On the non-relativistic limit of the spherically symmetric Einstein-Vlasov-Maxwell system

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Abstract

The Einstein-Vlasov-Maxwell (EVM) system can be viewed as a relativistic generalization of the Vlasov-Poisson (VP) system. As it is proved below, one of nice property obeys by the first system is that the strong energy condition holds and this allows to conclude that the above system is physically viable. We show in this paper that in the context of spherical symmetry, solutions of the perturbed (EVM) system by $\gamma := 1/c^2$, $c$ being the speed of light, exist and converge uniformly in the $L^\infty$-norm, as $c$ goes to infinity on compact time intervals to solutions of the non-relativistic (VP) system.

1 Introduction

The classical (VP) system models the time evolution of collisionless particles in the Newtonian dynamic setting, particles could be for instance atoms and molecules in neutral gas or electrons and ions in a plasma. In stellar dynamics, particles are either stars in galaxy or galaxies in a cluster of galaxies [1]. The global behaviour of solutions of the above system now is well understood (see [6], [11], [12], [16], [4]). In [13], the authors use an estimate to prove that the Newtonian limit of the spherically symmetric Vlasov-Einstein system is the classical (VP) system. This result is extended by Rendall in [14] to the general asymptotically flat Einstein-Vlasov system. The above investigation concerns the case where the charge of particles is small to be neglected.

We are inspired by what is done in [13] and we want to extend this result considering in this paper the (EVM) system that models the time evolution of self-gravitating collisionless charged particles in the general relativity setting. Firstly we discuss and obtain that the above system satisfies the strong energy condition, and then the system is physically viable. Secondly, we perturb the (EVM) system by a parameter $\gamma = 1/c^2$ and together with the assumption of spherical symmetry, we show using an estimate that for $\gamma$ small, the Cauchy problem associated with the obtained system admits a unique regular solution,
with time existence interval independent of \( \gamma \). Once this result is obtained we deduce that if \( c \) goes to infinity then the above solution converges uniformly to a solution of the classical (VP) system. To do so, we consider the new constraint equations and discuss the existence of solutions satisfying the constraints as we did in \([7]\). This gives rise to a new mathematical feature since the present equations contain two parameters \( \gamma \) and \( q \) (\( q \) being the charge of particle), rather than one as it is the case where \( \gamma = 1 \). We obtain that for given initial datum to the distribution function, solutions of the constraints exist and depend smoothly (i.e \( C^\infty \)) on parameter \( \zeta := (\gamma, q) \), for \( \zeta \) small, and this allows us to construct the set of initial data so that the initial datum for the metric function \( \lambda \) is bounded in the \( L^\infty \)-norm. So, with these initial data, we undertake the Cauchy problem for the modified (EVM) system and establish as we did in \([8]\), the local existence and uniqueness theorem and continuation criterion. The interest of this paper lies on the fact that the estimates we use are complicated to establish. In fact, contrary to the uncharged case, due to the presence of metric function \( \lambda \) in both side of equations, we have to estimate both the supremum of momenta on the support of the distribution function \( f \) and the supremum of \( e^{2\lambda \gamma} \). We are not aware that the above is already done.

It is appropriate at this point to put our investigation in the context of general relativity. The classical limit of the relativistic Vlasov-Maxwell system is studied by Jack Schaeffer \([17]\). Using and estimate, the author proves that this limit is the classical (VP) system. Also, the same result is established by Asano \([2]\) for the classical Vlasov-Maxwell system. The recent result in this field is that of S. Calogero and H.Lee \([3]\). They consider the relativistic Nordström-Vlasov system and establish that its non-relativistic limit is the classical (VP) system.

The paper is organized as follows. In Sect.2, we recall the classical (VP) system. In Sect.3, we introduce the general formulation of the (EVM) system and we discuss the energy conditions. In Sect.4, we introduce the perturbed spherically symmetric (EVM) system and state the main results of this paper.

## 2 The classical (non-relativistic) (VP) system

In the Newtonian dynamic, the time evolution of a set of collisionless particles is governed by the following equations known as the (VP) system:

\[
\partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_p f = 0 \tag{2.1}
\]

\[
\nabla_x U = 4\pi \eta M, \quad \eta = \pm 1, \quad M := \int_{\mathbb{R}^3} f(t, \tilde{x}, \tilde{p})d\tilde{p} \cdot \nabla_{\tilde{p}} f = 0 \tag{2.2}
\]

Here (2.1) is the Vlasov equation for the unknown \( f \), \( f \) being the distribution function that measures the probability density to find a particle (star) at time \( t \) with position \( \tilde{x} \) and with momentum \( \tilde{p} \), where \( t > 0, (\tilde{x}, \tilde{p}) \in \mathbb{R}^2 \). Note that \( f \) is defined on the mass shell. (2.2) is the Poisson equation for the unknown
\( U = U(t, \tilde{x}) \) that measures the Newtonian potential generated by stars. If \( \eta = -1 \) then (2.1) and (2.2) model the plasma physics case. In what follows, we consider the case where \( \eta = 1 \).

