Quintessence models of Dark Energy with non-minimal coupling

Tame Gonzalez, Genly Leon and Israel Quiros
Central University of Las Villas. Santa Clara. Cuba
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We explore quintessence models of dark energy which exhibit non-minimal coupling between the dark matter and the dark energy components of the cosmic fluid. The kind of coupling chosen is inspired in scalar-tensor theories of gravity. We impose a suitable dynamics of the expansion allowing to derive exact Friedmann-Robertson-Walker solutions once the coupling function is given as input. Coupling functions that lead to self-interaction potentials of the single and double exponential types are the target of the present investigation. The stability and existence of the solutions is discussed in some detail. The models are tested against the observational evidence.

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I. INTRODUCTION

At the time, all the available observational evidence suggest that the energy density of the universe today is dominated by a component with negative pressure of, almost, the same absolute value as its energy density. This mysterious component, that violates the strong energy condition (SEC), drives a present accelerated stage of the cosmic evolution and is called "dark energy" (DE). A pleiad of models have been investigated to account for this SEC violating source of gravity. Among them, one of the most successful models is a slowly varying scalar field, called 'quintessence' [1,2].

Many models of quintessence assume that the background and the dark energy evolve independently, so their natural generalization are models with non-minimal coupling between both components. Although experimental tests in the solar system impose severe restrictions on the possibility of non-minimal coupling between the DE and ordinary matter fluids [3], due to the unknown nature of the dark matter (DM) as part of the background, it is possible to have additional (non gravitational) interactions between the DE and the DM components, without conflict with the experimental data. In the present paper we will be concerned, precisely, with non-minimal coupling between the DM and DE components of the cosmological fluid. Since, models with non-minimal coupling imply interaction (exchange of energy) among the DM and the DE, these models provide new qualitative features for the coincidence problem [5,6]. It has been shown, in particular, that a suitable coupling, can produce scaling solutions. The way in which the coupling is approached is not unique. In reference [5], for instance, the coupling is introduced by hand. In [6] the type of coupling is not specified from the beginning. Instead, the form of the interaction term is fixed by the requirement that the ratio of the energy densities of DM and quintessence be a constant. In [7], a suitable interaction between the quintessence field and DM leads to a transition from the domination matter era to an accelerated expansion epoch in the model of Ref. [6]. The authors obtained new features for the coincidence problem. A model derived from the Dilaton is studied in [8]. In this model the coupling function is chosen as a Fourier expansion around some minimum of the scalar field.

A variety of self-interaction potentials have been studied. Among them, a single exponential is the simplest case. This type of potential leads to two possible late-time attractors in the presence of a barotropic fluid [9,10]: i) a scaling regime where the scalar field mimics the dynamics of the background fluid, i.e., the ratio between both DM and quintessence energy densities is a constant and ii) an attractor solution dominated by the scalar field. Given that single exponential potentials can lead to one of the above scaling solutions, then it should follow that a combination of exponentials should allow for a scenario where the universe can evolve through a radiation-matter regime (attractor i) and, at some recent epoch, evolve into the scalar field dominated regime (attractor ii)). Models with single and double exponential potentials has been studied also in references [11,12].

The aim of the present paper is to investigate models with non-minimal coupling among the components of the cosmic fluid, in which, single and double exponential self interaction potentials are involved. To specify the coupling, we take as a Lagrangian model, a scalar-tensor theory with Lagrangian written in Einstein frame variables. In this paper, additionally, we impose the dynamics of the expansion by exploring a linear relationship between the Hubble parameter and the time derivative of the scalar field. Using this relationship we can solve the field equations explicitly. A flat Friedmann-Robertson-Walker (FRW) universe filled with a mixture of two interacting fluids: DM and quintessence, is considered.

