SELF-ATTRACTING SELF-AVOIDING WALK

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Abstract. This article is concerned with self-avoiding walks (SAW) on \( \mathbb{Z}^d \) that are subject to a self-attraction. The attraction, which rewards instances of adjacent parallel edges, introduces difficulties that are not present in ordinary SAW. Ueltschi has shown how to overcome these difficulties for sufficiently regular infinite-range step distributions and weak self-attractions [1]. This article considers the case of bounded step distributions. For weak self-attractions we show that the connective constant exists, and, in \( d \geq 5 \), carry out a lace expansion analysis to prove the mean-field behaviour of the critical two-point function, hereby addressing a problem posed by den Hollander [2].

Key words and phrases. Self-interacting random walk, self-attracting walk, self-avoiding walk, linear polymers, lace expansion, critical phenomena, Hammersley-Welsh argument.

1. Introduction

1.1. Model definition. Let \( \mathbb{Z}^d \) denote the hypercubic lattice with nearest-neighbour edges, and assume \( d \geq 2 \). Let \( P \) be the law of a random walk on the vertices of \( \mathbb{Z}^d \) with i.i.d. increments distributed according to a step distribution \( D \). Letting \( \{ \pm e_i \}_{i=1}^d \) denote the standard generators of \( \mathbb{Z}^d \), a plaquette is a collection of vertices of the form \( \{ x, x + u, x + v, x + u + v \} \) where \( u \notin \{ \pm v \} \). Two nearest-neighbour edges \( \{ x_1, y_1 \} \) and \( \{ x_2, y_2 \} \) of \( \mathbb{Z}^d \) are adjacent if \( \{ x_1, y_1, x_2, y_2 \} \) is a plaquette.

A walk is a sequence of vertices in \( \mathbb{Z}^d \), and the edges of a walk \( \omega \) are the pairs \( \{ \omega_i, \omega_{i+1} \} \) of consecutive vertices. Note that, for general increment distributions \( D \), the edges of a walk in the support of \( P \) are not necessarily edges of \( \mathbb{Z}^d \). Define \( \text{adj}(\omega) \) to be the collection of pairs of edges of \( \omega \) that are adjacent edges of \( \mathbb{Z}^d \), and let \( |\text{adj}(\omega)| \) be the cardinality of this set. See Figure 1.

Let \( P_n \) denote the law induced by \( P \) on \( n \)-step walks that begin at the origin \( o \in \mathbb{Z}^d \), and recall that a walk is self-avoiding if it does not visit any vertex more than once (see Section 4.1 for a more precise definition). The models we are interested in are perturbations \( P_{n,\kappa} \) of \( P_n \) defined by

\[
P_{n,\kappa}(\omega) \propto 1_{\{\omega \in \Gamma_n\}} W_\kappa(\omega), \quad \kappa \geq 0,
\]

where \( \Gamma_n \) is the set of \( n \)-step self-avoiding walks with initial vertex \( \omega_0 = o \) and

\[
W_\kappa(\omega) = e^{-H_\kappa(\omega)} P_n(\omega), \quad e^{-H_\kappa(\omega)} = (1 + \kappa)^{|\text{adj}(\omega)|}.
\]

The law \( P_{n,\kappa} \) on \( n \)-step walks is called \((n\text{-step})\) attracting self-avoiding walk with attraction strength \( \kappa \), or \((n\text{-step})\) \( \kappa \)-ASAW. When the length of the walk is irrelevant the adjective \( n \)-step will be dropped. We think of the right-hand
side of (1.1) as defining the $\kappa$-ASAW weight of a walk $\omega$. The probability of a walk is proportional to its weight.

The law $P_{n,0}$ defined by (1.1) is the law of $n$-step self-avoiding walk (SAW) [3]. Physically, self-avoiding walk is a model of a linear polymer in a good solvent. The self-avoidance constraint represents the inability of two molecules in the polymer to occupy the same space. If $\kappa > 0$, walks under the $\kappa$-ASAW law are attracted to themselves. Physically, this is a model of a linear polymer in a poor solvent [4, Section 6.3] [2, Chapter 6]. The molecules huddle together to escape exposure to the surrounding solvent.

1.2. Lack of submultiplicativity. Before stating our results, we briefly discuss the central difficulty of the model. Let $c_n(\kappa)$ denote the normalization constant that makes $P_{n,\kappa}$ a probability measure, i.e.,

$$c_n(\kappa) = \sum_{\omega \in \Gamma_n} W_\kappa(\omega).$$

Note that $c_n(\kappa)$ is implicitly also a function of the step distribution $D$.

The first mathematical fact one learns about self-avoiding walk is that when $\kappa = 0$ the sequence $(c_n(\kappa))_{n \geq 1}$ is submultiplicative, i.e.,

$$c_{n+m}(0) \leq c_n(0)c_m(0).$$

This bound arises because any $(n+m)$-step self-avoiding walk can be split into an $n$-step self-avoiding walk and an $m$-step self-avoiding walk. Simple estimates and Fekete’s lemma ([5, Lemma 1.2.1]) on submultiplicative sequences imply that $(c_n(0))^{1/n}$ converges as $n \to \infty$: see [6, Section 1.2].

The basic difficulty in the study of $\kappa$-ASAW is that the sequence $(c_n(\kappa))_{n \geq 1}$ is generally not submultiplicative for $\kappa > 0$. To see that submultiplicativity cannot hold in general, consider the nearest-neighbour step distribution $D(x) = (2d)^{-1}1_{\{\|x\|_1=1\}}$. Submultiplicativity of the sequence $c_n(\kappa)$ would imply

$$c_n(\kappa) \leq c_1(\kappa)^n = 1,$$

which cannot hold for fixed $n \geq 3$ when $\kappa$ is sufficiently large, as the left-hand side is a polynomial in $\kappa$ of degree at least 1.

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1 Note that our definition of $c_n$ involves $D$, i.e., we are enumerating weighted self-avoiding walks.
2. Results

Henceforth it will be assumed that \(D(x)\) is invariant under the symmetries of \(\mathbb{Z}^d\) (namely, reflections in hyperplanes and rotations by \(\pi/2\) about coordinate axes), and that \(D(e_i) = p_1 > 0\). Thus, \(P[D = \pm e_i] = p_1\) for all choices of sign and \(i = 1, 2, \ldots, d\).

2.1. Connective constants. The limiting value \(\mu(\kappa)\) of \((c_n(\kappa))^{1/n}\), if the limit exists, is called the connective constant with self-attraction \(\kappa\). We will prove that the connective constant of \(\kappa\)-ASAW exists for \(\kappa\) sufficiently small despite not knowing if submultiplicativity holds.

**Theorem 2.1.** Let \(d \geq 2\). There exists a \(\kappa_0 = \kappa_0(D) > 0\) such that for \(0 < \kappa < \kappa_0\) the limit \(\mu(\kappa) = \lim_{n \to \infty} (c_n(\kappa))^{1/n}\) exists.

Note that the dependence of \(\kappa_0\) on \(D\) implicitly means that \(\kappa_0\) may depend on the dimension \(d\). The remainder of this section briefly describes the proof of Theorem 2.1, although we delay a discussion of how the lack of submultiplicativity is overcome to Section 2.3. The proof appears in Section 5.

An \(n\)-step self-avoiding walk \(\omega\) is a bridge if \(\pi_1(\omega_0) < \pi_1(\omega_j) \leq \pi_1(\omega_n)\), where \(\pi_1\) denotes projection onto the first coordinate. A key observation for the proof of Theorem 2.1 is that \(W_\kappa\) is supermultiplicative on bridges when \(\kappa \geq 0\). This implies the connective constant for bridges, \(\mu_B(\kappa)\), exists.

A classical argument due to Hammersley and Welsh shows that the number of \(n\)-step self-avoiding bridges is the same, up to sub-exponential corrections, as the number of \(n\)-step self-avoiding walks [6, Section 3.1]. An immediate consequence is that \(\mu_B(0) = \mu(0)\). To prove the existence of \(\mu(\kappa)\) we adapt the Hammersley-Welsh argument to \(\kappa > 0\), i.e., we prove that the difference in the \(\kappa\)-ASAW weight of \(n\)-step bridges and \(n\)-step walks is sub-exponential in \(n\).

The Hammersley-Welsh argument involves “unfolding” self-avoiding walks by reflecting segments of the walk through well chosen hyperplanes. It is during unfolding that the lack of submultiplicativity must be overcome.

2.2. Mean-field behaviour. To give a precise statement of our lace expansion results, we require further assumptions on the increment distribution \(D\).

**Definition 1.** Let \(L > 0\). A step distribution \(D\) is spread-out with parameter \(L\) if it has the form

\[
D(x) = \begin{cases} 
\frac{h(x/L)}{\sum_{x \in \mathbb{Z}^d(o)} h(x/L)} & x \neq o \\
0 & x = o,
\end{cases}
\]

where \(h: [-1,1]^d \to [0,\infty)\) is a piecewise continuous function such that \(h(0) > 0\),

(i) \(h\) is invariant under the symmetries of \(\mathbb{Z}^d\), and

(ii) \(\int h(x) \, dx = 1\).

In what follows when we consider a “spread-out step distribution” we mean the one-parameter family of step distributions obtained by choosing a single function \(h\). Note that \(h(0) > 0\) and \(h\) piecewise continuous implies that the denominator in (2.1) is positive and \(D(e_1) = p_1 > 0\) if \(L\) is taken sufficiently
large. We will implicitly assume \(L\) is at least this large in what follows. The variance \(\sigma^2\) of \(D\) will be denoted \(\sum_{x \in \mathbb{Z}^d} \|x\|_2^2D(x)\).

**Example 2.2.** Consider \(h(x) = 2^{-d}\) on \([-1,1]^d\). This leads to \(D(x)\) being uniformly distributed on vertices \(x \neq o\) with \(\|x\|_\infty \leq L\).

The proof of Theorem 2.1 relies in part on establishing that \(\kappa\)-ASAW is repulsive “on average”; roughly speaking, this means that a walk under the \(\kappa\)-ASAW law is not typically attracted to its earlier trajectory. This idea is explained more precisely in Section 2.3. The remainder of this section indicates further consequences of repulsion on average.

**Definition 2.** The susceptibility of \(\kappa\)-ASAW is the power series

\[
\chi_\kappa(z) = \sum_{n=0}^{\infty} c_n(\kappa)z^n.
\]

**Definition 3.** The critical point \(z_c = z_c(D, \kappa)\) of \(\kappa\)-ASAW is defined to be

\[
z_c = \sup\{z \geq 0 \mid \chi_\kappa(z) < \infty\}.
\]

For SAW, submultiplicativity implies that \(z_c(0) = \mu(0)^{-1}\). For \(\kappa\) small enough that Theorem 2.1 applies, it remains true that \(z_c(\kappa) = \mu(\kappa)^{-1}\). This is because \(\chi_\kappa(z)\) has non-negative coefficients, so \(z_c(\kappa)\) is the radius of convergence of \(\chi_\kappa(z)\) by Pringsheim’s theorem. At the same time \(z_c(\kappa)\) is also identified as the limit in Theorem 2.1 by the Cauchy-Hadamard characterization of the radius of convergence.

The two-point function of \(\kappa\)-ASAW is defined, for \(z \geq 0\) and \(x \in \mathbb{Z}^d\), by

\[
G_{z,\kappa}(x) = \sum_{n \geq 0} \sum_{\omega \in \Gamma_n(x)} z^nW_\kappa(\omega).
\]

Note that the inner sum is restricted to self-avoiding walks, so only \(n\)-step walks with positive probability under \(P_n,\kappa\) contribute.

The \(\kappa\)-ASAW two-point function should be compared with the simple random walk two-point function

\[
S_z(x) = \sum_{n \geq 0} \sum_{\omega \in \mathcal{W}_n(x)} z^n\mathbb{P}_n[\omega],
\]

in which the inner sum is over \(\mathcal{W}_n\), the set of all \(n\)-step walks with initial vertex \(o \in \mathbb{Z}^d\). The term “simple” is used to indicate that the associated law on \(n\)-step walks is \(\mathbb{P}_n\), although this may not be a nearest-neighbour walk. Let \(\|x\|\) denote \(\max\{\|x\|_2, 1\}\). Despite the notation, this is not a norm.

**Theorem 2.3.** Let \(d \geq 5\). For sufficiently spread-out step distributions, with parameter \(L \geq L_0(D)\), there is a \(\kappa_0\) such that if \(0 \leq \kappa \leq \kappa_0\) and \(\alpha > 0\) then

\[
G_{z_c}(x) = \frac{a_d}{\sigma^2} \left(1 + O(L^{\alpha-2}) + O\left(\frac{L^2}{\|x\|^2-\alpha}\right)\right),
\]

where \(a_d = 2^{-1}\pi^{-d/2}d\Gamma(d/2 - 1)\) and \(\sigma^2\) is the variance of the step distribution \(D\).
This theorem shows that the critical two-point function of $\kappa$-ASA\(W\) has the same asymptotics as the critical ($z = 1$) two-point function of simple random walk in $d \geq 5$. In the language of critical exponents, $\eta = 0$, i.e., this is a verification that $\kappa$-ASA\(W\) has mean-field behaviour. We have not attempted to optimize the relation between $\kappa_0$ and $L$ in our proof, as our primary interest is in the existence of $\kappa_0 > 0$ for finite $L$.

It is typically difficult to apply the lace expansion to models containing attracting interactions, as these attractions make it difficult to obtain what are known as diagrammatic bounds. We are able to overcome this difficulty as the on average repulsion that $\kappa$-ASA\(W\) satisfies is compatible with calculating diagrammatic bounds. This is discussed in more detail in Section 2.3. Once the diagrammatic bounds are obtained the remaining part of the lace expansion analysis is well understood and can be adapted from existing arguments [7]. We recall how this can be done in Appendix A. The proof of Theorem 2.3 is carried out in Section 7.

2.3. Main idea. The proofs of Theorems 2.1 and 2.3 are essentially independent, but they share a common idea which we explain here.

Let $\Gamma$ denote the set of all self-avoiding walks, not necessarily starting at the origin $o$. Writing a walk $\omega$ as a concatenation $\omega = \omega^1 \circ \omega^2$ of two subwalks determines an interaction conditional on $\omega^2$, i.e.,

\[(2.6) \quad \mathbb{1}_{\{\omega \in \Gamma\}} e^{-H_\kappa(\omega)} = \left( \mathbb{1}_{\{\omega^1 \in \Gamma\}} e^{-H_\kappa(\omega^1)} \right) \left( \mathbb{1}_{\{\omega^2 \in \Gamma\}} e^{-H_\kappa(\omega^2; \omega^1)} \right),\]

where this formula defines $H_\kappa(\cdot; \omega^1)$. Explicitly,

\[H_\kappa(\omega^2; \omega^1) = (1 + \kappa)|\text{adj}(\omega^2)| \left( 1 + \kappa \right)^{|\text{adj}(\omega^1; \omega^2)|},\]

where $\text{adj}(\omega^1; \omega^2)$ is the set of pairs of adjacent edges $\{f_1, f_2\}$ with $f_i \in \omega^i$, $i = 1, 2$.

