Partial least squares for sparsely observed curves with measurement errors

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Functional partial least squares (FPLS) is commonly used for fitting scalar-on-function regression models. For the sake of accuracy, FPLS demands that each realization of the functional predictor is recorded as densely as possible over the entire time span; however, this condition is sometimes violated in, e.g., longitudinal studies and missing data research. Targeting this point, we adapt FPLS to scenarios in which the number of measurements per subject is small and bounded from above. The resulting proposal is abbreviated as PLEASS. Under certain regularity conditions, we establish the consistency of estimators and give confidence intervals for scalar responses. Simulation studies help us test the accuracy and robustness of PLEASS. We finally apply PLEASS to clinical trial data and to medical imaging data.

KEYWORDS
Functional data analysis; Functional linear model; Partial least squares; Principal component analysis; Scalar-on-function regression

1 INTRODUCTION

Scalar-on-function (linear) regression (SoFR) is a category of elementary models in functional data analysis (FDA). They bridge a scalar response $Y$ to a functional predictor $X (= X(i))$, with the argument of $X$ often referred to as "time" and assumed to be confined to a bounded and closed interval $\mathbb{T} \subset \mathbb{R}$. (Without loss of generality, we take $\mathbb{T} = [0, 1]$ throughout this paper.) To be specific,

$$Y = \eta(X) = \mu_Y + \int_\mathbb{T} \beta(X - \mu_X).$$  

(1)
where $\mu_X$ (resp. $\mu_Y$) is the expectation of $X$ (resp. $Y$); the notation $\int f \, dt$ is the abbreviation of $\int f(t) \, dt$. Note that we do not include any error term on the right-hand side of (1); instead, we later merge this error with measurement error associated with $Y$ in (18). Functions in our discussion are assumed to belong to the $L^2$-space on $\mathbb{T}$ (or $\mathbb{T}^2$) with respect to (w.r.t.) Lebesgue measure, $L^2(\mathbb{T})$ (or $L^2(\mathbb{T}^2)$). Then the auto-covariance function

$$\nu_A = \nu_A(s, t) = \text{cov}\{X(s), X(t)\}$$

has countably many eigenvalues, say $\lambda_1 \geq \lambda_2 \geq \cdots$, with corresponding eigenfunctions $\phi_1, \phi_2, \ldots$. As well, it is standard in FDA to assume that $\sum_{j=1}^{\infty} \lambda_j < \infty$ and $\beta \in \text{span}(\phi_1, \phi_2, \ldots)$, where $\text{span}(\cdot)$ is the linear space spanned by items in the parentheses; the overline represents the closure. The auto-covariance operator $\mathcal{V}_A : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ is defined by

$$\mathcal{V}_A(f)(\cdot) = \int_{\mathbb{T}} f(t) \nu_A(t, \cdot) \, dt, \quad \forall f \in L^2(\mathbb{T}).$$

In this case, the Hilbert-Schmidt operator norm of $\mathcal{V}_A$ equals $\|\nu_A\|_2$, the $L^2$-norm of $\nu_A$. We abuse $\| \cdot \|_2$ for the matrix norm induced by Euclidean norm, i.e., for arbitrary $D \in \mathbb{R}^{p \times q}$ and $\alpha \in \mathbb{R}^{q \times 1}$, $\|D\|_2 = \sup_{\alpha: \|\alpha\|_2 = 1} \|D\alpha\|_2$. It is well known that $\|D\|_2$ is actually the largest eigenvalue of $D$ and reduces to the Euclidean norm for vectors.

People have applied SoFR to domains including chemometrics (e.g., Goutis 1998), food manufacturing (e.g., Aguilera et al. 2010), geoscience (e.g., Ballo 2009), medical imaging (e.g., Goldsmith et al. 2011) and many others. SoFR is favored for its natural interpretability: a time point corresponding to a higher absolute value of coefficient function contributes more to the response (Reiss et al. 2017). The typical first step in estimating $\beta$ is to expand it in terms of a set of basis functions either fixed (e.g., wavelets or splines) or data-driven (e.g., functional principal component (FPC) or functional partial least squares (FPLS)). Compared with FPC, FPLS is more adaptive to data, leading to a more parsimonious set of basis functions as well as more interpretability (Reiss et al. 2017). Formulated in Delaigle and Hall 2012 Appendix A.2), the standard iterative definition of FPLS starts with $X^{[1]} = X - \mu_X$, $Y^{[1]} = Y - \mu_Y$ and

$$w_1 = \arg \max_{\|w\|_2 = 1} \text{cov}^2\{Y, \int_{\mathbb{T}} Xw\} = \|v_C\|_2^{-1} v_C,$$

which involves the cross-covariance function

$$v_C = v_C(\cdot) = \text{cov}\{Y, X(\cdot)\}.$$ (5)

Subsequent FPLS basis functions are constructed in a successive way: given $w_1, \ldots, w_{j-1}$, assuming $v_C^{[1]} \neq 0$,

$$w_j = \arg \max_{\|w\|_2 = 1} \text{cov}^2\{Y^{[j]}, \int_{\mathbb{T}} X^{[j]}w\} = \|v_C^{[j]}\|_2^{-1} v_C^{[j]}.$$ (6)

where, for $j \geq 2$,

$$X^{[j]} = X^{[j]}(\cdot) = X^{[j-1]}(\cdot) - \left(\int_{\mathbb{T}} X^{[j-1]}w_{j-1}\right) \text{var}^{-1} \left(\int_{\mathbb{T}} X^{[j-1]}w_{j-1}\right) \text{cov}\{X^{[j-1]}(\cdot), \int_{\mathbb{T}} X^{[j-1]}w_{j-1}\},$$

$$Y^{[j]} = Y^{[j]} - \left(\int_{\mathbb{T}} X^{[j-1]}w_{j-1}\right) \text{var}^{-1} \left(\int_{\mathbb{T}} X^{[j-1]}w_{j-1}\right) \text{cov}\{Y^{[j-1]}, \int_{\mathbb{T}} X^{[j-1]}w_{j-1}\},$$
and
\[ Y_C^{[1]} = V_C^{[1]}(\cdot) = \text{cov} \left\{ Y^{[1]}, X_C^{[1]}(\cdot) \right\}. \tag{9} \]

The random variable \( Y_C^{[1]} \) (resp. \( X_C^{[1]}(t) \) for each \( t \in T \)) is the projection of \( Y^{[1]} \) (resp. \( X^{[1]}(t) \)) onto the orthogonal complement of \( \text{span} \left( \int_T X_1^{[1]}w_1, \ldots, \int_T X_{j-1}^{[1]}w_{j-1} \right) \).

**Remark 1**  FPLS basis functions \( w_j \) at \[6\] are orthogonal to each other. That is, if \( j_1 \geq 2 \), then, for all \( j_2 < j_1 \), we have \( \int_T w_{j_1}w_{j_2} = 0 \). This orthogonality is a functional counterpart of Bro and Eldén [2009 Proposition 1.1]. It is verified straightforwardly by noting that, from \[7\] and \[8\] (both of which are defined through least squares), the random variable
\[
\int_T X_j^{[1]}w_{j_2} = \int_T X_j^{[1]}w_{j_2} - \sum_{j=1}^{j_1-1} \left( \int_T X_j^{[1]}w_j \right) \left( \text{var}^{-1} \left( \int_T X_j^{[1]}w_j \right) \text{cov} \left( \int_T X_j^{[1]}w_{j_2}, \int_T X_j^{[1]}w_j \right) \right)
\]
is located in \( \text{span} \left( \int_T X_j^{[1]}w_{j_2}, \ldots, \int_T X_{j_1-1}^{[1]}w_{j_1-1} \right) \), while \( Y_C^{[1]} \) falls into its orthogonal complement. In this way, \[6\] and \[9\] jointly imply that, as long as \( Y_C^{[1]} \neq 0 \), \( \int_T w_{j_1}w_{j_2} = \| Y_C^{[1]} \|_2^{-1} \int_T X_j^{[1]}w_{j_2} = \| Y_C^{[1]} \|_2^{-1} \text{cov}(Y^{[1]}, \int_T X_j^{[1]}w_{j_2}) = 0 \).

The first \( p \) FPLS basis functions, \( w_1, \ldots, w_p \), span a linear space which turns out to be the functional version of the \( p \)-dimensional Krylov subspace (KS), namely,
\[ \text{KS}_p = \text{span}(w_1, \ldots, w_p) = \text{span} \{ V_A(\beta), \ldots, V_A^p(\beta) \} = \text{span} \{ V_C, V_A(V_C), \ldots, V_A^{p-1}(V_C) \}, \tag{10} \]
where \( V_A^j \) is the \( j \)-th power of \( V_A \); see (3.4) in Delaigle and Hall [2012b]. To be explicit, starting with \( V_A^1 = V_A \), recursively, \( V_A^j : L^2(T) \rightarrow L^2(T) \) satisfies the identity
\[ V_A^j(f)(\cdot) = \int_T V_A^{j-1}(f)(\cdot) V_A(\cdot, \cdot) dt, \quad \forall f \in L^2(T). \tag{11} \]
If the SoFR relation in \[1\] holds with \( \beta \in \text{span}(\phi_1, \phi_2, \ldots) \) (i.e., \( \beta \) is identifiable), then Delaigle and Hall [2012b Theorem 3.2] guarantees that \( \beta \) must be located in \( \text{KS}_\infty = \text{span}(w_1, w_2, \ldots) \) and is hence the limit (in the \( L^2 \) sense) of
\[ \beta_p = \arg \min_{\theta \in \text{KS}_p} \text{E} \left\{ Y - \mu_Y - \int_T \theta(X - \mu_X) \right\}^2 = c_p^\top \Lambda_p^{-1} [w_1, \ldots, w_p]^\top. \tag{12} \]
Here the notations \( c_p \) and \( \Lambda_p \) are respectively defined as
\[ c_p = \left[ \int_T w_1 V_A(\beta), \ldots, \int_T w_p V_A(\beta) \right]^\top = \left[ \int_T w_1 V_C, \ldots, \int_T w_p V_C \right]^\top = [\| V_C \|_2, 0, \ldots, 0]^\top \tag{13} \]
and
\[ \Lambda_p = \left[ \int_T w_{j_1} V_A(w_{j_2}) \right]_{1 \leq j_1, j_2 \leq p}. \tag{14} \]
The farthest right-hand side of \[13\] is derived from Eq. \[3\] and **Remark 1**.

In addition to estimation of the coefficient function, it is also of interest to predict, for a new trajectory \( X^* \) dis-
tributed as \(X\), the corresponding true response

\[
Y^* = \eta(X^*) = \mu_Y + \int_\Gamma \beta(X^* - \mu_X)
\]

which is the limit, as \(p \to \infty\), of

\[
\eta_p(X^*) = \mu_Y + \int_\Gamma \beta_p(X^* - \mu_X) = \mu_Y + c_p^T \Lambda_p^{-1} [\xi_1^p, \ldots, \xi_p^p]^T.
\]

Here the \(j\)th FPLS score (associated with \(X^*\)) is defined as

\[
\xi_j^* = \int_\Gamma w_j(X^* - \mu_X).
\]

Hereafter superscript "*" emphasizes items associated with the newly observed subject \(X^*\).

