DELTA SHOCK WAVE INTERACTIONS VIA WAVE FRONT TRACKING METHOD

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Abstract. In this paper we discuss delta shock interaction problem for a pressureless gas dynamics system with two different ways of approaching the subject. The first one is by using shadow wave solution concept. The result of two delta shock interactions is delta shock with non-constant speed in a general case. The second one is by perturbing the system with a small pressure term. The obtained perturbed system is strictly hyperbolic and its Riemann problem is solvable. We compare a limit of a numerical wave front tracking results as small pressure term vanishes with the shadow wave solution.

Key words: weighted shadow waves, delta shock waves, wave front tracking, Riemann problem, interactions

1. Introduction

Consider the one-dimensional Euler gas dynamics system given by

\[ \begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= 0, \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2 + p(\varepsilon, \rho)) &= 0,
\end{align*} \tag{1} \]

where \( \rho \) is the density, \( m = \rho u \) is the momentum, \( p(\varepsilon, \rho) = \varepsilon p_0(\rho) \) is the scalar pressure, \( \varepsilon << 1 \) and \( p_0(\rho) = \rho^\gamma / \gamma \). Taking \( \varepsilon \to 0 \) in (1), we obtain the pressure-less gas dynamics model (PGD model in the rest of the paper), also called sticky particles model (in [13]),

\[ \begin{align*}
\frac{\partial}{\partial t} \rho + \frac{\partial}{\partial x} (\rho u) &= 0, \\
\frac{\partial}{\partial t} (\rho u) + \frac{\partial}{\partial x} (\rho u^2) &= 0, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+.
\end{align*} \tag{2} \]

System (2) can be considered as a perturbation of system (2) which is weakly hyperbolic with a double eigenvalue \( \lambda_1 = \lambda_2 = u \). All entropy pairs \( (\eta, q) \) with a semiconvex function \( \eta \) are given by \( \eta := \rho S(u), \, q := \rho u S(u) \), where \( S'' \geq 0 \) (the entropy function \( \eta \) is semi-convex with respect to the variable \( (\rho, \rho u) \)). The Riemann problem

\[ \rho(x, 0) = \begin{cases}
\rho_0, & x < 0, \\
\rho_1, & x > 0,
\end{cases} \quad u(x, 0) = \begin{cases}
u_0, & x < 0, \\
u_1, & x > 0,
\end{cases} \tag{3} \]

has a classical entropy solution consisting of two contact discontinuities connected with the vacuum state \( (\rho = 0) \) if \( u_0 \leq u_1 \):

\[ (\rho(x,t), u(x,t)) = \begin{cases}
(\rho_0, u_0), & x < u_0 t, \\
(0, \psi(x/t)), & u_0 t < x < u_1 t, \\
(\rho_1, u_1), & x > u_1 t,
\end{cases} \]

where \( \psi(y) = y. \) We are now turning to the case \( u_0 > u_1 \) when there is no classical solution to the Riemann problem (2) [3].
Throughout this paper, the following constants will be fixed:

\[ \gamma = 1 + 2\epsilon, \quad 0 < \epsilon < \frac{1}{2}, \quad \kappa = \frac{\sqrt{\epsilon}}{\sqrt{\gamma}} \text{ and } p = \kappa^2 \rho^\gamma. \]  

(4)

2. Elementary waves of the perturbed system

The eigenvalues of system (1) are

\[ \lambda_1 = u - \kappa \sqrt{\gamma \rho^{\frac{2-1}{2}}}, \]

\[ \lambda_2 = u + \kappa \sqrt{\gamma \rho^{\frac{2-1}{2}}}, \]

(5)

and the corresponding eigenvectors are

\[ r_1 = (-1, -u + \kappa \sqrt{\gamma \rho^{\frac{2-1}{2}}})^T, \]

\[ r_2 = (1, u + \kappa \sqrt{\gamma \rho^{\frac{2-1}{2}}})^T. \]

(6)

We have chosen an orientation such that \( \nabla \lambda_i \cdot r_i > 0, \ i = 1, 2 \), since both fields are genuinely nonlinear. The corresponding Riemann invariants of system (1) are

\[ s = u + \frac{\kappa}{\sqrt{\epsilon}} (\rho^\epsilon - 1) : 1\text{-invariant}, \]

\[ r = u - \frac{\kappa}{\sqrt{\epsilon}} (\rho^\epsilon - 1) : 2\text{-invariant}. \]

(7)

The rarefaction curves through the point \((\rho_0, u_0)\) are given by

\[ u - u_0 = -\frac{\kappa}{\sqrt{\epsilon}} (\rho^\epsilon - \rho_0^\epsilon), \quad 0 \leq \rho \leq \rho_0 : 1\text{-rarefaction curve}, \]

\[ u - u_0 = \frac{\kappa}{\sqrt{\epsilon}} (\rho^\epsilon - \rho_0^\epsilon), \quad \rho \geq \rho_0 : 2\text{-rarefaction curve}, \]

(8)

while the shock curves through the point \((\rho_0, u_0)\) are given by

\[ u - u_0 = -\kappa \sqrt{\frac{\rho^\gamma - \rho_0^\gamma}{\rho_0 \rho (\rho - \rho_0)}} (\rho - \rho_0), \quad \rho > \rho_0 : 1\text{-shock curve}, \]

(9)

and

\[ u - u_0 = \kappa \sqrt{\frac{\rho^\gamma - \rho_0^\gamma}{\rho_0 \rho (\rho - \rho_0)}} (\rho - \rho_0), \quad 0 < \rho < \rho_0 : 2\text{-shock curve}. \]

(10)

With the Riemann invariants, shock curves starting from the point \((r_0, s_0)\) are

\[ S_1 : \begin{cases} r_0 - r = \kappa \rho_0^\gamma \left( \sqrt{\frac{(\alpha - 1)(\alpha \gamma - 1)}{\alpha}} + \sqrt{\gamma 1 - \alpha^\epsilon - 1} \right), \\ s_0 - s = \kappa \rho_0^\gamma \left( \sqrt{\frac{(\alpha - 1)(\alpha \gamma - 1)}{\alpha}} - \sqrt{\gamma 1 - \alpha^\epsilon - 1} \right), \end{cases} \]

(11)

where \( r_0 = r(\rho_0, u_0), \) \( s_0 = s(\rho_0, u_0) \) and \( \alpha = \rho/\rho_0 \geq 1, \) and

\[ S_2 : \begin{cases} s_0 - s = \kappa \rho_0^\gamma \left( \sqrt{\frac{(1 - \alpha)(1 - \alpha \gamma)}{\alpha}} + \sqrt{\gamma 1 - \alpha^\epsilon - 1} \right), \\ r_0 - r = \kappa \rho_0^\gamma \left( \sqrt{\frac{(1 - \alpha)(1 - \alpha \gamma)}{\alpha}} - \sqrt{\gamma 1 - \alpha^\epsilon - 1} \right). \]

(12)
where \( r_0 = r(\rho_0, u_0) \), \( s_0 = s(\rho_0, u_0) \) and \( 0 < \alpha = \rho/\rho_0 \leq 1 \). The corresponding rarefaction curves are given by

\[
R_1 : \quad r \geq r_0, \quad s = s_0, \quad (13)
\]

and

\[
R_2 : \quad s \geq s_0, \quad r = r_0. \quad (14)
\]

It is clear that from (11, 12) we have that \( r_0 - r \geq s_0 - s \) holds for \( S_1 \) curve and \( s_0 - s \geq r_0 - r \) holds for \( S_2 \) curve, respectively.

The Riemann problem for system (1) with initial data (3) was solved by Riemann [12], and the result is summarized in the following theorem (the proof can be found in Courant-Friedrichs [4] and Smoller [14]).

