Spectral Decomposition and Baxterisation of Exotic Bialgebras and Associated Noncommutative Geometries

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Abstract

We study the geometric aspects of two exotic bialgebras $S_{03}$ and $S_{14}$ introduced in \texttt{math.QA/0206053}. These bialgebras are obtained by the Faddeev-Reshetikhin-Takhtajan RTT prescription with non-triangular R-matrices which are denoted $R_{03}$ and $R_{14}$ in the classification of Hietarinta, and they are not deformations of either GL(2) or GL(1/1). We give the spectral decomposition which involves two, resp., three, projectors. These projectors are then used to provide the Baxterisation procedure with one, resp., two, parameters. Further, the projectors are used to construct the noncommutative planes together with the corresponding differentials following the Wess-Zumino prescription. In all these constructions there appear non-standard features which are noted. Such features show the importance of systematic study of all bialgebras of four generators.

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1 Introduction

Until very recently there was no complete list of the matrix bialgebras which are unital associative algebras generated by four elements. The list, of course, includes the four cases which are deformations of classical ones: two two-parameter deformations of each of $GL(2)$ and $GL(1|1)$, namely, the standard $GL_{pq}(2)$ [1], nonstandard (Jordanian) $GL_{gh}(2)$ [2], the standard $GL_{pq}(1|1)$ [3-5] and the hybrid (standard-nonstandard) $GL_{qh}(1|1)$ [6]. (Later, in [7] it was shown that there are no more deformations of $GL(2)$ or $GL(1|1)$.) The list includes also five exotic cases which are not deformations of the classical algebra of functions over the group $GL(2)$ or the supergroup $GL(1|1)$. These correspond to $4 \times 4$ $R$-matrices which are not deformations of the trivial $R$-matrix. In the classification of [8] there are altogether five nonsingular such $R$-matrices. The three triangular ones were introduced in [7] and their duals were found and studied in detail in [9]. The study of the two non-triangular cases was started in [10]. There the duals were found and their irreducible representations were constructed. In the present paper we continue the study of the non-triangular cases with the geometric aspects, which are very important also for the applications.

2 $S03$

2.1 Spectral decomposition

We start with the first (of two) nonsingular non-triangular $R$-matrix in [8] which is not a deformation of the unit matrix:

$$R_{s0,3} \equiv \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (2.1)$$

The actual tool for the spectral decomposition is the braid matrix $\hat{R} = PR$, where $P$ is the permutation matrix:

$$P \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.2)$$

With $R = R_{s0,3}$ we have for the braid matrix:

$$\hat{R} = PR = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix} \quad (2.3)$$
We need the minimal polynomial $\text{pol}(\cdot)$ in one variable such that $\text{pol}(\hat{R}) = 0$ is the lowest order polynomial identity satisfied by $\hat{R}$. In the case at hand, this identity is:

$$\hat{R}^2 - 2\hat{R} + 2I = 0 \quad (2.4)$$

or

$$(\hat{R} - (1 + i)I)(\hat{R} - (1 - i)I) = 0 \quad (2.5)$$

The last identity encodes the projectors we need. Indeed, define:

$$P_{(\pm)} \equiv \frac{1}{2}(I \pm i(\hat{R} - I)) \quad (2.6)$$

Then it is easy to see that $P_{(\pm)}$ satisfy the projector properties - orthogonality:

$$P_{(i)}P_{(j)} = \delta_{ij}P_{(i)}, \quad i, j = +, -,$$

and resolution of the identity:

$$P_{(+)} + P_{(-)} = I. \quad (2.7)$$

(In particular, $P_{(+)}P_{(-)} = 0$ is the same as $\text{(2.5)}$.) Thus, we obtain the spectral decomposition:

$$\hat{R} = (1 - i)P_{(+)}/2 + (1 + i)P_{(-)} = (1 + i)I - 2iP_{(+)} = (1 - i)I + 2iP_{(-)} \quad (2.8)$$

Note that although $\hat{R}$ is real, the roots in $\text{(2.5)}$ are complex and so are the projectors.

