Real-Variable Theory of Local Variable Hardy Spaces

Jian TAN

School of Science, Nanjing University of Posts and Telecommunications, Nanjing 210023, P. R. China
and
Department of Mathematics, Nanjing University, Nanjing 210093, P. R. China
E-mail: tanjian89@126.com

Abstract  In this paper, we give a complete real-variable theory of local variable Hardy spaces. First, we present various real-variable characterization in terms of several local maximal functions. Next, the new atomic and the finite atomic decomposition for the local variable Hardy spaces are established. As an application, we also introduce the local variable Campanato space which is showed to be the dual space of the local variable Hardy spaces. Analogous to the homogeneous case, some equivalent definitions of the dual of local variable Hardy spaces are also considered. Finally, we show the boundedness of inhomogeneous Calderón–Zygmund singular integrals and local fractional integrals on local variable Hardy spaces and their duals.

Keywords  Local Hardy space, atom, variable exponent analysis, local BMO-type space, inhomogeneous Calderón–Zygmund singular integrals, local fractional integrals

MR(2010) Subject Classification  42B30, 42B25, 42B35, 46E30

1 Introduction

The real-variable theory of classical global Hardy spaces $H^p(\mathbb{R}^n)$ in Euclidean spaces was developed by Stein and Weiss [33] and systematically developed by Fefferman and Stein [12]. It is well known that $H^p(\mathbb{R}^n)$ with $0 < p \leq 1$ is a good substitute of the Lebesgue space $L^p(\mathbb{R}^n)$ when studying the boundedness of classical operators in harmonic analysis. Moreover, the atomic characterization of $H^p(\mathbb{R}^n)$ is a very improtant tool for the study of function spaces and the operators acting on these spaces. The atomic charaterization of $H^p(\mathbb{R}^n)$ in one dimension is given by Coifman [4] in 1974 and later was extended to higher dimensions by Latter [23].

However, it is pointed out that $H^p(\mathbb{R}^n)$ is well suited only to the Fourier analysis, but is not stable under multiplication by Schwartz class. To circumvent the drawbacks, in 1979, Goldberg [13] introduced the theory of local Hardy space $h^p(\mathbb{R}^n)$ in the Euclidean spaces, which plays an important role in various fields of analysis and partial differential equations. Particularly, Goldberg [13] obtained the atomic decomposition characterization of $h^p(\mathbb{R}^n)$, introduced the
On the other hand, the study of variable Hardy spaces $H^{p(\cdot)}(\mathbb{R}^n)$ is inspired by the Lebesgue spaces with variable exponents $L^{p(\cdot)}(\mathbb{R}^n)$, which gain the attentions of many researchers. The theory of variable Hardy spaces was developed independently by Nakai and Sawano [26], Cruz-Uribe and Wang [9] by using different approaches. In [50], Zhuo et al. gave the equivalent theory of variable Hardy spaces was developed independently by Nakai and Sawano [26], Cruz-Uribe and Wang [9] by using different approaches. In [50], Zhuo et al. gave the equivalent characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$ in terms of several intrinsic square functions. In [45], Yang et al. obtained the Riesz transforms characterization for $H^{p(\cdot)}(\mathbb{R}^n)$. Zhuo et al. [49] also considered the variable Hardy spaces on RD-spaces. The atomic decomposition characterization of $H^{p(\cdot)}L^p(\mathbb{R}^n)$ is very useful when we consider the boundedness of operators on these spaces. The atomic decomposition of Hardy spaces with variable exponents $H^{p(\cdot)}L^p(\mathbb{R}^n)$ was established independently in [9, 26] by using maximal function characterization. Later, Sawano [29] refined the atomic decomposition characterizations of $H^{p(\cdot)}(\mathbb{R}^n)$, and obtained some applications to the boundedness of several operators on $H^{p(\cdot)}(\mathbb{R}^n)$. The author revisited the atomic decomposition of $H^{p(\cdot)}(\mathbb{R}^n)$ via Littlewood–Paley–Stein analysis and gave some applications to (sub)linear and (sub)multilinear operators in [34, 37, 38]. Ho [20] established the atomic decompositions for weighted variable Hardy spaces $H_w^{p(\cdot)}(\mathbb{R}^n)$. Moreover, the atomic decomposition characterization for $h^{p(\cdot)}(\mathbb{R}^n)$ have been studied in [35, 41]. Motivated by these results, we will focus on completing the real-variable theory of variable local Hardy spaces $h^{p(\cdot)}(\mathbb{R}^n)$, which includes that of the classical local Hardy space theory of Goldberg [13].

The main purpose of this paper is threefold. The first goal is to establish some real-variable characterization, including the atomic, the local vertical and the local non-tangential maximal functions, of local variable Hardy spaces $h^{p(\cdot)}(\mathbb{R}^n)$. The second goal is to introduce the local variable Campanato space $bmo^{p(\cdot)}(\mathbb{R}^n)$ and establish the duality between $h^{p(\cdot)}(\mathbb{R}^n)$ and $bmo^{p(\cdot)}(\mathbb{R}^n)$. The third goal is to show the boundedness of inhomogeneous Calderón–
Zygmund singular integrals and local fractional integrals on $h^{p(\cdot)}(\mathbb{R}^n)$ and $bmo^{p(\cdot)}(\mathbb{R}^n)$. The novelty of this paper can be summarized as follows: First, the approach of establishing atomic decompositions used in our paper is different from the constant exponent analogy. Indeed, we give a direct proof for the infinite atomic and finite atomic decomposition of $h^{p(\cdot)}(\mathbb{R}^n)$, by avoiding the atomic decompositions of $H^{p(\cdot)}(\mathbb{R}^n)$. Moreover, we do not require $p^+ \leq 1$ when we give the real-variable characterization of $h^{p(\cdot)}(\mathbb{R}^n)$ and obtain the boundedness of operators on $h^{p(\cdot)}(\mathbb{R}^n)$. Particularly, under certain conditions, if $f \in h^{p(\cdot)}(\mathbb{R}^n)$, then there exists a sequence of special local $(p(\cdot), q)$-atom $\{a_j\}_j$ with supports $\{Q_j\}_j$ and non-negative numbers $\{\lambda_j\}_j$ such that

$$f = \sum_j \lambda_j a_j$$

and

$$\|f\|_{h^{p(\cdot)}(\mathbb{R}^n)} \sim \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}.$$  

Finally yet importantly, we develop a complete dual spaces theory of $h^{p(\cdot)}(\mathbb{R}^n)$ for $0 < p^- \leq p^+ < \infty$. By products, we also establish the dual space for $H^{p(\cdot)}(\mathbb{R}^n)$ of $p^+ > 1$ and $p^- \leq 1$, which gives a complete answer to the open question proposed by Izuki et al. [21].

The remainder of this paper is organized as follows. In Section 2, we recall some precise definitions concerning variable Lebesgue spaces and state the necessary lemmas which is useful in the subsequent sections. In Section 3, we first recall the $h^{p(\cdot)}(\mathbb{R}^n)$ via the Littlewood–Paley–Stein theory. Next, we give the equivalent characterization via the local vertical and non-tangential maximal function. Then a new finite atomic decomposition for the local variable Hardy spaces is established in Section 4. In Section 5, we introduce a local variable Campanato space $bmo^{p(\cdot)}(\mathbb{R}^n)$ which is further proved to be the dual space of $h^{p(\cdot)}(\mathbb{R}^n)$. Finally, in Section 6, we show that inhomogeneous Calderón–Zygmund singular integrals and local fractional integrals are bounded on $h^{p(\cdot)}(\mathbb{R}^n)$ and their duals.

Throughout this paper, $C$ or $c$ denotes a positive constant that may vary at each occurrence but is independent to the main parameter, and $A \sim B$ means that there are constants $C_1 > 0$ and $C_2 > 0$ independent of the the main parameter such that $C_1 B \leq A \leq C_2 B$. To denote the dependence of the constants on some parameter $s$, we will write $C_S$. Given a measurable set $S \subset \mathbb{R}^n$, $|S|$ denotes the Lebesgue measure and $\chi_S$ means the characteristic function. Let $S$ be the space of Schwartz functions and let $S'$ denote the space of tempered distributions. We also use the notations $j \land j' = \min\{j, j'\}$ and $j \lor j' = \max\{j, j'\}$. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty]$ be a Lebesgue measurable function. We write $\mathbb{N} = \{1, 2, \ldots\}$. For a measurable subset $E \subset \mathbb{R}^n$, we denote $p^- (E) = \inf_{x \in E} p(x)$ and $p^+ (E) = \sup_{x \in E} p(x)$. Especially, we denote $p^- = p^- (\mathbb{R}^n)$ and $p^+ = p^+ (\mathbb{R}^n)$. We also write $p_- = p^- \land 1$. Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ be a measurable function with $0 < p^- \leq p^+ < \infty$ and $\mathcal{P}^0$ be the set of all these $p(\cdot)$. Let $\mathcal{P}$ be the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to [1, \infty)$ such that $1 < p^- \leq p^+ < \infty$.

## 2 Preliminaries

In this section, we will recall some definitions and basic results on the variable Lebesgue spaces. For brevity, we write $X(\mathbb{R}^n) = X$, where $X$ is some function space.
Definition 2.1 ([5, 10]) The variable Lebesgue space $L^{p(\cdot)}$ is defined as the set of all measurable functions $f$ for which the quantity $\int_{\mathbb{R}^n} |f(x)|^{p(x)} \, dx$ is finite for some $\varepsilon > 0$ and

$$
\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{|f(x)|}{\lambda} \right)^{p(x)} \, dx \leq 1 \right\}.
$$

As a special case of the theory of Nakano and Luxemberg, we see that $L^{p(\cdot)}$ is a quasi-normed space. Especially, when $p^* \geq 1$, $L^{p(\cdot)}$ is a Banach space. Note that the variable exponent spaces, such as the variable Lebesgue spaces and the variable Sobolev spaces, were studied by a substantial number of researchers (see, for instance, [6, 22]). In the study of variable exponent spaces it is common to assume that the exponent function $p(\cdot)$ satisfies the LH conditions. We say that $p(\cdot) \in LH$, if $p(\cdot)$ satisfies

$$
|p(x) - p(y)| \leq \frac{C}{-\log(|x - y|)}, \quad |x - y| \leq 1/2
$$

and

$$
|p(x) - p(y)| \leq \frac{C}{\log(|x| + e)}, \quad |y| \geq |x|.
$$

Let $\mathcal{B}$ be the set of $p(\cdot) \in \mathcal{P}$ such that the Hardy–Littlewood maximal operator $M$ is bounded on $L^{p(\cdot)}$. It is well known that $p(\cdot) \in \mathcal{B}$ if $p(\cdot) \in \mathcal{P} \cap LH$. Moreover, examples shows that the above LH conditions are necessary in certain sense, see Pick and Růžička [28] for details. For more information about the LH condition, we also refer the readers to [5, Chapter 2.1]. We also need the following boundedness of the vector-valued maximal operator $M$, whose proof can be found in [6, Corollary 2].

Lemma 2.2 Let $p(\cdot) \in \mathcal{P} \cap LH$. Then for any $q > 1$, $f = \{f_i\}_{i \in \mathbb{N}}$, $f_i \in L^{loc}$, $i \in \mathbb{N}$

$$
\|\|M(f_i)\|_{L^{q(\cdot)}} \leq C\|f\|_{L^{q(\cdot)}},
$$

where $M(f) = \{M(f_i)\}_{i \in \mathbb{N}}$.

Given a measurable function $w > 0$, for $1 < p < \infty$, it is said that $w \in A_p$ if

$$
[w]_{A_p} = \sup_B \left( \frac{1}{|B|} \int_B w(x) \, dx \right) \left( \frac{1}{|B|} \int_B w(x)^{1-p'} \, dx \right)^{p-1} < \infty,
$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$. Define the set

$$
A_\infty = \bigcup_{p \geq 1} A_p.
$$

The extrapolation in variable Lebesgue space is very useful when the corresponding weighted norm inequalities are known.

Lemma 2.3 ([6, Corollary 1.10]) $\mathcal{F}$ denote a family of ordered pairs of non-negative measurable functions $(f, g)$. Suppose that there exist some $p_0$ with $0 < p_0 < \infty$ and every weight $w_0 \in A_\infty$ such that

$$
\int_{\mathbb{R}^n} |f(x)|^{p_0} w_0 \, dx \leq \int_{\mathbb{R}^n} |g(x)|^{p_0} w_0 \, dx, \quad (f, g) \in \mathcal{F},
$$

for $f \in L^{p_0}$. If $p(\cdot) \in LH \cap \mathcal{P}^0$, then for any $(f, g) \in \mathcal{F}$ and $f \in L^{p(\cdot)}$, we have

$$
\|f\|_{L^{p(\cdot)}} \leq C\|g\|_{L^{p(\cdot)}}.
$$
The following generalized Hölder inequality on variable Lebesgue spaces has been proved in [5, Corollary 2.28].

**Lemma 2.4** Given the exponent function $p_i(\cdot) \in \mathcal{P}$, define $p(\cdot) \in \mathcal{P}$ by

$$\frac{1}{p(x)} = \sum_{i=1}^{2} \frac{1}{p_i(x)},$$

where $i = 1, 2$. Then for all $f_i \in L^{p_i(\cdot)}$ and $f_1 f_2 \in L^{p(\cdot)}$

$$\left\| \sum_{i=1}^{2} f_i \right\|_{p(\cdot)} \leq C \left\| \sum_{i=1}^{2} \| f_i \|_{p_i(\cdot)} \right\|_{p(\cdot)}. $$

**Lemma 2.5** ([5, Lemma 2.39]) Given an exponent function $p(\cdot) \in \mathcal{P}^0$ with $p^- \leq 1$, then for all $f, g \in L^{p(\cdot)}$,

$$\| f + g \|_{L^p(\cdot)}^p \leq \| f \|_{L^p(\cdot)}^p + \| g \|_{L^p(\cdot)}^p.$$ 

**Lemma 2.6** ([9, Lemma 2.2]) Given an exponent function $p(\cdot) \in \mathcal{P}$, if $E \subset \mathbb{R}^n$ is such that $|E| < \infty$, then $\chi_E \in L^{p(\cdot)}$ and

$$\| \chi_E \|_{L^p(\cdot)} \leq |E| + 1.$$ 

The following key lemma also plays a key role in the proofs of the main results. The Grafakos–Kalton lemma was first established in [14] when they considered the multilinear Calderón–Zygmund operators on Hardy spaces (also see [8] on the weighted extension). We need the variable exponent version as follows.

**Lemma 2.7** ([7, Lemma 4.8]) Let $q(\cdot) \in LH \cap \mathcal{P}^0$. Suppose that we are given a sequence of cubes $\{Q_j\}_{j=1}^{\infty}$ and a sequence of non-negative functions $\{F_j\}_{j=1}^{\infty}$. Then for any $q$ such that $(1 \cup p^+) < q < \infty$ we have

$$\left\| \sum_{j=1}^{\infty} \chi_{Q_j} F_j \right\|_{L^q(\cdot)} \leq C \left\| \sum_{j=1}^{\infty} \left( \frac{1}{Q_j} \int_{Q_j} F_j^q(y) dy \right)^{\frac{1}{q}} \chi_{Q_j} \right\|_{L^q(\cdot)}. $$

**Lemma 2.8** ([10, Corollary 4.5.9]) Suppose that $p(\cdot) \in LH$ and $0 < p^- \leq p^+ < \infty$.

1. For all cubes (or balls) $|Q| \leq 2^n$ and any $x \in Q$, we have

$$\| \chi_Q \|_{p(\cdot)} \sim |Q|^{\frac{1}{p^+}}.$$ 

2. For all cubes (or balls) $|Q| \geq 1$, we have

$$\| \chi_Q \|_{p(\cdot)} \sim |Q|^{\frac{1}{p_\infty}},$$

where $p_\infty = \lim_{x \to \infty} p(x)$.

### 3 Local Variable Hardy Spaces and Their Maximal Characterizations

In this section, we will give several equivalent characterization for local variable Hardy space. To state the results, we need some definitions. We write $\psi_t(x) = t^{-n} \psi(t^{-1} x)$ for all $x \in \mathbb{R}^n$. Let $\mathcal{D}$ denote the set of all $C^\infty$ functions on $\mathbb{R}^n$ with compact supports, equipped with the inductive limit topology, and $\mathcal{D}'$ its topological dual space, equipped with the weak-* topology. Write $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For $N \in \mathbb{N}_0, |\alpha| \leq N + 1$ and $R \in (0, \infty)$, let

$$\mathcal{D}_{N,R} = \left\{ \psi \in \mathcal{D} : \text{supp}(\psi) \subset B(0, R), \int \psi \neq 0, \| D^\alpha \psi \|_{\infty} \leq 1 \right\}.$$
First, we recall the local vertical, non-tangential grand maximal functions as follows.

