KOSZUL ALGEBRAS AND QUADRATIC DUALS
IN GALOIS COHOMOLOGY

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To the memory of Jean-Louis Koszul.

ABSTRACT. We investigate the Galois cohomology of finitely generated maximal pro-$p$ quotients of absolute Galois groups. Assuming the well-known conjectural description of these groups, we show that Galois cohomology has the PBW property. Hence in particular it is a Koszul algebra. This answers positively a conjecture by Positselski in this case. We also provide an analogous unconditional result about Pythagorean fields. Moreover, we establish some results that relate the quadratic dual of Galois cohomology with $p$-Zassenhaus filtration on the group. This paper also contains a survey of Koszul property in Galois cohomology and its relation with absolute Galois groups.

1. Introduction

Let $p$ be a prime number and $F$ be a field. The absolute Galois group $G_F = \text{Gal}(F_{\text{sep}}/F)$ is the Galois group of the maximal separable extension $F_{\text{sep}}$ of $F$. This group contains all the Galois-theoretic information about $F$, as any finite Galois group of an extension of $F$ is a quotient of $G_F$.

Despite such a central role in Galois theory, absolute Galois groups of fields remain rather mysterious. They are profinite groups, and one of the main problems in current research in Galois theory is to determine which profinite groups are absolute Galois groups of fields. One way to investigate the structure of profinite groups is Galois cohomology. This is a cohomology theory based on continuous cochains and coboundaries (cf. [Ser13, Ch. 2]). The Galois cohomology of $F$ with coefficients in the finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is a graded algebra

$$H^\bullet(G_F, \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(G_F, \mathbb{F}_p),$$

with respect to the homological degree and the graded-commutative cup product (cf. [NSW08, Ch. 1, § 4])

$$\cup : H^r(G_F, \mathbb{F}_p) \times H^s(G_F, \mathbb{F}_p) \to H^{r+s}(G_F, \mathbb{F}_p), \quad r, s \geq 0.$$

Throughout the paper, we adopt the following

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Standing Hypothesis 1.1. For $p$ odd, $F$ contains a primitive $p^{th}$ root of unity $\zeta_p$. For $p = 2$, $F$ contains $\sqrt{-1}$.

There will be few explicitly marked exceptions in which we consider $p = 2$ and fields that need not contain $\sqrt{-1}$.

Recently, M. Rost and V. Voevodsky completed the proof of the long-standing Bloch-Kato conjecture (cf. [SJ06, HW09, Voe10, Voe11]). Under our Hypothesis 1.1 this theorem implies that $H^\bullet(G_F, \mathbb{F}_p)$ is generated by elements of degree 1 and its relations are generated by homogeneous relations of degree 2. Such algebras are called quadratic algebras (cf. [PP05, § 1.2]).

Locally finite-dimensional (cf. [PP05, Page 1]) quadratic algebras come equipped with a duality. The quadratic dual of such an algebra $A_\bullet$ is a quadratic algebra generated over the same field by the dual space of the space of generators of $A_\bullet$. Its relators form the orthogonal complement of the space of the relators of $A_\bullet$ (see Definition 2.2). The double quadratic dual $(A_\bullet)^!$ is isomorphic to $A_\bullet$ as a graded algebra.

Among quadratic algebras, the significant class of Koszul algebras has been singled out in [Pri70]. These are the graded algebras for which the ground field has a linear free resolution as trivial module, that is a resolution of free graded modules $(P_i, d_i)$ with each $P_i$ generated in degree $i$. They are characterised by an uncommonly nice cohomological behaviour (cf. § 2).

For a general graded algebra $A_\bullet = \bigoplus_{n \geq 0} A_n$ over a field $k$, the $k$-cohomology is defined as the direct sum of the derived functors of the functor $\text{Hom}_{A_\bullet}(\_, k)$ evaluated on $k$, that is $\text{Ext}_{A_\bullet}(k, k) = \bigoplus_{i \in \mathbb{N}} \text{Ext}_{A_\bullet}^i(k, k)$. Finding an explicit description of the cohomology of an arbitrary graded algebra is generally a hopeless battle. But the cohomology of a Koszul $k$-algebra $A_\bullet$ is just its quadratic dual $A_\bullet^!$. This also means that Koszul algebras can be reconstructed from their cohomology. There is not the typical loss of information in passing from an algebraic or topological object to its cohomology.

Koszul algebras arise in various fields of mathematics, such as representation theory, noncommutative algebra, quantum groups, noncommutative geometry, algebraic geometry and topology (see, e.g., [BCS90, Frö75, Koh85, Koh83, LV12]). Moreover, Koszul algebras have been studied in the context of Galois cohomology by L. Positselski and A. Vishik (cf. [PV95]). This was considerably extended in further remarkable papers by Positselski (cf. [Pos05, Pos14]). In particular, in [Pos14, § 0.1] he explicitly formulated a general conjecture, of which the following is a special case.

Conjecture 1.2 (Positselski). Let $F$ be a field containing a primitive $p^{th}$ root of unity. Then the algebra $H^\bullet(G_F, \mathbb{F}_p)$ is Koszul.

In [Pos05 Theorem 1.3] it is shown that the Koszulity of the reduction modulo $p$ of the Milnor K-theory, along with the bijectivity of the corresponding norm residue map in degree 2 and the injectivity of the one in degree 3 would give an easier and natural proof of Bloch-Kato conjecture for the prime $p$. Positselski’s Conjecture can be seen as a strengthening of Bloch-Kato Conjecture.

In this paper we prove that in several cases the Galois cohomology and its quadratic dual have the stronger PBW property (cf. [LV12 Chapter 4]).

There is a conjectural description of maximal pro-$p$ quotients of absolute Galois groups that are finitely generated and satisfy our Hypothesis 1.1. This description
roughly says that all such groups can be constructed inductively from finitely generated free pro-$p$ groups and Galois groups of maximal $p$-extensions of local fields using free products and certain semidirect products. Slightly more general groups constructed this way are called elementary type pro-$p$ groups (see §4.3 for details) and the former conjecture is known as Elementary Type Conjecture, in short ETC (cf. [Efr95]).

Our main results are the following.

**Theorem A.** If $G$ is an elementary type pro-$p$ group, then $H^\bullet(G, \mathbb{F}_p)$ is PBW, and hence Koszul.

**Remark 1.3.** in Theorem A we can replace $G$ by any absolute Galois group $G_F$ such that its maximal pro-$p$ quotient $G_F(p)$ is of elementary type. This follows from the Hochschild-Serre spectral sequence associated to the exact sequence

$$1 \longrightarrow \text{Ker}(\pi) \longrightarrow G_F \longrightarrow G_F(p) \longrightarrow 1,$$

where $\pi$ is the canonical projection. Indeed, $\text{Ker}(\pi) \cong \text{Gal}(\text{F}_{\text{sep}}/F(p))$, where $F(p)$ is the compositum of all Galois extensions of $F$ of degree a power of $p$. By Bloch-Kato Conjecture, $H^n(\text{Gal}(\text{F}_{\text{sep}}/F(p)), \mathbb{F}_p) = 0$ for all $n > 0$, so Hochschild-Serre spectral sequence collapses at the second page. As a consequence, the inflation map $\inf : H^\bullet(G_F(p), \mathbb{F}_p) \to H^\bullet(G_F, \mathbb{F}_p)$ is an isomorphism.

The quadratic dual of cohomology is related to the successive factors of a descending series on $G$, the $p$-Zassenhaus filtration (see Definition 3.1). Let $\mathcal{U}(L(G))$ be the restricted universal enveloping algebra of the restricted Lie algebra $L(G)$ given by the $p$-Zassenhaus filtration on $G$ (see §3.3).

**Theorem B.** If $G$ is an elementary type pro-$p$ group, then the quadratic dual of $H^\bullet(G, \mathbb{F}_p)$ is $\mathcal{U}(L(G))$ and it is PBW, and hence Koszul.

**Remark 1.4.** Theorem B also admits an extension to absolute Galois groups $G_F$ with elementary type maximal pro-$p$ quotient $G_F(p)$. In fact, since the quotient of $G_F$ over any term of the $p$-Zassenhaus filtration is a finite $p$-group, the restricted Lie algebras $L(G_F(p))$ and $L(G_F)$ are isomorphic, and so are their respective restricted universal enveloping algebras $\mathcal{U}(L(G_F(p)))$ and $\mathcal{U}(L(G_F))$.

By a result of Jennings (Theorem 3.9), for a finitely generated pro-$p$ group $G$, $\mathcal{U}(L(G))$ can equivalently be described as the graded object $\text{gr}\mathbb{F}_p[G]$ associated to the filtration on the complete group algebra of $G$ by the successive powers of the complete augmentation ideal (see Definition 3.3). Thus, Theorem B gives a partial positive answer to a question of T. Weigel (cf. [Wei15]):

**Question 1.5** (Weigel). Let $F$ be a field containing a primitive $p^\text{th}$ root of unity and let $G_F(p)$ be the maximal pro-$p$ quotient of the absolute Galois group of $F$. Is the algebra $\mathcal{U}(L(G_F(p)))$ Koszul?

The next two results do not require Hypothesis 1.1. The first involves the class of Demushkin groups (see Section 4). In particular, it applies to maximal pro-$p$ quotients of absolute Galois groups of local fields that contain a primitive $p^\text{th}$ root of unity.

**Theorem C.** If $G$ is a Demushkin pro-$p$ group, then the algebras $H^\bullet(G, \mathbb{F}_p)$ and $\text{gr}\mathbb{F}_p[G]$ are quadratic dual to each other and PBW, hence Koszul.
We then use the results in [Min86] and [Jac81] to get an unconditional result for Pythagorean formally real fields. A field \( F \) is Pythagorean if \( F^2 + F^2 = F^2 \) and formally real if \(-1\) is not a sum of squares in \( F \).

**Theorem D.** Let \( F \) be a Pythagorean formally real field. If \( F \) has finitely many square classes, then the algebra \( H^*(G_F, \mathbb{F}_2) \) is PBW, and hence Koszul.

As Positselski proved in [Pos14], the cohomology of local and global fields also has PBW property. These facts corroborate a strengthening of Positselski’s Conjecture.

**Conjecture 1.6.** Let \( F \) be a field containing a primitive \( p \)th root of unity. Then the algebra \( H^*(G_F, \mathbb{F}_p) \) is PBW.

Koszul property of an algebra can be paraphrased as the fact that there are not surprising ways in which an element of the algebra can turn to be 0, or in other words “only what is expected to be 0 is actually 0”. Our next results go in the same philosophical direction. When a minimal presentation \( G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle \) of a finitely generated pro-\( p \) group is given (see Section 7), commutators and powers applied to a given relator \( r_i \) provide automatically some higher degree relators of \( \text{gr}\mathbb{F}_p[G] \). Let us say that these are the “predictable” higher degree relations. But a priori, \( \text{gr}\mathbb{F}_p[G] \) might be subject to other, “unpredictable” relations. Under the mild assumption that Galois cohomology is quadratic, the predictable relations are orthogonal to the relations in Galois cohomology:

**Theorem E.** Let \( G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle \) be a minimal presentation of a finitely generated pro-\( p \)-group \( G \), and assume that \( H^*(G, \mathbb{F}_p) \) is quadratic. Then there is an isomorphism of quadratic \( \mathbb{F}_p \)-algebras

\[
H^*(G, \mathbb{F}_p)^! \cong \frac{\mathbb{F}_p[X]}{R},
\]

where \( R \) is the two-sided ideal generated by the initial forms (see the end of Subsection 3.3) of the defining relations \( r_1, \ldots, r_m \).

In the same hypothesis, unpredictable relations cannot exist in lower degrees and do not exist at all for the class of quadratically defined pro-\( p \) groups introduced in [KLM11] (see Definition 7.8).

**Theorem F.** Let \( G \) be a finitely generated pro-\( p \) group with \( H^*(G, \mathbb{F}_p) \) quadratic. Then there is an epimorphism of graded \( \mathbb{F}_p \)-algebras

\[
H^*(G, \mathbb{F}_p)^! \longrightarrow \text{gr}\mathbb{F}_p[G]
\]

which is an isomorphism in degrees 0, 1, 2. Further, if \( G \) has a quadratically defined presentation, then the previous map is an isomorphism in all degrees.

Thus, the following question arises naturally.

**Question 1.7.** Does every maximal pro-\( p \) quotient of an absolute Galois group admit a quadratically defined presentation?

The structure of the paper is as follows. In Section 2 we provide a survey of quadratic algebras, focusing on Koszul and PBW properties. Section 3 is devoted to some preliminaries on filtrations and algebras associated to groups in general and pro-\( p \) groups in particular. In Section 4 we introduce the class of elementary type pro-\( p \) groups and we present the structure of maximal pro-2 quotients of absolute
Galois groups of Pythagorean fields. Theorems A, B and C are proved in Section 5. Theorem D in Section 6. Section 7 contains the relevant facts about pairings, quadratic duals and filtrations in Galois cohomology that lead to the proof of Theorems E and F. Finally, Section 8 deals with mild pro-$p$ groups, for which both the existence of a quadratically defined presentation and koszulity of Galois cohomology can be elementarily checked, and we provide a theorem in that direction.

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2. Quadratic algebras and the Koszul property

In this section $k$ denotes an arbitrary field.

Definition 2.1. An associative, unital $k$-algebra $A$ is graded if it decomposes as the direct sum of $k$-vector spaces $A_\bullet = \bigoplus_{i \in \mathbb{Z}} A_i$ such that, for all $i,j \in \mathbb{Z}$, $A_i A_j \subseteq A_{i+j}$. $A_i$ is the homogeneous component of degree $i$. A graded $k$-algebra $A_\bullet$ is connected if $A_i = 0$ for all $i < 0$ and $A_0 = k \cdot 1_A \cong k$; generated in degree 1 if all its (algebra) generators are in $A_1$; locally finite-dimensional if $\dim_k A_i < \infty$ for all $i \in \mathbb{Z}$ (another common terminology for this is finite-type). A module $M$ over a graded algebra $A_\bullet$ is graded if it decomposes as the direct sum of $k$-vector spaces $M_\bullet = \bigoplus_{i \in \mathbb{Z}} M_i$ s.t., for all $i,j \in \mathbb{Z}$, $A_i M_j \subseteq M_{i+j}$. A graded $A_\bullet$-module $M_\bullet$ is finite-type if $\dim_k M_i < \infty$ for all $i \in \mathbb{Z}$.

For a vector space $V$ over $k$, let $T_\bullet(V)$ denote the tensor $k$-algebra generated by $V$, i.e.,

$$T_\bullet(V) = \bigoplus_{n \geq 0} V^\otimes n,$$

with $V^\otimes 0 = k$.

It is a graded algebra with the tensor product as algebra multiplication. We will systematically omit tensor signs, and simply use juxtaposition, in writing the elements of a tensor algebra. In particular, if $V$ is finite-dimensional, with a basis $\{x_1, \ldots, x_n\}$, then $T_\bullet(V)$ can (and will) be identified with the algebra $k\langle x_1, \ldots, x_n \rangle$ of noncommutative polynomials in the variables $x_1, \ldots, x_n$, graded by polynomial degree.

Ignoring the grading, tensor algebras are the free objects in the category of (associative, unital) algebras (cf. [LV12 §1.1.3]), so any algebra is a quotient of a tensor algebra. When an algebra is presented as a quotient of a tensor algebra, we
will simply identify an element of the former with each of its representatives in the latter.

Henceforth we assume graded algebras to be locally finite-dimensional, connected and generated in degree 1, unless otherwise specified. In particular, every such algebra $A_\bullet$ is equipped with the augmentation map $\varepsilon: A_\bullet \to k$ that is the projection onto $A_0$. Its kernel is the augmentation ideal $A_+ = \bigoplus_{n \geq 1} A_n$. The augmentation map gives $k$ a canonical structure of graded $A_\bullet$-module concentrated in degree 0, via the action $a \cdot k = \varepsilon(a) k$. In the following, ground fields of algebras will be always understood to be equipped with this module structure.

2.1. Quadratic algebras. For a finite-dimensional $k$-vector space $V$, let $V^* = \text{Hom}_k(V, k)$ be the $k$-dual space of $V$. With a little abuse, we will then systematically identify $(V \otimes V)^* = V^* \otimes V^*$.

Definition 2.2. A quadratic algebra is a quotient

$$A_\bullet = \frac{T_\bullet(V)}{(\Omega)}$$

of the tensor algebra over some finite-dimensional $k$-vector space $V$, with $(\Omega)$ the two-sided ideal generated by a vector subspace $\Omega \subseteq V \otimes V$. In other words, it is a connected graded algebra that admits a presentation with finitely many degree-1 generators and homogeneous degree-2 relations. $V = A_1$ is the space of generators of $A_\bullet$, $\Omega$ is the space of relators, and we will use the notation $A = Q(V, \Omega)$.

