On some exceptional cases in the integrability of the
three–body problem

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Abstract

We consider the Newtonian planar three–body problem with positive masses $m_1$, $m_2$, $m_3$. We prove that it does not have an additional first integral meromorphic in the complex neighborhood of the parabolic Lagrangian orbit besides three exceptional cases $\sum m_im_j/(\sum m_k)^2 = 1/3, 2^3/3^3, 2/3^2$ where the linearized equations are shown to be partially integrable. This result completes the non–integrability analysis of the three–body problem started in papers [3], [5] and based of the Morales–Ramis–Ziglin approach.

Key words: meromorphic first integrals, non–integrability, Ziglin’s lemma, three–body problem

1 Introduction

Let $M \subset \mathbb{C}^n$ be a complex domain and let $\text{Hol}(M)$ be a set of functions $f : M \to \mathbb{C}$ holomorphic in $M$. We consider a system of $n$ ordinary differential equations written in the Pfaffian form

$$\frac{dx_1}{f_1(x)} = \frac{dx_2}{f_2(x)} = \cdots = \frac{dx_n}{f_n(x)}, \quad f_i \in \text{Hol}(M),$$

$$x = (x_1, x_2, \ldots, x_n) \in M.$$

As far as the dynamical properties of the flow are concerned it is more convenient to consider the trajectories of the vector field $F = (f_1, \ldots, f_2)$ as leaves of a codimension $n – 1$ foliation defined by (1.1) regardless the time parametrisation.

Let $h : \Gamma \to M$ be a particular leaf of (1.1) where $h$ is a certain holomorphic map (not necessary unique). We note that in many mechanical problems the surface $\Gamma$ is a punctured
Riemann sphere $\Gamma = \mathbb{CP}^1 \setminus \{z_1, \ldots, z_k\}$ and $h$ is rational. Let $e \in \Gamma$ be a fixed basepoint and $\{\gamma\}$ is the set of loops generating the fundamental group $\pi_1(e, \Gamma)$.

We take a particular closed path $\gamma_0 \in \{\gamma\}$ defined by a continuos map $\gamma_0 : [0, 1] \to \Gamma$, $\gamma(0) = \gamma(1) = e$. Each tangent vector $v \in T_e M$ can be transported along $\gamma_0$ following the neighborhood leafs of $h(\Gamma)$ back to the tangent space $T_e M$. One obtains a linear representation of $\pi_1(e, \Gamma)$ into $\text{GL}(n, \mathbb{C})$ called the monodromy group $G$.

This group measures the complexity of “enrolling” of the neighborhood to $h(\Gamma)$ leafs and usually contains strong obstacles to the existence of first integrals of (1.1) meromorphic in $M$. The following lemma of Ziglin [8] establishes the link between the integrability of (1.1) and rational invariants of $G$.

**Lemma 1.** Let $\Phi_1 \ldots \Phi_k$ be a set of functionally independent first integrals of the differential system (1.1) which are meromorphic in $M$. Then the monodromy group $G \subset \text{GL}(\mathbb{C}, n)$ admits $k$ functionally independent homogeneous rational invariants $I_1, \ldots, I_k$.

In many mechanical problems the previous lemma allows to reduce the initial integrability problem to the question from the theory of invariants of finitely generated linear groups $G = \langle G_1, \ldots, G_k \rangle \subset \text{GL}(n, \mathbb{C})$. Once it is shown that $G$ has no more than $p$ rational invariants one concludes that the system (1.1) can not have more than $p$ functionally independent first integrals meromorphic in $M$. If $G$ does have a rational invariant, the higher variational approach has to be applied (see [6]).

## 2 The planar three–body problem

It is natural to view the monodromy generators as linear transformations obtained through the solving of the normal variational equation of (1.1) along the particular solution $h(\Gamma)$ (see for definition [8]). This equation describes the linearization of the system (1.1) around the particular orbit after reduction of all already known first integrals.