In recent decades, there has been considerable work on FPC and FPLS; we refer readers to, e.g., [Febrerobande et al., 2017], for a general review of both techniques. Nevertheless, most of this work is designed for dense settings, i.e., realizations of \(X\) are supposed to be densely observed. This condition is not expected to be fulfilled in many cases. For example, in typical clinical trials, participants cannot be monitored 24/7; instead, they are required to visit the clinic repeatedly on specific dates. Due to cost and convenience, the scheduled visiting frequency is doomed to be sparse for essentially every subject. What is worse is that subjects tend to show up on their own basis with frequencies lower and more irregular than scheduled. Similar difficulties can arise in missing data problems where a number of recordings are lost for whatever reason.

To specify the sparsity discussed hereafter, we suppose the observable training sample consists of \(n\) two-tuples \((\tilde{X}_1, \tilde{Y}_1), \ldots, (\tilde{X}_n, \tilde{Y}_n)\) with scalar response \(\tilde{Y}_i\) and trajectory \(\tilde{X}_i = (\tilde{X}_i(t))\) measured at only \(L_i\) time points \(T_{i1}, \ldots, T_{iL_i}\). They are all observed with contamination in the sense that, for some positive \(\sigma_X\) and \(\sigma_Y\), we observe

\[
\tilde{X}_i(T_{ii}) = X_i(T_{ii}) + \sigma_X e_{i_{ii}} \quad \text{and} \quad \tilde{Y}_i = Y_i + \sigma_Y e_i,
\]

where \(X_i\) (resp. \(Y_i\)) are independently and identically distributed (iid) as \(X\) (resp. \(Y\)) and error terms \(e_{i_{ii}}\) and \(e_i\) are independent white noises of zero mean and unit variance. Conditional on \(L_i\) iid \(L_i\), the time points \(T_{i1}, \ldots, T_{iL_i}\) are assumed to be iid as random variable \(T\). As pioneers who extended classical FPC to this challenging setting [James et al., 2000] postulated a reduced rank mixed effects model fitted by the expectation-maximization algorithm and penalized least squares. In contrast, the proposal of [Yao et al., 2005a,b], abbreviated as PACE, introduced a local linear smoother (LLS) estimator for \(\nu_A\) followed by FPC scores (viz. \(\int_\Gamma \phi_j(X_i - \mu_X)\), \(i = 1, \ldots, n, j \in \mathbb{Z}^+\)) which are approximated by conditional expectations. To the best of our knowledge, there are still few extensions of FPLS applicable to such a scenario with sparsity. In this work, we attempt to fill in this blank by developing a new technique named Partial LEAst Squares for Sparsity (PLEASS), handling sparse observations and measurement errors simultaneously.

The overall procedure for PLEASS is sketched below. First, thanks to the iid assumption on subjects, we are able to pool together all the observations in order to recover the variance and covariance functions from which basis functions are extracted. Then, \(\beta\) is estimated by plugging empirical counterparts into \(\beta_p\) at 12. It is worth noting that, since \(X^*\) is not observed densely, PLEASS does not give a consistent prediction for \(\eta(X^*)\) at 15; instead it constructs a confidence interval (CI) for \(\eta(X^*)\) through conditional expectation.

The remainder of this paper is organized as follows. Section 2 details the implementation procedure for PLEASS. In Section 3, we present asymptotic results on the consistency of estimators and on the CI for \(\eta(X^*)\). Section 4 applies
PACE and PLEAS to both simulated and real datasets and compares their resulting performance. Concluding remarks are given in Section 5.

2 | METHODOLOGY

2.1 | Estimation and prediction

The first phase of PLEASS is to find estimators of $\mu_X$ at $[1]$, $\nu_A$ at $[2]$, $\nu_C$ at $[5]$, and $\sigma_X^2$ at $[18]$, respectively, say, $\hat{\mu}_X$, $\hat{\nu}_A$, $\hat{\nu}_C$ and $\hat{\sigma}_X^2$. In theory PLEASS is flexible as to how to estimate these four items, as long as $\|\hat{\mu}_X - \mu_X\|_\infty$, $\|\hat{\nu}_A - \nu_A\|_\infty$, $\|\hat{\nu}_C - \nu_C\|_\infty$ and $|\hat{\sigma}_X^2 - \sigma_X^2|$ all converge to zero as $n$ diverges, where $\| \cdot \|_\infty$ denotes the $L^\infty$-norm; that is, theoretically, the choice is not crucial to the consistency of estimators given by PLEASS. Existing methods for reconstructing the variance and covariance structure from sparse observations roughly fall into two categories: kernel smoothing (e.g., LLS in Yao et al. [2005a,b]; Li and Hsing [2010] and the modified kernel smoothing in Paul and Peng [2011] and maximum likelihood (ML, e.g., restricted ML in James et al. [2000]; Peng and Paul [2009] and quasi ML in Zhou et al. [2018]). Typically, the latter category requires initial values offered by the first one and is therefore more time-consuming. In the numerical study Section 4, we adopt LLS (whose details are relegated to Appendix A following Yao et al. [2005a,b]). The motivation for this choice is three-fold: first, it has nice asymptotic properties as stated by Hall et al. [2006]; second, implementation is convenient with the help of R-package `dapace` [Dai et al. 2018]; and third, it makes the comparison between PLEASS and PACE more fair, since PACE also exploits LLS. It is understood that LLS works in a pointwise manner, in the sense that, rather than giving analytical expressions for a whole function, it gives function values corresponding to a preset mesh grid. As a consequence, all the subsequent integrals involving $\hat{\nu}_C$ and $\hat{\nu}_A$ have to be approximated numerically by, e.g., Riemann sums or trapezoidal sums. Tasaki [2009] gave upper bounds on the (absolute) approximation errors for these two quadrature rules; these bounds tend to zero as the discretized grid for $\Gamma$ becomes dense. Because it is practical to implement LLS over a very dense grid, we continue to use the integral notation in the following for convenience.

Combined with the orthogonality of $\omega_j$, Eq. (10) inspires us to estimate $\omega_j$ by sequentially (Gram-Schmidt) orthonormalizing $j$ estimated functions $\hat{\nu}_C$, $\hat{\nu}_A^{-1}(\hat{\nu}_C)$, ..., $\hat{\nu}_A^{-j-1}(\hat{\nu}_C)$, i.e., subtracting projections onto previous functions and then scaling the remaining part. Here $\hat{\nu}_A^{-j}$ is an estimator for $\nu_A^{-j}$ which is constructed by substituting $\nu_A$ and $\nu_C$ at $[11]$ for corresponding empirical counterparts. To be precise,

$$
\hat{\nu}_A^{-j}(f)(\cdot) = \int_T \hat{\nu}_A^{-j-1}(f)(\cdot) \nu_A(t, \cdot) dt, \quad \forall f \in L^2(\Gamma).
$$

(19)

In particular, $\hat{\omega}_1 = \hat{\nu}_C / \|\hat{\nu}_C\|_2$ and, for $j \geq 2$, $\hat{\omega}_j$ are given successively by

$$
\hat{\omega}_j = \frac{\hat{\nu}_A^{-j-1}(\hat{\nu}_C) - \sum_{k=1}^{j-1} \hat{\nu}_k \int_T \hat{\nu}_k \hat{\nu}_A^{-j-1}(\hat{\nu}_C)}{\|\hat{\nu}_A^{-j-1}(\hat{\nu}_C) - \sum_{k=1}^{j-1} \hat{\nu}_k \int_T \hat{\nu}_k \hat{\nu}_A^{-j-1}(\hat{\nu}_C)\|_2}.
$$

(20)

Alternatively, the modified Gram-Schmidt procedure [Algorithm 1] gives mathematically equivalent but numerically more stable estimators for $\omega_j$ [Lang [2010], pp. 102]. Evidently a plug-in estimator for both $\hat{\beta}_p$ [12] and $\beta$ is then given by

$$
\hat{\beta}_p = \hat{\sigma}_p^2 \hat{\Lambda}_p^{-1}[\hat{\omega}_1, \ldots, \hat{\omega}_p]^T.
$$

(21)
Algorithm 1 Modified Gram-Schmidt orthonormalization to estimate $w_j$

\[
\text{for } j \text{ in } 1, \ldots, p \text{ do}
\]
\[\hat{w}_j^{[1]} \leftarrow \overline{V}_A^{j-1}(\hat{\nu}_C),\]
\[\text{if } j \geq 2 \text{ then}
\]
\[\text{for } i \text{ in } 1, \ldots, j - 1 \text{ do}
\]
\[\hat{w}_j^{[i+1]} \leftarrow \hat{w}_j^{[i]} - \hat{w}_j^{[i]} \int_\mathbb{T} \hat{w}_j^{[i]} \hat{w}_j^{[i]}.
\]
\[\text{end for}\]
\[\text{end if}\]
\[\hat{w}_j \leftarrow \hat{w}_j^{[j]} / \|\hat{w}_j^{[j]}\|_2.
\]
\[\text{end for}\]

These estimators converge to the true $\beta$ as $n$ and $p$, respectively, diverge at specific rates (see Theorem 1), where

\[\hat{e}_p = [\|\hat{\nu}_C\|_2, 0, \ldots, 0]^T\] (22)

and

\[\hat{\Lambda}_p = \left[\int_\mathbb{T} \hat{w}_j \overline{V}_A(\hat{w}_j)\right]_{1 \leq j_1, j_2 \leq p}\] (23)

are estimating [13] and [14], respectively.

Predicting $\eta(X^*)$ at (15) is a problem fairly different from estimating $\beta$. Since $\tilde{X}^*$ (i.e., the contaminated $X^*$) is only observed at $L^*$ time points (satisfying (C.1) in Appendix B), it is not practical to numerically integrate the product of $\hat{\beta}_p$ [21] and $\tilde{X}^*$. Instead we use conditional expectation and target the prediction of a surrogate for $\eta(X^*)$. That surrogate, denoted by $\tilde{\eta}_\infty(X^*)$, is defined at (25) below.

Write

\[\tilde{X}^* = [\tilde{X}^*(T^*_1), \ldots, \tilde{X}^*(T^*_L)]^T,\]
\[\mu_X^* = E(\tilde{X}^* | L^*, T^*_1, \ldots, T^*_L) = [\mu_X(T^*_1), \ldots, \mu_X(T^*_L)]^T,\]
\[\Sigma_{\tilde{X}}^* = \left[\nu_A(T^*_1, T^*_2)\right]_{1 \leq j_1, j_2 \leq L^*} + \sigma^2 X_{L^*},\]

and, for integer $j \in [1, p],$

\[h_j^* = [\nu_A(w_j(T^*_1), \ldots, \nu_A(w_j(T^*_L))]^T.\]

Given $L^*$ and $T^*_1, \ldots, T^*_L$, in view of the identity that

\[
\text{cov}(\tilde{X}^{*T}, \xi_1^*, \ldots, \xi_p^* | L^*, T^*_1, \ldots, T^*_L) =
\begin{bmatrix}
\Sigma_{\tilde{X}}^* & h_1^* & \cdots & h_p^* \\
h_1^T & \ddots & & \\
& & \ddots & \Lambda_p \\
h_p^T & & & \\
\end{bmatrix},
\]
the best linear unbiased predictor for $\xi_j^*$ is

$$\hat{\xi}_j^* = \mathbb{E}(\xi_j^* \mid \tilde{X}^*, L^*, T_{1}^*, \ldots, T_{L}^*) = h_j^*^\top \Sigma_{\tilde{X}^*}^{-1}(\tilde{X}^* - \mu_{\tilde{X}^*}).$$

which minimizes $\mathbb{E}( (\xi_j^* - f(\tilde{X}^*))^2 \mid L^*, T_{1}^*, \ldots, T_{L}^*)$ over all linear functions $f$ subject to $\mathbb{E}(\xi_j^* - f(\tilde{X}^*)) \mid L^*, T_{1}^*, \ldots, T_{L}^*) = 0$. It is even the best prediction over all measurable $f$, linear or not, as long as $\xi_1^*, \ldots, \xi_p^*$ and $\tilde{X}^*$ are jointly Gaussian; see, e.g., [Harville 1976] Theorem 1).

