THE METHOD OF AVERAGING FOR POISSON CONNECTIONS
ON FOLIATIONS AND ITS APPLICATIONS

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Abstract. On a Poisson foliation equipped with a canonical and cotangential
action of a compact Lie group, we describe the averaging method for Poisson
connections. In this context, we generalize some previous results on Hannay-
Berry connections for Hamiltonian and locally Hamiltonian actions on Poisson
fiber bundles. Our main application of the averaging method for connections is
the construction of invariant Dirac structures parametrized by the 2-cocycles
of the de Rham-Casimir complex of the Poisson foliation.

1. Introduction. In this paper we discuss some aspects of the averaging method
for Poisson connections on foliated manifolds with symmetry by generalizing the
previous results on Hannay-Berry connections on fibrations due to [12, 13]. Poisson
connections of such type play an important role in the normal form theory for
Hamiltonian systems of adiabatic type (see, for example, [1, 2]). One of our main
motivations is related to the further development of the averaging procedure for
Dirac structures with singular presymplectic foliations [20].

First of all, we describe the averaging procedure for Poisson connections on fo-
liations relative to a wide class of canonical (not necessarily Hamiltonian) actions
which appears in [8, 9] in the context of invariant Poisson cohomology. Then, these
results are applied in the construction of invariant Dirac structures.

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Our starting point is a Poisson foliation $(M, F, P)$ consisting of a regular foliation $F$ on a manifold $M$ and a vertical (leaf-tangent) Poisson tensor $P$ on $M$ characterized by the condition: each symplectic leaf of $P$ belongs to a leaf of $F$. We are interested in the set $\text{Conn}_H(M, F, P)$ of Ehresmann-Poisson connections $\gamma$ on $(M, F, P)$ satisfying the following condition: the curvature of $\gamma$ is Hamiltonian, that is, the curvature form takes values in Hamiltonian vector fields of $P$. Thus, we can assign to each connection $\gamma \in \text{Conn}_H(M, F, P)$ a horizontal 2-form called the Hamiltonian form of the curvature. This form is uniquely determined modulo a Casimir-valued horizontal 2-form. A Poisson connection $\gamma \in \text{Conn}_H(M, F, P)$ is said to be admissible if one can choose a Hamiltonian 2-form of the curvature to be $\gamma$-covariantly constant. Such class of Poisson connections with Hamiltonian curvature naturally arises in the context of the coupling method for Poisson and Dirac structures on fibered and foliated manifolds [18, 19, 21]. In particular, it is well-known that each coupling Dirac structure induces an admissible Poisson connection [7, 19, 22]. Conversely, the coupling procedure actually gives the conditions under which the vertical Poisson tensor $P$ can be extended to a special Dirac structure via a given connection $\gamma \in \text{Conn}_H(M, F, P)$. In general, the set $\text{Conn}_H(M, F, P)$ can be empty. In the case of fibrations, the question on the existence of Poisson connections with Hamiltonian curvature was discussed in [4].

Our purpose is to study the set $\text{Conn}_H(M, F, P)$ and the subset of admissible connections under the symmetry hypothesis that there exists an action on $M$ of a compact and connected Lie group $G$ which preserves the foliation $F$. In this situation, the natural question is to characterize the $G$-actions for which the averaging procedure preserves the set $\text{Conn}_H(M, F, P)$ and its subset of admissible Poisson connections. The fact that every canonical action with momentum map on a Poisson fiber bundle preserves $\text{Conn}_H(M, F, P)$ was originally stated in [12], and then extended to locally Hamiltonian actions on Poisson foliations in [20]. In the present paper, we observe that the averaging procedure preserves the Poisson connections with Hamiltonian curvature and admissible connections for the class of canonical $G$-actions which admit only pre-momentum map in the sense of [8, 9]. Our main application is that, starting with an admissible connection $\gamma \in \text{Conn}_H(M, F, P)$, we show how to construct a family of $G$-invariant Dirac structures on $(M, F, P)$ parametrized by 2-cocycles of the de Rham-Casimir complex associated to the Poisson foliation [14].

The paper is organized as follows. In Section 2, we recall some basic facts about Ehresmann connections on regular foliated manifolds. In Section 3, we describe an averaging procedure for vector-valued forms and connections relative to a foliation-preserving action of a compact and connected Lie group $G$. In Section 4, we study Poisson connections on Poisson foliated manifolds in the context of the averaging method. Our main result (Theorem 4.7) is related to the notion of Hannay-Berry connections for canonical and cotangential actions on Poisson foliated manifolds. In section 5, we prove that the set $\text{Conn}_H(M, F, P)$ is preserved by the averaging procedure (Theorem 5.4). Furthermore, we also prove that the subset of admissible Hamiltonian connections remains invariant. In Section 6, we extend the notion of canonical and cotangential actions to the case of Dirac manifolds. Finally, in Section 7, we apply our results to the construction of $G$-invariant coupling Dirac structures (Theorem 7.4).
2. Ehresmann connections on foliated manifolds. Let \((M, \mathcal{F})\) be a regular foliated manifold and \(\mathbb{V} := T \mathcal{F} \subset TM\) the tangent bundle called the \textit{vertical distribution}. A vector-valued 1-form \(\gamma \in \Omega^1(M; \mathbb{V})\) is said to be a \textit{connection} on \((M, \mathcal{F})\) if the vector bundle morphism \(\gamma : TM \rightarrow \mathbb{V}\) satisfies the conditions:

\[
\gamma \circ \gamma = \gamma \quad \text{and} \quad \text{Im} \gamma = \mathbb{V}.
\]

In fact, these conditions are equivalent to the following:

\[
Y \in \Gamma(\mathbb{V}) \implies \gamma(Y) = Y.
\]

Then, \(\mathbb{H} = \mathbb{H}^\gamma := \ker \gamma\) is a subbundle of \(TM\) normal to \(\mathcal{F}\), called the \textit{horizontal subbundle}. It is clear that \(id_{TM} - \gamma\) is just the projection to \(\mathbb{H}\) along \(\mathbb{V}\).

Conversely, given a normal bundle \(\mathbb{H}\) of \(\mathcal{F}\), one can define the associated connection as the projection \(\gamma = \gamma^\mathbb{H} := pr_2\) to \(\mathbb{V}\) according to the decomposition

\[
TM = \mathbb{H} \oplus \mathbb{V}.
\]

Then, the cotangent bundle splits as follows

\[
T^*M = \mathbb{V}^0 \oplus \mathbb{H}^0,
\]

where \(\mathbb{V}^0\) and \(\mathbb{H}^0\) are the annihilators of \(\mathbb{V}\) and \(\mathbb{H}\), respectively. These decompositions give rise to a \(\gamma\)-dependent bigrading of differential forms and tensor fields on \(M\). In particular, for any \(X \in \mathfrak{x}(M)\) and \(\alpha \in \Omega^1(M)\), we have

\[
X = X_{1,0} + X_{0,1} \quad \text{and} \quad \alpha = \alpha_{1,0} + \alpha_{0,1},
\]

where

\[
X_{1,0} = (id_{TM} - \gamma)(X) \in \Gamma(\mathbb{H})\quad \text{and} \quad \alpha_{1,0} = (id_{TM} - \gamma^*)(\alpha) \in \Gamma(\mathbb{V}^0)
\]

are the horizontal components while \(X_{0,1}\) and \(\alpha_{0,1}\) are the vertical ones, respectively. Here \(\gamma^* : T^*M \rightarrow T^*M\) is the adjoint of \(\gamma\).

