CROSSCAP NUMBER TWO KNOTS IN $S^3$ WITH (1,1) DECOMPOSITIONS

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Dedicated to Fico on the occasion of his 60th birthday.

Abstract. M. Scharlemann has recently proved that any genus one tunnel number one knot is either a satellite or 2-bridge knot, as conjectured by H. Goda and M. Teragaito; all such knots admit a (1,1) decomposition. In this paper we give a classification of the family of (1,1) knots in $S^3$ with crosscap number two (i.e., bounding an essential once-punctured Klein bottle).

1. Introduction

H. Goda and M. Teragaito classified in [6] the family of non-simple genus one tunnel number one knots, and conjectured that any genus one tunnel number one simple knot is a 2-bridge knot. This conjecture was shown by H. Matsuda [9] to be equivalent to the statement that any genus one tunnel number one knot in $S^3$ admits a (1,1) decomposition; it is in this form that M. Scharlemann has recently settled it in [13].

In this paper we explore the family of crosscap number two tunnel number one knots in $S^3$. Recall (cf. [1]) that a knot in $S^3$ has crosscap number two if it bounds a once-punctured Klein bottle but not a Moebius band; it was shown in [12] that a knot $K$ has crosscap number two iff its exterior contains a properly embedded essential (incompressible and boundary incompressible, in the geometric sense) once-punctured Klein bottle $F$, in which case $K$ is not a 2-torus knot, and $F$ has integral boundary slope by [8].

In contrast with genus one knots, a crosscap number two knot can bound once-punctured Klein bottles with distinct boundary slopes; however, as shown in [8] [12], such knots are all satellite knots, with the exception of the figure-8 knot and the Fintushel-Stern ($-2,3,7$) pretzel knot. Here we restrict our attention to the family of crosscap number two knots in $S^3$ which admit a (1,1) decomposition; the special cases of tunnel number one satellite knots, 2-bridge knots, and torus knots, are also discussed.

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In order to state our main result we need to define a particular family of \((1,1)\) knots in \(S^3\). Let \(S\) be a Heegaard torus of \(S^3\), and let \(S \times I\) be a product regular neighborhood of \(S\), with \(S\) corresponding to \(S \times \{1/2\}\). An arc \(\beta\) embedded in \(S \times I\) is called *monotone* if the natural projection map \(S \times I \to I\) is monotone on \(\beta\). For \(i = 0, 1\), let \(t_i\) be an embedded nontrivial circle in \(S \times \{i\}\); \(t_i^*\) will denote a \((\pm 1, 2)\) cable of \(t_i\) relative to \(S \times \{i\}\); that is, \(t_i^*\) is the boundary of a Moebius band \(B_i\) obtained by giving a half-twist to a thin annulus intersecting \(S \times \{i\}\) transversely in a core circle isotopic to \(t_i\). Let \(R = \beta \times I\) be a rectangle in \(S \times I\) such that \((B_0 \cup B_1) \cap R = (\partial B_0 \cup \partial B_1) \cap R = \partial \beta \times I\), and such that \(\beta\) is a monotone arc in \(S \times I\). Now let \(K(t_0^*, t_1^*, R)\) be the boundary of \(B_0 \cup R \cup B_1\) (see Fig. 1). With this notation, the following theorem summarizes our main result.

**Theorem 1.1.** Let \(K\) be a crosscap number two knot in \(S^3\). If \(K\) admits a \((1,1)\) decomposition, then \(K\) is either a torus knot, a 2-bridge knot, a satellite knot, or a knot of the form \(K(t_0^*, t_1^*, R)\).

The families of 2-bridge knots and tunnel number one satellite knots, both of which admit \((1,1)\) decompositions, are of independent interest, and we classify those having crosscap number two explicitly; we note here (see Section 3) that the exterior \(X_K\) of a tunnel number one satellite knot \(K \subset S^3\) can be decomposed as the union \(X_L \cup_T X_{K_0}\) for some 2-bridge link \(L\) and torus knot \(K_0\). We call any \((p,q)\) torus knot with \(|p| = 2\) or \(|q| = 2\) a 2-torus knot.

**Theorem 1.2.** Let \(K\) be a crosscap number two knot in \(S^3\); then,

(a) \(K\) is a 2-bridge knot iff \(K\) is a plumbing of an annulus and a Moebius band, i.e., iff \(K\) is of the form \((2m(2n+1)−1)/(2n+1)\) for \(m \neq 0\) (see Fig. 2(a));

(b) \(K\) is a tunnel number one satellite knot, with \(X_K = X_L \cup_T X_{K_0}\), iff, for some integer \(m\), either

(i) \(K_0\) is any nontrivial torus knot and \(L\) is the \(4(4m+2)/(4m+1)\) or \(8(m+1)/(4m+3)\) 2-bridge link (see Fig. 2(b),(c)),

(ii) \(K_0\) is any nontrivial 2-torus knot and \(L\) is the \((8m+6)/(2m+3)\) 2-bridge link (see Fig. 2(d)).
We remark that in Theorem 1.2(b) the knot $K$ is an iterated torus knot iff $m = 0$; in such case, combining the classifications of crosscap number two cable knots in [15] and of tunnel number one cable knots in [2], it follows that $K$ must be an iterated torus knot of the form $[(4pq \pm 1, 4), (p, q)]$ or $[(6p \pm 1, 3), (p, 2)]$ for some integers $p, q$.

Examples of $(1, 1)$ knots of the form $K(t^*_0, t^*_1, R)$ which are neither torus, 2-bridge, nor satellites are provided by the $(p, q, \pm 2)$ pretzel knots with $p, q$ odd integers distinct from $\pm 1$, as shown in Fig. 3; in fact, by [11], these are the only tunnel number one pretzel knots which are not 2-bridge.

