Electromagnetic waves along pseudo null curves in Minkowski space

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ABSTRACT

Considering the importance of Minkowski space in physics, it is an incomplete approach to deal with EM waves only in Euclidean space. For this reason, this paper deals with EM waves along pseudo null curves in Minkowski space. The main purpose of this study is to examine electromagnetic waves by defining an adapted orthogonal frame along the EM wave which contains both electric and magnetic fields. For this purpose, the extended derivative formulas of pseudo null frame are obtained. Depending on the values of Bishop curvatures, the linear transformations between the pseudo null frame and EM wave vector fields are described in two cases. For all these cases, the relations between these frames are stated, respectively. Moreover, the derivative formulas EM wave vectors are stated by means of geometric phase. Furthermore, the necessary and sufficient conditions provided by the geometric phase are expressed for EM wave vectors to be parallel transportation of the pseudo null frame. Finally, an application is given to investigate the obtained results.

1. Introduction

In physics, electromagnetic waves, also known as EM waves, are ones generated by vibrations between an electric field and a magnetic field. EM waves are emitted by accelerated electrically charged particles, which can interact with other charged particles. EM waves carry energy, momentum, and angular momentum from source particles and can impart these quantities to the matter with which they interact. Radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays and gamma rays are well-known examples of EM waves. It is known that EM waves appear physically with the oscillation of magnetic and electric fields. More precisely, EM waves arise when an electric field comes into contact with a magnetic field. The electric field and magnetic field are perpendicular to each other along an EM wave. However, they are also perpendicular to the direction of the EM wave. Because of the orthogonality, it is possible to define an adapted orthogonal frame along the EM wave which contains both electric and magnetic fields. This adapted orthogonal frame defined along the EM wave consists of $\mathbf{\hat{t}} = \mathbf{\hat{t}}(s,n,b)$ representing direction of EM wave, $\mathbf{\hat{E}} = \mathbf{\hat{E}}(s,n,b)$ representing electric field and $\mathbf{\hat{B}} = \mathbf{\hat{B}}(s,n,b)$ representing magnetic field. This is exactly where the study subjects of physics and geometry intersect. Differential geometric operations and interpretations along the frame $(\mathbf{\hat{t}},\mathbf{\hat{E}},\mathbf{\hat{B}})$ will reveal some new physical results. Because of this property of EM waves, we may investigate EM waves traveling through space curves. More precisely, $(\mathbf{\hat{t}},\mathbf{\hat{E}},\mathbf{\hat{B}})$ is calculated using the Frenet Serret frame of a space curve $\mathcal{C} = \mathcal{C}(s,n,b)$. There are many studies using these relationships between these frames, including other applications in the field of physics [1, 2, 3, 4, 5].

Let $\mathcal{C} = \mathcal{C}(s,n,b)$ be a given space curve. The distances among $t$-congruence, $n$-congruence, and $b$-congruence are represented by $s$, $n$, and $b$, respectively. The directional derivatives of the frame $(\mathbf{\hat{t}},\mathbf{\hat{E}},\mathbf{\hat{B}})$ are extensively detailed in [6]. The normal deformations of the vector-tube along $\mathbf{\hat{t}}$ and $\mathbf{\hat{b}}$

$$\beta_{ns} = g\left(\mathbf{\hat{n}}, \frac{d}{du} \mathbf{\hat{t}} \right), \quad \beta_{bs} = g\left(\mathbf{\hat{b}}, \frac{d}{du} \mathbf{\hat{t}} \right)$$

are indicated in [1, 2], respectively. Among the non-Euclidean geometries, Lorentzian geometry has the most well known applications [7, 8, 9, 10]. Lorentzian geometry is a branch of study that deals with soliton theory, fluid dynamics, integrable systems, field theories and other physical concerns [11, 12, 13, 14]. Null Cartan curves in space are related to the integrable systems in Minkowski space [15]. On the other hand, nonnull and null Cartan images of a given null Cartan curve under Backlund transformation are discussed in [16]. In addition, the geometric phases studied in the normal and binormal directions along the optical fiber with respect to the null Cartan frame are studied in [4] for anholonomic coordinates. The null Lorentz force equations in the normal and binormal directions, and even the null electromagnetic...
curves, are expressed according to the null Cartan frame in [4]. Then, a pseudo-null space curve in Minkowski 3-space is used to describe an optical fiber that is injected into monochromatic linear polarized light in [5].

