The effective action for edge states
in higher dimensional quantum Hall systems

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Abstract

We show that the effective action for the edge excitations of a quantum Hall droplet of fermions in higher dimensions is generically given by a chiral bosonic action. We explicitly analyze the quantum Hall effect on complex projective spaces CP^k, with a U(1) background magnetic field. The edge excitations are described by abelian bosonic fields on S^{2k-1} with only one spatial direction along the boundary of the droplet relevant for the dynamics. Our analysis also leads to an action for edge excitations for the case of the Zhang-Hu four dimensional quantum Hall effect defined on S^4 with an SU(2) background magnetic field, using the fact that CP^3 is an S^2 bundle over S^4.

To the memory of Professor Bunji Sakita

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1 Introduction

The study of quantum Hall effect in more general contexts than the classic two-dimensional plane, such as higher dimensions and different geometries, has recently been of some research interest, following the original analysis by Zhang and Hu [1]. They considered the Landau problem for charged fermions on $S^4$ with a background magnetic field which is the standard $SU(2)$ instanton. In ordinary quantum Hall effect a droplet of fermions occupying a certain area behaves as an incompressible fluid, the low energy excitations being area-preserving deformations which behave as massless chiral bosons. The motivation for considering $S^4$ was that the edge excitations could lead to higher spin massless fields, in particular the graviton, and might provide an approach to a quantum description of a graviton in four dimensions. In fact, in order to obtain a reasonable thermodynamic limit with a finite spatial density of particles and incompressibility properties, one has to consider that the fermions belong to a very large $SU(2)$ representation with infinite degrees of freedom. As a result one expects to find edge excitations of all values of spin present.

A number of papers have extended the original idea of Zhang and Hu [2]–[6]. In a previous paper [2] we studied the quantum Hall effect on even dimensional complex projective spaces $\mathbb{C}P^k$. This is an interesting case since one can obtain incompressible droplet states by coupling the fermions to a $U(1)$ background field, thus avoiding the infinite internal degrees of freedom. It was also suggested in [2] that the $S^4$ case with $SU(2)$ background could be understood in terms of a higher dimensional model, $\mathbb{C}P^3$ with a $U(1)$ background, using the fact that the latter is an $S^2$-bundle over $S^4$. This point of view has been explored further in [4, 5, 6].

Edge excitations for droplets in $R^4$ with a $U(1)$ and $SU(2)$ background, essentially the flat limit of the Zhang-Hu model, were studied in [7]. Since a droplet is confined to a finite volume and that is topologically a neighbourhood in $R^4$, the analysis in [7] yields many of the generic features of the edge excitations. For example, it was argued that the edge effective theory is essentially an infinite collection of one-dimensional theories and also lacks full Lorentz invariance.

The edge excitations of a quantum Hall droplet leads to a new class of field theories and so, the original motivation aside, some further analysis is warranted. What would be the spectrum of edge excitations on a general space? How is their dynamics related to the geometrical structures of the space on which the droplet is? These are some of the questions we take up in this paper. We find that the effective action for edge excitations is generically a chiral action in terms of an abelian bosonic field. Chirality here means that there is a preferred direction along the surface of the droplet that determines the dynamics and this is chosen by the geometry of the underlying space, where the nonrelativistic fermions are. The bosonic field itself can be expanded in terms of the surface oscillation modes of the incompressible droplet. This effectively provides a bosonic description of the underlying nonrelativistic fermionic theory, very similar in spirit with the collective variable approach.
introduced by Sakita [8].

The paper is organized as follows. In section 2 we present an approach for deriving a generic effective action for edge excitations for any quantum Hall system and outline the basic ingredients needed for such a construction. In section 3 we analyze in detail the case of \( \mathbb{CP}^k \) quantum Hall systems with a \( U(1) \) background field. Our analysis requires the notion of star products and we present a simple way of writing down star products for \( \mathbb{CP}^k \) spaces. In section 4 we derive, using this method, the well known one-dimensional chiral boson action for QHE on \( S^2 \). In section 5 we apply our approach along with the fact that \( \mathbb{CP}^3 \) can be viewed as an \( S^2 \)-bundle over \( S^4 \) to derive the effective edge action for the four dimensional quantum Hall system introduced by Zhang and Hu [1]. We conclude with a summary of our results.

2 The effective action for edge states

We will begin with a general discussion of the effective action for edge states for any quantum droplet. The strategy we follow is an adaptation of a method used by Sakita [8] to derive the edge effective action in the case of the conventional two-dimensional quantum Hall effect. The basic observation is very simple. The choice of the droplet we are considering is specified quantum mechanically by a density matrix \( \hat{\rho}_0 \). Once this state is chosen, time-evolution being unitary, any density matrix which is connected to \( \hat{\rho}_0 \) by time-evolution is of the form

\[
\hat{\rho} = \hat{U}\hat{\rho}_0\hat{U}^\dagger,
\]

where \( \hat{U} \) is a unitary operator. The entire dynamical information is in \( \hat{U} \) and we can write down an action for \( \hat{U} \) as

\[
S = \int dt \left[ iT\text{Tr}(\hat{\rho}_0\hat{U}^\dagger\partial_t\hat{U}) - \text{Tr}(\hat{\rho}_0\hat{U}^\dagger\mathcal{H}\hat{U}) \right]
\]

where \( \mathcal{H} \) is the Hamiltonian. Variation of \( \hat{U} \) leads to the extremization condition for \( S \) as

\[
\frac{i}{\hbar}\frac{\partial \hat{\rho}}{\partial t} = [\mathcal{H}, \hat{\rho}]
\]

which is the expected evolution equation for the density matrix. The strategy for obtaining an effective action for edge states is to evaluate various quantities in (1) as classical functions in a suitable ‘semiclassical’ approximation, in the limit of a large number of available states.

If we apply this in full generality to a quantum Hall droplet with \( K \) fermions, \( \hat{\rho}_0 \) should be specified as \( |\alpha\rangle \langle \alpha| \), where \( |\alpha\rangle \) is a state in the full Fock space which is of the form

\[
|\alpha\rangle = \int f(x_1, x_2, ..., x_K) \psi_1^\dagger(x_1)\psi_2^\dagger(x_2)\ldots\psi_K^\dagger(x_K) |0\rangle
\]

where \( \psi_i^\dagger \)'s are creation operators for fermions and \( f(x_1, x_2, ..., x_K) \) is the \( K \)-body wavefunction of the droplet state. The function \( f(x_1, x_2, ..., x_K) \) can include the effects of many-particle correlations. In practice, for a \( \nu = 1 \) droplet, one can take \( f(x_1, x_2, ..., x_K) \) to be
a product of effective one-particle functions suitably antisymmetrized, in other words, a 
Hartree-Fock approximation. If we further neglect interparticle interactions, concentrating 
on the edge dynamics, $\mathcal{H}$ is given in the form $\int \psi^\dagger \hat{h} \psi$ and the traces and operators in (10) reduce to the one-particle Hilbert space. In this case, $\hat{\rho}_0 = \sum_{i=0}^{K} \langle i | i \rangle$ where $| i \rangle$'s are the occupied one-particle states and $\hat{U}$ is now an operator on this one-particle Hilbert space. Since we will be applying this to quantum Hall droplets on compact spaces like $\mathbb{C}P^k$, we take the one-particle Hilbert space to have a finite dimension $N\times N$, so that $\hat{U}$ is a unitary ($N\times N$)-matrix. $K$ states in this Hilbert space are occupied and we will simplify all the expressions in the limit $N \to \infty$, and $K$ very large, but finite. For an abelian magnetic field, the large $N$-limit of $\hat{U}$ becomes an abelian field characterizing the surface deformations of the droplet.

There are basically two ingredients needed to extract an edge state action from (1).

1. We need $\hat{\rho}_0$, in the large $N$ limit, to be constant over the phase volume occupied by the droplet $D$, so that its derivative tends to a $\delta$-function with support at the edge of the droplet.

