ON THE RESIDUE CLASS DISTRIBUTION OF THE
NUMBER OF PRIME DIVISORS OF AN INTEGER

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Abstract. Let $\Omega(n)$ denote the number of prime divisors of $n$ counting multiplicity. One can show that for any positive integer $m$ and all $j = 0, 1, \ldots, m - 1$, we have

$$\# \{ n \leq x : \Omega(n) \equiv j \pmod{m} \} = \frac{x}{m} + o(x^\alpha),$$

with $\alpha = 1$. Building on work of Kubota and Yoshida, we show that for $m > 2$ and any $j = 0, 1, \ldots, m - 1$, the error term is not $o(x^\alpha)$ for any $\alpha < 1$.

§1. Introduction

The Liouville function, denoted $\lambda(n)$, is defined by $\lambda(n) := (-1)^{\Omega(n)}$, where $\Omega(n)$ is the number of prime divisors of $n$ counting multiplicity. The Liouville function is closely connected to the Riemann zeta function and hence to many results and conjectures in prime number theory. Recall from [5, pp. 617–621] that for $\Re s > 1$, we have

$$\sum_{n \geq 1} \frac{\lambda(n)}{n^s} = \frac{\zeta(2s)}{\zeta(s)},$$

so that $\zeta(s) \neq 0$ for $\Re s \geq \vartheta$, provided that $\sum_{n \leq x} \lambda(n) = o(x^{\vartheta})$. The prime number theorem allows the value $\vartheta = 1$, so that for $j = 0, 1$, we have

$$\# \{ n \leq x : \Omega(n) \equiv j \pmod{2} \} \sim \frac{x}{2}.$$

If the Riemann hypothesis holds, we even have, for $j = 0, 1$ and every $\alpha > 1/2$,

$$\# \{ n \leq x : \Omega(n) \equiv j \pmod{2} \} = \frac{x}{2} + o(x^\alpha).$$
Kubota and Yoshida [4] investigated whether similar asymptotic properties could hold in general for the functions

\[ N_{m,j}(x) := \# \{ n \leq x : \Omega(n) \equiv j \pmod{m} \}, \quad m \in \mathbb{Z}_{>0}, j = 0, 1, \ldots, m - 1. \]

To this end, they introduced and studied generalizations of the Liouville function.

The question of whether for all \( m \in \mathbb{Z}_{>0} \) and \( j = 0, 1, \ldots, m - 1 \), we have

(1) \[ N_{m,j}(x) = \frac{x}{m} + o(x^\alpha) \]

with \( \alpha = 1 \) left open by Kubota and Yoshida [4], but it turns out that this follows from a result of Rivat, Sárközy, and Stewart [6]. In Section 2, we show that this also follows very quickly from a result of Hall [3] on the mean values of multiplicative functions.

As for the question of whether (1) can hold with \( \alpha < 1 \) if \( m > 2 \), Kubota and Yoshida obtained the following surprising result.

**Theorem 1** ([4, Theorem 4]). Let \( m \in \mathbb{Z}_{>2} \), and let \( \alpha < 1 \). Then for at least one \( j = 0, 1, \ldots, m - 1 \), we have that (1) does not hold.

This is in striking contrast to the expected result for \( m = 2 \). The result of Kubota and Yoshida still leaves open the possibility that, for some \( m > 2 \) and some \( j = 0, 1, \ldots, m - 1 \), equation (1) holds with some \( \alpha < 1 \). Our main result is that this is impossible.

**Theorem 2.** Let \( m \in \mathbb{Z}_{>2} \), and let \( \alpha < 1 \). Then for all \( j = 0, 1, \ldots, m - 1 \), equation (1) does not hold.

A proof, building on the work of Kubota and Yoshida [4], is given in Section 3.

**§2. Generalizations of the Liouville function**

Let \( m \in \mathbb{Z}_{>0} \), and let \( \zeta_m := e^{2\pi i/m} \) be a primitive \( m \)th root of unity. As a generalization of Liouville’s function, define for \( k = 0, 1, \ldots, m - 1 \) the function

\[ \lambda_{m,k}(n) := \zeta_m^{k\Omega(n)}. \]

The functions \( \lambda_{m,k}(n) \) were introduced by Kubota and Yoshida [4] to study the asymptotics of \( N_{m,j}(x) \) for \( m > 2 \). To investigate the properties of
\[ N_{m,j}(x), \] it is natural to look at the partial sums
\[ S_{m,k}(x) := \sum_{n \leq x} \lambda_{m,k}(n). \]

