RIGOROUS ASYMPTOTICS OF A KDV SOLITON GAS

M. GIROTTI, T. GRAVA, AND K. D. T.-R. MCLAUGHLIN

Abstract. We analytically study the long time and large space asymptotics of a new broad class of solutions of the KdV equation introduced by Dyachenko, Zakharov, and Zakharov. These solutions are characterized by a Riemann–Hilbert problem which we show arises as the limit \( N \to +\infty \) of a gas of \( N \)-solitons. We establish an asymptotic description for large times that is valid over the entire spatial domain, in terms of Jacobi elliptic functions.

Contents

1. Introduction \hfill 1
   1.1. The soliton gas \hfill 3
   1.2. Statement of the results \hfill 3
2. Soliton gas as limit of \( N \) solitons as \( N \to +\infty \) \hfill 5
3. Behaviour of the potential \( u(x,0) \) as \( x \to -\infty \) \hfill 9
   3.1. Large \( x \) asymptotic \hfill 10
   3.2. Opening lenses \hfill 12
   3.3. The global parametrix \( S^{(\infty)} \) \hfill 14
   3.4. A second vector solution \hfill 16
   3.5. The global parametrix \( P^{(\infty)} \) \hfill 18
   3.6. The local parametrix \( P^{(x_j)} \) at the endpoints \hfill 18
   3.7. Small norm argument and determination of \( u(x,0) \) for large negative \( x \) \hfill 21
4. Behaviour of the potential \( u(x,t) \) as \( t \to +\infty \) \hfill 22
5. Super-critical case: the \( \alpha \)-dependency \hfill 23
   5.1. Opening lenses \hfill 27
   5.2. The global parametrix \( \tilde{P}^{(\infty)} \) \hfill 29
   5.3. The local parametrix \( P^{(x_\alpha)} \) \hfill 29
   5.4. Small norm argument and determination of \( u(x,t) \) as \( t \to +\infty \) \hfill 31
6. Sub-critical case \hfill 33
7. Conclusions \hfill 34
References \hfill 35

1. Introduction

This paper concerns the concept of a gas of solitons for the Korteweg-de Vries (KdV) equation,

\[
 u_t - 6uu_x + u_{xxx} = 0 .
\]  

It is well known that this nonlinear partial differential equation is integrable, arising as the compatibility condition of a Lax pair of linear differential operators. The compatibility condition
can be presented as the existence of a simultaneous solution to the pair of equations

\begin{align}
-\psi_{xx} + uu &= E \psi, \\
\psi_t - 4\psi_{xxx} + 6uu_x + 3uu_x &= 0,
\end{align}

where $E$ is the spectral parameter and $\psi = \psi(x,t)$. The Lax pair formulation yields a complete solution procedure for the initial value problem for (1.1) via the inverse scattering transform in the case of rapidly decaying or step-like initial data, and has led to large and ever-growing collection of results concerning the analysis of the initial value problem in many different asymptotic regimes, including the behaviour in the small dispersion limit, as well as a complete description of the long-time behaviour for fairly general decaying or step-like initial conditions. In the case of periodic boundary conditions as well, there have been many works that are aimed at understanding the behaviour of solutions as well as the geometry of the space of solutions. These works have all been driven by the physical origins of the KdV equation as a basic model for one-dimensional wave motion of the interface between air and water, and in particular the discovery of the soliton. The soliton is a rapidly decreasing travelling wave solution of the KdV equation, namely a solution of the form $u(x,t) = f(x - vt)$ and takes the form

\begin{equation}
\phi(x,t) = -2\eta^2 \text{sech}^2 \left(2\eta(x - 4\eta^2 t - x_0)\right)
\end{equation}
1.1. The soliton gas. Towards the goal of discovering new, broad families of solutions to integrable nonlinear PDEs, the “dressing method” as developed by Zakharov and Manakov [ZM85] has yielded some interesting new results in [DZZ16]. In that paper, the authors show how the dressing method can be used to produce a new family of solutions they refer to as “primitive potentials” which although are not random, can be naturally interpreted as a soliton gas. Cutting to the chase, the authors derive a Riemann–Hilbert problem which is to determine a vector \( \Xi = [\Xi_1, \Xi_2]^T \) satisfying a normalization condition at \( \infty \), and the jump relations

\[
\Xi_+(i\lambda) = J(\lambda)\Xi_-(i\lambda), \quad \Xi_+(-i\lambda) = J^T(\lambda)\Xi_-(-i\lambda), \quad \lambda \in (\eta_1, \eta_2)
\]

where the jump matrix \( J(\lambda) \) is given by

\[
J(\lambda) = \frac{1}{1 + r_1(\lambda)r_2(\lambda)} \begin{bmatrix} 1 - r_1(\lambda)r_2(\lambda) & 2ir_1(\lambda)e^{-2\lambda x} \\ 2ir_2(\lambda)e^{2\lambda x} & 1 - r_1(\lambda)r_2(\lambda) \end{bmatrix}
\]

The parameters \( \eta_1 \) and \( \eta_2 \) are taken to be real with \( 0 < \eta_1 < \eta_2 \), and the orientation of the jumps is the following: both the intervals \((i\eta_1, i\eta_2)\) and \((-i\eta_2, -i\eta_1)\) are oriented downwards.

The reflection coefficients \( r_1(\lambda) = r_1(\lambda;t) \) and \( r_2(\lambda) = r_2(\lambda;t) \) evolve in time according to

\[
r_1(\lambda;t) = r_1(\lambda;0)e^{(8\lambda^2 - 12\lambda)t}, \quad r_2(\lambda;t) = r_2(\lambda;0)e^{-(8\lambda^2 - 12\lambda)t}.
\]

The parameters \( \eta_1 \) and \( \eta_2 \) are taken to be real with \( 0 < \eta_1 < \eta_2 \), and the jump relations are oriented downwards.

The reflection coefficients \( r_1(\lambda) = r_1(\lambda;t) \) and \( r_2(\lambda) = r_2(\lambda;t) \) evolve in time according to

\[
r_1(\lambda;t) = r_1(\lambda;0)e^{(8\lambda^2 - 12\lambda)t}, \quad r_2(\lambda;t) = r_2(\lambda;0)e^{-(8\lambda^2 - 12\lambda)t}.
\]

The authors consider a number of different settings, and use a combination of analytical and computational methods to provide a description of the solutions of the KdV equation determined by this Riemann–Hilbert problem. In the case that \( r_2 \equiv 0 \), the potential is exponentially decaying as \( x \to \pm \infty \). But the behavior as \( x \) grows in the other direction (as well as the the asymptotic behavior for \( |x| \) large in the case that both reflection coefficients are nontrivial) was mentioned as a challenging problem for both analysis and computation.

The configuration of solitons considered in [DZZ16] is somewhat different than the solitonic gas configurations considered in [DP14] and [SP16], where they considered a large number of solitons that were spaced quite far apart from each other at \( t = 0 \). In other words, they considered a dilute gas of solitons that had enough space between them to evolve as isolated solitons until they interact, usually in a pair-wise fashion. In contrast, the soliton gas considered in [DZZ16] (and considered here as well) is a configuration that cannot be viewed as a collection of isolated solitons. Indeed, as we show, they are overlapping to the extent that, even at \( t = 0 \), they effectively behave as a genus-1 dispersive wave, and this poses a conceptual challenge in that finite-soliton interactions may or may not be relevant.

1.2. Statement of the results. In Section 2 we consider a sequence of Riemann–Hilbert problems, indexed by \( N \), for a pure \( N \)-soliton solution, with spectrum confined to the intervals \((-i\eta_2, -i\eta_1) \cup (i\eta_1, i\eta_2)\) for some \( \eta_2 > \eta_1 > 0 \) and show that for this sequence, as \( N \to +\infty \), the solution of the Riemann–Hilbert problem converges to the solution of the Riemann–Hilbert problem studied in [DZZ16], for the case \( r_2 \equiv 0 \).

Then in Section 3 (Theorem 3.6) we establish that the potential \( u(x) \) determined by this Riemann–Hilbert problem coincides with the Lamé potential as \( x \to -\infty \):

\[
u(x) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{\text{dn}^2}{(\eta_2(x + \phi) + K(m)|m) + O(x^{-1})}.
\]

The function \( \text{dn} (z|m) \) is the Jacobi elliptic function of modulus \( m = \eta_1/\eta_2 \). It is periodic with period \( 2K(m) \), and satisfies \( \text{dn}(0|m) = 1 \) and \( \text{dn}(K(m)|m) = \sqrt{1 - m^2} \). Here \( K(m) \) is the complete elliptic integral of the first kind of modulus \( m \): \( K(m) = \int_0^{\pi/2} \frac{\text{d}\phi}{\sqrt{1 - m^2 \sin^2 \phi}} \). Therefore the function \( (1.9) \) is periodic in \( x \) with period \( 2K(m)/\eta_2 \). The minimum amplitude of the oscillations
is $-\eta_2^2 - \eta_1^2$ and the maximum amplitude is $\eta_1^2 - \eta_2^2$ so that the amplitude of the oscillations is $2\eta_1^2$. The average value of $u(x)$ over an oscillation is equal to

$$\langle u(x) \rangle = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{E(m)}{K(m)},$$

where $E(m)$ is the complete elliptic integral of the second kind: $E(m) = \int_0^{\pi/2} \sqrt{1 - m^2 \sin^2 \theta} \, d\theta$. The phase $\phi$ in formula (1.10) depends on the coefficient $r_1(\lambda)$ that characterizes the continuum limit of the norming constants of the soliton gas and it is equal to

$$\phi = \int_{\eta_1}^{\eta_2} \frac{\log 2r_1(i\xi)}{\sqrt{(\zeta^2 - \eta_1^2)(\zeta^2 - \eta_2^2)}} \, d\zeta \in \mathbb{R}.$$  \hspace{1cm} (1.10)

A technical issue arises in the control of the error, and when $\frac{2y}{2K(m)}(x + \phi) = \frac{1}{2} + n, n \in \mathbb{Z}$, one requires a different local parametrix in the Riemann–Hilbert analysis or one needs consider, for this particular value of the parameter $x$, a vector-valued Riemann–Hilbert problem for the error, whereas for all other values of $x$ we essentially use an analysis based on matrix-valued Riemann–Hilbert problems. While the error term is expected to be uniform, in our analysis we explicitly assume that $\frac{2y}{2K(m)}(x + \phi) \neq \frac{1}{2} + n, n \in \mathbb{Z}$.

Finally in Sections 4 - 6 we provide a global long-time asymptotic description of the solution $u(x, t)$ to the KdV equation with this initial data $u(x)$. The asymptotic behaviour depends on the quantity $\xi = x/4t$. There are three main regions, (1) a constant region, (2) a region where the solution is approximated by a periodic traveling wave with constant coefficients specified by the spectral data, (3) a region where the solution is approximated by a periodic travelling wave with modulated coefficients see Figure 1. More precisely:

(1) for fixed $\xi > \eta_2^2$, there is a positive constant $C = C(\xi)$ so that

$$u(x, t) = O\left(e^{-Ct}\right).$$

(2) For $\xi < \xi_{\text{crit}}$ we have

$$u(x, t) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{d}{\eta_2^2} \left(\eta_2(x - 2(\alpha^2 + \eta_2^2)t + \phi) + K(m) | m \right) + O\left(t^{-1}\right), \hspace{1cm} (1.11)$$

with $m = \eta_1/\eta_2$ and $\phi$ as in (1.10). The critical value $\xi_{\text{crit}}$ is obtained from the equation

$$\xi_{\text{crit}} = \int_{\alpha}^{\eta_2} \frac{2r_1(i\xi)}{\sqrt{(\zeta^2 - \alpha^2)(\zeta^2 - \eta_2^2)}} \, d\zeta \in \mathbb{R}.$$  \hspace{1cm} (1.12)

(3) For $\xi_{\text{crit}} < \xi < \eta_2^2$ we have that

$$u(x, t) = \eta_2^2 - \alpha^2 - 2\eta_2^2 \frac{d}{\eta_2^2} \left(\eta_2(x - 2(\alpha^2 + \eta_2^2)t + \phi) + K(m, \alpha) | m \right) + O\left(t^{-1}\right),$$

where $dn(z | m, \alpha)$ is the Jacobi elliptic function of modulus $m, \alpha = \alpha/\eta_2$, and the coefficient $\alpha = \alpha(\xi)$ is determined from the Whitham modulation equation [Whi74]

$$\xi = \frac{x}{4t} = \frac{\eta_2^2}{2} W(m_{\alpha}),$$

where $W(m)$ has been defined in (1.12). Similar to our analysis for $x \to -\infty$ with $t = 0$, in each of these asymptotic regimes we exclude certain special values of the parameters, where the Riemann–Hilbert analysis requires additional care. So, in (1.11) we assume that $\frac{2y}{2K(m)}(x - 2(\eta_1^2 + \eta_2^2)t + \phi) = \frac{1}{2} + n, n \in \mathbb{Z}$, and in (1.13) we assume that $\frac{2y}{2K(m)}((x - 2(\alpha^2 + \eta_2^2)t + \phi)) = \frac{1}{2} + n, n \in \mathbb{Z}$.
The equation (1.14) was used by Gurevich and Pitaevskii \cite{GP73} to describe the modulation of the travelling wave that is formed in the solution of the KdV equation with a step initial data $u(x, 0) = -\eta_2^2$ for $x < 0$ and $u(x, 0) = 0$ for $x > 0$. Such a modulated travelling wave is also called a dispersive shock wave. The rigorous analysis of the dispersive shock wave emerging from step-like initial data problem was obtained via inverse scattering in \cite{Hru76} and more recently via Riemann–Hilbert methods in \cite{EGKT13}.

We observe that in this paper, if $\eta_1$ is set to 0, then $m = 0$ and $\lim_{m \to 0} dn(z|m) = 1$ so that our asymptotic description in (1.9) yields a potential that converges to $-\eta_2^2$ as $x \to -\infty$. The behavior for $t$ large agrees (globally) with the dispersive shock wave emerging from a step like initial data with reflection coefficient equal to zero on the real axis. In other words, in the case that $\eta_1 = 0$, the "primitive potential" identified in \cite{DZZ16} is in the class of step-like potentials.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{soliton_gas.png}
\caption{Soliton gas behaviour at $t = 10$ with endpoints $\eta_1 = 0.5$ and $\eta_2 = 1.5$ and reflection coefficient $r_1(\lambda) \equiv 1$.}
\end{figure}

\section{Soliton gas as limit of $N$ solitons as $N \to +\infty$}

The Riemann–Hilbert problem for a pure $N$-soliton solution (see for example \cite{GT09}) is described as follows: find a 2-dimensional row vector $M$ such that

(i) $M(\lambda)$ is meromorphic in $\mathbb{C}$, with simple poles at $\{\lambda_j\}_{j=1}^N$ in $i\mathbb{R}_+$, and at the corresponding conjugate points $\{\overline{\lambda}_j\}_{j=1}^N$ in $i\mathbb{R}_-;$

(ii) $M$ satisfies the residue conditions

\[
\text{res}_{\lambda=\lambda_j} M(\lambda) = \lim_{\lambda \to \lambda_j} M(\lambda) \begin{bmatrix} 0 \\ \frac{\overline{c_j} e^{2i\overline{\lambda}_j x}}{N} \\ 0 \end{bmatrix}, \quad \text{res}_{\lambda=\overline{\lambda}_j} M(\lambda) = \lim_{\lambda \to \overline{\lambda}_j} M(\lambda) \begin{bmatrix} 0 \\ \frac{-c_j e^{-2i\lambda_j x}}{N} \\ 0 \end{bmatrix},
\]

where $c_j \in i\mathbb{R}$;

(iii) $M(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + O\left(\frac{1}{\lambda}\right)$ as $\lambda \to \infty$. 
The $N$-soliton potential $u(x)$ is determined from $M$ via
\[
   u(x) = 2 \frac{d}{dx} \left( \lim_{\lambda \to \infty} \frac{\lambda}{i} (M_1(\lambda) - 1) \right),
\]
where $M_1(\lambda)$ is the first entry of the vector $M(\lambda)$. In particular, for a one soliton potential, namely $N = 1$, one recovers the expression (1.4) where the shift $x_0$ is given by
\[
   x_0 = \frac{1}{4\eta_1} \log \frac{c_1}{2i\eta_1}.
\]
We are interested in the limit as $N \to +\infty$, under the additional assumptions
\begin{enumerate}
   \item The poles $\{\lambda_j^{(N)}\}_{j=1}^N$ are sampled from a density function $\varrho(\lambda)$ so that $\int_{\eta_1}^{-i\lambda_j} \varrho(\eta)d\eta = j/N$, for $j = 1, \ldots, N$.
   \item The coefficients $\{c_j\}_{j=1}^N$ are purely imaginary (in fact $c_j \in i\mathbb{R}_+$) and are assumed to be discretizations of a given function:
   \[
      c_j = \frac{i(\eta_2 - \eta_1)}{\pi} r_1(\lambda_j) \quad j = 1, \ldots, N.
   \]
   \end{enumerate}

where $r_1(\lambda)$ is an analytic function for $\lambda$ near the intervals $(i\eta_1, i\eta_2)$ and $(-i\eta_2, -i\eta_1)$, with the symmetry $r_1(\overline{\lambda}) = r_1(\lambda)$, and is further assumed to be a real valued and non-vanishing function of $\lambda$ for $\lambda \in [i\eta_1, i\eta_2]$.

