A \textit{SO(1, 3)} gauge theory of quantum gravity: quantization of the non-interacting field

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Abstract
The non-interacting field belonging to a new \textit{SO(1, 3)} gauge field theory equivalent to general relativity is canonically quantized in the Lorentz gauge and the physical Fock space $\mathcal{F}_{\text{Phys}}$ for non-interacting gauge particles is constructed. To assure both Lorentz covariance and positivity of the norm and energy expectation value for physical states restrictions needed in the construction of the physical Fock space are put consequentially on state vectors, and not on the algebra of creation and annihilation operators as usual—alltogether providing the second step in consistently quantizing gravitation.

Keywords: canonical quantization of gravity, physical Fock space for quantum gravity, positive-norm, positive-energy quanta for quantum gravity, relativistic invariance of quantization of gravity

1. Introduction
Gravity has defied so far all attempts at consistent quantization. In fact, Einstein’s theory of general relativity (GR) and its generalizations turn out to be either not renormalizable or do not respect unitarity at the quantum level.

GR itself is not renormalizable in essence due to the fundamental field $g$, the metric, or equivalently $e$, the Vierbein, being of mass-dimension zero. Simple power-counting allows...
to demonstrate that the loop expansion of the quantum effective action contains divergent contributions of ever higher mass-dimension—destroying renormalizability [1].

All attempts to deal with this fundamental difficulty have failed essentially because all generalizations of GR analyzed in the literature offering a seeming way out still contain \( g \), or equivalently \( e \), as at least one dynamical field due to the local translation or coordinate invariance to which these theories are subject to. As a result in all cases one ends up with either a non-renormalizable or a non-unitary theory.

In [2] we have started a programme aiming to overcome those difficulties step-by-step.

Our first step has been to take a point of departure different from the literature at the classical level already: a theory for classical gravity which is not equal, but equivalent to GR in the sense established in [2], which allows for perturbatively renormalizable actions for the fundamental field. In fact in [2] we have developed a new classical gauge field theory of the Lorentz group \( SO(1,3) \) which (a) contains as the only dynamical field the dimension-one Lorentz gauge field in terms of which all else can be expressed and which allows for actions perturbatively renormalizable by power-counting, and (b) is equivalent to GR in a limiting case. Whilst a proof of perturbative renormalizability seems within reach for this theory [3], another difficulty has to be resolved first, which arises from the non-compactness of \( SO(1,3) \): establishing that the canonical quantization of the gauge fields allows for the definition of positive-norm, positive-energy states and a corresponding Lorentz-invariant physical Fock space for these fields - and that negative-norm states completely decouple.

The second step of our programme and the goal of this paper is exactly to provide that resolution. We do this by strictly adhering to our guiding principles of keeping to Lorentz and \( SO(1,3) \) gauge covariance, to treating the \( SO(1,3) \) gauge symmetry strictly as an inner symmetry, and to perturbative renormalizability which were at the origin of developing the \( SO(1,3) \) gauge field theory in the first place [2]. In addition in this paper we intertwine Lorentz covariance with positivity of the norm and energy expectation value for physical states, and consequentially put restrictions needed in the search of a physical Fock space on state vectors, and not on the algebra of creation and annihilation operators as usual. The restrictions found illuminate the conceptual difference to GR where the existence of decomposition of energy states in positive and negative ones hinges on the spacetime being stationary, whilst in algebraic QFT there are yet other criteria to be obeyed.

The next step of our programme will have to be the renormalizability proof of the full quantum theory [3] and the final one to establish the unitarity of the \( S \)-matrix on the physical Fock space constructed in this paper.

To establish that canonical quantization of the non-interacting gauge fields allows for the definition of positive-norm, positive-energy states and a corresponding Lorentz-invariant physical Fock space we have organized this paper as follows. In section 2 we revisit the fundamentals of the new Lorentz gauge field theory of gravitation which is perturbatively renormalizable by power-counting and equivalent to GR. In section 3 we derive the Lagrangian of the non-interacting Lorentz gauge field theory in the Lorentz gauge, i.e. the part of the perturbatively renormalizable Lagrangian which is quadratic in the fields. This serves as starting point for the Lorentz covariant canonical quantization of the non-interacting theory in terms of the gauge fields and their conjugate canonical momenta in section 4. In section 5 we recast the theory in terms of the creation and annihilation operators corresponding to the gauge field operators. In sections 6 and 7 we establish the one-particle subspace of states \( \mathcal{F}^{LG}_1 \) fulfilling
the Lorentz gauge condition and the one-particle subspace of states $F_1^{\text{P}}$ orthogonal to their dual state with positive semidefinite norm respectively which in combination allow for the definition of the one-particle subspace of states $F_1^{\text{LG}} \& F_1^{\text{P}}$ with positive semidefinite norm and energy expectation value in section 8. The final three sections are devoted to establish the physical one-particle space of states $F_1^{\text{Phys}}$ with positive norm and energy expectation value.

The physical one-particle space of states $F_1^{\text{Phys}}$ with positive norm and energy expectation value, to prove the covariance of the quantization approach on $F_1^{\text{Phys}}$ and to define the physical n-particle space of states $F_1^{\text{Phys}}$ and the physical Fock space $F_1^{\text{Phys}}$ for non-interacting gauge particles respectively.

2. Lorentz gauge field theory of gravitation equivalent to general relativity

In this section we revisit local Lorentz invariance and the corresponding classical gauge field theory equivalent to GR as developed in [2] to prepare for the derivation and perturbative quantization of the non-interacting Lorentz quantum gauge field theory.

Let us start with some notations around the Lorentz gauge group $\text{SO}(1,3)$ and algebra $\text{so}(1,3)$ at the core of the theory. We will mainly work with infinitesimal group elements $\mathbf{1} + \Theta_\omega(x)$, where

$$
\Theta_\omega(x) = \frac{i}{2} \omega^{\gamma\delta}(x)(L_{\gamma\delta} + \Sigma_{\gamma\delta}) \in \text{so}(1,3)
$$

(1)

is a local element of the gauge algebra $\text{so}(1,3)$, $x$ a point in Minkowski spacetime $\mathbf{M}^4 = (\mathbb{R}^4, \eta)$ given in Cartesian coordinates and $\eta = \text{diag}(-1, 1, 1, 1)$ is the flat spacetime metric with which indices are raised and lowered throughout this paper. Indices $\alpha, \beta, \gamma, ...$ denote quantities defined on $\mathbf{M}^4$ which transform Lorentz covariantly.

Above $\omega^{\gamma\delta} = -\omega^{\delta\gamma}$ are the six infinitesimal gauge parameters parametrizing the Lorentz algebra. Treating Lorentz transformations as ‘inner’ transformations [2] the spacetime-related algebra generators acting on field space are given by

$$
L_{\gamma\delta} = -L_{\delta\gamma} = -i(x_\gamma \partial_\delta - x_\delta \partial_\gamma)
$$

(2)

whilst $\Sigma_{\gamma\delta}$ denote the generators of an arbitrary finite-dimensional spin representation of the Lorentz algebra.

Generators of any representation obey the commutation relations

$$
[J_{\alpha\beta}, J_{\gamma\delta}] = i(\eta_{\alpha\gamma}J_{\beta\delta} - \eta_{\beta\gamma}J_{\alpha\delta} + \eta_{\beta\delta}J_{\alpha\gamma} - \eta_{\alpha\delta}J_{\beta\gamma})
$$

(3)

which in effect defines the Lorentz algebra $\text{so}(1,3)$.

The action of a local infinitesimal group element $\mathbf{1} + \Theta_\omega(x)$ on a field $\varphi(x)$ living in an arbitrary finite-dimensional spin representation of the Lorentz group is given by

$$
x^\alpha \longrightarrow x'^\alpha = x^\alpha
$$

$$
\varphi(x) \longrightarrow \varphi'(x) = \varphi(x) + \delta_\omega \varphi(x),
$$

(4)

where

$$
\delta_\omega \varphi(x) = (\Theta_\omega(x)\varphi)(x)
$$

$$
= -\omega^{\gamma\delta}(x)x_\delta \partial_\gamma \varphi(x) + \frac{i}{2} \omega^{\gamma\delta}(x)\Sigma_{\gamma\delta} \varphi(x).
$$

(5)
In order to define locally gauge-covariant expressions we need to introduce a covariant derivative \( \nabla_\alpha (x) \) as usual which obeys

\[
(\nabla_\alpha (x) \varphi)' = \nabla'_\alpha (x) \varphi'
\]

and a related gauge field \( B_\alpha (x) \)

\[
\nabla_\alpha^B = \partial_\alpha + B_\alpha (x)
\]

\[
B_\alpha = \frac{i}{2} B_\alpha^{\gamma\delta} (x) (L_{\gamma\delta} + \Sigma_{\gamma\delta})
\]

\[
= -B_\alpha^{\gamma\delta} (x) x_\delta \partial_\gamma + \frac{i}{2} B_\alpha^{\gamma\delta} (x) \Sigma_{\gamma\delta}
\]

living in the Lorentz algebra \( so(1,3) \). Above primed quantities refer to gauge-transformed quantities. For further details we refer to [2].