First of all, there is a simple but very useful linear relationship between \( S_{m,k}(x) \) and \( N_{m,j}(x) \). For \( k = 0, 1, \ldots, m - 1 \), we have
\[
\begin{align*}
S_{m,k}(x) &= \sum_{n \leq x} \zeta_m^{k \Omega(n)} = \sum_{j=0}^{m-1} \sum_{n \leq x} \zeta_m^{k \Omega(n)} \equiv j \pmod{m} \\
&= \sum_{\Omega(n) \equiv j \pmod{m}} \zeta_m^{k \Omega(n)} = \sum_{j=0}^{m-1} \zeta_m^{kj} N_{m,j}(x).
\end{align*}
\]

Conversely, for \( j = 0, 1, \ldots, m - 1 \), we have
\[
\begin{align*}
N_{m,j}(x) &= \sum_{n \leq x} \Omega(n) \equiv j \pmod{m} \\
&= \sum_{\Omega(n) \equiv j \pmod{m}} 1 = \sum_{n \leq x} 1 - \sum_{k=0}^{m-1} \zeta_m^{-jk} S_{m,k}(x).
\end{align*}
\]

Second, since \( \lambda_{m,k}(n) \) is a multiplicative function with values in the unit disk, we can apply the following theorem of Hall [3] to give an asymptotic bound of \( S_{m,k}(x) \).

**Theorem 3** (see [3]). Let \( D \) be a convex subset of the closed unit disk in \( \mathbb{C} \) containing zero with perimeter \( L(D) \). If \( f : \mathbb{Z}_{>0} \to \mathbb{C} \) is a multiplicative function with \( |f(n)| \leq 1 \) for all \( n \in \mathbb{Z}_{>0} \) and \( f(p) \in D \) for all primes \( p \), then
\[
\frac{1}{x} \left| \sum_{n \leq x} f(n) \right| \ll \exp \left( -\frac{1}{2} \left( 1 - \frac{L(D)}{2\pi} \right) \sum_{p \leq x} \frac{1 - \Re f(p)}{p} \right).
\]

**Lemma 4.** For every \( m \in \mathbb{Z}_{>0} \) there exists an \( A > 0 \) such that for all \( k = 1, 2, \ldots, m - 1 \), we have
\[
|S_{m,k}(x)| \ll \frac{x}{\log^A x}.
\]

**Proof.** Set \( D \) equal to the convex hull of the \( m \)th roots of unity, and set \( f(n) = \lambda_{m,k}(n) \). Because \( D \) is a convex subset strictly contained in the closed unit disk of \( \mathbb{C} \), we have \( L(D) < 2\pi \). This gives
\[
c := \frac{1}{2} \left( 1 - \frac{L(D)}{2\pi} \right) > 0.
\]
Applying Theorem 3 yields
\[ \frac{1}{x} \left| \sum_{n \leq x} \lambda_{m,k}(n) \right| \ll \exp\left(-c \sum_{p \leq x} \frac{1 - \Re \lambda_{m,k}(p)}{p}\right) = \exp\left(-c(1 - \Re \zeta^k_m) \sum_{p \leq x} \frac{1}{p}\right). \]

Since \( \sum_{p \leq x} p^{-1} = \log \log x + O(1) \), this quantity is
\[ \ll \exp\left(-c(1 - \Re \zeta^k_m) \log \log x\right) = \left( \frac{1}{\log x} \right)^{c(1 - \Re \zeta^k_m)}. \]

Noting that \( 0 < k < m \), we have \( c(1 - \Re \zeta^k_m) > 0 \). Set \( A := \min_{0 < k < m} \{ c(1 - \Re \zeta^k_m) \} \). Then \( A > 0 \), and we obtain
\[ \left| \sum_{n \leq x} \lambda_{m,k}(n) \right| \ll \frac{x}{\log Ax}. \]

As in the work of Rivat, Sárközy, and Stewart [6], this bound for the partial sums \( S_{m,k}(x) \) immediately leads to an asymptotics result for the counting functions \( N_{m,j}(x) \).

**Corollary 5.** Let \( m \in \mathbb{Z}_{>0} \). There exists an \( A > 0 \) (depending on \( m \)) such that for all \( j = 0, 1, \ldots, m - 1 \), we have
\[ N_{m,j}(x) = \frac{x}{m} + O\left( \frac{x}{\log A x} \right). \]

In particular, for all \( j = 0, 1, \ldots, m - 1 \), we have that (1) holds with \( \alpha = 1 \).