2. Commutators between operators tend to suitable Poisson brackets as $N \to \infty$ (which is not the usual $\hbar \to 0$ limit).

For the $\mathbb{C}P^k$ cases of interest, these results will be shown in the next section. The states will be characterized by an integer $n \sim BR^2$ where $B$ is the magnetic field and $R$ is the radius of the $\mathbb{C}P^k$; the dimension $N \sim n^k$ and scaling is done with $R \sim \sqrt{n}$. With this preparation, the effective action is easily obtained.

$$\begin{align*}
\text{iTr}(\hat{\rho}_0 \hat{U}^\dagger \partial_t \hat{U}) &= \text{iTr} \left[ \hat{\rho}_0 e^{-i\Phi} \frac{\partial}{\partial t} e^{i\Phi} \right] \\
&= -\int_0^1 d\alpha \text{Tr} \left[ \hat{\rho}_0 e^{-i\alpha\Phi} (\partial_t \hat{\Phi}) e^{i\alpha\Phi} \right] \\
&= -\int_0^1 d\alpha \text{Tr} \left[ e^{i\alpha\Phi} \hat{\rho}_0 e^{-i\alpha\Phi} \partial_t \hat{\Phi} \right] \\
&= -\int_0^1 d\alpha \text{Tr} \left[ (\hat{\rho}_0 + i\alpha [\Phi, \hat{\rho}_0] + \frac{i^2\alpha^2}{2} [\Phi, [\hat{\Phi}, \hat{\rho}_0]] + \cdots) \partial_t \hat{\Phi} \right]
\end{align*}$$

(4)

As $N \to \infty$,

$$
[\hat{\Phi}, \hat{\rho}_0] \approx \frac{i}{n} \{\Phi, \rho_0\}
$$

$$
[\hat{\Phi}, [\hat{\Phi}, \hat{\rho}_0]] \approx i^2 \left( \frac{c}{n} \right)^2 \{\Phi, \{\Phi, \rho_0\}\}, \quad \text{etc.}
$$

(5)

where $c$ is a constant of order one and $\Phi, \rho_0$ are the classical functions corresponding to $\hat{\Phi}, \hat{\rho}_0$. The Poisson brackets in (5) are calculated using rescaled, dimensionless coordinates.
Further,
\[ \text{Tr} \approx N \int d\mu \]  
(6)
where \( d\mu \) is the volume element on the phase space. The large \( N \) limit of (4) is thus
\[ i\text{Tr}(\hat{\rho}_0 U^\dagger \partial_t U) \approx \frac{Nc}{2n} \int d\mu \{ \Phi, \rho_0 \} \partial_t \Phi \]  
(7)
where we have dropped terms which are total time-derivative.

Let \( \Omega_{ij} \) be the symplectic two-form of the phase space corresponding to the Hilbert space of one-particle states. Then
\[ \{ \Phi, \rho_0 \} = (\Omega^{-1})^{ij} \partial\Phi / \partial \xi^i \partial \rho_0 / \partial \xi^j \]
\[ = (\Omega^{-1})^{ij} \hat{e}_j \partial\Phi / \partial \xi^i \partial \rho_0 / \partial r \]
\[ = (L\Phi) \left( \frac{\partial \rho_0}{\partial r^2} \right) \]  
(8)
where
\[ L\Phi = \left[ 2r \hat{e}_j (\Omega^{-1})^{ij} \frac{\partial \Phi}{\partial \xi^i} \right] \]
(9)
The coordinates \( \xi_i \) are dimensionless, measured in units of the radius \( R \) of the compact space, \( \xi_i = (x_i + iy_i)/R \). \( \hat{e}_j \) is a unit vector normal to the boundary \( \partial D \) of the droplet and \( r = \sqrt{\sum \xi_i \xi_j} \) is the dimensionless, normal coordinate. \( L\Phi \) involves only derivatives of \( \Phi \) with respect to a tangential direction. In deriving (8) we have used the fact that \( \rho_0 \) depends only on the normal coordinate \( r \). Thus
\[ i\text{Tr}(\hat{\rho}_0 U^\dagger \partial_t U) \approx \frac{Nc}{2n} \int d\mu \frac{\partial \rho_0}{\partial r^2} \left( \frac{\partial \Phi}{\partial t} L\Phi \right) \]  
(10)
The term involving the Hamiltonian can be simplified as follows. For the Landau level under consideration, we have \( N \) degenerate states of energy \( E_0 \) to begin with, the degeneracy being lifted by a potential \( \hat{V} \); thus \( \mathcal{H} = E_0 + \hat{V} \). The constant term \( E_0 \) is not of interest for edge dynamics, so we can write
\[ \text{Tr}(\hat{\rho}_0 U^\dagger \mathcal{H} U) = \text{constant} + \int_0^1 d\alpha \text{Tr}(\hat{\rho}_0 e^{-i\alpha \hat{\Phi}} [\hat{V}, \hat{\Phi}] e^{i\alpha \hat{\Phi}}) \]
\[ = \text{constant} + \text{Tr} \left( \hat{\rho}_0 [\hat{V}, \hat{\Phi}] + \frac{1}{2} \hat{\rho}_0 \{ \hat{V}, \hat{\Phi} \} + \cdots \right) \]
\[ = \text{constant} + i\text{Tr}(\{\hat{\rho}_0, \hat{V}\} \hat{\Phi}) + \frac{1}{2} \text{Tr}(\{\hat{\rho}_0, \hat{\Phi}\} [\hat{V}, \hat{\Phi}]) + \cdots \]
\[ \approx \text{constant} + \frac{Nc}{n} \int d\mu \{ V, \rho_0 \} \Phi - \frac{Nc^2}{2n^2} \int d\mu \{ \rho_0, \Phi \} \{ V, \Phi \} + \cdots \]  
(11)
We choose $\rho_0$ to minimize the potential energy, the particles being located at states of minimum available energy. $V$ is thus a constant along directions tangential to the boundary of the droplet $\partial D$, it depends only on the normal coordinate $r$. Further $\{V, \rho_0\} = 0$. The last term in (11) simplifies as

$$\{\rho_0, \Phi\}{\{V, \Phi\}} = \frac{\partial \rho_0}{\partial r^2} (L\Phi) \frac{\partial V}{\partial r^2} (L\Phi)$$

Putting all this together, we have, for the effective action for the edge states,

$$S \approx \frac{Nc}{2n} \int dt d\mu \left(\frac{\partial \rho_0}{\partial r^2}\right) \left[ \frac{\partial \Phi}{\partial t} (L\Phi) + \frac{c}{n} \frac{\partial V}{\partial r^2} (L\Phi)^2 \right] + \text{constant} + O(1/n)$$

where the operator $L$ in the above expression is the $n \to \infty$ limit of the expression (9).

