On the computation of star products

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Abstract
The problem of finding a generic star product on \( \mathbb{R}^n \) is reduced to the computation of a skew-symmetric biderivation.

1 Introduction
Let \( \alpha = \alpha^{ij}(x) \partial_i \wedge \partial_j \) be a Poisson bivector on \( \mathbb{R}^n \), where \( x = (x^1, \ldots, x^n) \), and \( x^i, i = 1, \ldots, n \), are coordinates. A star product is a \( \mathbb{R}[[t]] \) linear associative product on \( \mathbb{A}[[t]] \), defined for \( f, g \in \mathbb{A} = \mathbb{C}^\infty(\mathbb{R}^n) \) by

\[
f \ast g = \Pi_*(f, g), \quad \Pi_* = \sum_{k \geq 0} t^k \Pi_k,
\]

where \( \Pi_k \) are bidifferential operators, \( \Pi_0(f, g) = fg \), \( \Pi_1 = \frac{1}{2} \alpha \). The operators \( \Pi_k \) are determined by the associativity equation

\[
(f \ast g) \ast h = f \ast (g \ast h)
\]

for all \( f, g, h \in \mathbb{A}[[t]] \). The star product is the main ingredient of deformation quantization [1].

A construction providing a star product associated with any symplectic manifold was given in [2]. It allows to compute star products by a simple iterative procedure. An formula for the star product on an arbitrary Poisson manifold has been found by Kontsevich [3]. It looks like

\[
* = F(\alpha),
\]
where $F$ is described by a series of diagrams. However, the problem of finding the star product has not been solved, because there is no systematic method to compute weights of the corresponding graphs.

A third order contribution to the generic star product on $\mathbb{R}^n$ was explicitly computed in [4]. At present, a fourth order contribution to the star product is known [5].

The problem, which we discuss in the present paper, is related with computation of higher order contributions to star products. We construct an explicit function $G$ such that

$$* = G(\pi(\alpha)), \quad (4)$$

for some skew-symmetric biderivation $\pi$. This formula reduces the problem of finding the $*$-product to the computation of a skew-symmetric biderivation. Moreover, it clarifies (somewhat) eq. (3). To get (4) we consider a projection of the Maurer-Cartan equation on the image of the Hochschild coboundary operator and find a solution to the projected equation. It defines the function $G$. The complementary equation determines the part of deformation which depends only on a skew-symmetric biderivation.

The paper is organized as follows. In section 2, we introduce notations and decompose the Maurer-Cartan equation. In section 3, we construct the function $G(\pi)$.

## 2 Decomposition of the Maurer-Cartan equation

Let $C^p(A), p \geq 0$, be the space of differential $p$–cochains and let $\delta$ be the Hochschild coboundary operator

