Gerbes on orbifolds and exotic smooth $\mathbb{R}^4$

Torsten Asselmeyer-Maluga$^1$, Jerzy Król$^2$

$^1$ German Aerospace Center, Rutherfordstr. 2, 12489 Berlin, torsten.asselmeyer-maluga@dlr.de
$^2$ University of Silesia, ul. Uniwersytecka 4, 40-007 Katowice, iriking@wp.pl

Received: date / Accepted: date

Abstract By using the relation between foliations and exotic $\mathbb{R}^4$, orbifold $K$-theory deformed by a gerbe can be interpreted as coming from the change in the smoothness of $\mathbb{R}^4$. We give various interpretations of integral 3-rd cohomology classes on $S^3$ and discuss the difference between large and small exotic $\mathbb{R}^4$. Then we show that $K$-theories deformed by gerbes of the Leray orbifold of $S^3$ are in 1:1 correspondence with some exotic smooth $\mathbb{R}^4$'s. The equivalence can be understood in the sense that stable isomorphisms classes of bundle gerbes on $S^3$ whose codimension-1 foliations generates the foliations of the boundary of the Akbulut cork, correspond uniquely to these exotic $\mathbb{R}^4$'s. Given the orbifold $SU(2) \times SU(2) \xrightarrow{\sim} SU(2)$ where $SU(2)$ acts on itself by conjugation, the deformations of the equivariant $K$-theory on this orbifold by the elements of $H^3_{SU(2)}(SU(2), \mathbb{Z})$, correspond to the changes of suitable exotic smooth structures on $\mathbb{R}^4$.

Contents

1. Introduction ................................................................. 1
2. $S^1$- Gerbes on $S^3$ and exotic $\mathbb{R}^4$ .............................. 3
3. Abelian gerbes deforming $K$-theory of orbifolds and exotic $\mathbb{R}^4$ .... 8
4. Conclusion ................................................................. 13
A. Non-cobordant foliations of $S^3$ detected by the Godbillon-Vey class .. 14
B. Interpretations of the integer classes .................................. 14
C. Twisted $K$-theory over a groupoid .................................... 18

1. Introduction

This paper presents further results in recognizing exotic small $\mathbb{R}^4$'s as being relevant not only for classical GR but rather for the quantum version of it.
Moreover, non-standard $\mathbb{R}^4$ are also important for other QFT’s. As shown in [4], the exotic $\mathbb{R}^4$ can act like magnetic monopole to produce a quantization of the electric charge.

In our opinion, the possible modifications of physical theories are caused by exotic rather than standard smoothness of open 4-manifolds. In our previous paper [4] we represented an exotic $\mathbb{R}^4$’s by 3-rd real de Rham cohomology classes of a 3-sphere embedded in $\mathbb{R}^4$. Especially we observed that though the smooth structure on $S^3$ is unique up to isotopy in contrast to uncountable many non-diffeomorphic smoothings of the topological $\mathbb{R}^4$, one can detect these smoothings by considering the 3-sphere inside the $\mathbb{R}^4$. Then we have to consider other structures on the 3-sphere instead of the smoothness structure. Here we used codimension-1 foliations, $S^1$-gerbes and generalized Hitchin-Gualtieri structures.

The present paper extends the above program with a different view. Now we are mainly focused on the role played by the integral 3-rd cohomology classes of $S^3$ in recognizing exotic smoothness of $\mathbb{R}^4$. We will describe many different representations of the integral 3-rd cohomologies of $S^3$ with a strong physical motivation. Among them there is a direct relation between the classifying spaces of the foliation $B\mathbf{F}_1$ and of the bundle gerbes $B\mathbb{P}U(H)$ for some Hilbert space $H$. A more or less complete list of possible interpretations of the 3-rd integer cohomology classes can be found in the Appendix [4]. We’d like to direct the readers attention to the connection between the integer classes and $E_8$ bundles used in string or M-theory. In a future paper we will come back to this point. As we conjecture in sec. 2.3 the difference between small and large exotic $\mathbb{R}^4$’s is directly related to the difference between integer (or rational) numbers and real numbers. Then groupoids and deformations by gerbes as possible extensions of the integer classes $H^3(S^3, \mathbb{Z})$ will be crucial for the correct recognition of large exotic $\mathbb{R}^4$’s. This conjecture about the relation between small and large exotic $\mathbb{R}^4$ finishes section 2.

Then in subsection 3.1 we turn to orbifold constructions in the context of exotic smoothness, and show that an exotic $\mathbb{R}^4$ (given by an integral cohomology class in $H^3(S^3, \mathbb{Z})$) corresponds to the deformed K-theory on a certain orbifold. The deformation is performed via a gerbe on the orbifold which is (Morita-equivalent to) the Leray orbifold of $S^3$. This is described equivalently by bundle gerbes on $S^3$. By considering the groupoid $SU(2) \times SU(2) \twoheadrightarrow SU(2)$, where $SU(2)$ acts by conjugation on itself, we get in subsection 3.2 the deformation of the equivariant K-theory of the groupoid as coinciding with the changes of smoothings of $\mathbb{R}^4$. Then the deformation is performed via equivariant 3-rd cohomologies, $H^3_{SU(2)}(S^3, \mathbb{Z}) \simeq \mathbb{Z}$.

The analysis of small exotic smooth $\mathbb{R}^4$’s via groupoids and gerbes is interesting by itself, but our description may seem redundant or optional because the twisted K-theory on the Leray groupoid of $S^3$ is the twisted K-theory of $S^3$. However, groupoids and gerbes generalize the ordinary smooth manifolds from the point of view of K-theory, cohomology and geometry by introducing singular orbifold-like structures. Although the description on a manifold is local, the difference remains global. In this paper we show that these global structures and generalized cohomologies characterize small exotic $\mathbb{R}^4$’s as in our Th.’s [2, 3, 4].

The possibility of describing the cycles of the deformation explicitly is main advantage of the presented approach. It is an important step toward building...
an exotic smooth function on $\mathbb{R}^4$ with many physical applications. In particular, the results of this paper seem to be highly relevant in the process of further uncovering the meaning of exotic 4-smoothness in string topology and geometry as well string compactifications, i.e. in the final formulation of quantum gravity. We hope to address these important issues soon.

As we remark above, the usage of groupoids and gerbes generalize the concept of an ordinary smooth manifold to make room for slightly singular objects like orbifolds. In our approach it is possible to identify these objects as part of the ordinary spacetime (like $\mathbb{R}^4$): the main object is the Casson handle. That is a hierarchical object of wildly embedded disks having a tree-like (discrete) structure which is continuous at the same time. In this object, all specific properties of dimension 4 are concentrated. We will try to uncover some of its physical properties in our future work.

2. $S^1$- Gerbes on $S^3$ and exotic $\mathbb{R}^4$

In our previous paper [4] we uncover a relation between an exotic (small) $\mathbb{R}^4$ and a cobordism class of a codimension-1 foliation on $S^3$ classified by the Godbillon-Vey class as element of the cohomology group $H^3(\partial A, \mathbb{R})$. By using $S^1$-gerbes it was possible to interpret the integer elements $H^3(\partial A, \mathbb{Z})$ as characteristic class of a $S^1$-gerbe over $S^3$.

2.1. Exotic $\mathbb{R}^4$ and codimension-1 foliation. Here we present the main line of argumentation in our previous paper [4]:

1. In Bizacas exotic $\mathbb{R}^4$ one starts with the neighborhood $N(A)$ of the Akbulut cork $A$ in the K3 surface $M$. The exotic $\mathbb{R}^4$ is the interior of $N(A)$.
2. This neighborhood $N(A)$ decomposes into $A$ and a Casson handle representing the non-trivial involution of the cork.
3. From the Casson handle we constructed a grope containing Alexander's horned sphere.
4. The Akbulut construction gives a non-trivial involution, i.e. the double of that construction is the identity map.
5. From the grope we get a polygon in the hyperbolic space $\mathbb{H}^2$.
6. This polygon defines a codimension-1 foliation of the 3-sphere inside of the exotic $\mathbb{R}^4$ with an wildly embedded 2-sphere, Alexander's horned sphere (see [2]). This foliation agrees with the corresponding foliation of the homology 3-sphere $\partial A$. This codimension-1 foliations of $\partial A$ is partly classified by the Godbillon-Vey class lying in $H^3(\partial A, \mathbb{R})$ which is isomorphic to $H^3(S^3, \mathbb{R})$.
7. Finally we get a relation between codimension-1 foliations of the 3-sphere lying in $\partial A$ and exotic $\mathbb{R}^4$.

This relation is very strict, i.e. if we change the Casson handle then we must change the polygon. But that changes the foliation and vice versa. For the case of a codimension-1 foliation $\mathcal{F}$ we need an overall non-vanishing vector field or its dual, an one-form $\omega$. This one-form defines a foliation iff it is integrable, i.e.

\[ d\omega \wedge \omega = 0 \]

1 In short, a foliation of a smooth manifold $M$ is an integrable subbundle $N \subset TM$ of the tangent bundle $TM$. 
and the leaves are the solutions of the equation \( \omega = \text{const.} \). Now we define the one-forms \( \theta \) as the solution of the equation

\[
d\omega = -\theta \wedge \omega
\]

and consider the closed 3-form

\[
\Gamma_F = \theta \wedge d\theta
\]  (1)

associated to the foliation \( F \). As discovered by Godbillon and Vey \[15\], \( \Gamma_F \) depends only on the foliation \( F \) and not on the realization via \( \omega, \theta \). Thus \( \Gamma_F \), the Godbillon-Vey class, is an invariant of the foliation.