As we mentioned earlier, in the large $N$ limit, $\partial \rho_0/\partial r^2$ is a $\delta$-function with support only at the edge of the droplet; as a result the $r$-integration can be done and the action is entirely on the boundary of the droplet. Since the radius of the space scales like $\sqrt{n}$, the integral will give an extra factor of $n^{-k+1}$ for a compact $2k$ dimensional space. For the $\mathbb{C}P^k$ cases we analyze in detail in the next section, we show that $N \sim n^k$, so that the limit $n \to \infty$ gives a finite prefactor to the action. Also the potentials we choose will be such that $\partial V/\partial r^2 \sim n$ and so, $\omega = \frac{c}{n} (\partial V/\partial r^2)$ is finite as $n \to \infty$. Notice that if the potential does not have this property, then $\omega$ goes to zero or infinity as $n \to \infty$. In the first case, there are zero energy modes for the field $\Phi$, indicating that the potential has not uniquely confined the droplet to the shape initially chosen. In the second case, no fluctuations of the droplet are energetically allowed. It is thus physically reasonable to take the potential to scale as $n$ leading to a constant nonzero $\omega$ as $n \to \infty$. The final effective edge action (up to a multiplicative and additive constant) is of the form

$$S \approx -\int_{\partial D} dt \left[ \frac{\partial \Phi}{\partial t} (L\Phi) + \omega (L\Phi)^2 \right]$$

We see from (15) that the effective edge dynamics involves only the time-derivative of $\Phi$ and one tangential derivative given by $L\Phi$; the action is in this sense chiral. The field $\Phi$ can, and does indeed, depend on the remaining tangential directions, which we will denote by $\Sigma$, leading to a multiplicity of modes. We can expect the multiplicity to be parametrized by $\Sigma$, but this is not quite accurate. For a spherical droplet in even dimensions $2k$, $\partial D \sim S^{2k-1}$ and therefore the space $\Sigma$ is $\mathbb{C}P^{k-1}$. However, $S^{2k-1}$ is not the product $\mathbb{C}P^{k-1} \times S^1$ and so $\Phi$ cannot be expanded in terms of functions on $\mathbb{C}P^{k-1}$, but rather will involve sections of vector bundles on $\mathbb{C}P^{k-1}$. This will become clear from the examples in later sections.
3 Analysis on $\text{CP}^k$

Landau levels and wavefunctions

In the last section we derived the general form of the effective action for the edge states. To complete this derivation, we need to explicitly show the emergence of the Poisson brackets and also that the density $\rho_0$ is constant over the droplet $D$. We will do this now for $\text{CP}^k$ with a $U(1)$ background magnetic field. Our approach follows the group theoretic analysis used in our previous paper [2]. Since $\text{CP}^k = SU(k+1)/U(k)$, the required wavefunctions can be obtained as functions on $SU(k+1)$ which have a specific charge under the $U(1)$ subgroup.

Let $t_a$ denote the generators of $SU(k+1)$ as matrices in the fundamental representation; we normalize them by $\text{Tr}(t_a t_b) = \frac{1}{2} \delta_{ab}$. The generator corresponding to the $U(1)$ direction of the subgroup $U(k)$ will be denoted by $t_{k^2+2k}$. A basis of functions on $SU(k+1)$ is given by the Wigner $D$-functions which are the matrices corresponding to the group elements in a representation $J$

$$D_{L,R}^{(j)}(g) = \langle J, L_i | \hat{g} | J, R_i \rangle$$

(16)

where $L_i$, $R_i$ stand for two sets of quantum numbers specifying the eigenvalues of the diagonal generators for left and right $U(k)$ actions on $g$, respectively. On an element $g \in SU(k+1)$, considered as a $(k+1) \times (k+1)$-matrix, we can define left and right $SU(k+1)$ actions by

$$\hat{L}_a g = t_a g, \quad \hat{R}_a g = g t_a$$

(17)

As discussed in [2], the wavefunctions for the Landau levels on $\text{CP}^k$ with a $U(1)$ background field are singlets under the subgroup $SU(k)_R$ and carry $U(1)_R$ charge as specified by the background field. This implies that the quantum numbers $R_i$ in (16) are constrained to be

$$R_j = 0 \quad j = 1, \cdots, k^2 - 1$$

$$R_{k^2+2k} = -nk\frac{1}{\sqrt{2k(k+1)}}$$

(18)

$n$ goes like $BR^2$ where $B$ is the magnetic field and $R$ is the radius of the $\text{CP}^k$; $n$, upto factors involving the normalization of generators, is an integer in accordance with the Dirac quantization rule.

There are $2k$ right generators of $SU(k+1)$ which are not in $U(k)$; these can be separated into $t_{+,i}$ which are of the raising type and $t_{-,i}$ which are of the lowering type. The covariant derivatives on $\text{CP}^k$ can be identified with these $\hat{R}_{\pm i}$ right rotations on $g$. This is consistent with the fact that the commutator of covariant derivatives is the magnetic field. Since the Hamiltonian goes like $\hat{R}_{+,i} \hat{R}_{-,i}$, the lowest Landau level will also obey the condition $\hat{R}_{-,i} \Phi = 0$. One can therefore identify the lowest Landau level (LLL) states as the lowest weight vector for the representation corresponding to the right action.

The Landau levels correspond to the various possible irreducible $SU(k+1)$ representations $J$ which contain an $SU(k)$ singlet. Such representations can be labelled by two integers
\[ J = (p, q) \text{ such that } p - q = n. \] The lowest Landau level corresponds to \( p = n, \ q = 0 \). These are completely symmetric representations. The dimension of this representation \( J = (n, 0) \) is

\[
\text{dim} J = \frac{(n + k)!}{n!k!} \equiv N
\]

\( N \) expresses the number of states in the lowest Landau level.

The LLL wavefunctions on \( \mathbb{C}P^k \) can be written as

\[
\Psi^{(n)}_m(g) = \sqrt{N} D^{(n)}_{m;-n}(g)
\]

We denote the fixed state for the right action on the Wigner function above as \(-n\), indicating that the eigenvalue for \( \hat{R}_{k^2+2k} \) is \(-nk/\sqrt{2k(k+1)}\) as in (18). The index \( m \) specifies the state within the lowest Landau level.

The wavefunctions (20) are normalized by virtue of the orthogonality theorem

\[
\int d\mu(g) D^*(n)_{m;-n}(g) D^{(n)}_{m';-n}(g) = \frac{\delta_{mm'}}{N}
\]

where \( d\mu(g) \) is the Haar measure on \( SU(k+1)/U(k) \), normalized to unity \( (d\mu(g) \) can be taken as the Haar measure on the full group \( SU(k+1) \), properly normalized, since the integrand is \( U(k) \) invariant.\)

Since the LLL wavefunctions correspond to completely symmetric representations, a convenient way to represent them is in terms of the group elements

\[
g_{\alpha,k+1} \equiv u_\alpha \quad \alpha = 1, \ldots, k
\]

In terms of \( u_\alpha \)'s, the corresponding Wigner \( D \) functions are of the form \( D \sim u_{\alpha_1}u_{\alpha_2} \cdots u_{\alpha_n} \). Eq. (22) also implies that \( u^* \cdot u = 1 \) and so one may parametrize them as

\[
u_i = \frac{\xi_i}{\sqrt{1 + \bar{\xi} \cdot \xi}} \quad i = 1, \ldots, k
\]

\[
u_{k+1} = \frac{1}{\sqrt{1 + \bar{\xi} \cdot \xi}}
\]

\( \xi, \bar{\xi} \) can be thought of as the complex local coordinates for \( \mathbb{C}P^k \). With this parametrization, the wavefunctions for the lowest Landau level are

\[
\Psi^{(n)}_m = \sqrt{N} D^{(n)}_{m;-n}(g)
\]

\[
D^{(n)}_{m;-n}(g) = \left[ \frac{n!}{i_1!i_2! \cdots i_k!(n - s)!} \right]^{\frac{1}{2}} \frac{\xi_{i_1} \xi_{i_2} \cdots \xi_{i_k}}{(1 + \bar{\xi} \cdot \xi)^{\frac{s}{2}}}
\]

\( s = i_1 + i_2 + \cdots + i_k \)

(24)
The condition $\hat{R}_{-i}D_{m,-n}^{(n)} = 0$ is a holomorphicity condition and this is reflected in the fact that the wavefunctions for the lowest Landau level are holomorphic in $\xi$'s, apart from certain overall factors. The states (24) are coherent states for $\mathbb{C}P^k$. The inner product for the $\Psi$'s may be written in these coordinates as

$$
\langle \Psi | \Psi' \rangle = \int d\mu \, \Psi^* \Psi' \\
d\mu = \frac{k!}{\pi^k} \frac{d^k \xi d^k \bar{\xi}}{(1 + \xi \cdot \bar{\xi})^{k+1}}
$$