**Definition 3.1** For any $f \in \mathcal{D}'$, the local vertical grand maximal function $G_{N,R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^n$,$$
G_{N,R}(f)(x) \equiv \sup_{t \in (0,1)} \{|\psi_t \ast f(x)| : \psi \in \mathcal{D}_{N,R}\},$
and the local non-tangential grand maximal function $\tilde{G}_{N,R}(f)$ of $f$ is defined by setting, for all $x \in \mathbb{R}^n$,$$
\tilde{G}_{N,R}(f)(x) \equiv \sup_{|x-z|<t<1} \{|\psi_t \ast f(z)| : \psi \in \mathcal{D}_{N,R}\}.$$
For convenience, we write $G_{N,1}(f) = G_{N}^0(f)$ and $\tilde{G}_{N,1}(f) = \tilde{G}_{N}^0(f)$ and also write $G_{N,23(10+n)}(f) = G_N(f)$ and $\tilde{G}_{N,23(10+n)}(f) = \tilde{G}_N(f)$. Obviously,$$
G_{N}^0(f) \leq G_N(f) \leq \tilde{G}_N(f).
$$

Next, we also recall the local vertical, tangential and non-tangential maximal functions.

**Definition 3.2** Let $\psi_0 \in \mathcal{D}$ with $\int \psi_0(x) dx \neq 0$. The local vertical maximal function $M_{\psi_0}(f)$ is defined by$$
M_{\psi_0}(f)(x) \equiv \sup_{j \in \mathbb{N}_0} |(\psi_0)_j \ast f(x)|.
$$

For $j \in \mathbb{N}_0$, $A, B \geq 1$, the local tangential Peetre-type maximal function $\psi_{0,A,B}^*(f)$ is defined by$$
\psi_{0,A,B}^*(f)(x) \equiv \sup_{j \in \mathbb{N}_0, y \in \mathbb{R}^n} \frac{|(\psi_0)_j \ast f(x-y)|}{(1 + 2^j|y|)^A 2^B|y|}.
$$
The local non-tangential maximal function $M_{\psi_0}^*(f)$ is defined by$$
M_{\psi_0}^*(f)(x) \equiv \sup_{|x-y|<t<1} |(\psi_0)_t \ast f(y)|.
$$

Hereafter, $(\psi_0)_j(x) = 2^{jn}\psi_0(2^jx)$ and $(\psi_0)_t(x) = t^{-n}\psi_0(t^{-1}x)$.

We also introduce the local Littlewood–Paley–Stein square function below.

**Definition 3.3** Let $\varphi \in \mathcal{D}(\mathbb{R}^n)$ satisfy$$
\text{supp } \hat{\varphi} \subset \left\{ \xi : \frac{1}{2} < |\xi| \leq 2 \right\},
$$
and $\Phi$ whose Fourier transform does not vanish at the origin with$$
\text{supp } \hat{\Phi} \subset \{ \xi : |\xi| \leq 2 \}
$$
satisfy$$
|\hat{\Phi}(\xi)|^2 + \sum_{j=1}^{\infty} |\hat{\varphi}(2^{-j}\xi)|^2 = 1, \text{ for all } \xi \in \mathbb{R}^n.
$$

We denote $\Phi = \varphi_0$. For $f \in \mathcal{D}'$, we give the definition of local Littlewood–Paley–Stein square function$$
G_{\text{loc}}(f)(x) := \left( \sum_{j \in \mathbb{N}_0} |\varphi_j \ast f(x)|^2 \right)^{\frac{1}{2}},
$$
and the discrete Littlewood–Paley–Stein square function
\[ g_{\text{loc}}^d(f)(x) := \left( \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^n} |\varphi_j * f(2^{-j}k)|^2 \chi_{Q_{j,k}}(x) \right)^{\frac{1}{2}}, \]
where \( Q_{j,k} \) denote dyadic cubes in \( \mathbb{R}^n \) with side-lengths \( 2^{-j} \) and the lower left-corners of \( Q \) are \( 2^{-j}k \).

We now recall the local variable Hardy spaces via local Littlewood–Paley–Stein square function in [35].

**Definition 3.4** Let \( f \in \mathcal{D}' \) and \( p(\cdot) \in \mathcal{P}^0 \). The local Hardy space with variable exponent \( h^{p(\cdot)} \) is the set of all \( f \in \mathcal{D}' \) for which the quantity
\[ \|f\|_{h^{p(\cdot)}} = \| \Phi * f \|_{L^{p(\cdot)}} + \left\| \left\{ \sum_{j=1}^{\infty} |\varphi_j * f|^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}} < \infty. \]

**Remark 3.5** Letting all notation be as in Definition 3.4, observe that
\[ \|f\|_{h^{p(\cdot)}} \sim \left\| \left\{ \sum_{j=0}^{\infty} |\varphi_j * f|^2 \right\}^{\frac{1}{2}} \right\|_{L^{p(\cdot)}}. \]

Also, it is shown in [35, Theorem 1.3] that \( \|f\|_{h^{p(\cdot)}} \sim \|g_{\text{loc}}^d(f)\|_{L^{p(\cdot)}} \).

Now, let us state the main results in this section. We first obtain the equivalent characterizations of \( h^{p(\cdot)} \) as follows.

**Theorem 3.6** Let \( f \in \mathcal{D}' \), \( p(\cdot) \in \mathcal{P}^0 \cap \mathcal{L}H \). For a fixed large integer \( N_0 \), any \( N \geq N_0 \), the following statements are mutually equivalent:

1. \( f \in h^{p(\cdot)} \);
2. \( g_N(f) \in L^{p(\cdot)} \);
3. \( g^0_N(f) \in L^{p(\cdot)} \);
4. \( \tilde{g}_N(f) \in L^{p(\cdot)} \);
5. \( \tilde{g}^0_N(f) \in L^{p(\cdot)} \);
6. \( M_{\psi_0}(f) \in L^{p(\cdot)} \);
7. \( M^{\ast}_{\psi_0}(f) \in L^{p(\cdot)} \);
8. \( \psi_{0,A,B}^*(f) \in L^{p(\cdot)} \).

Moreover, for all \( f \in \mathcal{D}' \),
\[ \|f\|_{h^{p(\cdot)}} \sim \|g_N(f)\|_{L^{p(\cdot)}} \sim \|g^0_N(f)\|_{L^{p(\cdot)}} \sim \|\tilde{g}_N(f)\|_{L^{p(\cdot)}} \sim \|\tilde{g}^0_N(f)\|_{L^{p(\cdot)}} \sim \|M_{\psi_0}(f)\|_{L^{p(\cdot)}} \sim \|M^{\ast}_{\psi_0}(f)\|_{L^{p(\cdot)}} \sim \|\psi_{0,A,B}^*(f)\|_{L^{p(\cdot)}}, \]
where the implicit equivalent positive constants are independent of \( f \).

**Proof** The proof of equivalence for the first two norms can be found in [26, Section 9] and [35, Theorem 1.3]. We only need to prove that the rest of norms are equivalent. To end it, from [46, Theorem 3.14, Corollary 3.15], we know that
\[ \|g_N(f)\|_{L^w_{p(\cdot)}} \sim \|g^0_N(f)\|_{L^w_{p(\cdot)}} \sim \|\tilde{g}_N(f)\|_{L^w_{p(\cdot)}} \sim \|\tilde{g}^0_N(f)\|_{L^w_{p(\cdot)}} \sim \|M_{\psi_0}(f)\|_{L^w_{p(\cdot)}} \sim \|M^{\ast}_{\psi_0}(f)\|_{L^w_{p(\cdot)}} \sim \|\psi_{0,A,B}^*(f)\|_{L^w_{p(\cdot)}} \]
for every \( w \in A_{\infty} \). Notice that \( p(\cdot) \in \mathcal{P}^0 \cap \mathcal{L}H \), by Lemma 2.3, for \( (g_N(f)\chi_B(0,R), g^0_N(f)) \in \mathcal{F} \) and \( g_N(f)\chi_B(0,R) \in L^{p(\cdot)} \) with \( 0 < R < \infty \), we have
\[ \|g_N(f)\chi_B(0,R)\|_{L^{p(\cdot)}} \leq C \|g^0_N(f)\|_{L^{p(\cdot)}}. \]
If we take the limit as \( R \to \infty \), then by Fatou’s lemma
\[ \|g_N(f)\|_{L^{p(\cdot)}} \leq C \|g^0_N(f)\|_{L^{p(\cdot)}}. \]
Similarly, for $(G^0_N(f) \chi_{B(0,R)}, G_N(f)) \in \mathcal{F}$ and $G^0_N(f) \in L^p(\cdot)$, we have

$$
\|G^0_N(f)\|_{L^p(\cdot)} \leq C\|G_N(f)\|_{L^p(\cdot)}.
$$

Repeating the same argument, we can get the desired result.

\[\square\]

**Remark 3.7** If $f \in \mathcal{D}'$, $p(\cdot) \in \mathcal{P} \cap LH$, we want to stress that the function spaces $h^p(\cdot)$ and $L^p(\cdot)$ are isomorphic to each other. To see this, we only need to observe that in [42, Proposition 2.2], if $f \in A_p$ with $p \in (1, \infty)$, then $f \in L^p_p$ if and only if $f \in \mathcal{D}'$ and $G^0_N(f) \in L^p_p$ with $\|f\|_{L^p_p} \sim \|G^0_N(f)\|_{L^p_p}$. Then applying the [6, Corollary 1.11], we get that $\|f\|_{L^p(\cdot)} \sim \|G^0_N(f)\|_{L^p(\cdot)}$. We also remark that the $h^p(\cdot)$-norm is stronger than the topology of $\mathcal{D}'$; indeed, for any $\psi \in \mathcal{D}$ and supp $\psi \subset B_0 = B(0,1)$, by Lemma 2.4 and Lemma 2.6, we have

$$
|\langle f, \psi \rangle|^p = |f \ast \tilde{\psi}(0)|^p \leq C \inf_{y \in B_0} M^0_N(f)(y)^p \\
\leq C \frac{1}{|B_0|} \int_{B_0} M^0_N(f)(y)^p \, dy \\
\leq C \|M^0_N(f)\|_{L^p(\cdot)}^p \|\chi_{B_0}\|_{L^{p(\cdot)'}} \\
\leq C \|f\|_{h^p(\cdot)}^p,
$$

where $\tilde{\psi}(x) = \psi(-x)$. We also remark that, as a consequence of [35, Theorem 1.3], $L^q \cap h^p(\cdot)$ is dense in $h^p(\cdot)$ for $1 \leq q < \infty$.

We also obtain the completeness of $h^p(\cdot)$ that are of interest in their own right. We have proved it implicitly in [35] by applying the Littlewood–Paley–Stein theory. Here we give a different approach. We refer to [1, Theorem 1.6] for Banach function spaces and [9, Proposition 4.1] for $H^p(\cdot)$.

**Proposition 3.8** Given $p(\cdot) \in \mathcal{P} \cap LH$, the space $h^p(\cdot)$ is complete with respect to the quasi-norm $\|\cdot\|_{h^p(\cdot)}$.

**Proof** For any $\psi \in \mathcal{D}_{N,1}$, by the remark, if any sequence $\{f_k\}_{k=1}^\infty$ converges in $h^p(\cdot)$, then it also converges in $\mathcal{D}'$. We only need to consider the case $p^- \leq 1$, the other case is similar but easier. First we show that $h^p(\cdot)$ has the Riesz–Fisher property: given any sequence $\{g_k\}_{k=1}^\infty$ in $h^p(\cdot)$ fulfilling that

$$
\sum_{k=1}^\infty \|g_k\|_{h^p(\cdot)}^p < \infty,
$$

the series $\sum_{k=1}^\infty g_k$ converges in $h^p(\cdot)$. Let $G_j = \sum_{k=1}^j g_k$. Indeed, by Lemma 2.5, we obtain that the sequence $\{G_j\}_{j=1}^\infty$ is Cauchy in $h^p(\cdot)$ and so in $\mathcal{D}'$. Thus, it converges in $\mathcal{D}'$ to a distribution $g$. Furthermore, we have

$$
\|g\|_{h^p(\cdot)}^p = \left\| \sum_{k=1}^\infty g_k \right\|_{h^p(\cdot)}^p \leq \sum_{k=1}^\infty \|g_k\|_{h^p(\cdot)}^p < \infty.
$$

Therefore,

$$
\|g - G_j\|_{h^p(\cdot)}^p \leq \sum_{k \geq j+1} \|g_k\|_{h^p(\cdot)}^p \to 0.
$$
as $j$ tends to $\infty$. So the series converges to $g$ in $h^{p(\cdot)}$. To finish it, let $\{f_m\}_{m=1}^{\infty}$ a Cauchy sequence in $h^{p(\cdot)}$. Then we can find a sequence $\{f_{m_j}\}_{j=1}^{\infty}$ for every increasing sequence of positive integers $\{m_j\}_{j=1}^{\infty}$ such that

$$\|f_{m,j+1} - f_{m,j}\|_{h^{p(\cdot)}}^p \leq 2^{-j}.$$ 

Then by the Riesz–Fischer property, we see that the series $\sum_{j=1}^{\infty} (f_{m,j+1} - f_{m,j})$ converges in $h^{p(\cdot)}$. Thus, there exists an $f \in h^{p(\cdot)}$ such that $f = \lim_{j \to \infty} f_{m,j}$ in $h^{p(\cdot)}$. This finishes the proof. 

\[ \square \]

4 Atomic Characterization of $h^{p(\cdot)}$

In this section, we will give new atomic decompositions of $h^{p(\cdot)}$ using special local $(p(\cdot), q)$-atom. For one thing, we will discuss the infinite atomic characterization of $h^{p(\cdot)}$. For another, we will obtain the finite atomic decompositions of $h^{p(\cdot)}$.

4.1 Infinite Local $(p(\cdot), q)$-Atomic Decompositions

The first aim of this chapter is to sharpen the atomic decompositions theory of $h^{p(\cdot)}$, which have been established in [35, 41]. We mention that Wang et al. [43] established the atomic decompositions of local Hardy spaces $h_X$ associated with ball quasi-Banach function spaces $X$ by using the atomic decompositions of global Hardy spaces $H_X$, whereas we give the new atomic characterization of $h^{p(\cdot)}$, which is stronger than the previous one, by avoiding the atomic decomposition of $H^{p(\cdot)}$. In what follows, we introduce the new definitions for the special local $(p(\cdot), q)$-atom of $h^{p(\cdot)}$. Denote $Q(x, \ell(Q))$ the closed cube centered at $x$ and of sidelength $\ell(Q)$. Similarly, given $Q = Q(x, \ell(Q))$ and $\lambda > 0$, $\lambda Q$ means the cube with the same center $x$ and with sidelength $\lambda \ell(Q)$. We denote $\tilde{Q} = 2\sqrt{n}Q$ simply.

**Definition 4.1** Let $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, $p(\cdot) \in \mathcal{P}^0$ and $1 < q \leq \infty$. Fix an integer $d \geq d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} : p^-(n + d + 1) > n\}$. A function $a$ is said to be a special local $(p(\cdot), q)$-atom of $h^{p(\cdot)}$ if

(i) $\text{supp } a \subset Q$;
(ii) $\|a\|_{L^d} \leq |Q|^\frac{1}{d}$;
(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all $|\alpha| \leq d$, if $|Q| < 1$.

In Definition 4.1, if the condition (iii) is replaced by the condition (iii)’: $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all $|\alpha| \leq d$ and all cubes $Q \subset \mathbb{R}^n$, then the function $a$ is said to be a special $(p(\cdot), q)$-atom of $H^{p(\cdot)}$. Observe that for any $1 < q < \infty$, all $(p(\cdot), \infty)$-atoms are $(p(\cdot), q)$-atoms, since $|Q|^{-\frac{1}{d}}\|a\|_{L^d} \leq \|a\|_{L^\infty}$. The following theorems improve the previous atomic decomposition results in [35].