The paper has been organized as follow. In section 2

*It should be pointed out, however, that when the stability of dark energy potentials in quintessence models is considered, the coupling dark matter-dark energy is troublesome [4].
the details of the model are given. The method used to derive FRW solutions is explained in section 3. We use the approach of 'two fluids': a background fluid of DM and a self-interacting scalar field (quintessence). Two different couplings that lead to self-interaction potentials of single exponential and double exponential class are studied separately. In section 4 a study of existence and stability of the solutions is presented. The models studied in the former sections are tested against the observational evidence in section 5. Finally, in section 6, conclusions are given.

II. THE MODEL

We consider the following action inspired in a scalar-tensor theory written in Einstein frame, where the quintessence (scalar) field is coupled to the matter degrees of freedom:

\[ S = \int_{M_4} d^4x \sqrt{|g|} \left\{ \frac{1}{2} R - \frac{1}{2} (\nabla \phi)^2 - V(\phi) + C^2(\phi) \mathcal{L}_{\text{matter}} \right\}. \tag{1} \]

In this equation \( R \) is the curvature scalar, \( C^2(\phi) \) - the coupling function, and \( \mathcal{L}_{\text{matter}} \) is the Lagrangian density for the matter degrees of freedom.

We are concerned here with a flat FRW universe that is filled with a background pressureless dark matter and a quintessence field (the scalar field \( \phi \)). The line element is:

\[ ds^2 = -dt^2 + a(t)^2 \delta_{ij} dx^i dx^j, \tag{2} \]

where \( i,j = 1,2,3 \) and we use the system of units in which \( 8\pi G = c = h = 1 \). The field equations that are derived from the action (1) are:

\[ 3H^2 = \rho_m + \frac{1}{2} \dot{\phi}^2 + V, \tag{3} \]
\[ 2\dot{H} + 3H^2 = (1 - \gamma) \rho_m - \frac{1}{2} \ddot{\phi}^2 + V, \tag{4} \]
\[ \ddot{\phi} + 3H \dot{\phi} = -V' - (\ln X)' \rho_m, \tag{5} \]

where we have introduced the reduced notation \( X(\phi) \equiv C(\phi)^{(3\gamma - 4)/2} \). The "continuity" equations for background (DM) and the quintessence field are:

\[ \dot{\rho}_m + 3\gamma H \rho_m = -(\ln X)' \phi \rho_m, \tag{6} \]

or, after integration

\[ \rho_m = Ma^{-3\gamma} X^{-1}. \tag{7} \]

where \( M \) is a constant of integration. In the former equations the dot accounts for derivative in respect to the comoving time \( t \), while the comma denotes derivative in respect to \( \phi \).

III. DERIVING SOLUTIONS

Summing up equations (3) and (4) one obtains:

\[ \dot{H} + 3H^2 = \frac{2 - \gamma}{2} \rho_m + V. \tag{8} \]

Let us fix the dynamics of the cosmic evolution by introducing the following constrain

\[ \dot{\phi} = \lambda H, \tag{9} \]

which implies

\[ a = e^{\phi/\lambda}, \tag{10} \]

and rewriting of (7) in the form:

\[ \rho_m(\phi) = Me^{-\frac{M}{\lambda} \phi} X^{-1}(\phi). \tag{11} \]

If one substitutes Eqn.(9) in (5) and the resulting equation is compared with (8) then, one obtains a differential equation on the potential \( V \), that relates \( V \) and the coupling function \( X \) (and their derivatives):

\[ \frac{dV}{d\phi} + \lambda V(\phi) = \rho_m(\phi) \left( \frac{dX}{d\phi} - \frac{\lambda(\gamma - 2)}{2} \right). \tag{12} \]

