Determining the $\eta - \eta'$ mixing by the newly measured
$BR(D(D_s) \to \eta(\eta') + \bar{\ell} + \nu_\ell)$

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Abstract

The mixing of $\eta - \eta'$ or $\eta - \eta' - G$ is of a great theoretical interest, because it concerns many aspects of the underlying dynamics and hadronic structure of pseudoscalar mesons and glueball. Determining the mixing parameters by fitting data is by no means trivial. In order to extract the mixing parameters from the available processes where hadrons are involved, theoretical evaluation of hadronic matrix elements is necessary. Therefore model-dependence is somehow unavoidable. In fact, it is impossible to extract the mixing angle from a unique experiment because the model parameters must be obtained by fitting other experiments. Recently $BR(D \to \eta + \bar{\ell} + \nu_\ell)$ and $BR(D_s \to \eta(\eta') + \bar{\ell} + \nu_\ell)$ have been measured, thus we are able to determine the $\eta - \eta'$ mixing solely from the semileptonic decays of D-mesons where contamination from the final state interactions is absent. Thus we hope that the model-dependence of the extraction can be somehow alleviated. Once $BR(D \to \eta' + \bar{\ell} + \nu_\ell)$ is measured, we can further determine all the mixing parameters for $\eta - \eta' - G$. As more data are accumulated, the determination will be more accurate. In this work, we obtain the transition matrix elements of $D(s) \to \eta^{(0)}$ using the light-front quark model whose feasibility and reasonability for such processes have been tested.

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I. INTRODUCTION

The mixing among pseudoscalar mesons and glueballs is of great theoretical interests and significance for understanding the dynamics and hadronic structures. Study on the mixing has lasted for several decades, not only because of its importance, but also the difficulties caused by both theoretical and experimental aspects. As is well understood, the mixing is caused by the QCD anomaly and related to the chiral symmetry breaking \([1,2]\). Definitely, one would be able to gain a better insight into the dynamics, if the mixing parameters are more accurately determined. Many measurements on the processes where \(\eta\) and \(\eta'\) are involved, have been carried out to fix the mixing parameters. The mixing of \(\eta-\eta'\) is described in different forms, i.e. with different eigen-bases.

In the SU(3) quark model \(\eta\) and \(\eta'\) are the mixtures of \(\eta_0\) and \(\eta_8\) which are SU(3) singlet and octet states respectively \([3,4]\),

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix} \begin{pmatrix}
\eta_0 \\
\eta_8
\end{pmatrix},
\]

and the mixing angle \(\theta\) was fitted in a range of \(-10^\circ\) to \(-23^\circ\) \([4]\). Bhattacharya and Rosner’s recent work \([6]\) suggests that the data of the decays of \(D^0\), \(D^\pm\) and \(D_s\) into two light pseudoscalar mesons seem to favor a smaller octet-singlet mixing angle \(-11.7^\circ\).

In the quark content basis, \(\eta_0\) and \(\eta_8\) can be further written as mixtures of \(\eta_q\) and \(\eta_s\),

\[
\begin{pmatrix}
\eta_0 \\
\eta_8
\end{pmatrix} = \begin{pmatrix}
\sqrt{\frac{2}{3}} & -\frac{\sqrt{2}}{\sqrt{3}} \\
\frac{\sqrt{2}}{\sqrt{3}} & \frac{\sqrt{2}}{\sqrt{3}}
\end{pmatrix} \begin{pmatrix}
\eta_q \\
\eta_s
\end{pmatrix},
\]

where \(\eta_q = \frac{1}{\sqrt{2}}(u\bar{d} + d\bar{u})\) and \(\eta_s = s\bar{s}\). In Refs. \([7,8]\), the mixing of \(\eta\) and \(\eta'\) is written as

\[
\begin{pmatrix}
\eta \\
\eta'
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi \\
\sin \phi & \cos \phi
\end{pmatrix} \begin{pmatrix}
\eta_q \\
\eta_s
\end{pmatrix},
\]

where \(\cos \phi = \sqrt{\frac{2}{3}} \cos \theta - \sqrt{\frac{2}{3}} \sin \theta\) and \(\sin \phi = \sqrt{\frac{2}{3}} \cos \theta + \sqrt{\frac{2}{3}} \sin \theta\). We refer this mixing as scenario-I in this work. The mixing angle \(\theta = -11.7^\circ\) \([6]\) corresponds to \(\phi = 43.0^\circ\) in this scenario.

