Toric surfaces, $K$-stability and Calabi flow

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Abstract Let $X$ be a toric surface and $u$ be a normalized symplectic potential on the corresponding polygon $P$. Suppose that the Riemannian curvature is bounded by a constant $C_1$ and $\int_P u d\sigma < C_2$, then there exists a constant $C_3$ depending only on $C_1$, $C_2$ and $P$ such that the diameter of $X$ is bounded by $C_3$. Moreover, we can show that there is a constant $M > 0$ depending only on $C_1$, $C_2$ and $P$ such that Donaldson’s $M$-condition holds for $u$. As an application, we show that if $(X, P)$ is (analytic) relative $K$-stable, then the modified Calabi flow converges to an extremal metric exponentially fast by assuming that the Calabi flow exists for all time and the Riemannian curvature is uniformly bounded along the Calabi flow.

1 Introduction

Let $X$ be a polarized Kähler manifold with an ample line bundle $L$. The Yau [32], Tian [29] and Donaldson [11] conjecture says that the existence of cscK metrics is equivalent to that $(X, L)$ is $K$-stable. One way to understand this conjecture is through the geometrical flow. For example, Tosatti [30] shows that if the curvature along the Kähler Ricci flow is uniformly bounded, $(X, L)$ is $K$-stable and asymptotically Chow-semistable, then the Kähler Ricci flow converges to a Kähler-Einstein metric exponentially fast.

Geometrical flow not only can help us understand the Yau–Tian–Donaldson conjecture, but it can also help us understand other geometrical phenomena. For instance, Donaldson [13] conjectures that the infimum of the Calabi energy is equal to the supremum of all normalized Futaki invariants of all destabilizing test configurations. Assuming the long time existence of the Calabi flow, Székelyhidi [28] proves Donaldson’s conjecture when $X$ is a polarized toric variety.

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It is also expected that the Calabi flow can be related to the Yau–Tian–Donaldson conjecture. Chen [7] conjectures that the Calabi flow exists for all time and Donaldson [10] conjectures that if the Calabi flow exists for all time, then it converges to a cscK metric.

Chen’s conjecture is proved in [16] for the case \( X = \mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2) \). Moreover, the curvature is uniformly bounded along the Calabi flow. In the same paper, the authors also prove Donaldson’s conjecture when \( X = \mathbb{C}^2/(\mathbb{Z}^2 + i\mathbb{Z}^2) \). It is expected that the curvature is uniformly bounded along the Calabi flow assuming there exists an extremal metric. Thus we strengthen Chen’s conjecture and weaken Donaldson’s conjecture by adding that the curvature is uniformly bounded along the flow:

**Conjecture 1.1** Let \( [\omega] \) be a Kähler class of \( X \). Suppose that there exists an extremal metric in \( [\omega] \). Then for any Kähler metric \( \omega_1 \in [\omega] \), the Calabi flow starting from \( \omega_1 \) exists for all time and the Riemannian curvature is uniformly bounded along the flow.

**Conjecture 1.2** Let \( [\omega] \) be a Kähler class of \( X \). Suppose that there exists an extremal metric in \( [\omega] \). Let \( \omega_1 \in [\omega] \) be any Kähler metric invariant under the maximal compact subgroup of the identity component of the reduced automorphism group. If the Calabi flow starting from \( \omega_1 \) exists for all time and the Riemannian curvature is uniformly bounded along the flow, then the modified Calabi flow converges exponentially fast to an extremal metric in \( [\omega] \).

We want to link Chen and Donaldson’s conjectures to the Yau–Tian–Donaldson conjecture. Due to recent developments: [2,9] and [27], we only consider the case when \( X \) is a toric variety. We have the following conjectures:

**Conjecture 1.3** Suppose \((X, L)\) is relative K-stable, then for any Kähler metric \( \omega \in c_1(L) \), the Calabi flow starting from \( \omega \) exists for all time and the Riemannian curvature is uniformly bounded along the flow.

**Conjecture 1.4** Suppose \((X, L)\) is relative K-stable. Let \( \omega \in c_1(L) \) be any Kähler metric invariant under the toric action. If the Calabi flow starting from \( \omega \) exists for all time and the Riemannian curvature is uniformly bounded along the flow, then the modified Calabi flow converges exponentially fast to an extremal metric in \( c_1(L) \).

**Remark 1.5** In [33], Zhou–Zhu prove that the analytic relative K-stability is a necessary condition for the existence of extremal metrics on a toric surface. Their result implies that the statement of Conjecture (1.4) implies the one of Conjecture (1.2).

In this note, we prove Conjecture (1.4) when \( X \) is a toric surface and \((X, L)\) is (analytic) relative K-stable. First, we prove that

**Theorem 1.6** Suppose \( u \) is a normalized symplectic potential such that the Riemannian curvature is bounded by \( C_1 \) and there is a constant \( C_2 \) such that

\[
\int_{\partial P} ud\sigma < C_2.
\]

Then there is a constant \( C(C_1, C_2, P) \) depending only on \( C_1, C_2 \) and \( P \) such that the diameter of \( X \) and the maximal value of \( u \) is bounded by \( C(C_1, C_2, P) \).

\[1\] In fact, the initial metric needs to be invariant under the translation in the imaginary part of the complex variables.
We need to show that the complex structure does not jump when at the limit of the Calabi flow. In order to do that, we need Donaldson’s compactness result for toric surfaces [14], that the following theorem enables us to use.

**Theorem 1.7** Under the assumption of Theorem (1.6), there exists a constant $M > 0$ depending only on $C_1, C_2, P$ such that $u$ satisfies the $M$-condition.

Finally we have

**Theorem 1.8** Let $X$ be a toric surface with Kähler class $[\omega]$ and Delzant polygon $P$. Suppose $(X, P)$ is relative $K$-stable. Let $\omega_1 \in [\omega]$ be any toric invariant Kähler metric. If the Calabi flow starting from $\omega_1$ exists for all time and the Riemannian curvature is uniformly bounded along the flow, then the modified Calabi flow converges exponentially fast to an extremal metric in $[\omega]$.

