An asymmetric exclusion model with overtaking
a numerical and simulation study

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Abstract

An asymmetric exclusion model on an open chain with random
rates for hopping particles, where overtaking is also possible, is stud-
iected numerically and by computer simulation. The phase structure of
the model and the density profiles near the high and low density coex-
istence line are obtained. The effect of a site impurity is also studied.
1 Introduction

Over the past few years, a great deal of progress has been achieved in our understanding of boundary and disordered-induced phase transitions in driven diffusive systems. Our understanding of such macroscopic systems, which are far from equilibrium, is still in its infancy since we need in this case, to study in detail the stochastic macroscopic dynamics of the system. It is only in one dimension that various exact and approximate solutions (See [1, 2] for a review) have helped us to study the macroscopic stochastic dynamics of model systems, i.e. by writing the master equations of the underlying Markov process in suitable coordinates and solving these equations either exactly or by numerical-simulation techniques. The simplest such model is the Totally Asymmetric Simple Exclusion Process (TASEP), first introduced by Spitzer in 1970 [3], as an example of an interacting stochastic process [4, 5, 6]. This is a model incorporating a collection of random walkers, hopping with equal rates to their right empty site on a one-dimensional lattice and interacting via exclusion. There is now a rather extensive literature on ASEP and its various generalizations (see [1, 2] and references therein).

Among the generalizations, we mention one which is relevant to the present paper, namely the introduction of disorder in the hopping rates of particles [7, 8, 9, 10, 11, 12]. In [7, 8], a closed system is considered, there is no injection to or extraction of particles out of the system. Each particle $\mu$, has a hopping rate $p_\mu$, taken from a random distribution, the particles keep their initial order and no overtaking is possible. The bulk density of particles $\rho$, is an important control parameter, and it is shown that for $\rho > \rho_c$, where $\rho_c$ is a critical density, a segregated phase appears behind the slowest particle on the ring, a phenomenon shown [7] to be similar to Bose condensation in equilibrium statistical physics.

The same process has been simulated [13] on an open chain where each particle enters the left end of the system with rate $\alpha$ and leaves the right end with rate $\beta$, and the intrinsic hopping probability $p_\mu \in [c, 1]$ of a particle $\mu$, is drawn from the distribution

$$f(p) = \frac{n + 1}{(1 - c)^{n+1}} \cdot (p - c)^n$$

(1)

Using mostly the values $n = 1$ and $c = \frac{1}{2}$, it has been shown in [13] that the generic form of the phase diagram of the single species (pure) ASEP, prevails also in this case; that is depending on the value of alpha and beta, three phases develop in the system, namely low-density, high-density and maximal-current phases. Compared to the pure case, there is only a slight difference in the shape of the coexistence line between the low and high density phases.
The small effect of disorder on the phase diagram of the pure ASEP has been attributed [13] to insufficient time for density inhomogeneities (platoons) to develop up to the scale of system size. This in turn has been attributed to the average passing time of particles ($\tau \sim L$) and the approximate rate of growth of platoon sizes ($\xi(t) \sim t^{n+\frac{1}{2}}$) which is small compared with $L$, when $L \to \infty$, for any $n$.

The above results all apply to a model in which particles can not overtake each other. The appearance of high density phase and traffic jams are quite expected in such a model. One can however introduce models in which particles can overtake each other [9]. It’s natural to expect that fast particles have a possibility to overtake slow ones, and one may ask to what extend this new element may affect the nature of steady state and the phase structure of the system. The model of [9] admits an exact solution [11] by way of matrix product ansatz, where the phases can be determined, although the density profiles are still very difficult, if not impossible, to obtain analytically in this model. Also it is difficult to obtain any analytical result concerning other effects like the effect of impurities or blockage.

In this paper we study the model of [9] by computer simulations, and will obtain the phase structure of the model and density profile near the high-density/low-density coexistence line, and compare our results with those of [13]. We also study the effect of a fixed blockage in a simple two-species model.

