An Example in Complete Intersections and an Erratum

Satya Mandal
University of Kansas, Lawrence, Kansas 66045; mandal@ku.edu

31 January 2017

1 Introduction

The following is a version of the complete intersection conjecture of M. P. Murthy ([M], [M1, pp 85]).

**Conjecture 1.1.** Suppose $A = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring over a field $k$. Then, for any ideal $I$ in $A$, $\mu(I) = \mu(I/I^2)$, where $\mu$ denotes the minimal number of generators.

Recently, a solution of this conjecture (1.1) was claimed in [F], when $k$ is an infinite perfect field and $1/2 \in k$. Subsequently, using Popescu’s Desingularization Theorem ([P] [SwI]), the claim was strengthened in [M0], removing the perfectness condition. In deed, the main claims in [F], were further strengthened in [M0], for ideals $I$ in polynomial rings $A[X]$, over regular rings $A$ containing an infinite field $k$, with $1/2 \in k$, and $I$ containing a monic polynomial. It was latter established (see [MM]) that the methods in [F] [M0], would also work when $k$ is finite, with $1/2 \in k$.

The claimed proof of Conjecture 1.1 in [F] is a consequence of a stronger claim that, for integers $n \geq 2$, any set of $n$-generators of $I/I^2$ lifts to a set of $n$-generators of $I$. This question of liftability of generators of $I/I^2$ was considered in [MMu] and a counter example was given [MMu, Example 2.4],
when $n = 2$. In next section 2, we develop a larger class of examples, for all integers $n \geq 2$.

Due to the existence of such counter examples, some clarifications are needed regarding the published claims in $[F,M0]$. We provide the same in Section 3. We underscore that, there is no logical error in the methods in $[M0]$, barring the use of the claimed results in $[F]$.

## 2 The Example

The following two examples were worked out in collaboration with M. P. Murthy.

**Example 2.1.** Let $n \geq 3$ be any integer, and $A = k[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be a polynomial ring over any field $k$. Let $f = \sum_{i=1}^{n} X_i Y_i - 1 \in A$ and $I = Af$. Write $\overline{A} = \frac{A}{(f)}$. For elements in $A$ (respectively, in $I$), the images in $\overline{A}$ (respectively, in $\frac{I}{f^2}$) will be denoted by "overline". Then,

$$\overline{X_1f}, \overline{X_2f}, \ldots, \overline{X_nf}$$

generates $\frac{I}{f^2}$.

This set of generators of $\frac{I}{f^2}$ would not lift to a set of generators of $I$.

**Proof.** As in $[MMU]$, we have the commutative diagram

$$
\begin{array}{ccc}
A & \xrightarrow{f} & I \\
\downarrow & & \downarrow \\
\frac{A}{(f)} & \xrightarrow{\overline{f}} & \frac{I}{f^2}
\end{array}
$$

Suppose $\overline{X_1f}, \overline{X_2f}, \ldots, \overline{X_nf}$ lifts to a set of generators of $I$. Then, by the diagram above, the unimodular row

$$(\overline{X_1}, \overline{X_2}, \ldots, \overline{X_n})$$

of $\overline{A}$

lifts to a unimodular row

$$(F_1, F_2, \ldots, F_n)$$

of $A$. 

2
Since projective $A$-modules are free, there is a matrix $\sigma \in GL_n(A)$, whose first row is $(F_1, \ldots, F_n)$. Therefore, $(X_1, X_2, \ldots, X_n)$ is the first row of the image of $\sigma$ in $GL_n(A)$. So, the projective $A$-module defined by $(X_1, X_2, \ldots, X_n)$ is free. This is impossible, by the Theorem of N. Mohan Kumar and Madhav V. Nori (see [Sw1, Theorem 17.1]). The proof is complete. 

Example 2.2. Let $A = \mathbb{R}[X_0, X_1, \ldots, X_n]$ be a polynomial ring over the field of real numbers. Let $f = \sum_{i=0}^{n} X_i^2 - 1 \in \mathbb{R}$ and $I = Af$. Assume, $n \neq 0, 1, 3, 7$. Then, $X_0 f, X_1 f, \ldots, X_n f$ induce a set of generators for $I/I^2$, which would not lift to a set of generators of $I$.