### 2 Notations and setup

#### 2.1 Toric geometry

We review the basic theory of toric varieties from Abreu [1], Donaldson [11] and Guillemin [18, 19].

Let $X$ be a $n$-dimensional toric manifold and $L$ be an ample line bundle over $X$. Suppose $\omega \in c_1(L)$ is a toric invariant Kähler metric. We get a Delzant polytope $P$ through the moment map. The measure $dx$ on the interior of $P$ is the standard Lebesgue measure. The measure $d\sigma$ on the boundary of $P$ is a constant multiplying the standard Lebesgue measure. The constant is

$$\frac{1}{|\vec{n}_i|}$$

on each facet $P_i$ of $P$, where $\vec{n}_i \in \mathbb{Z}^n$ is an inward normal vector of $P_i$. The Delzant conditions for $P$ are: for any vertex $v$, there are exactly $n$ facets $P_{i_1}, \ldots, P_{i_n}$ at $v$ and $(\vec{n}_{i_1}, \ldots, \vec{n}_{i_n})$ is a basis of $\mathbb{Z}^n$.

Suppose $P$ has $d$ facets. Let $l_i(x) = \langle x, n_i \rangle - c_i$ and we choose $c_i$ properly such that $l_i = 0$ on $P_i$. Thus $P$ can be expressed as

$$P = \{x \in \mathbb{R}^n | l_i(x) > 0, i = 1, \ldots, d\}.$$ 

Let the set of symplectic potentials be $\mathcal{H}_T$. The Guillemin boundary conditions tell us that $u \in \mathcal{H}_T$ iff

- $u = f + \sum_{i=1}^d \frac{1}{2} l_i \ln l_i$, $f \in C^\infty(\bar{P})$.
- $u$ is a convex function in $P$.
- $u$ is also a convex function when restricted on the relative interior of each face of $P$.

The Abreu equation for the scalar curvature $R$ reads

$$R = -\sum_{ij} u_{ij}^{ij}.$$
2.2 Calabi flow and modified Calabi flow

By the $\partial \bar{\partial}$-lemma, any toric invariant Kähler metric $\omega_\varphi \in c_1(L)$ can be written as

$$\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi,$$

where $\varphi \in C^\infty(X)$ is a toric invariant function. The Calabi flow equation is

$$\frac{\partial \varphi}{\partial t} = R_\varphi - R,$$

where $R_\varphi$ is the scalar curvature of $\omega_\varphi$ and $R$ is the average of $R_\varphi$. By the proof of the short time existence of [6], if we start with a toric invariant Kähler metric, then the Kähler metric along the Calabi flow are all invariant under the toric action.

If we express the Calabi flow equation in $P$, it reads

$$\frac{\partial f}{\partial t} = R - R_f.$$

Notice that, apart for constant scalar curvature Kähler metrics, extremal metrics are not stationary points of the Calabi flow. The modified Calabi flow, introduced in [21], is more adequate to study extremal Kähler metrics. Moreover, it coincides with the Calabi flow whenever the Futaki invariant vanishes.

Let us recall the definition of the extremal vector field from [17]. In general, let $\omega$ be a Kähler metric which is invariant under a maximal compact group $G$ of the reduced automorphism group of $X$. Let $\mathcal{J}$ be the holomorphic structure compatible with $\omega$. A function $f$ is called a Killing potential if $\mathcal{J}(\nabla f)$ is a Killing vector field of the Riemannian metric $g_\omega$, or equivalently, $\nabla f$ is a real holomorphic vector field. A real holomorphic field $\mathcal{X}$ is called the extremal vector field if $\mathcal{J}(\mathcal{X})$ is a Killing vector field of the Riemannian metric $g_\omega$ and the potential of $\mathcal{X}$ is the $L^2_\omega$ projection of the scalar curvature $R_\omega$ to the sets of all Killing potentials of Killing vector fields corresponding to the Lie algebra of $G$.

Let $\theta$ be the real function on $X$ satisfying

$$L_{\mathcal{X}} \omega = \sqrt{-1} \partial \bar{\partial} \theta, \quad \int_X \theta \omega^n = \int_X R_\omega \omega^n.$$

Let $\varphi$ be a Kähler potential invariant under $G$. We define the modified Calabi flow starting from $\omega_\varphi = \omega + \sqrt{-1} \partial \bar{\partial} \varphi$ as:

$$\frac{\partial \varphi}{\partial t} = R_\varphi - \theta_\varphi,$$

where $\theta_\varphi = \theta + \mathcal{X}(\varphi)$.

When $X$ is a toric variety, the modified Calabi flow equation on $P$ reads

$$\frac{\partial f}{\partial t} = \theta_P - R_f,$$

where $\theta_P$ is an affine function such that for any affine function $u$

$$2 \int_P u d\sigma - \int_P u \theta_P dx = 0. \quad (c.f.[9])$$

We write $\theta$ instead of $\theta_P$ for convenience.

It is easy to check that when the Futaki invariant vanishes, the modified Calabi flow coincides with the Calabi flow.
Following [11], the modified Mabuchi energy is defined as

$$\mathcal{M}(f) = -\int_{\mathcal{P}} \log \det(D^2 u_f) dx + \mathcal{L}(u_f),$$

where

$$\mathcal{L}(u_f) = 2\int_{\partial \mathcal{P}} u_f d\sigma - \int_{\mathcal{P}} u_f \theta dx.$$ 

In fact, the modified Calabi flow is the downward gradient flow of the modified Mabuchi energy by the following calculations. Let $\delta f = h$, then

$$\delta \mathcal{M}(h) = -\int_{\mathcal{P}} u^{ij} h_{ij} dx + \mathcal{L}(h)$$

$$= -2\int_{\partial \mathcal{P}} h dx + \int_{\mathcal{P}} R_f h dx + 2\int_{\partial \mathcal{P}} h d\sigma - \int_{\mathcal{P}} h \theta dx$$

$$= \int_{\mathcal{P}} (R_f - \theta) h dx.$$ 

2.3 Relative $K$-stability

The algebraic relative $K$-stability for toric varieties in [11] states as follows:

**Definition 2.1** $(X, L)$ is algebraic relative $K$-stable if $\mathcal{L}(u) \geq 0$ for all rational piecewise linear function $u$. The equality holds if and only if $u$ is an affine function.

