LIMIT-POINT CRITERIA FOR THE MATRIX STURM-LIOUVILLE OPERATOR AND ITS POWERS

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Abstract. We consider matrix Sturm-Liouville operators generated by the formal expression
\[ l[y] = -(P(y' - Ry))' - R^* P(y' - Ry) + Qy, \]
in the space \( L^2_n(I), I := [0, \infty) \). Let the matrix functions \( P := P(x), Q := Q(x) \) and \( R := R(x) \) of order \( n \) \( (n \in \mathbb{N}) \) be defined on \( I \), \( P \) is a nondegenerate matrix, \( P \) and \( Q \) are Hermitian matrices for \( x \in I \) and the entries of the matrix functions \( P^{-1}, Q \) and \( R \) are measurable on \( I \) and integrable on each of its closed finite subintervals. The main purpose of this paper is to find conditions on the matrices \( P, Q \) and \( R \) that ensure the realization of the limit-point case for the minimal closed symmetric operator generated by \( l^k[y] \) \( (k \in \mathbb{N}) \). In particular, we obtain limit-point conditions for Sturm-Liouville operators with matrix-valued distributional coefficients.

Keywords: quasi-derivative, quasi-differential operator, matrix Sturm-Liouville operator, deficiency numbers, distributions.

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1. PRELIMINARIES

Let \( I := [0, +\infty) \) and let the complex-valued matrix functions \( P := P(x), Q := Q(x) \) and \( R := R(x) \) of order \( n \) \( (n \in \mathbb{N}) \) be defined on \( I \). Suppose that \( P \) is a nondegenerate matrix, \( P \) and \( Q \) are Hermitian matrices for \( x \in I \) and the entries of the matrix functions \( P^{-1}, Q \) and \( R \) are measurable on \( I \) and integrable on each of its closed finite subintervals (i.e. belong to the space \( L^1_{loc}(I) \)).

1.1. Let us consider the block matrix
\[ F = \begin{pmatrix} R & P^{-1} \\ Q & -R^* \end{pmatrix}, \]
where * is the conjugation symbol. Let $AC_{n,\text{loc}}(I)$ be the space of complex-valued $n$-vector functions $y(x) = (y_1(x), y_2(x), \ldots, y_n(x))'$, $t$ is the transposition symbol, with locally absolutely continuous entries on $I$. Using matrix $F$, we define quasi-derivatives $y^{[i]}$ ($i = 0, 1, 2$) of a given vector function $y \in AC_{n,\text{loc}}(I)$ by setting
\[
y^{[0]} := y, \quad y^{[1]} := P(y' - R y), \quad y^{[2]} := (y^{[1]})' + R^* y^{[1]} - Qy,
\]
provided that $y^{[1]} \in AC_{n,\text{loc}}(I)$ and a quasi-differential expression
\[
l[y](x) := -y^{[2]}(x), \quad x \in I.
\]
Thus,
\[
l[y] = -(P(y' - R y))' - R^* P(y' - R y) + Qy. \tag{1.2}
\]
The set of complex-valued vector functions $D := \{y(x) \mid y(x), y^{[1]}(x) \in AC_{n,\text{loc}}(I)\}$ is the domain of expression (1.2). For $y \in D$ the expression $l[y]$ exists a.e. on $I$ and locally integrable there.

We note here that for every pair of vector functions $f, g \in D$ and for every pair of numbers $\alpha$ and $\beta$ such that $0 \leq \alpha \leq \beta < \infty$ the following vector analogue of Green’s formula holds:
\[
\int_{\alpha}^{\beta} \{\langle l[f](x), g(x) \rangle - \langle f(x), l[g](x) \rangle\} \, dx = \langle f, g \rangle(\beta) - \langle f, g \rangle(\alpha), \tag{1.3}
\]
where $\langle u, v \rangle = v^* u = \sum_{s=1}^{n} u_s \overline{v_s}$ is the inner product of vectors $u$ and $v$ and the form $[f, g](x)$ is defined by
\[
[f, g](x) := \langle f(x), g^{[1]}(x) \rangle - \langle f^{[1]}(x), g(x) \rangle. \tag{1.4}
\]

Let $L^2_n(I)$ be the Hilbert space of equivalence classes of all complex-valued $n$-vector functions Lebesgue measurable on $I$ for which the sum of the squared absolute values of coordinates is Lebesgue integrable on $I$.

Let $D^*_0$ denote the set of all complex-valued vector functions $y \in D$ which vanish outside of a compact subinterval of the interior of $I$ (this subinterval may be different for different functions) and such that $l[y] \in L^2_n(I)$. This set is dense in $L^2_n(I)$. By formula $L^*_0 y = l[y]$ the expression $l$ on the set $D^*_0$ defines a symmetric (not necessary closed) operator in $L^2_n(I)$. Let $L_0$ and $D_0$ denote the closure of this operator and its domain, respectively. The operator $L_0$ and operators associated with it are called matrix Sturm-Liouville operators.