$$\delta : C^p(A)[[t]] \rightarrow C^{p+1}(A)[[t]]$$

which is defined by

$$\delta \Phi(f_1, f_2, \ldots, f_{p+1}) = f_1 \Phi(f_2, \ldots, f_{p+1}) +$$

$$+ \sum_{k=1}^{p} (-1)^k \Phi(f_1, \ldots, f_{k-1}, f_k f_{k+1}, f_{k+2}, \ldots, f_{p+1}) + (-1)^{p+1} \Phi(f_1, \ldots, f_p) f_{p+1}.$$
Equations (1) and (2) together are equivalent to the Maurer-Cartan equation
\[ \delta \Pi = \frac{1}{2} [\Pi, \Pi], \tag{5} \]
where \([ \ , \ ]\) is the Gerstenhaber bracket. For \(\Phi, \Psi \in C^2(A)\)
\[ [\Phi, \Psi](f, g, h) = \Phi(\Psi(f, g), h) - \Phi(f, \Psi(g, h)) + \Psi(\Phi(f, g), h) - \Psi(f, \Phi(g, h)). \]
In this case
\[ [\Phi, \Psi](f, g, h) = [\Psi, \Phi](f, g, h). \]
Let \(C^p_{nc}(A)\) denote the space of differential \(p\)-cochains vanishing on the constants. We shall seek a solution to eq. (5) satisfying
\[ \Pi \in C^2_{nc}(A). \]
Let \(\delta^+ : C^{p+1}(A)[[t]] \to C^p(A)[[t]]\) be a generalized inverse of \(\delta : \delta \delta^+ \delta = \delta, \quad \delta^+ \delta \delta^+ = \delta^+\).
The operator \(\delta^+\) exists [6]. It is defined by
\[ \delta^+ = \lim_{\epsilon \to 0} (\epsilon^2 I + \delta^T \delta)^{-1} \delta^T = \lim_{\epsilon \to 0} \delta^T (\epsilon^2 I + \delta^T \delta)^{-1}, \]
where \(I\) is the identity map, and \(\delta^T : C^{p+1}(A)[[t]] \to C^p(A)[[t]]\) is the transpose of \(\delta\). For
\[ \Psi = \Psi_{a_1 \ldots a_{p+1}}(x) X^{a_1} \otimes \ldots \otimes X^{a_{p+1}} \in C^{p+1}(A), \]
where
\[ X^a = \frac{1}{a!^1 a!^2 \ldots a!^n} \frac{\partial^{\vert a \vert}}{\partial (x^1)^{a^1} \partial (x^2)^{a^2} \ldots \partial (x^n)^{a^n}}, \]
\(a = (a^1, a^2, \ldots, a^n)\) is a multi-index, \(\vert a \vert = a^1 + a^2 + \ldots + a^n\), we have
\[ \delta^T \Psi = \Psi_{a_1 \ldots a_{p+1}}(x) \delta^T (X^{a_1} \otimes \ldots \otimes X^{a_{p+1}}), \]
\[ \delta^T(X^{a_1} \otimes \ldots \otimes X^{a_{p+1}}) = \sum_{k=1}^{p} (-1)^{k+1} X^{a_1} \otimes \ldots \otimes \delta^T(X^{a_k} \otimes X^{a_{k+1}}) \otimes \ldots \otimes X^{a_{p+1}}, \]

\[ \delta^T(X^{a} \otimes X^{b}) = -X^{a+b} + \delta^{a_0} X^{b} + X^{a} \delta^{b_0}. \]

The restriction of \( \delta^+ \) to \( C^3_{nc}(A)[[t]] \) is given by an infinite series

\[ \delta^+ = \lim_{\epsilon \to 0} K^{-1} \sum_{m=0}^{\infty} (UK^{-1})^m \delta^T, \]

where \( K = \epsilon^2 I + D, \)

\[ D(X^{a} \otimes X^{b}) = (\nu(a) + \nu(b)) X^{a} \otimes X^{b}, \]

\[ \nu(a) = \prod_{i=1}^{n} (a^i + 1) - 2, \]

\[ U(X^{a} \otimes X^{b}) = \sum_{s=0}^{a'} X^{a-s} \otimes X^{b+s} + \sum_{s=0}^{b'} X^{a+b-s} \otimes X^{s}, \]

\[ \sum_{s=0}^{a'} Y^{s} = \sum_{s=0}^{a} Y^{s} - Y^{a} - Y^{0}, \quad \sum_{s=0}^{a_1} \cdots \sum_{s_p=0}^{a_p} = \sum_{s_1=0}^{a_1} \cdots \sum_{s_p=0}^{a_p}. \]

In accordance with the decomposition

\[ C^3_{nc}(A)[[t]] = V_1 \oplus V_2, \]

where

\[ V_1 = PC^3_{nc}(A)[[t]], \quad V_2 = (I - P)C^3_{nc}(A)[[t]], \quad P = \delta \delta^+, \]

eq. (5) splits as

\[ \delta \Pi = \frac{1}{2} \delta \delta^+ [\Pi, \Pi], \quad (6) \]
\((I - \delta\delta^+)[\Pi, \Pi] = 0.\) \hspace{1cm} (7)

It follows from \((6)\) that

\[ \Pi = \Upsilon + \frac{1}{2}\delta^+ [\Pi, \Pi], \] \hspace{1cm} (8)

where \(\Upsilon \in C^2_{nc}(A)[[t]]\) is an arbitrary cocycle, \(\delta \Upsilon = 0\), subject only to the restriction

\[ \Upsilon = \frac{t}{2}\alpha + O(t^2). \]