Conditional on $L^*$ and $T_{1}^*, \ldots, T_{L}^*$, geometrically speaking, $\hat{\xi}_j^*$ at (24) is the (orthogonal) projection of $\xi_j^*$ at (17) onto span $\{\tilde{X}^*(T_{1}^*), \ldots, \tilde{X}^*(T_{L}^*)\}$. Define the $L^* \times p$ matrix

$$H_p = [h_1^*, \ldots, h_p^*].$$

Then the projection of $\eta(X^*) - \mu_X$ onto the same space is $\lim_{p \to \infty} c_p^\top \Lambda_p^{-1} H_p^\top \Sigma_{\tilde{X}^*}^{-1}(\tilde{X}^* - \mu_{\tilde{X}^*})$ which necessarily exists. So does

$$\hat{\eta}_p(X^*) := \lim_{p \to \infty} \hat{\eta}_p(X^*),$$

where

$$\hat{\eta}_p(X^*) = \mu_Y + c_p^\top \Lambda_p^{-1}[\hat{\xi}_1^*, \ldots, \hat{\xi}_p^*]^\top = \mu_Y + c_p^\top \Lambda_p^{-1} H_p^\top \Sigma_{\tilde{X}^*}^{-1}(\tilde{X}^* - \mu_{\tilde{X}^*})$$

is a natural surrogate for $\eta_p(X^*)$ at (16). A plug-in prediction for $\eta(X^*)$ at (13) is accordingly

$$\hat{\eta}_p(X^*) = \frac{1}{n} \sum_{i=1}^n \hat{Y}_i + c_p^\top \Lambda_p^{-1} H_p^\top \Sigma_{\tilde{X}^*}^{-1}(\tilde{X}^* - \mu_{\tilde{X}^*}),$$

following the replacement of $\Lambda_p$, $\Sigma_{\tilde{X}^*}$, $\mu_{\tilde{X}^*}$ and $H_p$ all at (26) with, respectively, $\tilde{\Lambda}_p$ at (23),

$$\hat{\Sigma}_{\tilde{X}^*} = \left[ \tilde{\varphi}_A(T_{h_1}^*, T_{h_2}^*) \right]_{1 \leq h_1, h_2 \leq L^*} + \tilde{\sigma}_X^2 I_{L^*},$$

$$\hat{\mu}_{\tilde{X}^*} = [\hat{\mu}_X(T_{1}^*), \ldots, \hat{\mu}_X(T_{L}^*)]^\top,$$

and

$$\hat{H}_p = [h_1^*, \ldots, h_p^*]$$

in which

$$h_j^* = [\tilde{\nu}_A(\hat{\xi}_j^*(T_{1}^*)), \ldots, \tilde{\nu}_A(\hat{\xi}_j^*(T_{L}^*))]^\top.$$
**Algorithm 2 PLEASS tuned through GCV**

Obtain \( \hat{\mu}_X, \hat{\nu}_A, \hat{\nu}_C \) and \( \hat{\sigma}_X^2 \) following Appendix A.

for \( j \) in \( 1, \ldots, p_{\text{max}} - 1 \) do

\[
\hat{\nu}_A^{(j)}(\nu_C) \leftarrow \int_0^\infty \hat{\nu}_A(t) \nu_C^{-1}(\nu_C)(t) \, dt.
\]

end for

Extract \( \hat{\omega}_j \) from \( \hat{\varphi}_C, \hat{\varphi}_A(\hat{\nu}_C), \ldots, \hat{\varphi}_A^{p_{\text{max}}-1}(\hat{\nu}_C) \) following Algorithm 1.

\( \hat{\mu}_0 \leftarrow 0. \)

\( \hat{\eta}_0(X^n) \leftarrow n^{-1} \sum_{i=1}^n \hat{Y}_i. \)

for \( p \) in \( 1, \ldots, p_{\text{max}} \) do

\[
\hat{\beta}_p \leftarrow \hat{e}_p^T \hat{\Lambda}_p \left[ \hat{\omega}_1, \ldots, \hat{\omega}_p \right]^T \text{ with } \hat{e}_p \text{ and } \hat{\Lambda}_p.
\]

\( \hat{\eta}_p(X^n) \leftarrow n^{-1} \sum_{i=1}^n \hat{Y}_i + \hat{e}_p^T \hat{\Lambda}_p^{-1} \hat{H}_p^T \hat{\Sigma}_X^{-1} (\hat{\nu}_X^* - \hat{\mu}_X^*) \text{ with } \hat{\Sigma}_X^*, \hat{\mu}_X^* \text{ and } \hat{H}_p^*. \)

end for

\( \rho_{\text{opt}} \leftarrow \arg\min_{0 \leq \rho \leq p_{\text{max}}} \text{GCV}(p). \)

\( (1 - \alpha) \text{ CI for } \eta(X^n) \leftarrow \hat{\eta}_{\text{opt}}(X^n) \pm \Phi^{-1}_{1-\alpha/2} \left\{ \hat{\Lambda}_{\text{opt}}^{-1} \hat{\mu}_{\text{opt}} \right\}^{1/2} \). 

\[
= \text{cov}([\xi_1^*, \ldots, \xi_p^*] \, | \, L^*, T^*_1, \ldots, T^*_L) - \text{cov}([\xi_1^*, \ldots, \xi_p^*] \, | \, L^*, T^*_1, \ldots, T^*_L) = \Lambda_p - H_p^T \Sigma_X^{-1} H_p^*.
\]

Under a Gaussian assumption (as in Corollary 1) and conditioning on \( L^* \) and \( T^*_1, \ldots, T^*_L \), the error \( \hat{\eta}_p(X^n) - \eta(X^n) \) is asymptotically normally distributed, as long as \( c_p \Lambda_p^{-1} (\Lambda_p - H_p^T \Sigma_X^{-1} H_p^*) \Lambda_p^{-1} c_p \) converges to a positive number as \( p \) goes to infinity. An asymptotic \( (1 - \alpha) \) (conditional) CI for \( \eta(X^n) \) at (15) is then

\[
\hat{\eta}_p(X^n) \pm \Phi^{-1}_{1-\alpha/2} \left\{ c_p^T \Lambda_p^{-1} \hat{\Lambda}_{\text{opt}} - H_p^T \Sigma_X^{-1} - \hat{H}_p \right\}^{1/2}.
\]

with the \( (1 - \alpha/2) \) standard normal quantile \( \Phi^{-1}_{1-\alpha/2} \).

2.2 | Selection of number of basis functions

Instead of one-curve-leave-out cross-validation or the Akaike information criterion ([Yao et al., 2005a, b], we adopt generalized cross validation (GCV: [Craven and Wahba, 1979]) to the context here, i.e., we choose an integer \( p \in [0, p_{\text{max}}] \) to minimize GCV\((p) = (n - p - 1)^{-2} \sum_{i=1}^n (\hat{Y}_i - \hat{\eta}_p(X_i))^2 \). This strategy is commonly adopted (e.g., [Cardot et al., 2007; Reiss and Ogden, 2010]) and, more importantly, less time-consuming than the one-curve-leave-out cross-validation.

Define the fraction of variance explained (FVE) by the first \( j \) FPCs as \( \text{FVE}(j) = \sum_{k=1}^j \lambda_k / \sum_{k=1}^{\infty} \lambda_k \) in which \( \lambda_k \) is replaced by empirical counterparts in practice. A heuristic \( p_{\text{max}} \) is then given by, e.g.,

\[
p_{\text{max}} = \min \{ j \in Z^+: \text{FVE}(j) \geq 95\% \}.
\]

which is justified in the following way: classical FPC takes \( p_{\text{max}} \) as one of the default truncation rules, while, as numerically illustrated by [Delaigle and Hall, 2012b] (Section 6), FPLS needs fewer terms than FPC to reach the same prediction accuracy. Another candidate for \( p_{\text{max}} \) is provided by [Delaigle and Hall, 2012a] (Section 3): \( p_{\text{max}} = n/2 \), which is acceptable for a small or moderate \( n \).
3 | ASYMPTOTIC PROPERTIES

Our theoretical results are established under assumptions [C1], [C15] in Appendix B. The remaining ones are used to show the consistency of LLS in Appendix A. For an arbitrarily fixed integer $p$, the consistency of $\hat{\beta}_p$ at (21) deduced immediately from Lemmas 1 and 2 in Appendix B by following the proof of Zhou (2019) Theorem 1. Unfortunately, this argument does not apply to the scenario with diverging $p = p(n)$, since the sequential construction in [20] tends to induce a bias accumulating with increasing $p$. As a result, it is indispensable to impose a sufficiently slow divergence rate on $p$, e.g., as required in (C15) or (C15').

Theorem 1 Assume that [C1]–[C15] all hold. As $n$ goes to infinity, $\|\hat{\beta}_p - \beta\|_2 \rightarrow P 0$. If we replace (C15) with stronger (C15') and, in addition, assume that $\|\beta_p - \beta\|_\infty \rightarrow p 0$, then the convergence of our estimator becomes uniform, i.e., $\|\hat{\beta}_p - \beta\|_\infty \rightarrow P 0$.

Analogous to PACE, our PLEASS results in an inconsistent prediction (see Theorem 2): the zero-convergent (unconditionally and in probability) discrepancy is $|\hat{\eta}_p(X^*) - \tilde{\eta}_\infty(X^*)|$, rather than $|\hat{\eta}_p(X^*) - \eta(X^*)|$, i.e., the limit for $\hat{\eta}_p(X^*)$ is just a surrogate for the desired $\eta(X^*)$ at (15). Nevertheless, this phenomenon is far from disappointing; one of its implications is that $\hat{\eta}_p(X^*) - \eta(X^*)$ is asymptotically distributed as $\tilde{\eta}_\infty(X^*) - \eta(X^*)$. An asymptotic $(1 - \alpha)$ conditional CI for $\eta(X^*)$ hence follows. In particular, the result for Gaussian cases is presented in Corollary 1.

Theorem 2 Under assumptions [C1], [C15] as $n$ goes to infinity, $\hat{\eta}_p(X^*) - \tilde{\eta}_\infty(X^*)$ converges to zero (unconditionally) in probability.

Corollary 1 Fix $L^*$ and $T^*_1, \ldots, T^*_n$. Assume [C1], [C15] and the following two extra conditions:

1) FPLS scores $\int w_j (X - \mu_X)$ and measurement errors $e_{ij}$ are jointly Gaussian.
2) $\lim_{p \rightarrow \infty} e_{ij}^T A_p^{-1} (A_p - H_p^T \Sigma^{-1} H_p) A_p^{-1} e_{ij} = \omega > 0$.

