FACTORIZATION, MAJORIZATION, AND DOMINATION
FOR LINEAR RELATIONS

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To our friend Zoltán Sebestyén on the occasion of his 70th birthday

Abstract. Let \( H_A, H_B, \) and \( H \) be Hilbert spaces. Let \( A \) be a linear relation from \( H \) to \( H_A \) and let \( B \) be a linear relation from \( H \) to \( H_B \). If there exists an operator \( Z \in B(H_B, H_A) \) such that \( ZB \subset A \), then \( B \) is said to dominate \( A \).

This notion plays a major role in the theory of Lebesgue type decompositions of linear relations and operators. There is a strong connection to the majorization and factorization in the well-known lemma of Douglas, when put in the context of linear relations. In this note some aspects of the lemma of Douglas are discussed in the context of linear relations and the connections with the notion of domination will be treated.

1. Introduction

Let \( A \) and \( B \) be a pair of linear relations with their domains of definition in the same Hilbert space \( H \) and their ranges in the Hilbert spaces \( H_A \) and \( H_B \), respectively. The relation \( B \) is said to dominate the relation \( A \) if there exists a bounded linear operator \( Z \) from \( H_B \) to \( H_A \) such that \( ZB \subset A \). Domination is preserved when the closures of \( A \) and \( B \) are considered. In the particular case that \( A \) and \( B \) are, not necessarily densely defined, operators this is equivalent to \( \text{dom } A \subset \text{dom } B \) and the existence of a constant \( c \geq 0 \) such that \( \|Af\| \leq c\|Bf\| \) holds for all \( f \in \text{dom } A \). The notion of domination, which is familiar from measure theory, plays an important role in the theory of Lebesgue type decompositions. This notion and its role in Lebesgue type decompositions for a pair of bounded operators go back to Ando [1]; it has a similar position when decomposing a nonnegative form with respect to another nonnegative form, see [11], or when decomposing an unbounded operator or a linear relation [12, 13, 14, 15], where some further history and references can be found.

In the present paper it will be shown that domination is closely related to the following well-known lemma of R.G. Douglas [6] when that lemma is put in the context of unbounded linear operators or, more generally, linear relations.

Lemma 1.1 (Douglas). Let \( A, B \in B(\mathcal{H}, \mathcal{K}) \), the bounded everywhere defined linear operators from a Hilbert space \( \mathcal{H} \) to a Hilbert space \( \mathcal{K} \). Then the following statements are equivalent:

(i) \( \text{ran } A \subset \text{ran } B \);
(ii) \( A = BW \) for some bounded linear operator \( W \in B(\mathcal{H}) \);
(iii) \( AA^* \leq \lambda BB^* \) for some \( \lambda \geq 0 \).

If the equivalent conditions (i) – (iii) hold, then there is a unique operator \( W \) such that
(a) \( \|W\|^2 = \inf \{ \mu : AA^* \leq \mu BB^* \} \);
(b) \( \ker A = \ker W \);
(c) \( \text{ran } W \subset \text{ran } B^* \).

In the literature one can find a statement which is equivalent to the three items in Lemma 1.1, namely

(iv) \( AA^* = BMB^* \), where \( M \in \mathcal{B}(\mathcal{A}) \) is nonnegative and \( \|M\| \leq \lambda \).

One may take \( \text{ran } M \subset \text{ran } B^* \). In addition to the results in the above lemma Douglas indicated some further results for the case when \( A \) and \( B \) are densely defined closed linear operators; see [6]. Various extensions of these basic results by Douglas can be found in the literature; see, for instance, [4, 7, 8]. The factorization aspect of the Douglas lemma was recently put in the context of linear relations by D. Popovici and Z. Sebestyén [16]; see also some refinements by A. Sandovici and Z. Sebestyén [17]. For the majorization aspect of the Douglas lemma, see [3].

The contents of the present paper are now briefly explained. For closed linear operators or relations \( A \) and \( B \) the following equivalence will be established in Theorem 3.4:

\[ A \subset BW \iff AA^* \leq c^2 BB^* , \]

where \( W \) is a bounded linear operator and \( c \geq 0 \), in fact \( \|W\| \leq c \). This result characterizes majorization in terms of a simple factorization type inclusion. Domination for a pair of closed linear operators or relations can be characterized in a similar way:

\[ ZB \subset A \iff A^* A \leq c^2 B^* B , \]

see Theorem 4.4. Some consequences of these results will be explored in Section 3 and Section 4. In particular, a characterization of the equalities \( A = BW \) and \( ZB = A \) is given. For bounded linear operators the factorization \( A = BW \) in the original Douglas lemma can be directly connected to the notion of domination for linear relations by means of the following observation:

\[ A = BW \iff WA^{-1} \subset B^{-1} , \]

see Lemma 5.1. This last equivalence, when combined with the two earlier equivalences, provides a simple proof for the characterization of the ordering of nonnegative selfadjoint relations in terms of resolvents; see Theorem 5.2. For the convenience of the reader some results concerning closed nonnegative forms and associated linear relations will be recalled in Section 2.

2. PRELIMINARIES

Let \( H \) be a linear relation from a Hilbert space \( \mathcal{H} \) to a Hilbert space \( \mathcal{K} \); i.e., \( H \) is a linear subspace of the product \( \mathcal{H} \times \mathcal{K} \). The domain, range, kernel, and multivalued part of \( H \) are denoted by \( \text{dom } H \), \( \text{ran } H \), \( \ker H \), and \( \text{mul } H \). The formal inverse \( H^{-1} \) of \( H \) is a relation from \( \mathcal{K} \) to \( \mathcal{H} \), defined by \( H^{-1} = \{ (f', f) : (f, f') \in H \} \), so that \( \text{dom } H^{-1} = \text{ran } H \), \( \text{ran } H^{-1} = \text{dom } H \), \( \ker H^{-1} = \text{mul } H \), and \( \text{mul } H^{-1} = \ker H \). For \( \mathcal{L} \subset \mathcal{H} \) the set \( H(\mathcal{L}) \) is a subset of \( \mathcal{K} \) defined by

\[ H(\mathcal{L}) = \{ h' : (h, h') \in H \text{ for some } h \in \mathcal{L} \} \).

In particular, \( H(\{0\}) = \text{mul } H \).

Let \( H_1 \) and \( H_2 \) be relations from a Hilbert space \( \mathcal{H} \) to a Hilbert space \( \mathcal{K} \). Then \( H_1 \) is a restriction of \( H_2 \) and \( H_2 \) is an extension of \( H_1 \) if \( H_1 \subset H_2 \).
Proposition 2.1. Let $H_1$ and $H_2$ be relations from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and assume that $H_1 \subset H_2$. Then the following statements are equivalent:

1. $\text{dom } H_1 = \text{dom } H_2$; 
2. $H_2 = H_1 + \{0\} \times \text{mul } H_2$.

Moreover, the following statements are equivalent:

1. $\text{ran } H_1 = \text{ran } H_2$; 
2. $H_2 = H_1 + (\ker H_2 \times \{0\})$.

Proof. By symmetry it suffices to show the equivalence between (i) and (ii).

