WHAT ARE WE QUANTIZING IN INTEGRABLE FIELD THEORY?

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Dedicated to my teacher Professor L.D. Faddeev on his forthcoming sixtieth birthday.

Abstract. We continue study of the connection of classical limit of integrable asymptotically free field theory to the finite-gap solutions of classical integrable equations. In the limit the momenta of particles turn into the moduli of Riemann surfaces while their isotopic structure is related to the period lattices. In this paper we explain that the classical limit of the local operators can be understood as a measure induced on the phase space by embedding into the projective space of "classical fields".

INTRODUCTION

The present paper develops the ideas of the paper [1]. Let us remind the basic points of [1]. One of the most important achievements in the understanding of the structure of integrable models in the last two years is in the realization of the fact that the bootstrap equation [2] for the form factors for certain models can be considered as deformed Knizhnik-Zamolodchikov equations [3,4]. Thus the very possibility of the exact solution can be considered as just a result of the possessing infinite dimensional quantum symmetry.

Certainly, we should try to develop deeper understanding of this situation. In particular it is important to realize what happens in the classical limit when the infinite-dimensional quantum symmetry turns into classical one. Formally the limit of the kind is possible for asymptotically free theories, it corresponds to the limit from Lüscher's nonlocal charges [5,6] into the dressing symmetries of classical integrable models [7]. As it is explained in [1] such a limit destroys the structure of space-time of the quantum theory. Effectively at every point of the space-time we get a hierarchy of classical finite-dimensional systems. These finite-dimensional systems correspond to $n$-particle subspaces. Different points are isolated from each other before the quantization, only after the quantization is performed they start to interact through the exchange of particles. It means that the relativistic space-time appears as a result of quantization which sounds as an amusing way to solve the main problem of quantum field theory i.e. to combine relativistic and quantum physics.

The classical analogues of momenta of particles in that approach are the moduli of classical solutions of the finite-dimensional systems in question which happen
to be different stationary (finite-gap) solutions related to Riemann surfaces [8] of classical integrable equations, and the isotopic structure is related to the period lattices. More complicated question is what is a classical analog of local operators of the quantum model (more exactly of generating operator of the local operators, see [1] for details). The answer proposed in [1] is not quite satisfactory, so we have to return to the question. In the present paper we shall argue that the local operators in classics provide a measure on classic trajectories (Jacobi varieties) induced by their embedding into projective spaces of classical fields.

1. **Theta Formula for the Solutions of KZ**

As it has been shown in [4] the form factors of the generating function of local fields (from which energy-momentum tensor and currents can be obtained) for \(SU(2)\) Thirring (chiral Gross-Neveu) model can be considered as invariant solutions of Yangian deformation of KZ equations for spin 1/2 vertex operators on level zero. Similar fact is true for other relativistic models with rich quantum symmetries [2] as well as for lattice models [9]. The appearance of zero central extension is absolutely universal in the context and deserves special attention.

Let us mention two basic points. First, the arguments of the equations are the rapidities of particles. Second, the inner automorphism of the quantum algebra (antipode square) which is the analog of \(L^{-1}\) for the usual equations is identified with the complete rotation of the space-time around the point where the local operator lives. That means that we are doing not the deformation of CFT, but study completely different application of KZ equations. Also that means that in the classical limit (Yangian double \(\hat{\mathfrak{sl}}(2)\)) the structure of the space-time is lost (antipode square which corresponds to topological operation turns into infinitesimal generator \(L_{-1}\)). These problems are discussed in details in [1]. The main conclusion is that in the limit the theory splits into finite-dimensional systems which are finite-gap solutions of KdV i.e. are related to different hyper-elliptic surfaces.

Let us remind the most important formula from [1] which presents the solutions of level zero KZ equations (into which the form factor equations turn in the classical limit) in terms of Riemann \(\theta\)-functions. The KZ equations in our case look as

\[
\left( \frac{d}{d\lambda_i} + \sum_{i \neq j} r_{i,j}(\lambda_i - \lambda_j) \right) f(\lambda_1, \cdots, \lambda_{2g+2}) = 0
\]

where \(r\) is the classical \(r\)-matrix:

\[
r_{i,j}(\lambda_i - \lambda_j) = \frac{\sigma^a_i \otimes \sigma^a_j}{\lambda_i - \lambda_j}
\]

