p| \to \infty$ sufficiently fast such that $h \in L^1(\Lambda) \cap L^\infty(\Lambda)$. For some $L/2 > R > R_0$, let

$$f_R(x) = \sup_{|y| \leq R} |h(x - y) - h(x)|,$$ \hspace{1cm} (2.6.2)

and

$$w_R(x) = \frac{2}{\pi^2} f_R(x) \int_\Lambda dy f_R(y).$$ \hspace{1cm} (2.6.3)

Note that $w_R$ is a periodic function on $\mathbb{R}^3$, with period $L$.

Let $U_R : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be a non-negative function that is supported in the interval $[R_0, R]$, and satisfies $\int_0^\infty dt t^2 U_R(t) \leq 1$. The following is a simple extension of Lemma 4 (and Corollary 1) in [7]. The proof follows closely the one in [7]. For completeness, we present it in the appendix.

**Lemma 2.** Let $y_1, \ldots, y_n$ denote $n$ points in $\Lambda$ and, for $x \in \Lambda$, let $y_{NN}(x)$ denote the nearest neighbor of $x$ among the points $y_j$, i.e., the $y_k$ minimizing $d(x, y_j)$ among all $y_j$. We then have, for any $\varepsilon > 0$,

$$-\nabla \chi(p)^2 \nabla + \frac{1}{2} \sum_{i=1}^{n} \bar{U}(d(x, y_i)) \geq (1 - \varepsilon)\bar{a}U_R(d(x, y_{NN}(x))) - \sum_{i=1}^{n} \frac{\varepsilon}{2} w_R(x - y_i).$$ \hspace{1cm} (2.6.4)

Here, the operator $-\nabla \chi(p)^2 \nabla$ stands for $\sum_p p^2 \chi(p)^2 |p\rangle \langle p|.$

We note that $y_{NN}(x)$ is well defined except on a set of measure zero. Compared with Lemma 4 in [7], the main differences are the boundary conditions used, and the fact that we do not demand a minimal distance between the points $y_i$. In [7], it was assumed that $d(y_i, y_j) \geq 2R$ for $i \neq j$, in which case $U_R(d(x, y_{NN}(x))) = \sum_i U_R(d(x, y_i))$. Note also that only the
inequality $\int dt t^2 U(t) \leq 1$ is needed for the estimate, not equality, as stated in [7].

We will use Lemma 2 for a lower bound to the operator $T + \mathbb{V}$ on $\mathcal{F}$. Note that the restriction of this operator to the sector of $n$ particles is just $\tilde{H}_n$, defined in (2.1.8). We write

$$\tilde{H}_n = \sum_{j=1}^{n} \left[ -\Delta_j + \frac{1}{2} \sum_{i \neq j} \tilde{v}(d(x_j, x_i)) \right], \quad (2.6.5)$$

and apply the estimate (2.6.4) to each term in square brackets, for fixed $j$ and fixed positions of the $x_i, i \neq j$. We want to keep a part of the kinetic energy for later use, however. To this end, we pick some $0 < \zeta < 1$, and write

$$p^2 = p^2(1 - (1 - \zeta)\chi(p)^2) + (1 - \zeta)p^2\chi(p)^2. \quad (2.6.6)$$

We split the kinetic energy in the Hamiltonian (2.6.5) accordingly, and apply (2.6.4) to the last part. Using also the positivity of the $\tilde{v}$, we thus obtain, for any subset $J_j \subseteq \{1, \ldots, j - 1, j + 1, \ldots, n\}$,

$$-\Delta_j + \frac{1}{2} \sum_{i \neq j} \tilde{v}(d(x_j, x_i))$$

$$\geq -\nabla_j(1 - (1 - \zeta)\chi(p_j)^2)\nabla_j$$

$$+ (1 - \varepsilon)(1 - \zeta)\tilde{a}U_R(d(x_j, x_{J_j}^\text{NN}(x_j))) - \frac{\tilde{a}}{\varepsilon} \sum_{i \in J_j} w_R(x_j - x_i). \quad (2.6.7)$$

Here we denoted by $x_{J_j}^\text{NN}(x_j)$ the nearest neighbor of $x_j$ among the points $x_i, i \in J_j$.

Our choice of $J_j$ will depend on the positions $x_i, i \neq j$. We want to choose it in such a way that $d(x_l, x_k) \geq R/5$ if $l \in J_j$ and $k \in J_j$. Moreover, we want the set to be maximal, in the sense that if $l \notin J_j$, then there exists a $k \in J_j$ such that $d(x_l, x_k) < R/5$. These properties of $J_j$ will be used in an essential way in Subsections 2.9 and 2.10 below.

There is no unique choice of $J_j$ satisfying these criteria. One way to construct it is the following. We first pick all $i$ corresponding to those $x_i$ whose distance to the nearest neighbor (among all the other $x_k, k \neq i, j$) is greater or equal to $R/5$. Secondly, going through the list $\{x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n\}$ one by one, we add $i$ to the list if $d(x_i, x_j) \geq R/5$ for all $j$ already in the list. This last procedure depends on the ordering of the $x_i$, and hence the resulting $J_j$ will depend on this ordering. The resulting interaction potential in (2.6.7) will thus be not symmetric in the particle coordinates. This is
of no importance, however, since we will take the expectation value of the resulting operator only in symmetric (bosonic) states anyway.

The set $J_j$ is chosen in order to satisfy the following requirements. On the one hand, we want the particles keep a certain minimal distance, $R/5$; this is necessary in order to control the error terms coming from the potentials $w_R$. We do not have sufficient control on the two-particle density to control these terms if all the particle configurations were taken into account. On the other hand, we want the balls of radius $R$ centered at the particle coordinates to be able to overlap sufficiently much, such that the desired lower bound can be obtained. We note that we want to derive a lower bound which is independent of $\vec{z}$; for certain values of $\vec{z}$, however, the system may be far from being homogeneous and particles may cluster in a relatively small volume. We want to ensure that there is still sufficient interaction among them.

### 2.7 Filling the Holes

One defect of Lemma 2 above is that the resulting interaction potential $U_R$ is supported outside a ball of radius $R_0$, which is the range of $\tilde{v}$. For our estimates in Subsection 2.9, it will be convenient to have a specific $U_R$ which is, in particular, positive definite and hence should not have a “hole” at the origin. We will show in this subsection that one can easily add the missing part to $U_R$, at the expense of only a small amount of kinetic energy.

We start with the description of our choice of $U_R$. Let $j : \mathbb{R}_+ \to \mathbb{R}_+$ denote the “hat function”

$$j(t) = \frac{144}{\pi} \int_{\mathbb{R}^3} dy \theta(\frac{1}{2} - |y|)\theta(\frac{1}{2} - |y - e t|) \quad (2.7.1)$$

for some unit vector $e \in \mathbb{R}^3$. Note that $j$ is supported in the interval $[0, 1]$, and $\int_0^1 dt t^2 j(t) = 1$. An explicit computation yields

$$j(t) = 12(t + 2)[1 - t]^2_+ . \quad (2.7.2)$$

Our desired interaction potential will be $\tilde{U}_R(t) = R^{-3} j(t/R)$. We will thus choose $U_R(t) = \tilde{U}_R(t) \theta(t - R_0)$ in (2.6.4).

In the following, it will be convenient to work with $\tilde{U}_R$ instead of $U_R$. I.e., we would like to add the missing part $\tilde{U}_R(\cdot) \theta(R_0 - \cdot)$ to the interaction. In order to achieve this, we use the following lemma. It is an easy consequence of the definition of the scattering length, given in (1.7).
Lemma 3. Let \( y_1, \ldots, y_n \) denote \( n \) points in \( \Lambda \), with \( d(y_i, y_j) \geq R/5 \) for \( i \neq j \). Let \( 0 \leq \lambda < \pi/2 \), and \( R_0 < R/10 \). Then

\[
- \Delta - \frac{\lambda^2}{R_0^2} \sum_{i=1}^{n} \theta(R_0 - d(x, y_i)) \geq - \frac{3R_0}{(R/10)^3 - R_0^3} \left( \frac{\tan \lambda}{\lambda} - 1 \right) \sum_{i=1}^{n} \theta(R/10 - d(x, y_i)).
\]  

(2.7.3)

Proof. It suffices to prove that

\[
\int_{|x| \leq R/10} |\nabla \phi(x)|^2 - \frac{\lambda^2}{R_0^2} \int_{|x| \leq R_0} |\phi(x)|^2 \\
\geq - \frac{3R_0}{(R/10)^3 - R_0^3} \left( \frac{\tan \lambda}{\lambda} - 1 \right) \int_{|x| \leq R/10} |\phi(x)|^2
\]

(2.7.4)

for any function \( \phi \in H^1 \). In fact, it is enough to prove (2.7.4) for radial functions. Note that the scattering length of the potential \( 2\lambda^2 R_0^{-2} \theta(R_0 \cdot \cdot) \) is given by \( R_0 (1 - \lambda^{-1} \tan \lambda) \). Hence, for any \( R_0 \leq s \leq R/10 \),

\[
\int_{|x| \leq s} |\nabla \phi(|x|)|^2 - \frac{\lambda^2}{R_0^2} \int_{|x| \leq R_0} |\phi(|x|)|^2 \\
\geq -4\pi R_0 \left( \frac{\tan \lambda}{\lambda} - 1 \right) |\phi(s)|^2.
\]

(2.7.5)

Eq. (2.7.4) follows by multiplying this inequality by \( s^2 \) and integrating \( s \) between \( R_0 \) and \( R/10 \). \( \blacksquare \)

Let \( \lambda = \pi/4 \) for concreteness. Recall that \( d(x_i, x_k) \geq R/5 \) for \( i, k \in J_j \). Since \( \tilde{U}_R(t) \leq j(0)/R^3 = 24/R^4 \), and \( \tilde{U}_R(t) \geq j(1/10)/R^3 \) for \( t \leq R/10 \), this lemma implies, in particular, that

\[
(\tilde{U}_R - U_R)(d(x_j, x_{JN}(x_j))) \leq \frac{24}{\pi^2} \frac{(4R_0)^2}{R^3} \Delta_j
\]

(2.7.6)

\[
+ \frac{18}{(\pi/4)^3 (4 - \pi)} \frac{R_0^3}{(R/10)^3 - R_0^3} \frac{1}{j(1/10)} \tilde{U}_R(d(x_j, x_{JN}(x_j))).
\]

Define

\[
a' \equiv a(1 - \varepsilon)(1 - \kappa) \left( 1 - \frac{18}{(\pi/4)^3 (4 - \pi)} \frac{R_0^3}{(R/10)^3 - R_0^3} \frac{1}{j(1/10)} \right)
\]

(2.7.7)

and

\[
\lambda' \equiv \kappa - \frac{24a}{\pi^2} \frac{(4R_0)^2}{R^3}.
\]

(2.7.8)
In the following, we will choose \( \kappa \gg a R_0^2 / R^3 \) and hence, in particular, \( \kappa' > 0 \). Combining the estimates (2.6.7) and (2.7.6) and applying them in each sector of particle number \( n \), we obtain the inequality

\[
T + V \geq T^c + W,
\]

where

\[
T^c = \sum_p \varepsilon(p) a_p^\dagger a_p, \quad \varepsilon(p) = \kappa' p^2 + (1 - \kappa) p^2 (1 - \chi(p)^2) - \mu_0,
\]

and \( W \) is, in each sector with particle number \( n \), given by the (symmetrization of the) multiplication operator

\[
\sum_{j=1}^n \left[ d\tilde{U}_R(d(x_j, x_{jN}(x_j))) - \tilde{a} \sum_{i \in J_j} w_R(x_j - x_i) \right].
\]

Note again that the set \( J_j \) depends on all the particle coordinates \( x_i, i \neq j \).

We now describe our choice of the kinetic energy cutoff \( \chi \). Let \( \nu : \mathbb{R}^3 \to \mathbb{R}_+ \) be a smooth radial function with \( \nu(p) = 0 \) for \( |p| \leq 1 \), \( \nu(p) = 1 \) for \( |p| \geq 2 \), and \( 0 \leq \nu(p) \leq 1 \) in-between. For some \( s \geq R \) we choose

\[
\chi(p) = \nu(sp).
\]

We will choose \( p_c \leq 1/s \) below. This implies, in particular, that \( \varepsilon(p) \), defined in (2.7.10) above, is equal to \( (1 - \kappa + \kappa') p^2 - \mu_0 \) for \( |p| \leq p_c \). Hence (compare with (2.5.7))

\[
\text{Tr}_F [T^c \Upsilon \vec{z}] = \text{Tr}_F [T^c \Upsilon \vec{z}] + \sum_{|p|<p_c} ((1 - \kappa + \kappa') p^2 - \mu_0) \pi_p.
\]

Using the fact that

\[
\text{Tr}_F [T^c \Upsilon \vec{z}] - \frac{1}{\beta} S(\vec{T} \vec{z}) \geq - \frac{1}{\beta} \ln \text{Tr}_{F\geq} \exp \left( - \beta T^c_\pi(\vec{z}) \right),
\]

we conclude from (2.5.8), (2.7.9) and (2.7.13) that

\[
F_\pi(\beta) \geq - \frac{1}{\beta} \ln \text{Tr}_{F\geq} \exp \left( - \beta T^c_\pi(\vec{z}) \right) + \text{Tr}_F [W \Upsilon \vec{z}] + \frac{1}{2} \text{Tr}_F [K \Upsilon \vec{z}]
\]

\[
- (\kappa - \kappa') \sum_{|p|<p_c} p^2 \pi_p - Z^{(2)}.
\]
Note that the first term on the right side of (2.7.15) can be computed explicitly. It is given by
\[
-\frac{1}{\beta} \ln \text{Tr}_{\Omega} \exp \left( -\beta T_s(z) \right) = \sum_{|p| < p_c} \left( (1 - x + x') p^2 - \mu_0 \right) |z_p|^2 \\
+ \frac{1}{\beta} \sum_{|p| \geq p_c} \ln \left( 1 - \exp \left( -\beta \varepsilon(p) \right) \right).
\]  (2.7.16)

In the following, we will derive a lower bound on the expectation value of \( W \) in the state \( \Upsilon_{\pi} \).

2.8 Localization of Relative Entropy

Our next task is to give a lower bound on \( \text{Tr}_F \left[ W \Upsilon_{\pi} \right] \). For that matter, we will show that we can replace the unknown state \( \Gamma_{\pi} \) in \( \Upsilon_{\pi} = U(\pi) \Pi U(\pi)^\dagger \otimes \Gamma \) by the quasi-free state \( \Gamma^0 \), which is the Gibbs state for the kinetic energy \( T_s(z) \). The error in doing so will be controlled by the upper bound on the relative entropy, Eq. (2.4.6). In order to do this, we have to obtain a “local” version of this bound.

