Coulomb branches for quaternionic representations

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Abstract

I describe the Chiral rings $\mathcal{R}_{3,4}$ for 3D, $N = 4$ supersymmetric $G$-gauge theory and matter fields in quaternionic representations $E$: first, by incorporating real structures in the construction of [BFN], and second, more explicitly, by Weyl group descent from the maximal torus. A topological obstruction is $w_4(E)$, modulo squares, for $\mathcal{R}_3$; a secondary obstruction $\eta \cdot E$ can appear for $\mathcal{R}_4$. Flatness over the Toda bases reduces calculations to the case of $SU_2$. For some representations, an Abelianization formula describes the $\mathcal{R}$ in terms of the maximal torus and the Weyl group. This provides an alternative and some corrections to a recent attempt [BDFRT].

Introduction

For a compact Lie group $G$ and a quaternionic representation $E$ there are expected to be (singular) hyper-Kähler Coulomb branches of the moduli of vacua for (3D, $N = 4$ supersymmetric) gauge theory, with matter fields in $E$. Arising, as they do, from four dimensions by dimensional reduction along a line or a circle, they come in two versions, $\mathcal{C}_3(G; E), \mathcal{C}_4(G; E)$, respectively. In a familiar pattern, they are associated to ordinary cohomology and to $K$-theory: the equivariant parameter for circle rotation becomes the (inverse of the) Bott periodicity generator.

The complex varieties $\mathcal{C}_{3,4}(G; 0)$ are the total spaces for the Toda integrable system and its finite difference version, respectively, with bases $\text{Spec } H^*_G, K^*_G$ of a point. They were thoroughly studied in [BFM], although not, at the time, related to 3D gauge theory. That connection was spelt out later [T1], but special cases were known in the physics literature [SW].

For polarized representations $E$ (symplectic doubles of complex ones), the first general construction of the Poisson algebras $\mathcal{R}_{3,4}(G; E)$ of algebraic functions on the modified varieties $\mathcal{C}_{3,4}(G; E)$ — called chiral rings in the physics literature — was given by Braverman, Finkelberg and Nakajima [BFN]; see also [BDG] for a physics perspective. These varieties should control gauge theory “with matter fields in $E$.” While a precise formulation of that goal is not yet at hand, their subsequent treatment in [T2] proposed the moral teaching (partly generalized in the present paper) that the Chiral rings with matter are formal consequences of the 2D gauged linear Sigma model (GLSM), viewed as a boundary condition.

Following that lead and the proposed relation with quantum and symplectic cohomology, a conceptual interpretation of that GLSM construction was recently established by Gonzalez, Mak and Pomerleano in [GMP]: the chiral ring $\mathcal{R}_3(G; E)$ is the subring of $\mathcal{R}_3(G; 0)$ which preserves the

1The term Coulomb branch is sometimes understood to incorporate the hyperkähler metric. That is known for $\mathcal{C}_{3,4}(G; 0)$, via the Nahm equations, using results of Bielawski [B], or for abelian $G$ [BDG]. A general construction in the polarized case was recently proposed in [BP].
equivariant quantum cohomology lattice of a Lagrangian half of $E$ within its symplectic cohomology. This may well be the first substantive theorem — as opposed to construction, or definition — relating Coulomb branches to pre-existing mathematics.

In this paper, I describe the Poisson rings $R_{3,4}(G; E)$ believed to underlie the $C_{3,4}$ in two different ways. First, I adapt the construction of [BFN] by exploiting real structures. Second, I adapt the “mirror B-model” construction from [T2]: the original version glues two copies of the Toda space along a vertical shift by the rational section $\exp(d\Psi)$, for the superpotential $\Psi$ of the GLSM. The analogue here is more complicated: simplistically, $\Psi$ is now Weyl-multi-valued.

Topologically, the first construction comes from an extension of the group $\Omega G$ by classes in the multiplicative monoid of $K$-theory, rather than by the group of units: the Euler class of an additive extension of $\Omega G$ by $KO$, built from the Atiyah index map of $E$. This suspends $\Omega G$ by a (locally defined) linear space $R_E$ with jumping fibers, a real form of the index sheaf. The $J$-homomorphism converts it into a constructible coefficient system for $(K)$-homology. Reality provides a ‘quantum square root’ substituting for a missing polar half of $E$. This could be extended to other generalized cohomology theories by incorporating an appropriate obstruction calculus; I noted at times the case of $KO$. With the obstructions in place, the outcome may be interpreted as a curved coefficient system, defining a ‘gerby space.’

The second, explicit construction realizes the $C_{3,4}(G; E)$ by symmetry-breaking to the maximal torus followed by Weyl descent. It is tempting to posit an elegant $A$-model interpretation, reflecting a boundary condition for the 3D theory built from the flag variety of $G$ and a polar half of $E$; this would be a worthy generalization of the main theorem in [GMP].

Alternative proposals. My construction is quite different from the suggested construction of the homological ring $R_3$ in [BDFRT]. While the outcomes ought to agree, comparison must await the announced paper [DLYZ], on which that other construction relies. Meanwhile, some distinctions are easily explained:

(i) The obstruction I give for $R_3$ is stronger: $w_4(E)$ must have a square root $\bar{r} \in H^2(BG; \mathbb{Z}/2)$ admitting an integral lift. The [BDFRT] condition can be shown to agree with this, minus the integral lifting condition. This is because I ask for a $\mathbb{Z}$-graded version of $R_3$. Without the integral lift, the homology $\mathbb{Z}$-grading is obstructed by the (big) Bockstein $B(\bar{r})$. The smallest example is $SU(2) \times \{\pm 1\} \otimes SO(6)$ with its standard representation $\mathbb{H} \otimes \mathbb{R}^6$ (Theorem A.4.ii in the Appendix). Here, $R_3$ may not be $\mathbb{Z}$-graded without breaking the Koszul sign rule.

(ii) The $\mathbb{Z}$-grading can be collapsed to $\mathbb{Z}/2$ in complex $K$-theory, so my obstruction could be loosened for $R_4$. Thus, the example in (i) above is unobstructed for Bott-periodic $K$-theory. It is obstructed for $KO$-theory or connective $kU$-theory, as are the Adams operations on $KU$. The significance of this is unclear, though.

(iii) A secondary obstruction to the existence of $R_4$ appears for certain groups, when $w_4(E) = 0$: see Theorem A.6. A typical example is $Sp(odd) \otimes \{\pm 1\} \otimes SO(4k)$ with the tensor product of standard representations.

Beyond this, some problems with the proposal in [BDFRT] must be addressed, before a comparison can be made.

(i) The argument offered for commutativity of $R_3$ in [BDFRT, §4.1] seems incomplete. A monoidal equivalence of categories—whose construction was, in any case, deferred to [DLYZ]—does not identify commutative algebra objects, as asserted; a braided equivalence is needed here. This must be refined to $E_3$, if one is to discuss the Poisson structure on the chiral ring.

An early version of the paper claimed that the second obstruction vanished for connected groups. Unfortunately, there was a mistake in one of the cases.
(ii) Closely related to this oversight is the incorrect obstruction calculation in the same section. The square root of the line bundle $\mathcal{L}$ may be forced to carry a $\mathbb{Z}/2$-grading (from the square root $\bar{r}$ above), as part of its $E_2$ structure. Without the $E_2$ requirement, the classification of multiplicative line bundles is false as asserted there.

Hopefully, these two problems in loc. cit. are simultaneously addressable.

**Improvements and future directions.**

(i) **Alternative descriptions.** The results of Theorem 1 below are likely not optimal. For example, the natural maps

$$\mathcal{C}_{3,4}(G; E) \times_{\mathcal{C}_{3,4}(G; 0)} \mathcal{C}_{3,4}(G; F) \to \mathcal{C}_{3,4}(G; E \oplus F),$$

described on ($K$-)homology coefficients in Part(v), are probably isomorphisms, if we interpret the left side as the affinized fibered quotient under the Toda group scheme.

(ii) **Disconnected groups.** This paper only treats connected groups explicitly: calculation and removal of obstructions is more involved in general. More significantly, twisted sectors appear for disconnected groups: the identity sector pertains to point defects in the 3D gauge theory, while the twisted sectors represent point defects embedded in topological ’t Hooft loops.

(iii) **Categorification of the Coulomb branch.** My $K$-theory coefficient systems $X_E$ can be replaced by the respective matrix factorization category (encountering the same obstructions). Dévissage equates the $K$-theory of the resulting category with the one here. It would be interesting to compare this categorification with the recent one in [CW].

(iv) Finally, the most interesting question was already mentioned: can we interpret the Weyl descent construction of the chiral rings in terms of boundary conditions for gauge theory with matter in the spirit of [T2], generalizing the theorem in [GMP]? The spaces $\mathcal{C}_{3,4}$ come with a Lagrangian multi-section, which could be the quantum cohomology of the total space of a vector bundle over the flag variety of $G$ with fiber a Lagrangian half of $E$.

**Organization.** The key results of the paper, Theorems 1–3, are stated in Section 1; the reader may need to refer to later sections for some details. In §2 we discuss two topological facts that underlie Theorems 1 and 3. Section 3 quickly reviews the construction [BFN] of the polarized case. Section 4 describes the changes required for the general case. In Section 5, we re-interpret the topological obstructions and their cancellation in terms of the Weyl group and maximal torus, preparing the global description of chiral rings in §6. Appendix A discusses obstructions and Appendix B collects some examples, or counterexamples to overly optimistic statements.

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1. Background and new results

(1.1) Pure gauge theory. The (K-)homology rings \( R_{3,4}(G;0) \) of the based loop group \( \Omega G \), equivariant with respect to the conjugation \( G \)-action, were analyzed in [BFM], and related to the Toda integrable systems. The double coset stack \( G \setminus LG / G \) of the free loop group \( LG \), homotopy equivalent to the free mapping stack from \( S^2 \) to \( BG \), is a more revealing model for the natural \( E_3 \) structure on the Pontryagin multiplication. This \( E_3 \) structure is shown in loc. cit. to define an algebraic symplectic form, while the (Hopf) algebra structures over the ground rings \( H^*_G, K^*_G \) of a point make the underlying spaces \( \mathcal{C}_{3,4} \) into relative abelian groups, which in addition admit integrable system structures. They are (fibre-wise group completions of) the classical Toda system and its finite-difference version, respectively. These spaces, which we now denote \( \mathcal{C}_{3,4}(G;0) \), were later interpreted in terms of gauge theory in the guise of classifying spaces for categories with topological \( G \)-action, with Gromov-Witten theory as motivating example: see [T1] or [T2, §2] for a summary.

(1.2) Polarizable matter. With a different motivation, a construction of the spaces \( \mathcal{C}_{3,4}(G;E) \) for polarized symplectic representations \( E = V \oplus V^\vee \) was provided by Braverman, Finkelberg and Nakajima [BFN], based on earlier ideas of Nakajima [N]. This involves the (K-)homology of a linear space \( L_V \) over \( \Omega G \), an algebraic fibration in vector spaces. The spaces \( \mathcal{C}_{3,4} \) can also be interpreted in 2-dimensional gauge theory. A polarization allows to couple a mass parameter to the matter fields: this means scaling the two polar summands by inverse actions of \( C^\times \), whose equivariant parameter becomes the ‘complex mass.’ This also defines two topological boundary conditions of the 3D gauge theory: the Gromov-Witten theories of the spaces \( V, V^\vee \), as real \( G \)-Hamiltonian manifolds. Each of them defines a regular Lagrangian section of \( \mathcal{C}_{3,4}(G;E) \) over the Toda base, and their ratio is a rational section of \( \mathcal{C}_{3,4}(G;0) \), which can be identified with the exponentiated derivative \( \exp(d\Psi) \) of the GLSM superpotential \( \Psi \). While the full structure of these physically inspired constructions has not been completely settled, it is proved in [T2] that we recover \( \mathcal{C}_{3,4}(G;E) \) from two copies of the Toda space glued after a relative shift by this section.

Both constructions require a polarization: without it, we seem to miss the space \( L_V \), and no gauge-invariant topological boundary conditions of geometric origin for the 3D theory are apparent that would reconstruct \( \mathcal{C}_{3,4} \).

(1.3) New results. The present paper overcomes these obstacles. First, I adapt the construction of [BFN] by exploiting a (twisted) reality structure on \( L_V \). Second, I polarize \( E \) after breaking the symmetry to the maximal torus and descend back under the Weyl group. The K-theoretic version \( \mathcal{C}_4 \) suggests the prospect of nice integral presentations, but those are not so obvious from my methods. (Integrality may be the shadow of a categorification of \( \mathcal{C}_4 \), [CW].) A topological subtlety requires removing two obstructions, to be discussed in detail in §4.4 below.

The constructions anchor a definition. In special cases, a one-step Abelianization (§7) gives a clean answer. In general, the answer is determined\(^3\) by reductions to the maximal torus and the Levi subgroups of semi-simple rank one, a simplification made possible by the freedom of the chiral rings over their Toda bases. The uniform algebraic description condenses this in §6. Computability seems to distinguish the methods presented here from other approaches.

(1.4) Statements. Postponing some details of the construction, here are the main theorems of the paper. Many proofs are repetitions of arguments in [BFN] or [T2]. Assume the removal of the first obstruction for \( \mathcal{C}_3 \), and also of the secondary one for \( \mathcal{C}_4 \). We also assume that \( G \) is connected\(^4\).

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\(^3\) After some loss of torsion information, in integral \( H_4 \).

\(^4\) Mainly, to avoid discussing the twisted sectors.
Theorem 1. There exist $G$-equivariant, $E_2$-multiplicative coefficient systems $\mathcal{H}_E, \mathcal{K}_E$ for $(K)$-homology over the based loop group $\Omega G$, constructible with respect to Bruhat stratification, such that:

(i) The equivariant homology $H^*_G(\Omega G; \mathcal{H}_E)$ is a $\mathbb{Z}$-graded, commutative algebra over $H^*_G$. With rational coefficients, it is a free $H^*_G$-module.

(ii) When $\pi_1 G$ has no torsion, this applies integrally to $K^*_G(\Omega G; \mathcal{H}_E)$ over $K^*_G$. (See Remark 1.5.iii below for the torsion case.)

(iii) These rings carry $E_3$ structures defined by Poisson structures (of homology degree 2). These are the leading terms of non-commutative deformations, constructed by incorporating the loop rotation action.

(iv) The Toda group schemes $\mathcal{C}_3,4(G; 0)$ act $E_3$-compatibly on $\mathcal{C}_3,4(G; E)$.

(v) More generally, there are multiplications $\mathcal{H}_E \odot \mathcal{H}_E \to \mathcal{H}_E \odot \mathcal{H}_E \to \mathcal{H}_E \odot \mathcal{H}_E$, compatible with all the listed structure, once the obstructions have been compatibly cancelled.

1.5 Remark (Complements to Theorem I).

(i) The coefficient systems are built from $G$-equivariant constructible sheaves of spectra, de-suspensions of $\Omega G$ by stratified virtual linear spaces $R_E \to \Omega G$ (§3.8, §4). They may be only locally defined, and up to suspension by real, oriented (respectively $Spin^c$) vector bundles. However, the $(K)$-homology sheaves $\mathcal{H}_E, \mathcal{K}_E$ are unambiguous and multiplicative, once the obstructions have been removed.

(ii) With the obstructions in place, we can interpret the $\mathcal{H}_E, \mathcal{K}_E$ as curved constructible coefficient systems: classes in $H^2(X; \mathbb{Z}/2)$ defines curved local systems for ordinary integral homology.