The vector function \(f(\lambda_1, \cdots, \lambda_{2g+2})\) belongs to \((\mathbb{C}^2)^{(2g+2)}\). We consider real \(\lambda_1, \cdots, \lambda_{2g+2}\) for they correspond to the limits of rapidities, also we require \(\lambda_1 < \cdots < \lambda_{2g+2}\). There are many solutions to the equations which are parametrized by different choices of the sets of \(g\) independent contours \((c_1, \cdots, c_g)\) on the hyper-elliptic surface \(\Sigma\) defined by the equation \(\tau^2 = \prod (\lambda - \lambda_i)\). Those solutions are of the main interest for which \(c_i \circ c_j = 0\) i.e. which can be used as half-bases of homology. For such solutions the components of the vector \(f_C(\lambda_1, \cdots, \lambda_{2n})\) appears to be the special values of one holomorphic function of \(g\) variables (function
on the Jacobi variety of $\Sigma$). This function is expressed in terms of Riemann $\theta$-function. Namely, consider $\theta$-function $\theta_C(z)$ constructed respectively to the half-basis $C = \{c_1, \ldots, c_g\}$, the variable $z \in \mathbb{C}^g$. Then one has

$$f_C(\lambda_1, \ldots, \lambda_{2g+2})^{\epsilon_1, \ldots, \epsilon_{2g+2}} = F(z)|_{z = a(\epsilon_1, \ldots, \epsilon_{2g+2})} \quad (1)$$

where $\epsilon_i = \pm$ is $\mathbb{C}^2$ index,

$$F(z) = D \frac{\theta_C(z)}{\theta(z)} \det \left[ \partial_z \partial_{z_j} \log \theta_C(z) \right]_{g \times g} \quad (2)$$

The constant $D$ in (2) depends on $\lambda_1, \ldots, \lambda_{2g+2}$ but does not depend on $\epsilon_1, \ldots, \epsilon_{2g+2}$. Finally, $a(\epsilon_1, \ldots, \epsilon_{2g+2})$ is the following half-period:

$$a(\epsilon_1, \ldots, \epsilon_{2g+2}) = \eta''(\epsilon_1, \ldots, \epsilon_{2g+2}) + \Omega_C \eta'(\epsilon_1, \ldots, \epsilon_{2g+2}) = \sum_{k=0}^{\lambda_k} \int_{\lambda_{2k+1}} \omega_C$$

where $\Omega_C$ is the period matrix constructed with respect to $C$, $\omega_C$ are corresponding normalized first kind differentials, $\{i_k\}_{k=0}^g$ are ordered numbers for which $\epsilon_{i_k} = +$, the vectors $\eta', \eta'' \in \frac{1}{2}\mathbb{Z}^g$.

Let us fix the canonical choice of the homology basis. We put the cuts on the plane between $\lambda_{2i-1}$ and $\lambda_{2i}$. Then the $a$-cycle $a_i$ starts from the upper bank of $(i + 1)$-th cut, goes to the upper bank of the $i$-th cut, then crosses it and returns to the starting point by another sheet. The cycle $b_i$ is taken as the sum of cycles around the $j$-th cuts for $1 \leq j \leq i$. This choice differs from one used in [1], but it is more appropriate for the connection with finite-gap integration: KdV angles vary over the product of $a$-cycles.

The solution to KZ which describes the asymptotics of the form factor corresponds to $C = \{b_1, \ldots, b_g\}$. Certainly, the formula (2) can be rewritten in terms of one canonical $\theta$-function, that defined respectively to $a$-cycles (corresponding $\Omega$ is pure imaginary). If $C$ is related to $A$ by

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g, \mathbb{Z})$$

then [10]

$$F_C(z) = \theta^4(z) \det \left[ (D + \Omega C)_{i,k} \partial_{z_k} \partial_{z_j} \log \theta(z) + 2\pi i \delta_{i,j} \right]_{g \times g}$$

In particular the limit of the form factor (which we denote by $f$ without index) corresponding to $b$-cycles is related to the following $F$:

$$F(z) = \theta^4(z) \det \left[ \Omega_{i,k} \partial_{z_k} \partial_{z_j} \log \theta(z) + 2\pi i \delta_{i,j} \right]_{g \times g} \quad (3)$$

$\delta_{i,j}$ in the last formula is the Kroneker symbol. It is clear that in order to understand the real meaning of the classical limit in question we have to understand the meaning of the function $F(z)$. We argue that the limit is connected with the finite-gap integration.

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Let us make several general remarks. The formulae (1),(2) describe the classical limit of the form factors for $SU(2)$ Thirring model in quite special terms. To every multiparticle state a hyper-elliptic surface is related such that the rapidities of particles are considered as the position of branching points (moduli) and the isotopic structure is given by even non-singular half-periods of the Jacobian variety. One can imagine that this situation is not restricted to the hyper-elliptic case, and that it might be possible to construct integrable theory for which the same amount of data (moduli and even nonsingular half-periods) taken for more general surfaces will describe the space of states, and the classical limit of matrix elements of local fields will be given by $F(z)$. The restrictions of such theory onto $Z_N$-invariant surfaces will give $SU(N)$-invariant Thirring models.