We consider three mass points $m_1 > 0, m_2 > 0, m_3 > 0$ in the plane which attract each other according to the Newtonian law. Using the Whittaker variables we the corresponding equations of motion can be written as a Hamiltonian system with 3 degrees of freedom

$$
\dot{q}_r = \frac{\partial H}{\partial p_r}, \quad \dot{p}_r = -\frac{\partial H}{\partial q_r}, \quad (r = 1, 2, 3),
$$

with the Hamiltonian function

$$
H = \frac{M_1}{2} \left\{ p_1^2 + \frac{1}{q_1^2} P^2 \right\} + \frac{M_2}{2} (p_2^2 + p_3^2) + \frac{1}{m_3} \left\{ p_1 p_2 - p_3 P \right\} - \frac{m_1 m_3}{r_1} - \frac{m_3 m_2}{r_2} - \frac{m_1 m_2}{r_3},
$$

$$
P = p_3 q_2 - p_2 q_3 - k, \quad M_1 = m_3^{-1} + m_1^{-1}, \quad M_2 = m_3^{-1} + m_2^{-1},
$$

where

$$
r_1 = q_1, \quad r_2 = \sqrt{q_2^2 + q_3^2}, \quad r_3 = \sqrt{(q_1 - q_2)^2 + q_3^2},
$$

are the mutual distances of the bodies; $q_1$ is the distance $m_3 m_1$; $q_2$ and $q_3$ are the projections of $m_2 m_3$ on, and perpendicular to $m_1 m_3$; $p_1$ is the component of momentum of $m_1$ along $m_3 m_1$; $p_2$ and $p_3$ are the components of momentum of $m_2$ parallel and perpendicular to $m_3 m_1$; $k$ is the constant of the angular momentum.

Let us denote $M(4, K)$ the set of square $n$ by $n$ matrices over a field $K$. In [5] we calculated the normal variational equation of (2.1) along the Lagrangian parabolic equilateral solution
for a fixed non-zero value of the angular momentum $k$

$$\frac{dx}{dz} = \left( \frac{A}{z - z_0} + \frac{B}{z - z_1} + \frac{C}{z - z_2} \right)x, \quad x \in \mathbb{C}^4, \quad z \in \Gamma, \quad A, B, C \in M(4, \mathbb{C}), \quad (2.2)$$

where

$$z_0 = \frac{\sqrt{3}m_1m_2}{2S_2}, \quad z_1 = \frac{m_1(\sqrt{3}m_2 + iS_3)}{2S_2}, \quad z_2 = \frac{m_1(\sqrt{3}m_2 - iS_3)}{2S_2}, \quad (2.3)$$

$$\Gamma = \mathbb{C} \setminus \{z_0, z_1, z_2\}$$

$$S_2 = m_1m_2 + m_2m_3 + m_3m_1, \quad S_3 = m_2 + 2m_3.$$

One verifies (see [3], Appendix A) that in (2.2)

$$\left\{ \begin{array}{l}
z_2 = \overline{z}_1, \\
A \in M(4, \mathbb{R}), \quad B = R + iJ, \quad C = \overline{B} = R - iJ \quad \text{with} \quad R, J \in M(4, \mathbb{R}).
\end{array} \right. \quad (2.4)$$

Therefore, the equations (2.2) are invariant under the complex conjugation fixing the time $t$. This is not surprising since the equations of the three–body problem (2.1) are real.

Let $\Sigma(z)$, $\Sigma(e) = \text{Id}$, $e \in \Gamma$ be the fundamental matrix solution of the linear differential equation (2.2). Continued along a closed path $\gamma \in \pi_1(e, \Gamma)$ the solution $\Sigma(z)$ gives a function $\hat{\Sigma}(z)$ which also satisfies (2.2). From linearity of (2.2) it follows that there exists $T_\gamma \in \text{GL}(4, \mathbb{C})$ such that $\hat{\Sigma}(z) = \Sigma(z)T_\gamma$. The set of matrices $G = \{T_\gamma\}$ corresponding to all paths from $\pi_1(e, \Gamma)$ form monodromy group of the linear system (2.2). Let $T_i$ be the elements of $G$ corresponding to loops around the singular points $z = z_i$, $i = 0, 1, 2$. Then $G$ is generated by $T_0, T_1, T_2$. Let $T_\infty \in G$ denotes the monodromy element around $z = \infty$.