For self-avoiding walk, i.e., $\kappa = 0$, the interaction is trivial: $e^{H_0(\omega)} = 1$. Submultiplicativity therefore follows from (2.6) and the observation that

\[\mathbb{1}_{\{\omega \in \Gamma\}} \leq \mathbb{1}_{\{\omega^1 \circ \omega^2 \in \Gamma\}} \leq \mathbb{1}_{\{\omega^2 \in \Gamma\}}.\]

For $\kappa > 0$ it is not generally true that $e^{-H_\kappa(\eta; \omega^1)} \leq e^{-H_\kappa(\eta)}$. See Figure 1 and consider splitting the walk at the indicated vertices.

Equation (2.6) highlights a tension between self-avoidance and self-attraction. Energetic rewards of $(1 + \kappa)$ due to the conditional interaction only occur if the walk $\omega^2$ has edges adjacent to edges in $\omega^1$. Such an edge in $\omega^2$ carries an entropic penalty, as the potential configurations of $\omega^2$ are reduced. Thus there is both an entropic benefit and an energetic penalty to dropping the conditional interaction due to $\omega^1$ when Equation (2.6) is summed over a suitable class of walks.

More explicitly, if $\omega^2$ contains an edge $\{x_1, y_1\}$ adjacent to an edge $\{x_2, y_2\}$ of $\omega^1$, then typically there is a self-avoiding modification of $\omega^2$ that traverses $\{x_2, y_2\}$ instead of $\{x_1, y_1\}$. The modified walk will be longer than the original, and will be assigned zero weight by the conditional interaction. However, it has positive $\kappa$-ASA\(W\) weight. The entropic gain of ignoring $\omega^1$ can therefore be estimated by considering the possible modifications to $\omega^2$ and estimating the
energetic cost of the modifications. The energetic cost decreases as \( \kappa \) decreases, and for \( \kappa \) sufficiently small we will show that the entropic benefit outweighs the energetic penalty. This idea, which involves a weighted version of the so-called multivalued map principle, has been fruitful in obtaining upper bounds on the number of self-avoiding polygons of given length [8, 9, 10, 11].

2.4. Discussion. Ueltschi [1] considered a model of SAWs with an attracting reward for pairs of nearest-neighbour vertices under the assumption that the step distribution \( D(x) \) and attraction strength \( \kappa \) satisfy

\[
(2.7) \quad \inf_{|x-y|=1, y \neq 0} \frac{D(y)}{D(x)} = \Delta > 0, \quad (1 + \kappa)^{2d} \leq 1 + \frac{\Delta^2}{2d(1 + \kappa)^{2d-1}}.
\]

Note that the condition on \( D(x) \) in (2.7) implies the step distribution has infinite range. Given (2.7) it can be shown that the entropic reward of ignoring \( \omega_1 \) outweighs the energetic cost. The fact that \( D(x) \) has infinite range and is “smooth” allows the use of a length preserving transformation to prove the model is submultiplicative. Using this idea Ueltschi also carries out a lace expansion analysis via the inductive approach of [12]; the length-preserving nature of the transformation is important for the application of the inductive method. Ueltschi’s result was significant for being the first application of the lace expansion to a self-attracting random walk. Self-attracting (also called non-repulsive) interactions are typically difficult to handle with lace expansion methods [4, Section 6.3].

The problem of analysing models of self-attracting self-avoiding walks under weaker hypotheses on the step distributions was raised by den Hollander [2, Chapter 4.8(5)], and our work addresses this question when \( d \geq 5 \). As described in Section 2.3 the main idea is to combine energy-entropy methods with classical techniques for self-avoiding walk.

Beyond our main theorems, an important aspect of this work is that it suggests that energy-entropy methods may be more generally useful in the context of the lace expansion. In particular there is no need to restrict to length-preserving transformations as in [1] (although length-preserving transformations do simplify technical aspects due to [12]). This is significant as energy-entropy methods should be a fairly robust way to overcome a lack of repulsion caused by weak attractions. Roughly speaking, the key step in such an argument is to first subdivide an object, and then to prove the gain in conformational freedom that arises when forgetting one part outweighs the loss of energetic attractions. Our proof implements this strategy for \( \kappa \)-ASAW, and it is plausible it could be implemented for other models, e.g., weakly self-attracting lattice trees in high dimensions via an adaptation of [13, 7].

It is worth noting that energy-entropy arguments are carried out by finding a transformation that estimates the number of new configurations that are available. Finding a transformation is a combinatorial and analytic problem, in contrast to other approaches to overcoming a lack of repulsion via correlation inequalities [14, 4], exact resummation identities [15], or underlying asymmetry assumptions [16].

We end this section by mentioning two recent related works on random walks subject to self-attraction. Firstly, there has been interesting progress on den
Hollander’s problem for weakly self-avoiding walk (WSAW) with a contact self-attraction when \( d = 4 \) [17]. The authors prove Gaussian decay of the critical two-point function when the self-attraction and self-repulsion strengths are sufficiently small by making use of a rigorous renormalization group analysis. Their techniques are wholly different than those of the present paper. An analysis of self-attracting WSAW when \( d \geq 5 \) via lace expansion techniques would be a very interesting complement to the results of [17]. Secondly, in [18] it has been shown that a related model known as prudent self-avoiding walk undergoes a collapse transition in \( d = 2 \) when the self-attraction is strong enough.

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4. Initial definitions, path transformations

4.1. Conventions. By a common abuse of notation \( \mathbb{Z}^d \) will denote the \( d \)-dimensional hypercubic lattice, i.e., the graph with vertex set \( \mathbb{Z}^d \) and edge set \( E(\mathbb{Z}^d) = \{\{x, y\} \mid \|x - y\|_1 = 1\} \). The standard generators of \( \mathbb{Z}^d \) will be denoted \( e_1, \ldots, e_d \).

For \( n \in \mathbb{N} = \{0, 1, 2, \ldots\} \), an \( n \)-step walk is a sequence \( (\omega_i)_{i=0}^n \), where \( \omega_i \in \mathbb{Z}^d \) for \( 0 \leq i \leq n \) and \( \omega_i \neq \omega_{i-1}, 1 \leq i \leq n \). For such a walk, let \( |\omega| = n \). Let \( W_n(x, y) \) be the set of \( n \)-step walks with \( \omega_0 = x \) and \( \omega_n = y \). We omit the first argument if \( x = o \), the origin of \( \mathbb{Z}^d \), and let \( W(x, y) = \bigcup_{n \geq 0} W_n(x, y) \).

A walk is self-avoiding if \( \omega_i \neq \omega_j \) for \( i \neq j \), and is a self-avoiding polygon if \( |\omega| > 2 \), \( \omega_i \neq \omega_j \) for \( 0 \leq i < j < |\omega| \), and \( \omega_0 = \omega_{|\omega|} \). Let \( \Gamma_n(x, y) \) denote the set of \( n \)-step self-avoiding walks from \( x \) to \( y \) and \( \tilde{\Gamma}_n(x) \) denote the set of \( n \)-step self-avoiding polygons with initial vertex \( x \). For self-avoiding walks let \( \Gamma_n(x) = \Gamma_n(o, x) \), and again we will omit the subscript \( n \) to indicate a union over \( n \). \( \Gamma \) and \( \tilde{\Gamma} \) denote the sets of all self-avoiding walks and polygons, respectively, with no restrictions on the initial vertex.

Let \( \text{adj}(A, B) \) denote the set of plaquettes spanned by pairs of adjacent edges \( e \in A, f \in B \) for subsets \( A, B \subseteq E(\mathbb{Z}^d) \), and let \( \text{adj}(A) = \text{adj}(A, A) \). Let \( E(\omega) = \{\{\omega_i, \omega_{i+1}\}\}_{i=0}^{|\omega|-1} \) be the set of edges traversed by a walk \( \omega \). By a slight abuse of notation we will write \( \text{adj}(\omega, \eta) \) in place of \( \text{adj}(E(\omega), E(\eta)) \), and similarly for \( \text{adj}(\omega) \). If \( A \) is a finite set let \( |A| \) denote its cardinality.

4.2. Transformations by symmetries of \( \mathbb{Z}^d \); basic path operations. For \( x \in \mathbb{Z}^d \), let \( \mathcal{T}_x \) denote the operator of translation by \( x \), i.e., \( \mathcal{T}_x f(y) = f(y - x) \) for \( f \) a function on \( \mathbb{Z}^d \). Translations will also act on subsets or collections of subsets of \( \mathbb{Z}^d \) by identifying sets with indicator functions. For example, if \( \omega \in \Gamma_n \), \( \mathcal{T}_x \omega \) is the \( n \)-step walk \( (\omega_0 + x, \omega_1 + x, \ldots, \omega_n + x) \).

The projection operator \( \pi_i : \mathbb{Z}^d \to \mathbb{Z} \) maps \( x = (x_1, \ldots, x_d) \) to \( x_i \). To lighten notation, let \( \pi_i^{-1}(x) = \pi_i^{-1}(\pi_i(x)) \) denote the hyperplane passing through \( x \) with
normal $e_i$. The reflection operator $\mathcal{R}_i: \mathbb{Z}^d \to \mathbb{Z}^d$ reflects $y$ in the coordinate hyperplane $\pi_i^{-1}(0)$.

If $\omega^1$ and $\omega^2$ are two walks of length $m$ and $n$ respectively, their concatenation $\eta = \omega^1 \circ \omega^2$ is the $(n + m)$-step walk with $\eta_i = \omega^1_i$ for $0 \leq i \leq m$, and $\eta_{m+i} = T_{(\omega^1_i, \omega^2_i)} \omega^2_i$ for $0 \leq i \leq n$. This translation moves the initial vertex of $\omega^2$ to the terminal vertex of $\omega^1$, so the concatenated walk continues from where $\omega^1$ ends. For $0 \leq i < j \leq |\omega|$, the walk $(\omega_i, \omega_{i+1}, \ldots, \omega_j)$ will be denoted $\omega[i,j]$.

4.3. Flips. Let $\omega$ be a walk and $P$ a plaquette such that (i) there is a unique $i$ such that $\omega_j \in P$ iff $j \in \{i, i + 1\}$ or (ii) there is a unique $i$ such that $\omega_j \in P$ iff $j \in \{i, i + 1, i + 2\}$. $P$ is called 1-flippable in the first case and 2-flippable in the second; these conditions say that $\omega$ has exactly 1 or 2 edges in the plaquette $P$ (and no other vertices in $P$). Otherwise $P$ is non-flippable. Since flippability is defined in terms of $\omega$ we will write, e.g., $1$-flippable for $\omega$ to indicate this dependence.

If $P$ is a 1-flippable plaquette for $\omega$, the flip of $\omega$ at $P$, denoted $\mathcal{F}_P(\omega)$, is the walk $\omega'$ that replaces $(\omega_i, \omega_{i+1})$ with the traversal of $P$ along the three edges distinct from $\{\omega_i, \omega_{i+1}\}$. See Figure 2. If $P$ is not 1-flippable, define $\mathcal{F}_P(\omega) = \omega$. Two plaquettes $P_1$ and $P_2$ are said to be disjoint if they have no vertices in common. Sets of disjoint 1-flippable plaquettes are what will be entropically important in what follows. The next three lemmas establish useful basic properties of $\mathcal{F}_P$.

**Lemma 4.1.** For any plaquette $P$ and vertices $x, y \in \mathbb{Z}^d$, $\mathcal{F}_P: \mathcal{W}(x, y) \to \mathcal{W}(x, y)$, and $\mathcal{F}_P: \Gamma(x, y) \to \Gamma(x, y)$. If $P'$ is disjoint from $P$, then $\mathcal{F}_{P'} \mathcal{F}_P = \mathcal{F}_P' \mathcal{F}_P$.

**Proof.** The only thing to prove for the first claim is that $\mathcal{F}_P$ does not change the endpoints of a walk. This is immediate as the first and last vertices of $\mathcal{F}_P(\omega)$ in $P$ are the same as the first and last vertices of $\omega$ in $P$.

If $\omega \in \Gamma(x, y)$ and $P$ is not 1-flippable for $\omega$, then $\mathcal{F}_P(\omega) = \omega$, so the image is in $\Gamma(x, y)$. If $P$ is 1-flippable, then $\omega$ contains one edge of $P$ and no other vertices; since $\mathcal{F}_P$ only modifies $\omega$ on $P$ the result is self-avoiding.

Lastly, commutativity holds as flips at disjoint plaquettes modify the walk on disjoint sets of edges. ■

**Lemma 4.2.** Let $\eta \circ \omega \in \Gamma$ and let $B \subseteq \text{adj}(\eta, \omega)$ be a disjoint set of plaquettes that are 1-flippable for $\omega$. Then $\omega$ is uniquely determined by $\mathcal{F}_B \omega$ and $\eta$.

**Proof.** Since the plaquettes in $B$ are disjoint, we can consider them separately. For each $P \in B$, there is a single edge in $\tilde{\omega} = \mathcal{F}_P(\omega)$ that is also in $\eta$. Suppose the edge is $(\tilde{\omega}_i, \tilde{\omega}_{i+1})$. Then $P$ is the plaquette $\{\tilde{\omega}_{i-1}, \tilde{\omega}_i, \tilde{\omega}_{i+1}, \tilde{\omega}_{i+2}\}$. Thus, given $\eta$ and $\tilde{\omega}$, we can determine $B$. By Lemma 4.1 $\mathcal{F}_B$ is an involution, so $\omega = \mathcal{F}_B \tilde{\omega}$ is uniquely determined. ■

Suppose a self-avoiding walk $\eta$ is composed of two subwalks $\omega^1$ and $\omega^2$, i.e., $\eta = \omega^1 \circ \omega^2$. It will be convenient to abuse notation and write $\text{adj}(\omega^1, \omega^2)$ in place of $\text{adj}(\omega^1, T_x \omega^2)$, where $T_x$ is the translation that takes the initial vertex of $\omega^2$ to the final vertex of $\omega^1$. As it will be contextually clear we are discussing
pairs of adjacent edges between the subwalks $\omega^1$ and $\omega^2$, this should not cause any confusion.

The next lemma shows that most plaquettes in $\text{adj}(\omega^1, \omega^2)$ are 1-flippable for $\omega^2$.

**Lemma 4.3.** If $\omega^1 \circ \omega^2 \in \Gamma$, there is at most one plaquette in $\text{adj}(\omega^1, \omega^2)$ that is not 1-flippable for $\omega^2$. If $\omega^1 \circ \omega^2 \in \tilde{\Gamma}$, there are at most two plaquettes in $\text{adj}(\omega^1, \omega^2)$ that are not 1-flippable for $\omega^2$.

**Proof.** Without loss of generality, assume that $\omega^2_0$ is the endpoint of $\omega^1$, and call the vertices in common to $\omega^2$ and $\omega^1$ points of concatenation. The proof characterises when $P \in \text{adj}(\omega^1, \omega^2)$ is not 1-flippable for $\omega^2$ case by case, depending on how many edges of $P$ are contained in $\omega^1 \circ \omega^2$. By the definition of $P \in \text{adj}(\omega^1, \omega^2)$ there are at least two such edges.