3 The (EVM) system

As we said before, we take fast moving collisionless particles with unit mass and charge \( q \). The gravitational constant and the speed of light are taken to be equal to unity. The basic spacetime is \((\mathbb{R}^4, g)\), with \( g \) a Lorentzian metric with signature \((-++,++)\). In the sequel, we assume that Greek indices run from 0 to 3 and Latin indices from 1 to 3. We adopt the Einstein summation convention. The (EVM) system reads in local coordinates at a given point \((x^\alpha) = (t, \tilde{x})\):

\[
R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = 8\pi(T_{\alpha\beta}(f) + \tau_{\alpha\beta}(F)) \tag{3.1}
\]

\[
\frac{\partial f}{\partial t} + p_i \frac{\partial f}{\partial x^i} - (\Gamma^i_{\beta\gamma}p_{\beta}p_{\gamma} + q p_{\beta}F_{\beta}^i) \frac{\partial f}{\partial p^i} = 0 \tag{3.2}
\]

\[
\nabla_{\alpha}F^{\alpha\beta} = J_{\beta}(f); \quad \nabla_{\alpha}F_{\beta\gamma} + \nabla_{\beta}F_{\gamma\alpha} + \nabla_{\gamma}F_{\alpha\beta} = 0, \tag{3.3}
\]

with:

\[
T_{\alpha\beta}(f) = -\int_{\mathbb{R}^3} p_{\alpha}p_{\beta}\omega_p; \quad \tau_{\alpha\beta} = \frac{g_{\alpha\beta}}{4}F_{\gamma\nu}F^{\gamma\nu} + F_{\beta\gamma}F_{\alpha}^{\gamma}
\]

\[
J^\beta(f)(x) = q \int_{\mathbb{R}^3} p^\beta f(x, p)\omega_p; \quad \omega_p = |g|^{1/2} \frac{dp^1dp^2dp^3}{p_0}, \quad p_0 = g_{00}p^0,
\]

where \( \Gamma^\alpha_{\beta\gamma} \) denote the Christoffel symbols. In the above, \( f \) stands for the distribution function of the charged particles defined on the mass shell:

\[
g_{\alpha\beta}p^{\alpha}p^{\beta} = -1,
\]

\( F \) stands for the electromagnetic field created by the charged particles. Here (3.1) is the Einstein equations for the metric tensor \( g = (g_{\alpha\beta}) \) with sources generated by both \( f \) and \( F \), that appear in the stress-energy tensor \( T_{\alpha\beta} + \tau_{\alpha\beta} \). Equation (3.2) is the Vlasov equation for \( f \) and (3.3) are the Maxwell equations.

3.1 The energy conditions

Physically, the quantity \( (T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha V^\beta \) represents the energy density of charged particles obtained by an observer whose 4-velocity is \((V^\alpha)\). So, for any physically viable theory, this quantity is nonnegative for every timelike vector \((V^\alpha)\), and the above assumption is known as the weak energy condition. We are going to show that in fact, the dominant energy condition holds, i.e: \( (T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha W^\beta \geq 0 \), for all future-pointing timelike vectors \((V^\alpha)\) and \((W^\alpha)\). This implies the weak energy condition. Note also that the above definition
of dominant energy condition is equivalent to that given by Hawking and Ellis in [5, p.91]. Once this is clarified, we show that since the Maxwell tensor is traceless, and the strong energy condition that is \( R_{\alpha\beta}V^\alpha V^\beta \geq 0 \), for every timelike vector \((V^\alpha)\), holds for the Einstein-Vlasov system, the same is true in our context.

**Lemma 3.1** The following assertions are equivalent:

1) For any two future-pointing timelike vectors \((V^\alpha), (W^\alpha)\), one has:
\[
(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha W^\beta \geq 0
\]

2) For every timelike vector \((V^\alpha)\), one has:
\[
(T_{\alpha\beta} + \tau_{\alpha\beta})V^\alpha V^\beta \geq 0, \text{ and } (T_{\alpha\beta} + \tau_{\alpha\beta})\text{ is a non-spacelike vector.}
\]

**Proof:** We first prove that 2) implies 1). If \((V^\alpha)\) and \((W^\alpha)\) are two future-pointing timelike vectors then 2) implies that \((T_{\alpha\beta} + \tau_{\alpha\beta})V^\beta\) is non-spacelike. This means by definition that it is either timelike or null. Using once again 2), its contraction with \((V^\alpha)\) is non-spacelike. To do this, let us assume that \((P_\alpha)\) is timelike vector orthogonal to \((V^\alpha)\) is past-pointing timelike, then \((P_\alpha)\) is non-spacelike and so is \((P_\alpha)\). In conclusion, we aim to show that \((P_\alpha)\) is non-spacelike. To do this, we assume that \((P_\alpha)\) satisfies \(P_\alpha W^\alpha \geq 0\) for every future-pointing timelike vector \((W^\alpha)\). We aim to show that \((P_\alpha)\) is non-spacelike. To do this, let us assume that \((P_\alpha)\) is spacelike, and get a contradiction. Set \(L := P_\alpha P^\alpha\). By assumption \(L > 0\). Let \((T^\alpha)\) be a future-pointing timelike vector orthogonal to \((P_\alpha)\) with \(T_\alpha T^\alpha = -L\) and \(T^0 > P^0\) (for the construction of vector \((T^\alpha)\), one can take for instance in normal coordinates: \(T^0 = \sqrt{\sum_{i=1}^{3} (P^i)^2}, T^i = P^0 P^i / T^0\)). Set \(W^\alpha = 2T^\alpha - P^\alpha\). Then \(W^\alpha W_\alpha = -3L < 0, W^0 = 2T^0 - P^0 > 0\) and \((W^\alpha)\) is a future-pointing timelike vector, and \(W^\alpha P_\alpha = -L < 0\). This is the desired contradiction. Now if \((V^\alpha)\) is past-pointing timelike, then \((-V^\alpha)\) is future-pointing and follow the first step of the proof in which \((P_\alpha)\) is replaced by \(-P_\alpha = (T_{\alpha\beta} + \tau_{\alpha\beta})(-V^\beta)\). Analogously, we are led to \((-P_\alpha)\) is non-spacelike and so is \((P_\alpha)\). In conclusion, the second part of condition 2) holds, for every timelike vector \((V^\alpha)\) and the proof is complete.