Suppose further that $\lambda \in \mathbb{C}$ and $\Im \lambda \neq 0$, $\Re \lambda$ is the imaginary part of the complex number $\lambda$. Denote by $R_\lambda$ and $R_{\overline{\lambda}}$ the ranges of $L_0 - \lambda I_n$ and $L_0 - \overline{\lambda} I_n$, $I_n$ is the $n \times n$ identity matrix, respectively, and by $N_\lambda$ and $N_{\overline{\lambda}}$ the orthogonal complements in $L^2_n(I)$ of $R_\lambda$ and $R_{\overline{\lambda}}$. The spaces $N_\lambda$ and $N_{\overline{\lambda}}$ are called deficiency spaces. The numbers $n_+$ and $n_-$ ($n_+ = \dim N_\lambda$, $n_- = \dim N_{\overline{\lambda}}$) are deficiency numbers of the operator $L_0$ in the upper-half or lower-half of the complex plane, respectively, moreover, the pair $(n_+, n_-)$ is called the deficiency index of $L_0$. 
As it was done, for example, in [1] and [18], it is possible to show that the deficiency numbers \( n_+ \) and \( n_- \) coincide with the maximum number of linearly independent solutions of the equation
\[
[l[y]] = \lambda y
\]
belonging to the space \( L^2_\infty(I) \), when \( \exists \lambda > 0 \) and \( \exists \lambda < 0 \), respectively. They also satisfy the double inequality
\[
n \leq n_+, n_- \leq 2n
\]
(1.5)
and, in addition, \( n_+ = 2n \) if and only if \( n_- = 2n \). Using the analogy of the spectral theory of scalar Sturm-Liouville operators on the half-axis, one may say that the expression \( l[y] \) (the operator \( L_0 \)) is in the limit-point case if \( n_+ = n_- = n \) or in the limit-circle case if \( n_+ = n_- = 2n \), (see, for example, [1]).

Let us consider the equation
\[
l[y](x) = f(x), \quad a \leq x \leq b,
\]
where \([a, b]\) is a finite real interval and \( f(x) \) some vector function in \( L^1_n[a, b] \), \( L^1_n[a, b] \) is the space of integrable \( n \)-vector functions on \([a, b]\).

Let vector function \( \phi(x) \) be such that
\[
\phi(x) \in AC_n[a, b], \quad \phi(a) = \phi(b) = 0.
\]
(1.7)
If we scalar multiply (1.6) by \( \phi(x) \), integrate over \([a, b]\) and integrate by parts on the left, we obtain
\[
\int_a^b \{\langle Py', \phi' \rangle - \langle PRy, \phi' \rangle - \langle R^* Py', \phi \rangle + \langle (R^* PR + Q)y, \phi \rangle\} = \int_a^b \langle f, \phi \rangle.
\]
(1.8)
If the equality (1.8) holds for all such functions \( \phi(x) \), then one may say that \( y \) is a weak solution of (1.6).

Thus, if \( y \) satisfies (1.6), we have (1.8) for all functions \( \phi(x) \) with (1.7). Conversely, one might ask whether if \( y \) satisfies (1.8) for all such \( \phi(x) \), then \( y \) satisfies (1.6).

Let \( P_0, Q_0 \) and \( P_1 \) be Hermitian matrix functions of order \( n \) with Lebesgue measurable entries on \( I \) such that \( P_0^{-1} \) exists and \( \|P_0^{-1}\|, \|P_0^{-1}\|P_1\|, \|P_0^{-1}\|Q_0\| \) are locally Lebesgue integrable. Let also \( \Phi := P_1 + iQ_0 \) and \( \tilde{\Phi} := P_1 - iQ_0 \). Assume that the block entries in the matrix (1.1) are represented as \( P := P_0, Q := -\tilde{\Phi}P_0^{-1}\Phi \) and \( R := P_0^{-1}\Phi \), then we obtain the block matrix
\[
F = \begin{pmatrix}
P_0^{-1}\Phi & P_0^{-1} \\
-\tilde{\Phi}P_0^{-1}\Phi & -\tilde{\Phi}P_0^{-1}
\end{pmatrix}.
\]
The conditions listed above on the matrix functions \( P_0, Q_0 \) and \( P_1 \) suggest that all entries of \( F \) belong to the space \( L^1_{loc}(I) \). Detailed justification of this fact is given in [17].
As above, using the matrix $F$, we can define the quasi-derivatives of given vector function $y \in AC_{n,loc}(I)$, assuming

$$y^0 := y, \quad y^1 := P_0 y' - \Phi y, \quad y^2 := (y^1)' + \Phi P_0^{-1} y^1 + \Phi P_0^{-1} \Phi y.$$  

