BRST-BFV and BRST-BV Lagrangians for Bosonic Fields with Continuous Spin on $R^{1,d-1}$

Č. Burdík$^a$, V.K. Pandey$^b$, A. Reshetnyak$^c$

$^a$Department of Mathematics, Czech Technical University, Prague 12000, Czech Republic,

$^b$Department of Physics, Banaras Hindu University, 221005 Varanasi, India,

$^c$Laboratory of Computer-Aided Design of Materials, Institute of Strength Physics and Materials Science SB RAS, 634055 Tomsk, Russia

Abstract

Gauge-invariant Lagrangian descriptions for a free bosonic scalar field of continuous spin in a $d$-dimensional Minkowski space-time using a metric-like formulation are constructed on the basis of a constrained BRST–BFV approach we propose. The resulting formulations contain different sets of auxiliary fields, depending on the manner of a partial gauge-fixing and a resolution of some of the equations of motion for a BRST-unfolded first-stage reducible gauge theory. To reach an equivalence of the resulting BRST-unfolded constrained Lagrangian equations of motion with the initial irreducible Poincare group conditions of a Bargmann–Wigner type, it is demonstrated that one should replace the field in these conditions by a class of gauge-equivalent configurations. Triplet-like, doublet-like and non-gauge constrained Lagrangian descriptions, as well as a quartet-like unconstrained Lagrangian formulation, are derived using both Fronsdal and new tensor fields, in particular, BRST–BV actions, in the minimal sector of the respective field and antifield configurations, are constructed in an explicit way.

1 Introduction

The Poincare group is a cornerstone of relativistic quantum field theories. For the first time, its representations in $\mathbb{R}^{1,3}$ were studied by E. Wigner [1]. The number of group representations describes the quantum states found in a local field theory, being some massless particles of fixed helicity (photon) and massive particles of integer (for vector and Higgs bosons) and half-integer (for quarks and leptons) spin. In higher space-time dimensions, the Poincare group $ISO(1,d-1)$ is shown to be useful in (super)string theories [2], [3], [4]. Until now, no examples have been found to realize any other representations that exist in the Nature. So, a tachyon representation of imaginary mass, which appears to be an excitation of the lowest
energy in the spectrum of bosonic string theories, is used as an indicator of instabilities, for instance, in spontaneous symmetry breaking. The other representations are known as continuous spin representations (CSR) which describe a massless object with an infinite number of helicities for which eigenstates of various helicities are mixed under the Lorentz transformations, in a way similar to the set of massive particles, leading to an infinite heat capacity of the vacuum, due to Wigner’s argumentation \[5\].

Numerous attempts have been undertaken to associate CSR with physical systems. It appears that the actual discovery of this procedure is yet to come. At the same time, it is obvious that single-valued (bosonic) and double-valued (fermionic) CSR with an infinite number of degrees of freedom (see, e.g., \[6\]) are contained in the respective spectra of second-quantized bosonic strings and superstrings, in addition to the massless higher-spin (HS) fields of all the integer (0, 1, 2,...) and half-integer (1/2, 3/2, 5/2, ... ) helicities (each having a finite number of degrees of freedom), so as to be extracted using the (super)string tensionless limit \[7\], \[8\].

The above property of CS particles is quite attractive nowadays due to an intense development of higher-spin theory \[9\], \[10\], \[11\]; see the reviews \[12\], \[13\], the discussion in the string-theory context \[14\] and references therein.

Unitary irreducible representations (UIR) using CS for the Poincare and super-Poincare groups in a \(d\)-dimensional Minkowski space-time with \(d > 4\) were first studied by the team of L. Brink and P. Ramond \[15\], and, in further detail, by X. Bekaert and N. Boulanger \[16\]. It was shown by A.M. Khan and P. Ramond \[17\] that it is possible to consider CSR with CS \(\Xi\) as a special limit for an HS particle of mass \(m\) and spin \(s\), when \(\lim_{m \to 0, s \to \infty} ms = \Xi\), used to derive the Fronsdal- and Fronsdal–Fang-like equations \[18\], \[19\], albeit having CSR in the limit corresponding to massive HS particles \[20\], and shown to be equivalent to the Wigner and Wigner–Bargmann equations \[21\] (for a review, see, e.g., \[22\]).

In turn, a search for Lagrangian formulations (LF) and forms of relativistic field equations, not necessarily Lagrangian ones, which are to equivalently reproduce the conditions selecting massless UIR with CS, has been variously developed for \(R^{1,d-1}\), in both \(d = 4\) and higher dimensions. So, a local covariant action for bosonic CSP formulated using an auxiliary Lorentz vector \(\eta_m\) and localized to the unit hyperboloid \(\eta^2 = -1\) has been presented by an integral over \(d^4xd^4\eta\) in \[23\] (see also \[24\]). An LF for a scalar bosonic CSR field in terms of an infinite set of (double-)traceless totally-symmetric tensor fields of any rank in constant-curvature \(d\)-dimensional spaces has been realized using an oscillator formalism by R. Metsaev \[25\], which was used in \[26\] to construct a quantum action for CSR field in \(R^{1,d-1}\), whereas a twistor description for massless particles with CS has been suggested in \[27\] (for relationship between the Fronsdal-like and Fang-Fronsdal-like equations \[28\] and ones obtained in \[25\] and for interactions, see as well, \[29\], \[30\], \[31\], \[32\]).

Some of the most efficient tools to reconstruct a local gauge-invariant LF from the initial UIR of the Poincare or anti-de-Sitter groups previously used merely for particles of discrete spin on a basis of the BRST–BFV approach originating from the BFV method \[33\], \[34\], invented to quantize dynamical constrained systems, and applied, nevertheless, to a solution of the inverse problem, in fact, to formulate an LF in terms of Hamiltonian-like objects using an auxiliary Hilbert space whose vectors consists of HS (spin)-tensor fields. It is not surprising that a first application in this way of the BRST–BFV method to CS fields in \(R^{1,3}\) has been recently proposed by A. Bengtsson \[35\], one of the inventors of the constrained BRST–BFV approach to lower-spin fields \[36\], \[37\]. An inclusion of holonomic (traceless and mixed-symmetry) constraints, together with differential ones, into a total system of constraints
which is to be closed with respect to Hermitian conjugation with an appropriate conversion
procedure for a subsystem with second-class constraints, has resulted in augmenting the
original method by an unconstrained BRST–BFV method, with no restrictions imposed on
the entire set of initial and auxiliary HS fields. The application of this method have been
initiated by A. Pashnev, M. Tsulaia [38], followed by C. B., I. Buchbinder, V. Krykhtin and
A R. [39], [40], [41], [42], for totally-symmetric HS fields and mixed-(anti)symmetric HS fields
in $R^{1,d-1}$ and AdS$_d$ [43], [44], [45], [46], [47], [48] (for a review and the interaction problem,
see [49]). A detailed correspondence between constrained and unconstrained BRST–BFV
methods for arbitrary massless and massive IR of the ISO(1, $d - 1$) group with a gener-
alized discrete spin has been recently studied in [50], where a constrained BRST–BFV LF
for fermionic HS fields subject to an arbitrary Young tableaux $Y(s_1, ..., s_k)$, $k \leq \lfloor d/2 \rfloor$ was
first suggested and an equivalence between the underlying constrained and unconstrained
LF was established. A development of this topic has resulted in an (un)constrained BRST–
BV method of finding minimal BV actions necessary to construct a quantum action within
the BV quantization [51] presented in [52] (for bosonic HS fields, also see [53], [54], [55]).
An application of the BRST–BFV method to a scalar bosonic CS field in $R^{1,3}$ on the basis
of a so-called four-constraint formalism [35] was recently proposed using the Weyl spinor
notation in [56] (for recent developments, see also [57], [58], [59], [60], [61]). A prescription
for a four-constraint formalism to derive an unconstrained BRST–BFV LF for a CS field
[35] is different from the one applied to HS fields of any discrete helicity, because the set
of conditions extracting a massless bosonic UIR of any integer spin and the one having CS
[21] contains the respective 2 and 4 equations, so that the “naive” numbers of the respective
constraints being linear in the ghost approximations of Hermitian BRST operators should
be 3 and 7.

Having in mind the equivalence between unconstrained and constrained BRST–BFV LF
for one and the same HS field of a generalized discrete spin in $R^{1,d-1}$ [50], we shall assume
that the same property is to be valid for unconstrained and constrained LF, we intend to
construct an LF for free massless CSR particles propagating in $R^{1,d-1}$. The article is devoted
to the following problems:

1. Derivation of a constrained BRST–BFV approach to constrained gauge-invariant LFs
   for a scalar CS field in $R^{1,d-1}$, with a compatible set of off-shell BRST-extended con-
   straints in the metric formulation;

2. Study of an equivalence between the resulting BRST–BFV Lagrangian equations of
   motion for a scalar CS field in $R^{1,d-1}$ with initial conditions extracting UIR of the
   ISO(1, $d - 1$) group with CS and making a comparison with a Fronsdal-like represen-
   tation;

3. Construction of constrained BRST–BV actions in the minimal sector of the field-
   antifield formalism on a basis of the suggested gauge-invariant constrained LFs for
   a scalar CS field in $R^{1,d-1}$;

4. Construction of an unconstrained gauge-invariant LF from a constrained BRST–BFV
   LF on a basis of additional compensating field.

The paper is organized as follows. In Section 2, we find an HS symmetry algebra for a
massless bosonic field with a given CS in $R^{1,d-1}$ and suggest (in Section 3) a constrained
BRST–BFV LF. In the latter point, we construct a constrained BRST operator with an
off-shell holonomic constraint, obtain a properly gauge-invariant LF, find its representations in terms of Fronsdal-like fields and resolve the problem 2 concerning an equivalence with the initial set of UIR CS conditions. BRST–BV minimal actions are derived in Section 4 and an unconstrained quartet-like LF is presented in Section 5. The short-list of the results is presented in the Conclusion. Finally, in Appendix A we construct auxiliary representation for HS symmetry algebra with additional pair of oscillators.

The convention \( \eta_{mn} = \text{diag}(+,-,...,-) \) for the metric tensor, with the Lorentz indices \( m,n = 0,1,...,d-1 \), and the notation \( \epsilon(A) \), \( |gh_H, gh_L, gh_{tot}|(A) \) for the respective values of Grassmann parity, BFV, \( gh_H \), BV, \( gh_L \) and total, \( gh_{tot} = gh_H + gh_L \), ghost numbers of a quantity \( A \) are used. The supercommutator \([A,B]\) of quantities \( A,B \) with definite values of Grassmann parity is given by \( |A,B| = AB - (-1)^{e(A)e(B)}BA \).

2 HS symmetry algebra \( A(\Xi; \mathbb{R}^{1,d-1}) \)

The irreducible Poincare group massless bosonic representation with CS in \( \mathbb{R}^{1,d-1} \) is described by the \( \mathbb{R} \)-valued function \( \Phi(x,\omega) \) of two independent variables \( x^m,\omega^m \) (being by scalar CS field [15], [23]) on which the quadratic \( C_2 = P^mP_m \) and quartic, \( C_4 = W_{m_1...m_{d-3}}W^{m_1...m_{d-3}} \) Casimir operators take the values

\[
C_2 \Phi(x,\omega) = 0, \quad C_4 \Phi(x,\omega) = \nu \Xi^2 \Phi(x,\omega), \quad \text{with } W_{m_1...m_{d-3}} = e^{m_1...m_d}P_{m_d-2}M_{m_d-1m_d}.
\]

\( W_{m_1...m_{d-3}} \) is the generalized Pauli-Lubanski \((d-3)\)-rank tensor\(^1\) with Levi-Civita tensor \( e^{m_1...m_d} \), momentum \( P_m = -i\frac{\partial}{\partial x^m} \), angular momentum \( M_{mn} = M_{mn} + S_{mn} \), for orbital and spin parts:

\[
\tilde{M}_{mn} = ix_m \frac{\partial}{\partial x^n} - ix_n \frac{\partial}{\partial x^m}, \quad S_{mn} = \omega_m \frac{\partial}{\partial \omega^n} - \omega_n \frac{\partial}{\partial \omega^m}.
\]

and with the real positive constant \( \Xi \), enumerating the value of CS in \( \mathbb{R}^{1,d-1} \) when \( \nu = 1 \). Explicitly, the field \( \Phi(x,\omega) \) should satisfy to the 4 relations (as it was suggested for \( d = 4 \) case by Wigner and Bargmann [21]) when \( \nu = 1 \) for the field \( \Phi(p,\xi) \) in momentum representation, being Fourier transform of \( \Phi(x,\omega) \):

\[
\hat{\Phi}(p,\xi) = (2\pi)^{-d/2} \int d^dxd^\omega \exp[ip_nx^m + i\xi_m\omega^n] \Phi(x,\omega),
\]

In terms of \( \Phi(p,\xi) \) and \( \Phi(x,\omega) \) the equations read:

\[
(\eta^{mn}p_np_n, \eta^{mn}\xi_mp_m, \eta^{mn}\frac{\partial}{\partial \xi^m}p_n - \Xi, \eta^{mn}\xi_m\xi_n + \nu)\tilde{\Phi}(p,\xi) = (0,0,0,0),
\]

\[
\eta^{mn} \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n} \Phi(x,\omega) = 0, \quad \eta^{mn} \frac{\partial}{\partial \omega^m} \frac{\partial}{\partial \omega^n} \Phi(x,\omega) = 0,
\]

\[
-\omega^m \frac{\partial}{\partial x^m} \Phi(x,\omega) = \Xi \Phi(x,\omega), \quad \eta^{mn} \frac{\partial}{\partial \omega^m} \frac{\partial}{\partial \omega^n} \Phi(x,\omega) = \nu \Phi(x,\omega),
\]

with some dimensionful parameter \( \nu \in \mathbb{R} \) (being the squared length for the space-like internal vector \( \xi^m, \xi^2 = -\nu \), expressing the fact of ambiguity in definition of internal variables \( \omega^m \) and determining the value of the quartic Casimir operator \( C_4 \) on the elements of IR space of the Poincare algebra iso\((1,d-1)\) as \( \nu \Xi^2 \)\(^2\). The equations (2.4), (2.5) are non-Lagrangian.

\(^1\)For \( d > 4 \) there exist additional Pauli-Lubanski tensors \( W^{m_1...m_e} = e^{m_1...m_d}P_{m_{e+1}}M_{m_{e+2}m_{e+3}} \times ... \times M_{m_{d-1}m_d} \), such that \( [P_m,W^{m_1...m_e}] = 0 \), thus providing for the operators \( C_{2e} = W_{m_1...m_e}W^{m_1...m_e} \), \( e = 1,3,...,d-3 \) for \( d = 2N \), \( e = 0,2,...,d-3 \) for \( d = 2N - 1 \) to be by Casimir operators [15] which are characterized by the parameters \( \nu, \Xi \) and integer spin-like parameter \( s_1,...,s_k \) for \( k = [(d - 4)/2] \).

\(^2\)For \( \Xi = 1, \nu = \mu^2 \) from above equations the relations given by (1.1)–(1.4) in [27] are obtained, whereas for \( \nu = 1 \) the Wigner and Bargmann equations [21] hold.
Expanding $\Phi(x, \omega)$ in powers of $\omega^m$ and its inverse degrees, $(\omega^m / \omega^2)^3$ in terms of independent tensor fields $\Phi_{(m)_k}(x)$ and $\hat{\Phi}_{(m)_k}(x)$:

$$\Phi(x, \omega) = \sum_{k \geq 0} \frac{1}{k!} \Phi_{(m)_k}(x) \omega^{m_1} \ldots \omega^{m_k} + \sum_{k > 0} \frac{1}{k!} \hat{\Phi}_{(m)_k}(x) \frac{\omega^{m_1}}{\omega^2} \ldots \frac{\omega^{m_k}}{\omega^2} \equiv (\Phi(+) + \Phi(-))(x, \omega), \quad (2.6)$$

(for $\omega^2 = \omega^m \omega_m$ and $\Phi_{(m)_0} = \Phi_{(m)_0} \equiv \Phi_0$) the relations (2.4), (2.5) take equivalent representation in powers of $\omega^m$ (2.7) and of $\omega^m / \omega^2$ (2.8) (with use of the notation for totally-symmetric set of indices $(m)_k \equiv m_1 \ldots m_k$):

$$\omega^{(m)_k} : \begin{cases} \eta^{mn} \frac{\partial}{\partial x^m} \Phi_{(m)_k}(x) = 0, & \partial x^{m_{k+1}} \Phi_{(m)_{k+1}}(x) = 0, \\ -i \frac{\partial}{\partial x^{m_k}} \Phi_{(m)_{k-1}}(x) = \Xi \Phi_{(m)_k}(x), & \eta^{m_{k+1}m_{k+2}} \Phi_{(m)_{k+2}}(x) = \nu \Phi_{(m)_k}(x); \end{cases} \quad (2.7)$$

$$\omega^{(m)_k} \frac{\omega^2}{\omega^k} : \begin{cases} \eta^{mn} \frac{\partial}{\partial x^m} \Phi_{(m)_k}(x) = 0, & \frac{\partial}{\partial x^{m_k}} \hat{\Phi}_{(m)_k}(x) = 0, \\ \eta_{(m_{k-1}m_k)} \frac{\partial}{\partial x^{m_{k-2}}} \hat{\Phi}_{(m)_{k-2}}(x) - 2 \frac{\partial}{\partial x^{m_k}} \hat{\Phi}_{(m)_{k-1}}(x) = 0, \\ \hat{\Phi}_{\{m_{k-1}k \}} n \eta_{m_{k-3}m_{k-2}} \eta_{m_{k-1}m_k} + 2(k - 2)(2 - d) \hat{\Phi}_{\{m_{k-2}k \}} n \eta_{m_{k-1}m_k} = \nu \hat{\Phi}_{(m)_k} \end{cases} \quad (2.8)$$

(for $k \in \mathbb{N}_0$) being respectively for each systems by D’Alambert, divergentless, gradient and generalized traceless equations. Note, first, that we have used the symmetrization in indices $m_{k+1}, (m)_k$: $\{m_{k+1}, (m)_k\}$; in $m_{k+1}m_{k+2}, (m)_k$: $\{m_{k+1}m_{k+2}, (m)_k\}$ and in 4 indices $m_{k-1}, \ldots, m_{k+2}$ with $(m)_{k-2}$ in (2.8) without numerical factor, second, the left-hand side of the last equation in (2.7) may be equivalently written as, $\eta_{m_{k+1}m_{k+2}} \Phi_{(m)_{k+2}}(x) = \Phi_{(m)_k} \omega^m(x)$ as it was done in the similar traceless equations in (2.8). The representation (2.6) leads to non-empty set of non-trivial solutions for the systems (2.7), (2.8). Third, from the system above it follows that the functions $\Phi_{(m)_k}(x)$ and $\hat{\Phi}_{(m)_k}(x)$ satisfy to different subsystems, with except for the third equation in (2.8) for $k = 2$ (with $\eta_{(m_{1}m_{2})} \hat{\Phi}_{(m)_0} \equiv \eta_{(m_{1}m_{2})} \hat{\Phi}_{0} = 2 \eta_{m_{1}m_{2}} \Phi_{0}$):

$$\frac{i}{\partial x^m} \hat{\Phi}_{m}(x) + 2 \eta_{nm} \Xi \Phi_{0}(x) = 0 \iff \begin{cases} i \frac{\partial}{\partial x^m} \hat{\Phi}_{m}(x) + \Xi \Phi_{0}(x) = 0, & \text{no sum in } n, \\ i \left( \delta_{\{m}^{\sigma} \delta_{n\}}^{\sigma} - \eta_{mn} \gamma_{\sigma} \right) \frac{\partial}{\partial x^m} \hat{\Phi}_{\sigma}(x) = 0. \end{cases} \quad (2.9)$$

The equation can be considered as the coupling equation among the systems (2.7) and (2.8). Note, the similar equivalent equations take place for the third equations in (2.8) for $k > 2$.4

3Expansion in terms of only non-negative degrees in $\omega^m$ in (2.6) leads to unique trivial solution, $\Phi(x, \omega) = 0$, for the first equation in (2.5) when solving it in powers of $\omega^m$, whereas the most general form for $\Phi(x, \omega)$ may be presented as, $\sum_{l \geq 0} \left( \sum_{k \geq 0} \frac{1}{k!} \Phi_{(m)_k}(x) \omega^{m_1} \ldots \omega^{m_k} \right) \omega^{-2l}$, for $\Phi^0_{(m)_k} \equiv \Phi_{(m)_k}$ in (2.6).