**Proposition 3.1**

1) For every two future-pointing vectors \((V^\alpha), (W^\alpha)\), one has:
\[
T_{\alpha\beta}V^\alpha W^\beta + \tau_{\alpha\beta}V^\alpha W^\beta \geq 0 \tag{3.4}
\]

2) For every timelike vector \((V^\alpha)\), one has:
\[
R_{\alpha\beta}V^\alpha V^\beta \geq 0.
\]
Proof: Consider part 1) of the above proposition. Since Penrose and Rindler state the dominant energy condition for the Maxwell equations in writing the Maxwell tensor $\tau_{\alpha\beta}$ as a quadratic form of spinor fields in [10], the second term in the left hand side of (3.4) is nonnegative and we just need to establish the same result for the first term in the left hand side of (3.4). Let $(V^\alpha)$, $(W^\alpha)$ be two future-pointing timelike vectors. Taking the first term in the left hand side of (3.4), we obtain, since $f \geq 0$, $(p^\alpha)$ is future-pointing timelike vector and $-p_0 > 0$, one has:

$$T_{\alpha\beta}V^\alpha W^\beta = \int_{\mathbb{R}^3} (p_\alpha V^\alpha)(p_\beta W^\beta)f \ g \left| \frac{1}{2} \ dp^1 dp^2 dp^3 - p_0 \right| \geq 0$$

So, (3.3) holds as announced. Now concerning part 2) of the above Proposition, the contraction of (3.1) gives, since $g_{\alpha\beta} \tau_{\alpha\beta} = 0$:

$$R = -8\pi T$$

where $T := g^{\alpha\beta}T_{\alpha\beta}$. Insertion of the above in (3.1) yields:

$$R_{\alpha\beta} = -4\pi T g_{\alpha\beta} + 8\pi (T_{\alpha\beta} + \tau_{\alpha\beta})$$

Next, let $(V^\alpha)$ be a timelike vector. Then

$$R_{\alpha\beta} V^\alpha V^\beta = 8\pi \tau_{\alpha\beta} V^\alpha V^\beta + 8\pi \left( T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta} \right) V^\alpha V^\beta$$

(3.5)

Since the Maxwell tensor satisfies the dominant energy condition and then the weak energy condition, we can deduce that the first term in the right hand side of (3.5) is nonnegative. Also, the strong energy condition holds for the Einstein-Vlasov system (for more details one can refer to [15], p.37-38). The latter shows that the second term in the right hand side of (3.5) is nonnegative and the strong energy condition holds in our context.

4 The perturbed spherically symmetric (EVM) system and the main results

We consider the Lorentzian spacetime $(\mathbb{R}^4, g_{\gamma})$ where $g_{\gamma}$ is obtained by scaling $g$ with $\gamma = 1/c^2$ in the spatial part. With the assumption of spherical symmetry, we take $g_{\gamma}$ of the following form:

$$ds^2 = -e^{2\mu}dt^2 + e^{2\lambda}dr^2 + r^2(\gamma d\vartheta^2 + \sin^2\vartheta d\varphi^2)$$

where $\mu = \mu(t, r); \lambda = \lambda(t, r); t \in \mathbb{R}; r \in [0, +\infty]; \vartheta \in [0, \pi]; \varphi \in [0, 2\pi]$.

Next, the assumption of spherical symmetry allows us to deduce the following perturbed (EVM) system in the $(t, x, v)$ coordinates (see [9]); where

$$v^i := p^i + (e^\lambda - 1) \frac{\hat{x} \cdot \hat{p} x^i}{r} :$$
\[ e^{-2\lambda}(2r\lambda' - 1) + 1 = 8\pi\gamma r^2 \rho \quad (4.1) \]

\[ e^{-2\lambda}(2r\mu' - 1) + 1 = 8\pi\gamma^2 r^2 \rho \quad (4.2) \]

\[ \frac{\partial f}{\partial t} + e^{\mu-\lambda} \frac{v}{\sqrt{1 + \gamma v^2}} \frac{\partial f}{\partial \tilde{x}} \left( e^{\mu-\lambda} \mu' \frac{1}{\gamma} \sqrt{1 + \gamma v^2} + \tilde{x} \cdot v - q e^{\lambda + \mu} e(t, r) \right) \frac{\tilde{x}}{r} \frac{\partial f}{\partial v} = 0 \quad (4.3) \]

\[ \frac{\partial}{\partial r} (r^2 e^\lambda) = qr^2 \gamma^{3/2} e^\lambda M \quad (4.4) \]

where \( \lambda' = \frac{d\lambda}{dr} \); \( \dot{\lambda} = \frac{d\lambda}{dt} \) and:

\[ \rho(t, \tilde{x}) := \int_{\mathbb{R}^3} f(t, \tilde{x}, v) \sqrt{1 + \gamma v^2} dv + \frac{1}{2} e^{2\lambda(t, \tilde{x})} e^2(t, \tilde{x}) \quad (4.5) \]

\[ p(t, \tilde{x}) := \int_{\mathbb{R}^3} \left( \frac{\tilde{x} \cdot v}{r} \right)^2 f(t, \tilde{x}, v) \frac{dv}{\sqrt{1 + \gamma v^2}} - \frac{1}{2} e^{2\lambda(t, \tilde{x})} e^2(t, \tilde{x}) \quad (4.6) \]

\[ M(t, \tilde{x}) := \int_{\mathbb{R}^3} f(t, \tilde{x}, v) dv. \quad (4.7) \]

Next, it is well known that in the context of spherical symmetry, the (VP) system reduces to:

\[ \partial_t f + v \cdot \partial_{\tilde{x}} f - K(t, \tilde{x}) \cdot \partial_v f = 0 \quad (4.8) \]

\[ K(t, \tilde{x}) = \frac{1}{r^2} \int_{|y| \leq r} M(t, y) dy. \quad (4.9) \]

