Quantum mechanical scattering by chaotic or disordered systems, has been a subject of a rather intensive research activity during the last ten years. This interest was motivated by various areas of physics, ranging from nuclear, atomic and molecular physics, to mesoscopies and classical wave scattering. The most fundamental object characterizing the process of quantum scattering is the unitary S-matrix relating the amplitudes of incoming waves to the amplitudes of outgoing waves. At present, there are two complementary theoretical tools employed to calculate statistical properties of the S-matrix, namely the semiclassical and the stochastic approach. The starting point of the first is a representation of the $S$-matrix elements in terms of a sum of classical orbits while the later exploits the similarity with ensembles of Random Matrices. Thus, for chaotic systems, many results are known. Namely, the distribution of the S-matrix was found to be described by the so-called Poisson Kernel. In the same framework, the Wigner delay time statistics has been studied intensively. This quantity captures the time-dependent aspects of quantum scattering. It is related to the time spent in the interaction region by a wavepacket of energy peaked at $E$. The delay time is not self-averaging and one must have its full probability distribution. For the one-channel case this was found for a 1D disordered sample of length $L$ with one semi-infinite perfect lead attached on the left side. The system is described by the tight-binding equation:

$$\psi_{n+1} + \psi_{n-1} = (E - V_n)\psi_n; \quad E = 2\cos k$$

where $\psi_n$ is the wavefunction amplitude at site $n$. We assume that for $0 \leq n \leq L$, $V_n$ is random delta-correlated given by a distribution $\mathcal{P}_V$ with mean zero and variance $\sigma^2$. For $n < 0$, $V_n = 0$ and we impose Dirichlet boundary conditions at the edge $\psi_{L+1} = 1$. Therefore, for $n \leq 0$, scattering states of the form $\psi_n = e^{ikn} + re^{-ikn}$ represent the superposition of an incoming and a reflected plane wave. Since there is only backscattering, the reflection coefficient $r(E) \equiv \frac{d\Phi(E)}{dE}$ is of unit modulus and the total information about the scattering is contained in the phase $\Phi(E)$. The Wigner delay time is given by $\tau(E) \equiv \frac{d\Phi(E)}{dE}$. Our aim is to find the probability distribution of the phases $\mathcal{P}_\Phi(\Phi)$ and of delay times $\mathcal{P}_\tau(\tau)$.

We study the distribution of phases and Wigner delay times for a one-dimensional Anderson model with one open channel. Our approach, based on classical Hamiltonian maps, allows us an analytical treatment. We find that the distribution of phases depends drastically on the parameter $\sigma_A = \sigma/sink$ where $\sigma^2$ is the variance of the disorder distribution and $k$ the wavevector. It undergoes a transition from uniformity to singular behaviour as $\sigma_A$ increases. The distribution of delay times shows universal power law tails $1/\tau^2$, while the short time behaviour is $\sigma_A$-dependent.

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Consequently we have the following relation for the total transfer matrix $P = \prod_{n=0}^{L} M_n$

$$
\begin{pmatrix}
0 \\
\psi_L
\end{pmatrix} = \begin{pmatrix}
P_{11} & P_{12} \\
P_{21} & P_{22}
\end{pmatrix} \begin{pmatrix}
1 + r e^{-ik} + r e^{ik} \\
1 - r e^{-ik} - r e^{ik}
\end{pmatrix}.
$$

Solving Eqn. \ref{eqn:psiL} for $r$ we get $r = \frac{P_{11} P_{22} - P_{12} P_{21}}{P_{21} + P_{12} e^{-ik}} = e^{2i\phi}$ where $\phi = \Phi/2$ is now given by

$$
tan\phi = \frac{P_{11} + P_{12} \cos k}{P_{12} \sin k}.
$$

As was indicated in \cite{13}, one can rewrite \eqref{eqn:PSI} in the form of a two-dimensional Hamiltonian map $Q_n$

$$
\begin{pmatrix}
x_{n+1} \\
p_{n+1}
\end{pmatrix} = Q_n \begin{pmatrix}
x_n \\
p_n
\end{pmatrix};
Q_n = \begin{pmatrix}
\cos k - A_n \sin k & \sin k \\
-A_n \cos k & \sin k
\end{pmatrix}.
$$