Equation \((8)\) can be iteratively solved as:

\[ \Pi = \Upsilon + \frac{1}{2}\delta^+ [\Upsilon, \Upsilon] + \ldots. \] \hspace{1cm} (9)

3 Partial deformation

To obtain an explicit expression for \(\Pi\) we introduce the functions

\[ (\ldots) : (C^2(A))^m \rightarrow C^2(A), \quad m = 1, 2, \ldots, \]

which recursively defined by

\[ \langle \Phi \rangle = \Phi, \quad \langle \Phi_1, \Phi_2 \rangle = \delta^+ [\Phi_1, \Phi_2], \]

\[ \langle \Phi_1, \ldots, \Phi_m \rangle = \frac{1}{2} \sum_{r=1}^{m-1} \sum_{1 \leq i_1 < \ldots < i_r \leq m} \langle \langle \Phi_{i_1}, \ldots, \Phi_{i_r} \rangle, \langle \Phi_1, \ldots, \hat{\Phi}_{i_1}, \ldots, \hat{\Phi}_{i_r}, \ldots, \Phi_m \rangle \rangle \] \hspace{1cm} (10)

if \(m = 3, 4, \ldots\), where \(\hat{\Phi}\) means that \(\Phi\) is omitted. Equation \((8)\) can be written as

\[ \Pi = \Upsilon + \frac{1}{2}\langle \Pi, \Pi \rangle. \] \hspace{1cm} (11)

Using induction on \(m\) one easily verifies that \(\langle \Phi_1, \ldots, \Phi_m \rangle\) is an \(m\)-linear symmetric function.

For \(m \geq 2, 1 \leq i, j \leq m,\) let

\[ P_{ij}^m : (C^2(A))^m \rightarrow (C^2(A))^{m-1} \]
be defined by

\[ P_{ij}^m(\Phi_1, \ldots, \Phi_m) = (\langle \Phi_i, \Phi_j \rangle, \Phi_1, \ldots, \hat{\Phi}_i, \ldots, \hat{\Phi}_j, \ldots, \Phi_m). \]

If \( \Phi \in C^2(A) \) is given by

\[ \Phi = P_{12}^2 P_{3}^3 P_{i,m-2j,m-2} \cdots P_{i_j}^{m-1} P_{i_j}^m (\Phi_1, \ldots, \Phi_m) \]

for some \((i_1, j_1), \ldots, (i_{m-2}, j_{m-2})\), we say that \( \Phi \) is a descendant of \((\Phi_1, \ldots, \Phi_m)\).

A descendant of \( \Phi \in C^2(A) \) is defined as \( \Phi \).

One can show that \( \langle \Phi_1, \ldots, \Phi_m \rangle \) equals the sum of all the descendants of \((\Phi_1, \ldots, \Phi_m)\) [7]. For example,

\[ \langle \Phi_1, \Phi_2, \Phi_3 \rangle = \langle \langle \Phi_1, \Phi_2 \rangle, \Phi_3 \rangle + \langle \langle \Phi_1, \Phi_3 \rangle, \Phi_2 \rangle + \langle \langle \Phi_2, \Phi_3 \rangle, \Phi_1 \rangle. \]

The function \( \langle \ldots \rangle : (C^2(A))^m \rightarrow C^2(A) \) can be uniquely extended to a function \( \langle \ldots \rangle : (C^2(A)[[t]])^m \rightarrow C^2(A)[[t]] \) by \( \mathbb{R}[[t]] \)-linearity.

Equation (9) can be written as [7]

\[ \Pi = \langle e^\Upsilon \rangle, \]

where

\[ \langle e^\Upsilon \rangle = \sum_{m \geq 0} \frac{1}{m!} \langle \Upsilon^m \rangle, \quad \langle \Upsilon^0 \rangle = 0. \]

Substituting (13) into (7), we get the equation determining \( \Upsilon \):

\[ (I - \delta \delta^*)[\langle e^\Upsilon \rangle, \langle e^\Upsilon \rangle] = 0. \]

The cocycle \( \Upsilon \) can be written as \( \Upsilon = \pi + \delta \lambda \), where \( \pi = \sum_{k \geq 1} t^k \pi_k \) is a skew-symmetric biderivation, and \( \lambda \) is an 1–cochain [8].

