Deformation of quantum mechanics in fractional-dimensional space
A. Matos-Abiague
Belasitsa 35, Gorna Orjahovitsa 5100, Bulgaria

ABSTRACT

A new kind of deformed calculus (the D-deformed calculus) that takes place in fractional-dimensional spaces is presented. The D-deformed calculus is shown to be an appropriate tool for treating fractional-dimensional systems in a simple way and quite analogous to their corresponding one-dimensional partners. Two simple systems, the free particle and the harmonic oscillator in fractional-dimensional spaces are reconsidered into the framework of the D-deformed quantum mechanics. Confined states in a D-deformed quantum well are studied. D-deformed coherent states are also found.

Keywords: fractional-dimensional space, deformed calculus

PACS numbers: 03.65.-w, 03.65.Ca, 03.65.Fd

*Present address: Max-Planck Institut für Mikrostrukturphysik, Weinberg 2, 06120 Halle, Germany
Email: amatos@mpi-halle.de
Phone: ++49-345-5582-537
I. INTRODUCTION

Fractional-dimensional space approaches have been shown to be useful in the study of several physical systems. Theoretical schemes dealing with non-integer space dimensionalities have frequently been considered in the study of critical phenomena (see for instance [1], [2]) and of fractal structures [3] or in modelling semiconductor heterostructure systems [4] - [8].

In the above mentioned schemes, the fractional dimensionality is not referred to the real space, but to an auxiliary effective environment used to describe the real system. Nevertheless, the idea of a real space-time having a dimension slightly different from four has also been considered by several authors [9] - [12]. Actually, the deviations of the space-time dimension from four have been found to be very small [9] - [12]. However, the question of whether the dimension of the space-time is an integer or a fractional number constitutes a basic problem not only for its conceptual significance but also because the possibility that the space-time dimension is different from four may lead to interesting consequences (e.g. it is well known that a deviation of the space-time dimension from the value four eliminates the logarithmic divergences of quantum electrodynamics, independently of how small the deviation from four may be [13]).

Recently, it has been shown the existence of some similarities between the so-called N-body Calogero models and the problem corresponding to a fractional-dimensional harmonic oscillator of a single degree of freedom [14]. This result together with the remarkable fact that fractional-dimensional bosons can be considered as generalized parabosons [14] suggests new potential applications of the non-integer-dimensional space approaches.

It was shown in [14] that the fractional-dimensional Bose operators together with the reflection operator form an R-deformed Heisenberg algebra with a deformation parameter depending on the dimension of the space. Deformations of the Heisenberg algebra leading to the so-called q-deformed quantum mechanics have been extensively investigated (see for instance [15] - [18]). Taking into account the results obtained in [14], we develop in the present paper a new deformed calculus (the D-deformed calculus) in analogy to the q-deformed calculus commonly treated in the literature [18] - [21]. The new calculus allows us to express and to solve problems concerning fractional-dimensional systems in a simple way and quite analogous to the corresponding undeformed (D = 1) problems. The paper is organized as follows. In section 2 we introduce the D-deformed calculus. The problems corresponding to the free particle and to the harmonic oscillator in fractional-dimensional space were studied in [14]. We reconsider these problems in sections 3 and 4 respectively, but now from the point of view of the D-deformed quantum mechanics. Of course, the final results in these sections coincide with the results obtained in [14]. However, in terms of the new D-deformed calculus, the above mentioned problems can be solved immediately and in an elegant way. In section 5 the single particle confined states in a fractional-dimensional quantum well are studied. The dimensional dependence of the eigenenergies corresponding to the ground and to the first excited states is shown. In both cases an increase of the energies as the dimension increases is observed. The probability density function describing the motion of the particle confined in the D-deformed quantum well is also studied for varying the dimensionality. The D-deformed coherent states are found in section 6 and conclusions are summarized in section 7.

