The meson $B_c$ annihilation to leptons and inclusive light hadrons

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Abstract

The annihilation of the $B_c$ meson to leptons and inclusive light hadrons is analyzed in the framework of nonrelativistic QCD (NRQCD) factorization. We find that the decay mode, which escapes from the helicity suppression, contributes a sizable fraction width. According to the analysis, the branching ratio due to the contribution from the color-singlet component of the meson $B_c$ can be of order $10^{-2}$. We also estimate the contributions from the color-octet components. With the velocity scaling rule of NRQCD, we find that the color-octet contributions are sizable too, especially, in certain phase space of the annihilation they are greater than (or comparative to) the color-singlet component. A few observables relevant to the spectrum of charged lepton are suggested, that may be used as measurements on the color-octet and color-singlet components in the future $B_c$ experiments. A typical long distance contribution in the annihilation is estimated too.

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I. INTRODUCTION

The observation of the $B_c$ meson by CDF collaboration at Fermilab$[^1]$ is one of the important discoveries in heavy quark physics and their results are consistent of the theoretical predictions$[^2, 3, 4, 5, 6, 7]$, when the theoretical uncertainties and experimental errors are concerned. More $B_c$ events in the new run of the Tevatron than those of the discovery and much more events by several orders at the hadron collider LHC are expected$[^8]$. Therefore, thorough experimental study of the $B_c$ meson and more precise comparisons with theoretical predictions on its properties, especially, on its various decay modes will be available in foreseen future.

One of the interesting decay modes is its pure leptonic decay. Since the decay amplitude is proportional to the decay constant $f_{B_c}$ and the CKM matrix element $V_{cb}$. In principle, it can be used to measure $f_{B_c}$ if we know the value of $V_{cb}$, or vice versa. However, the pure leptonic decay modes suffer by the the helicity suppression with small leptonic mass. The suppression is not severe for $B_c \rightarrow \tau \nu_{\tau}$ decay mode only, whereas, this decay mode is hard to be detected because its resultant state contains two neutrinos at least.

There is no helicity suppression if more particles are included in the decays. Such an example is the radiative leptonic decay, $B_c \rightarrow \gamma l^+\nu_l$, which has been analyzed in $[^8, 9, 10, 11, 12]$. The results show that the decay branching ratios are order of $10^{-5}$ for $l = (\mu, e)$, which is much larger than that of the pure leptonic decay of $e$-channel and comparable to that of $\mu$-channel. The relatively larger branching ratio may make them possibly to be measured at the LHC.

In this paper, we analyze a new annihilation decay mode of the meson $B_c$, i.e., the ‘semi-inclusive’ decay $B_c \rightarrow l^+\nu_l + \text{light hadrons}$. Here the light hadrons are produced from the emitted soft or/and hard gluons, i.e., we are not interested in any specific light hadron(s) but sum all of them. The annihilation process escapes from the the helicity suppression due to additional gluons in the final state. Since it is an annihilation of a pair heavy quarks essentially, so it can fully be analyzed in the framework of Nonrelativisitic QCD (NRQCD) factorization$[^13]$. According to the framework, the decay width can be factored into a sum of the products of short distance coefficients and NRQCD long distance matrix elements. The short distance coefficients can be expanded as power series of $\alpha_s$ at the energy scale of the heavy quark masses, thus they are calculable perturbatively. We may attract them
by matching the results obtained with QCD full theory calculations at the threshold to the ones obtained by NRQCD effective theory. Generally, the long distance matrix elements can be estimated only by means of the velocity power counting rule of the NRQCD effective theory.

The effective theory NRQCD considers a quarkonium state as an expansion of various Fock states. The annihilation may be via various Fock states and will depend on various long distance matrix elements. According to NRQCD, the annihilation $B_c \rightarrow l^+\nu_l + \text{light hadrons}$ can be carried out via leading Fock states $|c\bar{b}_1(^1S_0)\rangle$, $|c\bar{b}_8(^3S_1)g\rangle$ and $|c\bar{b}_8(^1P_1)g\rangle$ etc. With a naive order estimate, for the leading order the concerned annihilation may be the processes via the leading Fock state in velocity $v$ but an order higher in QCD, i.e., $(c\bar{b})_1(^1S_0) \rightarrow l^+\nu_l + gg$, and those via a higher Fock state but an order lower in QCD comparatively, such as $(c\bar{b})_8(^3S_1) \rightarrow l^+\nu_l + g$ and $(c\bar{b})_8(^1P_1) \rightarrow l^+\nu_l + g$ etc, because of the compensation for the Fock states and QCD couplings. Therefore, in the decay $B_c \rightarrow l^+\nu_l + \text{light hadrons}$ we should consider the various short distance processes. Potentially the annihilation can be used to probe the contributions from various components of the higher Fock states, and it is why we investigate the annihilation quantitatively in this paper in the framework of NRQCD factorization. Namely we estimate the decay rates accordingly by taking the potential model value for the color-singlet matrix element which relates to the wave function (derivative of the wave function) at origin directly and by means of the velocity scaling rule of NRQCD for the color-octet ones.

In conventional potential model, a heavy quarkonium just is a bound state of the heavy quark and antiquark in color singlet\cite{14}. Thus in the framework of potential model, when the annihilation of the meson $B_c$ is estimated, only the color-singlet component is taken into account. As a result, only color-singlet matrix elements in NRQCD may be related to the wave functions obtained in potential model.

In Ref.\cite{9}, some typical long distance contributions to the leptonic radiative decay have been taken into account, and results show that such contributions are negligible in comparison with the short distance contributions. In fact in the annihilation concerned here, there are also similar long-distance contributions. Therefore in this paper we also estimate the long distance contributions.

The paper is organized as follows: The first section contributes to the introduction. In section II, we briefly outline the formula for the annihilation within NRQCD framework first,
then in subsection A, we present the method to calculate the short distance coefficients for color-singlet one and final result, which corresponds to the process \((c\bar{b})_1 \rightarrow l^+\nu_{l}gg\); and in subsection B, we calculate the short distance coefficients for color-octet ones, which relates to \((c\bar{b})_8 \rightarrow l^+\nu_{l}g\) of higher Fock states (color octet ones). In section III, we estimate the typical long distance contributions of the color singlet leptonic decays using the so-called ‘generalized instantaneous approximation’\[3, 4\]. In section IV, we present the numerical results, a brief summary and discussions.

II. NRQCD FACTORIZATION ANALYSIS

In the non-relativistic heavy quark limit, there are several distinct energy scales in heavy quark and antiquark bound state system. The heavy quark masses \(m_c\) and \(m_b\) relate to the largest energy scale, while the off-shell 3-momentum \(mv\) and the typical binding energy \(mv^2\) are the small ones, where \(v \sim O(\alpha_s)\) is the relative velocity between the heavy quark and the antiquark. NRQCD effective theory is established by integrating out the energy scale effects at heavy quark mass. In the effective theory, the heavy quark and antiquark are described by non-relativistic two component fields. A physical state of \(B_c\) meson can be decomposed into a sum of a set of Fock states:

\[
|B_c\rangle = O(v^0)|c\bar{b}_1(^1S_0)\rangle + O(v^1)|c\bar{b}_8(^1P_1)g\rangle + O(v^1)|c\bar{b}_8(^3S_1)g\rangle + \ldots .
\]

According to the velocity scaling rule as indicated in the above expansion, the probability of each Fock state scales as a definite power of \(v\). Namely the leading Fock state of the \(B_c\) is \(|c\bar{b}_1(^1S_0)\rangle\), whose probability is of order \(O(v^0)\). The next leading Fock states are \(|c\bar{b}_8(^1P_1)g\rangle\) and \(|c\bar{b}_8(^3S_1)g\rangle\), whose probability is of order \(v^2\). In fact, ‘certain order’ in \(v\) comes from the integration for phase space, but here we restrict ourselves to consider the wave function expansion itself so we do not take into account those from relevant phase space integration. Throughout this paper we denote a state of \((c\bar{b})\) pair with a subscript for color: 1 for color singlet and 8 for color-octet, but for its angular momentum quantum number we put in the parentheses accordingly.

