Enumerative geometry and knot invariants

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Abstract

We review the string/gauge theory duality relating Chern-Simons theory and topological strings on noncompact Calabi-Yau manifolds, as well as its mathematical implications for knot invariants and enumerative geometry.
1 Introduction

Enumerative geometry and knot theory have benefitted considerably from the insights and results in string theory and topological field theory. The theory of Gromov-Witten invariants has emerged mostly from the consideration of topological sigma models and topological strings, and mirror symmetry has provided a surprising point of view with powerful techniques and deep implications for the theory of enumerative invariants.
On the other hand, the new invariants of knots and links that emerged in the eighties turned out to be deeply related to Chern-Simons theory, a topological gauge theory introduced by Witten in [92], which also provided a new family of invariants of three-manifolds. It is safe to say that these two topics, enumerative geometry and knot theory, have been deeply transformed through the emergence of these connections to physics.

A more recent surprise, however, is that, in many situations, knot invariants are related to enumerative invariants. The reason is that Chern-Simons gauge theory has a string description in the sense envisaged by 't Hooft [86], and this description turns out to involve topological strings, i.e. the physical counterparts of Gromov-Witten invariants. This relation between two seemingly unrelated areas of geometry is therefore based on a beautiful realization of the large $N$ string/gauge theory duality. The connection between Chern-Simons theory and topological strings was first pointed out by Witten in [95], and the current picture emerged in the works of Gopakumar and Vafa [37] and Ooguri and Vafa [76].

In this paper we have tried to review these developments. We have focused mostly in presenting results, general ideas and examples. Some of the physical arguments leading to these results are not covered in detail, mostly for reasons of space, but also with the hope that mathematicians will find this review more readable. Important related developments, like the interplay with mirror symmetry and the relation with M-theory on manifolds of $G_2$ holonomy, are only mentioned in the text. Other reviews of the topics covered here include [88, 57], and more recently [40], which provides extensive mathematical background.

The plan of this paper is the following. In section 2 we review some basic facts about open and closed topological strings and their structure in terms of integer invariants. In section 3, we give a quick review of Chern-Simons theory and knot and link invariants. In section 4, we state the basic ideas of string/gauge theory duality in the $1/N$ expansion, and we show, following Gopakumar and Vafa, that Chern-Simons theory has a description in terms of closed strings on the resolved conifold. In section 5 we show in detail how to incorporate Wilson loops in the duality. It turns out that the Chern-Simons/string duality can be extended to closed strings propagating in more complicated toric geometries, and we summarize some of the results in section 6. Finally, some conclusions and open problems are collected in section 7.
2 Topological strings

2.1 Topological sigma models

The starting point to construct topological strings is an $\mathcal{N} = (2, 2)$ superconformal field theory, the $\mathcal{N} = (2, 2)$ nonlinear sigma model. This model can be twisted in two ways in order to produce a topological field theory [91, 55, 93], which are usually called the A and the B model. We will focus here on the A-model.

The field content of this model is the following. First, we have a map $x : \Sigma_g \to X$ from a Riemann surface of genus $g$ to a target space $X$, that will be a Kähler manifold of complex dimension $d$. We also have fermions $\chi \in x^*(TX)$, which are scalars on $\Sigma_g$, and a fermionic one form $\psi_\alpha$ with values in $x^*(TX)$. This last field satisfies a selfduality condition which implies that its only nonzero components are $\psi_I^* \in x^*(T^{(1,0)}X)$ and $\psi_I \in x^*(T^{(0,1)}X)$, where $T^{(1,0)}X, T^{(0,1)}X$ denote, respectively, the holomorphic and the antiholomorphic tangent bundles, and $I, \bar{I}$ are the corresponding indices. The theory also has a BRST, or topological, charge $Q$ which acts on the fields according to

$$\{Q, x\} = i\chi,$$
$$\{Q, \chi\} = 0,$$
$$\{Q, \psi_I^*\} = -\partial_\bar{z}x^I - i\chi^J\Gamma^I_{JK}\psi^K_\bar{z},$$
$$\{Q, \psi_I \} = -\partial_z x^I - i\chi^J\Gamma^I_{JK}\psi^K \bar{z}. \tag{2.1}$$

The twisted Lagrangian turns out to be $Q$-exact, up to a topological term:

$$\mathcal{L} = i\{Q, V\} + \int_{\Sigma_g} x^*(\omega), \tag{2.2}$$

where $\omega = J + iB$ is the complexified Kähler class of $X$, and $V$ (sometimes called the gauge fermion) is given by

$$V = \int_{\Sigma_g} d^2 z \, G_{I\bar{J}}(\psi_I^* \partial_\bar{z} x^J + \partial_z x^I \psi_I^*). \tag{2.3}$$

In this equation, $G_{I\bar{J}}$ is the Kähler metric of $X$. Notice that the last term in (2.2) is a topological invariant characterizing the homotopy type of the map $x : \Sigma_g \to X$, therefore the energy-momentum tensor of this theory is given by:

$$T_{\alpha\beta} = \{Q, b_{\alpha\beta}\}, \tag{2.4}$$

where $b_{\alpha\beta} = \delta V/\delta g^{\alpha\beta}$. The fact that the energy-momentum tensor is $Q$-exact means that the theory is topological, and the fact that the Lagrangian is $Q$-exact up to a
The classical solutions of the sigma model action are holomorphic maps \( x : \Sigma_g \to X \), which are also known as worldsheet instantons, and the functional integral localizes to these configurations. The relevant operators in this theory, as in any topological theory of cohomological type, are the \( Q \)-cohomology classes. In this case they are given by operators of the form,
\[
O_\phi = \phi_{i_1 \cdots i_p} \chi^{i_1} \cdots \chi^{i_p},
\]
where \( \phi = \phi_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \) is a closed \( p \)-form representing a nontrivial class in \( H^p(X) \). Moreover, one can derive a selection rule for correlation functions of such operators: the vacuum expectation value \( \langle O_{\phi_1} \cdots O_{\phi_\ell} \rangle \) vanishes unless
\[
\sum_{k=1}^\ell \mathrm{deg}(O_{\phi_k}) = 2d(1 - g) + 2 \int_{\Sigma_g} x^*(c_1(X)),
\]
where \( \mathrm{deg}(O_{\phi_k}) = \mathrm{deg}(\phi) \). The right hand of this equation is nothing but the virtual dimension of the moduli space of holomorphic maps, \( \mathcal{M}_{\Sigma_g \to X} \). Since the operators \( O_{\phi_k} \) can be interpreted as differential forms on this moduli space, the above selection rule just says that we have to integrate top forms.

In the case of a Calabi-Yau manifold of complex dimension 3, we have \( c_1(X) = 0 \), and the selection rule says that at genus \( g = 0 \) (i.e. when the Riemann surface is a sphere \( S^2 \)) we have to insert three operators associated to 2-forms. The correlation functions can be evaluated by summing over the different topological sectors of holomorphic maps. These sectors can be labelled by “instanton numbers.” Let \( \Sigma_i \) denote a basis of \( H_2(X) \), with \( i = 1, \cdots, b_2 \). If the image of \( x(S^2) \) is in the homology class \( \beta = \sum_i n_i \Sigma_i \), then we will say that the worldsheet instanton is in the sector specified by \( \beta \), or equivalently, by the integers \( n_i \). The trivial sector corresponds to \( \beta = 0 \), i.e. the image of the sphere is a point in the target, and in this case the correlation function is just the classical intersection number \( D_1 \cap D_2 \cap D_3 \) of the three divisors \( D_i, i = 1, 2, 3 \), associated to the 2-forms, while the nontrivial instanton sectors give an infinite series. The final answer looks, schematically,
\[
\langle O_{\phi_1} O_{\phi_2} O_{\phi_3} \rangle = (D_1 \cap D_2 \cap D_3) + \sum_{\beta} I_{0,3,\beta}(\phi_1, \phi_2, \phi_3) q^\beta
\]
The notation is as follows: let \( \omega = \sum_{i=1}^{b_2} t_i \omega_i \) be the complexified Kähler form of \( X \), where \( \omega_i \) is a basis for \( H^2(X) \) dual to \( \Sigma_i \), and \( t_i \) are the complexified Kähler parameters. Set \( q_i = e^{-t_i} \). If \( \beta = \sum_i n_i \Sigma_i \), then \( q^\beta \) denotes \( \prod_i q_i^{n_i} \). The coefficients \( I_{0,3,\beta}(\phi_1, \phi_2, \phi_3) \) “count” in some appropriate way the number of holomorphic maps from the sphere to the Calabi-Yau, in the topological sector specified by \( \beta \), and in such a way that
the point of insertion of $O_{\phi_i}$ gets mapped to the divisor $D_i$. This is an example of a Gromov-Witten invariant, although to get the general picture we have to couple the model to gravity, as we will see very soon.

When $c_1(X) > 0$, correlation functions also have the structure of (2.7): the trivial sector gives just the classical intersection number of the cohomology ring, and then there are quantum corrections associated to the worldsheet instantons. One important aspect of the case $c_1(X) > 0$ is that the right hand side of (2.6) contains the positive integer $\sum_i n_i \int_{\Sigma} c_1(X)$, where $n_i$ are the instanton numbers labelling the topological sector of the holomorphic map. As the $n_i$ increase, it won’t be possible to satisfy the selection rule for the insertions. Therefore, only a finite number of topological sectors contribute to the correlation function, which will be given by the sum of a classical intersection number plus a finite number of “quantum” corrections. This is the starting point in the definition of the quantum cohomology of $X$, see [23] for details.

2.2 Closed topological strings

In the above considerations on topological sigma models we have focused on $g = 0$. For $g = 1$ and a Calabi-Yau manifold, the only vacuum expectation value (vev) that may lead to a nontrivial answer is that of the unit operator, i.e. the partition function itself, while for $g > 1$ the virtual dimension of the moduli space is negative and the above theory is no longer useful to study the enumerative geometry of the target space $X$. This corresponds mathematically to the fact that, for a generic metric on the Riemann surface $\Sigma_g$, there are no holomorphic maps at genus $g > 1$. In order to circumvent this problem, we have to couple the theory to two-dimensional gravity, which means considering all possible metrics on the Riemann surface. The resulting model is called a topological string theory. We will start by giving a general idea from a more mathematical point of view (see [23] for a rigorous discussion), and then we will present the physical construction.

The moduli space of possible metrics (or equivalently, complex structures) on a Riemann surface with punctures is the famous Deligne-Mumford space $\overline{M}_{g,n}$ of stable curves with $n$ marked points (the definition of what stable means can be found for example in [13]). The moduli space we have to consider in the theory of topological strings also involves maps. It consists on one hand of a point in $\overline{M}_{g,n}$, i.e. a Riemann surface with $n$ punctures, $(\Sigma_g, p_1, \ldots, p_n)$, and this involves a choice of complex structure on $\Sigma_g$. On the other hand, we have a map $x: \Sigma_g \to X$ which is holomorphic with respect to the choice of complex structure on $\Sigma_g$.

Let us now fix the topological sector of the holomorphic map, i.e. the homology class $\beta = x_*[\Sigma_g]$. In general, there will be many maps in this sector. The set given
by the possible data \((x, \Sigma_g, p_1, \ldots, p_n)\) associated to the class \(\beta\) can be promoted to a moduli space \(\overline{M}_{g,n}(X, \beta)\), provided a certain number of conditions are satisfied. This is the basic moduli space we will need in the theory of topological strings. Its (complex) virtual dimension is given by:

\[
(1 - g)(d - 3) + n + \int_{\Sigma_g} x^*(c_1(X)).
\] (2.8)

If we compare (2.8) to (2.6), we see that there is an extra \(3(g - 1) + n\) which comes from the Mumford-Deligne space \(\overline{M}_{g,n}\). The moduli space \(\overline{M}_{g,n}(X, \beta)\) comes equipped with the natural maps

\[
\begin{align*}
\pi_1 &: \overline{M}_{g,n}(X, \beta) \rightarrow X^n, \\
\pi_2 &: \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,n}.
\end{align*}
\] (2.9)

The first map is easy to define: given a point \((x, \Sigma_g, p_1, \ldots, p_n)\) in \(\overline{M}_{g,n}(X, \beta)\), we just compute \((x(p_1), \ldots, x(p_n))\). The second map sends \((x, \Sigma_g, p_1, \ldots, p_n)\) to \((\Sigma_g, p_1, \ldots, p_n)\), i.e. forgets the information about the map and leaves the punctured curve (there are some subtleties with this map, associated to the stability conditions; see [23]). We can now formally define the Gromov-Witten invariant \(I_{g,n,\beta}\) as follows. Let us consider cohomology classes \(\phi_1, \ldots, \phi_n\) in \(H^*(X)\). The map \(\pi_1\) induces a map \(\pi_1^* : H^*(X)^n \rightarrow H^*(\overline{M}_{g,n}(X, \beta))\), and we can pullback \(\phi_1 \otimes \cdots \otimes \phi_n\) to get a differential form on the moduli space of holomorphic maps. This form can be integrated as long as there is a well-defined fundamental class for this space, and the result is the Gromov-Witten invariant \(I_{g,n,\beta}(\phi_1, \ldots, \phi_n)\):

\[
I_{g,n,\beta}(\phi_1, \ldots, \phi_n) = \int_{\overline{M}_{g,n}(X, \beta)} \pi_1^*(\phi_1 \otimes \cdots \otimes \phi_n).\] (2.10)

By using the Gysin map \(\pi_2!\), one can reduce this to an integral over the moduli space of curves \(\overline{M}_{g,n}\). The Gromov-Witten invariant \(I_{g,n,\beta}(\phi_1, \ldots, \phi_n)\) vanishes unless the degree of the form equals the dimension of the moduli space. Therefore, we have the following selection rule:

\[
\frac{1}{2} \sum_{i=1}^{n} \text{deg}(\phi_i) = (1 - g)(d - 3) + n + \int_{\Sigma_g} x^*(c_1(X))
\] (2.11)

Notice that Calabi-Yau threefolds play a special role in the theory, since for those targets the virtual dimension only depends on the number of punctures, and therefore the above condition is always satisfied if the forms \(\phi_i\) have degree 2. The invariants (2.10) generalize the invariants obtained from topological sigma models. In particular, \(I_{0,3,\beta}\)
are the invariants involved in the evaluation of correlation functions of the topological sigma model with a Calabi-Yau threefold as its target in (2.7). When \( n = 0 \), one gets an invariant \( N_{g,\beta} = I_{g,0,\beta} \) which does not require any insertions. We will refer to this as the Gromov-Witten invariant of the Calabi-Yau threefold \( X \) at genus \( g \) and in the class \( \beta \). These are the only (closed) Gromov-Witten invariants that we will deal with here. It can be also shown that, for genus 0 \([23]\),

\[
I_{0,3,\beta}(\phi_1, \phi_2, \phi_3) = N_{0,\beta} \int_{\beta} \phi_1 \int_{\beta} \phi_2 \int_{\beta} \phi_3,
\]

so from these Gromov-Witten invariants one can recover as well the information about the three-point functions of the topological sigma model.

The physical point of view on the Gromov-Witten invariants \( N_{g,\beta} \) comes about as follows. It is clear that we have to couple the topological sigma model to two dimensional gravity in order to get nontrivial invariants. To do that, one realizes \([26, 15]\) that the structure of the twisted theory is tantalizingly close to that of the bosonic string. In the bosonic string, there is a nilpotent BRST operator, \( Q_{\text{BRST}} \), and the energy-momentum tensor turns out to be a \( Q_{\text{BRST}} \)-commutator: \( T(z) = \{ Q_{\text{BRST}}, b(z) \} \). This is precisely the same structure that we found in (2.4), so the field \( b_{\alpha\beta} \) plays the role of a ghost. Therefore, one can just follow the prescription of coupling to gravity for the bosonic string and define a genus \( g \) free energy as follows:

\[
F_g = \int_{\mathcal{M}_g} \left\langle \prod_{k=1}^{6g-6} (b, \mu_k) \right\rangle,
\]

where

\[
(b, \mu_k) = \int d^2 z (b_{zz}(\mu_k)_{\bar{z}}^z + b_{\bar{z}z}(\mu_k)_{\bar{z}}^z),
\]

and \( \mu_k \) are the usual Beltrami differentials. The vev in (2.13) refers to the path integral over the fields of the twisted sigma model. The result, which depends on the choice of complex structure of the Riemann surface, is then integrated over the moduli space \( \mathcal{M}_g \).

\( F_g \) can be evaluated again, like in the topological sigma model, as a sum over instanton sectors. It turns out \([15]\) that \( F_g \) is a generating functional for the Gromov-Witten invariants \( N_{g,\beta} \), or more precisely,

\[
F_g(t) = \sum_{\beta} N_{g,\beta} q^\beta.
\]

It is also useful to introduce a generating functional for the all-genus free energy:

\[
F(g_s, t) = \sum_{g=0}^{\infty} F_g(t) g_s^{2g-2}.
\]
The parameter $g_s$ can be regarded as a formal variable, but in the context of type II strings it is nothing but the string coupling constant.

The first term in (2.15) corresponds to the contribution of constant maps, with $\beta = 0$. It was shown in [15] (see also [35]) that, for $g \geq 2$, this contribution can be expressed as an integral over $\overline{M}_g$. The result is as follows: on $\overline{M}_g$ there is a complex vector bundle $E$ of rank $g$, called the Hodge bundle, whose fiber at a point $\Sigma$ is $H^0(\Sigma, K_\Sigma)$. The contribution of constant maps to $F_g$ is then given by

$$N_{g,0} = (-1)^g \frac{\chi(X)}{2} \int_{\overline{M}_g} c_{g-1}(E), \quad g \geq 2,$$

(2.17)

where $c_{g-1}$ is the $(g-1)$-th Chern class of $E$, and $\chi(X)$ is the Euler characteristic of the target space.

In general, Gromov-Witten invariants can be computed by using the localization techniques pioneered by Kontsevich [55]. These techniques are easier to implement in the case of non-compact Calabi-Yau manifolds (the so-called local case), where one can compute $N_{g,\beta}$ for arbitrary genus. For example, let us consider the non-compact Calabi-Yau manifold $O(-3) \to \mathbb{P}^2$. This is the total space of $\mathbb{P}^2$ together with its anticanonical bundle, and it has $b_2 = 1$, corresponding to the hyperplane class of $\mathbb{P}^2$. Therefore, the class $\beta$ is labelled by a single integer, the degree of the curve in $\mathbb{P}^2$. By using the localization techniques of Kontsevich, adapted to the noncompact case, one finds [20, 53]:

$$F_0(q) = -\frac{t^3}{18} + 3q - 45q^2 + 244q^3 - 12333q^4 \cdots$$

$$F_1(q) = -\frac{t}{12} + \frac{4q}{18} + \frac{3q^2}{16} + \frac{23q^3}{3} + \frac{3437q^4}{16} \cdots$$

$$F_2(q) = \frac{\chi(X)}{5720} + \frac{q}{80} + \frac{3q^3}{20} - \frac{514q^4}{5} \cdots$$

(2.18)

and so on. In (2.18), $t$ is the Kähler class of the manifold, $\chi(X) = 2$ is the Euler characteristic of the local $\mathbb{P}^2$, and $q = e^{-t}$. The first term in $F_2$ is the contribution of constant maps, and we will provide later on a universal expression for it.

