Thermodynamics of the $d = 3 + 1$ quantum XY model

Christoph P Hofmann

Facultad de Ciencias, Universidad de Colima, Bernal Díaz del Castillo 340, Colima C.P. 28045, Mexico
E-mail: christoph.peter.hofmann@gmail.com

Received 10 March 2016, revised 18 July 2016
Accepted for publication 10 August 2016
Published 14 September 2016

Abstract. Within effective field theory we explore the properties of the $d = 3 + 1$ quantum XY model at low temperatures and in weak magnetic or staggered fields. For this parameter regime only few results appear to be known, and furthermore are restricted to one-loop order. In the present study we systematically analyze the thermodynamics of the $d = 3 + 1$ quantum XY model up to three-loop order. In the low-temperature expansion of the free energy density, the free Bose gas term of order $T^4$ receives corrections of order $T^6$ and $T^8$. The discussion also includes the pressure, (staggered) magnetization and susceptibility. In particular, we show how these quantities are influenced by the spin-wave interaction.

Keywords: rigorous results in statistical mechanics, sigma models
1. Introduction

Whereas the finite-temperature properties of the \( d = 2 + 1 \) quantum XY model have received a lot of attention over the past few decades—presumably due to the occurrence of the Kosterlitz–Thouless phase transition—the same cannot be said about the quantum XY model in \( d = 3 + 1 \). Apart from some pioneering articles, no systematic studies of its thermodynamic properties at low temperatures, where the spin-wave picture applies, seem to be available. The present work closes this gap that apparently exists in the condensed matter literature.

As is well-known, the relevant low-energy excitations of the \( d = 3 + 1 \) quantum XY model are the spin waves. They emerge as a consequence of the spontaneously broken internal symmetry \( O(2) \to \mathbb{Z}_2 \). The low-energy behavior of the system can thus be captured by the physics of its Goldstone bosons. This is the effective-field theory point of view we pursue in the present study. In contrast to spin-wave theory or other microscopic approaches, the effective Lagrangian method allows one to study the low-temperature behavior of the \( d = 3 + 1 \) quantum XY model systematically and straightforwardly up to three loops, as we demonstrate below.

The properties of the \( d = 3 + 1 \) quantum XY model at zero temperature or near the critical temperature, have been addressed with spin-wave theory, high-temperature expansions, Monte Carlo simulations, and yet other methods [1–28]. However, regarding
its behavior at low temperatures—between the two extremes $T = 0$ and $T = T_c$—only little appears to be known. One result concerns the finite-temperature susceptibility in the presence of an external field $\mathcal{H}$, that diverges as $T:\sqrt{\mathcal{H}}$ in weak fields [12].

It should be stressed that the results presented in [12]—in effective field theory language—merely correspond to one-loop effects. Above all, the [1–27] do not address the manifestation of the spin-wave interaction in the thermodynamic behavior of the system. It is the objective of the present work to go up to three-loop order in the effective expansion and to systematically explore the impact of the spin-wave interaction at low temperatures and weak external fields.

Our study thus extends the knowledge on the low-temperature behavior of the $d = 3 + 1$ quantum $XY$ model substantially. It should be pointed out that in the context of Heisenberg ferromagnets in three spatial dimensions, more than a hundred publications on the manifestation of the spin-wave interaction in the spontaneous magnetization have appeared (see, e.g. [29–36] and references therein). We hence believe that we are dealing with an interesting and important question.

While the leading contribution in the free energy density of the $d = 3 + 1$ quantum $XY$ model is of order $T^4$, we find that subleading corrections are of order $T^6$ (two loops) and $T^8$ (three loops). The coefficient of the $T^6$-term can easily be expressed through microscopic quantities, since it only involves the leading-order effective constants $F$—where $F^2$ is the spin-wave stiffness—and $\Sigma$, i.e. the order parameter at zero temperature and infinite volume. At the three-loop level, the situation is more complicated, as the coefficients of the various $T^8$-terms involve next-to-leading order effective constants that are a priori unknown. Still, we can estimate their order of magnitude, which leads us to conclude that the three-loop corrections are small.

As it turns out, the spin-wave interaction is repulsive in the pressure at low temperatures and weak magnetic or staggered fields. Remarkably, regarding the order parameter and the susceptibility, the impact of the spin-wave interaction is rather counterintuitive: if the external field is weak, the temperature-dependent interaction contribution in the order parameter is positive, while in the susceptibility it is negative. It should be stressed that the present effective field theory analysis is completely systematic and goes up to three-loop order. Trying to reach the same level of accuracy with traditional microscopic methods such as spin-wave theory would be formidable.

Our results are valid at low temperatures and in weak fields, i.e. in a regime where both parameters $T$ and $|\mathcal{H}|$ are small with respect to the exchange integral $J$ that defines the natural scale of the system. Note that, on bipartite lattices, the field $\mathcal{H}$ can be interpreted as magnetic field coupled to the ferromagnetic quantum $XY$ model, or equivalently, as staggered field in connection with the antiferromagnetic $XY$ model. We should mention, however, that our analysis goes beyond the description of quantum spin models. Our results apply to any $d = 3 + 1$ (pseudo-)Lorentz-invariant system characterized by a spontaneously broken internal symmetry $\mathbb{O}_2$ [2].

Effective field theories based on Goldstone bosons are widely used and well established in particle physics. However, in condensed matter physics, systematic effective

\footnote{Although there is a residual discrete symmetry in the $XY$ model, $O(2) \rightarrow \mathbb{Z}_2$, in the effective field theory description $\mathbb{Z}_2$ is irrelevant. Therefore our effective analysis also applies to the case where the ground state does no longer exhibit any discrete symmetries: $O(2) \rightarrow 1$.}
Lagrangian techniques do not have the same status. But it is a fact that the low-energy behavior of many condensed matter systems is determined by Goldstone bosons—and the effective Lagrangian method is just designed for these systems [37, 38]. I would like to point out to the condensed matter community that in the past, various systems have successfully been analyzed within effective Lagrangian field theory. In particular, systems where the relevant Goldstone excitations are spin waves. These comprise ferromagnetic spin chains [39–41], ferromagnetic films [42–44], and ferromagnets in three spatial dimensions [32–35, 45–47]. They also include antiferromagnets and $XY$ models in two [48–52] and three [53–55] spatial dimensions. Apart from these systems whose physics is governed by Goldstone bosons, there are situations where additional excitations come into play: for instance high-temperature superconductors that also involve doped holes or electrons. Systematic effective field theories have also been constructed and applied to these remarkable systems, taking into account both square lattice [56–63] and honeycomb lattice [64, 65] geometries. Finally, we refer to [66–70], where the correctness and consistency of the effective field theory method has been demonstrated in high-accuracy Monte Carlo simulations.

The organization of the paper is as follows. In section 2 we discuss some basic aspects of the microscopic and the effective description of the $d = 3 + 1$ quantum $XY$ model. We then review the evaluation of the free energy density at low temperature up to three-loop order in section 3. The low-temperature expansions for the pressure, the (staggered) magnetization and (staggered) susceptibility are obtained in section 4. There we also discuss how the external field influences the spin-wave interaction in these quantities. Finally, in section 5 we present our conclusions. Technical details that concern the evaluation of the partition function Feynman graphs, the numerical evaluation of a specific three-loop graph, and the estimation of subleading effective constants are presented in three separate appendices.

2. Effective versus microscopic description

The microscopic Hamiltonian for the ferromagnetic $d = 3 + 1$ quantum $XY$ model is

$$\mathcal{H} = -J \sum_{\langle xy \rangle} (S_x^1 S_y^1 + S_x^2 S_y^2) - \vec{H} \cdot \sum_x \vec{S}_x, \quad J > 0. \tag{2.1}$$

Here $x$ and $y$ are nearest-neighbor pairs of lattice sites separated by the distance $\hat{a}$, the quantity $J$ is the exchange integral, and $\vec{H} = (0, H, 0)$ is a weak magnetic field in the $XY$-plane. The quantities $S_x^1$ and $S_x^2$ are the first two components of the vector spin operator $\vec{S}$, where the spin quantum number $S$ is arbitrary. For $S = \frac{1}{2}$, the first two components of the spin operator can be expressed by the first two Pauli matrices as $S_x^1 = \sigma_x^1, S_x^2 = \sigma_x^2$. We assume that the crystal lattice is bipartite, such that there is a mapping between the ferromagnetic $(J > 0)$ and the antiferromagnetic $(J < 0)$ model. Our formalism thus either describes the ferromagnetic quantum $XY$ model in an external magnetic field $\vec{H}$, or the antiferromagnetic quantum $XY$ model in an external magnetic field $-\vec{H}$. The organization of the paper is as follows. In section 2 we discuss some basic aspects of the microscopic and the effective description of the $d = 3 + 1$ quantum $XY$ model. We then review the evaluation of the free energy density at low temperature up to three-loop order in section 3. The low-temperature expansions for the pressure, the (staggered) magnetization and (staggered) susceptibility are obtained in section 4. There we also discuss how the external field influences the spin-wave interaction in these quantities. Finally, in section 5 we present our conclusions. Technical details that concern the evaluation of the partition function Feynman graphs, the numerical evaluation of a specific three-loop graph, and the estimation of subleading effective constants are presented in three separate appendices.