Finally, the crosscap number two torus knots are also classified in [15]; the crosscap number of torus knots in general are determined in [14].

**Theorem 1.3 ([15]).** A $(p, q)$ torus knot has crosscap number two iff $(p, q)$ or $(q, p)$ is of the form $(3, 5)$, $(3, 7)$, or $(2(2m + 1)n \pm 1, 4n)$ for some integers $m, n$, $n \neq 0$. 

We will work in the smooth category. In Section 2 we discuss $(g, n)$ decompositions for knots in $S^3$ and prove Theorem 1.2(a). This first case, involving 2-bridge knots, has a pleasant solution arising directly from the classification of $\pi_1$-injective surfaces in 2-bridge knot exteriors by Hatcher and Thurston [7]; we will follow and extend the...
basic ideas of [4, 5, 7] to handle the remaining cases along similar lines, via Morse position of essential surfaces relative to a Heegaard surface product structure. In the process it becomes necessary to deal with essential surfaces \( \Sigma \) in knot or link exteriors, all satisfying \( \chi(\Sigma) = -1 \). Section 3 improves slightly on the theme of [5] to allow for nonorientable essential surfaces in a 2-bridge link exterior; this is the content of Lemma 3.2 which leads to the proof of Theorem 1.2(b). To handle the case of knots with a \((1,1)\) decomposition along the same lines it is necessary to prove a statement similar to Lemma 3.2; this is done in Section 4, where Lemma 4.1 is established and which, along with some results from [9], leads to a proof of Theorem 1.1. Since a once-punctured torus \( \Sigma \) also satisfies \( \chi(\Sigma) = -1 \), the results of this paper can be modified to obtain the classification of genus one knots in \( S^3 \) with a \((1,1)\) decomposition as well.

We want to thank Mario Eudave-Muñoz for making his preprint [3] accessible to us, which motivated the line of argument used in Lemma 4.1.

2. \((g,n)\) decompositions and 2-bridge knots

A knot or link \( L \) in \( S^3 \) is said to be of type \((g,n)\) if there is a genus \( g \) Heegaard splitting surface \( S \) in \( S^3 \) bounding handlebodies \( H_0, H_1 \) such that, for \( i = 0,1 \), \( L \) intersects \( H_i \) transversely in a trivial \( n \)-string arc system. Let \( S \times I \) be a product regular neighborhood of \( S \) in \( S^3 \) and let \( h : S \times I \to I \) be the natural projection map. We denote the level surfaces \( h^{-1}(r) = S \times \{r\} \) by \( S_r \) for each \( 0 \leq r \leq 1 \), and assume that \( S_0 \subset H_0, S_1 \subset H_1 \), and that \( h|S \times I \cap L \) has no critical points (so \( S \times I \cap L \) consists of monotone arcs).

Let \( F \) be an essential surface properly embedded in the exterior \( X_L = S^3 \setminus \text{int} N(L) \) of \( L \); such a surface can always be isotoped in \( X_L \) so that:

1. \( F \) intersects \( S_0 \cup S_1 \) transversely; we denote the surfaces \( F \cap H_0, F \cap H_1, F \cap S \times I \) by \( F_0, F_1, \tilde{F} \), respectively;
2. each component of \( \partial F \) is either a level meridian circle of \( \partial X_L \) lying in some level set \( S_r \) or it is transverse to all the level meridians circles of \( \partial X_L \) in \( S \times I \);
3. for \( i = 0,1 \), any component of \( F_i \) containing parts of \( L \) is a \textit{cancelling disk} for some arc in \( L \cap H_i \) (see Fig. 4); in particular, such cancelling disks are disjoint from any arc of \( L \cap H \) other than the one they cancel;
4. \( h|\tilde{F} \) is a Morse function with a finite set \( Y(F) \) of critical points in the interior of \( \tilde{F} \), located at different levels; in particular, \( \tilde{F} \) intersects each noncritical level surface transversely.

We define the complexity of any surface \( F \) satisfying (M1)–(M4) as the number

\[ c(F) = |\partial F_0| + |\partial F_1| + |Y(F)|, \]

where \( |Z| \) stands for the number of elements in the finite set \( Z \), or the number of components of the topological space \( Z \).

We say that \( F \) is \textit{meridionally incompressible} if whenever \( F \) compresses in \( S^3 \) via a disk \( D \) with \( \partial D = D \cap F \) such that \( D \) intersects \( L \) transversely in one point
interior to $D$, then $\partial D$ is parallel in $F$ to some boundary component of $F$ which is a meridian circle in $\partial X_L$; otherwise, $F$ is meridionally compressible. Observe that if $F$ is essential and meridionally compressible then a ‘meridional surgery’ on $F$ produces a new essential surface in $X_L$.

In the sequel we will concentrate in the case of knots and 2-component links $L$ of types $(0, 2)$ or $(1, 1)$ and certain essential surfaces $F$ in $X_L$ with $\chi(F) = -1$. We close this section with a proof of the first part of Theorem 1.2.