4Another variant of solutions for (2.4), (2.5) due to the first equation in (2.5) can be chosen without poles in $\omega^m$ as its explicit solution:

$$\Phi(x, \omega) = \delta(\omega \rho - \Xi) \varphi(x, \omega); \quad \varphi(x, \omega) = \sum_{k \geq 0} \frac{1}{k!} \varphi_{(m)_k}(x) \omega^{m_1} \ldots \omega^{m_k}, \quad p_{mn} = -i \frac{\partial}{\partial x^m}.$$

To get the UIR with CS for the field $\varphi(x, \omega)$ one should to modify the rest equations in order to include the value of CS $\Xi$ in it that should provide the fulfillment of the equations on the Casimir operators (2.1) that was done, e.g. in 24, 50.
Considering the dynamics of the fields $\Phi^\pm$ and $\Phi^{-}$ jointly, we introduce a translational invariant vacuum vector: $|0\rangle: = \frac{d}{d\omega^m} |0\rangle$ or the Fock space $\mathcal{H}$, which is generated by one pair of the Grassmann-even bosonic oscillators: $(a_m, a^+_m) \equiv -i(\partial/\partial \omega^m, \omega^m)$ subject to the commutation relations:

$$
|\Phi\rangle = \sum_{k \geq 0} \frac{i^k}{k!} \Phi_{(m)k}(x) a^{+m_1}\ldots a^{+m_k} |0\rangle + \sum_{k \geq 0} \frac{(-i)^k}{k!} \Phi_{(m)l}(x) a^{+m_1} a^{+m_2} \ldots a^{+m_k} |0\rangle
$$

$$
= \Phi^+(x, a^+) |0\rangle + \Phi^-(x, a^+) |0\rangle \equiv |\Phi^+(\rangle + |\Phi^-(\rangle ,
$$

for square integrable component functions in $\Phi^+(x, a^+)$ and $\Phi^-(x, a^+)$ obtained from the decomposition (2.6). The Poincare group IR relations (2.4), (2.5) take the equivalent form in terms of the operators,

$$(l_0, l_1, m^+_1, m_{11}) |\Phi\rangle = 0 \iff \left\{ (l_0, l_1, m_{11}) |\Phi^\pm\rangle = 0 \text{ and } m^+_1 |\Phi\rangle = 0 \right\};$$

$$
(l_0, l_1, m^+_1, m_{11}) = \left( \eta^{m\bar{n}} \frac{\partial}{\partial x^m} \frac{\partial}{\partial \bar{x}^n}, \eta^{m\bar{n}} \frac{\partial}{\partial x^m} + i\Xi, a^m a_m + \nu \right).$$

To get Lagrangian form of the equations (2.12) we need real-valued Lagrangian action. Therefore, the set of initial constraints $\{a_{\alpha} \} = (l_0, l_1, m^+_1, m_{11})$ (2.13) should be closed with respect to $[\ , \ ]$-multiplication and Hermitian conjugation in $\mathcal{H}$. In spite of the fact, that the operators $\frac{a^m}{\alpha^\sigma}$ being Hermitian conjugated to $\frac{a^{+m}}{\alpha^\tau}$ with respect to the standard scalar product $\langle \cdot|\cdot \rangle$, can not be considered as "annihilation" operators for the latter: $[ \frac{a^m}{\alpha^\sigma}, \frac{a^{+m}}{\alpha^\tau} ] \neq -\eta^{m\bar{n}} C_1$, for some real constant $C_1$, we determine the scalar product as follows

$$
\langle \Psi|\Phi\rangle = \langle \Psi^+|\Phi^+\rangle + \langle \Psi^-|\Phi^-\rangle = \int d^4x \left\{ \sum_{k,p=0}^\infty \frac{i^k (-i)^p}{k!p!} \langle 0| \prod_{j=1}^p a^{m_j} \Psi^*_m(x)^p \times \sum_{k,p=0}^\infty \frac{i^k (-i)^p}{k!p!} \langle 0| \prod_{j=1}^p a^{m_j} \Psi^*_m(x)^p \times \right\}
$$

$$
= \sum_{k=0}^\infty \frac{(-1)^k}{k!} \int d^4x \left\{ \Psi^*_{(n)}(x) \Phi^{(n)}(x) \right\} + \theta_{k,0} \hat{\Psi}^*_{(n)}(x) K_{k,k} \hat{\Phi}^{(n)}(x) + \right\} K_{k,k+2l} \hat{\Phi}^{(n)}(m)(m)|c.c.\rangle ,
$$

with Heaviside symbol $\theta_{k,l} = 1(0)$, when integers $k > l(k \leq l)$ and some real numbers $K_{k,k}$ and $K_{k,k+2l}$, $K_{k+2l,k}$, $k \in \mathbb{N}$ for some $l$. Indeed, the orthogonality properties among the vectors $\langle 0|a^m, (a^{+m}) |0\rangle$ and $\langle 0|\frac{a^m}{\alpha^\sigma}, \frac{a^{+m}}{\alpha^\tau} |0\rangle$ take the form:

$$
\langle 0| \prod_{j=1}^q a^m_{i} a^{+m}_{i} |0\rangle = \delta_{pq} (1)^p p! S_{(n)}^{(m)} , \quad \langle 0| \prod_{j=1}^q a^m_{i} a^{+m}_{i} |0\rangle = 0,
$$

$$
\langle 0| \prod_{j=1}^k a^m |0\rangle = \delta_{pq} (1)^p p! S_{(n)}^{(m)} , \quad \langle 0| \prod_{j=1}^k a^m |0\rangle = 0,
$$

$$
\langle 0| \prod_{j=1}^k a^m a^{+m} |0\rangle = \left\{ \begin{array}{ll}
\delta_{p+2l,k} (1)^k a^m |0\rangle & \text{if } p > k \\
\delta_{p+2l,k} (1)^k a^m |0\rangle & \text{if } p \leq k
\end{array} \right.
$$

For quartic Casimir operator $C_4 = (M_{mn} P^m)^2$ evaluated for massless case on $\Phi(x, \omega)$ we have after explicit calculation with allowance made for the equations (2.4), (2.5) that: $C_4 \Phi(x, \omega) = (l_0^2 + \nu) \Phi(x, \omega) = \Xi^2 \nu \Phi(x, \omega)$, so that the relations (2.1) hold.
where \( K_{p,p+2l} = K_{p+2l,p} \) and

\[
K_{p,p} = \int_0^\infty dt_1 dt_2 \sum_{k=0}^\infty \frac{(t_1 t_2)^k}{(k!)^2} \prod_{j=1}^p 2j[d + 2(j + p - 1)],
\]

(2.17)

\[
K_{p,p+2l} = \int_0^\infty dt_1 dt_2 \sum_{M,N} \frac{(-1)^{M+N} \delta(kM, pN + l)}{M!N!} \prod_{j=1}^{kM} 2j[d + 2(j + p + 2l - 1)],
\]

if \( \exists N, M \in \mathbb{N} : pN - kM = l, \)

(2.18)

\( \forall p, k, q, l \in \mathbb{N} \) with the symmetrizer \( S_{(m)_k} \) and respective series in prime numbers \( N, M \), starting from \( N_{\text{min}} \) and \( M_{\text{min}} = (pN_{\text{min}} + l)/k \).

The first products in (2.15) are standard, whereas to prove the validity of the second ones, we apply the induction. For \( p = 1, \forall k \in \mathbb{N} \) we have

\[
\langle 0 | a^{m_1} \prod_{i=1}^k \frac{a^{+ni}}{(a^+)^2} | 0 \rangle = \langle 0 | (a^{m_1}, a^{+ni}) \frac{1}{(a^+)^2} - a^{+ni} [a^{m_1}, a^{+ni}] \frac{1}{(a^+)^4} \prod_{i=2}^k \frac{a^{+ni}}{(a^+)^2} | 0 \rangle = -\langle 0 | \left( \eta^{m_1}_{n_1} (a^+)^2 - 2a^{+ni} a^{m_1} \right) \frac{1}{(a^+)^4} \prod_{i=2}^k \frac{a^{+ni}}{(a^+)^2} | 0 \rangle = 0,
\]

(2.19)

due to \( \langle 0 | a^{+ni} = 0 \). Let for \( \forall p \leq p_0 \in \mathbb{N} \) the same equations as one (2.19) hold. Then, for \( p = p_0 + 1, \forall k \in \mathbb{N} \) it follows, with account of the relation above:

\[
\langle 0 | \left( \prod_{j=1}^{p_0} \left( \prod_{i=1}^k \frac{a^{+ni}}{(a^+)^2} \right) a^{m_{p_0+1}} \prod_{i=2}^k \frac{a^{+ni}}{(a^+)^2} \right) | 0 \rangle = -\langle 0 | \left( \prod_{j=1}^{p_0} a^{m_j} \right) \left( \eta^{m_{p_0+1} n_1} (a^+)^2 - 2a^{+ni} a^{m_{p_0+1}} \right) \frac{1}{(a^+)^4} \prod_{i=2}^k \frac{a^{+ni}}{(a^+)^2} \right) | 0 \rangle = -\langle 0 | \left( \eta^{m_{p_0+1} n_1} \eta^{m_{k+1} m_{k+2}} - \delta^{m_{p_0+1} n_1} \delta^{n_1}_{m_{k+3}} \right) \prod_{i=2}^{k+2} \frac{a^{+ni}}{(a^+)^2} | 0 \rangle = -\langle 0 | \left( \prod_{j=1}^{p_0} a^{m_j} \right) \left( \eta^{m_{p_0+1} n_1} \eta^{m_{k+1} m_{k+2}} - \delta^{m_{p_0+1} n_1} \delta^{n_1}_{m_{k+3}} \right) \prod_{i=2}^{k+2} \frac{a^{+ni}}{(a^+)^2} \right) | 0 \rangle = \ldots = -\langle 0 | \left( \prod_{j=1}^{p_0} a^{m_j} \right) \prod_{i=1}^k \frac{a^{+ni}}{(a^+)^2} a^{m_{p_0+1}} | 0 \rangle = 0,
\]

(2.20)
due to the repeated applying of the induction hypothesis, e.g. for the first summand in the relation before last (and for the second term in the previous relation), as well as with commutating of \( a^{m_{p_0+1}} \) with \( a^{+ni} \) for \( i = 3, \ldots, k \). The Hermitian conjugated quantities for ones in (2.15): \( \langle 0 | \prod_{j=1}^p a^{m_j} \prod_{i=1}^k \frac{a^{+ni}}{(a^+)^2} | 0 \rangle \), vanish as well. \( p \) To establish validity of (2.16) we should commute \( \frac{1}{(a^+)^{2p}} \) through \( \frac{1}{a^{2k}} \), which may be done with help of the integral representation for \( \frac{1}{(a^+)^{2k}} = \int_0^\infty dt \exp \{ -ta^{(+2k)} \} \), starting from the case, \( p = k \), by means of the
auxiliary relation:

\[ \langle 0 | \frac{1}{a^{2k}} \frac{1}{a^{(+)2k}} | 0 \rangle = \langle 0 | \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(-t_1)^e}{e!} \sum_{g \geq 0} \frac{(-t_2)^g}{g!} | 0 \rangle = \] (2.21)

\[ = \int_0^\infty dt_1 dt_2 \sum_{e,g \geq 0} \frac{(-t_1)^e(-t_2)^g}{e!g!} \{ \{ (a^2)^ke(a^2)^kg \} | 0 \} = \int_0^\infty dt_1 dt_2 \times \] (2.22)

with using of the expansion above in Taylor series for \( \exp \{-ta^{(+2k)}\} \), spectral properties:

\[ \langle 0 | (a^2)^ke(a^2)^kg | 0 \rangle \sim \delta_{ge} \ldots \text{, and that } \forall k \in \mathbb{N}_0: \]

\[ \langle 0 | (a^2)^k(a^{+2})^k | 0 \rangle = \prod_{j=1}^k \langle 0 | 4j(g_0 + j - 1) | 0 \rangle \text{ and } \langle 0 | (g_0 + j - 1) | 0 \rangle = (d/2 + j - 1). \] (2.23)

Therefore, we have respectively for \( p = k = 1 \) and \( p = 2, k = 1 \)

\[ \langle 0 | \frac{a^m}{a^2} \frac{a^m}{a^{(+)2}} | 0 \rangle = \langle 0 | \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} a^m(a^{2e}a^{+2e})a^+_n | 0 \rangle = -\delta^m_n \int_0^\infty dt_1 dt_2 \times \]

\[ \times \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} \prod_{j=1}^e \langle 0 | 4j(g_0 + j) | 0 \rangle = -\delta^m_n \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} \prod_{j=1}^e 2j[d + 2j]; \] (2.24)

\[ \langle 0 | \frac{a^m a^m}{a^2} \frac{a^m}{a^{(+)2}} | 0 \rangle = \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} (-1)^e \frac{(t_1 t_2)^e}{e!(2e)!} \langle 0 | a^m a^m a^{2e}a^{+4e}a^+_n | 0 \rangle = 0; \] (2.25)

so that, for any \( p = k + 1, p, k \in \mathbb{N} \) the presentation (2.16) is valid. Whereas for \( p = k, \forall p \in \mathbb{N} \) the average values in (2.16) calculated with account of (2.21), (2.22):

\[ \langle 0 | \prod_{j=1}^p a^{m_j} \prod_{i=1}^p \frac{a^{n_i}}{a^{(+)2}} | 0 \rangle = \int_0^\infty dt_1 dt_2 \sum_{e,g \geq 0} \frac{(-t_1)^e}{e!} \sum_{p \geq 0} \frac{(-t_2)^g}{g!} \{ \{ (a^2)^pe(a^{+2})^pg \} | 0 \rangle \prod_{j=1}^p a^{m_j} \prod_{i=1}^p a^{n_i} | 0 \rangle \]

\[ = \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} \left\{ \langle 0 | \prod_{j=1}^p a^{m_j} \prod_{j=1}^p [4j(g_0 + j - 1)] \prod_{i=1}^p a^{n_i} | 0 \rangle \right\} \]

\[ = \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} \left\{ \langle 0 | \prod_{j=1}^p [4j(g_0 + j + p - 1)] \prod_{j=1}^p a^{m_j} \prod_{i=1}^p a^{n_i} | 0 \rangle \right\} \]

\[ = (-1)^p! S_{(n)^p} \int_0^\infty dt_1 dt_2 \sum_{e \geq 0} \frac{(t_1 t_2)^e}{(e!)^2} \prod_{j=1}^p 2j[d + 2(j + p - 1)], \] (2.26)

that proves the validity of (2.16) with \( K_{p,p} \) in (2.17).

Turning to the case \( p \neq k \) in (2.16), from the equation in integers: \( pN - kM = l \) in (2.18) it follows that the scalar products vanish if \( p - k = 2L + 1 \). Moreover if for given \( p, k \) this equation has no any natural solutions that the respective vectors are orthogonal. E.g. it will take the place for \( p = 2P, k = KP, \forall P \in \mathbb{N} \) and any prime natural \( K \neq 2K_1 \), or product of prime numbers \( K_1, K_2, \ldots: K = \prod K_1 K_2 \ldots, K_i \neq 2 \). For instance, \( K_{2e,3e} = 0, \forall e \in \mathbb{N} \).
When the solution for the equation (2.18) exists, that means that, \( k - p = 2l \), the scalar product \((2.16)\) may be calculated as

\[
\langle 0 | \prod_{j=1}^{p} \frac{a_{m_j}^*}{a^2} \prod_{i=1}^{k} \frac{a_{n_i}^*}{(a_i^+)^2} | 0 \rangle = \int_0^\infty dt_1 dt_2 \sum_{e,g \geq 0} \frac{(-t_1)^e(-t_2)^g}{e!g!} \langle 0 | \prod_{j=1}^{p} a_{m_j} \left\{ (a^2)^e(a^+)^2)^g \right\} \prod_{i=1}^{k} a_{n_i}^+ | 0 \rangle
\]

\[
= \int_0^\infty dt_1 dt_2 \sum_{M,N} \frac{(-1)^{M+N} \delta(kM,pN+l)}{M!N!} \left\{ \langle 0 | \eta_{m_{p+1}m_{p+2}} \cdots \eta_{m_{2l+1}m_{2l+2}} \prod_{j=1}^{p} a_{m_j}^* \right\} = \int_0^\infty dt_1 dt_2 \sum_{M,N} \frac{(-1)^{M+N} \delta(kM,pN+l)}{M!N!} \left\{ \langle 0 | \eta_{m_{p+1}m_{p+2}} \cdots \eta_{m_{2l+1}m_{2l+2}} \prod_{j=1}^{p} a_{m_j}^* \right\}
\]

\[
\times \prod_{j=1}^{pN} \left[ 4j(g_0 + j - 1) \right] \prod_{i=1}^{k} a_{n_i}^+ | 0 \rangle = (-1)^k k! \sum_{(m)_k} \eta_{m_{p+1}m_{p+2}} \cdots \eta_{m_{2l+1}m_{2l+2}} K_{k-2l,k}\]

(2.27)

with using of the relation:

\[
\langle 0 | \prod_{j=1}^{k} a_{m_j} \prod_{j=1}^{pN} \left[ 4j(g_0 + j - 1) \right] \prod_{i=1}^{k} a_{n_i}^+ | 0 \rangle = \langle 0 | \prod_{j=1}^{pN} \left[ 4j(g_0 + j + k - 1) \right] \prod_{i=1}^{k} a_{m_j} \prod_{j=1}^{k} a_{n_i}^+ | 0 \rangle\]

(2.28)

and (2.23), that justify the representation \((2.16)\) with \( K_{k-2l,k} \) given in \((2.18)\).

The closedness of the set of operators \((2.13)\) with respect to the Hermitian conjugation in \( \mathcal{H} \) with a scalar product \((2.14)\) leads to its augmentation by the operators

\[
(l^+_1, \ m_1, \ m^+_1) = \left( -ia^{+m} \frac{\partial}{\partial x^m} , -ia^m \frac{\partial}{\partial x^m} - i\Xi, \ a^{+m} a^+_m + \nu \right) \]

(2.29)

and by the number particle operator,

\[
g_0 = \frac{1}{4} [\ m_{11}, \ m^+_1] : \ g_0 = -a^+_m a^m + \frac{d}{2} = \frac{1}{2} \{ a^m, a^+_m \},\]

(2.30)

being characterized by its action on \( |\Phi\rangle = \sum_s \{ |\Phi^{(+)}\rangle_s + |\Phi^{(-)}\rangle_s \} \)

\[
(g_0 - d/2) |\Phi\rangle = \sum_s \{ s |\Phi^{(+)}\rangle_s - s |\Phi^{(-)}\rangle_s \}.
\]

(2.31)

From the commutators:

\[
[g_0, \ m^+_1] = 2(\ m^+_1 - \nu) , \ \ [g_0, \ m_{11}] = -2(\ m_{11} - \nu)
\]

(2.32)

it follows, that the non-zero number \( \nu \) should be considered as a non-central charge.

---

\(^6\)The quantities \( K_{p,p}, K_{p+2l,p} \) \((2.17), (2.18)\) should be finite in order to make the scalar product \((2.14)\) and therefore Lagrangian to be well-defined. We will omitting the discussion of this point in the paper. For some another realization of HS symmetry algebra and the scalar product \((A.5)\) with additional pair of oscillators see Appendix A.
\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
& l_0 & m_1 & m_1^+ & l_1 & l_1^+ & m_{11} & m_{11}^+ \\
\hline
l_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m_1 & 0 & 0 & l_0 & 0 & 0 & -2l_1^+ & l_1 \\
m_1^+ & 0 & -l_0 & 0 & -l_0 & 0 & 2l_1 & 0 & -l_1^+ \\
l_1 & 0 & 0 & l_0 & 0 & 0 & -2l_1^+ & l_1 \\
l_1^+ & 0 & -l_0 & 0 & -l_0 & 0 & 2l_1 & 0 & -l_1^+ \\
m_{11} & 0 & 0 & -2l_1 & 0 & -2l_1 & 0 & 4g_0 & 2(m_{11} - \nu) \\
m_{11}^+ & 2l_1^+ & 0 & 2l_1^+ & 0 & -4g_0 & 0 & -2(m_{11} - \nu) & 0 \\
g_0 & 0 & -l_1 & l_1^+ & -l_1 & l_1^+ & -2(m_{11} - \nu) & 2(m_{11} - \nu) & 0 \\
\hline
\end{array}
\]

Table 1: HS symmetry algebra \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \).

Because of any linear combination of the constraints \( o_I = (o_\alpha, o_\alpha^+) \) should be constraint, we have, that

\[
m_1^+ - l_1^+ = i\Xi, \quad m_1 - l_1 = -i\Xi,
\]

(2.33)

\( \Xi \) should be considered as the non-central charge too, because of not extending the zero-mode constraint, \( l_0 \). Note, because of the operators \( l_1^+, m_1 \) can not be imposed as the constraints on \( |\Phi\rangle \) we could ignore the reducibility above.

Being combined, the total set of bosonic operators \( o_I = \{o_A, o_\alpha, o_\alpha^+: g_0, \Xi, \nu\} \), for \( \{o_A\} = \{l_0, l_1, l_1^+, m_1, m_1^+\} \), \( \{o_\alpha^+\} = \{m_{11}^+\} \) can be interpreted within the Hamiltonian analysis of the dynamical systems as the respective operator-valued first-class and second-class constraints subsystems among \( \{o_I\} \) for a topological gauge system, with additional operators \( g_0, \Xi, \nu \), which are not the constraints due to \( (2.30) \), and because of from the commutation relations for the operators \( o_I \) (forming a Lie algebra)

\[
[o_I, o_J] = f^K_{IJ} o_K, \quad f^K_{IJ} = -f^K_{JI},
\]

(2.34)

the following subsets can be extracted:

\[
[o_\alpha, o_\alpha^+] = f^{a\beta}_{ab} o_c + \Delta_{ab}(g_0), \quad [o_A, o_B] = f^{c}_{AB} o_C, \quad [o_A, o_B] = f^{c}_{ab} o_C.
\]

(2.35)

Here, the constants \( f^{a\beta}_{ab}, f^{c}_{AB}, f^{c}_{ab} \) are determined by the Multiplication Table II and possess the antisymmetry property with respect to permutations of lower indices, whereas the quantities \( \Delta_{ab}(g_0) \) form a non-degenerate \( (2 \times 2) \) matrix: \( |\Delta| = \text{antidiag}(-4g_0, 4g_0) \), in the Fock space \( \mathcal{H} \) on the surface \( \Sigma \subset \mathcal{H} \): \( \|\Delta\|_\Sigma \neq 0 \), which is determined by the equations \( o_A|\Phi\rangle = o_a|\Phi\rangle = 0 \).

