ON THE GENERAL SMARANDACHE’S SIGMA PRODUCT OF DIGITS

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ABSTRACT. This paper investigates the behaviour of one of the most famous Smarandache’s sequence given by A061076 on oeis [1]. In particular we first study the behaviour of two sequences (A061077 [2], A061078 [3]) strictly connected with the main Smarandache’s sigma product of digits. We’ll solve some open problems such as the determination of an upper bound for these sequences (which hold for all \( n \in \mathbb{N} \)) and the determination of a closed formula for each \( a(n) \) and \( b(n) \). Then combining these results it will be possible to understand the behaviour of the general sequence \( c(n) \). Every result will be accompanied by Wolfram Mathematica [4] scripts examples in order to support our thesis.

1. INTRODUCTION

Let us give the main definitions and the notations which we’ll use during the paper:

Definition. (1) Let \( n \in \mathbb{N} \). We define \( a(n) \) as the \( n \)th term of the Smarandache’s even sequence [2], or the sum of the products of the digits of the first \( n \) even numbers.

(2) Let \( n \in \mathbb{N} \). We define \( b(n) \) as the \( n \)th term of the Smarandache’s odd sequence [2], or the sum of the products of the digits of the first \( n \) odd numbers.

(3) Let \( n \in \mathbb{N} \). We define \( c(n) \) as the \( n \)th term of the general Smarandache’s general sequence [2], or the sum of the products of the digits of the first \( n \) numbers.

The first terms of \( a(n) \) are:

\[ 0, 2, 6, 12, 20, 22, 26, 32, 40, 40, \ldots \]

The first terms of \( b(n) \) are:

\[ 1, 4, 9, 16, 25, 29, 34, 41, 50, 52, \ldots \]

The first terms of \( c(n) \) are:

\[ 1, 3, 6, 10, 15, 21, 28, 36, 45, 45, 46, \ldots \]

Definition. (1) Let \( p \in \mathbb{N} \) be an even number. We define the contribution of \( p \) in \( a(n) \) as the product of its digits. We’ll indicate it using the notation \( C(p) \).

(2) Let \( p \in \mathbb{N} \) be an odd number. We define the contribution of \( p \) in \( b(n) \) as the product of its digits. We’ll indicate it using the same notation \( C(p) \).

For example:

\[ C(3688) = 1152 = 3 \cdot 6 \cdot 8 \cdot 8 \]

\[ C(37) = 21 = 3 \cdot 7 \]

Definition. (1) Let \( b, c \in \mathbb{N} \) even numbers and \( a(n) \) be the \( n \)th term of the Smarandache’s even sequence; we define \( C(b \to c) \) as the sum of the "contributes" of all the even numbers from \( b \) to \( c \). So:

\[ C(b \to c) = a\left(\frac{c}{2}\right) - a\left(\frac{b}{2} - 1\right) \]
Let $d, e \in \mathbb{N}$ even numbers and $b(n)$ be the $n$th term of the Smarandache’s odd sequence; we define $C(b \rightarrow c)$ as the sum of the "contributes" of all the odd numbers from $b$ to $c$. So:

$$C(d \rightarrow e) = b\left(\left\lceil \frac{e}{2} \right\rceil \right) - b\left(\left\lceil \frac{d}{2} \right\rceil - 1\right)$$

For example:

$$C(14 \rightarrow 22) = 1 \cdot 4 + 1 \cdot 6 + 1 \cdot 8 + 2 \cdot 0 + 2 \cdot 2 = a(11) - a(6)$$

$$C(3 \rightarrow 7) = 3 + 5 + 7 = a(4) - a(1)$$

2. **Smarandache’s sigma product of digits (even sequence)**

2.1. **First exploration of $a(n)$**. First of all note that:

$$a(5) = 2 + 4 + 6 + 8 + 1 \cdot 0 = 20 = a(4)$$

So $a(5)$ is the sum of the contributes of 5 numbers (2,4,6,8,10). But the contribute of a number which contains a 0 in its decimal representation is equal to 0. We can see that:

| 2 → 8 | 12 → 98 | 112 → 998 | 1112 → 9998 | ... |
|---|---|---|---|---|
| 20 | 900 | 40500 | 1822500 | ... |

**Table 1.** Sum of contributes of numbers from $a$ to $b$ ($a \rightarrow b$) in the sequence $a(n)$

In general it’s possible to prove by induction that:

$$C(\underbrace{11 \ldots 1}_{k \cdot 1} \rightarrow \underbrace{99 \ldots 9}_{k \cdot 9}) = 4 \cdot 5^{k+1} \cdot 9^k$$

Furthermore, note that:

$$\begin{cases} 
C(0 \rightarrow 10) = C(2 \rightarrow 8) \\
C(10 \rightarrow 100) = C(12 \rightarrow 98) \\
C(100 \rightarrow 1000) = C(112 \rightarrow 998) \\
\vdots & \vdots 
\end{cases}$$

Because, as said before, the contribute of a number which contains a 0 in its decimal representation is equal to 0.