### 2.2 Baxterisation

We now apply the Baxterisation procedure for our case. First we introduce the following Ansatz (choosing a convenient normalisation):

$$\hat{R}(x) = I + c(x)\hat{R} \quad (2.9)$$

and we try to find $c(x)$ such that $\hat{R}(x)$ would satisfy the parametrised Yang-Baxter equation:

$$\hat{R}_{(12)}(x)\hat{R}_{(23)}(xy)\hat{R}_{(12)}(y) - \hat{R}_{(23)}(y)\hat{R}_{(12)}(xy)\hat{R}_{(23)}(x) = 0 \quad (2.10)$$

With our Ansatz we actually have:

$$\hat{R}_{(12)}(x)\hat{R}_{(23)}(xy)\hat{R}_{(12)}(y) - \hat{R}_{(23)}(y)\hat{R}_{(12)}(xy)\hat{R}_{(23)}(x) =$$

$$(c(x) + c(y) - c(xy))(\hat{R}_{(12)} - \hat{R}_{(23)}) + c(x)c(y)(\hat{R}_{(12)}^2 - \hat{R}_{(23)}^2) \quad (2.11)$$
Using (2.4), i.e.,
\[ \hat{R}^2 = 2(\hat{R} - I) \]
we obtain:
\[ \hat{R}_{(12)}(x) \hat{R}_{(23)}(xy) \hat{R}_{(12)}(y) - \hat{R}_{(23)}(y) \hat{R}_{(12)}(xy) \hat{R}_{(23)}(x) = \]
\[ (c(x) + c(y) + 2c(x)c(y) - c(xy)) (\hat{R}_{(12)} - \hat{R}_{(23)}) \]  
(2.12)

Hence for Baxterisation one must have
\[ c(x) + c(y) + 2c(x)c(y) = c(xy) \]  
(2.13)
and then (2.10) holds.

The solution of (2.13) is:
\[ 2c(x) = x^p - 1 \]  
(2.14)

It is interesting to note that (see, for example, [11], Sec.3.5), the only change, as compared to $GL_q(N)$, is that one has a factor 2 on the left rather than $(q - q^{-1})$.

Setting $p = -2$ and absorbing a free overall factor one obtains, using also
\[ 2I = \hat{R} + 2\hat{R}^{-1} \]
the elegant, symmetric form
\[ \hat{R}(x) = (\sqrt{2}x)^{-1} \hat{R} + (\sqrt{2}x)\hat{R}^{-1} . \]  
(2.15)

Explicitly,
\[ \hat{R}(x) = \frac{1}{\sqrt{2}x} \left( \begin{array}{cccc}
  x+1 & 0 & 0 & 1-x \\
  0 & x+1 & x-1 & 0 \\
  0 & 1-x & x+1 & 0 \\
  x-1 & 0 & 0 & x+1 \\
\end{array} \right) . \]  
(2.16)

We shall also explore an Ansatz in terms of the projectors (instead of $\hat{R}$ as in (2.9)). Thus, for example, one may set
\[ \hat{R}(x) = I + a(x)P_+ \]  
(2.17)

In this case the Baxterisation constraint turns out to be (compare with (2.13)):
\[ a(xy) = \frac{a(x) + a(y) + a(x)a(y)}{1 - \frac{1}{2}a(x)a(y)} \]  
(2.18)

The relation between $c(x)$ of (2.9) and $a(x)$ of (2.17) can be shown to be:
\[ a(x) = \frac{2c(x)}{i + (i-1)c(x)} = \frac{1 + (1-i)c(x)}{1 + (1+i)c(x)} - 1 \]  
(2.19a)
\[ c(x) = \frac{ia(x)}{2 + (1-i)a(x)} \]  
(2.19b)
Eq. (2.18) is a special case of the functional equation:

\[ a(xy) = \frac{a(x) + a(y) + a(x)a(y)}{1 - k^2a(x)a(y)} \]  

(2.20)

which was studied in a more general context in [12] where also the solution of (2.20) was found:

\[ a(x) = \frac{f(x)}{f(x^{-1})} - 1 \]  

(2.21)

where

\[ f(x) = x^{-1} - x \pm \sqrt{1 - 4k^2(x + x^{-1})} \]  

(2.22)

One obtains complex \( a(x) \) due to the complex roots of (2.5) and the complex projectors in (2.6).