Consider the quasi-free state \( \Omega_{\pi} = \Pi \otimes \Gamma^0 \). Its one particle density matrix is given by
\[
\omega_{\pi} = \sum_p \frac{1}{e^{\ell(p)} - 1} |p\rangle \langle p|
\]  (2.8.1)

with \( \ell(p) = \beta(p^2 - \mu_0) \) for \( |p| \geq p_c \), and \( \ell(p) = \ln(1 + 1/\pi p) \) for \( |p| < p_c \).

Let \( \eta: \mathbb{R}^3 \rightarrow \mathbb{R} \) be a function with the following properties:

- \( \eta \in C^\infty(\mathbb{R}^3) \)
- \( \eta(0) = 1 \), and \( \eta(x) = 0 \) for \( |x| \geq 1 \)
- \( \hat{\eta}(p) = \int dx \eta(x)e^{-ipx} \geq 0 \) for all \( p \in \mathbb{R}^3 \).

An appropriate \( \eta \) can, for instance, we obtained by convolving a smooth function supported in a ball of radius \( \frac{1}{2} \) with itself. Given such a function \( \eta \), we define \( \eta_b(x) = \eta(x/b) \) for some \( b \leq L/2 \). Moreover, with a slight abuse of notation, we define a one-particle density matrix \( \omega_b \) on \( \mathcal{H} \) by the kernel
\[
\omega_b(x, y) = \omega_{\pi}(x, y) \eta_0(d(x, y)) \). \]  (2.8.2)

Note that this defines a positive operator, with plane waves as eigenstates. Note also that \( |\omega_b(x, y)| \leq |\omega_{\pi}(x, y)| \) since \( |\eta_b| \leq 1 \). We denote by \( \Omega_b \)
the corresponding (particle number conserving) quasi-free state on \( F \), and \( \Omega^\sharp_b = U(\vec{z})\Omega_b U(\vec{z}) \). Let also denote \( \rho = \omega_b(x, x) = \omega(x, x) \) the one-particle density of \( \Omega_b \) (which is independent of \( x \)). Abusing the notation even more, we shall sometimes write \( \omega_b(x, y) = \omega_b(x - y) \) if no confusion can arise.

For \( r < L/2 \), let \( \chi_{r, \xi}(\cdot) = \theta(r - d(\cdot, \xi)) \) denote the characteristic function of all ball of radius \( r \) centered at \( \xi \in \Lambda \). The function \( \chi_{r, \xi} \) defines a projection on the one-particle space \( \mathcal{H} = L^2(\Lambda) \), and hence the Fock space \( F \) over \( \mathcal{H} \) can be thought of as a tensor product of a Fock space over \( \chi_{r, \xi} \mathcal{H} \) and a Fock space over the complement. States on \( F \) can thus be restricted to the Fock space over \( \chi_{r, \xi} \mathcal{H} \), simply by taking the partial trace over the other factor. We denote such a restriction of a state \( \Gamma \) by \( \Gamma_{\chi_{r, \xi}} \).

For \( d(\xi, \zeta) \geq 2r \), \( \chi_{r, \xi} + \chi_{r, \zeta} \) defines a projection on \( \mathcal{H} \). Note that since \( \omega_b(x, y) \) vanishes if \( d(x, y) \geq b \), we have that \( \Omega_{b, \chi_{r, \xi} + \chi_{r, \zeta}} = \Omega_{b, \chi_{r, \xi}} \otimes \Omega_{b, \chi_{r, \zeta}} \) (2.8.3) if \( d(\xi, \zeta) \geq 2r + b \). This follows simply from the fact that the one-particle density matrix of \( \Omega_{b, \chi_{r, \xi} + \chi_{r, \zeta}} \) is given by \( (\chi_{r, \xi} + \chi_{r, \zeta})\omega_b(\chi_{r, \xi} + \chi_{r, \zeta}) = \chi_{r, \xi}\omega_b\chi_{r, \xi} + \chi_{r, \zeta}\omega_b\chi_{r, \zeta} \). The same factorization property (2.8.3) is obviously true with \( \Omega_b \) replaced by \( \Omega^\sharp_b = U(\vec{z})\Omega_b U(\vec{z}) \) since the unitary \( U(\vec{z}) \) has the same product structure. As in [17, Sect. 5.1], we have the following superadditivity property of the relative entropy.

**Lemma 4.** Let \( X_i \), \( 0 \leq i \leq k \), denote \( k \) mutually orthogonal projections on \( \mathcal{H} \). Let \( \Omega \) be a state on \( F \) which factorizes under restrictions as \( \Omega_{\sum_i X_i} = \otimes_i \Omega_{X_i} \). Then, for any state \( \Gamma \),

\[
S(\Gamma, \Omega) \geq \sum_i S(\Gamma_{X_i}, \Omega_{X_i}).
\] (2.8.4)

We note that the lemma applies, in particular, to a (particle number conserving) quasi-free state \( \Omega \) whose one-particle density matrix \( \omega \) satisfies \( X_i \omega X_j = 0 \) for \( i \neq j \). We emphasize that the factorization property of \( \Omega \) is crucial; in general, the relative entropy need not be superadditive. This is the reason for introducing the cutoff \( b \).

**Proof.** Let \( X \) denote the projection \( X = \sum_i X_i \). The relative entropy decreases under restrictions [6, 13], i.e.,

\[
S(\Gamma, \Omega) \geq S(\Gamma_X, \Omega_X) = S(\Gamma_X, \otimes_i \Omega_{X_i})
\]

\[
= \sum_i S(\Gamma_{X_i}, \Omega_{X_i}) + \sum_i S(\Gamma_{X_i}) - S(\Gamma_X).
\] (2.8.5)
The last two terms together are positive because of subadditivity of the von-Neumann entropy.

We take the $X_i$ to be the multiplication operators by characteristic functions of balls of radius $r$, separated a distance $2b$. By averaging over the position of the balls, Lemma 4 implies that, for any $b \geq 2r$ such that $L/(2b)$ is a positive integer, and for any state $\Gamma$,

$$S(\Gamma, \Omega_b^\xi) \geq \frac{1}{(2b)^3} \int_\Lambda d\xi S(\Gamma_{\chi_r,\xi}, \Omega_b^\xi_{\chi_r,\xi}). \quad (2.8.6)$$

We apply this inequality to the state $\Gamma = \Upsilon^\pi_\zeta = U(\zeta)\Pi U(\zeta)\dagger \otimes \Gamma_\zeta$.

We remark that that restriction of $L/(2b)$ being an integer will be of no concern to us, since we are interested in the thermodynamic limit $L \to \infty$, with $b$ independent of $L$.

We can now apply inequality (2.4.8) to the right side of (2.8.6). Using the Schwarz inequality for the integration over $\xi$, we thus obtain

$$\int_\Lambda d\xi \|\Upsilon^\pi_\chi_{\chi_r,\xi} - \Omega_b^\xi_{\chi_r,\xi}\|_1 \leq 4 \left(b^3 |\Lambda| S(\Upsilon^\pi_\chi, \Omega_b^\xi)\right)^{1/2} \quad (2.8.7)$$

for any $r \leq b/2$. Note that $S(\Upsilon^\pi_\chi, \Omega_b^\xi) = S(\Upsilon^\pi_\chi, \Omega_b^\pi)$ since the relative entropy is invariant under unitary transformations. Were it not for the cutoff $b$, we could use (2.4.6) to bound the right side of (2.8.7). We will estimate the effect of the cutoff in Subsection 2.13.

2.9 Interaction Energy, Part 1

The next step is to derive a lower bound on $\text{Tr}_F[\Upsilon^\pi_\zeta]$. The main input will be the bound (2.8.7) derived in the previous subsection. We split the estimate into three parts. First, we give a lower bound on the expectation value of the terms containing $\bar{U}_R$ in (2.7.11). In the next subsection, we bound the remaining energy containing the terms $w_R$. Finally, we combine the two estimates in Subsection 2.11. One of the difficulties in our estimates results from the fact that $\zeta$ is rather arbitrary, and hence the system can be far from being homogeneous.

From (2.7.11), we can write

$$\Upsilon = \Upsilon_1 - \Upsilon_2, \quad (2.9.1)$$

where

$$\Upsilon_1 = \bigoplus_{n=0}^\infty \sum_{j=1}^n a' \bar{U}_R(d(x_j, x'_{NN}(x_j))) \quad (2.9.2)$$

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and
\[ W_2 = \bigoplus_{n=0}^{\infty} \sum_{j=1}^{n} \sum_{\tilde{a}_j} w_R(x_j - x_i). \] (2.9.3)

We start by giving a lower bound on the expectation value of \( W_1 \) in the state \( \Upsilon_{\vec{z}} \). Recall that \( \Upsilon_{\vec{z}} \) is defined after Eq. (2.5.2) as \( \Upsilon_{\vec{z}} = U(\vec{z}) \Pi U(\vec{z})^\dagger \otimes \Gamma_{\vec{z}} \).

According to the decomposition (2.7.1), we can write \( \tilde{U}_R \) as
\[ \tilde{U}_R(d(x,y)) = \frac{144}{\pi R^6} \int_\Lambda d\xi \theta(R/2 - d(\xi,x)) \theta(R/2 - d(\xi,y)). \] (2.9.4)

This gives rise to a corresponding decomposition of \( W_1 \), which we write as
\[ W_1 = \frac{144d'}{\pi R^6} \int_\Lambda d\xi w(\xi). \] (2.9.5)

For \( r > 0 \), let \( n_{r,\xi} \) denote the operator that counts the number of particles inside a ball of radius \( r \) centered at \( \xi \in \Lambda \). It is the second quantization of the multiplication operator \( \chi_{r,\xi}(\cdot) = \theta(r - d(\xi,\cdot)) \) on \( L^2(\Lambda) \).

We claim that
\[ w(\xi) \geq n_{R/10,\xi} \theta(n_{R/10,\xi} - 2). \] (2.9.6)

This is just the second quantized version of the inequality
\[ \theta(R/2 - d(\xi,x_j)) \theta(R/2 - d(\xi,x_{j_{NN}(x_j)})) \]
\[ \geq \theta(R/10 - d(\xi,x_j)) \left( 1 - \prod_{i \neq j} \theta(d(\xi,x_i) - R/10) \right). \] (2.9.7)

To prove (2.9.7), we have to show that whenever \( x_j \) and some \( x_k, k \neq j \), are in a ball of radius \( R/10 \) centered at \( \xi \), then \( x_{j_{NN}(x_j)} \) is in a ball of radius \( R/2 \) (with the same center). Assume first that \( k \in J_j \). Then \( d(x_j, x_{j_{NN}(x_j)}) \leq d(x_j, x_k) \leq R/5 \), whence \( d(\xi, x_{j_{NN}(x_j)}) \leq 3R/10 \). If, on the other hand, \( k \notin J_j \), then there exists an \( l \in J_j \) such that \( d(x_l, x_k) < R/5 \). Hence \( d(x_j, x_{j_{NN}(x_j)}) \leq d(x_j, x_l) < 2R/5 \), and therefore \( d(\xi, x_{j_{NN}(x_j)}) < R/2 \). This proves (2.9.7).

Hence, in particular, we have that
\[ w(\xi) \geq \overline{w}(\xi) \equiv w(\xi) \theta(2 - n_{3R/2,\xi}) + n_{R/10,\xi} \theta(n_{R/10,\xi} - 2) \theta(n_{3R/2,\xi} - 3). \] (2.9.8)

We now claim that
\[ w(\xi) \theta(2 - n_{3R/2,\xi}) = n_{R/2,\xi} (n_{R/2,\xi} - 1) \theta(2 - n_{3R/2,\xi}). \] (2.9.9)
This implies, in particular, that the operator $\overline{w}(\xi)$ depends only on the Fock space restricted to a ball of radius $3R/2$ centered at $\xi$. Eq. (2.9.9) follows from the fact that if two particles with coordinates $x_i$ and $x_j$ are within a ball of radius $R/2$, and no other particle is in the bigger ball of radius $3R/2$, then the two particles must be nearest neighbors. Moreover, $j \in J_i$ and $i \in J_j$ by construction.

Note that (2.9.9) is a bounded operator, bounded by 2. Moreover, since $n_{R/10,\xi} \leq n_{3R/2,\xi}$, we also see that

$$|\overline{w}(\xi) - n_{R/10,\xi}| \leq 2. \quad (2.9.10)$$

Using (2.9.5), (2.9.8) and (2.9.10), we can estimate

$$\text{Tr}_{\mathcal{F}}[\mathcal{W}_1 \mathcal{T}_{\pi}^\circ] \geq \frac{144a'}{\pi R^6} \int_\Lambda d\xi \text{Tr}_{\mathcal{F}}[\overline{w}(\xi) \mathcal{T}_{\pi}^\circ]$$

$$\geq \frac{144a'}{\pi R^6} \int_\Lambda d\xi \text{Tr}_{\mathcal{F}}[\overline{w}(\xi) \mathcal{O}_{b} + n_{R/10,\xi}(\mathcal{T}_{\pi}^\circ - \mathcal{O}_{b})]$$

$$- 2 \frac{144a'}{\pi R^6} \int_\Lambda d\xi \|\mathcal{T}_{\pi,\chi_{3R/2,\xi}} - \mathcal{O}_{b,\chi_{3R/2,\xi}}\|_1. \quad (2.9.11)$$

Here we have also used that $\overline{w}(\xi)$ acts non-trivially only on the Fock space over $\chi_{3R/2,\xi} \mathcal{H}$. Note that the integral over the second term on the right side of (2.9.11) is equal to

$$\int_\Lambda d\xi \text{Tr}_{\mathcal{F}}[n_{R/10,\xi}(\mathcal{T}_{\pi}^\circ - \mathcal{O}_{b})] = \frac{4\pi}{3} \left(\frac{R}{10}\right)^3 \text{Tr}_{\mathcal{F}}[\mathcal{N}(\mathcal{T}_{\pi}^\circ - \mathcal{O}_{b})]. \quad (2.9.12)$$

Moreover, for the last term in (2.9.11), we can use (2.8.7) to estimate

$$\int_\Lambda d\xi \|\mathcal{T}_{\pi,\chi_{3R/2,\xi}} - \mathcal{O}_{b,\chi_{3R/2,\xi}}\|_1 \leq 4 \left(b^3 |\Lambda| S(\mathcal{T}_{\pi}^\circ, \mathcal{O}_{b})\right)^{1/2} \quad (2.9.13)$$
as long as $3R \leq b$.

