PROPER \(p\)-HARMONIC FUNCTIONS AND HARMONIC MORPHISMS ON THE CLASSICAL NON-COMPACT SEMI-RIEMANNIAN LIE GROUPS

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Abstract. We apply the method of eigenfamilies to construct new explicit complex-valued \(p\)-harmonic functions on the non-compact classical Lie groups, equipped with their natural semi-Riemannian metrics. We then employ this same approach to manufacture explicit complex-valued harmonic morphisms on these groups.

1. Introduction

For a positive integer \(p\), the complex-valued \(p\)-harmonic functions are solutions to a partial differential equation of order \(2p\). This equation arises in various contexts, see for example the extensive analysis in [5] and a historic account in [15]. The best known applications are in physics e.g. for \(p = 2\) in the areas of continuum mechanics, including elasticity theory and the solution of Stokes flows. The literature on 2-harmonic functions is vast, but until quite recently, the domains were either surfaces or open subsets of flat Euclidean space, with only very few exceptions. For this see the regularly updated online bibliography [8] maintained by the second author.

In their recent article [11], the authors produce \(p\)-harmonic functions on the classical Lie groups equipped with their standard Riemannian metrics. The primary goal of this work is to extend the study to the semi-Riemannian situation. By Theorem 3.2 we show how the problem can be reduced to finding an eigenfamily, i.e. a collection of complex-valued functions which are eigen both with respect to the Laplace-Beltrami operator \(\tau\) and the conformality operator \(\kappa\), on the semi-Riemannian manifolds involved. The main part of this paper is devoted to the construction of such families on the following classical Lie groups equipped with their natural semi-Riemannian metrics

\[
\begin{align*}
\text{GL}_n(\mathbb{C}),& \quad \text{GL}_n(\mathbb{R}), \quad \text{GL}_n(\mathbb{H}), \\
\text{SL}_n(\mathbb{C}),& \quad \text{SL}_n(\mathbb{R}), \quad \text{SL}_n(\mathbb{H}), \\
\text{SO}(n, \mathbb{C}),& \quad \text{Sp}(n, \mathbb{C}), \quad \text{Sp}(n, \mathbb{R}), \quad \text{SO}^*(2n), \quad \text{SU}(p, q), \quad \text{SO}(p, q), \quad \text{Sp}(p, q).
\end{align*}
\]

Our eigenfamilies can also be used to manufacture complex-valued harmonic morphisms on these manifolds as explained in Theorem 4.3. They

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can therefore be seen as an interesting byproduct of the process presented here.

For semi-Riemannian geometry we recommend O’Neill’s standard text [16]. Readers not familiar with harmonic morphisms are advised to consult the standard text [2] by Baird and Wood, [3], [4], [13] and the regularly updated online bibliography [7]. For the details from Lie theory, used in this paper, we refer the reader to [12] and [14].

2. Eigenfunctions and Eigenfamilies

In this paper we manufacture explicit complex-valued proper $p$-harmonic functions and harmonic morphisms on semi-Riemannian manifolds. For this we apply two different construction techniques which are presented in Theorem 3.2 and in Theorem 4.3, respectively. The main ingredients for both these recipes are the common eigenfunctions for the tension field $\tau$ and the conformality operator $\kappa$ which we now describe.

Let $(M,g)$ be an $m$-dimensional semi-Riemannian manifold and $T^CM$ be the complexification of the tangent bundle $TM$ of $M$. We extend the metric $g$ to a complex-bilinear form on $T^CM$. Then the gradient $\nabla \phi$ of a complex-valued function $\phi : (M,g) \to \mathbb{C}$ is a section of $T^CM$. In this situation, the well-known complex linear Laplace-Beltrami operator (alt. tension field) $\tau$ on $(M,g)$ acts locally on $\phi$ as follows

$$\tau(\phi) = \text{div}(\nabla \phi) = \sum_{i,j=1}^{m} \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x_j} \left( g^{ij} \sqrt{|g|} \frac{\partial \phi}{\partial x_i} \right).$$

For two complex-valued functions $\phi, \psi : (M,g) \to \mathbb{C}$ we have the following well-known fundamental relation

$$\tau(\phi \cdot \psi) = \tau(\phi) \cdot \psi + 2 \cdot \kappa(\phi, \psi) + \phi \cdot \tau(\psi),$$

where the complex bilinear conformality operator $\kappa$ is given by

$$\kappa(\phi, \psi) = g(\nabla \phi, \nabla \psi).$$

Locally this satisfies

$$\kappa(\phi, \psi) = \sum_{i,j=1}^{m} g^{ij} \cdot \frac{\partial \phi}{\partial x_i} \frac{\partial \psi}{\partial x_j}.$$

**Definition 2.1.** [10] Let $(M,g)$ be a semi-Riemannian manifold. Then a complex-valued function $\phi : M \to \mathbb{C}$ is said to be an eigenfunction if it is eigen both with respect to the Laplace-Beltrami operator $\tau$ and the conformality operator $\kappa$ i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$

A set $\mathcal{E} = \{\phi_i : M \to \mathbb{C} \mid i \in I\}$ of complex-valued functions is said to be an eigenfamily on $M$ if there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all
$\phi, \psi \in \mathcal{E}$ we have

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi \cdot \psi.$$  

The following Theorem 2.2 shows that, given an eigenfamily $\mathcal{E}$, one can employ this to produce an extensive collection $\mathcal{H}_d^\mathcal{E}$ of further such objects. The result is a semi-Riemannian version of Theorem 2.2 proven in [6] for the Riemannian case.

**Theorem 2.2.** Let $(M, g)$ be a semi-Riemannian manifold and the set of complex-valued functions

$$\mathcal{E} = \{ \phi_k : M \rightarrow \mathbb{C} \mid k = 1, 2, \ldots, n \}$$

be an eigenfamily on $M$ i.e. there exist complex numbers $\lambda, \mu \in \mathbb{C}$ such that for all $\phi, \psi \in \mathcal{E}$

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = \mu \cdot \phi \cdot \psi.$$  

Then the set of complex homogeneous polynomials of degree $d$

$$\mathcal{H}_d^\mathcal{E} = \{ P : M \rightarrow \mathbb{C} \mid P \in \mathbb{C}[\phi_1, \phi_2, \ldots, \phi_n], \ P(\alpha \cdot \phi) = \alpha^d \cdot P(\phi), \ \alpha \in \mathbb{C} \}$$

is an eigenfamily on $M$ such that for all $P, Q \in \mathcal{H}_d^\mathcal{E}$ we have

$$\tau(P) = (d \cdot \lambda + d(d - 1) \cdot \mu) \cdot P \quad \text{and} \quad \kappa(P, Q) = d^2 \cdot \mu \cdot P \cdot Q.$$  

**Proof.** The statement can be proven with exactly the same arguments as its Riemannian counterpart, see Theorem 2.2 in [6]. \qed

3. **Proper $p$-Harmonic Functions**

In this section we describe a method for manufacturing complex-valued proper $p$-harmonic functions on semi-Riemannian manifolds. This method was recently introduced for the Riemannian case in [11].

**Definition 3.1.** Let $(M, g)$ be a semi-Riemannian manifold. For a positive integer $p$, the iterated Laplace-Beltrami operator $\tau^p$ is given by

$$\tau^0(\phi) = \phi \quad \text{and} \quad \tau^p(\phi) = \tau(\tau^{p-1}(\phi)).$$  

We say that a complex-valued function $\phi : (M, g) \rightarrow \mathbb{C}$ is

(i) $p$-harmonic if $\tau^p(\phi) = 0$ and

(ii) proper $p$-harmonic if $\tau^p(\phi) = 0$ and $\tau^{p-1}(\phi)$ does not vanish identically.

The following Theorem 3.2 can be proven in exactly the same way as its Riemannian counterpart found as Theorem 3.1 in [11].

**Theorem 3.2.** Let $\phi : (M, g) \rightarrow \mathbb{C}$ be a complex-valued function on a semi-Riemannian manifold and $(\lambda, \mu) \in \mathbb{C}^2 \setminus \{0\}$ be such that the tension field $\tau$ and the conformality operator $\kappa$ satisfy

$$\tau(\phi) = \lambda \cdot \phi \quad \text{and} \quad \kappa(\phi, \phi) = \mu \cdot \phi^2.$$
Then for any positive integer \( p \in \mathbb{Z}^+ \) the non-vanishing function
\[
\Phi_p : W = \{ x \in M \mid \phi(x) \not\in (-\infty, 0] \} \to \mathbb{C}
\]
with
\[
\Phi_p(x) = \begin{cases}
  c_1 \cdot \log(\phi(x))^{p-1}, & \text{if } \mu = 0, \lambda \neq 0 \\
  c_1 \cdot \log(\phi(x))^{2p-1} + c_2 \cdot \log(\phi(x))^{2p-2}, & \text{if } \mu \neq 0, \lambda = \mu \\
  c_1 \cdot \phi(x)^{1-\frac{1}{p}} \log(\phi(x))^{p-1} + c_2 \cdot \log(\phi(x))^{p-1}, & \text{if } \mu \neq 0, \lambda \neq \mu
\end{cases}
\]
is a proper \( p \)-harmonic function. Here \( c_1, c_2 \) are complex coefficients not both zero.