**Theorem 4.2** Let $p(\cdot) \in \mathcal{P}^0 \cap LH$. Suppose that $p^+ < q \leq \infty$ when $p^+ \geq 1$ and $1 < q \leq \infty$ when $p^+ < 1$. Fix an integer $d \geq d_{p(\cdot)} \equiv \min\{d \in \mathbb{N} : p^-(n + d + 1) > n\}$. Given countable collections of cubes $\{Q_j\}_{j=1}^{\infty}$ of non-negative coefficients $\{\lambda_j\}_{j=1}^{\infty}$ and of the special local $(p(\cdot), q)$-atoms $\{a_j\}_{j=1}^{\infty}$, if

$$\left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}$$
is finite, then the series \( f = \sum_{j=1}^{\infty} \lambda_j a_j \) converges in \( h^{p(\cdot)} \) and satisfies
\[
\|f\|_{h^{p(\cdot)}} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.
\]

**Theorem 4.3** Let \( p(\cdot) \in P^0 \cap LH \), \( 1 < q \leq \infty \) and \( s \in (0, \infty) \). If \( f \in h^{p(\cdot)} \), then there exist non-negative coefficients \( \{\lambda_j\}_{j=1}^{\infty} \) and the special local \( (p(\cdot), q)\)-atoms \( \{a_j\}_{j=1}^{\infty} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j a_j \), where the series converges almost everywhere and in \( D' \), and that
\[
\left\| \left( \sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^s \right)^{1/s} \right\|_{L^{p(\cdot)}} \leq C_s \|f\|_{h^{p(\cdot)}}.
\]

As an immediate corollary, we will get the following atom decomposition for \( h^{p(\cdot)} \).

**Corollary 4.4** Let \( p(\cdot) \in P^0 \cap LH \). Suppose that \( p^+ < q \leq \infty \) when \( p^+ \geq 1 \) and \( 1 < q \leq \infty \) when \( p^+ < 1 \). Then \( f \in D' \) is in \( h^{p(\cdot)} \) if and only if there exists non-negative coefficients \( \{\lambda_j\}_{j=1}^{\infty} \) and the special local \((p(\cdot), q)\)-atoms \( \{a_j\}_{j=1}^{\infty} \) such that \( f = \sum_{j=1}^{\infty} \lambda_j a_j \), where the series converges in \( h^{p(\cdot)} \), and that
\[
\|f\|_{h^{p(\cdot)}} \sim \inf \left\{ \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\}
\]
\[
\sim \inf \left\{ \left\| \left( \sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^p \right)^{1/p} \right\|_{L^{p(\cdot)}} : f = \sum_{j=1}^{\infty} \lambda_j a_j \right\},
\]
where the infimum is taken over all expressions as above.

Now we are ready to prove Theorem 4.2.

**Proof of Theorem 4.2** Fix \( \psi \in D_{N,1} \) with \( \int \psi(x)dx \neq 0 \). By Theorem 3.6, we know that
\[
\|f\|_{h^{p(\cdot)}} \sim \|G_N^0(f)\|_{L^{p(\cdot)}} \equiv \left\| \sup_{0<t<1} |\psi_t \ast f| \right\|_{L^{p(\cdot)}}.
\]

Let every \( \{a_j\}_{j=1}^{\infty} \) be a special local \((p(\cdot), q)\)-atom which is supported in a cube \( Q_j \). Suppose that \( \{\lambda_j\}_{j=1}^{\infty} \) has only a finite number of non-zero entries. We consider the two cases for \( Q_j \) as follows.

**Case 1** \( |Q_j| < 1 \). In this case, we claim that
\[
G_N^0 \left[ \sum_{j=1}^{\infty} \lambda_j a_j \right](x) \leq C \sum_{j=1}^{\infty} \lambda_j \left( M(a_j)(x) \chi_{Q_j}(x) + M(\chi_{Q_j})(x) \frac{n+d+1}{n+d} \right).
\]

To prove it, we only need to observe that for any special \((p(\cdot), q)\)-atom \( a_j \) with \( Q_j = Q(x_j, \ell(Q_j)) \) and for all \( x \in Q_j^c \),
\[
G_N^0(a_j)(x) \leq C \frac{\ell(Q_j)^{n+d+1}}{(\ell(Q_j) + |x-x_j|)^{n+d+1}}.
\]

In fact, for any \( \psi \in D_{N,1} \) and \( 0 < t < 1 \), let \( P \) be the Taylor expansion of \( \psi \) at the point \( \frac{(x-x_j)}{t} \) with degree \( d \). By the Taylor remainder theorem, we have
\[
\left| \psi\left( \frac{x-y}{t} \right) - P\left( \frac{x-x_j}{t} \right) \right| \leq C \sum_{|\alpha|=d+1} \left| (D^{\alpha}\psi)\left( \frac{\theta(x-y) + (1-\theta)(x-x_j)}{t} \right) \right| \frac{|x_j-y|^{d+1}}{t^{d+1}},
\]
where multi-index $\alpha \in \mathbb{Z}_+^n$ and $\theta \in (0, 1)$. Since $0 < t < 1$ and $x \in \tilde{Q}_j^\circ$, then we notice that $\text{supp}(a_j * \psi_t) \subset B(x_j, 2\sqrt{n})$ and that $a_j * \psi_t(x) \neq 0$ implies that $t > \frac{|x-x_j|}{2}$. Thus, for all $x \in \tilde{Q}_j^\circ$, we have

$$
| (a_j * \psi_t)(x) | = \left| t^{-n} \int_{\mathbb{R}^n} a_j(y) \left( \psi \left( \frac{x-y}{t} \right) - P \left( \frac{x-x_j}{t} \right) \right) dy \right|
\leq C \chi_{\tilde{Q}_j^\circ}(x) |x-x_j|^{-(n+d+1)} \int_{\tilde{Q}_j} |a_j(y)||y-x_j|^{d+1} dy
\leq C \chi_{\tilde{Q}_j^\circ}(x) |x-x_j|^{-(n+d+1)} \ell(Q_j)^n+1.
$$

Hence, we have proved the claim. Then by the $L^q$-boundedness of the maximal operator $M$ we conclude

$$
\left( \frac{1}{|Q_j|} \int_{Q_j} |M(a_j)(x)|^q dx \right)^\frac{1}{q} \leq \frac{1}{|Q_j|^\frac{1}{q}} \|M(a_j)\|_{L^q}
\leq C \frac{1}{|Q_j|^\frac{1}{q}} \|a_j\|_{L^q}
\leq C.
$$

Choose $\tau$ such that $\tau p^* > 1$. Then by Lemma 2.7 and Lemma 2.2, we get that

$$
\left\| G_N^0 \left( \sum_{j=1}^\infty \lambda_j a_j \right) \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j M(a_j)\chi_{\tilde{Q}_j}\right\|_{L^p(\cdot)} + C \left\| \sum_{j=1}^\infty |\lambda_j (M\chi_{Q_j})_j|^{\frac{n+d+1}{n}} \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j \left( \frac{1}{|Q_j|} \int_{Q_j} |M(a_j)(x)|^q dx \right)^\frac{1}{q} \chi_{\tilde{Q}_j}\right\|_{L^p(\cdot)} + C \left\| \sum_{j=1}^\infty |\lambda_j (M\chi_{Q_j})_j|^{\frac{n+d+1}{n}} \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j \chi_{\tilde{Q}_j}| \right\|_{L^p(\cdot)}
\leq C \left\| \left( \sum_{j=1}^\infty |\lambda_j M^\tau (\chi_{Q_j})_j\right)^\frac{1}{\tau} \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j \chi_{Q_j}| \right\|_{L^p(\cdot)}.
$$

When $q = \infty$, it is easy to see that

$$
M(a_j)(x) \leq C \|a_j\|_{L^\infty} \leq C
$$

for $x \in \tilde{Q}_j$. Similarly, we have

$$
\left\| G_N^0 \left( \sum_{j=1}^\infty \lambda_j a_j \right) \right\|_{L^p(\cdot)} \leq C \left\| \sum_{j=1}^\infty |\lambda_j M(a_j)\chi_{\tilde{Q}_j}\right\|_{L^p(\cdot)} + C \left\| \sum_{j=1}^\infty |\lambda_j (M\chi_{Q_j})_j|^{\frac{n+d+1}{n}} \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j \chi_{\tilde{Q}_j}| \right\|_{L^p(\cdot)}
\leq C \left\| \sum_{j=1}^\infty |\lambda_j \chi_{Q_j}| \right\|_{L^p(\cdot)}.
Therefore, we have completed the proof of this theorem. □

Finally, by using similar but easier argument, we get that

\[ \left\| \mathcal{G}_N^{0} \left( \sum_{j=1}^{\infty} \lambda_j a_j \right) \right\|_{L^p(\cdot)} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j M(a_j) \chi_{Q_j} \right\|_{L^p(\cdot)} \]

\[ \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \left( \frac{1}{|Q_j|} \int_{Q_j} |M(a_j)(x)|^q \, dx \right)^{\frac{1}{q}} \chi_{Q_j} \right\|_{L^p(\cdot)} \]

\[ \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{100Q_j} \right\|_{L^p(\cdot)} \]

\[ \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)} . \]

When \( q = \infty \), we can get the desired result similarly.

Finally, we extend the result to the general case. Given countable collections of cubes \( \{Q_j\}_{j=1}^{\infty} \), of non-negative coefficients \( \{\lambda_j\}_{j=1}^{\infty} \) and of the special local \((p(\cdot), q(\cdot))\)-atoms \( \{a_j\}_{j=1}^{\infty} \), observe that

\[ \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)} < \infty \]

and that for \( 1 \leq m \leq n < \infty \)

\[ \left\| \mathcal{G}_N^{0} \left( \sum_{j=m}^{n} \lambda_j a_j \right) \right\|_{L^p(\cdot)} \leq C \left\| \sum_{j=m}^{n} \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)} . \]

Hence, the sequence \( \{\lambda_j a_j\}_{j=1}^{\infty} \) is Cauchy in \( h^{p(\cdot)} \) and converges to an element \( f \in h^{p(\cdot)} \).

From Remark 3.7, we know that the \( h^{p(\cdot)} \)-norm is stronger than the topology of \( \mathcal{D}' \). So the sequence \( \{\lambda_j a_j\}_{j=1}^{\infty} \) also converges to \( f \) in \( \mathcal{D}' \). By Fatou’s lemma, we obtain

\[ \|f\|_{h^{p(\cdot)}} \leq C \lim_{n \to \infty} \left\| \mathcal{G}_N^{0} \left( \sum_{j=1}^{n} \lambda_j a_j \right) \right\|_{L^p(\cdot)} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)} . \]

Therefore, we have completed the proof of this theorem. □

Before we prove the next theorem, we revisit the Calderón–Zygmund decomposition associated with the local grand maximal function on \( \mathbb{R}^n \). For more detail, we refer to [42, Section 4] (also see [2, Section 5] and [32, p. 102–105, p. 110–111]). Let \( d \in \mathbb{N} \cup \{0\} \) be some fixed integer.
and \( \mathcal{P}_d \) denote the linear space of polynomials in \( n \) variables of degrees no more than \( d \). For each \( i \) and \( P \in \mathcal{P}_d \), set

\[
\|P\|_i := \left[ \frac{1}{\int_{\mathbb{R}^n} |P(x)|^2 \eta_i(x) dx} \right]^{\frac{1}{2}}.
\]

Then \( (\mathcal{P}_d, \| \cdot \|_i) \) is a finite dimensional Hilbert space. Let \( f \in \mathcal{D}' \). Since \( f \) induces a linear functional on \( \mathcal{D}_d \) via \( Q \mapsto \int f(x) dx \langle f, \eta \rangle \), by Riesz lemma, there exists a unique polynomial \( P_i \in \mathcal{P}_d \) for each \( i \) such that for all \( Q \in \mathcal{P}_d \),

\[
\frac{1}{\int_{\mathbb{R}^n} \eta_i(x) dx} \langle f, \eta_i \rangle = \frac{1}{\int_{\mathbb{R}^n} \eta_i(x) dx} \langle P_i, \eta_i \rangle.
\]

**Lemma 4.5** Let \( d \in \mathbb{Z} \) and \( \lambda > 0 \). Suppose that \( f \in \mathcal{D}' \) and \( \Omega = \{ x \in \mathbb{R} : \mathcal{G}_N f(x) > \lambda \} \). Fix \( a = 1 + 2^{-(11+n)} \) and \( b = 1 + 2^{-(10+n)} \). Then there exist collections of closed cubes \( \{Q_k\} \) whose interiors distance from \( \Omega \) such that \( \Omega = \bigcup_k Q_k \) and \( Q_k \subset aQ_k \subset bQ_k \). Moreover, this gives us collections of \( \{Q_k^i\} \) and functions \( \{\eta_k\} \subset \mathcal{D} \), and a decomposition \( f = g + b, b = \sum_k b_k \), such that

(a) \( \bigcup Q_k^i = \Omega \) and the \( \{Q_k^i\} \) have the bounded interior property: every point is contained in at most a fixed number of the \( \{Q_k^i\} \).

(b) \( \chi_\Omega = \sum_k \eta_k \), with \( 0 \leq \eta \leq 1 \) and each function \( \eta_k \) is supported in \( Q_k^i \).

(c) The distribution function \( g \) fulfills the following inequality:

\[
\mathcal{G}_N^0 \langle g \rangle(x) \leq \chi_\Omega \mathcal{G}_N^0 (f)(x) + C \lambda \sum_i \left( \frac{\ell^i + 1}{\ell^i + |x - x_i|} \right)^{\frac{n}{n+1}}.
\]

(d) If \( f \in L^1_{\text{loc}} \), then \( g \in L^\infty \) with \( |g(x)| \leq C \lambda \) for a.e. \( x \in \mathbb{R}^n \).

(e) The distribution function \( b_i \) is defined by \( b_i = (f - P_i) \eta_i \) if \( \ell_i < 1 \), otherwise we set \( b_i = f \eta_i \) with a polynomial \( P_i \in \mathcal{P}_d \) such that \( \int_{\mathbb{R}^n} b_i(x) q(x) dx = 0 \) for all \( q \in \mathcal{P}_d \). Then if \( x \in Q_k^i \), \( \mathcal{G}_N^0 (b_i)(x) \leq C \mathcal{G}_N^0 (f)(x) \) and if \( x \in (Q_k^i)^c \),

\[
\mathcal{G}_N^0 (b_i)(x) \leq C \left( \frac{\ell^i + 1}{\ell^i + |x - x_i|} \right)^{\frac{n}{n+1}}.
\]

Hereafter, \( x_i \) and \( \ell_i \) denote the center and the sidelength of \( Q_i^k \), respectively.

In what follows, we denote \( E_1^k = \{ i \in \mathbb{N} : |Q_k^i| \geq 1/(2^4 n) \} \) and \( E_2^k = \{ i \in \mathbb{N} : |Q_k^i| \leq 1/(2^4 n) \} \), \( F_1^k = \{ i \in \mathbb{N} : |Q_k^i| \geq 1 \} \) and \( F_2^k = \{ i \in \mathbb{N} : |Q_k^i| \leq 1 \} \). Next, we will give the proof of Theorem 4.3.

**Proof of Theorem 4.3** First we assume that \( f \in h^{p(*)} \cap L^2 \). For each \( k \in \mathbb{Z} \), consider the level set \( \Omega_k = \{ x \in \mathbb{R}^n : \mathcal{G}(f)(x) > 2^k \} \). Then it follows that \( \Omega_{k+1} \subset \Omega_k \). By Proposition 4.5, \( f \) admits a Calderón–Zygmund decomposition of degree \( d \) and height \( 2^k \) associated with \( \mathcal{G}_N(f) \),

\[
f = g^k + \sum_i b_i^k, \quad \text{in} \mathcal{D}',
\]

where \( b_i^k = (f - P_i^k) \eta_i^k \) if \( \ell_i^k < 1 \) and \( b_i^k = f \eta_i^k \) if \( \ell_i^k \geq 1 \). We claim that \( g^k \to f \) in both \( h^{p(*)} \) and \( \mathcal{D}' \) as \( k \to \infty \). Indeed, applying Lemma 4.5

\[
\| f - g^k \|_{h^{p(*)}} \leq \sum_i \mathcal{G}_N^0 (b_i^k) \|_{L^{p(*)}}.
\]
Lemma 4.8

For any $P$ on $1242$ Tan J.

Lemma 4.7

If $L$ is bounded by $k$ as $P \rightarrow \infty$. We denote that a polynomial $H$ hence, where the series converges both in $D'$ and pointwise.