If one realizes that \( d\phi = \lambda d(\ln a) \) (see equation (10)), then the equation (12) can be rewritten as:

\[ V' + \lambda^2 V = \left( \frac{X'}{X} - \frac{\lambda(\gamma - 2)}{2} \right) \frac{M}{a^{3\gamma} X^{-1}}, \tag{13} \]

where now the comma denotes derivative in respect to the variable \( N = \ln a \). With the constrain (9) and the equation (11) we can integrate the equation (3) in quadratures:

\[ \int \frac{d\phi}{\sqrt{Me^{-\frac{M}{\lambda} \phi} X^{-1}(\phi) + V(\phi)}} = \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} (t + t_0), \tag{14} \]

or, if one introduces the time variable \( d\tau = e^{-\frac{M}{\lambda} \phi} X^{-1/2} dt \), (14) can be written as:

\[ \int \frac{d\phi}{\sqrt{M + e^{\frac{M}{\lambda} \phi} X(\phi)V(\phi)}} = \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} (\tau + \tau_0). \tag{15} \]

In consequence, once the function \( X(\phi) \) (or \( X(a) \)) is given as input, then one can solve equation (12) (or (13)) to find the functional form of the potential \( V(\phi) \) (or \( V(a) \)). The integral (14) (or (15)) can then be taken explicitly to obtain \( t = t(\phi) \) (or \( \tau = \tau(\phi) \)). By inversion we can obtain \( \phi = \phi(t) \) (or \( \phi = \phi(\tau) \)) so, the scale factor can be given as function of either the cosmic time \( t \) or the time variable \( \tau \) through Eqn.(10).
A. Particular Cases

In this section we study separately two different choices of the coupling function $X(\phi)$ that lead to double and single exponential potentials respectively. The importance of this class of potentials in cosmology has been already outlined in the introductory part of this paper.

1. Case A

Let us consider the simplest case when
\[
\frac{d}{d\phi} \ln X(\phi) = \text{const} = \alpha \Rightarrow X(\phi) = X_0 e^{\alpha \phi},
\]
where $X_0$ is an integration constant. In this case, by integration of (12) one obtains
\[
V(\phi) = V_0 e^{-\lambda \phi} + W_0 e^{-(\alpha + 3\gamma/\lambda) \phi},
\]
where the constant
\[
W_0 = \frac{2M}{X_0} \left( \frac{2\alpha - \lambda(2 - \gamma)}{\alpha + 3\gamma/\lambda - \lambda} \right).
\]
We are faced with a self-interaction potential that is a combination of exponentials with different constants in the exponent. The usefulness of this kind of potential has been already explained in the introduction and will be briefly discussed later within the frame of the present model, when we study the stability of the corresponding solution. If one introduce the variable $z = e^{-\frac{\lambda}{2} \phi}$, the equation (15) can be written as
\[
\int \frac{dz}{\sqrt{z^2 + a^2}} = -\frac{1}{2} \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} \frac{X_0 V_0}{a^2} (\tau + \tau_0)
\]
where $a^2 = (X_0 V_0)/(M + W_0 X_0)$ and $l = \alpha + 3\gamma/\lambda - \lambda$. Integration of (19) yields to:
\[
\phi(\tau) = \phi_0 + \ln (\sinh[\mu(\tau + \tau_0)])^{-2/l},
\]
and, consequently:
\[
a(\tau) = a_0 (\sinh[\mu(\tau + \tau_0)])^{-2/l},
\]
where $\mu = -\frac{1}{2} \sqrt{\frac{2\lambda^2}{6 - \lambda^2}} \frac{X_0 V_0}{a^2}$. On the other hand, by using the Friedmann equation (3) and inserting therein the potential (17), one obtains the Hubble parameter as function of the quintessence field $\phi$:
\[
H^2(\phi) = \frac{2V_0}{6 - \lambda^2} \left[ b^2 e^{-(\alpha + 3\gamma/\lambda) \phi} + e^{-\lambda \phi} \right],
\]
where we have considered Eqn.(11). In terms of the scale factor one has, instead,
\[
H^2(a) = \frac{2V_0}{6 - \lambda^2} \left[ b^2 a^{-(\alpha + 3\gamma)} + a^{-\lambda^2} \right].
\]
The former magnitudes can be given in terms of the redshift variable if one considers that $a(z) = a_0/(1 + z)$, where $a_0 \equiv a(z = 0)$ (for simplicity of the calculations we choose the normalization $a_0 = 1$):
\[
H^2(z) = \frac{2V_0}{6 - \lambda^2} \left[ b^2 (z + 1)^{-(\alpha + 3\gamma)} + a^{-\lambda^2} \right].
\]
In redshift variable the energy density of DM is $\rho_m(z) = (M/X_0)(z + 1)^{\alpha \lambda + 3\gamma} = \rho_0 (z + 1)^{\alpha \lambda + 3\gamma}$, so the DM dimensionless density parameter can be given as function of $z$ also:
\[
\Omega_m(z) = \frac{(6 - \lambda^2)\rho_0}{2V_0 b^2} \frac{(z + 1)^{\alpha \lambda + 3\gamma} - \lambda^2}{(z + 1)^{\alpha \lambda + 3\gamma} - \lambda^2 + 1/6^2}.
\]
In order to constrain the parameter space of the solution one should consider the model to fit the observational evidence about a universe with an early matter dominated period and a former transition to dark energy dominance. Therefore $\lambda^2 - \alpha \lambda - 3\gamma < 0$, besides, one can write $\Omega_m(\infty) = (1 - \epsilon)$, where $\epsilon$ is a small number (the small fraction of dark energy component during nucleosynthesis epoch). In correspondence $b^2 = (6 - \lambda^2)\rho_0 / (z + 1)^{\alpha \lambda + 3\gamma}$. Taking into account the observational fact that, at present, $(z = 0); \Omega_m(0) = 1/3$, then $1/b^2 = 2 - 3\epsilon$. After this Eqn.(25) can be rewritten in the following way:
\[
\Omega_m(z) = (1 - \epsilon) \frac{(z + 1)^{\alpha \lambda + 3\gamma} - \lambda^2}{(z + 1)^{\alpha \lambda + 3\gamma} - \lambda^2 + (2 - 3\epsilon)}
\]
Other physical magnitudes of observational interest are the quintessence equation of state (EOS) parameter and the deceleration parameter
\[
\omega_\phi = -1 + \frac{\lambda^2/3}{1 - \Omega_m},
\]
\[
q = -1 + \frac{\lambda^2}{2} + 3\gamma \Omega_m,
\]
respectively.