4. Complex-Valued Harmonic Morphisms

In this section we describe a method for constructing complex-valued harmonic morphisms \( \phi : (M, g) \to \mathbb{C} \) from semi-Riemannian manifolds. This is a special case of the much studied harmonic morphisms \( \phi : (M, g) \to (N, h) \) between semi-Riemannian manifolds. They are maps which pull back local real-valued harmonic functions on \((N, h)\) to harmonic functions on \((M, g)\).

The standard reference for the extensive theory of harmonic morphisms is the book [2], but we also recommend the updated online bibliography [7].

The following result is a direct consequence of Theorem 3 of the paper [4] by B. Fuglede.

**Proposition 4.1.** A function \( \phi : (M, g) \to \mathbb{C} \) from a semi-Riemannian manifold to the standard Euclidean complex plane, is a harmonic morphism if and only if it is harmonic and horizontally conformal i.e.
\[
\tau(\phi) = 0 \quad \text{and} \quad \kappa(\phi, \phi) = 0.
\]

The following Theorem 4.2 is a semi-Riemannian version of Theorem 5.2 of [1], see also [2]. It gives the theory of complex-valued harmonic morphisms a strong geometric flavour and provides a useful tool for the construction of minimal submanifolds of codimension two. This is our main motivation for studying these maps.

**Theorem 4.2.** Let \( \phi : (M, g) \to \mathbb{C} \) be a horizontally conformal map from a semi-Riemannian manifold to the standard Euclidean complex plane. Then \( \phi \) is harmonic if and only if its fibres are minimal at regular points of \( \phi \).

The next result shows that eigenfamilies can be utilised to manufacture a variety of harmonic morphisms.

**Theorem 4.3.** [10] Let \((M, g)\) be a semi-Riemannian manifold and
\[
\mathcal{E} = \{ \phi_k : M \to \mathbb{C} \mid k = 1, 2, \ldots, n \}
\]
be an eigenfamily of complex-valued functions on \( M \). If \( P, Q : \mathbb{C}^n \to \mathbb{C} \) are linearly independent homogeneous polynomials of the same positive degree
then the quotient
\[ \frac{P(\phi_1, \ldots, \phi_n)}{Q(\phi_1, \ldots, \phi_n)} \]
is a non-constant harmonic morphism on the open and dense subset
\[ \{ p \in M \mid Q(\phi_1(p), \ldots, \phi_n(p)) \neq 0 \} . \]

5. The Semi-Riemannian Lie Group \( \text{GL}_n(\mathbb{C}) \)

The complex general linear group \( \text{GL}_n(\mathbb{C}) \) of invertible \( n \times n \) matrices is given by
\[ \text{GL}_n(\mathbb{C}) = \{ z \in \mathbb{C}^{n \times n} \mid \text{det} z \neq 0 \} . \]
Its Lie algebra \( \mathfrak{gl}_n(\mathbb{C}) \) of left-invariant vector fields on \( \text{GL}_n(\mathbb{C}) \) can be identified with \( \mathbb{C}^{n \times n} \) i.e. the linear space of complex \( n \times n \) matrices. We equip \( \text{GL}_n(\mathbb{C}) \) with its natural semi-Riemannian metric \( g \) induced by the semi-Euclidean inner product \( \mathfrak{gl}_n(\mathbb{C}) \times \mathfrak{gl}_n(\mathbb{C}) \to \mathbb{R} \) on \( \mathfrak{gl}_n(\mathbb{C}) \) satisfying
\[ g(Z, W) \mapsto -\Re \text{trace}(Z \cdot W) . \]
For \( \mathfrak{gl}_n(\mathbb{C}) \) we then have the orthogonal decomposition
\[ \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{gl}_n^+(\mathbb{C}) \oplus \mathfrak{gl}_n^-(\mathbb{C}) , \]
where
\[ \mathfrak{gl}_n^+(\mathbb{C}) = u(n) = \{ Z \in \mathfrak{gl}_n(\mathbb{C}) \mid Z + \bar{Z}^t = 0 \} \]
is the set of skew-Hermitian matrices and
\[ \mathfrak{gl}_n^-(\mathbb{C}) = i \cdot u(n) = \{ Z \in \mathfrak{gl}_n(\mathbb{C}) \mid Z - \bar{Z}^t = 0 \} \]
the set of the Hermitian ones. Here \( u(n) \) is the Lie algebra of the unitary group \( U(n) \) which is the maximal compact subgroup of \( \text{GL}_n(\mathbb{C}) \) satisfying
\[ U(n) = \{ z \in \text{GL}_n(\mathbb{C}) \mid z \bar{z}^t = I_n \} . \]
For \( 1 \leq r, s \leq n \), we shall by \( E_{rs} \in \mathbb{R}^{n \times n} \) denote the matrix given by
\[ (E_{rs})_{\alpha\beta} = \delta_{r\alpha} \delta_{s\beta} \]
and for \( r < s \) let \( X_{rs}, Y_{rs} \) be the symmetric and skew-symmetric matrices
\[ X_{rs} = \frac{1}{\sqrt{2}}(E_{rs} + E_{sr}) , \quad Y_{rs} = \frac{1}{\sqrt{2}}(E_{rs} - E_{sr}) , \]
respectively. Further, let \( D_t \) be the diagonal elements with \( D_t = E_{tt} \). By \( \mathcal{B}^+ \) we denote the orthonormal basis for the Lie algebra \( u(n) \) satisfying
\[ \mathcal{B}^+ = \{ Y_{rs}, i X_{rs} \mid 1 \leq r < s \leq n \} \cup \{ i D_t \mid t = 1, 2, \ldots, n \} \]
and \( \mathcal{B}^- = i \mathcal{B}^+ \). Then \( \mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \) is an orthonormal basis for \( \mathfrak{gl}_n(\mathbb{C}) \) such that \( g(Z, Z) = 1 \) if \( Z \in \mathcal{B}^+ \) and \( g(Z, Z) = -1 \) for \( Z \in \mathcal{B}^- \). For later use we define the two standard matrices \( J_n \) and \( I_{pq} \) by
\[ J_n = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \quad \text{and} \quad I_{pq} = \begin{bmatrix} -I_p & 0 \\ 0 & I_q \end{bmatrix} . \]
Let $G$ be a subgroup of $\text{GL}_n(\mathbb{C})$ with Lie algebra $\mathfrak{g}$ inheriting the induced left-invariant semi-Riemannian metric from $g$. Then employing the Koszul formula for the Levi-Civita connection $\nabla$ on $(G, g)$, we see that for all $Z, W \in \mathfrak{g}$ we have

$$g(\nabla_Z Z, W) = g([W, Z], Z) = -\Re \text{tr}(WZ - ZW)Z = -\Re \text{tr} W(ZZ - ZZ) = 0.$$ 

If $Z \in \mathfrak{g}$ is a left-invariant vector field on $G$ and $\phi : U \to \mathbb{C}$ is a local complex-valued function on $G$ then the first and second order derivatives satisfy

$$Z(\phi)(p) = \left. \frac{d}{ds} (\phi(p \cdot \exp(sZ))) \right|_{s=0},$$

$$Z^2(\phi)(p) = \left. \frac{d^2}{ds^2} (\phi(p \cdot \exp(sZ))) \right|_{s=0}.$$

This implies that the tension field $\tau$ and the conformality operator $\kappa$ on $G$ fulfill

$$\tau(\phi) = \sum_{Z \in B_{\mathfrak{g}}} g(Z, Z) \cdot (Z^2(\phi) - \nabla_Z Z(\phi))$$

$$\kappa(\phi, \psi) = \sum_{Z \in B_{\mathfrak{g}}} g(Z, Z) \cdot Z(\phi) \cdot Z(\psi),$$

where $B_{\mathfrak{g}}$ is any orthonormal basis for the Lie algebra $\mathfrak{g}$.

The restriction of the semi-Riemannian metric $g$ on $\text{GL}_n(\mathbb{C})$ to its maximal compact subgroup $U(n)$ is its standard Riemannian metric. For this we have the following result, see Lemma 5.1 of [10].

**Lemma 5.1.** Let $z_{j\alpha} : U(n) \to \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the unitary group $U(n)$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $U(n)$ satisfy the following relations

$$\tau(z_{j\alpha}) = -n \cdot z_{j\alpha},$$

$$\kappa(z_{j\alpha}, z_{k\beta}) = -z_{k\alpha} \cdot z_{j\beta}.$$

With this at hand we yield the following statement.

