The variance and the asymptotic distribution of the length of longest $k$-alternating subsequences

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We obtain an explicit formula for the variance of the number of $k$-peaks in a uniformly random permutation. This is then used to obtain an asymptotic formula for the variance of the length of longest $k$-alternating subsequence in random permutations. Also a central limit is proved for the latter statistic.

Keywords: Alternating subsequences, $k$-alternating subsequences, Peak, central limit theorem

1 Introduction

Letting $(a_i)_{i=1}^n$ be a sequence of real numbers, a subsequence $a_{i_k}$, where $1 \leq i_1 < \ldots < i_k \leq n$, is called an alternating subsequence if $a_{i_1} > a_{i_2} < a_{i_3} > \ldots$. The length of the longest alternating subsequence of $(a_i)_{i=1}^n$ is defined to be the largest integer $q$ such that $(a_i)_{i=1}^n$ has an alternating subsequence of length $q$. Denoting the symmetric group on $n$ letters by $S_n$, an alternating subsequence of a permutation $\sigma \in S_n$ refers to an alternating subsequence corresponding to the sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$. See Stanley (2008) for a survey on the topic.

The purpose of this manuscript is to study a generalization of the length of longest alternating subsequences in uniformly random permutations. Letting $\sigma \in S_n$, a subsequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is said to be $k$-alternating for $\sigma$ if

$$\sigma(i_1) \geq \sigma(i_2) + k, \quad \sigma(i_2) + k \leq \sigma(i_3), \quad \sigma(i_3) \geq \sigma(i_4) + k, \ldots$$

In other words, the subsequence is $k$-alternating if it is alternating and additionally

$$|\sigma(i_j) - \sigma(i_{j+1})| \geq k, \quad j \in [t-1],$$

where we set $[m] = \{1, \ldots, m\}$ for $m \in \mathbb{N}$. Below the length of the longest $k$-alternating subsequence of $\sigma \in S_n$ is denoted by $a_{n,k}(\sigma)$, or simply $a_{n,k}$.

Let us also define $k$-peaks and $k$-valleys which will be intermediary tools to understand the longest $k$-alternating subsequences. Let $\sigma = \sigma(1) \ldots \sigma(n) \in S_n$. We say that a section $\sigma(i) \ldots \sigma(j)$ of the permutation $\sigma$ is a $k$-up ($k$-down, resp.) if $i < j$ and $\sigma(j) - \sigma(i) \geq k$ ($\sigma(i) - \sigma(j) \geq k$, resp.). We say that the section is $k$-ascending if it satisfies:

- $\sigma(i) = \min\{\sigma(i), \ldots, \sigma(j)\}$ and $\sigma(j) = \max\{\sigma(i), \ldots, \sigma(j)\}$, and
- the section $\sigma(i) \ldots \sigma(j)$ is a $k$-up, and
- there is no $k$-down in $\sigma(i) \ldots \sigma(j)$, i.e. for any $i \leq s < t \leq j$, we have $\sigma(s) - \sigma(t) < k$.

If also there is no $k$-ascending section that contains $\sigma(i) \ldots \sigma(j)$, it is called a maximal $k$-ascending section. In this case, $\sigma(i), \sigma(j)$ are called a $k$-valley and a $k$-peak of $\sigma$, respectively.

A maximal $k$-descending section $\sigma(i) \ldots \sigma(j)$ can be defined similarly, and this time $\sigma(i), \sigma(j)$ are called a $k$-peak and a $k$-valley of $\sigma$, respectively. An alternative description can be given as in Cai (2013).

**Proposition 1.1** Let $\sigma = \sigma(1) \sigma(2) \ldots \sigma(n) \in S_n$, $i \in [n]$ and $1 \leq k \leq n - 1$. Then $\sigma(i)$ is a $k$-peak if and only if it satisfies both of the following two properties:

(i) If there is an $s > i$ with $\sigma(s) > \sigma(i)$, then there is a $k$-down $\sigma(i) \ldots \sigma(j)$ in $\sigma(i) \ldots \sigma(s)$.