2 The model

The model which has been first introduced in [9] and shown to be exactly solvable via the matrix product ansatz, refers to a process in continuous time, in which each particle of type $\mu$, has an intrinsic hopping rate $v_\mu$ to its right empty site, as in the models considered in [7, 8, 13], but when this particle encounters a site already occupied by a particle of type $\mu'$ (with $v_{\mu'} < v_\mu$), the two particles exchange their sites with rate $v_\mu - v_{\mu'}$, as if the faster particle stochastically overtakes the slower one (Figure 1).

The need for exact solvability of the model, enforces the injection and extraction rates of particle $\mu$ to be fixed as

$$\alpha_\mu = \frac{\alpha v_\mu}{\bar{v}}$$  \hspace{1cm} (2)

and

$$\beta_\mu = \frac{v_\mu}{\bar{v}} + \beta - 1$$  \hspace{1cm} (3)
where $p$ is the number of type of particles and

$$\bar{v} := \frac{1}{p} \sum_{\mu=1}^{p} v_\mu. \quad (4)$$

It is assumed that there are $p$ type of particles with hopping rates $v_1 \leq v_2 \leq v_3 \leq \ldots \leq v_p$.

Here $\alpha$ and $\beta$ are representing the total injection and average extraction rates and $\beta > 1 - \frac{v_1}{\bar{v}}$, where $v_1$ is the slowest hopping rate. Note that for each particle $\mu$, $v_\mu$ is its velocity in an otherwise empty lattice, averaged over many realizations of the process. Thus, hereafter, we use the words intrinsic hopping rate and intrinsic velocity interchangeably. The model can be formulated for continuous distribution of intrinsic hopping rates which we denote by $\sigma(v)$. In this case the rate of injection of particles of intrinsic velocity between $v$ and $v + dv$ is given by $\alpha \sigma(v) dv$, where the extraction rate is given by $(\frac{v}{\bar{v}} + \beta - 1)dv$. By exactly solving this model [11], using the matrix product ansatz, the phase diagram of this model was determined. It was shown that besides $\alpha$ and $\beta$, the phase diagram depends crucially on a characteristic of the distribution $\sigma(v)$, denoted by $l[\sigma]$ and given by

$$l[\sigma] = \frac{\bar{v}^2}{v_1^2} - \left\langle \frac{v \bar{v}}{(v-v_1)^2} \right\rangle \quad (5)$$

where $v_1$ is the slowest velocity and the average is taken with respect to the distribution $\sigma(v)$.

If $\sigma(v)$ is such that $l[\sigma] < 0$, then the phase diagram is similar to the pure ASEP, although the shape of coexistence line and the position of the triple-point do have essential dependence on $\sigma$. However when $l[\sigma] > 0$, then there
is no high-density phase in the diagram (Figures 2a and 2b). Specifically for

\[ l < 0 \]

\[ l > 0 \]

Figure 2: Typical phase diagrams for two values of \( l \).

LD low density  HD high density  MC maximal current

distribution of the type (1), the size and in fact the existence or non-existence of the high-density phase depends crucially on the parameters \( c \) and \( \eta \). This means that if the chance of entrance of slow particles into the system (cars into a highway) is sufficiently low, and if there is a possibility for overtaking slow cars, there will not be a high-density phase (a traffic jam), regardless of the value of \( \alpha \) and \( \beta \). Note that in any case \( \beta \) is restricted by \( \beta > 1 - \frac{\alpha}{\nu_1} \), so that the extraction rates of all types of particles are positive. In case \( 1 - \frac{\nu_2}{\nu_1} < \beta \leq 1 - \frac{\nu_1}{\nu_1} \), the steady state will be trivial and the system will eventually be filled with particles of type 1, since in this case all types of particles except type 1 are extracted from the system. In any other case (e.g. \( 1 - \frac{\nu_1}{\nu_1} < \beta \leq 1 - \frac{\nu_1}{\nu_1} \)), the steady state depends on the initial conditions, since neither particles of type 1, nor type 2, are extracted and the steady state depends on the order of injection of these particles.