**Proposition 5.2.** Let $z_{j\alpha} : \text{GL}_n(\mathbb{C}) \to \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the general linear group $\text{GL}_n(\mathbb{C})$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $\text{GL}_n(\mathbb{C})$ fulfill the following relations

$$\tau(z_{j\alpha}) = -2n \cdot z_{j\alpha},$$

$$\kappa(z_{j\alpha}, z_{k\beta}) = -2z_{k\alpha} \cdot z_{j\beta}.$$
\[ \kappa(z_{j\alpha}, z_{k\beta}) = -2 \cdot z_{k\alpha} \cdot z_{j\beta}. \]

**Proof.** This is an immediate consequence of Lemma 5.1 and how the semi-Riemannian metric is defined on the complex Lie algebra

\[ \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{u}(n) \oplus i \cdot \mathfrak{u}(n). \]

\[ \square \]

**Theorem 5.3.** Let \( v \) be a non-zero element of \( \mathbb{C}^n \). Then the complex \( n \)-dimensional vector space

\[ E_v = \{ \phi_a : \text{GL}_n(\mathbb{C}) \to \mathbb{C} \mid \phi_a(z) = \text{trace}(v^taz^t), \ a \in \mathbb{C}^n \} \]

is an eigenfamily on \( \text{GL}_n(\mathbb{C}) \) such that for all \( \phi, \psi \in E_v \) we have

\[ \tau(\phi) = -2n \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = -2 \cdot \phi \cdot \psi. \]

**Proof.** This result can be proven in exactly the same way as Theorem 5.2 presented in [10]. \( \square \)

## 6. The Semi-Riemannian Lie Group \( \text{GL}_n(\mathbb{R}) \)

The real general linear group \( \text{GL}_n(\mathbb{R}) \) of invertible \( n \times n \) matrices is given by

\[ \text{GL}_n(\mathbb{R}) = \{ x \in \mathbb{R}^{n \times n} \mid \det x \neq 0 \}. \]

The Lie algebra \( \mathfrak{gl}_n(\mathbb{C}) \) of the complex general linear group \( \text{GL}_n(\mathbb{C}) \) has a natural orthogonal decomposition

\[ \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{R}) \oplus i \cdot \mathfrak{gl}_n(\mathbb{R}), \]

where \( \mathfrak{gl}_n(\mathbb{R}) \) is the Lie algebra of \( \text{GL}_n(\mathbb{R}) \) consisting of the real \( n \times n \) matrices.

**Proposition 6.1.** Let \( x_{j\alpha} : \text{GL}_n(\mathbb{R}) \to \mathbb{R} \) be the real-valued matrix elements of the standard representation of the general linear group \( \text{GL}_n(\mathbb{R}) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( \text{GL}_n(\mathbb{R}) \) satisfy the following relations

\[ \tau(x_{j\alpha}) = -n \cdot x_{j\alpha}, \]

\[ \kappa(x_{j\alpha}, x_{k\beta}) = -x_{k\alpha} \cdot x_{j\beta}. \]

**Proof.** This is an immediate consequence of Proposition 5.2 and how the semi-Riemannian metric is defined on the complex Lie algebra

\[ \mathfrak{gl}_n(\mathbb{C}) = \mathfrak{gl}_n(\mathbb{R}) \oplus i \cdot \mathfrak{gl}_n(\mathbb{R}). \]

\[ \square \]

As an immediate consequence of Proposition 6.1 we have the following result.
Theorem 6.2. Let \( v \) be a non-zero element of \( \mathbb{C}^n \). Then the complex \( n \)-dimensional vector space

\[
\mathcal{E}_v = \{ \phi_a : \text{GL}_n(\mathbb{R}) \to \mathbb{C} \mid \phi_a(x) = \text{trace}(v^t a x^t), \ a \in \mathbb{C}^n \}
\]

is an eigenfamily on \( \text{GL}_n(\mathbb{R}) \) such that for all \( \phi, \psi \in \mathcal{E}_v \) we have

\[
\tau(\phi) = -n \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = -\phi \cdot \psi.
\]

Proof. This is a direct consequence of Proposition 6.1 and Theorem 5.3 \( \square \)

7. The Semi-Riemannian Lie Group \( \text{GL}_n(\mathbb{H}) \)

In this section we consider the quaternionic general linear group \( \text{GL}_n(\mathbb{H}) \). Its standard complex representation \( \pi: \text{GL}_n(\mathbb{H}) \to \text{GL}_{2n}(\mathbb{C}) \) is given by

\[
\pi: (z + jw) \mapsto g = \begin{bmatrix}
z_{11} & \cdots & z_{1n} & w_{11} & \cdots & w_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{n1} & \cdots & z_{nn} & w_{n1} & \cdots & w_{nn} \\
-w_{11} & \cdots & -w_{1n} & \bar{z}_{11} & \cdots & \bar{z}_{1n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-w_{n1} & \cdots & -w_{nn} & \bar{z}_{n1} & \cdots & \bar{z}_{nn}
\end{bmatrix}.
\]

The Lie algebra \( \mathfrak{g}l_n(\mathbb{H}) \) of \( \text{GL}_n(\mathbb{H}) \) clearly satisfies

\[
\mathfrak{g}l_n(\mathbb{H}) = \left\{ \begin{bmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{bmatrix} \mid Z, W \in \mathfrak{g}l_n(\mathbb{C}) \right\}.
\]

As a subgroup of \( \text{GL}_{2n}(\mathbb{C}) \) the quaternionic general linear group \( \text{GL}_n(\mathbb{H}) \) inherits its natural semi-Riemannian metric \( g \) induced by the semi-Euclidean inner product \( \mathfrak{g}l_n(\mathbb{H}) \times \mathfrak{g}l_n(\mathbb{H}) \to \mathfrak{g}l_n(\mathbb{H}) \) on \( \mathfrak{g}l_n(\mathbb{H}) \) given by

\[
g(Z, W) = -9 \text{Re} \text{trace}(Z \cdot W).
\]

For the Lie algebra \( \mathfrak{g}l_n(\mathbb{H}) \) we have the orthogonal splitting

\[
\mathfrak{g}l_n(\mathbb{H}) = \mathfrak{g}l_n^+(\mathbb{H}) \oplus \mathfrak{g}l_n^-(\mathbb{H}),
\]

where

\[
\mathfrak{g}l_n^+(\mathbb{H}) = \mathfrak{sp}(n) = \left\{ \begin{bmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{bmatrix} \in \mathbb{C}^{2n \times 2n} \mid Z + Z^t = 0, W - W^t = 0 \right\}.
\]

By \( \mathcal{B}^+ \) we denote the following orthonormal basis for the Lie algebra \( \mathfrak{sp}(n) \) of the quaternionic unitary group \( \mathfrak{sp}(n) \) which is the maximal compact subgroup of \( \text{GL}_n(\mathbb{H}) \). This satisfies

\[
\mathcal{B}^+ = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & iX_r \\ iX_r & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & X_r \\ -X_r & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} iX_r & 0 \\ 0 & -iX_r \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} Y_r & 0 \\ 0 & Y_r \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & D_t \\ -D_t & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} iD_t & 0 \\ 0 & -iD_t \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & iD_t \\ iD_t & 0 \end{bmatrix} \mid 1 \leq r < s \leq n, 1 \leq t \leq n \right\}.
\]
For the orthogonal complement \( \mathfrak{gl}_n^- (\mathbb{H}) \) of \( \mathfrak{sp}(n) \) in \( \mathfrak{gl}_n (\mathbb{H}) \) we have the orthonormal basis
\[
B^- = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & Y_{rs} \\ -Y_{rs} & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & iY_{rs} \\ iY_{rs} & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} iY_{rs} & 0 \\ 0 & -iY_{rs} \end{bmatrix}, \right.
\[
\frac{1}{\sqrt{2}} \begin{bmatrix} X_{rs} & 0 \\ 0 & X_{rs} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} D_t & 0 \\ 0 & D_t \end{bmatrix} \mid 1 \leq r, s \leq n, 1 \leq t \leq n \right\}.
\]
Then \( B = B^+ \cup B^- \) is an orthonormal basis for \( \mathfrak{gl}_n (\mathbb{H}) \) such that \( g(Z, Z) = 1 \) if \( Z \in B^+ \) and \( g(Z, Z) = -1 \) if \( Z \in B^- \).

With this at our disposal we can now prove the following statement.

**Proposition 7.1.** Let \( z_{\alpha}, w_{\alpha} : GL_n (\mathbb{H}) \to \mathbb{C} \) be the complex-valued matrix elements of the standard representation of the quaternionic general linear group \( GL_n (\mathbb{H}) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( GL_n (\mathbb{H}) \) satisfy the following relations
\[
\tau(z_{\alpha}) = -2n \cdot z_{\alpha}, \quad \tau(w_{\alpha}) = -2n \cdot w_{\alpha},
\]
\[
\kappa(z_{\alpha}, z_{\beta}) = -z_{\alpha \beta} \cdot z_{\beta \alpha}, \quad \kappa(w_{\alpha}, w_{\beta}) = -w_{\alpha \beta} \cdot w_{\beta \alpha},
\]
\[
\kappa(z_{\alpha}, w_{\beta}) = -z_{\alpha \beta} \cdot w_{\beta \alpha}.
\]