It should be mentioned that there is of course a very powerful method to compute $F_g$, namely mirror symmetry (the B-model). In the B-model, the $F_g$ amplitudes are deeply related to the variation of complex structures on the Calabi-Yau manifold (Kodaira-Spencer theory) and can be computed through the holomorphic anomaly equations of [15]. B-model computations of Gromov-Witten invariants and $F_g$ amplitudes can be found for example in [15, 20, 44, 53, 50]. Finally, it should be mentioned that, when type II theory is compactified on a Calabi-Yau manifold, the $F_g$ appear naturally
as the couplings of some special set of F-terms of the low-energy supergravity action \[15, 7\]. This point of view has shown to be extremely important in understanding the properties of topological strings.

### 2.3 Open topological strings

Let us now consider open topological strings. The natural starting point is a topological sigma model in which the worldsheet is now a Riemann surface \( \Sigma_{g,h} \) of genus \( g \) with \( h \) holes. Such models were analyzed in detail in [95]. The main issue is of course to specify boundary conditions for the maps \( x : \Sigma_{g,h} \rightarrow X \). It turns out that, for the A-model, the relevant boundary conditions are Dirichlet, supported on Lagrangian submanifolds of the Calabi-Yau \( X \). If we denote by \( C_i, i = 1, \ldots, h \) the holes of \( \Sigma_{g,h} \) (i.e. the disconnected components of the boundary \( \partial \Sigma_{g,h} \)), we have to pick Lagrangian submanifolds \( L_i \), and consider maps such that

\[
x(C_i) \subset L_i.
\]

These boundary conditions are a consequence of requiring \( Q \)-invariance at the boundary. One also has boundary conditions on the fermionic fields of the theory, which require that \( \chi \) and \( \psi \) at the boundary \( C_i \) take values on \( x^*(TL_i) \). We can also couple the theory to Chan-Paton degrees of freedom on the boundaries, giving rise to a \( \otimes_i U(N_i) \) gauge symmetry. The model can then be interpreted as a topological open string theory in the presence of \( N_i \) topological D-branes wrapping the Lagrangian submanifolds \( L_i \). Notice that, in contrast to physical D-branes in Calabi-Yau manifolds, which wrap special Lagrangian submanifolds [13, 75], in the topological framework the conditions are relaxed to just Lagrangian.

Once boundary conditions have been specified, one can define the free energy of the topological string theory similarly to what we did in the closed case. Let us consider for simplicity the case in which one has a single Lagrangian submanifold \( L \), so that all the boundaries of \( \Sigma_{g,h} \) are mapped to \( L \). Now, in order to specify the topological sector of the map, we have to give two different kinds of data: the boundary part and the bulk part. For the bulk part, the topological sector is labelled by relative homology classes, since we are requiring the boundaries of \( x_*[\Sigma_{g,h}] \) to end on \( L \). Therefore, we will set

\[
x_*[\Sigma_{g,h}] = Q, \quad Q \in H_2(X,L)
\]

To specify the topological sector of the boundary, we will assume that \( b_1(L) = 1 \), so that \( H_1(L) \) is generated by a nontrivial one cycle \( \gamma \). We then have

\[
x_*[C_i] = w_i \gamma, \quad w_i \in \mathbb{Z}, \quad i = 1, \ldots, h,
\]
in other words, \( w_i \) is the winding number associated to the map \( x \) restricted to \( C_i \). We will collect these integers into a single vector \( h \)-uple denoted by \( w = (w_1, \ldots, w_h) \).

There are various generating functionals that we can consider, depending on the topological data that we want to keep fixed. It is very useful to fix \( g, h \) and the winding numbers, and sum over all bulk classes. This produces the following generating functional of open Gromov-Witten invariants:

\[
F_{w,g}(t) = \sum_Q F_{w,g}^Q e^{-Q \cdot t}. \tag{2.22}
\]

In this equation, we have labelled the relative cohomology classes \( Q \) of embedded Riemann surfaces by a vector \( Q \) of \( b_2(X) \) integers defined as

\[
\int_Q \omega = Q \cdot t, \tag{2.23}
\]

where \( t = (t_1, \cdots, t_{b_2(X)}) \) are the complexified Kähler parameters of the Calabi-Yau manifold. In many examples relevant to knot theory, the entries \( Q \) are naturally chosen to be half-integers. Finally, the quantities \( F_{w,g}^Q \) are the open string Gromov-Witten invariants, and they “count” in an appropriate sense the number of holomorphically embedded Riemann surfaces of genus \( g \) in \( X \) with Lagrangian boundary conditions specified by \( L \) and in the class represented by \( Q, w \). These are in general rational numbers.

We can now consider the total free energy, which is the generating functional for all topological sectors:

\[
F(V) = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} \sum_{w_1, \ldots, w_h} \frac{i^h}{h!} g_s^{2g-2+h} F_{g,w}(t) \operatorname{Tr} V^{w_1} \cdots \operatorname{Tr} V^{w_h}, \tag{2.24}
\]

where \( g_s \) is the string coupling constant, and \( V \) is a matrix source that keeps track of the topological sector at the boundary. The factor \( i^h \) is very convenient in order to compare to the Chern-Simons free energy, as we will see later. The factor \( h! \) is a symmetry factor which takes into account that the holes are indistinguishable (or one could have absorbed them into the definition of \( F_{g,w} \)).

In order to compare open Gromov-Witten invariants to knot invariants, it is useful to introduce the following notation. When all \( w_i \) are positive, one can label \( w \) in terms of a vector \( \vec{k} \). Given an \( h \)-uple \( w = (w_1, \ldots, w_h) \), we define a vector \( \vec{k} \) as follows: the \( i \)-th entry of \( \vec{k} \) is the number of \( w_j \)'s which take the value \( i \). For example, if \( w_1 = w_2 = 1 \) and \( w_3 = 2 \), this corresponds to \( \vec{k} = (2, 1, 0, \cdots) \). In terms of \( \vec{k} \), the number of holes and the total winding number are

\[
h = |\vec{k}| \equiv \sum_j k_j, \quad \ell = \sum_i w_i = \sum_j jk_j. \tag{2.25}
\]
Note that a given $\vec{k}$ will correspond to many $w$’s which differ by permutation of entries. In fact there are $h!/\prod_j k_j!$ $h$-tuples $w$ which give the same vector $\vec{k}$ (and the same amplitude). We can then write the total free energy for positive winding numbers as:

$$F(V) = \sum_{g=0}^{\infty} \sum_{\vec{k}} \frac{j[\vec{k}]}{\prod_j k_j!} g^2 g^{-2+2h} F_{g,\vec{k}}(t) \Upsilon_{\vec{k}}(V)$$  \hspace{1cm} (2.26)$$

where

$$\Upsilon_{\vec{k}}(V) = \prod_{j=1}^{\infty} (\text{Tr} V^j)^{k_j}.$$  \hspace{1cm} (2.27)$$

Although a rigorous theory of open Gromov-Witten invariants is not available, localization techniques make possible to compute them in various situations [51, 67, 39, 73, 16, 52]. It is also possible to use mirror symmetry to compute disc invariants (i.e. when $g = 0$, $h = 1$), as it was first shown in [4] and subsequently explored in [2, 72, 65, 47, 38]. Finally, we also mention that the open string amplitudes $F_{g,w}$ also appear as low-energy couplings of type II superstrings compactified on Calabi-Yau manifolds in the presence of D-branes [15, 89].

### 2.4 Integer invariants from topological strings

The closed and open Gromov-Witten invariants that have been introduced are both rational, due to the orbifold structure of the moduli spaces. On the other hand, these invariants are deeply related to questions in enumerative geometry, but the relation between the invariants and the number of holomorphic curves of a given genus and in a given homology class is far from being simple. An obvious reason for this is multicovering. Suppose you have found a holomorphic map $x : S^2 \to X$ in genus zero of degree $d$. Then, simply by composing this with a degree $k$ cover $S^2 \to S^2$, you get another holomorphic map of degree $kd$. Therefore, at every degree, in order to count the actual number of “primitive” holomorphic curves, one should subtract the contributions coming from multicovering of curves with lower degree. On top of that, the contribution of a $k$-cover appears in $N_{0,kd}$ with weight $k^{-3}$. Therefore, although in genus zero the Gromov-Witten invariants are not integer, this is due to the effects of multicovering, and once this has been taken into account one extracts integer numbers that correspond in many cases to actual numbers of rational curves. The multicovering phenomenon at genus 0 was found experimentally in [19] and later on derived in [8].

Another geometric effect that has to be taken into account is bubbling [14, 15]. Imagine that you found a map $x : \Sigma_g \to X$ from a genus $g$ surface to a Calabi-Yau threefold. By gluing to $\Sigma_g$ a small Riemann surface of genus $h$, and making it very
small, you get an approximate holomorphic map from a Riemann surface whose genus is topologically \( g + h \). This means that “primitive” maps at genus \( g \) contribute to all genera \( g' > g \), and in order to count curves properly we should take this effect into account.

These facts suggest that, although the Gromov-Witten invariants are not in general integer numbers, they have some hidden integral structure, and that one can extract from them integer invariants that are related to a counting problem. But it turns out that, instead of deriving the various effects of multicovering and bubbling from a geometrical point of view, the underlying integral structure of the Gromov-Witten invariants is better revealed when the \( F_g \) is regarded as a low-energy coupling in a compactification of type IIA theory on a Calabi-Yau manifold. Using this approach, Gopakumar and Vafa showed \[36\] that one can write the generating functional \( F(g_s, t) \) in terms of contributions associated to BPS states, and they used type IIA/M-theory duality to obtain a completely new point of view on topological strings. They showed in particular that Gromov-Witten invariants of closed strings can be written in terms of some new, integer invariants known as Gopakumar-Vafa invariants. These invariants count in a very precise way the number of BPS states that arise in the Calabi-Yau compactification of type IIA theory. We will now describe this result in some detail and provide some examples.

The result of Gopakumar and Vafa concerns the overall structure of \( F(g_s, t) \). According to \[36\], the generating functional \[2.16\] can be written as

\[
F(g_s, t) = \sum_{g=0}^{\infty} \sum_{\beta}^{\infty} \sum_{d=1}^{\infty} n_{g, \beta}^d \left( \frac{2 \sin \left( \frac{dg_s}{2} \right)}{2g_s} \right)^{2g_s^2 - 2} q^d, \tag{2.28}
\]

where \( n_{g, \beta}^d \), which are the Gopakumar-Vafa invariants, are integer numbers. In \[2.28\], \( t \) denotes the set of \( b_2(X) \) Kähler parameters, and \( q^d \) is defined as in \[2.7\]. It is very illuminating to expand \[2.28\] in powers of \( g_s \) and extract from it the structure of a given \( F_g \). One easily obtains, for \( g = 0 \), the well-known structure of the prepotential \[19\] \[8\]:

\[
F_0 = \frac{1}{3!} \int_X \omega^3 + \int_X c_2(X) \wedge \omega + \chi(X) \frac{\zeta(3)}{2} + \sum_{\beta} n_{\beta}^0 \text{Li}_3(q^\beta), \tag{2.29}
\]

up to the polynomial terms in \( t \). Here \( \chi(X) \), \( c_2(X) \) denote respectively the Euler characteristic and the second Chern class of the Calabi-Yau target. We recall that \( \text{Li}_j \) denotes the polylogarithm of index \( j \), which is defined by:

\[
\text{Li}_j(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^j}. \tag{2.30}
\]
Notice that \( \text{Li}_1(x) = -\log(1 - x) \), while for \( j \leq 0 \), \( \text{Li}_j(x) \) is a rational function of \( x \):

\[
\text{Li}_j(x) = \left( x \frac{d}{dx} \right)^{|j|} \frac{1}{1 - x} = |j|! \frac{x^{|j|}}{(1 - x)^{|j|+1}} + \cdots. \tag{2.31}
\]

For \( g = 1 \), one obtains:

\[
F_1 = \frac{1}{24} \int_X c_2(X) \wedge \omega + \sum_{\beta} \left( \frac{1}{12} n^0_\beta + n^1_\beta \right) \text{Li}_1(q^\beta). \tag{2.32}
\]

Finally, for \( g > 1 \), the Gopakumar-Vafa result gives:

\[
F_g(t) = \frac{(-1)^g \chi(X) |B_{2g}B_{2g-2}|}{4g(2g-2)(2g-2)!} + \sum_{\beta} \left( \frac{|B_{2g}| n^0_\beta}{2g(2g-2)!} + \frac{2(-1)^g n^2_\beta}{(2g-2)!} \pm \cdots - \frac{g - 2}{12} n^{-1}_\beta + n^g_\beta \right) \text{Li}_{3-2g}(q^\beta). \tag{2.33}
\]

In this equation, \( B_n \) denote the Bernoulli numbers. The first term in (2.33) is the contribution to \( F_g \) associated to maps from \( \Sigma_g \) to a single point. Comparing it with (2.17) we find that the Gopakumar-Vafa structure result predicts:

\[
\int_{\mathcal{M}_g} c^3_{g-1}(\mathcal{E}) = \frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!}. \tag{2.34}
\]

This expression was conjectured by Faber [30], derived in [70] from heterotic/type IIA duality, and proved in [31].

The polylogarithm in (2.33) indicates that the degree \( k \) multiover of a curve of genus \( g \) contributes with a factor \( k^{2g-3} \) to \( F_g \). This generalizes the results of [19] for genus 0 and results for genus 1 in [14]. The multiover contribution was also found in [70] by using heterotic/type II duality. But equation (2.33) also takes into account in a precise way the effect of bubbling on \( F_g \): at every genus \( g \), one has to take into account all the previous genera \( g' < g \) in order to extract the Gopakumar-Vafa invariants \( n^g_\beta \).

The Gopakumar-Vafa invariants contain all the information of the Gromov-Witten invariants, and vice versa: if one knows the Gopakumar-Vafa invariants \( n^g_\beta \) for all \( g \) and \( \beta \), one can deduce the \( N_{g,\beta} \), and the other way around. This follows just by comparing (2.28) with (2.10), and it is worked out in detail in [17], where explicit formulae for the relation between \( N_{g,\beta} \) and \( n^g_\beta \) are given. But one remarkable aspect of the Gopakumar-Vafa picture is that, in many situations, the integer invariants \( n^g_\beta \) can be computed much more easily than their Gromov-Witten counterparts [36, 50]. In fact, their computation involves in many cases just classical algebraic geometry, so one gets rid of the complications of the moduli space of maps. The physical reason
behind is that in the Gopakumar-Vafa picture one looks at worldsheet instantons using the physical gauge approach (in the terminology of [97]), i.e. one views the worldsheet instanton as a submanifold of the target, and not as a map embedding a Riemann surface $\Sigma_g$ inside a Calabi-Yau. Related developments can be found in [15].

Let us consider some simple examples of the Gopakumar-Vafa invariants. The simplest one refers to the noncompact Calabi-Yau manifold $O(-1) \oplus O(-1) \to \mathbb{P}^1$, also known as the resolved conifold, which will play an important role later on. This manifold is toric, and can be described as the zero locus of

$$|x_1|^2 + |x_4|^2 - |x_2|^2 - |x_3|^2 = s$$

(2.35)

quotiented by a $U(1)$ that acts as

$$x_1, x_2, x_3, x_4 \rightarrow e^{i\alpha} x_1, e^{-i\alpha} x_2, e^{-i\alpha} x_3, e^{i\alpha} x_4$$

(2.36)

This is the description that appears naturally in the linear sigma model of [96]. Notice that, for $x_2 = x_3 = 0$, (2.35) describes a $\mathbb{P}^1$ whose area is proportional to $s$. Therefore, $(x_1, x_4)$ can be taken as homogeneous coordinates of the $\mathbb{P}^1$ which is the basis of the fibration, while $x_2, x_3$ can be regarded as coordinates for the fibers. This manifold has $b_2(X) = 1$, corresponding to the $\mathbb{P}^1$ in the base, and its total free energy turns out to be

$$F(g_s, t) = \sum_{d=1}^{\infty} \frac{1}{d \left(2 \sin \frac{dg_s}{2}\right)^2} q^d,$$

(2.37)

where $q = e^{-t}$ and $t$ is the complexified area of the $\mathbb{P}^1$. We see that the only nonzero Gopakumar-Vafa invariant is $n_0^1 = 1$. On the other hand, this model already has an infinite number of nontrivial $N_{g,\beta}$ invariants, but these are all due to bubbling and multicovering: the model only has one true “primitive” curve, which is just $\mathbb{P}^1$, and this is what the Gopakumar-Vafa invariant is computing.

A more complicated example is the local $\mathbb{P}^2$ geometry considered before, which already has an infinite number of nontrivial Gopakumar-Vafa invariants. These have been computed in [53, 50, 3] using the A-model, the B-model, and the duality with Chern-Simons theory that we will explain in section 6. Some results are presented in Table 1. In this table, the integer $d$ labels the class $\beta$, and corresponds to the degree of the curve in $\mathbb{P}^2$. Notice that the first Gromov-Witten invariants are $N_{0,1} = 3$, and $N_{0,2} = -45/8$, as listed in (2.18), therefore using the multicovering/bubbling formula one finds $n_0^1 = N_{0,1} = 3$, and $N_{0,2} = n_0^1/8 + n_0^2$, which gives $n_0^2 = -6$.

For open topological strings one can derive a similar expression relating open Gromov-Witten invariants to a new set of integer invariants, that we will denote by $n_{w, g, Q}$. The
corresponding multicovering/bubbling formula was derived in [76, 63], following arguments similar to those in [36], and states that the free energies of open topological string theory in the sector labelled by \( w \) can be written in terms of the integer invariants \( n_{w,g,Q} \) as follows:

\[
\sum_{g=0}^{\infty} g_s^{2g-2} h^{2g-2} F_w(t) = \prod_i w_i \sum_{g=0}^{\infty} \sum_{d \mid w} (-1)^{h+g} n_{w/d,g,Q} d^{h-1} \left( 2 \sin \frac{dg_s}{2} \right)^{2g-2} \prod_i \left( 2 \sin \frac{w_i g_s}{2} \right) e^{-dQ \cdot t}. \tag{2.38}
\]

Notice there is one such identity for each \( w \). In this expression, the sum is over all integers \( d \) which satisfy that \( d \mid w_i \) for all \( i = 1, \cdots, h \). When this is the case, we define the \( h \)-uple \( w/d \) whose \( i \)-th component is \( w_i/d \). The expression (2.38) can be expanded to give a set of multicovering/bubbling formulae for different genera. Up to genus 2 one finds,

\[
F_{w,g=0}^Q = (-1)^h \sum_{d \mid w} d^{h-3} n_{w/d,0,Q/d},
\]

\[
F_{w,g=1}^Q = (-1)^h \sum_{d \mid w} \left( d^{h-1} n_{w/d,1,Q/d} - \frac{d^{h-3}}{24} (2d^2 - \sum_i w_i^2) n_{w/d,0,Q/d} \right),
\]

\[
F_{k,g=2}^Q = (-1)^h \sum_{d \mid w} \left( d^{h+1} n_{w/d,2,Q/d} + \frac{d^{h-1}}{24} n_{w/d,1,Q/d} \sum_i w_i^2 \right.
\]

\[+ \left. \frac{d^{h-3}}{5760} (24d^4 - 20d^2 \sum_i w_i^2 - 2 \sum_i w_i^4 + 5 \sum_{i_1,i_2} w_{i_1}^2 w_{i_2}^2) n_{w/d,0,Q/d} \right) \tag{2.39}.
\]

In these equations, the integer \( d \) has to divide the vector \( w \) (in the sense explained above) and it is understood that \( n_{w,d,Q/d} \) is zero if \( Q/d \) is not a relative homology class.

