The cascades route to chaos

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The presence of a period-doubling cascade in dynamical systems that depend on a parameter is one of the basic routes to chaos. It is rarely mentioned that there are virtually always infinitely many cascades whenever there is one. We report that for one- and two-dimensional phase space, in the transition from no chaos to chaos – as a parameter is varied – there must be infinitely many cascades under some mild hypotheses. Our meaning of chaos includes the case of chaotic sets which are not attractors. Numerical studies indicate that this result applies to the forced-damped pendulum and the forced Duffing equations, viewing the solutions once each period of the forcing. We further show that in many cases cascades appear in pairs (in joint parameter-state space) by an unstable periodic orbit. Paired cascades can be destroyed or created by perturbations, whereas unpaired cascades are conserved under even significant perturbations.

In Fig. 1 as \( \mu \) increases towards a value \( \mu_F \approx 3.57 \), one encounters a family of periodic orbits that undergo an infinite sequence of period doublings with the period of these orbits tending to \( \infty \). The appearance of infinitely-many such (period-doubling) cascades is one of the most prominent features observed in the study of parametrized maps. Cascades were first reported by Myrberg in 1962 [1] (cf. Fig. 1), and Robert May popularized their existence to a huge scientific audience [2]. They are found in a large variety of contexts: in Raleigh-Bernard convection [3], damped bouncing balls [4], Van der Pol oscillators [5] (cf. Fig. 2), vibratory ball milling [6], dust charge fluctuation in plasma systems [7], external optical injection in lasers [8], delay oscillators [12], vibrating damaged structures [13], glow discharge [14], combustion [15], very slow classical Cepheid stars [16], neuron and pancreatic cells [17], dripping faucets [18], bouncing droplets on soap films [19], Belousov-Zhabotinsky reactions [20], and in the cromorn, a medieval musical instrument [21].

We find that for many systems depending on a parameter, every periodic orbit is part of a cascade; for such systems, cascades are as fundamental as periodic orbits themselves. The scaling properties of individual cascades have been studied for cascades in nearly quadratic maps [22], but there are only a few results about the existence of cascades [23]. Furthermore, the mathematical and scientific literature focuses on the study of single cascades. In this Letter, we describe the results of a new general theory of cascades, which explains why cascades exist and why chaotic dynamical systems often have infinitely many cascades.

Two kinds of periodic orbits meet at period-doubling bifurcations. For a map \( F(\mu, x) \) that depends on a parameter \( \mu \), a point \((\mu, x_0)\) is a period-\( p \) point if \( F^p(\mu, x_0) = x_0 \), where \( p > 0 \) is chosen as small as possible. Writing \( x_{n+1} = F(\mu, x_n) \), its (periodic) orbit is the set of points \( \{(\mu, x_0), (\mu, x_1), (\mu, x_2), \cdots, (\mu, x_{p-1})\} \). By the eigenvalues of that orbit, we mean the eigenvalues of the

![FIG. 1: Cascades and their connected components. The attracting set for the logistic map \( F(\mu, x) = \mu x(1 - x) \) is shown in blue. There are infinitely-many cascades, each with infinitely-many period-doubling bifurcations. Each saddle-node bifurcation creates both a cascade and a path of unstable orbits, the latter shown as black curves for periods up to six. We call the paths of unstable orbits the stems of the cascades. Below the largest gaps or windows in the blue regions, we give the period of the associated cascade. For this particular map, the black curves of unstable orbits extend to \( \mu = -\infty \). Here, at each sufficiently large \( \mu \), there is exactly one orbit (\( k \) points) in each period-\( k \) stem.](image1)