The quadratic dual $A^!_\bullet$ of a quadratic $k$-algebra $A_\bullet$ is the quadratic $k$-algebra $A^!_\bullet = T_\bullet(V^*)/(\Omega^\perp)$, with

$$\Omega^\perp = \{ f \in (V \otimes V)^* \mid f(\omega) = 0 \text{ for all } \omega \in \Omega \} \subseteq V^* \otimes V^*.$$

Remark 2.3. For quadratic algebras, the natural grading on $T_\bullet(V)$ passes to the quotient thanks to the homogeneity of the relations. This is the way quadratic algebras are understood to be graded.

Thanks to finite-dimensionality assumptions, one has $(A^!_\bullet)^! = A_\bullet$.

Example 2.4. Let $V$ be a finite-dimensional $k$-vector space, with a basis $B = \{x_1, \ldots, x_d\}$.

(a) The tensor algebra $T_\bullet(V)$ is quadratic, with trivial space of relators. Its quadratic dual is the trivial algebra on $V^*$, that is $k \oplus V^*$ with trivial multiplication, whose space of relators is $V^* \otimes V^*$.

(b) The symmetric algebra

$$S_\bullet(V) = \frac{T_\bullet(V)}{(x_ix_j - x_jx_i \mid x_i, x_j \in B)}$$

and the exterior algebra

$$\Lambda_\bullet(V) = \frac{T_\bullet(V)}{(x_ix_i, x_ix_j + x_jx_i \mid x_i, x_j \in B)}$$

are quadratic and dual to each other. We shall identify the symmetric algebra $S_\bullet(V)$ with the commutative polynomial algebra $k[x_1, \ldots, x_d]$; the grading induced by $T_\bullet(V)$ coincides with polynomial degree.

Example 2.5 (Constructions of quadratic algebras). Let $A_\bullet$ and $B_\bullet$ be two quadratic algebras, with spaces of relators $\Omega_A$ and $\Omega_B$, respectively. From $A_\bullet$ and $B_\bullet$ one may construct new quadratic algebras as follows (cf. [PP03 § 3.1]).
(a) The direct sum \( C_\bullet = A_\bullet \sqcup B_\bullet \) of \( A_\bullet \) and \( B_\bullet \) is the connected quadratic algebra \( C_\bullet \) such that \( C_n = A_n \oplus B_n \) for every \( n \geq 1 \). In other words,
\[
C_\bullet = \frac{T_\bullet(A_1 \oplus B_1)}{(\Omega)}, \quad \text{with } \Omega = \Omega_A \oplus \Omega_B.
\]

(b) The free product \( C_\bullet = A_\bullet \sqcup B_\bullet \) of \( A_\bullet \) and \( B_\bullet \) is the connected quadratic algebra \( C_\bullet \) with \( C_1 = A_1 \oplus B_1 \) and
\[
C_\bullet = \frac{T_\bullet(A_1 \oplus B_1)}{(\Omega)}, \quad \text{with } \Omega = \Omega_A \oplus \Omega_B.
\]

(c) The symmetric tensor product \( C_\bullet = A_\bullet \otimes^1 B_\bullet \) of \( A_\bullet \) and \( B_\bullet \) is the quadratic algebra \( C_\bullet \) with \( C_1 = A_1 \oplus B_1 \) and
\[
C_\bullet = \frac{T_\bullet(A_1 \oplus B_1)}{(\Omega)}, \quad \text{with } \Omega = \Omega_A \oplus \Omega_B \oplus I,
\]
where \( I \) is the subspace generated by the elements \( ab - ba \) with \( a \in A_1 \) and \( b \in B_1 \).

(d) The skew-symmetric tensor product \( C_\bullet = A_\bullet \otimes^{-1} B_\bullet \) of \( A_\bullet \) and \( B_\bullet \) is the quadratic algebra \( C_\bullet \) with \( C_1 = A_1 \oplus B_1 \) and
\[
C_\bullet = \frac{T_\bullet(A_1 \oplus B_1)}{(\Omega)}, \quad \text{with } \Omega = \Omega_A \oplus \Omega_B \oplus I,
\]
where \( I \) is the subspace generated by the elements \( ab + ba \) with \( a \in A_1 \) and \( b \in B_1 \).

\[ \text{Remark 2.6.} \text{ The following De Morgan dualities hold (cf.} [PP05, Chapter 3, Corollary 1.2]):\]
\[
(A_\bullet \sqcup B_\bullet)^! &= A_\bullet^! \sqcup B_\bullet^!, \\
(A_\bullet \sqcap B_\bullet)^! &= A_\bullet^! \sqcap B_\bullet^!, \\
(A_\bullet \otimes^1 B_\bullet)^! &= A_\bullet^! \otimes^1 B_\bullet^!, \\
(A_\bullet \otimes^{-1} B_\bullet)^! &= A_\bullet^! \otimes^{-1} B_\bullet^!.
\]

2.2. Cohomology of algebras. Let \( L \) and \( M \) be modules over the graded \( k \)-algebra \( A_\bullet \). Let \( \text{Hom}_{A_\bullet}(L, M) \) denote the space of \( A_\bullet \)-module homomorphisms from \( L \) to \( M \). For every left \( A_\bullet \)-module \( M \), the contravariant functor \( \text{Hom}_{A_\bullet}(\cdot, M) \) is left-exact but in general not exact, hence it has right derived functors \( \text{Ext}^i_{A_\bullet}(\cdot, M) \). These are defined as follows: for another \( A_\bullet \)-module \( L \), choose a projective resolution \((P_\bullet, d_\bullet)\) of \( L \); then \( \text{Ext}^i_{A_\bullet}(L, M) \) is the \( i \)th cohomology group of the complex \((\text{Hom}_{A_\bullet}(P_\bullet, M), d_\bullet^*)\).

The cohomology \( H^\bullet(A_\bullet, k) \) of \( A_\bullet \) is defined to be the graded \( k \)-module
\[
\text{Ext}^\bullet_{A_\bullet}(k, k) = \bigoplus_{i \in \mathbb{N}} \text{Ext}^i_{A_\bullet}(k, k).
\]

The terms \( \text{Ext}^i_{A_\bullet}(k, k) \) may be computed as the cohomology of the \textit{normalized cobar complex}
\[
\text{Cob}^i(A_\bullet) = \text{Hom}_{A_\bullet}(A_\bullet \otimes A_\bullet^{\otimes^i} \otimes k, k) \cong (A_\bullet^{\otimes^i})^*
\]
(cf. [PP05, § 1.1]). For \( \varphi \in \text{Cob}^i(A_\bullet) \) and \( \psi \in \text{Cob}^j(A_\bullet) \), we can define the tensor product \( \varphi \otimes \psi \in \text{Cob}^{i+j}(A_\bullet) \) as the map
\[
(\varphi \otimes \psi)(a_1 \otimes \cdots \otimes a_{i+j}) = \varphi(a_1 \otimes \cdots \otimes a_i)\psi(a_{i+1} \otimes \cdots \otimes a_{i+j}).
\]
This tensor product satisfies \( d(\varphi \otimes \psi) = d\varphi \otimes \psi + (-1)^i \varphi \otimes d\psi \) and so induces a well-defined product on cohomology classes:

\[
\cup : \text{Ext}^i_{A_*}(k, k) \otimes \text{Ext}^j_{A_*}(k, k) \to \text{Ext}^{i+j}_{A_*}(k, k)
\]
given by \([\varphi] \cup [\psi] = [\varphi \otimes \psi]\). The map \( \cup \) is a special case of Yoneda product called the **cup product** and turns \( \text{Ext}^*_{A_*}(k, k) \) into a connected graded \( k \)-algebra.

Thanks to the finite-type assumption, the grading on \( A_* \) induces an additional grading on \( H^*(A_*) \), as follows. The isomorphisms

\[
\rho_{i,j} : A_i^* \otimes A_j^* \to (A_i \otimes A_j)^* \\
\rho_{i,j}(f \otimes g) : (a \otimes b) \mapsto f(a)g(b)
\]

define a map \( \Delta = \rho_{i,j}^{-1} \circ \mu^* : A^*_+ \to A_i^* \otimes A_j^* \) that is dual to the algebra multiplication \( \mu : A_+ \otimes A_+ \to A_+ \).

Then each \( \text{Cob}^i(A_*) \) decomposes as the direct sum of the submodules

\[
\text{Cob}^{i,j}(A_*) \cong (A^*_+)^J = \left( \bigoplus_{k_1 + \ldots + k_i = j, \ k_i \geq 1} A_{k_1} \otimes \cdots \otimes A_{k_i} \right)^* \\
\cong \bigoplus_{k_1 + \ldots + k_i = j, \ k_i \geq 1} A_{k_1}^* \otimes \cdots \otimes A_{k_i}^*
\]

(the condition that all \( k_s \geq 1 \) is required to stay in \( A_+ \)). In this description of the cobar complex, the differential reads

\[
d : \text{Cob}^{i,j} \to \text{Cob}^{i+1,j} \\
d(\varphi_1 \otimes \cdots \otimes \varphi_i) = \sum_{k=1}^i (-1)^k \varphi_1 \otimes \cdots \otimes \Delta \varphi_k \otimes \cdots \otimes \varphi_i.
\]

The differential respects the second grading, i.e. \( d(C^i,j) \subseteq C^{i+1,j} \), so this grading passes to cohomology. The cup product is compatible with both gradings:

\[
\cup : \text{Ext}^i_{A_*}(k, k) \otimes \text{Ext}^j_{A_*}(k, k) \to \text{Ext}^{i+j}_{A_*}(k, k).
\]

Hence \( \text{Ext}^*_{A_*}(k, k) \) becomes a **bigraded algebra**. The first grading is called the **homological grading**; the second one, induced by \( A_* \), is called the **internal grading**.

### 2.3. Koszul property.

**Definition 2.7.** A **Koszul algebra** is a quadratic algebra \( A_* \), such that the trivial \( A_* \)-module \( k \) has a graded projective resolution \( (P_*, d_*) \) with each \( P_i \) generated by its component of degree \( i \): \( P_i = A_*(P_i)_i \) (we require all the differentials \( d_i \) to have degree 0; gradings on the \( P_i \)s are set accordingly).

If, as in our case, the Ext algebra can be equipped with a well-defined internal grading (see Subsection 2.2), this is equivalent to \( \text{Ext}^*_{A_*}(k, k) \) being concentrated on the diagonal:

\[
\text{Ext}^i_{A_*}(k, k) = 0 \text{ for } i \neq j.
\]

**Example 2.8.**

1. Tensor algebras, trivial algebras, symmetric algebras and exterior algebras are Koszul (cf. [LVT12 § 3.4.5]).
2. Each of the direct sum, the free product, the symmetric tensor product and the skew-symmetric tensor product of two algebras \( A_* \) and \( B_* \) is Koszul if and only if both \( A_* \) and \( B_* \) are Koszul (cf. [PP05 § 3.1]).
The most interesting feature of Koszul algebras is that their cohomology admits a very explicit description.

**Theorem 2.9** (Priddy, [Pri70]). An algebra $A_*$ is Koszul if and only if there is a (degree-0) isomorphism of graded algebras

$$H^*(A_*) = \text{Ext}_{A_*}^*(k, k) \cong A_*^1$$

Also, a quadratic $k$-algebra $A_*$ is Koszul if and only if its quadratic dual $A_*^!$ is Koszul (cf. [PP05 Corollary 2.3.3]).

For further details we refer to [BGS96], [LV12, Chapter 3], [PP05 Chapters 1-2], [Pos05 Section 2], [Pri70].

### 2.4. PBW property.

Checking Koszulity with the definition is still a difficult task. PBW property is a strictly stronger condition, but it is generally easier to check.

Let $A_* = Q(V, \Omega)$ be a quadratic algebra. Suppose a totally ordered basis $\{x_1, \ldots, x_d\}$ of $V$ is given. This datum is equivalent to a total order $\leq_1$ on the set $I = \{1, \ldots, d\}$. The set $I = \bigcup_{n \in \mathbb{N}} I^n$ is the set of multiindices that uniquely identify each monomial in $T_*(V)$ (putting $I^0 = \{0\}$ and using it as the singleton set identifying the empty monomial 1). The concatenation of multiindices $(i_1, \ldots, i_k) \sim (j_1, \ldots, j_h) = (i_1, \ldots, i_k, j_1, \ldots, j_h)$ turns $I$ into a monoid with 0 as neutral element.

**Definition 2.10.** Let $V = \bigoplus_{i \in I} \langle x_i \rangle$, $\mathcal{I}$ and $\sim$ be as before. An admissible order (sometimes called \textit{monoid order}) on $\mathcal{I}$ is a total order $\leq$ such that

- $\leq$ extends $\leq_1$, that is, the two orders coincide on $I$;
- for all $s \in \mathcal{I}$, $0 \leq s$;
- for all $r, s, t \in \mathcal{I}$, $r \prec s \Rightarrow (r \sim t \prec s \sim t$ and $t \sim r \prec t \sim s$).

**Example 2.11.** Given any total order $\leq_1$ on $I$, the associated degree-lexicographic order $\preceq$ on $\mathcal{I}$ is the admissible order defined as follows. For $s = (i_1, \ldots, i_m) \in I^m$ and $t = (j_1, \ldots, j_n) \in I^n$, $s \preceq t$ if and only if either $m < n$ or $m = n$ and $\exists k \in \{1, \ldots, n\} : i_1 = j_1, i_2 = j_2, \ldots, i_{k-1} = j_{k-1}, i_k \leq j_k$.

If one accepts the axiom of countable choice, an admissible order $\preceq$ gives an order isomorphism $\text{index} : \mathbb{N} \rightarrow \mathcal{I}$, with $\text{index}(n)$ denoting the multiindex in position $n$, and in particular $\text{index}(0) = 0$.

We obtain an increasing and exhaustive filtration $F_\bullet T_\bullet(V)$ on $T_\bullet(V)$:

$$F_n T_\bullet(V) = \bigoplus_{s \in \mathcal{I}, s \leq \text{index}(n)} \langle \underline{x}_s \rangle,$$

where we use the multiindex notation $\underline{x}_s$ to denote the monomial identified by $s$. This in turn produces an increasing and exhaustive filtration $F_\bullet A$ on $A_*$, with $F_n A$ given by the image $F_n T_\bullet(V)$ under the canonical projection $T_\bullet(V) \rightarrow A_*$. The filtration on $A_*$ gives rise to a weight-graded module

$$\text{gr}_\bullet A = \bigoplus_{n \in \mathbb{N}} F_{n+1} A_*/F_n A_*,$$

which is a graded algebra with the product and the grading induced from $A_*$. We call it the \textit{weight-graded algebra of} $A_*$. The product on $\text{gr}_\bullet A$ is usually simpler than that on $A_*$, nevertheless, a spectral sequence argument shows that if the former is Koszul, the latter is Koszul as well (cf. [LV12 Proposition 4.2.1], [PP05 Section 4.7] or [Pri70 Section 5]).
The monomial identified by the $\preceq$-maximal multiindex within a given relator $r \in \Omega$ is called the leading monomial of $r$ and denoted $\text{lm}(r)$. Letting $\Omega_{\text{lead}}$ be the vector subspace generated by the leading terms of all elements of $\Omega$, a tentative presentation of $\text{gr}_r A$ is $T_r(V)/(\Omega_{\text{lead}})$. In fact, there is an epimorphism of graded algebras $\rho : T_r(V)/(\Omega_{\text{lead}}) \twoheadrightarrow \text{gr}_r A$ which is an isomorphism in degrees $0, 1, 2$. The Diamond Lemma (cf. the original reference [Ber78], or [LV12] § 4.2.4, § 4.3.5) asserts that the injectivity of $\rho$ in degree 3 and the Koszulity of $T_r(V)/(\Omega_{\text{lead}})$ are sufficient to get the isomorphism $T_r(V)/(\Omega_{\text{lead}}) \cong \text{gr}_r A$, and hence the Koszulity of $A_\bullet$. On the other side, $T_r(V)/(\Omega_{\text{lead}})$ is a monomial quadratic algebra, that is, the space of relators is linearly spanned by monomials of degree 2. Such algebras are always Koszul, for example as a consequence of Backelin’s Criterion, cf. [PP05 § 2.4].

The injectivity of $\rho$ in degree 3 can be algorithmically checked with the Rewriting Method (cf. [LV12] § 4.1).

First, one chooses a basis $B = \{r_1, \ldots, r_m\}$ of $\Omega$. By elementary linear algebra over a field, one can always suppose that the coefficients of $\text{lm}(r_1), \ldots, \text{lm}(r_m)$ are 1 and that, for each $i$, $\text{lm}(r_i)$ does not appear as a monomial in any other relator $r_j$, $j \neq i$. A basis with these additional properties is said to be normalized.

Each relation $r_i = 0$ can be interpreted as a rewriting rule

$$\text{lm}(r_i) \rightsquigarrow \text{(sum of non maximal terms)}.$$ 

Rewriting rules can be subsequently applied to an initial monomial until the result does not contain any more leading terms. This final shape is usually called a normal form of the initial monomial. The process can be depicted with a directed graph, whose vertices are the monomials obtained as steps of the aforementioned process and whose edges connect a monomial $x$ to a monomial $y$ that is obtained applying a single rewriting rule to $x$. Normal forms are then the terminal vertices of these rewriting graphs.