(i) $\Rightarrow$ (ii) It suffices to show that $H_2 \subset H_1 + \{0\} \times \text{mul } H_2$. Let $h, h' \in H_2$. Since $h \in \text{dom } H_2 \subset \text{dom } H_1$, there exists an element $k' \in \mathcal{K}$ such that $\{h, k'\} \in H_1$. Hence, with $\varphi' = h' - k'$, it follows that

$$\{h, h'\} = \{h, k'\} + \{0, \varphi'\},$$

and thus $\{0, \varphi'\} \in H_2$ or $\varphi' \in \text{mul } H_2$. Hence (ii) follows.

(ii) $\Rightarrow$ (i) This implication is trivial. \hfill $\Box$

The useful result in the following corollary can be found in [2].

Corollary 2.2. Let $H_1$ and $H_2$ be relations from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and assume that $H_1 \subset H_2$. Then the following statements are equivalent:

1. $H_1 = H_2$; 
2. $\text{dom } H_1 = \text{dom } H_2$ and $\text{mul } H_1 = \text{mul } H_2$; 
3. $\text{ran } H_1 = \text{ran } H_2$ and $\ker H_1 = \ker H_2$.

Corollary 2.3. Let $H_1$ and $H_2$ be relations from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$ and assume that $H_1 \subset H_2$. Then

1. $\text{dom } H_1 = \mathcal{K}$, $\text{mul } H_2 = \{0\}$ $\Rightarrow$ $H_1 = H_2$; 
2. $\text{ran } H_1 = \mathcal{K}$, $\ker H_2 = \{0\}$ $\Rightarrow$ $H_1 = H_2$.

The sum of two linear relations $H_1$ and $H_2$ from $\mathcal{H}$ to $\mathcal{K}$ is a linear relation defined by

$$H_1 + H_2 = \{(f, f') : \{f, f''\} \in H_1, \{f, f''\} \in H_2\},$$

while their componentwise sum is a linear relation defined by

$$H_1 + H_2 = \{(g + f', g' + f'') : \{f, f''\} \in H_1, \{g, g''\} \in H_2\}.$$

Let $H_1$ be a relation from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{M}$ and let $H_2$ be a relation from a Hilbert space $\mathcal{M}$ to a Hilbert space $\mathcal{K}$. The product $H_2 H_1$ is a linear relation from $\mathcal{Y}$ to $\mathcal{K}$ defined by

$$(2.1) \quad H_2 H_1 = \{(f, f') : \{f, f''\} \in H_1, \{f, f''\} \in H_2 \text{ for some } \varphi \in \mathcal{M}\}.$$

Observe, that

$$(2.2) \quad \ker (H_2 H_1) = H_1^{-1}(\ker H_2) = \{f \in \mathcal{H} : \{f, \varphi\} \in H_1 \text{ for some } \varphi \in \ker H_2\},$$

and

$$(2.3) \quad \text{mul } H_2 H_1 = H_2(\text{mul } H_1) = \{f' \in \mathcal{K} : \{\varphi, f'\} \in H_2 \text{ for some } \varphi \in \text{mul } H_1\}.$$

In particular, $\ker H_1 \subset \ker H_1 H_2$ and $\text{mul } H_2 \subset \text{mul } H_2 H_1$. The following identities are also easy to check:

$$(2.4) \quad H H^{-1} = I_{\text{ran } H} + (\{0\} \times \text{mul } H) \quad \text{and} \quad H^{-1} H = I_{\text{dom } H} + (\{0\} \times \ker H).$$
with both sums direct. Hence, in particular,

\[(2.5) \quad \text{mul} H = \{0\} \Rightarrow HH^{-1} = I_{\text{ran} H}; \quad \ker H = \{0\} \Rightarrow H^{-1} H = I_{\text{dom} H}.\]

The closure of a linear relation $H$ from $\mathfrak{H}$ to $\mathfrak{K}$ is the closure of the linear subspace in $\mathfrak{H} \times \mathfrak{K}$, when the product is provided with the product topology. The closure of an operator need not be an operator; if it is then one speaks of a closable operator. The relation $H$ is called closed when it is closed as a subspace of $\mathfrak{H} \times \mathfrak{K}$. In this case both $\ker H \subset \mathfrak{H}$ and $\text{mul} H \subset \mathfrak{K}$ are closed subspaces.

Let $H$ be a closed linear relation from $\mathfrak{H}$ to $\mathfrak{K}$. Then $H_{\text{mul}} = \{0\} \times \text{mul} H$ is a closed linear relation and $H_{s} = H \ominus H_{\text{mul}}$, so that $\text{dom} H_{s} = \text{dom} H$ is dense in $\text{dom} H = \mathfrak{H} \ominus \text{mul} H^\ast$, while $\text{ran} H_{s} \subset \text{dom} H^\ast = \mathfrak{K} \ominus \text{mul} H$. The operator part $H_{s}$ and $H_{\text{mul}}$ lead to the componentwise orthogonal decomposition

\[(2.6) \quad H = H_{s} \oplus H_{\text{mul}}.\]

The adjoint relation $H^\ast$ from $\mathfrak{K}$ to $\mathfrak{H}$ is defined by $H^\ast = JH^\perp = (JH)^\perp$, where $J\{f, f\}' = \{f', -f\}$. The adjoint is automatically a closed linear relation and the closure of $H$ is given by $H^{**}$. The operator part $(H^\ast)_{s}$ is densely defined in $\overline{\text{dom} H^\ast} = \mathfrak{H} \ominus \text{mul} H^{\ast}$ and maps into $\text{dom} H = \text{dom} H^{**} = \mathfrak{K} \ominus \text{mul} H^\ast$. When $H$ is closed the operator parts $H_{s}$ and $(H^\ast)_{s}$ are connected by

\[(2.7) \quad (H_{s})^{\times} = (H^\ast)_{s},\]

where $(H_{s})^{\times}$ denotes the adjoint of $H_{s}$ in the sense of the smaller spaces $\overline{\text{dom} H}$ and $\overline{\text{dom} H^\ast}$.

Let $H_{1}$ be a relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{M}$ and let $H_{2}$ be a relation from a Hilbert space $\mathfrak{M}$ to a Hilbert space $\mathfrak{K}$. The product satisfies

\[(2.8) \quad \text{mul} H_{1} H_{2} \subset (\text{mul} H_{2} H_{1})^{\ast}.\]

Moreover, if $H_{2} \in \mathbf{B}(\mathfrak{M}, \mathfrak{K})$ then there is actually equality

\[(2.9) \quad H_{1}^{\ast} H_{2}^{\ast} = (H_{2} H_{1})^{\ast},\]

see [10, Lemma 2.4], so that, in particular

\[H_{2} H_{1}^{**} \subset (H_{2} H_{1})^{**}.\]

Assume that the relations $H_{1}$ and $H_{2}$ are closed. In general the product $H_{2} H_{1}$ is not closed. However, if for instance $H_{1} \in \mathbf{B}(\mathfrak{H}, \mathfrak{M})$, then the product $H_{2} H_{1}$ is closed.

A linear relation $H$ in a Hilbert space $\mathfrak{H}$ is symmetric if $H \subset H^\ast$ and selfadjoint if $H = H^\ast$. If the relation $H$ is selfadjoint then $H_{s}$ is a selfadjoint operator in $\overline{\text{dom} H} = \mathfrak{H} \ominus \text{mul} H$. A linear relation $H$ in a Hilbert space $\mathfrak{H}$ is nonnegative if $(h', h) \geq 0$ for all $h, h' \in H$. In particular a nonnegative relation is symmetric.