Then, as $n \rightarrow \infty$,

$$\frac{\hat{\eta}_p(X^*) - \eta(X^*)}{\sqrt{e_{ij}^T A_p^{-1} (A_p - H_p^T \Sigma^{-1} H_p) A_p^{-1} e_{ij}}} \rightarrow d N(0, 1).$$

4 | NUMERICAL ILLUSTRATION

4.1 | Simulation study

Except in Corollary 1, we have not needed to specify a Gaussian assumption for the functional predictor $X$, the FPC scores $\rho_j = \int \phi_j (X - \mu_X)$ or the measurement errors $e_{ij}$ and $e_i$ in model (18). To illustrate the performance of PLEASS in non-normal cases, we generated iid normal samples (i.e., $\lambda_j^{1/2} \rho_j$, $e_{ij}$, $e_i$ were all iid as $N(0, 1)$) and exponential samples (i.e., $\lambda_j^{1/2} \rho_j$, $e_{ij}$, $e_i$ were all iid as $exp(1/1) - 1$, respectively, of size $R = 200$. Each sample consisted of $n = 300$ iid pairs of scalar responses and predictor trajectories. We took $\lambda_1 = 50$, $\lambda_2 = 10$ and $\lambda_3 = 1$ as the top three eigenvalues of the covariance function $\nu_A$ in [2], all other eigenvalues were set to zero. For the eigenfunctions, we considered a more realistic choice not as smooth as “man-made” functions like trigonometric functions or orthogonal polynomials. Specifically, the first three eigenfunctions of $\nu_A$ at (2) $\phi_1$, $\phi_2$ and $\phi_3$, were respectively set to be the first three (estimated) eigenfunctions of the dataset Canadianweather in R-package fda [Ramsay et al., 2017]. This dataset records the (base 10 logarithm of) precipitation at 35 different locations in Canada averaged over 1960 to 1994. The slope function $\beta$ was
respectively given by
\[ \beta = \phi_1 + \phi_2 \] (32)

and
\[ \beta = \phi_3. \] (33)

Moreover, we let \( \mu_Y = 0, \sigma^2_Y = \sum_{j=1}^{\infty} \lambda_j / 9 \) and \( \sigma^2_X = \text{var}(\int_0^1 \beta X) / 9. \) To embody the sparsity assumptions, in each sample, the \( i \)th subject was observed only at \( L_i \) points uniformly selected (without replacement) from an evenly spaced set \((0 \times 10^{-2}, \ldots, 100 \times 10^{-2})\). The total number of points \( L_i \), chosen in an iid way using the discrete uniform distribution supported on \( \{3, \ldots, 6\} \).

In each sample, we randomly took 60 (= \( n/5 \)) subjects for testing and used the remainder for training. After running through all 200 samples, we obtained (relative) integrated squared errors (ISE)
\[ \text{ISE}_r = \| \beta \|_2^2 \| \beta - \hat{\beta}(r) \|_2^2, \quad r = 1, \ldots, 200, \]
which reflected the global error of \( \hat{\beta}(r) \), the estimated coefficient function from the \( r \)th sample. Since neither PACE nor PLEASS lead to consistent predictions, it is better to evaluate the prediction quality via true coverage percentages (TCP) of corresponding 95% CIs constructed for training subjects, i.e.,
\[ \text{TCP}_r = (\#I_{\text{test},r})^{-1} \sum_{i \in I_{\text{test},r}} \mathbb{1}\{ \eta(X_i, r) \in \hat{C}_i, r \}, \quad r = 1, \ldots, 200, \]
where \( \mathbb{1}\{\cdot\} \) is the indicator function; \( \eta(X_i, r) \) (resp. \( \hat{C}_i, r \)) is the uncontaminated response (resp. the asymptotic 95% CI) corresponding to the \( i \)th subject of the \( r \)th sample; and \( I_{\text{test},r} \) is the index set for testing portion of the \( r \)th sample with its cardinality \( \#I_{\text{test},r} \).

We considered the Gaussian case first. We applied PACE and PLEASS to artificial normal samples with \( \beta \) as in (32). The resulting summary of ISEs (resp. TCPs) is presented in Figure 1 (resp. Table 1). In terms of both criteria, unsurprisingly, PACE outperformed PLEASS; but its advantage was not huge, especially, in terms of TCP (see the first row of Table 1). In this scenario, PACE was unlikely to miss vital components of the slope, while PLEASS possibly suffered more uncertainty and variation during the sequential construction of FPLS basis functions. PACE became unfavored if we changed \( \beta \) to be as in (33). In this case FVE(2) was already over 98% and hence PACE was likely to miss \( \phi_3 \), which would definitely worsen its performance; the mean TCP of CIs constructed through PLEASS was about 70%, over three times as high as the one from PACE; see the third row of Table 1. Nevertheless, we must admit that, when \( \beta \) was orthogonal to the first two eigenfunctions, the performance of PLEASS was also unsatisfactory. A potential cause lies in estimating \( \hat{\nu}_A \) at \( \nu_A \) through LLS; the estimate was sensitive to the selection of bandwidths: if the data-driven bandwidth selection was not close to optimal, the accuracy of the resulting estimate \( \hat{\nu}_A \) was worse. In contrast, small eigenvalues were more prone to being neglected by an ill-estimated \( \hat{\nu}_A \). Thus there is still room for improvement in PLEASS, despite its better performance than that of PACE with \( \beta \) at (33).

Besides the setting of \( \beta \), the normality assumption mattered too: only in the normal case with \( \beta \) as in (32) were TCPs of both techniques close enough to the stated level; fixing \( \beta \), compared with normal counterparts, the exponential cases were followed by a substantial loss in terms of both ISE and TCP for whichever technique. It seemed that neither
PACE nor PLEASS was robust to the normality assumption. This conclusion does not conflict with the one in [Yao et al. 2005b], because they took a different representative of non-normality and employed a different criterion as well.

### 4.2 Application to real datasets

We now apply PLEASS to two real datasets. The first comes from a typical clinical trial, whereas the second one is densely observed but recorded with missing values.

**Primary Biliary Cholangitis data.** Initially shared by [Therneau and Grambsch 2000], dataset `pbcseq` (accessible in
R-package survival created by [Herneau 2015] was collected in a randomized placebo controlled trial of D-penicillamine, a drug designed for primary biliary cholangitis (PBC, also known as primary biliary cirrhosis). PBC is a chronic disease with bile ducts in the liver slowly destroyed and can cause more serious problems including liver cancer. All the participants of the clinical trial were supposed to revisit the Mayo Clinic at six months, one year, and annually after their initial diagnoses. However, participants’ visiting frequencies, with an average of 6, varied among patients, ranging from 1 to 16 and leading to sparse and irregular recordings. Although the clinical trial lasted from January 1974 through May 1984, to satisfy the prerequisites of LLS, we merely included measurements within the first 3000 days and kicked out subjects with less than two visits. At each visit, several body indexes were measured and recorded, including alkaline phosphatase (ALP, in U/L) and aspartate aminotransferase (AST, in U/mL), both evaluating the health condition of liver. We focused on this pair of indicators and attempted to model a linear connection between participants’ latest AST measurements (response) and their ALP profiles (predictor).

**Diffusion tensor imaging data.** Fractional anisotropy (FA) is measured at a specific spot in the white matter, ranging from 0 to 1 and reflecting the fiber density, axonal diameter and myelination. Along a tract of interest, these values forms an FA tract profile. Collected at Johns Hopkins University and the Kennedy-Krieger Institute, dataset DTI (in R-package refund [Goldsmith et al. 2016]) contained FA tract profiles for the corpus callosum measured via the diffusion tensor imaging (DTI). Though these trajectories were not sparsely measured, a few of them confronted missing records which were able to be handled by PACE and PLEASS without presmoothing or interpolation. We bridged subjects’ FA tract profiles (predictor) to their Paced Auditory Serial Addition Test (PASAT) scores (response). Assessing impairments in the cognitive functioning, e.g., in multiple sclerosis research, PASAT score is most commonly reported as the number of correct answers (out of 60) given by one subject [Tombaugh 2006].

For each real dataset, 200 (independent) random splits were carried out. As we did in the simulation, (roughly) 80% of the subjects in each split were put into the training set and the remainder were kept for testing. Predicting responses in the testing set via PACE and PLEASS, we generated the relative mean squared prediction error (ReMSPE):

\[
\text{ReMSPE}_r = \frac{\sum_{i \in I_{test,r}} (\tilde{Y}_{i,r} - \hat{Y}_{i,r})^2}{\sum_{i \in I_{test,r}} (\tilde{Y}_{i,r} - \bar{Y}_{train,r})^2}, \quad r = 1, \ldots, 200,
\]

for each method and each split, where \( \tilde{Y}_{i,r} \) (resp. \( \hat{Y}_{i,r} \)) is the noisy observation (resp. the prediction) for the \( i \)th response of the \( r \)th split and \( \bar{Y}_{train,r} \) is the mean training response in the \( r \)th split. ReMSPE is indeed a comparison in terms of mean squared error between the proposed prediction on the above and the most naive prediction, i.e., the mean training response.

---

**TABLE 1**  Average TCPs of 95% CIs constructed through PACE and PLEASS under different simulation settings (with corresponding standard deviations in parenthesis). Numbers in bold are the ones closest to the stated level, 95%, in each row.

| \( \beta \) | Distribution of \( \lambda_j^{-1/2} \rho_j, e_{ij} \) and \( e_i \) | Mean TCP (%) |
|---|---|---|
| \( \phi_1 + \phi_2 \) | \( \mathcal{N}(0, 1) \) | **91.40** (16.28) |
| \( \exp(1) - 1 \) | 53.25 (31.20) |
| \( \phi_3 \) | \( \mathcal{N}(0, 1) \) | 20.03 (20.36) |
| \( \exp(1) - 1 \) | 13.00 (16.54) | **40.60** (30.49) |
FIGURE 2  Boxplots of ReMSPE for real data analysis. In each subfigure, the left (resp. right) boxplot corresponds to PACE (resp. PLEASS). Subfigures are both displayed with identical scales.

Boxplots of ReMSPE values were collected in Figure 2, which demonstrated a clear advantage of PLEASS over PACE: ReMSPEs belonging to PLEASS enjoyed lower averages and smaller standard deviations. Formally, the classical (one-sided) t-test suggested that, at the 5% level of significance, mean ReMSPEs of PLEASS were both significantly less than 1 (with corresponding p-values both almost 0), while those of PACE were not (with respective p-values 0.057 and 0.068); namely, in both cases PLEASS outperformed the mean training response, but PACE might not. This conclusion held even if we resorted to the corrected resampled t-test [Nadeau and Bengio, 2003] which was much more conservative (i.e., accompanied with lower Type I error rate) than the classical one.

5  |  DISCUSSIONS AND CONCLUSIONS

The main contributions of our work are summarized as follows. First, we propose PLEASS, a variant of FPLS modified for scenarios in which functional predictors are only observed at sparse time points. Secondly, we show that PLEASS is applicable to SoFR coupled with measurement errors, a setting more complex than the one in Delaigle and Hall [2012b]. Third, not only do we give estimators and predictions via PLEASS, we also construct CIs for mean responses. Assuming that ρ diverges to ∞ as a function of n, the consistency of our estimator is among the few asymptotic results available for FPLS and its variants. Fourth, we numerically illustrate the advantage of PLEASS in specific scenarios.