If to each site \( k \), we assign a random variable \( \tau^\mu(k) \) which takes the value 1 only if this site is occupied by a particle of type \( \mu \), then the average density of particle of type \( \mu \) at site \( k \), denoted by \( \rho^\mu(k) \) is

\[ \rho^\mu(k) = \langle \tau^\mu(k) \rangle \quad (6) \]

1The exact criteria is given in the main text.
where $\langle \rangle$ means the long time average, in the steady state. Note that in accordance with [13] and in contrast to the periodic lattice [7, 8], a separate disorder average is not necessary, since new particles are constantly injected into the system. Two other quantities are of interest, defined as:

$$\rho^\mu := \frac{1}{L} \sum_{k=1}^{L} \langle \tau^\mu(k) \rangle$$  \hspace{1cm} (7)

$$\rho(k) = \frac{1}{p} \sum_{\mu=1}^{p} \langle \tau^\mu(k) \rangle$$  \hspace{1cm} (8)

which are respectively the density of particles of type $\mu$, average over the lattice sites; and the total local density of particles, irrespective of their type. One can also define the current of type $\mu$ of particles, $J^\mu$, which in the steady state is independent of site and in view of the definition of the process is given by [3]

$$J^\mu = v^\mu \langle \tau^\mu_{k-1} e_k \rangle + \sum_{\mu' < \mu} (v^\mu - v^{\mu'}) \langle \tau^\mu_{k-1} \tau^{\mu'}_{k} \rangle - \sum_{\mu' > \mu} (v_{\mu'} - v_{\mu}) \langle \tau^\mu_{k-1} \tau^{\mu'}_{k} \rangle$$  \hspace{1cm} (9)

where $e_k := 1 - \sum_{\mu=1}^{p} \tau^\mu_k$. Note that $\langle e_k \rangle$ stands for the probability of site $k$ being empty.

The independence of this current of the site $k$, and its equality to the input and output current of particles of type $\mu$, given by

$$J_{in}^\mu = \frac{\alpha v^\mu}{p \tau} \langle e_1 \rangle$$  \hspace{1cm} (10)

$$J_{out}^\mu = (\beta + \frac{v^\mu}{\tau} - 1) \langle \tau^\mu_L \rangle$$  \hspace{1cm} (11)

gives us a set of equations, which in the mean field approximation

$$\langle \tau^\mu(k) \tau^{\nu}(k + 1) \rangle \simeq \langle \tau^\mu(k) \rangle \langle \tau^{\nu}(k + 1) \rangle$$

enables all the local densities $\rho^\mu(k)$ and hence all the currents $J^\mu$, to be evaluated numerically.

We also study the effect of a site impurity at a given site, say $k$, by multiplying all the hopping rates by $\lambda$ ($\lambda < 1$), when computing the currents of particles on the links $(k-1, k)$ and $(k, k+1)$. The densities and the current can again be computed for this case, for various $\lambda$’s and $k$’s.
3 Numerical solutions and simulation results

We consider a lattice of 200 lattice sites and simulate the following process with random updating on this lattice:

a) A two species process, where the hopping rates are taken to have the distinct values $v_1$ and $v_2$, with equal probabilities.

b) a multi-species process, where the hopping rates are taken from the distribution

$$\sigma(v) = \frac{n + 1}{(1 - c)^{n+1}}(v - c)^n, \quad c \leq v \leq 1$$  \hspace{1cm} (12)

c) a two species process, where there is a fixed impurity on one of the sites of the lattice. When the particles reach this site, they slow down their hopping rate by a factor $\lambda$. The position of the impurity is denoted by $k$, where $0 \leq k \leq L$.

In cases a and c, we complement our simulations by a mean field solution obtained by numerically solving the equations $J_{\mu}^{\text{in}} = \cdots = J_{(k)}^{\mu} = \cdots = J_{\text{out}}^{\mu}$.

Results

a) A two species process

Fixing the values of $v_1$ and $v_2$, we obtain the bulk density $\rho$, for different values of $\beta$, as function of the injection rate $\alpha$.

The results are shown in Figures 3a and 3b, for the two different choices of hopping rates.