**Proof of Theorem 1.2(a).** Suppose $K$ is a 2-bridge knot with a $(0, 2)$ decomposition relative to some 2-sphere $S$ in $S^3$. In this context, it is proved in [7, Lemma 2] that once $F$ has been isotoped so as to satisfy (M1)–(M4) with minimal complexity, then $F$ lies in $S \times I$ except for the cancelling disk components of $F_0 \cup F_1$, $h|\tilde{F}$ has only saddle critical points, $F \cap S_r$ has no circle components for any $r$, and each saddle joins distinct level arc components. As $\chi(F) = -1$ and $F_0$ consists of two cancelling disks only, $h|\tilde{F}$ has exactly three critical points, so $F$ is a plumbing of an annulus and a Moebius band by [7]; thus that $K$ must be a 2-bridge knot of the form $(2m(2n+1) - 1)/(2n+1)$ for $m \neq 0$ follows from Fig. 2(a), and the claim follows. □

3. Satellite knots

In this section we assume that $K$ is a tunnel number one satellite knot in $S^3$ of crosscap number two. By [2, 10], the exterior $X_K = S^3 \setminus \text{int} N(K)$ of $K$ can be decomposed as $X_L \cup_T X_{K_0}$, where $X_L = S^3 \setminus \text{int} N(L)$ is the exterior of some 2-bridge link $L \subset S^3$ other than the unlink or the Hopf link and $X_{K_0} = S^3 \setminus \text{int} N(K_0)$ is the exterior of some nontrivial torus knot $K_0 \subset S^3$, glued along a common torus boundary component $T$ in such a way that a meridian circle of $L$ in $T$ becomes a regular fiber of the Seifert fibration of $X_{K_0}$.

If $F$ is any once-punctured Klein bottle, then any orientation preserving nontrivial circle embedded in $F$ either cuts $F$ into a pair of pants, splits off a Moebius band from $F$, or is parallel to $\partial F$; in the first case we call such circle a meridian of $F$, while in the second case we call it a longitude (cf. [12, §2]). Notice any meridian and longitude circles of $F$ intersect nontrivially.

As mentioned in the Introduction, $K$ has crosscap number two iff its exterior $X_K$ contains a properly embedded essential once-punctured Klein bottle $F$, in which case $K$ is not a 2-torus knot and $F$ has integral boundary slope. We first show the existence of some once-punctured Klein bottle in $X_K$ which intersects the torus $T$ transversely in a simple way.
Lemma 3.1. Let $K$ be a tunnel number one satellite knot in $S^3$ of crosscap number two. Then there is an essential once-punctured Klein bottle $F \subset X_K = X_L \cup_T X_{K_0}$ which intersects $T$ transversely and such that either:

(i) $F$ lies in $X_L$, or

(ii) $F \cap X_L$ is a once-punctured Moebius band $F_L$ and $F \cap X_{K_0}$ is a Moebius band; in particular, $K_0$ is a 2-torus knot.

Proof. Let $F$ be an essential once-punctured Klein bottle in $X_K$, necessarily having integral boundary slope; we may assume $F$ has been isotoped so as to intersect $T$ transversely and minimally. Hence $T \cap F$ is a disjoint collection of circles which are nontrivial and orientation preserving in both $T$ and $F$, so each such circle is either a meridian or longitude of $F$, or parallel to $\partial F$ in $F$. Thus, the closure of any component of $F \setminus T$ is either an annulus, a Moebius band, a once-punctured Moebius band, a pair of pants, or a once punctured Klein bottle.

Suppose $\gamma \subset T \cap F$ is a component parallel to $\partial F$ in $F$; let $\rho$ denote the slope of a fiber of $X_{K_0}$ in $T$. Then the component of $F \cap X_L$ containing $\partial F$ is an annulus with the same boundary slope as $\gamma$ on $T$. If the slope of $\gamma$ on $T$ is integral then $K$ is isotopic to $K_0$, which is not the case; thus $\gamma$ has nonintegral slope on $T$ and so $K$ is a tunnel number one iterated torus knot with $\Delta(\gamma, \rho) = 1$ by [2, Lemma 4.6].

In particular, as any component of $F \cap X_{K_0}$ must be incompressible and not boundary parallel in $X_{K_0}$ by minimality of $T \cap F$, no such component can be an annulus, a Moebius band, or a pair of pants. Therefore, $F' = F \cap X_{K_0}$ is a once-punctured Klein bottle in $X_{K_0}$ with nonintegral boundary slope $\gamma$ on $T$, and so, by [12, Lemma 4.5], $F'$ must boundary compress in $X_{K_0}$ into a Moebius band $B$ such that $\Delta(\partial F', \partial B) = 2$. But then $K_0$ is a 2-torus knot and $\partial B$ is a fiber of $X_{K_0}$, so $\Delta(\gamma, \rho) = 2$, which is not the case. Therefore, no component of $F \cap T$ is parallel to $\partial F$ in $F$, so either $T \cap F$ is empty and (i) holds or its components are either all meridians or all longitudes of $F$. We now deal with the last two options.

Case 1. The circles $T \cap F$ are all meridians of $F$.

Then the component $P$ of $F \cap X_L$ containing $\partial F$ is a pair of pants with two boundary components $c_1, c_2$ on $T$. If $A$ is an annulus in $T$ cobounded by $c_1, c_2$, then $P \cup A$ is necessarily a once-punctured Klein bottle for $K$ which, after pushing slightly into $X_L$, satisfies (i).

Case 2. The circles $T \cap F$ are all longitudes of $F$.