**Theorem 1.** Let $a, b \in \mathbb{N}$ even numbers and $C(a \rightarrow b)$ be the sum of the contributes of the even numbers from $a$ to $b$ in $a(n)$. Then:

$$C(\underbrace{11 \ldots 1}_{k \cdot 1} \rightarrow \underbrace{99 \ldots 9}_{k \cdot 9}) = 4 \cdot 5^{k+1} \cdot 9^k$$

**Proof.** As said before we’ll prove this result by induction on $k$. The base of our induction argument is $k = 0$:

$$C(2 \rightarrow 8) = 20 = 4 \cdot 5^1 \cdot 9^0$$

Because:

$$C(2 \rightarrow 8) = a(4) = 2 + 4 + 6 + 8 = 20$$
Now suppose that:

\[ C(11 \cdots 12 \rightarrow 99 \cdots 9) = 4 \cdot 5^{k+1} \cdot 9^k \]

And we’ll prove that:

\[ C(11 \cdots 12 \rightarrow 99 \cdots 9) = 4 \cdot 5^{k+2} \cdot 9^{k+1} \]

Note that 11\ldots 12 is the first number larger than 10^{k+1} such that there are no zeros in its base 10 digit representation. Furthermore:

\[ C(11 \cdots 12 \rightarrow 99 \cdots 9) = 4 \cdot 5^{k+1} \cdot 9^k \]

In order to understand this fundamental concept of the proof, consider the following sub-case:

\[ C(12 \rightarrow 98) = 900 \land C(112 \rightarrow 998) = 40500 \]

Then, since the first digit of 112 is 1:

\[ C(112 \rightarrow 198) = C(12 \rightarrow 98) = 900 \]

But the first digit of 212 is 2, so:

\[ C(212 \rightarrow 298) = 2 \cdot C(12 \rightarrow 98) = 1800 \]

and so on until:

\[ C(912 \rightarrow 998) = 9 \cdot C(12 \rightarrow 98) = 8100 \]

So we’ll have that:

\[ C(11 \cdots 12 \rightarrow 99 \cdots 9) = \sum_{i=1}^{9} [4i \cdot 5^{k+1} \cdot 9^i] = 4 \cdot 5^{k+1} \cdot 9^k \sum_{i=1}^{9} i \]

And finally:

\[ C(11 \cdots 12 \rightarrow 99 \cdots 9) = 4 \cdot 5^{k+1} \cdot 9^k \cdot 45 = 4 \cdot 5^{k+2} \cdot 9^{k+1} \]

\[ \square \]
2.2. **A closed formula for** \( a(n) \). Let \( a_0 \) be an even digit and \( 1 \leq a_n \leq 9 \); it’s possible to note the following identities:

1. \( C(2 \rightarrow a_0) = \left( a_0^2 \cdot 4 + a_0 \right) \)
2. \( C(2 \rightarrow 10a_1 + a_0) = 20 + a_1 \left( a_0^2 \cdot 4 + a_0 \right) + 10a_1 \left( a_0 - 1 \right) \)
3. \( C(2 \rightarrow 100a_2 + 10a_1 + a_0) = 920 + a_1 a_2 \left( a_0^2 \cdot 4 + a_0 \right) + 10a_1 a_2 \left( a_0 - 1 \right) + 10a_1 a_2 \left( a_0 + 1 \right) + 10a_1 a_2 \left( a_0 + 1 \right) \)
4. \( C(2 \rightarrow 1000a_3 + 100a_2 + 10a_1 + a_0) = 41420 + a_1 a_2 a_3 \left( a_0^2 \cdot 4 + a_0 \right) + 10a_1 a_2 a_3 \left( a_0 - 1 \right) + 10a_1 a_2 a_3 \left( a_0 + 1 \right) + 10a_1 a_2 a_3 \left( a_0 + 1 \right) + \ldots \)

For example:

\[
a(3567) = C(2 \rightarrow 7134) = 41420 + 21 \cdot 6 + 10 \cdot 21 \cdot (3-1) + 450 \cdot 7 \cdot (1-1) + 20250 \cdot 7 \cdot (7-1) = 892466
\]

As you can see using this code:

```plaintext
LISTING 1. To compute the 3567-th term
Accumulate [Times @@ IntegerDigits [Range[2, 10000, 2]]] [[3567]]
```

The following theorem will generalize this recurrence.

**Theorem 2.** Let \( a(n) \) be the sum of the products of the digits of the first \( n \) even numbers. Then:

\[
a(n) = C(2 \rightarrow 2n) = C \left( 2 \rightarrow \sum_{k=0}^{m} a_k \cdot 10^k \right)
\]

\[
a(n) = \frac{5}{\Pi} \left( 45^m - 1 \right) + \sum_{k=1}^{m} \left[ \prod_{k < j \leq m} a_j \right] \left( 2 \cdot 5^k \cdot 9^{k-1} \cdot a_k \cdot (a_k - 1) \right) + \sum_{i=1}^{m} a_i \left( a_0^2 \cdot 4 + a_0 \right)
\]