2. Case B

A second very simple choice is to consider an exponential coupling function of the form:
\[
X = X_0 e^{\frac{\alpha \lambda + 3\gamma}{\lambda} \phi},
\]
In this case equation (12) simplifies to
\[
\frac{dV}{d\phi} + \lambda V = 0,
\]
which can be easily integrated to yield to a single exponential potential:
\( V = V_0 e^{-\lambda \phi}. \) \hspace{1cm} (31)

In consequence the equation (15) can be written as

\[
\int \frac{dw}{\sqrt{w^2 + A^2}} = \mu(\tau + \tau_0)
\] \hspace{1cm} (32)

where \( w = \exp\left(-\frac{2(6-\lambda^2)}{4\lambda} \phi\right), \ A^2 = V_0X_0/M \) and \( \mu = \gamma \sqrt{M(\lambda^2 - 6)/8} \), so we have the explicit solution

\[
\phi(\tau) = \phi_0 + \ln \left(\sinh \left[\mu(\tau + \tau_0)\right]^{\frac{1}{\gamma(6-\lambda^2)}}\right),
\] \hspace{1cm} (33)

and, consequently:

\[
a(\tau) = a_0 \sinh \left[\mu(\tau + \tau_0)\right]^{\frac{4\lambda}{\gamma(6-\lambda^2)}}.
\] \hspace{1cm} (34)

