A Continuous-Time Multi-Agent Systems Based Algorithm for Constrained Distributed Optimization

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To cite this article:
Ping Liu, Huaqing Li, Liping Feng. A Continuous-Time Multi-Agent Systems Based Algorithm for Constrained Distributed Optimization. Applied and Computational Mathematics. Vol. 5, No. 3, 2016, pp. 114-120. doi: 10.11648/j.acm.20160503.14

Received: May 25, 2016; Accepted: June 7, 2016; Published: June 18, 2016

Abstract: This paper considers a second-order multi-agent system for solving the non-smooth convex optimization problem, where the global objective function is a sum of local convex objective functions within different bound constraints over undirected graphs. A novel distributed continuous-time optimization algorithm is designed, where each agent only has an access to its own objective function and bound constraint. All the agents cooperatively minimize the global objective function under some mild conditions. In virtue of the KKT condition and the Lagrange multiplier method, the convergence of the resultant dynamical system is ensured by involving the Lyapunov stability theory and the hybrid LaSalle invariance principle of differential inclusion. A numerical example is conducted to verify the theoretical results.

Keywords: Distributed Optimization, Multi-Agent Network, Lyapunov Method, Bound Constraint, Continuous-Time

1. Introduction

The distributed optimization of a sum of local convex functions has been widely investigated in a variety of scenarios in recent years. Examples include multi-agents system, resource allocation in communication networks and localization in sensor networks [1-18], to just name a few. Numerous distributed optimization algorithms are designed to be in a discrete-time fashion to search the optimal solutions of the optimization problem in [3, 5, 8], while continuous-time strategies due to its relatively complete theoretical framework have been widely applied to the distributed optimization problems in [10-12].

Distributed algorithms are characterized by high reliability, scalability and reduced communication capabilities, which attract many researchers to intensively study the distributed optimization algorithms (see e.g. [13-24]). Nedić and Ozdaglar [25] was the first to systematically put forward the distributed optimization problems. A projection-based distributed algorithm was developed in [7], and the further investigations with respect to set constrained optimization were show in [26-27]. What is worth mentioning is that Bianchi and Jakubowicz [26] presented a distributed constraint non-convex optimization algorithm which consists of two steps: a local stochastic gradient descent at each agent and a gossip step that drives the network of agents to a consensus. Different from the above, a distributed optimization problem subject to the (in-)equality constraint or set constraint was investigated in [28]. The authors proposed two distributed subgradient algorithms for multi-agent optimization problems, where the goal of agents is to minimize a sum of local objective functions. Motived by [28], the primal-dual subgradient algorithm was studied in Yuan et al. [29] and Zhu et al. [24] for multi-agent optimization problems with set constraints. Furthermore, in order to solve an un-constrained optimization problem, where the objective function is formed by a sum of convex functions available to individual agent, a second-order distributed dynamic was given in [10], while a similar second-order continuous-time distributed algorithm was proposed to solve the convex
The function \( f : D \rightarrow \mathbb{R} \) is called Lipschitz near any point \( x \in D \), where \( L \) represents the Lipschitz constant. If \( f \) is Lipschitz near any point \( x \in D \), then \( f \) is also said to be locally Lipschitz in \( D \).

**Definition 2.2:** Assume that \( f : D \rightarrow \mathbb{R} \) is Lipschitz near \( x \). The generalized directional derivative of \( f \) at \( x \) in the direction \( v \in X \) is given by:

\[
f'(x; v) = \limsup_{y \to x, t \to 0} \frac{f(v + tv) - f(y)}{t}
\]

**Definition 2.3:** The generalized gradient of \( f \) is defined as:

\[
\partial f(x) = \{ \zeta \in X' \mid f(x + \zeta) = f(x), \forall \zeta \in X \}
\]

**Lemma 2.1:** If \( f \) is Lipschitz near any point \( x \in D \), then

\[
\partial f(x) = \text{co} \{ \lim_{\alpha \to 0} \nabla f(x) \mid x, x + \alpha \zeta \in \mathcal{O} \}
\]

where \( \mathcal{O} \) is the convex closed hull, and \( \mathcal{O} \) is the null measure set, which is composed by the undefined points of the generalized gradient of \( V \).

**C. Stability of Differential Inclusion**

For an autonomous differential inclusion system:

\[
\dot{x} \in F(x)
\]

where \( x \in \mathbb{R}^n, F : \mathbb{R}^n \rightarrow P(\mathbb{R}^n) \) is an upper semi-continuous set-valued mapping with compact convex values and 0 is a balance point of (1). That is to say, \( 0 \in F(0) \).