Let us assume by induction that \( \lambda = \sum_{k \geq n} t^k \lambda_k \). Then one can write

\[ \Pi_* = \Pi_0 + \langle e^\Upsilon \rangle = \sum_{k=0}^{n-1} t^k \Pi_k + t^n (\pi_n + \delta \lambda_n + \bar{\Pi}_n) + O(t^{n+1}) \]

where \( \bar{\Pi}_n \) depends only on \( \pi_k \) with \( k < n \). Define a map \( S : A[[t]] \rightarrow A[[t]] \) by

\[ S(f) = f - t^n \lambda_n(f) \]
with inverse
\[ S^{-1}(f) = f + \sum_{k \geq 1} t^k \lambda^k_n(f) \]
for any \( f \in A \). Then
\[ \Pi'_*(f, g) = S^{-1}(\Pi_*(S(f), S(g))) \] (16)
is a new equivalent star product
\[ \Pi'_* = \Pi_0 + \langle e^{\Upsilon'} \rangle, \quad \delta \Upsilon' = 0. \]

It follows from (15) and (16) that
\[ \Upsilon' = \sum_{k=1}^{n} t^k \pi_k + O(t^{n+1}). \]

It then follows from the vanishing of \( \delta \lambda_n \) that the coboundary \( \delta \lambda \) can be removed from (13) by a similarity transformation.

Finally, we get
\[ \Pi = \langle e^{\pi} \rangle, \quad \pi = \frac{t}{2} \alpha + \sum_{k \geq 2} t^k \pi_k, \] (17)
where \( \pi_k = \pi^i_j \partial_i \wedge \partial_j \). Equation (14) takes the form
\[ (I - \delta \delta^+)[\langle e^{\pi} \rangle, \langle e^{\pi} \rangle] = 0. \]
The cocycle \( \pi \) depends on \( \alpha^{ij} \) and the derivatives \( \partial_{k_1} \ldots \partial_{k_m} \alpha^{ij}, m = 1, 2 \ldots \).

In eq. (14)
\[ G(\pi) = \Pi_0 + \langle e^{\pi} \rangle. \]

4 Conclusions

In this paper we have studied a construction of the generic star product on \( \mathbb{R}^n \). Our approach is based on a decomposition of the Maurer-Cartan equation. In accordance with this decomposition the star product is represented as a composition of two functions. We show that one of these functions is a skew-symmetric biderivation and give an explicit expression for the other. It is an important problem to find a solution to the equation determining the skew-symmetric biderivation. It would also be interesting to reproduce decomposition (14) in the framework of the path integral approach [9].
References

[1] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerovich and D. Sternheimer, Deformation theory and quantization, *Ann. Phys.* **111** (1978), 61-110.

[2] B. Fedosov, A simple geometrical construction of deformation quantization, *J. Diff. Geom.* **40** (1994), 213-238.

[3] M. Kontsevich, Deformation quantization of Poisson manifolds, *Lett. Math. Phys.* **66** (2003), 157-216.

[4] M. Penkava and P. Vanhaecke, Deformation quantization of polynomial Poisson algebras, *J. Algebra* **227** (2000), 365-393.

[5] V. G. Kupriyanov and D. V. Vassilevich, Star products made (somewhat) easier, *Eur. Phys. J. C* **58** (2008), 627-637.

[6] A. V. Bratchikov, Generalized inversion of the Hochschild coboundary operator and deformation quantization, *Int. J. Geom. Meth. Mod. Phys.* **8** (2011), 99-106.

[7] A. V. Bratchikov, Explicit construction of the classical BRST charge for nonlinear algebras, *Central Eur. J. Phys.* **10** (2012), 61-65.

[8] S. Gutt and J. Rawnsley, Equivalence of star products on a symplectic manifold, *J. Geom. Phys.* **29** (1999), 347-392.

[9] A. C. Cattaneo and G. Felder, A path integral approach to the Kontsevich quantization formula, *Commun. Math. Phys.* **212** (2000), 591-611.