The dimensionless density parameter and the Hubble expansion parameter are given, as functions of the redshift, through:

\[
\Omega_m(z) = \frac{6 - \lambda^2}{6A^2} (1 + z)^{3\gamma + \gamma(6-\lambda^2)/2-\lambda^2}
\] \hspace{1cm} (35)

and

\[
H(z) = B \sqrt{\frac{1}{A^2} (1 + z)^{3\gamma + \gamma(6-\lambda^2)/2} + (1 + z)\lambda^2},
\] \hspace{1cm} (36)

respectively, where \( B = \sqrt{2V_0/(6-\lambda^2)}. \) Note that, \( \Omega_m \) is a maximum \( (\Omega_m^\infty = (6 - \lambda^2)/6A^2) \) at \( z = \infty. \) Taking into account that observations are not exact we admit that, at the epoch of nucleosynthesis \( \Omega_m(\infty) = (1 - \epsilon) \), where, as before, \( \epsilon \) is a very small number \( (\epsilon = [6(A^2 - 1) + \lambda^2]/6A^2) \). Then, if we assume (as before) that, at present, \( \Omega_m(0) = 1/3 \), then (35) can be rewritten as

\[
\Omega_m(z) = \frac{(1 - \epsilon)(1 + z)^{3\gamma + \gamma(6-\lambda^2)/2-\lambda^2}}{(1 + z)^{3\gamma + \gamma(6-\lambda^2)/2} - \lambda^2 + (2 - 3\epsilon)}.
\] \hspace{1cm} (37)

Other physical magnitudes of observational interest are the quintessence EOS parameter and the deceleration parameter, that are given in equations (27) and (28), respectively.

**IV. EXISTENCE AND STABILITY OF THE SOLUTIONS**

In this section we do not aim at a complete study of the stability in the case of the double exponential potential. Instead, we use the following simplifying fact. Note that, in terms of the scale factor, the potential (17) can be written as follows:

\[
V(a) = V_0 a^{-\lambda^2} + W_0 a^{-(\alpha\lambda + 3\gamma)},
\] \hspace{1cm} (38)

consequently, at early time (small \( a \)) the second term in the right hand side (RHS) of (38) prevails so, the first term can be neglected and we can talk about a single exponential potential with the constant \(-(\alpha + 3\gamma/\lambda)\) in the exponent (see Eqn.(17)). At late time \( (a \gg 1) \), instead, the first term in the RHS of (38) dominates, so we can talk about a single exponential with the constant \(-\lambda\) in the exponent. For this reason, and owing to the fact that in both cases (case A and case B), the coupling function is of an exponential form, we focuss (as first approximation) on the stability analysis of the model with a single exponential potential of the form:

\[
V = V_0 \exp(-\bar{a}\phi),
\] \hspace{1cm} (39)

and the following coupling function:

\[
X = X_0 \exp(b\phi).
\] \hspace{1cm} (40)

The constants \( \bar{a} \) and \( b \) depend on the model. Using the variables [9]

\[
x = \frac{\phi}{\sqrt{6H}}, \ y = \frac{\sqrt{V}}{\sqrt{3H}}
\] \hspace{1cm} (41)

the system of dynamical equations (3-5) can be arranged in the form of an autonomous dynamical system:

\[
x' = \bar{a} \sqrt{\frac{3}{2} y^2} - b \sqrt{\frac{3}{2}(1 - x^2 - y^2)}
\] \hspace{1cm} (42)

\[
y' = \frac{3}{2} y [2x^2 + \gamma(1 - x^2 - y^2)] - \bar{a} \sqrt{\frac{3}{2} y x}
\] \hspace{1cm} (43)

where, as before, the prime denotes derivative in respect to the variable \( N \equiv \ln(a) \). If in equations (42) and (43) we set \( b = 0 \), the case studied in Ref. [9] is recovered.