Another point concerns $\theta$-functions. Why is it important to have a description of the classical limit in terms of $\theta$-functions? The answer to this question becomes clear if we think about the structure of classical integrable models. The angle-variables should constitute real torus due to Liouville’s theorem. But the mechanism for the integrability in all interesting cases is Abel transformation. So, the torus in question allows extension to a complex torus. Doing any particular model we have to start with some not too complicated phase space, and then to embed the Liouville torus into this phase space. But in practice this construction allows complexification and the embedding happens to be holomorphic. The only object which can be used for the holomorphic embedding of complex torus into a reasonable phase space is Riemann $\theta$-function. For that reason doing quantization of integrable models sooner or later we have to come upon the quantization of $\theta$-function.

2. Connection with finite-gap solutions

The situation with which the finite-gap theory [8] deals can be summarized as follows. The infinite-dimensional system (KdV) has finite dimensional orbits. The motion on the latter is described as the motion of $g$ real points $P_1, \cdots, P_g$ situated on different segments $\lambda_{2i} \leq P_i \leq \lambda_{2i+1}$. In other words the divisor $P_1, \cdots, P_g$ runs over the product of $a$-cycles on the curve $\Sigma$. The dynamics is linearized by the Abel transformation which maps the motion above into the motion over the real $g$-dimensional subtorus ($J^R = a_1 \times \cdots \times a_g$) of the Jacobian ($J$) which is complex $g$-dimensional torus. The dynamical meaning of $J^R$ is clear: it is the torus of angles of the integrable system. Locally (with respect to actions) the phase space of integrable system with $2g$ degrees of freedom looks as $T^g \times \mathbb{R}^g$ where $T^g$ is the torus of angles and the actions vary over $\mathbb{R}^g$. In the theory of KdV the action variables depend on the moduli (positions of the branching points). Here it is quite undesirable for us to vary the moduli, there are two possibilities to avoid doing this. One way is to consider much bigger phase space, and then to apply constraints. Another way is to consider the phase space locally. Namely, consider not the phase space itself but the product of coordinate space ($J^R$) by the cotangent space in the direction of angle variables ($\mathbb{R}^g$ with the basis $\xi_1, \cdots, \xi_g$). This manifold is enough to write differential forms and things like that. Let us mention one important circumstance. We use the real part of the Jacobian variety, but it allows a natural complexification. One can try to use this circumstance in order to introduce certain structures on $J^R \times \mathbb{R}^g$ inducing them from the complexification. i.e. from $J = J^R \times J^I$, the imaginary part $J^I = b_1 \times \cdots \times b_g$.

So the space of the classical trajectories ($\mathfrak{M}$) is the same as the collection of
real parts of all the Jacobians of hyper-elliptic surfaces with real branching points. This space has singularities of double origin: first, the number of branching points can become infinite, second, singularities appear when two branching points coincide. It should be emphasized that we must not worry about these singularities on the classical level: the quantization takes care of them, they become irrelevant in quantum model.

Let us consider the affine model of $\mathfrak{N}$. Take the set of all $2 \times 2$ traceless, real, polynomial in additional real parameter $\lambda$ matrices with the leading coefficient fixed to be $\sigma_3$.

Factorize this set by the adjoint action of the matrices $\text{diag}(a, a^{-1})$. The set of such matrices splits into the orbits $O_{\lambda_1, \ldots, \lambda_{2g+2}}$: every such orbit consists of the matrices with fixed determinant

$$\det N(\lambda) = \prod_{i=1}^{2g+2} (\lambda - \lambda_i)$$

The affine model of $\mathfrak{N}$ (denoted by $\mathfrak{N}_a$) coincides with the joint of all such matrices with real $\lambda_1, \ldots, \lambda_{2g+2}$. $\mathfrak{N}_a$ is indeed a model for $\mathfrak{N}$ since the orbit $O_{\lambda_1, \ldots, \lambda_{2g+2}}$ is parametrized by the real part of the Jacobian associated to the curve with the branching points $\lambda_1, \ldots, \lambda_{2g+2}$. Exact description of that is given later ((4),(5),(6)).

From the point of view of finite-gap integration such matrix describes the $M$-operators associated with the stationary time.