First, note that a self-avoiding walk or polygon containing four edges in a single plaquette is a 4-step self-avoiding polygon. The claim is true in this case, as there are exactly two adjacent edges. Henceforth we may assume there are no plaquettes containing four edges.

Suppose that $\omega^1$ and $\omega^2$ each contain exactly one edge of $P$. If $P$ does not contain a point of concatenation, then $P$ is 1-flippable since the two vertices of $P$ in $\omega^1$ are not in $\omega^2$. If there is a single point of concatenation in $P$, then $P \notin \text{adj}(\omega^1, \omega^2)$. Containing two points of concatenation is impossible: it implies that $\omega^2$ and $\omega^1$ share an edge, contradicting $\omega^1 \circ \omega^2 \in \Gamma \cup \tilde{\Gamma}$. See Figure 3a.

Suppose that $\omega^1 \omega^2$ contains three edges of $P$. Note that the three edges must occur sequentially in $\omega^1 \circ \omega^2$ since this walk is self-avoiding or a self-avoiding polygon, and hence $P$ must contain a point of concatenation if it is to be in $\text{adj}(\omega^1, \omega^2)$. If two edges belong to $\omega^1$, then $P$ is 1-flippable for $\omega^2$. Otherwise, $P$ is not 1-flippable for $\omega^2$. As there are at most two points of concatenation, this verifies the claim. See Figure 3b.

The cost $p^2_P = \frac{F_{n+2}(\omega')}{F_n(\omega)}$ is the additional cost of the modified walk according to the \textit{a priori} measure $\mathbb{P}$. Recall the definition of $\mathcal{W}_\kappa$ in (1.2). The next lemma is our basic estimate for the energetic penalty of a flip.

**Lemma 4.4.** Let $\omega$ be a self-avoiding walk and $P$ a 1-flippable plaquette. Let $\omega' = F_P(\omega)$. Then

$$\frac{\mathcal{W}_\kappa(\omega')}{\mathcal{W}_\kappa(\omega)} \geq p^2_P(1 + \kappa)^{-2d-4}.$$
The factor of $p_2^2$ comes from comparing the a priori measures in the definition of $W_\kappa$. What remains is to bound the difference $|\text{adj}(\omega)| - |\text{adj}(\omega')|$.

The flip creates at least one pair of adjacent edges in $\omega'$ that was not present in $\omega$, namely the pair of adjacent edges in $P$. There are $2d - 2$ edges that are potentially adjacent to the unique edge of $\omega$ in $P$, and the hypothesis of $P$ being 1-flippable for $\omega$ implies at least one of these edges is not in $\omega$. Thus at most $2d - 3$ adjacent pairs of edges in $\omega$ are not present in $\omega'$. This proves (4.1). ■

Define $\alpha = \alpha(d)$ by

$$\alpha(d) = \frac{1}{1 + 8(d - 1)^2}. \quad (4.2)$$

In what follows it will be necessary to flip many plaquettes. Lemma 4.1 guarantees the result will be self-avoiding if the flipped plaquettes are disjoint. The next lemma guarantees that every collection of plaquettes has a positive density subset of disjoint plaquettes.

**Lemma 4.5.** Given a finite set $A$ of plaquettes in $\mathbb{Z}^d$, there exists a subset of pairwise disjoint plaquettes of $A$ of size $\lceil \alpha |A| \rceil$.

**Proof.** The constant $\alpha$ is $(1 + R)^{-1}$, where $R$ is the number of plaquettes $P' \neq P$ sharing a vertex with $P$. The remainder of the proof verifies the value of $R$ by performing inclusion-exclusion on the number of vertices $P' \neq P$ shares with $P$; note that this number is at most two.

Any vertex $x$ is contained in exactly $4 \binom{d}{2}$ plaquettes. To see this, note these plaquettes are in bijection with sets $\{(u_i, \sigma_i)\}_{i=1,2}$, where $u_1 \neq u_2$ are distinct generators of $\mathbb{Z}^d$ and $\sigma_i \in \{\pm 1\}$: these sets identify the unique plaquette containing $x, x + \sigma_1 u_1, x + \sigma_2 u_2$. Since every edge belongs to exactly $2(d - 1)$ plaquettes, the total number of plaquettes sharing a vertex with a plaquette $P$ is therefore

$$R = 4 \left(4 \binom{d}{2} - 1 - (2(d - 1) - 1)\right) = 8(d - 1)^2,$$

where in each term the factor of $-1$ corrects for presence of $P$ in our counts. ■

5. **Existence of the connective constant: proof of Theorem 2.1**

5.1. **Half-space walks and bridges.** We begin by recalling the basic definitions used in the Hammersley-Welsh argument.
Definition 4. The set $H_n$ of $n$-step half-space walks is the set of $\omega \in \Gamma_n$ such that

\begin{equation}
0 = \pi_1(\omega_0) < \pi_1(\omega_i), \quad 1 \leq i \leq n.
\end{equation}

The set $B_n$ of $n$-step bridges is the subset of $\omega \in H_n$ such that

\begin{equation}
0 = \pi_1(\omega_0) < \pi_1(\omega_i) \leq \pi_1(\omega_n), \quad 1 \leq i \leq n.
\end{equation}

Let $H = \bigcup_{n \geq 0} H_n$ and $B = \bigcup_{n \geq 0} B_n$. The masses $h_n(\kappa)$ of $n$-step bridges and $b_n(\kappa)$ of half-space walks are defined by

\begin{equation}
h_n(\kappa) = \sum_{\omega \in H_n} W(\omega), \quad b_n(\kappa) = \sum_{\omega \in B_n} W(\omega).
\end{equation}

Note that the inclusions $B_n \subset H_n \subset \Gamma_n$ imply that $b_n(\kappa) \leq h_n(\kappa) \leq c_n(\kappa)$.

While self-attraction on adjacent edges ruins the submultiplicativity of self-avoiding walks, it enhances the supermultiplicativity of bridges.

Proposition 5.1. Let $\kappa \geq 0$. The limit $\mu_B(\kappa) = \lim_{n \to \infty} (b_n(\kappa))^{1/n}$ exists and is equal to $\sup_{n \geq 1} (b_n(\kappa))^{1/n}$.

Proof. The definition of a bridge implies that the concatenation $\omega^1 \circ \omega^2$ of two bridges $\omega^1$ and $\omega^2$ is a bridge. Each pair of adjacent edges in $\omega^i$ remains adjacent in $\omega^1 \circ \omega^2$. Any other pair of adjacent edges in the concatenation receives weight $1 + \kappa \geq 1$, so

$$
\sum_{\omega^1 \in B_{n_1}} \sum_{\omega^2 \in B_{n_2}} W(\omega^1)W(\omega^2) \leq \sum_{\eta \in B_{n_1+n_2}} W(\eta)1_{\{\eta=\omega^1 \circ \omega^2, \omega^i \in B_{n_i}\}}
$$

for all choices of $n_1, n_2 \in \mathbb{N}$. The left-hand side is $b_{n_1}(\kappa)b_{n_2}(\kappa)$. Ignoring the indicator on the right-hand side gives an upper bound $b_{n_1+n_2}(\kappa)$. Thus $b_n(\kappa)$ is supermultiplicative, and the proposition follows by Fekete’s lemma. \hfill \square

5.2. Unfolding I. Classical unfolding. This section briefly recalls how half-space walks can be unfolded into a concatenation of bridges, as a multivalued extension of the unfolding procedure will be introduced in the next section. We omit the proofs of the facts that we recall; for more details, see [6].

Definition 5. Let $\omega \in H$. The (first) bridge point $\tau(\omega)$ of $\omega$ is the maximal index $i$ satisfying $\pi_1(\omega_i) = \max_j \pi_1(\omega_j)$.

Definition 6. The span of a self-avoiding walk is

\begin{equation}
\text{span}(\omega) = \max_j \pi_1(\omega_j) - \min_j \pi_1(\omega_j).
\end{equation}

Note that if $\omega$ is an $n$-step bridge then $\text{span}(\omega) = \pi_1(\omega_n)$.

Given a half-space walk $\omega \in H_n$, let $x = -\omega_{\tau(\omega)}$. Define the initial bridge $\omega^h = \omega_{[0, \tau(\omega)]}$ and the remainder $\omega^b = \tau(\omega, \omega^h]$. Given a half-space walk $\omega$, we will write $\omega = (\omega^h, \omega^b)$ in what follows to indicate this decomposition into an initial bridge and a remainder. The following properties of the decomposition are important.

(i) $\omega = \omega^b \circ \omega^h$;

(ii) $R_1(\omega^h)$ is a half-space walk,
(iii) $\text{span}(\omega^h) = \text{span}(\omega)$, and
(iv) $\text{span}(R_1(\omega^h)) < \text{span}(\omega^h)$, as $\omega$ never revisits the coordinate hyperplane $\pi_{1}^{-1}(o)$ after $\omega_0$.

**Definition 7.** The classical unfolding map $\Psi : H \to B$ is recursively defined as follows. $\Psi$ is the identity map on $B$. Otherwise, if $\omega \in H \setminus B$, let $\omega = (\omega^b, \omega^h)$ and define $\Psi(\omega) = \omega^b \circ \Psi(R_1(\omega^h))$.

In words, $\Psi$ reflects the remainder $\omega^h$ of the walk $\omega$ through $\pi_{1}^{-1}(\omega_{\tau(\omega)})$, the affine hyperplane with normal $e_1$ that contains the endpoint $\omega_{\tau(\omega)}$ of $\omega$. Since the reflection of $\omega^h$ is itself a half-space walk, this procedure can be iterated until the first bridge point of the newest half-space walk is also the endpoint of the walk. The recursion terminates at some depth $r = r(\omega)$ as the spans of the half-space walks produced are strictly decreasing. See Figures 4 and 5 for one step of this procedure; the meaning of the shaded plaquettes will be explained in the next section.

Thus, $\Psi$ produces a sequence of bridges $\omega^b_i$, $i = 1, \ldots, r$, and $\Psi(\omega) = \omega^b_1 \circ \cdots \circ \omega^b_r$. This is called the classical bridge decomposition of $\omega$. In what follows, the bridges in the classical bridge decomposition will always be denoted by $\{\omega^b_i\}_{i=1}^r$. Let us record some properties of this decomposition.

**Proposition 5.2.**
(i) $\text{span}(\Psi(\omega)) = \sum_{i=1}^{r} \text{span}(\omega^b_i)$,
(ii) the sequence $(\text{span}(\omega^b_i))_{i=1}^r$ of spans is strictly decreasing in $i$, and
(iii) given $\Psi(\omega)$ and $\{\text{span}(\omega^b_i)\}_{i=1}^r$, $\omega$ is uniquely determined.

Again we will not prove these claims, but let us remark that the third property holds as knowing the lengths of the spans indicates the locations at which to fold the bridge $\Psi(\omega)$ in order to undo the unfolding procedure.

### 5.3. Unfolding II. Multivalued unfolding

This section describes a multi-valued extension of the classical unfolding map. Roughly speaking, this multi-valued extension quantifies an entropic gain in unfolding a half-space walk $\omega$ into bridges. The gain arises because the unfolded walk may have fewer adjacent edges than does the original walk $\omega$.

**Definition 8.** The marked unfolding map $\bar{\Phi}$ is recursively defined on $H$ by

\[ \bar{\Phi}(\omega) = ((\omega^b, \text{adj}_1(\omega^b, \omega^h)), \bar{\Phi}(R_1(\omega^h))), \]

\[ \text{adj}_1(\omega^b, \omega^h) = \{ P \in \text{adj}(\omega^b, \omega^h) \mid P \text{ 1-flippable for } \omega^b \}. \]

Let $r = r(\omega)$ denote the number of bridges generated by this recursion. The image of $\bar{\Phi}(\omega)$ will be denoted $((\omega^b_i, \text{adj}_i))_{i=1}^r$, where $\text{adj}_r = \emptyset$.

The marked unfolding map is an extension of $\Psi$. It records the plaquettes at which $\omega^b_i$ is 1-flippable with respect to the remainder of the half-space walk with initial bridge $\omega^b_i$. Denote the set of discovered 1-flippable plaquettes by $\text{adj}_q(\omega)$, i.e.,

\[ \text{adj}_q(\omega) = \bigcup_i T_i \text{adj}_i, \]

where $T_i$ is the translation that translates the bridge $\omega^b_i$ to its location in the bridge $\Psi(\omega)$. 
For $k \in \mathbb{N}$, let $H_n^k \subset H_n$ be the set of $n$-step half-space walks with $|\text{adj}_\Phi(\omega)| = k$. To define the multivalued extension of $\Psi$ on $H_n^k$, we will make use of the following facts. First, Lemma 4.5 implies there exists a subset $\text{adj}_\Phi(\omega) \subset \text{adj}_\bar{\Phi}(\omega)$ of size $\lceil \alpha k \rceil$ such that the plaquettes in $\text{adj}_\Phi(\omega)$ are pairwise disjoint. Second, Lemma 4.1 implies if $B$ is a finite collection of pairwise vertex-disjoint plaquettes and $x \in \mathbb{Z}^d$, then $F_B = \prod_{P \in B} F_P$ is a well-defined map from $\Gamma(x)$ to $\Gamma(x)$.

**Definition 9.** Let $0 < \delta < \frac{1}{2}$, $n \in \mathbb{N}$, and $0 \leq k \leq n$. The multivalued unfolding map $\Phi: H_n^k \rightarrow 2^{B_{n+2\lceil \delta k \rceil}}$ is defined by

$$
\Phi(\omega) = \left\{ \omega' \mid \omega' = F_B(\Psi(\omega)), B \in \binom{\text{adj}_\Phi(\omega)}{\lceil \delta k \rceil} \right\},
$$

where $\binom{A}{k}$ denotes the $k$-element subsets of a set $A$.

The next lemma gives the basic properties of $\Phi$; in particular it verifies the stated codomain in the previous definition. To lighten the notation, define

$$
\alpha_k = \lceil \alpha k \rceil, \quad \delta_k = \lceil \delta k \rceil.
$$

**Lemma 5.3.** Let $\omega \in H_n^k$. Then

(i) Every $P \in \text{adj}_\Phi(\omega)$ is 1-flippable for $\Psi(\omega)$.

(ii) Each walk $\omega' \in \Phi(\omega)$ is a bridge in $B_{n+2\delta_k}$.

(iii) The half-space walk $\omega$ can be reconstructed from any $\omega' \in \Phi(\omega)$ given the spans $\text{span}(\omega^h)$.

Proof. Recall $\omega = \omega^b_1 \circ \omega^h$. Since $\omega$ is a half-space walk, no plaquette in $\text{adj}(\omega^b_1, \omega^h)$ contains vertices in the half-space $\pi_1^{-1}((\infty, 0))$. Similarly, no plaquette contains vertices in the half-space $\pi_1^{-1}(\lceil \text{span}(\omega^b_1) + 1, \infty \rceil)$, as $\omega^h$ is contained in $\pi_1^{-1}(\lceil 1, \text{span}(\omega^b_1) \rceil)$.