We recall the definition the analytic relative $K$-stability (c.f. [2, 33]):

**Definition 2.2** $(X, P)$ is analytic relative $K$-stable if $\mathcal{L}(u) \geq 0$ for all piecewise linear function $u$. The equality holds if and only if $u$ is an affine function.

Notice that the only difference between the two $K$-stabilities is that the second one does not require $u$ to be rational. Thus the latter implies the former. In this note, we use analytic relative $K$-stability as our relative $K$-stability.

Let $x_0$ be an interior point of $P$. A convex function $u$ is normalized (at $x_0$) if

$$u(x_0) = 0, \quad Du(x_0) = 0.$$ 

where $D$ is the Euclidean derivative. In fact, the way that we normalize a convex function $u$ is done by adding an affine function. This procedure does not affect the boundary conditions nor the metric. Moreover, the normalization corresponds to choosing the minimum point of $u$ to be $x_0$ and its minimal value to be zero. Also $\mathcal{M}(u)$ and $\mathcal{L}(u)$ do not change after the normalization.

Let $\mathcal{C}_\infty$ be the set of continuous convex functions on $\bar{P}$ which are smooth in the interior of $P$. Let $P^*$ be the union of $P$ and its facets and let $\mathcal{C}_1$ be the set of positive convex function $f$ on $P^*$ such that

$$\int_{\partial P} f d\sigma < \infty.$$
Proposition 5.2.2 in [11] shows that either there is a positive constant $\lambda > 0$ such that
\[
\mathcal{L}(f) \geq \lambda \int_{\partial \mathcal{P}} f \, d\sigma
\]
for all normalized functions $f$ in $C_\infty$ or there is a function $f$ in $C_1$ which is not an affine function and
\[
\mathcal{L}(f) \leq 0.
\]
Let us assume that $(X, P)$ is relative $K$-stable and $X$ is a toric surface. If there is a function $f \in C_1$ such that $\mathcal{L}(f) < 0$, then Corollary 4.1 in [31] tells us that $(X, P)$ is not relative $K$-stable, a contradiction. If for all functions $f \in C_1$, $\mathcal{L}(f) \geq 0$ and there is a not affine function $f \in C_1$, $\mathcal{L}(f) = 0$, then Theorem 4.1 in [31] shows that $(X, P)$ is not relative $K$-stable. Contradiction again. Thus we conclude the following proposition:

**Proposition 2.3** If $(X, P)$ is relative $K$-stable, then there is a constant $\lambda > 0$ such that
\[
\mathcal{L}(f) \geq \lambda \int_{\partial \mathcal{P}} f \, d\sigma
\]
for all normalized functions $f$ in $C_\infty$.

### 3 Regularity theorems in Calabi flow

The regularity theorem in Ricci flow is called Shi’s estimate [24]. Let $M$ be a Riemannian manifold and $g(t)$, $t \in [-1, 0]$ be a one parameter Riemannian metric satisfying the Ricci flow equation, i.e.,
\[
\frac{\partial g(t)}{\partial t} = -2Ric(t).
\]
Suppose $|Rm(g(t))|$ is bounded by $C_1$ for $t \in [-1, 0]$. Then for any integer $k > 0$, there is a constant $C(n, k, C_1)$ depending only on $n, k$ and $C_1$ such that
\[
|\nabla^k Rm(0, x)| < C(n, k, C_1),
\]
for any $x \in M$. The regularity theorem plays an important role in the singularity analysis in Hamilton-Perelman’s program, see e.g. [3,5,22,23].

Chen–He [8] develop the weak regularity theorem in the singularity analysis of the Calabi flow. Let $X$ be a Kähler manifold and $[\omega]$ be a Kähler class of $X$. Suppose that the Calabi flow $\omega(t) \in [\omega]$ exists for $t \in [-1, 0]$, the $L^2_{\omega(-1)}$ norm of the bisectional curvature $Rm(-1)$ is bounded by $C_0$ at $t = -1$ and the $L^\infty$ norm of the bisectional curvature $Rm(t)$ is bounded by $C_1$ for $t \in [-1, 0]$. Then for any integer $k > 0$, there is a constant $C(n, k, C_0, C_1)$ depending only on $n, k, C_0$ and $C_1$ such that
\[
\int_X |\nabla^k Rm(0, x)|^2 \omega_0^2 < C(n, k, C_0, C_1).
\]
Streets also develops similar estimates in [25].

In [20], the regularity theorem is shown to be one of the obstructions of the long time existence of the Calabi flow on toric varieties. Later, Streets [26] obtains the regularity
theorem: suppose the Calabi flow $\omega(t) \in [\omega]$ exists for $t \in [-1, 0]$ and the $L^\infty$ norm of the bisectional curvature $Rm(t)$ is bounded by $C_1$ for $t \in [-1, 0]$. Then for any integer $k > 0$, there is a constant $C(n, k, C_1)$ depending only on $n, k$ and $C_1$ such that

$$|\nabla^k Rm(0, x)| < C(n, k, C_1).$$

Notice that Streets proves the regularity theorem using the sectional curvature and the computations are done in real coordinates. In fact, he needs the following formula for the evolution equation of the sectional curvature:

$$\frac{\partial Rm}{\partial t} = -\Delta^2 Rm + \nabla^2 Rm \ast Rm + \nabla Rm \ast \nabla Rm.$$  

However, the evolution equation of the curvature in [8] is written in terms of bisectional curvature and holomorphic coordinates. Thus one needs to redo the calculations. But there is no essential difficulty to go through Streets' calculations.