**Proof.** For the tension field \( \tau \) on \( GL_n (\mathbb{H}) \) we have
\[
\tau(z_{\alpha}) = \sum_{Z \in B} g(Z, Z) \cdot Z^2(z_{\alpha})
\]
\[
= \sum_{Z \in B^+} Z^2(z_{\alpha}) - \sum_{Z \in B^-} Z^2(z_{\alpha})
\]
\[
= \frac{1}{2} e^j \left\{ 3 \sum_{r<s} \begin{bmatrix} X_{rs}^2 & 0 \\ 0 & -X_{rs}^2 \end{bmatrix} + \sum_{r<s} \begin{bmatrix} Y_{rs}^2 & 0 \\ 0 & -Y_{rs}^2 \end{bmatrix} \right.
\]
\[
+3 \sum_{t=1}^n \begin{bmatrix} -D_t^2 & 0 \\ 0 & -D_t^2 \end{bmatrix} \right\} e^t
\]
\[
- \frac{1}{2} e^j \left\{ \sum_{r<s} \begin{bmatrix} X_{rs}^2 & 0 \\ 0 & X_{rs}^2 \end{bmatrix} + \sum_{r<s} \begin{bmatrix} Y_{rs}^2 & 0 \\ 0 & -Y_{rs}^2 \end{bmatrix} \right.
\]
\[
+3 \sum_{t=1}^n \begin{bmatrix} D_t^2 & 0 \\ 0 & D_t^2 \end{bmatrix} \right\} e^t
\]
\[
= \frac{1}{2} e^j \left\{ -4 \sum_{r<s} \begin{bmatrix} X_{rs}^2 & 0 \\ 0 & X_{rs}^2 \end{bmatrix} + 4 \sum_{r<s} \begin{bmatrix} Y_{rs}^2 & 0 \\ 0 & -Y_{rs}^2 \end{bmatrix} \right.
\]
\[
-4 \sum_{t=1}^n \begin{bmatrix} D_t^2 & 0 \\ 0 & D_t^2 \end{bmatrix} \right\} e^t
\]
\[
= -2 e^j \begin{bmatrix} n \cdot I_n & 0 \\ 0 & n \cdot I_n \end{bmatrix} e^t
\]
\[
= -2n \cdot z_{\alpha}.
\]
For the conformal operation $\kappa$ on $\text{GL}_n(\mathbb{H})$ we similarly we yield

$$
\kappa(z_{j\alpha}, z_{k\beta}) = \sum_{Z \in B} g(Z, Z) \cdot Z(z_{j\alpha}) \cdot Z(z_{k\beta})
$$

$$
= \sum_{Z \in B^+} Z(z_{j\alpha}) \cdot Z(z_{k\beta}) - \sum_{Z \in B^-} Z(z_{j\alpha}) \cdot Z(z_{k\beta})
$$

$$
= e_j z \left\{ \sum_{Z \in B^+} Z \left[ \begin{array}{cc} E_{\alpha\beta} & 0 \\ 0 & 0 \end{array} \right] Z^t - \sum_{Z \in B^-} Z \left[ \begin{array}{cc} E_{\alpha\beta} & 0 \\ 0 & 0 \end{array} \right] Z^t \right\} z^t e_k^t
$$

$$
= e_j z \left\{ - \sum_{r<s} \left[ X_{rs} E_{\alpha\beta} X_{rs} \right] 0 + \sum_{r<s} \left[ Y_{rs} E_{\alpha\beta} Y_{rs} \right] 0 \right. 

\left. - \sum_{t=1}^n \left[ D_t E_{\alpha\beta} D_t \right] 0 \right\} z^t e_k^t
$$

$$
= -e_j z (E_{\beta\alpha}) z^t e_k^t
$$

$$
= -z_{k\alpha} \cdot z_{j\beta}.
$$

The other identities can be proven in exactly the same way. □

**Theorem 7.2.** Let $u, v$ be a non-zero elements of $\mathbb{C}^n$. Then the complex $2n$-dimensional vector space

$$
\mathcal{E}_{uv} = \{ \phi_{ab} : \text{GL}_n(\mathbb{H}) \rightarrow \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(u^t a z^t + v^t b w^t), \ a, b \in \mathbb{C}^n \}
$$

is an eigenfamily on $\text{GL}_n(\mathbb{H})$ such that for all $\phi, \psi \in \mathcal{E}_{uv}$ we have

$$
\tau(\phi) = -2n \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = -\phi \cdot \psi.
$$

8. The Semi-Riemannian Lie group $\text{SL}_n(\mathbb{C})$

In this section we construct eigenfamilies on the semisimple non-compact complex special linear group $\text{SL}_n(\mathbb{C}) = \{ z \in \text{GL}_n(\mathbb{C}) \mid \det z = 1 \}$ equipped with its semi-Riemannian metric inherited from $\text{GL}_n(\mathbb{C})$. For the Lie algebra

$$
\mathfrak{sl}_n(\mathbb{C}) = \{ Z \in \mathfrak{gl}_n(\mathbb{C}) \mid \text{trace } Z = 0 \}
$$

we have the orthogonal decomposition $\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{su}(n) \oplus i \cdot \mathfrak{su}(n)$. Here

$$
\mathfrak{su}(n) = \{ Z \in \mathfrak{u}(n) \mid \text{trace } Z = 0 \}
$$

is the Lie algebra of the special unitary group

$$
\text{SU}(n) = \{ z \in \mathfrak{U}(n) \mid \det z = 1 \}
$$

which is the maximal compact subgroup of $\text{SL}_n(\mathbb{C})$.

**Lemma 8.1.** Let $z_{j\alpha} : \text{SU}(n) \rightarrow \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the special unitary group $\text{SU}(n)$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $\text{SU}(n)$ satisfy the following relations

$$
\tau(z_{j\alpha}) = -\frac{(n^2 - 1)}{n} \cdot z_{j\alpha},
$$

$$
\kappa(z_{j\alpha}, z_{k\beta}) = \sum_{Z \in B} g(Z, Z) \cdot Z(z_{j\alpha}) \cdot Z(z_{k\beta})
$$

$$
= \sum_{Z \in B^+} Z(z_{j\alpha}) \cdot Z(z_{k\beta}) - \sum_{Z \in B^-} Z(z_{j\alpha}) \cdot Z(z_{k\beta})
$$

$$
= e_j z \left\{ \sum_{Z \in B^+} Z \left[ \begin{array}{cc} E_{\alpha\beta} & 0 \\ 0 & 0 \end{array} \right] Z^t - \sum_{Z \in B^-} Z \left[ \begin{array}{cc} E_{\alpha\beta} & 0 \\ 0 & 0 \end{array} \right] Z^t \right\} z^t e_k^t
$$

$$
= e_j z \left\{ - \sum_{r<s} \left[ X_{rs} E_{\alpha\beta} X_{rs} \right] 0 + \sum_{r<s} \left[ Y_{rs} E_{\alpha\beta} Y_{rs} \right] 0 \right. 

\left. - \sum_{t=1}^n \left[ D_t E_{\alpha\beta} D_t \right] 0 \right\} z^t e_k^t
$$

$$
= -e_j z (E_{\beta\alpha}) z^t e_k^t
$$

$$
= -z_{k\alpha} \cdot z_{j\beta}.
$$
\[ \kappa(z_{j\alpha}, z_{k\beta}) = -(z_{k\alpha} \cdot z_{j\beta} - \frac{1}{n} \cdot z_{j\alpha} \cdot z_{k\beta}). \]

Proof. For the Lie algebra \( u(n) \) of the unitary group \( U(n) \) we have the orthogonal splitting
\[ u(n) = su(n) \oplus 1, \]
where \( I \) is the real line generated by the unit vector \( E_n = i I_n/\sqrt{n} \). Hence the tension field \( \hat{\tau} \) on the unitary group \( U(n) \) satisfies
\[ \hat{\tau}(\phi) = \tau(\phi) + E_n^2(\phi), \]
so we have
\[ \tau(z_{j\alpha}) = \hat{\tau}(z_{j\alpha}) - E_n^2(z_{j\alpha}) = -(n \cdot z_{j\alpha} - \frac{1}{n} \cdot z_{j\alpha}) = -\left(\frac{n^2 - 1}{n}\right) \cdot z_{j\alpha}. \]

For the conformality operator \( \hat{\kappa} \) on \( U(n) \) we similarly yield
\[ \hat{\kappa}(\phi, \psi) = \kappa(\phi, \psi) + E_n(\phi) \cdot E_n(\psi). \]
Hence
\[ \kappa(z_{j\alpha}, z_{k\beta}) = \hat{\kappa}(z_{j\alpha}, z_{k\beta}) - E_n(z_{j\alpha}) \cdot E_n(z_{k\beta}) \]
\[ = -(z_{k\alpha} \cdot z_{j\beta} - \frac{1}{n} \cdot z_{j\alpha} \cdot z_{k\beta}). \]

\[ \quad \square \]

For the special linear group \( SL_n(\mathbb{C}) \) we have the following statement.