An important special case of a nonnegative selfadjoint relation appears when one considers relations of the form $T^\ast T$ where $T$ is a closed linear relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$; cf. [18].

**Lemma 2.4.** Let $T$ be a closed relation from a Hilbert space $\mathfrak{H}$ to a Hilbert space $\mathfrak{K}$. Then the product $T^\ast T$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. Furthermore,

\[(2.10) \quad T^\ast T = T^\ast T_{s} = (T_{s})^{\ast} T_{s},\]

so that in particular

\[(2.11) \quad \ker (T^\ast T) = \ker T = \ker T_{s}, \quad \text{mul} (T^\ast T) = \text{mul} T^\ast = \text{mul} (T^{\ast})^{\ast}.\]
The operator part of $T^*T$ can be rewritten as
\[(2.12)\quad (T^*T)_s = (T^*)_s T_s = (T_s)^* T_s.\]

Proof. It is clear from the definition that $T^*T$ is a nonnegative selfadjoint relation in $\mathfrak{H}$. In fact $T^*T$ is selfadjoint since $\text{ran}(T^*T + I) = \mathfrak{H}$, which follows from $\mathfrak{H}^2 = T \oplus T^* = T \oplus JT^*$.

Next, let $P$ be the orthogonal projection from $\mathfrak{H}$ onto $\text{dom}^* T$, so that the inclusions are both equalities. From (2.13) $T^*T$ is a nonnegative selfadjoint relation with $\text{mul} K = \text{mul} H$. Moreover, (2.14) leads to (2.10). Since $T_s$ is an operator, (2.11) is immediate from (2.10). Furthermore, (2.12) is clear from (2.7). \hfill \Box

Lemma 2.5. Let $H$ be a nonnegative selfadjoint relation in a Hilbert space $\mathfrak{H}$. Then there exists a unique nonnegative selfadjoint relation $K$ in $\mathfrak{H}$, denoted by $K = H^{\frac{1}{2}}$, such that $K^2 = H$. Moreover, $H^{\frac{1}{2}}$ has the representation
\[(2.13)\quad H^{\frac{1}{2}} = H_+^{\frac{1}{2}} \oplus H_{\text{mul}}.\]

Proof. It is clear that $K$ defined by the right hand side of (2.13) is a nonnegative selfadjoint relation with $\text{mul} K = \text{mul} H$. To see that $K^2 = H$, let $\{f, f'\} \in K^2$. Then $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$. Clearly,

$$\varphi = H^{\frac{1}{2}} f + \alpha, \quad f' = H^{\frac{1}{2}} \varphi + \beta,$$

with $\alpha \in \text{mul} H$ and $\beta \in \text{mul} H$. Since $\varphi \in \text{dom} H^{\frac{1}{2}}$, it follows that $\alpha = 0$ and $f' = H_s f + \beta$, so that $\{f, f'\} \in H$. It follows that $K^2 \subset H$, and since $K^2 = K^* K$ is selfadjoint, it follows that $K^2 = H$.

In order to show uniqueness, let $K$ be a nonnegative selfadjoint relation such that $K^2 = H$. Then

$$\text{mul} K = \text{mul} H.$$

To see this let $\{0, \psi\} \in K$, then clearly $\{0, \psi\} \in K^2 = H$, so that $\text{mul} K \subset \text{mul} H$. For the reverse inclusion, let $\{0, \psi\} \in H = K^2$. Then $\{0, \varphi\} \in K$ and $\{\varphi, \psi\} \in K$. Since $K$ is selfadjoint it follows that $\varphi = 0$, so that $\{0, \psi\} \in K$ and $\text{mul} H \subset \text{mul} K$. This implies that $K = K_s \oplus H_{\text{mul}}$, where $K_s$ is a nonnegative selfadjoint operator. It will now be shown that $H_s = (K_s)^2$, and since the square root of a nonnegative selfadjoint operator is uniquely determined it follows that $K_s = H_s^{1/2}$.

To see that $H_s = K_s^2$, let $\{f, f'\} \in H_s$. Then $\{f, f'\} \in H = K^2$ and $f' \perp \text{mul} H$. Now $\{f, \varphi\} \in K$ and $\{\varphi, f'\} \in K$ for some $\varphi \in \text{dom} K = \text{dom} H$. Hence $\{f, \varphi\} \in K_s$ and $\{\varphi, f'\} \in K_s$, so that $\{f, f'\} \in K_s^2$. Thus $H_s \subset (K_s)^2$. For the converse inclusion, let $\{f, f'\} \in (K_s)^2$. Then $\{f, \varphi\} \in K_s \subset K$, $\{\varphi, f'\} \in K_s \subset K_s$, so that $\{f, f'\} \in K^2 = H$. Since $f' \perp \text{mul} H$, it follows that $\{f, f'\} \in H_s$. \hfill \Box

Let $H$ be a nonnegative selfadjoint relation. Since Lemma 2.4 implies that $\text{mul} H^{\frac{1}{2}} = \text{mul} H$, it follows that

$$\text{dom} H \subset \text{dom} H^{\frac{1}{2}} \subset \overline{\text{dom} H^{\frac{1}{2}}} = \overline{\text{dom} H}.$$
Therefore the following statements are equivalent:

(2.16) \( \text{dom } H \) closed; \( \text{dom } H^+ \) closed; \( \text{dom } H = \text{dom } H^+ \).

Let \( H \) be a nonnegative selfadjoint relation. Then for each \( x > 0 \),

(2.17) \( \text{dom } (H + x)^{1/2} = \text{dom } H^{1/2} \),

and, moreover,

(2.18) \( \| (H + x)^{1/2} h \|^2 = \| (H^{1/2})_+ h \|^2 + x \| h \|^2, \quad h \in \text{dom } H^{1/2} \).

It is clear that the identity holds for \( h \in \text{dom } H \) and since \( \text{dom } H \) is a core for \( H^{1/2} \)

it holds for \( h \in \text{dom } H^{1/2} \).

There is a natural ordering for nonnegative selfadjoint relations in a Hilbert space \( \mathcal{H} \); it is inspired by the corresponding situation for selfadjoint operators \( H_1, H_1 \in \mathcal{B}(\mathcal{H}) \). Two nonnegative selfadjoint relations \( H_1 \) and \( H_2 \) are said to satisfy the inequality \( H_1 \leq H_2 \) if

(2.19) \( \text{dom } H^+_1 \subset \text{dom } H^+_2, \quad \| H^+_1 h \| \leq \| H^+_2 h \|, \quad h \in \text{dom } H^+_2. \)

It follows from (2.17) and (2.18) that \( H_1 \leq H_2 \) if and only if \( H_1 + x \leq H_2 + x \) for some (and hence for all) \( x > 0 \).

