On the Moebius deformable hypersurfaces

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Abstract

In the article [Deformations of hypersurfaces preserving the Möbius metric and a reduction theorem, Adv. Math. 256 (2014), 156–205], Li, Ma and Wang investigated the interesting class of Moebius deformable hypersurfaces, that is, the umbilic-free Euclidean hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that admit non-trivial deformations preserving the Moebius metric. The classification of Moebius deformable hypersurfaces of dimension $n \geq 4$ stated in the aforementioned article, however, misses a large class of examples. In this article we complete that classification for $n \geq 5$.

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1 Introduction

Let $f : M^n \to \mathbb{R}^m$ be an isometric immersion of a Riemannian manifold $(M^n, g)$ into Euclidean space with normal bundle-valued second fundamental
form $\alpha \in \Gamma(\text{Hom}(TM,TM;N_fM))$. Let $\|\alpha\|^2 \in C^\infty(M)$ be given at any point $x \in M^n$ by

$$\|\alpha(x)\|^2 = \sum_{i,j=1}^n \|\alpha(x)(X_i, X_j)\|^2,$$

where $\{X_i\}_{1 \leq i \leq n}$ is an orthonormal basis of $T_xM$. Define $\phi \in C^\infty(M)$ by

$$\phi^2 = \frac{n}{n-1}(\|\alpha\|^2 - n\|\mathcal{H}\|^2), \tag{1}$$

where $\mathcal{H}$ is the mean curvature vector field of $f$. Notice that $\phi$ vanishes precisely at the umbilical points of $f$. The metric

$$g^* = \phi^2 g,$$

defined on the open subset of non-umbilical points of $f$, is called the Moebius metric determined by $f$. The metric $g^*$ is invariant under Moebius transformations of the ambient space, that is, if two immersions differ by a Moebius transformation of $\mathbb{R}^m$, then their corresponding Moebius metrics coincide.

It was shown in [9] that a hypersurface $f: M^n \to \mathbb{R}^{n+1}$ is uniquely determined, up to Moebius transformations of the ambient space, by its Moebius metric and its Moebius shape operator $S = \phi^{-1}(A - HI)$, where $A$ is the shape operator of $f$ with respect to a unit normal vector field $N$ and $H$ is the corresponding mean curvature function. A similar result holds for submanifolds of arbitrary codimension (see [9] and Section 9.8 of [3]).

Li, Ma and Wang investigated in [8] the natural and interesting problem of looking for the hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that are not determined, up to Moebius transformations of $\mathbb{R}^{n+1}$, only by their Moebius metrics. This fits into the fundamental problem in Submanifold theory of looking for data that are sufficient to determine a submanifold up to some group of transformations of the ambient space.

More precisely, an umbilic-free hypersurface $f: M^n \to \mathbb{R}^{n+1}$ is said to be Moebius deformable if there exists an immersion $\tilde{f}: M^n \to \mathbb{R}^{n+1}$ that shares with $f$ the same Moebius metric and is not Moebius congruent to $f$ on any open subset of $M^n$. The first result in [8] is that a Moebius deformable hypersurface with dimension $n \geq 4$ must carry a principal curvature with multiplicity at least $n-2$. As pointed out in [8], for $n \geq 5$ this is already a consequence of Cartan’s classification in [1] (see also [4] and Chapter 17 of [3]) of the more general class of conformally deformable hypersurfaces. These
are the hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ that admit a non-trivial conformal deformation $\tilde{f}: M^n \to \mathbb{R}^{n+1}$, that is, an immersion such that $f$ and $\tilde{f}$ induce conformal metrics on $M^n$ and do not differ by a Moebius transformation of $\mathbb{R}^{n+1}$ on any open subset of $M^n$.

According to Cartan’s classification, besides the conformally flat hypersurfaces, which have a principal curvature with multiplicity greater than or equal to $n - 1$ and are highly conformally deformable, the remaining ones fall into one of the following classes:

(i) **conformally surface-like hypersurfaces**, that is, those that differ by a Moebius transformation of $\mathbb{R}^{n+1}$ from cylinders and rotation hypersurfaces over surfaces in $\mathbb{R}^3$, or from cylinders over three-dimensional hypersurfaces of $\mathbb{R}^4$ that are cones over surfaces in $\mathbb{S}^3$;

(ii) **conformally ruled hypersurfaces**, that is, hypersurfaces $f: M^n \to \mathbb{R}^{n+1}$ for which $M^n$ carries an integrable $(n - 1)$-dimensional distribution whose leaves are mapped by $f$ into umbilical submanifolds of $\mathbb{R}^{n+1}$;

(iii) hypersurfaces that admit a non-trivial conformal variation $F: (-\epsilon, \epsilon) \times M^n \to \mathbb{R}^{n+1}$, that is, a smooth map defined on the product of an open interval $(-\epsilon, \epsilon) \subset \mathbb{R}$ with $M^n$ such that, for any $t \in (-\epsilon, \epsilon)$, the map $f_t = F(t; \cdot)$, with $f_0 = f$, is a non-trivial conformal deformation of $f$;

(iv) hypersurfaces that admit a single non-trivial conformal deformation.

It was shown in [8] that, among the conformally surface-like hypersurfaces, the ones that are Moebius deformable are those that are determined by a Bonnet surface $h: L^2 \to \mathcal{Q}^3_\epsilon$ admitting isometric deformations preserving the mean curvature function. Here $\mathcal{Q}^3_\epsilon$ stands for a space form of constant sectional curvature $\epsilon \in \{-1, 0, 1\}$. It was also shown in [8] that an umbilic-free conformally flat hypersurface $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 4$ (hence with a principal curvature of constant multiplicity $n - 1$), admits non-trivial deformations preserving the Moebius metric if and only if it has constant Moebius curvature, that is, its Moebius metric has constant sectional curvature. Such hypersurfaces were classified in [5], and an alternative proof of the classification was given in [8]. They were shown to be, up to Moebius transformations of $\mathbb{R}^{n+1}$, either cylinders or rotation hypersurfaces over the so-called curvature spirals in $\mathbb{R}^2$ or $\mathbb{R}^2_+$, respectively, the latter endowed with the hyperbolic metric, or cylinders over surfaces that are cones over curvature spirals in $\mathbb{S}^2$. 

It is claimed in [8] that there exists only one further example of a Moebius deformable hypersurface, which belongs to the third of the above classes in Cartan’s classification of the conformally deformable hypersurfaces. Namely, the hypersurface given by

\[ f = \Phi \circ (\text{id} \times f_1) : M^n := \mathbb{H}^{n-3}_{-m} \times N^3 \to \mathbb{R}^{n+1}, \ m = \sqrt{\frac{n-1}{n}}, \]  

(2)

where id is the identity map of \( \mathbb{H}^{n-3}_{-m} \), \( f_1 : N^3 \to \mathbb{S}^4_m \) is Cartan’s minimal isoparametric hypersurface, which is a tube over the Veronese embedding of \( \mathbb{R}P^2 \) into \( \mathbb{S}^4_m \), and \( \Phi : \mathbb{H}^{n-3}_{-m} \times \mathbb{S}^4_m \subset L^{n-2} \times \mathbb{R}^5 \to \mathbb{R}^{n+1} \setminus \mathbb{R}^{n-4} \) is the conformal diffeomorphism given by

\[ \Phi(x, y) = \frac{1}{x_0}(x_1, \ldots, x_{n-4}, y) \]

for all \( x = x_0e_0 + x_1e_1 + \cdots + x_{n-3}e_{n-3} \in L^{n-2} \) and \( y = (y_1, \ldots, y_5) \in \mathbb{S}^4 \subset \mathbb{R}^5 \). Here \( \{e_0, \ldots, e_{n-3}\} \) denotes a pseudo-orthonormal basis of the Lorentzian space \( L^{n-2} \) with \( \langle e_0, e_0 \rangle = 0 = \langle e_{n-3}, e_{n-3} \rangle \) and \( \langle e_0, e_{n-3} \rangle = -1/2 \). The deformations of \( f \) preserving the Moebius metric have been shown to be actually compositions \( f_t = f \circ \phi_t \) of \( f \) with the elements of a one-parameter family of isometries \( \phi_t : M^n \to M^n \) with respect to the Moebius metric; hence all of them have the same image as \( f \).

The initial goal of this article was to investigate the larger class of infinitesimally Moebius bendable hypersurfaces, that is, umbilic-free hypersurfaces \( f : M^n \to \mathbb{R}^{n+1} \) for which there exists a one-parameter family of immersions \( f_t : M^n \to \mathbb{R}^{n+1} \), with \( t \in (-\epsilon, \epsilon) \) and \( f_0 = f \), such that the Moebius metrics determined by \( f_t \) coincide up to the first order, in the sense that \( \frac{\partial}{\partial t}|_{t=0} g_t^\ast = 0 \). This is carried out for \( n \geq 5 \) in the forthcoming paper [7].