Now let us map the space $\mathfrak{N}_a$ into a bigger space $\mathfrak{M}$. The points of the latter space are

$$\{\lambda_1, \ldots, \lambda_{2g+2}, \psi_1, \ldots, \psi_{2g+2}\}$$

where $\lambda_1, \ldots, \lambda_{2g+2} \in \mathbb{R}$, $\psi_i$ is vector from $\mathbb{C}^2$:

$$\psi_i = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}$$

Take some $N(\lambda) \in \mathfrak{N}_a$ with the determinant $\det N(\lambda) = \prod_i (\lambda - \lambda_i)$, and consider $N(\lambda_i)$ $(i = 1, \ldots, 2g + 2)$. Evidently being a matrix with zero trace and zero determinant it can be presented as

$$N(\lambda_i) = P'(\lambda_i)(\psi_i \otimes \bar{\psi}_i) = \begin{pmatrix} \alpha_i \beta_i, & -\alpha_i^2 \\ \beta_i^2, & -\alpha_i \beta_i \end{pmatrix},$$

$$\bar{\psi}_i = \psi_i c, \quad c = \begin{pmatrix} 0, & -1 \\ 1, & 0 \end{pmatrix}$$

for some $\psi_i$ ($P'(\lambda) \equiv \prod_k (\lambda_i - \lambda_k)$ is introduced for normalization). This describes (up to $\psi_i \rightarrow \pm \psi_i$) the map $\mathfrak{N}_a \rightarrow \mathfrak{M}$.

We consider the space $\mathfrak{M}$ as the phase space with canonical Poisson structure given on every finite-dimensional subspace by

$$\omega = \sum d\alpha_i \wedge d\beta_i$$

the variables $\{\lambda_1, \ldots, \lambda_{2g+2}\}$ are just constants (Poisson commute with everything). Now we want to proceed in opposite direction: to describe $\mathfrak{N}_a$ as join of trajectories of Hamiltonian systems defined on $\mathfrak{M}$. Consider the following momentum map

$$\{\lambda_1, \ldots, \lambda_{2g+2}, \psi_1, \ldots, \psi_{2g+2}\} \rightarrow N'(\lambda) = \sum_{i=1}^{2g+2} \frac{1}{\lambda_i - \lambda} \psi_i \otimes \bar{\psi}_i$$
The approach to integrable equations using this momentum map is presented in [10], see also the recent paper [11]. With the canonical Poisson structure for $\psi_i$ the matrices $N'$ satisfy r-matrix Poisson brackets [12] which provides that their determinants are in involution. The determinants can be written in the form

$$\det(N'(\lambda)) = \frac{P(\lambda)}{\prod(\lambda - \lambda_j)}, \quad \deg(P) \leq 2g$$

Generally for fixed values of $2g + 1$ integrals (fixed $P(\lambda)$) we have $2g$-dimensional torus of angles which coincides with the Jacobian of the surface

$$\tau^2 = P(\lambda) \prod(\lambda - \lambda_i)$$

But these are not generic orbits we are interested in. Oppositely, let us consider completely reduced orbits for which $P(\lambda) = 1$. Then the only special points of our matrix are $\lambda_1, \cdots, \lambda_{2g+2}$. If we do not impose this constraint every Jacobian will be counted many times. Now let us consider

$$N(\lambda) = \prod(\lambda - \lambda_i)N'(\lambda)$$

which is a polynomial matrix. Every completely reduced orbit is organized as ($J^R$ associated to the curve $\tau^2 = \prod(\lambda - \lambda_i)$) $\times$ (gauge transformations). Let us explain this. On every completely reduced orbit a non-degenerate subset exists of the matrices whose leading coefficient is not degenerate as $2 \times 2$ matrix (and, hence, stands before $\lambda^{g+1}$). This subset is parametrized by $J^R$. Other elements of the orbit are obtained from this subset by similarity transformations (gauge transformations) with polynomial matrix whose determinant equals 1, for example

$$\begin{pmatrix} 1, Q(\lambda) \\ 0, 1 \end{pmatrix}, \quad Q(\lambda) \text{ is polynomial}$$

So, to map the completely reduced orbit into $\mathfrak{M}_a$ one has to take the non-degenerate subset only (to consider one representative in every gauge class). This procedure allows usual Hamiltonian interpretation: in the polynomial $P(\lambda)$ there are $2g$ coefficients in involution, on the completely reduced orbit $g$ of them work as Hamiltonians (govern the motion along $J^R$) the remaining $g$ should be treated as first kind constraints.