Now we will discuss an important equivalence relation between foliations, cobordant foliations. Let \( M_0 \) and \( M_1 \) be two closed, oriented \( m \)-manifolds with codimension-\( q \) foliations. Then these foliated manifolds are said to be foliated cobordant if there is a compact, oriented \((m+1)\)-manifold with boundary \( \partial W = M_0 \sqcup M_1 \) and with a codimension-\( q \) foliation transverse to the boundary inducing the given foliation. The resulting foliated cobordism classes \( \mathcal{F} \Gamma_q \) form an abelian group under disjoint union. The Godbillon-Vey class \( \Gamma_F \) is also a foliated cobordism class and thus an element of \( \mathcal{F} \Gamma_1 \). In \[24\], Thurston constructed a codimension-1 foliation of the 3-sphere \( S^3 \) and calculated the Godbillon-Vey classes, see the Appendix A. According to Haefliger (see Lawson \[18\] section 5), non-cobordant, codimension-1 foliations of \( S^3 \) are classified by the elements of \( \pi_3(\mathcal{F} \Gamma_1) \). Thurston constructed in the work above a surjective homomorphism

\[
\pi_3(\mathcal{F} \Gamma_1) \to \mathbb{R}
\]

and by results of Mather etc. (see Lawson \[18\] section 5 for an overview) the classes \( \pi_k(\mathcal{F} \Gamma_1) = 0 \) for \( k < 3 \) vanish. By the Hurewicz isomorphism, the surjective homomorphism is now an element of \( H^3(S^3, \mathbb{R}) = Hom(\pi_3(\mathcal{F} \Gamma_1), \mathbb{R}) \). Then the Godbillon-Vey class is an element of \( H^3(S^3, \mathbb{R}) \) having values in the real numbers. Together with the results above we obtained: The exotic \( \mathbb{R}^4 \) (of Bizaca) is determined by the codimension-1 foliations with non-vanishing Godbillon-Vey class in \( H^3(S^3, \mathbb{R}) \) of a 3-sphere seen as submanifold \( S^3 \subset \mathbb{R}^4 \).

2.2. The reduction to integer classes and its interpretation. In this subsection we will discuss the interpretation of the integer classes in \( H^3(S^3, \mathbb{R}) \) and the transition to abelian gerbes. As discussed above, we have a partial classification of non-cobordant codimension-1 foliation \( F \) by Godbillon-Vey classes as elements of \( H^3(S^3, \mathbb{R}) \) and its relation to exotic \( \mathbb{R}^4 \)'s. The Godbillon-Vey class is a real 3-form

\[
\Gamma_F = \theta \wedge d\theta
\]

constructed from the one-form \( \theta \). Now we will discuss the reduction from the real classes in \( H^3(S^3, \mathbb{R}) \) to the integer classes in \( H^3(S^3, \mathbb{Z}) \). First of all, 3-rd integral cohomologies are isomorphism classes of projective, infinite dimensional, bundles and gerbes playing a distinguish role for twisting \( K \)-theory on manifolds and groupoids. This case is crucial for our following constructions. Twisted \( K \)-theories and the above interpretation are discussed briefly in the Appendix C.
Here we are interested in the interpretation of the integer classes in our context of non-cobordant foliations of the 3-sphere. A more or less complete list of possible interpretations for these classes can be found in the Appendix B. At first we remark that the cohomology class \([\Gamma_X]\) is unchanged by a shift \(\theta \rightarrow \theta + d\phi\) of the one-form \(\theta\) by an exact form, i.e. we have gauge invariance in the physical sense. Thus we can interpret the purely imaginary one-form \(A = i\theta\) as a connection of a complex line bundle over \(S^3\). Then the Godbillon-Vey class is related to the abelian Chern-Simons form with action integral

\[
S = \int_{S^3} A \wedge dA = \int_{S^3} \Gamma_X.
\]

But that is only the tip of the iceberg. Denote by \(\Gamma_q^r\) the set of germs of local \(C^r\)-diffeomorphisms of \(\mathbb{R}^q\) forming a smooth groupoid. A codimension-\(q\) Haefliger cocycle (over an open covering \(U(X) = \{O_i\}_{i \in I}\) of a space \(X\) is an assignment: one assigns to each pair \(i, j \in I\) a continuous map \(\gamma_{ij} : O_i \cap O_j \rightarrow \Gamma_q^r\) such that

\[
\gamma_{ij}(x) = \gamma_{ik}(x) \circ \gamma_{kj}(x)
\]

for all \(i, j, k \in I\) and \(x \in O_i \cap O_j \cap O_k\). Then the setting \(g_{ij} = d\gamma_{ij}\) in a neighborhood of \(x \in O_i \cap O_j\) defines a \(q\)-dimensional vector bundle with transition function \(g_{ij}\), called the normal bundle of the foliation. Two Haefliger structures \(\mathcal{H}_0, \mathcal{H}_1\) over \(X\) are equivalent if both are concordant (or cobordant), i.e. there is a Haefliger structure \(\mathcal{H}\) on \(X \times [0, 1]\), so that \(\mathcal{H}_k = i_k^* \mathcal{H}\) with \(i : X \rightarrow X \times [0, 1]\), \(i_k(x) = (x, k)\). Furthermore, it is known that to every topological groupoid \(\Gamma\) there is a classifying space \(BG\) (constructed for instance by Milnor’s join construction \([21, 22]\)). Then the equivalence classes of codimension-\(q\) Haefliger structures of class \(C^r\) over a manifold \(M\) is given by the set \([M, BG]\). Then a given map \(M \rightarrow BG\) determines a Haefliger structure. Now we will specialize to the (smooth) codimension-1 case over the 3-sphere, i.e. we consider maps \(S^3 \rightarrow BG\) (setting \(r = \infty\)). Given a constant map \(x_0 \rightarrow BG\) with \(x_0 \in S^3\), i.e. a map from 0-skeleton of \(S^3\) into \(BG\). Now we ask whether this can be extended over the other skeletons of \(S^3\) to get finally a map \(S^3 \rightarrow BG\). The question can be answered by obstruction theory to state that the elements of \(H^3(S^3, \pi_3(BG))\) label all possible extensions. Using Thurston’s surjective homomorphism we have uncountable infinite possible extensions, i.e. all elements of \(H^3(S^3, \mathbb{R})\).

By using that machinery, we define a codimension-1 foliation of \(S^3\) via a continuous function \(f : S^3 \rightarrow \mathbb{R}\) using the natural embedding \(i : \mathbb{R} \rightarrow \Gamma_1\) to obtain the Haefliger cocycle \(\gamma = i \circ f\). Alternatively we can also consider a function \(\tilde{f} : S^3 \rightarrow S^1 = U(1)\) with \(f = \tilde{f} \cdot \log(f)\) seen as a section \(f\) of some complex line bundle over \(S^3\). Every complex line bundle is given by a map into the classifying space \(BU(1)\), which is an Eilenberg-Mac Lane space \(K(2, \mathbb{Z})\). Thus, on the abstract level there is a map between the smooth groupoid \(\Gamma_1\) and the classifying space \(BU(1)\) of complex line bundles. Then we have shown

\[\text{Actually } [M, BG]\text{ has more structure than a set, i.e. it defines a generalized cohomology theory like } [M, BG]\text{ defines (complex) K theory for } G = SU\text{ or real K theory for } G = SO.\]

\[\text{An Eilenberg-Mac Lane space } K(n, G)\text{ is a topological space (unique up to homotopy) with the only non-vanishing homotopy group } \pi_n(K(n, G)) = G.\text{ The group } G\text{ has to be abelian for } n > 1.\]
Theorem 1. There is a natural map from the smooth groupoid $\Gamma_1$ to the classifying space $BU(1)$ of complex line bundles. Then every codimension-1 Haefliger structure over a 3-manifold $M$ as classified by $[M, B\Gamma_1]$ is canonically mapped via a surjection to

$[M, B(\text{map}BU(1))] = [M, BK(2,\mathbb{Z})] = [M, K(3,\mathbb{Z})] = [M, BPU(H)] = H^3(M,\mathbb{Z})$

The space $BU(1)$ is homotopy-equivalent to the infinite dimensional projective space $\mathbb{C}P^\infty = PU(H)$ which is the Eilenberg-MacLane space $K(2,\mathbb{Z})$ and we have $B(BU(1)) = BPU(H)$ where $PU(H)$ is the projective unitary group over some separable Hilbert space $H$.

The mapping above induces a mapping between $B\Gamma_1$ and the corresponding classifying space $BPU(H)$ of bundle gerbes.

The close relation between codimension-1 foliations and bundle gerbes together with the relation to (small) exotic $\mathbb{R}^4$ opens a new interpretation of the integer classes $H^3(S^3,\mathbb{Z})$. In Appendix A we will present the construction of uncountable infinite non-cobordant codimension-1 foliations of the 3-sphere $S^3$. Main part in the construction is the usage of a polygon $P$ in the hyperbolic space $\mathbb{H}$. The volume of $P$ is proportional to the Godbillon-Vey class of the foliation, i.e. one gets real numbers for this class. Thus, if we restrict ourselves to the integers, we will obtain integer values

$\text{vol}(P) = n \in \mathbb{Z}$

In the construction of the foliation, the polygon $P$ represents some leaves. Thus if we choose an integer Godbillon-Vey class for the foliation then these leaves have a quantized volume.

2.3. Small versus large exotic $\mathbb{R}^4$. In this subsection we will discuss the difference between small and large exotic $\mathbb{R}^4$ having omitted up to now. The non-interesting reader can switch to the next section without losing any substantial material.

