Berry phase and fidelity susceptibility of the three-qubit Lipkin–Meshkov–Glick ground state

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Abstract
Berry phases and quantum fidelities for interacting spins have attracted considerable attention, particularly in relation to entanglement properties of spin systems and quantum phase transitions. These efforts mainly focus either on spin pairs or the thermodynamic infinite spin limit, while studies of the multi-partite case of a finite number of spins are rare. Here we analyze Berry phases and quantum fidelities of the ground state of a Lipkin–Meshkov–Glick model consisting of three spin-\(\frac{1}{2}\) particles (qubits). We find explicit expressions for the Berry phase and fidelity susceptibility of the full system as well as the mixed-state Berry phase and partial-state fidelity susceptibility of its one- and two-qubit subsystems. We demonstrate a realization of a nontrivial magnetic monopole structure associated with local, coordinated rotations of the three-qubit system around the external magnetic field.

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(All figures in this article are in colour only in the electronic version)

1. Introduction

The relevance of the Berry phase [1] in different branches of physics has been demonstrated since its discovery. It is now well known that the Berry phase has applications in various fields ranging from high-energy to low-energy physical systems [2]. The Berry phase is a manifestation of the underlying geometry of state space [3]. In particular, when the external parameters of the Hamiltonian of a system are varied, the response of the ground state of the system can be studied in terms of the Berry phase as it shows nontrivial behavior related to points in parameter space where two or more energy levels become degenerate. In the vicinity
of a point where two energy levels cross, the Berry phase becomes proportional to the area (solid angle) enclosed in the parameter space of the system. This corresponds to a magnetic flux that originates from an effective magnetic monopole at the energy crossing. This result has opened up for simulations of magnetic monopoles in the laboratory [4]. The monopole structure of interacting spin pairs has been examined in several studies in the past [5–10].

Systems of interacting spin-\(\frac{1}{2}\) particles (qubits) are important tools to implement quantum information processing. This has triggered efforts to understand the behavior of the Berry phase for spin systems. Most of these works have been focused on spin-pairs, while studies of finite-size spin systems beyond the bipartite scenario are still quite rare. Extending the bipartite systems to the multi-spin case should be of interest as multi-spin systems may exhibit richer effective magnetic field configurations and subsystem Berry phases. Xing [11] examined a three-qubit model with uniaxial qubit–qubit interaction and demonstrated that the corresponding Berry phase admits a solid angle interpretation, provided the couplings are added to the underlying parameter space. Williamson and Vedral [12, 13] found a nontrivial relation between the Berry phase of translationally symmetric multi-qubit states and their multi-partite entanglement properties. The behavior of the Berry phase in the thermodynamic limit has been studied for XY spin-chains [14–16], the Dicke model [17] and the Lipkin–Meshkov–Glick (LMG) system [18] in the context of quantum phase transitions (QPT) [19].

Recently, the fidelity, an information theoretic measure, has been used [20] to analyze the quantal properties of the ground state of spin systems. The utility of this measure and the related fidelity susceptibility have been explored in a number of studies [21–23], in particular in relation to QPT [19]. The concept of partial-state Bures–Uhlmann fidelity [24, 25] has been developed, which measures the fidelity of a subsystem along with the associated notion of partial-state fidelity susceptibility [26, 27]. As described by the quantum geometric tensor [28, 29], it can be understood that the fidelity susceptibility and the Berry phase are two complementary manifestations of the inherent geometry of the state space.

Keeping all these features in mind, we intend to study the ground-state properties of a multi-spin system by examining the Berry phase of the system and its subsystems using pure-state [1] and mixed-state [30] Berry phases. Moreover, in this paper, we would also like to investigate the response of the ground state of the concerned system to parameter variation in terms of fidelity susceptibility and partial-state fidelity susceptibility. To this end, we use an analytically solvable, finite-size LMG-type model consisting of three spin-\(\frac{1}{2}\) particles.

The Lipkin–Meshkov–Glick (LMG) model, which was originally introduced in nuclear physics [31], has now become a basic exactly solvable [32, 33] model to describe magnetic properties of a collective spin system with long-range interactions. While changing the effective magnetic field, this model shows a rich phase diagram in both ground and excited states, independent of the system size [34]. In quantum information theory it can be used to test the fundamental relation between many-body entanglement and QPT [35]. It also captures the physics of interacting bosons in double-well-like structures [36, 37] and is thus relevant to two-mode Bose–Einstein condensates [38] as well as Josephson junctions. It has also been used to model a few trapped ions interacting with laser fields [39, 40]. Interestingly, in recent times, an experimentally feasible scheme has also been developed [41] to simulate a generalized LMG model in a Bose–Einstein condensate inside an optical cavity.

The outline of the paper is as follows. In the next section, the three-qubit LMG model is described, the corresponding energy eigenvalues and eigenvectors are found, and the ground states are identified. Sections 3 and 4 examine in detail the Berry phases and fidelity susceptibilities of the present LMG system and its subsystems. The paper ends with some conclusions.
2. The three-qubit LMG model