**Theorem 2.1.** [1] Consider system (1) with initial data (3). Suppose that

\[
u_1 - u_0 < \sqrt{\frac{2\sqrt{\gamma}}{\kappa}} \left( \rho_1^\epsilon + \rho_0^\epsilon \right),
\]
on equivalently \( s_0 - r_1 > -\frac{2\epsilon}{\sqrt{\gamma}} \). Then there exists a unique solution composed of constant states \( (\rho_0, u_0) = (r_0, s_0) \), \( (\rho_m, u_m) = (r_m, s_m) \) and \( (\rho_1, u_1) = (r_1, s_1) \) separated by centered rarefaction or shock waves satisfying the following estimates:

\[
\begin{align*}
\rho(x,t) &= \rho(\rho(x,t), u(x,t)) \geq \min\{r_0, r_1\}, \\
\rho(x,t) &= \rho(\rho(x,t), u(x,t)) \leq \max\{s_0, s_1\}.
\end{align*}
\]

(15)

The amplitude of the waves is denoted by

\[
\beta : = r_m - r_0 : \text{ amplitude of an } 1\text{-wave}, \quad \chi : = s_1 - s_m : \text{ amplitude of a } 2\text{-wave}.
\]

(16)

Here \( \beta, \chi \geq 0 \) for centered rarefaction waves and \( \beta, \chi < 0 \) for shock waves; absolute values \(|\beta|, |\chi|\) are called strengths of \( \beta \) and \( \chi \), respectively.

We shall use that notation throughout the rest of the paper.

### 3. Local Interactions Estimates

Our first task is to obtain a sharp estimate of wave strengths with respect to \( \epsilon \) as much as possible. In order to do that, we shall present some assertions from [11] together with modified proofs, since certain changes in estimates will be useful for our investigation.

**Theorem 3.1.** [11] The shock curve \( S_1 \) starting at the point \( (r_0, s_0) \) is given by

\[
s_0 - s = g_1(r_0 - r, \rho_0) = \int_0^{r_0 - r} h_1(\alpha) |_{\alpha = \alpha_1(\beta/\kappa \rho_0^\epsilon)} d\beta, \quad r < r_0,
\]

(17)

where \( 0 \leq g_1'(\beta, \rho_0) < 1 \) and \( g_1''(\beta, \rho_0) \geq 1 \). The shock curve \( S_2 \) starting at the point \( (r_0, s_0) \) is

\[
r_0 - r = g_2(s_0 - s, \rho_0) = \int_0^{s_0 - s} h_2(\alpha) |_{\alpha = \alpha_2(\chi/\kappa \rho_0^\epsilon)} d\chi, \quad s < s_0,
\]

(18)

where \( 0 \leq g_2'(\chi, \rho_0) < 1 \) and \( g_2''(\chi, \rho_0) \geq 0 \).

\(^1\)The primes denote differentiation with respect to the first argument.
\textbf{Proof.} We shall repeat the proof from [11] in order to fix the notation for the rest of the paper. Relation \( s_0 - s = g_1(r_0 - r, \rho_0) \) implies
\[
\frac{\partial(s_0 - s)}{\partial \alpha} = \frac{\partial g_1(r_0 - r, \rho_0)}{\partial(r_0 - r)} \cdot \frac{\partial(r_0 - r)}{\partial \alpha}, \quad \text{so} \quad \frac{\partial(s_0 - s) / \partial \alpha}{\partial(r_0 - r) / \partial \alpha} = g'_1(\beta, \rho_0). \tag{19}
\]
If
\[
h_1(\alpha) = \frac{\partial(s_0 - s) / \partial \alpha}{\partial(r_0 - r) / \partial \alpha},
\]
then one can easily see that
\[
h_1(\alpha) = \left( \frac{Y - 1}{Y + 1} \right)^2 \quad \text{with} \quad Y = \sqrt{\frac{\gamma \alpha^2 (\alpha - 1)}{\alpha^2 - 1}}, \quad \text{for} \quad \alpha > 1. \tag{20}
\]
From the first equation in (11), we have
\[
\frac{\beta}{\kappa \rho_0} = \sqrt{\frac{(\alpha - 1)(\alpha^2 - 1)}{\alpha}} + \sqrt{\gamma} \frac{\alpha^2 - 1}{\alpha} \varepsilon =: f(\alpha). \tag{21}
\]
Therefore
\[
f'(\alpha) > \frac{1}{2} \sqrt{\frac{\alpha}{(\alpha - 1)(\alpha^2 - 1)}} \cdot \frac{\alpha^2 - 1}{\alpha^2} + \sqrt{\gamma} \frac{\alpha^2 - 1}{\alpha} > 0
\]
since \( \alpha^2 > 1 \) for \( \alpha > 1 \) and \( \gamma > 1 \). Using the fact that \( f'(\alpha) > 0 \) and the Implicit Function Theorem yields that there exists \( \alpha = \alpha_1(\beta/\kappa \rho_0^2) \) such that
\[
g_1(r_0 - r, \rho_0) = \int_0^{r_0 - r} h_1(\alpha)_{|\alpha = \alpha_1(\beta/\kappa \rho_0^2)} \, d\beta. \tag{22}
\]
Since \( g'_1(\beta, \rho_0) = h_1(\alpha), \ g''_1(\beta, \rho_0) = h'_1(\alpha) \cdot \frac{d\alpha}{d\beta} \) and \( \frac{d\beta}{d\alpha} = \kappa \rho_0^2 \cdot f'(\alpha) > 0 \) it remains to prove that \( 0 \leq h_1(\alpha) < 1 \) and \( 0 \leq h'_1(\alpha) \). From (20) we have
\[
0 \leq h_1(\alpha) = \left( \frac{Y - 1}{Y + 1} \right)^2 < \left( \frac{Y + 1}{Y + 1} \right)^2 = 1,
\]
and
\[
0 \leq h'_1(\alpha) = 4 \cdot \frac{Y - 1}{(Y + 1)^3} \cdot Y', \tag{23}
\]
since \( Y \geq 1 \) and \( Y' \geq 0 \). The second part of the theorem can be proved using the same technique. \qed

\textbf{Lemma 3.2.} Let \( \rho_0 < \rho_1 \) and \( \beta/\kappa \rho_1^2 < \theta < \beta/\kappa \rho_0^2 \). Then
\[
\frac{d\alpha}{d\theta} = \frac{1}{f'(\alpha)} = \frac{2Y}{\sqrt{\gamma} \alpha^{2/2} (1 + Y)^2}. \tag{24}
\]

We would need an estimate of the difference of Riemann invariants across two shock waves which is more precise than the one in [11]. It is provided by the following theorem.

\textbf{Theorem 3.3.} Let \( 0 < \varepsilon < \frac{1}{2}, \ s_0 < s_1, \) and take two \( S_1 \) curves originating at the points \( (r_0, s_0) = (\rho_0, u_0) \) and \( (r_0, s_1) = (\rho_1, u_1) \), which are continued to the points \( (r, s) \) and \( (r, s_2) \), respectively. Then we have
\[
0 \leq (s_0 - s) - (s_1 - s_2) \leq C_* \sqrt{\varepsilon (r_0 - r)} (s_1 - s_0), \tag{25}
\]
where \( C_* \) is a constant independent of \( \varepsilon, \rho_0 \) and \( \rho_1 \).
**Proof.** Let \( z^0 = s_0 - s \), \( z^1 = s_1 - s_2 \) and \( w = r_0 - r \) (look at the diagram shown in Figure 1).

