HOMOLOGY OF SUBGROUPS OF RIGHT-ANGLED ARTIN GROUPS

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Abstract. We describe the (co)homology of a certain family of normal subgroups of right-angled Artin groups that contain the commutator subgroup, as modules over the quotient group. We do so in terms of (skew) commutative algebra of squarefree monomial ideals.

1. Introduction

Let $\Gamma$ be a simple graph on $n$ vertices $V = \{v_1, v_2, \ldots, v_n\}$. The right-angled Artin group (or graph group) $G_\Gamma$ is the group with generators $V$ and relations $v_iv_j = v_jv_i$ for each edge $v_iv_j$ in $\Gamma$. Charney and Davis [5] showed that such groups admit a finite classifying space, a subcomplex of the $n$-torus first introduced in [11]. It follows that the cohomology ring of $G_\Gamma$ has an elegant description as an exterior Stanley-Reisner ring.

Bestvina and Brady [2] make use of a particular subgroup of $G_\Gamma$ with infinite cyclic quotient as examples to distinguish finiteness properties: the kernel of the map to $\mathbb{Z}$ sending each $v_i$ to the generator 1 is finitely generated if and only if $\Gamma$ is connected and finitely presented if and only if $\Gamma$ is simply connected. Moreover, they show that this subgroup is $FP_k$ if and only if the flag complex $K_\Gamma$ of $\Gamma$ is homologically $k$-connected. The cohomology ring of this subgroup is computed in [12, 17] when it is finite-dimensional, by relating it to the simplicial topology of the flag complex $K_\Gamma$.

More generally, for an integer $m$ let $\rho: G_\Gamma \to \mathbb{Z}^m$ be a surjective group homomorphism with the property that $\rho(v_i)$, for each $i$, is a generator of the abelian group $\mathbb{Z}^m$. We shall call such a map $\rho$ a coordinate homomorphism, and denote its kernel by $N_\rho = N_\Gamma,\rho$, the coordinate subgroup. From Meier, Meinert and VanWyk’s calculation of the Bieri-Neumann-Strebel invariants of right-angled Artin groups [13], it follows that the homology groups of $N_\rho$ are not finitely generated except under very restrictive hypotheses.

The point of view of this paper is that, nevertheless, $H_i(N_\rho, \mathbb{Z})$ is a finitely-generated module over the group ring $\mathbb{Z}[G_\Gamma]$, and so is amenable to description in terms of the graph $\Gamma$ via combinatorial commutative algebra. Accordingly, we compute this module in terms of the exterior Stanley-Reisner ring of the clique complex of the graph $\Gamma$. In such terms, for example, one can determine the Krull dimension of each module $H_p(N_\rho, \mathbb{Z})$.

Under the additional hypothesis that the complex $K_\Gamma$ is Cohen-Macaulay, these results can be made more explicit via Bernstein-Gelfand-Gelfand duality (Section 5). In particular, we find that $H^q(G_\Gamma, \mathbb{Z}[G_\Gamma^{ab}])$ is zero except for $q = d + 1$ if and only if $K$

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is Cohen-Macaulay of dimension \(d\). This is an abelian version of a result of Brady and Meier [3]; in this case, we are also able to describe the dualizing module explicitly.

2. Classifying spaces

The construction used here is a generalization of constructions that appear independently in the work of various authors. In the context of right-angled Artin groups, the idea originates with Charney and Davis [5]. The language of partial product complexes is convenient, however; details and further references may be found in [6]. (These are also known as generalized moment-angled complexes; see [22, 4].)

2.1. Partial product complexes.

**Definition 2.1.** Let \(X\) be a space, and \(A \subset X\) a non-empty subspace. Given a simplicial complex \(K\) on vertex set \([n] = \{1, 2, \ldots, n\}\), define \(Z_K(X, A)\) to be the following subspace of the cartesian product \(X^{\times n}\):

\[
Z_K(X, A) = \bigcup_{\sigma \in K} (X, A)^\sigma,
\]

where \((X, A)^\sigma = \{x \in X^{\times n} | x_i \in A \text{ if } i \notin \sigma\}\).

If \(X\) is a pointed space, let \(Z_K(X) = Z_K(X, *)\). For example, \(Z_K(S^1)\) is a subcomplex of the \(n\)-torus \((S^1)^{\times n}\).