A sesquilinear form (or form for short) \( t[\cdot, \cdot] \) in a Hilbert space \( \mathcal{H} \) is a mapping from \( \mathcal{D} \times \mathcal{D} \) to \( \mathbb{C} \) where \( \mathcal{D} \) is a (not necessarily densely defined) linear subspace of \( \mathcal{H} \), such that it is linear in the first entry and anti-linear in the second entry. The domain \( \text{dom } t \) is defined by \( \text{dom } t = \mathcal{D} \). The corresponding quadratic form \( t[\cdot] \) is defined by \( t[\varphi] = t[\varphi, \varphi], \varphi \in \text{dom } t \). A sesquilinear form \( t \) is said to be nonnegative if

\[ t[\varphi] \geq 0, \quad \varphi \in \text{dom } t. \]

The semibounded form \( t \) in \( \mathcal{H} \) is said to be closed if for any sequence \( (\varphi_n) \) in \( \text{dom } t \)

one has

(2.20) \( \varphi_n \to \varphi, \quad t[\varphi_n - \varphi_m] \to 0, \quad \Rightarrow \quad \varphi \in \text{dom } t, \quad t[\varphi_n - \varphi] \to 0. \)

The inequality \( t_1 \leq t_2 \) for forms \( t_1 \) and \( t_2 \) is defined by

(2.21) \( \text{dom } t_2 \subset \text{dom } t_1, \quad t_1[h] \leq t_2[h], \quad h \in \text{dom } t_2. \)

In particular, \( t_2 \subset t_1 \) implies \( t_1 \leq t_2 \).

The theory of nonnegative forms can be found in [15]. The representation theorem gives a connection between nonnegative selfadjoint relations and nonnegative closed forms; see [9, 15].

**Theorem 2.6** (representation theorem). Let \( t \) be a closed nonnegative form in the Hilbert space \( \mathcal{H} \). Then there exists a nonnegative selfadjoint relation \( H \) in \( \mathcal{H} \) such that

(i) \( \text{dom } H \subset \text{dom } t \) and

(2.22) \( t[\varphi, \psi] = (\varphi', \psi) \)

for every \( (\varphi, \varphi') \in H \) and \( \psi \in \text{dom } t; \)

(ii) \( \text{dom } H \) is a core for \( t \) and \( \text{mul } H = (\text{dom } t)^\perp; \)

(iii) if \( \varphi \in \text{dom } t, \omega \in \mathcal{H}, \) and

(2.23) \( t[\varphi, \psi] = (\omega, \psi) \)

holds for every \( \psi \) in a core of \( t \), then \( \{\varphi, \omega\} \in H \).
The nonnegative selfadjoint relation $H$ is uniquely determined by \( i \).

The following result is a direct consequence of the representation theorem.

**Proposition 2.7.** Let $T$ be a closed linear relation from a Hilbert space $\mathcal{H}$ to a Hilbert space $\mathcal{K}$. The nonnegative selfadjoint relation $T^*T$ in the Hilbert space $\mathcal{H}$ corresponds to the closed nonnegative form

$$
(2.24) \quad t[h, k] = (T_s h, T_s k), \quad h, k \in \text{dom } t = \text{dom } T_s = \text{dom } T,
$$

and, in particular,

$$
(2.25) \quad t[h, k] = ((T_s)^* T_s h, k)_{\mathcal{K}}, \quad h \in \text{dom } T^*T, \quad k \in \text{dom } t.
$$

**Proof.** Since $T_s$ is a closed linear operator, it follows that the form in $(2.24)$ is closed. Clearly, if in $(2.24)$ one assumes that $h \in \text{dom } T^*T = \text{dom } (T_s)^* T_s$, see $(2.12)$, then $(2.25)$ follows. The result is now obtained from Theorem 2.6. \( \square \)

Proposition 2.7 combined with $(2.12)$ in Lemma 2.4 yields the so-called second representation theorem for closed forms.

**Corollary 2.8.** Let $t$ be a closed nonnegative form in the Hilbert space $\mathcal{H}$ and let $H$ be the corresponding nonnegative selfadjoint relation in $\mathcal{H}$ as in Theorem 2.6. Then

$$
(2.26) \quad \text{dom } t = \text{dom } H^\frac{1}{2}, \quad \text{and } \ t[\varphi, \psi] = (H^\frac{1}{2} \varphi, H^\frac{1}{2} \psi), \quad \varphi, \psi \in \text{dom } t.
$$

A subset of $\text{dom } t = \text{dom } H^\frac{1}{2}$ is a core of the form $t$ if and only if it is a core of the operator $H^\frac{1}{2}$. In particular, $\text{dom } H$ is a core of $H^\frac{1}{2}$.

As a straightforward consequence of the representation theorem one can state the following result which connects inequalities between nonnegative selfadjoint relations with inequalities between the corresponding nonnegative closed forms.

**Corollary 2.9.** Let $t_1$ and $t_2$ be closed nonnegative forms and let $H_1$ and $H_2$ be the corresponding nonnegative selfadjoint relations. Then

$$
(2.27) \quad t_1 \leq t_2 \quad \text{if and only if} \quad H_1 \leq H_2.
$$

**Corollary 2.10.** Let $\mathcal{H}$, $\mathcal{H}_1$, and $\mathcal{H}_2$ be Hilbert spaces. Let $T_1$ be a closed linear relation from $\mathcal{H}$ into $\mathcal{H}_1$ and let $T_2$ be a closed linear relation from $\mathcal{H}$ into $\mathcal{H}_2$. Then $T_1^* T_1 \leq T_2^* T_2$ if and only if

$$
\text{dom } T_2 \subset \text{dom } T_1 \quad \text{and} \quad \|(T_1)_s h\|_{\mathcal{H}_1} \leq \|(T_2)_s h\|_{\mathcal{H}_2}, \quad h \in \text{dom } T_2.
$$

**Proof.** Let $t_1$ and $t_2$ be the closed nonnegative forms in the Hilbert space $\mathcal{H}$ induced by $T_1^* T_1$ and $T_2^* T_2$. Hence by Corollary 2.1 one has $T_1^* T_1 \leq T_2^* T_2$ if and only if $t_1 \leq t_2$. By Proposition 2.7 $t_1 \leq t_2$ if and only if

$$
\text{dom } T_2 \subset \text{dom } T_1, \quad ((T_1)_s h, (T_1)_s h)_{\mathcal{H}_1} \leq ((T_2)_s h, (T_2)_s h)_{\mathcal{H}_2}, \quad h \in \text{dom } T_2. \quad \square
$$

3. The lemma of Douglas in the context of linear relations

In this section the lemma of Douglas, see Introduction, will be discussed in the context of linear relations. The first result to be presented is about range inclusion and factorisation. It goes back to D. Popovici and Z. Sebestyén [16], who stated it actually in the context of linear spaces. Some refinements can be found in [17].
Proposition 3.1. Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a linear relation from $\mathcal{H}_A$ to $\mathcal{H}$, and let $B$ be a linear relation from $\mathcal{H}_B$ to $\mathcal{H}$. Then $A \subset B$ if and only if there exists a linear relation $W$ from $\mathcal{H}_A$ to $\mathcal{H}_B$ such that $A \subset BW$.

Proof. $(\Rightarrow)$ Let the linear relation $W$ from $\mathcal{H}_A$ to $\mathcal{H}_B$ be defined by the product

$$W = B^{-1}A.$$ 

Then $f' \in \text{ran } A$, so that $f' \in \text{ran } B$ and there exists $\varphi \in \mathcal{H}_B$ such that $\{\varphi, f'\} \in B$ or $\{f', \varphi\} \in B$. Hence $\{f, \varphi\} \in W$ and $\{f, f'\} \in BW$.

