Extending the Born rule by phase-guided probability propagation

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Born’s rule is the critical link between theory and experiment in quantum mechanics because it connects the wave function to probability. I propose to put more focus on the phase of the wave function to determine probability. This can be done by using the phase to propagate an initial probability density. As an example, I consider a phase that penetrates deeper into a double-slit structure than the propagated probability density of particles which pass through the slits. The resulting interference pattern resembles the quantum mechanical prediction except for a discrepancy at the centre of the main maximum. An experimental verification of this discrepancy in high resolution should reveal whether the proposed extension of Born’s rule is justified.

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I. INTRODUCTION

Since history shows that every theory was eventually replaced by a more fundamental one, it is of key importance to challenge the foundations of quantum theory to possibly find deviations from what we presently expect. Because the Born rule is one of the fundamental postulates of quantum mechanics, testing possible generalizations of this rule is very much in the focus of recent research. I propose an extension of quantum mechanics which alters the fundamental role of Born’s rule.

The Born rule states that if the quantum mechanical state of a particle is described by the wave function $\psi(x,t)$, then the probability to find that particle at the position $x$ and time $t$ is described by the probability density $|\psi(x,t)|^2$.

$$P(x,t) = |\psi(x,t)|^2.$$ (1)

The recent interest in this postulate of quantum mechanics emerged due to the fact that this formula only supports second order interferences even in multi-slit experiments [2]. Based on this observation, theories involving higher order interferences were developed [3–8] and experiments conducted to check the existence of such terms [9–13]. However, recent experiments [13] provide a stringent constraint on possible higher order interferences which would violate Born’s rule.

I propose a different approach to calculate the probability density which does not include higher order interferences, but allows further modifications of the interference pattern. I do so by reducing the connection between the probability density to the wave function (1) to only a connection of the probability density to the wave function’s phase and an initial probability density. For this purpose, I rewrite the imaginary part of the Schrödinger equation in the form of a propagation equation. This equation can be used to propagate any probability density by using the phase of the quantum mechanical wave function.

More precisely, I extend Born’s rule (1) by a formalism which I use not only to obtain $P(x,t)$, but also a more general probability density $\rho(x,t)$ in three steps:

i) First, I solve the Schrödinger equation to obtain the wave function

$$\psi(x,t) \equiv \sqrt{P(x,t)} e^{iS(x,t)/\hbar}.$$ (2)

ii) From this wave function I extract the phase

$$S(x,t)/\hbar = \arg[\psi(x,t)].$$ (3)

iii) In order to calculate the probability density $\rho(x,t)$, I propagate an initial probability density $\rho_0 \equiv \rho(x,0)$ by using the phase (3) and the propagation equation which I derive in this letter.

If I propagate the initial probability density $P_0 \equiv P(x,0)$, my formalism results in the same density $P(x,t)$ as Born’s rule (1).

My approach is motivated by the finding of Schleich et al. [14], that a strong coupling between the probability density and phase of the wave function is at the heart of quantum theory. This means that, by reducing their coupling in Born’s rule, I directly challenge a very central element of quantum theory.

As an example for the application of my formalism, I describe a double-slit experiment, where the phase penetrates the double-slit structure deeper than the probability density of the particles which pass through the slits. The resulting interference pattern displays an irregularity at the main maximum which differs from the quantum mechanical prediction. An experimental analysis of this irregularity in high resolution should clarify if Born’s rule needs to be extended at this point.

In the next section I derive the propagation equation. In section III I apply this equation in the case of a double-slit experiment and section IV summarizes my results.

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II. PROPAGATION EQUATION

In order to derive the propagation equation, I use the wave function (2) in the Schrödinger equation

\[ i\hbar \frac{\partial}{\partial t} \psi(x, t) = \left( -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V \right) \psi(x, t) \]  

(4)

and separate the imaginary part

\[ \frac{\partial}{\partial t} P(x, t) = -\frac{\partial}{\partial x} [S_x(x, t) P(x, t)]. \]  

(5)

Here, I used the function

\[ S_x(x, t) \equiv \frac{1}{m} \frac{\partial}{\partial x} S(x, t). \]  

(6)

An integration of the imaginary part (5) leads us to

\[ P(x, t) = P_0 - \int_0^t dt' \frac{\partial}{\partial x} [S_x(x, t_1) P_0] \]

\[ + \int_0^t dt_1 \frac{\partial}{\partial x} S_x(x, t_2) \int_0^{t_2} dt_1 \frac{\partial}{\partial x} [S_x(x, t_1) P_0] - \ldots \]  

(8)

and finally to

\[ P(x, t) = \hat{T} \left[ \exp \left( -\int_0^t dt' \frac{\partial}{\partial x} S_x(x, t') \right) \right] P_0, \]  

(9)

where \( \hat{T} \) is the time-ordering operator, which is defined by

\[ \hat{T} \left[ \hat{A}(t) \hat{B}(t') \right] = \begin{cases} \hat{A}(t) \hat{B}(t') & \text{for } t \geq t' \\ \hat{B}(t') \hat{A}(t) & \text{for } t' \leq t \end{cases}. \]  

(10)

Equation (9) describes the propagation of an initial probability density \( P_0 \) to the final density \( P(x, t) \) by using the phase \( S(x, t)/\hbar \). However, it can not only be used to propagate the initial density \( P_0 \), but also to propagate a different density \( \rho_0 \neq P_0 \) to a final density

\[ \rho(x, t) = \hat{T} \left[ \exp \left( -\int_0^t dt' \frac{\partial}{\partial x} S_x(x, t') \right) \right] \rho_0. \]  

(11)

In contrast to \( P(x, t) \), the density \( \rho(x, t) \) might not agree with the quantum mechanical predictions. In this way, eq. (11) can be used to extend Born’s rule (1) to calculate the probability to find a particle at the position \( x \) and time \( t \).

