SOME BEURLING-FOURIER ALGEBRAS ON COMPACT GROUPS ARE OPERATOR ALGEBRAS

MAHYA GHANDEHARI, HUN HEE LEE, EBRAHIM SAMEI, AND NICO SPRONK

Abstract. Let $G$ be a compact connected Lie group. The question of when a weighted Fourier algebra on $G$ is completely isomorphic to an operator algebra will be investigated in this paper. We will demonstrate that the dimension of the group plays an important role in the question. More precisely, we will get a positive answer to the question when we consider a polynomial type weight coming from a length function on $G$ with the order of growth strictly bigger than the half of the dimension of the group. The case of $SU(n)$ will be examined, focusing more on the details including negative results. The proof for the positive directions depends on a non-commutative version of Littlewood multiplier theory, which we will develop in this paper, and the negative directions will be taken care of by restricting to a maximal torus.

1. Introduction

Group algebras $L^1(G)$ for locally compact groups $G$ are some of the most fundamental examples of Banach algebras, which are in some sense far away from $C^*$-algebras, or more generally (non-self-adjoint) operator algebras, i.e. closed subalgebras of $B(H)$ for some Hilbert space $H$. For example, $L^1(G)$ is not Arens regular for infinite group $G$ [34] whilst operator algebras are always Arens regular (see [9, Chapter 4] for example). By endowing an appropriate submultiplicative weight function $\omega : G \to [1, \infty)$ the weighted group algebra $L^1(G, \omega)$ could be closer to operator algebras in some cases. Indeed, if $G$ is a discrete countable group, then some of weighted group algebras $\ell^1(G, \omega)$ actually become Arens regular [9, Chapter 8]. Varopoulos proved that even more is true [32]. When $G = \mathbb{Z}$ and $\rho_\alpha$ is the polynomial type weight given by

$$\rho_\alpha(x) = (1 + |x|)^\alpha, \quad x \in \mathbb{Z}, \quad \alpha \geq 0,$$

$\ell^1(\mathbb{Z}, \rho_\alpha)$ is isomorphic to a $Q$-algebra if and only if $\alpha > \frac{1}{2}$. Recall that a $Q$-algebra is a quotient of a uniform algebra, a closed subalgebra of a commutative $C^*$-algebra $C(X)$ for some compact Hausdorff space $X$. Since a quotient algebra of an operator algebra is again an operator algebra (see [4, Proposition 2.3.4] for example), $Q$-algebras are always operator algebras.

The initial motivation of this paper was to consider a non-commutative version of Varopoulos’s result. The correct non-commutative analogue of group algebras are Fourier algebras $A(G)$. The discrete-compact duality suggests us that we might get weighted versions of Fourier algebras on certain compact groups which are isomorphic to operator algebras. Recently, the theory of weighted Fourier algebras have been developed by Spronk/Ludwig/Turowska [27] and Lee/Samei [25] under the name of Beurling-Fourier algebras, which we will use as our model of weighted Fourier algebras.

Key words and phrases. Weighted Fourier algebras, operator algebras, compact connected Lie groups, Littlewood multipliers.

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versions of Fourier algebras. Let $G$ be a compact group, and let $\hat{G}$ be the equivalence classes of irreducible unitary representations of $G$. We call a function $\omega : \hat{G} \to [1, \infty)$ a weight if

$$\omega(\sigma) \leq \omega(\pi)\omega(\pi')$$

for any $\pi, \pi' \in \hat{G}$ and $\sigma \in \hat{G}$ which appears in the irreducible decomposition of $\pi \otimes \pi'$. We define the Beurling-Fourier algebra $A(G, \omega)$ by

$$A(G, \omega) := \{ f \in C(G) : \| f \|_{A(G, \omega)} = \sum_{\pi \in \hat{G}} \omega(\pi) \| \hat{f}(\pi) \|_1 \},$$

where $\hat{f}(\pi)$ is the Fourier coefficient of $f$ at $\pi \in \hat{G}$ and $\| \cdot \|_1$ denotes the trace norm.

Note that the constant weight $\omega \equiv 1$ gives us the usual Fourier algebra. In this paper we will be mainly interested in the following weights. The first one is $\omega_\alpha$, $\alpha > 0$, the dimension weight of order $\alpha$ given by

$$\omega_\alpha(\pi) = d^\alpha_\pi, \quad \pi \in \hat{G}.$$  

If the compact group $G$ is a connected Lie group, then $\hat{G}$ is generated by a finite generating set $S$, so that we can consider the associated length function $\tau_S$. In this case we have the second kind of weight $\omega^\alpha_S$, which we call the polynomial weight of order $\alpha$, given by

$$\omega^\alpha_S(\pi) = (1 + \tau_S(\pi))^\alpha, \quad \pi \in \hat{G}.$$  

Note that the above weights are of polynomial type. One can also have exponential type weights; the main example be as follows. For $0 \leq \alpha \leq 1$, we define the exponential weight of order $\alpha$ as

$$\gamma_\alpha(\pi) = e^{\tau_S(\pi)}^\alpha, \quad \pi \in \hat{G}.$$  

See section 3 for the details of the above definitions.

As is usual in the theory of Fourier algebras, we will work in the category of operator spaces. This allows us to use D. Blecher’s completely isomorphic characterization of operator algebras requiring the algebra multiplication map to be completely bounded on the Haagerup tensor product [3]. Note that there is no such characterization of operator algebras in the category of Banach spaces [7].

We summarize the main results of this paper. It turns out that there is an interesting connection between the dimension of the Lie group and the property of being completely isomorphic to an operator algebra. Proofs of these results will be given in Section 4.

**Theorem 1.1.** Let $G$ be a compact connected Lie group. Then $A(G, \omega^\alpha_S)$ is completely isomorphic to an operator algebra if $\alpha > \frac{d(G)}{2}$ and fails to be completely isomorphic to an operator algebra if $G = SU(n)$ and $\alpha \leq \frac{d(G)}{2}$. Also $A(G, \gamma_\alpha)$ is completely isomorphic to an operator algebra if $0 < \alpha < 1$.

The situation for the dimension weights is more delicate. First we show that, if $G$ is not simple (as a compact, connected Lie group), then one can not get an operator algebra as an isomorphic image of a Beurling-Fourier algebras on $G$ coming from a dimension weight (see Theorem 4.7). Hence we need to restrict to compact, connected simple Lie groups to achieve positive result. However, even though we have developed the general theory, the computations become extremely technical as the dimension of the Lie group grows even for the most classical case of compact simple Lie group, namely $SU(n)$. Nonetheless, we have the following results which give some evidence that one may obtain various classes of Beurling-Fourier algebras coming from dimension weights which are completely isomorphic to operator algebras.
Theorem 1.2. $A(SU(n), \omega_\alpha)$, $2 \leq n \leq 5$ is completely isomorphic to an operator algebra if $\alpha > d(SU(n)) = \frac{n^2 - 1}{2}$ and fails to be completely isomorphic to an operator algebra if $\alpha \leq \frac{1}{2}$ for every $n \geq 2$.

It is natural to ask whether the exponent $\frac{d(G)}{2}$, obtained in the preceding theorems, is optimal. We could demonstrate the optimal exponent of $\frac{d(G)}{2}$ only for the $n$-dimensional torus (Theorem 1.3). On the other hand, the negative results obtained for $SU(n)$ are quite smaller than $\frac{d(G)}{2}$ and we are not aware of any means to improve this gap.

We would like to point out that we can not hope the Beurling-Fourier algebra on $SU(n)$ to be completely isomorphic to a $Q$-algebra since it may not be even completely isomorphic to a $Q$-space, a quotient operator space of a closed subspace of a commutative $C^*$-algebra (see Remark 1.6).

This paper is organized as follows. In section 2, we develop a non-commutative Littlewood multiplier theory, which is a main tool for the proof of positive results. This requires a heavy use of operator spaces, so that we collect the necessary background materials on operator spaces and operator algebras in the beginning of the section. Section 3 starts with a brief introduction of Beurling-Fourier algebras on compact groups and dimension weights. Then the definition of polynomial weights on connected compact Lie groups will follow after some preliminaries of corresponding Lie theory. We will close the section with a more detailed representation theory of $SU(n)$ and restriction results of weights to a maximal torus. In section 4 we present our main results starting with a complete solution of the problem in the case of $n$-dimensional torus, and then we focus on polynomial type weights and dimension weights. We will also prove positive results for exponential type of weights. We will study in details the case of $SU(n)$ as our main example of a compact connected, simple Lie group. In the appendix we present two technical proofs concerning estimates of the dimension weight and the exponential weight.

2. SOME NON-COMMUTATIVE LITTLEWOOD MULTIPLIERS

2.1. Preliminaries on operator spaces and operator algebras. We will assume that the reader is familiar with standard operator space theory including injective, projective and Haggerup tensor products of operator spaces. However, in this section we will recall some operator space theory which is somewhat less standard and will be used frequently later on.

The column and the row Hilbert spaces on a Hilbert space $H$ will be denoted by $H_c$ and $H_r$. Note that they are given by $H_c = B(C, H)$ and $H_r = B(H, C)$.

When $\dim H = n < \infty$, then $H_c$ and $H_r$ are usually denoted by $C_n$ and $R_n$.

For any operator space $E \subseteq B(H)$ and $T \in CB(C_n, E)$ we have the following concrete formula to calculate the completely bounded norm (shortly, cb-norm) of $T$.

$$
\|T\|_{cb} = \left\| \sum_{i=1}^{n} Te_i (Te_i)^* \right\|_{B(H)},
$$

where $\{e_i\}_{i=1}^{n}$ is an orthonormal basis of $C_n$. A similar formula for $T \in CB(R_n, E)$ is also available. For operator spaces $(F_i)_{i \in I}$ we have

$$
CB(E, \oplus_{i \in I} F_i) \cong \bigoplus_{i \in I} CB(E, F_i)
$$

completely isometrically via the following identification.

$$
T \mapsto \oplus_{i \in I} (P_i F_i \circ T),
$$
where $P^F_j : \bigoplus_{i \in I} F_i \to F_j$ is the canonical projection, which is a complete contraction.

The column and the row Hilbert spaces are closely related to the Haggerup tensor product. In this paper we will more concerned about its dual version, namely the extended Haggerup tensor product. The extended Haggerup tensor product of dual operator spaces $E^*$ and $F^*$ will be denoted by

$$E^* \otimes_{eh} F^*$$

which is given by $(E \otimes_h F)^*$ [5]. Then we have the following complete isometry [10, Lemma 13.3.1]

$$E^* \otimes_{eh} F^* \cong \Gamma^c_2(F, E^*)$$

via the map

$$A \otimes B \mapsto u, \text{ where } u(X) = \langle X, B \rangle A.$$

Here $\Gamma^c_2(E, F)$ is the space of mappings factorized through column Hilbert space [10, Section 13.3]. More precisely, $u \in \Gamma^c_2(E, F)$ if and only if there is a Hilbert space $H$ and $A \in CB(E, H_c), B \in CB(H_c, F)$ such that $u = B \circ A$. $\Gamma^c_2(E, F)$ is equipped with the norm $\gamma_2(u) = \inf \|A\|_{cb} \|B\|_{cb}$, where the infimum runs over all possible such factorization. We have a natural operator space structure on $\Gamma^c_2(E, F)$ given by the matricial norms

$$\gamma_{2, n}(u_{ij}) = \inf \|A\|_{cb} \|B\|_{cb}, (u_{ij}) \in M_n(\Gamma^c_2(E, F))$$

where the infimum runs over all possible

$$A \in CB(E, M_{1, n}(H_c)) \text{ and } B \in CB(H_c, M_{n, 1}(F))$$

with

$$u_{ij} = B_{1, n} \circ A.$$

Recall that for any linear map $T : E \to F$ between operator spaces we denote the amplified map $id_{M_{m, n}} \otimes T : M_{m, n}(E) \to M_{m, n}(F)$ simply by $T_{m, n}$.

There is one more tensor product we will use frequently later on, namely the normal spatial tensor product. For any two dual operator spaces $E^*$ and $F^*$ we define the normal spatial tensor product $E^* \otimes F^*$ by the weak$^*$-closure of the algebraic tensor product $E^* \otimes F^*$ in $CB(F, E^*)$ via the same identification map as above. Note that

$$E^* \otimes F^* = CB(F, E^*)$$

holds if $E$ satisfies the operator space approximation property (shortly OAP). See [10, chapter 11] for the details.

We close this section with some operator algebra related notations. Let $A$ be an operator algebra. We say that an operator space $E \subseteq B(H)$ is an (abstract) operator left $A$-module if $E$ is a left $A$-module with the left $A$-module map $\varphi : A \times E \to E$ and there are a complete isometry $j_E : E \to B(K)$ and a completely contractive map $j_A : A \to B(K)$ for some Hilbert space $K$ satisfying

$$j_A(X)j_E(Y) = j_E(\varphi(X, Y)), \quad X \in A, Y \in E.$$ 

We also say that $E$ is a left $h$-module over $A$ if the module map $\varphi$ (understood as the associated linear map) extends to a completely bounded map

$$\varphi : A \otimes_h E \to E.$$ 

It is straightforward to check that any left operator $A$-module is an left $h$-module over $A$. Note that operator right $A$-modules and right $h$-modules are similarly defined. Note also, in passing, that under mild assumptions, $h$-modules and operator modules are naturally isomorphic. See [3, Theorem 3.31] for example.
2.2. Non-commutative Littlewood multipliers. Let $\Sigma$ be a set for which we have a prescribed collection of natural numbers $(d_\sigma)_{\sigma \in \Sigma}$. If $1 \leq p < \infty$ and $d \in \mathbb{N}$ we let $S_d^p$ denote $M_d$ equipped with the Schatten $p$-norm $\| \cdot \|_p$.