**Proposition 8.2.** Let \( z_{j\alpha} : SL_n(\mathbb{C}) \to \mathbb{C} \) be the complex-valued matrix elements of the standard representation of the special linear group \( SL_n(\mathbb{C}) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( SL_n(\mathbb{C}) \) satisfy the following relations
\[ \tau(z_{j\alpha}) = -\frac{2(n^2 - 1)}{n} \cdot z_{j\alpha}, \]
\[ \kappa(z_{j\alpha}, z_{k\beta}) = -2(z_{k\alpha} \cdot z_{j\beta} - \frac{1}{n} \cdot z_{j\alpha} \cdot z_{k\beta}). \]

Proof. This is an immediate consequence of Lemma \ref{lem:8.1} and how the semi-Riemannian metric is defined on the complex Lie algebra \( sl_n(\mathbb{C}) = su(n) \oplus i \cdot su(n) \).

\[ \quad \square \]

Let \( P, Q : SL_n(\mathbb{C}) \to \mathbb{C} \) be homogeneous polynomials of the matrix elements \( z_{j\alpha} : SL_n(\mathbb{C}) \to \mathbb{C} \) of degree one i.e. of the form
\[ P(z) = \text{trace}(A \cdot z^t) = \sum_{j,\alpha=1}^{n} a_{j\alpha} z_{j\alpha}, \]
\[ Q(z) = \text{trace}(B \cdot z^t) = \sum_{k,\beta}^{n} b_{k\beta} z_{k\beta} \]
for some \( A, B \in \mathbb{C}^{n \times n} \). As a direct consequence of Proposition \ref{prop:8.2} we see that
\[ \tau(P) = -\frac{2(n^2 - 1)}{n} \cdot P, \quad \tau(Q) = -\frac{2(n^2 - 1)}{n} \cdot Q. \]
\[ \kappa(P, Q) + \frac{2(n-1)}{n} \cdot P \cdot Q = \sum_{j, \alpha, k, \beta = 1}^{n} a_{ja}b_{k\beta} \kappa(z_{ja}, z_{k\beta}) + \frac{2(n-1)}{n} \sum_{j, \alpha, k, \beta = 1}^{n} a_{ja}b_{k\beta} z_{ja}z_{k\beta} \]

\[ = -2 \sum_{j, \alpha, k, \beta = 1}^{n} a_{ja}b_{k\beta} z_{j\alpha}z_{k\alpha} + \frac{2}{n} \sum_{j, \alpha, k, \beta = 1}^{n} a_{ja}b_{k\beta} z_{ja}z_{k\beta} \]

\[ + \frac{2(n-1)}{n} \sum_{j, \alpha, k, \beta = 1}^{n} a_{ja}b_{k\beta} z_{ja}z_{k\beta} \]

\[ = 2 \sum_{j, \alpha, k, \beta = 1}^{n} (a_{ja}b_{k\beta} z_{ja}z_{k\beta} - a_{ja}b_{k\beta} z_{j\alpha}z_{k\alpha}) \]

\[ = 2 \sum_{j, \alpha, k, \beta = 1}^{n} (a_{ja}b_{k\beta} - a_{ka}b_{j\beta}) z_{ja}z_{k\beta}. \]

**Theorem 8.3.** Let \( v \) be a non-zero element of \( \mathbb{C}^n \). Then the complex \( n \)-dimensional vector space

\[ E_v = \{ \phi_a : SL_n(\mathbb{C}) \to \mathbb{C} \mid \phi_a(z) = \text{trace}(v^taz^t), a \in \mathbb{C}^n \} \]

is an eigenfamily on \( SL_n(\mathbb{C}) \) such that for all \( \phi, \psi \in E_v \) we have

\[ \tau(\phi) = -\frac{2(n^2 - 1)}{n} \cdot \phi, \quad \kappa(\phi, \psi) = -\frac{2(n-1)}{n} \cdot \phi \cdot \psi. \]

**Proof.** Assume that \( a, b \in \mathbb{C}^n \) and define \( A = v^ta \) and \( B = v^tb \). By construction any two columns of the matrices \( A \) and \( B \) are linearly dependent. This means that for all \( 1 \leq j, \alpha, k, \beta \leq n \)

\[ \det \begin{bmatrix} a_{ja} & b_{j\beta} \\ a_{ka} & b_{k\beta} \end{bmatrix} = a_{ja}b_{k\beta} - a_{ka}b_{j\beta} = 0. \]

The statement now follows from the calculation above. \( \square \)

**9. The Semi-Riemannian Lie Group \( SL_n(\mathbb{R}) \)**

In this section we construct eigenfamilies on the semisimple non-compact special linear group \( SL_n(\mathbb{R}) \) equipped with its semi-Riemannian metric inherited from \( GL_n(\mathbb{C}) \). The special linear group \( SL_n(\mathbb{R}) \) is the subgroup of \( GL_n(\mathbb{R}) \) satisfying

\[ SL_n(\mathbb{R}) = \{ x \in GL_n(\mathbb{R}) \mid \det x = 1 \} \]

with Lie algebra

\[ sl_n(\mathbb{R}) = \{ X \in gl_n(\mathbb{R}) \mid \text{trace } X = 0 \}. \]
**Proposition 9.1.** Let $x_{j\alpha} : \text{SL}_n(\mathbb{R}) \to \mathbb{R}$ be the real-valued matrix elements of the standard representation of the special linear group $\text{SL}_n(\mathbb{R})$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $\text{SL}_n(\mathbb{R})$ satisfy the following relations

$$
\tau(x_{j\alpha}) = -\frac{(n^2 - 1)}{n} \cdot x_{j\alpha},
$$

$$
\kappa(x_{j\alpha}, x_{k\beta}) = -(x_{j\beta} \cdot x_{k\alpha} - \frac{1}{n} \cdot x_{j\alpha} \cdot x_{k\beta}).
$$

**Proof.** The Lie algebra $\mathfrak{sl}_n(\mathbb{C})$ of the complex special linear group $\text{SL}_n(\mathbb{C})$ is the complexification of $\mathfrak{sl}_n(\mathbb{R})$ and we have the orthogonal decomposition

$$
\mathfrak{sl}_n(\mathbb{C}) = \mathfrak{sl}_n(\mathbb{R}) \oplus i \cdot \mathfrak{sl}_n(\mathbb{R}).
$$

Hence the statement is an immediate consequence of Lemma 8.2. \qed

The next result is a direct consequence of Theorem 8.3, Propositions 8.2 and 9.1.

**Theorem 9.2.** Let $v$ be a non-zero element of $\mathbb{C}^n$. Then the complex $n$-dimensional vector space

$$
E_v = \{ \phi_a : \text{SL}_n(\mathbb{R}) \to \mathbb{C} | \phi_a(x) = \text{trace}(v^t a x^t), a \in \mathbb{C}^n \}
$$

is an eigenfamily on $\text{SL}_n(\mathbb{R})$ such that for all $\phi, \psi \in E_v$ we have

$$
\tau(\phi) = -\frac{(n^2 - 1)}{n} \cdot \phi, \quad \kappa(\phi, \psi) = -\frac{(n - 1)}{n} \cdot \phi \cdot \psi.
$$

10. **The Semi-Riemannian Lie Group $\text{SL}_n(\mathbb{H}) \cong \text{SU}^*(2n)$**

In this section we construct eigenfamilies on the semisimple non-compact quaternionic special linear group $\text{SL}_n(\mathbb{H})$. This can be realised as

$$
\text{SU}^*(2n) = \left\{ \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \in \text{GL}_{2n}(\mathbb{C}) \mid (z + jw) \in \text{SL}_n(\mathbb{H}) \right\},
$$

with Lie algebra

$$
\text{su}^*(2n) = \left\{ \begin{bmatrix} Z & W \\ -W & Z \end{bmatrix} \in \mathfrak{sl}_n(\mathbb{H}) \mid \text{Re trace } Z = 0 \right\}.
$$

For the Lie algebra $\mathfrak{gl}_n(\mathbb{H})$ of $\text{GL}_n(\mathbb{H})$ we have the orthogonal decomposition

$$
\mathfrak{gl}_n(\mathbb{H}) = \text{su}^*(2n) \oplus \mathfrak{l}
$$

where $\mathfrak{l}$ is the real line in $\mathfrak{gl}_n(\mathbb{H})$ generated by the unit vector $E_{2n} = I_{2n}/\sqrt{2n}$.

**Proposition 10.1.** Let $z_{j\alpha}, w_{k\beta} : \text{SU}^*(2n) \to \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the Lie group $\text{SU}^*(2n)$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $\text{SU}^*(2n)$ satisfy the following relations

$$
\tau(z_{j\alpha}) = -\frac{(4n^2 - 1)}{2n} \cdot z_{j\alpha}, \quad \tau(w_{j\alpha}) = -\frac{(4n^2 - 1)}{2n} \cdot w_{j\alpha},
$$

$$
\kappa(z_{j\alpha}, z_{k\beta}) = -\frac{(4n - 1)}{n} \cdot z_{j\beta} \cdot z_{k\alpha} - \frac{1}{n} \cdot z_{j\alpha} \cdot z_{k\beta}.
$$
Lie group | Eigenfunctions $\phi$ | $\lambda$ | $\mu$ | Conditions
--- | --- | --- | --- | ---
$\text{GL}_n(\mathbb{C})$ | $\text{trace}(v^taz^t)$ | $-2n$ | $-2$ | $a \in \mathbb{C}^n$
$\text{GL}_n(\mathbb{R})$ | $\text{trace}(v^tax^t)$ | $-n$ | $-1$ | $a \in \mathbb{C}^n$
$\text{GL}_n(\mathbb{H})$ | $\text{trace}(u^taz^t + v^tbw^t)$ | $-2n$ | $-1$ | $a, b \in \mathbb{C}^n$
$\text{SL}_n(\mathbb{C})$ | $\text{trace}(v^taz^t)$ | $-\frac{2(n^2-1)}{n}$ | $-\frac{2(a-1)}{n}$ | $a \in \mathbb{C}^n$
$\text{SL}_n(\mathbb{R})$ | $\text{trace}(v^tax^t)$ | $-\frac{(n^2-1)}{n}$ | $-\frac{(n-1)}{n}$ | $a \in \mathbb{C}^n$
$\text{SL}_n(\mathbb{H})$ | $\text{trace}(u^taz^t + v^tbw^t)$ | $-\frac{(4n^2-1)}{2n}$ | $-\frac{(2n-1)}{2n}$ | $a, b \in \mathbb{C}^n$