A degree 3 monomial $x_a x_b x_c$ is critical if both $x_a x_b$ and $x_b x_c$ are leading terms of relators in $B$. This implies that there are two possible rewriting processes on it, starting with either the rewriting rule on $x_a x_b$ or that on $x_b x_c$. The resulting rewriting graph may then have two distinct terminal vertices, that is, the critical monomial may have two distinct normal forms. A critical monomial is confluent if its rewriting graph has only one terminal vertex.

The (cosets of) the terminal vertices of the rewriting graph of a critical monomial are equal in $\text{gr}_r A$, but distinct in $T_r(V)/(\Omega_{\text{lead}})$. Since this is the only obstruction to the injectivity of $\rho$ in degree 3, this injectivity is equivalent to the confluence of every critical monomial. $A_\bullet$ is called a PBW algebra if there exist bases for $V$ and $\Omega$ and an admissible order as above with respect to which every critical monomial is confluent. In that case, the above basis of $V$ is called a set of PBW generators of $A_\bullet$.

The whole discussion can be then summarised in the fact that a (quadratic) PBW algebra is Koszul.

**Remark 2.12.** PBW property is preserved under the quadratic dual construction. More precisely, if $\{x_1, \ldots, x_n\}$ is a set of PBW generators of the quadratic algebra $A_\bullet$ under the suitable order $\preceq$, then the dual basis $\{x_1^*, \ldots, x_n^*\}$ is a set of PBW generators of $A_\bullet^*$ under the opposite suitable order: $s \preceq^\text{op} t \iff t \preceq s$ (cf. [LV12] § 4.3.9).
Example 2.13. The following are PBW algebras that will be relevant subsequently.

1. All monomial algebras are PBW. Indeed, consider a set of algebra generators with respect to which relations are monomial. Then all the associated critical monomials reduce to 0 after one step in either way the rewriting rules are applied. In particular, tensor algebras and trivial algebras are PBW.

2. Symmetric and exterior algebras are PBW under the degree-lexicographic order.

3. Let $A \star = Q(V_A, \Omega_A)$ and $B \star = Q(V_B, \Omega_B)$ be two quadratic PBW algebras. Then the quadratic algebras $A \cap B \star$, $A \cup B \star$, $A \star \otimes 1 B \star$, and $A \star \otimes -1 B \star$ introduced in Example 2.5 are PBW (cf. [PP05, § 4.4]). In each case, a set of PBW generators is the union of the sets of PBW generators of $A \star$ and $B \star$.

Lemma 2.14. Let $A = k\langle x_1, \ldots, x_d \mid r \rangle = Q(\langle x_1, \ldots, x_d \rangle, \langle r \rangle)$ a quadratic algebra presented with a single relator $r$. Suppose that there is an index $i \in I = \{1, \ldots, d\}$ such that $x_i^2$ is not a monomial of $r$ but some $x_i x_j$, $j \neq i$, is. Then $A$ is PBW.

Proof. We can choose any total order $\preceq_1$ on $I$ with maximal element $i$ and second-to-maximal element $j$. As admissible order on $I$, we use the associated degree-lexicographic order $\preceq_{\text{deglex}}$ introduced in Example 2.11. Then the leading monomial $\text{lm}(r)$ is $x_i x_j$, and since $j \neq i$, there are no critical monomials at all. □

3. Filtrations on groups and associated algebras

In this section we define a filtration on an arbitrary group $G$. It provides a stratification of $G$ into layers in which the group product has a simpler description. The resulting graded object has a natural structure of a restricted Lie algebra (see Definition 3.6). This structure is intimately related to the ring-theoretic structure of the group algebra of $G$ over $\mathbb{F}_p$. Since group algebras are the basic ingredients from which group cohomology is defined, the restricted Lie algebra of a pro-$p$ group is expected to play a significant role in Galois cohomology. This will be addressed in the following sections.

Even though the contents of this section applies to any group, we are mostly interested in pro-$p$ or profinite groups. Exercise (6.d) in [Ser13, Section I.4.2] asks to prove that the topology of a finitely generated pro-$p$ group is determined by the group structure: more precisely, any subgroup of finite index is open. A full proof of this can be found in [DdSMS99]. The much more difficult statement that the same is true for every finitely generated profinite group was proved in [NS07a, NS07b]. However, it is not true that every subgroup of a finitely generated profinite (or pro-$p$) group is closed. We shall therefore adopt the usual

Convention. When dealing with profinite (or pro-$p$) groups, every generating set is to be intended in the topological sense, every subgroup is understood to be closed and every map is understood to be continuous.

3.1. Filtrations. Given two subgroups $C_1$ and $C_2$ of $G$, $[C_1, C_2]$ is the subgroup of $G$ generated by the commutators $[x, y] = (y^{-1})^x \cdot y = x^{-1} y^{-1} xy$, $x \in C_1, y \in C_2$.

Moreover, for $n \geq 1$, $C_1^n$ is the subgroup of $G$ generated by the elements $g^n$, $g \in C_1$. 
Definition 3.1. Let $G$ be a group.

(i) The *descending central series* $\{\gamma_i(G)\}$ of $G$ is defined by

\[
\gamma_1(G) = G, \\
\gamma_n(G) = [G, \gamma_{n-1}(G)], \quad n \geq 2.
\]

(ii) The *p-Zassenhaus filtration* (or simply *Zassenhaus filtration*, if the prime is clear from the context) of $G$ is defined by

\[ G^{(1)} = G, \quad G^{(n)} = G^{p([n/p])} \cdot \prod_{i+j=n} [G^{(i)}, G^{(j)}], \quad n \geq 2. \]

Here $\lceil n/p \rceil$ is the least integer $h$ such that $hp \geq n$.

Remark 3.2. The subgroups $G^{(i)}$ are often called the *modular dimension subgroups* of $G$, for example in [DdSMS99, Chapter 11]. Accordingly, the Zassenhaus filtration is also known as the *dimension series*.

The Zassenhaus filtration of $G$ is the fastest descending series starting at $G$ satisfying

\[ [G^{(i)}, G^{(j)}] \subseteq G^{(i+j)} \quad \text{and} \quad G^{(i)p} \subseteq G^{(ip)} \]

for every $i, j \geq 1$. Moreover, every quotient $G^{(n)}/G^{(n+1)}$ is a $p$-elementary abelian group, and thus a vector space over $\mathbb{F}_p$. M. Lazard established the useful formula

\[ G^{(n)} = \prod_{ip^h \geq n} \gamma_i(G)^{p^h}, \quad \forall n \geq 1 \]

(cf. [DdSMS99 Theorem 11.2]). In particular,

\[ G^{(3)} = \begin{cases} 
G^p[[G,G],[G]] & \text{if } p \neq 2 \\
G^4[G,G]^2 [[G,G],[G]] & \text{if } p = 2.
\end{cases} \]

3.2. Group algebras. Let $G$ be a group and let $\mathbb{F}_p[G]$ be its group algebra over the finite field $\mathbb{F}_p$. The *augmentation ideal* $I_G$ of $\mathbb{F}_p[G]$ is the kernel of the augmentation map $\epsilon: \mathbb{F}_p[G] \to \mathbb{F}_p$, given by $g \mapsto 1$ for every $g \in G$. The augmentation ideal induces a filtration $F^n\mathbb{F}_p[G] = I_G^{-n}$, and defines an associated graded object

\[ \text{gr}\mathbb{F}_p[G] = \bigoplus_{n \geq 0} I_G^{-n}/I_G^{-n+1}, \]

called the *graded group $\mathbb{F}_p$-algebra* of $G$. To avoid confusion with the previous notation for subgroups, we stress that $I_G^0 = \mathbb{F}_p[G]$ and that $I_G^{-n}$ denotes the usual $n$-fold ideal product of $I_G$.

If $G$ is a pro-$p$ group, there are completed versions of all the previous concepts (cf. [Koc02 Section 7.1]).

Definition 3.3. The *complete group algebra* of a pro-$p$ group $G$ is

\[ \mathbb{F}_p[G] = \lim_{\leftarrow U} \mathbb{F}_p[G/U] \]

where $U$ runs through all open normal subgroups of $G$. $\mathbb{F}_p[G]$ is a topological, compact $\mathbb{F}_p$-algebra. The closed two-sided ideal $\widehat{I}_G$ generated by the closure of the
kernel of the augmentation map $\epsilon : \mathbb{F}_p[G] \to \mathbb{F}_p$, is open and it is called the complete augmentation ideal.

The complete augmentation ideal induces a filtration $F^n\mathbb{F}_p[G] = \hat{I}_G^n$, and defines an associated graded object

$$\text{gr}\mathbb{F}_p[G] = \bigoplus_{n \geq 0} \hat{I}_G^n / \hat{I}_G^{n+1},$$

called the complete graded group $\mathbb{F}_p$-algebra of $G$. Here $\hat{I}_G^0 = \mathbb{F}_p[G]$ and $\hat{I}_G^n$ is the usual $n$-fold ideal product of $I_G$.

The Zassenhaus filtration and the graded group $\mathbb{F}_p$-algebra of a group are related by the following fundamental result due to S. A. Jennings (cf. [Jen41], [DdSMS99 § 12.2]).

**Proposition 3.4.** Let $G$ be a group. Then $G(n) = \{ g \in G \mid g - 1 \in I_G^n \}$ for all $n \geq 1$.

Again, if $G$ is a pro-$p$ group, a completed version of Jennings’s Theorem holds (cf. [Koc02] Section 7.4):

**Proposition 3.5.** Let $G$ be a pro-$p$ group. Then $G(n) = \{ g \in G \mid g - 1 \in \hat{I}_G^n \}$ for all $n \geq 1$.

### 3.3. Restricted Lie algebras.

**Definition 3.6.** A restricted Lie algebra (cf. [Jac79] Section V.5) $L$ over $\mathbb{F}_p$ is a Lie $\mathbb{F}_p$-algebra equipped with an additional unary operation $^p : L \to L$, called $p$-operation, satisfying

1. $(\alpha a)^p = \alpha^p a^p$ for all $\alpha \in \mathbb{F}_p, a \in L$;
2. $(a + b)^p = a^p + \sum_{i=1}^{p-1} s_i(a, b) + b^p$ for all $a, b \in L$, where $s_i(a, b)$ is the coefficient of $T_i(a, b)$ in $\text{ad}(T a + b)^{p-1}(a)$;
3. $\text{ad}(a^p) = \text{ad}(a)^p$ for all $a \in L$.

We recall that the map $\text{ad} : L \to \text{End}(L)$ is defined as $\text{ad}(a) = \text{ad}_a : b \mapsto [a, b]$.

A morphism of restricted Lie algebras is a morphism of Lie algebras that preserve the $p$-operations.

For any associative $\mathbb{F}_p$-algebra $A$, the bracket operation $[a, b] := ab - ba$ and the $p$-operation $a^p := a^p$ make the $\mathbb{F}_p$-module underlying $A$ into a restricted Lie algebra $A_L$. This defines a functor from associative algebras to restricted Lie algebras, which has a left adjoint functor $\mathcal{U}$. For a restricted Lie algebra $L$, $\mathcal{U}(L)$ is called the universal restricted enveloping algebra of $L$. Concretely, $\mathcal{U}(L)$ is the quotient of the tensor algebra $T(L)$ by the two-sided ideal generated by the elements $ab - ba - [a, b]$ and $a^p - a^p, a, b \in L$. It comes equipped with an injective function, the universal embedding $\vartheta : L \to \mathcal{U}(L)$, that is a monomorphism of restricted Lie algebras from $L$ to $\mathcal{U}(L)_L$ (cf. [Jac79] Section V.5). This makes precise the idea that the $p$-operation is the Lie analogue of the $p^k$th power map in the associative framework.

A restricted Lie subalgebra $M$, respectively a restricted ideal $\tau$, of a restricted Lie $\mathbb{F}_p$-algebra $L$ is a Lie subalgebra, respectively an ideal, of $L$ as Lie algebra with the further condition that $a^p \in M$ for every $a \in M$, respectively $a^p \in \tau$ for every $a \in \tau$. As remarked in [Gar15] Proposition 2.1, the well-known relationship between quotients of Lie algebras and universal enveloping algebras (cf. [Bou07 §I.2.3, Prop. 3]) remains true in the restricted framework:
Proposition 3.7. Let \( L \) be a restricted Lie algebra over \( \mathbb{F}_p \) and let \( \mathfrak{r} \subseteq L \) be a restricted ideal. Let \( \mathcal{R} \) be the left ideal of \( \mathcal{U}(L) \) generated by the image of \( \mathfrak{r} \) via the universal embedding \( \vartheta : L \to \mathcal{U}(L) \). Then \( \mathcal{R} \) is a two-sided ideal and the epimorphism \( L \to L/\mathfrak{r} \) induces a short exact sequence

\[
0 \to \mathcal{R} \to \mathcal{U}(L) \to \mathcal{U}(L/\mathfrak{r}) \to 0.
\]

The free product of two restricted Lie \( \mathbb{F}_p \)-algebras \( L_1 \) and \( L_2 \) is the coproduct of \( L_1 \) and \( L_2 \) in the category of restricted Lie \( \mathbb{F}_p \)-algebras. That is, \( L_1 \ast L_2 \) contains (isomorphic images of) \( L_1 \) and \( L_2 \) as restricted Lie subalgebras, and for any couple of morphisms of restricted Lie algebras, \( \psi_1 : L_1 \to H \) and \( \psi_2 : L_2 \to H \), there exists a unique morphism \( \psi : L_1 \ast L_2 \to H \) such that \( \psi_i = \psi|_{L_i} \) for \( i = 1, 2 \) (cf. [Lic80, Remark 1]). For the existence of the free product see [Lic80, Lemma 2].

The direct sum of two restricted Lie \( \mathbb{F}_p \)-algebras \( L_1 \) and \( L_2 \) is the quotient of \( L_1 \oplus L_2 \) by the restricted ideal generated by the Lie brackets \([a, b]\) with \( a \in L_1 \) and \( b \in L_2 \).

Proposition 3.8. Let \( L_1 \) and \( L_2 \) be two restricted Lie algebras over \( \mathbb{F}_p \).

(i) The universal restricted enveloping algebra of the free product \( L_1 \ast L_2 \) is the free product \( \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \).

(ii) The universal restricted enveloping algebra of the direct sum \( L_1 \oplus L_2 \) is the symmetric tensor product \( \mathcal{U}(L_1) \otimes^1 \mathcal{U}(L_2) \).

Proof. (i) This is asserted in [Lic80, Lemma 2], but there the proof is left to the reader. For the sake of completeness, we provide a proof.

First, we define a candidate universal embedding

\[
\vartheta : L_1 \ast L_2 \to \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2).
\]

Define the inclusion maps \( \eta_1 : \mathcal{U}(L_1) \to \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \), \( \eta_2 : \mathcal{U}(L_1) \to \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \) and \( \iota_1 : L_1 \to L_1 \ast L_2 \), \( \iota_2 : L_2 \to L_1 \ast L_2 \). There are universal embeddings \( \vartheta_1 : L_1 \to \mathcal{U}(L_1) \), \( \vartheta_2 : L_2 \to \mathcal{U}(L_2) \) which, composed with \( \eta_1, \eta_2 \), give two morphisms of Lie algebras \( L_1 \to (\mathcal{U}(L_1) \sqcup \mathcal{U}(L_2))_L \leftarrow L_2 \). The universal property of \( L_1 \ast L_2 \) as a coproduct gives a unique restricted Lie algebra morphism \( \varphi : L_1 \ast L_2 \to (\mathcal{U}(L_1) \sqcup \mathcal{U}(L_2))_L \) with the property that \( \varphi \iota_1 = \eta_1 \vartheta_1 \), \( \varphi \iota_2 = \eta_2 \vartheta_2 \).

Then we prove that for any associative algebra \( M \) and any restricted Lie morphism \( \varphi : L_1 \ast L_2 \to M \) there is a unique algebra morphism \( \alpha : \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \to M \) such that \( \alpha \vartheta = \varphi \). Let \( \varphi_1 = \varphi|_{L_1} \), \( \varphi_2 = \varphi|_{L_2} \). By the universal property of universal restricted enveloping algebras, there are unique algebra morphisms \( \alpha_1, \alpha_2 \) such that \( \alpha_1 \vartheta_1 = \varphi_1 \), \( \alpha_2 \vartheta_2 = \varphi_2 \). Since \( \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \), together with \( \eta_1, \eta_2 \), is a coproduct of unital, associative \( \mathbb{F}_p \)-algebras, there is a unique algebra morphism \( \alpha : \mathcal{U}(L_1) \sqcup \mathcal{U}(L_2) \to M \).
satisfying $\alpha \eta_1 = \alpha_1$, $\alpha \eta_2 = \alpha_2$.