We consider subspaces of the product space $\prod_{\sigma \in \Sigma} M_{d_\sigma}$ with associated norms for their elements:

$$
L^\infty = L^\infty(\Sigma) = \ell^\infty \bigoplus_{\sigma \in \Sigma} M_{d_\sigma}, \quad \|A\|_{L^\infty} = \sup_{\sigma \in \Sigma} \|A_\sigma\|_\infty
$$

$$
L^2 = L^2(\Sigma) = \ell^2 \bigoplus_{\sigma \in \Sigma} \sqrt{d_\sigma} S^2_d, \quad \|A\|_{L^2} = \left[ \sum_{\sigma \in \Sigma} d_\sigma \|A_\sigma\|_2^2 \right]^{\frac{1}{2}}
$$

$$
L^1 = L^1(\Sigma) = \ell^1 \bigoplus_{\sigma \in \Sigma} d_\sigma S^1_d, \quad \|A\|_{L^1} = \sum_{\sigma \in \Sigma} d_\sigma \|A_\sigma\|_1.
$$

The space $L^1$ is the predual of the space $L^\infty$ via the following standard duality bracket.

$$
\langle (A_\sigma), (B_\sigma') \rangle = \sum_{\sigma \in \Sigma} \text{Tr}(A_\sigma B_\sigma'), \quad (A_\sigma), (B_\sigma') \in L^1, (B_\sigma') \in L^\infty.
$$

Whenever we consider $L^\infty$ and $L^1$ as operator spaces, we assume their natural operator space structure as a von Neumann algebra and the predual of a von Neumann algebra, respectively.

**Proposition 2.1.** The formal identities $\text{id}_{2,\infty}^c : L^2_c \to L^\infty$, $\text{id}_{r,\infty}^c : L^r_c \to L^\infty$, $\text{id}_{1,2}^r : L^1 \to L^2_c$ and $\text{id}_{1,2}^c : L^1 \to L^2$ are complete contractions.

**Proof.** We will only check the case of $\text{id}_{2,\infty}^c$ since the case $\text{id}_{r,\infty}^c$ is similar. Moreover, $(\text{id}_{2,\infty}^c)^* = \text{id}_{r,\infty}^c$ and $(\text{id}_{1,2}^r)^* = \text{id}_{1,2}^c$.

Since $L^2_c = (\ell^2 \bigoplus_{\sigma \in \Sigma} \sqrt{d_\sigma} S^2_d)_{c}$, it is enough to show that the formal identity $\text{id}_n : (\sqrt{n} S^2_n)_{c} \to M_n$ is a complete contraction for any $n \geq 1$. Indeed, by (2.2) we have

$$
\|\text{id}_{2,\infty}^c\|_{cb} = \sup_{\sigma \in \Sigma} \|Q_\sigma \circ \text{id}_{2,\infty}^c\|_{cb},
$$

where

$$
Q_\sigma : \ell^\infty \bigoplus_{\rho \in \Sigma} M_{d_\rho} \to M_{d_\sigma}
$$

is the canonical projection, which is completely contractive. Moreover, we have

$$
Q_\sigma \circ \text{id}_{2,\infty}^c = \text{id}_n \circ P_\sigma, \quad \sigma \in \Sigma
$$

where $\text{id}_n : (\sqrt{n} S^2_n)_{c} \to M_n$ is the formal identity and

$$
P_\sigma : \ell^2 \bigoplus_{\rho \in \Sigma} \sqrt{d_\rho} S^2_{d_\rho} \to \sqrt{d_\sigma} S^2_{d_\sigma}
$$

is the canonical orthogonal projection, which explains that $\text{id}_{2,\infty}^c$ is completely contractive.

For the claim itself we let $\{e_{ij}\}_{i,j=1}^n$ be the matrix units in $M_n$. Since

$$
\|e_{ij}\|_{\sqrt{\pi S^2_n}} = \sqrt{n},
$$

(2.1) tells us that

$$
\|\text{id}_n\|_{cb} = \left\| \sum_{i,j=1}^n n^{-\frac{1}{2}} e_{ij} (n^{-\frac{1}{2}} e_{ij})^* \right\|_{M_n}^{\frac{1}{2}} = \left\| \sum_{i=1}^n e_{ii} \right\|_{M_n}^{\frac{1}{2}} = 1.
$$

We need to understand the $L^\infty$-module structure on $L^2$ as follows.
Proposition 2.2. \(L^2_c\) is a left operator \(L^\infty\)-module under the multiplication \(AB = (A_\sigma B_\tau)_{\sigma,\tau \in \Sigma}\) with \(A = (A_\sigma)_{\sigma \in \Sigma} \in L^\infty\) and \(B = (B_\sigma)_{\sigma \in \Sigma} \in L^2\), and \(L^2_r\) is a right operator \(L^\infty\)-module under a similar multiplication.

**Proof.** We only check the case of \(L^2_c\). Note that \(L^2_c = B(\mathbb{C}, L^2)\) can be completely isometrically embedded in \(B(L^2)\) by the embedding
\[
j_2 : L^2_c \hookrightarrow B(L^2), \quad Z \mapsto N_Z,
\]
where \(N_Z(A) = \langle A, \psi \rangle Z\) for some fixed unit vector \(\psi \in L^2\). If we consider the following standard representation of \(L^\infty\)
\[
j_\infty : L^\infty \hookrightarrow B(L^2), \quad X \mapsto M_X,
\]
where \(M_X\) is the left multiplication by \(X\). Then, we have \(j_\infty(X)j_2(Z) = j_2(XZ)\) for any \(X \in L^\infty\) and \(Z \in L^2\). Thus, \(L^2_c\) is a left operator \(L^\infty\)-module. \(\square\)

The above module structure can be easily extended to vector-valued cases.

**Lemma 2.3.** Let \(M\) and \(N\) be von Neumann algebras and \(E\) a dual left operator \(M\)-module, i.e. a dual operator space whose predual is a right operator \(M\)-module. Then \(E \overline{\otimes} N\) is a left operator \(M \overline{\otimes} N\)-module.

**Proof.** By [4, Theorem 3.8.3] we may assume that there is a Hilbert space \(H\) for which \(M, E \subset B(H)\) as weak*-closed subspaces, and \(E\) is a left \(M\)-module. We let \(x \in M \overline{\otimes} N\) and \(y \in E \overline{\otimes} N\) and find bounded nets \((x_\alpha) \subset M \otimes N\) and \((y_\beta) \subset E \otimes N\) which converge weak* to \(x\) and \(y\), respectively. Then \(x_\alpha y_\beta \in E \overline{\otimes} N\) and converges (with limit taken in either order) weak* to \(xy\) in \(B(H) \overline{\otimes} N\), and hence in the closed subspace \(E \overline{\otimes} N\). \(\square\)

Now we define non-commutative Littlewood multiplier spaces.

**Definition 2.4.** We define the spaces \(\mathcal{T}^2_c\) and \(\mathcal{T}^2_r\) by
\[
\mathcal{T}^2_c = \mathcal{T}^2_c(\Sigma) = L^2_c \overline{\otimes} L^\infty, \quad \mathcal{T}^2_r = \mathcal{T}^2_r(\Sigma) = L^\infty \overline{\otimes} L^2_c.
\]

**Remark 2.5.** In the classical case, namely when \(d_\sigma = 1\) for all \(\sigma\), we have \(L^p = \ell^p(\Sigma)\) for \(p = 1, 2, \infty\). It is straightforward to compute, via the identifications \(\mathcal{T}^2_c \cong CB(\ell^2_c, \ell^\infty) = B(\ell^2, \ell^\infty)\), and \(\mathcal{T}^2_r \cong CB(\ell^1, \ell^2_c) = B(\ell^1, \ell^2)\) that
\[
\mathcal{T}^2_c = \left\{[a_{\sigma, \tau}] : \sup_{\tau} \sum_{\sigma} |a_{\sigma, \tau}|^2 < \infty \right\}, \quad \mathcal{T}^2_r = \left\{[b_{\sigma, \tau}] : \sup_{\tau} \sum_{\sigma} |b_{\sigma, \tau}|^2 < \infty \right\}.
\]

Hence we recover the classical Littlewood function space \(\mathcal{T}^2(\mathbb{R})\) by the sum of the above two space \(\mathcal{T}^2_c + \mathcal{T}^2_r\). Note that the non-commutative setting above forces us to consider left (row) and right (column) cases separately. Note that we are using the term “Littlewood multipliers” instead of “Littlewood functions” as is used in the literature, which suits better in non-commutative contexts.

**Corollary 2.6.** \(\mathcal{T}^2_c\) is a left, and \(\mathcal{T}^2_r\) is a right, operator \(L^\infty \overline{\otimes} L^\infty\)-module.

**Proof.** We note that \(L^2_c\) is reflexive, and thus a dual \(L^\infty\)-module, and is a left operator \(L^\infty\)-module from Proposition 2.2. We thus apply Lemma 2.3 directly to see the result for \(\mathcal{T}^2_c\). The result for \(\mathcal{T}^2_r\) follows similarly. \(\square\)

**Proposition 2.7.** The following formal identities are complete contractions.
\[
I_c : \mathcal{T}^2_c \rightarrow L^\infty \otimes_{eh} L^\infty \quad \text{and} \quad I_r : \mathcal{T}^2_r \rightarrow L^\infty \otimes_{eh} L^\infty
\]
Proof. We again check the case of $I_c$ only. The other one follows similarly.

Note that $T_c^2 = L_c^2 \otimes L^\infty$ and $L^\infty \otimes_{cb} L^\infty$ can be identified with $CB(L^1, L^2_c)$ and $\Gamma_2(L^1, L^\infty)$ under the (essentially) same identification

$$A \otimes B \mapsto u, \text{ where } u(X) = \langle X, B \rangle A.$$ 

Thus it is enough to show that the map

$$CB(L^1, L^2_c) \to \Gamma_2(L^1, L^\infty), \quad X \mapsto \text{id}_{2,\infty} \circ X$$

is a complete contraction, where $\text{id}_{2,\infty} : L_c \to L^\infty$ is the formal identity. We start with an element

$$(u_{ij}) \in M_n(CB(L^1, L^2_c)),$$

where $u_{ij} : L^1 \to L^2_c$. We set

$$U : L^1 \to M_{1,n}(M_{n,1}(L^2_c)), \quad x \mapsto [(u_{ij}(x))]_{i=1}^n_{j=1}$$

and

$$V = \text{id}_{M_{1,n}} \otimes \text{id}_{2,\infty} : M_{n,1}(L^2_c) \to M_{n,1}(L^\infty).$$

Since

$$M_{1,n}(M_{n,1}(L^2_c)) \cong M_n(L^2_c)$$

naturally we have $\|U\|_{cb} = \|(u_{ij})\|_{cb}$, and clearly by Proposition 2.1 $\|V\|_{cb} = 1$.

Moreover, we have

$$(u_{ij}) = V_{1,n} \circ U$$

and $M_{n,1}(L^2_c) \cong C_n \otimes_h L^2_c$ is also a column Hilbert space. Thus

$$\gamma_{2,n}(u_{ij}) \leq \|V\|_{cb}\|U\|_{cb} = \|(u_{ij})\|_{M_n(CB(L^1, L^2_c))}.$$ 

This completes the proof. 

\[ \square \]

Combining Corollary 2.6 and Proposition 2.7 we get the following

**Theorem 2.8.** Every element in $T_c^2$ (resp. $T_c^\pi$) is a right (resp. left) cb-multiplier from $L^\infty \otimes L^\infty$ into $L^\infty \otimes_{cb} L^\infty$ with the same cb-norm.

### 3. Beurling-Fourier Algebras on Compact Groups

In this section we collect basic materials concerning Beurling-Fourier algebras on compact groups.

#### 3.1. Preliminary

Let $G$ be a compact group. We will use the notation

$$\sigma \subset \pi \otimes \pi', \pi, \pi' \in \hat{G}$$

which implies that $\sigma \in \hat{G}$ appears in the irreducible decomposition of $\pi \otimes \pi'$.