A small exotic $\mathbb{R}^4$ can be embedded smoothly into a 4-sphere whereas a large exotic $\mathbb{R}^4$ cannot. Thus the construction of both classes are rather different. As mentioned above, the small exotic $\mathbb{R}^4$ can be constructed by using the failure of the smooth h-cobordism theorem. For the large exotic $\mathbb{R}^4$, one considers non-smoothable, closed 4-manifolds and constructs an exotic $\mathbb{R}^4$ inside. Our result above uses extensively Bizaca’s construction of a small exotic $\mathbb{R}^4$ by using the Akbulut cork for a pair of non-diffeomorphic, but homeomorphic 4-manifolds. But what can we say about large exotic $\mathbb{R}^4$’s?

Given a compact, simply-connected, closed 4-manifold $M$. As shown by Freedman [14], this manifold is completely determined by a quadratic form, the intersection form, over the second homology group $H_2(M,\mathbb{Z})$. Lateron Donaldson [11] showed that not all 4-manifolds $M$ are smoothable. We don't want to speak about the details and refer the reader to the books [16, 3].

The diagonal values of the intersection form for a smooth 4-manifold have a simple interpretation as self-intersections of surfaces given by the square of the first Chern class. If that values is a non-quadratic number like 2 then a smooth structure don’t exists.
example we consider quadratic forms made from the parts\(^5\):

\[
E_8 = \begin{pmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1 \\
1 & 2 & 1
\end{pmatrix}, \quad H = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\]

The form \(E_8\) and every sum like \(E_8 \oplus E_8\) is not diagonalizable over \(\mathbb{Z}\). Everything changes if we add (1) or \(H\). As Freedman \([14]\) showed one can construct a manifold \(M\) by using every possible quadratic form over \(\mathbb{Z}\). Thus, there is a closed, compact, simply-connected 4-manifold \(|E_8|\) for \(E_8\) and \(H\) corresponds to \(S^2 \times S^2\). But \(|E_8|\) as well as \(|E_8 \oplus E_8|\) = \(|E_8| \# |E_8|\) with the connected sum \(#\) is not smoothable. Now we consider the form

\[E_8 \oplus E_8 \oplus H \oplus H \oplus H = 2E_8 \oplus 3H\]

corresponding to the K3 surface, which is a smooth 4-manifold but there is no smooth decomposition like

\[|2E_8| \# 3S^2 \times S^2\]

The sum \(3S^2 \times S^2\) represents the \(3H\) part in the intersection form. Now we consider the open manifold \(X = 3S^2 \times S^2 \setminus \text{int}(D^4)\) together with an embedding \(j : X \to K3\) in the K3 surface having a collar, i.e. a product neighborhood \(C(j) = j(\partial X) \times \mathbb{R}\) of \(j(\partial X)\). The open manifold \(W = 3S^2 \times S^2 \setminus j(X)\) is homeomorphic to \(\text{int}(D^4) = \mathbb{R}^4\) but not diffeomorphic, because of the non-smooth 4-manifold \(|2E_8| \setminus D^4\), i.e. there is no smoothly embedded 3-sphere!

Now we remark that \(X\) itself is a Casson handle. Thus in both cases of a small and a large exotic \(\mathbb{R}^4\), the central object is the Casson handle. But what is the difference in the usage of the Casson handle in both construction?

In Bizaca’s construction one glued the Casson handle along a 1-handle to the Akbulut cork and considers the interior of the resulting manifold. Then one needs a topological disk inside of the Casson handle to get the homeomorphism to the \(\mathbb{R}^4\). According to the reembedding theorems of Freedman \([14]\) such a disk exists after 6 stages of the Casson handle. The concrete realization of such an imbedding by Bizaca \([6]\) gives superexponential functions for the growth of the Casson handle. As Bizaca showed, all these handles can be considered to be equivalent for small exotic \(\mathbb{R}^4\). In contrast for large exotic \(\mathbb{R}^4\) we need the knowledge of the whole Casson handle because the construction don’t depend on the interior of the Casson handle (which is always diffeomorphic to the standard \(\mathbb{R}^4\)) but on the “boundary”. Of course there is no real boundary of the Casson handle \(CH\) but after a suitable compactification (like Shapiro-Bing or Freudenthal) one can define a substitute, the so-called frontier. The frontier is not a manifold but a so-called manifold factor, i.e. the factor \(W\) is not a manifold but \(W \times \mathbb{R}\) is one. For the simplest, non-trivial example of a Casson handle, the frontier is the Whitehead continuum \(Wh\), i.e. a 3-dimensional topological space not homeomorphic

\(^5\) The form \(E_8\) is the Cartan matrix of the semi-simple Lie group \(E_8\).
to $\mathbb{R}^3$ but where the product $W h \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. Sometimes one states that $W h$ is not simply-connected at infinity. That property characterizes also the large exotic $\mathbb{R}^4$: a smoothly embedded 3-sphere exists only “at infinity”. Of course this “3-sphere at infinity” can be also foliated to get a class in $H^3(S^3, \mathbb{R})$ but then we will get real numbers generated by sequences of super-exponential functions. This discussion supports the conjecture:

**Conjecture:** Small exotic $\mathbb{R}^4$ are characterized by rational numbers or by real numbers coming from at most exponential functions. Large exotic $\mathbb{R}^4$ are always characterized by real numbers generated by super-exponential functions.

### 3. Abelian gerbes deforming $K$-theory of orbifolds and exotic $\mathbb{R}^4$

In this section we will get a close connection between (small) exotic $\mathbb{R}^4$'s and twisted $K$-theory of orbifolds where the twisting is induced by a gerbe. The whole subject can be presented by using the concept of a groupoid which we will introduce now.

A groupoid $\mathcal{G}$ is a category where every morphism is invertible. Let $G_0$ be a set of objects and $G_1$ the set of morphisms of $\mathcal{G}$, then the structure maps of $\mathcal{G}$ read as:

\[
G_{1,t} \times_s G_{1} \overset{m}{\rightarrow} G_{1} \overset{i}{\rightarrow} G_{1} \overset{s}{\leftarrow} G_{0} \overset{e}{\rightarrow} G_{1}
\]

where $m$ is the composition of the composable two morphisms (target of the first is the source of the second), $i$ is the inversion of an arrow, $s, t$ the source and target maps respectively, $e$ assigns the identity to every object. We assume that $G_{0,1}$ are smooth manifolds and all structure maps are smooth too. We require that the $s, t$ maps are submersions [19], thus $G_{1,t} \times_s G_{1}$ is a manifold as well. These groupoids are called smooth groupoids. We will present a groupoid (2) by $G_{1,t} \rightarrow G_{0}$. In general when the source and target maps are local homeomorphisms (diffeomorphisms), the corresponding topological (smooth) groupoid is called an étale groupoid. A natural and important equivalence relation on groupoids is the Morita equivalence, see [19].

Following [20], let $\mathcal{G}$ be a proper étale, smooth groupoid $\mathcal{G}$. We denote the class of Morita equivalent groupoids of $\mathcal{G}$ as an orbifold $Ob$. Given a groupoid $\mathcal{G}$ we define $G_i$ by $G_i = G_{1,t} \times_s G_{1}\times_s ... \times_s G_{1}$, $i$ times, which are sets of composable arrays of morphisms, of the length $i$. A groupoid $\mathcal{G}$ is Leray when every $G_i, i = 0, 1...$ is diffeomorphic to a disjoint union of contractible open sets. As was shown by Moerdijk and Pronk [23] every orbifold can be represented by some Leray groupoid.

Given a smooth manifold $M$ we can attach to it a natural Leray groupoid $\mathcal{R} \rightrightarrows \mathcal{U}$ representing the manifold. Let $\{U_\alpha\}$ be an open cover of $M$. We take the disjoint union $\mathcal{U} = \bigsqcup_{\alpha} U_\alpha$ as the set of objects $G_0$ and $\mathcal{R} = \bigsqcup_{(\alpha,\beta)} U_\alpha \cap U_\beta$, $(\alpha, \beta) \neq (\beta, \alpha)$ as the set of morphisms. Next let us define $s, t, e, i$ and $m$ maps in a groupoid by the following natural maps:

\[
s|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_\alpha, \quad t|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_\alpha
\]

\[
e|_{U_{\alpha}} : U_{\alpha} \rightarrow U_{\alpha}, \quad i|_{U_{\alpha\beta}} : U_{\alpha\beta} \rightarrow U_{\beta}, \quad m|_{U_{\alpha\beta\gamma}} : U_{\alpha\beta\gamma} \rightarrow U_{\alpha\gamma}
\]
where $U_{\alpha \beta}$ is $U_\alpha \cap U_\beta$ and $U_{\alpha \beta \gamma}$ is $U_\alpha \cap U_\beta \cap U_\gamma$ as usual. This groupoid is constructed directly from the open cover of a manifold and is denoted by $\mathcal{M}(M, U_\alpha)$.

### 3.1. Bundle gerbes on $S^3$ and gerbes on groupoids.

Given a (differentiable, étale, proper) groupoid $G_1 \rightrightarrows G_0$ we can define a gerbe on it:

**Definition 1.** A gerbe over an orbifold $G_1 \rightrightarrows G_0$ (over a groupoid representing the orbifold) is a complex line bundle $L$ over $G_1$ provided

1. $i^* L \simeq L^{-1}$
2. $\pi_1^* L \otimes \pi_2^* L \otimes m^* i^* L \theta \simeq 1$
3. $\theta : G_1 \times_s G_1 \to U(1)$ is a 2-cocycle

$\pi_{1,2}$ are two projections from $G_1 \times_s G_1 \to G_1$ and $\theta$ is the trivialization of the line bundle $L$.

