Comparison of motivic Chern classes and stable envelopes for cotangent bundles

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Abstract
We consider a complex smooth projective variety equipped with an action of an algebraic torus with a finite number of fixed points. We compare the motivic Chern classes of Białynicki-Birula cells with the \( K \)-theoretic stable envelopes of a cotangent bundle. We prove that under certain geometric assumptions for example for homogenous spaces, these two notions coincide up to normalization.

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1 INTRODUCTION

A torus action on a smooth quasi-projective complex variety induces many cohomological structures. The Białynicki-Birula (BB) decomposition and the localization theorems are the most widely known examples. In this paper we aim to compare two families of \( K \)-theoretic characteristic classes induced by the BB-decomposition: the equivariant motivic Chern classes of BB-cells and the stable envelopes of cotangent bundles. For simplicity, we write BB-decomposition and BB-cells for Białynicki-Birula decomposition and Białynicki-Birula cells, respectively.

Stable envelopes are characteristic classes defined for symplectic varieties equipped with torus action. They are important objects in modern geometric representation theory (see [35] for a survey). Stable envelopes occur in three versions: cohomological [29], \( K \)-theoretic [34, 36] and elliptic [1]. In this paper we focus on the \( K \)-theoretic ones. They depend on a choice of a linearizable line bundle called slope. Their axioms define a unique class for general enough slope, yet the existence of elements satisfying the axioms is still unknown in many cases.

The motivic Chern class is an offshoot of the program of generalizing characteristic classes of tangent bundles to the singular case. It began with the construction of the Chern–Schwartz–MacPherson class in [28] and was widely developed (see, for example, [9, 10, 33] or [42] for...
The common point of many characteristic classes that have been defined is additivity properties with respect to the decomposition of a variety as a union of closed and open subvarieties. For example, the non-equivariant motivic Chern class $mC_y$ (cf. [9]) assigns to every map of varieties $f : X \to M$ a polynomial over the $K$-theory of coherent sheaves of $M$: an element $mC_y(X \to M) \in G(M)[y]$. Its additivity property states that

$$mC_y(X \to M) = mC_y(Z \to M) + mC_y(X \setminus Z \to M),$$

for every closed subvariety $Z \subset X$. Similar properties are satisfied by the Chern–Schwartz–MacPherson class and the Hirzebruch class.

Lately, equivariant versions of many such classes have been defined (for example, [4, 16, 32, 47]). For an algebraic torus $\mathbb{T} \simeq \mathbb{C}^r$, the $\mathbb{T}$-equivariant motivic Chern class (cf. [4, 16]) assigns to every $\mathbb{T}$-equivariant map of varieties $f : X \to M$ a polynomial over the $K$-theory of $\mathbb{T}$-equivariant coherent sheaves of $M$: an element $mC^T_y(X \to M) \in G^T(M)[y]$. It is uniquely defined by the following three properties [16, Section 2.3]:

1. **Additivity:** If a $\mathbb{T}$-variety $X$ decomposes as a union of closed and open invariant subvarieties $X = Y \sqcup U$, then

$$mC^T_y(X \to M) = mC^T_y(Y \to M) + mC^T_y(U \to M).$$

2. **Functoriality:** For an equivariant proper map $f : M \to M'$ we have

$$mC^T_y(X \to M') = f_* mC^T_y(X \to M).$$

3. **Normalization:** For a smooth $\mathbb{T}$-variety $M$ we have

$$mC^T_y(id_M) = \lambda_y(T^*M) := \sum_{i=0}^{\text{rank } T^*M} [\Lambda^i T^*M] y^i.$$

In many cases, one can directly compute this class using the Lefschetz–Riemann–Roch theorem (cf. [12, Theorem 5.11.7]) and the above properties. For examples of computations, see [13, 16, 25]. In [16] (see also [15]) it was found that for $G$-equivariant varieties with a finite number of orbits, the motivic Chern classes $mC^G_y$ of $G$-orbits satisfy axioms similar to those of the stable envelopes.

There is one more family of characteristic classes, called the weight functions, closely connected to the characteristic classes mentioned above. Their relations with other characteristic classes were widely studied, for example, in [16, 27, 38, 39, 41].

In this paper we consider a smooth projective variety $M$ equipped with an action of a torus $A$. Suppose that the fixed point set $M^A$ is finite. Our main result states that under some geometric assumptions on $M$, the stable envelopes for a small enough anti-ample slope and the motivic Chern classes coincide up to normalization. Namely:

**Theorem.** Let $M$ be as above. Consider the cotangent variety $X = T^*M$ with the action of the torus $\mathbb{T} = \mathbb{C}^* \times A$ (where the first factor $\mathbb{C}^*$ acts on fibres by scalar multiplication). Choose any weight chamber $\mathcal{C}$ of the torus $A$ and polarization $T^{1/2} := TM$. Suppose that the variety $M$ satisfies the
local product condition (see Definition 3.10). For any anti-ample \( A \)-linearizable line bundle \( s \) and a sufficiently large integer \( n \), the element

\[
\frac{mC_{-y}^A(M^+_F \rightarrow M)}{y^{\dim M^+_F}} \in K^A(M)[y, y^{-1}] \simeq K^T(M) \simeq K^T(T^n M)
\]

is equal to the \( K \)-theoretic stable envelope \( y^{-\frac{1}{2} \dim M^+_F} Stab_S^\pi_{\mathbb{C}^T} (1_F) \).

A similar comparison was done in [3, 14, 40] for the Chern–Schwartz–MacPherson class in the cohomological setting. The above theorem is a generalization of the previous results of [4, 16] where analogous equality is proved for the flag varieties \( G/B \). The approach of [4] is based on the study of the Hecke algebra action on \( K \)-theory of flag varieties, whereas our strategy is similar to that of [16]. First, we make a change in the definition of the stable envelope such that it coincides with Okounkov’s for general enough slope and is unique for all slopes (Section 3 and Appendix A). By direct check of axioms, we prove that the equality from the theorem holds for the trivial slope (Section 5). Then, we check that the stable envelopes for the trivial slope and a small anti-ample slope coincide (Section 6). Finally, in appendix B we check that the homogenous varieties \( G/P \) satisfy the local product condition mentioned in the theorem.

Our main technical tool is the study of limits of Laurent polynomials (of one or many variables). The limit technique was investigated for motivic Chern classes [16, 25, 48] as well as for stable envelopes [34, 45]. We use it mainly to prove various containments of Newton polytopes.

Homogenous varieties \( G/P \) are our main examples of varieties satisfying the local product condition. The study of characteristic classes and stable envelopes of such varieties is an important theme present in recent research (for example, [2, 37, 40, 44]). A priori the stable envelope is defined for symplectic varieties which admit a proper map to an affine variety. This condition is satisfied by the cotangent bundles to flag varieties of any reductive group and all homogenous varieties of \( GL_n \). There is a weaker condition (Section 3, condition (\( \star \))) which is sufficient to define stable envelope and holds for the cotangent bundle to any variety which satisfies the local product condition.

2 | TOOLS

This section gathers technical results, useful in the subsequent parts of this paper. All considered varieties are assumed to be complex and quasi-projective.

2.1 | Equivariant \( K \)-theory

Our main reference for the equivariant \( K \)-theory is [12]. Let \( X \) be a complex quasi-projective variety equipped with an action of a torus \( \mathbb{T} \). We consider the equivariant \( K \)-theory of coherent sheaves \( G^\mathbb{T}(X) \) and the equivariant \( K \)-theory of vector bundles \( K^\mathbb{T}(X) \). For a smooth variety these two
notions coincide. We use the lambda operations $\lambda_y : K^T(X) \to K^T(X)[y]$ defined by

$$\lambda_y(E) := \sum_{i=0}^{\text{rank } E} [\Lambda^i E] y^i.$$ 

The operation $\lambda_{-1} : K^T(X) \to K^T(X)$ applied to the dual bundle is the $K$-theoretic Euler class. Namely

$$eu(E) = \lambda_{-1}(E^*) .$$

Let $Y \subset X$ be an immersion of a smooth $\mathbb{T}$-invariant locally closed subvariety. Its normal bundle is denoted by

$$\nu(Y \subset X) = \ker(TY \to TX|_Y) \in K^T(Y).$$

We denote by $eu(Y \subset X)$ the Euler class of normal bundle. Namely

$$eu(Y \subset X) := \lambda_{-1}(\nu^*(Y \subset X)) \in K^T(Y).$$

**Definition 2.1.** Consider a $\mathbb{T}$-variety $X$. For an element $a \in G^T(X)$ and a closed invariant subvariety $Y \subset X$ we say that $\text{supp}(a) \subset Y$ if and only if $a$ lies in the image of a pushforward map

$$G^T(Y) \xrightarrow{i_*} G^T(X).$$

The short exact sequence (cf. [12, Proposition 5.2.14])

$$G^T(Y) \xrightarrow{i_*} G^T(X) \to G^T(X \setminus Y) \to 0$$

implies that $\text{supp}(a) \subset Y$ is equivalent to $a|_{X \setminus Y} = 0$.

**Remark 2.2.** Note that for an element $a \in K^T(X)$, the support of $a$ is not a well-defined subset of $X$. We can only define the notion $\text{supp}(a) \subset Y$ for a closed subvariety $Y \subset X$. The fact that $\text{supp}(a) \subset Y_1$ and $\text{supp}(a) \subset Y_2$ does not imply that $\text{supp}(a) \subset Y_1 \cap Y_2$.

**Proposition 2.3.** Consider a reducible $\mathbb{T}$-variety $X = X_1 \cup X_2$. Denote the inclusions of the closed subvarieties $X_1$ and $X_2$ by $i$ and $j$, respectively. Then the pushforward map

$$i_* + j_* : G^T(X_1) \oplus G^T(X_2) \to G^T(X)$$

is an epimorphism.

**Proof.** Denote by $U_1$ and $U_2$ the complements of the closed sets $X_1$ and $X_2$. Note that due to the exact sequence

$$G^T(X_1) \xrightarrow{i_*} G^T(X) \to G^T(U_1) \to 0$$
it is enough to prove that the composition

\[ \alpha : G^\top(X_2) \xrightarrow{j_*} G^\top(X) \to G^\top(U_1) \]

is an epimorphism. Note that \( U_1 \cap X_2 = U_1 \), and by pushforward–pullback argument the map \( \alpha \) is equal to the restriction to the open subset

\[ G^\top(X_2) \to G^\top(U_1). \]

Such restriction is an epimorphic map due to the exact sequence of a closed immersion. \( \square \)

Consider a \( \mathbb{C}^\times \)-variety \( F \) for which the action is trivial. Every equivariant vector bundle \( E \in \text{Vect}^{\mathbb{C}^\times}(F) \) decomposes as a sum of \( \mathbb{C}^\times \)-eigenspaces

\[ E = \bigoplus_{n \in \mathbb{Z}} E_n. \]

The sum \( E^+ = \bigoplus_{n > 0} E_n \) is called the attracting (or positive) part of \( E \) while the sum \( E^- = \bigoplus_{n < 0} E_n \) is called the repelling (or negative) part. The assignment of the positive (or negative) part induces a map of the equivariant \( K \)-theory. Namely:

**Proposition 2.4.** Let \( \sigma \subset \mathbb{T} \) be a one-dimensional subtorus. Suppose that \( F \) is a \( \mathbb{T} \)-variety for which the action of \( \sigma \) is trivial. Then taking the attracting part with respect to the torus \( \sigma \) induces a well-defined map

\[ K^\top(F) \to K^\top(F). \]

An analogous result holds for the repelling part. More generally, taking the direct summand corresponding to a chosen character of \( \sigma \) induces such a map.

**Proof.** \( \mathbb{T} \)-equivariant maps of vector bundles preserve the weight decomposition with respect to the torus \( \sigma \). Thus, any exact sequence of \( \mathbb{T} \)-vector bundles splits into a direct sum of sequences corresponding to characters of \( \sigma \). It follows that taking the part corresponding to any subset of characters preserves exactness. \( \square \)

**Remark 2.5.** Denote by \( R(\sigma) \) the representation ring of the torus \( \sigma \). Proposition 2.4 is a consequence of an isomorphism

\[ K^\top(F) \cong K^\top/\sigma(F) \otimes R(\sigma). \]

### 2.2 BB-decomposition

The BB-decomposition was introduced in [6] and further studied in [7] (see also [11] for a survey). We recall here its definition and fundamental properties. Consider a smooth Let \( \sigma = \mathbb{C}^\times \) be a one-dimensional torus and \( X \) a \( \sigma \)-variety.
Definition 2.6. Let $F$ be a component of the fixed point set $X^\sigma$. The positive BB-cell of $F$ is the subset

$$X_F^+ = \{ x \in X \mid \lim_{t \to 0} t \cdot x \in F \}.$$ 

Analogously the negative BB-cell of $F$ is the subset

$$X_F^- = \{ x \in X \mid \lim_{t \to \infty} t \cdot x \in F \}.$$ 

It follows from [6] that

Theorem 2.7.