The group von Neumann algebra $VN(G)$ of $G$ is defined by

$$VN(G) = \{\lambda(x) : x \in G\}'' \subset B(L^2(G)),$$

where $\lambda$ is the left regular representation of $G$. $VN(G)$ is equipped with the comultiplication

$$\Gamma : VN(G) \to VN(G) \otimes VN(G), \quad \lambda(x) \mapsto \lambda(x) \otimes \lambda(x).$$

Using representation theory of $G$ we have an equivalent formulation of $VN(G)$, namely

$$VN(G) \cong \ell^\infty \bigoplus_{\pi \in \hat{G}} M_{d_\pi}$$

under the $\ast$-isomorphism

$$\lambda(x) \mapsto (\bar{\pi}(x))_{\pi \in \hat{G}}, \quad x \in G.$$ 

We note that $VN(G)$ acting on $L^2(G)$ is unitarily equivalent to $\ell^\infty \bigoplus_{\pi \in \hat{G}} M_{d_\pi}$ as a von Neumann algebra acting on $\ell^2 \bigoplus_{\pi \in \hat{G}} S_{d_\pi}^2$ by left multiplication.
We will frequently use the above identification without further comment. For example, we will understand $(A(\pi))_\pi \in \ell^\infty - \bigoplus_{\pi \in \hat{G}} M_{d_\pi}$ as an element of $VN(G)$. By abuse of notation we will denote $\Gamma$ transferred to
\[
\ell^\infty - \bigoplus_{\pi \in \hat{G}} \bigl( \ell^\infty - \bigoplus_{\pi \in \hat{G}} M_{d_\pi} \bigr)
\]
again by $\Gamma$. The formula for the transferred one is the following, which is a folklore, but we include the proof for the convenience of the readers.

**Proposition 3.1.** For any $A = (A(\pi))_\pi \in \hat{G}$ we have
\[
\Gamma(A) = (X(\pi, \pi'))_{\pi, \pi' \in \hat{G}} \text{ with } X(\pi, \pi') \equiv \bigoplus_{\pi \subset \pi \otimes \pi'} A(\sigma)
\]
up to unitary equivalences.

**Proof.** It is straightforward to check that the formula holds for $\lambda(x) \equiv (\bar{\pi}(x))_{\pi \in \hat{G}}$ for any $x \in G$. Now we apply weak* -density of the linear span of $\{\lambda(x) : x \in G\}$ in $VN(G)$ to get the result in full generality. \(\square\)

### 3.2. Beurling-Fourier algebras
We refer the reader to [25, 27] for the details of this section.

Let $G$ be a compact group. We call a function $\omega : \hat{G} \to [1, \infty)$ a weight if
\[
\omega(\sigma) \leq \omega(\pi)\omega(\pi')
\]
for any $\pi, \pi' \in \hat{G}$ and $\sigma \in \hat{G}$ (see [25, Theorem 2.12] or [27, Section 3]). For any weight $\omega$, we set
\[
W = \bigoplus_{\pi \in \hat{G}} \omega(\pi)id_{M_{d_\pi}}.
\]
Note that $W$ is an unbounded operator in general, but $\Gamma(W)$ still can be well-defined also as an unbounded operator ([25, Section 2 and Theorem 2.12]). We may view $\Gamma(W)$ as a collection of matrices with possibly unbounded matrix norms. Moreover, $W^{-1} = \bigoplus_{\pi \in \hat{G}} \omega(\pi)^{-1}id_{M_{d_\pi}}$ is a bounded operator. Then, Proposition 3.1 tells us that
\[
\Gamma(W)(W^{-1} \otimes W^{-1})(\pi, \pi') \equiv \bigoplus_{\pi \subset \pi \otimes \pi'} \omega(\sigma) \omega(\pi)\omega(\pi')id_{M_{d_\pi}},
\]
so that the condition (3.1) can be restated as
\[
\Gamma(W)(W^{-1} \otimes W^{-1}) \leq 1_{VN(G)}.
\]

We define the **Beurling-Fourier algebra** $A(G, \omega)$ by
\[
A(G, \omega) := \{ f \in C(G) : \| f \|_{A(G, \omega)} = \sum_{\pi \in \hat{G}} d_\pi \omega(\pi) \| \hat{f}(\pi) \|_1 \},
\]
where
\[
\hat{f}(\pi) = \int_G f(x)\pi(x)dx \in M_{d_\pi}.
\]
$A(G, \omega)$ can be naturally identified with the space
\[
\ell^1 - \bigoplus_{\pi \in \hat{G}} d_\pi \omega(\pi)S^1_{d_\pi}.
\]
Thus the dual space is
\[
\ell^\infty - \bigoplus_{\pi \in \hat{G}} \omega(\pi)^{-1}M_{d_\pi}.
\]
Thus, we have a canonical isometry \( VN \rightarrow VN \) via the standard duality bracket, which we will denote by \( \bigcup \). Now we define \( \tau \) notation is justified by the fact that \( VN \rightarrow VN \) and the condition (3.1) or (3.3) implies that \( \tilde{\tau} \) is known to generate \( \hat{\tau} \). For Definition 3.2.

Clearly, the dimension weight of order \( \alpha \) is known to generate \( \hat{\alpha} \). More precisely, if we denote for every \( k \geq 0 \), we define

\[
\omega_\alpha(\pi) = d_\alpha^\alpha(\pi \in \hat{G}).
\]

Clearly \( \omega_\alpha \) satisfies the condition (3.1), and so, it defines a weight on \( \hat{G} \); it is called the dimension weight of order \( \alpha \).

We would like to point out that if \( G \) is abelian, then \( \omega_\alpha = 1 \). Hence the dimension weights are interesting only for compact groups that are far from being abelian.

3.3. Weights on the dual of compact connected Lie groups. When the group \( G \) is a connected compact Lie group, we have another fundamental example of weights on \( \hat{G} \) using the highest weight theory. See [33] or [27, section 5] for the details.

Let \( g \) be the Lie algebra of \( G \) with the decomposition \( g = \mathfrak{z} + \mathfrak{g}_1 \), where \( \mathfrak{z} \) is the center of \( g \) and \( \mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}] \). Let \( t \) be a maximal abelian subalgebra of \( \mathfrak{g}_1 \) and \( T = \text{ext}t \). Then there are fundamental weights \( \lambda_1, \cdots, \lambda_r, \Lambda_1, \cdots, \Lambda_l \in g^* \) with \( r = \dim \mathfrak{z} \) and \( l = \dim t \) such that any \( \pi \in \hat{G} \) is in one-to-one correspondence with its associated highest weight

\[
\Lambda_\pi = \sum_{i=1}^{r} a_i \lambda_i + \sum_{j=1}^{l} b_j \Lambda_j,
\]

which is parameterized by \( r \) integers \( (a_i)_{i=1}^{r} \) and \( l \) non-negative integers \( (b_j)_{j=1}^{l} \in \mathbb{Z}_+^l \). Note that we adapted the same notations for the weights \( \lambda_i \) and \( \Lambda_j \) from [27, section 5].

Let \( \chi_i \) be the character of \( G \) associated to the highest weight \( \lambda_i \) and \( \pi_j \) be the irreducible representation associated to the weight \( \Lambda_j \). Then,

\[
S = \{ \pm \chi_i, \pi_j : 1 \leq i \leq r, 1 \leq j \leq l \}
\]

is known to generate \( \hat{G} \). More precisely, if we denote for every \( k \geq 1 \),

\[
S^\otimes k = \{ \pi \in \hat{G} : \pi \subset \sigma_1 \otimes \cdots \otimes \sigma_k \text{ where } \sigma_1, \cdots, \sigma_k \in S \cup \{1\}\},
\]

then we have

\[
\bigcup_{k \geq 1} S^\otimes k = \hat{G}.
\]

Now we define \( \tau_S : \hat{G} \rightarrow \mathbb{N} \cup \{0\} \), the length function on \( \hat{G} \) associated to \( S \), by

\[
\tau_S(\pi) := k, \text{ if } \pi \in S^\otimes k \setminus S^\otimes (k-1).
\]
From the definition, we clearly have
\[ (3.6) \quad \tau_S(\sigma) \leq \tau_S(\pi) + \tau_S(\pi') \]
for any \( \pi, \pi' \in \widehat{G} \) and \( \sigma \subset \pi \otimes \pi' \). This fact allows us to use \( \tau_S \) to construct various weights on \( \widehat{G} \).

**Definition 3.3.** For \( \alpha \geq 0 \) and \( 1 \geq \beta \geq 0 \), we define \( \omega^\alpha_\mathcal{S}, \gamma^\beta_\mathcal{S} : \widehat{G} \to [1, \infty) \) by
\[
\omega^\alpha_\mathcal{S}(\pi) = (1 + \tau_S(\pi))^\alpha, \quad \gamma^\beta_\mathcal{S}(\pi) = e^{\tau_S(\pi)\beta} \quad (\pi \in \widehat{G}).
\]
Using (3.6), it follows routinely that both \( \omega^\alpha_\mathcal{G} \) and \( \gamma^\beta_\mathcal{G} \) satisfy (3.1), and hence, they define weights on \( \widehat{G} \); they are called the polynomial weight of order \( \alpha \) and the exponential weight of order \( \beta \), respectively. When \( \widehat{G} \) is abelian (e.g. \( G = T^n \)), then our definitions coincide with the classical polynomial and exponential weights on finitely generated abelian groups.

**Remark 3.4.**
1. We would like to highlight the fact that the above length function \( \tau_S \) is equivalent to the following 1-norm defined on \( \widehat{G} \):
\[
||\pi||_1 := \sum_{i=1}^{r} |a_i| + \sum_{j=1}^{l} b_j,
\]
where the integers \( a_i \) and \( b_j \) are defined in (3.5). Indeed, in the proof of [27, Theorem 5.4], it is proved that there is a constant \( C \) depending only on \( G \) such that
\[ (3.7) \quad \tau_S(\pi) \leq ||\pi||_1 \leq C \tau_S(\pi). \]
2. We may consider a variant of exponential weights of the form \( e^{D\tau_S(\pi)\beta} \) with an additional parameter \( D > 0 \). We note that all the results in this paper concerning the weight \( e^{\tau_S(\pi)\beta} \) still hold in the case of the weight \( e^{D\tau_S(\pi)\beta} \) with a minor modification of calculations.

### 3.4. Weights on the dual of \( SU(n) \) and its restriction to a maximal torus.

The classical group \( SU(n) \) is semisimple, so that we have \( \mathfrak{z} = 0 \). We denote the maximal torus of \( SU(n) \) consisting of diagonal matrices by \( H_n \cong \mathbb{T}^{n-1} \). Then \( \widehat{SU(n)} \) is in one-to-one correspondence with \( (n-1) \)-tuples
\[
(b_1 \cdot \cdot \cdot , b_{n-1}) \in \mathbb{Z}^{n-1}_+.
\]
Note that the canonical generating set is given by
\[
S = \{(1,0,\cdots,0), \cdots, (0,\cdots,0,1)\} \subset \mathbb{Z}^{n-1}_+.
\]
By setting
\[
\lambda_n = 0, \lambda_{n-1} = b_{n-1}, \lambda_{n-2} = b_{n-2} + b_{n-1}, \cdots, \lambda_1 = b_1 + \cdots + b_{n-1},
\]
we get a one-to-one correspondence between \( \widehat{SU(n)} \) and \( n \)-tuples \( \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n_+ \) satisfying
\[
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} \geq \lambda_n = 0.
\]
We will denote the \( n \)-tuple by \( \lambda = (\lambda_1, \cdots, \lambda_n) \), which is usually called a dominant weight in Lie theory. See [16] for the details of representation theory of \( SU(n) \). Let \( \pi_\lambda \) be the irreducible representation corresponding to \( \lambda = (\lambda_1, \cdots, \lambda_n) \). Then its length is
\[ (3.8) \quad \tau_S(\pi_\lambda) = ||\pi_\lambda||_1 = \lambda_1, \]
and its character function \( \chi_\lambda = \chi_{\pi_\lambda} \) has the following form when it is restricted to a maximal torus \( H_n = \{ \text{diag}(x_1, \cdots, x_n) \} \):
\[
\chi_\lambda(x_1, \cdots, x_n) = \sum T \cdot x_1^{t_1} \cdots x_n^{t_n},
\]
where $T$ runs through all the semistandard Young tableaux of shape $\lambda$ with parameters $t_1, \ldots, t_n$. Here the parameter $t_k$, $1 \leq k \leq n$, is the number of times $k$ appear in the tableau. Moreover, $\sum_{i=1}^n t_i = \sum_{i=1}^{n-1} \lambda_i$ and we have the following dimension formula:

$$d_\lambda = d_{\pi_\lambda} = \prod_{1 \leq i < j \leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i}.$$ 

Since $x_1 \cdots x_n = 1$, we may also write as follows

$$\chi_\lambda(x_1, \ldots, x_{n-1}) = \sum_{T} x_1^{t_1-\tau_n} \cdots x_{n-1}^{t_{n-1}-\tau_n}. \tag{3.9}$$

Now we turn our attention to the restriction of weights to subgroups. Let $H$ be a closed subgroup of $G$, and let $\omega : \hat{G} \to [1, \infty)$ be a weight. We define the restriction of $\omega$ on $\hat{H}$, $\omega_H : \hat{H} \to [1, \infty)$ by

$$\omega_H(\pi) = \inf \{ \omega(\bar{\pi}) \mid \pi \subset \bar{\pi} \} |_H. \tag{3.10}$$

Then it is shown in [25] Proposition 3.5 that $\omega_H$ is a weight on $\hat{H}$ and the restriction map

$$R_H : A(G, \omega) \to A(H, \omega_H), \ f \mapsto f|_H$$

is a complete quotient map.

We mainly focus on the case when $G = SU(n)$ and $H = H_n \cong \mathbb{T}^{n-1}$, a maximal torus of $SU(n)$. We first need to understand the decomposition of $\pi|_{H_n}$ for any $\pi \in \hat{G}$. The following proposition is an immediate consequences of the definition of $\pi_\lambda$.