Moreover, the exterior differential of forms on \(M\) has the following bigraded decomposition \(d = d_{1,0}^1 + d_{2,1}^1 + d_{0,1}^0\) associated with (2) (see, [17, 18]). The operator \(d_{1,0}^1\) is called the \textit{covariant exterior derivative} and is defined by

\[
d_{1,0}^1 \alpha(X_0, X_1, \ldots, X_k) := d\alpha((id - \gamma)X_0, \ldots, (id - \gamma)X_k),
\]

for all \(\alpha \in \Omega^k(M)\) and \(X_0, X_1, \ldots, X_k \in \Gamma(TM)\). In general, the covariant exterior derivative is not a coboundary operator.

The \textit{curvature} of a connection \(\gamma\) on \((M, \mathcal{F})\) is a vector-valued 2-form \(\text{Curv}^\gamma \in \Omega^2(M; \mathbb{V})\) on \(M\) given by

\[
\text{Curv}^\gamma := \frac{1}{2}[\gamma, \gamma]_{\text{FN}},
\]

here \([\cdot, \cdot]_{\text{FN}}\) denotes the Frölicher-Nijenhuis bracket [10]. Denote the space of all \textit{projectable vector fields} on \((M, \mathcal{F})\) by

\[
\mathfrak{x}_{pr}(M, \mathcal{F}) = \{ Z \in \mathfrak{x}(M) \mid [Z, \Gamma(\mathbb{V})] \subset \Gamma(\mathbb{V}) \}.
\]

The space of all (local) projectable vector fields which are tangent to the horizontal subbundle of a connection \(\gamma\) will be denoted by \(\Gamma_{pr}(\mathbb{H}^\gamma)\). It follows from (4) that

\[
\text{Curv}^\gamma(Z_1, Z_2) = \gamma([Z_1, Z_2])
\]

for any \(Z_1, Z_2 \in \Gamma_{pr}(\mathbb{H}^\gamma)\).

It is well-known that the set of all connections on a foliated manifold is an affine space. Indeed, fixing a connection \(\gamma\) on \((M, \mathcal{F})\), it is easy to see that any other connection \(\tilde{\gamma}\) is of the form \(\tilde{\gamma} = \gamma - \Xi\), where the vector bundle morphism
\( \Xi : TM \to TM \) is called the connection difference form and satisfies the conditions \( \text{im} \Xi \subseteq \nabla \subseteq \ker \Xi \). The horizontal subbundle associated with \( \tilde{\gamma} \) is given by

\[
\mathbb{H}^{\tilde{\gamma}} = (\text{Id} + \Xi)(\mathbb{H}^{\gamma})
\]

and hence

\[
\Gamma_{pr}(\mathbb{H}^{\tilde{\gamma}}) = \{ \tilde{Z} = Z + \Xi(Z) \mid Z \in \Gamma_{pr}(\mathbb{H}^{\gamma}) \}.
\]

From here and (4), one can deduce the transition rule for the curvature:

\[
\text{Curv}^{\tilde{\gamma}}(Z_1, Z_2) = \text{Curv}^{\gamma}(Z_1, Z_2) + [\Xi(Z_1), \Xi(Z_2)] + [\Xi(Z_1), Z_2] - [\Xi(Z_2), Z_1] - \Xi([Z_1, Z_2]),
\]

for \( Z_1, Z_2 \in \Gamma_{pr}(\mathbb{H}^{\gamma}) \).

3. The averaging procedure. First, we recall the averaging procedure for connections on a regular foliated manifold \((M, \mathcal{F})\).

Let \( G \) be a compact and connected Lie group and \( \mathfrak{g} \) its Lie algebra. Suppose that we are given an action \( \Phi : G \times M \to M \) of \( G \) which preserves the foliation, \( d_m \Phi_g \nabla_m = \nabla_{\Phi_g(m)} \), for all \( g \in G \). Equivalently,

\[
Y \in \Gamma(\mathcal{V}) \implies \Phi_g^* Y \in \Gamma(\mathcal{V}) \quad \forall g \in G.
\]

For every \( \xi \in \mathfrak{g} \), the corresponding infinitesimal generator \( \xi_M \in \Gamma(TM) \) of the \( G \)-action is defined by

\[
\xi_M(p) := \frac{d}{dt}_{|t=0} \Phi_{\exp(t\xi)}(p), \quad p \in M.
\]

Condition (7) implies that each infinitesimal generator is a projectable vector field,

\[
\xi_M \in \mathfrak{X}_{pr}(M, \mathcal{F}) \quad \forall \xi \in \mathfrak{g}.
\]

As a consequence, the \( G \)-action preserves the space of all projectable vector fields, \( \Phi_g^*(\mathfrak{X}_{pr}(M, \mathcal{F})) = \mathfrak{X}_{pr}(M, \mathcal{F}) \).

Let \( \Omega^k(M; TM) \) the space of vector-valued \( k \)-forms. For any \( K \in \Omega^k(M; TM) \), the \( G \)-average of \( K \) is the vector-valued form \( \langle K \rangle^G \in \Omega^k(M; TM) \) defined by the standard formula:

\[
\langle K \rangle^G := \int_M \Phi_g^* K dg.
\]

Here, the pull-back \( \Phi_g^* K \) of \( K \) is given by

\[
(\Phi_g^* K)(Y_1, ..., Y_k) = \Phi_g(K(\Phi_g Y_1, ..., \Phi_g Y_k))
\]

for \( Y_1, ..., Y_k \in \mathfrak{X}(M) \) and the integral is taken with respect to the normalized Haar measure \( dg \) on \( G \), \( \int_G dg = 1 \).

Recall that a vector-valued \( k \)-form \( K \) is said to be \( G \)-invariant if \( \Phi_g^* K = K \) for all \( g \in G \). Since the Lie group \( G \) is connected, this invariance condition can be expressed in infinitesimal terms: \( L_{\xi_M} K = 0 \) for all \( \xi \in \mathfrak{g} \). It is clear that the \( G \)-average \( \langle K \rangle^G \) is \( G \)-invariant for any \( K \). We have the following invariance criterion: \( K \) is \( G \)-invariant if and only if \( \langle K \rangle^G = K \).

Property (7) implies that the averaging operator preserves the set of all connections. In other words, for any connection \( \gamma \) on \((M, \mathcal{F})\), its \( G \)-average \( \tilde{\gamma} := \langle \gamma \rangle^G \) is a \( G \)-invariant vector-valued 1-form which again satisfies the conditions in (1). From the property that the Frölicher-Nijenhuis bracket is a natural operation with respect to the pull-back, it follows that the curvature form of \( \tilde{\gamma} \) is also \( G \)-invariant,

\[
\langle \text{Curv}^{\tilde{\gamma}} \rangle^G = \text{Curv}^{\tilde{\gamma}}.
\]
Indeed,
\[ \Phi^*_g \text{Curv} \tilde{\gamma} = \frac{1}{2} \Phi^*_g [\tilde{\gamma}, \tilde{\gamma}]_{FN} = \frac{1}{2} [\Phi^*_g \tilde{\gamma}, \Phi^*_g \tilde{\gamma}]_{FN} = \text{Curv} \tilde{\gamma}. \]

Now, consider the connection difference form
\[ \Xi^G := \gamma - \langle \gamma \rangle^G \in \Omega^1(M; \mathcal{V}). \]  

**Lemma 3.1.** We have the following representation
\[ \Xi^G = \int_G \int_0^1 \Phi^*_{\exp(t\xi)} [\gamma, \xi_M]_{FN} dt dg \quad (g = \exp \xi). \]  

**Remark 3.1.** The integral over \( G \) in the right hand side of (9) is well-defined because of the properties of the exponential map of a compact and connected Lie group (for details, see [20]).

**Proof of Lemma 3.1.** By the fundamental theorem of calculus, we obtain
\[ \Phi^*_{\exp(\xi)} \gamma - \gamma = \int_0^1 \Phi^*_{\exp(t\xi)} L_{\xi_M} \gamma dt. \]  

Integrating the equality (10) with respect to the Haar measure, we get
\[ \langle \gamma \rangle^G - \gamma = \int_G \int_0^1 \Phi^*_{\exp(t\xi)} L_{\xi_M} \gamma dt \, dg, \quad (g = \exp \xi). \]  

From (8) and the identity \( L_{\xi_M} \gamma = -[\gamma, \xi_M]_{FN} \), we obtain formula (9).