$(\Leftarrow)$ Let $f' \in \text{ran } A$, then for some $f \in \mathcal{H}_A$ one has $\{f, f'\} \in A$. Hence there is $\varphi \in \mathcal{H}_B$ such that $\{f, \varphi\} \in W$ and $\{\varphi, f'\} \in B$. This implies that $f' \in \text{ran } B$. □

For the next result, see [17, Proposition 2]; for completeness a short proof is presented.

Proposition 3.2. Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a linear relation from $\mathcal{H}_A$ to $\mathcal{H}$, and let $B$ be a linear relation from $\mathcal{H}_B$ to $\mathcal{H}$. Then there exists a linear relation $W$ from $\mathcal{H}_A$ to $\mathcal{H}_B$ such that $A = BW$ if and only if

$$\text{ran } A \subset \text{ran } B \quad \text{and} \quad \text{mul } B \subset \text{mul } A.$$ 

Proof. $(\Rightarrow)$ It follows from (2.3) that $\text{mul } B \subset \text{mul } BW = \text{mul } A$ while $\text{ran } A \subset \text{ran } B$ holds by Proposition 3.1.

$(\Leftarrow)$ As in the proof of Proposition 3.1 consider $W = B^{-1}A$ which satisfies $A \subset BW$. In view of (2.4) one can write

$$BW = BB^{-1}A = (\text{ran } B \hat{\oplus} (\{0\} \times \text{mul } B)) A.$$ 

Since $\text{ran } A \subset \text{ran } B$, it is clear from (3.1) that $\text{dom } BW = \text{dom } A$ and since $\text{mul } B \subset \text{mul } A$ one also concludes from (3.1) that $\text{mul } BW = \text{mul } A$. Therefore, the equality $BW = A$ holds by Corollary 2.2. □

Observe that if $W$ is a linear relation from $\mathcal{H}_A$ to $\mathcal{H}_B$, then the inclusion $A \subset BW$ shows that

$$\text{dom } A \subset \text{dom } W \quad \text{and} \quad \text{ran } A \subset \text{ran } B.$$ 

Furthermore, if $W$ is an operator, then the inclusion $A \subset BW$ is equivalent to:

$$\text{dom } A \subset \text{dom } W \quad \text{and} \quad \{Wf, f'\} \in B \quad \text{for all} \quad \{f, f'\} \in A,$$

so that in particular $W$ takes $A$ into $B$. Hence when the relation $W$ is a bounded operator then it may be assumed that $W \in B(\text{dom } A, \text{dom } B)$. In this case the zero continuation $W_c$ of $W$ to $(\text{dom } A)^\perp$ satisfies $A \subset BW_c$ and $\|W_c\| = \|W\|$, so that without loss of generality it may be assumed that $W \in B(\mathcal{H}_A, \mathcal{H}_B)$.

Lemma 3.3. Let $A \subset BW$ for some $W \in B(\mathcal{H}_A, \mathcal{H}_B)$. Then

$$W^*B^* \subset A^* \quad \text{and} \quad A^{**} \subset B^{**}.$$ 

Proof. Clearly $A \subset BW$ implies via (2.8) that

$$W^*B^* \subset (BW)^* \subset A^*.$$ 

This inclusion combined with $W^* \in B(\mathcal{H}_B, \mathcal{H}_A)$ and (2.9) in turn gives rise to

$$A^{**} \subset (W^*B^*)^* = B^{**}W^{**} = B^{**}W.$$ 

□
The main result in this section concerns factorization and majorization. If $A$ and $B$ are closed linear relations, then the case that $W \in B(\overline{\text{dom } A}, \overline{\text{dom } B})$ can be characterized as follows; see also \[3, 6\].

**Theorem 3.4.** Let $\mathcal{H}_A, \mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a closed linear relation from $\mathcal{H}_A$ to $\mathcal{H}$, and let $B$ be a closed linear relation from $\mathcal{H}_B$ to $\mathcal{H}$. Then there exists an operator $W \in B(\mathcal{H}_A, \mathcal{H}_B)$ (or equivalently an operator $W \in B(\overline{\text{dom } A}, \overline{\text{dom } B})$) such that

$$
\text{(3.2)} \quad A \subset BW, \\
\text{if and only if there exists } c \geq 0 \text{ such that} \quad AA^* \leq c^2 BB^*.
$$

One can take $\|W\| \leq c$.

**Proof.** ($\Rightarrow$) Let $A \subset BW$ with $W \in B(\overline{\text{dom } A}, \overline{\text{dom } B})$. By considering the zero continuation of $W$, again denoted by $W$, it may be assumed that $W \in B(\mathcal{H}_A, \mathcal{H}_B)$. Then

$$
\text{(3.4)} \quad W^*B^* \subset A^*,
$$

cf. Lemma 3.3. In particular this implies that $\text{dom } B^* \subset \text{dom } A^*$. Now let $\{f, f'\} \in (B^*)_s \subset B^*$. Then it follows from (3.4) that

$$
\{f, W^*f'\} \in A^*.
$$

Hence there is an element $\chi \in \text{mul } A^*$ such that

$$
W^*(B^*)_sf = (A^*)_sf + \chi.
$$

Observe that

$$
\|(A^*)_sf\|^2 \leq \|(A^*)_sf\|^2 + \|\chi\|^2 = \|W^*(B^*)_sf\|^2 \leq \|W\|^2\|\|(B^*)_sf\|^2.
$$

Together with $\text{dom } B^* \subset \text{dom } A^*$ this inequality proves (3.3); see Corollary 2.10.

($\Leftarrow$) Assume that (3.3) holds, in other words, assume that there exists $c \geq 0$ such that

$$
\text{(3.5)} \quad \text{dom } B^* \subset \text{dom } A^*, \quad c\|\|(B^*)_sf\| \geq \|(A^*)_sf\|, \quad f \in \text{dom } B^*.
$$

Consider $A_s$ as a densely defined operator from $\overline{\text{dom } A}$ to $(\text{mul } A)^\perp$ and $B_s$ as a densely defined operator from $\overline{\text{dom } B}$ to $(\text{mul } B)^\perp$. Then the assumption (3.5) is equivalent to

$$
\text{(3.6)} \quad \text{dom } B^* \subset \text{dom } A^*, \quad c\|\,(B_s)^xf\| \geq \|(A_s)^xf\|, \quad f \in \text{dom } B^*,
$$

where the adjoints $(A_s)^x$ and $(B_s)^x$ are with respect to these smaller spaces; see (2.7). Define the linear relation $D$ by

$$
D = \{ \langle (B_s)^x f, (A_s)^x f \rangle : f \in \text{dom } B^* \}.
$$

Then by (3.6) $D$ is a bounded operator from $\overline{\text{dom } B}$ to $\overline{\text{dom } A}$ with $\|D\| \leq c$. It has a unique extension, again denoted by $D$, from $\overline{\text{dom } B}$ to $\overline{\text{dom } A}$ with $\|D\| \leq c$, such that