**Proposition 3.5.** Let $\lambda$ and $\pi_\lambda$ be as above and $P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}$ be an $(n-1)$-tuple of integers. Then, the character $\chi_P$ of $\mathbb{T}^{n-1}$ associated to $P$ satisfies

$$\chi_P \subset \pi_\lambda|_{H_n}$$

if and only if there exists a semistandard Young tableau $T$ with parameters $t_1, \ldots, t_n$ such that $t_i - t_n = p_i$ for every $1 \leq i \leq n-1$.

We show in the following theorem that if we restrict dimension weights on $SU(n)$ down to $\hat{H}_n \cong \mathbb{Z}^{n-1}$, then we would again get polynomial weights on $\mathbb{Z}^{n-1}$. Recall the polynomial weight $\rho_\alpha$ of order $\alpha > 0$ on $\mathbb{Z}^{n-1}$ is given by

$$\rho_\alpha(P) = (1 + ||P||^\alpha = (1 + |p_1| + \cdots + |p_{n-1}|)^\alpha, \ P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}.$$ 

**Theorem 3.6.** Let $\alpha \geq 0$, and let $\omega_\alpha$ the dimension weight on $SU(n)$ defined in Definition 3.2. Then the restriction of $\omega_\alpha$ to $\hat{H}_n$ is equivalent to the polynomial weight $\rho_{(n-1)\alpha}$ on $\mathbb{Z}^{n-1}$ up to constants depending only on $n$ and $\alpha$. Moreover, $A(\mathbb{T}^{n-1}, \rho_{(n-1)\alpha})$ is completely isomorphic with the complete quotient of $A(SU(n), \omega_\alpha)$ coming from the restriction to $H_n$.

**Proof.** Fix $n \geq 2$ and $P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}$. By (3.10) and Proposition 3.5 we must estimate the infimum of $d_\lambda$ over all possible $\chi_P \subset \pi_\lambda|_{H_n} \in \hat{SU}(n)$. Without loss of generality, we can assume that $p_1 \geq p_2 \geq \cdots \geq p_{n-1}$ since the Schur polynomial is symmetric (so the rearrangement $t'_1 \geq t'_2 \geq \cdots \geq t'_{n-1}, t_n$ of $t_1, \ldots, t_{n-1}, t_n$ also appears in a semistandard Young tableau of shape $\lambda$), the latter is equivalent to $\chi_{P'} \subset \pi_\lambda|_{\mathbb{T}^{n-1}}$, where $P'$ is the rearrangement of $P$ in the non-increasing order.

We now consider a particular $\lambda = \lambda_P \in \mathbb{Z}^n$ such that $\chi_P \subset \pi_\lambda|_{\mathbb{T}^{n-1}}$ given by

$$\lambda_P : \lambda_1 = \sum_{i=1}^{n-1} p_i + n|p_{n-1}|, \ \lambda_2 = \cdots = \lambda_n = 0. \tag{3.12}$$
If we set the parameters $t_1, \ldots, t_n$ by

\begin{equation}
(3.13) \quad t_n = |p_{n-1}|, \\
t_i = p_i + |p_{n-1}|, \ 1 \leq i \leq n-1,
\end{equation}

then since $p_1 \geq p_2 \geq \ldots \geq p_{n-1}$, we have

$$t_1 \geq t_2 \geq \ldots \geq t_{n-1} = p_{n-1} + |p_{n-1}| \geq 0.$$ 

Therefore for each $1 \leq i \leq n$, we have $t_i \geq 0$. Note that $\lambda_P$ is a diagram with only one row, so that it is easy to find a semistandard Young tableau of shape $\lambda_P$ in which the weight of each integer $1 \leq j \leq n$ as follows:

\begin{equation*}
\begin{array}{c}
1 \quad 2 \quad \cdot \cdot \cdot \quad n \\
t_1 & t_2 & \ldots & t_n
\end{array}
\end{equation*}

Moreover,

$$d_{\lambda_P} = \frac{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i < j \leq n} (j - i)}$$

$$= \frac{1}{(n-1)!} \prod_{1 \leq i < j \leq n} \left( \sum_{i=1}^{n-1} p_i + n|p_{n-1}| + j - 1 \right)$$

$$\leq \frac{1}{(n-1)!} \prod_{1 \leq i < j \leq n} \left[ (n+1) \left( \sum_{i=1}^{n-1} |p_i| + 1 \right) \right]$$

$$= \frac{(n+1)^{n-1}}{(n-1)!} \left( \sum_{i=1}^{n-1} |p_i| + 1 \right)^{n-1}.$$ 

On the other hand, let $\lambda$ be any dominant weight such that $\chi_P \subset \pi_{\lambda} |_{\mathfrak{q}_{n-1}}$. Then there exist parameters $t_1, \ldots, t_n$ such that $p_i = t_i - t_n$ for every $1 \leq i \leq n - 1$. Moreover, we have

\begin{equation}
(3.14) \quad \sum_{i=1}^{n-1} |p_i| + 1 = \sum_{i=1}^{n-1} |t_i - t_n| + 1 \leq \sum_{i=1}^{n-1} (t_i + t_n) + 1
\end{equation}

$$\leq 1 + (n-1) \sum_{i=1}^{n} t_i = 1 + (n-1) \sum_{i=1}^{n-1} \lambda_i$$

$$\leq 1 + n(n-1)\lambda_1 \leq n^2(\lambda_1 + 1).$$

We also have

$$\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i)$$

$$\geq (\lambda_1 + 1)(\lambda_1 - \lambda_2 + 1)(\lambda_2 + 1) \cdots (\lambda_1 - \lambda_{n-1} + 1)(\lambda_{n-1} + 1)$$

$$\geq (\lambda_1 + 1) \left( \frac{\lambda_1 + 2}{2} \right)^{n-2} \geq \frac{1}{2^{n-2}} (\lambda_1 + 1)^{n-1},$$ 

where we used the fact that $ab \geq \frac{a+b}{2}$ whenever $a$ and $b$ are both at least 1. By combining the preceding inequality with (3.14), we get

$$d_{\lambda} = \frac{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i)}{\prod_{1 \leq i < j \leq n} (j - i)} \geq \frac{(\lambda_1 + 1)^{n-1}}{2^{n-2} \prod_{1 \leq i < j \leq n} (j - i)}$$

$$\geq c_n \left( \sum_{i=1}^{n-1} |p_i| + 1 \right)^{n-1},$$
where \(c_n = \frac{1}{(n^2)^{n-1}(n^2-1)\prod_{i<j} (j-i)}\). Consequently, we have

\[
c_n \left( \sum_{i=1}^{n-1} |p_i| + 1 \right)^{n-1} \leq \omega_1 |\hat{\mu}_n(\chi_P)| \leq d_n \left( \sum_{i=1}^{n-1} |p_i| + 1 \right)^{n-1},
\]

where \(d_n = \frac{(n+1)^{n-1}}{(n-1)!}\). Since \(n \geq 2\) and \(P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}\) are arbitrary, we conclude that for every \(\alpha \geq 0\),

\[
c_n^\alpha \rho_{(n-1)\alpha} \leq \omega_\alpha |\hat{H}_n| \leq d_n^\alpha \rho_{(n-1)\alpha}.
\]

The final result follows from (3.11) (see also [25, Proposition 3.5]).

We can also make similar estimation for the restriction of weights of polynomial type on \(SU(n)\) down to \(\hat{H}_n \cong \mathbb{Z}^{n-1}\). We will again obtain polynomial weights on \(\mathbb{Z}^{n-1}\). However, the order will be different and computation become more straightforward.

**Theorem 3.7.** Let \(\alpha \geq 0\), and let \(\omega_\alpha^S\) the polynomial weight on \(\widehat{SU(n)}\) defined in Definition 3.5. Then the restriction of \(\omega_\alpha^S\) to \(\hat{H}_n\) is equivalent to the polynomial weight \(\rho_\alpha\) on \(\mathbb{Z}^{n-1}\) up to constants depending only on \(n\) and \(\alpha\). Moreover, \(A(T^{n-1}, \rho_\alpha)\) is completely isomorphic with the complete quotient of \(A(SU(n), \omega_\alpha^S)\) coming from the restriction to \(\hat{H}_n\).

**Proof.** By Remark 3.4 and (3.8), we can assume that for every \(\pi_\lambda \in \widehat{SU(n)}\),

\[
\omega_\alpha^S(\pi_\lambda) = (1 + \lambda_1)^\alpha.
\]

Fix again \(n \geq 2\) and \(P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}\). As in the proof of Theorem 3.6 we can assume that

\[p_1 \geq p_2 \geq \cdots \geq p_{n-1},\]

and consider a particular \(\lambda = \lambda_P \in \mathbb{Z}_n^*\) such that \(\chi_P \subset \pi_\lambda|_{T^{n-1}}\) by assigning the same parameters \(\lambda_i\) and \(t_i\) as in (3.12) and (3.13). Then

\[1 + \lambda_1 = 1 + \sum_{i=1}^{n-1} p_i + n|p_{n-1}| \leq (n + 1) \left( \sum_{i=1}^{n-1} |p_i| + 1 \right).
\]

On the other hand, let \(\lambda\) be any dominant weight such that \(\chi_P \subset \pi_\lambda|_{T^{n-1}}\). Then we can use (3.14) to get

\[1 + \lambda_1 \geq \frac{1}{n^2} \left( \sum_{i=1}^{n-1} |p_i| + 1 \right).
\]

Putting together the preceding two inequalities, (3.10) and Proposition 3.5 we have

\[
\frac{1}{n^2} \left( \sum_{i=1}^{n-1} |p_i| + 1 \right) \leq \omega_1 |\hat{\mu}_n(\chi_P)| \leq (n + 1) \left( \sum_{i=1}^{n-1} |p_i| + 1 \right).
\]

Since \(n \geq 2\) and \(P = (p_1, \ldots, p_{n-1}) \in \mathbb{Z}^{n-1}\) are arbitrary, we conclude that for every \(\alpha \geq 0\),

\[
\frac{1}{n^{2\alpha}} \rho_\alpha \leq \omega_\alpha^S |\hat{H}_n| \leq (n + 1)^\alpha \rho_\alpha.
\]

The final result again follows from (3.11) (see also [25, Proposition 3.5]). □

In the above we get equivalence of weights since we are working on polynomial types of weights. When we deal with exponential type of weights the above restriction does not guarantee the equivalence of weights. However, restricting further down to 1-dimensional torus allows us to get an exact formula, which will help us later in section 4.5.
Theorem 3.8. Let $\gamma^1_S$ the exponential weight on $SU(n)$ defined in Definition 3.3. Let $T$ be the 1-dimensional torus in $H_n$ whose entries are all 1 except for the first two. Then the restriction of $\gamma^1_S$ to $\tilde{T}$ is exactly the same as the exponential weight $e^{\cdot 1}$ on $\mathbb{Z}$. Moreover, $\ell^1(\mathbb{Z}, e^{\cdot 1}$ is a complete quotient of $A(SU(n), \gamma^1_S)$.

Proof. The same approach as in Theorem 3.1 gives us the conclusion. Indeed, we begin with $P = (p_1, 0, \ldots, 0) \in \mathbb{Z}^{n-1}$. Now we set $t_1 = \lambda_1 = |p_1|$, $t_2 = \cdots = t_n = 0$, then it is easy to observe that this choice of parameters can be easily realized in a semistandard Young tableau of shape $\lambda^P$. Thus, we have

$$\gamma^1_S|_T(P) \leq e^{\|p\|_1}.$$  

For the converse direction we let $\lambda$ be any dominant weight such that $\chi_P \subset \pi_{\lambda}|_{\mathbb{T}^{n-1}}$. Then there exist parameters $t_1, \ldots, t_n$ such that $p_1 = t_1 - t_n$ and $0 = t_2 - t_n = \cdots = t_{n-1} - t_n$. Since the parameters should be realized a semistandard Young tableau of shape $\lambda$ we clearly have that $t_1, t_n \leq 1$, which implies that $|p_1| \leq 1$. Thus, we have

$$\gamma^1_S|_T(P) \geq e^{\|p\|_1}.$$ 

\hfill\Box

4. Beurling-Fourier algebras on compact groups which are operator algebras

In this section, we investigate when a Beurling-Fourier algebra on a compact connected Lie group can be completely boundedly isomorphic to an operator algebra. Throughout this section, we use the term “positive result” when such a thing happens and “negative result” when it does not.

Our approach for seeking Beurling-Fourier algebras as operator algebras is based on the following theorem of Blecher (3).

Theorem 4.1. Let $A$ be a completely contractive Banach algebra with the algebra multiplication $m : A \otimes A \to A$, where $\otimes$ is the projective tensor product of operator spaces. Then, $A$ is completely isomorphic to an operator algebra if and only if the multiplication map extends to a completely bounded map

$$m : A \otimes_h A \to A.$$ 

In the case of $A = A(G, \omega)$ with the operator space structure described in section 3, $A(G, \omega)$ is completely isomorphic to an operator algebra if and only if the following map is completely bounded.

$$\Gamma : VN(G) \to VN(G) \otimes_{ch} VN(G), \ A \mapsto \Gamma(\Lambda)(W^{-1} \otimes W^{-1}).$$

(4.1)

Since we already know $\Gamma : VN(G) \to VN(G) \otimes VN(G)$ is a complete contraction, we can get the positive direction (i.e. $A(G, \omega)$ being completely isomorphic to an operator algebra) if $\Gamma(W^{-1} \otimes W^{-1})$ can be split as a sum of right or left cb-multiplier from $VN(G) \otimes VN(G)$ into $VN(G) \otimes_{ch} VN(G)$, where we could apply non-commutative Littlewood multiplier theory we developed earlier.