To deal with the ever more involved algebraic expressions in the further development of the theory we rewrite the covariant derivative \( \nabla_\alpha (x) \)

\[
\nabla_\alpha^B = \partial_\alpha - B_\alpha^{\gamma\delta} x_\delta \partial_\gamma + \frac{i}{2} B_\alpha^{\gamma\delta} \Sigma_{\gamma\delta}
\]

\[
= (\eta_\alpha^{\gamma} - B_\alpha^{\gamma\delta} x_\delta) \partial_\gamma + \frac{i}{2} B_\alpha^{\gamma\delta} \Sigma_{\gamma\delta}
\]

\[
= d_\alpha^B + B_\alpha,
\]

and introduce the expression

\[
e^\alpha_\vartheta [B] \equiv \eta_\alpha^{\vartheta} - B_\alpha^{\vartheta\zeta} x_\zeta
\]

resembling a Vierbein which, however, is a functional of the dynamical gauge field \( B_\alpha^{\gamma\delta} (x) \) in our theory, and the short-hand notations

\[
d_\alpha^B \equiv e_\alpha^0 [B] \partial_0,
\]

\[
\bar{B}_\alpha \equiv \frac{1}{2} B_\alpha^{\gamma\delta} \Sigma_{\gamma\delta}.
\]

We have elaborated in depth why \( e_\alpha^{\eta} [B] \) not being a fundamental field is so crucial for the further development of the theory which turns out to be both equivalent to GR in the sense established in [2] and perturbatively renormalizable.

As usual we next define the field strength \( G \) operator acting on field space

\[
G_{\alpha\beta} [B] \equiv [\nabla_\alpha^B , \nabla_\beta^B ]
\]

and express it in terms of the gauge field \( B \)

\[
G_{\alpha\beta} [B] = [d_\alpha^B , d_\beta^B ] + d_\alpha^B \bar{B}_\beta - d_\beta^B \bar{B}_\alpha
\]

\[
+ [\bar{B}_\alpha , \bar{B}_\beta ] + (B_{\alpha\beta}^{\eta} - B_{\beta\alpha}^{\eta}) \nabla_\alpha^B
\]

To calculate

\[
[d_\alpha^B , d_\beta^B ] = (e_\alpha^{\zeta} [B] \partial_\zeta e_\beta^{\eta} [B] - e_\beta^{\zeta} [B] \partial_\zeta e_\alpha^{\eta} [B] ) \partial_\eta
\]
we assume that \( e_\alpha \gamma [B] \) is non-singular, i.e. \( \det e[B] \neq 0 \). Hence there is an inverse \( e^\gamma \eta [B] \) with \( e^\gamma \eta [B] e_\gamma \xi [B] = \delta_\eta ^\xi \), and we can write
\[
[d^\alpha _\alpha , d^\beta _\beta ] = H_{\alpha \beta } \gamma [B] d^\eta _\eta ,
\]
with
\[
H_{\alpha \beta } \gamma [B] \equiv e^\gamma \eta [B] \left( e_\alpha \gamma [B] \partial_\gamma e_\beta ^\eta [B] - e_\beta \gamma [B] \partial_\gamma e_\alpha ^\eta [B] \right).
\]
As a result we can express
\[
G_{\alpha \beta } [B] = ( H_{\alpha \beta } \gamma [B] + B_{\alpha \beta } \gamma - B_{\beta \alpha } \gamma ) \nabla_\gamma ^\delta + d^\eta _\alpha B_\beta - d^\eta _\beta B_\alpha + [B_\alpha , B_\beta ] - H_{\alpha \beta } \gamma [B] B_\gamma
\]
in terms of the covariant quantities
\[
T_{\alpha \beta } \gamma [B] \equiv -( B_{\alpha \beta } \gamma - B_{\beta \alpha } \gamma ) - H_{\alpha \beta } \gamma [B]
\]
and
\[
R_{\alpha \beta } [B] \equiv \frac{i}{2} R_{\alpha \beta } ^{\gamma \delta } [B] \Sigma_{\gamma \delta}
\]
\[
R_{\alpha \beta } ^{\gamma \delta } [B] = d^\eta _\alpha B_\beta \gamma ^\delta - d^\eta _\beta B_\alpha \gamma ^\delta + B_\alpha \gamma \eta B_\beta ^\eta \gamma ^\delta
\]
\[
- B_\beta \gamma \eta B_\alpha ^\eta \gamma ^\delta - H_{\alpha \beta } \gamma ^\eta [B] B_\eta ^\gamma .
\]
The geometrical significance of all: the gauge field \( B \), field strength \( G \) and \( T \) as well as \( R \) has been further illuminated in terms of a Banach fibre bundle structure with trivial base manifold \( M^4 \) and infinite-dimensional fibres for the various fields in [2].

There we also have given the transformation behaviour of the various quantities under local Lorentz gauge transformations.

Next we write down the most general gauge-invariant action for the gauge field \( B \) which is renormalizable by power-counting. It contains dimension-zero, -two and -four contributions \( S^{(0)}_G [B], S^{(2)}_G [B] \) and \( S^{(4)}_G [B] \) respectively which we cite from [2] without further details here starting with
\[
S^{(0)}_G [B] = \Lambda \int d^4 x \ det e^{-1} [B],
\]
where \( \Lambda \) is a constant of dimension \([\Lambda] = 4\). The most general dimension-two contribution reads
\[
S^{(2)}_G [B] = \frac{1}{\kappa} \int d^4 x \ det e^{-1} [B] \left\{ \alpha_1 R_{\alpha \beta } ^{\gamma \delta } [B] + \alpha_2 T_{\alpha \beta \gamma } [B] T^{\alpha \beta \gamma } [B] + \alpha_3 T_{\alpha \beta \gamma } [B] T^{\gamma \beta \alpha } [B] + \alpha_4 T_{\alpha \gamma } ^{\beta \alpha } [B] T^{\alpha \beta } [B] + \alpha_5 \nabla_\beta ^\alpha T^{\alpha \beta } [B] \right\}.
\]
\( \frac{1}{\kappa} = \frac{1}{4 \pi \Gamma} \) has mass-dimension \([\frac{1}{\kappa}] = 2\) with \( \Gamma \) denoting the Newtonian gravitational constant. The \( \alpha_i \) above are constants of dimension \([\alpha_i] = 0\).
Finally, the most general dimension-four contribution reads
\[ S_G^{(4)}[B] = \int d^4x \det e^{-1}[B] \left\{ \beta_1 R_{\alpha\beta}^{\gamma\delta}[B] R^{\alpha\beta}_{\gamma\delta}[B] \\
+ \beta_2 R_{\alpha\gamma}^{\alpha\delta}[B] R^{\beta\gamma}_{\beta\delta}[B] + \beta_3 R_{\alpha\beta}^{\alpha\beta}[B] R_{\gamma\delta}^{\gamma\delta}[B] \\
+ \beta_4 \nabla_{\beta}^{\alpha} \nabla_{\delta}^{\gamma} R_{\alpha\gamma}^{\alpha\delta}[B] + \beta_5 \nabla_{\beta}^{\gamma} \nabla_{\gamma}^{\delta} R_{\alpha\beta}^{\alpha\beta}[B] \\
+ \ldots \\
+ \gamma_1 \nabla_{\beta}^{\alpha} T_{\alpha\beta\gamma\delta}[B] \nabla_{\gamma}^{\alpha} T^{\alpha\beta}[B] + \gamma_2 \nabla_{\beta}^{\gamma} T_{\alpha\beta\gamma\delta}[B] \nabla_{\delta}^{\gamma} T^{\gamma\delta}[B] \\
+ \ldots \\
+ \delta_4 R T^4 - \text{terms} \\
+ \ldots \\
+ \delta_k R T^{2} - \text{terms}, R \nabla^{B} T - \text{terms} \\
+ \ldots \right\} \]  
(21)

with \( \beta_i, \gamma_j, \delta_k \) constants of dimension \( [\beta_i] = [\gamma_j] = [\delta_k] = 0 \).

By construction
\[ S_G[B] = S_G^{(0)}[B] + S_G^{(2)}[B] + S_G^{(4)}[B] \]  
(22)
is the most general action of dimension \( \leq 4 \) in the gauge fields \( B_{\alpha \beta}^{\gamma\delta} \) and their first and second derivatives \( \partial_\beta B_{\alpha \beta}^{\gamma\delta}, \partial_\alpha \partial_\beta B_{\alpha \beta}^{\gamma\delta} \) which is locally Lorentz invariant and—having consistent field quantization in mind—renormalizable by power-counting. The actual proof of perturbative renormalizability established in part 2 of [3] requires the much more involved demonstration that counterterms needed to absorb infinite contributions to the perturbative expansion of the effective action of the full quantum theory are again of the form equation (22) with possibly renormalized constants.

