Quantum theory of computation and relativistic physics

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In the e-print is discussed a few steps to introducing of "vocabulary" of relativistic physics in quantum theory of information and computation (QTI&C). The behavior of a few simple quantum systems those are used as models in QTI&C is tested by usual relativistic tools (transformation properties of wave vectors, etc.). Massless and charged massive particles with spin 1/2 are considered. Field theory is also discussed briefly.

Abstract

In the paper are described some steps for merger between relativistic quantum theory and theory of computation. The first step is consideration of transformation of qubit state due to rotation of coordinate system. The Lorentz transformation is considered after that. The some new properties of this transformation change usual model of qubit. The system of qubit seems more fundamental relativistic model. It is shown also that such model as electron is really such qubit system and for modelling of qubit is necessary to use massless particle like electron neutrino.

The quantum field theory (QFT) is briefly discussed further. The wave vectors of interacted particle now described by some operator and it can produce some multiparticle ('nonlinear') effects.

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1 Introduction

The paper describes some approaches to relativistic quantum theory of computation. The main purpose of the work is to consider essentially new properties of quantum computers due to relativistic phenomena rather than some small corrections to nonrelativistic formulae.

At first, in relativistic theory it is necessary to consider a qubit in different coordinate systems. In simplest case it may be 3D local rotations and SU(2) spinors.

For consideration of temporal coordinate it is necessary to use Lorentz transformations and 4D spinors. The more correct approach include full Poincare group and quantum field theory.

2 Qubit

A quantum two-state system is often called quantum bit or 'qubit' due to relativistic phenomena rather than some small corrections to nonrelativistic formulae.

At first, in relativistic theory it is necessary to consider a qubit in different coordinate systems. In simplest case it may be 3D local rotations and SU(2) spinors.

For consideration of temporal coordinate it is necessary to use Lorentz transformations and 4D spinors. The more correct approach include full Poincare group and quantum field theory.

The equations Eq. (3) can be used for demonstration of relation between SU(2) and SU(2). If we apply some
unitary transformation Eq. (4) $U : (c_0, c_1) \rightarrow (c'_0, c'_1)$ then $(X, Y, Z) \rightarrow (X', Y', Z')$. Unitary matrices do not change the norm Eq. (5) and length of the vector:

$$X^2 + Y^2 + Z^2 = (|c_0|^2 + |c_1|^2)^2$$

Angles between vectors also do not change. Unitary transformations of a state of the qubit correspond to rotations of the sphere (Fig. 1). Two matrices: $U$ and $-U$ produce the same rotation due to Eq. (6):

$$\langle \psi | U | \psi \rangle = \langle |U| |U| \rangle = \langle |U| |U| \rangle^*$$

The transformations of a state of $n$-qubits due to spatial rotation can be described by \textit{unitary} $2^{2n}$ matrices.

3 The relativistic consideration of a qubit

3.1 Lorentz transformation

For Lorentz transformation of coordinate system there is similar isomorphism between the group $SO(3,1)$ and the group $SL(2, \mathbb{C})$ of all complex $2 \times 2$ matrices with determinant unity. The group $SL(2, \mathbb{C})$ is isomorphic with Lorentz group in the same way as the group $SU(2)$ with group of 3D rotations \cite{3}. The group $SL(2, \mathbb{C})$ is a representation of Lorentz group $SO(3,1)$ in a space of 2D complex vectors.

On the other hand, we should not directly apply such representation of relativistic group $SL(2, \mathbb{C})$ to a qubit. Only the subgroup of unitary matrix saves the norm Eq. (6). The expression Eq. (3) in relativistic theory is not invariant scalar, but temporal part of 4–vector. Simple relation between transformations of coordinate system and unitary matrices is broken here.

