Finding Points in General Position

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Abstract

We study the General Position Subset Selection problem: Given a set of points in the plane, find a maximum-cardinality subset of points in general position. We prove that General Position Subset Selection is NP-hard, APX-hard, and present several fixed-parameter tractability results for the problem.

1 Introduction

For a set $P = \{p_1, \ldots, p_n\}$ of $n$ points in the plane, a subset $S \subseteq P$ is in general position if no three points in $S$ are collinear (that is, lie on the same line). A frequent assumption for point set problems in computational geometry is that the given point set is in general position. Nevertheless, the problem of computing a maximum-cardinality subset of points in general position from a given set of points has received little attention from the computational complexity perspective, although not from the combinatorial geometry perspective. In particular, to the best of our knowledge, the classical complexity of the aforementioned problem is unresolved. Formally, the decision version of the problem is:

\textbf{General Position Subset Selection}

\textbf{Input:} A set $P$ of points in the plane and $k \in \mathbb{N}$.

\textbf{Question:} Is there a subset $S \subseteq P$ in general position of cardinality at least $k$?

A well-known special case of General Position Subset Selection, referred to as the No-Three-In-Line problem, asks to place a maximum number of points in general position on an $n \times n$-grid. Since at most two points can be placed on any grid-line, the maximum number of points in general position that can be placed on an $n \times n$ grid is at most $2n$. Indeed, only for small $n$ it is known that $2n$ points can always be placed on the $n \times n$ grid. Erdős [24] observed that, for sufficiently large $n$, one can place $(1 - \epsilon)n$ points in general position on the $n \times n$ grid, for any $\epsilon > 0$. This lower bound was improved by Hall et al. [17] to $\left(\frac{2}{3} - \epsilon\right)n$. It was conjectured by Guy and Kelly [16] that, for sufficiently large $n$, one can place more than $\frac{\pi \sqrt{3}}{2} n$ many points in general position on the $n \times n$ grid. This conjecture remains unresolved, hinting at the challenging combinatorial nature of No-Three-In-Line, and hence of General Position Subset Selection as well.

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A problem closely related to General Position Subset Selection is Point Line Cover: Given a point set in the plane, find a minimum-cardinality set of lines, the size of which is called the line cover number, that cover all points. While Point Line Cover has been intensively studied, we aim to fill the existing gap for General Position Subset Selection by providing both computational hardness and fixed-parameter tractability results for the problem. In doing so, we particularly consider the parameters solution size $k$ (size of the sought subset in general position) and its dual $h := n - k$, and investigate their impact on the computational complexity of General Position Subset Selection.

**Related Work.** A general account on applying methods from parameterized complexity analysis to problems from computational geometry is due to Giannopoulos et al. [13]. Payne and Wood [23] provide improved lower bounds on the size of a point set in general position, a question originally studied by Erdős [9]. In his master’s thesis, Cao [4] gives a problem kernel of $O(k^4)$ points for General Position Subset Selection (there called Non-Collinear Packing problem) and a simple greedy $O(\sqrt{\text{opt}})$-factor approximation algorithm for the maximization version. He also presents an Integer Linear Program formulation and shows that it is in fact the dual of an Integer Linear Program formulation for Point Line Cover. As to results for the much more studied Point Line Cover, we refer to the work of Kratsch et al. [21] and the work cited therein.

**Our Contributions.** We show that General Position Subset Selection is NP-hard and APX-hard. Our main algorithmic results, however, concern the power of polynomial-time data reduction for General Position Subset Selection: We give an $O(k^3)$-point problem kernel and an $O(h^2)$-point problem kernel, and show that the latter kernel is asymptotically optimal under a reasonable complexity-theoretic assumption. Table 1 summarizes our results.

2 Preliminaries

**Geometry.** All coordinates of points are assumed to be represented by rational numbers. The collinearity of a set of points $P$ is the maximum number of points in $P$ that lie on the same line. A blocker for two points $p, q$ is a point on the open line segment $pq$.

**Graphs.** Let $G = (V(G), E(G))$ be an undirected graph. We write $|G|$ for $|V(G)| + |E(G)|$. A vertex $u \in V(G)$ is a neighbor of (or is adjacent to) a vertex $v \in V(G)$ if $\{u, v\} \in E(G)$. The degree of a vertex $v$ is the number of its neighbors.

An independent set of a graph $G$ is a set of vertices such that no two vertices in this set are adjacent. A maximum independent set is an independent set of maximum cardinality. A vertex cover of $G$ is a set of vertices such that each edge in $G$ is incident to at least one vertex in this set. The NP-complete Independent Set problem is: Given a graph $G$ and $k \in \mathbb{N}$, decide whether $G$ has an independent set of cardinality $k$. The Maximum Independent Set problem is the optimization version of Independent Set, which is the problem of computing an independent set of maximum cardinality in a given graph. The NP-complete Vertex Cover problem is: Given a graph $G$ and $k \in \mathbb{N}$, decide whether $G$ has a vertex cover of cardinality $k$. 

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Table 1: Overview of the results we obtain for General Position Subset Selection, where \( n \) is the number of input points, \( k \) is the parameter size of the sought subset in general position, \( h = n - k \) is the dual parameter, and \( \ell \) is the line cover number.

| Result | Reference |
|--------|-----------|
| | |
| Hardness | |
| NP-hard | Theorem 1 |
| APX-hard | Theorem 1 |
| no \( 2^{\Theta(n)} \cdot n^{O(1)} \)-time algorithm\(^a\) | Theorem 1 |
| no \( O(h^{2-\epsilon}) \)-point kernel\(^b\) | Theorem 5 |

\(^a\)Unless the Exponential Time Hypothesis fails. \(^b\)Unless coNP \( \subseteq \) NP/poly.

Parameterized Complexity. A parameterized problem is a set of instances of the form \((I, k)\), where \( I \in \Sigma^* \) for a finite alphabet set \( \Sigma \), and \( k \in \mathbb{N} \) is the parameter. A parameterized problem \( Q \) is fixed-parameter tractable, shortly FPT, if there exists an algorithm that on input \((I, k)\) decides whether \((I, k)\) is a yes-instance of \( Q \) in \( f(k)|I|^{O(1)} \) time, where \( f \) is a computable function independent of \( |I| \). A parameterized problem \( Q \) is kernelizable if there exists a polynomial-time algorithm that maps an instance \((I, k)\) of \( Q \) to another instance \((I', k')\) of \( Q \) such that:

1. \( |I'| \leq \lambda(k) \) for some computable function \( \lambda \),
2. \( k' \leq \lambda(k) \), and
3. \((I, k)\) is a yes-instance of \( Q \) if and only if \((I', k')\) is a yes-instance of \( Q \).