The LMG model of spin systems has found applications in Bose–Einstein condensates [38], statistical mechanics of mutually interacting spins [42] and entanglement theory [43, 44]. The LMG model describes a set of $N$ qubits (spin–$\frac{1}{2}$) mutually interacting through an XY-like term in the Hamiltonian and coupled to an external transverse magnetic field. The ferromagnetic version of the LMG Hamiltonian reads

$$H(\gamma, h; N) = -\frac{1}{N} \left( S^z_1 + \gamma S^z_3 \right) - h S_z^1,$$

$$= -\frac{1}{N} \left( \frac{1 + \gamma}{2} (S^z_2 - S^z_3) + \frac{1 - \gamma}{2} (S^z_2 + S^z_3) \right) - h S_z^1,$$  \hspace{1cm} (1)

where $\gamma$ is an anisotropy parameter ($\gamma = 1$ corresponds to the isotropic LMG model), $h$ is the strength of an external magnetic field in the $z$ direction and $S^\alpha_k = \sum_{\ell=1}^N \frac{1}{2} \sigma^\alpha_k$ is the $\alpha$th component of the total spin operator ($\ell = 1$ from now on) with $\sigma^x_k = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma^z_k = -i|0\rangle\langle 1| + i|1\rangle\langle 0|$, and $\sigma^y_k = |0\rangle\langle 0| - |1\rangle\langle 1|$ the Pauli operators of the $k$th qubit. $S^\pm_k = S^x_k \pm i S^y_k$ are the step operators.

The LMG Hamiltonian defines the constants of the motion $P^N_{\ell=1} \sigma^\pm_k$ (parity) and $S^2$ (total spin) [45], where the parity expresses the fact that only angular momentum states with total spin projections $\Delta M = \pm 2$ are coupled by the LMG Hamiltonian. In the two-qubit case ($N = 2$), the $M = \pm 1$ triplet states couple, leading to an effective magnetic monopole carrying half a unit magnetic charge [8]. Here, we analyze the three-qubit ($N = 3$) model, which comprises a ground state that exhibits a monopole of half a unit magnetic charge as well as a one- and two-qubit subsystem structure. The monopole structure is related to the fact that the $S = \frac{3}{2}$ states divides into two decoupled two-level sectors spanned by $M = \left( \frac{1}{2}, -\frac{1}{2} \right)$ and $M = \left( -\frac{3}{2}, \frac{1}{2} \right)$. We focus on the $N = 3$ LMG model as it is the only finite-size $N \geq 3$ case for which complete solutions can be written in compact form for all $\gamma$ and $h$.

The Hamiltonian for the three-qubit LMG system is given by

$$H(\gamma, h, 3) = H(\gamma, h, 3) = -\frac{1}{6} \left[ \sigma^x_1 \sigma^x_3 + \sigma^y_1 \sigma^y_3 + \sigma^z_1 \sigma^z_3 + \gamma (\sigma^x_1 \sigma^x_2 + \sigma^y_1 \sigma^y_2 + \sigma^z_1 \sigma^z_2) \right]$$

$$- \frac{h}{2} (\sigma^x_1 + \sigma^z_1 + \sigma^z_3),$$  \hspace{1cm} (2)

where we have ignored the unimportant constant term $-\frac{1}{3}(1 + \gamma)$. In the computational basis $\{|000\rangle, |011\rangle, |101\rangle, |110\rangle, |111\rangle, |100\rangle, |010\rangle, |001\rangle\}$, the Hamiltonian takes the block-diagonal form

$$H(\gamma, h) = \begin{pmatrix} P(\gamma, h) & 0 \\ 0 & P(\gamma, -h) \end{pmatrix},$$  \hspace{1cm} (3)

where

$$P(\gamma, h) = \begin{pmatrix} \frac{1}{6} h & -\frac{1}{6} (1 - \gamma) & -\frac{1}{6} (1 - \gamma) & -\frac{1}{6} (1 - \gamma) \\ -\frac{1}{6} (1 - \gamma) & -\frac{1}{6} h & -\frac{1}{6} (1 + \gamma) & -\frac{1}{6} (1 + \gamma) \\ -\frac{1}{6} (1 - \gamma) & -\frac{1}{6} (1 + \gamma) & -\frac{1}{6} h & -\frac{1}{6} (1 + \gamma) \\ -\frac{1}{6} (1 - \gamma) & -\frac{1}{6} (1 + \gamma) & -\frac{1}{6} (1 + \gamma) & -\frac{1}{6} h \end{pmatrix},$$  \hspace{1cm} (4)

and $0$ is the $4 \times 4$ null matrix. The block structure originates from the existence of the conserved parity $\sigma^x_1 \sigma^x_2 \sigma^x_3$, which displays the fact that only states with same spin parity interact [45]. Due to the $h \leftrightarrow -h$ symmetry between the two $P$ blocks of $H$, we may assume $h \geq 0$ without
loss of generality. To facilitate the diagonalization of $\mathbf{H}$, we define energy functions $\mathcal{E}_\pm$ and mixing angle $\Theta$ according to

$$\mathcal{E}_\pm (\gamma, h) = \frac{1}{6} (3h - 1 - \gamma \pm 2\sqrt{9h^2 + 3h(1 + \gamma) + 1 - \gamma + \gamma^2})$$

$$= \mathcal{E}_0 (\gamma, h) \pm \Delta \mathcal{E} (\gamma, h),$$

$$\tan \left[ \frac{1}{2} \Theta (\gamma, h) \right] = \frac{\sqrt{3}(\gamma - 1)}{6h + 1 + \gamma - 2\sqrt{9h^2 + 3h(1 + \gamma) + 1 - \gamma + \gamma^2}},$$

