On the Complexity of Connectivity in Cognitive Radio Networks Through Spectrum Assignment

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Abstract Cognitive Radio Networks (CRNs) are considered as a promising solution to the spectrum shortage problem in wireless communication. In this paper, we initiate the first systematic study on the algorithmic complexity of the connectivity problem in CRNs through spectrum assignments. We model the network of secondary users (SUs) as a potential graph, where two nodes having an edge between them are connected as long as they choose a common available channel. In the general case, where the potential graph is arbitrary and the SUs may have different number of antennae, we prove that it is NP-complete to determine whether the network is connectable even if there are only two channels. For the special case where the number of channels is constant and all the SUs have the same number of antennae, which is more than one but less than the number of channels, the problem is also NP-complete. For the special cases in which the potential graph is complete, a tree, or a graph with bounded treewidth, we prove the problem is NP-complete and fixed-parameter tractable (FPT) when parameterized by the number of channels. Exact algorithms are also derived to determine the connectability of a given cognitive radio network.

A preliminary version of this paper appears in ALGOSENSORS 2012. The main extensions are that we give more details of our results and investigate the special case where the potential graphs are of bounded treewidth.

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1 Introduction

Cognitive Radio is a promising technology to alleviate the spectrum shortage in wireless communication. It allows the unlicensed secondary users to utilize the temporarily unused licensed spectrums, referred to as white spaces, without interfering with the licensed primary users. Cognitive Radio Networks (CRNs) is considered as the next generation of communication networks and attracts numerous research from both academia and industry recently.

In CRNs, each secondary user (SU) can be equipped with one or multiple antennae for communication. With multiple antennae, a SU can communicate on multiple channels simultaneously (in this paper, channel and spectrum are used interchangeably). Through spectrum sensing, each SU has the capacity to measure current available channels at its site, i.e. the channels are not used by the primary users (PUs). Due to the appearance of PUs, the available channels of SUs have the following characteristics [1]:

- **Spatial Variation**: SUs at different positions may have different available channels;
- **Spectrum Fragmentation**: the available channels of a SU may not be continuous; and
- **Temporal Variation**: the available channels of a SU may change over time.

Spectrum assignment is to allocate available channels to SUs to improve system performance such as spectrum utilization, network throughput and fairness. Spectrum assignment is one of the most challenging problems in CRNs and has been extensively studied such as in [12,19,20,22,23].

Connectivity is a fundamental problem in wireless communication. Connection between two nodes in CRNs is not only determined by their distance and their transmission powers, but also related to whether the two nodes has chosen a common channel. Due to the spectrum dynamics, communication in CRNs is more difficult than in the traditional multi-channel radio networks studied in [4]. Authors in [14,15,16] investigated the impact of different parameters on connectivity in large-scale CRNs, such as the number of channels, the activity of PUs, the number of neighbors of SUs and the transmission power.

In this paper, we initiate the first systematic study on the complexity of connectivity in CRNs through spectrum assignment. We model the network as a potential graph and a realized graph before and after spectrum assignment respectively (refer to Section 2). We start from the most general case, where the network is composed of heterogenous SUs. SUs may be equipped with different number of antennae and the potential graph can be arbitrary (Figure 1). Then, we proceed to study the special case when all the SUs have the same number of antennae. If all the SUs are homogenous with transmission ranges large enough, the potential graph will be a complete graph. For some hierarchically organized networks, e.g. a set of SUs are connected to an access

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1 We assume two heterogenous SUs cannot communicate even when they work on a common channel and their distance is within their transmission ranges.
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Fig. 1 the general case. a) the potential graph: the set besides each SU is its available channels, and $\beta$ is its number of antennae. $u_2$ and $u_4$ are not connected directly because they are a pair of heterogeneous nodes or their distance exceeds at least one of their transmission ranges. b) the realization graph which is connected: the set beside each SU is the channels assigned to it.

point, the potential graph can be a tree. Therefore, we also study these special cases. Exact algorithms are also derived to determine connectivity for different cases. Our results are listed below. To the best of knowledge, this is the first work that systematically studies the algorithmic complexity of connectivity in CRNs with multiple antennae.

Our Contributions: In this paper we study the algorithmic complexity of the connectivity problem through spectrum assignment under different models. Our main results are as follows.

– When the potential graph is a general graph, we prove that the problem is NP-complete even if there are only two channels. This result is sharp as the problem is polynomial-time solvable when there is only one channel. We also design exact algorithms for the problem. For the special case when all SUs have the same number of antennae, we prove that the problem is NP-complete when $k > \beta \geq 2$, where $k$ and $\beta$ are the total amount of channels in the white spaces and the number of antennae on an SU respectively.

– When the potential graph is complete, the problem is shown to be NP-complete even if each node can open at most two channels. However, in contrast to the general case, the problem is shown to be polynomial-time solvable if the number of channels is fixed. In fact, we prove a stronger result saying that the problem is fixed parameter tractable when parameterized by the number of channels. (See [5] for notations in parameterized complexity.)

– When the potential graph is a tree, we prove that the problem is NP-complete even if the tree has depth one. Similar to the complete graph case, we show that the problem is fixed parameter tractable when parameterized by the number of channels. We then generalize this result, showing that the problem remains fixed parameter tractable when parameterized

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Fig. 1: The complete graph is a special case of disk graphs, which are commonly used to model wireless networks such as in [21].
by the number of channels if the underlying potential graph has bounded treewidth.

Paper Organization: In Section 2 we formally define our model and problems studied in this paper. We study the problem with arbitrary potential graphs in Section 3. The special cases where the potential graph is complete or a tree are investigated in Sections 4 and 5. The paper is concluded in Section 6 with possible future works.

2 Preliminaries

2.1 System Model and Problem Definition

We first describe the model used throughout this paper. A cognitive radio network is comprised of the following ingredients:

- \( U \) is a collection of secondary users (SUs) and \( C \) is the set of channels in the white spaces.
- Each SU \( u \in U \) has a spectrum map, denoted by \( \text{SpecMap}(u) \), which is a subset of \( C \) representing the available channels that \( u \) can open.
- The potential graph \( P\mathcal{G} = (U, E) \), where each edge of \( E \) is also called a potential edge. If two nodes are connected by a potential edge, they can communicate as long as they choose a common available channel.
- Each SU \( u \in U \) is equipped with a number of antennas, denoted as antenna budget \( \beta(u) \), which is the maximum number of channels that \( u \) can open simultaneously.