In the course of our investigation, however, we realized that the infinitesimally Moebius bendable hypersurfaces of dimension \( n \geq 5 \) in our classification that are not conformally surface-like are actually also Moebius deformable. Nevertheless, except for the example in the preceding paragraph, they do not appear in the classification of such hypersurfaces as stated in [8]. This has led us to revisit that classification under a different approach from that in [8].

To state our result, we need to recall some terminology. Let \( f : M^n \to \mathbb{R}^{n+1} \) be an oriented hypersurface with respect to a unit normal vector field \( N \). Then the family of hyperspheres \( x \in M^n \mapsto S(h(x), r(x)) \) with radius \( r(x) \)
and center \( h(x) = f(x) + r(x)N(x) \) is enveloped by \( f \). If, in particular, \( 1/r \) is the mean curvature of \( f \), it is called the \textit{central sphere congruence} of \( f \).

Let \( \mathbb{V}^{n+2} \) denote the light cone in the Lorentz space \( \mathbb{L}^{n+3} \) and let \( \Psi = \Psi_{v,w,C}: \mathbb{R}^{n+1} \to \mathbb{L}^{n+3} \) be the isometric embedding onto

\[
\mathbb{E}^{n+1} = \mathbb{E}_w^{n+1} = \{ u \in \mathbb{V}^{n+2} : \langle u, w \rangle = 1 \} \subset \mathbb{L}^{n+3}
\]

given by

\[
\Psi(x) = v + Cx - \frac{1}{2} \| x \|^2 w,
\]

in terms of \( w \in \mathbb{V}^{n+2} \), \( v \in \mathbb{E}^{n+1} \) and a linear isometry \( C: \mathbb{R}^{n+1} \to \{ v, w \}^\perp \).

Then the congruence of hyperspheres \( x \in M^n \mapsto S(h(x), r(x)) \) is determined by the map \( S: M^n \to \mathbb{S}_{1,1}^{n+2} \) that takes values in the Lorentzian sphere

\[
\mathbb{S}_{1,1}^{n+2} = \{ x \in \mathbb{L}^{n+3} : \langle x, x \rangle = 1 \}
\]

and is defined by

\[
S(x) = \frac{1}{r(x)} \Psi(h(x)) + \frac{r(x)}{2} w,
\]

in the sense that \( \Psi(S(h(x), r(x))) = \mathbb{E}^{n+1} \cap S(x)^\perp \) for all \( x \in M^n \). The map \( S \) has rank \( 0 < k < n \), that is, it corresponds to a \( k \)-parameter congruence of hyperspheres, if and only if \( \lambda = 1/r \) is a principal curvature of \( f \) with constant multiplicity \( n - k \) (see Section 9.3 of [3] for details). In this case, \( S \) gives rise to a map \( s: L^k \to \mathbb{S}_{1,1}^{n+2} \) such that \( S \circ \pi = s \), where \( \pi: M^n \to L^k \) is the canonical projection onto the quotient space of leaves of \( \ker(A - \lambda I) \).

\textbf{Theorem 1.} Let \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5 \), be a Moebius deformable hypersurface that is not conformally surface-like on any open subset and has a principal curvature of constant multiplicity \( n - 2 \). Then the central sphere congruence of \( f \) is determined by a minimal space-like surface \( s: L^2 \to \mathbb{S}_{1,1}^{n+2} \).

Conversely, any simply connected hypersurface \( f: M^n \to \mathbb{R}^{n+1}, n \geq 5 \), whose central sphere congruence is determined by a minimal space-like surface \( s: L^2 \to \mathbb{S}_{1,1}^{n+2} \) is Moebius deformable. In fact, \( f \) is Moebius bendable: it admits precisely a one-parameter family of conformal deformations, all of which share with \( f \) the same Moebius metric.

\textbf{Remarks 2.} 1) Particular examples of Moebius deformable hypersurfaces \( f: M^n \to \mathbb{R}^{n+1} \) that are not conformally surface-like on any open subset and have a principal curvature of constant multiplicity \( n - 2 \) are the minimal
hypersurfaces of rank two. These are well-known to admit a one-parameter associated family of isometric deformations, all of which are also minimal of rank two. The elements of the associated family, sharing with \( f \) the same induced metric, all have the same scalar curvature and, being minimal, also share with \( f \) the same Moebius metric. These examples are not comprised in the statement of Proposition 9.2 in [8] and, since the elements of the associated family of a minimal hypersurface of rank two do not have in general the same image, neither in the statement of Theorem 1.5 therein.

2) More general examples are the compositions \( f = P \circ h \) of minimal hypersurfaces \( h: M^n \to \mathbb{Q}^{n+1}_c \) of rank two with a "stereographic projection" \( P \) of \( \mathbb{Q}^{n+1}_c \) (minus one point if \( c > 0 \)) onto \( \mathbb{R}^{n+1} \). The latter are precisely the hypersurfaces \( f: M^n \to \mathbb{R}^{n+1} \) with a principal curvature of constant multiplicity \( n - 2 \) whose central sphere congruences are determined by minimal space-like surfaces \( s: L^2 \to S^{n+2}_{1,1} \subset \mathbb{L}^{n+3} \) such that \( s(L) \) is contained in a hyperplane of \( \mathbb{L}^{n+3} \) orthogonal to a vector \( T \in \mathbb{L}^{n+3} \) satisfying \( -\langle T, T \rangle = c \) (see, e.g., Corollary 3.4.6 in [4]).

3) The central sphere congruence of the hypersurface given by (2) is a Veronese surface in a sphere \( S^4 \subset S^{n+2}_{1,1} \).

4) The proof of Theorem 1 makes use of some arguments in the classification of the conformally deformable hypersurfaces of dimension \( n \geq 5 \) given in Chapter 17 of [3].

2  Preliminaries

In this short section we recall some basic definitions and state Wang’s fundamental theorem for hypersurfaces in Moebius geometry.

Let \( f: M \to \mathbb{R}^{n+1} \) be an umbilic-free immersion with Moebius metric \( g^* = \langle \cdot, \cdot \rangle^* \) and Moebius shape operator \( S \). The Blaschke tensor \( \psi \) of \( f \) is the endomorphism defined by

\[
\langle \psi X, Y \rangle^* = \frac{H}{\phi} \langle SX, Y \rangle^* + \frac{1}{2\phi^2} (\|\text{grad}^* \phi\|_2^2 + H^2) \langle X, Y \rangle^* - \frac{1}{\rho} \text{Hess}^* \phi (X, Y)
\]

for all \( X, Y \in \mathfrak{X}(M) \), where \( \text{grad}^* \) and \( \text{Hess}^* \) stand for the gradient and Hessian, respectively, with respect to \( g^* \). The Moebius form \( \omega \in \Gamma(T^*M) \) of
$f$ is defined by

$$\omega(X) = -\frac{1}{\phi}(\text{grad}^* H + S\text{grad}^* \phi, X)^*.$$  

The Moebius shape operator, the Blaschke tensor and the Moebius form of $f$ are Moebius invariant tensors that satisfy the conformal Gauss and Codazzi equations

$$R^*(X,Y) = SX \wedge^* SY + \psi X \wedge^* Y + X \wedge^* \psi Y$$  \hspace{1cm} (4)

and

$$(\nabla^*_X S)Y - (\nabla^*_Y S)X = \omega(X)Y - \omega(Y)X$$  \hspace{1cm} (5)

for all $X,Y \in \mathfrak{X}(M)$, where $\nabla^*$ denotes the Levi-Civita connection, $R^*$ the curvature tensor and $\wedge^*$ the wedge product with respect to $g^*$. We also point out for later use that the Moebius shape operator $S = \phi^{-1}(A - H I)$, besides being traceless, has constant norm $\sqrt{(n-1)/n}$.

The following fundamental result was proved by Wang (see Theorem 3.1 in [9]).

**Proposition 3.** Two umbilic-free hypersurfaces $f_1, f_2: M^n \rightarrow \mathbb{R}^{n+1}$ are conformally (Moebius) congruent if and only if they share the same Moebius metric and the same Moebius second fundamental form (up to sign).

## 3 Proof of Theorem 1

This section is devoted to the proof of Theorem 1. In the first subsection we use the theory of flat bilinear forms to give an alternative proof of a key proposition proved in [8] on the structure of the Moebius shape operators of Moebius deformable hypersurfaces. The proof of Theorem 1 is provided in the subsequent subsection.

### 3.1 Moebius shape operators of Moebius deformable hypersurfaces

The starting point for the proof of Theorem 1 is Proposition 6 below, which gives the structure of the Moebius shape operator of a Moebius deformable hypersurface of dimension $n \geq 5$ that carries a principal curvature of multiplicity $(n-2)$ and is not conformally surface-like on any open subset.
First we provide, for the sake of completeness, an alternative proof for \( n \geq 5 \), based on the theory of flat bilinear forms, of a result first proved for \( n \geq 4 \) by Li, Ma and Wang in [8] (see Theorem 6.1 therein) on the structure of the Moebius shape operators of any pair of Euclidean hypersurfaces of dimension \( n \geq 5 \) that are Moebius deformations of each other (see Proposition 5 below).