![Diagram of shock wave interactions](image)

**Figure 1.** Two 1-shock wave curves in \( r - s \) plane.

By Theorem 3.1 and the Mean Value Theorem we know that for \( \rho_1 > \rho_0 \), there exists \( \theta \) such that

\[
z^0 - z^1 = \int_0^w \frac{d h_1(\alpha)}{d \alpha} \bigg|_{\alpha=\alpha(\theta)} \cdot \alpha'(\theta) \left( \frac{\beta}{\kappa \rho_0^{\gamma}} - \frac{\beta}{\kappa \rho_1^{\gamma}} \right) d\beta ,
\]

where \( \theta \in \left( \frac{\beta}{\kappa \rho_1^{\gamma}}, \frac{\beta}{\kappa \rho_0^{\gamma}} \right) \). The definitions of \( h_1 \) and \( \alpha \) imply

\[
\frac{d h_1(\alpha)}{d \alpha} \geq 0, \quad \frac{d \alpha(\theta)}{d \theta} \geq 0 \quad \text{and} \quad \frac{\beta}{\kappa \rho_0^{\gamma}} - \frac{\beta}{\kappa \rho_1^{\gamma}} \geq 0
\]

so \( z^0 - z^1 \geq 0 \) for \( \rho_1 > \rho_0 \). We need to estimate the integrand in (26). By (20), we have

\[
\frac{d h_1(\alpha)}{d \alpha} \cdot \frac{d \alpha(\theta)}{d \theta} \leq \frac{4(Y - 1)(\gamma + 1)\alpha^{\frac{\gamma + \gamma}{\gamma}}}{\sqrt{7}(Y + 1)^3}.
\]

Thus,

\[
z^0 - z^1 \leq \frac{4(\gamma + 1)}{\kappa \sqrt{7} \rho_0^{\gamma} \rho_1^{\gamma}} (\rho_1^{\gamma} - \rho_0^{\gamma}) \int_0^w \beta \alpha^{\frac{\gamma + \gamma}{\gamma}} \bigg|_{\alpha=\alpha(\theta)} \frac{Y - 1}{(Y + 1)^3} d\beta
\]

and

\[
z^0 - z^1 \leq \frac{4(\gamma + 1)}{\kappa \sqrt{7} \rho_0^{\gamma} \rho_1^{\gamma}} (\rho_1^{\gamma} - \rho_0^{\gamma}) \int_0^w \beta \alpha^{\frac{\gamma + \gamma}{\gamma}} \bigg|_{\alpha=\alpha(\theta)} d\beta .
\]

From Lemma 3.2 we know that \( d\alpha/d\theta > 0 \) for \( \beta/\kappa \rho_1^{\gamma} < \theta < \beta/\kappa \rho_0^{\gamma} \). Hence,

\[
\alpha \left( \frac{\beta}{\kappa \rho_1^{\gamma}} \right) \leq \alpha(\theta) \leq \alpha \left( \frac{\beta}{\kappa \rho_0^{\gamma}} \right).
\]
Moreover, 
\[ \frac{\beta}{\kappa \rho_1} = f(\alpha) \leq 2 \sqrt{\frac{(\alpha - 1) \alpha^\gamma}{\alpha - 1}} = 2\alpha^{\gamma/2}, \text{ for } \alpha = \alpha \left( \frac{\beta}{\kappa \rho_1} \right). \]

At this point, we use a majorization of \( z^0 - z^1 \) different from the one in [11] in order to obtain bounds for \( C_* \), independent of \( \varepsilon \). By [24] and the above inequality, we obtain
\[ \frac{\beta}{2\kappa \rho_1} \leq \left( \alpha \left( \frac{\beta}{\kappa \rho_1} \right) \right)^{\gamma/2} \Rightarrow \left( \frac{\beta}{2\kappa \rho_1} \right)^{-\frac{\gamma+2}{2}} \geq \left( \alpha \left( \frac{\beta}{\kappa \rho_1} \right) \right)^{-\frac{\gamma+2}{4}}, \]

so
\[ \min \left\{ 1, \left( \frac{\beta}{2\kappa \rho_1} \right)^{-\frac{\gamma+2}{2}} \right\} \geq \left( \alpha \left( \frac{\beta}{\kappa \rho_1} \right) \right)^{-\frac{\gamma+2}{4}} \geq (\alpha(\theta))^{-\frac{\gamma+2}{4}}, \]

for \( \alpha \left( \frac{\beta}{\kappa \rho_1} \right) \geq 1 \) and \( \gamma > 1 \). Since \( \alpha \theta/d\theta > 0 \), it follows by [25] that
\[ z^0 - z^1 \leq \frac{4(\gamma + 1)}{\kappa \sqrt{\gamma} \rho_0 \rho_1} (\rho_1 - \rho_0) \int_0^w \beta \cdot \min \left\{ 1, \left( \frac{\beta}{2\kappa \rho_1} \right)^{-\frac{\gamma+2}{4}} \right\} \, d\beta \]
\[ \leq \frac{8(\gamma + 1)}{\sqrt{\gamma} \rho_0} \int_0^w \min \left\{ \frac{\beta}{2\kappa \rho_1}, \left( \frac{\beta}{2\kappa \rho_1} \right)^{-\frac{\gamma+2}{4}} \right\} \, d\beta \]
\[ \leq \frac{8(\gamma + 1)}{\sqrt{\gamma} \rho_0} (\rho_1^\gamma - \rho_0^\gamma) w. \]

Using \( \rho_1^\gamma - \rho_0^\gamma = \frac{\varepsilon}{2\kappa \sqrt{\gamma}} (s_1 - s_0) \)

and \( \kappa \sqrt{\gamma} = \sqrt{\varepsilon} \) together with [31] we finally get
\[ z^0 - z^1 \leq \frac{4(\gamma + 1)}{\sqrt{\gamma} \rho_0} \cdot \frac{\varepsilon}{\kappa \sqrt{\gamma}} (s_1 - s_0) \]
\[ = \frac{4(\gamma + 1)}{\sqrt{\gamma} \rho_0} \cdot \sqrt{\varepsilon} (s_1 - s_0) \, w. \]

(32)

Suppose that \( \rho \) is the first component of the solution of the Riemann problem [1] [3] for \( u_0 > u_1 \). Then Lemma 3.1 from [3] yields that for small \( \varepsilon > 0 \), there exists \( C > 0 \) independent of \( \varepsilon \), such that \( \rho \leq C/\varepsilon \). Now, using [32], if \( \rho_0 \sim 1/\varepsilon \) then \( \rho_0^\gamma \sim 1 \) as \( \varepsilon \to 0 \). For \( \varepsilon \) small enough we may write \( \rho_0^\gamma \geq C_1 \). Thus, there exists a constant \( C_* \), independent of \( \kappa, \rho_0, \rho_1 \) and \( \varepsilon \) such that
\[ z^0 - z^1 \leq C_* \sqrt{\varepsilon} (s_1 - s_0) w, \]

(33)

holds. This completes the proof of the theorem.

The theorem that follows can be proved in the same way.

**Theorem 3.4.** Let \( 0 < \varepsilon < \frac{1}{2}, r_0 > r_1, \) and take two \( S_2 \) curves originating at the points \( (r_0, s_0) = (\rho_0, u_0) \) and \( (r_1, s_0) = (\rho_1, u_1) \), which are continued to the points \( (r, s) \) and \( (r_2, s) \), respectively. Then we have
\[ 0 \leq (r_0 - r) - (r_1 - r_2) \leq C^* \varepsilon (s_0 - s) (r_0 - r_1), \]

(34)
where $C_{*,+}$ is a constant independent of $\varepsilon$, $\rho_0$ and $\rho_1$.

(A1) We shall use the following convention: $C_*$ denotes the maximum of the constants $C_*$ and $C_{*,+}$ from Theorems 3.3 and 3.4, respectively.

In the following theorem $\beta$ and $\chi$ denote $S_1$ and $S_2$, respectively, while $o$ and $\pi$ denote $R_1$ and $R_2$, respectively. The prime is reserved for after interaction waves. (For example, the interaction of $S_2$ and $S_1$ which produces $S_1$ and $S_2$ is denoted by $\chi + \beta \rightarrow \beta' + \chi'$.)

