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MAGNETIC DYNAMO ACTION IN HELICAL TURBULENCE
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ABSTRACT

We investigate magnetic field amplification in a turbulent velocity field with nonzero helicity, in the framework of the kinematic Kazantsev-Kraichnan model. We present the numerical solution of the model for the practically important case of Kolmogorov distribution of velocity fluctuations, with a large magnetic Reynolds number. We find that in contrast to the nonhelical case where growing magnetic fields are described by a few bound eigenmodes concentrated inside the inertial interval of the velocity field, in the helical case the number of bound eigenmodes considerably increases; moreover, new unbound eigenmodes appear. Both bound and unbound eigenmodes contribute to the large-scale magnetic field. This indicates a limited applicability of the conventional alpha model of a large-scale dynamo action, which captures only unbound modes.

Subject headings: magnetic fields — MHD — turbulence

Online material: color figure

1. INTRODUCTION

Magnetic fields in planets and stars, protogalaxies and galaxies, and possibly intergalactic medium are generated due to random stretching of magnetic field lines by turbulent motion of highly conducting fluids or plasmas in which these lines are frozen (e.g., Lynden-Bell 1994; Parker 1979; Moffatt 1978; Kulsrud 2005; Zweibel & Heiles 1997; Schekochihin & Cowley 2006; Kulsrud & Zweibel 2007). It is natural to expect that such a dynamo mechanism can amplify magnetic fields at scales smaller than the correlation scales of the velocity fields. However, magnetic fields observed in astrophysical systems often appear to be correlated at larger scales. Such magnetic fields can be explained if one assumes that the velocity field $\mathbf{v}(x, t)$ possesses nonzero kinetic helicity, that is, $H = \int (\mathbf{v} \times \nabla \mathbf{v}) \, dx \neq 0$. To describe the large-scale fields, one generally applies the alpha model (Steenbeck et al. 1966; Moffatt 1978; Kulsrud 2005). This model is obtained if one averages the induction equation

$$\partial_t \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \eta \nabla^2 \mathbf{B}, \tag{1}$$

where $\eta$ is resistivity or magnetic diffusivity, over small-scale fluctuations of the velocity and magnetic fields, assuming that these fluctuations are much weaker and concentrated at the scales much smaller than the scales of the growing large-scale field. As a result one obtains the alpha model equation for the large-scale magnetic field $\mathbf{B}(x, t)$, that is, the magnetic field averaged over the scales larger than the scales of velocity fluctuations,

$$\partial_t \mathbf{B} = \alpha \nabla \times \mathbf{B} + \beta \nabla^2 \mathbf{B}. \tag{2}$$

In this equation, $\alpha \sim v_t (\nabla \times \mathbf{v}) \tau_\alpha$, and $\beta \sim v_t / l_h$, where $v_t$ is a characteristic velocity, $l_h$ is the characteristic scale, and $\tau_\alpha \sim l_h / v_t$ is a characteristic eddy turnover time (decorrelation time) of fluid fluctuations. In astrophysical systems resistivity is very small, so that turbulent velocity field effectively generates small-scale magnetic fluctuations, which grow much faster than the large-scale field. In this case, the applicability of the mean-field equation (2) is questionable (see, e.g., the discussion in Vainshtein & Cattaneo 1992; Gruzinov & Diamond 1994; Field & Blackman 2002).

To investigate this question, we use a solvable model of kinematic dynamo, introduced by Kazantsev (1968) and Kraichnan (1968). In this model, velocity is assumed to be Gaussian, with zero mean, $\langle \mathbf{v} \rangle = 0$, and the covariance tensor

$$\langle \mathbf{v}(x, t) \mathbf{v}(x', t') \rangle = \kappa^H (|x - x'|) \delta(t - t'), \tag{3}$$

where $\kappa^H$ is an isotropic tensor,

$$\kappa^H(x) = \kappa_x \left( \delta^{\alpha} - \frac{x^\alpha x^{\alpha}}{x^2} \right) + \kappa_y \frac{x^\alpha x^{\alpha}}{x^2} + g e^{\alpha \beta} x^\alpha. \tag{4}$$