2. Effective versus microscopic description

The microscopic Hamiltonian for the ferromagnetic $d = 3 + 1$ quantum $XY$ model is

$$\mathcal{H} = -J \sum_{\langle xy \rangle} (S_x^1 S_y^1 + S_x^2 S_y^2) - \vec{H} \cdot \sum_x \vec{S}_x, \quad J > 0. \tag{2.1}$$

Here $x$ and $y$ are nearest-neighbor pairs of lattice sites separated by the distance $\hat{a}$, the quantity $J$ is the exchange integral, and $\vec{H} = (0, H, 0)$ is a weak magnetic field in the $XY$-plane. The quantities $S_x^1$ and $S_x^2$ are the first two components of the vector spin operator $\vec{S}$, where the spin quantum number $S$ is arbitrary. For $S = \frac{1}{2}$, the first two components of the spin operator can be expressed by the first two Pauli matrices as $S_x^1 = \sigma_x^1, S_x^2 = \sigma_x^2$. We assume that the crystal lattice is bipartite, such that there is a mapping between the ferromagnetic $(J > 0)$ and the antiferromagnetic $(J < 0)$ model. Our formalism thus either describes the ferromagnetic quantum $XY$ model in an external magnetic field $\vec{H}$, or the antiferromagnetic quantum $XY$ model in an external magnetic field $-\vec{H}$. The organization of the paper is as follows. In section 2 we discuss some basic aspects of the microscopic and the effective description of the $d = 3 + 1$ quantum $XY$ model. We then review the evaluation of the free energy density at low temperature up to three-loop order in section 3. The low-temperature expansions for the pressure, the (staggered) magnetization and (staggered) susceptibility are obtained in section 4. There we also discuss how the external field influences the spin-wave interaction in these quantities. Finally, in section 5 we present our conclusions. Technical details that concern the evaluation of the partition function Feynman graphs, the numerical evaluation of a specific three-loop graph, and the estimation of subleading effective constants are presented in three separate appendices.
staggered field $\hat{H}_s = (0, H_s, 0)$ [51, 71]. While the magnetic field is associated with the magnetization order parameter,

$$\hat{S} = \left( \sum_x S^1_x, \sum_x S^2_x, \sum_x S^3_x \right),$$  

the staggered field couples to the staggered magnetization order parameter,

$$\hat{S} = \left( \sum_x (-1)^{x_1+x_2+x_3} S^1_x, \sum_x (-1)^{x_1+x_2+x_3} S^2_x, \sum_x (-1)^{x_1+x_2+x_3} S^3_x \right).$$  

At zero temperature and infinite volume, the vacuum expectation values,

$$\Sigma = \langle 0 | \sum_x S^2_x | 0 \rangle / V,$$

$$\Sigma_s = \langle 0 | \sum_x (-1)^{x_1+x_2+x_3} S^2_x | 0 \rangle / V,$$  

are nonzero, signaling the spontaneous breakdown of the O(2) spin rotation symmetry. In the following, we will stick to the latter realization, bearing in mind that whenever we speak of the aniferromagnetic XY model in a staggered field, it can also be interpreted as ferromagnetic XY model in a magnetic field.

We now leave the microscopic description and consider the $d = 3 + 1$ quantum XY model within the effective field theory formalism. The essential point is that the system is characterized by a spontaneously broken continuous and global spin rotation symmetry: whereas the Hamiltonian is invariant under O(2), the ground state is not. As a consequence, a Goldstone boson excitation emerges—a magnon or spin wave—that dominates the low-temperature physics of the system. Note that these low-energy degrees of freedom obey a linear, i.e. relativistic dispersion relation. In the effective field theory description, one then defines the unit vector field $U^i(x)$,

$$U^i(x)U^i(x) = 1, \quad i = 1, 2,$$  

that corresponds to the magnetization (or staggered magnetization) order parameter field. In our convention the order parameter points along the second axis ($i = 2$): therefore the ground state of the quantum XY model is described by $\vec{U}_0 = (0, 1)$. The spin-wave field then corresponds to the transverse direction $U^1$ that represents the fluctuations of the order parameter vector $\vec{U}$ around the ground state.

The effective field theory approach is legitimate at low energies or low temperatures, and corresponds to a systematic derivative expansion of the effective Lagrangian. Clearly, the low-energy physics of the system is dominated by terms that contain just a few time or space derivatives. Higher-derivative terms are successively suppressed and hence are less important. Regarding the $d = 3 + 1$ quantum XY model, the leading piece in the effective Lagrangian is

$$\mathcal{L}_\text{eff}^2 = \frac{1}{2} F^2_{ij} \partial_0 U^i \partial_0 U^j - \frac{1}{2} F^2_{ij} \partial_0 U^i \partial_0 U^j + \Sigma_s H_s U^1,$$  

where $F_{ij}$ is the field strength tensor and $H_s$ is the staggered field.
and involves terms with two time ($\partial_0 \partial_0$) and two space ($\partial_i \partial_i$) derivatives. Both contributions are of order $p^2$. The leading term that contains the staggered field is linear in $H_s$. In the systematic derivative counting, the staggered field hence also counts as $p^2$, much like the terms with two time or two space derivatives: all terms in the leading-order effective Lagrangian $\mathcal{L}^2_{\text{eff}}$ are then of order $p^2$. At this point we have three effective constants, $F_1$, $F_2$, and $\Sigma$. Since time and space derivatives are on the same footing, the spin-wave dispersion relation can be written in a relativistic form,

$$\omega = \sqrt{v^2 k^2 + v^4 M^2}, \quad v = \frac{F_2}{F_1},$$

(2.7)

where $v$ is the spin-wave velocity, and the magnon ‘mass’—or energy gap—is identified with

$$M^2 = \frac{\Sigma H_s}{F^2}.$$  \hspace{1cm}  \text{(2.8)}

If one interprets the spin-wave velocity as the ‘velocity of light’—and furthermore sets $v = 1$—relativistic notation can be used, and the leading-order effective Lagrangian then takes the (pseudo-)Lorentz-invariant structure

$$\mathcal{L}^2_{\text{eff}} = \frac{1}{2} F^2 \partial_0 U^i \partial^0 U^i + \Sigma \partial_i U^j |U^i|, \quad F_1 = F_2 = F.$$  \hspace{1cm}  \text{(2.9)}

It is important to point out that we are not dealing with an approximation here. In the effective description, as is well-known \cite{49}, anisotropies due to the cubic lattice geometry only start to emerge at next-to-leading order in the derivative expansion of the effective Lagrangian. The leading piece $\mathcal{L}^2_{\text{eff}}$ is strictly (pseudo-)Lorentz-invariant. On the other hand, higher-order contributions in the effective Lagrangian do not share this accidental symmetry, and in principle all terms permitted by the discrete symmetries of the lattice have to be considered. In the present study, however, we assume that subleading contributions in the effective Lagrangian can also be written in a relativistic form. The qualitative behavior predicted by the (pseudo-)Lorentz-invariant theory is the same as that of the full theory where all possible terms consistent with the symmetries have been taken into account. However, (very small) quantitative differences are expected at three-loop order and beyond, as we discuss in the next section.

The only subleading piece in the effective Lagrangian that is explicitly needed for our calculation is the next-to-leading piece $\mathcal{L}^4_{\text{eff}}$. Assuming (pseudo-)Lorentz-invariance, it takes the form \cite{53, 54}

$$\mathcal{L}^4_{\text{eff}} = e_1 (\partial_0 U^i \partial^0 U^i)^2 + e_2 (\partial_0 U^i \partial^0 U^i)^2 + k_1 \frac{\Sigma}{F^2} (H_s^i U^i)(\partial^k U^j \partial^\mu U^k)$$

$$+ k_2 \frac{\Sigma^2}{F^4} (H_s^i U^i)^2 + k_3 \frac{\Sigma^2}{F^4} H_s^i H_s^i.$$  \hspace{1cm}  \text{(2.10)}

While there are two coupling constants ($F$, $\Sigma$) at leading order, at next-to-leading order five additional effective constants—$e_1$, $e_2$, $k_1$, $k_2$ and $k_3$—are required. Since the subsequent contributions $\mathcal{L}^6_{\text{eff}}$ ($\mathcal{L}_{\text{eff}}^8$) only show up in one-loop (tree) graphs, we do not need their explicit form (see next section).
The derivative structure of the terms in the effective Lagrangian is fixed by the symmetries of the underlying $d = 3 + 1$ quantum XY Hamiltonian. The effective constants $F, \Sigma, e_1, e_2, k_1, k_2, k_3$, however, are not determined by the symmetries of the system. One possibility to ascertain their numerical values is to match the effective calculation with the analogous microscopic calculation, provided the latter is available—this can be done for $F$ and $\Sigma$. Another possibility to fix the effective constants is by Monte Carlo simulation or by experiment. Unfortunately, for the $d = 3 + 1$ quantum XY model, Monte Carlo simulations, microscopic calculations or experiments that would allow one to determine next-to-leading order (NLO) effective constants, seem to be lacking. But we can still estimate their values.

The fact that the symmetries unambiguously determine the structure of the effective Lagrangian—irrespective of the concrete realization of the underlying microscopic system—is often referred to as ‘universality’ of the effective Lagrangian approach. Although two systems may be very different on the underlying level, the corresponding effective Lagrangians are the same: the difference between the two systems then only shows up in the specific numerical values of some low-energy effective constants that indeed depend on the actual microscopic details. We want to stress that the above use of the term ‘universal’ should not be associated with ‘universality’ class or with the classification of phase transitions. Actually, since the effective Lagrangian approach refers to low temperatures, our method is not designed to study physical systems at or near the phase transition.

3. Free energy density up to three-loop order

The perturbative evaluation of the partition function for (pseudo-)Lorentz-invariant systems that live in three spatial dimensions and that are characterized by a spontaneously broken global symmetry $O(N) \to O(N-1)$, has been presented in [54]. This effective field theory analysis was carried to three-loop order.

While our discussion is partially based on results derived in [54], we stress that here we go much beyond this reference. First, [54] focused on the Heisenberg antiferromagnet, and not on the quantum XY model where new subtleties in the renormalization process and the structure of the low-temperature expansion occur. Then, we numerically evaluate a rather complicated three-loop diagram. Moreover, the discussion in [54] concerned the limit of a zero external field, while here we explore the thermodynamics in presence of a weak external field. Finally, the susceptibility has not been considered in [54].

In the present section, and in appendix A, we provide a concise review of the basic formulas obtained in [54]. The evaluation of the partition function within effective field theory has been outlined in detail in [50] (section 2) and [33] (appendix A), much beyond the sketch we give below. The interested reader is referred to these references and to [72–74] that provide pedagogic introductions to the effective Lagrangian method.

The perturbative expansion of the partition function is based on the observation that each Goldstone-boson loop is suppressed by some power $n$ of temperature. The number $n$ depends both on the spatial dimension $d_s$ of the system and on the nature
Thermodynamics of the $d = 3 + 1$ quantum XY model

of the dispersion relation of its Goldstone bosons. Provided that the dispersion relation is linear, the suppression is $p^{d-1} T^{d-1}$ [53]. In the present case of the $d = 3 + 1$ quantum XY model, each loop is suppressed by two powers of momentum or temperature. This provides the basis to understand the organization of the Feynman diagrams of figure 1 that are relevant up to three-loop order.