Let us recall that a gerbe on a manifold $M$ can be defined via the following data [17]:

1. A line bundle $L_{\alpha \beta}$ on each double intersection $U_\alpha \cap U_\beta$
2. $L_{\alpha \beta} \simeq L_{\beta \alpha}^{-1}$
3. There exists a 2-cocycle $\theta_{\alpha \beta \gamma} : U_{\alpha \beta \gamma} \to U(1)$ which gives the trivialization of $L_{\alpha \beta} L_{\beta \gamma} L_{\gamma \alpha} \simeq 1$ on each triple intersections.

We see that in the case of the groupoid $\mathcal{M}(M, U_\alpha)$ representing a manifold $M$ and defining the gerbe on this groupoid as in Def. 1, we get exactly the gerbe on $M$ as above [19].

We can define yet another groupoid, $\mathcal{G}(Y, M)$, given a manifold $M$ and a surjective submersion $\pi : Y \to M$

1. $G_1 = Y \times_s Y =: Y^2$
2. $G_0 = Y$
3. $s = p_1 : Y^2 \to Y$, $s(y_1, y_2) = y_1$, $t = p_2 : Y^2 \to Y$, $t(y_1, y_2) = y_2$
4. $m((y_1, y_2), (y_2, y_3)) = (y_1, y_3)$
5. $(y_1, y_2)^{-1} = (y_2, y_1)$

**Definition 2.** A bundle gerbe over manifold $M$ is a pair $(L, Y)$ where $Y$ is a surjective submersion and $L \xrightarrow{p} Y^2$ is a line bundle such that

1. $L_{(y, y)} \simeq C$
2. $L_{(y_1, y_2)} \simeq L_{(y_2, y_1)}^*$
3. $L_{(y_1, y_2)} \otimes L_{(y_2, y_3)} \simeq L_{(y_1, y_3)}$

Now we state the following fact (Lemma 7.3.3. in [19]):

A bundle gerbe $(L, Y)$ over $M$ is the same as a gerbe over the groupoid $\mathcal{G}(Y, M)$, which is a direct consequence of Defs. 1 and 2.

Let $\mathcal{L}(Y, M)$ be a gerbe over the groupoid $\mathcal{G}(Y, M)$. This is the bundle gerbe $(L, Y)$ on $M$. In fact bundle gerbes over $M$ form a group with the tensor product

---

6 A surjective submersion $\pi : Y \to M$ is a map which allows local sections, i.e. locally split. This means that for every $x \in M$ there exists an open set $U$ containing $x$, $x \in U$ and a local section $s : U \to Y$, i.e. $s \circ \pi = \text{id}$. Locally split map is necessary surjective [7].
of bundles as the group operation \[19\]. This group is homomorphic with \(H^3(M, \mathbb{Z})\) where the homomorphism is defined via the Dixmier-Duady class \(d(L, Y)\) of the bundle gerbe \((L, Y)\) \[19\].

Let \(\text{Gb}(\mathcal{M}(M, U_\alpha))\) be the group of gerbes on the Leray groupoid \(\mathcal{M}(M, U_\alpha)\) of the manifold \(M\). In fact the group of bundle gerbes on \(M\) is isomorphic with the group of gerbes on \(\mathcal{M}(M, U_\alpha)\) \[19\], Corollary 7.3.5.\).

Gerbes on groupoids are naturally related via Morita equivalence similarly as groupoids are, where orbifolds represent their Morita equivalence class. In fact there is a bundle gerbe representing every Morita equivalence class of gerbes over \(M\). From the other side, a natural relation for bundle gerbes is the stable isomorphism of these, since they are defined via bundles. More precisely, given two bundle gerbes \((L_1, Y_1), (L_2, Y_2)\) on \(M\), we say that they are stable isomorphic if there exist trivial bundle gerbes on \(M\) given by bundles \(T_1, T_2\), such that the bundles

\[ L_1 \otimes T_1 \simeq L_2 \otimes T_2 \]  \hspace{1cm} (4)

are isomorphic. The trivial bundle gerbe is one whose Dixmier-Duady class in \(H^3(M, \mathbb{Z})\) is 0, i.e \(d(T_1) = d(T_2) = 0 \in H^3(M, \mathbb{Z})\). It holds \[19\], Corollary 7.3.10.:

Two bundle gerbes \((P, Y)\) and \((Q, Z)\) are stably isomorphic if and only if \(d(P) = d(Q)\).

Up to the stable isomorphisms the groups of bundle gerbes on \(M\) and \(H^3(M, \mathbb{Z})\) are in fact isomorphic \[19\], Theorem 7.3.13.:

There is a one-to-one correspondence between stably isomorphism classes of bundle gerbes over \(M\) and classes in \(H^3(M, \mathbb{Z})\). The category of bundle gerbes over \(M\) with stable isomorphisms is equivalent to the category of gerbes over \(M\) with Morita equivalences.

Thus an action of an element of \(H^3(M, \mathbb{Z})\) can be determined equivalently as the suitable action of the bundle gerbes whose Dixmier-Duady class in \(H^3(M, \mathbb{Z})\) is the element we began with. In \[4\] we assigned non-standard smoothings of \(\mathbb{R}^4\) to the elements from \(H^3(S^3, \mathbb{Z})\) hence the action of bundle gerbes on \(S^3\) can be correlated with the changes of the smoothings. In fact we are interested in twisting \(K\)-theories of the Leray groupoid of \(S^3\) by bundle gerbes on \(S^3\). Similarly as defining the \(K\)-theory for spaces and manifolds one can develop whole theory of bundles, cohomologies and \(K\)-theories on the groupoids representing orbifolds. This was performed by several authors (see e.g. \[19\], \[7\], \[1\], \[10\]). In fact mathematical development of the subject was motivated by the attempts in theoretical physics to formulate string theory on orbifolds and the need to use twisted \(K\)-theoretic classes of spacetime in order to classify the brane charges \[26\],[27\]. This is also one of the motivation for our approach to exotic smoothness by twisted (equivariant) cohomologies: they can uncover some fundamental relation of exotic with string theory hence QG. Besides these rather abstract constructions are possibly relevant for the large exotic smoothness of \(\mathbb{R}^4\) case. Both topics we want to present in a separate work.

Crucial for the twisted \(K\)-theory are bundle gerbe modules over \((L, Y)\). In fact given a bundle gerbe \(\mathcal{L}(L, Y)\) on \(M\) we can define the category of bundle gerbe modules over \((L, Y)\) (see the Appendix \[C\]). However, this category is equivalent to the category of \(\mathcal{L}(L, Y)\)-twisted vector bundles over \(G(Y, M)\).
isomorphism classes, completed by the Grothendieck procedure to a group, gives
rise to the twisted $K$-theory of the groupoid $G(Y, M)$, i.e. $\xi K_{\text{gpd}}(G(Y, M))$.

Gerbes on the orbifold $G(Y, S^3)$ are classified by $H^3(G(Y, S^3), \mathbb{Z})$ which is $H^3(S^3, \mathbb{Z})$. Thus from the above and the results of [1], it follows:

**Theorem 2.** Given an exotic $\mathbb{R}^4$, $e$, corresponding to some integral cohomology class $[e] \in H^3(S^3, \mathbb{Z})$ the change of the standard smoothing of $\mathbb{R}^4$ to the exotic one, $e$, determines the deformation $\delta_e$ of the $K$-theory of the Leray groupoid of $S^3$ by the bundle gerbe $\mathcal{L} \in [e]$, i.e.

$$\delta_e : K_{\text{gpd}}(G(Y, S^3)) \to \xi K_{\text{gpd}}(G(Y, S^3))$$

where $S^3$ lies at the boundary of the Akbulut cork of $e$.

We say that the exotic structure $e$ deforms the $K$-theory as above. Let us see how to construct the deformation from a given $e$. $e$ determines the codimension-1 foliation of $S^3$ and its Godbillon-Vey class $[e] \in H^3(S^3, \mathbb{R})$ which in our case is integral. From the class $[e] \in H^3(S^3, \mathbb{Z})$ we have the corresponding bundle gerbe $\mathcal{L} \in [e]$ representing the class, i.e. $d(\mathcal{L}) = [e]$. Now the deformation of the $K$-theory by $\mathcal{L}$ is well defined (see the Appendix C) and (5) expresses it.

We can be more explicit with the twisting of the $K$-theory of the Leray groupoid:

The twisted $K$-theory of the Leray groupoid of $S^3$ is the twisted $K$-theory of $S^3$, since $B(G(Y, S^3)) = S^3$ and gerbes on the orbifold $G(Y, S^3)$ are classified by $H^3(G(Y, S^3), \mathbb{Z})$ which is $H^3(S^3, \mathbb{Z})$.

Hence we can directly compute twisted cohomology $\xi K_{\text{gpd}}(G(Y, S^3))$ as $K^{\tau}(S^3)$ where $\tau = [\mathcal{L}] \in H^3(S^3, \mathbb{Z})$. This last, following [13], the example 1.4, reads as

$$K^k(S^3, n[]) = K^{\tau + k}(S^3) = \begin{cases} 0 & , k = 0 \\ \mathbb{Z}/n & , k = 1 \end{cases}$$

where $\tau = n[] \in H^3(S^3, \mathbb{Z})$ and $[]$ is the generator. This twisting is given by

$$\delta_n : K^1(S^3) = \mathbb{Z} \to \mathbb{Z}/n = K^{(n)+1}(S^3) = K^1(S^3, n[])$$

and reflects the effect of the change of the standard smooth $\mathbb{R}^4$ to the exotic one, corresponding to the integral class $n[] \in H^3(S^3, \mathbb{Z})$. We see that the effects are detectable in generalized twisted $K$-theory.