Suppose further that the elements of matrix function $P_0$ also belong to $L^1_{loc}(I)$, then the entries of $\Phi$ are locally integrable on $I$. Thus, if we interpret the derivative $'$ in the sense of distributions, then we can remove all the brackets in the expression $y^2$ and the quasi-differential expression $l[y]$ in terms of distributions can be written as

$$l[y] = -(P_0 y')' + i((Q_0 y)' + Q_0 y') + P_1 y.$$  

(1.9)

In particular, if $P_0(x) = I$, $Q_0(x) = O$, $O$ is the zero matrix and $P_1(x) = V(x)$, where $V(x)$ is a real-valued symmetric matrix function such that the entries of the matrix $V^2(x)$ are locally integrable on $I$, then the expression (1.9) takes the form

$$l[y] = -y'' + V'y.$$

Detailed description of scalar quasi-differential expressions of second order with generalized derivatives is given in [14] and matrix expressions in [15–17].

We note here that in this case the relation (1.8) takes the form

$$\int_a^b \{ \langle P_0 y', \phi' \rangle - \langle \Phi y, \phi \rangle - \langle \Phi y', \phi \rangle \} = \int_a^b \langle f, \phi \rangle.$$  

1.2. Let us consider the block matrix $F$ of order $2kn$ ($k \in \mathbb{N}, k > 1$):

$$F = \begin{pmatrix} R & P^{-1} & O & O & O & O & \ldots & O & O \\ Q & -R^* & I_n & O & O & O & \ldots & O & O \\ O & O & R & P^{-1} & O & O & \ldots & O & O \\ O & O & Q & -R^* & I_n & O & \ldots & O & O \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ O & O & O & O & O & O & \ldots & R & P^{-1} \\ O & O & O & O & O & O & \ldots & Q & -R^* \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity matrix and $P, Q, R$ satisfy the conditions listed in Subsection 1.1.

As above, using the matrix $F$, we define the quasi-derivatives $y[i]$ ($i = 0, 1, \ldots, 2k$) of a given vector function $y \in AC_{n,loc}(I)$ assuming

$$y^0 := y, \quad y^1 := P(y' - Ry), \quad y^2 := (y^1)' + R^* y^1 - Qy,$$

$$y^3 := P((y^2)' - Ry^2), \quad y^4 := (y^3)' + R^* y^3 - Qy^2, \ldots,$$

$$y^{[2k]} := P((y^{[2k-2]})' - Ry^{[2k-2]}), \quad y^{[2k]} := (y^{[2k-1]})' + R^* y^{[2k-1]} - Qy^{[2k-2]}.$$
provided that \( y^{[i]} \in AC_{n,loc}(I) \) \((i = 1, \ldots, 2k - 1)\) and a quasi-differential expression
\[
l^k[y](x) := (-1)^k y^{[2k]}(x), \quad x \in I. \tag{1.10}
\]
Note that the quasi-differential expression \( l^k[y] \) constructed in this way is a formal \( k \)-power of (1.2). The explicit form of this expression is too large, because of it we do not present it here.

The set of complex-valued vector functions
\[
D := \{ y(x) | y(x), y^{[i]}(x) \in AC_{n,loc}(I), i = 1, \ldots, 2k - 1 \}
\]
is the domain of (1.10). For \( y \in D \) the expression \( l^k[y] \) exists a.e. on \( I \) and locally integrable there.

Similarly as in Subsection 1.1, we can define a minimal closed symmetric operator \( L_0 \) generated by the expression (1.10) and introduce the concept of the deficiency numbers of this operator. And in this case, the numbers \( n_+ \) and \( n_- \) coincide with the maximum number of linearly independent solutions of the equation
\[
l^k[y] = \lambda y
\]
belonging to the space \( L^2_0(I) \) when \( \Im \lambda > 0 \) or \( \Im \lambda < 0 \). Moreover, they satisfy double inequality \( nk \leq n_+, n_- \leq 2kn \) and \( n_+ = 2kn \) if and only if \( n_- = 2kn \).

Additionally, assuming that the matrix functions \( P_0, P_1, Q_0 \) satisfy the conditions listed in Subsection 1.1, we can define a formal \( k \) power of the quasi-differential expression (1.9) where the derivatives are understood in the generalized sense.