The leading temperature-dependent term comes from diagram 4A, yielding a contribution of order $p^4 \propto T^4$ in the free energy density: this is the free Bose gas term. We then have a two-loop contribution (graph 6A) of order $p^6 \propto T^6$, and various three-loop contributions (graphs 8A–C) of order $p^8 \propto T^8$. Note that tree-level diagrams just correspond to $T$-independent contributions that can be absorbed into the vacuum energy density. The only $T$-dependent contribution that requires a piece of the effective Lagrangian beyond $L_{\text{eff}}^4$, is the one-loop diagram 8G that involves $L_{\text{eff}}^6$. However, this graph merely contributes to the renormalization of the Goldstone boson mass (see appendix A).

In three spatial dimensions, the loop-suppression by $T^2$ leads to the general pattern $T^4$, $T^6$, $T^8$ in the free energy density. Regarding the quantum XY model in two spatial dimensions, the situation is different since each loop is only suppressed by one power of temperature. This leads to the Feynman diagrams depicted in figure 2. Here the leading $T$-dependent term comes from diagram 3: the free Bose gas contribution of order $p^3 \propto T^3$. The two-loop (three-loop) corrections then yield terms of order $T^4$ ($T^5$) in the free energy density.

Figure 1. $D = 3 + 1$ quantum XY model: Feynman diagrams occurring in the low-temperature expansion of the partition function up to three-loop order $T^8$. Vertices involving the leading piece $L_{\text{eff}}^2$ of the effective Lagrangian correspond to a filled circle, while vertices associated with $L_{\text{eff}}^4$, $L_{\text{eff}}^6$, $L_{\text{eff}}^8$ are denoted by the numbers 4–8, respectively. Each loop is suppressed by $p^2 \propto T^2$. 

doi:10.1088/1742-5468/2016/09/093102
As described in appendix A, up to order $p T^8 \propto \sigma$ in the free energy density, the final result for general $N \geq 2$ amounts to

\[ z = z_0 - \frac{1}{2} (N-1) g_0 - 4 \pi a (g_1)^2 - \pi g \left[ b - \frac{j}{\pi^3 F^4} \right] + \mathcal{O}(p^{10}). \] (3.1)

This representation applies to any $d = 3 + 1$ (pseudo-)Lorentz-invariant system exhibiting the spontaneous symmetry breaking pattern $O(N) \to O(N-1)$. The various quantities are defined in appendix A. These include the $T = 0$ free energy density $z_0$ (A.28), the parameters $a$ and $b$ that contain NLO effective constants (A.42), the kinematical functions $g_r$ (A.25) and $g$ (A.41) that nontrivially depend on temperature and on the renormalized magnon mass $M_\pi$. The latter can be expressed in terms of the staggered field $H_s$ (A.29). Finally, $j$ is a dimensionless function defined in (A.44)—much like the kinematical functions $g_r$ and $g$, it depends on the dimensionless ratio $\tau$,

\[ \tau = \frac{T}{M_\pi}. \] (3.2)

The numerical evaluation of the three-loop integral $j$ is outlined in appendix B.

The structure of the low-temperature expansion becomes more transparent if the following dimensionless functions $h_0$, $h_1$ and $h$ are introduced:

\[ g_0(\sigma) = T^4 h_0(\sigma), \quad g_1(\sigma) = T^2 h_1(\sigma), \quad g(\sigma) = T^8 h(\sigma), \] (3.3)

where the dimensionless parameter $\sigma$ is

\[ \sigma = \frac{M_\pi}{2 \pi T} = \frac{1}{2 \pi \tau}. \] (3.4)

The kinematical functions $h_0, h_1, h_2, h_3$ are depicted in figure 3. The latter two are relevant in the magnetization and susceptibility—they scale like...
Using these representations, the low-temperature expansion for the free energy density of the $d = 3 + 1$ quantum XY model takes the form

$$z = z_0 - \frac{1}{2} h_0(\sigma) T^4 - \frac{1}{8 F^2 t^2} h_1(\sigma)^2 T^6 - \frac{3(e_1 + e_2)}{2} - \frac{1}{2} \frac{k}{256 \pi^2} h_1(\sigma)^2 T^8$$

$$- \frac{2(e_1 + e_2)}{F^4} h(\sigma) T^8 + \frac{1}{\pi^2 F^4} j(\sigma) h(\sigma) T^8 + \mathcal{O}(p^{10}), \quad (N = 2)$$

where $t$ is the dimensionless ratio

$$t = \frac{T}{M} = \frac{TF}{\sqrt{\Sigma_s H_s}}. \quad (3.7)$$

Up to two-loop order $T^6$, the coefficients merely involve the effective constants $F$ and $\Sigma_s$ from $L_{\text{eff}}^2$. NLO effective constants start showing up at three-loop order $T^8$: for the definition of $e_1$, $e_2$ and $k$ see equation (2.10) and equation (A.32)$^2$. The low-temperature expansion is characterized by even powers of the temperature. The free Bose gas term (order $T^4$) receives corrections of ascending powers $T^2$.

If one aims at three-loop accuracy, it is important to distinguish between $t$ and $\tau$: whereas the former involves the leading-order mass $M$, the latter involves the renormalized mass $M_s$ (see equation (A.29)). The difference between $1/t^2$ and $1/\tau^2$ is of order $M^4$. If one is only interested in two-loop accuracy, it is hence legitimate to replace $1/t^2$ by $1/\tau^2$ in the contribution of order $T^6$.

---

$^2$ Note that we use two-loop order (three-loop order) synonymous for order $T^6$ ($T^8$), since in our convention we only count loop-graphs that exclusively contain insertions from $L_{\text{eff}}^2$. It should be kept in mind that at order $T^6$ we also have a one-loop graph (6B), and that at order $T^8$ we have also have two-loop (8D, E) and one-loop (8F, G) graphs (see figure 1).

https://doi.org/10.1088/1742-5468/2016/09/093102
Here is the appropriate place to elaborate on (pseudo-)Lorentz-invariance where our approach is based upon. Abandoning (pseudo-)Lorentz-invariance can be done in two steps: (i) write down all terms that are still consistent with space-rotation invariance, or (ii) even abandon the idealization of space isotropy, by taking into account all terms that are consistent with the discrete symmetries of the lattice. How would the low-temperature series be affected? The point is that the temperature powers would not change, but the coefficients would be different: more subleading effective constants would show up. Also, it would no longer be possible to define a Goldstone boson mass $M_\pi$ via the relativistic dispersion relation

$$\omega = \sqrt{v^2 k^2 + v^4 M_\pi^2}.$$  \hfill (3.8)

It is not our intention, however, to explicitly incorporate all these subleading effective constants. Remember that Lorentz-symmetry breaking terms only start emerging in $\mathcal{L}_{\text{eff}}^4$. According to figure 1, the temperature-dependent interaction is thus only affected through the two-loop diagrams 8D and E, i.e. at next-to-next-to-leading order $T^8$ in the free energy density. For practical purposes it is legitimate to work within a (pseudo-) Lorentz-invariant formalism. After all, we are dealing with small effects, as we illustrate below.

### 4. Low-temperature series

The low-temperature representation for the free energy density, equation (3.6), provides the basis for our subsequent discussion—any other thermodynamic observable can be derived from there. But let us first clarify in which parameter range—defined by temperature and external field—our series are valid. The effective expansion is restricted to low energies. A natural energy scale is the exchange constant $J$ inherent in the underlying microscopic model, such that our series are valid as long as both $T$ and $H_s$ are small compared to $J$. Equivalently, we may consider the critical temperature $T_c$ where the order parameter drops to zero and the phase transition takes place—at or near this point, the spin-wave picture is no longer adequate. According to [1], for the simple-cubic quantum $XY$ model one has $T_c \approx 2.02 J$, and one may define low temperature and weak field as

$$T, H_s, M_\pi \lesssim 0.2 T_c \approx 0.4 J.$$  \hfill (4.1)

Furthermore, as we outline in appendix C, the connection between the microscopic scale $J$ and the effective constant $F$ is approximately $F \approx 0.4 J$ for the quantum $XY$ model on the simple cubic lattice. Accordingly, the parameter ranges translate into

$$T, H_s, M_\pi \lesssim F.$$  \hfill (4.2)

We now consider the low-temperature series for the pressure, order parameter, and susceptibility. One interesting topic is to explore the sign and strength of the spin-wave interaction in these quantities as a function of temperature and external field.
4.1. Pressure

The pressure can be extracted from the free energy density through

\[ P = z_0 - z. \]  

For the \( d = 3 + 1 \) quantum XY model we obtain

\[
P(T, H_0) = \frac{1}{2} h_0(\sigma) T^4 + \frac{1}{8 F^2 t^2} h_1(\sigma)^2 T^6 + \frac{3(e_1 + e_2)}{2} \frac{k}{F^4 T^4} - \frac{3}{256\pi^2} h_1(\sigma)^2 T^8
\]

\[ + \frac{2(e_1 + e_2)}{F^4} h(\sigma) T^8 - \frac{1}{\pi^2 F^4} j(\sigma) h(\sigma) T^8 + \mathcal{O}(T^{10}). \]  

The spin-wave interaction shows up at order \( T^6 \), subsequent corrections are of order \( T^8 \). If the external field is switched off, the pressure reduces to

\[ P(T, 0) = \frac{\pi^2}{90} T^4 + \frac{2\pi^4(e_1 + e_2)}{675F^4} T^8. \]  

There is no \( T^6 \)-contribution in this case.