We proceed with a lower bound on $\text{Tr}_{\mathcal{F}}[\overline{w}(\xi) \mathcal{O}_{b}^\circ]$. In fact, we will derive two different lower bounds on this expression. First, neglecting the last term in (2.9.8) and using (2.9.9),

$$\text{Tr}_{\mathcal{F}}[\overline{w}(\xi) \mathcal{O}_{b}^\circ] \geq \text{Tr}_{\mathcal{F}}[n_{R/2,\xi}(n_{R/2,\xi} - 1) \mathcal{O}_{b}^\circ]$$

$$- \text{Tr}_{\mathcal{F}}[n_{3R/2,\xi}(n_{3R/2,\xi} - 1)(n_{3R/2,\xi} - 2) \mathcal{O}_{b}^\circ]. \quad (2.9.14)$$
Since $\Omega_{b, z}^F$ is a combination of a coherent and quasi-free state, the last expression in (2.9.14) is, in fact, easy to estimate. Let $\Phi_{z}$ denote the one-particle state $|\Phi_{z}\rangle = \sum_{|p| < p_{c}} z_{p}|p\rangle$. We have

\[
\text{Tr}_F\left[n_{3R/2, \xi}(n_{3R/2, \xi} - 1)(n_{3R/2, \xi} - 2)\Omega_{b, z}^F\right] \\
= \left(\text{Tr}_F\left[n_{3R/2, \xi}\Omega_{b, z}^F\right]\right)^3 + 2 \text{tr} (\chi_{3R/2, \xi} \omega_b)^3 + 6 \langle \Phi_{z}|(\chi_{3R/2, \xi} \omega_b \chi_{3R/2, \xi})^2|\Phi_{z}\rangle \\
+ 3 \left(\text{Tr}_F\left[n_{3R/2, \xi}\Omega_{b, z}^F\right]\right) \left(2\langle \Phi_{z}|\chi_{3R/2, \xi} \omega_b \chi_{3R/2, \xi}|\Phi_{z}\rangle + \text{tr} (\chi_{3R/2, \xi} \omega_b)^2\right) \\
\leq 6 \left(\text{Tr}_F\left[n_{3R/2, \xi}\Omega_{b, z}^F\right]\right)^3. 
\] (2.9.15)

(Here, we use the symbol tr to denote the trace over the one-particle space $L^2(\Lambda)$, while Tr is reserved for the trace over the Fock space.)

A different lower bound can be obtained using

\[
\text{Tr}_F\left[w(\xi)|\Omega_{b, z}^F\rangle\right] \geq \text{Tr}_F\left[n_{R/10, \xi}\theta(n_{R/10, \xi} - 2)\Omega_{b, z}^F\right]. 
\] (2.9.16)

Eq. (2.9.16) follows easily from (2.9.8) and (2.9.9). The latter trace is non-trivial only over the Fock space over $\chi_{R/10, \xi} \mathcal{H}$. Denoting by $\Pi^F_0$ the vacuum on $\mathcal{F}$, we claim that

\[
\Omega_{b, \chi_{R/10, \xi}} \geq e^{-4\pi (R/10)^3 \theta / 3} \Pi^F_0, 
\] (2.9.17)

which implies that

\[
\Omega_{b, \chi_{R/10, \xi}}^F \geq e^{-4\pi (R/10)^3 \theta / 3} \left(U(\bar{z})^\dagger \Pi^F_0 U(\bar{z})\right)_{\chi_{R/10, \xi}}. 
\] (2.9.18)

Eq. (2.9.17) follows from the fact that $\Omega_{b, \chi_{R/10, \xi}}$ is a particle-number conserving quasi-free state, whose vacuum part is given by

\[
\exp \left(-\text{tr} \ln(1 + \chi_{R/10, \xi} \omega_b \chi_{R/10, \xi})\right) \geq \exp \left(-\text{tr} \chi_{R/10, \xi} \omega_b \chi_{R/10, \xi}\right) \\
= \exp \left(-4\pi (R/10)^3 \theta / 3\right). 
\] (2.9.19)

Eq. (2.9.18) implies, in particular, that

\[
(2.9.16) \geq e^{-4\pi (R/10)^3 \theta / 3} \text{Tr}_F\left[n_{R/10, \xi}\theta(n_{R/10, \xi} - 2)U(\bar{z})^\dagger \Pi^F_0 U(\bar{z})\right]. 
\] (2.9.20)

The state $U(\bar{z})^\dagger \Pi^F_0 U(\bar{z})$ is a coherent state on $\mathcal{F}$. Its restriction to the Fock space over $\chi_{R/10, \xi} \mathcal{H}$ is again a coherent state. In every sector of particle
number \( n \), it is given by the projection onto the \( n \)-fold tensor product of the wave function \( \chi_{R/10,\xi} \), appropriately normalized. Therefore

\[
\text{Tr}_F \left[ n_{R/10,\xi} \theta(n_{R/10,\xi} - 2) U(z)^\dagger \Pi_0 U(z) \right] = e^{-\langle \Phi_z | \chi_{R/10,\xi} \rangle} \sum_{n \geq 2} \frac{n^2}{n!} \langle \Phi_z | \chi_{R/10,\xi} \rangle^n (2.9.21)
\]

\[
= \langle \Phi_z | \chi_{R/10,\xi} \rangle \left( 1 - e^{-\langle \Phi_z | \chi_{R/10,\xi} \rangle} \right) \geq \langle \Phi_z | \chi_{R/10,\xi} \rangle^2 / (1 + \langle \Phi_z | \chi_{R/10,\xi} \rangle).
\]

The last inequality follows from the elementary estimate \( x(1-e^{-x}) \geq x^2/(1+x) \) for \( x \geq 0 \).

Summarizing the results of this subsection, we have shown that, for any \( 0 \leq \lambda \leq 1 \),

\[
\text{Tr}_F \left[ \mathbb{W}_1 \mathbb{Y}_z^2 \right] \geq \frac{24}{125} \frac{a'}{R^3} \text{Tr}_F \left[ \mathbb{N} \left( \mathbb{Y}_z - \Omega_\pi^z \right) \right] - 144 \frac{a'}{\pi R^6} \left( t^3 |\Lambda| S(\mathbb{Y}_\pi, \Omega_\pi^z) \right)^{1/2} + \lambda \frac{144}{\pi} \frac{a'}{R^6} \int_\Lambda d\xi \left[ \text{Tr}_F \left[ n_{R/2,\xi} (n_{R/2,\xi} - 1) \Omega_\pi^z \right] - 6 \left( \text{Tr}_F \left[ n_{3R/2,\xi} \Omega_\pi^z \right] \right)^3 \right] + (1 - \lambda) \frac{144}{\pi} \frac{a'}{R^6} e^{-4\pi(R/10)^3 \omega/3} \int_\Lambda d\xi \frac{(\Phi_z | \chi_{R/10,\xi} \rangle \Phi_z)^2}{1 + (\Phi_z | \chi_{R/10,\xi} \rangle \Phi_z)}. \quad (2.9.22)
\]

The choice of \( \lambda \) will depend on the function \( |\Phi_z| \). If it is approximately a constant (in a sense to be made precise in Subsection 2.11), we will take \( \lambda = 1 \), otherwise we choose \( \lambda = 0 \).

## 2.10 Interaction Energy, Part 2

Next we are going to give an upper bound on the expectation value of \( \mathbb{W}_2 \), defined in (2.9.3). To start, we claim that there exists a smooth function \( g \) of rapid decay (faster than any polynomial) such that

\[
w_R(x - y) \leq \frac{R^2}{s^5} g(d(x, y)/s) \quad (2.10.1)
\]

Although \( w_R \) depends on the box size \( L \), \( g \) can be chosen independent of \( L \) for large \( L \). This follows immediately from the following considerations. First of all, we have, from the definition (2.6.2) of \( f_R \) and because of \( R \leq s \),

\[
f_R(x) \leq R \sup_{d(x, y) \leq s} |\nabla h(y)|. \quad (2.10.2)
\]
Recall that $h(x) = |\Lambda|^{-1} \sum_p (1-\nu(sp)) e^{-ipx}$, where $1-\nu$ is a smooth function supported in a ball of radius 2. We need the following elementary lemma.

**Lemma 5.** Let $o : \mathbb{R}^3 \to \mathbb{C}$ be a smooth function, supported in a cube of side length 4, and let $u(x) = |\Lambda|^{-1} \sum_p o(sp)e^{-ipx}$. Then, for any non-negative integer $n$,

$$|u(x)| \leq \left( \frac{s}{16 d(x,0)} \right)^{2n} \|(-\Delta)^n o\|_\infty \left( \frac{2}{\pi s} + \frac{n + 1}{L} \right)^3.$$  \hspace{1cm} (2.10.3)

Here, $\Delta$ denotes the Laplacian on $\mathbb{R}^3$, not on $\Lambda$.

**Proof.** Introducing coordinates $x = (x_1, x_2, x_3)$, we can write

$$u(x) \left( 2L^2 \sum_{i=1}^3 (1 - \cos(2\pi x_i/L)) \right)^n = |\Lambda|^{-1} \sum_p (-\Delta) o(sp)e^{-ipx}.$$  \hspace{1cm} (2.10.4)

Here, $\Delta_d$ denotes the discrete Laplacian in momentum space, which acts as $L^{-2} (-\Delta_d) f(p) = 8 f(p) - \sum_{|\epsilon| = 1} f(p + 2\pi \epsilon/L)$. It is easy to see that the function $(-\Delta_d)^n f$ is bounded by $\|(-\Delta)^n f\|_\infty$. Moreover, if $f$ has support in a cube of side length $\ell$, then $(-\Delta_d)^n f$ is supported in a cube of length $\ell + 4\pi n/L$. This implies that

$$|(2.10.4)| \leq s^{2n} \|(-\Delta)^n o\|_\infty \left( \frac{2}{\pi s} + \frac{n + 1}{L} \right)^3.$$  \hspace{1cm} (2.10.5)

On the other hand, note that $1 - \cos(2\pi x_i/L) \geq 8L^{-2} \min_{k \in \mathbb{Z}} |x_i - kL|^2$, and hence

$$2L^2 \sum_{i=1}^3 (1 - \cos(2\pi x_i/L)) \geq 16 d(x,0)^2.$$  \hspace{1cm} (2.10.6)

This proves the lemma. $\blacksquare$

Applying the lemma to the function $\nabla h$ in (2.10.2), and using the definition (2.6.3) of $w_R$, we immediately conclude (2.10.1).

We now decompose the function $g$ into an integral over characteristic functions of balls. Such decompositions have been studied in detail in [3]. Recall that $j$ is defined in (2.7.1). According to [3, Thm. 1], we can write

$$g(t) = \int_0^\infty dr \ m(r) j(t/r),$$  \hspace{1cm} (2.10.7)
where
\[ m(r) = \frac{1}{72} r \left( g''(r) - rg'''(r) \right). \quad (2.10.8) \]

Note that \( m \) is a smooth function of rapid decay. Since \( j \) is monotone decreasing, we can estimate
\[ g(t) \leq j(t) \int_0^1 dr |m(r)| + \int_1^\infty dr |m(r)| j(t/r). \quad (2.10.9) \]

This estimate, together with (2.10.1), implies that
\[ \mathbb{W}_2 \leq \frac{144}{\pi} \tilde{a} R^2 \int_s^\infty dr \left( \delta(r-s) \int_0^1 dt |m(t)| + s^{-1}|m(r/s)| \right) \times \int_A d\xi \bigoplus_{n=0}^\infty \sum_{j=1}^n \sum_{i \in J_j} \chi_{r/2,\xi}(x_j) \chi_{r/2,\xi}(x_i). \quad (2.10.10) \]

Let \( v_r(\xi) \) denote the integrand in the last line in (2.10.10). Because \( d(x_i, x_k) \geq R/5 \) for \( i, k \in J_j \), the number of \( x_i \) inside a ball of radius \( r/2 \) is bounded from above by \( (1 + 5r/R)^3 \). Thus we have
\[ v_r(\xi) \leq n_{r/2,\xi}(1 + 5r/R)^3. \quad (2.10.11) \]

Moreover, we trivially have that \( v_r(\xi) \leq n_{r/2,\xi}(n_{r/2,\xi} - 1) \). By combining these two bounds, we obtain
\[ v_r(\xi) \leq f(n_{r/2,\xi}), \quad (2.10.12) \]

where
\[ f(n) = n \min\{n - 1, (1 + 5r/R)^3\}. \quad (2.10.13) \]

Proceeding similarly to (2.9.11), using that \( |f(n) - n(1 + 5r/R)^3| \leq (1 + (1 + 5r/R)^3)/4 \), we can estimate
\[ \text{Tr}_\mathcal{F}[v_r(\xi) \Upsilon^\xi_{\pi}] \leq \text{Tr}_\mathcal{F}[f(n_{r/2,\xi}) \Upsilon^\xi_{\pi}] \leq \text{Tr}_\mathcal{F}[f(n_{r/2,\xi}) \Omega^\xi_{\pi}] + (1 + 5r/R)^3 \text{Tr}_\mathcal{F}[n_{r/2,\xi}(\Upsilon^\xi_{\pi} - \Omega^\xi_{b})] \]
\[ + \frac{1}{4} (1 + (1 + 5r/R)^3)^2 \| \Upsilon^\xi_{\pi,\chi r/2,\xi} - \Omega^\xi_{b,\chi r/2,\xi} \|_1. \quad (2.10.14) \]

When integrating over \( \xi \), the last two terms can be handled in the same way as in the previous subsection, see Eqs. (2.9.12) and (2.9.13). We have to
assume that \( r \leq b \), however. For the first term on the right side of (2.10.14), we estimate

\[
\text{Tr}_\mathcal{F} \left[ f(n_{r/2,\xi})\Omega_b^\xi \right] \\
\leq \min \left\{ \text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}(n_{r/2,\xi} - 1)\Omega_b^\xi \right] , (1 + 5r/R)^3 \text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}\Omega_b^\xi \right] \right\} .
\]

(2.10.15)

Similarly to (2.9.15),

\[
\text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}(n_{r/2,\xi} - 1)\Omega_b^\xi \right] \leq 2 \left( \text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}\Omega_b^\xi \right] \right)^2 ,
\]

(2.10.16)

and hence

\[
\text{Tr}_\mathcal{F} \left[ f(n_{r/2,\xi})\Omega_b^\xi \right] \leq \frac{4 \left( \text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}\Omega_b^\xi \right] \right)^2}{1 + 2 \text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}\Omega_b^\xi \right] / (1 + 5r/R)^3} .
\]

(2.10.17)