II. D-DEFORMED CALCULUS

It is well known that the one-dimensional momentum operator is given by

\[ P = \frac{1}{i} \frac{d}{d\xi}, \tag{1} \]

where we have taken \( \hbar = 1 \). However, in a fractional-dimensional space, because of the inclusion of
the integration weight \[22\]
\[
\frac{\sigma(D)}{2} |\xi|^{D-1}, \quad \sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)},
\] (2)

this operator is no longer Hermitian. Therefore a more general momentum operator has to be defined for systems in fractional-dimensional spaces. Starting with the Wigner commutation relations for the canonical variables of a Bose-like oscillator of a single degree of freedom, the fractional-dimensional momentum operator has been found to be [14],
\[
P = \frac{1}{i} \frac{d}{d\xi} + \frac{(D-1)}{2\xi} R - i \frac{(D-1)}{2\xi},
\] (3)

where \(R\) is the reflection operator.

The momentum operator (equation (3)) suggests a deformation of quantum mechanics in fractional-dimensional spaces. Indeed, we can introduce a new D-deformed derivative operator
\[
\frac{d_D}{d_D\xi} = \frac{d}{d\xi} + \frac{(D-1)}{2\xi}(1 - R)
\] (4)

and then the fractional-dimensional momentum operator (equation (3)) can be rewritten in the standard form
\[
P = \frac{1}{i} \frac{d_D}{d_D\xi}.
\] (5)

Thus the D-deformed annihilation and creation operators can be defined in the following way
\[
a_D = \frac{1}{\sqrt{2}} \left( \xi + \frac{d_D}{d_D\xi} \right), \quad a_D^+ = \frac{1}{\sqrt{2}} \left( \xi - \frac{d_D}{d_D\xi} \right).
\] (6)

The action of these operators is given as follows [14]
\[
a_D|0\rangle = 0,
\] (7)
\[
a_D|2n\rangle = \sqrt{2n}|2n-1\rangle, \quad a_D^+|2n\rangle = \sqrt{2n+D}|2n\rangle,
\] (8)
\[
a_D^+|2n\rangle = \sqrt{2n+D}|2n+1\rangle, \quad a_D^+|2n+1\rangle = \sqrt{2n+2}|2n+2\rangle,
\] (9)

where \(n = 0, 1, 2, 3, ...\)

By now introducing the corresponding D-factor (analogue to the q-factor) as
\[
[n]_D = n + \frac{(D-1)}{2} (1 - (-1)^n),
\] (10)

the equations (8) and (9) can be rewritten in the usual form
\[
a_D|n\rangle = \sqrt{[n]_D}|n-1\rangle,
\] (11)

\[
a_D^+|n\rangle = \frac{\sqrt{[n]_D}}{\sqrt{[n+1]_D}} [n]_D^{-1/2} |n+1\rangle,
\] (12)

\[
[a_D^+]^2 |n\rangle = [n]_D^{-1} |n+1\rangle,
\] (13)
and
\[ a_D^\dagger |n\rangle = \sqrt{[n+1]_D} |n+1\rangle, \quad (12) \]
respectively.

Taking into account equation (10) and in analogy to the q-deformed standard procedures we can define a D-deformed factorial function as follows
\[
[n]_D! = [n]_D [n-1]_D ... [1]_D [0]_D! = \begin{cases} 
  \frac{2^n \left( \frac{n}{2} \right)! (\frac{n}{2}+D)}{1! (D/2)!} & \text{for } n \text{ even} \\
  \frac{2^n \left( \frac{n-1}{2} \right)! (\frac{n+1}{2}+D)}{1! (D/2)!} & \text{for } n \text{ odd}
\end{cases}.
\quad (13)
\]
This D-deformed factorial function is a particular case of the generalized factorial function [23].