For connections on foliated manifolds, we also have the following invariance criteria.

**Proposition 3.2.** For a given connection \( \gamma \) on \((M, \mathcal{F})\) and a foliation preserving action \( \Phi : G \times M \to M \) of a compact and connected Lie group \( G \), the following conditions are equivalent:

(i) \( \gamma \) is \( G \)-invariant;

(ii) \( \langle \gamma \rangle^G = \gamma \);

(iii) \( [\gamma, \xi_M]_{FN} = 0 \) for every \( \xi \in \mathfrak{g} \);

(iv) the horizontal distribution \( \mathbb{H}^\gamma \) is \( G \)-invariant,
\[ d_m \Phi_g (\mathbb{H}^\gamma_m) = \mathbb{H}^\gamma_{\Phi_g(m)} \forall g \in G; \]

(v) the connection difference form \( \Xi^G \) is zero.

**Proof.** The equivalence between the items (i) and (ii) follows by straightforward computations. The implication (i) \(\Rightarrow\) (iii) follows from the relations:
\[ \Phi^*_{\exp(t\xi)} [\gamma, \xi_M]_{FN} = -\Phi^*_{\exp(t\xi)} L_{\xi_M} \gamma = -\frac{d}{dt} \Phi^*_{\exp(t\xi)} \gamma. \]  

Conversely, condition (iii) together with (11) and the connectedness of \( G \) imply the invariance condition (i). Here we use the fact [23]: every element of a connected Lie group is the product of \( \exp(\xi_1) \) and \( \exp(\xi_2) \) for some \( \xi_1, \xi_2 \in \mathfrak{g} \). The equivalence between the items (i) and (iv) follows from the fact that, the \( G \)-invariance of \( \gamma \) is equivalent to the equation
\[ \gamma_{\Phi_g(m)} \circ T_m \Phi_g = T_m \Phi_g \circ \gamma_m. \]

Finally, the equivalence between the items (ii) and (v) follows directly from (8).

Next, we formulate some key properties of the averaged connection in the case of a leaf-tangent \( G \)-action.
Lemma 3.3. Assume that the $G$-action on $(M, \mathcal{F})$ is leaf-tangent,
\[ \xi_M \in \Gamma(\mathcal{V}) \forall \xi \in \mathfrak{g}. \] (12)
Then, for every connection $\gamma$ on $(M, \mathcal{F})$ the following assertions hold:
(a) $\gamma$ is $G$-invariant if and only if
\[ [Z, \xi_M] = 0 \forall Z \in \Gamma_{pr}(\mathbb{H}^\gamma), \quad \xi \in \mathfrak{g}. \] (13)
(b) The space of horizontal projectable vector fields associated with the averaged connection $\bar{\gamma} = (\gamma)^G$ is described as
\[ \Gamma_{pr}(\mathbb{H}^\gamma) = \{ (Z)^G \mid Z \in \Gamma_{pr}(\mathbb{H}^\gamma) \}. \] (14)
(c) For every $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$,
\[ \Xi^G(Z) = -\int G \int_0^1 \Phi^*_\exp(t\xi)[Z, \xi_M]dtdg. \] (15)

Proof. (a) First, we suppose that the $G$-action is arbitrary. For each $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$, we have
\[ [\gamma, \xi_M]_{FN}(Z) = [\gamma(Z), \xi_M] - \gamma([Z, \xi_M]) = -\gamma([Z, \xi_M]). \]
By item (iii) of Proposition 3.2, the $G$-invariance of $\gamma$ is equivalent to the condition that $[Z, \xi_M]$ is a horizontal vector field. If the action is leaf-tangent, then the vector field $[Z, \xi_M]$ is always vertical and hence it must be equals to zero. Conversely, if condition (13) holds, then $[\gamma, \xi_M]_{FN}(Z) = 0$ for all $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$ and $[\gamma, \xi_M]_{FN}(V) = 0$ for each $V \in \Gamma(\mathcal{V})$.

(b) Each vector field $\tilde{Z} \in \Gamma_{pr}(\mathbb{H}^\gamma)$, is of the form $\tilde{Z} = Z + \Xi^G(Z)$ with $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$. Moreover $\tilde{Z}$ is a $G$-invariant vector field by the item (a). From here and the fact that the average of $\Xi^G$ is zero, we get that $\tilde{Z} = (\tilde{Z})^G = (Z)^G$. This proves (14).

(c) For every $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$, and the $G$-invariant vector field $\bar{Z} := Z + \Xi^G(Z) \in \Gamma_{pr}(\mathbb{H}^\gamma)$ it follows from formula (9) that
\[ \Xi^G(Z) = \Xi^G(\bar{Z}) = \int G \int_0^1 \Phi^*_\exp(t\xi) \left( [\gamma, \xi_M]_{FN}(\bar{Z}) \right) dtdg. \] (16)

Then, formula (15) follows from (16) and the equality:
\[ [\gamma, \xi_M]_{FN}(\bar{Z}) = [\gamma, \xi_M]_{FN}(Z). \]

\hfill \Box

4. Hannay-Berry connections on foliations. We start by recalling some general facts and notions for Lie group actions on Poisson manifolds due to [8, 9, 11].

Canonical and cotangential Actions. Let $(M, P)$ be a Poisson manifold. Suppose that we are given an action $\Phi : G \times M \to M$ of a Lie group $G$ with Lie algebra $\mathfrak{g}$. Consider the infinitesimal action (anti-homomorphism) $g \ni \xi \mapsto \xi_M \in \mathfrak{X}(M)$. Recall that the $G$-action is said to be canonical or Poisson if the Lie group acts by Poisson diffeomorphisms, $\Phi^*_g = P$ for all $g \in G$. If the Lie group $G$ is connected, then this condition can be written in infinitesimal terms: $L_{\xi_M}P = 0$ for all $\xi \in \mathfrak{g}$. The property for the action $\Phi$ to be tangential means that all infinitesimal generators are tangent to the symplectic foliation of $P$, that is, $\xi_M(m) \in P^t(T^*_m M)$ for any $m \in M$ and $\xi \in \mathfrak{g}$. Follow [8, 9], we say that the $G$-action on the Poisson manifold is cotangential if there exists a linear map $\mu : \mathfrak{g} \to \Omega^1(M)$ such that
\[ \xi_M = P^\xi \mu^\xi \quad \forall \xi \in \mathfrak{g}, \]

where \( \mu^\xi := \mu(\xi) \). It is clear that a cotangential action is tangential, but the converse is not true, in general \([8]\). According to \([8, 9]\) the map \( \mu \) is called a cotangent lift or, a pre-momentum mapping as in \([11]\).

**Lemma 4.1.** A cotangential action \( \Phi \) of a connected Lie group \( G \) on a Poisson manifold \((M, P)\) is canonical if and only if for every \( \xi \in \mathfrak{g} \), the pull-back of the 1-form \( \mu^\xi \) to each symplectic leaf of \( P \) is closed,

\[ P^\xi d\mu^\xi = 0 \quad (17) \]

for all \( \xi \in \mathfrak{g} \).

**Proof.** By definition of the morphism \( P^\xi \) and the standard properties of the Lie derivative, we have \((P^\xi d\mu^\xi)(\alpha, \beta) = d\mu^\xi(P^\xi(\alpha), P^\xi(\beta))\) and

\[ (L_{\xi_M}P)(\alpha, \beta) = (L_{P^\xi}P)(\alpha, \beta) = d\mu^\xi(P^\xi(\alpha), P^\xi(\beta)) \]

for any \( \alpha, \beta \in \Omega^1(M) \). This proves the assertion. \( \square \)

Notice that a pre-momentum mapping \( \mu \) is determined uniquely modulo the transformation \( \mu \mapsto \mu + \nu \) for arbitrary \( \nu \in \text{Hom}(\mathfrak{g}, \Omega^1(M)) \) such that \( P^\xi \nu^\xi = 0 \) for all \( \xi \in \mathfrak{g} \). It is easy to see that the leafwise closedness condition \((17)\) is independent of this freedom.