The annihilation of the \(c\) and the \(\bar{b}\) quarks happens in short-distance comparing with perturbative QCD (pQCD) energy scale, because of \(c\) and \(\bar{b}\) being heavy. The characteristic space-time of the annihilation is of order \(O(1/m_c)\) or \(O(1/m_b)\). The size of the \(B_c\) meson is
of order $O(\frac{1}{v\mu})$, where $\mu = m_b m_c / (m_b + m_c)$ is the reduce mass of the $\bar{b}$ and $c$ system. They are distinctly separated before annihilating. Thus NRQCD factorization formula can also be used to analyze the decay of the $B_c$ meson. We adopt the formulism of NRQCD to do calculations in this paper without strong modification. In the annihilation $B_c \rightarrow l^+ \nu_l + light\ hadrons$, the $\bar{b}$, $c$ quarks annihilate by weak and strong interactions both, that, according to pQCD, is of the short-distance effects, because it happens in such a short space-time at an order $\ll 1/\lambda_{QCD}$. According to the NRQCD factorization formula, the decay width can be factored into

$$\Gamma = \frac{1}{2M} \sum_n C_n(B_c|\hat{O}_n|B_c),$$

(1)

where $C_n(n = 1, 2, \cdots)$ are the short distance coefficients, and $\langle B_c|\hat{O}_n|B_c\rangle(n = 1, 2, \cdots)$ are NRQCD long distance matrix elements. The short distance coefficients, being expanded in power of $\alpha_s$, can be calculated with pQCD. While the long distance matrix elements are the ‘average’ values of the operators $\hat{O}_n, n = 1, 2, \cdots$ which, being local gauge-invariant and relevant to the annihilation, consists of non-relativistic 4-quark operators. They measure the inclusive probability of finding a pair of $\bar{c}b$ being at the same point and with suitable color and angular-momentums being specified. By the the velocity scaling rule the possibility in the meson $B_c$ in order of $v$. The $(\bar{b}c)$ pair in the meson $B_c$, not only in color-singlet but also in color-octet, may be annihilated at short-distance. Correspondingly, the matrix element is regarded as a color-single one or a color-octet one. The lowest order process for the color-singlet, i.e. the short-distance annihilation of the meson $B_c$ to leptons and light-hadrons is $(\bar{b}c)_1 \rightarrow l\nu_l + gg$, while those for the color-octet ones are $(\bar{b}c)_s(^1P_1) \rightarrow l\nu_l + g$ and $(\bar{b}c)_s(^3S_1) \rightarrow l\nu_l + g$. Hence the factorization reads as the following form:

$$\Gamma = \frac{1}{2M} \left[ C_1(^1S_0) \frac{1}{M^2} \langle B_c|\psi_c^\dagger \chi_b \lambda_b^\dagger \psi_c|B_c\rangle 
+ C_8(^3S_1) \frac{1}{M^2} \langle B_c|\psi_c^\dagger \sigma^i T^a \chi_b \lambda_b^\dagger \sigma^i T^a \psi_c|B_c\rangle 
+ C_8(^1P_1) \frac{1}{M^4} \langle B_c|\psi_c^\dagger \left(-i\frac{1}{2} \slashed{D}\right) T^a \chi_b \lambda_b^\dagger \left(-i\frac{1}{2} \slashed{D}\right) T^a \psi_c|B_c\rangle \right],$$

(2)

where $C_1(^1S_0), C_8(^3S_1)$ and $C_8(^1P_1)$ are dimensionless short distance coefficients and they are of order $G_F \alpha_s^2$, $G_F \alpha_s$, and $G_F \alpha_s$ (the lowest order in weak interaction with suitable strong couplings) respectively. The matrix elements of the three term in the bracket scales as $v^3$, $v^5$ and $v^5$, respectively. Note that here we adopt the normalization for the meson
states as \( \langle B_c(P') | B_c(P) \rangle = 2 E_P (2\pi)^3 \delta^3(P' - P) \) and since we focus only those annihilations, in addition to the leptons, also to light hadrons, i.e. the pure leptonic annihilation of the meson \( B_c \) is not included, hence the contributions from pure leptonic annihilation should not be included in the coefficient \( C_1(\overset{1}{S}_0) \) appearing in Eq.(3).

Now let us to compute these short-distance coefficients. Namely we calculate the processes with pQCD precisely, and by matching the results with the NRQCD factorized formula, finally we will obtain the coefficients.

The matrix element for the decay \( B_c(P) \to l^+(k_4) + \nu_l(k_5) + X \) can generally be written as

\[
M = \frac{G_F V_{cb}}{\sqrt{2}} \langle \nu_l | j^\mu | 0 \rangle \langle X | J_\mu | B_c \rangle ,
\]

where \( V_{cb} \) is the CKM matrix element, and \( j^\mu \) and \( J_\mu \) are the weak currents for \( l\nu \) leptons and \( c\bar{b} \) quarks.

The decay width can then be expressed as

\[
\Gamma = \frac{G_F^2 V_{cb}^2}{4 M (2\pi)^6} \int \frac{d^3 \vec{k}_4}{2\varepsilon_4} \frac{d^3 \vec{k}_5}{2\varepsilon_5} L^{\mu\nu}(k_4, k_5) \cdot t_{\mu\nu}(P, k_4 + k_5) ,
\]

where

\[
L^{\mu\nu}(k_4, k_5) = \bar{u}(k_5) \gamma^\mu (1 - \gamma_5) v(k_4) \bar{v}(k_4) \gamma^\nu (1 - \gamma_5) u(k_5) ,
\]

\[
t_{\mu\nu}(P, k) = \text{Im} \int d^4 x e^{ikx} \langle B_c(P) | J^{w\mu}(x) J^{w\nu}(0) | B_c(P) \rangle ,
\]

are the leptonic tensor and the hadronic one, respectively. The \( c\bar{b} \) quark currents in the hadronic tensor precisely are \( J^{w\mu} = \bar{\psi}_c \gamma_\mu (1 - \gamma_5) \psi_b \) and \( J^{w\nu} = \bar{\psi}_b \gamma_\nu (1 - \gamma_5) \psi_c \).

The leptonic tensor \( L^{\mu\nu} \) can easily be calculated and reads:

\[
L^{\mu\nu} = 8(k_4^\mu k_5^\nu + k_5^\mu k_4^\nu - g^{\mu\nu}(k_4 \cdot k_5) - i\varepsilon^{\mu\nu\alpha\beta} k_4^\alpha k_5^\beta),
\]

where \( \varepsilon^{\mu\nu\alpha\beta} \) is the total antisymmetric tensor.

According to the factorization, the hadronic tensor contains both short-distance coefficients and long-distance matrix elements for suitable operators. Based on NRQCD, the hadronic tensor can be factored as:

\[
t_{\mu\nu}(P, k) \equiv \text{Im} \int d^4 x e^{ikx} \langle B_c(P) | J^{w\mu}(x) J^{w\nu}(0) | B_c(P) \rangle \\
= \left[ d^{\mu\nu}_1(1S_0; P, k) \frac{1}{M^2} \langle B_c | \psi_c \chi_b \chi_b^\dagger \psi_c | B_c \rangle \right]
\]
coefficients by matching relevant processes. In the following two subsections, we show how to determine these coefficients.

B. \( \bar{b}c \) process (\( \bar{c}b \) process \( \bar{c}b \)) is in a color-singlet state. At the lowest order, there are only six \( \bar{c} \bar{b}(\bar{s}s) \) pair near the threshold with specified quantum number are annihilated. These coefficients can be evaluated by using the threshold expansion method\[15\]. In the meantime, the real process of the \( \bar{b}c \) annihilation may be calculated in the framework of factorization formula. In the effective theory NRQCD and ‘suffers’ a similar factorization, in which the short distance coefficients are the same. By matching these two ways of calculations, the short distance coefficients can be ‘read off’.

More precisely the coefficients \( C_1(1S_0) \) and \( d_1^{\mu\nu}(1S_0; P,k) \) are determined by matching the process \( (\bar{c}b)_1(1S_0) \rightarrow l^+\nu_l + gg \), while \( C_8(3S_1) \) and \( d_8^{\mu\nu}(3S_1; P,k) \); \( C_8(1P_1) \) and \( d_8^{\mu\nu}(1P_1; P,k) \) are determined by matching the processes \( (\bar{b}c)_8(3S_1) \rightarrow l^+\nu_l + g \); \( (\bar{b}c)_8(1P_1) \rightarrow l^+\nu_l + g \) with the calculations on the annihilation of the meson \( B_c \) with the effective theory NRQCD respectively. In the following two subsections, we show how to determine these coefficients by matching relevant processes.

