Non-Linear Reinforcement Learning in Large Action Spaces: Structural Conditions and Sample-efficiency of Posterior Sampling

Alekh Agarwal
alekhagarwal@google.com
Google Research

Tong Zhang
tozhang@google.com
Google Research & HKUST

Abstract
Provably sample-efficient Reinforcement Learning (RL) with rich observations and function approximation has witnessed tremendous recent progress, particularly when the underlying function approximators are linear. In this linear regime, computationally and statistically efficient methods exist where the potentially infinite state and action spaces can be captured through a known feature embedding, with the sample complexity scaling with the (intrinsic) dimension of these features. When the action space is finite, significantly more sophisticated results allow non-linear function approximation under appropriate structural constraints on the underlying RL problem, permitting for instance, the learning of good features instead of assuming access to them. In this work, we present the first result for non-linear function approximation which holds for general action spaces under a linear embeddability condition, which generalizes all linear and finite action settings. We design a novel optimistic posterior sampling strategy, TS$^3$ for such problems, and show worst case sample complexity guarantees that scale with a rank parameter of the RL problem, the linear embedding dimension introduced in this work and standard measures of the function class complexity.

1 Introduction
Designing sample-efficient techniques for Reinforcement Learning (RL) settings with large state and action spaces is a key question at the forefront of RL research. A typical approach relies on the use of function approximation to generalize across the state and/or action spaces. When a sufficiently expressive feature embedding is available to the learner, and linear functions of these features are used for learning, a number of recent results provide computationally and statistically efficient techniques to handle continuous state, as well as action spaces with dependence only on an intrinsic dimensionality of the features [Yang and Wang, 2020, Jin et al., 2020, Agarwal et al., 2020a]. However, practitioners typically rely on neural networks to parameterize the learning agent, a case not covered by the linear results. A parallel line of work [Jiang et al., 2017, Sun et al., 2019, Du et al., 2021] studies more general settings that allow the use of non-linear function approximation, and provides sample complexity guarantees in terms of a structural parameter called the Bellman rank of the RL problem. The power of these conditions is elucidated in recent representation learning results [Agarwal et al., 2020b, Uehara et al., 2021, Modi et al., 2021], that leverage low Bellman rank to learn a good feature map that captures a near-optimal policy and/or the transition dynamics. However, these methods crucially rely on having a finite number of actions, and the guarantees scale with the cardinality of the action set. With this context, our paper asks the following question:

Can we design sample-efficient and non-linear RL approaches for large state and action spaces?
In this paper, we study this question in the framework of a Markov Decision Process (MDP), and focus on problems satisfying a generalization of the Bellman rank parameter introduced by Jiang et al. [2017]. While the definition of Bellman rank applies to both discrete and continuous action spaces, the OLIVE algorithm of Jiang et al. [2017] applies only to discrete action spaces. The presence of a small action set facilitates uniform exploration for one time step, which lets the agent collect valuable exploration data in the vicinity of states it has already explored, allowing the discovery of successively better states. When good features are available, like in a linear MDP, a basis in the feature space serves an analogous role and indeed recent works [Foster et al., 2021] show that experimental design in the right feature space can replace uniform exploration over discrete actions.

However, this strategy fails beyond (generalized) linear settings, where there is no easy mechanism for obtaining a good exploration basis over the action set apriori. Indeed, the results of Hao et al. [2021] imply that avoiding some dependence on the size of the action space in general is unavoidable (see Appendix C for further discussion). This situation motivates the investigation of additional structures between the hopeless worst-case result and the limiting small action settings. To this end, our approach is motivated by the recent work of Zhang [2021] on the Feel-Good Thompson Sampling (FGTS) algorithm, which they analyze for bandits and a class of RL problems under a linear embeddability assumption using a modified Thompson Sampling approach.

Our Contributions. With this context, our work makes the following contributions.

1. We introduce a new structural model for RL where a generalized form of Bellman rank is small, and further a linear embeddability assumption on the Bellman error is satisfied. We show that this setting subsumes prior works on both linear MDPs as well as finite action problems with a small Bellman rank. Crucially, in both linear MDPs and finite action problems, the embedding features in our definition are known apriori, while we also handle problems with an unknown linear embedding, which constitutes a significant generalization of the prior works. As an example, this allows us to generalize the combinatorial bandits setting to long horizon RL (Table 1).

2. We introduce a new algorithm, Two timeScale Two Sample Thompson Sampling (TS$^3$), which is motivated from FGTS [Zhang, 2021]. The algorithm design incorporates a careful two timescale strategy to
solve an online minimax problem for estimating Bellman residuals, and a decoupling between roll-in and roll-out policies crucial for the online minimization of these residuals. Both ideas appear novel relative to prior posterior sampling approaches in the literature. The use of nested posterior sampling to solve online minimax problems might be of independent interest.

3. We show that TS$^3$ solves all problems where the generalized Bellman rank and linear embeddability conditions hold, under additional completeness and realizability assumptions. The guarantees scale polynomially in $1/\epsilon$, horizon $H$, Bellman rank and linear embedding parameters, as well as the function class complexity. The result generalizes both guarantees for linear MDPs in terms of feature dimension with no dependence on action cardinality, as well as Bellman rank-based and representation learning guarantees to linearly embeddable infinite action spaces. Overall, our method provides the first approach to representation learning with continuous action spaces. We summarize the settings covered in this paper in Table 1.

4. The sample complexity of TS$^3$ scales as $1/\epsilon^4$, which is slower than the optimal $1/\epsilon^2$ scaling. We develop an experimental design based adaptation of TS$^3$ which does obtain the desired $1/\epsilon^2$ bound. However, in contrast with the general result of TS$^3$, the TS$^3$-D algorithm needs knowledge of the features in our linear embedding assumption, in order to carry out experimental design.

We remark that the convergence rate guarantees obtained here likely have room for improvement as they do not match the optimal rates for tabular and linear MDPs. We leave this as an important direction for future work, and comment on the potential sources of looseness in our analysis in the later sections of the paper. We now formalize the setting before discussing our structural assumptions and our approach. Detailed discussion of the related works can be found in Appendix [A].

2 Setting

We study RL in an episodic, finite horizon Contextual Markov Decision Process (MDP) that is parameterized as $(X, A, R, D, P)$, where $X$ is a state space, $A$ is an action space, $R$ is a distribution over rewards, $D$ and $P$ are distributions over the initial context and subsequent transitions respectively. An agent observes a context $x^1 \sim D$ for some fixed distribution $D$.

At each time step $h \in \{1, \ldots, H\}$, the agent observes the state $x^h$, chooses an action $a^h$, observes $r^h \sim R(\cdot \mid x^h, a^h)$ and transitions to $x^{h+1} \sim P(\cdot \mid x^h, a^h)$. We assume that $x^h$ for any $h > 1$ always includes the context $x^1$ to allow arbitrary dependence of the dynamics and rewards on $x^1$. Following prior work [e.g. Jiang et al., 2017], we assume that $r^h \in [0, 1]$ and $\sum_{h=1}^H r^h \in [0, 1]$ to capture sparse-reward settings [Jiang and Agarwal, 2018]. We make no assumption on the cardinality of the state and/or action spaces, allowing both to be potentially infinite. We use $\pi$ to denote the agent’s decision policy, which maps from $X \to \Delta(A)$, where $\Delta(\cdot)$ represents the space of all probability distributions over a set. The goal of learning is to discover an optimal policy $\pi^*_\star$, which is always deterministic and conveniently defined in terms of the $Q_\star$ function [see e.g. Puterman, 2014; Bertsekas and Tsitsiklis, 1996]

$$
\pi^*_\star(x^h) = \arg\max_{a \in A} Q^*_\star(x^h, a), \quad Q^*_\star(x^h, a^h) = \mathbb{E}[r^h + \max_{a' \in A} Q^*_{\star+1}(x^{h+1}, a') \mid x^h, a^h],
$$

where we define $Q^*_{\star+1}(x, a) = 0$ for all $x, a$.

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1We intentionally call $x^1$ a context and not an initial state of the MDP as we will soon make certain structural assumptions which depend on the context, but take expectation over the states.
In this work, we focus on value-based approaches, where the learner has access to a function class \( F \subseteq \{ X \times A \rightarrow [0,1] \} \), and uses this function class to approximate the optimal value function \( Q^*_h \). Let us denote \( [H] := \{1,2,\ldots,H\} \). We make two common expressivity assumptions on \( F \).

**Assumption 1 (Realizability)** For all \( h \in [H] \), we have \( Q^h \in F \).

There are well-known lower bounds on the sample complexity of RL [Sekhari et al., 2021], even under the structural assumptions we will make when realizability is not approximately satisfied. Let us use \( \pi_f(x) = \arg\max_{a \in A} f(x,a) \) and \( f(x) = \max_a f(x,a) \) for any \( f \in F \).

**Assumption 2 (Completeness)** For any function \( f \in F \) and \( h \in [H] \), \( \exists g \in F \) such that \( g(x^h, a^h) = T^h f(x^h, a^h) := \mathbb{E}[r^h + f(x^{h+1}, \pi_f(x^{h+1})) | x^h, a^h] \).

The completeness assumption is not essential information-theoretically and OLIVE [Jiang et al., 2017] does not require it. However, we make this assumption to facilitate scaling to infinite action sets, which seems challenging without completeness, as we discuss later. We now present the main structural conditions we impose on the environment, and motivate them with some examples.

### 3 Structural Conditions

In this section, we present the two main structural conditions, namely generalized Bellman rank and linear embeddability. We define the Bellman residual of \( f \in F \) using another \( g \in F \) as:

\[
\mathcal{E}^h(g, f, x^h, a^h) = g(x^h, a^h) - T^h f(x^h, a^h). \tag{2}
\]

We now state a generalized Bellman error decomposition for contextual setups.

**Assumption 3 (Generalized Bellman decomposition)** We assume that for all \( f, f' \in F \), and \( h \in [H] \), there exist (unknown) functions \( u^h, \psi^{h-1} \) and an inner product \( \langle \cdot, \cdot \rangle \) such that:

\[
\mathbb{E}_{x^h \sim \pi_f} \mathcal{E}^h(f, f'; x^h, \pi_f(x^h)) = \langle \psi^{h-1}(f', x^1), u^h(f, x^1) \rangle.
\]

We assume that \( \sup_{f \in F, x^1 \in X} \|u^h(f, x^1)\|_2 \leq B_1 \).

Notice that both factors depend on \( x^1 \), and a similar definition is recently considered in Foster et al. [2021] to cover contextual RL setups [Hallak et al., 2015; Abbasi-Yadkori and Neu, 2014; Modi et al., 2018]. The technical treatment of contextual dependency requires a conversion of RL problems into online learning, as in Foster et al. [2021] and in this paper. Techniques used by other earlier works on decomposition of average Bellman error cannot handle such a dependency. This definition allows mixtures of MDPs with a small Bellman rank for each context \( x^1 \), without any explicit scaling with the number of contexts in our results. We now make an effective dimension assumption on \( \phi^h, u^h \) instead of requiring them to be finite dimensional like Jiang et al. [2017].

**Definition 1 (Effective Bellman rank)** Given any probability measure \( p \) on \( F \). Let

\[
\Sigma^h(p, x^1) = \mathbb{E}_{f' \sim p} \psi^h(f', x^1) \otimes \psi^h(f', x^1), \quad K^h(\lambda) = \sup_p \text{trace} \left( (\Sigma^h(p, x^1) + \lambda I)^{-1} \Sigma^h(p, x^1) \right).
\]
For any \( \epsilon > 0 \), define the effective Bellman rank of the MDP as

\[
\br(\epsilon) = \sum_{h=1}^{H} \br^{h-1}(\epsilon), \quad \br^{h-1}(\epsilon) = \inf_{\lambda > 0} \left\{ K^{h-1}(\lambda) : \lambda K^{h-1}(\lambda) \leq \epsilon^2 \right\}.
\]

When \( \dim(\psi^{h-1}) = \dim(\psi^{h}) = d \), then \( K^{h}(\lambda) \leq d \) for all \( \lambda > 0 \) so that \( \br(0) \leq dH \). This might appear a factor of \( H \) larger than the Bellman rank, but note that we aggregate over the horizon, while taking a supremum like \cite{Jiang2017} simply incurs that factor in the sample complexity analysis instead. More generally, the following result gives the behavior of \( \br(\epsilon) \) under typical spectral decay assumptions on \( \Sigma^{h} \).

**Proposition 2 (Rank bounds under spectral decay)** Suppose that \( \|\psi(f, x^1)\| \leq 1 \) for all \( f \in \mathcal{F}, x^1 \in \mathcal{X} \). Let \( \lambda_{i}(A) \) denote the \( i \)th largest eigenvalue of a psd matrix \( A \). Suppose we have:

- Geometric decay: \( \sup_{h,p,x^1} \lambda_{i}(\Sigma^{h}(p, x^1)) \leq \alpha^{i} \), for \( \alpha \in (0, 1) \). Then \( \br(\epsilon) \leq 2H \cdot \log(2/\alpha(1-\alpha)) / \log(1/\alpha) \).

- Polynomial decay: \( \sup_{h,p,x^1} \text{trace}(\Sigma^{h}(p, x^1) \Sigma^{h}(p, x^1)^{\dagger}) \leq R_{q}, \text{for } q \in (0, 1) \). Then \( \br(\epsilon) \leq H \cdot \left( \frac{R_{q}}{\epsilon} \right)^{\frac{1}{1-q}} \).

We provide a proof of Proposition\footnote{2} in Appendix \footnote{3} \cite{Jin2021} study a related log-determinant based effective dimension for Bellman rank, but this definition is more natural in our analysis.

While Assumption\footnote{3} and Definition\footnote{1} facilitate scaling to infinite dimensional factorizations with a low intrinsic dimension, this is still not sufficient to succeed in problems with large action spaces due to the existing lower bounds on learning under sparse transitions (see \cite{Hao2021} and Appendix\footnote{4}). We now introduce the second structural condition, inspired by the recent work of \cite{Zhang2021} on contextual bandits and RL with deterministic dynamics, to handle this issue.

**Assumption 4 (Linearly embeddable Bellman error)** For all \( f \in \mathcal{F}, h \in [H], x^{h} \in \mathcal{X} \) and \( a^{h} \in \mathcal{A} \), there exist (unknown) functions \( w^{h}, \phi^{h} \) and an inner product \( \langle \cdot, \cdot \rangle \) such that

\[
\mathcal{E}^{h}(f, f; x^{h}, a^{h}) = \langle w^{h}(f, x^{h}), \phi^{h}(x^{h}, a^{h}) \rangle.
\]

We assume that \( \sup_{f \in \mathcal{F}, x^{h} \in \mathcal{X}} w^{h}(f, x^{h}) \leq B_{2} \).

Note that the function \( w^{h} \) is allowed to depend on \( x^{h} \), which makes this assumption significantly weaker than embedding Bellman errors in a fixed feature space. For instance, suppose \( |\mathcal{A}| = K \). Then we can always define \( \phi(x^{h}, a^{h}) = e_{a^{h}} \in \mathbb{R}^{K} \) to be the indicator of the action \( a^{h} \) and \( w^{h}(f, x^{h}) = (f(x^{h}, a_{1}), \ldots, f(x^{h}, a_{K})) \) to satisfy Assumption\footnote{4}. This shows that all finite action problems satisfy this assumption, and so do linear MDPs where \( w^{h}(f, x^{h}) \) only depends on \( f \) and \( \phi^{h}(x^{h}, a^{h}) \) are the features defining the transition dynamics \cite{Jin2020}. Similar to Assumption\footnote{3} we allow the embeddings \( w^{h}, \phi^{h} \) to have a small effective dimension.

**Definition 3 (Effective Embedding Dimension)** Given any probability measure \( p \) on \( \mathcal{F} \), define

\[
\Sigma^{h}(p, x) = \mathbb{E}_{f \sim p} \mathbb{E}_{a \sim \pi_{f}(x)} \phi^{h}(x, a) \otimes \phi^{h}(x, a), \quad \bar{K}^{h}(\lambda) = \sup_{p,x} \text{trace} \left( (\Sigma^{h}(p, x) + \lambda I)^{-1} \Sigma^{h}(p, x) \right).
\]
For any $\epsilon > 0$, define the effective embedding dimension of $\phi$ as

$$dc(\epsilon) = \sum_{h=1}^{H} dc^h(\epsilon), \quad \text{where} \quad dc^h(\epsilon) = \inf_{\lambda > 0} \left\{ \tilde{K}^h(\lambda) : \lambda \tilde{K}^h(\lambda) \leq \epsilon^2 \right\}. $$

The definition of $\tilde{K}^h(\lambda)$ measures the (approximate) condition number of the covariance for the worst distribution $p$. If $\dim(\phi^h) = d$, then $\tilde{K}^h(\lambda) \leq d$. In this case, we get $dc(0) \leq dH$. For infinite dimensional $\phi^h$, we obtain bounds similar to Proposition [2].

