Quaternion Mahalanobis Non-local Means for Color Image Denoising

Xudong Xu* and Jinling Zhao*
School of Mathematics and Physics, University of Science and Technology, Beijing, China
Email: xuxuduung@163.com; jlzhao@ustb.edu.cn

Abstract. By improving the quaternion non-local means (QNL) image denoising algorithm, the quaternion Mahalanobis non-local means (QMNLM) method is proposed. This method combines quaternion with non-local similarity priors and introduces quaternion Mahalanobis distance. Firstly, the Mahalanobis distance between the image patches is calculated in the eigenspace by singular value decomposition of quaternion since the Mahalanobis distance is not robust in the sample space. Secondly, the image data is analyzed with the principal component analysis method, thus the Mahalanobis distance equation is simplified. The image with noise affected by high noise was first processed with a Gaussian low pass filter (LPF). Experimental results show that the noise reduction effect of this method is better than the non-local means (NLM), generalized non-local Means (GNLM) and QNLM methods. The proposed algorithm can effectively eliminate image noise and efficiently improve the image visual effect.

Keywords. Non-local similarity; Quaternion Mahalanobis distance; Quaternion Mahalanobis non-local means (QMNLM); color image denoising.

1. Introduction
Image signals are often destroyed by various noises during the process of generation, transportation and processing, thus seriously destroying the visual effect of the image. Therefore, before the image is subjected to edge detection, image compression, feature extraction, image recognition, etc. it is an indispensable pre-processing step to use appropriate methods to reduce image noise damage to the image. Common noise preprocessing methods include patch-matching and 3-D filtering method (BM3D) [1, 2], generalized k-means clustering for singular value decomposition (K-SVD) [3, 4], image dictionary denoising based on sparse and structural clustering [5]. At the same time, the recommend of matching pursuit [6-8] and basic pursuit [2, 9] leads to the ability of image denoising as a direct sparse decomposition technique. Same dictionary learning algorithms have been proposed, including the optimal direction method [2, 10], online dictionary learning method [11]. The original non-local means (NLM) algorithm is proposed in [11, 12]. The main core problem of the non-local means filtering algorithm is how to determine the weight function. The NLM algorithm uses the direct-type function to weight [2, 12]. NLM has good results to eliminate the white noise, whereas it is not effective for the colored noise. Based on the above reasons, the literature [13] established a new generalized non-local means (GNLM) denoising model.

Some methods have good denoising effects on grayscale images [2, 3, 10, 13]. However, to extend these algorithms into color images, there is a dilemma between how to balance image noise reduction and preserve image red, green and blue (RGB) details. The easiest way is to apply the traditional noise reduction method to the channels of the RGB image separately, to filter the noise components separately,
and then integrate the three channels together. Although the effect of noise reduction can be achieved, it is unfortunate to ignore the interrelationship between multiple channels, which is likely to destroy the color details of the color image in the reconstruction result. Thus, how to construct a method that can maintain the image details has been a research focus.

The quaternion, as a super complex algebra, provides a useful mathematical object to process color image signals. Based on the imitation of human perception of the visual environment, a quaternion-based approach is proposed to process multi-channel information in a parallel manner. In recent years, several filtering methods for processing color images in view of quaternion theory have been proposed. These algorithms present new solutions to some basic problems, such as color image denoising [14-16], color image watermarking [2, 17], color image super-resolution [18], image colorization [2, 19], and color image segmentation [2, 20, 21]. The current method of processing color images based on quaternions lays the foundation for the algorithm of quaternion color image processing. Based on the original NLM algorithm [2, 10, 13], a quaternion non-local means (QNL) [22-24] algorithm is proposed. This paper applies the QNL algorithm to the color image denoising. The model fully considers the redundant information in the color image, but one of the disadvantages of QNL image denoising algorithm is the large amount of computation. It calculates the pixel value similarity and structure similarity of two image patches at the same time. It needs to consider multiple image patches in a certain area around two image patches. Another disadvantage comes from the use of Euclidean distance between two image patches to calculate the pixel weight. Euclidean distance is a similarity degree for different sample data. Although this method is simple, it also has obvious defects: it does not consider the different distribution of different samples, nor can it exclude the interference of data correlation, so it affects the QNL image denoising algorithm to obtain better image denoising effect [25-27]. In this paper, aiming at the second defect of QNL image denoising algorithm, an QMNLM algorithm based on Mahalanobis distance is proposed. QMNLM method is used to get the image denoising effect which is obviously better than the original QNL image denoising algorithm. QMNLM algorithm is mainly to find out the image patches with similar neighborhood structure of each image patch, and then calculate their weighted average value to get the estimated value of the pixel, so as to achieve denoising.

Extend the non-local similarity of gray image based on Mahalanobis distance to the quaternion color image processing. The Mahalanobis distance of quaternion is defined, and based on the prior similarity weight of quaternion similarity patch, the QMNLM method is proposed to keep the relevance between RGB channels well. We calculate the Mahalanobis distance between the image patches and replace the adverse of the covariance matrix with the Moore – Penrose adverse.

For color image denoising, it is divided into four cases: the std. of noise $\sigma$ are taken as 10, 30, 50 and 70. Firstly, we find the appropriate patch size for four cases, and then use QMNLM method to denoise the noisy image. The image with noise affected by high noise was first processed with a Gaussian low pass filter (LPF) [2], then denoise the filtered image with QMNLM method. Experimental results show that the noise reduction effect of this method is well, compared with the NLM, GNLM and QNL methods. This paper realizes various methods of color image denoising in MATLAB.

Other parts of the sentence are: Section II revisits some basic concepts and describes the notation used in the work, consisting of some theory of color image algebra and singular value decomposition of quaternion. The QMNLM model based on Mahalanobis distance and QNL algorithm based on Euclidean distance in the frame of non-local similarity of image sub-patches are proposed in section III. Section IV is the utilization of QMNLM method in RGB image denoising. Section V introduces the experiment of denoising and compares the four methods based on Euclidean distance and Mahalanobis distance in real number domain and quaternion domain respectively. Results in the future, there were no official affairs in this chapter VI. Finally, section VII summarizes our work.