Not every cotangential and canonical action is Hamiltonian, as we show in the following example.

**Example 4.2.** Let \( U = \mathbb{R}^3_+ \setminus \{(y_1 = y_2 = 0\} \) be the complement of the \( y_3 \)-axis. Consider the \( S^1 \)-action on \( U \) with infinitesimal generator

\[ \xi_U = y_3 \left( y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} \right) + g_0(y) \frac{\partial}{\partial y_3}, \]

where \( g_0(y) = -\frac{1}{2} \ln(y_1^2 + y_2^2) \). Now, consider the Poisson structure on \( U \)

\[ P_0 := (y_1g_0 + y_2h) \frac{\partial}{\partial y_2} \wedge \frac{\partial}{\partial y_3} + (y_2g_0 - y_1h) \frac{\partial}{\partial y_3} \wedge \frac{\partial}{\partial y_1} - (y_1^2 + y_2^2) y_3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial y_2}, \quad (18) \]

where \( h(y) := \frac{y_2}{\sqrt{y_1^2 + y_2^2}} \). Consider the closed 1-form \( \mu^\xi = \xi_U \) and the \( S^1 \)-action is cotangential, and canonical on \((U, P_0)\) but not Hamiltonian. The last assertion follows from the fact that there exists a symplectic leaf \( i : S \mapsto U \) of \( P_0 \) such that \( i^* \mu^\xi \) is not exact.

In the 3-dimensional case, to construct a new Poisson \( G \)-space with pre-momentum map from a given one, we can try to use the conformal invariance property of Poisson structures.

**Example 4.3.** Let \( M \) be a 3-dimensional oriented manifold equipped with a nowhere vanishing 3-tensor field \( \mathcal{T} \in \Gamma(\wedge^3 TM) \). Let \( P = i_3 \mathcal{T} \) be a Poisson tensor on \( M \), where the 1-form \( \beta \in \Omega^1(M) \) satisfies the integrability condition (Jacobi identity):

\[ d\beta \wedge \beta = 0. \]

Suppose that there exists a canonical and cotangential \( G \)-action on \( M \) with pre-momentum map \( \mu : \mathfrak{g} \to \Omega^1(M) \). Then, for any nowhere vanishing function \( k \in \]
the Poisson structures \( P \) and \( kP \) have one and the same symplectic foliation and hence, the \( G \)-action is also cotangential relative to \( kP \). Moreover, this action is canonical on \((M, kP)\) if and only if
\[
dk \wedge \mu \wedge \beta = 0.
\]

In general, the previous example gives rise to the question on the study of deformation of Poisson \( G \)-spaces with pre-momentum.

**The Case of Poisson Foliations.** Recall that a *Poisson foliation* is a triple \((M, F, P)\) consisting of a regular foliated manifold \((M, F)\) equipped with a vertical Poisson bivector field \( P \in \Gamma(\wedge^2 V) \), \([P, P]_{SCH} = 0\). Thus, the Poisson structure \( P \) is characterized by the property: every symplectic leaf of \( P \) is contained in a leaf of \( F \).

A connection \( \gamma \) on the foliated manifold \((M, F, P)\) is said to be *Poisson* if every (local) \( HH\)-tangent projectable vector field \( Z \in \Gamma_{pr}(HH) \) is Poisson on \((M, P)\), that is, \( L_Z P = 0 \). In this case, \( \text{Curv}^\gamma(Z_1, Z_2) \) is a vertical Poisson vector field, for every \( Z_1, Z_2 \in \Gamma_{pr}(HH) \).

**Example 4.4.** As is well known, every Poisson foliation which comes from a locally trivial fiber bundle, admits a Poisson connection. In particular, every trivial foliation defined by the product of arbitrary manifold \( B \) and a Poisson manifold \((N, P_0)\) carries a canonical Poisson connection \( \gamma_0 \). In fact, any Poisson connection on \( B \times N \) is given by \( \gamma = \gamma_0 + \Xi \) where \( \Xi(Z)|_{m \times N} \) is a Poisson vector field of \( P_0 \) for all \( Z \in \Gamma_{pr}(HH) \) and \( m \in B \).

The class of Poisson connections on foliated manifolds also appears in the context of semilocal Poisson geometry [18, 19, 21].

**Example 4.5.** Let \((M, P)\) be a Poisson manifold. Then, for every embedded symplectic leaf of \( P \) there exists a tubular neighborhood \( U \) of \( S \) which inherits a canonical structure of Poisson foliation with a Poisson connection.

Now, assume that we have on \((M, F, P)\) a foliation-preserving action \( \Phi : G \times M \to M \) of a connected and compact Lie group \( G \). The following result states conditions under which the \( G \)-averaging of a Poisson connection inherits the property of being Poisson.

**Lemma 4.6.** Let \((M, F, P)\) be a Poisson foliation. Suppose that the \( G \)-action is leaf-tangent (condition (12)) and canonical relative to \( P \),
\[
L_{\xi_M} P = 0 \quad \forall \ \xi \in g.
\]
Then, the \( G \)-average \( \bar{\gamma} \) of every Poisson connection \( \gamma \) on \((M, F, P)\) is again Poisson. Moreover, the curvature of \( \bar{\gamma} \) has the following property: if \( Z_1, Z_2 \in \Gamma_{pr}(HH) \) then \( \text{Curv}^\bar{\gamma}(Z_1, Z_2) \) is a vertical \( G \)-invariant Poisson vector field.

**Proof.** Taking into account that the action is canonical and \( \gamma \) is a Poisson connection, by standard properties of the averaging operator we obtain
\[
0 = \langle L_Z P \rangle^G = L_{\langle Z \rangle^G} P,
\]
for all \( Z \in \Gamma_{pr}(HH) \), that is, the average of a \( \gamma \)-horizontal projectable vector field is Poisson. Under the assumption that the action is leaf tangent, the item (b) of Lemma 3.3 implies that the \( \bar{\gamma} \)-horizontal projectable vector fields are Poisson and hence, \( \bar{\gamma} \) is a Poisson connection. The last assertion of the lemma follows directly from (5). \( \square \)
Now, for the class of canonical and cotangential actions on a Poisson foliated manifold, we get the following result.

**Theorem 4.7.** Suppose we have a $G$-action on the Poisson foliation $(M, F, P)$ which is canonical and cotangential with pre-momentum $\mu$. Let $\gamma$ be an arbitrary Poisson connection and $\bar{\gamma} := \langle \gamma \rangle^G$ its $G$-average. Then, the connection difference form $\Xi^G = \gamma - \bar{\gamma}$ takes values on Hamiltonian vector fields of the leaf-tangent Poisson structure $P$, that is

$$\Xi^G(Z) = P^i dQ(Z), \quad \forall Z \in \Gamma_{pr}(\mathbb{H}^\gamma),$$

where $Q \in \Gamma(\mathbb{V}^0)$ is a horizontal 1-form defined by

$$Q(Z) := - \int_G \int_0^1 \Phi_{exp(t\xi)}^* i_Z (\gamma^* \mu^\xi) dt dg \quad \forall Z \in \Gamma_{pr}(\mathbb{H}^\gamma).$$

Moreover, the curvature of the averaged connection $\bar{\gamma}$ is given by

$$\text{Curv} \bar{\gamma}(Z_1, Z_2) = \text{Curv} \gamma(Z_1, Z_2) + P^2 d \left( d_{1,0} Q(Z_1, Z_2) + \frac{1}{2} \{ Q(Z_1), Q(Z_2) \}_P \right),$$

for all $Z_1, Z_2 \in \Gamma_{pr}(\mathbb{H}^\gamma)$.