**Proof.** We’ll prove this result by induction on \( m \) (which is the number of digits of \( 2n \) in its decimal representation minus 1). From the definition of this sequence, \( a(n) \) is equal to the sum of the contributes of every even number from 2 to \( 2n \). The base case is when \( m = 0 \), so when there is only one (even) digit.

\[
2n = a_0 \in \{2, 4, 6, 8\} \implies n \in \{1, 2, 3, 4\}
\]

and:

\[
a(n) = C(2 \rightarrow a_0) = \begin{cases} 2 & \text{if } a_0 = 2 \\ 6 & \text{if } a_0 = 4 \\ 12 & \text{if } a_0 = 6 \\ 20 & \text{if } a_0 = 8 \end{cases}
\]

so:

\[
C(2 \rightarrow a_0) = \left[ a_0^2 \cdot 4 + a_0 \right]
\]

Suppose that the identity holds for \( m \) and we’ll prove that:

\[
C(2 \rightarrow 2n) = \frac{5}{\Pi} \left( 45^{m+1} - 1 \right) + \sum_{k=1}^{m+1} \left[ \prod_{k < j \leq m+1} a_j \right] \left( 2 \cdot 5^k \cdot 9^{k-1} \cdot a_k \cdot (a_k - 1) \right) + \sum_{i=1}^{m+1} a_i \left( a_0^2 \cdot 4 + a_0 \right)
\]
where \(2n\) has \(m + 2\) digits in base 10 and its representation is:

\[
2n = \sum_{k=0}^{m+1} a_k \cdot 10^k = \sum_{k=0}^{m} a_k \cdot 10^k + a_{m+1} 10^{m+1}
\]

So:

\[
C(2 \to 2n) = C(2 \to 10^{m+1}) + C(10^{m+1} \to 2n) = C(2 \to \underbrace{99 \ldots 9}_{m \text{ digits}}) + C(11 \ldots 12 \to 2n)
\]

Because the contribute of numbers which contain 0 in their decimal representation is equal to 0. Furthermore:

(1) \[
C(2 \to \underbrace{99 \ldots 9}_{m \text{ digits}}) = \frac{5}{11} (45^{m+1} - 1)
\]

And:

\[
C(11 \ldots 12 \to 2n) = C(11 \ldots 12 \to \underbrace{99 \ldots 9}_{m \text{ digits}}) + 2C(11 \ldots 12 \to \underbrace{99 \ldots 9}_{m - 1 \text{ digits}}) + \ldots + (a_{m+1} - 1)C(11 \ldots 12 \to \underbrace{99 \ldots 9}_{m - 1 \text{ digits}}) + a_{m+1} C(11 \ldots 12 \to 2n_0)
\]

So:

\[
C(11 \ldots 12 \to 2n) = \frac{a_{m+1}(a_{m+1} - 1)}{2} C(11 \ldots 12 \to \underbrace{99 \ldots 9}_{m - 1 \text{ digits}}) + a_{m+1} C(11 \ldots 12 \to 2n_0)
\]

But we proved before that:

(2) \[
C(11 \ldots 12 \to \underbrace{99 \ldots 9}_{m \text{ digits}}) = 4 \cdot 5^{m+1} \cdot 9^{m-1}
\]

\[
C(11 \ldots 12 \to 2n) = \frac{a_{m+1}(a_{m+1} - 1)}{2} \cdot 4 \cdot 5^{m+1} \cdot 9^{m-1} + a_{m+1} C(11 \ldots 12 \to 2n_0)
\]

(3) \[
C(11 \ldots 12 \to 2n) = a_{m+1}(a_{m+1} - 1) \cdot 2 \cdot 5^{m+1} \cdot 9^{m-1} + a_{m+1} C(11 \ldots 12 \to 2n_0)
\]

But from the induction hypothesis, since \(2n_0\) has \(m + 1\) digits:

\[
C(11 \ldots 12 \to 2n_0) = C(2 \to 2n_0) - C(2 \to \underbrace{99 \ldots 9}_{m \text{ digits}})
\]

\[
= \frac{5}{11} (45^{m+1} - 1) + \left[ \prod_{k=1}^{m} a_k \right] \left[ 2 \cdot 5^k \cdot g^{k-1} \cdot a_k \cdot (a_k - 1) \right] + \left[ \prod_{i=1}^{m} a_i \right] \left[ \frac{a_0^2}{4} + \frac{a_0}{2} \right] - \frac{5}{11} (45^{m+1} - 1)
\]

\[
= \sum_{k=1}^{m} \left[ \prod_{k=1}^{m} a_k \right] \left[ 2 \cdot 5^k \cdot g^{k-1} \cdot a_k \cdot (a_k - 1) \right] + \left[ \prod_{i=1}^{m} a_i \right] \left[ \frac{a_0^2}{4} + \frac{a_0}{2} \right] = S
\]