In figure 4 we compare the strength of the dominant interaction correction—the term of order \( T^6 \) in equation (4.4)—with the free Bose gas term (order \( T^4 \)), using the ratio

\[ \xi_p(T, H_0) = \frac{P_{\text{int}}(T, H_0)}{P_{\text{Bose}}(T, H_0)}, \]  

for the temperatures \( T/F = \{0.04, 0.07, 0.10, 0.13\} \) in presence of a weak external field, parametrized by \( M_p/F \). The sign of \( \xi_p \) is positive, meaning that the spin-wave interaction in the pressure is repulsive, irrespective of the temperature and strength of the external field. The interaction is maximal in an intermediate domain of staggered field strength. The corresponding maxima, however, are tiny—the spin-wave interaction in the \( d = 3 + 1 \) quantum XY model is very weak also in nonzero external field.
At order $T^8$ there are three terms that contribute to the interaction. The coefficients of the first two terms involve NLO effective constants that are \textit{a priori} unknown. They could be determined by matching our formulas with corresponding microscopic formulas, by Monte Carlo simulations of the $d = 3 + 1$ quantum $XY$ model at low temperatures, or by comparing our predictions with experiments. Unfortunately, none of these options seem to be available.

We can, however, estimate the order of magnitude of these NLO effective constants. The explicit steps can be found in appendix C. The outcome is

\[ \left| e_1 \right| \approx \left| e_2 \right| \approx \left| \vec{k} \right| \approx 0.001. \]  

This value corresponds to an estimate of the order of magnitude of these NLO effective constants. Given the ‘hand-waiving’ nature of the estimation method presented in appendix C, one may conclude that the actual values are within the range

\[ \left| e_1 \right| \approx \left| e_2 \right| \approx \left| \vec{k} \right| \approx 0.001 \ldots 0.01. \]

While the numerical values of these NLO effective constants are small, unfortunately, their sign still is inconclusive. It should be noted that the effective constant $\vec{k}$ in equation (4.4), unlike $e_1$ and $e_2$, requires logarithmic renormalization (see appendix A).

In figure 5 we display the ratio

\[ x_p(T, H) = \frac{P^{[8]}(T, H) + P^{[6]}(T, H)}{P_{\text{Bose}}(T, H)} \]

that measures the strength of the $T^8$- and $T^6$-interaction correction with respect to the Bose term. We depict two examples: on the one hand, $e_1$, $e_2$ and $\vec{k}$ all taking positive values, namely $0.003$, and, one the other hand $e_1 = e_2 = \vec{k} = -0.003$. One notices that the $T^8$-correction may be positive or negative, signaling that the repulsive $T^6$-contribution

\[ \text{Figure 5. Spin-wave interaction in the pressure of the } d = 3 + 1 \text{ quantum } XY \text{ model: the three-loop correction is small compared to the two-loop contribution, as measured by } x_p \text{. The curves refer to the temperatures } T/F = \{0.04,0.07,0.10,0.13\} \text{ from bottom to top in the figure.} \]
may be enhanced or weakened. In very weak staggered fields, the interaction among spin waves may even become attractive. In conclusion, the three-loop corrections are quite small and it is justified to only consider the leading correction of order $T^6$. In the ensuing plots, we will indeed restrict ourselves to two-loop accuracy.

Regarding the interpretation of the figures that show the effect of the spin-wave interaction, it is important to point out a subtlety related to $M$ and $M_\pi$. The crucial point is that the spin-wave interaction also manifests itself in temperature-independent quantities, in particular, in equation (A.29) that connects $M_\pi$ and $M$: the second (third) term on the RHS is the two-loop (three-loop) correction that the leading term $M^2$ receives—both terms originate from the spin-wave interaction at $T = 0$. What we have depicted in the figures is the finite-temperature interaction contribution that depends on $M_\pi$ which incorporates the interaction at $T = 0$. The interpretation of the figures presented in this section—and those presented below—hence is as follows: We start at zero temperature and switch on the external field. Afterwards we go from $T = 0$ to finite temperature—while keeping $H_s$ fixed—and study how this affects the interaction at finite temperature. In the pressure, the interaction is repulsive.

4.2. Order parameter

The staggered magnetization at finite temperature is defined by

$$
\Sigma_s(T, H_s) = -\frac{\partial z}{\partial H_s}.
$$

(4.10)

This is the order parameter, signaling that the internal symmetry O(2) is spontaneously broken.

With the representation for the free energy density, equation (3.6), the low-temperature series for the staggered magnetization amounts to

$$
\Sigma_s(T, H_s) = \Sigma_s(0, H_s) - \frac{\Sigma_0 \hat{b}}{2F^2} h_1(\sigma) T^2 + \frac{\Sigma_0}{8F^4} \left\{ h_1(\sigma)^2 - \frac{2\hat{b}}{t^2} h_1(\sigma) h_2(\sigma) \right\} T^4
$$

$$
+ \frac{2\Sigma_0}{t^2 F^6} \left\{ 3(e_1 + e_2) + \frac{1}{2} k - \frac{3}{256\pi^2} \right\} \left\{ h_1(\sigma)^2 - \frac{\hat{b}}{t^2} h_1(\sigma) h_2(\sigma) \right\} T^6
$$

$$
- \frac{3\Sigma_0 \hat{b}}{F^6} \left( 3(e_1 + e_2) - \frac{1}{\pi^2} j(\sigma) \right) \left\{ h_0(\sigma) h_1(\sigma) + \frac{h_1(\sigma)^2 + h_0(\sigma) h_2(\sigma)}{\tau^2} \right\} T^6
$$

$$
- \frac{3\Sigma_0 \hat{b}}{8\pi^4 F_0^2} \frac{\partial j(\sigma)}{\partial \sigma} \left\{ h_0(\sigma)^2 + \frac{h_0(\sigma) h_1(\sigma)}{\tau^2} \right\} T^6
$$

$$
- \frac{\Sigma_0}{64\pi^2 F_0^6 t^2} h_1(\sigma)^2 T^6 + O(T^8).
$$

(4.11)

The quantity $\hat{b}$ is
The spin-wave interaction manifests itself at order $T^4$ and $T^6$, both in presence or absence of the staggered field. At zero temperature, the staggered magnetization becomes

$$
\Sigma_s(0, H_s) = \Sigma_s \left\{ 1 + \tilde{k} \frac{\Sigma_s H_s}{F^4} - \frac{1}{64\pi^2} \frac{\Sigma_s H_s}{F^4} \right\} + O(H_s^2),
$$

(4.13)

where

$$
\Sigma_s = \Sigma_s(0, 0).
$$

(4.14)

In figure 6 we plot the ratio

$$
\xi_S(T, H_s) = \frac{\Sigma_{s(4)}(T, H_s)}{|\Sigma_{\text{Bose}}(T, H_s)|},
$$

(4.15)

that measures strength and sign of the spin-wave interaction in the leading interaction term—the term of order $T^4$ in equation (4.11)—with respect to the free Bose gas term ($T^2$). While the quantity $\xi_S$ is mainly negative in parameter space, interestingly, for very weak external fields (small $M_s$) it becomes positive.

It should be noted that these two-loop effects are quite small and that the properties of the order parameter are dominated by the (one-loop) free Bose gas contribution. As expected, this term ($\propto T^2$) is negative: the order parameter gradually decreases as the temperature rises. It is quite remarkable—or counterintuitive—that the spin-wave interaction not necessarily presents this behavior. If the temperature is low and the field $H_s$ is weak, the behavior is just the opposite: the interaction induces an increase of the order parameter, i.e. it tends to reinforce the (anti-)alignment of the spins and

Figure 6. Manifestation of the leading interaction contribution in the staggered magnetization of the $d = 3 + 1$ quantum $XY$ model, measured by $\xi_S$. The curves refer to the temperatures $T/F = \{0.04, 0.07, 0.10, 0.13\}$ from bottom to top in the figure (vertical cut at $M_s/F = 0$).
enhances the (staggered) magnetization. Keep in mind that we first switch on the field $H_s$ at zero temperature, and then go to finite $T$ while keeping $H_s$ fixed.

### 4.3. Staggered susceptibility

The staggered susceptibility corresponds to the derivative of the order parameter, equation (4.11), with respect to the staggered field,

$$\chi(T, H_s) = \frac{\partial \Sigma_\text{st}(T, H_s)}{\partial H_s} = \frac{\Sigma_s}{F^2} \frac{\partial \Sigma_\text{st}(T, M)}{\partial M^2}. \tag{4.16}$$

The low-temperature series exhibits the general structure

$$\chi(T, H_s) = \chi(0, H_s) + \chi_1(\tau) + \chi_2(\tau) T^2 + \chi_3(\tau) T^4 + O(T^6). \tag{4.17}$$

Since the expressions for the coefficients are rather lengthy, we do not display the full three-loop result that can trivially be obtained from equation (4.16). Here we provide explicit expressions up to two-loop order and furthermore work within the approximation $\hat{b} \approx 1$ (see equation (4.12)), which is sufficient for practical purposes. We then obtain

$$\chi_1(\tau) = \frac{\Sigma_s^2}{F^4} h_2,$$

$$\chi_2(\tau) = -\frac{\Sigma_s^2}{4F^6} \left\{ 2h_1 h_2 - \frac{1}{t^2} (h_2^2 + h_1 h_3) \right\}. \tag{4.18}$$

In figure 7 we plot the ratio

$$\xi_\text{st}(T, H_s) = \frac{\chi_\text{int}^2(T, H_s)}{\chi_\text{Bose}(T, H_s)}. \tag{4.19}$$

---

**Figure 7.** Manifestation of the leading interaction contribution in the staggered susceptibility of the $d = 3 + 1$ quantum $XY$ model, measured by $\xi_\text{st}$. The curves refer to the temperatures $T/F = \{0.04, 0.07, 0.10, 0.13\}$ from top to bottom in the figure (vertical cut at $M_s/F = 0$).
that measures strength and sign of the spin-wave interaction in the leading interaction term—the term $\chi_2(\tau) T^2$ in equation (4.17)—in the staggered susceptibility relative to the free Bose gas term $\chi_1(\tau)$ of order $T^0$.

Remarkably, $\xi \chi$ takes negative values if the staggered field is weak. Since the staggered field tends to reinforce the staggered spin pattern and to enhance the order parameter, one would expect $\chi(T, H_s)$ to take positive values. While this is indeed the case for the free Bose gas contribution, the effects concerning the spin-wave interaction are peculiar: negative values of $\xi \chi$ imply that the staggered field actually perturbs the antialignment of the spins, such that the order parameter decreases. It should be kept in mind, however, that we are dealing with rather weak effects—the free Bose gas term dominates and the overall staggered susceptibility is positive.