**Proof.** Observe that the map $\bar{\mu} : \mathfrak{g} \to \Omega^1(M)$ given by

$$\bar{\mu} := \bar{\gamma}^* \mu$$

is also a pre-momentum for the $G$-action. Indeed, since $P^2 \circ \bar{\gamma}^* = P^2$, we have

$$\xi_M = P^2 \mu^\xi = P^2 \bar{\gamma}^* \mu^\xi = P^2 \bar{\mu}^\xi.$$

Now, pick $Z \in \Gamma_{pr}(\mathbb{H}^\gamma)$. By item (c) of Lemma 3.3, taking into account that $Z \in \Gamma(\mathbb{H}^\gamma)$ is Poisson, and the fact that the $G$-action is canonical, we obtain

$$\Xi^G(Z) = - P^2 \left( \int_G \int_0^1 \Phi_{exp(t\xi)}^* L_Z \bar{\mu}^\xi dt dg \right),$$

$$= - \int_G \int_0^1 \Phi_{exp(t\xi)}^* P^2 i_Z d\bar{\mu}^\xi dt dg - P^2 d \left( \int_G \int_0^1 \Phi_{exp(t\xi)}^* i_Z \bar{\mu}^\xi dt dg \right).$$

So, by Lemma 4.1, $P^2 i_{\Xi \circ (\bar{\gamma}^*)} d\bar{\mu}^\xi = 0$. On the other hand, by the $G$-invariance of $\bar{\gamma}$ and item (a) of Lemma 3.3, we conclude $[(I - \bar{\gamma}^*)(Z), \xi_M] = 0$. Also, since the connection $\bar{\gamma}$ is Poisson, $[(I - \gamma^*)(Z), P^2 \bar{\mu}^\xi] = P^2 L_{(I - \gamma^*)(Z)} \bar{\mu}^\xi$. It follows from Cartan’s formula and the equality $(I - \gamma^*) \circ \bar{\mu} = 0$ that

$$P^2 L_{(I - \gamma^*)(Z)} \bar{\mu}^\xi = P^2 i_{(I - \gamma^*)(Z)} d\bar{\mu}^\xi + P^2 d i_{(I - \gamma^*)(Z)} \bar{\mu}^\xi$$

$$= P^2 i_{(I - \gamma^*)(Z)} d\bar{\mu}^\xi + P^2 i_{\Xi \circ (\bar{\gamma}^*)} d\bar{\mu}^\xi.$$  

Combining the above results, we get

$$0 = P^2 i_{(I - \gamma^*)(Z)} d\bar{\mu}^\xi = P^2 i_Z d\bar{\mu}^\xi + P^2 i_{\Xi \circ (\bar{\gamma}^*)} d\bar{\mu}^\xi = P^2 i_Z d\bar{\mu}^\xi.$$  

Therefore, the first term in (23) vanishes. This proves (19) and (20). Formula (21) follows from (6).

As consequences of the proof of Theorem 4.7, we have the following results.

**Corollary 4.8.** The following assertions hold.
(a) One can choose a pre-momentum map \( \bar{\mu} \) of the \( G \)-action such that
\[
d\bar{\mu}^\xi(P^\xi\alpha, Z) = 0,
\]
for all \( \xi \in \mathfrak{g}, \alpha \in \Omega^1(M) \) and \( Z \in \Gamma(H^\gamma) \). In terms of \( \bar{\mu} \), formula (20) for \( Q \) takes the form
\[
Q(Z) := -\int_G \int_0^1 \Phi^*_{\exp(\xi)} \bar{\gamma} Z(\bar{\mu}^\xi) dt dg \quad (g = \exp \xi).
\]
(b) Condition (25) is preserved under the following transformation of the connection: \( \gamma \mapsto \gamma = \gamma + P^\xi \circ \psi \) for arbitrary vector bundle morphism \( \psi : TE \rightarrow T^*E \).
(c) If \( \bar{\mu} \) satisfies (25), then \( i_Z \bar{\mu}^\xi \) is a Casimir function for all \( \xi \in \mathfrak{g} \) and \( Z \in \Gamma_{pr}(H^\gamma) \).

Proof. (a) Define the “new” pre-momentum map \( \bar{\mu} \) by (22). Then we one show that equation (24) implies that \( \bar{\mu} \) satisfies (25).
(b) For each \( \bar{Z} \in \Gamma(H^\gamma) \), there exists a \( Z \in \Gamma(H^\gamma) \) such that \( \bar{Z} = Z + P^\xi(\Psi(Z)) \).

By straightforward computations and using the canonicity of \( G \), we get
\[
d\bar{\mu}^\xi(P^\xi\alpha, \bar{Z}) = d\bar{\mu}^\xi(P^\xi\alpha, Z + P^\xi(\Psi(Z))) = (P^\xi d\bar{\mu}^\xi)(\alpha, \Psi(Z)) = 0.
\]
(c) Let \( Z \in \Gamma_{pr}(H^\gamma) \). Since \( Z \) is a \( G \)-invariant Poisson vector field, it follows that
\[
0 = [Z, \xi_M] = P^\xi(L_Z \bar{\mu}^\xi), \quad \text{for all } \xi \in \mathfrak{g}.
\]
From this fact and condition (25), we have
\[
0 = d\bar{\mu}^\xi(P^\xi\alpha, Z) = L_{P^\xi\alpha}(\bar{\mu}^\xi(Z)) - \bar{\mu}^\xi(P^\xi(L_Z \alpha)) + \bar{\mu}^\xi([Z, P^\xi\alpha]),
\]
for every \( \alpha \in \Omega^1(M) \). This implies that \( i_Z \bar{\mu}^\xi \) is a Casimir function.

By Lemma 3.1, we derive the following consequence of Theorem 4.7.

**Corollary 4.9.** The horizontal distribution of the averaged connection \( \bar{\gamma} \) is generated by the \( G \)-invariant Poisson vector fields of the form
\[
\langle Z \rangle^G = Z + P^\xi dQ(Z),
\]
where \( Z \) runs over \( \Gamma_{pr}(H^\gamma) \).

**Remark 4.1.** In the context of the Poisson cohomology of \( (M, P) \), one can derive from Corollary 4.9 the following fact [2]: for every \( \gamma \)-horizontal \( k \)-cocycle \( A \in \Gamma(\Lambda^k H^\gamma) \), \( [P, A]_{SCH} = 0 \), its Poisson cohomology class is represented by a \( G \)-invariant \( k \)-tensor. This partially recovers a result on the Poisson invariant cohomology due to [8].

To end this section, let us consider some special cases. It is clear that the hypotheses of Theorem 4.7 hold in the case when the \( G \)-action is *locally Hamiltonian* on \( (M; P) \), that is, the pre-momentum map \( \mu \) is closed-
\[
d\mu^\xi = 0 \quad \forall \xi \in \mathfrak{g}.
\]
In particular, in the standard case [12] of a *Hamiltonian \( G \)-action with momentum map* \( J : M \rightarrow g^* \),
\[
\xi_M = P^\xi d\bar{\gamma}^\xi,
\]
formula (26) for the horizontal 1-form \( Q \) reads
\[
Q = -\int_G \int_0^1 \Phi^*_{\exp(\xi)}(d_{1,0}^\xi) dt dg.
\]
Theorem 4.7 presents a generalized version of the results on Hannay-Berry connections obtained in [12] for the case of a Poisson fiber bundle equipped with Hamiltonian $G$-action with momentum map. Thus, in the case of a canonical and cotangential $G$-action with pre-momentum map $\mu$ on a Poisson foliation $(M, F, P)$, the averaged Poisson connection $\check{\gamma} = \langle \gamma \rangle^G$ can be also called a Hannay-Berry connection on $(M, F, P)$.