Considering applications of Minkowski space in physics, it is necessary to examine EM waves in Minkowski space as well. For this reason, it is possible to examine the physical and differential geometric properties EM waves in Minkowski space. EM waves can also be considered as occurring by the oscillation of magnetic and electric fields. Along a non-lightlike EM wave, electric and magnetic fields are Lorentzian orthogonal to each other. Similarly, the non-lightlike vector fields \( \vec{E} = \vec{E}(s, n, b) \) and \( \vec{B} = \vec{B}(s, n, b) \) are also orthogonal to the non-lightlike direction of EM wave. Therefore, an adapted non-lightlike orthogonal frame \( (\vec{t}, \vec{E}, \vec{B}) \) can be defined along EM wave. Then, EM wave vectors are examined by Frenet Serret frame of a non-lightlike curve [1].

Now, consider given non-lightlike unit speed curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \) in three-dimensional Minkowski space. The arc length parameter in the non-lightlike tangential direction is denoted by \( s \). Therefore, the unit non-lightlike tangent of \( s \) parameter curve is expressed as follows:

\[
\vec{t} = \frac{\partial \mathcal{Z}}{\partial s}
\]

The arc length parameter in the non-lightlike principal normal direction is denoted by \( n \) where unit non-lightlike tangent of \( n \) parameter curve is given as

\[
\vec{n} = \frac{\partial \mathcal{Z}}{\partial n}
\]

Moreover, the arc length parameter in non-lightlike binormal direction is denoted by \( b \). Thus, the unit non-lightlike tangent vector of \( b \) parameter curve is stated as

\[
\vec{b} = \frac{\partial \mathcal{Z}}{\partial b}
\]

[1, 2]. In Minkowski space, the intrinsic description of a vector field is more complicated. Anholonomic coordinates, which have eight parameters and are associated by three partial differential equations, can be used to describe a three-dimensional vector field [6]. The frame \( (\vec{t}, \vec{n}, \vec{b}) \) is called Frenet Serret frame of a given non-lightlike curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \) in Minkowski space. Although this frame is most commonly used to investigate the geometry of a non-lightlike curve, it is possible to consider the curves with different kinds of frames [17, 18, 19]. The non-lightlike EM wave vectors \( (\vec{t}, \vec{E}, \vec{B}) \) can be stated by Frenet Serret frame \( (\vec{t}, \vec{n}, \vec{b}) \) of a given non-lightlike curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \). The rotation takes place with in the normal plane of curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \) with the geometric phase (angle) \( \sigma = \sigma(s, n, b) \). The type of rotation transformation depends on the causal characterization of the curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \). Rotation transformations around timelike, spacelike, or null axis are defined very differently from one another. For further information about rotation transformations, see [20, 21, 22, 23, 24, 25]. In two different situations, the pseudo null frames \( (\vec{t}, \vec{n}, \vec{b}) \) and \( (\vec{t}, \vec{E}, \vec{B}) \) are described with a linear transformation of one another. Moreover, parallel transportation of vectors \( \vec{E} \) and \( \vec{B} \) along the tangent direction of the EM wave is called Rytov’s law. For more information about parallel transportation (Bishop) frame of curves in Minkowski spaces, see [26, 27].

This article consists of four sections. In section 1, a summary of the literature is stated to explain subject of paper. In section 2, extended derivative formulas of pseudo null frame are obtained. In section 3, there are two subsections, depending on the value of Bishop curvatures, which describe the linear transformations between the pseudo null frames \( (\vec{t}, \vec{n}, \vec{b}) \) and \( (\vec{t}, \vec{E}, \vec{B}) \). In the first subsection, the case of the first Bishop curvature function \( k_1 = 0 \) and in the second subsection, the case of the second Bishop curvature function \( k_2 = 0 \) are discussed, respectively. For all these cases, the relations between the frames \( (\vec{t}, \vec{n}, \vec{b}) \) and \( (\vec{t}, \vec{E}, \vec{B}) \) are stated. Moreover, the derivative formulas of EM wave vectors are obtained by means of geometric phase \( \sigma = \sigma(s, n, b) \). Furthermore, the necessary and sufficient conditions provided by the geometric phase \( \sigma = \sigma(s, n, b) \) are expressed for EM wave vectors to be parallel transportation of the pseudo null frame \( (\vec{t}, \vec{n}, \vec{b}) \). Finally, an application is presented to explain the results obtained in section 4.