In the limit $H_s \to 0$, the leading temperature-dependent term in the staggered susceptibility diverges according to

$$\lim_{H_s \to 0} \chi_1 \propto \frac{T}{\sqrt{H_s}}.$$ (4.20)

This corresponds to one of the few known results for the $d = 3 + 1$ quantum XY model at low temperatures where the spin-wave picture applies [12]. Note that the divergence originates from the free Bose gas term—in this sense it is a trivial one-loop result.

5. Conclusions

We have studied the low-temperature properties of the $d = 3 + 1$ quantum XY model in the regime where the physics is dominated by spin waves. To the best of our knowledge, all previous analyses were restricted to one-loop order in the low-temperature expansion. In particular, the manifestation of the spin-wave interaction in thermodynamic quantities has not been addressed before.

Here we have presented a systematic effective field theory analysis of the $d = 3 + 1$ quantum XY model at low temperatures in weak staggered (or magnetic) fields up to three-loop order. The basic quantity from which the thermodynamic properties of the system can be derived is the free energy density: the free Bose gas term in the low-temperature expansion is of order $T^4$, while two- and three-loop corrections yield interaction contributions of order $T^6$ and $T^8$, respectively.

The $T^6$-correction only involves the leading-order effective constants $F$ and $\Sigma_6$. On the other hand, at order $T^8$ in the free energy density, next-to-leading order effective constants show up that are a priori unknown. We have estimated their numerical values and concluded that these effects of order $T^8$ are small.

The spin-wave interaction in the pressure is repulsive at low temperatures. Only in very weak staggered fields $H_s$ it may become attractive—these small effects depend on the numerical values of NLO effective constants whose sign is not determined by the symmetries of the $d = 3 + 1$ quantum XY model. Remarkably, in the staggered magnetization (susceptibility) the sign of the temperature-dependent interaction contribution in weak fields becomes positive (negative).
Lattice anisotropies only start showing up in the subleading Lagrangian $\mathcal{L}_{\text{eff}}^4$ that hence depends on additional effective constants. These next-to-leading order effective constants slightly modify the coefficients of the low-temperature expansion beyond the free Bose gas term—but only if one aims at three-loop accuracy. These constants, however, do not alter the structure of the temperature powers. This all justifies the use of a (pseudo-)Lorentz-invariant framework.

There are materials that are believed to behave like $d = 3 + 1$ quantum XY ferromagnets [75–80]. Unfortunately, to experimentally detect the subtle two- and three-loop effects presented here, appears to be out of question. The behavior of any real material is more complicated than the simple quantum XY model system.

Still, the effective field theory predictions could be verified by simulating the ‘clean’ $d = 3 + 1$ quantum XY Hamiltonian. Therefore our analysis does not just represent some ‘academic’ exercise devoid of any impact on actual research—rather we believe that it has the potential to stimulate further studies on the $d = 3 + 1$ quantum XY model in the parameter regime we have considered here. In particular, numerical simulations have now reached a precision where even three-loop effects can be detected. As an example we mention the comprehensive numerical study of the $d = 2 + 1$ Heisenberg antiferromagnet presented in [81], where the three-loop predictions obtained earlier within effective field theory ([50]) were confirmed by Monte Carlo simulations. As we have seen, the spin-wave interaction in the $d = 3 + 1$ quantum XY model already starts manifesting itself at two-loop order: we are thus dealing with a promising candidate system where numerical simulations or even experimental studies may reveal the effect of the spin-wave interaction in the low-temperature regime.

Finally we find it instructive to briefly compare our three-loop results for the $d = 3 + 1$ quantum XY model with the analogous findings for the quantum XY model in two spatial dimensions. The organization of the loop expansion, as we have discussed in section 3, depends on the spatial dimension. The respective Feynman graphs are presented in figure 1 (three space dimensions) and figure 2 (two space dimensions). In the case of the $d = 2 + 1$ quantum XY model, the interaction diagrams only involve the leading piece $\mathcal{L}_{\text{eff}}^2$ of the effective Lagrangian. Regarding the $d = 3 + 1$ quantum XY model, however, we have two-loop diagrams (diagrams 8D, E in figure 1) that contain vertices from $\mathcal{L}_{\text{eff}}^4$. Accordingly, the coefficients of the interaction terms of order $T^8$ in the free energy density involve NLO effective constants. Note that, in $d = 2 + 1$, these two-loop diagrams are of order $T^6$, i.e. beyond three-loop level. One thus realizes that the spontaneously broken $O(2)$ symmetry is more restrictive in two spatial dimensions, in the sense that less effective constants are required, i.e. less information on the specific properties of the underlying microscopic XY model is needed.

Acknowledgments

The author would like to thank J Engels, E E Jenkins, H Leutwyler, A V Manohar, J Oitmaa, E Vicari and U-J Wiese for correspondence.
Appendix A. Evaluation of partition function diagrams

General aspects of the perturbative evaluation of the partition function within effective field theory have been discussed before (see, e.g. section 2 of [50], or appendix A of [33]). In the present appendix we focus on $d = 3 + 1$ (pseudo-)Lorentz-invariant systems with a spontaneously broken symmetry $O(N) \to O(N-1)$. Partial results have been presented in [54].

The basic object is the thermal Goldstone boson propagator $G(x)$,

$$G(x) = \sum_{n=-\infty}^{\infty} \Delta(\vec{x}, x_4 + n\beta), \quad \beta = \frac{1}{T},$$  \hspace{1cm} (A.1)

where $\Delta(x)$ is the Euclidean Goldstone boson propagator at zero temperature. Using dimensional regularization, it can be represented as

$$\Delta(x) = (2\pi)^{-d} \int d^d p e^{i p x} (M^2 + p^2)^{-1} = \int_0^{\infty} d\rho (4\pi \rho)^{-d/2} e^{-\rho M^2 - x^2/4\rho}. \hspace{1cm} (A.2)$$

Let us first list the contributions from the various graphs (see figure 1) that are relevant for the free energy density up to order $p^8$:

$$z_2 = -F^2 M^2. \hspace{1cm} (A.3)$$

$$z_{4A} = -\frac{1}{2} (N - 1) G_0. \hspace{1cm} (A.4)$$

$$z_{4B} = -(k_2 + k_3) M^4. \hspace{1cm} (A.5)$$

$$z_{6A} = \frac{1}{8} (N - 1)(N - 3) \frac{M^2}{F^2} (G_1)^2. \hspace{1cm} (A.6)$$

$$z_{6B} = (N - 1)(k_2 - k_1) \frac{M^4}{F^2} G_1. \hspace{1cm} (A.7)$$

$$z_{6C} = \hat{c}_1 M^6. \hspace{1cm} (A.8)$$

$$z_{8A} = \frac{1}{16} (N - 1)(N + 1)(N - 5) \frac{M^2}{F^4} (G_1)^3. \hspace{1cm} (A.9)$$

$$z_{8B} = -\frac{1}{4} (N - 1)(N - 3) \frac{M^2}{F^4} (G_1)^3 - \frac{1}{16} (N - 1)(N - 3)^2 \frac{M^4}{F^4} (G_1)^2 G_2. \hspace{1cm} (A.10)$$

$$z_{8C} = \frac{1}{6} N(N - 1) \frac{M^2}{F^4} (G_1)^3 + \frac{1}{48} (N - 1)(N - 3) \frac{M^4}{F^4} J_1$$

$$-\frac{1}{4} (N - 1)(N - 2) \frac{1}{F^4} J_2. \hspace{1cm} (A.11)$$
Thermodynamics of the $d = 3 + 1$ quantum XY model

$z_{8D} = -(N - 1)(2e_1 + Ne_2) \frac{1}{F^4} (G_{\mu\nu})^2 - (N - 1) \left[ e_1(N - 1) + e_2 - \frac{1}{2} k_1(N - 3) \right] \frac{M^4}{F^4} (G_1)^2.$  
(A.12)

$z_{8E} = -\frac{1}{2} (N - 1)[(N - 5)k_1 + 2k_2] \frac{M^4}{F^4} (G_1)^2 - \frac{1}{2} (N - 1)(N - 3)(k_2 - k_1) \frac{M^6}{F^4} G_1 G_2.$  
(A.13)

$z_{8F} = -2(N - 1)k_1(k_2 - k_1) \frac{M^6}{F^4} G_1 - (N - 1)(k_2 - k_1)^2 \frac{M^8}{F^4} G_2.$  
(A.14)

$z_{8G} = (N - 1) \hat{c}_0 \frac{M^6}{F^2} G_1.$  
(A.15)

$z_{8H} = \hat{d}_0 M^8.$  
(A.16)

Note that these expressions involve the bare Goldstone boson mass $M$ that can be translated into the external staggered field $H_s$ by

$M^2 = \frac{\Sigma H_s}{F^2}.$  
(A.17)

The thermodynamics of the $d = 3 + 1$ quantum XY model is contained in the functions $G_0, G_1, G_2, G_{\mu\nu}, J_1, J_2$: they all depend in a nontrivial way on the dimensionless ratio $M/T$. The quantity $G_1$ is the thermal propagator at the origin,

$G_1 \equiv G(x)|_{x=0},$  
(A.18)

while $G_2$ denotes the integral over the torus $T = \mathbb{R}^d \times S^1$ (where $S^1$ is the circle defined by $-\beta/2 \leq x_4 \leq \beta/2$, and $d$ is the spatial dimension), reading

$G_2 = \int_T d^4x \ (G(x))^2.$  
(A.19)

This quantity corresponds to the derivative of the thermal propagator at the origin with respect to the mass squared,

$G_2 = -\frac{dG_1}{dM^2}.$  
(A.20)

Then, $G_{\mu\nu}$ is the second derivative of the thermal propagator at the origin,

$G_{\mu\nu} = \partial_\mu \partial_\nu G(x)|_{x=0},$  
(A.21)

while $J_1$ and $J_2$ are the loop integrals
\[ J_1 = \int_T d^d x \{ G(x) \}^4, \]
\[ J_2 = \int_T d^d x \{ \partial_\mu G(x) \partial_\nu G(x) \}^2. \]  

(A.22)