In the regime $x \to +\infty$, it is easy to notice that all residue conditions contain only exponentially small terms and therefore, by a small norm argument, the potential is exponentially small.

On the other hand, for $x \to -\infty$ all of those terms are exponentially large. To show that the solution is also exponentially small in this latter case, one may reverse the triangularity of the residue conditions, by defining
\[
   A(\lambda) = M(\lambda) \prod_{j=1}^N \left( \frac{\lambda - \lambda_j}{\lambda - \overline{\lambda}_j} \right)^\sigma_j.
\]

Now the residue conditions are
\[
\begin{align*}
   \text{res}_{\lambda = \lambda_j} A(\lambda) &= \lim_{\lambda \to \lambda_j} A(\lambda) \left[ \begin{array}{c}
   0 \\
   0 \\
   \frac{N}{c_j} e^{-2i\lambda_j x} (\lambda_j - \overline{\lambda}_j)^2 \prod_{k \neq j} \left( \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda}_k} \right)^2
   \end{array} \right] \\
   \text{res}_{\lambda = \overline{\lambda}_j} A(\lambda) &= \lim_{\lambda \to \overline{\lambda}_j} A(\lambda) \left[ \begin{array}{c}
   0 \\
   0 \\
   \frac{N}{c_j} e^{2i\lambda_j x} (\overline{\lambda}_j - \lambda_j)^2 \prod_{k \neq j} \left( \frac{\overline{\lambda}_j - \lambda_k}{\overline{\lambda}_j - \lambda_k} \right)^2
   \end{array} \right]
\end{align*}
\]
while the potential $u(x)$ is still extracted from $A$ via the same calculation:
\[
   u(x) = 2 \frac{d}{dx} \left( \lim_{\lambda \to \infty} \frac{\lambda}{i} (A_1(\lambda) - 1) \right).
\]

The quantity $e^{-2i\lambda_j x}$ now decays exponentially as $x \to -\infty$, and this implies (again by a standard small-norm argument) exponential decay of the potential $u(x)$ for $x \to -\infty$. On the other hand, the product term is exponentially large in $N$:
\[
\prod_{k \neq j} \left( \frac{\lambda_j - \lambda_k}{\lambda_j - \overline{\lambda}_k} \right)^2 = O(e^{CN}) \quad \text{for each } j = 1, \ldots, N
\]
for some $C \in \mathbb{R}_+$. 

Therefore this exponential decay does not set in until $x$ is rather large. Indeed, in order for the residue conditions to all be exponentially small, it must be that $x \ll -CN$. In other words, the $N$-soliton solution that we are considering has very broad support, and in the large-$N$ limit, it is not exponentially decaying for $x \nearrow -\infty$.

We will show here how to derive the Riemann–Hilbert problem for a soliton gas with one reflection coefficient $r_1$ (as described in [DZZ16]) from a meromorphic Riemann–Hilbert problem for $N$ solitons in the limit as $N \nearrow +\infty$. First, we remove the poles by defining

$$Z(\lambda) = M(\lambda) \begin{bmatrix} 1 & \frac{1}{N} \sum_{j=1}^{N} c_j e^{2i\lambda x} \lambda - \lambda_j \end{bmatrix}$$

within a closed curve $\gamma_+$ encircling the poles counterclockwise in the upper half plane $\mathbb{C}_+$, and

$$Z(\lambda) = M(\lambda) \begin{bmatrix} 1 & \frac{1}{N} \sum_{k=1}^{N} c_j e^{-2i\lambda x} \lambda - \lambda_j \end{bmatrix}$$

within a closed curve $\gamma_-$ surrounding the poles clockwise in the lower half plane $\mathbb{C}_-$. Outside these two sets, we take $Z(\lambda) = M(\lambda)$.

Then the jumps are

$$Z_+(\lambda) = Z_-(\lambda) \begin{cases} 1 & \lambda \in \gamma_+ \\ \frac{1}{N} \sum_{j=1}^{N} c_j e^{2i\lambda x} \lambda - \lambda_j & \lambda \in \gamma_+ \\ 1 & \lambda \in \gamma_- \\ 1 & \lambda \in \gamma_- \end{cases}$$

where, for $\lambda \in \gamma_+$ or $\gamma_-$, the boundary values $Z_+(\lambda)$ are taken from the left side of the contour as one traverses it according to its orientation, and the boundary values $Z_-(\lambda)$ are taken from the left. The quantity $Z(\lambda)$ is normalized so that $Z(\lambda) = [1 \ 1] + O(\lambda^{-1})$ as $\lambda \to \infty$.

We assume now that in the limit as the number of poles goes to infinity, the poles are distributed according to some distribution $\varrho(\lambda)$ with density compactly supported in $(i\eta_1, i\eta_2)$ (and extended by symmetry on the corresponding interval in the lower half plane).

For the sake of simplicity, we can assume that the $N$ poles are equally spaced along $(i\eta_1, i\eta_2)$ with distance between two poles equal to $|\Delta \lambda| = \frac{2\pi}{N}$ and with (atomic) density:

$$\varrho_N(\lambda) = \frac{1}{Z_N} \sum_{j=1}^{N} c_j \delta_{\lambda_j}(\lambda) \quad \lambda \in (i\eta_1, i\eta_2),$$

for some normalization constant $Z_N$.

\textbf{Remark 2.1.} In the case where the poles are distributed according to a more general measure $\varrho(\lambda)$, the steps to follow are very similar. The entries of the jump matrices will carry the density function along, which can be eventually incorporated in the definition of the reflection coefficient $r_1$.

As the number of poles increases within the support of the measure, the following result holds.
Proposition 2.2. For any open set $K_+$ containing the interval $[i\eta_1,i\eta_2]$, and any open set $K_-$ containing the interval $[-i\eta_2,-i\eta_1]$, the following limit holds uniformly for all $\lambda \in \mathbb{C}\setminus K_+$:

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} \frac{c_j}{\lambda - \lambda_j} = \int^{i\eta_2}_{i\eta_1} \frac{2i\tau_1(\zeta)}{\lambda - \zeta} \frac{d\zeta}{2\pi i};$$

(2.12)

and the following limit holds uniformly for all $\lambda \in \mathbb{C}\setminus K_-$:

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} \frac{c_j}{\lambda - \lambda_j} = \int^{-i\eta_1}_{-i\eta_2} \frac{2i\tau_1(\zeta)}{\lambda - \zeta} \frac{d\zeta}{2\pi i},$$

(2.13)

where $\tau_1(\lambda)$ is an analytic function for $\lambda$ near the intervals $(i\eta_1,i\eta_2)$ and $(-i\eta_2,-i\eta_1)$, and it is assumed to be a real valued and non-vanishing function of $\lambda$ for $\lambda \in [i\eta_1,i\eta_2]$.

Proof. Using (2.3), the expressions in the jumps can be rewritten as

$$\frac{1}{N} \sum_{j=1}^{N} \frac{c_j}{\lambda - \lambda_j} = \frac{1}{N} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \left(\eta_2 - \eta_1\right) \frac{\tau_1(\lambda_j)}{\pi} = \frac{1}{2\pi i} \sum_{j=1}^{N} \frac{2\tau_1(\lambda_j)}{\lambda - \lambda_j} \Delta \lambda .$$

(2.14)

The convergence follows from the convergence of the Riemann sum to the Riemann–Stieltjes integral for $x \in K$ any compact subset of $\mathbb{R}$. 

Thanks to the proposition above and a small norm argument, we arrive at a limiting Riemann–Hilbert problem (which we still call $Z$ with abuse of notation)

$$Z_+(\lambda) = \begin{bmatrix} e^{2i\lambda x} \int^{i\eta_2}_{i\eta_1} \frac{2i\tau_1(\zeta)}{\zeta - \lambda} \frac{d\zeta}{2\pi i} & 0 \\ 1 & 1 \end{bmatrix} \quad \lambda \in \gamma_+,$$

(2.15)

$$Z_-(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + O\left(\frac{1}{\lambda}\right) \quad \lambda \to \infty .$$

(2.16)

At this point it is important to point out to the reader that the contour $(i\eta_1,i\eta_2)$ and $(-i\eta_2,-i\eta_1)$ are both oriented upwards.

Next, we define

$$X(\lambda) = Z(\lambda) \begin{bmatrix} e^{2i\lambda x} \int^{i\eta_2}_{i\eta_1} \frac{2i\tau_1(\zeta)}{\zeta - \lambda} \frac{d\zeta}{2\pi i} \\ 1 \end{bmatrix}$$

(2.17)

within the loop $\gamma_+$, and

$$X(\lambda) = Z(\lambda) \begin{bmatrix} 1 & e^{-2i\lambda x} \int^{-i\eta_1}_{-i\eta_2} \frac{2i\tau_1(\zeta)}{\zeta - \lambda} \frac{d\zeta}{2\pi i} \\ 0 \end{bmatrix}$$

(2.18)

within the loop $\gamma_-$. Outside these two curves, we define $X(\lambda) = Z(\lambda)$.

The jumps across the curves are no longer present, but there are jumps across $(i\eta_1,i\eta_2)$ and $(-i\eta_2,-i\eta_1)$ because the integrals have jumps across those intervals. Using Sokhotski-Plemelj
formula, we arrive at a Riemann–Hilbert problem for \(X\)

\[
X_+(\lambda) = X_-(\lambda) \begin{bmatrix}
1 & 0 \\
-2ir_1(\lambda)e^{2i\lambda x} & 1 \\
1 & 2ir_1(\lambda)e^{-2i\lambda x} \\
0 & 1
\end{bmatrix} \quad \lambda \in (i\eta_1, i\eta_2)
\]

\[
X(\lambda) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + O \left( \frac{1}{\lambda} \right) \quad \lambda \to \infty .
\]  

(2.19)

This Riemann–Hilbert problem is equivalent to the one described in [DZZ16] with \(r_2 = 0\), up to a transposition (\(X\) is a row vector here, while the solution of the Riemann–Hilbert problem in [DZZ16] is a column vector), and using the symmetry that \(r_1(\lambda) = r_1(\lambda)\) for \(\lambda \in (-i\eta_2, -i\eta_1)\).

We note that there is a sign discrepancy between this Riemann–Hilbert problem and the one appearing in [DZZ16], which is resolved by a careful interpretation of the sign conventions therein.

3. Behaviour of the potential \(u(x, 0)\) as \(x \searrow -\infty\)

We consider a soliton gas Riemann–Hilbert problem as in (2.19) – (2.20) with \(0 < \eta_1 < \eta_2\), and reflection coefficient \(r_1(\lambda)\) defined on \((i\eta_1, i\eta_2)\) such that it has an analytic extension to a neighbourhood of this interval. Furthermore we assume that \(r_1(-\lambda) = r_1(\lambda)\) on the imaginary axis. We set \(\Sigma_1 = (\eta_1, \eta_2)\) and \(\Sigma_2 = (-\eta_2, -\eta_1)\). The vector valued function \(X\) that will determine the KdV potential \(u(x)\) is the solution to the following Riemann–Hilbert problem:

\[
X(\lambda) \text{ analytic for } \lambda \in \mathbb{C} \setminus \{i\Sigma_1 \cup i\Sigma_2\}
\]

\[
X_+(i\lambda) = X_-(i\lambda) \begin{bmatrix}
1 & 0 \\
-2ir_1(i\lambda)e^{-2i\lambda x} & 1 \\
1 & 2ir_1(i\lambda)e^{2i\lambda x} \\
0 & 1
\end{bmatrix} \quad \lambda \in \Sigma_1
\]

\[
X(\lambda) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + O \left( \frac{1}{\lambda} \right) \quad \lambda \to \infty .
\]  

(3.1)

As explained in [DZZ16], we can recover the potential \(u(x)\) of the Schrödinger operator via the formula

\[
u(x) = 2 \frac{d}{dx} \lim_{\lambda \to \infty} \frac{\lambda}{i} (X_1(\lambda; x) - 1)
\]

(2.2)

where \(X_1(\lambda; x)\) is the first component of the solution vector \(X\).

We first perform a rotation of the problem in order to place the jumps on the real line, by setting

\[
Y(\lambda) = X(i\lambda), \quad r(\lambda) = 2r_1(i\lambda);
\]

(3.3)

the Riemann–Hilbert problem for \(Y\) reads as follows

\[
Y_+(\lambda) = Y_-(\lambda) \begin{bmatrix}
1 & 0 \\
-ir(\lambda)e^{-2i\lambda x} & 1 \\
1 & ir(\lambda)e^{2i\lambda x} \\
0 & 1
\end{bmatrix} \quad \lambda \in \Sigma_1
\]

\[
Y(\lambda) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + O \left( \frac{1}{\lambda} \right) \quad \lambda \to \infty .
\]  

(3.4)

(3.5)
The contour setting is shown in Figure 2. We can recover $u(x)$ from
\[ u(x) = 2 \frac{d}{dx} \left[ \lim_{\lambda \to \infty} \lambda (Y_1(\lambda; x) - 1) \right]. \]  

### 3.1. Large $x$ asymptotic.

Introduce the following new vector function
\[ T(\lambda) = Y(\lambda)e^{xg(\lambda)\sigma_3}f(\lambda)^{\sigma_3} \]
where $g(\lambda)$ and $f(\lambda)$ are scalar functions to be determined below. We require that

- $g(\lambda)$ is analytic in $\mathbb{C}\setminus[-\eta_2, \eta_2]$ and
  \[ g_+(\lambda) + g_-(\lambda) = 2\lambda \quad \lambda \in \Sigma_1 \cup \Sigma_2 \]  
  \[ g_+(\lambda) - g_-(\lambda) = \Omega \quad \lambda \in [-\eta_1, \eta_1] \]
  \[ g(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \lambda \to \infty, \]  
  where $\Omega$ is a constant independent of $x$ and needs to be determined.

- $f(\lambda)$ is analytic in $\mathbb{C}\setminus[-\eta_2, \eta_2]$ and
  \[ f(\lambda) = 1 + \mathcal{O}\left(\frac{1}{\lambda}\right). \]

In order to solve the scalar Riemann–Hilbert problem (3.7) – (3.9) for $g$ we observe that

- $g_+(\lambda) + g_-(\lambda) = 2$ \hspace{1cm} $\lambda \in \Sigma_1 \cup \Sigma_2$  
- $g_+(\lambda) - g_-(\lambda) = 0$ \hspace{1cm} $\lambda \in [-\eta_1, \eta_1]$  
- $g'(\lambda) = \mathcal{O}\left(\frac{1}{\lambda^2}\right) \quad \lambda \to \infty.$

From the above, we can write $g'(\lambda)$ as
\[ g'(\lambda) = 1 - \frac{\lambda^2 + \kappa}{R(\lambda)}, \]  

where
\[ R(\lambda) = \sqrt{(\lambda^2 - \eta_1^2)(\lambda^2 - \eta_2^2)}, \]

is real and positive on $(\eta_2, +\infty)$ with branch cuts on the contours $\Sigma_1$ and $\Sigma_2$ and $\kappa$ is a constant to be determined. By integration we obtain
\[ g(\lambda) = \lambda - \int_{\eta_2}^{\lambda} \frac{\zeta^2 + \kappa}{R(\zeta)} d\zeta. \]

The condition (3.7) implies that
\[ \int_{-\eta_1}^{\eta_1} \frac{\zeta^2 + \kappa}{R(\zeta)} d\zeta = 0. \]
and the condition (3.8) implies that

\[ \Omega = 2 \int_{\eta_1}^{\eta_2} \frac{\zeta^2}{R_+ (\zeta)} \, d\zeta. \]

This gives

\[ \Omega = \frac{2\pi i}{\int_{-\eta_1}^{\eta_1} \frac{d\zeta}{K(\zeta)}} = -i\pi \eta_2 \, e^{i\eta_2} \cdot m = \frac{\eta_1}{\eta_2}, \quad (3.17) \]

where \( K(m) = \int_0^\pi \frac{d\theta}{\sqrt{1 - m^2 \sin^2 \theta}} \) is the complete elliptic integral of the first kind with modulus \( m = \eta_1/\eta_2 \) and

\[ \kappa = -\int_{-\eta_1}^{\eta_1} \frac{\zeta^2 \Omega}{R(\zeta)} \frac{d\zeta}{2\pi i} = \eta_2 \left( \frac{E(m)}{K(m)} - 1 \right) \, e^{i\eta_2} \cdot \epsilon = \mathbb{R} \cdot (3.18) \]

where \( E(m) = \int_0^\pi \sqrt{1 - m^2 \sin^2 \theta} \, d\theta \) is the complete elliptic integral of the second kind.