2. Pseudo null frame

Minkowski space \( \mathbb{E}^3 \) is the real vector space \( \mathbb{R}^3 \) provided with standard indefinite flat metric \( (\cdot, \cdot)_L \) given by

\[
(\vec{a}, \vec{b})_L = -u_1v_1 + u_2v_2 + u_3v_3
\]

where \( \vec{a} = (u_1, u_2, u_3), \vec{b} = (v_1, v_2, v_3) \in \mathbb{R}^3 \). By definition, this product is not positive definite.

Assume that \( \alpha : I \to \mathbb{E}^3 \) is a pseudo-null curve which is given by the pseudo arc length parameter. And

\[
\vec{\ell}(s) = a'(s)
\]

tangent and \( \vec{n}(s) = a''(s) \) is null principal normal vector field. Then, \( \vec{b} \) is binormal vector field which is the unique null vector field orthogonal to \( \vec{t} \) satisfying the equation

\[
(\vec{n}(s), \vec{b}(s))_L = 1
\]

If \( a \) is straight line then \( \kappa(s) = 0 \), while \( \kappa(s) = 1 \) in other cases. The torsion function is expressed in the form

\[
\tau(s) = \begin{bmatrix} \vec{n}(s), \vec{b}(s) \end{bmatrix}_L.
\]

We have the following derivative formulas

\[
\frac{d}{ds} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ 0 & \tau(s) & 0 \\ -\kappa(s) & 0 & -\tau(s) \end{bmatrix} \begin{bmatrix} \vec{t}(s) \\ \vec{n}(s) \\ \vec{b}(s) \end{bmatrix}
\]

(1)

where

\[
\begin{align*}
&\langle \vec{n}(s), \vec{n}(s) \rangle_L = \langle \vec{n}(s), \vec{b}(s) \rangle_L = 0, \\
&\langle \vec{t}(s), \vec{n}(s) \rangle_L = \langle \vec{t}(s), \vec{b}(s) \rangle_L = 0, \\
&\langle \vec{t}(s), \vec{b}(s) \rangle_L = \langle \vec{n}(s), \vec{b}(s) \rangle_L = 1
\end{align*}
\]

[28, 29].

In this and following sections of the study, we will assume that \( \mathcal{Z} = \mathcal{Z}(s, n, b) \) curve is a pseudonull curve. We denote the distance along \( t\)-congruence by \( s \) where spacelike tangent vector of \( t\)-congruence is defined by

\[
\vec{t} = \frac{\partial \mathcal{Z}}{\partial s}
\]

Then \( n \) denotes the distance along \( n\)-congruence and null tangent vector of \( n\)-congruence is given by

\[
\vec{n} = \frac{\partial \mathcal{Z}}{\partial n}
\]

In addition, \( b \) denotes the distance along \( b\)-congruence and null tangent vector of \( b\)-congruence is stated as

\[
\vec{b} = \frac{\partial \mathcal{Z}}{\partial b}
\]

To investigate the intrinsic differential geometric structure of a pseudo null curve in Minkowski space, it is needed to know the pseudo arc length on the curve, curvature and torsion which are two independent parameters. The existence of a field of basis vectors such as \( (\vec{t}, \vec{n}, \vec{b}) \) to vector lines in three-dimensional Minkowski space does not imply the existence of a corresponding coordinate system in general. But as is known that a three-dimensional vector field can be described in terms of anholonomic coordinates which includes the parameters, related by three partial differential equations [30]. The following theorem gives the aforementioned description for pseudo null frame \( (\vec{t}, \vec{n}, \vec{b}) \) of the pseudo null curve \( \mathcal{Z} = \mathcal{Z}(s, n, b) \) in Minkowski space. These relations are also called extended derivative formulas of pseudo null frame.
Theorem 1. Directional derivatives of the moving trihedron of pseudo orthogonal vectors \((\vec{t}, \vec{n}, \vec{b})\) are stated as follows:

i)  
\[
\frac{\partial}{\partial s} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & \tau & 0 \\ -\kappa & -\tau & \vec{n} \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

ii)  
\[
\frac{\partial}{\partial n} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \frac{\mu_b}{\kappa} & \frac{\xi_{ns}}{\kappa} \\ -\frac{\xi_{ns}}{\kappa} & -((\vec{b} \times \kappa) \vec{n}) & 0 \\ -\frac{\mu_b}{\kappa} & 0 & -\vec{b} \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

iii)  
\[
\frac{\partial}{\partial b} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \xi_{bs} & -\mu_b \\ -\xi_{bs} & 0 & -\vec{b} \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

where  
\[
\xi_{ns} = \left( \frac{\partial \vec{t}}{\partial n} \right)_L \vec{n}, \quad \xi_{bs} = \left( \frac{\partial \vec{t}}{\partial b} \right)_L \vec{n}.
\]

Abnormality of \(\vec{n}\) and \(\vec{b}\) are  
\[
\mu_n = (\vec{c} \vec{n}, \vec{n})_L, \quad \mu_b = (\vec{c} \vec{b}, \vec{b})_L.
\]

respectively. The torsion and torsion functions are \(\kappa = \kappa(s, n, b)\) and \(\tau = \tau(s, n, b)\) respectively.

Proof. Proof of i) is derived directly from derivative formula given in Equation (1). Thus, the proof of ii) and iii) will be given. We know that for \(i = 1, 2, 3\) there exist smooth functions \(\eta_i\) and \(\gamma_i\) such that

\[
\frac{\partial}{\partial n} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \eta_1 & \eta_2 \\ 0 & \eta_3 & 0 \\ -\eta_1 & 0 & -\eta_2 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

and

\[
\frac{\partial}{\partial b} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix} = \begin{bmatrix} 0 & \gamma_1 & \gamma_2 \\ 0 & \gamma_3 & 0 \\ -\gamma_1 & 0 & -\gamma_2 \end{bmatrix} \begin{bmatrix} \vec{t} \\ \vec{n} \\ \vec{b} \end{bmatrix}.
\]

First of all, we have  
\[
\eta_2 = \left( \frac{\partial \vec{t}}{\partial n} \right)_L = \xi_{ns}, \quad \gamma_1 = \left( \frac{\partial \vec{t}}{\partial b} \right)_L = \xi_{bs}.
\]

by our assumptions. Then, other geometric quantities are calculated with vector analysis formulas as follows:

\[
\text{div}\vec{t} = \left( \frac{\partial \vec{t}}{\partial s} \right)_L + \left( \frac{\partial \vec{t}}{\partial n} \right)_L + \left( \frac{\partial \vec{t}}{\partial b} \right)_L
\]

\[
= (\vec{t}, \vec{n})_L + (\vec{n}, \vec{n}_L \vec{n})_L + (\vec{b}, \vec{t})_L
\]

\[
= \xi_{ns} + \xi_{bs}.
\]

\[
\text{div}\vec{n} = \left( \frac{\partial \vec{n}}{\partial s} \right)_L + \left( \frac{\partial \vec{n}}{\partial n} \right)_L + \left( \frac{\partial \vec{n}}{\partial b} \right)_L
\]

\[
= (\vec{n}, \vec{n}_L + (\vec{n}, \vec{n}_L \vec{n})_L + (\vec{b}, -\gamma_1 \vec{t} + \gamma_2 \vec{b})_L = \gamma_3
\]

and

\[
\text{div}\vec{b} = \left( \frac{\partial \vec{b}}{\partial s} \right)_L + \left( \frac{\partial \vec{b}}{\partial n} \right)_L + \left( \frac{\partial \vec{b}}{\partial b} \right)_L
\]

\[
= (\vec{b}, -\kappa \vec{t} - \tau \vec{b})_L + (\vec{n}, -\kappa \vec{n} - \eta_3 \vec{b})_L + (\vec{b}, -\gamma_1 \vec{t} + \gamma_2 \vec{b})_L = -\kappa - \eta_3.
\]