As example, we present here the explicit form of \( l^2[y] \) if the matrix \( F \) takes the form
\[
F = \begin{pmatrix}
  V(x) & I_n & O & O \\
  -V^2(x) & -V(x) & I_n & O \\
  O & O & V(x) & I_n \\
  O & O & -V^2(x) & -V(x)
\end{pmatrix},
\]
where \( V(x) \) is a matrix function with sufficiently smooth entries. In this case the quasi-differential expression \( l^2[y] \) has the form
\[
l^2[y] = y^{(4)} - 2(V'(x)y')' + ((V'(x))^2 - V^{(3)}(x))y.
\]

1.3. Let us mention here that one of the important problems in the spectral theory of the matrix Sturm-Liouville operators is to determine the deficiency numbers of the operator \( L_0 \). In particular, to find the conditions on the entries of the matrix function \( F \) that ensure the realization of the given pair \((n_-, n_+)\). One of the first works in this direction was a paper of V.B. Lidskii [12]. Later this problem for classical matrix Sturm-Liouville operators and operators with generalized coefficients was discussed in many works, see, for instance, [3–5, 9, 11–13, 15–17, 19–22] (and also the references therein). In particular, for example, in [17] the authors obtained the conditions of nonmaximality of deficiency numbers of operator \( L_0 \) generated by (1.2). M.S.P. Eastham in [4] investigated the values of the deficiency numbers depending on the
indices of power functions which are entries of the matrix coefficient of the second order differential operator. In [19] the method presented in [2] for scalar (quasi) differential operators was generalized to operators generated by the matrix expression $-y'' + P(x)y$.

In [13] the authors obtained several criteria for a matrix Sturm-Liouville-type equation of special form to have maximal deficiency indices. In [3] it is presented the conditions on the coefficients of the expression (1.2) such that the deficiency numbers of the operator $L_0$ are defined as the number of roots of a special kind polynomial lying in the left half-plane. The authors of [11] established a relationship between the spectral properties of the matrix Schrödinger operator with point interactions on the half-axis and block Jacobi matrices of certain class. In particular, they constructed examples of such operators with arbitrary possible equal values of the deficiency numbers. We also mention that in [1,23] the deficiency numbers problem for matrix operators generated by differential expressions of even order higher than the second is considered and in [6–8,10] this problem was discussed for powers of ordinary (quasi)differential expressions.

The main goal of this work is to obtain new sufficient conditions on the entries of the matrices $P, Q$ and $R$ when the limit-point case can be realized for the expressions $l[y]$ and $l^k[y]$ ($k > 1$) constructed above in Subsections 1.1 and 1.2 (Theorems 2.1 and 2.10). In particular, we apply these results to obtain new interval limit-point criteria (Corollary 2.11 and 2.12) and consider two examples of matrix Sturm-Liouville operators with minimal deficiency numbers. We also note here that our approach is based on the equality (1.8) and generalizes some results of [2] and [8] to the matrix case. This method allows to obtain the limit-point conditions for the operators with distributional coefficients and, in particular, for the matrix Sturm-Liouville operator with point interactions.

2. LIMIT-POINT CONDITIONS

One of the main theorem is the following:

**Theorem 2.1.** Let $w$ be a scalar non-negative absolutely continuous function on $I$, suppose that the $n \times n$ matrix functions $P, Q$ and $R$ satisfy the conditions listed above in Subsection 1.1 and there exist positive constants $K_1, K_2, K_3, K_4, K_5$ and $a$, such that for $x \geq a$

- (i) $P \geq K_1\|P\|I_n$,
- (ii) $\frac{w^2}{\|P\|} \leq K_2$,
- (iii) $\|P\| \left( \frac{w}{\|P\|^{\frac{1}{2}}} \right)^2 \leq K_3$,
- (iv) $w\|PR\| \leq K_4\|P\|$,
- (v) $w^2(R^*PR + Q) \geq -K_5\|P\|I_n$,
- (vi) $\int_a^\infty \frac{w}{\|P\|} = \infty$,

where $\| \cdot \|$ is the self-adjoint norm. Then the operator $L_0$ generated by (1.2) is in the limit-point case.
The proof of this theorem is established with the help of a few lemmas.

Let us mention that everywhere below the symbols \( K, K_1, K_2, \ldots \) denote various positive constants and \( \epsilon, \epsilon_1, \epsilon_2, \ldots \) denote “small” positive constants. These constants will not necessarily be the same on each occurrence. And we write \( K(\epsilon) \) when we indicate the dependence of \( K \) on \( \epsilon \).