For a set \( S \), let \( 2^S \) denote the power set of \( S \), i.e., the collection of all subsets of \( S \). A spectrum assignment is a function \( \mathcal{SA} : U \rightarrow 2^C \) satisfying

\[
\mathcal{SA}(u) \subseteq \text{SpecMap}(u) \text{ and } |\mathcal{SA}(u)| \leq \beta(u) \text{ for all } u \in U.
\]

Equivalently, a spectrum assignment is a way of SUs opening channels such that each SU opens at most \( \beta \) channels and can only open those in its spectrum map.

Given a spectrum assignment \( \mathcal{SA} \), a potential edge \( \{u, v\} \in E \) is called realized if \( \mathcal{SA}(u) \cap \mathcal{SA}(v) \neq \emptyset \), i.e., there exists a channel opened by both \( u \) and \( v \). The realization graph under a spectrum assignment is a graph \( R\mathcal{G} = (U, E') \), where \( E' \) is the set of realized edges in \( E \). Note that \( R\mathcal{G} \) is a spanning subgraph of the potential graph \( P\mathcal{G} \). A cognitive radio network is called connectable if there exists a spectrum assignment under which the realization graph is connected, in which case we also say that the cognitive radio network is connected under this spectrum assignment. Now we can formalize the problems studied in this paper.

The Spectrum Connectivity Problem. The Spectrum Connectivity problem is to decide whether a given cognitive radio network is connectable.
We are also interested in the special case where the number of possible channels is small and SUs have the same antenna budget. Therefore, we define the following subproblem of the Spectrum Connectivity problem:

**The Spectrum \((k, \beta)-Connectivity Problem.** For two constants \(k, \beta \geq 1\), the Spectrum \((k, \beta)-Connectivity\) problem is to decide whether a given cognitive radio network with \(k\) channels in which all SUs have the same budget \(\beta\) is connectable. For convenience we write \(\text{SpecCon}(k, \beta)\) to represent this problem.

Finally, we also consider the problem with special kinds of potential graphs, i.e. the potential graph is complete or a tree.

In the sequel, unless otherwise stated, we always use \(n := |U|\) and \(k := |C|\) to denote respectively the number of secondary users and channels.

### 2.2 Tree Decomposition

In this subsection we give some basic notions regarding the tree decomposition of a graph, which will be used later. The concept of treewidth was introduced by Robertson and Seymour in their seminal work on graph minors [17]. A tree decomposition of a graph \(G = (V, E)\) is given by a tuple \((T = (I, F), \{X_i \mid i \in I\})\), where \(T\) is a tree and each \(X_i\) is a subset of \(V\) called a bag satisfying that

- \(\bigcup_{i \in I} X_i = V\);
- For each edge \(\{u, v\} \in E\), there exists a tree node \(i\) with \(\{u, v\} \subseteq X_i\);
- For each vertex \(u \in V\), the set of tree nodes \(\{i \in I \mid u \in X_i\}\) forms a connected subtree of \(T\). Equivalently, for any three vertices \(t_1, t_2, t_3 \in I\) such that \(t_2\) lies in the path from \(t_1\) to \(t_3\), it holds that \(X_{t_1} \cap X_{t_3} \subseteq X_{t_2}\).

The width of the tree decomposition is \(\max_{i \in I} \{|X_i| - 1\}\), and the treewidth of a graph \(G\) is the minimum width of a tree decomposition of \(G\). For each fixed integer \(d\), there is a polynomial time algorithm that decides whether a given graph has treewidth at most \(d\), and if so, constructs a tree decomposition of width \(d\) [2]. Such a decomposition can easily be transformed to a nice tree decomposition \((T, \{X_i\})\) of \(G\) with the same width, in which \(T\) is a rooted binary tree with at most \(O(|V|)\) nodes (see e.g. [10]).

### 3 The Spectrum Connectivity Problem

In this section, we study the the Spectrum Connectivity problem from both complexity and algorithmic points of view.

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3 Commonly, the white spaces include spectrums from channel 21 (512Mhz) to 51 (608Mhz) excluding channel 37, which is totally 29 channels [3].
3.1 NP-completeness Results

We show that the Spectrum Connectivity problem is NP-complete even if the number of channels is fixed. In fact we give a complete characterization of the complexity of SpecCon\((k, \beta)\) by proving the following dichotomy result:

**Theorem 1** SpecCon\((k, \beta)\) is NP-complete for any integers \(k > \beta \geq 2\), and is in P if \(\beta = 1\) or \(k \leq \beta\).

The second part of the statement is easy: When \(\beta = 1\), each SU can only open one channel, and thus all SUs should be connected through the same channel. Therefore, the network is connectable if and only if there exists a channel that belongs to every SU’s spectrum map (and of course the potential graph must be connected), which is easy to check. When \(k \leq \beta\), each SU can open all channels in its spectrum map, and the problem degenerates to checking the connectivity of the potential graph.

In the sequel we prove the NP-completeness of SpecCon\((k, \beta)\) when \(k > \beta \geq 2\). First consider the case \(k = \beta + 1\). We will reduce a special case of the Boolean Satisfiability (SAT) problem, which will be shown to be NP-complete, to SpecCon\((\beta + 1, \beta)\), thus showing the NP-completeness of the latter.

A clause is called *positive* if it only contains positive literals, and is called *negative* if it only contains negative literals. For example, \(x_1 \lor x_3 \lor x_5\) is positive and \(x_2 \lor x_4\) is negative. A clause is called *uniform* if it is positive or negative. A uniform CNF formula is the conjunction of uniform clauses.

**Lemma 1** Uniform-SAT is NP-complete.

**Proof** Let \(F\) be a CNF formula with variable set \(\{x_1, x_2, \ldots, x_n\}\). For each \(i\) such that \(\overline{x_i}\) appears in \(F\), we create a new variable \(y_i\), and do the following:

- substitute \(y_i\) for all occurrences of \(\overline{x_i}\);
- add two clauses \(x_i \lor y_i\) and \(x_i \lor \overline{y_i}\) to \(F\). More formally, let \(F \leftarrow F \land (x_i \lor y_i) \land (\overline{x_i} \lor \overline{y_i})\). This ensures \(y_i = \overline{x_i}\) in any satisfying assignment of \(F\).