(iii) Given $G = G/\pi$, with $\pi$ finite, and a $G$-space $X$, the ring $K_G(X)$ is graded by the characters of $\pi$. The character pairing makes $\pi$ act by automorphisms of the ring. Taking $\pi$ to be the torsion subgroup of $\pi_1 G$, the statements of the theorem apply to the orbifolded $K$-theory $\pi \times K^*_G(\Omega G; \mathcal{H}_E \otimes \mathbb{C})$ over the orbifold Toda base $\pi \times K^*_G \otimes \mathbb{C}$.

The second construction of the chiral rings $\mathcal{C}_3,4(G; E)$ proceeds by breaking the symmetry to the maximal torus $H$, polarizing $E$ as $E_+ \oplus E_-$, and then descending back under the Weyl group $W$. We construct the associated chiral rings $\mathcal{R}(H; E)$ by a tweak the method of [12]. (Specifically, we convert the Lagrangian shift from loc. cit. into a charge conjugation automorphism, with the same effect of identifying the desired subring.) We then modify the Weyl action on $\mathcal{C}_3,4(H; 0)$ by suitable Euler class factors (see §5.6) to compensate for the broken symmetry. The Weyl quotients of the subrings surviving charge conjugation now give the (identity sectors of the) chiral rings for the normalizer $N(H)$. As in the original construction [BFM], we obtain the chiral ring for $G$ by an additional modification over the (affine) root hyperplanes; see §6.2 for details.

Theorem 2. The spaces $\mathcal{C}_3,4(G; E)$ are built from the affine quotients of the $\mathcal{C}_3,4(H; E)$ by the shifted Weyl action by explicit correction (§6.21) on the (affine) root hyperplanes.

1.6 Remark. It is worth explaining the added complexity in Theorem 2 versus the polarized case. The Euler class of the index bundle of $E$ defines a rational $\mathbf{5}$ torsor over the Toda spaces $\mathcal{C}_3,4(G; 0)$. Topologically, the torsor stems from the connecting map $\eta : Ksp \to \Sigma^3 K\Omega$, which obstructs the polarization lifting of $E$ to $K\Omega$ (cf. Wood’s sequence (2.2) below). This torsor has order 2, because $2\eta = 0$. (More intuitively but less accurately, the torsor is multiplicative with respect to the representation, and $E_+^{\otimes 2}$ is polarized, leading to a trivialization of the double.) There is some gymnastics involved in trivializing the torsor along the root hyperplanes, where the correction must be imposed to retrieve the Coulomb branches for $G$ from $H$.

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5In the sense of algebraic geometry, not rational homotopy...
The final formula is a reduction to the Cartan subgroup $H$ and Weyl group $W$ for certain representations $E$. This is an exact formula, not the usual, information-losing localization theorem.

**Theorem 3** (Abelianization). $\mathcal{C}_{3,4}(G; E) \cong \mathcal{C}_{3,4}(H; E \oplus gH)/W$, as soon as the roots of $g$ appear among the weights of $E$.

1.7 Remark (Complements to Theorem 3).

(i) Because $g$ is real, $E$ will contain $gH$ as a $H$-subspace, if the roots appear among the weights.

(ii) The $\mathcal{C}_{3,4}$ are determined by a collection of multiplicities associated to the $E$-weight hyperplanes [2, §5]. A particular root hyperplane can be abelianized if $E \geq gH$ in respect to that multiplicity. For SU(2), the only non-abelianizable $E$ are $0, H$ (which is obstructed) and $H \oplus H$ (which is polarized), allowing us to compute all the chiral rings.

(iii) For $\mathcal{C}_4$, the multiplicity condition in (ii) also applies to the affine hyperplanes. Half-integer multiples of the roots, which occur for symplectic groups, do not abelianize the space over the center of Sp. For instance, $\mathcal{C}_4$ does not abelianize the Chiral ring over the center of SU(2) for the irreducible 4-dimensional representation.

(iv) Over the base, the Chern character may be used to identify $\mathcal{C}_4$ locally (analytically) with spaces $\mathcal{C}_3$, for appropriate Levi subgroups of $G$. Generic reduction to the root hyperplanes on the Toda base determines every complexified $\mathcal{C}_{3,4}$.

2. Two topological facts

We review two topology constructions. The first one relies on Wood’s theorem, implicit in Bott’s periodicity of real K-theory; it underlies the construction of the $\mathcal{C}(G; E)$ by extracting a ‘quantum square root’ in place of the missing classical one. This is analogous to the Spinorial square root of the exterior power of a real vector space, which is only obstructed by $w_2$, rather than requiring a complex structure. The second fact, the stable splitting of a stratified de-suspension of a manifold, underlies the Abelianization of §7.

(2.1) First result: Wood’s theorem. Complexifying a real vector bundle defines a morphism of ring spectra $KO \to KU$. Wood’s theorem [W] identifies the resulting fibration sequence as the $KO$-module extension of $\Sigma^2 KO$ by $KO$ classified by $\eta \otimes : \Sigma^2 KO \to \Sigma^1 KO$ (the only interesting extension, since $\text{Ext}^1_{KO}(KO, KO) = \pi_1 KO = \{0, \eta\}$):

$$KO \xrightarrow{\text{C} \otimes \mathbb{R}} KU \xrightarrow{\Omega^2(\mathbb{H} \otimes \mathbb{C})} \Sigma^2 KO.$$  

The map $\Omega^2(\mathbb{H} \otimes \mathbb{C})$ is the double-looping of the quaternionization map $KU \to KSp = \Sigma^4 KO$, $V \mapsto \mathbb{H} \otimes \mathbb{C} V$. It is also the $\Sigma^2$ of the forgetful map $KU \to KO$; both times, we have implicitly used Bott periodicity on $KU$. A lift of $\mathbb{H} \otimes \mathbb{C}$ is the datum of a (stable) polarization on a quaternionic bundle $E$.

2.3 Remark. The result is not difficult: applying $KR$ to the two-point set $\{\pm\}$, swapped by the Real involution, leads to the fibration sequence

$$KO \to KR(\{\pm\}) \to ^cKO,$$

in which $KR(\{\pm\}) = KU$, while the twisting $^c$ of $KO$-theory is the suspension of $KR$ by the (reduced) sign representation of the Real involution. A Clifford algebra calculation of the relevant Thom isomorphism identifies the twisted $^cKO$ with $\Sigma^2 KO$. This argument has the merit of applying equivariantly as well.
This easy argument is not entirely honest, as it implicitly uses key properties of $K$-theory, including Bott periodicity, part of which identifies the complex Lagrangian Grassmannian $Sp/U \simeq B(U/O)$ with $\Omega Sp$; and this identification already describes Wood’s sequence.

(2.4) Application. A complex representation $E$ of $G$ gives rise to a complex, virtual index bundle $\text{Ind}_E$ over $\Omega G$, equivariant with respect to $G$-conjugation. The fiber of $\text{Ind}_E$ over a free loop $\gamma : S^1 \to G$ is the Dirac index of the $E$-bundle $\eta_\gamma : E(\gamma) \to \mathbb{P}^1$ defined by the equatorial transition function $\gamma$. This last description is equivariant for the left $\times$ right actions of $G \times G$ on the free loop group $LG$, and is compatible with loop rotation. A stricter, algebraic construction of $\text{Ind}_E$ arises by interpreting the $G$-equivariant homotopy type $\Omega G$ as that of the moduli stack of algebraic $G_C$-bundles on $\mathbb{P}^1$. The direct image of the associated $E$-bundles along $\mathbb{P}^1$, after a half-canonical twist, leads to a 2-term complex of coherent sheaves representing $\text{Ind}_E$.

An important feature of the index bundle, stemming from its doubly delooped origin, is its two-fold additivity; namely, $\text{Ind}_E : \Omega G \to KU$ is a $G$-equivariant $E_2$ map. Commutativity progresses by one step, to $E_3$, when we pass to fixed points, specifically the (geometric) fixed points of the stable homotopy or $K$-theory linearizations, giving $E_3$-compatible maps from $(\Sigma^\infty \Omega G_\gamma)^G$ or $(K \wedge \Omega G)^G$ to $KU^G$. The $E_2$ structure on the source is the sphere topology product alluded to in §1.1; this incorporates a wrong-way map, which is why stabilization (in the homotopical sense) is needed. We can build an analogous map in ordinary homology if we also change the codomain to $\Sigma^2 KO$.

A quaternionic structure on $E$ refines the index to a doubly-suspended real structure:

$$\text{Ind}_E : \Omega G \to \Omega Sp = \Sigma^2 KO.$$  \hspace{1cm} (2.6)

A polarization $E = V \oplus V^\vee$ supplies a lifting of this to $KU$ in Wood’s sequence (2.2):

$$\text{Ind}_V : \Omega G \to KU, \text{ with } \text{Ind}_E = \Omega^2(\mathbb{H} \otimes C) \circ \text{Ind}_V.$$ \hspace{1cm} (2.7)

This is used in [BFN] to construct the Coulomb branches $\mathcal{C}(G; E)$ (see §3 for a quick refresher). Applying the construction to $\text{Ind}_E$ instead of $\text{Ind}_V$ gives the Coulomb branch for the symplectically doubled representation $\mathbb{H} \otimes C E$. The following proposition is the key in extracting the square root of this construction in the absence of a polarization, as we shall do in §4.

2.8 Proposition. The map $\text{Ind}_E : \Omega G \to \Sigma^2 KO$ can be lifted locally, $G$-equivariantly to $KU$.

2.9 Remark.

(i) Liftings form a torsor over $KO_G(\Omega G)$; absent a polarization, there is no preferred lift.

(ii) Liftings need not be additive, let alone $E_2$. In fact, twice-delooing a global equivariant $E_2$ lifting would give a stable $G$-polarization of $E$. Complete reducibility of representations would lead to an actual $G$-polarization.

We will exploit the homotopy-equivalent Laurent polynomial subgroup $\Omega^4 G \subset \Omega G$. This is the quotient ind-variety $G_C((z))/G[[z]]$; it is stratified by $G[[z]]$-orbits, which are even-dimensional complex vector bundles over the $G$-orbits of the one-parameter subgroups in $G$. The latter are the generalized flag varieties $G/L$ of $G$, for various Levi subgroups $L$.

\[\text{Or rather, the double rotation, with its lift to spinors.}\]
Proof of Proposition 2.8. The obstruction \( \eta \otimes \text{Ind}_E \) to a local lifting lives in \( KO^1_L = 0 \).

For later use, we record the following.

2.10 Lemma. The stratification of \( \Omega G \) splits \( KO^G_\ast(\Omega G) \) into a sum of copies of \( KO, KSp \) and \( KU \).

Proof. The stratification assembles \( KO^G_\ast(\Omega G) \) from copies of the equivariant coefficient rings \( KO_L \), suspended by even-dimensional complex representations of the various \( L \). Each of these is a sum of copies of \( KO, KSp \) and \( KU \). Since \( \text{Hom}_{KO}(M, \Sigma N) = 0 \) for all listed \( KO \)-modules \( M, N \), there are no possible \( KO \)-linear connecting maps in the Gysin sequences for the strata and no \( KO \)-linear extensions.

(2.11) Second result: a stable splitting. Let \( M \) be a manifold equipped with a Morse function \( f \) whose Morse stratification satisfies the Whitney conditions. It is proved in [Ni] that the latter is ensured by the Smale transversality conditions.

Whitney’s Condition (A) asserts that the union \( N(f) \) of normals to the strata in the tangent bundle \( TM \) is closed. We form the Thom spectrum of \( N(f) \) and desuspend it by the tangent bundle, to obtain a spectrum \( \Sigma f \mathcal{M} := \Sigma \mathcal{N}^{-TM} \mathcal{M} \), sitting between the Spanier-Whitehead dual \( \Sigma^{-TM} \mathcal{M} \) and the suspension spectrum \( \Sigma^\infty \mathcal{M} \). The Morse stratification of \( M \) gives a filtration of \( \Sigma f \mathcal{M} \), with associated graded spectrum a sum of copies of the sphere \( S \), one for each critical point.

2.12 Proposition. \( \Sigma f \mathcal{M} \) is naturally a sum of copies of \( S \).

Proof. The filtration must split, since interesting extensions of \( S \) by a sum of copies of \( S \) are precluded by the absence of negative homotopy groups. To do better and select a splitting, we note a geometric splitting of the attaching maps: as we approach a lower stratum from a higher one, we find additional vertical directions in \( N(f) \) corresponding to the directions of approach, which we can use to shoot out the attaching map towards the base-point at \( \infty \).

There is a version of this result for Morse-Bott functions; the precise assumptions for the Whitney conditions have not been worked out, but they are expected to rely on nice enough behavior of the flow near the critical manifolds. The conclusion of Proposition 2.12 then applies equivariantly with respect to a compact group action [F]. We will use this in the algebraic case of the Lauernt polynomial loop group \( \Omega^a G \), where the Whitney property follows from homogeneity of the stratification under the subgroup \( G[[z]] \).

(2.13) Application: Abelianization of certain Coulomb branches. The \( G[[z]] \)-orbits in the subgroup \( \Omega^a G \subset \Omega G \) are the descending Morse-Bott strata for the \( G \)-invariant energy functional \( f : \Omega G \to \mathbb{R} \). Proposition 2.12 splits the spectrum \( \Sigma f \Omega^a \mathcal{G} \), \( G \)-equivariantly, into a sum of Spanier-Whitehead duals \( \Sigma^{-T(G/L)}(G/L) \) of flag varieties \( G/L \). The sum is labeled by Weyl orbits of co-characters of \( G \), each centralized by the respective \( L \).

We now apply equivariant \( K \)-homology and exploit the isomorphism of coefficient rings

\[
K^0_L = (K^0_H)^{W_L},
\]

for the Cartan subgroup \( H \subset G \) and Weyl group \( W_L \) of \( L \). This converts the \( K_G \) group to the \( W_G \)-invariant part of the sum, over all co-characters, of copies of \( K^0_H \). The ring structure may be

7Plus a technical clustering condition on the Hessian eigenvalues at critical points [Ni, Remark 4.3.4.b], which can be met by adjusting the metric at the critical points, and carries no topological content.
tracked by the ordinary localization theorem (see §4), and doing so recovers the computation in \[BFN\] for the adjoint Coulomb branch \((W = W_G)\):
\[
\mathcal{C}_4(G; g_H) \cong \mathcal{C}_4(H; 0)/W.
\]
We generalize this in \[Z\] to an Abelianization theorem
\[
\mathcal{C}_4(G; E) \cong \mathcal{C}_4(H; E \otimes g_H)/W,
\]
under the assumption that \(E\) should contain \(g_H\) as an \(H\)-representation.

3. Review of the polarized case

We recall the construction in \[BFN\] of the spaces \(\mathcal{C}_{3A}(G; E)\) for polarized representations \(E = V \oplus V^\vee\), before reframing it in terms of the equivariant Coulomb spectrum \(\Sigma(G; V)\). More details may be found the original paper, and a summary in \[T2, §3, §6\], from which the paragraphs below are excerpted. I will deviate from the sources by incorporating a Spin structure on the disk from the outset; while this clutters the notation with factors of \((dz)^{1/2}\), it avoids later redefinitions. Assume that our group \(G\) is connected; \(\pi_0 G\) leads to additional orbifolding.

