Towards a stochastic multi-point description of turbulence

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Abstract. In previous work it was found that the multi-scale statistics of homogeneous isotropic turbulence can be described by a stochastic “cascade” process of the velocity increment from scale to scale, which is governed by a Fokker-Planck equation. We now show how this description for increments can be extended in order to obtain the complete multi-point statistics in real space of the turbulent velocity field (Stresing & Peinke, 2010). We extend the stochastic cascade description by conditioning on an absolute velocity value itself, and find that the corresponding conditioned process is also governed by a Fokker-Planck equation, which contains as a leading term a simple additional velocity-dependent coefficient, $d_{10}$, in the drift function. Taking the velocity-dependence of the Fokker-Planck equation into account, the multi-point statistics in the inertial range can be expressed by two-scale statistics of velocity increments, which are equivalent to three-point statistics of the velocity field. Thus, we propose a stochastic three-point closure for the velocity field of homogeneous isotropic turbulence. Investigating the coefficient $d_{10}$ for different flows, we find clear evidence that the multipoint structure of small scale turbulence is not universal but depends on the type of the flow.

1. Introduction

Standard statistical analysis of small-scale turbulence is based on two-point correlations and their dependence on the distance $r$ between the two points. A central quantity is the longitudinal velocity increment $\xi(r)$,

$$\xi(r) = u(x + r) - u(x),$$

here $u$ denotes the component of the velocity in the direction defined by $x$ and $x + r$. The study of all so-called structure functions ($\xi(r)^N$) is equivalent to the study of the probability density function (PDF) $p(\xi(r))$. However, as there are infinite different possible interscale processes to generate the same one scale $p(\xi(r))$ statistics, a full multiscale characterization is needed in terms of the multiscale PDF $p(\xi_1, \xi_2, ..., \xi_N)$ where $\xi_i \equiv \xi(r_i)$. (Here we define the increments $\xi_i$ on different scales in such a way that they have one common value $u(x)$ at $x$. This multiscale PDF goes beyond the traditional analysis based on structure functions as $p(\xi(r))$ can be deduced by integrations. If the stochastic process for the evolution of the velocity increments in scale has the Markov property, i.e. if

$$p(\xi_i|\xi_{i+1}, \xi_{i+2}, ..., \xi_N) = p(\xi_i|\xi_{i+1}),$$

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where without loss of generality we assume \( r_1 < r_2 < \ldots < r_N \), the multiscale PDF \( p(\xi_1, \xi_2, \ldots, \xi_N) \) can be expressed by a product of conditional PDFs \( p(\xi_i|\xi_{i+1}) \),

\[
p(\xi_1, \xi_2, \ldots, \xi_N) = p(\xi_1|\xi_2) \cdot p(\xi_2|\xi_3) \cdot \ldots \cdot p(\xi_{N-1}|\xi_N) \cdot p(\xi_N).
\]  

(3)

The stochastic process for these conditional PDFs can be described by a Kramers-Moyal expansion. If the Kramers-Moyal coefficient of fourth order, \( D^{(4)} \), is zero, the expansion truncates after the second term and becomes a Fokker-Planck equation (also known as Kolomogorov equation) (Risken, 1996)

\[
-r \frac{\partial}{\partial r} p(\xi(r)|\xi_0(r_0)) = -\frac{\partial}{\partial \xi} \left[D^{(1)}(\xi, r) p(\xi(r)|\xi_0(r_0))\right] + \frac{\partial^2}{\partial \xi^2} \left[D^{(2)}(\xi, r) p(\xi(r)|\xi_0(r_0))\right],
\]

(4)

with \( r_0 > r \). The drift and diffusion functions \( D^{(1)} \) and \( D^{(2)} \) can be estimated point wise as Kramers-Moyal coefficients

\[
D^{(k)}(\xi, r) = \lim_{\Delta r \to 0} \frac{r}{k! \Delta r} \int_{-\infty}^{+\infty} (\hat{\xi} - \xi)^k p(\xi(r - \Delta r)|\xi(r)) \, d\hat{\xi},
\]

(5)

directly from the experimental data, as soon as one has estimated the conditioned probabilities \( p(\xi(r - \Delta r)|\xi(r)) \) for the two scales \( r \) and \( r - \Delta r \).