The Riemann–Hilbert problem for \( T(\lambda) \) is

\[ T_+ (\lambda) = T_- (\lambda) J_T (\lambda) \]

\[ J_T (\lambda) = \begin{bmatrix} e^{x(g_+ (\lambda) - g_- (\lambda))} f_+ (\lambda) & 0 & \lambda \in \Sigma_1 \\ -ir (\lambda) f_+ (\lambda) f_- (\lambda) & e^{-x(g_+ (\lambda) - g_- (\lambda))} f_- (\lambda) & \lambda \in \Sigma_2 \\ e^{x(g_+ (\lambda) - g_- (\lambda))} f_+ (\lambda) & \frac{ir (\lambda)}{f_- (\lambda) f_- (\lambda)} & \lambda \in [-\eta_1, \eta_1] \end{bmatrix} \]

\[ T(\lambda) = \begin{bmatrix} 1 & 1 \\ 0 & \frac{i}{\lambda} \end{bmatrix} \lambda \to \infty. \]

In order to solve the Riemann–Hilbert problem for \( T(\lambda) \) we wish to obtain a constant jump matrix \( J_T \). For this purpose we make the following ansatz on the function \( f \)

\[ f_+ (\lambda) f_- (\lambda) = \frac{1}{r (\lambda)} \lambda \in \Sigma_1 \]

\[ f_+ (\lambda) f_- (\lambda) = r (\lambda) \lambda \in \Sigma_2 \]

\[ f_+ (\lambda) \frac{f_- (\lambda)}{f_- (\lambda)} = e^\Delta \lambda \in [-\eta_1, \eta_1] \]

\[ f(\lambda) = 1 + O \left( \frac{1}{\lambda} \right) \lambda \to \infty. \]

It is easy to check that the function \( f(\lambda) \) is given by

\[ f(\lambda) = \exp \left[ \frac{R(\lambda)}{2\pi i} \left[ \int_{\Sigma_1} \frac{\log \frac{1}{r (\zeta)}}{R_+ (\zeta)} (\zeta - \lambda) \, d\zeta + \int_{\Sigma_2} \frac{\log r (\zeta)}{R_+ (\zeta)} (\zeta - \lambda) \, d\zeta + \int_{\eta_1}^{\eta_2} \frac{\Delta}{R(\zeta)} (\zeta - \lambda) \, d\zeta \right] \right]. \]

(3.26)
The constraint (3.25) gives $\Delta$ equal to

$$\Delta = \left[ \int_{\Sigma_1} \frac{\log r(\zeta)}{R_+} \, d\zeta - \int_{\Sigma_2} \frac{\log r(\zeta)}{R_+} \, d\zeta \right] \left[ \int_{\eta_1}^{\eta_2} \frac{\, d\zeta}{R(\zeta)} \right]^{-1} = -\frac{\eta_2 - \eta_1}{R(m)} \int_{\eta_1}^{\eta_2} \frac{\log r(\zeta)}{R_+} \, d\zeta , \quad (3.27)$$

where in the last equality in (3.27) we use the fact that $r(-\lambda) = r(\lambda)$. We remind the reader that we are assuming the function $r$ to be real positive and non-vanishing on $\Sigma_1$ and $\Sigma_2$.

### 3.2. Opening lenses.

We start by defining the analytic continuation $\tilde{r}(\lambda)$ of the function $r(\lambda)$ off the interval $(-\eta_2, -\eta_1) \cup (\eta_1, \eta_2)$ with the requirement that

$$\tilde{r}_+(\lambda) = \pm r(\lambda) , \quad \lambda \in (-\eta_2, -\eta_1) \cup (\eta_1, \eta_2) .$$

We can factor the jump matrix $J_T$ on $\Sigma_1$ as follows

$$\begin{bmatrix} e^{x(g_+(-\lambda)-g_-(\lambda))} f_+(\lambda) & 0 \\ -i & e^{-x(g_+(-\lambda)-g_-(\lambda))} f_-(\lambda) \end{bmatrix} = \begin{bmatrix} 1 - i e^{x(g_+(-\lambda)-g_-(\lambda))} \tilde{r}_-(\lambda) f_2(\lambda) \\ -i \tilde{r}_+(\lambda) f_2(\lambda) \end{bmatrix}$$

and on $\Sigma_2$ as

$$\begin{bmatrix} e^{x(g_+(-\lambda)-g_-(\lambda))} f_+(\lambda) & 0 \\ 0 & e^{-x(g_+(-\lambda)-g_-(\lambda))} f_-(\lambda) \end{bmatrix} = \begin{bmatrix} 1 f_2(\lambda) & 0 \\ -i e^{-x(g_+(-\lambda)-g_-(\lambda))} \tilde{r}_+(\lambda) f_2(\lambda) \end{bmatrix} = \begin{bmatrix} i e^{-x(g_+(-\lambda)-g_-(\lambda))} \tilde{r}_-(\lambda) f_2(\lambda) \\ 1 \tilde{r}_+(\lambda) e^{x(g_+(-\lambda)-g_-(\lambda))} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

We can now proceed with “opening lenses”. We define a new vector function $S$ as follows

$$S(\lambda) = \begin{bmatrix} T(\lambda) \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} & e^{-2x(g_+(-\lambda)-g_-(\lambda))} \\ T(\lambda) \begin{bmatrix} 1 & -i \\ 0 & 1 \end{bmatrix} & e^{-2x(g_+(-\lambda)-g_-(\lambda))} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$  

where the Riemann–Hilbert problem satisfied by $S$ is depicted in Figure 3. In order to proceed we need the following lemma
\[ \begin{bmatrix} \frac{f^2(\lambda)}{i(\lambda)} e^{i\pi g(\lambda) - \frac{1}{\lambda}} & 0 \\ \frac{f^2(\lambda)}{i(\lambda)} e^{i\pi g(\lambda) + \frac{1}{\lambda}} & 1 \end{bmatrix} \quad \begin{bmatrix} i & -i e^{-2\pi g(\lambda)/\lambda} \\ 0 & 1 \end{bmatrix} \]

Figure 3. Riemann–Hilbert problem for \( S(\lambda) \) defined in (3.28). Opening lenses: the entries in gray in the jump matrices on the contours \( C_1 \) and \( C_2 \) are exponentially small in the regime as \( x \searrow -\infty \).

**Lemma 3.1.** The following inequalities are satisfied
\[
\begin{align*}
\text{Re} \{ g(\lambda) - \lambda \} < 0, & \quad \lambda \in \mathcal{C}_1 \setminus \{ \eta_1, \eta_2 \} \quad (3.29) \\
\text{Re} \{ g(\lambda) - \lambda \} > 0, & \quad \lambda \in \mathcal{C}_2 \setminus \{ -\eta_1, -\eta_2 \}, \quad (3.30)
\end{align*}
\]
where \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are the contours defining the lenses as shown in Figure 3.

**Proof.** Given \( \lambda = x + iy \), we write \( g_\ast(\lambda) - \lambda = u(x, y) + iv(x, y) \). From the formula (3.16) for \( g \), it follows that \( g_\ast(\lambda) - \lambda \) is purely imaginary on \( \Sigma_1 \cup \Sigma_2 \); furthermore, for \( \lambda \in \Sigma_1 \)
\[
v_x = \text{Im} \{ g_\ast'(\lambda) - 1 \} = \frac{\lambda^2 + \kappa}{|R_\ast(\lambda)|} \frac{\Omega}{|R_\ast(\lambda)|} \int_{-\eta_1}^{\eta_1} \frac{\lambda^2 - \zeta^2}{\zeta R(\zeta)} \frac{d\zeta}{2\pi i} > 0. \quad (3.31)
\]
Using Cauchy–Riemann equation it follows that \( u_y = -v_x < 0 \) for \( \lambda \in \Sigma_1 \) and it follows that \( \text{Re} \{ g(\lambda) - \lambda \} < 0 \) for \( \lambda \) above \( \Sigma_1 \) and \( \lambda \in \mathcal{C}_1 \). Repeating the same reasoning for the function \( g_\ast(\lambda) - \lambda \) we obtain that \( \text{Re} \{ g(\lambda) - \lambda \} < 0 \) for \( \lambda \) below \( \Sigma_1 \) and \( \lambda \in \mathcal{C}_1 \). In a similar way the inequality (3.30) can be obtained. \( \square \)

Lemma 3.1 guarantees that the off diagonal entries of the jump matrices along the upper and lower lenses are exponentially small in the regime as \( x \searrow -\infty \), therefore those jump matrices are asymptotically close to the identity outside a small neighbourhood of \( \pm \eta_1 \) and \( \pm \eta_2 \). We are left with the model problem
\[
S_0(\infty)(\lambda) = S_0(\infty)(\lambda) = \begin{cases} 
\begin{bmatrix} e^{x\Omega + \Delta} & 0 \\ 0 & e^{-x\Omega - \Delta} \end{bmatrix} & \lambda \in [-\eta_1, \eta_1] \\
\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} & \lambda \in \Sigma_1 \\
\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} & \lambda \in \Sigma_2
\end{cases}
\]
\[
S(\infty)(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left( \frac{1}{\lambda} \right), \quad \lambda \to \infty. \quad (3.33)
\]

The Riemann–Hilbert problem for \( S(\infty) \) appeared already in the long time asymptotic for KdV with step-like initial data \([\text{EGKT13}]\). Below we follow the lines in \([\text{EGKT13}]\) to obtain the solution.
3.3. The global parametrix $S^{(\infty)}$. For solving the Riemann–Hilbert problem (3.32) and (3.33) we introduce a two-sheeted Riemann surface $X$ of genus 1 associated to the multivalued function $R(\lambda)$, namely
\[ X = \{ (\lambda, \eta) \in \mathbb{C}^2 \mid \eta^2 = R^2(\lambda) = (\lambda^2 - \eta_1^2)(\lambda^2 - \eta_2^2) \} . \]

The first sheet of the surface is identified with the sheet where $R(\lambda)$ is real and positive for $\lambda \in (\eta_2, +\infty)$. We introduce a canonical homology basis with the $B$ cycle encircling $\Sigma_1$ clockwise on the first sheet and the $A$ cycle going from $\Sigma_2$ to $\Sigma_1$ on the first sheet and coming back to $\Sigma_2$ on the second sheet. The points at infinity on the surface are denoted by $\infty^+$ where $\infty^+$ is on the first sheet and $\infty^-$ on the second sheet of $X$. See Figure 4. We introduce the holomorphic differential
\[ \omega = \frac{\Omega}{R(\lambda)} \frac{d\lambda}{4\pi i} \]
so that
\[ \oint_A \omega = 1 . \]

We also have
\[ \tau = \oint_B \omega = i \frac{K(\sqrt{1 - m^2})}{K(m)} , \quad m = \frac{\eta_1}{\eta_2} . \]

Next we introduce the Jacobi elliptic function
\[ \vartheta_3(z; \tau) = \sum_{n \in \mathbb{Z}} e^{2\pi i n z + \pi n^2 \tau} , \quad z \in \mathbb{C} , \]
which is an even function of $z$ and satisfies the periodicity conditions
\[ \vartheta_3(z + h + k; \tau) = e^{-\pi i k^2 \tau - 2\pi i kz} \vartheta_3(z; \tau) , \quad h, k \in \mathbb{Z} . \]

We also recall that the Jacobi elliptic function with modulo $\tau$ vanishes on the half period $\frac{\tau}{2} + \frac{1}{2}$. Finally, we define the integral
\[ w(\lambda) = \int_{\eta_2}^{\lambda} \omega \]
and we observe that
\[ w(+\infty) = -\frac{1}{4} , \quad w(\eta_1) = \frac{\tau}{2} , \quad w(-\eta_1) = -\frac{\tau}{2} - \frac{1}{2} . \]

We introduce the following functions
\[ \psi_1(\lambda) = \frac{\vartheta_3(2w(\lambda) + \frac{\pi i}{2\pi i} - \frac{1}{2}; 2\tau)}{\vartheta_3(2w(\lambda) - \frac{1}{2}; 2\tau)} \frac{\vartheta_3(0; 2\tau)}{\vartheta_3(\frac{\pi i}{2\pi i}; 2\tau)} , \]
\[ \psi_2(\lambda) = \frac{\vartheta_3(-2w(\lambda) + \frac{\pi i}{2\pi i} - \frac{1}{2}; 2\tau)}{\vartheta_3(-2w(\lambda) - \frac{1}{2}; 2\tau)} \frac{\vartheta_3(0; 2\tau)}{\vartheta_3(\frac{\pi i}{2\pi i}; 2\tau)} , \]
and we observe that
\[ \begin{cases} \vartheta_3(\pm 2w(\eta_1) - \frac{1}{2}; 2\tau) = \vartheta_3(\mp \tau - \frac{1}{2}; 2\tau) = 0 , \\ \vartheta_3(\pm 2w(-\eta_1) - \frac{1}{2}; 2\tau) = \vartheta_3(\mp \tau + 1 - \frac{1}{2}; 2\tau) = 0 . \end{cases} \]
Figure 4. Construction of the genus-1 Riemann surface $X$ and its basis of cycles.

It follows that the functions $\psi_1$ and $\psi_2$ have simple poles at $\lambda = \pm \eta_1$. Furthermore the following jump relations are satisfied:

\[
\begin{align*}
  w_+ (\lambda) - w_- (\lambda) &= 0, \quad \lambda \in [\eta_2, +\infty) \quad (3.38) \\
  w_+ (\lambda) + w_- (\lambda) &= 0, \quad \lambda \in \Sigma_1 \quad (3.39) \\
  w_+ (\lambda) - w_- (\lambda) &= -\tau, \quad \lambda \in (-\eta_1, \eta_1) \quad (3.40) \\
  w_+ (\lambda) + w_- (\lambda) &= -1, \quad \lambda \in \Sigma_2. \quad (3.41)
\end{align*}
\]

Therefore for $\lambda \in \Sigma_1 \cup \Sigma_2$ we have

\[
\begin{align*}
  \psi_{1+} (\lambda) &= \psi_{2-} (\lambda), \quad \psi_{2+} (\lambda) = \psi_{1-} (\lambda), \quad (3.42) \\
  \psi_{1+} (\lambda) &= \psi_{1-} (\lambda) e^{x \Omega + \Delta}, \quad \psi_{2+} (\lambda) = \psi_{2-} (\lambda) e^{-x \Omega - \Delta}. \quad (3.43)
\end{align*}
\]

Next we introduce the quantity

\[
\gamma (\lambda) = \left( \frac{\lambda^2 - \eta_1^2}{\lambda^2 - \eta_2^2} \right)^{1/4}.
\]

Then,

\[
\gamma_+ (\lambda) = -i \gamma_- (\lambda), \quad \text{for } \lambda \in \Sigma_1 \quad \text{and} \quad \gamma_+ (\lambda) = i \gamma_- (\lambda), \quad \text{for } \lambda \in \Sigma_2. \quad (3.44)
\]

We are now ready to construct the solution of the Riemann–Hilbert problem (3.32) – (3.33).

**Theorem 3.2.** The matrix $S^{(\infty)} (\lambda)$ given by

\[
S^{(\infty)} (\lambda) = \gamma (\lambda) \frac{\partial_3 (0; 2\tau)}{\partial_3 \left( \frac{x+i\lambda}{2\pi i}; 2\tau \right)} \left[ \begin{array}{cc}
  \vartheta_3 \left( 2w(\lambda) + \frac{x+i\lambda}{2\pi i}; 2\tau \right) & \vartheta_3 \left( -2w(\lambda) + \frac{x+i\lambda}{2\pi i}; 2\tau \right) \\
  \vartheta_3 \left( 2w(\lambda) - \frac{x+i\lambda}{2\pi i}; 2\tau \right) & \vartheta_3 \left( -2w(\lambda) - \frac{x+i\lambda}{2\pi i}; 2\tau \right)
\end{array} \right]
\]

solves the Riemann–Hilbert problem (3.32).

**Proof.** We observe that $S^{(\infty)} (\lambda)$ has at most fourth root singularities at the branch points and it is regular everywhere else on the complex plane. Because of (3.36) and (3.37) we have $S^{(\infty)} (\infty) = \left[ \begin{array}{cc} 1 & 1 \end{array} \right]$, namely the condition (3.33) is satisfied. Combining (3.42), (3.43) and (3.44), we conclude that the jump conditions (3.32) are satisfied. \(\square\)
This vector solution provides the asymptotic behavior of the solution $S$ to Riemann–Hilbert problem depicted in Figure 3, for all $z$ bounded away from the endpoints. However, in order to prove this, we need to construct a matrix solution to this Riemann–Hilbert problem, which we call $P^{(\infty)}$. This will be accomplished in the next two subsections, by creating a second, independent, vector solution.