The following lemma will be used frequently throughout this section.

Lemma 4.2. $\sum_{i \in \mathbb{Z}^n} \frac{1}{(1 + \|i\|^2)^\alpha} < \infty$ if and only if $\sum_{i \in \mathbb{Z}^n} \frac{1}{(1 + \|i\|^2)^\alpha} < \infty$ if and only if $\alpha > \frac{d}{2}$.

Proof. The above series is sometimes called an Epstein series ([17] p.277) for example. The results follows from a standard integral test argument. \hfill\Box
4.1. The case of $\mathbb{T}^n$ with polynomial weights. In this subsection, we consider the case of $G = \mathbb{T}^n$ with polynomial weights. Since $\hat{\mathbb{T}}^n = \mathbb{Z}^n$ and $A(\mathbb{T}^n, \omega) \cong \ell^1(\mathbb{Z}^n, \omega)$ we can reformulate our problem as follows.

The weighted convolution algebra $\ell^1(\mathbb{Z}^n, \omega)$ with the maximal operator space structure is completely isomorphic to an operator algebra if and only if the following map is completely bounded.

\[
\rho : \ell^\infty(\mathbb{Z}^n) \to \ell^\infty(\mathbb{Z}^n) \otimes _{eh} \ell^\infty(\mathbb{Z}^n), \quad (a_k)_{k \in \mathbb{Z}^n} \mapsto (T(i,j)a_{i+j})_{i,j \in \mathbb{Z}^n},
\]

where $T = (T(i,j))_{i,j \in \mathbb{Z}^n}$ is the matrix given by

\[
T(i,j) = \frac{\omega(i+j)}{\omega(i)\omega(j)}
\]

associated with the weight $\omega : \mathbb{Z}^n \to [1, \infty)$.

We will present a complete solution focusing on the case of polynomial weight $\rho_\alpha$. Note that the 1-dimensional case has already been established in [32] in the setting of Banach spaces. The authors thank Éric Ricard for providing the main idea of the proof.

For the proof we need some background material of harmonic analysis. Let

\[
Q : L^\infty(\mathbb{T}) \to B(\ell^2), \quad f \mapsto (\hat{f}(-i+j))_{i,j \in \mathbb{Z}}.
\]

According to Nehari’s theorem $Q$ is a contractive surjection onto the space of Hankelian matrices (see [28, Section 6] for example).

One more ingredient is the Rudin-Shapiro polynomials. Recall that the Rudin-Shapiro polynomials are defined in the following recursive way.

\[
P_0(z) := 1, \quad Q_0(z) := 1,
\]

and for $k \geq 0$,

\[
P_{k+1}(z) := P_k(z) + z^{2^k}Q_k(z), \quad Q_{k+1}(z) := Q_k(z) - z^{2^k}P_k(z).
\]

By doing an induction on $k$, it is straightforward to check that the coefficients of $P_k$ are $\pm 1$, $\deg P_k = \deg Q_k = 2^k - 1$ and

\[
|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} \quad (z \in \mathbb{T}).
\]

Hence

\[
\|P_k\|_{L^\infty(\mathbb{T})} \leq \sqrt{2^{k+1}}.
\]

Combining the above two ingredients we get a sequence of Hankelian matrices

\[
A_{2^k} = Q(\bar{P}_k), \quad k \geq 0,
\]

where $A_{2^k}$ is a $2^k \times 2^k$ matrix with entries $\pm 1$ satisfying

\[
\|A_{2^k}\|_\infty \leq \sqrt{2^{k+1}}.
\]

**Theorem 4.3.** The weighted convolution algebra $\ell^1(\mathbb{Z}^n, \rho_\alpha)$, $\alpha > 0$ with the maximal operator space structure is completely isomorphic to an operator algebra if and only if $\alpha > \frac{\log |\mathbb{T}^n|}{\log 2} = \frac{n}{2}$.

**Proof.** Let $T^\alpha$ be the matrix \[[139] \] associated to $\rho_\alpha$, which means

\[
T^\alpha(i,j) = \left(\frac{1 + \|i+j\|}{1 + \|i\| + \|j\|}\right)^\alpha.
\]

We need to determine for which value of $\alpha$, the mapping $\bar{\Gamma}$ defined in \[[122] \] is completely bounded. Clearly we have

\[
T^\alpha(i,j) \leq \left(\frac{1}{1 + \|i\|} + \frac{1}{1 + \|j\|}\right)^\alpha \leq 2^\alpha \left(\frac{1}{(1 + \|i\|)^\alpha} + \frac{1}{(1 + \|j\|)^\alpha}\right),
\]

where

\[
|P_k(z)|^2 + |Q_k(z)|^2 = 2^{k+1} \quad (z \in \mathbb{T}).
\]

Hence

\[
\|P_k\|_{L^\infty(\mathbb{T})} \leq \sqrt{2^{k+1}}.
\]

Combining the above two ingredients we get a sequence of Hankelian matrices

\[
A_{2^k} = Q(\bar{P}_k), \quad k \geq 0,
\]

where $A_{2^k}$ is a $2^k \times 2^k$ matrix with entries $\pm 1$ satisfying

\[
\|A_{2^k}\|_\infty \leq \sqrt{2^{k+1}}.
\]

**Theorem 4.3.** The weighted convolution algebra $\ell^1(\mathbb{Z}^n, \rho_\alpha)$, $\alpha > 0$ with the maximal operator space structure is completely isomorphic to an operator algebra if and only if $\alpha > \frac{\log |\mathbb{T}^n|}{\log 2} = \frac{n}{2}$.

**Proof.** Let $T^\alpha$ be the matrix \[[139] \] associated to $\rho_\alpha$, which means

\[
T^\alpha(i,j) = \left(\frac{1 + \|i+j\|}{1 + \|i\| + \|j\|}\right)^\alpha.
\]

We need to determine for which value of $\alpha$, the mapping $\bar{\Gamma}$ defined in \[[122] \] is completely bounded. Clearly we have

\[
T^\alpha(i,j) \leq \left(\frac{1}{1 + \|i\|} + \frac{1}{1 + \|j\|}\right)^\alpha \leq 2^\alpha \left(\frac{1}{(1 + \|i\|)^\alpha} + \frac{1}{(1 + \|j\|)^\alpha}\right),
\]

where

\[
\|P_k\|_{L^\infty(\mathbb{T})} \leq \sqrt{2^{k+1}}.
\]

Combining the above two ingredients we get a sequence of Hankelian matrices

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**Proof.** Let $T^\alpha$ be the matrix \[[139] \] associated to $\rho_\alpha$, which means

\[
T^\alpha(i,j) = \left(\frac{1 + \|i+j\|}{1 + \|i\| + \|j\|}\right)^\alpha.
\]

We need to determine for which value of $\alpha$, the mapping $\bar{\Gamma}$ defined in \[[122] \] is completely bounded. Clearly we have

\[
T^\alpha(i,j) \leq \left(\frac{1}{1 + \|i\|} + \frac{1}{1 + \|j\|}\right)^\alpha \leq 2^\alpha \left(\frac{1}{(1 + \|i\|)^\alpha} + \frac{1}{(1 + \|j\|)^\alpha}\right),
\]
so that \( T^\alpha = ST_1^\alpha + ST_2^\alpha \), where
\[
T_1^\alpha(i, j) = \frac{1}{(1 + \|i\|)^{2\alpha}}, \quad T_2^\alpha(i, j) = \frac{1}{(1 + \|j\|)^{2\alpha}}
\]
and \( S \in \ell^\infty(\mathbb{Z}^n \times \mathbb{Z}^n) \) with \( 0 < S \leq 2^\alpha \). Thus \( T^\alpha \in T^2 = T_r^2 + T_c^2 \) provided that
\[
\sum_{i \in \mathbb{Z}^n} \frac{1}{(1 + \|i\|)^{2\alpha}} < \infty.
\]
Now we have the positive result for \( \alpha > \frac{n}{2} \) by Theorem 2.8 and Lemma 4.2.

For the negative direction, we consider the restricted sequence \( T_d^\alpha \) of \( T^\alpha \) to the set of indices \( I_d^\alpha \times I_d^\alpha \), where
\[
I_d^\alpha = \{ i = (i_1, \cdots, i_n) : 1 \leq i_1, \cdots, i_n \leq d \}.
\]
Then we may regard \( T_d^\alpha \) as a matrix acting on \( \ell^2(I_d^\alpha) \), and the norm of \( T_d^\alpha \) in \( \ell^\infty(\mathbb{Z}^n) \otimes_{c_h} \ell^\infty(\mathbb{Z}^n) \) is exactly the Schur norm of \( T_d^\alpha \) by Theorem 3.1. Moreover, we have a lower estimate of the operator norm of \( T_d^\alpha \) as follows.

\[
\|T_d^\alpha\|_\infty \geq 2^{-\alpha} d^{\frac{n}{2}} \left( \sum_{i \in I_d^\alpha} \frac{1}{(1 + \|i\|)^{2\alpha}} \right)^{\frac{1}{2}}.
\]

Indeed, if we set \( v = \sum_{i \in I_d^\alpha} c_i \in \ell^2(I_d^\alpha) \), then \( \|v\|_2 = d^{\alpha} \) and
\[
\|T_d^\alpha v\|_2 = \left\| \sum_{j \in I_d^\alpha} \left( \frac{1 + \|i + j\|}{(1 + \|i\|)(1 + \|j\|)} \right) \right\|_{I_d^\alpha} \geq \left\| \sum_{j \in I_d^\alpha} \left( \frac{1 + \|i\|}{1 + \|j\|} \right) \right\|_{I_d^\alpha} \geq 2^{-\alpha} \left( \sum_{i \in I_d^\alpha} \frac{1}{(1 + \|i\|)^{2\alpha}} \right)^{\frac{1}{2}}.
\]

Now we recall a sequence of Hankelian matrices \( A_d \in M_d, \quad d = 2^k \geq 1 \) in (1.4).

Then we have
\[
A_d = (a_{i+j})_{i,j=1}^d
\]
with \( a_i \in \{ \pm 1 \} \). Let \( a = \sum_{i=1}^d a_i \delta_i \in \ell^\infty(I_d^1) \) and
\[
b = a \otimes \cdots \otimes a \in \ell^\infty(I_d^d) \subset \ell^\infty(\mathbb{Z}^n),
\]
i.e. \( b_{i_1, \cdots, i_n} = a_{i_1} \cdots a_{i_n} \). Then the associated Hankel matrix of \( b \), i.e.
\[
B = (b_{i+j})_{i,j \in I_d^\alpha},
\]
is nothing but \( B = A_d \otimes \cdots \otimes A_d \), the \( n \)-tensor power of \( A_d \). Since we have
\[
\tilde{B}(b) = [b_{i+j}T_d^\alpha(i, j)]_{i,j \in I_d^\alpha}
\]
and each \( b_i = \pm 1 \) we get
\[
\|T_d^\alpha\|_\infty \leq \|B\|_\infty \cdot \left\| [b_{i+j}T_d^\alpha(i, j)]_{i,j \in I_d^\alpha} \right\|_{\ell^\infty(I_d^\alpha) \otimes_{c_h} \ell^\infty(I_d^\alpha)}.
\]

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Then, by (4.4), \( \|B\|_{\infty} \leq \|A\|_{\infty} \leq (2d)^{\frac{\alpha}{2}} \), we get
\[
\| \Gamma \| \geq \left\| \left[ b_{i+j} T_d^c(i, j) \right]_{i, j \in T_2} \right\|_{\ell^\infty(T_2^c) \otimes \ell^\infty(T_2^c)} \\
\geq \|B\|_{\infty}^{-1} \| T_d^c \|_{\infty} \\
\geq d^{-\frac{\alpha}{2}} 2^{-\alpha - \frac{\alpha}{2}} d^{\frac{\alpha}{2}} \left( \sum_{i \in T_2^c} \frac{1}{(1 + ||i||)^{2\alpha}} \right) \left( \sum_{j \in T_2^c} \frac{1}{(1 + ||j||)^{2\alpha}} \right) \left( \sum_{i \in T_2^c} \frac{1}{(1 + ||i||)^{2\alpha}} \right)^{\frac{1}{2}} \text{ (by (4.5))}
\]
\[
= 2^{-\alpha - \frac{\alpha}{2}} \left( \sum_{i \in T_2^c} \frac{1}{(1 + ||i||)^{2\alpha}} \right)^{\frac{1}{2}}.
\]
If \( \alpha \leq \frac{3}{2} \), then the right-hand side grows without bounds when \( d \to \infty \) (Lemma 4.2), so that we have the negative result. \( \square \)

4.2. The general case of compact connected Lie groups with polynomial weights. When the group is a compact connected Lie group, then we have positive results if the order of the polynomial weight is strictly greater than the half of the dimension of the group.

**Theorem 4.4.** Let \( G \) be a compact connected Lie group, and let \( \omega_S^G \) be the polynomial weight of order \( \alpha \) on \( \hat{G} \) (Definition 4.3). Then \( A(G, \omega_S^G) \) is completely isomorphic to an operator algebra if \( \alpha > \frac{d(G)}{2} \).

