PATTERN RECOGNITION ON ORIENTED MATROIDS: 
THE EXISTENCE OF A TOPE COMMITTEE

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Abstract. Oriented matroids can serve as a tool of modeling of collective decision-making processes in contradictory problems of pattern recognition. We present a generalization of the committee techniques of pattern recognition to oriented matroids. A tope committee for an oriented matroid is a subset of its maximal covectors such that every positive halfspace contains more than half of the covectors from this subset. For a large subfamily of oriented matroids their committee structure is quite rich; for example, any maximal chains in their tope posets provide one with information sufficient to construct a committee.

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1. Introduction

In this paper we present a description of the pattern recognition problem in the language of oriented matroids, and show that for oriented matroids from a large family there exist subsets of their maximal covectors that can serve as building blocks of collective decision-making rules.

Book [10] is a standard text on pattern recognition.

In supervised learning to recognize, by means of synthesis of decision rules that make use of decision surfaces in the feature space, a teacher provides a class label for every pattern in the training set. The training patterns from the same class compose a training sample. The union of the training samples is called the training set. The effect of any decision rule is to divide the feature space into decision regions each of which contains training patterns of at most one class. A classifier, that is a pattern recognition system, relates a new unclassified pattern to a certain class partially presented by a training sample, on the basis of the inclusion of the new pattern into a decision region; the decision rule also can force the classifier to leave the new pattern unclassified.

If decision surfaces in the feature space $\mathbb{R}^n$ are hyperplanes, then decision regions are open convex polyhedra.

The theory of hyperplane arrangements is an area of an active cross-disciplinary study, see, e.g., [44, 50].

If a training set in $\mathbb{R}^n$ is composed of two samples which cannot be separated by one decision hyperplane then a classifier (called in the case of two classes a dichotomizer) operates with decision rules relying on infeasible linear inequality systems.

Various properties of infeasible linear constraint systems have been studied in depth, see, e.g., [3, 8, 12, 13, 14, 15, 16, 38, 45, 46].

A useful generalization of the notion of solution of a linear inequality system to the infeasible case is the notion of majority committee.

A committee for an infeasible system of strict linear inequalities over $\mathbb{R}^n$ is a finite subset of elements of $\mathbb{R}^n$ such that for every inequality more than half of the elements are its solutions.

Such committees were apparently first introduced in seminal notes [1, 2], where their application to the pattern recognition problem was discussed. Various committee constructions for a variety of contradictory problems have later been invented and explored in detail; some of the surveys in this subject are works [16, 27, 29, 38, 39, 40, 41].

Infeasible systems of strict linear inequalities over $\mathbb{R}^n$, real hyperplane arrangements for which the intersections of all positive halfspaces of $\mathbb{R}^n$ are empty, and realizable oriented matroids which are not acyclic, are closely connected mathematical objects that allow one to model mechanisms of collective decision-making in pattern recognition problems posed in terms of infeasible systems of linear constraints.
Oriented matroids are defined by various equivalent axiom systems, and they can be thought of as a combinatorial abstraction of point configurations over the reals, of real hyperplane arrangements, of convex polytopes, and of directed graphs.

Oriented matroids are reviewed and studied in detail, e.g., in [4, 5, 7, 48, 49, 55].

In the present paper, a direct generalization of the notion of committee for a linear inequality system, in terms of maximal covectors of an oriented matroid, is presented: A tope committee $K^*$ for an oriented matroid is a subset of its maximal covectors such that for every element of the ground set the corresponding positive halfspace of the oriented matroid contains more than half of the covectors from $K^*$.

One of the approaches to the study of committees consists in structural and combinatorial analysis of the family of maximal feasible subsystems of constraints, and in investigation of the properties of graphs which are naturally associated with those subsystems [23]; such graphs are an example of constructions dual to the Kneser graphs of set systems considered in [33, §3.3], [34]. The properties of the graphs associated with the maximal feasible subsystems, such as connectedness and the existence of an odd cycle, are important for graph-theoretic algorithms of synthesis of committees.

The present paper constitutes a review of central ideas from works [1] and [23], formulated in the language of oriented matroids.

In Section 2, some terminology of the theory of oriented matroids used in the paper is recalled. In Section 3 we review the setting of the pattern recognition problem, give the definitions of committees of maximal (co)vectors, and discuss committee decision rules. In central Section 4 it is shown that every oriented matroid without loops and antiparallel elements, has a tope committee of cardinality less than or equal to the cardinality of its ground set. The argument is based on analysis of consecutive reorientations of an initial acyclic oriented matroid. Section 5 is devoted to graphs which are naturally associated with the families of topes with inclusion-maximal positive parts. The sets of vertices of the odd cycles in such graphs are committees for the corresponding oriented matroids. We apply the graph-theoretic approach from [23] to a generalization of the basic construction of centrally-symmetric cycle of adjacent regions in a hyperplane arrangement from [1]; and vice versa, we use ‘symmetric cycles’ in the tope graph of an oriented matroid, inspired by the above-mentioned cycles of regions from [1], to prove several generalized graph-theoretic results from [23]. Section 6 mentions the link between the committees for an oriented matroid and blocker constructions in the Boolean lattice of subsets of the tope set.

2. Preliminaries

All oriented matroids considered in the paper are of rank that is greater than or equal to 2. We use quite nonstandard definitions of simple oriented
matroids and of graph homomorphisms: An oriented matroid is simple if it has no loops, parallel or antiparallel elements. A homomorphism of a graph to a graph is a mapping from the vertex set of the first graph to that of the second graph, such that either the image of any edge is an edge, or the images of the endvertices coincide.

\( E_m \) denotes the set \([1, m] := \{1, 2, \ldots, m\} \). \( T^{(+)} \) denotes the sign vector \((+ + \ldots +)\) whose components are all +; \( T^{(-)} := -T^{(+)} = (- - \ldots -) \).

See [51] Chapter 3, [5] §4.1 on posets. If \( X \) is a subset of a poset \( \mathcal{P} \), then \( \min X \) denotes the set of all minimal elements from \( X \). \( \mathcal{I}(X) \) and \( \mathcal{F}(X) \) denote the order ideal and filter in \( \mathcal{P} \) generated by \( X \), respectively. If \( \mathcal{P} \) is graded then \( \mathcal{P}(k) \) is the set of all its elements of poset rank \( k \).

For a set family \( \mathcal{F} := \{F_i : i \in [1, m]\} \), \( \min \mathcal{F} \) and \( \max \mathcal{F} \) denote the subfamilies of all inclusion-minimal and of all inclusion-maximal sets in \( \mathcal{F} \), respectively. The nerve of \( \mathcal{F} \) is an abstract simplicial complex on the vertex set \( \mathcal{F} \); a subset \( K \subseteq [1, m] \) is a face of the nerve iff \( \bigcap_{i \in K} F_i > 0 \), see, e.g., [5] §10, [47] §8.5.

See [9] on graphs. Throughout the paper, graphs are undirected; they have no loops and multiple edges. For a graph \( G \), its sets of vertices and of edges are denoted by \( \mathcal{V}(G) \) and \( \mathcal{E}(G) \), respectively. Cycles are regular subgraphs of valency 2; all vertices of paths in graphs are distinct.

The neighborhood complex \( \mathcal{N}(G) \) of a graph \( G \), defined in [30], is an abstract simplicial complex on the vertex set \( \mathcal{V}(G) \); a subset \( N := \{n_1, \ldots, n_k\} \subset \mathcal{V}(G) \) is a face of the complex iff there is \( v \in \mathcal{V}(G) \) such that \( \{n_1, v\}, \ldots, \{n_k, v\} \in \mathcal{E}(G) \).

Recall that for graphs \( G' \) and \( G'' \), a homomorphism of \( G' \) to \( G'' \), written as \( h : G' \rightarrow G'' \), is a mapping \( h : \mathcal{V}(G') \rightarrow \mathcal{V}(G'') \) such that \( \{u, v\} \in \mathcal{E}(G') \) implies \( \{h(u), h(v)\} \in \mathcal{E}(G'') \) or \( h(u) = h(v) \).

If \( \mathcal{F} \) is a set family then the Kneser graph \( KG(\mathcal{F}) \) of \( \mathcal{F} \), considered in [33] §3.3, [34], is the graph with \( \mathcal{V}(KG(\mathcal{F})) := \mathcal{F} \); if \( F', F'' \in \mathcal{F} \) then \( \{F', F''\} \in \mathcal{E}(KG(\mathcal{F})) \) iff \( |F' \cap F''| = 0 \).

We borrow almost all terminology concerning oriented matroids from [5] Chapters 3, 4, 7:

Let \( E \) be a finite set, \( \{-,0,+,\} \) the set of signs, and \( \{-,0,+,\}^E \) the set of sign vectors. The support of a sign vector \( X \in \{-,0,+,\}^E \) is \( \overline{X} := \{e \in E : X(e) \neq 0\} \); here \( X(e) \) denotes the \( e \)th component of \( X \). \( X^- := \{e \in E : X(e) = -\} \) denotes the set of negative elements of \( X \); \( X^+ := \{e \in E : X(e) = +\} \) is the set of positive elements of \( X \). Thus \( \overline{X} := X^- \cup X^+ \). \( X^- \) and \( X^+ \) are also called the negative and positive parts of \( X \), respectively. An inclusion \( e \in X \) means \( e \in X^- \). The zero sign vector \( (00 \ldots 0) \), with the empty support, is denoted by \( 0 \). The zero set \( z(X) \) of a sign vector \( X \) is the set \( \{e \in E : X(e) = 0\} \).

If \( \mathcal{P} \) is a set of sign vectors then \( \max^+(\mathcal{P}) := \{P \in \mathcal{P} : P^+ \in \max\{R^+ : R \in \mathcal{P}\}\} \); similarly, \( \min^+(\mathcal{P}) := \{P \in \mathcal{P} : P^+ \in \min\{R^+ : R \in \mathcal{P}\}\} = \{-P : P \in \max^+(\mathcal{P})\} \).
If $A \subseteq E$ then the sign vector $-_A X$ is defined by

$$(-_A X)(e) := \begin{cases} +, & \text{if } e \in A \text{ and } X(e) = -, \\ -, & \text{if } e \in A \text{ and } X(e) = +, \\ X(e), & \text{otherwise} \end{cases}$$

(if $e \in E$ then we write $-_e X$ instead of $-_\{e\} X$). In particular, the opposite of $X$ is $-X := -_E X$, that is,

$$(-X)(e) := \begin{cases} +, & \text{if } X(e) = -, \\ -, & \text{if } X(e) = +, \\ 0, & \text{if } X(e) = 0, \end{cases}$$

for all $e \in E$. If $\mathcal{F} \subseteq \{-, 0, +\}^E$ and $A \subseteq E$, then $-_A \mathcal{F} := \{-_A X : X \in \mathcal{F}\}$; in particular, $-_\mathcal{F} := -_E \mathcal{F} = \{-X : X \in \mathcal{F}\}$.

If $X \in \{-, 0, +\}^E$ then the sign vector $X$ is called nonpositive (resp., negative) if $X(e) \in \{-, 0\}$ (resp., $X(e) = -$), for all $e \in E$. Similarly, $X$ is nonnegative (resp., positive) if $-X$ is nonpositive (resp., negative).

The composition of two sign vectors $X$ and $Y$ is the sign vector $X \circ Y$ defined by

$$(X \circ Y)(e) := \begin{cases} X(e), & \text{if } X(e) \neq 0, \\ Y(e), & \text{otherwise}. \end{cases}$$

The separation set of $X$ and $Y$ is $S(X,Y) := \{e \in E : X(e) = -Y(e) \neq 0\}$. If $|S(X,Y)| = 0$ then one says that the sign vectors $X$ and $Y$ are conformal; in this case $X \circ Y = Y \circ X$. If sign vectors $X_1, X_2, \ldots, X_k \in \{-, 0, +\}^E$ are pairwise conformal then $\bigcirc_{i \in [1,k]} X_i$ is a short notation for the conformal composition $X_1 \circ X_2 \circ \cdots \circ X_k$.

The partial order on the set $\{-, 0, +\}$ is defined by the relations $0 < -$ and $0 < +$; the signs $-$ and $+$ are incomparable. This induces the product partial order on $\{-, 0, +\}^E$, in which sign vectors are compared componentwise. Thus $X \leq Y$ iff $X(e) \in \{0, Y(e)\}$ for all $e \in E$.

Oriented matroids are defined by several equivalent axiom systems.

Let $E$ be a finite set. If $\mathcal{C} \subseteq \{-, 0, +\}^E$, then $\mathcal{C}$ by definition is the set of circuits of an oriented matroid on $E$ iff it satisfies the following Circuit Axioms [5, Definition 3.2.1]:

(C0) $0 \not\in \mathcal{C}$;  
(C1) $X \in \mathcal{C}$ implies $-X \in \mathcal{C}$;  
(C2) $X, Y \in \mathcal{C}$ and $X \subseteq Y$ imply $X = Y$ or $X = -Y$;  
(C3) if $X, Y \in \mathcal{C}$, $X \neq -Y$, and $e \in X^+ \cap Y^-$, then there is $Z \in \mathcal{C}$ such that $Z^- \subseteq (X^- \cup Y^-) - \{e\}$ and $Z^+ \subseteq (X^+ \cup Y^+) - \{e\}$.