(ii) If there is an $s < i$ with $\sigma(s) > \sigma(i)$, then there is a $k$-up $\sigma(j) \ldots \sigma(i)$ in $\sigma(s) \ldots \sigma(i)$.
Considering the case where \( \sigma \) is a uniformly random permutation, our purpose in present paper is to study \( \text{Var}(a_{n,k}) \) and to show that \( a_{n,k} \) satisfies a central limit theorem. The statistic \( \text{Var}(a_{n,k}) \) is well understood for the case \( k = 1 \). Indeed, Stanley proved in \cite{Stanley2008} that
\[
\mathbb{E}[a_{n,1}] = \frac{4n + 1}{6} \quad \text{and} \quad \text{Var}(a_{n,1}) = \frac{8n}{45} + \frac{13}{180}.
\]
It was later shown in \cite{HoudreRestrepo2010} and \cite{Komik2011} that \( a_{n,1} \) satisfies a central limit theorem, and convergence rates for the normal approximation were obtained in \cite{Iskak2018}. All these limiting distribution results rely on the simple fact that \( a_{n,1} \) can be represented as a sum of \( m \)-dependent random variables (namely, the indicators of local extrema) and they then use the well-established theory of such sequences.

Regarding the general \( k \), Armstrong conjectured in \cite{Armstrong2014} that \( \mathbb{E}[a_{n,k}] = \frac{4(n-k)+5}{6} \). Pak and Pemantle \cite{PakPemantle2015} then used probabilistic methods to prove that \( \mathbb{E}[a_{n,k}] \) is asymptotically \( \frac{2(n-k)}{n} + O\left(n^{2/3}\right) \).

Let us very briefly mention their approach. For \( x \in (0,1) \), a vector \( y = (y_1, \ldots, y_n) \in [0,1]^n \) is said to be \( x \)-alternating if \((-1)^{j+1}(y_j - y_{j+1}) \geq x \) for all \( 1 \leq j \leq n - 1 \). Given a vector \( y = (y_1, \ldots, y_n) \in [0,1]^n \), a subsequence \( 1 \leq i_1 < i_2 < \ldots < i_r \leq n \) is said to be \( x \)-alternating for \( y \) if
\[
|y_{i_j} - y_{i_{j+1}}| \geq x, \quad j \in [r-1].
\]
Denoting the length of the longest \( x \)-alternating subsequence of a random vector \( y \), with Lebesgue measure on \([0,1]^n\), as its distribution, by \( a_{n,x}(y) \), their main observation was: If \( Z \) is a binomial random variable with parameters \( n \) and \( 1 - x \), then
\[
a_{n,x}(y) \overset{d}{=} a_{n,1}.
\]
(Here, \( d \) means equality in distribution). That is, they concluded that \( a_{n,x}(y) \) has the same distribution as the length of the longest ordinary alternating subsequence of a random permutation on \( S_n \). Afterwards, using \( \mathbb{E}[a_{n,1}] = \frac{4n+1}{6} \) and \( \text{Var}(a_{n,1}) = \frac{8n}{45} + \frac{13}{180} \), they proved
\[
\mathbb{E}[a_{n,x}] = \frac{2}{3}n(1-x) + \frac{1}{6} \quad \text{and} \quad \text{Var}(a_{n,x}) = (1-x)(2+5x)\frac{4n}{45}.
\]
Further, for suitable \( x_1 \) and \( x_2 \), they showed that \( \mathbb{E}[a_{n,x_2}] \leq \mathbb{E}[a_{n,k}] \leq \mathbb{E}[a_{n,x_1}] \) and in this way they are able to bound \( \mathbb{E}[a_{n,k}] \).