Here $\langle \cdot \rangle$ denotes ensemble average, $\epsilon^{\alpha \beta}$ is the unit antisymmetric pseudotensor, and summation over repeated indices is assumed. The first two terms on the right-hand side of equation (4) represent the mirror-symmetric, nonhelical part, while function $g(x)$ describes the helical part of the velocity fluctuations. For an incompressible velocity field (the only case we are considering here), we have $\kappa_x(x) = \kappa_y(x) + x \kappa'(x)/2$, where the prime denotes derivative with respect to $x = |x|$. Therefore, to describe the velocity field, we need to specify only two independent functions, say, $\kappa_y(x)$ and $g(x)$. The magnetic field correlator can similarly be expressed as

$$\langle B'(x, t) B'(0, 0) \rangle = M_y \left( \delta^{\alpha} - \frac{x^\alpha x^{\alpha}}{x^2} \right) + M_x \frac{x^\alpha x^{\alpha}}{x^2} + K e^{\alpha \beta} x^\alpha, \tag{5}$$

where the corresponding solenoidality constraint implies $M_x(x, t) = M_y(x, t) + (x/2) K(x, t)$. To describe the magnetic correlator we therefore need only two functions, $M_y(x, t)$ and $K(x, t)$, corresponding to magnetic energy and magnetic helicity.

Suppose that the velocity field (3) is given. The problem is then to find the correlation function (5) of the magnetic field. In the nonhelical case, $g(x) \equiv 0$, it was established by Kazan-
tsev (1968) that the problem is reduced to a quantum mechanical problem with imaginary time:

\[-\partial \psi = \hat{H} \psi.\]  

(6)

More precisely, given the velocity correlator (3) and magnetic resistivity \(\eta\), one constructs the self-adjoint Hamiltonian \(\hat{H}\). The magnetic correlator \(M_f(x, t)\) is then mapped to the “wave function” \(\psi(x, t)\). If equation (6) has growing solutions \(\psi(x, t)\), corresponding to negative eigenvalues of \(\hat{H}\), then dynamo action is possible. One can show that the smaller the resistivity \(\eta\), the deeper the effective potential in the quantum mechanical problem given in equation (6), and, therefore, nonhelical dynamo is always possible when the magnetic Reynolds number, \(Rm \propto \eta^{-1}\), is large enough (Vainshtein & Kichatinov 1986; Boldyrev & Cattaneo 2004; Iskakov et al. 2007). The quantum mechanical representation given in equation (6) is important since it ensures that the eigenvalues are real and the eigenfunctions are mutually orthogonal, which allows one to apply a variational principle for estimating dynamo growth rates. The Kazantsev-Kraichnan model of nonhelical dynamo action has been well investigated in the literature.

The situation is more complex when the velocity field possesses nonzero helicity, i.e., \(g(x) \neq 0\). In this case, given kinetic energy \(\kappa_p(x)\) and kinetic helicity \(h(x)\), one needs to solve two coupled partial differential equations for functions \(M_f(x, t)\) and \(K(x, t)\) related to magnetic energy and magnetic helicity. Such equations were first derived by Vainshtein & Kichatinov (1986). Due to their complexity, there have been relatively fewer results obtained for the helical case (e.g., Mezenguzzi et al. 1981; Kulsrud & Anderson 1992; Kim & Hughes 1997; Brandenburg 2001; Field & Blackman 2002; Brandenburg & Subramanian 2005). However, it is the helical case that is practically more important since astrophysical systems generally possess nonzero helicity. Moreover, while magnetic fields at small (velocity) scales are naturally expected on the base of equation (6), explanation of astrophysically observed large-scale magnetic fields commonly requires helicity effects, as, e.g., in equation (2).