**Definition 2.4:** Let \( x(t) \) be a solution of (1). If there is a sequence \( \{t_i\} \) satisfying:

\[
x(t_i) \rightarrow q \in \mathbb{R}^n, t_i \rightarrow +\infty
\]

then \( q \) is the \( \omega \)-limit point of the solution \( x(t) \) of (1). All the \( \omega \)-limit-points make up the limit-set, which is donated as \( \Omega(x) \).

**Definition 2.5:** For any point \( x_0 \) in \( \Omega \), if there exists a maximal solution of the system in \( \Omega \), then \( \Omega \) is called the weakly invariant set of the system (1).
Theorem 2.2: (Lasalle invariance principle of differential inclusion) Assuming \( \dot{V}: \mathbb{R}^n \rightarrow \mathbb{R} \) is a positive definite and locally Lipschitz regular function for almost all \( t \), it satisfies
\[
\frac{d}{dt}V(x) \leq 0
\]
If there exists a constant \( \ell > 0 \) such that \( L_t = \{ x \in \mathbb{R}^n \mid V(x) \leq \ell \} \) bounded, then for any solution \( x(t) \) passing through \( x_0 \in L_t \), we have that \( \text{dist}(x(t), M) \rightarrow 0, t \rightarrow +\infty \), where \( M \) is the largest weakly invariant subset of \( Z_\ell \). Here, \( Z_\ell \) is the closure of \( Z_\ell = \{ z \in \mathbb{R}^n \mid \bar{z} \in \mathbb{R}^m \mid 0 \in \mathcal{F}(\bar{z}, \bar{x}) \} \).

Remark 2.1: If all the conditions of Theorem 2.2 hold, and \( Z_\ell = \{ 0 \} \), then the trivial solution of the autonomous differential inclusion systems (1) is asymptotically stable.

3. Problem Formulation and Optimization Algorithm

A. Problem Formulation

Consider a network of \( n \) agents that interact with each other over a connected graph \( G \). Each agent has a local objective function \( f_i \) and a local bound constraint \( \Omega_i \) for all \( i \in \{1, \ldots, n\} \). The multi-agent group cooperatively solves the following distributed optimization problem:
\[
\begin{align*}
\text{minimize} & \quad f(x) = \sum_{i=1}^{n} f_i(x) \\
\text{subject to} & \quad x \in \cap_{i=1}^{n} \Omega_i
\end{align*}
\]
where \( \Omega_i \subseteq \mathbb{R}^n \) is the closed convex set, and \( \Omega = \cap_{i=1}^{n} \Omega_i \) has nonempty interior points. The local objective function \( f_i: \mathbb{R}^n \rightarrow \mathbb{R} \) is convex and not necessarily smooth. Here, \( x = [x_1, x_2, \ldots, x_n]^T \in \mathbb{R}^n \) is a column vector.

We give some meaningful results, which will be used in this paper:

Assumption 3.1. The optimization problem (2) has at least one finite optimal solution \( x^* \).

Assumption 3.2. At least one of the local objective functions \( f_i(x), i = 1, 2, \ldots, n \) has a positive definite Hessian matrix.

Assumption 3.3. The weighted graph \( G \) is undirected and connected.

Under Assumption 3.3, we have that 0 is a simple eigenvalue of Laplacian \( L \) and \( I_n \) is an eigenvector of \( L \), corresponding to the simple zero eigenvalue. Moreover, \( L_1 = 0, I_n^T L = 0_n \).

B. Optimization algorithm

Denote \( x_i \in \mathbb{R}^n, i = 1, 2, \ldots, n \), as an estimation of agent \( i \), \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^{mn} \) as a matrix with column vector \( x_i \), and \( \bar{x} = \text{vec}(x) = (x_1^T, x_2^T, \ldots, x_n^T)^T \in \mathbb{R}^{mn} \). The objective of optimization problem (2) is to achieve the global minimizer \( x^* = \arg \min f(x) \).

Next, we will provide an equivalent optimization problem of (2).

Lemma 3.1: Let \( L = L_n \otimes I_n \in \mathbb{R}^{mn \times mn} \), where \( L_n \in \mathbb{R}^{mn} \) is a symmetric and positive semi-definite Laplacian matrix of connected graph \( G \). \( \otimes \) is the Kronecker product and \( I_n \) denotes the identity matrix in \( \mathbb{R}^{mn} \). Then, the equivalence problem of (2) is described as:
\[
\begin{align*}
\text{minimize} & \quad \bar{f}(\bar{x}) = \sum_{i=1}^{n} f_i(x_i) \\
\text{subject to} & \quad \bar{L} \bar{x} = 0 \quad \bar{x} \in \Omega
\end{align*}
\]
where \( \Omega = \prod_{i=1}^{n} \Omega_i \) is the Cartesian product.