2. Quaternion Theory
Lower-case handwritten letters represent quaternion vectors, such as $a$; capitalized handwritten typeface represent quaternion matrices, such as $\mathcal{A}$; capitalized Roman typeface represent real matrices, such as $X$;
2.1. Quaternion Algebra

Definition 1. In 1832, quaternion was first proposed by Hamilton [27]. A quaternion \( q \) is consisted of one the actually number and three unreliable figure,

\[
q = a + bi + cj + dk
\]  

(1)

\( a, b, c, d \) are real numbers, \( i, j, k \) are imaginary unit vectors, and satisfy the following relations,

\[
i^2 = j^2 = k^2 = ijk = -1
\]  

(2)

If the quaternion \( q \) has no real part \( (a = 0) \), it is a pure quaternion. A matrix \( Q \) with quaternion elements is a quaternion matrix.

In 1999, Pei first proposed a quaternion model for color images [28]. Pixels in color images are composed of Red, Green, and Blue, three basic colors, in a certain proportion. Thus, it is only quaternion elements is a quaternion matrix.

Definition 2. [29, 30] Being \( a = a_0 + a_1 i + a_2 j + a_3 k \in \mathbb{H} \) and \( c = c_0 + c_1 i + c_2 j + c_3 k \in \mathbb{H} \) two quaternion numbers represented throughout the certain expression defined in (1), addition of \( a \) and \( c \) is a component-wise operation computed as

\[
a + c = (a_0 + c_0) + (a_1 + c_1)i + (a_2 + c_2)j + (a_3 + c_3)k
\]  

(4)

The product of \( a \) and \( c \) would be expressed as follow,

\[
a c = \left( a_0 c_0 - a_1 c_1 - a_2 c_2 - a_3 c_3 \right) + \left( a_0 c_1 + a_1 c_0 + a_2 c_3 - a_3 c_2 \right)i + \left( a_0 c_2 - a_1 c_3 + a_3 c_1 + a_1 c_3 - a_2 c_1 + a_3 c_0 \right)j + \left( a_0 c_3 + a_1 c_2 - a_2 c_1 + a_3 c_0 \right)k
\]  

(5)

Definition 3. [29, 30] Let \( q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} \) Quaternion conjugate \((q^H)\) is given by:

\[
q^H = q_0 - q_1 i - q_2 j - q_3 k
\]

Definition 4. [29, 30] N scalars form an N-dimensional quaternion vector. Each element is a quaternion: \( a = [a_1 \ a_2 \ldots \ a_n]^T \in \mathbb{H}^N, a_i \in \mathbb{H} \). In this vector space, the interior product of it can be expressed as:

\[
\langle a, c \rangle = a^H c = \sum_{i=1}^{N} a_i^H c_i
\]  

(6)

where \( a^H \) denotes the conjugate transposition of quaternion vector \( a \).

Definition 5. [29-31] The definition of the norm for quaternion vectors is

\[
\|a\| = \sqrt{\langle a, a \rangle} = \sqrt{a^H a}
\]  

(7)

Definition 6. [2, 31, 32] A metric is represented as measure the distance between two quaternion vectors \( a \) and \( c \),

\[
d(a, c) = \langle a - c, a - c \rangle = [(a - c)^H(a - c)]^{1/2}
\]  

(8)

The two matrices of quaternion are denoted as \( A = [a_1, a_2, \ldots, a_m, \ldots, a_M] \in \mathbb{H}^{N \times M}, a_m \in \mathbb{H}^N \)
and $\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, ..., \mathbf{b}_m, ..., \mathbf{b}_K] \in \mathbb{H}^{N \times K}$, $\mathbf{b}_m \in \mathbb{H}^N$.

**Definition 7.** [29, 30] The product of $a$ and $c$ would be expressed as $\mathbf{C} = \mathbf{A}^H \mathbf{B} \in \mathbb{H}^{M \times K}$, with the element $c_{mk} = \langle a_m, \mathbf{b}_k \rangle$.

The norm of matrix $\mathbf{C}$ is $\|\mathbf{C}\| = \sqrt{\text{Tr}(\mathbf{C}^H \mathbf{C})}$, where $\text{Tr}(\mathbf{C}^H \mathbf{C})$ is the trace of matrix.

### 2.2. Singular Value Decomposition of Quaternion Matrix (SVDQ)

**Definition 8.** [29-31] Given a quaternion matrix $\mathbf{I} \in \mathbb{H}^{N \times M}$. Its expression using Cayley-Dickson notation [32] is $\mathbf{I} = \mathbf{A} + \mathbf{B}j$, where $\mathbf{A}, \mathbf{B} \in \mathbb{C}^{N \times M}$ are complex matrices. One can then define the complex adjoint matrix [33], denoted by $\mathbf{X}_{\mathbf{I}} \in \mathbb{C}^{2N \times 2M}$, corresponding to the quaternion matrix $\mathbf{A}$, as follows,

$$
\mathbf{X}_{\mathbf{I}} = \begin{bmatrix}
\mathbf{A} & \mathbf{B} \\
-\mathbf{B}^\ast & \mathbf{A}^\ast
\end{bmatrix}_{2N \times 2M}
$$

**Definition 9.** [29-31] The rank of a quaternion matrix $\mathbf{A}$ is $r$ if and only if the rank of its complex adjoint $\mathbf{X}_{\mathbf{A}}$ is $2r$ [33]. In addition to this definition, we indicate that the rank of $\mathbf{A}$ is $r$ if $\mathbf{A}$ has $r$ non-zero singular values.

**Theorem 1** [31, 34] For any quaternion matrix $\mathbf{X} \in \mathbb{H}^{N \times M}$ with rank $r$, there are two quaternion matrices $\mathbf{U}, \mathbf{V}$,

$$
\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^H
$$

where $\Sigma$ is a diagonal matrix, and there are $r$ non-zero diagonal elements. $\mathbf{U} \in \mathbb{H}^{N \times N}$ and $\mathbf{V} \in \mathbb{H}^{M \times M}$ are orthogonal matrices.