We will also need the formulae for the left and right translations (17) as differential operators. The left rotations $\hat{L}_a$, $a = 1, \cdots, k^2 + 2k$ correspond to magnetic translations. One can easily write down the corresponding differential operator in terms of $u_\alpha$'s.

$$
\hat{L}_a = u_\beta (t^a)_{\alpha \beta} \frac{\partial}{\partial u_\alpha} - u_\alpha^* (t^a)_{\alpha \beta} \frac{\partial}{\partial u_\beta}
$$

The right generators cannot be easily written in terms of $u_\alpha$'s; we express them in terms of the group parameters $\varphi^i$

$$
\hat{R}_a = i(E^{-1})_a^i \frac{\partial}{\partial \varphi^i}
$$

where

$$
g^{-1} dg = -it_a E_i^a \, d\varphi^i
$$

**Star products, commutators and Poisson brackets**

Let $\hat{A}$ be a general matrix acting on the $N$-dimensional Hilbert space of lowest Landau level states with matrix elements $A_{ms}$. We define the symbol corresponding to $A$ as the function $^3$

$$
A(g) = A(\xi, \bar{\xi}) = \sum_{ms} D_{m,-n}^{(n)}(g) A_{ms} D_{s,-n}^{*(n)}(g)
$$

We are interested in the symbol corresponding to the product of two matrices $A$ and $B$. This can be written as

$$
(AB)(g) = \sum_r A_{mr} B_{rs} D_{m,-n}^{(n)}(g) D_{s,-n}^{*(n)}(g)
$$

$$
= \sum_{rr'} D_{m,-n}^{(n)}(g) A_{mr} D_{r,-n}^{*(n)}(g) D_{s,-n}^{(n)}(g) B_{r's} D_{s,-n}^{*(n)}(g)
$$

$^3$The symbol as defined in (29) is related to the expectation value of $A$ in the coherent state basis; it coincides with the expectation value in the large $n$ limit.
using $\delta_{rr'} = \sum_p D^{(n)}_{r,r'}(g)D^{(n)}_{r',r}(g)$. The term with $p = -n$ on the right hand side of (30) gives the product of the symbols for $A$ and $B$. The terms with $p > -n$ may be written using raising operators as

$$
D^{(n)}_{r,p}(g) = \frac{1}{n!i_1!i_2!\cdots i_k!} \left[ \frac{(n-s)!}{n!i_1!i_2!\cdots i_k!} \right]^{\frac{1}{2}} \hat{R}^{i_1}_{r+1} \hat{R}^{i_2}_{r+2} \cdots \hat{R}^{i_k}_{r+n}(g) \quad (31)
$$

Here $s = i_1 + i_2 + \cdots + i_k$ and the $t_{k+2} - \text{eigenvalue}$ for the state $|p\rangle$ is $(-nk + sk + s)/\sqrt{2k(k+1)}$. Since $\hat{R}_{r+i}D^{(n)}_{s,-n} = 0$, we can also write

$$
\left[ \hat{R}_{r+i}D^{(n)}_{s,-n}(g) \right] B_{r,s}D^{(n)}_{s,-n}(g) = \left[ \hat{R}_{r+i}D^{(n)}_{s,-n}B_{r,s}D^{(n)}_{s,-n}(g) \right] = \hat{R}_{r+i}B(g) \quad (32)
$$

Further keeping in mind that $\hat{R}_{r+i} = -\hat{R}_r$, we can combine (30) and (32) to get

$$
(AB)(g) = \sum_s (-1)^s \sum_{i_1+i_2+\cdots+i_k=s} \frac{s!}{i_1!i_2!\cdots i_k!} \hat{R}^{i_1}_{r+1} \hat{R}^{i_2}_{r+2} \cdots \hat{R}^{i_k}_{r+n}(g) \quad (33)
$$

The right hand side of this equation is what is known as the star product for functions on $\mathbb{C}P^k$. It has been written down in different forms in the context of noncommutative $\mathbb{C}P^k$ spaces [9]; our argument here follows the presentation in [10] which gives a simple and general way of constructing star products. The first term of the sum on the right hand side gives $A(g)B(g)$, successive terms involve derivatives and are down by powers of $n$, as $n \to \infty$. For the symbol corresponding to the commutator of $A$, $B$, we have

$$
([A,B])(g) = -\frac{1}{n} \left[ \hat{R}_{r-i}A \hat{R}_{r+i}B - \hat{R}_{r-i}B \hat{R}_{r+i}A \right] + \mathcal{O}(1/n^2) \quad (34)
$$

The Kähler two-form on $\mathbb{C}P^k$ may be written as

$$
\Omega = -i \sqrt{\frac{2k}{k+1}} \text{Tr}(t_{k+2}g^{-1}dg \wedge g^{-1}dg) = -\frac{1}{4} \sum_{i=1}^k \epsilon_{a_ib_i} E_{c_i}^{a_i} E_{d_i}^{b_i} d\varphi^c \wedge d\varphi^d \quad (35)
$$

Functions $A$, $B$ on $\mathbb{C}P^k$ obey the condition $\hat{R}_\alpha A = \hat{R}_\alpha B = 0$, where $\hat{R}_\alpha$ (with the index $\alpha$) is any generator of the subgroup $U(k)$. With this condition, we find

$$
i \sum_i \epsilon^{a_ib_i}(\hat{R}_\alpha A)(E^{-1})^c_i \Omega_{cd} = -\frac{\partial A}{\partial \varphi^d} \quad (36)
$$

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The symplectic form which via geometric quantization gives the states we are considering is proportional to the Kähler form.
This shows that $i \sum_i \epsilon^{a_i b_i} (\hat{R}_{a_i} A) (E^{-1})^a_b$ may be identified as the Hamiltonian vector field corresponding to $A$. The Poisson bracket of $A, B$, as defined by $\Omega$, is thus given by

$$\{A, B\} = (\Omega^{-1})^{ab} \frac{\partial A}{\partial \varphi^a} \frac{\partial B}{\partial \varphi^b}$$

$$= i \sum_{i=1}^k \left( \hat{R}_{-i} A \hat{R}_{+i} B - \hat{R}_{-i} B \hat{R}_{+i} A \right) \quad \text{(37)}$$

Combining with (34), we find

$$(A, B) \text{(g)} = \frac{i}{n} \{A, B\} + O(1/n^2) \quad \text{(38)}$$

This completes the demonstration of equation (5) with $c = 1$ for $\mathbb{C}P^k$. If desired, one can also write the Poisson bracket in terms of the local coordinates $\xi, \bar{\xi}$ introduced in (23). The relevant expressions are

$$\Omega = -i \left[ \frac{d\bar{\xi} \cdot d\xi}{(1 + \xi \cdot \bar{\xi})^2} - \frac{\xi \cdot d\bar{\xi} \cdot d\xi}{(1 + \xi \cdot \bar{\xi})^2} \right]$$

$$\{A, B\} = i(1 + \bar{\xi} \cdot \xi) \left( \frac{\partial A}{\partial \xi^i} \frac{\partial B}{\partial \bar{\xi}^i} - \frac{\partial A}{\partial \bar{\xi}^i} \frac{\partial B}{\partial \xi^i} \right)$$

$$+ \xi \cdot \frac{\partial A}{\partial \bar{\xi}^i} \frac{\partial B}{\partial \xi^i} - \bar{\xi} \cdot \frac{\partial A}{\partial \xi^i} \frac{\partial B}{\partial \bar{\xi}^i} \right) \quad \text{(39)}$$

The trace of an operator $\hat{A}$ may be written as

$$\text{Tr} \hat{A} = \sum_m A_{mm} = N \int d\mu(g) D_{m,\nu}^{(n)} A_{m,\nu} \ D_{m',\nu'}^{(n)}$$

$$= N \int d\mu(g) A(g) \quad \text{(40)}$$

which is consistent with (40). The trace of the product of two operators $A, B$ is given by

$$\text{Tr} \hat{A} \hat{B} = N \int d\mu(g) A(g) \ast B(g) \quad \text{(41)}$$

**The density matrix**

We now consider the density $\rho_0$ as we fill up states and make a droplet. If we decompose the representation of $SU(k+1)$ in terms of $SU(k)$ representations, we find that it is made up of the singlet of $SU(k)$, the fundamental of $SU(k)$, the rank two symmetric representation, etc., each occuring once. There are many ways to fill a part of these states, giving different $\hat{\rho}_0$’s; the results for the edge states will be different but qualitatively similar. The potentials
break translation symmetry and are functions of $\hat{L}_a$ in general. One can choose the potential appropriately to produce a chosen $\hat{\rho}_0$. The simplest case of a droplet would be to fill up a certain number of complete $SU(k)$ representations, preserving an $SU(k)$ symmetry. The potential which does this can be taken as a function of $\hat{L}_k^2 + 2k + nk$.