$$
D(B_s)^x \subset (A_s)^x,
$$

or taking adjoints, using (2.9),

$$
A_s = (A_s)^{xx} \subset (D(B_s)^x)^{xx} = (B_s)^{xx} D^x = B_s W_0,
$$

where $W_0$ is the operator of $W$.
where $W_0 = D^*$ is a bounded linear operator from $\overline{\text{dom } A}$ to $\overline{\text{dom } B}$, with $\|W_0\| = \|D\| \leq c$. Observe that the inclusion $\text{dom } B^* \subset \text{dom } A^*$ implies that
\begin{equation}
(3.8) \quad \text{mul } A \subset \text{mul } B.
\end{equation}
Now let $\{f, f'\} \in A$, so that $f' = A_s f + \varphi$ with $\varphi \in \text{mul } A$. By (3.8) one has $\varphi \in \text{mul } B$. By (3.7) the inclusion $\{f, A_s f\} \in A_s$ implies that $\{f, W_0 f\} \in W_0$,
\begin{equation}
\{W_0 f, A_s f\} \in B_s,
\end{equation}
and, hence $\{W_0 f, A_s f + \varphi\} \in B$.
Therefore one concludes that $\{f, f'\} \in BW_0$, i.e., $A \subset BW_0$ holds with $W_0 \in B(\overline{\text{dom } A}, \overline{\text{dom } B})$. Finally, let $W$ be the zero continuation of $W_0$ to $(\overline{\text{dom } A})^\perp$. Then $W \in B(\mathcal{H}_A, \mathcal{H}_B)$ with $\|W\| = \|W_0\|$ and, moreover, the inclusion $A \subset BW$ is satisfied. \[ \square \]

In particular, the equivalences $AA^* \leq BB^* \iff W^*B^* \subset A^* \iff A \subset BW$ with $\|W\| \leq 1$ can be found in [3, Proposition 2.2, Remark 2.3]. For densely defined operators $A$ and $B$ the implication $AA^* \leq BB^* \Rightarrow A \subset BW$, $\|W\| \leq 1$, can be found in [6, Theorem 2].

The following two corollaries are variations on the theme of Theorem 3.4.

**Corollary 3.5.** Let $A$ and $B$ be closed linear relations as in Theorem 3.4 and, in addition, let $T \in B(\mathcal{H}, \mathcal{H}_A)$ with $\mathcal{H}$ a Hilbert space. Then
\begin{equation}
AA^* \leq c^2 BB^* \quad \Rightarrow \quad AT^*A^* \leq c^2 \|T\|^2 BB^*,
\end{equation}
where $c \geq 0$. In particular,
\begin{equation}
BT^*B^* \leq \|T\|^2 BB^*
\end{equation}
holds for every $T \in B(\mathcal{H}, \mathcal{H}_A)$.

**Proof.** Assume that $AA^* \leq c^2 BB^*$, which by Theorem 3.4 is equivalent to the inclusion $A \subset BW$. Therefore it follows that
\begin{equation}
AT \subset BW.
\end{equation}
Observe that $AT$ is closed and that $WT$ is bounded. Hence again by Theorem 3.4 one obtains
\begin{equation}
AT(\overline{AT})^* \leq \|WT\|^2 BB^*.
\end{equation}
Now observe that (2.2) shows that
\begin{equation}
(\overline{AT})^* = (\overline{(T^*A^*)})^* = T^*A^*.
\end{equation}
Hence this leads to
\begin{equation}
AT^*A^* \leq c_T^2 BB^*,
\end{equation}
where one can take $c_T = \|WT\| \leq c\|T\|$. The last statement follows from the first one with the choices $A = B$ and $c = 1$. \[ \square \]

**Corollary 3.6.** Let $A$ and $B$ be closed linear relations as in Theorem 3.4 and let $T$ be a linear relation from the Hilbert space $\mathcal{H}$ to the Hilbert space $\mathcal{H}$. Then
\begin{equation}
(3.9) \quad AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{T}(\overline{AT})^* \leq c^2 T(\overline{TB})^*,
\end{equation}
where $c \geq 0$. In particular, if $T \in B(\mathcal{H}, \mathcal{H})$ then
\begin{equation}
AA^* \leq c^2 BB^* \quad \Rightarrow \quad \overline{TAA^*T^*} \leq c^2 TBB^*T^*.
\end{equation}
Proof. Assume that $AA^* \leq c^2 BB^*$. Then by Theorem 3.4 $A \subset BW$ for some $W \in B(H_A, H_B)$ with $\|W\| \leq c$. Hence it follows that

$TA \subset T(BW) = (TB)W \subset TBBW$.  

Due to (2.9) the following identity holds

$TBW = (W^*(TB)^*)^*$,

which implies that the relation $TBW$ is closed. Therefore one concludes from (3.10) that

$AA^* \leq c^2 BB^* \Rightarrow TA \subset TBW$.

By Theorem 3.4 this implication can be rewritten as the implication stated in (3.9). If $T \in B(H_A, H_B)$ the last statement is obtained by applying (2.9) to (3.9). □

The occurrence of the equality $A = BW$ in Theorem 3.4 can be characterized as follows.

**Proposition 3.7.** Let $H_A$, $H_B$, and $H$ be Hilbert spaces, let $A$ be a closed linear relation from $H_A$ to $H_B$, and let $B$ be a closed linear relation from $H_B$ to $H$. Then there exists a bounded (not necessarily closed) operator $W$ from $\text{dom } A$ into $\text{dom } B$ such that

$A = BW,$

if and only if the following conditions are satisfied

(i) the inequality (3.3) holds for some $c \geq 0$;

(ii) $\text{mul } A = \text{mul } B$.

**Proof.** ($\Rightarrow$) If $A = BW$ holds for some bounded operator $W$ from dom $A$ into dom $B$, then clearly $A \subset BW^*$ and here $W^* \in B(\text{dom } A, \text{dom } B)$, since dom $A \subset \text{dom } W$. Now the inequality (3.3) is obtained from Theorem 3.4. Since $W$ is an operator, one obtains $\text{mul } A = \text{mul } BW = \text{mul } B$; see (2.3).

($\Leftarrow$) The inequality (3.3) implies the existence of $W_0 \in B(\text{dom } A, \text{dom } B)$ such that $A \subset BW_0$ by Theorem 3.4. Then dom $A \subset \text{dom } W_0$ and the restriction $W := W_0 \upharpoonright \text{dom } A$ is a bounded operator such that $A \subset BW$ and dom $BW = \text{dom } A$. The second assumption implies that $\text{mul } BW = \text{mul } B = \text{mul } A$ and hence the equality $A = BW$ follows from Corollary 2.2. □

The following result concerns the alternative formulation of the Douglas lemma which is known in the literature, but now in the context of relations. The domain condition is a sufficient condition.

**Proposition 3.8.** Let $H_A$, $H_B$, and $H$ be Hilbert spaces, let $A$ be a closed linear relation from $H_A$ to $H$, and let $B$ be a closed linear relation from $H_B$ to $H$. Assume that $\text{dom } A^* = \text{dom } B^*$. Then the following statements are equivalent:

(i) $AA^* \leq c^2 BB^*$ for some $c \geq 0$;

(ii) $AA^* = BMB^*$ for some $0 \leq M \in B(H_B)$ with $\|M\| \leq c^2$.