In order to eventually remove the dimensional regularization parameter \( d \) in the above expressions, we decompose the thermal propagator—and all quantities obtained from there—into a temperature-independent and a temperature-dependent piece according to

\[ G(x) = \Delta(x) + \mathcal{G}(x). \]  

(A.23)

Especially, at the origin \( x = 0 \), we have

\[ G_0 = -\frac{4}{d} M^4 \lambda + g_0, \]
\[ G_1 = 2 M^2 \lambda + g_1, \]
\[ G_2 = (2 - d) \lambda + g_2, \]
\[ G_{\mu\nu} = 2 M^4 \delta_{\mu\nu} \frac{\lambda}{d} + \mathcal{G}_{\mu\nu}, \]
\[ \mathcal{G}_{\mu\nu} = -\frac{1}{2} \delta_{\mu\nu} g_0 + \delta_\mu^\lambda \delta_\nu^\lambda (\frac{1}{2} d g_0 + M^2 g_1). \]  

(A.24)

The kinematical functions \( g_r \) refer to the \( d \)-dimensional noninteracting Bose gas, and are defined by

\[ g_r(M, T) = 2 \int_0^\infty \frac{d^r \rho}{(4 \pi \rho)^{d/2}} \rho^{r-1} \exp(-\rho M^2) \sum_{n=1}^\infty \exp(-n^2/4\rho T^2). \]  

(A.25)

Note that the parameter \( \lambda \) in (A.24) is divergent in the limit \( d \to 4 \),

\[ \lambda = \frac{1}{2} (4\pi)^{-d/2} \Gamma(1 - \frac{1}{2} d) M^{d-4} \]
\[ = \frac{M^{d-4}}{16 \pi^2} \left[ \frac{1}{d-4} - \frac{1}{2} \left( \ln 4\pi + \Gamma'(1) + 1 \right) + \mathcal{O}(d-4) \right]. \]  

(A.26)

Finally, to remove the singularities contained in the loop integrals \( J_1 \) and \( J_2 \), we define the quantities \( \tilde{J}_1 \) and \( \tilde{J}_2 \) as,

\[ \tilde{J}_1 = J_1 - c_1 - c_2 g_1 + 6(d-2)\lambda(g_1)^2, \]
\[ \tilde{J}_2 = J_2 - c_3 - c_4 g_1 + \frac{1}{3} (d + 6)(d - 2)\lambda(\mathcal{G}_{\mu\nu})^2 + \frac{2}{3} (d - 2)\lambda M^4(g_1)^2. \]  

(A.27)

The explicit expressions for the temperature-independent counterterms \( c_1 \ldots c_4 \) are listed in [82].

Using the above decompositions and collecting first all terms that are independent of \( T \), we end up with the free energy density at zero temperature,

\[ \text{do}i:10.1088/1742-5468/2016/09/093102 \]
\[ z_0 = -F^2 M^2 + \frac{1}{2} (N - 1) M^4 \lambda - (k_2 + k_3) M^4 + \frac{1}{2} (N - 1) (N - 3) \frac{M^6}{F^2} \lambda^2 \\
+ 2(N - 1)(k_2 - k_1) \frac{M^6}{F^2} \lambda + \hat{c}_1 M^6 + \frac{1}{2} (N - 1)(N - 1) (N - 3) \frac{M^8}{F^4} \lambda^3 \\
- 2(N - 1)(N - 3) \frac{M^8}{F^4} \lambda^3 + \frac{1}{4} c_1 (N - 1)(N - 3) \frac{M^4}{F^4} - \frac{1}{4} c_3 (N - 1)(N - 2) \frac{1}{F^4} \\
+ \frac{4}{3} N(N - 1) \frac{M^8}{F^4} \lambda^3 + \frac{1}{4} (N - 1)(N - 3) M^8 \frac{(d - 2)}{d^4} \lambda^3 \\
- 4(N - 1)(2e_1 + Ne_2) \frac{\lambda^2}{d} \frac{M^8}{F^4} + 4(N - 1) \left[ e_1 (N - 1) + e_2 - \frac{1}{2} k_1 (N - 3) \right] \frac{M^8}{F^4} \lambda^2 \\
- 2(N - 1)((N - 5) k_1 + 2k_2) \frac{M^8}{F^4} \lambda^2 - (N - 1)(N - 3)(k_2 - k_1) \frac{M^8}{F^4} \lambda^2 \\
- 4(N - 1) k_1 (k_2 - k_1) \frac{M^8}{F^4} \lambda - (N - 1)(k_2 - k_1)^2 \frac{M^8}{F^4} \lambda \\
+ 2(N - 1) \hat{c}_0 \frac{M^8}{F^2} \lambda + \hat{d}_0 M^8 + \mathcal{O}(p^{10}). \tag{A.28} \]

Note that the leading contribution of order \( p^2 \) (the term \(-F^2 M^2\)) is finite, while all other terms are divergent as they contain \( \lambda \) as well as unrenormalized (infinite) subleading effective constants and the counterterms \( c_1 \) and \( c_3 \). However, these divergences can be ‘annihilated’ order-by-order in the effective expansion. For instance, at next-to-leading order \( p^4 \), the pole in \( \lambda \) is removed by renormalizing the combination \( k_2 + k_3 \) of effective constants from \( \mathcal{L}_{\text{eff}}^4 \). Then, the effective constants \( \hat{c}_0, \hat{c}_1 \) and \( \hat{d}_0 \) that originate from \( \mathcal{L}_{\text{eff}}^6 \) and \( \mathcal{L}_{\text{eff}}^8 \), absorb further infinities.

Next we consider all terms linear in the kinematical functions \( g_r \). Remarkably, all these contributions can be merged into a single kinematical function—namely \( g_0 \)—by expressing \( g_0 \) through the renormalized Goldstone boson mass \( M_\pi \),

\[ M_\pi^2 = \frac{\Sigma_\pi H_\pi}{F^2} + [2(k_2 - k_1) + (N - 3) \lambda] \left( \frac{\Sigma_\pi H_\pi}{F^6} + c \frac{(\Sigma_\pi H_\pi)^3}{F^{10}} + \mathcal{O}(H_\pi^4) \right), \tag{A.29} \]

rather than through the leading-order Goldstone boson mass \( M \),

\[ M^2 = \frac{\Sigma_\pi H_\pi}{F^2}. \]

In the course of this renormalization process, one Taylor expands \( g_0 \) according to

\[ g_0(M_\pi, T) = g_0(M, T) - \{ \epsilon_1 + \epsilon_2 + \mathcal{O}(M^8) \} g_1(M, T) + \frac{1}{2} \{ \epsilon_1^2 + \mathcal{O}(M^{10}) \} g_2(M, T), \]

\[ \epsilon_1 = [2(k_2 - k_1) + (N - 3) \lambda] \frac{M^4}{F^2}, \quad \epsilon_2 = \frac{M^6}{F^4}. \tag{A.30} \]

Note that the relation

\[ \text{doi:10.1088/1742-5468/2016/09/093102} \]
has been used. The constant $c$ in equation (A.29) corresponds to a rather lengthy expression containing terms involving $\lambda$, subleading effective constants, as well as the counterterms $c_2$ and $c_4$ (see equation (A.27)). While the explicit expression is not needed, we point out that the constant $c$—or $\varepsilon_2$—is finite. As far as $\varepsilon_1$ is concerned, there are only two terms: the pole in $\lambda$ can be absorbed by the combination $k_2 - k_1$ of next-to-leading order (NLO) effective constants. In the present case ($N = 2$), a renormalized effective constant $k$ can be defined as

$$k = 2(k_2 - k_1) - \lambda,$$  

(A.32)  

such that the renormalized Goldstone boson mass $M_\pi$ takes the form

$$M_\pi^2 = \frac{\Sigma_0 H_0}{F^2} + \frac{c(\Sigma_0 H_0)^2}{F^6} + \frac{c}{F^{10}} + O(H_0^4) \quad (N = 2).$$  

(A.33)  

After these manipulations, the final result for the contributions linear in the kinematical functions simply is

$$z^{[1]} = -\frac{1}{2} (N - 1) g_0(M_\pi, T).$$  

(A.34)  

Collecting all terms that are quadratic in the kinematical functions, we obtain

$$z^{[2]} = \frac{1}{8} (N - 1)(N - 3) \frac{M^2}{F^2} (g_1)^2 + C_1(N - 1) \frac{M^4}{F^4} (g_1)^2$$

$$+ C_2(N - 1) \frac{1}{F^4} (g_0)^2 + C_3(N - 1) \frac{M^2}{F^4} g_0 g_1,$$  

(A.35)  

with coefficients $C_1, C_2$ given by

$$C_1 = \frac{1}{2} (N - 1)^2 \lambda + \frac{1}{768\pi^2} (3N^2 + 32N - 67) - (N + 1)(\varepsilon_1 + \varepsilon_2) + k_1 - k_2;$$

$$C_2 = 5(N - 2)\lambda + \frac{3}{16\pi^2} (N - 2) - 3(2\varepsilon_1 + N\varepsilon_2).$$  

(A.36)  

In the kinematical functions we have again replaced the bare mass by the renormalized mass: $g_r(M, T) \rightarrow g_r(M_\pi, T)$, $r = 0, 1$. Note that in the present case ($N = 2$) the dependence on the singular quantity $\lambda$ drops out in $C_2$—one concludes that the sum $\varepsilon_1 + \varepsilon_2$ of NLO effective constants is finite. On the other hand, the pole in $\lambda$ contained in $C_1$ is absorbed by renormalizing the combination $k_2 - k_1$, as before in equation (A.32) that refers to mass renormalization. For $N = 2$, we thus have

$$z^{[2]} = -\frac{1}{8} \frac{M^2}{F^2} (g_1)^2 + \hat{C}_1 \frac{M^4}{F^4} (g_1)^2 + \hat{C}_2 \frac{1}{F^4} (g_0)^2 + \hat{C}_2 \frac{M^2}{F^4} g_0 g_1 \quad (N = 2),$$  

(A.37)  

with

doi:10.1088/1742-5468/2016/09/093102
\[ \hat{C}_1 = \frac{3}{256\pi^2} - 3(e_1 + e_2) - \frac{1}{2} \mathcal{F}, \]
\[ \hat{C}_2 = -6(e_1 + e_2). \] (A.38)

Note that in the terms proportional to \( C_1 \) and \( C_2 \) in equation (A.35), powers of \( M^2 \) can be replaced by powers of \( M_\pi^2 \); taking into account the difference \( M_\pi^2 - M^2 \) in these expressions is beyond order \( p^8 \) that we aim at in the present study. The exception is the first term in equation (A.35) where \( M^2 \) must be kept.