Here is an example of a cotangential and canonical $G$-action on a Poisson fibration which is not Hamiltonian.

**Example 4.10.** Consider the trivial bundle $M = B \times U$ over a manifold $B$ whose fiber is the complement $U \subset \mathbb{R}^3$ of the $y_3$-axis. We endow $M$ with the vertical Poisson structure $P$ given by (18) where $h(y) := \frac{y_2}{\sqrt{y_1^2 + y_2^2}}$ and $g_0$ is replaced by $g(x, y) := -\frac{1}{2} \ln(a(x)^2(y_1^2 + y_2^2))$ with a positive definite function $a \in C^\infty(B)$. Introduce the Ehresmann connection on $M$ given by $\gamma := \gamma_0 + \frac{1}{a} \partial a \otimes v$, where $\gamma_0$ is the canonical connection on $M \to B$, and $v := (y_1 + y_2 y_3)\frac{\partial}{\partial y_1} + (y_2 - y_1 y_3)\frac{\partial}{\partial y_2} + h \frac{\partial}{\partial y_3}$. Since $[v, P]_{SCBH} = (y_2 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial y_2}) \wedge \frac{\partial}{\partial y_3}$, one can verify without serious difficulty that $\gamma$ is a Poisson connection on $(M \to B, P)$. Consider the leaf-preserving $S^1$-action on $M$ given by the infinitesimal generator $\xi_U = y_3 \left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right) + g(x, y) \frac{\partial}{\partial y_3}$. It follows from Example 4.2 that this action is canonical and contangential with pre-momentum map $\mu^k = -\frac{y_2}{y_1^2 + y_2^2} \partial y_1 + \frac{y_1}{y_1^2 + y_2^2} \partial y_2$, but not Hamiltonian. By Theorem 4.7, the difference connection form $\Xi^{S^1}$ takes values in Hamiltonian vector fields. By (26) and straightforward computations, we get that

$$\Xi^{S^1} = -\frac{1}{a} \partial a \otimes \mathbf{P} \mathbf{d} g = \frac{1}{a} \partial a \otimes \left(y_2 y_3 \frac{\partial}{\partial y_1} - y_1 y_3 \frac{\partial}{\partial y_2} + h \frac{\partial}{\partial y_3}\right),$$

and the Hannay-Berry connection of $\gamma$ is

$$\check{\gamma} = \gamma - \Xi^{S^1} = \gamma_0 + \frac{1}{a} \partial a \otimes \left(y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2}\right).$$

5. Poisson connections with Hamiltonian curvature. Starting with a Poisson foliation $(M, F, P)$, let us denote by $\text{Conn}_H(M, F, P)$ the set of all Poisson connections $\gamma$ whose curvature form takes values in the space of Hamiltonian vector fields of the vertical Poisson structure $P$. More precisely,

$$\text{Curv}^\gamma(Z_1, Z_2) = -P^2 d\sigma^\gamma(Z_1, Z_2) \quad \forall Z_1, Z_2 \in \Gamma^h(\mathbb{H}^d), \quad (27)$$

for a certain horizontal 2-from $\sigma^\gamma \in \Gamma(\Lambda^2 \mathfrak{g}^0)$ which is called a Hamiltonian form of the curvature.

Denote by $\mathcal{C}^k := \mathcal{C}^k(M, F, P)$ the space of all horizontal $k$-forms $\beta \in \Gamma(\Lambda^k \mathfrak{g}^0)$ which take values in the space $\text{Casim}(M, P)$ of Casimir functions of $P$,

$$\beta(X_1, ..., X_k) \in \text{Casim}(M, P) \quad \forall X_i \in \mathfrak{X}_pr(M, F).$$

Then, it is clear that a Hamiltonian form $\sigma^\gamma$ of the curvature in (27) is unique up to the transformations

$$\sigma^\gamma \mapsto \sigma^\gamma + C \quad \forall C \in \mathcal{C}^2. \quad (28)$$

In particular, if $\sigma^\gamma \in \mathcal{C}^2$, then the connection is flat and the covariant exterior derivative $d^1_{\gamma,0}$ is a coboundary operator.
Definition 5.1. A Poisson connection $\gamma \in \text{Conn}_H(M,\mathcal{F},P)$ is said to be admissible if there exists a Hamiltonian form $\sigma = \sigma^i \in \Gamma(\Lambda^2\mathcal{V}(\mathcal{F}))$ of the curvature in (27) which satisfies the $\gamma$-covariantly constant condition:

$$d_{1,0}^\gamma \sigma = 0. \tag{29}$$

Notice that, in general, for a given $\gamma \in \text{Conn}_H(M,\mathcal{F},P)$, we have $d_{1,0}^\gamma \sigma \in \mathcal{C}^3$. In fact, for $u_1, u_2, u_3 \in \mathfrak{X}_{pr}(M,\mathcal{F})$, $P^d(d_{0,1}^\gamma \sigma(u_1,u_2,u_3))$ equals the cyclic sum

$$\sum_{(1,2,3)} -(L_{(1-\gamma)}u_1)P^d(\sigma(u_2,u_3)) + L_{(1-\gamma)}u_1(P^d(\sigma(u_2,u_3))).$$

The first term of each summand vanishes due to the Poisson property of $\gamma$. Taking into account (27), the fact $d_{1,0}^\gamma \sigma \in \mathcal{C}^3$ follows from the Bianchi identity:

$$P^d(d_{1,0}^\gamma \sigma(u_1,u_2,u_3)) = \frac{1}{2} [\gamma, \text{Curv}^\gamma]_{FN}(u_1,u_2,u_3) = 0. \tag{30}$$

On the other hand, one can also show from (27) that

$$(d_{1,0}^\gamma)^2 \sigma = L_{\text{Curv}^\gamma} \sigma = 0. \tag{31}$$

Observe that $d_{0,1}^\gamma(C^k) \subset C^{k+1}$. Hence, one can define the coboundary operator $\bar{d}^\gamma : C^k \to C^{k+1}$ just by $\bar{d}^\gamma := d_{0,1}^\gamma|_{C^k}$. Thus, one can associate to the setup $(M,\mathcal{F},P,\gamma)$ the cochain complex $(\bigoplus_{k=0}^\infty C^k, \bar{d}^\gamma)$ called the de Rham-Casimir complex [14, 21]. In terms of this complex, (30) and (31) say that $d_{1,0}^\gamma \sigma$ is a 3-cocycle. Taking into account that the freedom in the choice of $\sigma^\gamma$ is given by the transformation (28), we derive the following fact: the connection $\gamma$ is admissible if and only if the cohomology class of $d_{1,0}^\gamma \sigma$ relative to $\bar{d}^\gamma$ is trivial. Indeed, under the triviality property, we have $d_{1,0}^\gamma \sigma = d_{1,0}^\gamma c$ for some $c \in \mathcal{C}^2$. From here, we have that $\sigma - c$ is a Hamiltonian form for $\text{Curv}^\gamma$ satisfying (29).

We remark that if we have $\gamma, \tilde{\gamma} \in \text{Conn}_H(M,\mathcal{F},P)$ such that the connection difference form $\Xi := \gamma - \tilde{\gamma}$ takes values in Hamiltonian vector fields, then $\bar{d}^\gamma = \bar{d}^{\tilde{\gamma}}$. By Theorem 4.7, for a contangential and canonical $G$-action on $(M,\mathcal{F},P,\gamma)$ the de Rham-Casimir complex associated to the complex $(\bigoplus_{k=0}^\infty C^k, \bar{d}^\gamma)$ remains the same after averaging the connection form $\gamma$.