**Lemma 2.2.** Let \( w \) be as in Theorem 2.1 and let \( v \) be a scalar non-negative absolutely continuous function with support in a compact \( J \subset I \). Suppose that there exist positive constants \( K_i, (i = 1, 2, \ldots, 7) \) independent of \( J \) such that (i)–(v) in Theorem 2.1 are satisfied on \( J \) and also

\[
\begin{align*}
(a) \quad \|P\|v' & \leq K_6 w, \\
(b) \quad v & \leq K_7.
\end{align*}
\]

Let \( l[y](x) = f(x) \). Then, given any \( \epsilon > 0 \), there exists a positive constant \( K(\epsilon) \), independent of \( J \), such that

\[
\int_J v^{2+\alpha} w^2 y'(x)^2 dx \leq \epsilon \int_J v^\alpha \|y(x)\|^2 dx + K(\epsilon) \int_J v^{4+\alpha} l^2 \|y(x)\|^2 dx.
\]

**Proof.** The proof involves the use of (1.8) and the simple inequality

\[
2|ab| \leq \epsilon a^2 + (1/\epsilon)b^2
\]

which holds for arbitrary \( \epsilon > 0 \). All integrals are over \( J \) and we omit the \( dx \) symbol for brevity.

Using (1.8), we obtain

\[
\Re \int \langle Py', \phi' \rangle - \int |\langle PRy, \phi' \rangle + \langle R^* Py', \phi \rangle| + \Re \int \langle (R^* PR + Q)y, \phi \rangle \leq \int |\langle f, \phi \rangle|,
\]

where \( \Re f \) is a real part of function \( f \).

Assume that \( \phi = v^{2+\alpha} \frac{w^2}{\|P\|} y \).

Next, we note that

\[
\Re \int \langle Py', \left(v^{2+\alpha} \frac{w^2}{\|P\|} y\right)' \rangle \geq \int \{ \langle P \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' , \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \rangle \]

\[
- \langle P \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' , \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \rangle \}

\[
- \langle P \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y \right)' , \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \rangle \}

\[
- \langle P \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y \right)' , \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \rangle \}

Furthermore, using (i)–(iii) of Theorem 2.1, the Cauchy-Schwarz inequality and that \( P \) is Hermitian matrix, we get

\[
\Re \int \langle Py', \left(v^{2+\alpha} \frac{w^2}{\|P\|} y\right)' \rangle \geq K_1 \int \|P\| \left(v^{1+\alpha/2} \frac{w}{\|P\|^{1/2}} y\right)' \|^2 - K(\epsilon_1) \int v^\alpha \|y\|^2.
\]

(2.3)
Next, we estimate the expression
\[ \int \left| \left\langle PRy, \left( v^{2+\alpha} \frac{w^2}{\|P\| y} \right)' \right\rangle + \left\langle R^*Py', \left( v^{2+\alpha} \frac{w^2}{\|P\| y} \right) \right\rangle \right|. \]

Since the norm \( \| \cdot \| \) is self-adjoint, then \( \|PR\| = \|R^*P\| \). Using also the properties of inner products, norms and the condition (ii)–(iv) of Theorem 2.1 and (a),(b) of Lemma 2.2 we obtain
\[
\left| \left\langle PRy, \left( v^{2+\alpha} \frac{w^2}{\|P\| y} \right)' \right\rangle + \left\langle R^*Py', \left( v^{2+\alpha} \frac{w^2}{\|P\| y} \right) \right\rangle \right| \\
\leq \|PR\| \left( v^{1+\alpha/2} \frac{w}{\|P\|^{1/2} y} \right)' \left( v^{1+\alpha/2} \frac{w}{\|P\|^{1/2} y} \right)' \|y\| \\
+ \|PR\| \left( v^{1+\alpha/2} \frac{w}{\|P\|^{1/2} y} \right)' \|y\|^2 + \|PR\| \left( v^{2+\alpha} \frac{w^2}{\|P\|} \right) \|y'||\|y\| \tag{2.4}
\]

Furthermore, using (v), we obtain
\[
\Re \int \left\langle - (R^*PR + Q)y, v^{2+\alpha} \frac{w^2}{\|P\| y} \right\rangle \leq K \int v^\alpha \|y\|^2. \tag{2.5}
\]

Also we shall need the estimate
\[
\frac{1}{1 + \epsilon_3} v^{2+\alpha} w^2 \|y'||^2 \leq \|P\| \left( v^{1+\alpha/2} \frac{w}{\|P\|^{1/2} y} \right)' \|y\|^2 + K(\epsilon_3, \epsilon_4) v^\alpha \|y\|^2. \tag{2.6}
\]

This inequality immediately follows from the product rule for \( \left( v^{1+\alpha/2} \frac{w}{\|P\|^{1/2} y} \right)' \) and the conditions (ii), (iii) of Theorem 2.1 and (a), (b) of Lemma 2.2.