Proof: According to Assumption 3.3, let \( \bar{x} = 1_n \otimes x \), we have the following:
\[
\bar{L} \bar{x} = (L_n \otimes I_n) \bar{x} = (L_n \otimes I_n) (1_n \otimes x) = (L_n 1_n) \otimes (I_n x) = 0
\]

Denote A, B, and C as matrices with approximate dimensions. In virtue of the properties of Kronecker product vec(ABC) = (C^T \otimes A) vec(B), it is clearly that:
\[
\bar{L} \bar{x} = (L_n \otimes I_n) \bar{x} = (L_n \otimes I_n) \text{vec}(x) = \text{vec}(I_n x L_n)
\]

If \( \bar{L} \bar{x} = 0 \), then \( I_n x L_n = 0 \), which implies \( L_n x^* = 0 \). According to Assumption 3.3, we have \( x^* = \alpha \otimes 1_n \), where \( \alpha_i (i = 1, 2, \ldots, m) \) is any real number and \( \alpha = [\alpha_1, \alpha_2, \ldots, \alpha_m]^T \in \mathbb{R}^m \) is a column vector. Then,
\[
x = \alpha \otimes 1_n^T = \begin{pmatrix} \alpha_1 1_n^T \\ \alpha_2 1_n^T \\ \vdots \\ \alpha_m 1_n^T \end{pmatrix}
\]
where each column vector \( x_j \) of \( x \) satisfies \( x_j = \alpha \). We have \( \bar{x} = \text{vec}(x) = 1_n \otimes \alpha \).

Based on the above analysis, we have that \( \bar{L} \bar{x} = 0 \) if and only if \( \bar{x} = 1_n \otimes x \) for some \( x \in \mathbb{R}^n \). Since \( 1_n \otimes x \in \Omega \), we have \( x \in \cap_{i=1}^{n} \Omega_i \), which implies \( x \in \cap_{i=1}^{n} \Omega_i \) and \( \bar{f}(\bar{x}) = f(x) \). Therefore, the problem (2) is equivalent to the problem (3).

Remark 3.1: In this paper, we mainly consider the case that \( \bar{x} \in \Omega \) is the box constraint \( x^i_{\min} \leq x^i \leq x^i_{\max} \). Let \( x_i = (x^i_1, x^i_2, \ldots, x^i_n)^T \in \mathbb{R}^n \), \( x^i_{\min} = (x^i_{\min}, x^i_{\min}, \ldots, x^i_{\min})^T \in \mathbb{R}^n \), and \( x^i_{\max} = (x^i_{\max}, x^i_{\max}, \ldots, x^i_{\max})^T \in \mathbb{R}^n \) for \( i = 1, \ldots, n \).
Denote $\pi^{\text{min}} = [(x^{\text{min}}_1)^T, (x^{\text{min}}_2)^T, \ldots, (x^{\text{min}}_m)^T]^T \in \mathbb{R}^m$ and $\pi^{\text{max}} = [(x^{\text{max}}_1)^T, (x^{\text{max}}_2)^T, \ldots, (x^{\text{max}}_m)^T]^T \in \mathbb{R}^m$. Then the problem (3) can be rewritten as:

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad Lx = 0 \\
& \quad x_{i}^{\text{min}} \leq x_{i} \leq x_{i}^{\text{max}}, \quad k = 1, \ldots, m
\end{align*}
\]  

where $x_{i}$ stands for the matrix entry in the k-th row and i-th column of $x$.

Then, combining with the optimization problems (3), a new optimization problem based on the method of augmented Lagrangian is proposed as follow:

\[
\begin{align*}
\text{minimize} & \quad f(x) + \frac{1}{2} \lambda^T Lx \\
\text{subject to} & \quad Lx = 0 \\
& \quad x_{i}^{\text{min}} \leq x_{i} \leq x_{i}^{\text{max}}, \quad k = 1, \ldots, m
\end{align*}
\]  

where $\lambda \in \mathbb{R}^{mn\times n}$.