4 Diameter control

In this section, we prove Theorem (1.6).

Without loss of generality, we can assume that we work in a standard model, i.e., $O = (0, 0)$ is a vertex of $P$, $x_1, x_2$ are the edges of $P$ around $O$ and $P$ lies in the first quadrant. Guillemin's boundary conditions tell us that

$$u = \frac{1}{2} (x_1 \ln x_1 + x_2 \ln x_2) + f(x_1, x_2),$$

where $f$ is a smooth function up to the boundary.

Our first observation is the following lemma:

**Lemma 4.1** In edge $x_1$, let $V(x_1) = \frac{1}{2} x_1 \ln x_1 + f(x_1, 0)$, then

$$\left( \frac{1}{V''} \right)''(x_1) = u_{11}^{11}(x_1, 0).$$

**Proof** By direct computations, we have

$$u_{11}^{11} = \left( \frac{u_{221}}{\det(u_{ij})} - \frac{u_{22} \det(u_{ij})_1}{(\det(u_{ij}))^2} \right)_{11}$$

$$= \frac{u_{2211}}{\det(u_{ij})} - 2 \frac{u_{22} \det(u_{ij})_1}{(\det(u_{ij}))^2} - \frac{u_{22} \det(u_{ij})_{11}}{(\det(u_{ij}))^2} + 2 \frac{u_{22} \det(u_{ij})_1^2}{(\det(u_{ij}))^3}.$$  

Let $v(x_1) = V''(x_1)$. As $x \to (x_1, 0)$, we get

$$\lim_{x \to (x_1, 0)} u_{11}^{11}(x) = \left( -\frac{v''}{v^2} + 2 \frac{v'^2}{v^3} \right)(x_1)$$

$$= \left( -\frac{v'}{v^2} \right)'(x_1)$$

$$= \left( \frac{1}{v} \right)''(x_1).$$

Hence we obtain the desired result. $\Box$
It is shown in [12] that the norm of Riemannian curvature is expressed as

\[ |Rm|^2 = \sum u_{ij}^{kl} u_{ij}^{kl}. \]

By direction calculations, we have

**Lemma 4.2** All \( u_{ij}^{kl}(x_1, x_2) \) is finite and

\[ u_{11}^{22}(x_1, 0) = u_{12}^{22}(x_1, 0) = u_{11}^{12}(x_1, 0) = 0 \]

for all \( x_1, x_2 \in [0, 1] \).

**Proof** The proof follows directly from the Guillemin boundary condition expressed on \( (u_{ij}^{kl}) \) as stated in [2]. \( \square \)

As a corollary, we can simplify the expression of \( |Rm| \) in \((x_1, 0)\).

**Corollary 4.3**

\[ |Rm|^2(x_1, 0) = (u_{11}^{11})^2(x_1, 0) + 4(u_{12}^{12})^2(x_1, 0) + (u_{22}^{22})^2(x_1, 0) \]

**Proof** Notice that

\[ |Rm|^2(x_1, 0) = \sum_{i,j,k,l} u_{ij}^{kl} u_{ij}^{kl}(x_1, 0). \]

For each \( u_{ij}^{kl} \), if there are three 1 or 2 in \( i, j, k, l \), then by the above lemma, we have \( u_{ij}^{kl} u_{ij}^{kl}(x_1, 0) = 0 \). If there are exactly two 1 in \( i, j, k, l \) and \( i = j \), then we also have \( u_{ij}^{kl} u_{ij}^{kl}(x_1, 0) = 0 \). Thus we obtain the conclusion. \( \square \)

The integral bound of \( u \) on the boundary shows that: for every \( \epsilon > 0 \), there exists a constant \( C > 0 \) depending only on \( \epsilon, C_2 \) and \( P \) such that for every point \( x = (x_1, 0), \epsilon \leq x_1 \leq 1 \), we have

\[ u(x) < C, \quad \left| \frac{\partial u}{\partial x_1} (x) \right| < C. \]

Together with the above lemma, we have

**Proposition 4.4** There exists a constant \( C_3 > 0 \) depending only on \( C_1, C_2 \) and \( P \) such that

\[ u(0, 0) < C_3, \quad |\nabla f|(0, 0) < C_3. \]

**Proof** Let \( V(x_1) = \frac{1}{2} x_1 \ln x_1 + f(x_1, 0) \). Without loss of generality, we only need to show that there exists a constant \( C_3 > 0 \) depending only on \( C_1, C_2 \) and \( P \) such that

\[ V(0) < C_3, \quad \left| \frac{\partial f}{\partial x_1} (0) \right| < C_3. \]

It is easy to see that

\[ \left( \frac{1}{V''} \right)'(0) = 2. \]
Let us pick $\epsilon = \frac{1}{C_1}$, then for any $s \in (0, \epsilon]$, we have
\[
\left| \left( \frac{1}{V''} \right)'(s) - 2 \right| = \left| \int_0^s \left( \frac{1}{V''} \right)''(x) \, dx \right| \leq C_1 s.
\]
Hence
\[
2 - C_1 s \leq \left( \frac{1}{V''} \right)'(s) \leq 2 + C_1 s.
\]
Since
\[
\frac{1}{V''}(0) = 0,
\]
we have
\[
2s - \frac{C_1}{2} s^2 \leq \frac{1}{V''}(s) \leq 2s + \frac{C_1}{2} s^2.
\]
In terms of $f(x_1, 0)$, we have
\[
-\frac{C_1}{8 + 2C_1 s} \leq \frac{\partial^2 f}{\partial x_1^2}(s, 0) \leq \frac{C_1}{8 - 2C_1 s}.
\]
Since $|\nabla f(\epsilon, 0)| < C$, we conclude that $|\frac{\partial f}{\partial x_1}(s, 0)| < C$ for all $s \in [0, \epsilon]$. It is also easy to control the $C^0$ norm of $f$ at $(0, 0)$.