It is important to notice that the integer invariants \( n_{w,g,Q} \) are not the most fundamental ones. When all the winding numbers are positive, we can represent \( w \) by a
vector \( \vec{k} = (k_1, k_2, \cdots) \), as we explained in 2.3. Such a vector can be interpreted as a label for a conjugacy class \( C(\vec{k}) \) of the symmetric group \( S_\ell \), where \( \ell = \sum_j jk_j \) is the total winding number: \( C(\vec{k}) \) is the conjugacy class with \( k_1 \) one-cycles, \( k_2 \) two-cycles, and so on. The invariant \( n_{w,g,Q} \) will be denoted as \( n_{\vec{k},g,Q} \), and D-brane physics states that it can be written as

\[
n_{\vec{k},g,Q} = \sum_R \chi_R(C(\vec{k})) N_{R,g,Q}, \tag{2.40}
\]

where \( N_{R,g,Q} \) are integer numbers labelled by representations of the symmetric group, i.e. by Young tableaux, and \( \chi_R \) is the character of \( S_\ell \) in the representation \( R \). The above relation is invertible, since by orthonormality of the characters one has

\[
N_{R,g,Q} = \sum_{\vec{k}} \frac{\chi_R(C(\vec{k}))}{z_{\vec{k}}} n_{\vec{k},g,Q}, \tag{2.41}
\]

where

\[
z_{\vec{k}} = \frac{\ell!}{|C(\vec{k})|} = \prod k_j! \prod j^{k_j}. \tag{2.42}
\]

Notice that integrality of \( N_{R,g,Q} \) implies integrality of \( n_{\vec{k},g,Q} \), but not the other way around. In that sense, the invariants \( N_{R,g,Q} \) are more fundamental. We will further clarify this issue in section 4.

### 3 Chern-Simons theory and knot invariants

In this section we make a short review of Chern-Simons theory and its relations to knot invariants.

#### 3.1 Chern-Simons theory: basic ingredients

Chern-Simons theory, introduced by Witten in [92], provides a quantum field theory description of a wide class of invariants of three-manifolds and of knots and links in three-manifolds. The Chern-Simons action with gauge group \( G \) on a generic three-manifold \( M \) is defined by

\[
S = \frac{k}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \tag{3.1}
\]

Here, \( k \) is the coupling constant, and \( A \) is a \( G \)-gauge connection on the trivial bundle over \( M \). We will assume for simplicity that \( G \) is a simply-laced group, unless otherwise
stated. As noticed by Witten, since this action does not involve the metric, the resulting quantum theory is topological, at least formally. In particular, the partition function

$$Z_k(M) = \int [DA] e^{iS}$$  

(3.2)

should define a topological invariant of the manifold $M$. A detailed analysis [92] shows that this is in fact the case, with an extra subtlety: the invariant depends on the three-manifold and of a choice of framing, i.e. a choice of trivialization of the bundle $TM \oplus TM$ (this should be called, strictly speaking, a 2-framing, but we will refer to it as framing, following standard practice). As explained in [9], for every three-manifold there is a canonical choice of framing, and the different choices are labelled by an integer $s \in \mathbb{Z}$ in such a way that $s = 0$ corresponds to the canonical framing. In the following all the results will be presented in the canonical framing.

The partition function of Chern-Simons theory can be computed in a variety of ways. One can for example use perturbation theory and produce an asymptotic series in $k$ around a classical solution to the action. The classical solutions of Chern-Simons theory are just flat connections $F(A) = 0$ on $M$. Let us assume that these are a discrete set of points (this happens, for example, if $M$ is a rational homology sphere). In that situation, one expresses $Z_k(M)$ as a sum of terms associated to stationary points:

$$Z_k(M) = \sum_c Z_k^{(c)}(M),$$  

(3.3)

where $c$ labels the different flat connections $A^{(c)}$ on $M$. The structure of the perturbative series was analyzed in various papers [92, 83, 11] and is given by the following expression:

$$Z_k^{(c)}(M) = Z_{1-\text{loop}}^{(c)}(M) \exp \left\{ \sum_{\ell=1}^{\infty} S_{\ell}^{(c)} x^\ell \right\},$$  

(3.4)

In this equation, $x$ is the effective expansion parameter:

$$x = \frac{2\pi i}{k + y},$$  

(3.5)

where $y$ is the dual Coxeter of the group, and we will set $l = k + y$. For $G = SU(N)$, $y = N$. The one-loop correction $Z_{1-\text{loop}}^{(c)}(M)$ was first analyzed in [92], and studied in great detail since then. It involves some important normalization factors of the path-integral, and determinants of differential operators. After some work it can be written in terms of topological invariants of the three-manifold and the flat connection $A^{(c)}$,

$$Z_{1-\text{loop}}^{(c)}(M) = \frac{(2\pi x)^{\frac{1}{2}(\dim H^0 - \dim H^1)}}{\text{vol}(H_c)} e^{-\frac{i}{4}S_{\text{CS}}(A^{(c)})} \sqrt{\sum_{\tau(\ell)}},$$  

(3.6)
Figure 1: Heegard splitting of a three-manifold $M$ into two three manifolds $M_1$ and $M_2$ with a common boundary $\Sigma$.

where $H^{0,1}_{A^{(c)}}$ are the de Rham cohomology groups with values in the Lie algebra of $G$ and associated to the trivial connection $A^{(c)}$, $\tau^{(c)}_R$ is the Reidemeister-Ray-Singer torsion of $A^{(c)}$, $H_c$ is the isotropy group of $A^{(c)}$, and $\varphi$ is a certain phase. More details about the structure of this term can be found in [92, 32, 48, 83]. The terms $S^{(c)}_\ell$ in (3.4) correspond to connected diagrams at $\ell + 1$ loops, and since they involve evaluation of group factors of Feynman diagrams, they depend explicitly on the gauge group $G$ and the isotropy subgroup $H_c$. In the $SU(N)$ or $U(N)$ case, and for $A^{(c)} = 0$ (the trivial flat connection) they are polynomials in $N$. For the trivial flat connection, one also has that $\dim H^{0}_{A^{(c)}} = \dim G$, $\dim H^{1}_{A^{(c)}} = 0$, and $H_c = G$. The terms $S^{(c)}_\ell$ are also topological invariants associated to the three-manifold and the flat connection, and they emerge naturally from the perturbative analysis of Chern-Simons theory.

As Witten showed in [92], it is also possible to use nonperturbative methods to obtain a combinatorial formula for (3.2). This goes as follows. By canonical quantization, one associates a Hilbert space $\mathcal{H}(\Sigma)$ to any two-dimensional compact manifold that arises as the boundary of a three-manifold, so that the path-integral over a manifold with boundary gives a state in the corresponding Hilbert space. In order to compute the partition function of a three-manifold $M$, one can perform a Heegard splitting i.e. represent $M$ as the connected sum of two three-manifolds $M_1$ and $M_2$ sharing a common boundary $\Sigma$, where $\Sigma$ is a Riemann surface. If $f : \Sigma \to \Sigma$ is a homeomorphism, we will write $M = M_1 \cup_f M_2$, so that $M$ is obtained by gluing $M_1$ to $M_2$ through their common boundary by using the homeomorphism $f$. This is represented in Fig. 1. We can then compute the full path integral (3.2) over $M$ by computing first the path integral over $M_1$ and $M_2$. This produces two wavefunctions $|\Psi_{M_1}\rangle, |\Psi_{M_2}\rangle$ in $\mathcal{H}(\Sigma)$. On the other
hand, the homeomorphism $f : \Sigma \to \Sigma$ will be represented by an operator in the Hilbert space,

$$U_f : \mathcal{H}(\Sigma) \to \mathcal{H}(\Sigma)$$

and the partition function can then be evaluated as

$$Z_k(M) = \langle \Psi_{M_2} | U_f | \Psi_{M_1} \rangle.$$  \hspace{1cm} (3.8)

In order to use this method, we have to find first the Hilbert space associated to a boundary. There is one special case in which this can be done quite systematically, namely when $\Sigma = T^2$, a two-torus. As it was first shown in [22] (and worked out in detail in [28, 64, 59]), the states of the Hilbert space of Chern-Simons theory associated to the torus, $\mathcal{H}(T^2)$, are in one to one correspondence with the integrable representations of the Wess-Zumino-Witten (WZW) model with gauge group $G$ at level $k$. \footnote{We will use the following notations in the following: the fundamental weights of $G$ will be denoted by $\lambda_i$, the simple roots by $\alpha_i$, with $i = 1, \cdots, r$, and $r$ denotes the rank of $G$. The weight and root lattices of $G$ are denoted by $\Lambda_w$ and $\Lambda_r$, respectively, and $|\Delta_+|$ denotes the number of positive roots.}

A representation given by a highest weight $\Lambda$ is integrable if the weight $\rho + \Lambda$ is in the Weyl alcove $\mathcal{F}_l$, where $l = k + y$ and $\rho$ denotes as usual the Weyl vector, given by the sum of the fundamental weights. The Weyl alcove is given by $\Lambda \omega / l \Lambda$ modded out by the action of the Weyl group. For example, in $SU(N)$ a weight $p = \sum_{i=1}^r p_i \lambda_i$ is in $\mathcal{F}_l$ if

$$\sum_{i=1}^r p_i < l, \quad \text{and} \quad p_i > 0, \; i = 1, \cdots, r.$$  \hspace{1cm} (3.9)

In the following, the basis of integrable representations will be labelled by the weights in $\mathcal{F}_l$, and the states in the Hilbert state of the torus $\mathcal{H}(T^2)$ will be denoted by $|p\rangle = |\rho + \Lambda\rangle$ where $\Lambda$, as we have stated, is an integrable representation of the WZW model at level $l$. The states $|p\rangle$ can be chosen to be orthonormal [28, 64, 59].

There is a special class of homeomorphisms of $T^2$ that have a simple expression as operators in $\mathcal{H}(T^2)$. These are $\text{SL}(2, \mathbb{Z})$ transformations, whose generators $T$ and $S$ have the following simple matrix elements in the above basis:

$$T_{\alpha\beta} = \delta_{\alpha\beta} e^{2\pi i (h_{\alpha} - c/24)},$$

$$S_{\alpha\beta} = \frac{i^{|\Delta_+|}}{(k+y)^{r/2}} \left( \frac{\text{Vol} \Lambda_w}{\text{Vol} \Lambda_r} \right)^{\frac{1}{2}} \sum_{w \in \mathcal{W}} \epsilon(w) \exp \left( -\frac{2\pi i}{k+y} \alpha \cdot w(\beta) \right).$$  \hspace{1cm} (3.10)

In the first equation, $h_\alpha$ is the conformal weight of the primary field associated to $\alpha$:

$$h_\alpha = \frac{\alpha^2 - \rho^2}{2(k+y)}.  \hspace{1cm} (3.11)$$
and \( c \) is the central charge of the WZW model. In the second equation, the sum over \( w \) is a sum over the elements of the Weyl group \( \mathcal{W} \), and \( \epsilon(w) \) is the signature of \( w \). These explicit formulae allow us to compute the partition function of any three-manifold that admits a Heegard splitting along a torus, like for example a lens space. The case of \( S^3 \) is particularly simple. It is well-known that \( S^3 \) can be obtained by gluing two solid tori along their boundaries through an \( S^0 \) transformation. The wavefunction associated to the solid torus is simply the vacuum, which corresponds to \( |\rho\rangle \), and we find

\[
Z(S^3) = \langle \rho | S | \rho \rangle = S_{\rho\rho}.
\]

By using Weyl’s denominator formula,

\[
\sum_{w \in \mathcal{W}} \epsilon(w) e^{w(\rho)} = \prod_{\alpha > 0} \frac{2 \sinh \frac{\alpha}{2}}{\alpha},
\]

one finds

\[
Z(S^3) = \frac{1}{(k + y)^{r/2}} \left( \frac{\text{Vol} \Lambda^{w}}{\text{Vol} \Lambda^{r}} \right)^{\frac{r}{2}} \prod_{\alpha > 0} 2 \sin \left( \frac{\pi (\alpha \cdot \rho)}{k + y} \right).
\]

Besides providing invariants of three-manifolds, Chern-Simons theory also provides invariants of knots and links inside three-manifolds (for a survey of modern knot theory, see [68, 80]). Some examples of knots and links are depicted in Fig. 2. When dealing with knots, we will always consider that the Chern-Simons gauge group is \( G = SU(N) \) or \( U(N) \). Given a knot \( K \) in \( S^3 \), we can consider the trace of the holonomy of the gauge connection around \( K \) in a given irreducible representation \( R \) of \( SU(N) \), which gives the Wilson loop operator:

\[
W^K_R(A) = \text{Tr}_R \left( \text{P exp} \oint_{\gamma} A \right),
\]

where \( \text{P} \) denotes path-ordered exponential. This is a gauge invariant operator whose definition does not involve the metric on the three-manifold. The irreducible representations of \( SU(N) \) can be labelled by highest weights or equivalently by the lengths of rows in a Young tableau, \( l_i \), where \( l_1 \geq l_2 \geq \cdots \). If we now consider a link \( L \) with components \( K_i, i = 1, \cdots, L \), we can in principle compute the correlation function,

\[
W_{(R_1, \cdots, R_L)}(L) = \langle W^K_{R_1} \cdots W^K_{R_L} \rangle = \frac{1}{Z(M)} \int [DA] \left( \prod_{i=1}^{L} W^K_{R_i} \right) e^{iS}.
\]

The topological character of the action, and the fact that the Wilson loop operators can be defined without using any metric on the three-manifold, indicate that (3.16) is a topological invariant of the link \( L \). These correlation functions can be studied
Figure 2: Some knots and links. In the notation $x_n^L$, $x$ indicates the number of crossings, $L$ the number of components (in case it is a link with $L > 1$) and $n$ is a number used to enumerate knots and links in a given set characterized by $x$ and $L$. The knot $3_1$ is also known as the trefoil knot, while $4_1$ is known as the figure-eight knot. The link $2_1^2$ is called the Hopf link.
in a variety of ways. The nonperturbative approach pioneered by Witten in \[92\], by exploiting the relation with WZW model, shows that these correlation functions are rational functions of $q^{\pm \frac{1}{2}}, \lambda^{\pm \frac{1}{2}}$, where

\[ q = e^x = \exp\left(\frac{2\pi i}{k + N}\right), \quad \lambda = q^N. \quad (3.17) \]

It turns out that the correlation function (3.16) is the quantum group invariant of the link $L$ associated to the irreducible representations $R_1, \cdots, R_L$ of $U_q(su(N))$ (see for example \[82\] for a general definition of the quantum group invariant).

The invariants of knots and links obtained as correlation functions in Chern-Simons theory include and generalize the HOMFLY polynomial \[33\] (which is a generalization itself of the Jones polynomial). The HOMFLY polynomial of a link $L$, $P_L(q, \lambda)$, can be defined through the so-called skein relation. This goes as follows. Let $L$ be a link in $S^3$, and let us focus on one of the crossings in its plane projection. The crossing can be an overcrossing, like the one depicted in $L_+$ in Fig. 3, or an undercrossing, like the one depicted in $L_-$. If the crossing is $L_+$, we can form two other links either by undoing the crossing (and producing $L_0$ of Fig. 3) or by changing $L_+$ into $L_-$. In both cases the rest of the link is left unchanged. Similarly, if the crossing is $L_-$, we form two links by changing $L_-$ into $L_+$ or into $L_0$. The links produced in this way will be in general topologically inequivalent to the original one (they can even have a different number of components). The skein relation

\[ \lambda^{\frac{1}{2}}P_{L_+} - \lambda^{-\frac{1}{2}}P_{L_-} = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})P_{L_0} \quad (3.18) \]

expresses the HOMFLY polynomial of the original link in terms of the links that are obtained by changing the crossing. By using recursively this relation, one can undo all the crossings and express the polynomial in terms of its value on the unknot, or trivial knot. This value is usually taken to be $P = 1$. The HOMFLY polynomial corresponds to a Chern-Simons $SU(N)$ link invariant with all the components in the fundamental
representation $R_\alpha = \Box$:

$$W_{\Box, \ldots, \Box}(L) = \lambda^{\text{lk}(L)} \left( \frac{\lambda^{\frac{1}{2}} - \lambda^{-\frac{1}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} \right) P_L(q, \lambda)$$  \hspace{1cm} (3.19)

where $\text{lk}(L)$ is the linking number of $L$. This can be shown, as in [92], by proving that the vev in the fundamental representation satisfies the skein relation.