**Theorem 3.5.** If $0 < \varepsilon < \frac{1}{2}$, then the following estimates are valid for the corresponding interactions:

1. $S_2$ and $S_1$ interaction:
   - a. $\chi + \beta \rightarrow \beta' + \chi'$
     
     $$|\beta'| \leq |\beta| + C_* \sqrt{\varepsilon} |\beta|, \quad |\chi'| \leq |\chi| + C_* \sqrt{\varepsilon} |\beta|,$$

     there exist $\eta, \xi$ such that
   - b. $\chi + \beta \rightarrow \beta' + \chi'$
     
     $$0 \leq |\beta'| = |\beta| + \xi, \quad |\chi'| \leq |\chi| + C_* \sqrt{\varepsilon} |\beta| + \eta,$$

     where $0 \leq \eta \leq g_1'(|\beta|, \rho_0) \xi < \xi$, or

2. $S_2$ and $R_1$ (or $R_2$ and $S_1$) interaction:
   - a. $\chi + o \rightarrow o' + \chi'$
     
     $$|\chi'| = |\chi|, \quad |o'| \leq |o| + C_* \sqrt{\varepsilon} |\chi|.$$  
     
   - b. $\pi + \beta \rightarrow \beta' + \pi'$
     
     $$|\beta'| = |\beta|, \quad |\pi'| \leq |\pi| + C_* \sqrt{\varepsilon} |\beta|,$$

3. $S_2$ and $S_2$ (or $S_1$ and $S_1$) interaction:
   - a. $\chi_1 + \chi_2 \rightarrow o' + \chi'$
     
     $$|\chi'| = |\chi_1| + |\chi_2|, \quad |o'| \leq |\chi_1| + |\chi_2|.$$  
     
   - b. $\beta_1 + \beta_2 \rightarrow \beta' + \pi'$
     
     $$|\beta'| = |\beta_1| + |\beta_2|, \quad |\pi'| \leq |\beta_1| + |\beta_2|.$$  

4. $S_2$ and $R_2$ (or $R_1$ and $S_1$) interaction:
   - a. $1^o \chi + \pi \rightarrow \beta' + \chi'$: there exist 1-shock $\beta_0$ and 2-shock $\chi_0$ such that
     
     $$|\chi_0| = |\chi| - \xi, \quad |\beta_0| = \eta \quad \text{and} \quad \chi_0 + \beta_0 \rightarrow \beta' + \chi',$$

     where $0 < \eta \leq g_2'(|\chi|, \rho_1) \xi < \xi$.
     
     $$2^o \chi + \pi \rightarrow \beta' + \pi': \text{there exist} \eta, \xi \text{such that}$$

     $$|\pi'| \leq |\pi|, \quad |\beta'| = \eta < \xi = |\chi|,$$

     where $0 < \eta \leq g_2'(|\beta|, \rho_1) \xi < \xi$.
   - b. $1^o o + \beta \rightarrow \beta' + \chi'$: there exist 1-shock $\beta_0$ and 2-shock $\chi_0$ such that
     
     $$|\beta_0| = |\beta| - \xi, \quad |\chi_0| = \eta \quad \text{and} \quad \chi_0 + \beta_0 \rightarrow \beta' + \chi',$$

     where $0 < \eta \leq g_2'(|\beta|, \rho_2) \xi < \xi$.
     
     $$2^o o + \beta \rightarrow o' + \chi'$

     $$|o'| \leq |o|, \quad |\chi'| = \eta < \xi = |\beta|,$$

     where $0 < \eta \leq g_1'(|\beta|, \rho_0) \xi < \xi$.

   (5) $R_2$ and $S_2$ (or $S_1$ and $R_1$) interaction:
   - a. $1^o \pi + \chi \rightarrow \beta' + \chi'$: there exist $\eta, \xi$ such that
     
     $$|\chi'| = |\chi| - \xi, \quad |\beta'| = \eta,$$
where \( 0 < \eta \leq g'_1(|\chi|, \rho_2) \xi < \xi \).

2° \( \pi + \chi \rightarrow \beta' + \pi' \)
\[ |\pi'| = |\pi|, \quad |\beta'| = \eta < \xi = |\chi|, \]
where \( 0 < \eta \leq g'_2(|\chi|, \rho_0) \xi < \xi \).

(b) 1° \( \beta + o \rightarrow \beta' + \chi' \): there exist \( \eta, \xi \) such that
\[ |\beta'| = |\beta| - \xi, \quad |\chi'| = \eta, \]
where \( 0 < \eta \leq g'_1(|\beta|, \rho_1) \xi < \xi \).

2° \( \beta + o \rightarrow o' + \chi' \)
\[ |\beta'| = |\beta| - \xi, \quad |\chi'| = \eta < \xi = |\beta|, \]
where \( 0 < \eta \leq g'_1(|\beta|, \rho_1) \xi < \xi \).

(6) \( R_2 \) and \( R_1 \) interaction:
\[ \pi + o \rightarrow o' + \pi' \]
\[ |\pi'| = |\pi|, \quad |\pi'| = |\pi|. \]

Here \( C_* \) is a positive constant defined as in (A1).

**Proof.** This theorem can be proved using the same tools as in [11] and therefore will be omitted. The only differences are: the constant \( C_* \) is now independent of \( \varepsilon, \beta, \chi, \rho_0, \rho_1 \) and \( \rho_2 \), and we have \( \sqrt{\varepsilon} \) instead of \( \varepsilon \) in the estimates (1)(a) – (1)(c), (2)(a) and (2)(b).

The main part of the paper is the interaction problem of delta shocks via pressure perturbation. Thus, one needs to control shock and rarefaction strengths as \( \rho \) goes to infinity as \( \varepsilon \to 0 \) (more precisely, when \( \rho \) is bounded by \( \text{const}/\varepsilon \)). Because of that, we give their estimates in \( r - s \) plane based on Theorem 3.3 and Theorem 3.4. Let \( (\rho_0, u_0) \) be connected with \( (\rho, u) = (r, s) \) by a 1-rarefaction (or 1-shock) wave, while \( (\rho, u) = (r, s) \) be connected with \( (\rho_1, u_1) = (r_1, s_1) \) by a 2-rarefaction (or 2-shock) wave. Then the strength of 1-rarefaction wave is
\[ r - r_0 = \frac{2}{\sqrt{\varepsilon}}(\rho_0^* - \rho^*), \quad \rho < \rho_0, \quad (35) \]
and the strength of 2-rarefaction wave is
\[ s_1 - s = \frac{2}{\sqrt{\varepsilon}}(\rho_1^* - \rho^*), \quad \rho < \rho_1. \quad (36) \]

The strength of 1-shock wave is estimated by
\[ 2\rho_0^* \sqrt{\varepsilon} \ln \frac{\rho}{\rho_0} \leq r_0 - r \leq 2 \frac{\sqrt{\varepsilon}}{\sqrt{1 + 2\varepsilon}} \left( \frac{\rho}{\rho_0} \right)^{\gamma/2} \rho_0^*, \quad \rho > \rho_0, \quad (37) \]
while, the strength of 2-shock wave is estimated by
\[ 2\rho_1^* \sqrt{\varepsilon} \ln \frac{\rho}{\rho_1} \leq s_1 - s \leq 2 \frac{\sqrt{\varepsilon}}{\sqrt{1 + 2\varepsilon}} \left( \frac{\rho}{\rho_1} \right)^{\gamma/2} \rho_1^*, \quad \rho > \rho_1. \quad (38) \]

Let us estimate the upper bound of the 1-shock wave given in (37). For the function \( g_1 \) from [17] we have \( 0 \leq g'_1(\beta, \rho_0) < 1 \) and \( 0 \leq g''_1(\beta, \rho_0) \), so
\[ \lim_{|\beta| \to +\infty} g'_1(|\beta|, \rho_0) \leq 1. \]

Let us consider two special cases needed for our investigation. The first case: \( \rho > \rho_0 \) and \( \rho \sim 1/\varepsilon \). We have that there exist constants \( \tilde{C}, \tilde{C}^* \) and \( \tilde{C} \) independent of \( \varepsilon \) such
that
\[
\left( \frac{\rho}{\rho_0} \right)^{\gamma/2} \cdot \rho^\varepsilon = \sqrt{\frac{\rho}{\rho_0}} \cdot \rho^\varepsilon \leq \sqrt{\frac{\bar{C}}{\varepsilon}} \cdot \left( \frac{\bar{C}}{\varepsilon} \right)^\varepsilon \leq \sqrt{\frac{1}{\varepsilon}} \cdot \bar{C},
\]
so
\[
2 \frac{\sqrt{\varepsilon}}{\sqrt{1 + 2\varepsilon}} \left( \frac{\rho}{\rho_0} \right)^{\gamma/2} \cdot \rho^\varepsilon \leq 2 \frac{\sqrt{\varepsilon}}{\sqrt{1 + 2\varepsilon}} \cdot \bar{C} \leq \text{const}.
\]
It follows that there exists a constant \( C_2 \), independent of \( \varepsilon \) and \( \rho_0 \), such that
\[
\sup g_1'(\|\beta\|,\rho_0) := C_2 < 1.
\]
Hence,
\[
\frac{1 - g_1'(\|\beta\|,\rho_0)}{g_1'(\|\beta\|,\rho_0)} \geq \frac{1 - C_2}{C_2} =: C_3 > 0.
\]
The second case: \( \rho > \rho_0, \rho \sim 1/\varepsilon \) and \( \rho_0 \sim 1/\varepsilon \). Then
\[
\sqrt{\frac{\rho}{\rho_0}} \cdot \rho^\varepsilon \sim \text{const} \Rightarrow 2 \frac{\sqrt{\varepsilon}}{\sqrt{1 + 2\varepsilon}} \sqrt{\frac{\rho}{\rho_0}} \rho^\varepsilon \sim O(\sqrt{\varepsilon}).
\]
Again, \( |\beta| \to \infty \) is impossible and \( \text{[1]} \) holds. In order to estimate the strength of \( S_2 \), we can use the same arguments to prove
\[
\sup g_2'(\|\chi\|,\rho_0) := C_4 < 1,
\]
and
\[
\frac{1 - g_2'(\|\chi\|,\rho_0)}{g_2'(\|\chi\|,\rho_0)} \geq \frac{1 - C_4}{C_4} =: C_5 > 0.
\]
From now on, we shall put
\[
C_0 = \min\{C_3, C_5\}.
\]