Lemma 4.6

If $Q_i^{k+1} \cap Q_j^{(k+1)*} \neq \emptyset$, then $\ell_j^{k+1} \leq 2^4 \sqrt{n} \ell_i^k$ and $Q_i^{(k+1)*} \subset 2^n Q_i^{k+1} \subset \Omega_k$. Moreover, there exists a positive $L$ such that for each $j \in \mathbb{N}$ the cardinality of $\{i \in \mathbb{N} : Q_i^{k+1} \cap Q_j^{(k+1)*} \neq \emptyset\}$ is bounded by $L$.

Lemma 4.7

If $0 < \ell_j^{k+1} < 1$, $\sup_{y \in \mathbb{R}^n} |P_{ij}^{k+1}(y)\eta_j^{k+1}| \leq C2^{k+1}$.

Lemma 4.8

For any $k \in \mathbb{Z}$,

$$\sum_{i \in \mathbb{N}} \left( \sum_{j \in F_2^{k+1}} P_{ij}^{k+1} \eta_j^{k+1} \right) = 0,$$

where the series converges both in $D'$ and pointwise.

By Lemma 4.8 and $\sum_i \eta_i^k = \chi_{\Omega_k}$, we have

$$g^{k+1} - g^k = \left( f - \sum_j b_j^{k+1} \right) - \left( f - \sum_i b_i^k \right)
= \sum_i b_i^k - \sum_j b_j^{k+1}.$$
where the series converges both in \(D'\) and almost everywhere. For almost everywhere \(x \in (\Omega_{k+1})^c\),

\[|f(x)| \leq G_N(f)(x) \leq 2^{k+1}.
\]

By Lemma 4.5 and Lemma 4.8, for all \(i \in \mathbb{N}\)

\[||h^k||_{L^\infty} \leq C2^k.
\]

Next, we consider three cases as follows.

**Case 1** When \(i \in F_1^k\), we rewrite \(h^k_i\) into

\[h^k_i = f \eta^k_i - \sum_{j \in F_1^{k+1}} f \eta^k_{ij} \eta_i - \sum_{j \in F_2^{k+1}} (f - P^k_j) \eta^k_{ij} \eta_i + \sum_{j \in F_2^{k+2}} P^k_{ij} \eta^k_{ij} + \sum_{j \in F_2^{k+2}} P^k_{ij} \eta^k_{ij}.
\]

**Case 2** When \(i \in E_1^k \cap F_2^k\), we rewrite \(h^k_i\) into

\[h^k_i = (f - P^k_i) \eta^k_i - \sum_{j \in F_1^{k+1}} f \eta^k_{ij} \eta_i - \sum_{j \in F_2^{k+1}} (f - P^k_j) \eta^k_{ij} \eta_i + \sum_{j \in F_2^{k+2}} P^k_{ij} \eta^k_{ij}.
\]

**Case 3** When \(i \in E_2^k\), we rewrite \(h^k_i\) into

\[h^k_i = (f - P^k_i) \eta^k_i - \sum_{j \in F_1^{k+1}} (f - P^k_j) \eta^k_{ij} \eta_i + \sum_{j \in F_2^{k+2}} P^k_{ij} \eta^k_{ij}.
\]

In this case, we know that \(\sum_{j \in F_1^{k+1}} f \eta^k_{ij} = 0\). Indeed, when \(j \in F_1^{k+1}\), then \(\ell^k_i < \frac{1}{2^n} \ell^{k+1}\).

Then by Lemma 4.5 it follows that \(Q^k_i \cap Q^k_j \cap Q^k_j = \emptyset\).

Consider Case 1 and Case 2. \(h^k_i\) is supported in \(Q^k_i\) that contains \(Q^k_i\) and all the \(Q^k_j\) that intersect \(Q^k_i\). By Lemma 4.6, if \(Q^k_i \cap Q^k_j \cap Q^k_j \neq \emptyset\), then we have \(\ell^k_i \leq 2^k \sqrt{n} \ell^k_j\) and \(Q^k_j \subset 2^k nQ^k_i \subset \Omega_k\). Fixing \(\gamma = 1 + 2^{-12-n}\), if \(\ell^k_i < \frac{2}{(\gamma - 1)}\), we choose \(Q^k_i = 2^k nQ^k_i\).

Otherwise, we can choose \(Q^k_i = \gamma Q^k_i\). Observe that \(\ell^k_i < 1\) for \(j \in F_2^{k+1}\), then we get that \(Q^k_i \subset Q(x^k_i, a(\ell^k_i + 2))\) for \(j\) fulfilling \(Q^k_i \cap Q^k_j \cap Q^k_j \neq \emptyset\). Hence, \(\supp(h^k_i) \subset Q^k_i \subset \Omega_k\).

Consider Case 3. In this case, \(i \in E_2^k\) and \(j \in F_2^{k+1}\), then by Lemma 4.6, \(\supp(h^k_i) \subset Q^k_i \subset \Omega_k\), where \(Q^k_i = 2^k nQ^k_i\). Furthermore, we also obtain that \(h^k_i\) satisfies the moment conditions \(\int_{\mathbb{R}^n} h^k_i q(x)dx = 0\) for any \(q \in \mathcal{P}_s\). Indeed, from the constructions of \(P^k_i\) and \(P^k_{ij}\), \((f - P^k_i) \eta^k_i\) and \(\sum_{j \in F_2^{k+1}} (f - P^k_j) \eta^k_{ij} \eta_i - \sum_{j \in F_2^{k+2}} P^k_{ij} \eta^k_{ij} \eta_i\) both satisfy the moment conditions.

Let \(\lambda_{i,k} = C2^k\) and \(a_{i,k} = \frac{h^k_i}{\lambda_{i,k}}\). Then it follows that each \(a_{i,k}\) satisfies \(\supp(a_{i,k}) \subset Q^k_i\),

\[||a_{i,k}||_{L^q} \leq |Q^k_i|^\frac{1}{q}\] and the desired moment conditions for small cubes with

\[f = \sum_{i,k} \lambda_{i,k} a_{i,k}.
\]

For convenience, we only need to rearrange \(\{a_{i,k}\}\) and \(\{\lambda_{i,k}\}\).

\[f = \sum_{i,k} \lambda_{i,k} a_{i,k} \equiv \sum_{j=1}^\infty \lambda_j a_j.
\]
To prove the theorem, it remains to show the estimates of coefficients, for any fixed $s \in (0, \infty)\) 

$$ \left\| \left( \sum_{j=1}^{\infty} (\lambda_j \chi_{Q_j})^s \right)^{\frac{1}{s}} \right\|_{L^p(\cdot)} \leq C_s \| f \|_{h^p(\cdot)}.$$

To prove it, first observe that

$$ \left\| \left( \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} (\lambda_i^k \chi_{Q_i})^s \right)^{\frac{1}{s}} \right\|_{L^p(\cdot)} \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} (2^s \chi_{Q_k})^s \right\}^{\frac{1}{s}} \right\|_{L^p(\cdot)}.$$

Recall that $\Omega_{k+1} \subset \Omega_k$ and $\bigcap_{k=1}^{\infty} \Omega_k = 0$. Consequently for $a.e. \ x \in \mathbb{R}^n$, we have

$$ \sum_{k=-\infty}^{\infty} (2^k \chi_{\Omega_k}(x))^s = \sum_{k=-\infty}^{\infty} 2^{ks} \sum_{j=k}^{\infty} \chi_{\Omega_j \setminus \Omega_{j+1}}(x) = (1 - 2^{-s})^{-1} \sum_{j=-\infty}^{\infty} 2^{js} \chi_{\Omega_j \setminus \Omega_{j+1}}(x).$$

Therefore, by the definition of $\Omega_j$ we have

$$ \left\| \left\{ \sum_{k=-\infty}^{\infty} (2^k \chi_{\Omega_k})^s \right\}^{\frac{1}{s}} \right\|_{L^p(\cdot)} \leq C_s \left\| \left\{ \sum_{j=-\infty}^{\infty} (2^j \chi_{\Omega_j \setminus \Omega_{j+1}})^s \right\}^{\frac{1}{s}} \right\|_{L^p(\cdot)}$$

$$= C_s \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \sum_{j=-\infty}^{\infty} \frac{2^j \chi_{\Omega_j \setminus \Omega_{j+1}}}{\lambda} \right)^p(x) \ dx \leq 1 \right\}$$

$$= C_s \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{2^j}{\lambda} \right)^p(x) \ dx \leq 1 \right\}$$

$$\leq C_s \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} \left( \frac{\mathcal{G}_N f(x)}{\lambda} \right)^p(x) \ dx \leq 1 \right\}$$

$$\leq C_s \| f \|_{h^p(\cdot)}.$$

Finally, we prove that any $f \in h^p(\cdot)$ can be decomposed as in the theorem, since from Remark 3.7 we learn that $h^p(\cdot) \cap L^2$ is dense in $h^p(\cdot)$. Thus, we have completed the proof of Theorem 4.3. □

4.2 Finite Atomic Decompositions

The second goal of this section is to discuss that the atomic decomposition norm restricted to finite decompositions is equivalent to the $h^p(\cdot)$-norm on some subspace. For $(p^+ \vee 1) < q < \infty$, the function space $h_{\text{fin}}^{p(\cdot), q}$ is the subspace of $h^{p(\cdot)}$ consisting of all $f$ that have decompositions as finite sums of the special $(p(\cdot), q)$-atoms. By Corollary 4.4, we know that $h_{\text{fin}}^{p(\cdot), q}$ is dense in $h^{p(\cdot)}$.

**Theorem 4.9** Let $p(\cdot) \in \mathcal{P}^0 \cap LH$ and $(p^+ \vee 1) < q < \infty$. For $f \in h_{\text{fin}}^{p(\cdot), q}$, define

$$ \| f \|_{h_{\text{fin}}^{p(\cdot), q}} \equiv \inf \left\{ \left\| \sum_{j=1}^{M} \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)} : f = \sum_{j=1}^{M} \lambda_j a_j \right\},$$

and

$$ \| f \|_{h_{\text{fin}}^{p(\cdot), q}} \equiv \inf \left\{ \left\| \left( \sum_{j=1}^{M} (\lambda_j \chi_{Q_j})^p \right)^{\frac{1}{p^+}} \right\|_{L^p(\cdot)} : f = \sum_{j=1}^{M} \lambda_j a_j \right\},$$

where $0 < a_j < \infty$ for $N_{\text{fin}}$-a.e. $x$. Then

$$ \| f \|_{h_{\text{fin}}^{p(\cdot), q}} \leq C_f \| f \|_{h^{p(\cdot)}}$$

and

$$ \| f \|_{h^{p(\cdot)}} \leq C_f \| f \|_{h_{\text{fin}}^{p(\cdot), q}}.$$
Then, for any $x$ decomposition of $f$. Then
\[ \|f\|_{h^{p(\cdot)}_{\rm fin}} \sim \|f\|_{h^{p(\cdot)}_{\rm fin},q} \sim \|f\|_{h^{p(\cdot)}_{\rm fin},*}. \]

**Proof** We only need to prove that
\[ \|f\|_{h^{p(\cdot)}_{\rm fin}} \sim \|f\|_{h^{p(\cdot)}_{\rm fin},q}. \]
First, it is obvious that
\[ \|f\|_{h^{p(\cdot)}_{\rm fin}} \leq C\|f\|_{h^{p(\cdot)}_{\rm fin},q} \]
for $f \in h^{p(\cdot),q}_{\rm fin}$. Next we only need to prove that for any $f \in h^{p(\cdot),q}_{\rm fin}$,\[ \|f\|_{h^{p(\cdot)}_{\rm fin},q} \leq C\|f\|_{h^{p(\cdot)}_{\rm fin}}. \]
By homogeneity we can assume that $\|f\|_{h^{p(\cdot)}_{\rm fin}} = 1$. To prove this theorem, it suffices to show that $\|f\|_{h^{p(\cdot),q}_{\rm fin}} \leq C$. By Theorem 4.3, since $f \in h^{p(\cdot)}_{\rm fin} \cap L^q$, we can form the following decomposition of $f$ in terms of the special local $(p(\cdot),q)$-atoms:
\[ f = \sum_{i,k} \lambda_{i,k} a_{i,k}, \]
where the series converges in $\mathcal{D}'$ and almost everywhere. If $f \in h^{p(\cdot),q}_{\rm fin}$, then supp $f \subseteq Q(x_0,R_0)$ for fixed $x_0 \in \mathbb{R}^n$ and some $R_0 \in (1,\infty)$. We set $Q_0 = Q(x_0,\sqrt{n}R_0 + 2^{3(n+10)+1})$. Then for any $\psi \in \mathcal{D}_{N,1}$ and $x \in (Q_0)^c$, for $0 < t < 1$, we get that
\[ \psi_t \ast f(x) = \int_{Q(x_0,R_0)} \psi_t(x-y)f(y)dy = 0. \]
Then, for any $x \in (Q_0)^c$, it follows that $x \in (\Omega_k)^c$. Hence, $\Omega_k \subseteq Q_0$, and supp $\sum_{i,k} \lambda_{i,k} a_{i,k} \subseteq Q_0$. Now we claim that $\sum_{i,k} \lambda_{i,k} a_{i,k}$ converges to $f$ in $L^q$. In fact, for any $x \in \mathbb{R}^n$, we can find a $j \in \mathbb{Z}$ such that $x \in \Omega_j \setminus \Omega_{j+1}$. Since supp $a_{i,k} \subseteq Q_{i,k}^c \subseteq \Omega_k \subseteq \Omega_{j+1}$ for all $k > j$, then we have
\[ \left| \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} \lambda_{i,k} a_{i,k} \right| \leq \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{N}} |\lambda_{i,k} a_{i,k}| \leq \sum_{k \leq j} 2^k \leq C 2^j \leq C G_N f. \]
By the fact that $G_N f \in L^q$ and the Lebesgue dominated convergence theorem, we have proved the claim. For each integer $N > 0$, we write
\[ F_N = \{(i,k) : k \in \mathbb{Z}, i \in \mathbb{N}, |k| + i \leq N\}. \]
Then $f_N \equiv \sum_{(i,k) \in F_N} \lambda_{i,k} a_{i,k}$ is a finite combination of the special local $(p(\cdot),q)$-atoms with
\[ \|f_N\|_{h^{p(\cdot)}_{\rm fin},q} \leq \sum_{(i,k) \in F_N} \lambda_{i,k} \chi_{Q_{i,k}} \|_{L^q} \leq C \|f\|_{h^{p(\cdot)}_{\rm fin}} \leq C. \]
Since the series $\sum_{i,k} \lambda_{i,k} a_{i,k}$ converges absolutely in $L^q$, for any given $\epsilon \in (0,\infty)$ there exists $N$ such that $\|f - f_N\|_{L^q} < \frac{\epsilon (Q_0)^{1/q}}{\|\chi_{Q_0}\|_{L^p}}$. Meanwhile, supp $f_N \subseteq Q_0$ together with the support of $f$ implies that supp $(f - f_N) \subseteq Q_0$. So we can divide $Q_0$ into the union of cubes $\{Q_i\}_{i=1}^{N_0}$ with disjoint interior and sidelengths satisfying $\ell_i \in [1,2]$, where $N_0$ depends only on $R_0$ and
\(n\). Particularly, we know that \(g_{N,i} = \frac{\|\chi_{G_i}\|_{L_p}}{\epsilon} (f - f_N) \chi_{G_i}\) is a special local \((p(\cdot), q)\)-atom. Moreover,

\[
\|f - f_N\|_{\mathcal{H}_{p,q}^{\text{fin}}}^p = \left\| \sum_{i=1}^{N_0} \frac{\epsilon}{\|\chi_{G_i}\|_{L_p}} g_{N,i} \right\|_{\mathcal{H}_{p,q}^{\text{fin}}}^p \\
\leq C \sum_{i=1}^{N_0} \left\| \frac{\epsilon \chi_{G_i}}{\|\chi_{G_i}\|_{L_p}} \right\|_{L_p}^p \\
\leq C.
\]

Therefore, we conclude that

\[
f = \sum_{(i,k) \in F_N} \lambda_{i,k}a_{i,k} + \sum_{i=1}^{N_0} \frac{\epsilon}{\|\chi_{G_i}\|_{L_p}} g_{N,i}
\]

is the desired finite atomic decomposition. This finished the proof of Theorem 4.9. \(\square\)

5 Dual Spaces of \(h^{(p(\cdot))}\)

This section is devoted to giving a complete dual theory of \(h^{(p(\cdot))}\) for \(0 < p^- \leq p^+ < \infty\). First we consider the case when \(0 < p^- \leq p^+ \leq 1\) by introducing the local variable Campanato space \(bmo^{\text{loc}}\). Then we establish the local variable Campanato type space \(\widetilde{bmo}^{p(\cdot)}\) when we deal with duality of \(h^{(\cdot)}\) with \(p^+ > 1\) and \(p^- \leq 1\).