### 3.4. A second vector solution

In order to construct a second vector solution, note that for any complex numbers $a$ and $b$, we have that the following relations are satisfied:

$$
\begin{bmatrix}
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)} \\
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)}
\end{bmatrix}_+ = \begin{bmatrix}
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)} \\
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)}
\end{bmatrix}_- = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \lambda \in \Sigma_1, \quad (3.46)
$$

and

$$
\begin{bmatrix}
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)} \\
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)}
\end{bmatrix}_+ = \begin{bmatrix}
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)} \\
 a \gamma(\lambda) + b \frac{1}{\gamma(\lambda)} & a \gamma(\lambda) - b \frac{1}{\gamma(\lambda)}
\end{bmatrix}_- = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \lambda \in \Sigma_2. \quad (3.47)
$$

In addition, for any constants $D$, we have that the vector

$$
v(\lambda) = \begin{bmatrix}
 \vartheta_3 \left( w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right) \\
 \vartheta_3 \left( -w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right)
\end{bmatrix} = \begin{bmatrix}
 \vartheta_3 \left( w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right) \\
 \vartheta_3 \left( -w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right)
\end{bmatrix} \quad (3.48)
$$

satisfies the boundary value relations

$$
v_+ (\lambda) = v_- (\lambda) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \lambda \in \Sigma_1 \cup \Sigma_2, \quad (3.49)
$$

$$
v_+ (\lambda) = v_- (\lambda) \begin{bmatrix} e^{x_1 + \Delta} & 0 \\ 0 & e^{-x_1 + \Delta} \end{bmatrix}, \quad \lambda \in [-\eta_1, \eta_1]. \quad (3.50)
$$

We choose $D$ so that the quantity $\vartheta_3 \left( w(\lambda) + D; \tau \right)$ has a zero at the point $\infty^-$ on the second sheet of $\mathcal{X}$ namely

$$
D = \frac{1}{4} + \frac{\tau}{2}.
$$

In this way the quantity $\vartheta_3 \left( -w(\lambda) + D; \tau \right)$ has a simple zero at $\infty^+$ on the first sheet of the surface $\mathcal{X}$. Next we choose $a = b = 1$ so that the equation

$$
\gamma(\lambda) + \frac{1}{\gamma(\lambda)} = 0
$$

has a double zero at $\infty^-$ and the equation

$$
\gamma(\lambda) - \frac{1}{\gamma(\lambda)} = 0
$$

has a double zero at $\infty^+$. Therefore the vector $H(\lambda) = \begin{bmatrix} H_1 & H_2 \end{bmatrix}$:

$$
H_1 (\lambda) = \begin{bmatrix} \gamma(\lambda) + \frac{1}{\gamma(\lambda)} \end{bmatrix} \frac{\vartheta_3 \left( w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right)}{\vartheta_3 \left( w(\lambda) + D; \tau \right)},
$$

$$
H_2 (\lambda) = \begin{bmatrix} \gamma(\lambda) - \frac{1}{\gamma(\lambda)} \end{bmatrix} \frac{\vartheta_3 \left( -w(\lambda) + \frac{x_1 + \Delta}{2\pi i} + D; \tau \right)}{\vartheta_3 \left( -w(\lambda) + D; \tau \right)}, \quad (3.51)
$$
has no poles and at most fourth root singularity at the points $\lambda = \pm \eta_1$ and $\lambda = \pm \eta_2$ and satisfies the boundary value relations

\[
H_+(\lambda) = H_-(\lambda) \begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}, \quad \lambda \in \Sigma_1, \tag{3.52}
\]

\[
H_+(\lambda) = H_-(\lambda) \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \lambda \in \Sigma_2, \tag{3.53}
\]

\[
H_+(\lambda) = H_-(\lambda) \begin{bmatrix} e^{x\Omega + \Delta} & 0 \\ 0 & e^{-x\Omega - \Delta} \end{bmatrix}, \quad \lambda \in [-\eta_1, \eta_1]. \tag{3.54}
\]

Computing the behavior for $\lambda \to \infty$, we find

\[
\lim_{\lambda \to \infty} H(\lambda) = \begin{bmatrix} \frac{\partial_3(\frac{x\Omega + \Delta}{2\pi i} + \frac{\eta}{2}; \tau)}{\partial_3(\frac{\eta}{2}; \tau)} & 0 \end{bmatrix}. \tag{3.55}
\]

Such solution is well defined and linearly independent from the solution $S^{(\infty)}$ defined in (3.45) when

\[
\frac{x\Omega + \Delta}{2\pi i} \neq \frac{2n + 1}{2}, \quad n \in \mathbb{Z}.
\]

**Remark 3.3.** When $x\Omega + \Delta = \pi i + 2\pi i \mod n$, a technical / analytical issue arises concerning control of the error in the Riemann–Hilbert analysis. First, the solution $S^{(\infty)}$ vanishes at $\lambda = 0$. Indeed, at this special value, we have

\[
S^{(\infty)}(\lambda) = \gamma(\lambda) \frac{\partial_3(0; 2\tau)}{\partial_3\left(\frac{\eta}{2}; 2\tau\right)} \left[ \frac{\partial_3(2w(\lambda); 2\tau)}{\partial_3(2w(\lambda) - \frac{\eta}{2}; 2\tau)} \frac{\partial_3(-2w(\lambda); 2\tau)}{\partial_3(-2w(\lambda) - \frac{\eta}{2}; 2\tau)} \right], \tag{3.56}
\]

and, taking $\lambda = 0$, we see that both entries of $S^{(\infty)}$ vanish. Moreover, the second vector solution $H(\lambda)$ computed above vanishes identically as $\lambda \to \infty$. Indeed, at the special value $x\Omega + \Delta = \pi i$, we have

\[
H_1(\lambda) = \left(\gamma(\lambda) + \frac{1}{\gamma(\lambda)}\right) \frac{\partial_3\left(\frac{w(\lambda) + \frac{3}{4} + \frac{\eta}{2}}{\eta}; \tau\right)}{\partial_3\left(\frac{w(\lambda) + \frac{3}{4} + \frac{\eta}{2}}{\eta}; \tau\right)}, \tag{3.57}
\]

\[
H_2(\lambda) = \left(\gamma(\lambda) - \frac{1}{\gamma(\lambda)}\right) \frac{\partial_3\left(\frac{-w(\lambda) + \frac{3}{4} + \frac{\eta}{2}}{\eta}; \tau\right)}{\partial_3\left(\frac{-w(\lambda) + \frac{3}{4} + \frac{\eta}{2}}{\eta}; \tau\right)},
\]

and clearly both quantities converge to 0 as $\lambda \to \infty$. We see that for this special value of the parameters, the model Riemann–Hilbert problem defining $S^{(\infty)}$ is somewhat special in that there is a well-behaved vector-valued solution that vanishes identically as $\lambda \to \infty$, and the question of the existence of a suitably well-behaved matrix-valued solution is nontrivial.

We can construct a matrix-valued solution in this case, as follows. We take

\[
\tilde{H}(\lambda) = \begin{bmatrix} \frac{1}{2} \left(\frac{\gamma(\lambda) + 1}{\gamma(\lambda)}\right) & \frac{1}{2} \left(\frac{\gamma(\lambda) - 1}{\gamma(\lambda)}\right) \\ \frac{1}{2} \left(\frac{\gamma(\lambda) - 1}{\gamma(\lambda)}\right) & \frac{1}{2} \left(\frac{\gamma(\lambda) + 1}{\gamma(\lambda)}\right) \end{bmatrix}, \tag{3.58}
\]

where

\[
\tilde{\gamma}(\lambda) = \left(\frac{(\lambda - \eta_2)^3}{(\lambda^2 - \eta_1^2)(\lambda + \eta_2)}\right)^{\frac{1}{4}}
\]

It is straightforward to verify that

\[
\det \tilde{H} = 1. \tag{3.59}
\]
While this represents a reasonable solution to the outer model Riemann–Hilbert problem for 
\( \frac{x + \Delta}{2\pi i} = \frac{2n + 1}{2}, \ n \in \mathbb{Z} \), it raises the issue of the construction of a local parametrix near \( \lambda = \eta_2 \) which is new, open, and will not be addressed here.

3.5. The global parametrix \( P^{(\infty)} \). We consider the following matrix Riemann–Hilbert problem. Find a \( 2 \times 2 \) matrix-valued function \( P^{(\infty)}(\lambda) \) analytic in \( \mathbb{C} \backslash (-\eta_2, \eta_2) \) such that

\[
P^{(\infty)}(\lambda) = P_+^{(\infty)}(\lambda) \begin{bmatrix} e^{x\Omega + \Delta} & 0 \\ 0 & e^{-x\Omega - \Delta} \end{bmatrix} P_-^{(\infty)}(\lambda), \quad \lambda \in \left[-\eta_1, \eta_1\right]
\]

\[
P^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right), \quad \lambda \to \infty.
\]

We define \( M^{(\infty)}(\lambda) \) using \( S^{(\infty)}(\lambda) \) defined in (3.45) and \( H(\lambda) \), \( \hat{H}(\lambda) \) defined in (3.51) and (3.58) respectively

\[
M^{(\infty)}(\lambda) = \begin{bmatrix} S_1^{(\infty)}(\lambda) & S_2^{(\infty)}(\lambda) \\ \vartheta_3(\frac{z}{\tau}, \tau) H_1(\lambda) & \vartheta_3(\frac{z}{\tau}, \tau) H_2(\lambda) \end{bmatrix}, \quad \text{when } x\Omega + \Delta \neq (2n + 1)\pi i, \ n \in \mathbb{Z}
\]

\[
\hat{H}(\lambda), \quad \text{when } x\Omega + \Delta = (2n + 1)\pi i, \ n \in \mathbb{Z}
\]

which has the asymptotic behavior for \( \lambda \to \infty + \):

\[
M^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad \text{when } x\Omega + \Delta \neq (2n + 1)\pi i, \ n \in \mathbb{Z}
\]

\[
M^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{when } x\Omega + \Delta = (2n + 1)\pi i, \ n \in \mathbb{Z}.
\]

Theorem 3.4. The solution of the matrix Riemann–Hilbert problem (3.60) and (3.61) in given by the matrix

\[
P^{(\infty)}(\lambda) = (M^{(\infty)}(\lambda))^{-1} M^{(\infty)}(\lambda),
\]

where \( M^{(\infty)}(\lambda) \) and \( M^{(\infty)}(\lambda^+) \) are defined in (3.62) and (3.63) respectively.

3.6. The local parametrix \( P^{(\pm \eta_2)} \) at the endpoints. Thanks to Lemma 3.1, the off diagonal entries of the jump matrices for \( S \) exponentially vanish as \( x \searrow -\infty \) along the upper and lower lenses, while near the endpoints the \( g \) function has a square-root-vanishing behaviour

\[
g_+(\lambda) - g_-(\lambda) = \mathcal{O}\left(\sqrt{\lambda \mp \eta_2}\right), \quad \lambda \to \pm \eta_2,
\]

and

\[
g_+(\lambda) - g_-(\lambda) = \mathcal{O}\left(\sqrt{\lambda \mp \eta_1}\right), \quad \lambda \to \pm \eta_1.
\]

Therefore the jump matrices for \( S \) are bounded in a neighbourhood of those points (but they are not close to the identity).

On the other hand, the global parametrix \( P^{(\infty)} \) is a good approximation of the solution \( S \) to the Riemann–Hilbert problem away from the endpoints \( \lambda = \pm \eta_2, \pm \eta_1 \), where \( P^{(\infty)} \) exhibits a
fourth-root singularity. So, we need to introduce four local parametrices \( P^{(\pm \eta_j)} \) \((j = 1, 2)\) in a suitable neighbourhood of each endpoint.

We show here the construction of a (matrix) local parametrix \( P^{(\eta_2)} \) around \( \lambda = \eta_2 \). The constructions of the other local parametrices near \( \lambda = \pm \eta_1, -\eta_2 \) follow a similar argument.

In what follows, we will assume that \( \frac{\pi}{2n+1} \neq \frac{2n+1}{2}, n \in \mathbb{Z} \).

Performing the same calculations as in [KMAV04, Section 6], we will construct a local parametrix \( P^{(\eta_2)} \) with the help of modified Bessel functions. We fix a small neighbourhood \( B_\rho = \{ \lambda \in \mathbb{C} | |\lambda - \eta_2| < \rho \} \) of the endpoint \( \eta_2 \) and we define the (local) conformal map

\[
\zeta = \frac{1}{4} \left[ x \left( g(\lambda) - \lambda \right) \right]^2, \quad \lambda \in B_\rho.
\]

(3.67)

To define the local parametrix \( P^{(\eta_2)} \) in \( B_\rho \), we consider

\[
P(\lambda) = S(\lambda) \left( \frac{e^{i\pi/4}}{\sqrt{\pm f}} \right)^{\sigma_3} \quad \lambda \in B_\rho \cap \mathbb{C}_\pm,
\]

and then, using the inverse of the transformation \( \zeta(\lambda) \), we define

\[
P^{(1)}(\zeta) = P(\lambda(\zeta)) e^{-2\zeta^{\frac{1}{2}} \sigma_3} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \zeta \in \mathbb{C},
\]

with branch cut \((-\infty, 0]\). By construction, \( P^{(1)} \) satisfies a Riemann–Hilbert problem with jumps

\[
P^{(1)}_+(\zeta) = P^{(1)}_-(\zeta) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{on \{upper and lower lenses\} } \cap B_\rho,
\]

and

\[
P^{(1)}_+(\zeta) = P^{(1)}_-(\zeta) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{on } (-\infty, 0] \cap B_\rho.
\]

(3.68)

We introduce now the model parametrix \( \Psi_{\text{Bes}}(\zeta) \) as in [KMAV04, formulæ (6.16)–(6.20)]). The Riemann–Hilbert problem for \( \Psi_{\text{Bes}} \) is the following:

(a) \( \Psi_{\text{Bes}} \) is analytic for \( \zeta \in \mathbb{C} \setminus \Gamma_\Psi \), where \( \Gamma_\Psi \) is the union of the three contours \( \Gamma_+ = \{ \arg \zeta = \pm \frac{2\pi}{3} \} \) and \( \Gamma_0 = \{ \arg \zeta = \pi \} \);

(b) \( \Psi \) satisfies the following jump relations

\[
\Psi_{\text{Bes}+}(\zeta) = \Psi_{\text{Bes}-}(\zeta) \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{on } \Gamma_+ \cup \Gamma_-;
\]

(3.69)

(c) as \( \zeta \to 0 \)

\[
\Psi_{\text{Bes}}(\zeta) = \begin{bmatrix} O(\ln|\zeta|) & O(\ln|\zeta|) \\ O(\ln|\zeta|) & O(\ln|\zeta|) \end{bmatrix}.
\]

(3.70)
The solution is the following

\[
\Psi_{\text{Bes}}(\zeta) = \begin{cases} 
I_0(2\zeta^\frac{1}{2}) & \frac{i}{\pi} K_0(2\zeta^\frac{1}{2}) \\
2\pi i \zeta^\frac{1}{2} I'_0(2\zeta^\frac{1}{2}) & -2\zeta^\frac{1}{2} K_0(2\zeta^\frac{1}{2})
\end{cases} \quad \text{ arg } \zeta < \frac{2\pi}{3}
\]

\begin{align*}
|\arg \zeta| < \frac{2\pi}{3}
\end{align*}

with asymptotic behaviour at infinity

\[
\Psi_{\text{Bes}}(\zeta) = (2\pi \zeta^\frac{1}{2})^{-\frac{1}{2}\sigma_3} \frac{1}{\sqrt{2}} \left[ \begin{array}{c} 1 \\ i \\
\end{array} \right] \left( I + O\left( \frac{1}{\zeta^\frac{1}{2}} \right) \right) e^{2\sqrt{2} \zeta^\frac{1}{2}\sigma_3}
\]

uniformly as \( \zeta \to \infty \) everywhere in the complex plane aside from the jumps.

In the above formulae \( I_0(\zeta) \), \( K_0(\zeta) \) are the modified Bessel functions of first and second kind, respectively, and \( H^{(1)}(\zeta) \) the Hankel functions.