Proof. Same as the proof of (2.1), while we use the fact that tangent bundles over real $n$-spheres ($n \neq 0, 1, 3, 7$) are nontrivial (see [Sw1, Theorem 2.3]).

Remark 2.3. Note $I = Af$ in (2.1) is a principal ideal. So, Examples 2.1, 2.2 do not provide a counter example of the Complete Intersection Conjecture 1.1.

3 Erratum

The following list provides some clarifications regarding the inconsistencies in the literature [E, M0], at this time.

1. It was communicated by Mrinal K. Das that the proof of [E, Lemma 3.2.3] is not convincing. This may be the likely cause of all the inconsistencies in [E, M0], under discussion. Indeed, this creates an incompleteness in the proof of the key result [E, Theorem 3.2.7].

2. Theorem 3.2.8 in [E] does not have a valid proof in the literature. For a polynomial ring $A = k[X_1, \ldots, X_n]$, by [M0, Proposition 4.1], $Q_{2n}(A)$ is a singleton. Therefore, 2.1, 2.2 would be a counter example to [E, Theorem 3.2.8].

3. The claimed proof of [E, Theorem 3.2.9] is not valid, since it uses [E, Theorem 3.2.8]. Therefore, the Complete Intersection Conjecture 1.1 is
still open and the best result on this conjecture, at this time, remains those in [M1] and [M2].

4. There is no logical error in [M0]. However, since the main results in [M0] depends on the validity of the same in [F], they do not have any valid proofs, at this time. In particular,

(a) The main results [M0] Theorems 3.8, 4.2, 4.3] do not have valid proofs, at this time.
(b) The claimed proofs of Abhyankar’s epimorphism conjecture [M0, Thoerem 4.5, 4.6], are not valid, since they are routine consequence of [M0, Theorems 4.2, 4.3, 4.4].

5. The results in [M0] that are not dependent on results in [F] are valid. In particular,

(a) The [M0] Propositions 4.1], on triviality of homotopy obstructions, for ideals containing a monic polynomial, remain valid.
(b) Results in [M0] Section 5], on the alternate description of the Homotopy Obstruction set Q_{2n}(A), remain valid.

References

[F] Fasel, Jean Fasel, Jean On the number of generators of ideals in polynomial rings. *Ann. of Math. (2)* 184 (2016), no. 1, 315-331.; arXiv:1507.05734

[MM] Mandal Satya, Bibekananda Mishra The Homotopy Program in Complete Intersections, arXiv:1610.07495

[M0] Mandal, Satya On the complete intersection conjecture of Murthy. *J. Algebra* 458 (2016), 156?170.

[M1] Mandal, Satya Projective modules and complete intersections. Lecture Notes in Mathematics, 1672. *Springer-Verlag, Berlin*, 1997.

[M2] Mandal, Satya On efficient generation of ideals. Invent. Math. 75 (1984), no. 1, 59-67.
[MMu] Mandal, Satya; Pavaman Murthy, M. Ideals as sections of projective modules. *J. Ramanujan Math. Soc.* 13 (1998), no. 1, 51?62.

[Mk] Kumar, N. Mohan On two conjectures about polynomial rings. *Invent. Math.* 46 (1978), no. 3, 225-236.

[M] Murthy, M. Pavaman Complete intersections. *Conference on Commutative Algebra-1975* (Queen’s Univ., Kingston, Ont., 1975), pp. 196-211. Queen’s Papers on Pure and Applied Math., No. 42, Queen’s Univ., Kingston, Ont., 1975.

[P] Popescu, Dorin Letter to the editor: "General Néron desingularization and approximation" *Nagoya Math. J.* 118 (1990), 45-53.;

[S] Stavrova, A. Homotopy invariance of non-stable $K_1$-functors. *J. K-Theory* 13 (2014), no. 2, 199-248.

[Sw1] Swan, Richard G. Néron-Popescu desingularization. *Algebra and geometry (Taipei, 1995)*, 135-192, Lect. Algebra Geom., 2, *Int. Press, Cambridge, MA*, 1998.