Technical Report: Distributed Asynchronous Large-Scale Mixed-Integer Linear Programming via Saddle Point Computation

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Abstract

We solve large-scale mixed-integer linear programs (MILPs) via distributed asynchronous saddle point computation. This is motivated by the MILPs being able to model problems in multi-agent autonomy, e.g., task assignment problems and trajectory planning with collision avoidance constraints in multi-robot systems. To solve a MILP, we relax it with a nonlinear program approximation whose accuracy tightens as the number of agents increases relative to the number of coupled constraints. Next, we form an equivalent Lagrangian saddle point problem, and then regularize the Lagrangian in both the primal and dual spaces to create a regularized Lagrangian that is strongly-convex-strongly-concave. We then develop a parallelized algorithm to compute saddle points of the regularized Lagrangian. This algorithm partitions problems into blocks, which are either scalars or sub-vectors of the primal or dual decision variables, and it is shown to tolerate asynchrony in the computations and communications of primal and dual variables. Suboptimality bounds and convergence rates are presented for convergence to a saddle point. The suboptimality bound includes (i) the regularization error induced by regularizing the Lagrangian and (ii) the suboptimality gap between solutions to the original MILP and its relaxed form. Simulation results illustrate these theoretical developments in practice, and show that relaxation and regularization together have only a mild impact on the quality of solution obtained.

I. Introduction

Numerous problems in autonomy can be formulated into mixed-integer programs in which the integer constraints lead to a more realistic model of the system than a problem without integer constraints. In particular, mixed-integer programs have been used to model piece-wise affine functions which can approximate nonlinear dynamical systems for control [1], deep neural networks with ReLU activation functions to improve robustness [2], task assignment problems [3], and trajectory planning problems with collision avoidance [4]. In a task assignment problem, the integer decision variables model discrete decisions about which agents work on which tasks. For collision avoidance, the integer variables can model a halfspace avoidance region [4]. Mixed integer programs are useful in their modeling power, but scale poorly in comparison to non-discrete optimization methods due to the NP-Hardness of solving problems with integrality constraints [5]. Therefore, methods have been developed to computationally reach an approximate solution. Most existing methods are centralized, but as the size of problems increases parallelism and decentralized schemes are needed to partition the problems into smaller pieces and accelerate computations.

To help address this need, in this work we solve large-scale mixed-integer linear programs (MILPs) via a network of agents in a distributed fashion. This distributed form allows scaling to large networks of agents in which each agent performs computations locally and asynchronously shares the results with other agents. The goal is to minimize a linear cost while obeying individual constraints, which constrain a single agent, and shared coupling constraints, which jointly constrain all agents. This type of problem has been referred to in the literature as “constraint-coupled MILPs” [6]. For this class of problems, we seek to find a feasible approximate solution that has bounded, quantifiable suboptimality using a Lagrangian relaxation approach.

In particular, we solve these large-scale MILPs via distributed asynchronous computation of saddle points of the Lagrangian. First we relax the original MILP with a nonlinear program approximation whose suboptimality gap tightens as the number of agents increases relative to the number of coupled constraints. Then we form a Lagrangian saddle point problem equivalent to that nonlinear program and regularize in the primal and dual spaces to make the Lagrangian strongly-convex-strongly-concave. Regularization not only provides robustness to asynchronous updates, but also ensures the uniqueness of a solution; this uniqueness is not guaranteed in unregularized cases, unless it is assumed outright.

We develop a parallelized saddle point finding algorithm and apply it to the regularized Lagrangian. This algorithm partitions problems into blocks that are either scalars or sub-vectors of the primal or dual decision variables, and this algorithm is shown to be tolerant to asynchrony in computations and communications. The block structure simplifies the needed communications for our large-scale MILP problems because these problems are separable, which implies that agents

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can compute their local gradients without needing to share information with each other. We present convergence analysis for our algorithm, along with a suboptimality bound between the solution to the original MILP and the approximate solution obtained using our algorithm.

Our particular algorithm draws on recent developments in distributed primal-dual optimization and solving large-scale mixed-integer linear programs. Similar work appears in [6], [7], but one major distinction is that those works use a consensus-based method while we use a block-based method. Consensus methods require graph connectivity of some form over time, which typically cannot be verified locally by the agents in the network. Block-based methods, on the other hand, require only bounded delays along each communication channel, which are typically easier to verify. Block-based optimization can also lead to the communication of less information between agents and fewer required computations per agent, since each agent computes updates to and communicates only a (usually small) subset of the decision variables in a problem. This is in contrast to consensus-type methods, in which the entire decision vector is updated and communicated by each agent. These differences are particularly advantageous for multi-robot systems, especially in environments with unreliable communications, and in bandwidth-limited systems, such as the large-scale systems we consider.

Distributed convex constrained primal-dual optimization is a foundational element for the current work because it is used to solve the relaxed MILP [8]. Earlier works on decentralized primal-dual constrained optimization address the effect of delay bounds, but [9] did not address integer programming and [10] used consensus-based methods. In multi-agent optimization, few works solve large-scale mixed integer linear programs, but of those that do most recent works are based on the developments in [11]. That work was later improved for a tighter suboptimality bound in [10] and then it was subsequently distributed in [12]. Our algorithm differs from that in [12] because our algorithm is block-based and asynchronous while the algorithm in [12] is consensus-based and synchronous. Our algorithm converges with bounded delays, and the rate of convergence depends on the length of the delays and the regularization terms in a way that we quantify.

To summarize, the main contributions of this paper are the following:

- We are the first to solve large-scale MILPs with block-based asynchronous communications in a decentralized algorithm (Algorithm 1).
- We are the first to bound the suboptimality gap between the original MILP and its approximation by a regularized Lagrangian saddle point formulation (Theorem 1).
- We regularize to remove the unique linear program assumption present in all other large-scale MILP works.
- We present a rate of convergence for our algorithm under partial asynchrony (Theorem 3).

This paper is organized as follows. Preliminaries and problem statements are given in Section II. Section III presents the algorithm, convergence analysis, and suboptimality bounds. Section IV demonstrates the algorithm in simulation, and Section V concludes.

## II. PRELIMINARIES AND PROBLEM STATEMENTS

This section establishes the problems we solve, the assumptions on those problems, and the relationship between the relaxed and unrelaxed forms of the MILPs we solve. Below, we use the notation \( \text{conv}(\cdot) \) to denote the convex hull of a set, the set of indices \( [s] \triangleq \{1, ..., s\} \) for any \( s \in \mathbb{N} \), \( \mathbb{R} \) to denote real numbers, \( \mathbb{Z} \) to denote integers, \( \mathbb{R}_+ \) to denote positive real numbers, \( B_\mathbb{R}(r) \) to denote the closed ball of radius \( r \) centered on the origin, and \( \Pi_{S} [\cdot] \) to denote the Euclidean projection onto a compact, convex set \( S \).

We study the following MILP over a network of \( m \) agents.