1. The BB-cells are locally closed, smooth, algebraic subvarieties of $X$. Moreover, we have the equality of vector bundles

$$T(X_F^+) |_F = (TX |_F)^+ \oplus TF.$$ 

2. There exists an algebraic morphism

$$\lim_{t \to 0} : X_F^+ \to F.$$ 

3. Suppose that the variety $X$ is projective. Then there is a set decomposition (called BB-decomposition)

$$X = \bigsqcup_{i \in X^\sigma} X_F^+.$$ 

4. Suppose that the variety $X$ is projective. Then the morphism $\lim_{t \to 0} : X_F^+ \to F$ is an affine bundle.

5. Suppose that a bigger torus $\sigma \subset \mathbb{T}$ acts on $X$. Then the BB-cells (defined by the action of $\sigma$) are $\mathbb{T}$-equivariant subvarieties.

6. The BB-decomposition induces a partial order on the fixed point set $X^\sigma$, defined by the transitive closure of the relation

$$F_2 \in X_{F_1}^+ \Rightarrow F_1 > F_2.$$ 

Definition 2.8 (cf. [29, Paragraph 3.2.1]). Suppose that $X$ is a smooth $\mathbb{T}$-variety. Consider the space of cocharacters

$$t := \text{Hom}(\mathbb{C}^*, \mathbb{T}) \otimes_{\mathbb{Z}} \mathbb{R}.$$ 

For a fixed points component $F \subset X^\mathbb{T}$, denote by $v_1^F, \ldots, v_{\text{codim} F}^F$ the torus weights appearing in the normal bundle. A weight chamber is a connected component of the set

$$t \setminus \bigcup_{F \subset X^\mathbb{T}, i \leq \text{codim} F} \{ v_i^F = 0 \}. $$
Suppose that a torus $\mathbb{T}$ acts on a smooth variety $X$. For a one-dimensional subtorus $\sigma \subset \mathbb{T}$ and a weight chamber $\mathcal{C}$ we write $\sigma \in \mathcal{C}$ when the cocharacter of $\sigma$ belongs to the chamber $\mathcal{C}$.

**Proposition 2.9.** Let $X$ be a smooth $\mathbb{T}$-variety. Consider one-dimensional subtorus $\sigma \subset \mathbb{T}$ such that $\sigma \in \mathcal{C}$ for some weight chamber $\mathcal{C}$. Then the fixed point sets $X^\mathbb{T}$ and $X^\sigma$ are equal.

**Proposition 2.10.** Let $X$ be a smooth $\mathbb{T}$-variety. Choose a weight chamber $\mathcal{C}$. Consider one-dimensional subtori $\sigma_1, \sigma_2 \subset \mathbb{T}$ such that $\sigma_1, \sigma_2 \in \mathcal{C}$. Then the tori $\sigma_1$ and $\sigma_2$ induce the same decomposition of the normal bundle to the fixed point set into the attracting and the repelling part. Moreover, the BB-decompositions with respect to these tori are equal.

**Proof.** The only non-trivial part is the equality of the BB-decompositions. It is a consequence of [22, Theorem 3.5]. Alternatively, thanks to the Sumihiro theorem [46] it is enough to prove the lemma for $X$ equal to the projective space. In this case, the proof is straightforward. \qed

### 2.3 Symplectic varieties

**Definition 2.11** [36, Section 2.1.2]. Consider a smooth symplectic variety $(X, \omega)$ equipped with an action of a torus $\mathbb{T}$. Assume that the symplectic form $\omega$ is an eigenvector of $\mathbb{T}$, let $h$ be its character. A polarization is an element $T^{1/2} \in K^\mathbb{T}(X)$ such that

$$T^{1/2} \oplus C_{-h} \otimes (T^{1/2})^* = TX \in K^\mathbb{T}(X).$$

**Remark 2.12.** Note that according to the above definition, a polarization is an element of the $K$-theory, thus it may be a virtual vector bundle.

### 2.4 Newton polytopes

Let $R$ be a commutative ring with unit and $\Lambda$ a lattice of finite rank. Consider a polynomial $f \in R[\Lambda]$. The Newton polytope $N(f) \subset \Lambda \otimes_\mathbb{Z} \mathbb{R}$ is a convex hull of lattice points corresponding to the non-zero coefficients of the polynomial $f$. We recall elementary properties of Newton polytopes:

**Proposition 2.13.** Let $R$ be a commutative ring with unit. For any Laurent polynomials $f, g \in R[\Lambda]$.

(a) $N(fg) \subseteq N(f) + N(g)$.

(b) $N(fg) = N(f) + N(g)$ when the ring $R$ is a domain.

(c) $N(fg) = N(f) + N(g)$ when the coefficients of the class $f$ corresponding to the vertices of the polytope $N(f)$ are not zero divisors.

(d) $N(f + g) \subseteq \text{conv}(N(f), N(g))$.

(e) Let $\theta : R \rightarrow R'$ be a homomorphism of rings and $\theta' : R[\Lambda] \rightarrow R'[\Lambda]$ its extension. Then $N(\theta'(f)) \subseteq N(f)$. 

Consider a smooth variety $X$ equipped with an action of a torus $\mathbb{T}$. Choose a subtorus $A \subset \mathbb{T}$. For a fixed point set component $F \subset X$ and an element $a \in K^\mathbb{T}(X)$ we want to define the Newton polytope

$$N^A(a|_F) \subset \text{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R} := a^*.$$

It is possible due to the following proposition:

**Proposition 2.14.** Let $F$ be a smooth $\mathbb{T}$-variety. Let $A \subset \mathbb{T}$ be a subtorus which acts trivially on $F$. Any splitting of the inclusion $A \subset \mathbb{T}$ induces isomorphism

$$\alpha : K^\mathbb{T}(F) \approx K^{\mathbb{T}/A}(F) \otimes_{\mathbb{Z}} R(A) \approx K^{\mathbb{T}/A}(F)[\text{Hom}(A, \mathbb{C}^*)],$$

where $R(A)$ denotes the representation ring of the torus $A$. For an element $a \in K^{\mathbb{T}}(F)$ we consider the Newton polytope $N^A(a) \subset a^*$ defined by the polynomial $\alpha(a)$. The isomorphism $\alpha$ depends on the choice of splitting, yet the Newton polytope is independent of it.

**Proof.** Consider two splittings $s_1, s_2 : \mathbb{T}/A \rightarrow \mathbb{T}$. Denote by $\alpha_1, \alpha_2$ the induced isomorphisms

$$K^{\mathbb{T}}(F) \rightarrow K^{\mathbb{T}/A}(F)[\text{Hom}(A, \mathbb{C}^*)].$$

Note that the quotient $s_2^{-1} s_1$ induces a group homomorphism $h : \mathbb{T}/A \rightarrow A$. Consider an arbitrary class $E \in K^{\mathbb{T}/A}(F)$ and a character $\chi \in \text{Hom}(A, \mathbb{C}^*)$. Direct calculation provides us with the formula

$$\alpha_2 \circ \alpha_1^{-1}(Ee^\chi) = (E \otimes C_{\chi \circ h})e^\chi.$$

Thus, the Newton polytope is independent of the choice of splitting. \hfill \Box

**Remark 2.15.** Consider the situation as in Proposition 2.14. Let $E$ be a $\mathbb{T}$-vector bundle over $F$. Then the Euler class $eu(E)$ satisfies the assumption of Proposition 2.13(c). To see this use the weight decomposition of $E$ with respect to the torus $A$. Note that for a vector bundle $V$ with an action of $A$ given by a single character the Newton polytope $N^A(eu(V))$ is an interval. Moreover, the coefficients corresponding to the ends of this interval are invertible (they are equal to classes of the line bundles 1 and $\det V^*$).

At the end of this section we want to mention the behaviour of polytope $N^A$ after restriction to one-dimensional subtorus of $A$. Namely let $\sigma \subset A$ be a one-dimensional subtorus. Denote by

$$|_{\sigma} : K^A(pt) \rightarrow K^{\sigma}(pt)$$

the induced map on the $K$-theory and by

$$\pi_{\sigma} : \text{Hom}(A, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R} \rightarrow \text{Hom}(\sigma, \mathbb{C}^*) \otimes_{\mathbb{Z}} \mathbb{R}$$
the induced map on the characters.

**Proposition 2.16.** For an element \( a \in K^\T(F) \) there exists a finite union of hyperplanes \( K \) in the vector space of cocharacters such that for all one-dimensional subtori \( \sigma \) whose cocharacter does not belong to \( K \)

\[
N^\sigma(a|_\sigma) = \pi_\sigma(N^A(a)).
\]

**Proposition 2.17.** Let \( M \) be a smooth \( A \)-variety. Consider a one-dimensional subtorus \( \sigma \subset A \) such that \( \sigma \in \mathcal{C} \) for some weight chamber \( \mathcal{C} \). Let \( F \) be a component of the fixed point set \( M^A \). Then the point 0 is a vertex of the polytope \( N^A(\nu_F) \). Moreover, the point \( \pi_\sigma(0) \) is the minimal term of the line segment \( \pi_\sigma(N^A(\nu_F)) \).

### 3 STABLE ENVELOPES FOR ISOLATED FIXED POINTS

In this section we recall the definition of the \( K \)-theoretic stable envelope introduced in [34, 36] (see also [4, 39, 45] for special cases) in the case of isolated fixed points. In Appendix A we give a rigorous proof that such classes are unique (Proposition 3.6). Using Okounkov's definition it is true only for a general enough slope. We introduce a weaker version of the axioms to bypass this problem.

We use the following notations and assumptions

- \( \mathbb{T} \simeq \mathbb{C}^r \) is an algebraic torus.
- \( (X, \omega) \) is a smooth symplectic \( \mathbb{T} \)-variety.
- \( A \subset \mathbb{T} \) is a subtorus which preserves the symplectic form.
- We assume that the fixed point set \( X^A \) is finite (this implies that \( X^A = X^\T \)).
- We assume that \( \omega \) is an eigenvector of \( \mathbb{T} \) and denote its character by \( h \in \text{Hom}(\mathbb{T}, \mathbb{C}^*) \).
- \( \mathcal{C} \subset \mathfrak{a} \) is a weight chamber.
- For a fixed point \( F \in X^A \), we denote by \( X^+_F \) its positive BB-cell according to the chamber \( \mathcal{C} \) (cf. Definition 2.6 and Proposition 2.10).
- We denote by \( \geq \) the partial order on the fixed point set \( X^A \) induced by the chamber \( \mathcal{C} \) (cf. Theorem 2.7(6)).
- For a fixed point \( F \in X^A \), let

\[
\nu_F := \nu(F \subset X) \simeq TX|_F,
\]

be the normal bundle to the inclusion \( F \subset X \). Let \( \nu_F = \nu^+_F \oplus \nu^-_F \) be its decomposition induced by the chamber \( \mathcal{C} \) (cf. Proposition 2.10).
- \( T^{1/2} \in K^\T(X) \) is a polarization (cf. Definition 2.11).
- For a fixed point \( F \in X^A \), we denote by \( T^{1/2}_{F, > 0} \in K^\T(F) \) the attracting part of \( (T^{1/2})|_F \in K^\T(F) \) (cf. Proposition 2.4).
- \( s \in \text{Pic}(X) \otimes \mathbb{Q} \) is a rational \( A \)-linearizable line bundle which we call slope.
- For a fixed point \( F \in X^A \), we denote by \( 1_F \) the multiplicative unit in the ring \( K^\T(F) \) given by the class of the equivariant structure sheaf.

Moreover, we assume that the variety \( X \) satisfies the following condition:

\( (\star) \) The set \( \bigcup_{F \in X^A} X^+_F \) is closed.
Remark 3.1. The (⋆) condition implies that for any fixed point $F_0 \in X^A$ the set $\bigcup_{F \in X^A, F \leq F_0} X^+_F$ is closed.

In Okounkov’s papers stronger condition on $X$ is assumed. Namely it is required that $X$ admits an equivariant proper map to an affine variety cf. [29, Paragraph 3.1.1]. Existence of such a map implies the condition (⋆) cf. [29, Lemma 3.2.7]. It turns out that the condition (⋆) is sufficient to prove uniqueness of the stable envelope (cf. Proposition 3.6).