Recall that if \( W^{p,q} \) is a vector space of dimension \( p + q \) endowed with an inner product \( \langle \cdot, \cdot \rangle \) of signature \((p,q)\), and \( V, U \) are finite dimensional vector spaces, then a bilinear form \( \beta : V \times U \to W^{p,q} \) is said to be flat with respect to \( \langle \cdot, \cdot \rangle \) if
\[
\langle \langle \beta(X,Y), \beta(Z,T) \rangle \rangle - \langle \langle \beta(X,T), \beta(Z,Y) \rangle \rangle = 0
\]
for all \( X, Z \in V \) and \( Y, T \in U \). It is called null if
\[
\langle \langle \beta(X,Y), \beta(Z,T) \rangle \rangle = 0
\]
for all \( X, Z \in V \) and \( Y, T \in U \). Thus a null bilinear form is necessarily flat.

**Proposition 4.** Let \( f_1, f_2 : M^n \to \mathbb{R}^{n+1} \) be umbilic-free immersions that share the same Moebius metric \( \langle \cdot, \cdot \rangle^* \). Let \( S_i, \psi_i, i = 1,2, \) denote their corresponding Moebius shape operators and Blaschke tensors. Then, for each \( x \in M^n \), the bilinear form \( \Theta : T_x M \times T_x M \to \mathbb{R}^{2,2} \) defined by
\[
\Theta(X,Y) = \left( \langle S_1 X, Y \rangle^*, \frac{1}{\sqrt{2}} \langle \Psi_+ X, Y \rangle^* \langle S_2 X, Y \rangle^*, \frac{1}{\sqrt{2}} \langle \Psi_- X, Y \rangle^* \right),
\]
where \( \Psi_\pm = I \pm (\psi_1 - \psi_2) \), is flat with respect to the (indefinite) inner product \( \langle \cdot, \cdot \rangle \) in \( \mathbb{R}^{2,2} \). Moreover, \( \Theta \) is null for all \( x \in M^n \) if and only if \( f_1 \) and \( f_2 \) are Moebius congruent.

**Proof:** Using (1) for \( f_1 \) and \( f_2 \) we obtain
\[
\langle \langle \Theta(X,Y), \Theta(Z,W) \rangle \rangle - \langle \langle \Theta(X,W), \Theta(Z,Y) \rangle \rangle =
\langle \langle (S_1 Z \wedge^* S_1 X) Y, W \rangle^* - \langle \langle S_2 Z \wedge^* S_2 X Y, W \rangle^*
\]
\[
+ \langle \langle (\psi_1 - \psi_2) Z \wedge^* X Y, W \rangle^* + \langle \langle Z \wedge^* (\psi_1 - \psi_2) X Y, W \rangle^*
\]
\[
= 0
\]
for all \( x \in M^n \) and \( X, Y, Z, W \in T_x M \), which proves the first assertion.
Assume now that \( \Theta \) is null for all \( x \in M^n \). Then

\[
0 = \langle \Theta(X, Y), \Theta(Z, W) \rangle = \langle S_1X, Y \rangle^*\langle S_1Z, W \rangle^* - \langle S_2X, Y \rangle^*\langle S_2Z, W \rangle^* \\
+ \frac{1}{2}\langle (I + (\psi_1 - \psi_2))X, Y \rangle^*\langle (I + (\psi_1 - \psi_2))Z, W \rangle^* \\
- \frac{1}{2}\langle (I - (\psi_1 - \psi_2))X, Y \rangle^*\langle (I - (\psi_1 - \psi_2))Z, W \rangle^*
\]

for all \( x \in M^n \) and \( X, Y, Z, W \in T_xM \). This is equivalent to

\[
\langle S_1X, Y \rangle^*S_1 - \langle S_2X, Y \rangle^*S_2 + \frac{1}{2}\langle (I + (\psi_1 - \psi_2))X, Y \rangle^*(I + (\psi_1 - \psi_2)) \\
- \frac{1}{2}\langle (I - (\psi_1 - \psi_2))X, Y \rangle^*(I - (\psi_1 - \psi_2)) \\
= \langle S_1X, Y \rangle^*S_1 - \langle S_2X, Y \rangle^*S_2 + \langle X, Y \rangle^*(\psi_1 - \psi_2) + \langle (\psi_1 - \psi_2)X, Y \rangle^*I \\
= 0
\]

for all \( x \in M^n \) and \( X, Y \in T_xM \). Now we use that

\[
(n - 2)\langle \psi_1 X, Y \rangle^* = Ric^* (X, Y) + \langle S_2^2X, Y \rangle^* - \frac{n^2s^* + 1}{2n} \langle X, Y \rangle^*
\]

for all \( X, Y \in T_xM \), where \( Ric^* \) and \( s^* \) are the Ricci and scalar curvatures of the Moebius metric (see, e.g., Proposition 9.20 in \[3\]), which implies that

\[
tr \psi_1 = \frac{n^2s^* + 1}{2n} = tr \psi_2.
\]

Therefore, taking traces in \[6\] yields

\[
\langle (\psi_1 - \psi_2)X, Y \rangle^* = 0
\]

for all \( x \in M^n \) and \( X, Y \in T_xM \). Thus \( \psi_1 = \psi_2 \), and hence \( \langle S_1X, Y \rangle^*S_1 = \langle S_2X, Y \rangle^*S_2 \). In particular, \( S_1 \) and \( S_2 \) commute. Let \( \lambda_i \) and \( \rho_i \), \( 1 \leq i \leq n \), denote their respective eigenvalues. Then \( \lambda_i\lambda_j = \rho_i\rho_j \) for all \( 1 \leq i, j \leq n \) and, in particular, \( \lambda_i^2 = \rho_i^2 \) for any \( 1 \leq i \leq n \). If \( \lambda_1 = \rho_1 \neq 0 \), then \( \lambda_j = \rho_j \) for any \( j \), and then \( S_1 = S_2 \). Similarly, if \( \lambda_1 = -\rho_1 \neq 0 \), then \( S_1 = -S_2 \). Therefore, in any case, \( f_1 \) and \( f_2 \) are Moebius congruent by Proposition 3.

**Proposition 5.** Let \( f_1, f_2 : M^n \to \mathbb{R}^{n+1}, n \geq 5, \) be umbilic-free immersions that are Moebius deformations of each other. Then there exists a distribution \( \Delta \) of rank \( n - 2 \) on an open and dense subset \( \mathcal{U} \subset M^n \) such that, for each \( x \in \mathcal{U}, \Delta(x) \) is contained in eigenspaces of the Moebius shape operators of both \( f_1 \) and \( f_2 \) at \( x \) correspondent to a common eigenvalue (up to sign).
Proof: First notice that, for each trivial, for if \( Y \) is of the flat bilinear form \( \Theta: T_x M \times T_x M \to \mathbb{R}^{2,2} \) given by Proposition 4 is trivial, for if \( Y \in T_x M \) belongs to \( \mathcal{N}(\Theta) \), then \( \langle \Psi_+ Y, Y \rangle^* = 0 = \langle \Psi_- Y, Y \rangle^* \), which implies that \( \langle Y, Y \rangle = 0 \), and hence \( Y = 0 \).

Now, by Proposition 3 and the last assertion in Proposition 4, the flat bilinear form \( \Theta \) is not null on any open subset of \( M^n \), for \( f_1 \) and \( f_2 \) are not Moebius congruent on any open subset of \( M^n \). Let \( U \subset M^n \) be the open and dense subset where \( \Theta \) is not null. Since \( n \geq 5 \), it follows from Lemma 4.22 in [3] that, at any \( x \in U \), there exists an orthogonal decomposition \( \mathbb{R}^{2,2} = W_1^{1,1} \oplus W_2^{1,1} \) according to which \( \Theta \) decomposes as \( \Theta = \Theta_1 + \Theta_2 \), where \( \Theta_1 \) is null and \( \Theta_2 \) is flat with \( \dim \mathcal{N}(\Theta_2) \geq n - 2 \).

