Neutrino oscillations in medium with periodic square potential

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Abstract

We have investigated two flavor neutrino oscillations in medium with periodic step electron number density profile. An approximate analytical solution have been found when the length of the density fluctuation is smaller then the neutrino oscillation length.

Introduction

Neutrino oscillation mechanism gives an explanation for decreasing flux intensity of electronic neutrinos coming from the Sun. The problem was predicted in [1] and was solved via averaged vacuum oscillations with large mixing angles in [2]. In [3], [4], [5] solution for neutrino oscillations in matter with constant electron number density were obtained. An adiabatic neutrino oscillations were investigated in [6], [7]. An exact analytic expression for the probability amplitudes describing the oscillations of two (flavor) neutrinos in matter with exponentially varying density was obtained in [8]. This theoretical picture of such potential is in good agreement with the standard solar model according to which the matter density in the sun decreases exponentially with the distance from the center of the sun, except for the regions located close to the center and to the surface. But such a picture doesn’t take into account the unpredictable variations of electronic gas density. So it is naturally interesting to studying the potentials with variate density profile. A potentials with periodically variate profile were studied independently in [9], [10] in connection with a possibility of the parametric resonance in two neutrino system. In [9] an approximate solution for sinusoidal matter density profile was found. In [10] the exact analytic solution for the periodic step-function density profile was obtained. In [11] an analytic approximate solution for arbitrary electron density for small matter effects on the neutrino oscillations was wound.

In the present paper we have constructed toy model of electron density fluctuations. An approximate solution for two flavor neutrino oscillations...
probabilities was found in periodic square potential profile with suggestion
that the length of each square is smaller than the oscillation length.

**Neutrino oscillation in the periodic square potential**

In this section, we consider the mixing of two (active) neutrinos. \( \nu_f = U \nu_{\text{mass}} \), where \( \nu_f = (\nu_e, \nu_\alpha)^T \) are flavor states, \( \nu_\alpha \) is a linear combination of \( \nu_\mu \) and \( \nu_\tau \), \( \nu_{\text{mass}} = (\nu_1, \nu_2)^T \) are mass states with masses \( m_1 \) and \( m_2 \) respectively.

\[
U = \begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\]  

(1)

is a lepton mixing matrix in weak charged interactions. The equation for neutrino oscillations in matter is given by Schrodinger equation

\[
-\frac{i}{\hbar} \frac{d}{dx} \begin{pmatrix}
\nu_e \\
\nu_\alpha
\end{pmatrix} = \left[U \begin{pmatrix}
-\frac{\Delta m^2}{2E} & 0 \\
0 & 0
\end{pmatrix} U^T + \begin{pmatrix}
V & 0 \\
0 & 0
\end{pmatrix}\right] \begin{pmatrix}
\nu_e \\
\nu_\alpha
\end{pmatrix},
\]

(2)

Here \( V = \sqrt{2} G_F N_e \), \( N_e \) is electron number density, \( G_F \) is Fermi constant, \( E \) is the energy of a neutrino, \( \Delta m^2 = m_2^2 - m_1^2 \), \( x \) - the length of the neutrino beam path.

We will assume that the \( V \) is of the form periodic rectangularly shaped potentials, as shown in Fig.1.

![Figure 1: The dependence of the electronic potential \( V \) on distance \( L \). \( V_0 \) - is the averaged potential, \( \Delta V \) is the amplitude of the additional rectangularly shaped potential and \( \Delta x \) is the half-period of the potential.](image)

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The formal solution of eq. (2), that is the oscillation matrix for the flavor states from initial point $x_0$ to the final point $x_f$, can be written as

$$S_{x_0 \to x_f} = T e^{\int_{x_0}^{x_f} H(x) dx},$$

where $T$ means the chronological ordering.

Thus in order to get the oscillation matrix it is necessary to multiply consistently solutions on each period. Thus we get the oscillation matrix in the form of

$$S = (e^{iC})^n = e^{inC},$$

where $C$ is a hermitian matrix

$$C = \frac{1}{i} \ln \left( e^{iH^+ \Delta x} e^{iH^- \Delta x} \right).$$

Here

$$H^\pm = U \left( \begin{array}{cc} -\frac{\Delta m^2}{2E} & 0 \\ 0 & 0 \end{array} \right) U^T + \left( \begin{array}{cc} V_0 \pm \Delta V & 0 \\ 0 & 0 \end{array} \right)$$

and $\Delta x = \frac{x_f - x_0}{2n}$

is the half period of the potential, $n$ is the number of periods.

The $S$ matrix can be rewritten in the following form

$$S = e^{inC} = e^{inO^\dagger C^{\text{diag}} O} = O^\dagger e^{inC^{\text{diag}}} O = O^\dagger \begin{pmatrix} e^{inC_{11}^{\text{diag}}} & 0 \\ 0 & e^{inC_{22}^{\text{diag}}} \end{pmatrix} O,$$

where $C^{\text{diag}}$ is diagonalized $C$ matrix, $C_{11}^{\text{diag}}$ and $C_{22}^{\text{diag}}$ are its diagonal elements, $O$ is the matrix that diagonalize matrix $C$.

$$O = \begin{pmatrix} e^{-i\gamma} & 0 \\ 0 & e^{i\gamma} \end{pmatrix} \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}.$$

The phase $\gamma$ is unphysical and does not enter into the transition probabilities.