Finally collecting all terms cubic in the kinematical functions, along with the contributions involving the three-loop integrals \( \bar{J}_1 \) and \( \bar{J}_2 \), leads to

\[ z^{[3]} = \frac{1}{48} (N - 1)(N - 3)(3N - 7) M_\pi^2 \left( \frac{g_1}{\mathcal{F}} \right)^3 - \frac{1}{16} (N - 1)(N - 3)^2 M_\pi^4 \left( \frac{g_1}{\mathcal{F}} \right)^2 g_2 \]
\[ + \frac{1}{48} (N - 1)(N - 3) M_\pi^4 \bar{J}_1 - \frac{1}{4} (N - 1)(N - 2) \frac{1}{\mathcal{F}^4} \bar{J}_2. \] (A.39)

Here it is legitimate to express everything in terms of \( M_\pi \), rather than \( M \), since the error one introduces is beyond order \( p^8 \).

After this lengthy exercise, we obtain the following representation for the free energy density up to order \( p^8 \) (and for general \( N \geq 2 \)):

\[ z = z_0 - \frac{1}{2} (N - 1)g_0 - 4\pi a (g_1)^3 - \pi b g + \frac{1}{\mathcal{F}^4} I + \mathcal{O}(p^{10}). \] (A.40)

While the function \( g \) is a combination of the kinematical functions \( g_0 \) and \( g_1 \),

\[ g = 3g_0 (g_0 + M_\pi^2 g_1), \] (A.41)

the quantities \( a \) and \( b \) involve effective constants from the next-to-leading order piece \( \mathcal{L}_{\text{eff}}^4 \),

\[ a = -\frac{(N - 1)(N - 3) \Sigma_\pi H_\pi}{32\pi M_\pi^4} + \frac{N - 1}{4\pi} \frac{(\Sigma_\pi H_\pi)^2}{\mathcal{F}^8} \left\{ [(N + 1)(e_1 + e_2) + k_2 - k_1] \right. \]
\[ - \left. \frac{(N - 1)^2}{2} \lambda - \frac{3N^2 + 32N - 67}{768\pi^2} \right\}, \]

\[ b = \frac{N - 1}{\pi \mathcal{F}^4} \left\{ (2e_1 + Ne_2) - \frac{5(N - 2)}{3} \lambda - \frac{N - 2}{16\pi^2} \right\}. \] (A.42)

Finally, the function \( I \) reads

\[ I = \frac{1}{48} (N - 1)(N - 3) M_\pi^4 \bar{J}_1 - \frac{1}{4} (N - 1)(N - 2) \bar{J}_2 \]
\[ - \frac{1}{16} (N - 1)(N - 3)^2 M_\pi^4 (g_1)^2 g_2 + \frac{1}{48} (N - 1)(N - 3)(3N - 7) M_\pi^2 (g_1)^3. \] (A.43)

One notices that the evaluation of \( \bar{J}_2 \) is not needed for \( N = 2 \). Following the convention of [82], we rewrite \( I \) as

\[ \text{doi:10.1088/1742-5468/2016/09/093102} \]
which defines the dimensionless function $j$ that we have evaluated numerically (see appendix B). One then ends up with the representation

$$z = z_0 - \frac{1}{2} (N - 1) g_0 - 4 \pi a(g_1)^2 - \pi g \left[ b - \frac{j}{\pi^3 F^4} \right] + \mathcal{O}(\rho^{10}).$$

For the $d=3+1$ quantum $XY$ model ($N=2$), the quantities $I, a, b$ amount to

$$I = \frac{1}{48} M^2 g_1, \quad a = \frac{1}{32 \pi} \frac{\sum H_s}{F^4} + \frac{1}{16} \frac{\sum H_s}{F^8} \left\{ 3(e_1 + e_2) + k_2 - k_1 - \frac{1}{2} \lambda - \frac{3}{256 \pi^2} \right\},$$

$$b = \frac{2}{\pi F^4} (e_1 + e_2).$$

Note again that the combination $e_1 + e_2$ of NLO effective constants appearing in $b$ is finite. On the other hand, the pole in $\lambda$ showing up in $a$ can be absorbed into the combination $k_2 - k_1$ of bare NLO effective constants that get renormalized according to equation (A.32). The final expression for $a$ hence is

$$a = \frac{1}{32 \pi} \frac{\sum H_s}{F^4} + \frac{1}{4 \pi} \frac{\sum H_s}{F^8} \left\{ 3(e_1 + e_2) + \frac{1}{2} k - \frac{3}{256 \pi^2} \right\} \quad (N = 2).$$

**Appendix B. Evaluation of the cateye diagram in $d = 3 + 1$**

In this appendix we consider the renormalization and numerical evaluation of the cateye graph $\Sigma C$ of figure 1. The relevant contributions in $\Sigma C$, equation (A.11), are the singular three-loop integrals $J_1$ and $J_2$ that are defined in equation (A.22). How to extract the finite and physical pieces from these divergent expressions has been described in detail in [82]. The outcome is summarized in equation (A.27), where the quantities $J_1$ and $J_2$ are finite.

In the free energy density, $J_1$ and $J_2$ are contained in the function $I$ that we have defined in equation (A.43). In the present case of the $d = 3 + 1$ quantum $XY$ model ($N = 2$), only the first contribution ($J_1$) is relevant.

The explicit expression for the renormalized integral $J_1$ takes the form (for details see [82])

$$J_1 = \int_{\mathcal{T} \mathcal{S}} d^4 x \ U + \int_S d^4 x \ V - \int_{\mathcal{R} \mathcal{S}} d^4 x \ W,$$

$$U = G^4,$$

$$V = \mathcal{G}^4 + 4 \mathcal{G}^3 \Delta + 6 (\mathcal{G}^2 - g_1^2) \Delta^2,$$

$$W = 6 \mathcal{G}^2 \Delta^2 + 4 g_1 c h (M_X \Delta) \Delta^3 + \Delta^4.$$
The quantities $G$, $\bar{G}$, $\Delta$ and $g_1$ are defined in equations (A.1), (A.2), (A.23) and (A.25). All terms in the above representation for $\bar{J}_1$ are finite and refer to the limit $d \to 4$. Note that the case $d \to 3$ has been described in [51] which was devoted to the quantum $XY$ model in $d = 2 + 1$.

The functions $G(x)$, $\bar{G}(x)$ and $\Delta(x)$ only depend on the variables $r = |x|$ and $t = x_4$. The above integrals are therefore two-dimensional,

$$d^4x = 4\pi r^2 dr dt, \quad (B.2)$$

and can be evaluated straightforwardly.

The function $G(x)$ is a modified Bessel function of the second kind. In terms of the dimensionless variables $\xi$ and $\eta$,

$$\xi = T|x|, \quad \eta = Tx_4, \quad (B.3)$$

we have

$$G(x) = \sigma \sum_{n=-\infty}^{\infty} \frac{K_0(z \sigma)}{z}, \quad z = 2\pi \sqrt{(\eta + n)^2 + \xi^2}. \quad (B.4)$$

Notice that the radius of the sphere $|S| \leq \beta/2$ (with $\beta = 1/T$) that occurs in the representation for $\bar{J}_1$, is arbitrary. As a consequence, the result for $\bar{J}_1$ must be independent thereof. This provides us with a very useful consistency check regarding the numerical evaluation of the above integrals. By choosing different sizes of the sphere, we have checked that the final result for $\bar{J}_1$ indeed is independent of $|S|$.

As described in the previous appendix (and following [82]), the renormalized integral $\bar{J}_1$ is contained in the function $I (N = 2)$,

$$I = -\frac{1}{48} M^4_\pi \bar{J}_1 - \frac{1}{16} M^4_\pi (g_1)^2 g_2 + \frac{1}{48} M^2_\pi (g_1)^3,$$

that is rewritten by defining the dimensionless function $j = j(\sigma)$, $\sigma = M_\pi/2\pi T$, as
Thermodynamics of the $d = 3 + 1$ quantum $XY$ model

$$I = \frac{1}{\pi^2} g j.$$  

This function $j$ then appears in the free energy density and in all thermodynamic observables derived from there. A graph for the function $q(\sigma)$,

$$T^8 q(\sigma) = -\frac{1}{48} M_4^4 j_1,$$  

is depicted in figure B1.

**Appendix C. Estimation of NLO effective constants**

The leading-order effective Lagrangian involves $F^2$ and two derivatives, while $L^4_{\text{eff}}$ involves the NLO constants $e_i$ and four derivatives. The derivatives correspond to powers of momenta, where the momenta are small compared to a given underlying scale $\Lambda$. We thus have

$$L^2_{\text{eff}} \propto \frac{F^2}{2} p^2 = \frac{F^2}{2} \Lambda^2 \left( \frac{p^2}{\Lambda^2} \right),$$

$$L^4_{\text{eff}} \propto e_i p^4 = e_i \Lambda^4 \left( \frac{p^4}{\Lambda^4} \right).$$  

With respect to $L^2_{\text{eff}}$, contributions from $L^4_{\text{eff}}$ are suppressed by $p^2/\Lambda^2$, such that we obtain

$$\frac{F^2}{2} \Lambda^2 \approx e_i \Lambda^4,$$  

or

$$e_i \approx \frac{F^2}{2\Lambda^2}.$$  

The obvious question is which underlying scale $\Lambda$ we should choose. In analogy to quantum chromodynamics where this non-Goldstone boson scale can be identified with the mass of the $\rho$-resonance, let us consider the ferrimagnet. The spectrum of this condensed matter system is characterized by both acoustic and optical spin-wave excitations. While the former are Goldstone bosons, the latter are not and are characterized by an energy gap—a typical value is $\Delta E \approx 10J$ [83]. As a typical non-Goldstone boson scale in ferrimagnets one may thus choose $\Lambda = \Delta E \approx 10J$. Although there are no optical spin-wave branches in the quantum $XY$ model, by analogy, we may still choose the representative scale as $\Lambda = 10J$.