The instance \((I', k')\) is called problem kernel of \((I, k)\).

A parameterized problem is FPT if and only if it is kernelizable [3]. Refer to the books by Cygan et al. [5] and Downey and Fellows [7] for more details on parameterized complexity.

Approximation. A polynomial-time approximation scheme (PTAS) for a maximization problem \( Q \) is an algorithm \( A \) that takes an instance \( I \) and a constant \( \rho > 1 \) and returns in \( O(n^{f(\rho)}) \) time for some computable function \( f \) a solution of value at least \( \text{opt}(I)/\rho \), where \( \text{opt}(I) \) denotes the value of an optimal solution of \( I \). A maximization problem that is APX-hard does not admit a PTAS unless \( P = NP \). Refer to the books by Vazirani [26] and Williamson and Shmoys [28] for a more comprehensive discussion of approximation algorithms.

Exponential Time Hypothesis. The Exponential Time Hypothesis (ETH) [19] states that 3-SAT cannot be solved in \( 2^{o(n)} \cdot n^{O(1)} \) time, where \( n \) is the number of variables in the input formula.
3 Hardness Results

In this section, we prove that General Position Subset Selection is NP-hard, APX-hard, and presumably not solvable in subexponential time. Our hardness results follow from a transformation (mapping arbitrary graphs to point sets) that is based on a construction due to Ghosh and Roy [12, Section 5], which they used to prove the NP-hardness of the Independent Set problem on so-called point visibility graphs. This transformation, henceforth called \( \Phi \), allows us to transfer all the above-mentioned hardness results from NP-hard restrictions of Independent Set to General Position Subset Selection. Moreover, in Section 4.2, we will use \( \Phi \) to give a reduction from Vertex Cover to General Position Subset Selection in order to obtain kernel size lower bounds with respect to the dual parameter (see Theorem 4 and Theorem 5). We start by formally defining some properties that are required for the output point set of the transformation. As a next step, we prove that such a point set can be realized in polynomial time.

Let \( G \) be a graph with vertex set \( V(G) = \{v_1, \ldots, v_n\} \). Let \( C = \{p_1, \ldots, p_n\} \) be a set of points that are in strictly convex position (that is, the points in \( C \) are vertices of a convex polygon), where \( p_i \in C \) corresponds to \( v_i \), \( i = 1, \ldots, n \). For each edge \( e = \{v_i, v_j\} \in E(G) \), we place a blocker \( b_e \) on the line segment \( p_ip_j \) such that the following three conditions are satisfied:

(I) For any edge \( e \in E(G) \) and for any two points \( p_i, p_j \in C \), if \( b_e, p_i, p_j \) are collinear, then \( p_i, p_j \) are the points in \( C \) corresponding to the endpoints of edge \( e \).

(II) Any two distinct blockers \( b_e, b_e' \) are not collinear with any point \( p_i \in C \).

(III) The set \( B := \{b_e \mid e \in E(G)\} \) of blockers is in general position.

\[ \text{Proposition 1. There is a polynomial-time transformation } \Phi \text{ mapping arbitrary graphs to point sets that satisfy Conditions (I) to (III). Moreover, no four points in the point set } C \cup B \text{ produced by } \Phi \text{ are collinear.} \]

\[ \text{Proof. Given a graph } G, \text{ let } n = |V(G)| \text{ and let } C = \{p_1, \ldots, p_n\} \text{ be a set of rational points that are in a strictly convex position; for instance, let } p_j := \left( \frac{2j}{1+j^2}, \frac{1-j^2}{1+j^2} \right) \text{ for } j \in \{1, \ldots, n\} \text{ be } n \text{ rational points on the unit circle centered at the origin [25]. To choose the set } B \text{ of blockers, suppose (inductively) that we have chosen a subset } B' \text{ of blockers such that all blockers in } B' \text{ are rational points and satisfy Conditions (I) to (III). Let } b_e \not\in B' \text{ be a blocker corresponding to an edge } e = \{v_i, v_j\} \text{ in } G. \text{ To determine the coordinates of } b_e, \text{ we first mark the intersection points (if any) between the line segment } p_ip_j \text{ and the lines formed by every pair of distinct blockers in } B', \text{ every pair of distinct points in } C \setminus \{p_i, p_j\}, \text{ and every pair consisting of a blocker in } B' \text{ and a point in } C \setminus \{p_i, p_j\}. \text{ We then choose } b_e \text{ to be an interior point of } p_ip_j \text{ with rational coordinates that is distinct from all marked points. To this end, let } q \text{ be the first marked point on the segment } p_ip_j \text{ (starting from } p_i), \text{ and let } b_e \text{ be the midpoint of } p_iq. \text{ This point is rational since it is the midpoint of rational points. It is easy to see that } C \cup B \text{ can be constructed in polynomial time and that all points in } C \cup B \text{ are rational and satisfy Conditions (I) to (III). Moreover, it easily follows from the construction of } C \cup B \text{ that it satisfies Conditions (I) to (III) and that no four points in } C \cup B \text{ are collinear.} \]

In what follows, we will use transformation \( \Phi \) as a reduction from Independent Set to General Position Subset Selection in order to prove our hardness results. The following observation will be helpful in proving the correctness of the reduction.
Observation 1. Let $G$ be an arbitrary graph, and let $P := \Phi(G) = C \cup B$. For any point set $S \subseteq P$ that is in general position, there is a general position set of size at least $|S|$ that contains the set of blockers $B$.

Proof. Suppose that $S \subseteq P$ is in general position, and suppose that there is a point $b \in B \setminus S$. If $b$ does not lie on a line defined by any two points in $S$, then $S \cup \{b\}$ is in general position. Otherwise, $b$ lies on a line defined by two points $p, q \in S$. By Conditions (I) and (II), it holds that $p, q \in C$. Moreover, $p$ and $q$ are the only two points in $S$ that are collinear with $b$. Hence, we exchange one of them with $b$ to obtain a set of points in general position of the same cardinality as $S$. Since $b \in B$ was arbitrarily chosen, we can repeat the above argument to obtain a subset in general position of cardinality at least $|S|$ that contains $B$. 