A closely related problem to the longest alternating subsequence problem is that of calculating the longest zigzagging subsequence. For a given permutation \( \sigma \), denoting its vertical flip by \( \bar{\sigma} \), a subsequence is said to be zigzagging if it is alternating for either \( \sigma \) or \( \bar{\sigma} \). The \( k \)-zigzagging case is defined similarly. We will be using the notation \( z_{n,k} \) for the length of the longest \( k \)-zigzagging subsequence in the sequel. Note that in exactly half of the permutations, \( a_{n,k} \) and \( z_{n,k} \) are equal to each other, and in the other half the length of the \( k \)-zigzagging subsequence is exactly one more than the length of the \( k \)-alternating subsequence. This is seen via the involution map \( I: \sigma(1)\sigma(2)\ldots\sigma(n) \rightarrow (n+1-\sigma(1))(n+1-\sigma(2))\ldots(n+1-\sigma(n)) \) as noted in \cite{Cai2015b}. Therefore
\[
\mathbb{E}[z_{n,k}] = \mathbb{E}[a_{n,k}] + 1/2. \tag{1}
\]
Cai proved in 2015 that \( \mathbb{E}[z_{n,k}] = \frac{2(n-k)+4}{3} \), and then combining this with \( \eqref{1} \), solved the Armstrong conjecture \cite{Cai2015b}.

Our first result in this paper is an asymptotic formula for \( \text{Var}(a_{n,k}) \). Namely, we will prove
\[
\text{Var}(a_{n,k}) = \frac{8(n-k)}{45} + O(\sqrt{n}).
\]
In order to obtain this result, we first study the number of \( k \)-peaks \( P \) in random permutations and show that
\[
\text{Var}(P) = \frac{2(n-k) + 4}{45}.
\]
Our second result is a central limit theorem for \( a_{n,k} \):
\[
\frac{a_{n,k} - \mathbb{E}[a_{n,k}]}{\sqrt{\text{Var}(a_{n,k})}} \xrightarrow{d} G,
\]
where \( G \) is the standard normal distribution and where \( \xrightarrow{d} \) is used for convergence in distribution.

The rest of the paper is organized as follows. Next section proves our formulas for the variances of \( P \) and \( a_{n,k} \). In Section \[3\] we prove the central limit theorem for \( a_{n,k} \).
2 The variances of $P$ and $\text{as}_{n,k}$

Next result gives an exact formula for the variance of the number of $k$-peaks $P$ in a uniformly random permutation.

**Theorem 2.1** Let $P$ be the number of $k$-peaks in a uniformly random permutation in $S_n$. We have

$$\text{Var}(P) = \frac{2(n-k) + 4}{45}.$$

We will prove Theorem 2.1 after providing a corollary related to the length of longest $k$-alternating subsequence of a uniformly random permutation. Note that we have $\text{as}_{n,k} = 2P + E$ where $|E| \leq 1$ for any $n, k$. Thus, $\text{Var}(\text{as}_{n,k}) = 4\text{Var}(P) + \text{Var}(E) + 2\text{Cov}(P,E)$. Here, clearly $\text{Var}(E) \leq 1$ and by Cauchy-Schwarz inequality $|\text{Cov}(P,E)| \leq 2\sqrt{\text{Var}(P)\text{Var}(E)} \leq C_0\sqrt{n}$ where $C_0$ is a constant independent of $n$ and $k$. We now obtain the following.

**Corollary 2.1** Let $\text{as}_{n,k}$ be the length of longest $k$-alternating subsequence of a uniformly random permutation in $S_n$. Then,

$$\text{Var}(\text{as}_{n,k}) = \frac{8(n-k)}{45} + O(\sqrt{n}).$$

In particular, when $k = o(n)$, $\text{Var}(\text{as}_{n,k}) \sim \frac{8n}{45}$ as $n \to \infty$.

**Remark 2.1** In setting of Corollary 2.1 we conjecture that $\text{Var}(\text{as}_{n,k}) = \frac{8(n-k)}{45} + \frac{19}{180}$. Although we have a heuristic derivation of this equality, we were not able to justify it rigorously.

Now, let us proceed to the proof of Theorem 2.1.