Recently, it has been established in Boldyrev et al. (2005) that the Vainshtein & Kichatinov (1986) equations also possess a self-adjoint structure, and can be reduced to a quantum mechanical “spinor” form:

\[-\partial \psi^\alpha = \hat{H}^{\alpha\beta} \psi^\beta,\]  

(7)

where \(\alpha = \{1, 2\}\) and summation over repeated indices is assumed. Similarly to equation (6), the self-adjoint Hamiltonian \(\hat{H}^{\alpha\beta}\) depends on kinetic energy and kinetic helicity, \(\kappa_p(x)\) and \(g(x)\), and on magnetic resistivity \(\eta\). The two components of the \(\psi^\alpha(x, t)\) function are then related to functions \(M_f(x, t)\) and \(K(x, t)\) in the magnetic correlator (5).

Similar structures of the dynamo equations (6) and (7) allow one to investigate them on the same footing. In particular, one can compare the growth rates and the eigenvalues associated with helical and nonhelical dynamo action, and address the important question of whether the large-scale magnetic field generated due to helical dynamo action is described by the alpha model given in equation (2). This is the goal of the present work. We assume that the velocity field has the Kolmogorov spectrum and that the Reynolds number and the magnetic Reynolds number are large. Then we present the full numerical solution of the helical dynamo model given in equation (7), i.e., we find its eigenvalues and eigenfunctions. We demonstrate that at least at the kinematic stage, the alpha model (eq. [2]) may provide a nonadequate description of large-scale fields, since it misses the rapidly growing large-scale eigenmodes. In the next section we present our main results; a detailed discussion will be presented elsewhere.

2. NUMERICAL SOLUTION OF THE HELICAL DYNAMO MODEL

The self-adjoint equations for functions \(M_f(x, t)\) and \(K(x, t)\) were derived in Boldyrev et al. (2005). We will be interested in eigenmodes of these equations, and therefore assume that both functions depend on time as \(\exp(\lambda t)\). It is then convenient to introduce the auxiliary functions \(w_2(x)\) and \(w_1(x)\) defined as

\[M_f = \frac{\sqrt{2}e^{i\lambda}}{x^2}w_2(x), \quad K = -\frac{e^{i\lambda}}{\sqrt{2}x}[x^2w_1(x)]'.\]  

(8)

The eigenmode equation that we solve then takes the form

\[\left[ -\left(\mathcal{L}_x(\mathcal{L}_x) - \lambda \right) + \frac{\mathcal{L}_x}{\mathcal{L}_x}C(dddx)x^2 - x^2(dddx)C(\mathcal{L}_x) + \lambda \right]\left[ w_2 \right] = 0,\]  

(9)

where

\[\mathcal{E} = -\frac{1}{2}x \frac{d}{dx} B \frac{d}{dx} x + \frac{1}{2} (A - xA'),\]

\[A(x) = \sqrt{2}[\eta + \kappa_p(0) - \kappa_p(x)],\]

\[B(x) = 2\eta + \kappa_p(0) - \kappa_p(x),\]

\[C(x) = \sqrt{2}(g(0) - g(x)x),\]  

(10)

and primes denote derivatives with respect to \(x\). Equations (8)–(10) describe the growth of the magnetic field in the Kazantsev-Kraichnan model. Equation (9) is self-adjoint, which guarantees that all growth rates \(\lambda\) are real. The system given in equation (9) cannot be solved analytically for general velocity correlation functions \(\kappa_p(x)\) and \(g(x)\). Below we solve equation (9) numerically, and concentrate on functions \(w_2(x)\) and \(w_1(x)\) since they define the magnetic field correlator uniquely and contain all the information about magnetic energy and magnetic helicity.