We finally note that for the choice
\[ \alpha_1 = 1, \quad \alpha_2 = -\frac{1}{4}, \quad \alpha_3 = -\frac{1}{2}, \quad \alpha_4 = -1, \quad \alpha_5 = 2 \]  
(23)
\( S_G^{(0)}[B] + S_G^{(2)}[B] \) coupled to scalar matter is equivalent to GR with a cosmological constant term [2].

3. Non-interacting Lagrangian for the SO(1,3) gauge field theory in the Lorentz gauge

In this section, starting with the most general gauge-invariant Lagrangian \( S_G[B] \) renormalizable by power-counting, we introduce a coupling constant \( g \) and derive the truncated action \( S_G[B] \) quadratic in the gauge field \( B \) serving as the starting point for the gauge field quantization in the sections to follow.

Scaling the gauge field \( B \) as well as \( H, T \) and \( R \) by a dimension-less constant \( g \) as
\[ B_{\alpha}^{\gamma\delta} \rightarrow g B_{\alpha}^{\gamma\delta} \]
\[ H_{\alpha\beta}^{\gamma}[B] \rightarrow g H_{\alpha\beta}^{\gamma}[gB] \]
\[ T_{\alpha\beta}^{\gamma}[B] \rightarrow g T_{\alpha\beta}^{\gamma}[gB] \]
\[ R_{\alpha\beta}^{\gamma\delta}[B] \rightarrow g R_{\alpha\beta}^{\gamma\delta}[gB] \]  
(24)
and the action $S_G[B]$ as

$$S_G[B, g] = S_G^{(0)}[B, g] + \frac{1}{g} S_G^{(2)}[B, g] + \frac{1}{g^2} S_G^{(4)}[B, g]$$

we obtain

$$S_G[B, g] = \Lambda \int d^4x \, \det e \, \left( \frac{1}{g} \beta_1 R_{\alpha\beta} \gamma^\delta [gB] \right)$$

$$+ \int d^4x \, \det e \, \left( \frac{1}{g} \beta_2 R_{\alpha\gamma} \gamma^\delta [gB] \right)$$

$$+ \int d^4x \, \det e \, \left( \frac{1}{g} \beta_3 R_{\alpha\gamma} \gamma^\delta [gB] \right)$$

$$+ \cdots$$

Expanding equation (26) in orders of $g$ we get

$$S_G[B, g] = \Lambda \int d^4x$$

$$+ \frac{1}{\kappa} \int d^4x \, \det e \, \left( \frac{1}{g} \beta_1 R_{\alpha\beta} \gamma^\delta [gB] \right)$$

$$+ \int d^4x \, \det e \, \left( \frac{1}{g} \beta_2 R_{\alpha\gamma} \gamma^\delta [gB] \right)$$

$$+ \int d^4x \, \det e \, \left( \frac{1}{g} \beta_3 R_{\alpha\gamma} \gamma^\delta [gB] \right)$$

$$+ \cdots$$

$$+ \beta_4 \beta^\gamma \partial_\gamma R_{\alpha\beta} \alpha^\delta [gB]$$

$$+ \cdots$$

$$+ \beta_5 \beta^\gamma \partial_\gamma R_{\alpha\beta} \alpha^\delta [gB]$$

$$+ \cdots$$

$$+ O(g) - \text{terms.}$$
Taking the limit $g \to 0$ above we neglect the surviving dimension-zero and -one terms in this paper because the former is an infinite field-independent constant and the latter a total divergence. For the same reason we also neglect the $O(\frac{1}{g})$-terms in the dimension-four term which is crucial for a consistent free field limit. A refined treatment of both the infinite constant term and the pure divergence contributions might become important in the full quantum case.

Whilst there is more than one surviving term in the limit above we take
\[ S_0^G[B] = \int d^4x \mathcal{L}_G^\dag[B] \] (28)

with
\[ \mathcal{L}_G^\dag[B] = -\frac{1}{4} R_\alpha^\gamma \gamma^\delta [B] R_\beta^\alpha \gamma^\delta [B] - \frac{1}{2} \partial^\alpha B_\alpha \gamma^\delta \partial_\beta B^\beta \gamma^\delta \] (29)
as the non-interacting gauge field action with
\[ R_\alpha^\gamma \gamma^\delta [B] = \partial_\alpha B_\beta \gamma^\delta - \partial_\beta B_\alpha \gamma^\delta , \] (30)
for the sake of keeping calculations straightforward whilst all the other terms are to be treated as perturbations with small coupling constants $\beta_i, \gamma_j$ etc. Those terms should eventually be included.

Above we have normalized $\beta_1 = -\frac{1}{4}$ and introduced a Lorentz covariant gauge-fixing term $\partial^\alpha B_\alpha \gamma^\delta \partial_\beta B^\beta \gamma^\delta$ as usual.

### 4. Lorentz covariant canonical quantization of the non-interacting theory

In this section, in a fully Lorentz covariant way, we canonically quantize the gauge fields $B$ in the Lorentz gauge whose dynamics is governed by the non-interacting Lagrangian given by equation (29), and determine the expressions for the energy and momentum field operators in terms of $B$.

A usual we start with the gauge fields $B_\alpha \gamma^\delta (t, x)$ for a fixed time $t$ and first determine their canonically conjugate momenta
\[ \Pi^\alpha \gamma^\delta (t, x) = \frac{\partial \mathcal{L}_G^\dag[B]}{\partial (\partial_0 B_\alpha \gamma^\delta)(t, x)} = -R_\alpha^\gamma \gamma^\delta [B](t, x) - \eta^0 \partial_\alpha B^\alpha \eta^\gamma \eta^\delta \] (31)
which are well-defined due to the gauge-fixing term in equation (29).

We now demand the quantum fields and their conjugates to obey the Lorentz covariant equal-time canonical commutation relations
\[ [B_\alpha \gamma^\delta (t, x), \Pi^\alpha \gamma^\delta (t, y)] = i \eta^\alpha \beta \left( \eta^\gamma \eta^\delta \zeta - \eta^\gamma \zeta \eta^\delta \right) \delta^3(x - y) \] (32)
and
\[ [B_\alpha \gamma^\delta (t, x), B^\beta \eta^\delta \eta^\gamma \eta^\delta \eta^\gamma \zeta](t, y) = 0, \]
\[ [\Pi^\alpha \gamma^\delta (t, x), \Pi^\beta \eta^\delta \eta^\gamma \eta^\delta \eta^\gamma \zeta](t, y) = 0, \] (33)

where $\eta^\gamma \eta^\delta \zeta - \eta^\gamma \zeta \eta^\delta$ is the projection operator onto the space of tensors $B_\alpha \gamma^\delta$ antisymmetric in $\gamma$ and $\delta$. 
Using
\[ \dot{B}_0^\alpha \eta_\xi(t, y) = \Pi_0^\beta \eta_\xi(t, y) - \partial_\alpha B^\beta \eta_\xi(t, y) \]
\[ \dot{B}_0^\beta \eta_\xi(t, y) = \Pi_0^\gamma \eta_\xi(t, y) - \partial^\gamma B^\beta \eta_\xi(t, y) \]
(34)
a little algebra shows that equations (32) and (33) are equivalent to
\[ \dot{B}_0 \gamma^\alpha(t, x), B_0^\alpha \eta_\xi(t, y) \]
\[ = -i \eta_\alpha \beta \left( \eta^\gamma \eta^\delta \zeta \eta^\eta \zeta - \eta^\gamma \zeta \eta^\delta \eta \right) \delta^3(x - y) \]
(35)
and
\[ B_0 \gamma^\alpha(t, x), B_0^\beta \eta_\xi(t, y) \]
\[ = 0 \]
(36)
Equations (35) and (36) are nothing but canonical commutation relations for 24 non-interacting scalar fields written in compact form—however with only 12 of them coming with negative sign in equation (35) and 12 with positive.

In the remainder of the paper—generalizing the Lorentz covariant Gupta–Bleuler treatment of the photon field [4]—we will work our way step-by-step to a Lorentz invariant physical Fock space based on equations (35) and (36) which will display positive norm and energy for all its states, and in which only 6 from the original 24 degrees of freedom will survive.