**Proof of Theorem 2.1.** Below $P_i$ is the indicator of $i$ being a $k$-peak, i.e.

$$P_i := \begin{cases} 
1, & \text{if } i \text{ is a } k\text{-peak}, \\
0, & \text{otherwise}.
\end{cases}$$

In particular,

$$P = \sum_{i=1}^{n} P_i.$$

We are willing to compute

$$\text{Var}(P) = \text{Var} \left( \sum_{i=1}^{n} P_i \right) = \mathbb{E} \left[ \left( \sum_{i=1}^{n} P_i \right)^2 \right] - \left( \mathbb{E} \left[ \sum_{i=1}^{n} P_i \right] \right)^2.$$

Recall from Cai (2015) that

$$\mathbb{E} \left[ \sum_{i=1}^{n} P_i \right] = \mathbb{E}[P] = \frac{1}{2} \mathbb{E}[\text{zs}_k] = \frac{n-k+2}{3}. \quad (2)$$

Let us next analyze

$$\mathbb{E} \left[ \left( \sum_{i=1}^{n} P_i \right)^2 \right] = \sum_{i=1}^{n} \mathbb{E}[P_i^2] + 2 \sum_{i<j} \mathbb{E}[P_iP_j].$$

Denoting the probability that $i$ is a $k$-peak by $p_{n,k}(i)$ and the probability that both $i, j$ are $k$-peaks by $p_{n,k}(i,j)$, we may rewrite this last equation as

$$\mathbb{E} \left[ \left( \sum_{i=1}^{n} P_i \right)^2 \right] = \sum_{i=1}^{n} p_{n,k}(i) + 2 \sum_{i<j} p_{n,k}(i,j).$$

We already know from (2) that the first sum on the right-hand side is $\frac{n-k+2}{3}$. We are then left with calculating $p_{n,k}(i,j)$.

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(1) Note that when we say $i$ is a $k$-peak, we consider $i$ to be an element in the image of the permutation, not an element of the domain of the permutation. If the position $i$ is considered in domain of the permutation, we will be emphasizing it there.
With the definition of $k$-peaks in mind, for given $i$ and $j$, we can divide $[n] \setminus \{i\}$ and $[n] \setminus \{j\}$ into three sets according to the following partitions respectively. The first partition is with respect to $i$:

\[
A_i = \{\ell : 1 \leq \ell \leq i - k\},
B_i = \{\ell : i - k + 1 \leq \ell \leq i - 1\},
C_i = \{\ell : i + 1 \leq \ell \leq n\},
\]

and the second partition is with respect to $j$:

\[
A_j = \{\ell : 1 \leq \ell \leq j - k\},
B_j = \{\ell : j - k + 1 \leq \ell \leq j - 1\},
C_j = \{\ell : j + 1 \leq \ell \leq n\}.
\]

Assuming without loss of generality that $i < j$, observe

\[
i < j \implies A_i \subset A_j,
\]

\[
i < j \implies C_j \subset C_i.
\]

By Proposition 1.1, we observe that for $i$ to be a $k$-peak, there should be at least one element from $A_i$ between any element of $C_i$ and $i$, and similarly for $j$ to be a $k$-peak, there should be at least one element from $A_j$ between any element of $C_j$ and $j$. To ensure these two properties, we will place the elements accordingly.

Our procedure for placing the elements starts with placing $A_i \cup \{i\}$ in a row $a_1a_2\ldots a_{i-1}k+1$ arbitrarily. Leaving the insertion of the elements in $A_j \setminus A_i$ to the end of the argument, we will next focus on placing the elements of $C_i$ and $C_j$. Note that by the observation in previous paragraph, in order to have $i$ and $j$ as $k$-peaks, the two places next to $i$ are not available for the elements in $C_i \setminus C_j$, and the four places next to $i$ and $j$ are not available for the elements in $C_i \cap C_j = C_j$.

Now, let us focus on the elements of $C_i \setminus C_j = \{i + 1, \ldots, j\}$. There are $|A_i \cup \{i\}| = i - k + 1$ elements that are placed in a row. Thus, we have $i - k + 2$ vacant spots for the element $i - k + 2$ to be inserted into the row $a_1a_2\ldots a_{i-1}k+1$. Since the two places next to $i$ are prohibited, we see that

\[
\Pr(\{i + 1\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{i - k}{i - k + 2}.
\]

Now, we have $i + k + 3$ vacant spots for the element $i + 2$, and the two places next to $i$ are prohibited, and so,

\[
\Pr(\{i + 2\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{i - k + 1}{i - k + 3}.
\]