2.2. Right-angled Artin groups. If \(\Gamma\) is a graph with vertices \(V(\Gamma) = \{v_1, \ldots, v_n\}\) and edges \(E(\Gamma)\), recall the **right-angled Artin group** \(G_\Gamma\) is defined by the presentation

\[
G_\Gamma = \langle v_1, \ldots, v_n \mid v_i v_j = v_j v_i \text{ for each } v_i v_j \in E(\Gamma) \rangle
\]

Also recall that if \(\Gamma\) is a graph, its **clique complex** \(K_\Gamma\) is the simplicial complex with vertices \(V(\Gamma)\) and simplices \(\sigma\), for all \(\sigma \subseteq V(\Gamma)\) with the property that each pair of vertices of \(\sigma\) is connected by an edge. If a simplicial complex \(K\) is the clique complex of its 1-skeleton, \(K\) is called a flag complex. Then the construction of Definition 2.1 recovers the construction of [5]:

**Proposition 2.2 (5).** Let \(K\) be a simplicial complex and let \(\Gamma = K^{(1)}\), its 1-skeleton. Then \(G_\Gamma \cong \pi_1(Z_K(S^1), *)\). If, further, \(K\) is a flag complex, then \(Z_K(S^1)\) is an Eilenberg-Maclane space for \(G_\Gamma\).

For example, \(\pi_1(Z_K(S^1))\) is abelian if and only if the 1-skeleton of \(K\) is a complete graph. The clique complex of a complete graph is simply a full simplex \(K = \Delta^{n-1}\) on \(n\) vertices, in which case \(Z_K(S^1)\) is the \(n\)-torus \((S^1)^{\times n}\).

Since the main tool used here is the space \(Z_K(S^1)\) (rather than the group \(G_\Gamma\)), it will be natural to consider the homology of spaces, rather than groups; as the Proposition indicates, we will recover the case of groups by specializing to those \(K\) which are flag complexes. We shall correspondingly regard simplicial complexes as our primary objects, rather than graphs.
2.3. Coordinate homomorphisms. Here we identify the coordinate subgroups, defined in the Introduction, in terms of our topological construction.

Such subgroups are a special case of a more general construction. If $f : K \to L$ is a map of simplicial complexes sending vertices $[n]$ to vertices $[m]$, there is a natural map $Z_f : \pi_1(K(S^1)) \to \pi_1(L(S^1))$ obtained by restricting a map $\overline{f} : (S^1)^n \to (S^1)^m$. Here, $\overline{f}(x)_j = \prod_{i : f(i) = j} x_i$; see [6] Lemma 2.2.2 for details.

Using Proposition 2.2, it is routine to translate this to a statement about right-angled Artin groups.

**Proposition 2.3.** If $f : K \to L$ is a map of simplicial complexes, the induced map of fundamental groups $\bar{Z}_f : G_{K^{(1)}} \to G_{L^{(1)}}$ sends the $i$th generator of $G_{K^{(1)}}$ to the $f(i)$th generator in $G_{L^{(1)}}$.

In particular, if $\Gamma = K^{(1)}$ has $n$ vertices, then the abelianization of a right-angled Artin group $G_{\Gamma}$ is clearly $\mathbb{Z}^n$, and the abelianization map $G_{\Gamma} \to \mathbb{Z}^n$ is obtained by choosing $L = \Delta^{n-1}$, the full simplex on $n$ vertices. Moreover, we consider the following case:

**Definition 2.4.** Let $K$ be a simplicial complex on $n$ vertices, and let $f : [n] \to [m]$ be a surjective function on sets. Then $f$ extends uniquely to a map of simplicial complexes $f : K \to \Delta^{m-1}$. We will call $f$ a coordinate map.

It follows from Proposition 2.3 that a coordinate map $f : K \to [m]$ induces a coordinate homomorphism $\bar{Z}_f$, and this homomorphism factors through the abelianization of $G_{K^{(1)}}$:

\[ \bar{Z}_f : G_{K^{(1)}} \xrightarrow{\text{abelianization}} \mathbb{Z}^m \xrightarrow{\pi} \mathbb{Z}^n \]

Conversely, by comparing definitions one finds that all coordinate homomorphisms arise in this way.

**Example 1.** At the one extreme, we may take $f$ to be the identity map. Then the homomorphism $\bar{Z}_f : G_{K^{(1)}} \to \mathbb{Z}^n$ is the abelianization, and its kernel $N_f$ is the commutator subgroup of the right-angled Artin group.

**Example 2.** On the other hand, the (unique) map $f : [n] \to [1]$ induces a homomorphism $\bar{Z}_f : G_{K^{(1)}} \to \mathbb{Z}$ sending each $v_i$ to 1. The kernel is the Bestvina-Brady group $H_{\Gamma}$ considered in [2] [13] [14] [12] [17].

2.4. Abelian covers. Let $\pi : \mathbb{R} \to S^1$ be the universal cover of $S^1$, sending $\mathbb{Z}$ to the basepoint $\ast$. By [6] Lemma 2.9, the map of pairs $\pi : (\mathbb{R}, \mathbb{Z}) \to (S^1, \ast)$ induces a fibration

\[ \mathbb{Z}^n \to \mathbb{Z}_K(\mathbb{R}, \mathbb{Z}) \xrightarrow{\pi} \mathbb{Z}_K(S^1), \]

which is in fact the universal cover of $\mathbb{Z}_K(S^1)$.