It has been shown for several different flows that

- the process has the Markov property for step sizes \( \Delta r \geq l_{EM} \), where \( l_{EM} \), the so-called Einstein-Markov coherence length, is of the order of magnitude of the Taylor microscale, \( l_{EM} \approx 0.8 \lambda \);
- the fourth-order Kramers-Moyal coefficient \( D^{(4)} \) vanishes or is small enough to be neglected, and
- the experimental (conditional) PDFs of the velocity increments can be reproduced by integration of the Fokker-Planck equation, including intermittency effect,

for details see (Friedrich & Zeller & Peinke, 1998; Renner & Peinke & Friedrich, 2001; Renner & Peinke & Friedrich & Chanal & Chabaud, 2002; Lück & Renner & Peinke & Friedrich, 2006; Stresing & Peinke & Seoud & Vassilicos, 2010).

Nawroth & Peinke (2006) noticed that this multi-scale description can also be used for the generation of synthetic time series with the same statistical multi-scale properties as, for example, a given turbulent velocity time series, or for the prediction of financial time series (Nawroth & Friedrich & Peinke, 2010). They also noticed that it is necessary to take into account the statistics of the velocity itself in order to obtain stationary synthetic data. In the present work, we want to go beyond this approach and show precisely how the joint \( N \)-scale PDF \( p(\xi_1, \xi_2, \ldots, \xi_N) \) is related to the joint \((N+1)\)-point PDF \( p(u(x), u(x + r_1), \ldots, u(x + r_N)) \). As a consequence of the Markov property of the velocity increments, the multi-point statistics can be obtained based on the knowledge of the velocity field at three points. Thus we propose a stochastic three-point closure for homogeneous isotropic turbulence (for the above defined \( u \) component in one direction).

2. From multi-scale to multi-point statistics

We consider the \((N+1)\)-point statistics of the velocity values \( u_i \) at the location \( x_i \); \( u_{i+1} \) at \( x_{i+1} \)

\[
p(u_i, u_{i+1}, \ldots, u_{i+N}) = p(\xi_1, \ldots, \xi_N, u_i) = p(\xi_1, \ldots, \xi_N|u_i) \cdot p(u_i).
\]

(6)
Let us assume that the Markov property of the interscale process (2) is conserved when the process is conditioned on the velocity, i.e. let us assume that

\[ p(\xi_j|\xi_{j+1}, \xi_{j+2}, ..., \xi_N, u_i) = p(\xi_j|\xi_{j+1}, u_i). \]  

(7)

Then we obtain the following factorization of the multipoint joint PDF, similar to (3):

\[ p(u_i, u_{i+1}, ..., u_{i+N}) = p(\xi_1|\xi_2, u_i) \cdots p(\xi_{N-1}|\xi_N, u_i) p(\xi_N, u_i). \]  

(8)

If the fourth-order Kramers-Moyal coefficient vanishes \( D^{(4)}(\xi, r) = 0 \), we obtain a Fokker-Planck equation for the evolution of the conditional PDF \( p(\xi_j|\xi_k, u_i) \), with \( r_k > r_j \), similar to equ. (4):

\[ -r_j \frac{\partial}{\partial r_j} p(\xi_j|\xi_k, u_i) = -\frac{\partial}{\partial \xi_j} \left[ D^{(1)}(\xi_j, r_j, u_i)p(\xi_j|\xi_k, u_i) \right] + \frac{\partial^2}{\partial \xi_j^2} \left[ D^{(2)}(\xi_j, r_j, u_i)p(\xi_j|\xi_k, u_i) \right]. \]  

(9)

If this Fokker Planck equation is known, i.e. we know the drift and diffusion terms \( D^{(1)}(\xi, r) \) and \( D^{(2)}(\xi, r) \), all conditioned probabilities and thus the multipoint joint PDFs are known in the most general. Equ. (8) is the three point closure, too.

3. Experimental results

Next we present experimental evidence for the validity of condition (7) for step sizes \( \Delta r \equiv r_k - r_j \geq l_{EM} \) (Fig. 1) and show that the coefficient \( D^{(4)}(\xi_j, r_j, u_i) \) in fact can be neglected (Fig. 2c). Thus, equation (9) is valid.

We analyze hot-wire measurement data from a wake flow behind a cylinder with diameter \( D = 2 \) cm in a wind tunnel at downstream distance \( x = 100 \) D, for Taylor microscale Reynolds numbers \( R_\lambda = 338 \). This data is characterised by a dissipation scale \( \eta = 0.10 \) mm, Taylor microscale \( \lambda = 3.7 \) mm, and integral scale \( L = 119 \) mm. Further results for other experimental flow can be found in (Stresing & Peinke, 2010).