**Proof.** (i) $\Rightarrow$ (ii) By Theorem 3.4 it follows that $A \subset BW$, and that $W^*B^* \subset A^*$. Let $Q$ be the orthogonal projection onto $(\text{mul } A^*)^\perp$. Then clearly

$QW^*B^* \subset QA^*$. 

$\Box$
where $QA^*$ is an operator. The assumption $\text{dom } A^* = \text{dom } B^*$ implies that actually equality holds

$$QW^*B^* = QA^*.$$  

Therefore one obtains via $AA^* = QA^*$, see Lemma 2.4 that

$$AA^* = QA^* \subset BWQA^* = BWQW^*B^* = (BWQ)(QW^*)^*,$$

where the relation $BWQ$ is closed and

$$QW^*B^* = BWQW^*B^* = (BWQ)(QW^*)^* = BWQ.$$  

Hence the term $(BWQ)(QW^*)^*$ is selfadjoint and equality prevails:

$$AA^* = BWQW^*B^* = BM^* \quad \text{with } M = WQW^*.$$  

Note that $\|M\| \leq \|W\|^2 \leq c^2$.

(ii) $\Rightarrow$ (i) Since $M \geq 0$ is bounded one can rewrite (ii) in the form

$$(3.12) \quad AA^* = BM^*B^* = BM^1/2BM^1/2B^* \subset (BM^1/2)M^1/2B^*.$$  

Observe that by (2.9)

$$M^1/2B^* = (M^1/2B^*)^* = (BM^1/2)^*.$$  

This equality and the fact that $BM^1/2$ is closed together show that both sides in (3.12) are selfadjoint; see Lemma 2.4. Thus there is actually equality in (3.12):

$$AA^* = (BM^1/2)M^1/2B^*.$$  

Now Corollary 3.5 implies that

$$AA^* = BM^1/2M^1/2B^* \leq \|M\|BB^*,$$

so that (3.3) follows with $c^2 = \|M\|$. \hfill $\Box$

4. Domination of linear relations

The following notions and terminology are strongly influenced by the theory of Lebesgue type decompositions of linear relations and forms, cf. [11, 12, 19]. In fact in these papers the notion of domination is used for (mostly closable) operators. However domination can be defined also in the context of linear relations as follows.

**Definition 4.1.** Let $H_A$, $H_B$, and $H$ be Hilbert spaces, let $A$ be a linear relation from $H$ to $H_A$, and let $B$ be a linear relation from $H$ to $H_B$. Then $B$ dominates $A$ if there exists an operator $Z \in B(H_B, H_A)$ such that

$$ZB \subset A.$$  

Note that the inclusion $ZB \subset A$ in (4.1) means that

$$(4.2) \quad \{\{f, Zf'\} : \{f, f'\} \in B\} \subset A.$$  

This shows that dom $B \subset$ dom $A$ and that ker $B \subset$ ker $A$. Furthermore,

$$\text{mul } ZB = Z(\text{mul } B) \subset \text{mul } A.$$  

It follows from the definition that $Z$ takes ran $B$ into ran $A$; the boundedness implies that $Z$ takes ran $B$ into ran $A$. Hence one can assume that $(\text{ran } B)^\perp \subset$ ker $Z$, in which case $Z$ is uniquely determined. Domination is transitive: if $Z_1B \subset A$ and $Z_2C \subset B$ then

$$Z_1(Z_2C) \subset Z_1B \subset A,$$

so that $(Z_1Z_2)C \subset A.$
Let $A$ and $B$ be relations in a Hilbert space $\mathcal{H}$ which satisfy $B \subset A$. Then clearly $B$ dominates $A$ (with $Z = 1$). In particular, since $A \subset A^{**}$, it follows that $A$ dominates $A^{**}$.

In the particular case when $A$ and $B$ in the above definition are linear operators it is possible to give an equivalent characterization of domination.

**Lemma 4.2.** Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a linear operator from $\mathcal{H}$ to $\mathcal{H}_A$, and let $B$ be a linear operator from $\mathcal{H}$ to $\mathcal{H}_B$. Then $B$ dominates $A$ if and only if there exists $c \geq 0$ such that

\begin{equation}
\text{dom } B \subset \text{dom } A \quad \text{and} \quad \|Af\| \leq c\|Bf\|, \quad f \in \text{dom } B. \tag{4.3}
\end{equation}

**Proof.** Assume that $B$ dominates $A$. Then (4.1) shows that $\text{dom } B \subset \text{dom } A$ and that for all $f \in \text{dom } B$ one has $Z_Bf = Af$, which leads to

$$\|Af\| \leq \|Z\|\|Bf\|, \quad f \in \text{dom } B.$$  

The desired result follows from this with $c = \|Z\|$.

Conversely, assume that (4.3) holds. Define an operator $Z_0$ from $\text{ran } B$ to $\text{ran } A$ by $Z_0Bf = Af$, $f \in \text{dom } B$. It follows from (4.3) that the operator $Z_0$ is well defined and bounded with $\|Z_0\| \leq c$. Thus $Z_0$ can be continued to a bounded operator from $\text{ran } B$ to $\text{ran } A$ with the same norm. Let $Z$ be the extension of $\text{clos } Z_0$ obtained by defining $Z$ to be 0 on $(\text{ran } B)\perp$. Then clearly $Z : \mathcal{H}_B \rightarrow \mathcal{H}_A$ is bounded and $ZB \subset A$ holds.  

A weaker version of Lemma 4.2 with densely defined operators on a Banach space appears in [8, Theorem 2.8]; see also [4, 7].

**Lemma 4.3.** Let the relation $B$ dominate the relation $A$ as in (4.1), then

\begin{equation}
A^* \subset B^*Z^*, \tag{4.4}
\end{equation}

and, consequently

\begin{equation}
ZB^{**} \subset A^{**}. \tag{4.5}
\end{equation}

In other words, $B^{**}$ dominates $A^{**}$ with the same operator $Z$. In particular, if $B$ dominates $A$ then the following inclusions are valid

$$\text{dom } B \subset \text{dom } A, \quad \text{ran } A^* \subset \text{ran } B^*, \quad \text{and} \quad \text{dom } B^{**} \subset \text{dom } A^{**}.$$  

**Proof.** It follows from (4.1) and (2.9) that

$$A^* \subset (ZB)^* = B^*Z^*.$$  

Now taking adjoints again yields

$$Z^{**}B^{**} \subset (B^*Z^*)^* \subset A^{**},$$

and this proves (4.5). The remaining statement are clear from (4.4) and (4.5).  

So far domination has been defined for linear relations which are not necessarily closed. Due to Lemma 4.3 domination of closed linear relations can be characterized in terms of majorization.
Theorem 4.4. Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_A$, and let $B$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_B$. Then there exists an operator $Z \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that

$$ZB \subset A$$

if and only if there exists $c \geq 0$ such that

$$A^*A \leq c^2B^*B.$$  

One can take $\|Z\| \leq c$.

Proof. Since $A$ and $B$ are assumed to be closed the inclusions (4.6) and (4.4) are equivalent. Hence the result follows from Theorem 3.4.