If the circles $T \cap F$ are not all parallel in $F$ then there are two components of $F \setminus T$ whose closures are disjoint Moebius bands $B_1, B_2$ with boundaries on $T$. But then, if $A$ is an annulus in $T$ cobounded by $\partial B_1, \partial B_2$, the surface $B_1 \cup B_2 \cup A$ is a closed Klein bottle in $X_K \subset S^3$, which is not possible. Hence the circles $T \cap F \subset F$ are mutually parallel in $F$, and so the component of $F \cap X_L$ which contains $\partial F$ is a
once-punctured Moebius band $F_L$. Moreover, there is a component of $F \setminus T$ whose closure is a Moebius band $B$, properly embedded in $X_L$ or $X_{K_0}$. If $B$ lies in $X_L$ then $F \cap X_{K_0}$ is a nonempty collection of disjoint essential annuli in $X_{K_0}$, hence $\partial B$ is the meridian circle of a component of $L$, which implies that $B$ closes into a projective plane in $S^3$, an impossibility. Therefore $B$ lies in $X_{K_0}$, and if $A$ is an annulus in $T$ cobounded by $\partial F_L$ and $\partial B$ then $F_L \cup A \cup B$ can be isotoped into a once punctured Klein bottle for $K$ satisfying (ii).

Denote the components of $L$ by $K_1, K_2$, with $\partial F$ isotopic to $K_1$. We assume that a fixed 2-bridge presentation $L$ is given relative to some 2-sphere $S$ in $S^3$, and that $F$ has been isotoped so as to satisfy (M1)–(M4) and have minimal complexity. Notice that $H_0, H_1$ are 3-balls in this case. The next result will be useful in the sequel.

**Lemma 3.2.** Let $\Sigma'$ be a surface in $S^3$ spanned by $K_1$ (orientable or not) and transverse to $K_2$, such that $\Sigma = \Sigma' \cap X_L$ is essential and meridionally incompressible in $X_L$. If $\Sigma$ is isotoped so as to satisfy (M1)–(M4) with minimal complexity, then $|Y(\Sigma)| = 2 - (\chi(\Sigma) + |\partial \Sigma|)$, and

(i) each critical point of $h|\tilde{\Sigma}$ is a saddle,
(ii) for $0 \leq r \leq 1$ any circle component of $S_r \cap \Sigma$ is nontrivial in $S_r \setminus L$ and $\Sigma$, and
(iii) $\Sigma_0$ and $\Sigma_1$ each consists of one cancelling disk.

**Proof.** If $\Sigma$ is orientable the statement follows from the proof of [5, Theorem 3.1] without any constraints on the boundary of $\Sigma$. If $\Sigma$ is nonorientable, the given hypothesis on $\Sigma$ are sufficient for the arguments of [4, Proposition 2.1] and [7, Lemma 2] to go through and establish (i)–(iii); the meridional incompressibility condition is needed only for (iii), as in [5, Theorem 3.1], while the fact that any circle component of $S_r \cap \Sigma$ is nontrivial in $\Sigma$ follows by the argument of Lemma 4.1(ii). That $|Y(F)| = 2 - (\chi(\Sigma) + |\partial \Sigma|)$ follows now from (i) and (iii). □

**Proof of Theorem 1.2(b).** We will split the argument into several parts, according to Lemma 3.1

**Case (A):** $F \subset X_L$ and $F$ is meridionally incompressible.

In this case Lemma 3.2 applies with $\Sigma = F$, so $|Y(F)| = 2$ and $F \cap S_0, F \cap S_1$ have no circle components. Let $0 < r_1 < r_2 < 1$ be the levels at which the two saddles of $h|\tilde{F}$ are located, and let $\alpha_0, \alpha_1$ denote the arcs $F \cap S_0, F \cap S_1$, respectively. For any level $0 < r < 1$, any circle component of $F \cap S_r$ either separates or does not separate the points $S_r \cap K_2$; the first option is not possible by Lemma 3.2(ii) since $F$ is meridionally incompressible, while in the second option it is not hard to see that, with the aid of the cancelling disk $F_0$, $F$ compresses in $X_L$ along one such level circle (see Fig. 5).

Hence $S_r \cap F$ has no circle components for $0 \leq r \leq 1$, so the saddles, when seen from bottom to top and top to bottom, join the arcs $\alpha_0, \alpha_1$, respectively, in a nonorientable
fashion (see Fig. 5(a)) and so, for a sufficiently small \( \varepsilon > 0 \), \( B_1 = F \cap S \times [r_1 - \varepsilon, r_1 + \varepsilon] \) and \( B_2 = F \cap S \times [r_2 - \varepsilon, r_2 + \varepsilon] \) are Moebius bands in \( F \). For \( i = 1, 2 \), the core circle \( C_i \) of \( B_i \) in \( S_{r_i} \) necessarily separates the points \( K_2 \cap S_{r_i} \), else \( C_i \) bounds a disk \( D_i \) in \( S_{r_i} \) disjoint from \( K_2 \) as in Fig. 5(b), and a boundary compression disk for \( F \) can be constructed from the subdisk \( D'_i \) of \( D_i \) as in Fig. 5(c); also, \( \partial B_i \) is a \((\pm 1, 2)\) cable of \( C_i \). Let \( R \) be the rectangle \( F \cap S \times [r_1 + \varepsilon, r_2 - \varepsilon] \subset F \). As \( h \mid R \) has no critical points, there exists an embedded arc \( \beta \) in \( R \) with one endpoint in \( \partial B_1 \) and the other in \( \partial B_2 \), and such that \( h \mid N(\beta) \) has no critical points for some small regular neighborhood \( N(\beta) \) of \( \beta \) in \( R \); thus \( \beta \) is monotone. As the once-punctured Klein bottle \( F' = B_1 \cup N(\beta) \cup B_2 \) is isotopic in \( X_{K_2} \) to \( F \), it follows that the link \( L \) has the form of Fig. 5(a) up to isotopy (see Fig. 7), and hence that \( L \) is a \( 4(4m + 2)/(4m + 1) \) 2-bridge link.