**Proof.** From (3) we immediately get
\[ N_{m,j}(x) = \frac{1}{m} S_{m,0}(x) + \frac{1}{m} \sum_{k=1}^{m-1} \zeta^{-jk} S_{m,k}(x). \]

The first term of the right-hand side of (5) is
\[ \frac{1}{m} S_{m,0}(x) = \frac{1}{m} \sum_{n \leq x} 1 = \frac{x}{m} + O(1). \]

Applying the triangle inequality and Lemma 4, we get that the absolute value of the second term of the right-hand side of (5) is
\[ \left| \frac{1}{m} \sum_{k=1}^{m-1} \zeta^{-jk} S_{m,k}(x) \right| \ll \frac{x}{\log Ax} \]
for some \( A > 0 \). This gives us our desired result. \( \square \)
The constant $A$ in Corollary 5 can easily be made explicit, but it is not the purpose of this paper to determine a good value for $A$. Readers interested in the constant $A$ may wish to consult [6].

§3. Lower bounds for the error terms

Let $m \in \mathbb{Z}_{>0}$, and let $j = 0, 1, \ldots, m - 1$. We introduce the error term

$$R_{m,j}(x) := N_{m,j}(x) - \frac{x}{m}.$$ 

Our main result, Theorem 2, obviously translates as follows.

**Theorem 6.** Let $m \in \mathbb{Z}_{>2}$, and let $\alpha < 1$. None of $R_{m,0}, R_{m,1}, \ldots, R_{m,m-1}$ are $o(x^\alpha)$.

To prove Theorem 6, keeping with [4], we use the following results.

**Lemma 7.** Let $\{a_n\}_{n \in \mathbb{Z}_{>0}}$ be a sequence of complex numbers, and let $\alpha > 0$. If the partial sums satisfy $\sum_{n \leq x} a_n = o(x^\alpha)$, then the Dirichlet series $\sum_{n \geq 1} a_n n^{-s}$ converges for $\Re s > \alpha$ to a holomorphic (single-valued) function.

**Proof.** This follows directly from Perron’s formula (see [1, p. 243, Lemma 4]).

For $\Re s > 1$, denote

$$L_{m,k}(s) := \sum_{n \geq 1} \frac{\lambda_{m,k}(n)}{n^s}.$$ 

Kubota and Yoshida [4] introduced the function $L_{m,k}(s)$ and gave a multi-valued analytic continuation of $L_{m,1}(s)$ to the region $\Re s > 1/2$; their proof easily generalizes to give the result for all $k = 1, 2, \ldots, m - 1$; thus, we attribute to them the generalization as well.

**Theorem 8 (see [4]).** Let $m \in \mathbb{Z}_{>2}$, and let $k = 1, 2, \ldots, m - 1$. The Dirichlet series $L_{m,k}(s)$ can be analytically continued to a multivalued function on $\Re s > 1/2$ given by the product $\zeta(s)^{c_k} G_{m,k}(s)$, where $G_{m,k}(s)$ is a holomorphic function for $\Re s > 1/2$. In particular, if $k \neq m/2$, then for any $\alpha < 1$, the Dirichlet series $L_{m,k}(s)$ does not converge for all $s$ with $\Re s > \alpha$. 

Proof. The first part follows from (the proof of) [4, Theorem 1]. Note that \( \zeta^k_m \) is not rational for \( k \neq m/2 \). Since \( \zeta(s) \) has a pole at \( s = 1 \), this means that no branch of \( \zeta(s)^k_m \) is holomorphic in a neighborhood of \( s = 1 \).

Remark 9. Using these results, we can quickly obtain that if \( m > 2 \), at least two of the error terms are not \( o(x^\alpha) \) for any \( \alpha < 1 \). For \( k = 1, 2, \ldots, m - 1 \), using (2), we have

\[
S_{m,k}(x) = \sum_{j=0}^{m-1} \zeta^j_m R_{m,j}(x).
\]

By Lemma 7 and Theorem 8, \( S_{m,1}(x) \) is not \( o(x^\alpha) \) for any \( \alpha < 1 \), so that at least one of the error terms \( R_{m,j}(x) \) is not \( o(x^\alpha) \), which is the result of Kubota and Yoshida [4, Theorem 1]. From (2) with \( k = 0 \), we obtain

\[
\sum_{j=0}^{m-1} R_{m,j}(x) = S_{m,0}(x) - x = -\{x\},
\]

where \( \{x\} \) denotes the fractional part of \( x \). This shows that it is impossible that all but one of the error terms \( R_{m,j}(x) \) are \( o(x^\alpha) \) for an \( \alpha < 1 \).