The singular elements of a quaternion matrix $\mathbf{X}$ can be obtained from the SVD of its complex adjoint matrix $\mathbf{X}_X$ [31]. Firstly, we review the complex SVD of $\mathbf{X}_X$ as follows,

$$
\mathbf{X}_X = \mathbf{U}_X \Sigma_X \mathbf{V}_X^H
$$

where $\mathbf{U}_X$ and $\mathbf{V}_X$ are complex matrices, and they are composed of singular value vectors of $\mathbf{X}_X \mathbf{X}_X^H$ and $\mathbf{X}_X^H \mathbf{X}_X$, respectively. The relationship between SVDQ of quaternion matrix $\mathbf{X}$ and SVD of complex adjoint matrix $\mathbf{X}_X$ is as follows [33, 35],

$$
\Sigma = \text{row}_{odd}(\text{col}_{odd}(\Sigma_X))
$$

$$
\mathbf{U} = \text{col}_{odd}(\mathbf{U}^X_x) + \text{col}_{odd}(-\mathbf{U}^X_x^H)j
$$

$$
\mathbf{V} = \text{col}_{odd}(\mathbf{V}^X_x) + \text{col}_{odd}(-\mathbf{V}^X_x^H)j
$$

We get,

$$
\mathbf{X} = \mathbf{U} \Sigma \mathbf{V}^H
$$

where

$$
\mathbf{U}^X_x = \begin{bmatrix}
\mathbf{U}^X_{x1} & \mathbf{U}^X_{x2} \\
\mathbf{U}^X_{y1} & \mathbf{U}^X_{y2}
\end{bmatrix}_{N \times 2N}
$$

$$
\mathbf{V}^X_x = \begin{bmatrix}
\mathbf{V}^X_{x1} & \mathbf{V}^X_{x2} \\
\mathbf{V}^X_{y1} & \mathbf{V}^X_{y2}
\end{bmatrix}_{M \times 2M}
$$

Considering that the eigenvalues of complex adjoint matrix $\mathbf{X}_X$ appear as conjugate pairs along the diagonal line and that $\mathbf{X}_X \mathbf{X}_X^H$ is Hermitian [31], the eigenvalues of $\mathbf{X}$ are real and appear in pairs along the diagonal line [36].
3. Quaternion Non-local Means

3.1. QNLM Model

For color image processing, NLM [11-13] was realized in different channels. However, monochrome method and cascade method miss the relevance between RGB [2]. Since the quaternion-based method continue to have the all message of the 3D vector algebra [2], the NLM algorithm of matrix is extended to quaternion space [22-24], and then the noise image based on the quaternion is represented as:

\[ \mathbf{V} = \mathbf{X} + \mathbf{N} \]  

where \( \mathbf{V} \) is a quaternion matrix with noise, \( \mathbf{X} \) is the clear color image after denoising, and \( \mathbf{N} \) is the noise matrix. For any picture element \( i \) in the image field \( I \), the picture element number \( i \) is obtained in the weighted calculation in this specification with respect to the image block in the search window.

\[ \hat{x}_i = \frac{1}{z_i} \sum_{j \in I} \omega_{ij} \cdot v_j \]  

where \( \hat{x}_i \) is a quaternion matrix after denoising. The similarity weight \( \omega_{ij} \) represents the similarity between the pixel \( i \) and the pixel \( j \). Its value is determined by the domain patches \( C_i \) and \( C_j \) centered on \( i \) and \( j \),

\[ \omega_{ij} = \exp \left( -\frac{\|C_i - C_j\|^2}{h^2} \right) \]  

where \( C_i \) and \( C_j \) are respectively patch domains centered on \( i \) and \( j \), and \( \|C_i - C_j\| \) denotes the correlation between \( C_i \) and \( C_j \) in patch domains, and we know from (7) that it is \( d(C_i, C_j) \). The imaginary part of \( d(C_i, C_j) \) is 0, hence the real part is taken. The meaning of \( h \) is a smoothing parameter, which is a denoising parameter, which controls the attenuation degree of Gaussian function (refer to Algorithm 1).

The normalized parameter \( z_i \) is expressed as:

\[ z_i = \sum_{j \in I} \exp \left( -\frac{\|C_i - C_j\|^2}{h^2} \right) \]  

**Algorithm 1  QNLM method**

**Input:** Noisy image \( \mathbf{Y} \), search window size \( W \times W \), patch size \( w \times w \), similar matrix number \( n \), total patch number \( I \), denoising parameter \( h \), error \( \varepsilon \)

For \( i = 1: I \) do:

For each patch \( y_i \) in \( \mathbf{Y} \) do:

For each image patch \( y_j \) in the search window do:

\[ \|C_i - C_j\|^2 = (C_i - C_j)^H(C_i - C_j) \]

Calculate normalized parameter \( z_i \) by (15)

Estimate weight vector \( \omega_{ij} \) by (14)

End for

\[ \hat{y}_i = \frac{1}{z_i} \sum_{j \in I} \omega_{ij} \cdot y_j \]

End for

Calculate the average of repeated estimates for \( y_i \) and get the estimated clear image patch \( \hat{x}_i \)

End for

Aggregate all patches together, and get clear image \( \hat{X} \)

**Output:** Denoised image \( \hat{X} \)
3.2. Quaternion Mahalanobis Non-local Means (QMNLM) model

Mahalanobis distance is a more effective and reliable method to measure the similarity of sample data than Euclidean distance proposed by Indian statistician Mahalanobis. It overcomes the shortcomings of Euclidean distance. By standardizing the sample data and removing the interference of the correlation between the variables, the Mahalanobis distance measures the similarity between the sample data more effectively [37]. Similar to the Mahalanobis distance in real number field, we define the Mahalanobis distance of quaternion (refer to Algorithm 2).

**Definition 10.** Assuming that the column vectors composed of sample data \( i, j \) of two image patches are \( C_i \) and \( C_j \) respectively, the Mahalanobis distance between the two vectors is defined as:

\[
md(C_i, C_j) = \sqrt{(C_i - C_j)^T S^{-1}(C_i - C_j)}
\]

(17)

where \( S \) is the covariance matrix of vectors \( C_i \) and \( C_j \),

\[
S = (C_{ij} - C_m)(C_{ij} - C_m)^T
\]

(18)

\[
C_{ij} = (C_i, C_j), C_m = \frac{1}{2}((C_i, C_i) + (C_j, C_j))
\]

It can be seen from (16) that when \( S \) is a unit matrix, the Mahalanobis distance degenerates into Euclidean distance, so Mahalanobis distance is an extension of (7).