For example, we can take

$$\hat{V} = \sqrt{\frac{2k}{k+1}} \omega \left( \hat{L}_k^2 + \frac{nk}{\sqrt{2k(k+1)}} \right)$$

(42)

The normalization above has been chosen so that if $s$ is the rank of the symmetric representation $SU(k)$ then

$$\langle s|\hat{V}|s \rangle = \omega s$$

(43)

This will fill up states from $m = -n$ to the complete symmetric representation of $SU(k)$ of, say, rank $M$. This corresponds to $\hat{\rho}_0 = \sum_{s=0}^{M} \sum_{\lambda} |s\lambda\rangle \langle s\lambda|$ where $\lambda$ designates states within the rank $s$ representation of $SU(k)$. This $\hat{\rho}_0$ is the identity for the first $M$ $SU(k)$ representations and zero for the remainder. As a result, the commutator of $\hat{L}_a$ will have contributions only from states near the last rank-$M$ $SU(k)$ representation, i.e., near the edge. More specifically, the symbol for the density matrix for this choice is

$$\rho_0 = \sum_{s=0}^{M} \sum_{\lambda} \frac{n!}{s!(n-s)!} \left( \frac{\xi \cdot \bar{\xi}}{1 + \xi \cdot \bar{\xi}} \right)^s$$

(44)

Taking the derivative of $\rho_0(\xi, \bar{\xi})$ with respect to $\bar{\xi} \cdot \xi$ we find

$$\frac{\partial \rho_0}{\partial (\bar{\xi} \cdot \xi)} = -\frac{n!}{(1 + \xi \cdot \bar{\xi})^n (n-M-1)!} \frac{(\xi \cdot \bar{\xi})^M}{M!}$$

(45)

The coordinates $\xi, \bar{\xi}$ are in units of the radius $CP^k$, namely $\xi = (x + iy)/R$. Since $n \sim BR^2$ we define $v = n\xi \cdot \bar{\xi}$ where $v$ is $n$ independent, and take the $n \to \infty$ of (45). Using the fact that $(1 + \frac{v}{n})^n \to e^v$ as $n \to \infty$ and the Stirling’s formula

$$n! \sim \sqrt{2\pi e^{-n} n^{n+\frac{1}{2}} }$$

(46)

we find

$$\frac{\partial \rho_0}{\partial v} \approx -e^{-v} \frac{v^M}{M!}$$

(47)

For large $M$, (large number of fermions) the derivative of $\rho_0$ is sharply peaked at $v \approx M$. Expanding around $M$ and using Stirling’s formula again, we find

$$\frac{\partial \rho_0}{\partial v} \approx -\frac{e^{M-v}v^M}{M^M \sqrt{2\pi M}} \approx -\frac{1}{\sqrt{2\pi M}} \exp \left( -\frac{1}{2} M (1 - v/M)^2 \right)$$

$$\approx -\frac{1}{M} \delta \left( 1 - \frac{v}{M} \right) ,$$

(48)
as $M$ becomes very large. Thus

$$\rho_0 \approx \Theta \left(1 - \frac{v}{M}\right) = \Theta \left(1 - \frac{n\bar{\xi} \cdot \xi}{M}\right) \quad (49)$$

The radius of the droplet will be proportional to $\sqrt{M}$, in fact $r_D^2 = M/B$, where $B$ is the magnetic field. We have thus shown the required property of $\rho_0$.

**Calculation of the potential energy**

Finally we obtain the large $n$ limit of the potential energy $Tr(\rho_0 \hat{U} \hat{V} \hat{U})$ corresponding to the $SU(k)$ symmetric potential chosen in (42). In order to do this we have to calculate the classical function or symbol corresponding to $\hat{V}$ as outlined in (11). Using our definition (29) we get

$$V = \omega \sqrt{\frac{2k}{k+1}} \left[ \langle -n | \hat{L}_{k^2+2k} | -n \rangle + \frac{n k}{\sqrt{2k(k+1)}} \right] \quad (50)$$

Using

$$g^\dagger \hat{L}_a g = S_{ab} \hat{L}_b$$

$$S_{ab} = 2Tr(g^\dagger t_a g t_b) \quad (51)$$

we find that

$$V = \omega \sqrt{\frac{2k}{k+1}} \left[ \langle -n | \hat{L}_{k^2+2k} | -n \rangle S_{k^2+2k,k^2+2k} + \frac{n k}{\sqrt{2k(k+1)}} \right]$$

$$= \sqrt{\frac{2k}{k+1}} \omega \left( -\frac{k n}{\sqrt{2k(k+1)}} S_{k^2+2k,k^2+2k} + \frac{n k}{\sqrt{2k(k+1)}} \right) \quad (52)$$

A straightforward calculation gives

$$S_{k^2+2k,k^2+2k} = \frac{k - \bar{\xi} \cdot \xi}{1 + \xi \cdot \xi} \quad (53)$$

Combining (51), (52) we find

$$V = \omega n \left( \frac{\bar{\xi} \cdot \xi}{1 + \xi \cdot \xi} \right) \quad (54)$$

Notice that $\frac{\partial V}{\partial (\bar{\xi} \cdot \xi)} \sim n$ as indicated in section 2. Further as $n \to \infty$, $V$ is just a harmonic oscillator confining potential.

In a similar fashion one can calculate the classical function corresponding to any potential which is an arbitrary function of $\hat{L}_a$’s.
The effective action and nature of edge states

The formula (49) for the density shows that the boundary of the droplet is defined by $n\vec{\xi} \cdot \xi = M$; the radius of the droplet is $\sqrt{M/B}$. The boundary thus looks like a sphere $S^{2k-1}$. It is then convenient to go over to angular coordinates to evaluate the effective action. Changing to $\sqrt{n\xi}$ as the coordinate, we get a factor of $n^{-k}$ from the volume measure. Recalling that $N = (n+k)!/n!k! \sim n^k/k!$, we find from (13)

$$S_{\mathbb{C}P^k} \approx -\frac{1}{4\pi^k}M^{k-1} \int d\Omega_{S^{2k-1}} \left[ \frac{\partial \Phi}{\partial t} (L) + \omega (L\Phi)^2 \right]$$

(55)

where $d\Omega_{S^{2k-1}}$ denotes the volume element on the sphere $S^{2k-1}$; the factor $M^{k-1}$ is as expected for a droplet of radius $\sim \sqrt{M}$. The operator $L$ is identified in terms of the coordinates $\bar{\xi}, \xi$ as

$$L = i \left( \xi \cdot \frac{\partial}{\partial \xi} - \bar{\xi} \cdot \frac{\partial}{\partial \bar{\xi}} \right)$$

(56)

This operator is the one defined in (9); terms which vanish as $n \to \infty$ have been dropped.