In conclusion the local parametrix around the endpoint \( \lambda = \eta_2 \) is

\[
P^{(\eta_2)}(\lambda) = A(\lambda) \Psi_{\text{Bes}}(\zeta(\lambda)) \begin{bmatrix} 0 & 1 \\
1 & 0
\end{bmatrix} e^{2\zeta(\lambda)^\frac{1}{2}\sigma_3} \left( \frac{e^{i\pi/4}}{\sqrt{\pm \text{Re}(\lambda) f(\lambda)}} \right)^{-\sigma_3} \lambda \in B_\rho \cap \mathbb{C}_x,
\]

where \( A \) is a prefactor that is determined by imposing that

\[
P^{(\eta_2)}(\lambda) \left( P^{(\infty)}(\lambda) \right)^{-1} = I + O\left( |x|^{-1} \right) \quad \text{ as } x \searrow -\infty, \quad \text{ for } \lambda \in \partial B_\rho \setminus \Sigma_\Psi .
\]

Therefore, we set

\[
A(\lambda) = P^{(\infty)}(\lambda) \left( \frac{e^{i\pi/4}}{\sqrt{\pm \text{Re}(\lambda) f(\lambda)}} \right)^{-\sigma_3} \quad \frac{1}{\sqrt{2}} \left[ \begin{array}{c} -i \\ 1 \\
\end{array} \right] \left( 2\pi \zeta^\frac{1}{2} \right)^{\frac{1}{2}\sigma_3} \lambda \in B_\rho \cap \mathbb{C}_x.
\]

By construction, \( A \) is well-defined and analytic in a neighbourhood of \( \eta_2 \), minus the cut \( (-\infty, \eta_2] \); additionally, it is easy to see that \( A \) is invertible \( \det A(\lambda) = 1 \).

**Lemma 3.5.** \( A(\lambda) \) is analytic everywhere in the neighbourhood \( B_\rho \) of \( \eta_2 \).

**Proof.** To prove the statement, one needs to check that \( A \) has no jumps across the interval \([\eta_1, \eta_2] \cap B_\rho\) and that it has at most a removable singularity at \( \lambda = \eta_2 \). Careful computation (using the definition (3.75), with tears) reveals that \( A_+(\lambda) = A_-(\lambda) \) on \( (-\infty, \eta_2] \cap B_\rho \). Next, we notice that \( \zeta(\lambda) \) has a simple zero at \( \eta_2 \) by construction, thus \( \zeta(\lambda)^{\frac{1}{2}\sigma_3} \) has at most a fourth-root singularity at the point \( \lambda = \eta_2 \). Also the global parametrix \( P^{(\infty)}(\lambda) \) has at most a fourth-root singularity near \( \eta_2 \) and consequently all the entries of \( A(\lambda) \) have at most a square root singularity at \( \lambda = \eta_2 \).

On the other hand \( A(\lambda) \) is analytic in \( B_\rho \setminus \{\eta_2\} \), therefore the point \( \lambda = \eta_2 \) is a removable singularity and \( A(\lambda) \) is indeed analytic everywhere in \( B_\rho \). \( \square \)
3.7. Small norm argument and determination of \( u(x,0) \) for large negative \( x \). Consider the following “remainder” Riemann–Hilbert problem:

\[
\mathcal{E}(\lambda) = S(\lambda)U(\lambda)^{-1},
\]

(3.76)

where \( U \) is the (matrix) ensemble of the global parametrix \( P^{(\infty)} \) and the four local parametrices \( P^{(\pm n_i)}, j = 1, 2 \); thanks to the previous arguments, the vector-valued function \( \mathcal{E} \) satisfies a Riemann–Hilbert problem with jumps that are asymptotically close to the identity matrix:

\[
\mathcal{E}_+(\lambda) = \begin{cases} 
\mathcal{E}_-(\lambda) (I + \mathcal{O}(|x|^{-\infty})) & \text{on the upper and lower lenses} \\
\mathcal{E}_-(\lambda) (I + \mathcal{O}(|x|^{-1})) & \text{on the circles around the endpoints}
\end{cases}
\]

(3.77)

and \( \mathcal{E}(\lambda) = [1 \ 1] + \mathcal{O}(\lambda^{-1}) \) as \( \lambda \to \infty \).

Therefore, by a standard small norm argument (see, for example [Its11, Section 5.1.3]) the solution \( \mathcal{E} \) behaves as follows: \( \mathcal{E}(\lambda) = [1 \ 1] + \mathcal{O}(|x|^{-1}) \) as \( x \to -\infty \). We note in passing that the construction of a matrix-valued global approximation is very useful, in that we arrive directly at a small-norm Riemann–Hilbert problem. In addition, we note that the error term has not been established for \( x \) in a vicinity of those values where \( \frac{\Omega + \Delta}{2\pi i} = \frac{2n+1}{2}, \ n \in \mathbb{Z} \).

Keeping into account all the transformations we performed, we are now able to explicitly solve the original Riemann–Hilbert problem \( Y \) in the large negative \( x \) regime:

\[
Y(\lambda) = T(\lambda)e^{-xg(\lambda)\sigma_3} f(\lambda)^{-\sigma_3} = S(\lambda)e^{-xg(\lambda)\sigma_3} f(\lambda)^{-\sigma_3} = \left([1 \ 1] + \mathcal{O}(|x|^{-1})\right) U(\lambda)e^{-xg(\lambda)\sigma_3} f(\lambda)^{-\sigma_3},
\]

(3.78)

and in place of \( U(\lambda) \) we use the global or local parametrix. We recall that the potential \( u(x) \) can be calculated from the solution \( Y(\lambda) \) as

\[
u(x) = 2 \frac{d}{dx} \left[ \lim_{\lambda \to \infty} \lambda(Y_1(\lambda;x) - 1) \right],
\]

(3.79)

where \( Y_1(\lambda;x) \) is the first entry of the vector \( Y \).

**Theorem 3.6.** In the regime \( x \to -\infty \), with \( \frac{\Omega + \Delta}{2\pi i} = \frac{2n+1}{2}, \ n \in \mathbb{Z} \), the potential \( u(x) \) has the following asymptotic behaviour

\[
u(x) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{E(m)}{K(m)} - 2 \frac{\partial^2}{\partial x^2} \log \frac{\eta_2}{2K(m)} (x + \phi) - 2\tau + \mathcal{O}(|x|^{-1})
\]

(3.80)

where \( E(m) \) and \( K(m) \) are the complete elliptic integrals of the first and second kind respectively with modulus \( m = \eta_1/\eta_2 \), \( \phi \) is given by

\[
\phi = \int_{\eta_1}^{\eta_2} \frac{\log r(\zeta) \ d\zeta}{R_+(\zeta)} \pi i \in \mathbb{R}
\]

(3.81)

and \( 2\tau = i \frac{K(m')}{K(m)} \), \( m' = \sqrt{1 - m^2} \). The formula (3.80) can be written in the equivalent form

\[
u(x) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \frac{\text{dn}^2 \left| \frac{\eta_2 - K(m) m'}{K(m)} \right|}{\eta_2^2 + K(m) m'} + \mathcal{O}(|x|^{-1})
\]

(3.82)

where \( \text{dn}( \cdot | m) \) is the Jacobi elliptic function of modulus \( m \).

**Proof.** We are interested in the first entry of the vector \( Y(\lambda) \) (for \( \lambda \) large), and we have, from (3.64),

\[
\begin{bmatrix} 1 & 1 \end{bmatrix} P^{(\infty)}(\lambda) = \begin{bmatrix} P_{11}^{(\infty)}(\lambda) + P_{21}^{(\infty)}(\lambda) & P_{12}^{(\infty)}(\lambda) + P_{22}^{(\infty)}(\lambda) \end{bmatrix} = S^{(\infty)}(\lambda). \]

(3.83)

Hence

\[
Y_1(\lambda) = \left[ S_1^{(\infty)}(\lambda) + \mathcal{O}(|x|^{-1}) \right] \frac{e^{-xg(\lambda)}}{f(\lambda)},
\]

(3.84)
From the expression of the $g$ function (3.16), we have
\[ e^{-xg(\lambda)} = 1 - \frac{x}{\lambda} \left[ \frac{\eta^2 + \eta_t^2}{2} + \eta_2^2 \left( \frac{E(m)}{K(m)} - 1 \right) \right] + O\left( \frac{1}{\lambda^2} \right). \] (3.85)

From the formula of $f(\lambda)$ in (3.26) we have
\[ f(\lambda) = 1 + \frac{f_1}{\lambda} + O\left( \frac{1}{\lambda^2} \right), \]
where $f_1$ is independent of $x$. From the vector $S^{(\infty)}(\lambda)$ in (3.45) we have
\[ S_1^{(\infty)}(\lambda) = 1 - \frac{1}{\lambda} \left( \frac{\partial}{\partial x} \log \vartheta_3 \left( \frac{x + \Lambda}{2\pi i}; 2\tau \right) - \frac{\vartheta'_3(0)}{\vartheta_3(0)} \right) + O\left( \frac{1}{\lambda^2} \right), \]
where we have used the relations
\[ 2\int_{\infty}^{\Lambda} \omega = \frac{\eta_2}{2\lambda K(m)} + O\left( \frac{1}{\lambda^2} \right), \quad \frac{\Omega}{2\pi i} = -\frac{\eta_2}{2K(m)} \]
and the parity of the function $\vartheta_3(z + \frac{i}{2}; \tau)$. From the above expansions, using the explicit expression of $\Delta$ in (3.27), and the parity of $\vartheta_3(z; 2\tau)$ we obtain the expression of $u(x)$ in (3.79). In order to obtain the expression (3.82) we need the following identity ([Law89, pg.59])
\[ \frac{1}{4K^2(m)} \frac{d^2}{d^2x^2} \log \vartheta_3(z; 2\tau) = -\frac{E(m)}{K(m)} + \frac{dn^2(2K(m)z + K(m)|m)}{m}, \]
where $dn(z|m)$ is the Jacobi elliptic function of modulus $m$ and period $2K(m)$ and we recall that $2\tau = iK(m')/K(m)$. Then we can write
\[ \frac{\partial^2}{\partial x^2} \log \vartheta_3 \left( \frac{x + \Delta}{2\pi i}; 2\tau \right) = -\eta_2^2 \frac{E(m)}{K(m)} + \eta_2^2 \frac{dn^2(\eta_2(x + \phi) + K(m)|m)}{m}, \]
so that the expression for $u(x)$ in (3.80) can be written in the form (3.82). \( \square \)

**Remark 3.7.** While the above theorem requires $\frac{x + \Delta}{2\pi i} \neq \frac{2n + 1}{2}$, $n \in \mathbb{Z}$, the reason is because we have not constructed the requisite new parametrix near $\eta_2$ in this case. It is expected that the leading order behavior should not change for $x$ near such values, whereas the error term may or may not be altered.

4. **Behaviour of the potential $u(x,t)$ as $t \nearrow +\infty$**

Letting the potential $u(x,t)$ evolve in time according to the KdV equation, the reflection coefficient evolves as $r_1(\lambda; t) = r_1(\lambda)e^{-8\lambda^2t}$. This will lead to the study of a Riemann–Hilbert problem $Y$ for the soliton gas described as follows
\[ Y_+(\lambda) = Y_-(\lambda) \begin{bmatrix} 1 & i\lambda e^{-8\lambda^2(t)} \\ -ir(\lambda)e^{8\lambda^2(t)} & 1 \end{bmatrix} \lambda \in \Sigma_1 \]
\[ Y_+(\lambda) = Y_-(\lambda) \begin{bmatrix} 1 & 0 \\ i\lambda e^{-8\lambda^2(t)} & 1 \end{bmatrix} \lambda \in \Sigma_2 \]
\[ Y(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + O\left( \frac{1}{\lambda} \right) \lambda \to \infty. \] (4.1)
(4.2)

We are interested in the asymptotic behaviour of $Y(\lambda)$ in the long-time regime ($t \nearrow +\infty$).
The value of $g$ and $\xi$ where the value of $\xi$ wave parameters are changing slowly in time), while in the sub-critical case, the asymptotic description gives an asymptotic solution that is a modulated travelling wave (the contours in the following way: let $\alpha_1$ second, we will require monotonicity properties on $\Sigma$ (in $\Sigma_{\text{crit}}$ (the “super-critical” case) to $\xi \leq \xi_{\text{crit}}$ (the “sub-critical” case). In the first case the asymptotic description gives an asymptotic solution that is a modulated travelling wave (the wave parameters are changing slowly in time), while in the sub-critical case, the asymptotic solution is a travelling wave.

5. Super-critical case: the $\alpha$-dependency

We first consider the case

$$\xi_{\text{crit}} < \xi \leq \eta_2^2$$

(5.1)

where the value of $\xi_{\text{crit}} \in \mathbb{R}$ will be defined in (5.18).

In order to study the Riemann–Hilbert problem for $Y$ in this setting we need to split the contours in the following way: let $\alpha \in (\eta_1, \eta_2)$ and define the sub intervals

$$\Sigma_{1,\alpha} = (\alpha, \eta_2) \subseteq \Sigma_1$$

and

$$\Sigma_{2,\alpha} = (-\eta_2, -\alpha) \subseteq \Sigma_2.$$  

(5.2)

The value of $\alpha$ will be determined in Section 5 as a function of $\xi$.

We introduce again scalar functions $g(\lambda)$ and $f(\lambda)$ (in a slight abuse of notation, we are using the same letter $g$ and $f$ to denote these functions, though properly we should probably use $g_\alpha$ and $f_\alpha$). We make the first transformation $Y(\lambda) \rightarrow T(\lambda)$ given by

$$T(\lambda) = Y(\lambda)e^{ig(\lambda)\sigma_3f(\lambda)^{\sigma_3}}$$

(5.3)

such that

$$g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda = 0$$

$$\lambda \in \Sigma_{1,\alpha} \cup \Sigma_{2,\alpha}$$

(5.4)

$$g_+(\lambda) - g_-(\lambda) = \tilde{\Omega}$$

$$\lambda \in [-\alpha, \alpha]$$

(5.5)

$$g(\lambda) = \mathcal{O}\left(\frac{1}{\lambda}\right)$$

$$\lambda \rightarrow \infty.$$  

(5.6)

We further require that $g(\lambda) - 4\lambda^3 + 4\xi\lambda - \tilde{\Omega}$ behaves near $\lambda = \pm \alpha$ as $(\lambda \mp \alpha)^3$. In addition, there are two types of inequalities that must be satisfied by this function in order to have a successful Riemann–Hilbert analysis. First we will need inequalities satisfied on the complement (in $\Sigma_{1} \cup \Sigma_2$) of the sets $\Sigma_{1,\alpha}$ and $\Sigma_{2,\alpha}$:

$$\text{Re}[g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda] < 0$$

$$\lambda \in (\eta_1, \alpha)$$

(5.7)

$$\text{Re}[g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda] > 0$$

$$\lambda \in (-\alpha, -\eta_1).$$

(5.8)

Second, we will require monotonicity properties on $\Sigma_1$ and $\Sigma_2$:

$$-i(g_+(\lambda) - g_-(\lambda))$$

is purely real and monotonically decreasing on $(\alpha, \eta_2)$

(5.9)

$$-i(g_+(\lambda) - g_-(\lambda))$$

is purely real and monotonically increasing on $(-\eta_2, -\alpha).$  

(5.10)
It is well-known that there is a unique function \( g \) satisfying all these properties, which we will define explicitly here (we will actually define \( g' \), which of course determines \( g \)). We define

\[
g' (\lambda) = -12 \lambda^2 + 4 \xi + 12 \frac{Q_2(\lambda)}{R_\alpha(\lambda)} - 4 \xi \frac{Q_1(\lambda)}{R_\alpha(\lambda)},
\]

(5.11)

where

\[
R_\alpha(\lambda) = \sqrt{(\lambda^2 - \alpha^2)(\lambda^2 - \eta_2^2)}
\]

(5.12)

taken to be analytic in \( \mathbb{C} \setminus \{ \Sigma_1, \alpha \cup \Sigma_2, \alpha \} \) and real and positive on \( (\eta_2, +\infty) \); moreover, let

\[
Q_1(\lambda) = \lambda^2 + c_1, \quad \text{and} \quad Q_2(\lambda) = \lambda^4 - \frac{1}{2} \lambda^2 (\alpha^2 + \eta_2^2) + c_2.
\]

(5.13)

The constants \( c_1 \) and \( c_2 \) are chosen so that

\[
\int_{-\alpha}^\alpha \frac{Q_2(\zeta)}{R_{\alpha+}(\zeta)} \, d\zeta = 0, \quad \int_{-\alpha}^\alpha \frac{Q_1(\zeta)}{R_{\alpha+}(\zeta)} \, d\zeta = 0.
\]

(5.14)

Explicitly, we find

\[
c_1 = -\eta_2^2 + \eta_2 E(m_\alpha), \quad c_2 = \frac{1}{3} \alpha^2 \eta_2^3 + \frac{1}{6} (\eta_2^2 + \alpha^2) c_1, \quad m_\alpha = \frac{\alpha}{\eta_2},
\]

(5.15)

with \( K(m_\alpha) \) and \( E(m_\alpha) \) the complete elliptic integrals of the first and second kind respectively.