**Case (B):** \( F \subset X_L \) and \( F \) is meridionally compressible.

Observe that if \( F \) meridionally compresses along a circle \( \gamma \subset F \) then \( \gamma \) must be a meridian circle of \( F \): for if \( \gamma \) is trivial in \( F \) then a 2-sphere in \( S^3 \) can be constructed which intersects \( K_2 \) in one point, if \( \gamma \) is a longitude in \( F \) then \( S^3 \) contains \( RP^2 \), and if \( \gamma \) is parallel to \( \partial F \) then \( L \) is the Hopf link. Thus, \( F \) meridionally compresses into an essential pair of pants \( \Delta \) in \( X_L \), which is necessarily meridionally incompressible. By Lemma 5.2 we may therefore assume that \( \Delta \) satisfies (M1)–(M4) and lies within the region \( S \times I \) except for the cancelling disks \( \Delta_0, \Delta_1 \), and \( |Y(\Delta)| = 0 \).

Since \( \Delta \) is orientable, the saddles must join the corresponding arcs \( \alpha_0 = \Delta \cap S_0, \alpha_1 = \Delta \cap S_1 \) to themselves in an orientable fashion or to a level circle component, when seen from bottom to top and top to bottom, respectively. Let \( C_1, C_2 \) be the two level boundary circles of \( \Delta \), and let \( C_3, C_4 \) be the *limiting* circles in the saddle levels (see

![Figure 5.](image-url)

**Figure 5.**

![Figure 6.](image-url)

**Figure 6.** Boundary compression of \( F \).
Fig. 7: Isotoping the arc $\beta$.

Fig. 8.

Fig. 8: we assume that, for $1 \leq i \leq 4$, the $C_i$'s are located at distinct levels $r_i$, respectively. If $r_j$ and $r_k$ are the lowest and highest levels in this list, respectively, then there exists an embedded arc $\beta$ in $\tilde{\Delta}$ with one endpoint in $C_j$ and the other in $C_k$, such that $h|N(\beta)$ has no critical points for some small regular neighborhood $N(\beta)$ of $\beta$ in $\tilde{\Delta}$ (see Fig. 8). Then a small regular neighborhood $N(C_j \cup \beta \cup C_k)$ in $\Delta$ yields a 2-punctured disk with boundary isotopic to $K_1$ in $X_{K_2}$. As in Case (A), it follows that $L$ can be isotoped into the form of Fig. 2(b), so $L$ is a $8(m+1)/(4m+3)$ 2-bridge link. □ (Case (B))

Therefore part (i) holds when $F \subset X_L$. We now handle the last possible case.

Case (C): $F \cap X_L = F_L$.

As for any level $0 \leq r \leq 1$ each circle component of $F_L \cap S_r$ is either parallel to the boundary circle of $F_L$ isotopic to $K_1$, or parallel to the boundary circle of $F_L$ which is a level meridian of $K_2$, and $L$ is neither the unlink nor the Hopf link, it follows that $F_L$ is incompressible and meridionally incompressible, hence Lemma 3.2 applies.
Therefore, the method of proof used in Case (B) above immediately implies that \( L \) is isotopic to a link of the form of Fig. 2(c), hence (ii) holds in this case.

Since clearly any knot constructed as above has crosscap number two, the theorem follows. \( \square \)

4. Knots with \((1,1)\) decompositions

In this section we assume that \( K \) is a crosscap number two knot in \( S^3 \) admitting a \((1,1)\) decomposition relative to some Heegaard torus \( S \) of \( S^3 \). In this case the handlebodies \( H_0, H_1 \) are solid tori with meridian disks of slope \( \mu_0, \mu_1 \) in \( S_0, S_1 \), respectively. For \( \{i, j\} = \{0, 1\} \), we project \( \mu_j \) onto \( S_i \), continue to denote such projection by \( \mu_j \), and frame \( S_i \) via the circles \( \mu_i, \mu_j \), so that a \((p, q)\)-circle in \( S_i \) means a circle embedded in \( S_i \) isotopic to \( p\mu_i + q\mu_j \); thus \( S_i \) gets the standard framing as the boundary of the exterior of the core of \( H_i \), and a \((p, q)\)-circle in \( S_0 \) is isotopic in \( S \times I \) to a \((q, p)\)-circle in \( S_1 \).

Before studying the associated essential once-punctured Klein bottle for \( K \), we prove a statement similar to Lemma 3.2 in the present context.

**Lemma 4.1.** Suppose \( K \) is not a torus knot. Let \( \Sigma' \) be a spanning surface for \( K \) in \( S^3 \) (orientable or not) such that \( \Sigma = \Sigma' \cap X_K \) is essential in \( X_K \). If \( \Sigma \) is isotoped so as to satisfy (M1)–(M4) with minimal complexity, then \( |Y(\Sigma)| = 1 - \chi(\Sigma) \), and

(i) each critical point of \( h|\Sigma^0 \) is a saddle,

(ii) for \( 0 \leq r \leq 1 \) any circle component of \( S_r \cap \Sigma \) is nontrivial in \( S_r \setminus K \) and \( \Sigma \), and not parallel in \( \Sigma \) to \( \partial \Sigma \),

(iii) for \( i = 0, 1 \) \( \Sigma_i \) consists of one cancelling disk and either one Moebius band and some annuli components, or a collection of disjoint annuli each having boundary slope \( (p_i, q_i) \) in \( S_i \) with \( |q_i| \geq 2 \), and

(iv) the saddle closest to either the 0-level or 1-level does not join circle components.