It follows that $T_\omega(\omega) \mathcal{R}_1(\omega)$ does not contain any vertices of plaquettes in $\text{adj}(\omega^b_1, \omega^h)$, except possibly the terminal vertex $\omega^{\tau}(\omega)$. This implies each plaquette in $B \subset \text{adj}_1(\omega^b_1, \omega^h) \subset \text{adj}_\Phi(\omega)$ is 1-flippable for $\omega^b_1 \circ \mathcal{R}_1(\omega^h)$. This is because an adjacent edge between $\omega^b_1$ and the first edge of $\omega^h$ is necessarily...
2-flippable for $\omega_{h_1}$, as the adjacency cannot be with the last edge of $\omega_{h_1}$, but this final edge is contained in the plaquette containing the adjacency.

Iterating this argument for each bridge in the bridge decomposition of $\omega$ implies each plaquette in $\text{adj} \bar{\Phi}(\omega)$ is 1-flippable for $\Psi(\omega)$. This proves the first claimed property.

The preceding shows each $\omega' \in \Phi(\omega)$ is given by $\omega' = F_B(\omega) = F_B(\omega_{h_1}) \circ \cdots \circ F_B(\omega_{h_r})$ for $B \subset \text{adj}_g(\omega)$. This is because each plaquette in $B$ contains exactly one edge of $\Psi(\omega)$, and this edge is located in exactly one subwalk. By Lemma 4.1, each $\omega_j = F_B(\omega_{b_j})$ is a bridge with span $\text{span}(\omega_{b_j})$ for $j = 1, \ldots, r$. As a concatenation of bridges is a bridge and each flip adds exactly two edges, this proves the second claim, as each $\omega'$ results from applying $F_B$ with $|B| = \delta_k$.

By construction, the last bridge $\omega_{b_r}$ in the classical bridge decomposition has no flips applied to it in the formation of $\omega'$. Given span($\omega_j$) = span($\omega_{b_j}$) for each $j$, $\omega' = \omega_{b_r}$ is determined. Hence, by Lemma 4.2, $\omega_{b_{r-1}}$ can be reconstructed. Iterating this procedure reconstructs $\omega$ from $\omega'$, establishing the last claimed property. ■

Having established the basic properties of the multivalued unfolding map, we turn to estimating the weight of $n$-step half-space walks in terms of the weights of bridges. The next lemma gives the basic relation between these objects, which will then be analysed.

For $n \in \mathbb{N}$, let $P(n)$ denote the number of partitions of $n$ into distinct natural numbers. Let $B_n(\ell)$ be the set of $n$-step bridges $\omega$ with span($\omega$) $\leq \ell$.

**Proposition 5.4.** For $n, k \in \mathbb{N}$ and $0 \leq k \leq n$,

$$
(5.9) \quad \sum_{\omega \in H^k_n} W_k(\omega) \leq P(n) \left( \frac{\alpha_k}{\delta_k} \right)^{-1} \text{flip}_k^\delta \sum_{\omega' \in B_{n+2\delta_k}(n)} (1 + \kappa)^{k+\frac{3}{2}\sqrt{n}} W_k(\omega').
$$

**Proof.** We begin by comparing the weight of $\omega \in H^k_n$ to the weight of $\omega' \in \Phi(\omega)$. Let $r = r(\omega)$. 

**Figure 5.** The figure depicts the image in $\Phi(\omega)$ of the half-space walk $\omega$ depicted in Figure 5, when the subset $B$ of plaquettes at which flips occur is the set of crosshatched plaquettes.
Φ(H
This establishes Equation (5.9) if the sum on the right-hand side is replaced with

\[ \sum_{k \in \mathbb{H}_k^\delta} W_\kappa(\omega^k) \leq (1 + \kappa)^{k+r} W_\kappa(\omega^{h_1} \circ \ldots \circ \omega^{b_r}), \]

as there are \( r \) unfolding steps.

(ii) As \( \omega' = \mathcal{F}_B(\omega^{h_1} \circ \ldots \circ \omega^{b_r}) \) and \( |B| = \delta_k \), applying Lemma 4.4 \( \delta_k \) times implies

\[ W_\kappa(\omega^{h_1} \circ \ldots \circ \omega^{b_r}) \leq \text{flip}_k^\delta W_\kappa(\omega'). \]

This implies

\[ W_\kappa(\omega) \leq (1 + \kappa)^{k+r(\omega)} \text{flip}_k^\delta W_\kappa(\omega'). \]  

(5.10)

Next we apply the multivalued map principle (see [9, Section 2.0.1] for a general statement of this principle), and then use (5.10):

\begin{align*}
(5.11) & \quad \sum_{\omega \in \mathbb{H}_k^\delta} W_\kappa(\omega) |\Phi(\omega)| = \sum_{\omega \in \mathbb{H}_k^\delta} \sum_{\omega' \in \Phi(\omega)} W_\kappa(\omega) \\
(5.12) & \quad = \sum_{\omega' \in \Phi(\mathbb{H}_k^\delta)} \sum_{\omega \in \Phi^{-1}(\omega')} W_\kappa(\omega) \\
(5.13) & \quad \leq \sum_{\omega' \in \Phi(\mathbb{H}_k^\delta)} \sum_{\omega \in \Phi^{-1}(\omega')} (1 + \kappa)^{k+r(\omega)} \text{flip}_k^\delta W_\kappa(\omega').
\end{align*}

To make the inner sum uniform in \( \omega \), we use Proposition 5.2. The spans of the bridges in the bridge decomposition of \( \omega \) are distinct integers summing to \( \text{span}(\Psi(\omega)) \leq n \). This implies that \( r(\omega) \) is at most \( \frac{3}{2} \sqrt{n} \), and hence

\[ \sum_{\omega \in \mathbb{H}_k^\delta} |\Phi(\omega)| W_\kappa(\omega) \leq \sum_{\omega' \in \Phi(\mathbb{H}_k^\delta)} |\Phi^{-1}(\omega')| (1 + \kappa)^{k+\frac{3}{2} \sqrt{n}} \text{flip}_k^\delta W_\kappa(\omega'). \]  

(5.14)

Next we estimate \(|\Phi(\omega)|\) and \(|\Phi^{-1}(\omega')|\).

(i) Lemma 5.3 implies that

\[ |\Phi(\omega)| = \left( \frac{\alpha_k}{\delta_k} \right). \]  

(5.15)

Note that this is the cardinality of subsets \( B \) of \( \text{adj}_k^\delta(\omega) \) of size \( \delta_k \). The claim is true as (i) all plaquettes in \( B \) are 1-flippable for \( \Psi(\omega) \), and (ii) every distinct choice of a subset \( B \) in the definition of \( \Phi \) results in a distinct image.

(ii) By Lemma 5.3(iii) we can reconstruct \( \omega \) from an image \( \omega' \in \Phi(\omega) \) given the sequence of spans in the classical bridge decomposition of \( \omega \). The maximal possible span of \( \Psi(\omega) \), the bridge produced by the classical bridge decomposition, is \( n \), and by Proposition 5.2 the spans form a partition of \( \text{span}(\Psi(\omega)) \). Thus the number of preimages \(|\Phi^{-1}(\omega')|\) is at most \( P(n) \), the number of partitions of \( n \).

This establishes Equation (5.9) if the sum on the right-hand side is replaced with \( \Phi(\mathbb{H}_k^\delta) \). By Lemma 5.3(ii), \( \Phi(\mathbb{H}_k^\delta) \subset B_{n+2\delta_k}(n) \), as \( \text{span}(\omega') = \text{span}(\Psi(\omega)) \leq n \). The proposition follows by expanding the index of the sum, as each summand is non-negative. \( \blacksquare \)
The factor of $\mu_B^2$ on the left-hand side of the next lemma is present to simplify future calculations.

**Lemma 5.5.** For $\kappa$ sufficiently small, there are $\delta > 0$, $K > 0$, and $a > 0$ such that

$$
(5.16) \quad (\frac{\alpha_k}{\delta_k})^{-1} (\text{flip}_\kappa \mu_B^2 \delta_k (1 + \kappa))^k \leq Ke^{-ak}
$$

for all $k \in \mathbb{N}$.

**Proof.** We will show that for $0 < \delta < \frac{1}{2}$ small enough there is a $\kappa'$ such that for $\kappa \in [0, \kappa']$, there are $a, K > 0$ such that (5.16) holds. This implies the statement of the lemma. Since it suffices to prove the validity of (5.16) for $k > k_0 = (\delta \alpha)^{-1}$, we will restrict attention to such $k$.

We begin by estimating the combinatorial prefactor in (5.17). Recall the definition of $\alpha_k$ and $\delta_k$ in (5.8). By using (i) $(n \choose k) \geq (n/k)^k$ for $1 \leq k \leq n$, (ii) $\max\{1, x\} \leq \lceil x \rceil \leq x + 1$ for $x > 0$, and (iii) $k > k_0$ and $\delta < \frac{1}{2}$, we obtain

$$
(5.17) \quad \left( \frac{\alpha_k}{\delta_k} \right)^{-1} \leq \left( \frac{\delta_k}{\alpha_k} \right)^{\delta_k} \leq (\delta + (\alpha k)^{-1}) \delta_k \leq (2\delta)^{\delta \alpha k}.
$$

Thus, when $k > k_0$, the left-hand side of (5.16) is bounded above by

$$
(5.18) \quad \left[ 2\delta (\text{flip}_\kappa \mu_B^2 \delta k (1 + \kappa))^\delta \right]^{\delta \alpha k},
$$

and the claim will follow by showing that the quantity in square brackets is strictly less than one.

By Proposition 5.1, $\mu_B(\kappa) = \sup_n (b_n(\kappa))^{1/n}$, and this latter quantity is at most $\sup_n (c_n(\kappa))^{1/n}$.

This, in turn, is bounded above by $(1 + \kappa)^{2d - 2}$, as each edge in a self-avoiding walk can be adjacent to at most $2d - 2$ others. Since $\delta \alpha k > 1$ when $k > k_0$, we can bound above the bracketed quantity of (5.18) by

$$
(5.19) \quad 2\delta (\text{flip}_\kappa (1 + \kappa)^{4(d-1)}) (1 + \kappa)^{(\delta \alpha)^{-1}}.
$$

Recalling the definition of $\text{flip}_\kappa$, it follows that when $\kappa = 0$ the quantity in (5.19) is strictly less than one, for $\delta = \delta(p_1)$ sufficiently small. Since the expression in (5.19) is continuous in $\kappa$ for $\delta$ fixed, it is strictly less than one for small positive $\kappa$. This proves the claim. \[\blacksquare\]

The next theorem is a consequence of a much stronger result due to Hardy and Ramanujan; it will be needed to estimate the mass of $n$-step half-space walks.

**Theorem 5.6** (Hardy–Ramanujan [19]). Let $P(n)$ denote the number of partitions of $n$ into distinct parts. Then

$$
(5.20) \quad \log P(n) \sim \pi \sqrt{\frac{n}{3}}, \quad n \to \infty,
$$

meaning that the ratio of the two sides tends to 1 as $n \to \infty$. 
Proposition 5.7. For $\kappa$ sufficiently small, there are $a_1, K_1 > 0$ such that

$$h_n(\kappa) = \sum_{\omega \in \mathcal{H}_n} W_\kappa(\omega) \leq K_1 e^{a_1 \sqrt{n}} \mu_B^n.$$  

Proof. By Proposition 5.4 and Lemma 5.5,

$$\sum_{\omega \in \mathcal{H}_n^k} W_\kappa(\omega) \leq K e^{-ck} \mu_B^{-2k} (1 + \kappa)^{\frac{3}{2} \sqrt{n}} P(n) \sum_{\omega' \in B_{n+2h_k}(n)} W_\kappa(\omega').$$

By Proposition 5.1, $b_\ell(\kappa) \leq \mu_B^\ell(\kappa)$. Hence, dropping the constraint that the spans of bridges on the right-hand side of (5.3) are at most $n$ yields

$$\sum_{\omega \in \mathcal{H}_n^k} W_\kappa(\omega) \leq K e^{-ck} (1 + \kappa)^{\frac{3}{2} \sqrt{n}} P(n) \mu_B^n.$$  

The right-hand side of (5.22) is summable in $k$, and the left-hand side sums to $h_n(\kappa)$. By Theorem 5.6, $(1 + \kappa)^{\frac{3}{2} \sqrt{n}} P(n)$ is at most $K_1 e^{a_1 \sqrt{n}}$ for constants $K_1, a_1 > 0$; this completes the proof.

Proof of Theorem 2.1. The proof uses the ideas that led to Proposition 5.4 to bound $c_n(\kappa)$ by a sum over pairs of half-space walks. This sum will be estimated using Proposition 5.7 to show that, for some $d', K' > 0$,

$$b_n(\kappa) \leq c_n(\kappa) \leq K' e^{d' \sqrt{n}} \mu_B(\kappa)^n.$$  

From this $\mu(\kappa) = \mu_B(\kappa)$ follows. What remains is to prove the upper bound of (5.23).

For $\omega \in \Gamma_n$, let $m$ be the maximal $i$ such that $\pi_1(\omega_i)$ is minimized. Let $\omega^1 = \omega[0,m]$ and $\omega^2 = \mathcal{T}_{(-\omega_m)} \omega[m,n]$. The first part of the proof is to define a multivalued map $\Psi$ that assigns to $(\omega^1, \omega^2)$ a set of pairs of half-space walks $\{(\eta^1, \eta^2)\}$ in an injective way.

The reversal of a walk $\eta = (\eta_i)_{i=1}^m$ is the walk $(\eta_{m-i})_{i=0}^{m-1}$; this walk begins at the endpoint of $\eta$ and goes back to the initial vertex. Translate the reversal of $\omega^1$ so that the walk begins at the origin, i.e., consider $\mathcal{T}_{(-\omega_m)} \omega^1$. This is almost a half-space walk; the only problem is that it may visit the coordinate hyperplane $\pi_1^{-1}(o)$ repeatedly. Note that $\omega^2$ is a half-space walk.

We now define $\Psi$. Set $\eta^2 = \omega^2$. Define $\tilde{\omega}^1$ to be the concatenation of the one-step self-avoiding walk from $o$ to $e_1$ with the reversal of $\omega^1$. Let $x = e_1 - \omega_m$ denote the vector along which $\omega^1$ is translated when forming this concatenation. Note that $\tilde{\omega}^1$ is a half-space walk.

Let $\text{adj} = \text{adj}(\omega^1, \omega^2)$, and suppose this set contains $k$ 1-flippable plaquettes for $\omega^1$. Let $\text{adj}^* \subset \text{adj}$ be a subset of disjoint 1-flippable plaquettes of size $a_k$; such a subset exists by Lemma 4.5. Then

$$\Psi(\omega) = \left\{ (\eta^1, \eta^2) \mid \eta^1 = \mathcal{F}_{\mathcal{T}_k} B(\tilde{\omega}^1), B \in \left( \text{adj}^* \delta_k \right) \right\}.$$  

By an argument as in the proof of Lemma 5.3, $\eta^1$ is a half-space walk, as any flips do not change the minimal value of the first coordinate of $\tilde{\omega}^1$. Note that if $\eta^2 \in B_m$, then $\eta^1 \in B_{n-m-2h_k+1}$.