So the new weight defined in the QMNLM algorithm is

**Algorithm 2**  QMNLM method

**Input:** Noisy image \( \mathcal{Y} \), search window size \( W \times W \), patch size \( w \times w \), similar matrix number \( n \), total patch number \( I \), denoising parameter \( h \), error \( \varepsilon \)

For \( i = 1: I \) do:

For each patch \( y_i \) in \( \mathcal{Y} \) do:

For each image patch \( y_j \) in the search window do:

Calculate the covariance matrix \( S \) by (17)

\[
[U \Sigma V] = Q S V D(S)
\]

by (9)

Estimate Mahalanobis distance \( md(C_i, C_j) \)

Calculate normalized parameter \( z_i \) by (19)

Get weight vector \( w_{ij} \) by (14)

End for

\[
\hat{y}_i = \frac{1}{z_i} \sum_{j \in I} w_{ij} \ast y_j
\]

End for

Calculate the average of \( y_i \) and get the estimated clear image patch \( \hat{x}_i \)

End for

Aggregate all patches together, and get clear image \( \hat{X} \)

**Output:** Denoised image \( \hat{X} \)

\[
md_{ij} = \exp \left( -\frac{md^2_{ij}}{h^2} \right)
\]

(19)

The normalized parameter \( z_i \) is expressed as:

\[
z_i = \sum_{j \in I} \exp \left( -\frac{md^2_{ij}}{h^2} \right)
\]

(20)
4. Quaternion Mahalanobis Non-local Means

4.1. QMNLM Model for Color Image Denoising

Non-local self-similarity [1, 38, 39] denoising uses the redundancy of image information to remove image noise. This kind of algorithm originated from the NLM algorithm [11] proposed by Buades in 2005. Before that, image spatial domain algorithms used the similarity between neighboring pixels to remove noise, that is, for a pixel to be denoised. Because of its similarity with other pixels in the neighborhood, it can be de-noised by averaging or weighted averaging the pixels in the module. Non-local self-similarity is proposed to extend image similarity from neighborhood pixels to different image patches. Because image patch contains more information than a single pixel, it is more stable to use the similarity between patch as image denoising priori and get better denoising effect. Non-local self-similarity means that an image patch has many similar patches in other parts of the image [2]. Because the structure of a clean image is redundant, the similar patches of the image patch may exist in the surrounding position of the image.

The steps of QMNLM method are: Firstly, the image sub-patch $y_i(i \in [1, MN])$ of noise image $Y$ is selected, and the reference patch $y_i$ is arranged into a column vector, $\{y_i \in \mathbb{H}^w: i = 1,2,\ldots,\left(\frac{MN}{w}\right)\times \left(\frac{w}{w}\right)\}$ represents all the sub-patches of the image. Then, with reference patch $y_i$, as the hub, the search window with $W$ size is selected. Every $w \times w$ sub-patch in the search window is arranged into a column vector, and the set $P = \{y_j \in \mathbb{H}^w: j = 1,2,\ldots,\left(\frac{W}{w}\right)^2\}$ represents all image sub-patches contained in the search window. According to Mahalanobis distance, the similarity $md(y_i,y_j)$ between reference patch $y_i \in \mathbb{H}^w$ and image patch $y_j \in \mathbb{H}^w$ is calculated. The Mahalanobis distance is calculated by the formula sub-similarity between the pixels. After weighted average, the estimated value $\hat{x}_i$ of the pixel value is obtained (refer to Algorithm 2).

4.2. QMNLM Method

QMNLM algorithm calculates the similarity between the image patch centered on each pixel and the image patch in the search window, and then calculates the weighted average to get the estimated value. The key problem is how to calculate the weight $W$ [2]. Because weights represent similarities between image patches, proper weight calculation is important the proposed QMNLM model.

In order to calculate the similarity between image pixels, we define a neighborhood system on $I$.

**Definition 11.** (Neighborhoods) The neighborhood system on $I$ is the family $\mathcal{X} = \{X_i\}_{i \in I}$ of subsets of $I$, so that for all $i \in I$,

1. $i \in X_i$
2. $j \in X_i$ $\Rightarrow$ $i \in X_j$

The subset $X_i$ is called the neighborhood or similar window of $i$. We set $\bar{X}_i = X_i \setminus \{i\}$.

The restriction of $\nu$ to a neighborhood $X_i$ will be denoted by $\nu(X_i)$.

$$\nu(X_i) = (\nu(X_j), j \in \bar{X}_i)$$

The similarity between two patches $I$ and $J$ will depend on the similarity of quaternion vectors $\nu(X_i)$ and $\nu(X_j)$. Patches with color neighborhoods similar to $\nu(X_i)$ have larger weights.

To calculate the similarity between quaternion vectors $\nu(X_i)$ and $\nu(X_j)$, the Mahalanobis distance of quaternion vectors difference can be calculated. This measurement method is very suitable for additive Gauss noise. It measures the similarity between two image patches very well. Indeed,

$$E[md(\nu(X_i),\nu(X_j))] = E[md(\nu(X_i),\nu(X_j))] + 2\sigma^2S^{-1}$$
where $\mathbf{U}$ and $\mathbf{V}$ are original image and noise image respectively, $\mathcal{S}$ is the covariance matrix of vectors $\mathbf{U}(x_i)$ and $\mathbf{U}(x_j)$ and $\sigma^2$ is noise variance. Therefore, the most similar pixels to $i$ in $\mathbf{v}$ are also the most similar pixels to $i$ in $\mathbf{u}$. The weight calculated by the inner product is determined by

$$w_{ij} = \frac{1}{z_i} \exp \left( -\frac{\| \mathbf{V}(x_i) - \mathbf{V}(x_j) \|^2}{h^2} \right)$$

On the basis of the above discussion, the weighted vector $\mathbf{w}$ for estimating the denoised image $\mathbf{u}$ is calculated.

### 4.3. Calculation of Mahalanobis Distance

When we calculate the Mahalanobis distance, it appears that the inverse of (17) does not exist, thus the Mahalanobis distance defined in (16) is inaccessible. It is for this reason that the generalized inverse matrix of $\mathcal{S}$ is proposed.