By covering space theory, if $f : K \to [m]$ is a coordinate homomorphism, then $N_f$ is the fundamental group of the fibred coproduct $\mathbb{Z}_K(\mathbb{R}, \mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}^m$, where $\mathbb{Z}^n$ acts on $\mathbb{Z}_K(\mathbb{R}, \mathbb{Z})$ by deck transformations, and on $\mathbb{Z}^m$ by the induced map $\overline{f}$. 
2.5. CW-complexes. Combinatorially explicit cell structures are available for each complex. First, the standard cell structure on the torus \((S^1)^n\) restricts to give a cell structure for \(Z_K(S^1)\) (see [11]). From the definition (1), the cells are naturally labelled by the simplices of \(K\). For each simplex \(I \in K\) with \(k\) vertices, let \(\varepsilon_I\) denote the corresponding \(k\)-cell. Note that the complex is minimal in the sense that the attaching maps are zero.

We need the following notation. Let \(E = \mathbb{Z}[e_1, \ldots, e_n]\) denote the exterior algebra, which we regard as a graded-commutative Hopf algebra with generators in degree 1. Let \(\{\varepsilon_i\}\) denote the \(\mathbb{Z}\)-dual basis to the generators, so that \(E^* = \mathbb{Z}[\varepsilon_1, \ldots, \varepsilon_n]\) is also an exterior algebra.

Now let \(Z(K) = E/J_K\), the exterior Stanley-Reisner ring of \(K\), where \(J_K\) is the ideal generated by monomials indexed by nonfaces of \(K\). Then \(Z(K)^*\) is a sub-coalgebra of \(E^*\), spanned by monomials \(\varepsilon_I\), where we set \(\varepsilon_I := \varepsilon_{i_1} \cdots \varepsilon_{i_k}\) if \(I = \{i_1, \ldots, i_k\}\) is a simplex of \(K\). We identify \(Z(K)^*\) with the cellular chain complex of \(Z_K(S^1)\) (with zero differential). The reason for this notation is the following foundational result.

Theorem 1 ([11]). Let \(K\) be a simplicial complex. Then \(H^*(Z_K(S^1), \mathbb{Z}) \cong Z\langle K\rangle\) as graded rings.

Second, we require similar cell structure for the complex \(Z_K(\mathbb{R}, \mathbb{Z})\). It turns out that, in the flag-complex setting, the construction below is really the (abelianized) Salvetti complex for the right-angled Artin group, as explained and generalized by Charney and Davis [5].

To start, label the zero cells of \(\mathbb{R}\) by \(\{x^i : i \in \mathbb{Z}\}\), and the 1-cells by \(\{\varepsilon \cdot x^i : i \in \mathbb{Z}\}\) so that \(\partial(\varepsilon \cdot x^i) = x^{i+1} - x^i\) for each \(i\). Extending this to the product structure on \(\mathbb{R}^n\) and restricting to the subcomplex \(Z_K(\mathbb{R}, \mathbb{Z})\) gives a complex with \(k\)-cells labelled in a natural way by \(\{\varepsilon_I \cdot x^\alpha : I \in K, |I| = k, \alpha \in \mathbb{Z}^n\}\), where we regard \(x^\alpha\) equivalently as an element of \(\mathbb{Z}^n\) (written multiplicatively) or a Laurent monomial \(x^\alpha := x_1^{a_1} \cdots x_n^{a_n}\). It is straightforward to check that the differential is given by

\[
\partial(\varepsilon_I x^\alpha) = \sum_{i \in I} (-1)^{\sigma(I, i)} \varepsilon_{I - \{i\}} \cdot x^{\alpha + \chi_i} - \varepsilon_{I - \{i\}} \cdot x^\alpha,
\]

where \(\sigma(I, i)\) is the number of elements preceding \(i\) in \(I\) (in the standard order) and \(\chi_i\) is the \(i\)th coordinate vector in \(\mathbb{Z}^n\). Make the identification

\[
C^*_{\text{cell}}(Z_K(\mathbb{R}, \mathbb{Z})) \cong Z(K)^* \otimes_{\mathbb{Z}} \mathbb{Z}[\mathbb{Z}^n];
\]

then some useful properties follow immediately from the construction. (Recall \(v_i\) denotes the \(i\)th standard generator of \(\pi_1(Z_K(S^1))\) from [2].)

Proposition 2.5. For any \(K\),

1. The universal cover \(Z_\pi\) of cellular and satisfies \(Z_\pi(\varepsilon_I x^\alpha) = \varepsilon_I\) for all choices of \(I\) and \(\alpha\).
2. The cell structure \([5]\) is \(\mathbb{Z}^n\)-equivariant, and \(v_i\) acts by multiplication by \(x_i\) for \(1 \leq i \leq n\);
(3) The differential \( \partial \) is induced by \( \partial(\varepsilon_i) = x_i - 1 \) for each \( 1 \leq i \leq n \), together with \( \mathbb{Z}[\mathbb{Z}^n] \)-linearity and the Leibniz rule on \( \mathbb{Z}(K)^\ast \).