Comparison with Bellman-Eluder dimension [Jin et al., 2021]. Jin et al. [2021] present an alternative $Q$-type Bellman rank and its distributional generalization based on the Eluder dimension [Russo and Van Roy, 2013]. However, both Eluder dimension based results [Wang et al., 2021, Feng et al., 2021] and the $Q$-type Bellman-Eluder dimension do not capture all finite action, non-linear contextual bandit problems with a realizable reward (see Appendix [B]). In fact, the $Q$-type Bellman rank has a more linear-like structure, and is easier to deal with, because one only requires a single posterior sampling [Dann et al., 2021]. The $V$-type Bellman rank, which naturally captures non-linear setups like representation learning, requires a more sophisticated handling via double sampling from the posterior. Our setup subsumes the $V$-type Bellman rank of Jiang et al. [2017], but does not include all problems with a small $Q$-type Bellman rank. The most prominent example of linear MDP with infinite actions covered by the GOLF algorithm of Jin et al. [2021] is also captured by our assumptions. However, the setting of low inherent Bellman error using linear function approximation [Zanette et al., 2020b] is not covered by our assumptions.

So far, we have explained how both the settings of a bounded Bellman rank with a finite action set, as well as linear MDPs with infinite actions satisfy both of our assumptions with good bounds on $br(\epsilon)$ and $dc(\epsilon)$. Next we discuss some examples beyond these where our assumptions hold and which go beyond either of these two known special cases. For the examples, we assume that the underlying MDP has low-rank transition dynamics [Jiang et al., 2017, Jin et al., 2020] so that $Tf(x,a) = \langle w_f, \psi(x,a) \rangle$ for all functions $f : \mathcal{X} \times \mathcal{A} \to [0,1]$ with $\psi(x,a)$ being the state-action features which factorize the dynamics. For such problems, Assumption [B] always holds [Jiang et al., 2017] and Assumption [C] is equivalent to checking that $f(x,a)$ is linearly embeddable.

Example 1 (Linear embedding of combinatorial actions) Many recommendation settings consist of combinatorial action spaces such as lists, rankings and page layouts. While these problems are intractable in the worst-case, a line of work originating in combinatorial bandits [Cesa-Bianchi and Lugosi, 2012] assumes linear decomposition of rewards for tractability. For concreteness, let us consider a ranking scenario where the system observes some state features $x$ depending on the current user state and any other side information, and wants to choose a ranking $a$ of $L$ items $\alpha_1, \ldots, \alpha_L$ from a base set $\Omega$, with $|\Omega| = K$. We observe that $|A| = \binom{K}{L} \cdot L!$ in this example, which grows as $\mathcal{O}(K^L)$. Hence the sample complexity of direct exploration over rankings is $\mathcal{O}(K^L)$.

Mirroring the setup from combinatorial bandits and its contextual generalization [Swaminathan et al., 2017, Ie et al., 2019] propose the SlateQ model where the $Q_*$ function takes the form:

$$Q_*(x,a) = \sum_{\alpha \in \Omega} P(\alpha|x,a)g(x,\alpha). (3)$$

Here $P(\alpha|x,a)$ is the probability of a user (with features $x$) choosing the item $\alpha$ when the ranking $a$ is presented, and $g(x,\alpha)$ is an unknown value of recommending that item in state $x$ to the user. This assumption is motivated from typical approaches in the user-modeling literature in recommendation systems, and
effectively posits that the Q-value of a ranking a in a state depends on the unknown values of the base items in that ranking, weighted by a user interaction model P.

The user model, which encodes the likelihood of the user choosing a particular item in a ranking is often estimated separately from the RL task using a click probability model, or a cascade model [see e.g. Ie et al., 2019, for a discussion]. In such scenarios, we can define a class \( \mathcal{F} \) by parameterizing the function \( g \), and obtain linear embedding of Bellman residuals whenever the MDP also has low-rank dynamics in some features \( \psi \), using \( \phi(x, a) = (P(\cdot | x, a), \psi(x, a)) \) as the concatenation of the user model with dynamics features. Here \( \psi(x, a) \) intuitively encodes the information needed to describe how the state \( x \) of a user evolves across interactions, which could be low-dimensional, for instance, if there is a set of representative user types.

Notably, these choices lead to a linear embedding dimension which grows as \( K \). In many applications, the position of an item in the ranking is a dominant effect, in which case we can consider \( \alpha \) to consist of an item-position tuple, leading to an KL-dimensional factorization. In either case, this yields an exponential saving in the sample complexity compared with direct exploration in the ranking space, mirroring prior results [Cesa-Bianchi and Lugosi, 2012, Swaminathan et al., 2017, Ie et al., 2019], which either do not consider exploration or work in simpler bandit settings. We are not aware of other existing approaches that can handle rich observation RL and combinatorial action spaces simultaneously.

In Appendix D we give two further examples of using a basis expansion in the action space, as well as an infinite action version of representation learning for Block MDPs [Du et al., 2019], where Assumption 4 holds in a natural manner. We now proceed to describe our algorithm.

### 4 The Two timeScale Two Sample Thompson Sampling algorithm

Having presented our structural conditions, we now present our main algorithm in this section, which is based on Thompson Sampling [Thompson, 1933] and its FGTS adaptation in Zhang [2021]. To define the algorithm, we need some additional notation. At any round \( t \) of the algorithm, using the observed tuple \( (x^h_t, a^h_t, r^h_t, x^{h+1}_t) \), we define

\[
\hat{\Delta}^h_t(g, f) = g(x^h_t, a^h_t) - r^h_t - f(x^{h+1}_t),
\]

as a TD approximation of \( \mathcal{E}(g, f, x^h_t, a^h_t) \). The algorithm is presented in Algorithm 1.

At a high-level, the algorithm performs standard Thompson Sampling updates to the posterior given the observations, with three modifications. First is that we incorporate an optimistic bias in the distribution \( p_t \) over \( f \) (Line 6), similar to the FGTS approach. The second difference arises from the challenges in estimating the Bellman error, while the third is in how we sample from the posterior to obtain the agent’s policy at each time. We now explain the latter two issues in detail.

**Inner loop to estimate Bellman error.** Note that we would ideally define the likelihood function as \( \mathcal{E}(f, f, x, \pi_f(x))^2 \) for any \( x \), but this requires a conditional expectation over the sampling of the next state \( x' \sim P(\cdot | x, \pi_f(x)) \). Replacing the expectation with a single random sample suffers from bias due to the double sampling issue [Antos et al., 2008], which arises from the non-linearity of squared loss inside the expectation. An ideal solution to this, following [Antos et al., 2008] is to define the cumulative likelihood as:

\[
\tilde{L}_t(f) = \sum_{s=1}^t \hat{\Delta}^h_s(f, f)^2 - \min_{g \in \mathcal{F}} \sum_{s=1}^t \hat{\Delta}^h_s(g, f)^2.
\]
would draw the design of FGTS [Zhang, 2021] as well as OLIVE [Jiang et al., 2017]. At this point, a typical TS approach between the likelihood function we use to evaluate \( f \) as well, giving higher weight to functions \( p \) issue is most easily seen if we imagine the distribution \( p \) a Thompson sampling distribution \( h \) first demonstrations of the effectiveness of such a algorithm is to address in OLIVE by a one-step uniform exploration over a finite action space, which is also adopted by subsequent works. In this paper, we replace this one-step uniform exploration by a second, independent sample from the posterior, which allows us to perform one-step exploration in infinite action spaces without the need for our algorithm to know the underlying linear embedding. One main insight of this work is to demonstrate the effectiveness of such a two sample strategy (Line 8). Specifically, we execute \( \pi_t \) for the first \( h_t \) time steps, and then complete the roll-out using \( \pi_t' \) with \( h_t \in \{H\} \) chosen uniformly, and we use only the samples from step \( h_t \) in our likelihood at time \( t \). The decoupling of the two samples is crucial to our analysis, although it will be interesting to investigate whether a single sample strategy can be analyzed
5 Main Results

In this section, we present our main sample complexity guarantee for TS\(^3\). To do so, we need to introduce some measures of the complexity of our value function class \(\mathcal{F}\), which we do next.

**Definition 4** For any \(f \in \mathcal{F}\), we define the set \(\mathcal{F}(\epsilon, f) = \{g \in \mathcal{F} : \sup_{x,a,h} |E^h(g, f; x, a)| \leq \epsilon\}\) of functions that have a small Bellman error with \(f\) for all \(x, a\). Further assume that \(\mathcal{F}\) has an \(L_\infty\) cover \(f_1, \ldots, f_N \in \mathcal{F}\) for \(N = N(\epsilon)\), so that \(\forall f \in \mathcal{F}, \min_i \sup_{x,a} |f(x, a) - f_i(x, a)| \leq \epsilon\). Then we define \(\kappa(\epsilon) = \sup_{f \in \mathcal{F}} -\ln p_0(\mathcal{F}(\epsilon, f))\) and \(\kappa'(\epsilon) = \ln N(\epsilon)\).

If \(\mathcal{F}\) is finite with \(|\mathcal{F}| = N\), then we can choose \(p_0\) to be uniform over \(\mathcal{F}\) to get \(\kappa(\epsilon) = \kappa'(\epsilon) = \ln N\).

We state our guarantee in terms of the regret of the learned policies with respect to the optimal policy \(\pi^*_t\). We do this in order to simplify our result. We start with the well-studied setting of a finite function class, dimensions and function class complexity. However, the average regret decays at a rate of \(T^{-1/4}\). Consequently, the sample complexity to discover a policy with a regret at most \(\epsilon\) is \(O(H^4(d_1 \ln(NT))^2 |\mathcal{A}| \epsilon^{-4})\), using a different approach.

The following result gives our main sample complexity bound for TS\(^3\).

**Theorem 5** Under Assumptions\(\square\) suppose we run TS\(^3\) (Alg.\(\square\) with parameters \(\gamma = 0.1\) and \(\lambda \leq \eta \leq c/(\kappa(\epsilon) + \kappa'(\epsilon) + \ln T)\) for a universal constant \(c\). Then choosing any \(\epsilon = O(1/T)\), we have

\[
\mathbb{E} \sum_{t=1}^T \text{Regret}(f_t, x_t^1) = O\left(\frac{\kappa(\epsilon) + \kappa'(\epsilon)}{\lambda} + \lambda T + \bar{\epsilon}(\lambda/\eta)T\right),
\]

where \(\bar{\epsilon}(\lambda/\eta) = \inf_{\mu > 0} \left[8(\lambda/\eta)dc(\epsilon_2)H\mu^2 + 2\mu H\epsilon B_2 + \mu^{-1}br(\epsilon_1) + \epsilon_1 HB_1\right]\).

Note that the bound above is not a bound on the online regret of TS\(^3\), since the policy we execute at round \(t\) is not \(\pi_t\), but a mixture of \(\pi_t\) and \(\pi_t'\) (line \(\square\) of Algorithm \(\square\)). However, the bound implies a PAC guarantee as we can choose a policy \(\pi_1, \ldots, \pi_T\) uniformly, and use a standard online-to-batch conversion [Cesa-Bianchi et al.\(\square\)] to obtain a regret bound for the returned policy.

To interpret the general guarantee of Theorem 5, we now consider some special cases where the parameters can be set optimally to simplify our result. We start with the well-studied setting of a finite function class, with Bellman rank and linear embedding dimensions being finite as well.

**Corollary 6 (Finite dimensional embeddings and finite \(\mathcal{F}\))** Under conditions of Theorem 5, assume further that \(|\mathcal{F}| = N < \infty\) and \(\eta = O(1/\ln(NT))\) and \(\lambda = T^{-3/4}H(d_2^* d_2)^{-1/4}(\ln(NT))^{1/2}\) returns a policy with an average regret at most \(O\left(H^4(d_1^{1/4} \sqrt{d_1 \ln(NT)} T^{-1/4})\right)\).

The bound has a favorable dependence on all the complexity parameters, with mild scaling in the horizon, dimensions and function class complexity. However, the average regret decays at a rate of \(T^{-1/4}\).


which is suboptimal and slower than that of OLIVE in $\epsilon$ dependence. In the setting of a linear MDP, where $d_1 = d_2 = d$, the scaling with dimension is $d^3$.

The main source of the suboptimality in $\epsilon$ is that our analysis relies on a decoupling argument that leverages Assumption 3 with an extra Cauchy-Schwarz inequality than in the typical analysis. The importance weighting over a uniform distribution in OLIVE does not lose an exponent in this step, and improving the analysis for our general scenario is an open direction for future work. Note however that all prior works that studied both finite action and infinite action (generalized) linear settings employ different algorithms for the two cases [Jin et al., 2021, Du et al., 2021, Foster et al., 2021], unlike our unified approach. When the linear embedding features are known, this suboptimality can be removed through an explicit experimental design in the feature space, as we show in the next section.

As a first example of our general result, we can instantiate the setting of Example 1 where our sample complexity only scales with $O(K)$ instead of $O(K^L)$. In the setting of Example 2, we only depend on the size of the (potentially unknown) basis.

Under the infinite-dimensional examples of Proposition 2 we note that the geometric case is almost identical to the finite dimensional setting up to log factors, since the effective rank only scales as $\log(1/\epsilon)$, so that we can always make the additional terms coming from $\epsilon$ to be lower order. The polynomial case does not yield qualitatively different results, which we show next.

**Corollary 7 (Polynomial spectral decay and finite $F$)** Under conditions of Theorem 5 assume further that $|F| = N < \infty$, $B_1 = B_2 = B$ and $\lambda_\epsilon(\epsilon) \leq H d_3 \epsilon^{-2n/(1-q)}$ for some $q \in (0, 1)$. Then $TS^3$ with $p_0$ as uniform on $F$, $\gamma = 1/36$, along with appropriate settings of $\eta$ and $\lambda$ returns a policy with an average regret at most $O\left( H (\ln(NT))^{(1-q)/2} d^{(3-q)/4} T^{-3/4} B^{(3-q)/2} \right)$.

The result matches that of Corollary 6 for $q = 0$ corresponding to the case of finite dimensions. More generally, we allow scaling to infinite dimensional states and actions, as long as the Bellman rank and linear embedding assumptions have a low intrinsic dimension.

Our results have a qualitatively similar flavor to those of Zhang [2021], but for two important differences. Zhang [2021] do not need to account for the residual variance term in the likelihood due to the deterministic dynamics assumption, and hence they do not incur the loss in rates that we suffer. They also do not work under the low Bellman rank assumption on the problem, which significantly limits the class of problems where their guarantees hold.

We finally illustrate the benefits of our contextual formulation by studying mixtures of MDPs with a small Bellman rank.

**Corollary 8 (Mixture of low-rank MDPs)** Consider a collection $M$ different MDPs $\{M_i\}_{i=1}^M$ over the same state and action space, each with a Bellman rank at most $d_1$ and linear embedding dimension $d_2$. Let $x^1 \sim D$ where $D$ is uniform over $[M]$, with the subsequent transitions happening according $M_i$. Suppose $|F| = N$. Then under the parameter settings of Corollary 6 $TS^3$ returns a policy with a regret at most $O\left( H d_2^{1/4} \sqrt{d_1 \ln(NT)} T^{-1/4} \right)$.

Comparing Corollaries 8 and 6, we observe that the mixture setting poses no extra challenge. In contrast, the mixture problem has a Bellman rank scaling with $M$, as we need to concatenate the respective embeddings for each $M_i$, causing a naive application of OLIVE to incur an extra $M^2$ term in the sample complexity. We believe the difference arises since the average Bellman error of OLIVE mixes samples across different contexts, while we use the squared Bellman error which can leverage the structure for individual
Algorithm 2 Two timeScale Thompson Sampling with Design (TS$^2$-D)

**Require:** Function class $\mathcal{F}$, prior $p_0 \in \Delta(\mathcal{F})$, learning rates $\eta, \gamma$ and optimism coefficient $\lambda$.