By QSVD [34] of $\mathcal{S}$, we get

$$\mathcal{S} = \mathbf{U} \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \mathbf{V}^H$$

(23)

where $\mathbf{U}$ and $\mathbf{V}$ are the quaternion orthogonal matrixes, $\Sigma = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$, $\lambda_i (i = 1, 2, ..., r)$ is singular value of $\mathcal{S}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ and the rank of $\mathcal{S}$ is $r$; the superscript $H$ represents conjugate transpose of matrix. For (22), the Moore–Penrose inverse matrix [40] is solved,

$$\mathcal{S}^+ = \mathbf{V} \begin{bmatrix} \Sigma^{-1} & 0 \\ 0 & 0 \end{bmatrix} \mathbf{U}^H$$

(24)

where $\Sigma^{-1} = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1}, ..., \lambda_r^{-1})$, and (23) is a generalization of $\mathcal{S}^{-1}$ in the feature space. Using $\mathcal{S}^+$ to replace $\mathcal{S}^{-1}$ in (16), we get that the Mahalanobis distance of the feature space is,

$$md_{ij} = \sqrt{(\mathbf{C}_i - \mathbf{C}_j)^T \mathcal{S}^+(\mathbf{C}_i - \mathbf{C}_j)}$$

(25)

In $\Sigma = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_r)$, the value of $\lambda_i (i = 1, 2, ..., r)$ represent the amount of energy of each uncorrelated data component. For image data, $\lambda_i (i = 1, 2, ..., r)$ represent the variance of each component information in the image. In this paper, we find that $\lambda_1$ is always much larger than other values, that is $\lambda_1 \gg \lambda_i (\lambda_i \neq \lambda_1)$, it can be considered that $\lambda_1$ represents the variance of useful signals in the image block. Therefore, only $\lambda_1$ can be reserved while $\lambda_i (\lambda_i \neq \lambda_1)$ can be ignored, and other singular values are considered to be zero.

Based on the above analysis, we get

$$\Sigma = \lambda_1 \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$$

(26)

By inverting (25), $\Sigma^{-1} = \frac{1}{\lambda_1} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}$ is obtained. By substituting $\Sigma^{-1}$ into (23) and deducing it, we get

$$\mathcal{S}^+ = \frac{1}{\lambda_1} \mathbf{V} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{bmatrix} \mathbf{U}^H = \frac{1}{\lambda_1} \mathcal{S}$$

(27)

The improved Mahalanobis distance is obtained by replacing equation (16) with equation (26).
\[ m_{d_{ij}} = \frac{1}{\lambda_1} \sqrt{(c_i - c_j)^T S (c_i - c_j)} \]  

(28)

Equation (27) not only avoids the instability of Mahalanobis distance, but also simplifies the calculation and suppresses the interference of noise. Because of the symmetry of Mahalanobis distance, it can be used to reduce the calculation complexity by half without sacrificing the denoising performance.

5. Experiment

Twelve pictures were selected to prove the proposed method in this paper. As shown in figures 1a-1l, the pictures are all from the database in Ref. [41]. The pictures are cut to 128×128 size to facilitate calculation.

Since MATLAB does not directly support quaternion operations, we implement Q-Lib library [2] for quaternion vector and matrix operations. From [https://sourceforge.net/projects/qtfm/][2] download quaternion toolbox.

![Test image of color image denoising][1]

5.1. Image Quality Evaluation Criteria

The objective of image denoising is to reduce the influence of noise and to restore the original news of the image. The denoised image is not only close to the real value of the original image, but also can effectively maintain the texture, edge and other information in the image. Thus, the quality evaluation after image denoising is of great significance.

Mean squared error (MSE) [42] is an important index of image quality evaluation. This method judges image quality by pixel error of image. The definition of MSE is:

\[ \text{MSE} = \frac{1}{M \times N} \sum_{i=1}^{M} \sum_{j=1}^{N} (X(i,j) - Y(i,j))^2 \]  

(29)

The MSE is calculated according to the Euclidean distance of each point. The smaller the value, the closer the denoised image is to the real information of the real image.

Peak signal to noise ratio (PSNR(dB)) [42], just like the MSE, is calculated by the relative value of the MSE of the image. PSNR(dB) can be calculated in the following ways:

\[ \text{PSNR} = 10 \log_{10} \left( \frac{(2^k-1)^2}{\text{MSE}} \right) \]  

(30)
Each sample uses k-bit to represent the pixel, and generally takes the value of 8 [42]. PSNR(dB) is the standard used in this article to measure image quality.

5.2. QMNLM Model for Color Image Denoising
In image denoising, several parameters need to be fixed. The main parameters of the experiment include: noise variance $\sigma^2$, denoising parameter $h$, image sub-patch size $w \times w$ and search window size $W \times W$. Firstly, Gauss noise with average value of 0 and std. of $\sigma$ is added to the image. Then the proposed QMNLM method is used to remove the noise. The denoising results are compared with QNLMM, matrix NLM and matrix GNLM algorithm. The denoising effect is measured by PSNR(dB) value. The values of standard deviation $\sigma$ are 10, 30, 50 and 70. In this paper, 12 pictures are discussed under the influence of different variances of Gaussian noise.

For the proposed QMNLM method implementation, the experimental results are affected by parameters such as denoising parameter $h$, image sub-patch size $w$ and search window size. Buades et al. [11] gave the global estimation of $h$, which is regarded as $h$ has an approximate linear relationship with $\sigma$. In this paper, we assume $h=\sigma$ in low level noise. In high level noise, the image is preprocessed by Gaussian LPF, so when $\sigma=30$, $h=10$ and $\sigma=70$, $h=30$. Paper compare the denoising effects of different sub-patch sizes: $3 \times 3$, $5 \times 5$, $7 \times 7$, $9 \times 9$ in all noise levels. Using $56 \times 56$ image Lena as the test image. From figure 2, we can see the relevance among PSNR (dB) and sub block size.

![Figure 2](image1)

**Figure 2.** Effects of different image patch sizes on image denoising.

![Figure 3](image2)

**Figure 3.** Image Lena denoising (PSNR(dB)) with different image patch sizes ($\sigma = 30$).

To illustrate the effect, Gauss noise with mean value of 0 and std. of 30 is selected for Lena image. The contrast images of denoising effect using QMNLM algorithm for different image patch sizes are listed. In figure 3, enlarge the red calibration area in the lower left corner and compare the denoising effect of different image sub blocks. Figure 2 shows that no matter what noise level, $3 \times 3$ is the best image patch size. The situation is obvious at low noise levels. For high noise, although the optimal size of the image block sizes is $7 \times 7$, the change of the size of the image block has little effect on the denoising result, so the $3 \times 3$ image block with the highest efficiency can be selected.
5.3. Low Noise Level Results and Comparison
To prove the effectiveness and feasibility of the proposed algorithm, the algorithm 2 is applied to image denoising in figure 1. In the experiment, zero-mean additive white Gaussian noise with different noise levels $\sigma$ is added to the experimental image to simulate the noise [2].