### Problem 1. (Original MILP)

\[
\begin{align*}
\text{minimize} \quad & f(x) = \sum_{\ell=1}^{m} c^T \ell x_{\ell} \\
\text{subject to} \quad & g(x) = \sum_{\ell=1}^{m} A_\ell x_{\ell} \leq b \\
& x_{\ell} \in X_{\ell} \quad \text{for all } \ell \in [m],
\end{align*}
\]

where \( X_{\ell} \triangleq \{x_{\ell} \in \mathbb{Z}^{p_\ell} \times \mathbb{R}^{q_\ell} : S_\ell x_{\ell} \leq s_\ell \} \) for a given \( S_\ell \in \mathbb{R}^{r \times (p_\ell + q_\ell)} \) and \( s_\ell \in \mathbb{R}^r \), the linear cost is given by \( c_{\ell} \in \mathbb{R}^{p_\ell + q_\ell} \), the coupling constraints are given by \( A_\ell \in \mathbb{R}^{n \times (p_\ell + q_\ell)} \), and \( b \in \mathbb{R}^n \).

To solve Problem 1 branch-and-bound or cutting plane methods could be used to achieve an exact optimal solution. However, these methods are computationally expensive and can be too slow for many real-world applications, especially in large-scale instances. To this end, the following problem approximates Problem 1 and we take advantage of the separable
problem structure in a later proposition. Specifically, we obtain an approximate form of Problem 1 where \( \rho \) is a vector that tightens the coupling constraints in Problem 2, which will assist in reconstructing a feasible solution to Problem 1.

**Problem 2. (Relaxed MILP; approximation of Problem 1)**

\[
\begin{align*}
\text{minimize} \quad & f(z) = c^T z \\
\text{subject to} \quad & g(z) = Az \leq b - \rho \\
\end{align*}
\]

where \( z = [z_1^T, \ldots, z_m^T]^T \in \mathbb{R}^{p_1+q_1+\cdots+p_m+q_m}, \) \( Z = Z_1 \times \cdots \times Z_m \subseteq \mathbb{R}^{p_1+q_1+\cdots+p_m+q_m}, \) \( Z_\ell = \text{conv}(X_\ell), \) \( z_\ell \in \mathbb{R}^{p_\ell+q_\ell}, \) \( A = [A_1, A_2, \ldots, A_m] \in \mathbb{R}^{y \times (p_1+q_1+\cdots+p_m+q_m)}, \) \( A_\ell \in \mathbb{R}^{y \times (p_\ell+q_\ell)}, c = [c_1^T, \ldots, c_m^T]^T \in \mathbb{R}^{p_1+q_1+\cdots+p_m+q_m}, \) \( b \in \mathbb{R}^y, \) and \( \rho \in \mathbb{R}^y. \)

The symbol \( z \) is introduced to differentiate between the decision variables in Problem 1 and those in Problem 2. We are interested in large-scale problems, which are those with \( m \gg y. \) We make the following assumptions about Problem 2.

**Assumption 1 (Nonempty and Compact)** The set \( Z \) is nonempty and compact.

This assumption ensures the existence of a solution and allows for distributed projected update laws which will ensure satisfaction of the agents’ set constraints.

**Assumption 2 ( Slater’s Condition)** Slater’s condition holds for \( g, \) i.e., there exists \( \bar{z} \in Z \) such that \( g(\bar{z}) < 0. \)

This assumption will simplify our convergence analysis by allowing us to formulate a saddle point problem over a compact domain.

We solve Problem 2 using a primal-dual approach. Specifically, we will use a decentralized form of the classic Uzawa iteration, also called gradient descent-ascent, to find a saddle point of the Lagrangian associated with Problem 2. This Lagrangian is \( L(z, \lambda) = c^T z + \lambda^T (Az - b + \rho), \) which is affine in both \( z \) and \( \lambda. \) Thus it is convex in \( x \) and concave in \( \lambda. \) To ensure convergence of a decentralized asynchronous implementation of the Uzawa iteration, we will regularize \( L \) to make it strongly convex in \( x \) and strongly concave in \( \lambda. \) These properties have both been shown to improve convergence of decentralized algorithms. We apply a Tikhonov regularization in the primal and dual spaces to find

\[
L_\kappa(z, \lambda) = c^T z + \frac{\alpha}{2} ||z||^2 + \lambda^T (Az - b + \rho) - \frac{\delta}{2} ||\lambda||^2, \tag{1}
\]

where \( \lambda \in \mathbb{R}^y_+ \) is the dual variable, \( \alpha > 0 \) is the primal regularization term, \( \delta > 0 \) is the dual regularization term, and we use \( \kappa = (\alpha, \delta) \) to simplify notation.

Strong duality holds for Problem 2 [14, Section 5.2.3], which means it can be solved by computing the saddle point of the unregularized Lagrangian \( L. \) Instead of doing so exactly, we will compute a saddle point of the regularized Lagrangian \( L_\kappa \) with \( \alpha > 0 \) and \( \delta > 0 \) set to small values. As we show in Section IV, this regularization induces only minor error in the solutions we obtain. Another advantage of using this regularization is that we do not need to assume uniqueness of the solution to Problem 2 since we have strong convexity from regularization and thus \( L_\kappa \) has a unique saddle point. This is unlike existing large-scale MILP methods that rely on assuming that there is a unique solution to solve Problem 2 [6, 10, 11].

Formally, we will solve Problem 2 by finding a saddle point of \( L_\kappa, \) namely

\[
(\hat{z}_\kappa, \hat{\lambda}_\kappa) = \arg \min_{z \in Z} \arg \max_{\lambda \in \mathbb{R}^y_+} c^T z + \frac{\alpha}{2} ||z||^2 + \lambda^T (Az - b + \rho) - \frac{\delta}{2} ||\lambda||^2.
\]

Bounding the suboptimality gap between Problem 1 and the saddle point \((\hat{z}_\kappa, \hat{\lambda}_\kappa)\) is needed to show that solving the relaxed saddle point problem will approximately solve the original MILP. To that end, we first upper bound the dual variable, which will later allow us to bound the suboptimality that results from the relaxation and provide an upper bound on the maximum norm of the dual variable. We now explicitly define \( \rho \) as the contraction vector defined by \([11]\) as

\[
\rho_\zeta \triangleq y \cdot \max_{\ell \in [m]} \left( \max_{x_\ell \in X_\ell} A_{\text{row},\ell} x_\ell - \min_{x_\ell \in X_\ell} A_{\text{row},\ell} x_\ell \right), \tag{2}
\]

where \( \rho \in \mathbb{R}^y_+, \) \( A_{\text{row},\ell} \) is the \( \ell \)th row of \( A, \) and \( \rho_\zeta > 0 \) is the \( \zeta \)th entry of \( \rho \) for \( \zeta \in [y]. \) We next define a radius \( r \) to satisfy \( Z \subseteq B_0(r). \) This value of \( r \) is a bound on the maximum norm of \( z \in Z, \) and it can be found via

\[
r \triangleq \max_{z \in Z} ||z||. \tag{3}
\]
We will use the following proposition for the subsequent suboptimality theorem.