Since $(\alpha \vartheta)_j = \alpha \eta_j \vartheta_j = \alpha_j \vartheta_j = \varphi_j = \varphi v_j$ for $j = 1, 2$, by the universal property of $L_1 \ast L_2$ we have $\alpha \vartheta = \varphi$. If also $\beta : \mathcal{U}(L_1) \cup \mathcal{U}(L_2) \to M$ is an algebra morphism satisfying $\beta \vartheta = \varphi$, then for $j = 1, 2$

$$\begin{align*}
\beta \eta_j \vartheta_j &= \beta \vartheta j = \varphi \vartheta j = \varphi j \\
\Rightarrow \quad \beta \eta_j &= \alpha_j \\
\Rightarrow \quad \beta &= \alpha
\end{align*}$$

In particular, taking $M = \mathcal{U}(L_1 \ast L_2)$ and $\varphi$ the universal embedding, we see that $\vartheta$ is injective, as required in the definition of the universal restricted enveloping algebra. Summing up, $\mathcal{U}(L_1 \ast L_2) \cong \mathcal{U}(L_1) \cup \mathcal{U}(L_2)$.

(ii) By the definition of direct sum, Proposition [5.7] implies that

$$\mathcal{U}(L_1 \oplus L_2) = \frac{\mathcal{U}(L_1) \cup \mathcal{U}(L_2)}{R},$$

with $R$ the two-sided ideal generated by the commutators $[\vartheta(v_1), \vartheta(v_2)] = \vartheta(v_1) \vartheta(v_2) - \vartheta(v_2) \vartheta(v_1)$.

For a group $G$, let $L(G)$ be the graded object

$$L(G) = \bigoplus_{n \geq 1} G_{(n)}/G_{(n+1)}.$$

The group commutator and the $p$-power of $G$ induce the structure of graded restricted Lie algebra on $L(G)$.

The two versions of Jennings’s Theorem (Propositions [3.4] and [3.5] can be reformulated and completed in terms of $L(G)$ (cf. also [Qui68]).

**Theorem 3.9.** Let $G$ be a group, and for an element $g \in G_{(n)}$ set $\overline{g} = gG_{(n+1)} \in G_{(n)}/G_{(n+1)}$. Then the assignment

$$\vartheta(\overline{g}) = (g - 1) + I_{G}^{n+1} \in I_{G}^{n}/I_{G}^{n+1}$$

induces a monomorphism of restricted Lie algebras $\vartheta : L(G) \to (\text{grF}_p[G])_L$, such that $\text{grF}_p[G]$ endowed with $\vartheta$ is the universal restricted enveloping algebra of $L(G)$.

If $G$ is a pro-$p$ group, the same is true with $\text{F}_p[G]$ and $\widehat{I}_G$ in place of $\text{F}_p[G]$ and $I_G$.

We call $\overline{g} \in L(G)$ the initial form of $g \in G$. If $\overline{g} \in G_{(n)}/G_{(n+1)}$, we say that the initial form of $g$ has degree $n$. 

3.4. Cohomology of pro-$p$ groups. The natural cohomology theory to be used with profinite groups is Galois cohomology. It is defined in complete analogy with classical group cohomology, except for the requirement that all maps be continuous. Good accounts on Galois cohomology can be found for example in [NSW08, Ser13]. We content ourselves to quickly recall some useful facts, with a focus on finitely generated pro-$p$ groups, although the results are essentially valid in a broader context.

If $G$ is a pro-$p$ group, its Frattini subgroup is $\Phi(G) = G^p[G,G]$ and the quotient $G/G(2) = G/\Phi(G)$ is a compact elementary abelian $p$-group. Its Pontryagin dual is $H^1(G, \mathbb{F}_p)$ and the cardinality $d(G)$ of a minimal generating set of $G$ equals $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ (cf. [Ser13, I. § 4.2]). Thus, if $G$ is finitely generated, $G/\Phi(G) \cong (\mathbb{Z}/p\mathbb{Z})^{d(G)}$. Moreover, if $G$ is a finitely generated pro-$p$ group, and $\{x_1, \ldots, x_d\}$ is a minimal generating set of $G$, then the minimal number of defining relations between the $x_i$ is equal to $\dim_{\mathbb{F}_p} H^2(G, \mathbb{F}_p)$ (cf. [Ser13, I. § 4.3]). For a profinite group $G$, the space $H^\bullet(G, \mathbb{F}_p) = \bigoplus_{n \geq 0} H^n(G, \mathbb{F}_p)$ is a connected graded $\mathbb{F}_p$-algebra with respect to the cup product
\[(3.5) \quad \cup: H^s(G, \mathbb{F}_p) \times H^r(G, \mathbb{F}_p) \to H^{s+r}(G, \mathbb{F}_p), \quad r, s \geq 0\]
(cf. [NSW08 Proposition 1.4.4]). The cup product is graded-commutative, that is
\[a \cup b = (-1)^{rs} b \cup a \quad \text{for} \quad a \in H^r(G, \mathbb{F}_p), b \in H^s(G, \mathbb{F}_p).\]

This is of special interest when $G$ is a Galois group. If $F$ contains a primitive $p$th root of unity, the Kummer map $\kappa : F^\times \to H^1(G_F, \mathbb{F}_p)$ induces an isomorphism $F^\times/(F^\times)^p \to H^1(G_F, \mathbb{F}_p)$, thereby giving a concrete description of the first cohomology group of $G_F$. The possibility to extend this to higher cohomology groups was first suggested by J. Milnor for $p = 2$ (cf. [Mil70]) and then conjectured by S. Bloch and K. Kato for any prime (cf. [BK86]). In detail, the Milnor $K$-theory of $F$ is the quotient $K^{M}_n(F)$ of the tensor algebra (see §1.1) of the multiplicative group $F^\times$ by the two-sided ideal generated by the tensors $a \otimes (1-a)$, $a \in F \setminus \{0,1\}$. The tensor powers $\kappa \otimes \cdots \otimes \kappa$ factor through the aforementioned ideal (Steinberg relations, cf. [GST74 Proposition 4.6.1] or [Tat76 proof of Theorem 3.1]) and $H^\bullet(G_F, \mathbb{F}_p)$ is a $p$-torsion group. Hence for all $n \geq 1$ there is a well-defined norm residue map
\[K^M_n(F)/pK^M_n(F) \to H^n(G_F, \mathbb{F}_p).\]

The Bloch-Kato conjecture claims that all these maps are group isomorphisms. Relatively recently, M. Rost and V. Voevodsky completed the proof of the full Bloch-Kato conjecture, obtaining a very good insight into the structure of Galois cohomology (cf. [SJ06, HW09, Voe10, Voe11]). A relevant consequence of their achievement is that, if $F$ contains a primitive $p$th root of unity, then the algebra $H^\bullet(G_F, \mathbb{F}_p)$ is quadratic. From this and the graded-commutativity of the cup product, it follows that there is an epimorphism of quadratic $\mathbb{F}_p$-algebras $\Lambda_\bullet(H^1(G, \mathbb{F}_p)) \to H^\bullet(G, \mathbb{F}_p)$. In particular, for $p = 2$ this is the same as an epimorphism of quadratic algebras $S_\bullet(H^1(G, \mathbb{F}_p)) \to H^\bullet(G, \mathbb{F}_p)$.

4. Elementary type pro-$p$ groups

Few groups are known to be realizable as maximal pro-$p$ quotients of absolute Galois groups of fields satisfying Hypothesis [LS14] (in short: realizable). At present, all the finitely generated groups that are known to be realizable fall into a restricted...
class. This class is defined inductively, using two basic types of groups and two operations. We need a preliminary concept.

**Definition 4.1.** We denote the multiplicative group of units of $\mathbb{Z}_p$ by $U_p$. A cyclotomic pair is a couple $(G, \chi)$ made of a pro-$p$ group $G$ and a continuous homomorphism $\chi : G \to U_p$. We call $\chi$ a cyclotomic character (in [CMQ15] and [Qua14] $\chi$ is called an orientation).

**Remark 4.2.** Note that $U_p$ decomposes as the direct product of the pro-$p$ part $U_p(1)$ and a cyclic group of order $p-1$. The image of a continuous homomorphism from a pro-$p$ group to $U_p$ is included in $U_p(1)$ (cf. [Ser13, I. §4.5]).

This terminology is justified in view of the situation of a field $F$ satisfying Hypothesis 1.1. The action of $G_F(p)$ on the group $\mu_p(F_{\text{sep}})$ of the roots of unity of order a power of $p$ lying in a separable closure $F_{\text{sep}}$ induces a homomorphism $\chi_p : G \to U_p$. This homomorphism is known in the literature as the $(p)$-adic cyclotomic character.

**Definition 4.3.** A cyclotomic pair $(G, \chi)$ is realizable if there exits a field $F$ satisfying Hypothesis 1.1 for which $G = G_F(p)$ and $\chi = \chi_p$.

We extend the terminology relative to group properties to cyclotomic pairs, so that for example a finitely generated cyclotomic pair is a cyclotomic pair $(G, \chi)$ such that $G$ is finitely generated as a pro-$p$ group.

### 4.1. Basic groups.

**Definition 4.4.** A finitely generated pro-$p$ group is free on a set $I = \{x_1, \ldots, x_d\}$ if it is the inverse limit of the groups $L(I)/M$, where $L(I)$ is the discrete free group on $I$ and $M$ is a normal subgroup of $L(I)$ with index a (finite) power of $p$.

Free pro-$p$ groups satisfy the usual universal property of free objects (cf. [Ser13 I.1.5], where not necessarily finitely generated free pro-$p$ groups are introduced).

**Definition 4.5.** A pro-$p$ group is called a Demushkin group if $\dim_{\mathbb{F}_p} H^1(G, \mathbb{F}_p)$ is finite, $H^2(G, \mathbb{F}_p)$ is isomorphic to $\mathbb{F}_p$, and the cup product induces a non-degenerate skew-symmetric pairing

$$H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p) \cong \mathbb{F}_p.$$

By the previous discussion, Demushkin groups are finitely generated. The only finite Demushkin group is the group of order 2 (cf. [NSW08 Proposition 3.9.10]). It has a unique cyclotomic character with image $\{1, -1\}$. Infinite Demushkin groups are precisely the Poincaré duality groups of dimension 2 (cf. [Ser13 I. §4.5]). Infinite Demushkin pro-$p$ groups are classified by two invariants. The first is the minimal number of generators. As regards the second invariant, each infinite Demushkin group $G$ has exactly one cyclotomic character $\chi_G$ with the additional properties 1, 2 and 3 in [Lab67 Proposition 6] (cf. also [Ser95]). We then define the second invariant $q = q(G)$ to be 2 if $\text{Im}(\chi_G) = \{-1, +1\}$, and to be the maximal power $p^k$ such that $\text{Im}(\chi_G) \subseteq 1 + p^k \mathbb{Z}_p$ otherwise.

Using these two invariants, J. Labute completed the classification of Demushkin groups started by S. Demushkin and extended by J.-P. Serre (cf. [Lab67], [NSW08 § III.9], [Ser95]), summarized in the following.
Theorem 4.6. A finitely generated pro-$p$ group is an infinite Demushkin group if and only if it can be presented by a minimal set of generators $\{x_1, \ldots, x_d\}$, subject to one relator $r$ that has either of the following forms.

(i) If $q \neq 2$ (possibly $q = 0$), then

$$r = x_1^q[x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d],$$

with $d \geq 2$ necessarily even.

(ii) If $q = 2$ and $d \geq 3$ is odd, then

$$r = x_1^2x_2^f[x_2, x_3][x_4, x_5] \cdots [x_{d-1}, x_d]$$

with $f \in \{2, 3, \ldots\} \cup \{\infty\}$.

(iii) If $q = 2$, $d \geq 2$ is even and $[\Im \chi_G : (\Im \chi_G)^2] = 2$, then

$$r = x_1^{2+2^f}[x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d]$$

with $f \in \{2, 3, \ldots\} \cup \{\infty\}$.

(iv) If $q = 2$, $d \geq 4$ is even and $[\Im \chi_G : (\Im \chi_G)^2] = 4$, then

$$r = x_1^2[x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d]$$

with $f \in \{2, 3, \ldots\}$.

Here by convention $p^\infty = 0$.

Conversely, such presentations define an infinite Demushkin group for each value of the involved parameters. The last three cases describe pro-2 groups.

4.2. Operations.

Definition 4.7. The free product $G_1 \ast_p G_2$ of two pro-$p$ groups $G_1, G_2$ is the coproduct of $G_1$ and $G_2$ in the category of pro-$p$ groups. Explicitly, let $G$ be the discrete free product of $G_1$ and $G_2$ and let $N$ be the family of normal subgroups $N$ of $G$ such that $G/N$ is a finite $p$-group and $N \cap G_1, N \cap G_2$ are open subgroups of $G_1, G_2$ respectively. Then

$$G_1 \ast_p G_2 = \varprojlim_{N \in N} G/N$$

(cf. [BNW71]).

The free product of two cyclotomic pairs $(G_1, \chi_1)$ and $(G_2, \chi_2)$ is the cyclotomic pair $(G_1 \ast_p G_2, \chi_1 \ast_p \chi_2)$, where $\chi_1 \ast_p \chi_2$ is the character induced by $\chi_1$ and $\chi_2$ via the universal property of the coproduct (using Remark 4.2).

For properties of the free product in the category of pro-$p$ groups we refer to [RZ10] § 9.1.

Definition 4.8. Let $(G, \chi)$ be a cyclotomic pair and let $m \geq 1$ be an integer. The character $\chi$ induces an action of $G$ onto $\mathbb{Z}_p^m$, by $g \bullet (x_1, \ldots, x_m) = (\chi(g)x_1, \ldots, \chi(g)x_m)$ for $g \in G$ and $(x_1, \ldots, x_m) \in \mathbb{Z}_p^m$. This in turn defines the cyclotomic semidirect product $\mathbb{Z}_p^m \rtimes G$, by the rule $g(x_1, \ldots, x_m)g^{-1} = g\bullet(x_1, \ldots, x_m)$. The associated cyclotomic pair $(\mathbb{Z}_p^m \rtimes G, \chi \circ \pi)$, where $\pi : \mathbb{Z}_p^m \rtimes G \to G$ is the canonical projection, is also called the cyclotomic semidirect product.
4.3. **Elementary type groups.** Any free pro-$p$ group is realizable (this is a consequence of the proposition in [LvdD81, § 4.8]). Some are actually also maximal pro-$p$ quotients of absolute Galois groups of local fields not containing primitive $p^\text{th}$ roots of unity (cf. [Ser13, II.§ 5, Theorem 3]). On the other hand, maximal pro-$p$ quotients of absolute Galois groups of local fields that contain a primitive $p^\text{th}$ root of unity are Demushkin (cf. [Ser13, II.§ 5, Theorem 4]). It is not yet known whether all infinite Demushkin groups are realizable. Moreover, if $G_1$ and $G_2$ are realizable, so are also the free product $G_1 * G_2$ and all the semidirect products $\mathbb{Z}_m \rtimes G$, (cf. [EH94]).

I. Efrat conjectured that there are actually no other ways to obtain realizable finitely generated pro-$p$ groups (cf. [Efr95], [JW89]). The original formulation is in terms of cyclotomic pairs:

**Conjecture 4.9** (Elementary Type Conjecture (ETC)). The class $\mathcal{RFG}_p$ of realizable finitely generated pro-$p$ cyclotomic pairs is the smallest class of cyclotomic pairs such that

(a) any pair $(S, \chi)$, with $S$ a finitely generated free pro-$p$ group and $\chi$ an arbitrary cyclotomic character, is in $\mathcal{RFG}_p$; the pair $(1,1)$ consisting of the trivial group and the trivial character is not excluded;

(b) any pair $(G, \chi)$, with $G$ an actually realizable Demushkin group and $\chi$ its unique cyclotomic character, is in $\mathcal{RFG}_p$;

(c) if $(G_1, \chi_1), (G_2, \chi_2) \in \mathcal{RFG}_p$, then also the free product $(G_1 * G_2, \chi_1 * \chi_2)$ is in $\mathcal{RFG}_p$;

(d) if $(G, \chi) \in \mathcal{RFG}_p$, then for any positive integer $m$ also the cyclotomic semidirect product $(\mathbb{Z}_m \rtimes G, \chi \circ \pi)$ is in $\mathcal{RFG}_p$.

The conjecture parallels an analogue conjecture for Witt rings. For a general discussion we refer to [Mar04].

Demushkin groups that are not maximal pro-$p$ quotients of absolute Galois groups, provided they exist, have the same group-theoretic structure of realizable ones. Thus, as long as we focus on notions that can be formulated independently of number theory, we can work with a possibly bigger class:

**Definition 4.10.** The class $\mathcal{E}_p$ of elementary type cyclotomic pairs is the smallest class of cyclotomic pairs such that

(a) any pair $(S, \chi)$, with $S$ a finitely generated free pro-$p$ group and $\chi$ an arbitrary cyclotomic character, is in $\mathcal{E}_p$; the pair $(1,1)$ consisting of the trivial group and the trivial character is not excluded;

(b) any pair $(G, \chi)$, with $G$ a Demushkin group and $\chi$ its unique cyclotomic character, provided $\text{Im} \chi \subseteq 1 + p\mathbb{Z}_p$, is in $\mathcal{E}_p$;

(c) if $(G_1, \chi_1), (G_2, \chi_2) \in \mathcal{E}_p$, then also the free product $(G_1 * G_2, \chi_1 * \chi_2)$ is in $\mathcal{E}_p$;

(d) if $(G, \chi) \in \mathcal{E}_p$, then for any positive integer $m$ also the cyclotomic semidirect product $(\mathbb{Z}_m \rtimes G, \chi \circ \pi)$ is in $\mathcal{E}_p$.

