NONLINEAR INDENTATION OF SECOND-ORDER HYPERELASTIC MATERIALS

A PREPRINT

Yangkun Du
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Peter Stewart
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Nicholas A Hill
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Huabing Yin
Biomedical Engineering, School of Engineering
University of Glasgow
Glasgow G12 8LT, UK

Raimondo Penta
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Jakub Köry
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Xiaoyu Luo
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

Raymond Ogden
School of Mathematics and Statistics
University of Glasgow
Glasgow G12 8QQ, UK

September 14, 2022

ABSTRACT

The classical problem of indentation on an elastic substrate has found new applications in the field of the Atomic Force Microscopy. However, linearly elastic indentation models are not sufficiently accurate to predict the force-displacement relationship at large indentation depths. For hyperelastic materials, such as soft polymers and biomaterials, a nonlinear indentation model is needed. In this paper, we use second-order elasticity theory to capture larger amplitude deformations and material nonlinearity. We provide a general solution for the contact problem for deformations that are second-order in indentation amplitude with arbitrary indenter profiles. Moreover, we derive analytical solutions by using either parabolic or quartic surfaces to mimic a spherical indenter. The analytical prediction for a quartic surface agrees well with finite element simulations using a spherical indenter for indentation depths on the order of the indenter radius. In particular, the relative error between the two approaches is less than 1% for an indentation depth equal to the indenter radius, an order of magnitude less than that observed with models which are either first-order in indentation amplitude or those which are second-order in indentation amplitude but with a parabolic indenter profile.

Keywords Nonlinear indentation · Contact problem · Second-order elasticity · Hertz model · Hyperelasticity · Incompressibility

*corresponding author: yangkun.du@glasgow.ac.uk
1 Introduction

Understanding and quantifying the mechanical characteristics of soft materials, including elastomers and bio-tissues, is of great importance in many engineering applications [Chaudhuri et al., 2020, Gensbittel et al., 2021, Tian et al., 2020, Du et al., 2020]. Unlike hard materials, these soft materials, such as hydrogel and cells, are either too fragile or too small to implement traditional macroscopic stretch and compression tests. However, the emergence of Atomic Force Microscopy (AFM) (Figure 1) enables us to characterize the mechanical response of these soft materials through local nano-indentation tests [Krieg et al., 2019, Liang et al., 2020], measuring the force required to produce a given displacement. Material constants, such as Young’s modulus and the relaxation modulus, can be extracted by fitting the experimental data to theoretical indentation models [Chim et al., 2018, Efremov et al., 2017].

The classical Hertz model [Hertz, 1881] is one of the most widely used theoretical models for elucidating the load-displacement relationship in a frictionless indentation test [Johnson, 1982]. Assuming that the contact surface is a small elliptical region while the indentation depth is infinitesimal compared to the scale of the sample, Hertz solved the contact problem by applying the Boussinesq approximation with spatially distributed normal stress [Lai et al., 2009]. Over the years, a number of refinements to the Hertz model have been proposed for specific considerations, including substrate effects, friction, adhesive stress, viscoelasticity [Rheinlaender et al., 2020, Borodich, 2014, Spence, 1975, Storákers and Elaguine, 2005, Jin et al., 2013, Chim et al., 2018, Wang et al., 2020], and unknown contact conditions for nonlinear materials [Chang and Liu, 2018].

In a different approach, Sneddon [1965] put forward a general analytical solution to the indentation problem in terms of dual-integral functions with an arbitrary indenter profile. This solution is consistent with the Hertz model when the indenter is of paraboloid shape. In addition, Sneddon also provided an analytical expression for the load-displacement relationship in terms of the material constants and contact radius when the indenter has a hemispherical shape. However, both the Hertz and Sneddon approaches were focused on finding the solution to the Boussinesq problem when the deformations are infinitesimal and the substrate is a linearly elastic half-space. However, for soft materials, the displacements can become large under moderate loads, and the stress-strain relationship is unlikely to be linearly elastic. In addition, Zhang and Yang [2017] used finite element (FE) simulations to investigate the impact of large deformations and material nonlinearity on the indentation model of hyperelastic samples.

Based on their FE simulations, Zhang et al. [2014a] proposed explicit empirical load-displacement relationships for several hyperelastic materials through a dimensional analysis approach. Moreover, robust nonlinear indentation models have been applied to materials which exhibit a layered structure [Chen and Diebels, 2012], poroelasticity [Duan et al., 2012], and plasticity [Song and Komvopoulos, 2013] by fitting to FE simulations. However, these numerical-simulation-based finite indentation models are not universally applicable since they rely significantly on the particulars of the FE models, including the material, geometry, mesh, and boundary conditions. Therefore, a general theoretical nonlinear indentation model for hyperelastic materials is needed.
From the perspective of mathematical modelling, nonlinear indentation problems are significantly more localized and complicated. It seems unlikely that a fully nonlinear finite deformation model can be established for a general nonlinear (or even hyperelastic) material. However, we can include larger deformations by extending the linear elasticity approach to higher-order deformation amplitudes as in the weakly nonlinear procedure proposed by Rivlin [1953]. In particular, by assuming that the second-order terms in perturbation amplitude induced by products of first-order displacements can introduce additional body forces and surface tractions that can be satisfied by the second-order displacements, Rivlin showed that the second-order nonlinear boundary value problem could be reduced to two linear boundary value problems in classical elasticity theory.

By combining this method with the first-order analytical solution of Sneddon [1965], which indeed cannot always be used to describe nanoindentation of non-linear elastic materials, Zhang et al. [2014b], Sabin and Kaloni [1983] presented a general solution of the indentation problem up to second-order deformations, using a paraboloid to approximate to the analytical solution for a hemispherical indenter. Moreover, Giannakopoulos and Triantafyllou [2007] repeated the calculation of Sabin and Kaloni [1983], and obtained a different load-displacement relationship by specifying the third-order material constants in terms of the Lamé constants. However, both their analytical approaches exhibited an overestimation of the indentation force compared to FE simulations and experimental data [Liu et al., 2010].

In this paper, we revisit the second-order indentation problem of Sabin and Kaloni [1983], correcting several of their expressions and finding a result that agrees significantly better with the numerical calculations. In particular, the second-order nonlinear boundary value problem is reduced to two linear elastic boundary value problems. Based on the first-order solution constructed by Sneddon [1965], and introducing the integral transform method, the general solutions are expressed in the form of Hankel transforms of potential functions. To mimic the spherical indentation more accurately, we further provide asymptotic analytical solutions using a higher-order quartic surface to approximate the spherical indenter. We also implement FE simulations to verify our second-order indentation models for incompressible neo-Hookean and Mooney-Rivlin materials. Finally, we discuss the limitations of this current second-order elasticity method in accounting for more sophisticated incompressible hyperelastic materials.

The paper is organised as follows. First, in Section 2, we recap the method for expanding the governing equation to second-order in indentation depth. Next, in Section 3, we present the general mathematical modelling of the finite indentation problem. Furthermore, in Section 4, we provide the (corrected) solution up to second-order in indentation amplitude for both parabolic and quartic indenter profiles. Finally, we implement FE simulations to verify the second-order analytical results in Section 5, and make concluding remarks in Section 6.

## 2 Second-order elasticity method

Suppose that an isotropic elastic body undergoes a nonlinear deformation, so that the point \( x_i \) is moved to \( x_i + u_i \), where \( u_i \) is the displacement vector. We define the deformation gradient tensor

\[
g_{ik} = \left( \delta_{is} + \frac{\partial u_i}{\partial x_s} \right) \left( \delta_{ks} + \frac{\partial u_k}{\partial x_s} \right),
\]

and its corresponding scalar invariants

\[
I_1 = g_{ss}, \quad I_2 = G_{ss}, \quad I_3 = \det g_{ik}.
\]

Then the stress components can be obtained as

\[
t_{ik} = \frac{2}{\tau} \left[ g_{ik} \frac{\partial W}{\partial I_1} - G_{ik} \frac{\partial W}{\partial I_2} + \left( I_3 \frac{\partial W}{\partial I_3} + I_2 \frac{\partial W}{\partial I_2} \right) \delta_{ik} \right],
\]

where \( \tau = \det(\delta_{ik} + \partial u_i/\partial x_k) \), \( W \) is the strain-energy function, \( G_{ik} \) is the co-factor matrix of \( g_{ik} \), and \( \delta_{ik} \) is the Kronecker delta. We shall assume that the displacement gradients are asymptotically small i.e. \( |\partial u_i/\partial x_k| \sim \varepsilon \), say, where \( 0 < \varepsilon \ll 1 \), and that the strain energy function is given by the third-order Murnaghan [1937] expansion

\[
W = a_0 J_1 + a_1 J_2 + a_2 J_1^2 + a_3 J_1 J_2 + a_4 J_1^3 + a_5 J_3,
\]

where \( a_0, ..., a_5 \) are material constants, and \( J_1 = I_1 - 1, J_2 = I_2 - 2I_1 + 3, J_3 = I_3 - I_2 + I_1 - 1 \) are three other independent scalar invariants that are respectively \( O(\varepsilon), O(\varepsilon^2) \), and \( O(\varepsilon^3) \). In addition, \( a_0 = 0 \) if the undeformed configuration is stress-free, while \( a_1 \) and \( a_2 \) are related to the Lamé constants \( \lambda \) and \( \mu \) by

\[
a_1 = -\mu/2, \quad a_2 = (\lambda + 2\mu)/8,
\]

and to the Young modulus \( E_Y \) and Poisson’s ratio \( \eta \) by

\[
a_1 = -\frac{1}{4} \frac{E_Y}{1 + \eta}, \quad a_2 = \frac{1}{8} E_Y \frac{1 - \eta}{(1 + \eta)(1 - 2\eta)}.
\]
In \([6]\), we suppose that \(-1 < \eta < 1/2\) and will address the limit \(\eta \to 1/2\) for an incompressible material below. From equations (3) and (4), the stress components up to second order of quantities \(\partial u_i / \partial x_k\), i.e., \(O(\varepsilon^2)\), are

\[
t_{ik} = 2 \left\{ -a_1 e_{ik} + 2 (a_1 + 2 a_2) \Delta \delta_{ik} \right\} + \left\{ (4 a_2 - 2 a_3 + a_1) \Delta e_{ik} - a_1 \alpha_{ik} - (a_1 - a_3) E_{ik} \right\} + \left\{ (a_1 + 2 a_2) \alpha + (a_1 + a_3) E + 2 (6 a_4 + 2 a_3 - a_1 - 2 a_2) \Delta^2 \right\} \delta_{ik},
\]

where

\[
e_{ik} = \frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}, \quad \Delta = \varepsilon_{ss}/2, \quad \alpha_{ik} = \frac{\partial u_i}{\partial x_k} \frac{\partial u_k}{\partial x_i}, \quad \alpha = \alpha_{ss}, \quad E = E_{ss},
\]

and \(E_{ik}\) is the cofactor matrix of \(e_{ik}\).