Thus, we obtain  
\[
\gamma_3 = -\text{div}\vec{n}, \quad \gamma_3 = \text{div}\vec{b} + \kappa.
\]

We also get  
\[
\gamma_3 = \text{div}\vec{b} + \kappa.
\]
where $k_1 = 0$ and $k_2 = c_0 e^{i\sigma(s,n,b)}$, $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$ is the geometric phase and $c_0 \in \mathbb{R}^*_0$.

**Theorem 2.** The derivative formulas for EM wave vectors ($\vec{i}, \vec{E}, \vec{B}$) are given as follows:

1) 
\[
\frac{\partial}{\partial s} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

2) 
\[
\frac{\partial}{\partial n} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 0 & -k_1 & k_1 \\ -k_2 & 0 & 0 \\ 0 & 0 & k_2 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

3) 
\[
\frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 0 & k_2 \beta_k & 0 \\ 0 & 0 & \beta_k \\ -k_2 \beta_n & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

where $k_1 = 0$ and $k_2 = c_0 e^{i\sigma(s,n,b)}$, $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$ is the geometric phase and $c_0 \in \mathbb{R}^*_0$.

**Proof.** Proof of 1) is derived directly from derivative formula for Bishop frame of pseudo null curve. The readers are referred to 1) of Theorem 1 of the paper [31] for the proof. Therefore, we will only give the proof of ii) and iii). The following equations give the change of EM wave vectors ($\vec{i}, \vec{E}, \vec{B}$) between any two points on $a$ parameter curve along $\mathcal{S} = \mathcal{S}(s,n,b)$

\[
\frac{\partial}{\partial n} \vec{i} = \mu_\beta \vec{n} + \xi_\beta \vec{b} = \frac{\partial}{\partial n} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

\[
\frac{\partial}{\partial n} \vec{E} = \frac{\partial}{\partial n} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

and

\[
\frac{\partial}{\partial n} \vec{B} = \frac{\partial}{\partial n} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

by using Equations (3) and (5). Finally, following equations describe the change of EM wave vectors between any two points on $b$ parameter curve along $\mathcal{S}$

\[
\frac{\partial}{\partial b} \vec{i} = \xi_{\beta,b} \vec{n} - \mu_{\beta,b} \vec{b} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

\[
\frac{\partial}{\partial b} \vec{E} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

and

\[
\frac{\partial}{\partial b} \vec{B} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

by using Equations (3) and (5). Finally, following equations describe the change of EM wave vectors between any two points on $b$ parameter curve along $\mathcal{S}$

\[
\frac{\partial}{\partial b} \vec{i} = \xi_{\beta,b} \vec{n} - \mu_{\beta,b} \vec{b} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

\[
\frac{\partial}{\partial b} \vec{E} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

and

\[
\frac{\partial}{\partial b} \vec{B} = \frac{\partial}{\partial b} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

by using Equations (4) and (5).

The following theorem is clearly seen by the derivative formulas in Equations (6) and (7).

**Theorem 3.** EM wave vectors ($\vec{i}, \vec{E}, \vec{B}$) is parallel transportation of the pseudo null frame field ($\vec{n}, \vec{b}$) if and only if the geometric phase $\sigma = \sigma(s,n,b)$ satisfies the relations

\[
\frac{\partial \sigma}{\partial n} = \text{div} \, \vec{n} + 1, \quad \frac{\partial \sigma}{\partial b} = \text{div} \, \vec{b}
\]

where $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$. 