3. (Co)homology of abelian covers

In this section, we describe the homology of the coordinate subgroups in terms of commutative algebra. If \( f \) is a coordinate map (Def. 2.4), then \( \mathbb{Z}[\mathbb{Z}^m] \) is a \( \pi_1(\mathbb{Z}(S^1)) \)-module via the homomorphism \( \mathbb{Z}(S^1) \). Then

\[
H_\ast(\mathbb{Z}(S^1) \times_{\mathbb{Z}^n} \mathbb{Z}^m, \mathbb{Z}) \cong H_\ast(\mathbb{Z}(S^1), \mathbb{Z}[\mathbb{Z}^m]);
\]

in the case where \( K \) is a flag complex and \( \Gamma = K^{(1)} \), this is simply Shapiro’s Lemma:

\[
H_\ast(N_f, \mathbb{Z}) \cong H_\ast(G_\Gamma, \mathbb{Z}[\mathbb{Z}^m]).
\]

3.1. Linearization. Now fix a coordinate map \( f : [n] \to [m] \). Let \( S = \mathbb{Z}[t_1, \ldots, t_n] \), a (commutative) polynomial ring, and let \( R = \mathbb{Z}[s_1, \ldots, s_m] \). We regard \( R \) as a module over \( S \) via the ring homomorphism \( \tilde{f} : S \to R \) given by letting \( \tilde{f}(t_i) = s_{f(i)} \) for each \( i \). Let \( 1 \) denote the maximal ideal of \( S \) generated by \( \{t_1 + 1, t_2 + 1, \ldots, t_n + 1\} \). We will abuse notation and also write \( 1 \) for its image \( \tilde{f}(1) \) in \( R \). Then localize to invert the elements of \( 1 \), so that \( \mathbb{Z}[\mathbb{Z}^n] \cong S_1 \) and \( \mathbb{Z}[\mathbb{Z}^m] \cong R_1 \).

**Proposition 3.1.** For any \( K \), we have isomorphisms of complexes of \( \mathbb{Z}[\mathbb{Z}^n] \)-modules:

\[
\begin{align*}
C_p^{\text{cell}}(\mathbb{Z}_K(\mathbb{R}, \mathbb{Z})) & \cong (\mathbb{Z}(K)^\ast_p \otimes S, \partial)_1, \\
C_p^{\text{cell}}(\mathbb{Z}_K(\mathbb{R}, \mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}_m^m) & \cong (\mathbb{Z}(K)^\ast_p \otimes R, \partial)_1, \quad \text{and} \\
C_p^{\text{cell}}(\mathbb{Z}_K(\mathbb{R}, \mathbb{Z}) \times_{\mathbb{Z}^n} \mathbb{Z}_m^m) & \cong (\mathbb{Z}(K)^\ast_p \otimes R, \delta)_1,
\end{align*}
\]

for all \( p \geq 0 \), where the complex \( \mathbb{Z} \) is the cellular cochain complex with compact support.

The differential \( \partial \) is induced by \( \partial(\varepsilon_i) = t_i \), for all \( i \), and \( \overline{\partial} \) by \( \overline{\partial}(\varepsilon_i) = s_{f(i)} \). The differential \( \delta \) acts by left multiplication by the element \( \sum_{i=1}^n \varepsilon_i \otimes s_{f(i)} \).

**Proof.** The isomorphism \( (1) \) is obtained by localizing \( \mathbb{Z} \). To establish \( (2) \), we identify \( \mathbb{Z}[\mathbb{Z}^n] \) with the submodule of \( \text{Hom}_\mathbb{Z}(\mathbb{Z}[\mathbb{Z}^n], \mathbb{Z}) \) with finite supports. However, now \( v_i \) acts (contragrediently) by multiplication by \( x_i^{-1} \), for each \( i \). Putting this together with the dual of Proposition 2.5(3) gives the required isomorphism. In fact, the right-hand side of \( (3) \) is a complex of \( E \otimes S_1 \)-modules, since \( \delta \) commutes with multiplication by elements of \( E \).

Since localization is exact, it is equivalent (but more convenient) to regard the cellular (co)chain complexes above over the polynomial ring \( S \), and we shall do so in the rest of the paper.

**Remark 1.** Our restriction to coordinate subgroups in place of arbitrary subgroups with abelian quotient is needed for the isomorphisms of the Proposition above. More generally, the cellular (co)chain complexes are filtered (rather than graded) by powers of the augmentation ideal, and the isomorphisms \( (1) - (3) \) are merely isomorphisms of associated graded modules. As Stefan Papadima observes [16], the spectral sequence of the filtration fails to converge strongly even for very simple examples of non-coordinate homomorphisms.
3.2. Commutative algebra. Given the setup above, it is natural to interpret group (co)homology in terms of skew-commutative algebra. We follow the grading conventions of [8]; in particular, for a graded module $M$, let $M(r)_q = M_{r+q}$ for all $q$. For modules $M$ and $N$, the notation $\text{Ext}^{p,q}(M, N)$ refers to cohomological degree $p$ and polynomial degree $q$. Here, typically $q \leq -p$, so we write $\text{Ext}^p(M, N)_r = \text{Ext}^{p,q}(M, N)$, where $r = -p - q$.