1: Set $S_0 = \emptyset$.
2: for $t = 1, \ldots, T$ do
3:   Observe $x_t^h \sim \mathcal{D}$ and draw $h_t \sim \{1, \ldots, H\}$ uniformly at random.
4:   Define $q_t(g) = p(g|f, S_{t-1}) \propto p_0(g) \exp(-\gamma \sum_{s=1}^{t-1} \hat{\Delta}_s^h(g, f))$. \(\triangleright\) Inner loop TS update
5:   Define $L_t^h(f) = \eta \hat{\Delta}_t^h(f, f)^2 + \frac{2}{\gamma} \ln \mathbb{E}_{g \sim q_t} \left[ \exp(-\gamma \Delta_t^h(g, f)^2) \right]$. \(\triangleright\) Likelihood function
6:   Define $p_t(f) = p(f|S_{t-1}) \propto p_0(f) \exp(\sum_{s=1}^{t-1} (\lambda f(x_s^h) - L_{s}^{h+}(f)))$ as the posterior. \(\triangleright\) Outer loop Optimistic TS update
7:   Draw $f_t \sim p_t$ from the posterior. Let $\pi_t = \pi_{f_t}$ and execute $a_t^h = \pi_t(x_t^h)$ for $h = 1, \ldots, h_t - 1$ to observe $x_t^{h_t}$.
8:   Let $\rho_t \in \Delta(\mathcal{A})$ be a G-optimal design for $\phi(x_t^h, a)_{a \in \mathcal{A}}$ (Equation [8]). Draw $a_t^{h_t} \sim \rho_t$ and observe $r_t^{h_t}$ and $x_t^{h_t+1}$. \(\triangleright\) G-optimal design
9:   Update $S_t = S_{t-1} \cup \{x_t^h, a_t^h, r_t^h, x_t^{h+1}\}$ for $h = h_t$.
10: end for
11: return $(\pi_1, \ldots, \pi_T)$.

transitions more effectively. Of course, this comes at the price of an additional completeness assumption, which would be interesting to eliminate in future work.

We now describe an improvement to our results for the setting of known features in Assumption 4.

### 6 Experimental Design for Known Linear Embedding Features

In this section, we study a special case of our setting where the features $\phi(x, a)$ in Assumption 4 are known, and finite dimensional, with $\phi(x, a) \in \mathbb{R}^{d_2}$. For this setting, we consider an adaptation of TS$^3$ which replaces the two sample strategy with experimental design in the feature space. The algorithm is presented in Algorithm 2.

Before we delve into the pseudocode, recall that the G-optimal design, given a set of vectors $\{\phi(x, a)\}_{a \in \mathcal{A}}$ is a distribution $\rho(x) \in \Delta(\mathcal{A})$ over the action space, given by [see e.g. Fedorov, 2013, Kiefer and Wolfowitz, 1960].

$$
\rho(x) = \arg\min_{\rho \in \Delta(\mathcal{A})} \mathbb{E}_{a \sim \rho} \phi(x, a) \phi(x, a)^\top.
$$

If $\Sigma_x(\rho)$ is not full rank, then we can replace $\Sigma_x(\rho)^{-1}$ by the corresponding pseudo-inverse. Furthermore, by the Kiefer-Wolfowitz theorem [Kiefer and Wolfowitz, 1960], it is known that the design $\rho(x)$ satisfies $g(x, \rho(x)) = \text{rank}\{\phi(x, a) : a \in \mathcal{A}\} \leq d_2$. The criterion $g(\rho)$ corresponds to worst prediction variance at some action $a$, of an ordinary least squares estimator given samples drawn from $\rho(x)$. For intuition in the finite action setting, where $\phi(x, a) = e_a$, $\rho(x)$ corresponds to a uniform distribution over the action set and $d_2 = |\mathcal{A}|$ is the variance of importance sampling.

With this context, TS$^2$-D is a relatively natural adaptation of TS$^3$, when $\phi(x, a)$ is a known feature map. Concretely, we no longer use the two sample scheme. Instead, we only draw one function $f_t \sim p_t$ in Line 7.
and execute the first $h_t - 1$ actions using the corresponding greedy policy. Having observed $x_t^{h_t}$, we now choose the action at time $t$ using a $G$-optimal design in the feature space for this particular state in Line 8. That is, the action sampling distribution $\rho_t$ corresponds to $\rho(x_t^{h_t})$ from Equation 6 and the design is done individually at each state. We then observe the reward and next transition for this action, which gets recorded into our dataset as before.

We have the following general guarantee for TS$^2$-D.

**Theorem 9** Under Assumptions 24, suppose further that $dcl(0) = d_2$ and the features $\phi^h(x, a)$ are known. Suppose we run TS$^2$-D (Alg. 2) with parameters $\gamma = 0.1$ and $\lambda \leq \eta \leq c/(\kappa(\epsilon) + \kappa'(\epsilon) + \ln T)$ for a universal constant $c$. Then choosing any $\epsilon = O(1/T)$, we have

$$E \sum_{t=1}^{T} \text{Regret}(f_t, x_t^1) = O\left(\frac{\kappa(\epsilon) + \kappa'(\epsilon)}{\lambda} + \lambda T + T\left(\epsilon_1 HB_1 + \frac{2\lambda br(\epsilon_1)d_2H}{\eta}\right)\right).$$

Further optimizing over the choice of $\lambda$, we see that:

$$\frac{1}{T} E \sum_{t=1}^{T} \text{Regret}(f_t, x_t^1) = O\left(\epsilon_1 HB_1 + (\kappa(\epsilon) + \kappa'(\epsilon) + \ln T)\sqrt{\frac{br(\epsilon_1)d_2H}{T}}\right).$$

Thus, in the setting of Corollary 6, we immediately get a $1/\sqrt{T}$ rate in the above bound, leading to a $O(d_1d_2H\ln^2(NT)/\epsilon^2)$ sample complexity. This bound is superior or comparable to that for OLIVE in all the problem parameters, except for an extra dependency on $\ln N$. This extra dependency is caused by the two timescale learning rates (discrepancy between $\eta$ and $\gamma$) in the online minimax game analysis, which might be possible to improve in future work. For tabular problems with $S$ states and $A$ actions, where $d_1 = S$, $d_2 = A$ and $\ln N = \tilde{O}(SAH)$, the bounds for TS$^2$-D as well as OLIVE scale as $S^3A^2$, with an additional $H^2$ term in OLIVE compared to the bound of TS$^2$-D.

For linear MDPs with finite dimensional features, $d_1 = d_2 = d$ and $\ln N = O(d)$, so that our bound scales as $O(d^4)$, which is a factor of $d$ worse relative to LSVI-UCB [Jin et al., 2020], and a factor of $d^2$ worse relatively to [Zanette et al., 2020b]. One reason for the suboptimality in $d$ is due to our choice of $V$-type Bellman rank instead of $Q$-type Bellman rank to allow non-linear scenarios. For the nicer setting of $Q$-type Bellman rank, an analysis of a single sample based Thompson sampling is recently carried out in [Dann et al., 2021], leading to a result that matches that of [Zanette et al., 2020b] for linear MDPs. Another reason for the suboptimality is because our MDP model allows long range contextual dependency on $x_t$ and we do not consider it in this paper.

Note that the recent work of [Foster et al., 2021] also analyzes design-based approaches for model based RL, but both their problem setting and algorithmic details differ significantly from ours. The result of Theorem 9 demonstrates that the suboptimality in $\epsilon$ in our more general result of Theorem 5 stems purely from the harder setting of unknown linear embedding features. We leave the development of corresponding results for spectral decay and combinatorial action settings to the reader.

In terms of analysis, the only change is that we are able to do different handling of a decoupling step using the property of optimal design in the analysis of Theorem 9 and the error terms in the linear embedding
can be ignored due to the finite dimensional assumption. The rest of our arguments stay the same, and we provide a proof in Appendix I.

7 Proof Sketch of Theorem [5]

In this section, we provide the analysis for our main result on the sample complexity of TS³. To begin, we recall a standard result on the online regret of any value-based RL algorithm.

Proposition 10 (Jiang et al. [2017]) Given any \( f \in \mathcal{F} \) and \( x^1 \in \mathcal{X}^1 \), we have

\[
\text{Regret}(f, x^1) = \sum_{h=1}^{H} \mathbb{E}_{(x^h, a^h) \sim \pi_1^h} [\mathcal{E}^h(f, f, x^h, a^h) - \Delta f(x^1)], \quad \text{where } \Delta f(x^1) = f(x^1) - Q^*_1(x^1).
\]

The proposition is a consequence of a simple telescoping argument. The RHS looks related to the likelihood function in our update (line 6 in Algorithm 1), but there are a few important differences. First, we do not have access to the Bellman error, which is instead approximated through difference of the TD term \( \Delta_t(f, f)^2 \) and the residual variance measured using \( \Delta_t(g, f)^2 \). If the posterior of \( g \) concentrates around the Bellman projection of \( f \) onto \( \mathcal{F} \), then we can expect our likelihood to resemble \( \mathcal{E}^h(f, f, x^h, a^h)^2 \). The presence of a squared term instead of the linear dependence in Proposition 10 is typical to algorithms which use completeness [Jin et al., 2021]. However, there are two more significant issues. On the RHS of Proposition 10, the function \( f \) whose Bellman error is measured is identical to the one whose greedy policy picks all the actions. On the other hand, in our algorithm, we control the regression loss of a function of measure over both \( x \) and \( a \). For linear functions, change in measure over the states using the low-rank property, while the distribution of the final action is corrected by importance weighting over the finite action set. For linear functions, change of measure over both \( x^h \) and \( a^h \) can be done using a similar use of the low-rank property, without explicit weighting over the actions [Jin et al., 2020, Du et al., 2021, Jin et al., 2021].

In our case, we adopt a slightly different approach to analyze the Bellman error term in Proposition 10. We first apply the low-rank property to decouple the roll-in policy from the \( f \) being evaluated. We subsequently use the linear embeddability assumption to decouple the action selection at step \( h \). This part of our analysis resembles that of [Zhang, 2021] for the contextual bandit setting, which is the reason we adopt the name decoupling coefficient as their work, for the measure of the linear embeddability dimension. We call these two results decoupling lemmas, which can be found in Appendix I. Using the two decoupling coefficients, we can obtain the following result.

Proposition 11 (Decoupling) We have

\[
\lambda \mathbb{E} \text{Regret}(f_t, x^1_t) \leq \mathbb{E}_{f|S_{t-1}} \left[ -\lambda \Delta f(x^1_t) + 0.5 \eta \mathcal{E}^{h_t}(f, f, x^h_t, a^h_t)^2 \right] + \lambda \tilde{c}(\lambda/\eta),
\]

where \( \tilde{c}(\lambda/\eta) = \inf_{\mu > 0} \left[ 8(\lambda/\eta)dc(\epsilon_2)H \mu^2 + 2 \mu H \epsilon_2 B_2 + \mu^{-1}br(\epsilon_1) + \epsilon_1 HB_1 \right] \).

For optimal parameter settings, the bound scales with \( \sqrt{dc(\epsilon_2)\mathbb{E}_{f|S_{t-1}} \mathcal{E}(f, f; x^h_t, a^h_t)^2} \), and this square root is responsible for our \( O(\epsilon^{-4}) \) rate. In contrast with Proposition 10 Proposition 11 measures
the squared Bellman error of functions $f$ according to the states $x_t^h$ and actions $a_t^h$ observed during the algorithm’s execution, and which we can hope to control if the TS$^3$ updates converge to their respective optima for both the time scales. The remainder of our analysis focuses on this online learning component, details of which are presented in Appendix H. We now give our main result to control the regret of the online learning process.

**Proposition 12 (Outer loop convergence)** Assume that $\gamma = 0.1$ and $\eta \leq 0.01$. Then we have

$$
- \sum_{t=1}^{T} \mathbb{E}\mathcal{E}_{f|S_{t-1}}(\lambda \Delta f(x_t^1) + \eta(1 - 6\eta)e^{-3\eta(1 - 6\eta)} \sum_{t=1}^{T} \mathbb{E}\mathcal{E}_{f|S_{t-1}}^h(f, f; x_t^h, a_t^h)^2 \\
\leq \eta(1 + 6\eta)e^{3\eta(1 + 6\eta)} \sum_{t=1}^{T} \mathbb{E}\mathcal{E}_{f,g|S_{t-1}}^h(g, f; x_t^h, a_t^h)^2 + \lambda T + 4\eta T \epsilon^2 + \kappa(\epsilon) + 1.5\lambda^2 T.
$$

We prove Proposition 12 in Appendix H.2. The LHS of the proposition qualitatively resembles the RHS of Proposition 11. Indeed, we subsequently set the constants so that they match. This means that the regret in Proposition 11 can be further upper bounded by the RHS of Proposition 12. The first term in the bound is intuitive. It bounds the Bellman residual of $f$ in terms of quality of the functions $g \sim \pi(\cdot | f, S_{t-1})$ in capturing the Bellman operator applied to $f$. If the inner loop of TS$^3$ converges at a reasonable rate, we expect this error to be small by the completeness assumption. The next three terms on the RHS are a bound on the log-partition function in our outer loop updates, which are controlled by using typical potential function arguments in the analysis of multiplicative updates. The final term can be controlled by appropriate setting of the optimism parameter $\lambda$.

We now give the main bound on the convergence of our inner updates to control the first term on the RHS of Proposition 12.

**Proposition 13** Assume that $\gamma = 0.1$. Given any absolute constant $c_1 > 0$, there exists an absolute constant $c_0$ such that when $c_0(\kappa(\epsilon) + \kappa'(\epsilon) + \ln T) \leq 0.5c_1\gamma < 0.2\gamma$ and $\lambda \leq \eta$, then

$$
\mathbb{E}\sum_{t=1}^{T} \mathbb{E}\mathcal{E}_{f,g|S_{t-1}}^h(g, f; x_t^h, a_t^h)^2 \leq O\left(c_0(\epsilon T + \kappa(\epsilon) + \kappa'(\epsilon) + 1) + c_1(\lambda/\eta)\tilde{\epsilon}(\lambda/\eta)T
\right.
\left. + c_1 \sum_{t=1}^{T} \mathbb{E}\mathcal{E}_{f|S_{t-1}}^h(f, f; x_t^h, a_t^h)^2 + c_1(\lambda/\eta)\sum_{t=1}^{T} \mathbb{E}\text{Regret}(f_t, x_t^1)\right),
$$

where $\tilde{\epsilon}(\lambda/\eta)$ is defined in Proposition 17.

The proof of Proposition 13 is rather long and technical. For the reader’s convenience, we first prove a simpler result which yields a worse $O(\epsilon^{-8})$ sample complexity in Appendix H.3. We then show how to improve the bound to obtain Proposition 13 in Appendix H.4.

With these results, we set both $c_1$ and $\eta$ sufficiently small so that $(1 - 6\eta) \exp(-3(1 - 6\eta)) - O(c_1(1 + 6\eta)e^{3\eta(1 + 6\eta)}) \geq 0.5$, $O(c_1(1 + 6\eta)e^{3\eta(1 + 6\eta)}) \leq 0.5$, along with the stated values of $\epsilon$ and $\gamma$. Plugging these into the bounds of our intermediate results and simplifying gives the result of Theorem 5. Finally, we summarize the proof of Corollary 6 and defer that of Corollary 7 to Appendix H.4.

**Proof** [Proof of Corollary 6] We set the parameters as follows. Since $\epsilon_1 = \epsilon_2 = 0$, we minimize over $\mu$ to get $\mu = (d_1\eta/(d_2\lambda H))^{1/3}$. We now optimize over the choice of $\eta$, for which the leading order terms are $\eta^2 T \ln N/\lambda + T\tilde{\epsilon}(\lambda/\eta)$. Then optimizing for $\lambda$ by including the $\ln N/\lambda$ term yields the stated guarantee.
8 Conclusion and Discussion

In this paper, we combine the framework of low Bellman rank with a linear embedding assumption over the action space to introduce a new class of rich problems which encompasses all settings with finite actions and linear function approximation, and enables new ones such as combinatorial action spaces. We show that TS\(^3\) solves this class of problems under the usual completeness and realizability assumptions on the value function class. We believe that the identification of this linear embedding structure over actions as the key enabler of one-step exploration in the action space has the potential to apply to broader algorithmic approaches beyond TS\(^3\).

The most immediate direction for future work is to improve our sample complexity results by sharpening our decoupling results. Understanding if similar results are attainable without completeness is another challenge. Finally, it would be interesting to understand what structures beyond linear embeddability afford sample-efficiency, and study applications of these ideas to continuous control problems.