Finally we give the Noether expression for the energy-momentum four vector
\[ P^\beta = \int d^3x T_0^\beta \]
(37)derived from the canonical energy-momentum tensor
\[ T^{\alpha\beta} = \eta^{\alpha\beta} L_G - \frac{\partial L_G[B]}{\partial (\partial_\alpha B_\xi \eta^\xi)} \partial_\beta B_\xi \eta^\xi \]
(38)in terms of the fields and their first time derivative. We find for the energy operator
\[ P^0 = H = \int d^3x \left\{ \Pi_\alpha \gamma^\alpha(x) \dot{B}_\alpha \gamma^\delta(x) - L_G[B](x) \right\} \]
\[ = \frac{1}{2} \int d^3x \left\{ \dot{B}_\alpha \gamma^\delta(x) \dot{B}^\alpha \gamma^\delta(x) + \partial_\alpha B_\alpha \gamma^\delta(x) \partial^\gamma B^\alpha \gamma^\delta(x) \right\} \]
(39)and for the momentum operator
\[ P^\alpha = -\int d^3x \Pi_\alpha \gamma^\delta(x) \partial^\delta B_\alpha \gamma^\delta(x) = -\int d^3x \dot{B}_\alpha \gamma^\delta(x) \partial^\delta B^\alpha \gamma^\delta(x). \]
(40)

5. Creation and annihilation operators of the non-interacting theory

In this section we introduce the Lorentz covariant creation and annihilation operators $b^\dagger$ and $b$ respectively and calculate all: their commutation relations and the expressions for the energy and the momentum operators in terms of them. We then naively construct the corresponding one-particle pseudo Hilbert space of states, not yet taking into account neither the Lorentz gauge fixing nor the physical requirements of positive probability and positive energy.

As usual we first Fourier-transform the hermitean gauge field $B$ introducing $b^\dagger$ and $b$.
\[ B_\alpha^\gamma \delta(x) = \int d\tilde{k} \left( b_\alpha^\gamma \delta(k) e^{ikx} + b_\alpha^\gamma \delta(k) e^{-ikx} \right) = B_\alpha^\gamma \delta(x), \] (41)

where

\[ k^0 = \omega_k = \pm \sqrt{k^2} = |k| \]
\[ d\tilde{k} = \frac{d^4k}{(2\pi)^3} \frac{\theta(k^0)}{2\omega_k} \delta^4(k^2) \] (42)

expresses the on-shell condition \( k^2 = 0 \) assuring that \( B \) is a solution of the field equations derived from extremizing equation (29) and \( d\tilde{k} \) denotes the invariant on-shell volume element in momentum-space.

Next we invert equation (41) and find the time-independent expression

\[ b_\alpha^\gamma \delta(k) = i \int d^3x e^{-ikx} \partial_0 B_\alpha^\gamma \delta(t,x). \] (43)

It is now easy to show that the Lorentz covariant creation and annihilation operators obey the Lorentz covariant commutation relations

\[ \left[ b_\alpha^\gamma \delta(k), b_\beta^\eta \zeta(h) \right] = (2\pi)^3 2\omega_k \eta_\alpha\beta \left( \eta^\gamma \eta^\delta \zeta^\zeta - \eta^\gamma \zeta^\eta \zeta^\delta \right) \delta^3(k-h) \] (44)

and

\[ \left[ b_\alpha^\gamma \delta(k), b_\beta \eta \zeta(h) \right] = 0 \]
\[ \left[ b_\alpha^\gamma \delta(k), b_\beta^\eta \zeta(h) \right] = 0. \] (45)

Again we obtain 24 commutations relations for the creation and annihilation operators similar to those belonging to a scalar field—again with only 12 of them coming with positive sign in equation (44) and 12 with negative.

Next we express the energy operator in terms of the Lorentz covariant creation and annihilation operators

\[ H = \frac{1}{2} \int d^3x \left\{ \hat{B}_\alpha^\gamma \delta(x) \partial^\alpha \gamma \delta(x) + \partial_\alpha B_\alpha^\gamma \delta(x) \partial^\alpha \gamma \delta(x) \right\} \]
\[ = \frac{1}{2} \int d\tilde{k} \omega_k \left\{ b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) + b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) \right\}. \] (46)

As usual we introduce normal ordering denoted by \( :.:. \)—writing all the creation operators to the left of the annihilation operators—to normalize the energy of the vacuum to zero

\[ H = \frac{1}{2} : \int d\tilde{k} \omega_k \left\{ b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) + b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) \right\} : \]
\[ = \int d\tilde{k} \omega_k b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k). \] (47)

Similarly we find for the momentum operator

\[ P_i = - \int d^3x \hat{B}_\alpha^\gamma \delta(x) \partial^\alpha \gamma \delta(x) \]
\[ = \frac{1}{2} \int d\tilde{k} k^i \left\{ b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) + b_\alpha^\gamma \delta(k) b_\alpha^\gamma \delta(k) \right\} \] (48)
which needs no normal ordering. Normal ordering to remove the infinite energy of the vacuum state to be introduced later might need special care in the context of a full quantum theory of gravitation.

The \(1 + 3\) relations above can be combined into the Lorentz covariant expression for the hermitean energy-momentum operator

\[
P^\beta = \frac{1}{2} \int \tilde{k}^\beta \left\{ b^\alpha \gamma_\delta(\tilde{k}) b_\alpha^\dagger \gamma_\delta(\tilde{k}) + b_\alpha^\dagger \gamma_\delta(\tilde{k}) b^\alpha \gamma_\delta(\tilde{k}) \right\};
\]

\[
= \int \tilde{k}^\beta b_\alpha^\dagger \gamma_\delta(\tilde{k}) b^\alpha \gamma_\delta(\tilde{k}).
\]

(49)

Its commutator with a creation operator \(b^\dagger\) reads

\[
\left[ P^\beta, b^\dagger_\alpha \gamma_\delta(\tilde{k}) \right] = k^\beta b^\dagger_\alpha \gamma_\delta(\tilde{k})
\]

(50)

and it commutes with the energy operator, i.e. is time-independent

\[
\left[ H, P^\beta \right] = 0.
\]

(51)

Next we introduce the normalized vacuum state

\[|0\rangle\] with \(<0|0\rangle = 1 \]

(52)

which is annihilated by the annihilation operators \(b\)

\[b_\alpha \gamma_\delta(\tilde{k}) |0\rangle = 0 \quad \forall \tilde{k}; \alpha, \gamma, \delta \]

(53)

and out of which the creation operators \(b^\dagger\) create one-particle states

\[b^\dagger_\alpha \gamma_\delta(\tilde{k}) |0\rangle = \text{one-particle state} \]

(54)

with normalization

\[\langle0|b_\alpha \gamma_\delta(\tilde{k}) b^\dagger_\beta \eta\zeta(\tilde{h})|0\rangle = (2\pi)^3 2\omega_\eta \eta_\delta \gamma_\zeta(\gamma^\eta \eta^\delta - \gamma^\zeta \eta^\delta) \delta^3(\tilde{k} - \tilde{h}).\]

(55)

Let us look next at a general one-particle state defined by

\[|f\rangle = \frac{1}{\sqrt{2}} \int d\tilde{k} f^\alpha \gamma_\delta(\tilde{k}) b^\dagger_\alpha \gamma_\delta(\tilde{k}) |0\rangle, \]

(56)

where \(f^\alpha \gamma_\delta(\tilde{k}) = -f^{\alpha \gamma}(\tilde{k})\) is a generally complex polarization tensor antisymmetric in the indices \(\gamma\) and \(\delta\). \(f^\alpha \gamma_\delta(\tilde{k})\) characterizes a one-particle state completely which will be crucial in the further development of our thoughts.

The energy operator acts on \(|f\rangle\) as

\[H |f\rangle = \sqrt{2} \int d\tilde{k} \omega_\gamma f^\alpha \gamma_\delta(\tilde{k}) b^\dagger_\alpha \gamma_\delta(\tilde{k}) |0\rangle\]

(57)

and the norm of \(|f\rangle\) is calculated to be

\[\langle f | f \rangle = \int d\tilde{k} f^\alpha \gamma_\delta(\tilde{k}) f^\alpha \gamma_\delta(\tilde{k}) \]

(58)

whilst the energy expectation value for the state \(|f\rangle\) is

\[\langle f | H | f \rangle = \int d\tilde{k} \omega_\gamma f^\alpha \gamma_\delta(\tilde{k}) f^\alpha \gamma_\delta(\tilde{k}).\]

(59)
We again recognize the problem: whilst the quantization procedure is manifestly Lorentz covariant it yields 12 positive-norm, positive-energy states and 12 negative-norm, negative-energy states. In the next sections we turn to cure the flaws step-by-step.

6. One-particle subspace of states $\mathcal{F}^{1LG}$ fulfilling the Lorentz gauge condition

In this section we take the first step towards the identification of a viable physical Fock space by introducing the one-particle subspace of states $\mathcal{F}^{1LG}$ fulfilling the Lorentz gauge condition which is in essence a condition on the $\alpha$-index of the polarization tensor $f_{\alpha}\gamma^\delta(k)$.