(3.1) The Chiral rings \(\mathcal{R}_{3A}(G; 0)\) \[BFM\]. The space \(\mathcal{C}_3(G; 0) \coloneqq \text{Spec} H^*_c(\Omega G; \mathbb{C})\) is an affine symplectic resolution of singularities of the Weyl quotient \(T^v \mathcal{H}^v_C/W\). The homology grading represents the \(C^\times\)-scaling of the cotangent fibers. When \(G\) is simply connected, \(\text{Spec} K^*_c(\Omega G; \mathbb{C})\) is also a symplectic manifold, giving an affine resolution of \((H_C \times H^v_C)/W\); in general, it has quotient singularities under the torsion subgroup \(\pi \subset \pi_1 G\). To avoid those, we set \(G = \tilde{G}/\pi, H = \tilde{H}/\pi,\) and define \(\mathcal{C}_4(G; 0)\) as the smooth symplectic orbifold \(\pi \times \text{Spec} K^*_c(\Omega G; \mathbb{C})\).

The Hopf algebra structures of \(H^*_c(\Omega G), K^*_c(\Omega G)\) over the ground rings \(H^*_C, K^0_C\) of a point lead to relative abelian group structures
\[
\mathcal{C}_3(G; 0) \xrightarrow{\chi} \mathfrak{h}_C/W, \quad \mathcal{C}_4(G; 0) \xrightarrow{\kappa} \pi \times (\mathfrak{h}_C/W),
\]
which also define integrable systems: \(\chi\) is a fiberwise group completion of the classical Toda system \[BF\], \(\kappa\) is its finite-difference version. These groups act on all other Chiral rings.

The \(G\)-equivariant loop multiplication on \(\Omega G\) has an algebraic counterpart for \(\Omega^d G\), by means of the double coset stack \(\text{Spec} G[\mathbb{z}] \backslash \text{Spec} G((\mathbb{z})) / \text{Spec} G[\mathbb{z}]\) and the \(G[\mathbb{z}] \times G[\mathbb{z}]\)-equivariant correspondence diagram
\[
\Omega^d G \times G[\mathbb{z}] \backslash G((\mathbb{z})) \leftarrow G((\mathbb{z})) \times G[\mathbb{z}] G((\mathbb{z})) \to G((\mathbb{z})).
\]
In both cases, the underlying Poisson structure is the leading term of non-commutative deformations over \(\mathbb{C}[h] = H^*(BR)\) or \(\mathbb{C}[q^\pm] = K^0_R\), obtained by incorporating equivariance under the loop-rotation \((z\text{-scaling})\) circle \(R\). The analogue applies to all Coulomb branches below.

(3.4) The polarized case, \(E = V \oplus V^\vee\). The \(\mathcal{C}_{3A}(G; E)\) are the Specs of the \(G\)-equivariant \((K\text{-})\)homologies of a linear space \(L_V \to \Omega^d G\): a \(G[\mathbb{z}]\)-equivariant stratified space with vector space fibers. Namely, the fiber of \(L_V\) over a Laurent loop \(\gamma \in G((\mathbb{z}))\) is the kernel of the difference map
\[
L_V|_{\gamma} = \text{Ker} \left\{ V[\mathbb{z}] \oplus V[\mathbb{z}] \xrightarrow{\text{Id} - \gamma} V((\mathbb{z})) \right\} \otimes (dz)^{1/2}.
\]
This complex is equivariant under the left and right actions of \(G[\mathbb{z}]\) on the Laurent loop group, simultaneously acting on the respective factors \(V[\mathbb{z}]\), and with the left copy alone acting on \(V((\mathbb{z}))\).
Over any finite set of $G[z]$-orbits in $\Omega^a G$, projection to either summand $V[z](dz)^{1/2}$ embeds $L_V$ therein with bounded co-dimension. Moreover, $L_V$ also contains two sub-bundles of finite co-dimension, from a left and a right $z^n V[z]$, $n \gg 0$.

Stratified finiteness allows [BFN] to define the $G$-equivariant Borel-Moore $(K)$-homologies of $L_V$, after renormalising the homology grading as if dim $V[z]$ were zero. The normalised grading is compatible with the multiplication defined by the following correspondence diagram on $L_V$, living over the multiplication of two loops $\gamma, \delta \in G((z))$ in the correspondence (3.5):

$$L_V|_{\gamma} \oplus L_V|_{\delta} \leftrightarrow L_V|_{\gamma} \oplus L_V|_{\delta} \rightarrow L_V|_{\gamma \cdot \delta};$$  

the sum in the middle is fibered over the right component of $L_V|_{\gamma}$ and the left one of $L_V|_{\delta}$, while the right embedding projects to the outer $V[z]$ summands. The wrong-way map in homology along the first inclusion is defined after quotienting by a common sub-bundle $z^n V[z]$, and the result is independent of the sufficiently large $n$.

3.7 Remark. The twist in (3.5) by $(dz)^{1/2}$ is relevant to the loop rotation action and the non-commutative chiral rings; here, we only need it to make contact with the Dirac index bundle.

(3.8) The spectrum $\Sigma(G; V)$. We re-interpret the construction of $\mathcal{C}_{3,4}$, removing infinite dimensions and the consequent renormalization of homology degree, by "subtracting" the fiber $V[z](dz)^{1/2}$ over $1 \in \Omega^a G$ from the linear space $L_V$. That fiber being the largest of all, the transaction cost is passage to stable homotopy.

3.9 Definition. The Coulomb spectrum $\Sigma(G; V)$ is the de-suspension of the Thom spectrum of $L_V$ by the left bundle $V[z](dz)^{1/2}$.

This is a $G[z]$-equivariant stratified de-suspension of $\Omega^a G_+$. It generalizes the spectrum $\Sigma/\Omega^a G$ of §2.13 which we obtain for the adjoint representation $V = g$ (except for the half-integral correction $(dz)^{1/2}$ to loop rotation). The correspondence diagram (3.6) defines an $E_2$ multiplication on $\Sigma(G; V)$, compatible with its inclusion in the suspension spectrum $\Sigma^{\infty} \Omega^a G_+$. The latter is the group ring of $\Omega G$ over the sphere $S$, and we can think of $\Sigma(G; V)$ as a group ring with coefficients. The function rings of $\mathcal{C}_{3,4}(G; E)$ are the $G$-equivariant $(K)$-homologies of $\Sigma(G; V)$.

(3.10) Left versus right. Another version $\Sigma(G; V)_r$ of the Coulomb spectrum is obtained from the right factor of $V[z]$. The "left minus right" difference of bundles $V[z](dz)^{1/2}$ over $\Omega^a G$ is the index bundle $\text{Ind}_V$ [12 §6], so that the two versions are related by

$$\Sigma(G; V)_r = \Sigma^{\text{Ind}_V} \Sigma(G; V).$$

The $E_2$ property of the index bundle makes $\Sigma^{\text{Ind}_V} \Omega^a G_+$ into an $E_2$-ring spectrum: namely, the twisted group ring $\Omega G \ltimes_{\text{Ind}_V} S$, where $\Omega G$ acts on the ring spectrum $S$ by composing $\text{Ind}_V$ with the delooping $B J$ of the $J$-homomorphism:

$$\Omega G \xrightarrow{\text{Ind}_V} BU \xrightarrow{BJ} BGL_1(S).$$  

(3.11)

If $c_1 V \neq 0$, so that $\text{Ind}_V$ has non-zero rank, $\Omega G$ maps instead via $Z \times BU$ to Pic$(S)$. Thus, the left and right Coulomb spectra differ by a central extension of $\Omega G$ by $GL_1(S)$. Factorization through $BU$ in (3.11) makes the extension invisible when applied to a complex-oriented homology theory; however, when $c_1 V \neq 0$, the grading on the two versions will differ.
(3.12) **Crossed product interpretation.** Continuing this idea, denote by \( N_V := L_V \oplus V[z](dz)^{1/2} \) our (virtual) normalization of \( L_V \), and re-interpret \( \Sigma(G; V) \) as the crossed product \( \Omega G \ltimes_{N_V} S \), with \( N_V \) acting via \( f \). The jumps across strata lead to a constructible system instead of a bundle of coefficients.

We use this picture to summarize the construction of the next section. While the spectrum \( \Sigma(G; E) = \Omega G \ltimes_{N_E} S \) leads to the rings \( \mathcal{R}_{3,4}(G; H \otimes_C E) \) for the double of \( E \), we cut this in half by observing that the composition

\[
\mathbb{Z} \times BU \rightarrow \text{Pic}(S) \rightarrow \text{Pic}(KU)
\]

very nearly factors through the quotient \( \Omega^2Sp = BU/BO \): the Thom isomorphism may be used to factor out \( B\text{Spin}^c \). After removing the orientation and \( \text{Spin}^c \) obstructions, we will use the refined \( \Omega^2Sp \)-structure of \( \text{Ind}_E \) to construct \( \Omega G \ltimes_{N_E} KU \) for \( \mathcal{R}_4 \), and its homology version for \( \mathcal{R}_3 \).

4. **General case: real structure on the linear space**

Subject to obstructions and ambiguities to be discussed, we now construct the coefficient systems \( \mathcal{K}_E, \mathcal{K}_E \) over \( \Omega^gG \) replacing the use of \( \Sigma(G; V) \).

(4.1) **Stratified polarization of \( \text{Ind}_E \).** We can think of the linear space \( N_E := L_E \oplus E[z](dz)^{1/2} \) as a constructible lift of the index \( \text{Ind}_E \) to \( \mathbb{Z} \times BU \), with respect to Wood’s sequence \( (2,2) \) over the strata of \( \Omega^gG \). Stratum-by-stratum,

\[
\Omega^2(\mathbb{H} \otimes_C) \circ N_E = \text{Ind}_E. \tag{4.2}
\]

Indeed, near any one-parameter subgroup \( z^\gamma, \gamma \in \mathfrak{h}, E \) breaks up as \( E_+ \oplus E_0 \oplus E_- \) according to \( \gamma \)-eigenvalue and \( E_- \) polarizes the complementary representation \( E \oplus E_0 \) of \( L \). Realizing \( \text{Ind}_E \) as the \( \delta \)-cohomology of the bundle \( E(\gamma) \otimes \sqrt{K} \) over \( \mathbb{P}^1 \), \( H^0 \) comes from \( E_+ \) and \( H^1 \) from \( E_- \). (\( E_0 \) does not contribute.) Comparing with

\[
z^\gamma : E[z](dz)^{1/2} \rightarrow E([z]) / E[z](dz)^{1/2}
\]

identifies \( N_E \) with \((-H^1)\), and Serre duality plus the quaternionic structure on \( E \) give the anti-linear identification of \( H^0 \) and \( H^1 \) required for \( (4.2) \). The refined interpretation sees the left side of \( (4.2) \) as a degeneration of the right as a constructible coefficient systems: moving near the \( \gamma \)-stratum deforms \( \partial \) to give an extension class, which converts the symplectic double of \( N_E \) into \( \text{Ind}_E \).

(4.3) **Real structures.** A real structure on \( N_E \) would provide a de-suspension of \( \Omega G \) reaching halfway to \( \Sigma(G; E) \), analogous to the effect of a polar half \( V \). On neighborhoods \( U \) of \( G[z] \)-orbits, \( \eta \) can be trivialized (Proposition \( 2.8 \)) and a second, continuous \( G \)-invariant local lift \( S_E \) of \( \text{Ind}_E \) gives a stable real structure on \( N_E \oplus S_E \). The jumps in the fibers all come from \( N_E \). Denote the underlying real linear space by \( R_E \); writing

\[
\mathbb{C} \otimes R_E = N_E \oplus S_E
\]

makes the suspension \( \Sigma^{R_E}U \) into a real version of the de-suspension of \( \Sigma(G; E) \) by \( S_E \):

\[
\Sigma^{C_{R_E}}U = \left. \Sigma^{-S_E} \Sigma(G; E) \right|_{U^+}.
\]
Example: Polarized case. When $E = V \oplus V^\vee$, we can take $\text{Ind}_E$ for $S_E$. Denote by underlines the $\sqrt{K}$-twists of the associated bundles on $\mathbb{P}^1$. Then,

$$N_E - S_E = -H^1(E) - H^0(V^\vee) + H^1(V^\vee) = -H^1(V) - H^0(V^\vee) = -H^1(V) - H^1(V)^\vee;$$

the underlying real space is $-H^1(V) = L_V \oplus V [d(z)]^{1/2}$, and $\Sigma^{R_E} \Omega^{R} G_+ = \Sigma(G; V)$.

(4.4) Obstruction theory. The coefficient systems $H^*_G, K^*_G$ of $\Sigma^{R_E} U_+$ should give the constructible coefficient systems defining our spaces $\mathcal{C}_{3,4}(G; E)$. However, we meet two problems:

(i) The local spectra $\Sigma^{R_E} U_+$ depend on the auxiliary local section $S_E$; the ambiguity is a suspension by an arbitrary $G$-equivariant $KO$-class.

(ii) Even if $S_E$ were globally defined, there is no Pontryagin product if $S_E$ is not multiplicative.

The local ambiguity in (i) becomes a global obstruction if $\eta \otimes \text{Ind}_E \neq 0 \in KO^{[1]}_G(\Omega G)$. The multiplication in (ii) runs into the group extension of $\Omega G$ by $KO$, pulled from Wood’s sequence (2.2) by $\text{Ind}_E$. This gives a projective action of $\Omega G$ on the sphere $S$, pre-empting the crossed product ring $\Omega G \ltimes_{R_E} S$ of §3.10. An $E_2$ splitting of the extension is equivalent to a polarization of $E$ (Remark 2.9i). Without it, there is no stable homotopy chiral ring. Fortunately, the obstructions to building $\mathcal{C}_3$ and $\mathcal{C}_4$ are much milder:

- the primary obstruction is $w_4(E) \in H^4(BG; \mathbb{Z}/2)$ modulo squares $r^2, r \in H^2(BG; \mathbb{Z})$;
- the secondary obstruction $B \sigma$ arises after trivializing $w_4 - r^2$ by a cochain $c \in C^3(BG; \mathbb{Z}/2)$, where $\sigma := Sq^2 c \in H^5(BG; \mathbb{Z}/2)$ is defined up to $Sq^2 H^5(BG; \mathbb{Z}/2)$.

4.5 Theorem (Primary and Secondary obstructions).

(i) Construction of $\mathcal{C}_3(G; E)$ requires lifting the primary obstruction. The choices of $\mathcal{C}_3$ form a torsor over $H^3(BG; \mathbb{Z}/2)$.

(ii) Constructing $\mathcal{C}_4$ requires removing the secondary obstruction $B \sigma$, with a choices forming a torsor over the 2-torsion in $H^5(BG; \mathbb{Z})$.

(iii) Both obstructions vanish when $G$ is connected without symplectic factors. (See Theorems [A.4])

4.6 Remark (Complements).

(i) With complex (rather than integral) coefficients, the choices in $H^3$ affect of $\mathcal{C}_3$ only via its Bockstein image, the 2-torsion in $H^4(BG; \mathbb{Z})$. The latter vanishes for connected $G$.

(ii) When $\pi_0, \pi_1 G$ have no 2-torsion, $H^3(BG; \mathbb{Z}/2) = 0$, so a fortiori $\sigma$ and $B \sigma$ vanish.

(iii) The class $\sigma$ obstructs the $KO$-version of $\mathcal{C}_4$.

(iv) If we are willing to collapse the homology grading to $\mathbb{Z}/2$, the primary obstruction may lifted by a square root in $H^2(BG; \mathbb{Z}/2)$; see Remark 4.10

(v) When $E = V \oplus V^\vee$, the universal identity $c_2(E) = 2c_2(V) - c_1^2(V)$ cancels the obstructions. Conceptually, the lift of $E : BG \to BSp$ by $V : BG \to BU$ kills the obstruction source $\eta$.