Let us describe explicitly the map from $J^R$ into $\mathfrak{M}$ which is relevant for the construction above:

$$x \to \{\psi_j(x)\}_{j=1}^{2g+2}, \quad \psi_j(x) = \frac{1}{\theta(2x)} \left( \frac{\theta[\eta_j](r + 2x)}{\theta[\eta_j](r - 2x)} \right)$$

where $\eta_j$ is the half period defined by the integral from fixed branching point (say, $\lambda_1$) to the point $\lambda_j$,

$$r = \int_{\lambda_1}^{\infty^+} \omega$$

$\infty^+$ is one of the infinities on the surface, actually, any other real non Weierstrass point would do with minor changes, the thing which does matter is that the shift
of arguments in two components of (6) corresponds to a singular divisor i.e. two different points on the surface which project onto the same point on the complex plane. We put \(2x\) in the argument of \(\theta\)-functions in order that the map is properly defined on the Jacobian. Comments on the formulae (6) are given in Appendix.

Let us now, as in [1], using the solution of KZ construct the following homogeneous polynomials on \(2\mathcal{M}\):

\[
P(\psi) = \bar{\psi}_1, x_1 \otimes \cdots \otimes \bar{\psi}_{2g+2}, x_{2g+2} \ f(\lambda_1, \cdots, \lambda_{2g+2})^{x_1, \cdots, x_{2g+2}} \tag{7}
\]

which is just the scalar product of two vectors defined in \(\mathbb{C}^{\otimes (2g+2)}\). The formula (7) is constructed by analogy with the quantum case: it is similar to the contribution of \(2g+2\) particles with rapidities \(\lambda_1, \cdots, \lambda_{2g+2}\) into the generating function of local operators. The vectors \(\psi_i\) play role of creation operators of two-component particles. The idea of considering this object is the following. The space of particles in the quantum theory is “free” and rather huge (similar to classical \(2\mathcal{M}\)). The local operators cut some pieces from this space. In classics that should correspond to considering of (7) on the equation of motion i.e. we want to consider \(P(\psi(x))\) with \(\psi_i(x)\) given by (6).

So the problem is to calculate \(P(\psi(x))\) using the formulae (1),(2),(6). We expect the following result

\[
P(\psi(x)) = \text{Const} \ \det \left[ \Omega_{i,k} \partial_{x_i} \partial_{x_k} \log \theta(2x) + 2\pi i \delta_{i,j} \right]_{g \times g} \tag{8}
\]

The last formula is not easy to prove, however there is strong evidence in favor of it. First, considering the vectors \(f_C\) corresponding to all possible half-bases \(C\) one makes sure that the modular properties of RHS and LHS of (8) are the same. Second, it is easy to realize that \(P(\psi(x))\) does provide an interpolation for the RHS for all even nonsingular half-periods. As it has been said these half-periods correspond to partitions of \(\Lambda = \{\lambda_1, \cdots, \lambda_{2g+2}\}\) into \(\Lambda^+\) and \(\Lambda^-\) such that \(#\Lambda^+ = #\Lambda^- = g + 1\). The matrix \(N(\Lambda)\) for such half-period is given by

\[
U \left( \begin{array}{cc} 0, & \prod_{\lambda_i \in \Lambda^+} (\lambda - \lambda_i) \\ \prod_{\lambda_i \in \Lambda^-} (\lambda - \lambda_i), & 0 \end{array} \right) U^{-1} \tag{9}
\]

for some constant matrix \(U\) with \(\det U = 1\). So, the values of \(\psi_i\) at these points are quite simple, the matrix \(U\) can be omitted when substituting into (7) because \(f(\lambda_1, \cdots, \lambda_{2g+2})\) is singlet (spin zero) vector in the tensor product. It is explained in the Appendix how to get (9) from (6).

In a sense the formula (8) inverts the formula (1), it shows that not only \(f(\lambda_1, \cdots, \lambda_{2g+2})\) can be constructed via \(F(z)\) but the function from (8) which differs from \(\det F(2x)\) only by absence of \(\theta^4\) can be constructed via convolution of \(f(\lambda_1, \cdots, \lambda_{2g+2})\) with canonical vectors \(\psi_i\). So, in order to understand the classical limit of the generating function of local operators we have to understand the meaning of \(P(\psi(x))\). Let us think of the explicit formula (8). It looks as a measure on the phase space. Recall that locally (in action variables) we consider the phase space as angle variable multiplied by infinitesimal piece of the space of action variables: \(J^R \times \mathbb{R}^g\). The cotangent space has the basis \(dx_1, \cdots, dx_g; \xi_1 \cdots \xi_g\). We can write the formula:

\[
\det \left[ \Omega_{i,k} \partial_{x_i} \partial_{x_k} \log \theta(2x) + 2\pi i \delta_{i,j} \right]_{g \times g} dx_1 \wedge \cdots \wedge dx_g \wedge \xi_1 \wedge \cdots \wedge \xi_g = \wedge^g \omega
\]
where $\omega$ is the following 2-form:

$$\omega = (\Omega_{i,k} \partial_{x_k} \partial_{x_j} \log \theta(2x) + \delta_{i,j})dx_i \wedge \xi_j$$  \hspace{1cm} (10)

So, $P(\psi(x))$ is the maximal degree of the 2-form $\omega$. We have to understand the meaning of $\omega$. It looks quite symmetrical, for that reason it is natural to think of $\omega$ as of the form induced on the real space by some Kähler structure [13] on a complexification. That is what we shall explain in the next section.