In the history, there have been various ways to determine the mixing angle(s). Feldmann et al. \([10]\) summarized the issue and listed several possibilities to extract the mixing angle(s) from the experimental data. By fitting the ratio of nonleptonic decay widths \(\frac{\Gamma(J/\psi \rightarrow \eta'\rho)}{\Gamma(J/\psi \rightarrow \eta\rho)}\) the authors of Ref. \([11]\) obtained \(\phi = (39.9 \pm 2.9)^\circ\), while by fitting the decay widths of \(\Gamma(\eta' \rightarrow \rho\gamma)\) and \(\Gamma(\rho \rightarrow \eta\gamma)\), the value of \(\phi\) was set as \(\phi = (35.3 \pm 5.5)^\circ\). In terms of \(\frac{\Gamma(D_s \rightarrow \eta'\pi\pi)}{\Gamma(J/\psi \rightarrow \eta\pi\pi)}\) \(\phi = (43.1 \pm 3.0)^\circ\) was obtained. Using the ratio of decay widths of \(\frac{\Gamma(D_s \rightarrow \eta'\rho\gamma)}{\Gamma(D_s \rightarrow \eta\rho\gamma)}\) \([12]\) and the form factors given in Ref. \([13]\), \(\phi\) is fixed as \(\phi = (41.3 \pm 5.3)^\circ\) \([10]\). With the cross section of scattering processes \(\pi^-p \rightarrow \eta'n\) and \(\pi^-p \rightarrow \eta n\) one had \(\phi = (36.5 \pm 1.4)^\circ\) \([14]\).
and $\phi = (39.3 \pm 1.2)^\circ$ [15]. The Crystal Barrel Collaboration [16] measured the ratio of the annihilation processes $p\bar{p} \to \eta' + \text{meson}(\pi^0, \eta$ or $\omega)$ and $p\bar{p} \to \eta + \text{meson}(\pi^0, \eta$ or $\omega)$ and achieved $\phi = (37.4 \pm 1.8)^\circ$. Based on the ratio $\frac{B[\omega \to \eta'\gamma]}{B[\eta \to \pi^0\gamma]}$, the value of $\phi$ was obtained as $(39.0 \pm 1.6)^\circ$ [10]. The results seem a bit dispersive, but they are consistent with each other for the accuracy the present experiments can reach. Feldmann et al. obtained a weighted average of the $\phi$ value as $(39.3 \pm 1.0)^\circ$. Moreover, is not the end of the story, since the QCD anomaly, if it causes the mixing between $\eta$ and $\eta'$, also results in a mixing of $\eta$, $\eta'$ with gluonium of the quantum number $0^{-+}$.

As one extends the picture to involve glueballs, a new scenario which we refer as the scenario-II, was suggested in Refs. [13, 17–21] as

$$
\begin{pmatrix}
|\eta\rangle \\
|\eta'\rangle \\
|G\rangle
\end{pmatrix} =
\begin{pmatrix}
\cos \phi' & -\sin \phi' & 0 \\
\sin \phi' \cos \phi_G & \cos \phi' \cos \phi_G & \sin \phi_G \\
-\sin \phi' \sin \phi_G & -\cos \phi' \sin \phi_G & \cos \phi_G
\end{pmatrix}
\begin{pmatrix}
|\eta_q\rangle \\
|\eta_s\rangle \\
|g\rangle
\end{pmatrix},
$$

(4)

where $|g\rangle = |\text{gluonium}\rangle$ is a pure gluonium state and the physical state $G$ was identified as $\eta(1405)$ [19]. It is noted that the definition of the mixing angles in Ref. [19] is different from that adopted in Ref. [20]. In Ref. [22] the $\chi^2$ scheme was used to obtain $\phi_G$, $\phi'$ and other parameters by fitting the data of several radiative decays such as $\omega \to \eta\gamma$, $\rho \to \eta\gamma$ [23] and $\omega \to \pi^0\gamma$. The results are $\sin^2\phi_G = 0.115 \pm 0.036$ and $\phi' = (40.4 \pm 0.6)^\circ$ if one sets $\sin^2\phi_G$ as a free parameter.