An oriented matroid on $E$, with set of circuits $\mathcal{C}$, is denoted by $(E, \mathcal{C})$.

The circuit supports $\mathcal{C} := \{C : C \in \mathcal{C}\}$ in an oriented matroid $\mathcal{M} := (E, \mathcal{C})$ constitute the circuits of the underlying matroid of $\mathcal{M}$, denoted by $\mathcal{M}$. The rank of $\mathcal{M}$ by definition is the rank of $\mathcal{M}$. 

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A vector of an oriented matroid is any composition of its circuits. An oriented matroid on $E$, given by set of its vectors $\mathcal{V}$, is denoted by $(E, \mathcal{V})$. A maximal vector of an oriented matroid is a vector whose support is maximal with respect to inclusion. An oriented matroid on a set $E$, with set of maximal vectors $\mathcal{W}$, is denoted by $(E, \mathcal{W})$.

If $\mathcal{L} \subseteq \{-, 0, +\}^E$, then the pair $(E, \mathcal{L})$ is an oriented matroid on $E$, with the set of covectors $\mathcal{L}$, iff $\mathcal{L}$ satisfies the following Covector Axioms [5, Proposition 4.1.1]:

(L0) $0 \in \mathcal{L}$;
(L1) $X \in \mathcal{L}$ implies $-X \in \mathcal{L}$;
(L2) $X, Y \in \mathcal{L}$ implies $X \circ Y \in \mathcal{L}$;
(L3) if $X, Y \in \mathcal{L}$ and $e \in S(X, Y)$ then there exists $Z \in \mathcal{L}$ such that

$$Z(e) = 0 \text{ and } Z(f) = (X \circ Y)(f) = (Y \circ X)(f) \text{ for all } f \notin S(X, Y).$$

For an oriented matroid $(E, \mathcal{L})$ of rank $r$, the poset $\hat{\mathcal{L}} := \mathcal{L} \cup \{\hat{1}\}$, with a top element $\hat{1}$ adjoined, is a graded lattice (the so-called 'big' face lattice) of length $r + 1$, see [5, Theorem 4.1.14]; $\hat{0} = 0$ is the bottom element of $\hat{\mathcal{L}}$.

A maximal covector (a tope) of an oriented matroid is a covector whose support is maximal with respect to inclusion. An oriented matroid $\mathcal{M}$ on a set $E$, with set of topes $\mathcal{T}$, is denoted by $(E, \mathcal{T})$.

The set $\mathcal{C}^*$ of non-zero covectors of an oriented matroid $\mathcal{M}$, with inclusion-minimal supports, is the set of cocircuits of $\mathcal{M}$. An oriented matroid on $E$, with that set of cocircuits, is denoted by $(E, \mathcal{C}^*)$.

For every $e \in E$ the corresponding positive halfspace is the subset of topes $\mathcal{T}_e^+ := \{T \in \mathcal{T} : T(e) = +\}$; the negative halfspace $\mathcal{T}_e^-$ is the subset $-\mathcal{T}_e^+$. The set of vertices of the tope graph $\mathcal{T}(\mathcal{L})$ is the set of topes; two topes are connected by an edge if the topes are adjacent, that is, if they cover the same element (a subtope) of poset rank $r - 1$ in $\hat{\mathcal{L}}$, where $r$ is the rank of the oriented matroid.

If $B \subseteq T$ then the tope poset $\mathcal{T}(\mathcal{L}, B)$, based at $B$, is defined by the partial order on the set of topes: $T' \preceq T''$ iff $S(B, T') \subseteq S(B, T'')$.

A subset of topes $\mathcal{Q} \subseteq \mathcal{T}$ is $T$-convex if the following implication holds:

$$T', T'' \in \mathcal{Q}, \ T \in \mathcal{T}, \ |S(T', T'')| = |S(T', T)| + |S(T, T'')| \implies T \in \mathcal{Q},$$

that is, if $\mathcal{Q}$ contains every shortest path in the graph $\mathcal{T}(\mathcal{L})$ between any two of its members. The $T$-convex hull $\text{conv}_T(\mathcal{Q})$ of $\mathcal{Q} \subseteq \mathcal{T}$ is the intersection of all halfspaces that contain $\mathcal{Q}$.

All topes $T \in \mathcal{T}$ have the same support and the same zero set $E_0 := \text{z}(T)$. The elements in $E_0$ are called the loops of $\mathcal{M}$. Thus $e \in E$ is a loop of $\mathcal{M} := (E, \mathcal{C})$ iff there is a circuit $(0 \ldots 0 + 0 \ldots 0) \in \mathcal{C}$.

If $e \not\in C$ for every circuit $C \in \mathcal{C}$ of an oriented matroid $\mathcal{M} := (E, \mathcal{C})$, then $e$ is called a coloop of $\mathcal{M}$.

Elements $e, f \in E$, $e \neq f$, are called parallel if $X(e) = X(f)$ for all $X \in \mathcal{L}$; they are called antiparallel, if $X(e) = -X(f)$ for all $X \in \mathcal{L}$. 


As mentioned above, a simple oriented matroid means, throughout the paper, an oriented matroid without loops, parallel or antiparallel elements.

The restriction of a sign vector \( X \in \{-, 0, +\}^E \) to a subset \( A \subseteq E \) is the sign vector \( X|_A \in \{-, 0, +\}^A \) defined by \( (X|_A)(e) := X(e) \) for all \( e \in A \).

For an oriented matroid \( M := (E, \mathcal{C}) \), the oriented matroid \( (E - A, \mathcal{C}\setminus A) \) on \( E - A \), given by its set of covectors \( \mathcal{L}\setminus A := \{X|_{E-A} : X \in \mathcal{L}\} \subseteq \{-, 0, +\}^{E-A} \) is called the deletion \( M\setminus A \) or the restriction \( M|_{E-A} \). The oriented matroid \( (E, -A\mathcal{C}) \) on \( E \), given by its set of covectors \(-A\mathcal{C} \subseteq \{-, 0, +\}^E\) is called the reorientation \(-A M\); see [5] Lemma 4.1.8.

An oriented matroid \( M := (E, \mathcal{C}) = (E, \mathcal{T}) \) is acyclic if there is no nonnegative circuit in the set \( C \), or equivalently if there exists the nonnegative tope in \( \mathcal{T} \). A subset \( A \subseteq E \) is called acyclic if the restriction \( M|_A \) is acyclic. The oriented matroid \( M \) is totally cyclic if for each element \( e \in E \) there exists a nonnegative circuit \( C \in \mathcal{C} \) such that \( e \in C \). Recall that ‘most’ oriented matroids are neither acyclic nor totally cyclic [31, §6.3.1].

The circuits, vectors, and maximal vectors of an oriented matroid \( M \) are the cocircuits, covectors, and tope, respectively, of the oriented matroid \( M^* \), the dual (or orthogonal) of \( M \). The loops of \( M \) are the coloops of \( M^* \); \( M \) is acyclic if \( M^* \) is totally cyclic, see [31 Proposition 3.4.8].

For an oriented matroid \( M := (E, \mathcal{C}) \), a single element extension \( \tilde{M} := (\tilde{E}, \tilde{\mathcal{C}}^*) \) of \( M \) is an oriented matroid on a set \( \tilde{E} \) such that \( \tilde{E} = E \cup \{g\} \), with set of cocircuits \( \tilde{\mathcal{C}}^* \). If \( g \) is not a coloop of \( \tilde{M} \), then \( \tilde{M} \) is called a nontrivial extension of \( M \).

The set \( \tilde{\mathcal{C}}^* \) of cocircuits of an extension \( \tilde{M} \) is described in the following way [31, [5] Proposition 7.1.4]:

\begin{enumerate}
  \item Let \( \tilde{M} \) be a nontrivial single element extension of \( M := (E, \mathcal{C}^*) = (E, \mathcal{L}) \). Then for every cocircuit \( Y \in \mathcal{C}^* \) there is a unique way to extend \( Y \) to a cocircuit of \( \tilde{M} \); there is a unique function \( \sigma : \mathcal{C}^* \to \{-, 0, +\} \), called the localization, such that \( \{(Y, \sigma(Y)) : Y \in \mathcal{C}^*\} \subseteq \tilde{\mathcal{C}}^* \), that is, \( (Y, \sigma(Y)) \) is a cocircuit of \( \tilde{M} \) for every cocircuit \( Y \) of \( M \). Furthermore, this \( \sigma \) satisfies \( \sigma(-Y) = -\sigma(Y) \) for all \( Y \in \mathcal{C}^* \).

(\sigma) \tilde{M} \text{ is uniquely determined by } \sigma, with

\[
\tilde{\mathcal{C}}^* = \{ (Y, \sigma(Y)) : Y \in \mathcal{C}^* \} \\
\cup \{ (Y' \circ Y'', 0) : Y', Y'' \in \mathcal{C}^*, \sigma(Y') = -\sigma(Y'') \neq 0, \rho(Y' \circ Y'') = 2 \},
\]

where \( \rho \) denotes the poset rank function on \( \tilde{\mathcal{L}} \).
3. Pattern Recognition on Oriented Matroids

3.1. The Two-Class Pattern Recognition Problem. A training set is a simple oriented matroid $S$ on a ground set $E$, together with a mapping $\lambda : E \to \{-, +\}$ such that the training samples $\lambda^{-1}(-)$ and $\lambda^{-1}(+)$ are nonempty. The elements of $E$ are called the training patterns.

Classes $A$ and $B$ are disjoint sets such that $A \supseteq \lambda^{-1}(-)$ and $B \supseteq \lambda^{-1}(+)$. Thus, an element $e \in E$ a priori belongs to the class $A$ iff $\lambda(e) = -$; it a priori belongs to the class $B$ iff $\lambda(e) = +$.

Let $\tilde{S}$ denote the nontrivial single element extension of $S$ by a new unclassified pattern $g$ which is not a loop, and which is parallel or antiparallel to neither of the elements of $E$. A decision rule is any mapping $r : E \cup \{g\} \to \{-, 0, +\}$ such that $r(e) \mapsto \lambda(e)$, $e \in E$.

If $f \in E \cup \{g\}$ and $r(f) = -$, then a dichotomizer using the rule $r$ relates the pattern $f$ to the class $A$. If $r(f) = +$, then $f$ is classified by the dichotomizer as a pattern from $B$. If $r(f) = 0$ then the dichotomizer leaves the pattern $f$ unclassified.

3.2. Committees of Maximal (Co)Vectors for Oriented Matroids.

Definition 3.1. Let $p$ be a rational number such that $0 \leq p < 1$.

- Given an oriented matroid $M := (E, T)$, a subset $K^* \subset T$ is a tope $p$-committee (a $p$-committee of maximal covectors) for $M$ if for every $e \in E$ it holds
  $$|\{K \in K^* : K(e) = +\}| > p|K^*|.$$  
  A tope $\frac{1}{2}$-committee for $M$ is called a tope committee for $M$.

- Given an oriented matroid $M := (E, W)$, a subset $K \subset W$ is a $p$-committee of maximal vectors for $M$ if for every $e \in E$ it holds
  $$|\{K \in K : K(e) = +\}| > p|K|.$$  
  A $\frac{1}{2}$-committee of maximal vectors for $M$ is called a committee of maximal vectors for $M$.

Definition 3.1 implies that

- a set $K^*$ is a tope $p$-committee for $M$ iff $K^*$ is a $p$-committee of maximal vectors for the dual oriented matroid $M^*$;
- a subset $K^* \subset T$ is a committee for $M$ iff for every $e \in E$ it holds $|K^*| < 2|\{K \in K^* : K(e) = +\}|$;
- a subset $K^* \subset T$ is a committee for $M$ iff the set $\{-T : T \in T-K^*\}$ is;
- if $K^*$ is a tope committee for $M$, and if $\tilde{M}$ is a trivial single element extension of $M$ by a coloop, then the set $\{(T, +) : T \in K^*\}$ is a tope committee for $\tilde{M}$. 
Given an oriented matroid $M$, we denote by $K^*(M)$ the family of all tope committees for $M$.

**Definition 3.2.** Let $M := (E, T)$ be an oriented matroid, and $K^*$ a tope committee for $M$.

- $K^*$ is called minimal if any proper subset of the set $K^*$ is not a committee for $M$.
- If $K^*$ is minimal, then $K^*$ is called critical if
  
  $K \in K^*, \ T \in T, \ T^+ \not\subseteq K^+ \implies (K^* - \{K\}) \cup \{T\} \notin K^*(M)$.

- $K^*$ is called a minimum committee (a committee of minimal cardinality) if there is no committee $Q^*$ for $M$ such that $|Q^*| < |K^*|$.

If $M$ is simple and acyclic, then the one-element set $\{T^{(+)}\}$ is a critical tope committee for $M$.

It follows from the definition that minimal and minimum committees do not contain opposites.

**Example 3.3.** Consider the acyclic oriented matroid $N^0 := (E_6, T^0)$ given by the central hyperplane arrangement of Figure[1].