Proof. The Thom isomorphism in homology removes the local ambiguity §4.4i, provided we reduce the structure group $BO$ to $BSO$ in the sequence (2.2) over $\Omega G$:

$$BO \to BU \to \Sigma^2(Z \times BO).$$

---

8The vanishing of $Sq^2(r^2)$ and of $Sq^2(c_2)$ on $BSp$ converts $Sq^2 c$ into a co-cycle.
Doubly delooping such a reduction leads to an $E_2$ multiplication in §4.4.ii. A reduction to $B\text{Spin}^c$ accomplishes the same for complex $K$-theory.

Reducing the structure group meets the orientation obstruction $w_1$ and the $\text{Spin}^c$ obstruction $W_3 = Bu_2$. The two assemble to an exotic cohomology theory $\mathcal{W}_B$, co-fiber of the map $B\text{Spin}^c \to BO$, with homotopy groups $\pi_1 = Z/2$, $\pi_3 = Z$ and $k$-invariant $B \circ Sq^2$. Our chiral rings are then obstructed by the double delooping

$$BG \xrightarrow{E} BS\rho \xrightarrow{\eta \otimes} \Sigma^3 BO \to \Sigma^3 \mathcal{W}_B.$$ (4.7)

Specifically, $\mathcal{W}_3(G;E)$ is obstructed by $w_4(E) = c_2(E) \pmod{2}$. When that has been cancelled as $\delta c, c \in C^3(BG;Z/2)$, the remaining obstruction to defining $\mathcal{W}_4$ is the composition $\langle 4, 7 \rangle$, the integral Bockstein image $B\sigma = BSq^2(c) \in H^6(BG;Z)$, as claimed.

Reductive adjustment. A polarized representation of a reductive group may have odd $c_2$: such is the doubled standard representation of $U(1)$. This seems at odds with the contraction of $\mathcal{W}_3$ in that case. The hidden problem is that the fiber of the middle map $\eta \otimes : BS\rho \to \Sigma^3 BO$ in (4.7) is $BSU$, rather than $BU$, preventing lifts by a polar half with $c_1 \neq 0$.

Abandoning the $\Sigma^3 Z$ layer at the base of $BS\rho$, in favor of $\Sigma^3 BO$, came at cost. While $BS\rho$ has no interesting maps to $\Sigma^3 Z$, the trivial map there out of $BG$ has self-homotopies classified by $H^2(BG;Z)$. The $\Sigma^1 Z/2$ base of $\mathcal{W}_B$ is fibered over $Z$ by $Sq^2$, and the effect of an $h \in H^2(BG;Z)$ is to shift $w_4$ by $h^2 \pmod{2}$. The effect of $h$ on the full $\mathcal{W}_B$-obstruction can be determined from the unique lift of $Sq^2$ in the sequence

$$[HZ; \Sigma^3 Z] = 0 \to [HZ; \Sigma \mathcal{W}_B] \to \langle Sq^2 \rangle = [HZ; \Sigma^2 Z/2] \rightarrow^{BSq^2=0} [HZ; \Sigma^5 Z].$$

Alternatively, this newly-found freedom is explained by noting the formula

$$c_2(E \oplus L \oplus L^{-1}) = c_2(E) - c_1(L)^2$$

for a one-dimensional representation $L$. The polarized summand is killed by $\eta$, and does not change the obstruction $\eta \otimes E$ valued in $Sp$.

Finally, the multiplication improves to $E_3$ on equivariant ($K$-)homologies: the transgressive nature of the obstructions tracks their cancellation on the space of free maps $S^2 \to BG$. □

4.8 Remark (Real Spin obstruction). Just as $W_3 = Bu_2$, $\mathcal{W}_B$ is built from a spectrum $\mathcal{W}$ with $\pi_1 = \pi_2 = Z/2$ and $k$-invariant $Sq^2$, which contains the obstruction $\sigma$ to real Spin orientability. Incorporating the base $\mathcal{W}_0 \cong Z$ gives the 3-layer truncation $ko_{<3}$, or of $S^0_{<3}$.

One handle on $\sigma$ comes from the formula, shadowing the Adem relation $Sq^2Sq^2 = Sq^3$:

$$Sq^2 \sigma = Sq^3 c_2 - \frac{h^2}{2}$$ (4.9)

This stems from the restriction of $ko_{<3}$ to $2Z \subset \pi_0 ko$, which is built by stacking $\Sigma^2 Z/2$ over $2Z \times \Sigma Z/2$, with $k$-invariant $(x, y) \mapsto Sq^3(x/2) + Sq^2 y$.

4.10 Remark (Abandoning the homology grading). The classification in the Appendix show that, for connected $G$, the obstruction discussed in [BDFRT] is equivalent to the weaker requirement of a mod 2 square root of $w_4$. If so, only Case (i) of Theorem A.4 is obstructed. Now, the bottom two layers $Z, \Sigma Z/2$ of $ko$ may be collapsed to $Z/2, \Sigma Z/2$, and for $\mathcal{W}_3$, we gain the freedom of cancelling $w_4(E)$ by a square from $H^2(BG;Z/2)$. However, the bottom $Z$ represents the grading in homology: exploiting $H^2(BG;Z/2)$ collapses the homology $Z$-grading to $Z/2$.

$^9$There are now two lifts of $Sq^2$ in the extension group $[HZ; \Sigma \mathcal{W}] \cong Z/4$, but they differ by the shearing automorphism $Sq^1 : \Sigma Z/2 \to \Sigma^2 Z/2$ of $\mathcal{W}$. 

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4.11 Proposition (Parity check).

(i) If \( w_4(E) = r^2 \), with \( r \in H^2(BG; \mathbb{Z}/2) \), then \( R_E \) may be chosen with even-dimensional (real) fibers.

(ii) If \( \bar{r} \) has an integral lift, the even dimensions can be chosen to be additive for the Pontryagin product: the homology of its Thom spectrum is then (evenly) \( \mathbb{Z} \)-graded.

Proof. With notation as in §4.1,
\[
\dim_{\mathbb{R}} R_E|_{\gamma} = -\sum_{\nu} \langle \nu | \gamma \rangle - \dim_{\mathbb{C}} S_E,
\]
with \( \nu \) over \( \gamma \)-positive weights of \( E \) with their multiplicities. Call \( q \) the quadratic form on the Lie algebra associated to \( c_2 \). As \( k = k^2 \pmod{2} \), we may switch to the sum of squares \( \langle \nu | \gamma \rangle^2 \), which evaluates to \( q(\gamma) \).

For two co-weights \( \gamma, \gamma_0 \) in the same component of \( \Omega G \),
\[
q(\gamma) - q(\gamma_0) = \partial q(\gamma - \gamma_0) + q(\gamma - \gamma_0),
\]
with the associated bilinear form \( \partial q \). The latter is even, because \( c_2 \) is a sum of integral squares. Under our assumptions, \( q \) takes even values on co-roots, since \( \bar{r} = 0 \) there. Adjusting \( \dim_{\mathbb{C}} S_E \) on components can then render \( \dim R_E \) even, which proves (i).

With an integral lift \( r \), we may specifically choose \( \dim S_E = r(\gamma) \) settling Part (ii). \( \square \)

4.12 Corollary (Freedom over the Toda base). The rational homology chiral ring is free over the Toda base. The same applies to the integral K-theory chiral ring, when \( \pi_1 G \) is torsion-free. (See Remark 1.5.ii for torsion in \( \pi_1 \)).

Proof of the Corollary. The Bruhat stratification filters those groups with free subquotients in even degrees, ruling out connecting differentials. \( \square \)

4.13 Ambiguities. The choices in Proposition 4.5 are meaningful in TQFT. Thus, the ring of functions on \( \mathcal{C}_4(G; E) \) is expected to be the space associated to the sphere in a 3-dimensional gauge theory with group \( G \) and matter fields in \( E \).

The pure gauge theory, without matter fields, can be precisely, if incompletely\(^{10}\) defined as the “sphere K-theory” of the stack \( BG \). To a closed surface \( S \), this assigns the K-homology of the moduli \( \text{Bun}_G \) of topological or holomorphic \( G \)-bundles on \( S \) (both are homotopy equivalent to the mapping space of \( S \) to \( BG \)).

This theory admits discrete twists by \( \Sigma^2 \mathcal{W}(BG) \). For instance, such a twist transgresses over \( S \) to produce a class in \( \mathcal{W}(\text{Bun}_G(S)) \), which defines a (graded) twisting for K-theory over \( \text{Bun}_G(S) \). This modifies the K-theory space associated to \( S \).

5. Obstructions in terms of the Weyl group

I now review the topological obstructions in terms of the maximal torus of \( G \) and its normalizer. Their transgression to the free loop group of \( BG \) is related to a modified Weyl group action on the Toda space, to be used in the explicit construction of the chiral rings in the next section. One feature is to clarify the elusive second obstruction \( \sigma \): its transgression becomes a cohomology class.

---

\(^{10}\)In the sense that not all 3-dimensional operations are defined.
Notation: $H \subset G$ denotes the maximal torus, $H^\vee$ the dual torus, $\tilde{2}H^\vee \subset H^\vee$ the subgroup of 2-torsion points, $N(H)$ the normalizer in $G$, $W$ the Weyl group, $W_{\text{aff}}$ the affine Weyl group $W \times \Lambda^\vee$, with $\Lambda$ the weight lattice. Roots and coroots will be denoted by $\alpha$ and $h_\alpha$, weights of representations by $\nu$. A typical element of $H$ will be denoted by $x$, one of its Lie algebra $\mathfrak{h}$ by $\xi$. We will also choose a generic regular element $\xi_0 \in \mathfrak{h}$, splitting $E = E_+ \oplus E_0 \oplus E_-$ into positive, negative and zero weight spaces.

(5.1) Transgressed obstructions. The evaluation map $S^1 \times LB G \to BG$ on the loop space defines a transgression \[^{11}\] with the adjoint action of $G$ on itself on the right-hand side:

$$\tau : KS p^0_G \to KS p^{-1}_G(G)$$

Further restriction takes us to $KS p^{-1}_{N(H)}(H)$. Our two obstructions extracted from $\eta \otimes E$ lead to

$$\tau(w_4) \in H^3_{N(H)}(H; \mathbb{Z}/2), \quad \tau(\sigma) \in H^4_{N(H)}(H; \mathbb{Z}/2);$$

the second obstruction is in principle contingent on a cancellation of the first, but part of it turns out to be independently defined (Remark 5.3). We pare these down further by using the Leray spectral sequence $H^p_W(H^q(BH \times H)) \Rightarrow H^{p+q}_{N(H)}(H)$, to obtain classes in Weyl group cohomology:

5.2 Theorem.

(i) The Leray leading term of $\tau(w_4)$ is a class in $H^2_W(\Lambda/2) \subset H^2_W(H; \mathbb{Z}/2)$, given by

$$(u, v) \mapsto c(u, v) := \sum_{\nu > 0 \atop \nu v > 0} uv \in \Lambda/2.$$ 

(ii) The Leray leading term $\tau(\sigma)$ is a class in $H^1_W(\Lambda^{\otimes 2}/2) \subset H^1_W(BH \times H; \mathbb{Z}/2)$. It is always uniquely defined, and given by

$$w \mapsto s^{\otimes 2}(w) := \sum_{\nu > 0 \atop \nu v < 0} \nu v \otimes \nu v \in \Lambda^{\otimes 2}/2.$$ 

(iii) The big Bockstein $B \tau(\sigma) \in H^2_W(\Lambda^{\otimes 2})$ is

$$(u, v) \mapsto d(u, v) := \sum_{\nu < 0 \atop \nu v > 0} uv \otimes uv \in \Lambda^{\otimes 2}.$$ 

(iv) A trivialization of the topological obstructions $w_4$ and $\sigma$ leads to a trivialization of the Weyl cocycles in (i) and (ii), by means of co-chains valued in $\Lambda/2$, respectively in $\Lambda^{\otimes 2}/2$.

The cohomology degrees disambiguate the two copies, $H^1(H)$ and $H^2(BH)$, of $\Lambda$ in the formulas. 

5.3 Remark.

(i) The obstruction $\tau(\sigma)$ satisfies $\delta \tau(\sigma) = Sq^2 \tau(w_4)$, and will only define a cohomology class upon trivializing the latter. However, the natural cancellations we will find below for the $H^1_W$ components of $\tau(w_4)$ define $\tau(\sigma)$ over the 3-skeleton of $BW$, even when $w_4 \neq 0$.

(ii) An example with $w_4 = 0$ but $s^{\otimes 2} \neq 0$ is given in Appendix B.

(iii) These Weyl cocycles allow for more choices of trivializations than the topological obstructions for $G$. The ones coming from the stable splitting of $BG$ in $BN(H)$ should be used in the construction of $\varphi_{3,4}$ in the next section. An example of erroneous obstruction cancellation is found in Appendix B.

(iv) When the obstructions vanish, the cocycles $c$ and $d$ can be cancelled by Weyl 1-cochains of the form $\sum v$ and $\sum v \otimes v$. This follows from judicious choices of orientations (Remark 5.12).
(5.4) Preparation: symplectic case. The formulae are functorial under restriction from the representation $G \to \text{Sp}(E)$; this will reducing the calculations to $G = \text{Sp}(m)$ with its standard representation $E = \mathbb{H}^m$. Specifically, $H$ maps to the standard Cartan subgroup $H_{\text{Sp}} \cong \mathbb{U}(1)^{\times m} \subset \text{Sp}(E)$, giving a compatible map of normalizers, Weyl groups and dual map $H_{\text{Sp}}^\vee \to H^\vee$:

$$W_{\text{Sp}} = S_m \wr \mathbb{Z}/2, \quad N(H_{\text{Sp}}) = S_m \wr \text{Pin}^{-}, \quad \{v | v > 0\} = \{e_j\}, \quad H_{\text{Sp}} \cong H_{\text{Sp}}^\vee = C^m/\langle e_i \rangle.$$ 

Each $e_i$ flips signs under the $j$th factor of $\mathbb{Z}/2$ in $W$, and the last isomorphism identifies co-roots with fundamental weights.

Identify $\Lambda/2$ with the 2-torsion in $H_{\text{Sp}}$; then, $c$ defines the normalizer extension

$$1 \to \text{Spin}(2)^m \to S_m \wr \text{Pin}(2)^- \to S_m \wr \mathbb{Z}/2 \to 1.$$ 

More precisely, $c$ defines the Tits reduction to a 2-torsion extension $[\text{Ti}]$. An explicit account of Tits extensions can be found in [DW §5], but it is easy to check the symplectic case directly: the permutation group, and the $\mathbb{Z}/4$-extensions of the $\mathbb{Z}/2$ factors, all embed in Sp, and we can compare this extension with the formula for $c$.

**Proof of Theorem [5.2]** For Part (i), we show a stronger statement: over the 3-skeleton of $BW$,

$$w_4(E) = \sum_{\nu > 0} \nu^2 + c \in H^4(BN;\mathbb{Z}/2)/H^4(BW;\mathbb{Z}/2),$$

with $c \in H^2_W(\Lambda/2) \cong H^2_W(H^2(BH;\mathbb{Z}/2))$. The first term is unambiguous below $H^4(BW)$: the $\nu$ are defined over the 2-skeleton of $BW$, with ambiguity in $H^2(BW)$, so this splitting of $w_4$ is well-defined.