Let us start by introducing the negative-frequency part of the gauge field $B$ by

$$B^{(-)}_{\alpha}\gamma^\delta(x) = \int d\tilde{k} b_{\alpha}\gamma^\delta(\tilde{k}) e^{i\tilde{k}x}.$$  \hfill (60)

We then define a one-particle state $|f\rangle$ $\in \mathcal{F}^{1LG}$ to be in the one-particle subspace of states $\mathcal{F}^{1LG}$ fulfilling the Lorentz gauge condition if (i) holds

$$(i) : \partial^\alpha B^{(-)}_{\alpha}\gamma^\delta(x) |f\rangle = \sqrt{2} \int d\tilde{k} i\tilde{k} f_{\alpha}\gamma^\delta(\tilde{k}) e^{i\tilde{k}x} |0\rangle = 0$$ \hfill (62)

which assures that $|f\rangle$ fulfills the Lorentz condition in the mean

$$\langle f | \partial^\alpha B^{(-)}_{\alpha}\gamma^\delta(x) |f\rangle = 0.$$ \hfill (63)

To fulfill the condition above we have to have

$$k^\alpha f_{\alpha}\gamma^\delta(k) = k^0 f_0\gamma^\delta(k) + k^i f_i\gamma^\delta(k)$$

$$= \omega_k \left( f_0\gamma^\delta(k) + \frac{k^i}{\omega_k} f_i\gamma^\delta(k) \right) \frac{i}{\frac{1}{2}} = 0$$ \hfill (64)

for all $k^\alpha$ which implies

$$f_0\gamma^\delta(k) = -\frac{k^i}{\omega_k} f_i\gamma^\delta(k) = -f^0\gamma^\delta(k) \quad \forall k^\alpha.$$ \hfill (65)

Above we have implicitly introduced a coordinate system in Minkowski space singling out a time- or 0-axis and three space- or i-axes to which we will refer to later again. To be specific let one spatial axis be parallel and the two others be perpendicular to $\tilde{k}$.

Hence a general one-particle state $|f\rangle$ $\in \mathcal{F}^{1LG}$ can always be written as

$$|f\rangle = \sqrt{\frac{i}{2}} \int d\tilde{k} f^i\gamma^\delta(k) \left( b^\dagger_i\gamma^\delta(k) + \frac{k^i}{\omega_k} b^\dagger_0\gamma^\delta(k) \right) |0\rangle.$$ \hfill (66)

On such a state the energy operator acts as

$$H |f\rangle = \sqrt{2} \int d\tilde{k} \omega_k f^i\gamma^\delta(k) \left( b^\dagger_i\gamma^\delta(k) + \frac{k^i}{\omega_k} b^\dagger_0\gamma^\delta(k) \right) |0\rangle.$$ \hfill (67)

Finally the norm of a general one-particle state $|f\rangle$ $\in \mathcal{F}^{1LG}$ becomes
\[ \langle f | f \rangle = \int d\tilde{k} f^*_{\alpha} \gamma^\delta(k) f^j_{\gamma^\delta(k)} \left( \delta^j_j - \frac{k^j k^j}{k^2} \right) \] (68)

and the energy expectation value becomes
\[ \langle f | H | f \rangle = \int d\tilde{k} \omega f^*_{\alpha} \gamma^\delta(k) f^j_{\gamma^\delta(k)} \left( \delta^j_j - \frac{k^j k^j}{k^2} \right). \] (69)

Above \( \delta^j_j - \frac{k^j k^j}{k^2} \) is the projection operator onto the transversal part of \( f^j_{\gamma^\delta(k)} \). As a result both the norm and the energy expectation value of a general state \( |f\rangle \in \mathcal{F}^{LG}_1 \) depend only on the transversal parts of the one-particle state vector – in effect reducing the 24 degrees of freedom we have started with to \( 4 \times 3 = 12 \), however only 6 coming with positive-norm, positive-energy and another 6 still coming with negative-norm, negative-energy.

### 7. One-particle subspace of states \( \mathcal{F}^P_1 \) orthogonal to their dual state with positive semidefinite norm

In this section we take the second step towards the identification of a viable physical Fock space by introducing the one-particle subspace of states \( \mathcal{F}^P_1 \) orthogonal to their dual state with positive semidefinite norm which will turn out to be in essence a condition on the \( \gamma, \delta \)-indices of the polarization tensor \( f_{\alpha\gamma\delta}(k) \).

We start by defining a one-particle state \( |f\rangle \)
\[ |f\rangle \in \mathcal{F}^P_1 \] (70)
to be in the one-particle subspace of states \( \mathcal{F}^P_1 \) orthogonal to their dual state with positive semidefinite norm if the two conditions \((ii - a)\) and \((ii - b)\) hold
\[ (ii - a) : \quad \langle f | f \rangle \geq 0 \]
\[ (ii - b) : \quad \langle \hat{f} | f \rangle = 0. \] (71)

Above we have introduced the state \( |\hat{f}\rangle \) dual to \( |f\rangle \)
\[ |\hat{f}\rangle \equiv \sqrt{\frac{1}{2}} \int d\tilde{k} \varepsilon^{\gamma\delta} \eta^\zeta f^*_{\alpha} \gamma^\delta (k) \hat{b}^\dagger_{\alpha} \eta^\zeta (k) |0\rangle \] (72)
which plays a crucial role in the further development of the theory.

In addition to demanding \((ii - a)\) and \((ii - b)\) to hold for individual states we must require linearity
\[ |f\rangle, |g\rangle \in \mathcal{F}^P_1 \Rightarrow |f\rangle + |g\rangle = |f + g\rangle \in \mathcal{F}^P_1 \] (73)
so as to assure \( \mathcal{F}^P_1 \) is indeed a linear space and prove consistency of the three requirements alltogether.

To do so we note that the two requirements \((ii - a)'\) and \((ii - b)'\)
\[ (ii - a)' : \quad f^*_{\alpha} \gamma^\delta (k) f_{\alpha\gamma\delta}(k) \geq 0, \quad \text{no summation over } \alpha! \]
so that
\[ f^*_{\alpha} \gamma^\delta (k) f_{\alpha\gamma\delta}(k) \geq 0 \]
\[ (ii - b)' : \quad f^*_{\alpha} \gamma^\delta (k) f_{\alpha\gamma\delta}(k) = 0 \] (74)
are sufficient for \((ii - a)\) and \((ii - b)\) to hold.

On the other hand \((ii - a)'\) and \((ii - b)'\) imply for \(\alpha\) fixed the existence of a Lorentz transformation \(\Lambda^\gamma_{\eta}(f, \alpha, k)\) and an antisymmetric tensor \(f_{\alpha'}^{\eta'}(k)\) such that \(f_{\alpha}^{\gamma}(k)\) can be written

\[
\begin{align*}
    f_{\alpha}^{\gamma}(k) &= \Lambda^\gamma_{\eta}(f, \alpha, k) \Lambda^\delta_{\alpha}(f, \alpha, k) f_{\alpha}^{\eta'}(k) \\
    \text{with} \quad f_{\alpha}^{\eta'}(k) &= -f_{\alpha}^{\eta'}(k) = 0. \quad (75)
\end{align*}
\]

We note that both the Lorentz transformation and \(f^P\) are not uniquely defined at this point. To completely fix both \(\Lambda^\eta_{\gamma}(f, \alpha, k)\) and \(f^P\) we first perform a Lorentz transformation on \(f_{\alpha}^{\gamma}(k)\) to an arbitrary \(f_{\alpha}^{\eta'}(k)\) fulfilling \(f_{\alpha}^{\eta'}(k) = -f_{\alpha}^{\eta'}(k) = 0\). We then can still perform SO(3) rotations which respect \(f_{\alpha}^{\eta'}(k) = -f_{\alpha}^{\eta'}(k) = 0\). Let these rotations be parametrized by their rotation vectors \(\alpha\), where the direction of \(\alpha\) defines the rotation axis and its length the rotation angle. To fix \(\Lambda^\eta_{\gamma}(f, \alpha, k)\) and \(f^P\) completely we choose the rotation such that its rotation axis is parallel to \(k\) and the coordinate system of the plane perpendicular to \(k\) coincides with the one defined in the preceding section.

We finally note that at this point \(\Lambda^\eta_{\gamma}(f, \alpha, k)\) may depend on all: \(f\), \(\alpha\), and \(k\).

Let us turn the consideration above on its head and look at the linear space of tensors \(f_{\alpha}^{\eta'}(k)\) antisymmetric in \(\eta, \zeta\) with \(f_{\alpha}^{\eta'}(k) = -f_{\alpha}^{\eta'}(k) = 0\). It is then possible to demonstrate that having \(f_{\alpha}^{\gamma}(k)\) of the form

\[
\begin{align*}
    f_{\alpha}^{\gamma}(k) &= \Lambda_k^\gamma_{\eta}(k) \Lambda^{\delta}_{\alpha}(k) f_{\alpha}^{\eta'}(k) \quad (76)
\end{align*}
\]

with \(\Lambda^\gamma_{\eta}(k)\) depending only on \(k\), but neither on \(f\) nor \(\alpha\), is necessary and sufficient for \((ii - a)'\), \((ii - b)'\) and the linearity condition to hold which implies \((ii - a), (ii - b)\) and equation (73) to hold as well.