In our case (cf. (2.20)) we have

\[ k^2 = \frac{1}{2} \]

and hence

\[ f(x) = x^{-1} - x \pm i(x + x^{-1}) \]  

(2.23)

and it is sufficient to consider one, say the upper, sign - the lower sign will then correspond to the inversion \( x \to x^{-1} \). With this we obtain:

\[ a(x) = \frac{x^2 - 1}{x^4 + 1} (1 - x^2 + i(1 + x^2)) \]  

(2.24)

Note that \( a(x) \) is defined also for \( x = 0 \), although the auxiliary function \( f(x) \) is not.

Substituting this expression in (2.19b) we recover (2.14) for \( p = -2 \), i.e., the choice by which we obtained (2.15) is not only a consistent one but is distinguished.

On the other hand, if we replace \( x \) by \( x^{-p/2} \) in (2.21) and (2.19b) then we shall recover exactly (2.14). Such substitutions are possible since they do not change the character of equations (2.18) and (2.20).

2.3 Noncommutative Plane

Here we shall find the noncommutative plane for our case. We shall apply the standard approach (cf. [13], also [14]), however, not directly to the coordinates \((x_1, x_2)\) and the differentials \((dx_1, dx_2)\) denoted as \(\xi_1, \xi_2\), but to the following complex linear combinations:

\[ X_1 = (x_1 + ix_2) \quad X_2 = (x_1 - ix_2) \]
\[ Z_1 = (\xi_1 + i\xi_2) \quad Z_2 = (\xi_1 - i\xi_2) \]  

(2.25)
We introduce standard notation:

\[ X \otimes X = \begin{pmatrix} X_1X_1 \\ X_1X_2 \\ X_2X_1 \\ X_2X_2 \end{pmatrix} \]  \hspace{1cm} (2.26)

which we use also for \((X \otimes X) \mapsto (Z \otimes Z), (X \otimes X) \mapsto (X \otimes Z), (X \otimes X) \mapsto (Z \otimes X)\).

According to the mentioned prescription for the consistent covariant calculus satisfying Leibniz rule, the commutation relations between the coordinates and differentials are given as follows:

\[
\begin{align*}
(P - I) X \otimes X &= 0 \hspace{1cm} (2.27a) \\
(Q + I) Z \otimes Z &= 0 \hspace{1cm} (2.27b) \\
Q (Z \otimes X) - X \otimes Z &= 0 \hspace{1cm} (2.27c)
\end{align*}
\]

where \(P, Q\) are solutions of:

\[(P - I) (Q + I) = 0 \hspace{1cm} (2.28)\]

The last equality in our situation means that \(P - I, Q + I\) are proportional to the projectors \(P_+, P_-\) or \(P_-, P_+\), respectively. We choose:

\[
Q + I = \kappa P_+ , \hspace{1cm} P - I = P_- \hspace{1cm} (2.29)
\]

where \(\kappa\) is a complex proportionality constant and in the second relation we have set the proportionality constant equal to 1 due to the homogeneity of \((2.27c)\).

Now the constraints on \(x_i\) and \(\xi_i\) can easily be obtained. It turns out that the modular structure \((2.27)\), expressed in terms of \(x\) and \(\xi\), has real coefficients on choosing \(\kappa\) imaginary, i.e.,

\[
\kappa = ic , \hspace{1cm} c \in \mathbb{R} . \hspace{1cm} (2.30)
\]