Continuing in this manner, we see that when we arrive at $j$, which is the last element to be inserted in from the set $C_i \setminus C_j$, we have $i - k + (j - i + 1) = j - k + 1$ many vacant places, and the two places next to $i$ are prohibited, and then

\[
\Pr(\{j\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{j - k - 1}{j - k + 1}.
\]

More generally, for $t = 1, \ldots, j - i$, we have

\[
\Pr(\{i + t\} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{i - k + t - 1}{i - k + t + 1}.
\]

Therefore,

\[
\Pr(C_i \setminus C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \prod_{t=1}^{j-i} \Pr(\{i + t\} \text{ does not prevent } i, j \text{ being a } k\text{-peak})
\]

\[
= \prod_{t=1}^{j-i} \frac{i - k + t - 1}{i - k + t + 1}
\]

\[
= \frac{(i - k)(i - k + 1)}{(j - k)(j - k + 1)}.
\]
Now, let us focus on the elements of \( C_i \cap C_j = C_i \cap C_j = \{ j + 1, \ldots, n \} \). Recall that there are four prohibited places for these elements to be inserted. We have \( j - k + 2 \) many vacant places to insert \( j + 1 \) into but four of these are prohibited. Thus,

\[
\mathbb{P}(\{ j + 1 \} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{j - k - 2}{j - k + 2}.
\]

Similar to the analysis in \( C_i \setminus C_j \), continuing in this manner, we have \( n = j + (n - j) \), and in the end we will have \( j - k + (n - j + 1) = n - k + 1 \) many vacant places to insert \( n \), and four of these are prohibited. So,

\[
\mathbb{P}(n \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{n - k - 3}{n - k + 1}.
\]

We may generalize this to obtain

\[
\mathbb{P}(\{ j + t \} \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \frac{j - k + t - 3}{j - k + t + 1}
\]

for \( t = 1, \ldots, n - j \). We then obtain

\[
\mathbb{P}(C_i \cap C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak}) = \prod_{t=1}^{n-j} \mathbb{P}(\{ j + t \} \text{ does not prevent } i, j \text{ being a } k\text{-peak})
\]

\[
= \prod_{t=1}^{n-j} \frac{j - k + t - 3}{j - k + t + 1}
\]

\[
= \frac{(j - k - 2)(j - k - 1)(j - k)(j - k + 1)}{(n - k - 2)(n - k - 1)(n - k)(n - k + 1)}.
\]

Note that we can multiply the probabilities (here, and above in the case of \( C_i \setminus C_j \)), since in essence what we are doing is conditioning on the event that the previous added elements do not prevent \( i, j \) being a \( k\)-peak. Now, clearly, the elements of \( A_j \setminus A_i \) are in \( B_i \cup C_i \). Since the elements that are in \( C_i \) have been inserted, we will then be done once we insert the elements of \( B_i \) and \( B_j \). But the elements in the sets \( B_i \) and \( B_j \) have no effect on \( i \) and \( j \) being a \( k\)-peak (once the elements from \( C_i \) and \( C_j \) are placed), and so we may insert them in any place. Thus, overall, we have

\[
p_{n,k}(i, j) = \mathbb{P}(\text{the set } C_i \cup C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak})
\]

\[
= \mathbb{P}(C_i \setminus C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak})
\]

\[
\times \mathbb{P}(C_i \cap C_j \text{ does not prevent } i, j \text{ being a } k\text{-peak})
\]

\[
= \frac{(i - k)(i - k + 1)}{(j - k)(j - k + 1)} \frac{(j - k - 2)(j - k - 1)(j - k)(j - k + 1)}{(n - k - 2)(n - k - 1)(n - k)(n - k + 1)}
\]

\[
= \frac{(i - k)(i - k + 1)(j - k)(j - k + 1)}{(n - k - 2)(n - k - 1)(n - k)(n - k + 1)}.
\]