When $\sigma = 10$, we set the size of image sub-patch to $3 \times 3$, and we set the denoising parameter $h$ and the search window size to 5 and $5 \times 5$. When $\sigma = 30$, we set similar image sub-patches and search window sizes $w$ and $W$ to be 3 and 5 respectively, and the denoising parameters are 30.

Figures 4 and 5 show the visual quality of Voiture and Baboon images at noise levels of $\sigma = 10$ and $\sigma = 30$. Enlarging the red calibration area in the lower left corner, the denoising the effect of different image sub blocks is compared with that of the denoised image by NLM, QNLM and GNLM algorithms, and more details of the visual image are obtained. The QMNLM algorithm proposed by us can reconstruct images from noise observations very well. In addition, we found that QMNLM method maintains better local structure and produces the smallest visual artifacts in the restored image compared with others.

For eight low noise level ($0 < \sigma < 50$) test images, the results are given by using competing methods, and the results are table 1. From table 1, most of the images denoised by QMNLM algorithm under the influence of low noise get the highest PSNR (dB) and have the best denoising effect. For PSNR(dB), our method improves 0.1985-4.2949dB on average compared with others. Figures 4 and 5 also express that QMNLM algorithm gets better visual effect.

![Figures 4 and 5 showing visual quality of Voiture and Baboon images at noise levels of $\sigma = 10$ and $\sigma = 30$.](image)

**Figure 4.** Four methods for image Voiture denoising (PSNR(dB)) ($\sigma = 10$).

![Figures 4 and 5 showing visual quality of Voiture and Baboon images at noise levels of $\sigma = 10$ and $\sigma = 30$.](image)

**Figure 5.** Four methods for image Baboon denoising (PSNR(dB)) ($\sigma = 30$).

5.4. High Noise Level Results and Comparison
The image with noise affected by high noise was first processed with a Gaussian low pass filter (LPF) [2], then denoise the filtered image with QMNLM method. When $\sigma = 50$, we set the search window to
15, the denoising parameters $h$ and sub-patch size $w$ to 10 and 3, respectively. When $\sigma = 70$, in addition denoising parameters, we set the same parameters as when $\sigma = 50$. We set the search window $W$, denoising parameters $h$ and image patch size $w$ to 5, 30 and 3 respectively. In order to illustrate the effect, 0-mean Gaussian noise with std of 50 and 70 is added to image House and image Butterfly respectively, and the denoising effect comparison figures of QNLM, GNLM, NLM and QMNLM are listed, as shown in figures 6 and 7.

For eight high noise level ($\sigma \geq 50$) test images, the results are given by using multi-channel NLM, multi-channel GNLM, QNLM and our proposed QMNLM method, and the results are given in table 2. From table 2, for high noise level, we can find that the most of QMNLM algorithm proposed by the others achieves the highest PSNR(dB) and has the best denoising effect. For PSNR(dB), our method improves 0.192-0.9563 dB on average compared with other three methods. Figures 6 and 7 also show that our method improves the image visual effect.

Table 1. Denoising results at low noise levels (PSNR) using the four algorithms (the best result is bold).

| $\sigma = 10$ | Image    | Noisy image | NLM     | QNLM   | GNLM   | QMNLM |
|--------------|----------|-------------|---------|--------|--------|-------|
| Airplane     | 28.1358  | 28.8788     | 29.5411 | 29.7093| **30.1332** |
| Baboon       | 28.1358  | 27.8731     | 28.0010 | 27.4921| **28.2657** |
| Barbara      | 28.1667  | 29.6961     | 30.4227 | 30.8603| **31.4463** |
| Boats        | 28.1513  | 28.7298     | 29.2945 | 29.3050| **29.8343** |
| House        | 28.1027  | 31.2591     | 32.6727 | 34.1462| **34.3447** |
| Butterfly    | 28.1513  | 27.6572     | 28.0130 | 29.4995| **28.5349** |
| Lena         | 28.1248  | 30.0380     | 30.9135 | 31.4005| **31.9258** |
| Peppers      | 28.1555  | 29.6273     | 30.1948 | 31.0902| **31.1208** |
| Mare         | 28.1555  | 30.0826     | 30.8788 | 32.0678| **31.8689** |
| Voiture      | 28.1248  | 29.9377     | 30.7736 | 30.6220| **31.4877** |
| Tulips       | 28.1667  | 28.6702     | 28.6988 | **29.6013** | 29.4288 |
| Athens       | 28.1026  | 30.0561     | 31.0491 | 30.6810| **31.9452** |
| AVE.         | 28.1395  | 29.3755     | 29.3755 | 29.3755| **29.3755** |

| $\sigma = 30$ | Image    | Noisy image | NLM     | QNLM   | GNLM   | QMNLM |
|--------------|----------|-------------|---------|--------|--------|-------|
| Airplane     | 18.6147  | 22.0570     | 23.8732 | 24.6976| **25.7246** |
| Baboon       | 18.6147  | 21.6362     | 23.0400 | 23.5719| **24.2010** |
| Barbara      | 18.6130  | 22.0833     | 23.9499 | 24.8537| **26.0012** |
| Boats        | 18.5859  | 21.9506     | 23.6477 | 24.1250| **25.2513** |
| House        | 18.5588  | 22.6924     | 24.9788 | 27.1550| **27.8822** |
| Butterfly    | 18.5859  | 21.6288     | 22.9105 | 23.5691| **24.7321** |
| Lena         | 18.5867  | 22.3143     | 24.3106 | 25.7636| **26.6057** |
| Peppers      | 18.6166  | 22.2454     | 24.0193 | 25.7654| **26.2827** |
| Mare         | 18.6166  | 22.4606     | 24.3751 | 26.5290| **26.7555** |
| Voiture      | 18.5867  | 22.2488     | 24.2632 | 24.8576| **26.1427** |
| Tulips       | 18.6130  | 21.7979     | 23.1926 | 24.2454| **25.0104** |
| Athens       | 18.5588  | 22.3170     | 24.4139 | 25.7793| **26.4460** |
| AVE.         | 18.5960  | 22.1194     | 23.9146 | 25.0761| **25.9196** |
6. Discussions