Furthermore, up to \(O(\varepsilon^2)\), we can expand the displacement field as

\[
u_i = v_i + w_i,
\]

where the \(v_i = O(\varepsilon)\) and \(w_i = O(\varepsilon^2)\). Hence, the stress to \(O(\varepsilon^2)\) can be separated as

\[
t_{ik} = \tau_{ik} + \tau_{ik}' + \tau_{ik}''.
\]

where

\[
\tau_{ik} = 2 \left\{ -a_1 e_{ik}' + 2 (a_1 + 2 a_2) \Delta' \delta_{ik} \right\}
\]

is the first-order stress component,

\[
\tau_{ik}' = 2 \left\{ (4 a_2 - 2 a_3 + a_1) \Delta' e_{ik} - a_1 \alpha_{ik}' - (a_1 - a_3) E_{ik}' \right\} + \left\{ (a_1 + 2 a_2) \alpha' + (a_1 + a_3) E' + 2 (6 a_4 + 2 a_3 - a_1 - 2 a_2) \Delta'^2 \right\} \delta_{ik},
\]

\[
\tau_{ik}'' = 2 \left\{ -a_1 e_{ik}'' + 2 (a_1 + 2 a_2) \Delta'' \delta_{ik} \right\},
\]

are the second-order stress components, and

\[
e_{ik}' = \frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}, \quad \Delta' = \varepsilon_{ss}'/2, \quad e_{ik}'' = \frac{\partial w_i}{\partial x_k} + \frac{\partial w_k}{\partial x_i}, \quad \Delta'' = \varepsilon_{ss}''/2,
\]

\[
\alpha_{ik}' = \frac{\partial v_i}{\partial x_k} \frac{\partial v_k}{\partial x_i}, \quad \alpha' = \alpha_{ss}', \quad E = E_{ss}', \quad \text{and} \quad E_{ik}'\text{ is the cofactor matrix of } e_{ik}'.
\]

Then, the equilibrium equation and the boundary conditions to \(O(\varepsilon^2)\) are

\[
\frac{\partial \tau_{ik}}{\partial x_k} + \frac{\partial \tau_{ik}''}{\partial x_k} + \left[ \Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] \frac{\partial \tau_{ik}}{\partial x_s} + \left[ \Delta'' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] \frac{\partial \tau_{ik}''}{\partial x_s} + \rho_0 X_i = 0,
\]

\[
\mathcal{T}_i = \varepsilon l_k \tau_{ik} + \left[ \Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] l_s \tau_{ik} + l_k (\tau_{ik}' + \tau_{ik}''),
\]

where \(X_i\) and \(\mathcal{T}_i\) are the body force and surface traction associated with the first-order displacement \(v_i\), respectively. Following [Rivlin 1953], the second-order terms of the equilibrium equation and the boundary condition induced by the first-order displacement \(v_i\) can be considered as an additional body force \(X_i'\) and the surface traction \(\mathcal{T}_i\), respectively,

\[
\rho_0 X_i' = \left[ \Delta' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] \frac{\partial \tau_{ik}}{\partial x_s} + \left[ \Delta'' \delta_{sk} - \frac{\partial v_s}{\partial x_k} \right] \frac{\partial \tau_{ik}''}{\partial x_s},
\]

\[
\mathcal{T}_i' = - \left[ \Delta \delta_{sik} - \frac{\partial v_s}{\partial x_k} \right] l_s \tau_{ik} - l_k \tau_{ik}',
\]

where \(l_k\) are the direction-cosines of the normal to the deformed surface of the body. The additional body force and surface traction give rise to a second-order deformation. Thus, the equilibrium equation and the boundary condition of this second-order elastic problem can be reduced to two linear elastic problems of at \(O(\varepsilon)\) and \(O(\varepsilon^2)\), respectively,

\[
\frac{\partial \tau_{ik}}{\partial x_k} + \rho_0 X_i = 0 \quad \text{and} \quad \mathcal{T}_i = l_k \tau_{ik},
\]

\[
\frac{\partial \tau_{ik}''}{\partial x_k} + \rho_0 X_i' = 0 \quad \text{and} \quad \mathcal{T}_i' = l_k \tau_{ik}''.
\]
3 Mathematical modelling of the nonlinear indentation

As shown in Figure 2, suppose that a hyperelastic half-space body is approached by a rigid axisymmetric indenter with an arbitrary profile \( f(r) \). The deformed half-space is defined in terms of the cylindrical coordinates \(( r, \theta, z)\) centred with the indenter. The contact radius \( a \) is fixed, and we shall determine the corresponding at indentation depth \( D \), see Figure 2. In addition, we assume that there is no internal body force within the half-space, and that the interface is frictionless. Hence, at the contact surface \( z = 0 \) in the deformed configuration, we have the boundary conditions

\[
 u_z(r, 0) = D - f(r) \quad (0 \leq r \leq a), \\
 t_{zz}(r, 0) = 0 \quad (r > a), \\
 t_{rz}(r, 0) = 0 \quad (r \geq 0). \tag{17}
\]

We further assume that radius of curvature of the tip of the axisymmetric indenter is \( R \gg a \), and thus set \( \varepsilon = a/R \) as the small parameter. Similar to the expansion of the displacement and the stress fields, the indentation depth can be also expanded as

\[
 D = D_1 + D_2, \tag{18}
\]

where \( D_1 = O(\varepsilon) \) and \( D_2 = O(\varepsilon^2) \), and \( 0 \leq f(r) \leq D \). Hence, the boundary conditions \(17\) of the first-order \( O(\varepsilon) \) deformation are

\[
 v_z(r, 0) = D_1 - f(r) \quad (0 \leq r \leq a), \\
 \tau_{zz}(r, 0) = 0 \quad (r > a), \\
 \tau_{rz}(r, 0) = 0 \quad (r \geq 0), \tag{19}
\]

and those of the second-order \( O(\varepsilon^2) \) deformation are

\[
 w_z(r, 0) = D_2 \quad (0 \leq r \leq a), \\
 \tau_{zz}''(r, 0) = -\tau_z' \quad (r > a), \\
 \tau_{rz}''(r, 0) = -\tau_r' \quad (r \geq 0). \tag{20}
\]

Here we assume that the shape of the indenter (e.g. spherical) is such that \( f(r) \) does not contribute terms at \( O(\varepsilon^2) \). Following Rivlin [1953], we solve this second-order contact problem in two steps. First, we obtain the first-order solution that satisfies the boundary condition (19) based on classical elasticity theory. Second, having calculated additional second-order body force and surface traction from the first-order solution, we find the solution satisfies the boundary condition (20), also making use of classical elasticity theory.

In the following subsections, we present the complete derivation process for the general solution of the second-order indentation problem, referencing and correcting Sabin and Kaloni [1983]'s work. For better understanding and consistency, we adopt consistent notation and provide more details.
3.1 The first-order solution

Sneddon [1965] constructed a general analytical solution of the first-order contact problem with the equilibrium equation (16) and boundary condition (19), using an integral transform method. The solution of the first-order displacement is

\[ v_r(r, z) = -(1 - 2\eta)G_0(r, z) + zG_1(r, z), \]
\[ v_z(r, z) = 2(1 - \eta)F_0(r, z) + zF_1(r, z), \] (21)

where

\[ F_i(r, z) = H_0 \left[ \xi^{-1+i}\psi(\xi)e^{-\xi z}; \xi \to r \right], \quad G_i(r, z) = H_1 \left[ \xi^{-1+i}\psi(\xi)e^{-\xi z}; \xi \to r \right], \]
\[ \psi(\xi) = \frac{1}{2(1 - \eta)} \int_0^a \chi(t) \cos(\xi t)dt, \quad \chi(t) = \frac{2D_1}{\pi} - \frac{2t}{\pi} \int_0^r f'(x) dx, \] (22)

and \( H_j[g(x); x \to \zeta] = \int_0^\infty xg(x)J_j(\zeta x)dx \) is the \( j^{th} \)-order Hankel transform of function \( g(x) \). In addition, following Sneddon [1965]. \( \chi(a) = 0 \) is required to make sure that \( \tau_{zz}(a, 0) \) tends to a finite limit. Then, combining the equations (6), (11), (13), and (22), we obtain the first-order stress components

\[ \tau_{rr}(r, z) = -2\mu \left[ F_1(r, z) - zF_2(r, z) - (1 - 2\eta) \frac{G_0(r, z)}{r} + zG_1(r, z) \right], \]
\[ \tau_{\theta\theta}(r, z) = -2\mu \left[ 2\eta F_1(r, z) + (1 - 2\eta) \frac{G_0(r, z)}{r} - zG_1(r, z) \right], \]
\[ \tau_{zz}(r, z) = -2\mu \left[ F_1(r, z) + zF_2(r, z) \right], \quad \tau_{rz}(r, z) = -2\mu zG_2(r, z). \] (23)

Moreover, according to equation (12), the second-order stress components at the contact surface \( z = 0 \) induced by the first-order deformation are given by

\[ \tau_{rr}'(r, 0) = 2(1 - 2\eta)^2 \left[ a_1 + 4(a_2 + 3a_3 + 12a_4) \right] F_1^2(r, 0) \]
\[ - 8(1 - 2\eta)^2(a_1 + 5a_2 - 3a_3 - a_5) F_1(r, 0)G_0(r, 0) \]
\[ - 2(1 - 2\eta)^2(3a_1 - 4a_2 + 4a_3) \frac{G_0^2(r, 0)}{r^2} + 8(1 - \eta)^2(a_1 + 4a_2)G_1^2(r, 0), \]
\[ \tau_{\theta\theta}'(r, 0) = \frac{2(1 - 2\eta)^2}{r^2} \left[ 2(-3a_1 + 2(5a_3 + 12a_4) - 3a_2) \right] F_1^2(r, 0) \]
\[ + 2(5a_1 + 6a_2 - 2(a_3 + a_5))rF_1(r, 0)G_0(r, 0) \]
\[ + (4a_2 - 4a_3 - 3a_1)G_0^2(r, 0) \]
\[ \tau_{zz}'(r, 0) = 2(1 - 2\eta)^2 \left[ a_1 + 4(a_2 + 3a_3 + 12a_4) \right] F_1^2(r, 0) \]
\[ + 4(1 - 2\eta)^2(a_1 + 2a_2 - 2a_3 - 2a_5) \frac{G_0^2(r, 0)}{r^2} - rF_1(r, 0)G_0(r, 0) \]
\[ + 8(1 - \eta)^2(a_1 + 4a_2)G_1^2(r, 0), \]
\[ \tau_{rz}'(r, 0) = \frac{4a_3}{r} (1 - 2\eta)(1 - \eta)G_1(r, 0)G_0(r, 0). \] (24)

The additional body force, \( \rho_0 X_r' \), and the surface traction, \( T_r' \), are given by

\[ \rho_0 X_r' = \frac{2}{r} (1 - 2\eta)^2 \left[ a_1 - 4a_2 + 4a_3 \right] \left[ F_1(r, 0) - \frac{2}{r} G_0(r, 0) \right]^2 \]
\[ + 2(1 - 2\eta)G_2(r, 0) \left[ (1 - 2\eta)(3a_1 + 20a_2 - 12a_3 - 4a_5) \right] \]
\[ + 2(3a_1 - 2a_5) \frac{G_0(r, 0)}{r} + 2(a_1 + 8a_2 - 4a_3) \]
\[ - (1 - 2\eta)(a_1 + 4a_2 + 12a_3 + 48a_4) \frac{F_1(r, 0)}{r} \]
\[ - 8(1 - \eta)G_1(r, 0) \left[ a_1 - 4a_2 \right] F_2(r, 0) + \frac{4a_2}{r} G_1(r, 0) \], (25)
\[ T_r' = -4\mu(1 - \eta)G_1(r, 0) \left[ F_1(r, 0) - (1 - 2\eta) \frac{G_0(r, 0)}{r} \right] + \tau_{rz}'(r, 0), \]
\[ T_0' = 0, \quad T_z' = 2\mu(1 - 2\eta)F_1^2(r, 0) + \tau_{zz}'(r, 0). \]
3.2 The second-order solutions

We have calculated the additional body force and the surface traction \( (25) \) based on the first-order solutions. Now, we construct the solution of the second linear problem with the equilibrium equation \((16)_2\) and boundary condition \((20)\).