Let \( m > 2 \), and let \( j = 0, 1, \ldots, m - 1 \). From (3) we get

\[
R_{m,j}(x) = \frac{1}{m} \sum_{k=1}^{m-1} \zeta^{-jk}_m S_{m,k}(x) - \frac{\{x\}}{m}.
\]

In light of Lemma 7, to obtain that \( R_{m,j}(x) \) is not \( o(x^\alpha) \) for any \( \alpha < 1 \), it suffices to show that the generating function of \( R_{m,j}(x) + \{x\}/m \), which is

\[
\frac{1}{m} \sum_{k=1}^{m-1} \zeta^{-jk}_m L_{m,k}(s),
\]

cannot be analytically continued to a holomorphic (single-valued) function in the half-plane \( \Re s > \alpha \).

We now proceed with the proof of Theorem 6.

Proof of Theorem 6. Let \( 1/2 < \alpha < 1 \), and let \( c_1, c_2, \ldots, c_{m-1} \in \mathbb{C}^* \). We will prove that the linear combination

\[
f(s) := \sum_{k=1}^{m-1} c_k L_{m,k}(s)
\]
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cannot be analytically continued to a holomorphic (single-valued) function in the half-plane $\Re s > \alpha$. Suppose, to the contrary, that it can, and assume for now that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$, which will be shown later. Let $C$ denote a smooth path in the half-plane $\Re s > \alpha$, starting and ending in an $s_0$ with $\Re s_0 > 1$, winding around $s = 1$ once in the positive direction and not winding around (and not passing) any zeros of $\zeta(s)$. (One way to obtain rigorous statements below is to consider all linear combinations of $L_{m,k}(s)$ and analytic continuations along $C$ thereof as single-valued holomorphic functions in the half-plane $\Re s > 1$.) By Theorem 8, as pointed out in [4, Remark 1], the analytic continuation of $L_{m,k}(s)$ along $C$ gives us $\exp(-2\pi i \zeta_m^k) L_{m,k}(s)$. From the holomorphicity assumption on $f(s)$, it follows that the analytic continuation of $f(s)$ along $C$ is $f(s)$ itself. So we have

$$\sum_{k=1}^{m-1} c_k L_{m,k}(s) = \sum_{k=1}^{m-1} c_k \exp(-2\pi i \zeta_m^k) L_{m,k}(s),$$

and from the linear independence over $\mathbb{C}$ of the functions $L_{m,k}(s)$, we obtain that $\exp(-2\pi i \zeta_m^k) = 1$ for $k = 1, 2, \ldots, m - 1$. This means that $\zeta_m^k \in \mathbb{Z}$ for $k = 1, 2, \ldots, m - 1$, a contradiction if $m > 2$.

We are left with proving that $L_{m,1}(s), L_{m,2}(s), \ldots, L_{m,m-1}(s)$ are linearly independent over $\mathbb{C}$. By the uniqueness of Dirichlet series (see, e.g., [1, Theorem 11.3]), this would follow from the linear independence over $\mathbb{C}$ of the functions $\lambda_{m,k}(n) = \zeta_m^k \Omega(n)$ for $k = 1, 2, \ldots, m - 1$. To prove the latter, suppose that for some $d_1, d_2, \ldots, d_{m-1} \in \mathbb{C}$, we have that $\sum_{k=1}^{m-1} d_k \zeta_m^k \Omega(n) = 0$ for all $n \in \mathbb{Z}_{>0}$. Then, in particular, $\sum_{k=1}^{m-1} d_k (\zeta_m^k)^i = 0$ for $i = 0, 1, \ldots, m - 2$. This defines a system of linear equations in the $d_k$ with matrix $M$ of Vandermonde type. The values $\zeta_m^k$ for $k = 1, 2, \ldots, m - 1$ are all distinct, so $\det M \neq 0$. Therefore, $d_1, d_2, \ldots, d_{m-1}$ must all be zero; that is, $\lambda_{m,1}(n), \lambda_{m,2}(n), \ldots, \lambda_{m,m-1}(n)$ are linearly independent over $\mathbb{C}$. This completes the proof.

Remark 10. In the spirit of prime number races, it seems fitting that further study should be taken to investigate the sign changes of $N_{m,j}(x) - N_{m,j'}(x)$ for $j \neq j'$. For the case $m = 2$, some such investigations have been undertaken (see [2] and the references therein).

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