**Proof.** Suppose that \( \alpha > \frac{d(G)}{2} \). For simplicity, we write \( \omega \) instead of \( \omega_S^G \). We set \( W = \bigoplus_{\pi \in \hat{G}} \omega(\pi)\text{id}_{M_{d_\pi}} \in VN(G) \) be the operator associated to \( \omega \) and
\[
T = \Gamma(W)(W^{-1} \otimes W^{-1}) \in VN(G) \otimes VN(G).
\]

Then, by (3.2), we have
\[
T(\pi, \pi') = \bigoplus_{\sigma \subset \pi \otimes \pi'} \left( \frac{1 + \tau_S(\sigma)}{(1 + \tau_S(\pi))(1 + \tau_S(\pi'))} \right)^{\alpha} \text{id}_{M_{d_\sigma}}.
\]

By (4.1), we need to show that \( \widehat{\Gamma} : A \mapsto \Gamma(A)T \) is well-defined and completely bounded. To achieve this, we will apply the non-commutative Littlewood machinery developed in Section 2 with \( L^\infty = VN(G) \) to get the decomposition of the operator \( T \) into \( T = T_1 + T_2 \) with \( T_1 \in T_e^2 \) and \( T_2 \in T_0^2 \). In order to do so, we first need to estimate each component of \( T \) as follows.

Let \( \pi, \pi' \in \hat{G} \) and \( \sigma \subset \pi \otimes \pi' \). Then, by (5.7), we have
\[
\left( \frac{1 + \tau_S(\sigma)}{(1 + \tau_S(\pi))(1 + \tau_S(\pi'))} \right)^{\alpha} \leq (1 + C)^{-2\alpha} \left( \frac{1 + \||\sigma||_1}{(1 + \||\pi||_1)(1 + \||\pi'||_1)} \right)^{\alpha}
\]
\[
\leq (1 + C)^{-2\alpha} \left( \frac{1 + \||\pi||_1 + \||\pi'||_1}{(1 + \||\pi||_1)(1 + \||\pi'||_1)} \right)^{\alpha}
\]
\[
\leq \left( \frac{2}{(1 + C)^{\alpha}} \right)^{\alpha} \left( \frac{1}{(1 + \||\pi||_1)^\alpha} + \frac{1}{(1 + \||\pi'||_1)^\alpha} \right).
\]

Hence
\[
T = S(\tilde{T}_1 + \tilde{T}_2),
\]
where \( \tilde{T}_1, \tilde{T}_2 \in VN(G) \otimes VN(G) \) are positive and central elements defined by
\[
\tilde{T}_1(\pi, \pi') = \frac{1}{(1 + \||\pi||_1)^\alpha} \text{id}_{M_{d_\pi}} \otimes \text{id}_{M_{d_{\pi'}}}
\]
and
\[ \tilde{T}_2(\pi, \pi') = \frac{1}{(1 + \|\pi'\|_1)^\alpha} \text{id}_{M_{d_\pi}} \otimes \text{id}_{M_{d_{\pi'}}}, \]
and \( S \in VN(G) \otimes VN(G) \) is some positive element with \( \|S\| \leq \left( \frac{2}{1 + 1/2} \right) \alpha \). We claim that \( \tilde{T}_1 \in \mathcal{T}_\pi^2, \tilde{T}_2 \in \mathcal{T}_\pi^2 \). Indeed
\[ \tilde{T}_1 = \left( \bigoplus_{\pi \in G} \frac{1}{(1 + \|\pi\|_1)^\alpha} \text{id}_{M_{d_\pi}} \right) \otimes 1_{VN(G)} \]
and
\[ \tilde{T}_2 = 1_{VN(G)} \otimes \left( \bigoplus_{\pi' \in G} \frac{1}{(1 + \|\pi'\|_1)^\alpha} \text{id}_{M_{d_{\pi'}}} \right). \]

Hence
\[ \left\| \tilde{T}_1 \right\|_{T_2} = \left\| \bigoplus_{\pi \in G} \frac{1}{(1 + \|\pi\|_1)^\alpha} \text{id}_{M_{d_\pi}} \right\|_{L^2} = \left( \sum_{\pi \in G} \frac{d_\pi^2}{(1 + \|\pi\|_1)^{2\alpha}} \right)^{1/2}, \]
so that \( \tilde{T}_1 \in \mathcal{T}_\pi^2 \) since \( \alpha > \frac{d(G)}{2} \) (see Lemma 5.6.7]). Similarly, \( \tilde{T}_2 \in \mathcal{T}_\pi^2 \). Now, by the centrality of \( \tilde{T}_2 \), we have that for any \( A \in VN(G) \),
\[ \tilde{\Gamma}(A) = \Gamma(A)T = \Gamma(A)S\tilde{T}_1 + \Gamma(A)S\tilde{T}_2 = \Gamma(A)S\tilde{T}_1 + \tilde{T}_2 \Gamma(A)S. \]

Since the maps
\[ VN(G) \otimes VN(G) \to VN(G) \otimes_{ch} VN(G), \ X \mapsto XST_1 \]
and
\[ VN(G) \otimes VN(G) \to VN(G) \otimes_{ch} VN(G), \ X \mapsto \tilde{T}_2 XS \]
are completely bounded by Corollary 2.9 and Theorem 2.8, we can conclude that \( \tilde{\Gamma} \) is also completely bounded.

\[ \square \]

In general, we were not able to obtain the negative result for \( A(G, \omega_3^2) \). In fact, we believe this to be very difficult. However, in the special case when \( G = SU(n) \), we have the following:

**Theorem 4.5.** \( A(SU(n), \omega_3^2) \) is not completely isomorphic to an operator algebra if \( \alpha \leq \frac{n-1}{2} \).

**Proof.** It follows from Theorem 4.3 that \( A(T^{n-1}, \rho_\alpha) \) is not completely isomorphic to an operator algebra if \( \alpha \leq \frac{n-1}{2} \). On the other hand, by Theorem 3.7, \( A(T^{n-1}, \rho_\alpha) \) is completely isomorphic to a complete quotient of \( A(SU(n), \omega_3^2) \). Hence the result follows from the fact that a complete quotient of an operator algebra is again an operator algebra [3, Proposition 2.3.4]. \[ \square \]

**Remark 4.6.**
1. Theorem 4.3 tells us that the exponent \( \frac{d(G)}{2} \) is optimal when \( G = T^n \) whilst by comparing Theorem 4.4 and Theorem 4.5 we see that we have a rather big gap for the case of \( SU(n) \).
2. Varapolous showed that \( A(T, \rho_1) \) is a Q-algebra if and only if \( \alpha > 1/2 \) [32]. However, in general we can not expect \( A(G, \omega_3^2) \) to be completely isomorphic to a Q-algebra since it may not be even completely isomorphic to a Q-space. Recall that an operator space \( E \) is called a Q-space if it is a operator space quotient of a minimal operator space. More generally, the cb-distance of \( E \) from a Q-space is defined by
\[ d_Q(E) = \inf \{ \|T\|_{cb} \|T^{-1}\|_{cb} \}, \]
where the infimum runs over all possible complete isomorphism \( T : E \to F \) for some \( Q \)-space \( F \). Clearly, \( Q \)-algebras are \( Q \)-spaces. Moreover, we have the following estimates ([4, Proposition 5.4.16]).

\[
d_Q(C_n) = \sqrt{n}.
\]

Indeed, \( A(SU(n), \omega_{\alpha}^Q) \) contains row Hilbert spaces of arbitrarily large dimensions so that \( A(SU(n), \omega_{\alpha}^Q) \) is not completely isomorphic to a \( Q \)-space.

### 4.3. The case of compact connected non-simple Lie groups with dimension weights

In this section, we show that, for a non-simple compact Lie group \( G \), one cannot find a Beurling-Fourier algebra on \( G \) which is isomorphic to an operator algebra. Hence we need to restrict our attention to simple cases (such as \( SU(n) \)) to obtain operator algebra for dimension weights.

**Theorem 4.7.** Let \( G \) be a compact connected non-simple Lie group and \( \alpha \geq 0 \). Then \( A(G, \omega_{\alpha}) \) is not isomorphic to an operator algebra.

**Proof.** Since \( G \) is not simple, by [29, 6.5.6], \( G \cong (P \times T)/A \), where \( P \) is a product of compact connected simple Lie groups, \( T \) is an infinite compact connected abelian group and \( A \) is a central subgroup of \( P \times T \). This, in particular, implies that \( G' \neq G \), where \( G' \) is the derived subgroup of \( G \). Hence \( G/G' \) is an infinite compact connected abelian group. On the other hand, since \( G' \) is compact, we can view \( C(G/G') \) as a subalgebra of \( C(G) \). With this identification, a straightforward computation shows that, for every \( f \in C(G/G') \subset C(G) \) and \( \pi \in \hat{G} \),

\[
\hat{f}(\pi) = 0 \text{ if } d_{\pi} > 1.
\]

Thus

\[
\|f\|_{A(G, \omega_{\alpha})} = \sum_{\pi \in \hat{G}} d_{\pi}^{\alpha+1} \|\hat{f}(\pi)\|_1
\]

\[
= \sum_{d_{\pi} = 1} d_{\pi}^{\alpha+1} \|\hat{f}(\pi)\|_1
\]

\[
= \sum_{\chi \in \hat{G}/\hat{G'}} |\hat{f}(\chi)|.
\]

Thus the commutative group algebra \( \ell^1(\hat{G}/\hat{G'}) \) is a closed subalgebra of \( A(G, \omega_{\alpha}) \). Hence if \( A(G, \omega_{\alpha}) \) is isomorphic to an operator algebra, then so is \( \ell^1(\hat{G}/\hat{G'}) \). However, this is impossible because \( \ell^1(\hat{G}/\hat{G'}) \) is not Arens regular [44], and so, \( A(G, \omega_{\alpha}) \) is not isomorphic to an operator algebra.

**4.4. The case of SU\((n)\) with dimension weights.** Let \( \pi_{\lambda}, \pi_{\mu}, \pi_{\nu} \in \hat{SU}(n) \) with \( \pi_{\nu} \subset \pi_{\lambda} \otimes \pi_{\mu} \).

**Conjecture 1.** There is a constant \( C(n) \) depending only on \( n \) such that

\[
\frac{d_{\nu}}{d_{\lambda} d_{\mu}} \leq C(n) \left( \frac{1}{\lambda_1 + 1} + \frac{1}{\mu_1 + 1} \right).
\]

**Theorem 4.8.** Conjecture [2] is true for \( 2 \leq n \leq 5 \).

**Proof.** The case \( n = 2 \) is trivial. The proof for \( 3 \leq n \leq 5 \) will be presented in the appendix. \( \square \)

Now we consider the case of \( SU(n) \) with the dimension weights.

**Theorem 4.9.** Let \( \omega_{\alpha} \) be the dimension weight of order \( \alpha \) on \( SU(n) \) (Definition [23]). Then:

(i) \( A(SU(n), \omega_{\alpha}) \) is completely isomorphic to an operator algebra if \( 2 \leq n \leq 5 \) and
\[ \alpha > \frac{d(SU(n))}{2} = \frac{\nu^2 + 1}{2}. \]

(iii) \( \Lambda(SU(n), \omega_\alpha) \) is not completely isomorphic to an operator algebra if \( n \geq 2 \) and \( \alpha \leq \frac{1}{2} \).

**Proof.** For (i), since by Theorem 4.8 Conjecture 1 holds, we have

\[
\frac{\omega_\alpha(\pi_\nu)}{\omega_\alpha(\pi_\lambda) \omega_\alpha(\pi_\mu)} \leq (2C(n))^{\alpha} \left( \frac{1}{(\lambda_1 + 1)^\alpha} + \frac{1}{(\mu_1 + 1)^\alpha} \right)
\]

for any \( \pi_\nu \subset \pi_\lambda \otimes \pi_\mu \). The rest of the argument goes exactly the same as the one presented in the proof of Theorem 4.4. Part (ii) follows from Theorem 3.6 Theorem 4.3 and the fact that a complete quotient of an operator algebra is again an operator algebra [4, Proposition 2.3.4]. \( \square \)

4.5. **The case of compact connected Lie groups with exponential weights.**

Let \( G \) be a compact connected Lie group. For \( 0 \leq \alpha \leq 1 \), we recall the exponential weight \( \gamma_\alpha = \gamma^G_\alpha \) in Definition 3.3. In this section, we will study when the Beurling-Fourier algebra \( A(G, \gamma_\alpha) \) is completely isomorphic to an operator algebra. If we want to apply the same approach as before we need to find an appropriate decomposition of the function

\[ \frac{\gamma_\alpha(\sigma)}{\gamma_\alpha(\pi) \gamma_\alpha(\pi')} \]

for any \( \pi, \pi', \sigma \in \hat{G} \) with \( \sigma \subset \pi \otimes \pi' \). However, the lack of subadditivity of the function \( e^{\tau_\alpha(\pi)} \) makes the problem more complicated. Instead, we use the following estimate.