Moreover,

\[
\text{Tr}_\mathcal{F} \left[ n_{r/2,\xi}\Omega_b^\xi \right] = \frac{\pi}{6} r^3 \theta_\omega + \langle \Phi_\xi | \chi_{r/2,\xi} | \Phi_\xi \rangle .
\]

(2.10.18)

Using convexity of the function \( x \mapsto x^2/(1 + x) \), we obtain the bound

\[
\text{Tr}_\mathcal{F} \left[ f(n_{r/2,\xi})\Omega_b^\xi \right] \leq 8 \left( \frac{\pi}{6} r^3 \theta_\omega \right)^2 + \frac{8 \langle \Phi_\xi | \chi_{r/2,\xi} | \Phi_\xi \rangle^2}{1 + 4 \langle \Phi_\xi | \chi_{r/2,\xi} | \Phi_\xi \rangle / (1 + 5r/R)^3} .
\]

(2.10.19)

We use (2.10.19) in (2.10.14) and integrate over \( \xi \). We obtain (assuming \( r \leq b \), as mentioned above)

\[
\int_\Lambda d\xi \text{Tr}_\mathcal{F} \left[ n_{r}(\xi)\Omega_\pi^\xi \right] \leq (1 + 5r/R)^3 \frac{\pi}{6} r^3 \text{Tr}_\mathcal{F} \left[ N(\Omega_\pi^\xi - \Omega_b^\xi) \right] + 8|\Lambda| \left( \frac{\pi}{6} r^3 \theta_\omega \right)^2 \\
+ (1 + (1 + 5r/R)^3)^2 \left( b^3 |\Lambda| S(\Omega_\pi^\xi, \Omega_b^\xi) \right)^{1/2} \\
+ \int_\Lambda d\xi \frac{8 \langle \Phi_\xi | \chi_{r/2,\xi} | \Phi_\xi \rangle^2}{1 + 4 \langle \Phi_\xi | \chi_{r/2,\xi} | \Phi_\xi \rangle / (1 + 5r/R)^3} .
\]

(2.10.20)

In order to be able to compare the last term with the last term in (2.9.22), we note that

\[
\chi_{r/2,\xi} \leq \left( 1 + \frac{5r}{R} \right)^3 \int_{|a| \leq r/2 + R/10} da \chi_{R/10,\xi + a} .
\]

(2.10.21)
Here, we denote by \( f \) the normalized integral, i.e., we divide by the volume of the ball of radius \( r/2 + R/10 \). Using monotonicity and convexity of the map \( x \mapsto x^2/(1 + x) \), we thus have the upper bound

\[
\frac{\langle \Phi \mid \chi_{r/2} \mid \Phi \rangle^2}{1 + 4\langle \Phi \mid \chi_{r/2} \mid \Phi \rangle/(1 + 5r/R)^3}
\leq \left( 1 + \frac{5r}{R} \right)^6 \int_{|a| \leq r/2 + R/10} da \frac{\langle \Phi \mid \chi_{R/10, a} \mid \Phi \rangle^2}{1 + 4\langle \Phi \mid \chi_{R/10, a} \mid \Phi \rangle}.
\] (2.10.22)

After integration over \( \xi \), the right side of (2.10.22) simply becomes

\[
\left( 1 + \frac{5r}{R} \right)^6 \int_{\Lambda} d\xi \frac{\langle \Phi \mid \chi_{R/10, \xi} \mid \Phi \rangle^2}{1 + 4\langle \Phi \mid \chi_{R/10, \xi} \mid \Phi \rangle} \leq \left( \frac{6r}{R} \right)^6 \int_{\Lambda} d\xi \frac{\langle \Phi \mid \chi_{R/10, \xi} \mid \Phi \rangle^2}{1 + \langle \Phi \mid \chi_{R/10, \xi} \mid \Phi \rangle}.
\] (2.10.23)

Here we have used the fact that \( r \geq s \geq R \) for the relevant values of \( r \).

As noted above, the estimates leading to (2.10.20) are only valid for \( r \leq b \). To bound the expectation value of \( \mathbb W_2 \) in (2.10.10) we have to consider all \( r \geq s \), however. For \( r \geq b \), we use (2.10.11) to obtain the simple estimate

\[
\int_{\Lambda} d\xi \text{Tr}_{\mathcal F} [v_r(\xi) \Upsilon_{\pi}^z] \leq \left( 1 + \frac{5r}{R} \right)^3 \int_{\Lambda} d\xi \text{Tr}_{\mathcal F} [n_{r/2} \Upsilon_{\pi}^z] \\
\leq \left( \frac{6r}{R} \right)^3 \frac{\pi}{6} r^3 \text{Tr}_{\mathcal F} [N \Upsilon_{\pi}^z].
\] (2.10.24)

The contribution of \( r \geq b \) to the integral in (2.10.10) is thus bounded from above by

\[
\frac{1}{s} \int_{b}^{\infty} dr \ |m(r/s)| \int_{\Lambda} d\xi \text{Tr}_{\mathcal F} [v_r(\xi) \Upsilon_{\pi}^z] \\
\leq \frac{\pi}{6} s^3 \left( \frac{6s}{R} \right)^3 \text{Tr}_{\mathcal F} [N \Upsilon_{\pi}^z] \int_{b/s}^{\infty} dr r^6 |m(r)|.
\] (2.10.25)

Since \( |m| \) is a function that decays faster than any polynomial, the last integral is bounded above by any power of the (small) parameter \( s/b \).

Let \( c \) denote the constant

\[
c = \int_0^1 dr |m(r)| + \int_1^{\infty} dr r^6 |m(r)|.
\] (2.10.26)
To summarize, we have derived the upper bound
\[
\text{Tr}_F [\mathcal{W} \mathbb{Y}_2] \leq 6^3 \frac{24\tilde{a}}{\varepsilon R s^2} \text{Tr}_F [\mathcal{N}(\mathbb{Y}_{\pi} - \Omega)^2] + 32\pi |\Lambda| a_\xi^2 \frac{R^2}{\varepsilon s^2} \\
+ \frac{144}{\pi} (1 + 6^3)^2 \frac{\tilde{a}}{\varepsilon R^4 s^2} c \left( b^3 |\Lambda| S(\mathbb{Y}_{\pi}, \Omega)^2 \right)^{1/2} \\
+ 6^3 \frac{24\tilde{a}}{\varepsilon R s^2} \text{Tr}_F [\mathcal{N} \Omega_b^2] \int_{b/s}^{\infty} dr r^6 |m(r)| \\
+ 8 \frac{144}{\pi} \frac{6^6 \tilde{a}}{\varepsilon s^2 R^4} c \int_{\Lambda} d\xi \frac{\langle \Phi | \chi_{R/10, \xi} | \Phi \rangle^2}{1 + \langle \Phi | \chi_{R/10, \xi} | \Phi \rangle}. \quad (2.10.27)
\]

2.11 Interaction Energy, Part 3

We now put the bounds of the previous two subsections together in order to obtain our final lower bound on \( \text{Tr}_F [\mathcal{W} \mathbb{Y}_2] \). We will distinguish two cases, depending on the value of a certain function of \( |\Phi| \), given in (2.11.1) below.

Assume first that
\[
\int_{\Lambda} d\xi \frac{\langle \Phi | \chi_{R/10, \xi} | \Phi \rangle^2}{1 + \langle \Phi | \chi_{R/10, \xi} | \Phi \rangle} \geq \frac{\pi^2}{18} |\Lambda| (R^3) \chi_0^2. \quad (2.11.1)
\]

This condition means, essentially, that \( |\Phi| \) is far from being a constant. In this case, we choose \( \lambda = 0 \) in (2.9.22), and find that the contribution of the last terms in (2.9.22) and (2.10.27), respectively, is bounded from below by

\[
8\pi |\Lambda| a_\xi^2 \left( \frac{a'}{a} - \frac{4\pi}{3} \left( \frac{R}{10} \right)^3 \chi_\omega - 8c \frac{6^6 R^2}{\varepsilon s^2} \right). \quad (2.11.2)
\]

Next, consider the case when (2.11.1) is false. In this case, using (2.10.21) for \( r = 3R \), as well as convexity of \( x \mapsto x^2/(1 + x) \), we find that

\[
\int_{\Lambda} d\xi \frac{\langle \Phi | \chi_{3R/2, \xi} | \Phi \rangle^2}{1 + 16^{-3} \langle \Phi | \chi_{3R/2, \xi} | \Phi \rangle} \leq 16^6 \frac{\pi^2}{18} |\Lambda| (R^3) \chi_0^2. \quad (2.11.3)
\]

Pick some \( D > 0 \), and let \( \mathcal{B} \subset \Lambda \) denote the set

\[
\mathcal{B} = \{ \xi \in \Lambda : \langle \Phi | \chi_{3R/2, \xi} | \Phi \rangle \leq 16^3 D R^3 \chi_0 \}. \quad (2.11.4)
\]

Using (2.11.3), as well as monotonicity of \( x \mapsto x/(1 + x) \), we find that

\[
\int_{\mathcal{B}} d\xi \langle \Phi | \chi_{3R/2, \xi} | \Phi \rangle \leq \frac{16^3 \pi^2}{D} \frac{\chi_0}{18} |\Lambda| (R^3) \chi_0 (1 + D R^3 \chi_0). \quad (2.11.5)
\]
Similarly, we have the estimate

\[ |B| \leq |\Lambda| \frac{1}{D^2} \frac{\pi^2}{18} (1 + D R^3 \varrho) . \]  

(2.11.6)

We choose \( \lambda = 1 \) in (2.9.22) and estimate the relevant term from below by

\[
\int_{\Lambda} d\xi \left[ \text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b^2 \right] - 6 \left( \text{Tr}_F \left[ n_{3R/2,\xi} \Omega_b^2 \right] \right)^3 \right] \\
\geq \int_{\Lambda \setminus B} d\xi \left( \text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b^2 \right] - 6 \left( \text{Tr}_F \left[ n_{3R/2,\xi} \Omega_b^2 \right] \right)^3 \right). 
\]  

(2.11.7)

Using \( \text{Tr}_F \left[ n_{3R/2,\xi} \Omega_b^2 \right] = 9\pi R^3 \varrho / 2 + \langle \Phi_2 | \chi_{3R/2,\xi} | \Phi_2 \rangle \), the definition of \( B \) in (2.11.4) and convexity of \( x \mapsto x^3 \), we can bound the last term as

\[
\int_{\Lambda \setminus B} d\xi \left( \text{Tr}_F \left[ n_{3R/2,\xi} \Omega_b^2 \right] \right)^3 \leq 4|\Lambda| \left( \frac{9\pi}{2} R^3 \varrho \right)^3 + 18\pi |z|^2 R^3 (16^3 D R^3 \varrho)^2 . 
\]  

(2.11.8)

We now investigate the first term on the right side of (2.11.7). A simple calculation shows that

\[
\text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b^2 \right] \\
= \text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b \right] + 2 \langle \Phi_2 | \chi_{R/2,\xi} \omega_b \chi_{R/2,\xi} | \Phi_2 \rangle \\
+ \frac{\pi}{3} R^3 \varrho \omega \langle \Phi_2 | \chi_{R/2,\xi} | \Phi_2 \rangle + \langle \Phi_2 | \chi_{R/2,\xi} | \Phi_2 \rangle^2 . 
\]  

(2.11.9)

Here, we have used again the translation invariance of \( \Omega_b \). Note that this invariance also implies that the first term on the right side of (2.11.9) is independent of \( \xi \). Since \( \Omega_b \) is a quasi free state, it can be rewritten in terms of the one-particle density matrix \( \omega_b \) as

\[
\text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b \right] = (\text{tr} \chi_{R/2,\xi} \omega_b)^2 + \text{tr} \chi_{R/2,\xi} \omega_b \chi_{R/2,\xi} \omega_b . 
\]  

(2.11.10)

The first term is just \( (\pi R^3 \varrho / 6)^2 \), and the second is bounded from above by this expression. Therefore,

\[
\int_B d\xi \text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1) \Omega_b \right] \leq 2|B| \left( \frac{\pi}{6} R^3 \varrho \right)^2 . 
\]  

(2.11.11)

Note also that \( \langle \Phi_2 | \chi_{R/2,\xi} \omega_b \chi_{R/2,\xi} | \Phi_2 \rangle \leq \text{tr} \chi_{R/2,\xi} \omega_b \langle \Phi_2 | \chi_{R/2,\xi} | \Phi_2 \rangle \), and thus

\[
\int_B d\xi \left( 2 \langle \Phi_2 | \chi_{R/2,\xi} \omega_b \chi_{R/2,\xi} | \Phi_2 \rangle + \frac{\pi}{3} R^3 \varrho \omega \langle \Phi_2 | \chi_{R/2,\xi} | \Phi_2 \rangle \right) \\
\leq \frac{2\pi}{3} R^3 \varrho \omega \int_B d\xi \langle \Phi_2 | \chi_{R/2,\xi} | \Phi_2 \rangle . 
\]  

(2.11.12)
The last expression can be bounded from above using (2.11.5). For the last term in (2.11.9), we use Schwarz’s inequality, together with (2.11.5), to estimate

\[
\int_{\Lambda \setminus B} d\xi \left( \Phi_{\vec{z}} \right)_{\chi_R/2,\xi} \left( \Phi_{\vec{z}} \right)^2 \geq \frac{1}{|\Lambda|} \left( \int_{\Lambda \setminus B} d\xi \left( \Phi_{\vec{z}} \right)_{\chi_R/2,\xi} \right)^2 
\geq |\Lambda| \frac{\pi^2 R^6}{36} \left( g_\omega^2 - \frac{2\pi}{3} g_\omega \frac{16^3}{D} (1 + DR^3 g) \right).
\] (2.11.13)

Here we set again \(g_\omega = |\vec{z}|^2 / |\Lambda|\).

Putting all these estimates together, we have thus derived the lower bound

\[
\int_{\Lambda \setminus B} d\xi \text{Tr}_F \left[ n_{R/2,\xi}(n_{R/2,\xi} - 1)\Omega_b \right] \geq |\Lambda| \frac{\pi^2 R^6}{36} \left( g_\omega^2 - \frac{2\pi}{3} g_\omega \frac{16^3}{D} (1 + DR^3 g) \right) + 2 \int_{\Lambda} d\xi \langle \Phi_{\vec{z}} | \chi_R/2,\xi \omega_b | \chi_R/2,\xi \omega_b \rangle - |\Lambda| \frac{\pi R^3}{144} \int_{\Lambda} dx |\omega_b(x)|^2 j(d(x, 0)/R) \geq |\Lambda| \frac{\pi^2 R^6}{36} \gamma_b^2,
\] (2.11.14)

where we introduced the notation

\[
\gamma_b = \frac{1}{4\pi R^3} \int_{\Lambda} dx \omega_b(x) j(d(x, 0)/R).
\] (2.11.16)

Eq. (2.11.15) follows by applying Schwarz’s inequality to the integration over \(\Lambda\), noting that \(\int_{\Lambda} dx j(d(x, 0)/R) = 4\pi R^3\).