It is seen that for every $\beta$, there is a jump discontinuity in the bulk density at a critical value of $\alpha_c = \alpha_c(\beta)$. The value of the jump vanishes at a certain point $(\alpha_c, \beta_c)$, which makes the triple point in the phase diagram. The corresponding phase diagrams are shown in Figure 4.

b) A multi species process

In this case we consider a continuous distribution of hopping rates. For the sake of comparison with [13], we take the distribution (12).

For this distribution $\mathcal{L}[\sigma]$ is calculated to be [10]:
\[ l[\sigma] = \begin{cases} \frac{1}{n-1} - (n+1)\sigma, & \text{if } 0 \leq n \leq 1 \\ \frac{2}{(n+1)(1-c)}^2 - (n+1)\sigma, & \text{if } 1 < n \end{cases} \]

where \( \bar{v} := \int_c^1 v \sigma(v) dv = c + \frac{n+1}{n+2}(1-c) \).

For \( n = 1 \), we expect from the above formula that, since \( l[\sigma] < 0 \), we have the usual three phases seen in the pure case, namely the low density, the high density and the maximal current phases. This expectation is borne out in our simulation, where we have obtained the average density as a function of \( \alpha \), for various values of \( \beta \) (Figure 5a).

As far as \( c = \frac{1}{2} \), one can see after a simple calculation that \( l[\sigma] \) can never be positive. To obtain a positive \( l \), we take \( n = 2 \) and \( c = \frac{1}{3} \) (a broader spectrum of hopping rates) for which \( l \) is found to be equal to 3. In this case the results of simulations are shown in Figure 5b.

It is seen that when \( l[\sigma] > 0 \), no jump discontinuity exists in density, implying that there is no high density phase in the phase diagram. For all \( \beta \), there is a critical \( \alpha_c \) independent of \( \beta \), after which we enter the maximal current phase; this is in accord with the exact analysis of [11]. The corresponding phase diagram is shown in Figures 6a and 6b.

c) The effect of a site impurity

In this problem, our aim is to see what effect a site impurity has on the density profiles of particles, when the particles have the possibility of overtaking each other. For simplicity, we consider a two species model.

It is expected that such a blockage will mostly affect slow particles, since fast particles have a chance of passing by the particle accumulated behind the blockage. This is indeed seen in our numerical solution. Figure 7a and 7b depict the density profiles of both types of particles (fast and slow) for two different values of \( v_1 \) and \( v_2 \). Density profiles changes versus the strength of impurity \( \lambda \) (the slowing down factor) is shown in Figure 10.
Figure 3: The bulk density as function of $\alpha$ for various values of $\beta$, in the two species model.
Figure 4: The phase diagram for problem a.
Figure 5: Bulk density versus $\alpha$ for the distribution (12)\((type number = 200)\).
(a) $n = 2, c = \frac{1}{2}$

(b) $n = 2, c = \frac{1}{3}$

Figure 6: The phase diagram for problem b (type number = 200).
Figure 7: Density profile of slow and fast particles in the presence of an impurity ($\lambda = 0.5, \alpha = 0.8, \beta = 0.8$).
Figure 8: Total density profile versus the strength of impurity ($v_1 = 0.8, v_2 = 1.2, \alpha = 0.8, \beta = 0.8$).
4 Conclusion

In this paper we have studied an asymmetric exclusion model on an open chain with disorder in particle hopping rates. Following the method of [13], and guided by the analysis of [11, 9], we have shown how the distribution of hopping rates affects the generic phase diagram of the single species ASEP. In particular we have taken the hopping rates from the distribution

\[ \sigma(v) = \frac{n + 1}{(1 - c)^{n+1}}(v - c)^{n}, \quad c \leq v \leq 1 \] (14)

also studied in [13], and have shown that for \( n = 2 \), the high density phase may or may not exist depending on the value of \( c \); i.e. the width of the distribution. We have also studied how in a single two species process, with hopping rates \( v_1 \) and \( v_2 \), a single site impurity affects the density profiles of fast and slow particles.

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