Call the new formula \(F'\). For example, if \(F = (x_1 \lor \overline{x_2}) \land (\overline{x_1} \lor x_3)\), then \(F' = (x_1 \lor y_2) \land (y_1 \lor x_3) \land (x_1 \lor y_1) \land (\overline{x_1} \lor \overline{y_1}) \land (x_2 \lor y_2) \land (\overline{x_2} \lor \overline{y_2})\).

It is easy to see that \(F'\) is a uniform CNF formula, and that \(F\) is satisfiable if and only if \(F'\) is satisfiable. This constitutes a reduction from SAT to Uniform-SAT, which concludes the proof. \(\square\)

**Theorem 2** SpecCon\((\beta + 1, \beta)\) is NP-complete for any integer \(\beta \geq 2\).

**Proof** The membership of SpecCon\((\beta + 1, \beta)\) in NP is clear. In what follows we reduce Uniform-SAT to SpecCon\((\beta + 1, \beta)\), which by Lemma 1 will prove the NP-completeness of the latter.

Let \(c_1 \land c_2 \land \ldots \land c_m\) be an input to Uniform-SAT where \(c_j, 1 \leq j \leq m,\) is a uniform clause. Assume the variable set is \(\{x_1, x_2, \ldots, x_n\}\). We construct an instance of SpecCon\((\beta + 1, \beta)\) as follows.
Channels: There are $\beta + 1$ channels $\{0, 1, 2, \ldots, \beta\}$.

SUs:
- For each variable $x_i$, there is a corresponding SU $X_i$ with spectrum map $\text{SpecMap}(X_i) = \{0, 1, 2, \ldots, \beta\}$ (which contains all possible channels);
- for each clause $c_j$, $1 \leq j \leq m$, there is a corresponding SU $C_j$ with $\text{SpecMap}(C_j) = \{p_j\}$, where $p_j = 1$ if $c_j$ is positive and $p_j = 0$ if $c_j$ is negative;
- there is an SU $Y_2$ with $\text{SpecMap}(Y_2) = \{2\}$. For every $1 \leq i \leq n$ and $2 \leq k \leq \beta$, there is an SU $Y_{i,k}$ with $\text{SpecMap}(Y_{i,k}) = \{k\}$; and
- all SUs have the same antenna budget $\beta$.

Potential Graph: For each clause $c_j$ and each variable $x_i$ that appears in $c_j$ (either as $x_i$ or $\overline{x_i}$), there is a potential edge between $X_i$ and $C_j$. For each $1 \leq i \leq n$ and $3 \leq k \leq \beta$, there is a potential edge between $X_i$ and $Y_{i,k}$. Finally, there is a potential edge between $Y_2$ and every $X_i$, $1 \leq i \leq n$.

Denote the above cognitive radio network by $\mathcal{I}$, which is also an instance of $\text{SpecCon}(\beta + 1, \beta)$. We now prove that $c_1 \land c_2 \land \ldots \land c_m$ is satisfiable if and only if $\mathcal{I}$ is connectable.

First consider the “only if” direction. Let $A : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ be a satisfying assignment of $c_1 \land c_2 \land \ldots \land c_m$, where 0 stands for FALSE and 1 for TRUE. Define a spectrum assignment as follows. For each $1 \leq i \leq n$, let user $X_i$ open the channels $\{2, 3, \ldots, \beta\} \cup \{A(i)\}$. Every other SU opens the only channel in its spectrum map.

We verify that $\mathcal{I}$ is connected under the above spectrum assignment. For each $1 \leq i \leq n$, $X_i$ is connected to $Y_2$ through channel 2. Then, for every $2 \leq l \leq \beta$, $Y_{i,l}$ is connected to $X_i$ through channel $l$. Now consider SU $C_j$ where $1 \leq j \leq m$. Since $A$ satisfies the clause $c_j$, there exists $1 \leq i \leq n$ such that: 1) $x_i$ or $\overline{x_i}$ occurs in $c_j$; and 2) $A(x_i) = 1$ if $c_j$ is positive, and $A(x_i) = 0$ if $c_j$ is negative. Thus $X_i$ and $C_j$ are connected through channel $A(x_i)$. Therefore the realization graph is connected, completing the proof of the “only if” direction.

We next consider the “if” direction. Suppose there is a spectrum assignment that makes $\mathcal{I}$ connected. For every $1 \leq i \leq n$ and $2 \leq l \leq \beta$, $X_i$ must open channel $l$, otherwise $Y_{i,l}$ will become an isolated vertex in the realization graph. Since $X_i$ can open at most $\beta$ channels in total, it can open at most one of the two remaining channels $\{0, 1\}$. We assume w.l.o.g. that $X_i$ opens exactly one of them, which we denote by $a_i$.

Now, for the formula $c_1 \land c_2 \land \ldots \land c_m$, we define a truth assignment $A : \{x_1, \ldots, x_n\} \rightarrow \{0, 1\}$ as $A(x_i) = a_i$ for all $1 \leq i \leq n$. We show that $A$ satisfies the formula. Fix $1 \leq j \leq m$ and assume that $c_j$ is negative (the case where $c_j$ is positive is totally similar). Since the spectrum map of SU $C_j$ only contains channel 0, some of its neighbors must open channel 0. Hence, there exists $1 \leq i \leq n$ such that $\overline{x_i}$ appears in $c_j$ and the corresponding SU $X_i$ opens channel 0. By our construction of $A$, we have $A(x_i) = 0$, and thus the clause $c_j$ is satisfied by $A$. Since $j$ is chosen arbitrarily, the formula
\[ c_1 \land c_2 \land \ldots \land c_m \] is satisfied by \( A \). This completes the reduction from \textsc{Uniform-SAT} to \textsc{SpecCon}(\beta, \beta + 1), and the theorem follows. \( \Box \)

**Corollary 1** \textsc{SpecCon}(\( k, \beta \)) is \( \text{NP} \)-complete for any integers \( k > \beta \geq 2 \).

*Proof* By a simple reduction from \textsc{SpecCon}(\( \beta + 1, \beta \)): Given an instance of \textsc{SpecCon}(\( \beta + 1, \beta \)), create \( k - \beta - 1 \) new channels and add them to the spectrum map of an (arbitrary) SU. This gives an instance of \textsc{SpecCon}(\( k, \beta \)). Since the new channels are only contained in one SU, they should not be opened, and thus the two instances are equivalent. Hence the theorem follows. \( \Box \)