### 3.2. The deformation of the $K$-theory of the groupoid $SU(2) \times SU(2) \cong SU(2)$ and exotic $\mathbb{R}^4$.

Consider the hyperbolic plane $\mathbb{H}^2$ and its unit tangent bundle $T_1 \mathbb{H}^2$, i.e the tangent bundle $T \mathbb{H}^2$ where every vector in the fiber has norm 1. Thus the bundle $T_1 \mathbb{H}^2$ is a $S^1$-bundle over $\mathbb{H}^2$. There is a foliation $\mathcal{F}$ of $T_1 \mathbb{H}^2$ invariant under the isometries of $\mathbb{H}^2$ which is induced by bundle structure and by a family of parallel geodesics on $\mathbb{H}^2$. The foliation $\mathcal{F}$ is transverse to the fibers of $T_1 \mathbb{H}^2$. Let $P$ be any convex polygon in $\mathbb{H}^2$. We will construct a foliation $\mathcal{F}_P$ of the three-sphere $S^3$ depending on $P$. Let the sides of $P$ be labelled $s_1, \ldots, s_k$ and let the angles have magnitudes $\alpha_1, \ldots, \alpha_k$. Let $Q$ be the closed region bounded by $P \cup P'$, where $P'$ is the reflection of $P$ through $s_1$. Let $Q_e$, be $Q$ minus an open $\epsilon$-disk about each vertex. If $\pi : T_1 \mathbb{H}^2 \to \mathbb{H}^2$ is the projection of the bundle
Given the conjugation classes of $SU(2)$ on $SU(2)$ (these are 2-spheres, $S^2$, and 2 poles) the natural $\mathbb{Z}_2$-involution changes the classes and fixes the equator $S^2$. As follows from [4] such an involution determines the standard smooth structure on $\mathbb{R}^4$ whereas non-zero 3-rd integral cohomologies $H^3(S^3, \mathbb{Z})$ correspond to some exotic smooth $\mathbb{R}^4$s, $R_k$, $k \in \mathbb{Z}$. Now we change slightly the view and consider the involution induced by an action of the $SU(2)$ on itself. Then we obtain elements in the equivariant cohomology $H^3_{SU(2)}(SU(2), \mathbb{Z})$. By using that idea we will get an unexpected relation to the Verlinde algebra where the level is determined by an element in $H^4(S^3, \mathbb{Z})$. As we discussed in subsection 2.2 that level can be interpreted as a surface in the hyperbolic space with quantized volume.

Similarly as elements of $H^3(S^3, \mathbb{Z})$ can twist the ordinary $K$-theory, the elements of equivariant cohomologies $H^3_{SU(2)}(S^3, \mathbb{Z})$ can be used to twist equivariant $K$-theory. The untwisted equivariant case as above corresponds to the standard $\mathbb{R}^4$ (0-twist). The twisted equivariant cohomologies by non-zero 3-rd integral cohomologies correspond to the exotic smooth $\mathbb{R}^4$s. This is because there exists a canonical map $e : H^*_G(S^3) \to H^*_H(S^3)$ where $H \subset G$ is a subgroup of $G$. Taking $H = \{1\}$ we have $H^*_G(S^3) \to H^*(S^3)$. In the case of $SU(2)$, $H^*_Z(S^3, \mathbb{Z}) \simeq H^*(S^3, \mathbb{Z}) \simeq \mathbb{Z}$, thus the equivariant twisting corresponds to the non-equivariant by $e_k : k[\cdot] \to k[\cdot]_{eq}$ where $[\cdot]$ and $[\cdot]_{eq}$ mean the generators of $H^3(S^3, \mathbb{Z})$ and $H^3_{SU(2)}(S^3, \mathbb{Z})$ correspondingly. We say that a (bundle) gerbe $d(\mathcal{L}) = [e] \in H^3(S^3, \mathbb{Z})$ twists the equivariant $K$-theory of $SU(2)$ acting on $SU(2)$ by conjugation when the equivariant class $e_k([e]) \in H^3_{SU(2)}(S^3, \mathbb{Z})$ twists the equivariant cohomology. Again, assigning to gerbes some non-standard small smoothings of $\mathbb{R}^4$, where $S^3 \subset \mathbb{R}^4$, the discussion above, the result of [12, 13] and the example 7.2.17 in [19] give the following correspondence:

**Theorem 3.** Given an exotic $\mathbb{R}^4$, $e$, corresponding to some integral cohomology class $[e] = k[\cdot] \in H^3(S^3, \mathbb{Z})$, the change of the standard smoothing of $\mathbb{R}^4$ to the exotic one, $e$, determines the twisting $\delta_e$ of the equivariant $K$-theory of the groupoid $SU(2) \times SU(2) \rightrightarrows SU(2)$ by the gerbe $\mathcal{L}_k$ over this groupoid where $d(\mathcal{L}_k) = e_k([e]) \in H^3_{SU(2)}(S^3, \mathbb{Z})$, i.e. the twisting of the equivariant $K$-theory of $SU(2)$ acting on itself by conjugation

$$\delta_e : K_{SU(2)}(SU(2)) \to \mathcal{L}_k K_{SU(2)}(SU(2))$$

(7)

where $S^3 \simeq SU(2)$ is the 3-sphere lying at the boundary of the Akbulut cork of $e$ and the codimension-1 foliations of this 3-sphere generates the codimension-1 foliations of the boundary.

Following [13] Ex. 1.7, we can explicitly compute the ,,exotic twisting” of the equivariant $K$-theory:
\[ K^n_{SU(2)}(SU(2)) = K^{(0)+n}_{SU(2)}(SU(2)) \to K^{\tau+n}_{SU(2)}(SU(2)) = \begin{cases} 0 & , n = 0 \\ R(SU(2))/\langle \rho_{k-1} \rangle & , n = 1 \end{cases} \]

where \( \tau = k[|] \in H^3(S^3, \mathbb{Z}) \) twists the ordinary equivariant \( K \)-groups, in fact \( R^0_{SU(2)} = K_{SU(2)}^{0,1} \) by Bott’s periodicity, and determines exotic \( \mathbb{R}^4 \), \( (\rho_l) \) are up to \( l \)-dimensional representations of \( SU(2) \) and \( R(SU(2)) \) is the ring of the representations of \( SU(2) \).

Composing the Theorem 3 with the result of Freed, Hopkins and Teleman [13], we arrive at the following formulation:

**Theorem 4.** Given two exotic \( \mathbb{R}^4 \)'s, \( e_k, e_k' \) corresponding to the integral cohomology classes \( [e_k] = (5+k)[|] \) and \( [e_k'] = (5+k')[|] \in H^3(S^3, \mathbb{Z}) \) where \( [|] \) is the generator of \( H^3(S^3, \mathbb{Z}) \), the change of the smoothing of \( \mathbb{R}^4 \) from \( e_k \) to \( e_k' \), determines the shift of the Verlinde algebra of \( SU(2) \) from the level \( k \) to \( k' \):

\[ V_k(SU(2)) \to V_{k'}(SU(2)) \] (8)

This is based on the relation \( R(SU(2))/\langle \rho_{k-1} \rangle = V_k(SU(2)) \). Here, one has \( [e_k] = (3 + 2 + k)[|] \) where 2 is the Dual Coxeter number for \( SU(2) \) and \( \dim(SU(2)) = 3 \) [13]. It is understood that the 3-spheres lying at the boundaries of the Akbulut corks are both the same \( S^3 = SU(2) \), and the difference between smoothings of \( \mathbb{R}^4 \) is seen as the shift of the levels of \( V_k(SU(2)) \) as in (8).

It follows that the changes between smoothings of some exotic \( \mathbb{R}^4 \)'s can be described in terms of 2-dimensional CFT or \( SU(2) \) WZW models (cf. [4]).

### 4. Conclusion

This paper is a natural enhancement of our previous work [3]. Here we concentrated on the integer classes \( H^3(S^3, \mathbb{Z}) \) which we interpreted as bundle gerbes. Then the full approach including the relation to twisted \( K \)-theory of orbifolds was worked out to show a relation to the Verlinde algebra. This result based on the work in [13] is not fully unexpected. In a ground-breaking paper of Witten [25], he related the theory of 3-manifolds to conformal field theory by using Chern-Simons theory. As we mentioned in subsection 2.2 (see also the appendix [3] below), the corresponding 3-form of Chern-Simons is the Godbillon-Vey invariant. But this invariant is the key to understand exotic smoothness on 4-manifolds. Thus we obtain a dimension ladder: a conformal field theory in 2 dimensions determines via the level the Godbillon-Vey invariant of a codimension-1 foliation of the 3-spheres which determines the smoothness structure on the 4-space \( \mathbb{R}^4 \) and vice versa.

The whole bunch of connections and relations in this paper are partly related to quantum field theory. Then we may ask: Is it possible to understand the quantization procedure in terms of exotic smoothness? We will answer this question in the next paper by analyzing the codimension-1 foliation on the 3-sphere \( S^3 \) more carefully.
Appendix

A. Non-cobordant foliations of $S^3$ detected by the Godbillon-Vey class

In [24], Thurston constructed a foliation of the 3-sphere $S^3$ depending on a polygon $P$ in the hyperbolic plane $\mathbb{H}^2$ so that two foliations are non-cobordant if the corresponding polygons have different areas. We will present this construction now.