It is useful to note the Fourier transformed version of equation (4):

\[k^i(k) = F(k) \left( \hat{\delta}^0 - \frac{k^i k^i}{k^2} \right) + iG(k)\epsilon^{ijk}k^j,\]  

(11)

where functions \(F(k)\) and \(G(k)\) can be expressed in terms of the three-dimensional Fourier transforms of \(\kappa_p(x)\) and \(g(x)\) (Monin & Yaglom 1971), and

\[\langle B'(k, t)B'(k, t) \rangle = F_\delta(k, t) \left( \hat{\delta}^0 - \frac{k^i k^i}{k^2} \right) - i \frac{H_\delta(k, t)}{2k^2} \epsilon^{ijk}k^j,\]  

(12)

where \(F_\delta(k, t)\) is the magnetic energy spectral function, \(\langle |B(k, t)|^2 \rangle = 2F_\delta(k, t)\), and \(H_\delta(k, t)\) is the spectral function
of the electric current helicity, \( \langle B^2(\mathbf{k}, t) \rangle^{\frac{1}{2}} k^2 B^2(\mathbf{k}, t) = H_e(k, t) \).

First, as easily derived from equation (9) the asymptotic behavior of functions \( w_2(x) \) and \( w_1(x) \) when \( x \ll [\eta/k_i^2(0)]^{1/2} \) is

\[
\begin{align*}
0 & \quad \text{where, without loss of generality, we use scaling \( w_2 x \to 1 \) as} \\
\frac{w_2}{w_3} & \quad \text{and \( w_1(x) \) as} \\
& \quad \text{coefficient} \ \xi \ \text{is a free parameter, related to the average} \\
\frac{w_2}{w_3} & \quad \text{helicity of the electric current,} \\
\xi & \quad \text{The linear-scale plots} A \ \text{and} \ B \ \text{correspond to} \\
\xi & \quad \text{as} \ \lambda > \lambda_0, \ \text{and for} \ \lambda \ll \lambda_0 \ \lambda \text{-dependent on the magnetic field growth rate} \ \lambda. \\
\xi & \quad \text{If} \ \lambda > \lambda_0 \ \text{is} \ \lambda \text{-dependent on the magnetic field growth rate} \ \lambda. \\
\xi & \quad \text{and} \ \lambda \ll \lambda_0 \ \text{is} \ \lambda \text{-dependent on the magnetic field growth rate} \ \lambda. \\
\xi & \quad \text{This asymptotic behavior suggests an analogy between} \\
\xi & \quad \text{the Kazantsev-Kraichnan model (eq. [3]), the velocity field enters the eigenfunction} \\
\xi & \quad \text{this results in the condition in eq. (15).}
\end{align*}
\]

To study a realistic case, we consider velocity correlation tensor (11) with the Kolmogorov power spectrum. It is important to note that in the Kazantsev-Kraichnan model (eq. [3]), the velocity field enters the eigenfunction equations (6), (7), and (9) only in the form of turbulent diffusivity \( \kappa''(x-x') = |(v'(x, t) v'(x', t)) dt| \). In the Kolmogorov turbulence, the latter scales as \( v_L \sim 1/3 \) \( \sqrt{r \cdot l} \). (15)\(^3\) \( \kappa(x) = \kappa(0)(xL)^{\frac{3}{2}} \approx v_L(xl)^{\frac{3}{2}} \) \( l \). Without loss of generality we take \( l \sim 1, v_L \sim 1, \) and therefore