Recently, the branching ratios of $D_s \to \eta(\eta')\ell^+\nu_\ell$ and $D^+ \to \eta\ell^+\nu_\ell$ have been measured [24, 25] and the collaboration [25] obtains a new upper limit of $\text{BR}(D^+ \to \eta'\ell^+\nu_\ell)$. Even though the missing energy of the produced neutrino may bring up certain uncertainties, semi-leptonic decay modes have an obvious advantage over the radiative and non-leptonic decays. Since the contamination from the final state interaction is absent in semi-leptonic decays, the theoretical calculation is more reliable and the physical quantities extracted from data, such as the mixing angles, would be closer to reality. This is an ideal opportunity to determine the structure of $\eta(\eta')$, e.g. extract the mixing angle(s) of $\eta - \eta'$ ($\eta - \eta' - G$) from data. This advantage motivates us to study the mixing via the semi-leptonic decays alone.

As was indicated above, in most cases, the mixing angle and the model parameters were not simultaneously determined by fitting solely one type of data. Since all the relevant processes involve hadron transitions, one needs to evaluate the hadronic transition matrix elements which are fully governed by the non-perturbative QCD, in terms of phenomenological models. Therefore extraction of the mixing is somehow model dependent. However, if one can determine the mixing angle(s) solely from one type of processes, the model dependence would be alleviated because he can just employ one phenomenological model to deal with all the concerned reactions, e.g. the model parameters and the mixing angles are fixed altogether. Indeed, in this way, we can expect that the model dependence which is unavoidable, could be reduced to minimum.
Our strategy is following. We will concentrate on the study of the semi-leptonic decays of $D$ and $D_s$. At the tree-level, only the $d\bar{d}$ ($s\bar{s}$) component of $\eta, \eta'$ contributes to the transition $D^+ \rightarrow \eta(\eta')(D_s \rightarrow \eta(\eta'))$. The amplitude at the quark level can be obtained in terms of the weak effective theory, so the key point is to calculate the hadronic transition matrix elements. Concretely, we are going to use the light front quark model (LFQM) to evaluate the hadronic transition matrix elements. It is believed that the LFQM is a relativistic model which has obvious advantages for dealing with hadronic transitions where light hadrons are involved [26, 27]. The light-front wave function is manifestly Lorentz invariant and expressed in terms of the fractions of internal momenta of the constituents which are independent of the total hadron momentum. Applications of this approach have been discussed in some details by the authors of Ref. [29] and their results are in good agreement with data. Moreover, in Ref. [30] we explored the structure of $f_0(980)$ via the transition $D_s \rightarrow f_0(980)$ in the LFQM. However, in the approach, there are a few parameters to be fixed: the $\beta$ values in the hadron wave-functions.

In our scheme, we let the mixing angle and $\beta$ be free parameters, when we fit the data, we obtain the mixing angle and the $\beta$ values simultaneously. In this way, we “extract” the mixing angle directly from the data on the semileptonic decays of $D$ and $D_s$. At the present $\mathcal{BR}(D^+ \rightarrow \eta e^+\nu_e)$ and $\mathcal{BR}(D_s^+ \rightarrow \eta(\eta')e^+\nu_e)$ have been measured, but the data are not enough to determine mixing angles of $\eta, \eta'$ and glueball. Once $\mathcal{BR}(D^+ \rightarrow \eta' e^+\nu_e)$ is measured in the future at BES, it would be optimistic that one will be able to further determine the mixing structure which is extremely valuable for getting insight into the physics picture of light hadrons and probably the glueballs. Thus so far, we only concern the scheme I where only $\eta - \eta'$ mixing exists.

This paper is organized as follows: after the introduction, in section II we present the form factors of $D^+ \rightarrow \eta(\eta')(D_s \rightarrow \eta(\eta'))$ which are evaluated in the LFQM, then we calculate the corresponding decay rates and by fitting the data we gain the mixing angle and the model parameters altogether. Section III is devoted to our conclusion and discussions.