We claim that \( \Delta = \mathcal{N}(\Theta_2) \) is contained in eigenspaces of both \( S_1 \) and \( S_2 \) at any \( x \in U \). In order to prove this, take any \( T \in \Gamma(\Delta) \), so that \( \Theta(X, T) = \Theta_1(X, T) \) for any \( X \in T_x M \), and hence \( \langle \Theta(X, T), \Theta(Z, Y) \rangle = 0 \) for all \( X, Y, Z \in T_x M \). Equivalently,

\[
\langle S_1 X, T \rangle^* S_1 - \langle S_2 X, T \rangle^* S_2 + \langle (\psi_1 - \psi_2) X, T \rangle^* I + \langle X, T \rangle^* (\psi_1 - \psi_2) = 0 \tag{8}
\]

for any \( X \in T_x M \). In particular, for \( X \) orthogonal to \( T \),

\[
\langle S_1 X, T \rangle^* S_1 - \langle S_2 X, T \rangle^* S_2 + \langle (\psi_1 - \psi_2) X, T \rangle I = 0.
\]

Assume that \( T \) is not an eigenvector of \( S_1 \). Then there exists \( X \) orthogonal to \( T \) such that \( \langle S_1 X, T \rangle \neq 0 \). Since \( f_1 \) is umbilic-free, we must have \( \langle S_2 X, T \rangle \neq 0 \). Thus \( S_1 \) and \( S_2 \) are mutually diagonalizable. Let \( X_1, \ldots, X_n \) be an orthonormal diagonalizing basis of both \( S_1 \) and \( S_2 \) with respective eigenvalues \( \lambda_i \) and \( \rho_i \), \( 1 \leq i \leq n \). Since \( T \) is not an eigenvector, there are at least two distinct eigenvalues, say, \( 0 \neq \lambda_1 \neq \lambda_2 \), with corresponding eigenvectors \( X_1 \) and \( X_2 \), such that \( \langle X_1, T \rangle \neq 0 \neq \langle X_2, T \rangle \). Thus (8) yields

\[
\lambda_1 \langle X_1, T \rangle^* S_1 - \rho_1 \langle X_1, T \rangle^* S_2 + \langle (\psi_1 - \psi_2) X_1, T \rangle^* I + \langle X_1, T \rangle^* (\psi_1 - \psi_2) = 0
\]

and

\[
\lambda_2 \langle X_2, T \rangle^* S_1 - \rho_2 \langle X_2, T \rangle^* S_2 + \langle (\psi_1 - \psi_2) X_2, T \rangle^* I + \langle X_2, T \rangle^* (\psi_1 - \psi_2) = 0.
\]

It follows from (7) that \((n - 2)(\psi_1 - \psi_2) = S_1^2 - S_2^2 \). Hence

\[
\lambda_1 S_1 - \rho_1 S_2 + \frac{1}{n - 2} (\lambda_1^2 - \rho_1^2) I + (\psi_1 - \psi_2) = 0
\]

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and
\[ \lambda_2 S_1 - \rho_2 S_2 + \frac{1}{n-2}(\lambda_2^2 - \rho_2^2)I + (\psi_1 - \psi_2) = 0. \]

Taking traces in the above expressions we obtain
\[ \lambda_1^2 - \rho_1^2 = \lambda_2^2 - \rho_2^2. \]

On the other hand, the above relations also yield
\[ \lambda_1 \lambda_i - \rho_1 \rho_i + \frac{1}{n-2}(\lambda_i^2 - \rho_i^2) = 0 \]
and
\[ \lambda_2 \lambda_i - \rho_2 \rho_i + \frac{1}{n-2}(\lambda_i^2 - \rho_i^2) = 0 \]
for any \( 1 \leq i \leq n \). Assume first that \( \lambda_1 = \rho_1 \), and hence \( \lambda_2 = \rho_2 \). Then the preceding expressions become
\[ (\lambda_i - \rho_i) \left( \lambda_j + \frac{1}{n-2}(\lambda_i + \rho_i) \right) = 0 \]
for \( j = 1, 2 \) and \( 1 \leq i \leq n \). Since \( S_1 \neq S_2 \) and both tensors have vanishing trace, there must exist at least two directions for which \( \lambda_i - \rho_i \neq 0 \). For such a fixed direction, say \( k \), we have
\[ \lambda_j + \frac{1}{n-2}(\lambda_k + \rho_k) = 0, \]
with \( j = 1, 2 \). Thus \( \lambda_1 = \lambda_2 \), which is a contradiction.

Similarly, if we assume \( \lambda_1 = -\rho_1 \), we obtain that \( \lambda_2 = -\rho_2 \), and then
\[ (\lambda_i + \rho_i) \left( \lambda_j + \frac{1}{n-2}(\lambda_i - \rho_i) \right) = 0 \]
for \( j = 1, 2 \) and \( 1 \leq i \leq j \). By the same argument as above, we see that \( \lambda_1 = \lambda_2 \), reaching again a contradiction. Therefore \( T \) must be an eigenvector of \( S_1 \). Since \( S_2 \) is not a multiple of the identity, taking \( X \) orthogonal to \( T \) we see from [\( S \)] that \( T \) must also be an eigenvector of \( S_2 \). Given that \( T \in \Gamma(\Delta) \) was chosen arbitrarily, we conclude that \( \Delta \) is contained in eigenspaces of both \( S_1 \) and \( S_2 \).
Let $\mu_1$ and $\mu_2$ be such that $S_1|_{\Delta} = \mu_1 I$ and $S_2|_{\Delta} = \mu_2 I$. By (8) we have
\[ \mu_1^2 - \mu_2^2 + \frac{2}{n-2}(\mu_1^2 - \mu_2^2) = 0. \]
Thus $\mu_1^2 - \mu_2^2 = 0$, and hence $\mu_1 = \pm \mu_2$.

It remains to argue that $\dim \Delta = n - 2$. After changing the normal vector of either $f_1$ or $f_2$, if necessary, one can assume that $\mu_1 = \mu_2 := \mu$. Since $S_1|_{\Delta} = \mu I = S_2|_{\Delta}$, if $\dim \Delta = n - 1$ then the condition $\text{tr}(S_1) = 0 = \text{tr}(S_2)$ would imply that $S_1 = S_2$, a contradiction. 

Now we make the extra assumptions that $f$ is not conformally surface-like on any open subset of $M^n$ and has a principal curvature with constant multiplicity $n - 2$.

**Proposition 6.** Let $f_1: M^n \to \mathbb{R}^{n+1}$, $n \geq 5$, be a Moebius deformable hypersurface with a principal curvature $\lambda$ of constant multiplicity $n - 2$. Assume that $f_1$ is not conformally surface-like on any open subset of $M^n$. If $f_2: M^n \to \mathbb{R}^{n+1}$ is a Moebius deformation of $f_1$, then the Moebius shape operators $S_1$ and $S_2$ of $f_1$ and $f_2$, respectively, have constant eigenvalues $\pm \sqrt{(n-1)/2n}$ and 0, and the eigenspace $\Delta$ correspondent to $\lambda$ as a common kernel. In particular, $\lambda$ and the corresponding principal curvature of $f_2$ coincide with the mean curvatures of $f_1$ and $f_2$, respectively. Moreover, the Moebius forms of $f_1$ and $f_2$ vanish on $\Delta$.

For the proof of Proposition 6, we will make use of Lemma 7 below (see Theorem 1 in [2] or Corollary 9.33 in [3]), which characterizes conformally surface-like hypersurfaces among hypersurfaces of dimension $n$ that carry a principal curvature with constant multiplicity $n - 2$ in terms of the splitting tensor of the corresponding eigenbundle. Recall that, given a distribution $\Delta$ on a Riemannian manifold $M^n$, its splitting tensor $C: \Gamma(\Delta) \to \Gamma(\text{End}(\Delta^\perp))$ is defined by
\[ C_T X = -\nabla^h_X T \]
for all $T \in \Gamma(\Delta)$ and $X \in \Gamma(\Delta^\perp)$, where $\nabla^h_X T = (\nabla_X T)|_{\Delta^\perp}$.

**Lemma 7.** Let $f: M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, be a hypersurface with a principal curvature of multiplicity $n - 2$ and let $\Delta$ denote its eigenbundle. Then $f$ is conformally surface-like if and only if the splitting tensor of $\Delta$ satisfies $C(\Gamma(\Delta)) \subset \text{span}\{I\}$. 

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Proof of Proposition 6: Since $f_1$ has a principal curvature $\lambda$ of constant multiplicity $n - 2$, it follows from Proposition 5 that, after changing the normal vector field of either $f_1$ or $f_2$, if necessary, we can assume that the Moebius shape operators $S_1$ and $S_2$ of $f_1$ and $f_2$ have a common eigenvalue $\mu$ with the same eigenbundle $\Delta$ of rank $n - 2$.

Let $\lambda_i$, $i = 1, 2$, be the eigenvalues of $S_1|_{\Delta^\perp}$. In particular, $\lambda_1 \neq \mu \neq \lambda_2$. The conditions $\text{tr}(S_1) = 0 = \text{tr}(S_2)$ and $\|S_1\|^2 = (n - 1)/n = \|S_2\|^2$ imply that $S_1$ and $S_2$ have the same eigenvalues. Then we must also have $\lambda_1 \neq \lambda_2$, for otherwise $S_1$ and $S_2$ would coincide.