The parameter \( \alpha \) is determined by requiring that the function \( g(\lambda) - 4 \lambda^3 + 4 \xi \lambda - \tilde{\Omega} \) has a zero at \( \lambda = \pm \alpha \), which yields the equation

\[
\xi = 3 \frac{Q_2(\pm \alpha)}{Q_1(\pm \alpha)} - \frac{1}{2} (\alpha^2 + \eta_2^2) + \frac{\alpha^2 (\alpha^2 - \eta_2^2)}{\alpha^2 - \eta_2^2 + \eta_2^2 \frac{E(m_\alpha)}{K(m_\alpha)}},
\]

(5.16)

determining the constant \( \alpha \) implicitly as a function of \( \xi \).

Before continuing our analysis we want to comment on equation (5.16). We can rewrite it in the form

\[
\xi = \frac{x}{4t} = \frac{\eta_2^2}{2} W(m_\alpha), \quad W(m_\alpha) = \left[ 1 + m_\alpha^2 + 2 \frac{m_\alpha^2 (1 - m_\alpha^2)}{1 - m_\alpha^2 - \frac{E(m_\alpha)}{K(m_\alpha)}} \right].
\]

(5.17)

This relation describes the modulation of the parameter \( \alpha \) as a function of \( \xi \). The quantity \( \eta_2^2 W(m_\alpha) \) was derived by Whitham in his modulation theory of the traveling wave solution of the KdV equation [Whi74]. In the general theory there are three parameters involved, while in our case, two parameters are fixed, one being zero and the other one \( \eta_2 \). This specific case gives a self-similar solution to the Whitham equations. This solution was derived and used by Gurevich-Pitaevskii [GP73] to describe the modulation of the travelling wave that is formed in the solution of the KdV equation with step initial data \( u(x) = -\eta_2^2 \) for \( x < 0 \) and \( u(x) = 0 \) for \( x > 0 \) and was called a dispersive shock wave in analogy with the shock wave that is formed in the solution of the Hopf equation \( u_t + 6uu_x = 0 \) for step initial data.

Using the expansion of the elliptic functions one has

\[
\frac{E(m_\alpha)}{K(m_\alpha)} = 1 - \frac{1}{2} m_\alpha^2 + O(m_\alpha^4), \quad \text{as} \quad m_\alpha \to 0 \quad \text{and} \quad \frac{E(m_\alpha)}{K(m_\alpha)} = \frac{2}{\log (8/(1-m_\alpha))}, \quad \text{as} \quad m_\alpha \to 1,
\]

so that

\[
\lim_{\alpha \to 0} \frac{3Q_2(\alpha)}{Q_1(\alpha)} = -\frac{3\eta_2^2}{2}, \quad \text{and} \quad \lim_{\alpha \to \eta_2} \frac{3Q_2(\alpha)}{Q_1(\alpha)} = \eta_2^2.
\]
The Whitham equations are strictly hyperbolic ([Lev88]), so that \( \frac{\partial}{\partial \alpha} W(m_\alpha) > 0 \) for \( 0 < \alpha < \eta_2 \). Hence by the implicit function theorem, the equation (5.17) defines \( \alpha \) as a monotone increasing function of \( \xi \) for \( \xi \in [\xi_{\text{crit}}, \eta_2^2] \) where \( \xi_{\text{crit}} \) is given by

\[
\xi_{\text{crit}} = 3Q_2(\eta_1) \frac{Q_1(\eta_1)}{Q_1(\eta_2)} = \frac{1}{2} (\eta_1^2 + \eta_2^2) + \frac{\eta_1^2 (\eta_1^2 - \eta_2^2)}{\eta_1^2 - \eta_2^2 + \eta_2^2 \frac{\kappa(m)}{\kappa(m_\alpha)}}, \quad m = \frac{\eta_1}{\eta_2}.
\]

(5.18)

Then, clearly \( \xi_{\text{crit}} > \frac{3\eta_2^2}{2} \).

From \( g'(\lambda) \), we also have a representation of \( g(\lambda) \):

\[
g(\lambda) = -4\lambda^3 + 4\xi \lambda + 12 \int_{\eta_2}^{\lambda} \frac{Q_2(\zeta)}{Q_1(\zeta)} d\zeta - 4\xi \int_{\eta_2}^{\lambda} \frac{Q_1(\zeta)}{Q_1(\zeta)} d\zeta.
\]

(5.19)

This, together with (5.5), yields the formula

\[
\tilde{\Omega} = 24 \int_{\eta_2}^{\alpha} \frac{Q_2(\zeta) - Q_2(\alpha)}{R_{\alpha}(\zeta)} d\zeta - 8\xi \int_{\eta_2}^{\alpha} \frac{Q_1(\zeta) - Q_1(\alpha)}{R_{\alpha}(\zeta)} d\zeta,
\]

(5.20)

which gives, using the Riemann bilinear relations,

\[
\tilde{\Omega} = 2\pi i \frac{4\xi - 2(\alpha^2 + \eta_2^2)}{\int_{-\alpha}^{\alpha} \frac{d\zeta}{R_{\alpha}(\zeta)}} = 2\pi i \eta_2 \frac{\alpha^2 + \eta_2^2 - 2\xi}{K(m_\alpha)} \in i\mathbb{R}, \quad m_\alpha = \frac{\alpha}{\eta_2}.
\]

(5.21)

**Lemma 5.1.** The following identities are satisfied

\[
\frac{\partial}{\partial x} tg'(\lambda) = 1 - \frac{Q_1(\lambda)}{R_\alpha(\lambda)},
\]

(5.22)

\[
\frac{\partial}{\partial x} \tilde{t}\Omega = -\frac{\pi i \eta_2}{K(m_\alpha)}.
\]

(5.23)

**Proof.** We observe that \( g'(\lambda)d\lambda \) defined in (5.11) is a meromorphic one-form on the Riemann surface \( \mathfrak{X}_\alpha \) defined as

\[
\mathfrak{X}_\alpha = \{(\eta, \lambda) \in \mathbb{C}^2 \mid \eta^2 = R_\alpha^2(\lambda) = (\lambda^2 - \alpha^2)(\lambda^2 - \eta_2^2)\}.
\]

We define the homology basis on \( \mathfrak{X}_\alpha \) in the following way: the \( B \) cycle encircles the cut \([\alpha, \eta_2]\) clockwise and the \( A \) cycle starts on the cut \([-\eta_2, -\alpha]\) on the upper semi-plane, goes to the cut \([\alpha, \eta_2]\) and then goes back to \([-\eta_2, -\alpha]\) on the second sheet of \( \mathfrak{X}_\alpha \). Then we have

\[
\oint_A g'(\zeta) d\zeta = 0, \quad \oint_B g'(\zeta) d\zeta = -\tilde{\Omega}.
\]

(5.24)

Regarding the first relation in (5.22) we have

\[
\frac{\partial}{\partial x} tg'(\lambda)d\lambda = \frac{\partial}{\partial x} \left[ -12t\lambda^2 d\lambda + xd\lambda + 12t \frac{Q_2(\lambda)}{R_\alpha(\lambda)} d\lambda - x \frac{Q_1(\lambda)}{R_\alpha(\lambda)} d\lambda \right]
\]

(5.25)

\[
= d\lambda - \frac{Q_1(\lambda)}{R_\alpha(\lambda)} d\lambda + \frac{\partial}{\partial \alpha} \left[ 12t \frac{Q_2(\lambda)}{R_\alpha(\lambda)} d\lambda - x \frac{Q_1(\lambda)}{R_\alpha(\lambda)} d\lambda \right] \frac{\partial t\alpha}{\partial x}
\]

(5.26)

\[
= d\lambda - \frac{Q_1(\lambda)}{R_\alpha(\lambda)} d\lambda,
\]

(5.27)
since the term \( \frac{\partial}{\partial \alpha} \left[ 12\pi \frac{Q_2(\lambda)}{R_\alpha(\lambda)} d\lambda - x \frac{Q_1(\lambda)}{R_\alpha(\lambda)} d\lambda \right] \) vanishes because it is a holomorphic one-form, (no singularity at \( \pm \alpha \) or infinity) and it is normalized to zero on the \( A \) cycle because of (5.24) and therefore it is identically zero \([\text{Kri88}]\) (see also \([\text{Gra02}]\)). Another alternative proof is to calculate the derivative and use the explicit formulæ of the constants \( c_1 \) and \( c_2 \) in (5.15). We conclude that

\[
\frac{\partial}{\partial x} e^{-tg(\lambda)} = -\frac{1}{\lambda} \left[ \frac{\alpha^2 + \eta^2}{2} + \eta^2 \left( \frac{E(m_\alpha)}{K(m_\alpha)} - 1 \right) \right] + \mathcal{O} \left( \frac{1}{\lambda^2} \right).
\]

Regarding the relation (5.23), by (5.22) and (5.24) we have

\[
\frac{\partial}{\partial x} \left( \frac{1}{\Omega} \right) = -\frac{\partial}{\partial x} \int_B t g'(\lambda) d\lambda = -\int_B \frac{\partial}{\partial x} (tg'(\lambda) d\lambda) = -\frac{\pi i \eta^2}{K(m_\alpha)}.
\]

\[\square\]

As we did in Section 3, we choose the function \( f \) to simplify the jumps on \( \Sigma_{1,\alpha} \) and \( \Sigma_{2,\alpha} \) via

\[
\begin{align*}
 f_+(\lambda) f_-(\lambda) &= \frac{1}{r(\lambda)} & \lambda \in \Sigma_{1,\alpha} \\
 f_+(\lambda) f_-(\lambda) &= r(\lambda) & \lambda \in \Sigma_{2,\alpha} \\
 f_+(\lambda) &= e^{\Delta} & \lambda \in [-\alpha, \alpha] \\
 f(\lambda) &= 1 + \mathcal{O} \left( \frac{1}{\lambda} \right) & \lambda \to \infty.
\end{align*}
\]

(5.28, 5.29, 5.30, 5.31)

It is easy to check that the function \( f(\lambda) \) is given by

\[
f(\lambda) = \exp \left\{ \frac{R_\alpha(\lambda)}{2\pi i} \left[ \int_{\Sigma_{1,\alpha}} \frac{\log r(\zeta)}{R_{\alpha}(\zeta)} (\zeta - \lambda) d\zeta + \int_{\Sigma_{2,\alpha}} \frac{\log r(\zeta)}{R_{\alpha}(\zeta)} (\zeta - \lambda) d\zeta + \int_{-\alpha}^{\alpha} \frac{\Delta}{R_\alpha(\zeta) (\zeta - \lambda)} d\zeta \right] \right\}.
\]

(5.32)

The constraint (5.31) yields a formula for \( \Delta \):

\[
\Delta = \left[ \int_{\Sigma_{1,\alpha}} \frac{\log r(\zeta)}{R_{\alpha}(\zeta)} d\zeta - \int_{\Sigma_{2,\alpha}} \frac{\log r(\zeta)}{R_{\alpha}(\zeta)} d\zeta \right] \left[ \int_{-\alpha}^{\alpha} \frac{d\zeta}{R_\alpha(\zeta)} \right]^{-1} = 2 \left[ \int_{\Sigma_{1,\alpha}} \frac{\log r(\zeta)}{R_{\alpha}(\zeta)} d\zeta \right] \left[ \int_{-\alpha}^{\alpha} \frac{d\zeta}{R_\alpha(\zeta)} \right]^{-1}
\]

(5.33)

where in the last relation in (5.33) we use the fact that \( r(-\lambda) = r(\lambda) \).
As a consequence, \( T \) satisfies the following Riemann–Hilbert problem:

\[
T_+ (\lambda) = T_- (\lambda)
\]

\[
\begin{bmatrix}
  e^{i (g_+ (\lambda) - g_- (\lambda))} \frac{f_+ (\lambda)}{f_- (\lambda)} & 0 \\
  -i & e^{-t (g_+ (\lambda) - g_- (\lambda))} \frac{f_- (\lambda)}{f_+ (\lambda)} \\
  e^{\eta t + \Delta} & 0 \\
  0 & e^{\eta t - \Delta}
\end{bmatrix}
\]

\( \lambda \in \Sigma_{1,\alpha} \)

\( \lambda \in \Sigma_{2,\alpha} \)

where \( \eta = \frac{1}{\lambda} \), \( \alpha \), and \( \Omega \) are defined as before.

\( T (\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O} \left( \frac{1}{\lambda} \right) \quad \lambda \to \infty \).  \( (5.35) \)

5.1. Opening lenses. It is useful to provide representations of the entries appearing in the jump matrix for \( T (\lambda) \) in either \( \Sigma_{1,\alpha} \) or \( \Sigma_{2,\alpha} \), that clearly demonstrate their analytic continuation off of these intervals, as was done in Section 3. The following formulæ are valid on both intervals:

\[
g_+ (\lambda) - g_- (\lambda) = 2g_+ (\lambda) + 8\lambda^3 - 8\xi \lambda, \quad (5.36)
\]

\[
g_+ (\lambda) - g_- (\lambda) = - (2g_+ (\lambda) + 8\lambda^3 - 8\xi \lambda). \quad (5.37)
\]

The following formulæ are valid on \( \Sigma_{1,\alpha} \):

\[
\frac{f_+ (\lambda)}{f_- (\lambda)} = - \frac{1}{f_+ (\lambda) \hat{r}_- (\lambda)} \quad \text{and} \quad \frac{f_- (\lambda)}{f_+ (\lambda)} = \frac{1}{f_- (\lambda) \hat{r}_+ (\lambda)}. \quad (5.38)
\]

And the following ones are valid on \( \Sigma_{2,\alpha} \):

\[
\frac{f_- (\lambda)}{f_+ (\lambda)} = - \frac{f_-^2 (\lambda)}{\hat{r}_+ (\lambda)} \quad \text{and} \quad \frac{f_+ (\lambda)}{f_- (\lambda)} = \frac{f_+^2 (\lambda)}{\hat{r}_- (\lambda)}. \quad (5.39)
\]

As was done in Section 3, we can factor the jump matrix on \( \Sigma_{1,\alpha} \) as follows:

\[
\begin{bmatrix}
  e^{i (g_+ (\lambda) - g_- (\lambda))} \frac{f_+ (\lambda)}{f_- (\lambda)} & 0 \\
  -i & e^{-t (g_+ (\lambda) - g_- (\lambda))} \frac{f_- (\lambda)}{f_+ (\lambda)} \\
  1 - ie^{-(2g_+ (\lambda) + 8\lambda^3 - 8\xi \lambda)} & 0 \\
  0 & 1
\end{bmatrix}
\]

and on \( \Sigma_2 \) as:

\[
\begin{bmatrix}
  e^{i (g_+ (\lambda) - g_- (\lambda))} \frac{f_+ (\lambda)}{f_- (\lambda)} & i \\
  0 & e^{-t (g_+ (\lambda) - g_- (\lambda))} \frac{f_- (\lambda)}{f_+ (\lambda)} \\
  f_-^2 (\lambda) \hat{r}_+ (\lambda) & 0 \\
  0 & f_+^2 (\lambda) \hat{r}_- (\lambda)
\end{bmatrix}
\]

\[
\begin{bmatrix}
  1 & 0 \\
  0 & 1 \\
  0 & 0 \\
  0 & 0
\end{bmatrix}
\]

These factorizations permit us to open lenses as shown in Figure 5.
Lemma 5.2. The following inequalities are satisfied:

\[
\begin{align*}
\text{Re} \left[ 2g(\lambda) + 8\lambda^3 - 8\xi\lambda \right] &> 0 \quad \text{for } \lambda \in \mathcal{C}_1 \setminus \{\alpha, \eta_2\} , \\
\text{Re} \left[ 2g(\lambda) + 8\lambda^3 - 8\xi\lambda \right] &< 0 \quad \text{for } \lambda \in \mathcal{C}_2 \setminus \{-\eta_2, -\alpha\} , \\
\text{Re} \left[ g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda \right] &< 0 \quad \text{for } \lambda \in [\eta_1, \alpha] , \\
\text{Re} \left[ g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda \right] &> 0 \quad \text{for } \lambda \in (-\alpha, -\eta_1] .
\end{align*}
\]

**Proof.** Using (5.16) the function \( g'(\lambda) \) in (5.11) can be written in the form

\[
g'(\lambda) = -12\lambda^2 + 4\xi + 12 \frac{Q_2(\lambda) - Q_2(\alpha)}{R_\alpha(\lambda)} - 4\xi \frac{Q_1(\lambda) - Q_1(\alpha)}{R_\alpha(\lambda)} ,
\]

so that we have

\[
g'_+(\lambda) - g'_-(\lambda) = -i24\sqrt{\frac{\lambda^2 - \alpha^2}{\eta_2^2 - \lambda^2}} \left[ \lambda^2 - \left( \frac{\eta_2^2 - \alpha^2}{2} + \frac{\xi}{3} \right) \right] .
\]

and from (5.14) we deduce that the quadratic polynomial has one root \( \rho_+ \) in the interval \([0, \alpha]\) and it is positive for \( \lambda > \alpha \). Therefore, for \( \lambda \in \Sigma_{1,\alpha} \)

\[
\text{Im} \left[ g'_+(\lambda) - g'_-(\lambda) \right] = -24\sqrt{\frac{\lambda^2 - \alpha^2}{\eta_2^2 - \lambda^2}} \left[ \lambda^2 - \left( \frac{\eta_2^2 - \alpha^2}{2} + \frac{\xi}{3} \right) \right] < 0 .
\]