II. CALCULATION OF THE WIDTHS OF $D^+ \rightarrow \eta(\eta')e^+\nu_e$ AND $D_s \rightarrow \eta(\eta')e^+\nu_e$ IN LFQM

In this section we are going to calculate the decay widths of $D^+ \rightarrow \eta(\eta')e^+\nu_e$ and $D_s \rightarrow \eta(\eta')e^+\nu_e$ in terms of the LFQM. The crucial task is to evaluate the form factors of $D^+ \rightarrow \eta(\eta')(D_s \rightarrow \eta(\eta'))$. In Ref. [31], we studied $D_s \rightarrow \eta(\eta')$ in the LFQM, thus the corresponding formulas can be directly used in this work. The transition diagram is shown in Fig. 1.
A. Formulations

For being self-content, we list some key formulae given in Refs. [26, 27] here. The form factors for $P \rightarrow P$ transition are defined as

$$
\langle P(P'')|A_\mu|P(P')\rangle = f_+(q^2)P_\mu + f_-(q^2)q_\mu.
$$

(5)

It is convenient to redefine them as

$$
\langle P(P'')|A_\mu|P(P')\rangle = \left(P_\mu - \frac{M'^2 - M''^2}{q^2}q_\mu\right)F_1(q^2) + \frac{M'^2 - M''^2}{q^2}q_\mu F_0(q^2),
$$

(6)

where $q = P' - P''$ and $P = P' - P''$. The relations among the quantities are

$$
F_1(q^2) = f_+(q^2), \quad F_0(q^2) = f_+(q^2) + \frac{q^2}{q \cdot P}f_-(q^2).
$$

(7)

Functions $f_\pm(q^2)$ can be calculated in the LFQM and their explicit expressions were presented as [27],

$$
f_+(q^2) = \frac{N_c}{16\pi^3} \int dx_2 d^2p'_\perp \frac{h'_p h''_p}{x_2 N'_1 N''_1} \left[ -x_1(M'^2_0 + M''^2_0) - x_2 q^2 + x_1(m'_1 - m''_1) \right] + x_1(m'_1 - m_2)^2 + x_1(-m''_2 + m_2)^2,
$$

$$
f_-(q^2) = \frac{N_c}{16\pi^3} \int dx_2 d^2p'_\perp \frac{2h'_p h''_p}{x_2 N'_1 N''_1} \left[ x_1 x_2 M'^2 + p'_\perp + m'_1 m_2 + (-m''_1 + m_2)(x_2 m'_1 + x_1 m_2) \right] - 2 \frac{q \cdot P}{q^2} \left( p'^2_\perp + 2 \frac{(p'_\perp \cdot q_\perp)^2}{q^2} - 2 \frac{(p'_\perp \cdot q_\perp)^2}{q^2} + \frac{(p'_\perp \cdot q_\perp)}{q^2} \right) \left[ M''^2 - x_2(q^2 + q \cdot P) \right] - (x_2 - x_1)M'^2 + 2x_1 M'^2_0 - 2(m'_1 - m_2)(m'_1 + m''_1)].
$$

(8)

where $m'_1$, $m''_1$, and $m_2$ are the corresponding quark masses, $M'$ and $M''$ are the masses of the initial and final mesons respectively. All other notations can be found in the Appendix.

The form factors are parameterized in a three-parameter form as

$$
F(q^2) = \frac{F(0)}{1 - a \left( \frac{q^2}{M^2} \right) + b \left( \frac{q^2}{M^2} \right)^2}.
$$

(9)
where $F(q^2)$ represents the form factors $F_1$, $F_0$, and $F(0)$ is the form factor at $q^2 = 0$; $M$ is the mass of the initial meson. The three parameters $F(0)$ and $a$, $b$ are fixed by performing a three-parameter fit to the form factors which are calculated in the space-like region and then extended to the physical time-like region.

For semileptonic decay of a pseudoscalar meson ($D$ or $D_s$) into a pseudoscalar meson, i.e. $P(P') \rightarrow P(P'') l \nu$, the differential width is

$$
\frac{d\Gamma}{dq^2}(P \rightarrow P l \nu) = \frac{G_F^2|V_{CKM}|^2 p^3}{24\pi^3} |F_1(q^2)|^2,
$$

where $q = P' - P''$ is the momentum transfer and $q^2$ is the invariant mass of the lepton-neutrino pair; $p$ is the final meson momentum in the $D$ or $D_s$ rest frame and

$$
p = |\vec{P}''| = \frac{\sqrt{(M^2 - (M_f - \sqrt{q^2})^2) (M^2 - (M_f + \sqrt{q^2})^2)}}{2M},
$$

$M_f$ denotes the mass of the produced meson. It is noted that the differential width is governed by only one form factor $F_1(q^2)$ because we neglect the light lepton masses.