Theorem 2 indicates that the Spectrum Connectivity problem is \( \text{NP} \)-complete even if the cognitive radio network only has three channels. We further strengthen this result by proving the following theorem:

**Theorem 3** The Spectrum Connectivity problem is \( \text{NP} \)-complete even if there are only two channels.

*Proof* We present a reduction from \textsc{Uniform-SAT} similar as in the proof of Theorem 2. Let \( c_1 \land c_2 \land \ldots \land c_m \) be a uniform CNF clause with variable set \( \{ x_1, x_2, \ldots, x_n \} \). Construct a cognitive radio network as follows: There are two channels \( \{ 0,1 \} \). For each variable \( x_i \) there is a corresponding SU \( X_i \) with spectrum map \( \text{SpecMap}(X_i) = \{ 0, 1 \} \) and antenna budget \( \beta(X_i) = 1 \). For each clause \( c_j \) there is a corresponding SU \( C_j \) with \( \text{SpecMap}(C_j) = \{ p_j \} \) and \( \beta(C_j) = 1 \), where \( p_j = 1 \) if \( c_j \) is positive and \( p_j = 0 \) if \( c_j \) is negative. There is an SU \( Y \) with \( \text{SpecMap}(Y) = \{ 0, 1 \} \) and \( \beta(Y) = 2 \). Note that, unlike in the case of \textsc{SpecCon}(\( k, \beta \)), SUs can have different antenna budgets. Finally, the edges of the potential graph include: \( \{ X_i, C_j \} \) for all \( i, j \) such that \( x_i \) or \( \overline{x_i} \) appears in \( c_j \), and \( \{ Y, X_i \} \) for all \( i \). This completes the construction of the cognitive radio network, which is denoted by \( I \). By an analogous argument as in the proof of Theorem 2, \( c_1 \land c_2 \land \ldots \land c_m \) is satisfiable if and only if \( I \) is connectable, concluding the proof of Theorem 3. \( \Box \)

Theorem 3 is sharp in that, as noted before, the problem is polynomial-time solvable when there is only one channel.

### 3.2 Exact Algorithms

In this subsection we design algorithms for deciding whether a given cognitive radio network is connectable. Since the problem is \( \text{NP} \)-complete, we cannot expect a polynomial time algorithm.

Let \( n, k, t \) denote the number of SUs, the number of channels, and the maximum size of any SU’s spectrum map, respectively (\( t \leq k \)). The simplest idea is to exhaustively examine all possible spectrum assignments to see if there exists one that makes the network connected. Since each SU can have at most \( 2^t \) possible ways of opening channels, the number of assignments is at most \( 2^{tn} \). Checking each assignment takes \( \text{poly}(n, k) \) time. Thus the running
time of this approach is bounded by $2^{tn(nk)^{O(1)}}$, which is reasonable when $t$ is small. However, since in general $t$ can be as large as $k$, this only gives a $2^{O(kn)}$ bound, which is unsatisfactory if $k$ is large. In the following we present another algorithm for the problem that runs faster than the above approach when $k$ is large.

**Theorem 4** There is an algorithm that decides whether a given cognitive radio network is connectable in time $2^{O(k + n \log n)}$.

**Proof** Let $\mathcal{I}$ be a given cognitive radio network with potential graph $PG$. Let $n$ be the number of SUs and $k$ the number of channels. Assume that $\mathcal{I}$ is connected under some spectrum assignment. Clearly the realization graph contains a spanning tree of $PG$, say $T$, as a subgraph. If we change the potential graph to $T$ while keeping all other parameters unchanged, the resulting network will still be connected under the same spectrum assignment. Thus, it suffices to check whether there exists a spanning tree $T$ of $G$ such that $\mathcal{I}$ is connectable when substituting $T$ for $PG$ as its potential graph. Using the algorithm of [7], we can list all spanning trees of $PG$ in time $O(Nn)$ where $N$ is the number of spanning trees of $PG$. By Cayley’s formula [3,18] we have $N \leq n^{n-2}$. Finally, for each spanning tree $T$, we can use the algorithm in Theorem 9 (which will appear in Section 5) to decide whether the network is connectable in time $2^{O(knO(1))}$. The total running time of the algorithm is $O(n^{n-2})2^{O(knO(1))} = 2^{O(k + n \log n)}$. $\square$

Combining Theorem 4 with the brute-force approach, we obtain:

**Corollary 2** The Spectrum Connectivity problem is solvable can be solved in time $2^{O(\min\{kn, k + n \log n\})}$.

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**4 Spectrum Connectivity with Complete Potential Graphs**

In this section we consider the special case of the Spectrum Connectivity problem, in which the potential graph of the cognitive radio network is complete. We first show that this restriction does not make the problem tractable in polynomial time.

**Theorem 5** The Spectrum Connectivity problem is NP-complete even when the potential graph is complete and all SUs have the same antenna budget $\beta = 2$.

**Proof** The membership in NP is trivial. The hardness proof is by a reduction from the Hamiltonian Path problem, which is to decide whether a given graph contains a Hamiltonian path, i.e., a simple path that passes every vertex exactly once. The Hamiltonian Path problem is well-known to be NP-complete [8]. Let $G = (V, E)$ be an input graph of the Hamiltonian Path problem. Construct an instance of the Spectrum Connectivity problem as follows: The collection of channels is $E$ and the set of SUs is $V$; that
is, we identify a vertex in $V$ as an SU and an edge in $E$ as a channel. For every $v \in V$, the spectrum map of $v$ is the set of edges incident to $v$. All SUs have antenna budget $\beta = 2$. Denote this cognitive radio network by $\mathcal{I}$. We will prove that $G$ contains a Hamiltonian path if and only if $\mathcal{I}$ is connectable.

First suppose $G$ contains a Hamiltonian path $P = v_1v_2\ldots v_n$, where $n = |V|$. Consider the following spectrum assignment of $\mathcal{I}$: for each $1 \leq i \leq n$, let SU $v_i$ open the channels corresponding to the edges incident to $v_i$ in the path $P$. Thus all SUs open two channels except for $v_1$ and $v_n$, each of whom opens only one. For every $1 \leq i \leq n - 1$, $v_i$ and $v_{i+1}$ are connected through the channel (edge) $\{v_i, v_{i+1}\}$. Hence the realization graph of $\mathcal{I}$ under this spectrum assignment is connected.