We now verify that it is possible to reconstruct $(\omega^1, \omega^2)$, and hence $\omega$ itself, from any image $\{(\eta^1, \eta^2)\}$. First observe that $\omega^1$ is determined by $\eta^1$. Given $\eta^1$,
the translation applied to \(\omega^1\) is determined, as flips do not change the endpoints of walks. The translation applied to \(\omega^1\) determines the translation applied to \(\eta^2\), and hence determines \(\omega^2\). Hence, by Lemma 4.2, \(\omega^1\) is determined since we know \(\omega^2\) and \(\eta^1\).

Note \(\Gamma_n = \bigcup_{k=0}^n \Gamma_n^k\), where \(\Gamma_n^k\) is the set of self-avoiding walks such that \(|\text{adj}(\omega^1, \omega^2)| = k\) and where we have used the notation \(\bigcup\) to indicate a union of disjoint sets. Proceeding as in the proof of Proposition 5.4 yields

\[
c_n(\kappa) \leq \sum_{m=0}^n \sum_{k \leq m} \left( \frac{c_k}{\delta_k} \right)^{-1} (\text{flip}_k)^{1/2} (1 + \kappa)^{k+1} h_m(\kappa) h_{n-m+2\delta_k+1}(\kappa).
\]

The exponent is \(k + 1\) as we used one unfolding step to form the pair of half-space walks, and this may have removed one 2-flippable plaque in \(\text{adj}(\omega^1, \omega^2)\). Using Proposition 5.7 to estimate the factors \(h_k(\kappa)\) and Lemma 5.5 to estimate the remaining terms yields

\[
c_n(\kappa) \leq K''(n+1)e^{d'\sqrt{n}/\mu_B^n}
\]

for some \(K'', d' > 0\). As this establishes (5.23), the proof is complete.

6. AVERAGED SUBMULTIPLICATIVITY

The transformation used to estimate entropic gains in Section 5 flipped a fixed fraction \(\delta\alpha\) of 1-flippable plaquettes, and this led to \(p_1\)-dependent constants in estimates. This was convenient as it gave us precise control over the length added to a walk, but, for a lace expansion analysis, the lack of uniformity in \(p_1\) complicates matters.

This section defines a greedier transformation that enables estimates uniform in \(p_1\) when \(\kappa = \kappa(p_1)\) is chosen correctly. The price to pay is less control over the increase in length of a walk.

6.1. \(\kappa\)-ASA W with a memory. We first define memories, which will play the role of boundary conditions for self-avoiding walks.

**Definition 10.** A memory is a self-avoiding walk \(\eta \in \bigcup_x \Gamma(x, o)\).

For \(n \in \mathbb{N}\), \(\kappa \geq 0\), and a memory \(\eta\), define \(n\)-step \(\kappa\)-ASA W with memory \(\eta\) to be the law \(P_n^{\eta,\kappa}\) on \(\Gamma_n\) given by

\[
P_n^{\eta,\kappa}(\omega) \propto W_{\kappa}(\omega; \eta) \mathbb{1}_{\{\eta \cup \omega \in \Gamma\}},
\]

where

\[
W_{\kappa}(\omega; \eta) = e^{-H_{\kappa}(\omega; \eta)} W_n(\omega) \mathbb{1}_{\{\eta \cup \omega \in \Gamma\}},
\]

\[
e^{-H_{\kappa}(\omega; \eta)} = (1 + \kappa)^{|\text{adj}(\eta, \omega)|}.
\]

Thus \(P_n^{\eta,\kappa}\) is supported on self-avoiding extensions of \(\eta\). The factor \(W_n(\omega)\) gives a reward of \((1 + \kappa)\) for each pair of adjacent edges in \(\omega\), while \(e^{-H_{\kappa}(\omega)}\) gives such a factor whenever an edge of \(\omega\) is adjacent to an edge in \(\eta\).
6.2. Averaged submultiplicativity. Let \( \eta \) be a memory, and let \( \#_1 \text{adj}(\eta, \omega) \) be the number of 1-flippable plaquettes for \( \omega \) in \( \text{adj}(\eta, \omega) \). Define

\[
\begin{align*}
\Gamma^\eta_{n,k}(x) &= \{ \omega \in \Gamma_n(x) \mid \eta \circ \omega \in \Gamma, \#_1 \text{adj}(\eta, \omega) = k \}, \\
\tilde{\Gamma}^\eta_{n,k} &= \{ \omega \in \Gamma_n(x) \mid \omega \circ \eta \in \tilde{\Gamma}, \#_1 \text{adj}(\eta, \omega) = k \}.
\end{align*}
\]

The \((n, k)\) two-point function with memory \( \eta \) is

\[
(c^\eta_{n,k}(x) = \sum_{\omega \in \Gamma^\eta_{n,k}(x)} W_k(\omega; \eta).\tag{6.3}
\]

The dependence of \( c^\eta_{n,k}(x) \) on \( \kappa \) is left implicit. Let \( c^\eta_n(x) = \sum_{k \geq 0} c^\eta_{n,k}(x) \).

Define \( q^\eta_{n,k} \) by summing over \( \tilde{\Gamma}_n \) instead of \( \Gamma_n \) in (6.3), and let \( q^\eta_n = \sum_k q^\eta_{n,k} \).

**Definition 11.** Let \( z \geq 0, \kappa \geq 0 \), and \( x \in \mathbb{Z}^d \). The two-point function \( G^\eta_{z,\kappa}(x) \) with memory \( \eta \) is defined to be

\[
G^\eta_{z,\kappa}(x) = \sum_{n \geq 0} z^n c^\eta_{n,k}(x) \tag{6.4}
\]

When \( \eta = \emptyset \), this is called the 2-point function of \( \kappa\)-ASAW. Define \( \tilde{G}^\eta_{z,\kappa} \) by replacing \( c^\emptyset_n(x) \) with \( q^\emptyset_n \).

The next proposition will imply \( G^\eta_{z,\kappa} \) is well-defined when \( z < z_c(\kappa) \), where we recall \( z_c(\kappa) \) was defined in Definition 3.

The choice of \( z_0 \) in the next proposition is for convenience when we apply this result in Section 7.5.

**Proposition 6.1.** Suppose that \( \kappa_0(d, p_1) \) is sufficiently small and that \( z \) is at least \( z_0(\kappa) = (1 + \kappa)^{-2(d-1)} \). Then

\[
G^\eta_{z,\kappa}(x) \leq (1 + \kappa)G_{z,\kappa}(x). \tag{6.5}
\]

In particular, \( G^\eta_{z,\kappa}(x) \) is well-defined as a convergent power series when \( z_0 \leq z < z_c(\kappa) \).

**Proof.** We begin by defining a multivalued map \( \Phi \) that will quantify the entropic gain of forgetting a memory. Let \( \omega \in \Gamma^\eta_{n,k} \). By Lemma 4.5, there exists a set \( \text{adj}^* \subset \text{adj}(\eta, \omega) \) of \( \alpha_k = [\alpha k] \) disjoint 1-flippable plaquettes for \( \omega \). Define \( \Phi \) by

\[
\Phi(\omega) = \{ \omega' \mid \omega' = \mathcal{F}_B(\omega), B \subset \text{adj}^* \}. \tag{6.6}
\]

To begin, we compare the weight of \( \omega \in \Gamma^\eta_{n,k}(x) \) under \( W_\kappa(\cdot; \eta) \) to the \( W_\kappa(\cdot) \) weight of its image \( \Phi(\omega) \). We claim that

\[
z^m W_\kappa(\omega; \eta) \leq (1 + \kappa) \left( \frac{1 + \kappa}{1 + z^2 p_1^2 (1 + \kappa)^{-(2d-4)}} \right)^{\alpha_k} \sum_{\omega' \in \Phi(\omega)} z^{||\omega'||} W_\kappa(\omega') \tag{6.7}
\]

To see this, note that at each \( P \in \text{adj}^* \) a flip may or may not occur. Hence the definition of \( \Phi \) implies

\[
\sum_{\omega' \in \Phi(\omega)} z^{||\omega'||} W_\kappa(\omega') \geq \prod_{j=1}^{[\alpha k]} (1 + z^2 p_1^2 (1 + \kappa)^{-(2d-4)}) z^m W_\kappa(\omega). \tag{6.8}
\]
Formally, this bound arises by applying Lemma 4.4 \(|B|\) times to \(ω' = F_B ω\), and then summing over all \(B \subset \text{adj}^∗\). As \(ω ∈ Λ^n_{m,k}(x)\), there is a \(σ ∈ \{0, 1\}\) such that \(W_k(ω) = (1 + \kappa)^{−k−σ}W_k(ω; η)\). The value of \(σ\) can only take these values by Lemma 4.3, as there is at most one 2-flippable plaquette in \(\text{adj}(ω, η)\). Inserting this formula for \(W_k(ω)\) into (6.8) and rearranging gives (6.7).

Since \(κ ≥ 0\),

\[
(6.9) \quad \frac{(1 + \kappa)^{−k}}{1 + z^2 p_1^2 (1 + \kappa)^{−(2d−4)}} ≤ \frac{(1 + \kappa)^{−1}}{1 + z^2 p_1^2 (1 + \kappa)^{−(2d−4)}}.
\]

When \(κ = 0\) the right-hand side of (6.9) is \((1 + z^2 p_1^2)^{−1} ≤ (1 + z_0^2 p_1^2)^{−1} < 1\). The right-hand side of (6.9) is continuous in \(κ\) and hence is strictly less than one for small positive \(κ\). Thus, for \(κ\) sufficiently small and \(z ≥ z_0\), (6.8) may be rewritten

\[
(6.10) \quad z^m W_k(ω; η) ≤ (1 + \kappa) \sum_{ω' ∈ Φ(ω)} z^{∥ω'∥} W_k(ω').
\]

Sum (6.10) over \(ω ∈ Γ^n_n(x) = \cup_{k>0} Γ^n_{m,k}(x)\). To conclude the proof what must be shown is that \(Φ(ω) ∈ Γ(x)\), and that if \(ω^i ∈ Γ^n_n(x), i = 1, 2\), then

\[
(6.11) \quad ω^1 ≠ ω^2 \implies Φ(ω^1) ∩ Φ(ω^2) = \emptyset.
\]

The first claim follows from Lemma 4.1. To prove the second claim, suppose \(γ^i ∈ Φ(ω^i), i = 1, 2\). Then as \(γ^i\) is the result of flipping \(ω^i\) at a disjoint set \(B\) of 1-flippable plaquettes, Lemma 4.2 implies \(ω^i\) is determined by \(γ^i\) and \(η\). Hence if \(γ^1 = γ^2\), then \(ω^1 = ω^2\).

**Corollary 6.2.** Proposition 6.1 holds for \(G^n_{z, κ}\) in place of \(G^n_{z, κ}\) after changing \((1 + \kappa)\) to \((1 + \kappa)^2\). Moreover, both of these inequalities continue to hold if we restrict the two-point functions to sums of walks of length at least \(m\) for \(m ∈ \mathbb{N}\).

**Proof.** The proof for \(G^n_{z, κ}\), i.e., self-avoiding polygons, is mutatis mutandis the proof of Proposition 6.1. The extra factor of \(1 + \kappa\) arises as Lemma 4.3 only guarantees there are at most two 2-flippable plaquettes in \(\text{adj}(η, ω)\), as opposed to one. ■

**Corollary 6.3.** Both Proposition 6.1 and Corollary 6.2 hold when the two-point functions are restricted to sums over walks of length at least \(m\).

**Proof.** The map \(Φ\) used in the proof of Proposition 6.1 only increases the length of a walk. ■

### 6.3. First consequence of averaged submultiplicativity.

It will be necessary to temporarily consider \(κ\)-ASAW in finite volume. Precisely, let \(Λ = (\mathbb{Z}/L\mathbb{Z})^d\) be a torus of side length \(L\). Define

\[
P_{n, κ, Λ}(ω) ∝ \left( \prod_{i=0}^{n} \mathbb{1}_{\{ω_i ∈ Λ\}} \right) P_{n, κ}(ω).
\]

Let \(χ_Λ(z)\) be the susceptibility associated to \(κ\)-ASAW on \(Λ\), i.e.,

\[
(6.12) \quad χ_Λ(z) = \sum_{n ≥ 0} z^n \sum_{ω ∈ Γ_n} W_k(ω) \prod_{i=0}^{n} \mathbb{1}_{\{ω_i ∈ Λ\}}.
\]
Note that $\chi_{\Lambda,\kappa}(z)$ is a polynomial in $z$ whenever $\Lambda$ is finite.

**Lemma 6.4.** Fix $\kappa < \kappa_0(d, p_1)$ and $z \geq z_0$. On a finite torus $\Lambda$,

$$-\frac{d}{dz} \chi_{\Lambda,\kappa}^{-1}(z) \leq (1 + \kappa)z_0^{-1}.$$

**Proof.** The proof is the standard one for self-avoiding walk, with averaged submultiplicativity (Proposition 6.1) replacing submultiplicativity. While Proposition 6.1 is for $\kappa$-ASAW on $\mathbb{Z}^d$, the proof applies immediately to $\Lambda$ as well: the only property of the graph that was used in the proof is that flips of 1-flippable plaquettes are well-defined.

As $\chi_{\Lambda,\kappa}(z)$ is a polynomial in $z$, we can compute

$$\frac{d}{dz} \left[ z \chi_{\Lambda,\kappa}(z) \right] = \sum_{y \in \Lambda} \sum_{\omega \in \Gamma(y)} (|\omega| + 1) z^{||\omega||} W_{\kappa}\omega,$$

and then splitting $\omega$ into $\eta = \omega_{[0,1]}$ and $\omega'$. The second equality follows from the definition of the conditional weight $W_{\kappa}(\cdot; \eta)$ and the translation invariance of $G_{\eta,\kappa}^y(x, y)$ on a torus.

By (the finite-volume version of) Proposition 6.1, the memory $\eta$ can be ignored to obtain an upper bound. Summing over $y$ results in a factor $\chi_{\Lambda,\kappa}(z)$, as does the sum over $x$. The result is

$$\frac{d}{dz} \left[ z \chi_{\Lambda,\kappa}(z) \right] \leq (1 + \kappa)(\chi_{\Lambda,\kappa}(z))^2.$$

Computing the derivative on the left-hand side, rearranging, and using $\chi_{\Lambda,\kappa}(z) \geq 0$ proves the lemma.

The validity of Lemma 6.4 for all $z \geq z_0$ and all finite $\Lambda$ implies the continuity of the phase transition for $\kappa$-ASAW and a mean-field lower bound on the critical exponent $\gamma$. The next proposition is a formal statement of these facts; we include the proof for completeness, although the implication given Lemma 6.4 is well-known [20].