(5)

where $\mathcal{E}_0 (\gamma, h) = \frac{1}{6} (3h - 1 - \gamma)$ and $\Delta \mathcal{E} (\gamma, h) = \frac{1}{6} \sqrt{9h^2 + 3h(1 + \gamma) + 1 - \gamma + \gamma^2}$. Note, in particular, that $\tan \left[ \frac{1}{2} \Theta (\gamma, h) \right]$ diverges (tends to zero) in the isotropic limit $\gamma \to 1$. Thus, $\Theta (1, h) = \pi$ and $\Theta (1, -h) = 0$. We may now write the eigenvalues $E$ and orthonormalized eigenvectors $|V\rangle$ of $H$ in terms of $\mathcal{E}_\pm$, $\mathcal{E}$ and $\Theta$ as

$$E_+ (\gamma, h) = \mathcal{E}_+ (\gamma, h) : |V_+^{(\gamma,h)}\rangle = -\sin \left[ \frac{1}{2} \Theta (\gamma, h) \right] |000\rangle + \cos \left[ \frac{1}{2} \Theta (\gamma, h) \right] |W\rangle,$$

$$E_- (\gamma, h) = \mathcal{E}_- (\gamma, h) : |V_-^{(\gamma,h)}\rangle = \cos \left[ \frac{1}{2} \Theta (\gamma, h) \right] |111\rangle + \sin \left[ \frac{1}{2} \Theta (\gamma, h) \right] |W\rangle,$$

$$E_0 (\gamma, h) = \mathcal{E}_0 (\gamma, h) : |V_0^{(\gamma,h)}\rangle = \frac{1}{\sqrt{2}} \left( |011\rangle + |101\rangle \right),$$

$$|V_1^{(\gamma,h)}\rangle = \frac{1}{\sqrt{6}} \left( |010\rangle + |100\rangle \right),$$

where $|W\rangle = \sigma_x \otimes \sigma_x \otimes \sigma_x |W\rangle = \frac{1}{\sqrt{2}} \left( |011\rangle + |101\rangle \right)$.

Alternatively, we may use that the total spin $S^2$ commutes with the LMG Hamiltonian [45], which implies that the eigensolutions may be labeled by the total spin. For instance, we may write the two types of ground states as $|V_0^{(\gamma,h)}\rangle = \cos \left[ \frac{1}{2} \Theta (\gamma, \pm h) \right] |\frac{1}{2}, \pm \frac{1}{2}\rangle + \sin \left[ \frac{1}{2} \Theta (\gamma, \pm h) \right] |\frac{1}{2}, \mp \frac{1}{2}\rangle$ with $|S, M\rangle$ being the common eigenvectors of $S^2$ and $S_z$. Thus, $|\frac{1}{2}, \frac{1}{2}\rangle = |000\rangle$, $|\frac{1}{2}, \frac{1}{2}\rangle = |111\rangle$, $|\frac{1}{2}, \frac{1}{2}\rangle = |W\rangle$ and $|\frac{1}{2}, \frac{1}{2}\rangle = |\bar{W}\rangle$.

The lowest energy state is $V_0^{(+)}$ or $V_0^{(-)}$ associated with energies $E_0^{(+)}$ and $E_0^{(-)}$, respectively, shown in figure 1. The crossing points lie along the lines $\gamma \mapsto h C = h_0 (\gamma)$, which are determined by

$$\mathcal{E}_0 (\gamma, h_0 (\gamma)) = \mathcal{E}_- (\gamma, -h_0 (\gamma)).$$

(7)

This yields the following two classes of solutions:

$$h_0^{(1)} = 0, \quad \gamma \text{ arbitrary}$$

(8)

and

$$\left( h_0^{(2)} \right)^2 = \frac{4}{9} \gamma.$$

(9)

The ground state in the low-field-strength regime, corresponding to $h < h_0^{(1)}$, is $V_0^{(+)}$; in the high-field-strength regime, corresponding to $h > h_0^{(2)}$, it is $V_0^{(-)}$. 

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Figure 1. Energy levels $E^{(+)}_1$ and $E^{(-)}_2$ of the two potential ground states $V^{(+)}_1$ and $V^{(-)}_2$ as a function of the isotropy parameter $\gamma$ and the magnetic field strength $h$.

The ground state may take any of the three main three-qubit forms: $W$, GHZ and product states. Here, we identify the corresponding mixing angles $\Theta$ defined in equation (5) and delineate the exact form of these ground states. The $W$ and product forms are obtained for mixing angle being an integer multiple of $\pi$, which may happen only in the isotropic case $\gamma = 1$. Indeed, we found above that $\Theta(1, h) = \pi$ and $\Theta(1, -h) = 0$, which implies the ground state

\[ |V^{(+)}_1\rangle = |W\rangle, \quad h^{(1)}_1 < h < h^{(2)}_1, \]

in the isotropic LMG model. The GHZ form $U_1 \otimes U_2 \otimes U_3 \frac{1}{\sqrt{2}}(|000\rangle \pm |111\rangle)$, $U_1$, $U_2$, $U_3$ being any one-qubit unitary operators, requires mixing angles that satisfy $\tan \left(\frac{1}{2} \Theta_1\right) = \pm \sqrt{3}$. This happens at $\Theta = \Theta(0, 0) = \frac{2\pi}{3}$ and $\Theta = \Theta(2, -\frac{1}{3}) = \frac{4\pi}{3}$. Here, $\Theta = \frac{2\pi}{3}$ corresponds to the two-fold degenerate ground state

\[ |V^{(\pm)}_1\rangle = e^{-i\frac{\pi}{2} \sigma_x} \otimes e^{-i\frac{\pi}{2} \sigma_y} \otimes e^{-i\frac{\pi}{2} \sigma_z} |000\rangle \pm |111\rangle, \]

i.e. the GHZ form with $U_1 = U_2 = U_3 = \sigma_x$. Thus, the ground state tends to a GHZ when approaching the origin in the $(\gamma, h)$ plane. The angle $\Theta = \frac{2\pi}{3}$ yields a GHZ with $U_1 = U_2 = U_3 = e^{i\frac{\pi}{3} \sigma_x}$. However, the corresponding states at $(\gamma, h) = (2, \pm \frac{1}{3})$ are $V^{(\pm)}_1$, neither of which being the ground state. In other words, the ground state may be of GHZ form only at $(\gamma, h) = (0, 0)$.