**Proposition 1** ([3]). The following assertions about the monodromy group $G$ hold

a) The singularity $z_0$ is an apparent one i.e $T_0 = \text{Id}$ and

$$T_1T_2 = T_\infty^{-1}. \quad (2.5)$$

b) The generators $T_1, T_2$ are unipotent trnsformations. Moreover, there exist $U, V \in \text{GL}(4, \mathbb{C})$ such that

$$U^{-1}T_1U = V^{-1}T_2V = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}. \quad (2.6)$$

c) $\text{Spectr}(T_\infty) = \{e^{2\pi i \lambda_1}, e^{2\pi i \lambda_2}, e^{-2\pi i \lambda_1}, e^{-2\pi i \lambda_2}\}$

where

$$\lambda_1 = \frac{3}{2} + \frac{1}{2} \sqrt{13 + \sqrt{\theta}}, \quad \lambda_2 = \frac{3}{2} + \frac{1}{2} \sqrt{13 - \sqrt{\theta}}, \quad (2.7)$$

and

$$\theta = 144 \left(1 - \frac{3S_2}{S_1^2}\right). \quad (2.8)$$
Let $U_{\Gamma}$ be a connected neighborhood of the Lagrangian parabolic equilateral solution of the three–body problem (2.1) defined in [3]. Below we summarize the results concerning the integrability of (2.1) obtained in our previous papers.

**Theorem 1.** ([2], [3]) For arbitrary $m_i > 0$, $i = 1, 2, 3$ the monodromy group $G$ of (2.2) does not have two independent rational invariants. As a result, the three–body problem (2.1) never has two additional independent first integrals meromorphic in $U_{\Gamma}$.

**Corollary 1.** The three–body problem (2.1) is not completely meromorphically integrable in the sense of Liouville–Arnold.

We introduce the parameter

$$
\sigma = \frac{m_1m_2 + m_2m_3 + m_3m_1}{(m_1 + m_2 + m_3)^2} .
$$

(2.10)

The following two theorems concern the non–existence of one additional meromorphic first integral.

**Theorem 2.** ([4], [5]) Let $\sigma \notin \left\{ \frac{1}{3}, \frac{5}{27}, \frac{2}{9}, \frac{7}{48}, \frac{5}{24} \right\}$. Then the three–body problem (2.1) does not have an additional first integral meromorphic in $U_{\Gamma}$.

**Theorem 3.** ([4], [5]) If $\sigma \in \left\{ \frac{1}{3}, \frac{5}{27} \right\}$ then $G$ has a polynomial invariant and the normal variational equation (2.2) admits a first integral $I(x, z)$ which is a quadratic polynomial with respect to $x$ and which is a rational function with respect to $z$.

The proof of these results is based on the following result from [8]: to every additional first integral of (2.1) independent with $H$ and meromorphic in $U_{\Gamma}$ corresponds a rational invariant of the monodromy group $G$. We used also the infinitesimal techniques from the Morales–Ramis differential Galois approach [6].

As follows from Theorems 1–3, the only remaining values of $\sigma$ for those the integrability property was not clear are

$$
\sigma \in \left\{ \frac{2}{9}, \frac{7}{48}, \frac{5}{24} \right\} .
$$

(2.11)

Our main result makes more precise the integrability property for these values of $\sigma$.

**Theorem 4.** If $\sigma = \frac{7}{48}, \frac{5}{24}$ then the three–body problem (2.1) does not have an additional first integral meromorphic in $U_{\Gamma}$. If $\sigma = \frac{2}{9}$ then $G$ has a polynomial invariant of degree 1 or 2 so that the normal variational equation (2.2) admits a first integral $I(x, z)$ which is a linear or quadratic polynomial with respect to $x$ and which is a rational function with respect to $z$.

The proof is contained in the next section.

### 3 The reflection symmetry of the monodromy group

As shown in [4], [5] the monodromy group $G$ always possesses a centralizer in GL(4, $\mathbb{C}$) given explicitly by

$$
T = T_{\infty} + T_{\infty}^{-1} - 2 \text{Id} .
$$

(3.1)
Proposition 2. Let 
\[ \sigma \in \left\{ \frac{7}{48}, \frac{5}{24} \right\}. \] \tag{3.2}
Then \( \text{Spectr}(T) = \{ \sigma_1, \sigma_1, \sigma_2, \sigma_2 \} \) where \( \sigma_1 \neq \sigma_2 \).