**Proposition 6.5.** Let $z_0 = (1 + \kappa)^{-2(d-1)}$ and fix $\kappa_0(d, p_1)$ small enough such that for $\kappa \leq \kappa_0(d, p_1)$ Proposition 6.1 holds. Then for $0 < \kappa < \kappa_0$ and $z \geq z_0$

$$\chi_{\kappa}(z) \geq \frac{(1 + \kappa)z_0^{-1}}{z_c - z}.$$

**Proof.** Note that $\chi_{\Lambda,\kappa}(z) \geq 1$ due to the contribution of the zero-step walk. Fixing $\epsilon > 0$ and integrating (6.13) from $z > z_0$ to $z_c + \epsilon$ implies

$$\chi_{\Lambda,\kappa}^{-1}(z) - \chi_{\Lambda,\kappa}^{-1}(z_c + \epsilon) \leq (1 + \kappa)z_0^{-1}(z_c - z) + \epsilon(1 + \kappa)z_0^{-1}.$$
The quantity $\chi_{\Lambda,\kappa}(z)$ is an increasing function of $\Lambda$ and hence has a limit in $[1, \infty]$ as $\Lambda \uparrow \mathbb{Z}^d$. Take $\Lambda \uparrow \mathbb{Z}^d$ in (6.19). We have that $\chi_{\Lambda,\kappa}^{-1}(z_c + \epsilon)$ tends to zero by the definition of $z_c$, so letting $\epsilon \to 0$ proves the result.

7. A Lace Expansion for $\kappa$-ASAW

This section derives a lace expansion for $\kappa$-ASAW based on Lemma 7.1 below. Our derivation proceeds along the same lines as the algebraic approach to the lace expansion for SAW [4]. As SAW is the special case $\kappa = 0$ of $\kappa$-ASAW, many aspects of the derivation mirror the SAW case. In what follows we therefore focus on the details specific to $\kappa \neq 0$, and give precise statements and references for the standard details that we omit. A similar expansion for a vertex attraction was derived in [1].

7.1. Explicit $\kappa$-ASAW Interaction. An algebraic formulation of the $\kappa$-ASAW weight will be needed. Let

$$U_{ij}(\omega) = 1_{\{\omega_i = \omega_j\}} - \kappa 1_{\{(\omega_i, \omega_{i+1}, \omega_j) \text{ is a plaquette}\}}.$$  

**Lemma 7.1.** If $\omega \in W_n$ is an $n$-step walk, then $P_{n,\kappa}(\omega)$ is proportional to

$$W_\kappa(\omega) 1_\omega \in \Gamma = \prod_{i=0}^{n-1} D(\omega_{i+1} - \omega_i) \prod_{0 \leq i < j \leq n} (1 - U_{ij}(\omega)).$$

**Proof.** The first indicator in (7.1) encodes the self-avoidance constraint. The second indicator encodes the self-attraction between edges. Since $\omega_i = \omega_j$ precludes $\{\omega_i, \omega_{i+1}, \omega_{j-1}, \omega_j\}$ being a plaquette, the lemma follows. ■

7.2. Derivation of the Lace Expansion. Let $a < b$ be integers, and $[a, b] = \{a, a + 1, \ldots, b\}$. An edge $\{i, j\}$ is an element of $\left(\begin{array}{c}a \vdots b \end{array}\right)$ with $|i - j| > 1$; $\{i, j\}$ will be abbreviated $ij$. A graph $G$ on $[a, b]$ is a set of edges, and $G$ is connected if for all $j \in [a + 1, b - 1]$ there are $i < j < k$ such that $ik$ is an edge in $G$. Let $G_{[a, b]}$ denote the set of graphs on $[a, b]$, and $G^e_{[a, b]}$ the set of connected graphs. Let $\omega$ be a walk and define

$$K_{[a, b]}(\omega) = \sum_{G \in G_{[a, b]}} \prod_{ij \in G} -U_{ij}(\omega), \quad \text{and}$$

$$J_{[a, b]}(\omega) = \sum_{G \in G^e_{[a, b]}} \prod_{ij \in G} -U_{ij}(\omega).$$

These definitions imply $K_{[a,a+1]} = J_{[a,a+1]} = 0$, as the concerned sums have empty index sets.

Expanding the product of $(1 - U_{ij})$ over $ij$ in (7.2) and using (6.4) with $\eta = \emptyset$ implies the two-point function can be written in terms of graphs:

$$G_{z,\kappa}(x) = \sum_{n=0}^{\infty} \sum_{\omega \in W_n(x)} z^n P_n(\omega) K_{[0,n]}(\omega).$$

In what follows, we manipulate (7.5) by rewriting the term $K_{[0,n]}(\omega)$ to obtain a convolution equation for $G_{z,\kappa}$. The first step is the next well-known lemma, whose proof is based on asking, for each $G \in G_{[a,b]}$, if $a$ is in an edge of $G$. 
Lemma 7.2 (Lemma 5.2.2 of [6]). For any walk $\omega$ and $a < b$,

\[(7.6) \quad K_{[a,b]}(\omega) = K_{[a+1,b]}(\omega) + \sum_{j=a+2}^{b} J_{[a,j]}(\omega)K_{[j,b]}(\omega).\]

For $x \in \mathbb{Z}^d$, define $\Pi_{x,\kappa}(x)$ by

\[(7.7) \quad \Pi_{x,\kappa}(x) = \sum_{n=2}^{\infty} \sum_{\omega \in W_n(x)} z^n J_{[0,n]}(\omega)\mathbb{P}_n(\omega).\]

It is not a priori clear that the series defining $\Pi_{x,\kappa}$ is convergent. When convergence is unknown we will interpret the series and the formulas in which it occurs as formulas relating formal power series in $z$.

For $f, g : \mathbb{Z}^d \rightarrow \mathbb{R}$, denote the convolution of $f$ and $g$ by $(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(y)g(x-y)$.

Proposition 7.3 (Theorem 5.2.3 of [6]). As formal power series in $z$,

\[(7.8) \quad G_{x,\kappa}(x) = \mathbb{1}_{\{x=0\}} + (zD * G_{x,\kappa})(x) + (\Pi_{x,\kappa} * G_{x,\kappa})(x).\]

This is an equality between functions if both $z < z_c(\kappa)$ and $\Pi_{x,\kappa}(x)$ converges absolutely for all $x$.

Proof. If $z < z_c$, then $G_{x,\kappa}$ is an absolutely convergent series, so if $\Pi_{x,\kappa}$ is also absolutely convergent, the validity of (7.8) as a relation between formal power series implies it also holds as an equality of functions.

What remains is to prove the formal equality. Consider the contribution of $n$-step walks to $G_{x,\kappa}(\omega)$, i.e.,

\[(7.9) \quad \sum_{\omega \in W_n(x)} z^n W_n(\omega) = \sum_{\omega \in W_n(x)} z^n K_{[0,n]}(\omega).\]

Since $U_{ij}(\omega) only depends on those $\omega_i$ with $i \leq \ell \leq j$, $K_{[0,j]}(\omega)J_{[j\omega],[j\omega]}(\omega)$ is equal to $K_{[0,j]}(\omega_{[0,j]}^\ell)J_{[j\omega],[j\omega]}(\omega_{[0,j]}^\ell)$. Hence, by using (7.6) to rewrite the factor $K_{[0,n]}$ in (7.9), the right-hand side of (7.9) can be rewritten as

\[(7.10) \quad \sum_{\omega \in W_n(x)} zD(\omega_1 - \omega_0)z^{n-1}\mathbb{P}(\omega_{[1,n]})K_{[1,n]}(\omega)\]

\[+ \sum_{\omega \in W_n(x)} \sum_{j=2}^{n} z^j\mathbb{P}(\omega_{[0,j]}) J_{[0,j]}(\omega_{[0,j]})z^{n-j}\mathbb{P}(\omega_{[j,n]}) K_{[j,n]}(\omega_{[j,n]}).\]

To conclude, sum (7.10) over all $n$. The left-hand side is $G_{x,\kappa}(x)$. For $n = 0$, the right-hand side is $\mathbb{1}_{\{x=0\}}$, the contribution of the zero-step walk. The factors of $G_{x,\kappa}$ arise by (7.5) and the translation invariance of $W_n$. The term $\Pi_{x,\kappa}$ arises from the sum over $j$ by (7.7). This proves (7.8) in the sense of formal power series, as the coefficient of $z^n$ consists of only finitely many terms for all $n$. ■

7.3. Representation of $\pi_{x,\kappa}^{(m)}(x)$. To make use of Proposition 7.3 as an equality between functions, it is necessary to understand when (7.7) is an absolutely convergent series. This requires further definitions.
Then (i) \( L \) encode by the vector \( n \) and \( L \) are called the edges \( G \).

These definitions imply that, as formal power series,

\[
\text{Lemma 7.4 (p.126-p.128 of [6]). For } G \in G_{a,b}, \text{ let } \mathcal{L}(G) \text{ be the graph with edges } \{s_i t_i \} \text{ defined by } s_1 = a, t_1 = \max \{t \mid s_1 t_1 \in G\}, \text{ and }
\]

\[
t_{i+1} = \max \{t \mid \exists s < t_i \text{ such that } st \in G\}, \quad s_{i+1} = \min \{s \mid st_{i+1} \in G\}.
\]

Then (i) \( \mathcal{L}(G) \) is a lace and (ii) if \( \mathcal{L}(G) = \mathcal{L}(H) \) then \( \mathcal{L}(G \cup H) = \mathcal{L}(G) \).

See Figures 6 and 7 for an illustration of \( \mathcal{L}(G) \).

\( G_{a,b} \) is partially ordered by inclusion, and Lemma 7.4 implies that for every lace \( G \) there is a maximal graph \( G \) such that \( \mathcal{L}(G) = L \). The edges of the maximal graph \( G \) are \( G \cup C(L) \), where \( C(L) \) are called the compatible edges.

Explicitly, these are the edges \( ij \in C(L) \) if and only if \( ij \notin L \) and \( \mathcal{L}(L \cup ij) = L \).

It follows that

\[
J_{a,b}(\omega) = \sum_{L \in L_{a,b}} \prod_{ij \in L} -U_{ij}(\omega) \prod_{i' j' \in C(L)} (1 - U_{i' j'}(\omega)). \tag{7.11}
\]

Define \( J_{a,b}^{(m)} \) to be the contribution to (7.11) given by \( L \in L_{a,b}^{(m)} \), and

\[
\pi_{z,\kappa}^{(m)}(x) = \sum_{\omega \in \mathcal{W}_{n}(x)} z^n \mathbb{P}_{n}(\omega) J_{a,b}^{(m)}(0, n)(\omega). \tag{7.12}
\]

These definitions imply that, as formal power series,

\[
\Pi_{z,\kappa}(x) = \sum_{m \geq 1} \pi_{z,\kappa}^{(m)}(x), \tag{7.13}
\]

and our approach to understanding the convergence of \( \Pi_{z,\kappa} \) will be to estimate the terms \( \pi_{z,\kappa}^{(m)} \). To do this, we rewrite \( \pi_{z,\kappa}^{(m)} \) in terms of walks subject to conditional interactions \( \mathcal{W}_{n}(\cdot ; \eta) \). As in Section 7.1, the arguments are similar to standard arguments for self-avoiding walk [6].

A lace \( L \in L_{0,n}^{(m)} \) partitions \( [0, n] \) into \( 2m - 1 \) intervals with disjoint interiors, and this induces a composition of \( n \) into \( 2m - 1 \) parts. This composition is encoded by the vector \( n = (n_1, n_2, \ldots, n_{2m-1}) \) of interval lengths, and has the properties

\[
n_j \geq 0, \quad j \in \{3, 5, 7, \ldots, 2m - 3\} \tag{7.14}
\]

\[
n_j \geq 1, \quad j \in [2m - 1] \setminus \{3, 5, 7, \ldots, 2m - 3\}.
\]
By definition, \( n = \sum_{j=1}^{2m-1} n_j \). Let \( M_i = \sum_{j=1}^{i} n_j \), \( I_1 = [0, M_1] \) and \( I_j = (M_{j-1}, M_j) \) for \( j = 2, 3, \ldots, 2m-1 \). Further, let \( C_i(L) = \{ i'j' \mid j' \in I_i \} \) be the set of compatible edges whose right endpoint is in \( I_i \). These definitions imply that, for any lace \( L \in \mathbb{L}_{[0,n]}^{(m)} \) and walk \( \omega \in \mathbb{W}_n \),

\[
(7.15) \quad \prod_{ij \in C(L)} (1 - U_{ij}(\omega)) = \prod_{i=1}^{2m-1} \prod_{i'j' \in C_i(L)} (1 - U_{i'j'}(\omega)),
\]

as \( C(L) = \bigsqcup_{i=1}^{2m-1} C_i(L) \) since the intervals \( I_i \) are a partition of \([0, n]\).

For \( \omega \in \mathbb{W}_n \), let \( \omega \) be the vector of \( 2m - 1 \) walks defined by \( \omega^{(i)} = \omega_{[M_{i-1}, M_i]} \).

Re-expressing the inner product in (7.15) in terms of the \( \omega^{(i)} \) requires characterising \( C(L) \). Let \( k \geq 1 \) and \( i < j \). For \( m \geq 2 \), the compatible edges are given by

(i) If \( j \in I_1 \cup I_2 \) then \( i \in I_1 \cup I_2 \), and \( j = M_2 \) implies \( i > 0 \).
(ii) If \( j \in I_{2k} \) then \( i \in \bigcup_{\ell=2k-3}^{2k-1} I_\ell \), and \( j = M_{2k} \) implies \( i > M_{2k-3} \).
(iii) If \( j \in I_{2k+1} \) then \( i \in \bigcup_{\ell=2k-1}^{2k+1} I_\ell \), and \( j = M_{2k+1} \) implies \( i > M_{2k} \).

When \( m = 1 \) all edges but 0\( n \) are compatible. To denote the effect of compatible edges in terms of walks, let \( \eta^{(0)} = \emptyset \), \( \eta^{(1)} = \omega^{(1)} \), and, for \( k \geq 1 \), define \( \eta^{(2k)} \) to be \( \omega^{(2k-3)} \circ \omega^{(2k-3)} \circ \omega^{(2k-1)} \) and \( \eta^{(2k+1)} \) to be \( \omega^{(2k-1)} \circ \omega^{(2k)} \). By identifying the \( i \)th term of the product in (7.15) as the weight of the walk \( \omega^{(i)} \), we obtain the following proposition, where \( \text{SA} = \Gamma \cup \tilde{\Gamma} \).

**Proposition 7.5.** For \( L \in \mathbb{L}^{(m)} \) and \( i \in [2m-1] \), let \( A_i \) be the event the last edge of \( \omega^{(i)} \) is adjacent to the first edge of the memory \( \eta^{(i)} \). Then

\[
(7.16) \quad \prod_{i'j' \in C_i(L)} (1 - U_{i'j'}(\omega)) = \mathbb{1}_{\{ \eta^{(i)} \circ \omega^{(i)} \in \text{SA} \}} W_\kappa(\omega^{(i)}; \eta^{(i)})(1 + \kappa)^{-3} A_i.
\]

**Proof.** The proposition is essentially a translation of the definition of a compatible edge and the definitions of the walks \( \eta^{(i)} \). Constraints that vertices be distinct, or that there be a factor of \((1 + \kappa)\) included due to an adjacent pair of edges, are encoded by the factors of \((1 - U_{ij}(\omega))\) associated to compatible edges when the vertices of \( \omega \) relevant for computing \( U_{ij}(\omega) \) are contained in \( \eta^{(i)} \) and \( \omega^{(i)} \).