Let us denote:

$$T = ||\psi||^2 \equiv \psi^* \psi = c_0 \tau_0 + c_1 \tau_1$$

We can write\footnote{In the matrix notation $\psi^* \psi$ is scalar and $\psi \psi^*$ is $2 \times 2$ matrix (with Dirac notation: $\langle \psi | \phi \rangle$ and $|\psi \rangle \langle \phi|$ respectively).}, using equations Eq. (3), Eq. (6):

$$V \equiv \begin{pmatrix} T + Z & X - iY \\ X + iY & T - Z \end{pmatrix} = 2 \begin{pmatrix} c_0 \tau_0 & c_0 \tau_1 \\ c_1 \tau_0 & c_1 \tau_1 \end{pmatrix}$$

$$\frac{1}{2} V = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \begin{pmatrix} \tau_0 & \tau_1 \end{pmatrix} = \psi \psi^*$$

$$\text{det } V = T^2 - X^2 - Y^2 - Z^2 =$$

$$= 2c_0 \tau_0^2 - 2c_1 \tau_0 \tau_1 - 2c_1 \tau_0 \tau_1 - 2c_0 \tau_0 \tau_1 = 0$$

Linear transformations with determinant unity of a qubit correspond to Lorentz transformation of the vector $(T, X, Y, Z)$:

$$\psi' = A \psi; \quad \text{det } A = 1$$

$$V' = 2A \psi \psi^* A^* = AVA^*$$

$$\text{det } V' = T'^2 - X'^2 - Y'^2 - Z'^2 =$$

$$= \text{det } V = T^2 - X^2 - Y^2 - Z^2$$

Figure 2: Null vector $(T, X, Y, Z)$

Only if the matrix $A$ is unitary, $AVA^* = AVA^{-1}$ and $\text{Trace } V$ i.e. the norm Eq. (6) does not change. Otherwise Eq. (6) should be considered as the ‘$T$–component’ of a 4–vector.

The relation between $SL(2, \mathbb{C})$ and Lorentz group Eq. (3) is valid not only for null vectors. Any vector is a sum of two null vectors and

$$A(V + U)A^* = AVA^* + AUA^*.$$ 

The qubit is described by two-component complex vector or \textit{Weyl spinor}. It corresponds to massless particle with spin $1/2$. Such particle always moves with the speed of light. The equations Eq. (7) show a correspondence between such spinor and 4D null vector (Fig. 2). This
vector can be also rewritten by using Pauli matrices:
\[
\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]
\[
V = T_1 + X\sigma_x + Y\sigma_y + Z\sigma_z,
\]
\[
V_i = \frac{1}{2} Tr(\sigma_i V) = Tr(\sigma_i \psi \psi^*) = \psi^* \sigma_i \psi; \quad \sigma = \{\sigma_x, \sigma_y, \sigma_z\} = \{T, \{X, Y, Z\}\} = (\psi^* \psi, \psi \sigma \psi)
\]  

### 3.2 Massive particle

Massive charged particle with spin 1/2 like an electron is described by two Weyl spinors and has four complex components:

\[
\psi = \begin{pmatrix} \varphi_R \\ \varphi_L \end{pmatrix}, \quad \varphi_R, \varphi_L \in \mathbb{C}^2; \quad \psi = \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}
\]  

It is possible to consider such massive particle as two qubits:

\[
\psi = c_{00}|00\rangle + c_{01}|01\rangle + c_{10}|10\rangle + c_{11}|11\rangle
\]  

The first index is similar to $|\uparrow\rangle$ and $|\downarrow\rangle$ for each $\varphi_R, \varphi_L$. The other one corresponds to discrete coordinate transformation like spatial reflection: $P: (t, \vec{x}) \rightarrow (t, -\vec{x})$.

It is also possible to build a vector by using the 4D spinor and $4 \times 4$ Dirac matrices $\gamma^\mu$. It is 4D vector of current Fig. 3:

\[
j^\mu = \psi^* \gamma^0 \gamma^\mu \psi
\]

\[
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & -\sigma \\ \sigma & 0 \end{pmatrix}
\]

with always positive:

\[
j^0 = \psi^* \psi = \sum_i |\psi_i|^2 \leq ||\varphi_R||^2 + ||\varphi_L||^2
\]  

but $j^0$ is not Lorentz invariant. The Lorentz invariant scalar is

\[
\psi^* \gamma^0 \psi = \varphi_R^* \varphi_L + \varphi_L^* \varphi_R
\]  

### 3.3 Representations of Lorentz group

We have used very simple construction of a qubit, but any other constructions also have limitations because a representation of Lorentz group cannot satisfy contemporary two following conditions:

- The representation is finite dimensional.
- The representation is unitary in a definite norm.

It can be considered as some mathematical reasons for:

- Using of quantum field theory (QFT) instead of systems with finite number of states.
- Necessity of a consideration of different kinds of interacting quantum fields.