As a result a general one-particle state \(|f\rangle \in F^P_T\) is written as

\[
\begin{align*}
    |f\rangle &= \sqrt{\frac{T}{2}} \int d\tilde{k} \Lambda^\gamma_{\eta}(k) \Lambda^{\delta}_{\alpha}(k) f_{\alpha}^{\eta'}(k) \Lambda^\alpha_{\gamma}(k) \Lambda^\alpha_{\delta}(k) b^P_{\alpha} |0\rangle \\
    &= \sqrt{\frac{T}{2}} \int d\tilde{k} f^P_{\alpha} |\alpha\rangle b^\dagger P_{\gamma}(k) |0\rangle \\
    &= \sqrt{\frac{T}{2}} \int d\tilde{k} f^P_{\alpha} |\alpha\rangle b^\dagger P_{\gamma}(k) |0\rangle = |f^P\rangle
\end{align*}
\]

with \(b^P_{\alpha} |\alpha\rangle\), \(b^\dagger P_{\gamma}(k)\) fulfilling the same commutation relations equations (44) and (45) as \(b^\dagger P_{\gamma}(k) = \Lambda^\gamma_{\eta}(k) \Lambda^{\delta}_{\alpha}(k) b^P_{\alpha} |\alpha\rangle\) and \(b^\dagger P_{\gamma}(k)\) do due to the covariance of the commutation relations.

Finally the energy operator acts on a one-particle state \(|f\rangle \in F^P_T\) as

\[
H |f\rangle = \sqrt{2} \int d\tilde{k} \omega f^P_{\alpha} |\alpha\rangle b_{\alpha}^P |\alpha\rangle |0\rangle = H |f^P\rangle.
\]

(78)

The norm of such a state becomes

\[
\langle f | f \rangle = \int d\tilde{k} f^P_{\alpha}^\dagger |\alpha\rangle f^P_{\alpha} |\alpha\rangle = \langle f^P | f^P \rangle \geq 0
\]

(79)

and its energy expectation value

\[
\langle f | H | f \rangle = \int d\tilde{k} \omega f^P_{\alpha}^\dagger |\alpha\rangle f^P_{\alpha} |\alpha\rangle = \langle f^P | H | f^P \rangle \geq 0.
\]

(80)
As a result both the norm and the energy expectation value of a general state \(|f\rangle \in \mathcal{F}_1^{\text{PLG}}\) depend only on the \(ij\)-components of the one-particle state vector—in effect reducing the 24 degrees of freedom we have started with to \(4 \times 3 = 12\). Note that both the norm and the energy expectation value are positive semidefinite due to the equation (71) defining \(\mathcal{F}_1^{\text{PLG}}\).

8. One-particle subspace of states \(\mathcal{F}_1^{\text{PLG}} \& P\) with positive semidefinite norm and energy expectation value

In this section we take the third and crucial step towards the identification of a viable physical Fock space by introducing the one-particle subspace of states \(\mathcal{F}_1^{\text{PLG}} \& P\) with positive semidefinite norm and energy expectation value which combines the conditions on the \(\alpha\)-index and the \(\gamma, \delta\)-indices of the polarization tensor \(f^{\gamma\delta}_{\alpha}(k)\) formulated in the two preceding sections.

We start by defining a one-particle state \(|f\rangle\) to be in the one-particle subspace of states \(\mathcal{F}_1^{\text{PLG}} \& P\) with positive semi-definite norm and energy expectation value if the three conditions (i), (ii - a) and (ii - b) plus the linearity condition hold

\[
\begin{align*}
(i): \quad & \partial^\alpha B_\alpha\gamma\delta (x) \langle f | f \rangle = 0 \\
\text{or} \quad & k^\alpha f^{\gamma\delta}_{\alpha}(k) = 0 \\
(ii - a): \quad & \langle f | f \rangle \geq 0 \\
(ii - b): \quad & \langle \hat{f} | f \rangle = 0.
\end{align*}
\]

(82)

A general one-particle state \(|f\rangle \in \mathcal{F}_1^{\text{PLG}} \& P\) can always be written as

\[
|f\rangle = \sqrt{\frac{1}{2}} \int \tilde{d}k \ f^{P^j}_{\delta}(k) \left( b^\dagger_j (k) + \frac{k^j}{\omega_k} b^\dagger_0 \delta_j (k) \right) |0\rangle = |f^P\rangle.
\]

(83)

On such a state the energy operator acts as

\[
H |f\rangle = \sqrt{2} \int \tilde{d}k \ \omega_k f^{P^j}_{\delta}(k) \left( b^\dagger_j (k) + \frac{k^j}{\omega_k} b^\dagger_0 \delta_j (k) \right) |0\rangle = H |f^P\rangle.
\]

(84)

Finally the norm of a general one-particle state \(|f\rangle \in \mathcal{F}_1^{\text{PLG}} \& P\) becomes

\[
\langle f | f \rangle = \int \tilde{d}k \ f^{P^j}_{\delta}(k)f^{P_j}_{\delta}(k) \left( \delta^j_j - \frac{k^j}{k^2} \right) = \langle f^P | f^P \rangle \geq 0
\]

(85)

and the energy expectation value becomes

\[
\langle f | H | f \rangle = \int \tilde{d}k \ \omega_k f^{P^j}_{\delta}(k)f^{P_j}_{\delta}(k) \left( \delta^j_j - \frac{k^j}{k^2} \right) = \langle f^P | H | f^P \rangle \geq 0.
\]

(86)

The positive-semi-definiteness becomes obvious if we write

\[
f^{P^j}_{\delta}(k)f^{P_j}_{\delta}(k) = \left| f^{P^j}_{\delta}(k) \right| \left| f^{P_j}_{\delta}(k) \right| \cos \theta(k),
\]

(87)

where \(\theta(k)\) is the angle between \(f^{P^j}_{\delta}(k)\) and \(k_j\). We can now re-write
\[ (f | f) = \int d\tilde{k} \left| P^{* kl}(k) \right| P^{l k}(k) \left| 1 - \cos^2 \vartheta(k) \right| \geq 0 \]  

as well as
\[ (f | H | f) = \int d\tilde{k} \omega_k \left| P^{* kl}(k) \right| P^{l k}(k) \left| \sin^2 \vartheta(k) \right| = \langle f|H|f \rangle \geq 0. \]  

As a result both the norm and the energy expectation value of a general state \(| f \rangle \in \mathcal{F}^{1}_{LG \& P}
 depend only on the transversal parts of the one-particle state vector—in effect reducing the 24 degrees of freedom we have started with to \(2 \times 3 = 6\), however only 3 coming with positive-norm, positive-energy and another 3 still coming with negative-norm, negative-energy. Note that the 2 in \(2 \times 3 = 6\) comes from the transversality condition as discussed in section 6.

9. Physical one-particle space of states \(\mathcal{F}^{1}_{\text{Phys}}\) with positive norm and energy expectation value

In this section we take the fourth and final step towards the identification of a physical one-particle space of states \(\mathcal{F}^{1}_{\text{Phys}}\) with positive norm and energy expectation value by identifying all states \(| f^P \rangle \in \mathcal{F}^{1}_{LG \& P}\) whose difference has norm zero.

As noted in the preceding section a general one-particle state \(| f^P \rangle \in \mathcal{F}^{1}_{LG \& P}\) can be written as
\[ | f^P \rangle = \sqrt{\frac{1}{2}} \int d\tilde{k} f^{P,i \, jk}(k) \left( b_{i \, jk}(k) + \frac{k^i}{\omega_k} b^i_{0 \, jk}(k) \right) | 0 \rangle \]

with norm
\[ \langle f^P | f^P \rangle = \int d\tilde{k} f^{P, \ast \, kl}(k) f^{P, l k}(k) \left( \delta^i_j - \frac{k^i k^j}{k^2} \right) \geq 0. \]

From equation (91) it becomes obvious that longitudinal states with \(f^{P,i \, jk}(k) \sim k^i h_{j k}(k)\) are non-zero states with norm zero and energy expectation value zero—so in some sense equivalent to the zero state, but non-zero.

Separating \(| f^P \rangle = | f^P_L \rangle + | f^P_T \rangle\) into its transversal and longitudinal parts the longitudinal part
\[ | f^P_L \rangle = \sqrt{\frac{1}{2}} \int d\tilde{k} \frac{k^i k^j}{k^2} f^{P, i \, k l}(k) \left( b^i_{\, jk}(k) + \frac{k^i}{\omega_k} b^i_{\, jk}(k) \right) | 0 \rangle \]

is an example of such a non-zero state with norm zero, \(\langle f^P_L | f^P_L \rangle = 0\), and energy expectation value zero, \(\langle f^P_L | H | f^P_T \rangle = 0\).

To deal with these non-zero states we next define equivalence classes of states with two states being equivalent
\[ | f^P \rangle \sim | g^P \rangle \quad \text{for} \quad | f^P \rangle, | g^P \rangle \in \mathcal{F}^{1}_{LG \& P} \]
\[ \text{if} \quad \langle f^P - g^P | f^P - g^P \rangle = 0 \]  

(93)
if their difference has norm zero [4].