Next, we note here that
\[
\int |\langle f, \phi \rangle| = \int \left| \left\langle f, v^{2+\alpha} \frac{w^2}{\|P\| y} \right\rangle \right| \leq \epsilon \int v^{4+\alpha} \|f\|^2 + K(\epsilon) \int v^\alpha \|y\|^2. \tag{2.7}
\]

Substitute now (2.3)–(2.7) into (2.2) and choose \( \epsilon_1, \epsilon_2, \epsilon_3 \) sufficiently small so that \( (K_1 - \epsilon_1 K_3/2)(1 + \epsilon_3)^{-1} - \epsilon_2/2 > 0 \) we obtain the inequality (2.1). \( \square \)

From the Green’s formula (1.3) we obtain the following lemma.

**Lemma 2.3.** If \( y_1, y_2 \) are solutions of
\[
l[y_1](x) = f_1(x), \quad l[y_2](x) = f_2(x) \tag{2.8}
\]
and \( y_1, y_2, f_1, f_2 \in L^2_n(I) \) then the form \( [y_1, y_2](x) \) (see (1.4)) tends to a finite limit as \( x \to \infty \).
Moreover, we get the ensuing lemma.

**Lemma 2.4.** If $f_1, f_2$ in $L_n^2(I)$ and for every pair of solutions $y_1, y_2 \in L_n^2(I)$ of (2.8)

$$[y_1, y_2](x) \to 0, \ x \to \infty,$$

then the set of such solutions has dimension at most $n$.

**Proof of Theorem 2.1.** Here we apply the ideas of [8] to the matrix case. From (vi) it follows that, for some $b > a$, $w(b) > 0$ and hence, since $w$ is continuous, there is a $\delta > 0$ such that $\frac{w}{\|P\|} > 0$ on $[b, b + \delta]$. Define

$$\theta(x) = \int_b^x \frac{w}{\|P\|}, \ x \geq b,$$

$$v(x) = \begin{cases} 1 - \exp(\theta(x) - \theta(X)), & b + \delta \leq x \leq X, \\ 0, & x \geq X, \end{cases}$$

and in $[b, b + \delta)$ choose $v$ such that it vanishes in a right neighborhood of $b$, $0 \leq v(x) \leq 1$ and $v$ has a continuous derivative in $[b, b + \delta]$. Then from (ii)

$$v' = O\left(\frac{w}{\|P\|}\right).$$

We also choose $X$ such that $\theta(X) > \ln 2$ and $T$ such that $\theta(T) = \theta(X) - \ln 2$. Then

$$v(x) \geq \frac{1}{2}, \ b + \delta \leq x \leq T. \tag{2.9}$$

Let us consider

$$\left|\int_b^x \frac{vw}{\|P\|}[f, g]\right| \leq \int_b^x \frac{vw}{\|P\|}\left\{|\langle f, g^{[1]} \rangle| + |\langle f^{[1]}, g \rangle|\right\}.$$ 

Using now the properties of inner products, norms and (2.1) we obtain that

$$\left|\int_b^x \frac{vw}{\|P\|}[f, g]\right| \leq K \int_b^x \|f\|^2 + \|g\|^2 + \|l[f]\|^2 + \|l[g]\|^2. \tag{2.10}$$

By Lemma 2.3, we know that $[f, g]$ tends to a finite limit. Assume that this limit is $c \neq 0$ and show that this leads to a contradiction with (vi).

Supposing that $[f, g](x) \geq c$ for large $x$, say $x \geq \gamma$ and choosing $a > \gamma$. For $f, g$ satisfying (2.8) of Lemma 2.3 we have from (2.9) and (2.10) that

$$\frac{c}{2} \int_{b+\delta}^T \frac{w}{\|P\|} \leq \int_b^x \frac{vw}{\|P\|}[f, g] \leq K.$$
It leads to a contradiction with (vi). Therefore, $[f,g] \to 0$ when $x \to \infty$. Using now Lemma 2.4 and the inequality (1.5) we obtain that the operator $L_0$ generated by (1.2) is in the limit-point case.

Corollary 2.5. Let $w$ be a scalar non-negative absolutely continuous function on $I$, suppose that the $n \times n$ matrix functions $P_0, P_1$ and $Q_0$ satisfy the conditions listed above in Subsection 1.1 and there exist positive constants $K_1, K_2, K_3, K_4$ and $a$, such that for $x \geq a$

(i) $P_0 \geq K_1 \|P_0\| I_n$,
(ii) $\frac{w}{\|P_0\|} \leq K_2$,
(iii) $\|P_0\| \left( \frac{w}{\|P_0\|} \right)^2 \leq K_3$,
(iv) $w \|P_1 + iQ_0\| \leq K_4 \|P_0\|$,
(v) $\int_a^\infty \frac{w}{\|P_0\|} = \infty$.

where $\| \cdot \|$ is the self-adjoint norm. Then the operator $L_0$ generated by (1.9) is in the limit-point case.