Note, first, that we omitted in the Table I the center of \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \) consisting from \( \Xi, \nu \). Second, the linear dependence of \( o_k = (m_1, l_1, \Xi) \) and \( o_k^+ = (m_1^+, l_1^+, \Xi) \) for \( k = 1, 2, 3 \) means the existence of independent bosonic proper zero eigen-vectors \( Z^k, Z^{k+} \):

\[
o_k Z^k = 0, \quad o_k^+ Z^{k+} = 0, \quad \text{for} \quad Z^k = \beta(1, -1, \nu), \quad Z^{k+} = \beta(1, -1, -\nu), \quad \forall \beta \in \mathbb{R} \setminus \{0\},
\]

(2.36)

whose set is linear independent. Third, because of the oscillators \( a^m, a^{-m} \) transfer the vectors \( |\Phi^{(+)}\rangle_s, s \geq 0 \) (see (2.31)) into the vectors \( \theta_s, s_0 |\Phi^{(+)}\rangle_{s-1}, |\Phi^{(+)}\rangle_{s+1} \), the elements \( o_I \) from \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \) have the same property, whereas for their actions on the the vectors \( |\Phi^{(-)}\rangle_s, s > 0 \) the only \( a^m \) obeys by the similar property:

\[
a_m |\Phi^{(-)}\rangle_s = -sc^{(-)} |\Phi^{(-)}\rangle_{s+1}, \quad a^{+m} |\Phi^{(-)}\rangle_s = c^{(+)} \eta^{mn} |\Phi^{(-)}\rangle_{s+1} + |\Phi^{(-)}\rangle_{s-1},
\]

(2.37)

\[
|\Phi^{(-)}\rangle_{s-1} = \left\{ (a^{-m} a^n)/a^2 - c^{(+)} \eta^{mn} \right\} |\Phi^{(-)}\rangle_{s-1},
\]

(2.38)
with some constants \( c^{(+)} \), \( c^{(-)} \). Thus, the only non-constant operators: \( l_0, l_1, m_1, l_{11}, g_0 \), when acting on \( |\Phi\rangle \), preserve the grading in \( \mathcal{H} \), induced by the decomposition of \( |\Phi\rangle \) by \( g_0 \): \( (2.31) \), whereas \( l_1^+, m_1^+ \) leave the transformed vector \( |\Phi\rangle \) inside \( \mathcal{H} \) with only the restrictions on non-graded vectors: \( \tilde{\Phi}^{(-)} s_{-1} = 0 \), equivalently presented for \( s = 1 \) in \( (2.9) \).

### 3 Constrained BRST-BFV Lagrangian formulations

To construct constrained LF we extend the results of general research \([50]\), realized there for HS fields with generalized integer and half-integer spins on \( \mathbb{R}^{1,d-1} \), for CS case.

#### 3.1 Constrained BRST operator, BRST-extended constraints

We consider the set of the first-class constraints \( \{ o_A \} \) as the dynamical one with the element \( \Xi \), and the off-shell algebraic constraint (one from the second-class constraints) \( m_{11} \). Due to the fact that the operator \( g_0 \) does not now relate to CS value \( \Xi \), as it was for the case of discrete spin \([50]\), we introduce generating equation for superalgebra of the Grassmann-odd constrained BRST operator, \( Q_C \), and extended in the Fock space \( \mathcal{H}_C: \mathcal{H}_C = \mathcal{H} \otimes H^0_{gh} \), off-shell constraint \( \tilde{M}_{11} \) in the form:

\[
[Q_C, Q_C] = 0, \quad [Q_C, \tilde{M}_{11}] = 0, \quad \text{for} \quad gh_H(Q_C, \tilde{M}_{11}) = \epsilon(Q_C, \tilde{M}_{11}) = (1, 0), \quad (3.1)
\]

with boundary conditions for \( Q_C, \tilde{M}_{11} \):

\[
\left( \frac{\delta}{\delta C^A}, \frac{\delta}{\delta \bar{A}}, \frac{\delta}{\delta \eta_{\Xi}}, \frac{\delta}{\delta \eta_{Z}}, \frac{\delta}{\delta \eta_{\Xi}} \right) Q_C \big|_{\epsilon = 0} = \left( o_A, \Xi, \sum_k Z^k P_k, \sum_k Z^{+k} P^+_k \right), \quad \tilde{M}_{11} \big|_{\epsilon = \bar{P} = 0} = m_{11}, \quad (3.2)
\]

when vanishing ghost coordinates, momenta \( (C^A, P_A; \eta_{\Xi}, \eta_{Z}) \) for constraints \( (o_A, \Xi) \) and ones for eigen-vectors \( Z^k, Z^{+k} \), \( (\eta_{\Xi}, \eta_{Z}^+) \), being by the generating elements for Fock space \( H^0_{gh} \).

The solution for the system \((3.1)\) is sought in powers series in ghost operators with choice of some \((CP)\)-ordering for \( [C^A, P_B] = \delta_{AB} \), which satisfy to the Grassmann, ghost number distributions and respective non-vanishing (anti)commutator relations:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
C^A & P_A & \eta_{\Xi} & \eta_{Z}^+ & P_{\Xi}^+ & P_{\Xi}^+ \\
\hline
\epsilon & \bar{\eta} & 1 & 1 & 1 & 0 & 0 \\
gh & 1 & -1 & 1 & -1 & 2 & 2 \\
\hline
\end{array}
\quad (3.3)
\]

\[
\begin{align*}
\{ \eta_0, P_0 \} = \{ \eta_{\Xi}, P_{\Xi} \} = \epsilon, & \quad \{ \eta_{\Xi}, P_{\Xi}^+ \} = \{ \eta_{\Xi}, P_{\Xi}^+ \} = 1, \quad (3.4) \\
\end{align*}
\]

In \((3.3), (3.4)\) the Hermitian conjugation for the zero-mode ghosts is determined by the rule: \( (\bar{\eta}_0, P_0, \bar{P}, \eta_{\Xi}, -P_{\Xi}) = (\bar{\eta}_0, -P_0, \eta_{\Xi}, -P_{\Xi}) \) for hermitian operators \( l_0, \Xi \) from the center of \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \) with the rest ghost operators, which form the Wick pairs.

The BRST operator, \( Q' \), for the Lie algebra \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \) of the constraints \( o_I \) \((2.34)\), whose linear dependence means the presence of proper zero eigen-vectors \( Z_{l}^I \): \( o_I Z_{l}^I = 0, \quad \epsilon(Z_{l}^I) = 0 \), such that they supercommute with \( o_I \): \( [o_I, Z_{l}^I] = 0 \), should be found from the equation: \( [Q', Q'] = 2(Q')^2 = 0 \), and has the form:

\[
Q' = C' o_I + \frac{1}{2} C^j f_{jkl} P_k + C^l Z_{l}^I P_I. \quad (3.5)
\]
Here the set of fermionic ghost operators \((C^f, P_f)\) for the bosonic constraints \(o_f\) and bosonic ghost coordinates and momenta \((C^h, P_h)\) for \(Z^h_i\) corresponds to the minimal sector of BRST-BFV method [34] for the topological (i.e. without Hamiltonian) first-stage reducible dynamical system with the first-class constraints.

For the case of the constraints \(o_A \equiv (2.35)\) with multiplication table \(I\) and constant proper zero eigen-vectors \(Z^h_i = (Z^k, Z^{+k})\), the solution for \(Q_C\) in (3.1) follows from the general anzatz (3.5) in the form:

\[
\tilde{Q}_C = C^A \left( o_A + \frac{1}{2} g_B f^B_A P_D \right) + \eta_\Xi \Xi + \eta_Z \sum_k Z^{+k} P^+_k + \eta^Z_\Xi \sum_k Z^k P_k, \tag{3.6}
\]

for \(P_k = (P^+_m, P_1, P_\Xi)\) and \(P^+_k = (P^+_m, P^+_1, -P_\Xi)\). Explicitly, we have

\[
\tilde{Q}_C = \eta_0 0 + \eta_1^+ l_1 + l_1^+ \eta + m_1 \eta_1^+ + \eta_1^+ m^+_1 + i(\eta_1^+ \eta_1^+ m^+_1 + \eta_1^+ \eta^+_m + \eta^+_m \eta_1^+) P_0 + \eta_\Xi \Xi + \eta_Z (P^+_m - P^+_1 - P_\Xi) + \eta^Z_\Xi (P^+_m - P_1 + P_\Xi). \tag{3.7}
\]

Because of, first, the physical space (as the set of states being equivalent to one described by the equations (2.4) and the first from (2.5)), in fact, should be extracted by imposing of linear in ghost \(C^A, \Xi\) terms from \(\tilde{Q}_C\) (see, e.g. Statement 2 in [40]), second, the operator \(\Xi\) cannot be imposed as the constraint on the vectors from \(\mathcal{H}\), we instead consider another variant of inclusion of the term, \(\eta_\Xi \Xi\), when calculating of zero ghost number cohomology of \(Q_C\) in \(\mathcal{H}_C\).

To do so, we define the representation in \(\mathcal{H}_C\) for the vacuum vector \(|0\rangle\):

\[
(\eta_1, \eta_1^+, P_1, P_1^+, P_0, \eta_\Xi, \eta_Z, P_\Xi) |0\rangle = 0, \quad |0\rangle \in \mathcal{H}_C, \tag{3.8}
\]

such that the requirement \(\eta_\Xi \Xi |\chi_C\rangle = 0\), for arbitrary physical vector \(|\chi_C\rangle\): \(|\chi_C\rangle \in \mathcal{H}_C, gh_H(|\chi_C\rangle) = 0\) to be not depending on \(\eta_\Xi\) and \(P_\Xi\) \((P_\Xi = i(\partial / \partial \eta_\Xi))\) means that we, in fact, extract only linear independent constraints, when acting on arbitrary \(|\chi_C\rangle\):

\[
|\bar{\chi}_C\rangle = \sum_n n_0^{f_0} (\eta_1^+)^{f_1} (\eta_1^+)^{m_1} (P_1^+)^{n_1} (P_1^1)^{n_{m_1}} (\eta_Z)^{f_z} (P_Z^+)^{n_{f_z}} (P_\Xi)^{n_{\Xi}}, \tag{3.9}
\]

where \(n_{f_z}, n_{p_z}\) are running from 0 to \(\infty\), whereas the rest \(n\)'s from 0 to 1. The ghost-independent vectors \(|\Phi(a^+)_{n_{f_0}...}\rangle\) have the dependence in \(a^+\) according to (2.10). Thus, we resolved the linear dependence problem for the sets: \(\{l_1, m_1, \Xi\}\) and \(\{l_1^+, m_1^+, \Xi\}\) on the space of \(P_\Xi\)-independent vectors (3.9) and should remove dependence on proper zero eigen-vectors and respective ghosts \(\eta_\Xi^+, P_\Xi^+\) in \(\tilde{Q}_C\) and \(|\bar{\chi}_C\rangle\) turning to

\[
(Q_C, |\chi_C\rangle) = (\tilde{Q}_C, |\bar{\chi}_C\rangle) |(\eta_\Xi^+ = P_\Xi^+ = P_\Xi = 0), \tag{3.10}
\]

It provides that from \(Q_C |\chi_C\rangle = 0\) it follows, due to the choice of (3.8), the equations in power of ghosts:: \((l_0 + O(C^A), l_1 + O(C^A), m_1^+ + O(C^A)) |\chi_C\rangle = 0\), being compatible with (2.12). It is in the agreement with the observation that the operators \(l_1^+, m_1\) can not be imposed as the constraints on \(|\Phi\rangle\), so that among the operators \(l_1^+, m_1^+ (l_1, m_1)\) the only \(m_1^+\) \((l_1)\) is the constraint. The solution for the second equation in (3.1) can be found in the form

\[
\mathcal{M}_{11} = m_{11} + 2 \eta_1 P_1 + 2 \eta_1^m P_1. \tag{3.11}
\]
The respective BRST-extended number particle operator \( \hat{\sigma}_C(g) \), \( \epsilon((\hat{\sigma}_C(g)) = 0 \) (known for the discrete spin as the spin operator \([50]\)), which should satisfy to the additional equations

\[
[Q_C, \hat{\sigma}_C(g)] = 0, \quad [\hat{M}_{11}, \hat{\sigma}_C(g)] = 2(\hat{M}_{11} - \nu),
\]

is uniquely determined in the form

\[
\hat{\sigma}_C(g) = g_0 + \eta_1^+ P_1 - \eta_1 P_1^+ + \eta_1^m P_1^+ - \eta_1^m P_1^+.
\]

### 3.2 Constrained Lagrangian dynamics

To derive LFs we should solve spectral problem for the vectors \( |\chi_C^\ell\rangle \in \mathcal{H}_C' \) due to existence of \( \mathbb{Z} \)-grading in \( \mathcal{H}_C: \mathcal{H}_C = \bigoplus_k \mathcal{H}_C^k \) for \( gh_H(|\chi_C^k\rangle) = -k, \ k \in \mathbb{N}_0 \):

\[
Q_C|\chi_C^0\rangle = 0, \quad \hat{M}_{11}|\chi_C^0\rangle = 0, \quad (\epsilon, gh_H(|\chi_C^0\rangle) = (0, 0),
\]

\[
\delta|\chi_C^0\rangle = Q_C|\chi_C^1\rangle, \quad \hat{M}_{11}|\chi_C^1\rangle = 0, \quad (\epsilon, gh_H(|\chi_C^1\rangle) = (1, -1),
\]

\[
\delta|\chi_C^1\rangle = Q_C|\chi_C^2\rangle, \quad \hat{M}_{11}|\chi_C^2\rangle = 0, \quad (\epsilon, gh_H(|\chi_C^2\rangle) = (0, -2).
\]

The closedness of the superalgebra of \( Q_C, \hat{M}_{11} \) guarantees the joint set of solution for the system (3.14)–(3.16). Thus, the physical state \( |\chi_C\rangle \equiv |\chi_C^0\rangle \) for the vanishing of all ghost variables \( \eta_0, \eta_1, \eta_1^m, P_1^+, P_1^m \), contains only the physical string-like vector \( |\Phi\rangle = |\Phi(a^+)_0^0, 0^f_1, 0^m_1, 0^p_1, 0^m_0, 0^f_2, 0^m_2\rangle \) (2.10), so that

\[
|\chi_C^0\rangle = |\Phi\rangle + |\Phi_{aux}\rangle, \quad |\Phi_{aux}\rangle(\eta_0, \eta_1^+, \eta_1^m, P_1^+, P_1^m) = 0. \quad (3.17)
\]

The vectors \( |\chi_C^k\rangle \) inherit by the construction the decomposition (2.10): \( |\chi_C^k\rangle = |\chi_C^{(+)k}\rangle + |\chi_C^{(-)k}\rangle \), in sum of the vectors with positive and negative degrees in \( a_\pm \). The equations of motion: \( Q_C|\chi_C\rangle = 0 \) (\( |\chi_C\rangle \equiv |\chi_C^0\rangle \)) in (3.14) obtained at independent degrees in powers of the ghost oscillators are Lagrangian and can be derived from the Lagrangian action (determined with accuracy up to the numerical factor) to be invariant with respect to reducible gauge transformations with off-shell constraints

\[
S_C|\Xi\rangle = \int d\eta_0 \langle \chi_C^0|Q_C|\chi_C^0\rangle, \quad \delta|\chi_C^0\rangle = Q_C|\chi_C^1\rangle, \quad \delta|\chi_C^1\rangle = Q_C|\chi_C^2\rangle, \quad \delta|\chi_C^2\rangle = 0, (3.18)
\]

\[
\hat{M}_{11}|\chi_C^k\rangle = 0, \ k = 0, 1, 2. \quad (3.19)
\]

The vanishing of all \( |\chi_C^k\rangle \), for \( l \geq 3 \) is due to the possible maximal ghost momenta degree: \( P_1^+ P_1^m \) to be realized for only \( |\chi_C^2\rangle \):

\[
|\chi_C^2\rangle = P_1^+ P_1^m |\varpi(a^+)\rangle \quad \text{for} \quad |\varpi(a^+)\rangle \equiv |\Phi(a^+_0^0, 0^f_1, 0^m_0, 0^p_1, 0^m_0, 0^f_2, 0^m_2)\rangle. \quad (3.20)
\]

Thus, we constructed the constrained gauge-invariant Lagrangian formulation of the first-stage reducibility for the massless scalar bosonic field with CS \( \Xi \) for \( \nu = 1 \).
Having the decomposition in ghost oscillators for the field and first level gauge parameters \( |\chi^l_C\rangle, l = 0, 1 \) with \( \mathbb{R} \)-valued coefficient functions (as well as for \( |\omega(a^+)\rangle\)):

\[
|\chi^0_C\rangle = |\Phi\rangle + \eta^+_0 \left( |P^+_1|\chi_1(a^+)\rangle + |P^+_1|\chi_1^m(a^+)\rangle \right) + \eta^-_m \left( |P^+_1|\chi_2(a^+)\rangle + |P^+_1|\chi_2^m(a^+)\rangle \right) + \eta^+_0 \left( |P^+_1|\chi_0(a^+)\rangle + |P^+_1|\chi_0^m(a^+)\rangle \right) + \eta^+_1 \left( |P^+_1|\chi_0(a^+)\rangle + |P^+_1|\chi_0^m(a^+)\rangle \right)
\]

\[
|\chi^1_C\rangle = |P^+_1|\varsigma(a^+)\rangle + |P^+_1|\varsigma^m(a^+)\rangle + \eta^+_1 |P^+_1|\varsigma_0(a^+)\rangle + \eta^+_1 |P^+_1|\varsigma^m_0(a^+)\rangle.
\]

From the BRST-extended constraints (3.19) and structure of operator \( \hat{M}_{11} \) (3.11) it follows in powers of independent ghost monomials for the gauge parameters and field vectors:

\[
l = 2: \quad m_{11}|\varpi\rangle = 0, \quad m_{11}|\varsigma\rangle = 0, \quad m_{11}|\varsigma^m\rangle = 0, \quad m_{11}|\varsigma_0\rangle = 0, \quad m_{11}|\varsigma^m_0\rangle = 0.
\]

The \( \eta_0 \)-independent equivalent representation for the Lagrangian action, equations of motion and gauge transformations (3.18) in the supermatrix form look:

\[
\begin{align*}
\mathcal{S}_{C/E} &= \left( \langle S^0_C |, \langle B^0_C | \right) \left( \begin{array}{c}
l_0 \\
-\Delta Q_C \\
-\Delta Q_C \\
\end{array} \right) \left( \begin{array}{c}
\eta^-_1 + \eta^+_0 \\
\eta^-_1 + \eta^+_0 \\
\end{array} \right) \left( \begin{array}{c}
|S^0_C \rangle \\
|B^0_C \rangle \\
\end{array} \right), \\
\delta \left( \begin{array}{c}
|S^l_C \rangle \\
|B^l_C \rangle \\
\end{array} \right) &= \left( \begin{array}{c}
\Delta Q_C \\
l_0 \\
-\Delta Q_C \\
\end{array} \right) \left( \begin{array}{c}
\eta^-_1 + \eta^+_0 \\
\eta^-_1 + \eta^+_0 \\
\end{array} \right) \left( \begin{array}{c}
|S^{l+1}_C \rangle \\
|B^{l+1}_C \rangle \\
\end{array} \right) \theta_{2l}, \quad l = 0, 1, 2,
\end{align*}
\]

\[
\Delta Q_C = \eta^-_1 l_1 + \eta^+_0 m_1^+ + l_1^+ \eta_1 + m_1 \eta^+_1
\]

for \( |\chi^l_C\rangle = |S^l_C \rangle + \eta_0 |B^l_C \rangle, |\chi^1_C\rangle = |B^1_C \rangle \equiv 0 \), with equations of motion in (3.29) for \( l = -1 \) and with the representations for dual (bra-) vectors:

\[
\langle \chi^0_C | = \langle \Phi | + \langle \chi_0(a) |P_1 + \langle \chi_0^m(a) |P_1^m \rangle \eta_1 + \langle \chi_2(a) |P_1 + \langle \chi_2^m(a) |P_1^m \rangle \eta_1^m \rangle \eta_1^m + \langle \chi_0(a) |P_1 + \langle \chi_0^m(a) |P_1^m \rangle \eta_1 + \langle \chi_0^m(a) |P_1^m \rangle \eta_1^m \rangle \eta_0 + \langle \chi_1(a) |P_1^m + \langle \chi_1^m(a) |P_1^m \rangle \eta_1 + \langle \chi_1^m(a) |P_1^m \rangle \eta_1^m \rangle \eta_1 \\
\langle \chi^1_C | = \langle \varsigma(a) |P_1 + \langle \varsigma^m(a) |P_1^m + \langle \varsigma_0(a) |P_1^m + \langle \varsigma_0(a) |P_1^m \rangle \eta_1 + \langle \varsigma_0(a) |P_1^m \rangle \eta_1^m \rangle \eta_1 + \langle \varsigma(a) |P_1^m + \langle \varsigma(a) |P_1^m \rangle \eta_1 + \langle \varsigma(a) |P_1^m \rangle \eta_1^m \rangle \eta_0 + \langle \varsigma^m(a) |P_1^m + \langle \varsigma^m(a) |P_1^m \rangle \eta_1 + \langle \varsigma^m(a) |P_1^m \rangle \eta_1^m \rangle \eta_1 + \langle \varsigma(a) |P_1^m + \langle \varsigma(a) |P_1^m \rangle \eta_1 + \langle \varsigma(a) |P_1^m \rangle \eta_1^m \rangle \eta_1
\]

\[
\langle \chi^2_C | = \langle \varpi(a) |P_1^m + P_1.
\]