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A  Related Work

Settings for sample-efficient RL. There is substantial recent literature [Jiang et al., 2017, Sun et al., 2019, Jin et al., 2020, Yang and Wang, 2020, Du et al., 2019, 2021, Jin et al., 2021, Foster et al., 2021] on structural conditions that enable sample-efficient RL. Of these, the Bellman rank framework and its recent generalization in the Bilinear Classes model remain the broadest known frameworks for which provably sample-efficient methods are known. While Bellman rank itself makes no assumptions on the complexity of the action space [Jiang et al. [2017] show small Bellman rank for LQRs), the algorithm OLIVE developed for this setting crucially relies on importance sampling over a discrete action set. Ideas from these works have further been developed to a representation learning setting, where the transition dynamics of the MDP are linear in some unknown features and the agent learns this map, given a class of candidate features [Agarwal et al., 2020b, Uehara et al., 2021, Modi et al., 2021], which captures rich non-linear function approximation in the original state and action. Jin et al. [2021] developed the Bellman-Eluder dimension notion to better capture infinite action sets using a notion they call $Q$-type Bellman rank, while the original version of Jiang et al. [2017] is termed the $V$-type Bellman rank. Note, however, that the GOLF algorithm of Jin et al. [2021] for problems with a small $Q$-type Bellman rank, which scales to infinite action spaces, cannot be used for feature learning and does not capture all contextual bandit problems (see Section B for a lower bound), showing the limitations of this approach in terms of the non-linearity it permits. For $V$-type Bellman-Eluder dimension, Jin et al. [2021] also rely on uniform exploration over actions similar to OLIVE. The techniques developed here do scale to feature learning, capture contextual bandits fully and apply to infinite action scenarios satisfying the linear embedding assumption. At the same time, our assumptions do not capture all problems with a small $Q$-type Bellman rank, so there are problems which GOLF can handle which we do not. That said, perhaps the most prominent example for GOLF in the infinite action setting is that of linear MDPs, where the linear embedding assumption made here holds.

Continuous control. Large action spaces are the standard framing in continuous control, where the action is typically a vector in $\mathbb{R}^d$ for some control input dimension $d$. While there has been a number of recent results at the intersection of learning and control [see e.g., Dean et al., 2020, Mania et al., 2019, Agarwal et al., 2019, Simchowitz and Foster, 2020], a large body of work typically focuses on highly structured settings such as the Linear Quadratic Regulator (LQR), where online exploration is straightforward due to the presence of a Gaussian noise in the dynamics. More recent results [Kakade et al., 2020, Mania et al., 2020] do combine online control and exploration, they typically focus on model-based settings and still rely on access to good features. We note that Mhammedi et al. [2020] carry out feature learning for continuous control, but their setting does not have a small Bellman rank and hence is not admissible in our conditions either.
Posterior sampling. Posterior sampling methods for RL, motivated by Thompson sampling \cite{thompson1933likelihood}, have been extensively developed and analyzed in terms of their expected regret under a Bayesian prior by many authors \cite[e.g.][]{osband2013near,russo2017information,osband2016td} and are often popular as they offer a simple implementation heuristic through approximation by ensembles trained on random subsets of data. Worst-case analysis of TS in RL settings has also been done for both tabular \cite{russo2019learning,agrawal2017thompson} and linear \cite{zanette2020near} settings. Our work is most closely related to the recent Feel-Good Thompson Sampling strategy proposed and analyzed in \cite{zhang2021feel}, primarily for bandits but also for RL problems with deterministic dynamics. They study problems with a similar linear embeddability assumption, but the absence of any further structure like a small Bellman rank precludes their techniques for application to general stochastic dynamics. We also observe that FGTS retains the significant optimistic component of other exploration techniques like LSVI-UCB \cite{jin2020provably} and OLIVE \cite{jiang2017near}, which partly explains its success in worst-case settings. On the other hand, the approach has the remarkable property of working for both linear bandits and non-linear bandits with finite action sets with an identical algorithm and analysis, and we extend that benefit to RL in this work.

Minimax objectives in RL. FGTS algorithm uses a likelihood term for Thompson Sampling based on the TD error, which is well-known to have a bias in estimating the Bellman error \cite{antos2008minimax,sutton1998between} for stochastic dynamics. The usual technique of removing the residual variance from \cite{antos2008minimax} creates a minimax objective, which we also use in this paper. Other minimax formulations \cite{dai2018minimax} to remove this bias are also possible in general, but we find that the one from \cite{antos2008minimax} is the most natural under our structural assumptions. We also note that approach of keeping two timescales \cite{borkar2009stochastic} used here has been used previously in offline RL for TD learning methods \cite{sutton2009between}, but its online analysis in TS appears novel, and different from the analysis in \cite{dann2021minimax}, which cannot handle general nonlinear feature learning considered here.

B Lower bound for $Q$-type Bellman rank

We now instantiate a contextual bandit problem where the realizability assumption holds, but the $Q$-type Bellman rank grows linearly in the number of contexts. The construction is due to \cite{dann2018minimax}, but has not appeared in the literature. Note that the $V$-type Bellman rank of \cite{jiang2017near}, which we further generalize in this work, is always 1 for a contextual bandit problem. The lower bound is demonstrated using a typical hard instance for contextual bandit problems. Let us consider a context distribution which is uniform on $[N]$, where we have $N$ unique contexts. We have two actions $\{a_1, a_2\}$. We also have $|\mathcal{F}| = N + 1$ with the following structure.

$$f^*(x, a_1) = f_{N+1}(x, a_1) = 0, \quad \text{and} \quad f^*(x, a_2) = f_{N+1}(x, a_2) = 0.5.$$

For $i < N + 1$, we have $f_i(x, a) = f^*(x, a)$ when $x \neq i$, and $f_i(x, a_1) = 1$, $f_i(x, a_2) = 0.5$ so that $f_i$ makes incorrect predictions on the context $i$ for action $a_1$. Notice that the design of $\mathcal{F}$ also ensures that for each context $i$, there is a policy $\pi_*$ (greedy wrt $f^*$) which picks the action $a_2$, while another policy $\pi_i$ (greedy wrt $f_i$) which picks action $a_1$. Since this is a problem with horizon $H = 1$, the Bellman error is simply equal to $f(x, a) - f^*(x, a)$ for any $x, a$. Let $\mathcal{E}^Q(f, \pi) = \mathbb{E}_{x, a \sim \pi} \mathcal{E}(f, f, x, a)$ be the $Q$-type Bellman error. Then, we observe that for $1 \leq i, j \leq N$:

$$\mathcal{E}^Q(f_i, \pi_j) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{E}_{a \sim \pi_j} [f(x, a) - f^*(x, a)] = \frac{1(j = i)}{N},$$
where $1(\cdot)$ is an indicator function. Hence, we see that the Bellman error matrix contains an identity sub-matrix of size $N$, so that its rank is at least $N$.

## C Connection with sparse RL

In this section, we formalize the relationship between representation learning and RL with sparsity. Concretely, let us consider two formulations.

$$P(x' \mid x, a) = \phi^{lr}(x, a)^\top \mu^{lr}(x'), \quad \text{where} \quad (\phi^{lr}, \mu^{lr}) \in \Omega^{lr},$$

is the standard model-based representation learning formulation for RL [Agarwal et al., 2020b, Uehara et al., 2021]. Here we consider a slightly generalized setup where the feature maps $\phi$ correspond to the index of $\Omega^{lr}$ instead of separate classes for $\phi$ and $\mu$, which reduces to the more typical framing with separate classes for $\phi$ and $\mu$ when we set $\Omega^{lr}$ to be the cartesian product of their respective sets. However, all existing representation learning algorithms can handle this setup for a general $\Omega^{lr}$ without any extra difficulty. Let us denote $d^{lr} = \dim(\phi^{lr})$, where we use the superscript $lr$ to denote the low-rank problem $P1$.

The second formulation is the sparse linear MDP setup of Hao et al. [2021].

$$P(x' \mid x, a) = \sum_{i \in I} \phi^s(x, a)^\top x_i \mu^s(x'), \quad \text{where} \quad |I| = k \ll d^s = \dim(\phi^s),$$

with the feature maps $\phi^s, \mu^s$ considered known in $P2$, but the index set $I$ is unknown. We now show that the two problems are equivalent when $d^{lr} \cdot |\Omega^{lr}| = d^s$ and $k = d^{lr}$, in that any solution to $P1$ can solve $P2$ at no additional sample cost, and vice versa.

In one direction, we define the concatenated feature map for all $x, a$ and $x'$:

$\phi^{all}(x, a) = (\phi_1(x, a), \phi_2(x, a), \ldots, \phi_N(x, a))$, where $N = |\Omega^{lr}|$, and

$\mu^{all}(x') = (\mu_1(x'), \mu_2(x'), \ldots, \mu_N(x'))$,

where $\phi_i, \mu_i$ refer to the $i$th feature maps in $\Omega^{lr}$. Clearly, the assumption $\phi^{lr}, \mu^{lr} \in \Omega$ guarantees that $P(x' \mid x, a) = \phi^{all}(x, a)^\top \mu^{all}(x')$ with $\dim(\phi^{all}) = d^{lr} |\Omega^{lr}| = d^s$, by construction. However, the transitions are also sparse in the features $(\phi^{all}, \mu^{all})$, in that we can choose the $d^{lr}$ coordinates of $\phi^{all}, \mu^{all}$ which correspond to the index of $(\phi^{lr}, \mu^{lr})$ in $\Omega^{lr}$. Consequently, any solution to $P2$ for $k = d^{lr}$ which uses samples that are $O(poly(k \log d^s))$ can be used to solve $P1$ with a sample complexity of $O(poly(d^{lr} \log(|\Omega^{lr}|)))$, which is considered the standard goal of the representation learning problem $P1$.

In the other direction, let us say we have a solution to $P1$ with a sample complexity that is $O(poly(d^{lr} \log(|\Omega^{lr}|)))$. Then we give a pair of high-dimensional feature maps $\phi^s, \mu^s$, we define a class $\Omega^{lr}$ as follows:

$$\Omega^{lr} = \{(\phi, \mu) : \phi(x, a) = \phi^s(x, a)_{\mathcal{J}}, \mu(x') = \mu^s(x')_{\mathcal{J}}, \quad \text{where} \quad |\mathcal{J}| = k\}.$$

In words, we add features corresponding to every subset of size $k$ from the original $d^s$ features as a candidate representation to $\Omega^{lr}$. Then $|\Omega^{lr}| = \binom{d^s}{k} = O((d^s)^k)$ and each feature map in $\Omega^{lr}$ has $d^{lr} = k$. Clearly the sparse linear MDP assumption in the definition of $P2$ implies that $\Omega^{lr}$ contains a pair $(\phi, \mu)$ under which the MDP is linear. Furthermore, our method for representation learning applied to the sparse linear MDP has a sample complexity of $O(poly(k \log d^s))$.

The above reduction show that any obstacle to sparse linear MDP learning also results in a lower bound for representation learning. In particular, the construction of Hao et al. [2021] precludes learning sparse linear MDPs where the action set has a cardinality $O(d^s)$, so that we cannot expect to carry out representation learning in arbitrarily large action spaces without further assumptions.
D Examples satisfying Assumption 4

Example 2 (Basis expansions in action space) For continuous action problems, Asadi et al. [2021] study the class of deep radial basis value functions, where any function class takes the form

\[
f(x, a; \theta) = \sum_{i=1}^{N} \frac{\exp(-\beta \| a - a_i(x; \theta) \|)}{\sum_{i=1}^{N} \exp(-\beta \| a - a_i(x; \theta) \|)} v_i(x; \theta),
\]

where \( a_i(x; \theta) \) are a (state and \( \theta \)-dependent) basis, while \( v_i(x; \theta) \) are some reference values at these points. If we consider the case where the \( a_i(x; \theta) \) are only dependent on the state \( x \), but fixed across \( \theta \), then this definition satisfies Assumption 4 with \( w(f, x) = v_i(x; \theta) \) and \( \phi(x, a)_i = \exp(-\beta \| a - a_i(x; \theta) \|) / \sum_{i=1}^{N} \exp(-\beta \| a - a_i(x; \theta) \|) \). More generally, whenever there is a basis set of actions such that \( f(x, a) = \sum_{i=1}^{N} \alpha_i(x; a) g_f(x, a_i) \) for some functions \( \alpha_i \) and \( g_f \), then the assumption holds. Such a basis can be obtained, for instance, by standard finite element approximation techniques such as triangulation in an appropriate metric or other finite element techniques in the action space [Zienkiewicz et al., 2005], as long as \( f \) is sufficiently smooth in \( a \).

Example 3 (Block MDP with latent state, action features) Several authors study the Block MDP model [Du et al., 2019; Misra et al., 2019], which is a special case of MDPs with a small Bellman rank. In a block MDP, each observation \( x \in \mathcal{X} \) is emitted a unique latent state \( z \in \mathcal{Z} \), so that there is an unknown one-to-one mapping \( g^* : \mathcal{X} \rightarrow \mathcal{Z} \) which maps an observed state to the latent state which generated it. Let \( \psi(x^h, a^h) \in \mathbb{R}^{|\mathcal{Z}| \times |\mathcal{A}|} \) be an indicator vector \( e_{(g^*(x^h),a^h)} \) with a one corresponding to the entry \( (g^*(x^h),a^h) \) and 0 everywhere else. Then the MDP dynamics are linear in these features [see e.g. Zhang et al., 2022], and OLIVE solves this model using poly\(|\mathcal{Z}|,|\mathcal{A}|\) samples with no explicit dependence on \( |\mathcal{X}| \) when the actions are finite, with computationally efficient approaches provided in later works [Du et al., 2019; Misra et al., 2019]. In the setup of this work, let us consider a more expressive model where \( |\mathcal{A}| \) can be infinite, but we are given a feature map \( \zeta(z, a) \in \mathbb{R}^d \) such that \( P(z^{h+1}|z^h, a^h) \) and the reward function are both linear in \( \zeta \), that is:

\[
r(x, a) = w_r^\top \zeta(g^*(x), a), \quad \text{and} \quad P(z^{h+1}=z_i|z^h, a^h) = w_p(i)^\top \zeta(z^h, a^h),
\]

for some \( w_r, w_p(1), \ldots, w_p(|\mathcal{Z}|) \in \mathbb{R}^d \). Then it is easily shown that for any function \( v : \mathcal{X} \rightarrow \mathbb{R} \), we have \( \mathbb{E}[r(x^h, a^h) + v(z^{h+1} | x^h, a^h)] = w_r^\top \zeta(x^h, a^h) \) for some \( w_v \in \mathbb{R}^d \). Consequently, Assumption 4 is satisfied when \( f(x^h, a^h) \) is linearly embeddable in some features \( \zeta'(x^h, a^h) \) and we set \( \phi(x^h, a^h) = (\zeta'(x^h, a^h), \zeta(x^h, a^h)) \) to be the concatenation of the two feature sets. Note that we require \( \mathcal{F} \) to satisfy completeness and realizability assumptions, so a natural choice is to pick \( \zeta' = \zeta \) when \( \zeta \) is known. Then our function class takes the form \( f(x^h, a^h) = w(f, x^h)^\top \zeta(x^h, a^h) \) for some choice of the mapping \( w \). For instance, we may pick \( w(f, x) = w_f^\top M(x) \), where \( w_f \in \mathbb{R}^{|\mathcal{Z}|} \) is a vector and \( M(x) \in \mathbb{R}^{|\mathcal{Z}| \times d} \) is a matrix which encodes \( g^*(x) \) as a non-zero row in the matrix. As before, this generalizes the finite action scenario where \( \zeta \) can be chosen to be indicator features of actions. More generally, this corresponds to problems where we have a reasonable embedding for actions in each latent state. This might happen, for instance, in control problems where given the agent’s physical state, reasonable control policies can be designed by using action embeddings that are either random, or learned apriori [see e.g. Rajeswaran et al., 2017; Kakade et al., 2020], but the physical position might not be available when the observation is based on vision or other sensory measurements.

E Proof of Proposition 2

We start by reviewing the bound on \( \text{br}(\epsilon) \) for the finite dimensional case. In this case, let \( \{ \lambda_i \}_{i=1}^d \) be the eigenvalues of \( \Sigma^h(p, x^1) \) in decreasing order, for some distribution \( p \in \Delta(\mathcal{F}) \) and \( x^1 \in \mathcal{X} \), where we recall
that $\lambda_d \geq 0$ since $\Sigma^h(p, x^1)$ is a covariance matrix. Then we have

$$\text{trace}\left( (\Sigma^h(p, x^1) + \lambda I)^{-1}\Sigma^h(p, x^1) \right) = \sum_{i=1}^{d} \frac{\lambda_i}{\lambda + \lambda_i} \leq \sum_{i=1}^{d} 1 \leq d.$$ 

**Proof for geometric decay case.** This follows effectively by reducing to the finite dimensional case. For any positive integer $n$, we have

$$\text{trace}\left( (\Sigma^h(p, x^1) + \lambda I)^{-1}\Sigma^h(p, x^1) \right) \leq n + \frac{\alpha^n}{\lambda(1 - \alpha)}.$$ 

Choosing $n = n_0$ such that $\frac{\alpha^{n_0}}{(1 - \alpha)} = \epsilon^2/2$, we see that

$$\lambda K^h(\lambda) \leq n\lambda + \frac{\epsilon^2}{2},$$
so that we can set $\lambda = \epsilon^2/2n_0$. With these settings, we have

$$K^h(\lambda) \leq n_0 + \frac{2n_0\alpha^{n_0}}{\epsilon^2(1 - \alpha)} = 2n_0 = 2 \log \frac{\epsilon^2}{2(1 - \alpha)} \log \frac{1}{\alpha},$$
where the last equality follows from our setting of $n_0$.