The link invariants defined in (3.16) can be computed in many different ways. A particularly useful framework is the formalism of knot operators [59]. In this formalism, one constructs operators that “create” knots wrapped around a Riemann surface in the representation $R$ of the gauge group associated to the highest weight $\Lambda$:

$$W^K_\Lambda: \mathcal{H}(\Sigma) \rightarrow \mathcal{H}(\Sigma).$$  \hspace{1cm} (3.20)

Notice that the topology of $\Sigma$ restricts the type of knots that one can consider. So far these operators have been constructed in the case when $\Sigma = \mathbb{T}^2$. The knots that can be put on a torus are called torus knots, and they are labelled by two integers $(n, m)$ that specify the number of times that they wrap the two cycles of the torus. Here, $n$ refers to the winding number around the noncontractible cycle of the solid torus, while $m$ refers to the contractible one. The trefoil knot $3_1$ in Fig. 2 is the $(2, 3)$ torus knot, and the knot $5_1$ is the $(2, 5)$ torus knot. The operator that creates the $(n, m)$ torus knot will be denoted by $W^{(n, m)}_\Lambda$, and it has a fairly explicit expression:

$$W^{(n, m)}_\Lambda |\rho\rangle = e^{2\pi i nm h_{\rho+\Lambda}} \sum_{\mu \in M_\Lambda} \exp \left[ -i \pi \mu^2 \frac{nm}{k+N} - 2\pi i \frac{m}{k+N} p \cdot \mu \right] |p + n\mu\rangle.$$  \hspace{1cm} (3.21)

In this equation, $h_{\rho+\Lambda}$ is the conformal weight, and $M_\Lambda$ is the set of weights corresponding to the irreducible representation with highest weight $\Lambda$. This equation allows us to compute the vev of the Wilson loop around a torus knot in $\mathbb{S}^3$ as follows: first of all, one makes a Heegard splitting of $\mathbb{S}^3$ into two solid tori, as we explained before. Then, one puts the torus knot on the surface of one of the solid tori by acting with the knot operator (3.21) on the vacuum $|\rho\rangle$. Finally, one glues together the tori by performing an $S$-transformation. The normalized vev of the Wilson loop is then given by:

$$\langle W^{(n, m)}_\Lambda \rangle = \frac{\langle \rho | SW^{(n, m)}_\Lambda | \rho \rangle}{\langle \rho | S | \rho \rangle}.$$  \hspace{1cm} (3.22)

One can show that [59]

$$W^{(1, 0)}_\Lambda |\rho\rangle = |\rho + \Lambda\rangle.$$  \hspace{1cm} (3.23)

On the other hand, the operator $W^{(1, 0)}_\Lambda$ clearly creates a trivial knot, or unknot, on the torus, therefore the states $|\rho + \Lambda\rangle$ are obtained by doing the path integral over
the solid torus with an insertion of a Wilson loop around the noncontractible loop in the representation $\Lambda$, as shown in [92]. We can now evaluate easily the corresponding Chern-Simons invariant. Using the explicit expression in (3.10), we find:

$$W_{R_\Lambda}(\text{unknot}) = \langle \rho | S \rho \rangle = \frac{\sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+N} \rho \cdot w(\Lambda+\rho)}}{\sum_{w \in W} \epsilon(w) e^{-\frac{2\pi i}{k+N} \rho \cdot w(\rho)}}. \quad (3.24)$$

Using Weyl’s denominator formula, the vacuum expectation value can be written as a character

$$W_{R_\Lambda}(\text{unknot}) = \text{ch}_{\Lambda} \left[ -\frac{2\pi i}{k+N} \rho \right]. \quad (3.25)$$

Moreover, using (3.13), we can finally write

$$W_{R_\Lambda}(\text{unknot}) = \prod_{\alpha>0} \frac{\sin \left( \frac{\pi}{k+N} \alpha \cdot (\Lambda + \rho) \right)}{\sin \left( \frac{\pi}{k+N} \alpha \cdot \rho \right)}. \quad (3.26)$$

Notice that, in the limit $k+N \to \infty$ (i.e. in the semiclassical limit), this becomes the dimension of the representation $R$. For this reason, the above quantity is called the quantum dimension of $R$, denoted by $\text{dim}_q R$. It can be explicitly written as follows. Define the $q$-numbers:

$$[x] = q^{\frac{x}{2}} - q^{-\frac{x}{2}}, \quad [x]_\lambda = \lambda^{\frac{1}{2}} q^{\frac{x}{2}} - \lambda^{-\frac{1}{2}} q^{-\frac{x}{2}}. \quad (3.27)$$

If $R$ has a Young tableau with $c_R$ rows of lengths $l_i$, $i = 1, \cdots, c_R$, then the quantum dimension can be explicitly written as:

$$\text{dim}_q R = \prod_{1 \leq i < j \leq c_R} \frac{[l_i - l_j + j - i]}{[j - i]} \prod_{i=1}^{c_R} \frac{\prod_{v=-i+1}^{l_i} [v]_\lambda}{\prod_{v=1}^{l_i} [v - i + c_R]} \prod_{i=1}^{c_R} [v]_\lambda. \quad (3.28)$$

This gives the Chern-Simons invariant of the unknot in the representation $R$.

What about other torus knots? When acting with the knot operator (3.21) on the vacuum, we get the set of weights $\rho + n\mu$, where $\mu \in M_\Lambda$. These weights will have representatives in the Weyl alcove $F_i$, which can be obtained by a series of Weyl reflections. The set of representatives in $F_i$ will be denoted by $\mathcal{M}(n, \Lambda)$, and it depends on the irreducible representation with highest weight $\Lambda$, and on the integer number $n$. Using the fact that $\rho + n\mu = w(\rho + \xi)$ for some $w \in W$, we conclude that the Chern-Simons invariant of a torus knot $(n, m)$ can be written as:

$$e^{2\pi imh_{\rho+\Lambda}} \sum_{\rho+\xi \in \mathcal{M}(n, \Lambda)} \exp \left[ -\frac{i\pi m}{n(k+N)} \xi \cdot (\xi + 2\rho) \right] \text{ch}_{\xi} \left[ -\frac{2\pi i}{k+N} \rho \right]. \quad (3.29)$$
Notice that, since the representatives $\rho + \xi$ live in $F_i$, the weights $\xi$ can be considered as highest weights for a representation, hence (3.29) makes sense. As an example of this procedure, one can compute the invariant in the fundamental representation. By performing Weyl reflections, one can show that $\mathcal{M}(n, \lambda_1)$ is given by the following weights [60]:

$$\rho + (n - i)\lambda_1 + \lambda_i, \quad i = 1, \cdots, N.$$  (3.30)

The computation of the characters is now straightforward (they are just the quantum dimensions of the weights (3.30)), and one finally obtains:

$$W_{(n,m)} = t^{\frac{n}{2}}\lambda^{-\frac{1}{2}}\left(\frac{(\lambda t - 1)^{(n-1)(n-1)}}{t^n - 1}\right) \sum_{p+i+1=n \atop p,i\geq 0} (-1)^{i}t^{-mi+\frac{1}{2}p(p+1)}\frac{\prod_{j=-p}^{-1}(\lambda - t^j)}{(i)!p!}$$  (3.31)

This is in fact the unnormalized HOMFLY polynomial of an $(n, m)$ torus knot. If we divide by the vev of the unknot, we find the expression for the HOMFLY polynomial first obtained in [49]. For the trefoil one has for example:

$$W_{(1,1)} = \frac{1}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}(-2\lambda^{\frac{1}{2}} + 3\lambda^{\frac{3}{2}} - \lambda^{\frac{5}{2}}) + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})(-\lambda^{\frac{1}{2}} + \lambda^{\frac{3}{2}}).$$  (3.32)

With more effort one can obtain invariants of torus knots and links in arbitrary representations [60, 61, 63]. For the trefoil in representations with two boxes one finds:

$$W_{(2,1)} = \frac{(\lambda - 1)(\lambda q - 1)}{\lambda(q^{\frac{3}{2}} - q^{-\frac{3}{2}})^2(1 + q)}\left((\lambda q - 1)^2(1 - \lambda q^2 + q^3 - \lambda q^4 + q^5 + \lambda q^6)\right)$$

$$W_{(2,2)} = \frac{(\lambda - 1)(\lambda - q)}{\lambda(q^{\frac{3}{2}} - q^{-\frac{3}{2}})^2(1 + q)}\left((\lambda q^{-2})(1 - \lambda - \lambda q + \lambda q^3 + \lambda q^5 + \lambda q^7)\right)$$

$$+ \lambda^2 q + q^2 + q^3 - \lambda q^4 - \lambda q^6\right)$$  (3.33)

For the Hopf link, one finds:

$$W_{(2,2)} = \left(\frac{\lambda^{\frac{3}{2}} - \lambda^{-\frac{3}{2}}}{q^{\frac{3}{2}} - q^{-\frac{3}{2}}}\right)^2 - \lambda^{-1}(\lambda - 1),$$  (3.34)

which can be also easily obtained using the skein relations of the HOMFLY polynomial (3.18) together with (3.19).

### 3.2 Framing dependence

In the above discussion on the correlation functions of Wilson loops we have missed an important ingredient. We mentioned that, in order to define the partition function
of Chern-Simons theory at the quantum level, one has to specify a framing of the three-manifold. It turns out that the evaluation of correlation functions like (3.16) also involves a choice of framing of the knots, as Witten discovered in [92]. Since this is important in the duality with topological strings, we will explain it in some detail.

A good starting point to understand the framing is to take Chern-Simons theory with gauge group $U(1)$. This is also useful to understand $U(N)$ versus $SU(N)$ Chern-Simons theory, and to get a concrete feeling of how to deal with correlation functions like (3.16). The Abelian Chern-Simons theory turns out to be extremely simple, since the cubic term in (3.1) drops out, and we are left with a Gaussian theory [79]. The different representations are labelled by integers, and in particular the vevs of Wilson loop operators can be computed exactly. In order to compute them, however, one has to choose a framing for each of the knots $K_i$. This arises as follows: in evaluating the vev, contractions of the holonomies corresponding to different $K_i$ produce the following integral:

$$\text{lk}(K_i, K_j) = \frac{1}{4\pi} \oint_{K_i} dx^\mu \oint_{K_j} dy^\nu \epsilon_{\mu\nu\rho}(x - y)^\rho/|x - y|^3.$$  \hspace{1cm} (3.35)

This is in fact a topological invariant, i.e. it is invariant under deformations of the knots $K_i, K_j$, and it is in fact their linking number $\text{lk}(K_i, K_j)$. On the other hand, contractions of the holonomies corresponding to the same knot $K$ involve the integral

$$\phi(K) = \frac{1}{4\pi} \oint_K dx^\mu \oint_K dy^\nu \epsilon_{\mu\nu\rho}(x - y)^\rho/|x - y|^3.$$  \hspace{1cm} (3.36)

This integral is well-defined and finite (see, for example, [42]), and it is called the cotorsion of $K$. The problem is that the cotorsion is not invariant under deformations of the knot. In order to preserve topological invariance one has to choose another definition of the composite operator $(\int_K A)^2$ by means of a framing. A framing of the knot consists of choosing another knot $K_f$ around $K$, specified by a normal vector field $n$. The cotorsion $\phi(K)$ becomes then

$$\phi_f(K) = \frac{1}{4\pi} \oint_K dx^\mu \oint_{K_f} dy^\nu \epsilon_{\mu\nu\rho}(x - y)^\rho/|x - y|^3 = \text{lk}(K, K_f).$$  \hspace{1cm} (3.37)

The correlation function that we obtain in this way is a topological invariant (a linking number) but the price that we have to pay is that our regularization depends on a set of integers $p_i = \text{lk}(K_i, K_f)$ (one for each knot). The vev (3.16) in the Abelian case can now be computed, after choosing the framings, as follows:

$$\langle \prod_i \exp(n_i \int_{\gamma_i} A) \rangle = \exp \left( \frac{\pi i}{\hbar} \sum_i n_i^2 p_i + \frac{\pi i}{\hbar} \sum_{i \neq j} n_i n_j \text{lk}(K_i, K_j) \right).$$  \hspace{1cm} (3.38)
This regularization is nothing but the ‘point-splitting’ method familiar in the context of QFT’s.

Let us now consider Chern-Simons theory with gauge group $SU(N)$, and suppose that you want to compute a correlation function like (3.16). If you try to do it in perturbation theory, for example, you will find very soon that self-contractions of the holonomies lead to the same kind of ambiguities that we found in the Abelian case, i.e. you will have to make a choice of framing for each knot $K_i$. The only difference is that the self contraction comes with a group factor $\text{Tr}_{R_i}(T_aT_a)$ for each knot $K_i$, where $T_a$ is a basis of the Lie algebra $[42]$. The precise result can be better stated as the effect on the correlation function (3.16) under a change of framing, and it says that, under a change of framing of $K_i$ by $p_i$ units, the vev of the product of Wilson loops changes as follows $[92]$:

$$W(R_1, \ldots, R_L) \rightarrow \exp \left[ 2\pi i \sum_i p_i h_{R_i} \right] W(R_1, \ldots, R_L),$$

(3.39)

In this equation, $h_R$ is the conformal weight of the WZW primary field corresponding to the representation $R$. In (3.11) we labelled $R$ through $\alpha = \rho + \Lambda$, where $\Lambda$ is the highest weight of $R$. In fact, one can write (3.11) as

$$h_R = \frac{C_R}{2(k+N)},$$

(3.40)

where $C_R = \text{Tr}_R(T_aT_a)$ is the quadratic Casimir in the representation $R$. For $SU(N)$, one has

$$C_R^{SU(N)} = N\ell + \kappa_R - \frac{\ell^2}{N},$$

(3.41)

where $\ell$ is the total number of boxes in the tableau, and

$$\kappa_R = \ell + \sum_i (l_i^2 - 2il_i).$$

(3.42)

We then see that the evaluation of vacuum expectation values of Wilson loop operators in Chern-Simons theory depends on a choice of framing for knots. It turns out that for knots and links in $S^3$, there is a standard or canonical framing, defined by requiring that the self-linking number is zero. The expressions listed in (3.33) and (3.34) are all in the standard framing, and the skein relations for the HOMFLY polynomial produce invariants in the standard framing as well. Once the value of the invariant is known in the standard framing, the value in any other framing specified by nonzero integers $p_i$ can be easily obtained from (3.39).

Let us now consider a $U(N)$ Chern-Simons theory. The $U(1)$ factor decouples from the $SU(N)$ theory, and all the vevs factorize into an $U(1)$ and an $SU(N)$ piece. Representations of $U(N)$ are also labelled by Young tableaux, and they decompose into a
representation of $SU(N)$ corresponding to that tableau, and a representation of $U(1)$ with charge:
\[ n = \frac{\ell}{\sqrt{N}}, \]  
(3.43)
where $\ell$ is the number of boxes in the Young tableau. In order to compute the vevs associated to the $U(1)$ of $U(N)$, one has to take also into account that the coupling constant $k$ is shifted as $k \rightarrow k + N$. We then find that the vev of a product of $U(N)$ Wilson loops in representations $R_i$ is given by:
\[ W_{U(N)}^{(R_1, \ldots, R_L)} = \exp\left(\frac{\pi i}{N(k+N)} \sum_i \ell_i^2 p_i + \frac{\pi i}{N(k+N)} \sum_{i \neq j} \ell_i \ell_j \text{lk}(K_i, K_j)\right) W_{SU(N)}^{R_i}, \]  
(3.44)
where the $SU(N)$ vev is computed in the framing specified by $p_i$. Notice that, in the case of knots, the $SU(N)$ and $U(N)$ computations differ in a factor which only depends on the choice of framing, while for links the answers also differ in a topological piece involving the linking numbers. The change of framing for vacuum expectation values in the $U(N)$ theory is again governed by (3.39) and (3.40), but now the quadratic Casimir is given by
\[ C_{R}^{U(N)} = N\ell + \kappa_{R}, \]  
(3.45)
Notice that the difference between the change of $SU(N)$ and $U(N)$ vevs under the change of framing is consistent with (3.44). In terms of the variables (3.17) we see that $U(N)$ vevs change, under the change of framing, as
\[ W_{(R_1, \ldots, R_L)} \rightarrow q^{\frac{1}{2} \sum_i \kappa_{R_i} p_i} \lambda^{\frac{1}{2} \sum_i \ell_i p_i} W_{(R_1, \ldots, R_L)}. \]  
(3.46)

### 3.3 Generating functionals for Wilson loops

As we will see, the relation between Chern-Simons theory and string theory involves the vacuum expectation values for arbitrary irreducible representations of $U(N)$, so it is convenient to have a generating functional that encodes all the information about them. We will for simplicity consider the case in which one has just a single knot. We then have to find a suitable basis for the Wilson loop operators. There are two natural basis for the problem: the basis labelled by representations $R$, and the basis labelled by conjugacy classes $C(\vec{k})$ of the symmetric group. Let $U$ be the holonomy of the gauge connection around the knot $K$, and consider the operator $\Upsilon_{\vec{k}}(U)$ defined as in (2.27). The vevs of these operators give the “$\vec{k}$-basis” for the vacuum expectation values of the Wilson loops:
\[ W_{\vec{k}} = \langle \Upsilon_{\vec{k}}(U) \rangle = \sum_R \chi_R(C(\vec{k})) W_R \]  
(3.47)
where $\chi_R$ are characters of the permutation group $S_\ell$ in the representation $R$, and we have used Frobenius formula

$$\text{Tr}_R(U) = \sum_{\vec{k}} \frac{\chi_R(C(\vec{k}))}{z_{\vec{k}}} \Upsilon_{\vec{k}}(U),$$

(3.48)

and $z_{\vec{k}}$ has been defined in (2.42). If $V$ is a $U(M)$ matrix (a “source” term), one can define the following operator, which was introduced in [76] and is known sometimes as the Ooguri-Vafa operator:

$$Z(U, V) = \exp \left[ \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} U^n \text{Tr} V^n \right].$$

(3.49)

When expanded, this operator can be written in the $k$-basis as follows,

$$Z(U, V) = 1 + \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \Upsilon_{\vec{k}}(U) \Upsilon_{\vec{k}}(V).$$

(3.50)

We see that $Z(U, V)$ includes all possible Wilson loop operators $\Upsilon_{\vec{k}}(U)$ associated to a knot $K$. One can also use Frobenius formula to show that

$$Z(U, V) = \sum_R \text{Tr}_R(U) \text{Tr}_R(V),$$

(3.51)

where the sum over representations starts with the trivial one. In $Z(U, V)$ we assume that $U$ is the holonomy of a dynamical gauge field and that $V$ is a source. The vacuum expectation value $Z(V) = \langle Z(U, V) \rangle$ has then information about the vevs of the Wilson loop operators, and by taking its logarithm one can define the connected vacuum expectation values $W^{(c)}_{\vec{k}}$:

$$F_{CS}(V) = \log Z(V) = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}!} W^{(c)}_{\vec{k}} \Upsilon_{\vec{k}}(V)$$

(3.52)

One has, for example:

$$W^{(c)}_{(2,0,\cdots)} = \langle (\text{Tr} U)^2 \rangle - \langle \text{Tr} U \rangle^2 = W_{\Box} + W_{\Box} - W_{\Box}.$$

The free energy $F_{CS}(V)$, which is a generating functional for connected vevs $W^{(c)}_{\vec{k}}$, will be the relevant object for the duality with topological strings.
4 Chern-Simons theory and large $N$ transitions

4.1 The $1/N$ expansion

As ’t Hooft pointed out in [86] (see [21] for a nice review), given a theory with $U(N)$ or $SU(N)$ gauge symmetry one can always perform a $1/N$ expansion of the free energy and the correlation functions. To do that, one writes the Feynman diagrams of the theory as “fatgraphs” or ribbon graphs. The amplitude associated to these ribbon graphs depends on the coupling constant $x$ and on the rank of the gauge group (through its group factor). Let us consider for example the expansion of the free energy. This will involve connected vacuum bubbles with $V$ vertices, $E$ propagators and $h$ loops of internal indices, and therefore will have a factor

$$x^{E-V}N^h = x^{2g-2+h}N^h = x^{2g-2}t^h,$$

(4.1)

where $t = Nx$ is the so called ’t Hooft parameter. In writing this equation we regard the fatgraph as a Riemann surface with holes, i.e. each internal loop represents the boundary of a hole, and we used Euler’s relation $E - V + h = 2g - 2$. In Fig. 4 we show a fatgraph with $g = 1$ and $h = 9$, and in Fig. 5 the Riemann surface that can be associated to it. We can then write,

$$F^p = \sum_{g=0}^{\infty} \sum_{h=1}^{\infty} F^p_{g,h}x^{2g-2}t^h.$$

(4.2)

The superscript $p$ means that this is the perturbative contribution to the free energy. The full free energy may also have a nonperturbative contribution. This is easily seen, in the case of Chern-Simons theory, in (3.4): the free energy has a perturbative contribution coming from the $S_\ell$, but there is a nonperturbative contribution due to the one-loop prefactor (which also depends on $N$, $x$) and involves one-loop determinants.