An **elementary type** pro-$p$ group (**ET group** for short) is a group $G$ appearing in a pair in $\mathcal{E}_p$.

The condition on cyclotomic characters in (b) is due to the adoption of Hypothesis 1.1.

4.4. **Pythagorean formally real fields.** In this subsection we do not assume Hypothesis 1.1 but we still insist that fields contain a primitive $2^\text{nd}$ root of 1,
which is equivalent to having characteristic different from 2. We point out the unconditional result analogous to ETC that holds for \( p = 2 \) and the family \( \mathcal{PFR} \) of Pythagorean formally real fields with finitely many square classes (cf. [Min80, Jac81]). We recall that a field \( F \) belongs to \( \mathcal{PFR} \) if \( F^2 + F^2 = F^2, -1 \) is not a sum of squares in \( F \) and \( |F^\times/(F^\times)^2| < \infty \). Particular elements of \( \mathcal{PFR} \) are Euclidean fields, that is fields \( F \) such that \((F^\times)^2 = (F^\times)^2 \) and \( F^\times = (F^\times)^2 \cup -(F^\times)^2 \) (cf. [Bec74]).

**Theorem 4.11.** The family of maximal pro-2 quotients of absolute Galois groups of fields in \( \mathcal{PFR} \) is described inductively as follows.

(i) For any Euclidean field \( E \in \mathcal{PFR} \) (for instance, \( E = \mathbb{R} \)), \( G_E(2) = \mathbb{Z}/2\mathbb{Z} \).

(ii) For any two fields \( F_1, F_2 \in \mathcal{PFR} \) there exists \( F \in \mathcal{PFR} \) such that \( G_{F_1}(2) \ast_2 G_{F_2}(2) \cong G_F(2) \).

(iii) For any field \( F_0 \in \mathcal{PFR} \) and any finite product \( Z = \prod_{i=1}^m \mathbb{Z}/2 \) there exists \( F \in \mathcal{PFR} \) such that \( Z \times G_{F_0}(2) \cong G_F(2) \), where the action of \( G_{F_0}(2) \) on \( Z \) is given by

\[
\sigma^{-1} z \sigma = z^{-1} \quad \text{for any } \sigma \in G_{F_0}(2) \setminus \{1\}, \sigma^2 = 1 \text{ and } z \in Z
\]

Moreover, each maximal pro-2 quotient of the absolute Galois group of a field in \( \mathcal{PFR} \) can be obtained inductively from Galois groups of Euclidean fields using the two operations above.

Observe that the action in (iii) is analogous to the cyclotomic action mentioned at the beginning of Section 4. In fact, a property of Pythagorean fields \( F \) is that their extension \( F(\mu_{2\infty}) \) with all roots of unity of order a power of 2 coincides with \( F(\sqrt{-1}) \). Each \( \sigma \) as in (iii) is sent to \(-1 \) by the cyclotomic character

\[
\chi : G_{F_0}(2) \to \{-1, +1\} \cong \text{Gal}(F(-1)/F) = \text{Gal}(F(\mu_{2\infty})/F).
\]

Pythagorean fields \( F \) which are not formally real are uninteresting from our point of view. In fact, each element \( a \) in a field of characteristic not 2 is the difference of two squares, namely

\[
a = \frac{(a + 1)^2}{4} - \frac{(a - 1)^2}{4}
\]

But if \(-1 \) is a sum of squares in \( F \), then \(-1 \) is a square by the Pythagorean condition. Then \( a = (a + 1)^2/4 + (\sqrt{-1}(a - 1))^2/4 \) is the sum of two squares, and hence a square by the Pythagorean condition. In other words, \( F^\times = (F^\times)^2 \), so \( G_F(2) = \{1\} \).

**5. Proof of Theorems A, B and C**

The proof of Theorems A and B proceeds in accordance with the inductive definition of the class of ET groups (Definition 4.10). In other words, we prove first that the theorems hold for free and Demushkin groups, and then that their validity is preserved under free and cyclotomic semidirect products. Theorem C will be proved as an intermediate step in Subsection 6.2.

### 5.1. Free pro-\( p \) groups

Let \( S \) be a finitely generated free pro-\( p \) group, with minimal generating set \( \{x_1, \ldots, x_d\} \). Let \( \mathbb{F}_p\langle X \rangle \) be the algebra of formal power series in the non-commuting indeterminates \( X = \{X_1, \ldots, X_d\} \) over \( \mathbb{F}_p \). It can be described as the completion of the free algebra \( \mathbb{F}_p\langle X \rangle \) with respect to the maximal
two-sided ideal \( I(X) = (X_1, \ldots, X_d) \). As such, \( \mathbb{F}_p \langle \langle X \rangle \rangle \) is a topological, compact \( \mathbb{F}_p \)-algebra, and the closed ideal \( I(X) \) is open. Let \( \mathbb{F}_p \langle \langle X \rangle \rangle^\times \) denote the group of units of \( \mathbb{F}_p \langle \langle X \rangle \rangle \). The Magnus morphism, given by \( x_i \mapsto 1 + X_i \), induces a monomorphism of profinite groups

\[
(5.1) \quad \mu : S \to 1 + I(X) \subseteq \mathbb{F}_p \langle \langle X \rangle \rangle^\times,
\]

and an isomorphism of compact \( \mathbb{F}_p \)-algebras \( \mathbb{F}_p[S] \cong \mathbb{F}_p \langle \langle X \rangle \rangle \). Moreover, one has \( S_{(n)} = \{ g \in S \mid \mu(g) - 1 \in I(X)^n \} \) for every \( n \geq 1 \) (cf. [Koc02] Sections 4.2, 7.6).

The graded object

\[
gr \mathbb{F}_p \langle \langle X \rangle \rangle = \bigoplus_{n \geq 0} I(X)^n / I(X)^{n+1}, \quad \text{with} \ I(X)^0 = \mathbb{F}_p \langle \langle X \rangle \rangle,
\]

is isomorphic to the non-commutative polynomial algebra \( \mathbb{F}_p \langle X \rangle = T_\ast \text{span}(X) \). Then the embedding \( S \hookrightarrow \mathbb{F}_p[S] \), given by \( g \mapsto g - 1 \), induces a monomorphism of restricted Lie \( \mathbb{F}_p \)-algebras \( L(S) \to gr \mathbb{F}_p \langle \langle X \rangle \rangle_L \), mapping the initial form \( \mathfrak{m}_i \) of \( x_i \) to \( X_i \). Since the image of \( L(S) \) is the free restricted Lie algebra generated by \( X \), and \( \mathbb{F}_p \langle X \rangle \) is its universal restricted enveloping algebra, we get an isomorphism of graded \( \mathbb{F}_p \)-algebras \( U(L(S)) \cong \mathbb{F}_p \langle X \rangle \).

On the other hand, the \( \mathbb{F}_p \)-cohomology of a free \( p \)-group is the trivial algebra on a basis dual to \( \{ x_1, \ldots, x_d \} \). Therefore, Examples 2.4 and 2.13 imply the following.

**Theorem 5.1.** Let \( S \) be a finitely generated free \( p \)-group. Then the algebras \( H^\ast(S, \mathbb{F}_p) \) and \( \text{gr} \mathbb{F}_p[S] \) are quadratic dual to each other and PBW.

**5.2. Demushkin groups.** We stress that in the treatment of Demushkin groups in this subsection we allow full generality. That is, we do not assume \( \text{Im} \chi \subseteq 1 + p\mathbb{Z}_p \), notwithstanding Hypothesis [1.1]. If \( G \) is a cyclic group of order 2, then its \( \mathbb{F}_2 \)-cohomology is isomorphic to the polynomial \( \mathbb{F}_2 \)-algebra in one variable, i.e., \( H^1(G, \mathbb{F}_2) \) has order 2 and

\[
(5.2) \quad H^\ast(G, \mathbb{F}_2) = T_\ast(H^1(G, \mathbb{F}_2)) \cong \mathbb{F}_2[X_1].
\]

On the other hand, \( G_{(n)} \) is trivial for \( n \geq 2 \), so \( \text{gr} \mathbb{F}_p[G] \) is the trivial \( \mathbb{F}_2 \)-algebra on a 1-dimensional vector space:

\[
gr \mathbb{F}_p[G] \cong \mathbb{F}_2 \oplus G,
\]

Hence by Examples 2.4 and 2.13 the two \( \mathbb{F}_2 \)-algebras are quadratic dual to each other and PBW.

Assume now that \( G \) is infinite. By definition, \( H^\ast(G, \mathbb{F}_p) \) is concentrated in degrees 0, 1 and 2. More specifically, \( H^1(G, \mathbb{F}_p) \) has a \( \mathbb{F}_p \)-basis \( \{ \chi_1, \ldots, \chi_d \} \), with \( d \) equal to the minimal number of generators of \( G \), and \( H^2(G, \mathbb{F}_p) \) has a \( \mathbb{F}_p \)-basis with just one element \( \xi \). The generator \( \xi \) and the cup product on \( H^\ast(G, \mathbb{F}_p) \) depend on the shape of a presentation of \( G \) in accordance with Theorem 4.6. Namely (cf. [NSW08] Proposition 3.9.13),

\[
\xi = \chi_1 \cup \chi_2 = -\chi_2 \cup \chi_1 = \chi_3 \cup \chi_4 = -\chi_4 \cup \chi_3 = \cdots = \chi_{d-1} \cup \chi_d = -\chi_d \cup \chi_{d-1}
\]

and \( \chi_i \cup \chi_j = 0 \) in any other case for presentation [1.1];

\[
\xi = \chi_1 \cup \chi_1 = \chi_2 \cup \chi_3 = \chi_3 \cup \chi_2 = \chi_4 \cup \chi_5 = \chi_5 \cup \chi_4 = \cdots = \chi_{d-1} \cup \chi_d = \chi_d \cup \chi_{d-1}
\]

and \( \chi_i \cup \chi_j = 0 \) in any other case for presentation [1.2];

\[
\xi = \chi_1 \cup \chi_1 = \chi_1 \cup \chi_2 = \chi_2 \cup \chi_1 = \chi_3 \cup \chi_4 = \chi_4 \cup \chi_3 = \cdots = \chi_{d-1} \cup \chi_d = \chi_d \cup \chi_{d-1}
\]
and \( \chi_i \cup \chi_j = 0 \) in any other case for presentations (1.3) and (1.4).

On the other hand, by \( \text{G"ar15} \) Theorem 6.3, \( G \) is mild (see Section 3) with respect to the \( p \)-Zassenhaus filtration. Then by \( \text{G"ar15} \) Theorem 2.12

\[
\text{gr} \mathbb{F}_p[G] \cong \frac{\mathbb{F}_p(X_1, \ldots, X_d)}{\mathcal{R}},
\]

where \( \mathcal{R} \) is the two-sided ideal generated by the image of the initial form of the relator \( r \) in \( \mathbb{F}_p(X_1, \ldots, X_d) \cong \text{gr} \mathbb{F}_p[\langle X_1, \ldots, X_d \rangle] \). This image is

\[
\begin{align*}
[X_1, X_2] + [X_3, X_4] + \cdots + [X_{d-1}, X_d] & \quad \text{for presentation (4.1)} \\
X_1^2 + [X_2, X_3] + \cdots + [X_{d-1}, X_d] & \quad \text{for presentation (4.2)} \\
X_1^2 + [X_1, X_2] + \cdots + [X_{d-1}, X_d] & \quad \text{for presentations (4.3) and (4.4)}.
\end{align*}
\]

In each of the 3 cases, by construction, \( \{\chi_1, \ldots, \chi_d\} \) and \( \{X_1, \ldots, X_d\} \) are dual bases. Then the quadratic dual of \( \text{gr} \mathbb{F}_p[\mathcal{G}] \) is seen to be \( H^*(G, \mathbb{F}_p) \) by explicit computation. Now \( \text{gr} \mathbb{F}_p[G] \) is PBW by Lemma 2.14 and consequently so is \( H^*(G) \) by Remark 2.12. This completes the proof of the following.

**Theorem 5.2.** Let \( G \) be a finitely generated Demushkin group. Then the algebras \( H^*(G, \mathbb{F}_p) \) and \( \text{gr} \mathbb{F}_p[G] \) are quadratic dual to each other and PBW.

### 5.3. Free products

The \( \mathbb{F}_p \)-cohomology of the free pro-\( p \) product of two pro-\( p \) groups is described by the following (cf. \([\text{NSW08} \) Theorem 4.1.4]).

**Proposition 5.3.** Let \( G_1 \) and \( G_2 \) be finitely generated pro-\( p \) groups, and set \( G = G_1 \ast_p G_2 \). Then the inclusions \( G_i \hookrightarrow G \) induce an isomorphism of graded \( \mathbb{F}_p \)-algebras

\[
\text{res}^*_{G_i, G} \oplus \text{res}^*_{G, G_2} : H^*(G, \mathbb{F}_p) \xrightarrow{\cong} H^*(G_1, \mathbb{F}_p) \cap H^*(G_2, \mathbb{F}_p).
\]

In particular, if both \( H^*(G_1, \mathbb{F}_p) \) and \( H^*(G_2, \mathbb{F}_p) \) are quadratic, so is \( H^*(G, \mathbb{F}_p) \).

On the other hand, the graded \( \mathbb{F}_p \)-group algebra of the free pro-\( p \) product of two pro-\( p \) groups is described by the following (see also \([\text{Lic80} \) Section 3]).

**Proposition 5.4.** Let \( G_1, G_2 \) and \( G \) be as above. Then there is an isomorphism of graded \( \mathbb{F}_p \)-algebras

\[
U(L(G_1)) \cup U(L(G_2)) \xrightarrow{\cong} U(L(G)).
\]

**Proof.** Let \( G^\text{abs} \) denote the abstract free product of \( G_1 \) and \( G_2 \) as abstract, discrete groups. Then \( G \) is the completion of \( G^\text{abs} \) with respect to the pro-\( p \) topology and one has the chain of inclusions \( G_i \hookrightarrow G^\text{abs} \hookrightarrow G \) for both \( i = 1, 2 \). By \([\text{Lic80} \) Theorem 2] the restricted Lie algebra of \( G^\text{abs} \) is the free product of restricted Lie algebras

\[
L(G^\text{abs}) = L(G_1) \ast L(G_2).
\]

Since \( G \) is a finitely generated pro-\( p \) group, every subgroup \( G(n) \) is open in \( G \), and thus the Zassenhaus filtration \( (G(n))_{n \geq 1} \) is a basis of open normal subgroups for \( G \). Therefore, the inclusion \( G^\text{abs} \hookrightarrow G \) induces isomorphisms of finite pro-\( p \)-groups \( G^\text{abs} / G^\text{abs}(n) \cong G / G(n) \) for every \( n \geq 1 \), since \( G \) is the pro-\( p \) completion of \( G^\text{abs} \). Thus

\[
L(G_1) \ast L(G_2) = L(G^\text{abs}) \xrightarrow{\cong} L(G).
\]

Now (5.6) follows from Proposition 3.8. \( \square \)
Proposition 5.5. Let $G_1$ and $G_2$ be two finitely generated pro-$p$ groups, and set $G = G_1 \times_p G_2$. Assume that $H^\bullet(G_1, \mathbb{F}_p)$ and $H^\bullet(G_2, \mathbb{F}_p)$ are quadratic $\mathbb{F}_p$-algebras.

(i) If $i = 1, 2$, then $H^\bullet(G_1, \mathbb{F}_p) \cong \text{gr} \mathbb{F}_p[G_1]$, and $H^\bullet(G_2, \mathbb{F}_p) \cong \text{gr} \mathbb{F}_p[G_2]$.

(ii) If $i = 1, 2$, then $H^\bullet(G_1, \mathbb{F}_p)$, resp. $\text{gr} \mathbb{F}_p[G_1]$, are $\text{PBW}$ algebras, then also $H^\bullet(G, \mathbb{F}_p)$, resp. $\text{gr} \mathbb{F}_p[G]$, is $\text{PBW}$.

Proof. (i) follows from Proposition 5.3, Proposition 5.4, and Remark 3.5. (ii) follows from (i) and Example 2.13.

5.4. Semidirect products. Let $(G_0, \chi_0)$ be a finitely generated cyclotomic pair. The $\mathbb{F}_p$-cohomology of a semidirect product $(\mathbb{Z}^m_p \rtimes G_0, \chi_0 \circ \sigma)$ as in Definition 4.8 is described in [Wad83, Theorem 3.1, Corollary 3.4, Theorem 3.6]. Actually, in that paper the author uses valuations on fields, but a careful checking reveals that one can extract from his arguments a proof of the following proposition that relies only on group-theoretic data we have and Hochschild-Serre spectral sequence.

The following definition is more general than we need for Theorems A and B, but we will use it in this form to prove Theorem D.