The respective gauge transformations in the ghost independent form follow from (3.29) for the first level gauge parameters:

\[
\delta \left( |\varsigma\rangle, |\varsigma^m\rangle, |\varsigma_0\rangle, |\varsigma_1\rangle, |\varsigma_0\rangle \right) = \left( - m_1, l_1^+, l_1, m_1^+, l_0 \right) |\varpi\rangle,
\]

(3.34)
and for the field vectors
\[ \delta|\Phi\rangle = l_1^+|\sigma\rangle + m_1|s^m\rangle - |s_0\rangle, \quad \delta|\chi_1\rangle = l_1|\sigma\rangle + m_1|s_{01}\rangle, \quad \delta|\chi_1^m\rangle = l_1|s^m\rangle - l_1^+|s_{01}\rangle - |s_0\rangle, \]
(3.35)
\[ \delta|\chi_2\rangle = m_1^+|\sigma\rangle + m_1|s_{11}\rangle - |s_0\rangle, \quad \delta|\chi_2^m\rangle = m_1^+|s^m\rangle - l_1^+|s_{11}\rangle, \quad \delta|\chi_{11}\rangle = l_1|s_{11}\rangle - m_1^+|s_{01}\rangle - |s_0\rangle, \]
(3.36)
\[ \delta|\chi_0\rangle = m_1|s_{0}\rangle + l_0|\sigma\rangle, \quad \delta|\chi_0^m\rangle = -l_1^+|s_{0}\rangle + l_0|s^m\rangle, \]
(3.37)
\[ \delta|\chi_{01}\rangle = -l_1|s_{0}\rangle + l_0|s_{01}\rangle, \quad \delta|\chi_{01}^m\rangle = -m_1^+|s_{0}\rangle + l_0|s_{11}\rangle. \]
(3.38)

The respective to (3.28) ghost-independent Lagrangian action takes the form
\[
S_{C|\Xi} = \left[ \langle \Phi | \left\{ \frac{1}{2} l_0 | \Phi \rangle - l_1^+ | \chi_0 \rangle - m_1 | \chi^m_{0} \rangle \right\} + \langle \chi_1 | \left\{ \frac{1}{2} l_0 | \chi_1 \rangle + l_1 | \chi_0 \rangle + m_1 | \chi_{01} \rangle \right\} \right] + h.c.
\]
(3.39)

Thus, the relations (3.34)-(3.39) determine the constrained gauge theory of the first stage reducibility for the massless free function \( \Phi(x, \omega) \) of CS \( \Xi \) in \( \mathbb{R}^{1,d-1} \) subject to the constraints (3.23)-(3.27) with 9 auxiliary tensor fields.

The special structure of the constraints permits to make gauge-fixing procedure starting from the lowest gauge parameter \( |\varpi\rangle \) which together with the linear combination of the gauge parameters \( (|s^m\rangle - |s_{11}\rangle) \) (that follows from (3.24)) belongs to the set of ker \( m_{11} \). After invertible change of the basis of the first-level gauge parameters:
\[
\{ |\sigma\rangle, |s_{01}\rangle, |s_{11}\rangle, |s_0\rangle \} \rightarrow \{ |\tilde{\sigma}\rangle, |\tilde{s}^m\rangle, |s_{01}\rangle, |s_{11}\rangle, |s_0\rangle \} \text{ for } (\tilde{s}^m, s_{11}) = \frac{1}{2} (s^m \mp s_{11}) \quad (3.40)
\]
we may gauge away the parameter \( |\tilde{s}^m\rangle \) from \( \delta|\tilde{s}^m\rangle = -\frac{1}{2} \Xi|\varpi\rangle \) by means of complete using of \( |\varpi\rangle \). Now, the theory becomes by irreducible gauge theory with independent gauge-invariant parameters \( |\tilde{\sigma}\rangle, |s_{01}\rangle, |s_{11}\rangle, |s_0\rangle \) for \( m_{11}^2|s_{11}\rangle = 0 \) and with the rest parameters satisfying to the first constraints in (3.24).

Turning to the field vectors we replace the parameters \( |s^m\rangle, |s_{11}\rangle \) in (3.35)-(3.38) on \( |\tilde{s}_{11}\rangle \) see, that two pairs of the fields \( |\chi^m_{01}\rangle, |\chi^m_1\rangle \) and \( |\chi^m_{11}\rangle, |\chi^m_{01}\rangle \) obey to similar constraints in (3.25)-(3.27) as the parameters \( (|s^m\rangle - |s_{11}\rangle) \). Making invertible change of the basis of the fields:
\[
\{ |\chi^m_{0}\rangle, |\chi^m_{01}\rangle, |\chi^m_1\rangle, |\chi^m_{11}\rangle \} \rightarrow \{ |\tilde{\chi}^m_{0}\rangle, |\tilde{\chi}^m_{01}\rangle, |\tilde{\chi}^m_1\rangle, |\tilde{\chi}^m_{11}\rangle \}
\]
(3.41)
for \( (\tilde{\chi}^m_{0}, \tilde{\chi}^m_{01}; \tilde{\chi}^m_1, \tilde{\chi}^m_{11}) = \frac{1}{2} (\chi^m_{0} \mp \chi^m_{01}; \chi^m_1 \mp \chi^m_{11}) \),
with untouched rest fields: \( |\Phi\rangle, |\chi_0\rangle, |\chi_01\rangle, |\chi_1\rangle, |\chi_2\rangle, |\chi_2^m\rangle \), we may gauge away the fields \( |\tilde{\chi}^m_{0}\rangle \) from \( \delta|\tilde{\chi}^m_{0}\rangle = \frac{1}{2} \Xi|s_0\rangle \) and \( |\tilde{\chi}^m_{11}\rangle \) from \( \delta|\tilde{\chi}^m_{11}\rangle = \frac{1}{2} \Xi|s_{01}\rangle \) by means of complete using of \( |s_0\rangle \) and \( |s_{01}\rangle \) respectively, in view of theirs satisfaction to the same constraint. Then, from the gauge transformation, \( \delta|\chi^m_{2}\rangle = i \Xi|s_{11}\rangle \) we gauge away the field \( |\chi^m_{2}\rangle \), which now obeys, together with \( |s_{11}\rangle \), to the constraints: \( m_{11}|\chi^m_{2}\rangle = m_{11}|s_{11}\rangle = 0 \) (3.26) after using \( |s_{01}\rangle \). Thus, the
following 7 fields with gauge transformations survive after partial gauge-fixing with unique gauge parameter \( |\varsigma\rangle \), \( m_{11}|\varsigma\rangle = 0:

\[
\delta \left( |\Phi\rangle, |\chi_1\rangle, |\chi_2\rangle, |\tilde{\chi}^m_{11}\rangle, |\chi_0\rangle, |\chi_{01}\rangle, |\tilde{\chi}^m_{01}\rangle \right) = \left( l^+_1, l_1, m^+_1, 0, l_0, 0, 0 \right) |\varsigma\rangle \tag{3.42}
\]

and the action transforms to the functional

\[
\mathcal{S}_{C|\Xi} = \left[ \langle \Phi | \left\{ \frac{1}{2} l_0 |\Phi\rangle - l^+_1 |\chi_0\rangle - m_1 |\tilde{\chi}^m_{01}\rangle \right\} + \langle \chi_1 | \left\{ - \frac{1}{2} l_0 |\chi_1\rangle + l_1 |\chi_0\rangle + m_1 |\chi_{01}\rangle \right\} \right] (3.43)
\]

\[
+ \langle \tilde{\chi}^m_{11} | \left\{ l_0 \left( |\tilde{\chi}^m_{11}\rangle - |\chi_2\rangle \right) + m^+_1 \left( |\chi_0\rangle + |\chi_{01}\rangle \right) - i \Xi |\tilde{\chi}^m_{01}\rangle \right\} \right] + \langle \chi_2 | \left\{ l^+_1 |\tilde{\chi}^m_{01}\rangle - l^+_1 |\chi_{01}\rangle \right\} \right]
\]

\[
+ \langle \chi_0 | \left\{ \frac{1}{2} |\chi_0\rangle + |\chi_{01}\rangle \right\} + \frac{1}{2} \langle \chi_{01} |\chi_0\rangle - \frac{1}{2} \langle \tilde{\chi}^m_{01} |\tilde{\chi}^m_{01}\rangle \right\} \right] + h.c.,
\]

which may be considered as the \textit{triplet-like Lagrangian formulation} for scalar bosonic field with CS, due to presence of the fields \( |\tilde{\chi}^m_{01}\rangle, |\chi_0\rangle, |\chi_{01}\rangle \) by analogy with case of HS fields with integer spin \([62]\). After resolution of the algebraic equations of motion formally for \( \delta \mathcal{S}^{-1}_{C} \) in \( (3.29) \) with respect to \( |\tilde{\chi}^m_{01}\rangle, |\chi_0\rangle \) and from the latter equation in \( (3.25) \) in terms of \( |\Phi\rangle, |\chi_2\rangle, |\tilde{\chi}^m_{11}\rangle, |\chi_1\rangle \):

\[
|\tilde{\chi}^m_{01}\rangle = i \Xi \left( |\tilde{\chi}^m_{11}\rangle - |\Phi\rangle \right) + l^+_1 \left( |\chi_2\rangle - |\Phi\rangle \right), \tag{3.44}
\]

\[
|\chi_{01}\rangle = - \frac{1}{2} m_{11} |\tilde{\chi}^m_{01}\rangle = - \frac{1}{2} m_{11} \left( i \Xi \left( |\tilde{\chi}^m_{11}\rangle - |\Phi\rangle \right) + l^+_1 \left( |\chi_2\rangle - |\Phi\rangle \right) \right), \tag{3.45}
\]

\[
|\chi_0\rangle = \frac{1}{2} m_{11} |\tilde{\chi}^m_{01}\rangle - m_1 |\tilde{\chi}^m_{11}\rangle - l^+_1 |\chi_1\rangle + l_1 |\Phi\rangle \tag{3.46}
\]

we obtain the gauge-invariant action, \( \mathcal{S}_{C|\Xi}^0 = \mathcal{S}_{C|\Xi} \big|_{\left( \mathcal{B}^0 = \mathcal{B}^0(\mathcal{S}) \right)} \) given in the configuration space \( \mathcal{M}_{cl} \) parameterized by \( |\Phi\rangle, |\chi_1\rangle, |\chi_2\rangle, |\tilde{\chi}^m_{11}\rangle \) subject to the gauge transformations \( (3.42) \):

\[
\mathcal{S}_{C|\Xi}^0 = \left[ \langle \Phi | \left\{ \frac{1}{2} \left( l_0 - l^+_1 l_1 + \Xi^2 \right) |\Phi\rangle + \left( \frac{1}{2} \Xi^2 m^+_1 + (l^+_1)^2 \right) |\chi_1\rangle + \left( l^+_1 m_1 - \Xi^2 \right) |\tilde{\chi}^m_{11}\rangle \right\} \right]
\]

\[
+ \left( \chi_1 | \left\{ - \frac{1}{2} \left( l_0 - l^+_1 l_1 \right) |\chi_1\rangle - \left( l^+_1 m_1 + \frac{1}{2} \Xi^2 m_{11} \right) |\tilde{\chi}^m_{11}\rangle + \frac{i}{2} \Xi m_{11} l^+_1 \left( |\chi_2\rangle - |\Phi\rangle \right) \right\} \right]
\]

\[
+ \left( \tilde{\chi}^m_{11} | \left\{ \frac{1}{2} \left( l_0 - m^+_1 m_1 + \Xi^2 \right) |\tilde{\chi}^m_{11}\rangle - \frac{i}{2} \Xi \left( 2 l^+_1 + m_{11} l^+_1 \right) \left( |\chi_2\rangle - |\Phi\rangle \right) - l_0 |\chi_2\rangle \right\} \right]
\]

\[
+ \frac{1}{4} \left( \langle \chi_2 | - \langle \Phi | \right) \left( l^+_1 m^+_1 l_1 + l^+_1 m_1 l^+_1 + 2 l^+_1 l^+_1 \right) \left( |\chi_2\rangle - |\Phi\rangle \right) \right] + h.c.
\]

In \( (3.47) \) we have singled out the combination of fields, \( \langle \chi_2 | - \langle \Phi | : m_{11} \langle \chi_2 | - \langle \Phi | = 0 \). The LF given by the action \( \mathcal{S}_{C|\Xi}^0 \) may be considered as \textit{dual-like Lagrangian formulation} for scalar bosonic field with CS by analogy with totally-symmetric HS field with integer spin. The independent holonomic constraints take the form:

\[
m_{11}|\chi_1\rangle = m_{11}|\tilde{\chi}^m_{11}\rangle = m_{11}|\varsigma\rangle = 0, \quad m_{11}|\chi_2\rangle + 2|\chi_1\rangle = 0, \quad m_{11}|\Phi\rangle + 2|\chi_1\rangle = 0. \tag{3.48}
\]

Note, first, that in terms of the functions \( (\Phi, \chi_1, \chi_2, \tilde{\chi}^m_{11}, \varsigma) \langle x, a^+ \rangle = (\Phi, \chi_1, \chi_2, \tilde{\chi}^m_{11}, \varsigma) (x, \omega) \) according to correspondence \( (2.6), (2.10) \) both the initial \( (3.39) \) and triplet-like \( (3.43) \) as
well as the duplet-like LFs \[ (3.47) \] with the respective gauge transformations \[ (3.34)–(3.38) \], may be easily rewritten by omitting the vacuum vector from the left- and right-hand sides in the above expressions and with change: \( (a^m, a^{+m}) \equiv -i(\partial/\partial \omega^m, \omega^m) \) in all functions, including theirs duals, and operators. Second, the kinetic operator for the initial field \( \Phi \): 

\[
\left( \frac{\delta}{\delta (\Phi)} \right) s^0_{C|\Xi} \left( \frac{\delta}{\delta (\Phi)} \right)
\]

contains the massive-like term \( \Xi^2 \).

In the tensor form, the constrained LF with the action \( S^{0}_{C|\Xi} \) \[ (3.47) \], with account for the representation \( (2.10) \) being valid for the rest fields, reads

\[
S^{0}_{C|\Xi} = S^{(+0)}_{C|\Xi} + \tilde{S}^{0}_{C|\Xi} : \left\{ \right. S^{(\pm)}_{C|\Xi}, S^{(0)}_{C|\Xi} \left. \right\} \left[ \Phi(\ldots), \chi(\ldots), \chi^{(\ldots)} \right] \equiv \left\{ S^{(+0)}_{C|\Xi} : 0 \right\};
\]

\[
S^{(+0)}_{C|\Xi} (\Phi^{(+)}, \chi^{(+)}, \chi^{(-)}, \chi^{(0)m}) = \sum_{s \geq 0} (-1)^s \int d^4x \left[ \Phi^{(n)s} \left( \partial^2 + \Xi^2 \right) \Phi^{(n)s} - s \partial^{m} \partial^{n} \Phi^{(n)s-1} \right]
\]

\[
- s(s - 1) \left[ \Xi^2 \eta_{n+1} \eta_{n} - 2 \partial^{n-1} \eta_{n} \right] \chi^{(n)s-2} + 2 \Xi^2 \left[ \nu/2 \chi^{(n)s} - \chi^{(n)s}_{111} \right] + 2i \int \text{d} \left[ \chi^{(n)s} \chi^{(n)s} + \nu \chi^{(n)s} \right] + i \Xi \partial^{n} \left( \Phi^{(n)s} - \chi^{(n)s} \right)
\]

\[
- i \Xi(s - 1) \eta_{n+1} \eta_{n} \partial^{n} \left( \Phi^{(n)s} - \chi^{(n)s} \right)
\]

whereas the gauge transformations and holonomic constraints take the form (for \( s > 0 \)):

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s}, \chi^{(0)m} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right), \quad \text{with } \left. \eta^{(m)s} \right|_{m=0} = 0.
\]

\[
\delta \left( \Phi^{(0)}, \chi^{(0)} \right) = - \left( \partial^{(0)}, \partial^{(0)} \right) - i \Xi \left( \eta^{(0)} \right)
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]

\[
\delta \left( \Phi^{(m)s}, \chi^{(m)s} \right) = - \left( \partial^{(m)s}, \partial^{(m)s} \right) - i \Xi \left( \eta^{(m)s} \right) + \nu \chi^{(m)s} + \nu \chi^{(m)s}
\]
where \( \Lambda_j \in \{\chi_1, \chi_1^m, \varsigma\} \) and \( W_i \in \{\Phi, \chi_2\} \) and the part of the action \( \tilde{S}_C^0|_{\Xi} \) should be calculated according to (2.14). Note, the gauge parameters with negative ”spin” values \( \varsigma_{(m)_s} \) are subject to the first-order differential constraints (3.53) with symmetrizers \( S_{mn(n)_s} \); \( S_{(m)_s-1;ij} \) without indices \( m_i, n_i, m_j, n_j \) in the latter symmetrizer. The tensor representation (3.49)–(3.55) will be used to make comparison with the LF for HS fields with integer spin in the Subsection 3.3.

Expressing the field \(|\chi_1\rangle\) as the generalized trace of the basic field \(|\Phi\rangle\) from (3.48) we present the gauge-invariant Lagrangian in terms of the generalized traceless, \(|\tilde{\chi}_1^m\rangle\), and doubly generalized traceless fields, \(|\Phi\rangle, |\chi_2\rangle\), fields as:

\[
S^0_{C|\Xi}(\Phi, \chi_2, \tilde{\chi}_1^m) = S^0_{C|\Xi}|_{|\chi_1|=-\frac{1}{2}m_{11}\Phi} \delta\left(|\Phi\rangle, |\chi_2\rangle, |\tilde{\chi}_1^m\rangle\right) = \left(l_1^+, m_1^+, 0\right)|\varsigma\rangle. \tag{3.56}
\]

Let us consider another variant to reduce the spectrum of fields for the LF (3.47) and therefore to simplify the dynamics. To this end let us make following change of field variables:

\[
\begin{align*}
\{\Phi, |\chi_2\rangle\} &\rightarrow \{|\tilde{\Phi}\rangle, |\tilde{\chi}_2\rangle\} = \frac{1}{2}\{|\Phi\rangle \pm |\chi_2\rangle\}, \delta\left(|\tilde{\Phi}\rangle, |\tilde{\chi}_2\rangle\right) = \left(l_1^+ + \frac{i}{2}\Xi, -\frac{i}{2}\Xi\right)|\varsigma\rangle, \tag{3.57} \\
m_{11}|\tilde{\chi}_2\rangle = 0, \quad m_{11}|\tilde{\Phi}\rangle + 2|\chi_1\rangle = 0, \tag{3.58}
\end{align*}
\]

due to the same generalized tracelessness of the fields \(|\Phi\rangle, |\chi_2\rangle\). Because of the vector \(|\tilde{\chi}_2\rangle\) satisfies the same constraint as the parameter \(|\varsigma\rangle\) we may gauge away \(|\tilde{\chi}_2\rangle\) completely by using \(|\varsigma\rangle\). The resulting LF, after expressing, first, \(|\chi_1\rangle\) as \(-\frac{1}{2} m_{11}\tilde{\Phi}\rangle\) from (3.58), second, of \(|\Phi\rangle, |\chi_2\rangle\) as \((|\tilde{\Phi}\rangle + |\tilde{\chi}_2\rangle), (|\tilde{\Phi}\rangle - |\tilde{\chi}_2\rangle)\) respectively, and, third, after vanishing of \(|\tilde{\chi}_2\rangle\), takes the form, \(\tilde{S}^0_{C|\Xi} = S^0_{C|\Xi}(\Phi, \chi_1, \chi_2, \tilde{\chi}_1^m) \rightarrow (\Phi, \chi_1 = -\frac{1}{2} m_{11}\tilde{\Phi}, \tilde{\chi}_2, \tilde{\chi}_1^m)\):

\[
\begin{align*}
\tilde{S}^0_{C|\Xi} &= \langle \Phi \left| \left[ l_0 - l_1^+ l_1 + \Xi^2 - \frac{1}{2} \left( l_1^+ \right)^2 m_{11} + m_{11}^+ (l_1)^2 \right] - \frac{1}{4} m_{11}^+ \left( l_0 + l_1 l_1^+ + 2\Xi^2 \right) m_{11} \right| \tilde{\Phi} \right\rangle \\
+ &\langle \tilde{\chi}_1^m \left| \left[ l_0 + m_1^+ m_{11} \right] m_{11}^+ \right| \tilde{\Phi} \right\rangle \tag{3.59}
\end{align*}
\]