An alternative to the iterative algorithm in Section 1 provides another definition of FPLS basis functions, say $w_1^A, w_2^A, \ldots$. They are now described via a constrained optimization problem. Let $w_1^A = w_1 = v_C / \|v_C\|_2$. Subsequently, for integer $j \geq 2$, $w_j^A = \arg \max \{ Y - \eta_{p-1}(X), \int_X X w \} \text{ subject to } \var(\int_X X w_j^A) = \int_X w_j^A V_A(w_j^A) = 1$ and $\cov(\int_X X w_j^A, \int_X X w_k^A) = \int_X w_j^A V_A(w_k^A) = 0, 1 \leq k < j$. For $p \geq 2$, $(w_1^A, \ldots, w_p^A)$ typically differs from $(w_1, \ldots, w_p)$; however, they span the same linear space. As a result this alternative definition does not affect $\beta_p$ at (12) and $\eta_p(X^*)$ at (16). Moreover, the empirical counterparts of these quantities are also invariant, as long as we slightly modify the definition of inner product in Algorithm 1 by substituting $\int_X f \tilde{V}_A(g)$ for $\int f g$.

Estimators for the variance and covariance structure could be further revised. When estimating the value at a
specific point, LLS borrows strength from a neighbourhood whose bandwidth turns out to impact PLEASS to a certain extent. Competitors of LLS are more or less haunted too by the challenging bandwidth selection problem whose solution remains an open question. The data-driven selection adopted by Appendix A may not be optimal in practice or perhaps not even close to optimal. If trajectories are no longer independent of each other, the proposal of [Paul and Peng, 2011] is more competitive. One more limitation concerns the nature of the missingness: sparse observations (and missing values) are assumed independent of trajectories and measurement errors, as in condition [C3]. Once the missingness is permitted to be correlated with unobserved time points, we speculate that, after necessary modifications, estimates of the ML type would be still promising in estimating components of covariance structure.

In contrast with PLEASS, which is for now concentrated on SoFR only, PACE is more versatile: aside from handling even function-on-function regression (FoFR, with response and predictor both curves), PACE is capable as well of recovering predictor trajectories. We are still working on enlarging the application scope of PLEASS, e.g., by adapting it to FoFR.

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A | LOCAL LINEAR SMOOTHER

Let $\kappa = \kappa(\cdot)$ be a function on $\mathbb{R}$ satisfying [C8] [C10] in Appendix B; examples include the symmetric Beta family (Fan and Gijbels, 1996) Eq. 2.5) that has the Epanechnikov kernel $\kappa(t) = 0.75(1 - t^2)1\{|t| \leq 1\}$ as a special case. LLS actually falls into the framework of weighted least squares (WLS) (Fan and Gijbels, 1996 pp. 58–59). Given $1_M$ (viz. the $M$-vector of ones), $u$ (viz. an $M$-vector), $T$ (viz. an $M \times m$ matrix) and $W$ (viz. an $M \times M$ non-negative definite matrix), one solves

$$\min_{a_0, \alpha} (u - a_01_M - Ta)^T W (u - a_01_M - Ta)$$

for the scalar $a_0 \in \mathbb{R}$ and an $m$-vector $\alpha = [a_1, \ldots, a_m]^T$. In fact, LLS only exploits the WLS solution for $a_0$, i.e.,

$$\hat{a}_0 = \left(1_M^TW^{1/2}P_w^{1/2}W^{1/2}1_M + T^TW + T^TW^T\right)^{+}1_M^TW^{1/2}P_w^{1/2}W^{1/2}u$$

$$= \left(1_M^T(W - WT(T^TW)^*T^TW)1_M + T^TW^T\right)^{+}1_M^T(W - WT(T^TW)^*T^TW)u$$

in which the Moore-Penrose generalized inverse is denoted by $^{+}$ and $P_w^{1/2} = I - W^{1/2}(T^TW)^*T^TW^{1/2}$. In particular, four different combinations of $u$, $T$ and $W$ yield estimates of the four targets of interest, namely, $\mu_X$, $\nu_C$, $\nu_A$, and $\gamma$ as follows:

(i) Given $t \in T$, estimate $\mu_X(t)$ by $\hat{\mu}_X(t) = \hat{a}_0$ from \[34\] with $\sum_{1 \leq i \leq n} L_i$-vectors

$$u \equiv \left[\tilde{X}_1(T_{11}), \ldots, \tilde{X}_1(T_{1L_1}), \ldots, \tilde{X}_n(T_{n1}), \ldots, \tilde{X}_n(T_{nL_n})\right]^T$$

\[34\]
and $T = \left[ t - T_{11}, \ldots, t - T_{1L_1}, \ldots, t - T_{n1}, \ldots, t - T_{nL_n} \right]^\top$ and $\sum_{1 \leq i \leq n} L_i \times \sum_{1 \leq i \leq n} L_i$ matrix

$$W = \text{diag} \left\{ \kappa \left( \frac{t - T_{11}}{h_\mu} \right), \ldots, \kappa \left( \frac{t - T_{1L_1}}{h_\mu} \right), \ldots, \kappa \left( \frac{t - T_{n1}}{h_\mu} \right), \ldots, \kappa \left( \frac{t - T_{nL_n}}{h_\mu} \right) \right\}.$$

(ii) Write $\tilde{Y} = n^{-1} \sum_{1 \leq i \leq n} \tilde{Y}_i$. For arbitrary $t \in \mathbb{T}$, $\hat{v}_C(t)$ is $\hat{a}_0$ from (34) with $\sum_{1 \leq i \leq n} L_i$-vectors

$$u = \begin{bmatrix} (\tilde{X}_1(T_{11}) - \hat{\mu}_X(T_{11}))(\tilde{Y}_1 - \tilde{\gamma}), \ldots, (\tilde{X}_1(T_{1L_1}) - \hat{\mu}_X(T_{1L_1}))(\tilde{Y}_1 - \tilde{\gamma}), \ldots, \\
(\tilde{X}_n(T_{n1}) - \hat{\mu}_X(T_{n1}))(\tilde{Y}_n - \tilde{\gamma}), \ldots, (\tilde{X}_n(T_{nL_n}) - \hat{\mu}_X(T_{nL_n}))(\tilde{Y}_n - \tilde{\gamma}) \end{bmatrix}^\top$$

and $T = \left[ t - T_{11}, \ldots, t - T_{1L_1}, \ldots, t - T_{n1}, \ldots, t - T_{nL_n} \right]^\top$ as well as $\sum_{1 \leq i \leq n} L_i \times \sum_{1 \leq i \leq n} L_i$ matrix

$$W = \text{diag} \left\{ \kappa \left( \frac{t - T_{11}}{h_C} \right), \ldots, \kappa \left( \frac{t - T_{1L_1}}{h_C} \right), \ldots, \kappa \left( \frac{t - T_{n1}}{h_C} \right), \ldots, \kappa \left( \frac{t - T_{nL_n}}{h_C} \right) \right\}.$$

(iii) Fix $s, t \in \mathbb{T}$. Then $\hat{v}_A(s, t)$ is $\hat{a}_0$ from (34) with $\sum_{1 \leq i \leq n} L_i(L_i - 1)$-vector

$$u = \begin{bmatrix} \ldots, (\tilde{X}_i(T_{i1}) - \hat{\mu}_X(T_{i1}))(\tilde{Y}_i(T_{i1}) - \tilde{\gamma}), \ldots, (\tilde{X}_i(T_{iL_i}) - \hat{\mu}_X(T_{iL_i})) \end{bmatrix}^\top \begin{bmatrix} (\tilde{X}_i(T_{i1}) - \hat{\mu}_X(T_{i1}))(\tilde{Y}_i(T_{i1}) - \tilde{\gamma}), \ldots, (\tilde{X}_i(T_{iL_i}) - \hat{\mu}_X(T_{iL_i})) \end{bmatrix} \ldots,$$

$I_{1 \leq i \leq n} L_i(L_i - 1) \times 2$ matrix

$$T = \begin{bmatrix} \ldots & s - T_{i1} & \ldots & s - T_{i1} & \ldots & s - T_{i1} & \ldots & s - T_{i1} & \ldots \\ \ldots & t - T_{i1} & \ldots & t - T_{iL_i} & \ldots & t - T_{iL_i} & \ldots & t - T_{iL_i} & \ldots \end{bmatrix}^\top$$

and $\sum_{1 \leq i \leq n} L_i(L_i - 1) \times \sum_{1 \leq i \leq n} L_i(L_i - 1)$ matrix

$$W = \text{diag} \left\{ \kappa \left( \frac{s - T_{i1}}{h_A} \right), \kappa \left( \frac{t - T_{i1}}{h_A} \right), \ldots, \kappa \left( \frac{s - T_{iL_i - 1}}{h_A} \right), \kappa \left( \frac{t - T_{iL_i - 1}}{h_A} \right), \ldots, \kappa \left( \frac{s - T_{i1}}{h_A} \right), \kappa \left( \frac{t - T_{i1}}{h_A} \right), \ldots \right\}.$$

(iv) Rotate the two tuple $(T_{i11}, T_{i12})$ to become

$$\begin{bmatrix} T_{i11}' \\ T_{i12}' \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix} \begin{bmatrix} T_{i11} \\ T_{i12} \end{bmatrix}.$$  

For arbitrarily fixed $t \in \mathbb{T}$, let $\hat{v}(t)$ equal $\hat{a}_0$ from (34) with $\sum_{1 \leq i \leq n} L_i$-vector

$$u = \begin{bmatrix} (\tilde{X}_1(T_{11}) - \hat{\mu}_X(T_{11}))^2 & \ldots & (\tilde{X}_1(T_{1L_1}) - \hat{\mu}_X(T_{1L_1}))^2 & \ldots & (\tilde{X}_n(T_{n1}) - \hat{\mu}_X(T_{n1}))^2 & \ldots & (\tilde{X}_n(T_{nL_n}) - \hat{\mu}_X(T_{nL_n}))^2 \end{bmatrix}^\top.$$


\[ \sum_{1 \leq i \leq n} L_i \times 2 \text{ matrix} \]

\[
\mathbf{T} = \begin{bmatrix}
-T'_{11} & \cdots & -T'_{1n} \\
-t/\sqrt{2} - T'_{11} & \cdots & t/\sqrt{2} - T'_{1n} \\
-t/\sqrt{2} - T'_{n1} & \cdots & t/\sqrt{2} - T'_{nn}
\end{bmatrix}^T
\]

and \( \sum_{1 \leq i \leq n} L_i \times \sum_{1 \leq i \leq n} L_i \text{ matrix} \)

\[
\mathbf{W} = \text{diag}\left( \kappa \left( t/\sqrt{2} - T'_{11} / h_\sigma \right), \ldots, \kappa \left( t/\sqrt{2} - T'_{n1} / h_\sigma \right), \ldots, \kappa \left( t/\sqrt{2} - T'_{nn} / h_\sigma \right) \right).
\]

Then, as suggested in Yao et al. [2005a], \( \sigma_X^2 \) is estimated by averaging \( \hat{\varphi}(t) - \hat{\varphi}(t, t) \) over a truncated version of \( \mathbb{T} \), say \( \mathbb{T}_1 = [1/4, 3/4] \), i.e., \( \hat{\sigma}_X^2 = 2 \int_{\mathbb{T}_1} (\hat{\varphi}(t) - \hat{\varphi}(t, t)) \, dt \).