Definition 5.6. Let $A_\bullet$ be a graded-commutative quadratic $\mathbb{F}_p$-algebra, with space of generators $V = \text{span}_{\mathbb{F}_p} \{t, a_1, \ldots, a_d\}$, space of relations $\Omega$ and a distinguished element $t$ such that $t + t = 0$. Let $\{x_j \mid j \in J\}$ be a set of distinct symbols not in $A$. The twisted extension of $A$ by $J$ is the quadratic $\mathbb{F}_p$-algebra $A[J,t]$, with space of generators $\text{span}_{\mathbb{F}_p}(t, a_1, \ldots, a_d, x_j \mid j \in J)$ and space of relations $\text{span}_{\mathbb{F}_p}(\Omega \cup \{x_j + x_j x_i, x_j t + tx_j, x_i a_k + a_k x_j, x_j^2 - tx_j \mid i, j \in J, k = 1, \ldots, d\})$.

Proposition 5.7. Let $(G_0, \chi_0)$ be a finitely generated cyclotomic pair and $m$ be a positive integer, and set $(G, \chi) = (\mathbb{Z}^m_p \rtimes G_0, \chi_0 \circ \pi)$. Then the inflation map $H^\bullet(G_0, \mathbb{F}_p) \rightarrow H^\bullet(G, \mathbb{F}_p)$ and the restriction map $H^\bullet(G, \mathbb{F}_p) \rightarrow H^\bullet(\mathbb{Z}^m_p, \mathbb{F}_p)$ induce an isomorphism of $\mathbb{F}_p$-algebras

\begin{equation}
H^\bullet(G, \mathbb{F}_p) \xrightarrow{\cong} H^\bullet(G_0, \mathbb{F}_p)[J,t],
\end{equation}

with $J = \{1, \ldots, m\}$. In particular, if $H^\bullet(G_0, \mathbb{F}_p)$ is quadratic, then so is $H^\bullet(G, \mathbb{F}_p)$.

If $G_0$ is the maximal pro-$p$ quotient of the absolute Galois group of a field $F$ satisfying Hypothesis 1.1, the element $t$ corresponds to the square class of $-1 \in F$ in Bloch-Kato isomorphism, so $t = 0$ (cf. [Wad83, Examples 1.12]). More generally, $t = 0$ for all cyclotomic pairs in the class $\mathcal{F}_p$ (see Definition 1.10) and so

\begin{equation}
H^\bullet(G, \mathbb{F}_p) \xrightarrow{\cong} A_\bullet(V) \otimes^{-1} H^\bullet(G_0, \mathbb{F}_p),
\end{equation}

where $V \cong \mathbb{F}^m_p$.

Let $k \in \mathbb{N} \cup \{\infty\}$ be such that $\text{Im}(\chi_0) = 1 + p^k \mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$, with the convention that $p^\infty = 0$. We set $Z = \mathbb{Z}_p^m$, and we shall use the multiplicative notation for it.

Lemma 5.8. Let $(G_0, \chi_0)$, $(G, \chi)$ and $Z$ be as above.

(i) For every $s \geq 1$ one has $(ZG_0)^p^s = Z^p^s G_0^p^s$.

(ii) For every $n \geq 2$ one has $\gamma_n(G) = Z_p^{p^{n-1}k} \cdot \gamma_n(G_0)$.

(iii) For every $n \geq 1$, $G(n) = Z_p^{(\log_p(n))_+} \times (G_0)(n)$.
Proof. Every element $g$ of $G$ can be written as $g = z \cdot w$, with $z \in Z$ and $w \in G_0$. Clearly, for any $s \geq 1$ one has the inclusions $Z^{p^s} \subseteq (ZG_0)^{p^s} \supseteq G^{p^s}$, so that $(ZG_0)^{p^s} \supseteq Z^{p^s}G_0^{p^s}$. On the other hand, for any $z \in Z$ and $w \in G_0$ one has

$$(zw)^{p^s} = z^{1+\chi_0(w)+\ldots+\chi_0(w)^{p^s-1}} \cdot w^{p^s} = z^{\frac{\chi_0(w)^{p^s-1}}{\chi_0(w)-1}} \cdot w^{p^s},$$

and since $\chi_0(w) \in 1+p^kZ_p$, the exponent $\frac{\chi_0(w)^{p^s-1}}{\chi_0(w)-1}$ is divisible by $p^s$, thus $(zw)^{p^s} \in Z^{p^s}G_0^{p^s}$. This yields statement (i).

Elementary commutator calculus implies that $\gamma_2(G) = Z^{p^k} \cdot \gamma_2(G_0)$. Note that $\gamma_n(G_0) \subseteq \text{Ker}(\chi)$ for every $n \geq 2$, so that the action of $\gamma_n(G_0)$ on $Z$ is trivial. Now statement (ii) follows by induction on $n$, since assuming $\gamma_n(G) = Z^{p^{(n-1)k}} \cdot \gamma_n(G_0)$ yields

$$
\gamma_{n+1}(G) = [G, Z^{p^{(n-1)k}} \cdot \gamma_n(G_0)] = [G, \gamma_n(G_0)] \cdot [G, Z^{p^{(n-1)k}}] = [G_0, \gamma_n(G_0)] \cdot (Z^{p^{(n-1)k}})^{p^k}.
$$

Finally, by Equality (5.2), one has

$$G(n) = G^{p^{[\log_p(n)]}} \cdot \prod_{i \geq 2, i p^h \geq n} \gamma_i(G)^{p^h} = \left(Z^{p^{[\log_p(n)]}} \cdot G_0^{p^{[\log_p(n)]}}\right) \cdot \prod_{i \geq 2, i p^h \geq n} \left(Z^{p^{(i-1)k}} \cdot \gamma_i(G_0)^{p^h}\right) = Z^{p^{[\log_p(n)]}} \cdot G_0^{p^{[\log_p(n)]}} \cdot Z^{p^\eta} \cdot \prod_{i \geq 2, i p^h \geq n} \gamma_i(G_0)^{p^h},$$

where $\eta = \min\{i - 1k + h \mid i \geq 2, ip^h \geq n\}$, as $Z^{p^{(i-1)k}} \subseteq Z^{p^\eta}$ and therefore $\prod_{i \geq 2, i p^h \geq n} Z^{p^{(i-1)k}} = Z^{p^\eta}$. Moreover, $p^h \geq n$ for every $n \geq 2$, thus $\eta \geq [\log_p(n)]$, and $Z^{p^\eta} \subseteq Z^{p^{[\log_p(n)]}}$, so that

$$Z^{p^{[\log_p(n)]}} \cdot G_0^{p^{[\log_p(n)]}} \cdot Z^{p^\eta} \cdot \prod_{i \geq 2, i p^h \geq n} \gamma_i(G_0)^{p^h} = Z^{p^{[\log_p(n)]}} \cdot G_0^{p^{[\log_p(n)]}} \prod_{i \geq 2, i p^h \geq n} \gamma_i(G_0)^{p^h},$$

which gives statement (iii). \qed

Proposition 5.9. Let $(G_0, \chi_0)$, $(G, \chi)$ and $Z$ be as above. One has an isomorphism of graded $\mathbb{F}_p$-algebras

$$\mathbb{F}_p[X] \otimes^L \mathfrak{gr} \mathbb{F}_p[G_0] \xrightarrow{\cong} \mathfrak{gr} \mathbb{F}_p[G],$$

with $X = \{X_1, \ldots, X_m\}$.

Proof. The restricted Lie algebra $L(Z)$ is a free abelian restricted Lie algebra on $m$ generators. Thus, $\mathcal{U}(L(Z))$ is isomorphic to the commutative polynomial algebra $\mathbb{F}_p[X]$, with $X = \{X_1, \ldots, X_m\}$. 

For $n \geq 1$, Lemma 5.8 implies that
\[
G(n)/G(n+1) = \begin{cases} 
Z^h/Z^{h+1} \times (G_0)/(G_0(n)_(n+1)), & \text{for } n = p^h, h \geq 0 \\
(G_0)/(G_0(n)_(n+1)), & \text{otherwise}
\end{cases}
\]
Therefore, the restricted Lie algebra $L(G)$ is the direct sum $L(Z) \oplus L(G_0)$ of restricted Lie algebras. Hence,
\[
\text{gr}F_p[G] = U(L(Z) \oplus L(G_0)) \cong U(L(Z)) \otimes^1 U(L(G_0)) = F_p[X] \otimes^1 \text{gr}F_p[G_0]
\]
by Proposition 5.8.

Summing up, Propositions 5.7 and 5.9 and Exercise 5.13 imply the following.

**Theorem 5.10.** Let $(G_0, \chi_0)$ be a finitely generated cyclotomic pair and $(G, \chi) = (\mathbb{Z}_p^m \rtimes G_0, \chi_0 \circ \pi)$ for some positive integer $m$. Assume that $H^\bullet(G_0, F_p)$ is a quadratic $F_p$-algebra.

(i) If $H^\bullet(G_0F_p)^l \cong \text{gr}F_p[G_0]$, then $H^\bullet(G, F_p)^l \cong \text{gr}F_p[G]$.

(ii) If $H^\bullet(G_0, F_p)$, resp. $\text{gr}F_p[G_0]$, is a PBW algebra, then also $H^\bullet(G, F_p)$, resp. $\text{gr}F_p[G]$, is PBW.

**Example 5.11** (p-rigid fields). For a field $F$ let $\sqrt[p]{F}$ denote the extension of $F$ obtained by adjoining $\sqrt[p]{a}$ for all $k \geq 1$ and $a \in F$. A field $F$ containing a primitive $p^\text{th}$ root of unity and such that $F^\times/(F^\times)^p$ is finite is said to be $p$-rigid if $F(p) = \sqrt[p]{F}$ (cf. [CMQ15, Section 4]). In particular, the following three statements are equivalent (cf. [CMQ15, Section 3]):

(i) $F$ is $p$-rigid;

(ii) $(G_F(p), \chi_{F,p}) \cong \mathbb{Z}_p^m \rtimes (G_0, \chi_0)$ for some $m \geq 0$ and some cyclotomic pair $(G_0, \chi_0)$ with $\chi_0$ injective;

(iii) $H^\bullet(G_F(p), F_p) \cong \Lambda_\bullet(H^1(G_F(p), F_p))$.

Then by Proposition 5.9 for $F$ $p$-rigid one has $\text{gr}F_p[G_F(p)] \cong F_p[X]$ with $X = \{X_1, \ldots, X_d\}$ for some $d \geq 1$. In fact

**Corollary 5.12.** Let $p$ an odd prime number and let $F$ be a field containing a primitive $p^\text{th}$ root of unity and such that $F^\times/(F^\times)^p$ is finite. Then $F$ is $p$-rigid if, and only if, $\text{gr}F_p[G_F(p)]$ is an abelian graded $F_p$-algebra.

**Proof.** We have already observed that if $F$ is $p$-rigid, then $\text{gr}F_p[G_F(p)]$ is isomorphic to a symmetric algebra. For the converse, if $\text{gr}F_p[G_F(p)]$ is abelian, then $L(G_F(p))$ is an abelian restricted Lie algebra. In particular, the surjective map
\[
[\_, \_] : \frac{G_F(p)}{G_F(p)_{(2)}} \times \frac{G_F(p)}{G_F(p)_{(2)}} \to \frac{G_F(p)_{(2)}}{G_F(p)_{(3)}}
\]
is the zero map, so $G_F(p)_{(2)}/G_F(p)_{(3)} = 0$. But the $F_p$-dual of $G_F(p)_{(2)}/G_F(p)_{(3)}$ is the kernel of the cup product $\cup : H^1(G_F(p), F_p) \wedge H^1(G_F(p), F_p) \to H^2(G_F(p), F_p)$ (cf. [Hir85, page 268]), so the assertion follows by Condition (iii) above.

6. Proof of Theorem D

Using the inductive description of the family $\mathcal{PFR}$ given by Theorem 4.11 we can take advantage of previous results. In particular, $H^\bullet(\mathbb{Z}/2\mathbb{Z}, F_2)$ has been studied at the beginning of Subsection 5.2 and the cohomology of free products has been addressed in Subsection 5.3.
As regards semidirect products, for any field \( F_0 \in \mathcal{PFR} \) and any finite product 
\( Z = \prod_{i=1}^{m} \mathbb{Z}_2 \), the \( \mathbb{F}_2 \)-cohomology of \( Z \rtimes G_{F_0}(2) \) is a twisted extension (see Definition 5.6) \( H^\bullet(G_{F_0}(2), \mathbb{F}_2)[J, t] \), with \( J = \{1, \ldots, m\} \) and \( t \) corresponding to the square class of \(-1\) in Bloch-Kato isomorphism. Then the proof of Theorem D is completed by the following result of independent interest.

**Proposition 6.1.** Let \( A_\bullet \) be a graded-commutative quadratic \( \mathbb{F}_2 \)-algebra endowed with space of generators \( V = \text{span}_{\mathbb{F}_2}\{t, a_1, \ldots, a_d\} \) and space of relators \( \Omega \), let \( J = \{1, \ldots, m\} \subset \mathbb{N} \) and let \( \{x_j \mid j \in J\} \) be a set of distinct symbols not in \( A \). Suppose that \( \{t, a_1, \ldots, a_d\} \) is a set of PBW generators of \( A_\bullet \) with respect to the degree-lexicographic order associated to \( t \preceq_1 a_1 \preceq_1 \cdots \preceq_1 a_d \). Then \( \{t, a_1, \ldots, a_d, x_1, \ldots, x_m\} \) is a set of PBW generators of the Wadsworth extension \( A[J; t]_\bullet \), with respect to the degree-lexicographic order associated to 
\( t \preceq_1 a_1 \preceq_1 \cdots \preceq_1 a_d \preceq_1 x_1 \preceq_1 \cdots \preceq_1 x_m \).

**Proof.** We can always choose a normalized basis of \( \Omega \) containing all the commutativity relators \( a_it + ta_i \) and \( a_ia_j + a_ja_i \) \((1 \leq i < j \leq d)\), plus possibly other relators. The rewriting rules in \( A[J; t]_\bullet \) are those in \( A_\bullet \) and

- \( x_ja_k \rightarrow a_kx_j \) \((j \in J, k \in \{1, \ldots, d\})\);
- \( x_jt \rightarrow tx_j \) \((j \in J)\);
- \( x_jx_i \rightarrow x_ix_j \) \((i, j \in J, i < j)\);
- \( x_j^2 \rightarrow tx_j \) \((j \in J)\).

The critical monomials in \( A[J; t]_\bullet \) are those in \( A_\bullet \) and

1. \( x_j^3 \);
2. \( x_j^2x_i \) \((i < j)\);
3. \( x_jx_i^2 \) \((i < j)\);
4. \( x_jx_ix_h \) \((h < i < j)\);
5. \( x_j^2t \);
6. \( x_jxt \) \((i < j)\);
7. \( x_j^2a_k \);
8. \( x_jx_ia_k \) \((i < j)\);
9. \( x_jb_1b_2 \) \((b_1b_2 \text{ a leading monomial in } \Omega)\).

The critical monomials in \( A_\bullet \) are confluent by hypothesis. The rewriting graphs of the new ones follow:

**Type (1):**

```
                x_jx_jx_j
                   ↓
                  x_jtx_j
                  ↓
                 tx_jx_j
define critical monomials
```

**Type (2):**

```
x_jx_jx_i
```

The critical monomials in \( A_\bullet \) are confluent by hypothesis. The rewriting graphs of the new ones follow:
The dotted arrows in Types (1) and (5) denote a further reduction in case \(tt \in \Omega\). The label \((\Omega)\) in Types (7) and (9) denotes the application of a rewriting rule in \(A_\ast\). In particular, in (7) \(a_k t \leadsto ta_k\) is always a rewriting rule according to our choice of a normalized basis of \(\Omega\). In (9), the rewriting rule associated to the leading monomial \(b_1 b_2\) is supposed to be \(b_1 b_2 \leadsto (\sum p_r q_r)\); here the possibility of a monomial rewriting rule \(b_1 b_2 \leadsto 0\) is not excluded: in this case the graph can be simplified, but confluency still holds. 

7. Pairings and quadratic duals

Most results of this section are valid in general, but for simplicity and coherence with the rest of the paper we assume groups to be finitely generated.
Let $G$ be a finitely generated pro-$p$ group. A minimal presentation of $G$ is a short exact sequence of pro-$p$ groups

$$
1 \longrightarrow R \longrightarrow S \longrightarrow G \longrightarrow 1,
$$

where $S$ is a free pro-$p$ group and $R \subseteq S_{(2)}$, that is, the epimorphism $S \rightarrow G$ induces an isomorphism $S/S_{(2)} \cong G/G_{(2)}$. For such a presentation, a set of defining relations is a minimal subset of $R$ which generates $R$ as a normal subgroup of $S$. If $S$ is free on the set $\{x_1, \ldots, x_d\}$ and $\{r_1, \ldots, r_m\}$ is a set of defining relations, we also write

$$
G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle.
$$

The sequence (7.1) induces the five-term exact sequence in cohomology

$$
0 \longrightarrow H^1(G, \mathbb{F}_p) \xrightarrow{\text{inf}^1_{G,S}} H^1(S, \mathbb{F}_p) \xrightarrow{\text{res}^1_{S,R}} H^1(R, \mathbb{F}_p)^G \xrightarrow{\text{trg}} H^2(G, \mathbb{F}_p) \longrightarrow H^2(S, \mathbb{F}_p).
$$

As $S$ is free, $H^2(S, \mathbb{F}_p) = 0$ (cf. [Ser13, § I.3.4]). As $S$ and $G$ have the same minimal number of generators, the map $\inf^1_{G,S}: H^1(G, \mathbb{F}_p) \rightarrow H^1(S, \mathbb{F}_p)$ is an isomorphism. Then $\text{trg}: H^1(R, \mathbb{F}_p)^G \rightarrow H^2(G, \mathbb{F}_p)$ is an isomorphism as well. We set $V = H^1(G, \mathbb{F}_p)$.