It remains to integrate the last term in (2.11.14). We claim that

\[
\int_{\Lambda} d\xi \langle \Phi_{\vec{z}} | \chi_R/2,\xi \omega_b | \chi_R/2,\xi \left( \Phi_{\vec{z}} \right) \rangle \geq |\vec{z}|^2 \frac{\pi^2 R^6}{36} \left( \gamma_b - R p_c g_\omega \right). \tag{2.11.17}
\]
To see this, we write
\[ \frac{144}{\pi R^3} \int_{\Lambda} d\xi \langle \Phi_\xi | \chi_{R/2} \chi_{R/2} | \Phi_\xi \rangle - |\xi|^2 \int_{\Lambda} dx \omega_b(x) j(d(x,0)/R) \]
\[ = \int_{\Lambda \times \Lambda} dx dy (\Phi_\xi(x + y)^n - \Phi_\xi(y)^n) \Phi_\xi(y) \omega_b(x) j(d(x,0)/R) \]
\[ \geq -\| \Phi_\xi \|_2 \int_{\Lambda} dx \| \Phi_\xi(x + \cdot) - \Phi_\xi(\cdot) \|_2 \omega_b(x) j(d(x,0)/R). \quad (2.11.18) \]

We can estimate \( |\omega_b(x)| \leq \omega_b(0) = \theta_\omega \). Moreover, writing the 2-norm as a sum in momentum space, and using the fact that \( \Phi_\xi \) has non-vanishing Fourier coefficients only for \(|p| < p_c\), it is easy to see that \( \| \Phi_\xi(x + \cdot) - \Phi_\xi(\cdot) \|_2 \leq \| \Phi_\xi \|_2 p_c d(x,0) \). Since the range of \( j(\cdot / R) \) is \( R \), the integral over \( \Lambda \) can be estimated as \( \int_{\Lambda} dx \ j(d(x,0)/R) \leq R \int_{\Lambda} dx \ j(d(x,0)/R) = 4\pi R^4 \). This yields (2.11.17).

Collecting all the estimates above, we conclude the following lower bound on the expectation value of \( \mathcal{W} \):
\[ \text{Tr}_{\mathcal{F}} [\mathcal{W} \mathcal{Y}_\pi^z] \geq 24 \frac{\bar{a}}{R^3} \text{Tr}_{\mathcal{F}} [N(\mathcal{Y}_\pi^z - \Omega_b^z)] \left( \frac{1}{125} \frac{a'}{a} - 6^3 \frac{c R^2}{\varepsilon s^2} \right) \]
\[ - \frac{144}{\pi} \frac{\bar{a}}{R^6} \left( b^3 |\Lambda| S(\mathcal{Y}_\pi^z, \Omega_b^z) \right)^{1/2} \left( 8 + (1 + 6^3)\frac{c R^2}{\varepsilon s^2} \right) \]
\[ - 4\pi \bar{a} |\Lambda| \left( 8 \theta_\omega^2 \frac{R^2}{\varepsilon s^2} + 6^4 \frac{\theta_\omega \theta_\varepsilon}{\bar{a}} + \theta_\varepsilon \int_{|r|/b}^{\infty} dr r^6 |m(r)| \right) \]
\[ + 4\pi a' |\Lambda| \min \{ A_1, A_2 \}. \quad (2.11.19) \]

Here we have used the simple bound \( a' \leq \bar{a} \), and we have set
\[ A_1 = 2\theta_\varepsilon^2 \left( 1 - \frac{4\pi}{10} \left( \frac{R}{10} \right)^3 \theta_\omega - 8c \frac{6^6 R^2}{\bar{a}^2 \varepsilon s^2} \right) \quad (2.11.20) \]

and
\[ A_2 = (\theta_\varepsilon^2 + 2 \theta_\varepsilon \gamma_0 + \gamma_0^2) + 2 \theta_\varepsilon \theta_\varepsilon (1 - R p_c) \]
\[ + \theta_\varepsilon \left( 1 - \frac{\pi^2}{D^2} \frac{R^3}{3} \right) (1 + DR^3 \theta_\varepsilon) - 2\pi 3^3 R^3 \theta_\varepsilon \]
\[ - \theta_\varepsilon \theta_\varepsilon \left( 4 \pi \frac{16^3}{3} \frac{R^3}{D} \right) - 6^3 \frac{\bar{a}}{\varepsilon s^2} \]
\[ - \varepsilon \theta_\varepsilon \left( \frac{2^3 3^4}{\pi} (16^3 D)^2 R^3 \theta_\varepsilon + 2 \frac{16^3}{3} \frac{R^3}{D} (1 + DR^3 \theta_\varepsilon) \right). \quad (2.11.21) \]
We will choose $D = (R^3 \varrho)^{-1/3}$ in order to minimize the error terms in $A_2$. Moreover, since $a'/\bar{a}$ contains a factor $(1-\varepsilon)$ (see (2.7.7)), it is best to choose $\varepsilon = R/s$. We note that one can also use the simple bound (2.4.10) in order to estimate $\rho_\varepsilon$ in the error terms.

Since $R_0 \ll R \ll s$, the term in round brackets in the first line of (2.11.19) is non-negative and, therefore, we will need a lower bound on $\text{Tr}_F[N(\Upsilon_\pi^\varepsilon - \Omega_0^\varepsilon)]$. Moreover, we will need an upper bound on the relative entropy $S(\Upsilon_\pi^\varepsilon, \Omega_0^\varepsilon)$. Appropriate bounds will be derived in the next two subsections.

### 2.12 A Bound on the Number of Particles

Our lower bound on the expectation value of $W$ in the previous subsection contains the expression $\text{Tr}_F[N(\Upsilon_\pi^\varepsilon - \Omega_0^\varepsilon)]$, multiplied by a positive parameter. Hence we need a lower bound on this expression in order to complete our bound. In fact, we will combine the first term on the right side of (2.11.19) with the last term $\frac{1}{2} \text{Tr}_F[N \Upsilon_\pi^\varepsilon]$ in (2.7.15), which we have not used so far. I.e., we seek a lower bound on

$$24 \frac{\bar{a}}{R^3} \text{Tr}_F[N(\Upsilon_\pi^\varepsilon - \Omega_0^\varepsilon)] \left( \frac{1}{125} \frac{a'}{\bar{a}} - 6^3 \frac{R}{s} \right) + 2 \frac{2 \pi C}{|\Lambda|} \text{Tr}_F[N - \Upsilon_\pi^\varepsilon].$$

(2.12.1)

(Here we have used that $\varepsilon = R/s$, as mentioned at the end of the previous subsection.) First, note that $\text{Tr}_F[N \Upsilon_\pi^\varepsilon] = |\varepsilon|^2 + \text{Tr}_F[N \Omega_0^\varepsilon] = |\varepsilon|^2 + \text{Tr}_F[N \Omega_\pi^\varepsilon]$ and $\text{Tr}_F[N \Upsilon_\pi^\varepsilon] = |\varepsilon|^2 + \text{Tr}_F[N \Upsilon_\pi^\varepsilon]$. Let $N^\varepsilon = \sum_{|\mu| \geq p_c} a^\dagger_\mu a_\mu$ denote the number operator on $\mathcal{F}_\varepsilon$. Using that $\Omega_\pi = \Pi \otimes \Gamma_0$ and that $\Upsilon_\pi = \Pi \otimes \Gamma_\varepsilon$, we can thus write

$$\text{Tr}_F[N(\Upsilon_\pi^\varepsilon - \Omega_0^\varepsilon)] = \text{Tr}_F[N^\varepsilon (\Gamma_\varepsilon - \Gamma_0)].$$

(2.12.2)

For the second term in (2.12.1), we use

$$(N - N)^2 \geq (|\varepsilon|^2 + \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon \Gamma_0^\varepsilon] - N)^2$$

(2.12.3)

$$+ 2 (|\varepsilon|^2 + \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon \Gamma_0^\varepsilon] - N)(N - |\varepsilon|^2 - \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon \Gamma_0^\varepsilon]),$$

and hence

$$\text{Tr}_F[(N - N)^2 \Upsilon_\varepsilon] \geq (|\varepsilon|^2 + \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon \Gamma_0^\varepsilon] - N)^2$$

(2.12.4)

$$+ 2 (|\varepsilon|^2 + \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon \Gamma_0^\varepsilon] - N) \text{Tr}_{\mathcal{F}_\varepsilon}[N^\varepsilon (\Gamma_\varepsilon - \Gamma_0^0)].$$
Thus, we conclude that the expression (2.12.1) is bounded from below by

\[\frac{2\pi \tilde{a}C}{|\Lambda|} \left( |z|^2 + \text{Tr}_{F^\to} [N^\to \Gamma^0] - N \right)^2 + \text{Tr}_{F^\to} \left[ (N^\to - N^0) \Gamma_z \right] \]

\[\times \left[ \frac{24}{R^3} \left( \frac{a'}{125} - 6^3 \frac{\tilde{a}R}{s} \right) + \frac{4\pi \tilde{a}C}{|\Lambda|} \left( |z|^2 + \text{Tr}_{F^\to} [N^\to \Gamma^0] - N \right) \right]. \tag{2.12.5}\]

We will choose \( R, s \) and \( C \) below in such a way that \( R^3 \theta \ll 1/C \) and \( R \ll s \). The last term in square brackets is thus positive, irrespective of the value of \( |z| \). Hence we need to derive a lower bound on \( \text{Tr}_{F^\to} [N^\to (\Gamma_z - \Gamma^0) \Gamma_z] \).

To this end, we note that, for any \( \mu \leq 0 \),

\[S(\Gamma_z, \Gamma^0) - \beta \mu \text{Tr}_{F^\to} [N^\to \Gamma_z] \geq \beta (\tilde{f}(\mu) - \tilde{f}(0)) \geq \beta \left( \tilde{f}(\mu) - \tilde{f}(0) \right). \tag{2.12.6}\]

Here, we denoted

\[\tilde{f}(\mu) = \frac{1}{\beta} \sum_{|p| \geq p_c} \ln \left( 1 - e^{-\beta(p^2 - \mu_0 - \mu)} \right). \tag{2.12.7}\]

Eq. (2.12.6) is simply the variational principle for the free energy \( \tilde{f}(\mu) \). Since \( \tilde{f} \) is completely monotone (i.e., all derivatives are negative), we can estimate \( \tilde{f}(\mu) \geq \tilde{f}(0) + \mu \tilde{f}'(0) + \frac{1}{2} \mu^2 \tilde{f}''(0) \). But \( \tilde{f}'(0) = -\text{Tr}_{F^\to} [N^\to \Gamma] \) and hence, optimizing over all (negative) \( \mu \),

\[\text{Tr}_{F^\to} [N^\to (\Gamma_z - \Gamma^0) \Gamma_z] \geq - \left( S(\Gamma_z, \Gamma^0) \sum_{|p| \geq p_c} \frac{1}{\cosh(\beta(p^2 - \mu_0)) - 1} \right)^{1/2} \geq - \left( \sum_{|p| \geq p_c} \frac{1}{\cosh(\beta(p^2 - \mu_0)) - 1} \right)^{1/2}. \tag{2.12.8}\]

We can use (2.4.6) to estimate the relative entropy as \( S(\Gamma_z, \Gamma^0) \leq 8\pi |\Lambda| \tilde{a} \beta \theta^2 \).

For the sum over \( p \), we use that \( \cosh(x) - 1 \geq x^2/2 \). In the thermodynamic limit, we can replace the sum over \( p \) by an integral. We thus have to bound

\[\frac{2 |\Lambda|}{\beta^2 (2\pi)^3} \int_{|p| \geq p_c} dp \frac{1}{(p^2 - \mu_0)^2}. \tag{2.12.9}\]

We use two different bounds. On the one hand,

\[\int_{|p| \geq p_c} dp \frac{1}{(p^2 - \mu_0)^2} \leq \int_{\mathbb{R}^3} dp \frac{1}{(p^2 - \mu_0)^2} = \frac{\pi^2}{\sqrt{-\mu_0}}. \tag{2.12.10}\]

On the other hand, since \( \mu_0 \leq 0 \),

\[\int_{|p| \geq p_c} dp \frac{1}{(p^2 - \mu_0)^2} \leq \int_{|p| \geq p_c} dp \frac{1}{(p^2)^2} = \frac{4\pi}{p_c}. \tag{2.12.11}\]
In combination, these bounds imply that
\[
(2.12.9) \leq \frac{|\Lambda|}{\pi^2 \beta^2} \left( \frac{1}{\max\{p_c, 4\pi^{-1}\sqrt{-\mu_0}\}} \right),
\]  
\[ (2.12.12) \]

Using (2.12.12) in (2.12.8), we obtain the lower bound
\[
\text{Tr}_{\mathcal{F}} \left[ N^> (\Gamma_{\vec{z}} - \Gamma^0) \right] \geq -\text{const.} \frac{\beta \tilde{a}^{1/2}}{\beta^{1/2}} (p_c + \sqrt{-\mu_0})^{-1/2} - o(|\Lambda|).
\]
\[ (2.12.13) \]

We apply this bound in (2.12.5), noting again that the last term in square brackets is positive. We conclude that
\[
(2.12.1) \geq \frac{2\pi \tilde{a} C}{|\Lambda|} (|\vec{z}|^2 + \text{Tr}_{\mathcal{F}} \left[ N^> \Gamma^0 \right] - N)^2 - Z^{(3)} - o(|\Lambda|),
\]
\[ (2.12.14) \]

with
\[
Z^{(3)} = \text{const.} |\Lambda| \frac{\beta^{\alpha^{3/2}}}{\beta^{1/2}} (p_c + \sqrt{-\mu_0})^{-1/2} \left[ \frac{1}{R^2} + C \left( \rho \left( 1 + 2\sqrt{C} \right) + \varrho_\omega \right) \right].
\]
\[ (2.12.15) \]

Here we have used (2.4.10) to bound $|\vec{z}|^2$ from above, as well as the fact that $\text{Tr}_{\mathcal{F}} \left[ N^> \Gamma^0 \right] \leq \text{Tr}_{\mathcal{F}} \left[ N\Omega \right] = |\Lambda| \varrho_\omega$.