These add up to

\[
\sum_{i<j} p_{n,k}(i, j) = \sum_{i=k+1}^{n} \sum_{j=i+1}^{n} \frac{(i - k)(i - k + 1)(j - k)(j - k + 1)}{(n - k - 2)(n - k - 1)(n - k)(n - k + 1)}
\]

\[
= \frac{1}{90} (5k - 5n + 3)(k - n - 2),
\]

where the sum is computed fairly easily noting that essentially we are summing the consecutive integers and squares of consecutive integers. Therefore we obtain

\[
\mathbb{E} \left[ \left( \sum_{i=1}^{n} P_i \right)^2 \right] = \sum_{i=1}^{n} p_{n,k}(i) + 2 \sum_{i<j} p_{n,k}(i, j) = \frac{n - k + 2}{3} + \frac{1}{45} (5k - 5n + 3)(k - n - 2)
\]

\[
= \frac{n - k + 2}{3} \left( 1 + \frac{1}{15} (5n - 5k - 3) \right) = \frac{1}{45} (n - k + 2)(5n - 5k + 12).
\]
Using this we arrive at

\[
\text{Var}(P) = \mathbb{E} \left[ \left( \sum_{i=1}^{n} P_i \right)^2 \right] - \left( \mathbb{E} \left[ \sum_{i=1}^{n} P_i \right] \right)^2 \\
= \frac{1}{45} (n - k + 2)(5n - 5k + 12) - \left( \frac{n - k + 2}{3} \right)^2 = \frac{2(n - k) + 4}{45}
\]

as asserted in Theorem \ref{thm:clt}.

3. A Central Limit Theorem

In this section, we will prove the following central limit theorem.

**Theorem 3.1** Let \( k \) be a fixed positive integer. Then the length of the longest \( k \)-alternating subsequence \( s_{n,k} \) of a uniformly random permutation satisfies a central limit theorem,

\[
\frac{s_{n,k} - \mathbb{E}[s_{n,k}]}{\sqrt{\text{Var}(s_{n,k})}} \rightarrow_{d} \mathcal{G},
\]

where \( \mathcal{G} \) is the standard normal distribution.

The proof involves a suitable truncation argument that allows us to reduce the problem to proving a central limit theorem for sums of locally dependent random variables for which a theory is already available. Since the length of the longest \( k \) alternating sequence differs from twice the number of \( k \) peaks by at most 1, we may focus on the number of peaks. For any \( i \), let \( P_i \) be the random variable that is 1 if the value \( i \) is a \( k \)-peak and zero otherwise as before. Also recall \( P = P_1 + \cdots + P_n \). We know that \( P_i = 1 \) precisely when

- Scanning to the right of the value \( i \), we encounter an element in \([i-k, i]\) before we encounter an element in \([i+1, n]\). It is permitted that we do not encounter an element from \([i+1, n]\) at all.
- Scanning to the left of the value \( i \), we encounter an element in \([i-k, i]\) before we encounter an element in \([i+1, n]\). It is permitted that we do not encounter an element from \([i+1, n]\) at all.

Our approach to getting a central limit theorem is to define a suitable truncation that can be computed using local data. There are a number of theorems that establish central limit behaviour for variables with only local correlations and this approach has been employed in a number of situations.

Note that the condition on \( P_i = 1 \) can be restated as

- There is an index \( j > \sigma^{-1}(i) \) such that \( i - k \geq \sigma(j) \) and such that
  \[
i = \max_{s \in [\sigma^{-1}(i), j]} \sigma(s), \quad \sigma(j) = \min_{s \in [\sigma^{-1}(i), j]} \sigma(s).
\]

- There is an index \( j < \sigma^{-1}(i) \) such that \( i - k \geq \sigma(j) \) and such that
  \[
i = \max_{s \in [j, \sigma^{-1}(i)]} \sigma(s), \quad \sigma(j) = \min_{s \in [j, \sigma^{-1}(i)]} \sigma(s).
\]

Note that we might need to scan far to the left and right in order to determine whether a value is a \( k \)-peak or not and thus we will have long range dependence. We will show that ignoring long range interactions does not change the statistic very much.