The action (3.59) has not possess any gauge symmetry and describes Lagrangian dynamics of free scalar bosonic field \(\Phi(x, \omega)\) (2.6) with CS \(\Xi\) (for \(\nu = 1\)) in terms of modified doubly generalizred traceless field \(\tilde{\Phi}(x, \omega)\) (3.57) and auxiliary generalized traceless field \(\tilde{\chi}_1^m(x, \omega)\). It seems to be unexpected result, however the structure of the initial IR conditions with divergentless (2.7) and gradient (2.8) equations (equivalently (2.12)) unambiguously tell us that these constraints play the role of gauge conditions for each other. There exists LF in terms of only (modified) CS field \(\tilde{\Phi}(x, \omega)\) to be obtained by means of non-local expression of the field \(\tilde{\chi}_1^m(x, \omega)\) from the equation of motion:

\[
\frac{\delta \tilde{S}_{C|\Xi}}{\delta \tilde{\chi}_1^m} = K(l_0, \Xi)|\tilde{\chi}_1^m\rangle + \left[ m_{11}^+ l_1 - l_0 - \Xi^2 + \frac{1}{2} \left( m_{11}^+ l_1^+ + \frac{1}{2} \Xi^2 m_{11}^+ \right) \left( l_0 + l_1 l_1^+ + 2\Xi^2 \right) m_{11} \right]|\tilde{\Phi}\rangle = 0 \tag{3.60}
\]

for \(K(l_0, \Xi) \equiv l_0 - m_1^+ m_{11} + \Xi^2 = K^+(l_0, \Xi), \quad K^{-1}(l_0, \Xi)K(l_0, \Xi) = 1, \tag{3.61}\)

\[
|\tilde{\chi}_1^m\rangle = \left[ 1 - K^{-1}(l_0, \Xi)X \right]|\tilde{\Phi}\rangle \Rightarrow \langle \tilde{\chi}_1^m \rangle = \langle \tilde{\Phi} \left| \left[ 1 - X + K^{-1}(l_0, \Xi) \right] \right. \rangle, \tag{3.62}
\]

\footnote{We may choose another change of variables, with unit Jacobian in the respective path integral: \(\{\Phi, |\chi_2\rangle\} \rightarrow \{|\tilde{\Phi}\rangle, |\tilde{\chi}_2\rangle\} = \{\Phi, |\chi_2\rangle - |\Phi\rangle\} \) and the similar ones for the rest fields, e.g. for \(\tilde{\chi}_1^m\), \(\tilde{\chi}_0^m\) and \(\tilde{\chi}_0^m\), \(\tilde{\chi}_0^m\) instead of (3.41)–(3.57) without changing of the resulting triplet-, duplet-like and non-gauge constrained LF but now given with non-modified fields \(|\Phi\rangle, |\chi_1^m\rangle, |\chi_0^m\rangle\).}
where we have introduced the notations for the operators \( X, X^+ \) and inverse operator \( K^{-1} = K^{-1}(l_0, \Xi) \):

\[
X \equiv i\Xi m_1^+ + \frac{1}{2} \left( (m_1^+)^2 - i\Xi m_1^+ + \frac{1}{2} \Xi^2 m_{11}^+ \right) m_{11}, \quad K^{-1} = \left( l_0 - m_1^+ m_1 + \Xi^2 \right)^{-1}, \quad (3.63)
\]

\[
X^+ = -i\Xi m_1 + \frac{1}{2} m_{11}^+ \left( m_1^2 + i\Xi m_1 + \frac{1}{2} \Xi^2 m_{11} \right). \quad (3.64)
\]

The action can be presented with use of the identity, \( l_0 - l_1^+ l_1 = K(l_0, \Xi) - i\Xi (l_1^+ - l_1) \), as follows:

\[
\bar{S}^0_{\{\Xi}(\Phi) = \langle \Phi | \left\{ K(l_0, \Xi) + \Xi^2 - i\Xi (l_1^+ - l_1) - i\Xi m_1 + \frac{1}{2} \Xi^2 m_{11} - \frac{1}{2} \left( (l_1^+)^2 m_{11} + m_{11}^+ (l_1)^2 \right) \right\} - \left( 1 - X^+ K^{-1}(l_0, \Xi) \right) K(l_0, \Xi) \left( 1 - K^{-1}(l_0, \Xi) X \right) | \Phi \rangle \quad (3.65)
\]

\[
= \langle \Phi | \left\{ - X^+ K^{-1}(l_0, \Xi) X - \Xi^2 + i\Xi \left( l_1^+ m_{11} - m_1^+ l_1 \right) - \frac{1}{4} m_{11}^+ \left( l_0 + m_1 m_1^+ \right) \right\} \rangle \quad (3.66)
\]

To derive (3.66) the easily verified expressions were used

\[
X^+ + X - i\Xi (l_1^+ - l_1) - \frac{1}{2} \left( (l_1^+)^2 m_{11} + m_{11}^+ (l_1)^2 \right) = \frac{1}{2} m_{11}^+ \Xi^2 m_{11} - 2\Xi^2 + \frac{i\Xi}{2} \left( l_1^+ m_{11} - m_1^+ l_1 \right), \quad l_0 + l_1^+ l_1 + 2\Xi^2 = l_0 + m_1 m_1^+ + \Xi^2 + i\Xi (l_1^+ - l_1). \quad (3.67)
\]

Equivalently, the action (3.66) due to the formal representation:

\[
K^{-1} = \left( (K + \Xi^2) - \Xi^2 \right)^{-1} = - \sum_{n \geq 0} \Xi^{-2(n+1)} (K + \Xi^2)^n, \quad (3.68)
\]

takes the local-like form

\[
\bar{S}^0_{\{\Xi}(\Phi) = \langle \Phi | \left\{ X^+ \sum_{n \geq 0} \left[ \Xi^{-2(n+1)} (K + \Xi^2)^n \right] X - \Xi^2 + \frac{i\Xi}{2} \left( l_1^+ m_{11} - m_1^+ l_1 \right) \right\} \langle \Phi | \right\} \quad (3.69)
\]

We see, that the action (3.59) for scalar bosonic field with CS does not possesses by any gauge symmetry as in case of integer spin 0 field. At the same time the LF for non-scalar field \( \Psi(x, \omega)(\mu^1_{s_1} ... (\mu^k_{s_k})^\dagger \) with CS, \( \Xi, \) and integer generalized spin, \( s = (s_1, ..., s_k), \) \( k \leq [d - 4/2] \) [15], should necessary be by gauge-invariant with reducible gauge symmetry, being by the subject for further research.

The situation with \( ISO(1, d - 1) \) representations with integer spin looks completely another [38, 40, 50].

### 3.3 Comparison with dynamics for higher integer spin fields

First of all, let us present the result for the LFs for scalar CS field obtained in the Section 3.2 in terms of the bosonic fields being subject to the usual traceless or double traceless condition, generated by the operator \( l_{11} = m_{11}\big|_{\nu = 0} \), following to Fronsdal suggestion for (half-)integer
HS fields \[18, 19\]. To this end we present the fields \(|\Phi| = \sum_{n \geq 0} \left\{ |\Phi^{(+)}|_n + |\Phi^{(-)}|_n \right\}, |\varsigma| = \sum_{n \geq 0} \left\{ |\varsigma^{(+)}|_n + |\varsigma^{(-)}|_n \right\}\) satisfying to:

\[ m^2_{11} |\Phi| = 0, \quad m_{11} |\varsigma| = 0 \]  

(3.70)

in according with the decomposition \((2.10)\) in terms of Fronsdal-like double traceless \(|\phi^{(\pm)}|_n\) and traceless \(|\psi^{(\pm)}|_n\) fields and new fields \(|\phi^{(-)}|_n, |\psi^{(-)}|_n, \forall n \in \mathbb{N}_0\), (for fixed degree in powers of \(a^{+m}, \text{deg}_{a^+} |\phi^{(\pm)}(\psi)|_n = \pm n\)) as\(^8\)

\[|\Phi| = \sum_{n \geq 0} \sum_{k \geq 0} \gamma_{k,n}(l^+_{11})^k |\phi|_{n-2k}, \quad |\phi|_n = \sum_{k \geq 0} \frac{n!}{k!} \phi(m)_k(x) a^{+m_1} \ldots a^{+m_k} |0\rangle, \quad l^2_{11} |\phi|_n = 0, \]  

(3.71)

\[|\varsigma| = \sum_{n \geq 0} \sum_{k \geq 0} \delta_{k,n}(l^+_{11})^k |\psi|_{n-2k}, \quad |\psi|_n = \sum_{k \geq 0} \frac{n!}{k!} \psi(m)_k(x) a^{+m_1} \ldots a^{+m_k} |0\rangle, \quad l_{11} |\psi|_n = 0. \]  

(3.72)

Here, the unknown rational coefficients \(\gamma_{k,n}, \delta_{k,n}\) are determined from \((3.70)\) as the solutions of the system of recursive equations at each fixed monomial \((l^+_{11})^k |\phi|_{n-2k}, (l^+_{11})^{k+1} l_{11} |\phi|_{n-2k}\) and \((l^+_{11})^k |\psi|_{n-2k}, k = 0, \ldots, [n/2], n \in \mathbb{N}_0:\)

\[\left\{ \nu^2 \gamma_{k,n}(l^+_{11})^k + 2\nu \gamma_{k+1,n+2} l_{11} (l^+_{11})^{k+1} + \gamma_{k+2,n+4} \left[ l^2_{11}, (l^+_{11})^{k+2} \right] \right\} |\phi|_{n-2k} = 0, \]  

(3.73)

\[\left\{ \nu \delta_{k,n}(l^+_{11})^k + \delta_{k+1,n+2} l_{11} (l^+_{11})^{k+1} \right\} |\psi|_{n-2k} = 0, \]  

(3.74)

as follows

\[\gamma_{k,n} = \frac{(-\nu)^k \gamma_{0,n-2k}}{4^k k! \prod_{i=1}^{k} \left[ (n - 2k + d/2 - 1) + i - 1 \right]} = \frac{(-\nu)^k \gamma_{0,n-2k}}{4^k k! (n - 2k + d/2 - 1)_k}, \]  

(3.75)

\[\delta_{k,n} = \frac{(-\nu)^k \delta_{0,n-2k}}{4^k k! \prod_{i=1}^{k} \left[ (n - 2k + d/2) + i - 1 \right]} = \frac{(-\nu)^k \delta_{0,n-2k}}{4^k k! (n - 2k + d/2)_k} \]  

(3.76)

with arbitrary constants \(\delta_{0,n}, \gamma_{0,n}\), concrete choice of which depends on \(n, d\) and with \((x)_n\) being by the Pochhammer symbol. The coefficients related as: \((\delta_{k,n-1}/\delta_{0,n-1-2k}) = (\gamma_{k,n}/\gamma_{0,n-2k})\). The solution \((3.75)\) for \((3.73)\) follows from the recursive relations

\[\nu^2 \gamma_{k,n} + 8(k+1)\nu \gamma_{k+1,n+2} \left[ n - k + d/2 \right] + 4^2 \gamma_{k+2,n+4} \prod_{i=k}^{k+1} [i+1] [i+n-2k+d/2] = 0, \]  

(3.77)

\[\left\{ 2
\nu \gamma_{k+1,n+2} + 8(k+2)\gamma_{k+2,n+4} [k+n-2k+d/2] \right\} l_{11} = 0, \]  

(3.78)

with account for: \(g_0 |\psi^{(\pm)}|_{n-2k} = (\pm(n-2k+d/2) |\psi^{(\pm)}|_{n-2k} - 8)\). Substituting \(\gamma_{k+2,n+4}\) expressed from \((3.78)\) in terms of \(\gamma_{k+1,n+2}\) in \((3.77)\) we get \((3.75)\).

Now, substituting, instead of \(|\Phi|, |\chi|, |\chi_{11}\rangle, |\varsigma|\), \(\bar{i} = 1, 2\) theirs presentations in terms of series of respective traceless: \(|\lambda_i|_n, n \in \mathbb{N}_0\): \(\lambda_i \in \{|\chi_{F1,1}; \bar{x}_F|_{11}\} \) and double traceless tensor

\(^{8}\text{We restrict ourselves by the case of standard fields } |\Phi^{(+)}| \text{ and theirs Fronsdal-like fields leaving the solution of the same problem for additional fields } |\Phi^{(-)}|, |\phi^{(-)}|_n, |\psi^{(-)}|_n \text{ outside of the paper scope, with omitting the sign } \gamma^{(+)}_n \text{ at the vectors in this Subsection.}\)
fields: \( |w_i\rangle_n, w_i \in \{ \phi, \chi F2 \}, \ i = 1, 2 \), as well as the gauge parameters \( |\epsilon\rangle_n \): 

\[
|W_i\rangle = \sum_{n \geq 0} \sum_{k \geq 0} \frac{(-\nu)^k \gamma_{0,n-2k}}{4^k k!(n-2k+d/2-1)_k} (l^+_{11})^k |w_i\rangle_{n-2k}, \ l^2_{11} |w_i\rangle_n = 0, \ W_i \in \{ \Phi, \chi_2 \}, \ (3.79)
\]

\[
|\Lambda_i\rangle = \sum_{n \geq 0} \sum_{k \geq 0} \frac{(-\nu)^k \delta_{0,n-2k}}{4^k k!(n-2k+d/2)_k} (l^+_{11})^k |\Lambda_i\rangle_{n-2k}, \ l_{11} |\Lambda_i\rangle_n = 0, \ \Lambda_i \in \{ \chi_1, \tilde{\chi}_{11} \}, \ (3.80)
\]

\[
|\varsigma\rangle = \sum_{n \geq 0} \sum_{k \geq 0} \frac{(-\nu)^k \delta_{0,n-2k}}{4^k k!(n-2k+d/2)_k} (l^+_{11})^k |\varsigma\rangle_{n-2k}, \ l_{11} |\varsigma\rangle_n = 0, \ (3.81)
\]

and doing so with negative spin values fields \( |W_{i}^{(-)}\rangle_{\ell}, \ |\Lambda_{i}^{(-)}\rangle_{\ell}, \ |\varsigma^{(-)}\rangle_{\ell} \), we get gauge-invariant duplet-like \((3.47)\) and non-gauge \((3.59)\) (with fields \( |\Phi\rangle \) and \( |\phi\rangle_n = (1/2)(|\phi\rangle_n + |\chi_2\rangle_n) \), see as well the footnote 7) LFs in terms of Fronsdal-like totally-symmetric fields. To compare these results with triplet, duplet and Fronsdal LFs for the fields with all integer spins \( s \in \mathbb{N}_0 \) we remind that the latters are encoded by only the D'Alambert, divergentless and usual traceless (for \( \nu = 0 \)) equations \((2.7)\) for the basic field \( |\phi\rangle_s \) of any integer spin \( s \): \( g_0 |\phi\rangle_s = (s + \frac{d}{2}) |\phi\rangle_s \), without presence of new fields \( |\phi^{(-)}\rangle_s \).

The respective constrained gauge-invariant LFs for the massless field, \( |\phi\rangle_s \) of integer spin \( s \) in terms, first, of triplet: \( |\phi\rangle_s, |\chi_0\rangle_{s-1}, |\chi_1\rangle_{s-2} \), second, of duplet \( |\phi\rangle_s, |\chi_1\rangle_{s-2} \) (having expressed of \( |\chi_0\rangle_{s-1} \), being similar to \( |\chi_0\rangle \) \((3.21)\), from triplet formulation through algebraic equation of motion with indices \( s, s-1, s-2 \) meaning the rank of the component Lorentz tensors, i.e. \( \deg_s |\phi(\chi)\rangle_s = s \ (3.71) \), and, third, in terms of unique field \( |\phi\rangle_s \) look as

\[
S_{C|s}(\phi, \chi_0, \chi_1) = (s|\phi\rangle_{s-1} |\chi_0\rangle_{s-2} |\chi_1\rangle) \begin{pmatrix} l_0 & -l^+_1 & 0 & 0 \\ -l^-_1 & 1 & l^+_1 & 0 \\ 0 & 0 & l_1 & -l_0 \end{pmatrix} \begin{pmatrix} |\phi\rangle_s \\ |\chi_0\rangle_{s-1} \\ |\chi_1\rangle_{s-2} \end{pmatrix}, \ (3.82)
\]

\[
\delta (|\phi\rangle_s, |\chi_0\rangle_{s-1}, |\chi_1\rangle_{s-2}) = (l^+_1, l_0, l_1) |\epsilon\rangle_s, \quad \text{and} \quad l_{11} (|\phi\rangle, |\chi_0\rangle, |\chi_1\rangle, |\epsilon\rangle) = (-2 |\chi_0\rangle, 0, 0, 0); \ (3.83)
\]

\[
\hat{S}_{C|s}(\phi, \chi_1) = (s|\phi\rangle_{s-2} |\chi_1\rangle) \begin{pmatrix} l_0 - l^+_1 l^-_1 & (l^+_1)^2 \\ l^+_1 & -l_0 - l^+_1 l^-_1 \end{pmatrix} \begin{pmatrix} |\phi\rangle_s \\ |\chi_1\rangle_{s-2} \end{pmatrix}; \ (3.84)
\]

\[
\hat{S}^0_{C|s}(\phi) = s|\phi\rangle \left( l_0 - l^+_1 l^-_1 - \frac{1}{2} (l^+_1)^2 l_{11} - \frac{1}{2} l^+_1 l^-_1 l^+_1 l^-_1 - \frac{1}{4} l^+_1 (l_0 + l^+_1 l^-_1) l_{11} \right) |\phi\rangle_s; \quad \delta |\phi\rangle_s = l^+_1 |\epsilon\rangle_{s-1} \quad \text{and} \quad l^+_{11} |\phi\rangle_s = l_{11} |\epsilon\rangle_{s-1} = 0, \quad (3.85)
\]

for \( \hat{S}_{C|s} = S_{C|s}|_{\chi_0 = \chi_0(\phi, \chi_1)} \) and \( \hat{S}^0_{C|s} = \hat{S}_{C|s}|_{\chi_1 = -(1/2) l_{11} \phi} \). Thus, the gauge-invariant actions

\[
\left( S_{C|\infty}(\phi, \chi_0, \chi_1), \ \hat{S}_{C|\infty}(\phi, \chi_1), \ \hat{S}^0_{C|\infty}(\phi) \right) = \sum_{s \geq 0} \left( S_{C|s}, \hat{S}_{C|s}, \hat{S}^0_{C|s} \right), \quad (3.87)
\]

with \( \left( |\phi\rangle, |\chi_0\rangle, |\chi_1\rangle \right) = \left( \sum_{s \geq 0} |\phi\rangle_s, \sum_{s \geq 1} |\chi_0\rangle_{s-1}, \sum_{s \geq 2} |\chi_1\rangle_{s-2} \right) \)

for massless fields of all spins \( s = 0, 1, 2, \ldots \) take in the ghost-independent vector-like notations the respective forms: \((3.82), (3.84), (3.85)\) with allowance made for the changes \( (|\phi\rangle_s, |\chi_0\rangle_{s-1}, |\chi_1\rangle_{s-2}) \to (|\phi\rangle, |\chi_0\rangle, |\chi_1\rangle) \). The corresponding gauge transformations \((3.83)\),
are now written for the fields of all integer spins with the gauge parameter: $| \epsilon | = \sum_{s \geq 1} | \epsilon |_{s-1}$, with the same forms of the traceless constraints.