The definitions \((2.25)\) were introduced to assure this feature. The final results are:

\[
x_1^2 = x_1 x_2 , \hspace{1cm} x_2^2 = -x_2 x_1 \hspace{1cm} (2.31)
\]

\[
\xi_1^2 = -\xi_1 \xi_2 , \hspace{1cm} \xi_2^2 = \xi_2 \xi_1 \hspace{1cm} (2.32)
\]

and

\[
x_1 \xi_1 = (c - 1) \xi_1 x_1 + c \xi_1 x_2 \\
x_1 \xi_2 = (c - 1) \xi_1 x_2 + c \xi_1 x_1 \\
x_2 \xi_1 = (c - 1) \xi_2 x_1 - c \xi_2 x_2 \\
x_2 \xi_2 = (c - 1) \xi_2 x_2 - c \xi_2 x_1 \hspace{1cm} (2.33)
\]

For \(c = 1\) there is a supplementary simplification.
3 S14

3.1 Spectral decomposition

We take up now the second (of two) nonsingular non-triangular \( R \)-matrix in [8] which is not a deformation of the unit matrix:

\[
R_{S14} = \begin{pmatrix}
0 & 0 & 0 & q \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
q & 0 & 0 & 0
\end{pmatrix}
\]  
(3.34)

where \( q^2 \neq 1 \) (the case \( q^2 = 1 \) turned out [10] to be equivalent to a special case of \( GL_{p,q}(2) \)). Here the braid matrix is

\[
\hat{R} = PR = \begin{pmatrix}
0 & 0 & 0 & q \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
q & 0 & 0 & 0
\end{pmatrix}
\]  
(3.35)

The minimal polynomial identity \( \hat{R} \) satisfies is:

\[
(\hat{R} - I)(\hat{R} - qI)(\hat{R} + qI) = 0
\]  
(3.36)

The projectors are:

\[
P_{(0)} = \frac{1}{1 - q^2}(\hat{R} - qI)(\hat{R} + qI) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]  

\[
P_{(+)} = \frac{1}{2q(q-1)}(\hat{R} + qI)(\hat{R} - I) = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}
\]  
(3.37)

\[
P_{(-)} = \frac{1}{2q(q+1)}(\hat{R} - qI)(\hat{R} - I) = \frac{1}{2} \begin{pmatrix}
1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 1
\end{pmatrix}
\]  
(3.38)

satisfying

\[
P_{(0)}P_{(j)} = \delta_{ij}P_{(i)}
\]

and

\[
P_{(0)} + P_{(+)} + P_{(-)} = I
\]  
(3.39)
Note that the projectors are independent of $q$. This leads to important simplifications. The spectral decomposition is:

$$\hat{R} = \hat{R}(q) = P(0) + qP(+) - qP(-) = I + (q - 1)P(+) - (q + 1)P(-)$$  \hspace{0.5cm} (3.40)

from which it is obvious that:

$$\hat{R}^{-1} = \hat{R}(q^{-1})$$  \hspace{0.5cm} (3.41)

### 3.2 Baxterisation

Here we introduce an Ansatz in terms of the projectors:

$$\hat{R}(v, w) = I + vP(+) + wP(-)$$  \hspace{0.5cm} (3.42)

from which follows:

$$\left(\hat{R}(v, w)\right)^{-1} = I - \frac{v}{1 + v}P(+) - \frac{w}{1 + w}P(-)$$  \hspace{0.5cm} (3.43)

The presence of two projectors above (as compared to one for $SO3$, cf. \(2.17\)) leads to a more elaborate structure analogous to that for $SO_q(N)$ (as compared to $GL_q(N)$). We briefly display the analogy to the $SO_q(N)$ case [12, 15] along with the drastic simplifications due to the special features of the projectors noted above.

Let us also introduce the following notations:

$$X_1 \equiv P(+) \otimes I, \quad X_2 \equiv I \otimes P(+), \quad Y_1 \equiv P(-) \otimes I, \quad Y_2 \equiv I \otimes P(-).$$  \hspace{0.5cm} (3.44)

In which terms we have:

$$\hat{R}_{(12)}(v, w) = I \otimes I \otimes I + vX_1 + wY_1, \quad \hat{R}_{(23)}(v, w) = I \otimes I \otimes I + vX_2 + wY_2$$  \hspace{0.5cm} (3.45)