Definition 3.2 (cf. [36, Chapter 2]). The stable envelope is a morphism of $K^\square(pt)$-modules

$$\text{Stab}^s_{\mathbb{G},T^{1/2}} : K^\square(X^A) \to K^\square(X)$$

satisfying three properties:

(a) for any fixed point $F$

$$\text{supp} \left( \text{Stab}^s_{\mathbb{G},T^{1/2}}(1_F) \right) \subseteq \bigcup_{F' \leq F} X^+_F ;$$

(b) for any fixed point $F$

$$\text{Stab}^s_{\mathbb{G},T^{1/2}}(1_F) \mid_F = e^u(v^{-F}) \frac{(-1)^{\text{rank}_F T^{1/2}_{F,>0}} h^{1/2 \cdot \text{rank}_F T^{1/2}_{F,>0}}}{\text{det} T^{1/2}_{F',>0}} ;$$

(c) choose any $A$-linearization of the slope $s$. For a pair of fixed points $F', F$ such that $F' < F$ we demand containment of Newton polytopes

$$N^A \left( \text{Stab}^s_{\mathbb{G},T^{1/2}}(1_F) \mid_{F'} \right) + s_{|F'} \subseteq \left( N^A(e^u(v^{-F'})) \setminus \{0\} \right) - \text{det} T^{1/2}_{F',>0} + s_{|F'} ,$$

where the Newton polytopes are defined as in Proposition 2.14. An addition of the restriction of a line bundle means translation by its character.

For a comparison of the above definition with the one given in [34, 36], see Appendix A.

Remark 3.3. To define the element $h^{1/2 \cdot \text{rank}_F T^{1/2}_{F,>0}}$ in axiom (b) one may need to pass to the double cover of the torus $T$ (cf. [36, Paragraph 2.1.4]). To avoid this problem we consider normalized version of the stable envelope (see expression (1)).

Remark 3.4. Okounkov’s definition differs from the one given above in axiom (c). In the paper [36] weaker set containment is required

$$N^A \left( \text{Stab}^s_{\mathbb{G},T^{1/2}}(1_F) \mid_{F'} \right) + s_{|F} \subseteq N^A(e^u(v^{-F})) - \text{det} T^{1/2}_{F',>0} + s_{|F'} .$$

It defines the stable envelope uniquely only for a general enough slope (cf. [34, Proposition 9.2.2; 36, Paragraph 2.1.8] and Example A.1). With our version of axioms the stable envelope is unique for any choice of slope and coincides with Okounkov’s for general enough slope.
For simplicity, we omit the weight chamber and polarization in the notation. The stable envelope is determined by the set of elements

\[ \text{Stab}^s(F) := h^{-\frac{1}{2} \text{rank} T_{F,>0}} \text{Stab}(1_F), \]

indexed by the fixed point set \( X^A \). It leads to the following equivalent definition.

**Definition 3.5.** The \( K \)-theoretic stable envelope is a set of elements \( \text{Stab}^s(F) \in K^T(X) \) indexed by the fixed point set \( X^A \), such that

(a) for any fixed point \( F \)

\[ \text{supp}(\text{Stab}^s(F)) \subset \bigcup_{F' \leq F} X^A_{F'} ; \]

(b) for any fixed point \( F \)

\[ \text{Stab}^s(F)|_F = eu(y_F^-)\left(\frac{-1}{\text{rank} T_{F,>0}}\right) ; \]

(c) choose any \( A \)-linearization of the slope \( s \). For a pair of fixed points \( F', F \) such that \( F' < F \) we demand containment of Newton polytopes

\[ N^A(\text{Stab}^s(F)|_{F'}) + s|_F \subseteq \left( N^A(eu(y_{F'}^-)) \setminus \{0\} \right) - \text{det} T_{F',>0}^{1/2} + s|_{F'} . \]

**Proposition 3.6.** Under the assumptions given at the beginning of this section the stable envelope (Definitions 3.2, 3.5) is unique.

The simplest example of a symplectic variety is the cotangent bundle to a smooth variety. It is natural to ask whether such a variety satisfies the assumptions needed to define \( K \)-theoretic stable envelope. Consider a smooth \( A \)-variety \( M \) with a finite number of fixed points. Consider the cotangent variety \( X = T^*M \) with the action of the torus \( T = A \times \mathbb{C}^* \) such that \( \mathbb{C}^* \) acts on the fibres by scalar multiplication. The fixed point set of this action is finite. In fact, we have equalities

\[ X^T = X^A = M^A. \]

The variety \( X \) is equipped with the canonical symplectic non-degenerate form \( \omega \). This form is preserved by the torus \( A \) and it is an eigenvector of the torus \( T \) with character corresponding to the projection on the second factor. Denote this character by \( y \). The subbundle \( TM \subset TX \) satisfies the polarization condition (see Definition 2.11). Choose a weight chamber \( \mathcal{C} \) of the torus \( A \) and a one-dimensional subtorus \( \sigma \subset A \) such that \( \sigma \in \mathcal{C} \). The BB-cells of \( \sigma \) in \( X \) are the conormal bundles to the BB-cells of \( \sigma \) in \( M \). The condition (\( \star \)) means that the set

\[ \bigcup_{F \in M^A} y^*(M^+_F) \]

is a closed subset of \( T^*M \). This condition is not always satisfied.
Remark 3.7. Consider a smooth manifold $M$ decomposed as a disjoint union of smooth locally closed submanifolds

$$M = \bigsqcup_i S_i.$$  

The disjoint union of the conormal bundles to strata is a closed subset of the cotangent bundle $T^*M$ if and only if the decomposition satisfies the Whitney condition (A) (cf. [18, Exercise 2.2.4]). Thus, the $(\star)$ condition can be seen as an algebraic counterpart of the Whitney condition (A).

Example 3.8. Consider the projective space $\mathbb{P}^2$ with the action of the diagonal torus of $GL_3(\mathbb{C})$. Choose a general enough subtorus $\sigma$. Let $x$ be the middle fixed point in the Bruhat order. Then the cotangent bundle to the blow-up $T^*(Bl_x \mathbb{P}^2)$ does not satisfy the $(\star)$ condition.

Example 3.9.

(1) The cotangent bundle to any flag variety $G/B$ admits an equivariant projective morphism to an affine variety (cf. [12, Section 3.2]) so it satisfies $(\star)$ condition.

(2) The cotangent bundle to any $GL_n$ homogenous variety is a Nakajima quiver variety (cf. [30, Section 7]) so it admits an equivariant projective morphism to an affine variety [17, Section 5.2; 31, Section 3].

We introduce a stronger condition on the BB-cells of $M$ which implies the $(\star)$ condition for $X = T^*M$ and is satisfied by the homogenous varieties (cf. Theorem B.1).

Definition 3.10. Consider a projective smooth variety $M$ equipped with an action of a torus $A$ with a chosen one-dimensional subtorus $\sigma$. Suppose that the fixed point set $M^\sigma$ is finite. We say that $M$ satisfies the local product condition if for any fixed point $x \in M^\sigma$ there exists an $A$-equivariant Zariski open neighbourhood $U$ of $x$ and $A$-variety $Z_x$ such that

(1) there exists an $A$-equivariant isomorphism

$$\theta : U \simeq (U \cap M^+_x) \times Z_x;$$

(2) for any fixed point $y \in M^\sigma$ there exists a subvariety $Z'_{x,y} \subset Z_x$ such that $\theta$ induces isomorphism:

$$U \cap M^+_y \simeq (U \cap M^+_x) \times Z'_{x,y}.$$  

Proposition 3.11. Suppose that a projective smooth $\mathbb{C}^*$-variety $M$ satisfies the local product condition. Then the cotangent variety $T^*M$ satisfies the $(\star)$ condition.

Proof. It is enough to prove that for every fixed point $F_0 \in M^\sigma$ there is an inclusion

$$\nu^*(M^+_F \subset M) \subset \bigsqcup_{F \in M^\sigma} \nu^*(M^+_F \subset M).$$

It is equivalent to claim that for arbitrary fixed points $F_0, F$

$$\nu^*(M^+_F \subset M) \cap T^*M_{|M^+_F} \subset \nu^*(M^+_F \subset M).$$
Denote by $U$ the neighbourhood of $F$ from the definition of the local product condition. All of the subsets in the above formula are $\sigma$ equivariant. Thus, it is enough to prove that

$$\nu^*(M^+_F \cap U \subset U) \cap T^*U \subset \nu^*(M^+_F \cap U \subset U).$$

The local product property implies existence of isomorphisms

$$U \simeq M^+_F \times Z, \quad M^+_F \simeq M^+_F \times \{pt\}, \quad M^+_F \simeq M^+_F \times Z',$$

for some subvariety $Z' \subset Z$ and point $pt \in Z$. Denote by $E$ the subbundle

$$M^+_F \times T^*Z \subset T^*U.$$

Note that

$$\nu^*(M^+_F \cap U \subset U) = E_{|M^+_F}$$

$$\nu^*(M^+_F \cap U \subset U) = M^+_F \times \nu^*(Z' \subset Z) \subset E_{|M^+_F}$$

Thus

$$\nu^*(M^+_F \cap U \subset U) \cap T^*U_{|M^+_F} \subset \overline{E_{|M^+_F}} \cap T^*U_{|M^+_F}$$

$$
\subset E \cap T^*U_{|M^+_F} = E_{|M^+_F} = \nu^*(M^+_F \cap U \subset U). \quad \square
$$

4 | MOTIVIC CHERN CLASS

The motivic Chern class is defined in [9]. The equivariant version is due to [4, Section 4] and [16, Section 2]. Here we recall the definition of the torus equivariant motivic Chern class. Consult [4, 16] for a detailed account.

**Definition 4.1** (After [16, Section 2.3]). Let $A$ be an algebraic torus. The motivic Chern class assigns to every $A$-equivariant map of quasi-projective $A$-varieties $f : X \rightarrow M$ an element

$$mC_y^A(f) = mC_y^A(X \rightarrow M) \in G^A(M)[y]$$

such that the following properties are satisfied

1. **Additivity**: If a $A$-variety $X$ decomposes as a union of closed and open invariant subvarieties $X = Y \cup U$, then

$$mC_y^A(X \rightarrow M) = mC_y^A(Y \rightarrow M) + mC_y^A(U \rightarrow M).$$

2. **Functoriality**: For an equivariant proper map $f : M \rightarrow M'$ we have

$$mC_y^A(X \rightarrow M') = f_* mC_y^A(X \rightarrow M).$$
3. Normalization: For a smooth $A$-variety $M$ we have

$$mC^A_y(id_M) = \lambda_y(T^*M) := \sum_{i=0}^{\text{rank } T^*M} [\Lambda^i T^*M] y^i.$$ 

The motivic Chern class is the unique assignment satisfying the above properties.

5 | COMPARISON WITH THE MOTIVIC CHERN CLASSES

In this section we aim to compare the stable envelopes for the trivial slope with the motivic Chern classes of BB-cells. Our main results are

**Proposition 5.1.** Let $M$ be a projective, smooth variety equipped with an action of an algebraic torus $A$. Suppose that the fixed point set $M^A$ is finite. Consider the variety $X = T^*M$ equipped with the action of the torus $T = \mathbb{C}^* \times A$. Choose any weight chamber $\mathcal{C}$ of the torus $A$, polarization $T^{1/2} = TM$ and the trivial line bundle $\theta$ as a slope. Then, the elements

$$mC^A_y(M^+_F \to M) \frac{y^{\dim M^+_F}}{y^{\dim M^+_F}} \in K^A(X)[y, y^{-1}] \simeq K^T(X) \simeq K^T(T^*X)$$

satisfy axioms (b) and (c) of the stable envelope $Stab^\theta(F)$.

**Remark 5.2.** In this proposition we do not assume that $M$ satisfies the local product condition or even that $X$ satisfies the $(\star)$ condition.

**Theorem 5.3.** Consider the situation such as in Proposition 5.1. Suppose that the variety $M$ with the action of a one-dimensional torus $\sigma \in \mathcal{C}$ satisfies the local product condition (Definition 3.10). Then the element

$$mC^A_y(M^+_F \to M) \frac{y^{\dim M^+_F}}{y^{\dim M^+_F}} \in K^A(X)[y, y^{-1}] \simeq K^T(X) \simeq K^T(T^*X)$$

is equal to the $K$-theoretic stable envelope $Stab^\theta(F)$.

Our main examples of varieties satisfying the local product property are homogenous spaces (see Appendix B). Let $G$ be a reductive, complex Lie group with a chosen maximal torus $A$. Let $B$ be a Borel subgroup and $P$ a parabolic subgroup. We consider the action of the torus $A$ on the variety $G/P$.

A choice of weight chamber $\mathcal{C} \subset \mathfrak{a}$ induces a choice of Borel subgroup $B_\mathcal{C} \subset G$. Let $F \in (G/P)^A$ be a fixed point. It is a classical fact that the BB-cell $(G/P)^+_F$ (with respect to the chamber $\mathcal{C}$) coincides with the $B_\mathcal{C}$-orbit of $F$. These orbits are called Schubert cells.