Figure 4: This figure, taken from [77], shows a fatgraph with $h = 9$ and $g = 1$. 
as well as the precise normalization of the path integral. In (4.2) we have written the diagrammatic series as an expansion in \( x \) around \( x = 0 \), keeping \( t \) fixed. Equivalently, we can regard it as an expansion in \( 1/N \) for fixed \( t \), and then the \( N \) dependence appears as \( N^{2g-2} \). The above expansion can be interpreted as the perturbative expansion of an open string theory, where \( F^p_{g,h} \) corresponds to some amplitude on a Riemann surface of genus \( g \) with \( h \) holes like the one depicted in Fig. 5. If we now introduce the function

\[
F^p_g(t) = \sum_{h=1}^{\infty} F^p_{g,h} t^h, \tag{4.3}
\]

the total perturbative free energy can be written as

\[
F^p(x, t) = \sum_{g=0}^{\infty} x^{2g-2} F_g^p(t), \tag{4.4}
\]

which looks like a closed string expansion where \( t \) is some modulus of the theory. Notice that in writing (4.3) we have grouped together all open Riemann surfaces with the same bulk topology but with different number of holes, so by “summing over all holes” we “fill up the holes” to produce a closed Riemann surface. This leads to ’t Hooft’s idea [50] that, given a gauge theory, one should be able to find a string theory interpretation in the way we have described, namely, the fatgraph expansion of the free energy is resummed to give a function of the ’t Hooft parameter \( F_g(t) \) at every genus that is then interpreted as a closed string amplitude.

We can now ask what is the interpretation of the vacuum expectation values of Wilson loop operators in this context. Using standard large \( N \) techniques (as reviewed for example in [21]), it is easy to see that the vevs that have a well-defined behavior in the \( 1/N \) expansion are the connected vevs \( W_k^{(c)} \) introduced in (3.32). One finds that
these vevs admit an expansion of the form,

\[ W^{(c)}_{\vec{k}} = \sum_{g=0}^{\infty} W_{g,\vec{k}}(t) x^{2g-2+|\vec{k}|}. \]  

(4.5)

This can be regarded as an open string expansion, where \( W_{g,\vec{k}}(t) \) are interpreted as amplitudes in an open string theory at genus \( g \) and with \( h = |\vec{k}| \) holes. The vector \( \vec{k} \) specifies the winding numbers of the holes around a one-cycle in the target space of the theory, according to the rule we gave in subsection 2.3. We could say that the Wilson loop “creates” a one-cycle in the target space where the boundaries of Riemann surfaces can end, and the generating functional for connected vevs (3.52) is interpreted as the total free energy of an open string, as in (2.24). These open strings shouldn’t be confused with the ones that we associated to the expansion (4.2). The open strings underlying (4.5) should be regarded as an open string sector in the closed string theory associated to the resummed expansion (4.4).

This is then the program to interpret gauge theories with \( U(N) \) or \( SU(N) \) symmetry in terms of a string theory. So far this program has been led to completion in just a few examples. A first example is a class of gauge theories in zero dimensions, the matrix models of Kontsevich, which are equivalent to topological minimal matter in two dimensions coupled to gravity [54], i.e. to topological strings in \( d < 1 \) dimensions. Another example is Yang-Mills theory in two dimensions, which also has a string theory description [41, 22]. Finally, \( N = 4 \) supersymmetric Yang-Mills theory is equivalent to type IIB string theory on \( S^5 \times \text{AdS}_5 \) [6]. The last example shows very clearly that the target of the string theory is not necessarily the spacetime where the gauge theory lives, and that the string description may need “extra” dimensions. The question we want to address now is the following: is there a string description of Chern-Simons theory? As we will see, at least for Chern-Simons on the three-sphere, the answer is yes. The resulting description provides a very nice realization of ’t Hooft ideas, and as we will show, leads to new insights on knot and link invariants\(^2\).

### 4.2 Chern-Simons theory as an open string theory

In order to give a string theory interpretation of Chern-Simons theory on \( S^3 \), a good starting point is to give an open string interpretation to the \( 1/N \) expansion of the free energy (4.2). This was done by Witten in [95], and we will summarize here the main points of the argument.

\(^2\)Other attempts to find a string theory interpretation of Chern-Simons theory can be found in [78, 27].
First of all, we have to recall that open bosonic strings have a spacetime description in terms of the cubic open string field theory introduced in [90]. The action of this theory is given by

\[ S = \frac{1}{g_s} \int \left( \frac{1}{2} \Psi \star Q_{\text{BRST}} \Psi + \frac{1}{3} \Psi \star \Psi \star \Psi \right). \] (4.6)

In this equation, \( \Psi \) is the string field, \( \star \) is the associative, noncommutative product obtained by gluing strings, and the integration is a map \( \int : \Psi \to \mathbb{R} \) that involves the gluing of the two halves of the string field (more details can be found in [90]). If we add Chan-Paton factors, the string field is promoted to a \( U(N) \) matrix of string fields, and the integration includes \( \text{Tr} \). This action has all the information about the spacetime dynamics of open bosonic strings, with or without D-branes. In particular, one can derive the Born-Infeld action describing the dynamics of D-branes from this cubic string field theory (see for example [85]).

Consider now a three-manifold \( M \). The total space of its cotangent bundle \( T^*M \) is a noncompact Calabi-Yau manifold. Moreover, it is easy to see that \( M \) is a Lagrangian submanifold in \( T^*M \). We can then consider a system of \( N \) topological D-branes wrapping \( M \), thus providing Dirichlet boundary conditions for the open strings. We want to obtain a spacetime action describing the dynamics of these topological D-branes. To do this, we can exploit again the analogy between open topological strings and the open bosonic string that we used to define the coupling of topological sigma models to gravity (i.e., that both have a nilpotent BRST operator and an energy-momentum tensor that is \( Q_{\text{BRST}} \)-exact). Using the fact that both theories have a similar structure, one can argue [95] that the dynamics of topological D-branes in \( T^*M \) is governed as well by (4.6). However, one has to work out what is exactly the string field, the \( \star \) algebra and so on in the context of topological open strings. It turns out that the string field is simply a \( U(N) \) gauge connection \( A \) on \( M \), the integration of string functionals becomes ordinary integration of forms on \( M \), and the star product becomes the usual wedge product of forms. We then have the following dictionary:

\[ \Psi \to A, \quad Q_{\text{BRST}} \to d \]

\[ \star \to \wedge, \quad \int \to \int_M. \] (4.7)

The resulting action (4.6) is then the usual Chern-Simons action, and we have the following relation between the string coupling constant and the Chern-Simons coupling

\[ g_s = \frac{2\pi}{k+N}, \] (4.8)

after accounting for the usual shift \( k \to k+N \). Notice that, in the open bosonic string, the string field involves an infinite tower of string excitations. For the open
topological string, the topological character of the model implies that all excitations are $Q$-exact (and therefore decouple), except for the lowest lying one, which is a $U(N)$ gauge connection. In other words, the usual reduction to a finite number of degrees of freedom that occurs in topological theories downsizes the string field to a single excitation.

The topological open string theory that we are obtaining has some important differences with the one that we described in section 2. As Witten pointed out in [95], there are no honest worldsheet instantons in this geometry! To be precise, worldsheet instantons whose boundaries lie in $M$ must have zero area, and one would then conclude that the only contributions come from constant maps. A detailed analysis shows however that there are nontrivial worldsheet instantons contributing to the path integral, but they are degenerate and belong to the boundary of the moduli space of holomorphic maps. These degenerate instantons look just like fatgraphs, and in fact they correspond to the Feynman diagrams of the $1/N$ expansion of Chern-Simons theory! In particular, to characterize topologically these degenerate instantons we just need their genus $g$ and number of holes $h$, which are of course the same ones of the associated fatgraph. There are no winding numbers to specify.

The outcome of this discussion is that, for topological open strings on noncompact Calabi-Yau manifolds of the form $T^*M$, the dynamics is governed by the usual Chern-Simons action on $M$. In particular, the coefficient $F_{g,h}^p$ in (4.2) can be interpreted as the free energy of an open string of genus $g$ and $h$ holes propagating on $T^*M$ and with Lagrangian boundary conditions specified by $M$.

This result can be extended [95], and the more general picture will be extremely useful later on. Consider a Calabi-Yau manifold $X$ together with some Lagrangian submanifolds $M_i \subset X$, with $N_i$ D-branes wrapped over $M_i$. In this case the topological open strings will have contributions from degenerate holomorphic curves, which are captured by Chern-Simons theories in the way we explained for $T^*M$, as well as some honest holomorphic curves. As shown in [95], these honest holomorphic curves are open Riemann surfaces whose boundaries are embedded knots inside the three-manifolds $M_i$ and give rise to Wilson loops. Each holomorphic curve with area $A$ ending on the knot $K_i$ will contribute $e^{-A} \prod_i \text{Tr} U_{K_i}$ to the free energy, where $U_{K_i}$ denotes the holonomy of the Chern-Simons $U(N_i)$ gauge connection $A_i$ around the knot $K_i$. We can then take into account the contributions of all curves by including the corresponding Chern-Simons theories $S_{CS}(A_i)$, which account for the degenerate curves, coupled in an appropriate way to the honest holomorphic curves. The spacetime action will then have the form

$$S(A_i) = S_{CS}(A_i) + F_{ndg}(U_{K_i}) \quad (4.9)$$
Figure 6: This figure shows a partially degenerated worldsheet instanton of genus $g = 0$ and with $h = 3$ ending on an unknot. The instanton is made out of a honest holomorphic disk and the degenerate piece, which is a fatgraph.

where

$$F_{\text{ndg}} = \sum_{\text{instantons}} e^{-A} \prod_i \text{Tr} U_{K_i}$$

(4.10)

denotes the contribution of the non-degenerate holomorphic curves, and it is a sum over honest open worldsheet instantons. Notice that all the Chern-Simons theories $S_{\text{CS}}(A_i)$ have the same coupling constant, equal to the string coupling constant. More precisely,

$$\frac{2\pi}{k_i + N_i} = g_s.$$  

(4.11)

In the action (4.10), the honest holomorphic curves are put “by hand” in $F_{\text{ndg}}$, and in principle one has to solve a nontrivial enumerative problem to find them. Once they are included in the action, the path integral over the Chern-Simons connections will join degenerate instantons to these honest worldsheet instantons: if we have a nondegenerate worldsheet instanton ending on a knot $K$, it will give rise to a Wilson loop operator in (4.10), and the evaluation of the vacuum expectation value will generate, in the $1/N$ expansion, all possible fatgraphs $\Gamma$ joined to the knot $K$, as it is well-known in Chern-Simons perturbation theory in the presence of Wilson loops (see for example [56]). These fatgraphs are interpreted as degenerate instantons. Therefore, the path integral with the action (4.9) will be a sum of contributions coming from partial degenerations of Riemann surfaces, in which a surface $\Sigma_{g,h}$ degenerates to another surface $\Sigma'_{g',h'}$ whose boundary ends on a knot $K$, together with a fatgraph whose external legs end in $K$ as well. An example of this situation is depicted in Fig. 6, where a disc ends on an unknot, and the fatgraph generated by Chern-Simons perturbation theory gives in the end a Riemann surface of $g = 0$ and $h = 3$. This more complicated scenario was explored in [5, 24, 25, 8], and we will provide examples of (4.9) in section 6.
4.3 The conifold transition

We have learned that Chern-Simons theory on $S^3$ is a topological open string theory on $T^*S^3$. Notice that the target of the string theory is different from (and has higher dimensionality than) the spacetime of the gauge theory, as in the string description of $\mathcal{N} = 4$ Yang-Mills theory. The next step is to see if there is a closed string theory leading to the resummation (4.4). As shown by Gopakumar and Vafa in an important paper [37], the answer is yes.

One way to motivate their result is as follows: since the holes of the Riemann surfaces are due to the presence of D-branes, “filling the holes” to get the closed strings means getting rid of the D-branes. But this is precisely what happens in the AdS/CFT correspondence [8], where type IIB theory in flat space in the presence of D-branes is conjectured to be equivalent to type IIB theory in AdS$_5 \times S^5$ with no D-branes. The reason for that is that, at large $N$, the presence of the D-branes can be traded by a deformation of the background geometry, and the radius of the $S^5$ is related to the number of D-branes. In other words, we can make the branes disappear if we change the background geometry at the same time. As Gopakumar and Vafa have pointed out, large $N$ dualities relating open and closed strings should involve transitions in the geometry. This reasoning suggests to look for a transition involving the background $T^*S^3$. It turns out that such a transition is well-known in the physical and the mathematical literature, and it is called the conifold transition (see for example [18]). Let us explain this in detail.

Although we have regarded $T^*S^3$ as the total space of the cotangent space bundle of the three-sphere, this background can be also regarded as the deformed conifold geometry, which is usually described by the algebraic equation

$$\sum_{\mu=1}^{4} \eta^2_{\mu} = a. \quad (4.12)$$

To see this equivalence, let us write $\eta_{\mu} = x_{\mu} + ip_{\mu}$, where $x_{\mu}, p_{\mu}$ are real coordinates. We find the two equations

$$\sum_{\mu=1}^{4} (x^2_{\mu} - p^2_{\mu}) = a,$$

$$\sum_{\mu=1}^{4} x_{\mu} p_{\mu} = 0. \quad (4.13)$$

The first equation indicates that the locus $p_{\mu} = 0, \mu = 1, \cdots, 4$, describes a sphere $S^3$ of radius $R^2 = a$, and the second equation shows that the $p_{\mu}$ are coordinates for the cotangent space. Therefore, (4.12) is nothing but $T^*S^3$. 

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It is useful to rewrite the deformed conifold in yet another way. Introduce the following complex coordinates:

\[ x = \eta_1 + i\eta_2, \quad v = i(\eta_3 - i\eta_4), \]
\[ u = i(\eta_3 + i\eta_4), \quad y = \eta_1 - i\eta_2. \]  
(4.14)

The deformed conifold can be now written as

\[ xy = uv + a. \]  
(4.15)

Notice that in this parameterization the geometry has a \( T^2 \) fibration

\[ x, y, u, v \rightarrow xe^{i\theta_a}, ye^{-i\theta_a}, ue^{i\theta_b}, ve^{-i\theta_b} \]  
(4.16)

where the \( \theta_a \) and \( \theta_b \) actions above can be taken to generate the (1, 0) and (0, 1) cycles of the \( T^2 \). The \( T^2 \) fiber can degenerate to \( S^1 \) by collapsing one of its one-cycles. In the equation above, for example, the \( U(1)_a \) action fixes \( x = 0 = y \) and therefore fails to generate a circle there. In the total space, the locus where this happens, i.e. the \( x = 0 = y \) subspace of \( X \), is a cylinder \( uv = -a \). Similarly, the locus where the other circle collapses, \( u = 0 = v \), gives another cylinder \( xy = a \). Therefore, we can regard the whole geometry as a \( T^2 \times \mathbb{R} \) fibration over \( \mathbb{R}^3 \): if we define \( z = uv \), the \( \mathbb{R}^3 \) of the base is given by \( \text{Re}(z) \) and the axes of the two cylinders. The fiber is given by the circles of the two cylinders, and by \( \text{Im}(z) \). It is very useful to represent the above geometry by depicting the singular loci of the torus action in the base \( \mathbb{R}^3 \). The loci where the cycles of the torus collapse, which are cylinders, project to lines in the base space. Notice that \( S^3 \) can be regarded as a torus fibration over an interval, with singular loci at the endpoints. In Fig. 7 the three-sphere of the deformed conifold geometry is represented by a dashed line in the \( z \)-plane between \( z = 0 \) and \( z = -a \), together with the \( \theta_a \) and the \( \theta_b \) circles that degenerate over the endpoints.

The conifold singularity appears when \( a = 0 \) and the three-sphere collapses. This is described by the equation:

\[ xy = uv. \]  
(4.17)

In algebraic geometry, singularities can be avoided in two ways, in general. The first way is to deform the complex geometry. This leads in our case to the deformed conifold (4.12). The other way is to resolve the singularity, for example by performing a blow up, and this leads to the resolved conifold geometry (see for example [18]). The resolution of the geometry can be explained as follows. When \( a = 0 \), (4.15) says that \( xy = uv \). We can solve (4.17) by setting

\[ x = \lambda v, \quad u = \lambda y \]  
(4.18)
where $\lambda$ is regarded as an inhomogeneous coordinate in $\mathbb{P}^1$. The space described by the complex coordinates $x, y, \lambda, u, v$ together with the relations (4.18) is the resolved conifold, and it turns out to be the bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1$, as one can see from (4.18) [18]. To make contact with the toric description given in (2.35), we put $x = x_1 x_2$, $y = x_3 x_4$, $u = x_1 x_3$ and $v = x_2 x_4$. We then see that $\lambda = x_1 / x_4$ is the inhomogeneous coordinate for the $\mathbb{P}^1$ described in (2.35) by $|x_1|^2 + |x_4|^2 = s$. It is instructive to represent the resolved conifold by solving the constraint (2.35) in the first octant of $\mathbb{R}^3$, and depicting the fixed point locus of the isometries above. In terms of the coordinates $x_1, \ldots, x_4$, the $T^2$ action (4.16) is given by

$$x_1, x_2, x_3, x_4 \rightarrow e^{i(\theta_a + \theta_b)} x_1, e^{-i\theta_a} x_2, e^{-i\theta_b} x_3, x_4,$$  

and the fixed loci are depicted in Fig. 8. In the conifold transition, the three-sphere of the deformed conifold shrinks to zero size as $a$ goes to zero, and then a two-sphere of size $s$ blows up giving the resolved conifold.

We know that Chern-Simons theory is an open topological string on the deformed conifold geometry with $N$ topological D-branes wrapping the three-sphere. The conjecture of Gopakumar and Vafa is that at large $N$ the D-branes induce a conifold transition in the background geometry, so that we end up with the resolved conifold and no D-branes. But in the absence of D-branes that enforce boundary conditions we just have a theory of closed topological strings. Therefore, Chern-Simons theory on $S^3$ is equivalent to closed topological string theory on the resolved conifold.
This conjecture has been proved by embedding the duality in type II superstring theory [89] and lifting it to M-theory [1, 10], and more recently a worldsheet derivation has been presented in [77]. In the remaining of this section, we will give evidence for the conjecture at the level of the free energy.

### 4.4 First test of the duality: the free energy

A nontrivial test of the duality advocated by Gopakumar and Vafa is to verify that the free energy of $U(N)$ Chern-Simons theory on the sphere agrees with the free energy of closed topological strings on the resolved conifold. The partition function of CS with gauge group $U(N)$ on the sphere is a slight modification of (3.14):

$$Z = \frac{1}{(k+N)^{N/2}} \prod_{\alpha>0} 2 \sin \left( \frac{\pi (\alpha \cdot \rho)}{k+N} \right).$$

and differs from it in an overall factor $N^{1/2}/(k+N)^{1/2}$ which is the partition function for the $U(1)$ factor (recall that $U(N) = U(1) \otimes SU(N)/\mathbb{Z}_N$). Using the explicit description of the positive roots of $SU(N)$, one gets

$$F = \log Z = -\frac{N}{2} \log(k+N) + \sum_{j=1}^{N-1} (N-j) \log \left[ 2 \sin \frac{\pi j}{k+N} \right].$$
We can now write the sin as
\[ \sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right), \] (4.22)
and we find that the free energy is the sum of two pieces. One of them is the nonperturbative piece:
\[ F_{np} = -\frac{N^2}{2} \log(k + N) + \frac{1}{2} N(N - 1) \log 2\pi + \sum_{j=1}^{N-1} (N - j) \log j, \] (4.23)
and the other piece is the perturbative one:
\[ F_p = \sum_{j=1}^{N-1} (N - j) \sum_{n=1}^{\infty} \log \left[1 - \frac{j^2 g_s^2}{4\pi^2 n^2}\right], \] (4.24)
where \( g_s \) corresponds to the open string coupling constant and it is given by (4.8). To see that (4.23) corresponds to the nonperturbative piece of the free energy, we notice that the volume of \( U(N) \) can be written as (see for example [77]):
\[ \text{vol}(U(N)) = \left(\frac{2\pi}{g_s}\right)^{\frac{1}{2} N(N+1)} G_2(N+1) \] (4.25)
where \( G_2(z) \) is the Barnes function, defined by
\[ G_2(z + 1) = \Gamma(z) G_2(z), \quad G_2(1) = 1. \] (4.26)
It is now easy to see that
\[ F_{np} = \log \left(\frac{\left(2\pi g_s\right)^{\frac{1}{2} N^2}}{\text{vol}(U(N))}\right) \] (4.27)
so it is given by the log of [3.6], where \( A^{(c)} \) is in this case the trivial flat connection. Therefore, \( F_{np} \) is the log of the prefactor associated to the normalization of the path integral, which is not captured by Feynman diagrams.