The difference of two general states is the state

$$| f^P - g^P \rangle = \sqrt{\frac{1}{2}} \int \! d\tilde{k} \; (f^{P \dagger}_{i, k} (\tilde{k}) - g^{P \dagger}_{i, k} (\tilde{k})) \left( b^{\dagger}_{i, k} (\tilde{k}) + \frac{k^j}{\omega_k} b^{\dagger}_{0, j} (\tilde{k}) \right) | 0 \rangle.$$  \hfill (94)

Demanding its norm to vanish

$$\langle f^P - g^P | f^P - g^P \rangle = \int \! d\tilde{k} \; (f^{P \dagger}_{i, k} (\tilde{k}) - g^{P \dagger}_{i, k} (\tilde{k})) \cdot (f^{P}_{i, k} (\tilde{k}) - g^{P}_{i, k} (\tilde{k})) \left( \delta^j - \frac{k^j k_i}{E^2} \right) \geq 0$$ \hfill (95)

we find that

$$f^{P \dagger}_{i, k} (\tilde{k}) = g^{P \dagger}_{i, k} (\tilde{k}),$$ \hfill (96)

or that the transversal parts \( | f^P_T \rangle = | g^P_T \rangle \) of the respective polarization tensors have to be equal.

Put it differently the longitudinal parts of states are ‘invisible’ or equivalent to a zero state

$$| f^P_L \rangle = \sqrt{\frac{1}{2}} \int \! d\tilde{k} \; k^j k_i f^{P}_{i, k} (\tilde{k}) \left( b^{\dagger}_{i, k} (\tilde{k}) + \frac{k^j}{\omega_k} b^{\dagger}_{0, j} (\tilde{k}) \right) | 0 \rangle \sim 0$$ \hfill (97)

whilst a one-particle state is equivalent to its transversal part

$$| f^P \rangle \sim | f^P_T \rangle.$$ \hfill (98)

Finally we are in a position to define a physical one-particle state as the following equivalent class of states

$$| f^{\text{Phys}} \rangle \equiv \left( | f^P \rangle \right) = \left\{ | g^P \rangle \in F^1_{\text{L & P}} \; | g^P \rangle \sim | f^P \rangle \right\}.$$ \hfill (99)

There is always one distinguished element in each of the equivalence classes, namely the transversal part of a general state

$$| f^{\text{Phys}}_T \rangle = \sqrt{\frac{1}{2}} \int \! d\tilde{k} \; f^{P \dagger}_{i, k} (\tilde{k}) \left( b^{\dagger}_{i, k} (\tilde{k}) + \frac{k^j}{\omega_k} b^{\dagger}_{0, j} (\tilde{k}) \right) | 0 \rangle.$$ \hfill (100)

with

$$f^{P \dagger}_{i, k} (\tilde{k}) = \left( \delta^j - \frac{k^j k_i}{E^2} \right) f^{P}_{i, k} (\tilde{k}).$$ \hfill (101)

On a general physical one-particle state the energy operator acts as

$$H | f^{\text{Phys}} \rangle = \sqrt{2} \int \! d\tilde{k} \; \omega_k f^{P \dagger}_{i, k} (\tilde{k}) \left( b^{\dagger}_{i, k} (\tilde{k}) + \frac{k^j}{\omega_k} b^{\dagger}_{0, j} (\tilde{k}) \right) | 0 \rangle,$$ \hfill (102)

where the polarization tensor \( k f^{P \dagger}_{i, k} (\tilde{k}) = 0 \) is transversal.

Finally the norm of a physical one-particle state \( | f^{\text{Phys}} \rangle \in F^1_{\text{Phys}} \) becomes

$$\langle f^{\text{Phys}} | f^{\text{Phys}} \rangle = \int \! d\tilde{k} \; f^{P \dagger \star}_{i, k} (\tilde{k}) f^{P}_{i, k} (\tilde{k}) \geq 0$$ \hfill (103)
where \( k f_P^{\mu j}(k) = 0 \), and the energy expectation value becomes

\[
\langle f | H | f \rangle = \int \tilde{d}k \omega_k f_P^{\mu j}(k) f_P^{\mu j}(k) \geq 0,
\]

where again \( k f_P^{\mu j}(k) = 0 \).

10. Lorentz Covariance of the quantization approach on \( \mathcal{F}_1^\text{Phys} \)

In this section we demonstrate the Lorentz covariance of our approach to canonically quantizing the Lorentz gauge fields on the physical one-particle space of states \( \mathcal{F}_1^\text{Phys} \) with positive norm and energy expectation value.

Before starting with the demonstration of the covariance we note that our approach allows to consistently represent the creation and annihilation operator algebra given by equations (44) and (45) on the physical one-particle space of states \( \mathcal{F}_1^\text{Phys} \) with positive norm and energy expectation value and—as we will show in the next section—on the physical Fock space of states \( \mathcal{F}^\text{Phys} \). This is an extension of the Gupta–Bleuler approach to the quantization of free gauge fields for compact gauge groups, where positivity of the norm and energy expectation value of states is assured by the positive definiteness of the Cartan metric on the gauge algebra, and is in itself an interesting result.

Let us turn now to the demonstration of the Lorentz covariance of our approach to canonically quantize the Lorentz gauge fields. We will split this demonstration into two steps.

The first step starts with a physicist who works in the primed reference frame to write down a general physical one-particle state

\[
| f^\text{Phys} \rangle = \sqrt{\frac{1}{2}} \int \tilde{d}k \b' f_P^{\mu j}(k') \left( b'^{\dagger \mu j}(k') + \frac{k'^i}{\omega_{k'}} b'^{\dagger 0 j}(k') \right) | 0 \rangle.
\]

Writing down the state vector above is equivalent to writing down a general one-particle state

\[
| f^\text{Phys} \rangle = \sqrt{\frac{1}{2}} \int \tilde{d}k \b' f_P^\gamma j^\gamma_{\alpha}(k') b'^{\dagger \alpha j}(k') | 0 \rangle
\]

subject to the conditions

\[
(i) ': \quad \partial'^\alpha B^{(-) \alpha \gamma j}(x') | f' \rangle = 0
\]

or

\[
k'^\alpha f_P^\gamma j^\gamma_{\alpha}(k') = 0
\]

\[
(ii - a) ': \quad \langle f^\text{Phys} | f^\text{Phys} \rangle \geq 0
\]

\[
(ii - b) ': \quad \langle f^\text{Phys} | f^\text{Phys} \rangle = 0
\]

in the primed frame. Hence we have \( f_P^\gamma j^\gamma_{\alpha}(k') \) obeying

\[
f_0^\gamma j^\delta(k') = -\frac{k'^i}{\omega_{k'}} f_P^\gamma j^\delta(k')
\]

\[
f_a^\gamma j^\delta(k') = -f_a^\gamma j^\delta(k') = 0.
\]

Note that the state equation (105) does contain a longitudinal part in general.
Next we look at another physicist who works in the unprimed reference frame related to the primed frame by a Lorentz transformation \( \Lambda \). This Lorentz transformation relates the unprimed to the primed quantities as

\[
\begin{align*}
k'^{\alpha} &= \Lambda^{\alpha}_{\beta} k^\beta \\
f^{P}_{\beta \eta \zeta}(\mathbf{k}') &= \Lambda_{\beta}^{\eta} \Lambda_{\gamma}^{\eta} \Lambda_{\delta}^{\eta} f_{\alpha}^{\gamma\delta} (\mathbf{k}) \\
b^{\dagger \beta \eta \zeta}(\mathbf{k}') &= \Lambda_{\beta}^{\alpha} \Lambda_{\eta}^{\gamma} \Lambda_{\zeta}^{\delta} b^{\dagger \alpha \beta \gamma} (\mathbf{k})
\end{align*}
\]

and we find

\[
| f^{\text{phys}} \rangle = \sqrt{\frac{1}{2}} \int d\mathbf{k} f^{\gamma \delta} (\mathbf{k}) b^{\dagger \alpha \beta \gamma} (\mathbf{k}) | 0 \rangle = | f \rangle
\]

with the unprimed \( b^{\dagger}, b \) obeying the same creation and annihilation operator algebra given by equations (44) and (45) as do the primed creation and annihilation operators. In addition we have

\[
\begin{align*}
(i) : & \quad k^{\alpha} f_{\alpha}^{\gamma \delta} (\mathbf{k}) = 0 \\
(ii - a) : & \quad \langle f | f \rangle = \langle f^{\text{phys}} | f^{\text{phys}} \rangle \geq 0 \\
(ii - b) : & \quad \langle \hat{f} | f \rangle = \langle \hat{f}^{\text{phys}} | f^{\text{phys}} \rangle = 0.
\end{align*}
\]

We note that in general the \( f^{P}_{\alpha \gamma \delta}(\mathbf{k}) \) are not subject to the unprimed equation (108)

\[
f_{0}^{\gamma \delta}(\mathbf{k}) = - \frac{k^{i}}{\omega k} f_{i}^{\gamma \delta}(\mathbf{k})
\]

but \( f_{\alpha}^{0 \beta}(\mathbf{k}) \neq 0 \) (112) potentially destroying the covariance of our approach.