The sets of topes $T^0$ and $T^2$ of the oriented matroids $N^0$ and $N^2 := -[1,2]N^0 = (E_6, T^2)$, respectively, are as follows:

$T^0 := \{ + + + + + +, - - + + + +, - - + - + +, + - + - + +, + - + + - +, + + - + - +, + + - - + +, + + - - - +, - - - + + +, - - - - + +, - - - - - +, - - - - - + \}$

$T^2 := \{ - - - - - +, - - - - - +, - - - - + - +, - - - - + - +, - - - - + - +, - - - + - - +, - - - + - - +, - - - - - +, - - - - - +, - - - - - +, - - - - - +, - - - - - + \}$

The tope committee

$\{ + + + - - +, - - + - + +, + - + + + +, + - + - - +, + - - + + +, + - - - - +, + + + + + - \}$
Figure 1. A central hyperplane arrangement that realizes a simple acyclic oriented matroid $\mathcal{N}^0 := (E_6, T^0)$ with 28 topes. The positive halfspaces of $\mathbb{R}^3$ are marked by arrows.

for $\mathcal{N}^2$, of even cardinality, is not minimal; indeed, it splits up into a disjoint union of two critical committees,

$$\left\{ \begin{array}{c}
+ & + & - & + & - \\
- & + & + & - & + \\
+ & - & - & + & + \\
\end{array} \right\} \quad \text{and} \quad \left\{ \begin{array}{c}
+ & + & - & - & + \\
- & - & - & - & + \\
- & + & + & - & - \\
\end{array} \right\} ,$$

of minimal cardinality, because $\mathcal{N}^2$ is not acyclic.

As the topes of the committee

$$\left\{ \begin{array}{c}
+ & + & - & + & - \\
- & - & + & - & + \\
+ & - & - & + & + \\
+ & - & - & - & + \\
\end{array} \right\}$$
have the positive parts which are maximal with respect to inclusion, it is, intuitively, of 'higher quality' than the committee

\[
\begin{pmatrix}
- & + & - & + \\
- & + & + & - \\
+ & + & - & + \\
+ & - & + & + \\
\end{pmatrix}
\]

3.3. Committee Decision Rules. Let \( S \) be a training set on a ground set \( E \). Denote by \( \mathcal{M} \) the reorientation

\[
\mathcal{M} := -\lambda^{-1}(-)S.
\]

Let \( \widetilde{\mathcal{M}} := (E \cup \{g\}, \widehat{C}^*) \) denote a nontrivial single element extension of \( \mathcal{M} = (E, C^*) = (E, \mathcal{L}) \) by a new pattern \( g \), such that \( \widetilde{\mathcal{M}} \) is simple. Let \( \sigma : C^* \to \{-, 0, +\} \) denote the corresponding localization.

If \( K^* \) is a tope committee for \( \mathcal{M} \), then assign to every tope \( K \in K^* \) the set of cocircuits

\[
C_{K^*} := \{D \in C^* : D \text{ conforms to } K\}.
\]

\[\begin{aligned}
\bullet \quad & \text{If the sets } \{(D, \sigma(D)) : D \in C_{K^*}\} \text{ are conformal, for all } K \in K^*, \\
& \text{then define a subset of topes } \widetilde{K}^* \text{ of } \widetilde{\mathcal{M}} \text{ in the following way:} \\
& \quad \widetilde{K}^* := \left\{ \bigcap_{D \in C_K^*} (\lambda^{-1}(-)D, \sigma(D)) : K \in K^* \right\}.
\end{aligned}\]

The committee decision rule corresponding to \( K^* \) is the mapping \( \tau : E \cup \{g\} \to \{-, 0, +\} \) such that

\[
\tau : f \mapsto \begin{cases} -, & \text{if } \left| \{\widetilde{K} \in \widetilde{K}^* : \widetilde{K}(f) = -\} \right| > \left| \{\widetilde{K} \in \widetilde{K}^* : \widetilde{K}(f) = +\} \right|, \\
+ , & \text{if } \left| \{\widetilde{K} \in \widetilde{K}^* : \widetilde{K}(f) = -\} \right| < \left| \{\widetilde{K} \in \widetilde{K}^* : \widetilde{K}(f) = +\} \right|, \\
0 , & \text{otherwise}.
\end{cases}
\]

\[\begin{aligned}
\bullet \quad & \text{If } \{(D, \sigma(D)) : D \in C_{K^*}\} \text{ is not conformal, for some } K \in K^*, \text{ then the committee decision rule } \tau \text{ corresponding to } K^* \text{ is defined by} \\
& \quad \tau : e \mapsto \begin{cases} - , & \text{if } \left| \{K \in K^* : (\lambda^{-1}(-)K)(e) = -\} \right| > \left| \{K \in K^* : (\lambda^{-1}(-)K)(e) = +\} \right|, \\
+ , & \text{if } \left| \{K \in K^* : (\lambda^{-1}(-)K)(e) = -\} \right| < \left| \{K \in K^* : (\lambda^{-1}(-)K)(e) = +\} \right|, \\
\end{cases}
\end{aligned}\]

for all \( e \in E \). By convention, \( \tau : g \mapsto 0 \).

**Example 3.4.** Figure 2(a) depicts a realization of a rank 2 training set \( S \) on the ground set \( E_1 \). A realization of its reorientation \( \mathcal{M} := -\lambda^{-1}(-)S = -4S \) is shown in Figure 2(b). The set of topes

\[
K_1 := + - - - \\
K_2 := - + + + \\
K_3 := + + + -
\]

is a committee for \( \mathcal{M} \).
Let $\tilde{M}$ be a nontrivial single element extension of $M$ by the pattern 5, as shown in Figure 2(c).

Each of the sets of cocircuits

$$\{(D, \sigma(D)) : D \in C_{K_1^*}\} = \{0 - - + + + - 0 + \},$$

$$\{(D, \sigma(D)) : D \in C_{K_2^*}\} = \{- + 0 + - - + 0 - \},$$

$$\{(D, \sigma(D)) : D \in C_{K_3^*}\} = \{0 + + - - + 0 - \}.$$
is conformal. The set of topes \( \tilde{K}^* \), defined by (3.1), is
\[
\tilde{K}^* = \{ + - - - + , - + + - - , + + + + - \}.
\]
Therefore the decision rule \( r \), corresponding to \( K^* \) and defined by (3.2), recognizes the pattern 5 as an element of the class \( A \).

4. The Existence of a Tope Committee: Reorientations

In this section we show that every simple oriented matroid has a tope committee. In fact, a critical tope committee for such an oriented matroid can be built based on information on an arbitrary maximal chain in the tope poset.

Our argument relies on the mechanism of consecutive reorientations of an initial acyclic oriented matroid.

The main construction which we make use of is a direct generalization of centrally-symmetric cycles of regions in a central hyperplane arrangement from [1]:

Remark 4.1. Let \( \mathcal{M} := (E_m, \mathcal{L}) = (E_m, \mathcal{T}) \) be a simple oriented matroid. Let \( \mathcal{R} := (T^0, T^1, \ldots, T^{2m-1}, T^0) \) be a cycle in the tope graph \( \mathcal{T}(\mathcal{L}) \) such that
\[
T^{k+m} = -T^k , \quad 0 \leq k \leq m - 1.
\]
We call such a cycle symmetric.

(i) For every \( e \in E_m \) the set of topes
\[
\{ T \in \mathfrak{Y}(\mathcal{R}) : T(e) = + \}
\]
is the set of vertices of a path of length \( m - 1 \) in \( \mathcal{R} \); if a tope \( T \) is an endvertex of the path then the other endvertex is the tope \( -e(T) \).

(ii) Let \( (T^{k_1}, T^{k_2}, T^{k_3}) \) be a 2-path in \( \mathcal{R} \). We have
\[
T^{k_2} \in \max^+(\mathfrak{Y}(\mathcal{R}))
\]
iff \( (T^{k_1})^+ \subseteq (T^{k_2})^+ \supseteq (T^{k_3})^+ \) or, equivalently, \( |(T^{k_1})^+| = |(T^{k_3})^+| = |(T^{k_2})^+| - 1 \).

In other words, let \( \{ f \} := S(T^{k_1}, T^{k_2}) \) and \( \{ g \} := S(T^{k_2}, T^{k_3}) \); then inclusion (4.1) holds iff \( T^{k_2}(f) = T^{k_2}(g) = + \).
4.1. Rank 2. In the theory of oriented matroids the rank 2 case is instructive.

**Lemma 4.2.** Let \( N^0 := (E_m, T^0) \) be a simple acyclic oriented matroid of rank 2, on the ground set \( E_m \), with set of topes \( T^0 \).

Let \(( j_1, \ldots, j_s)\) be a nonempty sequence of integers with \( j_i \in E_m \), \( 1 \leq i \leq s \). Define the reorientation \( N^i := (E_m, T^i) := -j_i N^{i-1} \) whose set of topes is denoted by \( T^i \).

The reorientation \( N^i \) of \( N^0 \) has a critical tope committee.

Three types of transformations of committees are carried out by Algorithm 4.3 that underlies the proof below. We illustrate the transformations by considering several rank 2 oriented matroids represented by central line arrangements in the plane. The regions corresponding to the topes from committees will be marked in figures by discs or circles. Let \( K_0^* := \{T^{(+)}\} \), and let \( K_i^* \) denote the tope committee built by Algorithm 4.3 for the reorientation \( N^i \).

1) If there is a tope \( K \in K_{i-1}^* \) such that
\[ K(j_i) = + \quad \text{and} \quad -j_i(-K) \in T^{i-1} \]
(or, equivalently, \( K(j_i) = + \), and there is a subtope \( H < K \) such that \( z(H) = \{j_i\} \)), but there is no tope \( S \) in \( K_{i-1}^* \) such that \( S(j_i) = + \) and \( -j_i S = -K \), then the set
\[ K_i^* := -j_i(K_{i-1}^* - \{K\}) \cup \{K\} \]
is a tope committee for \( N^i \); see Figure 3.

2) If there is no tope \( K \in K_{i-1}^* \) such that \( K(j_i) = + \) and \( -j_i(-K) \in T^{i-1} \), then pick the pair of topes \( \{T'', T'''\} \subseteq T^{i-1} \) such that
\[ T''(j_i) = T'''(j_i) = - \quad \text{and} \quad -j_i T'' = -T''' \]. The set
\[ K_i^* := -j_i(K_{i-1}^* \cup \{T'', T'''\}) \]

**Figure 3.** A transformation of a tope committee under a reorientation: \( K_i^* := -j_i(K_{i-1}^* - \{K\}) \cup \{K\} \); here \( |K_i^*| = |K_{i-1}^*| \).
is a tope committee for $N^1$; see Figure 4.

3) If there is a tope $K \in K_{i-1}^*$ such that $K(j_i) = +$ and $-j_i(-K) \in T^{i-1}$, and if there is a tope $S \in K_{i-1}^*$ such that $-j_i S = -K$, then the set

$$K_i^* := -j_i(K_{i-1}^* - \{K, S\})$$

is a tope committee for $N^i$; see Figure 5.

**Proof.** The set $K_{i}^* \subset T^s$ built by means of Algorithm 4.3 is a tope committee for $N^s$:

Claim 1. For any $i$, $1 \leq i \leq s$, $K_i^*$ is of odd cardinality, and for any $e \in E_m$, it holds

$$|\{K \in K_i^*: K(e) = +\}| = \left\lceil \frac{|K_i^*|}{2} \right\rceil.$$  \hfill (4.2)
Claim 2. For any $i$, $1 \leq i \leq s$, we have

\[ \mathcal{K}_i^* = \{ K \in \mathcal{T}^i : T \in \mathcal{T}^i, \mathbf{S}(K, T) = \{ e \} \implies K(e) = + \}. \]  

(4.3)

As a consequence, the committee $\mathcal{K}_i^*$ is minimal.

\[ \text{Indeed, let } i = 1. \]

- If conditions (4.1) hold for $K := T^{(+)}$ in $\mathcal{N}^{i-1}$, then pick the tope $T \in \mathcal{T}^{i-1}$ such that $\mathbf{S}(K, T) = \{ j_i \}$ in $\mathcal{N}^{i-1}$; the one-element set $\mathcal{K}_i^* := \{ -j_i T \} = \{ K \} = \{ T^{(+)} \}$, formed at Step 10 of the algorithm, is a tope committee for $\mathcal{N}^i$.

- If (4.1) do not hold for $K := T^{(+)}$, then the three-element set of covectors $\mathcal{K}_i^* := \{ K', K'', K''' \}$, built at Step 14 of the algorithm, where

\[ K' := -j_i K, \quad K'' := -j_i T'', \quad K''' := -j_i T''', \]

is a tope committee for $\mathcal{N}^i$ because we have

\[ \mathcal{K}_i^* = \{ K' = + \ldots + \uparrow (j-1) \uparrow j \uparrow (j+1) \ldots +, K'' = ? \ldots ? + ? \ldots ? , K''' = -(K''(1)) \ldots -(K''(j-1)) + -(K''(j+1)) \ldots -(K'''(m)) \}, \]

that is, for every $e \in E_m$, it holds $|\{ K \in \mathcal{K}_i^* : K(e) = + \}| = 2 = \left\lfloor \frac{|\mathcal{K}_i^*|}{2} \right\rfloor$.

Note that (4.2) and (4.3) hold for $i = 1$.

Now, let $i > 1$.