Using the new variables, the Friedmann equation looks like:

\[
\frac{\rho_m}{3H^2} = (1 - x^2 - y^2),
\] \hspace{1cm} (44)

otherwise:

\[
\Omega_\phi = x^2 + y^2.
\] \hspace{1cm} (45)

Since \( \rho_m \geq 0 \), the new variables are confined to the compact region (see [9]), \( 0 \leq x^2 + y^2 \leq 1 \) so, the evolution of the system under investigation is completely described by trajectories within the unit disc. The lower half-disc, \( y < 0 \), corresponds to contracting universes, but we are interest in expanding universes. On the other hand, due to the following discrete symmetries : \( (x, y) \rightarrow (x, -y) \) and \( t \rightarrow -t \), we consider the upper half-disc, \( y \geq 0 \) in the subsequent discussion [9].

The effective EOS parameter for the quintessence field at any point is given by

\[
\gamma_\phi \equiv \frac{\rho_\phi + p_\phi}{\rho_\phi} = \frac{\dot{\phi}^2}{V(\phi) + \dot{\phi}^2/2} = \frac{2x^2}{x^2 + y^2}.
\] \hspace{1cm} (46)

We are able to find up to five fixed points (critical points) where \( x' = 0 \) and \( y' = 0 \), which are listed in Table I.
| $x$ | $y$ | Existence | Stability | $\Omega_\phi$ | $\gamma_\phi$ |
|-----|-----|-----------|-----------|--------------|--------------|
| -1  | 0   | All $\bar{a}$ and $\bar{b}$ | Stable node for $\bar{a} < -\sqrt{6}$ and $\bar{b} < \sqrt{2} (\gamma - 2)$ | 1             | 2             |
|     |     |           | Unstable node for $\bar{a} < -\sqrt{6}$ and $\bar{b} < \sqrt{2} (\gamma - 2)$ |               |               |
|     |     |           | Saddle point otherwise |               |               |
| 1   | 0   | All $\bar{a}$ and $\bar{b}$ | Stable node for $\bar{a} > \sqrt{6}$ and $\bar{b} > \sqrt{2} (2 - \gamma)$ | 1             | 2             |
|     |     |           | Unstable node for $\bar{a} < \sqrt{6}$ and $\bar{b} < \sqrt{2} (2 - \gamma)$ |               |               |
|     |     |           | Saddle point otherwise |               |               |
| $\bar{a} \sqrt{6}$ | $\sqrt{1 - \frac{a^2}{a^2}}$ | All $\bar{a}$ and $\bar{b}$ | Stable node for $\bar{a} < \frac{2\sqrt{(2 - \gamma)}}{26}$ and $\bar{b} > \frac{3\sqrt{(2 - \gamma)}}{8}$ | $\frac{2\sqrt{2}}{3(\gamma - 2)^2}$ | 2             |
|     |     |           | Saddle point for $\bar{a} > \frac{3\sqrt{(2 - \gamma)}}{8}$ |               |               |
| $\frac{1}{2} \sqrt{\frac{3 - \gamma}{a - b}}$ | $\sqrt{1 - \frac{a^2}{a^2}} + \frac{3\gamma}{(a - b)^2} - \frac{3v^2}{2(a - b)^2}$ | All $\bar{a}$ and $\bar{b}$ | Stable node or Saddle point (see analysis in Table II) | $\frac{3\gamma^2 - 6\bar{b}}{(a - b)^2}$ | $\frac{3\gamma^2}{3\gamma + b^2 - 6\bar{b}}$ |

**TABLE I.** Existence and stability of the critical points.

**A. Double exponential potential**

In the case $\bar{a} = \alpha$ is fixed. Since we have used the constrain (9), then the dynamical variable $x$ is fixed to be $x = \frac{1}{\sqrt{6}}$ so, depending on $\bar{a}$, we are led with two possibilities: i) $\bar{a} = \lambda$: the point 4 in table I is an attractor quintessence dominated regime (Fig.1) and ii) $\bar{a} = \alpha + 3\gamma/\lambda$; the point 5 in table I is a matter-scaling attractor (Fig.2). Therefore, we can trace the evolution of the universe from a matter-scaling attractor in the past into a late time, DE dominated attractor.