The phase which is needed in order to propagate the probability density \( \rho_0 \) is obtained by first solving the Schrödinger equation (4) and then extracting the phase \( S(x, t) \) from the wave function \( \psi(x, t) \).

Note that inserting the propagation equation (9) into the real part of the Schrödinger equation leads formally to an equation which may be used to determine the function \( S_x(x, t) \). This function can then be used to propagate \( \rho_0 \) via eq. (11). However, analysing the applicability of this idea in detail goes beyond the scope of this letter and will be the topic of a future publication.

III. DOUBLE-SLIT EXPERIMENT

In the following, I show how the propagation equation (11) can be applied in a double-slit experiment.

Since the goal of this letter is to analyse the consequences of decoupling the probability density from the phase of the quantum mechanical wave function, the question arises: Which parameter in the double-slit experiment has a different influence on the probability density than on the phase?

I decided to analyse a scenario where the width of the slits influences both differently. One motivation for this decision was the picture of particles dissolved in a fluid, where the distribution of particles corresponds to the probability density and the fluid distribution to the spreading of the phase. In this picture, a situation is possible where the fluid penetrates the walls of a double-slit deeper than the particles. In order to analyse if such a picture is applicable in quantum mechanics, I assume in the following that the phase penetrates deeper into the double-slit borders than the probability density.

First, I describe the solution of the Schrödinger equation (4) in the case of a double-slit experiment by the wave function

\[ \psi(x, t, \sigma) = \frac{\psi_0(x, t, \sigma) + \psi_1(x, t, \sigma)}{\sqrt{2}}, \]  

(12)

which describes the possibility of a particle to enter both slits. It consists of the sum of the wave functions \( \psi_0(x, t, \sigma) \) at the upper and \( \psi_1(x, t, \sigma) \) at the lower slit. The width \( \sigma \) describes how much the wave function expands in a single slit.

For simplicity, I represent the wave functions

\[ \psi_{0,1}(x, t, \sigma) = \frac{1}{\sqrt{2\pi}\sqrt{\sigma(1+i\gamma)}} \exp \left[ -\frac{(x \mp X)^2}{4\sigma^2(1+i\gamma)} \right] \]  

(13)

by Gaussian distributions which evolve freely. Here, I defined the parameter \( \gamma \equiv \hbar t/(2m\sigma^2) \) and \( 2X \) is the slit separation.
From the wave function (12), I extract the phase

\[ S(x, t)/\hbar = \arg[\psi(x, t, \sigma_S)]. \]  

(14)

Here, the parameter \( \sigma_S \) describes how much the phase expands in a single slit.

In order to describe a probability density with a smaller expansion \( \sigma_\rho < \sigma_S \) in a single slit, I define the initial density

\[ \rho_0 \equiv |\psi(x, 0, \sigma_\rho)|^2 \]  

(15)

by using the wave function (12) in Born’s rule (1) only at \( t = 0 \). I calculate the probability density \( \rho(x, t) \) after a time \( t > 0 \) numerically by propagating the initial density (15) with the help of the phase (14) and the propagation equation (11).

In fig. 1, I depicted \( \rho(x, t_1) \) at the time \( t_1 = 2 \) ns for parameters similar to the ones of the experiment performed by Jönsson [16]. The slit separation is taken to be \( 2X = 1 \) \( \mu \)m and the slit width \( \sigma_\rho = 100 \) nm. This width does not change in fig. 1, but the length scale \( \sigma_S \), on which the phase (14) penetrates the double-slit changes from \( \sigma_S = \sigma_\rho = 100 \) nm (light grey) to \( \sigma_S = 101.5 \) nm (black).

The case \( \sigma_S = \sigma_\rho \) describes the situation, where my results agree with the quantum mechanical prediction. However, with increasing \( \sigma_S \), higher order maxima change slightly while the centre of the main maximum changes drastically. It seems as if the main maximum gets split in two parts with increasing \( \sigma_S \).

To my knowledge, the currently published data on particle double-slit experiments are insufficient to distinguish between curves similar to the ones I show in fig. 1. This is because many experiments [16–19] mainly focused on positions of maxima and minima of the interference patterns. Even data of double-slit experiments [20–24] which were fitted to theoretical predictions are not sufficient to exclude effects of a phase that penetrates deeper into the double-slit material than the particles. Therefore, new high-resolution particle double-slit experiments are required to investigate the main maximum of the interference pattern in more detail to decide whether Born’s rule needs to be extended at this point.

IV. SUMMARY AND CONCLUSION

In summary, I have derived a propagation equation which allows to use the phase of a quantum mechanical wave function to propagate any initial probability density to a final one. If the initial probability density is obtained by applying the Born rule on the wave function which is used to obtain the phase, my formalism gives the same results as quantum mechanics. However, by choosing a different initial probability, my theory extends Born’s rule and can hereby break the limits of quantum physics.

As an example, I have applied the propagation equation in the case of a double-slit experiment. Here, I used a phase which penetrates deeper into the double-slit struc-
ture than the probability density. This resulted in interference patterns which mostly agree with the quantum mechanical predictions except for a discrepancy at the top of the main maximum. To my knowledge, no particle double-slit experiment was performed up to now with a resolution which is high enough to uncover this discrepancy. In order to resolve this issue, new experiments are required to reveal the possibility of such a process and thereby test whether the proposed extension of Born’s rule is justified.

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