**Theorem 2.** For any simplicial complex $K$ and for all $q \geq 0$,

$$H_q(Z_K(\mathbb{R}, \mathbb{Z}), \mathbb{Z}) \cong \text{Ext}^q_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z})$$

as $S$-modules. Moreover, for all $p \geq 0$,

$$(9) \quad \text{Gr}_p H_*(Z_K(\mathbb{R}, \mathbb{Z}), \mathbb{Z}) \cong \text{Ext}^p_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z})$$

where $\text{Gr}$ denotes the grading associated to the filtration by powers of the augmentation ideal of $\mathbb{Z}^m$.

Before beginning the proof, we note that the (left) $S$-module structure on $\text{Ext}^p_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z})$ here comes the identification $S \cong \text{Ext}^0_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z})$ and its natural action: see [20].

**Proof.** We compute $\text{Ext}^p_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z})$ by the standard injective resolution of $\mathbb{Z}$:

$$0 \to \mathbb{Z} \to I^0 \to I^1 \to \cdots$$

Recall that $E$ is self-injective. Then $\mathbb{Z}$ maps to the generator of $E_n$, and $I^q = E(n + q) \otimes S^q$, with differential induced $E \otimes S$-linearly by multiplication by $\sum_{i=1}^n e_i \otimes t_i$. Then

$$\text{Ext}^p_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z}) = H \text{Hom}_{\mathbb{Z}}(\mathbb{Z}(K), I^p) \cong H (\text{Hom}_{\mathbb{Z}}(\mathbb{Z}(K), \mathbb{Z}) \otimes S)$$

since $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}) = \text{Hom}_{\mathbb{Z}}(-, E)(n)$.

Now we may compare this with Proposition 3.1(6) to obtain the cellular homology of $Z_K(\mathbb{R}, \mathbb{Z})$. □

Now fix a simplicial complex $K$ with $n$ vertices, and a coordinate map $f: [n] \to [m]$. Let $A = A_f$ denote the kernel of the homomorphism $\tilde{f}: S \to R$. Let $a = a_f$ denote the ideal of $E$ generated in degree 1 by functionals that vanish on $A_1$. Let $\text{ann} a$ denote its annihilator in $E$. These ideals have the following properties.

**Lemma 3.2.** For any $f: [n] \to [m]$,

1. the ideal $A$ is generated in degree 1, and $A_1$ is a free $\mathbb{Z}$-module of rank $n - m$.
2. the ideal $a_f$ is generated by $m$ elements, for $1 \leq j \leq m$:

   $$h_j := \sum_{i: f(i) = j} e_i,$$

3. the ideal $\text{ann} a_f$ is principal, generated by $h_1 h_2 \cdots h_m$.

**Proof.** The first two assertions come from the definitions. The third follows from the fact that $\{h_1, \ldots, h_m\}$ are linearly independent in $E_1$. □
In view of Lemma 3.2.13, let \( a_f = h_1 h_2 \cdots h_m \), the generator of \( \text{ann} a_f \). To avoid complications, in what follows we will work over a coefficient field \( \k \). In order to state the next result, recall the following definition. The (combinatorial) Alexander dual of a simplicial complex \( K \) on \( [n] \) is a complex \( K^* \) on \( [n] \). By definition,

\[
K^* = \{ \sigma \subseteq [n] : [n] - \sigma \not\subseteq K \}.
\]

**Theorem 3.** Let \( \k \) be a field, \( K \) a simplicial complex, and \( f : [n] \to [m] \) a coordinate map. Then, for all \( q \geq 0 \),

\[
\begin{align*}
H_q(Z_K(S^1), k[\mathbb{Z}^m]) &\cong \text{Ext}_E(k(K), (a_f))_{q-n} \quad \text{and} \\
H^q(Z_K(S^1), k[\mathbb{Z}^m]) &\cong \text{Ext}_E(J_{K^*}, (a_f))_{q-n},
\end{align*}
\]

where \( J_{K^*} \) is the exterior monomial ideal associated with \( K^* \) (see [2, A]), and \( (a_f) \) is the principal ideal defined above.