3. The Berry phase

3.1. Full system

Here, we examine Berry phases [1] arising in adiabatic variation of the LMG Hamiltonian. For given $\gamma$ and $h$, let us consider the isospectral one-parameter Hamiltonian family

\[ H(\gamma, h; \phi) = e^{-i\phi S_z} H(\gamma, h) e^{i\phi S_z}, \]
where $\phi$ is slowly varying. Note that the unitary operator $e^{-i\phi \mathbb{S}}$, corresponding to coordinated rotation of the system around the $z$-axis by an angle $\phi$, preserves the $4 \times 4$ block structure of $H(\gamma, h)$. Thus, by preparing the system in the ground state and by varying $\phi$ slowly, the system remains in the corresponding two-dimensional subspace. The state of the system may be represented by one of the double-valued eigenvectors

$$\left| V(\pm) \right\rangle = e^{-i\phi \mathbb{S}} \left| V(\pm) \right\rangle.$$  

(13)

After completion of a $2\pi$ rotation around the $z$-axis, corresponding to increasing $\phi$ from 0 to $2\pi$, we obtain the Berry phase in cyclic adiabatic evolution as \cite{46, 47}

$$\beta_g = \begin{cases} \beta^+ = \arg\langle V^+(0) | V^+(2\pi) \rangle + i \int_0^{2\pi} \left( V^+(\phi) \left| \frac{\partial}{\partial \phi} V^+(\phi) \right| \right) d\phi & h^{-1} < h < h_c^{(2)} \\ \beta^- = \arg\langle V^-(0) | V^-(2\pi) \rangle + i \int_0^{2\pi} \left( V^-(\phi) \left| \frac{\partial}{\partial \phi} V^-(\phi) \right| \right) d\phi & h > h_c^{(2)} \end{cases}$$  

(14)

The absolute value $|\beta_g|$ of the ground-state Berry phase $\beta_g$ is shown in figure 2. It should be noted that the Berry phase is the defined modulus $2\pi$, which implies that the $4\pi$ jump at the crossing point $h_c^{(2)} = \frac{2}{3}$ in the isotropic ($\gamma = 1$) LMG model that is visible in figure 2 cannot be detected experimentally.

In order to understand the origin of the nontrivial two-level type $\beta_g$ in equation (14), we project the Hamiltonian $H(\gamma, h; \phi)$ onto two-dimensional subspaces spanned by $\{\left| 000 \right\rangle, \left| \bar{W} \right\rangle \}$ for $h^{-1} < h < h_c^{(2)}$ and $\{\left| 111 \right\rangle, \left| W \right\rangle \}$ for $h > h_c^{(2)}$. Let $P^{(+)} = \left| 000 \right\rangle \langle 000 \right| + \left| \bar{W} \right\rangle \langle \bar{W} \right|$ and $P^{(-)} = \left| 111 \right\rangle \langle 111 \right| + \left| W \right\rangle \langle W \right|$ be the corresponding projection operators. Furthermore, we define $\Sigma^{(+)} = \left| 000 \right\rangle \langle 000 \right| + \left| \bar{W} \right\rangle \langle \bar{W} \right|$, $\Sigma^{(-)} = -i\left| 000 \right\rangle \langle \bar{W} \right| + i\left| \bar{W} \right\rangle \langle 000 \right|$ and $\Sigma^{(k)} = \left| 000 \right\rangle \langle 000 \right| - \left| \bar{W} \right\rangle \langle \bar{W} \right|$, as well as $\Sigma^{(k)} = \sigma_k \otimes \sigma_k \otimes \sigma_k + \sigma_k \otimes \sigma_k \otimes \sigma_k$, for $k = x, y, z$. 

![Figure 2](image-url)
This yields the effective projected two-level ground-state Hamiltonian
\[
H_{\text{eff}}(\gamma, h, \phi) = \begin{cases}
P^{(+)} H(\gamma, h, \phi) P^{(+)} = E_0(\gamma, h) P^{(+)} + \Delta E(\gamma, h) \left( \sin[\Theta(\gamma, h)] \cos(2\phi) \Sigma_{z}^{(+)} \right. \\
+ \sin[\Theta(\gamma, h)] \sin(2\phi) \Sigma_{x}^{(+)} + \cos[\Theta(\gamma, h)] \Sigma_{y}^{(+)}, \\
& h^1 < h < h^2(1), \\
+ \Delta E(\gamma, -h) \sin(\Theta(\gamma, -h)) \cos(2\phi) \Sigma_{x}^{(-)} - \sin[\Theta(\gamma, -h)] \sin(2\phi) \Sigma_{y}^{(-)} \\
+ \cos[\Theta(\gamma, -h)] \Sigma_{z}^{(-)}, \\
& h > h^2(2).
\end{cases}
\]

This describes a two-level system exposed to an effective magnetic field with strength \(\Delta E(\gamma, \pm h)\) that takes the form
\[
B^{(\pm)} = \Delta E(\gamma, \pm h) \left\{ \sin[\Theta(\gamma, \pm h)] \cos(2\phi), \pm \sin[\Theta(\gamma, \pm h)] \sin(2\phi), \cos[\Theta(\gamma, \pm h)] \right\}
\]
\[
= \Delta E(\gamma, \pm h) \mathbf{n}(\Theta(\gamma, \pm h), \phi).
\]