**B. Calculating the decay rates of $D^+(D_s) \rightarrow \eta(\eta')e^+\nu_e$ and determining the mixing angle**

Now, let us extract the mixing angle in scenario-I by means of the experimental data of $\mathcal{B}R(D^+ \rightarrow \eta e^+\nu_e) = (13.3 \pm 2.0 \pm 0.6) \times 10^{-4}$, $\mathcal{B}R(D_s \rightarrow \eta e^+\nu_e) = (2.48 \pm 0.29 \pm 0.13) \times 10^{-2}$ and $\mathcal{B}R(D_s \rightarrow \eta' e^+\nu_e) = (0.91 \pm 0.33 \pm 0.05) \times 10^{-2}$.

First we need to calculate the form factors $F_1^{D^+(D_s)\eta}(q^2)$ and $F_1^{D^+(D_s)\eta'}(q^2)$ using the formulas given in section II A. Since $d$ quark in $D^+$ appears in the final meson, only the $d\bar{d}$ component of $\eta(\eta')$ contributes to the transition of $D^+ \rightarrow \eta(\eta')\bar{l}l\nu$ i.e. the mixing angle is included in the form factors. The similar situation for the transition $D_s \rightarrow \eta(\eta')\bar{l}l\nu$ was discussed in [31] but we let the mixing angle be free.

In this work, the input parameters for quark masses are directly taken from [27] as $m_u = 0.26$ GeV, $m_s = 0.37$ GeV, $m_c = 1.4$ GeV. From the experimental results for decay constants $f_D^{exp} = 0.221$ MeV, $f_{D_s}^{exp} = 0.27$ MeV, the parameters for $\beta_{D_s}$ and $\beta_D$ are fixed to be $\beta_{D_s} = 0.592$ GeV, $\beta_D = 0.499$ GeV [31].

Totally, there are five free parameters $\beta^q_{\eta}, \beta^q_{\eta'}, \beta^s_{\eta}, \beta^s_{\eta'}$ and $\phi$ to be fixed. It seems that we do not have enough equations to determinate all these parameters. However, the $\beta^q_{\eta,\eta'}$ is a parameter in the Gaussian wave function whose value only depends on the quark flavor (q or s), but not on the characters of the hadron, therefore two relations $\beta^q_{\eta} = \beta^q_{\eta'}$ and $\beta^s_{\eta} = \beta^s_{\eta'}$ hold. The relations are also deduced in the Appendix B where we do not make any special assumption on them. By these relations we reduce the number of unknowns to three. From the three decay modes, we obtain the values of $\phi$ and $\beta^q_{\eta(\eta')}, \beta^s_{\eta(\eta')}$. The mixing angle is fit
FIG. 2: The dependence of the form factor $F_1(q^2)$ on $q^2$ at $\phi = 39.9^\circ$. (a) $D^+ \to \eta$ (the solid line) and $D_s \to \eta$ (the dotted line); (b) $D^+ \to \eta'$ (the solid line) and $D_s \to \eta'$ (the dotted line)

to be $\phi = (39.9 \pm 2.6)^\circ$ where the errors are from the experiment. The parameters $\beta$ are $\beta_{\eta(q'q')}^q = 0.398 \text{ GeV}, \beta_{\eta(q'q')}^s = 0.453 \text{ GeV}$.

Now, let us discuss the theoretical uncertainties in our used light front quark model. Here, we only address the errors coming from the phenomenological parameters. The numerical results are not sensitive to the variance of $\beta_{\eta(q'q')}^q$ and $\beta_{\eta(q'q')}^s$. In [31], it is found that the variations of decay constants of $f_D$ and $f_{D_s}$ with parameters $\beta$ is nearly a linear relation. One can ascribe the uncertainties from $\beta_D, \beta_{D_s}$ to experimental errors of $f_D, f_{D_s}$.

For the quark masses, we made a variation of their values by 10% and explore the uncertainties of predictions. We found that the numerical numbers are not sensitive to $m_d, m_s$. In particular, with variations of $m_d, m_s$, the errors of $\phi$ is only 0.7°. The uncertainties from charm quark mass $m_c$ is in a controllable region, $\Delta \phi = 2.1^\circ$. Combing them, we have an error 2.3° in determination of $\phi$, this is the main error of our approach. After considering the experimental error, the mixing angle is determined to be $\phi = (39.9 \pm 2.6(\text{exp}) \pm 2.3(\text{the}))^\circ$.