Table 1. Eigenfunctions on classical non-compact Lie groups.

\[
\kappa(z_{j\alpha}, z_{k\beta}) = -(z_{k\alpha} \cdot z_{j\beta} - \frac{1}{2n} \cdot z_{j\alpha} \cdot z_{k\beta}),
\]
\[
\kappa(z_{j\alpha}, w_{k\beta}) = -(z_{k\alpha} \cdot w_{j\beta} - \frac{1}{2n} \cdot z_{j\alpha} \cdot w_{k\beta}),
\]
\[
\kappa(w_{j\alpha}, w_{k\beta}) = -(w_{k\alpha} \cdot w_{j\beta} - \frac{1}{2n} \cdot w_{j\alpha} \cdot w_{k\beta}).
\]

**Proof.** Let $\hat{\tau}$ and $\hat{\kappa}$ denote the tension field and the conformality operator on $\text{GL}_n(\mathbb{H})$, respectively. Then it follows from Proposition 7.1 and the orthogonal decomposition $\mathfrak{g}_{n}(\mathbb{H}) = \mathfrak{su}^*(2n) \oplus \mathbb{R}$ that
\[
\tau(z_{j\alpha}) = \hat{\tau}(z_{j\alpha}) + E_n^2(z_{j\alpha})
\]
\[
= -2n \cdot z_{j\alpha} + \frac{1}{2n} \cdot z_{j\alpha}
\]
\[
= -\frac{(4n^2-1)}{2n} \cdot z_{j\alpha}.
\]
Similarly, we have
\[
\kappa(z_{j\alpha}, z_{k\beta}) = \hat{\kappa}(z_{j\alpha}, z_{k\beta}) + E_n(z_{j\alpha}) \cdot E_n(z_{k\beta})
\]
\[
= -(z_{k\alpha} \cdot z_{j\beta} - \frac{1}{2n} \cdot z_{j\alpha} \cdot z_{k\beta}).
\]

$\square$
Theorem 10.2. Let $u, v$ be a non-zero elements of $\mathbb{C}^n$. Then the complex $2n$-dimensional vector space

$$\mathcal{E}_{uv} = \{ \phi_{ab} : SU^*(2n) \to \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(u^taz^t + v^tbw^t), a, b \in \mathbb{C}^n \}$$

is an eigenfamily on $SU^*(2n)$ such that for all $\phi, \psi \in \mathcal{E}_{uv}$ we have

$$\tau(\phi) = -\frac{(4n^2 - 1)}{2n} \cdot \phi \quad \text{and} \quad \kappa(\phi, \psi) = -\frac{(2n - 1)}{2n} \cdot \phi \cdot \psi.$$ 

Proof. Here the statement can be proven in the exactly the same way as that of Theorem 8.3. \hfill \Box

11. The Semi-Riemannian Lie Group $SO(n, \mathbb{C})$

The semisimple complex special orthogonal group $SO(n, \mathbb{C})$ is the subgroup of $GL_n(\mathbb{C})$ defined by

$$SO(n, \mathbb{C}) = \{ z \in SL_n(\mathbb{C}) \mid z \cdot z^t = I_n \}.$$ 

Its Lie algebra

$$so(n, \mathbb{C}) = \{ Z \in gl_n(\mathbb{C}) \mid Z + Z^t = 0 \}$$

has the orthogonal decomposition

$$so(n, \mathbb{C}) = so(n) \oplus i \cdot so(n),$$

where $so(n)$ is the Lie algebra of the special orthogonal group $SO(n)$ consisting of the real skew-symmetric $n \times n$ matrices. The restriction of the semi-Riemannian metric $g$ on $SO(n, \mathbb{C})$ to its maximal compact subgroup $SO(n)$ is its standard Riemannian metric. For this we have the following result, see Lemma 4.1 of [10].

Lemma 11.1. Let $x_{j\alpha} : SO(n) \to \mathbb{R}$ be the real-valued matrix elements of the standard representation of the special orthogonal group $SO(n)$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $SO(n)$ satisfy the following relations

$$\tau(x_{j\alpha}) = -\frac{(n - 1)}{2} \cdot x_{j\alpha},$$

$$\kappa(x_{j\alpha}, x_{k\beta}) = -\frac{1}{2} \cdot (x_{j\beta} \cdot x_{k\alpha} - \delta_{jk} \cdot \delta_{\alpha\beta}).$$

Proposition 11.2. Let $z_{j\alpha} : SO(n, \mathbb{C}) \to \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the complex special orthogonal group $SO(n, \mathbb{C})$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $SO(n, \mathbb{C})$ satisfy the following relations

$$\tau(z_{j\alpha}) = -(n - 1) \cdot z_{j\alpha},$$

$$\kappa(z_{j\alpha}, z_{k\beta}) = -(z_{j\beta} \cdot z_{k\alpha} - \delta_{jk} \cdot \delta_{\alpha\beta}).$$
Proof. This is an immediate consequence of Lemma 11.1 and the fact how the semi-Riemannian metric is defined on the complex Lie algebra
\[\mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n) \oplus i \cdot \mathfrak{so}(n).\]

\[\Box\]

Theorem 11.3. Let \(v \in \mathbb{C}^n\) be a non-zero isotropic element i.e. \((v, v) = 0\), then the complex \(n\)-dimensional vector space

\[\mathcal{E}_v = \{\phi_a : \text{SO}(n, \mathbb{C}) \to \mathbb{C} \mid \phi_a(z) = \text{trace}(v^t a z^t), \ a \in \mathbb{C}^n\}\]

is an eigenfamily on \(\text{SO}(n, \mathbb{C})\) such that for all \(\phi, \psi \in \mathcal{E}_v\) we have

\(\tau(\phi) = -(n-1) \cdot \phi, \ \kappa(\phi, \psi) = -\phi \cdot \psi.\)

Proof. The Lie algebra \(\mathfrak{so}(n, \mathbb{C})\) of the complex special linear group \(\text{SO}(n, \mathbb{C})\) is the complexification of \(\mathfrak{so}(n)\) and we have the orthogonal decomposition
\[\mathfrak{so}(n, \mathbb{C}) = \mathfrak{so}(n) \oplus i \cdot \mathfrak{so}(n).\]

Hence the statement is an immediate consequence of Theorem 4.3 of [10].

\[\Box\]

12. The Semi-Riemannian Lie Group \(\textbf{Sp}(n, \mathbb{C})\)

The semisimple complex symplectic group \(\textbf{Sp}(n, \mathbb{C})\) is the subgroup of \(\text{SL}_{2n}(\mathbb{C})\) with

\[\textbf{Sp}(n, \mathbb{C}) = \{z \in \text{SL}_{2n}(\mathbb{C}) \mid z J_n z^t = J_n\}\]

and Lie algebra

\[\mathfrak{sp}(n, \mathbb{C}) = \{Z \in \mathfrak{sl}_{2n}(\mathbb{C}) \mid Z J_n + J_n Z^t = 0\}.\]

The maximal compact subgroup of \(\textbf{Sp}(n, \mathbb{C})\) is the quaternionic unitary group

\[\textbf{Sp}(n) = \{z \in \textbf{Sp}(n, \mathbb{C}) \mid z z^t = I_{2n}\}\]

with Lie algebra

\[\mathfrak{sp}(n) = \{Z \in \mathfrak{sp}(n, \mathbb{C}) \mid Z + Z^t = 0\}.\]

For \(\mathfrak{sp}(n, \mathbb{C})\) we have the orthogonal decomposition

\[\mathfrak{sp}(n, \mathbb{C}) = \mathfrak{sp}(n) \oplus i \cdot \mathfrak{sp}(n).\]

The restriction of the semi-Riemannian metric \(g\) on \(\textbf{Sp}(n, \mathbb{C})\) to \(\textbf{Sp}(n)\) is its standard Riemannian metric. For this we have the following result, see Lemma 6.1 of [10] and Lemma 6.1 of [9].