![FIG. 2: Cascades in the double-well Duffing equation. The attracting sets (in black) and periodic orbits up to period ten (in red) for the time-2\( \pi \) map of the double-well Duffing equation: \( x''(t) + 0.3 x'(t) - x(t) + (x(t))^2 + (x(t))^3 = \mu \sin t \). Numerical studies show regions of chaos interspersed with regions without chaos, indicating that our Off-On-Off Chaos Theorem applies to this situation.](image2)
we require only one aspect of chaos: we say that there is chaos at a particular $\mu$ if there are infinitely many regular periodic orbits. For example, there is chaos whenever there is a transverse homoclinic point. That is equivalent to having a horseshoe for some iterate of the map, and it implies there are infinitely many regular saddles in two dimensions, and infinitely many regular unstable orbits in dimension one. This definition of chaos is sufficiently general as to include having one or multiple coexisting chaotic attractors, as well as the case of transient chaos. We say a map has no chaos at a particular $\mu$ if there are at most finitely many regular periodic orbits.

**Cascades Route to Chaos Theorem.** Assume the parametrized map $F(\mu, \cdot)$ has a one- or two-dimensional phase space. If at parameter value $\mu_1$ there is no chaos, and at $\mu_2$, there is chaos and there are at most finitely many attracting orbits (and finitely many repelling orbits in dimension 2), then $F$ has infinitely many cascades between $\mu_1$ and $\mu_2$.

Thus for one- and two-dimensional smooth families, the only route to chaos is through infinitely many cascades. It is for instance impossible to get chaos at $\mu_2$ from a single cascade.

If we were to omit the assumption that there are infinitely many regular orbits, the conclusion would be false. For example, in the quadratic map let $\mu_F$ denote the Feigenbaum parameter value, i.e., the first parameter value where the period doublings accumulate. At $\mu_F$ there are infinitely many periodic orbits, and all but one are flip orbits. Furthermore $\mu_F$ is preceded by a single cascade.

Each cascade will have some minimum period $p$ and will have periodic orbits of periods $p, 2p, 4p, 8p, \ldots$. If the map is one-dimensional or is two-dimensional and dissipative in the sense that there are no repelling periodic orbits, then each of these periods is the period of some attracting orbits in the path, though not all orbits will necessarily be attracting.

Based on numerical studies, a number of maps appear to satisfy the conditions of the Cascades Route to Chaos Theorem. Note that these numerical verifications involve significantly more work than just plotting the attracting sets for each parameter, since we are concerned about both the stable and the unstable behavior to determine whether there is chaos. Examples include the time-$2\pi$ maps for the double-well Duffing (Fig. 2), the triple-well Duffing, and forced-damped pendulum (Fig. 4), as well as the Ikeda map used to describe the field of a laser cavity, and the pulsed damped rotor map.

We now discuss the creation of a path $(\mu(\psi), [Y(\psi)])$ of regular orbits. Consider the procedure of starting at a regular periodic orbit and following the path of periodic orbits containing it in $(\mu, y)$-space, following only regular orbits. Follow the path – such as numerically – through saddle-node bifurcations by reversing the direction in the parameter space. When period-$p$ orbits reach

![FIG. 3: Bounded paired cascades. The attracting set for the function $F(\mu, x) = h(\mu)x(1-x)$, where $h(\mu) = \mu(1.18 + 0.17\cos(2.4\mu))$. Cascades $C_1$ and $C_2$ constitute a pair of cascades, since they are connected by a family of orbits (black curves). Cascade $C_3$ is unbounded and unpaired, as indicated by the black curve continuing to the right to $\mu = \infty$.](image-url)
We call an entire connected path of regular orbits a \(\mu, y\) component. Each cascade in the following one-dimensional maps contains a cascade that is between \(\mu_1\) and \(\mu_2\), and distinct regular orbits at \(\mu_2\) are in different components. Starting at each regular orbit at \(\mu_2\), there is a unique path of regular orbits – initially in the direction of \(\mu_1\) – ending in a cascade. Thus the number of cascades hitting the boundary of the parameter interval depends only on the number of regular orbits at \(\mu_2\), independent of the behavior of the map between \(\mu_1\) and \(\mu_2\).

This leads to a heuristic conservation principle for \(\mu, y\) cascades: if \(F(\mu, y)\) satisfies property (i) for \(\mu\) sufficiently negative and property (ii) for \(\mu\) sufficiently positive, then any perturbation \(F(\mu, y) + g(\mu, y)\) such that \(F\) dominates \(g\) for \(|\mu| \to \infty\) has the same set of \(\mu, y\) cascades.