The equilibrium equation \((16)_2\) can be expanded to give

\[
\begin{align*}
\frac{\partial \tau''_r}{\partial r} + \frac{\partial \tau''_z}{\partial z} + \frac{\tau''_r - \tau''_0}{r} + \varrho_0 X'_r &= 0, \\
\frac{\partial \sigma''_r}{\partial r} + \frac{\partial \sigma''_z}{\partial z} + \frac{\sigma''_r}{r} + \varrho_0 X'_z &= 0.
\end{align*}
\]

(26)

Since the system \((26)\) is linear, the solution can be decomposed into the sum of three separate linear problems in terms of the stresses \(\sigma''_{ik}, \sigma''_{ik}, \sigma''_{ik}\) and the displacements \(w'_i, w''_i, w''''_i\), written as

\[
\tau''_{ik} = \sigma''_{ik} + \sigma''_{ik} + \sigma''_{ik}, \quad w_i = w'_i + w''_i + w''''_i.
\]

(27)

Thus, the equilibrium equation \((26)\) and the boundary condition \((20)\) are equivalent to the sum of linear problem \((i)\),

\[
\begin{align*}
\frac{\partial \sigma'_{rr}}{\partial r} + \frac{\partial \sigma'_{rz}}{\partial z} + \frac{\sigma'_{rr} - \sigma'_{0r}}{r} + \varrho_0 X'_r &= 0 & (w'_r(r,0) = 0 \quad (0 \leq r \leq a)) \\
\frac{\partial \sigma'_{rz}}{\partial r} + \frac{\partial \sigma'_{zz}}{\partial z} + \frac{\sigma'_{rz}}{r} + \varrho_0 X'_z &= 0 & (\sigma'_r(r,0) = -T'_r \quad (r > a)).
\end{align*}
\]

(28)

linear problem \((ii)\),

\[
\begin{align*}
\frac{\partial \sigma''_{rr}}{\partial r} + \frac{\partial \sigma''_{rz}}{\partial z} + \frac{\sigma''_{rr} - \sigma''_{0r}}{r} &= 0 & (w''_r(r,0) = 0 \quad (0 \leq r \leq a)) \\
\frac{\partial \sigma''_{rz}}{\partial r} + \frac{\partial \sigma''_{zz}}{\partial z} + \frac{\sigma''_{rz}}{r} &= 0 & (\sigma''_r(r,0) = -T'_r \quad (r > a)).
\end{align*}
\]

(29)

and linear problem \((iii)\),

\[
\begin{align*}
\frac{\partial \sigma'''_{rr}}{\partial r} + \frac{\partial \sigma'''_{rz}}{\partial z} + \frac{\sigma'''_{rr} - \sigma'''_{0r}}{r} &= 0 & (w'''_r(r,0) = D_2 \quad (0 \leq r \leq a)) \\
\frac{\partial \sigma'''_{rz}}{\partial r} + \frac{\partial \sigma'''_{zz}}{\partial z} + \frac{\sigma'''_{rz}}{r} &= 0 & (\sigma'''_r(r,0) = 0 \quad (r > a)).
\end{align*}
\]

(30)

This separation helps to simplify the calculations. Furthermore, these linear elastic problems can be solved by using Papkovitch–Neuber potential function method [Lai et al., 2009]. The general solution of the displacement vector for linear elastostatic problems is given by

\[
w = \nabla (\varphi + x \cdot \psi) - 4(1-\eta)\psi,
\]

(31)

where \(\varphi\) is a scalar function, and \(\psi\) is a vector function. With the decomposition (31), the general equilibrium equation \((16)_2\) can be rewritten as

\[
\frac{2\mu(1-\eta)}{(1-2\eta)} \left[ x_n \frac{\partial \nabla^2 \psi_n}{\partial x_i} + \nabla^2 \psi_1 + \frac{\partial \nabla^2 \varphi}{\partial x_i} \right] - 4\mu(1-\eta)\nabla^2 \psi_1 + g' X_i = 0.
\]

(32)

For this axisymmetric problem in cylindrical coordinates, the two potential functions can be specified as \(\varphi = \varphi(r, z)\) and \(\psi = (0, 0, \psi(r, z))\). Hence, the equilibrium equation \((16)_2\) and (32) are equivalent to

\[
\nabla^2 \psi = \frac{\varrho_0}{4\mu(1-\eta)} \left[ X'_z + \frac{\partial S}{\partial z} \right] = K_1(r, z),
\]

\[
\nabla^2 \varphi = \frac{\varrho_0}{4\mu(1-\eta)} \left[ -z \frac{\partial S}{\partial z} + 2(1-2\eta)S - z X'_z \right] = K_2(r, z),
\]

(33)

where \(S(r, z) = \int_r^\infty X'_i(r', z)dr'\). Moreover, from Sabin and Kaloni [1983], the two potential functions are given by

\[
\psi(r, z) = H_0 \left[ (B(\xi) + K'_1(\xi, z)) e^{-\xi z}; \xi \to r \right],
\]

\[
\varphi(r, z) = H_0 \left[ (A(\xi) + K'_2(\xi, z)) e^{-\xi z}; \xi \to r \right],
\]

(34)
where $A(\xi)$ and $B(\xi)$ are arbitrary functions that need to be determined from the boundary conditions, and

\[
K_1^*(\xi, z) = \int_0^z e^{2\xi_z} \int_0^{z_1} e^{-\xi_z} \int_0^{r_1} r K_1(r, z_1) J_0(\xi r) \, dr \, dz_1 \, dz_2,
\]

\[
K_2^*(\xi, z) = \int_0^z e^{2\xi_z} \int_0^{z_1} e^{-\xi_z} \int_0^{r_1} r K_2(r, z_1) J_0(\xi r) \, dr \, dz_1 \, dz_2.
\]

(35)

Then, combining Eqs. (11), (12), (13), and (31), we obtain the general solutions of the second-order displacement $w$ and the corresponding stress components as

\[
w_z = \frac{\partial \varphi}{\partial z} + z \frac{\partial \psi}{\partial z} - (3 - 4\eta) \psi, \quad \tau''_{zz} = 2\mu \frac{\partial \varphi}{\partial r} \left[ \frac{\partial \varphi}{\partial z} + z \frac{\partial \psi}{\partial z} - (1 - 2\eta) \psi \right],
\]

\[
\tau''_{zz} = 2\mu \left[ \frac{\partial^2 \varphi}{\partial z^2} + z \frac{\partial^2 \psi}{\partial z^2} - 2(1 - \eta) \frac{\partial \psi}{\partial z} + \frac{\eta}{1 - 2\eta} \left( \nabla^2 \varphi + z \nabla^2 \psi \right) \right].
\]

(36)

Note that (36) is the general solution for the three separate linear problems (28), (29), and (30). Next, we derive the functions $A(\xi)$ and $B(\xi)$ in (34) by applying the boundary conditions to each of the systems (i), (ii), and (iii).

### 3.2.1 Solving linear problem (i)

First, based on the equation (34), we construct the partial derivatives of the potential functions $\varphi(r, z)$ and $\psi(r, z)$ at the contact surface $z = 0$ given by

\[
\left. \frac{\partial \psi}{\partial z} \right|_{z=0} = - H_0 [\xi B(\xi); \xi \to r], \quad \left. \frac{\partial \psi}{\partial r} \right|_{z=0} = - H_1 [\xi B(\xi); \xi \to r],
\]

\[
\left. \frac{\partial^2 \psi}{\partial r^2} \right|_{z=0} = - H_0 \left[ \xi^2 B(\xi); \xi \to r \right] + \frac{1}{r} H_1 [\xi B(\xi); \xi \to r],
\]

\[
\left. \frac{\partial^2 \psi}{\partial z^2} \right|_{z=0} = H_0 \left[ \xi^2 B(\xi); \xi \to r \right] + H_0 \left[ \int_0^r r' K_1(r', 0) J(\xi r') \, dr', \xi \to r \right],
\]

\[
\left. \frac{\partial^2 \psi}{\partial r \partial z} \right|_{z=0} = H_1 \left[ \xi^2 B(\xi); \xi \to r \right],
\]

and

\[
\left. \frac{\partial \varphi}{\partial z} \right|_{z=0} = - H_0 [\xi A(\xi); \xi \to r], \quad \left. \frac{\partial \varphi}{\partial r} \right|_{z=0} = - H_1 [\xi A(\xi); \xi \to r],
\]

\[
\left. \frac{\partial^2 \varphi}{\partial r^2} \right|_{z=0} = - H_0 \left[ \xi^2 A(\xi); \xi \to r \right] + \frac{1}{r} H_1 [\xi A(\xi); \xi \to r],
\]

\[
\left. \frac{\partial^2 \varphi}{\partial z^2} \right|_{z=0} = H_0 \left[ \xi^2 A(\xi); \xi \to r \right] + H_0 \left[ \int_0^r r' K_2(r', 0) J(\xi r') \, dr', \xi \to r \right],
\]

\[
\left. \frac{\partial^2 \varphi}{\partial r \partial z} \right|_{z=0} = H_1 \left[ \xi^2 A(\xi); \xi \to r \right].
\]

(37)

(38)

For the linear problem (i) of (28), based on (36), (37), and (38), we have

\[
\left. w'_{zz} \right|_{z=0} = - H_0 [\xi A(\xi) + (3 - 4\eta) B(\xi); \xi \to r] = 0 \quad (0 \leq r \leq a),
\]

\[
\left. \sigma'_{zz} \right|_{z=0} = 2\mu H_0 \left[ \xi^2 A(\xi) + 2(1 - \eta) \xi B(\xi); \xi \to r \right] + \phi_0 \int_0^\infty X_1' (r', 0) \, dr'
\]

\[
= - T' \quad (r > a),
\]

\[
\left. \sigma'_{rzz} \right|_{z=0} = 2\mu \xi \{ H_1 [\xi A(\xi) + (1 - 2\eta) B(\xi); \xi \to r] \} = 0 \quad (r \geq 0).
\]

(39)

The Hankel transform of (39) yields

\[
B(\xi) = \frac{\xi A(\xi)}{1 - 2\eta}.
\]

(40)
Then, the boundary condition of displacement \( w'_z \) and the normal stress \( \sigma'_{zz} \) in (39) can be rewritten as

\[
\begin{align*}
\left. w'_z \right|_{z=0} &= 2(1 - \eta) \frac{H_0}{1 - 2\eta} H_0[\xi A(\xi); \xi \rightarrow r] = 0 \quad (0 \leq r \leq a), \\
\left. \sigma'_{zz} \right|_{z=0} &= -\frac{2\mu}{1 - 2\eta} H_0 \left[ \xi^2 A(\xi); \xi \rightarrow r \right] + \varrho_0 \int_r^\infty X'_r (r', 0) \, dr \\
&= -T'_z \quad (r > a),
\end{align*}
\]

to which we can further derive the governing equation of \( A(\xi) \) in the form

\[
\begin{align*}
H_0[\xi A(\xi); \xi \rightarrow r] &= 0 \quad (0 \leq r \leq a), \\
H_0 \left[ \xi^2 A(\xi); \xi \rightarrow r \right] &= \frac{1 - 2\eta}{2\mu} \left[ T'_z + \varrho_0 \int_r^\infty X'_r (r', 0) \, dr \right] \quad (r > a).
\end{align*}
\]

Following Sneddon [1960], Eq. (42) can be satisfied if

\[
A(\xi) = \xi^{-2} \int_a^\infty \beta(t) \cos(\xi t) \, dt,
\]
\[
\beta(t) = \frac{2}{\pi} \frac{1 - 2\eta}{2\mu} \int_0^t \frac{r T'_z(r, 0)}{\sqrt{t^2 - r^2}} + \varrho_0 X'_r(r, 0) \sqrt{r^2 - t^2} \, dr.
\]

Hence, by combining Eqs. (39), (40), and (43), the final solutions for \( w'_z \) and \( \sigma'_{zz} \) of the linear problem (i) in Eq. (28) are

\[
\begin{align*}
\left. w'_z \right|_{z=0} &= \begin{cases} 0, & (0 \leq r \leq a), \\
\frac{2(1 - \eta)}{1 - 2\eta} \int_a^r \frac{\beta(t)}{\sqrt{t^2 - r^2}} \, dt, & (r > a),
\end{cases} \\
\left. \sigma'_{zz} \right|_{z=0} &= \begin{cases} \frac{2\mu}{1 - 2\eta} \int_a^\infty \frac{t \beta(t)}{(t^2 - r^2)^{3/2}} \, dt + \varrho_0 \int_r^\infty X'_r (r', 0) \, dr', & (0 \leq r \leq a), \\
\int_r^\infty \frac{\beta'(t)}{\sqrt{t^2 - r^2}} \, dt = -T'_z, & (r > a).
\end{cases}
\end{align*}
\]

### 3.2.2 Solving linear problem (ii)

The linear problem (ii) of Eq. (29) can be solved by the same approach as in Section 3.2.1 for linear problem (i). First, based on (36), (37), and (38), we have

\[
\begin{align*}
\left. w''_z \right|_{z=0} &= -H_0[\xi A(\xi) + (3 - 4\eta)B(\xi); \xi \rightarrow r] = 0 \quad (0 \leq r \leq a), \\
\left. \sigma''_{zz} \right|_{z=0} &= 2\mu H_0 \left[ \xi^2 A(\xi) + 2(1 - \eta)B(\xi); \xi \rightarrow r \right] = 0 \quad (r > a), \\
\left. \sigma'_{zz} \right|_{z=0} &= 2\mu \xi \left( H_1[\xi A(\xi) + (1 - 2\eta)B(\xi); \xi \rightarrow r] \right) = -T'_z \quad (r \geq 0).
\end{align*}
\]