**Proof.** The idea of a parallel frame of a curve is based on the theory that partial derivatives of the frame depend on the tangent vector, while the tangent vector field does not change and other vector fields are taken in the plane perpendicular to the tangent vector field. By Equations (6) and (7), it is seen that partial derivatives of ($\vec{i}, \vec{E}, \vec{B}$) have only tangent component if and only if

\[
\frac{\partial}{\partial n} \left( \frac{1}{k_2} \right) k_2 - \text{div} \, \vec{b} - 1 = \frac{\partial k_1}{\partial n} \frac{1}{k_2} + \text{div} \, \vec{n} + 1 = 0, \\
\frac{\partial}{\partial b} \left( \frac{1}{k_2} \right) k_2 + \text{div} \, \vec{n} = \frac{\partial k_1}{\partial b} \frac{1}{k_2} - \text{div} \, \vec{n} = 0.
\]

On the other hand, we have

\[
\frac{\partial}{\partial n} \left( \frac{1}{k_2} \right) k_2 = -\frac{\partial \sigma}{\partial n} \frac{1}{k_2} - \frac{\partial k_1}{\partial n} \frac{1}{k_2} = -\frac{\partial \sigma}{\partial n}, \\
\frac{\partial}{\partial b} \left( \frac{1}{k_2} \right) k_2 = -\frac{\partial \sigma}{\partial b} \frac{1}{k_2} - \frac{\partial k_1}{\partial b} \frac{1}{k_2} = -\frac{\partial \sigma}{\partial b}
\]

where $k_1 = c_0 e^{i\sigma(s,n,b)}$, $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$ is the geometric phase and $c_0 \in \mathbb{R}^*_0$. Therefore, we get the proof.

### 3.2. Parallel transportation law of electric wave vector for $k_2 = 0$

In this subsection, we consider that the Bishop curvature functions $k_1(s,n,b) = c_e^{i\sigma(s,n,b)}$ and $k_2 = 0$ where $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$ and $c_0 \in \mathbb{R}^*_0$. Similarly, the frames ($\vec{i}, \vec{n}, \vec{b}$) and ($\vec{E}, \vec{B}$) are described in terms of one another by a linear transformation using Bishop frame of pseudo null curves discussed in [31]. The relations between the frames ($\vec{i}, \vec{n}, \vec{b}$) and ($\vec{E}, \vec{B}$) are stated as

\[
\begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{k_1} & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

and

\[
\begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{k_1} & 0 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{n} \\ \vec{b} \end{bmatrix}
\]

where $k_1 = c_0 e^{i\sigma(s,n,b)}$, $\sigma(s,n,b) = \int \tau(s,n,b) \, ds$ is the geometric phase and $c_0 \in \mathbb{R}^*_0$.

**Theorem 4.** The derivative formulas of EM wave vectors are stated as follows:

1) 
\[
\frac{\partial}{\partial s} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix} = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & 0 & k_1 \end{bmatrix} \begin{bmatrix} \vec{i} \\ \vec{E} \\ \vec{B} \end{bmatrix}
\]

2)
Proof. Proof of i) is derived directly from derivative formula of pseudo null frame of the curve. The results are referred to ii) of Theorem 1 in [31] for the proof of i). The following equations state the change of EM wave vectors $\vec{t}, \vec{E}, \vec{B}$ between any two points on the $n$ parameter curve along $\mathcal{S}$

$$\frac{\partial}{\partial n} \vec{t} = \mu_n \vec{n} + \xi_n \vec{b} = \mu_n(-k_1 \vec{B}) + \xi_n (-\frac{1}{k_1} \vec{E}) = -\xi_n \frac{1}{k_1} \vec{E} - \mu_n k_1 \vec{B},$$

$$\frac{\partial}{\partial n} \vec{E} = \frac{\partial}{\partial n} (-k_1 \vec{B}) = -\frac{\partial k_1}{\partial n} \vec{B} - k_1 (\frac{\partial}{\partial n} (\vec{E} + 1)) \vec{B}$$

$$= -k_1 \mu_e + (\frac{\partial}{\partial n} (\frac{1}{k_1} \vec{B} + \frac{1}{k_1} \vec{E})) \vec{E}$$

and

$$\frac{\partial}{\partial n} \vec{B} = \frac{\partial}{\partial n} (\frac{1}{k_1} \vec{n}) = \frac{\partial}{\partial n} (\frac{1}{k_1} (\frac{1}{k_1} \vec{E} - (\vec{E} + 1)) \vec{n}$$