Here, we have introduced the unit vectors \(\mathbf{n}(\Theta(\gamma, \pm h), \phi)\) that rotate with twice the rotation angle \(\phi\) and make polar angles \(\Theta(\gamma, \pm h)\) with the effective \(z\)-axis. Thus, \(\mathbf{n}\) rotates twice around the effective \(z\)-axis when the system is rotated once around the magnetic field \(h\), in real space; a feature that explains the extra factor 2 in the pure-state Berry phases in equation (14). The sign difference in the expression for the two types of ground-state Berry phases in equation (14) originates from that \(B^{(\pm)}\) rotate in the opposite direction. It is visible that the origin of the nontrivial ground-state Berry phase is a monopole carrying half a unit magnetic charge sitting at the point where \(\Delta E = 0\). This happens at \((\gamma, h) = (1, -\frac{1}{2})\) in the low-field regime \((h^1 < h < h^2(1))\) and at \((\gamma, h) = (1, \frac{1}{2})\) in the high-field regime \((h > h^2(2))\); for \(\gamma \neq 1\) there is an avoided crossing at \(h = -\frac{1}{2}(1 + \gamma)\) \((h = \frac{1}{2}(1 + \gamma))\) corresponding to a minimal energy difference \(2\Delta E = \frac{1}{\sqrt{3}}|1 - \gamma|\) in the low- (high-) field regime. The Berry effective gauge field takes the magnetic monopole form
\[
B_{\text{eff}}^{(\pm)} = \mp \frac{1}{2} \frac{\mathbf{n}(\Theta(\gamma, \pm h), \phi)}{[\Delta E(\gamma, h)]^2}
\]
and the Berry phase shown in figure 2 is the flux of \(B_{\text{eff}}^{(\pm)}\) through any surface enclosed by the curve traversed in the parameter space \((\Delta E, \Theta, \phi)\), where \(\Delta E\) and \(\Theta\) are determined by \(\gamma\) and \(h\). We may therefore interpret the jump across the crossing line \(\gamma \mapsto h^2(1)\) as an interplay between a jump in the polar angle \(\Theta\) and that the two types of ground states feel monopoles sitting at different points in the \((\gamma, h)\) plane.

3.2. Subsystems

An interferometer experiment to detect the Berry phase could be set up for one or two of the qubits. As the states of the subsystems in general are mixed, the corresponding Berry phases would coincide with the mixed-state geometric phase in [30], applied to adiabatic evolution. Here, we examine the behavior of these mixed-state Berry phases in the LMG system.

We calculate the subsystem Berry phases under slow rotation around the \(z\)-axis. To this end, we need the reduced ground states \(\rho^{(\pm)}\) and \(\varrho^{(\pm)}\) of the one- and two-qubit subsystem, respectively. Taking into account the translational symmetry of the ground states \(V^{(\pm)}\), these marginal states for any of the qubit or qubit pair read
\[
\rho^{(\pm)} = \frac{1}{2}[1 + r(\gamma, \pm h)\sigma_z]
\]
and
\[
\varrho^{(\pm)} = \frac{1}{2}[1 + r(\gamma, \pm h)]|\psi_1^{(\pm)}\rangle\langle\psi_1^{(\pm)}| + \frac{1}{2}[1 - r(\gamma, \pm h)]|\psi_2^{(\pm)}\rangle\langle\psi_2^{(\pm)}|.
\]
respectively. Here,

\[ r(\gamma, h) = \frac{1}{3} \left[ 1 + 2 \cos \Theta(\gamma, h) \right], \]

\[ |\psi_i^{(+)}\rangle = \frac{1}{\sqrt{2 + \cos[\Theta(\gamma, h)]}} \left\{ \sqrt{3} \cos \left[ \frac{1}{2} \Theta(\gamma, h) \right] |00\rangle + \sin \left[ \frac{1}{2} \Theta(\gamma, h) \right] |11\rangle \right\}, \]

\[ |\psi_i^{(-)}\rangle = \frac{1}{\sqrt{2 + \cos[\Theta(\gamma, -h)]}} \left\{ \sqrt{3} \cos \left[ \frac{1}{2} \Theta(\gamma, -h) \right] |11\rangle + \sin \left[ \frac{1}{2} \Theta(\gamma, -h) \right] |00\rangle \right\}, \]

\[ |\psi_2^{(\pm)}\rangle = \frac{1}{\sqrt{2}} (|01\rangle \pm |10\rangle), \quad (20) \]

which define the eigenvectors \( e^{-i\phi_{(\sigma_3, \otimes 1+1\otimes \sigma_3)}^{(\pm)} (\gamma)} |\psi_{\mu}^{(\pm)}\rangle \) of \( \phi^{(\pm)}(\phi) \) corresponding to its nonzero eigenvalues.

We note that the one-qubit Berry phases vanish since the corresponding reduced density operators are diagonal in the \(|0\rangle, |1\rangle\) basis and thereby commute with \( e^{-i\phi_{(\sigma_3, \otimes 1+1\otimes \sigma_3)}^{(\pm)} (\gamma)} \). On the other hand, the two-qubit Berry phases may be non-vanishing. To see this, we note that the reduced two-qubit density operator reads \( \rho(\gamma, h) = e^{-i\phi_{(\sigma_3, \otimes 1+1\otimes \sigma_3)}^{(\pm)} (\gamma)} e^{-i\phi_{(\sigma_1, \otimes 1+1\otimes \sigma_1)}^{(\pm)} (\gamma)} \). We see that \( \phi^{(\pm)}(\phi) \neq \phi^{(\pm)}(\phi) \), which opens up for nontrivial mixed-state Berry phase \( \Gamma^{(\pm)} \) of the two-qubit states.