4. Global interaction estimates

This section contains all the necessary assertions from [1] with several changes in constants. All changes are similar to those from the previous section.

**Definition 4.1.** [1] A Lipschitz curve \( J \) defined by \( t = T(x) \), \( x \in \mathbb{R} \) is called an \( I \)-curve, if \( |T'(x)| < 1/\lambda \). We denote \( J_2 > J_1 \), if \( T_1 \neq T_2 \) and \( T_2(x) \geq T_1(x), \forall x \in \mathbb{R} \). Denoting by \( S_j(J) \) the set of \( j \)-shock waves crossing \( J \) and \( S(J) = S_1(J) + S_2(J) \), we define
\[
L^-(J) = \sum_{\alpha \in S(J)} |\alpha|, \quad Q(J) = \sum_{\beta \in S_1(J), \chi \in S_2(J), \beta, \chi \text{ approach}} |\beta||\chi|.
\]
Set \( F(J) = L^-(J) + \bar{K} \cdot Q(J) \), where \( \bar{K} := 4C_\varepsilon \sqrt{\varepsilon} \). A space-like line lying between the initial line and the first interaction point is denoted with \( O \).

**Lemma 4.2.**
\[
Q(O) \leq L^-(O)^2.
\]
**Proof.** The proof follows straightforward from Definition [4.1]

**Lemma 4.3.** Assuming \( 4C_\varepsilon \sqrt{\varepsilon} L^-(O) \leq 1 \), we have
\[
F(O) \leq 2L^-(O).
\]
Proof. 
\[ F(O) = L^-(O) + \bar{K} Q(O) \leq L^-(O) + \bar{K} L^-(O)^2 \] (by (43))
\[ = L^-(O)(1 + \bar{K} L^-(O)) = L^-(O)(1 + 4 C_* \sqrt{\varepsilon} L^-(O)) \]
\[ \leq L^-(O)(1 + 1) = 2L^-(O). \]

As in [2], consider an interval \( J \subset \mathbb{R} \) and a map \( a : J \to \mathbb{R}^n \). The total variation (TV) of \( a \) is then defined as
\[ TV(a) := \sup \left\{ \sum_{j=1}^{N} |a(x_j) - a(x_{j-1})| \right\}, \]
where the supremum is taken over all \( N \geq 1 \) and all \((N+1)\)-tuples of points \( x_j \in J \) such that \( x_0 < x_1 < \cdots < x_N \). Now, we give a new estimate for \( L^+(O) \). Here, \( L^+(O) \) denotes the sum of the rarefaction waves strengths which cross the line \( O \).

**Lemma 4.4.** We have \( L^-(O) \leq TV(r_0(x), s_0(x)) \) and \( L^+(O) \leq TV(r_0(x), s_0(x)) \). \((47)\)

The estimates in previous Lemma can easily be verified. The uniform bounds of \( F(J) \) follows from the following theorem.

**Theorem 4.5.** If \( C_* \sqrt{\varepsilon} F(O) \leq \min \left\{ 1, \frac{C_0}{4} \right\} \), then \( F(J_2) \leq F(J_1) \) for \( J_2 > J_1 \).

**Proof.** This theorem can be proved in the same way as Lemma 5 from [11] and hence the proof will be omitted. One has just to substitute a constant \( K \) from the original proof with the determined value \( \tilde{K} \) here. \( \square \)

**Lemma 4.6.** Assume that \( \bar{K} L^-(O) \leq 1 \) and that
\[ \sqrt{\gamma - 1} TV(r_0(x), s_0(x)) \leq \frac{1}{C_*} \cdot \min \left\{ \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8} C_0 \right\}. \] (48)

Then \( \bar{K} F(O) \leq \min \{ 2, C_0 \} \).

**Proof.** Using Lemma 4.3 and 4.4 we have
\[ \frac{\sqrt{2}}{2} \sqrt{\varepsilon} F(O) \leq \sqrt{2} \sqrt{\varepsilon} L^-(O) \leq \sqrt{\gamma - 1} TV(r_0(x), s_0(x)) \leq \frac{1}{C_*} \cdot \min \left\{ \frac{\sqrt{2}}{4}, \frac{\sqrt{2}}{8} C_0 \right\}. \]

Multiplying it with \( 8 C_* / \sqrt{2} \), one gets
\[ 4 C_* \sqrt{\varepsilon} F(O) = \bar{K} F(O) \leq \min \{ 2, C_0 \}. \]

which proves the claim. \( \square \)

The right hand side of (48) does not depend on \( \varepsilon \), and then one can say that \( TV(r_0(x), s_0(x)) \) may be arbitrarily large since we can always choose \( \varepsilon \) small enough in order to fulfill (48) with \( \gamma = 1 + 2\varepsilon \). Then we can apply wave front tracking procedure from [1] for each such \( \varepsilon \), and obtain a sequence of step functions converging to the entropic solution. One only needs to replace \( C_\varepsilon F(O) \) and \( \frac{1 - \delta}{8} \) from [1] with \( C_* \sqrt{\varepsilon} F(O) \) and \( C_0 \), respectively.
5. Approximate delta shock solutions to pressureless gas dynamics

Our main task is to solve delta shock interaction problem for pressureless gas dynamics model. Accordingly, we will introduce a solution concept from \[9\] (somewhat simplified) and check consistency of theoretical and numerical wave front tracking results by letting \( \varepsilon \to 0 \).

5.1. Basic notions. In this section we shall use the notions and assertions from \[9\]. It contains results for a \( 3 \times 3 \) system with energy conservation law added, but all the results can also be applied to system \( (\mathcal{U}) \), too. Let us start with the basic definitions. Vector valued function of the form

\[
U_\varepsilon(x,t) = \begin{cases} 
U_0, & x < c(t) - a_\varepsilon(t) \\
U_{1,\varepsilon}(t), & c(t) - a_\varepsilon(t) < x < c(t) \\
U_{2,\varepsilon}(t), & c(t) < x < c(t) + b_\varepsilon(t) \\
U_1, & x > c(t) + b_\varepsilon(t)
\end{cases}
\]  

(49)

is called weighted shadow wave (weighted SDW, for short). Here, \( U := (\rho, u) \). The functions \( a_\varepsilon, b_\varepsilon \) are continuous functions satisfying \( a_\varepsilon(0) = x_{1,\varepsilon} \) and \( b_\varepsilon(0) = x_{2,\varepsilon} \). The SDW is constant if \( U_{1,\varepsilon} \) and \( U_{2,\varepsilon} \) are just constants. If, in addition, \( x_{1,\varepsilon} = x_{2,\varepsilon} = 0 \), then the wave is called simple.