Here, $(x_n,p_n)$ play the role of a position and momentum of a parametric linear oscillator subjected to periodic kicks of strength $A_n = \frac{\sqrt{\sigma}}{\pi}$ and period $T = 1$. Between two successive kicks, there is a free rotation in the phase space which is determined by the eigenenergy $E$ of our initial equilibrium $\psi_0$. In such a representation, amplitudes $\psi_n$ of a specific eigenstate correspond to positions of the oscillator at times $t_n = n$. $Q_n$ is related to the transfer matrix $M_n$ through a similarity transformation \cite{13}:

$$
Q_n = RM_n R^{-1};
\begin{pmatrix}
1 & 0 \\
\cos k & \sin k
\end{pmatrix}.
$$

As a result, the total transfer matrix $P$ is related to the map $F = \prod_{n=0}^{L} Q_n$ through the similarity transformation $P = R^{-1} F R$. Using this, together with \eqref{eqn:PSI}, we write $\phi$ in terms of our Hamiltonian map $F$ as:

$$
tan\phi = \frac{F_{11}}{F_{12}}.
$$

We can give a geometrical interpretation for \eqref{eqn:PSI}. Consider the time evolution of the vector $v(t) = (0,1)^T$ under the inverse map $F^{-1}$. For time $t_n = L$ we have

$$
v(t = L) = F^{-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -F_{12} \\ F_{11} \end{pmatrix}.
$$

Then $\phi$ is exactly the angle between the vector $v(t = L)$ and the x-axis. It is convenient therefore to pass to polar coordinates $(r_n,\theta_n)$, using the transformation $x = r \cos \theta$ and $p = r \sin \theta$. Then \eqref{eqn:PSI} is written in the form \cite{13}:

$$
\begin{align*}
t_n^2 &= \frac{1}{\sigma} D_n; \\
D_n &= 1 + A_n^2 \cos^2 \theta_n - A_n \sin 2\theta_n \\
\cos \theta_{n+1} &= D_n^{-1} [\cos (\theta_n + k) + A_n \cos \theta_n \sin k] \\
\sin \theta_{n+1} &= D_n^{-1} [\sin (\theta_n + k) - A_n \cos \theta_n \cos k].
\end{align*}
$$

The relation between $\phi_{n+1}$ and $\phi_n$ is found after inversion of \eqref{eqn:PSI} and has the form \cite{17}:

$$
tan(\phi_{n+1}) = tan(\phi_n - k) + A_n.
$$

From \eqref{eqn:PSI}, we get the following equation for the stationary ($n \to \infty$) distribution of phases $P_\phi(\phi)$:

$$
P_\phi(\phi) = \int ds P_\phi(\phi + k) P_s(\phi).
$$

In the $\sigma_A \ll 1$ limit, we can approximate $P_s(\phi)$ by a $\delta$-function. Then from \eqref{eqn:3} we have for the $P_\phi(\phi)$

$$
P_\phi(\phi) = \frac{1}{2\pi};
\phi \notin Q
$$

in agreement with previous numerical results \cite{10}. We understand \cite{13} in the following way: For $\sigma_A \ll 1$, the particle travels long distances inside the sample and undergoes many scattering events which leads to randomization of the phase. We should stress that for $k$ equals to rational values of $\pi$, the distribution is non-uniform. For example, for $k = \pi/2$ we obtain \cite{14, 17, 18}

$$
P_\phi(\phi) = \left(2\sqrt{2} \frac{1}{\sqrt{3} + \cos (2\phi)}\right)^{-1};
k = \frac{\pi}{2}
$$

where $K$ is the complete elliptic integral of the first kind. A similar anomaly at the band center governs the distribution of the angles $\theta$ of the direct map \cite{13}. The above expressions \cite{13, 14} describe well the corresponding numerical results presented in Figs. 1a,b.

In the $\sigma_A \gg 1$ limit, we distinguish between two parts of the spectrum. Namely, $k$ near the band edges (i.e. $k \simeq 0, \pi$) and far away from them. For the latter case and for $V_n$ distributed uniformly between $[-\frac{1}{2}, \frac{1}{2}]$, one can derive an analytical expression for $P_\phi$ by rewriting \eqref{eqn:PSI} in terms of the probability distributions $P_{\tan(\phi)}$.

$$
P_{\tan(\phi)}(y) = (\sqrt{2} \sigma_A)^{-1} \Theta \left(\sqrt{3} \sigma_A - |y - cot k|\right).
$$

where $\Theta(x)$ is the Heavyside function. The distribution of phases $P_\phi(\phi)$ can be derived easily from \eqref{eqn:13}:

$$
P_\phi(\phi) = \frac{0.5}{\sqrt{3} \sigma_A} \Theta \left(\sqrt{3} \sigma_A - \frac{1}{2} \tan(\frac{\phi}{2}) - cot k\right).
$$