Now we prove the other direction. Assume that $\mathcal{I}$ is connectable. Fix a spectrum assignment under which the realization graph of $\mathcal{I}$ is connected, and consider this particular realization graph $RG = (V, E')$. Let $\{v_i, v_j\}$ be an arbitrary edge in $E'$. By the definition of the realization graph, there is a channel opened by both $v_i$ and $v_j$. Thus there is an edge in $E$ incident to both $v_i$ and $v_j$, which can only be $\{v_i, v_j\}$. Therefore $\{v_i, v_j\} \in E$. This indicates $E' \subseteq E$, and hence $RG$ is a connected spanning subgraph of $G$. Since each SU can open at most two channels, the maximum degree of $RG$ is at most 2. Therefore $RG$ is either a Hamiltonian path of $G$, or a Hamiltonian cycle which contains a Hamiltonian path of $G$. Thus, $G$ contains a Hamiltonian path.

The reduction is complete and the theorem follows. \qed

Notice that the reduction used in the proof of Theorem 5 creates a cognitive radio network with an unbounded number of channels. Thus Theorem 5 is not stronger than Theorems 1 or 3. Recall that Theorem 3 says the Spectrum Connectivity problem is NP-complete even if there are only two channels. In contrast we will show that, with complete potential graphs, the problem is polynomial-time tractable when the number of channels is small.

**Theorem 6** The Spectrum Connectivity problem with complete potential graphs can be solved in $2^{2k} + O(k) \cdot n^{O(1)}$ time.

**Proof** Consider a cognitive radio network $\mathcal{I}$ with SU set $U$, channel set $C$ and a complete potential graph, i.e., there is a potential edge between every pair of distinct SUs. Recall that $n = |U|$ and $k = |C|$. For each spectrum assignment $\mathcal{SA}$, we construct a corresponding *spectrum graph* $\mathcal{G}_{\mathcal{SA}} = (V, E)$ where $V = \{C' \subseteq C \mid \exists u \in U \text{ s.t. } \mathcal{SA}(u) = C'\}$ and $E = \{\{C_1, C_2\} \mid C_1, C_2 \in V; C_1 \cap C_2 \neq \emptyset\}$. Thus, $V$ is the collection of subsets of $C$ that is opened by some SU, and $E$ reflects the connectivity between pairs of SUs that open the corresponding channels. Since each vertex in $V$ is a subset of $C$, we have $|V| \leq 2^k$, and the number of different spectrum graphs is at most $2^{2^k}$.

We now present a relation between $\mathcal{G}_{\mathcal{SA}}$ and the realization graph of $\mathcal{I}$ under $\mathcal{SA}$. If we label each vertex $u$ in the realization graph with $\mathcal{SA}(u)$, and contract all edges between vertices with the same label, then we obtain precisely the spectrum graph $\mathcal{G}_{\mathcal{SA}} = (V, E)$. Therefore, in the language of graph theory, $\mathcal{G}_{\mathcal{SA}} = (V, E)$ is a minor of the realization graph under $\mathcal{SA}$.\[\]
Since graph minor preserves connectivity, \( \mathcal{I} \) is connectable if and only if there exists a connected spectrum graph. Hence we can focus on the problem of deciding whether a connected spectrum graph exists.

Consider all possible graphs \( G = (V, E) \) such that \( V \subseteq 2^C \), and \( E = \{ \{C_1, C_2\} \mid C_1, C_2 \in V; C_1 \cap C_2 \neq \emptyset \} \). There are \( 2^{2^k} \) such graphs each of which has size \( 2^{O(k)} \). Thus we can list all such graphs in \( 2^{2^k + O(k)} \) time. For each graph \( G \), we need to check whether it is the spectrum graph of some spectrum assignment of \( \mathcal{I} \). We create a bipartite graph in which nodes on the left side are the SUs in \( \mathcal{I} \), and nodes on the right side all the vertices of \( G \). We add an edge between an SU \( u \) and a vertex \( C' \) of \( G \) if and only if \( C' \subseteq \text{SpecMap}(u) \) and \( |C'| \leq \beta(u) \), that is, \( u \) can open \( C' \) in a spectrum assignment. The size of \( H \) is \( \text{poly}(n, 2^k) \) and its construction can be finished in \( \text{poly}(n, 2^k) \) time.

Now, if \( G \) is the spectrum graph of some spectrum assignment \( \mathcal{SA} \), then we can identify \( \mathcal{SA} \) with a subgraph of \( H \) consisting of all edges \( (u, \mathcal{SA}(u)) \) where \( u \) is an SU. In addition, in this subgraph we have

- every SU \( u \) has degree exactly one; and
- every node \( C' \) on the right side of \( H \) has degree at least one.

Conversely, a subgraph of \( H \) satisfying the above two conditions clearly induces a spectrum assignment whose spectrum graph is exactly \( G \). Therefore it suffices to examine whether \( H \) contains such a subgraph. Furthermore, the above conditions are easily seen to be equivalent to:

- every SU \( u \) has degree at least one in \( G \); and
- \( G \) contains a matching that includes all nodes on the right side.

The first condition can be checked in time linear in the size of \( H \), and the second one can be examined by any polynomial time algorithm for bipartite matching (e.g., \[9\]). Therefore, we can decide whether such subgraph exists (and find one if so) in \( \text{time poly}(n, 2^k) \). By our previous analyses, this solves the Spectrum Connectivity problem with complete potential graphs. The total running time of our algorithm is \( 2^{2^k + O(k)} \text{poly}(n, 2^k) = 2^{2^k + O(k)}n^O(1) \).

\[ \square \]

**Theorem 7** The Spectrum Connectivity problem with complete potential graphs is fixed parameter tractable (FPT) when parameterized by the number of channels.

## 5 Spectrum Connectivity on Trees and Bounded Treewidth Graphs

In this section, we study another special case of the Spectrum Connectivity problem where the potential graph of the cognitive radio network is a tree. We will also investigate the problem on the class of bounded-treewidth graphs. Many NP-hard combinatorial problems become easy on trees, e.g., the dominating set problem and the vertex cover problem. Nonetheless, as indicated by the following theorem, the Spectrum Connectivity problem remains hard on trees.
5.1 Trees

We state the complexity of the spectrum connectivity problem with trees as the potential graph in the following theorem.