- If there is no tope $K$ in $\mathcal{K}_{i-1}^*$ such that (4.1) hold, then for the tope $T''$ and $-j_i T'''$ added to the set $\mathcal{K}_i^*$ at Step 14 of the algorithm, we have $(-j_i T')(j_i) = (-j_i T''')(j_i) = +$. Assume that for the tope $T \in \mathcal{T}^{i-1}$ such that $\mathbf{S}(T'', T) = \{ f \}$ and $f \neq j_i$, it holds $T''(f) = -$. Then the tope $K := -T''$, with $K(j_i) = K(f) = +$, must belong to the committee $\mathcal{K}_{i-1}^*$ and satisfy conditions (4.1), but this contradicts the negative decision made at Step 06. Hence $T''(e) = +$. For the tope $Q \in \mathcal{T}^i$ such that $\mathbf{S}(T''', Q) = \{ g \}$ and $g \neq j_i$, we also have $T'''(g) = +$. As a result, (4.3) and (4.2) hold; hence $\mathcal{K}_i^*$ is a tope committee for $\mathcal{N}^i$, of cardinality $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}^*| + 2$.

- If there are topes $K, S \in \mathcal{K}_{i-1}^*$ such that conditions (4.4) and (4.5) hold, then the algorithm excludes them from consideration at Steps 09 and 12, and we obtain $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}^*| - 2$.

If there is a tope $K \in \mathcal{K}_{i-1}^*$ such that (4.4) hold, but there is no tope $S$ in $\mathcal{K}_{i-1}^*$ satisfying (4.5), then we have $|\mathcal{K}_i^*| = |\mathcal{K}_{i-1}^*|$.

In any case, (4.2) holds, for all $e \in E_m$; therefore $\mathcal{K}_i^*$ is a tope committee for $\mathcal{N}^i$. 
If $T(+) \not\in \mathcal{K}_s^*$ then assume that $\mathcal{K}_s^*$ is not minimal, that is, there is a proper subset $Q^*$ of the set $\mathcal{K}_s^*$ such that $\mathcal{K}_s^* - Q^*$ is a committee for $\mathcal{M}$. Since $T(+) \not\in \mathcal{K}_s^*$, we have $|\mathcal{K}_s^* - Q^*| > 1$.

Denote by $R := (T^0, T^1, \ldots, T^{2m-1}, T^0)$ the cycle which is the tope graph for $\mathcal{N}^s$, and without loss of generality suppose that $T^0 \in \mathcal{K}_s^*$ and $T^0 \not\in Q^*$. Recall that $\mathcal{K}_s^*$ is precisely the set $\text{max}^+(\mathcal{M}(R)) = \text{max}^+(T^s)$, see Remark 4.1(ii).

Let $\{g\} := S(T^0, T^1)$ and $\{f\} := S(T^{2m-1}, T^0)$; note that $f \neq g$. We have $T^0(f) = T^0(g) = +$. For every $k$, $0 < k < m$, we have $T^k(f) = +$; for every $l, m < l < 2m$, we have $T^l(g) = +$, see Remark 4.1(i).

Claim 1 implies that for every $e \in \{f, g\}$ it holds $|\{Q \in Q^* : Q(e) = -\}| = |\{R \in Q^* : R(e) = +\}|$, and we have

$$|\{T^1, T^2, \ldots, T^{m-1}\} \cap Q^*| = |\{T^{m+1}, T^{m+2}, \ldots, T^{2m-1}\} \cap Q^*| = \frac{|Q^*|}{2};$$

in particular, $Q^*$ is of even cardinality.

Let $T^{k_2}$ be a tope from the set $Q^*$, and $(T^{k_1}, T^{k_2}, T^{k_3})$ a 2-path in $R$. If $\{p\} := S(T^{k_1}, T^{k_2})$ and $\{q\} := S(T^{k_2}, T^{k_3})$, then there is an element $h \in \{p, q\}$ such that $|\{K \in \mathcal{K}_s^* - Q^* : K(h) = +\}| = \left\lfloor \frac{|\mathcal{K}_s^* - Q^*|}{2} \right\rfloor$, that is, the set of topes $\mathcal{K}_s^* - Q^*$ is not a committee for $\mathcal{M}$, a contradiction. Thus, $\mathcal{K}_s^*$ is minimal.

Claims 1 and 2 show that the committee $K^*$ is critical.

\[\square\]

Algorithm 4.3.
\begin{enumerate}
    \item $\mathcal{K}_0^* \leftarrow \{T(+)\}$;
    \item for $i \leftarrow 1$ to $s$
        \begin{enumerate}
            \item $\mathcal{K}_i^* \leftarrow \emptyset$;
            \item $\text{FOUND} \leftarrow \text{false}$;
            \item while $|\mathcal{K}_{i-1}^*| > 0$
                \begin{enumerate}
                    \item do pick a tope $K \in \mathcal{K}_{i-1}^*$;
                    \item if $K(j_i) = +$ and $-j_i(-K) \in T^{i-1}$ (4.4) then $\text{FOUND} \leftarrow \text{true}$;
                    \item if there is a tope $S \in \mathcal{K}_{i-1}^*$ such that $S(j_i) = +$ and $-j_iS = -K$ (4.5) then $\mathcal{K}_{i-1}^* \leftarrow \mathcal{K}_{i-1}^* \cup \{S\}$;
                    \item else $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \cup \{K\}$;
                    \item $\mathcal{K}_{i-1}^* \leftarrow \mathcal{K}_{i-1}^* \cup \{-j_iK\}$;
                    \item $\mathcal{K}_{i-1}^* \leftarrow \mathcal{K}_{i-1}^* \cup \{-j_iK\}$;
                \end{enumerate}
        \end{enumerate}
    \item if $\text{FOUND} = \text{false}$
        \begin{enumerate}
            \item pick the pair of topes $\{T''', T''''\} \subset T^{i-1}$ such that $T''''(j_i) = T''''(j_i) = -$ and $-j_iT'''' = -T''''$;
            \item $\mathcal{K}_i^* \leftarrow \mathcal{K}_i^* \cup \{-j_iT'', -j_iT'''\}$;
        \end{enumerate}
\end{enumerate}
Proposition 4.4. Let \( \mathcal{M} := (E, \mathcal{T}) \) be a simple oriented matroid of rank 2.

(i) The set 
\[ K^* := \{ K \in \mathcal{T} : T \in \mathcal{T}, \quad S(K, T) = \{ e \} \implies K(e) = + \} \tag{4.6} \]
is a critical tope committee for \( \mathcal{M} \).

(ii) Committee \( \text{(4.6)} \) is the set \( K^* = \max^+(\mathcal{T}) \).

Proof. (i) If \( \mathcal{M} \) is acyclic, then the one-element set \( \{ \mathcal{T}(+) \} \) is its critical tope committee.

Suppose that \( \mathcal{M} \) is not acyclic. If \( \mathcal{N}^0 \) is an acyclic reorientation of \( \mathcal{M} \) and \( J := (j_1, \ldots, j_s) \subset E_m \) is an ordered set of integers such that \( \mathcal{M} = _{-J}\mathcal{N}^0 \), then the proposition follows from Lemma 4.2.

Assertion (ii) follows from (i) and from Remark 4.1(ii). \( \square \)

4.2. Rank \( \geq 2 \). We now show that a simple oriented matroid of arbitrary rank has a tope committee. We again use the technique of reorientations of an initial acyclic oriented matroid \( \mathcal{N}^0 \). To simplify our presentation, we will suppose below that the \( i \)th reorientation of \( \mathcal{N}^0 \) is the oriented matroid \( -[1, i]\mathcal{N}^0 \).

Lemma 4.5. Let \( \mathcal{N}^0 := (E_m, \mathcal{L}^0) = (E_m, \mathcal{T}^0) \) be a simple acyclic oriented matroid on the ground set \( E_m \), with set of covectors \( \mathcal{L}^0 \) and set of topes \( \mathcal{T}^0 \).

Let \( s \) be an integer, \( s \leq m \). For every \( i, 1 \leq i \leq s \), define the reorientation \( \mathcal{N}^i := -[1, i]\mathcal{N}^0 \).

The reorientation \( \mathcal{N}^s \) of \( \mathcal{N}^0 \) has a tope committee.

Proof. The set of covectors \( K_s^* \subset \mathcal{T}^s \) built by Algorithm 4.6 is a tope committee for \( \mathcal{N}^s \):

Claim. The set \( K_s^* \) is of odd cardinality, and for any \( e \in E_m \), it holds
\[ |\{ K \in K_s^* : K(e) = + \}| = \left\lceil \frac{|K_s^*|}{2} \right\rceil. \tag{4.7} \]

Fix a maximal chain \( \mathbf{m} := (R^0 := \mathcal{T}(+) \prec \cdot \cdot \cdot \prec R^m := \mathcal{T}(-)) \) in the tope poset \( \mathcal{T}^0(\mathcal{L}^0, \mathcal{T}(+)) \). Then assign to every tope \( R^i \in \{ R^1, \ldots, R^m \} \) a label \( i \in [1, m] \) defined by
\[ \{ i \} := S(R^{i-1}, R^i). \tag{4.8} \]

Note that

- \( \{ R^i : i \in [1, m-1] \} \cap \{ -i(-R^i) : i \in [2, m] \} = 0 \) because in the poset \( \mathcal{T}^0(\mathcal{L}^0, \mathcal{T}(+)) - \{ \mathcal{T}(+), \mathcal{T}(-) \} \) its maximal chains
  \( \mathbf{m} - \{ R^0, R^m \} \)
and
  \( \{ -i_0(-R^m) \prec -i_{m-1}(-R^{m-1}) \prec \cdot \cdot \cdot \prec -i_2(-R^2) \} \)
are disjoint;
• for every $i \in [1, m-1]$ it holds 
  
  \[-R^i = -t_{i+1}(-R^{i+1})\]

as a consequence, the set of topes 
\[\{R^i : i \in [1, m-1]\} \cup \{-t_i(-R^i) : i \in [2, m]\}\]

contains precisely $m-1$ pairs of opposites, see Figure 6.

• the multiset \{$R^0, -t_1(-R^1), R^m$\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\} \{T(+)\)
Algorithm 4.6.

01 $K_0^* \leftarrow \{T(+)\}$;

$m \leftarrow \text{maximal chain } (R_0^0 \prec R_1^1 \prec \cdots \prec R_m^m) \text{ in } T^0(L^0, T(+));$

$(\{l_1\}, \ldots, \{l_m\}) \leftarrow (S(R_0^0, R_1^1), \ldots, S(R_{m-1}^m, R_m^m));$

02 for $i \leftarrow 1 \text{ to } s$

03 do $K_i^* \leftarrow -i(K_{i-1}^*)$;

$R \leftarrow R^k \text{ such that } l_k = i$;

$K_i^* \leftarrow K_i^* \cup \{-[l_i, l_{i-1}]R, -[l_{i-1}, l_i]R\}$;

04 while $K_i^* \supset \{K, T\}$ such that $T = -K$

05 do $K_i^* \leftarrow K_i^* - \{K, T\}$;

Algorithm 4.7.

01 $K_0^* \leftarrow \{T(+)\}$;

$m \leftarrow \text{maximal chain } (R_0^0 \prec R_1^1 \prec \cdots \prec R_m^m) \text{ in } T^0(L^0, T(+));$

$(\{l_1\}, \ldots, \{l_m\}) \leftarrow (S(R_0^0, R_1^1), \ldots, S(R_{m-1}^m, R_m^m));$

02 for $i \leftarrow 1 \text{ to } s$

03 do multiset $K_i^* \leftarrow \{-iK : K \in K_{i-1}^*\}$;

$R \leftarrow R^k \text{ such that } l_k = i$;

$K_i^* \leftarrow \{K_i^*, -[l_i, l_{i-1}]R, -[l_{i-1}, l_i]R\}$;

04 while $K_i^* \supset \{K, T\}$ such that $T = -K$

05 do $K_i^* \leftarrow K_i^* - \{K, T\}$;

Proposition 5.9 of Section 5.1 asserts that Algorithm 4.6 in fact constructs critical committees.

Example 4.8. Consider the acyclic oriented matroid $N^0 := (E_6, T^0)$ given by the central hyperplane arrangement of Figure 1. If the maximal chain

$$
(\begin{array}{ccccccc}
R_0^0 & := & + & + & + & + & + \\
R_1^1 & := & + & + & - & + & + \\
R_2^2 & := & - & + & + & + & + \\
R_3^3 & := & - & + & - & - & + \\
R_4^4 & := & - & - & - & - & + \\
R_5^5 & := & - & - & - & - & - \\
R_6^6 & := & - & - & - & - & - \\
\end{array})
$$

in its tope poset $T^0(L^0, T(+))$ is chosen at Step 01 of Algorithm 4.6 with

$I_1 = 3, \ I_2 = 1, \ I_3 = 4, \ I_4 = 6, \ I_5 = 2, \ I_6 = 5,$
then the algorithm produces, when applying to the reorientation \( N^6 := -[1,6]N^0 \), the following sequence of critical tope committees:

\[
\begin{align*}
K_1^* &= \{ - - + + + + + + + + - - - - - + \} \\
K_2^* &= \{ + + + - - - - - - + + \} \\
K_3^* &= \{ + + + - - - + + + \} \\
K_4^* &= \{ + + + + + + + + + \} \\
K_5^* &= \{ - + + - - - + + + + + \} \\
K_6^* &= \{ + + + + + + + + + \}
\end{align*}
\]

Algorithm 4.6 does not necessarily construct a tope committee of minimal cardinality. For example, it builds for the reorientation \( N^5 := -[1,5]N^0 \) of the oriented matroid \( N^0 \) (which is realized by the hyperplane arrangement of Figure 1) a three-element committee, while \( N^5 \) is acyclic, that is, the one-element set \( \{ T^{(+)} \} \) is a committee for \( N^5 \).