In the tensor notations the latter duplet and Fronsdal LF read (up to the common factor (1/2)):

$$
\tilde{S}^0_{C|\infty}(\phi, \chi_1) = \sum_{s \geq 0} \frac{(-1)^s}{s!} \int d^dx \left\{ \phi(m)_s \left( \partial^2 \phi(m)_s - s \partial^{m+} \partial^n \phi(m)_{s-1} + s(s-1) \partial^{m-1} \partial^{m+} \chi_1^{(m)} \right) 
- s(s-1) \chi_1^{(m)} \frac{1}{2} \left( 2 \partial^2 \chi_1^{(m)-2} + (s-2) \partial^{m-2} \partial^m \chi_1^{(m)-3} - \partial^{m-1} \partial_m \phi(m)_s \right) \right\},
$$

(3.88)

and the double traceless basic field: $\sum_{s \geq 0} \phi(m)_s$: $\phi(m)_s = 2 \chi_1^{(m)-2}$, providing the standard form of the gauge transformations:

$$
\delta \left( \sum_{s \geq 0} \phi(m)_s ; \sum_{s \geq 2} \chi_1^{(m) s-2} \right) = - \sum_{s \geq 0} \left( \partial_{(m)} \epsilon(m)_{s-1} ; \partial^{m-1} \epsilon(m)_{s-1} \right),
$$

(3.90)

from (3.83), (3.86). To be complete, note the constrained BRST-BFV LF for HS field, $| \phi |_s$, of integer spin $s$ are given by the relations:

$$
S_{C|s}(\phi, \chi_0, \chi_1) = \int d\eta \phi_0 |Q_C^{(1)}| \chi_1^0 |C^{(1)}| \phi_0 |s, \delta |\chi_1^0 |_s = Q_C^{(1)} |\chi_1^0 |_s, \delta |\chi_1^1 |_s = 0,
$$

(3.91)

\[ L_{11} |\chi_1^k |_s = \left( l_{11} + 2 \eta P \right) |\chi_1^k |_s = 0, \delta C^{(1)} |\chi_1^k |_s = \left( s + \frac{d}{2} \right) |\chi_1^k |_s, \ k = 0, 1, (3.92) \]

which are related with ones (3.18), (3.19) for CS field $| \Phi |$ as follows:

$$
\left( Q_C^{(1)}, |\chi_1^0 |_s, \ L_{11} + \nu, \delta C^{(1)} |g \right) = \left( Q_C, |\chi_1^{(k)} C \right), \ M_{11}, \delta C^{(g)} \right) |_{\eta = \eta_1 = \eta_2 = \eta_3 = \eta_4 = 0},
$$

(3.93)

Explicit comparison of the duplet LF (3.88), (3.90) for HS fields of all integer spins and duplet-like LF (3.50) for CS field shows its difference both by the contents of the configuration spaces, due to presence, first, of "negative spin value" fields: $(\Phi^{(-)}, \chi_1^{(-)}, \chi_2^{(-)}, \chi_1^{(-) m})$, second, by $(\sum_{s \geq 0} \chi_1^{(m) n})$ in the latter, by the structure of the constraints among the fields and by the relation for the actions:

$$
S_{C|\infty}^{(+)}(\Phi, \chi_1, \chi_2, \chi_{11}^{(m)}) = \tilde{S}^0_{C|\infty}(\Phi, \chi_1) + \Xi \Omega_0 + \Omega_1, \ \Omega_1 = \sum_{s \geq 0} \int d^dx \chi_{11}^{(m) s}(x) X^{(m)}(x),
$$

(3.94)

with local bosonic functionals $\Omega_k = \Omega_k(\Phi, \chi_1, \chi_2, \chi_{11}^{(m)})$, $k = 0, 1$ and tensor functions $X^{(m)}$: $X^{(m)}(\Phi, \chi_1, \chi_2, \chi_{11}^{(m)})$ depending on all fields. In (3.94) the form of the action $\tilde{S}^0_{C|\infty}(\Phi, \chi_1)$ coincides with one (3.88) for Fronsdal fields, but written for double and single $m_{11}$-traceless fields to be expressed in terms of the Fronsdal-like ones according to (3.79)–(3.81).
Thus, we see that in the cases of all integer spins the both LFs remain by gauge-invariant with traceless parameter in comparison with respective LFs for CS field. Indeed, the only triplet-like (3.43) and duplet-like LFs (3.47) are the gauge-invariant ones, whereas the Lagrangian action \( S_{C|\Xi}^0 \) (3.59) appears by the non-gauge theory. The same properties remain valid for scalar case with vanishing CS, \( \Xi \), when \( \nu = 1 \). We stress, that the main difference concerns the presence of infinite number of new tensor fields with “negative spin values”.

### 3.4 Equivalence to Initial Irreducible Relations

Let us preliminarily consider the problem of establishing of the equivalence of the Lagrangian equations of motion for massless totally-symmetric field \( \phi_{(m)}(x) \) with integer spin \( s \), in the triplet formulation (3.82), (3.83), which have the form, when expanding the equations of motion for massless totally-symmetric field \( \phi \)

\[
\eta_0 : \ l_0|\phi\rangle_s - l_1^+|\chi_0\rangle_{s-1} = 0, \quad \eta_1^+ : \ l_1|\phi\rangle - l_1^+|\chi_1\rangle_{s-2} - |\chi_0\rangle_{s-1} = 0, \tag{3.95}
\]

\[
\eta_0\eta_1^+D_\pm^+ : \ l_0|\chi_1\rangle_{s-2} - l_1|\chi_0\rangle_{s-1} = 0, \quad l_{11}(|\phi\rangle_s, |\chi_0\rangle_{s-1}, |\chi_1\rangle_{s-2}, |\epsilon\rangle_{s-1}) = (-2|\chi_1\rangle_{s-2}, 0, 0, 0) \tag{3.96}
\]

with non-Lagrangian conditions which should extract the massless UIR of the Poincare group \( ISO(1, d - 1) \) with discrete spin \( s \) in terms of tensor fields:

\[
(l_0, l_1, l_{11})|\phi\rangle_s = (0, 0, 0). \tag{3.97}
\]

The conditions (3.97) do not fix completely an ambiguity in the definition of \(|\phi\rangle_s \) as a representative of the UIR space of \( ISO(1, d - 1) \) group due to existence of a residual gauge symmetry, which we intend to determine. First of all, we use part of the degrees of freedom from the gauge parameter \(|\epsilon\rangle_{s-1} \) to gauge away the field \(|\chi_1\rangle_{s-2} \) by means of the gauge transformations (3.83). For \( s = 0 \) the equivalence is trivial, whereas for \( s = 1 \) there is no field \(|\chi_1\rangle_{s-1} \equiv 0 \). To do so, we expand \(|\epsilon\rangle_{s-1} \) into sum of longitudinal, \(|\epsilon^L\rangle_{s-1} \), and transverse, \(|\epsilon^\perp\rangle_{s-1} \), components:

\[
|\epsilon\rangle_{s-1} = |\epsilon^L\rangle_{s-1} + |\epsilon^\perp\rangle_{s-1} \equiv \sum_{k=1}^{s-1} (-1)^{k-1} (l_1^+)^k l_1^k |\epsilon\rangle_{s-1} + \left(1 + \sum_{k=1}^{s-1} (-1)^{k-1} (l_1^+)^k l_1^k \right) |\epsilon\rangle_{s-1}, \tag{3.98}
\]

so that \( l_1|\epsilon^\perp\rangle_{s-1} \equiv 0 \) and both of the components are traceless: \( l_{11}|\epsilon^L\rangle = l_{11}|\epsilon^\perp\rangle = 0 \). Thus, first, we use only part of \(|\epsilon^L\rangle_{s-1} \) from the parameter \(|\epsilon^L\rangle_{s-1} = |\epsilon^L\rangle + |\epsilon^\perp\rangle \) for \( s \geq 2 \) to gauge away the field \(|\chi_1\rangle_{s-2} \) completely. So we have, from the stability of the solution \(|\chi_1\rangle_{s-2} \equiv 0 \) under the gauge symmetry, that \( \delta|\chi_1\rangle_{s-2} = 0 \Leftrightarrow l_1|\epsilon^{L}_{s-1} = 0 \Rightarrow l_1|\epsilon^L\rangle_{s-1} = -l_1|\epsilon^\perp\rangle_{s-1} \). Second, from the first equation in (3.96) we observe that the field \(|\chi_0\rangle_{s-1} \) is the transverse one and we may therefore use the unused parameter \(|\epsilon^\perp\rangle_{s-1} \) choosing \(|\epsilon^\perp\rangle_{s-1} = -l_0^{-1}|\chi_0\rangle_{s-1} \) to have gauge away this field completely, so that the stability of the solution \(|\chi_0\rangle_{s-1} = 0 \) under the gauge transformations means, that \( \delta|\chi_0\rangle_{s-1} = 0 \Leftrightarrow l_0|\epsilon\rangle_{s-1} = 0 \).

\[9\]The realization of the first step allows one to get Maxwell-like LF with traceless field \(|\phi\rangle_s \), when having substituted \(|\chi_0\rangle_{s-1} \) being expressed from the second equation in (3.95) into the first one, as follows: \( (l_0 - l_1^+ l_1)|\phi\rangle_s = 0 \), so that \( \delta|\phi\rangle_s = s|\phi\rangle \) \( (l_0 - l_1^+ l_1)|\phi\rangle_s \) with \( \delta|\phi\rangle_s = l_1^+|\epsilon\rangle_{s-1}, l_1|\epsilon\rangle_{s-1} = 0 \). The equivalent reducible respective LF for the latter with elimination of the differential constraint on \(|\epsilon\rangle \) were considered in [64] among them for AdS space.
As the result, from the equations \(3.95\), \(3.96\) it follows the validity of the system \(3.97\) with residual gauge transformations determined by the longitudinal gauge parameter, \(\epsilon_s\), which satisfy to the same restrictions as the field \(|\phi\rangle_s\) in \(3.97\). Therefore, the conditions which should select the (tensor) field of any spin \(s \in \mathbb{N}_0\) as the element of irreducible massless unitary representation must be determined as:

\[
(l_0, l_1, l_{11})|\phi\rangle_s = (0,0,0), \quad \delta|\phi\rangle_s = l_1^+|\epsilon\rangle_{s-1}, \quad (l_0, l_1, l_{11})|\epsilon\rangle_{s-1} = (0,0,0).
\]  

(3.99)

The latter equations on \(|\epsilon\rangle_{s-1}\) means that the parameter may be considered as the element of massless UIR of \(ISO(1,d-1)\) of spin \(s-1\), but without own gauge symmetry\(^{10}\). Note, first that the dimensional reduction procedure being applied to massless UIR conditions \(3.99\) of \(ISO(1,d)\) in \(\mathbb{R}^{1,d}\) space-time permits one explicitly derive the massive UIR conditions of \(ISO(1,d-1)\) in \(\mathbb{R}^{1,d-1}\) with the same spin, as follows: \((l_0+m, l_1, l_{11})|\phi\rangle_s = (0,0,0)\) without any gauge symmetry. Second, the independent counting of the numbers of the physical degrees of freedom being extracted by \(3.99\) and by the equations \(3.95\), \(3.96\) with the gauge symmetry transformations \(3.83\) shows their coincidence.

Having in mind, the above analysis for HS field with integer spin, let us consider lagrangian equations of motion for the basic field \(|\Phi\rangle\) with CS, which follow from BRST-BFV equation \(3.14\) (or from \(3.29\)), as well as the holonomic constraints in powers of ghost monomials \(C(C\Phi)^k\), \(k = 0, 1, 2\):

\[
\eta_0 : \ l_0|\Phi\rangle - l_1^+|\chi_0\rangle - m_1|\chi_0^m\rangle = 0, \quad (3.100)
\]

\[
\eta_1^+ : \ l_1|\Phi\rangle - l_1^+|\chi_1\rangle - m_1|\chi_1^m\rangle - |\chi_0\rangle - |\chi_{01}\rangle = 0, \quad (3.101)
\]

\[
\eta_1^m : \ m_1^+|\Phi\rangle - l_1^+|\chi_2\rangle - m_1|\chi_2^m\rangle + |\chi_0\rangle - |\chi_1^m\rangle = 0, \quad (3.102)
\]

\[
\ 
\]

\[
\eta_0 l_1^+ P_1^+ : \ l_0|\chi_1\rangle - l_1|\chi_0\rangle - m_1|\chi_{01}\rangle = 0, \quad (3.104)
\]

\[
\eta_0 l_1^+ P_1^m : \ l_0|\chi_1^m\rangle - l_1|\chi_0^m\rangle + l_1^+|\chi_{01}\rangle = 0, \quad (3.105)
\]

\[
\eta_0 l_1^m P_1^+ : \ l_0|\chi_2\rangle - m_1^+|\chi_0\rangle - m_1|\chi_{01}\rangle = 0, \quad (3.106)
\]

\[
\eta_0 l_1^m P_1^m : \ l_0|\chi_2^m\rangle - m_1^+|\chi_0^m\rangle + l_1^+|\chi_{01}\rangle = 0, \quad (3.107)
\]

\[
\eta_0 l_1^m P_1^+ P_1^m : \ l_0|\chi_{11}\rangle + m_1^+|\chi_{01}\rangle - l_1|\chi_{01}^m\rangle = 0, \quad (3.108)
\]

\[
\eta_1^+ \eta_1^m P_1^+ : \ -m_1^+|\chi_1\rangle + l_1|\chi_2\rangle - m_1|\chi_{11}\rangle - |\chi_0\rangle - |\chi_{01}\rangle = 0, \quad (3.109)
\]

\[
\eta_1^+ \eta_1^m P_1^m : \ -m_1^+|\chi_1^m\rangle + l_1|\chi_2^m\rangle + l_1^+|\chi_{11}^m\rangle - |\chi_0^m\rangle = 0, \quad (3.110)
\]

\(^{10}\text{For the case of mixed-symmetric massless HS field with generalized integer spin } s = (s_1, ..., s_k) \text{ given on } \mathbb{R}^{1,d-1} \text{ the conditions of extraction of only UIR of Poincare group } ISO(1,d-1) \text{ in the space of tensor fields } \phi((m^1)_s, ..., (m^k)_s) \in Y(s, ..., s_k), k \leq [d/2]: (l_0, l_1, l_2) |\phi\rangle = 0 \text{ being initial in } Y \text{ should be augmented according to } 3.99 \text{ by adding the reducible gauge symmetry: } \delta|\phi\rangle = l_1^+|\epsilon\rangle_{(m^1)s, ..., (m^k)s}, ..., \text{ subject to the same requirements as for the tensor field itself. For the half-integer totally- and mixed-symmetric massless HS fields the situation with the exact formulation of the UIR is the same, e.g. one can show that for totally-symmetric case it is necessary to add the gauge transformations of the same form with gauge spin-tensor of rank } (n-1), \text{ but for basic spin-tensor field } \psi(m)_s \text{ of spin, } n+1/2, \text{ with suppressed Dirac indice and being subject to the same conditions: Dirac and } \gamma \text{-traceless constraints. Thus, the theorem in } 50 \text{ concerning the equivalence of the solutions of the equations of motion from the respective constrained BRST-BFV LF and ones for UIR conditions will be guaranteed, because of the latter solutions contains some gauge identities due to residual gauge symmetry presence.}\)
Again, the conditions (2.12) do not fix completely an ambiguity in the definition of \(|\Phi\rangle\) as a representative of the CS UIR space of \(ISO(1,d-1)\) group, due to existence of a residual gauge symmetry, which we should to determine. We will call the equations (3.100)–(3.110) as the **BRST-unfolded equations**, due to appearance of any field variable there with a coefficient being, at most the first degree in powers of the symmetry algebra \(A(\Xi; \mathbb{R}^{1,d-1})\) elements of \(\Re^1\).

First of all, we repeat the procedure from the Section 3.2 of gauge fixing up to surviving of only the fields \((\Phi, |\chi_1\rangle, |\chi_2\rangle, |\bar{\chi}_1\rangle, |\chi_0\rangle, |\bar{\chi}_0\rangle, |\bar{\chi}_1\rangle)\) with the gauge transformations (3.42) with unique independent gauge parameter \(|\varsigma\rangle\). Therefore, in the equations (3.100)–(3.110) the fields \(|\chi_2\rangle, (|\chi_1\rangle - |\bar{\chi}_1\rangle)\), \(i = 0,1\) vanish, whereas the \((|\chi_1\rangle + |\bar{\chi}_1\rangle)\) are changed on \(|\chi_1\rangle\).

Second, we expand \(|\varsigma\rangle\) into sum of longitudinal, \(|\varsigma^L\rangle\), and transverse, \(|\varsigma^\perp\rangle\), components:

\[
|\varsigma\rangle = |\varsigma^L\rangle + |\varsigma^\perp\rangle = \sum_{k=1}^\infty (-1)^{k-1} \frac{(l_1^+)^k k!}{k!(l_0)^k} |\varsigma\rangle + \sum_{k=0}^\infty (-1)^k \frac{(l_1^+)^k k!}{k!(l_0)^k} |\varsigma\rangle,
\]

so that \(l_1|\varsigma^\perp\rangle \equiv 0\) and both of the components are generalized traceless: \(m_{11}|\varsigma^L\rangle = m_{11}|\varsigma^\perp\rangle = 0\). Then, we use a part \(|\varsigma^L\rangle\) of the longitudinal gauge parameter:

\[
|\varsigma^L\rangle = |\varsigma_{\chi_1}^L\rangle + |\varsigma_{\chi_2}^L\rangle + |\varsigma_{\Phi}^L\rangle
\]

(3.112)
to gauge away the field \(|\chi_1\rangle\) completely. From the stability: \(\delta|\chi_1\rangle = 0\), of the solution \(|\chi_1\rangle = 0\) under the gauge symmetry, it follows the relations:

\[
\delta|\chi_1\rangle = 0 \Leftrightarrow l_1|\varsigma^L\rangle = l_1|\varsigma\rangle = 0 \Rightarrow l_1|\varsigma_{\chi_1}^L\rangle + l_1|\varsigma_{\chi_2}^L\rangle = -l_1|\varsigma_{\Phi}^L\rangle\text{ and } m_{11}|\Phi\rangle = 0.
\]

(3.113)

Third, from the equations (3.107), (3.110) we get \((l_1^+ - m_1^+)|\bar{\chi}_1\rangle = -i\Xi|\bar{\chi}_1\rangle = 0\) for \(i = 0,1\) and therefore the non-gauge fields \(|\bar{\chi}_1\rangle, |\bar{\chi}_0\rangle\) vanish. The analogous solution we obtain for the field \(|\chi_0\rangle\) from the difference of the equations (3.105) and (3.108).

Fourth, from the equation (3.109) we obtain that: \(l_1|\chi_2\rangle = |\chi_0\rangle\) and thus the field \(|\chi_2\rangle\) is double transverse, due to \(l_1|\chi_0\rangle = 0\) from (3.104). Then, we use the remaining degrees of freedom from the parameter \(|\varsigma\rangle\) (both \(|\varsigma^L\rangle\) and \(|\varsigma^\perp\rangle\)) to gauge away the field \(|\chi_2\rangle\) completely. Then, the requirement \(\delta|\chi_2\rangle = 0\) leads to:

\[
m_1^+(|\varsigma_{\chi_2}^L\rangle + |\varsigma_{\Phi}^L\rangle + |\varsigma^\perp\rangle) = m_1^+|\varsigma\rangle = 0 \Rightarrow m_1^+|\varsigma_{\chi_2}^L\rangle = -m_1^+(|\varsigma_{\chi_2}^L\rangle + |\varsigma^\perp\rangle)\text{ and } |\chi_0\rangle = 0.
\]

(3.114)

As the result, the only initial field \(|\Phi\rangle\) survives after the procedure above, satisfying to the relations (2.12) with residual gauge transformations:

\[
(l_0, l_1, m_1^+, m_{11})|\Phi\rangle = (0,0,0,0)\text{, } \delta|\Phi\rangle = l_1^+|\varsigma\rangle, \text{ } (l_0, l_1, m_1^+, m_{11})|\varsigma\rangle = (0,0,0,0)
\]

(3.115)

with taken into account of the fact, that \(l_0|\varsigma\rangle = [l_1, m_1^+]|\varsigma\rangle = 0\). The latter equations on \(|\varsigma\rangle\) mean that the parameter may be considered as the element of massless UIR of \(ISO(1,d-1)\) of CS \(\Xi\), but without own gauge symmetry. Again, we suppose, that the dimensional reduction when applied to massless CS UIR conditions (3.115) in \(\mathbb{R}^{1,d}\) can be used to derive massive-like CS UIR relations in \(\mathbb{R}^{1,d-1}\) for the same value of CS \(\Xi\).

Thus, we show, that the equations of motion (2.12) [or, equivalently, (2.4), (2.5)], can be achieved by using the action (3.28) after gauge-fixing and removing the auxiliary fields by using a total set of the equations of motion.