**B. Single exponential potential**

In this case two critical points emerge. Actually, although $\bar{a}$ and $\bar{b}$ are fixed (we have a single exponential in this case): $\bar{a} = \lambda$ and $\bar{b} = \lambda/2(2 - \gamma)$, the points 3 and 4 in table I are consistent with the constrain (9). The point 3 is a scaling solution and the point 4 is a solution dominated by the scalar field.

As in the former case, the evolution of the universe transits from a scaling attractor into a scalar field dominated attractor in the future. This fact makes possible to deal with the fine tuning and the coincidence problems [5,6].
V. OBSERVATIONAL TESTING

Although we cannot talk literally about observational testing of the solutions found (we point out that the observational facts are not yet conclusive in some cases and are not accurate enough in others), in this section we shall study whether our solutions agree with some observational constrains that are more or less well established. The main observational facts we consider are the following [13]:

1.- At present \((z = 0)\) the expansion is accelerated \((q(0) < 0)\).

2.- The accelerated expansion is a relatively recent phenomenon. Observations point to a decelerated phase of the cosmic evolution at redshift \(z = 1.7\). There is agreement in that transition from decelerated into accelerated expansion occurred at a \(z \approx 0.5\) [14].

3.- The equation of state for the scalar field at present \(\omega_\phi(0) \approx -1\) (it behaves like a cosmological constant). With a 95% confidence limit \(\omega < -0.6\) [15].

4.- Although, at present, both the scalar (quintessence) field and the ordinary matter have similar contributions in the energy content of the universe \((\Omega_m(0) = 1/3 \Rightarrow \Omega_\phi(0) = 2/3)\), in the past, the ordinary matter dominated the cosmic evolution, \(^\dagger\) meanwhile, in the future, the quintessence field will dominate (it already dominates) and will, consequently, determine the destiny of the cosmic evolution.

5.- As stated before [18], nucleosynthesis predictions claims that at 95% confidence level, \(\Omega_\phi(1MeV \simeq z = 10^{10}) \leq 0.045\).

6.- During galaxy formation epoch [19], around \(z \approx 2-4\), the value of quintessence density parameter is \(\Omega_\phi < 0.5\).

Now we proceed to "observationally" test the solutions found in the cases studied in the former sections.

A. Double exponential potential

In this case the solution depends on two parameters, \(\lambda\) and \(\varepsilon\) (\(\alpha\) and \(\gamma\) are fixed: \(\gamma = 1\), meaning cold dark matter dominance at present, and \(\alpha = 1\)). To constrain the values of these parameters we use the aforementioned facts (1-6). Using these constraints we bound the parameter space \((\lambda \text{ and } \varepsilon)\) by means of a computing code. These parameters range as \(0 \leq \varepsilon \leq 0.045\) and \(0 \leq \lambda \leq 0.38\). Then we select a pair from the above interval in order to compare our model with the supernova data. We follow the procedure used in [20] to further constrain the parameters in the model.\(^5\) For instance, if we consider \(\varepsilon = 0.01\), the \(\chi^2\) distribution has a minimum in the \(m_0\) direction\(^*\) at \(m_0 = 24\); however, it has no minimum in the \(\lambda\)-direction, meaning that the parameter \(\lambda\) can take any values in the interval \(0 \leq \lambda \leq 0.38\). In the Figure 3, \(\chi^2\) is plotted as a function of the free parameters \(\lambda\) and \(m_0\) (we chose \(\varepsilon = 0.01\), \(\lambda\) could be any value in the physically meaningful range). Although the model fulfils the observational requirements (1-6) given at the beginning of this section, we point out that other observations should be considered in order to further constraint the space of parameter.

\(^{1}\)Corasaniti and Copeland [17] found that the determination of the third peak in the BOOMERANG data limits the value today of equation of state \(-1 \leq \omega_\phi(0) \leq -0.93\).