The eigenstates \(|n\rangle\) of the operator
\[ N_D |n\rangle = n |n\rangle; \quad N_D = \frac{1}{2} \{a_D^\dagger, a_D\} - D/2 \quad (14) \]
may be obtained by repeated applications of \(a_D^\dagger\) on the vacuum state \(|0\rangle\)
\[ |n\rangle = \left( a_D^\dagger \right)^n |0\rangle \sqrt{[n]_D!} \quad (15). \]

It is easy to prove that in this Fock space, the relations
\[ a_D^\dagger a_D = [n]_D; \quad a_D a_D^\dagger = [n+1]_D \quad (16) \]
take place.

From the definition of the D-deformed derivative (equation (4)) we can introduce a D-deformed integration, so that if
\[ \frac{dDf(\xi)}{dD\xi} = F(\xi) \quad (17) \]
then
\[ f(\xi) = \int F(\xi) dD\xi + \text{const.} \quad (18) \]

With the aim to find the appropriate expression for the D-deformed integration, we observe that
\[ \frac{dDf(\xi)}{dD\xi} = \left[ 1 + \frac{(D-1)}{2}\xi (1 - R) \int d\xi \right] \frac{df(\xi)}{d\xi} = F(\xi) \quad (19) \]
and hence
\[ \frac{df(\xi)}{d\xi} = \left[ 1 + \frac{(D-1)}{2}\xi (1 - R) \int d\xi \right]^{-1} F(\xi) \quad (20) \]
From the equation above it follows
\[
 f(\xi) = \int F(\xi) d_{D}\xi = \sum_{n=0}^{\infty} \left[ -\int d\xi \left( \frac{(D-1)}{2\xi} (1-R) \right) \right]^{n} \int d\xi F(\xi) .
\] (21)

This expression may be rewritten as
\[
 \int F(\xi) d_{D}\xi = \sum_{n=0}^{\infty} (-1)^{n} I_{n} ,
\] (22)

where the terms \( I_{n} \) satisfy the following recurrence formula
\[
 I_{n+1} = \int \left( \frac{D-1}{2\xi} (1-R) \right) I_{n} d\xi ; \quad I_{0} = \int F(\xi) d\xi .
\] (23)

With respect to the D-deformed calculus induced by the fractional-dimensional integration weight (equation (2)), the following identities can be easily demonstrated
\[
 \frac{d_{D}[f(\xi)g(\xi)]}{d_{D}\xi} = g(\xi) \frac{d_{D}f(\xi)}{d_{D}\xi} + \frac{d_{D}g(\xi)}{d_{D}\xi} Rf(\xi) + \frac{dg(\xi)}{d\xi} (1-R) f(\xi)
\] (24)

and after integrating the equation above
\[
 \int g(\xi) \frac{d_{D}f(\xi)}{d_{D}\xi} d_{D}\xi = f(\xi)g(\xi) - \int \frac{d_{D}g(\xi)}{d_{D}\xi} Rf(\xi) d_{D}\xi - \int \frac{dg(\xi)}{d\xi} (1-R) f(\xi) d_{D}\xi .
\] (25)

One should notice that if either \( f(\xi) \) or \( g(\xi) \) is an even function of \( \xi \), equations (24) and (25) reduce to a D-deformed Leibnitz rule and to a D-deformed formula of integration by parts, respectively. This is a consequence of the fact that the D-deformed derivative acts on even functions as the ordinary derivative.

III. D-DEFORMED FREE PARTICLE

The eigenstates of the momentum operator corresponding to a free particle of a single degree of freedom in a fractional-dimensional space can be found now in terms of the D-deformed calculus introduced in the previous section. Thus, the eigenstates of the fractional-dimensional momentum operator are determined by the following equation
\[
 P\Psi_{p} = -i \frac{d_{D}\Psi_{p}}{d_{D}\xi} = p\Psi_{p} .
\] (26)