Next we want to show that the diameter of $(X, u)$ is bounded by a constant $C$ depending only on $C_1$, $C_2$ and $P$. Without loss of generality, we only need to show for a special case: for any point $x = (x_1, x_2)$ with $x_1, x_2 \geq 0$, $x_1 + x_2 = 1$, the Riemannian distance $d_u(O, x) < C$. The general case can be easily converted to the special case.

Let the vector $\mathbf{v} = (a, b)$, $a, b \geq 0$, $a + b = 1$ be a vector pointing from $O$ to $x$. We can parametrize the line interval from $O$ to $x$ as following:
\[
x_1(t) = at, \quad x_2(t) = bt, \quad t \in [0, 1].
\]
Let $V(t)$ be the restriction of $u$ on the above line interval. Then
\[
V(t) = \frac{1}{2} (at \ln(at) + bt \ln(bt)) + f(at, bt) = \frac{1}{2} t \ln t + g(t),
\]
where $g(t) = \frac{1}{2} (at \ln a + bt \ln b) + f(at, bt)$.

Notice that since
\[
|\nabla f \cdot \mathbf{v}|(0) < C,
\]
we have $|g'(0)| < C$. In fact, we can prove the following lemma.

**Lemma 4.5** There is a constant $C > 0$ depending only on $C_1$, $C_2$ and $P$ such that for every $t \in [0, 1]$.
\[
|g'(t)| < C.
\]
Proof Pick a small constant $\epsilon > 0$, we only need to control $g'(t)$ for $t \in [0, \epsilon]$. Since $V(t)$ is a convex function and $V(0), V(1) \geq 0$, $V(0), V(1) \leq C(C_1, C_2, P)$, we can control $g'(\epsilon)$ by a constant depending only on $\epsilon, C_1, C_2$ and $P$.

By Lemma 3 in [14], we have

$$\left(\frac{1}{V''(t)}\right)'' \leq |Rm|(x(t), y(t)).$$

The arguments in Proposition (4.4) show that

$$g''(t) > -\frac{C_1}{8 + 2C_1 t}.$$ 

Together with the fact $g'(0), g'(\epsilon)$ are bounded, we conclude that $g'(t)$ is bounded by a constant depending only on $C_1, C_2$ and $P$. \qed

As a result, we have

**Corollary 4.6** There exists a constant $C > 0$ depending only on $C_1, C_2$ and $P$ such that

$$d_u(O, x) < C(C_1, C_2, P).$$

*Proof* We split the interval $[0, 1]$ into infinite many intervals as

$$\left[\frac{1}{2^i}, \frac{1}{2^{i+1}}\right], \left[\frac{1}{2^i}, \frac{1}{2^{i+1}}\right], \ldots, \left[\frac{1}{2^i}, \frac{1}{2^{i+1}}\right], \ldots$$

Then

$$d_u(O, x) \leq \int_0^1 \sqrt{V''(t)} dt$$

$$= \lim_{N \to \infty} \sum_{i=0}^{N} \int_{\frac{1}{2^i+1}}^{\frac{1}{2^i}} \sqrt{V''(t)} dt$$

$$\leq \lim_{N \to \infty} \sum_{i=0}^{N} \sqrt{\frac{1}{2^i+1} \int_{\frac{1}{2^i+1}}^{\frac{1}{2^i}} V''(t) dt}$$

$$= \lim_{N \to \infty} \sum_{i=0}^{N} \sqrt{\frac{1}{2^i+1} \left(\ln 2 + g'(\frac{1}{2^i}) - g'(\frac{1}{2^i+1})\right) dt}$$

$$\leq C(C_1, C_2, P) \sum_{i=0}^{\infty} \frac{1}{(\sqrt{2})^{i+1}}$$

$\leq C(C_1, C_2, P)$ \qed

*Proof of Theorem 1.6* By Proposition (4.4) and Corollary (4.6), we obtain the result. \qed
5 M-condition

We recall the definition of $M$-condition from [14]. Let $x_1 x_4$ be any segment of $P$ and $x_2, x_3$ be two points in $x_1 x_4$ such that $|x_1 x_2| = |x_2 x_3| = |x_3 x_4|$. Let $\vec{v}$ be an unit vector pointing from $x_1$ to $x_4$. $u$ satisfies the $M$-condition on $x_1 x_4$ if

$$|Du \cdot \vec{v}(x_2) - Du \cdot \vec{v}(x_3)| < M.$$ 

**Definition 5.1** $u$ satisfies the $M$-condition on $P$ if for any segment $l \subset P$, $u$ satisfies the $M$-condition on $l$.

Let us write $u(x_1, x_2) = \frac{1}{2}(x_1 \ln x_1 + x_2 \ln x_2) + f(x_1, x_2)$. Proposition (4.4) shows that

$$\left| \frac{\partial f}{\partial x_1} (x_1, 0) \right|, \left| \frac{\partial f}{\partial x_2} (0, x_2) \right| < C(C_1, C_2, P),$$

for any $x_1, x_2 \in [0, 1]$. Our next few lemmas show that we can also control $|\frac{\partial f}{\partial x_1} (0, x_2)|$.

**Lemma 5.2**

$$u_{12}^{12}(0, x_2) = -\frac{\partial}{\partial x_2} \frac{\frac{2f_{12}}{f_{22} + \frac{1}{x_2}}}{f_{22} + \frac{1}{x_2}}(0, x_2),$$

where $x_2 \in (0, 1]$.