**Theorem 8** The Spectrum Connectivity problem is NP-complete even if the potential graph is a tree of depth one.

*Proof* We give a reduction from the Vertex Cover problem which is well known to be NP-complete \([72x-250]3\). Given a graph \(G = (V, E)\) and an integer \(r\), the Vertex Cover problem is to decide whether there exists \(r\) vertices in \(V\) that cover all the edges in \(E\). Construct a cognitive radio network \(\mathcal{I}\) as follows. The set of channels is \(C = \{c_v \mid v \in V\}\). For each edge \(e = \{u, v\} \in E\) there is an SU \(U_e\) with \(\text{SpecMap}(U_e) = \{c_u, c_v\}\) and antenna budget 2. There is another SU \(M\) with \(\text{SpecMap}(M) = C\) and antenna budget \(r\). The potential graph is a star centered at \(M\), that is, there is a potential edge between \(M\) and \(U_e\) for every \(e \in E\). This finishes the construction of \(\mathcal{I}\).

We prove that \(G\) has a vertex cover of size \(r\) if and only if \(\mathcal{I}\) is connectable. First assume \(G\) has a vertex cover \(S \subseteq V\) with \(|S| \leq r\). Define a spectrum assignment \(A(S)\) as follows: \(M\) opens the channels \(\{c_v \mid v \in S\}\), and \(U_e\) opens both channels in its spectrum map for all \(e \in E\). Since \(S\) is a vertex cover, we have \(u \in S\) or \(v \in S\) for each \(e = \{u, v\} \in E\). Thus at least one of \(c_u\) and \(c_v\) is opened by \(M\), which makes it connected to \(U_e\). Hence the realization graph is connected. On the other hand, assume that the realization graph is connected under some spectrum assignment. For each edge \(e = \{u, v\} \in E\), since the potential edge \(\{M, U_e\}\) is realized, \(M\) opens at least one of \(c_u\) and \(c_v\). Now define \(S = \{v \in V \mid c_v\) is opened by \(M\}\). It is clear that \(S\) is a vertex cover of \(G\) of size at most \(\beta(M) = r\). This completes the reduction, and the theorem follows. \(\Box\)

We next show that, in contrast to Theorems \([2, 3]\) and \([8]\) this special case of the problem is polynomial-time solvable when the number of channels is small.

**Theorem 9** Given a cognitive radio network whose potential graph is a tree, we can check whether it is connectable in \(2^{O(kn)}\) time, where \(t\) is the maximum size of any SU’s spectrum map. In particular, this running time is at most \(2^{O(kn)}\).

*Proof* Let \(\mathcal{I}\) be a given cognitive radio network whose potential graph \(\mathcal{PG} = (V, E)\) is a tree. Root \(\mathcal{PG}\) at an arbitrary node, say \(r\). For each \(v \in V\) let \(\mathcal{PG}_v\) denote the subtree rooted at \(v\), and let \(\mathcal{I}_v\) denote the cognitive radio network obtained by restricting \(\mathcal{I}\) on \(\mathcal{PG}_v\). For every subset \(S \subseteq \text{SpecMap}(v)\), define \(f(v, S)\) to be 1 if there exists a spectrum assignment that makes \(\mathcal{I}_v\) connected in which the set of channels opened by \(v\) is exactly \(S\); let \(f(v, S) = 0\) otherwise. For each channel \(c \in C\), define \(g(v, c)\) to be 1 if there exists \(S, \{c\} \subseteq S \subseteq \text{SpecMap}(v)\), for which \(f(v, S) = 1\); define \(g(v, c) = 0\) otherwise. Clearly \(\mathcal{I}\) is connectable if and only if there exists \(S \subseteq \text{SpecMap}(r)\) such that \(f(r, S) = 1\).
We compute all \( f(v, S) \) and \( g(v, c) \) by dynamic programming in a bottom-up manner. Initially all values to set to 0. The values for leaf nodes are easy to obtain. Assume we want to compute \( f(v, S) \), given that the values of \( f(v', S') \) and \( g(v', c) \) are all known if \( v' \) is a child of \( v \). Then \( f(v, S) = 1 \) if and only if for every child \( v' \) of \( v \), there exists \( c \in S \) such that \( g(v', c) = 1 \) (in which case \( v \) and \( v' \) are connected through channel \( c \)). If \( f(v, S) \) turns out to be 1, we set \( g(v, c) \) to 1 for all \( c \in S \). It is easy to see that \( g(v, c) \) will be correctly computed after the values of \( f(v, S) \) are obtained for all possible \( S \). After all values have been computed, we check whether \( f(r, S) = 1 \) for some \( S \subseteq \text{SpecMap}(r) \).

Recall that \( n = |V|, \ k = |C|, \) and denote \( t = \max_{v \in V} |\text{SpecMap}(v)| \). There are at most \( n(2^t + k) \) terms to be computed, each of which takes time \( \text{poly}(n, k) \) by our previous analysis. The final checking step takes \( 2^t \text{poly}(n, k) = 2'(kn)^O(1) \), which is at most \( 2^O(k)n^O(1) \) since \( t \leq k \). Finally note that it is easy to modify the algorithm so that, given a connectable network it will return a spectrum assignment that makes it connected. 

**Corollary 3** The **Spectrum Connectivity** problem with trees as potential graphs is fixed parameter tractable when parameterized by the number of channels.

### 5.2 Bounded Treewidth Graphs

In this part we deal with another class of potential graphs, namely the class of graphs with bounded treewidth. Our main result is the following theorem, which generalizes Theorem 9 as a tree has treewidth one.

**Theorem 10** There is an algorithm that, given a cognitive radio network whose potential graph has bounded treewidth, checks whether it is connectable in \( 2^O(k)n^O(1) \) time.

**Proof** Suppose we are given a cognitive radio network \( I \) with potential graph \( G = (V, E) \), which has treewidth \( \text{tw} = O(1) \). Let \( (T = (I, F), \{X_i \mid i \in I\}) \) be a nice tree decomposition of \( G \) of width \( \text{tw} \) (see Section 2.2 for the related notions). Recall that \( T \) is a rooted binary tree with \( O(|V|) \) nodes and can be found in polynomial time. Let \( r \) be the root of \( T \). For every non-leaf node \( i \) of \( T \), let \( i_L \) and \( i_R \) be the two children of \( i \). (We can always add dummy leaf-nodes to make every non-leaf node have exactly two children, which at most doubles the size of \( T \).)

