Abstract

Covariant stochastic partial (pseudo)-differential equations are studied in any dimension. In particular a large class of covariant interacting local quantum fields obeying the Morchio-Strocchi system of axioms for indefinite quantum field theory is constructed by solving the analysed equations. The associated random cosurface models are discussed and some elementary properties of them are outlined.

1 Introduction

The strict connection of the Euclidean (bosonic) quantum field theory with (infinite-dimensional) Stochastic Analysis objects and concepts \cite{21, 22} is well known in different situations. Let us recall few of them.

Example 1 Scalar fields as solutions of the Ito-type SDEs

Let $B_t$ be a cylindric version of $\mathcal{S}'(\mathbb{R}^d)$-valued Brownian motion (where $\mathcal{S}'(\mathbb{R}^d)$ stands for the space of real tempered distributions) i.e. for any $f \in \mathcal{S}(\mathbb{R}^d)$ the coordinate
process $b_t^f \equiv (B_t, f)$ is a version of one-dimensional Brownian motion. The linear Itô equation
\[ d\xi_t^0 = \sqrt{1 - \Delta} \xi_t^0 dt + dB_t \tag{1} \]
has a stationary solution $\xi_t^0$ which has a law $L(\xi_0^0) = \mu_0^N$ easily identified with the free Nelson field, i.e. $\mu_0^N$ is a Gaussian probability measure on $(S'(R^{d+1}), \mathcal{B}(S'(R^{d+1}))) \equiv$ cylinder $\sigma$ -algebra) ($\equiv$ Gaussian generalized random field) with the mean equal to zero and covariance
\[ \mathbb{E} \xi_t^0(x) \xi_t^0(y) = \frac{e^{-|t\sqrt{1 - \Delta}|}}{2\sqrt{1 - \Delta}} (x - y) = \int_{S'(R^{d+1})} d\mu_0^N(\varphi) \varphi(0, x) \varphi(t, y). \tag{2} \]
In particular, the invariant measure $\nu_0^{(N)}$ of the $S'(R^d)$-valued Markov diffusion $\xi_t^0(x_0)$ (where $x_0 \in S'(R^d)$ is the initial condition) corresponding to $\xi_t^0$ is easily computable with the result that $\nu_0^{(N)}$ is centered Gaussian measure on $(S'(R^d), \mathcal{B}(S'(R^d)))$ with the following characteristic functional:
\[ \int_{S'(R^d)} d\rho_0^{(N)}(\varphi) e^{i(\varphi, f)} = e^{-\frac{1}{2} ||f||^2}; \tag{3} \]
where
\[ ||f||^2 = \int f(x)(1 - \Delta)^{-\frac{1}{2}}(x - y)f(y)dxdy. \]
For a various aspects of the process(es) $\xi_t^0$ (resp. $\xi_t^0(x_0)$) we refer to [7, 24, 1, 2].

Example 2 Scalar fields as invariant measures of the Itô-type SDEs.
Let us consider the following stochastic differential equations:
\[ dL_t^0 = (1 - \Delta)\xi_t^0 dt + (1 - \Delta)^{-\frac{1}{2}} dB_t \tag{4} \]
where $dB_t$ is the cylindrical $S'(R^{d+1})$-valued Brownian motion and the (regularizing) parameter $\epsilon \in (0, 1]$. By simple computations it follows that the invariant measure for all of the equations (4) is equal to the free Nelson field $\mu_0^{(N)}$. The law $L(L_t^1, 0) \equiv \mu_0^{(L)}$ of the stationary solution of (4), called the free Langevin field, is easily seen as centered Gaussian probability measure on $(S'(R^{d+1}), \mathcal{B}(S'(R^{d+1})))$ characterized by
\[ \int_{S'(R^{d+1})} d\mu_0^{(L)}(\varphi) e^{i(\varphi, f)} = e^{-\frac{1}{2} ||f||^2_{1, -2}}; \tag{5} \]
where
\[ ||f||^2_{1, -2} = \int f(s, x)(-\partial_x^2 + (1 - \Delta)^2)^{-1}(s - t, x - y)f(t, y)dsdxdtdy. \]
The corresponding to $d\mu_0^{(L)}$ generalized random field is (sharp)-Markov in the computer time direction $t$ and (germ)-Markov in the other directions.