In the sequel we denote by \((EV\gamma M)\) the system \( (4.1), (4.2), (4.3) \) and \((4.4)\). It is prescribed on \((EV\gamma M)\) the initial data \( \lambda(0) = \tilde{\lambda} \) and \( f(0) = \tilde{f} \), where \( \tilde{\lambda} \) is nonnegative, spherically symmetric function i.e invariant when one applies any rotation of \( \mathbb{R}^3 \) on both the variables \( \tilde{x} \) and \( v \), \( \tilde{\lambda} \) being a given function. Concerning the system \((4.8)\) and \((4.9)\) only the initial condition \( f(0) = \tilde{f} \) is needed. Also, we are interesting with the asymptotic flatness of spacetime and a regular center at \( r = 0 \). So, we have the following boundary conditions:

\[ \lim_{r \to +\infty} \lambda(t, r) = \lim_{r \to +\infty} \mu(t, r) = \lambda(0, r) = 0, \quad t \geq 0. \quad (4.10) \]

Now, the assumption that the electric field \( E \) defined by \( E(t, \tilde{x}) = e(t, r) \tilde{\tilde{x}} \), is spherically symmetric and the fact that at the spatial infinity there is no charged particle lead to the following boundary condition (for more details, see [9]):

\[ \lim_{r \to +\infty} e(t, r) = e(0, r) = 0, \quad t \geq 0. \quad (4.11) \]
Before stating the main result of this section, we recall the following result whose proof is made by straightforward calculation (see [9]). The concept of regularity of solutions we use is the same as in [9].

**Lemma 4.1** Take $\gamma = 1$. Let $(f, \lambda, \mu, e)$ be a regular solution of $\text{EV}_M$ satisfying (4.10) and (4.11) on some interval $I$. Then for every $a \in \mathbb{R} \setminus \{0\}$,

$$
\begin{align*}
   f_a(t, \tilde{x}, v) &:= a^2 f(at, a\tilde{x}, v); \quad \lambda_a(t, r) := \lambda(at, ar) \\
   \mu_a(t, r) &:= \mu(at, ar); \quad e_a(t, r) := ae(at, ar)
\end{align*}
$$

defines another regular solution of this system on the interval $a^{-1}I$.

Another thing to discuss is the existence of initial data satisfying the constraints. The constraint equations are obtained by setting $t = 0$ in (4.1), (4.2) and (4.4).

As we said in [7], these constraints reduce to

$$
\begin{align*}
   e^{-2\lambda}(2r\lambda - 1) + 1 &= 8\pi r^2 \rho \\
   \frac{d}{dr}(r^2 e^\lambda \tilde{e}) &= qr^2 \gamma^{3/2} e^\lambda \tilde{M}
\end{align*}
$$

where

$$
\begin{align*}
   \tilde{e}(r) &:= e(0, r); \quad \tilde{\rho}(r) := \rho(0, r); \quad \tilde{M}(r) := M(0, r)
\end{align*}
$$

If $\gamma \in ]0, 1]$ and

$$
8\pi \int_0^r s^2 ds \int_{\mathbb{R}^3} \tilde{f}(s, v) \sqrt{1 + v^2} dv < r, \quad r > 0, \quad (4.12)
$$
then

$$
8\pi \gamma \int_0^r s^2 ds \int_{\mathbb{R}^3} \tilde{f}(s, v) \sqrt{1 + \gamma v^2} dv < r, \quad r > 0. \quad (4.14)
$$

So, under the assumption (4.14), we can apply the results obtained in [7] to (4.12) and (4.13), and deduce the following

**Proposition 4.1** Let $\tilde{f} \in C^\infty(\mathbb{R}^6)$ be nonnegative, compactly supported, spherically symmetric and satisfies (4.14). Take $\gamma \in ]0, 1]$. Then, for $\zeta = (\gamma, q)$ small enough, the equations (4.12) and (4.13) have a unique global solution $(\lambda_\gamma, \tilde{e}_\gamma) \in (C^\infty([0, +\infty[))^2$ such that $\lambda_\gamma(0) = \tilde{e}_\gamma(0) = 0$. Moreover, this solution depends smoothly (i.e. $C^\infty$) on the parameter $\zeta$.

Next, with the help of Proposition 4.1, for $\gamma \in ]0, 1]$ and $\Lambda > 0$, one can define the set of initial data as:

$$
D := \{(\tilde{f}, \lambda_\gamma, \tilde{e}_\gamma) \in C^\infty(\mathbb{R}^6) \times (C^1([0, +\infty[))^2, \quad \tilde{f} \geq 0, \text{spherically symmetric}, \text{ and satisfies (4.14) and } (\lambda_\gamma, \tilde{e}_\gamma) \text{ is a regular solution of the constraints with } \| \lambda_\gamma \|_{L^\infty} \leq \Lambda\}.
$$
We now study the Cauchy problem given by (EV \( M_1 \)), the initial data, (4.10) and (4.11).

**Theorem 4.1** There exists \( T > 0 \) and a continuous function \( u : [0,T] \rightarrow \mathbb{R}_+ \) such that for \( \gamma \in [0,1] \) and \((\tilde{f}, \tilde{\lambda}, \tilde{\mu}, \tilde{\epsilon}) \in D\), the system (EV \( M_1 \)) has a unique regular solution \((f, \lambda, \mu, \epsilon)\) on the interval \([0,T]\) with initial data \((f, \lambda, \mu, \epsilon)\) and

\[
f_\gamma(t, \tilde{x}, v) = 0, \quad |v| > u(t), \quad \tilde{x} \in \mathbb{R}^3, t \in [0,T],
\]

where \( \tilde{\mu}_\gamma := \mu_\gamma(0,r) \).