What needs to be explained is why \( \eta^{(i)} \circ \omega^{(i)} \in \tilde{\Gamma} \) is possible, and where \( A_i \) comes from. The explanation is that the term \( U_{ij} \) controlling these factors is not associated to a compatible edge, since it is an edge of the lace graph. Hence

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lace_graph}
\caption{A lace graph with edges \( \{\{0, 5\}, \{4, 8\}, \{5, 10\}\} \) and the intervals it determines. Here \( n_1 = 4, n_2 = 1, n_3 = 0, n_4 = 3, \) and \( n_5 = 2 \).}
\end{figure}
the endpoint of $\omega^{(i)}$ is permitted to be the initial vertex of $\eta^{(i)}$, and if the final edge of $\omega^{(i)}$ is adjacent to the first edge of $\eta^{(i)}$, there should not be a reward $(1 + \kappa)$. □

7.4. Diagrammatic bounds. Proposition 7.5 leads to an explicit formula for $\pi_{z,\kappa}^{(m)}(x)$ in terms of a sum over collections $\omega$ of interacting walks. This will be used to estimate the size of $|\Pi_{z,\kappa}(x)|$ in terms of convolutions of $G_{z,\kappa}(x)$.

Define $r_\kappa(x) = 1_{\{x = o\}} + \kappa 1_{\{\|x\|_\infty = 1\}}$.

Lemma 7.6. $|U_{ij}(\omega)| \leq r_\kappa(\omega_j - \omega_i)$ for all walks $\omega$.

Proof. This is immediate by considering separately $\|\omega_j - \omega_i\|$ being 0, 1, or at least 2. □

In what follows, $\kappa = \kappa(p_1)$ will be a function of $p_1$, chosen such that $\kappa \leq \kappa_0(d, p_1)$, so that Proposition 6.1 applies with $z_0 = (1 + \kappa)^{-2(d-1)}$. Let $H_{z,\kappa}(x) = G_{z,\kappa}(x) - \delta_{x,o}$ be the sum of contributions to $G_{z,\kappa}(x)$ due to non-trivial walks.

Proposition 7.7. For $\kappa \leq \kappa_0$ and $z_0 < z < z_c$, the following bounds hold

\begin{align}
|\pi_{z,\kappa}^{(1)}(x)| &\leq (1 + \kappa)^2 z r_\kappa(x)(D * H_{z,\kappa})(x) \\
|\pi_{z,\kappa}^{(m)}(x)| &\leq (1 + \kappa)^{4m-2} \sum_x H_{z,\kappa}(x_2 - x_0) r_\kappa(x_2m+1 - x_2m-2) H_{z,\kappa}(x_2m+1 - x_2m-1)
\end{align}

\[
\prod_{j=1}^{m-1} r_\kappa(x_{2j+1} - x_{2j-2}) H_{z,\kappa}(x_{2j+1} - x_{2j}) G_{z,\kappa}(x_{2j+2} - x_{2j+1}),
\]

where $x_0 = x_1 = o$, $x_{2m+1} = x_{2m} = x$, and the sum is over vectors $x = (x_2, \ldots, x_{2m-1})$ of vertices in $\mathbb{Z}^d$.

Proof. Let $W_{z,\kappa}(\omega; \eta) = z^{|\omega|} W_{\kappa}(\omega; \eta)$, where $|\omega| = n$ if $\omega \in W_n$. By (7.12), each term of $\pi_{z,\kappa}^{(m)}(x)$ results from applying the weight of a lace $L$ to a walk $\omega \in W(x)$. By the discussion in Section 7.3, if $L \in L^{(m)}$ then $\omega$ is the concatenation of $2m - 1$ walks $\omega^{(i)}$, $\omega^{(i)}$ has length $n_i$, and the vector $n$ of lengths satisfies Equation (7.14). If $|\omega| = n$, distribute the factor $z^n$ so that there are $n_i$ factors associated to the walk $\omega^{(i)}$. Each factor of $U_{ij}(\omega)$ for $ij \in L$ can be replaced by a factor $r_\kappa$ by Lemma 7.6. The result of this, by Proposition 7.5, is

\[
\pi_{z,\kappa}^{(1)}(x) = -\sum_y z D(y) \sum_{\omega \in W(y,x)} r_\kappa(o, \omega_{|\omega|}) 1_{\{(o,y) : o \in SA\}} W_{z,\kappa}(\omega^{(i)}; (o, y))(1 + \kappa)^{-\|A_i\|}.
\]

\[
\pi_{z,\kappa}^{(m)}(x) = (-1)^m \sum_x \sum_\omega \sum_{\omega^{(1)}} r_\kappa(\omega^{(1)}, \omega^{(2)}) \sum_{\omega^{(2m-2)}} r_\kappa(\omega^{(2m-2)}, \omega^{(2m-1)})
\]

\[
\prod_{j=1}^{m-1} r_\kappa(\omega^{(2j)}, \omega^{(2j+2)}) \prod_{j=1}^{2m-1} 1_{\{\eta^{(i)} : o \in SA\}} W_{z,\kappa}(\omega^{(i)}; \eta^{(i)})(1 + \kappa)^{-\|A_i\|},
\]
where the outer sum is over \(x = (x_2, \ldots, x_{m-1}) \in \mathbb{Z}^{d(m-2)}\) and the inner sum is over \(\omega = (\omega^{(1)}, \ldots, \omega^{(2m-1)})\) such that \(\omega^{(i)} \in W(x_i, x_{i+1})\). For \(\pi_{Z,\kappa}^{(1)}\), we split the walk after the first step, in anticipation of the desired upper bound.

We now iteratively sum over the subwalks \(\omega^{(i)}\), starting with \(i = 2m - 1\). Since \(\omega^{(j)}\) is fixed for \(j = 1 \ldots, 2m - 2\), the memory \(\eta^{(2m-1)}\) is determined. Temporarily letting \(\tilde{n} = n_{2m-1}\), \(\tilde{\omega} = \omega^{(2m-1)}\), \(\tilde{\eta} = \eta^{(2m-1)}\), \(y_1 = x_{2m-1}\), and \(y_2 = x_{2m}\), this yields the upper bound

\[
\sum_{\tilde{\omega} \in W_n(y_1, y_2)} \mathbb{I}_{\{\tilde{\eta} \in \omega \in \text{SA}\}} W_{Z,\kappa}(\tilde{\omega}; \tilde{\eta}) \leq (1 + \kappa)^2 \sum_{\tilde{\omega} \in W_n(y_1, y_2)} \mathbb{I}_{\tilde{\omega} \in \Gamma} W_{Z,\kappa}(\tilde{\omega})
\]

by Proposition 6.1, Corollary 6.2 (as \(\tilde{\eta} \circ \tilde{\omega} \in \text{SA}\) includes the possibility of it being a self-avoiding polygon), and Corollary 6.3. The sum on the right-hand side of this equation is \(H_{Z,\kappa}(x_{2m+1} - x_{2m-1})\), as there is no contribution from zero-step walks due to the constraint \(\tilde{m} \geq 1\).

Repeating this procedure for \(2m - 2, 2m - 3, \ldots, 1\) gives the two-point functions in the upper bounds of the proposition. Factors of \(H_{Z,\kappa}\) arise for edges on which \(n_i \geq 1\), and factors of \(G_{Z,\kappa}\) for those with \(n_i \geq 0\).

7.5. Proof of Gaussian decay. We will now prove Theorem 2.3 by making use of Theorem A.6. Recall that \(\|x\| = \max\{\|x\|_2, 1\}\).

Proposition 7.8 (Prop. 1.7 of [7]). If \(f, g: \mathbb{Z}^d \to \mathbb{R}\) satisfy \(|f(x)| \leq \|x\|^{-a}\) and \(|g(x)| \leq \|x\|^{-b}\) with \(a \geq b > 0\), there exists \(C = C(a, b, d)\) such that

\[
|f \ast g(x)| \leq \begin{cases} C \|x\|^{-b} & a > d, \\ C \|x\|^{d-(a+b)} & a < d \text{ and } a + b > d. \end{cases}
\]

Proposition 7.9. Let \(d > 4\). Suppose \(\beta > 0\), \(\kappa \leq \min\{\kappa_0, \beta\}\), \(z_0 \leq z \leq 2\), and that

\[
G_{Z,\kappa}(x) \leq \beta \|x\|^{-(d-2)}, \quad x \neq o.
\]

If \(\beta \leq \beta_0(D)\), then there is a \(c = c(d) > 0\) such that

\[
|\Pi_{Z,\kappa}(x)| \leq c\beta \mathbb{1}_{\{x = o\}} + \frac{c\beta^2}{\|x\|^{3(d-2)}}.
\]
Proof. $G_{z,\kappa}(o) = 1$ as only the trivial self-avoiding walk ends at the origin, so (7.20) implies $G_{z,\kappa}(x) \leq \|x\|^{-(d-2)}$ and $H_{z,\kappa}(x) \leq \beta \|x\|^{-(d-2)}$ for all $x \in \mathbb{Z}^d$. In particular, $\|H_{z,\kappa}\|_\infty \leq \beta$.

First consider $\pi^{(1)}_{z,\kappa}(x)$. If $x = o$ then $r_\kappa = 1$, so (7.17), $z \leq 2$, and the inequality $\|f \ast g\|_1 \leq \|f\|_\infty \|g\|_1$ imply

$$\left| \pi^{(1)}_{z,\kappa}(o) \right| \leq (1 + \kappa)^2 z \sum_{y \in \mathbb{Z}^d} D(y) H_{z,\kappa}(-y) \leq 2(1 + \kappa)^2 \beta$$

(7.22) as $\sum_y D(y) = 1$. If $x \neq o$ then $|r_\kappa(x)| \leq \kappa 1_{\{|x| = 1\}}$, and an argument as above shows that

$$\left| \pi^{(1)}_{z,\kappa}(x) \right| \leq (1 + \kappa)^2 \kappa z 1_{\{|x| = 1\}} \sum_{y \in \mathbb{Z}^d} D(y) H_{z,\kappa}(x - y) \leq 2(1 + \kappa)^2 \beta^2 1_{\{|x| = 1\}}$$

(7.23) where we have used $\kappa \leq \beta$.

Next consider $\pi^{(m)}_{z,\kappa}(x)$ for $m \geq 2$. The factors $r_\kappa$ in (7.18) imply the collections $x$ of vertices that give a non-zero contribution satisfy

$$\|x_{2j+1} - x_{2j-2}\|_\infty \leq 1, \quad j = 1, \ldots, 2m - 2.$$  

(7.24) Given $x$ satisfying (7.24), define $\rho$ to be the collection of vectors

$$\rho_{2j+1} = x_{2j+1} - x_{2j-2}, \quad j = 1, \ldots, m.$$  

(7.25) Each $\rho_j$ satisfies $\|\rho_j\|_\infty \leq 1$. The sum over $x$ in (7.18) can be replaced by a sum over $x_i$ for $i = 2, 4, \ldots, 2m - 2$ and a sum over the possible $\rho$. Formally, letting $\pi^{(m)} = \pi^{(m)}_{z,\kappa}$, we re-express Equation (7.18) as

$$\pi^{(m)}(x) = (1 + \kappa)^{4m-2} \sum_{x'} \sum_{\rho} \pi^{(m)}_{x',\rho},$$

(7.26) where the sum over $x'$ is over tuples $(x_2, x_4, \ldots, x_{2m-2})$. We have abused notation in (7.26), but the bold subscripts will ensure that $\pi^{(m)}_{x',\rho}$ is distinguished from $\pi^{(m)}_{z,\kappa}$.

We will now show the proposition follows from the estimate

$$\sum_{x'} \pi^{(m)}_{x',\rho} \leq \beta^m C^m \|x\|^{-3(d-2)}, \quad m \geq 2,$$

(7.27) for a constant $C > 0$ independent of $\beta$. Equation (7.27) is uniform in $\rho$, so with (7.26) it implies

$$\left| \pi^{(m)}_{z,\kappa}(x) \right| \leq (1 + (3^d - 1)\kappa)^m (C\beta)^m \|x\|^{-3(d-2)}, \quad m \geq 2.$$  

(7.28) The factor of $(1 + (3^d - 1)\kappa)^m$ arises as (i) each $\rho_i$ has $3^d - 1$ non-zero possibilities, (ii) each $i$ with $\|\rho_i\|_\infty = 1$ has absolute value $\kappa$, and (iii) $r_\kappa(o) = 1$. Summing (7.28) over $n$ and combining it with the bounds (7.22) and (7.23) for $m = 1$ implies the proposition. The remainder of the proof establishes (7.27).
With these definitions, we obtain we will be somewhat brief. See, e.g., [7, Prop. 1.8(a)] for more details.

\[ T = 2 \]

where in the case \( m = 2 \) we have degraded the bound slightly by using the estimate \( H \leq G \). Let \( c = (c')^2 \), and note that

\[ A(u, v, x, y) \leq \frac{c \beta}{\| v - u \|^{d-2} \| y - u \|^{d-2}} \mathbb{1}_{\{ v = x \}}. \]
Define
\[ \mathcal{S} = \sup_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \frac{1}{\|y\|^{d-2} \|x - y\|^{d-2}}. \]

When \( d > 4 \), \( \mathcal{S} \) is finite by an elementary convolution estimate [7, Proposition 1.7]. By induction on \( m \) using (7.33), there is a \( C = C(d) \) such that
\[ (7.34) \quad M^{(m)}(x, y) \leq (c\beta)^m (C\mathcal{S})^{m-2} \frac{1}{\|x\|^{2(d-2)} \|y\|^{d-2}}, \quad m \geq 2, \]
which proves Equation (7.27).

\[ \blacksquare \]

**Proof of Theorem 2.3.** To prove Theorem 2.3, it suffices to verify that there is a \( \kappa_0 \) such that, if \( \kappa \leq \kappa_0 \), the hypotheses of Appendix A.2 on \( D, G_{z,\kappa} \), and \( \Pi_{z,\kappa} \) are satisfied.

Hypothesis A.1 is trivially satisfied. Hypothesis A.3 is satisfied for \( \kappa \leq \kappa_0(L) \) by Theorem 2.1 which ensures the critical point exists, and Proposition 6.5, which ensures the divergence of the susceptibility.

We now verify (i), (ii), and (iv) of Hypothesis A.4. Since \( G_{z,\kappa}(x) \) is an absolutely convergent power series with positive coefficients when \( z < z_c \), it is monotone and continuous for \( z < z_c \). Since the summation defining \( G_{z,\kappa}(x) \) contains only walks of length at least \( \|x\|_\infty \), \( G_{z,\kappa}(x) \) decays exponentially in \( \|x\|_\infty \) for \( z \in [0, z_c - t) \) for any \( t > 0 \) (for more details, see [4, (2.20)–(2.23)].