The value

\[
\sigma_\varepsilon(t) := a_\varepsilon(t) U_{1,\varepsilon}(t) + b_\varepsilon(t) U_{2,\varepsilon}(t)
\]

is called the strength and \( c'(t) \) is called the speed of the shadow wave. We assume that \( \lim_{\varepsilon \to 0} \sigma_\varepsilon(t) = \sigma(t) \in \mathbb{R}^n \) exists for every \( t \geq 0 \) and

\[
\lim_{\varepsilon \to 0} \int U_\varepsilon(x,t) \phi(x,t) \, dx \, dt = \langle U_0 + (U_1 - U_0) \theta(x - c(t)) + \sigma(t) \delta(x - c(t)), \phi(x,t) \rangle,
\]

for \( t \geq 0 \), where \( \theta \) is a Heaviside function. The SDW central line is given by \( x = c(t) \), while \( x = c(t) - a_\varepsilon(t) \) and \( x = c(t) + b_\varepsilon(t) \) are called the external SDW lines. The values \( x_{1,\varepsilon} \) and \( x_{2,\varepsilon} \) are called the shifts, while \( U_{1,\varepsilon}(t) \) and \( U_{2,\varepsilon}(t) \) are called the intermediate states of a given SDW.

Let \( i \in \{1, 2, \ldots, n\} \). We assume \( \|U_i^\varepsilon\|_{L^\infty} = \mathcal{O}(\varepsilon^{-1}) \), if \( f \) and \( g \) have at most a linear growth with respect to \( i \)-th component, or otherwise \( \|U_i^\varepsilon\|_{L^\infty} = o(\varepsilon^{-1}) \). The components of the first kind are called major ones, while the ones of the second kind are called minor ones.

A delta shock is a SDW associated with a \( \delta \) distribution with all minor components having finite limits as \( \varepsilon \to 0 \).

The following lemma is the base of all calculations involving SDWs.

**Lemma 5.1.** Let \( f, g \in C(\Omega : \mathbb{R}^n) \) and \( U : \mathbb{R}_+^2 \to \Omega \subset \mathbb{R}^n \) be a piecewise constant function for every \( t \geq 0 \). Let us also suppose that \( f \) and \( g \) satisfy

\[
\max_{i=1,2} \{\|f(U_{1,\varepsilon})\|_{L^\infty}, \|g(U_{1,\varepsilon})\|_{L^\infty}\} = \mathcal{O}(\varepsilon^{-1}).
\]

(50)
Then
\[
\langle \partial_t f(U_\varepsilon), \phi \rangle \approx \int_0^\infty \lim_{\varepsilon \to 0} \frac{d}{dt} \left( a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t)) \right) \phi(c(t), t) \, dt \\
- \int_0^\infty c'(t) \left( f(U_1) - f(U_0) \right) \phi(c(t), t) \, dt \\
+ \int_0^\infty \lim_{\varepsilon \to 0} c'(t) \left( a_\varepsilon(t) f(U_{1,\varepsilon}(t)) + b_\varepsilon(t) f(U_{2,\varepsilon}(t)) \right) \partial_x \phi(c(t), t) \, dt
\]
and
\[
\langle \partial_x g(U_\varepsilon), \phi \rangle \approx \int_0^\infty \left( g(U_1) - g(U_0) \right) \phi(c(t), t) \, dt \\
- \int_0^\infty \lim_{\varepsilon \to 0} \left( (a_\varepsilon(t) g(U_{1,\varepsilon}(t)) + (b_\varepsilon(t) g(U_{2,\varepsilon}(t))) \right) \partial_x \phi(c(t), t) \, dt.
\]

5.2. Entropy conditions. Let \( \eta(U) \) be a semi-convex entropy function for \( \mathcal{E} \), with entropy-flux function \( q(U) \). We shall use entropy condition in the following form. A weak or approximate solution \( U_\varepsilon = (\rho_\varepsilon, u_\varepsilon) \) to system \( \mathcal{E} \) with initial data \( U|_{t=0} = U_{0,\varepsilon} \) is admissible provided that for every \( T > 0 \) we have
\[
\lim_{\varepsilon \to 0} \int_0^T \int_{\mathbb{R}} \eta(U_\varepsilon) \partial_t \phi + q(U_\varepsilon) \partial_x \phi \, dx \, dt + \int_{\mathbb{R}} \eta(U_{0,\varepsilon}(x, 0)) \phi(x, 0) \, dx \geq 0,
\]
for all non-negative test functions \( \phi \in C_0^\infty(\mathbb{R} \times (-\infty, T)) \).

Using Lemma 5.1 with \( f \) substituted by \( \eta \) and \( g \) by \( q \) and the fact that the delta function is a non-negative distribution, the first condition for SDW \( U_\varepsilon \) from (49) to be admissible is given by
\[
-c'(t)(\eta(U_1) - \eta(U_0)) + (q(U_1) - q(U_0)) \\
+ \lim_{\varepsilon \to 0} \frac{d}{dt} (\eta(U_{1,\varepsilon}(t)) a_\varepsilon + \eta(U_{2,\varepsilon}(t)) b_\varepsilon) \leq 0.
\]
The derivative of delta function changes the sign, so \( U_\varepsilon \) has to satisfy
\[
\lim_{\varepsilon \to 0} c'(t)(\eta(U_{1,\varepsilon}(t)) a_\varepsilon + \eta(U_{2,\varepsilon}(t)) b_\varepsilon) \\
- q(U_{1,\varepsilon}(t)) a_\varepsilon(t) - q(U_{2,\varepsilon}(t)) b_\varepsilon(t) = 0
\]
in addition.

These conditions are much simpler in the case of simple SDW when \( U_0, U_1, U_{1,\varepsilon} \) and \( U_{2,\varepsilon} \) are constants:
\[
\lim_{\varepsilon \to 0} -c(\eta(U_1) - \eta(U_0)) + a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon}) + q(U_1) - q(U_0) \leq 0
\]
and
\[
\lim_{\varepsilon \to 0} -c(a_\varepsilon \eta(U_{1,\varepsilon}) + b_\varepsilon \eta(U_{2,\varepsilon})) + a_\varepsilon q(U_{1,\varepsilon}) + b_\varepsilon q(U_{2,\varepsilon}) = 0.
\]
In most of the papers with delta or singular shock solution, the authors use over-compressibility as the admissibility condition. A wave is called the overcompressive one if all characteristics from both sides of the SDW line run into a shock curve, i.e.
\[
\lambda_i(U_0) \geq c'(t) \geq \lambda_i(U_1), \quad i = 1, \ldots, n,
\]
where \( c \) is a shock speed and \( x = \lambda_i(U)t, \quad i = 1, \ldots, n \) are the characteristics of the system. One will see that these notations coincide with our model case.
The entropy condition is connected with the problem of uniqueness for a weak solution of the conservation law system. We give a definition of weak (distributional) uniqueness and some results about it afterward.

**Definition 5.2.** An SDW solution is called weakly unique if its distributional image is unique. More precisely, a speed $c$ of the wave has to be unique as well as the limit
\[
\lim_{\varepsilon \to 0} a_{\varepsilon} U_{1,\varepsilon} + b_{\varepsilon} U_{2,\varepsilon}.
\]
Let $i \in \{1, \ldots, n\}$. If a limit $\lim_{\varepsilon \to 0} a_{\varepsilon} U_{1,i,\varepsilon} + b_{\varepsilon} U_{2,i,\varepsilon}$ is unique, then we say that the $i$-th component is unique.

Note that all minor components of $U_\varepsilon$ are unique by default.

### 5.3. Entropy solutions to Riemann problem for pressureless gas dynamics model.

The proof for the following theorem in the case of $3 \times 3$ PGD model is given in [9]. Its restriction to a $2 \times 2$ system is straightforward and therefore not discussed here.

**Theorem 5.3.** Suppose that $u_0 > u_1$. Then there exists a unique shadow wave solution of the form (49) to the Riemann problem (2, 3) satisfying the entropy inequality (53) with $\eta$ and $q$ as defined above.

Moreover, the validity of (53) for all semi-convex entropies $\eta$ are equivalent to the overcompressibility of the shadow wave.