**Proof:** Take \( \gamma \in [0,1] \) and \((f, \lambda, \mu, \epsilon) \in D\). For \( \gamma = 1 \), it is shown in [8] that the system (EV \( M_1 \)) admits a unique regular local solution \((f, \lambda, \mu, \epsilon)\) defined on a maximal existence interval \([0,T_1]\), with initial data \((f, \tilde{\lambda}, \tilde{\mu}, \tilde{\epsilon})\), and using Lemma 4.1, one deduces that \((f, \lambda, \mu, \epsilon) \in \mathcal{M}(\gamma^{1/2}, \gamma^{1/2}), \lambda := \lambda(\gamma^{1/2}, \gamma^{1/2}), \mu := \mu(\gamma^{1/2}, \gamma^{1/2}), \epsilon := \gamma^{1/2} \epsilon(\gamma^{1/2}, \gamma^{1/2})\) solves (EV \( M_1 \)) on \([0,T_\gamma]\), where \( T_\gamma := \gamma^{-1/2} T_1 \). Set for every \( t \in [0,T_\gamma] \)

\[
U_\gamma(t) := \sup \{ |v| \mid (\tilde{x}, v) \in \text{supp} f_\gamma(t) \}
\]

\[
Q_\gamma(t) := \sup \{ e^{2 \lambda_\gamma(t,r)} \mid r \geq 0 \}.
\]

We now look for an estimate for \( U_\gamma \) and \( Q_\gamma \). Let \( s \mapsto (X_\gamma(s,t,\tilde{x},v), V_\gamma(s,t,\tilde{x},v)) \) be the solution of characteristic system

\[
\dot{x} = e^{\mu_\gamma - \lambda_\gamma} \frac{v}{\sqrt{1 + \gamma v^2}}
\]

\[
\dot{v} = - \left( \tilde{\lambda}, \frac{x}{r} \right) \frac{v}{r} + e^{\mu_\gamma - \lambda_\gamma} \frac{1}{\gamma} \left( \frac{\mu'}{\mu} \right) \frac{\dot{x}}{r} - q e^{\mu_\gamma + \lambda_\gamma} \epsilon_\gamma \frac{\dot{x}}{r}
\]

with \( X_\gamma(t,t,\tilde{x},v) = x \) and \( V_\gamma(t,t,\tilde{x},v) = v \). It is well known that the solution of (4.13) with initial datum \( \tilde{f} \) is given by

\[
f_\gamma(t, \tilde{x}, v) = \tilde{f}(X_\gamma(0,t,\tilde{x},v), V_\gamma(0,t,\tilde{x},v))
\]

and \( \|
\tilde{f}(t)\|_{L^\infty} = \|
\tilde{f}(t)\|_{L^\infty}, t \geq 0 \). Along a characteristic, we obtain since \( \mu_\gamma - \lambda_\gamma \leq 0 \): \( |\dot{x}| \leq \sqrt{q} v \) \( |\sqrt{\gamma v^2} \leq \frac{1}{\sqrt{\gamma}} \) and one deduces from the above by integration on \([0,t]\) that \( r := |\tilde{x}| \leq r_0 + \frac{1}{\sqrt{\gamma}}, \)

where \( r_0 := \sup \{ |\tilde{x}| \mid (\tilde{x}, v) \in \text{supp} \tilde{f} \} \)

The integration of (4.14) yields:

\[
e_\gamma(t,r) = \frac{q}{r^2} \gamma^{3/2} e^{-\lambda_\gamma(t,r)} \int_0^t s^2 e^{\lambda_\gamma(t,s)} M_\gamma(t,s) ds
\]

(4.15)
where $M_\gamma$ is deduced from (4.17), replacing $f$ by $f_\gamma$. Now, since $-\lambda_\gamma \leq 0$, distinguishing the cases $r < r_0 + \frac{t}{\sqrt{r}}$ and $r \geq r_0 + \frac{t}{\sqrt{r}}$, one obtains the following estimate for $e_\gamma$, since $\gamma \in [0, 1]$:

$$|e_\gamma(t, r)| \leq C(1 + r_0 + t)Q^{1/2}_\gamma(t)U^3_\gamma(t), \quad r \geq 0$$

(4.16)

where $C = C(q, \| f \|_{L^\infty})$ is a constant. Using (4.16), we deduce

$$(e^{2\lambda_\gamma} e^2_\gamma)(t, r) \leq C(1 + r_0 + t)^2Q^{3/2}_\gamma(t)U^6_\gamma(t), \quad r \geq 0$$

(4.17)

We use (4.17) and estimates obtained in the proof of Theorem 1 in [13] to deduce and estimates for $\rho_\gamma$ and $p_\gamma$:

$$\| \rho_\gamma(t) \|_{L^\infty} \leq C U^3_\gamma(t) \sqrt{1 + U^2_\gamma(t) + C(1 + r_0 + t)^2Q^{3/2}_\gamma(t)U^6_\gamma(t)},$$

$$\| p_\gamma(t) \|_{L^\infty} \leq C \min \left\{ \frac{1}{\gamma} U^4_\gamma(t), U^6_\gamma(t) \right\} + C(1 + r_0 + t)^2Q^{3/2}_\gamma(t)U^6_\gamma(t).$$

Here $\rho_\gamma$ and $p_\gamma$ are deduced from (4.16) and (4.17) replacing $f$, $\lambda$ and $e$ by $f_\gamma$, $\lambda_\gamma$ and $e_\gamma$ respectively. We now introduce the mass function given by:

$$m_\gamma(t, r) := 4\pi \int_0^r s^2 \rho_\gamma(t, s)ds = \int_{|y| \leq r} \rho_\gamma(t, y)dy$$

and with the help of the estimate of $\rho_\gamma$, one obtains, after distinguishing the cases $r \leq r_0 + \frac{t}{\sqrt{r}}$ and $r > r_0 + \frac{t}{\sqrt{r}}$, and using the relation

$$(e^{2\lambda_\gamma} e_\gamma)(t, r) = \left( \frac{r_0 + t/\sqrt{r}}{r} \right)^2 (e^{2\lambda_\gamma} e_\gamma)(t, r_0 + t/\sqrt{r}), \quad r \in [r_0 + t/\sqrt{r}, +\infty[$$

Now, one has:

$$\frac{m_\gamma(t, r)}{r^2} \leq 4\pi \int_0^r s\rho_\gamma(t, s)ds$$

$$\leq C \left( r_0 + \frac{t}{\sqrt{r}} \right)^2 U^3_\gamma(t) \sqrt{1 + U^2_\gamma(t)} + C \left( r_0 + \frac{t}{\sqrt{r}} \right)^2 (1 + r_0 + t)^2Q^{3/2}_\gamma(t)U^6_\gamma(t)$$