**Proof for polynomial decay case.** In this case, the assumption on trace guarantees that

$$\sum_i \lambda_i^q \leq R_q.$$ 

Now since $q \in (0, 1)$, we have

$$\text{trace}\left( (\Sigma^h(p, x^1) + \lambda I)^{-1}\Sigma^h(p, x^1) \right) \leq \sum_i \frac{\lambda_i}{\lambda + \lambda_i} \leq \sum_i \left( \frac{\lambda_i}{\lambda + \lambda_i} \right)^q \leq \sum_i \frac{\lambda_i^q}{\lambda^q} \leq \frac{R_q^q}{\lambda^q}.$$

With this, we have $\lambda K^h(\lambda) \leq \lambda^{1-q} R_q^q$, so that we choose

$$\lambda_0 = \left( \epsilon^2 R_q^{-q} \right)^{1/(1-q)}.$$ 

With this choice, we get

$$K^h(\lambda_0) \leq \frac{R_q^q}{\lambda_0^q} = \left( \epsilon^{-2} R_q \right)^{q/(1-q)}.$$ 

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We now set $\eta$ as in Theorem 5 and optimize to get $\lambda = O \left(T^{-3/4}(1+q)/(\ln N)\right)$, where $c_q = d_q^{(1-q)/(1+q)} H^{1/((1-q)(1+q))} B^{2q/(1+q)}$. Substituting this back into Theorem 5 completes the proof. }

### G Proofs of decoupling results

In this section, we prove Lemma 11. This will be done across two smaller lemmas which provide one level of decoupling using the Bellman rank and another using the linear embedding. The first lemma’s proof technique resembles the recently used “one-step back” inequalities in low-rank MDP literature [Agarwal et al., 2020b, Uehara et al., 2021]. Throughout this section, we use $u^\top v$ to denote $\langle u, v \rangle$ even for infinite dimensional vectors with a slight abuse of notation to improve readability.

**Lemma 14 (Bellman Rank Decoupling)** Consider any distribution $p$ over $F$. The following inequality holds for any $\epsilon, \mu_1 > 0$.

$$
\mathbb{E}_{f \sim p} \mathbb{E}_{(x, h^*) \sim \pi_f} |E^h(f, f, x^h, a^h)| \\
\leq \sqrt{b h^{-1}(\epsilon)} \mathbb{E}_{f \sim p} \mathbb{E}_{(x, h^*) \sim \pi_f} \left| \mathbb{E}_{f \sim p} E^h(f, f, x^h, \pi_f(x^h)) \right|^2 + \epsilon B_1 \\
\leq \mu_1 \mathbb{E}_{f \sim p} \mathbb{E}_{(x, h^*) \sim \pi_f} \left| \mathbb{E}_{f \sim p} E^h(f, f, x^h, \pi_f(x^h)) \right|^2 + \mu_1^{-1} \epsilon h^{-1}(\epsilon) + \epsilon B_1.
$$

**Proof** We have

$$
\mathbb{E}_{f \sim p} \mathbb{E}_{(x, h^*) \sim \pi_f} |E^h(f, f, x^h, a^h)| \\
= \mathbb{E}_{f \sim p} \mathbb{E}_{(x, h^*) \sim \pi_f} \left| \mathbb{E}_{x \sim h^{-1}, a \sim h^{-1}} E^h(f, x^h, \pi_f(x^h)) \right| \\
= \mathbb{E}_{f \sim p} u^h(f, x^1)^\top \psi^{-1}(f, x^1) \\
\leq \sqrt{K h^{-1}(\lambda) \mathbb{E}_{f \sim p} u^h(f, x^1)^\top \left( \mathbb{E}_{f \sim p} \psi^{-1}(f', x^1) \psi^{-1}(f', x^1)^\top \right) u^h(f, x^1)} \\
\leq \sqrt{K h^{-1}(\lambda) \mathbb{E}_{f \sim p} u^h(f, x^1)^\top \left( \mathbb{E}_{f \sim p} \psi^{-1}(f', x^1) \psi^{-1}(f', x^1)^\top \right) u^h(f, x^1)} \\
+ \sqrt{\lambda K h^{-1}(\lambda) \mathbb{E}_{f \sim p} \|u^h(f, x^1)\|^2} \\
\leq \sqrt{K h^{-1}(\lambda) \mathbb{E}_{f \sim p} u^h(f, x^1)^\top \psi^{-1}(f', x^1) u^h(f', x^1)} \\
+ \sqrt{\lambda K h^{-1}(\lambda) \mathbb{E}_{f \sim p} \|u^h(f, x^1)\|^2} \\
\leq \sqrt{K h^{-1}(\lambda) \mathbb{E}_{f \sim p} u^h(f, x^1)^\top E(f, f, x^h, \pi_f(x^h))} \\
+ \sqrt{\lambda K h^{-1}(\lambda) \mathbb{E}_{f \sim p} \|u^h(f, x^1)\|^2} \\
\leq \sqrt{K h^{-1}(\lambda) \mathbb{E}_{f \sim p} u^h(f, x^1)^\top \psi^{-1}(f', x^1) u^h(f', x^1)} \\
+ \sqrt{\lambda K h^{-1}(\lambda) \mathbb{E}_{f \sim p} \|u^h(f, x^1)\|^2}.
$$
Here (a) holds by using Assumption 3 once more to rewrite the inner product as a Bellman error, while (b) is a consequence of Jensen’s inequality. Now we recall that

$$\|u^h(f, x^h)\|_2 \leq B_1.$$  

By taking the largest $\lambda$ so that $\lambda \sup_p K^{-1}(\lambda, p) \leq \epsilon^2$, we obtain the desired bound. \hfill \blacksquare

The next decoupling lemma separates the sampling of the action $a^h$ from the function $f$ whose Bellman error is being evaluated by using Assumption 2, which is crucial for the analysis as explained in Section 7. Note that in this particular derivation, we reduce squared Bellman error to the square root of a decoupled squared Bellman error, which loses rate. It may be possible to improve this reduction by a more careful analysis in future work.

**Lemma 15 (Linear Embedding Decoupling)** We have for all $x^h \in X^h$ and probability measures $p$ on $F$ and $\mu_2, \epsilon_2 > 0$:

$$\mathbb{E}_{f \sim p}[\mathcal{E}^h(f, f; x^h, \pi_f(x^h))^2] \leq 2 \sqrt{d\mathcal{E}^h(\epsilon)\mathbb{E}_{f \sim p, f' \sim p, a \sim \pi, f^h, x^h \sim \pi}[\mathcal{E}^h(f, f; x^h, a^h)^2] + 2\epsilon B_2}$$

$$\leq 2\mu_2 \mathbb{E}_{f \sim p, f' \sim p, a \sim \pi}[\mathcal{E}^h(f, f; x^h, a^h)^2 + 2\mu_2^{-1}d\mathcal{E}^h(\epsilon) + 2\epsilon B_2].$$

**Proof** We have

$$\mathbb{E}_{f \sim p}[\mathcal{E}^h(f, f; x^h, \pi_f(x^h)) = \mathbb{E}_{f \sim p}[w^h(f, x^h)^T \phi^h(x^h, \pi_f(x^h))].$$

(Assumption 4)

$$\leq \left[\mathbb{E}_{f \sim p}w^h(f, x^h)^T(\tilde{\Sigma}^h + \tilde{\lambda} I)w^h(f, x^h)\right]^{1/2} \left[\mathbb{E}_{f \sim p}(\phi^h(x^h, \pi_f(x^h))^T(\tilde{\Sigma}^h + \tilde{\lambda} I)^{-1}\phi^h(x^h, \pi_f(x^h))\right]^{1/2},$$

(Cauchy-Schwarz)

where $\tilde{\Sigma}^h$ is short for $\tilde{\Sigma}^h(p, x^h)$. Therefore we have

$$\mathbb{E}_{f \sim p}[\mathcal{E}^h(f, f; x^h, \pi_f(x^h))]^2$$

\leq 2 \mathbb{E}_{f \sim p}[\mathcal{E}^h(f, f; x^h, \pi_f(x^h))]^2

\leq 2 \mathbb{E}_{f \sim p, f' \sim p, a \sim \pi, f^h, x^h \sim \pi}(w^h(f, x^h)^T\phi^h(x^h, a^h)\phi^h(x^h, a^h)^T + \tilde{\lambda} I)w^h(f, x^h))^{1/2}\tilde{K}^h(\tilde{\lambda})

\leq 2 \mathbb{E}_{f \sim p, f' \sim p, a \sim \pi, f^h, x^h \sim \pi}(\mathcal{E}^h(f, f; x^h, a^h)^2 + \tilde{\lambda}\|w^h(f, x^h)\|_2^2)^{1/2}\tilde{K}^h(\tilde{\lambda})

\leq 2 \mathbb{E}_{f \sim p, f' \sim p, a \sim \pi, f^h, x^h \sim \pi}(\mathcal{E}^h(f, f; x^h, a^h)^2)\tilde{K}^h(\tilde{\lambda}) + 2 \sqrt{\lambda}\tilde{\lambda}\tilde{K}^h(\tilde{\lambda})\mathbb{E}_{f \sim p}\|w^h(f, x^h)\|_2^2.

By taking the largest $\tilde{\lambda}$ so that $\lambda \sup_p K^{-1}(\lambda, p) \leq \epsilon^2$, we obtain the first desired bound. The second bound follows by Cauchy-Schwarz inequality. \hfill \blacksquare
Proof of Proposition 11. Using the two lemmas above, it is easy to establish Proposition 11. 

Proof [Proof of Proposition 11] Since $x^1_t$ is randomly drawn from $D$, it is independent of $f_t$. Therefore we have

$$
\lambda \mathbb{E} \text{Regret}(f_t, x^1_t) + \lambda \mathbb{E} \mathbb{E}_{f|S_{t-1}} \Delta^1 f(x^1_t) = \lambda \mathbb{E} \text{Regret}(f_t, x^1_t) + \lambda \mathbb{E} \Delta^1 f(x^1_t)
$$

$$
= \lambda \sum_{h=1}^{H} \mathbb{E} \mathbb{E}_{(x,a) \sim \pi_f} \mathcal{E}^h(f_t, f_t, x^h, a^h) = \lambda H \mathbb{E} \mathcal{E}^h(f_t, f_t, x^h, a^h)
$$

$$
\leq \lambda \left[ \epsilon_1 HB_1 + \mu_1^{-1} \text{br}(\epsilon_1) + H \mu_1 \mathbb{E} \mathbb{E}_{f|S_{t-1}} \mathcal{E}^h(f, f, x^h, \pi_f(x^h))^2 \right]
$$

$$
\leq \lambda \left[ 2 \mu_1 \mu_2 H \mathbb{E} \mathbb{E}_{f|S_{t-1}} \mathcal{E}^h(f, f, x^h, \pi_f(x^h))^2 + 2 \mu_1 \mu_2^{-1} \text{dc}(\epsilon_2) + 2 \epsilon_2 \mu_1 H B_2 + \epsilon_1 HB_2 + \mu_1^{-1} \text{br}(\epsilon_1) \right].
$$

The first equality used the independence of $x^1_t$ and $f_t$. The second equality used Proposition 10. The third equality used the fact that $h_t$ is uniformly drawn from $[H]$. The first inequality used Lemma 14. The second inequality used Lemma 15. Note that the last two steps crucially use that the function $f_t$ used to draw $x^h_t$, $f'_t$ to draw $a^h_t$, and $f \sim p$ being evaluated are all mutually independent. Taking $\mu_1 = \mu$ and $\mu_2 = \eta/(4\lambda \mu_1 H)$, we obtain the desired result.

H Convergence of the online learning in TS$^3$

In this section, we study the convergence of the online learning updates for the $f$ and $g$ functions. To do so, it is helpful to define some additional notation.

Define $\sigma = 2$, and let

$$
\varepsilon^h_t(f) = T^h f(x^h_t, a^h_t) - (r^h_t + f^{h+1}(x^{h+1}_t))
$$

be the noise in the Bellman residuals. We observe that $|\varepsilon^h_t(f)| = \sigma$ for all $t \in [T], h \in [H]$ and $f \in F$ under our normalization assumptions. For convenience, we use $\sigma$ to denote this upper bound on the $\varepsilon^h_t(f)$ to avoid carrying constants. We also have the following observations about the noise.

For each $h \geq 1$, we also define:

$$
\delta^h_t(g, f) = \hat{\Delta}^h(g, f)^2 - \varepsilon^h_t(f)^2,
$$

$$
\delta^h_t(f) = \mathbb{E}_{g \sim p \mid g(f, S_{t-1})} \delta^h_t(g, f), \quad \text{and}
$$

$$
\hat{Z}^h_t(f) = -\frac{1}{\gamma} \ln \mathbb{E}_{g \sim p \mid g(S_{t-1})} \exp(-\gamma \delta^h_t(g, f)). \quad (7)
$$

Here $\delta^h_t(g, f)$ measures how well $g$ captures the Bellman residual of $f$ and $\delta^h_t(f)$ measures the quality of our posterior distribution over $g$ in doing so. Finally $\hat{Z}^h_t(f)$ is a log-partition function for the posterior. We further define the log-partition function for the posterior over $f$:

$$
Z(S_t) = -\ln \mathbb{E}_{f \sim p_0} \exp \left( \sum_{s=1}^{t} \lambda \Delta f(x^s_s) - \eta \sum_{s=1}^{t} [\delta^h_s(f, f) - \hat{Z}^h_s(f)] \right),
$$

where

$$
\Delta f(x^1) = f(x^1) - Q^1(x^1).
$$
We also introduce the following definition
\[ \hat{Z}_t = \hat{Z}_t^{h_t} = -\frac{1}{\eta} \ln \mathbb{E}_{f|S_{t-1}} \exp \left( \lambda \Delta f(x_t^1) - \eta [\hat{\delta}_t^h (f, f) - \hat{Z}_t^h (f)] \right), \]
which is the normalization factor of \( p(f|S_t)/p(f|S_{t-1}) \).

Let \( \psi(z) = (e^z - z - 1)/z^2 \). It is known that \( \psi(z) \) is an increasing function of \( z \). We organize the rest of this section as follows. We begin with some basic properties of Bellman residuals, then analyze the convergence of outer and inner updates respectively.

**H.1 Properties of Bellman residuals**

**Lemma 16** For any \( f \) that may depend on \( S_{t-1} \):
\[ \mathbb{E}_{x_{t+1}^h, x_t^h | x_t^h, a_t^h, h_t = h} \hat{c}_t^h (f) = 0, \]
and for any constant \( b_t \) independent of \( \hat{c}_t^h (f) \):
\[ \mathbb{E}_{x_{t+1}^h, x_t^h | x_t^h, a_t^h, h_t = h} \exp(b_t \hat{c}_t^h (f)) \leq \exp(b_t^2 \sigma^2 / 2). \]

The first equality follows since the conditional expectation only acts over the MDP rewards and dynamics, which are conditionally independent of any \( f \) that even depends on \( S_{t-1} \). The second bound is a consequence of the sub-Gaussian bound for bounded random variables in the proof of Hoeffding’s inequality.