Bandwidths \( h_\mu, h_C, h_A \) and \( h_\sigma \) are all tuned through GCV, i.e., they are chosen to minimize

\[
\frac{\mathbf{u}^T \mathbf{W}^{1/2} \mathbf{P} \mathbf{W}^{1/2} \mathbf{T} \mathbf{W}^{1/2} \mathbf{u}}{( \sum_{i=1}^n \mathcal{L}_i - \text{tr}(\mathbf{P} \mathbf{W}^{1/2} \mathbf{T} \mathbf{W}^{1/2} \mathbf{T} \mathbf{W}^{1/2} \mathbf{P}) / 2) \mathbf{u}}
\]

with their respective corresponding \( \mathbf{u}, \mathbf{T} \) and \( \mathbf{W} \). Fan and Gijbels [1996] Eq. 4.3) suggested a rule of thumb which is a good starting point in determining candidate pools for bandwidths.

**B | TECHNICAL DETAILS: ASSUMPTIONS, LEMMAS, AND PROOFS**

Recall that trajectories are observed at time points \( T_{ij} \overset{iid}{\sim} T \), the numbers of observations are \( L_i \overset{iid}{\sim} L \), predictor trajectories are \( X_i \overset{iid}{\sim} X \), scalar responses are \( Y_i \overset{iid}{\sim} Y \) and measurement errors are \( e_{ij} \overset{iid}{\sim} e \) and \( \varepsilon_{ij} \overset{iid}{\sim} \varepsilon \). Write \( f_1, f_2 \) and \( f_3 \) as the respective pdfs of \( T_{ij}, (T_{ij}, X_i(T_{ij})) \) and \( (T_{ij1}, T_{ij2}, X_i(T_{ij1}), X_i(T_{ij2})) \) in which \( X_i(T_{ij}) = X_i(T_{ij1}) + e_{ij} \) are noisy measurements. Regularity conditions [C1]–[C7] are imposed on the above random variables and pdfs. Hyper-parameters in LLS are restricted by conditions [C8]–[C14]. The first three of these are imposed on the kernel function \( \kappa \) and the last four, taking \( h_\mu, h_A, h_\sigma, h_C \) as functions of \( n \), ensure proper convergence rates of these four bandwidths as \( n \) diverges.

Condition [C15] (resp. [C15]) ensures the consistency of \( \hat{\beta}_p \) at [21] in the \( L^2 \) (resp. \( L^\infty \)) sense. These conditions limit the highest divergence rate of \( \rho = p(n) \): as \( n \) diverges, \( \rho \) is chosen to diverge not faster than \( O(\zeta_X^2 A) = O(n^{1/2} h_A^2) \), where \( \zeta_X \) is defined as in [C12]. This restriction on \( \rho \) is fairly close to the setting in Dellaigle and Hall [2012b] Theorem 5.3) who limited discussion to the case with \( ||\nu_A||_2 < 1 \) (reachable by changing the scale on which \( X_i \) is measured). In detail our assumptions are listed as below.

(C1) \( E(L) < \infty \) and \( \text{Pr}(L > 1) > 0 \).

(C2) \( \mu \) and \( \nu_C \) are both continuous on \( \mathbb{T} \), and \( \nu_A \) is continuous on \( \mathbb{T}^2 \). Hence \( ||\mu||_{\infty}, ||\nu_C||_{\infty} \) and \( ||\nu_A||_{\infty} \) are all finite.

(C3) \( X_i, T_{ij1}, \ldots, T_{ijL} \) and \( e_{ij1}, \ldots, e_{ijL} \) are all independent of \( L_i \) in the sense that, given \( L_i = \ell \), \( X_i, T_{ij1}, \ldots, T_{ij\ell} \) and \( e_{ij1}, \ldots, e_{ij\ell} \) are all independent and the conditional laws are those of \( X, T, \) and \( e \).

(C4) \( E \{ (X(T) - \mu_X(T)) + \sigma_X e \}^4 < \infty \).

(C5) \( \frac{d^2}{dt^2} f_1 \) exists and is uniformly continuous on \( \mathbb{T} \) with pdf \( f_1 \) supported on \( \mathbb{T} \).

(C6) \( \frac{d^2}{dt^2} f_2 \) exists and is uniformly continuous on \( \mathbb{T} \times \mathbb{R} \).

(C7) \( \frac{d^2}{dt_1 dt_2} f_2 \), \( \frac{d^2}{dt_1 dt_2} f_3 \) and \( \frac{d^2}{dz_1 dz_2} f_3 \) all exist and are uniformly continuous on \( \mathbb{T}^2 \times \mathbb{R}^2 \).

(C8) The kernel function \( \kappa \) in Appendix A is symmetric (w.r.t. the \( y \) axis) and nonnegative on \( \mathbb{R} \) such that \( \int_{\mathbb{R}} \kappa(t) \, dt = 1 \),
\[ \int_\mathbb{R} \tau^2 \kappa(t)dt < \infty, \text{ and } \int_\mathbb{R} \kappa^2(t)dt < \infty. \]

(C9) The kernel function \( \kappa \) is compactly supported, i.e., it has a bounded support.

(C10) The Fourier transform of \( \kappa \), \( \int_\mathbb{R} e^{-i \tau t} \kappa(s)ds \), is absolutely integrable, i.e., \( \int_\mathbb{R} \left| \int_\mathbb{R} e^{-i \tau t} \kappa(s)ds \right| dt < \infty. \)

(C11) \( h_\mu \to 0, nh_\mu^4 \to \infty, \) \( nh_\mu^6 = O(1), \) and \( \xi_\mu = n^{-1/2} h_\mu^{-1} = o(1), \) as \( n \to \infty. \)

(C12) \( h_A \to 0, nh_A^4 \to \infty, \) \( nh_A^6 = O(1), \) and \( \xi_A = n^{-1/2} h_A^{-2} = o(1), \) as \( n \to \infty. \)

(C13) \( h_\sigma \to 0, nh_\sigma^4 \to \infty, \) \( nh_\sigma^6 = O(1), \) and \( \xi_\sigma = n^{-1/2} (h_\sigma^2 + h_\sigma^{-1}) = o(1), \) as \( n \to \infty. \)

(C14) \( h_C \to 0, nh_C^4 \to \infty, \) \( nh_C^6 = O(1), \) and \( \xi_C = n^{-1/2} (h_C^2 + h_C^{-1}) = o(1), \) as \( n \to \infty. \)

(C15) As \( n \to \infty, \) \( p = p(n) = O(\xi_A^{-1}). \) Additional requirements on \( p \) vary with the magnitude of \( \|v_A\|_2; \) they also depend on \( \tau_p, \) the smallest eigenvalue of \( D_\tau \), which is defined at (35).

- If \( \|v_A\|_2 \geq 1, \) then \( \Omega(\tau_p^{-1} p \|v_A\|_2^2 \xi_C \max(1, \tau_p^{-1} p \|v_A\|_2^2)) \) and \( \Omega(\tau_p^{-1} p^2 \|v_A\|_2^2 \xi_A \max(1, \tau_p^{-1} p \|v_A\|_2^2)) \) are both of order \( o(1); \)
- if \( \|v_A\|_2 < 1, \) then \( \tau_p^{-2} \max(\xi_A, \xi_C) \) and \( \tau_p^{-1} \max(\xi_A, \xi_C) \) are both of order \( o(1). \)

(C15') [C15] holds with the \( L^2 \)-norm \( \|\cdot\|_2 \) replaced by the infinity norm \( \|\cdot\|_\infty \).

The first fourteen of the conditions above are inherited from Yao et al. [2005a]. So is Lemma 1 which states the convergence rate of LLS estimators. We then extend (35) to a more general version (see Lemma 2).

Lemma 1 [Yao et al., 2005a Theorem 1 and Corollary 1, Yao et al., 2005b Lemma A.1] Under assumptions (C1)–(C14) as \( n \to \infty, \)

\[ \|\hat{\mu}_X - \mu_X\|_\infty = O_p(\xi_\mu) = o_p(1), \]

\[ \|\hat{v}_A - v_A\|_\infty = O_p(\xi_A) = o_p(1), \]

\[ |\hat{\sigma}_X^2 - \sigma_X^2| = O_p(\xi_\sigma) = o_p(1), \]

and

\[ \|\hat{v}_C - v_C\|_\infty = O_p(\xi_C) = o_p(1), \]

where \( \xi_\mu, \xi_A, \xi_\sigma \), and \( \xi_C \) are respectively defined as in (C11)–(C14).

Lemma 2 Assume (C1)–(C14) and that there is a \( C > 0 \) such that for all \( n \) we have \( p \in [1, C\xi_A^{-1}] \). Then, for each \( \epsilon > 0, \) there are positive constants \( C_1 \) and \( C_2 \) and an integer \( n_0 > 0 \) such that, for each \( n > n_0, \)

\[ \Pr \left[ \int_1^n \left( \|V_A(\beta) - \hat{V}_A(\beta)\|_2 \leq C_1 \|v_A\|_2^{-1} \xi_C + C_2 (u - 1) \|v_A\|_2^{-1} \xi_A \right) \right] \geq 1 - \epsilon. \]

and

\[ \Pr \left[ \int_1^n \left( \|V_A(\beta) - \hat{V}_A(\beta)\|_\infty \leq C_1 \|v_A\|_\infty^{-1} \xi_C + C_2 (u - 1) \|v_A\|_\infty^{-1} \xi_A \right) \right] \geq 1 - \epsilon. \]

Proof of Lemma 2 Recall the definitions of \( V_A \) in (3) and of \( \hat{V}_A \) in (19). Since \( V_A(\beta) = v_C \) and \( \hat{V}_A(\beta) = \hat{v}_C \), Lemma 2 reduces to (35) when \( j = 1 \). For integer \( j \geq 2 \) and each \( t \in \mathbb{T}, \) the identity

\[ |\hat{V}_A(\beta)(t) - V_A(\beta)(t)| \]
$$\|V_A^j(\beta) - V_A^j(\beta)\|_2 \leq \| \hat{v}_A\|_2 \| V_A^j(\beta) - V_A^j(\beta)\|_2 + \| V_A(\beta)\|_2 v_A - \hat{v}_A, \tag{36}$$

and

$$\|V_A^j(\beta) - V_A^j(\beta)\|_\infty \leq \| \hat{v}_A\|_\infty \| V_A^j(\beta) - V_A^j(\beta)\|_\infty + \| V_A^j(\beta)\|_2 v_A - \hat{v}_A. \tag{37}$$