The study of the equations (4) is a part of the so called stochastic quantisation programme [20, 17, 13, 23, 27, 28].
Example 3 Free quantum fields as stochastic integrals.

3.A Let $\eta$ be a Gaussian white noise on the space $S'(\mathbb{R}^d)$, i.e. $\eta$ is distributed according to the Gaussian measure $d\mu^{GN}$ characterized by:

$$
\int_{S'(\mathbb{R}^d)} d\mu^{GN}(\eta) e^{i(\eta,f)} = e^{-\frac{1}{2}|f|^2}.
$$

For $\lambda \in (0, \frac{1}{2}]$, let us consider the following partial (pseudo)-differential stochastic equation

$$(1-\Delta)^{\lambda} \varphi_{\lambda} = \eta.$$

The solution of (7), given by the stochastic integral

$$\varphi_{\lambda} = (1-\Delta)^{-\lambda} * \eta$$

is easily recognized as a generalized free (Euclidean) quantum field with the two-point Schwinger function

$$E_{\varphi_{\lambda}}(x)\varphi_{\lambda}(y) = (1-\Delta)^{-2\lambda}(x-y).$$

In particular, for $\lambda = \frac{1}{2}$, $\varphi_{\lambda}$ is identical to the free Nelson field.

3.B Now, let $\eta$ be a $S'(\mathbb{R}^d) \otimes \mathbb{R}^N$-valued Gaussian white noise and let $\tau$ be some real (orthogonal) representation of rotation group $(S)O(d)$ in the space $\mathbb{R}^N$ and let $D$ be $\tau$-covariant differential operator. Providing that $D$ is such that corresponding Green function $D^{-1}$ can be properly defined, it follows that the stochastic integral $A \equiv D^{-1} * \eta$ gives the solution of the following covariant stochastic differential equation

$$\tilde{D}A = \eta.$$

where $\tilde{D}$ is the adjoint of $D$ in the canonical pairing $S' \langle \cdot, \cdot \rangle_S$. In particular, taking $d = 3$, $\tau = D_1 \oplus D_1$ and

$$
D = \begin{pmatrix}
m & 0 & 0 & 0 & b\partial_z & -b\partial_y \\
0 & m & 0 & -b\partial_z & 0 & b\partial_x \\
0 & 0 & m & b\partial_y & -b\partial_x & 0 \\
0 & c\partial_z & -c\partial_y & m & 0 & 0 \\
-c\partial_z & 0 & c\partial_y & 0 & m & 0 \\
c\partial_y & -c\partial_x & 0 & 0 & 0 & m
\end{pmatrix}
$$

with $b^2 = c^2 = 1$, and $bc = -1$ it follows that the stochastic integral $D^{-1} * \eta$ for $\eta$ being pure Gaussian white noise gives two independent copies of two real massive (with the mass $m$) Euclidean Proca fields. For a systematic approach to such constructions see [12] and for a particular application to the free EM$_4$ fields [8, 4, 5].

The particular features of the above listed examples are: the linearity of the corresponding equations and Gaussianity of the corresponding noise. These features lead to the Gaussian solutions, therefore not very interesting from the point of view of physics. The interesting physics seems to be described by non-Gaussian examples.
In order to get them two different approaches were introduced. The first approach is to perturb the (linear) drifts by adding some nonlinear perturbation. However the main difficulty here is that the typical realisations (sample paths) of the underlying solutions are generically distributions (not functions!). The second approach to the problem of constructing non-Gaussian examples is to change the Gaussian noise into some tractable non-Gaussian noise. The simplest possibility is to (perturb/exchange) the Gaussian white-noise by Poisson noise.

Example 1 (continuation).
It is well known [7, 10, 27, 28] that at least for \( d = 1 \) there exist nonlinear measurable perturbations of the linear drift in (\( \mathbb{I} \)) which lead to the stationary solutions of the corresponding Itô SDE being identical (on the level of laws) to the interacting models of scalar fields constructed in the so called Constructive Quantum Field Theory ([26, 13] and references therein).

However the problem is that the explicit form of the corresponding perturbations is not known. Nevertheless this shows that there exist nonlinear Itô SDEs on \( S'(\mathbb{R}^d) \) that lead to nontrivial quantum fields obeying all Wightman axioms. The challenging problem to describe such possibilities in an explicit (or constructive) form is still open.