**Corollary 5.4.** In the situation presented above the stable envelopes for the trivial slope are equal to the motivic Chern classes of Schubert cells

$$mC^A_y((G/P)^+_F \to G/P) \frac{y^{\dim(G/P)^+_F}}{y^{\dim(G/P)^+_F}} = y^{-\frac{1}{2} \dim(G/P)^+_F} Stab^\theta_{\mathcal{C}, T(G/P)}(1_F).$$
Proof. Theorem B.1 implies that homogenous varieties satisfy the local product condition. Thus, the corollary follows from Theorem 5.3.

Remark 5.5. In the case of flag varieties $G/B$, our results for the trivial slope agree with the previous results of [4, Theorem 8.5] for a small anti-ample slope up to a change of $y$ to $y^{-1}$. This difference is a consequence of the fact that in [4] the inverse action of $\mathbb{C}^*$ on the fibres of cotangent bundle is considered.

Before the proof of Theorem 5.3 we make several simple observations. Let $\mathring{\nu}_F \cong TM|_F$ denote the normal space to the fixed point $F$ in $M$. Denote by $\mathring{\nu}_F^-$ and $\mathring{\nu}_F^+$ its decomposition into the positive and negative parts induced by the weight chamber $\mathcal{C}$. Let

$$\nu_F \cong TX|_F \cong TM|_F \oplus (T^*M|_F \otimes \mathbb{C}_y)$$

denote the normal space to the fixed point $F$ in the variety $X = T^*M$. It is a straightforward observation that

$$\nu_F^- \cong \mathring{\nu}_F^- \oplus y(\mathring{\nu}_F^+)^*,$$
$$\nu_F^+ \cong \mathring{\nu}_F^+ \oplus y(\mathring{\nu}_F^-)^*,$$
$$T_{1/2}^{1/2}F,>0 = \mathring{\nu}_F^+.$$ 

In the course of proofs we use the following computation:

**Lemma 5.6.** Let $V$ be a $\mathbb{T}$-vector space. We have an equality

$$\frac{\lambda_{-1}(y^{-1}V)}{\det V} = \frac{\lambda_{-y}(V^*)}{(-y)^{\dim V}}$$

in the $\mathbb{T}$-equivariant $K$-theory of a point.

Proof. Both sides of the formula are multiplicative with respect to the direct sums of $\mathbb{T}$-vector spaces. Every $\mathbb{T}$-vector space decomposes as a sum of one-dimensional spaces, so it is enough to check the equality for $\dim V = 1$. Then it simplifies to trivial form:

$$\frac{1 - (\alpha/y)}{\alpha} = \frac{1 - (y/\alpha)}{-y},$$

where $\alpha$ is the character of the action of the torus $\mathbb{T}$ on the linear space $V$.

Proof of Proposition 5.1. We start the proof by checking axiom (b). We need to show that

$$\frac{mC^A_{-y}(M_F^+ \to M)|_F}{y^{\dim M_F^+}} = eu(\nu_F^-)(-1)^{\rank T_{1/2}^{1/2}F,>0} \det T_{1/2}^{1/2}F,>0.$$
which is equivalent to

\[
\frac{mC_A^{y}(M_F^+ \to M)_{|F'}}{(-y)^{\dim \varphi_F^+}} = \text{eu}(\varphi_{F'}) \frac{\lambda_{-1}(y^{-1} \varphi_{F'}^+)}{\det \varphi_{F'}^+}.
\]

(2)

The BB-cell $M_F^+$ is a locally closed subvariety. Choose an open neighbourhood $U$ of the fixed point $F$ in $M$ such that the morphism $M_F^+ \cap U \subset U$ is a closed immersion. The functorial properties of the motivic Chern class (cf. [4, Theorem 4.2; 16, Paragraph 2.3]) imply that

\[
mC_A^{y}(M_F^+ \to M)_{|F'} = mC_A^{y}(M_F^+ \cap U \to U)_{|F'} = i_* mC_A^{y}(id_{M_F^+ \cap U})_{|F'} = i_* (\lambda_{-y} (T^* (M_F^+ \cap U)))_{|F'} = \lambda_{-y} (\varphi_{F'}^+) \text{eu}(\varphi_{F'}). 
\]

So the left-hand side of expression (2) is equal to

\[
\text{eu}(\varphi_{F'}) \frac{\lambda_{-y} (\varphi_{F'}^+)}{(-y)^{\dim \varphi_{F'}^+}}.
\]

Lemma 5.6 implies that the right-hand side is also of this form.

We proceed to axiom (c). Consider a pair of fixed points $F, F'$ such that $F' < F$. We need to prove the inclusion

\[
N^A \left( \frac{mC_A^{y}(M_F^+ \to M)_{|F'}}{y^{\dim M_F^+}} \right) \subseteq N^A(\text{eu}(\nu_{F'}^-)) - \det T^{1/2}_{F',>0}
\]

and take care of the distinguished point

\[
- \det T^{1/2}_{F',>0} \notin N^A \left( \frac{mC_A^{y}(M_F^+ \to M)_{|F'}}{y^{\dim M_F^+}} \right).
\]

(4)

Let us concentrate on the inclusion (3). There is an equality of polytopes

\[
N^A(\text{eu}(\nu_{F'}^-)) - \det T^{1/2}_{F',>0} = N^A \left( \frac{mC_A^{y}(M_F^+ \to M)_{|F'}}{y^{\dim M_F^+}} \right) = N^A \left( \frac{\lambda_{-1}(y^{-1} \varphi_{F'}^+)}{\det \varphi_{F'}^+} \right) = N^A \left( \lambda_{-y} \left( \varphi_{F'}^+ \right) \right),
\]

where the second equality follows from Lemma 5.6. After substitution of $y = 1$ into the class $\text{eu}(\varphi_{F'}^-) \lambda_{-y} (\varphi_{F'}^+)$ we obtain the class $\text{eu}(\varphi_{F'})$. Thus, Proposition 2.13(e) implies that

\[
N^A(\text{eu}(\nu_{F'}^-)) \subseteq N^A \left( \lambda_{-y} \left( \varphi_{F'}^+ \right) \right).
\]

Moreover

\[
N^A \left( \frac{mC_A^{y}(M_F^+ \to M)_{|F'}}{y^{\dim M_F^+}} \right) = N^A \left( mC_A^{y}(M_F^+ \to M)_{|F'} \right) = N^A \left( mC_A^{y}(M_F^+ \to M)_{|F'} \right).
\]
Theorem 4.2 from [16] implies that there is an inclusion

\[ N^A\left(mC_y^A(M_F^+ \to M)_{|F'}\right) \subseteq N^A(eu(\bar{\nu}_{F'})). \]

To conclude, we have proven inclusions

\[ N^A\left(mC_y^A(M_F^+ \to M)_{|F'}\right) \subseteq N^A(eu(\bar{\nu}_{F'})) \]

\[ \subseteq N^A\left(eu(\bar{\nu}_{F'})\lambda_{-y}\left(\bar{\nu}_{F'}\right)\right) = N^A(eu(\nu_{F'}^+)) - \det T_{F', > 0}^{1/2}. \]

The next step is the proof of the formula (4). We proceed in a manner similar to the proof of [16, Corollary 4.5]. Consider a general enough one-dimensional subtorus \( \sigma \in \mathbb{C} \). Proposition 2.16 implies that

\[ N^\sigma\left(mC_y^A(M_F^+ \to M)_{|F'}\right) = \pi_\sigma\left(N^A\left(mC_y^A(M_F^+ \to M)_{|F'}\right)\right). \] (5)

Theorem 4.2 from [16] (cf. also [48, Theorem 10]), with the limit in \( \infty \) changed to the limit in 0 to get positive BB-cell instead of negative one, implies that

\[ \lim_{\xi \to 0}\left(\frac{mC_y^A(M_F^+ \subset M)_{|F'}}{eu(\bar{\nu}_{F'})}\right) = \chi_y(M_F^+ \cap M_F^+) = \chi_y(\emptyset) = 0, \]

where \( \xi \) is the chosen primitive character of the torus \( \sigma \) and the class \( \chi_y \) is the Hirzebruch genus (cf. [9, 21]).

Thus, the lowest term of line segment \( N^\sigma(mC_y^A(M_F^+ \to X)_{|F'}|_\sigma) \) is greater than the lowest term of line segment \( N^\sigma(eu(\bar{\nu}_{F'})|_\sigma) \), which is equal to \( \pi_\sigma(-\det T_{F', > 0}^{1/2}) \) by Proposition 2.17. Thus

\[ \pi_\sigma(-\det T_{F', > 0}^{1/2}) \notin \pi_\sigma\left(N^A\left(mC_y^A(M_F^+ \to X)_{|F'}\right)\right). \]

This implies

\[ -\det T_{F', > 0}^{1/2} \notin N^A\left(\frac{mC_y^A(M_F^+ \to X)_{|F'}}{y^{\dim M_F^+}}\right), \]

as demanded in (4).

\[ \square \]

To prove Theorem 5.3 we need the following technical lemma.

**Lemma 5.7** (cf. [38, Remark after Theorem 3.1 39, Lemma 5.2-4]). Let \( M \) and \( X \) be varieties such as in Proposition 5.1. Suppose that \( X \) satisfies the \((\lambda)\) condition. Consider a fixed point \( F \in M^A \). Suppose that an element \( a \in K^T(X) \) satisfies two conditions:

1. \( \text{supp}(a) \subset \bigcup_{F' \in F} T^*M_{|M_F^+}; \)
2. \( \lambda_{-y}(T^*M_{F_i}^+) |_{F_i} \) divides \( a |_{F_i} \) for any fixed points \( F_i \in M^A \).

Then \( \text{supp}(a) \subset \bigcup_{F' \in F} v^y(M_{F'}^+, \subset M). \)
Proof. Consider the set of positive BB-cells of $M$ corresponding to fixed points $F' \leq F$. Arrange them in a sequence $B_1, \ldots, B_k$ in such a way that for every $t \leq k$ the sum $\bigcup_{i=1}^{t} B_i$ is a closed subset of $M$ (cf. [7, Theorem 3]). Denote by $F_i$ the fixed point corresponding to the BB-cell $B_i$. Denote by

$$E_t = \bigcup_{i=1}^{t} T^* M_{|B_i}$$

and by $V = \bigcup_{i=1}^{k} \nu^*(B_i \subset M)$.

Note that the $(\star)$ condition implies that the sets $E_t \cup V$ are closed. Our goal is to prove by induction that

$$\text{supp}(a) \subset E_t \cup V.$$ 

The first condition implies this containment for $t = k$, which allows to start induction. To prove the proposition we need this containment for $t = 0$.

Assume that $\text{supp}(a) \subset E_t \cup V$ for some $t \geq 1$. We want to prove that $\text{supp}(a) \subset E_{t-1} \cup V$. Denote by $t$ the inclusion

$$t : E_t \cup V \subset X.$$ 

Element $a$ is equal to $t \circ \alpha$ for some $\alpha \in G^\top(E_t \cup V)$. Denote by $U$ the variety $T^* M_{|B_t} \setminus \nu^* B_t$. Using the equality of the complements

$$(E_t \cup V) \setminus (E_{t-1} \cup V) = T^* M_{|B_t} \setminus \nu^* B_t = U,$$

we get an exact sequence

$$G^\top(E_{t-1} \cup V) \rightarrow G^\top(E_t \cup V) \rightarrow G^\top(U) \rightarrow 0.$$ 

So it is enough to show that $\alpha$ restricted to the open subset $U$ vanishes. The variety $E_t \cup V$ is reducible. Denote by $i$ and $j$ the inclusions $E_t \subset E_t \cup V$ and $V \subset E_t \cup V$. Proposition 2.3 implies that the map

$$i_\ast + j_\ast : G^\top(E_t) \oplus G^\top(V) \rightarrow G^\top(E_t \cup V)$$

is epimorphic. Choose any decomposition

$$\alpha = i_\ast \alpha_E + j_\ast \alpha_V$$

such that $\alpha_E \in G^\top(E_t)$ and $\alpha_V \in G^\top(V)$. The subsets $V$ and $U$ have empty intersection so

$$(j_\ast \alpha_V)|_U = 0.$$ 

Thus, it is enough to show that $i_\ast \alpha_E$ also vanishes after restriction to $U$.