The corresponding eigenfunctions are immediately found to be
\[
 \Psi_{p} = A_{p} E_{D}(ip\xi) ,
\] (27)

where \( E_{D}(x) \) represents the D-deformed exponential function (see Appendix A). The normalization factor \( A_{p} \) can be found from the orthonormalization condition
\[ \langle \Psi_p | \Psi_{p'} \rangle = \frac{\sigma(D)}{2} \lim_{\gamma \to 0} \int_{-\infty}^{\infty} e^{-\gamma \xi^2} \Psi_p^*(\xi) \Psi_p(\xi) |\xi|^{D-1} d\xi = \delta(p-p') \quad (\gamma > 0), \quad (28) \]

in a similar way as in [14]. After the corresponding calculations we arrive to the following expression

\[ A_p = \frac{1}{2^{D/2-1} \Gamma(D/2)} \sqrt{\frac{p^{D-1}}{2\sigma(D)}}. \quad (29) \]

One should notice that the eigenfunctions \( \Psi_p \) describing the motion of a free particle of a single degree of freedom in a fractional-dimensional space can be considered as D-deformed plane waves and they reduce to the ordinary plane de Broglie waves when \( D = 1 \).

### IV. D-DEFORMED HARMONIC OSCILLATOR

The eigenfunctions in coordinate representation corresponding to the D-deformed harmonic oscillator can be derived from equation (15) without much difficulty. First we consider the vacuum state \( |0\rangle \) which satisfies equation (7). Then, using the expression of \( a_D \) in coordinate representation (equation (3)) we have the following D-deformed differential equation

\[ \left( \frac{d}{d\xi} + \xi \right) \chi_0 = 0 \quad , \quad (30) \]

where \( \chi_0 = \langle \xi |0\rangle \) represents the eigenfunction of the ground state of the D-deformed harmonic oscillator. By now solving equation (30) we found that

\[ \chi_0 = C_0 \exp[-\xi^2/2] \quad , \quad (31) \]

where \( C_0 \) is a normalization factor. From the normalization condition

\[ \frac{\sigma(D)}{2} \int_{-\infty}^{\infty} |\chi_0|^2 |\xi|^{D-1} d\xi = 1 \quad , \quad (32) \]

the normalization constant is found to be

\[ C_0 = \frac{1}{\pi^{D/4}}. \quad (33) \]

Once the ground state has been found, the excited states may be calculated from equation (15). Thus the excited states are determined by

\[ \chi_n = \langle \xi |n\rangle = \frac{C_0}{\sqrt{|n|}!} \left[ \frac{1}{\sqrt{2}} \left( \xi - \frac{d}{d\xi} \right) \right]^n \exp[-\xi^2/2] \quad . \quad (34) \]

If we now take into account that

\[ (-1)^n \exp[\xi^2/2] \left( \frac{d}{d\xi} \right)^n \exp[-\xi^2] = \left( \xi - \frac{d}{d\xi} \right)^n \exp[-\xi^2/2] \quad , \quad (35) \]
a relation that can be demonstrated by induction, the excited states in coordinate representation can be written as

\[ \chi_n = \langle \xi | n \rangle = \frac{[n]_D! \exp[-\xi^2/2]}{n! \sqrt{\pi D/2}^n [n]_D!} H_n^D(\xi) \]  

(36)

where

\[ H_n^D(\xi) = \frac{n!}{[n]_D!} (-1)^n \exp[\xi^2] \left( \frac{d_D}{d_D \xi} \right)^n \exp[-\xi^2] \]  

(37)

can be understood as D-deformed Hermite polynomials. In fact the polynomials \( H_n^D(\xi) \) above defined are a particular case of the generalized Hermite polynomials studied in [23].

It is worth remarking that if \( D = 1 \) the results obtained in the present section reduce to the well known results corresponding to the undeformed one-dimensional case.