And finally combining together equations (1), (2), (3)

\[
C(2 \to 2n) = \frac{5}{11} (45^{m+1} - 1) + a_{m+1}(a_{m+1} - 1) \cdot 2 \cdot 5^{m+1} \cdot 9^m + S
\]
But $a_{m+1}(a_{m+1} - 1) \cdot 2 \cdot 5^{m+1} \cdot 9^m$ represents the $m+1$ term of $S$. So:

$$C(2 \rightarrow 2n) = \frac{5}{\Pi}(45^{m+1}-1)+\sum_{k=1}^{m+1} \left[ \prod_{k<j \leq m+1} a_j \right]\left[ 2 \cdot 5^k \cdot 9^{k-1} \cdot a_k \cdot (a_k-1) \right] + \prod_{i=1}^{m+1} a_i \left[ \frac{a_0^2}{4} + \frac{a_0}{2} \right]$$

which is our thesis.

\[\square\]

This formula could be implemented in Mathematica [5] using this code:

**Listing 2.** To compute the first 10000 terms

```mathematica
Table[5/11*(45^(Length[IntegerDigits[2 n]]-1)-1) + Sum[(Product[IntegerDigits[2 n][[j]], {j, 1, Length[IntegerDigits[2 n]]-k-1}])*(2*(5^k)*9^(k-1))*(Product[IntegerDigits[2 n][[Length[IntegerDigits[2 n]]-k]], {k, 1, Length[IntegerDigits[2 n]]-1}]) + (Product[IntegerDigits[2 n][[i]], {i, 1, Length[IntegerDigits[2 n]]-1}])^2/4 + (Product[IntegerDigits[2 n][[Length[IntegerDigits[2 n]]]]])^2/2), {n, 1, 10000}]
```

Which is equivalent to:

**Listing 3.** To compute the first 10000 terms

```mathematica
Accumulate[Times @@ IntegerDigits[Range[2, 20000, 2]]]
```

2.3. **Upper bound for $a(n)$**. We’ll prove an important inequality between the $n$th term of $a(n)$ and a function which depends on $n$. Look at the following graphs:

![Graphs](image.png)

(A) Graph of $a(n)$ (red) and $f(n)$ (black) where $n \in [1, 50]$

(B) Graph of $a(n)$ (red) and $f(n)$ (black) where $n \in [1, 500]$

**Figure 1.** Comparison between $a(n)$ and $f(n)$ in different intervals
Theorem 3. Let \( a(n) \) be the sum of the products of the digits of the first \( n \) even numbers. Then:
\[
a(n) \leq \frac{5}{11} \left[ 4^{5 \log_{10} \left( \frac{n+1}{2} \right)} + 1 \right] = f(n) \quad \forall n \in \mathbb{N}
\]
And we have the equality if and only if \( n = 5 \cdot 10^k - 1 \) for some \( k \in \mathbb{N} \).

Proof. First of all note that:

\[
\frac{5}{11} \left( 4^{5 \log_{10} \left( \frac{n+1}{2} \right)} + 1 \right) \in \mathbb{N} \iff n = 5 \cdot 10^k - 1
\]

But furthermore \( a(5 \cdot 10^k - 1) = a(5 \cdot 10^k), \forall k \in \mathbb{N} \). In fact:
\[
a(5 \cdot 10^k - 1) = C(2 \rightarrow 10^{k+1} - 2) = C(2 \rightarrow 10^{k+1}) = a(5 \cdot 10^k)
\]

But in theorem 2 we derived a closed formula for \( a(n) \) which depends on the digits in the decimal representation of \( 2n \). So:
\[
a(5 \cdot 10^k - 1) = a(5 \cdot 10^k) = C(2 \rightarrow 1 \underbrace{00 \ldots 0}_{k+1 	ext{ zeros}}) = \frac{5}{11} \left( 4^{5^{k+1}} - 1 \right) = f(5 \cdot 10^k - 1)
\]

So \( a(n) = f(n) \) if and only if \( n = 5 \cdot 10^k - 1 \) for some \( k \in \mathbb{N} \).

Now we want to prove that:
\[
\frac{5}{11} \left( 4^m - 1 \right) + \sum_{k=1}^{m} \prod_{k<j \leq m} a_j \left[ 2 \cdot 5^k \cdot 9^{k-1} \cdot a_k \cdot (a_k-1) \right] + \sum_{i=1}^{m} a_i \left[ \frac{a_i^2}{4} + \frac{a_i}{2} \right] \leq \frac{5}{11} \left[ 4^{5 \log_{10} \left( \frac{m+1}{2} \right)} + 1 \right]
\]

\( \forall n \in \mathbb{N} \). Note that here \( m+1 \) is the number of digits of \( 2n \).