Let us work out the perturbative piece (4.24). By expanding the log, using that
\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \zeta(2k), \] and the formula
\[ \sum_{j=1}^{N-1} j^{2k} = -\frac{N^{2k}}{2} + \sum_{l=0}^{k} \binom{2k + 1}{2l} \frac{B_{2l}}{2k + 1} N^{2k+1-2l} \] (4.28)
we find that (4.24) can be written as
\[ F_p = \sum_{g=0}^{\infty} \sum_{h=2}^{\infty} F_{g,h} \left(\frac{g_s}{g_s}\right)^{2g-2+h} N^h, \] (4.29)
where $F_{g,h}^p$ is given by:

\[
F_{0,h}^p = -\frac{2\zeta(h - 2)}{(2\pi)^h - 2(h - 2)h(h - 1)},
\]

\[
F_{1,h}^p = \frac{1}{6} \frac{\zeta(h)}{(2\pi)^h},
\]

\[
F_{g,h}^p = 2 \frac{\zeta(2g - 2 + h)}{(2\pi)^{2g - 2 + h}} \left(\frac{2g - 3 + h}{h}\right) \frac{B_{2g}}{2g(2g - 2)}, \quad g \geq 2.
\] (4.30)

This gives the contribution of connected diagrams with two loops and beyond to the free energy of Chern-Simons on the sphere, so we can write

\[
\sum_{\ell=1}^{\infty} S_{\ell}(N)x^\ell = \sum_{g=0}^{\infty} \sum_{h=2}^{\infty} (-1)^{g-1+h/2} F_{g,h}^p x^{2g-2+h} N^h,
\] (4.31)

where $x$ is given by (3.5), and we have explicitly indicated the dependence of $S_{\ell}$ on $N$. Notice that the only nonzero $F_{g,h}^p$ have $h$ even. One can check that the $F_{g,h}^p$ that we have obtained in (4.30) are in agreement with known results of perturbative Chern-Simons theory on the sphere (see for example [12, 69]). The nonperturbative piece also admits an expansion that can be easily worked out from the asymptotics of the Barnes function [78, 77]. One finds:

\[
F^{np} = \frac{N^2}{2} \left(\log(N g_s) - \frac{3}{2}\right) - \frac{1}{12} \log N + \zeta'(-1) + \sum_{g=2}^{\infty} \frac{B_{2g}}{2g(2g - 2)} N^{2g-2},
\] (4.32)

So far, what we have uncovered is the open string expansion of Chern-Simons theory, which is (order by order in $x$) determined by the perturbative expansion. In order to find a closed string interpretation, we have to sum over the holes, as in (4.3). Define the 't Hooft parameter $t$ as:

\[
t = ig_s N = xN,
\] (4.33)

then

\[
F_g^p(t) = \sum_{h=1}^{\infty} F_{g,h}^p (-it)^h.
\] (4.34)

We will now focus on $g \geq 2$. To perform the sum explicitly, we write again the $\zeta$ function as $\zeta(2g - 2 + 2p) = \sum_{n=1}^{\infty} n^{2-2g-2p}$, and use the binomial series,

\[
\frac{1}{(1 - z)^q} = \sum_{n=0}^{\infty} \binom{q + n - 1}{n} z^n
\] (4.35)

to obtain:

\[
F_g^p(t) = \frac{(-1)^g |B_{2g} B_{2g-2}|}{2g(2g - 2)(2g - 2)!} + \frac{B_{2g}}{2g(2g - 2)} \sum_{n \in \mathbb{Z}} \frac{1}{(-it + 2\pi n)^{2g-2}},
\] (4.36)

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where ′ means that we omit \( n = 0 \). Now we notice that, if we write
\[
F^{np} = \sum_{g=0}^{\infty} F^{np}_g(t) g^{2g-2}_{s}
\] (4.37)
then for, \( g \geq 2 \),
\[
F^{np}_g(t) = \frac{B_{2g}}{2g(2g-2)}(-it)^{2-2g},
\]
which is precisely the \( n = 0 \) term missing in (4.36). We then define:
\[
F_g(t) = F_p^{g}(t) + F^{np}_g(t). \tag{4.38}
\]
Finally, since
\[
\sum_{n \in \mathbb{Z}} \frac{1}{n + z} = \frac{2\pi i}{1 - e^{-2\pi iz}}, \tag{4.39}
\]
by taking derivatives w.r.t. \( z \) we can write
\[
F_g(t) = \frac{(-1)^g |B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!} + \frac{|B_{2g}|}{2g(2g-2)!} \text{Li}_{3-2g}(e^{-t}), \tag{4.40}
\]
again for \( g \geq 2 \). If we now compare (4.40) to (2.33), we see that it has precisely the structure of the free energy of a closed topological string, with \( n_1^0 = 1 \), and the rest of the Gopakumar-Vafa invariants being zero. Also, from the first term, which gives the contribution of the constant maps, we find that \( \chi(X) = 2 \). In fact, (4.40) is the \( F_g \) amplitude of the resolved conifold. One can also work out the expressions for \( F_0(t) \) and \( F_1(t) \) and find agreement with the corresponding results for the resolved conifold \[37\]. This is a remarkable check of the conjecture.

5 Wilson loops and large \( N \) transitions

5.1 Incorporating Wilson loops

How do we incorporate Wilson loops in the large \( N \) duality for Chern-Simons theory? As we discussed in the previous section, once one has a closed string description of the \( 1/N \) expansion, Wilson loops are related to the open string sector in the closed string geometry. Since the string description involves topological strings, it is natural to assume that Wilson loops are going to be described by open topological strings in the resolved conifold, and this means that we need a Lagrangian submanifold specifying boundary conditions.

These issues were addressed in an important paper by Ooguri and Vafa \[76\]. In order to give boundary conditions for the open strings in the resolved conifold, Ooguri
and Vafa constructed a Lagrangian submanifold $\hat{C}_K$ in $T^*S^3$ for any knot $K$ in $S^3$. This Lagrangian is rather canonical, and it is called the conormal bundle of $K$. The details are as follows: suppose that the knot is parameterized by a curve $q(s)$, where $s \in [0, 2\pi)$, for example. The conormal bundle of $K$ is then the space

$$\hat{C}_K = \left\{ (q(s), p) \in T^*S^3 \mid \sum_i p_i \dot{q}_i = 0, \ 0 \leq s < 2\pi \right\} \quad (5.1)$$

where $p_i$ are coordinates for the cotangent bundle, and $\dot{q}_i$ denote the derivatives w.r.t. $s$. This space is an $\mathbb{R}^2$-fibration of the knot itself, where the fiber on the point $q(s)$ is given by the two-dimensional subspace of $T_q^*S^3$ of planes orthogonal to $\dot{q}_i(s)$. $\hat{C}_K$ has in fact the topology of $S^1 \times \mathbb{R}^2$, and intersects $S^3$ along the knot $K$.

One can now consider, together with the $N$ branes wrapping $S^3$, a set of $M$ probe branes wrapping $\hat{C}_K$, and study the effective theory that one obtains in this way. On the $N$ branes wrapping $S^3$ we have $U(N)$ Chern-Simons theory. But the strings stretched between the $N$ branes and the $M$ branes give an extra state in topological string field theory, which turns out to be a massless complex scalar field $\phi$ in the bifundamental representation $(N, M)$, and living in the intersection of the two branes, $K$. If $A$ denotes the $U(N)$ gauge connection on $S^3$, and $\tilde{A}$ denotes the $U(M)$ gauge connection on $\hat{C}_K$, the action for the scalar is given by

$$\oint_K \text{Tr} \tilde{A} D\phi,$$

where $D = d + A - \tilde{A}$. Here we regard $\tilde{A}$ as a source. We can now proceed to integrate out $\phi$. This is just a one loop computation giving

$$\exp \left[ -\log \det D \right] = \exp \left[ -\text{Tr} \log \left( U^{-\frac{1}{2}} \otimes V^\frac{1}{2} - U^{\frac{1}{2}} \otimes V^{-\frac{1}{2}} \right) \right]. \quad (5.3)$$

In this equation, $U, V$ are the holonomies of $A, \tilde{A}$ around the knot $K$. To obtain this equation, we have diagonalized $A, \tilde{A}$ and taken into account that

$$\log \det \left[ \frac{d}{ds} + i\theta \right] = \sum_{n=-\infty}^{\infty} \log(n + \theta) = \log(\sin(\pi \theta)) + \text{const.}, \quad (5.4)$$

where use has been made of (4.22). In this way we obtain the effective action for the $A$ field

$$S_{\text{CS}}(A) + \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} U^n \text{Tr} V^{-n} \quad (5.5)$$

where $S_{\text{CS}}(A)$ is the Chern-Simons action for $A$ associated to the $N$ branes on the three-sphere $^3$. Therefore, in the presence of the probe branes, the action gets deformed by

$^3$In the above equation we have factored out a contribution involving the $U(1)$ pieces of $U(N), U(M)$. These can be reabsorbed in a change of framing.

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the Ooguri-Vafa operator that we introduced in (3.49). Since we are regarding the $M$ branes as a probe, the holonomy $V$ is an arbitrary source, and we will put $V^{-1} \rightarrow V$.

Let us now follow this system through the geometric transition. The $N$ branes disappear, and the background geometry becomes the resolved conifold. However, the $M$ probe branes are still there. The first conjecture of Ooguri and Vafa is that these branes are wrapping a Lagrangian submanifold $C_K$ of $O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1$ that can be obtained from $\hat{C}_K$ through the geometric transition. The final outcome is therefore the existence of a map

$$\{\text{knots in } S^3\} \rightarrow \{\text{Lagrangian submanifolds in } O(-1) \oplus O(-1) \rightarrow \mathbb{P}^1\} \quad (5.6)$$

which sends

$$K \rightarrow C_K. \quad (5.7)$$

Moreover, one has $b_1(C_K) = 1$. This conjecture is clearly well-motivated in the physics of the problem, and some aspects of the map (5.6) are already well understood: in [76] Ooguri and Vafa constructed $C_K$ explicitly when $K$ is the unknot, and [83] proposed Lagrangian submanifolds for certain algebraic knots and links (including torus knots). Taubes has generalized this proposal [84] and constructed in detail a map from a wide class of knots to Lagrangian submanifolds in the resolved conifold. Later on we will discuss the case of the unknot.

The resulting Lagrangian submanifold $C_K$ in the resolved geometry provides boundary conditions for open strings, and therefore it gives the open string sector that is needed in order to extend the large $N$ duality to Wilson loops. The second conjecture of [76] states that the free energy of open topological strings (2.24) with boundary conditions specified by $C_K$ is identical to the free energy of the deformed Chern-Simons theory with action (5.5), which is nothing but (3.52):

$$F_{\text{string}}(V) = F_{\text{CS}}(V). \quad (5.8)$$

Notice that, since $b_1(C_K) = 1$, the topological sectors of maps with positive winding numbers correspond to vectors $\vec{k}$ labelling the connected vevs, and one finds

$$\sum_{g=0}^\infty F_{g,\vec{k}}(t) g^{2g-2+|\vec{k}|} = \frac{1}{\prod j} \sum_{\vec{k}} W^{(c)}_{\vec{k}}. \quad (5.9)$$

Of course, $F_{g,\vec{k}}(t)$ are (up to constants) the functions of the 't Hooft parameter that appeared in (4.5). The variable $\lambda$ defined in (3.17) that appears in the Chern-Simons invariants of knots and links is related to the 't Hooft parameter through

$$\lambda = e^t.$$
Notice that the Chern-Simons invariants are labelled by vectors $\vec{k}$, therefore they only give rise to positive winding numbers in the string side. At the same time, they involve both positive and negative powers of $\lambda$, while in the string side we only have negative powers. Therefore, in order to make (5.8) precise, we further need some sort of analytic continuation that gives an appropriate matching of the variables. In the cases where both sides of the equality are known, there is such an analytic continuation, and it is expected that this will be the case in more general situations. Up to these subtleties, (5.9) tells us that the Chern-Simons invariant in the left-hand side is a generating function for open Gromov-Witten invariants, for all degrees and genera, but with fixed boundary data (i.e. the number of holes and the winding numbers). To extract a particular open Gromov-Witten invariant from the Chern-Simons invariant, we consider the connected vev labelled by the vector $\vec{k}$ associated to the boundary data, we write it in terms of $\lambda = e^t$ and $q = e^x$, and then we expand the result in powers of $x = ig_s$. The coefficients of this series, which are polynomials in $\lambda$, are then equated to the generating function of open Gromov-Witten invariants at fixed genus $g$.

We should mention that, although we have focused on knots for simplicity, all these results can be extended to links, as shown in [63].

### 5.2 BPS invariants for open strings from knot invariants

In section 2 we have learned that Gromov-Witten invariants can be written in terms of integer, or BPS invariants. We will now find what is the precise relation between Chern-Simons invariants and these integer invariants. This will lead to some surprising structure results for the Chern-Simons invariants of knots.

The first step is to introduce the so-called $f$-polynomials, through the relation:

$$ F_{\text{CS}}(V) = \sum_{n=1}^{\infty} \sum_{R} \frac{1}{n} f_R(q^n, \lambda^n) \text{Tr}_R V^n. \tag{5.10} $$

As shown in [61, 62], the $f_R$ polynomials are completely determined by this equation, and can be expressed in terms of the usual vevs of Wilson loops $W_R$ by:

$$ f_R(q, \lambda) = \sum_{d,m=1}^{\infty} (-1)^{m-1} \frac{\mu(d)}{dm} \sum_{\vec{k}_1, \ldots, \vec{k}_m} \sum_{R_1, \ldots, R_m} x_R(C((\sum_{j=1}^{l} \vec{k}_j)_d)) $$

$$ \times \prod_{j=1}^{m} \frac{\chi_{R_j}(C(\vec{k}_j))}{z_{\vec{k}_j}} W_{R_j}(q^d, \lambda^d), \tag{5.11} $$

where $\vec{k}_d$ is defined as follows: $(\vec{k}_d)_d = k_d$ and has zero entries for the other components.
Therefore, if $\vec{k} = (k_1, k_2, \cdots)$, then
\[ \vec{k}_d = (0, \cdots, 0, k_1, 0, \cdots, 0, k_2, 0, \cdots) \]
where $k_1$ is in the $d$-th entry, $k_2$ is in the $2d$-th entry, and so on. The sum over $\vec{k}_1, \cdots, \vec{k}_m$ is over all vectors with $|\vec{k}_j| > 0$. In (5.11), $\mu(d)$ denotes the Moebius function. Recall that the Moebius function is defined as follows: if $d$ has the prime decomposition $d = \prod_{i=1}^{a} p_i^{m_i}$, then $\mu(d) = 0$ if any of the $m_i$ is greater than one. If all $m_i = 1$ (i.e. $d$ is square-free) then $\mu(d) = (-1)^{a}$. Some examples of (5.11) are
\[
\begin{align*}
  f_{\square}(q, \lambda) &= W_{\square}(q, \lambda), \\
  f_{\boxtimes}(q, \lambda) &= W_{\boxtimes}(q, \lambda) - \frac{1}{2} (W_{\square}(q, \lambda)^2 + W_{\boxtimes}(q^2, \lambda^2)), \\
  f_{\mathcal{R}}(q, \lambda) &= W_{\mathcal{R}}(q, \lambda) - \frac{1}{2} (W_{\square}(q, \lambda)^2 - W_{\boxtimes}(q^2, \lambda^2)).
\end{align*}
\]
(5.12)

Therefore, given a representation $R$ with $\ell$ boxes, the polynomial $f_R$ is given by $W_R$, plus some “lower order corrections” that involve $W'_R$ where $R'$ has $\ell' < \ell$ boxes. One can then easily compute these polynomials starting from the results for vevs of Wilson loops in Chern-Simons theory. Although we are calling $f_R$ polynomials, they are not, strictly speaking. In fact, it follows from the multicovering/bubbling formula that the $f_R$ have the structure
\[ f_R(q, \lambda) = \frac{P_R(q, \lambda)}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}. \]
(5.13)

But we can be more precise about the structure of $f_R$. As shown in [63], one can write the $f_R$ in terms of even more basic objects, that were denoted by $\hat{f}_R$. The precise relation between them is
\[ f_R = \sum_{R'} M_{RR'} \hat{f}_{R'} \]
(5.14)

where the sum in $R'$ runs over all representations with the same number of boxes than $R$, and the matrix $M_{RR'}$ is given by:
\[ M_{RR'} = \sum_{R''} C_{RR'R''} S_{R''}(q). \]
(5.15)

In this equation, $C_{RR'R''}$ are the Clebsch-Gordon coefficients of the symmetric group. They can be explicitly written in terms of characters [34]:
\[ C_{RR'R''} = \sum_{\vec{k}} \frac{|C(\vec{k})|}{\ell!} \chi_R(C(\vec{k})) \chi_{R'}(C(\vec{k})) \chi_{R''}(C(\vec{k})). \]
(5.16)
The $S_R(q)$ are monomials defined as follows. If $R$ is a hook or L-shaped representation of the form

\[
\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\vdots \\
\end{array}
\]

with $\ell$ boxes in total, and $\ell - d$ boxes in the first row, then

\[
S_R(q) = (-1)^d q^{-\frac{\ell^2}{2} + d},
\]

and $S_R(q) = 0$ for the rest of the representations. For example, for the case of two boxes one has that $S_{\square}(q) = q^{-1/2}$ and $S_{\Box}(q) = -q^{1/2}$; while for $\ell = 3$ one has

\[
S_{\square}(q) = q^{-1}, \quad S_{\Box}(q) = -1, \quad S_{\Box}(q) = q.
\]

The square matrix $M_{RR'}$ that relates $f_R$ to $\hat{f}_R$ is invertible. This can be easily seen: define the polynomials $P_{\vec{k}}(q)$, labelled by conjugacy classes, as the character transforms of the monomials $S_R(q)$:

\[
P_{\vec{k}}(q) = \sum_R \chi_R(C(\vec{k})) S_R(q).
\]

It can be seen that

\[
P_{\vec{k}}(q) = \prod_j (q^{-\frac{j^2}{2}} - q^{\frac{j^2}{2}})^{k_j}/(q^{-\frac{j^2}{2}} - q^{\frac{j^2}{2}}).
\]

In terms of these polynomials, the matrix $M_{RR'}$ is written as

\[
M_{RR'} = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_R(C(\vec{k})) \chi_{R'}(C(\vec{k})) P_{\vec{k}}(q),
\]

and using the orthogonality of the characters one can see that

\[
M_{RR'}^{-1} = \sum_{\vec{k}} \frac{1}{z_{\vec{k}}} \chi_R(C(\vec{k})) \chi_{R'}(C(\vec{k}))(1/P_{\vec{k}}(q)).
\]

Therefore, one can obtain the polynomials $\hat{f}_R$ from the $f_R$, i.e. one can obtain the polynomials $\hat{f}_R$ from the knot invariants of Chern-Simons theory. The claim is now that the $\hat{f}_R$ are generating functions for the BPS invariants $N_{R,g,Q}$ that were introduced in (2.40). More precisely, one has

\[
\hat{f}_R(q, \lambda) = \sum_{g \geq 0} \sum_Q N_{R,g,Q}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{2g-1}\lambda^Q
\]

Therefore, this gives a very precise way to compute the BPS invariants $N_{R,g,Q}$ from Chern-Simons theory: compute the usual vevs $W_R$, extract $f_R$ through the relation (5.11), compute $\hat{f}_R$, and expand them as in (5.24).
We would like to point out two important things. First, the fact that one can extract the integer invariants $N_{R,g,Q}$ from Chern-Simons theory in the way we have just described is by no means obvious and constitutes a strong check of the large $N$ duality between Chern-Simons theory and topological strings. We will see examples of this in the next subsection. Another important comment is that the statement that $\hat{f}_R$ have the structure predicted in (5.24) is equivalent to the multicovering/bubbling formula for open string invariants (2.38) (more precisely, it is equivalent to the strong version of this formula, which says that in addition to (2.38) one can write the $n_{k,g,Q}$ in terms of integer $N_{R,g,Q}$ through (2.40)). This is easily seen by noticing that, according to (5.14) and (5.15), $f_R$ is given by

$$f_R(q,\lambda) = \sum_{g \geq 0} \sum_Q \sum_{R',R''} C_{R'R''} S_{R''}(q) N_{R'',g,Q}(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2 g - \frac{1}{2} \lambda Q. \quad (5.25)$$

If we now write the exponent in the r.h.s. of (5.10) in the $\vec{k}$ basis, it is easy to see that one obtains precisely (2.38), after making use of (5.21).