### 2.13 Relative Entropy, Effect of Cutoff

Next, we are going to give an estimate on the relative entropy of the two states $\Upsilon_{\vec{z}}^{\pi}$ and $\Omega_{\vec{z}}^{b}$. This is needed for the lower bound on the expectation value of $W$ obtained in (2.11.19). As already noted, the relative entropy is invariant under unitary transformations, and hence
\[
S(\Upsilon_{\vec{z}}^{\pi}, \Omega_{\vec{z}}^{b}) = S(\Pi \otimes \Gamma_{\vec{z}}, \Omega_{\vec{b}}).
\]
\[ (2.13.1) \]

We want to bound this expression by the relative entropy of $\Pi \otimes \Gamma_{\vec{z}}$ with respect to $\Omega_{\vec{z}} = \Pi \otimes \Gamma^0$, which satisfies
\[
S(\Pi \otimes \Gamma_{\vec{z}}, \Omega_{\vec{z}}) = S(\Gamma_{\vec{z}}, \Gamma^0) \leq 8\pi \tilde{a} \beta |\Lambda| \varrho^2
\]
\[ (2.13.2) \]

according to (2.4.6). I.e., we want to estimate the effect of the cutoff $b$ on the relative entropy $S(\Pi \otimes \Gamma_{\vec{z}}, \Omega_{\vec{b}})$. Here it will be important that $\Pi$ is not the vacuum state. The cutoff $b$ corresponds to a mollifying of the one-particle density matrix in momentum space, and the error in doing so would not be small enough if this one-particle density matrix is strictly zero for $|p| < p_c$. 

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This is the reason for replacing the vacuum state $\Pi_0$ by a more general quasi-free state $\Pi$ in Subsection 2.5.

If $\Omega_\omega$ denotes a general (particle number conserving) quasi-free state with one-particle density matrix $\omega$, it is easy to see that $S(\Gamma, \Omega_\omega)$ is convex in $\omega$ for an arbitrary state $\Gamma$. The one-particle density matrix of $\Omega_b$ can be written as

$$\omega_b = \frac{1}{|\Lambda|} \sum_q \tilde{\eta}_b(q) \sum_p \frac{1}{2} (\omega_\pi(p + q) + \omega_\pi(p - q)) \langle p | p \rangle.$$

(2.13.3)

Therefore,

$$S(\Pi \otimes \Gamma, \Omega_b) \leq \frac{1}{|\Lambda|} \sum_q \tilde{\eta}_b(q) S(\Pi \otimes \Gamma, \Omega_q),$$

(2.13.4)

where $\Omega_q$ is the quasi-free state corresponding to the one-particle density matrix with eigenvalues $\frac{1}{2} (\omega_\pi(p + q) + \omega_\pi(p - q))$. (This is the same estimate as in [17, Sect. 5.2].) Moreover, as in [17, Eq. (5.14)], simple convexity arguments yield

$$S(\Pi \otimes \Gamma, \Omega_q) \leq (1 + t^{-1}) S(\Pi \otimes \Gamma, \Omega_\pi)$$

(2.13.5)

$$+ \sum_p (h_q(p) - h_0(p)) \left( \frac{1}{e^{h_0(p) + t(h_0(p) - h_q(p))} - 1} - \frac{1}{e^{h_0(p)} - 1} \right)$$

for any $t > 0$. Here

$$h_q(p) = \ln \frac{2 + \omega_\pi(p + q) + \omega_\pi(p - q)}{\omega_\pi(p + q) + \omega_\pi(p - q)}.$$

(2.13.6)

Note that $h_0(p) = \ell(p)$, defined in (2.8.1). To estimate the expression (2.13.5) from above, we need the following lemma.

**Lemma 6.** Let $\ell : \mathbb{R}^3 \to \mathbb{R}_+$, and let $L_\pm = \pm \sup_p \sup_{\|q\| = 1} \pm (q \nabla)^2 \ell(p)$ denote the supremum (infimum) of the largest (lowest) eigenvalue of the Hessian of $\ell$. Let $\omega_\pi(p) = [e^{\ell(p)} - 1]^{-1}$, and let $h_q(p)$ be given as in (2.13.6). Then

$$h_q(p) - h_0(p) \leq L_+ q^2,$$

(2.13.7)

and

$$h_q(p) - h_0(p) \geq q^2 L_+ + q^2 \min\{L_-, 0\} - 4q^2 \sup_p [\nabla \ell(p)]^2 \omega_\pi(p)$$

$$- 2q^2 (|q| + |p|) \sup_p [\nabla \ell(p)]^2 / p^2.$$

(2.13.8)
Proof. By convexity of \( x \mapsto \ln(1 + 1/x) \),

\[
h_q(p) \leq \frac{1}{2} (\ell(p + q) + \ell(p - q)) \leq \ell(p) + L_+ q^2,
\]

proving the (2.13.7). To obtain the lower bound in (2.13.8), we proceed similarly to [17, Lemma 5.2]. We can write

\[
h_q(p) - h_0(p) = \int_0^1 d\lambda \left( 1 - \lambda \right) \frac{\partial^2}{\partial \lambda^2} h_{\lambda q}(p).
\]

Denoting \( p_{\pm} = p + \lambda q \) and \( \omega_{\pm} = \omega_{\pi}(p_{\pm}) \), we can write the second derivative of \( h_{\lambda q}(p) \) as

\[
\frac{\partial^2}{\partial \lambda^2} h_{\lambda q}(p) = \left[ \frac{1}{(\omega_+ + \omega_-)^2} - \frac{1}{(2 + \omega_+ + \omega_-)^2} \right] \left( \frac{\partial}{\partial \lambda} (\omega_+ + \omega_-) \right)^2
\]

\[
- \frac{2}{(\omega_+ + \omega_-)(2 + \omega_+ + \omega_-)} \frac{\partial^2}{\partial \lambda^2} (\omega_+ + \omega_-).
\]

The first term is positive and can thus be neglected for a lower bound. For the second term, we use

\[
\frac{\partial^2}{\partial \lambda^2} \omega_+ = \omega_+(1 + \omega_+) \left[ (1 + 2\omega_+) (q\nabla \ell_+)^2 - (q\nabla)^2 \ell_+ \right],
\]

where we denoted \( \ell_+ = \ell(p + \lambda q) \). The last term in the square brackets is bounded above by \( -q^2 L_- \). Moreover,

\[
(1 + 2\omega_+) (q\nabla \ell_+)^2 \leq q^2 (|p| + |q|)^2 \sup_p \left[ |\nabla \ell(p)|^2 / p^2 \right] + 2q^2 \sup_p \left[ \omega_\pi(p) |\nabla \ell(p)|^2 \right].
\]

The same bounds hold with + replace by −. Using in addition that

\[
\frac{1}{2} \leq \frac{\omega_+(1 + \omega_+) + \omega_-(1 + \omega_-)}{(\omega_+ + \omega_-)(2 + \omega_+ + \omega_-)} \leq 1,
\]

we arrive at (2.13.8).

Let \( g : \mathbb{R}^3 \mapsto [0, 1] \) be a smooth radial function supported in the ball of radius 1. We assume that \( g(p) \geq \frac{1}{2} \) for \( |p| \leq \frac{1}{2} \). We then choose

\[
\ell(p) = \beta(p^2 - \mu_0) + \beta p_c^2 g(p/p_c).
\]

Since \( \ell(p) = \ln(1 + 1/\pi_p) \) for \( |p| < p_c \) by definition, this corresponds to the choice \( \pi_p = (\exp(\beta(p^2 - \mu_0) + \beta p_c^2 g(p/p_c)) - 1)^{-1} \). Note that, in particular,
\[ \pi_p \leq \text{const.}/(\beta p_c^2) \] and hence \( P \leq \text{const.} M/(\beta p_c^2) \sim p_c|A|/\beta. \) This bound is important for estimating the error term \( Z^{(2)} \) in (2.5.6).

For the \( \ell \) given in (2.13.15), both \( \beta^{-1}L_+ \) and \( \beta^{-1}L_- \) are bounded independent of all parameters. Moreover \( |\nabla \ell(p)| \leq \text{const.} |\beta|p| \). Using that \( \omega_\pi(p) \leq \ell(p)^{-1} \leq (\beta p^2)^{-1} \), the bounds in Lemma 6 imply that

\[ -Dq^2 1 + q^2 \leq h_q(p) - h_0(p) \leq Dq^2 \]  \hspace{2cm} (2.13.16)

for some constant \( D > 0 \).

Using the fact that \( \sinh(x)/x \leq \cosh(x) \) for all \( x \in \mathbb{R} \), we can estimate

\[ (h_q(p) - h_0(p)) \left( \frac{1}{e^{h_0(p)+t(h_q(p)-h_0(p))}} - \frac{1}{e^{h_0(p)}} \right) \left( \frac{1}{e^{-h_0(p)+t(h_q(p)-h_0(p))}} \right) \]  \hspace{2cm} (2.13.17)

\[ \leq \frac{1}{2}(1 + t)(h_q(p) - h_0(p))^2 \left( \frac{e^{-h_q(p)} + e^{-h_0(p)+t(h_q(p)-h_0(p))}}{1 - e^{-h_0(p)+t(h_q(p)-h_0(p))}} \right) . \]

The estimate (2.13.16) implies the bound

\[ (h_q(p) - h_0(p))^2 \leq D^2(\beta q^2)^2 (1 + \beta(p + |q|)^2)^2 . \]  \hspace{2cm} (2.13.18)

Moreover, using the upper bound in (2.13.16) to estimate \( h_q(p) - h_0(p) \) from above, we obtain

\[ e^{-h_q(p)} + e^{-h_0(p)+t\beta q^2} \]  \hspace{2cm} \[ \frac{1}{(1 - e^{-h_0(p)+t\beta q^2})(1 - e^{-h_q(p)})} \]

\[ = \omega^f(p) + \frac{1}{2}(\omega_\pi(p + q) + \omega_\pi(p - q)) (1 + 2\omega^f(p)) \]  \hspace{2cm} (2.13.19)

as an upper bound to the last fraction in (2.13.17). Here, we denoted \( \omega^f(p) = [e^{h_0(p) - D\beta q^2} - 1]^{-1} \), assuming \( t \) to be small enough such that \( h_0(p) - D\beta q^2 \geq 0 \) for all \( p \). Recall that \( h_0(p) = \ell(p) \) is given in (2.13.15).

In the thermodynamic limit, the sum over \( p \) in (2.13.5) converges to the corresponding integral and, hence, we are left with bounding

\[ \int_{\mathbb{R}^3} dp \ (1 + \beta(|p| + |q|)^2)^2 \omega^f(p) \left( \frac{1}{2}(\omega_\pi(p + q) + \omega_\pi(p - q)) (1 + 2\omega^f(p)) \right) \]  \hspace{2cm} (2.13.20)

from above. We can replace \( \omega_\pi(p - q) \) by \( \omega_\pi(p + q) \) in (2.13.20) without changing the value of the integral. Using Schwarz’s inequality, the fact that \( \omega_\pi(p) \leq \omega^f(p) \), and changing variables \( p \rightarrow p - q \), we see that (2.13.20) is bounded from above by

\[ 2 \int_{\mathbb{R}^3} dp \ (1 + \beta(|p| + 2|q|)^2)^2 \omega^f(p)(1 + 2\omega^f(p)) \]  \hspace{2cm} (2.13.21)
It remains to bound \( \omega^t(p) \). For this purpose, we need a bound on \( \ell(p) - D\beta tq^2 \) for an appropriate choice of the parameter \( t \). We will choose \( t = \min\{1, (b^2q^2)^{-1}\} \). We then have \( tq^2 \leq b^{-2} \), and it is easy to see that

\[
\ell(p) - D\beta tq^2 \geq \beta \left[ \frac{1}{2}p^2 - \mu_0 + p_c^2 \left( \frac{1}{8} - \frac{D}{b^2p_c^2} \right) \right]
\]

in this case. Since \( \ell(p) \geq \beta(p^2 - \mu_0) \), this can be seen immediately in the case \( |p| \geq \frac{1}{2}p_c \). For \( |p| \leq p_c/2 \) even a slightly better bound holds, this time using the fact that \( g(p) \geq 1/2 \) for \( |p| \leq 1/2 \).

We will choose \( b \) and \( p_c \) below in such a way that \( bp_c \gg 1 \). Denoting by \( \tau \) the (positive) number

\[
\tau = -\beta \mu_0 + \beta p_c^2 \left( \frac{1}{8} - \frac{D}{b^2p_c^2} \right),
\]

we thus have the bound

\[
\omega^t(p) \leq \left[ e^{-\tau} e^{\frac{1}{2}b \beta p^2 - 1} \right]^{-1} = e^{-\tau} e^{-\frac{1}{2}b \beta p^2} \left( 1 + \frac{1}{\tau + \frac{1}{2}b \beta p^2} \right).
\]

The last bound follows from the elementary inequality \( (e^x - 1)^{-1} \leq e^{-x} (1 + 1/x) \) for all \( x > 0 \). We insert this bound for \( \omega^t \) into (2.13.21). Simple estimates then yield

\[
(2.13.21) \leq \text{const.} \frac{e^{-\tau}}{b^{\beta/2}} \left( 1 + \tau^{-1/2} \right) \left( 1 + (\beta q^2)^2 \right). \tag{2.13.25}
\]

Combining (2.13.5), (2.13.17), (2.13.18) and (2.13.25), and using that \( t^{-1} \leq 1 + b^2q^2 \), we have thus shown that

\[
S(\Pi \otimes \Gamma_{\vec{z}}, \Omega_{\vec{q}}) \leq \left( 2 + b^2q^2 \right) S(\Pi \otimes \Gamma_{\vec{z}}, \Omega_{\pi})
+ \text{const.} \, |\Lambda| \beta^{1/2} q^4 \tau^{-1/2} \left( 1 + (\beta q^2)^2 \right) + o(|\Lambda|). \tag{2.13.26}
\]

After inserting this estimate in (2.13.4) and summing over \( q \), this yields the bound

\[
S(\Upsilon_{\pi'} \otimes \Omega_{\vec{q}'}, \Omega_{\vec{q}}) \leq \text{const.} \, |\Lambda| \left( \frac{\alpha b q^2 + \beta^{1/2} \tau^{-1/2}}{b^4} \right) + o(|\Lambda|). \tag{2.13.27}
\]

Here, we have used the assumed smoothness of \( \eta \) to estimate \( \sum_q \hat{\eta}_q(q)|q|^n \leq \text{const.} \, |\Lambda| b^{-n} \) for integers \( n \leq 8 \). We have also used the fact that we will
choose $b^2 \gg \beta$ and hence, in particular, $\beta b^{-2} \leq \text{const.}$ Moreover, the assumption (2.4.6) has been used to bound $S(\Pi \otimes \Gamma_{\bar{z}}, \Omega_\mu) = S(\Gamma_{\bar{z}}, \Gamma^0)$.