**Proposition 1.** Let Problem 2 have a feasible solution and let \( \bar{z} \) be a vertex of its feasible set. Then, there exists an index set \( I_\ell \subseteq [m] \) with cardinality \( |I_\ell| \geq m - y \) such that \( z_{\ell} \in X_{\ell} \) for all \( \ell \in I_\ell \).

**Proof:** See [11, Theorem 3.3].

This proposition means the optimal solution to Problem 2 is partially mixed-integer and there is a bound on the number of agents whose solution may not be mixed-integers for Problem 1. In particular, as the size of problems grows such that \( m \gg y \), we find that the integrality constraints become approximated better. In the following lemma, we bound the dual multipliers to be used in a later theorem.

**Lemma 1.** Let Assumptions 1 and 2 hold. Then

\[
\hat{\lambda}_\kappa \in M \triangleq \left\{ \lambda \in \mathbb{R}^m_+ : ||\lambda||_1 \leq \frac{c^T \bar{z} + \frac{\alpha}{2} ||\bar{z}||^2 - ||c|| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{z} + \rho_j - b_j} \right\},
\]

where \( \bar{z} \) is a Slater point of \( g \), \( c^T \bar{z} \) is the cost at the Slater point, \( \alpha \) is the primal regularization term, \( r \) is the radius found from (3). \( A_{row,j} \) denotes row \( j \) of \( A \), and \(-A_{j} \bar{z} + \rho_j - b_j \) is the \( j \)-th entry of \( g(\bar{z}) \).

**Proof:** As discussed in [15, Section II.C] for any Slater point \( \bar{z} \) of \( g \) we have

\[
\hat{\lambda}_\kappa \in M \triangleq \left\{ \lambda \in \mathbb{R}^m_+ : ||\lambda||_1 \leq \frac{f(\bar{z}) + \frac{\alpha}{2} ||\bar{z}||^2 - \min_{z \in Z} f(z)}{\min_{1 \leq j \leq m} - g_j(\bar{z})} \right\}.
\]

We expand \( g(\bar{z}) = A \bar{z} + \rho - b \), which implies \( g_j(\bar{z}) = A_j \bar{z} + \rho_j - b_j \). Then

\[
\min_{1 \leq j \leq m} - g_j(\bar{z}) = \min_{1 \leq j \leq m} - A_{row,j} \bar{z} - \rho_j + b_j.
\]

Next, we find a lower bound for \( \min_{z \in Z} f(z) \). It follows that

\[
\min_{z \in Z} c^T z \geq \min_{z \in B_o(r)} c^T z = \bar{c}^T \left( -\frac{c}{||c||} \right) \cdot r = -||c|| \cdot r.
\]

Substituting (3) and (7) into (4) gives the desired bound. ■

We now bound the suboptimality in the primal component of the saddle point of \( L_\kappa \) that is due to the regularization of \( L \). Below, we will use this lemma to bound the suboptimality of the primal component of the saddle point of \( L_\kappa \), namely \( \hat{z}_\kappa \), relative to the optimal solution to Problem 1.

**Lemma 2.** Let Assumptions 1 and 2 hold. Then

\[
c^T \hat{z}_\kappa \leq c^T z^* + ||c|| \cdot \left( \frac{c^T \bar{z} + \frac{\alpha}{2} ||\bar{z}||^2 - ||c|| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{z} + \rho_j - b_j} \right) \cdot \sqrt{\frac{\delta}{2 \alpha}} + \frac{\alpha}{2} \cdot r,
\]

where \( \hat{z}_\kappa \) is the primal component of the saddle point of the regularized Lagrangian, \( z^* \) is the primal component of the saddle point of the unregularized Lagrangian \( L \), \( \bar{z} \) is a Slater point for \( g \), \( \alpha \) is the primal regularization term, \( \delta \) is the dual regularization term, \( r \) is found from (3), and \(-A_{j} \bar{z} - \rho_j + b_j \) is the explicit form of \(-g_j(\bar{z}) \).

**Proof:** From the general form presented in [13, Lemma 3.3], we find an upper bound between the costs at \( \hat{z}_\kappa \) and \( z^* \) via

\[
|f(\hat{z}_\kappa) - f(z^*)| \leq \max_{z \in Z} ||\nabla f(z)|| \cdot \max_{\lambda \in M} ||\lambda|| \cdot \sqrt{\frac{\delta}{2 \alpha}} + \frac{\alpha}{2} \cdot \max_{z \in Z} ||z||.
\]

From Problem 2, we find

\[
\max_{z \in Z} ||\nabla f(z)|| = ||c||,
\]

since \( \nabla f = c \). Then we find an upper bound for \( \max_{z \in Z} ||z|| \) as

\[
\max_{z \in Z} ||z|| \leq \max_{z \in B_o(r)} ||z|| = r,
\]

and

\[
|f(\hat{z}_\kappa) - f(z^*)| \leq \max_{z \in Z} ||\nabla f(z)|| \cdot \max_{\lambda \in M} ||\lambda|| \cdot \sqrt{\frac{\delta}{2 \alpha}} + \frac{\alpha}{2} \cdot r.
\]

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\[
\max_{z \in Z} ||z|| \leq \max_{z \in B_o(r)} ||z|| = r,
\]
which can be found explicitly from (3). Since $\lambda$ is a vector, we always have $\max_{\lambda \in \mathcal{M}} \|\lambda\|_1 \leq \max_{\lambda \in \mathcal{M}} \|\lambda\|_2$. From Lemma 1, we can therefore upper bound $\|\lambda\|_2$ as

$$\max_{\lambda \in \mathcal{M}} \|\lambda\|_2 \leq \max_{\lambda \in \mathcal{M}} \left( \frac{c^T \bar{\lambda} + \frac{\alpha}{2} \|\bar{\lambda}\|^2 - \|c\| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{\lambda} + \rho_j - b_j} \right)$$

Then, using (8) and (10) in (7) we find

$$|c^T \bar{z} - c^T z^*| \leq \|c\| \cdot \left( \frac{c^T \bar{z} + \frac{\alpha}{2} \|\bar{\lambda}\|^2 - \|c\| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{\lambda} + \rho_j - b_j} \right) \cdot \sqrt{\frac{\delta}{2\alpha} + \frac{\alpha}{2} \cdot r}.$$

Rearranging terms we have an upper bound on $c^T \bar{z} - c^T z^*$ via

$$c^T \bar{z} \leq c^T z^* + \|c\| \cdot \left( \frac{c^T \bar{z} + \frac{\alpha}{2} \|\bar{\lambda}\|^2 - \|c\| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{\lambda} + \rho_j - b_j} \right) \cdot \sqrt{\frac{\delta}{2\alpha} + \frac{\alpha}{2} \cdot r}.$$

Now that we have Lemma 1, Lemma 2, and Proposition 1, we construct the suboptimality bound on the difference in optimal costs between Problem 1 and the primal component of the regularized Lagrangian, $\bar{z}$.