Using 1, we can prove a polynomial-time many-one reduction from Independent Set to General Position Subset Selection based on transformation $\Phi$.

Lemma 1. There is a polynomial-time many-one reduction from Independent Set to General Position Subset Selection. Moreover, each instance of General Position Subset Selection produced by this reduction satisfies the property that no four points in the instance are collinear.

Proof. Let $(G, k)$ be an instance of Independent Set, where $k \in \mathbb{N}$. The General Position Subset Selection instance is defined as $(P := \Phi(G), k + |E(G)|)$. Clearly, by Proposition 1, the set $P$ can be computed in polynomial time, and no four points in $P$ are collinear. We show that $G$ has an independent set of cardinality $k$ if and only if $P$ has a subset in general position of cardinality $k + |E(G)|$.

Suppose that $I \subseteq V(G)$ is an independent set of cardinality $k$, and let $S := \{p_i \mid v_i \in I\} \cup B$, where $B$ is the set of blockers in $P$. Since $|B| = |E(G)|$, we have $|S| = k + |E(G)|$. Suppose towards a contradiction that $S$ is not in general position, and let $q, r, s$ be three distinct collinear points in $S$. By Conditions (II) and (III), and since the points in $C$ are in a strictly convex position, it follows that exactly two of the points $q, r, s$ must be in $C$. Suppose, without loss of generality, that $q = p_i, r = p_j \in C$ and $s \in B$. By Condition (I), there is an edge between the vertices $v_i$ and $v_j$ in $G$ that correspond to the points $p_i, p_j \in C$, contradicting that $v_i, v_j \in I$. It follows that $S$ is a subset in general position of cardinality $k + |E(G)|$.

Conversely, assume that $S \subseteq P$ is in general position and that $|S| = k + |E(G)|$. By 1, we may assume that $B \subseteq S$. Let $I$ be the set of vertices corresponding to the points in $P \setminus B$, and note that $|I| = k$. Since $B \subseteq S$, no two points $v_i, v_j$ in $I$ can be adjacent; otherwise, their corresponding points $p_i, p_j$ and the blocker of edge $\{v_i, v_j\}$ would be three collinear points in $S$. It follows that $I$ is an independent set of cardinality $k$ in $G$.

Lemma 1 implies the NP-hardness of General Position Subset Selection. Furthermore, a careful analysis of the proof of Lemma 1 reveals the intractability of an extension variant of General Position Subset Selection, where as an additional input to the problem a subset $S' \subseteq P$ of points in general position is given and the task is to find $k$ additional points in general position, that is, one looks for a point subset $S \subseteq P$ in general position such that $S' \subset S$ and $|S| \geq |S'| + k$. By 1, we can assume for the instance created by transformation $\Phi$ that $B$ (the set of blockers) is contained in a maximum-cardinality point subset in general position. Thus, we can set $S' := B$. The proof of Lemma 1 then shows that $k$ points can be added to $S'$ if and only if the graph $G$ contains an independent set of size $k$. Since Independent Set is $W[1]$-hard with respect to the solution size [7], it follows
that the described extension variant of General Position Subset Selection is W[1]-hard with respect to the solution size \(k\), and this extension is not fixed-parameter tractable with respect to \(k\), unless W[1] = FPT. The reader may want to contrast the W[1]-hardness result for the aforementioned extension version of General Position Subset Selection with the fixed-parameter tractability results for General Position Subset Selection shown in Section 4.1.

Next, we turn our attention to approximation. A closer inspection of transformation \(\Phi\) reveals that we can obtain a PTAS-reduction from Maximum Independent Set to the optimization version of General Position Subset Selection.

**Definition 1.** Given two maximization problems \(Q\) and \(Q'\), a PTAS-reduction from \(Q\) to \(Q'\) consists of three polynomial-time computable functions \(h, h'\) and \(\alpha : Q \to (1, \infty)\) such that:

1. For any instance \(I\) of \(Q\) and for any constant \(\rho > 1\), \(h\) produces an instance \(I' = h(I, \rho)\) of \(Q'\).
2. For any solution \(x'\) of \(I'\) and for any \(\rho > 1\), \(h'\) produces a solution \(x = h'(I, x', \rho)\) of \(I\) such that:
   \[
   \frac{\text{opt}(I')}{|x'|} \leq \alpha(\rho) \Rightarrow \frac{\text{opt}(I)}{|x|} \leq \rho.
   \]

By IS-3 we denote the Maximum Independent Set problem restricted to graphs of maximum degree at most 3. By Maximum General Position Subset Selection we denote the optimization version of General Position Subset Selection in which one seeks to compute a largest subset of points in general position in a given point set.

**Lemma 2.** There is a PTAS-reduction from IS-3 to Maximum General Position Subset Selection.

**Proof.** Let \(G\) be an instance of IS-3, and note that \(|E(G)| \leq 3|V(G)|/2\). It is easy to see that \(G\) has an independent set of cardinality at least \(|V(G)|/4\) that can be obtained by repeatedly selecting a vertex in \(G\) of minimum degree and discarding all its neighbors until the graph is empty. We define the computable functions \(h, h'\) and \(\alpha\) in Definition 1 as follows. The function \(h\), on input \((G, \rho)\), outputs \(P := \Phi(G)\); by Proposition 1, \(h\) is computable in polynomial time. Let \(S\) be a subset in general position in \(P\). By 1, there is a subset in general position \(S'\) of cardinality at least \(|S|\) that contains \(B\). We may assume that \(|S'| \geq |B| + |V(G)|/4\); this assumption is justified because \(G\) has an independent set of cardinality at least \(|V(G)|/4\), and hence, by the proof of Lemma 1, \(P\) has a subset in general position of cardinality at least \(|B| + |V(G)|/4\), which we may assume to contain \(B\) by 1. By the proof of Lemma 1, \(G\) has an independent set \(I\) of cardinality \(|S'\) − \(|B| \geq |V(G)|/4\). We define \(h'(G, S', \rho) := I\), which is clearly polynomial-time computable. Finally, we define \(\alpha(\rho) := (\rho + 6)/7\).