**Proof.** Since \( A_S \) is generated in degree 1 and a \( \k \)-basis for \( A_S^1 \) is a regular sequence in \( S \), its Koszul complex is a linear, free resolution of \( R \) over \( S \). That is, \( R \) is a Koszul module over \( S \), so as left \( E \)-modules, \( E/a_f \cong \text{Ext}_S(R, \k) \). Koszul duality is an involution, so \( R \cong \text{Ext}_E(E/a_f, \k) \). This is to say that \( E/a_f \) has a linear, free resolution

\[
0 \leftarrow E/a_f \leftarrow (E \otimes R^*, d)
\]

with a Koszul differential \( \delta \) given by \( \delta(1 \otimes s^*_j) = \sum i : f(i) = j e_i \otimes 1 \), extending \( E \)-linearly and by the Leibniz rule on \( R^* \). Now apply \( \text{Hom}_E(-, E) \). Since \( \text{Hom}_E(E/a_f, E) \) is naturally identified with \( \text{ann} a_f = (a_f) \), this ideal has an injective resolution \( E \otimes R \) with differential given by \( \sum_{i=1}^n e_i \otimes s_{f(i)} \). The proof of isomorphism (10) concludes as in Theorem 2 using Proposition 3.1.7.

The isomorphism (11) is analogous: using the fact that \( J_{K^*} = \text{ann} J_K = \text{Hom}_E(k(K), k) \), we see that the complex \( (k(K) \otimes_k R, \delta) \) of Proposition 3.1.8 actually computes \( \text{Ext}_E(J_{K^*}, (a_f)) \). \( \square \)

**Example 1** (continued). Here, the coordinate map \( f \) is an isomorphism, so \( a_f = E^\geq 1 \), and \( \text{ann} a \cong k(-n) \), the socle of \( E \). Thus Theorem 3 reduces to Theorem 2 (with coefficients in \( k \)). If \( K \) is the clique complex of a graph \( \Gamma \), then Theorem 2 says

\[
H_q(G_\Gamma, \mathbb{Z}[\mathbb{Z}^n]) \cong \text{Ext}_E(\mathbb{Z}(K), \mathbb{Z})_{q,1}
\]

for all \( q \geq 0 \), as modules over \( \mathbb{Z}[\mathbb{Z}^n] \). In general, the homology of the universal abelian cover of the torus complex is encoded in the minimal resolution of a monomial ideal over an exterior algebra.

**Example 2** (continued). In the case of the Bestvina-Brady group, the kernel of the map \( \bar{f} : S \to k[s_1] \) is generated by \( \{ t_i - t_j : 1 \leq i < j \leq n \} \). Then the ideal \( a \) is principal, generated by \( a = \sum_{i=1}^n e_i \), and \( \text{ann} (a) \). So

\[
H_q(H_K, \k) = \text{Ext}_E(k(K), (a))_{q,1}
\]
4. Applications

A main result from the work of [2, 13] is that the Bestvina-Brady group $H_K$ is $FP_n$ iff the flag complex $K$ is $n$-acyclic, yet $H_K$ is finitely presented iff $K$ is simply connected. Then any acyclic, noncontractible flag complex $K$ gives rise to a group which is not finitely presented, yet has finite-dimensional (co)homology. By regarding the homology groups as modules over the group algebra of the abelianization, we can measure (in terms of Krull dimension) how far they are from having finite rank.

4.1. Dimension calculations. Returning to Example 1, recall Proposition 2.1 of [1] provides a description of the bigraded Betti numbers of (9): for $A = E$ or $A = S$, let $eta_{pq}^A(M) = \dim_k \text{Tor}_p^A(M, k)$, for an $A$-module $M$, and let $P_M(t, u) = \sum_{p,q} \beta_{pq}^A(M) t^p u^q$. Let $I = I_K$ denote the squarefree monomial ideal of $K$ in $S$. Then

$$P_{E/J}(t, u) = \sum_{p,q} \beta_{pq}^S(S/I) \frac{t^p u^q}{(1-t)^{p+q}},$$

and

$$\beta_{pq}^S(S/I) = \sum_{\mathcal{I} \subseteq [n]} \dim_k \tilde{H}^{q-1}(K_{\mathcal{I}}, k),$$

by Hochster’s formula.

Then Theorem 2 has the following corollaries.

**Corollary 4.1.** For $q > 0$, the Krull dimension of $H_q(Z_K(\mathbb{Z}, \mathbb{R}), k)$ as a $k[\mathbb{Z}^n]$-module is equal to the size of the largest set $\mathcal{I} \subseteq [n]$ for which $\tilde{H}^{q-1}(K_{\mathcal{I}}, k) \neq 0$. (If there is no such set, then $H_q(Z_K(\mathbb{R}, \mathbb{Z}), k) = 0$.)

**Proof.** It follows from [1, Theorem 4.2, Corollary 3.8] that the dimension of $\text{Ext}_E(k(K), k)_q$ is $p + q$, where $p$ is the largest integer for which $\beta_{pq}^S(S/I) \neq 0$. Now use formula (13). Since the submodule of a graded $S$-module annihilated by the ideal 1 is zero, localization preserves dimension, and our conclusion follows by Theorem 2. 

We may also make use of Alexander duality. In order to draw a parallel with the main result of [10], this result is stated in terms of group cohomology.