Any simple oriented matroid has a tope committee of cardinality that is less than or equal to the cardinality of its ground set:

**Theorem 4.9.** Let \( N^0 := (E_m, T^0) \) be a simple acyclic oriented matroid on the ground set \( E_m \), with set of topes \( T^0 \). Let \( s \) be an integer, \( s \leq m \). Define the reorientation \( N^s := -[1,s]N^0 \).

(i) \( N^s \) has a tope committee \( K_s^* \) such that \( |K_s^*| \leq m \) when \( m \) is odd, and \( |K_s^*| \leq m - 1 \) when \( m \) is even.

As a consequence, any simple oriented matroid \( M \) on the set \( E_m \) has a tope committee \( K^* \) with

\[
|K^*| \leq \begin{cases} m, & \text{if } m \text{ is odd,} \\ m - 1, & \text{if } m \text{ is even.} \end{cases}
\]

(ii) If \( s \leq \lfloor m/2 \rfloor \) then \( N^s \) has a tope committee \( K_s^* \) with \( |K_s^*| \leq 1 + 2s \).

**Proof.** Apply Algorithm 4.6 to \( N^s \); this algorithm constructs a tope committee for \( N^s \), see the proof of Lemma 4.5.

(i) \( \bullet \) If \( m \) is odd, then the tope committee of maximal cardinality which can be constructed by Algorithm 4.6 is the set

\[
K^*_s = \{ -[1,s]R^0 \} \cup \{ -[1,s]R^k, -[s,m]R^k : 2 \leq k \leq m - 1, k \text{ even} \}
\]

of cardinality \( m \).

Figure 7 depicts such an arrangement of topes, cf. Figure 6.

\( \bullet \) If \( m \) is even, then the tope committee of maximal cardinality which can be constructed by Algorithm 4.6 is either the set

\[
K^*_s = \{ -[1,s]R^k, -[s,m]R^k : 2 \leq k \leq m - 2, k \text{ even} \} \cup \{ -[s,m]R^m \}
\]
or the set
\[ \mathcal{K}_s^* = \{-[1,s]R^1\} \cup \{-[1,s]R^K, -[s,m]R^K : 3 \leq k \leq m - 1, k \text{ odd}\}, \]
(4.11)
see Figures 8(a) and (b), respectively, cf. Figure 6. We have \(|\mathcal{K}_s^*| = m - 1\).

Figure 8. The tope committees of maximal cardinality which can be constructed by Algorithm 4.6 in the case of \(m\) even: (4.10) on the left, and (4.11) on the right; cf. Figure 6.

Figure 6 suggests that if Algorithm 4.6 builds for \(s = \lfloor m/2 \rfloor\) a tope committee of the form (4.9), (4.10) or (4.11), then for every \(s\)
such that \( s > \lfloor m/2 \rfloor \), the cardinality of \( K_s^* \) will decrease because pairs of opposites will be removed.

(ii) The committee constructions of maximal cardinality which we have considered in part (i) were built under \( s = \lfloor m/2 \rfloor \). One can argue in an analogous way to prove assertion (ii).

\[ \square \]

5. Graphs Related to Tope Committees

From the point of view of modeling of decision-making procedures, naturally associated with a simple oriented matroid \( M := (E_m, T) \) is a certain graph \( \Gamma(M) \) that is isomorphic to the Kneser graph \( KG(\{ T^- : T \in T \}) \) of the family of the negative parts of the topes of \( M \); the vertex sets of odd cycles in \( \Gamma(M) \) are tope committees:

**Lemma 5.1.** Let \( M := (E_m, \mathcal{L}) = (E_m, \mathcal{T}) \) be a simple oriented matroid. Consider the graph \( \Gamma := \Gamma(M) \) defined by

\[
\mathfrak{V}(\Gamma) := \mathcal{T}, \\
\{ T^k, T^l \} \in \mathfrak{E}(\Gamma) \iff (T^k)^+ \cup (T^l)^+ = E_m. \tag{5.1}
\]

If \( C \) is an odd cycle in \( \Gamma \), then the set of its vertices \( \mathfrak{V}(C) \) is a tope committee for \( M \).

**Proof.** Assume that there is an element \( e \in E_m \) such that \( |\{ K \in \mathfrak{V}(C) : K(e) = -\} | \geq \left\lceil \frac{|C|}{2} \right\rceil \). Then there exists an edge \( \{ T', T'' \} \in \mathfrak{E}(C) \) with \( (T')^+ \cup (T'')^+ \not\supseteq e \); hence \( (T')^+ \cup (T'')^+ \not= E_m \), a contradiction. \( \square \)

**Example 5.2.** Consider the reorientation \( N^3 := -[1,3]N^0 \) of the oriented matroid \( N^0 \) which is realized by the hyperplane arrangement of Figure 1. The set of vertices of the 5-cycle in \( \Gamma(N^3) \), shown in Figure 9, is a tope committee for \( N^3 \).

5.1. Symmetric Cycles in the Tope Graph. We now show that a direct graph generalization of centrally-symmetric cycles of adjacent regions in hyperplane arrangements from [II] leads to constructions of odd cycles in subgraphs of the graphs \( \Gamma \) defined by (5.1).

**Proposition 5.3.** Let \( M := (E_m, \mathcal{L}) = (E_m, \mathcal{T}) \) be a simple oriented matroid. Let \( R := (T^0, T^1, \ldots, T^{2m-1}, T^0) \) be a symmetric cycle (that does not contain the positive tope \( T^{(+)} \)) in the tope graph \( \mathcal{T}(\mathcal{L}) \).

Consider the graph \( \mathfrak{G} \) defined by

\[
\mathfrak{V}(\mathfrak{G}) := \mathfrak{V}(R) = \{ T^0, T^1, \ldots, T^{2m-1} \}, \\
\{ T^k, T^l \} \in \mathfrak{E}(\mathfrak{G}) \iff (T^k)^+ \cup (T^l)^+ = E_m. \tag{5.2}
\]

The set \( \text{max}^+ (\mathfrak{V}(R)) \) is the vertex set of an odd cycle in \( \mathfrak{G} \).
Proof. We without loss of generality suppose that \(T^0 \in \max^+ (\mathfrak{M}(R))\).

Example 5.4 and Figure [1] illustrate the proof.

Note that the path \((T^1, T^2, \ldots, T^m)\) contains at least one vertex \(T^j\) such that \(T^j \in \max^+ (\mathfrak{M}(R))\). This follows from the observation that \(|(T^0)^+| < m = |S(T^0, T^m)|\) and \(|(T^m)^+| > 0\), and from Remark [4.i].

Let \(T^l\) be a vertex of \(R\) such that \(1 < l < m\), \(T^l \in \max^+(\mathfrak{M}(R))\), and \((T^l)^+ \supseteq (T^m)^+ = (-T^0)^+\). The pair \(\{T^0, T^l\}\) is an edge of \(G\).

We have \(\{T^0, T^l\} \in \mathcal{E}(G)\) for all \(j, l \leq j \leq m\).

On the contrary, if \(0 < j < l\) then \(\{T^0, T^j\} \notin \mathcal{E}(G)\). Indeed, let \(\{e\}\) be the one-element separation set for the topes \(T^l\) and \(T^{l-1}\). Then we have \(e \notin (T^j)^+\) and \(e \notin (T^0)^+\).

Similarly, there is a unique vertex \(T^p\) of the cycle \(R\) such that \(p > m\), \(T^p \in \max^+ (\mathfrak{M}(R))\), and \(\{T^0, T^p\} \in \mathcal{E}(G)\). Here \((T^p)^+ \supseteq (T^m)^+ = (-T^0)^+\). For all \(j, m \leq j \leq p\), we have \(\{T^0, T^j\} \in \mathcal{E}(G)\). If \(p < j \leq 2m - 1\) then \(\{T^0, T^j\} \notin \mathcal{E}(G)\).

Thus,

\[
\{\{T', T''\} \in \mathcal{E}(G) : \{T', T''\} \supseteq T^0\} = \{\{T^0, T^l\}, \{T^0, T^{l+1}\}, \ldots, \{T^0, T^{m-1}\}, \{T^0, T^m\}, \{T^0, T^{m+1}\}, \ldots, \{T^0, T^{p-1}\}, \{T^0, T^p\}\} .
\]

Note that for all \(j, l < j < p\), we have \(T^j \notin \max^+ (\mathfrak{M}(R))\).

Let \(T^i\) be a vertex of \(R\) such that \(1 < i \leq l\), \(T^i \in \max^+(\mathfrak{M}(R))\), and \((T^{i-1})^+ \supseteq (T^0)^+\). The pair \(\{T^0, T^i\}\) is an edge of \(G\).

We have the inclusion

\[
\{T^0, T^i\}, \{T^0, T^p\}, \{T^p, T^i\} \in \mathcal{E}(G) .
\]
If \( i = l \) then the sequence of vertices \((T^0, T^l, T^p, T^0)\) is a triangle in \( G \) with the property \( \{T^0, T^l, T^p\} = \text{max}^+(\mathcal{U}(R)) \).

In the general case, when \( i \leq l \), consider successively all the vertices \( T^j \) with \( i \leq j \leq l \) to see that the set \( \text{max}^+(\mathcal{U}(R)) \) is of odd cardinality because

\[
| \{ i : 1 < i < m, \ T^i \in \text{max}^+(\mathcal{U}(R)) \} | = | \{ i : m < i < 2m - 1, \ T^i \in \text{max}^+(\mathcal{U}(R)) \} | ,
\]

and this set is the vertex set \( \mathcal{U}(C) \) of a cycle \( C \) in \( G \): if

\[
\text{max}^+(\mathcal{U}(R)) = \{T^0, T^{k_1}, \ldots, T^{k_d}, T^{k_{d+1}}, \ldots, T^{k_{2d}}\}, \quad 0 < k_1 < \cdots < k_{2d},
\]

then the family of edges of this cycle is

\[
\mathcal{E}(C) = \{ \{T^0, T^{k_d}\}, \{T^0, T^{k_{d+1}}\}, \{T^{k_1}, T^{k_{d+1}}\}, \{T^{k_{d+1}}, T^{k_{d+2}}\}, \ldots, \\
\{T^{k_{d-1}}, T^{k_{2d-1}}\}, \{T^{k_{d-1}}, T^{k_{2d}}\}, \{T^{k_d}, T^{k_{2d}}\} \}.
\]

Example 5.4. Consider the reorientation \( \mathcal{M} := -[1,2]N^0 \) of an oriented matroid \( N^0 \), where \( N^0 \) is realized by the hyperplane arrangement of Figure 1. Figure 10 depicts the set of vertices of a symmetric cycle \( R \) in the tope graph of \( \mathcal{M} \), and the corresponding graph \( G \) defined by (5.2). The set \( \text{max}^+(\mathcal{U}(R)) \) is the vertex set of the odd cycle in \( G \).

\[\text{Figure 10. The graph } G, \text{ defined by (5.2), that corresponds to a symmetric cycle } R := (T^0, T^1, \ldots, T^{2m-1}, T^0) \text{ in the tope graph of the reorientation } \mathcal{M} := -[1,2]N^0, \text{ where the oriented matroid } N^0 \text{ is realized by the hyperplane arrangement of Figure 1; } m = 6. \text{ The edges that connect opposites are not depicted. The set } \text{max}^+(\mathcal{U}(R)) \text{ is the vertex set of the 5-cycle in } G.\]
Lemma 5.5. Let $\mathcal{M} := (E_m, \mathcal{L}) = (E_m, T)$ be a simple oriented matroid. Let $R := (T^0, T^1, \ldots, T^{2m-1}, T^0)$ be a symmetric cycle (that does not contain the positive tope $T^{(+)}$) in the tope graph $T(\mathcal{L})$. For every $e \in E_m$, we have

$$|\{T \in \max^+(\mathfrak{B}(R)) : T(e) = +\}| = \left\lceil \frac{\max^+(\mathfrak{B}(R))}{2} \right\rceil.$$ 

Proof. Let $G$ and $C$ be the graph and the odd cycle, respectively, which were constructed in the proof of Proposition 5.3; see descriptions (5.2) and (5.4).

Let $P := (T^{l_0}, \ldots, T^{l_{m-1}})$ be the $(m-1)$-path in $R$ such that $T^{l_0}(e) = \ldots = T^{l_{m-1}}(e) = +$; see Remark 4.1(i).