The mixed-state geometric phase proposed in [30] for a cyclic adiabatic evolution of the ground-state candidates reads

\[ \Gamma^{(\pm)} = \arg \left( \sum_{\mu} p_{\mu}^{(\pm)} e^{i\beta_{\mu}^{(\pm)}} \right), \quad (21) \]

where \( p_{\mu}^{(\pm)} \) and \( \beta_{\mu}^{(\pm)} \) are the reduced density operator’s eigenvalues and eigenstate Berry phases, respectively. We obtain from equation (21) the two-qubit geometric phase \( \Gamma_\gamma \) of the ground state as

\[
\Gamma_\gamma = \begin{cases}
\Gamma^{(+)} = \arg \left\{ (2 + \cos[\Theta(\gamma, h)]) |e^{i\beta_1^{(+)}(\gamma)}| + (1 - \cos[\Theta(\gamma, h)]) |e^{i\beta_2^{(+)}(\gamma)}| \right\} & \text{for } h_{c}^{(1)} < h < h_{c}^{(2)} \\
= \arctan \left( \frac{2 + \cos[\Theta(\gamma, h)]}{1 - \cos[\Theta(\gamma, h)] + (2 + \cos[\Theta(\gamma, h)]) \cos \beta_1^{(+)}} \right) \cos \beta_1^{(+)} & \text{for } h > h_{c}^{(2)}
\end{cases}
\]

\[
\Gamma^{(-)} = \arg \left\{ (2 + \cos[\Theta(\gamma, -h)]) |e^{i\beta_1^{(-)}(\gamma)}| + (1 - \cos[\Theta(\gamma, -h)]) |e^{i\beta_2^{(-)}(\gamma)}| \right\} & \text{for } h_{c}^{(1)} < h < h_{c}^{(2)} \\
= \arctan \left( \frac{2 + \cos[\Theta(\gamma, -h)]}{1 - \cos[\Theta(\gamma, -h)] + (2 + \cos[\Theta(\gamma, -h)]) \cos \beta_1^{(-)}} \right) \cos \beta_1^{(-)} & \text{for } h > h_{c}^{(2)}
\end{cases}
\]

(22)

where we have used that \( \beta_2^{(\pm)} = 0 \) since \( |\psi_{1}^{(\pm)}\rangle \) are eigenvectors of \( \sigma_z \otimes \hat{1} + \hat{1} \otimes \sigma_z \). Here,

\[ \beta_1^{(\pm)} = \mp 2\pi \frac{1 - \cos[\Theta(\gamma, \pm h)]}{2 + \cos[\Theta(\gamma, \pm h)]}, \quad (23) \]

which is \( \beta^{(\pm)} \) quenched by a factor \( (2 + \cos[\Theta(\gamma, \pm h)])^{-1} \). The absolute value \( |\Gamma_\gamma| \) of the two-qubit Berry phase \( \Gamma_\gamma \) of the ground state is shown in figure 3.

Note that the relative phase \( \arg \text{Tr}(e^{-i\phi_{(\sigma_3, \otimes 1+1\otimes \sigma_3)}^{(\pm)} (\gamma)} \rho^{(\pm)}) \) [30] between the initial and final two-qubit states is measurable for all \( (\gamma, h) \), as the visibility \( |\text{Tr}(e^{-i\phi_{(\sigma_3, \otimes 1+1\otimes \sigma_3)}^{(\pm)} (\gamma)} \rho^{(\pm)})| \) cannot
vanish for these states. On the other hand, the geometric part $\Gamma_g$ of this relative phase is not always well defined; in fact, it is undefined if $\rho^{(\pm)}$ has nonzero degenerate eigenvalues [30], i.e. when $\cos \Theta = -\frac{1}{2}$. Thus, the mixed-state Berry phase is undefined precisely when the three-qubit ground state is of GHZ form, i.e., at $(\gamma, h) = (0, 0)$, see equation (11).

Note also the oscillatory behavior of $\Gamma_g$ that is visible in figure 3 when approaching the degeneracy. These oscillations may be understood from the fact that the mixed-state Berry phase in unitary evolution is known to vary more rapidly in the vicinity of a degeneracy point of the corresponding density operator [48]. Finally, just as the three-qubit Berry phase factor, the two-qubit Berry phase factor is smooth across the crossing point at $h = \frac{2}{3}$ in the isotropic LMG model. Indeed, $\Gamma_g$ vanishes for all $h$ as $\beta_1^{(\pm)}$ is an integer multiple of $2\pi$ when $\gamma = 1$.

4. Fidelity

4.1. Fidelity susceptibility

Fidelity is a measure of similarity between different quantum states and is therefore expected to be sensitive to abrupt changes in the ground-state properties in many-body systems. This has triggered work to use fidelity measures in the context of quantum critical phenomena [20–23, 28, 29]. Here, we analyze the fidelity susceptibility [21] in the present three-qubit LMG system.

An abrupt change in the LMG model system can be induced by slowly tuning the external magnetic field $h$ across the crossing value $h_c^{(2)} = \frac{2}{3} \sqrt{\gamma}$ at fixed $\gamma$. The fidelity susceptibility $\chi_h^g(\gamma, h)$ is taken to measure the response of the ground state to such variations in $h$. A convenient form for this $\chi_h^g(\gamma, h)$ may be found by writing the LMG Hamiltonian as

$$H(\gamma, h) = H_0 + h H_1,$$

(24)
where $H_0 = -\frac{1}{6} (S_x^2 + 2S_y^2)$ is independent of $h$ and $H_I = -S_z$ is the driving Hamiltonian, the relative strength of $H_0$ and $H_I$ being controlled by $h$. Let $\ket{V_g(\gamma, h)} = \ket{V^{(\pm)}}$ be the normalized ground state of $H(\gamma, h)$ and $E_g(\gamma, h)$ the corresponding ground-state energy. The fidelity susceptibility $\chi^h_g(\gamma, h)$ of $V_g$ is defined as the leading nontrivial contribution in $\delta h$ to the fidelity $F_g$ between the ground states $\ket{V_g(\gamma, h)}$ and $\ket{V_g(\gamma, h + \delta h)}$, according to