In contrast with the $\sigma_A \ll 1$ limit \cite{13}, here the distribution of phases \cite{10} is highly non-uniform. By increasing the disorder strength $\sigma$, two peaks appear in the neighbourhood of $\Phi = \pi$ (see Fig. 1c) while a gap is created.
between them. As \( \sigma_A \) increases further, the two peaks move closer to one another. Now the scattering is so strong that most of the particles are reflected back from near the surface, thus being scattered only from a few sites. In the limit \( \sigma_A \to \infty \), the distribution becomes a \( \delta \) function centered at \( \Phi = \pi \). (10) is in very good agreement with our numerical data presented in Fig. 1c. The behavior near \( \Phi = \pi \) is more subtle than the one given above; nevertheless, (11) gives the correct scale on which the distribution vanishes near \( \Phi = \pi \). Distribution (10) agrees quite nicely also for other disorder potentials [15].

Near the band edge \( k \approx 0, \pi \), our detail numerical analysis showed that \( \mathcal{P}_\phi(\Phi) \) is again highly non-uniform. Namely the distribution becomes narrower and centered (although with a slight asymmetry) at \( \Phi = \pi \). We notice that such a choice of the parameters, can be realized even for weak disorder \( \sigma \ll 1 \). In Fig.1d we present a representative case corresponding to \( k = 10^{-3}\sqrt{\pi} \) and \( \sigma = 0.0577 \).

Let us now turn to the analysis of delay times. From (11) we get the following iteration relation for \( \tau_n \)

\[
\tau_{n+1} = G_n^{-1} \left( \tau_n + \frac{1}{\text{sink}} \right) + \frac{A_n}{1 + (\text{tanh}(\phi_n - k) + A_n)^2 \text{sink}} \cot k
\]

\[
G_n = 1 + A_n \text{sin}(2(\phi_n - k)) + A_n^2 \cos^2(\phi_n - k)
\]

which proves to be very convenient for numerical calculations since it anticipates the numerical differentiation which is a rather unstable operation.

For \( \sigma_A \ll 1 \) and \( k \) equals to irrational multiples of \( \pi \), we obtain an analytical expression for \( \mathcal{P}_\tau(\tau) \). To this end, we first derive from (11) and (17) a stochastic equation for the phases \( \phi \) and rescale delay times \( \tilde{\tau} = \sigma^2 \tau \) (up to \( \sigma_A^2 \)), respectively [13]:

\[
\frac{d\phi}{dt} \approx -k - \sigma_A^2 \sin(\phi - k)\cos^3(\phi - k) + \cos^2(\phi - k)A
\]

\[
\frac{d\tilde{\tau}}{dt} \approx -\sigma_A^2 \left( \tilde{\tau} \cos^2(\phi - k) - \sin^2 2(\phi - k) \right) - \tilde{\tau} \sin(\phi - k)A.
\]

Using the van Kampen lemma, from (13) we obtain the Fokker-Planck equation for the joint probability distribution of \( \phi \) and \( \tilde{\tau} \). Using the fact that \( \phi \) follows the uniform distribution (see Eqn. (13)) and assuming that the variables \( \phi \) and \( \tilde{\tau} \) are statistically independent, we obtain the Fokker-Planck equation for \( \mathcal{P}_\tau(\tilde{\tau}) \)

\[
\frac{\partial \mathcal{P}_\tau(\tilde{\tau}, t)}{\partial t} = \frac{\sigma_A^2}{4} \left[ \frac{\partial}{\partial \tilde{\tau}} \left( \tilde{\tau} - 4 \text{sink} \right) \mathcal{P}_\tau(\tilde{\tau}, t) \right]
\]

\[
+ \frac{\partial}{\partial \tilde{\tau}} \left( \tilde{\tau} \frac{\partial}{\partial \tilde{\tau}} \left( \mathcal{P}_\tau(\tilde{\tau}, t) \right) \right).
\]