Without loss of generality suppose that $(T^0)^+ \supseteq (T^{l_0})^+ \supseteq (T^{k_3})^{-}$. Then we have $T^{k_3} \in \mathfrak{B}(P)$, and the assertion follows from (5.3) because $\{T \in \max^+(\mathfrak{B}(R)) : T(e) = +\} = \mathfrak{B}(P) \cap \max^+(\mathfrak{B}(R)) = \{T^0, T^{k_1}, T^{k_2}, \ldots, T^{k_3}\}$.

The higher-rank analogue of Proposition 4.4 is as follows:

Proposition 5.6. Let $\mathcal{M} := (E_m, \mathcal{L}) = (E_m, T)$ be a simple oriented matroid. Let $R := (T^0, T^1, \ldots, T^{2m-1}, T^0)$ be a symmetric cycle in the tope graph $T(\mathcal{L})$. The set

$$\mathcal{K}^* := \max^+(\mathfrak{B}(R))$$

or, equivalently,

$$\mathcal{K}^* = \{T \in \mathfrak{B}(R) : S \in \mathfrak{B}(R), S(S, T) = \{e\} \Rightarrow T(e) = +\}$$

is a critical tope committee for $\mathcal{M}$, that satisfies the equality

$$|\{K \in \mathcal{K}^* : K(e) = +\}| = \left\lceil \frac{\mathcal{K}^*}{2} \right\rceil,$$

for every $e \in E$.

Proof. Descriptions (5.5) and (5.6) are equivalent by Remark 4.1(ii).

If $T^{(+)} \in \mathfrak{B}(R)$ then (5.5) is the one-element set $\{T^{(+)}\}$, that is a critical committee for $\mathcal{M}$; we are done.

If $T^{(+)} \not\in \mathfrak{B}(R)$ then Lemma 5.5 implies that $\mathcal{K}^*$ is a tope committee that satisfies (5.7), for all $e \in E_m$. We have to show that $\mathcal{K}^*$ is critical.

Assume that there is a proper subset $Q^*$ of the set $\mathcal{K}^*$ such that $\mathcal{K}^* - Q^*$ is a committee for $\mathcal{M}$. Since $T^{(+)} \not\in \mathfrak{B}(R)$, we have $|\mathcal{K}^* - Q^*| > 1$.

We without loss of generality suppose that $T^0 \in \max^+(\mathfrak{B}(R))$ and $T^0 \not\in Q^*$.

Let $\{g\} := S(T^0, T^1)$ and $\{f\} := S(T^{2m-1}, T^0)$; note that $f \neq g$. We have $T^0(f) = T^0(g) = +$. For every $k$, $0 < k < m$, we have $T^k(f) = +$; for every $l, m < l < 2m$, we have $T^l(g) = +$, see Remark 4.1(i).

By Lemma 5.5 for every $e \in \{f, g\}$ it holds $|\{Q \in \mathcal{Q}^* : Q(e) = -\}| = |\{R \in \mathcal{Q}^* : R(e) = +\}|$, and we have

$$|\{T^1, T^2, \ldots, T^{m-1}\} \cap \mathcal{Q}^*| = |\{T^{m+1}, T^{m+2}, \ldots, T^{2m-1}\} \cap \mathcal{Q}^*| = \frac{|\mathcal{Q}^*|}{2}.$$
$Q^*$ is of even cardinality. Let $T^{k_2}$ be a tope from the set $Q^*$, and $(T^{k_1}, T^{k_2}, T^{k_3})$ a 2-path in $R$. If \( \{p\} := S(T^{k_1}, T^{k_2}) \) and \( \{q\} := S(T^{k_2}, T^{k_3}) \), then there is an element \( h \in \{p, q\} \) such that \( |\{K \in K^* - Q^* : K(h) = +\}| = \left\lfloor \frac{|K^* - Q^*|}{2} \right\rfloor \), that is, the set of topes $K^* - Q^*$ is not a committee for $M$, a contradiction. Thus, $K^*$ is minimal and, as a consequence, it is critical, in view of (5.7).

Note that under the hypothesis of the Proposition 5.6, the set $-\{ \mathfrak{G}(R) - \text{max}^+(\mathfrak{G}(R)) \}$ is a tope committee for $M$ as well.

We now discuss some poset-theoretic properties of topes which are useful for analysis of the coverings of the ground sets of oriented matroids by pairs of the positive parts of topes.

**Corollary 5.7.** Let $M := (E_m, \mathcal{L}) = (E_m, T)$ be a simple oriented matroid that is not acyclic. Let $m$ be an arbitrary maximal chain in the tope poset $T(\mathcal{L}, B)$ with base tope $B \in \text{max}^+(T)$.

(i) Let \( c := \max\{|T^+| : T \in T\} \).

The subchain $\text{max}^+(m)$ contains a unique tope $K$ such that $B^+ \cup K^+ = E_m$. The poset rank $\rho(K)$ of $K$ satisfies the inequality

$$\rho(K) \geq 2m - c - |B^+| .$$  

(5.8)

For a tope $R \in m$, we have

$$B^+ \cup R^+ = E_m \iff R \supseteq K .$$  

(5.9)

(ii) For all $T', T'' \in m - \{B\}$, it holds $(T')^+ \cup (T'')^+ \neq E_m$.

(iii) The subposet

$$O(B) := \{T \in T(\mathcal{L}, B) : B^+ \cup T^+ = E_m\} = \bigcap_{e \in B^-} T_e^+$$  

(5.10)

is an order filter in the tope poset $T(\mathcal{L}, B)$, with

$$\min O(B) = \mathcal{G}(B) ,$$

where the antichain $\mathcal{G}(B)$ is defined by

$$\mathcal{G}(B) := \{T \in T(\mathcal{L}, B) : T \in \text{max}^+(T), B^+ \cup T^+ = E_m\} .$$  

(5.11)

Furthermore, if $M$ is totally cyclic, then it holds

$$O(B) = \text{conv}_T(\mathcal{G}(B)) .$$

- The union $\bigcup_{B \in \text{max}^+(T)} O(B)$ covers the set of topes $T$ of $M$.
- For any topes $B', B'' \in \text{max}^+(T)$, we have

$$|O(B') \cap O(B'')| > 0 \iff |\mathcal{G}(B') \cap \mathcal{G}(B'')| > 0 .$$

**Proof.** (i) The uniqueness of the tope $K \in \text{max}^+(\mathfrak{G}(R))$ such that $B^+ \cup K^+ = E_m$, and relation (5.9) are discussed in the proof of Proposition 5.3 (substitute $T^0$, $T'$ and $T''$ in that proof by $B$, $K$ and $-B$, respectively).

We have $|B^+| = m - |B^+|$ and $|K^+| = |(-B)^+| + (m - \rho(K))$; hence $|K^+| = 2m - |B^+| - \rho(K) \leq c$, and (5.8) follows.
(ii) This assertion is also inspired by the proof of Proposition 5.3 if \( m = (T^0, T^1, \ldots, T^m) \), then for all \( T', T'' \in m - \{B\} \), we have \((T')^+ \cup (T'')^+ \neq e\), where \( \{e\} := S(T^0, T^1) \).

(iii) The assertion follows from (i). \( \square \)

Example 5.8. Figure 11 depicts the Hasse diagram of a subposet \( O(B) \) from Corollary 5.7(iii) related to a reorientation of an oriented matroid realized by the hyperplane arrangement of Figure 1.

![Hasse diagram](image)

**Figure 11.** The Hasse diagram of the order filter \( O(B) \) in the tope poset \( T(L, B) \), defined by (5.10), for the reorientation \( M := -[1, 2] N^0 \); the oriented matroid \( N^0 \) is realized by the hyperplane arrangement of Figure 1, \( B := +--++ \).

The antichain \( G(B) := \min O(B) \) is \( \{+++--++, +++++-+-, -+-+-++-\} \subset \max^+(T) \).

The following assertion (a proof of which we sketch in the Appendix) shows that Algorithm 4.6 always constructs critical committees.

**Proposition 5.9.** Let \( N^0 := (E_m, L^0) = (E_m, T^0) \) be a simple acyclic oriented matroid whose sets of covectors and of topes are denoted by \( L^0 \) and \( T^0 \), respectively.

Let \( m := (R^0 := T^+ \prec R^1 \prec \cdots \prec R^m := T^-) \) be a maximal chain in the tope poset \( T(L^0, T^+) \).

Let \( s \) be an integer, \( 1 \leq s \leq m \). Denote by \( L^s \) and \( T^s \) the sets of covectors and of topes, respectively, of the reorientation \( N^s := -[1, s] N^0 \).

Let \( R := (T^0, T^1, \ldots, T^{2m-1}, T^0) \) be a symmetric cycle in the tope graph \( T^s(L^s) \), such that

\[
T^k := -[1, s] R^k, \quad 0 \leq k \leq m.
\]

Algorithm 4.6 builds the set \( \max^+ (\mathcal{U}(R)) \) which is a critical tope committee for \( N^s \).

5.2. The Graph of Topes with Maximal Positive Parts. Let \( M := (E_m, L) = (E_m, T) \) be a simple oriented matroid. Choose in the graph \( \Gamma(M) \), defined by (5.11), the subgraph of topes with inclusion-maximal positive parts \( \Gamma^+_{\max} := \Gamma^+_{\max}(M) \) for which

\[
\mathcal{U}(\Gamma^+_{\max}) := \max^+(T), \quad \{T^k, T^i\} \in \mathcal{E}(\Gamma^+_{\max}) \iff (T^k)^+ \cup (T^i)^+ = E_m. \quad (5.12)
\]
Example 5.10. The graph $\Gamma_{\text{max}}^+$ associated with a reorientation of an oriented matroid, that is realized by the hyperplane arrangement of Figure 1, is given in Figure 12.

The graph $\Gamma_{\text{max}}^+$, defined by (5.12), is a direct generalization of the graph of maximal feasible subsystems of an infeasible linear inequality system which has been studied in works [17, 18, 19, 21, 23].

The hypergraph of maximal feasible subsystems of an infeasible linear inequality system is discussed, e.g., in [26, 28, 29, 40]. An analogous construction for oriented matroids can be defined in the following way: given a simple oriented matroid $M = (E, T)$, the set of vertices of the hypergraph of topes with maximal positive parts $\Xi_{\text{max}}^+ := \Xi_{\text{max}}^+(M)$ is the set $\text{max}^+(T)$; a subset $H \subseteq \text{max}^+(T)$ is a hyperedge of $\Xi_{\text{max}}^+$ if $\bigcup_{T \in H} T^+ = E_m$; thus the family of hyperedges $\mathcal{E}(\Xi_{\text{max}}^+)$ is an order filter in the Boolean lattice of subsets of the set $\text{max}^+(T)$, with $\text{min} \mathcal{E}(\Xi_{\text{max}}^+) \supset \mathcal{E}(\Gamma_{\text{max}}^+)$.

A construction that is closely related to $\Gamma_{\text{max}}^+(M)$ for realizable coloopless simple oriented matroids $M$ is the graph of diagonals of a convex polytope. If $P$ is a convex polytope with vertex set $\text{vert} P$ then a diagonal of $P$ is a subset $D \subseteq \text{vert} P$ such that the convex hull $\text{conv} D$ of $D$ is not a proper face of $P$, but $\text{conv}(D - \{v\})$ lies in a proper face of $P$, for all vertices $v \in D$, see, e.g., [20, 22, 52]; if, furthermore, $\text{conv}(V)$ is a face of $P$, for any proper subset $V \subset D$, then $D$ is called an empty simplex (a missing face) of $P$, see, e.g., [21, 25, 42]. The graph of diagonals of $P$ is defined as the Kneser graph of the family of diagonals of $P$, see, e.g., [38, Definition 2.2.9].

Many properties of $\Gamma_{\text{max}}^+$, among which the most important are connectedness and the existence of an odd cycle, are inherited from the realizable case and lay the foundation of graph-theoretic procedures of constructing tope committees of ‘high quality’.
5.2.1. **General Properties of** \( \Gamma^\max \):

**Proposition 5.11.** Let \( \mathcal{M} := (E_m, \mathcal{L}) = (E_m, T) \) be a simple oriented matroid that is not acyclic.

(i) The graph \( \Gamma^\max := \Gamma^\max (\mathcal{M}) \) is connected. The degree of every its vertex is at least two. Any edge of \( \Gamma^\max \) is an edge of a cycle.

(ii) If for any 2-path \( (R, B, S) \) in \( \Gamma^\max \) there exists topes \( R', S' \in T \) such that

\[
(R')^+ \subseteq R^+ , \quad (S')^+ \subseteq S^+ ,
\]

\[
B^+ \cup (R')^+ = B^+ \cup (S')^+ = E_m ,
\]

\[
|(R')^+ \cap B^+ \cap (S')^+| = 0 ,
\]

(5.13)

then \( B \) is not a cutvertex in \( \Gamma^\max \).

(iii) \( \Gamma^\max \) contains an odd cycle.

**Proof.** (i) Let \( B \) and \( R \) be any distinct topes from the set \( \max^+(T) \).

Let \( R := (T^0 := B, T^1, \ldots , T^k := R, \ldots , T^{2m-1}, T^0) \) be a symmetric cycle in the tope graph \( T(\mathcal{L}) \). By Proposition 5.3, the set \( \max^+(\mathfrak{B}(R)) \) is the set of vertices of an odd cycle \( \mathcal{C} \), defined in the following way: for \( T', T'' \in \max^+(\mathfrak{B}(R)) \), we have \( (T', T'') \in \mathfrak{E}(\mathcal{C}) \) iff \( (T')^+ \cup (T'')^+ = E_m \).