**Proposition 4.10.** Let \( 0 < \alpha < 1 \) and \( \beta \geq \max\{1, \frac{6}{\alpha(1 - \alpha)}\} \). There is a constant \( M \) depending only on \( \alpha, \beta \), and the group \( G \) such that

\[
\frac{\gamma_\alpha(\sigma)}{\gamma_\alpha(\pi) \gamma_\alpha(\pi')} \leq M^2 \frac{\omega_\beta(\sigma)}{\omega_\beta(\pi) \omega_\beta(\pi')}
\]

for any \( \pi, \pi', \sigma \in \hat{G} \) with \( \sigma \subset \pi \otimes \pi' \).

**Proof.** We present the proof in the appendix. \( \square \)

Now we have the results for exponential weights. Note that in the case of exponential weights we have a better understanding of the negative results.

**Theorem 4.11.** Let \( G \) be a compact connected Lie group and \( 0 < \alpha < 1 \). Then:

(i) \( A(G, \gamma_\alpha) \) is completely isomorphic to an operator algebra if \( 0 < \alpha < 1 \).

(ii) If \( G \) is not simple, then \( A(G, \gamma_1) \) is not isomorphic to an operator algebra.

(iii) If \( G = SU(n) \), then \( A(G, \gamma_1) \) is not isomorphic to an operator algebra.

**Proof.** First assume that \( 0 < \alpha < 1 \) and take \( \beta \geq \max\left\{1, \frac{6}{\alpha(1 - \alpha)}\right\} \). Then, by Proposition 4.10 we have

\[
\frac{\gamma_\alpha(\sigma)}{\gamma_\alpha(\pi) \gamma_\alpha(\pi')} \leq M^2 \frac{\omega_\beta(\sigma)}{\omega_\beta(\pi) \omega_\beta(\pi')} \leq M^2 \left( \frac{2}{(1 + C)} \right)^\beta \left( \frac{1}{(1 + \|\pi\|_1)^\beta} + \frac{1}{(1 + \|\pi'\|_1)^\beta} \right).
\]

If we take \( \beta \) large enough, then by a similar argument to the one presented in the proof of Theorem 4.4 we can conclude that \( A(G, \gamma_\alpha) \) is completely isomorphic to an operator algebra. This proves (i).

For part (ii), similar to the proof of Theorem 4.7 we can show that the commutative Beurling algebra \( \ell^1(G/G', \gamma_\alpha) \) is a closed subalgebra of \( A(G, \gamma_1) \), where \( \gamma_\alpha = (\gamma_\alpha)_{G/G'} \). Hence if \( A(G, \gamma_1) \) is isomorphic to an operator algebra, then so is \( \ell^1(G/G', \gamma_\alpha) \). However it follows routinely from the definition of \( \tau_S \) (preceding to
Definition 3.3) and the fact that $\hat{\mathcal{G}}/\hat{\mathcal{G}}'$ has a copy of $\mathbb{Z}$ that $\ell^1(\hat{\mathcal{G}}/\hat{\mathcal{G}}', \gamma_\alpha)$ is a closed subalgebra of $\ell^1(\hat{\mathcal{G}}', \gamma_\alpha)$. But $\ell^1(\hat{\mathcal{G}}, e^{|\cdot|})$ is not Arens regular by [9, Theorem 8.11], and so, it can not be isomorphic to an operator algebra. Therefore $\ell^1(\hat{\mathcal{G}}/\hat{\mathcal{G}}', \tilde{\gamma}_\alpha)$ can not be isomorphic to an operator algebra. This completes the proof of (ii).

Finally, if $G = SU(n)$, then by Theorem 3.8 we have that $\ell^1(\mathbb{Z}, e^{|\cdot|})$ is a complete quotient of $A(SU(n), \gamma_1)$. Hence $A(SU(n), \gamma_1)$ is not isomorphic to an operator algebra since quotients of operator algebras are again operator algebras (see [4, Proposition 2.3.4] for example). □

Remark 4.12. We note that when $\alpha = 0$ we have $A(G, \gamma_0) = A(G)$, which is not Arens regular by [15]. Hence it can not be isomorphic to an operator algebra.

APPENDIX A. Solution of the conjecture for $3 \leq n \leq 5$

Let $\pi_\lambda, \pi_\mu, \pi_\nu \in \widehat{SU(n)}$ with $\pi_\nu \subset \pi_\lambda \otimes \pi_\mu$. The Littlewood-Richardson rule tells us that $\nu_1 \leq \lambda_1 + \mu_1$.

Now we have

$$\frac{d_\nu}{d_\lambda d_\mu} = \frac{\prod_{1 \leq i < j \leq n} (\nu_i - \nu_j + j - i) \prod_{1 \leq i < j \leq n} (j - i)}{\prod_{1 \leq i < j \leq n} (\lambda_i - \lambda_j + j - i) \prod_{1 \leq i < j \leq n} (\mu_i - \mu_j + j - i)} = I \cdot II$$

with

$$I = \frac{\nu_1 + n - 1}{(\lambda_1 + n - 1)(\mu_1 + n - 1)}$$

and $II$ is the rest of the factors. For $I$ we clearly have

$$I \leq \frac{\lambda_1 + \mu_1 + n - 1}{(\lambda_1 + n - 1)(\mu_1 + n - 1)} \leq \frac{1}{\lambda_1 + 1} + \frac{1}{\mu_1 + 1}.$$ 

Thus, we can prove the conjecture once we get the following estimate.

$$(A.1) \quad II \leq C(n).$$

We will introduce the following notations for simplicity.

$$\begin{align*}
\lambda_{ij} &:= \lambda_i - \lambda_j, \\
\lambda_{\mu_i} &:= \lambda_i + \mu_i, \\
\lambda_{\mu_{ij}} &:= \lambda_i - \lambda_j + \mu_i - \mu_j.
\end{align*}$$

(A.2)

Note that it is enough to check the following to get (A.1).

$$II' = \prod_{1 \leq i < j \leq n} (\lambda_{ij} \cdot \frac{\nu_{ij}}{1 \leq i < j \leq n}(i,j) \neq (1,n) \prod_{1 \leq i < j \leq n}(i,j) \neq (1,n)) \leq C'(n)$$

(A.3)

for non-zero integers $\lambda_{ij}$ and $\mu_{ij}$.

Note also that it is not clear whether the case of $SU(n - 1)$ is included in the case of $SU(n)$.
A.1. **The case of SU(3).** Let \( \lambda = (\lambda_1, \lambda_2, \lambda_3 = 0) \) and \( \mu = (\mu_1, \mu_2, \mu_3 = 0) \). From the Littlewood-Richardson rule any \( \pi_{\nu} \subset \pi_{\lambda} \otimes \pi_{\mu} \) must be of the following form:

\[
\begin{align*}
\nu_1 &= \lambda \mu_1 - \alpha_1 - \alpha_2 \\
\nu_2 &= \lambda \mu_2 + \alpha_1 - \beta = \lambda \mu_2 + A \quad \text{(where } A = \alpha_1 - \beta) \\
\nu_3 &= \lambda \mu_3 + \alpha_2 + \beta,
\end{align*}
\]

where \( \alpha_1, \alpha_2 \geq 0 \) are the numbers of “new” boxes with 1 in the second and the third row, respectively, and \( \beta \geq 0 \) is the number of “new” boxes with 2 in the third row.

\[SU(3):\]

\[
\begin{array}{ccccccc}
& & & & & & \\
& \lambda_1 & & & & & \\
& & \lambda_2 & & & & \\
& & & \mu_1 - \alpha_1 - \alpha_2 & & & \\
& & & & \mu_2 - \beta & & \\
1 & \ldots & 1 & 2 & \ldots & 2 & \\
\alpha_2 & & 1 & \alpha_1 & 2 & \ldots & 2 \\
\end{array}
\]

\[
\begin{align*}
\nu_1 &= \lambda_1 + \mu_1 - \alpha_1 - \alpha_2 \\
\nu_2 &= \lambda_2 + \mu_2 + \alpha_1 - \beta \\
\nu_3 &= \alpha_2 + \beta \\
\end{align*}
\]

\[
\begin{align*}
\nu_1' &= \lambda_1 + \mu_1 - \alpha_1 - \alpha_2 - (\alpha_2 + \beta) \\
\nu_2' &= \lambda_2 + \mu_2 + \alpha_1 - \beta - (\alpha_2 + \beta)
\end{align*}
\]

Note that \( \nu = (\nu_1, \nu_2, \nu_3) \cong (\nu_1', \nu_2', 0) \), where \( \nu_1' = \nu_1 - \nu_3 \) and \( \nu_2' = \nu_2 - \nu_3 \).

The Littlewood-Richardson rule tells us that there are two kinds of restrictions on the parameters \( \alpha_1 \) and \( \beta \). The first one comes from “Pieri’s formula”, which says that no two boxes in the same column can have the same number, so that we have

\[
\begin{align*}
\alpha_1 &\leq \lambda_{12}, & \alpha_2 + \beta &\leq \lambda_{23} + \alpha_1 \\
\alpha_2 &\leq \lambda_{23}.
\end{align*}
\]

The second one goes as follows. When we list the new boxes from right to left, starting with the top row and working down, say from 1 to \( k \)-th boxes, the number \( i \) should appear no less than the number \( i + 1 \), so that we have

\[
\begin{align*}
\alpha_1 + \alpha_2 &\leq \mu_{12} + \beta, & \beta &\leq \mu_{23} \\
\alpha_2 &\leq \mu_{12}.
\end{align*}
\]

Now we can extract constraints for \( A = \alpha_1 - \beta \) as follows.

(A.4) \( \left\{
\begin{align*}
A &\leq \min(\lambda_{12}, \mu_{12}) \\
-A &\leq \min(\lambda_{23}, \mu_{23})
\end{align*}
\right\} \).

Thus, the conjecture (A.3) boils down to finding an upper bound (independent of \( \lambda \) and \( \mu \)) of

\[
\begin{align*}
II' &= \frac{(\lambda \mu_{12} - \alpha_1 - \alpha_2 - A)(\mu_{23} + A - (\alpha_2 + \beta))}{\lambda_{12}\lambda_{23}\mu_{12}\mu_{23}},
\end{align*}
\]

where \( A = \alpha_1 - \beta \). Now we make a simple estimate.

\[
\begin{align*}
II' &\leq \frac{(\lambda \mu_{12} - A)(\mu_{23} + A)}{\lambda_{12}\lambda_{23}\mu_{12}\mu_{23}} = II''.
\end{align*}
\]
We will frequently use the following inequality.

\[(A.5) \quad \frac{(a + b) \min(a, b)}{ab} \leq 2, \ a, b > 0\]

Now we divide the cases into two parts, namely (1) \(A \geq 0\), (2) \(A < 0\).

(1) When \(A \geq 0\) by \((A.4)\) and \((A.5)\) we have

\[II'' \leq \frac{\lambda_{12}(\lambda_{23} + A)}{\lambda_{12}\lambda_{23}\mu_{12}\mu_{23}} = \frac{\lambda_{12}\lambda_{23}A}{\lambda_{12}\lambda_{23}\mu_{12}\mu_{23}} \leq 4.\]

(2) When \(A < 0\) we similarly have \(II'' \leq 4\). Note that \(II''\) is symmetric in \((A, \lambda_{12}, \mu_{12})\) and \((-A, \lambda_{23}, \mu_{23})\).

**A.2. The case of \(SU(4)\).** Let \(\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4 = 0)\) and \(\mu = (\mu_1, \mu_2, \mu_3, \mu_4 = 0)\). From the Littlewood-Richardson rule any \(\pi_\nu \subset \pi_\lambda \otimes \pi_\mu\) must be of the following form.

\[
\begin{cases}
\nu_1 = \lambda_1 - \alpha_1 - \alpha_2 - \alpha_3 \\
\nu_2 = \lambda_2 + \alpha_1 - \beta_1 - \beta_2 = \lambda_2 + A \quad (\text{where } A = \alpha_1 - \beta_1 - \beta_2) \\
\nu_3 = \lambda_3 + \alpha_2 + \beta_1 - \gamma = \lambda_3 + B \quad (\text{where } B = \alpha_2 + \beta_1 - \gamma) \\
\nu_4 = \lambda_4 + \alpha_3 + \beta_2 + \gamma,
\end{cases}
\]

where \(\alpha_1, \alpha_2, \alpha_3 \geq 0\) are the numbers of “new” boxes with 1 in the second, the third and the fourth row, \(\beta_1, \beta_2 \geq 0\), the numbers of “new” boxes with 2 in the third and the fourth row, and \(\gamma\), the number of “new” boxes with 3 in the fourth row.