The integration of (4.17) on $[0, r]$ yields $e^{-2\lambda_\gamma}$ as a function of $m_\gamma$, and the insertion of this relation in (4.2) gives $\mu'_\gamma$. So multiplying the obtained equation by $e^{\mu_\gamma - \lambda_\gamma}$ yields:

$$(e^{\mu_\gamma - \lambda_\gamma} \mu'_\gamma)(t, r) = e^{\mu_\gamma + \lambda_\gamma} \left( \gamma \frac{m_\gamma(t, r)}{r^2} + 4\pi r^2 \rho_\gamma(t, r) \right).$$
Since $\mu_\gamma + \lambda_\gamma \leq 1$, one obtains the following estimate, after distinguishing the cases $r \leq r_0 + t/\sqrt{\gamma}$ and $r > r_0 + t/\sqrt{\gamma}$ for the second term in the right hand side of the expression above:

$$\left| \frac{1}{\gamma} e^{\mu_\gamma - \lambda_\gamma} \mu_\gamma' \right| \leq C(1 + r_0 + t)U_\gamma^3(t) \sqrt{1 + U_\gamma^2(t)} + C(1 + r_0 + t)^3 Q_\gamma^{3/2}(t) U_\gamma^6(t)$$

$$+ C(1 + r_0 + t)U_\gamma^5(t) + (1 + r_0 + t)^3 Q_\gamma^2(t) U_\gamma^6(t). \quad (4.18)$$

Another part of the Einstein equations is (see [8]):

$$\dot{\lambda}_\gamma(t, r) = -4\pi\gamma r e^{\mu_\gamma + \lambda_\gamma} k_\gamma(t, r),$$

where

$$k_\gamma(t, r) = k_\gamma(t, \tilde{x}) := \int_{\mathbb{R}^3} \frac{\tilde{x} \cdot v}{r} f_\gamma(t, \tilde{x}, v) dv,$$

from which one deduces the following estimate for $\dot{\lambda}_\gamma$:

$$|\dot{\lambda}_\gamma(t, r)| \leq C(1 + r_0 + t)U_\gamma^5(t) \quad (4.19)$$

Inserting (4.16), (4.18) and (4.19) in the characteristic system, one has:

$$|\dot{v}| \leq C(1 + r_0 + t)U_\gamma^6(t) + C(1 + r_0 + t)^3 Q_\gamma^{1/2}(t) U_\gamma^3(t).$$

Since

$$Q_\gamma^{1/2}(t) \leq \sqrt{1 + Q_\gamma(t)} \leq 1 + Q_\gamma(t),$$

$$Q_\gamma^{1/2}(t) \leq (1 + Q_\gamma(t))^3 \leq (1 + Q_\gamma(t) + U_\gamma(t))^3,$$

one deduces from the above that

$$|\dot{v}| \leq C(1 + r_0 + t)^3 (1 + Q_\gamma(t) + U_\gamma(t)).$$

So, the integration of the above inequality on $[0, t]$ gives:

$$U_\gamma(t) \leq U_0 + C \int_0^t (1 + r_0 + s)^3 (1 + Q_\gamma(s) + U_\gamma(s))^{11} ds, \quad (4.20)$$

where

$$U_0 := \sup \{|v| \mid (\tilde{x}, v) \in \text{supp} f \}.$$
We recall that in the above, the constant $C$ depends on $q$ and $\int f \, \|_{L^\infty}$ and does not depend neither on $t$ and nor on $\gamma$. (4.20) shows that we need an estimate for $Q_\gamma(t)$. We will proceed exactly like we did in [8] when showing that the sequence of iterates is bounded in the $L^\infty$-norm. So, we differentiate $e^{-\lambda \gamma}$ w.r.t. $t$ and obtain:

$$\left| \frac{\partial}{\partial t} e^{2\lambda \gamma(t,r)} \right| \leq 2\gamma Q_\gamma^2(t) \frac{\dot{m}_\gamma(t,r)}{r}.$$  

We see from the above inequality that an estimate for $\dot{m}_\gamma$ is needed. Using the Gauss theorem after insertion of the Vlasov equation, we deduce:

$$\frac{\dot{m}_\gamma}{r} = -4\pi \rho e^{\mu_\gamma - \lambda_\gamma} k_\gamma + \frac{1}{r} \int_{|y| \leq \gamma} \int_{\mathbb{R}^3} \frac{y \cdot v}{|y|} (\mu'_{\gamma} - \lambda'_{\gamma}) e^{\mu_{\gamma} - \lambda_{\gamma}} f_\gamma dy$$

$$- \frac{1}{r} \int_{|y| \leq \gamma} \int_{\mathbb{R}^3} \left( \lambda_{\gamma} \sqrt{1 + \gamma v^2} + \lambda_{\gamma} \left( \frac{y \cdot v}{|y|} \right)^2 \frac{1}{\sqrt{1 + \gamma v^2}} \right) f_\gamma dy$$

$$- \frac{2}{r} \int_{|y| \leq \gamma} \int_{\mathbb{R}^3} e^{\mu_{\gamma} - \lambda_{\gamma}} \frac{\mu'_{\gamma}}{|y|} f_\gamma dy$$

$$\quad - \frac{\rho}{r} \int_{|y| \leq \gamma} \int_{\mathbb{R}^3} e^{\mu_{\gamma} + \lambda_{\gamma} \gamma} \frac{y \cdot v}{\sqrt{1 + \gamma v^2}} f_\gamma dy$$

$$\quad + \frac{2\pi}{r} \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda \gamma} e_\gamma^2) ds.$$  

(4.21)