**Theorem 1.** (Suboptimality Bound) Let Assumptions 1 and 2 hold. Let $x^*$ denote the solution to Problem 1 and let $\bar{z}$ denote the primal component of the saddle point of $I_{\mathcal{M}}$ with $\rho$ as defined in (2). Let $\eta_\ell = \max_{x\in X_\ell} c^T x - \min_{x\in X_\ell} c^T x$ and let $\bar{z} \in Z$ denote a Slater point for $g$. Then

$$c^T \bar{z} - c^T x^* \leq y \cdot \max_{\ell \in [m]} \eta_\ell + \|c\| \cdot \left( \frac{c^T \bar{z} + \frac{\alpha}{2} \|\bar{\lambda}\|^2 - \|c\| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{\lambda} + \rho_j - b_j} \right) \cdot \sqrt{\frac{\delta}{2\alpha} + \frac{\alpha}{2} \cdot r},$$

where $\alpha > 0$ is the primal regularization term, $\delta > 0$ is the dual regularization term, and $r$ is from (3).

**Proof:** Let $z^*$ denote the solution to Problem 2 with $\rho$ as defined in (2). Expanding $c^T \bar{z} - c^T x^*$, we have

$$c^T \bar{z} - c^T x^* = (c^T \bar{z} - c^T z^*) + (c^T z^* - c^T x^*).$$

According to Proposition 1 there exists $I^*_z$ such that $|I^*_z| \geq m - y$ and $z^*_j \in X_\ell$ for each $\ell \in I^*_z$. This means $z^*_j = x^*_\ell$ for all $\ell \in I^*_z$. Define $I^*_\mathcal{M} = [m] \setminus I^*_z$, which contains indices such that $z^*_j \in conv(X_\ell) \setminus X_\ell$. We can simplify the difference between $c^T z^*$ and $c^T x^*$ as

$$c^T z^* - c^T x^* = \sum_{\ell \in I^*_z} (c^T x^*_\ell - c^T x^*_\ell) + \sum_{j \in I^*_\mathcal{M}} (c^T z^*_j - c^T x^*_j)$$

$$= \sum_{j \in I^*_\mathcal{M}} (c^T z^*_j - c^T x^*_j).$$

We bound $c^T z^* - c^T x^*$ in terms of the remaining $I^*_\mathcal{M}$ terms by upper and lower bounding each term in (13). Because Problem 2 is a relaxation of Problem 1 we have $c^T x^*_\ell \leq c^T x_\ell$ for every $x_\ell \in X_\ell$. Then, for any $x_\ell \in X_\ell$, we have $c^T z^*_j \leq c^T x_\ell \leq \max_{x_\ell \in X_\ell} c^T x_\ell$. Next, for every $j \in I^*_\mathcal{M}$, for $x^*$ we have $\min_{x_\ell \in X_\ell} c^T x_\ell \leq c^T x^*_j$. Then we find

$$c^T z^*_j - c^T x^*_j \leq \max_{x_\ell \in X_\ell} c^T x_\ell - \min_{x_\ell \in X_\ell} c^T x_\ell.$$

We thus find an upper bound on the distance between $c^T z^*$ and $c^T x^*$ as

$$\sum_{j \in I^*_\mathcal{M}} c^T z^*_j - c^T x^*_j \leq \sum_{j \in I^*_\mathcal{M}} \left( \max_{x_\ell \in X_\ell} c^T x_\ell - \min_{x_\ell \in X_\ell} c^T x_\ell \right) = |I^*_\mathcal{M}| \cdot \max_{\ell \in [m]} \left( \max_{x_\ell \in X_\ell} c^T x_\ell - \min_{x_\ell \in X_\ell} c^T x_\ell \right) \leq y \cdot \max_{\ell \in [m]} \eta_\ell,$$

which follows from the definition of $\eta_\ell$ and the fact that $|I^*_\mathcal{M}| \leq y$. Next, we use Lemma 2 to relate $c^T z^*$ to $c^T \bar{z} - c^T x^*$ via
\[ c^T \hat{z}_c - c^T z^* \leq \|c\| \cdot \left( \min_{1 \leq j \leq m} \frac{c^T z + \frac{\alpha}{2} \|z\|^2 - \|c\| \cdot \rho_j - b_j}{-A_{row,j}} \right) \cdot \sqrt{\frac{\delta}{2\alpha} + \frac{\alpha}{2} \cdot r}. \] (15)

Using the results of (13), (14) and (15) in (12) completes the proof.

Now that we have a suboptimality bound between the solution to Problem 1 and the primal component \( \hat{z}_c \) of the saddle point of the regularized Lagrangian \( L_c \), we will find the saddle point of \( L_c \) in a parallelized way, and the next section develops the algorithm for doing so.

III. DISTRIBUTED ALGORITHM

In this section we define the main algorithm we use to find the saddle point of \( L_c \), analyze its convergence, and describe how its output can be mapped back to an approximate solution to Problem 1.

A. Algorithm Definition

The distributed algorithm we develop consists of three types of operations: (i) a primal variable update, (ii) a dual variable update, and (iii) communication of these updates. Recall from Problem 2 that \( z = (z_T^1, \ldots, z_T^m)^T \), which is partitioned into \( m \) blocks. We assume that there are \( H > m \) agents to find the saddle point \( (\hat{z}_c, \lambda_c) \), of which there are \( m \) primal agents indexed over the set \( A_P = [m_1] \) and \( H - m \) dual agents, indexed over the set \( A_D = [H \setminus [m]] \). Each primal and dual agent has partial knowledge of Problem 2 and only updates its assigned block of the decision variables. We partition the dual variable \( \lambda \in \mathbb{R}_+^H \) into blocks via

\[ \lambda = (\lambda_T^1, \ldots, \lambda_T^{H-m})^T, \] (16)

where \( \lambda_c \in \mathbb{R}^H_c \) for \( c \in A_D \) and \( \sum_{c \in A_D} r_c = y \).

Computations and communications are asynchronous, which leads to disagreements among the values of the decision variables that agents store onboard. Therefore, there is a need to compute the saddle point \( (\hat{z}_c, \lambda_c) \) with an update law that is robust to asynchrony. We use a decentralized form of the classic Uzawa iteration, also called gradient descent-ascent, to find a saddle point of the regularized Lagrangian \( L_c \) associated with Problem 2.

Each primal agent updates a block of the primal variables it stores onboard. Primal agent \( i \) stores onboard itself a local copy of the vector of primal decision variables, denoted \( z^i \in Z \), and local copy of the vector of dual decision variables, denoted \( \lambda^i \in M \). Dual agent \( q \) stores a local primal variable onboard, denoted \( z^q \in Z \), and local dual vector, denoted \( \lambda^q \in M \). Each dual agent updates its block of the dual variable in the copy it stores onboard. Each dual variable corresponds to an inequality constraint on the primal variables, and each dual agent sends values of its updated block to all primal agents whose decision variables appear in the constraints that correspond to those dual variables.