Now we consider two codimension-1 foliations $F_1, F_2$ depending on the convex polygons $P_1, P_2$ in $\mathbb{H}^2$. As mentioned above, these foliations $F_1, F_2$ are defined by two one-forms $\omega_1$ and $\omega_2$ with $d\omega_a \wedge \omega_a = 0$ and $a = 0, 1$. Now we define the one-forms $\theta_a$ as the solution of the equation

$$d\omega_a = -\theta_a \wedge \omega_a$$

and consider the closed 3-form

$$\Gamma_{F_a} = \theta_a \wedge d\theta_a$$

(9)

associated to the foliation $F_a$. As discovered by Godbillon and Vey [15], $\Gamma_F$ depends only on the foliation $F$ and not on the realization via $\omega, \theta$. Thus $\Gamma_F$, the Godbillon-Vey class, is an invariant of the foliation. Let $F_1$ and $F_2$ be two cobordant foliations then $\Gamma_{F_1} = \Gamma_{F_2}$. In case of the polygon-dependent foliations $F_1, F_2$, Thurston [24] obtains

$$\Gamma_{F_a} = \text{vol}(\pi^{-1}(Q)) = 4\pi \cdot \text{Area}(P_a)$$

and thus

- $F_1$ is cobordant to $F_2$ $\implies$ $\text{Area}(P_1) = \text{Area}(P_2)$
- $F_1$ and $F_2$ are non-cobordant $\iff$ $\text{Area}(P_1) \neq \text{Area}(P_2)$

We note that $\text{Area}(P) = (k - 2)\pi - \sum k_i \alpha_k$. The Godbillon-Vey class is an element of the deRham cohomology $H^3(S^3, \mathbb{R})$ which will be used later to construct a relation to gerbes. Furthermore we remark that the classification is not complete. Thurston constructed only a surjective homomorphism from the group of cobordism classes of foliation of $S^3$ into the real numbers $\mathbb{R}$. We remark the close connection between the Godbillon-Vey class (9) and the Chern-Simons form if $\theta$ can be interpreted as connection of a suitable line bundle.

B. Interpretations of the integer classes

Apart from the bundle gerbe interpretation, we will present different interpretations for the reduction of $H^3(S^3, \mathbb{R})$ to the integer classes in $H^3(S^3, \mathbb{Z})$:

1. as intersection numbers in abelian Chern-Simons theory
2. as volume quantization of some leaves in the foliation
3. as obstruction cocycle to extend a section of a Haefliger structure
4. as parity anomaly of a $SU(2)$ gauge theory coupled to a Dirac field over the Alexander sphere
5. as equivalence classes of loop group $\Omega E_8$ bundles over the 3-sphere
6. as differential character a la Cheeger-Simons

ad 1. The cohomology class $[\Gamma_F]$ is unchanged by a shift $\theta \to \theta + d\phi$ of the one-form $\theta$ by an exact form, i.e. gauge invariance in the physical sense. Thus we can interpret the purely imaginary one-form $A = i\theta$ as a connection of a complex line bundle over $S^3$. Then the Godbillon-Vey class is the abelian Chern-Simons form with action integral

$$S = \int_{S^3} A \wedge dA .$$

To get any restrictions for that integral, we have to consider a 4-manifold with boundary $S^3$ which by using cobordism theory always exists. There are many models for such a 4-manifold. We start with a closed 4-manifold $M$, i.e. $\partial M = \emptyset$, and cut a 4-disk $D^4$ with $\partial D^4 = S^3$ off. Then we obtain the desired 4-manifold $N = M \setminus D^4$ with $\partial N = S^3$ and for the integral

$$\int_{\partial N} A \wedge dA = \int_N dA \wedge dA .$$

Now we will follow our interpretation above, that $A$ is the connection of a complex line bundle $L$. The first Chern class $c_1(L)$ of that bundle is given by $c_1(L) = \frac{i}{2\pi} dA$ classifying the complex line bundles over $N$. Then we obtain for the integral

$$S = \int_{\partial N} A \wedge dA = -4\pi^2 \int_N c_1 \wedge c_1$$

as the number of self-intersections of the 2-complex $PD(c_1(L))$ ($PD$ Poincare dual). Thus we obtain one possible interpretation of the integer classes $H^3(S^3, \mathbb{Z})$ as self-intersections in $N$ or as intersection between a 1-complex $PD(A)$ and $PD(dA)$ in the 3-sphere $S^3$. As example we can use the 4-manifold $M = \mathbb{C}P^2$ and construct $N = \mathbb{C}P^2 \setminus D^4$. Inside of $N$ there is a $\mathbb{C}P^1 = S^2 \subset N$ having a canonical complex line bundle. Then the integral $S$ is the self-intersection of $\mathbb{C}P^1$, i.e. the intersection form of $\mathbb{C}P^2$ having one single self-intersection with $S = 1$.

ad 2. A second interpretation is given by Thurston’s construction (see Appendix A) of non-cobordant, codimension-1 foliations on $S^3$. He used a polygon $P$ in the hyperbolic 2-space $\mathbb{H}^2$ to construction such a foliation. In the construction of the foliation, this polygon represents some of the leaves whereas the other are given by the Reeb components to fill in the punctures. Then the Godbillon-Vey invariant is proportional to the volume of the polygon $P$. Thus we obtain that the integer classes are equivalent to a quantization of the volumes of polygons and therefore to a quantization of the leaves of the foliation.

ad 3. The third interpretation used a slightly generalized version of a foliation, the Haefliger structure. The main idea was motivated by the observation that homotopy-theoretic properties of a foliation are similar to a bundle. But a $G$–principal bundle over $M$ is classified by the homotopy classes $[M, BG]$. Thus, one defines a Haefliger structure of codimension $q$ over $M$ which is classified by $[M, B\Gamma_q]$. Denote by $\Gamma_q$ the set of germs of local $C^r$-diffeomorphisms of $\mathbb{R}^q$ forming a topological groupoid. A codimension-$q$ Haefliger cocycle over an
open covering \( U(X) = \{ O_i \}_{i \in I} \) of a space \( X \) is an assignment to each pair of \( i, j \in I \) of a continuous map \( \gamma_{ij} : O_i \cap O_j \to \Gamma_q \) such that for all \( i, j, k \in I \)

\[
\gamma_{ij}(x) = \gamma_{ik}(x) \circ \gamma_{kj}(x)
\]

for all \( x \in O_i \cap O_j \cap O_k \). If we set \( g_{ij} = d\gamma_{ij} \) in a neighborhood of \( x \in O_i \cap O_j \) then one defines a \( q \)-dimensional vector bundle with transition function \( g_{ij} \), the normal bundle of the foliation. Two Haefliger structures \( \mathcal{H}_0, \mathcal{H}_1 \) over \( X \) are equivalent if both are concordant (or cobordant), i.e. there is a Haefliger structure \( \mathcal{H} \) on \( X \times [0, 1] \), so that \( \mathcal{H}_k = i_k^* \mathcal{H} \) with \( i : X \to X \times [0, 1] \), \( i_k(x) = (x, k) \). Furthermore, it is known that to every topological groupoid \( I \) there is a classifying space \( B\Gamma \). Now we specialize to the codimension-1 case over the 3-sphere, i.e. we consider maps \( SU(2) \) bundles are trivial and we have the gauge group \( G = SU(2) \) agreeing with the transition function of a bundle over \( S^3 \). The question can be answered by obstruction theory to state that the elements of \( H^3(S^3, \pi_3(B\Gamma_1)) \) are all possible extensions. Using Thurston’s surjective homomorphism we have uncountable infinite possible extensions, i.e. all elements of \( H^3(S^3, \mathbb{R}) \).

4. Finally we consider a Dirac theory coupled to a \( SU(2) \) gauge field defined over the “spacetime” \( \mathbb{R} \times S^2 \) or \( [0, 1] \times S^2 \). The classical action is constructed to be parity-invariant but the quantized theory fails to have that symmetry. The problem is the appearance of a symmetry-braking phase

\[
\Phi = \exp \left( \frac{i}{2} \eta \right)
\]

with the \( \eta \) invariant of the Dirac operator. As stated by Yoshida [28], this invariant is related to the Chern-Simons functional

\[
CS(A) = \frac{1}{8\pi^2} \int_{S^2 \times [0, 1]} tr \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right)
\]

of the \( SU(2) \) gauge field. A bundle over \( S^2 \times [0, 1] \) is determined by the transition function \( S^2 \to SU(2) \) agreeing with the transition function of a bundle over \( S^3 \). Both \( SU(2) \) bundles are trivial and we have the gauge group \( \mathcal{G} = \text{Map}(S^3, SU(2)) = \Omega^3 SU(2) \). The isotopy group \( \pi_0(\mathcal{G}) \) or the number of connecting components is given by

\[
\pi_0(\mathcal{G}) = \pi_0(\Omega^3 SU(2)) = \pi_3(SU(2)) = \mathbb{Z}
\]

determines the group \( H^3(S^3, \mathbb{Z}) = H^3(S^3, \mathbb{Z}) \) via duality and Hurewicz isomorphism. The elements \( g \) of the isotopy group \( g \in \pi_0(\mathcal{G}) \) are global gauge transformations

\[
A \to A + g^{-1}dg
\]

Actually \([M, BI_q]\) has more structure than a set, i.e. it defines a generalized cohomology theory like \([M, BG]\) defines (complex) K theory for \( G = SU \) or real K theory for \( G = SO \).
changing the Chern-Simons functional to

$$CS(A) \rightarrow CS(A + g^{-1} dg) = CS(A) + \frac{1}{8\pi^2} \int_{S^2 \times [0,1]} (g^{-1} dg)^3$$

The last expression, the WZW functional, admits integer values so that the 3-form $(g^{-1} dg)^3$ can be seen as element of $H^3(S^3, \mathbb{Z})$ via the isomorphism $S^3 = SU(2)$.