Fix a number \( m \) that we will specify later. Let \( Y_i = 1 \) if we can determine that \( i \) is a \( k \)-peak by only looking at \( m \) positions to the left and right of \( i \). Precisely, let \( Y_i = 1 \) if

- There is an index \( j \in [\sigma^{-1}(i), \sigma^{-1}(i) + m] \) such that \( i - k \geq \sigma(j) \) and such that
  \[
i = \max_{s \in [\sigma^{-1}(i), j]} \sigma(s), \quad \sigma(j) = \min_{s \in [\sigma^{-1}(i), j]} \sigma(s).
\]

- There is an index \( j \in [\sigma^{-1}(i) - m, \sigma^{-1}(i)] \) such that \( i - k \geq \sigma(j) \) and such that
  \[
i = \max_{s \in [j, \sigma^{-1}(i)]} \sigma(s), \quad \sigma(j) = \min_{s \in [j, \sigma^{-1}(i)]} \sigma(s).
\]
If $Y_i = 1$, we call it a local $k$-peak (suppressing the reference to $m$). Note that any local $k$-peak is a $k$-peak and thus, $Y_i \leq P_i$. We should next understand the case where $Y_i = 0$ and $P_i = 1$. Note that if $i \leq k$, then $P_i = Y_i = 0$.

If $\sigma^{-1}(i) \in [m+1]$, there is no issue when scanning to the left. However, if we scan to the right and this event happens, then the $m$ indices to the right should have values in $[i-k+1, i-1]$. The probability of this is at most $\left(\frac{k-1}{n-1}\right)^m$. Similarly, the probability of this event when $\sigma^{-1}(i) \in [n-m, n]$ is at most $\left(\frac{k-1}{n-1}\right)^m$.

If $\sigma^{-1}(i) \in [m+2, n-m-2]$, the event can only happen if the $2m$ positions, $m$ to the left and $m$ to the right take values in $[i-k-1, i-1]$ and the probability of this is at most $\left(\frac{k-1}{n-1}\right)^{2m}$.

Putting these together, recalling $Y_i \leq P_i$, and denoting the total variation distance by $d_{TV}$, we see that

$$d_{TV}(P_i, Y_i) = \frac{1}{2} \sum_{j=0}^{1} |\mathbb{P}(P_i = j) - \mathbb{P}(Y_i = j)|$$

$$= \frac{1}{2} (\mathbb{P}(Y_i = 0) - \mathbb{P}(P_i = 0) + \mathbb{P}(P_i = 1) - \mathbb{P}(Y_i = 1))$$

$$= \frac{1}{2} (2(\mathbb{P}(P_i = 1) - \mathbb{P}(Y_i = 1)))$$

$$= \mathbb{P}(Y_i = 0), P_i = 1$$

$$\leq \frac{2m + 2}{n} \left(\frac{k-1}{n-1}\right)^m + \frac{n-2m-2}{n} \left(\frac{k-1}{n-1}\right)^{2m}.$$

This implies

$$d_{TV}(P_1 + \ldots + P_n, Y_1 + \ldots + Y_n) \leq \frac{(2m + 2)(n)}{n} \left(\frac{k-1}{n-1}\right)^m$$

$$+ \frac{(n-2m-2)(n)}{n} \left(\frac{k-1}{n-1}\right)^{2m},$$

$$\leq (2m + 2) \left(\frac{k}{n}\right)^m + n \left(\frac{k}{n}\right)^{2m},$$

$$\leq 3n \left(\frac{k}{n}\right)^m. \tag{3}$$

When $k$ is fixed, taking $m = 3$ suffices for our purpose. Note in particular that

$$\mathbb{P}(Y_1 + \ldots + Y_n < P_1 + \ldots + P_n) = o \left(\frac{1}{n}\right), \tag{4}$$

when $m$ is chosen appropriately.

Next we will show that $Y = Y_1 + \ldots + Y_n$ satisfies a central limit theorem. Let $Z_i = 1$ if the position $i$ is a local $k$-peak and 0 otherwise. It is immediate that $Z = Z_1 + \ldots + Z_n$ and $Y$ have the same distribution. We let $Z$ be such a random variable for which $(P, Z)$ and $(P, Y)$ have the same distribution. Further, note that the variables $Z_i$ have the property that $Z_i$ and $Z_j$ are independent if $|i - j| > 2m$.