Let \(\text{opt}(G)\) denote the cardinality of a maximum independent set in \(G\), and let \(\text{opt}(P)\) be the cardinality of a largest subset in general position in \(P\). From Lemma 1, it follows that \(\text{opt}(P) = |B| + \text{opt}(G)\). Let \(S\) be an approximate solution to \(P\), and by the discussion above, we may assume that \(S\) contains \(B\) and is of cardinality at least \(|B| + |V(G)|/4\). Let \(I = h'(G, S, \rho)\), and note that \(|I| \geq |V(G)|/4\). To finish the proof, we need to show that if \(\text{opt}(P)/|S| \leq (\rho + 6)/7\), then \(\text{opt}(G)/|I| \leq \rho\). In effect, after noting that \(|B| = |E(G)| \leq \)
3|V(G)|/2 and |I| ≥ |V(G)|/4, we have:

\[
\frac{\text{opt}(P)}{|S|} \leq \frac{\rho + 6}{7} \\
\iff \frac{|B| + \text{opt}(G)}{|B| + |I|} \leq \frac{\rho + 6}{7} \\
\iff \frac{\text{opt}(G)}{|I|} \leq \frac{\rho + 6}{7} \cdot \frac{|B| + |I|}{|I|} = \frac{\rho + 6}{7} \cdot \left( \frac{|B|}{|I|} + 1 \right) - \frac{|B|}{|I|} \\
= \frac{\rho - 1}{7} \cdot |B| + \frac{\rho + 6}{7} \\
\leq \frac{\rho - 1}{7} \cdot 6 + \frac{\rho + 6}{7} \\
= \rho.
\]

Finally, we prove that transformation Φ yields a polynomial-time reduction from IS-3 to Maximum General Position Subset Selection, where the number of points in the point set depends linearly on the number of vertices in the graph. This implies an exponential-time lower bound based on the Exponential Time Hypothesis (ETH) [18].

**Lemma 3.** There is a polynomial-time reduction from IS-3 to General Position Subset Selection mapping a graph G to a point set P of size \(O(|V(G)|)\).

**Proof.** For an instance \(G\) of IS-3, the set \(P := \Phi(G)\) is of cardinality \(|P| = |V(G)| + |E(G)| \leq |V(G)| + 3|V(G)|/2 \in O(|V(G)|)\). By the proof of Lemma 1, Φ is a polynomial-time reduction.

We summarize the consequences of Lemmas 1 to 3 in the following theorem:

**Theorem 1.** The following are true:

(a) General Position Subset Selection is NP-complete.
(b) Maximum General Position Subset Selection is APX-hard.
(c) Unless ETH fails, General Position Subset Selection is not solvable in \(2^{o(n)} \cdot n^{O(1)}\) time.

We note that parts (a)–(c) above even hold for the restriction of General Position Subset Selection to instances in which no four points are collinear.

**Proof.** Part (a) follows from the NP-hardness of Independent Set [11], combined with Proposition 1 and Lemma 1 (membership in NP trivially holds). Part (b) follows from the APX-hardness of IS-3 [1], combined with Proposition 1 and Lemma 2. Concerning Part (c), it is well known that, unless ETH fails, Maximum Independent Set is not solvable in subexponential time [19], and the same is true for IS-3 by the results of Johnson and Szegedy [20]. Hence, by the reduction in Lemma 3, General Position Subset Selection cannot be solved in subexponential time since this would imply a subexponential-time algorithm for IS-3.
Parts (a)–(c) remain true for the restriction of General Position Subset Selection to instances in which no four points are collinear because the point set produced by transformation $\Phi$ satisfies this property (see Proposition 1).

Currently, the best approximation result for Maximum General Position Subset Selection is due to Cao [4], who provided a simple greedy $\sqrt{\text{opt}}$-factor approximation algorithm. Therefore, a large gap remains between the proven upper and the lower bound on the approximation factor.

4 Fixed-Parameter Tractability

In this section, we prove several fixed-parameter tractability results for General Position Subset Selection. In Section 4.1 we develop cubic problem kernels with respect to the parameter size $k$ of the sought subset in general position, and with respect to the line cover number $\ell$. In Section 4.2, we show a quadratic problem kernel with respect to the dual parameter $h := n - k$, that is, the number of points whose deletion leaves a set of points in general position. Moreover, we prove that this problem kernel is essentially optimal, unless an unlikely collapse in the polynomial hierarchy occurs.

4.1 FPT Results for the Parameter Solution Size $k$

Let $(P, k)$ be an instance of General Position Subset Selection, and let $n = |P|$. Cao [4] gave a problem kernel for General Position Subset Selection of size $O(k^4)$ based on the following idea. Suppose that there is a line $L$ containing at least $\left(\frac{k-2}{2}\right) + 2$ points from $P$. For any subset $S' \subset P$ in general position with $|S'| = k - 2$, there can be at most $\left(\frac{k-2}{2}\right)$ points on $L$ such that each is collinear with two points in $S'$. Hence, we can always find at least two points on $L$ that together with the points in $S'$ form a subset $S$ in general position of cardinality $k$. Based on this idea, Cao [4] introduced the following data reduction rule:

**Rule 1 ([4]).** Let $(P, k)$ be an instance of General Position Subset Selection. If there is a line $L$ that contains at least $\left(\frac{k-2}{2}\right) + 2$ points from $P$, then remove all the points on $L$ and set $k := k - 2$.

Cao showed that Rule 1 can be exhaustively applied in $O(n^3)$ time ([4, Lemma B.1.]), and he showed its correctness, that is, an instance $(P', k')$ that is reduced with respect to Rule 1 is a yes-instance of General Position Subset Selection if and only if $(P, k)$ is ([4, Theorem B.2.]). Using Rule 1, he gave a kernel for General Position Subset Selection of size $O(k^4)$ that is computable in $O(n^3)$ time ([4, Theorem B.3.]). We shall improve on Cao’s result, both in terms of the kernel size and the running time of the kernelization algorithm. We start by showing how, using a result by Guibas et al. [15, Theorem 3.2], Rule 1 can be applied exhaustively in $O(n^2 \log n)$ time.

**Lemma 4.** Given an instance $(P, k)$ of General Position Subset Selection where $|P| = n$, in $O(n^2 \log n)$ time we can compute an equivalent instance $(P', k')$ such that either $(P', k')$ is a trivial yes-instance, or the collinearity of $P'$ is at most $\left(\frac{k-2}{2}\right) + 1$.

**Proof.** Let $\lambda = \left(\frac{k-2}{2}\right) + 2$. We start by computing the set $L$ of all lines that contain at least $\lambda$ points from $P$. By a result of Guibas et al. [15, Theorem 3.2], this can be performed in
Theorem 2. General Position Subset Selection admits a problem kernel containing at most $15k^3$ points that is computable in $O(n^2 \log n)$ time.