by using Equations (3) and (8). Finally, the change of EM wave vectors between any two points on $b$ parameter curve along $\mathcal{S} = \mathcal{S}(s, n, b)$ is stated as follows

$$\frac{\partial}{\partial b} \vec{t} = \xi_b \vec{n} - \mu_n \vec{b} = \mu_n \frac{1}{k_1} \vec{B} - \xi_b k_1 \vec{B},$$

$$\frac{\partial}{\partial b} \vec{E} = \frac{\partial}{\partial b} (-k_1 \vec{B}) = -\frac{\partial k_1}{\partial b} \vec{B} - k_1 (-\xi_b \vec{E} - (\div \vec{n} \vec{b}))$$

$$= k_1 \xi_b \vec{E} + (\frac{\partial}{\partial b} (\frac{1}{k_1} \vec{B} + \frac{1}{k_1} \vec{E})) \vec{E}$$

and

$$\frac{\partial}{\partial b} \vec{B} = \frac{\partial}{\partial b} (\frac{1}{k_1} \vec{n}) = \frac{\partial}{\partial b} (\frac{1}{k_1} (\frac{1}{k_1} \vec{E} - (\vec{E} + 1)) \vec{n} - \frac{1}{k_1} (\mu_e \vec{E} + (\div \vec{n} \vec{b}))$$

$$= -\frac{1}{k_1} \mu_e \vec{E} + (\frac{\partial}{\partial b} (\frac{1}{k_1} \vec{E} + \frac{1}{k_1} \vec{B})) \vec{B}$$

by using Equations (4) and (8).

The following theorem is clearly seen by the derivative formulas in Equations (9) and (10).

Theorem 5. EM wave vectors $(\vec{t}, \vec{E}, \vec{B})$ is parallel transportation of the pseudo null frame field $(\vec{t}, \vec{n}, \vec{b})$ if and only if the geometric phase $\sigma = \sigma(s, n, b)$ satisfies the relations

$$\frac{\partial \sigma}{\partial n} = -\div \vec{b} - 1, \quad \frac{\partial \sigma}{\partial b} = \div \vec{n}$$

where $\sigma(s, n, b) = \int \tau(s, n, b) ds$. 

Proof. By Equations (9) and (10), it is seen that partial derivatives of $(\vec{t}, \vec{E}, \vec{B})$ have only tangent component if and only if

$$\frac{\partial k_1}{\partial n} + \div \vec{b} + 1 = \frac{\partial}{\partial n} (\frac{1}{k_1} \vec{k} - \div \vec{b} - 1 = 0,$$

$$\frac{\partial}{\partial b} (\frac{1}{k_1}) \vec{k} + \div \vec{n} = \frac{\partial}{\partial b} (\frac{1}{k_1}) \vec{k} + \div \vec{n} = 0.$$

On the other hand, we have

$$\frac{\partial}{\partial n} (\frac{1}{k_1}) k_1 = -\frac{\partial}{\partial n} (\frac{1}{k_1}) k_1 \quad \frac{\partial}{\partial b} (\frac{1}{k_1}) k_1 = -\frac{\partial}{\partial b} (\frac{1}{k_1}) k_1$$

where $k_1 = c_0 \sigma(s, n, b)$, $\sigma(s, n, b) = \int \tau(s, n, b) ds$ is the geometric phase and $c_0 \in \mathbb{R}_0^+$ and $k_2 = 0$.

4. A physical and geometric application

In this section, a geometric and physical application of the obtained results is presented. Let the surface $M : \mathcal{S} = \mathcal{S}(s, n)$ be given with following parametric representation

$$\mathcal{S}(s, n) = \left( \frac{s^3}{3} + \frac{s^2}{2} + 2sn + n, \frac{s^3}{3} + \frac{s^2}{2} + 2sn + n, s \right)$$

where $2s + 1 \neq 0$. The unit tangent vector is obtained as

$$\frac{\partial}{\partial s} \mathcal{S}(s, n) = \vec{t}(s, n) = (s^2 + s + 2s^2 + s + 2s + 1, 1)$$

which corresponds to the direction EM wave along the surface $M$. Then, we also find the null principal normal vector field as follows

$$\frac{\partial}{\partial n} \mathcal{S}(s, n) = \vec{n}(s, n) = (2s + 1, 2s + 1, 0).$$