Example 2 (continuation).
At least for \( d = 2 \) the nonlinear perturbations of \( \mathbb{I} \) of gradient type (but see also [16] for nongradient type case) were studied intensively [20, 17, 15, 23, 27, 28]. The typical, two dimensional quantum field theory models like \( P(\Phi)^2 \) are again obtained as stationary distributions of the underlying Markov diffusions. A new approach to these equations, based on the idea of the "ground state transformation" together with the methods of the Constructive Quantum Field Theory [13, 26] was recently invented in [3].

After preparing this report we get a copy of [1] in which some of the results presented here are also obtained.

In the present exposition we shall focus our attention on a recent progress connected to the perturbation of the noise in the part 3B of Examples 3.

2 Interacting local covariant quantum fields from Covariant SPDEs

Let \( d\mu^{(P, \tau)} \) be a regular Poisson \( \tau \)-covariant noise on the space \( S'(\mathbb{R}^d) \otimes \mathbb{R}^{dim\tau} \), where \( \tau \) is some real representation of the rotation group \( SO(d), d \geq 2 \). The noise \( d\mu^{(P, \tau)} \) is characterized by

\[
\int_{S'(\mathbb{R}^d) \otimes \mathbb{R}^{dim\tau}} e^{i(\varphi, f)} d\mu^{(P, \tau)}(\varphi) = e^{\int dx \int d\lambda(\alpha)(e^{i<\alpha, f(x)>} - 1)}
\]
where we assume that the so-called Levy measure \( d\lambda \) on \( \mathbb{R}^{\dim \tau} \) is such that

(i) \( d\lambda \) has all moments

(ii) \( d\lambda \) is \( \tau \)-invariant

Let \( \tau' \) be an another real representation of the group \( SO(d) \) in \( \mathbb{R}^{\dim \tau'} \). Here, for simplicity we assume that \( \dim \tau = \dim \tau' \) referring to a more general case \( \dim \tau \neq \dim \tau' \) to our (forthcoming) paper \([14]\). The action \( \tau \) (resp. \( \tau' \)) of the group \( SO(d) \) can be naturally lifted to the action \( T_{\tau} \) (resp. \( T_{\tau'} \)) of \( SO(d) \) in the space \( \mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau} \).

Recall that a first order (for simplicity again) differential operator \( D = \sum_{i=1}^{d} B_i \partial_i + M, \) where \( B_i, M \in \text{Hom}(\mathbb{R}^{\dim \tau}, \mathbb{R}^{\dim \tau'}) \) is called \( (\tau, \tau') \)-covariant iff the following diagram

\[
\begin{array}{c}
\mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau} \\
\downarrow T_{\tau}
\end{array}
\begin{array}{c}
\mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau'} \\
\downarrow T_{\tau'}
\end{array}
\begin{array}{c}
\mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau'} \\
\downarrow \mathcal{D}
\end{array}
\begin{array}{c}
\mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau'}
\end{array}
\tag{13}
\]

do commute.

Let us denote by \( \text{Cov}(\tau, \tau') \) the set of all such \( (\tau, \tau') \)-covariant differential operators. The complete description of the sets \( \text{Cov}(\tau, \tau') \) is given in the paper \([14]\). From the definition of \( \mathcal{D} \in \text{Cov}(\tau, \tau') \) it follows that the symbol \( \sigma_{\mathcal{D}} \) of \( \mathcal{D} \) defined as \( \sigma_{\mathcal{D}}(p) \equiv i \sum_{j=1}^{d} B_j p_j \) has the property:

\[
\det(\sigma_{\mathcal{D}}(p) + m1) = c \prod_{k=1}^{n} (p_1^2 + ... p_d^2 + m_k^2)
\]

where \( m_k \in \mathbb{C}, k = 1, ..., n, n \leq N/2, c \) is a complex number. If all \( m_k \) are real and \( c \neq 0 \) the operator \( \mathcal{D} \) is invertible on suitably chosen function space and in this case we shall call it \textit{admissible}. If additionally all \( m_k \neq 0 \), operator \( \mathcal{D} \) is said to have a strictly positive mass spectrum.