Note that Lemma 5.6 implies the following equality in $K^\top(F_t)$:

$$\lambda_{-y}(T^* B_t) = \frac{(-y)^{\dim B_t}}{\det T^* B_t} \lambda_{-1}(-y^{-1} T B_t) = \frac{(-y)^{\dim B_t}}{\det T^* B_t} e_u(\nu^* B_t \subset T^* M_{|B_t}).$$
Moreover, the first map in the exact sequence of closed immersion

\[ K^\mathbb{T}(\nu^* B_i) \xrightarrow{i_*} K^\mathbb{T}(T^* M|_{B_i}) \rightarrow K^\mathbb{T}(U) \rightarrow 0 \]

is multiplication by the Euler class \( eu(\nu^* B_i \subset T^* M|_{B_i}) \). It follows that for an arbitrary element \( b \in K^\mathbb{T}(T^* M|_{B_i}) \) the restriction of \( b \) to the set \( U \) is trivial if and only if \( \lambda_{-y}(T^* B_i)|_{F_i} \) divides \( b|_{F_i} \).

The second assumption and the fact that \( (t_* j_* \alpha_V)|_{U} = 0 \) imply that the element

\[ (t_* i_* \alpha_E)|_{F_i} = a|_{F_i} - (t_* j_* (\alpha_V))|_{F_i} \]

is divisible by \( \lambda_{-y}(T^* B_i)|_{F_i} \). The pushforward–pullback argument shows that

\[ \left( eu(T^* M|_{B_i} \subset X) \alpha_E \right)|_{F_i} = (t_* i_* \alpha_E)|_{F_i}. \]

It follows that \( \lambda_{-y}(T^* B_i)|_{F_i} \) divides \( \alpha_E|_{F_i} \). We use the following simple algebra exercise:

**Exercise.** Let \( R \) be a domain and \( R[y, y^{-1}] \) the ring of Laurent polynomials. Assume that \( A(y) \in R[y, y^{-1}] \) is a monic Laurent polynomials and \( r \in R \) a non-zero element. Then for any polynomial \( B(y) \in R[y, y^{-1}] \)

\[ A(y)|B(y) \iff A(y)|rB(y). \]

The ring \( K^\mathbb{T}(F_i) \) is isomorphic to \( K^A(F_i)[y, y^{-1}] \), the polynomial \( \lambda_{-y}(T^* B_i)|_{F_i} \) is monic (the smallest coefficient is equal to 1) and the Euler class \( eu(T^* M|_{B_i} \subset X)|_{F_i} \) belongs to the subring \( K^A(F_i) \). The exercise implies that \( \lambda_{-y}(T^* B_i) \) divides \( \alpha_E|_{F_i} \). So the class \( \alpha_E \) vanishes on \( U \). Proof of the lemma follows by induction. \( \square \)

**Proof of Theorem 5.3.** Proposition 5.1 implies that axioms (b) and (c) hold. It is enough to check the support axiom.

Chose a fixed point \( F \in M^A \). Functorial properties of the motivic Chern class imply that the support of class

\[ mC^A_{-y}(M^+_F \to M) \in K^A(M)[y] \subset K^\mathbb{T}(M) \]

is contained in the closure of \( M^+_F \) which is contained in the closed set \( \bigcup_{F' \leq F} M^+_F \). It follows that the support of the pullback element

\[ mC^A_{-y}(M^+_F \to M) \in K^\mathbb{T}(T^* M) \]

is contained in restriction of the cotangent bundle \( T^* M \) to the subset \( \bigcup_{F' \leq F} M^+_F \). To prove the support axiom we need to check that it is contained in the smaller subset

\[ \bigcup_{F' \leq F} \nu^*(M^+_F \subset M). \]
Thus, it is enough to check the assumptions of Lemma 5.7 for \( a \) equal to the class \( mC_{-y}^A(M^+_{F'} \to M) \). We know that the first assumption holds. The local product condition (Definition 3.10) implies that for any fixed point \( F' \)

\[
mC_{-y}^A(M^+_{F'} \to M)\mid_{F'} = mC_{-y}^A(M^+_F \cap U \to U)\mid_{F'}
\]

\[
= mC_{-y}^A(Z'_{F', F} \times M^+_F \to Z_{F'} \times M^+_F)\mid_{F'}
\]

\[
= mC_{-y}^A(Z'_{F', F} \subset Z_{F'})\mid_{F'} mC_{-y}^A(M^+_F \to M^+_F)\mid_{F'}
\]

\[
= mC_{-y}^A(Z'_{F', F} \subset Z_{F'})\mid_{F'} \lambda_{-y}(T^*M^+_F)\mid_{F'}
\]

The second equality follows from the local product condition and the third from [4, Theorem 4.2(3)]. Hence the class \( mC_{-y}^A(M^+_{F'} \to M) \) satisfies the assumptions of Lemma 5.7.

\[\square\]

Remark 5.8. In the proof of Theorem 5.3 we need the local product condition only to get divisibility demanded in Lemma 5.7. Namely for a pair of fixed points \( F, F' \) such that \( F > F' \) we need divisibility of \( mC_{-y}^A(M^+_F \subset M)\mid_{F'} \) by \( \lambda_{-y}(T^*M^+_F)\mid_{F'} \). If the closure of the BB-cell of \( F \) is smooth at \( F' \) and the BB-cells form a stratification of \( M \) then the divisibility condition automatically holds. In general case one can assume the existence of a motivically transversal slice instead of the local product condition. Namely suppose there is a smooth locally closed subvariety \( S \subset M \) such that

- \( F' \in S \) and \( S \) is transversal to \( M^+_F \) at \( F' \);
- \( S \) is of dimension complementary to \( M^+_F \);
- \( S \) is motivically transversal (cf. [15, Section 8]) to \( M^+_F \).

Then [15, Theorem 8.5], or reasoning analogous to the proof of [16, Lemma 5.1] proves the desired divisibility condition. In the case of homogenous varieties, divisibility can be also acquired using [16, Theorem 5.3] for the Borel group action.

6 | OTHER SLOPES

Computation of the stable envelopes for the trivial slope allows one to easily get formulas for all integral slopes.

Corollary 6.1. Consider a situation described in Theorem 5.3. For a \( A \)-linearizable line bundle \( s \in \text{Pic}(X) \) the element

\[
\frac{s}{s\mid_F} \cdot \frac{mC_{-y}^A(M^+_{F'} \to M)}{y^{\dim M^+_F}} \in K^A(M)[y, y^{-1}] \simeq K^T(M) \simeq K^T(T^*M)
\]

is equal to the \( K \)-theoretic stable envelope \( \text{Stab}^s(F) \).

In this section we aim to prove that the stable envelope for the trivial slope coincides with the one for a sufficiently small anti-ample slope. Namely:
Theorem 6.2. Let $M$ be a projective, smooth $A$-variety. Suppose that the fixed point set $M^A$ is finite. Consider the variety $X = T^*M$ with the action of the torus $\mathbb{T} = \mathbb{C}^* \times A$. Choose any weight chamber $C$ of the torus $A$ and polarization $T^{1/2} = TM$. Suppose that $M$ satisfies the local product condition. For any anti-ample $A$-linearizable line bundle $s$ and a sufficiently big integer $n$, the element

$$mC_y^A(M^+_F \to M) \frac{y^{\dim M^+_F}}{y^{\dim M^+_F}} \in K^A(M)[y, y^{-1}] \simeq K^T(M) \simeq K^T(T^*M)$$

is equal to the $K$-theoretic stable envelope $\text{Stab}^\frac{1}{n}(F)$.

Proof. Theorem 5.3 implies that the considered element satisfies axioms (a) and (b) of stable envelope. It is enough to check axiom (c). Namely for a fixed point $F' \leq F$ we need to show that

$$N^A\left(\frac{mC_y^A(M^+_F \to M)_{|F'}}{s_F - s_{F'}} \frac{y^{\dim M^+_F}}{n} \subseteq N^A(eu(v^-_{F'})) - \{0\} - \det T^{1/2}_{F',>0} \right).$$

Note that a part of Theorem 5.3 is an analogous inclusion for the trivial slope. It implies that the point $-\det T^{1/2}_{F',>0}$ does not belong to the Newton polytope of motivic Chern class. The Newton polytope is a closed set, thus for a small enough vector $v \in a^*$ its translation by $v$ also does not contain the point $-\det T^{1/2}_{F',>0}$. So it is enough to prove that there exists an integer $n$ such that

$$N^A\left(\frac{mC_y^A(M^+_F \to M)_{|F'}}{s_F - s_{F'}} \frac{y^{\dim M^+_F}}{n} \subseteq N^A(eu(v^-_{F'})) - \det T^{1/2}_{F',>0} \right).$$

Moreover, in the course of proof of Proposition 5.1 we showed containment of polytopes

$$N^A(eu(\tilde{v}_F^-)) \subseteq N^A(eu(v^-_{F'})), \det T^{1/2}_{F',>0}.$$

Therefore, it is enough to show that for a big enough integer $n$ there is an inclusion

$$N^A\left(\frac{mC_y^A(M^+_F \to M)_{|F'}}{s_F - s_{F'}} \frac{y^{\dim M^+_F}}{n} \subseteq N^A(eu(\tilde{v}_F^-)) \right).$$

Consider a lattice polytope $N \subset a^*$. Define a facet as a codimension one face. Let integral hyperplane denote an affine subspace of codimension one, spanned by lattice points. Suppose that the whole interior of the polytope $N$ lies on one side of an integral hyperplane $H$. Denote by $E_H$ half-space which is the closure of the component of complement of $H$ which contains the interior of $N$.

Suppose that the affine span of a lattice polytope $N$ is the whole ambient space. For a facet $\tau$ of $N$, let $H_\tau$ be an integral hyperplane which is the affine span of the face $\tau$. Note that

$$N = \bigcap_{\tau} E_{H_\tau}$$

where the intersection is indexed by the set of codimension one faces.

A similar argument can be applied to any lattice polytope, not necessarily spanning the whole ambient space. Denote by $\text{aff}(\cdot)$ the affine span operator. For any facet $\tau$ choose an integral hyperplane $H_\tau$ such that

$$H_\tau \cap \text{aff}(N) = \text{aff}(\tau).$$
Then

\[
N = \text{aff}(N) \cap \bigcap_{\tau} E_{H_{\tau}}
\]

Thus, to check containment in the polytope \( N \) it is enough to check containment in finitely many integral half-spaces \( E_{H_{\tau}} \) and the affine span of \( N \). We use this observation for \( N \) equal to the Newton polytope \( N^A(eu(\tilde{\varphi}_F^-)) \).

Let \( H \) be an integral hyperplane. We say that a vector \( v \in a^* \) points to \( E_H \) when addition of \( v \) preserves \( E_H \). Our strategy of the proof is to show that for an integral hyperplane \( H_{\tau} \) corresponding to a facet \( \tau \) of the polytope \( N^A(eu(\tilde{\varphi}_F^-)) \) at least one of the following conditions holds:

- The intersection \( N^A(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'}) \cap H_{\tau} \) is empty.
- The vector \( s_F - s_{F'} \) points to \( E_{H_{\tau}} \).

Moreover, if an integral hyperplane \( H \) contains the whole polytope \( N^A(eu(\tilde{\varphi}_F^-)) \) then the addition of the vector \( s_F - s_{F'} \) preserves \( H \).

Note that the above facts are sufficient to prove the theorem. Namely, Proposition 5.1 shows that the polytope \( N^A(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'}) \) is contained in \( N^A(eu(\tilde{\varphi}_F^-)) \). It follows that it lies inside \( E_{H_{\tau}} \) for every facet \( \tau \). If the vector \( s_F - s_{F'} \) points to \( E_{H_{\tau}} \) then for every integer \( n \in \mathbb{N} \)

\[
N^A\left(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'} \right) + \frac{s_F - s_{F'}}{n} \subset E_{H_{\tau}}.
\]

On the other hand if the intersection \( N^A(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'}) \cap H_{\tau} \) is empty then translation of the polytope \( N^A(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'}) \) by a sufficiently small vector still lies in \( E_{H_{\tau}} \). There are only finitely many facets of the polytope \( N^A(eu(\tilde{\varphi}_F^-)) \) so there exists an integer \( n \) such that

\[
N^A\left(mC^-_y(M^+_F \rightarrow M)_{|\tilde{F}'} \right) + \frac{s_F - s_{F'}}{n} \subset \bigcap_{\tau} E_{H_{\tau}}.
\]

Moreover, addition of the vector \( s_F - s_{F'} \) preserves the affine span of \( N^A(eu(\tilde{\varphi}_F^-)) \). It follows that for a sufficiently big integer \( n \) the desired inclusion holds.