**\(SU(4)\):**

\[
\begin{array}{c}
\lambda_1 \\
\hline
\lambda_2 \\
\hline
\lambda_3 \\
\hline
\mu_1-\alpha_1-\alpha_2-\alpha_3 \\
\hline
\mu_2-\beta_1-\beta_2 \\
\hline
\mu_3-\gamma \\
\end{array}
\]

\[
\begin{cases}
\nu_1 = \lambda_1 + \mu_1 - \alpha_1 - \alpha_2 - \alpha_3 \\
\nu_2 = \lambda_2 + \mu_2 + \alpha_1 - \beta_1 - \beta_2 \quad \Rightarrow \quad \nu'_2 = \nu_{14} \\
\nu_3 = \lambda_3 + \mu_3 + \alpha_2 + \beta_1 - \gamma \quad \Rightarrow \quad \nu'_3 = \nu_{24} \\
\nu_4 = \alpha_3 + \beta_2 + \gamma \quad \Rightarrow \quad \nu'_4 = \nu_{34}
\end{cases}
\]

The Littlewood-Richardson rule tells us that there are two kinds of restrictions on the parameters \(\alpha_i, \beta_i\) and \(\gamma\). The first one coming from “Pieri’s formula” is

\[
\begin{align*}
\alpha_1 & \leq \lambda_{12} \\
\alpha_2 + \beta_1 & \leq \lambda_{33} + \alpha_1 \\
\alpha_2 & \leq \lambda_{23} \\
\alpha_3 + \beta_2 & \leq \lambda_{34} + \alpha_2 \\
\alpha_3 & \leq \lambda_{34}
\end{align*}
\]
The second one by counting is
\[ \begin{align*}
\alpha_1 + \alpha_2 + \alpha_3 & \leq \mu_{12} + \beta_1 + \beta_2, \\
\alpha_2 + \alpha_3 & \leq \mu_{12} + \beta_2, \\
\alpha_3 & \leq \mu_{12}.
\end{align*} \]
Now we can extract constraints for \( A = \alpha_1 - \beta_1 - \beta_2 \) and \( B = \alpha_2 + \beta_1 - \gamma \) as follows.
\[
(A.6) \quad \begin{cases}
A \leq \min(\lambda_{12}, \mu_{12}) \\
B \leq \min(\lambda_{13}, \mu_{13}), \quad \lambda_{23}
\end{cases}
\]
\[
(A.7) \quad \begin{cases}
-B \leq \min(\lambda_{34}, \mu_{34}) \\
-A \leq \min(\lambda_{24}, \mu_{24}), \quad \lambda_{23}
\end{cases}
\]
Note that \( B \) and \(-A\) have two different upper bounds, respectively.
As before we will find an upper bound (independent of \( \lambda \) and \( \mu \)) of II'. We have one principle to follow.

(*) If there is a term which is negative always, then we ignore it.

Following (*) we make a simple estimate.
\[
II' \leq \frac{(\lambda_{12} - A)(\lambda_{13} - B)(\lambda_{23} + A - B)(\lambda_{24} + A)(\lambda_{34} + B)}{\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{24}\lambda_{34}} = II''.
\]
Now we divide the cases into 4 parts, namely (1) \( A, B \geq 0 \), (2) \( A < 0, B \geq 0 \), (3) \( A \geq 0, B < 0 \) and (4) \( A, B < 0 \). We will apply (*) for each cases, and we expand all the factors and estimate term by terms using the above constraints.

(1) When \( A, B \geq 0 \) we have
\[
II'' \leq \frac{\lambda_{12}\lambda_{13}A^2\lambda_{34}}{\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{24}\lambda_{34}} = \frac{A\lambda_{12}}{\lambda_{12}\lambda_{13}}, \quad \frac{A\lambda_{13}}{\lambda_{12}\lambda_{13}}, \quad \frac{\lambda_{34}}{\lambda_{23}\lambda_{24}\lambda_{34}} \leq 8.
\]
Now the most worried case of \( A, B \) remained, namely the following.
\[
\frac{\lambda_{12}\lambda_{13}A^2B}{\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{24}\lambda_{34}} = \frac{A\lambda_{12}}{\lambda_{12}\lambda_{13}}, \quad \frac{A\lambda_{13}}{\lambda_{12}\lambda_{13}}, \quad \frac{B}{\lambda_{23}\lambda_{24}\lambda_{34}} \leq 8.
\]
Note that we have used \( B \leq \lambda_{23} \).

(2) When \( A < 0, B \geq 0 \) we similarly have
\[
II'' \leq \frac{(\lambda_{12} - A)(\lambda_{13} - B)(\lambda_{23} + A - B)(\lambda_{24} + A)(\lambda_{34} + B)}{\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{24}\lambda_{34}}.
\]
Note that the factor \( (\lambda_{12} - A)\lambda_{13} \) can cover up to 2 factors of \( A \) and \( B \). Indeed, the factor \( \lambda_{12}\lambda_{13} \) is the same situation as before and the factor \((-A)\lambda_{13}\) can also cover up to 2 factors of \( A \) and \( B \), since now the term \( \lambda_{12} \) is absent. Similarly, the factor \( \lambda_{24}(\lambda_{34} + B) \) can cover up to 2 factors of \(-A\) and \( B \). Thus, this case is easy since all the terms have \leq 1 factors of \( B \) and \(-A\), respectively.

(3) When \( A \geq 0, B < 0 \) we similarly have
\[
II'' \leq \frac{(\lambda_{24} + A)(\lambda_{34} + 1)(\lambda_{12} - B)(\lambda_{23} + A - B)}{\lambda_{24}\lambda_{34}\lambda_{12}\lambda_{13}\lambda_{23}\lambda_{24}\lambda_{34}}.
\]
All the terms have \leq 2 factors of \( A \) and \(-B\), respectively, so the same approach as in (2) can be applied.
(4) When $A, B < 0$ we can use symmetry between $(A, B, \lambda_{12}, \lambda_{13}, \mu_{12}, \mu_{13})$ and $(-A, -B, \lambda_{24}, \lambda_{34}, \mu_{24}, \mu_{34})$.

A.3. The case of $SU(5)$. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 = 0)$ and $\mu = (\mu_1, \mu_2, \mu_3, \mu_4, \mu_5 = 0)$. By the same approach we end up with finding an upper bound (independent of $\lambda$ and $\mu$) of the following

$$II'' = \frac{III}{\Pi \lambda \cdot \Pi \mu},$$

where

$$III = (\lambda_{12} - A)(\lambda_{13} - B)(\lambda_{14} - C)(\lambda_{23} + A - B)(\lambda_{24} + A - C)(\lambda_{25} + A) \times (\lambda_{34} + B - C)(\lambda_{35} + B)(\lambda_{45} + C)$$

and

$$\Pi \lambda = \lambda_{12}\lambda_{13}\lambda_{14}\lambda_{23}\lambda_{24}\lambda_{25}\lambda_{34}\lambda_{35}\lambda_{45}.$$

$\Pi \mu$ means a similar product for $\mu$.

The constraints are

$$A \leq \min(\lambda_{12},\mu_{12})$$
$$B \leq \min(\lambda_{13},\mu_{13}), \lambda_{23}$$
$$C \leq \min(\lambda_{14},\mu_{14}), \lambda_{34} + \min(\lambda_{23},\mu_{23})$$

and

$$-C \leq \min(\lambda_{45},\mu_{45})$$
$$-B \leq \min(\lambda_{35},\mu_{35}), \lambda_{34}$$
$$-A \leq \min(\lambda_{25},\mu_{25}), \lambda_{23} + \min(\lambda_{34},\mu_{34})$$

Now we divide the cases into 8 parts according to the signs of $A, B$ and $C$. Our basic strategy is the same. We will apply $(*)$ for each cases, and we expand all the factors and estimate term by terms using the above constraints.

(1) When $A, B, C \geq 0$ we have

$$III \leq 
\lambda_{12}\lambda_{13}\lambda_{14}\lambda_{23}\lambda_{24}\lambda_{25} + A(\lambda_{24} + A)(\lambda_{25} + A) \times (\lambda_{34} + B)(\lambda_{35} + B)(\lambda_{45} + C).$$

Note that the factor $\lambda_{12}\lambda_{13}\lambda_{14}$ in the numerator can cover up to 3 factors of $A, B$ and $C$. Actually, much more is possible. When a factor $A$ appears in a specific term, then $\lambda_{23}, \lambda_{24}$ or $\lambda_{25}$ is absent in the same term. Thus, we can use the constraint $B \leq \lambda_{23}$ to cover 1 factor of $B$ for free by (A.3). Moreover, if we use the constraint $B \leq \lambda_{23}$, then in turn $\lambda_{34}$ or $\lambda_{35}$ is absent in the term, so that we can use the constraint $C \leq \lambda_{34} + \min(\lambda_{23},\mu_{23})$ to cover 1 factor of $C$ for free. Note that we again need to split the term. When $\geq 2$ factors of $A$ appear, then we can cover 1 factor of $C$ without using the constraint $B \leq \lambda_{23}$. This observation leads us to the boundedness of each terms.

(2) When $A, B, C < 0$ we can use symmetry between

$$(A, B, C, \lambda_{12}, \lambda_{13}, \lambda_{14}, \mu_{12}, \mu_{13}, \mu_{14})$$

and

$$(-A, -B, -C, \lambda_{25}, \lambda_{35}, \lambda_{45}, \mu_{25}, \mu_{35}, \mu_{45}).$$

(3) For the remaining cases we note the following. The term

$$(\lambda_{12} - A)(\lambda_{13} - B)(\lambda_{14} - C)$$

can cover up to 3 factors of $A, B$ and $C$ and the term

$$(\lambda_{25} + A)(\lambda_{25} + B)(\lambda_{45} + C)$$
can cover up to 3 factors of $-A$, $-B$ and $-C$. Moreover, if we count the number of $A$, $B$ and $C$ appearing and the number of $-A$, $-B$ and $-C$ appearing, then at least one of them is $\leq 3$. Thus, we can apply the same scheme as in (1) and (2).

**Appendix B. The proof of Proposition 4.10**

First, we recall the following lemma from [26].

**Lemma B.1.** Let $0 < \alpha < 1$ and take $\beta \geq \max \left\{ 1, \frac{6}{\alpha(1-\alpha)} \right\}$. Define the functions $p : [0, \infty) \to \mathbb{R}$ and $q : \mathbb{R}^+ \to \mathbb{R}$ by

\[
p(x) = Cx^\alpha - \beta \ln(1+x), \quad q(x) = \frac{p(x)}{x}.
\]

Then $p$ is increasing and $q$ is decreasing on \( \left( \frac{\beta^2}{\alpha(1-\alpha)} \right)^{1/\alpha}, \infty \).

Now we present the second lemma.

**Lemma B.2.** Let $0 < \alpha < 1$, $\beta \geq \max \left\{ 1, \frac{6}{\alpha(1-\alpha)} \right\}$, and let $G$ be a compact group. Suppose that $\tau : \hat{G} \to [0, \infty)$ is a function satisfying

\[
(\text{B.2}) \quad \tau(\sigma) \leq \tau(\pi) + \tau(\pi').
\]

for every $\pi, \pi' \in \hat{G}$ and $\sigma \subset \pi \otimes \pi'$. Let $p$ and $q$ be the functions defined in \( \text{(B.1)} \) and consider the function $\omega : \hat{G} \to [1, \infty)$ defined by

\[
(\text{B.3}) \quad \omega(\pi) = e^{p(\tau(\pi))} = e^{\tau(\pi)q(\tau(\pi))} \quad (\pi \in \hat{G}).
\]

Then, for every $\pi, \pi' \in \hat{G}$ and $\sigma \subset \pi \otimes \pi'$,

\[
\omega(\sigma) \leq M^2 \omega(\pi)\omega(\pi'),
\]

where

\[
(\text{B.4}) \quad M = \max\{e^{p(t)-p(s)-p(r)} : t, s, r \in [0, 2K] \cap \mathbb{Z}\}.
\]

and

\[
K = \left( \frac{\beta^2}{\alpha(1-\alpha)} \right)^{1/\alpha}.
\]

**Proof.** By Lemma \[\text{B.1}\], $p$ is increasing and $q$ is decreasing on $[K, \infty)$. Let $\pi, \pi' \in \hat{G}$ and $\sigma \subset \pi \otimes \pi'$ be a subrepresentation of $\pi \otimes \pi'$. We will prove the statement of the theorem by considering various cases:

**Case I:** $\max\{\tau(\pi), \tau(\pi')\} \leq K$. In this case, $\tau(\sigma) \leq \tau(\pi) + \tau(\pi') \leq 2K$. Hence

\[
\frac{\omega(\sigma)}{\omega(\pi)\omega(\pi')} = e^{p(\tau(\sigma))-p(\tau(\pi))-p(\tau(\pi'))} \leq M.
\]

**Case II:** $\max\{\tau(\pi), \tau(\pi')\} > K$, $\min\{\tau(\pi), \tau(\pi')\} \leq K$, and $\tau(\sigma) < K$. Without loss of generality, we can assume that $\tau(\pi) > K$ and $\tau(\pi') \leq K$. Thus, by Lemma
Case IV: \[\min\{\tau(\pi), \tau(\pi')\} \geq K, \quad \text{and} \quad \tau(\sigma) \leq K.\]

Therefore by comparing the above five cases and considering the fact that \(M \geq e^{-p(0)} = 1\), it follows that

\[
\omega(\sigma) \leq M^2 \omega(\pi)\omega(\pi').
\]

**Proof of Proposition** If we set the function \(\omega\) by

\[
\omega(\pi) = \frac{\gamma_\alpha(\pi)}{\omega_\alpha(\pi)} = e^{\tau_\pi^\alpha - \beta \ln(1 + \tau_\pi^\alpha)} \quad (\pi \in SU(n)),
\]
then by Lemma B.2 for every $\pi, \pi' \in SU(n)$ and $\sigma \subset \pi \otimes \pi'$,
\[
\omega(\sigma) \leq M^2 \omega(\pi) \omega(\pi'),
\]
where $M$ is the constant defined in (B.3). This gives us the conclusion we wanted.

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