Furthermore, the null binormal vector field is found as follows

$$\vec{b}(s, n) = -\frac{1}{2s + 1} (2n + s + s^2)^2 + 1, (2n + s + s^2)^2 - 1, 2(s + s^3))$$

Thus, we obtain

$$\kappa(s, n) = 1, \quad \tau(s, n) = \frac{2}{2s + 1}$$

by Equation (2). According to the derivative formulas in Equation (3), we also find

$$\mu_b(s, n) = \frac{2}{2s + 1}, \quad \xi_b(s, n) = 0, \quad \div \vec{b}(s, n) = -1.$$

Let us consider that the first Bishop curvature function $k_1 = 0$. Then we get the geometric phase

$$\sigma(s, n) = \ln |2s + 1|$$

according to the Theorem 3. By assuming $c_0 = 1$, the second Bishop curvature function is obtained as follows

$$k_2(s, n) = |2s + 1|.$$

Since

$$\frac{\partial \sigma}{\partial n} = \div \vec{b} + 1 = 0,$$

then EM wave vectors $(\vec{t}, \vec{E}, \vec{B})$ is parallel transportation of the $(\vec{t}, \vec{n}, \vec{b})$ by Theorem 3. Therefore, we obtain the electric vector field as follows

$$\vec{E}(s, n) = \frac{1}{|2s + 1|} (2s + 1, 2s + 1, 0).$$

It is seen that the direction electric vector field is constant. Finally we find magnetic vector field as

$$\vec{B}(s, n) = -\frac{|2s + 1|}{4s + 2} (2n + s + s^2)^2 + 1, (2n + s + s^2)^2 - 1, 2(2n + s + s^2)).$$

Then, the geometric phase $\sigma = \sigma(s, n)$ and the direction of magnetic field vectors $\vec{B}$ are both illustrated in Fig. 1 and Fig. 2.
5. Conclusion

Vector analysis is a helpful factor in approaching many applications in the field of geometry. The emergence of important theorems in fields such as physics, classical mechanics and field theory is realized with the help of vector analysis. Also, there is a strong relationship between geometry and physics. Vector analysis is needed to explain this relationship. Curves are an important subject of geometry as they provide many important applications in physics. Especially the curves in Minkowski space bring us closer to physical interpretations. Investigation of geometric structure of the curve is the most preferable method from this approach. For this reason, vector analysis of pseudo-null curves is preferred as a subject worth investigating in this study.

When an electric field comes into contact with a magnetic field, electromagnetic waves are generated. They’re called ‘electromagnetic’ waves because of this. The difference of this study from others is the investigation of relationship between electromagnetic field theory, which is studied in the field of physics, and pseudo null curves in Minkowski space. Thus, this study will accompany the scientists who will conduct new studies on similar subjects as an important resource since it is one of the first studies on this subject. In this context, the current research gives a new approach on the pseudo null curve. Moreover, examination of electromagnetic waves in different geometric structures will make a great contribution to the field. So, depending on the values of Bishop curvatures, which describe the linear transformations between the pseudo null frames (\(\mathbf{i}, \mathbf{n}, \mathbf{h}\)) and (\(\mathbf{i}, \mathbf{\tilde{n}}, \mathbf{\tilde{h}}\)) are investigated in different subsection. For all cases, the relations between the frames (\(\mathbf{i}, \mathbf{n}, \mathbf{h}\)) and (\(\mathbf{i}, \mathbf{\tilde{n}}, \mathbf{\tilde{h}}\)) are given. Moreover, the derivative formulas EM wave vectors are stated by means of geometric phase. Furthermore, the necessary and sufficient conditions provided by the geometric phase are expressed for EM wave vectors to be parallel transportation of the pseudo null frame by Rytov’s law. Finally, the data visualizations of the obtained results are presented in the form of a physical and geometric application.

Declarations

Author contribution statement

Melek Erdoğdu: Conceived and designed the experiments; Performed the experiments; Analyzed and interpreted the data; Contributed reagents, materials, analysis tools or data; Wrote the paper.

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