V. CONFINED STATES IN A D-DEFORMED QUANTUM WELL

In the present section we will study the motion of a particle confined in a fractional-dimensional quantum well defined by the following potential

\[ V(\xi) = \begin{cases} 0 & \text{if } |\xi| < 1/2 \\ \infty & \text{otherwise} \end{cases} \]  

(38)

where we have taken an unitary well width \((L = 1)\). The corresponding Schrödinger equation may be written as

\[ \left[ -\frac{1}{2} \frac{d_D^2}{d_D \xi^2} + V(\xi) \right] \Psi_n(\xi) = E_n \Psi_n(\xi) \]  

(39)

In terms of the introduced D-deformed calculus, the equation above can be immediately solved. The wavefunctions are given by

\[ \Psi_n(\xi) = \begin{cases} A_n^{even} \text{COS}_D k_n \xi & \text{for } n \text{ even} \\ A_n^{odd} \text{SIN}_D k_n \xi & \text{for } n \text{ odd} \end{cases} \]  

(40)

where

\[ k_n = \sqrt{2E_n} \]  

(41)

and \( \text{COS}_D X, \text{SIN}_D X \) represent the D-deformed cosine and sine functions respectively (see Appendix). The constants \( A_n^{even}, A_n^{odd} \) are normalization factors corresponding to even and odd states respectively.

The eigenenergies can be now easily computed from the boundary conditions

\[ \text{COS}_D \frac{k_n}{2} = 0 \]  

(42)
for even states and
\[ \sin_D \frac{k_n}{2} = 0, \]  \hspace{1cm} (43)

for odd states.

One should notice that the D-deformed calculus allow us to express the fractional-dimensional problems in a very simple way and quite analogous to the corresponding undeformed (D = 1) problems.

The D-dependence of the energies corresponding to the ground state (n = 0) and to the first excited state (n = 1) is shown in figure 1. The eigenenergies increase as the dimension increases. This behaviour has also been observed in other fractional-dimensional systems (see for instance [4], [14]).

In figure 2 we present the probability density
\[ \rho_n(\xi) = \frac{\sigma(D)}{2} |\xi|^{D-1} |\Psi_n|^2, \]  \hspace{1cm} (44)
corresponding to n = 0 (a) and to n = 1 (b) as a function of the pseudocoordinate \( \xi \) for different values of the dimensionality. In both cases one can appreciate the existence of compression (spreading) of the probability density when \( D < 1 \) (\( D > 1 \)). We remark that this behaviour is not a consequence of the form of the wavefunctions but of the presence of the integration weight in the probability density. Indeed, the integration weight acts as an attractive (repulsive) barrier when \( D < 1 \) (\( D > 1 \)). It diverges at the origin of pseudocoordinates when \( D < 1 \) (i.e. when \( D < 1 \) all the volume of the space is almost concentrated around \( \xi = 0 \) ) and causes a strong localization of the probability density (see figure 2(a)). In the case of the first excited state, however, because of the odd parity the probability density becomes zero at the origin of pseudocoordinate and there is no longer localization in the central region.

It is worth noting that in the present case when \( D > 1 \), the maximum of the probability density increases as the dimension increases (see figure 2), contrary to the behaviour observed in the fractional-dimensional harmonic oscillator (see figures 2(b) and 3 of [14]). This is because as the dimensionality increases, the integration weight becomes more and more repulsive favouring the tunnelling through the harmonic barriers. Consequently, the particle becomes more delocalized and the maximum of the probability density decreases. In the present case, however, the quantum well is considered infinitely deep and the tunnelling is suppressed. The particle is then compressed between the well barriers and the repulsive integration weight leading to an increase in the maximum of the probability density.