Before checking (5.5) for Sp, let us see how it implies Part (i). We have $\tau Sq^2\nu = Sq^2(\tau \nu) = 0$; furthermore, $\tau$ vanishes identically after restriction to $N(H)$. The $H^0$ and $H^1$ components of $\tau(w_4)$ thus vanish, and our leading term for $\tau(w_4)$ is indeed in $H^2_W(\Lambda/2)$, where it arises from the identification $\tau : H^2(BH) \tilde{\to} H^1(H)$.

Returning to $\text{Sp}(m)$, the product decomposition and its strict $S_m$ equivariance reduces us to checking the claim on the group $\text{Pin}^{-} \subset \text{SU}(2)$, with Weyl group $\mathbb{Z}/2$. Call $x$ the generator of $H^1(B\mathbb{Z}/2;\mathbb{Z}/2)$ and $\nu$ that of $H^2(B\text{Spin}_2)$. The extension class in $\text{Pin}^{-}(2)$ leads to a non-zero Leray differential $d_3(\nu) = x^3 \in H^3(\mu_2;\mathbb{Z}/2)$, so that

$$d_3(Sq^2\nu - w_4) = d_3 Sq^2\nu = Sq^2 d_3(\nu) = Sq^2(x^3) = x^5.$$ 

which implies $\nu^2 - w_4 = x^2\nu$, confirming (5.5).

For Part (ii), note first that the transgression to $BH \times H$ is linear on $H$; but the $H$-linear part of $H^2_W(BH \times H)$ vanishes, so the leading term is indeed in $H^1_W$. Similarly, ambiguities from $H^3(BG;\mathbb{Z}/2)$ transgress to a leading term in $H^1_W(H^1(H))$, whose $Sq^2$ vanishes over the 1-skeleton of $BW$ and cannot alter the leading component of $\tau(\sigma)$.

Returning to $\text{Sp}(m)$ for the calculation, note that the cocycle $s \otimes x^2$ is strictly invariant under $S_m \subset W$; the same applies to the class $\tau(\sigma)$, so we are again reduced to a check for $\text{SU}(2)$, when $s \otimes x^2$ reduces to the unique non-trivial class in $H^1_{\text{Z}/2}(\mathbb{Z}/2)$. With $x, \nu, \tau(\nu)$ as before, we must show that $\tau(\sigma) = xv \tau(\nu) \in H^1_{\text{Z}/2}(H \times BH)$. Working over the 4-skeleton of $B\mathbb{Z}/2$,

$$\delta(\eta \cdot w_4) = Sq^2(w_4), \text{ so that }$$

$$\delta(\tau(\sigma)) = Sq^2(\tau(w_4)) = Sq^2(x^2 \tau(\nu) + O(x^3)) = x^4 \tau(\nu) + O(x^5).$$

The only class in the Leray sequence cancelling this (via $d_3$) is the advertised class $xv \cdot \tau(\nu)$.

Part (iii) is a direct calculation of the coboundary on group cochains.

Part (iv) follows by tracking a topological trivialization under the transgression $\tau$. 

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(5.6) **Modified Weyl action.** We now interpret the transgressed obstructions in terms of a modified rational action of the Weyl group on the Toda space. Choose $E_+ \subset E$ to be the sum of positive weight spaces in the standard representation. To an element $w \in W$, assign the rational sections

$$
\chi_w : \mathfrak{h} \to H^\vee, \quad \zeta \mapsto \prod_{\nu > 0, \nu \varpi < 0} \langle \nu \varpi | \zeta \rangle^{w_{\nu \varpi}},
$$



$$
\kappa_w : H \to H^\vee, \quad x \mapsto \prod_{\nu > 0, \nu \varpi < 0} \left(1 - \chi^{-w_{\nu \varpi}}\right)^{w_{\nu \varpi}},
$$

and modify the action of $w$ on Toda spaces by $\chi$- and $\kappa$ shifts, as follows:

$$(\zeta, h) \mapsto (w \zeta, \chi_w(w \zeta) \cdot wh),
$$

$$(x, h) \mapsto (wx, \kappa_w(wx) \cdot wh).$$

This Weyl action may not quite close; to see this, identify $\Lambda/2 \cong \mathbb{Z} H^\vee$ and $\operatorname{Hom}(H; H^\vee) \cong \Lambda^{\otimes 2}$, and recall the Weyl cocycles $c(u, v)$ and $d(u, v)$ from Theorem 5.2. The product $c \cdot d$ can then be viewed as an affine map $H \to H^\vee$.

**5.8 Proposition** (Projective obstructions).

$$
\begin{align*}
\delta \chi &= c : \quad \chi_{w_{uv}} = \chi_u \cdot [\chi_v] \cdot c(u, v), \\
\delta \kappa &= c \cdot d : \quad \kappa_{w_{uv}} = \kappa_u \cdot [\kappa_v] \cdot c(u, v) \cdot d(u, v)
\end{align*}
$$

**Proof.** We check the formula for $\chi$. Denoting by $\varphi_v = \langle v | \zeta \rangle^v$, we have

$$
\varphi_{-v} = (-1)^v \varphi_v^{-1}, \quad \varphi_{wv} = w [\varphi_v] \text{ for } w \in W,
$$

$$
\prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v = \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v
$$

whereas

$$
\prod_{v > 0, \nu \varpi < 0} u \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v = \prod_{v > 0, \nu \varpi < 0} u \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v
$$

$$
= \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v^{-1}, \quad \prod_{v > 0, \nu \varpi < 0} (-1)^{u \nu \varpi} \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v = \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} u \nu \varphi_v \cdot \prod_{v > 0, \nu \varpi < 0} (-1)^{u \nu \varpi},
$$

where in the second line we changed the sign of the label $v$. The check for $\kappa$ uses the relation $\psi_{-v} = (-x^v)^v \cdot \psi_v^{-1}$ for the corresponding factors, leading to the extra factor $d(u, v)$. $\square$

**5.10 Corollary.** A cancellation of the obstructions $\omega_4$ and $\sigma$ leads to a cancellation of the projective Weyl cocycles, by means of 1-cochains in $\mathbb{Z} H^\vee$ for $c$, and by the Bockstein of an element in $\Lambda^{\otimes 2}/2$ for $d$. $\square$

(5.11) **Interpretation.** In Weyl group cohomology, the cocycle $\chi$ represents the connecting image (generalized Bockstein) of the $H$-equivariant Euler class of $\text{Ind}(E)$, with respect to the fibration sequence

$$
BSO \to BU \to \Omega Sp^{e\nu}
$$

(in which $Sp^{e\nu} \to Sp$ is defined by the evenness condition on $c_2$). Specifically, $\chi$ is the Weyl coboundary of the Euler class of the $(H$-equivariant) lift $\text{Ind}_{E_+}$ of $\text{Ind}_E$ to $BU$. The analogue holds for $\kappa$, $BSpin^c$ and the $K$-theory Euler classes. (See the relations in 6.18 in the next section.)
(5.12) **Obstruction-removal.** To reconcile the Weyl and topological stories of obstruction removal, consider, for \( w \in W \), the \( H \)-equivariant Dirac index over \( \mathbb{P}^1 \):

\[
\langle D(w) | \gamma \rangle := \text{Ind}_{w\mathbb{P}^1}^{E_4(\gamma)} \otimes \text{Ind}_{E_4(\gamma)}^{H_3(\gamma)}.
\]

These have a real structure, as they stem from the difference of two polarizations of \( E \) (cf. 4). The \( D(w) \) carry torus actions with no invariant lines, and are thus always individually \((K)\)-orientable. The Weyl cocycles \( c \) and \( B_s^{\otimes 2} \) stem from inconsistent local orientation choices.

More precisely, \((w, \gamma) \mapsto \langle D(w) | \gamma \rangle\) defines a class in \( H^2_{Waff}(kO_H) \), with leading component in \( H^1_W(\Lambda \otimes kO_H) \) because of the linearity in \( \gamma \). Therein,

- The leading term in \( H^2_W(\Lambda) \) is the Bockstein of the Weyl-invariant element \( w_2^H(E_+) \in \Lambda/2 \).
  This vanishes when \( w_4(E) = 0 \); otherwise, it leads to the appearance of some 2-torsion points in the calculus of the next section, e.g. Remark 6.8.
- The orientation obstruction \( w_1 \in \mathbb{Z}/2 \) takes us to a class in \( H^1_W(\Lambda/2) \), represented by the Weyl co-cycle \( c \); vanishing of \( w_4(E) \) allows a choice of consistent orientations.
- The Spin obstruction, \( w_2^H \in \Lambda/2 \), leads to the class in \( H^1_W(\Lambda^{\otimes 2}/2) \) represented by \( s^{\otimes 2} \); when \( B_0 \) vanishes, we can choose consistent Spin* orientations.

5.13 **Remark** (Spin case). Even when \( c \) and \( s^{\otimes 2} \) are trivializable, the individual \( D(w) \) need not be spinnable. Their \( KO \) Euler classes (Spin determinants) are then naturally sections of \( H^* \)-bundles of order 2 over \( H \), rather than maps to \( H^* \) (Remark 6.15 below). Obstructions are cancelled by coherent identifications between the Weyl-transformed bundles and their determinant sections.

6. **Construction by Weyl descent**

We move to the explicit algebraic construction of the \( \mathcal{C}_3(G; E) \) extending the polarized case of [12, Theorem 2]. One tweak is the use of a charge conjugation symmetry \( C \), replacing the vertical shift of \( \text{loc. cit.} \); this effects a good clean-up of signs in the formulae. The main result, Theorem 2, is restated in precise form at the end of this section; however, a brief navigational chart should help:

- §6.1 recalls the GLSM construction for the polarized case;
- §6.5 reformulates that in terms of \( C \);
- Remark 6.8 (optional and partly speculative) discusses the relation to boundary theories;
- §6.9 outlines the general construction;
- §6.16 describes the (the identity component of) the chiral ring for the normalizer \( N(H) \) by Weyl descent;
- Finally, §6.21 completes the construction of \( \mathcal{R}(G; E) \).

(6.1) **Review of the polarized case.** We “couple a complex mass term” to \( E = V \oplus V^\vee \); this means scaling \( V \) and \( V^\vee \) under opposite actions of \( S^1 \) and adjoining the respective equivariant parameters, \( \mu \in H^2(BS^1) \) or \( e^{\pm \mu} = m^{\pm 1} \in k^0_{S^1} \), to the bases of the respective Toda systems. We then introduce the rational Lagrangian Euler sections \( \varepsilon_V, \lambda_V \) of the Toda projections,

\[
\varepsilon_V : \xi \in \mathfrak{h} \mapsto \prod_v (\langle \xi | \xi \rangle + \mu)^v,
\]

\[
\lambda_V : x \in H \mapsto \prod_v \left( 1 - m^{-1} x^{-v} \right)^v.
\]

\[\text{Coulomb branches for disconnected groups have twisted sectors, from the components of the group.}\]
with \( \nu \) ranging over the weights of \( V \) (multiplicities included). In parsing these formulae, note the two uses of \( \nu \): as an infinitesimal character of \( H \), and as a lattice vector in \( h^\vee = \text{Lie}(H^\vee) \) (so that \( a^\nu \) is a point of \( H^\vee \) when \( a \in \mathbb{C}^\times \)). Remarks 6.13–6.15 below review the meaning of these formulae.

**Theorem** ([T2], Theorems 1 and 2). The (massive) chiral rings \( \mathcal{R}_3(G; E)[\mu] \) and \( \mathcal{R}_4(G; E)[m^\pm] \) comprise those regular functions on the Toda space (with \( \mu, m^\pm \) adjoined) which remain regular under vertical shift by \( \epsilon_V \), respectively \( \lambda_V \). Specializing to \( \mu = 0 \), respectively \( m = 1 \), gives the massless versions.

6.3 Remark (Role of mass terms). The auxiliary step of adding \( \mu, m \) is unnecessary when the group \( G \) contains a circle acting on \( V \) with strictly positive weights. Absent such a circle, we enlarge \( G \) to include the ‘mass circle’ \( M \), scaling \( V \) and \( V^\vee \) with opposite weights. From the Coulomb branch for \( G \times M \), we get to the massive Coulomb branch for \( G \) by collapsing the Toda fibers corresponding to \( M \).

6.4 Remark. The affine spaces \( \mathcal{C}(G; E) \) submerge onto their Toda bases away from co-dimension 2 (in the total spaces); they are thus determined by their regular local sections. For instance, following the description in [BFM], regular are those sections of \( \mathcal{C}(G; 0) \) which pull back to sections \( s \) of \( \mathcal{C}(H; 0) \) satisfying the evaluation conditions \( \exp(h_a) \circ s = 1 \) over the root hyperplanes \( a \neq 0 \), or respectively \( \epsilon^a = 1 \). To construct \( \mathcal{C}(G; E) \), we simply include \( \epsilon_V \) or \( \lambda_V \), along with their Toda group translates, as regular sections, and affinize the resulting space.

(6.5) Charge conjugation \( C \). It will prove useful to modify the Lagrangian shift, combining it with an automorphism of the Toda space \( \mathcal{C}(G; 0) \), which dualizes the \( G \)-representations and simultaneously changes orientation on the 2-sphere. The former is realized by inversion on the Toda base: \( \xi \leftrightarrow -\xi \), or \( x \leftrightarrow x^{-1} \); the latter is inversion in \( \Omega G \), and thus on the fibers \( H^\vee \). This gives the automorphisms of order 2,

\[
C : (\xi, h) \leftrightarrow (-\xi, h^{-1}), \quad (x, h) \leftrightarrow (x^{-1}, h^{-1}).
\]

Clearly, functions in \( \mathcal{R}_3(G; E) \) are equally characterized by regularity under the modified charge conjugations \( C_V \) combining the shift with \( C \):

\[
C_V : (\xi, h) \mapsto (-\xi, \epsilon_V^{-1} \cdot h^{-1}), \quad (x, h) \mapsto (x^{-1}, \lambda_V^{-1} \cdot h^{-1}).
\]

The \( C_V \) square to the vertical shifts by the sections

\[
\xi \mapsto \prod_v (-1)^v, \quad \text{respectively} \quad x \mapsto \prod_v (-x^v)^{-v},
\]

which are automorphisms of \( \mathcal{R}_3(G; E) \). The first one gives the \( \mathbb{Z}/2 \)-grading on \( \mathcal{R}_3(G; E) \) by the class \( w_2(V) \in H^2(BG; \mathbb{Z}/2) \), leading to a sign automorphism. The second shift acts on \( \mathcal{R}_4(G; E) \) by tensoring with the (possibly graded) line bundle on \( \Omega G \) transgressed from a half of \( c_2(E) \).

6.8 Remark (Interpretation: boundary theories). The first section in (6.7) corresponds to an invertible topological boundary theory for (the homological) 3D gauge theory. Algebraically, this stems from a topological action of \( G \) on the category \( \text{Vect} \) of vector spaces: the action is determined by the \( \mu_2 \)-extension of \( G \) classified by the equivariant \( w_2(V) \). The interpretation of boundary conditions as Lagrangian sections of the Toda space is the one outlined in [T1].

Similarly, \( \frac{1}{2} c_2(E) \) defines a 2-extension of \( G \) by \( \mathbb{C}^\times \), and should define an invertible boundary theory for 4D topological gauge theory. Transgressing it to a central \( \mathbb{C}^\times \)-extension of the free loop group \( LG \) defines topological action of the latter on \( \text{Vect} \), and thus an invertible boundary theory.
for 3D LG-gauge theory. The spaces $\mathcal{C}_4$ may be considered as Coulomb branches for the loop group, and the second section in [6.7] is the Lagrangian associated to this boundary theory.