**Lemma 17** Given any \( g, f \). We have
\[ -\sigma^2 \leq \hat{\delta}_t^h (g, f) = \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2 + 2\epsilon_t^h (f) \mathcal{E}_t^h (g, f; x_t^h, a_t^h) \leq 1 + 2\sigma. \]

Therefore
\[ \max (|\hat{\delta}_t^h (f, f) - \hat{Z}_t^h (f)|, |\hat{\delta}_t^h (f, f) - \hat{\delta}_t^h (f)|) \leq (1 + \sigma)^2. \]

Moreover for any \( t \in [T], h \in [H], x \in \mathcal{X} \) and \( a \in \mathcal{A} \), we have
\[ \mathbb{E}[\hat{\delta}_t^h (g, f) \mid x_t^h, a_t^h] = \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2, \quad \mathbb{E}[\hat{\delta}_t^h (g, f)^2 \mid x_t^h, a_t^h] \leq (1 + 4\sigma^2) \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2, \]
and for any \( c > 0 \),
\[ \mathbb{E}[\exp(c \hat{\delta}_t^h (g, f) \mid x_t^h, a_t^h] \leq \exp(c(1 + 2c\sigma^2) \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2), \]
\[ \mathbb{E}[\exp(-c \hat{\delta}_t^h (g, f) \mid x_t^h, a_t^h] \leq \exp(-c(1 - 2c\sigma^2) \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2) \]

**Proof** We note that
\[ \hat{\delta}_t^h (g, f) = \mathcal{E}_t^h (g, f; x_t^h, a_t^h)^2 + 2\epsilon_t^h (f) \mathcal{E}_t^h (g, f; x_t^h, a_t^h). \]
This implies the first result. We note that Lemma \[ \[16] \] implies that
\[ \mathcal{E}_t^h (g, f; x, a) = \mathbb{E}_{x_{t+1}^h \mid x_t^h = x, a_t^h = a} \hat{\delta}_t^h (g, f). \]
This implies the second result. Moreover, Lemma 16 implies that
\[
\mathbb{E}_{t, x_t, a_t \mid x_t, a_t} \hat{\delta}_t^h(g, f)^2 = \mathcal{E}_t^h(g, f; x, a)^4 + 4 \mathbb{E}_{t, x_t, a_t \mid x_t, a_t} \hat{\delta}_t^h(g, f)^2 \mathcal{E}_t^h(g, f; x, a)^2 \\
\leq (1 + 4\sigma^2)\mathcal{E}_t^h(g, f; x, a)^2.
\]
This implies the third result. The last two inequalities follow by using the first part of the lemma and then using the sub-Gaussian exponential bound from Lemma 16 with \( b_t = 2c\mathcal{E}(g, f; x^h_t, a^h_t) \) and \( b_t = -2c\mathcal{E}(g, f; x^h_t, a^h_t) \) respectively for the two bounds.

**H.2 Convergence of outer updates**

In this section, we build the necessary results to prove Lemma 12. We begin with the following bound for \( \hat{Z}_t^h(f) \).

**Lemma 18** We have
\[
\hat{\delta}_t^h(f) - \gamma \psi(\gamma \sigma^2)\mathbb{E}_{g, f, S_{t-1}} \hat{\delta}_t^h(g, f)^2 \leq \hat{Z}_t^h(f) \leq \hat{\delta}_t^h(f).
\]
Assume that
\[
\alpha' = \exp(\gamma(1 - 2\gamma \sigma^2))/(1 - 2\gamma \sigma^2) > 0.
\]
Then for any \( f \) that may depend on \( S_{t-1} \):
\[
\mathbb{E}_{x_t, a_t, r_t \mid h_t = h} \mathbb{E}_{g \mid f, S_{t-1}} \mathcal{E}_t^h(g, f; x^h_t, a^h_t)^2 \leq \alpha' \mathbb{E}_{x_t, a_t, r_t \mid h_t = h} \hat{Z}_t^h(f).
\]

**Proof** We have
\[
-\gamma \hat{Z}_t^h(f) = \ln \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} \exp(-\gamma \hat{\delta}_t^h(g, f)) \\
\leq \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} \exp(-\gamma \hat{\delta}_t^h(g, f)) - 1 \quad \text{(ln } z \leq z - 1) \\
= \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} \left[ -\gamma \hat{\delta}_t^h(g, f) + \gamma^2 \psi(-\gamma \hat{\delta}_t^h(g, f)) \hat{\delta}_t^h(g, f)^2 \right] \quad \text{(} \psi(z) \text{ is increasing in } z) \\
\leq \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} \left[ -\gamma \hat{\delta}_t^h(g, f) + \gamma^2 \psi(\gamma \sigma^2) \hat{\delta}_t^h(g, f)^2 \right]. \quad \text{(Jensen’s inequality)}
\]
Now conditioned on \( h_t = h \), we have
\[
- \gamma \mathbb{E}_{x_t, a_t, r_t} \hat{Z}_t^h(f) \\
\leq \mathbb{E}_{x_t, a_t, r_t} \ln \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} \mathbb{E}_{x_t+1, a_t+1 \mid x_t, a_t, r_t} e^{-\gamma \hat{\delta}_t^h(g, f)} \quad \text{(Jensen’s inequality)} \\
\leq \mathbb{E}_{x_t, a_t, r_t} \ln \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} e^{-\gamma(1 - 2\gamma \sigma^2) \mathcal{E}_t^h(g, f, x_t, a_t)^2} \quad \text{(Lemma 17)} \\
\leq \mathbb{E}_{x_t, a_t, r_t} \left[ \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} e^{-\gamma(1 - 2\gamma \sigma^2) \mathcal{E}_t^h(g, f, x_t, a_t)^2} - 1 \right] \quad \text{(ln } z \leq z - 1) \\
\leq - \mathbb{E}_{x_t, a_t, r_t} \mathbb{E}_{g \sim p(g \mid f, S_{t-1})} e^{-\gamma(1 - 2\gamma \sigma^2) \gamma(1 - 2\gamma \sigma^2) \mathcal{E}_t^h(g, f, x_t, a_t)^2}.
\]
Here the last inequality used \( e^{-z} - 1 \leq -e^{-z'}z \) for \( 0 \leq z \leq z' \). This implies the desired bound.
Lemma 19 We have
\[ \hat{Z}_t \leq \mathbb{E}_{f | S_{t-1}} \left[ -\frac{\lambda}{\eta} \Delta f(x^1_t) + \delta^{ht}_t(f, f) - \hat{Z}^{ht}_t(f) \right], \]
and
\[ |\hat{Z}_t| \leq (\lambda/\eta) + (1 + \sigma)^2. \]

Proof The first inequality follows from Jensen’s inequality. The second inequality follows from Lemma 17.

We also require a bound on the log-partition function.

Lemma 20 If \(2\gamma\sigma^2 < 1\), then for all \(t \leq T\):
\[ \mathbb{E} Z(S_t) \leq \lambda \epsilon T + 4\eta T \epsilon^2 + \kappa(\epsilon). \]

Proof For any probability distribution \(p\) on \(F\), we have
\[ \mathbb{E}_{S_t} Z(S_t) \leq \mathbb{E}_{f \sim p} \mathbb{E}_{S_t} \left( \sum_{s=1}^{t} -\lambda \Delta f(x^1_s) + \eta \sum_{s=1}^{t} \delta^{ht}_s(f, f) - \hat{Z}^{ht}_s(f) \right) + \mathbb{E}_{f \sim p} \ln \frac{p(f)}{p_0(f)} \]
\[ \leq \mathbb{E}_{f \sim p} \mathbb{E}_{S_t} \left( \sum_{s=1}^{t} -\lambda \Delta f(x^1_s) + \eta \sum_{s=1}^{t} \sum_{h=1}^{H} \delta^{h}(f, f) \right) + \mathbb{E}_{f \sim p} \ln \frac{p(f)}{p_0(f)}. \]

The first inequality used the fact that \(p(f | S_t)\) is the minimizer of the right hand side over \(p\). The second inequality used Lemma 18 and \(2\gamma\sigma^2 < 1\). Using Lemma 17 we further obtain
\[ \mathbb{E}_{S_t} Z(S_t) \leq \mathbb{E}_{f \sim p} \left[ \mathbb{E}_{S_t} \left( \sum_{s=1}^{t} -\lambda \Delta f(x^1_s) + \eta \sum_{s=1}^{t} \mathcal{E}^{h}(f, f, x_s, a_s)^2 \right) + \mathbb{E}_{f \sim p} \ln \frac{p(f)}{p_0(f)} \right]. \]

We now recall our definition of the set \(\mathcal{F}(f, \epsilon)\) for any \(f \in \mathcal{F}\) from Definition 4 as the set of all functions which capture the Bellman error of \(f\) up to an error \(\epsilon\). Then we have \(\forall f \in \mathcal{F}(Q*, \epsilon) = \prod_h \mathcal{F}(Q^h_*, \epsilon)\)
\[ |\Delta f(x^1_s)| \leq \epsilon, \quad \mathcal{E}^{h}(f, f, x_t, a_t) \leq 2\epsilon. \]

We can now take
\[ p(f) = \frac{p_0(f) I(f \in \mathcal{F}(Q*, \epsilon))}{p_0(\mathcal{F}(Q*, \epsilon))}. \]

It implies the desired bound.

We are now ready to prove Proposition 12.
Proof of Proposition 12

Proof Let us define $\alpha = 6\eta \sigma^2 < 1$ Lemma 18 implies that

$$\hat{Z}^h_t(f) \leq \hat{\delta}^h_t(f).$$

Therefore

$$\mathbb{E}[Z(S_{t-1}) - Z(S_t)] = \mathbb{E} \left[ \ln \mathbb{E}_{f|S_{t-1}} \exp \left( \lambda \Delta f(x_t^1) - \eta \hat{\delta}^h_t(f, f) - \hat{Z}^h_t(f) \right) \right]$$

$$\leq \mathbb{E} \left[ \ln \mathbb{E}_{f|S_{t-1}} \exp \left( \lambda \Delta f(x_t^1) - \eta \hat{\delta}^h_t(f, f) - \hat{\delta}^h_t(f) \right) \right]$$

$$\leq \frac{1}{3} \left[ \ln \mathbb{E}_{f|S_{t-1}} \exp \left( 3 \lambda \Delta f(x_t^1) \right) + \ln \mathbb{E}_{f|S_{t-1}} \exp \left( -3 \eta \hat{\delta}^h_t(f, f) \right) + \ln \mathbb{E}_{f|S_{t-1}} \exp \left( 3 \hat{\delta}^h_t(f) \right) \right],$$

where the last inequality follows from Jensen’s inequality to show that $\mathbb{E}[XY] \leq \sqrt{\mathbb{E}[X^2] \mathbb{E}[Y^2]}$ for non-negative random variables $X, Y, Z$. Now we further simplify each term by using the last two bounds in Lemma 17 by taking a conditional expectation with respect to $x_t^{h+1}, r_t^h$, conditioned on $x_t^h, a_t^h$ to obtain

$$\mathbb{E}[Z(S_{t-1}) - Z(S_t)]$$

$$\leq \frac{1}{3} \mathbb{E}_{f|S_{t-1}} \left( 3 \lambda \Delta f(x_t^1) + 9 \lambda^2 / 2 \right) + \frac{1}{3} \ln \mathbb{E}_{f|S_{t-1}} \exp \left( -3 \eta - 18 \eta^2 \sigma^2 \right)$$

$$+ \frac{1}{3} \ln \mathbb{E}_{f|S_{t-1}} \exp \left( 3 \lambda \Delta f(x_t^1) \right)$$

$$\leq \frac{1}{3} \left[ \mathbb{E}_{f|S_{t-1}} \left( 3 \lambda \Delta f(x_t^1) + 9 \lambda^2 \right) - \eta (1 - \alpha) e^{-3 \eta (1 - \alpha)} \mathbb{E}_{f|S_{t-1}} \mathcal{E}^h_t(f, f, x_t, a_t) \right]$$

$$+ \eta (1 + \alpha) e^{3 \eta (1 + \alpha)} \mathbb{E}_{f|S_{t-1}} \mathcal{E}_t(f, f, x_t, a_t).$$

The last inequality used $\alpha = 6\eta \sigma^2 < 1$ and

$$\ln \mathbb{E}_\xi \exp(-z(\xi)) \leq \mathbb{E}_\xi \exp(-z(\xi)) - 1 \leq - \exp(-\max_{\xi} z(\xi)) \mathbb{E}_\xi z(\xi),$$

again using $e^{-z} - 1 \leq -ze^{-z'}$ for $0 \leq z \leq z'$. Similarly, we also get $\ln \mathbb{E}_\xi \exp(z(\xi)) \leq \exp(\max_{\xi} z(\xi)) \mathbb{E}_\xi z(\xi)$ for $z(\xi) \geq 0$. By summing over $t = 1$ to $t = T$, and note that $Z(S_0) = 0$, we obtain the desired bound.

H.3 Simplified convergence analysis of the inner updates

In this section, we establish the following result, which is simpler to prove than Proposition 13.

Proposition 21 (Inner loop convergence) Suppose $\gamma < 1/2$ and $\lambda + \eta (1 + \sigma^2) \leq 0.1$. Then we have for all $\epsilon > 0$:

$$\frac{1}{2} \mathbb{E}_{f, g|S_{t-1}} \mathcal{E}^h_t(g, f; x_t^h, a_t^h)^2 \leq \epsilon T (\epsilon + 3)(6 + 10\lambda T + 90\eta T) + 25(\lambda + 9\eta) T$$

$$+ \frac{1}{3} \left[ \mathbb{E}_{f, g|S_{t-1}} \left( 3 \lambda \Delta f(x_t^1) + 9 \lambda^2 \right) - \eta (1 - \alpha) e^{-3 \eta (1 - \alpha)} \mathbb{E}_{f|S_{t-1}} \mathcal{E}^h_t(f, f, x_t, a_t) \right]$$

$$+ \eta (1 + \alpha) e^{3 \eta (1 + \alpha)} \mathbb{E}_{f|S_{t-1}} \mathcal{E}_t(f, f, x_t, a_t).$$
We first show how this immediately yields an $O(T^{-1/8})$ sample complexity of our algorithm, before proving the statement through a series of lemmas. Though we eventually supercede this analysis with a sharper one, several intermediate results will be reused and we believe that the simpler analysis of this proposition illustrates the main ideas of solving the nested minimax setup.

We now state a form of Theorem \ref{thm:main} using Proposition \ref{prop:potential}.

**Theorem 22** Under Assumptions \ref{assump:general} suppose we run TS$^3$ (Algorithm \ref{alg:ts3}) with some parameters $\gamma \leq 1/36$ and $\eta \leq 0.01$. Then choosing any $\epsilon \leq 0.6/T^2$, we have

$$
\mathbb{E} \sum_{t=1}^{T} \text{Regret}(f_t, x^*_t) = O \left( \frac{\eta}{\lambda} (1 + \lambda T + \eta T) (\kappa(\epsilon) + \kappa'(\epsilon)) + \lambda T + \frac{\kappa(\epsilon)}{\lambda} + \bar{c}(\lambda/\eta) T \right),
$$

where $\bar{c}(\lambda/\eta) = \inf_{\mu > 0} \left[ 8(\lambda/\eta)dc(\epsilon_2)H\mu^2 + 2\mu H\epsilon_2B_2 + \mu^{-1}br(\epsilon_1) + \epsilon_1 HB_1 \right]$. 

To get this result, we set $\eta < 0.01$ so that $(1 - 6\eta) \exp(-3(1 - 6\eta)) \geq 0.5$ along with the stated values of $\epsilon$ and $\gamma$. Plugging these into the bounds of Propositions \ref{prop:main} \ref{prop:potential} \ref{prop:second} \ref{prop:third} and \ref{prop:potential} and simplifying gives the result of Theorem \ref{thm:main}. Further assuming the conditions of Corollary \ref{cor:main} we can choose $\mu = (d_1\eta/(d_2\lambda H^2))^{1/3}$, $\eta = 1/\sqrt{T}$, $\gamma = 1/36$ and $\lambda = T^{-7/8}(d_1^2d_2H^2)^{-1/4}(\ln N)^{3/4}$ to get a sample complexity of $O(H\ln N^{1/4}(d_1^2d_2H)^{1/4}T^{-1/8})$.

We begin the proof of Proposition \ref{prop:potential} with a result that carries out a potential function analysis for the inner updates. We recall our definition of $\tilde{q}_t = p(g|f, S_t)$ in line 4 of Algorithm \ref{alg:ts3} which will be repeatedly used in this section.