We will now consider the nature of the edge excitations. We can expand $\Phi$ in powers of $z_i = \sqrt{n} \xi_i$ and $\bar{z}_i = \sqrt{n} \bar{\xi}_i$. Since $\bar{z} \cdot z$ is fixed at the droplet boundary, $\Phi$ is obviously a function on $S^{2k-1}$. We can write $z = hz_0$, where $z_0 = (0,0,...,0,\sqrt{M})$, $h \in SU(k)$. This naturally leads to $SU(k)/SU(k-1) = S^{2k-1}$. Powers of $z, \bar{z}$ are of the form $z^{i_1}z^{i_2}...z^{i_q}z_{j_1}...z_{j_p}$; the irreducible representations are thus of the tensorial type $T_{p}^{q}$ with $p$ symmetric lower indices, $q$ symmetric upper indices and the contraction (or trace) of any $p$-type index with any $q$-type index must vanish. On these, $L$ has the value $il$, $l = p - q$. Each such representation contains a unique $SU(k-1)$-invariant state, which is of the form $|0, p, q\rangle \sim \bar{z}^q z^p_k$. Comparing this with $z_i = h_{i,k^{2k-1}}$ we see that $l$ may be identified as the eigenvalue of $t_{k^{2k-1}}$ of $SU(k)$ acting on the right of $h$. The mode expansion of the field $\Phi$ may therefore be written as

$$\Phi = \sum_{l} \sum_{p,q|p-q=l} c_{m_{(0,p,q)}}^{p,q}(h) D_{m_{(0,p,q)}}^{(p,q)}(h)$$

(57)

Notice that $D_{m_{(0,p,q)}}^{(p,q)}$ are sections of $U(1)$-bundles on $\mathbb{C}P^{k-1}$ and are therefore similar to wavefunctions of a reduced Landau problem on $\mathbb{C}P^{k-1}$ with different choices of magnetic field.

4 CP$^1$ case

We now illustrate briefly how our approach works for the simple case of quantum Hall effect on $\mathbb{C}P^1 = S^2 = SU(2)/U(1)$. 

14
In this case the $SU(2)$ group element $g$ can be parametrized in terms of the complex variables $u_1, u_2$, where $u^* \cdot u = 1$

$$g = \begin{pmatrix} u_2^* & u_1 \\ -u_1^* & u_2 \end{pmatrix}$$

and

$$u = \frac{1}{\sqrt{1 + \bar{\xi}\xi}} \begin{pmatrix} \xi \\ 1 \end{pmatrix}$$

where $\xi, \bar{\xi}$ are the local complex coordinates for $S^2$.

Given (58), (59) one can derive expressions for both the left and right generators in terms of $u$'s. In particular

$$\hat{R}_+ = -\epsilon_{\alpha\beta}u_\alpha^{\ast} \frac{\partial}{\partial u_\beta} = (1 + \bar{\xi}\xi) \frac{\partial}{\partial \xi}$$

$$\hat{R}_- = \epsilon_{\alpha\beta}u_\alpha \frac{\partial}{\partial u_\beta^{\ast}} = -(1 + \bar{\xi}\xi) \frac{\partial}{\partial \bar{\xi}}$$

Using (60) we find

$$([A, B])(g) = -\frac{1}{n}(1 + \bar{\xi}\xi)^2 \left( \partial_\xi A \partial_\xi B - \partial_\xi A \partial_\bar{\xi} B \right)$$

Comparing this with (39) we easily verify that (38) holds.

The $U(1)$ invariant confining potential that allows the formation of a droplet with density of the form (44) is

$$\hat{V} = \hat{L}_3 + \frac{n}{2}$$

with the corresponding function

$$V(\xi, \bar{\xi}) = \omega n \frac{\xi\bar{\xi}}{1 + \xi\bar{\xi}}$$

This is essentially a harmonic oscillator potential for large $n$. Further

$$\{ \Phi, \rho_0 \} = i(1 + \xi\bar{\xi})^2 \left( \partial_\xi \Phi \partial_\xi \rho_0 - \partial_\xi \Phi \partial_\xi \rho_0 \right)$$

$$= i(1 + \xi\bar{\xi})^2 \left( \xi \partial_\xi - \bar{\xi} \partial_\xi \right) \Phi \frac{\partial \rho_0}{\partial r^2}$$

which identifies the operator in (14) as $\hat{L} = \partial_\theta$, where $\xi = re^{i\theta}$.

Using (63), (64) in (13) we find the well known one-dimensional chiral bosonic action describing edge excitations for $\nu = 1$ two-dimensional QHE

$$S = -\frac{1}{4\pi} \int d\theta \left( \partial_t + \omega \partial_\theta \right) \Phi(\theta, t) \partial_\theta \Phi(\theta, t)$$
5 Edge states for \( S^4 \) from \( \text{CP}^3 \)

As mentioned in section 3, even for the case of a \( U(1) \) background magnetic field on \( \text{CP}^k \), one can get somewhat different results depending on the choice of potential and how the states are filled to form the droplet. Among these possibilities, the case of \( \text{CP}^3 \) is especially interesting because of the fact that it is an \( S^2 \) bundle over \( S^4 \). Edge states for the Hall effect on \( S^4 \) with an instanton background field can be obtained using a \( U(1) \) background field on \( \text{CP}^3 \). Here we will obtain the edge effective action by reducing our general action for the case of \( \text{CP}^3 \) to \( S^4 \).

The space \( \text{CP}^3 \) is realized as \( SU(4)/U(3) \), so that one can use the group elements as local coordinates, as we have done so far. But one can also use the homogeneous coordinates and this is more convenient for the reduction to \( S^4 \). We can describe \( \text{CP}^3 \) by the four complex coordinates \( Z_i, i = 1, ..., 4 \), with the identification \( Z_i \sim \lambda Z_i \) where \( \lambda \) is any complex number except zero, \( \lambda \in \mathbb{C} - \{0\} \). We write \( Z_i \) in terms of two two-component spinors \( w, \pi \)

\[
(Z_1, Z_2, Z_3, Z_4) = (w_1, w_2, \pi_1, \pi_2)
\]

Coordinates \( x_\mu \) on \( S^4 \) are then defined by

\[
w = (x_4 - i \sigma \cdot x) \pi
\]

The scale invariance \( Z \sim \lambda Z \) can be realized as the scale invariance \( \pi \sim \lambda \pi \); the \( \pi \)'s then describe a \( \text{CP}^1 = S^2 \). This will be the fiber space. The coordinates \( x_\mu \) are the standard stereographic coordinates for \( S^4 \); one can in fact write

\[
y_0 = \frac{1 - x^2}{1 + x^2}, \quad y_\mu = \frac{2x_\mu}{1 + x^2},
\]

(68)

to realize the \( S^4 \) as embedded in \( \mathbb{R}^5 \). The definition of \( x_\mu \) in terms of \( w \) may be solved as

\[
x_4 = \frac{1}{2} \frac{\bar{\pi} w + \bar{w} \pi}{\bar{\pi} \pi}
\]

\[
x_i = \frac{1}{2} \frac{i \bar{\pi} \sigma_i w - \bar{w} \sigma_i \pi}{\bar{\pi} \pi}
\]

The Kähler two-form on \( \text{CP}^3 \) is given as

\[
\Omega = -i \left[ \frac{d\bar{Z} \cdot dZ}{Z \cdot \bar{Z}} - \frac{d\bar{Z} \cdot Z \cdot d\bar{Z}}{(Z \cdot \bar{Z})^2} \right]
\]

(70)

Notice that this is invariant under \( Z \rightarrow \lambda Z \), and \( \bar{Z} \rightarrow \lambda \bar{Z} \). We can reduce this using (66), (69) to get

\[
\Omega_{\text{CP}^3} = \Omega_{\text{CP}} - F
\]

\[
F = dA + A \cdot A
\]

(71)

\[
A = \frac{i N^\alpha \eta^\mu_\nu x^\mu dx^\nu}{(1 + x^2)}
\]
Here $\eta_{\mu\nu}^a$ is the 't Hooft tensor

$$\eta_{\mu\nu}^a = \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{4\nu} - \delta_{a\nu}\delta_{4\mu}$$

(72)

and

$$N^a = \bar{\pi}\sigma^a\pi/\bar{\pi}\pi$$

(73)

$N^a$ is a unit three-vector, which may be taken as parametrizing the fiber $\mathbb{C}P^1 \sim S^2$. The field $F$ is the instanton field. We see that we can get an instanton background on $S^4$ by taking a $U(1)$ field on $\mathbb{C}P^3$ proportional to the Kähler form.