**Proof.** Part (i) follows from the argument of [11 Proposition 2.1].

Suppose now that \( \gamma \) is a circle component of \( S_r \cap \Sigma \) for some level \( 0 \leq r \leq 1 \). If \( \gamma \) bounds a disk \( D \) in \( S_r \setminus K \) then \( \gamma \) bounds a disk \( D' \) in \( \Sigma \), since \( \Sigma \) is incompressible in \( X_K \). Construct a surface \( \Sigma'' \) isotopic to \( \Sigma \) from \( (\Sigma \setminus D') \cup D \) by pushing \( D \) slightly above or below \( S_r \), so that \( \Sigma'' \) satisfies (M1)–(M4) and the singularities of \( h|\Sigma'' \) are exactly those of \( \Sigma \setminus D' \) with an additional local extremum in the interior of \( D \); thus, \( h|\Sigma'' \) has at most \( |Y(\Sigma)| + 1 \) critical points.

If \( D' \) is disjoint from \( S_0 \cup S_1 \) then \( D' \) lies in \( S \times I \) and, since \( \partial D' \) is level, \( h|D' \) has a local extremum in \( \partial D' \), contradicting (i). If \( D' \) intersects \( S_0 \cup S_1 \) then \( |\partial \Sigma_0''| + |\partial \Sigma_1''| < |\partial \Sigma_0| + |\partial \Sigma_1| \) while \( |Y(\Sigma'')| \leq |Y(\Sigma)| + 1 \), hence \( c(\Sigma'') \leq c(\Sigma) \) and so \( c(\Sigma'') = c(\Sigma) \) by minimality of \( c(\Sigma) \), again contradicting (i). Therefore, \( \gamma \) is nontrivial in \( S_r \setminus K \) and, since \( K \) is not a torus knot, \( \gamma \) is not parallel in \( \Sigma \) to \( \partial \Sigma \). Thus it only remains
to verify that $\gamma$ is nontrivial in $\Sigma$ for (ii) to hold, which we will do by the end of the proof.

If some component of $\Sigma_0$, other than the cancelling disk, compresses in $H_0$, then there is one such component $\sigma$ which compresses in $H_0$ via a disk $D$ disjoint from all other components of $\Sigma_0$. Since $\Sigma$ is essential in $X_K$, $\partial D$ bounds a disk $D'$ in $\Sigma$. Let $\Sigma'' = (\Sigma \setminus D') \cup D$. Then $h|\Sigma''$ has at most $|Y(\Sigma)|$ singular points and, since $\text{int} D'$ necessarily intersects $S_0 \cup S_1$, $|\partial \Sigma''| < |\partial \Sigma_0| + |\partial \Sigma_1|$, and so $\sigma(\Sigma'') < c(\Sigma)$, an impossibility. Therefore, any component of $\Sigma_0$ is incompressible in $H_0$, hence it must be either an annulus, a Moebius band, or a disk; since $H_0$ is a solid torus, $\Sigma_0$ may have at most one Moebius band component.

Suppose $\Sigma_0$ has an annulus component $\sigma$; then $\sigma$ separates $H_0$ into two pieces, one of which contains the cancelling disk component of $\Sigma_0$. If $\sigma$ is parallel in $H_0$ into $S_0$ away from all other components of $\Sigma_0$, then $\sigma$ can be pushed into the region $S \times I$; notice this is the case if the slope of $\sigma$ in $S_0$ is of the form $(p_0, q_0)$ with $|q_0| = 1$. It is then possible to isotope $\sigma$ and $\Sigma$ appropriately, so that $h|\sigma$ has one saddle and one local minimum and $\Sigma$ continues to satisfy (M1)–(M4); hence $|\partial \Sigma_0|$ will decrease by two while $|Y(\Sigma)|$ will increase by two, and so $c(\Sigma)$ will remain minimal. However, this time $h|\Sigma$ has a local minimum critical point in $\sigma$, contradicting (i). Therefore, since $\sigma$ is incompressible in $H_0$, any boundary component of $\sigma$ must be nontrivial in $S_0 \setminus K$ and distinct from $\mu_0$, so it follows that the boundary slope of $\sigma$ in $S_0$ is of the form $(p_0, q_0)$ with $|q_0| \geq 2$.

Consider the first saddle above level 0; if it joins a circle component $\gamma$ of $\Sigma \cap S_0$ to itself or to another such circle component then it is possible to lower the saddle below level $S_0$ while satisfying (M1)–(M4), thus reducing the value of $c(\Sigma)$, which is not possible. Hence (iv) holds, and the first saddle above level 0 joins the arc component $\alpha_0$ of $S_0 \cap \Sigma$ to itself or to a circle component.

Suppose now that $\sigma$ is a disk component of $\Sigma_0$ other than the cancelling disk; then $\sigma$ is either a trivial disk or a meridian disk of $H_0$. In the first case, $\sigma$ separates $H_0$ into a 3-ball $B^3$ and a solid torus, with the cancelling disk of $\Sigma_0$ contained in $B^3$ by the first part of (ii); we may further assume that $\partial \sigma$ and $\alpha_0$ are adjacent in $S_0$. Consider the first saddle above level 0. If it joins the arc component $\alpha_0$ of $\Sigma \cap S_r$ to itself then either a Moebius band is created by the saddle with core a circle bounding a disk in the saddle level, so $\Sigma$ is boundary compressible (see Fig. 10), or a trivial circle component is created in a level slightly above the saddle level, contradicting the first part of (ii). If the saddle joins $\partial \sigma$ to $\alpha_0$ then pushing down the saddle slightly below level 0 isotopes $\Sigma$ so as to still satisfy (M1)–(M4) but lowers its complexity. Since by (iv) these are the only possibilities for the first saddle, if $\Sigma_0$ contains any disk components other than the cancelling disk then all such components are meridian disks of $H_0$. The analysis of the possible scenarios for the first saddle above level 0 is similar to that of the previous cases, except for when the saddle joins $\alpha_0$ to itself as in Fig. 11(a). In such case, if $r$ is the level of the first saddle above level 0, the Moebius band created by the saddle has as core a circle in $S_r$ which bounds a meridian disk of the solid torus bounded by $S_r$ below the level $S_r$ (see Fig 11(b)). The situation is
similar to that of Fig. 6, so Σ is boundary compressible, which is not the case. Hence Σ₀, and similarly Σ₁, has no such disk components and (iii) holds.