Example 5.2. Let $\pi : N \to B$ be a principal $G$-bundle and $(F,P_F)$ a Poisson $G$-space with pre-momentum map $\mu_F : \mathfrak{g} \to \Omega^1(F)$. Consider the associated bundle $M = N \times_G F$ which is a locally trivial fiber bundle $(M,P)$ over $B$ with typical fiber bundle $(F,P_F)$. Here, $P$ is a vertical Poisson tensor on $M$ defined by $P = (\pi_{N\times F},P_F)$, where $\pi_{N\times F} = N \times F \to N \times_G F$ is the canonical projection. A given connection $\theta \in \Omega^1(N;\mathfrak{g})$ on the principal bundle $N$ induces an Ehresmann connection $\gamma$ on $M$ with horizontal lift

$$\text{hor}^\gamma(u)(b,y) = (T_{b,y}\pi_{N\times F})(\text{hor}^\theta(u)(b) \oplus 0_y)$$

for $b \in B$, $y \in F$, $u \in \mathfrak{X}(B)$. Then, $\gamma$ is a Poisson connection on $(M,P)$ [22]. Moreover, the curvature form of $\gamma$ is given by

$$\text{Curv}^\gamma(\text{hor}^\gamma(u),\text{hor}^\gamma(v)) = P^d \left( (\pi_{N\times F})_*\mu_F ((\text{Curv}^\theta)(\text{hor}^\theta(u),\text{hor}^\theta(v))) \right)$$

for $u,v \in \mathfrak{X}(B)$. According to [22], the connection $\gamma$ is admissible in the sense of Definition 5.1, if $(F,P_F)$ is a Hamiltonian $G$-space with a momentum map.
Now, suppose that we are given an action \( \Phi \) on \( M \) of a connected and compact Lie group \( G \) which is canonical and cotangential with a pre-momentum map \( \mu \).

Since all infinitesimal generators \( \xi_M \) of the \( G \)-action are tangent to the symplectic foliation of \( P \), we have

\[
k \in \text{Casim}(M, P) \implies L_{\xi_M}k = 0 \quad \forall \xi \in \mathfrak{g}.
\]

Hence, any horizontal 2-form \( C \in \mathbb{C}^2 \) is \( G \)-invariant, \( L_{\xi_M}C = 0 \). It follows that the \( G \)-invariance of a Hamiltonian form \( \sigma^\gamma \) is preserved under transformation (28).

Furthermore, the \( G \)-invariance of Poisson connection \( \gamma \) implies the \( G \)-invariance of any Hamiltonian form \( \sigma \).

**Proposition 5.3.** Under the above assumptions, if \( \gamma \in \text{Conn}_H(M, \mathcal{F}, P) \) is \( G \)-invariant, then any Hamiltonian form \( \sigma \) of the curvature \( \text{Curv}^\gamma \) is \( G \)-invariant.

**Proof.** First, we shall prove that \( L_{\xi_M}\sigma \) is \( G \)-invariant for all \( \xi \in \mathfrak{g} \). Since the Frölicher-Nijenhuis bracket is natural with respect to the pull-back, the curvature \( \text{Curv}^\gamma \) is \( G \)-invariant. So, for every \( \xi \in \mathfrak{g} \) and \( Z_0, Z_1 \in \Gamma_p(H^n) \), we have

\[
0 = L_{\xi_M}(\text{Curv}^\gamma(Z_0, Z_1)) = -P^t dL_{\xi_M}(\sigma(Z_0, Z_1)) = -P^t d[(L_{\xi_M}\sigma)(Z_0, Z_1)].
\]

Thus, \( L_{\xi_M}\sigma \in \mathbb{C}^2 \). It follows from here that \( L_{\xi_M}(L_{\xi_M}\sigma) = 0 \) for all \( \xi \in \mathfrak{g} \) and hence \( L_{\xi_M}\sigma \) is \( G \)-invariant. Due to the compactness of \( G \) [15, Theorem 5.18] \( \mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{j}(\mathfrak{g}) \), where \( \mathfrak{g}' \), \( \mathfrak{j}(\mathfrak{g}) \) denote the derived algebra and the center of \( \mathfrak{g} \), respectively. The \( G \)-invariance of \( L_{\xi_M}\sigma \) for each \( \xi \in \mathfrak{g} \) implies that \( L_{\xi_M}\sigma = 0 \) for all \( \xi \in \mathfrak{g}' \). On the other hand, \( \mathfrak{j}(\mathfrak{g}) \) is generated by elements \( \xi \) such that \( \exp t\xi \) is a closed curve. Then, the flow of \( \xi_M \) is periodic with period, say \( T_\xi > 0 \). The \( G \)-invariance of \( L_{\xi_M}\sigma \) implies that \( \frac{d}{dt}(\text{Fl}_{t\xi}^\gamma)^*\sigma = L_{\xi_M}\sigma \). By integrating in \( t \), we obtain

\[
T_\xi L_{\xi_M}\sigma = \int_0^{T_\xi} L_{\xi_M}\sigma dt = (\text{Fl}_{T_\xi}^\gamma)^*\sigma - \sigma = 0.
\]

Since \( T_\xi > 0 \), we conclude that \( L_{\xi_M}\sigma = 0 \) for all \( \xi \in \mathfrak{j}(\mathfrak{g}) \). Therefore, \( \sigma \) is \( G \)-invariant. \( \square \)

Since the \( G \)-action is canonical relative to \( P \) and preserves the vertical distribution \( V \), it is easy to see that the group \( G \) naturally acts on the set of Poisson connections \( \text{Conn}_H(M, \mathcal{F}, P) \), \( \gamma \mapsto \Phi_{\gamma}^*\gamma \). For every \( \beta \in \Gamma(\Lambda^q V^0) \) and \( Q \in \Gamma(\Lambda^1 V^0) \), denote by \( \{Q \wedge \beta\}_P \) the element of \( \Gamma(\Lambda^{q+1} V^0) \) given by

\[
\{Q \wedge \beta\}_P(Z_0, Z_1, ..., Z_q) := \sum_{i=0}^{q} (-1)^i \{Q(Z_i), \beta(Z_0, Z_1, ..., \hat{Z}_i, ..., Z_q)\}_P
\]

where \( \{,\} \) is the Poisson bracket associated to \( P \).

As a consequence of Theorem 4.7, we get the following fact.

**Theorem 5.4.** Suppose that we are given an action \( \Phi \) on \( M \) of a connected and compact Lie group \( G \) which is canonical and cotangential with a pre-momentum map \( \mu \). The averaging procedure preserves the set \( \text{Conn}_H(M, \mathcal{F}, P) \), that is,

\[
\gamma \in \text{Conn}_H(M, \mathcal{F}, P) \implies \langle \gamma \rangle \in \text{Conn}_H(M, \mathcal{F}, P).
\]

Moreover, if \( \sigma \) is a Hamiltonian of the curvature of \( \gamma \), then a Hamiltonian form of the curvature of \( \langle \gamma \rangle \) is given by

\[
\bar{\sigma} = \sigma^{\langle \gamma \rangle} := \sigma^\gamma - \left( d_{\gamma,0}^\gamma Q + \frac{1}{2} \{Q \wedge Q\}_P \right),
\]