We define \( D \subseteq \mathbb{N} \) as the set of times when all dual agents compute updates to their decision variables, and we define \( K^t \subseteq \mathbb{N} \) as the set of times at which primal agent \( i \in A_P \) computes updates to its decision variables. To state an algorithm, we use \( k \) as the iteration count used by all the primal agents, and \( t \) as the iteration count used by all dual agents. The sets \( K^t \) and \( D \) are tools for analysis and discussion, and they need not be known by any agent. Additionally, within the vector \( z^i \), the notation \( z^i_{[j]} \) indicates agent \( i \)'s value for the primal block \( j \), \( z^i_{[i]} \) indicates agent \( i \)'s onboard value for the primal block \( i \), \( \lambda^i_{[j]} \) indicates agent \( i \)'s onboard value for the dual block \( j \), \( \nabla z^i_{[j]} := \frac{\partial}{\partial z^i_{[j]}} \) is the derivative with respect to the \( j^{th} \) block of \( z \), and \( \nabla \lambda^i_{[q]} := \frac{\partial}{\partial \lambda^i_{[q]}} \) is the derivative with respect to the \( q^{th} \) block of \( \lambda \).

Assumption 3 (Bounded Delays)

Let \( K^t \) be the set of times when primal agent \( i \) performs updates and let \( \tau^i_j(k) \) be the time when primal agent \( j \) originally computed the value of \( z^i_{[j]}(k) \) onboard by agent \( i \) at time \( k \). Then there exists a positive integer \( B \) such that

1) For every \( i \in A_P \), at least one of the elements of the set \( \{k, k + 1, \ldots, k + B - 1\} \) is in \( K^t \).
2) There holds \( k - B < \tau^i_j(k) \leq k \), for all \( i, j \in [m], j \neq i \), and all \( k \in K^t \).

This assumption ensures that no primal block onboard any agent was computed more than \( B \) iterations prior to the current time, and it ensures that all primal agents perform at least one computation every \( B \) timesteps.

Primal agent \( i \) updates its block \( z^i_{[j]} \) via projected gradient descent at each time \( k \in K^t \). If \( k \notin K^t \), then agent \( i \) does not update and \( z^i_{[i]} \) is held constant. If a communication of an updated block from primal agent \( j \) is received, then agent \( i \) stores it in the block \( z^i_{[j]} \) by overwriting the previous value of \( z^i_{[j]} \) that it had onboard. At times at which agent \( i \) does

One can use \( H = m \) agents and simply have each agent perform the duties of both a primal agent and a dual agent, though here we assume that \( H > m \) to allow primal and dual agents to be separate, which simplifies our discussion.
not receive a communication from primal agent $j$, $z^i_{[j]}$ is held constant. At each $t \in D$, dual agents update via projected gradient ascent onto the set $M_q$, which we define as

$$
M_q := \left\{ \lambda_c \in \mathbb{R}_{+}^{r_c} : ||\lambda_c||_1 \leq \frac{c^T \bar{z} + \frac{n}{2} ||\bar{z}||^2 - ||c|| \cdot r}{\min_{1 \leq j \leq m} - A_{row,j} \bar{z} + \rho_j - b_j} \right\}.
$$

We note that the upper bound in $M_q$ is the same as that in $M$ in Lemma 1 and thus is a bound on each individual block of $\lambda$ as well.

Primal agents $i$’s computations take the form

$$
z^i_{[i]}(k+1) = \begin{cases} \Pi_{Z^i_{[i]}}(z^i_{[i]}(k) - \gamma \nabla_{z^i_{[i]}} L_{\kappa}(z^i(k), \lambda^i(k))) & k \in K^i \\ z^i_{[i]}(k) & \text{otherwise.} \end{cases}
$$

For all $q \in A_D$, dual agent $q$ only performs an update occasionally. Specifically, it only performs an update after the counter $k$ has increased by some amount that is divisible by $B$. When it updates, dual agent $q$’s computations take the form

$$
\lambda_q^q(tB) = \Pi_{M_q}[\lambda_q^q(tB - 1) + \beta \nabla_{\lambda_q^q} L_{\kappa}(z^q(tB), \lambda^q(tB - 1))],
$$

and between updates it holds the value of $\lambda_q^q$ constant. After updating, dual agent $q$ sends the new value of $\lambda_q^q$ to all primal agents that need it in their computations, and we note that a primal agent only needs to receive a dual block if that primal agent’s decision variables appear in the constraints corresponding to those dual variables. Similarly, a dual agent only needs to receive communications from a primal agent if that primal agent’s decision variables appear in the constraints encoded in the dual variables that the dual agent updates.

It is required that all primal agents use the same values of the dual blocks in their computations. This has been shown to be necessary [8] for convergence; any mechanism to enforce this agreement can be used. No communication is required between dual agents since updates to $\lambda_q^q$ do not depend on the values of other blocks of $\lambda$.

We present the full parallelized block gradient-based saddle point algorithm in Algorithm 1.
Algorithm 1

Initialize: All primal variables $z(0) \in Z$, all dual variables $\lambda(0) \in M$, and the tightening vector $\rho$ as defined in (2).

for $k \in \mathbb{N}$ do
  for $i \in A_P$ do
    if $k \in K^i$
      $z^i_{[1]}(k + 1) = \Pi_{Z^i_{[1]}}[z^i_{[1]}(k) - \gamma \nabla z_{[i]} L_\kappa(z^i(k), \lambda^i(k))]$
      Communicate Agent $i$ sends its updated block to other primal agents, but it may not arrive for some time due to asynchrony.
    else
      $z^i_{[1]}(k + 1) = z^i_{[1]}(k)$
    end
    if $i$ receives $z^j_{[j]}$ at time $k$
      $z^j_{[j]}(k + 1) = z^j_{[j]}(\tau^j_{[j]}(k + 1))$
    else
      $z^j_{[j]}(k + 1) = z^j_{[j]}(k)$
    end
  end
  for $q \in A_D$
    if $k = tB$ with $t \in D$
      Communicate All primal agents send their most recent iterate to all dual agents that need it
      for $q \in A_D$ do
        $\lambda^q(tB) = \Pi_{M_q}[\lambda^q(tB - 1) + \beta \nabla \lambda^q_{[q]} L_\kappa(z^q(tB), \lambda^q(tB - 1))]$
      end
      Communicate Agent $q$ sends its updated block to the primal agents, but it may not arrive for some time due to asynchrony.
    end
  end

B. Convergence Analysis

We next need to verify that Algorithm 1 will converge to the saddle point of $L_\kappa$. We start with a proof of convergence for the primal agents when they have a fixed dual variable onboard. In it, we define the primal agents’ true iterate for all $k \in \mathbb{N}$ as

$$ z(k) = (z^1_{[1]}(k)^T, z^2_{[2]}(k)^T, \ldots, z^m_{[m]}(k)^T). \quad (19) $$

Lemma 3. (Primal Convergence) Let Assumptions 1-3 hold, and consider using Algorithm 1 to find a saddle point of $L_\kappa$. Fix $t \in D$ and let $t'$ be the smallest element of $D$ that is greater than $t$. Fix $\lambda(tB) \in M$ and define $\hat{z}(tB) = \arg \min_{z \in Z} L_\kappa(z, \lambda(tB))$ as the point that the primal agents would converge to with $\lambda(tB)$ held fixed. For agents executing Algorithm 1 at times $k \in \{tB, tB + 1, \ldots, t'B\}$, there exists a scalar $\gamma > 0$ such that for $\gamma \in (0, \gamma_1)$ we have

$$ \|z(tB') - \hat{z}(tB)\| \leq (1 - \theta \gamma)^{t' - t}\|z(tB) - \hat{z}(tB)\|, $$

where $\theta$ is a positive constant, and $(1 - \theta \gamma) \in [0, 1)$.