Now let 0 ≤ r ≤ 1 and γ be any circle component of (S₀ ∪ Sᵣ ∪ S₁) ∩ Σ. If γ is trivial and innermost in Σ then it bounds a subdisk D in Σ with interior disjoint from S₀ ∪ Sᵣ ∪ S₁, hence D lies either in Σ₀, Σ₁, or S × I. But, as shown above, neither Σ₀ nor Σ₁ have disk components other than the cancelling disks, and if D lies in S × I then, as ∂D = γ is level, h|D must have a local extremum in int D, contradicting (i). Hence γ is nontrivial in Σ and so the proof of (ii) is complete. That |Y(Σ)| = 1 − χ(Σ) now follows from (i) and (iii).

In preparation for the proof of Theorem 1.1, the following result specializes Lemma 4.1 to the case when Σ is a once punctured Klein bottle F; its first part is a slight generalization of a construction by Matsuda in [9, pp. 2161–2162]. We will say that an essential annulus A properly embedded in S × I is an F-spanning annulus if A can be isotoped so as to be disjoint from the component of ˜F = F ∩ S × I containing parts of K, and its boundary slope in S₀ is of the form (p, q) for some |p|, |q| ≥ 2. Notice that an F-spanning annulus A is isotopic in S × I to the annulus (∂A ∩ S₀) × I, and its boundary component in S₁ has slope (q, p).

Lemma 4.2. Let F be an essential once-punctured Klein bottle spanned by K which has been isotoped so as to satisfy (M1)–(M4) with minimal complexity. If there is an F-spanning annulus in S × I having boundary slope (p, q) in S₀ then K is either a (p, q) torus knot or a satellite of a (p, q)-torus knot; otherwise, F ∩ (S₀ ∪ S₁) has at most two circle components.

Proof. Let F′ denote the component of ˜F = F ∩ S × I containing parts of K. Let A be an F-spanning annulus with boundary slope (p, q) in S₀, and suppose K is not a (p, q) torus knot. By Lemma 4.1(ii),(iii), F′ is either a once-punctured Moebius band or a pair of pants embedded in the solid torus V = S × I \ int N(A), where N(A) is a small regular neighborhood of A in S × I. In either case, ∂F′ has one component K′ ⊂ int V which is isotopic to K in S³, and one or two more components embedded in ∂V, each running once around V. Notice that V is a regular neighborhood of a (p, q) torus knot, so K′ is not a core of V.
If $F'$ is a once-punctured Moebius band then $K'$ is a nontrivial knot in $V$ with odd winding number. If $F'$ is a pair of pants then, by Lemma 4.1(ii),(iii), the closure of $F \setminus F'$ consists either of two Moebius bands or an annulus with core a meridian circle of $F$. In the first case $F_0$ and $F_1$ each have a Moebius band component which, due to the presence of the spanning annulus $A$, have boundary slopes $(p, q)$ and $(q, p)$, respectively, an impossibility since then $|p| = |q| = 2$; in the latter case, the closure of the annulus $F \setminus F'$ intersects $V$ in annuli running once around $V$, thus it can be isotoped in $S^3$, away from $F'$, into $S^3 \setminus \text{int } V$, and so the components of $\partial F'$ other than $K'$ must be coherently oriented in $\partial V$; therefore $K'$ has winding number two in $V$ and hence it is a nontrivial satellite of the core of $V$. The first part of the lemma follows.

Suppose now that $F \cap (S_0 \cup S_1)$ has at least three circle components; if, say, three such components lie in $S_0$, or at least two lie in $S_0$ and at least one in $S_1$, then, since $|Y(F)| = 2$ by Lemma 4.1 and the saddles do not join circle components, at least one of the circle components of $F \cap S_0$ must flow along an annulus component of $\tilde{F}$ from $S_0$ to $S_1$ without interacting with the saddles. Thus $\tilde{F}$ has at least one annulus component which, by Lemma 4.1(iii), has boundary slope of the form $(p, q)$ in $S_0$ for some $|p|, |q| \geq 2$, and so must be an $F$-spanning annulus. Thus the second part of the lemma follows.

**Proof of Theorem 1.1**. Let $K$ be a crosscap number two knot in $S^3$, and let $F$ be an essential once-punctured Klein bottle spanned by $K$; we assume $F$ has been isotoped so as to satisfy (M1)–(M4) with minimal complexity. To simplify notation, let $F'_0, F'_1$ denote the components of $F_0, F_1$, respectively, other than the cancelling disks. By Lemma 4.2 if $K$ is neither a torus nor a satellite knot then $S \times I$ contains no $F$-spanning annuli and $F \cap (S_0 \cup S_1)$ has at most two circle components; thus, without loss of generality, $F'_0$ and $F'_1$ fit in one of the following cases.