**Lemma 23** Assume that $2\gamma \sigma^2 \leq 1$. Let

$$
\tilde{p}(g|f) = \frac{p_0(g)I(g \in \mathcal{F}(\epsilon, f))}{p_0(\mathcal{F}(\epsilon, f))}, \quad \text{and} \quad \hat{H}_t(f) = \mathbb{E}_{g \sim \tilde{p}(\cdot|f)} \ln \frac{\tilde{p}(g|f)}{p(g|f, S_t)}.
$$

Then for all $t$:

$$
\mathbb{E}_{S_t} \sup_f \hat{H}_t(f) \leq \ln \mathbb{E}_{S_t} \sup_f \exp(\tilde{H}_t(f)) \leq \kappa(\epsilon) + \kappa'(\epsilon) + 4\gamma(\epsilon + 1 + \sigma)t.
$$

**Proof** Let

$$
A_t(g, f) = \gamma \sum_{s=1}^{t} \delta_s^{h_s}(g, f).
$$

Then for each $g \in \mathcal{F}(\epsilon, f)$, we have:

$$
-2\gamma t \epsilon \sigma \leq A_t(g, f) \leq \gamma(\epsilon^2 + 2\epsilon \sigma)t.
$$

Let $f_1, \ldots, f_N$ be a cover of $\mathcal{F}$ so that for any $f \in \mathcal{F}$, $\exists j$ such that $|f_h(x) - f_j^h(x)| \leq \epsilon$ for all $x$. We know that $\ln N \leq \kappa'(\epsilon)$ (Definition \ref{defn:count}). This implies that for all $S_t$, $\exists j \in [N]$:

$$
-A_t(g, f) \leq -A_t(g, f_j) + \gamma(\epsilon^2 + 2\epsilon(1 + \sigma))t.
$$
It follows that with \( p_0(g) = \prod_{h=1}^{H} p_0^h(g^h) \), we obtain
\[
\ln E_{S_t} \sup_{f \in \mathcal{F}} \mathbb{E}_{g \sim p_0} \exp \left( -A_t^h(g, f) \right)
\]
\[
\leq \gamma \epsilon (\epsilon + 2 + 2\sigma) + \ln E_{S_t} \sup_{f \in \mathcal{F}} \exp \left( -\gamma \sum_{s=1}^{t} \hat{\delta}_s^h(g, f_j) \right)
\]
\[
\leq \gamma \epsilon (\epsilon + 2 + 2\sigma) + \ln \sum_{j} \mathbb{E}_{g \sim p_0} \mathbb{E}_{S_t} \exp \left( -\gamma \sum_{s=1}^{t} \hat{\delta}_s^h(g, f_j) \right)
\]
\[
\leq \gamma \epsilon (\epsilon + 2 + 2\sigma) + \ln \sum_{j} \mathbb{E}_{g \sim p_0} \mathbb{E}_{S_t} \exp \left( -\gamma (1 - 2\gamma \sigma^2) e^h(g, f_j, x_t^{h_t}, a_t^{h_t})^2 - \gamma \sum_{s=1}^{t-1} \hat{\delta}_s^h(g, f_j) \right)
\]
\[
\leq \gamma \epsilon (\epsilon + 2 + 2\sigma) + \ln \sum_{j} \mathbb{E}_{g \sim p_0} \mathbb{E}_{S_t} \exp \left( -\gamma \sum_{s=1}^{t-1} \hat{\delta}_s^h(g, f_j) \right)
\]
\[
\leq \cdots
\]
\[
\leq \gamma \epsilon (\epsilon + 2 + 2\sigma) + \ln N.
\]

The first inequality used covering property. Inequality (a) uses the last bound in Lemma \( 17 \). We thus have
\[
E_{S_t} \sup_{f \in \mathcal{F}} \hat{H}_t(f) \leq \ln E_{S_t} \sup_{f \in \mathcal{F}} \exp(\hat{H}_t(f))
\]
\[
= \ln E_{S_t} \sup_{f \in \mathcal{F}} \left( \mathbb{E}_{g \sim \tilde{p}(\cdot|f)} \ln \frac{\tilde{p}(g|f)}{p(g|f, S_t)} \right)
\]
\[
= \ln E_{S_t} \sup_{f \in \mathcal{F}} \left[ E_{g \sim \tilde{p}(\cdot|f)} \ln \frac{p_0(g)}{p_0(g) \exp(-A_t(g, f))} + \ln \frac{1}{p_0(F, f)} + \ln \mathbb{E}_{g \sim p_0} \exp (-A_t(g, f)) \right]
\]
\[
\leq \gamma (\epsilon^2 + 2\epsilon \sigma) t + \kappa(\epsilon) + 2\gamma \epsilon (\epsilon + 2 + 2\sigma) + \kappa'(\epsilon).
\]

The first inequality used Jensen’s inequality. This implies the result.

We give an upper bound on the log-partition function \( \hat{Z}_t^h(f) \). Note that this is the part of our analysis which relies on \( \eta \) being smaller than \( \gamma \), and leads to a loss in rates. It is possible to sharpen this analysis through a more careful self-bounding argument. The calculation will be more complex, and we leave the refined analysis to the next section.

**Lemma 24** Let
\[
\alpha'' = 2(\lambda/\eta + (1 + \sigma)^2) \exp(2\lambda + 2\eta(1 + \sigma)^2).
\]

Then
\[
E \sum_{t=1}^{T} \mathbb{E}_{f|S_{t-1}} \hat{Z}_t^h(f) \leq \epsilon T(\epsilon + 1 + \sigma)(6 + 4\eta\alpha''T) + \eta T(1 + \sigma)^2 \alpha'' + (\gamma^{-1} + \eta\alpha''T/\gamma)(\kappa(\epsilon) + \kappa'(\epsilon)).
\]

**Proof** We know that
\[
p(f|S_t) = p(f|S_{t-1}) \exp(\lambda \Delta f(x_t^h) - \eta[\tilde{\delta}_t^h(f, f) - \hat{Z}_t^h(f)]) - \tilde{Z}_t).
\]

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Let $p_t = p(f|S_t)$. We have
\[
\mathbb{E}_{f \sim p_t} \hat{H}_t(f) - \mathbb{E}_{f \sim p_{t-1}} \hat{H}_{t-1}(f) \\
= \mathbb{E}_{f \sim p_t} \mathbb{E}_{g \sim \tilde{p}(|f|)} \ln \frac{\tilde{p}(|f|)}{p(g|f, S_{t-1})} - \mathbb{E}_{f \sim p_{t-1}} \mathbb{E}_{g \sim \tilde{p}(|f|)} \ln \frac{p(g|f, S_t)}{p(g|f, S_{t-1})}
\]
Now we observe that
\[
p(g|f, S_t) = p(g|f, S_{t-1}) \exp(-\gamma (\hat{\delta}_t^h(g, f) - \hat{Z}_t^h(f))),
\]
which allows us to further rewrite
\[
\mathbb{E}_{f \sim p_t} \hat{H}_t(f) - \mathbb{E}_{f \sim p_{t-1}} \hat{H}_{t-1}(f) \\
= \mathbb{E}_{f \sim p_t} \left[ e^{\lambda \Delta f(x_t)} - \eta (\hat{\delta}_t^h(f, f) - \hat{Z}_t^h(f)) - 1 \right] \mathbb{E}_{g \sim \tilde{p}(|f|)} \ln \frac{\tilde{p}(|f|)}{p(g|f, S_{t-1})}
\]
\[
- \gamma \mathbb{E}_{f \sim p_t} \left[ e^{\lambda \Delta f(x_t)} - \eta (\hat{\delta}_t^h(f, f) - \hat{Z}_t^h(f)) - 1 \right] \mathbb{E}_{g \sim \tilde{p}(|f|)} [\hat{\delta}_t^h(g, f) + \hat{Z}_t^h(f)]
\]
\[
- \gamma \mathbb{E}_{f \sim p_{t-1}} \mathbb{E}_{g \sim \tilde{p}(|f|)} [\hat{\delta}_t^h(g, f) + \hat{Z}_t^h(f)]
\]
\[
\leq \eta \alpha'' \mathbb{E}_{f \sim p_{t-1}} \hat{H}_{t-1}(f) + \eta \gamma (1 + \sigma)^2 \alpha'' - \gamma \mathbb{E}_{f|S_{t-1}} \mathbb{E}_{g \sim \tilde{p}(|f|)} [\hat{\delta}_t^h(g, f) + \hat{Z}_t^h(f)].
\]
The last inequality used $|\hat{\delta}_t^h(g, f) + \hat{Z}_t^h(f)| \leq (1 + \sigma)^2$, along with our earlier inequality (8) and the observation that $\mathbb{E}_{g \sim \tilde{p}(|f|)} \ln \frac{\tilde{p}(|f|)}{p(g|f, S_{t-1})}$ is a KL divergence, and hence non-negative. By rearranging the terms, we obtain
\[
\mathbb{E}_{f \sim p_{t-1}} \hat{Z}_t^h(f) \leq \mathbb{E}_{f \sim p_{t-1}} \mathbb{E}_{g \sim \tilde{p}(|f|)} \hat{\delta}_t^h(g, f) + \eta (1 + \sigma)^2 \alpha'' + \frac{1}{\gamma} (1 + \eta \alpha'') \mathbb{E}_{f \sim p_{t-1}} \hat{H}_{t-1}(f) - \mathbb{E}_{f \sim p_t} \hat{H}_t(f).
\]
Note that for $g \in F$, we have $\hat{\delta}_t^h(g, f) \leq \epsilon^2 + 2\epsilon \sigma$. By summing over $t$, we obtain
\[
\mathbb{E} \sum_{t=1}^T \mathbb{E}_{f|S_{t-1}} \hat{Z}_t^h(f) \leq \epsilon (\epsilon + 2\sigma) T + \eta T (1 + \sigma)^2 \alpha'' + \frac{1}{\gamma} (1 + \eta \alpha'') T \mathbb{E} \sup_{t \leq T, f \in F} \hat{H}_t(f).
\]
We can now obtain the desired bound using Lemma [23].

We are now ready to prove Proposition [21].
Proof of Proposition 21

Proof

The proof essentially follows from Lemma 24. We have from Lemma 18

\[ \sum_{t=1}^{T} \mathbb{E} \mathbb{E}_{f,g \mid S_{t-1}} \mathcal{E}^{h_t}(g, f; x_t, a_t h_t)^2 \]

\[ \leq \alpha' \sum_{t=1}^{T} \mathbb{E} \mathbb{E}_{f,g \mid S_{t-1}} \hat{Z}^{h_t} \]

\[ \leq \alpha' \left[ eT(\epsilon + 1 + \sigma)(6 + 4\eta\alpha''T) + \eta T(1 + \sigma)^2 \alpha'' + (\gamma^{-1} + \eta\alpha''T/\gamma)(\kappa(\epsilon) + \kappa'(\epsilon)) \right] \]

\[ \leq 2 \left[ eT(\epsilon + 3)(6 + 10\lambda T + 90\eta T) + 25(\lambda + 9\eta T + (1/\gamma)(1 + 3\lambda T + 25\eta T)(\kappa(\epsilon) + \kappa'(\epsilon)) \right] \]

The last inequality used our assumptions on the various parameters, which imply that \( \alpha < 0.25, \alpha' < 2, \) and \( \alpha'' < 2.5((\lambda/\eta) + 9) \).

H.4 Proof of Proposition 13

In the following, we derive a refinement of Lemma 24, which allows us to prove Proposition 13. In order to avoid complex constant calculations, we will use the \( O(\cdot) \) notation that hides absolute constants. Here we take \( \sigma = O(1), \eta = O(1), \gamma = O(1), \) and \( \epsilon = O(1) \). Consequently, we also have that \( \alpha = O(1) \) and \( \alpha' = O(\gamma) = O(1) \). We also repeatedly use that for \( b = O(1), \exp(b) \leq 1 + \theta b \) for \( \theta \leq b \) by the intermediate value theorem, so that \( \exp(b) - 1 = O(b) \) when \( b = O(1) \). In particular, for the function \( \psi(z) \) defined at the start of this section, we have

\[ z^2 \psi(z) = e^z - z - 1 = O(z^2), \quad \text{when} \quad z = O(1). \]  

We have the following high probability bound for the entropy considered in Lemma 25.

Lemma 25 Under the Assumption of Lemma 23 for each \( t \), event \( A_t \), defined below, holds with probability at least \( 1 - 1/T^2 \) over \( S_t \):

\[ A_t = \left\{ \sup_{f \in \mathcal{F}} \hat{H}_t(f) \leq \kappa(\epsilon) + \kappa'(\epsilon) + 6\gamma \epsilon (\epsilon + 1 + \sigma)t + 2 \ln T \right\}. \]

Proof From Lemma 24 we obtain for each \( t \leq T \):

\[ \ln \mathbb{E}_{S_t} \sup_{f \in \mathcal{F}} \exp(\hat{H}_t(f)) \leq \kappa(\epsilon) + \kappa'(\epsilon) + 6\gamma \epsilon (\epsilon + 1 + \sigma)t. \]

Using Markov’s inequality, we obtain for each \( t \leq T \), with probability \( 1 - 1/T^2 \),

\[ \sup_{f} \exp(\hat{H}_t(f)) \leq \kappa(\epsilon) + \kappa'(\epsilon) + 6\gamma \epsilon (\epsilon + 1 + \sigma)t + 2 \ln T. \]

This leads to the bound.

We also have the following uniform bound for the entropy considered in Lemma 25.
Lemma 26  Under the Assumption of Lemma 23 for all \( t \leq T \) and \( f \):

\[
\hat{H}_t(f) = O(\kappa(\epsilon) + T).
\]

**Proof**  We note that in the proof of Lemma 23, we can simply bound

\[
|A_t(g, f)| = O(T).
\]

We thus have

\[
\hat{H}_t(f) = \mathbb{E}_{g \sim \tilde{p}(|f|)} \ln \frac{\tilde{p}(g|f)}{p(g|f, S_t)} + \ln \frac{1}{p_0(\mathcal{F}(\epsilon, f))} + \ln \mathbb{E}_{g \sim p_0} \exp(-A_t(g, f)) = \kappa(\epsilon) + O(T).
\]

This implies the result.

Lemma 27  We have

\[
\mathbb{E}_{x_t, a_t, r_t|h_t=h} |\hat{Z}_t^h(f)|^2 = O(\mathbb{E}_{x_t, a_t, r_t|h_t=h} \hat{Z}_t^h(f)).
\]

**Proof**  From Lemma 18 along with (9), we obtain

\[
|\hat{Z}_t^h(f)| = O(|\mathbb{E}_{g|f,S_t-1} \hat{\delta}_t^h(g, f)| + \gamma \mathbb{E}_{g|f,S_t-1} \hat{\delta}_t^h(g, f)^2) = O(\mathbb{E}_{g|f,S_t-1} |\hat{\delta}_t^h(g, f)|).
\]

(since \( \gamma = O(1) \) by assumption)

It follows that

\[
\mathbb{E}_{x_t, a_t, r_t|h_t=h} |\hat{Z}_t^h(f)|^2 = \mathbb{E}_{x_t, a_t, r_t|h_t=h} O(\mathbb{E}_{g|f,S_t-1} |\hat{\delta}_t^h(g, f)|^2)
\]

\[
= \mathbb{E}_{x_t, a_t, r_t|h_t=h} O(\mathbb{E}_{g|f,S_t-1} \mathcal{E}^h(g, f; x_t^h, a_t^h)^2) \quad \text{(Lemma 17)}
\]

\[
= \mathbb{E}_{x_t, a_t, r_t|h_t=h} O(\mathbb{E}_{g|f,S_t-1} \hat{Z}_t^h(f)). \quad \text{(Lemma 18)}
\]

This proves the desired result.

Lemma 28  We have

\[
\mathbb{E}_{x_t, a_t, r_t|h_t=h} |\hat{Z}_t^h(f)|^2 = \mathbb{E}_{x_t, a_t, r_t|h_t=h} \mathbb{E}_{f|S_t-1} O((\lambda/\eta)^2 |\Delta f(x_t^1)| + |\mathcal{E}^h(f, f; x_t^h, a_t^h)|^2 + \hat{Z}_t^h(f)).
\]
Here the second inequality follows from the intermediate value theorem, since
\[ c = 5\eta(1 + 10\eta^2)E_{g|f,S_{t-1}}\mathcal{E}^h(g, f, x^h_t, a^h_t)^2 = \mathcal{O}(1), \]
so that \[ e^c \leq 1 + \mathcal{O}(c). \] In the last inequality, we use \[ \alpha' = \mathcal{O}(1) \] in Lemma 18 since \[ \gamma = \mathcal{O}(1). \]
where the last inequality assumed that we choose $\eta$ small enough so that $O(\eta^2) \leq 5\eta$, and $E_{x,t,a_r,r_t} \hat{Z}_t^h(f') \geq 0$, which follows from Lemma 18.