By construction, $\epsilon_V$, $\lambda_V$ become regular Lagrangian sections of the Coulomb branches $\mathcal{C}_{3,4}$. The shifted automorphism $C_V$ of $\mathcal{C}_3$ cycles between the unit section, $\epsilon_V^{-1}$ and their translates by the section $\prod (1)^{1/2}$. One expects these sections to correspond to the 2-dimensional boundary theories for $E/G$ associated to the two polar halves $V, V'$. Strictly speaking, the preferred “unit” section is a feature (or bug) of the construction of the Coulomb branch, not intrinsic to it. When $w_2(V) \neq 0$, there is a choice in gauging the linear Sigma model, related to the role of Spin structures in defining Floer and symplectic cohomology; this is reflected in the torsion sections $\prod (1)^{1/2}$.

As defined, $C_V$ has infinite order on $\mathcal{C}_4$. It can be modified to have finite order, but this may involve a torsor of order 2 over the Toda space (the one appearing in Remark 6.15 below). The latter becomes necessary when one aims to build a KO version of $\mathcal{C}_4$. I shall not work out the details for KO here.

(6.9) The general case in outline. Polarize $E$, after first reducing the symmetry to $H$. Weyl symmetry must be initially broken. The construction by charge conjugation is then quotiented out by the latter becomes necessary when one aims to build a KO version of $\mathcal{C}_4$. I shall not work out the details for KO here.

(6.10) Euler Lagrangians. Recall the splitting $E = E_+ \oplus E_0 \oplus E_-$; the $H$-invariant part $E_0$ of $E$ will not contribute in what follows. Consider the maps to $H^\nu$

$$\epsilon_+ : \xi \in \mathfrak{h} \mapsto \prod_{\nu > 0} \langle \nu | \xi \rangle^\nu, \quad \lambda_+ : x \in H \mapsto \prod_{\nu > 0} (1 - x^{-\nu})^\nu. \quad (6.11)$$

The graphs of these maps are Lagrangian (see below); their closures are smooth away from an indeterminacy locus of co-dimension 2 over the bases $\mathfrak{h}, H$.

6.12 Remark (Polarized case). When $E = V \oplus V'$, we can choose $\xi_0$ along the line of the mass parameter $\mu$ in the Toda base. Then, $E_+ = V, E_- = V'$, recovering the ‘massive Lagrangians’ $\epsilon_V, \lambda_V$ of (6.2). In this case, $W$-invariance of the splitting will reduce the modified Weyl action below to the natural one, recovering the construction for the polarized case.

6.13 Remark (GLSM). $\epsilon^+$ and $\lambda^+$ are the exponentiated differentials of the superpotentials for the mirror of the GLSM for $E_+$ and $H$:

$$\xi \mapsto \Psi(\xi) = \text{Tr}_{E_+}(\xi (\log \xi - 1)), \quad x \mapsto \Psi(x) = \text{Tr}_{E_+} \text{Li}_2(x).$$

6.14 Remark (Index interpretation). A co-weight $\gamma$ of $H$ defines a character $\exp(2\pi i \gamma)$ of $H^\nu$, as well as an $E_+$-fiber bundle $E^\gamma_+ \to \mathbb{P}^1$. Then, $\exp(2\pi i \gamma) \circ \epsilon_+$ and $\exp(2\pi i \gamma) \circ \lambda_+$ are the equivariant Euler classes, in cohomology and $K$-theory, of the Dirac index of $E^\gamma_+$ over $\mathbb{P}^1$.

6.15 Remark (Spin orientation). More canonical than $\lambda_+$ is the Euler class in KO-theory \footnote{This would be needed for the KO-version of $\mathcal{C}_4$.}

$$\lambda^\nu_0 : x \in H \mapsto \prod_{\nu > 0} \left( x^{\nu/2} - x^{-\nu/2} \right)^\nu.$$

which is a section of the $H^\nu$-torsor of order 2 over $H$, defined by the bilinear form

$$\sum_{\nu > 0} \nu \otimes \nu : \pi_1 H \to \mathbb{Z}/2.$$

This represents a real central extension of $H$ by $H^\nu$, which can be non-trivial even in unobstructed situations, such as for $\text{SU}(6)$ acting on $\Lambda^3 \mathcal{C}^6$. \footnote{This would be needed for the KO-version of $\mathcal{C}_4$.}
(6.16) **Construction of \( \mathcal{C}_{\mathbb{A}}^3(N(H); E) \).** Recall the modified Weyl action of §5, adjusted to cancel the projective cocycle. We have the

6.17 **Proposition** (Modified charge conjugation). The modified Weyl action commutes with the following rational “\( C_+ \) automorphisms” of Toda spaces:

\[
\begin{align*}
(\xi, h) &\mapsto \left( -\xi, e_+^{-1}(\xi) \cdot h^{-1} \right), \\
(x, h) &\mapsto \left( x^{-1}, \lambda_+^{-1}(x) \cdot h^{-1} \right).
\end{align*}
\]

**Proof.** We write out the check for \( \chi \) (\( \kappa \) is analogous):

\[
\begin{align*}
(\xi, h) &\xrightarrow{C_+} \left( -\xi, e_+^{-1}(\xi) \cdot h^{-1} \right) \xrightarrow{\text{Weyl}} \left( -w\xi, \chi_w(-w\xi) \cdot w \left[ e_+^{-1}(\xi) \right] \cdot wh^{-1} \right) \\
(\xi, h) &\xrightarrow{\text{Weyl}} \left( w\xi, \chi_w(w\xi) \cdot wh \right) \xrightarrow{C_+} \left( -w\xi, e_+^{-1}(w\xi) \cdot \chi_w(w\xi)^{-1} wh^{-1} \right),
\end{align*}
\]

and the equality of right-hand sides amounts to the (easily checked) relations

\[
\begin{align*}
\chi(\xi)\chi(-\xi) &= \delta e_+(\xi), \\
\kappa(x)\kappa(x^{-1}) &= \delta \lambda_+(x),
\end{align*}
\]

or more precisely,

\[
\begin{align*}
\chi_w(\xi)\chi_w(-\xi) &= w \left[ e_+(\xi) \right] \cdot e_+^{-1}(w\xi), \\
\kappa_w(x)\kappa_w(x^{-1}) &= w \left[ \lambda_+(x) \right] \cdot \lambda_+^{-1}(wx).
\end{align*}
\]

(6.18)

The identities persist after correcting \( \chi \) and \( \kappa \) by 1-cochains, valued in \( {}^2H^\vee \) or in \( \Lambda^\otimes 2 \).

6.19 **Theorem.** The identity components of \( \mathcal{C}_{\mathbb{A}}^3(N(H); E) \) comprise precisely the modified-Weyl invariant functions on \( \mathfrak{h} \times H^\vee \), respectively \( H \times H^\vee \) which remain regular under the automorphisms \( C_+ \).

**Proof.** The topological construction proceeds from the linear spaces \( H^0(E^\vee(-1)) \) over \( \Omega H^0(N(H)) \): we subtract the Dirac index bundles of local polar halves of \( E \) and define real structures and (K-theory) orientations on the resulting spaces. Interpreting ‘local’ as ‘on the Weyl cover’ and using the polarization \( E_+ \) leads to

\[
H^0(E^\vee(-1)) \oplus H^0(E^\vee_+(1)) \oplus H^1(E^\vee_+(1)) = H^0(E^\vee(-1)) \oplus H^0(E^\vee_+(1))^\vee.
\]

The rings \( \mathcal{C}_{\mathbb{A}}^3(H; E) \) are then spanned, over their Toda bases, by the Fourier modes labeled by the co-weights of \( H \), coupled to the (K-theory) Euler classes of the (underlying real) bundles \( H^0(E^\vee_+(-1)) \):

\[
\begin{align*}
(\xi, h) &\mapsto h^\vee \cdot \prod_{\nu \in \mathbb{Z}_{<0}} (\nu|\xi)^{(-\nu|\gamma)}, \\
(x, h) &\mapsto h^\vee \cdot \prod_{\nu \in \mathbb{Z}_{<0}} \left( 1 - x^{-\nu} \right)^{(-\nu|\gamma)}.
\end{align*}
\]

(6.20)

We can see that including these Euler classes

- is compatible with the modified Weyl action of §5.6 and
- converts the charge conjugation \( C \) into the \( C_+ \) Proposition 6.17.

The characterization of the spans of (6.20) given in Theorem 6.16 should now be clear. 

\[ \square \]
(6.21) From $R^1 (N(H))$ to $R(G)$. Just as in the polarized case, the $R(G; E)$ will be super-rings of the $R^1 (N(H))$, corrected over the (affine) root hyperplanes $\alpha = 0 (e^\alpha = 1)$. This will match the answer for Levi subgroups of semi-simple rank one and settles the construction as in [BFN §6], due to the freedom of the $R$ as modules over the Toda base. The correction can be described in several ways:

- by allowing controlled poles in the functions;
- by an affine blow-up;
- (as we shall do) by restricting the space of sections of $C(N(H); E)$ over the Toda bases on which the sections functions required to be regular.

(6.22) Root hyperplanes. The classes of $\chi$ and $\kappa$ need not vanish, leading to non-trivial rational $H^\vee$-torsors over $h/W$ and $H/W$. The torsors are trivial for $SU(2)$ when $w_4 = 0$. More generally, we can find sections of the torsors near the affine root hyperplanes; as sections of the Coulomb branches, they will even regular.

6.23 Lemma (Sections). Assume as before that $G$ is connected. Cancelling the obstructions for connected $G$ allows us to choose trivializations $r_\alpha, q_\alpha$ of the corrected co-cycles $\chi_\alpha(s_\alpha)$ and $\kappa(s_\alpha)$ near the (affine) $\alpha$-root hyperplanes which are invariant under the Weyl centralizer of $s_\alpha$:

$$r_\alpha(\xi) \cdot s_\alpha r_\alpha^{-1} (s_\alpha \xi) = \chi_\alpha(\xi),$$
$$q_\alpha(\xi) \cdot s_\alpha q_\alpha^{-1} (s_\alpha \xi) = \kappa(\xi).$$

6.24 Remark. We must choose the obstruction cancellations coming from $G$ and not just from the normalizer $N(H)$. A counterexample for a ‘bad’ choice of cancellation is found in §?? below.

Proof. The centralizer of the $\alpha$-root subgroup and its hyperplane Levi subgroup have the forms

$$Z \times SU(2), \quad Z \times SO(3), \quad or \ Z \times \mu_2 \ SU(2),$$
$$H' \times SU(2), \quad H' \times SO(3), \quad or \ H' \times \mu_2 \ SU(2)$$

for a subgroup $Z$ of $G$ and subtorus $H' \subset H$. In the last two cases, symplectic representations are polarizable; the Coulomb branches can be constructed from a polar decomposition and have global regular sections over the base. (The possible ambiguities for the space $C_4 (L_{\alpha}, E)$, in the Bockstein image $BH^1 (BG; \mathbb{Z}/2) \subset H^3 (BG; \mathbb{Z})$, vanish in all cases.) In the first case $Z \times SU(2)$, the same applies for portion of $E$ which does not come from $SU(2)$ representations of odd spin. For the latter, the sum of positive multiples of the fundamental weight $\omega$ must be even, and their contribution to Euler classes $\chi_\alpha, \kappa_\alpha$

$$\prod_n (\omega|\xi|^n \psi), \quad \prod_n (1 - x^{-n\omega})^{\psi},$$

allow (when $\sum n$ is even) for straightforward construction of $r_\alpha$ and $q_\alpha$ near the (affine) root hyperplanes $\langle \alpha | \xi \rangle = 0$ and $x^\alpha = \pm 1$, respectively.

We are left, in all cases, to show that requisite sections can be made invariant under the Weyl group of $Z$. For this, I claim that that, on the root hyperplane, the sections are ‘valued’ in the subgroup $\alpha^a \subset H$ — where ‘value’ designates the leading Laurent term, in case when the Euler class is singular on the hyperplane. Invariance then follows, because the Weyl cocycles form $Z$ and their correction only involve $H^\vee$-translation spanned by weights normal to $\alpha$; it thus preserves the ‘value’ on the root hyperplane, and the section can be made invariant, locally in the normal directions, by averaging.
Symmetry under $s_{\alpha}$ and duality implies the claim up to 2-torsion points in $H^\vee$. In the first two cases in (6.25), we can be more precise. Choose a polar decomposition of $E$ so that $v$ and $s_{\alpha}v$ are always in opposite polar halves when $\langle v|h_{\alpha}\rangle \neq 0$. We have

$$\text{Ind}_{E_+}(h) \otimes \text{Ind}_{s_{\alpha}E_+}(h) = \bigoplus_{\nu>0} \left( C_{\nu}^{\oplus (\nu|h)} \oplus C_{s_{\alpha}\nu}^{\oplus (\nu|h)} \right) .$$

When $\langle \alpha|h \rangle = 0$, we can rotate the polar half $s_{\alpha}E_+$ continuously by SU(2) into $E_+$, leading to a cancellation of the Euler class, evident on the left side, and proving our claim.

In the third case, the representation is polarizable, and the SU(2)-invariant polar decomposition leads to trivial cocycles $\chi_\alpha$, $\kappa_\alpha$.

**Theorem 2.** Choose $r_\alpha$, $q_\alpha$ as in Lemma 6.23. The Coulomb branches $\mathcal{C}_{3,4}(G;E)$ are the affinizations of the spaces generated by those regular sections $s$ of the $\mathcal{C}_{3,4}(N(H);E)$ satisfying, for every (affine) root $\alpha$, the condition

$$\exp(h_{\alpha}) \circ (s \cdot r_\alpha) = O(\alpha),$$

$$\exp(h_{\alpha}) \circ (s \cdot q_\alpha) = O(\alpha^2 - 1).$$

**Proof.** Away from root hyperplanes, this is just equivariant localization. Generically on an $\alpha$-root hyperplane, the $\mathcal{C}_{3,4}$ are controlled by restriction to the Levi subgroup $L_\alpha$. Now, with respect to the classification (6.25), the representation $E$ is polarizable on $L_\alpha$ in the last two cases, and the result follows from the analysis in [BFN] or [T2, §5]: the regularity conditions for $\mathcal{C}_{3,4}(G;E)$ are weakened precisely by the evaluation condition on sections of the $\mathcal{C}(G;0)$.

The same applies to $L_\alpha = H' \times SU(2)$ for the non-multiples of $\alpha$ among the weights. So we can factor out $H'$ and need only check the theorem for SU(2). In that case, the $G[z]$-orbits in $\Omega G$ pass through positive integer multiples $n$ of the coroot. The polarized example in Section 4.3 shows that $E$ effects a de-suspension by the real space underlying

$$\bigoplus_k \left( C_{k\alpha} \oplus C_{-k\alpha} \right)^{\oplus 2kn}$$

where $k$ ranges over the (positive half-integer) multiples of $\alpha$ appearing in $E$. The sum $S := \sum k$ is even, and the homological Euler class is a multiple of $S\alpha$, the agreeing with the one coming from the representation $(H \oplus H)^{\oplus S}$. For $\mathcal{C}_4$, we must separate the half-integral $k$, which impose no additinoal constraint on sections at the central point $-1$, from the integral ones which impose the same condition as for $\mathcal{C}_3$. In either case, the answer is seen, from the explicit case of SU(2) as in [T2, §5], to confirm the theorem.