For each $\epsilon > 0$, there is $n_0 > 0$ such that, for all $n > n_0$, we have

$$1 - \epsilon/2 \leq \text{Pr}(\|v_c - v_A\|_2 \leq C_0 \xi_A) \leq \text{Pr}(\|v_A\|_2 \leq v_A + C_0 \xi_A),$$

$$1 - \epsilon/2 \leq \text{Pr}(\|v_c - v_A\|_\infty \leq C_0 \xi_A) \leq \text{Pr}(\|v_A\|_\infty \leq v_A + C_0 \xi_A),$$

$$1 - \epsilon/2 \leq \text{Pr}(\|v_C - v_A\|_2 \leq C_0 \xi_C),$$

and

$$1 - \epsilon/2 \leq \text{Pr}(\|v_C - v_A\|_\infty \leq C_0 \xi_C).$$

with constant $C_0 > 0$, by \textbf{Lemma 1}. It follows from (34) that

$$1 - \epsilon \leq \frac{1}{p} \sum_{j=1}^p \left( \|V_A^j(\beta) - V_A^j(\beta)\|_2 \leq \left( \|v_A\|_2 + C_0 \xi_A \right) \| V_A(\beta)\|_2 + C_0 \xi_A \sum_{k=1}^{j-1} \| v_A \|_2 \left( \|v_A\|_2 + C_0 \xi_A \right)^{j-k-1} \right) \right)$$

$$\leq \frac{1}{p} \sum_{j=1}^p \left( \|V_A^j(\beta) - V_A^j(\beta)\|_2 \leq C_0 \xi_A / \|v_A\|_2 \| V_A(\beta)\|_2 \| v_A \|_2 \left( \|v_A\|_2 + C_0 \xi_A \right)^{j-1} \right)$$

$$\leq \frac{1}{p} \sum_{j=1}^p \left( \|V_A^j(\beta) - V_A^j(\beta)\|_2 \leq C_1 \|v_A\|_2 \| v_A \|_2 \| v_A \|_2 \left( \|v_A\|_2 + C_0 \xi_A \right)^{j-1} \right), \quad (p \leq C_0 \xi_A^{j-1} \text{ with arbitrarily fixed } C > 0)$$

where $C_1 = C_0 \exp(C_0 / \|v_A\|_2) \geq C_0 \exp(C_0 / \|v_A\|_\infty)$ and $C_2 = \|v\|_2 \|v\|_2$. It is worth noting that we have assumed
that the range of \( p \) is constrained in \([1, C \xi_A^{-1}]\); the quantity \((1 + C_0 \xi_A / \|v_A\|_2)^p\) may not be bounded if \( p \) diverges too fast. Similarly, inequality \([37]\) implies that, for \( 1 \leq p \leq C \xi_A^{-1} \),

\[
\Pr \left[ \left\| \frac{1}{p} \sum_{j=1}^{p} \left( \|V_A^j(\beta) - \hat{V}_A^j(\beta)\|_\infty \right) \right\| \leq C_1 \|v_A\|_\infty ^{-1} \xi_C + C_2 (j - 1) \|v_A\|_\infty ^{-1} \xi_C \right] \geq 1 - \epsilon.
\]

**Proof of Theorem 1.** The following alternative expression for \( \beta_p \) \([12]\), drawn from [Delagile and Hall] \([2012b] \) Eq. (3.6), dramatically facilitates our further moves:

\[
\beta_p = \beta_p(\cdot) = [V_A^1(\beta)(\cdot), \ldots, V_A^p(\beta)(\cdot)] \mathbf{D}_p^{-1} \alpha_p .
\]

where

\[
\mathbf{D}_p = [d_{j_1,j_2}]_{1 \leq j_1,j_2 \leq p}
\]

and

\[
\alpha_p = [\alpha_1, \ldots, \alpha_p]^T.
\]

with \( d_{j_1,j_2} = \int_T V_A^j(\beta)^* V_A^j(\beta) = \int_T V_A^j(\beta)^* V_A^{j_2}(\beta) \) and \( \alpha_j = \int_T V_A(\beta)^* V_A^j(\beta) = \int_T \hat{V}_C(\beta)^* V_A^j(\beta) \). As is known, \( \mathbf{D}_p^{-1} \) and \( \alpha_p \) are bounded, respectively, as

\[
\| \mathbf{D}_p^{-1} \|_2 = r_p^{-1}
\]

and

\[
\| \alpha_p \|_2 = \left[ \sum_{j=1}^{p} \left( \int_T \|C - \hat{V}_C(\beta)\|_2 \right)^2 \right]^{1/2} \leq \left[ \sum_{j=1}^{p} \|v_C\|_2^2 \|\hat{V}_A(\beta)\|_2^2 \right]^{1/2} \quad \text{(Cauchy-Schwarz)}
\]

\[
= \begin{cases} O(p^{1/2} \|v_A\|_2^2) & \text{if } \|v_A\|_2 \geq 1 \\ O(1) & \text{if } \|v_A\|_2 < 1. \end{cases}
\]

Corresponding to \([38]\), \( \hat{\beta}_p \) at \([21]\) is rewritten as

\[
\hat{\beta}_p = \hat{\beta}_p(\cdot) = [\hat{V}_A(\beta)(\cdot), \ldots, \hat{V}_A^p(\beta)(\cdot)] \mathbf{\hat{D}}_p^{-1} \alpha_p .
\]

in which \( \mathbf{\hat{D}}_p = [\hat{d}_{j_1,j_2}]_{1 \leq j_1,j_2 \leq p} \) and \( \hat{\alpha}_p = [\hat{\alpha}_1, \ldots, \hat{\alpha}_p]^T \) are respective empirical counterparts of \( \mathbf{D}_p \) at \([39]\) and \( \alpha_p \) at \([40]\), with \( \hat{d}_{j_1,j_2} = \int_T \hat{V}_A^j(\beta)^* \hat{V}_A^j(\beta) \) and \( \hat{\alpha}_j = \int_T \hat{V}_C(\beta)^* \hat{V}_A^j(\beta) = \int_T \hat{V}_C^j(\beta) \).

Observe that, by the Cauchy-Schwarz inequality,

\[
|\alpha_j - \hat{\alpha}_j| = \left| \int_T (C - \hat{V}_C(\beta))^2 V_A^j(\beta) \right| + \left| \int_T \hat{V}_C(\beta)^* \left[ (\hat{V}_A(\beta) - V_A(\beta))^2 \right] \right| \leq \|\beta\|_2 \|v_A\|_2^2 \|C - \hat{V}_C\|_2 + \|\hat{V}_C\|_2 \|\hat{V}_A(\beta) - V_A(\beta)\|_2.
\]
For every $\epsilon > 0$ and $1 \leq p \leq C\zeta^{-1}$, there is $n_0 > 0$ such that, $\forall n > n_0$, 

$$1 - \epsilon \leq \Pr \left[ \bigcap_{j=1}^{p} \{ |\alpha_j - \hat{d}_j| \leq C_3 \|v_A\|_2^{-1} \zeta_c + C_4(j-1) \|v_A\|_2^{-1} \zeta_A \} \right],$$

(by Lemmas 1 and 2)

with constants $C_3, C_4 > 0$. Analogously, writing $\Delta_{jk} = \hat{d}_{jk} - d_{jk}$, the Cauchy-Schwarz inequality implies that

$$|\Delta_{jk}| \leq \|\tilde{\nu}_j^{i+1}(\beta) - \nu_j^{i+1}(\beta)\|_2 \|\tilde{\nu}_k^{\beta+1}(\beta)\|_2 + \|\tilde{\nu}_k^{\beta}(\beta) - \nu_k^{\beta}(\beta)\|_2 \|\tilde{\nu}_j^{i+1}(\beta)\|_2 \leq \|\tilde{\nu}_A^{i+1}(\beta) - \nu_A^{i+1}(\beta)\|_2 \|\tilde{\nu}_A^{\beta+1}(\beta)\|_2 + \|\tilde{\nu}_A^{\beta}(\beta) - \nu_A^{\beta}(\beta)\|_2 \|\tilde{\nu}_A^{i+1}(\beta)\|_2,$$

and further, by Lemmas 1 and 2, as long as $1 \leq p \leq C\zeta^{-1}$,

$$1 - \epsilon \leq \Pr \left[ \bigcap_{j,k=1}^{p} \{ |\Delta_{jk}| \leq C_5 \|v_A\|_2^{j+k} \zeta_c + C_6 \max(j,k-1) \|v_A\|_2^{j+k} \zeta_A \} \right].$$

where $C_5$ and $C_6$ are positive constants. Thus, if $\Delta_p = |\Delta_{jk}|_{pxp} = \hat{D}_p - D_p$, then

$$\|\Delta_p\|_2^2 \leq \sum_{1 \leq j,k \leq p} \Delta_{jk}^2 = O_p \left( \frac{\epsilon^2 c}{\epsilon} \sum_{1 \leq j,k \leq p} \|v_A\|_2^{2+j+k} \right) + O_p \left( \frac{\epsilon^2 c}{\epsilon} \sum_{1 \leq j,k \leq p} \max(j^2, (k-1)^2) \|v_A\|_2^{2+j+k} \right)$$

$$= \begin{cases} O_p(p^2 \|v_A\|_2^{2p} \zeta_c^2) + O_p(p^4 \|v_A\|_2^{2p} \zeta_A^2) & \text{if } \|v_A\|_2 \geq 1 \\ O_p(\epsilon^2 c) + O_p(\epsilon^2 c) & \text{if } \|v_A\|_2 < 1. \end{cases}$$

(44)

In a similar manner, one proves that

$$\|\hat{\alpha}_p - \alpha_p\|_2^2 = \sum_{1 \leq j \leq p} |\hat{d}_j - \alpha_j|^2 = O_p \left( C_1 \frac{\epsilon^2 c}{\epsilon} \sum_{1 \leq j \leq p} \|v_A\|_2^{2j-2} \right) + O_p \left( \frac{\epsilon^2 c}{\epsilon} \sum_{1 \leq j \leq p} (j-1)^2 \|v_A\|_2^{2j-2} \right)$$

$$= \begin{cases} O_p(p \|v_A\|_2^{2p} \zeta_c^2) + O_p(p^3 \|v_A\|_2^{2p} \zeta_A^2) & \text{if } \|v_A\|_2 \geq 1 \\ O_p(\epsilon^2 c) + O_p(\epsilon^2 c) & \text{if } \|v_A\|_2 < 1. \end{cases}$$

(45)

Denote by $\tau_p$ the smallest eigenvalue of $D_p$. Notice that, for $p = \rho(n) = O(\zeta^{-1})$,

$$\|D_p^{-1}\Delta_p\|_2 \leq \tau_p^{-1} \|\Delta_p\|_2 \leq \begin{cases} O_p(\tau_p^{-1} p \|v_A\|_2^{2p} \zeta_c) + O_p(\tau_p^{-1} p^2 \|v_A\|_2^{2p} \zeta_A) & \text{if } \|v_A\|_2 \geq 1 \\ O_p(\tau_p^{-1} \zeta_c) + O_p(\tau_p^{-1} \zeta_A) & \text{if } \|v_A\|_2 < 1. \end{cases}$$

(41) and (44)