According to Eq. (45), we have

\[
B = -\frac{\xi A}{(1 - 2\eta)} - \frac{\xi^{-1}}{2\mu(1 - 2\eta)} Q(\xi),
\]

where \( Q(\xi) = H_1 \left[ T'_z; r \rightarrow \xi \right] \). Then, the boundary condition for the displacement \( w''_z \) and the normal stress \( \sigma''_{zz} \) in Eq. (45) can be rewritten as

\[
\begin{align*}
\left. w''_z \right|_{z=0} &= \frac{2(1 - \eta)}{1 - 2\eta} H_0[\xi A(\xi); \xi \rightarrow r] - \frac{3 - 4\eta}{2\mu(1 - 2\eta)} H_0 \left[ \xi^{-1} Q(\xi); \xi \rightarrow r \right] = 0 \quad (0 \leq r \leq a), \\
\left. \sigma''_{zz} \right|_{z=0} &= -\frac{2\mu}{1 - 2\eta} H_0 \left[ \xi^2 A(\xi); \xi \rightarrow r \right] + \frac{2(1 - \eta)}{1 - 2\eta} H_0[Q(\xi); \xi \rightarrow r] = 0 \quad (r > a),
\end{align*}
\]

from which we can further derive the governing equation of \( A(\xi) \) as

\[
\begin{align*}
H_0[\xi A(\xi); \xi \rightarrow r] &= \frac{3 - 4\eta}{4\mu(1 - \eta)} H_0 \left[ \xi^{-1} Q(\xi); \xi \rightarrow r \right] \quad (0 \leq r \leq a), \\
H_0 \left[ \xi^2 A(\xi); \xi \rightarrow r \right] &= \frac{1 - \eta}{\mu} H_0[Q(\xi); \xi \rightarrow r] \quad (r > a).
\end{align*}
\]
However, it is not straightforward to explicitly obtain the solution \( A(\xi) \) from (48). Instead, if we assume that \( \xi^2 A(\xi) = \frac{1 - \eta}{\mu} Q(\xi) + T(\xi) \), then the governing equation (48) can be further reduced to

\[
H_0 \left[ \xi^{-1} T(\xi); \xi \to r \right] = -\frac{(1 - 2\eta)^2}{4\mu(1 - \eta)} Q(\xi) H_0 \left[ \xi^{-1} Q(\xi); \xi \to r \right], \quad (0 \leq r \leq a),
\]

\[
H_0[T(\xi); \xi \to r] = 0, \quad (r > a).
\]

Then, following Sneddon [1960], Eq. (49) is satisfied if

\[
T(\xi) = \int_0^\infty \gamma(t) \cos(\xi t) dt,
\]

\[
\gamma(t) = \frac{(1 - 2\eta)^2}{2\pi \mu(1 - \eta)} \left[ \int_0^\infty \widetilde{T}_r'(r, 0) dr - \int_0^t \frac{t T_0'(r, 0)}{\sqrt{t^2 - r^2}} dr \right].
\]

Hence, by combining Eqs. (45), (46), and (50), the final solutions for \( w''_z \) and \( \sigma''_{zz} \) of the linear problem (ii) in Eq. (29) are given by

\[
w''_z|_{z=0} = \begin{cases} 
0, & (0 \leq r \leq a), \\
\frac{2(1 - \eta)}{1 - 2\eta} \int_0^a \gamma(t) dt - \frac{(1 - 2\eta)}{2\mu} \int_0^\infty \widetilde{T}_r'(0, 0) d\xi, & (r > a),
\end{cases}
\]

\[
\sigma''_{zz}|_{z=0} = \begin{cases} 
-\frac{2\mu}{1 - 2\eta} \left( \frac{\gamma(a)}{\sqrt{a^2 - r^2}} - \int_0^a \frac{\gamma(t)}{\sqrt{t^2 - r^2}} dt \right), & (0 \leq r \leq a), \\
0, & (r > a).
\end{cases}
\]

3.2.3 Solving linear problem (iii)

The linear problem (iii) of Eq. (30) corresponds to the well-known linear contact problem with a flat indenter and zero body force. Hence, we can adopt the solution of Sneddon [1965], which gives

\[
w''_z|_{z=0} = \begin{cases} 
\frac{D_2}{\pi} \arcsin(a/r), & (0 \leq r \leq a), \\
\frac{2D_2}{\pi} \arcsin(a/r), & (r > a),
\end{cases}
\]

\[
\sigma''_{zz}|_{z=0} = \begin{cases} 
-\frac{2\mu D_2}{\pi(1 - \eta)} (a^2 - r^2)^{-1/2}, & (0 \leq r \leq a), \\
0, & (r > a).
\end{cases}
\]

3.3 Closing the second-order elastic problem

In the proceeding subsection, we have derived the required second-order solutions at the contact surface \( z = 0 \). By linear superposition, the final solutions of this second-order contact problem are the sums of these separate solutions. For the displacement \( u_z \) and \( t_{zz} \), these are given by

\[
u_z = v_z + w'_z + w''_z + w'''_z,
\]

\[
t_{zz} = \tau_{zz} + \sigma'_zz + \sigma''_{zz} + \sigma'''_{zz}.
\]

In addition, the applied force \( P \) is

\[
P = -2\pi \int_0^a r t_{zz}|_{z=0} dr.
\]

4 Asymptotic solution for spherical indentation

In this section, we focus on one of the most common indentation problems using a spherical indenter. For the rigid spherical indenter with radius \( R \), the profile function is

\[
f(r) = R - \sqrt{R^2 - r^2}.
\]

Sneddon [1965] obtained the first-order analytical solution, but in this case, we have not been able to find the second-order analytical solution with this spherical function. The explicit integrals could not be found for Eq. (22). In the following subsections, we derive two asymptotic analytical solutions using parabolic and quartic surfaces.
Asymptotic solutions using a parabolic surface

As a simplification of the indenter profile adopted by Hertz [1881], we use an axisymmetric paraboloid to obtain an asymptotic analytical solution of spherical indentation up to the second-order in perturbation amplitude. As shown in Figure 3B, the spherical profile function \( f(r) \) can be approximated by a parabolic function

\[
f(r) = \frac{r^2}{2R} + O \left( \frac{r^4}{R^2} \right).
\]  

(56)

Furthermore, given \( \varepsilon = a/R \) and substituting (56) into (22), we obtain

\[
\chi(t) = \frac{2}{\pi} \left( D_1 - \varepsilon \frac{t^2}{a} \right),
\]  

(57)

by which, recalling that \( \chi(a) = 0 \) is required, we find

\[
D_1 = \varepsilon a.
\]  

(58)

Hence, according to (23), we have

\[
\psi(\xi) = \frac{2\varepsilon (\sin(a\xi) - a\xi \cos(a\xi))}{\pi a (1 - \eta) \xi^3}.
\]  

(59)

Based on (22) and (22), \( F_i(r, z) \) and \( G_i(r, z) \) are given by

\[
F_i(r, z) = \int_0^\infty \xi^i \psi(\xi) e^{-\xi z} J_0(\xi r) d\xi,
\]

\[
G_i(r, z) = \int_0^\infty \xi^i \psi(\xi) e^{-\xi z} J_1(\xi r) d\xi.
\]  

(60)

At \( z = 0 \), (60) can be rewritten in the form of the Weber-Sonin-Schafheitlin integral [Korenev 2002]:

\[
F_i(r, 0) = \sqrt{\frac{2a}{\pi}} \frac{\varepsilon}{(1 - \eta)} \int_0^\infty \frac{J_{3/2}(a\xi) J_0(\xi r)}{\xi^{3/2 - 1}} d\xi,
\]

\[
G_i(r, 0) = \sqrt{\frac{2a}{\pi}} \frac{\varepsilon}{(1 - \eta)} \int_0^\infty \frac{J_{3/2}(a\xi) J_1(\xi r)}{\xi^{3/2 - 1}} d\xi.
\]  

(61)

See Appendix A for the details. Recalling Eqs. (53), for \( r > a \), we have the displacement field

\[
u_z(r, 0) = \frac{\varepsilon}{\pi a} \left( \left( r^2 - 2a^2 \right) \arcsin \left( \frac{a}{r} \right) - \frac{a \sqrt{r^2 - a^2}}{r} \right) + \nu'_z(r, 0) + \nu''_z(r, 0) + \frac{2D_2}{\pi} \arcsin \left( \frac{a}{r} \right),
\]  

(62)
where

\[
w'_z(r, 0) = -\frac{8\varepsilon^2 \sqrt{r^2 - a^2}}{3\pi^2 a_1} - \frac{(3a_1 - 4a_5) (1 - 2\eta) \varepsilon^2 a_4^4}{9\pi^2 a_1 (1 - \eta)} \frac{r \sqrt{r^2 - a^2}}{a^2} + \\
\ln \left( \frac{r + \sqrt{r^2 - a^2}}{a} \right) + \frac{16\varepsilon^2}{3\pi^2 a} \int_a^r \left[ 2a \left( t^2 - a^2 \right) I_1 + (3a^2 - 2t^2) I_2 \right] \frac{dt}{\sqrt{t^2 - r^2}},
\]

\[
w''_z(r, 0) = \frac{(1 - 2\eta) \varepsilon^2}{3\pi^2 (1 - \eta)} \left( \frac{a^2 \sqrt{r^2 - a^2}}{r^2} + a \arcsin \left( \frac{a}{r} \right) \right) - \frac{2(1 - 2\eta) \varepsilon^2 a}{3\pi^2 (1 - \eta)} I_4 \\
- \frac{(1 - 2\eta)^2 \varepsilon^2}{18\pi^2 a^2 (1 - \eta)^2} \left\{ 3a^3 (1 - 2\eta) (1 - 2 \ln a) \arcsin \left( \frac{a}{r} \right) \right. \\
\left. + \left[ (2\eta + 5) a^2 \sqrt{r^2 - a^2} + 6a^3 (1 - 2\eta) I_3 \right. \\
\left. + \sqrt{r^2 - a^2} \left( (8\eta + 2) a^2 - (2\eta + 5) r^2 \right) \arcsin \left( \frac{a^2 - r^2}{a^2 - a^2} \right) \right] \right\},
\]

and \( I_1, I_2, I_3, \) and \( I_4 \) are listed in Appendix C. Similarly, for \( r < a \), we have the stress component

\[
t_{zz}(r, 0) = \frac{8a_1 \varepsilon \sqrt{a^2 - r^2}}{\pi a (1 - \eta)} + t'_z(r, 0) + t'_z(r, 0) + \sigma''_{zz}(r, 0) \\
- \frac{2\mu D_2}{(1 - \eta) \pi \sqrt{a^2 - r^2}},
\]