We shall consider SPDEs of the type:

\[
\mathcal{D} \varphi = \eta
\]

where: \( \eta \) is given (regular) \( \tau \)-covariant noise on \( \mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau} \), \( \mathcal{D} \in \text{Cov}(\tau, \tau') \) is such that the Green function \( \mathcal{D}^{-1} \) of \( \mathcal{D} \) can be defined as a continuous imbedding of some nuclear space \( \mathcal{F} \otimes \mathbb{R}^{\dim \tau'} \) into the space \( \mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{\dim \tau} \) (such operators are called regular).

A generalized random field \( \varphi \) indexed by \( \mathcal{F} \otimes \mathbb{R}^{\dim \tau'} \) is called weak solution of \([14]\) iff \( (\varphi, f) \cong (\eta, \mathcal{D}^{-1} f) \) for all \( f \in \mathcal{F} \otimes \mathbb{R}^{\dim \tau'} \) where \( \cong \) means the equality inlaw. Let \( \Gamma_\eta \) denote the characteristic functional of the field \( \eta \), then the characteristic functional \( \Gamma_\varphi \) of the weak solution of \([14]\) is given by: \( \Gamma_\varphi = \Gamma_\eta(\mathcal{D}^{-1} f) \). In particular, if \( \eta \) is a \( \tau \)-covariant regular white noise then the characteristic functional of \( \varphi \) is given by:

\[
\Gamma_\varphi(f) = e^{-\frac{i}{2} \langle \mathcal{D}^{-1} f | \mathcal{D}^{-1} f \rangle} e^{\int dx \int d\lambda(\alpha)(\text{exp}(\alpha \cdot \mathcal{D}^{-1} f(x))) - 1},
\]

(15)

From the assumption that \( d\lambda \) has all moments it follows that all Schwinger functions of the field \( \varphi \) do exist.
The main observation is the following:

**Theorem 1 (Existence of Wightman functions)**

Let $\eta$ be a regular, $\tau$-covariant white noise and let $\mathcal{D} \in \text{Cov}(\tau, \tau')$ has an admissible mass spectrum. By $\ctau'$ we denote the analytically continued real representation $\tau'$ to the corresponding (real) representation of the special orthochronous Lorentz group. Then, there exists a system of tempered distributions $\mathcal{W}_{\tau'}_n$ which is: local, covariant (with respect to $\ctau'$), spectral and such that restrictions of the moments of the field $\varphi$ being a weak solution of $\mathcal{D}\varphi = \eta$ to the set $x^0_1 < ... < x^0_{n+1}$ are equal to the Laplace-Fourier transformations of certain linear combinations $\mathcal{W}_{\tau'}_n$ of $\mathcal{W}_{\ctau'}_n$, i.e.: for $x^0_1 < ... < x^0_{n+1}$

\[
\mathbf{E}\varphi(x^0_1, x_1) \cdots \varphi(x^0_{n+1}, x_{n+1}) = \\
\int e^{-\sum_{j=1}^{n} p_j^0 (x^0_{j+1} - x^0_j)} e^{i\sum_{j=1}^{n} p_j (x_{j+1} - x_j)} \mathcal{W}_{\tau'}_n(p_1, ..., p_n) \otimes_{j=1}^{n} dp_j
\]

**Proof:** See [12], [6].

The very interesting question whether there exist non-Gaussian examples of such equations which lead to reflection positive solutions is still unsolved (although there are strong negative indications). However, in the context of nonpositive quantum field theory axiomatized by Morchio-Strocchi in [19] the constructed models may lead to interesting new examples of interacting quantum fields with nontrivial scattering matrices.

**Theorem 2 (Hilbert Space Structure Condition)**

Assume that $\eta$ as in Theorem 1, $\mathcal{D} \in \text{Cov}(\tau, \tau')$ with the admissible mass spectrum $\{m_1, ..., m_n\}$ such that $m_l \neq m_j$ for $l \neq j$. Then there exists a sequence $\{|| \cdot ||_n\}$ of Hilbert norms on $\mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{dim\ctau'}$ which are continuous in the Schwartz topology and such that

\[
||\mathcal{W}_{m+n}(f_m \otimes g_n)|| \leq ||f||_m ||g||_n
\]

for all $f \in \mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{dim\tau'}$, $g \in \mathcal{S}'(\mathbb{R}^d) \otimes \mathbb{R}^{dim\ctau'}$ and where $f^*$ is an appropriate conjugation of $f$.

**Proof:** It follows by elaborating on the explicit form of the corresponding Wightman functions $\{\mathcal{W}_{\tau'}_n\}$ as obtained in [12]. For details we refer to [14].