We obtain:

$$\hat{R}_{(12)}(v, w)\hat{R}_{(23)}(v', w')\hat{R}_{(12)}(v'', w'') - \hat{R}_{(23)}(v'', w'')\hat{R}_{(12)}(v', w')\hat{R}_{(23)}(v, w) =$$

$$= (v + v'' + vv'' - v')S_1 + (w + w'' + ww'' - w')S_2 +$$

$$+ vv'v''S_5 + ww'w''S_6 +$$

$$+ (vv' - v'w)J_1 + (v''w' - v''w')J_2 +$$

$$+ vv'w''K_1 + vv'v''K_2 + vv''K_3 +$$

$$+ ww'v''L_1 + ww''L_2 + vv'w''L_3$$  \hspace{0.5cm} (3.46)

where

$$S_1 = X_1 - X_2, \quad S_2 = Y_1 - Y_2, \quad J_1 = X_1Y_2 - Y_1X_2, \quad J_2 = Y_2X_1 - X_2Y_1$$

$$S_5 = X_1X_2X_1 - X_2X_1X_2, \quad S_6 = Y_1Y_2Y_1 - Y_2Y_1Y_2$$

$$K_1 = X_1X_2Y_1 - Y_2X_1X_2, \quad K_2 = X_1Y_2X_1 - X_2Y_1X_2, \quad K_3 = Y_1X_2X_1 - X_2X_1Y_1$$

$$L_1 = Y_1X_2Y_1 - X_2Y_1Y_2, \quad L_2 = Y_1X_2Y_1 - Y_2X_1Y_2, \quad L_3 = X_1Y_2X_1 - Y_2X_1X_2$$
First we note that from (3.40) and (3.41) it follows that the right hand side of (3.46) vanishes on setting
\[ v = v' = v'' = q^{\pm 1} - 1, \quad w = w' = w'' = -(q^{\pm 1} + 1) \quad (3.48) \]
However, not all these relations are necessary for the vanishing of the RHS of (3.46). More than this, the successful Baxterisation would mean that this vanishing is achieved with pairs \((v, w), (v', w'), (v'', w'')\) which are not identical (though satisfying some constraints in general).

To accomplish this we first obtain a set of constraints relating the members of (3.47). For this we shall use the fact that for any function \(f(x)\), \(\hat{R}_\epsilon\) denoting for \(\epsilon = \pm 1\) the matrices (3.40) and (3.41) respectively, one has the well-known relations [16] (cf. also [11]):
\[
\begin{align*}
  f(\hat{R}_{(12)}^\epsilon)\hat{R}_{(23)}^\epsilon\hat{R}_{(12)}^\epsilon &= \hat{R}_{(23)}^\epsilon\hat{R}_{(12)}^\epsilon f(\hat{R}_{(23)}^\epsilon) \\
  \hat{R}_{(12)}^\epsilon\hat{R}_{(23)}^\epsilon f(\hat{R}_{(12)}^\epsilon) &= f(\hat{R}_{(23)}^\epsilon)\hat{R}_{(12)}^\epsilon\hat{R}_{(23)}^\epsilon 
\end{align*}
\quad (3.49)
\]
For \(f(\hat{R}_{(12)}^\epsilon)\) and \(f(\hat{R}_{(23)}^\epsilon)\) one can choose \((X_1, Y_1)\) and \((X_2, Y_2)\) respectively and apply them in (3.49) successively.

Exploiting systematically all the constraints implied by (3.49) one obtains for the members of (3.47):
\[
\begin{align*}
  S_5 &= \frac{1}{4} S_1, \quad S_6 = \frac{1}{4} S_2 \quad (3.50) \\
  K_1 &= -\frac{1}{4}(S_2 + 2J_2), \quad K_3 = -\frac{1}{4}(S_2 + 2J_1), \quad K_2 = \frac{1}{4}(S_2 + 2(J_1 + J_2)) \\
  L_1 &= -\frac{1}{4}(S_1 - 2J_2), \quad L_3 = -\frac{1}{4}(S_1 - 2J_1), \quad L_2 = \frac{1}{4}(S_1 - 2(J_1 + J_2))
\end{align*}
\]
Hence the right hand side of (3.46) becomes
\[
a_1 S_1 + a_2 S_2 + b_1 J_1 + b_2 J_2 \quad (3.51)
\]
where
\[
\begin{align*}
  4a_1 &= 4(v + v'' + vv'' - v') + vv''v' - w'wv'' + v'wuw'' - w''wv' \\
  4a_2 &= 4(w + w'' + ww'' - w') + ww''w' - v'ww'' + w'vv'' - v''w' \\
  2b_1 &= (w'v - v'w)(v'' + w'' + 2) \\
  2b_2 &= (w'v'' - v''w)(v + w + 2) \quad (3.52)
\end{align*}
\]
The equations
\[
b_1 = 0, \quad b_2 = 0
\]