**7. Abelianization**

When the $H$-restriction of $E$ contains the doubled representation $g_H$, we can build the Coulomb branches in two steps. The roots of $g$ will be contained in a polarized part of $E$, because of orthogonality of $g$; so we can use the polarized construction there. Preliminary de-suspension by $N_\beta$ disconnects the Bruhat strata of $\Omega G$, as in Proposition 2.12. If $g_H$ is a $G$-subrepresentation, this disconnection can happen $G$-equivariantly; in general, we can only effect it equivariantly for $N(H)$, but this is sufficient.

This gives an (additive) identification of the resulting spectrum with the disjoint union of stabilizers $BL$ of one-parameter subgroups. Subsequent de-suspension by $E \ominus g_H$ leads to an additive equivalence advertised in Theorem 2,

$$\mathcal{C}_{3,4}(G;E) \cong \mathcal{C}_{3,4}(H;E \ominus g_H)/W.$$
In the homology statement, we must invert the order of the Weyl group to relate $L$-equivariant homology with the Weyl invariants in $BH$.

Equality of the multiplicative structures is enforced by localizing away from the root hyperplanes on the Toda base, and by the standard localization theorem to the maximal torus.

(See also [12] §5, Example 5.4] for the Weyl descent presentation.)

**A. Appendix: Obstructions for connected groups**

We discuss the obstructions to the construction of chiral rings for a connected compact group $G$ with quaternionic representation $E$. The main results are Theorems [A.4] and [A.6].

(A.1) Quaternionic irreducibles. Quaternionic representations are self-dual over $\mathbb{C}$, and an irreducible self-dual complex representation is either orthogonal or quaternionic. The simply connected simple groups carrying complex-irreducible quaternionic representations are:

\[ \text{Sp, } \text{SU}(2 \text{ mod } 4), \text{ Spin}(\pm 3 \text{ mod } 8), \text{ Spin}(4 \text{ mod } 8), \text{ and } E_7. \]  

(A.2)

Except for SU, all complex representations of the listed groups are self-dual. For Spin groups, quaternionic are precisely those complex-irreducibles that do not factor through SO; for the other groups, the test is that the unique\(^{14}\) central element of order 2 should acts as $(-1)$.

**A.3 Proposition.** Let $E$ be a quaternionic representation of a simple group $G$.

(i) If $w_4(E) \neq 0$, then $G$ is a symplectic group.

(ii) If $E$ is complex-irreducible with $G = \text{Sp}(m)$, then $\dim_H E = m \cdot c_2(E) \pmod{4}$.

(iii) If $E$ is complex-irreducible, $\dim_H E$ is odd iff $G = \text{Sp}(m)$ with $m$ and $c_2(E)$ both odd.

Moving to general connected groups, the first obstruction, $w_4$ modulo integral squares, is additive on symplectic representations and vanishes for polarized ones; so it suffices to understand complex-irreducibles. Those factor through the quotient by the connected part of the center, so we may assume that $G$ is semi-simple.

**A.4 Theorem.** For a complex-irreducible quaternionic representation $E$ of $G$, $w_4(E) = 0$ except in one of the following (mutually exclusive) cases:

(i) $G = G_o \times \text{Sp}(m)$ and $E = R \otimes S$, with an odd-dimensional orthogonal representation $R$ of $G_o$ and a symplectic representation $S$ of $\text{Sp}(m)$ with odd $c_2$; $w_4(E) \neq 0$ on $\text{Sp}$ and vanishes on $G_o$.

(ii) $G = G_o \times \mu_2 \text{Sp}(m)$ and $E = R \otimes S$, with an orthogonal, $(4n + 2)$-dimensional representation $R$ of $G_o$ and an odd $\mathbb{H}$-dimensional symplectic representation $S$ of $\text{Sp}(m)$.

In either case, $R$ is orthogonal on each simple factor of $G_o$, so the factorization is unique.

(A.5) Remark. In case (ii), $m$ is necessarily odd, and $\mu_2$ acts via the sign on both $R$ and $S$. Furthermore, $w_4(E)$ has a square root $\tilde{r} \in H^2(BG; \mathbb{Z}/2)$ which lifts mod 4, because $R \otimes S$ comes from $\text{Spin}(4n + 2) \times \mu_2 \text{Sp}(m)$, where that is the case. The dichotomy in the theorem shows that such a lift always exists, once the square root does. We can then define a secondary obstruction $\sigma := Sq^2 c$ from a trivialization $w_4 - \tilde{r}^2 = \delta c$:

\[
\delta Sq^2 c = Sq^2 \delta c = Sq^2 w_4 - (Sq^1 r)^2 = 0;
\]

$Sq^2 w_4$ vanishes universally on $BSp$, while $Sq^1 r$ is killed by a lift mod 4. In case (ii), $\sigma$ vanishes because its home $H^3(BG; \mathbb{Z}/2) / Sq^2 H^3(BG; \mathbb{Z}/2)$ is zero.

\(^{14}\)The center of $\text{Spin}(4 \text{ mod } 8)$ is $\mu_2^2$, with the two factors interchanged by the outer automorphism; SO is the quotient by the diagonal $\mu_2$.  

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A.6 Theorem. Let $G$ be connected and $E$ an irreducible quaternionic representation for which $w_4(E)$ has a square root in $H^2(BG;\mathbb{Z}/2)$. Then, $\sigma \in H^5(BG;\mathbb{Z}/2)/Sq^2 H^3$ vanishes, except possibly when

$$G = G_o \times \mu_4 \text{Sp}(m) \text{ and } E = R \otimes S,$$

with a $4k$-dimensional orthogonal representation $R$ of $G_o$ and an odd $\mathbb{H}$-dimensional representation $S$ of $\text{Sp}(m)$. In this case, $\sigma = w_3(R) \cup x$, with the generator $x \in H^2(B^2\mu_2;\mathbb{Z}/2)$.

In the exceptional case, $m$ must be odd. The obstruction $\sigma$ and its Bockstein $B\sigma$ are then non-zero for $G_o = \text{SO}(4k)$ and the standard representations $R, S$.

(A.7) Technicalities on $H^4(BG)$. Classes $x \in H^4(BG;\mathbb{Z})$ are represented by invariant quadratic forms $q(x)$ on the Lie algebra $\mathfrak{g}$ which are integer-valued on the co-weights. We study the Leray spectral sequence for the fibration

$$BG \hookrightarrow BG \to B^2 \pi,$$

with $\pi$ finite and $G = \tilde{G}/\pi$. We will use the isomorphism

$$H^5(B^2\pi;\mathbb{Z}) \cong H^4(B^2\pi;\mathbb{Q}/\mathbb{Z});$$

the right group classifies $\mathbb{Q}/\mathbb{Z}$-valued homogeneous quadratic forms on the ratio $\pi$ of the co-weight lattices [LM].

When $\tilde{G}$ is simply connected, the leading differential $d_5 x \in H^5(B^2\pi;\mathbb{Z})$ gives the restriction of $q(x)$ (mod $\mathbb{Z}$) to $\pi$. In general, there are prior Leray differentials $d_2 x \in H^2(B^2\pi;H^3(B\tilde{G};\mathbb{Z}))$ and $d_3$ to $H^3(B^2\pi;H^2(B\tilde{G};\mathbb{Z}))$, representing the $\mathbb{Q}/\mathbb{Z}$-valued bilinear pairing defined from $q(x)$ on $\pi \times (\pi_1 G)_{\text{tors}}$, respectively $(\pi_1 G)_{\text{free}} \times \pi$. Should these two vanish, $d_5 x$ is again represented by the (now well-defined) restriction of $q$ (mod $\mathbb{Z}$) to $\pi$.

(a) For $G = \mathbb{PSp}(m) = \text{Sp}(m)/\{\pm 1\}$, with co-root lattice $\langle \pm e_i \rangle$, $q(c_2) = -\sum x_i^2$ and

$$d_5(c_2) = -\frac{m}{4} \in \frac{1}{4} \mathbb{Z}/\mathbb{Z} = H^5(B^2\pi;\mathbb{Z}/2).$$

(b) For $G = \mathbb{PSU}(n)$, $d_5(c_2)$ is the generator $\frac{1-n}{2n}$ of $H^5(B^2\mu_n) \cong \mathbb{Z}/(n \cdot \text{gcd}(2, n))$.

(c) In type $D_l$, the co-roots are $\{\pm e_i \pm e_j\}_{i < j}$ and $q(p_1)$ is the sum-of-squares; the generating class for $\text{Spin}(2l)$ is $p_1/2$, while its center is $\mu_4$ for $l$ odd and $\mu_2^{l/2}$ for $l$ even. Generators are $b_\pm := [\pm 1/2, \ldots, 1/2]$; let also $a := b_- - b_+ = [1, 0, \ldots, 0]$. We have

$$q\left(\frac{p_1}{2}\right): b_\pm \mapsto \frac{1}{8}, \quad a \mapsto \frac{1}{2} \mod \mathbb{Z}.$$

(d) For $G = \text{SO}(2l) = \text{Spin}(2l)/\langle a \rangle$, $p_1$ is the surviving generator.

(e) For $G = \mathbb{PSO}(2l) = \text{SO}(2l)/\{\pm 1\}$, $q(p_1)$ sends each generator $b_\pm$ to $l/4 \mod \mathbb{Z}$ and pairs it integrally with $a$, so that

$$d_2 p_1 = 0 \quad \text{and} \quad d_5 p_1 = \frac{l}{4} \mod \mathbb{Z}.$$

The surviving $H^4$ generators are $p_1, 2p_1$ and $4p_1$, respectively, for $l = 0, 2$ and $\pm 1 \mod 4$.

(f) For $G = \text{Spin}(4k)/\langle b_+ \rangle, d_5(p_1/2) = k/4 \mod \mathbb{Z}$; same for $b_-$. 

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(g) In $G = \text{PSO}(4k)$ with the generating classes $u_\pm \in H^2(B^2\pi_1;\mathbb{Z}/2)$, we have

$$H^5(BG;\mathbb{Z}) = \mathbb{Z}/4 \oplus \mathbb{Z}/4 \oplus \mathbb{Z}/2;$$

the first two summands are generated by the Bocksteins $B_4 : \mathbb{Z}/4 \to \Sigma\mathbb{Z}$ of the Pontrjagin squares $\varphi(u_\pm) \in H^4(B^2\mu_2;\mathbb{Z}/4)$. Matching the quadratic form in (c),

$$d_5(p_1/2) = k \cdot B_4\varphi(u_+ + u_-) + B(u_+u_-) \in H^5(B^2\pi_1;\mathbb{Z}).$$

Reducing $B_4\varphi(x)$ mod 2 gives $xSq^4x + Sq^2Sq^1x$.

**Proof of Proposition A.3.** For Part (i), note that $Sq^2 = 0$ on $H^4(B\mathbb{P})$, whereas I claim that $Sq^2 \neq 0$ for the generators of $H^4$ in the other Lie types in (A.2), forcing $w_4(E)$ to vanish.

If $\pi_2 G$ has odd order, $BG$ is equivalent at the prime 2 to a simply connected type in the list, where the non-vanishing of $Sq^2$ is known. The remaining possibility is $G = \text{Spin}(8n + 4)/\langle b_+ \rangle$ (or $b_-$). Then, [A.7] shows that pull-back from the base $B^2\mu_2$ induces an isomorphism

$$H^4(BG;\mathbb{Z}/2) \cong H^4(B^2\mu_2;\mathbb{Z}/2);$$

$d_5(p_1/2) \in H^5(B^2\mu_2;\mathbb{Z})$ is a generator, so that $p_1/2$ does not survive in the mod 2 Leray sequence. Now, $Sq^2H^4(B^2\mu_2) \to H^6(B^2\mu_2;\mathbb{Z}/2)$ is injective, and no degree 5 class is present to give a kernel when mapping to $H^6(BG)$.

In Part (ii), $(-1) \in \text{Sp}(m)$ maps to $(-I) \in \text{Sp}(E)$, identifying the two $B^2\mu_2$ bases in the fibrations [A.7]a. The transgressions $m \cdot c_2(E)$ and $\dim_{\mathbb{H}} E$ are thereby equated mod 4.

Finally, in Part (iii), $G$ has a central element mapping to $(-I) \in \text{Sp}(E)$. As $\dim_{\mathbb{H}}(E)$ is odd, $c_2(E)$ transgresses to a generator of $H^5(B^2\mu_2;\mathbb{Z})$: in particular, it is odd in $H^4(BG)$, and then $G = \text{Sp}(m)$ with $m$ odd, as per Parts (i) and (ii).

**Proof of Theorem A.4.** A finite cover of $G$ splits into simple factors and a torus, over which $E$ factors as a tensor product of irreducible representations. Self-duality forces the torus to act trivially. Choose a simple factor $G_s$ which comes with a quaternionic representation $S$; the others combined carry an orthogonal representation $R$, with $E = R \otimes S$. Matching this factorization, we write $G = G_o \times_F G_s$, for a finite central subgroup $F$ of $G_o \times G_s$ which embeds in $G_s$.

**Case (i)** $F$ has odd order. For 2-primary questions, we lift to the product $G_o \times G_s$, where

$$c_2(E) = c_2(S) \dim_{\mathbb{C}} R - p_1(R) \dim_{\mathbb{C}} S.$$  \hspace{1cm} (A.8)

As $\dim_{\mathbb{C}} S$ is even, we need both $\dim_{\mathbb{C}} R$ and $c_2(S)$ to be odd, and Proposition [A.3] settles this case.

**Case (ii)** $F = \mu_2 \times F'$, with $F'$ odd. Now, $\mu_2$ acts via the sign on $R$ and $S$: else, we could descend $E$ to a product group in Case (i), giving the contradiction $\text{Sp}(m) = G_s/\langle \mu_2 \rangle$. In particular, $R$ has even dimension $2l$. Passing to $G' := \text{SO}(2l) \times_{\mu_2} \text{Sp}(S)$, Leray gives an exact sequence

$$0 \to H^4(BG';\mathbb{Z}) \to H^4(B\mathbb{SO} \times B\mathbb{Sp};\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{d_5} \mathbb{Z}/4 = H^5(B^2\mu_2;\mathbb{Z}),$$

$$d_5 : p_1(R) \mapsto l, \quad c_2(S) \mapsto (- \dim_{\mathbb{H}} S) \pmod{4}. \hspace{1cm} (A.9)$$

(As in Aside [A.7] $d_2 = 0$ because the dot product pairing on $\mu_2 \times \pi_1\text{SO}(2l)$ is integral.) From [A.8],

$$\frac{d_5 c_2(E)}{2} = d_5 (c_2(S) \cdot l - p_1(R) \cdot \dim_{\mathbb{H}} S) = -2l \cdot \dim_{\mathbb{H}} S \pmod{4}, \hspace{1cm} (A.10)$$

which is non-zero precisely when both $\dim_{\mathbb{H}} S$ and $l$ are odd. In particular, $G_s = \text{Sp}(m)$ with $m$ odd, as per Proposition [A.3]iii, and $F'$ is trivial.

**Case (iii)** If $G_s = \text{Spin}(8k + 4)$ and $F = \mu_2^2$, then one of the two $\pm 1$ factors must act trivially on $R$, thus also on $S$, and dividing it out in both factors gets a contradiction with Cases (i) or (ii).

In the list (A.2), $G_s$ never contains a central $\mathbb{Z}/4$, so we have covered all cases. \qed
Proof of Theorem A.6. After removing all polarizable representations from $E$, remaining complex-irreducible summands factor as $R \otimes S$ over $G = G_o \times_f G_s$. Any central torus in $G$ must act trivially and may be quotiented out. We also ignore the odd part of $F$. On $R$ and $S$, $F$ acts either trivially, or else by the same sign.