3. Tau-function and Lefschetz embedding

Let us start this section with one formula from the paper [14]:

$$\tau(x + y)\tau(x - y) = \sum F_i(x)G_i(y)$$ \hspace{1cm} (11)

where $\tau$ is KdV $\tau$-function, $x = \{x_1, x_2, \cdots\}$ is an infinite set of times, the functions $G_i$ in the RHS are taken in "minimal" way (will be clarified soon). In the approach of [14] $\tau$ is associated with the level 1 representation of $\hat{sl}(2)$ denoted by $V(\Lambda)$. The LHS of (11) can be thought about as the tensor product of two such representations. Due to the complete reducibility the linear hull of LHS for different $y$ is level 2 representation ($V(2\Lambda)$). This linear hull is kept in mind when we talk about minimality of the set of $G_i$. For the minimal choice of $G_i$ the functions $F_i(x)$ constitute the basis of the representation $V(2\Lambda)$. The latter is realized in the Fock space associated to one massless Bose field (Heisenberg algebra) and one massless Majorana fermion (Clifford algebra) which is a special case of general parafermionic picture [15]. It is instructive to consider in this situation the Virasoro central charges which, roughly speaking, count the number of states in different modules. For the tensor product of two representations $V(\Lambda)$ the central charge equals 2, For the representation $V(2\Lambda)$ the central charge is $\frac{3}{2}$ ($1$ for boson and $\frac{1}{2}$ for fermion). So, for the orthogonal complement of $V(2\Lambda)$ in the tensor product of two $V(2\Lambda)$ the central charge is $\frac{1}{2}$, this subspace gives rise to all the Hirota equations.

Let us return to the basis in $V(2\Lambda)$ given by $F_i(x)$. The dependence on $x = \{x_1, x_2, \cdots\}$ is understood due to bosonic structure while the index $i$ corresponds to decomposition in fermions. On the other hand the index $i$ counts all the nonzero Hirota derivatives of the $\tau$ function. The space of these derivatives can be considered as the space of different KdV fields, but not exactly, due to Sato we know that more adequate understanding of the space of different KdV fields is the projective space associated to the space of Hirota derivatives. For example consider the KdV field $u$ itself. The corresponding second Hirota derivative (the coefficient before $(y_1)^2$ in Taylor decomposition of (11)) and the function $\tau^2(x)$ both are coordinates in our space. So, $u$ being equal to this second Hirota derivative divided by $\tau^2$ is the projective coordinate. Hence, we think of the map $x \rightarrow \{F_i(x)\}$ as of a mapping of the infinite dimensional abelian group into the projective space constructed from the fermionic Fock space.

Let us consider the finite dimensional version of this construction corresponding to the finite-gap solutions. For them infinite-dimensional group of $x$ reduces to finite-dimensional one ($J^R$), and the $\tau$-function is $\theta$-function (usually it is multiplied by some exponent of quadratic form, but we omit this multiplier which anyway would disappear from our final formulae). Not only the group of times but also
the space of KdV fields reduces to finite-dimensional in this limit because for the decomposition (11) we can use the classical formula:

\[
\theta(x + y|\Omega)\theta(x - y|\Omega) = \sum_{a \in \frac{1}{2}Z^g} \theta'_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2x|2\Omega)\theta'_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2y|2\Omega)
\]

(12)

So, the finite-dimensional version of the space of KdV fields is the projective space associated to the \(2^g\) dimensional space which evidently can be related to Clifford algebra with \(g\) generators. The group \(J^R\) is mapped into the space via

\[
x \rightarrow \theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2x|2\Omega)
\]

This mapping allows natural complexification which is the mapping of whole complex torus \(J\) into the complex projective space \(CP^{2^g-1}\):

\[
z \rightarrow w_a(z) = \theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2z|2\Omega)
\]

This is a classical mapping considered by Lefschetz [see 10].