We have thus shown that the effect of the cutoff $b$ on the relative entropy can be estimated by $|\Lambda|/(\beta/\tau)^{1/2}b^{-4}$. We note that the exponent $-4$ of $b$ is important, since the relative entropy has to be multiplied by $b^3$ in the estimate (2.11.19).

2.14 Final Lower Bound on $F_{\bar{z}}(\beta)$

We have now obtained all the necessary estimates to complete our lower bound on $F_{\bar{z}}(\beta)$. For this purpose, we put all the bounds from Subsections 2.7, 2.11, 2.12 and 2.13 together. In fact, from (2.7.15), (2.11.19), (2.12.14) and (2.13.27), we have the following lower bound on $F_{\bar{z}}(\beta)$:

$$F_{\bar{z}}(\beta) \geq -\frac{1}{\beta} \ln \text{Tr}_{F_{\bar{z}}} \exp \left( -\beta T^c_s(\bar{z}) \right) - Z^{(2)} - Z^{(3)} - Z^{(4)}$$

$$= - (\varepsilon - \varepsilon') \sum_{|p| < p_c} p^2 \pi_p - o(|\Lambda|)$$

$$+ \frac{2\pi a C}{|\Lambda|} (|\bar{z}|^2 + \text{Tr}_{\nabla}[\mathcal{N}^> - \mathcal{N}])^2$$

$$+ 4\pi a |\Lambda| \min \{ 2\varepsilon^2, \varepsilon^2 + 2\varepsilon(\gamma_b + \omega) + \varepsilon^2 + \gamma^2_b \} , \quad (2.14.1)$$

where we denoted

$$Z^{(4)} = \text{const.} |\Lambda| \left[ \varepsilon^2 \left( \varepsilon + \frac{R}{s} + Rp_c + (R^3 \varepsilon)^{1/3} + \left( \frac{R_0}{R} \right)^3 \right) \right.$$

$$+ \frac{\varepsilon}{R^2 s} \int_{b/s}^{\infty} dr \, r^6 |m(r)| + \frac{1}{R^6} \left( b^2 \varepsilon a \frac{\varepsilon^2}{\beta} + b \right) \left( \frac{\beta^{1/2}s^{-1/2}}{b} \right)^{1/2} \left( \varepsilon^2 + \gamma^2_b \right) \right] . \quad (2.14.2)$$

Here, we have used the definition (2.7.7) of $a'$, (2.4.10) to bound $\varepsilon$ in the error terms, together with $\gamma_b \leq \omega$ and $|\Lambda| \to \infty \omega \leq \varepsilon$. This last estimate follows from $\ell(p) \geq \beta(p^2 - \mu_0)$ and (1.6). The error terms $Z^{(2)}$ and $Z^{(3)}$ are defined in (2.5.6) and (2.12.15), respectively.

Using (2.7.8), the term in the second line of (2.14.1) can be estimated by $(\varepsilon - \varepsilon') \sum_{|p| < p_c} p^2 \pi_p \leq (\varepsilon - \varepsilon') p^2 \pi p \leq \text{const.} \left( \frac{a}{R} \right)^3 \rho^2 |\Lambda|/\beta$. Here, we have also used the bound on $P = \sum_{|p| < p_c} p_\pi$ derived after Eq. (2.13.15).

The two terms in the third and fourth line of (2.14.1) can be bounded from below independently of $\bar{z}$, simply using Schwarz’s inequality. More
precisely, introducing $\varrho^0 \equiv |\Lambda|^{-1} \text{Tr}_{F>}[N^>,\Gamma^0] = \varrho_\omega - P/|\Lambda|$, we obtain

$$
\frac{2\pi \tilde{\alpha} C}{|\Lambda|} ((\tilde{\varrho}^2 + \text{Tr}_{F>} [N^>,\Gamma^0] - N)^2 + 4\pi \tilde{\alpha}|\Lambda| (\varrho_\omega^2 + 2\varrho_\omega \gamma_b + \varrho_\omega + \gamma_b^2) \\
\geq \frac{4\pi \tilde{\alpha}|\Lambda|}{1 + 2/C} ((\varrho - \varrho^0)^2 + 2(\varrho - \varrho^0)(\varrho_\omega + \gamma_b) + \varrho_\omega^2 + \gamma_b^2 - \frac{2}{C}(\varrho_\omega + \gamma_b)^2).
$$

(2.14.3)

We note that

$$
\varrho^0 = \frac{1}{(2\pi)^3} \int_{|p| \geq p_c} dp \frac{1}{e^{\beta(p^2 - \mu_0)} - 1} + o(1)
$$

(2.14.4)

in the thermodynamic limit. Hence, from (1.6),

$$
\varrho^0 = \min \{ \varrho, \varrho_c(T) \} - \frac{1}{(2\pi)^3} \int_{|p| \leq p_c} dp \frac{1}{e^{\beta(p^2 - \mu_0)} - 1} + o(1)
\\
\geq \min \{ \varrho, \varrho_c(T) \} - \frac{1}{2\pi^2} \frac{p_c}{\beta} + o(1).
$$

(2.14.5)

This estimate is obtained by using $e^{\beta(p^2 - \mu_0)} - 1 \geq \beta p^2$ in the denominator of the integrand. Moreover, $\varrho^0 \leq \varrho_\omega \leq \min \{ \varrho, \varrho_c(T) \} + o(1)$.

It remains to give a lower bound on $\gamma_b$. According to (2.11.16) and (2.8.2),

$$
\gamma_b = \frac{1}{4\pi R^3} \int_{\Lambda} dx \omega_\pi(x) \eta(d(x,0)/b) j(d(x,0)/R).
$$

(2.14.6)

We note that, since $\omega_\pi(x)$ is real,

$$
\omega_\pi(x) - \varrho_\omega = \frac{1}{|\Lambda|} \sum_p \frac{1}{e^{\ell(p)} - 1} (\cos(px) - 1) \geq - \frac{d(x,0)^2}{2|\Lambda|} \sum_p \frac{p^2}{e^{\ell(p)} - 1}.
$$

(2.14.7)

We can estimate $d(x,0) \leq R$ in the integrand in (2.14.6). Since $\ell(p) \geq \beta |p|^2$, $|\eta| \leq 1$ and $\int_{\Lambda} dx j(d(x,0)/R) = 4\pi R^3$, the contribution of the last term to (2.14.6) is bounded by

$$
\frac{R^2}{2} \frac{1}{(2\pi)^3 \beta^{5/2}} \int_{R^3} dp \frac{p^2}{e^{p^2} - 1} + o(1)
$$

in the thermodynamic limit. Moreover, we can bound $\eta$ from below as $\eta(t) \geq 1 - \text{const.} t^2$, and hence

$$
\varrho_\omega \geq \gamma_b \geq \varrho_\omega \left(1 - \text{const.} \frac{R^2}{b^2}\right) - \text{const.} R^2 \beta^{-5/2} - o(1).
$$

(2.14.9)
Using the bounds on $\varrho^0$ and $\gamma_b$ just derived, we have

\begin{equation}
(2.14.3) \geq 4\pi |\Lambda| \left( 2\varrho^2 - [\varrho - \varrho_c(T)_+^2] \right) - \text{const.} |\Lambda| \left( \varrho^2 \left( \frac{1}{C} + \frac{R^2}{b^2} \right) + \varrho \left( \frac{p_c}{\beta} + \frac{R^2}{\beta^5/2} \right) \right) - o(|\Lambda|).
\end{equation}

In particular, the terms in the third and fourth line of (2.14.1) are bounded from below by the right side of (2.14.10).

### 2.15 The “Free” Free Energy

We now insert the lower bound on $F_\tilde{z}(\beta)$ derived in the previous subsection into (2.3.9). We note that the only $\tilde{z}$-dependence left is in the first term in (2.14.1), which is the “free” part of the free energy. Taking also the constant $\mu_0 N$ in (2.3.9) into account, we are thus left with evaluating

\begin{equation}
\mu_0 N - \frac{1}{\beta} \ln \int_{C_M} d^Mz \operatorname{Tr}_{\mathcal{F}_>} \exp \left( - \beta T_\Lambda(\tilde{z}) \right)
= \mu_0 N + \frac{1}{\beta} \left( \sum_{|p| < p_c} \ln (\beta \varepsilon(p)) + \sum_{|p| \geq p_c} \ln \left( 1 - e^{-\beta \varepsilon(p)} \right) \right). \quad (2.15.1)
\end{equation}

Using $x \geq (1 - e^{-x})$ for non-negative $x$, (2.15.1) becomes, in the thermodynamic limit,

\begin{equation}
(2.15.1) \geq N \mu_0 + \frac{|\Lambda|}{\beta (2\pi)^3} \int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta \varepsilon(p)} \right) - o(|\Lambda|). \quad (2.15.2)
\end{equation}

Recall that $\varepsilon(p)$ is defined in (2.7.10). It is given by $\varepsilon(p) = (1 - \kappa + \kappa')p^2 - \mu_0$ for $|p| \leq 1/s$, and satisfies the bound $\varepsilon(p) \geq \kappa'p^2$ for $|p| \geq 1/s$. Hence

\begin{equation}
\int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta \varepsilon(p)} \right) \geq (1 - \kappa + \kappa')^{-3/2} \int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta (p^2 - \mu_0)} \right) + \frac{1}{(\kappa' \beta)^{3/2}} \int_{|p| \geq \kappa' \beta / s} dp \ln \left( 1 - e^{-p^2} \right). \quad (2.15.3)
\end{equation}

We will choose $s^2 \ll \kappa' \beta$ below, in which case the last integral is exponentially small in the parameter $s^2 / (\kappa' \beta)$. Inserting the definition (1.4) of $f_0(\beta, \varrho)$, we have thus shown that

\begin{equation}
(2.15.1) \geq |\Lambda|(1 - \kappa + \kappa')^{-3/2} f_0(\beta, \varrho) \quad (2.15.4)
+ \frac{|\Lambda|}{\beta^{5/2} \kappa'^{3/2} (2\pi)^3} \int_{|p| \geq \kappa' \beta / s} dp \ln \left( 1 - e^{-p^2} \right) - o(|\Lambda|).
\end{equation}
Here, we have also used that $\mu_0 \leq 0$.

### 2.16 Choice of Parameters

We have now essentially finished our lower bound on $f(\beta, \varrho)$. It remains to collect all the error terms, and choose the various parameters in an appropriate way. All the error terms we have to take into account are given in (2.1.1), (2.3.9), (2.14.1), (2.14.10) and (2.15.4).

We will choose the various parameters in our estimates as follows:

$$
R = a\varrho^{-1/3}(a\varrho^2\beta^{5/2})^{3/403}, \quad b = \beta^{1/2}(a\varrho^2\beta^{5/2})^{-121/403},
$$

$$
s = (\beta\varrho^{-1/3})^{1/3}(a\varrho^2\beta^{5/2})^{1/403}.
$$

(2.16.1)

Moreover,

$$
\varphi = a(a\varrho^2\beta^{5/2})^{-A}, \quad C = (a\varrho^2\beta^{5/2})^{-B}
$$

for $4/403 \leq A \leq 79/403$ and $2/403 \leq B \leq 161/403$. Depending on $\mu_0$, we choose

$$
p_c = \begin{cases} 
\beta^{-1/2}(a\varrho^2\beta^{5/2})^{81/403} & \text{if } \beta |\mu_0| \leq (a\varrho^2\beta^{5/2})^{162/403} \\
0 & \text{otherwise}.
\end{cases}
$$

(2.16.2)

Finally, we choose $\kappa = s^2\beta^{-1}(a\varrho^2\beta^{5/2})^{-\delta}$ for some $\delta > 0$. Our estimates then imply that

$$
f(\beta, \varrho) \geq f_0(\beta, \varrho) + 4\pi a\left(2\varrho^2 - [\varrho - \varrho_c(\beta)]^2\right)(1 - o(1)),
$$

(2.16.3)

with

$$
o(1) \leq C_\delta((\beta\varrho^2)^{-1})(a\varrho^2\beta^{5/2})^{2/403-\delta}
$$

(2.16.4)

for some function $C_\delta$, depending on $\delta$, that is uniformly bounded on bounded intervals.

The choice of the parameters $p_c$, $b$, $s$, $R$ and $\kappa$ is determined by minimizing the sum of all the error terms. The main terms to consider are, in fact, the terms $M\tilde{p}_c^2 \sim |\Lambda|p_c^2$ in $Z^{(1)}$ in (2.3.7), and $|\Lambda|\tilde{a}^2(\kappa + R/s)$ as well as $|\Lambda|\tilde{a}R^{-6}(b^2a\beta\varrho^2 + (p^2 - \mu_0)^{-1/2}b^{-1})^{1/2}$ in $Z^{(4)}$ in (2.14.2). Moreover, we have to take the restriction $s^2 \ll \kappa\beta$ in (2.15.3) into account. This leads to the choice of parameters above.

All other error terms are of lower order for small $a\varrho^2\beta^{5/2}$. This is true, in particular, for all the terms containing $\varphi$ and $C$, which explains the freedom in their choice above.
2.17 Uniformity in the Temperature

Our final result, Eq. (2.16.4), does not have the desired uniformity in the temperature. It is only useful in case the dimensionless parameter \( a \beta^{5/2} \) is small. In particular, one can not take the zero-temperature limit \( \beta \to \infty \).

The reason for this restriction is that our argument was essentially perturbative, using that the correction term we want to prove is small compared to the main term, i.e., \( a \ll f_0(\beta, \varrho) \). Below the critical temperature, \( f_0(\beta, \varrho) = \text{const.} \beta^{-5/2} \), hence the assumption is only satisfied if \( a \beta^{5/2} \ll 1 \).

If the temperature is smaller, we can use a different argument for a lower bound on \( f \), which uses in an essential way the result in [11]. There, a lower bound in the zero temperature case was derived.