A. Short distance coefficient for color-singlet

In this subsection, we use the threshold expansion method to determine the short distance coefficients \( C_1(1S_0) \) and \( d_1^{\mu\nu}(1S_0; P,k) \) by matching the process \( c(p_1)\bar{b}(p_2) \rightarrow l^+(k_4)\nu_l(k_5) + g(k_1)g(k_2) \), where \( (\bar{c}b) \) is in a color-singlet state. At the lowest order, there are only six

\[
\begin{align*}
&+ d_8^{\mu\nu}(3S_1; P,k) \frac{1}{M^2} \langle B_c|\bar{c}c(\sigma^i T^a \chi_b \chi_b^\dagger T^a \psi_c)|B_c \rangle \\
&+ d_8^{\mu\nu}(1P_1; P,k) \frac{1}{M^2} \langle B_c|\bar{c}c(-\frac{1}{2} i \overleftrightarrow{D})T^a \chi_b \chi_b^\dagger (-\frac{1}{2} i \overleftrightarrow{D})T^a \psi_c|B_c \rangle
\end{align*}
\]

(7) here \( d_1^{\mu\nu}(1S_0; P,k) \), \( d_8^{\mu\nu}(3S_1; P,k) \) and \( d_8^{\mu\nu}(1P_1; P,k) \) are the factors of the short-distance coefficients which we need to compute out precisely. Comparing Eq. (4) with Eq. (2), they are related to the coefficients \( C_1(1S_0), C_8(3S_1) \) and \( C_8(1P_1) \) in Eq. (3) precisely as follows:

\[
\begin{align*}
C_1(1S_0) &= \frac{G_F^2 V_{cb}^2}{2(2\pi)^6} \int \frac{d^3 k_4}{2\varepsilon_4} \frac{d^3 k_5}{2\varepsilon_5} L_{\mu\nu}(k_4, k_5) \cdot d_1^{\mu\nu}(1S_0; P,k) , \\
C_8(3S_1) &= \frac{G_F^2 V_{cb}^2}{2(2\pi)^6} \int \frac{d^3 k_4}{2\varepsilon_4} \frac{d^3 k_5}{2\varepsilon_5} L_{\mu\nu}(k_4, k_5) \cdot d_8^{\mu\nu}(3S_1; P,k) , \\
C_8(1P_1) &= \frac{G_F^2 V_{cb}^2}{2(2\pi)^6} \int \frac{d^3 k_4}{2\varepsilon_4} \frac{d^3 k_5}{2\varepsilon_5} L_{\mu\nu}(k_4, k_5) \cdot d_8^{\mu\nu}(1P_1; P,k)
\end{align*}
\]

(8) (9) (10)

with \( k = k_4 + k_5 \).
Feynman diagrams contributing to the matrix element. Three of them are shown in Fig.[1] and the other three can be obtained by exchanging the gluons. The amplitude is given by:

\[ M = \frac{G_F V_{cb}}{\sqrt{2}} \langle l \nu_l | j_\mu | 0 \rangle A^\mu , \]  

where \( A^\mu \) is defined by

\[ A_\mu \equiv \langle gg | J_\mu | (\bar{b}c)_1 (^1 S_0) \rangle = \sum_{i=1,2,3} A_{\mu,i} \]  

The six terms in the amplitude correspond to six Feynman diagrams, respectively. The first three, which correspond to the diagrams in Fig.[1], are given by:

\[ A_{\mu,1} = \frac{g^2 s_{ab}}{2\sqrt{3}} \tilde{b}(p_2) \gamma^5(k_2) \frac{1}{\kappa_2 - \kappa_1 - m_b} \gamma^\mu(1 - \gamma_5) \frac{1}{p_1 - \kappa_1 - m_c} \tilde{c}(p_1) , \]  

\[ A_{\mu,2} = \frac{g^2 s_{ab}}{2\sqrt{3}} \tilde{b}(p_2) \gamma^\mu(k_1) \frac{1}{\kappa_2 - \kappa_1 - m_b} \frac{1}{p_1 - \kappa_1 - m_c} \gamma^\nu(1 - \gamma_5) c(p_1) , \]  

\[ A_{\mu,3} = \frac{g^2 s_{ab}}{2\sqrt{3}} \tilde{b}(p_2) \frac{1}{\kappa_2 - \kappa_1 - m_b} \tilde{c}(p_1) \gamma^\mu(1 - \gamma_5) \frac{1}{p_1 - \kappa_1 - m_c} \frac{1}{\kappa_2 - \kappa_1 - m_c} \gamma^\nu(k_1) c(p_1) , \]  

where \( \tilde{b}(p_2) \), \( c(p_1) \) are the Dirac four-component spinors of the antiquark \( \bar{b} \) and the quark \( c \) respectively. \( \tilde{\varepsilon}^a(k_1) \) and \( \tilde{\varepsilon}^b(k_2) \) are the polarization vectors of the two gluons with color indices \( a, b \) respectively. The other three terms of the amplitude can be obtained by exchanging the gluon vertices.

The momenta of \( \bar{b}, c \) quarks are related to their total and relative momenta \( P \) and \( q \) as follows:

\[ p_1 = \mu_c P + q , \quad p_2 = \mu_b P - q \]  

here \( \mu_c \equiv \frac{m_c}{m_c + m_b} \) and \( \mu_b \equiv \frac{m_b}{m_c + m_b} \). In the case of the lowest order calculations for the \( S \)-wave, that is we are considering, the relative momentum \( q \) can be set to null, i.e. \( q = 0 \), hence the momenta can be further simplified as

\[ p_1 = \mu_c P , \quad p_2 = \mu_b P . \]  

Then the hadronic tensor \( t^{\mu\nu}(P,k) \) for the annihilation \( [(c\bar{b})_1(^1 S_0)] \rightarrow l^+ + \nu_l + g + g \) can be expressed as

\[ t^{\mu\nu}(P,k) = \frac{1}{(2\pi)^2} \int_0^\infty \frac{d^3 k_1}{2\varepsilon_1} \frac{d^3 k_2}{2\varepsilon_2} A^{\mu*} A^\nu \delta^4(P - k_1 - k_2) \]  

\[ = a^{\mu\nu}_1(1^4 S_0; P,k) \frac{1}{M^2} \langle (c\bar{b})_1(1^1 S_0)|\psi_c^\dagger \chi_b \lambda_b^\dagger \psi_c|(c\bar{b})_1(1^1 S_0) \rangle + \cdots , \]  

where \( a^{\mu\nu}_1 \) is the one-gluon exchange contribution and \( M \) is the mass of the quark-antiquark system.
where “⋯” denotes those terms, which, as the ones corresponding to the pure leptonic decays and those corresponding to the \((c\bar{b})\) with other quantum numbers, we are not interested in this paper.

The integration over the gluon phase space can be performed. The details of the calculations are given in the Appendix A, but we outline the basic steps here. By introducing three Lorentz invariant variables: 
\[ z = \frac{(k_1 + k_2)^2}{M^2}, \quad y = \frac{k^2}{M^2}, \quad \text{and} \quad x_l = \frac{2k_4 \cdot P}{M^2}, \]
the hadronic tensor \( t^{\mu\nu}(P, k) \) can be expressed in terms of them. The short distance coefficients \( d_1^{\mu\nu}(1S_0; P, k) \) can then be obtained by matching the Eq.\( (17) \) and Eq.\( (16) \).

The most general form of the tensor \( d_1^{\mu\nu}(1S_0; P, k) \) can be expressed as:
\[
\begin{align*}
d_1^{\mu\nu}(1S_0; P, k) & = A g^{\mu\nu} + B P^\mu P^\nu + C P^\mu k^\nu + D k^\mu P^\nu + E k^\mu k^\nu.
\end{align*}
\]
However, when we contract it with the leptonic tensor Eq.\( (4) \), the contribution from the last three terms of Eq.\( (18) \) vanish due to the fact that we ignore the lepton masses totally, so the leptonic weak current (even including the axial component) are conserved. Namely we have \( k_\mu L^{\mu\nu}(k_4, k_5) = 0 \). Thus only the first two terms are effective. The detail calculations and the expressions of the coefficients \( A \) and \( B \) are put in Appendix A.