$$F_g(\gamma, h, \delta h) = |\langle V_g(\gamma, h) | V_g(\gamma, h + \delta h) \rangle| = 1 - \frac{1}{2} \chi^h_g(\gamma, h) \delta h^2 + \ldots , \tag{25}$$

where we may note that $\chi^h_g$ is independent of the arbitrary parameter $\delta h$. By expanding to second order in $\delta h$ and using the form of $H(\gamma, h)$, we obtain [21]

$$\chi^h_g(\gamma, h) = \sum_{n \neq g} \frac{|\langle V_n(\gamma, h) | H_I | V_g(\gamma, h) \rangle|^2}{[E_n(\gamma, h) - E_g(\gamma, h)]^2} , \tag{26}$$

where $\ket{V_n(\gamma, h)}$ and $E_n(\gamma, h)$ are eigenvectors and eigenvalues, respectively, of $H(\gamma, h)$. By inserting equation (6) into equation (26) and using $H_I = -S_z$, we obtain

$$\chi^h_g(\gamma, h) = \begin{cases} \chi(\gamma, h) = \frac{|\langle V_+^{(\pm)} | (-S_z) | V_+^{(\pm)} \rangle|^2}{[\Delta E(\gamma, h)]^2} = \frac{\sin^2[\Theta(\gamma, h)]}{[\Delta E(\gamma, h)]^2} , & h^{(1)}_c < h < h^{(2)}_c , \\ \chi(\gamma, -h) = \frac{|\langle V_-^{(\pm)} | (-S_z) | V_-^{(\pm)} \rangle|^2}{[\Delta E(\gamma, -h)]^2} = \frac{\sin^2[\Theta(\gamma, -h)]}{[\Delta E(\gamma, -h)]^2} , & h > h^{(2)}_c . \end{cases} \tag{27}$$

We may note that $S_z$ does not couple the degenerate states $V_1^{(\pm)}$ and $V_2^{(\pm)}$ to any of the two candidate ground states, since $V_1^{(\pm)}$ and $V_2^{(\pm)}$ belong to different $S_z^2$ eigenvalues. The fidelity susceptibility $\chi^h_g(\gamma, h)$ is shown in figure 4.

The fidelity susceptibility of the LMG ground state vanishes identically for $\gamma = 1$ as $\Theta(1, h) = 0$ or $\pi$. $\chi^h_g$ therefore shares the behavior of the Berry phase factors $e^{i\beta_r}$ and $e^{i\Gamma_s}$.
in that it is smooth across the crossing point in the isotropic ($\gamma = 1$) case. Furthermore, $\chi^{h}(\gamma, h)$ decreases monotonically toward zero as a function of $\gamma$ when $\gamma$ increases. This can be explained by noting that $\sin^{2}(\Theta)$ decreases monotonically when $\Theta$ increases from $\Theta(0, 0) = \frac{\pi}{2}$ to $\Theta(1, h)$ = $\pi$. On the other hand, $\chi^{h}(\gamma, h)$ has a local maximum since to decrease $\Theta(0, 0) = \frac{\pi}{2}$ to $\Theta(1, h)$ = $\pi$ one must pass the intermediate angle $\Theta = \frac{\pi}{2}$ at which $\sin \Theta$ has its maximum. Furthermore, we note that the fidelity susceptibility is singular close to the degeneracies at $(\gamma, h) = \{1, \pm \frac{1}{2}\}$ which corresponds to the locations of the effective magnetic monopoles and where the adiabatic approximation breaks down. The singular behavior expresses the fact that small variations in the parameters may cause transitions between the two orthogonal states that cross at these points.

4.2. Partial-state fidelity susceptibility

Partial-state fidelity susceptibility has been developed to deal with the response of a subsystem $s$ to the driving Hamiltonian [26, 27]. It is defined as the leading nontrivial contribution of the Bures–Uhlmann fidelity $F_{s,g}$ [24, 25] of two marginal ground states $\rho_{s}(\gamma, h) = \text{Tr}_{p}|\psi_{s}(\gamma, h)⟩⟨\psi_{s}(\gamma, h)|$ and $\rho_{s}(\gamma, h+\delta h) = \text{Tr}_{p}|\psi_{s}(\gamma, h+\delta h)⟩⟨\psi_{s}(\gamma, h+\delta h)|$, $\text{Tr}_{p}$ being partial trace over one or two of the qubits. Explicitly,

$$F_{s,g}(\gamma, h, \delta h) = \text{Tr}_{p}\left[\sqrt{\rho_{s}(\gamma, h)}\rho_{s}(\gamma, h+\delta h)\sqrt{\rho_{s}(\gamma, h)}\right]$$

$$= 1 - \frac{1}{2} \chi^{h}(\gamma, h)|\delta h|^{2} + \cdots$$

defines the partial-state susceptibility $\chi^{h}$ of the ground state [26, 27].