To obtain the desired bound for very low temperature, it is possible to skip steps 1–5 entirely, and start immediately with the Dyson lemma, Lemma 2, applied to the original potential \( v \). Using this lemma, we have that

\[
H_N \geq \sum_{j=1}^{N} \left[ -\nabla_j (1 - (1 - \kappa) \chi(p_j)^2) \nabla_j + (1 - \varepsilon)(1 - \kappa) a U_R (d(x_j, x^J(x_j))) - \frac{a}{\varepsilon} \sum_{i \in J_j} w_R (x_j - x_i) \right].
\]

Since \( d(x_i, x_k) \geq R/5 \) for \( i, k \in J_j \), we can estimate (using (2.10.1))

\[
\sum_{j=1}^{N} a \varepsilon \sum_{i \in J_j} w_R (x_j - x_i) \leq \text{const.} \frac{a N \varepsilon R}{\varepsilon R s^2}.
\]

Moreover, the calculation in [11] shows that, for the choice \( \kappa = (a^3 \varrho)^{1/17} \) and \( R = a(a^3 \varrho)^{-5/17} \),

\[
\sum_{j=1}^{N} \frac{\kappa}{2} \Delta_j + (1 - \varepsilon)(1 - \kappa) a U_R (d(x_j, x^J(x_j))) \geq 4\pi a N \varrho \left( 1 - \varepsilon - \text{const.} (a^3 \varrho)^{1/17} \right).
\]

(Strictly speaking, this result was derived in [11] for Neumann boundary conditions, and with \( J_j = \{1, \ldots, N\} \) independent of all the particle coordinates. It is easy to see, however, that the same result applies to our
Hamiltonian, being defined with periodic boundary conditions, and having a slightly smaller interaction. 

For this choice of \( \kappa \) and \( R \), we thus have

\[
H_N \geq \sum_{j=1}^{N} l(\sqrt{-\Delta j}) + 4\pi a N \varrho \left( 1 - \varepsilon - \text{const.} \left( a^3 \varrho \right)^{1/17} - \text{const.} \frac{1}{\varepsilon R s^2 \varrho} \right),
\]

(2.17.4)

with \( l(|p|) = p^2(1 - \chi/2 - (1 - \chi) \chi(p)^2) \). To obtain a lower bound on the free energy for this Hamiltonian, we can go to grand-canonical ensemble, introducing a chemical potential in the usual way. Taking the thermodynamic limit, this yields

\[
f(\beta, \varrho) \geq \sup_{\mu \leq 0} \left\{ \mu \varrho + \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta(l(p) - \mu)} \right) \right. \\
+ 4\pi a \varrho^2 \left( 1 - \varepsilon - \text{const.} \left( a^3 \varrho \right)^{1/17} \right) - \text{const.} \frac{\varrho}{\varepsilon s^2} \left( a^3 \varrho \right)^{5/17}.
\]

(2.17.5)

Recall that \( \chi(p) = \nu(sp) \), where \( \nu \) is a function with \( 0 \leq \nu(p) \leq 1 \) that is supported outside the ball of radius 1. This implies that \( l(p) = (1 - \chi/2)p^2 \) for \( |p| \leq 1/s \), and \( l(p) \geq (1 - \chi)p^2 \) for \( |p| \geq 1/s \). Hence

\[
\int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta(l(p) - \mu)} \right) \geq (1 - \chi/2)^{-3/2} \int_{\mathbb{R}^3} dp \ln \left( 1 - e^{-\beta(p^2 - \mu)} \right) \\
+ \frac{1}{(\chi \beta)^{3/2}} \int_{|p|^2 \geq \chi \beta/s^2} dp \ln \left( 1 - e^{-\beta p^2} \right).
\]

(2.17.6)

The last expression is exponentially small in the (small) parameter \( s^2/(\chi \beta) \). We choose, for some \( \delta > 0 \),

\[
s^2 = \beta \left( a^3 \varrho \right)^{1/17 + \delta} , \quad \varepsilon^2 = \left( \frac{a^3 \varrho}{a^2 \beta^{3/2}} \right)^{3/85} \left( \frac{a^2 \beta^{3/2}}{a \varrho^2 \beta^{5/2}} \right)^{2/5},
\]

(2.17.7)

and obtain

\[
f(\beta, \varrho) \geq f_0(\beta, \varrho) + 4\pi a \varrho^2 \left( 1 - o(1) \right),
\]

(2.17.8)

with

\[
o(1) = \text{const.} \left( a^3 \varrho \right)^{1/17} \left( 1 + \frac{1}{a^2 \varrho^2 \beta^{5/2}} \right) + \left( \frac{a^3 \varrho}{a^2 \beta^{3/2}} \right)^{3/2} \left( \frac{a^2 \beta^{3/2}}{a \varrho^2 \beta^{5/2}} \right)^{1/5}.
\]

(2.17.9)
Compared with our desired lower bound, we also have to take into account
the missing term \( a(G^2 - [\theta - \theta_c(\beta)]_+^2) \), which can be bounded as
\[
a(G^2 - [\theta - \theta_c(\beta)]_+^2) \leq \text{const.} \, a \theta^3 \beta^{-3/2}.
\] (2.17.10)

In combination, the estimates (2.16.4) and (2.17.8) provide the desired
uniform lower bound on the free energy. Depending on the value of \( a \theta^2 \beta^{5/2} \),
one can apply either one of them. To minimize the error, one has to apply
(2.16.4) for \( a \theta^2 \beta^{5/2} \leq (a^3 \theta)^{403/6885} \), and (2.17.8) otherwise. This yields our
main result, Theorem 1, for \( \alpha = 2/2295 - \delta \).

### A Appendix: Proof of Lemma 2

For simplicity, we drop the \( \tilde{\cdot} \) on \( v \) and \( a \) in our notation in this appendix.

We start by dividing up \( \Lambda \) into Voronoi cells
\[
B_j = \{ x \in \Lambda : d(x, y_j) \leq d(x, y_k) \forall k \neq j \}.
\] (A.1)

For a given \( \psi \in H^1(\Lambda) \), let \( \xi \) be the function with Fourier coefficients \( \hat{\xi}(p) = \chi(p) \hat{\psi}(p) \). We thus have to show that
\[
\int_{B_j} dx \left[ |\nabla \xi(x)|^2 + \frac{1}{2} v(d(x, y_j))|\psi(x)|^2 \right] \geq (1 - \varepsilon) a \int_{B_j} dx U(d(x, y_j))|\psi(x)|^2
\[ - \frac{a}{\varepsilon} \int_{\Lambda} dx w_R(x - y_j)|\psi(x)|^2.
\] (A.2)

The statement of the lemma then follows immediately by summing over \( j \)
and using the positivity of \( v \).

We will actually show that (A.2) holds even when the integration region
\( B_j \) on the left side of the inequality is replaced by the smaller set \( B_R \equiv B_j \cap \{ x \in \Lambda : d(x, y_j) \leq R \} \). Note that the first integral on the right side
of (A.2) is also over this region, since the range of \( U \) is supposed to be less
than \( R \).

As in Subsection 1.3, let \( \phi_v \) denote the solution to the zero-energy scat-
ering equation
\[
-\Delta \phi_v(x) + \frac{1}{2} v(|x|) \phi_v(x) = 0
\] (A.3)
subject to the boundary condition \( \lim_{|x| \to \infty} \phi_v(x) = 1 \). Let \( \nu \) be a complex-valued function on the unit sphere \( S^2 \), with \( \int_{S^2} |\nu|^2 = 1 \). We use the same
symbol for the function on \( \mathbb{R}^3 \) taking values \( \nu(x/|x|) \). For \( \psi \) and \( \xi \) as above, consider the expression

\[
A \equiv \int_{B_R} dx \nu(x - y_j) \nabla \xi^*(x) \cdot \nabla \phi_v(x - y_j)
+ \frac{1}{2} \int_{B_R} v(d(x, y_j)) \psi(x) \phi_v(x - y_j) \nu(x - y_j).
\] (A.4)

By using the Cauchy-Schwarz inequality, we can obtain the upper bound

\[
|A|^2 \leq \int_{B_R} dx \left[ |\nabla \xi(x)|^2 + \frac{1}{2} v(d(x, y_j)) |\psi(x)|^2 \right]
\times \int_{B_R} dx \left[ |\nabla \phi_v(x - y_j)|^2 + \frac{1}{2} v(d(x, y_j)) |\phi_v(x - y_j)|^2 \right] |\nu(x - y_j)|^2.
\] (A.5)

For an upper bound, we can replace the integration region \( B_R \) in the second integral by \( \mathbb{R}^3 \). Since \( \phi_v(x) \) is a radial function, the angular integration in then can be performed by using \( \int_{S^2} |\nu|^2 = 1 \). The remaining expression is then bounded by \( a \) because of \( \int_{\mathbb{R}^3} dx \left[ |\nabla \phi_v(x)|^2 + \frac{1}{2} v(|x|) |\phi_v(x)|^2 \right] = 4\pi a \).

Hence we arrive at

\[
\int_{B_R} dx \left[ |\nabla \xi(x)|^2 + \frac{1}{2} v(d(x, y_j)) |\psi(x)|^2 \right] \geq \frac{|A|^2}{a}
\] (A.6)
for any choice of \( \nu \) as above. It remains to derive a lower bound on \( |A|^2 \).

By partial integration,

\[
\int_{B_R} dx \nu(x - y_j) \nabla \xi^*(x) \cdot \nabla \phi_v(x - y_j)
= - \int_{B_R} dx \xi^*(x) \nu(x - y_j) \Delta \phi_v(x - y_j)
+ \int_{\partial B_R} d\omega_R \xi^*(x) \nu(x - y_j) n \cdot \nabla \phi_v(x - y_j),
\] (A.7)

where \( d\omega_R \) denotes the surface measure of the boundary of \( B_R \), denoted by \( \partial B_R \), and \( n \) is the outward normal unit vector. Here we used the fact that \( \nabla \nu(x) \cdot \nabla \phi_v(x) = 0 \). Now, by definition of \( h(x) \), \( \xi(x) = \psi(x) - (2\pi)^{-3/2} h * \psi(x) \), where \( * \) denotes convolution, i.e., \( h * \psi(x) = \int_{\Lambda} dy h(x - y) \psi(y) \). Using
the zero-energy scattering equation (A.3) for $\phi_v$, we thus see that

$$A = \int_{\partial B_R} d\omega_R \psi^*(x) \nu(x - y_j) n \cdot \nabla \phi_v(x - y_j)$$

$$- (2\pi)^{-3/2} \int_{\partial B_R} d\omega_R (h * \psi)^*(x) \nu(x - y_j) n \cdot \nabla \phi_v(x - y_j)$$

$$+ (2\pi)^{-3/2} \int_{B_R} dx (h * \psi)^*(x) \nu(x - y_j) \Delta \phi_v(x - y_j). \quad (A.8)$$

The last two terms on the right side of (A.8) can be written as

$$\frac{1}{(2\pi)^{-3/2}} \int_{\Lambda} dx \psi^*(x) \left( \int_{B_R} d\mu(y) h(y - x) \right), \quad (A.9)$$

where $d\mu$ is a (non-positive) measure supported in $B_R$. Explicitly, $d\mu(x) = \nu(x - y_j) \Delta \phi_v(x - y_j) dx - n \cdot \nabla \phi_v(x - y_j) d\omega_R$, the second part being supported on the boundary $\partial B_R$. Note that $\int_{B_R} d\mu = 0$, and also $\int_{B_R} d|\mu| \leq 2a \int_{S^2} |\nu| \leq 2a \sqrt{4\pi}$ (by Schwarz’s inequality). Hence

$$\left| \int_{B_R} d\mu(y) h(y - x) \right| \leq 2a \sqrt{4\pi} f_R(x - y_j), \quad (A.10)$$

with $f_R$ defined in (2.6.2). The expression (A.9) is thus bounded from below by

$$(A.9) \geq -\frac{1}{(2\pi)^{-3/2}} 2a \sqrt{4\pi} \int_{\Lambda} dx |\psi(x)| f_R(x - y_j)$$

$$\geq -a \left( \int_{\Lambda} dx |\psi(x)|^2 w_R(x - y_j) \right)^{1/2}. \quad (A.11)$$

Here, we used Schwarz’s inequality as well as the definition of $w_R$ (2.6.3) in the last step. Note that this last expression is independent of $\nu$.

The only place where $\nu$ still enters is the first term on the right side of (A.8). By construction, $\nu$ depends only on the direction of the line originating from $y_j$, which hits the boundary of $B_R$ at a distance not greater than $R$. We distinguish two cases. First, assume that the line hits the boundary at a distance $R$. In this case, we choose $\nu$ to be equal to the value of $\psi$ at this boundary point. Secondly, if the length of the line is strictly less than $R$, we then choose $\nu$ to be zero. Of course we also have to normalize $\nu$ appropriately. The integrals are thus only over the part of the boundary
of $\mathcal{B}_j$ which is at a distance $R$ from $y_j$. Let us denote this part of $\partial \mathcal{B}_R$ by $\tilde{\partial} \mathcal{B}_R$, assuming for the moment that it is not empty. We then have

$$\int_{\partial \mathcal{B}_R} d\omega_R \psi^*(x) \nu(x-y_j)n \cdot \nabla \phi_v(x-y_j) = R \frac{\int_{\tilde{\partial} \mathcal{B}_R} d\omega_R |\psi(x)|^2 n \cdot \nabla \phi_v(x-y_j)}{\left(\int_{\tilde{\partial} \mathcal{B}_R} d\omega_R |\psi(x)|^2\right)^{1/2}}. \quad (A.12)$$

We note that $n \cdot \nabla \phi_v(x-y_j) = a/R^2$ on $\tilde{\partial} \mathcal{B}_R$. We thus obtain from (A.8)–(A.12)

$$A \geq \frac{a}{R} \left(\int_{\partial \mathcal{B}_R} d\omega_R |\psi(x)|^2\right)^{1/2} - a \left(\int_{\Lambda} dx |\psi(x)|^2 w_R(x-y_j)\right)^{1/2}. \quad (A.13)$$

With the aid of Schwarz’s inequality, we see that, for any $\varepsilon > 0$,

$$|A|^2 \geq \frac{a^2}{R^2} \left(1 - \varepsilon\right) \int_{\partial \mathcal{B}_R} d\omega_R |\psi(x)|^2 - \frac{a^2}{\varepsilon} \int_{\Lambda} dx |\psi(x)|^2 w_R(x-y_j). \quad (A.14)$$

At this point we can also relax the condition that $\tilde{\partial} \mathcal{B}_R$ be non-empty; in case it is empty, (A.14) holds trivially.

In combination with (A.6), (A.14) proves the desired result (A.2) in the special case when $U(|x|)$ is a radial $\delta$-function sitting at a radius $R$, i.e., $U(|x|) = R^{-2} \delta(|x|-R)$. The case of a general potential $U(|x|)$ follows simply by integrating this result (i.e., Ineq. (A.2) for this special $U(|x|)$) against $U(R)R^2 dR$, noting that $\int dR U(R) R^2 \leq 1$ and that $w_R(x)$ is pointwise monotone increasing in $R$.

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