where

\[
t'_z(r, 0) = \frac{2\varepsilon^2}{9a^2 (1 - \eta)^2} \left[ 36 (1 - 2\eta)^2 (a_1 + 4a_2 + 12a_3 + 48a_4) \left( a^2 - r^2 \right) \right. \\
\left. - 24 (1 - 2\eta)^2 (a_1 + 2a_2 - 2a_3 - 2a_5) \sqrt{a^2 - r^2} \left( a^3 - (a^2 - r^2)^{3/2} \right) \right] \\
- \frac{24 (1 - 2\eta)^2 (a_1 + 2a_2 - 2a_3 - 2a_5) \sqrt{a^2 - r^2} \left( a^3 - (a^2 - r^2)^{3/2} \right)}{\pi^2 r^2} \\
+ \frac{8(1 - 2\eta)^2 (a_1 + 2a_2 - 2a_3 - 2a_5) \left( a^3 - (a^2 - r^2)^{3/2} \right)^2}{\pi^2 r^4} \\
+ 9(1 - \eta)^2 r^2 (a_1 + 4a_2) \right],
\]
\[ \sigma'_{zz}(r,0) = \frac{4a^3(1-2\eta)^2\varepsilon^2(3a_1-4a_5)}{9\pi^2(1-\eta)^2(2a^3 + 2a^2\sqrt{a^2-r^2} - r^2\sqrt{a^2-r^2} - 2ar^2)} + \frac{16\mu\varepsilon^2I_7}{\pi^2a(1-\eta)} + \frac{8\mu\varepsilon^2}{3\pi^2(1-\eta)\sqrt{a^2-r^2}} + \frac{4\mu\varepsilon^2}{\pi^2(1-\eta)} + \frac{4\varepsilon^2}{9\pi^2(1-\eta)^2} \left[ 2a_1(\eta-2)(4\eta-5) - 8a_2(1-2\eta)^2 + 9\pi^2a_2(\eta-1) + 8a_3(1-2\eta)^2 \right] + \frac{48(1-2\eta)\varepsilon^2}{9\pi^2(1-\eta)^2} \left[ 4\eta(a_1 + 4a_2 - 3a_3 - a_5) - 5a_1 + 4(a_3 - 2a_2 + a_5) \right] \ln \left(1 + \sqrt{1 - \frac{r^2}{a}} \right) + \frac{8(1-2\eta)^2\varepsilon^2(a_1 - 4a_2 + 4a_3)(2a^4 - ar^2\sqrt{a^2-r^2} - 2a^3\sqrt{a^2-r^2})}{9\pi^2r^4(1-\eta)^2} + \frac{2\varepsilon^2[9\pi^2a_1(\eta-1) - 4a_1(28\eta^2 + 8\eta - 11)]}{9\pi^2(1-\eta)^2} + \frac{32(1-2\eta)\varepsilon^2[a_3(68\eta - 43) + 108a_4(2\eta - 1) + 3a_5(\eta-1)]}{9\pi^2(1-\eta)^2} - \frac{4a_2\varepsilon^2[9\pi^2(\eta-1) - 8(4\eta^2 - 40\eta + 19)]}{9\pi^2(1-\eta)^2} + \frac{2a_1\varepsilon^2r^2[80\eta^2 + 64\eta - 9\pi^2(\eta-1) - 52]}{9\pi^2a_2(1-\eta)^2} + \frac{32(2\eta-1)\varepsilon^2r^2[a_3(64\eta - 41) + 108a_4(2\eta - 1) + 3a_5(\eta-1)]}{9\pi^2a^2(1-\eta)^2} + \frac{4a_2\varepsilon^2r^2[8(4\eta^2 + 32\eta - 17) + 9\pi^2(\eta-1)]}{9\pi^2a^2(1-\eta)^2}, \]  

(67)

\[ \sigma''_{zz}(r,0) = \frac{2a_1(2\eta-1)^2\varepsilon^2}{9\pi^2a(1-\eta)^2\sqrt{a^2-r^2}} \left\{ a^2 \left[ \eta(22 - 12\ln 2) + 37 + 6\ln 2 \right] + 9a\sqrt{a^2-r^2}\left[ (2\eta - 5)I_6 - 2I_5 \right] - 9(2\eta + 5)r^2 \right\}, \]  

(68)

and \( I_5, I_6, I_7, \) are listed in Appendix C. To avoid the singularity of \( t_{zz} \) induced by the terms with \( (a^2 - r^2)^{-1/2} \) at \( r = a \), \( D_2 \) can be chosen as

\[ D_2 = \frac{\varepsilon D_1 (2\eta - 1)^2}{9\pi a_1 (\eta-1)} (3a_1 \ln 2 - 2a_1 + 4a_5) = \frac{\varepsilon D_1 (2\eta - 1)}{3\pi (\eta-1)} + \frac{4\varepsilon D_1 (\ln 2 - 1)}{3\pi}. \]  

(69)

Hence, the total indentation depth is given by

\[ D = D_1 + \frac{\varepsilon D_1 (2\eta - 1)^2}{9\pi a_1 (\eta-1)} (3a_1 \ln 2 - 2a_1 + 4a_5) - \frac{\varepsilon D_1 (2\eta - 1)}{3\pi (\eta-1)} + \frac{4\varepsilon D_1 (\ln 2 - 1)}{3\pi}. \]  

(70)

Based on Eq. (54), the total force \( P \) is given by

\[ P = \frac{16\mu a^2 a_1 \varepsilon}{3(\eta-1)} - \frac{a^2 \varepsilon^2}{9\pi(1-\eta)^2} \left\{ 9\pi^2a_1(1-\eta)^2 + 12a_1\eta(10\eta - 37) + 174a_1 
- (1 - 2\eta) \left[ 288(a_3 - 2a_2)(1-\eta) - 18a_2\pi^2(1-\eta) - 16a_5(\eta - 5) \right] \right\}. \]  

(71)

For incompressible materials \( \eta = 1/2 \), and the Lamé constants behave as

\[ \lambda = \frac{2\mu\eta}{1-2\eta} \to \infty, \quad \mu = \frac{E}{2(1+\eta)} \to \frac{E}{3}, \]  

(72)

which indicates that

\[ a_1 = \frac{\mu}{2} = -E/6, \quad a_2 = \frac{\lambda + 2\mu}{8} = O(\lambda/\mu) \to \infty. \]  

(73)
Moreover, according to Destrade and Ogden [2010], for incompressible material, we have

\[ a_3 = \mathcal{O}(\lambda/\mu), \quad a_4 = \mathcal{O}(\lambda^2/\mu^2), \quad a_5 = \mathcal{O}(\mu), \]

\[ a_3 - 2a_2 = a_1 - a_5 = \mathcal{O}(\mu). \] (74)

Note that the material constants \(a_3, a_4, a_5\) used in Giannakopoulos and Triantafyllou [2007] do not satisfy these constraints, since \(a_4\) was wrongly set to be of order \(\mathcal{O}(\lambda/\mu)\) and \(a_3 - 2a_2 \neq a_1 - a_5\).

Next, according to (73) and (74), for incompressible materials, the corresponding results reduce to

\[ u_z(r, 0) = \frac{\varepsilon}{\pi a} \left( (r^2 - 2a^2) \arcsin \left( \frac{a}{r} \right) - a \sqrt{r^2 - a^2} \right) - \frac{8\varepsilon^2 r^2 - a^2}{3\pi^2} + \frac{2D_2}{\pi} \arcsin \left( \frac{a}{r} \right) + \frac{16\varepsilon^2}{3\pi^3 a} \int_a^r [2a (t^2 - a^2) I_1 + (3a^2 - 2t^2) I_2 - (t^2 - a^2) \ln (t^2 - a^2) \arcsin \left( \frac{a}{t} \right)] d\frac{dt}{\sqrt{r^2 - t^2}}, (r > a) \] (75)

\[ t_{zz}(r, 0) = 2\mu \varepsilon^2 \left( \frac{a^2 - r^2}{a^2} + \frac{16I_9}{\pi^3 a^2} \right) - \frac{8\mu \varepsilon \sqrt{a^2 - r^2}}{\pi a}, (0 \leq r \leq a) \]

\[ D = \varepsilon a + \frac{4\varepsilon^2 a (\ln 2 - 1)}{3\pi} \]

\[ P = \frac{16}{3} \mu \varepsilon a^2 - \frac{4\mu \varepsilon^2 a^2}{\pi}. \]

It is worth noting that the analytical solutions (75) correct those given by both Giannakopoulos and Triantafyllou [2007] and Sabin and Kaloni [1983]. Giannakopoulos and Triantafyllou [2007] used inappropriate material constants which do not satisfy the incompressibility constraints. In addition, we found that Giannakopoulos and Triantafyllou [2007] used the same expression for the applied force \(P\) as Sabin and Kaloni [1983]. Furthermore, it has been found that compared to the numerical simulation results, the applied forces predicted by both Giannakopoulos and Triantafyllou [2007] and Sabin and Kaloni [1983] are significantly overestimated.

### 4.2 Asymptotic solutions using quartic surface

As shown in Figure 3D, the parabolic surface is not sufficiently accurate to approximate the spherical indenter as the indentation depth increases. Alternatively, following Liu et al. [2010], we can further expand the profile function \(f(r)\) up to quartic surface \(O(r^4/R^3)\), which gives

\[ f(r) = \frac{r^2}{2R} + \frac{r^4}{8R^3} + O \left( \frac{r^6}{R^5} \right). \] (76)

Substituting (76) into (22), we obtain

\[ \chi(t) = \frac{2D_1}{\pi} - \frac{2t \left( 3R^2 t + t^3 \right)}{3\pi R^3}. \] (77)

Recalling that the boundary conditions require \(\chi(a) = 0\), we get

\[ D_1 = \varepsilon a \left( 1 + \frac{\varepsilon^2}{3} \right). \] (78)

Note that following earlier authors, we include \(O(\varepsilon^3)\) terms in \(D\), instead of pursuing a formal expansion of \(D\) to \(O(\varepsilon^3)\) i.e. \(D = \varepsilon D_1 + \varepsilon^2 D_2 + \varepsilon^3 D_3 + ...\). Next, according to (22), we have

\[ \psi(\xi) = \frac{2\varepsilon}{3\pi a} (2\varepsilon^2 + 3) \sin(\alpha \xi) - 2\varepsilon \cos(\alpha \xi) \frac{\xi^3}{(1 - \eta)^3} + \frac{8\varepsilon^3}{3\pi a^3} \sin(\alpha \xi) - 3 \sin(\alpha \xi) + 3\alpha \xi \cos(\alpha \xi) \] (79)
Based on (79), \( F_i(r, z) \) and \( G_i(r, z) \) in terms of (60) at \( z = 0 \) can be rewritten as

\[
F_i(r, 0) = \sqrt{\frac{2a \varepsilon (3 + 2\varepsilon^2)}{3(1 - \eta)}} \int_0^\infty J_{3/2}(a \xi) J_0(r \xi) \frac{d\xi}{\xi^{3/2-i}} - \frac{2}{\pi a} \frac{4\varepsilon^3}{3(1 - \eta)} \int_0^\infty J_{5/2}(a \xi) J_0(r \xi) \frac{d\xi}{\xi^{5/2-i}}
\]

\[
G_i(r, 0) = \sqrt{\frac{2a \varepsilon (3 + 2\varepsilon^2)}{3(1 - \eta)}} \int_0^\infty J_{3/2}(a \xi) J_1(r \xi) \frac{d\xi}{\xi^{3/2-i}} - \frac{2}{\pi a} \frac{4\varepsilon^3}{3(1 - \eta)} \int_0^\infty J_{5/2}(a \xi) J_1(r \xi) \frac{d\xi}{\xi^{5/2-i}}
\]

(80)

See Appendix B for the details about \( F_i(r, 0) \) and \( G_i(r, 0) \), \( i = 0, 1, 2 \).

Next, to simplify the calculation in this case, we shall only provide the solution for incompressible materials. For \( 0 \leq r \leq a \), the stress component \( t_{zz} \) is given by

\[
t_{zz}(r, 0) = \frac{8a_1 \varepsilon \sqrt{a^2 - r^2}}{9\pi a^3(1 - \eta)} \left[ 2a^2(2\varepsilon^2 + 9) + 4r^2\varepsilon^2 \right] + \frac{r^2(a_1 + 4a_2)(2a^2\varepsilon + r^2\varepsilon^3)}{2a^6} \]

\[+ \frac{128a_4(1 - 2\eta)^2\varepsilon^2}{27\pi^2a^6(\eta - 1)^2} \left[ 2a^2(2\varepsilon^2 + 9) + 4r^2\varepsilon^2 \right] + \frac{a}{(1 - \eta)\pi\sqrt{a^2 - r^2}} \]

(81)

where

\[
\sigma'_{zz}(r, 0) = I_8 + \frac{e^2}{945\pi^2} \left[ 70r^6 \varepsilon^4 \left[ 9\pi^2(a_1 - 3a_2) + 4096a_4(1 - 2\eta)^2 \right] \right. \]

\[+ \frac{315r^4\varepsilon^2}{\pi a^4} \left[ 3\pi^2(a_1 - 2\varepsilon^2) + 27\pi^2a_4(1 - 2\eta)^2 \right] \]

\[- \frac{420r^2\varepsilon^4}{a^2} \left[ 128a_4(1 - 2\eta)^2(4e^4 + 9\varepsilon^2 + 27) - 9\pi^2(a_1 - 2\varepsilon^2) \right] + \frac{24a_1 \varepsilon^4}{\sqrt{a^2 - r^2}}(96\ln 2 - 38) + 21e^2(3 + 4\ln 2) - 420(2\ln 2 - 1) \]

\[\left. - 35 \left[ 9\pi^2 a_1(2\varepsilon^4 + 9\varepsilon^2 + 12) + 512a_4(1 - 2\eta)^2 (2\varepsilon^2 + 9)^2 \right] \right], \]

and \( I_8 \) is given in Appendix C. To avoid the singularity of \( t_{zz} \) at \( r = a \), \( D_2 \) can be chosen as

\[
D_2 = \frac{4\varepsilon^2 a(\ln 2 - 1)}{3\pi} - \frac{\varepsilon^4 a(4\ln 2 + 3)}{15\pi} - \frac{\varepsilon^6 a(96\ln 2 - 38)}{315\pi}
\]