The complex projective space is Kähler manifold. It allows hermitian, riemannian, symplectic structures which are related in different ways. In particular the symplectic form is given by

\[
\omega = \partial_{w_a}\partial_{\bar{w}_c}\log\left(\sum_a |w_a|^2\right)dw_b \wedge d\bar{w}_c
\]

This form induces symplectic form on \(J\):

\[
\omega = \partial_{z_i}\partial_{\bar{z}_j}\log\left(\sum_a \theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2z|2\Omega)^2\right)dz_i \wedge d\bar{z}_j
\]

The last formula can be simplified using one more time the addition theorem (12):

\[
\sum_a \theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2z|2\Omega)^2 = \sum_a \theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2z|2\Omega)\theta_{\left[\begin{array}{c} a \\ 0 \end{array}\right]}(2\bar{z}|2\Omega) = \theta(2x|\Omega)\theta(2y|\Omega)
\]

(13)

where \(x = \frac{1}{2}(z + \bar{z}), y = \frac{1}{2i}(z - \bar{z})\). Let us emphasize that the fact that \(\Omega\) is pure imaginary (related to \(\lambda_1, \cdots, \lambda_{2g+2} \in \mathbb{R}\) and proper choice of homology basis) is important in this calculation. Using (13) the symplectic form on \(J\) can be rewritten as

\[
\omega = \left(\partial_{x_i}\partial_{x_j}\log\theta(2x|\Omega) + \partial_{y_i}\partial_{y_j}\log\theta(2y|\Omega)\right)dx_i \wedge dy_j
\]

(14)

The last formula has much in common with (10) but it belongs to \(\wedge^2 T^*(J)\) for the complex torus \(J\) while the form (10) belongs to \(T^*(J^R) \times \mathbb{R}^g\). So, to obtain the form (10) from (14) one has to map \(T^*(J^R)\) into \(\mathbb{R}^g\). This is done by the period map which relates to any 1-form \(\omega^1 \in T^*(J^R)\) the following vector from \(\mathbb{R}^g\) with basis \(\{\xi_i\}\):

\[
\omega^1 \rightarrow \sum_i \left(\int_0^{\Omega_{\xi_i}} \omega^1\right)\xi_i
\]
where $\Omega$ is the period matrix, $\{e_i\}$ are basic vector of the lattice $\mathbb{Z}^g$. Using the properties of $\theta$-function one shows that under this mapping the form (14) turns into

$$\omega = (\Omega_{i,k} \partial_{x_k} \partial_{x_j} \log(2x) + 2\pi i \delta_{i,j}) dx_i \wedge \xi_j$$

which coincides with (10). Thus we have shown that the form (10) is induced on the local (with respect to actions) phase space by period mapping from the complex Jacobian, on which the symplectic form is induced by the canonical mapping into the complex projective space.

Returning to the general formula (11) we can say the following. It looks as if our real goal was to introduce a measure on the embedding of the infinite-dimensional group of times into the projective space related to the fermionic part of $V(2\Lambda)$ (space of classical fields). But we can not do it directly, so, we split the infinite-dimensional orbit into finite-dimensional ones and work with them. This is the very idea of Fock space. It is remarkable that in this way we are able to perform the exact quantization. It is also important that after the quantization the contributions from different Jacobians are connected through the residue equation for form factors [2].

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APPENDIX.

Here we provide useful information about $\theta$-functions which can be found in the books [10,16].

1. Definition of theta-function.

$$\theta(z|\Omega) = \sum_{m \in \mathbb{Z}^g} \exp\{\pi i m^t \Omega m + 2\pi i z^t m\}$$

$$\theta[\eta](z|\Omega) = \exp\{\pi i \eta^t \Omega \eta' + 2\pi i (z + \eta'')^t \eta'\} \theta(z + \eta'' + \Omega \eta'|\Omega)$$

where $z \in \mathbb{C}^g$, $\eta = \eta', \eta'' \in \mathbb{Q}^g$.

2. Riemann theorem for theta-function on hyper-elliptic surface.

$$\theta(\int_{P_1}^{Q_1} - \int_{\lambda_3}^{Q_3} - \cdots - \int_{\lambda_{2g+1}}^{Q_{2g+1}})$$

is either identically zero or has simple zeros only at the points $P = Q_1, \cdots, Q_g$, we use Fay’s notations [16]: $\int = \int \omega$.

3. Divisor of meromorphic function.

$$\sum_{P_i}^{Q_i} \int = 0$$

if and only if there is a meromorphic function with simple zeros at $\{P_i\}$ and simple poles at $\{Q_i\}$. 