Let $X, Y \in \Gamma(\Delta^\perp)$ be an orthonormal frame of eigenvectors of $S_1|_{\Delta^\perp}$ with respect to $g^*$. Then $S_1X = \lambda_1X$, $S_1Y = \lambda_2Y$, $S_2X = b_1X + cY$ and $S_2Y = cX + b_2Y$ for some smooth functions $b_1$, $b_2$ and $c$. Since $\text{tr}(S_1) = 0 = \text{tr}(S_2)$ and $\|S_1\|^2 = (n - 1)/n = \|S_2\|^2$, we have

\begin{align*}
\lambda_1 + \lambda_2 + (n - 2)\mu &= 0, \\
\lambda_1^2 + \lambda_2^2 + (n - 2)\mu^2 &= \frac{n - 1}{n}, \\
b_1 + b_2 + (n - 2)\mu &= 0, \\
b_1^2 + b_2^2 + 2c^2 + (n - 2)\mu^2 &= \frac{n - 1}{n}.
\end{align*}

(9) \hspace{1cm} (10) \hspace{1cm} (11) \hspace{1cm} (12)

Thus the first assertion in the statement will be proved once we show that $\mu$ vanishes identically. The last assertion will then be an immediate consequence of (5).

The umbilicity of $\Delta$, together with (5) evaluated in orthonormal sections $T$ and $S$ of $\Delta$ with respect to $g^*$, imply that $\omega_1(T) = T(\mu) = \omega_2(T)$, where $\omega_i$ is the Moebius form of $f_i$, $1 \leq i \leq 2$. Taking the derivative of (9) and (10) with respect to $T \in \Gamma(\Delta)$, we obtain

$$T(\lambda_1) = \frac{(n - 2)(\mu - \lambda_2)}{\lambda_2 - \lambda_1}T(\mu) \hspace{0.5cm} \text{and} \hspace{0.5cm} T(\lambda_2) = \frac{(n - 2)(\lambda_1 - \mu)}{\lambda_2 - \lambda_1}T(\mu).$$

The $X$ and $Y$ components of (5) for $S_1$ evaluated in $X$ and $T \in \Gamma(\Delta)$ give, respectively,

$$\langle \mu - \lambda_1 \rangle \langle \nabla_X^* T, X \rangle^* = T(\lambda_1) - T(\mu) = -\frac{n\lambda_2}{\lambda_2 - \lambda_1}T(\mu) \hspace{1cm} (13)$$

and

$$\langle \mu - \lambda_2 \rangle \langle \nabla_X^* T, Y \rangle^* = (\lambda_1 - \lambda_2)\langle \nabla_Y^* X, Y \rangle^*. \hspace{1cm} (14)$$
Similary, the \( X \) and \( Y \) components of (5) for \( S_1 \) evaluated in \( Y \) and \( T \) give, respectively,

\[
(\mu - \lambda_1) \langle \nabla^*_Y T, X \rangle^* = (\lambda_2 - \lambda_1) \langle \nabla^*_T Y, X \rangle^* \quad (15)
\]

and

\[
(\mu - \lambda_2) \langle \nabla^*_Y T, Y \rangle^* = T(\lambda_2) - T(\mu) = \frac{n\lambda_1}{\lambda_2 - \lambda_1} T(\mu). \quad (16)
\]

We claim that \( S_1 \) and \( S_2 \) do not commute, that is, that \( c \neq 0 \). Assume otherwise. Then Eqs. (9) to (12) imply that \( S_2 X = \lambda_2 X \) and \( S_2 Y = \lambda_1 Y \). Hence, the \( X \) and \( Y \) components of (5) for \( S_2 \) evaluated in \( X \) and \( T \in \Gamma(\Delta) \) give, respectively,

\[
(\mu - \lambda_2) \langle \nabla^*_X T, X \rangle^* = T(\lambda_2) - T(\mu) \quad (17)
\]

and

\[
(\mu - \lambda_1) \langle \nabla^*_X T, Y \rangle^* = (\lambda_2 - \lambda_1) \langle \nabla^*_T X, Y \rangle^*. \quad (18)
\]

Similary, the \( X \) and \( Y \) components of (5) for \( S_2 \) evaluated in \( Y \) and \( T \) give, respectively,

\[
(\mu - \lambda_2) \langle \nabla^*_Y T, X \rangle^* = (\lambda_1 - \lambda_2) \langle \nabla^*_T Y, X \rangle^* \quad (19)
\]

and

\[
(\mu - \lambda_1) \langle \nabla^*_Y T, Y \rangle^* = T(\lambda_1) - T(\mu). \quad (20)
\]

Adding (14) and (18) yields

\[
(2\mu - \lambda_1 - \lambda_2) \langle \nabla^*_X T, Y \rangle^* = 0.
\]

Similarly, Eqs (15) and (19) give

\[
(2\mu - \lambda_1 - \lambda_2) \langle \nabla^*_Y T, X \rangle^* = 0.
\]

If \( 2\mu - \lambda_1 - \lambda_2 \) does not vanish identically, there exists an open subset \( U \subset M^n \) where \( \langle \nabla^*_X T, Y \rangle^* = 0 = \langle \nabla^*_Y T, X \rangle^* \). Now, from (13) and (17) we obtain

\[
(\lambda_2 - \lambda_1) \langle \nabla^*_X T, X \rangle^* = T(\lambda_1 - \lambda_2).
\]

Similarly, using (16) and (20) we have

\[
(\lambda_1 - \lambda_2) \langle \nabla^*_Y T, Y \rangle^* = T(\lambda_2 - \lambda_1).
\]
The preceding equations imply that the splitting tensor $C^*$ of $\Delta$ with respect to the Moebius metric satisfies $C^*_T \in \text{span}\{I\}$ for any $T \in \Gamma(\Delta|_U)$. From the relation between the Levi-Civita connections of conformal metrics we obtain

$$C^*_T = C_T - T(\log \phi) I,$$  \hspace{1cm} (21)

where $\phi$ is the conformal factor of $g^*$ with respect to the metric induced by $f_1$ and $C$ is the splitting tensor of $\Delta$ corresponding to the latter metric. Therefore, we also have $C^*_T \in \text{span}\{I\}$ for any $T \in \Gamma(\Delta|_U)$, and hence $f_1|U$ is conformally surface-like by Lemma 7, a contradiction. Thus $(2\mu - \lambda_1 - \lambda_2)$ must vanish everywhere, which, together with (9), implies that also $\mu$ is everywhere vanishing. Hence $\lambda_1 = -\lambda_2$, and therefore $S_1 = -S_2$, which is again a contradiction, and proves the claim.

Now we compute

$$\langle (\nabla^*_T S_2)X, X \rangle^* = \langle \nabla^*_T(b_1X + cY), X \rangle^* - \langle S_2\nabla^*_T X, X \rangle^*$$

$$= T(b_1) + c\langle \nabla^*_T Y, X \rangle^* - c\langle \nabla^*_T X, Y \rangle^*$$

$$= T(b_1) + 2c\langle \nabla^*_T Y, X \rangle^*.$$  \hspace{1cm} (22)

In a similar way,

$$\langle (\nabla^*_T S_2)Y, Y \rangle^* = T(b_2) + 2c\langle \nabla^*_T X, Y \rangle^*.$$  \hspace{1cm} (22)

Adding the preceding equations and using (11) yield

$$\langle (\nabla^*_T S_2)X, X \rangle^* + \langle (\nabla^*_T S_2)Y, Y \rangle^* = (2 - n)T(\mu).$$  \hspace{1cm} (22)

From (5) we obtain

$$\langle (\nabla^*_T S_2)X, X \rangle^* = \langle (\nabla^*_X S_2)T, X \rangle^* + T(\mu)$$

$$= \mu\langle \nabla^*_X T, X \rangle^* - \langle \nabla^*_X T, S_2X \rangle^* + T(\mu)$$

$$= (\mu - b_1)\langle \nabla^*_X T, X \rangle^* - c\langle \nabla^*_X T, Y \rangle^* + T(\mu),$$

and similarly,

$$\langle (\nabla^*_T S_2)Y, Y \rangle^* = (\mu - b_2)\langle \nabla^*_Y T, Y \rangle^* - c\langle \nabla^*_Y T, X \rangle^* + T(\mu).$$

Substituting the preceding expressions in (22) gives

$$nT(\mu) + (\mu - b_1)\langle \nabla^*_X T, X \rangle^* + (\mu - b_2)\langle \nabla^*_Y T, Y \rangle^* = c\langle \nabla^*_X T, Y \rangle^* + c\langle \nabla^*_Y T, X \rangle^*. $$  \hspace{1cm} (23)
Let us first focus on the terms on the left-hand side of the above equation. Using (13) and (16) we obtain

\[
\begin{align*}
n^T(\mu) + (\mu - b_1)\langle \nabla^*_X T, X \rangle^* + (\mu - b_2)\langle \nabla^*_Y T, Y \rangle^* \\
= n^T(\mu) - \frac{n\lambda_2(\mu - b_1)}{(\mu - \lambda_1)(\lambda_2 - \lambda_1)}T(\mu) + \frac{n\lambda_1(\mu - b_2)}{(\mu - \lambda_2)(\lambda_2 - \lambda_1)}T(\mu) \\
= \frac{(n - 1)(\lambda_1 - b_1)}{(\mu - \lambda_2)(\lambda_2 - \lambda_1)}T(\mu).
\end{align*}
\]