**VI. D-DEFORMED COHERENT STATES**

We now observe the spectrum problem corresponding to a fractional-dimensional annihilation operator \( a_D \) by using the rules of the D-deformed calculus. The eigenstates of \( a_D \):
\[ a_D |\alpha\rangle = |\alpha\rangle \]  \hspace{1cm} (45)

are a D-deformation of the usual coherent states. The solution of equation \( (45) \) is given by

\[ E_n = -\frac{4}{(2n + D - 2)^2}, \]  \hspace{1cm} (46)

The energies of the fractional-dimensional harmonic oscillator and the hydrogenic atom are given by \( E_n = n + D/2 \) and \( E_n = \frac{4}{(2n + D - 2)^2} \), respectively. In both cases the energy increases when the dimension increases.
\[ |\alpha\rangle = A_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{|n|_D!}} |n\rangle . \] (46)

From the normalization condition \( \langle \alpha | \alpha \rangle = 1 \) the normalization constant is found to be

\[ A_\alpha = \frac{1}{\sqrt{E_D(|\alpha|^2)}} . \] (47)

As usually, the equation (46) can be written in term of the vacuum state as follows

\[ |\alpha\rangle = \frac{1}{\sqrt{E_D(|\alpha|^2)}} E_D(\alpha a_D^\dagger) |0\rangle . \] (48)

Once we have found the expression of the D-deformed coherent states in the Fock representation, we can easily obtain its expression in coordinate representation by using the relation

\[ \Phi_\alpha(\xi) = \langle \xi | \alpha \rangle = \sum_{n=0}^{\infty} \langle \xi | n \rangle \langle n | \alpha \rangle . \] (49)

By now considering equations (49) and (46) we arrive to the following result

\[ \Phi_\alpha(\xi) = \exp[-\xi^2/2] \frac{1}{\sqrt{E_D(|\alpha|^2) \pi^{D/2}}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{2^n n!}} H_n^D(\xi) . \] (50)

From equation (46), the probability distribution of a D-deformed coherent state in Fock representation is found to be

\[ |\langle n | \alpha \rangle|^2 = \frac{1}{E_D(|\alpha|^2)} \frac{(|\alpha|^2)^n}{|n|_D!} . \] (51)

i. e. a D-deformation of the Poisson distribution.

**VII. CONCLUSIONS**

Summing up, taking into account recent developments in the mathematical physics of the fractional-dimensional space and in analogy to the q-deformed calculus we have developed a new deformed calculus that we have called D-deformed calculus. Some simple fractional-dimensional systems, the free particle, the harmonic oscillator, and the particle confined in a quantum well have been studied into the framework of the D-deformed quantum mechanics. Finally, the D-deformed coherent states are found.

**APPENDIX:**

Here we will study the properties of some D-deformed functions. From the definition of the D-deformed derivative (equation (4)) it is easy to find that
\[
\frac{d_D \xi^n}{d_D \xi} = [n]_D \xi^{n-1}. \tag{A1}
\]

On the other hand, from the definition of the D-deformed integration (equation (22)), we also found

\[
\int \xi^n d_D \xi = \frac{\xi^{n+1}}{[n+1]_D} + \text{const.} \tag{A2}
\]

In this way, one can introduce the D-deformed exponential function as follows

\[
E_D(\xi) = \sum_{n=0}^{\infty} \frac{\xi^n}{[n]_D}! . \tag{A3}
\]

From equation (A3) and making use of the equations (A1) and (A2) one can straightforwardly demonstrate that

\[
\frac{d_D E_D(\lambda \xi)}{d_D \xi} = \lambda E_D(\lambda \xi) ; \quad \lambda = \text{const.} \tag{A4}
\]

and consequently

\[
\int E_D(\lambda \xi) d_D \xi = \frac{E_D(\lambda \xi)}{\lambda} + \text{const.} \tag{A5}
\]

Actually, the D-deformed exponential function is a particular case of the generalized exponential function defined in [23] and can be represented as follows

\[
E_D(\xi) = \exp[\xi \Phi(D-1/2, D, -2\xi)] , \tag{A6}
\]

or

\[
E_D(\xi) = \Gamma(D/2) \left( \frac{\xi}{2} \right)^{1-D/2} \left[ I_{D/2-1}(\xi) + I_{D/2}(\xi) \right] , \tag{A7}
\]

where \( \Phi(a, b, x) \) and \( I_\nu(x) \) are the confluent hypergeometric function and the modified Bessel function respectively.