Since we are interested in the energy spectrum of the charged lepton which is measurable, so we also give the coefficients before carrying out the integration over the leptonic spectrum \( x_l \), and those before the integration over the \( z, y \) and \( x_l \):
\[
C_1(1S_0) = \frac{G_F^2 V_{cb}^2 M^4}{2^9 \pi^4} \int dy dz dx_l (d_1^{\mu\nu}(1S_0; P, k)L^{\mu\nu}) \vartheta(x_l) \vartheta(1 + y - z - x_l) \vartheta(Y),
\]
where the new variable
\[
Y = \frac{(1 + y - z - x_l)^2 x_l^2}{4} - (y - \frac{(1 + y - z - x_l)x_l}{2})^2
\]
is introduced. By setting \( Y = 0 \) and drawing the Dalitz diagram for the process, we may determine the integration area for \( y, z \) and \( E_l \) (or \( x_l \)). The result about the integration area is also put in Appendix. Since the integration is quite complicated so we carry out it(them) numerically.

If one carry out the integration in the order: to integrate \( y \) and \( z \) first, then \( x_l \), then, before doing the last integration for \( x_l \), the measurable energy spectrum for the charged lepton \( \frac{d\Gamma}{dx_l} \) can be obtained. Thus we will do the integration in this order and discuss the obtained energy spectrum in Section IV.
In order to check the numerical results, we also take a different order to do the integration. Namely we try to make the integration of the variable of the leptons \( y \) at the last, but try to integrate the other variables of the leptons and those of gluons. First of all, we need to change the lepton tensor from \( L^{\mu \nu} \) to \( N^{\mu \nu} \):

\[
N^{\mu \nu} \equiv \frac{4\pi}{3} (k^\mu k^\nu - k^2 g^{\mu \nu}).
\]

To complete the phase space integration in this order, and having all the other variables, such as angles etc, of the phase space integrated out, we reach to the step that only three independent integration variables: \( y \), the sum of the gluons’ energies \( \varepsilon_s \) and the difference of the gluons’ energies \( \varepsilon_d \) need to be integrated out. Then to carry out the integration further, we need to determine the integration area for these three variables by the Dalitz diagram, and the obtained result about the integration area is

\[
0 \leq y \leq 1, \quad -\varepsilon_{d\text{max}} \leq \varepsilon_d \leq \varepsilon_{d\text{max}} \quad \text{and} \quad -\varepsilon_{s\text{min}} \leq \varepsilon_s \leq \varepsilon_{s\text{max}},
\]

where

\[
\varepsilon_{d\text{max}} = \sqrt{(M - \varepsilon_s)^2 - M^2 y},
\]
\[
\varepsilon_{s\text{max}} = M(1 - \sqrt{y}),
\]
\[
\varepsilon_{s\text{min}} = \frac{M(1 - y)}{2}.
\]

So the coefficient \( C_1(1S_0) \) is written:

\[
C_1(1S_0) = \int dy \int_{\varepsilon_{s\text{min}}}^{\varepsilon_{s\text{max}}} \int_{-\varepsilon_{d\text{max}}}^{\varepsilon_{d\text{max}}} f_l(\varepsilon_s, \varepsilon_d, y) d\varepsilon_d d\varepsilon_s,
\]

where the integrand \( f_l(\varepsilon_s, \varepsilon_d, y) \) is obtained straightforwardly in this order step by step as described above. Since the results for each step are very tedious so we do not present here, but we have done the computations carefully and really have the check for the phase space integration numerically. Indeed we find that the final results for the annihilation rate obtained by numerical integration in these two orders are the same, so we are sure that the numerical results very well. In fact, as a semi-finished result in this integration order, the ‘spectrum’ \( \frac{d\Gamma}{d\varepsilon_s dy} \) on \( y = \frac{k^2}{M^2} \) may be obtained too, but it is not easy to measure experimentally so we will not present the curves of the spectrum here.

**B. Short distance coefficients for color-octet**

In this subsection, we use the threshold expansion method to determine the short distance coefficients \( C_8(3S_1), d_s^{\mu \nu}(3S_1; P, k) \) and \( C_8(1P_1), d_s^{\mu \nu}(1P_1; P, k) \) by matching the process
\(c(p_1) \bar{b}(p_2) \rightarrow l^+(k_3) \nu_l(k_5) + g(k_1)\) where \(c \bar{b}\) is in a color-octet state obviously with the one computed by the effective theory NRQCD. At the lowest order, there are only two Feynman diagrams contributing to the amplitude of the process. They are shown in Fig.2. The amplitude is given by:

\[
M^{a'} = \frac{G_F V_{cb}}{\sqrt{2}} \langle l \nu_l | J^{\mu}_a | 0 \rangle A^{a'}_\mu ,
\]

where \(A^{a'}_\mu\) is defined by

\[
A^{a'}_\mu \equiv \langle g | J^a_\mu (\bar{b}c) \rangle = \sum_{i=1,2} A^{a'}_{\mu,i} ,
\]

where \(a = 1, \ldots 8\) are the color indices. The two terms of the amplitude, corresponding to the two Feynman diagrams shown in Fig.2, are given by:

\[
A^{a'}_{\mu,1} = \frac{1}{2} Tr[T^a T^b] g_{\mu} \tilde{b}(p_2) \gamma_\mu (1 - \gamma_5) \frac{1}{p_1 - \bar{k}_1 - m_c} \bar{c}(p_1) ,
\]

\[
A^{a'}_{\mu,2} = \frac{1}{2} Tr[T^a T^b] g_{\mu} \tilde{b}(p_2) \gamma_\mu (1 - \gamma_5) \frac{1}{p_2 - \bar{k}_1 - m_b} \bar{c}(p_1) .
\]

In the cases of the \(S\)-wave and \(P\)-wave \(c \bar{b}\) that we are considering and to the lowest order approximation, we expand the expression in power of \(q\), the relative momentum, and keep the terms only up-to linear ones of \(q\) (because we are doing the leading order calculations only).

Then the hadronic tensor \(t^{\mu \nu}(P, k)\) for the annihilation \((c \bar{b})_1(l^1 S_0) \rightarrow l^+ \nu_l g\) can be expressed as

\[
t^{\mu \nu}(P, k) = \int \frac{d^3 k_1}{2 \varepsilon_1} A^{a'\mu \ast} A^{a' \nu} \delta^4(P - k - k_1)
\]

\[
= \frac{1}{M^2} \delta^{\mu \nu}(3 S_1; P, k) \langle (c \bar{b})_8^3 S_1 | \psi_c^\dagger \sigma^i T^a \chi_b \chi_b^\dagger \sigma^i T^a \psi_c | (c \bar{b}_8^3 S_1) \rangle
\]

\[
+ \frac{1}{M^4} \delta^{\mu \nu}(1 P_1; P, k) \langle (c \bar{b}_8 P_1 | \psi_c^\dagger (-i \frac{1}{2} \not{D}) T^a \chi_b \chi_b^\dagger (-i \frac{1}{2} \not{D}) T^a \psi_c | (c \bar{b}_8 P_1) \rangle + \cdots (27)
\]

where \(\cdots\) denotes the terms corresponding to the \(c \bar{b}\) being in the other states, which we are not interested in here.