Let \( H \subset a^* \) be any integral hyperplane. Denote by \( \tilde{H} \) the vector space parallel to \( H \) (that is, a hyperplane passing through 0). Consider the one-dimensional subspace

\[
\mathfrak{h} = \text{ker}(a \mapsto \tilde{H}^*).
\]

The hyperplane \( H \) is integral so \( \mathfrak{h} \) corresponds to a one-dimensional subtorus \( \sigma_H \subset A \). Choose an isomorphism \( \sigma_H \simeq \mathbb{C}^* \) such that the induced map

\[
\pi_H : a^*/H \simeq \mathfrak{h}^* \simeq \mathbb{R},
\]

sends the vectors pointing to \( E_H \) to the non-negative numbers. Thus, the vector \( s_F - s_{F'} \) points to \( E_H \) if and only if

\[
\pi_H(s_F - s_{F'}) \geq 0.
\]
The choice of isomorphism \( \sigma_H \cong \mathbb{C}^* \) corresponds to the choice of primitive character \( \mathfrak{t} \) of the torus \( \sigma_H \). To study the intersection with hyperplane \( H \) we use the limit technique with respect to the torus \( \sigma_H \). We use the definition of limit map from [25, Definition 4.1]. It is a map defined on a subring of the localized \( K \)-theory:

\[
\lim_{t \to 0} : S_A^{-1} K^A(pt)[y,y^{-1}] \to S_{A/\sigma_H}^{-1} K^{A/\sigma_H}(pt)[y,y^{-1}].
\]

The multiplicative system \( S_A \) (respectively, \( S_{A/\sigma_H} \)) consists of all non-zero elements of \( K^A(pt) \) (respectively, \( K^{A/\sigma_H}(pt) \)). We present a sketch of construction of the above map. Choose an isomorphism of tori \( A \cong \sigma_H \times A/\sigma_H \). It induces an isomorphism \( K^A(pt) \cong K^{A/\sigma_H}(pt)[\mathfrak{t},t^{-1}] \). Then the limit map is defined on the subring \( K^{A/\sigma_H}(pt)[\mathfrak{t}][y,y^{-1}] \) by killing all positive powers of \( \mathfrak{t} \). For technical details and extension to the localized \( K \)-theory see [25, Section 4].

Let \( H \) be a hyperplane corresponding to a facet of the polytope \( N^A(\text{eu}(\tilde{\nu}^-)) \subset E_H \). Thus (for a more detailed discussion, see Remark 6.5)

\[
N^A \left( mC_A^{-y}(M^+_F \to M)|_{F'} \right) \cap H = \emptyset \iff \lim_{t \to 0} \frac{mC_A^{-y}(M^+_F \to M)|_{F'}}{\text{eu}(\tilde{\nu}^-)} = 0.
\]

Let \( F \subset X^\sigma_H \) be a component of the fixed point set which contains \( F' \). Proposition 4.3 and [25, Theorem 4.4] imply that

\[
\lim_{t \to 0} \frac{mC_A^{-y}(M^+_F \to M)|_{F'}}{\text{eu}(\tilde{\nu}_F^-)} = \lim_{t \to 0} \left( \frac{mC_A^{-y}(M^+_F \to M)|_{F'}}{\text{eu}(F \to M)\lambda^{-1}(T^*F)} \right)|_{F'} = \left( \frac{1}{\lambda^{-1}(T^*F)} \lim_{t \to 0} \frac{mC_A^{-y}(M^+_F \to M)|_{F'}}{\text{eu}(F \to M)} \right)|_{F'}
\]

\[
= \left( \frac{mC_A^{-y}(M^+_F \cap M^{\sigma_H,+}_F \to F)|_{F'}}{\lambda^{-1}(T^*F)} \right)|_{F'},
\]

where \( M^{\sigma_H,+}_F \) is the positive BB-cell of \( F \) with respect to the torus \( \sigma_H \). It follows that if the intersection \( M^+_F \cap M^{\sigma_H,+}_F \) is empty then the intersection \( N^A(mC_A^{-y}(M^+_F \to M)|_{F'}) \cap H \) is also empty. The closure of the set \( M^{\sigma_H,+}_F \) is \( A \)-equivariant, thus \( M^+_F \cap M^{\sigma_H,+}_F \) can be non-empty only if \( F \) belongs to the closure of \( M^{\sigma_H,+}_F \). To conclude it is enough to prove that \( \pi_H(s_F - s_{F'}) \geq 0 \) whenever \( F \) belongs to the closure of \( M^{\sigma_H,+}_F \).

For \( F \in M^{\sigma_H,+}_F \) there exists a finite number of points \( A_1, \ldots, A_{m-1}, B_1, \ldots, B_m \in M^{\sigma_H} \) such that

- \( F = B_1 \) and \( B_m \in F \);
- for every \( i \) points \( A_i \) and \( B_i \) lies in the same component of the fixed point set \( M^{\sigma_H} \);
- there exists one-dimensional \( \sigma_H \)-orbit from point \( B_i \) to \( A_{i-1} \);

(see [7, Lemma 9] for a proof in the case of isolated fixed points). For a fixed point \( B \in M^{\sigma_H} \) the fibre \( s_{|B} \) is a \( \sigma_H \)-representation, denote by \( \tilde{s}_B \) its character. If \( B \in M^A \) then there is an equality
\( \tilde{s}_B = \pi(s_B) \). The line bundle \( s \) is anti-ample, so its restriction to every one-dimensional \( \sigma_H \) orbit is also anti-ample. For every anti-ample line bundle on \( \mathbb{P}^1 \), weight on the repelling fixed point is greater or equal than weight on the attracting one. Thus

\[
\tilde{s}_{B_i} = \tilde{s}_{A_i} \geq \tilde{s}_{B_{i+1}}.
\]

It follows that

\[
\pi(s_F) = \tilde{s}_{B_1} \geq \tilde{s}_{B_m} = \pi(s_{F'})
\]

Assume now that an integral hyperplane \( H \) contains the whole polytope \( N^A(eu(\tilde{\nu}^-_{F'})) \). Then the torus \( \sigma_H \) acts trivially on the tangent space \( \tilde{\nu}^-_{F'} \). Consider the fixed point set component \( F_H \subset M^{\sigma_H} \) which contains \( F \). It is a smooth closed subvariety of \( M \) whose tangent space at \( F' \) is the whole tangent space \( T_{F'}M \). Thus, it contains the connected component of \( F' \). It follows that \( \sigma_H \) weights of \( s \) restricted to \( F \) and \( F' \) coincide, thus \( \pi(s_F - s_{F'}) = 0 \).

\[\square\]

**Remark 6.3.** Let \( H \) be a hyperplane corresponding to a facet of the polytope \( N^A(eu(\tilde{\nu}^-_{F'})) \) and let \( \sigma_H \subset A \) be the corresponding one-dimensional subtorus. The fixed point set \( M^{\sigma_H} \) may be non-isolated.

**Remark 6.4.** The inequality about weights of an anti-ample line bundle on \( \mathbb{P}^1 \) can be checked directly using the fact that all anti-ample line bundles on the projective line \( \mathbb{P}^1 \) are of the form \( \mathcal{O}(n) \) for \( n < 0 \). It can be also derived from the localization formula in equivariant cohomology and the fact that the degree of an anti-ample line bundle on \( \mathbb{P}^1 \) is negative (cf. [29, Paragraph 3.2.4]).

**Remark 6.5.** Consider the subtorus \( \sigma_H \subset A \), corresponding to some integral hyperplane \( H \), with chosen primitive character \( t \). The choice of \( t \) induces a choice of a half-space \( E_H \). Consider classes \( a, b \in K^A(pt) \) such that

\[
N^A(a), N^A(b) \subset E_H \text{ and } N^A(b) \cap H \neq \emptyset.
\]

Intuitively, the limit of fraction \( \lim_{t \to 0} \frac{a}{b} \) takes into account the parts of classes \( a, b \) that correspond to the intersections \( N^A(a) \cap H \) and \( N^A(b) \cap H \). More formally, a choice of splitting \( A \simeq \sigma_H \times A/\sigma_H \) corresponds to a choice of integral character \( \gamma \in a^* \) whose restriction to the subtorus \( \sigma_H \) is equal to \( t \). There exists an integer \( m \) such that \( 0 \in \gamma^mH \) (thus \( \gamma^mH = \tilde{H} \)). It follows that under the isomorphism \( K^A(pt) \simeq K^{A/\sigma_H}(pt)[\gamma, \gamma^{-1}] \) we have

\[
\gamma^m a, \gamma^m b \in K^{A/\sigma_H}(pt)[\gamma].
\]

Moreover, \( \gamma^m b \) has non-trivial coefficient corresponding to \( \gamma^0 \). It is equal to the part of class \( b \) corresponding to the intersection \( H \cap N^A(b) \). Denote by \( q \) the projection

\[
q : K^{A/\sigma_H}(pt)[\gamma] \to K^{A/\sigma_H}(pt)
\]
defined by \( q(\gamma) = 0 \). It follows that the limit map is defined on the element \( \frac{a}{b} \) as

\[
\lim_{t \to 0} \frac{a}{b} = \lim_{t \to 0} \frac{\gamma^m a}{\gamma^m b} = \frac{q(\gamma^m a)}{q(\gamma^m b)}.
\]

**Remark 6.6.** Using limit techniques one usually restricts to \( K^G(pt) \cong \mathbb{Z}[t, t^{-1}] \) for a general enough subtorus \( \sigma \) and then consider limits (cf. [16, 45]). In our case, we consider a chosen subtorus \( \sigma_H \) so we cannot proceed in this manner. It may happen that after restriction to \( K^{G_H}(pt) \) denominator vanishes.

**Remark 6.7.** For a generalization of Theorem 6.2 to the case of arbitrary slope see our next paper [26].

### 7 EXAMPLE: THE PROJECTIVE PLANE

In this section we aim to illustrate the proof of Theorem 6.2 by presenting explicit computations in the case of projective plane. We consider (using notation from Theorem 6.2)

- the torus \( A = (\mathbb{C}^*)^2 \) acting on the projective plane \( M = \mathbb{P}^2 \) by

\[
(t_1, t_2)[x : y : z] = [x : t_1 y : t_2 z];
\]

- the weight chamber corresponding to the one-dimensional subgroup \( \sigma(t) = (t, t^2) \);

- the anti-ample line bundle \( s = \mathcal{O}(-1) \) as a slope;

- the fixed points \( F = [0 : 1 : 0] \) and \( F' = [0 : 0 : 1] \).

Denote by \( \alpha \) and \( \beta \) characters of the torus \( \mathbb{T} \) given by projections to the first and the second coordinates of \( A \), respectively. Local computation leads to formulas:

\[
eu(v(F' \to M))_{|F'} = \left(1 - \frac{\beta}{\alpha}\right) (1 - \beta)
\]

\[
mC_{-y} (M^+_F \to M)_{|F'} = (1 - y) \frac{\beta}{\alpha} (1 - \beta) \]

\[
det T^{1/2}_{F', > 0} = 0
\]

\[
v := s_{|F} - s_{|F'} = \frac{\alpha}{\beta}
\]

Denote by \( B := N^A(mC_{-y} (M^+_F \to M)_{|F'}) \) (blue interval), \( C = N^A(eu(v(F' \to M))_{|F'}) \) (yellow parallelogram) and \( D = det T^{1/2}_{F', > 0} = 0 \) (red point). Denote the facets of polytope \( C \) by \( \tau_1, \tau_2, \tau_3, \tau_4 \) according to the picture.