Now in order to estimate the value of the effective constant $F$ that also appears in equation (C.3), we invoke the critical temperature $T_c$ where the order parameter drops
to zero. The low-temperature expansion of the order parameter, equation (4.11), at leading order and in the absence of an external field, reduces to

$$\Sigma_s(T) = \Sigma_s(1 - \frac{1}{24F^2T^2}).$$  \hspace{1cm} (C.4)

The condition $\Sigma_s(T) = 0$ then leads to

$$\frac{F^2}{T_c^2} = \frac{1}{24},$$  \hspace{1cm} (C.5)

or

$$F = 0.20\ T_c.$$  \hspace{1cm} (C.6)

For the simple cubic lattice we have $T_c \approx 2.02J$ [1], such that

$$F \approx 0.41\ J.$$  \hspace{1cm} (C.7)

Hence for the NLO effective constants $e_i$ we obtain the estimate\(^5\)

$$e_i \approx \frac{F^2}{2\Lambda^2} \approx 0.001.$$  \hspace{1cm} (C.8)

In contrast to $e_1$ and $e_2$, the combination $k$ of NLO effective constants requires logarithmic renormalization. However our convention equation (A.32) also leads to

$$k \approx \frac{F^2}{2\Lambda^2} \approx 0.001.$$  \hspace{1cm} (C.9)

Since the above reasoning is based on a comparison with the ferrimagnet, the arguments may be considered as ‘hand-waiving’. Still, it is reasonable to conclude that the actual values of these NLO effective constants are within the range $10^{-3} \ldots 10^{-2}$.

References

[1] Betts D D, Elliott C J and Lee M H 1970 Can. J. Phys. 48 1566
[2] Betts D D and Lothian J R 1973 Can. J. Phys. 51 2249
[3] Lee M H 1973 Phys. Rev. B 8 1203
[4] Tsai J T and Elliott C J 1973 Phys. Lett. A 45 295
[5] Dekeyser R and Rogiers J 1975 Physica A 81 72
[6] Lee M H 1975 Phys. Rev. B 12 276
[7] Betts D D 1977 Physica B 86-8 556
[8] Oitmaa J and Betts D D 1978 Phys. Lett. A 68 450
[9] Rogiers J, Betts D D and Lookman T 1978 Can. J. Phys. 56 420
[10] Uchinami M, Takada S and Takano F 1979 J. Phys. Soc. Japan 47 1047
[11] Kim I M and Lee M H 1979 Phys. Rev. B 19 5815
[12] Aoki T, Homma S and Nakano H 1980 Prog. Theor. Phys. 64 448
[13] Brahmachari R 1980 Phys. Lett. A 76 165

\(^5\) It should be noted that we estimate the magnitude of these NLO effective constants—their signs remain inconclusive.
Thermodynamics of the $d = 3 + 1$ quantum XY model

[14] Aoki T, Homma S and Nakano H 1981 Prog. Theor. Phys. 66 861
[15] Lee M H 1983 Physica A 119 504
[16] Gomez-Santos G and Joannopoulos J D 1987 Phys. Rev. B 36 8707
[17] Hasenbusch M and Meyer S 1990 Phys. Lett. B 241 238
[18] Thoma S, Frey E and Schwabl F 1991 Phys. Rev. B 43 5831
[19] WeiHong Z, Oitmaa J and Hamer C J 1991 Phys. Rev. B 44 11869
[20] Gottlob A P, Hasenbusch M and Meyer S 1993 Nucl. Phys. B 30 838 (Proc. Suppl.)
[21] Zhang L 1993 Phys. Rev. 47 14364
[22] Oitmaa J, Hamer C J and WeiHong Z 1994 Phys. Rev. B 50 3877
[23] Tominaiga S and Yoneyama H 1995 Phys. Rev. B 51 8243
[24] Gouvea M E and Pires A S T 1996 Phys. Rev. B 54 14907
[25] Betts D D and Stewart G E 1997 Can. J. Phys. 75 47
[26] Schulenburg J, Flynn J S, Betts D D and Richter J 2001 Eur. Phys. J. B 21 191
[27] Cucchieri A, Engels J, Holtmann S, Mendes T and Schulze T 2002 J. Phys. A: Math. Gen. 35 6517
[28] Kleinert H 1989 Gauge Fields in Condensed Matter vol 1 (Singapore: World Scientific)
[29] Dyson F J 1956 Phys. Rev. 102 1217
[30] Dyson F J 1956 Phys. Rev. 102 1230
[31] Zittartz J 1965 Z. Phys. 184 506
[32] Hofmann C P 2002 Phys. Rev. B 65 094430
[33] Hofmann C P 2011 Phys. Rev. B 84 064414
[34] Radolevič S M, Pantić M R, Pavković-Hrvojević M V and Kapor D V 2013 Ann. Phys. 339 382
[35] Radolevič S M 2015 Ann. Phys. 362 336
[36] Hofmann C P arXiv:1510.08930
[37] Leutwyler H 1994 Phys. Rev. D 49 3033
[38] Andersen J O, Brauner T, Hofmann C P and Vuorinen A 2014 J. High Energy Phys. JHEP08(2014)088
[39] Gerber U, Hofmann C P, Kämpfer F and Wiese U-J 2010 Phys. Rev. B 81 064414
[40] Hofmann C P 2013 Phys. Rev. B 87 184420
[41] Hofmann C P 2014 Physica B 442 81
[42] Hofmann C P 2012 Phys. Rev. B 86 054409
[43] Hofmann C P 2012 Phys. Rev. B 86 184409
[44] Hofmann C P 2014 ISRN Thermodyn. 2014 546198
[45] Hofmann C P 1999 Phys. Rev. B 60 388
[46] Roman J M and Soto J 1999 Int. J. Mod. Phys. B 13 755
[47] Roman J M and Soto J 1999 Ann. Phys. 273 37
[48] Hasenfratz P and Niedermayer F 1991 Phys. Lett. B 268 231
[49] Hasenfratz P and Niedermayer F 1993 Z. Phys. B 92 91
[50] Hofmann C P 2010 Phys. Rev. B 81 014416
[51] Hofmann C P 2014 J. Stat. Mech. P02006
[52] Hofmann C P 2016 Nucl. Phys. B 904 348
[53] Hasenfratz P and Leutwyler H 1990 Nucl. Phys. B 343 241
[54] Hofmann C P 1999 Phys. Rev. B 60 406
[55] Roman J M and Soto J 2000 Phys. Rev. B 62 3390
[56] Kämpfer F, Moser M and Wiese U-J 2005 Nucl. Phys. B 729 317
[57] Brügger C, Kämpfer F, Moser M, Pepe M and Wiese U-J 2006 Phys. Rev. B 74 224432
[58] Brügger C, Kämpfer F, Pepe M and Wiese U-J 2006 Eur. Phys. J. B 53 433
[59] Brügger C, Hofmann C P, Kämpfer F, Pepe M and Wiese U-J 2007 Phys. Rev. B 75 014421
[60] Brügger C, Hofmann C P, Kämpfer F, Moser M, Pepe M and Wiese U-J 2007 Phys. Rev. B 75 214405
[61] Brügger C, Hofmann C P, Kämpfer F, Pepe M and Wiese U-J 2008 Physica B 403 1447
[62] Jiang F-J, Kämpfer F, Hofmann C P and Wiese U-J 2009 Eur. Phys. J. B 69 473
[63] Vlasii N D, Hofmann C P, Jiang F-J and Wiese U-J 2012 Phys. Rev. B 86 155113
[64] Kämpfer F, Bessiere B, Wirtz M, Hofmann C P, Jiang F-J and Wiese U-J 2012 Phys. Rev. B 85 075123
[65] Vlasii N D, Hofmann C P, Jiang F-J and Wiese U-J 2015 Ann. Phys., NY 354 213
[66] Wiese U-J and Ying H P 1994 Z. Phys. B 93 147
[67] Gerber U, Hofmann C P, Jiang F-J, Nyfeler M and Wiese U-J 2009 J. Stat. Mech. P03021
[68] Jiang F-J and Wiese U-J 2011 Phys. Rev. B 83 155120
[69] Jiang F-J 2011 Phys. Rev. B 83 024419
[70] Gerber U, Hofmann C P, Jiang F-J, Palma G, Stebler P and Wiese U-J 2011 J. Stat. Mech. P06002
[71] Hamer C J, Oitmaa J and WeiHong Z 1991 Phys. Rev. B 43 10789

doi:10.1088/1742-5468/2016/09/093102
Thermodynamics of the $d = 3 + 1$ quantum XY model

[72] Brauner T 2010 Symmetry 2 609
[73] Burgess C P 2007 Annu. Rev. Nucl. Part. Sci. 57 329
[74] Leutwyler H 1995 Hadron Physics 94—Topics on the Structure and Interaction of Hadronic Systems ed V E Herscovitz et al (Singapore: World Scientific) p 1
[75] Wielinga R F, Lubbers J and Huiskamp W J 1967 Physica 37 375
[76] Algra H A, de Jongh L J, Huiskamp W J and Carlin R L 1976 Physica B 83 71
[77] Algra H A, de Jongh L J, Huiskamp W J and Reedijk J 1977 Physica B 86–8 737
[78] Bartolome J, Algra H A, de Jongh L J and Carlin R L 1978 Physica B 94 60
[79] van der Bilt A, Joung K O, Carlin R L and de Jongh L J 1981 Phys. Rev. B 24 445
[80] Carlin R L, Carnegie D W Jr, Bartolome J, Gonzalez D and Floria L M 1985 Phys. Rev. B 32 7476
[81] Johnston D C, McQueeney R J, Lake B, Honecker A, Zhitomirsky M E, Nath R, Furukawa Y, Antropov V P and Singh Y 2011 Phys. Rev. B 84 094445
[82] Gerber P and Leutwyler H 1989 Nucl. Phys. B 321 387
[83] Brooks Harris A 1963 Phys. Rev. 132 2398