For the right-hand side of (23), using (14) and (15) we have

\[
\begin{align*}
c(\langle \nabla^*_X T, Y \rangle^* + \langle \nabla^*_Y T, X \rangle^*) &= c\left(\frac{\lambda_1 - \lambda_2}{\mu - \lambda_2}\langle \nabla^*_T X, Y \rangle^* + \frac{\lambda_2 - \lambda_1}{\mu - \lambda_1}\langle \nabla^*_T Y, X \rangle^*\right) \\
&= c\frac{(\lambda_1 - \lambda_2)(\mu - \lambda_1 + \mu - \lambda_2)}{(\mu - \lambda_1)(\mu - \lambda_2)}\langle \nabla^*_T X, T \rangle^* \\
&= c\frac{n\mu(\lambda_1 - \lambda_2)}{(\mu - \lambda_1)(\mu - \lambda_2)}\langle \nabla^*_T X, Y \rangle^*.
\end{align*}
\]

Therefore (22) becomes

\[
(n - 1)(b_1 - \lambda_1)T(\mu) = nc\mu(\lambda_1 - \lambda_2)^2\langle \nabla^*_T X, T \rangle^*. \tag{24}
\]

Now evaluate (5) for $S_2$ in $X$ and $T$. More specifically, the $Y$ component of that equation is

\[
T(c) = (\mu - b_2)\langle \nabla^*_X T, Y \rangle^* - c\langle \nabla^*_X T, X \rangle + (b_2 - b_1)\langle \nabla^*_T Y, X \rangle^*.
\]

Substituting (13) and (14) in the above equation, and using (9) and (11), we obtain

\[
\begin{align*}
T(c) &= \frac{(\mu - b_2)(\lambda_1 - \lambda_2)}{\mu - \lambda_2}\langle \nabla^*_T X, Y \rangle^* + \frac{c\lambda_2}{(\mu - \lambda_1)(\lambda_2 - \lambda_1)}T(\mu) \\
&\quad + (b_2 - b_1)\langle \nabla^*_T X, Y \rangle^* \\
&\quad + \frac{\mu\lambda_1 - \mu\lambda_2 - b_2\lambda_1 + b_2\lambda_2 + \mu b_2 - \mu b_1 - \lambda_2 b_2 + b_1\lambda_2}{\mu - \lambda_2}\langle \nabla^*_T X, Y \rangle^* \\
&\quad + \frac{c\lambda_2}{(\mu - \lambda_1)(\lambda_2 - \lambda_1)}T(\mu) \\
&\quad = \frac{n\mu(\lambda_1 - b_1)}{\mu - \lambda_2}\langle \nabla^*_T X, Y \rangle^* + \frac{c\lambda_2}{(\mu - \lambda_1)(\lambda_2 - \lambda_1)}T(\mu). \tag{25}
\end{align*}
\]
Similarly, the $X$ component of (5) for $S_2$ evaluated in $Y$ and $T$ gives

\[ T(c) = (\mu - b_1)\langle \nabla^*_Y T, X \rangle^* - c\langle \nabla^*_Y T, Y \rangle^* + (b_2 - b_1)\langle \nabla^*_T X, Y \rangle^*. \]

Substituting (15) and (16) in the above equation we obtain

\[ T(c) = -\frac{cn\lambda_1}{(\mu - \lambda_2)(\lambda_2 - \lambda_1)}T(\mu) + \frac{n\mu(\lambda_1 - b_1)}{\mu - \lambda_1}\langle \nabla^*_T X, Y \rangle^*. \]  
(26)

Using (10), it follows from (25) and (26) that

\[ (n - 1)cT(\mu) = n\mu(\lambda_1 - b_1)(\lambda_1 - \lambda_2)^2\langle \nabla^*_T X, Y \rangle^*. \]

(27)

Comparing (24) and (27) yields

\[ \mu((\lambda_1 - b_1)^2 + c^2)\langle \nabla^*_T X, Y \rangle^* = 0. \]

Since $(\lambda_1 - b_1)^2 + c^2 \neq 0$, for otherwise the immersions would be Moebius congruent, then $\mu\langle \nabla^*_T X, Y \rangle^* = 0$.

If $\mu$ does not vanish identically, then there is an open subset $U$ where $\langle \nabla^*_T X, Y \rangle^* = 0$ for any $T \in \Gamma(\Delta)$. Then (14) and (15) imply that the splitting tensor of $\Delta$ with respect to the Moebius metric satisfies $C^*_T \in \text{span}\{I\}$ for any $T \in \Gamma(\Delta)$. As before, this implies that the splitting tensor of $\Delta$ with respect to the metric induced by $f_1$ also satisfies $C^*_T \in \text{span}\{I\}$ for any $T \in \Gamma(\Delta)$, and hence $f_1|_U$ is conformally surface-like by Lemma 7, a contradiction. Thus $\mu$ must vanish identically.

### 3.2 Proof of Theorem 1

In this subsection we prove Theorem 1. First we recall one further definition.

Let $f: M^n \to \mathbb{R}^{n+\perp}$, $n \geq 3$, be a hypersurface that carries a principal curvature of constant multiplicity $n - 2$ with corresponding eigenbundle $\Delta$. Let $C: \Gamma(\Delta) \to \Gamma(\text{End}(\Delta^\perp))$ be the splitting tensor of $\Delta$. Then $f$ is said to be *hyperbolic* (respectively, *parabolic* or *elliptic*) if there exists $J \in \Gamma(\text{End}(\Delta^\perp))$ satisfying the following conditions:

- (i) $J^2 = I$ and $J \neq I$ (respectively, $J^2 = 0$, with $J \neq 0$, or $J^2 = -I$),
- (ii) $\nabla^*_T J = 0$ for all $T \in \Gamma(\Delta)$,
- (iii) $C(\Gamma(\Delta)) \subset \text{span}\{I, J\}$, but $C(\Gamma(\Delta)) \not\subset \text{span}\{I\}$.

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Proof of Theorem 7. Let \( f_2 : M^n \to \mathbb{R}^{n+1} \) be a Moebius deformation of \( f_1 := f \). By Proposition 6 the Moebius shape operators \( S_1 \) and \( S_2 \) of \( f_1 \) and \( f_2 \), respectively, share a common kernel \( \Delta \) of dimension \( n-2 \). Let \( S_i, i = 1, 2 \), denote also the restriction \( S_i|_{\Delta^\perp} \) and define \( D = \Gamma(\text{End}(\Delta^\perp)) \) by
\[
D = S_1^{-1}S_2.
\]

It follows from Proposition 6 that \( \det D = 1 \) at any point of \( M^n \), while Proposition 3 implies that \( D \) cannot be the identity endomorphism up to sign on any open subset \( U \subset M^n \), for otherwise \( f_1|_U \) and \( f_2|_U \) would be Moebius congruent by Lemma 7. Therefore, we can write \( D = aI + bJ \), where \( b \) does not vanish on any open subset of \( M^n \) and \( J \in \Gamma(\text{End}(\Delta^\perp)) \) satisfies \( J^2 = \epsilon I \), with \( \epsilon \in \{1, 0, -1\} \), \( J \neq I \) if \( \epsilon = 1 \) and \( J \neq 0 \) if \( \epsilon = 0 \).

From the symmetry of \( S_2 \) and the fact that \( b \) does not vanish on any open subset of \( M^n \), we see that \( S_1J \) must be symmetric. Moreover, given that \( \text{tr} S_1 = 0 = \text{tr} S_2 \), also \( \text{tr} S_1J = 0 \).

Assume first that \( J^2 = 0 \). Let \( X, Y \in \Gamma(\Delta^\perp) \) be orthogonal vector fields, with \( Y \) of unit length (with respect to the Moebius metric \( g^* \)), such that \( JX = Y \) and \( JY = 0 \). Replacing \( J \) by \( \|X\|^*J \), if necessary, we can assume that also \( X \) has unit length. Let \( \alpha, \beta, \gamma \in C^\infty(M) \) be such that \( S_1X = \alpha X + \beta Y \) and \( S_1Y = \beta X + \gamma Y \), so that \( S_1JX = \beta X + \gamma Y \) and \( S_1JY = 0 \). From the symmetry of \( S_1J \) and the fact that \( \text{tr} S_1J = 0 \) we obtain \( \beta = 0 = \gamma \), and hence \( \alpha = \text{tr} S_1 = 0 \). Thus \( S_1 = 0 \), which is a contradiction.

Now assume that \( J^2 = I \), \( J \neq I \). Let \( X, Y \) be a frame of unit vector fields (with respect to \( g^* \)) satisfying \( JX = X \) and \( JY = -Y \). Write \( S_1X = \alpha X + \beta Y \) and \( S_1Y = \gamma X + \delta Y \) for some \( \alpha, \beta, \gamma, \delta \in C^\infty(M) \), so that \( S_1JX = \alpha X + \beta Y \) and \( S_1JY = -\gamma X - \delta Y \). Since \( \text{tr} S_1J = 0 = \text{tr} S_1 \), then \( \alpha = 0 = \delta \). The symmetry of \( S_1 \) and \( S_2J \) implies that \( \beta = 0 = \gamma \), which is again a contradiction.