each has two factors that can be zero. Setting the first factors equal to zero do not give satisfactory results when $a_1$ and $a_2$ are equated to zero. So we set the second factors of both $b_1$ and $b_2$ to zero:

\[ v + w + 2 = 0, \quad v'' + w'' + 2 = 0, \quad \text{(3.53)} \]

which is consistent with (3.48). Taking into account (3.53) leads to:

\[ 2a_1 = 2a_2 = (v' + w' + 2)(v + v'' + vv'') \quad \text{(3.54)} \]

Consistently with (3.48) the conditions $a_1 = a_2 = 0$ lead to:

\[ v' + w' + 2 = 0 \quad \text{(3.55)} \]

Thus, summarising, we have:

\[ v + w = v' + w' = v'' + w'' = -2 \quad \text{(3.56)} \]

But apart from this the pairs $(v, w), (v', w'), (v'', w'')$ are mutually independent of each other.

Thus denoting

\[ \hat{R}(q) = I + (q - 1)P_+ - (q + 1)P_- \quad \text{(3.57)} \]

satisfying

\[ v + w = (q - 1) - (q + 1) = -2 \]

one obtains

\[ \hat{R}_{(12)}(q)\hat{R}_{(23)}(q')\hat{R}_{(12)}(q'') = \hat{R}_{(23)}(q'')\hat{R}_{(12)}(q')\hat{R}_{(23)}(q) \quad \text{(3.58)} \]

without further restrictions on the triplet $(q, q', q'')$. Explicitly, $\hat{R}(q)$ is given by formula (3.55).

One may choose

\[ q = f(x), \quad q'' = f(y), \quad q' = f(xy) \quad \text{(3.59)} \]

to obtain a conventional Baxterisation. But more freedom is implied in (3.58).

Let us consider some special cases. First we note that for

\[ w = w'' = w' = 0, \quad a_2 = b_1 = b_2 = 0 \]

and $a_1 = 0$ leads to

\[ v' = \frac{v + v'' + vv''}{1 - \frac{1}{4}vv''} \quad \text{(3.60)} \]

Similarly for $v = v'' = v' = 0$

\[ w' = \frac{w + w'' + ww''}{1 - \frac{1}{4}ww''} \quad \text{(3.61)} \]
Finally we note an amusing point. The general solution for the diagonaliser of $\hat{R}$ contains arbitrary parameters. They can be so chosen that the $\hat{R}$ of $S_{03}$ can be implemented to diagonalise the $\hat{R}$ of $S_{14}$. Thus, setting (cf. (2.3):\)

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$  

(3.62)

which fulfils:

$$M^t M = M M^t = I$$

one obtains:

$$M \hat{R} \hat{R}^t M = \text{diag}(q, 1, 1, -q)$$  

(3.63)

where $\hat{R}$ is from (3.35). (The mutually orthogonal rows of $M$ may be permuted to reorder the diagonal elements of (3.63).)

The situation is not reciprocal. Fixing suitably arbitrary parameters one may construct a simple unitary matrix for diagonalisation which we present for comparison. One can use:

$$M' = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1 \end{pmatrix}$$  

(3.64)

which fulfils:

$$M' M'^t = M'^t M' = I$$

to obtain:

$$M' \hat{R}' M'^t = \text{diag}(1 - i, 1 - i, 1 + i, 1 + i)$$  

(3.65)

where $\hat{R}'$ is from (2.3).