To solve the original optimization problem (2), the dynamics of multi-agent network is designed as:

\[
\begin{align*}
\dot{x}_{i} & = - \frac{\partial f(x)}{\partial x_{i}} - \sum_{j=1}^{n} a_{ij} (x_{j} - x_{i}) - \sum_{k=1}^{m} \theta (\ln [x_{i}^{\text{min}} - x_{i}] + \ln [x_{i}^{\text{max}} - x_{i}] ) + \theta (\frac{1}{x_{i}^{\text{min}} - x_{i}} - \frac{1}{x_{i}^{\text{max}} - x_{i}}) \\
\dot{\lambda}_{k} & = \sum_{i=1}^{n} a_{ij} (x_{j} - x_{i}), \quad i=1, \ldots, n, \quad k = 1, \ldots, m
\end{align*}
\]
Let \( \bar{x} = \theta(\mathbf{1} - x_{i_{\min}} - x_{i_{\max}}) + (\mathbf{1} - x_{i_{\min}} - x_{i_{\max}}) \). Then (11) can be written in a compact form:
\[
\bar{x} \in -\nabla f(x) - L\bar{x} - \bar{\tau} + \bar{\lambda}
\]
\[
\bar{\lambda} = L\bar{x}
\]

4. The Convergence Analysis

In this section, a complete convergence proof of dynamic system (11) [or (12)] is provided in the following theorems.

Lemma 4.1: Under the Assumptions 1-3, the trajectories of dynamic system (11) [or (12)] with any finite initial points are bounded.

Proof: In order to prove the stability of the dynamic system (11) [or (12)], we construct a Lyapunov function as follows:
\[
W(\bar{x}, \bar{\lambda}) = \frac{1}{2} \| \bar{x} - \bar{x}^* \|^2 + \frac{1}{2} \| \bar{\lambda} - \bar{\lambda}^* \|^2
\]

Obviously, \( W(\bar{x}, \bar{\lambda}) \geq 0 \). In view of the chain rule, the time derivative of \( W(\bar{x}, \bar{\lambda}) \) along the trajectories of dynamic system (11) [or (12)] is:
\[
\frac{dW(\bar{x}, \bar{\lambda})}{dt} = \dot{\bar{x}}^T (\bar{x} - \bar{x}^*) + \dot{\bar{\lambda}}^T (\bar{\lambda} - \bar{\lambda}^*)
\]
\[
= -\nabla L(\bar{x}, \bar{\lambda})^T (\bar{x} - \bar{x}^*) + \nabla \bar{L}(\bar{x}, \bar{\lambda})^T (\bar{\lambda} - \bar{\lambda}^*)
\]

Since \( L(\bar{x}, \bar{\lambda}) \) is convex in \( \bar{x} \), and concave in \( \bar{\lambda} \), combining with the properties of convex function, we have:
\[
[\nabla L(\bar{x}, \bar{\lambda})] = L(\bar{x}^*, \bar{\lambda}) - L(\bar{x}, \bar{\lambda})
\]
\[
\frac{dV(\bar{x}, \bar{\lambda})}{dt} = \frac{1}{2} < \dot{\bar{x}}, \bar{x}^* > + \frac{1}{2} < \dot{\bar{\lambda}}, \bar{\lambda}^* >
\]
\[
= \dot{\bar{x}}^T \text{diag}[-\nabla^2 f(x)] \bar{x} - L\bar{x} - \bar{L}\bar{\lambda} + \text{diag}[\nabla z] \bar{\lambda} + \bar{\lambda}^T L\bar{x}
\]
\[
= \dot{\bar{x}}^T \text{diag}[-\nabla^2 f(x)] \bar{x} - \bar{\lambda}^T L\bar{x} + \text{diag}[\nabla z] \bar{\lambda} + \bar{\lambda}^T L\bar{x}
\]
\[
= \dot{\bar{x}}^T \text{diag}[-\nabla^2 f(x)] \bar{x} - \bar{\lambda}^T L\bar{x} + \text{diag}[\nabla z] \bar{\lambda} + \bar{\lambda}^T L\bar{x}
\]
\[
< 0
\]

where \( \text{diag}[\nabla z] = \text{diag}[\frac{\theta}{(x_{i_{\min}} - x_{i_{\max}})^2} - \frac{\theta}{(x_{i_{\min}} - x_{i_{\max}})^2} - \frac{\theta}{(x_{i_{\min}} - x_{i_{\max}})^2} - \frac{\theta}{(x_{i_{\min}} - x_{i_{\max}})^2}] \).