There are a number of related theorems that guarantee central limit behaviour for sums of locally dependent variables. A result due to Rinott will suffice for our purpose. The version we give is a slight variation of the one discussed in [1994].

**Theorem 3.2** Let $U_1, \ldots, U_n$ be random variables such that $U_i$ and $U_j$ are independent when $|i - j| > 2m$. Setting $U = U_1 + \ldots + U_n$, we have

$$d_K \left(\frac{U - \mathbb{E}[U]}{\sqrt{\text{Var}(U)}}, G\right) \leq C(2m + 1) \sqrt{\frac{\sum_{i=1}^{n} |\mathbb{E}[U_i]|^3}{(\text{Var}(U))^{3/2}},$$

where $d_K$ is the Kolmogorov distance.

We will now apply this result for $Z = Z_1 + \ldots + Z_n$. For this purpose we need a lower bound on the variance of the random variable $Z$. Recall that the variance of $P$ is $\Omega(n)$ and let us show that the same holds for $Z$. 
We have

\[
\sqrt{\text{Var}(Z)} \geq \sqrt{\text{Var}(P)} - \sqrt{\text{Var}(P - Z)} \\
\geq \sqrt{\text{Var}(P)} - \sqrt{\mathbb{E}[(P - Z)^2]} \\
\geq \sqrt{\text{Var}(P)} - \sqrt{n \mathbb{E}|P - Z|} \\
\geq \sqrt{\text{Var}(P)} - \sqrt{n \mathbb{E}|P - Z| \mathbb{P}(P \neq Z)} \\
\geq \sqrt{\text{Var}(P)} - \sqrt{n \mathbb{E}|P - Z|^2} \\
= \Omega(\sqrt{n}) - o(\sqrt{n}) \\
= \Omega(\sqrt{n}).
\]

Also, the \(Z_i\) are Bernoulli random variables and thus \(\sum_{i=1}^{n} \mathbb{E}|Z_i| \leq O(n)\). This shows that

\[
d_K \left( \frac{Z - \mathbb{E}[Z]}{\sqrt{\text{Var}(Z)}}, \mathcal{G} \right) \leq O \left( \frac{m}{n^{1/4}} \right),
\]

proving that when \(k\) is fixed, we have a central limit theorem,

\[
\frac{Z - \mathbb{E}[Z]}{\sqrt{\text{Var}(Z)}} \rightarrow_d \mathcal{G}.
\]

Together with the total variation distance bound between \(P\) and \(Z\), and noting that convergence in \(d_{TV}\) implies convergence in \(d_K\), we conclude that \(P\) satisfies a central limit theorem. Since \(as_{n,k}\) differs from \(2P\) by at most 1, the same holds for it as well after proper centering and scaling.

**Remark 3.1** The arguments given in this section carry over to certain cases where \(k\) grows with \(n\). For example, considering the case \(k = \gamma n\) for constant \(\gamma\), the quantity \(3n \left( \frac{k}{n} \right)^m\) in (3) can be made \(o(1/n)\) by choosing \(m\) suitably. To see this, letting \(\alpha > 1\), suppose \(\frac{1}{n^\alpha} = 3n \left( \frac{k}{n} \right)^m\). Since \(\gamma = \frac{k}{n}\), we then have

\[
n^{-1-\alpha} = 3(\gamma)^m,
\]

and then \(m = \frac{(-1-\alpha) \log(n) - \log(3)}{\log(\gamma)}\). We can choose \(\alpha = 2\) so that \(m = \frac{-3 \log(n)}{\log(\gamma)}\). Note that \(m > 0\) since \(\log \left( \frac{k}{n} \right) < 0\).

**Remark 3.2** In notation of the Introduction, if we were to prove a central limit theorem for \(as_{n,\infty}\), then that would be straightforward. This is thanks to the fact that it can be written as a random sum (where the number of summands is binomial) of locally dependent variables, and that central limit theorem for such cases are already available. See, for example, Islak (2016).

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