SU(3) breaking is a substantial effect in $\eta - \eta'$ mixing. In our approach, we have included their effects in differences of quark masses $m_d, m_s$; $\beta$ parameter differences in $\beta_{\eta(q')}^q$ and $\beta_{\eta(q')}^s$. Using these values, we plot the dependence of the form factor $F_1(q^2)$ on $q^2$ in the physical region of $q^2 \geq 0$ in Fig. 2 and we estimate $\mathcal{BR}(D^+ \to \eta' e^+ \nu_e) = (2.12 \pm 0.23(\text{exp}) \pm 0.20(\text{the})) \times 10^{-4}$ which is lower than the experimental upper bound $\mathcal{BR}(D^+ \to \eta' e^+ \nu_e) \leq 3.5 \times 10^{-4}$ [25].

Using the formula of Eq. (B1) in Appendix and the above parameters we calculate the central value of the decay constants $f_{\eta}^q = 80 \text{ MeV}, f_{\eta}^s = 113 \text{ MeV}, f_{\eta'}^q = 67 \text{ MeV}$ and $f_{\eta'}^s = 145 \text{ MeV}$. These values are obtained by fit to the experiment. They are very close to the results $f_{\eta}^q = 78 \text{ MeV}, f_{\eta}^s = 113 \text{ MeV}, f_{\eta'}^q = 64 \text{ MeV}$ and $f_{\eta'}^s = 141 \text{ MeV}$ given in
Ref. [32] where two mixing angles $\theta_0 = -9.1^\circ$ and $\theta_8 = -22.2^\circ$ are used for the mixing of $\eta$ and $\eta'$. It is noted that our result extracted solely from the data on the semi-leptonic decays of $D$ and $D_s$ is consistent with the previously determined weighted value $39.3^\circ$ [10] and $41.4^\circ$ [22] within a reasonable error tolerance. In Refs. [10, 13] the authors also used the data $\Gamma(D_s \rightarrow \eta' e\nu)/\Gamma(D_s \rightarrow \eta e\nu)$ to determine the mixing angle. In this work, we employ an alternative model to evaluate the hadronic transition matrix elements. Moreover, nowadays there are more data available, which enable us to make more accurate estimation on the $\eta - \eta'$ mixing.

In principle we will be able to fix the mixing angles $\phi'$ and $\phi_G$ in Eq. (4) simultaneously as long as a sufficiently large database on the semileptonic decay modes is available.

III. CONCLUSION

Because of absence of the final state interactions, the semileptonic decays have obvious advantages for determining the properties of the produced light hadrons, such as the structure of $f_0(980)$, $\eta - \eta'$ mixing and even a mixing of pseudoscalar mesons with glueball, over other modes from the theoretical aspect. Indeed, moreover, for the semileptonic decays, one only needs to calculate the hadronic transition elements where only one final hadron is involved. Instead, besides the complication caused by the final state interaction, the evaluation of hadronic matrix elements of the non-leptonic decays is much more difficult because there are two hadrons in the final state. To do the job, the factorization scheme may be invoked and more ambiguities are raised by the scheme.

In this work we extract the mixing angle of $\eta$ and $\eta'$ by the semileptonic decays $D \rightarrow \eta e^+\nu_e$ and $D_s \rightarrow \eta(\eta')e^+\nu_e$. At first we calculate the form factors of $D \rightarrow \eta$ and $D_s \rightarrow \eta(\eta')$ in the LFQM where the mixing angle of $\eta$ and $\eta'$ and the parameters $\beta$s were free. Then we compute the rates of the semileptonic decays $D \rightarrow \eta e^+\nu_e$ and $D_s \rightarrow \eta(\eta')e^+\nu_e$. Equating our theoretical derivations with the data we obtain $\phi = (39.9 \pm 2.6(\text{exp}) \pm 2.3(\text{the}))^\circ$, $\beta_{\eta(\eta')} = 0.398$ GeV and $\beta_{\eta(\eta')}^s = 0.453$ GeV. We estimate $\mathcal{B}R(D^+ \rightarrow \eta' e^+\nu_e) = (2.12 \pm 0.23(\text{exp}) \pm 0.20(\text{the})) \times 10^{-4}$.