The relativistic physics have both these properties. We can consider Quantum Electrodynamics (QED) as an example.

It is not quite compatible with such properties of usual model of quantum computation as fixed size of registers and gates, one kind of qubits, etc..

### 4 Quantum field theory and computations

In articles about quantum computers Feynman has used one of usual tools of a QFT — annihilation and creation operators. The “no-go” result for bounded quantum networks.
creation operators $a$ and $a^*$:

$$a = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$
with Fermi relation for the anticommutator:

$$\{a^*, a\}_{+} ≡ a^* a + a a^* = 1$$

These operators are used for describing usual quantum gate in Ref.

### 4.1 Secondary quantization

In a QFT wave functions are operators \[\hat{\psi}\]. Let us consider photons as an example:

$$\hat{\psi}_p = c_p e^{-ipx} + c^*_p e^{ipx}$$

There $c_p$ and $c^*_p$ are operators of annihilation and creation of the particle with 4-momentum $p$ and so $\hat{\psi}$ is an operator. There is Bose relation for the commutator:

$$[c^*, c]_- ≡ c^* c - c c^* = 1$$

### 4.2 States and operators

The operators $c_p$ and $c^*_p$ act in some auxiliary Hilbert space and functions like Eq. (18) have more direct physical meaning than states in this space. The quantum field of electrons is described by some expression similar to Eq. (15).

The matrices Eq. (16) are used for presentation of quantum gates in Ref., but it should be mentioned that in relativistic physics there is no sharp division between q-gates and qubits due to formulae like Eq. (18).

This property of a QFT has some analogy with functional style of programming in modern computer science Ref. In both cases there is no essential difference between data (states) and functions (operators). A function can be used as data for some other function.

For example, let us consider an electron as the model of a qubit. In nonrelativistic quantum theory of computation a q-gate can change state of the qubits $\psi' = U \psi$ (Fig. 4). Here $\psi, \psi'$ are wave vectors of quantum system (‘qubits’) and $U$ is an operator of the gate.

Figure 4: Nonrelativistic gate

The gate can be built as some electro-magnetic device. From point of view of QED it is described as an interaction of two quantum fields and we should not split the processes on q-gates and qubits. The usual formula of secondary quantization is $\Psi' = U_{\hat{\psi}, \hat{A}} \Psi$ (Fig. 5). Here $\Psi, \Psi'$ describe occupation numbers, and $\hat{\psi}$ is wave operator for electron (positron), and $\hat{A}$ for photons. The wave operators for particles are included in $U$ and can form many nonlinear expressions. They correspond to Feynman diagrams. Such description is linear in respect of $\Psi, \Psi'$, but not on $\hat{\psi}, \hat{A}$.

Figure 5: Relativistic gate

### 4.3 Algebraic and matrix notation

The relations Eq. (17) and Eq. (19) describe one particle. If we have a few particles then the full set of relations is:

$$\{a_k, a_{k'}\}_+ = \{a^*_k, a^*_{k'}\}_+ = 0 \quad \{a_k, a^*_{k'}\}_+ = \delta_{kk'}$$

(20)

for particles like electrons (Fermi statistic, half-integer spin) and

$$[c_k, c_{k'}]_- = [c^*_k, c^*_{k'}]_- = 0 \quad [c_k, c^*_{k'}]_- = \delta_{kk'}$$

(21)

for particles like photons (Bose statistic, integer spin).
The equations Eq. (16), Eq. (17) show representation of operators with Fermi relations for one particle. The matrix representations of Eq. (20) for many particles are more complicated.

The relations for Bose particles Eq. (19), Eq. (21) are impossible to express by using finite-dimensional matrices because for any two matrices $A, B$:

$$\text{Trace}(AB - BA) = 0 \implies [A, B] \neq 1$$  \hspace{1cm} (22)

Due to such properties of algebras of commutators the presentation by using formal expressions with operators of annihilation and creation $\hat{a}$ instead of matrices can be more convenient in quantum theory of computation from the point of view of relativistic physics.

## 5 Conclusion

In nonrelativistic quantum theory of computation it was necessary only to point number of states $2^n$ for description of $q^n$bit. In relativistic theory there are many special cases. The charged and neutral, massive and massless particles etc. should be described differently.

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