Let us first consider the one-qubit partial-state fidelity $\chi^{h}_{1,g}$ with respect to variations of $h$. The marginal ground state $\rho^{(\pm)}$ of any of the three qubits is diagonal in the fixed $|0\rangle, |1\rangle$ basis. This implies that only changes in the purity parameter $r(\gamma, h)$ contribute to $\chi^{h}_{1,g}$. An explicit calculation yields

$$\chi^{h}_{1,g}(\gamma, h) = \begin{cases} 
\chi_{1}(\gamma, h) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 - r(h)}^{2} \left[ \frac{\partial r(h)}{\partial h} \right]^{2} \sin^{2} \frac{\Theta(h)}{2}, \\
\chi_{1}(\gamma, -h) = \frac{1}{2} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{1 - r(-h)}^{2} \left[ \frac{\partial r(-h)}{\partial h} \right]^{2} \sin^{2} \frac{\Theta(-h)}{2}. 
\end{cases}$$

The one-qubit partial-state fidelity susceptibility $\chi^{h}_{1,g}(\gamma, h)$ is shown in figure 5.

The two-qubit partial-state fidelity $\chi^{h}_{2,g}$ originates from changes in the purity of $\rho^{(\pm)}$ and in the parameter-dependent eigenvector $|\psi_{2}^{(\pm)}⟩$ of $\rho^{(\pm)}$. The relevant purity parameter $r(\gamma, h)$ implies that the contribution to $\chi^{h}_{2,g}$ from the purity coincides with the one-qubit partial-state fidelity susceptibility $\chi^{h}_{1,g}$. The additional contribution related to the change of
Figure 5. One-qubit partial-state fidelity susceptibility $\chi_{1,\gamma}^h$ of the LMG ground state as a function of the isotropy parameter $\gamma$ and the magnetic field strength $h$.

$|\psi_1^{(+)}\rangle$ equals the corresponding pure-state fidelity susceptibility, weighted by the probability $\frac{1}{2}[1 + r(\gamma, \pm h)]$. Explicitly, we have

$$
\chi_{1,\gamma}^h = \frac{1}{2} \left[ 1 - r(\gamma, h)^2 \right] \left[ \frac{\partial}{\partial h} r(\gamma, h) \right]^2 + [1 + r(\gamma, h)] \left( \frac{\partial}{\partial h} |\psi_1^{(+)}\rangle \left( \frac{\partial}{\partial h} |\psi_1^{(+)}\rangle \right) \right.
$$

$$
- \left. \left( \frac{\partial}{\partial h} |\psi_1^{(+)}\rangle \langle \psi_1^{(+)} | \frac{\partial}{\partial h} |\psi_1^{(+)}\rangle \right) \right)

= \chi_{1,\gamma}^h + \frac{1}{2} \left( \frac{\partial}{\partial h} \Theta(\gamma, h) \right)^2,
$$

$$
h_c^{(1)} < h < h_c^{(2)}.
$$

The two-qubit partial-state fidelity susceptibility $\chi_{2,\gamma}^h$ is shown in figure 6.
Both the one- and two-qubit partial-state fidelity susceptibilities behave similarly as that of the full ground state $V_g$: both $\chi_{1,g}^b(\gamma, h)$ and $\chi_{2,g}^b(\gamma, h)$ vanish in the isotropic ($\gamma = 1$) case and there is a similar dependence on $\gamma$ close to the crossing line $\gamma \mapsto h(2)$. Furthermore, by comparing figures 4, 5 and 6, we note that the pure-state fidelity susceptibilities is typically larger than the partial-state fidelity susceptibilities. It is apparent that $\chi_{2,g}^b(\gamma, h) \geq \chi_{1,g}^b(\gamma, h)$ since the second term on the right-hand side of equation (30) is non-negative. This may be interpreted to be a consequence of the loss of purity for each qubit that is traced out. It may be noted in relation to this result that a rigorous inequality between fidelity and reduced fidelity in the context of unitary dynamics of composite quantum systems has been demonstrated in [49, 50].

5. Conclusions

A detailed characterization of the ground state of a three-qubit Lipkin–Meshkov–Glick (LMG)-type model has been given. We have calculated Berry phases for the three-qubit state as well as for the reduced two-qubit state in the case of local, coordinated $2\pi$ rotation around the axis of the external magnetic field. We have identified an underlying two-level structure of the three-qubit Berry phase and found the relevant magnetic monopole distribution. The ground state of the model is of GHZ-type if the external field and the isotropy parameter both vanish. The reduced two-qubit state at this point in parameter space is two-fold degenerate and separable, from which follows that the corresponding mixed-state Berry phase is undefined. The three- and two-qubit Berry phases vanish modulus $2\pi$ in the isotropic LMG model.

We have calculated the fidelity susceptibility and the one- and two-qubit partial-state fidelity susceptibility for the LMG model. These fidelity susceptibilities all behave similarly, but decrease in size for each qubit being traced out. We have found that the fidelity susceptibilities all vanish in the isotropic LMG model. Analogously to the fidelity susceptibility and Berry phase in the pure-state case, the partial-state fidelity susceptibility
and the Uhlmann holonomy [51] measure the geometry of the space of mixed quantum states. This observation makes it natural to ask whether the Uhlmann holonomy may yield further insights into the ground-state properties of interacting spin models. Paunković and Rocha Vieira [52] have found a rich structure in the Uhlmann holonomy for thermal states in the Stoner–Hubbard and BCS models. A similar calculation of the partial-state holonomy seems pertinent in relation to the present work.

By increasing $N$, additional subsystem structures are introduced associated with higher spin and a richer Berry phase structure related to more than two coupled spin states. For instance, for $N = 4$, the $S = 2$ states with a total spin projection $M = -2, 0, 2$ form an effective three-level system which should comprise an $SU(3)$ Berry phase [53, 54]. An interesting extension of this work would be to study these structures in $N \geq 4$ LMG systems.

We hope that the analysis presented in this work may initiate investigations of few-qubit models to explore further their effective magnetic structure and its associated state space geometry.

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