Our aim is to show the structure of a solution in order to be able to compare it with a numerical approximation described above. For our purposes it is safe to take $a_\varepsilon = b_\varepsilon = \varepsilon$ in the sequel. In the proof of Theorem 5.3 we showed that a SDW solution with $U = (\rho, u)$ to (2) and initial data (3), with $u_0 > u_1$, had to satisfy
\[
c = u_s = \lim_{\varepsilon \to 0} u_\varepsilon = \left[\frac{\rho u}{\rho} - \frac{[u]}{[\rho]}\right] \sqrt{\rho_0 \rho_1} (u_s \text{ does not depend on } \varepsilon)
\]
\[
\lim_{\varepsilon \to 0} \varepsilon \rho_\varepsilon = c[\rho] - [pu] = (u_0 - u_1) \sqrt{\rho_0 \rho_1},
\]
if $\rho_0 \neq \rho_1$, and $c = u_s = (u_0 + u_1)/2$, if $\rho_0 = \rho_1$. That defines a weakly unique SDW solution to the problem.

### 5.4. Two SDWs interaction.

The main advantage of using weighted SDWs (intermediate states vary with $t$ in addition) is for solving SDW interaction problem. Then we can proceed with the main part of the paper by showing numerically that such a solution can be viewed as a limit of gas dynamics model with a vanishing pressure as perturbation. Note that verification of delta shock existence has already been obtained in [3] (see [6] for a somewhat general model).

Suppose that two SDWs interact in a point $(X,T)$. The superscript 1 is used for data in the left wave while the superscript 2 is used for the right one. The first SDW connects the states $U_0 = (\rho_0, u_0)$ with $U_1 = (\rho_1, u_1)$, while the second one connects the states $U_1 = (\rho_1, u_1)$ with $U_2 = (\rho_2, u_2)$.

Again, the following theorem has been proved in [9] for the extended PGD system, and the proof can easily be adopted for the present one (2).

**Theorem 5.4.** The result of two SDW interactions for the pressureless system (2) is a weakly unique single entropic weighted SDW.
We use the following notation: \([x]_1 := x_1 - x_0, \ [x]_2 := x_2 - x_1\) and \([x] := x_2 - x_0\). The weighted SDW solution from the above theorem satisfies the following: The speed is given by \(c'(t) = u_s(t) := \lim_{\varepsilon \to 0} u_{\varepsilon}(t)\), while \(u_s(t)\) and \(\xi(t) := \lim_{\varepsilon \to 0} \varepsilon \rho(t)\) satisfy the following ODEs system

\[
\begin{align*}
\xi'(t) &= u_s(t)[\rho] - [\rho u]_1, \\
(\xi(t)u_s(t))' &= u_s(t)[\rho u] - [\rho u^2]
\end{align*}
\]

with the initial data

\[
\xi(T) = (\xi^1 + \xi^2)T = (-[u]_1 \sqrt{\rho_0 \rho_1} - [u]_2 \sqrt{\rho_1 \rho_2})T,
\]

\[
\xi(T)u_s(T) = (c^1 \xi^1 + c^2 \xi^2)T = \left(-\frac{[pu]_1 - [u]_1 \sqrt{\rho_0 \rho_1}}{[\rho]_1} \cdot [u]_1 \sqrt{\rho_0 \rho_1}ight) + \left(-\frac{[pu]_2 - [u]_2 \sqrt{\rho_1 \rho_2}}{[\rho]_2} \cdot [u]_2 \sqrt{\rho_1 \rho_2}\right)T.
\]

Here are some facts regarding the solution \((\xi(t), u_s(t))\), \(t \geq T\) to the above initial data problem (see [9]):

1. \(\xi(t)\), for \(t > T\), is an increasing function when exists. The initial data \(\xi(T) > 0\) and \(\xi(t)\) is always positive function for \(t > T\) (when exists), since \(u_0 > u_1 > u_2\).
2. From the system (58) we have

\[
u_s'(t) = -\frac{1}{\xi(t)}([\rho]u_s^2(t) - 2[pu]u_s(t) + [pu^2]).
\]

The value \(-1/\xi(t)\) is now always negative for \(t > T\). The roots of the right-hand side of the above ODE are denoted as \(A_1 < A_2\). Then, for \([\rho] \neq 0\),

\[
A_{1,2} = \frac{[pu] \pm |u_0 - u_2| \sqrt{\rho_0 \rho_2}}{|\rho|}.
\]

Assume that \([\rho] > 0\). If \(u_s(t) \in (A_1, A_2)\), then \(u_s(t)\) increases, and if \(u_s(t) \in (-\infty, A_1) \cup (A_2, +\infty)\), then \(u_s(t)\) decreases. The opposite holds if \([\rho] < 0\). There are two possible cases:

- If \(\rho_0 > \rho_2\), then \(u_2 \leq A_1 \leq u_0 \leq A_2\). If \(u_s(T) \in (u_2, A_1)\), then \(u_s(t)\) increases for \(t > T\) but stays bellow \(A_1\). If \(u_s(T) \in (A_1, u_0)\), then \(u_s(t)\) decreases for \(t > T\) but stays above \(A_1\).
- If \(\rho_2 > \rho_0\), then \(A_1 \leq u_2 \leq A_2 \leq u_0\). Again, if \(u_s(T) \in (u_2, A_2)\), then \(u_s(t)\) increases for \(t > T\) but stays bellow \(A_2\). If \(u_s(T) \in (A_2, u_0)\), then \(u_s(t)\) decreases for \(t > T\) but stays above \(A_2\).

This implies \(u_0 \geq u_s(t) \geq u_2\) (the SDW is overcompressive). Also, one will see that numerical examples resemble these asymptotic properties of \(u_s(t)\) as \(t \to \infty\).

6. Numerical results

In this section one can find numerical results which show a consistency of theoretical (in the sense of SDWs) and numerical results. Consider system (11) with the initial data

\[
(\rho, u)|_{t=0} = \begin{cases} 
(\rho_0, u_0), & x < a_1 \\
(\rho_1, u_1), & a_1 < x < a_2 \\
(\rho_2, u_2), & x > a_2
\end{cases}
\]  

(60)
Second component of the intermediate state of the solution for (1, 60).

Adiabatic constant defined in (4).

Speed of the last right shock.

Left hand side of the integral on the second equation in (1).

Where \( a_1 < a_2 \), \( u_0 > u_1 > u_2 \). Then (see [3]), for \( \varepsilon \) small enough, there exist \((\rho_1, u_0) \in \mathbb{R}_+ \times \mathbb{R}\) and \((\rho_2, u_2) \in \mathbb{R}_+ \times \mathbb{R}\), so that:

- \((\rho_0, u_0)\) is connected with \((\rho_1, u_1)\) by an 1-shock, and \((\rho_1, u_1)\) is connected with \((\rho_1, u_1)\) by a 2-shock,
- \((\rho_1, u_1)\) is connected with \((\rho_2, u_2)\) by an 1-shock, while \((\rho_2, u_2)\) is connected with \((\rho_2, u_2)\) by a 2-shock.

A numerical solution is obtained by wave front tracking algorithm described in [1]. In order to verify two delta shocks interaction, we shall consider two cases.

**Case A.** Suppose that \((\rho_0, u_0)\) is connected with \((\rho_1, u_1)\) by a single delta shock and \((\rho_1, u_1)\) is connected with \((\rho_2, u_2)\) by a single delta shock, too. Assume that \((\rho_0, u_0)\) can be connected with \((\rho_2, u_2)\) by a single delta shock (so-called simple SDW, see [3]). The resulting SDW has a constant speed as a consequence. That can be done by choosing a special value for \( \rho_2 \) provided that \( \rho_0, u_0, \rho_1, u_1 \) and \( u_2 \) are already given.

**Case B.** We choose arbitrarily \( \rho_2 \), i.e. the resulting SDW has a variable speed (a central SDW curve is no longer a line). The numerical results are given in Tables [2] [3] and [4].

### 6.1. Case A.

**Example 6.1.** Let \( a_1 = 0, a_2 = 2, (\rho_0, u_0) = (1, 1), (\rho_1, u_1) = (1.2, 0.8) \) and \( u_2 = 0.7 \). Now, for \( \rho_2 = 1.14286 \), there exists a single simple SDW as a solution to the interaction problem.