The physical origin of the structure of $f_R$ (and therefore of the multicovering/bubbling formula for open Gromov-Witten invariants) can be easily understood in physical terms. We will give a short account, referring the reader to [63] for more details. In the D-brane approach to open string instantons, one regards the open Riemann surfaces ending on a Lagrangian submanifold as D2-branes ending on $M$ D4-branes wrapping the Lagrangian submanifold. Following the approach of [36], we have to study the moduli space of D2-branes ending on D4-branes. This moduli space is the product of three factors: the moduli of Abelian gauge fields on the worldvolume of the D2 brane, the moduli of geometric deformations of the D2’s in the ambient space, and finally the Chan-Paton factors associated to the boundaries of the D2 which appear in the D4 as magnetic charges [76]. If the D2’s are genus $g$ surfaces with $\ell$ holes in the relative cohomology class labelled by $Q$, the moduli space of Abelian gauge fields gives rise to the Jacobian $J_{g,\ell} = T^{2g+\ell-1}$, and the moduli of geometric deformations will be a manifold $\mathcal{M}_{g,\ell,Q}$. Finally, for the Chan-Paton degrees of freedom we get a factor of $F$ (the fundamental representation of $SU(M)$) from each hole. The Hilbert space is obtained by computing the cohomology of these moduli, and we obtain

$$F^{\otimes \ell} \otimes H^*(J_{g,\ell}) \otimes H^*(\mathcal{M}_{g,\ell,Q}). \quad (5.26)$$

An important point is that this Hilbert space is associated with the moduli space of $\ell$ distinguished holes, which is not physical, and we have to mod out by the action of the permutation group $S_\ell$. We can factor out the cohomology of the Jacobian $T^{2g}$ of the “bulk” Riemann surface, $H^*(T^{2g})$, since the permutation group does not act on it. The projection onto the symmetric piece can be easily done using the Clebsch-Gordon
coefficients $C_{RR'RR''}$ of the permutation group $S_\ell$:

$$\text{Sym} \left( F^{\otimes\ell} \otimes H^*((S^1)^{\ell-1}) \otimes H^*(\mathcal{M}_{g,\ell,Q}) \right) = \sum_{RR'RR''} C_{RR'RR''} S_R(F^{\otimes\ell}) \otimes S_{R'}(H^*((S^1)^{\ell-1})) \otimes S_{R''}(H^*(\mathcal{M}_{g,\ell,Q}))$$

(5.27)

where $S_R$ is the Schur functor that projects onto the corresponding subspace. The space $S_R(F^{\otimes\ell})$ is nothing but the vector space underlying the irreducible representation $R$ of $SU(M)$. $S_{R'}(H^*((S^1)^{\ell-1}))$ gives the hook Young tableau, and the Euler characteristic of $S_{R''}(H^*(\mathcal{M}_{g,\ell,Q}))$ is the integer invariant $N_{R'',g,Q}$. Therefore, the above decomposition corresponds very precisely to (5.25).

All the results above have been stated for knot invariants in the canonical framing. The situation for arbitrary framing was analyzed in detail in [71]. Suppose that we consider a knot in $S^3$ in the framing labelled by an integer $p$ (the canonical framing corresponds to $p = 0$). Then, the integer invariants $N_{R,g,Q}(p)$ are obtained from (5.11) but with the vevs

$$W^p_R(q,\lambda) = (-1)^p q^{\frac{1}{2}p \kappa_R} W_R(q,\lambda),$$

(5.28)

where $\kappa_R$ is defined in (3.42). One has, for example,

$$f^{(p)}(\square,q,\lambda) = (-1)^p W_{\square}(q,\lambda),$$

$$f^{(p)}(\Box,q,\lambda) = q^p W_{\Box}(q,\lambda) - \frac{1}{2} (W_{\Box}(q,\lambda)^2 + (-1)^p W_{\Box}(q^2,\lambda^2)), $$

$$f^{(p)}(\Box,q,\lambda) = q^{-p} W_{\Box}(q,\lambda) - \frac{1}{2} (W_{\Box}(q,\lambda)^2 - (-1)^p W_{\Box}(q^2,\lambda^2)),$$

(5.29)

and so on. Notice that the right framing factor in order to match the topological string theory prediction is $3.42$, and not $3.41$. This is yet another indication that the duality of [30] involves the $U(N)$ gauge group, not the $SU(N)$ group. The rationale for introducing the extra sign $(-1)^p$ is not completely clear in the context of Chern-Simons theory, and it was introduced by consistency with the results for the B-model in [2]. This sign is crucial for integrality of $N_{R,g,Q}(p)$.

All the above results on $f$-polynomials, integer invariant structure, etc., can be extended to links, see [63, 62].

### 5.3 Tests involving Wilson loops

There are two types of tests of the large $N$ duality involving Wilson loops: a test in the strong sense, in which one verifies that the open Gromov-Witten invariants agree with the Chern-Simons amplitude, and a test in the weak sense, in which one verifies that
the Chern-Simons knot invariants satisfy the integrality properties that follow from the conjectured dual description.

The only test so far of the duality in the strong sense is for the framed unknot. In this case, we know both sides of the duality in detail and we can compare the results. Let us start with the string description. The first thing we need is a construction of the Lagrangian submanifold $C_k$ that corresponds to the unknot in $S^3$. This was done by Ooguri and Vafa in [76]. The construction goes as follows. Let us start with $T^*S^3$ expressed as (4.12), and consider the following anti-holomorphic involution on it.

$$\eta_{1,2} = \bar{\eta}_{1,2}, \quad \eta_{3,4} = -\bar{\eta}_{3,4}. \tag{5.30}$$

The symplectic form $\omega$ changes its sign under the involution, therefore its fixed point set is a Lagrangian submanifold of $T^*S^3$. If we write $\eta_\mu = x_\mu + ip_\mu$, the invariant locus of the action (5.30) is

$$p_{1,2} = 0, \quad x_{3,4} = 0 \tag{5.31}$$

and intersects the deformed conifold at

$$x_1^2 + x_2^2 = a + p_3^2 + p_4^2. \tag{5.32}$$

Therefore, the fixed point locus intersects $S^3$ along the equator, which is an unknot described by the equations

$$x_1^2 + x_2^2 = a, \quad x_3 = x_4 = 0.$$

We conclude that, if we denote by $U$ the unknot in $S^3$, the above fixed point locus defined by (5.30) is the Lagrangian submanifold $\hat{C}_U$. Now we want to construct the Lagrangian submanifold $C_U$, obtained from $\hat{C}_U$ after the conifold transition. To do that, we continue to identify it with the invariant locus of the anti-holomorphic involution. We can describe this explicitly by using the coordinates $(x, u, z)$ or $(y, v, z^{-1})$ defined in (4.14) and (4.18). In these coordinates, $\hat{C}_U$ is characterized by

$$x = \bar{y}, \quad u = \bar{v}, \tag{5.33}$$

and the conifold equation (4.17) restricted to $\hat{C}_U$ becomes

$$x\bar{x} = uu. \tag{5.34}$$

The complex coordinate on the base $\mathbb{P}^1$ defined by (4.18) is

$$z = \frac{x}{u}. \tag{5.35}$$

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Figure 9: This figure represents the Lagrangian submanifold in \( O(-1) \oplus O(-1) \to \mathbb{P}^1 \) that corresponds to the unknot in \( S^3 \). The notation is as in [76], and is related to ours by \( u \to x \) and \( v \to -u \).

but since \(|x| = |u|\), \( z \) is a phase. We then find that \( \mathcal{C}_U \) is a line bundle over the equator \(|z| = 1\) of \( \mathbb{P}^1 \), and the fiber over \( z \) is the subspace of \( O(-1) + O(-1) \) given by \( x = z \bar{u} \) (remember that \( x, u \) are complex coordinates for the fibers). In particular, \( \mathcal{C}_U \) intersects with the \( \mathbb{P}^1 \) at the base along \(|z| = 1\), see Fig. 5.3.

The open Gromov-Witten invariants associated to open strings in \( O(-1) \oplus O(-1) \to \mathbb{P}^1 \) whose boundaries end in the above Lagrangian submanifold have been computed in [51, 67, 73] (see also [39]). The procedure relies on localization formulae, as in the closed string case. However, in the open string case, it has been realized that the open invariants depend on an extra choice of an integer (the calculation depends on the weights on the localizing torus action). This is precisely the dependence we expect on Chern-Simons theory, since there is a choice of framing also labelled by an integer. This framing ambiguity in the context of open strings was first discovered in the B-model [2], and subsequently confirmed in the A-model computation of [51] as well as in other examples [39, 73]. Let us now make a detailed comparison of the answers. Katz and Liu [51] compute the open Gromov-Witten invariants \( F^Q_{w,g} \) for \( Q = \ell/2 \), where \( \ell = \sum_i w_i \), and obtain:

\[
F^\ell/2_{w,g} = (-1)^{p+1}(p(p+1))^{h-1} \left( \prod_{i=1}^{h} \frac{\prod_{j=1}^{w_i-1}(j+w_ip)}{(w_i-1)!} \right)
\]
\[ \text{Res}_{u=0} \int_{\overline{M}_{g,h}} \frac{c_g(\mathcal{E}^\vee(u))c_g(\mathcal{E}^\vee((-p-1)u))c_g(\mathcal{E}^\vee(pu))u^{2h-4}}{\prod_{i=1}^h (u - w_i \psi_i)}. \] (5.36)

In this formula, \( \overline{M}_{g,h} \) is the Deligne-Mumford moduli space of genus \( g \) stable curves with \( h \) marked points, \( \mathcal{E} \) is the Hodge bundle over \( \overline{M}_{g,h} \), and its dual is denoted by \( \mathcal{E}^\vee \). The Chern classes of the Hodge bundle will be denoted by:

\[ \lambda_j = c_j(\mathcal{E}). \] (5.37)

In (5.36), we have written

\[ c_g(\mathcal{E}^\vee(u)) = \sum_{i=0}^g c_{g-i}(\mathcal{E}^\vee)u^i, \] (5.38)

and similarly for the other two factors. The integral in (5.36) also involves the \( \psi_i \) classes of two-dimensional topological gravity, which are constructed as follows. We first define the line bundle \( \mathcal{L}_i \) over \( \overline{M}_{g,h} \) to be the line bundle whose fiber over each stable curve \( \Sigma \) is the cotangent space of \( \Sigma \) at \( x_i \) (where \( x_i \) is the \( i \)-th marked point). We then have,

\[ \psi_i = c_1(\mathcal{L}_i), \quad i = 1, \ldots, h. \] (5.39)

The integrals of the \( \psi \) classes can be obtained by the results of Witten and Kontsevich on 2d topological gravity \([94, 54]\), while the integrals involving \( \psi \) and \( \lambda \) classes (the so-called Hodge integrals) can be in principle computed by reducing them to pure \( \psi \) integrals \([29]\). Explicit formulae for some Hodge integrals can be found in \([35]\).

In the above formula (5.36), \( p \) is an integer that parameterizes the ambiguity in the open string calculation. A particularly simple case of the above expression is when \( p = 0 \), i.e. the standard framing. The only contribution comes from \( h = 1 \), and the above integral boils down to

\[ \text{Res}_{u=0} \int_{\overline{M}_{g,1}} \frac{\lambda_g c_g(\mathcal{E}^\vee(u))c_g(\mathcal{E}^\vee(-u))u^{2h-4}}{(u - w \psi_1)}, \] (5.40)

where \( w \) is the winding number. The Mumford relations \([74]\) give \( c(\mathcal{E})c(\mathcal{E}^\vee) = 1 \), which implies

\[ c_g(\mathcal{E}^\vee(u))c_g(\mathcal{E}^\vee(-u)) = (-1)^g u^{2g} \] (5.41)

After taking the residue, we end up with the following expression for the open Gromov-Witten invariant:

\[ G_{w, g}^{w/2} = -w^{2g-2} \int_{\overline{M}_{g,1}} \psi_1^{2g-2} \lambda_g, \] (5.42)
The above Hodge integral has been computed in [31], and it is given by $b_g$, where $b_g$ is defined by the generating functional

$$\sum_{g=0}^{\infty} b_g x^g = \frac{x/2}{\sin(x/2)}. \quad (5.43)$$

We can now sum over all genera and all positive winding numbers to obtain [51]

$$F(V) = -\sum_{d=1}^{\infty} \frac{e^{dt/2}}{2d \sin \left(\frac{d\psi}{2}\right)} \text{Tr} V^d. \quad (5.44)$$

Notice that the above open Gromov-Witten invariants correspond to a disk instanton wrapping the northern hemisphere of $\mathbb{P}^1$, with its boundary on the equator, together with all the multicoveryings and bubblings at genus $g$. Let us now compare to the Chern-Simons computation. In the case of the unknot in the canonical framing, Ooguri and Vafa showed [76] that the generating function (3.52) can be explicitly computed to all orders. The reason is that the quantum dimension in the representation $R$ can be regarded as the trace in the representation $R$ of an $N \times N$ diagonal matrix $U_0$ whose $i$-th diagonal entry is

$$\exp \left[ -\frac{\pi i}{k+N} (N - 2i - 1) \right]. \quad (5.45)$$

This is easily seen by remembering that $\rho$ lives in the dual of the Cartan subalgebra $H$, and by using the natural isomorphism between $H$ and $H^*$ induced by the Killing form we obtain the above result from (3.25). Notice that $U_0$ is like a “master field” that gives the right answer by evaluating a “classical” trace. Therefore, one can compute $F_{CS}(V)$ by substituting $\text{Tr} U_0^d$ in (3.49), to obtain

$$F(V) = -i \sum_{d=1}^{\infty} \frac{e^{dt/2} - e^{-dt/2}}{2d \sin \left(\frac{d\psi}{2}\right)} \text{Tr} V^d. \quad (5.46)$$

The answer from Chern-Simons theory contains the contribution given in (5.44), together with a similar contribution (with $e^{\psi/2}$) that corresponds to holomorphic maps wrapping the southern hemisphere of the $\mathbb{P}^1$.