(83)

and, therefore, we obtain the total indentation depth as

\[
D = a\varepsilon + \frac{4\varepsilon^2 a(\ln 2 - 1)}{3\pi} + \frac{\varepsilon^3 a}{3} - \frac{\varepsilon^4 a(4\ln 2 + 3)}{15\pi} - \frac{\varepsilon^6 a(96\ln 2 - 38)}{315\pi}
\]

(84)

In addition, according to (83), and (73), (74) for incompressible materials, \( t_{zz}(r, 0) \) in (81) can be further simplified to

\[
t_{zz}(r, 0) = I_8 - \frac{8\mu e\sqrt{a^2 - r^2}}{\pi a} + \frac{2\mu e^2}{a^2} (a^2 - r^2) - \frac{16\mu e^3}{9\pi a^3} \left( a^2 + 2r^2 \right) \sqrt{a^2 - r^2}
\]

\[+ \frac{3\mu e^4}{2a^4} (a^4 - r^4) + \frac{\mu e^6}{3a^6} (a^6 - r^6)
\]

(85)

Finally, based on Eq. (54), the total force \( P \) is given by

\[
P = \frac{16}{3} \mu a^2 e - \frac{4}{\pi} \mu a^2 e^2 + \frac{32}{15} \mu a^2 e^3 - \frac{16}{3\pi} \mu a^2 e^4 - \frac{96}{45\pi} \mu a^2 e^5
\]

(86)

where the higher-order terms are generated by the quartic profile function (76) and needed to avoid the singularity of \( \sigma'_{zz}(r, 0) \) that would otherwise appear at \( r = a \). Equation (79)–(86) contain terms that are \( O(\varepsilon^3) \) or higher, beyond the \( O(\varepsilon^2) \) expansion of the deformation field. Retaining these extra terms that arise from the indenter shapes greatly improves the agreement with the FE simulations.
Table 1: Summary of the indentation models for incompressible materials

| Name of model (Theoretical method used) | Analytical solutions of the force and displacement | Profile of the indenter $f(r)$ used in calculation |
|----------------------------------------|---------------------------------------------------|--------------------------------------------------|
| Sneddon model (First-order elasticity) | $D_S = \frac{1}{2} a \ln \left( \frac{R + a}{R - a} \right)$ | $f(r) = R - \sqrt{R^2 - r^2}$ |
|                                        | $P_S = 2\mu \left( a^2 + R^2 \right) \ln \left( \frac{R + a}{R - a} \right) - 2aR$ | |
| Hertz model (First-order elasticity)   | $D_H = \varepsilon a$ | $f(r) = \frac{r^2}{2R}$ |
|                                        | $P_H = \frac{16}{3} a^2 \varepsilon \mu$ | |
| Liu’s model (First-order elasticity)   | $D_L = \varepsilon a + \frac{\varepsilon^3 a}{3}$ | $f(r) = \frac{r^2}{2R} + \frac{r^4}{8R^3}$ |
|                                        | $P_L = \frac{16}{3} a^2 \varepsilon \mu + \frac{32}{15} a^2 \varepsilon^3 \mu$ | |
| Analytical parabolic (Second-order elasticity) | $D_P = \varepsilon a - \frac{4a\varepsilon^2}{3\pi} (1 - \ln 2)$ | $f(r) = \frac{r^2}{2R}$ |
|                                        | $P_P = \frac{16}{3} a^2 \varepsilon \mu - \frac{4a^2 \varepsilon^2 \mu}{\pi}$ | |
| Analytical quartic (Second-order elasticity) | $D_Q = \varepsilon a + \frac{\varepsilon^3 a}{3} - \frac{4a\varepsilon^2}{3\pi} (1 - \ln 2)$ | $f(r) = \frac{r^2}{2R} + \frac{r^4}{8R^3}$ |
|                                        | $- \frac{a^4}{15\pi} (3 + 4\ln 2) + \frac{a^6}{315\pi} (38 - 96 \ln 2)$ | |
|                                        | $P_Q = \frac{16}{3} a^2 \varepsilon \mu - \frac{4a^2 \varepsilon^2 \mu}{\pi} + \frac{32}{15} a^2 \varepsilon^3 \mu$ | $- \frac{16a^2 \varepsilon^4 \mu \ln 2}{3\pi} + \frac{4a^2 \varepsilon^6 \mu}{45\pi} (7 - 24 \ln 2)$ |

5 Results and discussion

Table 1 shows a summary of both first- and second-order indentation models and their analytical solutions for the indentation force and displacement. The first-order indentation models include the Hertz model, Liu’s model [Liu et al. 2010], and Sneddon’s model [Sneddon 1965], that are derived using the parabolic, quartic, and spherical profile functions, respectively. The second-order indentation models, analytical parabolic and analytical quartic are derived using the parabolic and quartic profile functions. In the following subsection, we verify these indentation models by comparison with finite element (FE) simulations.

5.1 Finite element simulations

ABAQUS (2017) [Smith 2017] is used to simulate the indentation problem with nonlinear deformation. To simplify the calculation, we establish axisymmetric models for both the indenter and substrate. The indenter is assumed to be a rigid body with radius $R = 3\text{mm}$, while the half-space substrate is assumed to be an incompressible neo-Hookean solid and mimicked by a finite cylinder with appropriate scale and boundary conditions. For the incompressible neo-Hookean solid, the energy function is

$$W_{NH} = C_{10}(\tilde{I}_1 - 3), \quad (87)$$

where $C_{10}$ is the material constant, $\tilde{I}_1 = \text{tr} \left( F^T F \right)$ is the first invariant, and $F$ is the deformation gradient tensor. Up to third-order, the energy function of an incompressible neo-Hookean solid can be expanded as

$$W_{NH}^* = -C_{10} J_2 + \frac{C_{10}}{2} J_2^2 - C_{10} J_1 J_2 + C_{10} J_3^3 + C_{10} J_3,$$ \quad (88)

so $C_{10} = -\alpha_4$. Here, we specify the nonlinear material constant $C_{10} = \mu/2 = 0.15\text{MPa}$. The maximum finite indentation depth is set to be the same as the indenter radius $R$, that is, $D/R = 1.0$. In addition, we use 2-node linear
axisymmetric rigid elements to discretise the indenter, and 4-node axisymmetric reduced integration hybrid elements (CAX4RH) and some 3-node bilinear axisymmetric hybrid elements (CAX3H) to discretise the half-space body.

Figure 4: The FE models established in ABAQUS, including parabolic, quartic, and spherical indentations ($R = 3$mm), showing the mesh and the indenter shapes. Axisymmetric models are established for simplifying the calculation. The maximum finite indentation depth is set to be the same as the indenter radius $R$, that is, $D/R = 1.0$. 2-node linear axisymmetric rigid elements are used to discretise the indenter, and 4-node axisymmetric reduced integration hybrid elements (CAX4RH) and some 3-node bilinear axisymmetric hybrid elements (CAX3H) are used to discretise the half-space body.

Compared to an infinite half-space, using a finite scale cylinder requires that we should specify additional information, including its size and the boundary conditions. Therefore, we should very carefully consider these two influential factors. With reference to Appendix D, we studied their impact by setting control groups and then determining an appropriate scale and boundary conditions. The results show that establishing the FE model for a cylinder with radius and height of 270mm and boundary condition $u_{Bz} = 0$ (i.e. the displacement of the bottom surface along the $z$-direction is constrained) to represent the half-space substrate is appropriate.

Influence of the indenter shapes

In Figure 5A, we display the comparison of the force-displacement curves obtained from FE simulations with different indenter shapes. Figure 5B shows the percentage differences for the parabolic and quartic indentations relative to the spherical indentation, where both the parabolic and quartic indenters provide an overestimate of the applied force. This is explained by Figure 5B, where we see that, at the same indentation depth, both the parabolic and quartic indenters have larger contact areas. According to Figure 5B, using the parabolic indenter would make a 4% difference when the indentation depth $D/R \approx 0.4$, while using the quartic indenter would only exhibit the same difference when $D/R \approx 1.0$. Moreover, at the maximum indentation depth of $D/R = 1.0$, the difference in using the quartic indenter is three times smaller than that using the parabolic indenter. Therefore, the quartic indenter is indeed a much better approximation of the spherical indenter than the parabolic indenter, as expected.
5.2 Verification of indentation models

In this section, we use FE simulations, including parabolic, quartic, and spherical indenter profiles as benchmark results, and verify the indentation models in Table 1 by comparing them to their corresponding FE results.

Parabolic indentation

Figure 6A shows the force-displacement curves from the first-order Hertz model, the second-order analytical parabolic solution (75), and the FE computation of parabolic indentation. Figure 6B displays the differences between the Hertz model and the second-order analytical parabolic solution compared to the FE computation with a parabolic indentation. As shown in Figure 6A, the classical Hertz model overestimates the external force. In contrast, the second-order analytical parabolic solution exhibits very well-matched results over the entire indentation process. In particular, according to Figure 6B, the percentage difference of the analytical parabolic solution is only slightly more than 1% at the maximal indentation depth of $D/R = 1.0$. As a comparison, the percentage difference of the first-order solution Hertz model is over 1% at the indentation depth $D/R \approx 0.05$, and over 6% at the maximal indentation depth $D/R = 1.0$.

Figure 6: Comparisons among the first-order Hertz model, the second-order analytical parabolic solution, and the FE simulation result of the parabolic indentation. (A) The corresponding force-displacement curves; (B) the percentage difference of Hertz model and second-order analytical parabolic solution to the FE parabolic indentation, where $\bar{P}$ is the force by FE parabolic indentation.
Quartic indentation

Similarly, we show the force-displacement curves of the first-order model of Liu et al. [2010], our second-order analytical quartic solution (86), and the FE computation of quartic indentation in Figure 7A. Moreover, in Figure 7B, we present the percentage difference between both the model of Liu et al. [2010] and our second-order analytical quartic solution compared to the FE computation with a quartic indentation. We note that Liu’s model overestimates the external force compared to the FE simulations, while the second-order analytical parabolic solution slightly underestimates the external force when the indentation depth increases but still closely matches the FE result. In particular, according to Figure 7B, the percentage difference of the second-order analytical quartic solution only becomes more than 2% at the indentation depth $D/R \approx 0.8$. As a comparison, the first-order solution Liu’s model exhibits the same level of difference at the indentation depth $D/R \approx 0.1$. In addition, at the maximal indentation depth $D/R = 1.0$, Liu’s model has more than a 7% difference, which is twice as big as the difference found than when using our second-order analytical quartic solution.

![Quartic indenter (R=3)](image)

Figure 7: Comparisons among the first-order Liu’s model, the second-order analytical quartic solution, and the FE quartic indentation. (A) The corresponding force-displacement curves; (B) the percentage difference of Liu’s model and second-order analytical quartic solution to the FE quartic indentation, where $\bar{P}$ is the force by FE quartic indentation.

Spherical indentation

Finally, for the more common indentation problem involving a spherical indenter, we show the force-displacement curves of all indentation models listed in Table 1 and the FE spherical indentation in Figure 8A. In this case, all the indentation models significantly overestimate the applied forces, except for the second-order analytical quartic solution which gives a nearly perfect prediction. In Figure 8B, we present the percentage differences of all indentation models listed in Table 1 compared to the FE spherical indentation. The first-order Hertz model with parabolic indenter exhibits the biggest difference over the whole indentation process. The difference is more than 5% for the indentation depth $D/R \approx 0.2$ and over 20% at the maximal indentation depth $D/R = 1.0$. Liu’s model and the second-order analytical parabolic solution make similar predictions. Their differences are more than 5% when the indentation depth $D/R \gtrsim 0.4$ and over 12% at the maximal indentation depth $D/R = 1.0$. Due to using the accurate spherical indenter profile function, the first-order Sneddon model still provides a better prediction, but the difference is over 5% when the indentation depth $D/R \gtrsim 0.5$ and is around 9% at the maximal indentation depth $D/R = 1.0$. Finally, our second-order analytical quartic solution makes the best prediction, where the difference is less than 1% over the whole indentation process. This improvement is partly due to our consideration of second-order deformations but also due to the retention of higher order ($O(\varepsilon^3)$) terms in the expansion of the shape of the indenter tip. According to Figures 5A and 7A, such close agreement is due to the fact that the second-order analytical quartic solution slightly underestimates the FE quartic simulation, while the FE spherical indentation also slightly underestimates the fourth-order indentation.