We’ll prove this fact by contradiction. We know that \( a(n) \) and \( f(n) \) are monotone increasing functions and \( a(n) = f(n) \) if and only if \( n = 5 \cdot 10^k - 1 \); so if \( \exists n \in \mathbb{N} \) such that \( a(n) > f(n) \), then the inequality must hold in a closed interval of the form: \((5 \cdot 10^k - 1, 5 \cdot 10^{k+1} - 1) \subset \mathbb{N} \). That’s because \( a(n) \) and \( f(n) \) intersect each other only when \( n = 5 \cdot 10^k - 1 \) for some \( k \in \mathbb{N} \). So:
\[
a(n) > f(n) \quad \forall n \in (5 \cdot 10^k - 1, 5 \cdot 10^{k+1} - 1)
\]

But now we know that:
\[
a(5 \cdot 10^k - 1) = f(5 \cdot 10^k - 1) \quad \text{and} \quad a(5 \cdot 10^k) > f(5 \cdot 10^k)
\]

And this isn’t true because \( a(5 \cdot 10^k) = a(5 \cdot 10^k - 1) = f(5 \cdot 10^k) \). We arrived at a contradiction caused by supposing that \( a(n) > f(n) \) for some \( n \in \mathbb{N} \). \( \square \)
### 3. Smarandache’s Sigma Product of Digits (Odd Sequence)

#### 3.1. First Exploration of \( b(n) \)

First of all note that as in the first sequence, the contribute of a number which contains a 0 in its decimal representation is equal to 0. We can see that:

\[
\begin{align*}
1 & \rightarrow 9 \\
11 & \rightarrow 99 \\
111 & \rightarrow 999 \\
1111 & \rightarrow 9999 \\
\cdots & \\
25 & \rightarrow 1125 \\
50 & \rightarrow 50625 \\
225 & \rightarrow 2278125 \\
\cdots &
\end{align*}
\]

Table 2. Sum of contributes of numbers from \( a \) to \( b \) (\( a \rightarrow b \)) in the sequence \( b(n) \)

In general it’s possible to prove by induction (using the same technique of the proof of theorem [1]) that:

\[
C(1 \rightarrow 9_{\underbrace{1 \ldots 1}_{k \times k}}) = 5^{k+1} \cdot 9^{k-1}
\]

Furthermore, note that:

\[
\begin{align*}
C(1 \rightarrow 9) &= C(1 \rightarrow 9) \\
C(11 \rightarrow 99) &= C(11 \rightarrow 99) \\
C(101 \rightarrow 999) &= C(111 \rightarrow 999) \\
\vdots & \vdots 
\end{align*}
\]

Because, as said before, the contribute of a number which contains a 0 in its decimal representation is equal to 0.

#### 3.2. A Closed Formula for \( b(n) \)

Let \( a_0 \) be an odd digit and \( 1 \leq a_n \leq 9, 0 \leq a_{n-1}, \ldots, a_1 \leq 9 \); it’s possible to note the following identities:

1. \( C(2 \rightarrow a_0) = \left(\frac{a_0+1}{2}\right)^2 \)
2. \( C(2 \rightarrow 10a_1 + a_0) = 25 + a_1 \left(\frac{a_0+1}{2}\right)^2 + 25a_1(a_1-1) \)
3. \( C(2 \rightarrow 100a_2 + 10a_1 + a_0) = 1150 + a_1a_2\left(\frac{a_0+1}{2}\right)^2 + 25a_1a_2(a_1-1) + 1125a_2(a_2-1) \)
4. \( C(2 \rightarrow 1000a_3 + 100a_2 + 10a_1 + a_0) = 51775 + a_1a_2a_3\left(\frac{a_0+1}{2}\right)^2 + 25a_2a_3(a_1-1) + 1125a_3(a_3-1) \)

(5) …

For example:

\[
b(4637) = C(1 \rightarrow 9273) = 51775 + 504 + 9450 + 10125 + 1822500 = 1894354
\]

**Theorem 4.** Let \( b(n) \) be the sum of the products of the digits of the first \( n \) even numbers. Then:

\[
b(n) = C(1 \rightarrow 2n - 1) = C\left(1 \rightarrow \sum_{k=0}^{n} a_k \cdot 10^k \right)
\]

\[
b(n) = \frac{25}{44} \left[ 45^{m-1} \right] + \sum_{k=1}^{m} \left[ \prod_{1 \leq j \leq k} a_j \right] \left[ 5^{k+1} \cdot 9^{k-1} \cdot \frac{a_k \cdot (a_k - 1)}{2} \right] + \sum_{i=1}^{m} \left[ \frac{a_0 + 1}{2} \right] ^2
\]
Proof. We’ll prove this result by induction on \( m \) (which is the number of digits of \( 2n - 1 \) in its decimal representation minus 1). From the definition of this sequence, \( b(n) \) is equal to the sum of the contributes of every even number from 1 to \( 2n - 1 \). The base case is when \( m = 0 \), so when there is only one (odd) digit.

\[
2n - 1 = a_0 \in \{1, 3, 5, 7, 9\} \implies n \in \{1, 2, 3, 4, 5\}
\]

and:

\[
b(n) = C(1 \rightarrow a_0) = \begin{cases} 
1 & \text{if } a_0 = 1 \\
4 & \text{if } a_0 = 3 \\
9 & \text{if } a_0 = 5 \\
16 & \text{if } a_0 = 7 \\
25 & \text{if } a_0 = 9 
\end{cases}
\]

so:

\[
C(1 \rightarrow a_0) = \left( \frac{a_0 + 1}{2} \right)^2
\]