**Table 2.** \((\rho_0, u_0) = (1, 1), (\rho_1, u_1) = (1.2, 0.8), (\rho_2, u_2) = (1.14286, 0.7)\)

| \( \gamma \) | \( \kappa \) | \( \varepsilon \) | \( \rho_0 \) | \( u_0 \) | \( \rho_1 \) | \( u_1 \) | \( c_1 \) | \( c_2 \) | \( |E_{q_1}| \) | \( |E_{q_2}| \) |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 0.5 | 0.5 | 1.29979 | 0.80062 | 0.13553 | 1.55337 | 2 \cdot 10^{-5} | 3 \cdot 10^{-5} |
| 1.2 | 0.29 | 0.1 | 1.68612 | 0.82806 | 0.57746 | 1.09746 | 1 \cdot 10^{-4} | 2 \cdot 10^{-4} |
| 1.02 | 0.099 | 0.01 | 4.22718 | 0.84163 | 0.79256 | 0.89411 | 2 \cdot 10^{-3} | 3 \cdot 10^{-3} |
| 1.01 | 0.071 | 0.005 | 6.71337 | 0.84311 | 0.81565 | 0.87247 | 8 \cdot 10^{-3} | 2 \cdot 10^{-3} |
| 1.006 | 0.055 | 0.003 | 9.95729 | 0.84379 | 0.82636 | 0.86254 | 1 \cdot 10^{-2} | 2 \cdot 10^{-3} |

After interaction, the speed of the resulting wave is \( c_3 = 0.844994 \). Two SDWs will interact in a point \((X, T) = (12.365, 13.8088)\) with such data. Now, we are going to
explain Figures 2, 7, and 12 which are illustrations of appropriate numerical results. For each \( \varepsilon \) we have two piecewise linear half-lines. The left one originates from the point \((x,t) = (a_1,0)\), while the right one originates from the point \((x,t) = (a_2,0)\). The \( i \)-th linear segment of these half-lines can be written in the form \( x = c_{i,j} (t-t_i) + x_i, i \geq 1, j = 1,2, x_i \leq x \leq x_{i+1}, t_i \leq t \leq t_{i+1} \), where \( c_{i,1} \) stands for the speed of the first \( (S_1) \) wave on the left hand side in phase plane, while \( c_{i,2} \) stands for the speed of the last \( (S_2) \) wave on the left hand side in phase plane at each \( i \)-th segment. Interactions of the waves occur at the points \((x_i,t_i), i \geq 1\). After two delta shock interaction, the resulting delta shock central line in Figure 2 (dashed line) starts from \((X,T)\) and it is calculated explicitly from system (2).

6.2. Case B.

Example 6.2. Let \( a_1 = 0, a_2 = 2, (\rho_0, u_0) = (1,1), (\rho_1, u_1) = (1.2,0.8) \) and \((\rho_2, u_2) = (1.3,0.7)\). Two SDWs will interact in a point \((X,T) = (12.2291,13.657)\) with such data. After two delta shock interaction, the resulting delta shock central lines in Figures 7 and 12 (dashed lines) start from \((X,T)\), too. Here, \[
x(t) = \int_t^\infty u_s(p) \, dp + X, \quad t \geq T,
\]
while \( u_s(t) \) represents the second component of the solution \((\xi(t), u_s(t))\) of system (45) with initial conditions (59).

**Table 3.** \((\rho_0, u_0) = (1,1), (\rho_1, u_1) = (1.2,0.8), (\rho_2, u_2) = (1.3,0.7)\)

| \( \gamma \) | \( \kappa \) | \( \varepsilon \) | \( \rho_x \) | \( u_x \) | \( c_1 \) | \( c_2 \) | \( |E_{Q1}| \) | \( |E_{Q2}| \) |
|---|---|---|---|---|---|---|---|---|
| 2  | 0.5 | 0.5 | 1.38131 | 0.74968 | 0.09312 | 1.54395 | 3 \( \times \) 10^{-4} | 5 \( \times \) 10^{-4} |
| 1.2 | 0.29 | 0.1 | 1.79287 | 0.80661 | 0.56268 | 1.08778 | 6 \( \times \) 10^{-4} | 6 \( \times \) 10^{-4} |
| 1.02 | 0.099 | 0.01 | 4.49667 | 0.83356 | 0.78596 | 0.88787 | 3 \( \times \) 10^{-3} | 3 \( \times \) 10^{-3} |
| 1.01 | 0.071 | 0.005 | 7.40338 | 0.83646 | 0.80983 | 0.86684 | 3 \( \times \) 10^{-3} | 3 \( \times \) 10^{-3} |
| 1.006 | 0.055 | 0.003 | 10.5884 | 0.83782 | 0.82091 | 0.85711 | 2 \( \times \) 10^{-4} | 2 \( \times \) 10^{-4} |

Example 6.3. Let \( a_1 = 0, a_2 = 2, (\rho_0, u_0) = (1,1), (\rho_1, u_1) = (0.8,0.9) \) and \((\rho_2, u_2) = (0.9,0.7)\). Two SDWs will interact in a point \((X,T) = (12.2364,12.8427)\) with such data.

**Table 4.** \((\rho_0, u_0) = (1,1), (\rho_1, u_1) = (0.8,0.9), (\rho_2, u_2) = (0.9,0.7)\)

| \( \gamma \) | \( \kappa \) | \( \varepsilon \) | \( \rho_x \) | \( u_x \) | \( c_1 \) | \( c_2 \) | \( |E_{Q1}| \) | \( |E_{Q2}| \) |
|---|---|---|---|---|---|---|---|---|
| 2  | 0.5 | 0.5 | 1.16621 | 0.88674 | 0.20529 | 1.51813 | 8 \( \times \) 10^{-5} | 4 \( \times \) 10^{-5} |
| 1.2 | 0.29 | 0.1 | 1.50419 | 0.86711 | 0.60356 | 1.11606 | 2 \( \times \) 10^{-4} | 1 \( \times \) 10^{-4} |
| 1.02 | 0.099 | 0.01 | 3.75748 | 0.85659 | 0.80459 | 0.90592 | 1 \( \times \) 10^{-3} | 2 \( \times \) 10^{-3} |
| 1.01 | 0.071 | 0.005 | 5.96447 | 0.85544 | 0.82632 | 0.88306 | 1 \( \times \) 10^{-3} | 8 \( \times \) 10^{-4} |
| 1.006 | 0.055 | 0.003 | 8.66370 | 0.85341 | 0.83808 | 0.86341 | 2 \( \times \) 10^{-4} | 3 \( \times \) 10^{-4} |
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7. Appendix

Figure 2. Phase $x - t$ plane, Case A, Example 6.1.

Figure 3. Speed of delta shock formed after double delta shock interaction, Case A, Example 6.1.
Figure 4. Speed of the first ($S_1$) wave on the left hand side and the last ($S_2$) wave on the right hand side for various $\varepsilon$, Case A, Example 6.1.

Figure 5. Solution $\rho(x,t)$ for various $\varepsilon$ at $t = 15000$, Case A, Example 6.1.
Figure 6. Solution $u(x,t)$ for various $\varepsilon$ at $t = 15000$, Case A, Example 6.1.

Figure 7. Phase $x-t$ plane, Case B, Example 6.2.
Figure 8. Speed of delta shock formed after double delta shock interaction, Case B, Example 6.2.

Figure 9. Speed of the first ($S_1$) wave on the left hand side and the last ($S_2$) wave on the right hand side for various $\varepsilon$, Case B, Example 6.2.
Figure 10. Solution $\rho(x,t)$ for various $\varepsilon$ at $t = 15000$, Case B, Example 6.2.

Figure 11. Solution $u(x,t)$ for various $\varepsilon$ at $t = 15000$, Case B, Example 6.2.
Figure 12. Phase $x - t$ plane, Case B, Example 6.3.

Figure 13. Speed of delta shock formed after double delta shock interaction, Case B, Example 6.3.

1. $\varepsilon = 0.5$
2. $\varepsilon = 0.1$
3. $\varepsilon = 0.01$
4. $\varepsilon = 0.005$
5. $\varepsilon = 0.003$
**Figure 14.** Speed of the first \((S_1)\) wave on the left hand side and the last \((S_2)\) wave on the right hand side for various \(\epsilon\), Case B, Example 6.3.

**Figure 15.** Solution \(\rho(x,t)\) for various \(\epsilon\) at \(t = 15000\), Case B, Example 6.3.
Figure 16. Solution $u(x,t)$ for various $\varepsilon$ at $t = 15000$, Case B, Example 6.3.