From the formula (5.19) for \( g \) we also have that for \( \lambda \in [\eta_1, \alpha] \)

\[
g_+(\lambda) + g_-(\lambda) + 8\lambda^3 - 8\xi\lambda = -24 \int_\alpha^\lambda \sqrt{\frac{\lambda^2 - \zeta^2}{\eta_2^2 - \zeta^2}} \left[ \zeta^2 - \left( \frac{\eta_2^2 - \alpha^2}{2} + \frac{\xi}{3} \right) \right] d\zeta .
\]

Setting

\[
h_{\alpha,\xi}(\zeta) = \sqrt{\frac{\lambda^2 - \zeta^2}{\eta_2^2 - \zeta^2}} \left[ \zeta^2 - \left( \frac{\eta_2^2 - \alpha^2}{2} + \frac{\xi}{3} \right) \right] ,
\]
we need to show that the function
\[ H_{\alpha, \xi}(\lambda) = \int_{\lambda}^{\alpha} -h_{\alpha, \xi}(\zeta) \, d\zeta < 0 \quad \text{for} \ \lambda \in [\eta_1, \alpha]. \] (5.48)

It is easy to check that \( H_{\alpha, \xi}(\alpha) = 0 \) and \( H_{\alpha, \xi}(0) = 0 \) (see 5.14). Next, \( H'_{\alpha, \xi}(\lambda) = h_{\alpha, \xi}(\lambda) \) is negative on \([0, \rho_*] \) and positive on \([\rho_*, \alpha] \). This implies that indeed the inequality (5.48) is satisfied on \([\eta_1, \alpha] \). □

Because of lemma 5.2, letting \( t \not\to +\infty \), the jump matrices (as depicted in the Figure 5) will converge to constant jumps exponentially fast outside neighbourhoods of \( \pm \alpha \) and \( \pm \eta_2 \). We then obtain the following model Riemann–Hilbert problem for \( \tilde{S}^{(\infty)} \):
\[
\tilde{S}^{(\infty)}_+(\lambda) = \tilde{S}^{(\infty)}_-(\lambda) \begin{bmatrix} e^{i\Omega + \Delta} & 0 \\ 0 & e^{-i\Omega - \Delta} \end{bmatrix} \begin{cases} \lambda \in [-\alpha, \alpha] \\ 0 & -i \\ -i & 0 \end{cases} \lambda \in \Sigma_{1,\alpha} \\
\begin{cases} 0 & i \\ i & 0 \end{cases} \lambda \in \Sigma_{2,\alpha}
\]
\[
\tilde{S}^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right), \ \lambda \to \infty.
\] (5.50)

5.2. The global parametrix \( \tilde{P}^{(\infty)} \). Along the same lines as we did in Section 3, we construct a (matrix) model problem whose solution will yield a solution of the above (vector) Riemann–Hilbert problem. Since the solution of this model problem will be invertible, one is able to arrive at a small-norm Riemann–Hilbert problem for the error in the large-time regime, more directly than if one considers only vector Riemann–Hilbert problems. And, as with Section 3, we build this local parametrix under the assumption that \( \frac{\Omega + \Delta}{2\pi i} \neq \frac{2n+1}{2} \), \( n \in \mathbb{Z} \).

We want to determine the matrix valued function \( \tilde{P}^{(\infty)} \) that is analytic in \( \mathbb{C} \backslash (-\eta_2, \eta_2) \) and satisfies the following Riemann–Hilbert problem
\[
\tilde{P}^{(\infty)}_+(\lambda) = \tilde{P}^{(\infty)}_-(\lambda) \begin{bmatrix} e^{i\Omega + \Delta} & 0 \\ 0 & e^{-i\Omega - \Delta} \end{bmatrix} \begin{cases} \lambda \in [-\alpha, \alpha] \\ 0 & -i \\ -i & 0 \end{cases} \lambda \in \Sigma_{1,\alpha} \\
\begin{cases} 0 & i \\ i & 0 \end{cases} \lambda \in \Sigma_{2,\alpha}
\]
\[
\tilde{P}^{(\infty)}(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \mathcal{O}\left(\frac{1}{\lambda}\right), \ \lambda \to \infty.
\] (5.52)

The solution is obtained from the solution of \( P^{(\infty)} \) in (3.64) by replacing \( \eta_1 \) with \( \alpha \).

5.3. The local parametrix \( P^{(\pm \alpha)} \). We will construct now a (matrix) local parametrix around the point \( \lambda = -\alpha \). The construction of the other local parametrix near the endpoint \( \lambda = \alpha \) is analogous, while the construction of the local parametrix near \( \lambda = \pm \eta_2 \) is the same one as in the Section 3.6 (again, under the assumption \( \frac{\Omega + \Delta}{2\pi i} \neq \frac{2n+1}{2} \), \( n \in \mathbb{Z} \)).
We focus again on a small but fixed neighbourhood \( B_\rho = \{ \lambda \in \mathbb{C} \mid |\lambda + \alpha| < \rho \} \) of the endpoint \( \lambda = -\alpha \). We define the conformal map as

\[
\zeta = \left( \frac{3}{4} \right)^2 \left[ t \int_{-\alpha}^\lambda g'_s(s) - g'_t(s) \, ds \right]^{\frac{2}{3}} = \left[ 18t \int_{-\alpha}^\lambda \left( \frac{\sqrt{\alpha^2 - s^2}}{\sqrt{\eta_2^2 - s^2}} \right) \left( s^2 - \frac{\eta_2^2 - \alpha^2}{2} - \frac{\xi}{3} \right) \, ds \right]^{\frac{2}{3}}
\]

locally in \( B_\rho \).

To define the local parametrix \( P^{(-\alpha)} \) in \( B_\rho \), we consider

\[
P(\lambda) = S(\lambda) e^{\frac{2i}{3} \sigma_3} \left( \frac{\sqrt{\pm i(\lambda)}}{f(\lambda)} \right)^{\sigma_3} e^{\frac{i}{2} (\Omega t + \Delta) \sigma_3} , \quad \lambda \in B_\rho \cap \mathbb{C}_+ .
\]

and then, using the inverse of the transformation \( \zeta(\lambda) \), we define

\[
P^{(1)}(\zeta) = P(\lambda(\zeta)) e^{-\frac{2}{3} \zeta^2 \sigma_3} , \quad \zeta \in \mathbb{C}
\]

with branch cut \((-\infty, 0]\). By construction, \( P^{(1)} \) satisfies a Riemann–Hilbert problem with jumps in a neighbourhood of \( \zeta = 0 \) as shown in Figure 6.

![Figure 6: The contour setting under the conformal map \( \zeta \) in a neighbourhood of 0.](image)

We introduce the (local) Airy parametrix (see [Dei99] and [DKM'99]): let \( \Psi_{Ai}(\zeta) \) the solution to the following Riemann–Hilbert problem

(a) \( \Psi_{Ai} \) is analytic for \( \zeta \in \mathbb{C} \setminus \Gamma_\Psi \), where the contours \( \Gamma_\Psi \) are defined as \( \Gamma_\pm = \{ \arg \zeta = \pm \frac{\pi}{3} \} \), \( \Gamma_{0,-} = \{ \arg \zeta = \pi \} \) and \( \Gamma_{0,+} = \{ \arg \zeta = 0 \} \);

(b) \( \Psi \) satisfies the following jump relations

\[
\Psi_{Ai+}(\zeta) = \Psi_{Ai-}(\zeta) \begin{cases} 
1 & 0 \\
1 & 1 \\
0 & 1 \\
-1 & 0 \\
1 & 1 \\
0 & 1 
\end{cases}
\]

on \( \Gamma_+ \) and \( \Gamma_- \) on \( \Gamma_{0,-} \) ;

\[
\Psi_{Ai+}(\zeta) = \Psi_{Ai-}(\zeta) \begin{cases} 
1 & 0 \\
1 & 1 \\
0 & 1 \\
-1 & 0 \\
1 & 1 \\
0 & 1 
\end{cases}
\]

on \( \Gamma_{0,+} \)

(c) as \( \zeta \to \infty \)

\[
\Psi_{Ai}(\zeta) = \zeta^{-\frac{2}{3} \sigma_3} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \left( I + \mathcal{O}\left( \frac{1}{\zeta^\frac{1}{2}} \right) \right) e^{-\frac{2}{3} \zeta^2 \sigma_3} .
\]
(d) $\Psi_{\text{Ai}}$ remains bounded as $\zeta \to 0$, $\zeta \in \mathbb{C}\setminus \Gamma_\Phi$.

The solution to this Riemann–Hilbert problem is constructed out of Airy functions: setting

$$\omega = e^{\frac{2\pi i}{3}},$$

then

$$\Psi_{\text{Ai}}(\zeta) = \sqrt{2\pi} \begin{bmatrix} \text{Ai}(\zeta) & -\omega^2 \text{Ai}(\omega^2 \zeta) \\ -i \text{Ai}'(\zeta) & i\omega \text{Ai}'(\omega^2 \zeta) \end{bmatrix}$$

for $0 < \arg \zeta < \frac{2\pi}{3}$ (5.56)

$$\Psi_{\text{Ai}}(\zeta) = \sqrt{2\pi} \begin{bmatrix} -\omega \text{Ai}(\omega \zeta) & -\omega^2 \text{Ai}(\omega^2 \zeta) \\ i\omega \text{Ai}'(\omega^2 \zeta) & i\omega \text{Ai}'(\omega^2 \zeta) \end{bmatrix}$$

for $\frac{2\pi}{3} < \arg \zeta < \pi$ (5.57)

$$\Psi_{\text{Ai}}(\zeta) = \sqrt{2\pi} \begin{bmatrix} -\omega^2 \text{Ai}(\omega^2 \zeta) & \omega \text{Ai}(\omega \zeta) \\ i\omega \text{Ai}'(\omega^2 \zeta) & -i\omega^2 \text{Ai}'(\zeta) \end{bmatrix}$$

for $-\pi < \arg \zeta < -\frac{2\pi}{3}$ (5.58)

$$\Psi_{\text{Ai}}(\zeta) = \sqrt{2\pi} \begin{bmatrix} \text{Ai}(\zeta) & \omega \text{Ai}(\omega \zeta) \\ -i \text{Ai}'(\zeta) & -i\omega^2 \text{Ai}'(\zeta) \end{bmatrix}$$

for $-\frac{2\pi}{3} < \arg \zeta < 0,$ (5.59)

where $\text{Ai}(\zeta)$ is the Airy function.

In conclusion, our local parametrix is then defined as

$$P^{(-\alpha)}(\zeta(\lambda)) = A(\lambda) \Psi_{\text{Ai}}(\zeta(\lambda)) e^{\frac{i}{2} \zeta \sigma_3 e^{\frac{i}{2}(\bar{\eta} + \bar{\Xi})} \left( \frac{f(\lambda)}{\sqrt{\pm f(\lambda)}} \right)^{\sigma_3} e^{-\frac{\pi i}{2} \sigma_3}, \lambda \in B_\rho \cap \mathbb{C}_+,$$ (5.60)

where $A$ is an analytic prefactor whose expression is determined by imposing that

$$P^{(-\alpha)}(\lambda) (\bar{P}^{(\infty)}(\lambda))^{-1} = I + \mathcal{O} \left( t^{-1} \right) \quad \text{as } t \to +\infty, \text{ for } \lambda \in \partial B_\rho \setminus \Gamma_\Phi.$$ (5.61)

In light of this asymptotic behaviour we set

$$A(\lambda) = \bar{P}^{(\infty)}(\lambda) e^{\frac{i}{2} \bar{\Xi} \sigma_3 e^{\frac{i}{2}(\bar{\eta} + \bar{\Xi})} \left( \frac{\sqrt{\pm f(\lambda)}}{f(\lambda)} \right)^{\sigma_3} \frac{1}{\sqrt{2}} \left[ 1 \begin{array}{c} 1 \\ -i \end{array} \right] \zeta(\lambda) \right]^{\frac{1}{2} \sigma_3}, \quad \text{for } \lambda \in B_\rho \cap \mathbb{C}_+.$$ (5.62)

By construction, $A$ is defined and analytic in a neighbourhood of $-\alpha$, minus the cuts $(-\infty, -\alpha] \cup [-\alpha, +\infty)$; moreover, $A$ is invertible (det $A(\lambda) \equiv 1$).

**Lemma 5.3.** $A(\lambda)$ is analytic in a whole neighbourhood of $-\alpha$.

**Proof.** The proof entails verifying that $A$ has no jumps across the interval $(-\alpha - \epsilon, -\alpha + \epsilon)$ and that it has at most a removable singularity at $\lambda = -\alpha$. We leave the verification that $A_+ = A_-$ across the interval $(-\alpha - \epsilon, -\alpha + \epsilon)$ to the reader, using the jump relations satisfied by $\bar{P}^{(\infty)}$ and the above definitions.

The conformal map $\zeta(\lambda)$ has a simple zero at $\lambda = -\alpha$ (by construction), therefore $\zeta(\lambda)^{-\frac{1}{2} \sigma_3}$ has at most a fourth-root singularity at $-\alpha$. Similarly, $\bar{P}^{(\infty)}(\lambda)$ has a fourth-root singularity at $-\alpha$, as well; therefore, all the entries of $A(\lambda)$ have at most a square root singularity at $\lambda = -\alpha$, and $A(\lambda)$ is analytic in $B_\rho \setminus \{-\alpha\}$. The point $\lambda = -\alpha$ is a removable singularity. This implies that $A(\lambda)$ is indeed analytic everywhere in $B_\rho$. \(\square\)

5.4. **Small norm argument and determination of** $u(x,t)$ **as** $t \to +\infty$. If we call $U$ the (matrix) model problem given by the global parametrix and the four local parametrices, then we define the following “remainder” Riemann–Hilbert problem:

$$\mathcal{E}(\lambda) = S(\lambda) U(\lambda)^{-1},$$ (5.63)

where the vector $\mathcal{E}$ satisfies

$$\mathcal{E}_+(\lambda) = \begin{cases} \mathcal{E}_+(\lambda) (I + \mathcal{O} \left( t^{-\infty} \right)) & \text{on the upper and lower lenses, outside the discs} \\ \mathcal{E}_-(\lambda) (I + \mathcal{O} \left( t^{-1} \right)) & \text{on the circles around the endpoints} \end{cases}$$ (5.64)
and
\[ \mathcal{E}(\lambda) = [1 \quad 1] + O\left(\frac{1}{\lambda}\right) \quad \text{as } \lambda \to \infty . \] (5.65)

Therefore, by a small norm argument (see [Its11, Section 5.1.3]) the solution satisfies
\[ \mathcal{E}(\lambda) = [1 \quad 1] + O\left(t^{-1}\right) \quad \text{in the regime } t \nearrow +\infty . \] (5.66)

Unraveling the transformations, we can again get back to the potential. Our original Riemann–Hilbert problem was for the unknown \( Y \), which satisfies
\[
Y(\lambda) = T(\lambda)e^{-t g(\lambda)\sigma_3} f(\lambda)^{-\sigma_3} = S(\lambda)e^{-t g(\lambda)\sigma_3} f(\lambda)^{-\sigma_3}
\]
and in particular we are interested in the first entry of the vector \( Y(\lambda) \) for large \( \lambda \):
\[ Y_1(\lambda) = \left[ \widetilde{S}^{(\infty)}(\lambda) + O\left(t^{-1}\right) \right] e^{-t g(\lambda)} f(\lambda)^{-1} \]

since
\[ u(x, t) = 2 \frac{d}{dx} \left[ \lim_{\lambda \to \infty} (\lambda Y_1(\lambda; x, t) - 1) \right] . \] (5.67)

(We note that \( \widetilde{S}^{(\infty)} \) refers to the the first vector solution \( S^{(\infty)} \), with \( \eta_1 \) replaced by \( \alpha \).)