Suppose that the identity holds for \( m \) and we’ll prove that:

\[
C(1 \rightarrow 2n-1) = \frac{25}{44} \left( 45^{m+1} - 1 \right) + \sum_{k=1}^{m+1} \left( \prod_{k<j \leq m+1} a_j \right) \left[ 5^{k+1} \cdot 9^{k-1} \cdot \frac{a_k \cdot (a_k - 1)}{2} \right] + \left[ \prod_{i=1}^{m+1} a_i \right] \left( \left( \frac{a_0 + 1}{2} \right)^2 \right)
\]

where \( 2n - 1 \) has \( m + 2 \) digits in base 10 and its representation is:

\[
2n - 1 = \sum_{k=0}^{m+1} a_k \cdot 10^k = \sum_{k=0}^{m} a_k \cdot 10^k + a_{m+1} \cdot 10^{m+1}
\]

So:

\[
C(1 \rightarrow 2n-1) = C(1 \rightarrow 10^{m+1}) + C(10^{m+1} \rightarrow 2n-1) = C(1 \rightarrow \underbrace{99 \ldots 9}_{m+1\text{ digits}}) + C(1 \rightarrow \underbrace{99 \ldots 9}_{m+2\text{ digits}})
\]

Because the contribute of numbers which contain 0 in their decimal representation is equal to 0. Furthermore:

\[
(5) \quad C(1 \rightarrow \underbrace{99 \ldots 9}_{m+1\text{ digits}}) = \frac{25}{44} \left( 45^{m+1} - 1 \right)
\]

And:

\[
C(1 \rightarrow \underbrace{99 \ldots 9}_{m+2\text{ digits}}) = C(1 \rightarrow \underbrace{99 \ldots 9}_{m+1\text{ digits}}) + 2C(1 \rightarrow \underbrace{99 \ldots 9}_{m\text{ digits}})
\]

\[
+ \cdots + (a_{m+1} - 1)C(1 \rightarrow \underbrace{99 \ldots 9}_{m\text{ digits}}) + a_{m+1}C(1 \rightarrow 2n-1)
\]

So:

\[
(6) \quad C(1 \rightarrow \underbrace{99 \ldots 9}_{m+2\text{ digits}}) = \frac{a_{m+1}(a_{m+1} - 1)}{2} C(1 \rightarrow \underbrace{99 \ldots 9}_{m+1\text{ digits}}) + a_{m+1}C(1 \rightarrow 2n-1)
\]

But we proved before that:

\[
(7) \quad C(1 \rightarrow \underbrace{99 \ldots 9}_{m\text{ digits}}) = 5^{m+2} \cdot g^m
\]
But from the induction hypothesis, since $2n_0 - 1$ has $m + 1$ digits:

$$C(\overbrace{1, \ldots, 1}^{m + 1} \rightarrow 2n_0 - 1) = C(1 \rightarrow 2n_0 - 1) - C(1 \rightarrow \overbrace{9, \ldots, 9}^m)$$

$$= \frac{25}{44} \left( 45^m - 1 \right) + \sum_{k=1}^{m} \prod_{k < j \leq m} a_j \left[ 5^{k+1} \cdot 9^{k-1} \cdot \frac{a_k \cdot (a_k - 1)}{2} \right] + \left[ \prod_{i=1}^{m} a_i \right] \left[ \left( \frac{a_0 + 1}{2} \right)^2 \right] - \frac{25}{44} \left[ 45^m \right]$$

And finally combining equations 5, 6, 7:

$$C(1 \rightarrow 2n - 1) = \frac{25}{44} \left( 45^m + 1 \right) + \frac{a_{m+1}(a_{m+1} - 1)}{2} \cdot 5^{m+1} \cdot 9^m + S$$

But $\frac{a_{m+1}(a_{m+1} - 1)}{2} \cdot 5^{m+2} \cdot 9^m$ represents the $m + 1$ term of $S$. So:

$$C(1 \rightarrow 2n - 1) = \frac{25}{44} \left( 45^m + 1 \right) + \frac{a_{m+1}(a_{m+1} - 1)}{2} \cdot 5^{m+2} \cdot 9^m + S$$

which is our thesis. □

3.3. **Upper bound for $b(n)$**. As for the sequence $a(n)$ we’ll prove an important inequality between the $n$th term of $b(n)$ and a function which depends on $n$. Look at the following graphs:

![Graphs of $b(n)$ and $g(n)$](image)

**Figure 3.** Comparison between $b(n)$ and $g(n)$ in different intervals

**Theorem 5.** Let $b(n)$ be the sum of the products of the digits of the first $n$ odd numbers. Then:

$$b(n) \leq \frac{25}{44} \left[ 45^{\log_{10}(\frac{n}{2}) + 1} \right] - 1 = g(n) \forall n \in \mathbb{N}$$

And we have the equality if and only if $n = 5 \cdot 10^k$ for some $k \in \mathbb{N}$.