5.1 Duality of \(h^{(p(\cdot))}\) with \(p^+ \leq 1\)

In this subsection, we first introduce the local variable Campanato space \(bmo^{\text{loc}}\) and show that the dual space of \(h^{(p(\cdot))}\) is \(bmo^{\text{loc}}\). Furthermore, we also give some equivalent characterization of the dual of local variable Hardy spaces. Before we state the main results in this subsection, it is worth to point out that Nakai and Sawano [27] introduced the generalized local Campanato space and studied that the duals of local Orlicz–Hardy space. We also remark that, very recently, Sun et al. [31] introduced the local John–Nirenberg–Campanato space and show that the local Campanato space is the limit case of this space. Now we begin with some notions and definitions.

Let \(g \in L_{\text{loc}}\), and \(Q\) be the cube in \(\mathbb{R}^n\). Let us point out the fact that there exists a unique \(P \in \mathcal{P}_d\) with the degree not greater than \(d\), such that

\[
\int_Q (g(x) - P(x))q(x)\,dx = 0,
\]

for any \(q \in \mathcal{P}_d\). Denote this unique \(P\) by \(P_{Qg}\).

**Definition 5.1** Suppose that \(p(\cdot) \in LH\), \(0 < p^- \leq p^+ \leq 1 < q < \infty\). Let

\[
\|f\|_{bmo^{p(\cdot)}} = \sup_{|Q| < 1} \frac{|Q|}{\|\chi_Q\|_{L_p}} \left( \frac{1}{|Q|} \int_Q |f(x) - P_Q f(x)|^q \,dx \right)^{\frac{1}{q}} \\
+ \sup_{|Q| \geq 1} \frac{|Q|}{\|\chi_Q\|_{L_p}} \left( \frac{1}{|Q|} \int_Q |f(x)|^q \,dx \right)^{\frac{1}{q}},
\]

where the suprema are taken over all the cubes \(Q \subset \mathbb{R}^n\). Then the function spaces

\[
bmo^{p(\cdot)}(\mathbb{R}^n) = \{ f \in L_{\text{loc}} : \|f\|_{bmo^{p(\cdot)}} < \infty \}
\]
are called the local variable Campanato spaces.

We also introduce the local variable Lipschitz spaces as follows.

**Definition 5.2** Suppose that \( p(\cdot) \in LH, \) \( 0 < p^- \leq p^+ \leq 1 < q < \infty. \) Let
\[
\|f\|_{lip_p(\mathbb{R}^n)} \equiv \sup_{Q \subset \mathbb{R}^n} \sup_{1 \leq |Q| < 1} \frac{|Q| |f(x) - f(y)|}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}} + \sup_{|Q| \geq 1} \frac{|Q| \|f\|_{L^\infty(\mathbb{R}^n)}}{\|\chi_Q\|_{L^p(\mathbb{R}^n)}},
\]
where the first supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) with \( |Q| < 1 \) and the second supremum is taken over all cubes \( Q \subset \mathbb{R}^n \) with \( |Q| \geq 1. \) Then the local variable Lipschitz spaces are defined by setting
\[
lip_p(\mathbb{R}^n) = \{ f \in L_{\text{loc}} : \|f\|_{lip_p(\mathbb{R}^n)} < \infty \}.
\]

**Theorem 5.3** Suppose that \( p(\cdot) \in LH, 0 < p^- \leq p^+ \leq 1 < q < \infty. \) The dual space of \( h^p(\cdot) \) is \( bmo^p(\cdot) \) in the following sense:

1. For any \( g \in bmo^p(\cdot), \) the linear functional \( l_g, \) defined initially on \( h^p_{\text{fin}}(\cdot)^q \) has a unique extension to \( h^p(\cdot) \) with \( \|l_g\| \leq C \|g\|_{bmo^p(\cdot)} \).

2. Conversely, for any \( l \in (h^p(\cdot))^!, \) there exists a unique function \( g \in bmo^p(\cdot) \) such that \( l_g(f) = \langle g, f \rangle \) holds true for all \( f \in h^p_{\text{fin}}(\cdot)^q \) with \( \|g\|_{bmo^p(\cdot)} \leq C \|l\|. \)

Before we prove Theorem 5.3, we need to obtain several lemmas.

**Lemma 5.4** Suppose that \( p(\cdot) \in LH, 0 < p^- \leq p^+ \leq 1 < q < \infty. \) Then, for all \( g \in bmo^p(\cdot) \) and all special local \( (p(\cdot), q)\)-atoms \( a \) of \( h^p(\cdot), \)
\[
\left| \int_{\mathbb{R}^n} a(x)g(x)dx \right| \leq \|\chi_Q\|_{L^p(\cdot)} \|g\|_{bmo^p(\cdot)}.
\]

**Proof** Suppose that \( a \) is any special local \( (p(\cdot), q)\)-atom with \( \text{supp}(a) \subset Q. \) When \( |Q| < 1, \) by the cancellation of \( a \) and Hölder’s inequality, we conclude that
\[
\left| \int_{\mathbb{R}^n} a(x)g(x)dx \right| \leq \int_{\mathbb{R}^n} a(x)(g(x) - P_Qg(x))\chi_Qdx \leq \|a\|_{L^q(\mathbb{R}^n)} \left( \int_{\mathbb{R}^n} |g(x) - P_Qg(x)|^q\chi_Qdx \right)^{\frac{1}{q}} = \|\chi_Q\|_{L^p(\cdot)} \left( \frac{|Q|}{|Q|} \int_Q |g(x) - P_Qg(x)|^qdx \right)^{\frac{1}{q}} \leq \|\chi_Q\|_{L^p(\cdot)} \|g\|_{bmo^p(\cdot)}.
\]

Similarly, when \( |Q| \geq 1, \) then we obtain that
\[
\left| \int_{\mathbb{R}^n} a(x)g(x)dx \right| \leq \|a\|_{L^q(\mathbb{R}^n)} \left( \int_Q |g(x)|^qdx \right)^{\frac{1}{q}} = \|\chi_Q\|_{L^p(\cdot)} \left( \frac{|Q|}{|Q|} \int_Q |g(x)|^qdx \right)^{\frac{1}{q}} \leq \|\chi_Q\|_{L^p(\cdot)} \|g\|_{bmo^p(\cdot)}. \]

**Lemma 5.5** Assume that \( p^+ \leq 1. \) For sequences of numbers \( \{\lambda_j\}_{j=1}^\infty \) and cubes \( \{Q_j\}_{j=1}^\infty, \) we have
\[
\sum_{j=1}^\infty |\lambda_j| \|\chi_{Q_j}\|_{L^p(\cdot)} \leq \left\| \left( \sum_{j=1}^\infty (\lambda_j \chi_{Q_j})^{p^-} \right)^{\frac{1}{p^-}} \right\|_{L^p(\cdot)}. \]
Proof. Write $\kappa = \sum_{j}^{\infty} |\lambda_j|\|\chi_{Q_j}\|_{L^p}$. Since $p^+ \leq 1$, then we obtain that
\[
\int_{\mathbb{R}^n} \left( \sum_{i=1}^{\infty} \left( \frac{|\lambda_i|\chi_{Q_i}(x)}{\kappa} \right)^{p^+} \right)^{\frac{1}{p^+}} dx \geq \int_{\mathbb{R}^n} \sum_{i=1}^{\infty} \left( \frac{|\lambda_i|\chi_{Q_i}(x)}{\kappa} \right)^{p(x)} dx \\
\geq \sum_{i=1}^{\infty} \frac{|\lambda_i|\|\chi_{Q_i}\|_{L^p}}{\kappa} \\
= 1.
\]
Therefore, we obtain the desired result. \hfill \square

Lemma 5.6. For any special local $(p(\cdot), q)$-atom $a$ with $\text{supp}(a) \subset Q$, we have
\[
\|a\|_{L^p} \leq C\|\chi_Q\|_{L^p}.
\]
Proof. Let $\varphi \in \mathcal{D}$ be a nonnegative and radial function supported on $Q(0, 1/2)$ with $\int \varphi(x)dx \neq 0$. Applying Lemma 2.8, the boundedness of the maximal operator $M$ and the fact that $M_a(x) \leq C M_a(x)$ yield that
\[
\|(M\varphi a)\chi_Q\|_{L^p} \leq C\|(Ma)\chi_Q\|_{L^p} \leq C\|\chi_Q\|_{L^p},
\]
where $\tilde{q}()$ is defined by $\frac{1}{p(\cdot)} = \frac{1}{q} + \frac{1}{\tilde{q}(\cdot)}$. Next we need to show that $\|(M\varphi a)\chi_{\tilde{Q}}\|_{L^p} \leq C\|\chi_Q\|_{L^p}$. When $|Q| \leq 1$, by following the standard argument in [26, p. 3682], we obtain that
\[
|(a \ast \varphi_j)(x)| \leq C\frac{\ell(Q)^{n+d+1}}{|x - c_Q|^{n+d+1}}.
\]
where $x \in (\tilde{Q})^c$ and $c_Q$ is the center of the cube $Q$. Hence, we get that
\[
\|(M\varphi a)\chi_{\tilde{Q}}\|_{L^p} \leq C\left\|\frac{\ell(Q)^{n+d+1}}{|x - c_Q|^{n+d+1}}\right\|_{L^p} \\
\leq C\|\chi_Q\|^{(n+d+1)/n}_{L^p} \\
\leq C\|\chi_Q\|_{L^p}.
\]
When $|Q| > 1$, observe that $j \geq 0$, for any special local $(p(\cdot), q)$-atom $a$ with $\text{supp} a \subset Q$ and $x \in (\tilde{Q})^c$, we have
\[
|(a \ast \varphi_j)(x)| \leq \int_{Q} |a(y)\varphi_j(x - y)|dy \\
\leq \sup_{y \in Q} |\varphi_j(x - y)| \int_{Q} |a(y)|dy \\
\leq C\frac{2^{jn}}{(1 + 2^j|x - c_Q|)^M} \|a\|_{L^q} |Q|^\frac{1}{n} \\
\leq C\frac{2^{jn-M}(|\ell(Q)|)^M}{|x - c_Q|^M} \leq C\frac{(|\ell(Q)|)^M}{|x - c_Q|^M}
\]
for any sufficiently large $M > n > 0$. We choose $M$ such that $\frac{Mn}{n} > 1$. Similarly, we can obtain that
\[
\|(M\varphi a)\chi_{\tilde{Q}}\|_{L^p} \leq C\|(MX\tilde{Q})\|_{L^p} \leq C\|\chi_Q\|_{L^p}.
\]
This proves Lemma 5.6.

Now we return to the proof of Theorem 5.3.

Proof of Theorem 5.3 Let \( g \in \text{bmo}^{p'} \) and \( f \in \text{h}_\text{fin}^{p,q} \). Then for some non-negative coefficients \( \{\lambda_j\}_{j=1}^M \) and the special local \((p\cdot),q\)-atoms \( \{a_j\}_{j=1}^M \) with \( \text{supp}(a_j) \subset Q_j \), we have

\[
f = \sum_{j=1}^M \lambda_j a_j
\]
in \( \mathcal{D}' \). By Lemma 5.4 and Lemma 5.5, we conclude that

\[
\left| \int_{\mathbb{R}^n} f(x)g(x)dx \right| = \left| \sum_{j=1}^M \lambda_j \int_{\mathbb{R}^n} a_j(x)g(x)dx \right|
\leq \sum_{j=1}^M |\lambda_j| \left| \int_{\mathbb{R}^n} a_j(x)g(x)\chi_{Q_j}dx \right|
\leq \left( \sum_{j=1}^M |\lambda_j| \|\chi_{Q_j}\|_{L^p(\cdot)} \right) \|g\|_{\text{bmo}^{p'}}
\leq C \|f\|_{\text{h}_\text{fin}^{p,q}} \|g\|_{\text{bmo}^{p'}}.
\]

By Theorem 4.9, we conclude that

\[
|l_g(f)| := |\langle g, f \rangle| \leq C \|f\|_{\text{h}_\text{fin}^{p,q}} \|g\|_{\text{bmo}^{p'}}.
\]
This shows that \( l_g \) can be extended uniquely to a bounded linear functional \( \text{h}_\text{fin}^{p,q} \) with

\[
\|l_g\| \leq C \|g\|_{\text{bmo}^{p'}}.
\]
Thus, the former conclusion of the theorem holds.

Now we prove (2). Fix a cube \( Q \) with \( \ell(Q) \geq 1 \). For any given \( f \in L^q(Q) \) with \( \|f\|_{L^q(Q)} > 0 \), set

\[
a(x) \equiv \frac{|Q|^{\frac{1}{q}} f(x)\chi_{Q}(x)}{\|f\|_{L^q(Q)}}.
\]
Then \( a \) is obviously a special local \((p\cdot),q\)-atom. By Lemma 5.6, we have

\[
|l(a)| \leq |l||\|a\|_{\text{h}_\text{fin}^{p,q}} \leq C \|l\|\|\chi_{Q}\|_{L^p(\cdot)}.
\]
It follows that for any \( l \in (\text{h}_\text{fin}^{p,q})' \),

\[
|l(f)| \leq |l||\|f\|_{\text{h}_\text{fin}^{p,q}} \leq C \|l\|\|f\|_{L^q(Q)}\|\chi_{Q}\|_{L^{p'}(\cdot)}|Q|^{-\frac{1}{q}}.
\]
Hence, \( l \in (L^q(Q))' \) and \( (\text{h}_\text{fin}^{p,q})' \subset (L^q(Q))' \). Since \( 1 < q < \infty \), using the duality \( L^q(Q) - L^q(Q) \), we find that there exists a \( g^Q \in L^{q'}(Q) \) such that for all \( f \in L^q(Q) \),

\[
l(f) = \int_{Q} f(x)g^Q(x)dx,
\]
and \( \|g^Q\|_{L^{q'}(Q)} \leq C \|l\|\|\chi_{Q}\|_{L^{p'}(\cdot)}|Q|^{-\frac{1}{q}} \). Take a sequence \( \{Q_j\}_{j \in \mathbb{N}} \) of cubes such that \( Q_j \subset Q_{j+1} \), \( \bigcup_{j \in \mathbb{N}} Q_j = \mathbb{R}^n \) and \( \ell(Q_1) \geq 1 \). Similarly, we know that there exists a \( g^{Q_j} \in L^{q'}(Q_j) \) such that for each \( Q_j \),

\[
l(f) = \int_{Q_j} f(x)g^{Q_j}(x)dx,
\]
and \( g^{Q_j} \|_{L^{q'}(Q_j)} \leq C \| l \| \| \chi_{Q_j} \|_{L^{q}(\cdot)|Q_j}^{-1/q} \). Then we can construct a function \( g \) such that for all \( f \in L^{q}(Q_j) \)

\[
l(f) = \int_{Q_j} f(x) g(x) dx.
\]

Assume that \( f \in L^{q}(Q_1) \). We have that there exists a \( g^{Q_1} \in L^{q'}(Q_1) \) such that

\[
l(f) = \int_{Q_1} f(x) g^{Q_1}(x) dx.
\]

Observe that \( f \in L^{q}(Q_1) \subset L^{q}(Q_2) \) and it follows that there exists a \( g^{Q_2} \in L^{q'}(Q_2) \) such that

\[
l(f) = \int_{Q_1} f(x) g^{Q_1}(x) dx = \int_{Q_2} f(x) g^{Q_2}(x) dx.
\]

Therefore, for all \( f \in L^{q}(Q_1) \):

\[
\int_{Q_1} f(x)(g^{Q_1}(x) - g^{Q_2}(x)) dx = 0,
\]

which implies that \( g^{Q_1}(x) = g^{Q_2}(x) \), where \( x \in Q_1 \). Set \( g(x) = g^{Q_1}(x) \) when \( x \in Q_1 \) and \( g(x) = g^{Q_2}(x) \) when \( x \in Q_2 \setminus Q_1 \). Then for all \( f \in L^{q}(Q_j) \) with \( j = 1, 2 \), we have

\[
l(f) = \int_{Q_j} f(x) g(x) dx.
\]

Repeating the similar argument, we can obtain a \( g(x) \) such that the above equality holds for all \( j \in \mathbb{N} \), which implies that

\[
l(f) = (g, f) = \int_{\mathbb{R}^n} f(x) g(x) dx
\]

also holds for all \( f \in h_{in}^{p(\cdot),q} \). Then by the equality \( l(a) = \int_{Q} a(x) g(x) dx \) and \( l \in (h_{in}^{p(\cdot)})' \), we obtain that

\[
|l(a)| = \left| \int_{Q} a(x) g(x) dx \right| \leq C \| l \| \| \chi_{Q} \|_{L^{p}(\cdot)}.
\]

The above inequality implies that

\[
\| \chi_{Q} \|_{L^{p}(\cdot)}^{-1} |Q|^{-\frac{1}{q}} \left| \int_{Q} f(x) g(x) dx \right| \leq C \| l \|.
\]

It follows that

\[
\| \chi_{Q} \|_{L^{p}(\cdot)}^{-1} \left( \frac{1}{|Q|} \int_{Q} |g(x)|^{q'} dx \right)^{\frac{1}{q'}} \leq C \| l \|.
\]

When \( |Q| \leq 1 \), by using the similar argument in [26, pp. 3724–p. 3725], we can deduce that

\[
\| \chi_{Q} \|_{L^{p}(\cdot)}^{-1} \left( \frac{1}{|Q|} \int_{Q} |g(x) - P_{Q} g(x)|^{q'} dx \right)^{\frac{1}{q'}} \leq C \| l \|.
\]

Therefore, we have shown that \( g(x) \) belongs to \( bmo^{p(\cdot)} \). This finishes the proof of Theorem 5.3. \( \square \)

Repeating the almost same argument in the proof of Theorem 5.3, we immediately deduce the following theorem and we omit the details.
The local variable Carleson measure space

For more details on Carleson measure spaces, see [15–17, 24, 36].