Proof. By Lemma 4, after $O(n^2 \log n)$ preprocessing time, we can either return an equivalent yes-instance of $(P, k)$ of constant size, or obtain an equivalent instance for which the collinearity of the point set is at most $\left(\frac{k-2}{2}\right) + 1$. Therefore, without loss of generality, we can assume in what follows that the collinearity of $P$ is at most $\lambda = \left(\frac{k-2}{2}\right) + 1$.

Payne and Wood [23, Theorem 2.3] showed that any set of $n$ points whose collinearity is at most $\lambda$ contains a subset of points in general position of size at least $\alpha n/\sqrt{n \ln \lambda + \lambda^2}$, for some constant $\alpha \in R$. A lower bound of $\alpha \geq \sqrt{6}/72$ can be computed based on Payne [22, Lemmas 4.1, 4.2, and Theorems 2.2, 2.3, 4.3]. Since $\lambda \leq \left(\frac{k-2}{2}\right) + 1$, we can compute a value of $n$, as a function of $k$, above which we are guaranteed to have a subset in general position of cardinality at least $k$. We do this by solving for $n$ in the inequality $\alpha n/\sqrt{n \ln \lambda + \lambda^2} \geq k$ after substituting $\lambda$ with $\left(\frac{k-2}{2}\right) + 1$ and $\alpha$ with $\sqrt{6}/72$. We obtain that if $n \geq 15k^3$, then the aforementioned inequality is satisfied for all $k \geq 29337$. The kernelization algorithm distinguishes the following three cases: First, if $k < 29337$, then the algorithm decides the instance in $O(1)$ time, and returns an equivalent instance of $O(1)$ size. Second, if $k \geq 29337$ and $n \geq 15k^3$, then the algorithm returns a trivial yes-instance of constant size. Third, if none of the two above cases applies, then it returns the (preprocessed) instance $(P, k)$ which satisfies $|P| \leq 15k^3$.

We can derive the following result by a brute-force algorithm on the above problem kernel:

Corollary 1. General Position Subset Selection can be solved in $O(n^2 \log n + 41k \cdot k^{2k})$ time.

Proof. Let $(P, k)$ be an instance of General Position Subset Selection. By Theorem 2, after $O(n^2 \log n)$ preprocessing time, we can assume that $|P| \leq 15k^3$. We enumerate every subset of size $k$ in $P$, and for each such subset, we use the result of Guibas et al. [15, Theorem 3.2] to check in $O(k^2 \log k)$ time whether the subset is in general position. If we find such

\[O(n^2 \log (n/\lambda)/\lambda)\] time. We then iterate over each line $L \in \mathcal{L}$, checking whether $L$, at the current iteration, still contains at least $\lambda$ points; if it does, we remove all points on $L$ from $P$ and decrement $k$ by 2. For each line $L$, the running time of the preceding step is $O(\lambda)$, which is the time to check whether $L$ contains at least $\lambda$ points. Additionally, we might need to remove all points on $L$. If $k$ reaches zero, we can return a trivial yes-instance $(P', k')$ of General Position Subset Selection in constant time. Otherwise, after iterating over all lines in $\mathcal{L}$, by Rule 1, the resulting instance $(P', k')$ is an equivalent instance to $(P, k)$ satisfying that no line in $P'$ contains $\lambda$ points, and hence the collinearity of $P'$ is at most $\left(\frac{k'-2}{2}\right) + 1$. Overall, the above can be implemented in time $O((n^2 \log (n/\lambda)/\lambda) \cdot \lambda) = O(n^2 \log n)$.

We move on to improving the size of the problem kernel. Payne and Wood [23, Theorem 2.3] proved a lower bound on the maximum cardinality of a subset in general position when an upper bound on the collinearity of the point set is known. We show next how to obtain a kernel for General Position Subset Selection of cubic size based on this result of Payne and Wood [23].

Proof. By Lemma 4, after $O(n^2 \log n)$ preprocessing time, we can either return an equivalent yes-instance of $(P, k)$ of constant size, or obtain an equivalent instance for which the collinearity of the point set is at most $\left(\frac{k-2}{2}\right) + 1$. Therefore, without loss of generality, we can assume in what follows that the collinearity of $P$ is at most $\lambda = \left(\frac{k-2}{2}\right) + 1$.

Payne and Wood [23, Theorem 2.3] showed that any set of $n$ points whose collinearity is at most $\lambda$ contains a subset of points in general position of size at least $\alpha n/\sqrt{n \ln \lambda + \lambda^2}$, for some constant $\alpha \in R$. A lower bound of $\alpha \geq \sqrt{6}/72$ can be computed based on Payne [22, Lemmas 4.1, 4.2, and Theorems 2.2, 2.3, 4.3]. Since $\lambda \leq \left(\frac{k-2}{2}\right) + 1$, we can compute a value of $n$, as a function of $k$, above which we are guaranteed to have a subset in general position of cardinality at least $k$. We do this by solving for $n$ in the inequality $\alpha n/\sqrt{n \ln \lambda + \lambda^2} \geq k$ after substituting $\lambda$ with $\left(\frac{k-2}{2}\right) + 1$ and $\alpha$ with $\sqrt{6}/72$. We obtain that if $n \geq 15k^3$, then the aforementioned inequality is satisfied for all $k \geq 29337$. The kernelization algorithm distinguishes the following three cases: First, if $k < 29337$, then the algorithm decides the instance in $O(1)$ time, and returns an equivalent instance of $O(1)$ size. Second, if $k \geq 29337$ and $n \geq 15k^3$, then the algorithm returns a trivial yes-instance of constant size. Third, if none of the two above cases applies, then it returns the (preprocessed) instance $(P, k)$ which satisfies $|P| \leq 15k^3$.

We can derive the following result by a brute-force algorithm on the above problem kernel:

Corollary 1. General Position Subset Selection can be solved in $O(n^2 \log n + 41k \cdot k^{2k})$ time.

Proof. Let $(P, k)$ be an instance of General Position Subset Selection. By Theorem 2, after $O(n^2 \log n)$ preprocessing time, we can assume that $|P| \leq 15k^3$. We enumerate every subset of size $k$ in $P$, and for each such subset, we use the result of Guibas et al. [15, Theorem 3.2] to check in $O(k^2 \log k)$ time whether the subset is in general position. If we find such

From Theorems 2.2 and 2.3 we can deduce that Lemma 4.1 holds for the constant $c = 128$. Plugging $c = 128$ into the proof of Theorem 4.3 gives the desired lower bound $\sqrt{6}/72$ for $\alpha$. 