Provided that (C15) holds, for sufficiently large $n$, one has $\tau_p^{-1} \|\Delta_p\|_2 < \gamma$, for some $\gamma \in (0, 1)$. In this case, Delaigle and Hall (2012) argued that, as $n$ goes to infinity,

$$\hat{D}_p^{-1} = (1 - D_p^{-1} \Delta_p + O_p(\tau_p^{-2} \|\Delta_p\|_2^2))D_p^{-1},$$
which can be rewritten as
\[
\|\hat{D}_p^{-1} - D_p^{-1}\|_2 = \| (O_p(\tau_p^{-2} \| A_p \|_2^2) - D_p^{-1} A_p) D_p^{-1} \|_2 \\
= \begin{cases}
\tau_p^{-1} O_p(\tau_p^{-2} p^2 \| v_A \|_2^{Ap} \xi_C^2) + O_p(\tau_p^{-2} p^4 \| v_A \|_2^{Ap} \xi_A^2) - D_p^{-1} A_p \|_2 & \text{if } \| v_A \|_2 \geq 1 \\
\tau_p^{-1} O_p(\tau_p^{-2} \xi_C^2) + O_p(\tau_p^{-2} \xi_A^2) - D_p^{-1} A_p \|_2 & \text{if } \| v_A \|_2 < 1
\end{cases}
\] (by (44))
\[
= \begin{cases}
O_p(\tau_p^{-2} p \| v_A \|_2^{2P} \xi_C^2) + O_p(\tau_p^{-2} p^2 \| v_A \|_2^{2P} \xi_A^2) & \text{if } \| v_A \|_2 \geq 1 \\
O_p(\tau_p^{-2} \xi_C^2) + O_p(\tau_p^{-2} \xi_A^2) & \text{if } \| v_A \|_2 < 1.
\end{cases}
\] (by condition [C15]) (46)

Combining (41), (42), (45) and (46), one obtains
\[
\|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \\
\leq \|\hat{D}_p^{-1} - D_p^{-1}\|_2 \|\alpha_p\|_2 + \|\hat{D}_p^{-1}\|_2 \|\hat{\alpha}_p - \alpha_p\|_2 \\
= \begin{cases}
O_p(\tau_p^{-2} p^{3/2} \| v_A \|_2^{2P} \xi_C) + O_p(\tau_p^{-2} p^{5/2} \| v_A \|_2 \xi_A) + O_p(\tau_p^{-1} p^{1/2} \| v_A \|_2^2 \xi_C) + O_p(\tau_p^{-2} p^{3/2} \| v_A \|_2 \xi_A) & \text{if } \| v_A \|_2 \geq 1 \\
O_p(\tau_p^{-2} \xi_C) + O_p(\tau_p^{-2} \xi_A) + O_p(\tau_p^{-1} \xi_C) + O_p(\tau_p^{-1} \xi_A) & \text{if } \| v_A \|_2 < 1
\end{cases}
\] (47)

Next, for each \( t \in \mathbb{T} \), we have
\[
|\hat{\beta}_p(t) - \beta_p(t)|^2 \\
= |\hat{V}_A(\beta)(t), \ldots, \hat{V}_A^p(\beta)(t)|\hat{D}_p^{-1} \hat{\alpha}_p - |V_A(\beta)(t), \ldots, V_A^p(\beta)(t)|D_p^{-1} \alpha_p|^2 \\
\leq \|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \left[ \sum_{j=1}^p (|\hat{V}_A^j(\beta)(t)|^2)^{1/2} \right]^{1/2} + \|D_p^{-1} \alpha_p\|_2 \left[ \sum_{j=1}^p (|V_A^j(\beta)(t)|^2 - |\hat{V}_A^j(\beta)(t)|^2)^2 \right]^{1/2} \\
\leq 2\|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \left[ \sum_{j=1}^p (|\hat{V}_A^j(\beta)(t)|^2)^{1/2} \right] + 2\|D_p^{-1} \alpha_p\|_2 \left[ \sum_{j=1}^p (|V_A^j(\beta)(t)|^2 - |\hat{V}_A^j(\beta)(t)|^2)^2 \right] \\
(\text{Cauchy-Schwarz})
\]

Thus \( \|\hat{\beta}_p - \beta_p\|_2 \) is bounded as below:
\[
\|\hat{\beta}_p - \beta_p\|_2 \leq 2\|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \sum_{j=1}^p \| V_A^j(\beta)(t) \|_2^2 + 2\|D_p^{-1} \alpha_p\|_2 \sum_{j=1}^p \| V_A^j(\beta)(t) - \hat{V}_A^j(\beta)(t) \|_2^2 \\
\leq 2\|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \sum_{j=1}^p \| V_A^j(\beta)(t) \|_2^2 + 2\tau_p^{-2} \sum_{j=1}^p \| V_A^j(\beta)(t) - \hat{V}_A^j(\beta)(t) \|_2^2 \\
+ 2\sum_{j=1}^p \| V_A^j(\beta)(t) - \hat{V}_A^j(\beta)(t) \|_2^2 \\
(48)
\]

Owing to (47),
\[
\|\hat{D}_p^{-1} \hat{\alpha}_p - D_p^{-1} \alpha_p\|_2 \leq \begin{cases}
O_p(\tau_p^{-4} p^4 \| v_A \|_2^{2P} \xi_C^2) + O_p(\tau_p^{-4} p^6 \| v_A \|_2^{2P} \xi_A^2) + O_p(\tau_p^{-2} p^2 \| v_A \|_2^{2P} \xi_C^2) + O_p(\tau_p^{-2} p^4 \| v_A \|_2 \xi_A^2) & \text{if } \| v_A \|_2 \geq 1 \\
O_p(\tau_p^{-2} \xi_C^2) + O_p(\tau_p^{-2} \xi_A^2) + O_p(\tau_p^{-2} \xi_C^2) + O_p(\tau_p^{-2} \xi_A^2) & \text{if } \| v_A \|_2 < 1.
\end{cases}
\] (49)
the rate of \(49\) is given by \(42\) and \[2\] jointly, i.e.,

\[
49 = \begin{cases} O_p(\tau_p^2 p^2 \| \nu_A \|_2^{8p} \xi_c^2) + O_p(\tau_p^2 p^4 \| \nu_A \|_2^{8p} \xi_A^2) & \text{if } \| \nu_A \|_2 \geq 1 \\
O_p(\tau_p^2 \xi_c^2) + O_p(\tau_p^2 \xi_A^2) & \text{if } \| \nu_A \|_2 < 1.
\end{cases}
\]

In this way we deduce

\[
\| \hat{\beta}_p - \beta_p \|_2^2 = \begin{cases} O_p(\tau_p^2 p^2 \| \nu_A \|_2^{8p} \xi_c^2) + O_p(\tau_p^4 p^6 \| \nu_A \|_2^{8p} \xi_c^2) + O_p(\tau_p^2 p^2 \| \nu_A \|_2^{8p} \xi_A^2) + O_p(\tau_p^4 p^6 \| \nu_A \|_2^{8p} \xi_A^2) & \text{if } \| \nu_A \|_2 \geq 1 \\
O_p(\tau_p^2 \xi_c^2) + O_p(\tau_p^2 \xi_A^2) + O_p(\tau_p^2 \xi_A^2) + O_p(\tau_p^2 \xi_A^2) & \text{if } \| \nu_A \|_2 < 1.
\end{cases}
\]

(50)

Condition \([C15]\) then implies that both \(48\) and \(49\) converge to 0 in probability. The consistency of PLEASS estimators in the \(L^2\) sense follows, from the \(L^2\) convergence of \(\beta_p\) to \(\beta\) [Delaigle and Hall 2012b Theorem 3.2].

Finally, we bound the estimation error in the supremum metric:

\[
\| \hat{\beta}_p - \beta_p \|_\infty = \left\| \left[ \hat{\nu}_A(\beta), \ldots, \hat{\nu}_A'(\beta) \right] \right\|_{\infty} \leq \left\| \tilde{\nu}_A(\beta) \right\|_{\infty} + \left\| \nu_A(\beta) - \hat{\nu}_A(\beta) \right\|_{\infty}.
\]

That is, the upper bound for \(\| \hat{\beta}_p - \beta_p \|_\infty\) can be obtained from \([50]\) by replacing \(\| \nu_A \|_2\) with \(\| \nu_A \|_\infty\). Condition \([C15]\) completes the proof for the zero-convergence of \(\| \hat{\beta}_p - \beta \|_\infty\), as long as we assume \(\| \beta_p - \beta \|_\infty \to 0 \) as \(p \to \infty\).

**Proof of Theorem 2.** Recall the definitions of \(\beta_p\) at \([12]\) and \(\hat{\beta}_p\) at \([21]\). Introduce \(s_p = [\nu_A(\mathbf{w}_1), \ldots, \nu_A(\mathbf{w}_p)]^T\) and its empirical version \(\hat{s}_p = [\hat{\nu}_A(\mathbf{w}_1), \ldots, \hat{\nu}_A(\mathbf{w}_p)]^T\). Note the identities that \[c_p^T \Lambda_p^{-1} s_p = \nu_A(\beta_p)\] and \[c_p^T \Lambda_p^{-1} \hat{s}_p = \hat{\nu}_A(\hat{\beta}_p)\].

Thus, conditions \([C1]\)–\([C15]\) jointly ensure that, for arbitrarily given \(L^*, T_1^*, \ldots, T_L^*\),

\[
\| \mathbf{H}_p \Lambda_p^{-1} \mathbf{e}_p - \hat{\mathbf{H}}_p \Lambda_p^{-1} \mathbf{e}_p \|_2^2 \\
\leq L^* \| \mathbf{H}_p \Lambda_p^{-1} \mathbf{e}_p - \hat{\mathbf{H}}_p \Lambda_p^{-1} \mathbf{e}_p \|_\infty^2 \\
= L^* \sup_{t \in [0, T]} \int_T \nu_A(s, t) (\beta_p - \hat{\beta}_p, s) ds + \int_T (\nu_A - \hat{\nu}_A)(s, t) \hat{\beta}_p(s) ds \\
\leq L^* \sup_{t \in [0, T]} \left( \int_T \nu_A(s, t) ds \right)^{1/2} \| \beta_p - \hat{\beta}_p \|_2 + \left( \int_T (\nu_A - \hat{\nu}_A)^2(s, t) ds \right)^{1/2} \| \beta_p \|_2^2
\] (Cauchy-Schwarz)
\[ \leq \mathcal{L}^*(\|v_A\|_{\infty} \|\beta - \hat{\beta}\|_2 + \|v_A - \hat{v}_A\|_{\infty} \|\hat{\beta} - \beta\|_2)^2 \]

\( \rightarrow_p 0. \) (by Lemma 1 and Theorem 1)

The convergence to 0 (in probability and conditional on \( L^* \) and \( T_1^*, \ldots, T_{L^*} \)) of \( \tilde{\eta}_p(X^*) - \eta_p(X^*) \) (with \( \tilde{\eta}_p(X^*) \) at 27 and \( \tilde{\eta}_\infty(X^*) \) at 25) follows from Lemma 1 and the continuous mapping and Slutsky’s theorems. Since \( L^* \) and \( T_1^*, \ldots, T_{L^*} \) are arbitrary the dominated convergence theorem enables us to drop the conditioning. This completes the proof of Theorem 2.

**Proof of Corollary 1** Recall \( \eta_p(X^*) \) at 16, \( \tilde{\eta}_p(X^*) \) at 26 and \( \tilde{\eta}_\infty(X^*) \) at 25. As discussed in the last paragraph of Section 2.1, \( [\xi_1^* - \xi_1^*, \ldots, \xi_p^* - \xi_p^*] \sim N(0, A_p - H_p^\top \Sigma_p^{-1} H_p) \). It follows that \( \tilde{\eta}_p(X^*) - \eta_p(X^*) \sim N(0, c_p \Sigma_p^{-1}(A_p - H_p^\top \Sigma_p^{-1} H_p)\Lambda_p^{-1} c_p) \) and further that \( \tilde{\eta}_p(X^*) - \eta_p(X^*) \) converges (in distribution) to \( N(0, \omega) \) as \( n \to \infty \), by Theorem 2. Slutsky’s theorem now completes the proof.

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