Therefore, the only possible case is that \( J^2 = -I \). Let \( X, Y \in \Gamma(\Delta^\perp) \) be a frame of unit vector fields such that \( JX = Y \) and \( JY = -X \). Write as before \( S_1X = \alpha X + \beta Y \) and \( S_1Y = \gamma X + \delta Y \) for some \( \alpha, \beta, \gamma, \delta \in C^\infty(M) \). Then \( S_1JX = \gamma X + \delta Y \) and \( S_1JY = -\alpha X - \beta Y \), hence \( \beta = \gamma \), for \( \text{tr} S_1J = 0 \). From the symmetry of \( S_1 \) we obtain
\[
\langle S_1JX, Y \rangle = \langle JX, S_1Y \rangle = \langle Y, S_1Y \rangle = \gamma \langle X, Y \rangle + \delta = \beta \langle X, Y \rangle + \delta,
\]
and similarly,
\[
\langle S_1JY, X \rangle = -\alpha - \beta \langle X, Y \rangle.
\]
Comparing the two preceding equations, taking into account the symmetry of $S_1 J$ and the fact that $\text{tr} S_1 = 0$, yields $\beta \langle X, Y \rangle = 0$. If $\beta$ is nonzero, then $X$ and $Y$ are orthogonal to each other. This is also the case if $\beta$, hence also $\gamma$, is zero, for in this case $X$ and $Y$ are eigenvectors of $S_1$. Thus, in any case, we conclude that $J$ acts as a rotation of angle $\pi/2$ on $\Delta^\perp$.

Eq. (5) and the fact that $\omega_i|_{\Delta} = 0$ imply that the splitting tensor of $\Delta$ with respect to the Moebius metric satisfies

$$\nabla^*_T S_i = S_i C^*_T$$

for all $T \in \Gamma(\Delta)$ and $1 \leq i \leq 2$, where

$$(\nabla^*_T S_i) X = \nabla^*_T S_i X - S_i \nabla^*_T X$$

for all $X \in \Gamma(\Delta^\perp)$ and $T \in \Gamma(\Delta)$. Here $\nabla^*_T X = (\nabla^*_T X)_{\Delta^\perp}$. In particular,

$$S_i C^*_T = C^*_T S_i, \; 1 \leq i \leq 2.$$

Therefore

$$S_1 D C^*_T = S_2 C^*_T = C^*_T S_2 = C^*_T S_1 D = S_1 C^*_T D,$$

and hence

$$[D, C^*_T] = 0.$$ 

This implies that $C^*_T$ commutes with $J$, and hence $C^*_T \in \text{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$. It follows from [21] that also the splitting tensor $C$ of $\Delta$ corresponding to the metric induced by $f$ satisfies $C_T \in \text{span}\{I, J\}$ for any $T \in \Gamma(\Delta)$. Moreover, by Lemma 7 and the assumption that $f$ is not surface-like on any open subset, we see that $C(\Gamma(\Delta)) \not\subset \text{span}\{I\}$ on any open subset. Now, since $J$ acts as a rotation of angle $\pi/2$ on $\Delta^\perp$, then $\nabla^*_T J = 0$. We conclude that $f$ is elliptic with respect to $J$.

By Proposition 6 the central sphere congruence $S: M^n \to S^{n+2}_{1,1}$ of $f$ is a two-parameter congruence of hyperspheres, which therefore gives rise to a surface $s: L^2 \to S^{n+2}_{1,1}$ such that $s = S \circ \pi$, where $\pi: M^n \to L^2$ is the (local) quotient map onto the space of leaves of $\Delta$. Since $\nabla^*_T J = 0 = [C_T, J]$ for any $T \in \Gamma(\Delta)$, it follows from Corollary 11.7 in [3] that $J$ is projectable with respect to $\pi$, that is, there exists $\bar{J} \in \text{End}(TL)$ such that $\bar{J} \circ \pi_* = \pi_* \circ J$. In particular, the fact that $J^2 = -I$ implies that $\bar{J}^2 = -I$, where we denote also by $I$ the identity endomorphism of $TL$.

Now observe that, since $f_2$ shares with $f_1$ the same Moebius metric, its induced metric is conformal to the metric induced by $f_1$. Moreover, $f_2$ is not
Moebius congruent to $f_1$ on any open subset of $M^n$ and $f_1$ has a principal
curvature of constant multiplicity $(n - 2)$. Thus $f_1$ is a so-called Cartan
hypersurface. By the proof of the classification of Cartan hypersurfaces given
in Chapter 17 of [3] (see Lemma 17.4 therein), the surface $s$ is elliptic with
respect to $\bar{J}$, that is, for all $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$ we have

$$\alpha^s(\bar{J}\bar{X}, \bar{Y}) = \alpha^s(\bar{X}, \bar{J}\bar{Y}).$$

We claim that $\bar{J}$ is an orthogonal tensor, that is, it acts as a rotation of
angle $\pi/2$ on each tangent space of $L^2$. The minimality of $s$ will then follow
from this, the fact that $\bar{J}^2 = -I$ and (28).

In order to show the orthogonality of $\bar{J}$, we use the fact that the metric
$\langle \cdot, \cdot \rangle'$ on $L^2$ induced by $s$ is related to the metric of $M^n$
by

$$\langle \bar{Z}, \bar{W} \rangle' = \langle (A - \lambda I)Z, (A - \lambda I)W \rangle$$

for all $\bar{Z}, \bar{W} \in \mathfrak{X}(L)$, where $A$ is the shape operator of $f$, $\lambda$ is the principal
curvature of $f$ having $\Delta$ as its eigenbundle, which coincides with the mean
curvature $H$ of $f$ by Proposition 6, and $Z, W$ are the horizontal lifts of $\bar{Z}$
and $\bar{W}$, respectively. Notice that $(A - \lambda I)$ is a multiple of $S_1$. Since $S_1J$ is
symmetric, then also $(A - \lambda I)J$ is symmetric. Therefore, given any $\bar{X} \in \mathfrak{X}(L)$
and denoting by $X \in \Gamma(\Delta^\perp)$ its horizontal lift, we have

$$\langle \bar{X}, \bar{J}\bar{X} \rangle' = \langle (A - \lambda I)X, (A - \lambda I)JX \rangle$$

$$= \langle (A - \lambda I)J(A - \lambda I)X, X \rangle$$

$$= \langle J(A - \lambda I)X, (A - \lambda I)X \rangle$$

$$= 0,$$

where in the last step we have used that $J$ acts as a rotation of angle $\pi/2$ on
$\Delta^\perp$. Using again the symmetry of $(A - \lambda I)J$, the proof of the orthogonality
of $\bar{J}$ is completed by noticing that

$$\langle \bar{J}\bar{X}, \bar{J}\bar{X} \rangle' = \langle (A - \lambda I)JX, (A - \lambda I)JX \rangle$$

$$= \langle J(A - \lambda I)JX, (A - \lambda I)X \rangle$$

$$= \langle JJ^t(A - \lambda I)X, (A - \lambda I)X \rangle$$

$$= -\langle J^2(A - \lambda I)X, (A - \lambda I)X \rangle$$

$$= \langle \bar{X}, \bar{X} \rangle'.$
Conversely, assume that the central sphere congruence of \( f: M^n \to \mathbb{R}^{n+1} \), with \( M^n \) simply connected, is determined by a space-like minimal surface \( s: L^2 \to S^{n+2}_{1,1} \). Let \( \bar{J} \in \Gamma(\text{End}(TL)) \) represent a rotation of angle \( \pi/2 \) on each tangent space. Then \( \bar{J}^2 = -I \) and the second fundamental form of \( s \) satisfies the minimality of \( s \) by the minimality of \( s \). In particular, \( s \) is elliptic with respect to \( \bar{J} \). By Lemma 17.4 in [3], the hypersurface \( f \) is elliptic with respect to the lift \( J \in \Gamma(\text{End}(\Delta^+)) \) of \( \bar{J} \), where \( \Delta \) is the eigenbundle correspondent to the principal curvature \( \lambda \) of \( f \) with multiplicity \( n - 2 \), which coincides with its mean curvature. Therefore, the splitting tensor of \( \Delta \) satisfies the Codazzi equation, and \( C(\Gamma(\Delta)) \not\subset \text{span}\{I,J\} \) on any open subset, for \( f \) is not conformally surface-like on any open subset, then \( (A - \lambda I)J \) is also symmetric.