In this subsection, we introduce a kind of local variable Campanato type space $h_{\text{lip}}^p$. Inspired by [19], we show that the dual space of $h^p$ is $\tilde{bmo}^p$, for all $p(\cdot) \in LH$ fulfilling $0 < p^- \leq p^+ < \infty$. To prove it, we also need to prove that a kind of variable Campanato type space $BMO$ is the dual space of the global variable Hardy space $H^p$, which gives a complete answer to the open question proposed by Izuki et al. in [21]. We begin with some definitions.

**Theorem 5.7** Suppose that $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1 < q < \infty$. The dual space of $h^p$ is $\text{lip}_p$ in the following sense:

1. For any $g \in \text{lip}_p$, the linear functional $l_g$, defined initially on $h^p_{\text{fin}}$, has a unique extension to $h^p$ with $\|l_g\| \leq C\|g\|_{\text{lip}_p}$.

2. Conversely, for any $l \in (h^p)'$, there exists a unique function $g \in \text{lip}_p$ such that $l_g(f) = \langle g, f \rangle$ holds true for all $f \in h^p_{\text{fin}}$ with $\|g\|_{\text{lip}_p} \leq C\|l\|$.

Next, we also state the Carleson measure characterization for the dual of $h^p$. We see that the local variable Carleson measure space $cmo^p$ is the dual space of the local variable Hardy space $h^p$ in [40]. We need some notations. Denote by $\ell(Q) = 2^{-j}$ the side length of $Q = Q_{j,k}$, $k \in \mathbb{Z}^n$. Denote by $z_Q = 2^{-j}k$ the left lower corner of $Q$ and by $x_Q$ is any point in $Q$ when $Q = Q_{j,k}$. Denote $\Pi_j = \{Q : Q = Q_{j,k}\}$ and $\Pi = \bigcup_{j \in \mathbb{N}} \Pi_j$. For any function $\psi$ defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$, and $Q = Q_{j,k}$, set

$$\psi_j(x) = 2^{jn}\psi(2^jx), \quad \psi_Q(x) = |Q|^{\frac{1}{p}}\psi_j(x - z_Q).$$

For more details on Carleson measure spaces, see [15–17, 24, 36].

**Definition 5.8** The local variable Carleson measure space $cmo^p(\mathbb{R}^n)$ is the collection of all $f \in \mathcal{D}'$ fulfilling

$$\|f\|_{cmo^p} \equiv \sup_{P \in \Pi} \left\{ \frac{|P|}{\|\chi_P\|_{p(\cdot)}^2} \int P \sum_{j \in \mathbb{N}} \sum_{Q \in \Pi_j, Q \subset P} |Q|^{-1}|\langle f, \psi_Q \rangle|^2 \chi_Q(x) \, dx \right\}^{\frac{1}{p}} < \infty.$$

**Theorem 5.9** ([40]) Suppose that $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1$. The dual space of $h^p$ is $cmo^p$ in the following sense.

1. For $g \in cmo^p$, the linear functional $l_g$, defined initially on $\mathcal{D}$, extends to a continuous linear functional on $h^p$ with $\|l_g\| \leq C\|g\|_{cmo^p}$.

2. Conversely, every continuous linear functional $l$ on $h^p$ satisfies $l = l_g$ for some $g \in cmo^p$ with $\|g\|_{cmo^p} \leq C\|l\|$.

From Theorems 5.3, 5.7 and 5.9, we immediately deduce the following equivalent definitions of the dual of local variable Hardy spaces.

**Corollary 5.10** Let $p(\cdot) \in LH$ and $0 < p^- \leq p^+ \leq 1 < q < \infty$. Then local variable Campanato spaces $bmo^p$, local variable Lipschitz spaces $\text{lip}_p$ and local variable Carleson measure spaces $cmo^p$ coincide as sets and

$$\|f\|_{bmo^p} \sim \|f\|_{H^p} \sim \|f\|_{cmo^p}.$$
Definition 5.11 Suppose that $p(\cdot) \in LH$, $0 < p^− \leq p^+ < \infty$ and $1 < q < \infty$. Let
\[
\|f\|_{\widehat{BMO}^{p(\cdot)}} \equiv \sup\left\{\sum_{i=1}^{M} \lambda_i \chi_{Q_i} \left\|\chi_{Q_i}\right\|_{L^{p(\cdot)}}^{-1} \left\{\sum_{j=1}^{M} \left|\lambda_j\right| \chi_{Q_j}\left(\frac{1}{|Q_j|} \int_{Q_j} |f(x) - P_{Q_j} f(x)|^q dx\right)^{\frac{1}{q'}}\right\}\right\},
\]
where the supremum is taken over all $M \in \mathbb{N}$, the cubes $Q_j \subset \mathbb{R}^n$, and non-negative numbers $\{\lambda_j\}_{j=1}^{M}$ satisfying $\sum_{j=1}^{M} \lambda_j \|\chi_{Q_j}\|_{L^{p(\cdot)}} \neq 0$. Then the function spaces
\[
\widehat{BMO}^{p(\cdot)}(\mathbb{R}^n) = \{f \in L_{\text{loc}} : \|f\|_{\widehat{BMO}^{p(\cdot)}} < \infty\}
\]
are called the variable Campanato type spaces.

Remark 5.12 Repeating the similar argument in [19, Proposition 2.9 and Corollary 3.10], we find that if $p(\cdot) = p \in (0, 1]$, then the variable Campanato type spaces are just classical Campanato type spaces and that if $p(\cdot) = p \in (1, \infty)$, then the variable Campanato type spaces are just classical Lebesgue spaces.

Definition 5.13 Suppose that $p(\cdot) \in LH$, $0 < p^− \leq p^+ < \infty$ and $1 < q < \infty$. Let
\[
\|f\|_{\widehat{bmo}^{p(\cdot)}} \equiv \sup_{|Q| < 1} \left\{\sum_{i=1}^{M} \lambda_i \chi_{Q_i} \left\|\chi_{Q_i}\right\|_{L^{p(\cdot)}}^{-1} \left\{\sum_{j=1}^{M} \left|\lambda_j\right| \chi_{Q_j}\left(\frac{1}{|Q_j|} \int_{Q_j} |f(x) - P_{Q_j} f(x)|^q dx\right)^{\frac{1}{q'}}\right\}\right\}
\]
+ \sup_{|Q| \geq 1} \left\{\sum_{i=1}^{M} \lambda_i \chi_{Q_i} \left\|\chi_{Q_i}\right\|_{L^{p(\cdot)}}^{-1} \left\{\sum_{j=1}^{M} \left|\lambda_j\right| \chi_{Q_j}\left(\frac{1}{|Q|} \int_{Q} |f(x)|^q dx\right)^{\frac{1}{q'}}\right\}\right\},
\]
where the first supremum is taken over all $M \in \mathbb{N}$, the cubes $Q_j \subset \mathbb{R}^n$ with $|Q| < 1$, and non-negative numbers $\{\lambda_j\}_{j=1}^{M}$ satisfying $\sum_{j=1}^{M} \lambda_j \|\chi_{Q_j}\|_{L^{p(\cdot)}} \neq 0$ and the second supremum is taken over all $M \in \mathbb{N}$, the cubes $Q_j \subset \mathbb{R}^n$ with $|Q| \geq 1$, and non-negative numbers $\{\lambda_j\}_{j=1}^{M}$ satisfying $\sum_{j=1}^{M} \lambda_j \|\chi_{Q_j}\|_{L^{p(\cdot)}} \neq 0$. Then the function spaces
\[
\widehat{bmo}^{p(\cdot)}(\mathbb{R}^n) = \{f \in L_{\text{loc}} : \|f\|_{\widehat{bmo}^{p(\cdot)}} < \infty\}
\]
are called the local variable Campanato type spaces.

Next, we obtain the duality between $H^{p(\cdot)}(\mathbb{R}^n)$ and $\widehat{BMO}^{p(\cdot)}(\mathbb{R}^n)$ with $0 < p^− \leq p^+ < \infty$.

Theorem 5.14 Suppose that $p(\cdot) \in LH \cap P^0$, $(p^+ \vee 1) < q < \infty$. The dual space of $H^{p(\cdot)}$ is $\widehat{BMO}^{p(\cdot)}$ in the following sense:

1. Let $g \in \widehat{BMO}^{p(\cdot)}$. Then the linear functional $L_g : f \rightarrow L_g(f) = (g, f)$, defined initially on $H^{p(\cdot),q}_{\text{fin}}$ has a bounded extension to $H^{p(\cdot)}$ with $\|L_g\| \leq C \|g\|_{\widehat{BMO}^{p(\cdot)}}$.

2. Conversely, for any $L \in (H^{p(\cdot)})'$, there exists a unique function $g \in \widehat{BMO}^{p(\cdot)}$ such that $L_g(f) = (g, f)$ holds true for all $f \in H^{p(\cdot),q}_{\text{fin}}$ with $\|g\|_{\widehat{BMO}^{p(\cdot)}} \leq C \|L\|$. 

Proof To prove Theorem 5.14, we first recall some known results on atomic decomposition characterization of $H^{p(\cdot)}$, which can be found in [7, Proposition 2.4] (also see [9, Theorem 7.8]). Fix $(p^+ \vee 1) < q < \infty$. Given countable collections of cubes $\{Q_j\}^\infty_{j=1}$, of non-negative coefficients $\{\lambda_j\}^\infty_{j=1}$ and of the special $(p(\cdot), q)$-atoms $\{a_j\}^\infty_{j=1}$, if
\[
\left\|\sum_{j=1}^{\infty} \lambda_j \chi_{Q_j}\right\|_{L^{p(\cdot)}} < \infty,
\]
then the series $\sum_{j=1}^{\infty} \lambda_j a_j$ converges in $H^{p(\cdot)}$ and satisfies
\[
\left\| \sum_{j=1}^{\infty} \lambda_j a_j \right\|_{H^{p(\cdot)}} \leq C \left\| \sum_{j=1}^{\infty} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.
\]
Furthermore, the finite atomic variable Hardy spaces $H^{p(\cdot),q}_{\text{fin}}$ are defined to be the set of all $f$ fulfilling that there exists an $M \in \mathbb{N}$, such that $f = \sum_{j=1}^{M} \lambda_j a_j$ with the quasi-norm
\[
\|f\|_{H^{p(\cdot),q}_{\text{fin}}} \equiv \inf \left\{ \sum_{j=1}^{M} \lambda_j \chi_{Q_j} \right\}_{L^{p(\cdot)}} < \infty,
\]
where the infimum is taken over all decompositions of $f$ as above. Then $\|f\|_{H^{p(\cdot),q}_{\text{fin}}}$ and $\|f\|_{H^{p(\cdot)}}$ are equivalent quasi-norms on $H^{p(\cdot),q}_{\text{fin}}$. Now we prove (1). Let $g \in \widetilde{BMO}^{p(\cdot)}$ and $f \in H^{p(\cdot),q}_{\text{fin}}$. Then for some numbers $\{\lambda_j\}_{j=1}^{M}$ and the special $(p(\cdot),q)$-atoms, we have $f = \sum_{j=1}^{M} \lambda_j a_j$. By the atomic decomposition results of $H^{p(\cdot)}$, the cancellation of $a_j$, Hölder’s inequality and the size condition of $a_j$, we get that
\[
|L_g(f)| = \int_{\mathbb{R}^n} f(x)g(x)dx \\
\leq \sum_{j=1}^{M} \lambda_j \int_{\mathbb{R}^n} a_j(x)\left|g(x) - P_{Q_j}g(x)\right|dx \\
\leq \sum_{j=1}^{M} |\lambda_j||Q_j|^{q'(p'-1)} \frac{1}{|Q_j|} \int_{Q_j} \left|g(x) - P_{Q_j}g(x)\right|^q dx,
\]
that is,
\[
|L_g(f)| \leq \left\| \sum_{j=1}^{M} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}} \left\| g \right\|_{\widetilde{BMO}^{p(\cdot)}} \\
\leq C\|f\|_{H^{p(\cdot),q}_{\text{fin}}}, \left\| g \right\|_{\widetilde{BMO}^{p(\cdot)}} \sim C\|f\|_{H^{p(\cdot)}} \left\| g \right\|_{\widetilde{BMO}^{p(\cdot)}},
\]
which implies that (1) holds true. It remains to prove (2). For any $M \in \mathbb{N}$, any cubes $Q_j \subset \mathbb{R}^n$, and non-negative numbers $\{\lambda_j\}_{j=1}^{M}$ with $\sum_{j=1}^{M} \lambda_j \|\chi_{Q_j}\|_{L^{p(\cdot)}} \neq 0$, let $f_j \in L^q(Q_j)$ with $\|f_j\|_{L^q(Q_j)} = 1$ satisfying
\[
\left[ \int_{Q_j} \left|g(x) - P_{Q_j}g(x)\right|^q dx \right]^{\frac{1}{q}} = \int_{Q_j} \left|g(x) - P_{Q_j}g(x)\right| f_j(x) dx
\]
and, for any $x \in \mathbb{R}^n$, define
\[
a_j(x) \equiv \frac{|Q_j|^{1/q}(f_j(x) - P_{Q_j}f_j(x))\chi_{Q_j}}{\|f_j - P_{Q_j}f_j(x)\|_{L^q(Q_j)}}.
\]
Then from the definition of the atom, it follows that $a_j$ is a special $(p(\cdot),q)$-atom and $\sum_{j=1}^{M} \lambda_j a_j \in H^{p(\cdot)}$. For any $L \in (H^{p(\cdot)})'$, repeating the similar argument to that used in the proof of Theorem 5.3, we find that there exists a unique $g \in \widetilde{BMO}^{p(\cdot)}$ such that
\[
L_g(f) = \int_{\mathbb{R}^n} f(x)g(x)dx.
\]
In fact, if $L \in (H^{p(\cdot)})'$, then we know that
\[
L \left( \sum_{j=1}^{M} \lambda_j a_j \right) \leq \|L\| \left\| \sum_{j=1}^{M} \lambda_j a_j \right\|_{H^{p(\cdot)}} \leq C \|L\| \left\| \sum_{j=1}^{M} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.
\]
Also, we have
\[
\sum_{j=1}^{M} \lambda_j |Q_j| \left( \frac{1}{|Q_j|} \int_{Q_j} |g(x) - P_{Q_j}g(x)|^q dx \right)^{\frac{1}{q}}
\]
\[
= \sum_{j=1}^{M} \lambda_j |Q_j|^{\frac{1}{q}} \int_{Q_j} |g(x) - P_{Q_j}g(x)| f_j(x) dx
\]
\[
= \sum_{j=1}^{M} \lambda_j |Q_j|^{\frac{1}{q}} \int_{Q_j} [f_j(x) - P_{Q_j}f_j(x)] g(x) \chi_{Q_j} dx.
\]

Hence, by using the fact that $\|P_{Q_j} f_j(x)\|_{L^q} \leq C \|f_j\|_{L^q}$, we conclude that
\[
\sum_{j=1}^{M} \lambda_j |Q_j| \left( \frac{1}{|Q_j|} \int_{Q_j} |g(x) - P_{Q_j}g(x)|^q dx \right)^{\frac{1}{q}}
\]
\[
\leq C \sum_{j=1}^{M} \lambda_j \int_{Q_j} a_j(x) g(x) dx \sim \sum_{j=1}^{M} \lambda_j L(a_j) \sim L \left( \sum_{j=1}^{M} \lambda_j a_j \right)
\]
\[
\leq C \|L\| \left\| \sum_{j=1}^{M} \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}},
\]
which implies that $g \in \widetilde{BMO}^{p(\cdot)}$ with $\|g\|_{\widetilde{BMO}^{p(\cdot)}} \leq C \|L\|$. Therefore, we have completed the proof of Theorem 5.14. \qed

**Remark 5.15** Very recently, Zhang et al. [48] introduced a new ball Campanato-type function spaces $L_{X,q,d,s}$ associated with $X$ and proved that these spaces are the dual spaces of Hardy spaces $H_X$ associated with $X$ under some additional assumptions, where $X$ is a ball quasi-Banach function space on $\mathbb{R}^n$. When $X := L^{p(\cdot)}$, [48, Theorem 3.14] is essentially equivalent to Theorem 5.14. In particular, here we can choose $s = 1$ with the help of the atomic decomposition of $H^{p(\cdot)}$ in [7, Proposition 2.4].