Let \( \phi : \mathfrak{B}(\mathcal{C}) \rightarrow \mathfrak{B}(\Gamma^\max) := \max^+(T) \) be any mapping such that \( T^+ \subseteq (\phi(T))^+ \), for all \( T \in \mathfrak{B}(\mathcal{C}) \). This mapping is a graph homomorphism from \( \mathcal{C} \) to \( \Gamma^\max \). Since \( \mathcal{C} \) is (2-)connected, there is a path in \( \Gamma^\max \) between the vertices \( \phi(T') \) and \( \phi(T'') \), for all \( T', T'' \in \mathfrak{B}(\mathcal{C}) \). In particular, there is a path in \( \Gamma^\max \) between \( B \) and \( R \) because \( \phi(B) = B \) and \( \phi(R) = R \).

Now suppose that \( B^+ \cup R^+ = E_m \), that is, \( \{B, R\} \in \mathfrak{E}(\mathcal{C}) \). Let \( \{B, T\} \) be the edge of \( \mathcal{C} \) such that \( T \neq R \); then \( \{B, R\}, \{B, \phi(T)\} \in \mathfrak{E}(\Gamma^\max) \), where \( \phi(T) \neq B \) and \( \phi(T) \neq R \), therefore the degree of \( B \) in \( \Gamma^\max \) is greater than one.

Let \( D \) denote the path in the cycle \( \mathcal{C} \) between the vertices \( B \) and \( R \), such that \( T \in \mathfrak{B}(D) \). The image of \( D \) under the homomorphism \( \phi \) is a connected subgraph of \( \Gamma^\max \) whose set of edges does not contain the edge \( \{B, R\} = \{\phi(B), \phi(R)\} \). Hence, the edge \( \{B, R\} \in \mathfrak{E}(\Gamma^\max) \) is an edge of a cycle.

(ii) Assume that \( R \) and \( S \) belong to different blocks of the graph \( \Gamma^\max \), that is, \( B \) is a cutvertex.

Since, by the hypothesis of the assertion, it holds \( B^+ \cup (R')^+ = B^+ \cup (S')^+ = E_m \), condition \((5.13)\) implies

\[
\mathcal{S}(B, -R') \subset \mathcal{S}(B, S') , \quad \mathcal{S}(B, -S') \subset \mathcal{S}(B, R')
\]

and, as a consequence,

\[
S' \supset -R' , \quad R' \supset -S'
\]

in the tope poset \( T(\mathcal{L}, B) \). This implies that in the tope graph \( T(\mathcal{L}) \) there exists a symmetric cycle \( \mathcal{R} \) such that \( \{R', B, S'\} \subset \mathfrak{B}(R) \). Let \( T'' \) and \( S'' \)
be the topes with \((R')^+ \subseteq (R'')^+\) and \((S')^+ \subseteq (S'')^+\), such that \(R'', S'' \in \text{max}^+(\mathfrak{V}(R))\).

Let \(\mathcal{C} \text{ and } \phi : \mathfrak{V}(\mathcal{C}) \to \mathfrak{V}(\Gamma^\text{max}_+)\) be an odd cycle and a graph homomorphism, respectively, which were defined in the proof of assertion (i), with \(\phi(R'') := R\) and \(\phi(S'') := S\).

By Proposition 5.3, the sets \(\{R'', B\}\) and \(\{B, S''\}\) are edges of \(\mathcal{C}\) and, as a consequence, \((\phi(R''), \phi(B), \phi(S'')) = (R, B, S)\) is a 2-path in \(\Gamma^\text{max}_+\). Let \(\mathcal{D}\) denote the path in the cycle \(\mathcal{C}\) between the vertices \(R''\) and \(S''\), that does not contain \(B\). The image of \(\mathcal{D}\) under the homomorphism \(\phi\) is a connected subgraph of \(\Gamma^\text{max}_+\) whose set of vertices does not contain the tope \(B = \phi(B)\).

Hence, there is a cycle in \(\Gamma^\text{max}_+\) such that \(\{R, B, S\}\) is a subset of its vertices. This contradicts our assumption that \(B\) is a cutvertex in \(\Gamma^\text{max}_+\).

(iii) Let \(\mathcal{R} := (T^0, T^1, \ldots, T^{2m-1}, T^0)\) be a symmetric cycle in the tope graph \(\mathcal{T}(\mathcal{L})\), such that \(T^0 \in \text{max}^+(\mathcal{T})\). Let \(\mathcal{G}\) be the graph defined by (5.2). Recall that the set \(\text{max}^+(\mathfrak{V}(\mathcal{R}))\) is the set of vertices \(\mathfrak{V}(\mathcal{C})\) of an odd cycle \(\mathcal{C}\) in \(\mathcal{G}\), see Proposition 5.3. The set \(\mathcal{E}(\mathcal{C})\) of edges of \(\mathcal{C}\) is described by (5.13).

Assume that \(\Gamma^\text{max}_+\) is bipartite, with partition classes \(\mathcal{V}'\) and \(\mathcal{V}''\). Suppose that \(T^0 \in \mathcal{V}'\). Then, for a homomorphism

\[
\phi : \mathfrak{V}(\mathcal{C}) := \text{max}^+(\mathfrak{V}(\mathcal{R})) \to \mathfrak{V}(\Gamma^\text{max}_+) := \text{max}^+(\mathcal{T})
\]

from \(\mathcal{C}\) to \(\Gamma^\text{max}_+\), such that \(T^+ \subseteq (\phi(T))^+\) for all \(T \in \mathfrak{V}(\mathcal{C})\), we have

\[
\phi(T^0) = T^0 \in \mathcal{V}' , \quad \{T^0, T^{k_d+1}\}, \{T^0, T^{k_d}\} \in \mathcal{E}(\mathcal{C}) \implies \phi(T^{k_d+1}) \in \mathcal{V}' , \quad \phi(T^{k_d}) \in \mathcal{V}'' ;
\]

\[
\phi(T^{k_d+1}) \in \mathcal{V}' , \quad \{T^{k_1}, T^{k_d+1}\} \in \mathcal{E}(\mathcal{C}) \implies \phi(T^{k_1}) \in \mathcal{V}' ;
\]

\[
\phi(T^{k_1}) \in \mathcal{V}' , \quad \{T^{k_1}, T^{k_d+2}\} \in \mathcal{E}(\mathcal{C}) \implies \phi(T^{k_d+2}) \in \mathcal{V}'' ;
\]

\[
\vdots
\]

\[
\phi(T^{k_d}) \in \mathcal{V}' , \quad \{T^{k_d}, T^{k_d+2}\} \in \mathcal{E}(\mathcal{C}) \implies \phi(T^{k_d}) \in \mathcal{V}' .
\]

Since (5.15) contradicts (5.14), \(\Gamma^\text{max}_+\) is not bipartite; as a consequence, it contains an odd cycle, see, e.g., [21 Proposition 1.6.1].

Suppose that \(\Gamma^\text{max}_+\) has no cutvertices, that is, \(\Gamma^\text{max}_+\) is 2-connected. Since \(\Gamma^\text{max}_+\) contains an odd cycle then the general properties of 2-connected graphs [43 §5.4] imply that every vertex of \(\Gamma^\text{max}_+\) is contained in an odd cycle.

5.2.2. The Neighborhood Complex of \(\Gamma^\text{max}_+\). Let \(\mathcal{M} := (E_m, \mathcal{L}) = (E_m, \mathcal{T})\) be a simple oriented matroid that is not acyclic. Recall that the neighborhood of a vertex \(B\) in \(\Gamma^\text{max}_+ := \Gamma^\text{max}_+(\mathcal{M})\) is the set \(\{T \in \text{max}^+(\mathcal{T}) : B^+ \cup T^+ = E_m\}\). Equivalently, the neighborhood of \(B\) is the antichain \(\mathcal{G}(B)\) in the tope poset \(\mathcal{T}(\mathcal{L}, B)\), defined by (5.11).
According to the Folkman-Lawrence Topological Representation Theorem [5, §5.2], one can consider a representation of $\mathcal{M}$ by an arrangement of oriented pseudospheres

$$\{S_e : e \in E_m\}$$

lying on the standard $(r(\mathcal{M}) - 1)$-dimensional sphere $\mathbb{S}^{r(\mathcal{M})-1}$, where $r(\mathcal{M})$ denotes the rank of $\mathcal{M}$, see [11].

Recall that the simplicial complex of acyclic subsets of $E_m$, denoted by $\Delta_{\text{acyclic}}(\mathcal{M})$, is the nerve of the family

$$\{S_e^+ : e \in E_m\},$$

where $S_e^+$ denotes the open positive hemisphere corresponding to the pseudosphere $S_e$, that is, the positive side of $S_e$, see [11].

Suppose $\mathcal{M}$ is totally cyclic. Recall that in this case the complex $\Delta_{\text{acyclic}}$ is homotopy equivalent to $\mathbb{S}^{r(\mathcal{M})-1}$ because the union of the sets from (5.16) is an open cover of the sphere by subspaces whose nonempty intersections are contractible:

$$\bigcup_{e \in E_m} S_e^+ = \mathbb{S}^{r(\mathcal{M})-1},$$

see [11]. Combinatorial homotopy is discussed, e.g., in [6, §10].

By Corollary 5.7(iii), the neighborhood complex $\mathcal{NC}(\Gamma_{\text{max}}^+)$ of $\Gamma_{\text{max}}^+$ is the nerve of the family of open subspaces

$$\left\{ \bigcap_{e \in B^-} S_e^+ : B \in \text{max}^+(\mathcal{T}) \right\}$$

(5.17)
such that every their nonempty intersection is contractible.

The Nerve Theorem [6, Theorem 10.7] implies that $\mathcal{NC}(\Gamma_{\text{max}}^+)$ is homotopy equivalent to the subspace of $\mathbb{S}^{r(\mathcal{M})-1}$ covered by family (5.17).

Suppose $\mathcal{M}$ is neither acyclic nor totally cyclic. It is shown in [11] that there exists a unique non-negative covector $F \in \mathcal{L}$ with inclusion-maximum positive part. Denote by $\Gamma_{\text{max}}^+(\mathcal{M}\setminus F^+)$ the graph of topes with maximal positive parts, which is associated with the (totally cyclic) deletion $\mathcal{M}\setminus F^+$. The graph $\Gamma_{\text{max}}^+(\mathcal{M})$ is isomorphic to $\Gamma_{\text{max}}^+(\mathcal{M}\setminus F^+)$. 

6. Committees, Halfspaces, and Relatively Blocking Elements

Let $\mathcal{M} = (E_m, \mathcal{T})$ be a simple oriented matroid with set of topes $\mathcal{T}$.

Denote by $\mathcal{I}^+(\mathcal{T}_1^+, \ldots, \mathcal{T}_m^+)$ the family of all set-theoretic committees for the family of positive halfspaces $\{\mathcal{T}_1^+, \ldots, \mathcal{T}_m^+\}$: by definition, a set $\mathcal{P} \subset \mathcal{T}$ is a committee for the family of positive halfspaces $\{\mathcal{T}_1^+, \ldots, \mathcal{T}_m^+\}$ iff it holds

$$|\mathcal{P} \cap \mathcal{T}_e^+| > \frac{|\mathcal{P}|}{2},$$

for all $e \in E_m$. For every $e \in E_m$, we have $|\mathcal{T}_e^+| = \frac{|\mathcal{T}|}{2}$; the way of computing the cardinality of the tope set $\mathcal{T}$ is well-known [5, Theorem 4.6.1], [32, 53, 54].
We have

\[ K^*(\mathcal{M}) = I^\bot_2(T_1^+, \ldots, T_m^+) \],

that is, a set \( K^* \subseteq T \) is a tope committee for \( \mathcal{M} \) if \( K^* \) is a set-theoretic committee for the family \( \{T_1^+, \ldots, T_m^+\} \).

Let \( \mathbb{B}(T) \) denote the Boolean lattice of all subsets of \( T \). For any \( e \in E_m \), we relate to the set \( T_e^+ \) the element \( \nu_e := \bigvee_{T \in T_e^+} T \in \mathbb{B}(T) \), the join of those atoms of \( \mathbb{B}(T) \) that compose \( T_e^+ \).

Consider every tope committee for \( \mathcal{M} \) as an element of the Boolean lattice \( \mathbb{B}(T) \). Then \( K^*(\mathcal{M}) \) is precisely the subposet \( I^\bot_2(\mathbb{B}(T), \{v_1, \ldots, v_m\}) \) of all relatively \( \frac{1}{2} \)-blocking elements for the antichain \( \{v_1, \ldots, v_m\} \) in \( \mathbb{B}(T) \), with respect to the poset rank function on \( \mathbb{B}(T) \). Relative blocking in posets is discussed in [37]. The antichain \( \eta^\bot_2(\mathbb{B}(T), \{v_1, \ldots, v_m\}) := \min I^\bot_2(\mathbb{B}(T), \{v_1, \ldots, v_m\}) \), called in [37] the relative \( \frac{1}{2} \)-blocker of \( \{v_1, \ldots, v_m\} \) in \( \mathbb{B}(T) \), is the family of all minimal committees for \( \mathcal{M} \).