To prove the theorem about deficiency numbers of the operator generated by $l^k[y]$, $k > 1$ we need some additional lemma.

Lemma 2.6. Suppose that all hypothesis of Lemma 2.2 are satisfied. Then, given any $\epsilon > 0$, there exists a positive constant $K(\epsilon)$, independent of $J$, such that

$$
\int_J v^{4j} \|l^j[y]\|^2 dx \leq \epsilon \int_J v^{4(j+1)} \|l^{j+1}[y]\|^2 dx + K(\epsilon) \int_J v^{4(j-1)} \|l^{j-1}[y]\|^2 dx.
$$

(2.11)

Proof. In the proof all integrals are over $J$ and we omit $dx$ symbol for brevity. Put $f = l^{j-1}[y]$, $g = l[f] = l^j[y]$. Then

$$
\int v^{4j} \langle l^{j-1}[y], l^{j+1}[y] \rangle = \int v^{4j} \langle f, l[g] \rangle = \int v^{4j} \langle l[f], g \rangle + \int (v^{4j})' \langle Pf, g' \rangle - \int (v^{4j})' \langle R^* Pf, g \rangle - \int (v^{4j})' \langle P f', g \rangle + \int (v^{4j})' \langle PRf, g \rangle.
$$

(2.12)

Using (a) of Lemma 2.2, we note that

$$(v^{4j})' \leq K v^{4j-1} \frac{w}{\|P\|}.$$ 

Therefore, we obtain

$$
\left| \int (v^{4j})' \langle Pf, g' \rangle \right| \leq \int |(v^{4j})'| \|P\| \|f\| \|g'\| \leq K \int v^{4j-1} w \|f\| \|g'\|.
$$
From (2.1) with $\alpha = 4(j - 1)$ we have

$$
\left| \int (v^j)'(Pf, g') \right| \leq K_1(\epsilon_1, \epsilon_2) \int v^{4(j+1)}[f]^2 + K_2(\epsilon_1, \epsilon_2) \int v^{4j}[f]^2 + K_3(\epsilon_1) \int v^{4(j-1)}[f]^2.
$$

(2.13)

And

$$
\left| \int (v^j)'(Pf', g) \right| \leq K_4(\epsilon_3) \int v^{4j}[f]^2 + K_5(\epsilon_3) \int v^{4(j-1)}[f]^2.
$$

(2.14)

Similarly, using (iv) of Theorem 2.1, we get

$$
\left| \int (v^j)'(R^*Pf, g) \right| \leq K_6(\epsilon_4) \int v^{4(j-1)}[f]^2 + K_7 \int v^{4j}[l[f]]^2.
$$

(2.15)

Therefore, substituting (2.13)–(2.15) into (2.12), we obtain (2.11).

\begin{lemma}
Under the hypothesis of Lemma 2.2, given $\epsilon > 0$ there exists a $K(\epsilon) > 0$, independent of $J$, such that

$$
\int_j v^j[l^j[y]]^2 dx \leq \epsilon \int_j v^{4k}[l^k[y]]^2 dx + K(\epsilon) \int y^2 dx
$$

(2.16)

for $j = 1, 2, \ldots, k - 1$.
\end{lemma}

\begin{proof}
The proof is by induction on $k$ and almost exactly the same as the proof of Lemma 2.4 in [10, p. 91].
\end{proof}

\begin{definition}[see [10]]
Let $l[y]$ be a symmetric differential expression and let $k \in \mathbb{N}, k > 1$. We say that $l^k[y]$ is partially separated if $y$ and $l^k[y]$ in $L_n^2(I)$ together imply that $l^r[y]$ is in $L_n^2(I)$ for $r = 1, 2, \ldots, k - 1$.
\end{definition}

The next lemma follows from [10, Corollary 5.3.6].

\begin{lemma}
If $l[y]$ is limit-point then $l^k[y], k > 1$ is limit-point if and only if $l^k[y]$ is partially separated.
\end{lemma}

\begin{theorem}
Suppose the hypothesis of Theorem 2.1 hold. Then $l^k[y]$ is limit-point for any $k \in \mathbb{N}$.
\end{theorem}

\begin{proof}
Let us show that the expression $l^k[y]$ is partially separated.

Using the definition of $v$ given in the proof of Theorem 2.1, Lemma 2.7 and (2.16) we get

$$
\left( \frac{1}{2} \right)^{4j} \int_{b+\delta}^{t} \|l^j[y]\|^2 \leq \sum_{b+\delta}^{X} \int v^{4j}||l^j[y]\|^2 \leq K \int \{||l^k[y]\|^2 + \|y\|^2\}.
$$

Since $t \to \infty$ as $X \to \infty$ we can conclude that $l^j[y]$ is in $L_n^2(I)$ for $j = 1, 2, \ldots, k - 1$ and that $l^k[y]$ is partially separated. Therefore, the statement of Theorem 2.10 follows from Lemma 2.9.
\end{proof}
Now we give some applications of Theorems 2.1 and 2.10.