What happens for $p \neq 0$? In that case, it is no longer possible to sum up all the correlation functions, but we can still compute the connected vevs $W_k^{(c)}$ at arbitrary framing [71]. To do that, remember that the $W_R$ for the unknot in the canonical framing are just the quantum dimensions of the representation $R$ given in (3.28). We

4In this equation we have chosen the sign for the instantons wrapping the northern hemisphere in such a way that one has $e^{dt}$ in the generating function, in order to compare to the results in [71].
have to correct them with the framing factor as prescribed in (5.28), compute the $W_{\vec{k}}$ with Frobenius formula, and then extract the connected piece by using:

$$\frac{1}{z_{\vec{k}}} W_{\vec{k}}^{(c)} = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \sum_{\vec{k}_1, \ldots, \vec{k}_n} \delta_{\sum_{i=1}^{n} \vec{k}_i, \vec{k}} \prod_{i=1}^{n} \frac{W_{\vec{k}_i}}{z_{\vec{k}_i}}. \quad (5.47)$$

In this equation, the second sum is over $n$ vectors $\vec{k}_1, \ldots, \vec{k}_n$ such that $\sum_{i=1}^{n} \vec{k}_i = \vec{k}$ (as indicated by the Kronecker delta), and therefore the right hand side of (5.47) involves a finite number of terms. The generating functional for the open Gromov-Witten invariants is then explicitly given by

$$
\sum_{Q} \sum_{g=0}^{\infty} F_{\vec{k},g}^{Q} g^{2g-2+|\vec{k}|} e^{Qt} = \sum_{\ell} \prod_{j} k_{j} \sum_{n \geq 1} \frac{(-1)^{n}}{n} \sum_{\vec{k}_1, \ldots, \vec{k}_n} \delta_{\sum_{\sigma=1}^{n} \vec{k}_{\sigma}, \vec{k}} \prod_{\sigma} \chi_{R_{\sigma}}(C(\vec{k}_{\sigma})) 
\prod_{1 \leq i < j \leq c_{R_{\sigma}}} \sin \left[ \frac{(l_{i}^{\sigma} - l_{j}^{\sigma} + j - i) g_{s}/2}{2} \right] \prod_{i} c_{R_{\sigma}}^{i} \prod_{v=1}^{c_{R_{\sigma}}} 2 \sin \left[ \frac{(v - i + c_{R_{\sigma}}) g_{s}/2}{2} \right].
(5.48)
$$

Let us compare this expression with the result of Katz and Liu in some simple examples with $h = 1$. Notice that the Chern-Simons result is slightly more general, since it gives the answer for any $Q$, while (5.36) only computes $Q = \ell/2$. For Riemann surfaces with one hole the homotopy class of the map is given by a single winding number $w$. For $g = 1$, one finds from (5.36):

$$F_{w,1}^{w/2} = \frac{(-1)^{pw}}{(w - 1)!} \prod_{l=1}^{w-1} (l + w p) \left( \int_{\mathcal{M}_{1,1}} \lambda_{1} - w \psi_{1} \right) p(p + 1) + \int_{\mathcal{M}_{1,1}} \lambda_{1}, \quad (5.49)$$

and for $g = 2$,

$$F_{w,2}^{w/2} = \frac{(-1)^{pw}}{(w - 1)!} \prod_{l=1}^{w-1} (l + w p) \left( \int_{\mathcal{M}_{2,1}} w^{2} \psi_{1}^{4} - w \psi_{1}^{3} \lambda_{1} + \psi_{1}^{2} \lambda_{2} \right) w^{2} p^{3} (p + 2)
+ \left( \int_{\mathcal{M}_{2,1}} w^{3} \psi_{1}^{4} - 2w^{2} \psi_{1}^{3} \lambda_{1} - \psi_{1} \lambda_{1} \lambda_{2} + 3w \psi_{1}^{2} \lambda_{2} \right) w p^{2}
+ \left( \int_{\mathcal{M}_{2,1}} -w^{2} \psi_{1}^{3} \lambda_{1} - \psi_{1} \lambda_{1} \lambda_{2} + 2w \psi_{1}^{2} \lambda_{2} \right) w p + w^{2} \int_{\mathcal{M}_{2,1}} \psi_{1}^{2} \lambda_{2}. \quad (5.50)$$

To obtain this expression, we have used the Mumford relation, which implies in particular $\lambda_{2}^{2} = 0$ and $\lambda_{1}^{2} = 2 \lambda_{2}$. On the other hand, the Chern-Simons answer for the
connected vevs when $w = 1$ and $w = 2$ is:

$$iW_1^{(c)}(g_s) = \frac{(-1)^p}{g_s} \left( 1 + \frac{1}{24} g_s^2 + 7 \frac{g_s^4}{5760} + O(g_s^6) \right),$$

$$\frac{i}{2}W_2^{(c)}(g_s) = \frac{1 + 2p}{g_s} \left( \frac{1}{4} - \frac{1}{24} (p^2 + p - 1) g_s^2 \right.$$

$$\left. + \frac{1}{1440} (7 - 11p - 8p^2 + 6p^3 + 3p^4) g_s^4 + O(g_s^6) \right), \quad (5.51)$$

and so on. By using now the following values of the Hodge integrals for $g = 1$

$$\int_{\overline{M}_{1,1}} \psi_1 = \int_{\overline{M}_{1,1}} \lambda_1 = \frac{1}{24}, \quad (5.52)$$

and for $g = 2$

$$\int_{\overline{M}_{2,1}} \psi_1^4 = \frac{1}{1152}, \quad \int_{\overline{M}_{2,1}} \psi_1^3 \lambda_1 = \frac{1}{480},$$

$$\int_{\overline{M}_{2,1}} \psi_1^2 \lambda_2 = \frac{7}{5760}, \quad \int_{\overline{M}_{2,1}} \psi_1 \lambda_1 \lambda_2 = \frac{1}{2880},$$

we find perfect agreement between (5.49) and (5.50) for $w = 1, 2$, and the Chern-Simons answer. Moreover, it is in principle possible to compute all the integrals over $\overline{M}_{g,h}$ that appear in (5.36) from the explicit expression (5.48). These Hodge integrals include an arbitrary number of $\psi$ classes and up to three $\lambda$ classes. Therefore, all correlation functions of two-dimensional topological gravity can in principle be extracted from (5.48). It should be noted, however, that some of the simple structural properties of (5.36) are not at all obvious from (5.48). For example, for $g = 0, h = 1$, (5.36) gives a fairly compact expression for the open Gromov-Witten invariant, and the fact that this equals the Chern-Simons answer amounts to a rather nontrivial combinatorial identity. It is also possible to check that the open Gromov-Witten invariants obtained in this way can be expressed in terms of BPS invariants, see [71] for more details.

Unfortunately, although there are proposals for the Lagrangian submanifolds that should correspond to other knots [63, 84], the associated open Gromov-Witten invariants have not been computed yet, so one is forced to test the conjecture in the “weak” sense of showing that one can extract integer invariants from the Chern-Simons invariants in the way described before. This was done in [61, 81, 63] for various knots and links and it was shown in all cases that indeed such invariants can be extracted in a highly nontrivial way. We will give a simple example of this, involving the trefoil knot. By using the known values for the Chern-Simons invariants [83, 88], and the defining relations for the $f$-polynomials (5.12), one can easily obtain:

$$f_{\overline{M}}(q, \lambda) = \frac{q^{-\frac{1}{2}} \lambda (\lambda - 1)^2 (1 + q^2) (q + \lambda^2 q - \lambda (1 + q^2))}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

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Table 2: BPS invariants for the trefoil knot in the symmetric representation.

| g | Q = 1 | 2 | 3 | 4 | 5 |
|---|------|---|---|---|---|
| 0 | -2   | 8 | -12| 8 | -2|
| 1 | -1   | 6 | -10| 6 | -1|
| 2 | 0    | 1 | -2 | 1 | 0|

Table 3: BPS invariants for the trefoil knot in the antisymmetric representation.

| g | Q = 1 | 2 | 3 | 4 | 5 |
|---|------|---|---|---|---|
| 0 | -4   | 16| -24| 16| -4|
| 1 | -4   | 20| -32| 20| -4|
| 2 | -1   | 8 | -14| 8 | -1|
| 3 | 0    | 1 | -2 | 1 | 0|

\[ f_0(q, \lambda) = -\frac{1}{q^2} f_0(q, \lambda). \]  
(5.53)

Notice that, although the Chern-Simons invariants have complicated denominators, the \( f \)-polynomials have indeed the structure (5.13). One can go further and extract the BPS invariants \( N_{g,q}^{0}, Q, \), \( N_{g,q}^{0}, Q \), from (5.53), by using (5.14). The results are presented in Table 2 and Table 3 respectively. The above results have been obtained in the canonical framing. Some integer invariants for the trefoil knot in arbitrary framing are listed in [71]. Results for the BPS invariants of other knots and links can be found in [63].

6 Large \( N \) transitions and toric geometry

The duality between Chern-Simons on \( S^3 \) and closed topological strings on the resolved conifold gives a surprising point of view on Chern-Simons invariants of knots and links. However, from the “gravity” point of view we do not learn much about the closed string geometry, since the resolved conifold is quite simple (remember that it only has one nontrivial Gopakumar-Vafa invariant). It would be very interesting to find a topological gauge theory dual to more complicated geometries, in such a way that we could use our knowledge of the gauge theory side to learn about enumerative invariants of closed strings, and about closed strings in general.

Such a program was started by Aganagic and Vafa in [41]. Their basic idea is to construct geometries that locally contain \( T^*S^3 \)'s, and then follow geometric transitions to dual geometries where the “local” deformed conifolds are replaced by resolved coni-
Figure 10: This shows a Calabi-Yau which is a $T^2 \times \mathbb{R}$ fibration of $\mathbb{R}^3$, where the $\alpha$, $\beta$, and $\alpha + \beta$ cycles of the torus degenerate at three lines.

folds. Remarkably, a large class of non-compact toric manifolds can be realized in this way, as it was made clear in [3]. Let us consider in detail an example that allows one to recover the local $\mathbb{P}^2$ geometry.

Recall from our discussion in section 4 that the deformed conifold can be represented by a graph where one indicates the degeneration loci of the cycles of the torus fiber. Following this idea, one can construct more general $T^2 \times \mathbb{R}$ fibrations of $\mathbb{R}^3$ by specifying degeneration loci in the basis. An example of this is shown in Fig. 10. Notice that this geometry contains three $S^3$'s, represented as dashed lines in Fig. 11. One can then think about a geometric transition where the three-spheres go to zero size, and then the corresponding singularities are blown-up to give a resolved geometry, as shown in Fig. 11. The resolved geometry turns out to be toric, and in fact it can be obtained by three blowups of the Calabi-Yau manifold $\mathcal{O}(-3) \to \mathbb{P}^2$. Up to flops of the three $\mathbb{P}^1$'s, the resulting geometry is the noncompact Calabi-Yau manifold given by the del Pezzo surface $\mathbb{B}_3$ together with its canonical bundle. To recover the local $\mathbb{P}^2$ geometry, one just sends the sizes of the three $\mathbb{P}^1$'s to infinity. The remaining “triangle” is the toric diagram for the local $\mathbb{P}^2$ geometry, see [6, 2].

Let us now wrap $N_i$ branes, $i = 1, 2, 3$, around the three $S^3$'s of the deformed geometry depicted in Fig. 10. What is the effective topological action describing the resulting open strings? For open strings with both ends on the same $S^3$, the dynamics is described by Chern-Simons theory with gauge group $U(N_i)$, therefore we will have three Chern-Simons theories with groups $U(N_1)$, $U(N_2)$ and $U(N_3)$. However, there are new sectors of open strings stretched between two spheres, giving the nondegenerate
Figure 11: This shows the geometric transition of the Calabi-Yau in the previous figure. In the leftmost geometry there are three minimal 3-cycles. The lengths of the dashed lines are proportional to their sizes. The intermediate geometry is singular, and the figure on the right is the base of the smooth toric Calabi-Yau after the transition. This Calabi-Yau is related to $\mathbb{P}_3$ by flopping three $\mathbb{P}^1$’s.

instantons that we described in 4.2, following [95]. Instead of describing these open strings in geometric terms, it is better to use the spacetime physics associated to these strings. In fact, a similar situation was considered when we analyzed the incorporation of Wilson loops in the large $N$ duality. There we had two sets of intersecting D-branes, giving a massless complex scalar field living in the intersection and in the bifundamental representation of the gauge groups. Now, if we focus, say, on the $N_1$, $N_2$ branes, we will get again a complex scalar $\phi$ in $(N_1, N_2)$. This complex scalar is generically massive, and its mass is proportional to the “distance” between the two three-spheres, and it is given by a complexified Kähler parameter that will be denote by $r$. We can now integrate out this complex scalar field to obtain the correction to the Chern-Simons actions on the three-spheres due to the presence of the new sector of open strings, which is given by:

$$
O(U_1, U_2; r) = \exp \left[ -\text{Tr} \log(e^{r/2}U_1^{-1/2} \otimes U_2^{1/2} - e^{-r/2}U_1^{1/2} \otimes U_2^{-1/2}) \right]
$$

$$
= \exp \left\{ \sum_{n=1}^{\infty} \frac{e^{-nr}}{n} \frac{\text{Tr}U_1^n \text{Tr}U_2^{-n}} \right\},
$$

(6.1)

where $U_{1,2}$ are the holonomies of the corresponding gauge fields around a loop. Note that the operator $O$ is the amplitude for a primitive annulus of size $r$ together with its multicovers, as one can see from the first equation of (2.39) for $h = 2$. This annulus “connects” the two $S^3$’s, i.e. one of its boundaries is in one three-sphere, and the other boundary is in the other sphere. The exponent in (6.1) is the contribution to $F_{\text{ndg}}$ in
due to these configurations of open strings, and $r$ is the complexified area of the annulus.

The problem now is to determine how many configurations like this one contribute to the full amplitude. It turns out that the only contributions come from open strings stretching along the degeneracy locus. This was found by Diaconescu, Florea and Grassi [25] using localization arguments, and derived in [3] by exploiting invariance under deformation of complex structures. This result simplifies the problem enormously, and gives a precise description of all the nondegenerate instantons contributing in this geometry: they are annuli stretching along the fixed lines of the $T^2$ action, together with their multicoverings. This is illustrated in Fig. 12. The action describing the dynamics of topological D-branes is then:

$$S = \sum_{i=1}^{3} S_{CS}(A_i) + \sum_{n=1}^{\infty} \frac{1}{n} \left( e^{-nr_1} \text{Tr} U_1^n \text{Tr} U_2^{-n} + e^{-nr_2} \text{Tr} U_2^n \text{Tr} U_3^{-n} + e^{-nr_3} \text{Tr} U_3^n \text{Tr} U_1^{-n} \right),$$

(6.2)

where the $A_i$ are $U(N_i)$ gauge connections on each of the $S^3$'s, $i = 1, 2, 3$, and $U_i$ are the corresponding holonomies around loops. There is a very convenient way to write the free energy of the theory with the above action. First notice that, by following the...
same steps that led to (3.51), one can write the operator (6.1) as

$$O(U_1, U_2; r) = \sum_R \text{Tr}_R U_1 e^{-\ell_r} \text{Tr}_R U_2^{-1},$$

(6.3)

where $\ell$ denotes the number of boxes of the representation $R$. In the situation depicted in Fig. 12 we see that there are two annuli ending on each three-sphere. The boundaries of these annuli give knots, so we have a two-component link in each $S^3$. The holonomies around the components of these links will be in different representations of $U(N)$, as indicated in Fig. 12. Therefore, the free energy will be given by:

$$F = \sum_{i=1}^3 F_{\text{CS}}(N_i, g_s) + \log \left\{ \sum_{R_1, R_2, R_3} e^{-\sum_{i=1}^3 \ell_{r_i}} W_{R_1, R_2}(\mathcal{L}_1) W_{R_2, R_3}(\mathcal{L}_2) W_{R_3, R_1}(\mathcal{L}_3) \right\},$$

(6.4)

where $\ell_i$ is the number of boxes in the representation $R_i$, and $F_{\text{CS}}(N_i, g_s)$ denotes the free energy of Chern-Simons theory with gauge group $U(N_i)$. These correspond to the degenerate instantons that come from each of the three-spheres.

Of course, in order to compute (6.4) we need some extra information: we have to know what are, topologically, the links $\mathcal{L}_i$, and also if there is some framing induced by the geometry. It turns out that these questions can be easily answered by looking at the geometry of the degeneracy locus. The key point is to note that in this geometry the three-spheres represented by dashed lines between two degeneracy loci have natural Heegard splittings into two tori, and the gluing instructions are determined by the $\text{Sl}(2, \mathbb{Z})$ transformation that maps the degenerating cycle at the end of the corresponding three-sphere, to the degenerating cycle at the other end [3]. For example, the three-sphere between the $\alpha$ and the $\beta$ degenerating loci in Fig. 12 comes from gluing two tori with an $S^{-1}$ transformation, which maps the $\alpha$ cycle into the $\beta$ cycle. Following this procedure (see [3] for details) one finds that the $\mathcal{L}_i$ are all Hopf links (see Fig. 2), and that some of the components do actually have nontrivial framing. If we denote the components of $\mathcal{L}_i$ by $\mathcal{K}_i$ and $\mathcal{K}'_i$, $i = 1, 2, 3$, the framings turn out to be the following: $\mathcal{K}_1$, $\mathcal{K}'_1$ and $\mathcal{K}_3$ have framing zero, while the remaining knots have framing $p = 1$. This means that $\mathcal{L}_1$ is in the canonical framing, in $\mathcal{L}_2$ both components are framed, while in $\mathcal{L}_3$ only one of the components, $\mathcal{K}'_3$, is framed. This is depicted in Fig. 13.

What happens now if we go through the geometric transition of Fig. 11? As in the case originally studied by Gopakumar and Vafa, the string coupling constant gives the Chern-Simons “effective” coupling constant $g_s = 2\pi/(k_i + N_i)$ (which is the same for the three theories, see (4.11)), while the ’t Hooft parameters $t_i = g_s N_i$ correspond to the sizes of the three outward legs of the toric diagram on the right side of Fig. 11. The free energy (6.4) is, due to the large $N$ transition, the free energy of topological
Figure 13: The figure shows the Hopf links $L_i$, $i = 1, 2, 3$. The numbers indicate the framing of each knot.

closed strings propagating in that toric geometry. In order to recover just local $\mathbb{P}^2$, we have to take the 't Hooft parameters to infinity, and “tune” the sizes of the annuli at the same time. It turns out that one has to perform a double scaling limit, taking both $t_i$ and $r_i$ to infinity in such a way that

$$r = r_1 - \frac{t_1 + t_3}{2} = r_2 - \frac{t_1 + t_2}{2} = r_3 - \frac{t_2 + t_3}{2}$$

remains finite. Then, $r$ can be identified with the complexified Kähler parameter of local $\mathbb{P}^2$. We refer again to [3] for details. The free energy has in this limit the structure:

$$F = \log \left\{ 1 + \sum_{\ell=1}^{\infty} a_{\ell}(q)e^{-\ell r} \right\} = \sum_{\ell=1}^{\infty} a_{\ell}^{(c)}(q)e^{-\ell r}$$

where $q = e^{ig_s}$. The coefficients $a_{\ell}(q)$, $a_{\ell}^{(c)}(q)$ can be easily obtained in terms of the invariants of the Hopf link in arbitrary representations. One finds, for example [3],

$$a_1(q) = -\frac{3}{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2},$$

$$a_2^{(c)}(q) = \frac{6}{(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^2} + \frac{1}{2}a_1(q^2).$$

If we compare to (2.28) and take into account the effects of multicovery, we find the following values for the Gopakumar-Vafa invariants of $O(-3) \to \mathbb{P}^2$:

$$n_1^0 = 3, \quad n_1^g = 0 \text{ for } g > 0,$$

$$n_2^0 = -6, \quad n_2^g = 0 \text{ for } g > 0,$$

in agreement with the results listed in Table 1. In fact, one can go much further with this method and compute the Gopakumar-Vafa invariants to high degree. The advantage of this procedure is that, in contrast to both the A and the B model computations,
one gets the answer for all genera, see [3] for a complete listing of the invariants up to degree 12.

Although we have focused here on local $\mathbb{P}^2$, one can analyze in a similar way other toric geometries, including local $\mathbb{P}^1 \times \mathbb{P}^1$ and other local del Pezzo surfaces (see also [25, 46]). In fact, one can in principle recover all local toric geometries in this way. We then see that large $\mathcal{N}$ transitions produce gauge theory duals of topological strings propagating on various toric backgrounds. The gauge theory dual is given in general by a product of Chern-Simons theories together with complex scalars in bifundamental representations, and moreover the gauge theory data are nicely encoded in the toric diagram. Other aspects of these dualities for toric manifolds can be found in [3, 25, 46].

7 Conclusions

The remarkable connections between enumerative geometry and knot invariants that have been reviewed in this paper certainly deserve further investigation. Some directions for further research are the following:

1) The correspondence between knot invariants and open Gromov-Witten invariants has been tested only for the unknot. It would be very interesting to test nontrivial knots and improve our understanding of the map relating knots and links in $S^3$ to Lagrangian submanifolds in the resolved conifold. This will certainly open new perspectives in the study of Chern-Simons knot invariants.

2) Another direction to explore is the correspondence between coupled Chern-Simons systems and closed string invariants that we explained in section 6. Extensions to more general local toric geometries, and even to compact geometries, would give a fascinating new point of view on the enumerative geometry of Calabi-Yau threefolds.

3) The “unreasonable effectiveness of physics in solving mathematical problems” [57] has given again surprising results connecting two seemingly unrelated areas of geometry, and we need a deeper mathematical understanding of these connections. For example, the results of section 6 may be understood in terms of the localization techniques introduced in [55], as suggested in [3].

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