It can be verified directly with help of the following formulas obtained in [5]
\[ \text{Spectr}(T) = \{ \sigma_1, \sigma_1, \sigma_2, \sigma_2 \}, \quad \sigma_i = 2(\cos(2\pi \lambda_i) - 1), \quad i = 1, 2, \] \tag{3.3}
where \( \lambda_i \) are defined by (2.8).

Thus, if (3.2) holds, then the Jordan canonical form of \( T \) always contains two 2 \times 2 blocks (either diagonal or not) corresponding to two different eigenvalues \( \sigma_1 \) and \( \sigma_2 \). Then, as follows from the solution of the Frobenius problem (see e.g. [1]), the relations \( [T_i, T_j] = 0, \quad i = 1, 2 \) imply the existence of a linear basis in which the monodromy generators \( T_{1,2} \) have the same block–diagonal form
\[ T_1 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \quad T_2 = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \] \tag{3.4}
with unipotent blocks \( T_{i,kk} \in \text{GL}(2, \mathbb{C}) \).

Proposition 3. Under the condition (3.2) the group \( G \) does not have a rational invariant.

Proof. The proof is exactly the same as in [3], pp. 243-244. Since \( T \neq \alpha \text{Id}, \quad \alpha \in \mathbb{C} \) it is sufficient to verify that \( 1 \notin \text{Spectr}(T_{\infty}) \) (*). For \( \sigma = 7/48 \) one obtains \( \lambda_1 = 3/2 + \sqrt{22}/2, \quad \lambda_2 = 5/2 \) and for \( \sigma = 5/24 \) respectively \( \lambda_1 = 7/2, \quad \lambda_2 = 3/2 + \sqrt{10}/2 \). In both cases \( \lambda_i \notin \mathbb{Z} \) and the condition (*) follows from (2.7). \( \square \)

We note that the result of Proposition 3 was not contained in our work [3] where \( T_{\infty} \) was supposed to be diagonalizable.

Lemma 2. If \( \sigma = \frac{2}{9} \) then \( G \) has a polynomial invariant.

Proof. Below we assume that the monodromy group \( G = < T_1, T_2 > \) of (2.2) is defined for the basepoint \( e = 0 \). One significant problem in the analysis of \( G \) is that the Jordan canonical form of \( T_{\infty} \) and whose of the centralizer \( T \) depend on the masses \( m_i \) in a quite complicated way. At least no elementary algebraic description of this dependence is known. This difficulty can be overcome using the “reflection” symmetry of \( G \) given by the following lemma.

Lemma 3. The monodromy transformations \( T_1 \) and \( T_2 \) are related by \( T_1^{-1} = T_2 \).

Proof. Let \( \Sigma(z), \Sigma(0) = \text{Id} \) be the normalised fundamental matrix solution of (2.2) and let \( G \) be the corresponding monodromy group. For a function \( f(z) \) defined by its Taylor expansion \( f(z) = \sum a_n z^n, \quad a_n \in \mathbb{C} \) we define the operator of complex conjugation \( f(z) \mapsto \overline{f(z)} \) according to \( \overline{f(z)} = \sum \overline{a_n} z^n \). One can always represent locally \( \Sigma(z) \) as a power series convergent in a small neighborhood of the regular point \( z = 0 \). The symmetry conditions (2.4) imply then \( \overline{\Sigma(z)} = \Sigma(z) \).

Let \( \gamma_1, \gamma_2 \) be two loops starting from \( z = 0 \) and going in the counter clock–wise direction around the singular points \( z_1 \in \mathbb{C}_+ \) and \( z_2 \in \mathbb{C}_- \) respectively. By definition of \( T_1 \) we have \( (A): \Sigma(z) \overset{\gamma_1}{\to} \Sigma(z)T_1 \) after the analytic continuation of \( \Sigma(z) \) along \( \gamma_1 \). Let \( \overline{\gamma}_1 \) denote the loop symmetric to \( \gamma_1 \) with respect to the real axis \( \text{Im} z = 0 \) ( \( \overline{\gamma}_1 \) has the clock–wise orientation). According to (2.4) and (A) one will have \( \Sigma(z) \overset{\overline{\gamma}_1}{\to} \Sigma(z)T_1 \) after the analytic continuation
along $\gamma_1$. At the same time we have (B): $\Sigma(z) \subset \Sigma(z)T_2$ with $\Sigma(z)$ continued along the loop $\gamma_2$. The curves $\gamma_2$ and $\gamma_1$ are homotopic of opposite orientations. Hence, comparing (A) and (B) we obtain $T_2 = T_1^{-1}$ that achieves the proof of Lemma 3.