By Theorem 17.5 in [3], the set of conformal deformations of \( f \) is in one-to-one correspondence with the set of tensors \( \bar{D} \in \Gamma(\text{End}(TL)) \) with \( \det \bar{D} = 1 \) that satisfy the Codazzi equation

\[
(\nabla'_X \bar{D}) \bar{Y} - (\nabla'_Y \bar{D}) \bar{X} = 0
\]

for all \( \bar{X}, \bar{Y} \in \mathcal{X}(L) \), where \( \nabla' \) is the Levi-Civita connection of the metric induced by \( s \). For a general elliptic hypersurface, this set either consists of a one-parameter family (continuous class) or of a single element (discrete class; see Section 11.2 and Exercise 11.3 in [3]). The surface \( s \) is then said to be of the complex type of first or second species, respectively. For a minimal surface \( s: L^2 \to S^{n+2}_{1,1} \), each tensor \( \bar{J}_\theta = \cos \theta I + \sin \theta \bar{J} \), \( \theta \in [0,2\pi) \), satisfies both the condition \( \det \bar{J}_\theta = 1 \) and the Codazzi equation, since it is a parallel tensor in \( L^2 \). Thus \( \{\bar{J}_\theta\}_{\theta \in [0,2\pi)} \) is the one-parameter family of tensors in \( L^2 \) having determinant one and satisfying the Codazzi equation.

In particular, the surface \( s \) is of the complex type of first species. Therefore, the hypersurface \( f \) admits a one-parameter family of conformal deformations, each of which determined by one of the tensors \( \bar{J}_\theta \in \text{End}(TL), \theta \in [0,2\pi) \). The proof of Theorem (11) will be completed once we prove that any of such conformal deformations shares with \( f \) the same Möbius metric.

Let \( f_\theta: M^n \to \mathbb{R}^{n+1} \) be the conformal deformation of \( f \) determined by \( \bar{J}_\theta \). Let \( F_\theta: M^n \to \mathbb{V}^{n+2} \) be the isometric light-cone representative of \( f_\theta \), that is, \( F_\theta \) is the isometric immersion of \( M^n \) into the light-cone \( \mathbb{V}^{n+2} \subset \mathbb{L}^{n+3} \) given by \( F_\theta = \varphi^{-1}_\theta(\Psi \circ f_\theta) \), where \( \varphi_\theta \) is the conformal factor of the metric \( \langle \cdot, \cdot \rangle_\theta \) induced by \( f_\theta \) with respect to the metric \( \langle \cdot, \cdot \rangle \) of \( M^n \), that is, \( \langle \cdot, \cdot \rangle_\theta = \varphi^2_\theta \langle \cdot, \cdot \rangle \).
$\Psi : \mathbb{R}^n \to \mathbb{V}^{n+2}$ is the isometric embedding of $\mathbb{R}^n$ into $\mathbb{V}^{n+2}$ given by (3). As shown in the proof of Lemma 17.2 in [3], as part of the proof of the classification of Cartan hypersurfaces of dimension $n \geq 5$ given in Chapter 17 therein, the second fundamental form of $F_\theta$ is given by

$$\alpha^{F_\theta}(X,Y) = \langle AX,Y \rangle \mu - \langle (A - \lambda I)X,Y \rangle \zeta + \langle (A - \lambda I)J_\theta X,Y \rangle \bar{\zeta}$$

(30)

for all $X,Y \in \mathfrak{X}(M)$, where $\{\mu, \zeta, \bar{\zeta}\}$ is an orthonormal frame of the normal bundle of $F_\theta$ in $\mathbb{L}^{n+3}$ with $\mu$ space-like, $\lambda = -\langle \mu, F_\theta \rangle^{-1}$ and $\zeta = \lambda F_\theta + \mu$ (hence $\langle \zeta, \zeta \rangle = -1$). Here $J_\theta$ is the horizontal lift of $\bar{J}_0$, which has been extended to $TM$ by setting $J_\theta|_\Delta = I$.

Let $\bar{X}, \bar{Y} \in \mathfrak{X}(L)$ be an orthonormal frame such that $\bar{J}\bar{X} = \bar{Y}$ and $\bar{J}\bar{Y} = -\bar{X}$, and let $X,Y \in \Gamma(\Delta^\perp)$ be the respective horizontal lifts. It follows from (29) that $\{(A - \lambda I)X, (A - \lambda I)Y\}$ is an orthonormal frame of $\Delta^\perp$. From the symmetry of $(A - \lambda I)J$ and $(A - \lambda I)$ we have

$$\langle J(A - \lambda I)X, (A - \lambda I)X \rangle = \langle (A - \lambda I)J(A - \lambda I)X, X \rangle = \langle (A - \lambda I)X, (A - \lambda I)JX \rangle = \langle \bar{X}, J\bar{X} \rangle'$$

$$= 0.$$

In a similar way one verifies that $\langle J(A - \lambda I)Y, (A - \lambda I)Y \rangle = 0$ and

$$\langle J(A - \lambda I)Y, (A - \lambda I)X \rangle = 1 = -\langle J(A - \lambda I)X, (A - \lambda I)Y \rangle.$$

Thus $J$ acts on $\Delta^\perp$ as a rotation of angle $\pi/2$. The symmetry of both $(A - \lambda I)J$ and $(A - \lambda I)$ implies that $\text{tr} (A - \lambda I) = 0 = \text{tr} (A - \lambda I)J$, hence

$$\text{tr} (A - \lambda I)J_\theta = 0$$

(31)

for all $\theta \in [0, 2\pi)$.

Now we use the relation between the second fundamental forms of $f_\theta$ and $F_\theta$, given by Eq. 9.32 in [3], namely,

$$\alpha^{F_\theta}(X,Y) = \langle \varphi(A_\theta - H_\theta I)X, Y \rangle_2 \tilde{N} - \psi(X,Y)F_\theta - \langle X,Y \rangle \zeta_2,$$

(32)

where $\langle \cdot, \cdot \rangle_\theta = \varphi^2_\theta(\cdot, \cdot)$ is the metric induced by $f_\theta$, $A_\theta$ and $H_\theta$ are its shape operator and mean curvature, respectively, $\psi$ is a certain symmetric bilinear form, $\tilde{N} \in \Gamma(N_F M)$, with $\langle \tilde{N}, F_\theta \rangle = 0$, is a unit space-like vector field, and
ζ_2 ∈ Γ(N_F,M) satisfies ⟨̃N, ζ_2⟩ = 0 = ⟨ζ_2, ζ_2⟩ and ⟨F_θ, ζ_2⟩ = 1. Eqs. (30) and (32) give

\[(A - λI)J_θX, Y⟩ = \langle \alpha F_θ(X, Y), \zeta \rangle = \varphi_θ⟨(A_θ - H_θ I)X, Y⟩⟨\tilde{N}, \tilde{\zeta}⟩ - ⟨X, Y⟩⟨ζ_2, \tilde{\zeta}⟩\]

for all X, Y ∈ X(M), or equivalently,

\[(A - λI)J_θ = \varphi_θ⟨\tilde{N}, \tilde{\zeta}⟩(A_θ - H_θ I) - ⟨ζ_2, \tilde{\zeta}⟩I. \tag{33}\]

Using that

\[\text{tr}((A - λI)J_θ) = 0 = \text{tr}((A_θ - H_θ I),\]

we obtain from the preceding equation that ⟨ζ_2, ζ⟩ = 0. Thus ζ ∈ span{F_θ, ζ_2}⊥, and hence ζ = ±̃N. Therefore, (33) reduces to

\[(A - λI)J_θ = ±\varphi(A_θ - H_θ I). \tag{34}\]

In particular, \((A_θ - H_θ I)|_Δ = 0\), hence also \(S_θ|_Δ = 0\), where \(S_θ = \phi_θ^{-1}(A_θ - H_θ I)\) is the Moebius shape operator of \(f_θ\), with \(φ_θ\) given by (11) for \(f_θ\). Since the Moebius shape operator of an umbilic-free immersion is traceless and has constant norm \(\sqrt{(n-1)/n}\), then \(S_θ\) must have constant eigenvalues \(\sqrt{(n-1)/2n}, -\sqrt{(n-1)/2n}\) and 0. The same holds for the Moebius second fundamental form \(S_1\) of \(f\), which has also \(Δ\) as its kernel. We conclude that the eigenvalues of \((A_θ - H_θ I)|_Δ\) are

\[δ_1 = φ_θ\sqrt{(n-1)/2n} \quad \text{and} \quad δ_2 = -φ_θ\sqrt{(n-1)/2n}\]

and, similarly, the eigenvalues of \((A - λI)|_Δ\) are

\[λ_1 = φ_1\sqrt{(n-1)/2n} \quad \text{and} \quad λ_2 = -φ_1\sqrt{(n-1)/2n},\]

where \(φ_1\) is given by (11) with respect to \(f\). On the other hand, since

\[\det((A - λI)J_θ) = \det((A - λI),\]

for \(\det J_θ = 1\), and both \((A - λI)\) and \((A - λI)J_θ\) are traceless (see (31)), it follows that \((A - λI)\) and \((A - λI)J_θ\) have the same eigenvalues. This and (34) imply that

\[φ_1^2 = ϕ_θ^2 φ_θ^2,\]

hence the Moebius metrics of \(f\) and \(f_θ\) coincide. \[\square\]
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