The integration over the gluon phase space can be performed easily. Thus with Eq.(9) and Eq.(10), the short distance coefficients can be computed easily.

\[
C_8(3 S_1) = \int dx_l \frac{\alpha_s M^6 (m_b^2 + m_c^2) G_F^2 V_{bc}^2}{24 \pi^2 m_b^2 m_c^2} x_l (x_l - 4 (1 - x_l) \log(1 - x_l))
\]

(28)
for the component \(((c\bar{b})_s(3S_1))\), and

\[
C_{8}(1P_1) = \int dx_i \frac{\alpha_s M^6 G_F^2 V_{bc}^2}{12\pi^2 m_b^4 m_c^4(1-x_i)} \left\{ x_i \left[ M^2 \{-3m_b m_c^3(x_i-1) + m_b^4(x_i-1)^2 + m_c^4(x_i-1)^2 \ight. \\
- m_b^3 m_c(x_i-1)(8x_i-7) + 2m_b^2 m_c^2[3 + 2(x_i-2)x_i] \right] \\
+ 2Mm_b m_c \{- m_c^3(x_i-1)(2x_i-7) + m_b^2 m_c^2(x_i-1)(1+2x_i) \\
+ m_b^3[(7-2x_i)x_i-5] + m_b^2 m_c[1+x_i(6x_i-11)] \} \\
+ 2m_b^2 m_c^2 \{m_b^2[11+(x_i-6)x_i] + m_c^2[(x_i-2)x_i-1] + 2m_b m_c(x_i^2-3) \} \\
- (1-x_i) \ln(1-x_i) \left[ M^2 \{m_b^3 m_c(7-6x_i) + m_b^4(x_i-1) + m_c^4(x_i-1) \\
+ 2m_b^2 m_c^2(2x_i-3) + m_b m_c^3(2x_i-3) \} \\
- 2Mm_b m_c \{5m_b^3(x_i-1) + m_b^2 m_c^2(x_i-1) + m_b^2 m_c(1-7x_i) + m_c^3(5x_i-7) \} \\
+ 2m_b^2 m_c^2 \{m_b^2(x_i+1) - 2m_b m_c(x_i-3) + m_b^2(5x_i-11) \} \right]\}
\]

(29)

for the component \(((c\bar{b})_s(1P_1))\).

### III. THE LONG DISTANCE EFFECTS

Besides those between \(b\) and \(\bar{c}\) quark pair in the initial state (the interactions make the two quarks into a bound state:the meson \(B_c\)), there are also multi soft-gluon interactions between the \(c\) and \(\bar{c}\) pair when the \(\bar{b}\)-quark has decayed into \(\bar{c}\) and the pair of \(c\) and \(\bar{c}\) has not annihilated yet. Since the interactions owes to multi soft-gluon exchange, so people generally attribute them as to long distance effects. In this section, we consider this kind of effects in the annihilation for the color singlet component \(B_c \rightarrow l^+ \nu gg\) only, namely the possible ones, where the interactions make the \(c\bar{c}\) pair to form a bound state \(\eta_c\) with suitable quantum numbers, because they may be sizable based on naive order counting but for the concerned octet components the effects cannot be very great.

As a rough estimate, it is enough to consider the dominant effects just to take into account the possible intermediate meson state \(\eta_c\). Now what we are considering is described in Fig.\[\text{III}\]. The amplitude corresponding to Fig.\[\text{III}\] is

\[
M = -\frac{G_F V_{cb} g_s^2}{2\sqrt{6}}(2\pi)^4 \delta(P - Q - k) \bar{u}_\nu(k_3) \gamma^\mu(1 - \gamma_5)v_k(k_4) \sum_{Q = k_1 + k_2} \frac{\delta_{ab}\epsilon_{\mu}(k_1)\epsilon_{\nu}(k_2)}{2Q_0(Q_0 - \omega_Q + i\epsilon)} \\
\cdot \left( Tr[\chi_Q(r)\gamma_m \frac{1}{2}Q + \frac{i}{2} - \frac{Q}{2} - m_c] + Tr[\chi_Q(r)\gamma_n \frac{1}{2}Q + \frac{i}{2} - \frac{Q}{2} - m_c] \right)
\]

12
\[ \cdot \int \frac{d^4 q}{(2\pi)^4} Tr[\chi_P(q)\Gamma_\mu \bar{\chi}_Q(q')(\mu_c P + \hat{q} + m_c)] , \]  

(30)

where \( \Gamma_\mu = \gamma_\mu (1 - \gamma_5) \), \( \epsilon^a_m(k_1) \) and \( \epsilon^b_n(k_2) \) are the polarization vectors of the two real gluons, 
\( q = \mu_b p_1 - \mu_c p_2 \) and \( q' = q + \frac{1}{2}[(\mu_c - \mu_b) P + k] \) and \( r = \frac{1}{2}(p_1 - p_2 + k) \) are the relative momenta between the two constitute quarks of the \( B_c \) meson and the bound state \( \eta_c \) correspondingly. Noting that the equations 
\( P = p_1 + p_2 , \quad k = k_4 + k_5 , \quad Q = k_1 + k_2 \)

have been used in Eq.(30). \( \omega_\vec{Q} \equiv \sqrt{\vec{Q}^2 + (M')^2} \) (\( M' \) is the mass of \( \eta_c \)). Note that for the time-component, usually off mass shell we have \( Q_0 \neq \omega_\vec{Q} \), but in the present case, the charged lepton energy may be not very high, i.e., \( E_l \leq \frac{M^2 - (M')^2}{2M} \), so the intermediate bound state \( \eta_c \) may reach to its mass-shell, that is \( Q_0 = \omega_\vec{Q} \), and leads to a singularity in Eq.(30). This difficulty happens is due to the fact that we ignore the width of \( \eta_c \). Thus here the width of \( \eta_c \) should be considered i.e. to replace \( i\epsilon \) with \( i\frac{M' Q_0}{Q_0} \Gamma_{\eta_c} \) in the relevant propagator, where \( i\epsilon \) is the ‘tiny’ imaginary quantity but \( \Gamma_{\eta_c} \) is total width of the bound state \( \eta_c \) of \( c\bar{c} \).

Then under the non-relativstic approximation, we have
\[ \frac{1}{2Q_0(Q_0 - \omega_\vec{Q} + i\epsilon)} = \frac{1}{Q^2 - (M')^2 + iM'\Gamma_{\eta_c}}. \]

Since we consider \( \eta_c \) as the only one intermediate bound state as indicated by Eq.(30), so only the ‘weak current matrix element’ corresponding to Fig.3 and the amplitude for \( \eta_c \) annihilation are needed to be computed:
\[ \langle \eta_c | \Gamma_\mu | P \rangle = i \int \frac{d^4 q}{(2\pi)^4} Tr[\chi_{Bc}(q)\Gamma_\mu \bar{\chi}_{\eta_c}(q')(\mu_c P + \hat{q} + m_c)] , \]  

(31)

and
\[ \langle gg | \eta_c \rangle = -\frac{g_s^2 \delta_{ab} \epsilon^*_m(k_1) \epsilon^b_n(k_2)}{2\sqrt{3}} \int \frac{d^4 r}{(2\pi)^4} (Tr[\chi_{\eta_c}(r)\gamma_m] \frac{1}{p_1 - k_2 - m_c}) \gamma_n \]
\[ + Tr[\chi_{\eta_c}(r)\gamma_n] \frac{1}{p_1 - k_1 - m_c}) \gamma_m \]  

(32)

In fact it is easy to compute the second matrix elements, while to compute the first one, that corresponds to the Feynman diagram Fig.4, we need to pay more care to take into account the effects of recoil for the intermediate state. We adopt the approach, i.e. the so-called generalized instantaneous approximation, which was proposed firstly in Ref. [3],
to deal with the recoil effects in the first current matrix element. The main points of
the approach are to ‘extend’ the potential model, which is based on Schrödinger equation,
into the one on Bethe-Salpeter (B.S.) equation for the non-relativistic binding systems,
and then, according to Mandelstam method [10], to formulate the current matrix element,
so the current matrix element is set on a full relativistic formulation, finally by making
the ‘generalized instantaneous approximation’ on the whole relativistic matrix element, i.e.,
integrating out the ‘time’ component of the relative momentum in the formulation of the
matrix element by a contour integration, the matrix element turns back to be formulated by
means of the Schrödinger wave functions sandwiched by proper operators. The Schrödinger
wave functions are just those of the original potential model for each system and have
direct relation to the B.S. wave functions through the original potential model for each
system and have direct relation to the B.S. wave functions through the original instantaneous
approximation proposed by Salpeter. Since the approach at a middle stage has a fully
relativistic formulation for the weak current matrix (the Mandelstam formulation of the
matrix element), so the final formulation surely takes the recoil effects into account properly.
Since one may find the details of the approximation in several references [3, 4, 9], and so we
do not repeat it here.