Functions $A$ on $\mathbb{C}P^3$ can be considered as functions of the four variables $Z_i$, but because of the homogeneity conditions, they must obey

$$Z \cdot \frac{\partial A}{\partial Z} = 0, \quad \bar{Z} \cdot \frac{\partial A}{\partial \bar{Z}} = 0$$

(74)

The Kähler form can be inverted on functions obeying this condition to get the Poisson bracket as

$$\{A, B\} = i(\bar{Z} \cdot Z) \left( \frac{\partial A}{\partial Z_\alpha} \frac{\partial B}{\partial Z_\alpha} - \frac{\partial A}{\partial Z_\alpha} \frac{\partial B}{\partial \bar{Z}_\alpha} \right)$$

(75)

It is now straightforward to simplify this in terms of the $S^4$ coordinates, using (66), (69) to get

$$\{A, B\}_{\mathbb{C}P^3} = (1 + x^2)K_{\mu\nu}^A \frac{\partial A}{\partial x^\mu} \frac{\partial B}{\partial x^\nu} + (1 + x^2)\{A, B\}_{\mathbb{C}P^1}$$

$$K_{\mu\nu} = -\frac{1}{2}N^a\eta_{\mu\nu}^a$$

(76)

The second term is the Poisson bracket for $\mathbb{C}P^1$. $K_{\mu\nu}$ defines a local complex structure, a way of combining the $x_\mu$ into complex coordinates; for every point on $S^2$ given by $N^a$ we have a corresponding $K_{\mu\nu}$. This is in accordance with $\mathbb{C}P^3$ being a bundle of local complex structures on $S^4$.

The Poisson bracket, split as in (76), is one of the ingredients for the effective action. We will now turn to the density matrix. The required density should depend only on the coordinates of $S^4$, since we want to interpret this as a droplet in $S^4$. We take it to be of the form

$$\rho_0 = \sum_{s=0}^{M} \frac{n!}{s!(n-s)!} \frac{(x^2)^s}{(1 + x^2)^n}$$

(77)

This will behave like a step function of radius $\sim \sqrt{M}$ and so corresponds to a droplet on $S^4$. In terms of the homogeneous coordinates $Z = (w, \pi)$ we find that this is

$$\rho_0 = \sum_{s=0}^{M} \frac{n!}{s!(n-s)!} \frac{\bar{Z}_3Z_3 + \bar{Z}_4Z_4)^{n-s}(\bar{Z}_1Z_1 + \bar{Z}_2Z_2)^s}{(Z \cdot Z)^n}$$

(78)
By expanding out the numerator, identifying $\xi_i = Z_i/Z_4$, $i = 1, 2, 3$, and comparing with the coherent states defined earlier in (24), we can identify the filled states. We define an $SU(2)$ subgroup of $SU(4)$ as the group acting on $\xi_1, \xi_2$. The $SU(4)$ representation of the lowest Landau level can be split into multiplets of this $SU(2)$, which we call the right isospin with $I_R = \frac{1}{2}s$, where $s$ is the number of $\xi_1$ or $\xi_2$ as in (78). (The right isospin will correspond to $SU(2)_R$ in a decomposition of the $O(4)$ algebra of $S^4$ as $O(4) \sim SU(2)_L \times SU(2)_R$.) The density (77) is produced when all the $SU(2)_R$ isospin multiplets up to $s = M$ are filled. The potential which will lead to filling the various isospin multiplets, from $s = 0$ to $s = M$, can be taken as proportional to $s$ itself, namely

$$\langle s|\hat{V}|s\rangle = \omega s$$

(79)

Its expression in terms of Lie algebra generators is

$$\hat{V} = \frac{\omega}{6} (t_{15} + 2t_8 + 3n)$$

(80)

The calculation of the corresponding classical function as in (50)-(53) gives

$$V = \omega n \frac{\bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2}{1 + \xi \cdot \xi} = \omega n \frac{x^2}{1 + x^2}$$

(81)

With this potential, we then get the density in (77).

The density does not depend on the $\mathbb{CP}^1$ coordinates. As a result, the contribution from the $\mathbb{CP}^1$ Poisson brackets will integrate to zero in the action. The action then becomes

$$S = -\frac{M}{4\pi^2} n \int d\mu_{\mathbb{CP}^1} \int d\Omega_3 \left[ \frac{\partial \Phi}{\partial t} (L\Phi) + \omega (L\Phi)^2 \right]$$

(82)

where $L\Phi = 2\pi^\nu K^\mu_\nu \partial_\mu \Phi$. $\Phi$’s are to be expanded in terms of harmonics on $\mathbb{CP}^1$ which correspond to the representations of $SU(2)_L$ in $O(4) \sim SU(2)_L \times SU(2)_R$. (This is the subgroup corresponding to the instanton gauge group.) Recall that the inner product for states on $\mathbb{CP}^1$ has a factor which is the dimension of the representation in it, and so we must interpret the prefactor $n$ for the integral over $d\mu_{\mathbb{CP}^1}$ as this quantity. As $n \to \infty$, this becomes very large; thus the number of left isospin states becomes very large, in agreement with the analysis of [1].

We will conclude this section with a few remarks on the nature of the excitations in (82). The equation of motion for $\Phi$ is of the form $L (i\partial_t \Phi + i\omega L\Phi) = 0$. The zero momentum state with $L\Phi = 0$ does not give a deformation of the boundary and so, for states of interest, the equation of motion implies

$$i\partial_t \Phi = -i\omega L\Phi$$

(83)

To understand the nature of the states, we need to seek eigenstates of $L$. By direct application on the $w, \bar{w}$ in (77), we see that

$$-iL \ w_\tilde{A} = w_\tilde{A}$$

$$-iL \ \bar{w}_\tilde{A} = -\bar{w}_\tilde{A}$$

(84)
where \( \tilde{w}_\tilde{A} = \epsilon_{\tilde{A}B} \tilde{w}_B \). Eigenstates of \( L \) are thus given by a monomial with a certain number of \( w \)'s and \( \tilde{w} \)'s and an arbitrary number of \( \pi \)'s and \( \tilde{\pi} \)'s, where \( \tilde{\pi}_C = \epsilon_{CD} \pi_D \)

\[
- iL \tilde{w}_{\tilde{A}_1} \cdots \tilde{w}_{\tilde{A}_m} w_{\dot{B}_1} \cdots w_{\dot{B}_l} \tilde{\pi}_{C_1} \cdots \tilde{\pi}_{C_q} \pi_{D_1} \cdots \pi_{D_k} \\
= (l - m) \tilde{w}_{\tilde{A}_1} \cdots \tilde{w}_{\tilde{A}_m} w_{\dot{B}_1} \cdots w_{\dot{B}_l} \tilde{\pi}_{C_1} \cdots \tilde{\pi}_{C_q} \pi_{D_1} \cdots \pi_{D_k} \tag{85}
\]