The important consequence of this Theorem is that the corresponding GNS inner-product space obtained from $\{\mathcal{W}_{\tau'}_n\}$ has a natural structure of a Krein space. This means that the underlying infrared singularities are not so bad; see [19]; see also [4, 5] for a related models.

**Remark**

The assumption $m_l \neq m_j$ for $l \neq j$ in the Theorem 2 is not necessary. In general, in the presence of the Poisson part in the noise one can impose an algebraic condition (see [14]) (on the covariant operator $\mathcal{D}$) which is sufficient to get the HSSC.
3 Random cosurfaces

Let \( C_k \) denote the set of \( C^1 \)-piecewise cocycles in \( \mathbb{R}^4 \), i.e., elements of \( C_k \) are \( k \)-dimensional (\( k = 1, 2, 3 \)) \( C^1 \)-piecewise boundaryless compact submanifolds of \( \mathbb{R}^4 \). Let \( D \in \text{Cov}(\tau, \tau') \), where we assume that \( \tau' \) contains for a fixed \( k = 1, 2, 3 \) at least one subrepresentation \( \tau(k) \subset \tau' \) which is: for \( k = 1 \) of vector type (\( \tau(1) \simeq (0, 2) \)), for \( k = 2 \) of skew-symmetric tensor type, i.e. \( \tau(2) \simeq (-1, 2) \oplus (1, 2) \) and \( \tau(3) \) is of skew-symmetric tensor type i.e. \( \tau(3) \simeq (0, 2) \).

For regular \( D \) as above we consider the equation 14 and let \( A(k) \) be a part of the multiplet \( \varphi \) transforming covariantly under the subrepresentation \( \tau(k) \). Then, for a fixed collection \( \Gamma_1, \ldots, \Gamma_n \in C_k \) we would like to give a mathematical meaning to the following random map:

\[
S'(\mathbb{R}^4) \otimes \mathbb{R}^{\text{dim} \tau(k)} \ni A(k) \mapsto e^{iA(k)(\Gamma_1)} \cdots e^{iA(k)(\Gamma_n)}
\]

where

\[
A(k)(\Gamma_j) = \oint_{\Gamma_j} A(k)
\]

where the right-hand side of (17) is understood in the sense of differential forms calculus. The map (17) is called random cosurface connected to the field \( A(k) \).

**Proposition 1**

Let \( (\tau', \tau) \) be such that \( \tau' \supset \tau(k) \) for some some \( k \in \{1, 2, 3\} \), \( \eta \) is \( \tau \)-covariant regular pure Poisson noise; \( D \in \text{Cov}(\tau, \tau') \) is regular and such that

\[
D^{-1}(x) \sim \frac{1}{|x|^{4+\delta}} \quad \text{for} \quad |x| \to +\infty
\]

with \( \delta > 0 \). Fix a collection \( \{\Gamma_1, \ldots, \Gamma_n\} \subset C_k \). Then for almost every realisation of the field \( A(k) \in S'(\mathbb{R}^d) \otimes \mathbb{R}^{\text{dim} \tau(k)} \) the random cosurface map \( \prod_{p=1}^n e^{iA(k)(\Gamma_p)} \) is well defined and moreover the following a.s. version of the Stokes Theorem is valid:

\[
A(k)(\Gamma) = dA(k)(\delta \Gamma)
\]

where \( \delta \Gamma \) is the coboundary of \( \Gamma \) and the equality (19) holds for almost every realisation of \( A(k) \).

The proof of this Proposition follows straightforwardly from the following two technical lemmas (the proofs of which are contained in [14]). To formulate them let us recall that the set

\[
\{A(k) = \sum_j \alpha_j D^{-1}|_{(k)}(x - x_j) \mid \{x_j\} \text{is locally finite subset of } \mathbb{R}^d; \alpha \in \text{supp}d\lambda\}
\]

is of measure 1.
Lemma 1
Let $\Sigma \subset \mathbb{R}^4$ be of Lebesgue measure zero and let $A^{(k)}$, $\{\Gamma_1, ..., \Gamma_n\}$ be as in Theorem 1. Then the set

$$\{ A^{(k)} = \sum_j \alpha_j D^{-1}|_{(k)}(x - x_j) \mid \{x_j\} \cap (\Gamma_1 \cup ... \cup \Gamma_n) \neq \emptyset\}$$

is of measure zero.