**Proof.** First of all note that:

$$\frac{25}{44} \left[ 45^{\log_{10}(\frac{n}{2}) + 1} \right] - 1 \in \mathbb{N} \iff n = 5 \cdot 10^k$$
Furthermore:

\[ b(5 \cdot 10^k) = C(1 \rightarrow 10^k + 1 - 1) = C(1 \rightarrow 99 \ldots \underbrace{9}_{n + 1} \cdot \underbrace{9}_{k + 1} \cdot 9) \]

But from Theorem 4 we know that:

\[ C(1 \rightarrow 99 \ldots \underbrace{9}_{n + 1} \cdot \underbrace{9}_{k + 1} \cdot 9) = \frac{25}{44} \left[ 45^{k+1} - 1 \right] \]

So \( b(n) = g(n) \) if and only if \( n = 5 \cdot 10^k \) for some \( k \in \mathbb{N} \).

As in theorem 3, in order to prove the second part we proceed by contradiction; We know that \( b(n) \) and \( g(n) \) are monotone increasing functions and \( b(n) = g(n) \) if and only if \( n = 5 \cdot 10^k \); so if \( \exists n \in \mathbb{N} \) such that \( b(n) > g(n) \), then the inequality must hold in an open interval of the form: \( (5 \cdot 10^k, 5 \cdot 10^{k+1}) \subset \mathbb{N} \). That’s because \( b(n) \) and \( g(n) \) intersect each other only when \( n = 5 \cdot 10^k \) for some \( k \in \mathbb{N} \). So:

\[ a(n) > f(n) \quad \forall n \in (5 \cdot 10^k, 5 \cdot 10^{k+1}) \]

But now we know that:

\[ b(5 \cdot 10^k) = g(5 \cdot 10^k) \text{ and } b(5 \cdot 10^k + 1) > g(5 \cdot 10^k + 1) \]

And this isn’t true because \( b(5 \cdot 10^k + 1) = b(5 \cdot 10^k) = g(5 \cdot 10^k) \). We arrived at a contradiction caused by supposing that \( b(n) > g(n) \) for some \( n \in \mathbb{N} \). \( \square \)

4. The general Smarandache’s sigma product of digits

4.1. First exploration of \( c(n) \). First of all note that:

\[ c(2n) = a(n) + b(n) \quad \forall n \in \mathbb{N} \]

In fact \( a(n) \) gives the sum of the contributes of the even numbers less than or equal to \( n \) while \( b(n) \) gives the contributes of the odd numbers less than or equal to \( n \).

4.2. A closed formula for \( c(n) \). From theorems 2, 4 and from equation 8 we can easily compute the \( n \)th term of the general sequence \( c(n) \). In fact:

\[
\begin{cases}
  c(2n) = a(n) + b(n) \\
  c(2n + 1) = a(n) + b(n) + C(2n + 1)
\end{cases}
\]

Where \( C(2n + 1) \) is the contribute of \( 2n + 1 \) in \( c(n) \) (or simply the product of the digits of \( 2n + 1 \).

4.3. Upper bound for \( c(n) \).

**Lemma 6.** Let \( c(n) \) be the sum of the products of the digits of the first \( n \) numbers and \( C(n) \) the product of the digits of \( n \). Then:

\[
c(n) \leq \begin{cases}
  25 \left\lfloor \frac{45^{\log_{10}(n)} - 1}{44} \right\rfloor + 5 \left\lfloor \frac{45^{\log_{10}(n+2)} - 1}{44} \right\rfloor & \text{if } n \text{ is even} \\
  25 \left\lfloor \frac{45^{\log_{10}(n)} - 1}{44} \right\rfloor + 5 \left\lfloor \frac{45^{\log_{10}(n+2)} - 1}{44} \right\rfloor + C(n) & \text{if } n \text{ is odd}
\end{cases}
\]

\( \forall n \in \mathbb{N} \)

**Proof.** The result is simply obtained by combining the previous theorems regarding the upper bounds of \( a(n) \) and \( b(n) \). If \( n \) is an even number, then \( n = 2n_1 \) for some \( n_1 \in \mathbb{N} \). Furthermore, from equation 8 we know that:

\[ c(2n_1) = a(n_1) + b(n_1) = a\left(\frac{n_1}{2}\right) + b\left(\frac{n_1}{2}\right) \]

But combining theorem 3 and theorem 5 we’ll have the following inequality:

\[ c(n) = a\left(\frac{n}{2}\right) + b\left(\frac{n}{2}\right) \leq \frac{25}{44} \left[ 45 \log_{10}(n) - 1 \right] + \frac{5}{11} \left[ 45 \log_{10}(\frac{n+2}{10})+1 - 1 \right] \]

Instead for odd numbers we’ll have the same inequality except that we must add the contribute of the argument \( C(n) \) (or simply the product of its digits). In fact if \( n \) is odd, then \( n = 2n_2 + 1 \) for some \( n_2 \in \mathbb{N} \). So:

\[ c(n) = c(2n_2 + 1) = c(2n_2) + C(2n_2) = a(n_2) + b(n_2) + C(2n_2) \]

And finally:

\[ c(n) \leq \frac{25}{44} \left[ 45 \log_{10}(n) - 1 \right] + \frac{5}{11} \left[ 45 \log_{10}(\frac{n+2}{10})+1 - 1 \right] + C(n) \]