Theorem 5.3 implies that the blue interval \( B \) is contained in the yellow polytope \( C \) and the red point \( D \) does not belong to the interval \( B \). It is enough to prove that for a sufficiently big integer \( n \)
and every facet $\tau$

$$B + \frac{\mathbf{v}}{n} \subset E_{H_\tau}.$$  

Let us compute half-planes and subtori associated with the facets:

| Facet $\tau$ | $E_{H_\tau}$ | $H_\tau$ | $\sigma_\tau \subset A$ | $\pi_{H_\tau}$ | Character $t$ |
|--------------|--------------|----------|-----------------------|---------------|----------------|
| $\tau_1$    | $\{x\alpha + y\beta | x \leq 0\}$ | $\text{span}(\beta)$ | $(t, 0)$ | $x\alpha + y\beta \to -x$ | $(t, 0) \to \frac{1}{t}$ |
| $\tau_2$    | $\{x\alpha + y\beta | x + y \leq 1\}$ | $\text{span}(\alpha - \beta)$ | $(t, t)$ | $x\alpha + y\beta \to -x - y$ | $(t, t) \to \frac{1}{t}$ |
| $\tau_3$    | $\{x\alpha + y\beta | x \geq -1\}$ | $\text{span}(\beta)$ | $(t, 0)$ | $x\alpha + y\beta \to x$ | $(t, 0) \to t$ |
| $\tau_4$    | $\{x\alpha + y\beta | x + y \geq 0\}$ | $\text{span}(\alpha - \beta)$ | $(t, t)$ | $x\alpha + y\beta \to x + y$ | $(t, t) \to t$ |

Here $\text{span}$ denotes the linear span. A choice of splitting $A \simeq \sigma_\tau \times A/\sigma_\tau$ corresponds to a choice of integral character $\gamma \in a^*$ whose restriction to the subtorus $\sigma_\tau$ is equal to $t$. For $\tau_1$ and $\tau_3$ let us choose $\gamma = (\frac{\alpha}{\beta^2})^{\pm 1}$. It induces a splitting of cohomology $K^A(pt) = \mathbb{Z}[\beta, \beta^{-1}][\alpha, \frac{\beta^2}{\alpha}, \frac{\beta}{\alpha}]$. Using this splitting we can compute limits

$$\tau_1 : \lim_{t \to 0} (1 - y) \frac{\beta}{1 - \frac{\beta}{\alpha}} = (1 - y) \lim_{t \to 0} \frac{\beta^2}{\alpha} - \frac{\beta^2}{\alpha} = (1 - y) \frac{0}{\beta} = 0,$$

$$\tau_3 : \lim_{t \to 0} (1 - y) \frac{\beta}{1 - \frac{\beta}{\alpha}} = (1 - y) \lim_{t \to 0} \frac{1}{\beta} \frac{\alpha}{\beta^2} - \frac{1}{\beta} = (1 - y) \frac{1}{\beta} = y - 1.$$

For $\tau_2$ and $\tau_4$ let us choose $\gamma = \alpha^{\pm 1}$. It induces a splitting $K^A(pt) = \mathbb{Z}[\frac{\beta}{\alpha}, \frac{\alpha}{\beta}][\alpha, \alpha^{-1}]$ (the character $\frac{\beta}{\alpha}$ is a basis of $H_\tau$) and

$$\tau_2, \tau_4 : \lim_{t \to 0} (1 - y) \frac{\beta}{1 - \frac{\beta}{\alpha}} = (1 - y) \frac{\beta}{1 - \frac{\beta}{\alpha}}.$$

These calculations imply that

$$B \cap \tau_i \neq \emptyset \iff i \in \{2, 3, 4\}.$$

We want to show that the addition of the vector $\mathbf{v}$ preserves half-plane $E_{H_\tau}$ for these three facets by proving that

$$\pi_{H_{\tau_i}}(\mathbf{v}) > 0 \text{ for } i \in \{2, 3, 4\}.$$  

For $\tau_2, \tau_4$ the points $F', F$ belong to the same fixed point set component of the torus $\sigma_\tau$. It implies that $\pi_{H_\tau}(\mathbf{v}) = 0$. This agrees with direct computation

$$\pi_{H_\tau}(\mathbf{v}) = \pi_{H_\tau}(\alpha - \beta) = \pm(1 + (-1)) = 0.$$
Moreover, for \( \tau_3 \) there is one-dimensional \( \sigma_H \)-orbit \([0 : x : y]\) from \( F \) to \( F' \). It implies that 
\[
\pi_{H\tau_3}(v) > 0.
\]
This agrees with direct computation
\[
\pi_{H\tau_3}(v) = \pi_{H\tau_3}(\alpha - \beta) = 1.
\]
To conclude, for every \( n \in \mathbb{N} \) the interval \( B + \sum_{n}^{\infty} E_{H\tau_i} \) is contained in the intersection of half-planes \( \bigcap_{i=2}^{4} E_{H\tau_i} \). Moreover, it is also contained in \( E_{H\tau_1} \) for a sufficiently big integer \( n \).

**APPENDIX A: UNIQUENESS OF THE STABLE ENVELOPES**

In [34, 36] the stable envelope was defined for an action of a reductive group \( G \). In this Appendix we show that for the group \( G \) equal to a torus and a general enough slope our Definition 3.2 of the stable envelope coincides with Okounkov’s. Moreover, we prove the uniqueness of stable envelopes for an arbitrary slope.

We use the notations and assumptions from the beginning of Section 3. According to [34, 36] the stable envelope is a map:
\[
K^T(X^A) \rightarrow K^T(X)
\]
given by a correspondence
\[
Stab \in K^T(X^A \times X),
\]
which satisfies three properties (cf. [34, Paragraph 9.1.3]). For a \( \mathbb{T} \)-variety \( X \) and a finite set \( F \) (with the trivial \( \mathbb{T} \)-action) any map of \( K^T(pt) \) modules
\[
f : K^T(F) \rightarrow K^T(X)
\]
is determined by a correspondence \( G \in K^T(F \times X) \) such that for any \( x \in F \)
\[
G_{x \times X} = f(1_x).
\]
We denote both morphism and correspondence by \( Stab \). The main ingredient in Okounkov’s definition are attracting sets. For a one parameter subgroup \( \sigma : \mathbb{C}^* \rightarrow A \) it is defined as (cf. [34, Paragraph 9.1.2; 36, Paragraph 2.1.3]):
\[
Attr = \{(y, x) \in X^A \times X| \lim_{t \to 0} \sigma(t)x = y\}.
\]
Moreover, for a fixed point \( F \in X^A \) we define
\[
Attr(F) = \{x| \lim_{t \to 0} x \in F\} \subset X.
\]
The straightforward comparison of definitions shows that the attracting sets coincide with the BB-cells
\[
Attr(F) = X^+_F.
\]
Support condition [29, Theorem 3.3.4, point (i); 34, Paragraph 9.1.3, point 1; 36, Paragraph 2.1.1]: In Okounkov’s papers it is required that

$$\text{supp}(\text{Stab}) \subset \bigsqcup_{F \in X^A} \left( F \times \bigsqcup_{F' < F} \text{Attr}(F') \right)$$

which means exactly that for any fixed point $F \in X^A$

$$\text{supp}(\text{Stab}(1_F)) \subset \bigsqcup_{F' < F} \text{Attr}(F').$$

The attracting sets coincide with the BB-cells, so this is an equivalent formulation of axiom (a).

Normalization condition [34, Paragraph 9.1.5; 36, Paragraph 2.1.4]: Consider any fixed point $F \in X^A$. In Okounkov’s papers it is required that the correspondence $\text{Stab} \in K^T(X^A \times X)$ inducing the stable envelope morphism satisfies

$$\text{Stab}|_{F \times F} = (-1)^{\text{rank } T^1_{F > 0}} \left( \frac{\det \nu_F^-}{\det T^1_{F > 0}} \right)^{1/2} \otimes \mathcal{O}_{\text{Attr}}|_{F \times F}.$$

After substitutions

$$\mathcal{O}_{\text{Attr}}|_{F \times F} = \mathcal{O}_{\text{diag } F} \otimes eu(\nu_F^-),$$

$$\frac{\det \nu_F^-}{\det T^1_{F > 0}} = h^\text{rank } T^1_{F > 0} \left( \frac{1}{\det T^1_{F > 0}} \right)^2$$

as noted in [36, Paragraph 2.1.4] we obtain

$$\text{Stab}|_{F \times F} = eu(\nu_F^-) \left( -1 \right)^{\text{rank } T^1_{F > 0}} \otimes h^\text{rank } T^1_{F > 0} \otimes \mathcal{O}_{\text{diag } F}.$$

Changing correspondence to a morphism we get an equivalent condition

$$\text{Stab}(1_F)|_F = eu(\nu_F^-) \left( -1 \right)^{\text{rank } T^1_{F > 0}} \otimes h^\text{rank } T^1_{F > 0},$$

which is exactly our axiom (b).

Smallness condition [34, Paragraph 9.1.9; 36, Paragraph 2.1.6]: In the case of isolated fixed points, the last axiom of stable envelope from Okounkov’s papers states that for any pair of fixed points $F_1, F_2 \in X^A$

$$N^A\left( \text{Stab}|_{F_1 \times F_2} \otimes s_{|F_1} \right) \subseteq N^A\left( \text{Stab}|_{F_2 \times F_2} \otimes s_{|F_2} \right).$$
The support condition implies that this requirement is non-trivial only when $F_1 > F_2$. Changing correspondence to a morphism we get an equivalent form

$$N^A \left( \text{Stab}(1_{F_1} | F_2) \right) + s_{|F_1} \subseteq N^A \left( \text{Stab}(1_{F_2} | F_2) \right) + s_{|F_2}.$$ 

The normalization axiom implies that

$$N^A \left( \text{Stab}(1_{F_2} | F_2) \right) = N^A \left( \text{Stab}(1_{F_2} | F_2) \right) + \begin{pmatrix} e^{(v_{-F_2})^T} & -1 & \frac{\text{rank } T_{F_2,>0}^{1/2}}{\text{det } T_{F_2,>0}^{1/2}} h_{\frac{1}{2} \text{rank } T_{F_2,>0}^{1/2}} \\ e^{(v_{F_2})^T} & 1 & \frac{\text{rank } T_{F_2,>0}^{1/2}}{\text{det } T_{F_2,>0}^{1/2}} \end{pmatrix}.$$ 

Note that the torus $A$ preserves the symplectic form $\omega$, thus multiplication by $h$ does not change Newton polytope $N^A$. Thus, we get an equivalent formulation

$$N^A \left( \text{Stab}(1_{F_1} | F_2) \right) + s_{|F_1} \subseteq N^A \left( \text{Stab}(1_{F_1} | F_2) \right) - \text{det } T_{F_2,>0}^{1/2} + s_{|F_2},$$

which is very similar to axiom (c). The only difference is that we additionally require

$$-\text{det } T_{F_2,>0}^{1/2} + s_{|F_2} \notin N^A \left( \text{Stab}(1_{F_1} | F_2) \right) + s_{|F_1}.$$ 

For a general enough slope this requirement automatically holds because the point

$$-\text{det } T_{F_2,>0}^{1/2} + s_{|F_2} - s_{|F_1}$$

is a vertex of polytope which is not a lattice point. The addition of this assumption is necessary to acquire uniqueness of the stable envelopes for all slopes.

**Example A.1.** Consider the variety $X = T^* \mathbb{P}^1$ equipped with the action of the torus $\mathbb{T} = \mathbb{C}^* \times A$ where $A$ is the one-dimensional torus acting on $\mathbb{P}^1$ by

$$\alpha[a:b] = [\alpha a : b]$$

and $\mathbb{C}^*$ acts on the fibres by scalar multiplication. Denote by $\alpha$ and $y$ characters of $\mathbb{T}$ corresponding to projections to the tori $A$ and $\mathbb{C}^*$. The action of the torus $\mathbb{T}$ has two fixed points $e_1 = [1:0]$ and $e_2 = [0:1]$. The variety $X$ satisfies the condition ($\star$) in a trivial way.

Consider the stable envelope for the positive weight chamber (such that $\alpha$ is positive), the tangent bundle $T \mathbb{P}^1$ as polarization and the trivial line bundle $\delta$ as a slope. If we omit the point zero in axiom (c) then both

$$\text{Stab}^\delta(e_1) = 1 - O(-1), \quad \text{Stab}^\delta(e_2) = \frac{1}{y} - \frac{O(-1)}{\alpha},$$

and

$$\text{Stab}^\delta(e_1) = 1 - O(-1), \quad \text{Stab}^\delta(e_2) = \frac{O(-1)}{y} - \frac{O(-2)}{\alpha},$$

satisfy the axioms of stable envelope.
The rest of this Appendix is devoted to the proof of uniqueness of the stable envelope (Proposition 3.6). For a general enough slope it was proved in [34, Proposition 9.2.2]. For the sake of completeness, we present it with all necessary technical details omitted in the original. The proof is a generalization of the proof of uniqueness of cohomological envelopes [29, Paragraph 3.3.4]. We need the following lemma.

**Lemma A.2.** Choose a set of vectors $l_F \in \text{Hom}(A, C^*) \otimes \mathbb{Q}$ indexed by the fixed point set $X^A$. Suppose that an element $a \in K^T(X)$ satisfies conditions

1. $\text{supp}(a) \subset \bigsqcup_{F \in X^A} X_F^+$;
2. for any fixed point $F \in X^A$ we have containment of the Newton polytopes

$$N^A(a|_F) \subseteq \left(N^A(eu(v_F^-)) \setminus \{0\}\right) + l_F. \tag{A.1}$$

Then $a = 0$.

**Proof.** We proceed by induction on the partially ordered set $X^A$. Suppose that the element $a$ is supported on the closed set $Y = \bigsqcup_{F \in Z} X_F^+$ for some subset $Z \subset X^A$. Choose a BB-cell $X_F^+$, corresponding to fixed point $F_1 \in X^A$, which is an open subvariety of $Y$. We aim to show that $a$ is supported on the closed subset $\bigsqcup_{F \in (Z - F_1)} X_F^+$. By induction it implies that $a = 0$.