\[
\begin{align*}
F(k) & \quad \text{Here the minimal cutoff wavevumber} k_{\min} = 2, \ \text{the maximal} \\
F(k) & \quad \text{and} \ \kappa_{\max} = 2[\kappa(0)/\nu^{3/4} \approx 4\nu^{3/4} \text{is determined by the plasma kinematic viscosity} \ \nu, \ \text{and the helicity parameter} \\
F(k) & \quad \text{must satisfy the realizability condition} \ -1 \leq h \leq 1; \ \text{the velocity} \\
F(k) & \quad \text{field is maximally helical when} \ |h| = 1. \ \text{The resulting} \\
F(k) & \quad \text{growth rates} \ \lambda_v \ \text{of the bound (localized) eigenmodes,} \ \lambda_v \leq \lambda_0, \ \text{are shown in Figure 1, where the magnetic diffusivity is} \\
F(k) & \quad \text{chosen to be} \ \eta = 10^{-6}. \ \text{The growth rates are measured in units of} \\
F(k) & \quad \text{large-scale eddy turnover rate} \ \tau \sim 1/H. \ \text{The linear-scale plots} A \ \text{and} \\
F(k) & \quad \text{B correspond to} \ h = 1 \ \text{and} \ h = 0.1, \ \text{respectively, while} \\
F(k) & \quad \text{with} \ k_{\max} = 3, \ \text{which is consistent with the Reynolds number being of order unity (i.e., a single-scale velocity field).} \\
F(k) & \quad \text{The logarithmic-scale plots} C \ \text{and} \ D \ \text{correspond to} \ h = 1 \ \text{and} \ h = 0.1, \ \text{where} \\
F(k) & \quad \text{with} \ k_{\max} = 3000, \ \text{which is consistent with large Reynolds} \\
F(k) & \quad \text{and large magnetic Prandtl numbers. Finally, the logarithmic-} \\
F(k) & \quad \text{scale plots} E \ \text{and} \ F \ \text{correspond to} \ h = 1 \ \text{and} \ h = 0.1, \ \text{while} \\
F(k) & \quad \text{with} \ k_{\max} = 3 \times 10^7, \ \text{which is consistent with a very large Reynolds} \\
F(k) & \quad \text{number and a small Prandtl number.}
\end{align*}
\]

3. DISCUSSION AND CONCLUSION

We find that when the Reynolds number is of order unity (a single-scale velocity field), the magnitude of the kinetic helicity parameter \( h \) does not have much effect on the bound magnetic eigenmodes (plots A and B in Figure 1). However, in the case when the Reynolds number is large and the velocity fluctuations extend over a large range of scales, the bound eigenmodes are significantly affected by kinetic helicity at scales larger than the viscous scale. (Magnetic fluctuations at the scales much smaller than the viscous scale, which are excited in the case

\[
\begin{align*}
\text{Our results do not depend on the exact choice of the boundaries of the computational interval} \ x_{\min} \leq x \leq x_{\max}. \\
\text{Given} \ M_f(x, t), \ \text{the function} \ K_i(x, t) \ \text{cannot be chosen arbitrarily; its Fourier} \\
\text{image must satisfy the realizability condition} \ |H_i(k, t) \leq F(k, t). \ \text{Analogously,} \\
\text{given} \ \eta(x, t), \ \text{the function} \ g(x, t) \ \text{must also satisfy a similar realizability condition} \ (Moffatt 1978). \ \text{The latter results in the condition} \ -1 \leq h \leq 1 \ \text{in eq. (15).}
\end{align*}
\]
Pr >> 1, are not significantly affected by magnetic helicity, which is consistent with previous considerations; e.g., Kulsrud & Anderson [1992].) The number of bound eigenmodes increases considerably when the kinetic helicity increases, and the corresponding eigenvalues, \( \lambda_n \), become strongly concentrated near the eigenvalue of the fastest unbound eigenmode, \( \lambda_{\infty} \). This result follows from a nearly uniform distribution of \( \lambda_n \) on the logarithmic-scale plots in Figure 1.

Moreover, we observe an important fact that in all the cases in Figure 1 the growth rate of the first bound eigenmode, \( \lambda_1 \), happens to be very close to the growth rate of the fastest growing unbound eigenmode, \( \lambda_{\infty} \). We conjecture that for high Reynolds numbers the potential in equation (9) always has a shallow bound state \( \lambda_1 \) (such that \( \lambda_1 - \lambda_\infty \ll \lambda_\infty \)) whose characteristic scale is much larger than the velocity correlation scale, due to equation (14). This has an important consequence for the dynamo mechanism. To understand it, we note that the

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