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a subset, then we answer positively; otherwise (no such subset exists), we answer negatively. The number of enumerated subsets is

\[
\frac{|P|}{k!} \leq \frac{15k^3}{k!} \leq \frac{(15k^3)^k}{(k/e)^k} = (15ek^3/k)^k \leq (40.78)^k k^{2k},
\]

where \( e \) is the base of the natural logarithm and \( k! \geq (k/e)^k \) follows from Stirling’s formula. Putting everything together, we obtain an algorithm for General Position Subset Selection that runs in \( O(n^2 \log n + (40.78)^k \cdot k^{2k} \cdot k^2 \log k) = O(n^2 \log n + 41^k \cdot k^{2k}) \) time.

Let 3-General Position Subset Selection denote the restriction of General Position Subset Selection to instances in which the point set contains no four collinear points. By Theorem 1, 3-General Position Subset Selection is NP-complete. Füredi [10, Theorem 1] showed that every set \( P \) of \( n \) points in which no four points are collinear contains a subset in general position of size \( \Omega(\sqrt{n \log n}) \). Based on Füredi’s result and the idea in the proof of Theorem 2, we get:

**Corollary 2.** 3-General Position Subset Selection admits a problem kernel containing \( O(k^2/\log k) \) points that is computable in \( O(n) \) time.

Cao [4] made the following observation on the relation between the cardinality of a maximum-cardinality point subset in general position and the line cover number, that is, the minimum number of lines that cover all points in the point set. For the sake of self-containment, we also give a short proof.

**Observation 2 ([4]).** For a set \( P \) of points let \( S \subseteq P \) be a maximum subset in general position and let \( \ell \) be the line cover number of \( P \). Then, \( \sqrt{\ell} \leq |S| \leq 2\ell \).

**Proof.** For the first inequality, note that \( |S| \) points in general position define \( \binom{|S|}{2} \leq |S|^2 \) lines. Since all other points in \( P \) have to lie on a line defined by two points in \( S \), it follows \( \ell \leq |S|^2 \). The second inequality clearly holds since any maximum subset in general position can contain at most two points on any line. \( \square \)

As a consequence of 2, we can assume that \( k \leq 2\ell \) and, thus, we can transfer our results for the parameter \( k \) to the parameter \( \ell \).

**Corollary 3.** General Position Subset Selection can be solved in time \( O(n^2 \log n + 41^{2\ell} \cdot \ell^{4\ell}) \), and there is a kernelization algorithm that, given an instance \((P, k)\) of General Position Subset Selection, computes an equivalent instance containing at most \( 120\ell^3 \) points in time \( O(n^2 \log n) \).

### 4.2 FPT Results for the Dual Parameter

In this section we consider the dual parameter number \( h := n - k \) of points that have to be deleted (i.e., excluded from the sought point set in general position) so that the remaining points are in general position. We show a problem kernel containing \( O(h^2) \) points for General Position Subset Selection. Moreover, we show that this problem kernel is essentially tight, that is, there is presumably no problem kernel with \( O(h^{2 - \epsilon}) \) points for any \( \epsilon > 0 \).

We start with the problem kernel that relies essentially on a problem kernel for the 3-Hitting Set problem:
**3-Hitting Set**

**Input:** A universe $U$, a collection $C$ of size-3 subsets of $U$, and $h \in \mathbb{N}$.

**Question:** Is there a subset $H \subseteq U$ of size at most $h$ containing at least one element from each subset $S \in C$?

There is a close connection between General Position Subset Selection and 3-Hitting Set: For any collinear triple $p, q, r \in P$ of distinct points, one of the three points has to be deleted in order to obtain a subset in general position. Hence, the set of deleted points has to be a hitting set for the family of all collinear triples in $P$. Since 3-Hitting Set can be solved in $O(2^{0.8h} + |C| + |U|)$ time [27], we get:

**Proposition 2.** General Position Subset Selection can be solved in $O(2^{0.8h} + n^3)$ time.

3-Hitting Set is known to admit a problem kernel with a universe of size $O(h^2)$ computable in $O(|U| + |C| + h^{1.5})$ time [2]. Based on this, one can obtain a problem kernel of size $O(h^2)$ computable in $O(n^3)$ time. The bottleneck in this running time is listing all collinear triples. We can improve the running time of this kernelization algorithm by giving a direct kernel exploiting the simple geometric fact that two non-parallel lines intersect in one point. We first need two reduction rules.

**Rule 2.** Let $(P, k)$ be an instance of General Position Subset Selection. If there is a point $p \in P$ that is not collinear with any two other points in $P$, then delete $p$ and decrease $k$ by one.

Clearly, Rule 2 is correct since we can always add a point which is not lying on any line defined by two other points to a general position subset. The next rule deals with points that are in too many conflicts. The basic idea here is that if a point lies on more than $h$ distinct lines defined by two other points of $P$, then it has to be deleted. This is generalized in the next rule.

**Rule 3.** Let $(P, k)$ be an instance of General Position Subset Selection. For a point $p \in P$, let $L(p)$ be the set of lines containing $p$ and at least two points of $P \setminus \{p\}$, and for $L \in L(p)$ let $|L|$ denote the number of points of $P$ on $L$. Then, delete each point $p \in P$ satisfying $\sum_{L \in L(p)}(|L| - 2) > h$.

**Lemma 5.** Rule 3 is correct.

**Proof.** Let $(P, k)$ be an instance of General Position Subset Selection and let $(P', k) := (P \setminus D, k)$ be the reduced instance, where $D \subseteq P$ denotes the set of removed points. We show that $(P, k)$ is a yes-instance if and only if $(P', k)$ is a yes-instance.

Clearly, if $(P', k)$ is a yes-instance, then also $(P, k)$ is one. For the converse, we show that any size-$k$ subset of $P$ in general position does not contain any point $p \in D$: For each line $L \in L(p)$, all but two points need to be deleted. If a subset $S \subseteq P$ in general position contains $p$, then the points that have to be deleted on the lines in $L(p)$ are all different since any two of these lines only intersect in $p$. This means that $\sum_{L \in L(p)}(|L| - 2)$ points need to be deleted. However, since this value is by assumption larger than $h$, the solution $S$ is of size less than $k = |P| - h$.