From Theorem 5.3 and Theorem 5.14, by applying nearly identical method to the above proofs, we can deduce the duality between $h^{p(\cdot)}(\mathbb{R}^n)$ and $\widetilde{bmo}^{p(\cdot)}(\mathbb{R}^n)$ with $0 < p^- \leq p^+ < \infty$.

**Theorem 5.16** Suppose that $p(\cdot) \in \mathcal{L}H \cap \mathcal{P}^0$, $(p^+ \vee 1) < q < \infty$. The dual space of $h^{p(\cdot)}$ is $\widetilde{bmo}^{p(\cdot)}$ in the following sense:

1. Let $g \in \widetilde{bmo}^{p(\cdot)}$. Then the linear functional $L_g : f \to l_g(f) = \langle g, f \rangle$, defined initially on $h^{p(\cdot),q}_{\text{lin}}$ has a bounded extension to $h^{p(\cdot)}$ with \( \|l_g\| \leq C\|g\|_{\widetilde{bmo}^{p(\cdot)}}. \)

2. Conversely, for any $L \in (h^{p(\cdot),q})'$, there exists a unique function $g \in \widetilde{bmo}^{p(\cdot)}$ such that $L_g(f) = \langle g, f \rangle$ holds true for all $f \in h^{p(\cdot),q}_{\text{lin}}$ and $\|g\|_{\widetilde{bmo}^{p(\cdot)}} \leq C\|L\|$.

6 Boundedness of Some Operators

In this section, we will consider the boundedness of the inhomogeneous Calderón–Zygmund singular integrals and the local fractional integrals. First we recall the inhomogeneous Calderón–
Zygmund singular integrals in [11, 25]. Precisely, the operator $T$ is said to be an inhomogeneous Calderón–Zygmund singular integral if $T$ is a continuous linear operator from $\mathcal{D}$ to $\mathcal{D}'$ defined by

$$\langle T(f), g \rangle = \int K(x, y) f(y) g(x) \, dx \, dy$$

for all $f, g \in \mathcal{D}(\mathbb{R}^n)$ with disjoint supports, where $K(x, y)$, the kernel of $T$, satisfies the conditions as follows.

$$|K(x, y)| \leq C \min \left\{ \frac{1}{|x-y|^{n-\delta}}, \frac{1}{|x-y|^\delta} \right\} \quad \text{for some } \delta > 0 \text{ and } x \neq y$$

and for $\epsilon \in (0, 1)$

$$|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \leq C \frac{|y-y'|^\epsilon}{|x-y|^{n+\epsilon}},$$

when $|y-y'| \leq \frac{1}{2}|x-y|.$

The first result of this section is as follows.

**Theorem 6.1** Suppose that $p(\cdot) \in \text{LH}$ and \{\frac{n}{n+\epsilon} \leq p^- \leq p^+ < \infty. Let $T$ be an inhomogeneous Calderón–Zygmund singular integral. If $T$ is a bounded operator on $L^2$, then $T$ can be extended to an $(h^{p(\cdot)} - L^{p(\cdot)})$ bounded operator. That is, there exists a constant $C$ such that

$$\|T(f)\|_{L^{p(\cdot)}} \leq C \|f\|_{h^{p(\cdot)}}.$$

**Proof** Recalling the atomic decomposition of local Hardy space $h^{p(\cdot)}$ in Theorem 4.3, we know that if $f \in h^{p(\cdot)}$, then there exist non-negative coefficients $\{\lambda_j\}_{j=1}^\infty$ and the special local $(p(\cdot), q)$-atoms $\{a_j\}_{j=1}^\infty$ such that $f = \sum_{j=1}^\infty \lambda_j a_j$ in $h^{p(\cdot)} \cap L^q$, and that

$$\left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}} \leq C \|f\|_{h^{p(\cdot)}}$$

for $0 < q < \infty$. Then for $x \in \mathbb{R}^n$, we have

$$|T(f)(x)| \leq \sum_j |\lambda_j| |T(a_j)(x)| \chi_{Q_j}(x) + \sum_j |\lambda_j||T(a_j)(x)||\chi_{\tilde{Q}_j}(x)| =: I + II.$$

To prove the theorem, it will suffice to prove that

$$\|T(f)\|_{L^{p(\cdot)}} \leq C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.$$

First we need prove that

$$\|I\|_{L^{p(\cdot)}} \leq C \left\| \sum_{j=1}^\infty \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.$$

Observe that $K(x, y)$, the kernel of $T$, satisfies the following conditions:

$$|K(x, y)| \leq C \min \left\{ \frac{1}{|x-y|^{n-\delta}}, \frac{1}{|x-y|^\delta} \right\} \leq C \frac{1}{|x-y|^{n}} \quad \text{for some } \delta > 0 \text{ and } x \neq y,$$

and for $\epsilon \in (0, 1)$

$$|K(x, y) - K(x, y')| + |K(y, x) - K(y', x)| \leq C \frac{|y-y'|^\epsilon}{|x-y|^{n+\epsilon}},$$
when \(|y - y'| \leq \frac{1}{2}|x - y|\). Also, \(T\) is a bounded operator on \(L^2\). From the Calderón–Zygmund real method in [25, Section 7.3], we get that \(T\) is also bounded on \(L^q\) for any \(1 < q < \infty\). Fix atoms \(a_j\) supported in cubes \(Q_j\). For any \((p^+ + 1) < q < \infty\). Then we have

\[
\left( \frac{1}{|Q_j|} \int_{Q_j} |T(a_j)(x)|^q dx \right)^{\frac{1}{q}} \leq \frac{1}{|Q_j|^{\frac{1}{p^+}}} \|a_j\|_{L_p} \leq C.
\]

Applying Lemma 2.7, we get that

\[
\|I\|_{L^p(\cdot)} \leq \left\| \sum_j |\lambda_j||T(a_j)||\chi_{Q_j}||_{L^p(\cdot)} \right\|
\]

\[
\leq C \left\| \sum_j |\lambda_j| \left( \frac{1}{|Q_j|} \int_{Q_j} |T(a_j)(x)|^q dx \right)^{\frac{1}{q}} \chi_{Q_j} \right\|_{L^p(\cdot)}
\]

\[
\leq C \left\| \sum_j |\lambda_j| \chi_{Q_j} \right\|_{L^p(\cdot)}
\]

\[
\leq C \left\| \sum_j |\lambda_j| \chi_{Q_j} \right\|_{L^p(\cdot)}.
\]

To estimate the term \(II\), we will divide in the following two cases.

**Case 1** \(|Q_j| \leq 1\). In this case, \(a_j\) satisfies the vanishing moment condition. Noting that \(x \in \tilde{Q}_j\) and \(c_{Q_j}\) is the center of \(Q_j\), we have \(|x - c_{Q_j}| \geq 2|y - c_{Q_j}|\) and \(|y - c_{Q_j}| \leq \ell(Q)\). By using the smooth condition of the kernel \(\mathcal{K}\) we obtain that

\[
|T(a_j)(x)| = \left| \int_{Q_j} \mathcal{K}(x, y)a_j(y)dy \right|
\]

\[
\leq \int_{Q_j} |\mathcal{K}(x, y) - \mathcal{K}(x, c_{Q_j})||a_j(y)|dy
\]

\[
\leq C \int_{Q_j} \frac{|y - c_{Q_j}|^\epsilon}{|x - c_{Q_j}|^{n+\epsilon}}|a_j(y)|dy
\]

\[
\leq C \|a_j\|_{L^\infty} \frac{\ell(Q_j)^{n+\epsilon}}{|x - c_{Q_j}|^{n+\epsilon}}
\]

\[
\leq C \left( \frac{\ell(Q_j)^n}{|x - c_{Q_j}|^n} \right)^{\frac{n+\epsilon}{n}}.
\]

**Case 2** \(|Q_j| \geq 1\). In this case, we have \(|x - y| \sim |x - c_{Q_j}|\) and \(|x - y| \geq 1/2\) where \(x \in \tilde{Q}_j\) and \(y \in Q_j\). By using the size condition of \(\mathcal{K}\), for any \(x \in \tilde{Q}_j\) we obtain that

\[
|T(a_j)(x)| = \left| \int_{Q_j} \mathcal{K}(x, y)a_j(y)dy \right|
\]

\[
\leq \int_{Q_j} |\mathcal{K}(x, y)||a_j(y)|dy
\]

\[
\leq C \frac{|Q_j|}{|x - c_{Q_j}|^{n+\delta}}
\]

\[
\leq C \frac{\ell(Q_j)^{n+\delta}}{|x - c_{Q_j}|^{n+\delta}}.
\]
\[ \sim (M(\chi_{Q_j})(x))^\frac{n+d}{n}. \]

We denote \( \gamma = \left\{ \frac{n+d}{n+\varepsilon} \right\}. \) The condition \( \left\{ \frac{n+d}{n+\varepsilon} \lor \frac{n+d}{n+\delta} \right\} < p^- \) means that \( \gamma p^- > 1. \) Then we conclude that
\[
\| II \|_{L^p(\cdot)} \leq C \left\| \sum_j |\lambda_j| M^\gamma(\chi_{Q_j}) \right\|_{L^p(\cdot)}
\leq C \left( \sum_j |\lambda_j| M^\gamma(\chi_{Q_j}) \right)^\frac{1}{\gamma} \| T \|_{L^\gamma p(\cdot)}
\leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)}.
\]

Therefore, together with Remark 3.7 and a density argument, we finish the proof of Theorem 6.1. \( \square \)

Next, we will establish mapping properties from the variable local Hardy space \( h^p(\cdot) \) into itself for the inhomogeneous Calderón–Zygmund singular integral \( T. \) To state it, we also need to assume one additional condition on \( T, \int_{\mathbb{R}^n} T(a)(x)dx = 0 \) for the special local \( (p(\cdot), q) \)-atoms \( a \) and \( \text{supp} a \subset Q \) with \( |Q| < 1. \) For convenience, we write \( T^p_{\text{loc}}(1) = 0, \) if \( T \) satisfies the above moment condition.

**Theorem 6.2** Suppose that \( p(\cdot) \in LH \) and \( \left\{ \frac{n+d}{n+\varepsilon} \lor \frac{n+d}{n+\delta} \right\} < p^- \leq p^+ < \infty. \) Let \( T \) be an inhomogeneous Calderón–Zygmund singular integral. If \( T \) is bounded operator on \( L^2 \) and \( T^p_{\text{loc}}(1) = 0, \) then \( T \) has a unique extension on \( h^p(\cdot) \) and, moreover, there exists a constant \( C \) such that
\[ \| T(f) \|_{h^p(\cdot)} \leq C \| f \|_{h^p(\cdot)}, \]
for all \( f \in h^p(\cdot). \)

**Proof** By the argument similar to that used in the above proof, it will suffice to prove that, for \( h^p(\cdot) \cap L^q \) with \( (p^+ \lor 1) < q < \infty, \)
\[
\| G^0_N T(f) \|_{L^p(\cdot)} \leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^p(\cdot)}.
\]

We claim that for \( x \in \mathbb{R}^n, \) we have
\[
\sup_{0 < t < 1} |t^{-n} \psi(t^{-1} \cdot) \ast T(f)(x)| \leq \sum_j \lambda_j (M(T(a_j))(x)\chi_{2\sqrt{n}Q_j}(x) + (M(\chi_{Q_j})(x))^\gamma),
\]
where \( \gamma = \left\{ \frac{n+d}{n} \lor \frac{n+d}{n+1} \right\}. \) Assuming the claim for the moment and repeating the nearly identical argument to the proof of Theorem 6.1, we can obtain the desired result. To this end, we only need to show the claim. When \( x \in 2\sqrt{n}Q_j, \) we just need the pointwise estimate \( G^0_N T(f)(x) \leq C \sum_j \lambda_j M(T(a_j))(x). \) When \( x \in (2\sqrt{n}Q_j)^c, \) we will consider two cases: \( |Q_j| < 1 \) and \( |Q_j| \geq 1. \)

Consider the case \( |Q_j| \geq 1. \) In this case, \( \ell(Q_j) \geq 1, \) then we have
\[
|\psi_t \ast T(a_j)(x)| = \left| \int_{\mathbb{R}^n} \psi_t(x-y)T(a_j)(y)dy \right|
\leq t^{-n} \int_{B(x,t)} |T(a_j)(y)|dy \leq \sup_{y \in B(x,t)} |T(a_j)(y)|.
\]
Noting that $0 < t < 1 \leq \ell(Q_j)$ and $x \in (2\sqrt{n}Q_j)^c$, it implies that $y \in (\tilde{Q})^c$. From the proof of Theorem 6.1, we conclude that

$$\sup_{y \in B(x,t)} |T(a_j)(y)| \leq C(M(\chi_{Q_j})(x))^\gamma.$$ 

Consider the case $|Q_j| < 1$. If $0 < t \leq |x - c_{Q_j}|/2$, then together with $x \in (2\sqrt{n}Q_j)^c$ we can get that $y \in (\tilde{Q})^c$. Thus, repeating the same argument as used above, then we have

$$|\psi_t \ast T(a_j)(x)| \leq C(M(\chi_{Q_j})(x))^\gamma.$$ 

Finally, we consider the case that $t > |x - c_{Q_j}|/2$. Observing that $\ell(Q_j) < 1$, it follows that $a_j$ has the vanishing moment condition and $T_\epsilon^\alpha[Q_j](1) = 0$. We write $\psi \equiv (\epsilon \wedge \delta)$. For any $x \in (2\sqrt{n}Q_j)^c$, then by using the mean value theorem together with Hölder’s inequality yield that

$$|\psi_t \ast T(a_j)(x)| = \left| \int_{\mathbb{R}^n} (\psi_t(x) - \psi_t(x - c_{Q_j}))T(a_j)(y)dy \right|$$

$$\leq t^{-n} \int_{\mathbb{R}^n} \left| \frac{y - c_{Q_j}}{t} \right| |\psi'((x - c_{Q_j} + \theta(c_{Q_j} - y))/t)||T(a_j)(y)||dy$$

$$\leq C|x - c_{Q_j}|^{-n-1} \left( \int_{Q_j} |y - c_{Q_j}||T(a_j)(y)||dy + \int_{(Q_j)^c} |y - c_{Q_j}||T(a_j)(y)||dy \right)$$

$$\leq C|x - c_{Q_j}|^{-n-1} \left( \ell(Q_j)^\frac{n}{n+1} ||T(a_j)||_{L^s} + \int_{(Q_j)^c} \frac{\ell(Q_j)^{n+\eta}}{|y - c_{Q_j}|^{n+\eta-1}}dy \right)$$

$$\leq C|x - c_{Q_j}|^{-n-1} \ell(Q_j)^{n+1}$$

where $\theta \in (0,1)$ and $s \in (1,\infty)$.