With Assumption 3.2, at least one of the local objective functions \( f^*(x), i = 1, \ldots, n \) of \( f(x) \) has a positive definite Hessian matrix. Then \( \text{diag}[-\nabla^2 f(x)] \) is negative definite, which implies \( \bar{x}^T \text{diag}[-\nabla^2 f(x)] \bar{x} < 0 \). Since \( L \) is a positive semi-definite matrix, then \( \bar{x}^T L\bar{x} \leq 0 \). In addition, due to \( \text{diag}[-\nabla z] < 0 \), the last step of (14) holds.

Next, we shall prove \( \Omega(\bar{x}, \bar{\lambda}) \subset Z_v \cap L_\lambda \). According to the definition of \( L_\lambda \), we have \( \bar{\Omega}(\bar{x}, \bar{\lambda}) \subset L_\lambda \). Noting that \( W(\bar{x}(t), \bar{\lambda}(t)) \) is monotonic, non-increasing and has a lower bound, so \( \lim dW(\bar{x}(t), \bar{\lambda}(t)) = c \geq 0 \). According to the continuity of \( W(\bar{x}, \bar{\lambda}) \), we have \( W(\bar{x}, \bar{\lambda}) = c \). Denoting \( \psi(\bar{x}, \bar{\lambda}, t) \) as a solution of \( \Omega(\bar{x}, \bar{\lambda}) \), which satisfies \( \psi(\bar{x}, \bar{\lambda}, 0) = y, \forall y \in \Omega(\bar{x}, \bar{\lambda}) \), then \( W(\psi(\bar{x}, \bar{\lambda}, t)) = c \) for any \( t \). Thus, it gives
\[
\frac{d}{dt} W(\psi(\bar{x}, \bar{\lambda}, t)) = 0
\]

For almost all \( t, 0 \in \dot{\psi}(\psi(\bar{x}, \bar{\lambda}, t)) \). It can be concluded that there exists \( \{t_i\}, t_i \to 0 \) satisfying \( \psi(\bar{x}, \bar{\lambda}, t_i) \in Z_v \). By the
continuity of $\psi$, it yields $\lim\psi(\bar{x},\bar{t},t_j) = \psi(\bar{x},\bar{t},0) = j \in Z$. Hence, we have $\Omega(\bar{x},\bar{t}) \subset Z$. Noting that $\Omega(\bar{x},\bar{t}) \subset M$ is weakly invariant set, then $\text{dist}(\bar{x},\bar{t}), M) \to 0, t \to \infty$. From the above, according to Remark 2.1, we have that the trivial solution of the system (11) [or (12)] is asymptotically stable and it is also the optimal solution of (2).

5. Simulation

In this section, a simulation examples are presented to verify the theoretical analysis of the proposed second-order algorithm (11) [or (12)].

Example: Consider optimization problem (2) with $f^i(x) = ix_i - 2ix_i + \left|x_i + x_j + x_k\right|$, $x \in \mathbb{R}^3$ and $\Omega = \{x \in \mathbb{R}^3 : -22 \leq x_i \leq i + 10, i = 1, 2, ..., 12\}$, where $f^i(x), (i = 1, ..., 12)$ are non-smooth objective functions.

For the above optimization problem, we first assume that the network topology $G$ is a cyclic connected network, as shown in Fig. 1(a). The connection weight is set to 1 if there exists a path between agent $i$ and $j$, otherwise 0. The trajectories of twelve agents are shown in Fig. 1(b). It can be seen that all the agents converge to the same optimal solution $x^* = (-10, 7.5, 13)^T$ (approximate solution). Next, supposing that the network topology $G$ is fully connected, as shown in Fig. 2(a), and the simulation results are shown in Fig. 2(b). It is clearly that the tighter the network connection, the faster the convergence rate is.

6. Conclusions

In this paper, a novel distributed continuous-time algorithm based on the KKT condition and the Lagrange multiplier method has been proposed for a distributed convex optimization problem. It aims to minimize the sum of the non-smooth local objective functions with local bound constraints over an undirected graph. Furthermore, the convergence analysis of the dynamical system is accomplished by using the Lyapunov stability theory and the hybrid LaSalle invariance principle of differential inclusion. The numerical simulation shows the performance of the proposed algorithm. In the future, our works may turn to the optimization problem with respect to directed topology and equality constraint, meanwhile analysis of its convergence speed.

Acknowledgements

This work described in this paper was supported in part by the Visiting Scholarship of State Key Laboratory of Power Transmission Equipment & System Security and New Technology (Chongqing University) under Grant 2007DA10512716421, in part by the Fundamental Research Funds for the Central Universities under Grant XDJK2016B016, in part by the Natural Science Foundation...
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