As indicated above, there are three free parameters including the $\eta - \eta'$ mixing in our model-dependent calculations, and we fix them simultaneously by the data of $\mathcal{B}R(D \rightarrow \eta e^+\nu_e)$ and $\mathcal{B}R(D_s \rightarrow \eta(\eta')e^+\nu_e)$ in our scheme I. If in the future $\mathcal{B}R(D^+ \rightarrow \eta' e^+\nu_e)$ can be accurately measured (not just an upper bound), we will be able to test our theoretical prediction of $(2.12\pm0.23(\text{exp})\pm0.20(\text{the})) \times 10^{-4}$ which is made in our scheme I. If the data are consistent with our estimation, it implies that our scheme I is valid, e.g, the mixing between $\eta - \eta'$ with gluonium states is small. Instead, if there is an observable discrepancy between our theoretical prediction on $\mathcal{B}R(D^+ \rightarrow \eta' e^+\nu_e)$ and the data, one needs to further consider the scheme II where the mixing between $\eta - \eta'$ is accounted, and by the data, we can determine...
the mixing angles $\phi'$ and $\phi_G$ in eq.(4). We wish our experimental colleagues to carry out measurements on $D^+ \to \eta' e^+ \nu_e$ and further improve the accuracy of the measurements of semileptonic decays.

Of course, the branching ratios of semileptonic decay modes are smaller than that of non-leptonic modes, and due to the existence of a neutrino (antineutrino), the missing energy would make the event reconstruction more difficult. However, the obvious advantage of the semileptonic decays for determining the properties of the light hadrons, would be the reason to urge our experimental colleagues to conduct more precise measurements. It is worthwhile.

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Appendix A: Notations

Here we list some variables appearing in the context. The incoming (outgoing) meson in Fig. 1 has the momentum $P_1' = p_1' + p_2$ where $p_1' \text{ and } p_2$ are the momenta of the off-shell quark and antiquark and

$$p_1'^+ = x_1 P_1'^+ , \quad p_2^+ = x_2 P_1'^+ ,$$

$$p_1'^\perp = x_1 P_1'^\perp + p_1'^\perp , \quad p_2^\perp = x_2 P_1'^\perp - p_1'^\perp , \quad (A1)$$

with $x_i$ and $p_i'^\perp$ are internal variables and $x_1 + x_2 = 1$.

The variables $M_0'$, $\tilde{M}_0'$, $h_p'$ and $\hat{N}_1'$ are defined as

$$M_0'^2 = \frac{p_1'^2 + m_1'^2}{x_1} + \frac{p_2'^2 + m_2^2}{x_2} ,$$

$$\tilde{M}_0' = \sqrt{M_0'^2 - (m_1' - m_2)^2} , \quad (A2)$$

$$h_p' = (M'^2 - M_0'^2) \sqrt{\frac{x_1 x_2}{N_c}} \frac{1}{\sqrt{2 M_0'}} \varphi' , \quad (A3)$$

where

$$\varphi' = 4 \left( \frac{\pi}{\beta' q^2} \right)^{3/4} \sqrt{\frac{d p_z'}{d x_2}} \exp \left( \frac{-p_z'^2 + p_\perp'^2}{2 \beta'^2} \right) , \quad (A4)$$

with $p_z' = \frac{x_1 M_0'}{2} - \frac{m_1'^2 + m_2^2}{2 x_2 M_0'}$.

$$\hat{N}_1' = x_1 (M'^2 - M_0'^2) . \quad (A5)$$
Appendix B: the relations between $\beta_{\eta(\eta')}^q$ and $\beta_{\eta(\eta')}^s$

For a pseudoscalar meson the decay constant can be evaluated
\[ f_P = \frac{\sqrt{N_c}}{16\pi^3} \int dx_2 d^2 p_\perp' \frac{\phi'}{\sqrt{2x_1 x_2 M_0'}} 4(m'_1 x_2 + m_2 x_1), \] (B1)

so the parameters $\beta_{\eta(\eta')}^q$, $\beta_{\eta(\eta')}^s$ can be determined by the decay constants of $f_{\eta(\eta')}^q$ and $f_{\eta(\eta')}^s$, which are defined in Ref. [10]

\[ f_q^q = f_q \cos \phi , \quad f_s^q = -f_s \sin \phi , \]
\[ f_{q'}^q = f_q \sin \phi , \quad f_{s'}^s = f_s \cos \phi . \] (B2)

where $q$ represents $u$ or $d$ quark.

Since $\eta = \eta_q \cos \phi - \eta_s \sin \phi$ and $\eta' = \eta_q \sin \phi + \eta_s \cos \phi$, we get the relations $\beta_{\eta}^{q(s)} = \beta_{\eta'}^{q(s)}$.
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