By applying the result of the generalized instantaneous approximation on the current
matrix element to computing the matrix elements, one may obtain the final formula straight-
forwardly and estimate the long distance effects.

IV. NUMERICAL RESULTS AND DISCUSSIONS

We take the parameters as in Refs. [4, 3, 13, 14]: \( \alpha_s = 0.24 \), \( \Gamma_{B_c} = 2.714 \text{ps}^{-1} \), \( \psi(0) = 0.350 GeV^{\frac{3}{2}} \), \( \psi'(0) = 0.250 GeV^{\frac{5}{2}} \), in numerical calculations. For comparison, let us list here
the branching ratios of the pure leptonic decays for \( B_c \) meson

\[
Br(B_c \to e\nu_e) = 1.89 \times 10^{-9}, \quad Br(B_c \to \mu\nu_\mu) = 7.57 \times 10^{-5}, \quad Br(B_c \to \tau\nu_\tau) = 1.95 \times 10^{-2}.
\]

To obtain more reliable values, we should be careful to take the quarks’ masses. According
to the discussions in Ref. [7], here we take the effective masses of \( c \) and \( b \) quarks to be
\( m_c^{\text{eff}} = 1.5 GeV \), \( m_b^{\text{eff}} = 4.9 GeV \). For the mass of \( B_c \) meson, we take pole masses of the
quarks $m_c^{pole} = 1.88 GeV$, $m_b^{pole} = 5.02$ \cite{7, 13, 19}, first, and then being consistent with that of potential model, the value of $B_c$ mass $M = 6.352 GeV$ is taken.

For the short distance contributions of the process $B_c \rightarrow ggl\nu$, we find that when $E_l \leq \frac{m_b^2 - m_c^2}{2m_b}$, where $E_l$ is the energy of the charged lepton, being very different from that of the one photon radiative correction \cite{9}, the $\bar{c}$ quark from the decay of $\bar{b}$ may reach to mass-shell. Thus to obtain meaningful result, here we should also keep the ‘width’ of $c$ quark in its propagator, namely we should make the following replace:

\[
\frac{\not{q} + m_c}{q^2 - m_c^2 + i\epsilon} \rightarrow \frac{\not{q} + m_c}{q^2 - m_c^2 + i(4m_c\Gamma_c)},
\]

where $\Gamma_c$ is the total width of an on-shell $c$-quark and from $D$ decays its value should be $\Gamma_c \simeq 1.229 ps^{-1}$ \cite{3}.

For the annihilation of color singlet component, $B_c \rightarrow ggl\nu$, since the color singlet matrix element $\langle B_c | \psi^\dagger_c \chi^\dagger_b \psi^\dagger_b | B_c \rangle$ may be related to the wave function at origin squared $|\psi(0)|^2$ and if only the short distance contributions are taken into account, the branching ratio can be computed out quite precisely:

\[
Br^{short}(B_c \rightarrow l\nu gg) = 2.71 \times 10^{-2}.
\]

Whereas if only the long distance contributions are concerned as in Sec.III, the branching ratio may be computed out too:

\[
Br^{long}(B_c \rightarrow l\nu gg) = 4.45 \times 10^{-3}.
\]

From the above values one may see that the long distance effects are quite great here. Note that there is slight overlapping for the short distance contributions and the long distance contributions and these two kinds of contributions should have interference, but here we ignore all of them.

For the annihilation of color octet components, $B_c \rightarrow lvg$, there is no reliable way to calculate the color octet matrix elements $\langle B_c | \psi^\dagger_c \sigma^i T^a \chi^\dagger_b \chi^\dagger_b \psi^\dagger_b | B_c \rangle$ and $\langle B_c | \psi^\dagger_c (-i\frac{1}{2} \not{D}^a T^a \chi^\dagger_b \chi^\dagger_b (-i\frac{1}{2} \not{D}) T^a \psi^\dagger_b | B_c \rangle$ which are necessary when computing the annihilation.

In order to have order estimate roughly, we, based on the velocity scale rules of NRQCD, try to assume them to be smaller by certain order $O(v)$ than the S-wave wave functions at the origin $|\psi(0)|^2$ for color singlet, and the derivative of the P-wave function at origin
\[ |\psi'(0)|^2, \text{ which can be computed by potential model. Namely based on the velocity scale rule, we assume} \]

\[ \langle B_c | \psi_c^\dagger \sigma^i T^a \chi_b \chi_b^\dagger T^a \psi_c | B_c \rangle \simeq \Delta_S^2 \cdot \langle B_c | \psi_c^\dagger \sigma^i T^a \chi_b \chi_b^\dagger \psi_c | B_c \rangle \]

and

\[ \langle B_c | \psi_c^\dagger \left(-i \frac{1}{2} \hat{T} D\right)^a \chi_b \chi_b^\dagger \left(-i \frac{1}{2} \hat{T} D\right)^a \psi_c | B_c \rangle \simeq \Delta_P^2 \cdot \langle B_c | \psi_c^\dagger \left(-i \frac{1}{2} \hat{T} D\right)^a \chi_b \chi_b^\dagger \left(-i \frac{1}{2} \hat{T} D\right)^a \psi_c | B_c \rangle \]

with \( \Delta_S, \Delta_P \) being \( O(v) \) constants. So with the assumption we may evaluate the color octet contributions accordingly.

In order to explore the characteristics of the color octet components in the decay for the meson \( B_c \) and to have a comparison with those of the color singlet, we try two possible choices for the S-wave color octet wave function and the derivative of the P-wave color octet wave function at the origin: **CASE A**: \( \Delta_S \simeq \Delta_P \simeq 0.1 \) and **CASE B**: \( \Delta_S \simeq \Delta_P \simeq 0.3 \), because we think that according to the velocity scale rule the range from **CASE A** to **CASE B** is reasonable and it is helpful to see the possibility if the color octet components of the meson \( B_c \) can be detectable or not experimentally. The branching ratio of the color octet component annihilations for **CASE A**:

\[ Br(B_c(c\bar{b}_8(3S_1)) \rightarrow l\nu g) = 1.73 \times 10^{-4} , \quad Br(B_c(c\bar{b}_8(1P_1)) \rightarrow l\nu g) = 2.24 \times 10^{-5} \]  \hspace{1cm} (35)

and for **CASE B**:

\[ Br(B_c(c\bar{b}_8(3S_1)) \rightarrow l\nu g) = 1.55 \times 10^{-3} , \quad Br(B_c(c\bar{b}_8(1P_1)) \rightarrow l\nu g) = 2.02 \times 10^{-4} . \]  \hspace{1cm} (36)

From the values above, we see that the helicity suppression in the annihilation processes of \( B_c \) is released. In total, the color singlet annihilation to light hadrons is bigger than those of color octet. Furthermore, when there is one more gluon bremsstrahlung than the pure leptonic decay, there are more freedoms in energy and quantum number carried by the gluon(s), that the intermediate, \( \bar{c} \) quark, even a relevant bound state such as \( \eta_c \) etc (compounded by the produced \( \bar{c} \)-quark and the ‘original’ \( c \)-quark in the meson \( B_c \)) may become on shell, thus the width of the concerned annihilations may be so great as that of the semi-leptonic decays \( b \rightarrow cl\nu \) or \( B_c \rightarrow \eta_c l\nu \) correspondingly.

In Fig. 5, we show the lepton energy spectrum of the annihilation to leptons and light hadrons. The dashed line dictates that of the spectrum for the color singlet components,
where the short distance and long distance contributions are combined. The dotted and solid lines, which correspond to the contributions from the color octet components \((c\bar{b})_8(3S_1)\) and \((c\bar{b})_8(1P_1)\) in \(B_c\) meson respectively, dictate the spectra for the color octet components. It is interesting to point out from Fig. 5 that because the meson \(B_c\) the wave function of the color octet components are suppressed, the concerned annihilations due to the color octet components (one-gluon bremsstrahlung) even though are smaller comparatively in the most region than that due to the color singlet component (two-gluon bremsstrahlung), it may become greater in certain region of the spectrum, so that it is possible to see the color octet contributions experimentally though studying the changed lepton energy spectrum of the inclusive decay \(B_c \rightarrow l^+\nu_l \cdots\) carefully, especially, around the end point of spectrum.