ad 5. Consider the semi-simple Lie group $E_8$ as 248-dimensional, smooth manifold. If we introduce a twisted product $P = M \ast G$ to express that $P$ is a $G$-principal bundle over $M$ then we have the splitting (see [8] (IV.1) on page 154)

$$E_8 = S^3 \ast S^{15} \ast S^{23} \ast S^{27} \ast S^{35} \ast S^{39} \ast S^{47} \ast S^{59}$$

Thus we can immediately write down the first homotopy groups:

$$\pi_i(E_8) = 0 \quad i < 3$$
$$\pi_3(E_8) = \mathbb{Z}$$
$$\pi_k(E_8) = 0 \quad 4 < k < 15$$

Thus the $E_8$ is an Eilenberg-MacLane space

$$E_8 \sim K(\mathbb{Z}, 3) \quad \text{up to 14-skeleton}$$

Now we consider an $E_8$ bundle over a manifold $M$ of dimension $\dim M < 15$. Such bundles are classified by the (abelian group of) homotopy classes $[M, BE_8]$. The space $BE_8$ is then given by

$$BE_8 \sim K(\mathbb{Z}, 4) \quad \text{up to 14-skeleton}$$

and thus

$$[M, BE_8] = [M, K(\mathbb{Z}, 4)] = H^4(M, \mathbb{Z}) \quad \text{up to 14-skeleton.}$$

Every $E_8$ bundle over the 3-sphere $S^3$ is then classified by $[S^3, BE_8] = H^4(S^3, \mathbb{Z}) = 0$, i.e. every $E_8$ bundle is trivial over any 3-manifold. But by using the path fibration of $BE_8$ we get the homotopy equivalence

$$\Omega BE_8 \sim E_8$$

where $\Omega BE_8$ is the mapping space of maps $S^1 \rightarrow BE_8$. Both functors can be interchanged to get

$$B(\Omega E_8) \sim E_8$$

To proof this we consider the path fibration

$$\Omega E_8 \rightarrow PE_8 \rightarrow E_8$$

where $PE_8$ are all maps $[0, 1] \rightarrow E_8$ and construct via the functor $B$ another fibration

$$B(\Omega E_8) \rightarrow B(PE_8) \rightarrow BE_8$$
The space $B(PE_8)$ is contractable and we obtain the desired result. Then we can construct a bundle with structure group $\Omega E_8$ classified by

$$[M, B(\Omega E_8)] = [M, E_8] = [M, K(\mathbb{Z}, 3)] = H^3(M, \mathbb{Z})$$

up to 14-skeleton.

Then a $\Omega E_8$ principal bundle over $S^3$ is classified by $[S^3, B(\Omega E_8)] = H^3(S^3, \mathbb{Z}) = \mathbb{Z}$. We remark that these classes are canonical isomorphic to classes of $E_8$ bundles over the 4-sphere $S^4$ via the isomorphism $[S^3, B(\Omega E_8)] = [S^3, \Omega B E_8] = [S^4, BE_8]$.

ad 6. Last but not least we consider the isomorphism $H^2(M, S^1) = H^3(M, \mathbb{Z})$ induced by the exact sequence

$$1 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} = S^1 \to 1$$

(in the sheaf-theoretic sense) and ask for the realization of classes in $H^2(M, S^1)$. In short, Cheeger and Simons [9] studied special homomorphism from the abelian group of (integer) cycles $Z_k(M, \mathbb{Z})$ into the circle group $S^1$. Now we consider a 3-cycle $c \in Z_3(M, \mathbb{Z})$ together with a 3-form $\omega$ and define the homorphism as

$$\chi(\partial c) = \exp \left( 2\pi i \int_c \omega \right)$$

A concrete realization of the 3-form is given by the Chern-Simons form.

C. Twisted $K$-theory over a groupoid

It was shown in [7] that the twisted $K$-theory of a pair $(M, [H])$, where $M$ is a manifold and $[H]$ is an integral Čech class - the curvature of a gerbe, can be obtained from the $K$-theory of the bundle gerbes representing the Morita equivalence class of the gerbe. The twisting of the $K$-theory contains 2 different, though related via gerbes, situations. This is the case of $[H]$ which is a torsion class in $H^3(M, \mathbb{Z})$ and non-torsion $[H]$. The first case was considered by Witten [26, 27] in the context of string theory and charges of $Dp$-branes. The second is important for us since $H^3(S^3, \mathbb{Z})$ is torsionless. Non-torsion elements twisting the $K$-theory were found to be important in string theory as well. Both cases appeared as being uniformly described via bundle gerbes [19, 7].

One can define the $K$-theory of bundle gerbes as the Grothendieck group of the semi-group of bundle gerbe modules. These bundle gerbes modules are finite dimensional. When $[H]$ is torsion in $H^3(M, \mathbb{Z})$, bundle gerbe $K$-theory is isomorphic to twisted $K$-theory $K(M, [H])$. When $[H]$ is not a torsion class one should consider the lifting bundle gerbe associated to a projective $PU(H)$-bundle (which will be explain below) with Dixmier-Douady class $[H]$. In this case the twisted $K$-theory is the Grothendieck group of the semi-group of, the discussed later on, „$U_K$-bundle gerbe modules”. These are infinite dimensional bundle gerbe modules.

Let $L = (L, Y)$ be a bundle gerbe over a manifold $M$, as in Sec. 3.1 Def. 2, and let $E \to Y$ be a finite rank, hermitian vector bundle. If $\phi$ is the isomorphism
of hermitian bundles $\phi : L \otimes \pi_1^{-1}E \to \pi_2^{-1}E$ it can be \textit{compatible with the bundle gerbe multiplication} in the sense that the map given by:

$$L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes E_{y_3}) \to L_{(y_1,y_2)} \otimes E_{y_2} \to E_{y_1}$$

is equal to the following map:

$$L_{(y_1,y_2)} \otimes (L_{(y_2,y_3)} \otimes E_{y_3}) \to L_{(y_1,y_2)} \otimes E_{y_2} \to E_{y_1}.$$

We say that this bundle gerbe \textit{acts} on $E$.

**Definition 3.** A hermitian vector bundle $E$ over $M$ is a bundle gerbe module if the isomorphism $\phi : L \otimes \pi_1^{-1}E \to \pi_2^{-1}E$ is compatible with the bundle gerbe multiplication as above.

Two bundle gerbe modules are \textit{isomorphic} if they are isomorphic as vector bundles and the isomorphism preserves the action of the bundle gerbe. Let $\text{Mod}(\mathcal{L})$ be the set of all isomorphism classes of bundle gerbe modules for $\mathcal{L}$. The set $\text{Mod}(\mathcal{L})$ is a semi-group [7]. Recall that $d(\mathcal{L}) \in H^3(M,\mathbb{Z})$ is the Dixmier-Douady class of a bundle gerbe $\mathcal{L} = (L,Y)$ as in Sec. 3.1.

**Definition 4.** Given a bundle gerbe $\mathcal{L} = (L,Y)$ with torsion Dixmier-Douady class, $d(\mathcal{L}) \in H^3(M,\mathbb{Z})$, the Grothendieck group of the semi-group $\text{Mod}(\mathcal{L})$, is the $K$ group of the bundle gerbe and is denoted as $K(\mathcal{L})$.

The group $K(\mathcal{L})$ depends only on the class $d(\mathcal{L}) \in H^3(M,\mathbb{Z})$ since every stable isomorphism between bundle gerbes $\mathcal{L}$ and $\mathcal{J}$ defines a canonical isomorphism $K(\mathcal{L}) \simeq K(\mathcal{J})$. For any class $[H]$ in $H^3(M,\mathbb{Z})$ we can define a bundle gerbe $\mathcal{L}$ with $d(\mathcal{L}) = [H]$ and its group $K(\mathcal{L})$. Due to the dependence on $[H]$ and the relation of bundle gerbes with the manifold $M$ this group is sometimes denoted by $K_{bg}(M,[H])$.

In particular it holds [7]:

1. If $\mathcal{L} = (L,Y)$ is a trivial bundle gerbe then $K_{bg}(\mathcal{L}) = K(M)$ where $K(M)$ is the untwisted $K$-theory of the manifold $M$.
2. $K_{bg}(\mathcal{L})$ is a module over $K(M)$.

We have made use of the point 1. above for the case $M = S^3$ in Sec. 3.2.

As was explained in Sec. 13 given a class $[H] \in H^3(M,\mathbb{Z})$ we can represent it by a projective $PU(\mathcal{H})$ bundle $Y$ whose class is $[H]$. Here $\mathcal{H}$ is some separable, possibly infinite dimensional, Hilbert space and $U(\mathcal{H})$ is the group of unitary operators on $\mathcal{H}$. This is because the classifying space of the third cohomology group of $M$ is the Eilenberg-Maclane space $K(\mathbb{Z},3)$. The projective unitary group on $\mathcal{H}$, $PU(\mathcal{H}) = U(\mathcal{H})/U(1)$, can be defined. A model for $K(\mathbb{Z},3)$ is the classifying space of $PU(\mathcal{H})$, i.e., $K(\mathbb{Z},3) = BPU(\mathcal{H})$. This means that $H^3(M,\mathbb{Z}) = [M,K(\mathbb{Z},3)] = [M,BPU(\mathcal{H})]$, where $[X,Y]$ denotes the homotopy classes of continuous maps from $X$ to $Y$. Thus we obtain the realization of $H^3(M,\mathbb{Z})$:

\textit{Isomorphism classes of principal $PU(\mathcal{H})$ bundles over $M$ are parametrized by $H^3(M,\mathbb{Z})$}.