For color image denoising, QMNLM algorithm is superior to QNLM, multi-channel NLM and GNLM algorithm regardless of the noise level. The basic principles of QMNLM and QNLM are the same, but when QNLM is used to process the image, the Euclidean distance between the reference patch and the surrounding image patches is used as the weight to reconstruct the image. The ability to describe the structural features of the image is limited, especially for the large-scale image with complex structure. QMNLM takes the quaternion Mahalanobis distance between the reference patch and the surrounding image patches as the weight of the surrounding image patch. In addition, the difference between QMNLM and NLM and GNLM is that the proposed algorithm is implemented in quaternion field. For color image processing, the key and difficulty is to maintain the relationship between multi-channels [2]. The appearance of quaternion solves this problem well. When image denoising is implemented in quaternion domain, the details of RGB can be well preserved. Therefore, the PSNR(dB) of QMNLM algorithm is efficiently higher than the other three algorithms.

In the high noise level, QMNLM algorithm overemphasizes the similarity information between patches in the image and ignores the information of the patch itself, so after denoising, false texture and edge loss will appear in the smooth region. So this is the main problem we need to solve in the future, in order to get better denoising effect.

In this article, QMNLM algorithm is applied to color image denoising. This method can also be applied to other image processing fields [2], such as color image segmentation [43], saliency detection [44] and deburring.
Table 2. Denoising results at high noise levels (PSNR) using the four algorithms (The best effect is bold).

| Image  | Noisy image | NLM    | QNLM   | GNLM   | QMNLM  |
|--------|-------------|--------|--------|--------|--------|
|        |             | 22.1296| 22.2400| 22.3089| 22.4238|
| Airplane| 14.1865     |        |        |        |        |
| Baboon | 14.1865     | 21.8714| 21.9545| 21.9847| 22.0723|
| Barbara| 14.1494     | 22.5245| 22.6543| 22.6725| 22.8461|
| Boats  | 14.1503     | 21.7973| 21.9070| 21.8736| 22.0397|
| House  | 14.1399     | 23.7503| 23.9079|        | 24.3650|
| Butterfly| 14.1503   | 19.5087| 19.5855| 19.4806| 19.6695|
| Lena   | 14.1450     | 22.8700| 22.9890| 23.1996| 23.2633|
| Peppers| 14.1410     | 22.5909| 22.6943| 22.8778| 22.9220|
| Mare   | 14.1410     | 23.4031| 23.5199|        | 23.8979|
| Voiture| 14.1450     | 22.8416| 22.9816| 23.0643| 23.2009|
| Barbara| 14.1494     | 21.2103| 21.2688| 21.2475| 21.3757|
| House  | 14.1399     | 23.9025| 24.0590| 24.3185| 24.3854|
| AVE.   | 14.1520     | 22.3667| 22.4802| 22.6076| 22.7003|

σ = 50

| Image  | Noisy image | NLM    | QNLM   | GNLM   | QMNLM  |
|--------|-------------|--------|--------|--------|--------|
|        |             | 22.1296| 22.2400| 22.3089| 22.4238|
| Airplane| 14.1865     |        |        |        |        |
| Baboon | 14.1865     | 21.8714| 21.9545| 21.9847| 22.0723|
| Barbara| 14.1494     | 22.5245| 22.6543| 22.6725| 22.8461|
| Boats  | 14.1503     | 21.7973| 21.9070| 21.8736| 22.0397|
| House  | 14.1399     | 23.7503| 23.9079|        | 24.3650|
| Butterfly| 14.1503   | 19.5087| 19.5855| 19.4806| 19.6695|
| Lena   | 14.1450     | 22.8700| 22.9890| 23.1996| 23.2633|
| Peppers| 14.1410     | 22.5909| 22.6943| 22.8778| 22.9220|
| Mare   | 14.1410     | 23.4031| 23.5199|        | 23.8979|
| Voiture| 14.1450     | 22.8416| 22.9816| 23.0643| 23.2009|
| Barbara| 14.1494     | 21.2103| 21.2688| 21.2475| 21.3757|
| House  | 14.1399     | 23.9025| 24.0590| 24.3185| 24.3854|
| AVE.   | 14.1520     | 22.3667| 22.4802| 22.6076| 22.7003|

σ = 70

7. Conclusion
We propose a new QMNLM model and algorithm based on non-local similarity framework, and apply it to color image to remove noise. Firstly, the quaternion matrix is used to represent the color image, so as to maintain the correlation between the color channels. Then, we use the nonlocal self-similarity of color image blocks and the newly defined quaternion Mahalanobis distance to give different weights to similar image blocks to calculate each reference image block. The experimental simulation of multi-channel NLM, GNLM, QNLM and QMNLM methods color image denoising using software platform proves that the algorithm in this paper can achieve better denoising effect than the other three image denoising algorithms. When the noise variance is more than 50, Gaussian LPF is used to preprocess the noise image, and then the filtered image is processed in color. From the perspective of visual standards, it can well retain the inherent color structure. After the experiment, we found that after LPF preprocessing, the QMNLM algorithm enhances the denoising effect of PSNR (dB) and visual perception.
Acknowledgment
The writers were grateful for all Professor Zhengwei Shen’s helpful advices. The National Natural Science Foundation of China, under grant 11101029, and the Fundamental Research Funds for the Central Universities, under grant FRF-DF-19-004, and National Key R&D Program of China, under grant 2017YFF0207401 held out this paper.