So let us turn to the second step. We have demonstrated in section 7 that equation (111) ensure the existence of both a unique Lorentz transformation \( \Lambda_{\gamma} (k', k) \) and an \( f^{P} \) such that

\[
\begin{align*}
f_{\alpha}^{\gamma \delta}(\mathbf{k}) &= \Lambda^{\rho}_{\gamma} (\mathbf{k}') \Lambda^{\rho}_{\delta} (\mathbf{k}') \Lambda_{\alpha \rho}(\mathbf{k}) f^{P}_{\alpha \gamma \delta}(\mathbf{k}) \\
b^{\dagger \alpha \beta \gamma}(\mathbf{k}) &= \Lambda_{\alpha \eta}(\mathbf{k}) \Lambda_{\beta \zeta}(\mathbf{k}) \Lambda_{\gamma \delta}(\mathbf{k}) b^{\dagger \alpha \beta \gamma}(\mathbf{k})
\end{align*}
\]

so that \( f^{P}_{\alpha \gamma \delta}(\mathbf{k}) \) is subject to

\[
\begin{align*}
f^{P}_{0 \gamma \delta}(\mathbf{k}) &= - \frac{k^{i}}{\omega k} f^{P}_{i \gamma \delta}(\mathbf{k}) \\
f^{P}_{\alpha 0 \beta}(\mathbf{k}) &= - f^{P}_{\alpha 0 \beta}(\mathbf{k}) = 0.
\end{align*}
\]

Obviously we still have

\[
\begin{align*}
(i) : & \quad k^{\alpha} f^{P}_{\alpha \gamma \delta}(\mathbf{k}) = 0 \\
(ii - a) : & \quad \langle f^{\text{phys}} | f^{\text{phys}} \rangle = \langle f | f \rangle \geq 0 \\
(ii - b) : & \quad \langle \hat{f}^{\text{phys}} | f^{\text{phys}} \rangle = \langle \hat{f} | f \rangle = 0
\end{align*}
\]

for the resulting state

\[
| f \rangle = \sqrt{\frac{1}{2}} \int d\mathbf{k} f^{P}_{\alpha \gamma \delta}(\mathbf{k}) b^{\dagger \alpha \beta \gamma}(\mathbf{k}) | 0 \rangle = | f^{\text{phys}} \rangle
\]
with $b^{\dagger i}$, $b^i$ obeying the same creation and annihilation operator algebra given by equations (44) and (45) as do the creation and annihilation operators $b^{\dagger 1}$, $b^1$.

Hence—leaving aside the superscript $P$ in the creation and annihilation operators—we have

$$|f^{\text{Phys}}\rangle = \sqrt{\frac{1}{2}} \int \frac{d\tilde{k}}{f_{\text{Phys}}^{0 \gamma \delta}(\tilde{k})} \left( b^{\dagger i} \tilde{k}^{\gamma \delta}(\tilde{k}) + \frac{k^i}{\omega_k^t} b^{\dagger 0} \tilde{k}^{\gamma \delta}(\tilde{k}) \right) |0\rangle = |f^{\text{Phys}}\rangle$$

which is what we wanted to demonstrate.

In addition it is easy to show that

$$P^\alpha|f^{\text{Phys}}\rangle = \Lambda^\alpha_{\beta} P^\beta|f^{\text{Phys}}\rangle$$

and

$$\langle f^{\text{Phys}}|f^{\text{Phys}}\rangle = \langle f^{\text{Phys}}|f^{\text{Phys}}\rangle \geq 0$$

as well as

$$\langle f^{\text{Phys}}|P^\alpha|f^{\text{Phys}}\rangle = \Lambda^\alpha_{\beta} \langle f^{\text{Phys}}|P^\beta|f^{\text{Phys}}\rangle.$$  \hspace{1cm} (120)

Specifically we have

$$\langle f^{\text{Phys}}|H'|f^{\text{Phys}}\rangle = \Lambda^0_{\beta} \langle f^{\text{Phys}}|P^\beta|f^{\text{Phys}}\rangle \geq 0$$

$$\Rightarrow \langle f^{\text{Phys}}|H|f^{\text{Phys}}\rangle \geq 0.$$  \hspace{1cm} (121)

11. Physical n-particle space of states $F^{\text{Phys}}_n$ and physical Fock space $F^{\text{Phys}}$

In this section we extend the construction of the physical one-particle space of states $F^{\text{Phys}}_1$ with positive norm and energy expectation value to the physical n-particle space of states $F^{\text{Phys}}_n$ with positive norm and energy expectation value and finally to the full physical Fock space $F^{\text{Phys}}$.

To keep notations simple let us first define the one-particle creation operator $b^{\dagger [f^{\text{Phys}}]}$

$$b^{\dagger [f^{\text{Phys}}]} \equiv \sqrt{\frac{1}{2}} \int \frac{d\tilde{k}}{f^{P \gamma \delta}(\tilde{k})} \left( b^{\dagger i} \tilde{k}^{\gamma \delta}(\tilde{k}) + \frac{k^i}{\omega_k^t} b^{\dagger 0} \tilde{k}^{\gamma \delta}(\tilde{k}) \right)$$

the application of which to the vacuum state $|0\rangle$ generates a physical one-particle state $|f^{\text{Phys}}\rangle$—with an analogous expression for the one-particle annihilation operator $b[f^{\text{Phys}}]$. Its application to a general state destroys a physical one-particle state $|f^{\text{Phys}}\rangle$.

Then the one-particle space of states $F^{\text{Phys}}_1$ is given by

$$F^{\text{Phys}}_1 = \left\{ |f^{\text{Phys}}\rangle \mid \langle f^{\text{Phys}}|f^{\text{Phys}}\rangle = b^{\dagger [f^{\text{Phys}}]}|0\rangle \right\}$$

with

$$f^{\alpha \gamma \delta}(\tilde{k}) = -\frac{k^\alpha}{\omega_k^t} f^{P \gamma \delta}(\tilde{k})$$

and

$$f^{0 \gamma \delta}(\tilde{k}) = -f^{\gamma 0 \delta}(\tilde{k}) = 0.$$  \hspace{1cm} (123)

Analogously the physical two-particle states are given by

$$\frac{1}{\sqrt{2}} b^{\dagger [f^{\text{Phys}}]} b^{\dagger [f^{\text{Phys}}]}|0\rangle \sim \text{two-particle state}$$

and the physical n-particle states by
\[ \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} b^{\dagger}_{i} [f_{i}^{\text{Phys}}] \mid 0 \rangle \sim n\text{-particle state.} \quad (125) \]

It is easy to demonstrate the \( n \)-particle states above have positive norm and positive energy expectation value.

Finally the following set of \( n \)-particle states forms a dense set in \( \mathcal{F}^{\text{Phys}} \)

\[ \mathcal{F}_{\text{Phys}}^{\text{Basis}} = \left\{ \frac{1}{\sqrt{n!}} \prod_{i=1}^{n} b^{\dagger}_{i} [f_{i}^{\text{Phys}}] \mid 0 \rangle \; \forall n, \; \forall f_{i}^{\text{Phys}} \right\} \quad (126) \]

and serves as a basis of the physical Fock space with states having positive norm and positive energy expectation values.

12. Conclusions

In this paper we have first derived the non-interacting \( \text{SO}(1,3) \) gauge field theory in the Lorentz gauge by truncating the full Lagrangian and retaining only terms quadratic in the fields. We then have canonically quantized the non-interacting theory and step-by-step established the Lorentz invariant physical Fock space \( \mathcal{F}^{\text{Phys}} \) for non-interacting gauge particles which contains only states with positive norm and energy expectation value.

As an encouraging result we have at this point successfully completed two steps out of our four-step programme aiming to establish a perturbatively renormalizable quantum theory of gravity—the first step being to establish a theory for classical gravity which is not equal, but equivalent to GR as established in [2], and which in return allows for perturbatively renormalizable actions for the dynamical field, the second to show that a proper canonical quantization of the non-interacting gauge fields allows for the definition of positive-norm, positive-energy states and a corresponding Lorentz invariant physical Fock space for these fields.

Obviously much remains to be done—first and foremost the third step of our programme which is to prove the perturbative renormalizability of the full quantum theory [3] and the final one which is to establish the unitarity of the \( S \)-matrix on the physical Fock space constructed in this paper. Then there is the question about the \( \beta \)-functions for both the pure gravitational theory and the gravitational theory coupled to all other fields in the standard model of particle physics.

In addition a deeper understanding is needed of the Lorentz invariant mirror Fock space of negative-norm, negative-energy states which can be constructed in exactly the same way as its positive-norm, positive-energy Fock space cousin, especially around the question what protects against transitions from one to the other if the interactions are activated.

Finally as already mentioned in [2] the explanation of phenomena such as the accelerated expansion of the universe or the galaxy rotation curves not compatible with the observed matter distribution might after all not be linked to dark energy or dark matter, but rather to a refined theory of gravitation at both the classical and the quantum level—the free-field limit of which we have established in this paper. Further progress is possible.

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