Proof: We show that Assumptions A and B in [17] are satisfied, which enables the use of Proposition 2.2 from the same reference to show convergence. Assumption A requires that

1) $L_\kappa(\cdot, \lambda(tB))$ is bounded from below on $Z$,  
2) the set $Z$ contains at least one point $y$ such that $y = \Pi_Z[y - \nabla z L_\kappa(y, \lambda(tB))]$, and
3) $\nabla_z L_\kappa(\cdot, \lambda(tB))$ is Lipschitz on $Z$.

First, we see that for all $z \in Z$ we have

$$ \nabla_z L_\kappa(z, \lambda(tB)) = c + \alpha z + A^T \lambda(tB), \quad (20) $$

which is Lipschitz in $z$ with constant $\alpha$. Then Assumption A.3 is satisfied.
Next, for every choice of \( \lambda(tB) \in M \), the function \( L_\kappa(\cdot, \lambda(tB)) \) is bounded from below because it is continuous and its domain \( Z \) is compact. Then Assumption A.2 is satisfied. For every \( t \in \mathbb{N} \) and every fixed \( \lambda(tB) \in M \), the strong convexity of \( L_\kappa(\cdot, \lambda(tB)) \) implies that is has a unique minimum over \( Z \), denoted \( \hat{z}(tB) \). This point is the unique fixed point of the projected gradient descent mapping and Assumption A.1 is satisfied. Then all conditions of Assumption A in \( [17] \) are satisfied.

Assumption B in \( [17] \) requires
1) the isocost curves of \( L_\kappa(\cdot, \lambda(tB)) \) to be separated, and
2) the “error bound condition”, stated as (2.5) in \( [17] \), is satisfied.

It is observed in \( [17] \) that both criteria are satisfied by functions that are strongly convex with Lipschitz gradients. In this work, \( L_\kappa(\cdot, \lambda(tB)) \) has both of these properties. Then Assumption B is satisfied as well, and an application of Proposition 2.2 \( [17] \) completes the proof.

The following lemma is stated here to later be used in the main convergence theorem for the dual variable.

**Lemma 4.** Let \( \lambda_1, \lambda_2 \in \mathbb{R}_+^\mathbb{Y} \). Then for any points \( z_1 \) and \( z_2 \) such that

\[
z_1 = \arg \min_{z \in Z} L_\kappa(z, \lambda_1) \quad \text{and} \quad z_2 = \arg \min_{z \in Z} L_\kappa(z, \lambda_2)
\]

the pairs \((z_1, \lambda_1)\) and \((z_2, \lambda_2)\) satisfy

\[
(\lambda_2 - \lambda_1)^T (-g(z_2) + g(z_1)) \geq \frac{\alpha}{\|A\|^2_2} \|g(z_2) - g(z_1)\|^2.
\]

Moreover, they satisfy

\[
\|\lambda_2 - \lambda_1\| \geq \frac{\alpha}{\|A\|^2_2} \|z_2 - z_1\|.
\]

**Proof:** See \( [13] \) Lemma 4.1.

Next, we derive a proof of convergence for dual agents across two sequential timesteps.

**Lemma 5.** (Dual Convergence Between Two Time Steps) Let Assumptions 1-3 hold and consider the use of Algorithm 4 to find the saddle point of \( L_\kappa \). Let the dual step size be \( 0 < \beta < \min \left\{ \frac{2\alpha}{\|A\|^2 + 2\alpha \delta + \frac{1}{2d^2}}, \frac{2\beta}{\|A\|^2} \right\} \), and let \( t_1 \) and \( t_2 \) denote two consecutive times that dual updates have occurred, with \( t_1 < t_2 \) and \( t_1, t_2 \in D \). Then Algorithm 1 produces dual variables \( \lambda(t_1 B) \) and \( \lambda(t_2 B) \) that satisfy

\[
\|\lambda(t_2 B) - \hat{\lambda}_\kappa\|^2 \leq q_d \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + 4\alpha^{2}q_d q_p^{2(t_2-t_1)} \cdot \|A\|^{2}_2 + 8\beta^2 q_p^{(t_2-t_1)} \cdot \|A\|^{2}_2
\]

where \( q_d \triangleq (1 - \beta\delta)^2 + \beta^2 \in \{0, 1\} \), \( q_p \triangleq (1 - \theta\gamma) \in \{0, 1\} \), \( \gamma \) is from \( [3] \), and \( \hat{\lambda}_\kappa \) is the dual component of the unique saddle point of \( L_\kappa \).

**Proof:** Define \( \hat{z}_\kappa(t_1 B) \triangleq \arg \min_{z \in Z} L_\kappa(z, \lambda(t_1 B)) \) and recall \( \hat{z}_\kappa = \arg \min_{z \in Z} L_\kappa(z, \hat{\lambda}_\kappa) \). Given that all dual agents use the same primal variables in their computations, we analyze all dual agents’ computations simultaneously with the combined dual update law

\[
\lambda(t_2 B) = \Pi_M \left[ \lambda(t_1 B) + \beta \nabla_{\lambda} L_\kappa(z(t_2 B), \lambda(t_1 B)) \right],
\]

where \( z(t_2 B) \) is the common primal variable among dual agents. Then expanding the dual update law and using the non-expansiveness of \( \Pi_M \) gives

\[
\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| = \|\Pi_M [\lambda(t_1 B) + \beta (g(z(t_2 B) - \delta \lambda(t_1 B))] - \Pi_M [\hat{\lambda}_\kappa + \beta (g(\hat{z}_\kappa) - \delta \hat{\lambda}_\kappa)]\|^{2} = (1 - \beta\delta)^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + \beta^2 \|g(z(t_2 B) - g(\hat{z}_\kappa))\|^2 - 2\beta(1 - \beta\delta)(\lambda(t_1 B) - \hat{\lambda}_\kappa)^T (g(\hat{z}_\kappa) - g(z(t_2 B))).
\]