Therefore, we complete the proof of Theorem 6.2. 

We now show that the local fractional integrals are bounded from $h^{p(\cdot)}$ to $L^{q(\cdot)}$ when $q^- > 1$, and from $h^{p(\cdot)}$ to $h^{q(\cdot)}$ when $q^- \leq 1$. The following local fractional integral is introduced by D. Yang and S. Yang [46].

**Definition 6.3** Let $\alpha \in [0, n)$ and let $\phi_0 \in D$ be such that $\phi_0 \equiv 1$ on $Q(0,1)$ and supp $(\phi_0) \subset Q(0,2)$. The local fractional integral $I_{\alpha}^\text{loc}(f)$ of $f$ is defined by

$$I_{\alpha}^\text{loc}(f)(x) \equiv \int_{\mathbb{R}^n} \frac{\phi_0(y)}{|y|^{n-\alpha}} f(x - y)dy.$$ 

**Theorem 6.4** Let $0 < \alpha < n$. Suppose that $p(\cdot) \in LH \cap P^0$, and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ for any $x \in \mathbb{R}^n$. Then $I_{\alpha}^\text{loc}$ admits a bounded extension from $h^{p(\cdot)}$ to $L^{q(\cdot)}$ when $1 < q^- \leq q^+ < \infty$. Furthermore, when $0 < q^- \leq q^+ \leq 1$, $I_{\alpha}^\text{loc}$ admits a bounded extension from $h^{p(\cdot)}$ to $h^{q(\cdot)}$.

**Proof** The proof of this theorem is similar to the proof of Theorems 6.1 and 6.2 and so we only need to concentrate on the differences. First we consider the case when $1 < q^- \leq q^+ < \infty$. By the atomic decomposition of $h^{p(\cdot)}$ and a dense argument, in order to show $I_{\alpha}^\text{loc}$ admits a bounded extension from $h^{p(\cdot)}$ to $L^{q(\cdot)}$, we only need to prove that

$$\left\| \sum_j \lambda_j I_{\alpha}^\text{loc}(a_j) \right\|_{L^{q(\cdot)}} \leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.$$ 

To prove it, we will consider two cases for $\ell(Q_j)$. 

---

*Note: The above text is rendered with clear and consistent formatting, ensuring readability and coherence.*
Case 1  \( \ell(Q_j) > 1 \). In this case, from the definition of \( I^\text{loc}_\alpha(a_j) \), we know that

\[
\text{supp}(I^\text{loc}_\alpha(a_j)) \subset Q_j(c_{Q_j}, \ell(Q_j) + 4) \subset 10\ell(Q_j).
\]

Then we have

\[
\left\| \sum_j \lambda_j I^\text{loc}_\alpha(a_j) \right\|_{L^{q}(\cdot)} = \left\| \sum_j \lambda_j |I^\text{loc}_\alpha(a_j)| \chi_{10Q_j} \right\|_{L^{q}(\cdot)}
\]

\[
\leq C \left\| \sum_j \lambda_j \left( \frac{1}{Q_j} \int_{Q_j} |I^\text{loc}_\alpha(a_j)|^q \, dx \right)^{\frac{1}{q}} \chi_{10Q_j} \right\|_{L^{q}(\cdot)}
\]

\[
\leq C \left\| \sum_j \lambda_j \ell(Q_j)^{\alpha} \chi_{10Q_j} \right\|_{L^{q}(\cdot)}
\]

\[
\leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^{p}(\cdot)},
\]

where the second inequality follows from the boundedness of \( I^\text{loc}_\alpha \) on classical Lebesgue spaces ([46, Lemma 8.9]) and the last inequality follows from [29, Lemma 5.2].

Case 2  \( \ell(Q_j) \leq 1 \). For any special \((p(\cdot), q)-\text{atom} \ a_j\), we have the following pointwise estimates:

\[
|I^\text{loc}_\alpha(a_j)(x)| \leq C \frac{\ell(Q_j)^{n+d+1}}{(\ell(Q_j) + |x - c_{Q_j}|)^{n+d+1-\alpha}} \leq C \ell(Q_j)^{\alpha} (M \chi_{Q_j}(x))^r,
\]

where \( r = \frac{n+d+1-\alpha}{n+1} \). In fact, when \(|x - c_{Q_j}| \leq \ell(Q_j)\), by using the size condition of \( a_j \), we obtain that

\[
|I^\text{loc}_\alpha(a_j)(x)| \leq C \int_{Q_j} \frac{1}{|x-y|^{n-\alpha}} |a_j(y)| \, dy \leq C \ell(Q_j)^{\alpha}.
\]

Let \( P_N(y) \) be the Taylor polynomial of degree \( d \) of the kernel of \( I^\text{loc}_\alpha \) centered at \( c_{Q_j} \). When \(|x - c_{Q_j}| > \ell(Q_j)\), by using the moment condition of \( a_j \) and the Taylor expansion theorem, we have

\[
|I^\text{loc}_\alpha(a_j)(x)| \leq C \int_{Q_j} \frac{1}{|x-y|^{n-\alpha}} - P_N(y) \, |a_j(y)| \, dy
\]

\[
\leq C \int_{Q_j} \frac{1}{|x-c_{Q_j}|^{n+d+1-\alpha}} |y - c_{Q_j}|^{d+1} \, dy
\]

\[
\leq C \frac{\ell(Q_j)^{n+d+1}}{|x-c_{Q_j}|^{n+d+1-\alpha}}.
\]

Similarly, we have

\[
\left\| \sum_j \lambda_j I^\text{loc}_\alpha(a_j) \right\|_{L^{q}(\cdot)} \leq C \left\| \sum_j \lambda_j \ell(Q_j)^{\alpha} (M \chi_{Q_j})^r \right\|_{L^{q}(\cdot)}
\]

\[
\leq C \left\| \sum_j \lambda_j \ell(Q_j)^{\alpha} \chi_{Q_j} \right\|_{L^{q}(\cdot)}
\]

\[
\leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^{p}(\cdot)}.
\]
Thus, we have proved the first part of the theorem. Now we consider the boundedness of $I_{\alpha}^0$ from $h^{p(\cdot)}$ to $h^{q(\cdot)}$. To end this, we need to prove that
\[
\left\| \sum_j \lambda_j \mathcal{G}_N(I_{\alpha}^0(a_j)) \right\|_{L^{p(\cdot)}} \leq C \left\| \sum_j \lambda_j \chi_{Q_j} \right\|_{L^{p(\cdot)}}.
\]
Observe that when $\ell(Q_j) > 1$, from the definition of $\mathcal{G}_N(I_{\alpha}^0(a_j))$, we see that
\[
\text{supp} (I_{\alpha}^0(a_j)) \subset Q_j (cQ_j, \ell(Q_j) + 8) \subset 20\ell(Q_j),
\]
which yields that $\| I_{\alpha}^0(f) \|_{L^{p(\cdot)}} \leq C \| \sum_j \lambda_j \chi_{Q_j} \|_{L^{p(\cdot)}}$. When $\ell(Q_j) \leq 1$, we follow the same proof as in [7, Theorem 1.5] (also see [39, Proposition 3.1]), then we can obtain the desired results.

Therefore, the proof of Theorem 6.4 is completed. \hfill \Box

Finally, we present the boundedness of the inhomogeneous Calderón–Zygmund operators and the local fractional integrals on the duals of $h^{p(\cdot)}$. Precisely, we can obtain that the inhomogeneous Calderón–Zygmund operators are bounded on $bmo^{p(\cdot)}$ and the local fractional integrals are bounded from $bmo^{p(\cdot)}$ to $bmo^{p(\cdot)}$ and bounded from $L^{q(\cdot)'}$ to $bmo^{p(\cdot)}$ under some conditions.

By Theorems 6.2, 6.4, Corollary 5.10 and [40, Proposition 3.2], we have the following results.

**Corollary 6.5** Suppose that $p(\cdot) \in LH$ and $\max\left\{ \frac{n}{n+\alpha}, \frac{n}{n+\beta} \right\} < p^- \leq p^+ \leq 1$. Let $T$ be an inhomogeneous Calderón–Zygmund singular integral. If $T$ is a bounded operator on $L^2$ and $T^{loc}(1) = 0$, then there exists a constant $C$ such that
\[
\| T(f) \|_{bmo^{p(\cdot)}} \leq C \| f \|_{bmo^{p(\cdot)}},
\]
for all $f \in bmo^{p(\cdot)}$.

**Corollary 6.6** Let $0 < \alpha < n$. Suppose that $p(\cdot) \in LH$, $0 < p^- \leq p^+ \leq 1$ and $\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{n}$ for any $x \in \mathbb{R}^n$. For $0 < q^- \leq q^+ \leq 1$, there exists a constant $C$ such that
\[
\| I_{\alpha}^{loc}(f) \|_{bmo^{p(\cdot)}} \leq C \| f \|_{bmo^{p(\cdot)}},
\]
for all $f \in bmo^{p(\cdot)}$. For $p^+ < 1 < q^- \leq q^+ < \infty$, there exists a constant $C$ such that
\[
\| I_{\alpha}^{loc}(f) \|_{bmo^{p(\cdot)}} \leq C \| f \|_{L^{q(\cdot)'}},
\]
for all $f \in L^{q(\cdot)'}$.

**Acknowledgements** The author wishes to express his heartfelt thanks to the anonymous reviewers for their carefully reading and so valuable comments which significantly improve the quality of the paper.

**References**

[1] Bennett, C., Sharpley, R.: Interpolation of Operators, Pure and Applied Mathematics, Vol. 129, Academic Press, Inc., Boston, MA, 1988

[2] Bownik, M.: Anisotropic Hardy spaces and wavelets. *Mem. Amer. Math. Soc.*, **164**(781), 1–122 (2003)

[3] Bui, H.: Weighted Hardy spaces. *Math Nachr.*, **103**, 45–62 (1981)

[4] Coifman, R.: A real variable characterization of $Hp$. *Studia Math.*, **51**, 269–274 (1974)

[5] Cruz-Uribe, D., Fiorenza, A.: Variable Lebesgue Spaces: Foundations and Harmonic Analysis, Birkhäuser, Basel, 2013
[6] Cruz-Uribe, D., Fiorenza, A., Martell, J., et al.: The boundedness of classical operators on variable $L^p$ spaces, *Ann. Acad. Sci. Fenn. Math.*, 31, 239–264 (2006)

[7] Cruz-Uribe, D., Moen, K., Nguyen, H. V.: A new approach to norm inequalities on weighted and variable Hardy spaces. *Ann. Acad. Sci. Fenn. Math.*, 45, 175–198 (2020)

[8] Cruz-Uribe, D., Moen K., Nguyen, H. V.: The boundedness of multilinear Calderón–Zygmund operators on weighted and variable Hardy spaces. *Publ. Mat.*, 63(2), 679–713 (2019)

[9] Cruz-Uribe, D., Wang, L.: Variable Hardy spaces. *Indiana Univ. Math. J.*, 63, 447-493 (2014)

[10] Ding, W., Han Y., Zhu, Y.: Boundedness of singular integral operators on local Hardy spaces and dual spaces. *Potential Anal.*, 55(3), 419–441 (2021)

[11] Grafakos, L., Kalton, N.,: Multilinear Calderón–Zygmund operators on Hardy spaces. *Collect. Math.*, 52(2), 169–179 (2001)

[12] Han, Y.-C., Han, Y.-S., Li, J.: Criterion of the boundedness of singular integrals on spaces of homogeneous type. *J. Funct. Anal.*, 271(12), 3423–3464 (2016)

[13] Han, Y.-C., Han, Y.-S., Li, J., et al.: Hardy and Carleson measure spaces associated to operators on spaces of homogeneous type. *Math. Nachr.*, 294(5), 900–955 (2021)

[14] Kováčik, O., Rákosník, J.: On spaces $L^p(x)$ and $W^{k,p(x)}$, *Czechoslovak Math. J.*, 41, 592–618 (1991)

[15] Lee, M.-Y., Lin, C.-C.: Carleson measure spaces associated to para-accretive functions, *Commun. Contemp. Math.*, 14(1), 1250002, 1–19 (2012)

[16] Meyer, Y.: Ondelettes et opérateurs. II. (French) OPCA, Hermann, Paris, 1990.

[17] Nakai, E., Sawano, Y.: Hardy spaces with variable exponent, In: Harmonic Analysis and Nonlinear Partial Differential Equations, RIMS Kôkyûroku Bessatsu B42, Res. Inst. Math. Sci., Kyoto, 2013, 109–136.

[18] Sawano, Y.: Atomic decompositions of Hardy spaces with variable exponents and generalized Campanato spaces. *J. Funct. Anal.*, 262, 3665–3748 (2012)

[19] Sawano, Y.: Orlicz–Hardy spaces and their duals, *Sci. China Math.*, 57(5), 903–962 (2014)

[20] Stein, E. M.: Harmonic Analysis: Real-Variables Methods. Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton, 1993

[21] Stein, E. M., Weiss, G.: On the theory of harmonic functions of several variables. I. The theory of $H^p$-spaces. *Acta Math.*, 103, 25–62 (1960)

[22] Tan, J.: Atomic decomposition of variable Hardy spaces via Littlewood–Paley–Stein theory, *Ann. Funct. Anal.*, 9(1), 87–100 (2018)
[35] Tan, J.: Atomic decompositions of localized Hardy spaces with variable exponents and applications, *J. Geom. Anal.*, **29**(1), 799–827 (2019)

[36] Tan, J.: Carleson measure spaces with variable exponents and their applications, *Integr. Equat. Oper. Th.*, **91**(5), Paper No. 38, 27 pp. (2019)

[37] Tan, J.: Boundedness of multilinear fractional type operators on Hardy spaces with variable exponents. *Anal. Math. Phys.*, **10**(4), Paper No. 70, 16 pp. (2020)

[38] Tan, J.: Boundedness of maximal operator for multilinear Calderón–Zygmund operators on products of variable Hardy spaces. *Kyoto J. Math.*, **60**(2), 561–574 (2020)

[39] Tan, J.: Weighted Hardy and Carleson measure spaces estimates for fractional integrations. *Publ. Math. Debrecen*, **98**(3–4), 313–330 (2021)

[40] Tan, J., Wang, H., Liao, F.: The continuity of pseudo-differential operators on local variable Hardy spaces and their dual spaces, preprint (2021)

[41] Tan, J., Zhao, J.: Multilinear pseudo-differential operators on product of local Hardy spaces with variable exponents *J. Pseudo-Differ. Oper. Appl.*, **10**(2), 379–396 (2019)

[42] Tang, L.: Weighted local Hardy spaces and their applications, *Illinois J. Math.*, **56**(2), 453–495 (2012)

[43] Wang, F., Yang, D., Yang, S.: Applications of Hardy spaces associated with ball quasi-Banach function spaces. *Results Math.*, **75**(1), Paper No. 26, 58 pp. (2020)

[44] Yang, D., Liang, Y., Ky, L. D.: Real-Variable Theory of Musielak–Orlicz Hardy Spaces. Lecture Notes in Mathematics, Vol. 2182. Springer, Cham, 2017

[45] Yang, D., Zhuo, C., Nakai, E.: Characterizations of variable exponent Hardy spaces via Riesz transforms, *Rev. Mat. Complut.*, **29**(2), 245–270 (2016)

[46] Yang, D., Yang, S.: Weighted local Orlicz–Hardy spaces with applications to pseudo-differential operators. *Dissertationes Math.*, **478**, 1–78 (2011)

[47] Yang, D., Yang, S.: Local Hardy spaces of Musielak–Orlicz type and their applications. *Sci. China Math.*, **55**(8), 1677–1720 (2012)

[48] Zhang, Y., Huang, L., Yang, D., et al.: New ball Campanato-type function spaces and their applications. *J. Geom. Anal.*, **32**(3), Paper No. 99, 42 pp. (2022)

[49] Zhuo, C., Sawano, Y., Yang, D.: Hardy spaces with variable exponents on RD-spaces and applications. *Dissertationes Math.*, **520**, 1–74 (2016)

[50] Zhuo, C., Yang, D., Liang, Y.: Intrinsic square function characterizations of Hardy spaces with variable exponents, *Bull. Malays. Math. Sci. Soc.*, **2**(4), 1541–1577 (2016)