 Comments on New representations of the Hecke algebra and algebraic Bethe Ansatz for an integrable generalized spin ladder, by H.-Q. Zhou, H. Frahm and M.D. Gould, cond-mat/9911072

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Abstract
The authors of cond-mat/9911072 claim to introduce “new representations of the Hecke algebra.” These representations are shown to be the XXC models introduced two years ago in solv-int/9712008, and repeatedly studied and referred to in subsequent papers. They are in fact representations of the Temperley-Lieb algebra. Several other remarks are made and mistakes are pointed out.

1) The authors of [1] claim to have recently constructed a “novel class of representations of the Hecke algebra.” It is shown here that these representations were already known. They were discovered two years ago and named XXC models [2].

Start from the matrix $\tilde{R}(x)$ defined by (2) and (4) in [1]:

$$\tilde{R}(x) = \tilde{R} - xq^2\tilde{R}^{-1},$$

where

$$\tilde{R} = q \sum_{\alpha \in A, \beta \in B} \left( X^{\alpha\beta} \otimes X^{\beta\alpha} + X^{\beta\alpha} \otimes X^{\alpha\beta} \right) + \left( q^2 + 1 \right) \left( \frac{1}{2} - \sum_{\alpha \in A} X^{\alpha\alpha} \right) \otimes \left( \frac{1}{2} - \sum_{\alpha \in A} X^{\alpha\alpha} \right)$$

$$+ \frac{q^2 - 1}{2} \sum_{\alpha \in A} \left( X^{\alpha\alpha} \otimes I - I \otimes X^{\alpha\alpha} \right) + \left( -\frac{1}{4}q^2 + 3 \right) I \otimes I$$

and the index sets $A$ and $B$ take values from $\{1, 2, \ldots, m\}$ and $\{m+1, m+2, \ldots, n\}$, respectively. The $n \times n$ matrix $X^{\alpha\beta}$ has only one non-vanishing element, a 1 at row $a$ and column $b$. The matrix $X^{\alpha\beta}$ is denoted by $E^{\alpha\beta}$ below.

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Now consider the operators defined in [2]. Let \( n, n_1 \) and \( n_2 \) be three positive integers such that \( n_1 + n_2 = n \), and \( A, B \) be two disjoint sets whose union is the set of basis states of \( \mathbb{C}^n \), with \( \text{card}(A) = n_1 \) and \( \text{card}(B) = n_2 \). Let also

\[
P^{(3)} = \sum_{a \in A} \sum_{\beta \in B} \left( E^{\beta a} \otimes E^{a\beta} + E^{a\beta} \otimes E^{\beta a} \right) \quad (3)
\]

\[
P^{(1)} = P^{(+)} + P^{(-)} = \sum_{a \in A} \sum_{\beta \in B} \left( E^{aa} \otimes E^{\beta\beta} + E^{\beta\beta} \otimes E^{aa} \right) \quad (4)
\]

\[
P^{(2)} = \sum_{a, a' \in A} E^{aa'} \otimes E^{a'a'} + \sum_{\beta, \beta' \in B} E^{\beta\beta'} \otimes E^{\beta'\beta'} \quad (5)
\]

The \( n_1 n_2 \) parameters \( x_{a\beta} \) of [2] were taken all equal to one. Latin indices belong to \( A \) while Greek indices belong to \( B \).

With obvious identifications and \( m \to n_1 \) and \( n_2 = n - m \), a simple and short calculation allows one to find \( \tilde{R}^{-1} \), and to rewrite (1) as follows:

\[
\tilde{R}(x) = q(1 - x)P^{(3)} + (1 - q^2)(xP^{(+)} + P^{(-)}) + (1 - xq^2)P^{(2)} \quad (6)
\]

With the usual passage to additive variable, one obtains the XXC models in their asymmetric form i.e. their Hecke algebra form [3]. A gauge transformation (a special type of similarity transformation) gives the ‘deformed free-fermion’ and Temperley-Lieb forms [2, 3].

2) The XXC models are not just representations of Hecke algebra, they are more accurately representations of the Temperley-Lieb algebra [3]. They have an underlying \( sl(2) \) structure. The natural generalizations to \( sl(m + 1) \) are the multiplicity \( A_m \) models, which include the XXC models [3]. They are representations of the Hecke algebra.

3) The matrix (1), at \( x = 1 \), is more accurately equal to \( (1 - q^2)\mathbb{I} \), and not \( \mathbb{I} \).

4) The Hamiltonian (5) in [4] is exactly the XXC Hamiltonian (18) in [2]. (The boundary terms do not contribute under periodic boundary conditions.) See also (22) in [3]. The \( J \)-term was added by hand, and commutes with all the conserved quantities of the model [3].

5) In equation (13) of [3], \( \Lambda^{(1)} \) is independent of its arguments. After equation (13): “Unfortunately, it \( \tau^{(1)} = tr_s[P_1 \cdots P_{M_s}] \) can not be diagonalized directly in terms of the algebraic Bethe Ansatz.” Rather fortunately, any such unit-shift operator is automatically diagonalized by algebraic Bethe Ansatz (for periodic boundary conditions) every time \( \tilde{R}(1) \) is proportional to the identity operator (i.e. \( \tilde{R} \) is regular). This is one of the most basic aspects of the algebraic Bethe Ansatz in the framework of the QISM. It is enough to consider any regular \( \tilde{R} \)-matrix with 3 states per site. Examples include the spin-1 \( sl(2) \) matrix and the \( sl(3) \) matrix, both trigonometric or rational.

6) Footnote 1: The thermodynamics may be affected by the increased degeneracy of all the states. The phases of certain roots of unity with order proportional to the number of sites, may affect the finite-size corrections. All this requires careful checks.

7) The “mapping” of [3] is misleading and meaningless. Most Bethe Ansatz equations look alike and are determined by the Dynkin diagram of the Lie algebra and weight of the representation at hand. It necessary to complement them with the eigenvalues and eigenvectors.
to which they refer, and to take into account the degeneracies and the specific features of the model. The mapping to the Perk-Shultz models is false.

Case in point: The Bethe Ansatz equations (16) in [1] are simply wrong.

References

[1] H.-Q. Zhou, H. Frahm and M.D. Gould, *New representations of the Hecke algebra and algebraic Bethe Ansatz for an integrable generalized spin ladder*, cond-mat/9911072.

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[3] Z. Maassarani, Eur. Phys. J. B 7 (1999) 627–633, solv-int/9805009. Erratum 9 (1999) 371.