1. If $F$ acts by a sign, our summand comes from $\text{SO}(R) \times_\mu_2 \text{Sp}(S)$, with $\dim R$ even. If $4 \nmid \dim R$, $w_4(E)$ is the square of a class which lifts mod 4 (because $\pi_1 = \mu_4$). In that case, $\text{Im} \, d_5$ and $Sq^2 H^3$ span $H^5(BG; \mathbb{Z}/2)$: so there is no home for $\sigma$. The case $4 \mid \dim R$ will be discussed below.

2. If $F$ acts trivially, we can factor through $F \times F$:

$$G \to G_o / F \times G_s / F \to \text{SO}(R) \times \text{Sp}(S).$$

If $\dim R$ is even, then so is $c_2(R \otimes S)$ on the right-hand group, where $\sigma$ vanishes. ($H^5$ comes from $\text{SO}(R)$, on which $R \otimes S$ is polarizable.) If, instead, $\dim R$ is odd, and $c_2(S)$ is also odd on $G_s / F$, then $G_s = \text{Sp}(m)$ and $F$ is trivial (Theorem A.3(ii)). Another summand $R' \otimes S'$ of $E$ is needed to make $c_2$ even, and the $\sigma$ of the combined summands vanishes on $G_o \times G_s$, for the reason above.

More generally, this argument shows the vanishing of $\sigma$ when $G_s / F$ has no 2-torsion and $c_2(S)$ is even. The classification in §A.7 leaves only the possibility $G_s / F = \text{Spin}(8k + 4) / \langle b \pm \rangle$, for one of the generators in §A.7(c).

4. To handle the remaining groups, we need the 7-skeleton truncation of $R \otimes S : BG \to BS_p$. The space $O$ splits as $\mathbb{Z}/2 \times \Sigma \mathbb{Z}/2 \times \Sigma^2 \mathbb{Z}$ in that range, with generators $\eta, \eta^2$ and $\alpha$. Incorporating Bott periodicity, $\alpha$ will also denote the generator of $\pi_0 KS_p$ and $\mathbf{1}$ that of $\pi_4 BS_p$.

For $G = \text{SO}(4k) \times_\mu_2 \text{Sp}(m)$, we have

$$c_2(E) = 4k \cdot c_2 - 2m \cdot p_1, \quad d_5 c_2(E) = 8kmB_{4\theta} (x) + 4mB_2(u_+ u_-) \in H^5(BG; \mathbb{Z}),$$

with the $H^2$ generators $u_+, u_-$ of §A.7, and the generator $x := u_+ + u_-$ of $H^2(B^2 \mu_2)$. In particular, $c_2(E)$ is even (and divisible by 4 if $m$ is even).

The representations $R, S$ now live in the $x$-twisted $KO$-groups $^x KO(\text{PSO}(4k)), ^x KS_p (\text{PSp}(m))$. They multiply naturally to untwisted $KS_p(BG)$. Since $\dim R$ is now even,

$$\eta \otimes R = w_3(R) \alpha \in ^x KO^{-1}(BG_{\leq 6}).$$

This is well-defined: the ambiguity $Bx$ of $w_3$ on $B\text{PSO}$ is killed by $d_3 \eta^2$ in the twisted Atiyah-Hirzebruch sequence for $KO$. This case, the only obstructed one, is settled by the Lemma that follows.

A.11 Lemma. We have $\eta \otimes R \otimes S = mx \cup w_3(R) \cdot \eta^2$ in

$$H^5(BG; \pi_5 Sp) / Sq^2 H^3 \to KS_p^{-1}(BG_{\leq 6}; \mathbb{Z}/2).$$

Proof. First, this is indeed the leading term (and the only one, in the truncated range): $ma \cdot \alpha$ vanishes ($a^2 = 4 \cdot 1$), and the $\eta$-term couples to $c_2(E)$, which is even. The ambiguity $Sq^2 H^3$ has the same source as in $\sigma$. 

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The leading term is best detected after 3-fold looping. Consider the multiplication

\[ xKSp^0(BPSp(n)) \otimes xKSp^0(BPSp(m)) \to KO^0(B(Sp(n) \times_{\mu_2} Sp(m))). \]

With mod 2 coefficients for $BPSp(n)$ and for the right-hand side, taking the standard representations $H^n, H^m$ as factors will match our desired calculation, but 3 dimensions down, in $\pi_2(BO/2)$: the term $w_3(R)\alpha$ from $\eta \otimes R$ is replaced by $n \cdot \alpha$. The product $H^n \otimes H^m$ is the standard representation pulled back under the map

\[ Sp(n) \times_{\mu_2} Sp(m) \to SO(4mn). \]

On the right, the leading term $w_2 \cdot \eta^2$ generates $H^2(B^2\pi_1SO;\mathbb{Z}/2)$. However, its pull-back vanishes if either $m$ or $n$ are even: we can see this by restricting to the diagonal copies of $Sp(1) = SU(2)$ on the left factors, and the corresponding $SO(4)$ on the right; the logarithm of the central generator of $\mu_2$ maps to $mn$ times the one in $so(4mn)$. \hfill \Box

### B. Appendix: Examples

(B.1) Obstructed $\mathcal{C}_4$ but not $\mathcal{C}_3$. We work out the second obstruction for the smallest connected example exhibiting a secondary but no primary obstruction for $\mathcal{C}_4$, but not for $\mathcal{C}_3$ (see Theorem 5.2.ii). Specifically, we show that the Weyl cocycle $s^{\otimes 2}$ of $\mathbb{S}^5$ is non-zero.

\[ G = SU(2)^{\times 3}/S(\mu_2^{\times 3}). \quad E = H^{\otimes 3} \]

We use mod 2 cohomology in the fibration

\[ B\tilde{H} \hookrightarrow BN(H) \twoheadrightarrow BW \times B^2S(\mu_2^{\times 3}) \]

with the Weyl group $W = \mu_2^{\times 3}$ with $H^1$ generators $r_{1,2,3}$, the cover $\tilde{H} = Spin(2)^{\times 3}$ of the maximal torus $H$ with $H^2$ generators $\omega_{1,2,3}$ and additional base $B^2S(\mu_2^{\times 3})$ with $H^2$ generators $u_{1,2,3}$ and relation $u_1 + u_2 + u_3 = 0$. The projection is induced from the maps $(w_1, w_1^2 + w_2)$ in each $O(2)$ projection of $N(H)$. The shortest Leray differential is

\[ d_3 : \omega_i \mapsto Sq^1u_i + r_i u_i \]

because of the relation $Sq^1w_1 + w_1w_2 = w_3 = 0$ in $BO(2)$.

The topological calculation in Theorem 5.2.ii gives $\sigma = Sq^1u_1 \cdot u_2$, becoming here $r_1 u_1 u_2$. The transgression is $r_1 \cdot \tau(u_2 u_3)$. This represents the Weyl co-cycle in $\Lambda^{\otimes 2}/2$ which is zero on Weyl triples, unless the first group element is $-1$, in which case it takes the value

\[ 4\omega_1 \otimes \omega_2 + 4\omega_2 \otimes \omega_1. \quad (B.2) \]

The weights of $E$ are $\pm \omega_1 \pm \omega_2 \pm \omega_3$ (independent signs). For our Weyl calculation, choose $E_+$ defined by the positive sign on $\omega_1$. The cocycle we get from Theorem 5.2.ii again vanishes on Weyl triples, unless the first element is $(-1)$, in which case we get

\[ (\omega_1 + \omega_2 + \omega_3)^{\otimes 2} + (\omega_1 + \omega_2 - \omega_3)^{\otimes 2} + (\omega_1 - \omega_2 + \omega_3)^{\otimes 2} + (\omega_1 - \omega_2 - \omega_3)^{\otimes 2} = 4 \sum \omega_1^{\otimes 2} = 4 (\omega_1 \otimes \omega_2 + \omega_2 \otimes \omega_1) + 8 (\omega_1^{\otimes 2} + \omega_2^{\otimes 2}). \]

Since $2\omega_i \in \Lambda$, this agrees with (B.2) mod 2, without vanishing.
(B.3) A disconnected group with $B\sigma \neq 0$. Abandoning connectivity makes second obstructions easier to find. Let $G$ be the extension of $\mu_2$ by $H := U(1)^{×3}$, with $\mu_2$ inverting each factor and extended via the tri-diagonal class in $H^2(B\mu_2; H)$. Equivalently, the non-trivial component of $G$ restricts from the representation $H \mapsto \text{tridiagonal class in } \omega$ restricts to 3 becomes a homomorphism to $\text{Pin}^{-2}$, and $u$ the generator of $H^1(B\mu_2; \mathbb{Z}/2)$. I claim that

(i) $H^4(BG; \mathbb{Z})$ injects into $H^4(BH; \mathbb{Z})$;

(ii) $h := \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3$ is in the image. (More precisely, $H^4(BG) = \langle h, \omega_2^2 \rangle$.)

Both claims are seen from the Leray sequence for $BH \rightarrow BG \rightarrow BG$. With $\mathbb{Z}/2$ coefficients, $d_3[\omega_i] = u^3$, from the central extension. Item (i) holds because $H^2(B\mu_2; \Lambda) = 0$, while $H^4(B\mu_2; \mathbb{Z})$ is killed by $d_3[B^2]\langle \omega_i \rangle = B(u^3)$. Part (ii) holds because $h$ survives in the integral Leray sequence: reducing mod 2 shows that

$$d_3(\omega_i) = u^2 \cdot B([\omega_i] + [\omega_i]),$$

so that $d_3h = 0$, and there is no landing place for $d_5$. We conclude that

$$c_2(E) = \omega_1^2 + \omega_2^2 + \omega_3^2 + (\omega_1 + \omega_2 + \omega_3)^2 = 2(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) + 2\sum \omega_i^2 = 2(h + \sum \omega_i^2)$$

and is even in $H^4(BG; \mathbb{Z})$.

Now, we have $Sq^1[\omega_i] = u \cdot [\omega_i]$ and $Sq^2[\omega_i] = [\omega_i]^2 \pmod{u^4}$; in particular, this holds over the 2-skeleton $\mathbb{R}P^2$ of the base. We compute

$$Sq^2h = Sq^2(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) = \sum_i [\omega_i]^2 \cdot [\omega_j] + u^2h$$

$$Sq^3h = Sq^1Sq^2h = u \cdot \sum_{i \neq j} [\omega_i]^2 \cdot [\omega_j]$$

$$Sq^3(\omega_i^2) = 0$$

Since $Sq^2\sigma = Sq^3h$, we conclude that $\sigma = u \cdot h + u \cdot \varphi$, with $Sq^2(u\varphi) = 0$ and $\varphi$ quadratic in the $[\omega_i]$, because $u^3 = 0$. Symmetry allows only 0 or $\sum \omega_i^2$ as options. Then,

$$B\sigma = u^2 \cdot (h + \varphi) \in H^6(BG; \mathbb{Z}).$$

Moreover, $H^3(BG; \mathbb{Z}/2)$ is spanned by the $u \cdot [\omega_i + \omega_j]$, and the class above is not a sum of their squares, so the obstruction class $B\sigma$ does not vanish modulo the ambiguity $BSq^3H^3$.

**B.4 Remark.** We can determine $\sigma$ by restricting to the diagonal $\text{Pin}_2^{-2} \subset H$. The representation becomes

$$L^{×3} \oplus L^{×3} \oplus (\text{dual})$$

which is restricted from the representation $C^2 \oplus C^2 \oplus C^4$ of SU(2), and has no obstruction. Since $h$ restricts to $3\omega^2$, we need $\varphi = \sum \omega_i^2 = 3\omega^2$ to cancel $B\sigma$. Thus, $\sigma = uh + u \sum [\omega_i^2]$.
(B.5) A non-trivial Toda torsor. When the Weyl cocycle $\chi$ is not exact, the space $C_3$ becomes is a non-trivial rational torsor over the Toda group scheme. The smallest example for a connected group may be $G = \text{Sp}(2)$ with the irreducible component $E \subset \mathbb{R}^5 \otimes \mathbb{H}^2$ complementary to the standard representation $\mathbb{H}^2$.

Under the normalizer $N(H)$, $E$ splits into two copies of $\mathbb{H}^2$ plus the sum of the eight weight spaces for
$$v_{i,3,1} = \pm 3\xi_1 \pm \xi_2 \quad \text{and} \quad v_{i,1,3} = \pm \xi_1 \pm 3\xi_2,$$
with independent choices of sign. (Recall that the co-roots are $\pm e_i$ and $\pm e_i \pm e_j$, $i \neq j$.) Polarize by one copy of $\mathbb{H}^2$, together with $v_{3,1,1}$ and $v_{1,1,3}$ and focus on the inversion $(\xi \leftrightarrow -\xi, h \leftrightarrow h^{-1})$. The relevant $\chi$ is
$$\chi(\xi_1, \xi_2) = \left[\frac{v_{1,3} \cdot v_{3,1} \cdot v_{1,1,-3} \cdot v_{3,1}^3}{v_{1,3}^3 \cdot v_{3,1} \cdot v_{1,1,-3} \cdot v_{3,1}^3}\right] \in \left[\frac{\mathbb{C}^\times}{\mathbb{C}^\times}\right] = H^\vee$$
up to correction by a constant 2-torsion point in $H^\vee$. This $\chi$ cannot be expressed as
$$\delta \varphi := \varphi(-\xi)^{-1}/\varphi(\xi)$$
for any rational map $\varphi : h \to H^\vee$; such $\delta \varphi$ must have even valuation at any of the lines $v_{i,j} = 0$. The torsor $C_3(G; E)$ is therefore rationally non-trivial.

(B.6) Erroneous removal of the Weyl obstruction. Consider the group $G = \text{SU}(2) \times \text{U}(1)$, with representation $E = \mathbb{C}^2 \otimes (\mathbb{C} \oplus \overline{\mathbb{C}})$. The natural polarization supplies two Lagrangian sections of $C_3(G; E)$, giving birational equivalences to the Toda space.

Let us instead choose $E_+ := \langle e_1, e_2 \rangle$, the highest-weight space for $\text{SU}(2)$. Up to possible adjustment by a 2-torsion point in $H = \mathbb{C}^\times \times \mathbb{C}^\times$, the Weyl reflection (acting on the first coordinates of $h$ and $H$) now gets multiplied by the factor
$$\chi(\xi_1, \xi_2) = \left[\frac{(\xi_1 + \xi_2)(\xi_1 - \xi_2)}{(\xi_1 + \xi_2)(\xi_1 - \xi_2)^{-1}}\right] \in \left[\frac{\mathbb{C}^\times}{\mathbb{C}^\times}\right] \quad \text{(B.7)}$$
This modified reflection squares to 1, so it seems that no correction is needed. However, after restricting to the root hyperplane $\xi_1 = 0$, the natural Weyl action gets multiplied by $[-\xi_2^2, -1]$. This gives the non-trivial class in $H^1_{\mathbb{Z}/2}(C^\times)$ on the second $\mathbb{C}^\times$ factor, and can thus not be used to define the trivial torsor $C_3(G; E)$.

To apply the Weyl descent construction, we must change the sign in the bottom entry of (B.7). This exploits the ambiguity $\mathbb{Z}/2$ in the construction of $C_3(N(H); E)$, the integral Bockstein image of $H^2_{\mathbb{N}(H)}(\mathbb{Z}/2)$. By contrast, $C_3(G; E)$ is unambiguous: see Remark 4.6.1.

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