\(^{\dagger}\)A sufficiently long matter dominated decelerated phase is needed for the observed structure to develop from the density inhomogeneities [16]

\(^{5}\)In fact this procedure cannot be used to further constrain the space of parameter, since the free parameters in our model are not very sensitive to these observations.

\(^{*}\)\(m_0\) is a parameter connected to the absolute magnitude and the Hubble parameter.
FIG. 3. The function $\chi^2$ ($C$ in the figure) is plotted vs the free parameters $m$ and $\lambda$ ($L$ in the figure), for the model with double exponential potential. We chose the parameter $\varepsilon = 0.01$. Note that $\lambda$ could be any value in the physically meaningful range so, SNIa luminosity observations do not allow for further constrain of the parameter space. Other observations could be considered for this purpose.

B. Single exponential potential

In this case our solution depends on two parameters also $\lambda$ and $\varepsilon$ (recall that we fix $\gamma$ to be unity). We apply exactly the same procedure as that in the former subsection to constraint the parameter space. We found that the physically meaningful region in the parameter space is bounded by $0 \leq \varepsilon \leq 0.045$ and $0 \leq \lambda^2 \leq 0.135$.

We want to stress that, as in the former case, the free parameters of the model are not very sensitive to the SNIa data, so, the standard used, for instance in [20], is not suitable for further constraining the space of parameters of the model. Other observational evidence could be considered for this purpose.

It should be pointed out that, in two cases (A and B), the meaningful region in parameter space is chosen such that the main observational facts (1-6) explained at the beginning of the section are fulfilled. As an illustration, in Figure 4, we show the evolution of both dimensionless DM and scalar-field energy densities $\Omega_m$ and $\Omega_\phi$ respectively vs $z$ for the single exponential potential (the Case A is very similar). In Figure 5, the evolution of the equation of state is shown vs $z$, while in Figure 6, we plot the deceleration parameter to show the transition redshift when the expansion turns from decelerated into accelerated ($z \approx 0.55$).

VI. CONCLUSIONS

We have found a new parametric class of exact cosmological scaling solutions in a theory with general non-minimal coupling between the components of the cosmic mixture: the cold dark matter and the dark energy (the quintessence field). To specify the general form of the coupling we were inspired in a scalar tensor theory of gravity written in the Einstein frame. We have studied particular coupling functions that lead to self-interaction potentials of the following class: i) double exponential potential, ii) single exponential potential. In order to derive exact flat FRW solutions we have assumed a linear relationship between the Hubble expansion parameter and the time derivative of the scalar field. The stability and existence of these solutions have been studied.

In Case A (double exponential potential) the universe evolves from an attractor regime, characterized by the scaling of matter and dark energy, into a late time, dark energy dominated attractor regime. In Case B we see that a single exponential can lead to one of the following scaling attractor solutions [21]: either 1) a late-time attractor is a state when the scalar field mimics the evolution of the background (DM) fluid, or 2) the late-time attractor is the scalar field dominated solution. We show that the assumed linear relationship between the Hubble parameter and the time derivative of the scalar field
FIG. 6. We plot the evolution of the deceleration parameter $q$ vs $z$ for the Model B (single exponential potential). As before, the following values of the free parameters $\varepsilon = 0.01$ and $\lambda = 0.3$ have been chosen. The transition redshift when the expansion turns from decelerated into accelerated occurred at $z_T \approx 0.54 - 0.56$.

always leads to attractor scaling solutions.

In all cases, the models were tested against the widely accepted observational facts. The space of parameters is then reduced to a compact region whenever it is possible. It is shown that the existing observational evidence coming from SNIa data is not enough to further constrain the space of parameters. In all cases the models fit with enough accuracy the observational data.

We conclude that models with non-minimal coupling between the dark energy and the dark matter are easy to handle mathematically if one assumes a suitable dynamics. Since, in the cases studied, the coupling function is an exponential (these differ only in the constant in the exponent), we point out that the resulting theory belongs in the class of Brans-Dicke-type of theory, written in the Einstein frame.

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