**Corollary 4.2.** Let $\Gamma$ be a graph not isomorphic to a complete graph, and $K$ its clique complex. Then the dimension of the $k[\mathbb{Z}^n]$-module $H^q(G_{\Gamma}, k[\mathbb{Z}^n])$ equals the largest integer $r$ for which there exists a simplex $\sigma \in K$ with $n-r$ vertices satisfying $\tilde{H}_{r-q-1}(\text{link}_K(\sigma), k) \neq 0$. (If there is no such simplex, $H^q(G_{\Gamma}, k[\mathbb{Z}^n]) = 0$.)

**Proof.** Since $k(K^*) = E/J_{K^*}$, it follows from the long exact sequence for $\text{Ext}_E$ and Theorem 3 that, for all $q > 0$,

$$H^q(G_{\Gamma}, k[\mathbb{Z}^n]) \cong \text{Ext}_E(J_{K^*}, k)_q \cong \tilde{H}_{q-1}(Z_{K^*}(S^1), k[\mathbb{Z}^n]).$$
Now we use the Alexander dual formulation of Hochster’s formula, for which we refer to [15]: if \( \sigma \in K \) and \( I = [n] - \sigma \), then for all \( q \),
\[
\bar{H}^{q-2}(K_*^I, k) \cong \bar{H}_{|I|-q-1}(\text{link}_K(\sigma), k).
\]
Then formula (13) becomes
\[
\beta_{p,q-1}^\mathbb{Z}(S/I_K^\bullet) = \sum_{\sigma \in K: ||\sigma| = n-(p+q-1)} \dim_k \bar{H}_{p-2}(\text{link}_K(\sigma), k),
\]
from which the result follows as in Corollary 4.1, by letting \( r = p + q - 1 \). \( \square \)

Recall that \( \mathbb{Z}^n \) is the abelianization of \( G_\Gamma \). The Corollary shows (by comparing with [10]) that \( H'(G_\Gamma, k[G_\Gamma]) \) and \( H'(G_\Gamma, k[G_\Gamma^{ab}]) \) depend on \( \Gamma \) in the same way.

4.2. The rank variety of \( k\langle K \rangle \). Recall that if \( M \) is a graded \( E \)-module and \( a \in E^1 \), then \( M \) may be regarded as a chain complex with differential given by multiplication by \( a \). Aramova, Avramov, and Herzog [11] define an element \( a \in E^1 \) to be \( M \)-singular if the cohomology of \( (M, a) \) is nonzero, and let \( V(M) \) be the variety of all \( M \)-singular elements. Say \( a \in E^1 \) is \( M \)-regular if \( a \notin V(M) \).

If \( a = \sum_{i=1}^n \alpha_i e_i \), let \( \text{supp}(a) = \{ i : \alpha_i \neq 0 \} \), and consider the cohomology of \( k\langle K \rangle = E/J \), regarded as a chain complex with differential given by multiplication by the element \( a \). We recall the following Proposition 4.3 of [11] (with indexing corrected). (See also [15], Theorem 5.5.)

**Proposition 4.3.** The cohomology of \( (E/J, a) \) depends only on \( \text{supp}(a) \). Let \( I = \text{supp}(a) \). Then
\[
H^q(E/J, a) \cong \bigoplus_{\sigma \in K: |\sigma| \cap I = \emptyset} \bar{H}^{q-|\sigma|-1}(\text{link}_{K^I_\tau}(\sigma), k),
\]
where by definition \( \text{link}_{K^I_\tau}(\sigma) = \{ \tau \in K^I_\tau : \tau \cup \sigma \in K \} \).

In particular, this characterizes the \( E/J \)-singular elements.

A sequence of elements \( a_1, \ldots, a_r \in E^1 \) is called \( M \)-regular if \( a_1 \) is \( M \)-regular and \( a_i \) is \( M/(a_1, \ldots, a_{i-1}) \)-regular for each \( i \), \( 2 \leq i \leq r \).

**Proposition 4.4.** For any coordinate map \( f : [n] \to [m] \), \( H_*(Z_K(S^1), k[Z^m]) \) is finite-dimensional (over \( k \)) if and only if the elements \( \{ h_1, \ldots, h_m \} \) defined in Lemma 4.2(2) form a \( k\langle K \rangle \)-regular sequence.

**Proof.** If \( m = 1 \), \( h_1 = a = \sum_{i=1}^n e_i \). Then the ideal \( (a) \) has an injective resolution
\[
0 \rightarrow (a) \rightarrow E \xrightarrow{a} E \xrightarrow{a} \cdots
\]
so \( \text{Ext}_E(k\langle K \rangle, (a)) \) is the cohomology of the infinite complex
\[
E/J \xrightarrow{a} E/J \xrightarrow{a} \cdots
\]
It follows \( \text{Ext}^0_E(k\langle K \rangle, (a)) = E/(J + (a)) \), and
\[
\text{Ext}^p_E(k\langle K \rangle, E/(a))_q \cong H^q(E/J, a)
\]
for all \( p \geq 1 \). In particular, \( \text{Ext}_E(k\langle K \rangle, (a)) \) is finite-dimensional iff \( a \) is \( E/J \)-regular.
The general result follows from induction on \( m \), using the K"unneth formula. \( \Box \)

**Example 2** (continued). For the Bestvina-Brady group (where \( f: [n] \rightarrow [1] \)), the only summand in Proposition \[4.3\] is indexed by the empty simplex, so \( H^q(E/J, a) = H^{q-1}(K, k) \).