**Corollary 2.11.** Let
\[ [a_m, b_m], \ m = 1, 2, \ldots \]
be a sequence of intervals such that
\[ 0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots \]
and \( M_1, M_2, \ldots \) a sequence of positive numbers such that
\[ \sum_{m=1}^{\infty} \frac{(b_m - a_m)^2}{M_m} = \infty. \]  
(2.17)

For some fixed \( K > 0 \) suppose that in each \([a_m, b_m]\) we have

(i) \( P(x) \geq M_m I_n, \quad \|P(x)\| \leq K M_m, \)
(ii) \( (b_m - a_m)\|PR\| \leq K M_m, \)
(iii) \( (b_m - a_m)^2 (R^* PR + Q) \geq -K M_m I_n, \)

Then the operator \( L_0 \) generated by (1.2) and all its powers \( l^k[y], \ k = 2, 3, \ldots \) are in the limit-point case.

*Proof.* Taking
\[ w(x) = \begin{cases} 
  x - a_m, & a_m \leq x \leq (a_m + b_m)/2, \\
  b_m - x, & (a_m + b_m)/2 \leq x \leq b_m, \\
  0, & \text{otherwise}
\end{cases} \]

in Theorem 2.1 and applying Theorem 2.10 we get the corollary. \( \square \)

**Corollary 2.12.** Let \([a_m, b_m]\) and \( M_m, \ m = 1, 2, \ldots \) be sequences of intervals and positive numbers satisfying (2.17) as in Corollary 2.11. And for some fixed \( K > 0 \) suppose that in each \([a_m, b_m]\) we have

(i) \( P_0(x) \geq M_m I_n, \quad \|P_0\| \leq K M_m, \)
(ii) \( (b_m - a_m)\|P_1 + iQ_0\| \leq K M_m, \)

Then the operator \( L_0 \) generated by (1.9) and all its powers \( l^k[y], \ k = 2, 3, \ldots, \) are in the limit-point case.

3. **EXAMPLES**

3.1. Let us consider the differential expression
\[ l[y] = -(P_0 y')' + P_1'^* y \]
(3.1)
on \( I := [a, +\infty), a > 0, \) where \( P_0 = x^\alpha I_n, \ P_1 = x^{-\beta} Q(x^\gamma), \ \alpha \in [0, 2], \ \beta \geq 0 \) and \( Q(x^\gamma) \) is \( n \times n \) periodic matrix function with continuous entries. Applying Corollary 2.5 with
\( w = x^{\alpha-1} \) to this expression and observing that \( x^{-\beta-1}Q(x^{\gamma})y \) is a boundary operator, we obtain that the operator, generated by

\[-(x^{\alpha}y')' + x^\delta Q'(x^{\gamma})y, \quad \delta \leq \gamma \]

is in the limit-point case and all its powers are also limit-point.

**Remark 3.1.** We note here that the expression \(-y'' + x^\delta Q(x^{\gamma})y, Q \) is \( n \times n \) periodic matrix function with continuous entries is discussed in detail in [19].

3.2. Let us consider the differential expression (3.1). Suppose that \( 0 = x_0 < x_1 < x_2 < \ldots \) and \( \lim_{m \to \infty} x_m = \infty \). Assume that \( P_1(x) \) is a piecewise continuously differentiable matrix function on \( I \) and \( x_m (m = 0, 1, 2 \ldots) \) are points of discontinuity of the first kind of \( P_1(x) \). Suppose also that \( P_1(x) = Q_m(x), (x_m - x_{m-1})\|Q_m\| \leq k \) \((k > 0)\) on \((x_{m-1}, x_m] \) and

\[ \mathcal{H}_m = (h_{ij}^m)_{i,j=1}^n := Q_{m+1}(x_m + 0) - Q_m(x_m - 0) \]

is a jump of the matrix function \( P_1(x) \) in \( x_m \). Assume also

\[ \sum_{m=1}^{\infty} (x_m - x_{m-1})^2 = \infty. \]

Then, applying Corollary 2.12, we obtain that the operator, generated by

\[-y'' + (P'_1(x) + \sum_{k=1}^{\infty} \mathcal{H}_m \delta(x - x_m))y, \]

here \( \delta(x) \) is the Dirac \( \delta \)-function and \( P'_1(x) \) is a derivative of \( P_1(x) \) when \( x \neq x_m (m = 0, 1, 2 \ldots) \) is in the limit-point case and all its powers are also limit-point.

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