To verify (iii) of Hypothesis A.4, let \( z_0 = (1 + \kappa)^{-2(d-1)} \). When \( z \leq z_0 \),
\[ (7.35) \quad G_{z,\kappa}(x) = \sum_n \sum_{\omega \in \Gamma_n(x)} z^n W_\kappa(\omega) \leq \sum_n \sum_{\omega \in \Gamma_n(x)} (1 + \kappa)^{-2(d-1)n} W_\kappa(\omega) \leq \sum_n \sum_{\omega \in \Gamma_n(x)} \mathbb{P}_n(\omega) = G_{1,0}(x). \]

This implies \( G_{z_0,\kappa}(x) \leq S_1(x) \), as \( S_1(x) \) is clearly an upper bound for the SAW two-point function.

For any \( D \), Hypothesis A.5 follows for \( G_{z,\kappa} \) by Proposition 7.9 when \( \kappa \) is small enough, with \( \beta_0 \) uniform in \( \kappa \). Thus, for \( \kappa \leq \kappa_0(L_0) \), with \( L_0 \) the constant of Theorem A.6, we can apply Theorem A.6 by the discussion of Appendix A.4. This proves the theorem.

\[ \blacksquare \]

**Appendix A. Gaussian Asymptotics**

This appendix reviews [7, Theorem 1.2], which derives Gaussian asymptotics for critical two-point functions. Our motivation is that the presentation in [7] is, at places, dependent on the particular models being studied. The proofs, however, apply essentially verbatim to other models. Our review axiomatizes sufficient assumptions for models similar to self-avoiding walk. We indicate where these assumptions are used in proofs, but omit the portions of the proofs that purely replicate [7]. We emphasise that the result and techniques are those of [7], and our presentation is primarily for the benefit of the reader who is not familiar with [7].
A.1. Setup. Let $\mathbb{R}_{\geq 0}$ denote the non-negative reals. For $z \in \mathbb{R}_{\geq 0}$, $G_z: \mathbb{Z}^d \rightarrow \mathbb{R}_{\geq 0}$, $\tilde{\Pi}_z: \mathbb{Z}^d \rightarrow \mathbb{R}$, and $D$ a probability distribution on $\mathbb{Z}^d$, we consider the convolution equation

\begin{equation}
G_z(x) = \delta_{0,x} + \tilde{\Pi}_z(x) + (zD * (\delta + \tilde{\Pi}_z) * G_z)(x).
\end{equation}

We will further assume that $G_z$, $\tilde{\Pi}_z$, and $D$ are all $\mathbb{Z}^d$-symmetric. We will see in Appendix A.4 that the analysis of (A.1) also applies to the convolution equation derived for $\kappa$-ASAW in the main body of the text.

The critical point $z_c$ is $z_c = \sup \{ z \in \mathbb{R}_{\geq 0} \mid \chi(z) < \infty \}$, where the susceptibility $\chi(z)$ is defined by

\begin{equation}
\chi(z) = \sum_{x \in \mathbb{Z}^d} G_z(x).
\end{equation}

A.2. Hypotheses and Theorem.

Hypothesis A.1. Assume that $D$ is a spread-out step distribution as defined in Definition 1.

Let $X_n$ be a discrete time simple random walk with step distribution $D$. Let $\sigma^2 = \sum_{x \in \mathbb{Z}^d} D(x) \|x\|^2$. Note that $\sigma^2$ is comparable to the spread-out parameter $L^2$. The non-interacting two-point function $S_\mu$ is defined by

\begin{equation}
S_\mu(x) = \sum_{n=0}^{\infty} \mu^n \mathbb{P}_0 [X_n = x].
\end{equation}

An important consequence of the form of $D$ is the following proposition. Let $a_d = \frac{d \Gamma(d/2 - 1)}{2 \pi^{d/2}}$, where $\Gamma$ is Euler's gamma function.

Proposition A.2 ([7, Prop. 1.6]). Suppose $d > 2$ and Hypothesis A.1 holds. For $L$ sufficiently large, $\alpha > 0$, $\mu \leq 1$, and $x \in \mathbb{Z}^d$,

\begin{equation}
S_\mu(x) \leq \delta_{0,x} + O \left( \frac{1}{L^{2-\alpha} \|x\|^{d-2}} \right)
\end{equation}

\begin{equation}
S_1(x) \leq \frac{a_d}{\sigma^2 \|x\|^{d-2}} + O \left( \frac{1}{\|x\|^{d-\alpha}} \right).
\end{equation}

The implicit constants may depend on $\alpha$, but not on $L$.

Note that, for fixed $d$, the leading coefficient in (A.5) is proportional to $L^{-2}$. The next two hypotheses deal with the critical point and behaviour of $G_z$ for $z_0 \leq z < z_c$, where $z_0 > 0$ is a chosen value of the parameter $z$.

Hypothesis A.3. The critical point $z_c$ satisfies $z_0 < z_c < \infty$. The susceptibility diverges as the critical point is approached from below: $\lim_{z \uparrow z_c} \chi(z) = \infty$.

Hypothesis A.4. $G_z$ is well-defined and not identically zero. For $z_0 \leq z < z_c$ and for each $x \in \mathbb{Z}^d$,

(i) $G_z(x)$ is monotone increasing in $z$,
(ii) $G_{z_0}(x) \leq S_1(x)$,
(iii) $G_z(x)$ is continuous for $z \in [z_0, z_c)$, and
(iv) for \( t > 0 \) and \( z \in [z_0, z_c - t) \) there are constants \( c(t), C(t) > 0 \) such that

\[
G_z(x) \leq C(t)e^{-c(t)\|x\|}.
\]

The most substantial hypothesis is the next one.

**Hypothesis A.5.** Assume

\[
G_z(x) \leq \beta \|x\|^{-d+2}, \quad x \neq 0.
\]

Suppose also that \( z_0 \leq z \leq 2 \). If \( \beta < \beta_0 \), there is a constant \( c = c(d) > 0 \) such that

\[
\left| \Pi_z(x) \right| \leq c\beta \delta_{0,x} + \frac{c\beta^2}{\|x\|^{3(d-2)}}.
\]

**Theorem A.6** ([7, Theorem 1.2]). Assume \( D, G_z, \) and \( \Pi_z \) satisfy the hypotheses of Appendix A.2. Choose \( 0 < \alpha < 2 \). Let \( \beta_0 \) be the constant of Hypothesis A.5.

There is an \( L_0(d, \alpha, \beta_0) \) such that, for \( L \geq L_0 \), the function \( G_{z_c} : \mathbb{Z}^d \to \mathbb{R} \) is well-defined, and there is an \( A > 0 \) such that

\[
G_{z_c}(x) \sim \frac{a_d A}{\sigma^2 \|x\|^{2-\alpha}} \left( 1 + O \left( \frac{L^2}{\|x\|^{\alpha-2}} \right) \right).
\]

The implicit constants are uniform in \( x \) and \( L \). The values of \( z_c \) and \( A \) are \( 1 + O(L^{\alpha-2}) \).

**A.3. Proof.** The next proposition is the heart of the analysis. In what follows we assume the hypotheses of Theorem A.6; in particular, \( \beta_0 \) is given.

**Proposition A.7.** Fix \( \alpha > 0 \). There is an \( L_0 = L_0(\beta_0, d, \alpha, z_0) \) such that, for \( L \geq L_0 \),

\[
G_{z_c}(x) \leq \frac{\text{const}}{L^{2-\alpha}\|x\|^{d-2}}, \quad x \neq 0,
\]

and \( z_c \leq 1 + O(L^{2+\alpha}) \).

**Lemma A.8** (Lemma 2.1 [7]). Let \( f : [z_1, z_c) \to \mathbb{R} \), and \( a \in (0, 1) \). Suppose

(i) \( f \) is continuous on \( [z_1, z_c) \),

(ii) \( f(z_1) \leq a \), and

(iii) for \( z \in [z_1, z_c) \) the inequality \( f(z) \leq 1 \) implies the inequality \( f(z) \leq a \).

Then \( f(z) \leq a \) for all \( z \in [z_1, z_c) \).

**Proof of Proposition A.7.** The proof is essentially that in [7]. We present the steps in which our hypotheses, as opposed to model-specific facts, are used.

Note that it suffices to prove that (A.10) holds for \( \alpha < \frac{1}{2} \), as the right-hand side is increasing in \( \alpha \). By Hypothesis A.4 and the monotone convergence theorem, it is enough to prove this for all \( z_0 < z < z_c \).

Let \( K \) be the optimal constant for the error bound in Proposition A.2:

\[
K = \sup_{L \geq 1, x \neq 0} L^{2-\alpha}\|x\|^{d-2}S_1(x).
\]

Define

\[
g_z(x) = (2K)^{-1}L^{2-\alpha}\|x\|^{d-2}G_z(x),
\]
and let \( g(z) = \sup_{x \neq o} g_x(z) \). To prove (A.10), we will use Lemma A.8 with \( f(z) = \max\{g(z), \frac{z}{z_0}\} \), \( z_1 = z_0 \), and \( a \in \left( \frac{1}{2}, 1 \right) \) arbitrary. The claim that \( z_c = 1 + O(L^{-2+\alpha}) \) will be established in the course of the argument.

**Claim:** Hypothesis 1 holds.

**Proof:** For \( x \in \mathbb{Z}^d \), \( g_x(z) \) is continuous on \([z_0, z_c)\) by Hypothesis A.4. It suffices to show \( \sup_{x \neq o} g_x(z) \) is continuous on \([z_0, z_c - t)\) whenever \( z_c - t > z_0 \).

By Hypothesis A.4, \( g_x(z) \) decays exponentially in \( \|x\|_2 \) with decay rate independent of \( t \). Therefore, \( \sum_{x \in \mathbb{Z}^d} g_x(z) \) converges exponentially fast with rate independent of \( t \). It follows that the supremum of \( g_x(z) \) occurs on \( \mathcal{B}_R(o) \), the ball of radius \( R \) about the origin, for some \( R = R(L) > 0 \). This proves the claim as the supremum of a finite set of continuous functions is continuous. 

**Claim:** Hypothesis 2 holds.

**Proof:** By Hypothesis A.4 and the definition of \( K \), \( g_x(z_0) \leq \frac{1}{2} \) for all \( x \). Since \( a > \frac{1}{2} \), this proves the claim. 

**Claim:** Hypothesis 3 holds.

**Proof:** Fix \( z_0 < z < z_c \) and suppose \( f(z) \leq 1 \). Then \( z \) is at most \( 2z_0 \), and

\[
G_z(x) \leq 2z_0KL^{-2+\alpha}\|x\|^{2-d}, \quad x \neq o.
\]

Let \( \beta = 2z_0KL^{-2+\alpha} \). By Hypothesis A.5, when \( L^{-2+\alpha} \) is sufficiently small there is a \( c > 0 \) such that

\[
\left| \tilde{\Pi}_z(x) \right| \leq c\beta \delta_{0,x} + c\beta^2 \|x\|^{-3(d-2)} \leq \frac{c\beta}{\|x\|^{3(d-2)}}.
\]

By Hypothesis A.4, \( G_z \) is not identically zero. Thus \( \chi(z) > 0 \), and the sum of (A.1) over all \( x \in \mathbb{Z}^d \) can be rearranged to give

\[
\chi(z) = \frac{1 + \sum_x \tilde{\Pi}_z(x)}{1 - z - \sum_x \tilde{\Pi}_z(x)} > 0.
\]

By (A.12), \( \|\tilde{\Pi}_z(x)\|_1 < 1 \) for \( L \) large enough. This implies the numerator, and hence the denominator, of (A.13) is strictly positive. Since \( f(z) \leq 1 \), this implies that

\[
z < 1 - z \sum_{x \in \mathbb{Z}^d} \tilde{\Pi}_z(x) \leq 1 + O(z_0L^{-2+\alpha}).
\]

Thus \( z \) is bounded above by \( a \) for \( a \in \left( \frac{1}{2}, 1 \right) \), provided that \( L \) is large enough.

What remains is to prove \( g(z) \leq a \) for \( a \in \left( \frac{1}{2}, 1 \right) \) when \( L \) is large enough. This exactly follows the presentation in [7, p.364], and hence we omit it. 

This proves the desired bounds, as we have proven that \( f(z) \leq a \) for \( z_0 \leq z < z_c \). The bound on \( z_c \) follows from (A.14), which holds as it was derived under the hypothesis that \( f(z) \leq 1 \). 

**Proof of Theorem A.6.** This follows [7, Theorem 1.2]. The only model specific step in the cited proof is showing that an auxiliary parameter \( \mu_z \) increases to
\( \mu_{z_c} = 1 \) as \( z \uparrow z_c \). We define this parameter below and show that it takes the desired value by Hypothesis A.3.

By (A.12), \( \tilde{\Pi}_z(x) \) has a finite second moment when \( L \) is large enough. It therefore makes sense to define

\[
\lambda_z = \frac{1}{1 + z \sigma^2 \sum_x \|x\|^2 \tilde{\Pi}_z(x)}.
\]

We define this parameter below and show that it takes the desired value by Hypothesis A.3.

By (A.12), \( \tilde{\Pi}_z(x) \) has a finite second moment when \( L \) is large enough. It therefore makes sense to define

\[
\lambda_z = 1 - \frac{1}{1 + z \sigma^2 \sum_x \|x\|^2 \tilde{\Pi}_z(x)}.
\]

Equation (A.12) implies \( \lambda_z \to 1 \) as \( L \to \infty \) uniformly in \( z \in [z, z_c] \). By Equation (A.13) and Hypothesis A.3, as \( z \uparrow z_c \), the quantity in brackets in (A.16) tends to zero. Thus, \( \mu_{z_c} \uparrow 1 \) as \( z \uparrow z_c \).

A.4. Other convolution equations. Consider the equation

\[
G_z = \delta + z (D \ast G_z) + (\Pi \ast G_z).
\]

If \( \Pi \) satisfies Hypothesis A.5, it is possible to manipulate (A.17) into the form (A.1). To see this, rewrite (A.17) as

\[
G = \delta + \Pi + zD \ast (\delta + \Pi) \ast G - \Pi \ast (\delta + zD \ast G - G)
\]

where, in the second equality, we have used (A.17) to rewrite the term in parentheses, and the subscripts \( z \) have been omitted. Rewriting the last factor of \( G \) using (A.17) yields

\[
G = \delta + \Pi + \Pi^2 \ast zD \ast (\delta + \Pi + \Pi^2) \ast G \ast \Pi^3 \ast G,
\]

where \( A^{*k} \) is the \( k \)-fold autoconvolution of \( A \). Iterating this procedure yields (A.1) with

\[
\tilde{\Pi}_z = \sum_{k \geq 1} \Pi^* \ast k,
\]

since \( \lim_{n \to \infty} \Pi^{*n} = 0 \) under the assumption that \( \Pi \) satisfies Hypothesis A.5. Finally, [7, Proposition 1.7] implies that, if \( \Pi_z \) satisfies Hypothesis A.5, then \( \tilde{\Pi}_z \) defined by (A.18) satisfies Hypothesis A.5, for possibly different constants. The change in constants depends only on \( d \). See [7, Section 4.1] for a further discussion of this point. Thus to apply Theorem A.6 to the convolution equation (A.17), it suffices to verify the hypotheses of Appendix A.2 for \( G_z, D, \) and \( \Pi \).

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