By the argument above, \( H^q(Z_K(S^1), k[2]) \) is finite-dimensional if and only if \( \tilde{H}^{q-1}(K, k) = 0 \), in which case \( H^q(Z_K(S^1), k[2]) \cong E/(J + (a))^q \). This recovers the additive part of a result due to Leary and Saadeto"glu \[12\].

5. **Duality**

We single out the case of Cohen-Macaulay complexes \( K \) for their particularly nice properties. First, recall the Theorem of Eagon and Reiner in \[7\]. For this, let \( I_K \) denote the defining ideal of the Stanley-Reisner ring of a complex \( K \), and write \( k[K] = S/I_K \).

**Theorem 4** (\[7\]). The ideal \( I_K \) has a linear, free resolution over \( S \) if and only if the complex \( K \) is Cohen-Macaulay.

If \( K \) is a Cohen-Macaulay simplicial complex of dimension \( d \), consider the Cartan complex: this is the complex \( (k(K) \otimes S, \omega) \), whose differential is given by multiplication by the element \( \omega = \sum_{i=1}^{n} e_i \otimes u_i \). Let \( F_K = H^{d+1}(k(K) \otimes S, \omega) \).

The next Lemma is a basic consequence of Bernstein-Gelfand-Gelfand or Koszul duality for the module \( F_K \). We refer to \[8, 9, 19\] for background.

**Lemma 5.1.** The following are equivalent.

1. \( K \) is Cohen-Macaulay of dimension \( d \).
2. \( H^p(k(K) \otimes S, \omega) = 0 \) for \( p \neq d + 1 \).
3. \( \text{Tor}^S(F_K, k) \cong k(K)^n \otimes (n - d + 1) \cong J_K, (n - d + 1) \) as graded \( E \)-modules.

An interesting special case is that of \( K \) a homology sphere (Gorenstein* complex). From \[7\], it follows that:

**Lemma 5.2.** \( F_K \cong I_K \) as \( S \)-modules.

Then BGG duality give the following reformulation of Theorem \[8\]

**Theorem 5.** For any Cohen-Macaulay complex \( K \) of dimension \( d \) and coordinate map \( f: [n] \rightarrow [m] \), there is an isomorphism of \( S \)-modules for \( q \geq 0 \):

\[
H_q(Z_K(S^1), k[Z^m]) \cong E_{d+1-q}(F_K, R).
\]

Dually,

\[
H^q(Z_K(S^1), k[Z^m]) \cong \text{Tor}^S_{d+1-q}(F_K, R).
\]

**Proof.** By Lemma \[5.1.2\], the Cartan complex is a free resolution of \( F_K \) over \( S \). By applying \( \text{Hom}_S(-, R) \), we obtain the cellular chain complex for \( Z_K(S^1) \times_{Z^m} Z^m \) of Proposition \[3.1.7\], and the result follows.

The dual statement is proven in the same way. \( \Box \)

**Corollary 5.3.** Suppose \( K \) is the clique complex of a graph \( \Gamma \) and \( K \) is Cohen-Macaulay of dimension \( d \). For any coordinate map \( f: [n] \rightarrow [m] \), we have

\[
H^q(G_\Gamma, k[Z^m]) \neq 0 \quad \text{for} \quad q \leq m - n + d \quad \text{and} \quad q > d + 1.
\]
Proof. Since \( \text{pd}_S R = n - m \), \( \text{Tor}^S_i(F_K, R) = 0 \) for \( i > n - m \). Now apply (14). \( \square \)

Example 1 (continued). Suppose \( K \) is the clique complex of \( \Gamma \), and \( K \) is Cohen-Macaulay of dimension \( d \). Then the cohomology with compact support of the universal abelian cover is concentrated in dimension \( d + 1 \); more precisely, as modules over \( k[G^\text{ab}_\Gamma] \),

\[
H^q(G_\Gamma, k[G^\text{ab}_\Gamma]) \cong \begin{cases} (F_K)_1 & \text{for } q = d + 1; \\ 0 & \text{otherwise}. \end{cases}
\]

It should be noted, however, that the vanishing of cohomology here and in the Corollary above also follow from the paper of Brady and Meier [3]: they establish the more general result that \( G_\Gamma \) is a duality group if and only if \( K \) is a Cohen-Macaulay complex.

Example 2 (continued). Under the same hypotheses, the homology of the Bestvina-Brady group is given for \( q \geq 0 \) by

\[
H_q(H_K, k) \cong \text{Ext}_{S}^{d+1-q}(I_K^*, k[s]).
\]

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