In order to describe the difference quantitatively in the spectrum of the charged lepton near the end point, where the color octet contributions may become dominant over those of the color singlet ones, we introduce the ratios of the integrated partial decay widths of \(B_c\) meson for the color-singlet and the color-octet,

\[
R_S = \frac{\Gamma(c\bar{b}_1(1S_0), x_{cut})}{\Gamma(c\bar{b}_1(1S_0), x_{cut}) + \Gamma(cb_8(3S_1), x_{cut})}
\]

for the S-wave color-octet component \(|c\bar{b}_8(3S_1)\rangle\), and

\[
R_P = \frac{\Gamma(c\bar{b}_1(1S_0), x_{cut})}{\Gamma(c\bar{b}_1(1S_0), x_{cut}) + \Gamma(cb_8(1P_1), x_{cut})}
\]

for P-wave color-octet state \(|c\bar{b}_8(1P_1)\rangle\), which depend on the cut of the lepton energy \(x_{cut}\).

Here

\[
\Gamma(c\bar{b}_1(1S_0), x_{cut}) \equiv \int_{x_{cut}}^{1-\delta} dx \frac{d\Gamma(c\bar{b}_1(1S_0))}{dx},
\]

\[
\Gamma(cb_8(3S_1), x_{cut}) \equiv \int_{x_{cut}}^{1-\delta} dx \frac{d\Gamma(cb_8(3S_1))}{dx}
\]

and

\[
\Gamma(c\bar{b}_8(1P_1), x_{cut}) \equiv \int_{x_{cut}}^{1-\delta} dx \frac{d\Gamma(c\bar{b}_8(1P_1))}{dx}
\]

with \(\delta = \frac{m_g}{M}\) and a giving tiny gluon mass \(m_g = 0.2 GeV\). We evaluate them and put result in the Table I.

From Fig. 5 and Table I, one may see clearly that there is a possibility to verify experimentally if the color octet components in the meson \(B_c\) play some roles in the \(B_c\) annihilation \(B_c \rightarrow l^+\nu_l + \text{hadrons}\). It is known that NRQCD is a very absorbing theory, whereas it needs
to verify widely thus we thank that our estimates are very preliminary but may addresses experimentalists’ attention on this subject. It is worth further to study the possibility in the hadronic collision environments more quantitatively and the first is Monte Carlo simulation and then carrying on experimental analysis for the verification.

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APPENDIX: INTEGRATION FORMULA OF THE COLOR SINGLET $B_c$ MESON

A number of kinematic variables will appear repeatedly in the discussion. For clarity all of them will be collected first. $P, k_1, k_2, k_4, k_5$ denote the four-momenta of the various particles, $Q = k_1 + k_2$, and $k = k_4 + k_5$ characterize the gluon-gluon system and the virtual $W$ respectively.

The scaled masses and lepton energies

$$y = \frac{k^2}{M^2}, \quad z = \frac{Q^2}{M^2}, \quad x_l = \frac{2E_l}{M}, \quad x_\nu = \frac{2E_\nu}{M}$$

(37)

vary in the region

$$0 \leq x_l \leq 1,$$

$$0 \leq y \leq x_l,$$

$$0 \leq z \leq z_{max} = (1 - x_l)(1 - \frac{y}{x_l}),$$

(38)

where the leptons’ mass is ignored. Frequently used kinematical variables which characterize the gluon-gluon system are

$$R_0 = \frac{(1 - y + z)}{2},$$

$$R_3 = \frac{\sqrt{(1 - y + z)^2 - 4z}}{2},$$

$$Y_P = \ln \left( \frac{(R_0 + R_3)^2}{z} \right).$$

(39)

All of the variables are chosen in such a scaled form on the $B_c$-meson’s mass so as to make no explicit mass dimension left for convenience in the following defined coefficients $c_i$. 
To calculate the annihilation, one observes that the squared matrix element can be ‘factorized’ into the leptonic tensor \( L_{\mu\nu}(k_4, k_5) \) and the hadronic tensor \( T_{\mu\nu}(P, k_1, k_2) \equiv A_\mu^* \cdot A_\nu \), \( (A_\mu \) is the amplitude). To perform integration over the gluon-gluon space, the resultant integration \( t_{\mu\nu} \) depends on \( P \) and \( Q = k_1 + k_2 \) only:

\[
t_{\mu\nu}(P, Q) = \int dR_2(Q; k_1, k_2) T_{\mu\nu}(P, k_1, k_2),
\]

\[
= d_{1,\mu\nu}(1S_0; P, k) \frac{1}{M^2} \langle B_c|\bar{\psi}_c \chi_b \chi_b^\dagger \psi_c|B_c\rangle
\]

(40)

where the phase space \( dR_2 \) is defined by:

\[
dR_2(Q; k_1, k_2) = (2\pi)^4 \delta(Q - k_1 - k_2) \frac{d^3k_1}{2(2\pi)^3 E_1} \frac{d^3k_2}{2(2\pi)^3 E_2},
\]

(41)

and the tensor \( d_{1,\mu\nu}(1S_0; P, k) \) has the general structure:

\[
d_{1,\mu\nu}(1S_0; P, k) = A g_{\mu\nu} + B P_\mu P_\nu + C Q_\mu Q_\nu + D P_\mu Q_\nu + E Q_\mu P_\nu.
\]

(42)

For the massless lepton pair using the lepton pair current conservation relation, \( k_\mu L^{\mu\nu} = k_\nu L^{\mu\nu} = 0 \) where \( k = P - Q \) is the total momentum of the lepton pair. With these equations the above equation Eq.(12) can be further simplified as

\[
d_{1,\mu\nu}(1S_0; P, k) = A g_{\mu\nu} + (B + C + D + E) P_\mu P_\nu = A g_{\mu\nu} + B' P_\mu P_\nu.
\]

(43)

Instead of integrating the tensor \( T_{\mu\nu} \), it is sufficient to integrate the following scalar projections

\[
c_1 = \frac{M^2}{\langle B_c|\bar{\psi}_c \chi_b \chi_b^\dagger \psi_c|B_c\rangle} \int dR_2 T_{\mu\nu} P^2 g^{\mu\nu},
\]

\[
c_2 = \frac{M^2}{\langle B_c|\bar{\psi}_c \chi_b \chi_b^\dagger \psi_c|B_c\rangle} \int dR_2 T_{\mu\nu} P_\mu P_\nu,
\]

(44)

and the coefficients \( A, B', \cdots \) can be expressed in terms of the scalar functions \( c_i, i = 1, 2 \) as follows:

\[
A = \frac{c_1 - c_2}{3P^2}, \quad B' = \frac{4c_2 - c_1}{3P^4}.
\]

(45)

The numerators of \( c_i \) are given by polynomials in \( (P k_1) \) and \( (P k_2) \) with coefficients depending on \( y, z, (QP) = M^2 R_0 \) and \( (Q k_1) = (Q k_2) = \frac{M^2 z}{2} \). In fact, all of the phase space integrations may be attributed to the following integrations:

\[
I_{m,n} = \int dR_2(Q; k_1, k_2)(P k_1)^m(P k_2)^n \quad (-2 \leq m, n \leq 0).
\]

(46)
Because of the Lorentz invariance, we evaluate the integral in the \( Q = k_1 + k_2 \) rest system and the results show

\[
\begin{align*}
I_{0,0} & = \frac{1}{8\pi}, \\
I_{-1,0} & = I_{0,-1} = \frac{Y_P}{8\pi M^2 R_3}, \\
I_{-2,0} & = I_{0,-2} = \frac{1}{2\pi M^4 z}, \\
I_{-2,-1} & = I_{-1,-2} = \frac{1}{4\pi M^6} \left( \frac{Y_P}{R_3 R_0^2} + \frac{2}{z R_0} \right), \\
I_{-1,-1} & = \frac{Y_P}{4\pi M^4 R_0 R_3}, \\
I_{-2,-2} & = \frac{1}{4\pi M^8} \left( \frac{Y_P}{R_3 R_0^3} + \frac{2}{z R_0^2} \right) .
\end{align*}
\]