**Theorem 5.4.** Given \( \xi = \frac{\pi}{4} \), in the region \( \xi_{crit} < \xi < \eta_2^2 \) the solution of the KdV equation in the large time limit is
\[
u(x, t) = \eta_2^2 - \alpha^2 - 2m_2 \frac{E(m_\alpha)}{K(m_\alpha)} - \frac{2}{d^2 x^2} \log \vartheta_2 \left( \frac{\eta_2}{2K(m_\alpha)}(x - 2(\alpha^2 + \eta_2^2)t + \widetilde{\phi}); 2\tau_{\alpha} \right) + O(t^{-1}) \] (5.68)
where \( E(m_\alpha) \) and \( K(m_\alpha) \) are the complete elliptic integrals of first and second kind respectively, with modulus \( m_\alpha = \frac{\alpha}{\eta_2} \); \( 2\tau_{\alpha} = i \frac{K(m_\alpha')}{K(m_\alpha)} \), with \( m_\alpha' = \sqrt{1 - m_\alpha^2} \),
\[
\widetilde{\phi} = \int_\alpha^{m_2} \frac{\log r(\zeta) d\zeta}{R_{\alpha}(\zeta)} \frac{\pi i}{2} \in \mathbb{R}
\]
and the parameter \( \alpha = \alpha(\xi) \) is determined from the equation
\[ \xi = \frac{\eta_2^2}{2} \left[ 1 + m_\alpha^2 + 2 \frac{m_\alpha^2(1 - m_\alpha^2)}{1 - m_\alpha^2 - E(m_\alpha) / K(m_\alpha)} \right]. \]
The error term \( O(t^{-1}) \) is uniform provided \( \frac{m_2 + \bar{m}_2}{2n + 1} = \frac{2n + 1}{2} \), \( n \in \mathbb{Z} \).

Alternatively,
\[
u(x, t) = \eta_2^2 - \alpha^2 - 2m_2 \frac{d^2}{d^2 x^2} \left( \eta_2(x - 2(\alpha^2 + \eta_2^2)t + \widetilde{\phi}) + K(m_\alpha) m_\alpha \right) + O(t^{-1}) \] (5.69)
where \( d^2 n(z \mid m) \) is the Jacobi elliptic function.

**Proof.** Expanding each term of \( Y_1(\lambda) \) in (5.4) in a neighbourhood of infinity gives the following. Regarding \( f(\lambda) \) defined in (5.32) we have
\[
f(\lambda) = 1 + \frac{f_1(\alpha, \eta_2)}{\lambda} + O\left(\frac{1}{\lambda^2}\right),
\]
where
\[
f_1(\alpha, \eta_2) = \left[ \int_{\alpha}^{m_2} \frac{\zeta^2 \log r(\zeta) d\zeta}{R_{\alpha}(\zeta)} \frac{\pi i}{2} - \Delta \int_{-\alpha}^{\alpha} \frac{\zeta^2}{R_{\alpha}(\zeta)} \frac{d\zeta}{2\pi i} \right].
\]
Regarding $e^{-t \varphi(\lambda)}$ we are interesting in the $x$ derivative of this expression. Using (5.22) we have

$$\frac{\partial}{\partial x} e^{-t \varphi(\lambda)} = -\frac{1}{\lambda} \left[ \frac{\alpha^2 + \eta_2^2}{2} + \eta_2 \left( \frac{E(m_\alpha)}{K(m_\alpha)} - 1 \right) \right] + O\left( \frac{1}{\lambda^2} \right).$$

Regarding $\tilde{S}_1^{(\infty)}(\lambda)$, we have

$$\tilde{S}_1^{(\infty)}(\lambda) = 1 + \frac{1}{\lambda} \left[ \left( \log \vartheta_3 \left( \frac{t\tilde{\varphi} + \tilde{\Delta}}{2\pi i} ; 2\tau \right) \right) ^\prime - \frac{\eta_2}{2K(m_\alpha)} \right] + O\left( \frac{1}{\lambda^2} \right),$$

where $'$ stands for the derivative with respect to the argument of the theta-function. By (5.23) we have

$$\frac{\partial}{\partial x} \tilde{S}_1^{(\infty)}(\lambda) = \frac{1}{\lambda} \left[ \log \vartheta_3 \left( \frac{t\tilde{\varphi} + \tilde{\Delta}}{2\pi i} ; 2\tau \right) \right] ^{\prime\prime} - \frac{\eta_2}{2K(m_\alpha)} \frac{\partial}{\partial x} \tilde{\Delta} + O\left( \frac{1}{\lambda^2} \right),$$

where $^{\prime\prime}$ stands for second derivative with respect to the argument. Taking into account that for any smooth function $F(\alpha(\xi), \eta_2)$, $\frac{\partial}{\partial x} F(\alpha(\xi), \eta_2) = O(t^{-1})$ by (5.16), we can write the above expression in the form

$$\frac{\partial}{\partial x} \tilde{S}_1^{(\infty)}(\lambda) = -\frac{1}{\lambda} \left[ \frac{\partial}{\partial x^2} \log \vartheta_3 \left( \frac{t\tilde{\varphi} + \tilde{\Delta}}{2\pi i} ; 2\tau \right) + O(t^{-1}) \right] + O\left( \frac{1}{\lambda^2} \right).$$

Gathering the above expansions and using the explicit expression of $\tilde{\varphi}$ and $\tilde{\Delta}$ in (5.21) and (5.33) respectively we obtain (5.68). Also in this case, using theorem 3.6 we can reduce the expression of $u(x,t)$ to the form (5.69).

6. SUB-CRITICAL CASE

As the parameter $\xi \leq \eta_2^2$ decreases, we proved that there is a critical value $\xi_{\text{crit}}$ (see Section 3) such that

$$\alpha(\xi_{\text{crit}}) = \eta_1.$$

For $\xi < \xi_{\text{crit}}$, we define

$$g'(\lambda) = -12\lambda^2 + 4\xi + 12 \frac{Q_2(\lambda)}{R(\lambda)} - 4\xi \frac{Q_1(\lambda)}{R(\lambda)},$$

where $R$ is defined in (3.15), specifically $R(\lambda) = \sqrt{(\lambda^2 - \eta_1^2)(\lambda^2 - \eta_2^2)}$, and

$$Q_1(\lambda) = \lambda^2 + c_1, \quad Q_2(\lambda) = \lambda^4 - \frac{1}{2} \lambda^2 (\eta_1^2 + \eta_2^2) + c_2,$$

with the constants $c_1$ and $c_2$ chosen so that

$$\int_0^\eta \frac{Q_2(\zeta)}{R(\zeta)} d\zeta = 0, \quad \int_0^\eta \frac{Q_1(\zeta)}{R(\zeta)} d\zeta = 0. \quad (6.4)$$

Integration yields

$$g(\lambda) = -\frac{4\lambda^3}{2 \lambda^2 - 8\xi \lambda} + \int_0^\lambda \frac{12Q_2(\zeta) - 4\xi Q_1(\zeta)}{R(\zeta)} d\zeta.$$

By construction, $g$ satisfies the following constraints:

$$g_+ (\lambda) + g_- (\lambda) + 8\lambda^2 - 8\xi \lambda = 0 \quad \lambda \in \Sigma_1 \cup \Sigma_2 \quad (6.6)$$

$$g_+ (\lambda) - g_- (\lambda) = \Lambda \quad \lambda \in [-\eta_1, \eta_1] \quad (6.7)$$

$$g(\lambda) = O \left( \frac{1}{\lambda} \right) \quad \lambda \rightarrow \infty \quad (6.8)$$
with
\[ \Omega = 2\pi i \eta_2 \frac{2\xi - (\eta_1^2 + \eta_2^2)}{K(m)} \in i\mathbb{R}. \] (6.9)

**Remark 6.1.** The reader may verify that for \( \xi = \xi_{\text{crit}} \) the above function \( g(\lambda; \eta_1, \eta_2) \) in (6.2) agrees with the function \( g(\lambda; \alpha = \eta_1, \eta_2) \) in (5.19).

In order to show that the usual contour deformations can be carried out, as they were in Sections 3 and 4, we need to verify that the quantity \( \text{Re} \left[ 2g(\lambda) + 8\lambda^3 - 8\xi^2\lambda \right] \) is positive on the contour \( \mathcal{C}_1 \), and negative on the contour \( \mathcal{C}_2 \), where these contours are as shown in Figure 3.

To accomplish this, we consider the quadratic polynomial
\[ q(r; \xi) = 12 \left( r^2 - \frac{1}{2} r (\eta_1^2 + \eta_2^2) + c_2 \right) - 4\xi (r + c_1), \] (6.10)
with \( r \in [0, \eta_1^2] \). A quick inspection shows that \( q(\eta_1^2; \xi_{\text{crit}}) = 0 \) and \( q(0; \xi_{\text{crit}}^2) > 0 \), and moreover, for all \( \xi \in \mathbb{R} \)
\[ \frac{\partial q}{\partial \xi}(0; \xi) > 0 \quad \text{and} \quad \frac{\partial q}{\partial \xi}(\eta_1^2; \xi) < 0; \] (6.11)
therefore, \( 0 = q(\eta_1^2; \xi_{\text{crit}}) < q(\eta_1^2; \xi) \) for all \( \xi < \xi_{\text{crit}} \). So, for all \( \xi < \xi_{\text{crit}} \), there are two roots of \( q(r; \xi) \) within \((0, \eta_1^2)\), and the polynomial is strictly positive on \([\eta_1^2, \eta_2^2]\).

This in turn implies, using arguments nearly identical to those used to prove Lemma 5.2, that
\[ \text{Re} \left[ 2g(\lambda) + 8\lambda^3 - 8\xi^2\lambda \right] > 0 \quad \text{for} \quad \lambda \in \mathcal{C}_1 \setminus \{\eta_1, \eta_2\}, \] (6.12)
\[ \text{Re} \left[ 2g(\lambda) + 8\lambda^3 - 8\xi^2\lambda \right] < 0 \quad \text{for} \quad \lambda \in \mathcal{C}_2 \setminus \{-\eta_1, -\eta_2\}. \] (6.13)

The use of this function, and the sequence of steps in the Riemann–Hilbert analysis which have been carried out for \( t = 0 \) in Section 3, may be applied directly to the present situation, and we use the same outer model problem as was used in Section 3, along with the same local parametrices near each of the endpoints \( \pm \eta_1, \pm \eta_2 \). Therefore we arrive at the following result.

**Theorem 6.2.** In the regime \( t \searrow +\infty, \xi \leq \xi_{\text{crit}}, \frac{\eta_1 + \lambda}{2n+1} \neq \frac{2n+1}{2}, n \in \mathbb{Z} \), the potential \( u(x,t) \) has the following asymptotic expansion
\[ u(x,t) = \eta_2^2 - \eta_1^2 - 2\eta_2^2 \text{dn}^2(\eta_2 \xi (x - 2(\eta_1^2 + \eta_2^2)t + \phi) + K(m) \mid m) + \mathcal{O} \left( t^{-1} \right), \] (6.14)
where \( m = \eta_1 / \eta_2 \), and
\[ \phi = \int_{\eta_1}^{\eta_2} \frac{\log r(\xi) \, d\xi}{R_+ (\xi) / \pi i}. \] (6.15)

**7. Conclusions**

In this paper we have considered the Riemann–Hilbert problem of \([DZZ16]\) in the case of one non-trivial reflection coefficient. We have shown how this Riemann–Hilbert problem describes a soliton gas as the limit of a finite \( N \)-soliton configuration as \( N \) tends to \( +\infty \). Then we established rigorous asymptotics of the KdV potential in several different regimes. First, for the initial configuration, we studied the challenging behaviour as \( x \searrow -\infty \), and obtained a universal asymptotic description in terms of the periodic travelling wave solution of KdV. Then, we provided a complete analysis of the long-time behavior of the solution of the KdV equation determined by the Riemann–Hilbert problem of \([DZZ16]\). For large \( t \), there are three fundamental spatial domains, in which the solution \( u(x,t) \) displays different asymptotic behaviour, either
exponentially small, or in terms of the periodic travelling wave solution of KdV with slowly varying parameters, or the periodic travelling wave solution of KdV with fixed parameters.

Several challenges remain, like the asymptotic analysis when there are two nontrivial reflection coefficients or when the case where the spectral parameters of the soliton gas accumulates in disconnected components of the imaginary axis. Beyond these, it is enticing to consider the interaction of one large soliton with this gas like in [CDE16] or the interaction between two such soliton gases.

Acknowledgements. T.G. and M.G. acknowledges the support of the H2020-MSCA-RISE-2017 PROJECT No. 778010 IPADEGAN. K.M. was supported in part by the National Science Foundation under grant DMS-1733967. Part of the work of M.G. and K.M. was done during their visits at SISSA; we acknowledge SISSA for excellent working conditions and generous support. T.G. wish to thank Roberto Camassa and Marco Bertola for useful feedback.

References

[CDE16] F. Carbone, D. Dutykh, and G. A. El. Macroscopic dynamics of incoherent soliton ensembles: Soliton gas kinetics and direct numerical modelling. *EPL*, 113(3):30003, 2016.

[Dei99] P. Deift. *Orthogonal Polynomials and Random Matrices: A Riemann–Hilbert Approach*, volume 3 of Courant Lecture Notes. New York University, 1999.

[DKM+99] P. Deift, T. Kriecherbauer, K.T-R McLaughlin, S. Venakides, and X. Zhou. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Math. Phys.*, 2:1491–1552, 1999.

[DP14] D. Dutykh and E. Pelinovsky. Numerical simulation of a solitonic gas in KdV and KdV-BBM equations. *Phys. Lett. A*, 378(42):3102–3110, 2014.

[DZZ16] S. Dyachenko, D. Zakharov, and V. Zakharov. Primitive potentials and bounded solutions of the KdV equation. *Phys. D*, 333:148–156, 2016.

[EGKT13] I. Egorova, Z. Gładka, V. Kotlyarov, and G. Teschl. Long-time asymptotics for the Korteweg-de Vries equation with step-like initial data. *Nonlinearity*, 26(7):1839–1864, 2013.

[EK05] GA El and A.M. Kamchatnov. Kinetic equation for a dense soliton gas. *Phys. Rev. Lett.*, 95:204101, 2005.

[EKPZ11] G. A. El, A. M. Kamchatnov, M. V. Pavlov, and S. A. Zykov. Kinetic equation for a soliton gas and its hydrodynamic reductions. *J. Nonlinear Sci.*, 21(2):151–191, 2011.

[EL16] G. A. El. Critical density of a soliton gas. *Chaos*, 26(2):023105, 6, 2016.

[GPT73] A.V. Gurevich and L.P. Pitaevskii. Decay of initial discontinuity in the Korteweg de Vries equation. *JETP Letters*, 17:193–195, 1973.

[Grava] T. Grava. Riemann-Hilbert problem for the small dispersion limit of the KdV equation and linear overdetermined systems of Euler-Poisson-Darboux type. *Comm. Pure Appl. Math.*, 55(4):395–430, 2002.

[GT09] K. Grunert and G. Teschl. Long-time asymptotics for the Korteweg-de Vries equation via nonlinear steepest descent. *Math. Phys. Anal. Geom.*, 12(3), 2009.

[Hru76] E. Ja. Hruslov. Asymptotic behavior of the solution of the Cauchy problem for the Korteweg-de Vries equation with steplike initial data. *Mat. Sb. (N.S.),* 99(141)(2):261–281, 296, 1976.

[Its11] A. Its. Large $N$ asymptotics in random matrices. In J. Harnad, editor, *Random Matrices, Random Processes and Integrable Systems*, CRM Series in Mathematical Physics. Springer, 2011.

[KAV04] A. Kuijlaars, K. McLaughlin, W. Van Assche, and M. Vanlessen. The Riemann-Hilbert approach to strong asymptotics for orthogonal polynomials on $[-1,1]$. *Adv. Math.*, 188(2):337–398, 2004.

[Kri88] I. M. Krichever. The averaging method for two-dimensional “integrable” equations. *Punktstional. Anal. i Prilozhen.,* 22(3):37–52, 96, 1988.

[Law89] D. F. Lawden. *Elliptic functions and applications*, volume 80. Springer-Verlag., applied mathematical sciences edition, 1989.

[Lev88] C. D. Levermore. The hyperbolic nature of the zero dispersion KdV limit. *Comm. Partial Differential Equations*, 13(4):495–514, 1988.

[LL83a] P. D. Lax and C. D. Levermore. The small dispersion limit of the Kortewegde Vries equation. I. *Comm. Pure Appl. Math.*, 36(3):253–290, 1983.

[LL83b] P. D. Lax and C. D. Levermore. The small dispersion limit of the Kortewegde Vries equation. II. *Comm. Pure Appl. Math.*, 36(5), 1983.
[LL83c] P. D. Lax and C. D. Levermore. The small dispersion limit of the Korteweg-de Vries equation. III. *Comm. Pure Appl. Math.*, 36(6), 1983.

[SP16] E. G. Shurgalina and E. N. Pelinovsky. Nonlinear dynamics of a soliton gas: modified Korteweg-de Vries equation framework. *Phys. Lett. A*, 380(24):2049–2053, 2016.

[Whi74] G. B. Whitham. *Linear and nonlinear waves*. Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974. Pure and Applied Mathematics.

[Zak71] V. Zakharov. Kinetic equation for solitons. *Sov. Phys. -JETP*, 33(3):538–541, 1971.

[Zak09] V. Zakharov. Turbulence in integrable systems. *Stud. Appl. Math.*, 122(3):89–101, 2009.

[ZM85] V. Zakharov and S. Manakov. Construction of higher-dimensional nonlinear integrable systems and of their solutions. *Funct. Anal. Appl.*, 19(2):89–101, 1985.