5.3 Limitations

In the previous subsection, we have shown that the second-order quartic solution makes a very close prediction of the finite indentation for an incompressible neo-Hookean solid. However, there is still a limitation of this current second-order indentation model in accounting for other incompressible hyperelastic materials, such as the incompressible Mooney-Rivlin solid.
Consider the energy function of an incompressible Mooney-Rivlin solid

$$ W_{MR} = C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3), \quad (89) $$

where $C_{10}$ and $C_{01}$ are material constants; $\bar{I}_1 = \text{tr} \left( F^T F \right)$, $\bar{I}_2 = \text{tr} \left( (F^T F)^{-1} \right)$. Up to the third-order elasticity, it can be expanded as

$$ W_{MR}^* = - (C_{10} + C_{01}) J_2 + \frac{C_{10} + C_{01}}{2} J_1^2 - (C_{10} + 2C_{01}) J_1 J_2 + \frac{C_{10} + 2C_{01}}{3} J_3^3 + (C_{10} + 2C_{01}) J_3, \quad (90) $$

where $a_1 = -(C_{10} + C_{01})$. Hence, the two material constants can be represented as

$$ C_{10} = -(1 - T)a_1, \quad C_{01} = -Ta_1, \quad (91) $$

where $0 \leq T \leq 1$, and for $T = 0$ the energy function reduces to the incompressible neo-Hookean material.

In Figure 9, we present a comparison of the force-displacement curves among the FE simulation results for a Mooney-Rivlin solid (with $T = 0, 0.1, 0.3, 0.5, 0.7, 1.0$) and the second-order analytical quartic solution. The force increases as $T$ increases, while the theoretical prediction does not account for the variation of $T$. According to the equations (84) and (86), the final expressions of the total displacement and force for an incompressible solid $\eta = 0.5$ only involves one material constant $\mu$ or $a_1$. In addition, recall that $a_1 = -(C_{10} + C_{01})$. It is, thus, obvious that this second-order solution would only reflect the sum of $C_{10}$ and $C_{01}$ and can not reflect the separate variation of these two constants.

In other words, the second-order solutions are not available to account for incompressible hyperelastic solids with more than one material constant. Further investigation of this problem shows that the first-order incompressible condition $\Delta = 0$ causes the first-order deformation to go to zero, which further suppresses the material constants $a_3$, $a_4$, $a_5$ in the second-order terms and the final expression of the total force. Hence, we need to consider higher-order elasticity models to overcome this limitation.

## 6 Conclusions

In this paper, we re-present the general solution of the indentation problem using second-order elasticity. Going beyond the first-order analytical solution provided by Sneddon, the second-order solution can account for the nonlinear deformation and stress during the indentation process. It provides a significant step towards theoretically modelling the indentation problem for hyperelastic materials, especially for the common nano-indentation test on bio-materials.

We have identified and corrected mistakes made by Sabin and Kaloni [1983] and Giannakopoulos and Triantafyllou [2007]. We have not only corrected their second-order solutions for parabolic indentation but also have provided...
the second-order analytical solution for quartic indentation, which is a much better approximation to the spherical indentation (Figure 8). We verify the indentation models by comparing them to FE simulations. For all indentation models, the second-order solutions show improved predictions compared to the first-order solutions. In addition, for spherical indentation with an incompressible neo-Hookean material, our second-order quartic solution only exhibits less than 1% difference for all indentation depths $D$ less than or equal to the indenter radius $R$. As a comparison, when the indentation depth $D$ equals to the indenter radius $R$, the classic Hertz model exhibits more than 20% difference, while the corrected second-order parabolic solution exhibits 12% difference and the first-order Sneddon model nearly 10% difference. Consequently, we believe the second-order solutions, especially the second-order quartic solution, should be widely adopted in future experimental and theoretical studies. The second-order indentation models have limitations in accounting for some high-order incompressible hyperelastic materials. Refining this methodology by introducing the higher-order elasticity and using a better approximation of the spherical indenter profile should further improve prediction.

**Acknowledgement**

This research was funded by EPSRC grants EP/S030875/1 and EP/S020950/1. We also thank Professor Michel Destrade (NUI Galway) for his help with the incompressible limit. Raimondo Penta conducted the research according to the inspiring scientific principles of the national Italian mathematics association Indam (“Istituto nazionale di Alta Matematica”).
Appendix A

For the parabolic indenter, according to Weber-Sonin-Schafheitlin integral \cite{Korenev2002}, the expressions for $F_i(r, 0)$ and $G_i(r, 0)$ ($i = 0, 1, 2$) are

\begin{align*}
F_0(r, 0) &= \begin{cases} 
\varepsilon \left(2a^2 - r^2\right), & (0 \leq r \leq a) \\
\frac{4a(1-\eta)}{\varepsilon (2a^2 - r^2)}, & (r > a)
\end{cases} \\
F_1(r, 0) &= \begin{cases} 
\frac{2\varepsilon \sqrt{a^2 - r^2}}{\pi a (1-\eta)}, & (0 \leq r \leq a) \\
0, & (r > a)
\end{cases} \\
F_2(r, 0) &= \begin{cases} 
\frac{\varepsilon}{a(1-\eta)}, & (0 \leq r \leq a) \\
\frac{2\varepsilon}{\pi a (1-\eta)} \left(\arcsin \left(\frac{a}{r}\right) - \frac{a}{\sqrt{r^2 - a^2}}\right), & (r > a)
\end{cases} \\
G_0(r, 0) &= \begin{cases} 
\frac{2\varepsilon a^3}{3\pi r a(1-\eta)}, & (0 \leq r \leq a) \\
\frac{3\pi r a(1-\eta)}{2\varepsilon a^3}, & (r > a)
\end{cases} \\
G_1(r, 0) &= \begin{cases} 
\frac{\varepsilon r}{\pi r a(1-\eta)}, & (0 \leq r \leq a) \\
\frac{2a(1-\eta)}{\varepsilon} \left(\frac{r^2 \arcsin \left(\frac{a}{r}\right) - a\sqrt{r^2 - a^2}}{\pi r a(1-\eta)}\right), & (r > a)
\end{cases} \\
G_2(r, 0) &= \begin{cases} 
\frac{2\varepsilon r}{\pi a(1-\eta) \sqrt{a^2 - r^2}}, & (0 \leq r \leq a) \\
0, & (r > a)
\end{cases}
\end{align*}

(A1)
Appendix B

For the quartic indenter, according to Weber-Sonin-Schafheitlin integral [Korenev, 2002], the expressions for \( F_i(r, 0) \) and \( G_i(r, 0) \) \((i = 0, 1, 2)\) are

\[
F_0(r, 0) = \begin{cases} 
\frac{8a^4 \varepsilon (\varepsilon^2 + 3) - 12a^2 r^2 \varepsilon - 3r^4 \varepsilon^3}{48a^3(1 - \eta)}, & (0 \leq r \leq a) \\
\frac{\varepsilon}{24\pi a^3(1 - \eta)} \left\{ a \sqrt{r^2 - a^2} \left[ 2a^2 (\varepsilon^2 + 6) + 3r^2 \varepsilon^2 \right] + \left[ 8a^4 (\varepsilon^2 + 3) - 12a^2 r^2 - 3r^4 \varepsilon^2 \right] \arcsin \left( \frac{\varepsilon}{r} \right) \right\}, & (r > a)
\end{cases}
\]

\[
F_1(r, 0) = \begin{cases} 
\frac{2\varepsilon \sqrt{a^2 - r^2} \left[ a^2 (2\varepsilon^2 + 9) + 4r^2 \varepsilon^2 \right]}{9\pi a^3(1 - \eta)}, & (0 \leq r \leq a) \\
0, & (r > a)
\end{cases}
\]

\[
F_2(r, 0) = \begin{cases} 
\frac{\varepsilon (a^2 + r^2 \varepsilon^2)}{a^3(1 - \eta)}, & (0 \leq r \leq a) \\
\frac{3 \varepsilon}{3\pi a^3(1 - \eta)} \left\{ 3 \left( a^2 + r^2 \varepsilon^2 \right) \arcsin \left( \frac{\varepsilon}{r} \right) + a \left( r^2 - a^2 \right)^{-1/2} \left[ a^2 (\varepsilon^2 - 3) - 3r^2 \varepsilon^2 \right] \right\}, & (r > a)
\end{cases}
\]

\[
G_0(r, 0) = \begin{cases} 
\frac{\varepsilon}{144(1 - \eta)} \left[ \frac{32 (2\varepsilon^2 + 3)}{\pi ar} \left[ a^3 - (a^2 - r^2)^{3/2} \right] + \frac{3\varepsilon^2 (8a^2 r^2 - 8a^4 - 3r^4)}{a^3} \right], & (0 \leq r \leq a) \\
\frac{\varepsilon}{72\pi a^3(1 - \eta) r} \left\{ 16a^5 (2\varepsilon^2 + 3) + 9ar \varepsilon^2 \sqrt{r^2 - a^2} (r^2 - 2a^2) - 3r^2 \varepsilon^2 (8a^4 - 8a^2 r^2 + 3r^4) \arcsin \left( \frac{\varepsilon}{r} \right) \right\}, & (r > a)
\end{cases}
\]

\[
G_1(r, 0) = \begin{cases} 
\frac{4a^3 \varepsilon}{\varepsilon^3 (1 - \eta)}, & (0 \leq r \leq a) \\
\frac{6\pi a^3 r (1 - \eta)}{3r^2 (2a^2 + r^2 \varepsilon^2) \arcsin \left( \frac{\varepsilon}{r} \right) - 3ar^2 \varepsilon^2 \sqrt{r^2 - a^2}} - 2a^3 (\varepsilon^2 + 3) \sqrt{r^2 - a^2}, & (r > a)
\end{cases}
\]

\[
G_2(r, 0) = \begin{cases} 
\frac{8r^3 \varepsilon^3 - a^2 r (4\varepsilon^3 - 6\varepsilon)}{3\pi a^3 (1 - \eta) \sqrt{a^2 - r^2}}, & (0 \leq r \leq a) \\
0, & (r > a)
\end{cases}
\]
Appendix C

\[ I_1 = \int_{t}^{\infty} \ln \left[ \sqrt{\xi^2 - a^2} + \sqrt{\xi^2 - t^2} \right] \frac{d\xi}{\xi \sqrt{\xi^2 - a^2}}, \quad t \geq a. \]

\[ I_2 = \int_{t}^{\infty} \frac{\sqrt{\xi^2 - t^2}}{\xi \sqrt{\xi^2 - a^2}} \arcsin \left( \frac{a}{\xi} \right) \, d\xi, \quad t \geq a. \]

\[ I_3 = \int_{0}^{a} \frac{\sqrt{a^2 - t^2}}{\sqrt{r^2 - t^2}} \, dt \quad t \leq a, \quad I_4 = \int_{r}^{\infty} \frac{1}{\xi} \arcsin \left( \frac{a}{\xi} \right) \, d\xi. \]

\[ I_5 = \int_{r}^{a} \sqrt{\frac{x}{\xi^2 - a^2}} \frac{dt}{\sqrt{t^2 - r^2}} \]

\[ I_6 = \frac{1}{2} \int_{r}^{a} \left( 1 - \frac{t^2}{a^2} \right) \ln \left( \frac{a + t}{a - t} \right) \frac{dt}{\sqrt{t^2 - r^2}} \]

\[ I_7 = \int_{a}^{\infty} \left[ \frac{x}{a} \arcsin(a/x) - \frac{\sqrt{x^2 - a^2}}{x} \right] \times \left[ \arcsin(a/x) - \frac{a}{\sqrt{x^2 - a^2}} \right] \arcsin \left( \frac{\sqrt{a^2 - r^2}}{x^2 - r^2} \right) \, dx, \quad r \leq a. \]

\[ I_8 = \int_{a}^{\infty} \left\{ \frac{32a_1 \varepsilon^2 \arcsin \left( \frac{\sqrt{r^2 - a^2}}{\pi^2 - \xi^2} \right)}{\pi^2 \varepsilon^2 (\eta - 1) \xi} \left[ \frac{a^3 - 2a_1 \varepsilon^2 \arcsin \left( \frac{a}{\xi} \right) + a^2 + \xi^2 \arcsin \left( \frac{a}{\xi} \right)}{\sqrt{\xi^2 - a^2}} \right] \right. \]

\[ + \frac{16a_1 \varepsilon^4 \arcsin \left( \frac{\sqrt{r^2 - a^2}}{\pi^2 - \xi^2} \right)}{3\pi^2 \varepsilon^4 (\eta - 1) \xi} \left[ 9a^2 \xi^2 + 9\xi^4 \arcsin \left( \frac{a}{\xi} \right) \right] \frac{2a^5 + 9a^3 \xi^2 - 18a_1 \varepsilon^4 \arcsin \left( \frac{a}{\xi} \right)}{\sqrt{\xi^2 - a^2}} \right. \]

\[ \left. + \frac{16a_1 \varepsilon^6 \arcsin \left( \frac{\sqrt{r^2 - a^2}}{\pi^2 - \xi^2} \right)}{9\pi^2 \varepsilon^6 (\eta - 1) \xi} \left[ 9\xi^6 \arcsin \left( \frac{a}{\xi} \right) \right] - a^2 \left( 2a^4 - 3a^2 \xi^2 - 9\xi^4 \right) \right) \]

\[ + 6a_1 \varepsilon^2 \frac{\pi^2 a^4 + a^2 \xi^2 - 3\xi^4 \arcsin \left( \frac{a}{\xi} \right)}{\sqrt{\xi^2 - a^2}} \right\} \, d\xi. \]
Appendix D: FEM validation