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4. Two formulae. From Riemann theorem and the property of the divisor of meromorphic function one gets

\[
\frac{\theta[\delta](\int_{\lambda_1}^P \frac{\partial}{\partial \theta} \theta[\delta](\int_{\lambda_1}^P + f_{\lambda_1}^S))}{\theta[\delta](\int_{\lambda_1}^P - gr) \theta[\delta](\int_{\lambda_1}^P + gr)} = C(S) \prod_{i \in S} (x - \lambda_i) \quad \#S = g ;
\]

\[
\frac{\theta[\delta](\int_{\lambda_1}^P \frac{\partial}{\partial \theta} \theta[\delta](\int_{\lambda_1}^P + f_{\lambda_1}^S))}{\theta[\delta](\int_{\lambda_1}^P - gr) \theta[\delta](\int_{\lambda_1}^P + gr)} = C(S)(\prod_{i \in S} (x - \lambda_i) - \prod_{i \in \bar{S}} (x - \lambda_i))
\]

\#S = g + 1

where \(x\) is the projection of \(P\) onto the complex plane, \(\Lambda_S\) is a subset of the set of branching points, \(\delta\) is Riemann constant:

\[\delta'' + \Omega \delta' = \sum_{i=1}^{g+1} \int_{\lambda_i} \]

finally

\[r = \int_{\lambda_1}^{\infty^+} = - \int_{\lambda_1}^{\infty^-} \]

5. Theta constants. The formulae above allow to calculate certain special values of \(\theta\)-function. First type of them is given by

\[\theta[\eta_T](0) = C_1 \prod_{i > j \in T} (\lambda_i - \lambda_j)^\frac{1}{4} \prod_{i > j \in T} (\lambda_i - \lambda_j)^\frac{1}{4}, \quad \#T = g + 1, \quad \bar{T} = B \setminus T\]

where \(T \in \{1, 2, \ldots, 2g + 2\}\),

\[\eta''_T + \Omega \eta'_T = \int_{\Lambda_T} \]

the subset \(U = \{1, 3, 5, \ldots\}\) corresponds to Riemann constants. The positive constant \(C_1\) depends on \(\{\lambda_i\}\) but does not depend on the partition \(T\), the exact value of \(C_1\) is given by Tomae formula. Another type of \(\theta\)-constants of interest is given by

\[\theta[\eta_S](r) = C_2 \prod_{i > j \in S} (\lambda_j - \lambda_i)^\frac{1}{4} \prod_{i > j \in S} (\lambda_j - \lambda_i)^\frac{1}{4} \]

where \(\#S = g\), the characteristic is given by

\[\eta''_S + \Omega \eta'_S = \int_{\Lambda_S} \]

\[C_1 \text{ is a positive constant. The second relation allows to prove that the matrix } N(\lambda, x) \text{ defined by (4),(5),(6) takes at even non-singular half-periods values (9). The first formula is used when substituting corresponding } \psi_i \text{ into (7).}

6. Frobenius formulae.
There is the following nice addition formula on hyper-elliptic surfaces:

\[ 2g + 2 \sum_{j=1}^{2g+2} (-1)^j \theta[\zeta_1 + \eta_j](x_1) \theta[-\zeta_1 + \eta_j](x_2) \theta[\zeta_2 + \eta_j](x_3) \theta[-\zeta_2 + \eta_j](x_4) = 0 \]

for arbitrary characteristics \( \zeta_1, \zeta_2, x_1 + x_2 + x_3 + x_4 = 0 \), the characteristic \( \eta_j \) is associated to \( j \)-th branching point:

\[ \eta_j'' + \Omega \eta_j' = \int_{x_1}^{x_4} \lambda_j \]

Using these formula for \( x_1 = x_2 = 0, x_3 = -x_4 = 2x \) and for \( x_1 = x_2 = -r, x_3 = r-2x, x_4 = r+2x \) and for proper half-periods \( \zeta_1, \zeta_2 \) one gets after some calculations using the theta constants above:

\[ \sum_{j=1}^{2g+2} \lambda_j^p \alpha_j^2(x) = 0 , \quad p = 0, \ldots, g \]

\[ \sum_{j=1}^{2g+2} \lambda_j^p \beta_j^2(x) = 0 , \quad p = 0, \ldots, g \]

\[ \sum_{j=1}^{2g+2} \lambda_j^p \alpha_j(x) \beta_j(x) = 0 , \quad p = 0, \ldots, g - 1 \]

\[ \sum_{j=1}^{2g+2} (2\lambda_j^g + 1 - \sum_{i=1}^{g} \lambda_i^g) \alpha_j(x) \beta_j(x) = 0 \]

which is needed for proof that the formula (6) gives parametrization of \( \mathcal{M}_a \).

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