### 7.1. Pairings

Recall that a pairing $A \times B \rightarrow \mathbb{F}_p$ of vector spaces over $\mathbb{F}_p$ is said to be perfect if it induces isomorphisms $A \cong B^*$ and $B \cong A^*$. The transgression map induces the natural perfect pairing

$$
\langle \cdot, \cdot \rangle_R: \frac{R}{R \cap [R,S]} \times H^2(G) \longrightarrow \mathbb{F}_p,
$$

which is given by $\langle \hat{r}, \hat{a} \rangle = (\text{trg}^{-1}(a))(r)$ (cf. [NSW08, § III.9, page 233]).

On the other hand, one has an isomorphism of finite $\mathbb{F}_p$-vector spaces $G/G_{(2)} \cong S/S_{(2)} \cong V^*$. Let $\{x_1, \ldots, x_d\}$ be a minimal generating system for $S$. Recall from Subsection 5.1 that we may identify the tensor algebra $T_n(V^*)$ with the graded free algebra $\mathbb{F}_p(X)$, with $X = \{X_1, \ldots, X_d\}$, and that the quotient $S_{(2)}/S_{(3)}$ embeds in $(V \otimes V)^* = T^2(V^*)$.

Recall that we have

$$
gr(S) \hookrightarrow \mathbb{F}_p(X) \cong T_1 V^*.
$$

Using these maps, we will identify $S_{(2)}/S_{(3)}$ with its image in $V^* \otimes V^* = (V \otimes V)^*$. Here the latter identification is via the pairing

$$
(V^* \otimes V^*) \times (V \otimes V) \rightarrow \mathbb{F}_p, (f \otimes g, u \otimes v) = (-1)f(u)g(v)
$$

in accordance with Koszul sign convention (cf. [LV12, § 1.5.3]). Thus, one has the pairing

$$
\langle \cdot, \cdot \rangle_S: S_{(2)}/S_{(3)} \times (V \otimes V) \longrightarrow \mathbb{F}_p
$$

(note that this pairing is not perfect).

Since $R \subseteq S_{(2)}$, one has that $R^p[R, S] \subseteq S_{(3)}$, and one may define the morphism $f: R/R^p[R, S] \rightarrow S_{(3)}/S_{(3)}$ induced by the inclusion. Then the following holds, where we denote by $\cup$ the natural map $V \otimes V \rightarrow H^2(G, \mathbb{F}_p)$ induced by the cup product.
Proposition 7.1. Let $G$ be a finitely generated pro-$p$ group with minimal presentation \([\mathbb{A}_i]\). The diagram of pairings
\[
\begin{array}{c}
S_{(2)}/S_{(3)} \times (V \otimes V) & \longrightarrow & \mathbb{F}_p \\
\downarrow f & \uparrow & \\
R/R^p[R,S] \times H^2(G,\mathbb{F}_p) & \longrightarrow & \mathbb{F}_p
\end{array}
\]
is commutative, i.e., \(\langle f(\bar{r}), \alpha \rangle_S = \langle \bar{r}, \cup(\alpha) \rangle_R\), for every \(r \in R, \alpha \in V \otimes V\).

Proof. Let \(\{\chi_1, \ldots, \chi_d\} \subseteq H^1(S,\mathbb{F}_p)\) be a basis dual to \(\{x_1, \ldots, x_d\}\), so \(\chi_i(x_j) = \delta_{ij}\).
Then \(\{\chi_i \cup \chi_j, 1 \leq i, j \leq d\}\) is a basis for \(H^1(G) \otimes H^1(G)\). Therefore, it is enough to show that \(\langle f(\bar{r}), \chi_k \otimes \chi_l \rangle = \langle \bar{r}, \chi_k \cup \chi_l \rangle\) for every \(r \in R\) and \(1 \leq k, l \leq d\).

Note that \(r\) may be uniquely written as
\[
r = \begin{cases} 
\prod_{i=1}^{d} x_i^{2a_i} \prod_{i<j}^{d} [x_i, x_j]^{b_{ij}} \cdot r', & \text{if } p = 2, \\
\prod_{i<j}^{d} [x_i, x_j]^{b_{ij}} \cdot r', & \text{if } p \neq 2,
\end{cases}
\]
where \(a_i, b_{ij} \in \{0, 1, \ldots, p-1\}\) and \(r' \in S_{(3)}\) (cf. \cite{Vog04} Proposition 1.3.2 and \cite{NSW08} Proposition 3.9.13). Then one can see that
\[
f(\bar{r}) = \begin{cases} 
\sum_{i} a_i \bar{x}_i^2 + \sum_{i<j}^{d} b_{ij} (\bar{x}_i \bar{x}_j - \bar{x}_j \bar{x}_i) & \text{if } p = 2, \\
\sum_{i<j}^{d} b_{ij} (\bar{x}_i \bar{x}_j - \bar{x}_j \bar{x}_i) & \text{if } p \neq 2.
\end{cases}
\]

Hence
\[
\langle f(\bar{r}), \chi_k \otimes \chi_l \rangle = \begin{cases} 
-b_{kl} & \text{if } k < l, \\
b_{kl} & \text{if } k > l, \\
\left(-\frac{p}{2}\right) a_k & \text{if } k = l.
\end{cases}
\]

On the other hand
\[
\langle \bar{r}, \cup(\chi_k \otimes \chi_l) \rangle = \text{trg}^{-1}(\chi_k \cup \chi_l)(r).
\]

The result then follows from \cite{Vog04} Proposition 1.3.2. \(\square\)

Proposition 7.2. The pairing \([\mathbb{A}_i]\) and the commutative diagram \((\mathbb{A}_i)\) induce the commutative diagrams of perfect pairings
\[
\begin{array}{c}
S_{(2)}/S_{(3)} \times \Lambda_2(V) & \longrightarrow & \mathbb{F}_p \\
\downarrow f & \uparrow & \\
R/R^p[R,S] \times H^2(G,\mathbb{F}_p) & \longrightarrow & \mathbb{F}_p
\end{array}
\]
\[
\begin{array}{c}
S_{(2)}/S_{(3)} \times S_2(V) & \longrightarrow & \mathbb{F}_2 \\
\downarrow f & \uparrow & \\
R/R^p[R,S] \times H^2(G,\mathbb{F}_2) & \longrightarrow & \mathbb{F}_2
\end{array}
\]
(the left-hand side one if \(p\) is odd, the right-hand side one if \(p = 2\)).

Proof. The identification \(T_\bullet (V^\ast) = \mathbb{F}_p(\langle X \rangle)\) induces an embedding of the quotient \(S_{(2)}/S_{(3)}\) in the space of homogeneous polynomials of degree two.

Assume first that \(p\) is odd. It is well known (cf. \cite{Gar15} Lemma 5.3) that the set of commutators \(\{[X_i, X_j], 1 \leq i < j \leq d\}\) is a basis for \(S_{(2)}/S_{(3)}\). On the other hand, the set
\[
\{\chi_i \otimes \chi_j - \chi_j \otimes \chi_i \mid 1 \leq i < j \leq d\} \cup \{\chi_i \otimes \chi_j + \chi_j \otimes \chi_i, 1 \leq i \leq j \leq d\}
\]
is a basis for $V \otimes V$. Since 
\[
\langle [X_i, X_j], \chi_k \otimes \chi_k - \chi_j \otimes \chi_i \rangle_S = 2 \delta_{i,0} \delta_{jk},
\]
\[
\langle [X_i, X_j], \chi_k \otimes \chi_i + \chi_k \otimes \chi_j \rangle_S = 0
\]
for every $1 \leq i < j \leq d$ and $1 \leq h \leq k \leq d$, the orthogonal space $(S_2/S_3)^\perp$ in $V \otimes V$ is the subspace generated by $\chi_i \otimes \chi_j + \chi_j \otimes \chi_i$, $1 \leq i \leq j \leq d$, and the pairing (7.1) induces the perfect pairing 
\[
S_2/S_3 \times \frac{V \otimes V}{(S_2/S_3)^\perp} = S_2(S_3) \times \Lambda_2(V) \longrightarrow F_p.
\]
Note that $\dim(S_2/S_3) = \dim(\Lambda_2(V)) = d(d - 1)/2$.

If $p = 2$ then $\{[X_i, X_j], X_i^2, 1 \leq i < j \leq d\}$ is a basis for $S_2/S_3$ (cf. [G\" ar15, Lemma 5.3]). Since 
\[
\langle [X_i, X_j], \chi_k \otimes \chi_i + \chi_k \otimes \chi_j \rangle_S = 0,
\]
\[
\langle X^2_i, \chi_k \otimes \chi_k + \chi_k \otimes \chi_k \rangle_S = 0
\]
for every $1 \leq i < j \leq d$ and $1 \leq h \leq k \leq d$, the subspace generated by $\chi_i \otimes \chi_j + \chi_j \otimes \chi_i$, $1 \leq i \leq j \leq d$, is contained in $(S_2/S_3)^\perp$. On the other hand, one has $\dim(S_2/S_3) = \dim(S_2(V)) = d(d + 1)/2$. Thus, the pairing (7.4) induces the perfect pairing 
\[
S_2/S_3 \times \frac{V \otimes V}{(S_2/S_3)^\perp} = S_2(S_3) \times S_2(V) \longrightarrow F_2.
\]

Since the cup product is graded-commutative, the commutativity of diagram (7.6) follows by the commutativity of diagram (7.5). \qed

**Theorem 7.3.** Let $G$ be a finitely generated pro-$p$ group with minimal presentation (7.7). The following are equivalent.

(i) The pairing (7.3) induces a perfect pairing $RS_3/S_3 \times H^2(G, F_p) \longrightarrow F_p$.

(ii) The cup product $: H^1(G, F_p) \otimes H^1(G, F_p) \longrightarrow H^2(G, F_p)$ is surjective.

(iii) One has the equality $R \cap S_3 = R^p[R,S]$.

**Proof.** Let $W$ be the subspace of $H^2(G, F_p)$ generated by the cup products of elements of $V$. From (7.6) one obtains the diagram of pairings 
\[
\begin{array}{cccc}
S_2/S_3 \times \Lambda_2(V) & \longrightarrow & F_p \\
\downarrow & & \downarrow \\
RS_3/S_3 \times W & \longrightarrow & F_p \\
\downarrow & & \downarrow \\
R/R^p[R,S] \times H^2(G, F_p) & \longrightarrow & F_p
\end{array}
\]
(with $S_2(V)$ instead of $\Lambda_2(V)$ if $p = 2$) which commutes by Proposition (7.2). Also, the top and bottom rows are perfect pairings. Therefore, by linear algebra considerations, also the middle row is a perfect pairing, and the claim follows by dimension counting. \qed

The above result has the following consequence.

**Corollary 7.4.** Let $G$ be a finitely generated pro-$p$ group with minimal presentation (7.1) and with quadratic $F_p$-cohomology. Then $R \cap S_3 = R^p[R,S]$. 

Remark 7.5. In the case $G$ is the maximal pro-$p$ Galois group $G_F(p)$ of a field $F$ containing a root of unity of order $p$ then the three equivalent statements of Theorem 7.3 hold. Moreover, in this case Corollary 7.4 is a well-known result (cf. [EM17, Examples 8.3(1)], [MS96] for $p = 2$, [CEM12] Theorem 8.3 and [NQD12] Corollary 3.5).

Corollary 7.6. Let $G$ be a finitely generated pro-$p$ group with minimal presentation (7.1). Then one has the equality $R \cap S(3) = R^p[R,S]$ if, and only if, all initial forms $\rho_i$ of the defining relations are of degree two.

Proof. Consider the quotients $RS(3)/S(3)$ and $R/R^p[R,S]$ as $\mathbb{F}_p$-vector spaces. In particular, the former is a quotient space of the latter, and one has
\[
\dim_{\mathbb{F}_p}(R/R \cap S(3)) = \dim_{\mathbb{F}_p}(RS(3)/S(3)) \leq \dim_{\mathbb{F}_p}(R/R^p[R,S]) = m,
\]
where $\dim_{\mathbb{F}_p}(RS(3)/S(3))$ is the number of defining relations whose initial forms are of degree two. Thus all such initial forms are of degree two if, and only if, $\dim_{\mathbb{F}_p}(R/R^p[R,S]) = \dim_{\mathbb{F}_p}(RS(3)/S(3))$. $\square$

7.2. Quadratic duals. Let $G$ be a finitely generated pro-$p$ group, and assume that its $\mathbb{F}_p$-cohomology algebra is quadratic. Then we may write $H^\bullet(G, \mathbb{F}_p)$ as the quotient

\[
H^\bullet(G, \mathbb{F}_p) = \frac{T_\bullet(V)}{\langle \Omega \rangle}, \quad V = H^1(G, \mathbb{F}_p), \quad \Omega \subseteq V \otimes V.
\]

In this case it is possible to describe the quadratic dual of $H^\bullet(G, \mathbb{F}_p)$ in terms of the initial forms of the relations.

Theorem 7.7. Let (7.1) be a minimal presentation of a finitely generated pro-$p$-group $G$, and assume that $H^\bullet(G, \mathbb{F}_p)$ is quadratic as in (7.8). Then one has the equality $\Omega^\perp = RS(3)/S(3)$. In particular, if $d$ is the minimal number of generators of $G$ and $X = \{X_1, \ldots, X_d\}$, one has an isomorphism of quadratic $\mathbb{F}_p$-algebras

\[
H^\bullet(G, \mathbb{F}_p)^\perp \cong \frac{\mathbb{F}_p[X]}{\mathcal{R}},
\]

where $\mathcal{R}$ is the two-sided ideal generated by the initial forms of the defining relations of $G$.

Proof. Recall that one has the chain of inclusions
\[
RS(3)/S(3) \subseteq S(2)/S(3) \subseteq (V \otimes V)^*.
\]
By Proposition 7.1 and Corollary 7.3 the diagram of pairings
\[
\begin{array}{ccc}
\langle \_ , \_ \rangle_1 : & (V \otimes V)^* \times (V \otimes V) & \longrightarrow \mathbb{F}_p \\
\downarrow & & \\
\langle \_ , \_ \rangle_2 : & RS(3)/S(3) \times H^2(G) & \longrightarrow \mathbb{F}_p
\end{array}
\]
commutes, and by Theorem 7.3 also the bottom row is a perfect pairing. If $\alpha \in \Omega$ then one has $\langle \bar{r}, \alpha \rangle_1 = \langle \bar{r}, \cup(\alpha) \rangle_2 = \langle \bar{r}, 0 \rangle_2 = 0$ for every $\bar{r} \in RS(3)/S(3)$. Thus, $RS(3)/S(3)$ is contained in the orthogonal space $\Omega^\perp$. On the other hand, one has
\[
\dim(RS(3)/S(3)) = \dim(H^2(G)) = \dim(V \otimes V) - \dim(\Omega),
\]
so that $RS(3)/S(3) = \Omega^\perp$. Then the isomorphism (7.9) follows by the identification $T_\bullet(V^*) = \mathbb{F}_p[X]$. $\square$
For the following definition we borrow the terminology from [KLM11 § 4].

**Definition 7.8.** Let $G$ be a finitely generated pro-$p$ group with minimal presentation (7.2).

(i) The presentation is called **quadratic** (with respect to the Zassenhaus filtration) if the initial forms $\rho_1, \ldots, \rho_m$ of the defining relations are of degree two.

(ii) The presentation is called **quadratically defined** if it is quadratic and $\text{gr}\mathbb{F}_p[G]$ is isomorphic to $\mathbb{F}_p(X)/\mathcal{R}$, with $X = \{X_1, \ldots, X_d\}$ and $\mathcal{R} = (\rho_1, \ldots, \rho_m)$.

**Remark 7.9.** (a) By Corollary 7.6, a pro-$p$ group has a quadratic presentation if, and only if, it satisfies the three equivalent conditions of Theorem 7.3. In particular, a finitely generated pro-$p$ group with quadratic $\mathbb{F}_p$-cohomology ring has a quadratic presentation.

(b) By Proposition 8.3, if a mild presentation of a mild pro-$p$ group is quadratic, then it is also quadratically defined.