Proposition 4.5. Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_A$, and let $B$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_B$. Then there exists an operator $Z \in \mathcal{B}(\mathcal{H}_B, \mathcal{H}_A)$ such that

$$A = ZB$$

if and only if the following three conditions are satisfied:

(i) the inequality (4.7) holds for some $c \geq 0$;

(ii) $\text{dom } A = \text{dom } B$;

(iii) $\dim (\text{mul } A) \leq \dim (\text{mul } B)$.

Proof. ($\Rightarrow$) Property (i) follows directly from Theorem 4.4. Since $\text{dom } Z = \mathcal{H}_B$, the equality (4.8) implies (ii). Finally, it follows from (2.3) and (4.8) that $\text{mul } A = \text{mul } ZB = Z(\text{mul } B)$, i.e. $Z$ maps $\text{mul } B$ onto $\text{mul } A$, and hence (iii) holds.

($\Leftarrow$) Decompose $A$ and $B$ via their operator parts:

$$A = A_s \oplus A_{\text{mul}}, \quad B = B_s \oplus B_{\text{mul}}.$$  

By Lemma 2.4 the condition (4.7) is equivalent to $(A_s)^*A_s \leq c^2(B_s)^*B_s, \ c \geq 0$. Now by Theorem 4.4 there exists $Z_0 \in \mathcal{B}(\mathcal{H}_B \oplus \text{mul } B, \mathcal{H}_A \oplus \text{mul } A)$ such that

$$Z_0B_s \subset A_s.$$  

By the condition (ii) $\text{dom } A_s = \text{dom } B_s$ and hence, in fact, the equality $Z_0B_s = A_s$ prevails. Moreover, the condition (iii) guarantees the existence of a surjective operator $Z_m \in \mathcal{B}(\text{mul } B, \text{mul } A)$. Finally, by taking $Z = Z_0 \oplus Z_m$ one gets the desired identity $ZB = A$.

Finally note that the result in Proposition 3.8 has a counterpart in the setting of Theorem 4.4. Let $\mathcal{H}_A$, $\mathcal{H}_B$, and $\mathcal{H}$ be Hilbert spaces, let $A$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_A$, and let $B$ be a closed linear relation from $\mathcal{H}$ to $\mathcal{H}_B$. If $\text{dom } A = \text{dom } B$, then the following statements are equivalent:

(i) $A^*A \leq c^2B^*B, \ c \geq 0$;

(ii) $A^*A = B^*MB$ for some $0 \leq M \in \mathcal{B}(\mathcal{H}_B)$ with $\|M\| \leq c^2$.

5. Majorization and domination

There is a direct connection between the majorization of bounded operators as in the original Douglas lemma and the notion of domination of linear relations as in Definition 4.1.
Lemma 5.1. Let $\mathfrak{A}$, $\mathfrak{B}$, and $\mathfrak{S}$ be Hilbert spaces, let $A \in \mathcal{B}(\mathfrak{A}, \mathfrak{S})$, $B \in \mathcal{B}(\mathfrak{B}, \mathfrak{S})$, and $W \in \mathcal{B}(\mathfrak{A}, \mathfrak{B})$. Then
\[(5.1) \quad A = BW \iff WA^{-1} \subset B^{-1}.
\]

Proof. First observe that if $H \in \mathcal{B}(\mathfrak{A}, \mathfrak{S})$, then
\[(5.2) \quad HH^{-1} = I_{\text{ran} H} \subset I_{\mathfrak{S}}, \quad H^{-1}H = I_{\mathfrak{A}} \oplus \left(\{0\} \times \ker H\right) \supset I_{\mathfrak{A}},
\]
as is clear from (2.4) and (2.5).
\[(\Rightarrow) \quad \text{Assume that } A = BW. \text{ Then by } (5.2) \text{ it follows that}
\[WA^{-1} \subset B^{-1}BW A^{-1} = B^{-1}AA^{-1} \subset B^{-1}.
\]
\[(\Leftarrow) \quad \text{Assume that } WA^{-1} \subset B^{-1}. \text{ Then by } (5.2) \text{ it follows that}
\[BW \subset BW A^{-1}A \subset BB^{-1}A \subset A,
\]
so that $BW \subset A$. Actually equality $BW = A$ prevails here, since both $BW$ and $A$ are everywhere defined operators. \hfill \Box

In other words, the lemma expresses the fact that when $A$ and $B$ are bounded operators, then $B$ majorizes $A$ in the sense of $AA^* \leq \lambda BB^*$ (cf. Lemma 1.1) if and only if the relation $A^{-1}$ dominates the relation $B^{-1}$ in the sense of Definition 1.1.

The connection in Lemma 5.1 is useful as it yields a particularly simple proof for the characterization of the ordering of nonnegative selfadjoint relations as in (2.19). For earlier treatments of the ordering, see [5, 9].

Theorem 5.2. Let $H_1$ and $H_2$ be nonnegative selfadjoint relations in a Hilbert space $\mathfrak{S}$. Then the following statements are equivalent:
\[(i) \quad H_1 \preceq H_2; \quad (ii) \quad (H_1 + x)^{-1} \succeq (H_2 + x)^{-1} \text{ for some and hence for every } x > 0; \quad (iii) \quad H_1^{-1} \succeq H_2^{-1}.
\]

Proof. (i) $\iff$ (ii) Recall that $H_1 \preceq H_2$ if and only if for some (and hence for all) $x > 0$
\[H_1 + x \preceq H_2 + x,
\]
and note that for $x > 0$ the inverses $(H_1 + x)^{-1}$ and $(H_2 + x)^{-1}$ belong to $\mathcal{B}(\mathfrak{S})$. By Theorem 1.4 $H_1 + x \preceq H_2 + x$ is equivalent to the existence of $Z \in \mathcal{B}(\mathfrak{S})$ such that
\[(5.3) \quad Z(H_2 + x)^{1/2} \subset (H_1 + x)^{1/2}, \quad \|Z\| \leq 1;
\]
cf. Corollary 2.10 Now an application of Lemma 5.1 shows that (5.3) is equivalent to
\[(5.4) \quad (H_2 + x)^{-1/2} = (H_1 + x)^{-1/2}Z.
\]
Finally, by Lemma 1.1 (or Theorem 3.4) (5.4) is equivalent to
\[(H_2 + x)^{-1/2}(H_2 + x)^{-1/2} \leq (H_1 + x)^{-1/2}(H_1 + x)^{-1/2},
\]
since $\|Z\| \leq 1$.

(ii) $\iff$ (iii) Let $H$ be a nonnegative selfadjoint relation. Then clearly also $H^{-1}$ is a nonnegative selfadjoint relation and it is connected to $H$ via
\[(5.5) \quad (H + x)^{-1} = \frac{1}{x} - \frac{1}{x^2} \left(H^{-1} + \frac{1}{x}\right)^{-1},
\]
where \( x > 0 \). Hence for a pair of nonnegative selfadjoint relations \( H_1 \) and \( H_2 \) one obtains for each \( x > 0 \):

\[
(H_2 + x)^{-1} - (H_1 + x)^{-1} = \frac{1}{x^2} \left[ \left( H_1^{-1} + \frac{1}{x} \right)^{-1} - \left( H_2^{-1} + \frac{1}{x} \right)^{-1} \right].
\]

Now the equivalence is obtained from (i) ⇔ (ii).

\[\Box\]

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