Solving these equation we find that

$$\sin^2 2\varphi = \frac{4 |C_{12}|^2}{(C_{11} - C_{22})^2 + 4 |C_{12}|^2},$$

$$C_{22}^{\text{diag}} - C_{11}^{\text{diag}} = \sqrt{(C_{11} - C_{22})^2 + 4 |C_{12}|^2}.$$
Now we can write the probability of $\nu_e - \nu_\alpha$ transition:

$$P_{\nu_e \rightarrow \nu_\alpha} = |S_{\nu_e \nu_\alpha}|^2 = \sin^2 2\varphi \sin^2 n \frac{C_{22}^{\text{diag}} - C_{11}^{\text{diag}}}{2}. \quad (11)$$

We assume that the period of the potential, $2\Delta x$, is smaller than the neutrino oscillation length in matter with average potential $V_0$ and the amplitude of the additional rectangularly shaped potential $\Delta V$ is smaller than $(\Delta x)^{-1}$. In that case we may perform series expansion for (5) (according to the Hausdorff formula)

$$C = C^{(1)} + C^{(2)} + C^{(3)} + C^{(4)} + O(\Delta x^5), \quad (12)$$

where

$$C^{(1)} = \left(H^+ + H^\right) \Delta x,$$

$$C^{(2)} = \frac{i}{2} \left[H^+, H^\right] \Delta x^2, \quad (13)$$

$$C^{(3)} = \frac{1}{12} \left[(H^- - H^+), [H^+, H^-]\right] \Delta x^3,$$

$$C^{(4)} = \frac{i}{24} \left[H^{-2}H^+ - H^2H^{-2} + 2 \left(H^+H^-H^+ - H^-H^+H^+\right)\right] \Delta x^4.$$

The hermitian matrix $C$ has following elements

$$C_{22} - C_{11} = \frac{\Delta m^4}{E} \cos 2\theta_m \Delta x + O(\Delta x^5),$$

$$\text{Re } C_{12} = \frac{\Delta m^4}{2E} \sin 2\theta_m \Delta x - \frac{\Delta m^4}{12E} \sin 2\theta_m \Delta V^2 \Delta x^3 + O(\Delta x^5), \quad (14)$$

$$\text{Im } C_{12} = \frac{\Delta m^4}{4E} \sin 2\theta_m \Delta V \Delta x^2 + \frac{\Delta m^4}{48E} \sin 2\theta_m \Delta V \left[\Delta m^2 \frac{2 \Delta V^2}{E} - \Delta V^2\right] \Delta x^4 + O(\Delta x^5).$$

Here

$$\sin 2\theta_m = \frac{\sin 2\theta}{\sqrt{(\cos \theta - \frac{2EV_0}{\Delta m^2})^2 + \sin^2 \theta}}, \quad (15)$$

$$\Delta m^2_m = \Delta m^2 \sqrt{\left(\cos \theta - \frac{2EV_0}{\Delta m^2}\right)^2 + \sin^2 \theta} \quad (16)$$

are the mixing angle and the effective mass square difference in the medium with the potential $V_0$. 

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Inserting (14) into (9) and (11) we obtain an approximate solution for the $\nu_e \rightarrow \nu_\alpha$ transition probability:

$$P_{\nu_e \rightarrow \nu_\alpha} \simeq \sin^2 2\theta_m (1 + \beta_1) \sin^2 \left( \frac{\Delta m^2}{4E} (x_f - x_0) (1 + \beta_2) \right),$$ \hspace{1cm} (17)

where

$$\beta_1 = -\frac{1}{12} \Delta V^2 \Delta x^2 \cos^2 2\theta_m + O(\Delta x^4),$$ \hspace{1cm} (18)

$$\beta_2 = -\frac{1}{24} \Delta V^2 \Delta x^2 \sin^2 2\theta_m + O(\Delta x^4).$$ \hspace{1cm} (19)

As it follows from eqs. (17)- (19) in the lowest approximation the $\nu_e \rightarrow \nu_\alpha$ transition probability is given by the MSW equation - the oscillations in the matter with the average potential $V_0$.

In the first approximation, by the length of the period of the potential, the oscillation amplitude gets an additional factor (18) and the oscillation phase shifts according to eqs.(17) and (19). Thus in that approximation only $\Delta V^2 \Delta x^2$ enters in the corrections. According to (14), in the next approximation the corrections (to the amplitude and to the phase of the oscillation probability) have forms $(\Delta V \frac{\Delta m^2}{2E} \Delta x^2)^2$ and $(\Delta V \Delta x)^4$.

**Conclusion**

An approximate analytic solution for two flavor neutrino oscillations probabilities was found in the periodic square potential profile with suggestion that the length of each square is smaller than the oscillation length in the matter with averaged potential and is smaller than the inverse amplitude ($(\Delta V)^{-1}$) of the additional rectangularly shaped potential.

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