**Theorem 3.** General Position Subset Selection admits a problem kernel containing at most $2h^2 + h$ points that is computable in $O(n^2)$ time.

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Proof. Let \((P, k)\) be a \textsc{General Position Subset Selection} instance. We first show that applying Rule 2 exhaustively and then applying Rule 3 once indeed gives a small instance \((P', k')\). Note that each point \(p \in P'\) is “in conflict” with at least two other points, that is, \(p\) is on at least one line containing two other points in \(P'\), since the instance is reduced with respect to Rule 2. Moreover, since the instance is reduced with respect to Rule 3, it follows that each point is in conflict with at most \(2h\) other points. Thus, deleting \(h\) points can give at most \(h \cdot 2h\) points in general position. Hence, if \(P'\) contains more than \(2h^2 + h\) points, then the input instance is a no-instance.

We next show how to apply Rules 2 and 3 in \(O(n^2)\) time. To this end, we follow an approach described by Edelsbrunner et al. \([8]\) and Gómez et al. \([14]\) which uses the dual representation and line arrangements. The dual representation maps points to lines as follows: \((a, b) \mapsto y = ax + b\). A line in the primal representation containing some points of \(P\) corresponds in the dual representation to the intersection of the lines corresponding to these points. Thus, a set of at least three collinear points in the primal corresponds to the intersection of the corresponding lines in the dual. An \textit{arrangement} of lines in the plane is, roughly speaking, the partition of the plane formed by these lines. A representation of an arrangement of \(n\) lines can be computed in \(O(n^2)\) time \([8]\). Using the algorithm of Edelsbrunner et al. \([8]\), we compute in \(O(n^2)\) time the arrangement \(A(P^*)\) of the lines \(P^*\) in the dual representation of \(P\).

Rule 2 is now easily computable in \(O(n^2)\) time: Initially, mark all points in \(P\) as “not in conflict”. Then, iterate over the vertices of \(A(P^*)\) and whenever the vertex has degree six or more (each line on an intersection contributes two to the degree of the corresponding vertex) mark the points corresponding to the intersecting lines as “in conflict”. In a last step, remove all points that are marked as “not in conflict”.

Rule 3 can be applied in a similar fashion in \(O(n^2)\) time: Assign a counter to each point \(p \in P\) and initialize it to zero. We want this counter to store the number \(\sum_{L \in \mathcal{L}(P)}(|L| - 2)\) on which Rule 3 is conditioned. To this end, we iterate over the vertices in \(A(P^*)\) and for each vertex of degree six or more we increase the counter of each point corresponding to a line in the intersection by \(d/2 - 2\) where \(d\) is the degree of the vertex. After one pass over all vertices in \(A(P^*)\) in \(O(n^2)\) time, the counters of the points store the correct values and we can delete all points whose counter is more than \(h\).

We remark that the results in Proposition 2 and Theorem 3 also hold when replacing the parameter \(h\) by the “number \(\gamma\) of inner points”, where we call a point an inner point if it is not a corner point of the convex hull of \(P\). The reason is that in all non-trivial instances we have \(h \leq \gamma\) since removing all inner points yields a set of points in general position.

We can prove a matching (conditional) lower bound on the problem kernel size for \textsc{General Position Subset Selection} via a reduction from \textsc{Vertex Cover}. Given an undirected graph \(G\) and \(k \in \mathbb{N}\), \textsc{Vertex Cover} asks whether there is a subset \(C\) of at most \(k\) vertices such that every edge is incident to at least one vertex in \(C\). Using a lower bound result by Dell and van Melkebeek \([6]\) for \textsc{Vertex Cover} (which is based on the common assumption in complexity theory that \(\text{coNP} \not\subseteq \text{NP/poly}\) since otherwise the polynomial hierarchy collapses to its third level), we obtain the following:

\textbf{Theorem 4.} Unless \(\text{coNP} \subseteq \text{NP/poly}\), for any \(\epsilon > 0\), \textsc{General Position Subset Selection} admits no problem kernel of size \(O(h^{2-\epsilon})\).
Proof. We give a polynomial-time reduction from VERTEX COVER, where the resulting dual parameter \( h \) equals the size of the sought vertex cover. The claimed lower bound then follows because, unless \( \text{coNP} \subseteq \text{NP/poly} \), for any \( \epsilon > 0 \), VERTEX COVER admits no problem kernel of size \( O(k^{2-\epsilon}) \), where \( k \) is the size of the vertex cover [6].

Given a VERTEX COVER instance \((G, k)\), we first reduce it to the equivalent INDEPENDENT SET instance \((G, |V(G)| - k)\). We then apply transformation \( \Phi \) (see Section 3) to \( G \) to obtain a set of points \( P \), where \(|P| = |V(G)| + |E(G)|\); we set \( k' := |V(G)| + |E(G)| - k \), and consider the instance \((P, k')\) of GENERAL POSITION SUBSET SELECTION. Clearly, \( G \) has a vertex cover of cardinality \( k \) if and only if \( G \) has an independent set of cardinality \(|V(G)| - k\), which, by Lemma 1, is true if and only if \( P \) has a subset in general position of cardinality \(|E(G)| + |V(G)| - k\). Hence, the dual parameter \( h = |P| - k' \) equals the sought vertex cover size.

Note that Theorem 4 gives a lower bound only on the total size (i.e., instance size) of a problem kernel for GENERAL POSITION SUBSET SELECTION. We can show a stronger lower bound on the number of points contained in any problem kernel using ideas from Kratsch et al. [21], which are based on a lower bound framework by Dell and van Melkebeek [6]. Kratsch et al. [21] showed that there is no polynomial-time algorithm that reduces a POINT LINE COVER instance \((P, k)\) to an equivalent instance with \( O(k^{2-\epsilon}) \) points for any \( \epsilon > 0 \) unless \( \text{coNP} \subseteq \text{NP/poly} \). The proof is based on a result by Dell and van Melkebeek [6] who showed that VERTEX COVER does not admit a so-called oracle communication protocol of cost \( O(k^{2-\epsilon}) \) for \( \epsilon > 0 \) unless \( \text{coNP} \subseteq \text{NP/poly} \). An oracle communication protocol is a two-player protocol, in which one player is holding the input and is allowed polynomial (computational) time in the length of the input, and the second player is computationally unbounded. The cost of the communication protocol is the number of bits communicated from the first player to the second player in order to solve the input instance.