Adding \( g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa(t_1 B)) \) inside the last set of parentheses gives

\[
\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| = (1 - \beta\delta)^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + \beta^2 \|g(z(t_2 B) - g(\hat{z}_\kappa))\|^2 - 2\beta(1 - \beta\delta)(\lambda(t_1 B) - \hat{\lambda}_\kappa)^T (g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa(t_1 B))) - 2\beta(1 - \beta\delta)(\lambda(t_1 B) - \hat{\lambda}_\kappa)^T (g(\hat{z}_\kappa(t_1 B)) - g(z(t_2 B))).
\]

Applying Lemma 4 to the pairs \((\hat{z}_\kappa, \hat{\lambda}_\kappa)\) and \((\hat{z}_\kappa(t_1 B), \lambda(t_1 B))\) results in
\[(\lambda(t_1 B) - \hat{\lambda}_\kappa)^T (-g(z_\kappa) + g(\hat{z}_\kappa(t_1 B))) \geq \frac{\alpha}{\|A\|^2_2} \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2,\]

which we apply to the third term on the right-hand side of (25) to find

\[\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| \leq (1 - \beta \delta)^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + \beta^2 \|g(z(t_2 B) - g(\hat{z}_\kappa))\|^2
- 2\beta(1 - \beta \delta) \frac{\alpha}{\|A\|^2_2} \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2
+ (1 - \beta \delta)^2 \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2 + \beta^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2.\]

To bound the last term, we can see that

\[-2\beta(1 - \beta \delta)\lambda(t_1 B) - \hat{\lambda}_\kappa)^T (g(\hat{z}_\kappa(t_1 B)) - g(z(t_2 B))) \leq (1 - \beta \delta)^2 \|g(\hat{z}_\kappa(t_1 B)) - g(z(t_2 B))\|^2 + \beta^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2.\]

Then we can apply (28) to the last term in (27) to obtain

\[\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| \leq (1 - \beta \delta)^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + \beta^2 \|g(z(t_2 B) - g(\hat{z}_\kappa))\|^2
- 2\beta(1 - \beta \delta) \frac{\alpha}{\|A\|^2_2} \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2
+ (1 - \beta \delta)^2 \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2 + \beta^2 \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2.\]

Next, add and subtract \(g(\hat{z}_\kappa(t_1 B))\) inside the norm in the second term of (29) to obtain

\[\beta^2 \|g(z(t_2 B)) - g(\hat{z}_\kappa)\|^2 \leq \beta^2 \|g(z(t_2 B)) - g(\hat{z}_\kappa(t_1 B))\|^2 + \beta^2 \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2
+ 2\beta^2 \|g(z(t_2 B)) - g(\hat{z}_\kappa(t_1 B))\| \cdot \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|.

Applying this to the second term in (29) and grouping terms gives

\[\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| \leq ((1 - \beta \delta)^2 + \beta^2) \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + \left(\beta^2 - 2\beta(1 - \beta \delta) \frac{\alpha}{\|A\|^2_2}\right) \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|^2
+ ((1 - \beta \delta)^2 + \beta^2) \|g(\hat{z}_\kappa(t_1 B)) - g(z(t_2 B))\|^2
+ 2\beta^2 \|g(z(t_2 B)) - g(\hat{z}_\kappa(t_1 B))\| \cdot \|g(\hat{z}_\kappa(t_1 B)) - g(\hat{z}_\kappa)\|.

By hypothesis \(0 < \beta < \frac{2\alpha}{\|A\|^2_2},\) which makes the second term negative. Dropping this negative term and applying the Lipschitz property of \(g\) give the upper bound

\[\|\lambda(t_2 B) - \hat{\lambda}_\kappa\| \leq ((1 - \beta \delta)^2 + \beta^2) \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + ((1 - \beta \delta)^2 + \beta^2) \|A\|^2_2 \cdot \|\hat{z}_\kappa(t_1 B) - z(t_2 B)\|^2
+ 2\beta^2 \cdot \|A\|^2_2 \cdot \|z(t_2 B) - \hat{z}_\kappa(t_1 B)\| \cdot \|\hat{z}_\kappa(t_1 B) - \hat{\lambda}_\kappa\|^2.

Lemma 3 can be applied to \(\|\hat{z}_\kappa(t_1 B) - z(t_2 B)\|^2\) to give an upper bound of

\[\|\hat{z}_\kappa(t_1 B) - z(t_2 B)\|^2 \leq q_p^2(t_2 - t_1) \|z(t_1 B) - \hat{z}_\kappa(t_1 B)\|^2,
\]

where \(q_p = (1 - \theta \gamma) \in [0, 1].\) Next, the maximum distance between any two primal variables is bounded by \(\max_{z, y E \mathbb{R}^d} \|z - y\| \leq \max_{z, y E \mathbb{R}^d} \|z\| + \|y\| \leq 2r.\) Using this maximum distance and setting \(q_d = ((1 - \beta \delta)^2 + \beta^2),\) it follows that \(q_d \in [0, 1]\) because \(\beta < \frac{2\alpha}{\|A\|^2_2}.\) Finally, we come to the result that

\[\|\lambda(t_2 B) - \hat{\lambda}_\kappa\|^2 \leq q_d \|\lambda(t_1 B) - \hat{\lambda}_\kappa\|^2 + 4r^2 q_d q_p^2(t_2 - t_1) \cdot \|A\|^2_2 + 8r^2 \beta^2 q_p^2(t_2 - t_1) \cdot \|A\|^2_2.
\]
Theorem 2. (Dual Convergence to Optimum) Let Assumptions 1-3 hold and consider the use of Algorithm \([7]\) to find the saddle point of \(L_n\). Let the dual step size be \(0 < \beta \leq \min \left\{ \frac{2\alpha}{\|A\|_2 + 2\beta^3}, \frac{2\delta}{1 + \delta^2} \right\}\). Additionally, let \(t_n\) denote the \(n^{th}\) entry in \(D\) where \(t_1 < t_2 < \ldots < t_n\), which means \(t_n B\) is the time when the \(n^{th}\) dual update occurs for all dual agents. Then the dual agents executing Algorithm \([7]\) generate dual variables that satisfy

\[
||\lambda(t_n B) - \hat{\lambda}_n||^2 \leq q_d^2 ||\lambda(0) - \hat{\lambda}_n||^2 + (4r^2 q_d q_p^2 ||A||^2 + 8r^2 \beta^2 q_p ||A||^2)^n \sum_{i=0}^{n} q_d^i,
\]

where \(q_d = (1 - \beta \delta)^2 + \beta^2 \in [0, 1)\), \(q_p = (1 - \theta \gamma) \in [0, 1)\), \(r\) is from \([3]\), and \(\hat{\lambda}_n\) is the dual component of the unique saddle point of \(L_n\).