If $Fred$ is the space of Fredholm operators on $\mathcal{H}$, then, non-twisted $K$-theory of $M$ is determined by $[5]$.
\[ K(M) = [M, \text{Fred}] \quad \text{(10)} \]

We can associate to a class \([H]\) (torsion or not) \(PU(H)\) bundle \(Y\) representing the class. Let \(PU(H)\) acts by conjugations on \(\text{Fred}\). We can form an associated bundle

\[ Y(\text{Fred}) = Y \otimes_{PU(H)} \text{Fred} = Y \sim \text{Fred} \]

Let \([M, Y(\text{Fred})]\) denote the space of all homotopy classes of sections of the bundle \(Y(\text{Fred})\). Then one can define the twisted \(K\)-theory:

**Definition 5.** The twisted by \([H] \in H^3(M, \mathbb{Z})\) \(K\)-theory of \(M\), i.e. \(K(M, [H])\) is given by the homotopy classes \([M, Y(\text{Fred})]\) of the sections of \(Y(\text{Fred})\), i.e.

\[ K(M, [H]) = [M, Y(\text{Fred})] \quad \text{(11)} \]

It holds:

**Theorem 5.** For a torsion class \([H] \in H^3(M, \mathbb{Z})\) the twisted \(K\)-theory and bundle gerbe \(K\)-theory of \(L = (M, L)\), coincide, i.e.

\[ K(M, [H]) = K_{bg}(M, L) \quad \text{(12)} \]

where \(d(L) = [H]\).

In general \([H]\) need not be torsion. One can still relate the twisted \(K\)-theory of a groupoid with the classes of gerbes over groupoids, such that in the particular case of manifolds one yields the twisted \(K\)-theory of these by (bundle) gerbes over the manifolds, and the Dixmier-Duady class of the bundle gerbe is the twisting non-torsion 3-rd integral cohomology class \([H]\) [19].

Let us recall that in the case of a smooth manifold \(M\) the set of isomorphism classes of gerbes, \(Gb(M)\), is the set of the homotopy classes \([X, BU(\mathcal{H})]\) for some Hilbert space \(\mathcal{H}\). Let \(PU(\mathcal{H})\) denotes the groupoid \(\star \times PU(\mathcal{H}) \to \star\). Then one can prove (Proposition 6.2.5. in [19]):

**For an orbifold \(Ob_a\) given by a groupoid \(G\) we have \(Gb(G) = [X, PU(\mathcal{H})]\) where \([X, PU(\mathcal{H})]\) represents the Morita equivalence classes of morphisms from \(G\) to \(PU(\mathcal{H})\). For a manifold \(M\) we obtain that \([X, PU(\mathcal{H})] = H^3(M, \mathbb{Z}) = Gb(M)\) where \(X\) is the groupoid representing \(M\). In what follows we will not distinguish between groupoid, say \(PU(\mathcal{H})\), and the space, say \(PU(\mathcal{H})\), when the meaning of their use is fixed by the context.**

In the case of a non-torsion class \(\alpha\) on an orbifold \(Ob_a\) which is represented by the morphism \(\alpha : \mathcal{G} \to PU(\mathcal{H})\) one should somehow deal with infinite dimensional vector spaces. Following [19] let \(K\) be the space of compact operators of a Hilbert space \(\mathcal{H}\). Let \(U_K\) be the subgroup of \(U(\mathcal{H})\) consisting of unitary operators \(I + K\) where \(I\) is the identity operator and \(K \in K\). If \(h \in PU(\mathcal{H})\) and \(g \in U_K\) then \(hgh^{-1} \in U_K\) and the semiproduct \(U_K \otimes PU(\mathcal{H})\) is defined.

Now the \(K\)-theory for an orbifold \(Ob_a\) represented by the groupoid \(\mathcal{G}\), twisted by a gerbe \(L\) with non-torsion class \(\alpha : \mathcal{G} \to PU(\mathcal{H})\) can be defined (Def. 7.2.15 in [19]):
Definition 6. Let us consider the set of isomorphisms classes of groupoid homomorphisms $f: G \to U_K \otimes PU(H)$ which are lifts of the homomorphisms $\alpha: G \to PU(H)$ such that these are compatible with the projection $q_2: U_K \otimes PU(H) \to PU(H)$, i.e. $q_2 \circ f = \alpha$. This set is the groupoid $K$-theory twisted by the gerbe $L$ and is denoted by $\mathcal{L}K_{gpd}(G)$.

One advantage dealing with gerbes on groupoids is that this includes twisted equivariant $K$-theory on manifolds automatically. When the groupoid is $G := SU(2) \otimes SU(2) \Rightarrow SU(2)$ as was the case in Sec. 3.2., and $SU(2)$ acts on itself by conjugation, one can reformulate the remarkable result of Freed, Hopkins and Teleman in terms of ,,twistings by gerbes”, i.e. the twisted $K$-theory on the groupoid by the class $d(L)$ from $H^3(S^3, \mathbb{Z})$, or twisted equivariant $K$-theory on $S^3$ by non-torsion $[H]$, is precisely $\mathcal{L}K_{gpd}(G)$ where $d(L) = (\dim(SU(2) + 2 + k)[] \in H^3(S^3, \mathbb{Z})$. Moreover $\mathcal{L}K_{gpd}(G) = V_k(SU(2))$ is the Verlinde algebra of $SU(2)$ at level $k$.

Acknowledgment

T.A. wants to thank C.H. Brans and H. Rosé for numerous discussions over the years about the relation of exotic smoothness to physics. J.K. thanks for the explanations and suggestions given to him by Robert Gompf several years ago, and for the influential discussions with Jan Sladkowski.

References

1. A. Adem and Y. Ruan. Twisted orbifold K-theory. Comm. Math. Phys., 237:533, 2003. arXiv: math/0107108
2. J.W. Alexander. An example of a simple-connected surface bounding a region which is not simply connected. Proceedings of the National Academy of Sciences of the United States, 10:8 – 10, 1924.
3. T. Asselmeyer-Maluga and C.H. Brans. Exotic Smoothness and Physics. WorldScientific Publ., Singapore, 2007.
4. T. Asselmeyer-Maluga and J. Krol. Gerbes, SU(2) WZW models and exotic smooth $\mathbb{R}^4$. submitted to Comm. Math. Phys., arXiv: 0904.1276.
5. M.F. Atiyah and G.B. Segal. Twisted K-theory. Ukrainian Math. Bull., 1, 2004.
6. B. Bizaca. A reimbedding algorithm for Casson handles. Trans. Amer. Math. Soc., 345:435–510, 1994.
7. P. Bouwknegt, A.L. Carey, V. Mathai, M.K. Murray, and D. Stevenson. Twisted K-theory and K-theory of bundle gerbes. Comm. Math. Phys., 228:17–45, 2002. arXiv: hep-th/0106194.
8. L.J. Boya. The geometry of compact lie groups. Rep. Math. Phys., 30:149–161, 1992.
9. J. Cheeger and J. Simons. Differential characters and geometric invariants. In Geometry and Topology, Lect. Notes in Math. No. 1167, pages 50–80, New York, 1985. Springer-Verlag.
10. W. Chen and Y. Ruan. A new cohomology theory for orbifolds. Comm. Math. Phys., 248:1, 2004. arXiv: math/0004129
11. S. Donaldson. An application of gauge theory to the topology of 4-manifolds. J. Diff. Geom., 18:269–316, 1983.
12. D. Freed. The verlinde algebra is twisted equivariant K-theory. Turk. J. Math., 25:159–167, 2001. arXiv: math.RT/0101035
13. D. Freed, M.J. Hopkins, and C. Teleman. Loop groups and twisted K-theory. arXiv: 0711.1906, 2002.
14. M.H. Freedman. The topology of four-dimensional manifolds. J. Diff. Geom., 17:357 – 454, 1982.
15. C. Godbillon and J. Vey. Un invariant des feuilletages de codimension. *C. R. Acad. Sci. Paris Ser. A-B*, 273:A92, 1971.
16. R.E. Gompf and A.I. Stipsicz. *4-manifolds and Kirby Calculus*. American Mathematical Society, 1999.
17. N. Hitchin. Lectures on Special Lagrangian Submanifolds. In *Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, volume 23 of *AMS/IP Stud. Adv. Math.*, pages 151 – 182, Providence, RI, 2001. AMS.
18. H.B. Lawson. Foliations. *BAMS*, 80:369 – 418, 1974.
19. E. Lupercio and B. Uribe. Gerbes over orbifolds and twisted K-theory. arXiv: math/0105039, 2001.
20. E. Lupercio and B. Uribe. An introduction to gerbes over orbifolds. *Ann. Math. Blaise Pascal*, 11:155, 2004.
21. J. Milnor. Construction of universal bundles I. *Ann. Math.*, 63:272 – 284, 1956.
22. J. Milnor. Construction of universal bundles II. *Ann. Math.*, 63:430 – 436, 1956.
23. I. Moerdijk and D.A. Pronk. Simplicial cohomology of orbifolds. *Indag. Math. (N.S.)*, 10:269, 1999.
24. W. Thurston. Noncobordant foliations of $S^3$. *BAMS*, 78:511 – 514, 1972.
25. E. Witten. Quantum field theory and the Jones polynomial. *Comm. Math. Phys.*, 121:351–400, 1989.
26. E. Witten. D-branes and K-theory. *J. High Energy Phys.*, 9812:019, 1998.
27. E. Witten. Overview of K-theory applied to strings. *Int. J. Mod. Phys.*, A16:693–706, 2000.
28. T. Yoshida. The $\eta$-invariant of hyperbolic 3-manifolds. *Inv. Math.*, 81:473–514, 1985.

Communicated by name