**Case (A):** $F'_0$ is an annulus and $F'_1$ is empty.

Fig. 10(a) shows the only possible construction (abstractly) of the surface $F$, starting from $F_0$, via the two saddles of $h|\tilde{F}$. By Lemma 4.1(iii), the boundary slope of the annulus $F'_0$ in $S_0$ is of the form $(p, q)$ with $|q| \geq 2$. It is not hard to see that the boundary circle $C$ of the annulus $\partial F'_0$ in Fig. 10(a) bounds an essential annulus $A$ in $S \times I \setminus F$, hence $|p| = 1$ since $S \times I$ has no $F$-spanning annuli, so $K$ is a 2-bridge knot by the argument of [9, pp. 2161–2162].

**Case (B):** Both $F'_0$ and $F'_1$ are Moebius bands.

The only possibility in this case is the one shown (abstractly) in Fig. 10(b): for otherwise, by Lemma 4.1(iv), the first saddle above the 0-level would join the arc component of $S_0 \cap F$ with itself, necessarily in an orientable fashion, and so $S_r \cap F$ would have two circle components for any level $r$ in between the saddle levels; but then the first saddle below the 1-level must join the circle component of $S_1 \cap F$ with itself, contradicting Lemma 4.1(iv).
Hence \( \tilde{F} \) is a pair of pants and, since all the critical points of \( h|\tilde{F} \) are saddles, there exists an embedded arc \( \beta \) in \( \tilde{F} \) with one endpoint in \( \partial F'_0 \) and the other in \( \partial F'_1 \) which is monotone in \( S \times I \) and such that \( h|R \) has no critical points for some small regular neighborhood \( R \) of \( \beta \) in \( \tilde{F} \). Observe that, for \( i = 0, 1 \), if \( \partial F'_i \) is a \((p_i, 2)\)-circle in \( S_i \), then \( F'_i \) is isotopic in \( S^3 \) to a Moebius band \( B_i \) which is a \((1, 2)\) cable of a \((p_i, 1)\)-circle \( t_i \) in \( S_i \). Therefore the once-punctured Klein bottle \( F'_0 \cup R \cup F'_1 \) can be isotoped into \( B_0 \cup R' \cup B_1 \) for some monotone subrectangle \( R' \) of \( R \). As \( F'_0 \cup R' \cup F'_1 \) is isotopic to \( F \) in \( S^3 \), it follows that \( K \) is a knot of the form \( K(t_0^*, t_1^*, R') \).

**Case (C):** Both \( F'_0 \) and \( F'_1 \) are empty.

In this case the saddles, when read from bottom to top and top to bottom, must join the arcs \( S_0 \cap F, S_1 \cap F \) with themselves, respectively, both in an orientable fashion or both in a nonorientable fashion; the possible cases are described (abstractly) in Fig. 11. In the case of Fig. 11(a), if the level circle \( C \) has slope \((p, q)\) relative to \( S_0 \), then there is an essential annulus in \( S \times I \setminus \tilde{F} \) with boundary slope \((p, q)\) in \( S_0 \). Hence \(|p| = 1 \) or \(|q| = 1 \) since \( S \times I \) has no \( F \)-spanning annuli, so \( K \) is a 2-bridge knot by the argument of [9, pp. 2161–2162].

In the case of Fig. 11(b) let \( 0 < r_1 < r_2 < 1 \) be the saddle levels and, for \( i = 1, 2 \), let \( B_i \) be the Moebius band \( F \cap S \times [r_i - \varepsilon, r_i + \varepsilon] \) for a sufficiently small \( \varepsilon > 0 \). Then \( F \cap S \times [r_1 + \varepsilon, r_2 - \varepsilon] \) is a rectangle \( R \), and \( B_1 \cup R \cup B_2 \) is a once-punctured Klein bottle isotopic to \( F \) in \( S^3 \). Hence \( K \) is a knot of the form \( K(t_0^*, t_1^*, R') \), where \( t_i \) is the core of the Moebius band \( B_i \) in the level \( r_i \) and \( R' \) is a monotone subrectangle of \( R \).

**Case (D):** \( F'_0 \) is a Moebius band and \( F'_1 \) is empty.
Suppose the first saddle below the 1-level joins the arc component of $F \cap S_1$ with itself in an orientable fashion; then the first saddle above the 0-level necessarily joins the arc component of $F \cap S_0$ with itself in a nonorientable fashion. The situation here is similar to that of Case (A): the circle $\partial F'_0$ bounds an annulus $A$ in $S \times I$ which can be isotoped away from $\tilde{F}$ (see Fig. 10(a), with $C = \partial F'_0$), hence the slope of $\partial A$ in $S_1$ must be integral and so $K$ is a 2-bridge knot.

Otherwise, the first saddle below the 1-level, say at level $0 < r_1 < 1$, joins the arc component of $F \cap S_1$ with itself in a nonorientable fashion, while the first saddle above the 0-level joins the arc component of $F \cap S_0$ with the circle $\partial F'_0$. This time the situation is similar to that of Cases (B) and the second part of (C): for a small $\varepsilon > 0$, if $B_1$ is the Moebius band $F \cap S \times [r_1 - \varepsilon, r_1 + \varepsilon]$, then $R = F \cap S \times [0, r_1 - \varepsilon]$ is a rectangle and $F'_0 \cup R \cup B_1$ is a once-punctured Klein bottle isotopic to $F$ in $S^3$, hence $K$ is a knot of the form $K(t_0^*, t_1^*, R')$, where $R'$ is a monotone subrectangle of $R$ and $t_0, t_1$ can be described as in Cases (B) and (C), respectively. □

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