The Zassenhaus filtration of the free pro-$p$ group $S$ induces the filtration $\{R \cap S_{(n)}\}$, with $n \geq 2$. Let $I(R)_{\text{ind}}$ be the graded object induced by such filtration — i.e.,

$$I(R)_{\text{ind}} = \bigoplus_{n \geq 2} (R \cap S_{(n)})/(R \cap S_{(n+1)}).$$

Then $I(R)_{\text{ind}}$ is a restricted ideal of the restricted Lie algebra $L(F)$, and one has the short exact sequence of restricted Lie algebras

$$0 \rightarrow I(R)_{\text{ind}} \rightarrow L(S) \rightarrow L(G) \rightarrow 0,$$

i.e., $G(n) = S_{(n)}/R \cap S_{(n)}$ for each $n \geq 1$ (cf. [Gar15 Proof of Theorem 2.12]). In general for a finitely generated pro-$p$ group $G$ with a quadratic presentation the induced ideal $I(R)_{\text{ind}}$ and the restricted ideal $\mathfrak{r}$ generated by the initial forms of the relations may be different.

**Lemma 7.10.** Let $G$ be a finitely generated pro-$p$ group with quadratic presentation (7.1). Then $I(R)_{\text{ind}} \supseteq \mathfrak{r}$.

**Proof.** Since $G$ has a quadratic presentation, the initial forms $\mathfrak{r}_i$ of the defining relations of $G$ are of degree 2. In particular, $\mathfrak{r}_i \in R/R \cap S_{(3)}$ for every $i \in \{1, \ldots, m\}$. Since $R/R \cap S_{(3)} = (I(R)_{\text{ind}})_{2}$, and since $\mathfrak{r}$ is the restricted ideal generated by the $\mathfrak{r}_i$'s, the claim follows. \hfill $\Box$

Thus, a finitely presented pro-$p$ group $G$ with a quadratic presentation is quadratically defined if, and only if, $\mathfrak{r} = I(R)_{\text{ind}}$.

Theorem 7.7 has the following consequence.

**Proposition 7.11.** Let $G$ be a finitely generated pro-$p$ group with $H^\bullet(G, \mathbb{F}_p)$ quadratic. Then there is an epimorphism of graded $\mathbb{F}_p$-algebras

$$H^\bullet(G, \mathbb{F}_p) \rightarrow \text{gr}\mathbb{F}_p[G]$$

which is an isomorphism in degrees 0, 1, 2. In particular, if $G$ has a quadratically defined presentation, then (7.11) is an isomorphism.
Proof. Let \( R' \) and \( R \) be the two-sided graded ideals of \( \mathbb{F}_p(X) \) generated by \( I(R)_{\text{ind}} \) and \( r \) respectively. By Theorem 7.7

\[
H^\bullet(G, \mathbb{F}_p) = \frac{\mathbb{F}_p(X)}{R}.
\]

On the other hand, by Proposition 5.7 applied to the sequence 7.10,

\[
\text{gr}\mathbb{F}_p[G] \cong \frac{\mathbb{F}_p(X)}{R'}.
\]

Since \( r \subseteq I(R)_{\text{ind}} \), so there is a graded algebra epimorphism

\[
H^\bullet(G, \mathbb{F}_p) = \frac{\mathbb{F}_p(X)}{R} \rightarrow \frac{\mathbb{F}_p(X)}{R'} \cong \text{gr}\mathbb{F}_p[G].
\]

Since the elements of degree 2 in \( r \) and \( I(R)_{\text{ind}} \) coincide, being the elements of \( RS(3)/S(3) \), the previous epimorphism is an isomorphism in degrees 0, 1 and 2.

If \( G \) has a quadratically defined presentation, then by definition \( R = R' \) and \( \text{gr}\mathbb{F}_p[G] \cong \mathbb{F}_p(X)/R \).

\[ \square \]

Remark 7.12. As a consequence, by [PP05, Proposition 3.1] one has an isomorphism of quadratic \( \mathbb{F}_p \)-algebras

\[
\bigoplus_{i \geq 0} \text{Ext}^{i,i}_{\text{gr}\mathbb{F}_p[G]}(\mathbb{F}_p, \mathbb{F}_p) \cong H^\bullet(G).
\]

Remark 7.13. Let \( p \) be an odd prime number and \( G \) a finitely generated pro-\( p \) group. We have \( H^1(G, \mathbb{F}_p) = H^1(G/G(2), \mathbb{F}_p) \) and we shall identify (see Subsection 3.4)

\[
H^1(G, \mathbb{F}_p)^* = G/G(2).
\]

Let \( \text{lie}(H^1(G, \mathbb{F}_p)^*) \) be the free restricted Lie algebra on the \( \mathbb{F}_p \)-vector space \( H^1(G, \mathbb{F}_p)^* \). Then we have a natural restricted Lie algebra epimorphism

\[
\varphi_G : \text{lie}(H^1(G, \mathbb{F}_p)^*) \rightarrow L(G),
\]

which restricts to the identity in degree 1, and to the commutator map

\[
[\cdot, \cdot] : G/G(2) \wedge G/G(2) \rightarrow G(2)/G(3)
\]

in degree 2. Let

\[
\delta_G : H^2(G, \mathbb{F}_p)^* \rightarrow H^1(G, \mathbb{F}_p)^* \wedge H^1(G, \mathbb{F}_p)^*
\]

be the map dual to the cup product \( \cup : H^1(G, \mathbb{F}_p) \wedge H^1(G, \mathbb{F}_p) \rightarrow H^2(G, \mathbb{F}_p) \). From [Hil85, page 268], we have the following exact sequence

\[
1 \rightarrow (G(2)/G(3))^* \xrightarrow{d} H^1(G, \mathbb{F}_p) \wedge H^1(G, \mathbb{F}_p) \xrightarrow{\cup} H^2(G, \mathbb{F}_p),
\]

where \( d \) is dual to the commutator map \( [\cdot, \cdot] : G/G(2) \wedge G/G(2) \rightarrow G(2)/G(3) \).

Thus \( \text{Im}\delta_G = \text{Ker}([\cdot, \cdot]) \). Hence we have a natural epimorphism

\[
\Phi_G : \text{lie}(H^1(G, \mathbb{F}_p)^*)/(\text{Im}\delta_G) \rightarrow L(G).
\]

Following [SW], the *holonomy restricted Lie algebra* \( \mathfrak{h}_G(G) \) of \( G \) is the holonomy restricted Lie algebra of the cohomology ring \( H^\bullet(G, \mathbb{F}_p) \), that is,

\[
\mathfrak{h}_G(G) = \mathfrak{h}_G(H^\bullet(G, \mathbb{F}_p)) := \text{lie}(H^1(G, \mathbb{F}_p)^*)/(\text{Im}\delta_G).
\]

We also say that \( G \) is \((p-)\text{graded-formal if the natural epimorphism}\)

\[
\Phi_G : \mathfrak{h}_G(G) \rightarrow L(G)
\]
is an isomorphism of graded restricted Lie algebras. Note also that, by [SW, Proposition 3.4], one has
\[ U(H(G)) = \frac{H^\bullet(G, \mathbb{F}_p)}{\text{Ker} \cup}, \]
where \( H^\bullet(G, \mathbb{F}_p) \) is the quadratic closure of \( H^\bullet(G, \mathbb{F}_p) \), defined as \( H^\bullet(G, \mathbb{F}_p) = \Lambda^\bullet(H^1(G, \mathbb{F}_p)) / (\text{Ker} \cup) \).

By Jennings’s Theorem [3.9, Theorem 3.9] the natural epimorphism \( \Phi_G \) induces a natural epimorphism \( \frac{H^\bullet(G, \mathbb{F}_p)}{\text{Ker} \cup} \rightarrow \text{gr} \frac{\mathbb{F}_p[[G]]}{\mathbb{F}_p[G]} \).

Our Theorem B says that any elementary type pro-\( p \) group \( G \) is \( p \)-graded formal.

7.3. A refinement of \( \{ R \cap S(n) \} \). The content of the present subsection is a minor adaptation of the results in [KLM11] to our situation: there the authors deal with the descending \( p \)-central series, here we deal with the Zassenhaus filtration, but the proofs are essentially the same. Mimicking the definition of the Zassenhaus filtration, one may define the following refinement of the filtration \( \{ R \cap S(n) \} \). Set \( R(1, S) = R \) and
\[ R(n, S) = R_{\lceil n/p \rceil, S} \prod_{i+j=n} [R(i, S), S(j)] \]
for every \( n \geq 2 \).

**Lemma 7.14.** Let \( n \geq 1 \) be an integer.
(i) \( R(n, S) \subseteq S_{n+1} \).
(ii) \( R(n, S) \subseteq R \cap S_{n+1} \).

**Proof.** We prove (i) by induction on \( n \). For \( n = 1 \), \( R(1, S) = R \subseteq S(2) \). Assume that (i) holds for \( n - 1 \). We note that \( \lceil n + 1/p \rceil \leq \lceil n/p \rceil + 1 \), hence
\[ R_{\lceil n/p \rceil, S} \subseteq S_{\lceil n/p \rceil + 1} \subseteq S_{\lceil n+1/p \rceil}. \]

Then
\[ R(n, S) = R_{\lceil n/p \rceil, S} \prod_{i+j=n} [R(i, S), S(j)] \]
\[ \subseteq S_{\lceil n+1/p \rceil} \prod_{i+j=n} [S(i+1), S(j)] \]
\[ = S_{n+1}, \]
as desired. Statement (ii) follows immediately from (i). \( \square \)

The filtration \( \{ R(n, S) \} \), \( n \geq 1 \), yields the restricted Lie algebra
\[ L(R, F) = \bigoplus_{n \geq 1} R_{n, F} / R_{n+1, F}. \]

By Lemma [3.14] one has a homomorphism of \( \mathbb{F}_p \)-vector spaces
\[ \iota_n : R(n, S) / R_{n+1, S} \rightarrow R \cap S_{n+1} / R \cap S_{n+2}, \]
and hence a restricted homomorphism \( \iota : L(R, S) \rightarrow I(R)_{\text{ind}} \). The following analogues of [KLM11, Theorem 4.2 and Theorem 4.4] hold.

**Proposition 7.15.** The following statements are equivalent.

(1) \( R(n, F) = R \cap S_{n+1} \) for every \( n \geq 1 \).
(2) The homomorphism \( \iota \) is injective.

(3) The homomorphism \( \iota \) is surjective.

**Proposition 7.16.** Let \( G \) be a finitely generated pro-\( p \) group with minimal presentation (7.1). Then \( \iota \) is bijective if and only if \( R = 1 \) or \( G \) is quadratically defined.

8. Mild pro-\( p \) Groups

The concept of mild group has been introduced by J. Labute in [Lab85] and D. Anick in [Ani87] as groups whose relators are as independent as possible with respect to the lower central series. Mild pro-\( p \) groups were subsequently defined for several filtrations in [Lab06], for \( p \) odd, and in [LM11] and [For11], for \( p = 2 \). It came as a surprise when J. Labute discovered that these groups have cohomological dimension 2 and some occur naturally as Galois groups of maximal \( p \)-extensions with restricted ramification. Here we present the definition with respect to Zassenhaus filtration.

Recall that the Hilbert series of a locally finite-dimensional graded algebra \( A_* \) over an arbitrary field \( k \) is the formal power series

\[
h_{A_*}(z) = \sum_{n=0}^{\infty} (\dim_k A_n) z^n.
\]

We fix a set \( X = \{X_1, \ldots, X_d\} \) of indeterminates and denote by \( I = (X_1, \ldots, X_d) \) the augmentation ideal of \( \mathbb{F}_p \langle X \rangle \).

**Definition 8.1.** Let \( \rho_1, \ldots, \rho_m \in I \) be homogeneous elements of \( I \) of degrees \( s_i = \deg \rho_i \) and let \( \mathcal{R} \) be the ideal of \( \mathbb{F}_p \langle X \rangle \) generated by \( \{\rho_1, \ldots, \rho_m\} \). The sequence \( (\rho_1, \ldots, \rho_m) \) is strongly free if the Hilbert series of the quotient algebra \( A_* = \mathbb{F}_p \langle X \rangle / \mathcal{R} \) is

\[
h_{A_*}(z) = \frac{1}{1 - d z + (t^{s_1} + \cdots + t^{s_m})},
\]

or equivalently (cf. [For11 Section 1]), if the left \( A_* \)-module \( \mathcal{R}/\mathcal{R}I \) is free on the images of \( \rho_1, \ldots, \rho_m \). The empty sequence is considered to be strongly free.

**Definition 8.2.** A minimal presentation \( G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle \) of a finitely generated pro-\( p \) group is called a strongly free presentation (with respect to the Zassenhaus filtration) if the sequence \( (\overline{r_1}, \ldots, \overline{r_m}) \) of the initial forms of the relators is strongly free. Here we use the embedding \( L(S) \hookrightarrow \text{gr}\mathbb{F}_p[S] \cong \mathbb{F}_p(X) \), where \( S \) is the free pro-\( p \)-group on the generators \( x_1, \ldots, x_d \), to see \( \overline{r_1}, \ldots, \overline{r_m} \in L(S) \) as elements of \( \mathbb{F}_p(X) \).

A finitely generated pro-\( p \) group that has a strongly free presentation is called a mild group (with respect to the Zassenhaus filtration).

For such pro-\( p \) groups one can compute the completed graded group algebra directly from their defining relations, as shown by the following (cf. [Gar15 Theorem 2.12]).

**Proposition 8.3.** Let \( G \) be a mild pro-\( p \) group, with a strongly free presentation \( G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle \). Then

(i) for all \( n > 2 \), \( H^n(G, \mathbb{F}_p) = 0; \)
(ii) one has an isomorphism of graded $\mathbb{F}_p$-algebras $\text{gr}\mathbb{F}_p[G] \cong \mathbb{F}_p(X)/\mathcal{R}$, where $\mathcal{R}$ is the two-sided ideal generated by the initial forms of the defining relations $\{r_1, \ldots, r_m\}$.

As a consequence, if a strongly free presentation of a mild pro-$p$ group is quadratic, then it is also quadratically defined.

**Theorem 8.4.** Let $G$ be a mild pro-$p$ group, with strongly free presentation $G = \langle x_1, \ldots, x_d \mid r_1, \ldots, r_m \rangle$. Assume that the initial forms of the defining relations are of degree two and that $H^*(G, \mathbb{F}_p)$ is quadratic. Then the algebras $H^*(G, \mathbb{F}_p)$ and $\text{gr}\mathbb{F}_p[G]$ are quadratic dual to each other and Koszul.

**Proof.** By Proposition 8.3 and Theorem 7.7 one has the isomorphisms
$$\text{gr}\mathbb{F}_p[G] \cong \mathbb{F}_p(X)/\mathcal{R} \cong H^*(G)^1,$$
where $\mathcal{R}$ is the two-sided ideal of $\mathbb{F}_p(X)$ generated by the initial forms $\overline{r_1}, \ldots, \overline{r_m}$, and this yields the first claim. On the other hand, the quadratic $\mathbb{F}_p$-algebras $\text{gr}\mathbb{F}_p[G]$ and $H^*(G, \mathbb{F}_p)$ have Hilbert series
$$h_{\text{gr}\mathbb{F}_p[G]}(z) = \frac{1}{1 - dz + mz^2} \quad \text{and} \quad h_{H^*(G, \mathbb{F}_p)}(z) = 1 + dz + mz^2.\]$$
Since $h_{\text{gr}\mathbb{F}_p[G]}(z) \cdot h_{H^*(G, \mathbb{F}_p)}(-z) = 1$, both algebras are Koszul by [PP05] § 2.2, Corollary 2.4. □

Mildness is a very strict restriction on the structure of a pro-$p$ group, as it implies in particular a $\mathbb{F}_p$-cohomological dimension less or equal to 2. On one side, Example 5.11 provides realizable pro-$p$ groups of arbitrarily large $\mathbb{F}_p$-cohomological dimension. On the other side, there are mild pro-$p$ groups that are not realizable:

**Example 8.5.** Let $G$ be the pro-$p$ group with presentation
$$G = \langle x_1, x_2, x_3, x_4 \mid [x_1, x_2], [x_2, x_3], [x_3, x_4], [x_4, x_1] \rangle.$$ 

By [Gär15] Example 2.10-(i) $G$ is mild, and thus
$$\text{gr}\mathbb{F}_p[G] \cong \frac{\mathbb{F}_p(X)}{[X_1, X_2], \ldots, [X_4, X_1]}.\]$$
Moreover, [NSW08] Proposition 3.9.13, along with the results in Subsection 3.4 implies that $H^2(G, \mathbb{F}_p) = \text{span}_{\mathbb{F}_p} \{\chi_1 \cup \chi_2, \ldots, \chi_4 \cup \chi_1\}$, where $\chi_i$ denotes the dual of $x_i$ for each $i \in \{1, 2, 3, 4\}$. On the other hand, by [Qua14] Theorem 5.6 $G$ does not occur as maximal pro-$p$ quotient of the absolute Galois group of a field containing a root of unity of order $p$.

**Remark 8.6.** Infinite Demushkin groups are mild (cf. [Gär15] Theorem 6.3), so Theorem 8.3 gives another proof of the fact that the cohomology and the complete graded group algebra of an infinite Demushkin groups are quadratic dual to each other and Koszul.

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