For each \( i \in I \), define
\[
Y_i := \{ v \in X_j \mid j = i \text{ or } j \text{ is a descendent of } i \},
\]
and let \( I_i \) be a new instance of the problem that is almost identical to \( I \) except that we replace the potential graph with \( G[Y_i] \), i.e., the subgraph of \( G \) induced on the vertex set \( Y_i \subseteq V \).
For each \( i \in I \), suppose \( X_i = \{v_1, v_2, \ldots, v_t\} \) where \( t = |X_i| \) and \( v_j \in V \) for all \( 1 \leq j \leq t \). For each tuple \((S_1, S_2, \ldots, S_t)\) such that \( S_j \subseteq \text{SpecMap}(v_j) \) for all \( 1 \leq j \leq t \), we use a Boolean variable \( B_i(S_1, S_2, \ldots, S_t) \) to indicate whether there exists a spectrum assignment \( \mathcal{SA}_i \) that makes \( I_i \) connected such that \( \mathcal{SA}_i(v_j) = S_j \) for all \( 1 \leq j \leq t \). Notice that for each \( i \), the number of such variables is at most \((2k)^{|X_i|} \leq 2^{k\cdot tw}\), and we can list them in \( 2^{O(k\cdot tw)} \) time. Initially all variables are set to FALSE. Assume \( X_T = \{w_1, w_2, \ldots, w_{|X_T|}\} \) (recall that \( r \) is the root of \( T \)). Then, clearly, deciding whether \( I \) is connectable is equivalent to checking whether there exists \((S_1, S_2, \ldots, S_{|X_T|})\), where \( S_j \subseteq \text{SpecMap}(w_j) \) for all \( 1 \leq j \leq |X_T| \), such that \( B_i(S_1, S_2, \ldots, S_{|X_T|}) \) is TRUE.

We will compute the values of all possible \( B_i(S_1, S_2, \ldots, S_t) \) by dynamic programming. For each leaf node \( i \), we can compute the values of all the variables related to \( I_i \) in time \( 2^{O(k\cdot tw)}\) by the brute-force approach.

Now suppose we want to decide the value of \( B_i(S_1, S_2, \ldots, S_{|X_i|}) \) for some non-leaf node \( i \), provided that the variables related to any children of \( i \) have all been correctly computed. Recall that \( i_L \) and \( i_R \) are the two children of \( i \). We define:

- \( NEW = X_i \setminus (Y_{i_L} \cup Y_{i_R}) \);
- \( OLD = X_i \setminus NEW = X_i \cap (Y_{i_L} \cup Y_{i_R}) \);
- \( Z_L = Y_{i_L} \setminus X_i \), and \( Z_R = Y_{i_R} \setminus X_i \).

It is clear that \( Y_i = NEW \cup OLD \cup Z_L \cup Z_R \). By using the properties of a tree decomposition, we have the following fact:

**Lemma 2** \( NEW \), \( Z_L \), and \( Z_R \) are three pairwise disjoint subsets of \( V \), and there is no edge of \( G \) whose endpoints lie in different subsets.

**Proof** Since \( NEW \subseteq X_i \) and \( Z_L = Y_{i_L} \setminus X_i \), we have \( NEW \cap Z_L = \emptyset \), and similarly \( NEW \cap Z_R = \emptyset \). Assume that \( Z_L \cap Z_R \neq \emptyset \), and let \( v \in Z_L \cap Z_R \). Since \( Z_L \subseteq Y_{i_L} \) and \( Z_R \subseteq Y_{i_R} \), we have \( v \in Y_{i_L} \cap Y_{i_R} \). By the definition of a tree decomposition, \( v \in X_i \), so \( v \in X_i \cap Z_L = X_i \cap (Y_{i_L} \setminus X_i) = \emptyset \), a contradiction. Therefore \( Z_L \cap Z_R = \emptyset \). This proves the pairwise disjointness of the three sets.

Now assume that there exists an edge \( e = (u, v) \in E \) such that \( u \in Z_L \) and \( v \in Z_R \). Then, by the definition of a tree decomposition, there exists \( p \in I \) such that \( \{u, v\} \subseteq X_p \). We know that \( p \neq i \). So there are three possibilities: \( p \) lines in the subtree rooted at \( X_{i_L} \), or in the subtree rooted at \( X_{i_R} \), or it is not in the subtree rooted at \( X_i \). It is easy to verify that, in each of the three cases, we can find a path that connects two tree nodes both containing \( u \) (or \( v \)) and goes through \( i \), which implies \( u \in X_i \) or \( v \in X_i \) by the property of a tree decomposition. This contradicts our previous result. Thus there is no edge with one endpoint in \( Z_L \) and another in \( Z_R \). Similarly, we can prove that there exists no edge with one endpoint in \( NEW \) and another in \( Z_L \) or \( Z_R \). This completes the proof of the lemma. \( \square \)

We now continue the proof of Theorem 10. Recall that we want to decide \( B_i(S_1, S_2, \ldots, S_{|X_i|}) \), i.e., whether \( I_i \), the network with \( G[Y_i] \) as the potential...
graph, is connectable under some spectrum assignment $\mathcal{S}_A$ such that $\mathcal{S}_A(v_j) = S_j$ for all $1 \leq j \leq |X_i|$ (we assume that $X_i = \{v_1, v_2, \ldots, v_{|X_i|}\}$). Note that $Y_i = NEW \cup OLD \cup Z_L \cup Z_R$. Due to Lemma 2, the three subsets $NEW$, $Z_L$ and $Z_R$ can only be connected through $OLD$ (or, we can think $OLD$ as an “intermediate” set). Therefore, for any spectrum assignment $\mathcal{S}_A$ such that $\mathcal{S}_A(v_j) = S_j$ for all $j$, $T_i$ is connected under $\mathcal{S}_A$ if and only if the following three things simultaneously hold:

- $G[X_i]$ is connected under $\mathcal{S}_A$;
- $B_L(S_1', \ldots, S_{|X_i|}')$ is TRUE for some $(S_1', \ldots, S_{|X_i|}')$ that accords with $(S_1, \ldots, S_{|X_i|})$, i.e., the two vectors coincide on any component corresponding to a vertex in $X_i \cap X_L$;
- $B_R(S_1', \ldots, S_{|X_i|}')$ is TRUE for some $(S_1', \ldots, S_{|X_i|}')$ that accords with $(S_1, \ldots, S_{|X_i|})$, i.e., the two vectors coincide on any component corresponding to a vertex in $X_i \cap X_R$.