Lemma 2
Let $\{\Gamma_1, ..., \Gamma_n\}; \eta; (\tau, \tau')$ be as in Proposition 1; $D \in \text{Cov}(\tau, \tau')$ be such as in (18). Then

$$\Pr\{ A^{(k)} = \sum_j \alpha_j D^{-1}(x - x_j) \mid \lim_{n \to \infty} \sum_{|x_j| \leq n} |\alpha_j||D^{-1}|_{(k)}(x - x_j)| > 0\} = 0$$

In particular it follows that for any fixed configuration $\{\Gamma_1, ..., \Gamma_n\} \subset C_k$ the function (called $k$-cocycles Schwinger function)

$$S(\Gamma_1, ..., \Gamma_n) \equiv E e^{iA^{(k)}(\Gamma_1)} ... e^{iA^{(k)}(\Gamma_n)}$$

is well defined. For $k = 1$, the corresponding 1-cocycles Schwinger function are known as Wilson loops (Schwinger) functions and as is well known they play an important role in different physical theories, see i.e. [25, 11]. However, the almost sure results presented here are not very satisfactory due to the problem of exceptional sets. To provide a computable approach to $k$-cocycles Schwinger functions $L^p$-version of the cosurface map [18] should be given. We illustrate this in the case of random loop variables.

Let $\eta \in C^\infty_0(\mathbb{R}^4)$ be such that: $\eta \geq 0$, supp $\eta \subset [-1, 1]^4$ and $f \eta(x) dx = 1$ and let $\eta'(x) \equiv \epsilon^{-4} \eta(\epsilon^{-1} x)$. For a given loop $\Gamma \in C_1$ we define a family of test functions $\rho_{\Gamma,k}(x) = \int_{\Gamma} \eta'(x - z) dz^k$. Then we define (the regularized) random loop variable:

$$\mathcal{L}(\Gamma) = e^{i(A(\Gamma), \rho_{\Gamma})}$$

Theorem 3
Let: $k = 1; \eta$ be a regular $\tau'$-covariant Poisson noise as above. Let $\tau'$ be such that $\tau' \supset \tau(1)$ and let a regular $D \in \text{Cov}(\tau, \tau')$ be given. Assume that:

1. $|D_{kl}^{-1}(x)| \leq \frac{c_{kl}}{|x|^k + \epsilon_{kl}^2}$ for $c_{kl} > 0$, $\epsilon_{kl} > 0$
2. $\int d\lambda(\alpha)(e^{\lambda x, y} - 1) \leq c|y|^{1+\eta}$ for $|y| \to 0$, $\eta \in (\frac{1}{3+\min(\epsilon_{kl})}, 1]$.
3. $\int d\lambda(\alpha)(e^{\lambda x, y} - 1) \leq c|y|^{1+\eta}$ for $|y| \to +\infty$, $\eta \in (-1, \frac{\max(\epsilon_{kl})}{3+\max(\epsilon_{kl})})$.

Then for any collection $\{\Gamma_1, ..., \Gamma_n\}$ of loops and $p \in [1, +\infty)$ there exists (in the norm $L^p(d\mu_A)$sense) limit $\lim_{\epsilon \to 0^+} \prod_{i=1}^n \mathcal{L}(\Gamma_i)$ and then:

$$S(\Gamma_1, ..., \Gamma_n) \equiv \lim_{\epsilon \to 0^+} E \prod_{i=1}^n \mathcal{L}(\Gamma_i) =$$
exp \int dx \int d\lambda(\alpha) \{ \exp(i \sum_{l=1}^{n} \oint_{\Gamma_l} < \alpha, D^{-1}(x - \cdot) > ) - 1 \}

Moreover, the functionals \( S(\Gamma_1, ..., \Gamma_n) \) obey a system of axioms as proposed in [25] with the exception of reflection positivity.

Remark
In the context of reflection positive Wilson loops functions suitable technique for the reconstruction of quantum mechanical dynamics in the real (Minkowski) time out of them was presented in [25]. The interesting problems here are to find a convenient substitutes of Laplace-Fourier property and HSSC in the context of not-necessarily reflection positive k- cocycles Schwinger functions that enables us to describe the corresponding real time dynamics.

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