**Lemma 30** For each $t \leq T$, we have

\[
E\ E_{f|S_{t-1}} \left[ e^{\lambda \Delta f(x_t^1)-\eta[(\hat{Z}_t^h(f,f)-\hat{Z}_{t+1}^h(f))] - 1} \right] \tilde{H}_{t-1}(f)
\]

\[
= \eta E\ E_{f|S_{t-1}} O \left( \frac{\lambda}{\eta} |\Delta f(x_t^1)| + |\hat{Z}_t^h(f) + \varepsilon_t^h(f,f,x_{t+1}^h,\alpha_t^h)|^2 \right) (\kappa(\epsilon) + \kappa'(\epsilon) + \ln T) + O \left( \frac{(\eta + \lambda)}{T} (\kappa(\epsilon) + 1) \right).
\]

**Proof** Let the indicator $I(A_t)$ denotes that the event of Lemma 25 holds. By using the short hand notation

\[
w_t(f) = E_{x_t,a_t,r_t} \frac{\lambda}{\eta} |\Delta f(x_t^1)| + |\Delta f'(x_t^1)|,
\]

we have

\[
E\ E_{f|S_{t-1}} \left[ e^{\lambda \Delta f(x_t^1)-\eta[(\hat{Z}_t^h(f,f)-\hat{Z}_{t+1}^h(f))] - 1} \right] \tilde{H}_{t-1}(f)
\]

\[
= E\ E_{f|S_{t-1}} \left[ E_{x_t,a_t,r_t} \left[ E_{f|S_{t-1}} e^{\lambda \Delta f(x_t^1)-\eta[(\hat{Z}_t^h(f,f)-\hat{Z}_{t+1}^h(f))] - 1} \right] \tilde{H}_{t-1}(f) \right]
\]

(Since $x_t, a_t, r_t$ are independent of $f$)

\[
= \eta E\ E_{f|S_{t-1}} O \left( w_t(f) + E_{x_t,a_t,r_t} \hat{Z}_t^h(f) + E_{x_t,a_t,r_t} e^{\varepsilon_t^h(f,f,x_{t+1}^h,\alpha_t^h)} \tilde{H}_{t-1}(f) \right)
\]

(Lemma 29)

\[
= \eta \left[ E\ E_{f|S_{t-1}} O \left( w_t(f) + E_{x_t,a_t,r_t} \hat{Z}_t^h(f) + E_{x_t,a_t,r_t} e^{\varepsilon_t^h(f,f,x_{t+1}^h,\alpha_t^h)} \tilde{H}_{t-1}(f) \right) \right] (1 - I(A_t))
\]

\[
+ \eta E\ E_{f|S_{t-1}} O \left( w_t(f) + E_{x_t,a_t,r_t} \hat{Z}_t^h(f) + E_{x_t,a_t,r_t} e^{\varepsilon_t^h(f,f,x_{t+1}^h,\alpha_t^h)} \tilde{H}_{t-1}(f) \right) I(A_t)
\]

(Lemma 26)

\[
= \eta O \left( E_{f|S_{t-1}} w_t(f) + E_{f|S_{t-1}} \hat{Z}_t^h(f) + E_{f|S_{t-1}} e^{\varepsilon_t^h(f,f,x_{t+1}^h,\alpha_t^h)} \tilde{H}_{t-1}(f) \right) O(\kappa(\epsilon) + \kappa'(\epsilon) + \ln T)
\]

(Definition of $A_t$ in Lemma 25 and $w_t(f) \geq 0, E_{x_t,a_t,r_t} \hat{Z}_t^h(f) \geq 0$)

\[
+ (\eta + \lambda) O(\kappa(\epsilon) + T) E(1 - I(A_t))
\]

(Probability of $A_t$ in Lemma 25)
This implies the desired bound by noticing that conditioned on \( S_{t-1}, f \) and \( x_t \) are independent.

**Lemma 31** We have

\[
\mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \left[ \left( e^{\lambda \Delta f(x_t^1)} - \eta (\hat{\delta}_t^{h_t}(f,f) - \hat{Z}_t^{h_t}(f)) - 1 \right) \mathbb{E}_{g \sim \tilde{p}(.|f)} [ - \hat{\delta}_t^{h_t}(g,f) + \hat{Z}_t^{h_t}(f)] \right] = \eta \mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \mathcal{O} \left( (\lambda/\eta)^2 |\Delta f(x_t^1)| + |\mathcal{E}^{h_t}(f,f;x_t^h,a_t^h)|^2 + \hat{Z}_t^{h_t}(f) \right) + (\lambda + \eta)\mathcal{O}(\epsilon).
\]

**Proof** We have

\[
|\lambda \Delta f(x_t^1) - \eta (\hat{\delta}_t^{h_t}(f,f) - \hat{Z}_t^{h_t}(f)) - 1| = \mathcal{O}(\lambda + \eta),
\]

and \( \forall g \in \mathcal{F}(f), \) we have \( \hat{\delta}_t^{h_t}(g,f) = \mathcal{O}(\epsilon) \). Therefore by the definition of \( \tilde{p}(.)|f) \) in Lemma 23 we have

\[
\left| e^{\lambda \Delta f(x_t^1)} - \eta (\hat{\delta}_t^{h_t}(f,f) - \hat{Z}_t^{h_t}(f)) - 1 \right| \mathbb{E}_{g \sim \tilde{p}(.)|f} \hat{\delta}_t^{h_t}(g,f) = (\lambda + \eta)\mathcal{O}(\epsilon).
\]

Moreover,

\[
\mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \left[ \left( e^{\lambda \Delta f(x_t^1)} - \eta (\hat{\delta}_t^{h_t}(f,f) - \hat{Z}_t^{h_t}(f)) - 1 \right) \mathbb{E}_{g \sim \tilde{p}(.)|f} [ - \hat{\delta}_t^{h_t}(g,f) + \hat{Z}_t^{h_t}(f)] \right]
\]

\[
= \eta \mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \mathcal{O} \left( (|\lambda/\eta| |\Delta f(x_t^1)| + |\hat{\delta}_t^{h_t}(f,f)| + |\hat{Z}_t^{h_t}(f)| + |\hat{Z}_t^{h_t}||\hat{Z}_t^{h_t}(f)|) \right)
\]

\[
= \eta \mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \mathcal{O} \left( (\lambda/\eta)^2 |\Delta f(x_t^1)|^2 + |\hat{\delta}_t^{h_t}(f,f)|^2 + |\hat{Z}_t^{h_t}(f)|^2 + |\hat{Z}_t^{h_t}|^2 \right) \quad \text{(Cauchy-Schwarz)}
\]

\[
= \eta \mathbb{E}_{x_t,a_t,r_t|h_t=h}\mathbb{E}_{f|S_{t-1}} \mathcal{O} \left( (\lambda/\eta)^2 |\Delta f(x_t^1)|^2 + |\mathcal{E}_t^{h_t}(f,f;x_t^h,a_t^h)|^2 + |\hat{Z}_t^{h_t}(f)|^2 \right) + (\lambda + \eta)\mathcal{O}(\epsilon)
\]

where the second to the last equation was obtained by taking conditional expectation conditioned on \((x_t^h,a_t^h)\) with respect to the transition and reward, followed by Lemma 17 The last equation used Lemma 27 and Lemma 28.

**Lemma 32** We have

\[
\lambda \mathbb{E}_{f|S_{t-1}} |\Delta f^1(x_t^1)| \leq \mathbb{E}_{f|S_{t-1}} \left[ \lambda \text{Regret}(f,x_t^1) + 0.5\eta \mathcal{E}_t^{h_t}(f,f;x_t^h,a_t^h)^2 \right] + \lambda \tilde{\epsilon}(\lambda/\eta),
\]

where \( \tilde{\epsilon}(\lambda/\eta) \) is defined in Proposition 11.
Proof The proof is similar to that of Proposition 11

\[ \lambda E_{f|S_{t-1}} |\Delta f^1(x_t^1)| \]
\[ \leq \lambda E_{f|S_{t-1}} \text{Regret}(f_t, x_t^1) + \lambda \sum_{h=1}^{H} E_{E(x,a) \sim \pi_f(x_t^1)} |\mathcal{E}_h^i(f_t, f_t, x_t^h, a_t^h)| \quad \text{(Proposition 10)} \]
\[ = \lambda E_{f|S_{t-1}} \text{Regret}(f_t, x_t^1) + \lambda HE |\mathcal{E}_h^i(f_t, f_t, x_t^h, a_t^h)| \quad (h_t \text{ is uniformly drawn from } [H]) \]
\[ \leq \lambda E_{f|S_{t-1}} \text{Regret}(f_t, x_t^1) + \lambda \left[ \epsilon_1 HB_1 + \mu_1^{-1} \text{br}(\epsilon_1) + H \mu_1 E_{f|S_{t-1}} \mathcal{E}_h^i(f, f, x_t^h, \pi_f(x_t^h))^2 \right] \quad \text{(Lemma 14)} \]
\[ \leq \lambda E_{f|S_{t-1}} \text{Regret}(f_t, x_t^1) \]
\[ + \lambda \left[ 2\mu_1 \mu_2 H E_{f|S_{t-1}} \mathcal{E}_h^i(f, f; x_t^h, a_t^h)^2 + 2\mu_1 \mu_2^{-1} \text{dc}(\epsilon_2) + 2\epsilon_2 \mu_1 HB_2 + \epsilon_1 HB_1 + \mu_1^{-1} \text{br}(\epsilon_1) \right]. \quad \text{(Lemma 15)} \]

Taking \( \mu_1 = \mu \) and \( \mu_2 = \eta/(4\lambda \mu_1 H) \) as in the proof of Proposition 11 we obtain the desired result. \( \blacksquare \)

We are now ready to prove Proposition 13.

Proof of Proposition 13

From the proof of Lemma 24 we obtain

\[ E[E_{f \sim p_t} \hat{H}_t(f) - E_{f \sim p_{t-1}} \hat{H}_{t-1}(f)] \]
\[ = E \left[ e^{\lambda \Delta f(x_t^1)} - \eta E_{\hat{f}_t^i(h_t^i, f)} - \dot{Z}_t^{h_t} - 1 \right] \hat{H}_{t-1}(f) \]
\[ - \gamma E \left[ e^{\lambda \Delta f(x_t^1)} - \eta E_{\hat{f}_t^i(h_t^i, f)} - \dot{Z}_t^{h_t} - 1 \right] E_{g \sim \bar{p}^i(f)}[\hat{f}_t^{h_t} - \bar{Z}_t^{h_t} + \hat{H}_t(f)] \]
\[ \leq \eta \epsilon_0 \left[ E_{f \sim p_{t-1}} O \left( \frac{\lambda}{\eta} |\Delta f(x_t^1)| + |\mathcal{E}_h^i(f, f; x_t^h, a_t^h)|^2 + \hat{Z}_t^{h_t}(f) \right) (\kappa(\epsilon) + \kappa'(\epsilon) + \ln T) + \frac{\eta}{T} \epsilon_0 (\kappa(\epsilon) + 1) \right] \quad \text{(Lemma 30)} \]
\[ + \gamma \eta \epsilon_0 \left[ E_{f \sim p_{t-1}} O \left( \frac{\lambda}{\eta} |\Delta f(x_t^1)| + |\mathcal{E}_h^i(f, f; x_t^h, a_t^h)|^2 + \hat{Z}_t^{h_t}(f) \right) \right] \quad \text{(Lemma 31)} \]
\[ - \lambda E \left[ f \sim p_{t-1} \hat{Z}_t^{h_t}(f) + O(\epsilon) \right] \quad (\forall g \in \mathcal{F}(\epsilon, f), \text{ we have } \hat{f}_t^{h_t}(g, f) = O(\epsilon)) \]
\[ \leq c_1 \gamma (\lambda/\eta) E \left[ |\Delta f(x_t^1)| + c_1 \gamma E_{f \sim p_{t-1}} O(|\mathcal{E}_h^i(f, f; x_t^h, a_t^h)|^2) - 0.5 \gamma E \left[ f \sim p_{t-1} \hat{Z}_t^{h_t}(f) \right. \right. \]
\[ + c_0 O(\epsilon + \eta(\kappa(\epsilon) + 1)/T). \]

The last inequality used the condition of the Lemma concerning the choice of \( \eta \). Now by summing over \( t = 1 \) to \( t = T \), and apply Lemma 23 to bound \( E \hat{H}_0(f) \), we obtain when \( c_0 \) is sufficiently large.

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\[
\sum_{t=1}^{T} \mathbb{E}_{f \sim p_{t-1}} \hat{Z}_{t}^{h}(f)
\]
\[
\leq c_0 \gamma O(\epsilon T + \kappa(\epsilon) + \kappa'(\epsilon) + 1) + 2 c_1 \gamma \sum_{t=1}^{T} \mathbb{E}_{f \sim p_{t-1}} O \left[ |\mathcal{E}_{h}^{*}(f, f; x_{t}^{h}, a_{t}^{h})|^2 \right] + (\lambda/\eta) \hat{\epsilon}(\lambda/\eta) O(T).
\]

The proof is now completed by recalling Lemma 18 with \(\alpha' < 6\).

## I Proof of Theorem 9

In this section, we provide a proof for Theorem 9. We focus on the differences from the proof of Theorem 5, which are limited to our decoupling analysis. In particular, in the setting of Theorem 9, we have the following improved analog for Lemma 15.

**Lemma 33 (Design based decoupling)** We have for all \(x^h \in \mathcal{X}^h\) and \(f \in \mathcal{F}\):

\[
\mathcal{E}^h(f, f ; x^h, \pi_f(x^h))^2 \leq \mathcal{E}^h(f, f , x^h, \rho^h(x^h))^2,
\]

where \(\rho^h(x^h)\) is the G-optimal design (6) at the state \(x^h\).

**Proof** Let \(\Sigma^*_h(x^h)\) be the covariance \(\mathbb{E}_{a^h \sim \rho^h(x^h)} [\phi^h(x^h, a^h) \phi^h(x^h, a^h)^\top]\). The definition of G-optimal design implies that

\[
\sup_{a^h \in \mathcal{A}} \|\phi^h(x^h, a^h)\|_{\Sigma^*_h(x^h)^{-1}} \leq d_2. \tag{10}
\]

By Assumption 4 we have

\[
\mathcal{E}^h(f, f ; x^h, \pi_f(x^h))^2 = \left\langle w^h(f, x^h), \phi^h(x^h, \cdot) \right\rangle^2 \leq \|w^h(f, x^h)\|_{\Sigma^*_h(x^h)^{-1}} \|\phi^h(x^h, a^h)\|_{\Sigma^*_h(x^h)^{-1}}^2 \leq d_2 \mathbb{E}_{a^h \sim \rho^h(x^h)} \left\langle w^h(f, x^h), \phi^h(x^h, \cdot) \right\rangle^2 = d_2 \mathcal{E}^h(f, f , x^h, \rho^h(x^h))^2.
\]

The first inequality used Cauchy-Schwarz. The second inequality used (10).

Combining this with the Bellman rank decoupling in Lemma 14 and Proposition 10, we get

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\[ \lambda \mathbb{E} \text{Regret}(f_t, x_t^1) + \lambda \mathbb{E} \mathbb{E}_{f_t \mid S_{t-1}} \Delta f^1(x_t^1) = \lambda \mathbb{E} \mathcal{E}^{h_t}(f_t, f_t, x_t^{h_t}, a_t^{h_t}) \leq \lambda \left[ \epsilon_1 H B_1 + \mu_1^{-1} \text{br}(\epsilon_1) + H \mu_1 \mathbb{E} \mathbb{E}_{f_t \mid S_{t-1}} \mathcal{E}^{h_t}(f, f, x_t^{h_t}, \pi_f(x_t^{h_t}))^2 \right] \leq \lambda \left[ \epsilon_1 H B_1 + \mu_1^{-1} \text{br}(\epsilon_1) + H \mu_1 d_2 \mathbb{E} \mathbb{E}_{f_t \mid S_{t-1}} \mathcal{E}^{h_t}(f, f, x_t^{h_t}, \rho^{h_t}(x_t^{h_t}))^2 \right] = \lambda \left[ \epsilon_1 H B_1 + \mu_1^{-1} \text{br}(\epsilon_1) + H \mu_1 d_2 \mathbb{E} \mathbb{E}_{f_t \mid S_{t-1}} \mathcal{E}^{h_t}(f, f, x_t^{h_t}, a_t^{h_t})^2 \right] \]

Further choosing \( \lambda H \mu_1 d_2 = 0.5 \eta \), we get the following analog of Proposition 11:

\[ \lambda \mathbb{E} \text{Regret}(f_t, x_t^1) \leq \mathbb{E} \mathbb{E}_{f_t \mid S_{t-1}} \left[ -\lambda \Delta f(x_t^1) + 0.5 \eta \mathcal{E}^{h_t}(f, f, x_t^{h_t}, a_t^{h_t})^2 \right] + \lambda \left( \epsilon_1 H B_1 + \frac{2 \lambda \text{br}(\epsilon_1) H_1 d_2}{\eta} \right). \]

Now we combine this result with Proposition 12 (the result still hold without modification, because the proofs did not rely on how \( a_t^{h_t} \) is generated given \( x_t^1 \)), and a slight modification of Proposition 13 with \( \tilde{e}(\lambda/\eta) \) replaced by \( \tilde{e}'(\lambda/\eta) \) (we just need to replace the decoupling used in Lemma 32 by the new decoupling result used here), leading to

\[ \sum_{t=1}^{T} \lambda \mathbb{E} \text{Regret}(f_t, x_t^1) \leq O \left( \lambda \epsilon T + 4 \eta T \epsilon^2 + \kappa(\epsilon) + 1.5 \lambda^2 T \right) + O(\epsilon T + \kappa(\epsilon) + \kappa'(\epsilon) + 1) + O \left( \lambda T \tilde{e}'(\lambda/\eta) \right). \]

This implies the result of Theorem 9.