The explicit form of \( c_i \) are shown in the following, where a common factor \( \frac{2\pi}{3} \alpha_s^2 \) is contracted out for convenience and the terms that can be obtained by interchanging \( m_b \) and \( m_c \) is not shown explicitly, that is, the actual value of each \( c_i \) equals \((c_i + c_i(r_1 \leftrightarrow r_2))\), where \( r_1 = \frac{m_b}{M} \) and \( r_2 = \frac{m_c}{M} \)

\[
c_1 = \frac{16}{f_1^2 f_2 z r_1^2 r_2^2 R_0^2} \left( -8 r_1^3 r_2 (3 f_2 z + 4 (f_2 - f_1) r_2^2) R_0^2 + f_1^2 f_2 (y z + 4 R_0^2) \right) - 4 f_1 r_1 r_2 (f_1 f_2 + 2 f_1 (f_1 - z) R_0 + (2 f_1 f_2 + z - 4 f_2 z + 4 f_2 r_2) R_0^2) + 8 r_1^2 (f_2 z (-f_1 + z) R_0^2 + f_2 r_2 R_0^2 (3 z + (4 f_1 - 2 z) R_0) - 4 f_2 r_2^3 R_0 (2 f_1 + R_0 + R_0^2) + r_2^2 (2 f_1^2 f_2 + R_0^2 (4 f_2 - 4 f_1 f_2 + 5 f_1 z - 2 f_1 R_0 + 4 f_2 R_0^2)))) + \frac{8 Y_P}{f_1^2 f_2 z r_1^2 r_2^2 R_3 R_0} \left( 16 f_2 z r_1^4 R_0^2 + f_1^2 f_2 z (y - 2 R_0^2) \right) + 4 f_1 r_1 (- (f_1 f_2 z r_2) - 2 f_2 (f_1 - z) r_2 R_0 + r_2 (2 f_1 f_2 + z (z - 1 - 2 f_2)) + 4 f_2 (3 - 2 r_2) R_0^2 + 2 (f_2 z - 2 z r_2 + 2 f_2 r_2^2) R_0^3 + 4 r_2 R_0^4) + 8 r_1^3 r_2 R_0^2 (3 f_2 z + 4 r_2 (f_1 r_2 + f_2 R_0)) + 8 r_1^2 (- 2 f_2 z R_0^2 + 4 f_2 r_2^4 R_0^2 + f_2 (- 2 f_1 + z) R_0^4 - 4 f_2 r_2^3 R_0 (2 f_1 + R_0^2) + f_2 r_2 R_0^2 (z - 2 R_0 (2 - 2 f_1 + z + 2 R_0^2)) + r_2^2 (2 f_1^2 f_2 + R_0^2 (4 f_1 f_2 + 3 f_2 z - 4 (f_1 - f_2) R_0 (1 + R_0))))) ,
\]

\[
c_2 = \frac{8}{f_1^2 f_2 z r_1^2 r_2^2 R_0^2} \left( f_1^2 f_2 (z^2 + 8 r_1 r_2 (- z + 4 r_1 r_2)) + 4 f_1 r_1 R_0 (r_2 (4 f_1 - z R_0) + r_1 (z R_0 + 16 r_2 (- 2 r_2 + R_0))) + 2 R_0^2 (f_1^2 f_2 z + 2 r_1 (f_1 f_2 z - 2 f_1 r_2 (z + 3 f_2 z + 4 f_2 r_2)) -
\]

\]

20
\[ 4r_1^2 r_2^2(f_2 + 2(-f_1 + f_2)r_2 - 2f_2 R_0) + r_1(f_2 z^2 + 2r_2^2(5f_1 z + 2f_2(y + z) + 2f_2 r_2(1 + 2r_2 - 2R_0) - 4f_1 R_0)) \] 
\[
\frac{8Y_p}{f_1 f_2 r_1^2 r_2^3 R_0 R_3}(f_1^2 f_2 z^2 + 8f_1^2 f_2 r_1 r_2(-z + 4r_1 r_2) - 16f_1 f_2 r_1 r_2(f_1 - 8r_1^2 R_0) + 2(f_1^2 f_2(1 - 2z) + 2r_1(f_2 z(f_1 + (-2 - 3f_1 + 4y)r_1) + f_1(4f_1 f_2 + (-3 + f_2 + y)z)r_2 - 4(2f_1 f_2 + r_1(8f_1 f_2 - (3f_1 + f_2)z + 2f_1(-1 + r_1) r_1) r_2^2 + 8f_1 r_1^2 r_2^3 + 16 f_2 r_1 r_2^4)) R_0^2 - 16 r_1(-f_1 r_2(1 + f_2 - y + 2f_2 r_2) - r_1^2(f_2 z + r_2(f_2 - (f_1 - 2f_2) r_2)) + r_1 r_2(f_2(-4f_1 + y + z) + r_2(2f_1 + f_2 + (f_1 + 6 f_2) r_2))) R_0^3 - 16 r_1^2 r_2(2f_2 r_1 + (f_1 - 3 f_2) r_2) R_0^4, \]

where \( f_1 = -1 + y + 2r_1 R_0, \) \( f_2 = -1 + y + 2r_2 R_0. \)

After contraction of the hadronic tensor with \( L_{\mu\nu}(k_4, k_5), \) replacing \( Q \) by \( P - k \) and substituting

\[
(k_4 k_5) = \frac{M^2 y}{2}, (kk_4) = (kk_5) = \frac{M^2 y}{2}, (P k_4) = \frac{M^2 x_l}{2},
\]
\[
(P k_5) = \frac{M^2 x_\nu}{2}, (P k) = \frac{M^2 (1 + y - z)}{2}.
\]

One is left with the task to integrate the function of \( x_l, y \) and \( z, \) which can be done numerically.

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TABLE I: The ratios of the integrated partial width $R_S$ and $R_P$ for **CASE A** ($\Delta_S = 0.1, \Delta_P = 0.1$) and **CASE B** ($\Delta_S = 0.3, \Delta_P = 0.3$) (The definition about $R_S, R_P, \Delta_S, \Delta_P$ and $x^\text{cut}_l$ is in text).

|       | CASE A: ($\Delta_S = 0.1$) | CASE B: ($\Delta_S = 0.3$) |
|-------|----------------------------|----------------------------|
| $x^\text{cut}_l$ | 0.80 0.85 0.90 0.95 | 0.80 0.85 0.90 0.95 |
| $R_S$ | 0.257 0.191 0.122 0.047 | 0.037 0.026 0.015 0.005 |

|       | CASE A: ($\Delta_P = 0.1$) | CASE B: ($\Delta_P = 0.3$) |
|-------|----------------------------|----------------------------|
| $x^\text{cut}_l$ | 0.90 0.92 0.94 0.96 | 0.80 0.85 0.90 0.95 |
| $R_P$ | 0.280 0.212 0.154 0.089 | 0.132 0.081 0.041 0.014 |

FIG. 1: Three typical Feynman diagrams for the decay $B_c \rightarrow l^+\nu gg$. 

\[\text{FIG. 1: Three typical Feynman diagrams for the decay } B_c \rightarrow l^+\nu gg.\]
FIG. 2: The diagrams for the annihilation width of $B_c \rightarrow l^+\nu g$, where $B_c$ is indicated in a color octet Fock state. The vertical curcul line in the diagrams is understood the according imaginary part being taken.

FIG. 3: The diagrams for long distance effects.
FIG. 4: Current matrix element for the long distance effects.

FIG. 5: The energy spectra of the charged lepton with different ‘color-octet matrix elements’. The left figure is that for **CASE A**: $\Delta_S = 0.1$ and $\Delta_P = 0.1$. The right figure is that for **CASE B**: $\Delta_S = 0.3$ and $\Delta_P = 0.3$. The dashed line in the figures stands for the summed effects of short distance contributions and the long distance one with $\eta_c$ as an intermediate bound state in the color singlet decay $B_c \to l^+ \nu gg$. The dotted and the solid lines for the color octet annihilations $B_c \to l^+ \nu g$ with $c\bar{b}_8(^3S_1)$ and $c\bar{b}_8(^1P_1)$ components in $B_c$ meson correspondingly.