Choose an open subset $U \subset X$ such that $U \cap Y = X_F^+$. Consider the diagram

$$
\begin{array}{ccc}
U & \xrightarrow{i} & X \\
\downarrow{j} & & \downarrow{j} \\
F_1 & \xrightarrow{s_0} & X_F^+ & \xrightarrow{i} & Y.
\end{array}
$$

The square in the diagram is a pullback. The BB-cells are smooth, locally closed subvarieties, so the map $\tilde{j}$ is an inclusion of a smooth subvariety. There exists an element $\alpha \in G^T(Y)$ such that $j_*(\alpha) = a$. It follows that

$$a|_{F_1} = (j_*\alpha)|_{F_1} = s_0^*\tilde{j}_*i_*j_*\alpha = s_0^*\tilde{j}_*\tilde{i}_*\alpha = eu(v_F^-)\alpha|_{F_1},$$

which implies

$$N^A(eu(v_F^-)\alpha|_{F_1}) = N^A(a|_{F_1}) \subseteq \left(N^A\left(eu(v_F^-)\right) \setminus \{0\}\right) + l_{F_1}. \tag{A.1}$$

Assume that $\alpha|_{F_1}$ is a non-zero element. Then the Newton polytope $N^A(\alpha|_{F_1})$ is non-empty. The ring $K^T/A(F_1)$ is a domain so Proposition 2.13(b) implies that

$$N^A\left(eu(v_F^-)\right) \subseteq N^A\left(eu(v_{F_1}^-)\right) + N^A(\alpha|_{F_1}) = N^A\left(eu(v_{F_1}^-)\alpha|_{F_1}\right). \tag{A.2}$$

The inclusions (A.1) and (A.2) imply that

$$N^A\left(eu(v_F^-)\right) \subseteq \left(N^A\left(eu(v_{F_1}^-)\right) \setminus \{0\}\right) + l_{F_1}.$$
But no polytope can be translated into a proper subset of itself. This contradiction proves that the element $\alpha_{|F_1}$ is equal to 0. The map $s_0$ is a section of an affine bundle so it induces an isomorphism of the algebraic $K$-theory. It follows that $\alpha_{|X^+_F} = 0$. Thus, the element $\alpha$ is supported on the closed set $\bigsqcup_{F \in (Z-F_1)} X^+_F$. It follows that $a$ is also supported on this set.

**Proof of Proposition 3.6.** Let $\{Stab(F)\}_{F \in X^A}$ and $\{\overline{Stab}(F)\}_{F \in X^A}$ be two sets of elements satisfying the axioms of stable envelope. It is enough to show that for any fixed point $F \in X^A$ the element $Stab(F) - \overline{Stab}(F)$ satisfies conditions of Lemma A.2 for the set of vectors

$$l_F = s_F - s_F - \det T^{1/2}_{F',>0}.$$

The support condition follows from axiom (a). Let us focus on the second condition. The only non-trivial case is $F' < F$. In the other cases axioms (a) and (b) imply that

$$Stab(F)_{|F'} - \overline{Stab}(F)_{|F'} = 0.$$

When $F' < F$ axiom (c) implies that the Newton polytopes $N^A(Stab(F)_{|F'})$ and $N^A(\overline{Stab}(F)_{|F'})$ are contained in the convex set (cf. Proposition 2.17)

$$\big( N^A(eu(\nu_F^-) \setminus \{0\}) \big) + l_F.$$

Thus

$$N^A \left( Stab(F)_{|F'} - \overline{Stab}(F)_{|F'} \right) \subseteq \text{conv} \left( N^A(Stab(F)_{|F'}), N^A(\overline{Stab}(F)_{|F'}) \right) \subseteq \left( N^A(eu(\nu_F^-)) \setminus \{0\} \right) + l_{F'}.$$

□

**Remark A.3.** In this paper we always assume that the fixed point set $X^A$ is finite. In the case of non-isolated fixed points, our Definition 3.5 is not equivalent to Okounkov’s definition. For a component of the fixed point set $F \subset X^A$, the morphism

$$K^T(F) \to K^T(X)$$

is not determined by its value on the element $1_F$. However, even in this case, there is at most one element satisfying the axioms of Definition 3.5. The proofs of analogues of Proposition 3.6 and Lemma A.2 are almost identical to those presented above. The only difference is that the ring $K^T(F)$ may not be a domain. Thus, in the proof of Lemma A.2, one needs to use Proposition 2.13(c) (for the class $eu(\nu_{F_1}^-)$, see Remark 2.15) instead of 2.13(b).

**APPENDIX B: THE LOCAL PRODUCT PROPERTY OF SCHUBERT CELLS**

Let $G$ be a reductive, complex Lie group with chosen maximal torus $T$ and Borel subgroup $B^+$. Any one-dimensional subtorus $\sigma \subset T$ induces a linear functional

$$\varphi_\sigma : t^* \to \mathbb{C}.$$
For a general enough subtorus $\sigma$ we can assume that no roots belong to the kernel of this functional. Consider the Borel subgroups $B^+_\sigma$ such that the corresponding Lie algebra is the union of these weight spaces whose characters are positive with respect to $\varphi_\sigma$. Denote its unipotent subgroup by $U^+_\sigma$. Analogously one can define groups $B^-_\sigma$ and $U^-_\sigma$.

For a parabolic group $B^+ \subset P \subset G$ consider the BB-decomposition of the variety $G/P$ with respect to the torus $\sigma$. It is a classical fact that the positive (respectively, negative) BB-cells are the orbits of group $B^+_\sigma$ (respectively, $B^-_\sigma$). We prove that the stratification of $G/P$ by BB-cells of the torus $\sigma$ behaves like a product in a neighbourhood of a fixed point of the torus $\mathbb{T}$ (see Definition 3.10).

**Theorem B.1.** Consider the situation described above. Any fixed point $x \in (G/P)^\mathbb{T}$ has an open neighbourhood $U$ such that

1. there exists a $\mathbb{T}$-equivariant isomorphism
   $$\vartheta : U \simeq (U \cap (G/P)^+_x) \times (U \cap (G/P)^-_x);$$
2. for any fixed point $y \in (G/P)^\mathbb{T}$ the isomorphism $\vartheta$ induces isomorphism:
   $$U \cap (G/P)^+_y \simeq (U \cap (G/P)^+_x) \times \left( U \cap (G/P)^-_x \cap (G/P)^+_y \right).$$

In the course of proof we use the following interpretation of classical notions of the theory of Lie groups in the language of BB-decomposition. Note that we consider BB-cells in smooth quasi-projective varieties.

**Lemma B.2.** Consider the action of the torus $\sigma$ on the group $G$ defined by conjugation. Denote by $F$ the component of the fixed point set which contains the identity. For a subset $Y \subset F$ we use abbreviations

$$G^+_Y = \{ x \in G | \lim_{t \to 0} x \in Y \} \text{ and } G^-_Y = \{ x \in G | \lim_{t \to \infty} x \in Y \}$$

for the fibres of projections $G^+_F \to F$ and $G^-_F \to F$ over $Y$.

1. The Borel subgroup $B^+_\sigma$ (respectively, $B^-_\sigma$) is the positive (respectively, negative) BB-cell of the maximal torus $\mathbb{T}$, that is,
   $$B^+_\sigma = G^+_\mathbb{T}.$$
2. The unipotent subgroup $U^+_\sigma$ (respectively, $U^-_\sigma$) is the positive (respectively, negative) BB-cell of the identity element, that is,
   $$U^+_\sigma = G^+_\text{id}.$$

**Proof.** We prove only the first case for the positive Borel subgroup. The other cases are analogous. It is enough to show that $G^+_\mathbb{T}$ is a connected subgroup of $G$ whose Lie algebra coincides with the Lie algebra of $B^+_\sigma$. 
The variety $G_\mathbb{T}$ is a subgroup because the maximal torus $\mathbb{T}$ is a group and the limit map preserves multiplication. Namely for $g, h \in G_\mathbb{T}$ such that $\lim_{t \to 0} g = a \in \mathbb{T}$ and $\lim_{t \to 0} h = b \in \mathbb{T}$ it is true that

$$\lim_{t \to 0} g^{-1} h = a^{-1} b \in \mathbb{T}.$$ 

The variety $G_\mathbb{T}$ is connected because the maximal torus $\mathbb{T}$ is connected. So it is enough to compute the tangent space to $G_\mathbb{T}$ at identity. The exponent map is an isomorphism in some neighbourhood of zero so we can limit ourselves to computation in the Lie algebra $\mathfrak{g}$. The action of the torus $\sigma$ on $\mathfrak{g}$ is given by differentiation of the action on $G$. The tangent space $T_0G_\mathbb{T}$ is equal to the BB-cell $q_\mathbb{T}^+$. Consider the weight decomposition

$$\mathfrak{g} = \bigoplus_{h \in \mathfrak{t}^*} V_h.$$ 

Differentiation of the action of $\sigma$ on $\mathfrak{g}$ is equal to the Lie bracket so

$$\mathfrak{g}_\mathbb{T}^+ = \bigoplus_{h \in \mathfrak{t}^*, \varphi_\sigma(h) \geq 0} V_h.$$ 

This is exactly the tangent space to the Borel subgroup $B_\sigma^+$. □

**Proof of the Theorem** B.1. Note that the Weyl group acts transitively on the fixed point set $(G/P)^\mathbb{T}$. Thus, replacing the torus $\sigma$ by its conjugate by a Weyl group element we may assume that a fixed point $x$ is equal to the class of identity.

Denote the Lie algebra of the parabolic subgroup $P$ by $\mathfrak{p} \subset \mathfrak{g}$. Denote by $\mathfrak{u}_P$ the Lie subalgebra consisting of the root spaces which do not belong to $\mathfrak{p}$. Let $U_P$ be the corresponding Lie group. The group $U_P$ is unipotent (as a subgroup of the unipotent group $U^-$). Consider the action of the torus $\mathbb{T}$ on $U_P$ by conjugation. Let us note two facts from the theory of Lie groups.

1. $U_P$ is isomorphic to its complex Lie algebra as a complex $\mathbb{T}$-variety (cf. [8, Paragraph 15.3b]; [24, Paragraph 8.0]).
2. The quotient map $G \to G/P$ induces $\mathbb{T}$-equivariant isomorphism from $U_P$ to some open neighbourhood of identity.

Choose $U_P$ as a neighbourhood $U$ of identity. The second observation and the second point of Lemma B.2 imply that

$$X_+ := U_P \cap (G/P)_{id}^+ \simeq U_P \cap G_{id}^+ \simeq U_P \cap U_\sigma^+,$$

analogously

$$X_- := U_P \cap (G/P)_{id}^- \simeq U_P \cap U_\sigma^-.$$ 

Both isomorphisms are given by the quotient morphism $G \to G/P$. We define morphism

$$\vartheta : X_+ \times X_- \to U_P$$
We aim to prove that this is an isomorphism. We start by showing injectivity on points. Both varieties $X_+$ and $X_-$ are subgroups of $U_P$. So to prove injectivity it is enough to show that $X_+ \cap X_- = \{id\}$. But $X_+$ is contained in the positive unipotent group and $X_-$ in the negative unipotent group, so their intersection must be trivial.

As a variety $U_P$ is isomorphic to an affine space — its Lie algebra $\mathfrak{u}_P$. The induced action of $\mathbb{T}$ on the linear space $\mathfrak{u}_P$ is linear — it is a part of the adjoint representation of $G$. It follows that both $X_+$ and $X_-$ are BB-cells of a linear action on a linear space and therefore linear subspaces. Thus, the product $X_+ \times X_-$ is isomorphic to an affine space of dimension equal to dimension of $U_P$. Thus, the map $\vartheta$ is an algebraic endomorphism of an affine space which is injective on points. The Ax–Grothendieck theorem (cf. [5, 20, Theorem 10.4.11]) implies that it is bijective on points. Affine space is smooth and connected so the Zariski main theorem [19, Theorem 4.4.3] implies that $\vartheta$ is an algebraic isomorphism.

To prove the second property, it is enough to show the containment

$$\vartheta \left( X_+ \times (U_P \cap (G/P)_y^+) \right) \subset (G/P)_y^+,$$

for any fixed point $y$. Note that

$$X_+ \subset U_\sigma^+ \subset B_\sigma^+.$$

Moreover, the BB-cell $(G/P)_y^+$ is an orbit of the group $B_\sigma^+$ and the morphism $\vartheta$ coincides with the action of $B_\sigma^+$. So the desired inclusion holds.

**Remark B.3.** One can alternatively show that $X_-$ and $X_+$ are isomorphic to affine spaces using [23, Theorem 1.5]. It is also possible to omit the Ax–Grothendieck theorem using classical results of the theory of Lie groups. Namely, [43, Lemma 17] implies that the map $\vartheta$ is bijective.

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