Kratsch et al. [21] devise an oracle communication protocol of cost \( O(n \log n) \) for deciding instances of POINT LINE COVER with \( n \) points. Thus, a problem kernel for POINT LINE COVER with \( O(k^{2-\epsilon'}) \) points implies an oracle communication protocol of cost \( O(k^{2-\epsilon'}) \) for some \( \epsilon' > 0 \) since the first player could simply compute the kernelized instance in polynomial time and subsequently apply the protocol yielding a cost of \( O(k^{2-\epsilon'} \cdot \log(k^{2-\epsilon'})) \), which is in \( O(k^{2-\epsilon'}) \) for some \( \epsilon' > 0 \). This implies an \( O(k^{2-\epsilon''}) \)-cost oracle communication protocol for VERTEX COVER for some \( \epsilon'' > 0 \) (via a polynomial-time reduction with a linear parameter increase [21, Lemma 6]). We show that there exists a similar oracle communication protocol of cost \( O(n \log n) \) for GENERAL POSITION SUBSET SELECTION. The protocol is based on order types of point sets. Let \( P = \langle p_1, \ldots, p_n \rangle \) be an ordered set of points and denote by \( \binom{[n]}{3} \) the set of ordered triples \( \langle i, j, k \rangle \) where \( i < j < k, i, j, k \in [n] := \{1, \ldots, n\} \). The order type of \( P \) is a function \( \sigma : \binom{[n]}{3} \to \{-1, 0, 1\} \), where \( \sigma((i, j, k)) \) equals 1 if \( p_i, p_j, p_k \) are in counter-clockwise order, equals -1 if they are in clockwise order, and equals 0 if they are collinear. Two point sets \( P \) and \( Q \) of the same cardinality are combinatorially equivalent if there exist orderings \( P' \) and \( Q' \) of \( P \) and \( Q \) such that the order types of \( P' \) and \( Q' \) are identical.

A key step in the development of an oracle communication protocol is to show that two instances of POINT LINE COVER with combinatorially equivalent point sets are actually equivalent [21, Lemma 2]. We can prove an analogous result for GENERAL POSITION SUBSET SELECTION:

Observation 3. Let \((P, k)\) and \((Q, k)\) be two instances of GENERAL POSITION SUBSET SELECTION.
Selection. If the point sets \( P \) and \( Q \) are combinatorially equivalent, then \((P, k)\) and \((Q, k)\) are equivalent instances of General Position Subset Selection.

Proof. Let \( P \) and \( Q \) be combinatorially equivalent point sets with \(|P| = |Q| = n\) and let \( P' = \langle p_1, \ldots, p_n \rangle \) and \( Q' = \langle q_1, \ldots, q_n \rangle \) be orderings of \( P \) and \( Q \), respectively, having the same order type \( \sigma \).

Now, a subset \( S \subseteq P' \) is in general position if and only if no three points in \( S \) are collinear, that is, \( \sigma(\langle p_i, p_j, p_k \rangle) \neq 0 \) holds for all \( p_i, p_j, p_k \in S \). Consequently, it holds that \( \sigma(\langle q_i, q_j, q_k \rangle) \neq 0 \), and thus the subset \( \{ q_i \mid p_i \in S \} \subseteq Q' \) is in general position. Hence, \((P, k)\) is a yes-instance if and only if \((Q, k)\) is a yes-instance.

Based on \(3\), we obtain an oracle communication protocol for General Position Subset Selection. The proof of the following lemma is completely analogous to the proof of Lemma 7 in \(21\):

**Lemma 6.** There is an oracle communication protocol of cost \( O(n \log n) \) for deciding instances of General Position Subset Selection with \( n \) points.

The basic idea is that the first player only sends the order type of the input point set so that the computationally unbounded second player can solve the instance (according to \(3\) the order type contains enough information to solve a General Position Subset Selection instance). We conclude with the following lower bound result:

**Theorem 5.** Let \( \epsilon > 0 \). Unless \( \text{coNP} \subseteq \text{NP/poly} \), there is no polynomial-time algorithm that reduces an instance \((P, k)\) of General Position Subset Selection to an equivalent instance with \( O(h^{2-\epsilon}) \) points.

Proof. Assuming that such an algorithm exists, the oracle communication protocol of Lemma 6 has cost \( O(h^{2-\epsilon'}) \) for some \( \epsilon' > 0 \). Since the reduction from Vertex Cover in Theorem 4 outputs a General Position Subset Selection instance where the dual parameter \( h \) equals the size \( k \) of the vertex cover sought, we obtain a communication protocol for Vertex Cover of cost \( O(k^{2-\epsilon'}) \), which implies that \( \text{coNP} \subseteq \text{NP/poly} \) \([6, Theorem 2]\).

**Remark on the Kernel Lower Bound Framework of Kratsch, Philip and Ray.** As a final extract, we mention that the framework of Kratsch et al. \([21]\) only relies on the equivalence of instances with respect to order types of point sets. Hence, we observe that for every decision problem on point sets for which

1. two instances with combinatorially equivalent point sets are equivalent (cf. \(3\)), and
2. there is no oracle communication protocol of cost \( O(k^{2-\epsilon}) \) for some parameter \( k \) and any \( \epsilon > 0 \) unless \( \text{coNP} \subseteq \text{NP/poly} \),

there is no problem kernel with \( O(k^{2-\epsilon'}) \) points for any \( \epsilon' > 0 \) unless \( \text{coNP} \subseteq \text{NP/poly} \).

5 Conclusion and Outlook

The intent of our work is to stimulate further research on the computational complexity of General Position Subset Selection. The kernelization results we presented rely mostly on combinatorial arguments; the main geometric property we used is that two distinct lines intersect in at most one point. Therefore, a natural question to ask is whether there are further geometric properties that can be exploited in order to obtain improved algorithmic
results for General Position Subset Selection. We conclude with the following concrete open questions: Can the \((15k^3)\)-point kernel (Theorem 2) for General Position Subset Selection be asymptotically improved? Or can we derive a cubic, or even a quadratic, lower bound on the (point) kernel size of General Position Subset Selection? Can the FPT algorithm (cf. Corollary 1) for General Position Subset Selection be (significantly) improved? With respect to approximation, we could only show the APX-hardness of Maximum General Position Subset Selection. It remains open whether Cao’s \(O(\sqrt{\text{opt}})\)-factor approximation can be improved.

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