Influence of the size effect

As shown in Figure 4, we establish FE models for parabolic, quartic, and spherical indentations with \( R = 3 \text{mm} \) and different scales of the substrate \( r = h = 60\text{mm}, 90\text{mm}, 180\text{mm}, 270\text{mm}, \) respectively. At the same time, we set the same boundary conditions \( u_{Bz} = 0 \) for all FE models, where \( u_{Bz} \) represents the displacement of the bottom surface along the \( z \) direction.

Figure A1: The force-displacement curves obtained from the FE simulations of parabolic, quartic, and spherical indentations, with different sizes.

Figure A1 shows the force-displacement curves obtained from the FE simulations of parabolic, quartic, and spherical indentation, respectively. For all subfigures, there are small but visible differences between curves for the minimal size \( r = h = 60\text{mm} \) and maximal size \( r = h = 270\text{mm} \). As the size increases, the curves for the size \( r = h = 180\text{mm} \) and \( r = h = 270\text{mm} \) nearly overlap, which indicates that the simulation for this case would approach a convergent result as the cylinder size \( r = h \geq 180\text{mm} \). Further increasing the size of the cylinder will not create a significant difference compared to the real half-space. Hence, for this indentation problem, a cylinder with the size \( r = h = 270\text{mm} \) should be big enough to mimic the half-space body.

Influence of the boundary conditions

Then, we establish FE models for parabolic, quartic, and spherical indentations with indenter radius \( R = 3\text{mm} \) and cylinder size \( r = h = 270\text{mm} \) obtained previously. To study the influence of the boundary conditions, we set different boundary conditions at the bottom surface and lateral surface of the cylindrical substrate, which are \( u_{Bz} = 0 \), \( u_{Bz} = u_{Lr} = 0 \), \( u_{Bz} = u_{Lz} = 0 \), \( u_{Bz} = u_{Lr} = u_{Lz} = 0 \), \( u_{Bz} = u_{Lr} = u_{Bz} = 0 \), \( u_{Bz} = u_{Bz} = u_{Lz} = 0 \), \( u_{Bz} = u_{Lr} = u_{Lz} = 0 \), \( u_{Bz} = u_{Lr} = u_{Lz} = 0 \), and \( u_{Bz} = u_{Bz} = u_{Lz} = 0 \), where the subscript \( u_{Bz} \) and \( u_{Bz} \) indicate the displacement of the bottom surface along the \( r \) and \( z \) directions, \( u_{Lr} \) and \( u_{Lz} \) indicate the displacement of the lateral surface along the \( r \) and \( z \) directions.

Figure A2: The force-displacement curves obtained from the FE simulations of parabolic, quartic, and spherical indentations, with different boundary conditions.

Figure A2 shows the force-displacement curves obtained from FE simulations of parabolic, quartic, and spherical indentation, respectively. As shown in all subfigures, the force-displacement curves under different boundary conditions...
are not significantly different. In other words, as there are no boundary conditions to be specified for the real half-space body when the cylinder size is big enough, the influence of the boundary conditions on this indentation problem is negligible.

Hence, based on the studies of these two influential factors above, we could then trust the FE simulations that use a cylinder with the size \( r = h = 270 \text{mm} \) and the boundary condition \( u_{Bz} = 0 \) to represent the half-space substrate.

References

Ovijit Chaudhuri, Justin Cooper-White, Paul A Janmey, David J Mooney, and Vivek B Shenoy. Effects of extracellular matrix viscoelasticity on cellular behaviour. *Nature*, 584(7822):535–546, 2020.

Valentin Gensbittel, Martin Kräter, Sébastien Harlepp, Ignacio Busnelli, Jochen Guck, and Jacky G Goetz. Mechanical adaptability of tumor cells in metastasis. *Developmental Cell*, 56(2):164–179, 2021.

Fang Tian, Tsung-Cheng Lin, Liang Wang, Sidong Chen, Xingxiang Chen, Pak Man Yiu, Ophelia KC Tsui, Jun Chu, Ching-Hwa Kiang, and Hyokeun Park. Mechanical responses of breast cancer cells to substrates of varying stiffness revealed by single-cell measurements. *The Journal of Physical Chemistry Letters*, 11(18):7643–7649, 2020.

Yangkun Du, Yipin Su, Chaofeng Lü, Weiqiu Chen, and Michel Destade. Electro-mechanically guided growth and patterns. *Journal of the Mechanics and Physics of Solids*, 143:104073, 2020.

Michael Krieg, Gotthold Fläschner, David Alsteens, Benjamin M Gaub, Wouter H Roos, Gijs JL Wuite, Hermann E Gaub, Christoph Gerber, Yves F Dufresne, and Daniel J Müller. Atomic force microscopy-based mechanobiology. *Nature Reviews Physics*, 1(1):41–57, 2019.

Wenfeng Liang, Haohao Shi, Xieliu Yang, Junhai Wang, Wenguang Yang, Hemin Zhang, and Lianqing Liu. Recent advances in AFM-based biological characterization and applications at multiple levels. *Soft Matter*, 16(39):8962–8984, 2020.

Ya Hua Chim, Louise M Mason, Nicola Rath, Michael F Olson, Manlio Tassieri, and Huabing Yin. A one-step procedure to probe the viscoelastic properties of cells by atomic force microscopy. *Scientific Reports*, 8(1):1–12, 2018.

Yuri M Efremov, Wen-Horng Wang, Shana D Hardy, Robert L Geahlen, and Arvind Raman. Measuring nanoscale viscoelastic parameters of cells directly from AFM force-displacement curves. *Scientific Reports*, 7(1):1–14, 2017.

Johannes Rheinlaender, Andrea Dimitracopoulos, Bernhard Wallmeyer, Niels M Kronenberg, Kevin J Chalut, Malte C Gather, Timo Betz, Guillaume Charras, and Kristian Franze. Cortical cell stiffness is independent of substrate mechanics. *Nature Materials*, 19(9):1019–1025, 2020.

Heinrich Hertz. On the contact of elastic solids. *Zeitschrift für Reine und Angewandte Mathematik*, 92:156–171, 1881.

Kenneth L Johnson. One hundred years of Hertz contact. *Proceedings of the Institution of Mechanical Engineers*, 196(1):363–378, 1982.

W Michael Lai, David H Rubin, David Rubin, and Erhard Kreml. *Introduction to Continuum Mechanics*. Butterworth-Heinemann, 2009.

Feodor M Borodich. The Hertz-type and adhesive contact problems for depth-sensing indentation. *Advances in Applied Mechanics*, 47:225–366, 2014.

DA Spence. The Hertz contact problem with finite friction. *Journal of Elasticity*, 5(3-4):297–319, 1975.

Bertil Storåkers and Denis Elaguine. Hertz contact at finite friction and arbitrary profiles. *Journal of the Mechanics and Physics of Solids*, 53(6):1422–1447, 2005.

Fan Jin, Xu Guo, and Huajian Gao. Adhesive contact on power-law graded elastic solids: The JKR-DMT transition using a double-Hertz model. *Journal of the Mechanics and Physics of Solids*, 61(12):2473–2492, 2013.

Ming Wang, Shaobao Liu, Zhimin Xu, Kai Qu, Moxiao Li, Xin Chen, Qing Xue, Guy M Genin, Tian Jian Lu, and Feng Xu. Characterizing poroelasticity of biological tissues by spherical indentation: an improved theory for large relaxation. *Journal of the Mechanics and Physics of Solids*, 138:103920, 2020.

Alice Chingshuang Chang and Bernard Haochih Liu. Modified flat-punch model for hyperelastic polymeric and biological materials in nanoindentation. *Mechanics of Materials*, 118:17–21, 2018.

Ian N Sneddon. The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile. *International Journal of Engineering Science*, 3(1):47–57, 1965.

Qiang Zhang and Qing-Sheng Yang. Effects of large deformation and material nonlinearity on spherical indentation of hyperelastic soft materials. *Mechanics Research Communications*, 84:55–59, 2017.
Man-Gong Zhang, Yan-Ping Cao, Guo-Yang Li, and Xi-Qiao Feng. Spherical indentation method for determining the constitutive parameters of hyperelastic soft materials. *Biomechanics and Modeling in Mechanobiology*, 13(1):1–11, 2014a.

Zhaoyu Chen and Stefan Diebels. Nanoindentation of hyperelastic polymer layers at finite deformation and parameter re-identification. *Archive of Applied Mechanics*, 82(8):1041–1056, 2012.

Zheng Duan, Yonghao An, Jiaping Zhang, and Hanqing Jiang. The effect of large deformation and material nonlinearity on gel indentation. *Acta Mechanica Sinica*, 28(4):1058–1067, 2012.

Z Song and K Komvopoulos. Elastic–plastic spherical indentation: deformation regimes, evolution of plasticity, and hardening effect. *Mechanics of Materials*, 61:91–100, 2013.

RS Rivlin. The solution of problems in second order elasticity theory. *Journal of Rational Mechanics and Analysis*, 2:53–81, 1953.

Man-Gong Zhang, Jinju Chen, Xi-Qiao Feng, and Yanping Cao. On the applicability of sneddon’s solution for interpreting the indentation of nonlinear elastic biopolymers. *Journal of Applied Mechanics*, 81(9), 2014b.

GCW Sabin and PN Kaloni. Contact problem of a rigid indenter in second order elasticity theory. *Zeitschrift für angewandte Mathematik und Physik ZAMP*, 34(3):370–386, 1983.

AE Giannakopoulos and A Triantafyllou. Spherical indentation of incompressible rubber-like materials. *Journal of the Mechanics and Physics of Solids*, 55(6):1196–1211, 2007.

DX Liu, ZD Zhang, and LZ Sun. Nonlinear elastic load–displacement relation for spherical indentation on rubber-like materials. *Journal of Materials Research*, 25(11):2197–2202, 2010.

Francis Dominic Murnaghan. Finite deformations of an elastic solid. *American Journal of Mathematics*, 59(2):235–260, 1937.

Ian N Sneddon. The elementary solution of dual integral equations. *Glasgow Mathematical Journal*, 4(3):108–110, 1960.

Boris Grigorevich Korenev. *Bessel functions and their applications*. CRC Press, 2002.

Michel Destrade and Raymond W Ogden. On the third-and fourth-order constants of incompressible isotropic elasticity. *The Journal of the Acoustical Society of America*, 128(6):3334–3343, 2010.

Michael Smith. *ABAQUS/Standard User’s Manual, Version 2017*. Dassault Systèmes Simulia Corp, United States, 2017.