Lemma 12.1. Let \(z_{j\alpha}, w_{j\beta} : \textbf{Sp}(n) \to \mathbb{C}\) be the complex-valued matrix elements of the standard representation of the quaternionic unitary group \(\textbf{Sp}(n)\). Then the tension field \(\tau\) and the conformality operator \(\kappa\) on \(\textbf{Sp}(n)\) satisfy the following relations

\[\tau(z_{j\alpha}) = -\frac{2n+1}{2} \cdot z_{j\alpha}, \ \tau(w_{k\beta}) = -\frac{2n+1}{2} \cdot w_{k\beta},\]

\[\kappa(z_{j\alpha}, z_{k\beta}) = -\frac{1}{2} z_{k\alpha} \cdot z_{j\beta}, \ \kappa(w_{j\alpha}, w_{k\beta}) = -\frac{1}{2} w_{k\alpha} \cdot w_{j\beta}.\]
\[ \kappa(z_{j\alpha}, w_{k\beta}) = -\frac{1}{2} \cdot z_{k\alpha} \cdot w_{j\beta}. \]  

With this at hand we then yield the following result.

**Proposition 12.2.** Let \( z_{j\alpha}, w_{j\alpha} : \text{Sp}(n, \mathbb{C}) \to \mathbb{C} \) be the complex-valued matrix elements of the standard representation of the quaternionic unitary group \( \text{Sp}(n) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( \text{Sp}(n) \) satisfy the following relations

\[
\tau(z_{j\alpha}) = -(2n + 1) \cdot z_{j\alpha}, \quad \tau(w_{k\beta}) = -(2n + 1) \cdot w_{k\beta},
\]

\[
\kappa(z_{j\alpha}, z_{k\beta}) = -z_{k\alpha} \cdot z_{j\beta}, \quad \kappa(w_{j\alpha}, w_{k\beta}) = -w_{k\alpha} \cdot w_{j\beta},
\]

\[
\kappa(z_{j\alpha}, w_{k\beta}) = -z_{k\alpha} \cdot w_{j\beta}.
\]

**Proof.** This is an immediate consequence of Lemma 12.1 and how the semi-Riemannian metric is defined on the complex Lie algebra \( \text{sp}(n, \mathbb{C}) = \text{sp}(n) \oplus i \cdot \text{sp}(n). \)

**Theorem 12.3.** Let \( u, v \in \mathbb{C}^n \) be a non-zero elements of \( \mathbb{C}^n \), then the complex \( 2n \)-dimensional vector space \( E_{uv} = \{ \phi_{ab} : \text{Sp}(n, \mathbb{C}) \to \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(u^taz^t + v^tbw^t), \ a, b \in \mathbb{C}^n \} \) is an eigenfamily on \( \text{Sp}(n, \mathbb{C}) \) such that for all \( \phi, \psi \in E_{uv} \) we have

\[
\tau(\phi) = -(2n + 1) \cdot \phi, \quad \kappa(\phi, \psi) = -\phi \cdot \psi.
\]

**Proof.** The statement is an immediate consequence of Proposition 12.2.

\[ \square \]

13. **The Semi-Riemannian Lie Group \( \text{Sp}(n, \mathbb{R}) \)**

The semisimple real symplectic group \( \text{Sp}(n, \mathbb{R}) \) is the subgroup of the complex symplectic group \( \text{Sp}(n, \mathbb{C}) \) given by

\[ \text{Sp}(n, \mathbb{R}) = \{ x \in \text{SL}_{2n}(\mathbb{R}) \mid x J_n x^t = J_n \} \]

with Lie algebra

\[ \text{sp}(n, \mathbb{R}) = \{ X \in \text{gl}_{2n}(\mathbb{R}) \mid X J_n + J_n X^t = 0 \}. \]

For the Lie algebra \( \text{sp}(n, \mathbb{C}) \) we have the orthogonal decomposition

\[ \text{sp}(n, \mathbb{C}) = \text{sp}(n, \mathbb{R}) \oplus i \cdot \text{sp}(n, \mathbb{R}). \]

**Theorem 13.1.** Let \( v \) be a non-zero element of \( \mathbb{C}^n \), then the complex \( n \)-dimensional vector space

\[ E_v = \{ \phi_a : \text{Sp}(n, \mathbb{R}) \to \mathbb{C} \mid \phi_a(x) = \text{trace}(v^t a x^t), \ a \in \mathbb{C}^n \} \]

is an eigenfamily on \( \text{Sp}(n, \mathbb{R}) \) such that for all \( \phi, \psi \in E_v \) we have

\[
\tau(\phi) = -\frac{2n + 1}{2} \cdot \phi, \quad \kappa(\phi, \psi) = -\frac{1}{2} \cdot \phi \cdot \psi.
\]

**Proof.** The result follows directly from Theorem 12.3 and the fact that \( \text{sp}(n, \mathbb{C}) = \text{sp}(n, \mathbb{R}) \oplus i \cdot \text{sp}(n, \mathbb{R}). \)

\[ \square \]
14. The Semi-Riemannian Lie Group $\text{SO}^*(2n)$

In this section we construct eigenfamilies of complex-valued functions on the semisimple non-compact Lie group

$$\text{SO}^*(2n) = \{g \in \text{SU}(n,n) \mid g \cdot I_{nn} \cdot J_n \cdot g^t = I_{nn} \cdot J_n\},$$

where

$$\text{SU}(n,n) = \{z \in \text{SL}_{2n}(\mathbb{C}) \mid z \cdot I_{n,n} \cdot z^* = I_{n,n}\}.$$

For the Lie algebra

$$\mathfrak{so}^*(2n) = \left\{ \begin{bmatrix} Z & W \\ -\bar{W} & \bar{Z} \end{bmatrix} \in \mathbb{C}^{2n \times 2n} \mid Z + Z^* = 0 \text{ and } W + W^t = 0 \right\}$$

of $\text{SO}^*(2n)$ we have the orthogonal splitting $\mathfrak{so}^*_+(2n) \oplus \mathfrak{so}^*_-(2n)$, where the subspaces $\mathfrak{so}^*_+(2n)$ and $\mathfrak{so}^*_-(2n)$ have the orthonormal basis $\mathcal{B}^+$ and $\mathcal{B}^-$, respectively, with

$$\mathcal{B}^+ = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} Y_{rs} & 0 \\ 0 & Y_{rs} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} iX_{rs} & 0 \\ 0 & -iX_{rs} \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} iD_t & 0 \\ 0 & -iD_t \end{bmatrix} \mid 1 \leq r < s \leq n \text{ and } 1 \leq t \leq n \right\}$$

and

$$\mathcal{B}^- = \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & Y_{rs} \\ -Y_{rs} & 0 \end{bmatrix}, \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & iY_{rs} \\ iY_{rs} & 0 \end{bmatrix} \mid 1 \leq r < s \leq n \right\}.$$

Here $g(Z, Z) = 1$ for all $Z \in \mathcal{B}^+$ and $g(Z, Z) = -1$ for all $Z \in \mathcal{B}^-.$

**Proposition 14.1.** Let $z_{ja}, w_{k\beta} : \text{SO}^*(2n) \to \mathbb{C}$ be the complex-valued matrix coefficients of the standard representation of $\text{SO}^*(2n).$ Then the tension field $\tau$ and the conformality operator $\kappa$ on $\text{SO}^*(2n)$ satisfy the following relations

$$\tau(z_{ja}) = -\frac{2n - 1}{2} \cdot z_{ja}, \quad \tau(w_{ja}) = -\frac{2n - 1}{2} \cdot w_{ja},$$

$$\kappa(z_{ja}, z_{k\beta}) = -\frac{1}{2} \cdot z_{ka} \cdot z_{j\beta}, \quad \kappa(w_{ja}, w_{k\beta}) = -\frac{1}{2} \cdot w_{ka} \cdot w_{j\beta},$$

$$\kappa(z_{ja}, w_{k\beta}) = -\frac{1}{2} \cdot z_{ka} \cdot w_{j\beta}.$$

**Proof.** Here we can apply exactly the same strategy as for the proof of Proposition 7.1. \qed

**Theorem 14.2.** Let $u, v \in \mathbb{C}^n$ be a non-zero elements of $\mathbb{C}^n,$ then the complex $2n$-dimensional vector space

$$\mathcal{E}_{uv} = \{ \phi_{ab} : \text{SO}^*(2n) \to \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(u^t a z^t + v^t b w^t), \ a, b \in \mathbb{C}^n \},$$

is an eigenfamily on $\text{SO}^*(2n)$ such that for all $\phi, \psi \in \mathcal{E}_{uv}$ we have

$$\tau(\phi) = -\frac{2n - 1}{2} \cdot \phi, \quad \kappa(\phi, \psi) = -\frac{1}{2} \cdot \phi \cdot \psi.$$

**Proof.** The statement is an immediate consequence of Proposition 14.1. \qed
15. The Semi-Riemannian Lie Group $SU(p,q)$

In this section we construct eigenfamilies on the non-compact semisimple Lie group

$$SU(p,q) = \{ z \in SL_{p+q}(\mathbb{C}) \mid z \cdot I_{p,q} \cdot z^* = I_{p,q} \}.$$ 

For its Lie algebra

$$su(p,q) = \{ Z \in sl_{p+q}(\mathbb{C}) \mid Z \cdot I_{p,q} + I_{p,q} \cdot Z^* = 0 \}$$

we have a natural orthogonal splitting

$$su(p,q) = \mathfrak{s}(u(p) + u(q)) \oplus i \cdot \mathfrak{m}$$

such that

$$su(p+q) = \mathfrak{s}(u(p) + u(q)) \oplus \mathfrak{m}$$

is the Lie algebra of the special orthogonal group $SU(p+q)$.