Another part of the Maxwell equations is given by:

$$\frac{\partial}{\partial t} (e^{\lambda \gamma} e_\gamma) = -\frac{\rho}{r} \gamma^{3/2} e^{\mu_{\gamma}} N_\gamma,$$

where

$$N_\gamma(t,r) = N_\gamma(t,\tilde{x}) := \int_{\mathbb{R}^3} \frac{\tilde{x} \cdot v}{\sqrt{1 + \gamma v^2}} f_\gamma(t,\tilde{x},v) dv.$$  

So, the last term in the right hand side of (4.21) yields:

$$\frac{2\pi}{r} \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda \gamma} e_\gamma^2) ds = -4\pi \gamma^{3/2} \frac{\rho}{r} \int_0^r s e^{\lambda_{\gamma} + \mu_{\gamma}} e_\gamma N_\gamma ds,$$

and since

$$|N_\gamma(t,r)| \leq \left\{ \begin{array}{l} C(r_0/\sqrt{\gamma} + t/\gamma) U_\gamma^3(t) \\ C(r_0 + t/\sqrt{\gamma}) U_\gamma^4(t), \end{array} \right.$$  

we deduce the following estimate:

$$\left| \frac{2\pi}{r} \int_0^r s^2 \frac{\partial}{\partial t} (e^{2\lambda \gamma} e_\gamma^2) ds \right| \leq C(1 + r_0 + t)^3 Q_\gamma^{3/2}(t) U_\gamma^6(t).$$  

(4.21a)

Next, we give an estimate for the first term in the right hand side of (4.21), after distinguishing the cases $r \leq r_0 + t/\sqrt{\gamma}$ and $r > r_0 + t/\sqrt{\gamma}$:

$$| -4\pi \rho e^{\mu_{\gamma} - \lambda_{\gamma}} k_\gamma | \leq C(1 + r_0 + t) U_\gamma^2(t).$$  

(4.21b)
Next, from (4.1), we deduce the following
\[ e^{\mu \gamma - \lambda \gamma} \gamma = e^{\mu \gamma + \lambda \gamma} \left( -\gamma \frac{m_{\gamma}}{r^2} + 4\pi \gamma r \rho \right) \]
and then
\[ e^{\mu \gamma - \lambda \gamma} \left( \mu' \gamma - \lambda' \gamma \right) = e^{\mu \gamma + \lambda \gamma} \left( 2\gamma \frac{m_{\gamma}}{r^2} + 4\pi \gamma^2 r \rho - 4\pi \gamma r \rho \right), \]
and using the above, one has:
\[ | e^{\mu \gamma - \lambda \gamma} \left( \mu' \gamma - \lambda' \gamma \right) | (t, r) \leq C(1 + r_0 + t) U_1^3(t) \sqrt{1 + U_1^2(t)} \]
\[ + C(1 + r_0 + t)^3 Q_1^{3/2}(t) U_1^6(t) \]
\[ + C(1 + r_0 + t) U_1^2(t). \]

So, with the help of this, and denoting by \( A \) the second term in the right hand side of (4.21), one deduces the following estimate for \( A \):
\[ | A | \leq C(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^{11}(t) \]
\[ + C(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^8(t) \sqrt{1 + U_1^2(t)} \]
\[ + C(1 + r_0 + t)^3 \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 Q_1^{3/2} U_1^{12}(t). \] (4.21c)

Using (4.19), we obtain the following estimate for the third term, say \( B \), in the right hand side of (4.21):
\[ | B | \leq C(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^8(t) \sqrt{1 + U_1^2(t)} \]
\[ + C(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^{10}(t). \] (4.21d)

Using (4.18), we deduce an estimate for the fourth term, say \( C \), in the right hand side of (4.21):
\[ | C | \leq C\gamma(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^7(t) \sqrt{1 + U_1^2(t)} \]
\[ + C\gamma(1 + r_0 + t)^3 \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 Q_1^{3/2}(t) U_1^{10}(t) \]
\[ + C\gamma(1 + r_0 + t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 U_1^9(t) \]
\[ + C\gamma(1 + r_0 + t)^3 \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^2 Q_1^2(t) U_1^{10}(t). \] (4.21e)
Using (4.15), the fifth term, say $D$ in the right hand side of (4.21) can be estimated by:

$$
|D| \leq C\gamma^{3/2} \left(r_0 + \frac{t}{\sqrt{\pi}}\right)^4 Q_\gamma^{1/2}(t)U_\gamma^7(t). \quad (4.21f)
$$

So, taking into account (4.21a), (4.21b), (4.21c), (4.21d), (4.21e), (4.21f), one obtains the following estimate:

$$
\left| \frac{\partial}{\partial t} e^{2\lambda_\gamma(t,r)} \right| \leq C(1 + r_0 + t)^3 Q_\gamma^{5/2}(t)U_\gamma^6(t) + C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^3(t)
$$

$$
+ C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^3(t)\sqrt{1 + U_\gamma^2(t)} + C(1 + r_0 + t)^5 Q_\gamma^{5/2}(t)U_\gamma^{12}(t) + C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^{11}(t)
$$

$$
C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^5(t)\sqrt{1 + U_\gamma^2(t)} + C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^{10}(t)
$$

$$
+ C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^3(t)\sqrt{1 + U_\gamma^2(t)} + C(1 + r_0 + t)^5 Q_\gamma^{5/2}(t)U_\gamma^{10}(t) + C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^9(t)
$$

$$
+ C(1 + r_0 + t)^3 Q_\gamma^2(t)U_\gamma^7(t) + C(1 + r_0 + t)^4 Q_\gamma^{5/2}(t)U_\gamma^7(t),
$$

thus,

$$
\left| \frac{\partial}{\partial t} e^{2\lambda_\gamma(t,r)} \right| \leq C(1 + r_0 + t)^5(1 + Q_\gamma(t) + U_\gamma(t))^{17}.
$$