We mention only briefly here the utilities of explicit construction of diagonalisers [12], which here are $M$ and $M'$. They yield easily and directly the eigenvectors of $\hat{R}$, the latter being significant in related statistical mechanical models. They also furnish new insights concerning related noncommutative spaces.

### 3.3 Noncommutative planes

Using the notations and conventions used for $S_{03}$, but using $x$ and $\xi$ (not passing via $X$ and $Z$ as before), we use (2.27) with $(X, Z) \rightarrow (x, \xi)$. As in (2.28) we choose

$$\mathcal{P} - I = P_- \Rightarrow P_-(x \otimes x = 0$$

Following (2.28) we should have:

$$(Q + I)P_- = 0$$  

(3.66)
from which follows that:

\[ Q = -I + 2k_+ P_{(+)} + k_0 P_{(0)} \]  

(3.67)

where \( k_+, k_0 \) are free parameters. Thus, finally

\[ P_{(-)} x \otimes x = 0 \]  

(3.68a)

\[ (2k_+ P_{(+)} + k_0 P_{(0)}) \xi \otimes \xi = 0 \]  

(3.68b)

\[ x \otimes \xi = (-I + 2k_+ P_{(+)} + k_0 P_{(0)}) \xi \otimes x \]  

(3.68c)

Note that from (3.68b) follows:

\[ P_{(+)} \xi \otimes \xi = 0, \quad P_{(0)} \xi \otimes \xi = 0 \]

as \( P_{(+)} \) and \( P_{(0)} \) are orthogonal.

The relations resulting from (3.68) are:

\[ x_1^2 - x_2^2 = 0 \]  

(3.69)

\[ \xi_1^2 + \xi_2^2 = 0, \quad \xi_1 \xi_2 = \xi_2 \xi_1 = 0 \]  

(3.70)

\[ x_1 \xi_1 = (k_+ - 1) \xi_1 x_1 + k_+ \xi_2 x_2 \]

\[ x_2 \xi_2 = (k_+ - 1) \xi_2 x_2 + k_+ \xi_1 x_1 \]

\[ x_1 \xi_2 = (k_0 - 1) \xi_1 x_2 \]

\[ x_2 \xi_1 = (k_0 - 1) \xi_2 x_1 \]  

(3.71)

Note that for \( k_+ = 1 \) and/or \( k_0 = 1 \) there are significant (even drastic for \( k_0 = 1 \)) simplifications.

4 Conclusions

In our first paper on exotic bialgebras [9] we used triangular \( R \)–matrices which had multiple roots for \( \hat{R} \) except for one case when it was no longer ”exotic” but the non-standard Jordanian one. Various features of the Jordanian case have been studied (in a generalised biparametric form) elsewhere [17]. For the remaining cases mentioned above the multiple roots of \( \hat{R} \) prevent straightforward spectral decomposition.

In the \( S_{03}, S_{14} \) cases studied here though there are no multiple roots, one encounters in each case remarkable special features.

For \( S_{03} \) the projectors are complex. With the Ansatz for Baxterisation formulated directly in terms of \( \hat{R} \), (cf. (2.23)), a remarkable analogy with the \( GL_q \) case emerged. Following the crucial equations (2.13) and (2.14), we noted that a coefficient \((q - q^{-1})\) for the latter case is replaced by 2 in ours. Starting from an alternative Ansatz using
a (complex) projector it was shown again (cf. (2.20), (2.21), (2.22)) how a particular case of a functional equation arising quite generally [12] led to the solution.

For $S14$ one needs three projectors for resolution of $\hat{I}$ and the spectral decomposition of $\hat{R}$. This introduces features analogous to the more complicated $SO_q$ (and $Sp_q$) cases. One could have arrived at our final results by working directly with the explicit, numerical $8 \times 8$ matrices for the tensored projectors. Indeed, we have computed these matrices. But the algebraic approach preferred here gives a deeper understanding of the structure, emphasising both, the analogies and the differences with the case of $SO_q(N)$.

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