**Proposition 15.1.** Let $z_{\alpha} : SU(p,q) \rightarrow \mathbb{C}$ be the complex-valued matrix elements of the standard representation of the special unitary group $SU(p,q)$. Then the tension field $\tau$ and the conformality operator $\kappa$ on $SU(p,q)$ satisfy the following relations

$$\tau(z_{\alpha}) = - (p+q)^2 - 1 \cdot z_{\alpha},$$

$$\kappa(z_{\alpha}, z_{\beta}) = -(z_{\alpha} \cdot z_{\beta} - \frac{1}{(p+q)} \cdot z_{\alpha} \cdot z_{\beta}).$$

**Proof.** This is an immediate consequence of Lemma [8.1], the above relationship between the Lie algebras $su(p+q)$ and $su(p,q)$ and how the semi-Riemannian metric $g$ is defined on the complex Lie algebra $gl_{p+q}(\mathbb{C})$. □

**Theorem 15.2.** Let $v$ be a non-zero element of $\mathbb{C}^{p+q}$. Then the complex $(p+q)$-dimensional vector space

$$\mathcal{E}_v = \{ \phi_a : SU(p,q) \rightarrow \mathbb{C} \mid \phi_a(z) = \text{trace}(v^t a z^t), \ a \in \mathbb{C}^{p+q} \}$$

is an eigenfamily on $SU(p,q)$ such that for all $\phi, \psi \in \mathcal{E}_v$ we have

$$\tau(\phi) = - \frac{(p+q)^2 - 1}{(p+q)} \cdot \phi, \quad \kappa(\phi, \psi) = - \frac{(p+q-1)}{(p+q)} \cdot \phi \cdot \psi.$$ 

**Proof.** The statement follows directly from Proposition [15.1]. □

16. The Semi-Riemannian Lie Group $SO(p,q)$

In this section we construct eigenfamilies on the non-compact semisimple Lie group

$$SO(p,q) = \{ x \in SL_{p+q}(\mathbb{R}) \mid x \cdot I_{p,q} \cdot x^t = I_{p,q} \}.$$ 

For its Lie algebra

$$so(p,q) = \{ X \in sl_{p+q}(\mathbb{R}) \mid X \cdot I_{p,q} + I_{p,q} \cdot X^t = 0 \}$$

we have a natural orthogonal spilling

$$so(p,q) \cong (so(p) \oplus so(q)) \oplus i \cdot \mathfrak{m}$$
such that

\[ \mathfrak{so}(p+q) = (\mathfrak{so}(p) \oplus \mathfrak{so}(q)) \oplus \mathfrak{m} \]

is the Lie algebra of the special orthogonal group \( \text{SO}(p+q) \).

**Proposition 16.1.** Let \( x_{j\alpha} : \text{SO}(p,q) \to \mathbb{R} \) be the real-valued matrix elements of the standard representation of the special orthogonal group \( \text{SO}(p,q) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( \text{SO}(p,q) \) satisfy the following relations

\[
\tau(x_{j\alpha}) = -\frac{(p+q-1)}{2} \cdot x_{j\alpha},
\]

\[
\kappa(x_{j\alpha}, x_{k\beta}) = -\frac{1}{2} \cdot (x_{j\beta} \cdot x_{k\alpha} - \delta_{jk} \cdot \delta_{\alpha\beta}).
\]

**Proof.** This is an immediate consequence of Lemma 11.1, the above relationship between the Lie algebras \( \mathfrak{so}(p+q) \) and \( \mathfrak{so}(p,q) \) and how the semi-Riemannian metric \( g \) is defined on the complex Lie algebra \( \mathfrak{gl}_{p+q}(\mathbb{R}) \). \( \square \)

**Theorem 16.2.** Let \( v \in \mathbb{C}^{p+q} \) be a non-zero isotropic element i.e. \( (v,v) = 0 \), then the complex \( (p+q) \)-dimensional vector space

\[ \mathcal{E}_v = \{ \phi : \text{SO}(p,q) \to \mathbb{C} | \phi(a) = \text{trace}(v^t a z^t), a \in \mathbb{C}^{p+q} \} \]

is an eigenfamily on \( \text{SO}(p,q) \) such that for all \( \phi, \psi \in \mathcal{E}_v \) we have

\[
\tau(\phi) = -\frac{(p+q-1)}{2} \cdot \phi, \quad \kappa(\phi, \psi) = -\frac{1}{2} \cdot \phi \cdot \psi.
\]

**Proof.** The statement follows directly from Proposition 16.1. \( \square \)

17. The Semi-Riemannian Lie Group \( \text{Sp}(p,q) \)

In this section we construct eigenfamilies on the non-compact semisimple Lie group

\[ \text{Sp}(p,q) = \{ g \in \text{SL}_{p+q}(\mathbb{H}) | g \cdot I_{p,q} \cdot g^* = I_{p,q} \}. \]

For its Lie algebra

\[ \mathfrak{sp}(p,q) = \{ Z \in \mathfrak{sl}_{p+q}(\mathbb{H}) | Z \cdot I_{p,q} + I_{p,q} \cdot Z^* = 0 \} \]

we have a natural orthogonal decomposition

\[ \mathfrak{sp}(p,q) = (\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)) \oplus i \cdot \mathfrak{m} \]

such that

\[ \mathfrak{sp}(p+q) = (\mathfrak{sp}(p) \oplus \mathfrak{sp}(q)) \oplus \mathfrak{m} \]

is the Lie algebra of the quaternionic unitary group \( \text{Sp}(p,q) \).

**Proposition 17.1.** Let \( z_{j\alpha}, w_{j\alpha} : \text{Sp}(p,q) \to \mathbb{C} \) be the complex-valued matrix elements of the standard representation of the quaternionic unitary group \( \text{Sp}(p,q) \). Then the tension field \( \tau \) and the conformality operator \( \kappa \) on \( \text{Sp}(p,q) \) satisfy the following relations

\[
\tau(z_{j\alpha}) = -\frac{2(p+q) + 1}{2} \cdot z_{j\alpha}, \quad \tau(w_{j\beta}) = -\frac{2(p+q) + 1}{2} \cdot w_{j\beta},
\]
Table 2. Eigenfunctions on classical non-compact Lie groups.

| Lie group  | Eigenfunctions $\phi$ | $\lambda$  | $\mu$  | Conditions |
|------------|-----------------------|------------|--------|------------|
| $\text{SO}(n, \mathbb{C})$ | $\text{trace}(v^t az^t)$ | $-(n-1)$  | $-1$    | $a \in \mathbb{C}^n, \ (v,v) = 0$ |
| $\text{Sp}(n, \mathbb{C})$ | $\text{trace}(u^t az^t + v^t bw^t)$ | $-(2n+1)$ | $-1$    | $a, b \in \mathbb{C}^n$ |
| $\text{Sp}(n, \mathbb{R})$ | $\text{trace}(v^t ax^t)$ | $-\frac{2n+1}{2}$ | $-\frac{1}{2}$ | $a \in \mathbb{C}^n$ |
| $\text{SO}^*(2n)$ | $\text{trace}(u^t az^t + v^t bw^t)$ | $-\frac{2n-1}{2}$ | $-\frac{1}{2}$ | $a, b \in \mathbb{C}^n$ |
| $\text{SU}(p,q)$ | $\text{trace}(v^t az^t)$ | $-\frac{(p+q)^2-1}{(p+q)}$ | $-\frac{(p+q-1)}{(p+q)}$ | $a \in \mathbb{C}^n$ |
| $\text{SO}(p,q)$ | $\text{trace}(v^t ax^t)$ | $-\frac{(p+q-1)}{2}$ | $-\frac{1}{2}$ | $a \in \mathbb{C}^n$ |
| $\text{Sp}(p,q)$ | $\text{trace}(u^t az^t + v^t bw^t)$ | $-\frac{2(p+q)+1}{2}$ | $-\frac{1}{2}$ | $a, b \in \mathbb{C}^n$ |

Proof. This is an immediate consequence of Lemma 12.1, the above relationship between the Lie algebras $\mathfrak{sp}(p+q)$ and $\mathfrak{sp}(p,q)$ and how the semi-Riemannian metric $g$ is defined on the complex Lie algebra $\mathfrak{gl}(p+q, \mathbb{H})$. □

Theorem 17.2. Let $u, v \in \mathbb{C}^{p+q}$ be a non-zero element of $\mathbb{C}^{p+q}$, then the complex $(p+q)$-dimensional vector space

$$\mathcal{E}_{uv} = \{\phi_{ab} : \text{Sp}(p,q) \to \mathbb{C} \mid \phi_{ab}(g) = \text{trace}(u^t az^t + v^t bw^t), \ a, b \in \mathbb{C}^{p+q}\}$$

is an eigenfamily on $\text{Sp}(p,q)$ such that for all $\phi, \psi \in \mathcal{E}_{uv}$ we have

$$\tau(\phi) = -\frac{2(p+q)+1}{2} \cdot \phi, \ \kappa(\phi, \psi) = -\frac{1}{2} \cdot \phi \cdot \psi.$$ 

Proof. The statement follows directly from Proposition 17.1. □

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