---

11The analogous type of the BRST-unfolded equations were written in (3.95), (3.96) for totally-symmetric integer spin case.
4 BRST-BV minimal Lagrangian actions

To construct a quantum action being sufficient for determination of the non-degenerate path integral within conventional BV quantization method [51], one necessary to derive preliminarily the so called BV action in the minimal sector of field variables organized in terms of respective vectors on a Fock space $H$, when considering instead of the field vector $|\chi^0_C\rangle \in H_C$ the generalized field-antifield vector $|\chi_{g(C)}\rangle \in H_{g(C)}$:

$$ H_{g(C)} := H_g \otimes H_{gh}^A \quad \text{with \, Z - grading} \quad H_{g(C)} = \lim_{M \to \infty} \oplus_{l=-M}^{M} H_{g(C)}^l $$

(4.1)

for $gh_{tot}(|\chi^l_{g(C)}\rangle) = -l$, $|\chi^l_{g(C)}\rangle \in H_{g(C)}^l$. The total configuration space for initial first-stage reducible gauge constrained LF in the minimal sector, $\mathcal{M}_{\min} = \{\Phi^A_{\min}(x,\omega)\}$, contains, in addition to the field $|\chi^0_C\rangle$, the 0-level ghost field vector, $|C^0_C\rangle$, and 1-st level ghost field one, $|C^1_C\rangle$, introduced by the rule:

$$ \varpi(x,\omega) = C^1(x,\omega)\mu_0\mu_1 \implies |\chi^2_C\rangle = |C^1_C\rangle\mu_0\mu_1, \quad |\chi^1_C\rangle = |C^0_C\rangle\mu_0, \quad |\chi^0_C\rangle = |C^{(+)i}_C\rangle + |C^{(-)i}_C\rangle, $$

(4.2)

$|\chi^1_{\text{gen}(C)}\rangle = |\chi^{(+0)}_{\text{gen}(C)}\rangle + |\chi^{(-0)}_{\text{gen}(C)}\rangle = |\chi^0_C\rangle + \sum_{i=0}^{1} |C^i_C\rangle, \quad \{\epsilon, gh_{tot}\}|\chi^0_{\text{gen}(C)}\rangle = (0, 0).$ (4.3)

The corresponding (according to (3.9)) antifields $\Phi^*_{\text{A}_{\min}}(x,\omega) = (\Phi^*_{n_0,0,n_1,0,n_{p1},n_{p2},0,0,0,0}; C^0_k, C^{*1})(x,\omega)$ and respective Fock space vectors from $H^0_{g(C)}$ with $Z_2, Z$-gradings

$$ \begin{array}{c|c|c|c|c|c|c}
\epsilon & \epsilon & C^A & C^1(x,\omega) & C^0_k(x,\omega) & C^*_{k1}(x,\omega) & \mu_i \\
\hline
gh_H & 1 & 1 & 0 & 1 & 0 & 1 \\
gh_L & 1 & -1 & 0 & 0 & -1 - i & 0, \quad i = 0, 1; \\
g_{\text{tot}} & 1 & -1 & 2 & 1 & i + 1 & -1 \\
\end{array} $$

(4.4)

are combined into generalized antifield vector as follows:

$$ |\chi^{*0}_{\text{gen}(C)}\rangle = |\chi^0_C\rangle + \sum_{i=0}^{1} |C^i_C\rangle + \left\{ |B^0_C\rangle + |B^0_{\text{c}(C)}\rangle \right\} + \eta_0 \left\{ |S^0_C\rangle + \sum_{i=0}^{1} |S^i_C\rangle \right\}, $$

(4.5)

$$ |\chi^{*0}_C\rangle = \eta^+_1 |\chi_0^{(a^+)}\rangle + \eta^+_m |\chi_0^{*m}(a^+)\rangle + \eta^+_1 P^+_1 \eta^+_1 |\chi_0^{*1}(a^+)\rangle + \eta^+_m P^+_1 m |\chi_0^{*m}(a^+)\rangle + \eta^+_1 \eta^+_1 \sum_{i=0}^{1} \sum_{j=0}^{1} \eta^+_i \eta^+_1 |\chi^{*i}_1\rangle $$

(4.6)

$$ + \eta^+_1 \sum_{i=0}^{1} \sum_{j=0}^{1} \eta^+_i |\chi^{*i}_1\rangle + \eta^+_m \sum_{i=0}^{1} \sum_{j=0}^{1} \eta^+_i |\chi^{*m}_1\rangle $$

(4.7)

where under $\varsigma_k$ and $C^0_k$ we mean all component fields in $|\chi^1_C\rangle$ (3.22) and with constant $\mu_i$: $\{\mu_i, \mu_j\} = 0, i, j = 0, 1$, which due to the vanishing of the total ghost number and Grassmann parity may be combined with $|\chi^0_C\rangle$ into generalized field vector:

(4.8)
off-shell BRST extended constraints

HS fields with integer spin in \( R \) for the CS field, being interacting both with itself, or with another scalar CS fields and with HS fields with "negative spin values" in their antifields according to the representations (3.79)–(3.81). The presence of the ghost formalisms, as well as the formulation in terms of the Fronsdal-like (double) traceless fields and can be immediately obtained according to the receipt above but for irreducible gauge theoretical fields. Note, the respective constrained BRST-BV minimal actions in the ghost-independent, \( x,\omega \), fields. Explicitly, the action

\[
|C^{s0}_C| = \eta_0 \left( \eta_1^+ |C^{s0}_C(a^+)\rangle + \eta_1^m |C^{s0}_C(a^+)\rangle + \eta_1^+ P_1^+ \eta_1^m |C^{s0}_C| (a^+)\rangle \right)
\]

(4.9)

\[
|C^{s1}_C| = \eta_0 \eta_1^+ \eta_1^m |C^{s1}_C(a^+)\rangle,
\]

(4.10)

for \( |B^{s1}_C| \equiv 0 \) and \( |x_0^{0\text{gen}}_C| = [x_0^{(0\text{gen})}_C] + [x_0^{(0\text{gen})}_C] \). The ghost independent antifield vectors have the decompositions in powers of \( a_m^+ \) similar to (2.10) as for the respective field vectors. The generalized field (4.5) and antifield (4.7) vectors are uniquely written in terms of the generalized field-antifield vector:

\[
|x_0^{0\text{gen}}_C| = \left| x_0^{(+0\text{gen})}_C \right| + \left| x_0^{(-0\text{gen})}_C \right| = \left| x_0^{0\text{gen}}_C \right| + \left| x_0^{0\text{gen}}_C \right|, \quad \left( \epsilon, gh_{\text{tot}} \right) |x_0^{0\text{gen}}_C| = (0,0).
\]

(4.11)

The constrained minimal BV action, \( S_{\text{min}} \equiv S_{C|\Xi} \), for the free massless field \( \Phi(x, \omega) \) of CS \( \Xi \) (for \( \nu = 1 \) in \( \mathbb{R}^{1,d-1} \)) is given according to the general prescription: \( S_{\text{min}} = S_0 + \Phi^{\text{gen}} A_{\text{min}} \Xi \Phi^{\text{gen}}_0 \) [51], with account for specific of Fock space \( g_{\text{gen}}^{0\text{g}}_C \) and reality of \( S_{C|\Xi} \):

\[
S_{C|\Xi} = S_{C|\Xi} + \int d\eta_0 \sum_{i=0} \left\{ \langle x_0^{0\text{gen}}_C \rangle \langle s | C^{i-1}_C \rangle + \langle C^{i-1}_C | s \langle x_0^{0\text{gen}}_C \rangle \right\},
\]

(4.12)

with right \( \langle s_0 \rangle \) (left \( \langle s \rangle \)) generator of Lagrangian BRST-like transformations in the minimal sector of the fields combined within the generalized field \( |x_0^{0\text{gen}}_C| \):

\[
\delta_B |x_0^{0\text{gen}}_C| = \mu \langle s_0 | x_0^{0\text{gen}}_C \rangle = \mu Q_C |x_0^{0\text{gen}}_C|.
\]

(4.13)

For dual vector, \( \langle x_0^{0\text{gen}}_C \rangle \), the transformation (4.13) with account of hermiticity \( Q_C, \mu \) looks as:

\[
\delta_B \langle x_0^{0\text{gen}}_C \rangle = \left( \delta_B |x_0^{0\text{gen}}_C| \right)^* : \delta_B |x_0^{0\text{gen}}_C| = \langle x_0^{0\text{gen}}_C \rangle \langle s \rangle \mu = \langle x_0^{0\text{gen}}_C \rangle |Q_C \mu|.
\]

(4.14)

Explicitly, the action \( S_{C|\Xi} \) and its BRST-like invariance transformations can be given in the form

\[
S_{C|\Xi} = \int d\eta_0 \langle x_0^{0\text{gen}}_C \rangle |Q_C |x_0^{0\text{gen}}_C| \rangle, \quad \delta_B S_{C|\Xi} = 0.
\]

(4.15)

Here, both the generalized field, \( |x_0^{0\text{gen}}_C| \), and antifield, \( |x_0^{0\text{gen}}_C| \), vectors are subject to the off-shell BRST extended constraints \( \overline{M}_{11} \) (3.11):

\[
\overline{M}_{11} |x_0^{0\text{gen}}_C| = 0 \iff \overline{M}_{11} |x_0^{0\text{gen}}_C| = 0, \quad \overline{M}_{11} |x_0^{0\text{gen}}_C| = 0.
\]

(4.16)

Thus, we have derived the constrained BRST-BV minimal action for the first-stage reducible gauge theory with constrained BRST-BFV LF (3.18) for free CS UIR of \( ISO(1,d-1) \) group described by the field \( \Phi(x, \omega) \) and auxiliary classical and ghost fields and theirs antifields. Note, the respective constrained BRST-BV minimal actions in the ghost-independent, triplet- and duplet-like forms with respective BRST-BFV formulations (3.39), (3.43), (3.47) can be immediately obtained according to the receipt above but for irreducible gauge theories, as well as the formulation in terms of the Fronsdal-like (double) traceless fields and their antifields according to the representations (3.79)–(3.81). The presence of the ghost fields with ”negative spin values” in \( \langle C^{(0)}_C \rangle \) in the spectrum of the variables of BRST-BV formulation, e.g. within duplet-like form, means that the component ghost tensor fields should be subject to the differential constraints (3.51), (3.53).

Different BRST-BV minimal actions may be used as the starting points to construct a LF for the CS field, being interacting both with itself, or with another scalar CS fields and with HS fields with integer spin in \( R^{1,d-1} \) on a base of preservation underlying master equation.
5 Generalized quartet-like unconstrained Lagrangians

To solve the problem, beyond of the extension of the constrained BRST-BFV approach to unconstrained one, it is sufficient to start from the triplet-like LF (3.43) we may obtain unconstrained quartet-like LF (following to idea of [63] for the case of integer spin) by introducing a compensator field \(|\vartheta\rangle\): \(\delta|\vartheta\rangle = m_{11}|\varsigma\rangle\). Then we should enlarge the constraints (3.25)–(3.27) on the fields \(|\Phi\rangle, |\chi_1\rangle, |\chi_2\rangle, |\bar{\chi}_{11}^m\rangle, |\chi_0\rangle, |\chi_{01}\rangle, |\bar{\chi}_{01}^m\rangle\) with only nontrivial gauge transformations (3.42) leaving by invariant the action \(\mathcal{S}_{C|\Xi}(3.43)\) up to the gauge-invariant equations as follows:

\[
\begin{align*}
    m_{11}|\chi_0\rangle - l_0|\vartheta\rangle &= 0, \\
    m_{11}|\chi_2\rangle + 2|\chi_1\rangle - m_1^+|\vartheta\rangle &= 0, \\
    m_{11}|\chi_1\rangle - l_1|\vartheta\rangle &= 0,
\end{align*}
\]

Introducing seven new bosonic (one-valued) fields \(|\lambda_i\rangle, i = 1, ..., 7\), playing the role of the lagrangian multipliers for the modified (5.1), (5.2) and rest untouched constraints for the fields \((\bar{\chi}_{11}^m), |\chi_{01}\rangle, |\bar{\chi}_{01}^m\rangle\), we get an unconstrained Lagrangian formulation with the action:

\[
\mathcal{S}_\Xi = \mathcal{S}_{C|\Xi} + \left[ \langle \lambda_1 | \left\{ m_{11} |\Phi\rangle + 2|\chi_1\rangle - l_1^+|\vartheta\rangle \right\} + \langle \lambda_2 | \left\{ m_{11} |\chi_2\rangle + 2|\chi_1\rangle - m_1^+|\vartheta\rangle \right\} \right] (5.3)
\]

\[
\begin{align*}
    &+ \langle \lambda_3 | \left\{ m_{11} |\chi_1\rangle - l_1|\vartheta\rangle \right\} + \langle \lambda_4 | \left\{ m_{11} |\chi_0\rangle - l_0|\vartheta\rangle \right\} + \langle \lambda_5 | m_{11} \bar{\chi}_{11}^m \rangle \\
    &+ \langle \lambda_6 | m_{11} |\chi_{01}\rangle + \langle \lambda_7 | m_{11} \bar{\chi}_{01}^m \rangle + h.c. \right],
\end{align*}
\]

which is invariant with respect to the gauge transformations with unconstrained gauge parameter \(|\varsigma\rangle\)

\[
\delta \left( |\Phi\rangle, |\chi_1\rangle, |\chi_2\rangle, |\bar{\chi}_{11}^m\rangle, |\chi_0\rangle, |\chi_{01}\rangle, |\bar{\chi}_{01}^m\rangle, |\vartheta\rangle \right) = \left( l_1^+, l_1, m_1^+, 0, l_0, 0, 0, m_{11} \right)|\varsigma\rangle. (5.4)
\]

Enlarging the terminology from the HS fields with discrete spin we will call the obtained irreducible gauge-invariant LF as the quartet-like unconstrained formulation for scalar bosonic field with CS \(\Xi\) on \(\mathbb{R}^{1,d-1}\).

The unconstrained LF given by the relations (5.3), (5.4) presents the basic result of the section.

Both the equivalent formulation in terms of Fronsdal-like fields and the unconstrained BRST-BV minimal action may be derived explicitly following to the prescriptions of the Sections 3.3–4.

6 Conclusion

In this paper, we have developed a constrained BRST-BFV approach to construct gauge-invariant Lagrangian descriptions of free scalar CSR for the Poincare group, with an arbitrary fixed continuous spin \(\Xi\) (when parameter \(\nu = 1\) in Minkowski space-time \(\mathbb{R}^{1,d-1}\) of any dimension in a “metric-like” formulation. The final constrained LF, given by equations (3.18) and (3.19), in fact, determined in terms of Wigner fields of two space-time variables \(x^m, \omega^m\), represents a first-stage reducible gauge theory and contains an auxiliary set of fields providing a BRST-unfolded form (in a ghost-independent representation), of both the field
equations \((3.100)\)–\((3.110)\) and the gauge transformations \((3.34)\), \((3.35)\)–\((3.38)\) preserving the invariance of the action \((3.39)\).

To construct a constrained BRST–BFV LF, we started by transforming the Bargmann–Wigner equations in the coordinate form \((2.4)\), \((2.5)\) with new original ansatz for their non-trivial solution in powers of direct and inverse degrees in the oscillators \((2.6)\), in terms of infinite set of independent usual \(\Phi_{(m)_k}(x)\) and new \(\Phi_{(m)_k}(x)\) tensor fields, into four constraints imposed on the respective Fock-space vector \((2.10)\). The vector contains standard part with usual massless HS fields with rank \(s = 0, 1, 2, \ldots\) and new one, on which the number particle operator extract the vectors with ”negative spin value” \(-n\): \(n = 1, 2, 3, \ldots\). The realization of the CSR on the former \((2.7)\) and latter \((2.8)\) ones are different but not independent due to coupling equation \((2.9)\). The scalar product in the respective Fock space \((2.14)\)

provides the standard realization of the Hermitian conjugation, but in the subspace with new vectors acquires non-standard form with non-trivial entanglement. The closure of the algebra of these constraints under the commutator multiplication and Hermitian conjugation generates an HS symmetry algebra \(\mathcal{A}(\Xi; \mathbb{R}^{1, d-1})\) given by Table \(1\) with two center elements: the parameter \(\nu\) and the value of CS, \(\Xi\) for \(\nu = 1\), since any linear combination of the constraints should also be a constraint. Extracting the second-class constraint subsystem: the generalized trace, \(m_{11}\), its dual, \(m_{11}^+\), and the particle number, \(g_0\), operators from the remaining \((4 + 1)\) first-class differential constraints, i.e., the divergence, \(l_1\), the generalized divergence, \(m_1\), their duals, \(l_1^\dagger, m_1^+\), and the D’Alambert operator, we construct, with respect to the reducible set of first-class constraints (considering \(m_1 - l_1 = -i\Xi\) a constraint), a constrained BRST operator, \(\tilde{Q}_C\) \((3.7)\), and a BRST-extended off-shell constraint, \(\tilde{M}_{11} = m_{11} + \ldots\), in an enlarged Fock space, \(\mathcal{H}_C\). They are found as a solution of the generating equations \((3.1)\) with the boundary conditions \((3.2)\). A correct calculation of \(Q_C\)-cohomology in the ghost number zero subspace of \(\mathcal{H}_C\), which should lead to the Bargmann–Wigner equations, fixes in a unique way the representation \((3.8)\) in \(\mathcal{H}_C\), which allows one to select only an independent set of constraints, and then to reduce \(\tilde{Q}_C\) to the constrained BRST operator \(Q_C\) \((3.10)\), without the first-stage reducible ghost operators, and to determine the off-shell constraint \(\tilde{M}_{11}\) \((3.11)\). The well-known application of the spectral problem, with a BRST equation \(Q_C|\chi_C^0\rangle = 0, \; (3.14)\)–\((3.16)\), albeit with no spin condition, as in the case of HS fields with discrete spin \([50]\), leads to the constrained BRST–BFV LF in question.

A specific structure of the constraints has permitted to realize a partial gauge-fixing and a resolution of some of the Lagrangian equations of motion and to obtain from the constrained LF the \textit{triplet-like} \((3.42)\), \((3.43)\) and \textit{doublet-like} \((3.47)\) LFs for CSR fields being irreducible gauge theories, respectively, with six and three additional auxiliary fields, by analogy with the triplet and doublet LFs for an HS field of an integer spin \(s\) \([62]\). The fields and the gauge parameter \(|\varsigma\rangle\) satisfy the generalized traceless (or, simply, \(m_{11}\)-traceless) conditions \((3.25)\)–\((3.27)\). A linear combination of the initial field \(\Phi(x, \omega)\) with the auxiliary field \(\chi_2(x, \omega)\) having similar gauge transformations \(\delta(|\Phi\rangle, |\chi_2\rangle) = (l_1^\dagger, m_1^+)\varsigma\) allows one to derive a non-gauge LF with the action \((3.59)\) in terms of the initial double \(m_{11}\)-traceless field \(\Phi(x, \omega)\) and the auxiliary \(m_{11}\)-traceless field \(\chi_{11}^m(x, \omega)\). This result seems unexpected, but the structure of the initial UIR conditions with the divergentless \((2.4)\) and gradient \((2.5)\) equations (equivalently, \((2.12)\)) informs us unambiguously that these constraints play the role of gauge conditions for each other. The LF \((3.66)\) (equivalently, \((3.69)\)) merely with the initial field of CS \(\Xi\), for \(\nu = 1\), after expressing \(\chi_{11}^m(x, \omega)\) through its equations of motion, does not possess any gauge symmetry and is non-local. By the characteristic feature of the
constrained BRST–BFV LF and their derivative LFs is the presence of respective sets of new infinite set of tensor fields with "negative spin values".

We have found, first, the interrelations of the resulting BRST–BFV LF for a scalar CSR field given in the basis of $m_{11}$-traceless fields with those for totally-symmetric HS fields with any integer spin $s = 0, 1, 2, \ldots$ in terms of Fronsdal-like (traceless) fields, and, second, have found the correspondence of the (double-) $m_{11}$-traceless fields with the usual Fronsdal-like (double) traceless fields in (3.79)–(3.81). The latter allows one to present, the parts of all the constrained LF which contain usual tensor and auxiliary fields for an CSR field only in terms of Fronsdal-like fields.

We presented the another way of HS symmetry algebra $\mathcal{A}(\Xi; \mathbb{R}^{1,d-1})$ realization in the sector of new tensor fields $\hat{\Phi}_{(m)_k}(x)$ without explicit using of the inverse degrees in the oscillators, but with endowing of the Fock space $\mathcal{H}$ with a new scalar product by its presenting as $\mathcal{H} = \mathcal{H}^+ + \mathcal{H}^-$, which is generated by two pairs of the Grassmann-even bosonic (dependent) oscillators with further application for BRST LFs for CSR field.

We have established an equivalence of Lagrangian equations of motion in the BRST unfolded form (3.100)–(3.110) of the suggested constrained BRST–BFV LF with the irreducible CSR relations (2.4), (2.5) and have found that the latter should be subject to the residual gauge transformations (3.115) with their parameter considered as an element of the same massless CS UIR of $ISO(1,d-1)$. Incidentally, we have clarified the form of conditions necessary to select UIR of $ISO(1,d-1)$ with integer spin (3.99) and with residual gauge transformations, thus determining a class of gauge equivalent configurations instead of its unique representative. Note, that the constraints in the respective conditions that select massless UIR both with CS and with integer spin are sufficient (without use of the residual gauge transformations) to construct the constrained BRST operators and to derive the respective BRST–BFV LFs.

We have developed a BRST–BV approach to the suggested constrained BRST–BFV gauge-invariant LF for a CSR field in a $\mathbb{R}^{1,d-1}$ space-time and explicitly constructed the BRST–BV action (4.15) with a corresponding BRST-like invariance in the minimal set of constrained field-antifield configurations. The crucial point here is that all the fields, ghost fields and their antifields are combined within a unique generalized field-antifield vector (4.11). The actions serve, first, to construct quantum actions under an appropriate choice of gauge conditions, second, to develop a construction of theories interacting with the CS field. The consistency of deformation for a free LF is to be controlled by a master equation for the deformed action with the interaction terms, thus producing a sequence of relations for these terms. We stress, that the construction of the minimal BRST–BV actions is differed from the procedure of finding BRST-BV minimal and quantum actions developed in [26] for the scalar CS field.

An unconstrained quartet-like LF (similar to the one for the integer spin case [63]) has also been found in (5.3) by including a compensator field to remove the $m_{11}$-tracelessness of the gauge parameter and by introducing to the action for a triplet-like LF of the augmented gauge-invariant constraint conditions (5.1), (5.2) with seven ungauged and unconstrained Lagrangian multipliers.

There are numerous ways to elaborate the suggested constrained BRST–BFV and BRST–BV approaches so as to study the Lagrangian dynamics of CSR in $\mathbb{R}^{1,d-1}$ in the case of arbitrary one-valued mixed-symmetric UIR with CS and to construct unconstrained BRST–BFV and BRST–BV approaches, as well as to adapt the formalism to accommodate two-valued CSR in $\mathbb{R}^{1,d-1}$.
Acknowledgements A.R. is grateful to I. Buchbinder, A. Isaev, Yu. Zinoviev, A. Sharapov, K. Stepanyantz, P. Moshin and to the participants of the International Conference “QFTG’2018”, which originated the idea of this paper, for valuable discussions as well as to G. Bonelli, R. Metsaev, M. Najafizadeh, B. Mischuk and to V. Krykhtin for important clarifying comments and discussions. The paper was supported by the Program of Fundamental Research sponsored by the Russian Academy of Sciences, 2013-2020.