**Proof** This is done by direct calculations.

$$u_{12}^{12}(0, x_2) = -\lim_{x_1 \to 0} \frac{\partial^2}{\partial x_1 \partial x_2} \frac{f_{12}}{u_{11} u_{22} - f_{12}^2}(x_1, x_2)$$

$$= -\lim_{x_1 \to 0} \frac{\partial}{\partial x_2} \frac{2f_{12}}{f_{22} + \frac{1}{x_2}}(x_1, x_2)$$

$$= -\frac{2f_{12}}{f_{22} + \frac{1}{x_2}}(0, x_2)$$

$$\square$$

**Proposition 5.3** There exists a constant $C$ depending only on $C_1, C_2$ and $P$ such that for any $x_2 \in (0, 1]$, we have

$$\left| \frac{\partial f}{\partial x_1} (0, x_2) \right| < C.$$

**Proof** Combining the previous results, we have

$$\left| \frac{\partial}{\partial x_2} \frac{\frac{2f_{12}}{f_{22} + \frac{1}{x_2}}}{f_{22} + \frac{1}{x_2}}(0, x_2) \right| = |u_{12}^{12}(0, x_2)| < C_1.$$

Notice that

$$\lim_{x_2 \to 0} \frac{2f_{12}}{f_{22} + \frac{1}{x_2}}(0, x_2) = 0.$$
Then
\[ \left| \frac{2f_{12}}{f_{22} + \frac{1}{2x_2}}(0, x_2) \right| < Cx_2. \]

When \( x_2 \) is close to 0, Proposition (4.4) tells us that \( f_{22}(0, x_2) \) is bounded. When \( x_2 \) is away from 0, Lemma 4 of [14] tells us that \( f_{22} + \frac{1}{2x_2} \) is bounded. So we conclude that for \( x_2 \in [0, 1] \),
\[ f_{12}(0, x_2) < C. \]

Together with the fact that \( f_1(0, 0) \) is bounded, we obtain the conclusion. \( \square \)

**Proposition 5.4** For any \( x_1, x_2 \in [0, 1] \), we have
\[ |\nabla f(x_1, x_2)| < C(C_1, C_2, P). \]

**Proof** Without loss of generality, we only need to control \( \frac{\partial f}{\partial x_1}(x_1, x_2) \) for any \( x_1, x_2 \in (0, 1] \).

In fact, we only need to consider the case when \( x_1 \) is small. Let us pick a constant \( \epsilon > x_1 \). It is easy to see that \( \frac{\partial f}{\partial x_1}(\epsilon, x_2) \) is controlled by a constant depending only on \( \epsilon, C_1, C_2 \) and \( P \).

Lemma 3 in [14], together with our assumption on \( |Rm| \), shows that
\[ \left| \frac{1}{u_{11}}(s, x_2) \right| < C_1 \]
for all \( s \in [0, \epsilon] \). Let \( V(t) = u(t, x_2), t \in [0, \epsilon] \), we have
\[ \left( \frac{1}{V''} \right)(t) < C. \]

The calculations in Proposition (4.4) show that
\[ f_{11}(s, x_2) > C \]
for all \( s \in [0, \epsilon] \) since \( \frac{1}{x_2} \ln x_2 \) has no contribution. Together with the fact that \( \frac{\partial f}{\partial x_1}(0, x_2) \) and \( \frac{\partial f}{\partial x_1}(\epsilon, x_2) \) are controlled, we obtain that \( \frac{\partial f}{\partial x_1}(s, x_2) \) is controlled for all \( s \in (0, \epsilon) \). \( \square \)

As a consequence, we have proved Theorem (1.7).

### 6 Compactness

Let \( u^{(\alpha)}(t, x), t \in [-1, 0], x \in P \) be a sequence of modified Calabi flows on \( P \) satisfying:

- For any \( \alpha \), the Riemannian curvature of \( u^{(\alpha)}(t, x), t \in [-1, 0], x \in P \) is bounded by \( C_1 \) and \( u^{(\alpha)}(0, x) \) is normalized.
- For any \( \alpha \) and any \( t \in [-1, 0] \), let \( \tilde{u}^{(\alpha)}(t, x) \) be the normalization of \( u^{(\alpha)}(t, x) \), then
  \[ \int_P \tilde{u}^{(\alpha)}(t, x)d\sigma < C_2. \]

By the results of the previous section, we conclude that \( \tilde{u}^{(\alpha)}(t, x) \) satisfies the \( M \)-condition for all \( \alpha \) and \( t \in [-1, 0] \). Thus the injectivity radius of \( (X, u^{(\alpha)}(t, x)) \) is bounded from below.
by Proposition 4 of [14]. By applying the weak regularity theorem or the regularity theorem to the modified Calabi flow (see Remark (6.1)), we obtain
\[ |\nabla^k Rm^{(\alpha)}|(t, x) \leq C(k, C_1, C_2, P), \]
for all \( \alpha, x \in P \) and \( t \in [-\frac{1}{2}, 0] \).

**Remark 6.1** According to [21], the modified Calabi flow is just the pull back of the Calabi flow by a one parameter group diffeomorphisms generating by the real extremal vector field. We apply the weak regularity theorem or the regularity theorem to the corresponding Calabi flow to get the estimates. Then we get the same estimates for the modified Calabi flow. Notice that by Calabi-Chen [4], the Calabi flow decreases the distance. We can also conclude that the modified Calabi flow decreases the distance.

For any \( \epsilon > 0 \), let \( P_\epsilon \) be the region in \( P \) whose point is away from \( \partial P \) with Euclidean distance at least \( \epsilon \). By Lemma 4 and Lemma 6 of [14], we can control \( \left( \frac{\partial^2 u^{(\alpha)}(t, x)}{\partial x_i \partial x_j} \right) \) for any \( \alpha, t \in [-\frac{1}{2}, 0] \) and \( x \in P_\epsilon \).