Proof: We apply Lemma \([5]\) twice to find

\[
||\lambda(t_n B) - \hat{\lambda}_n||^2 \leq q_d^2 ||\lambda(t_{n-1} B) - \hat{\lambda}_n||^2 + 4r^2 q_d q_p^2 (t_{n-1} - t_n) \cdot ||A||^2 + 8r^2 \beta^2 q_d q_p ||A||^2 + 4r^2 q_d q_p^2 \cdot ||A||^2 + 8r^2 \beta^2 q_p \cdot ||A||^2 \sum_{i=0}^{n} q_d^i,
\]

where the second inequality follows because \(t_n - t_{n-1} \geq 1\). Recursively applying Lemma \([5]\) then

\[
||\lambda(t_n B) - \hat{\lambda}_n||^2 \leq q_d^2 ||\lambda(0) - \hat{\lambda}_n||^2 + (4r^2 q_d q_p^2 \cdot ||A||^2 + 8r^2 \beta^2 q_p \cdot ||A||^2)^n \sum_{i=0}^{n} q_d^i,
\]

as desired.

Next we use this bound to formulate a convergence rate for the primal variables in Algorithm \([1]\).

Theorem 3. (Overall Primal Convergence) Let Assumptions 1-3 hold and the dual stepsize satisfy \(0 < \beta < \min\{\frac{2\alpha}{\|A\|_2 + 2\alpha^3}, \frac{2\delta}{1 + \delta^2}\}\). Consider using Algorithm \([7]\) to compute the saddle point of \(L_n\), and let \(t_n\) denote the \(n^{th}\) entry in \(D\) where \(t_1 < t_2 < \ldots < t_n\). That is, \(t_n B\) is the time at which the \(n^{th}\) dual update occurs across all dual agents. Then primal agents executing Algorithm \([7]\) generate primal iterates that satisfy

\[
||z(t_n B) - \hat{z}_n|| \leq 2q_p^{-n-1} + \frac{\|A\|_2}{\alpha} \left( q_p^{-n} \|\lambda(0) - \hat{\lambda}_n\|^2 + (4r^2 q_d q_p^2 \cdot ||A||^2 + 8r^2 \beta^2 q_p \cdot ||A||^2)^n \sum_{i=0}^{n-1} q_d^i \right)^{1/2},
\]

where \(q_p = (1 - \theta \gamma) \in [0, 1)\), \(q_d = (1 - \beta \delta)^2 + \beta^2 \in [0, 1)\), and \(\hat{\lambda}_n\) is the dual at the optimum of \(L_n\).

Proof: Using the triangle inequality, we have

\[
||z(t_n B) - \hat{z}_n|| = ||z(t_n B) - \hat{z}_n(t_{n-1} B) + \hat{z}_n(t_{n-1} B) - \hat{z}_n|| 
\leq ||z(t_n B) - \hat{z}_n(t_{n-1} B)|| + ||\hat{z}_n(t_{n-1} B) - \hat{z}_n||.
\]

From Lemma \([3]\) we see that

\[
||z(t_n B) - \hat{z}_n(t_{n-1} B)|| \leq (1 - \theta \gamma)^{t_{n-1}} ||z(t_{n-1} B) - \hat{z}_n(t_{n-1} B)|| \leq 2q_p^{-t_{n-1}} r,
\]

where the second inequality follows from the definition of \(r\) in \([3]\). Next, from Lemma \([4]\) we see that

\[
||\hat{z}_n(t_{n-1} B) - \hat{z}_n|| \leq \frac{\|A\|_2}{\alpha} ||\lambda(t_{n-1} B) - \hat{\lambda}_n||.
\]

Using \([34]\) and \([35]\) in \([32]\) gives

\[
||z(t_n B) - \hat{z}_n|| \leq 2q_p^{-t_{n-1}} r + \frac{\|A\|_2}{\alpha} ||\lambda(t_{n-1} B) - \hat{\lambda}_n||,
\]

and bounding the last term using Theorem \([2]\) completes the proof.
C. Mixed-Integer Solution Recovery

Due to Theorem 1, agents computing the saddle point of $L_\kappa$ will compute a point that is within a bounded distance from the solution to Problem 1. With the convergence results in Theorem 3, we have a rate of convergence to the saddle point and an upper bound on how far the iterates of Algorithm 1 are from the saddle point at any given time. We round the final output of Algorithm 1 to achieve a feasible mixed integer solution, i.e., a solution that obeys all integrality constraints in Problem 1. Rounding does not drastically affect the feasibility of the solution, and this will be demonstrated in Section IV.

An existing error bound on the rounding of relaxed MILP solutions suggests that a similar theoretical bound is likely to exist for our saddle point formulation [18]. Given the extensive theoretical developments that such a bound would require, we defer its development to a future publication. We also observe that, for fast computation of MILP solutions, MILP solvers often use heuristics to speed up exact methods [19] and recent applications have shown effective results from heuristic methods [20], [21]. Therefore, a simple rounding heuristic is used on the final output generated by Algorithm 1 to obtain a fast solution. One advantage of this rounding heuristic is that it can be executed onboard each agent individually, without any communication among agents. In the next section, we also show that it only introduces slight suboptimality in the mixed-integer solutions that it produces.

IV. Simulation

Random MILPs are used to test Algorithm 1 in MATLAB for varying communication rates when computing the saddle point of $L_\kappa$. For Algorithm 1, we simulate random communications between agents using a probability drawn from [0,1]. The regularization parameters are $\alpha = 10^{-4}$ and $\delta = 10^{-3}$, and for 300 agents with 30 coupling constraints there are 285 primal agents and 15 dual agents. For the MILPs themselves, the local constraints take the form $-s_\ell \leq S_\ell x_\ell \leq s_\ell$, where $S_\ell = 1$ and $s_\ell = 80$ for all $\ell \in [285]$. The coupling constraint terms $A_\ell$ for $\ell \in [285]$ and $b$ are random numbers on the intervals [0,1] and [20 120], respectively, and the linear cost term $c_\ell$ is a random vector on the interval [0 5] for all $\ell \in [285]$. Finally, the local decision vector is $x_\ell \in \mathbb{R}^3 \times \mathbb{Z}^5$, with a primal step size of $\gamma = 0.1$. One hundred random MILPs were run in a Monte Carlo simulation, and the suboptimality in the rounded solutions was computed via $\frac{|c^T z^* - c^T x^*|}{|c^T z^*|}$. For clarity, the convergence of four specific, representative runs is shown in Figure 1 and the suboptimality of all runs is shown in Figure 2.

![Distance Between Iterates Convergence Comparison](image)

Figure 1. Four specific representative runs are compared based on communication rates for a random MILP with 300 agents and 30 coupling constraints where there are 285 primal agents and 15 dual agents. The distance between iterates is compared with respect to the communication rates.
V. Conclusion

This paper presents an approximate solution to large-scale mixed-integer linear programs (MILPs). The MILPs are solved via a distributed saddle point finding algorithm robust to asynchrony. Theoretical bounds on the suboptimality gap between the original MILP and the relaxed version are presented along with convergence rates for the distributed algorithm. A numerical example verifies the convergence analysis. Future work will apply the work to large-scale assignment problems and explore the effect of updating the tightening vector within the algorithm in an asynchronous setting.

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