□

**Conjecture 7.** Let \( c(n) \) be the sum of the products of the digits of the first \( n \) numbers. Then:

\[ c(n) \leq \frac{25}{44} \left[ 45 \log_{10}(n) - 1 \right] + \frac{5}{11} \left[ 45 \log_{10}(\frac{n+2}{10})+1 - 1 \right] = h(n) \ \forall n \in \mathbb{N} \setminus \{10^k - 1\}_{k \in \mathbb{Z}^+} \]

We really think that the equality above holds without the \( C(n) \) part for all natural numbers not equal to \( 10^k - 1 \) for some positive integer \( k \). Look at the following graphs:

![Graphs](image1.png)

**Figure 4.** Comparison between \( c(n) \) and \( h(n) \) in different intervals

![Graphs](image2.png)

**Figure 5.** Comparison between \( c(n) \) and \( h(n) \) in different intervals

As you can see \( c(n) \leq h(n) \) except for some particular values. The only ones which I found are in fact of the form \( 10^k - 1 \). Look at the following table:
\[ c(9) = 45 \quad c(99) = 2070 \quad c(999) = 93195 \quad \ldots \]
\[ h(9) \approx 44.4 \quad h(99) \approx 2066.3 \quad h(999) = 93177 \quad \ldots \]

| \begin{tabular}{lcc}
      \hline
      $c(n)$ & $h(n)$ & \hline
      $c(9)$ & $h(9)$ & \approx 44.4 \\
      $c(99)$ & $h(99)$ & \approx 2066.3 \\
      $c(999)$ & $h(999)$ & = 93177 \\
      \hline
\end{tabular} |

**Table 3.** Comparison between $c(n)$ and $h(n)$ for particular values of $n$

5. Final Considerations

In this section we’ll analyze the obtained results and we’ll combine them together in order to prove some interesting corollaries. In particular we want to study the behaviour of the sequence \( \frac{a(n)}{b(n)} \). Surprisingly this sequence is bounded as suggested from these plots:

![Plots](image)

(A) Plot of \( \frac{a(n)}{b(n)} \) where \( n \in [1, 5] \)  
(B) Plot of \( \frac{a(n)}{b(n)} \) where \( n \in [1, 500] \)

**Figure 6.** Plot of \( \frac{a(n)}{b(n)} \) in different intervals

**Conjecture 8.** Let \( a(n) \) and \( b(n) \) be defined as before. Then:

\[
\frac{4}{5} \leq \frac{a(n)}{b(n)} \leq 2 \quad \forall n \in \mathbb{N}
\]
\[
\frac{4}{5} \leq \frac{a(n)}{b(n)} < 1 \quad \forall n \geq 5
\]

It’s easy to see that \( \frac{a(n)}{b(n)} < 1 \), \( \forall n \geq 5 \). In fact:

\[ a(4) = 2 + 4 + 6 + 8 > 1 + 3 + 5 + 7 = b(4) \]

But from \( n = 5 \):

\[ a(5) = 2 + 4 + 6 + 8 + 1 \cdot 0 < 1 + 3 + 5 + 7 + 9 = b(5) \]

Note that the contribution in \( a(n) \) of terms like 10, 20, 30, . . . is equal to 0, while in \( b(n) \) such numbers can not exists because it counts the contributions only from odd numbers. Using this argument we can see that there are more “zero-contributions” in \( a(n) \) than in \( b(n) \), and this is sufficient to understand the first inequality. In order to prove the second inequality it’s sufficient to show that:

\[ 5a(n) - 4b(n) \geq 0 \quad \forall n \in \mathbb{N} \]

The conjecture is probably true as suggested from the plot of \( 5a(n) - 4b(n) \).

**Corollary 9.** Let \( a(n) \) and \( b(n) \) be defined as before. Then:

\[
\lim_{n \to +\infty} \frac{a(n)}{b(n)} = \frac{4}{5}
\]
Proof. From theorems 3, 5 we know that:

\[ \frac{a(n)}{b(n)} \leq \frac{5}{11} \frac{45\log_{10}(n^{-1}) + 1}{45\log_{10}(n^{-1}) + 1 - 1} = \frac{4}{5} \frac{45\log_{10}(n^{-1}) + 1 - 1}{45\log_{10}(n^{-1}) + 1 - 1} \xrightarrow{n \to +\infty} \frac{4}{5} \]

Hence \( \forall \varepsilon > 0 \exists n_0 \in \mathbb{N} \) such that:

\[ |r(n) - 1| < \varepsilon \quad \forall n > n_0 \]

So:

\[ \frac{a(n)}{b(n)} - \frac{4}{5} \leq \frac{4}{5} r(n) - \frac{4}{5} = \frac{4}{5} (r(n) - 1) < \frac{4}{5} \varepsilon \quad \forall n > n_0 \]

And assuming conjecture 8, we’ll have:

\[ \frac{a(n)}{b(n)} - \frac{4}{5} \geq 0 > -\varepsilon \quad \forall n \in \mathbb{N} \]

Which concludes the proof. \( \square \)

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