Since \( \epsilon_{\tilde{A}B} \tilde{w}_{\tilde{A}} w_B = x^2 \pi \cdot \tilde{\pi} \) and \( x^2 \) is fixed on the boundary of the droplet, such contractions may be removed from \( L \). One may consider all the \( \{ \tilde{A}, \dot{B} \} \) indices to be symmetric and similarly all the \( \{ C, D \} \) indices to be symmetric, making the eigenstates of \( L \) irreducible tensors of \( SU(2)_R \times SU(2)_L \). Further, since \( \Phi \) is a function on \( \mathbb{CP}^1 \), we must have invariance under the scaling \( \pi \to \lambda \pi \). The mode expansion for \( \Phi \) in terms of eigenstates of \( L \) is given by

\[
\Phi = \sum_{l \geq m} C(\tilde{A})_m(\dot{B})_l(\tilde{C})_{l-m+k}(D)_k f(\tilde{A})_m(\dot{B})_l(\tilde{C})_{l-m+k}(D)_k \\
+ \sum_{l < m} \tilde{C}(\tilde{A})_m(\dot{B})_l(\tilde{C})_k(D)_{m-l+k} \tilde{f}(\tilde{A})_m(\dot{B})_l(\tilde{C})_k(D)_{m-l+k} \\
f(\tilde{A})_m(\dot{B})_l(\tilde{C})_{l-m+k}(D)_k = \frac{1}{(\pi \cdot \pi)^{l+k}} \tilde{w}_{\tilde{A}_1} \cdots \tilde{w}_{\tilde{A}_m} w_{\dot{B}_1} \cdots w_{\dot{B}_l} \tilde{\pi}_{C_1} \cdots \tilde{\pi}_{C_{l-m+k}} \pi_{D_1} \cdots \pi_{D_k} \\
\tilde{f}(\tilde{A})_m(\dot{B})_l(\tilde{C})_k(D)_{m-l+k} = \frac{1}{(\pi \cdot \pi)^{m+k}} \tilde{w}_{\tilde{A}_1} \cdots \tilde{w}_{\tilde{A}_m} w_{\dot{B}_1} \cdots w_{\dot{B}_l} \tilde{\pi}_{C_1} \cdots \tilde{\pi}_{C_k} \pi_{D_1} \cdots \pi_{D_{m-l+k}} \tag{86}
\]

where \( (\tilde{A})_m = \tilde{A}_1 \cdots \tilde{A}_m \), \( (C)_k = C_1 \cdots C_k \) and similarly for the other indices. Each function \( f \) (\( \tilde{f} \)) transforms as an irreducible representation of \( SU(2)_L \times SU(2)_R \), with the \( j \)-values \( \frac{1}{2} |l - m| + k \) and \( \frac{1}{2} (l + m) \) respectively.

In comparing with the equation of motion, \( \Box \), we see that all these states satisfy the dispersion relation \( E = -i\omega L \sim P_3 \), where \( P_3 \) is the linear momentum corresponding to \( -iL \). All states with the same value for \( -iL \), namely fixed \( |m - l| \), are degenerate in terms of energy. Out of these, there is a subset of states, which satisfy the relativistic dispersion relation \( E \sim P_3 = \sqrt{P^2} \). These are the states with \( P_1 = P_2 = 0 \), where \( P_1, P_2 \) are the other two directions of momentum on the boundary of the droplet. The three momentum operators do not commute with each other.

From \( \Box \) the operator \( -iL \) can be written as

\[
- iL = w_\tilde{A} \frac{\partial}{\partial w_\tilde{A}} - \tilde{w}_\tilde{A} \frac{\partial}{\partial \tilde{w}_\tilde{A}} \tag{87}
\]

We now define the operators

\[
L_+ = \tilde{w}_\tilde{A} \frac{\partial}{\partial \tilde{w}_\tilde{A}} \\
L_- = w_\tilde{A} \frac{\partial}{\partial w_\tilde{A}} \tag{88}
\]

19
These form an $SU(2)$ algebra along with $-iL$ and correspond to translations along the two other directions (1 and 2) on the boundary of the droplet. Among the modes in (86), there are special states, the highest (lowest) weight states which obey the condition

$$L_+ f = 0 \quad \text{or} \quad L_- \tilde{f} = 0$$

(89)

They are of the form

$$f_{B_1 \cdots B_l C_1 \cdots C_k D_1 \cdots D_k} = \frac{1}{(\bar{\pi} \cdot \pi)^{l+k}} w_{B_1} \cdots w_{B_l} \bar{\pi}_{C_1} \cdots \bar{\pi}_{C_k} \pi_{D_1} \cdots \pi_{D_k}$$

$$\tilde{f}_{\tilde{A}_1 \cdots \tilde{A}_l \tilde{C}_1 \cdots \tilde{C}_k \tilde{D}_1 \cdots \tilde{D}_l+k} = \frac{1}{(\bar{\pi} \cdot \pi)^{l+k}} \tilde{w}_{\tilde{A}_1} \cdots \tilde{w}_{\tilde{A}_l} \bar{\pi}_{\tilde{C}_1} \cdots \bar{\pi}_{\tilde{C}_k} \pi_{\tilde{D}_1} \cdots \pi_{\tilde{D}_l+k}$$

(90)

For these states, the expectation value of $P_1^2 + P_2^2 = (L_- L_+ + L_+ L_-)/r_D^2$ tends to zero as the size of the droplet becomes very large, compared to $P_3$ which remains finite. Their momentum is essentially all given by $P_3 = -iL/r_D$. These states therefore satisfy the relativistic dispersion relation $E = -i\omega L \sim P_3 \sim \sqrt{P^2}$. (These are related to the extremal dipole states in [1].) For states which are not of highest (or lowest) weight, $E \sim P_3$ but this is no longer the square root of $P^2$ since $P_2$ are not zero.

Since the modes transform as irreducible representations of $SU(2)_L \times SU(2)_R$, a consistent definition of helicity would be the difference of the $j$-values for the two $SU(2)$’s. For the highest (lowest) weight states, which satisfy the relativistic dispersion relation, this value is $k$. These results are in general agreement with the results on edge states in [1, 7].

We have found that the effective edge action describing the particle-hole like excitations of the Zhang-Hu four dimensional quantum Hall droplet does not describe a relativistic theory. Whether and how one can consistently truncate the spectrum to keep only the highest (lowest) weight states in order to obtain a relativistic theory is not clear.

6 Conclusions

We have presented a method for obtaining the general form of the effective action for edge excitations of a quantum Hall droplet in higher dimensions. We analyzed in detail the case of $\nu = 1$ QHE on even dimensional $\mathbb{C}P^k$ spaces with a $U(1)$ background magnetic field, which admit droplet configurations in the presence of a confining potential. We find that the edge excitations can be described, in the limit of large number of fermions, in terms of a chiral action for an abelian bosonic field. The chirality here is expressed by the fact that there is a preferred direction along the boundary of the droplet which is relevant for the dynamics of the edge excitations. This tangential direction is the Poisson conjugate to the normal to the droplet, with the Poisson structure given by the Kähler form of the underlying space where the fermions live, thus it is determined by the geometry of the underlying space. We have also obtained, starting from $\mathbb{C}P^3$, the edge action for QHE droplets on $S^4$, the original model.
of Zhang and Hu [1]. We have clarified the kind of potential needed and the corresponding
density function for the reduction to $S^4$ to go through. The corresponding effective edge
action does not describe a relativistic theory, although there are energy eigenmodes in the
decomposition of the bosonic field $\Phi$ that satisfy the relativistic dispersion relation.

We emphasize that there is a general pattern to all these cases. For $\mathbb{CP}^k$, the Kähler
structure determines the form of the action. $S^4$ does not have a complex structure, but $\mathbb{CP}^3$
can be viewed as the bundle of local complex structures on $S^4$; this is the twistor space
approach. And not surprisingly, the edge action for $S^4$ is determined by the local complex
structure. In reference [4], an effective description of the Hall droplet in terms of a Chern-
Simons theory is obtained. It would be interesting to see if our action for edge excitations
can be related to this bulk theory via gauge anomaly arguments.

The analysis in this paper has been for abelian background magnetic fields and for one
species of fermions. If we have a nonabelian background field, or if we have more than one
species of fermions, the edge excitations are described by an element of a nonabelian group.
The resulting edge action will be related to a chiral Wess-Zumino-Witten theory. This will
be described in more detail elsewhere [12].

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