So, the integration of the above inequality on $[0, t]$ yields:

$$
e^{2\lambda_\gamma(t,r)} \leq e^{2\lambda_\gamma(r)} + C\int_0^t (1 + r_0 + s)^5(1 + Q_\gamma(s) + U_\gamma(s))^{17}ds
$$

and bearing in mind that $(\bar{f}, \bar{\lambda}, \bar{\varepsilon}) \in D$, one has an estimate for $Q_\gamma$:

$$
Q_\gamma(t) \leq C + C \int_0^t (1 + r_0 + s)^5(1 + Q_\gamma(s) + U_\gamma(s))^{17}ds, \quad (4.22)
$$

where $C$ is a constant which depends on $\bar{f}$, $q$ and $\Lambda$, and not on $\gamma$ and $t$. Next, adding (4.21a) and (4.22), one has:

$$
Q_\gamma(t) + U_\gamma(t) \leq U_0 + C + C\int_0^t (1 + r_0 + s)^5(1 + Q_\gamma(s) + U_\gamma(s))^{17}ds,
$$

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for $t \in [0, T]$. Let $u : [0, T] \to \mathbb{R}_+$ be the maximal solution of the following integral equation:

$$u(t) = U_0 + C + C \int_0^t (1 + r_0 + s)^5(1 + u(s))^{17} ds.$$ 

Then

$$U_\gamma(t) \leq Q_\gamma(t) + U_\gamma(t) \leq u(t), \quad t \in [0, T \cap [0, T]$$

and with the continuation criterion proved in $[3]$, one concludes that $U_\gamma(t) \leq u(t)$ for $t \in [0, T]$ and this ends the proof of Theorem 4.1.

We now state and prove the essential result of this paper. This is concerned with the convergence of a solution $(f_\gamma, \lambda_\gamma, \mu_\gamma, e_\gamma)$ of $(EVM_\gamma)$ to a solution $f$ for the (VP) system given by $(4.8)$ and $(4.9)$, as $\gamma$ tends to 0.

**Theorem 4.2** Let $0 < T \leq \infty$ be such that for every $\gamma \in [0, 1]$ and $(\tilde{f}, \tilde{\lambda}_\gamma, \tilde{\mu}_\gamma) \in D$, the solution $(f_\gamma, \lambda_\gamma, \mu_\gamma, e_\gamma)$ of $(EVM_\gamma)$ exists on $[0, T]$ and

$$f_\gamma(t, \tilde{x}, v) = 0, \quad | v | > u(t), \quad \tilde{x} \in \mathbb{R}^3, \quad t \in [0, T],$$

where $u : [0, T] \to \mathbb{R}_+$ is a continuous function. Let $f \in C^1([0, +\infty] \times \mathbb{R}^3)$ be the solution of (VP) with $f(0) = \tilde{f}$. Then for every $T' \in [0, T]$ there exists a constant $C > 0$ such that for any $\gamma \in [0, 1]$ the following estimate holds:

$$\| f_\gamma(t) - f(t) \|_{L^\infty} + \| \lambda_\gamma(t) \|_{L^\infty} + \| \mu_\gamma(t) \|_{L^\infty} + \| \dot{\lambda}_\gamma(t) \|_{L^\infty} + \| \dot{\lambda}_\gamma(t) \|_{L^\infty} \leq C\gamma,$$

for every $t \in [0, T']$.

**Proof:** Take $0 < T' < T$. Let $C_1$ be the upper bound of $u$ on $[0, T']$. Then one obtains, using the estimates above:

$$U_\gamma(t), \quad Q_\gamma(t), \quad \frac{m_\gamma(t, r)}{r}, \quad \frac{m_\gamma(t, r)}{r^2} \leq C, \quad t \in [0, T'], \quad r > 0.$$

Using (1.6), we obtain

$$| e_\gamma(t, r) | \leq C\gamma^{3/2} \left( r_0 + \frac{t}{\sqrt{\gamma}} \right) \leq C\gamma, \quad r \geq 0, \quad t \in [0, T'], \quad \gamma \in [0, 1].$$

The remaining terms in (1.28) can be estimated exactly as in the proof of Theorem 2 of [14]. But the sole change is on estimate concerning $\frac{1}{2} \partial_2 \mu_\gamma(t, \tilde{x}) - K(t, \tilde{x})$ that involves the quantity $\left( \frac{m_\gamma}{r^2} - \frac{m_\gamma}{r^2} \right)(t, r)$. In our case, we have after distinguishing the cases $r \leq r_0 + t/\sqrt{\gamma}$ and $r > r_0 + t/\sqrt{\gamma}$:

$$\left| \frac{m_\gamma}{r^2} - \frac{m_\gamma}{r^2} \right|(t, r) \leq C \| f_\gamma(t) - f(t) \|_{L^\infty}$$

$$+ C \int_{r_0 + t/\sqrt{\gamma}}^{r_0} e^{2\lambda_\gamma(t, s)} c_\gamma^2(t, s) ds$$

(4.24)
and using once again (4.15), the integral that appears in (4.24) can be estimated as:
\[
\int_0^{r_0 + t/\sqrt{\gamma}} e^{2\lambda_\gamma(t,s)} e_{\gamma}^2(t, s) ds = q^2 \gamma^3 \int_0^{r_0 + t/\sqrt{\gamma}} \frac{1}{s^4} ds \left( \int_0^s \tau^2 e^{\lambda_\gamma(t, \tau)} M_\gamma(t, \tau) d\tau \right)^2 \\
\leq C \gamma^3 \int_0^{r_0 + t/\sqrt{\gamma}} ds \left( e^{\lambda_\gamma(t, \tau)} M_\gamma(t, \tau) d\tau \right)^2 \\
\leq C \gamma^3 Q_\gamma(t) U_\gamma(t) \left( r_0 + \frac{t}{\sqrt{\gamma}} \right)^3 \\
\leq C \gamma.
\]
So, the proof of Theorem 4.2 is complete.

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