Let us temporarily suppress \( \alpha \) and consider the Abreu equation for \( t \in [-\frac{1}{2}, 0] \) and \( x \in P_\epsilon \):
\[ -U^{ij}\left( \frac{1}{\det(D^2u(t, x))} \right) = R(t, x). \]

By (1) and Corollary 5.2 of [16], we can control the \( C^\infty \) norm of \( R(t, x) \), where the \( C^\infty \) norm is for \( x \) coordinates and is measured in terms of the standard Euclidean metric. Thus we can control the \( C^\infty \) norm of \( u(t, x) \) by Shauder’s estimates. Again, the \( C^\infty \) norm is for \( x \) coordinates and is measured in terms of the standard Euclidean metric. In order to control the derivatives of \( u(t, x) \) in terms of \( t \) and the mixed derivatives of \( t \) and \( x \), one can use Proposition 5.4 of [16]. Notice that the Euclidean \( C^0 \) and \( C^1 \) norm of \( u^{(\alpha)}(t, x) \) in space direction are controlled for \( t \in [-\frac{1}{2}, 0], x \in P_\epsilon \) because \( R(t, x) \) and \( D(R(t, x)) \) are controlled, where \( D(R(t, x)) = (\frac{\partial R}{\partial x_1}, \frac{\partial R}{\partial x_2}) \) is the Euclidean derivative in the space direction. Our discussions lead to the following result:

**Theorem 6.2** By passing to a subsequence, \( u^{(\alpha)}(t, x) \) converges to a smooth function \( u(t, x), t \in [-\frac{1}{2}, 0], x \in P \). Moreover \( u(t, x) \) are symplectic potentials on \( P \) satisfying the modified Calabi flow equation.

**Proof** The only thing that we need to prove is that \( u(t, x) \) satisfies the Guillemin boundary conditions. The proof is exactly as the proof of Proposition 8 of [14]. \( \square \)

### 7 Relative \( K \)-stability

By Proposition (2.3) in Sect. 2.3 and Proposition 5.1.2 of [11], we know that the modified Mabuchi energy is bounded from below in \( C^\infty \). Moreover, if \( u^{(\alpha)} \) is any sequence of normalized functions in \( C^\infty \), which is a minimizing sequence of the modified Mabuchi energy, Proposition 5.1.8 of [11] shows that
\[ \int_{\partial P} u^{(\alpha)} \leq C_2. \]

Now we suppose that the Calabi flow \( u(t, x) \) exists for all time and the Riemannian curvature is uniformly bounded by \( C_1 \). Then the corresponding modified Calabi flow exists for all time.
and the Riemannian curvature is also uniformly bounded by $C_1$. We still denote the modified Calabi flow as $u(t, x)$. Let us take a sequence of $t_i \to \infty$ and define a sequence of the modified Calabi flow by

$$u^{(\alpha)}(t, x) = u(t_\alpha + t, x), \quad t \in [-1, 0], \ x \in P.$$  

For each $\alpha$, we add some affine function $l^{(\alpha)}(x)$ to $u^{(\alpha)}(t, x)$ such that $u^{(\alpha)}(0, x)$ is normalized. Notice that the modified Mabuchi energy does not change if we add $l^{(\alpha)}(x)$ to $u^{(\alpha)}(t, x)$. With the fact that the modified Calabi flow decreases the modified Mabuchi energy, we conclude

$$\int \tilde{u}^{(\alpha)}(t, x)dx \leq C_2,$$

where $\tilde{u}^{(\alpha)}(t, x)$ is the normalization of $u^{(\alpha)}(t, x)$ for any $\alpha$ and any $t \in [-1, 0]$.

Theorem (6.2) tells us that $u^{(\alpha)}(t, x)$ converges to $u^{(\infty)}(t, x), \ t \in [-\frac{1}{2}, 0], \ x \in P$ by passing to a subsequence. Since the modified Mabuchi energy is bounded from below, the modified Mabuchi energy of $u^{(\infty)}(0, x)$ is the infimum of the modified Mabuchi energy of $u(t, x)$. We can also see that the modified Mabuchi energy of $u^{(\infty)}(t, x), \ t \in [-\frac{1}{2}, 0]$ is the infimum of the modified Mabuchi energy of the original modified Calabi flow. Thus the scalar curvature of $u^{(\infty)}(t, x)$ is $\theta$.

In order to show that $u(t_i, x)$ converges to an extremal metric, we only need to show that $l_\alpha(x)$ is bounded. Calabi and Chen show that the Calabi flow decreases the geodesic distance in $[4]$. By Remark (6.1), this result also holds for the modified Calabi flow, i.e.,

$$\int_P (u(t, x) - u^{(\infty)}(0, x))^2dx$$

is decreasing as $t$ increases. It tells us that

$$\int_P u^2(t, x)dx$$

is bounded. Together with the fact that

$$\int_P (u^{(\alpha)}(0, x))^2dx$$

is bounded, we conclude that the affine function $l^{(\alpha)}(x)$ is bounded. Thus by passing to a subsequence, $u(t_i, x)$ converges to an extremal metric $u^{(\infty)}(x)$. Let us write the Kähler metric of $u(t_i, x), u^{(\infty)}(x)$ as $\omega(t_i, x), \omega^{(\infty)}(x)$ respectively. Let $\varphi_i$ be the difference of the Legendre transform of $u(t_i, x)$ and $u^{(\infty)}(x)$. We obtain

$$\omega(t_i) = \omega^{(\infty)} + \sqrt{-1}\partial\bar{\partial}\varphi_i.$$  

In page 118 of [15], Donaldson shows that

$$||\varphi_i||_L^\infty = ||u(t_i) - u^{(\infty)}||_L^\infty.$$  

Thus the $L^\infty$ norm of $\varphi_i$ is independent of $i$. Since the $L^\infty$ norm of $Ric(\varphi_i)$ is also independent of $i$, we conclude that the $C^{3,\alpha}$ norm of $\varphi_i$ is also independent of $i$ by Theorem 5.1 of [6]. Notice that the distance between $u(t_i, x)$ and $u^{(\infty)}(x)$ goes to 0 as $i$ goes to infinity. Applying the results of [21], we obtain the exponential convergence of the modified Calabi flow. Hence we complete the proof of Theorem (1.8).
8 Discussion

It is easy to see that the analytic relative $K$-stability is stronger than the algebraic relative $K$-stability. But Donaldson [11] shows that for toric surfaces with vanishing Futaki invariant, the algebraic relative $K$-stability implies the analytic relative $K$-stability. More precisely, he shows that there exists a constant $\lambda > 0$ such that for all normalized convex function $f$ in $C_{\infty}$, we have

$$\mathcal{L}(f) > \lambda \int_{\partial P} f \, d\sigma.$$ 

Thus we obtain the following stronger result when the Futaki vanishes.