The resulting stationary distribution is obtained by setting \( \frac{d\mathcal{P}_\tau}{dt} = 0 \) and has the form

\[
\mathcal{P}(\tau) = \frac{l_\infty}{\nu^2} e^{-l_\infty/\nu}; \quad \sigma_A \ll 1
\]

where \( l_\infty = 2(4 - E^2)/\sigma^2 \) is the localization length and \( \nu = |\partial E/\partial k| \) is the group velocity. We note that (20) is independent of the nature of the disorder; it only depends on its second moment through the localization length. (20) takes its maximum value at \( \tau_{\max} = 0.5 l_\infty/\nu \), indicating that the most probable trajectory that an electron travels (forth and back) before it scatters outside the sample is the mean free path \( l_M = l_\infty/4 \). Our numerical results (see Fig. 2a) are in perfect agreement with (20). An expression similar to (20) was obtained in [14] and more recently in [17] using totally different approaches. It is important to stress here that the latter analytical results refer to continuous disordered models.

Finally we discuss the distribution of delay times \( \mathcal{P}(\tau) \) for \( \sigma_A \gg 1 \). In this limit, we were not able to derive any analytical expression. Our iteration relation (11) however, has proven very efficient for numerical investigations. In Fig. 2b we show the distribution of the delay times for a uniform and a Gaussian on-site potential distribution with the same variance \( \sigma^2 = 10 \). It is clear that the short time distribution differs considerably in the two cases and also with respect to the theoretical prediction (20). On the other hand, the distribution of large delay times, reflecting the times for which the wave penetrates deeper into the sample shows the same \( 1/\tau^2 \) behavior independent of the form of the disorder potential. It is this part of the reflected wave, and not the prompt part that is expected to behave in a universal way [18]. The asymptotics of the distribution for large \( \tau \) is presented in the inset of Fig. 2b, where we plot the integrated distribution

\[
I(\gamma) = \int_0^\tau \mathcal{P}(\gamma')d\gamma'
\]

of the inverse delay time \( \gamma = 1/\tau \) in
log-log scale. To this end, we calculated $10^7$ delay times using the iteration relation (17). In both cases (Gaussian and uniform distribution) presented in the inset of Fig. 2b we collected at least $10^4$ delay times that were larger than $\tau > 150$. Our numerical data clearly show that $I(\tau) \sim \gamma$ for $\gamma \ll 1$. Thus $P(\tau) \sim 1/\tau^2$ in agreement with the above mentioned universal behavior (10).\[P(\tau)\]

![FIG. 2. (a) Distribution of the delay times $P(\tau)$ for on-site potential, uniformly distributed between $[-0.1;0.1]$ and wavenumber $k = \sqrt{\pi}$. The dashed line corresponds to (10). (b) Distribution of the delay times $P(\tau)$ for uniform and Gaussian $P(\gamma)$. In both cases the variance is $\sigma^2 = 10$ and the wavenumber is $k = \sqrt{\pi}$. The dashed line corresponds to (10). The inset shows the integrated distributions of inverse delay time $\gamma$. The dashed line has slope 1 and is drawn to guide the eye.]

To summarize, we analyzed the distributions of phases $P(\Phi)$ and delay times $P(\tau)$ for a 1D Anderson model with one channel in the localized regime. For $P(\Phi)$ we found that it undergoes a transition from uniform ($\sigma_A < 1$) to non-uniform behaviour ($\sigma_A > 1$), where $\sigma_A = \sigma/sink$. The latter limit can be achieved either by decreasing the disorder strength $\sigma$, or by taking $k \to 0, \pi$. This has further implications for the so-called one-parameter scaling hypothesis. The scaling theory of disordered systems (15) assumes that $P(\Phi)$ is always uniform in the weak disorder limit $\sigma \ll 1$ leading to the one parameter scaling. Violation of the latter is always associated with an increase of $\sigma$. Our analytical and numerical results showed however that a strong violation of the phase uniformity occurs even in the case of weak disorder $\sigma \ll 1$, in an apparent contradiction with the existing picture. In this limit, the states near the band edge $k \approx 0, \pi$ never obey the single-scaling hypothesis since $\sigma_A \gg 1$. Thus the spectrum of the system is divided into two groups with different scaling properties, which coexist at the same strength of disorder (16). For $P(\tau)$ we found that only the short time behaviour is affected as we are increasing $\sigma_A$. For $\tau \to \infty$, $P(\tau) \sim 1/\tau^2$ independent of the value of $\sigma_A$. This leads to a logarithmic divergence of the average value of $\tau$ indicating the possibility of the particle traversing the infinite sample before being totally reflected. As was indicated in (13) this is another manifestation of the fact that in the localized regime the conductance shows lognormal distribution due to the presence of Ahbel resonances.

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