Three rigorous examples of this idea are encapsulated in the following one-dimensional maps

\[
F(\mu, y) = \mu - x^2 + g(\mu, y) \quad \text{(quadratic)},
\]

\[
F(\mu, y) = \mu x - x^3 + g(\mu, y) \quad \text{(cubic)},
\]

\[
F(\mu, y) = x^4 - 2\mu x^2 + \mu^2/2 + g(\mu, y) \quad \text{(quartic)},
\]

where for some real positive \(\beta\),

\[
|g(\mu, 0)| < \beta \quad \text{for all } \mu, \quad \text{and}
\]

\[
|g_x(\mu, y)| < \beta \quad \text{for all } \mu, x.
\]

For such \(g\), each of the three maps each has no regular periodic orbits for \(\mu\) sufficiently negative, and for \(\mu\) sufficiently large has a one-dimensional horseshoe map. The quartic map was chosen so that when \(g\) is identically \(0\) and \(\mu > 0\), the graph has two minima, \((\pm \sqrt{\mu}, -\mu^2/2)\), and the local maximum is at \((0, +\mu^2/2)\). The conditions on \(g\) guarantee that it does not significantly affect the periodic orbits when \(|\mu|\) is sufficiently large; in particular it does not affect their eigenvalues, so it does not affect the number of period-\(p\) regular periodic orbits for large \(|\mu|\). Hence one can check that all three maps have no regular periodic orbits for \(\mu\) very negative and have no attracting periodic orbits for \(\mu\) very positive, and for sufficiently large \(|\mu|\), all periodic orbits are contained in the set \([-2\sqrt{\mu}, 2\sqrt{\mu}]\). Thus the following holds.

![The forced-damped pendulum.](image-url)

FIG. 4: The forced-damped pendulum. For this figure, periodic points with periods < 10 were plotted in red for the time-\(2\pi\) map of the forced-damped pendulum equation: \(x''(t) + 0.2x'(t) + \sin(x(t)) = \omega \cos(t)\), indicating the general areas with chaotic dynamics for this map. Then the attracting sets were plotted in black, hiding some periodic points. Much more detailed calculations confirm that interspersed with the chaos, there are some parameter ranges without chaos.
| $k$ | $Quad(k)$ | $Cubic(k)$ | $Quart(k)$ |
|-----|-----------|------------|-----------|
| 1   | 1 2       | 2          |           |
| 2   | 0 1 2     |            |           |
| 3   | 1 4 10    |            |           |
| 4   | 1 8 28    |            |           |
| 5   | 3 24 102  |            |           |
| 6   | 4 56 330  |            |           |
| 7   | 9 156 1152|            |           |
| 8   | 14 400 4064|           |           |
| 9   | 28 1092 14560|        |           |

$k \gg 10 \sim 2^k/2k \sim 3^k/2k \sim 4^k/2k$

TABLE I: The number of unbounded cascades of stem period $k$ for any large-scale perturbation of the quadratic, cubic, and quartic maps, respectively labeled $Quad(k)$, $Cubic(k)$, and $Quart(k)$.

**Conservation of cascades theorem.** For any $F$ of the functions in Eqn. [1] with $g$ chosen as in Eqn. [3] the number of stem-period-$k$ unbounded cascades is independent of the choice of $g$.

The number of unbounded cascades in each of these three cases is summarized in our Table. The relative sizes reflect the complexity of the three horseshoe maps; the quadratic map has the complexity of a standard two-branch horseshoe, whereas the cubic has a horseshoe with three branches, and the quartic has four branches.

The conservation principle works for two- and higher-dimensional maps as well. We have shown in [25] that large-scale perturbations of the two-dimensional Hénon map conserve unbounded cascades. In [24], we have shown that there is conservation of unbounded cascades for a coupled system of $N$ quadratic maps.

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