References
[1] Dabov K, Foi A, Katkovnik V and Egiazarian K 2007 Image denoising by sparse 3-D transform domain collaborative filtering IEEE Transactions on Image Processing 16 2080-2095.
[2] Yu Y, Zhang Y and Yuan S 2018 Quaternion-based weighted nuclear norm minimization for color image denoising Neurocomputing.
[3] Elad M and Aharon M 2006 Image denoising via sparse and redundant representations over learned dictionaries IEEE Transactions on Image Processing 15 3736-3745.
[4] Aharon M, Elad M and Bruckstein A 2006 K-SVD: An algorithm for designing over complete dictionaries for sparse representation IEEE Trans. Signal Process 54 4311-4322.
[5] Dong W, Li X, Zhang L and Shi G 2011 Sparsity-based image denoising via dictionary learning and structural clustering IEEE Conference Computer Vision and Pattern Recognition (CVPR) pp. 457-464.
[6] Stéphane G, Mallat and Zhang Z 1993 Matching pursuits with time-frequency dictionaries IEEE Trans. on Signal Processing 41 3397-3415.
[7] Pati Y C, Rezaifar R and Krishnaprasad P 1993 Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition Record of the Twenty-Seventh Asilomar Conference Signals Systems and Computers pp. 40-44.
[8] Chen S S and Saunders D M A 2001 Atomic decomposition by basis pursuit SIAM Review 43 129-159.
[9] Engan K, Aase S O and Husoy J H 1999 Method of optimal directions for frame design IEEE International Conference on Acoustics 2443-2446.
[10] Mairal J, Bach F, Ponce J and Sapiro G 2009 Online dictionary learning for sparse coding The International Conference Machine Learning (ICML) pp. 689-696.
[11] Buades A, Coll B and Morel J M 2005 A review of image denoising algorithms with a new one Multiscale Modeling and Simulation 4 490-530.
[12] Tao G H, Wei W X and Qin Z Y 2015 Generalized non-local means algorithm for image denoising Application Research of Computers 7 304-307.
[13] Buades A, Coll B and Morel J M 2010 Image denoising methods: A new nonlocal principle SIAM Review 52 113-14.
[14] Moxey C E, Sangwine S J and Ell T A 2003 Hyper complex correlation techniques for vector images IEEE Transactions on Signal Processing 51 1941-1953.
[15] Traversoni L 2001 Image analysis using quaternion wavelets geometric Algebra with Applications in Science and Engineering.
[16] Bas P, Bihan N L and Chassery J 2007 Color image watermarking using quaternion fourier transform IEEE International Conference on Acoustics.
[17] Yu M, Xu Y and Sun P 2014 Single color image super-resolution using quaternion-based sparse representation IEEE International Conference Acoustics Speech and Signal Processing (ICASSP) pp. 5804-5808.
[18] Teschke G and Ramlau R 2007 An iterative algorithm for nonlinear inverse problems with joint sparsity constraints in vector-valued regimes and an application to color image inpainting Inverse Problems 23 1851-1870.
[19] Shi L and Funt B 2007 Quaternion color texture segmentation Computer Vision & Image Understanding 107 88-96.
[20] Tang T 2012 Invariant texture retrieval based on quaternion wavelet and log-polar Computer Engineering 38 191-193.
[21] Subakan Ö N and Vemuri B C 2011 A quaternion framework for color image smoothing and segmentation International Journal of Computer Vision 91 233-250.
[22] Jin L, Liu H, Xu X and Song E 2010 Quaternion-based color image filtering for impulsive noise suppression Journal of Electronic Imaging 20 43-55.
[23] Hua W G, Yang L and Zhou Z T 2014 A quaternion-based switching filter for colour image denoising Signal Processing 102 216-225.
[24] Sheng L Q, Shu H, Sun X, Chen B and Ming L 2014 Removing Gaussian noise for color images by quaternion representation and optimization of weights in non-local means filter IET Image Processing 8 591-600.
[25] Reddy B D V and Prasad D T J 2010 Frequency domain filtering of colour images using quaternion Fourier transforms International Journal of Computer Science and Technology 1 6-52.
[26] Khalil M I 2012 Applying quaternion Fourier transforms for enhancing color images International Journal of Image, Graphics and Signal Processing 4 9-15.
[27] Hamilton and William Rowan 1846 On quaternions, or on a new system of imaginaries in algebra Philosophical Magazine 25 10-13.
[28] Zhang F 1997 Quaternions and Matrices of Quaternions Linear Algebra Applications 251 21-57.
[29] Girard P R 2007 Quaternions, Clifford algebras and relativistic physics SIAM Review 50 382-385.
[30] Altmann S L 1986 Rotations, quaternions, and double groups Acta Crystallographica 44 298-303.
[31] Le Bihan N and Mars J 2004 Singular value decomposition of quaternion matrices: a new tool for vector-sensor signal processing Signal Process 84 1177-1199.
[32] Li M and Xu Y 2019 Improved non-local means algorithm for image denosing 2019 IEEE 11th International Conference on Communication Software and Networks (ICCSN).
[33] Pei S-C, Chang J H and Ding J J 2003 Quaternion matrix singular value decomposition and its applications for color image processing image processing ICIP 2003 pp 801-805.
[34] Sangwine S J and Bihan N L 2006 Quaternion singular value decomposition based on bidiagonalization to a real or complex matrix using quaternion householder transformations Applied Mathematics and Computation 182 727-738.
[35] Xu Y, Yu L, Xu H and Zhang H 2015 Vector sparse representation of color image using quaternion matrix analysis IEEE Trans. Image Process 24 1315-1329.
[36] Zhang F 1997 Quaternions and matrices of quaternions Linear Algebra Abd Its Applications 251 21-57.
[37] Yurong L and Guobo X 2006 A linear discriminant analysis classification algorithm based on Mahalanobis distance Computer Simulation 23 86-88.
[38] Dong W, Shi G and Li X 2013 Nonlocal image restoration with bilateral variance estimation: a low-rank approach IEEE Trans. Image Process 22 700-711.
[39] Dong W, Zhang L and Shi G 2011 Centralized sparse representation for image restoration Proceedings of the IEEE International Conference Computer Vision (ICCV) pp 1259-1266.
[40] Kyrchei I 2017 Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse Applied Mathematics and Computation 309 1-16.
[41] Pei S C and Cheng C M 1999 Color image processing by using binary quaternion-moment-preserving thresholding technique Transactions on Image Processing a Publication of the IEEE Signal Processing Society 8 614-628.
[42] Martens J B and Meesters L 1998 Image dissimilarity Signal Process 70 155-176.
[43] Shen J, Peng J, Dong X, Shao L and Porikli F Higher order energies for image segmentation IEEE Trans. Image Process 26 4911-4922.
[44] Wang W, Shen J and Shao L 2018 Video salient object detection via fully convolutional networks IEEE Trans. Image Process 27 38-49.