Appendix

A On auxiliary representation for HS symmetry algebra \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \)

In this appendix, we describe another way to present the algebra \( \mathcal{A}(\Xi; \mathbb{R}^{1,d-1}) \) in the sector of new tensor fields \( \tilde{\Phi}_{(m)k}(x) \) without explicit using of the inverse degrees in \( a^{+m} \) oscillators. To this end, we endow \( \mathcal{H} \) with a new scalar product by presenting \( \mathcal{H} = \mathcal{H}^+ + \mathcal{H}^- \), which is generated by two pairs of the Grassmann-even bosonic (dependent) oscillators with modification for only the vectors \( \Phi^-(x, ia^+)|0\rangle \) (2.11):

\[
\mathcal{H}^+ = \left\{ \Phi^+(x, ia^+)|0\rangle \left| (a_m, a^{+n}) \equiv -i(\partial/\partial \omega_m, \omega^n) \right. \right\}, \tag{A.1}
\]

\[
\mathcal{H}^- = \left\{ \Phi^-(x, ia^+)|0\rangle \left| (a_{m+}, a^{+n} = -b_{m+}) = \Phi^-(x, ib^+)|0\rangle \right| (b_m, b^{+n}) \equiv -\left( \frac{\partial}{\partial b_{m+}}, \frac{\omega_m}{\omega^2} \right) \right\}; \tag{A.2}
\]

\[
|\Phi(x, a^+, b^+)| = \left( \Phi^+(x, ia^+) + \Phi^-(x, ib^+) \right)|0\rangle, \text{ with } [b_m, b^{+n}] = -\eta^{mn}, b_m|0\rangle = 0. \tag{A.3}
\]

The different pairs of the oscillators are not independent, because of,

\[
a^{+m}b^+_m = -1 \implies a^{+2}b^{+2} = 1, \quad b^{+m} = -a^{+m}/a^{+2} \quad \text{and} \quad a^{+m} = -b^{+m}/b^{+2}, \tag{A.4}
\]

so that they both look as the inverse-like operators for each other, creating ”particle” and ”antiparticle” respectively. Now, the scalar product in the Fock space \( \mathcal{H} \) with the realization (A.1)-(A.3) has the standard form:

\[
\langle \Psi | \Phi \rangle = \langle \Psi^+ | \Phi^+ \rangle + \langle \Psi^- | \Phi^- \rangle = \int d^d x \left\{ \sum_{k,p=0}^{\infty} \frac{(-1)^p}{k! p!} \langle 0 | \prod_{j=1}^p a^{m_j} \Psi_0^* (m)_p \right.
\]

\[
\times \Phi_{(n)_k} \prod_{i=1}^k a^{+n_i}|0\rangle + \sum_{k,p>0} \frac{(-1)^k p}{k! p!} \langle 0 | \prod_{j=1}^p b^{m_j} \tilde{\Psi}_0^* (m)_p \tilde{\Phi}_{(n)_k} \prod_{i=1}^k b^{+n_i}|0\rangle \left\} \right.
\]

\[
= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int d^d x \left\{ \tilde{\Psi}_{(n)_k}^* \Phi_{(n)_k} + \theta_{k,0} \tilde{\Psi}_{(n)_k}^* \tilde{\Phi}_{(n)_k} \right\}. \tag{A.5}
\]

The operators \( \alpha_l \) act on the subspace \( \mathcal{H}^+ \) as it was determined explicitly in (2.13), (2.29) whereas on \( \mathcal{H}^- \) they should be expressed in terms of \( b, b^+ \) according to the natural prescription:

\[
\alpha_l |\Phi\rangle|_{\mathcal{H}^+} = \alpha_l^a |\Phi^+\rangle, \quad \alpha_l |\Phi\rangle|_{\mathcal{H}^-} = \alpha_l^b |\Phi^-\rangle, \quad \alpha_l^b = \alpha_l |(a,a^+) = f(b,b^+)\rangle. \tag{A.6}
\]
The explicit form of $\phi_1^b$ relates both with the rule (A.3) and with the transformation rule (for the right derivative in $\omega^m$) to present the equations (2.4), (2.5):

$$\frac{\partial}{\partial \omega^m} = \frac{\partial b^+}{\partial \omega^m} \frac{\partial}{\partial b^+} = -i \frac{\partial \omega^m/\omega^2}{\partial \omega^m} \frac{\partial}{\partial b^+}$$

$$= i \left( \frac{\delta_m^{\omega^2}}{\omega^2} - \frac{2 \omega^m \omega_m}{\omega^2} \right) b^{-m} = i \left( \frac{\delta_m^{\omega_k} \omega_k}{\omega^2} - \frac{2 \omega^m \omega_m}{\omega^2} \right) b_m = i b^+ \left[ 2 b^+_m b_n - b^+_n b_m \right],$$

As the result the list of $\phi_1^b$ takes the form:

$$l_1^b = -i \left( b^+ \left[ 2 b^+_m b_n - b^+_n b_m \right] \frac{\partial}{\partial x^m} \right), \quad m_1^b = -i \left( b^+ \left[ 2 b^+_m b_n - b^+_n b_m \right] \frac{\partial}{\partial x^m} + \Xi \right),$$

$$l_1^{b^+} = i \left( b^+ \frac{\partial}{\partial x^m} \right), \quad m_1^{b^+} = i \left( b^+ \frac{\partial}{\partial x^m} + \Xi \right),$$

$$m_{11}^{b^+} = \left( b^+_m b^+_m \right)^2 b^+_m b_n - 2(d - 2) b^+_m b^+_m b^+_m b_k + \nu, \quad m_{11}^{b^+} = \frac{1}{b^+} + \nu, \quad$$

$$g_0^b = b^+ b_n + (d/2).$$

Note, the only operators $g_0^b, l_0$ are Hermitian with respect to the scalar product (2.14), whereas the $l_1^b, l_1^{b^+}, m_{11}^b, m_{11}^{b^+}$ as the rational functions in the oscillator variables $(b, b^*)$ do not obey the usual properties

$$(l_1^b)^+ \neq l_1^b, \quad (m_1^b)^+ \neq m_1^b, \quad (m_{11}^{b^+})^+ \neq m_{11}^{b^+}$$

if one should use the standard rules of Hermitian conjugation for the new creation and annihilation operators: $(b)^+ = b^+$. To restore the proper Hermitian conjugation properties for the $\phi_1^b$, we change the scalar product in the Fock space $\mathcal{H}^-$ as follows:

$$\langle \Phi_1^(-) | \Phi_2^(-) \rangle_{\text{new}} = \langle \Phi_1^(-) | K | \Phi_2^(-) \rangle,$$

for any vectors $|\Phi_1^(-)\rangle, |\Phi_2^(-)\rangle$ with some, yet unknown, operator $K$. The operator $K$ is determined by the condition that all the operators of the algebra should have the proper Hermitian properties with respect to the new scalar product:

$$\langle \Phi_1^(-) | KE^{-\alpha} | \Phi_2^(-) \rangle = \langle \Phi_2^(-) | KE^\alpha | \Phi_1^(-) \rangle^*, \quad \langle \Phi_1^(-) | K g_0^b \Phi_2^(-) \rangle = \langle \Phi_2^(-) | K g_0^b \Phi_1^(-) \rangle^*,$$

for $(E^\alpha; E^{-\alpha}) = (l_1^b, m_1^b; l_1^{b^+}, m_1^{b^+}, m_{11}^b, m_{11}^{b^+})$. The relations (A.14) permit one to determine the operator $K$, Hermitian with respect to the usual scalar product $\langle \ | \rangle$, in the form:

$$K = Z^+ Z, \quad Z = \sum_{k=0}^{\infty} \left( \frac{-1}{2} \right)^k \prod_{i=1}^{k} \left( \frac{a^{n_i}}{(a^+)^2} \right) |0\rangle \langle 0| b_{m_i},$$

$$K = \sum_{l=0}^{\infty} \left( \frac{-1}{2} \right)^l \prod_{j=1}^{l} \left( b^+_m |0\rangle \langle 0| \right)^m \left( \frac{a^{n_i}}{(a^+)^2} \right) \sum_{k=0}^{\infty} \left( \frac{-1}{2} \right)^k \prod_{i=1}^{k} \left( \frac{a^{n_i}}{(a^+)^2} \right) |0\rangle \langle 0| b_{n_i},$$

$$= |0\rangle \langle 0| + \left\{ \sum_{p \geq 1} \sum_{l \geq 1} K_{p+2l} \frac{\eta_{m_p+1m_{p+1}} \cdots \eta_{m_p+2l-1m_{p+2l}}}{p! (p + 2l)!} \prod_{h=1}^{2l} b_{m_{p+h}}^+ \prod_{j=1}^{p} b_{m_j}^+ |0\rangle \langle 0| b_{m_j} \right\} + h.c.$$
where the sign "h.c" means the standard Hermitian conjugation in $\mathcal{H}^-$. The real numbers $K_{p,p}$ and $K_{p,p+2l}$, $p,l \in \mathbb{N}$ are determined in (2.17), (2.18). The total scalar product in $\mathcal{H}$, which respect the Hermitian conjugation, may be constructed with use of initial scalar product $\langle \cdot | \cdot \rangle$ as:

$$
\langle \Phi_1 | \Phi_2 \rangle_{\text{new}} = \langle \Phi_1^{(+)} | \Phi_2^{(+)} \rangle + \langle \Phi_1^{(-)} | K | \Phi_2^{(-)} \rangle.
$$

(A.17)

The respective BRST operator (3.7), (3.10) $Q_C = Q_{C}(\sigma_A) + Q_{C}(\sigma_A^\dagger)$, where $Q_{C}(\sigma_A) = Q_{C}|_{\sigma_A \rightarrow \sigma_A^\dagger}$, should be hermitian in $\mathcal{H}_C$ according to the rule

$$
Q_C^{+}K_C = K_CQ_C, \quad \text{for } K_C = (\hat{1} \oplus K) \otimes \hat{1}_{gh},
$$

(A.18)

with the tensor product of the operator $(\hat{1} \oplus K)$ in $(\mathcal{H}^+ + \mathcal{H}^-)$ and the unit operator in $\mathcal{H}_{gh}^{\sigma_A}$. The constrained gauge-invariant BRST-BFV Lagrangian action, now may be determined in the form:

$$
S_{C|\Xi} = \int d\eta \langle \chi_C^0 | K_CQ_C | \chi_C^0 \rangle, \quad \delta |\chi_C^0\rangle = Q_C |\chi_C^1\rangle, \quad \delta |\chi_C^1\rangle = Q_C |\chi_C^2\rangle, \quad \delta |\chi_C^2\rangle = 0, \quad (A.19)
$$

$$
\hat{M}_{11} |\chi_C^k\rangle = 0, \quad k = 0, 1, 2, \quad \text{for } |\chi_C^k\rangle = |\chi_C^{(+)}|^k + |\chi_C^{(-)}|^k. \quad (A.20)
$$

References

[1] E.P. Wigner, On unitary representations of the inhomogeneous Lorentz group, Annals Math. 40 (1939) 149.

[2] P. Goddard, J. Goldstone, C. Rebbi, and C. B. Thorn, Nucl. Phys. B56 (1973) 109.

[3] L. Brink, M. Henneaux, Principles of String Theory, Plenum Press, New York, 1988.

[4] M. Green, J. Schwarz, E. Witten, Superstring Theory, Vols 1 and 2, Cambridge, 1987.

[5] E. Wigner in Theoretical Physics, International Atomic Energy Agency, Vienna, 1963.

[6] G.J. Iverson, C. Mack, Annals Phys. 64 (1971) 253.

[7] G. Bonelli, Nucl. Phys. B 669 (2003) 159, [arXiv: hep-th/0305155].

[8] A. Sagnotti, M. Tsulaia, Nucl. Phys. B 682 (2004) 83, [arXiv:hep-th/0311257].

[9] M.A. Vasiliev, Annals Phys. 190 (1989) 59.

[10] M.A. Vasiliev, Phys. Lett. B257 (1991) 111.

[11] M.A. Vasiliev, Phys. Lett. B285 (1992) 225.

[12] M.A. Vasiliev, Relativity, causality, locality, quantization and duality in the Sp(2M) invariant generalized spacetime, in Multiple Facets of Quantization and Supersymmetry, Michael Marinov Memorial Volume, Eds. M.Olshanetsky and A.Vainshtein, World Scientific, 2002, 826-872, [arXiv:hep-th/0111119].
[13] X. Bekaert, S. Cnockaert, C. Iazeolla, M.A. Vasiliev, Nonlinear higher spin theories in various dimensions, Proceedings of the 1st Solvay Workshop on Higher Spin Gauge Theories, 12-14 May 2004. Brussels, Belgium, Eds. R. Argurio, G. Barnich, G. Bonelli, M. Grigoriev, Int. Solvay Institutes, 2006, 132-197, [arXiv:hep-th/0503128].

[14] M.A. Vasiliev, JHEP 1808 (2018) 051, [arXiv:1804.06520 [hep-th]].

[15] L. Brink, A.M. Khan, P. Ramond X.-Z. Xiong, J. Math. Phys. 43 (2002) 6279, [arXiv:hep-th/0205145].

[16] X. Bekaert, N. Boulanger, The Unitary representations of the Poincare group in any spacetime dimension, [arXiv:hep-th/0611263].

[17] A.M. Khan, P. Ramond, J. Math. Phys. 46 (2005) 053515 [J. Math. Phys. 46 (2005) 079901], [arXiv:hep-th/0410107].

[18] C. Fronsdal, Phys. Rev. D18 (1978) 3624.

[19] J. Fang, C. Fronsdal, Phys. Rev. D18 (1978) 3630.

[20] X. Bekaert, J. Mourad, JHEP 0601 (2006) 115, [arXiv:hep-th/0509092].

[21] E.P. Wigner, Relativistische Wellengleichungen, Z. Physik 124 (1947) 665; V. Bargmann, E.P. Wigner, Group theoretical discussion of relativistic wave equations, Proc. Nat. Acad. Sci. US 34 (1948) 211.

[22] D. Sorokin, AIP Conf. Proc. 767 (2005) 172, [arXiv:hep-th/0405069].

[23] P. Schuster, N. Toro, Phys. Rev. D91 (2015) 025023, [arXiv:1404.0675 [hep-th]].

[24] V.O. Rivelles, Phys. Rev. D 91 (2015) 125035, [arXiv:1408.3576 [hep-th]].

[25] R.R. Metsaev, Phys. Lett. B 767 (2017) 458, [arXiv:1610.00657 [hep-th]].

[26] R.R. Metsaev, Phys. Lett. B 781 (2018) 568, [arXiv:1803.08421 [hep-th]].

[27] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, A. Rusnak, JHEP 1807 (2018) 031, [arXiv:1805.09706 [hep-th]].

[28] M. Najafizadeh, Phys.Rev. D97 (2018), 065009, [arXiv:1708.00827 [hep-th]].

[29] R.R. Metsaev, JHEP 1711 (2017) 197, [arXiv:1709.08596 [hep-th]].

[30] X. Bekaert, J. Mourad, M. Najafizadeh, JHEP 1711 (2017) 113, [arXiv:1710.05788 [hep-th]].

[31] V.O. Rivelles, A Gauge Field Theory for Continuous Spin Tachyons, [arXiv:1807.01812 [hep-th]].

[32] R.R. Metsaev, JHEP 1812 (2018) 055, [arXiv:1809.09075 [hep-th]].

[33] E.S. Fradkin, G.A. Vilkovisky, Phys. Lett. B55 (1975) 224; I.A. Batalin, G.A. Vilkovisky, Phys.Lett. B69 (1977) 309; M. Henneaux, Phys. Reports 126 (1985) 1.
[34] I.A. Batalin, E.S. Fradkin, Phys. Lett. B128 (1983) 303.
[35] A.K.H. Bengtsson, JHEP 1310 (2013) 108, [arXiv:1303.3799[hep-th]].
[36] A.K.H. Bengtsson, Phys.Lett. B 182 (1986) 321.
[37] S. Ouvry, J. Stern, Phys.Lett. B177 (1986) 335; W. Siegel, B. Zwiebach, Nucl. Phys. B 282 (1987) 125; W. Siegel, Nucl. Phys. B284 (1987) 632.
[38] A. Pashnev, M. Tsulaia, Mod. Phys. Lett. A13 (1998) 1853, [arXiv:hep-th/9803207].
[39] C. Burdik, A. Pashnev, M. Tsulaia, Mod. Phys. Lett. A15 (2000) 281, [arXiv:hep-th/0011195].
[40] I.L. Buchbinder, A. Pashnev, M. Tsulaia, Phys. Lett. B523 (2001) 338, [arXiv:hep-th/0109067]; Massless Higher Spin Fields in the AdS Background and BRST Constructions for Nonlinear Algebras, [arXiv:hep-th/0206026]; X. Bekaert, I.L. Buchbinder, A. Pashnev, M. Tsulaia, Class.Quant.Grav. 21 (2004) S1457, [arXiv:hep-th/0312252].
[41] I.L. Buchbinder, V.A. Krykhtin, A. Pashnev, Nucl. Phys. B711 (2005) 367, [arXiv:hep-th/0410215]; I.L. Buchbinder, V.A. Krykhtin, L.L. Ryskina, H. Takata, Phys.Lett. B 641 (2006) 386, [arXiv:hep-th/0603212].
[42] I.L. Buchbinder, V.A. Krykhtin, A.A. Reshetnyak, Nucl. Phys. B787 (2007) 211, [arXiv:hep-th/0703049]; I.L. Buchbinder, V.A. Krykhtin, P.M. Lavrov, Nucl. Phys. B762 (2007) 344, [arXiv:hep-th/0608005].
[43] C. Burdik, A. Pashnev, M. Tsulaia, Mod.Phys.Lett. A16 (2001) 731, [arXiv:hep-th/0101201]; The Lagrangian description of representations of the Poincare group, Nucl. Phys. Proc. Suppl. 102 (2001) 285, [arXiv:hep-th/0103143].
[44] A.A. Reshetnyak, P.Yu. Moshin, JHEP 10 (2007) 040 [arXiv:0707.0386[hep-th]]; Russ. Phys. Journal 56 (2013) 307, [arXiv:1304.7327[hep-th]]; I.L. Buchbinder, V.A. Krykhtin, , H. Takata, Phys.Lett. B 856 (2007) 253, [arXiv:0707.2181[hep-th]].
[45] C. Burdik, A. Reshetnyak, On representations of Higher Spin symmetry algebras for mixed-symmetry HS fields on AdS-spaces. Lagrangian formulation, J. Phys. Conf. Ser. 343 (2012) 012102, [arXiv:1111.5516[hep-th]].
[46] I.L. Buchbinder, A.A. Reshetnyak, Nucl. Phys. B 862 (2012) 270, [arXiv:1110.5044[hep-th]].
[47] A.A. Reshetnyak, Nucl. Phys. B 869 (2013) 523, [arXiv:1211.1273[hep-th]].
[48] A.A. Reshetnyak, Phys. Part. Nucl. Lett. 14 (2017) 411, [arXiv:1604.00620[hep-th]].
[49] A. Fotopoulos, M. Tsulaia, Int. J. Mod. Phys A24 (2008) 1, [arXiv:0805.1346[hep-th]].
[50] A.A. Reshetnyak, JHEP 1809 (2018) 104, [arXiv:1803.04678[hep-th]].
[51] I.A. Batalin, G.A. Vilkovisky, Phys.Lett. B102 (1981) 27; ibid B120 (1983) 166; Phys.Rev. D28 (1983) 2567.
[52] A.A. Reshetnyak, Phys.Part.Nucl. 49 (2018) 952, [arXiv:1803.05173 [hep-th]].

[53] G. Barnich, M. Grigoriev, A. Semikhatov, I. Tipunin, Comm. Math. Phys. 260 (2005) 147, [arxiv:hep-th/0406192]; G. Barnich, M. Grigoriev, JHEP 0608 (2006) 013, [arxiv:hep-th/0602166].

[54] K.B. Alkalaev, M. Grigoriev, I.Yu. Tipunin, Nucl. Phys. B. 823 (2009) 509, [arXiv:0811.3999 [hep-th]].

[55] R.R. Metsaev, Phys. Lett. B720 (2013) 237, [arXiv:1205.3131 [hep-th]].

[56] I.L. Buchbinder, V.A. Krykhtin, H. Takata, Phys.Lett. B 785 (2018) 315, [arXiv:1806.01640 [hep-th]].

[57] R.R. Metsaev, Phys. Lett. B773 (2017) 135, [arXiv:1703.05780 [hep-th]].

[58] X. Bekaert, E.D. Skvortsov, Int. J. Mod. Phys. A32 (2017) 1730019, [arXiv:1708.01030 [hep-th]].

[59] M.V. Khabarov, Yu.M. Zinoviev, Nucl. Phys. B928 (2018) 182, [arXiv:1711.08223 [hep-th]].

[60] K.B. Alkalaev, M.A. Grigoriev, JHEP 1803 (2018) 030, [arXiv:1712.02317 [hep-th]].

[61] I.L. Buchbinder, S. Fedoruk, A.P. Isaev, Nucl. Phys. B945 (2019) 114660, [arXiv:1903.07947 [hep-th]].

[62] D. Francia, A. Sagnotti, Class. Quant. Grav. 20 (2003) S473, Comment.Phys.Math.Soc.Sci.Fenn. 166 (2004) 165, PoS JHW2003 (2003) 005, [arXiv:hep-th/0212185].

[63] I.L. Buchbinder, A.V. Galajinsky, V.A. Krykhtin, Nucl. Phys. B779 (2007) 155, [arXiv:hep-th/0702161].

[64] A. Campoleoni, D. Francia, JHEP 1303 (2013) 168, [arXiv:1206.5877 [hep-th]]; D. Francia, S.L. Lyakhovich, A.A. Sharapov, Nucl. Phys. B881 (2014) 248, [arXiv:1310.8589 [hep-th]].