We can also introduce D-deformed cosine and sine functions through the following definitions

\[
COS_D \xi = \sum_{n=0}^{\infty} \frac{(-1)^n \xi^{2n}}{[2n]_D!} = \Gamma(D/2) \left( \frac{\xi}{2} \right)^{1-D/2} J_{D/2-1}(\xi) , \tag{A8}
\]

and

\[
SIN_D \xi = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \xi^{2n-1}}{[2n-1]_D!} = \Gamma(D/2) \left( \frac{\xi}{2} \right)^{1-D/2} J_{D/2}(\xi) , \tag{A9}
\]

where \( J_\nu(x) \) represents the Bessel function. Thus, the following identities can be easily verified
$$E_D(\pm i\xi) = \cos D\xi \pm i \sin D\xi, \quad (A10)$$

and

$$\frac{d_D\cos D\xi}{d_D\xi} = -\sin D\xi; \quad \frac{d_D\sin D\xi}{d_D\xi} = \cos D\xi \quad (A11)$$

It is straightforwardly to check that all the definitions and equations given in this appendix recover the corresponding undeformed expressions when $D = 1$, as it must.

References

1. Ma S 1973 Rev. Mod. Phys. 45 589
2. Fisher M E 1974 Rev. Mod. Phys. 46 597
3. Mandelbrot B B 1989 The Fractal Geometry of Nature (San Francisco: Freeman)
4. He X F 1991 Phys. Rev. B 43 2063
5. Mathieu H, Lefebvre P and Christol P 1992 Phys. Rev. B 46 4092
6. Christol P, Lefebvre P and Mathieu H 1993 J. Appl. Phys. 74 5626
7. Matos-Abiague A, Oliveira L E and de Dios-Leyva M 1998 Phys. Rev. B 58 4072
8. Reyes-Gómez E, Matos-Abiague A, Perdomo-Leiva C A, de Dios-Leyva M and Oliveira L E 2000 Phys. Rev. B 61 13104
9. Zeilinger A and Svozil K 1985 Phys. Rev. Lett. 54 2553
10. Jarlskog C and Ynduráin F J 1986 Europhys. Lett. 1 51
11. Schäfer A and Müller B 1986 J. Phys. A: Math. Gen. 19 3981
12. Torres J L and Ferreira Herrejón P 1989, Rev. Mex. Fís. 35 97
13. Weisskopf V F 1939 Phys. Rev. 56 72
14. Matos-Abiague A 2001 J. Phys. A: Math. Gen. 34 3125
15. Macfarlane A J 1989 J. Phys. A: Math. Gen. 22 4581
16. Sun C P and Fu H C 1989 J. Phys. A: Math. Gen. 22 L983
17. Chang Z, Chen W, Guo H Y and Yan H 1990 J. Phys. A 23 5371
18. Biedenharn L C 1989 J. Phys. A: Math. Gen. 22 L873
19. Li Y and Sheng Z 1992 J. Phys. A: Math. Gen. 25 6779
20. Chung W, Chung K, Nam S and Um C 1993 Phys. Lett. 183A 363
21. Chung K and Chung W 1994 J. Phys. A: Math. Gen. 27 5037
22. Stillinger F H 1977 J. Mat. Phys. 18 1224
23. Rosenblum M 1994 Operator Theory: Advances and Applications vol. 73 (Basel: Birkhäuser) pp 369-396
**Figure captions**

Figure 1. The eigenenergies corresponding to the ground \((n = 1)\) and to the first exited \((n = 2)\) states of a particle confined in a D-deformed quantum well as a function of the dimensionality.

Figure 2. Position dependence of the probability density \(\rho_n\) corresponding to a particle confined in a D-deformed quantum well and for different values of the dimensional parameter. (a) For the ground state \((n = 1)\) and (b) for the first exited state \((n = 2)\).