For \( k \in [1, |\mathcal{T}| - 1] \), the subposet \( I^\bot_{2,k}(\mathbb{B}(T), \{v_1, \ldots, v_m\}) \) of elements of rank \( k \) from \( I^\bot_2(\mathbb{B}(T), \{v_1, \ldots, v_m\}) \) is

\[ I^\bot_{2,k}(\mathbb{B}(T), \{v_1, \ldots, v_m\}) = \mathbb{B}(T)^{(k)} \cap \bigcap_{e \in E_m} \mathfrak{F}(\mathcal{J}(v_e) \cap \mathbb{B}(T)^{([k+1]/2)}) \],

according to [37, Proposition 5.1(ii)]. Layers \( I^\bot_{2,k}(\mathbb{B}(T), \{v_1, \ldots, v_m\}) \) are discussed in [36], and Farey subsequences appearing in their analysis are considered in [35] and [36].

**Appendix**

We sketch here a proof of Proposition 5.9.

**Sketch of proof.** Argue by induction on \( i \), \( 1 \leq i \leq s \).

The following observation is useful:

**Claim.** For any \( t' \) and \( t'' \) such that \( 0 \leq t' < t'' \leq m \), it holds

\[
((-1,i] R^{t''})|_{[1,i]}^+ \subseteq ((-1,i] R^{t''})|_{[1,i]}^+,
\]

\[
((-1,i] R^{t'})|_{[i+1, m]}^+ \supseteq ((-1,i] R^{t''})|_{[i+1, m]}^+.
\]

For any \( t' \) and \( t'' \) such that \( 1 \leq t' < t'' \leq m \), it holds

\[
((-i,m] R^{t''})|_{[1,i]}^+ \supseteq ((-i,m] R^{t''})|_{[1,i]}^+,
\]

\[
((-i,m] R^{t'})|_{[i+1, m]}^+ \subseteq ((-i,m] R^{t''})|_{[i+1, m]}^+.
\]

Assign to every tope \( R^t \in \{R^1, \ldots, R^m\} \) the label \( l_t \) defined by (4.8).

Suppose \( i = 1 \).
If \( l_1 = i \) then we have \( \mathcal{K}^*_i = \{-1,i\}R^1 \} = \{T(+)\} \) because at Step 03 the algorithm builds the set \( \mathcal{K}^*_i = \{-1,i\}R^1, -\{1,i\}R^0, -\{1,i\}R^1 \} \), and the pair of opposites \( \{-1,i\}R^0, -\{1,i\}R^1 \} \) is removed at Steps 04-05.

If \( l_m = i \) then we have \( \mathcal{K}^*_i = \{-\{i\}m\}R^m \} = \{T(+)\} \) because at Step 03 the algorithm builds the set \( \mathcal{K}^*_i = \{-\{i\}m\}R^m, -\{1,i\}R^0, -\{1,i\}R^m \} \), and the pair of opposites \( \{-1,i\}R^0, -\{1,i\}R^m \} \) is removed at Steps 04-05.

If \( l_j = i \), for some \( j \), \( 1 < j < m \), then we have \( \mathcal{K}^*_i = \{-1,i\}R^0, -\{1,i\}R^k, -\{i,m\}R^k \} \). Since

\[
(-l_m(-R^m))(i) = R^0(i) = R^1(i) = +, \quad (-l_m(-R^m))^+ \subseteq (R^0)^+ \supseteq (R^1)^+, \quad R^j(i) = R^j+1(i) = -, \quad (R^j-1)^+ \supseteq (R^j)^+ \supseteq (R^j+1)^+, \quad
\]

and

\[
(-l_{j-1}(-R^j-1))(i) = (-l_{j}(-R^j))(i) = -, \quad (-l_{j+1}(-R^j+1))(i) = +, \quad (-l_{j-1}(-R^j-1))^+ \subseteq (-l_{j}(-R^j))^+ \subseteq (-l_{j+1}(-R^j+1))^+, \quad
\]

we have

\[
(-\{i,m\}R^m)^+ \subseteq (R^0)^+ \supseteq (R^1)^+, \quad (-\{1,i\}R_{j-1})^+ \subseteq (-\{1,i\}R_{j})^+ \supseteq (-\{1,i\}R_{j+1})^+, \quad (-\{i,m\}R^m)^+ \subseteq (R^0)^+ \supseteq (R^1)^+, \quad
\]

therefore

\[
(R^0)^+, (-\{1,i\}R^j)^+, (-\{i,m\}R^j)^+ \in \max\{(\{-1,i\}R_k^k)^+ : 0 \leq k \leq 2m-1\}, \quad
\]

see Remark 4.4(ii).

One can show by means of a similar argument that for every \( R \in \Omega(R) \) such that \( R \not\in \{R^0, R^j, -l_j(-R^j)\} \), it holds \( R^+ \not\in \max\{(\{-1,i\}R_k^k)^+ : 0 \leq k \leq 2m-1\} \).

Thus,

\[
\{K^+ : K \in \mathcal{K}^*_i\} = \max\{(\{-1,i\}R^k)^k^+ : 0 \leq k \leq 2m-1\}. \quad
\]

If \( i = s \) then we are done.

Suppose \( i > 1 \).

By the induction hypothesis, we have

\[
\{K^+ : K \in \mathcal{K}^*_i\} = \max\{(\{-1,1,i\}R_k^k)^k^+ : 0 \leq k \leq 2m-1\}. \quad
\]

Suppose that \( l_j = i \), for some \( j \), \( 1 \leq j \leq m \).

Consider the set

\[
\overline{\mathcal{K}^*_i} := \{-iK : K \in \mathcal{K}^*_i\} \cup \{-1,i\}R^j, -\{1,i\}R^j \} . \quad
\]

• Suppose \( j = 1 \).
  • Suppose that
    \[
    \{-1,i\}R^0 \not\in \overline{\mathcal{K}^*_i}, \quad [-i,m]R^j+1 \not\in \overline{\mathcal{K}^*_i}. \quad (6.1)
    \]
For $i > 1$, we only describe the induction step for the case where $j = 1$ and \((6.1)\) holds. Analysis of other situations is completely analogous.

Condition \((6.1)\) implies that
\[-[1,i]R^{m-1} \not\subseteq K_i^\ast, \quad -[1,i]R^m \not\subseteq K_i^\ast, \quad -[1,i]R^{i+1} \not\subseteq K_i^\ast,\]
and it holds
\[-[1,i-1]mR^m+ \not\subseteq (-[1,i-1]mR^0)+ \not\subseteq (-[1,i-1]mR^1)+ \not\subseteq (-[1,i-1]mR^i)+;\]
\[-[1,i-1]R^{m-1}+ \subseteq (-[1,i-1]mR^m)+ = ([1,i-1]mR^j)+ \subseteq (-[1,i-1]mR^i)+\]
\([-[1,i]R^0)(e) = - , 1 \leq e \leq i-1 ; \quad (-[1,i]R^0)(e) = + , i \leq e \leq m ;\]
\([-[1,i]R^j)(e) = - , 1 \leq e \leq i ; \quad (-[1,i]R^j)(e) = + , i + 1 \leq e \leq m ;\]
\([-[1,i]R^{i+1})(e) = - , 1 \leq e \leq i .\]

As a consequence, we have
\([-[1,i]R^0)+ \subseteq (-[1,i]mR^j)+ \supseteq (-[1,i]R^{i+1})+,\]
that is,
\([-[1,i]R^j)+ \in \max\{(-[1,i]R^k)+ : 0 \leq k \leq 2m - 1\} , \quad (6.2)\]
see Remark \(4.1(ii)\).

Since
\([-[1,i-1]R^{m-1})(e) = - , 1 \leq e \leq m ;\]
\([-[1,i-1]R^m)(e) = + , 1 \leq e \leq i-1 ; \quad (-[1,i-1]R^m)(e) = - , i \leq e \leq m ,\]
and \([-[i,m]R^{i+1}) = (-[1,i]R^j)+ \in \min\{(-[1,i]R^k)+ : 0 \leq k \leq 2m - 1\} , \quad (6.2)\]
we obtain
\([-[1,i]R^{m-1}) \supseteq (-[1,i]mR^m)+ = (-[i,m]R^j)+ \supseteq (-[i,m]R^{i+1})+,\]
that is,
\([-[i,m]R^j)+ \in \max\{(-[1,i]R^k)+ : 0 \leq k \leq 2m - 1\} ,\]
by Remark \(4.1(ii)\).

Note that for all $k$, $1 \leq k \leq m$, it holds \((-[1,i]R^k)(i) = +\), cf. Remark \(4.1(i)\).

Let \(-iK \in K_i^\ast - \{-[1,i]R^j, -[i,m]R^j\}\), that is, $K \in K_{i-1}^\ast$. Let \((R', K, R'')\) be a 2-path in the cycle \([-[1,i-1]R^0, -[1,i-1]R^1, \ldots, -[1,i-1]R^{2m-1} \subseteq [1,i-1]R^0\); by Remark \(4.1(ii)\), the vertices of the path satisfy \((R')^+ \subseteq K^+ \not\subseteq (R'')^+.\) Remark \(4.1(i)\) implies that \((R'(i) = K(i) = R''(i)\). As a consequence, the equality \((-iK)'(i) = (-iK)'(i) = (-iK'')(i)\) holds as well, and we have \((-iR')^+ \subseteq (-iK)^+ \not\subseteq (-iR'')^+.\) Thus, we have \(\{(-iK)^+ : K \in K_{i-1}^\ast\} \subseteq \max\{(-[1,i]R^k)+ : 0 \leq k \leq 2m - 1\}\).
In a similar manner, one can show that for any $R \in \{-[1,i]R^0, -[1,i]R^1, \ldots, -[1,i]R^{2m-1}\}$ such that $-[1,i]R \not\in K_i^*$ it holds $(-[1,i]R)^+ \not\in \max\{(-[1,i]R^k)^+ : 0 \leq k \leq 2m-1\}$.

The algorithm builds the set

$$K_i^* = \overline{K_i^*};$$

we have seen that

$$\{K^+ : K \in K_i^*\} = \max\{(-[1,i]R^k)^+ : 0 \leq k \leq 2m-1\} .$$

⋄ If $-[1,i]R^0 \in K_i^*$, $-[1,i]R^j+1 \not\in K_i^*$, then

$$K_i^* = \overline{K_i^*} - \{-[1,i]R^0, -[1,i]R^j\}$$

$$= \{-iK : K \in K_{i-1}^*\} - \{-[1,i]R^0\} \cup \{-[1,i]R^j\} .$$

⋄ If $-[1,i]R^0 \not\in K_i^*$, then

$$K_i^* = \overline{K_i^*} - \{-[1,i]R^0, -[1,i]R^j, -[1,i]R^j+1\}$$

$$= \{-iK : K \in K_{i-1}^*\} - \{-[1,i]R^0, -[1,i]R^j+1\} .$$

If $i = s$ then we are done.

• Suppose $j = m$.

⋄ If $-[1,i]R^0 \not\in K_i^*$, then

$$K_i^* = \overline{K_i^*} .$$

⋄ If $-[1,i]R^0 \in K_i^*$, $-[1,i]R^j-1 \not\in K_i^*$, then

$$K_i^* = \overline{K_i^*} - \{-[1,i]R^0, -[1,i]R^j\}$$

$$= \{-iK : K \in K_{i-1}^*\} - \{-[1,i]R^0\} \cup \{-[1,i]R^j\} .$$

⋄ If $-[1,i]R^0 \not\in K_i^*$, then

$$K_i^* = \overline{K_i^*} - \{-[1,i]R^0, -[1,i]R^j-1\}$$

$$= \{-iK : K \in K_{i-1}^*\} - \{-[1,i]R^0, -[1,i]R^j-1\} .$$
Suppose $1 < j < m$.

- If $-[1,i]R^j - 1 \notin \mathcal{K}_i^*$, $-[i,m]R^j + 1 \notin \mathcal{K}_i^*$,
  then $\mathcal{K}_i^* = \mathcal{K}_i^*$.

- If $-[1,i]R^j - 1 \in \mathcal{K}_i^*$, $-[i,m]R^j + 1 \notin \mathcal{K}_i^*$,
  then $\mathcal{K}_i^* = \mathcal{K}_i^* - \{-[1,i]R^j - 1, -[i,m]R^j\}$
  $= \{-K : K \in \mathcal{K}_{i-1}^*\} - \{-[1,i]R^j - 1\} \cup \{-[1,i]R^j\}$.

- If $-[1,i]R^j - 1 \in \mathcal{K}_i^*$, $-[i,m]R^j + 1 \in \mathcal{K}_i^*$,
  then $\mathcal{K}_i^* = \mathcal{K}_i^* - \{-[1,i]R^j - 1, -[1,i]R^j, -[i,m]R^j, -[i,m]R^j + 1\}$
  $= \{-K : K \in \mathcal{K}_{i-1}^*\} - \{-[1,i]R^j - 1, -[i,m]R^j + 1\}$.

By induction, we have $\mathcal{K}_s^* = \max^+(\mathfrak{R}(R))$.

According to Proposition 5.6, this is a critical committee for $\mathcal{N}^s$. \qed

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