The first condition above can be checked in polynomial time, and the last two conditions can be verified in $2^{O(k \cdot tw)} n^{O(1)}$ time. Thus the time spent on determining $B_i(S_1, \ldots, S_{|X_i|})$ is $2^{O(k \cdot tw)} n^{O(1)}$. After all such terms have been computed, we can get the correct answer by checking whether there exists $(S_1, \ldots, S_{|X_i|})$ such that $B_r(S_1, \ldots, S_{|X_i|})$ is TRUE, which costs another $2^{O(k \cdot tw)} n^{O(1)}$ time. Since there are at most $O(|V|) = O(n)$ nodes in $T$, the total running time of the algorithm is $2^{O(k \cdot tw)} n^{O(1)} = 2^{O(k)} n^{O(1)}$ as $tw = O(1)$. The proof is complete.

\textbf{Corollary 4} The Spectrum Connectivity problem on bounded treewidth graphs is fixed parameter tractable when parameterized by the number of channels.

6 Conclusion and Future Work

In this paper, we initiate a systematic study on the algorithmic complexity of connectivity problem in cognitive radio networks through spectrum assignment. The hardness of the problem in the general case and several special cases are addressed, and exact algorithms are also derived to check whether the network is connectable.

In some applications, when the given cognitive radio network is not connectable, we may want to connect the largest subset of the secondary users. This optimization problem is NP-hard, since the decision version is already NP-complete on very restricted instances. Thus it is interesting to design polynomial time approximation algorithms for this optimization problem.

In some other scenarios, we may wish to connect all the secondary users but keep the antenna budget as low as possible. That is, we want to find the smallest $\beta$ such that there exists a spectrum assignment connecting the graph in which each SU opens at most $\beta$ channels. It is easy to see that this problem generalizes the minimum-degree spanning tree problem [8], which
asks to find a spanning tree of a given graph in which the maximum vertex degree is minimized. The latter problem is NP-hard, but there is a polynomial time algorithm that finds a spanning tree of degree at most one more than the optimum [6]. It would be interesting to see whether this algorithm can be generalized to the min-budget version of our connectivity problem, or whether we can at least obtain constant factor approximations.

Another meaningful extension of this work is to design distributed algorithms to achieve network connectivity. Moreover, due to interference in wireless communications, the connected nodes using the same channel may not be able to communicate simultaneously. Therefore, it is also interesting to investigate distributed algorithms with channel assignment and link scheduling jointly considered to achieve some network objective such as connectivity and capacity maximization, especially under the realistic interference models.

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References

1. P. Bahl, R. Chandra, T. Moscibroda, R. Murty, and M. Welsh. White space networking with Wi-Fi like connectivity. In ACM SIGCOMM, 2009.

2. H. L. Bodlaender. A linear-time algorithm for finding tree-decompositions of small treewidth. SIAM J. Comput., 25(6):1305–1317, 1996.

3. A. Cayley. A theorem on trees. Quarterly Journal on Pure and Applied Mathematics, 23:376–378, 1889.

4. S. Dolev, S. Gilbert, R. Guerraoui, and C. C. Newport. Gossiping in a multi-channel radio network. In DISC, 2007.

5. R. Downey and M. Fellows. Parameterized Complexity. Springer, 1998.

6. M. Fürer and B. Raghavachari. Approximating the minimum-degree steiner tree to within one of optimal. J. Algorithms, 17(3):409–423, 1994.

7. H. Gabow and E. Myers. Finding all spanning trees of directed and undirected graphs. SIAM J. Comput., 7(3):280–287, 1978.

8. M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.

9. J. Hopcroft and R. Karp. An n5/2 algorithm for maximum matchings in bipartite graphs. SIAM J. Comput., 2(4):225–231, 1973.

10. T. Kloks. Treewidth: Computations and Approximations, volume 842 of LNCS. Springer, 1994.

11. F. Kuhn, R. Wattenhofer, and A. Zollinger. Ad hoc networks beyond unit disk graphs. Wireless Networks, 14(5):715–729, 2008.

12. X.-Y. Li, P. Yang, Y. Yan, L. You, S. Tang, and Q. Huang. Almost optimal accessing of nonstochastic channels in cognitive radio networks. In IEEE INFOCOM, 2012.

13. H. Liang, T. Lou, H. Tan, Y. Wang, and D. Yu. Complexity of connectivity in cognitive radio networks through spectrum assignment. In ALGOSENSORS’12, volume 7718 of LNCS, pages 108–119, 2013.
14. D. Lu, X. Huang, P. Li, and J. Fan. Connectivity of large-scale cognitive radio ad hoc networks. In *IEEE INFOCOM*, 2012.
15. W. Ren, Q. Zhao, and A. Swami. Connectivity of cognitive radio networks: Proximity vs. opportunity. In *ACM CoRoNet*, 2009.
16. W. Ren, Q. Zhao, and A. Swami. Power control in cognitive radio networks: How to cross a multi-lane highway. *IEEE JSAC*, 27(7):1283–1296, 2009.
17. N. Robertson and P. D. Seymour. Graph minors. ii. algorithmic aspects of treewidth. *J. Algorithms*, 7:309–322, 1986.
18. A. Shukla. A short proof of Cayley’s tree formula. *arXiv:0908.2324v2*.
19. B. Wang and K. J. R. Liu. Advances in cognitive radio networks: A survey. *IEEE J. Selected Topics in Signal Processing*, 5(1):5–23, 2011.
20. C. Xu and J. Huang. Spatial spectrum access game: nash equilibria and distributed learning. In *ACM MobiHoc*, 2012.
21. J. Yu, H. Roh, W. Lee, S. Pack, and D.-Z. Du. Topology control in cooperative wireless ad-hoc networks. *IEEE JSAC*, 30(9):1771–1779, 2012.
22. Y. Yuan, P. Bahl, R. Chandra, T. Moscibroda, and Y. Wu. Allocating dynamic time-spectrum blocks in cognitive radio networks. In *ACM MobiCom*, 2007.
23. X. Zhou, S. Gandhi, S. Suri, and H. Zheng. ebay in the sky: Strategyproof wireless spectrum auctions. In *ACM MobiCom*, 2008.