**Theorem 8.1** Let $X$ be a toric surface with ample line bundle $L$. Suppose $(X, L)$ is algebraic $K$-stable with vanishing Futaki invariant. Let $\omega \in c_1(L)$ be any toric invariant Kähler metric. If the Calabi flow starting from $\omega$ exists for all time and the Riemannian curvature is uniformly bounded along the flow, then the Calabi flow converges exponentially fast to a csc$K$ metric in $c_1(L)$.

It is an interesting question whether the algebraic $K$-stability implies the analytic $K$-stability when $X$ is a toric surface.

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**References**

1. Abreu, M.: Kähler geometry of toric varieties and extremal metrics. Int. J. Math. 9, 641–651 (1998)
2. Apostolov, V., Calderbank, D.M.J., Gauduchon, P., Tonnesen-Friedman, C.W.: Hamiltonian 2-forms in Kähler geometry III: extremal metrics and stability. Invent. Math. 173, 547–601 (2008)
3. Bessières, L., Besson, G., Boileau, M., Maillot, S., Porti, J.: The Geometrisation of 3-Manifolds, EMS Tracts in Mathematics, volume 13. European Mathematical Society, Zurich (2010)
4. Calabi, E., Chen, X.X.: Space of Kähler metrics and Calabi flow. J. Differ. Geom. 61(2), 173–193 (2002)
5. Cao, H.D., Zhu, X.P.: A complete proof of the Poincaré and geometrization conjectures—application of the Hamilton–Perelman theory of the Ricci flow. Asian J. Math. 10(2), 165–492 (2006)
6. Chen, X.X., He, W.Y.: On the Calabi flow. Am. J. Math. 130(2), 539–570 (2008)
7. Chen, X.X., He, W.Y.: The Calabi flow on toric Fano surface. Math. Res. Lett. 17(2), 231–241 (2010)
8. Chen, X.X., He, W.Y.: The Calabi flow on Kähler surface with bounded Sobolev constant-(I). Math. Ann. 354(1), 227–261 (2012)
9. Donaldson, S.K.: b-Stability and blow-ups. arXiv:1107.1699
10. Donaldson, S.K.: Conjectures in Kähler geometry. Strings and geometry, 71–78, Clay Math. Proc., 3, Am. Math. Soc., Providence, RI (2004)
11. Donaldson, S.K.: Scalar curvature and stability of toric varieties. J. Differ. Geom. 62, 289–349 (2002)
12. Donaldson, S.K.: Interior estimates for solutions of Abreu’s equation. Collect. Math. 56, 103–142 (2005)
13. Donaldson, S.K.: Lower bounds on the Calabi functional. J. Differ. Geom. 70(3), 453–472 (2005)
14. Donaldson, S.K.: Extremal metrics on toric surfaces: a continuity method. J. Differ. Geom. 79(3), 389–432 (2008)
15. Donaldson, S.K.: Constant scalar curvature metrics on toric surfaces. Geom. Funct. Anal. 19(1), 83–136 (2009)
16. Feng, R.J., Huang, H.N.: The global existence and convergence of the Calabi flow on $C^n = \mathbb{Z}^n + i\mathbb{Z}^n$. J. Funct. Anal. 263(4), 1129–1146 (2012)
17. Futaki, A., Mabuchi, T.: Bilinear forms and extremal Kähler vector fields associated with Kähler classes. Math. Ann. 301, 199–210 (1995)
18. Guillemin, V.: Moment maps and combinatorial invariants of Hamiltonian $T^n$-spaces. Birkhauser (1994)
19. Guillemin, V.: Kähler structures on toric varieties. J. Differ. Geom. 40, 285–309 (1994)
20. Huang, H.N.: On the extension of the Calabi flow on Toric varieties. Ann. Glob. Anal. Geom. 40(1), 1–19 (2011)
21. Huang, H.N., Zheng, K.: Stability of Calabi flow near an extremal metric. Ann. Sc. Norm. Super. Pisa Cl. Sci. 11(1), 167–175 (2012)
22. Kleiner, B., Lott, J.: Notes on Perelman’s papers. Geom. Topol. 12, 2587–2855 (2008)
23. Morgan, J., Tian, G.: Ricci flow and the Poincaré Conjecture. Clay Mathematics Monographs, 3. American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge (2007)
24. Shi, W.X.: Ricci deformation of the metric on complete noncompact Riemannian manifolds. J. Differ. Geom. 30(2), 303–394 (1989)
25. Streets, J.: The gradient flow of $\int_M |Rm|^2$. J. Geom. Anal. 18(1), 249–271 (2008)
26. Streets, J.: The long time behavior of fourth-order curvature flows. Calc. Var. Partial Differ. Equ. 46(1–2), 39–54 (2013)
27. Székelyhidi, G.: Filtrations and test-configurations. arXiv:1111.4986
28. Székelyhidi, G.: Optimal test-configurations for toric varieties. J. Differ. Geom. 80, 501–523 (2008)
29. Tian, G.: Kähler–Einstein metrics of positive scalar curvature. Invent. Math. 130, 1–57 (1997)
30. Tosatti, V.: Kähler–Ricci flow on stable Fano manifolds. J. Reine Angew. Math. 640, 67–84 (2010)
31. Wang, X., Zhou, B.: On the existence and nonexistence of extremal metrics on toric Kähler surfaces. Adv. Math. 226, 4429–4455 (2011)
32. Yau, S.T.: Review of Kähler–Einstein metrics in algebraic geometry. Israel Math. Conf. Proc. Bar-Ilan Univ. 9, 433–443 (1996)
33. Zhou, B., Zhu, X.H.: $K$-stability on toric manifolds. Proc. Am. Math. Soc. 136(9), 3301–3307 (2008)