If $\sigma = 2/9$ then $\text{Spectr}(T_\infty) = (p, p^{-1}, p^{-1})$, $p = e^{2\pi i/\sigma}$, $p \neq p^{-1}$ in view of (2.7). We denote by $L_p$ and $L_{p^{-1}}$ the eigenspaces of $T_\infty$ corresponding to the eigenvalues $p$ and $p^{-1}$ respectively. Firstly, we consider the case dim$(L_p) = 2$ (the case dim$(L_{p^{-1}}) = 2$ is similar).

From Lemma 3 and (2.5) we deduce $T_\infty = T_1^{-1}$. Therefore, if $v \in L_p$ then $v \in L_{p^{-1}}$ and hence dim$(L_p) = \text{dim}(L_{p^{-1}}) = 2$. Thus, $T_\infty \in M(4, \mathbb{C})$ is diagonalizable. In this case, as shown in [5], pp. 245-246, the group $G$ possesses a quadratic polynomial invariant that proves the statement. Let us consider the remaining case dim$(L_p) = \text{dim}(L_{p^{-1}}) = 1$ with $L_p = < v >$ and $L_{p^{-1}} = < \tau >$ spanned by the linearly independent vectors $v$ and $\tau$ respectively. Then the following proposition holds.

**Proposition 4.** There exist two linearly independent vectors $w, \bar{w} \in \mathbb{C}^4$ such that the dual transformations $T_1 = T_1^T$, $T_2 = T_2^T$ act on $w, \bar{w}$ by permutations:

$$T_1 w = \bar{w}, \quad T_1 \bar{w} = w, \quad T_2 w = \bar{w}, \quad T_2 \bar{w} = w.$$  

(3.5)

**Proof.** Let $T_\infty = T_1^T$ then $\text{Spectr}(T_\infty) = \text{Spectr}(T_\infty)$. Since the geometric multiplicity of any eigenvalue in $T_\infty$ is the same as in $T_\infty$, we define $w$ and $\bar{w}$ as the only eigenvectors of $T_\infty$ corresponding to the eigenvalues $p$ and $p^{-1}$ respectively. The dual centralizer $T = T^T$ commutes with $T_1$, $T_2$ and its only eigendirections are $w$ and $\tau$ as follows from (3.1). The transformations $T_1, T_2$ preserve the eigendirections of $T$. So, in view of (2.6) and the identity $T_2 = T_1^{-1}$, either the relations (3.5) take place or we have

$$T_1 w = w, \quad T_1 \bar{w} = \bar{w}, \quad T_2 w = \bar{w}, \quad T_2 \bar{w} = w.$$  

(3.6)

One defines $< u, l > = \sum_{i=1}^n u_i l_i$ for $u, l \in \mathbb{C}^4$. In the case (3.5) the monodromy group $G$ has two independent polynomial invariants: $I(x) = < w, x >$ and $I(x) = < \bar{w}, x >$. But it is impossible according to Theorem 1. The proposition is proved.

In the remaining case (3.6) the group $G$ has a linear polynomial invariant given by $I(x) = < w + \bar{w}, x >$. That achieves the proof of Lemma 2.

Theorem 4 follows immediately from Proposition 3, Lemma 2, Lemma 1 and the fact (see [5], p. 246) that to every rational invariant of $G$ corresponds a single–valued first integral of the linear differential system (2.2). We believe that these integrals, existing for $\sigma = 1/3, 2/3^3, 2/3^2$ may contribute towards a better understanding of the dynamics of the three–body problem in the vicinity of parabolic Lagrangian orbits.

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