First we show evidence of the Markov properties. In order to verify the validity of condition (7), we apply the (Mann-Whitney-)Wilcoxon test (Wilcoxon, 1945; Mann & Whitney, 1947; Renner & Peinke & Friedrich, 2001), which tests whether or not two samples of different sizes have the same statistical distribution. Since the two distributions of (7) are necessarily of different sizes, the Wilcoxon test is an appropriate method to estimate its validity. Fig. 1 shows the results of the Wilcoxon test for different values of \( u_i \) and \( \xi_j \). In the present implementation of the Wilcoxon test, a statistical test value \( \langle \Delta Q^* \rangle \) is computed, which must be close to one for acceptance of the hypothesis expressed by (7). For sufficient large scale distances \( \Delta r > l_{EM} \approx 0.6 \lambda \), the values \( \langle \Delta Q^* \rangle \) in Fig. 1 are one, except for some inevitable scattering. Thus, the Markov property of the interscale process is valid for the additional condition of \( u_i \), and (7) and (8) do apply.

We examine the empirical dependence of the new drift and diffusion functions, \( D^{(1)}(\xi, r, u) \) and \( D^{(2)}(\xi, r, u) \), on the velocity \( u \) (we skip the indices of \( \xi, r, \) and \( u \)). As shown in Fig. 2, \( D^{(1)} \) is basically shifted vertically as a function of \( u \), and \( D^{(2)} \) depends very little on \( u \). The drift and diffusion functions can be approximated by simple polynomials in \( \xi \), with \( r \)-dependent coefficients \( d_{ij} \)

\[ D^{(1)}(\xi, r, u) \approx d_{10}(r, u) - d_{11}(r)\xi, \quad D^{(2)}(\xi, r, u) \approx d_{20}(r) - d_{21}(r)\xi + d_{22}(r)\xi^2. \]  

(10)

Only the coefficient \( d_{10} \) depends on \( u \) and can be approximated by a second-order polynomial in \( r \),

\[ d_{10}(r, u) = d_{101}(u) \frac{r}{\lambda} + d_{102}(u) \left( \frac{r}{\lambda} \right)^2. \]  

(11)
It was found in previous works that the coefficients of the Fokker-Planck equation for the interscale process are not universal, but depend on the Reynolds number (Renner & Peinke & Friedrich & Chanal & Chabaud, 2002) and/or flow geometry (Stresing & Peinke & Seoud & Vassilicos, 2010). Based on the analysis of data from different flows, namely free jet flows and the wake flows behind a cylinder and a fractal grid, we find a similar result for the velocity-
dependent coefficient $d_{10}$, which does depend on the flow type as shown in Fig. 3 (for further details see (Stresing & Peinke, 2010)). (Typically the diffusion term $D^{(2)}$ shows the Reynolds number dependency.)

\begin{align*}
R_\lambda &= 86 \\
R_\lambda &= 163 \\
R_\lambda &= 260 \\
R_\lambda &= 338 \\
R_\lambda &= 175 \\
R_\lambda &= 366 \\
R_\lambda &= 740
\end{align*}

**Figure 3.** Coefficient $d_{10}$ according to (11), at different Reynolds numbers $R_\lambda$ given in the legends. (a): cylinder wake, (b): free jet, (c): fractal square grid. The horizontal error bars shown for the highest Reynolds number data represent the range (bin size) of values of $u$ used for the estimation of the drift. The vertical error bars are the errors of the estimated parameter $d_{10}$. After (Stresing & Peinke, 2010)

4. Conclusion

As a consequence of the Markov property (7), the multi-point joint PDF of the velocity, $p(u(x), u(x + r_1), \ldots, u(x + r_N))$, can be expressed as a product of three-point PDFs $p(u(x + r_1)|u(x + r_2), u(x))$, which are equivalent to the conditional increment PDFs $p(\xi(r_1)|\xi(r_2), u(x))$. Thus, a stochastic three-point closure for the turbulent velocity is given. The PDFs $p(\xi(r_1)|\xi(r_2), u(x))$, in turn, are described by the Fokker-Planck equation (9), which displays a simple dependence on the velocity $u$. Details of the Fokker-Planck equation show experimental evidence that also in small scale turbulence some informations of the large scale structures are present. Thus with respect to higher order and higher point statistics turbulence seems not to be universal.

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