(32)
where the horizontal 1-form $Q \in \Gamma(\mathcal{V}^0)$ is defined in terms of $\mu$ by formula (20). Furthermore, the admissibility of $\gamma$, implies that $\langle \gamma \rangle$ is also admissible,

$$d_1^\gamma \sigma = 0 \implies d_1^{(\gamma)} \sigma = 0.$$  

**Proof.** The first part of the theorem is a direct consequence of Theorem 4.7. In particular, the formula for the Hamiltonian form of $\gamma$ follows from equation (21). So, it remains to prove that the averaging procedure preserves the admissibility property. Assume that $d_1^\gamma \sigma = 0$. Since $\sigma \in \Gamma(\mathcal{V}^0)$, the relation (8) implies the equality

$$d_1^\gamma \sigma = d_1^0 \sigma + \{Q \wedge \sigma\}_P.$$  

Recall that the exterior differential has the following bigraded decomposition $d = d_1 + d_0 + d_{-1}$ depending on the connection $\gamma$. Taking account that $d^2 = 0$, we obtain the following identity $(d_1^\gamma)^2 = -[d_0^\gamma, d_{-1}^\gamma]$. In particular, for $Q \in \Omega^{1,0}(M)$, the equation (27) implies that

$$(d_1^\gamma)^2 Q = \{Q \wedge \sigma\}_P.  \quad (34)$$  

On the other hand, since $\gamma$ is a Poisson connection, by straightforward computation, we obtain that

$$d_1^0 \frac{1}{2} \{Q \wedge \sigma\}_P = -\{Q \wedge d_1^0 Q\}_P \quad (35)$$  

From relations (32)-(35), it follows that $d_1^\gamma \sigma = 0$. \hfill \Box

### 6. Cotangential actions on Dirac manifolds.

First, we recall some facts from the theory of Dirac structures (for more details, see [5, 6, 7]). A subbundle $D \subset TM := TM \oplus T^*M$ is said to be a Dirac structure if $D$ is maximally isotropic with respect to the natural pairing $\langle \langle X, \alpha \rangle, \langle Y, \beta \rangle \rangle := \beta(X) + \alpha(Y)$ and involutive with respect to the Courant bracket

$$\llbracket \langle X, \alpha \rangle, \langle Y, \beta \rangle \rangle := \left( [X,Y], L_X \beta - L_Y \alpha + \frac{1}{2}d(\alpha(Y) - \beta(X)) \right).$$  

Every Dirac structure $D$ induces a presymplectic (singular) foliation $(\mathcal{S}, \omega)$ on $M$, where $TS = \text{pr}_TM(D)$ with $\text{pr}_TM : TM \to TM$ the natural projection onto the first factor, and $\omega$ is a (smooth) leafwise presymplectic form. Recall that $\omega$ is defined as follows: at each point $m \in M$, we define $\omega_m(X,Y) := \alpha(Y)$ for all $X,Y \in T_m\mathcal{S}$, with $\alpha \in T^*_mM$ such that $\langle X, \alpha \rangle \in D_m$. On the contrary, each presymplectic foliation $(\mathcal{S}, \omega)$ on $M$, induces a Dirac structure $D$ given on $m \in M$ by

$$D_m := \{ \langle X, \alpha \rangle \mid X \in T_m\mathcal{S}, \alpha \mid_{T_m\mathcal{S}} = -i_X \omega \},$$

provided that $D$ is smooth. Suppose we are given an action $\Phi : G \times M \to M$ of a connected Lie group $G$ on the Dirac manifold $(M, D)$. The $G$-action is said to be Dirac or canonical if

$$(\Phi^*_g X, \Phi^*_g \alpha) \in \Gamma(D) \quad (36)$$

for all $\langle X, \alpha \rangle \in \Gamma(D)$ and $g \in G$. In other words, this condition is just the invariance of the distribution $D$ with respect to the $G$-action. Infinitesimally, (36) reads

$$(X, \alpha) \in \Gamma(D) \implies (L_{\xi_m} X, L_{\xi_m} \alpha) \in \Gamma(D) \forall \xi \in g. \quad (37)$$

Conversely, if the Lie group $G$ is connected, condition (37) means that the $G$-action is Dirac.
Definition 6.1. A $G$-action on the Dirac manifold $(M, D)$ is said to be cotangential if there exists a $\mathbb{R}$-linear mapping $\mu \in \text{Hom}(\mathfrak{g}, \Omega^1(M))$ such that

$$(\xi_M, \mu^\xi) \in \Gamma(D) \quad \forall \xi \in \mathfrak{g}.$$  \hfill (38)

In this case, we say that $\mu$ is a pre-momentum map for the cotangential $G$-action on $(M, D)$.

By (38), the infinitesimal generators of the $G$-action are tangent to the presymplectic foliation $(\mathcal{S}, \omega)$ on $M$ induced by the Dirac structure $D$, if the action is cotangential. Definition 6.1 is a generalization of the notion of a cotangential $G$-action on a Poisson Manifold $(M, \Pi)$. If we consider the Dirac structure $D_\Pi = \text{Graph}(\Pi)$, then one can easily verify that a $G$-action on $M$ is cotangential on the Poisson manifold $(M, \Pi)$ if and only if the $G$-action is cotangential on the Dirac manifold $(M, D_\Pi)$.

Remark 6.1. Definition 6.1 actually coincides with the notion of compatible $G$-action on the Dirac manifold $(M, D)$ introduced in [20].

In terms of the presymplectic form, condition (38) can be rewritten as follows

$$i_{\xi_M} \omega_S = -i_{\xi}^* d\mu^\xi,$$

for any presymplectic leaf $S$ and $\xi \in \mathfrak{g}$.

Denote by

$$I^1_{T_S}(M) := \{ \alpha \in \Omega^1(M) \mid (0, \alpha) \in \Gamma(D) \},$$

the set of all 1-forms $\alpha$ on $M$ vanishing along the characteristic foliation $\mathcal{S}$, $i_{\xi}^* \alpha = 0$ for each leaf $S$ of $\mathcal{S}$. Then, $\mu$ is uniquely determined by (38) modulo elements $\kappa \in \text{Hom}(\mathfrak{g}, I^1_{T_S}(M))$, $\mu \mapsto \mu + \kappa$. Since $i_{\xi}^* d\kappa^{\xi} = 0$, the pull-back $i_{\xi}^* d\mu^\xi$ is independent of the freedom in the choice of $\mu^\xi$.

Our next result states necessary and sufficient conditions under which a cotangential $G$-action of $(M, D)$ is canonical.

Lemma 6.2. A cotangential $G$-action on the Dirac manifold $(M, D)$ is canonical if and only if

$$i_{\xi_M}^* d\mu^\xi = 0 \quad \forall \xi \in \mathfrak{g}$$ \hfill (39)

that is, $\mu^\xi$ is closed along every presymplectic leaf $S$.

Proof. Computing the Courant bracket between $(\xi_M, \mu^\xi)$ and a section $(X, \alpha) \in \Gamma(D)$ and using the isotropy property, we get

$$\Gamma(D) \ni [(\xi_M, \mu^\xi), (X, \alpha)] = (L_{\xi_M} X, L_{\xi_M} \alpha) - (0, i_X d\mu^\xi).$$

It follows that condition (37) is equivalent to (39). \hfill $\square$

We remark that Lemma 6.2 can be reformulated in the following way: the Dirac structure $D$ is canonical with respect to a cotangential $G$-action if and only if the pull-back of $\mu$ to each presymplectic leaf is closed.

In particular, if $\mu$ in (38) is exact-valued, $\mu = d \circ J$, for some $\mathbb{R}$-linear function $J : \mathfrak{g} \to C^\infty(M)$, then one says that we have a Hamiltonian $G$-action on the Dirac manifold with momentum map $J$ (see, also [3]).

Finally, we observe that if a cotangential $G$-action on $(M, D)$ is locally Hamiltonian, $d\mu^\xi = 0$ for all $\xi \in \mathfrak{g}$, then this action is canonical.
7. Averaging the coupling Dirac structures. Our point is to construct $G$-invariant Dirac structures on $(M, F, P)$ by combining the averaging procedure for Poisson connections in $\text{Conn}_H(M, F, P)$ with the so-called coupling method (see also [7, 19, 20]).

**Dirac structures from admissible Poisson connections.** Now, pick an admissible Poisson connection $\gamma \in \text{Conn}_H(M, F, P)$, and fix a Hamiltonian 2-form of the curvature $\sigma = \sigma^\gamma$ in (27). Then, one can introduce the distribution $D^{\gamma, \sigma}$ given by the following subbundle of $TM \oplus T^*M$:

$$D^{\gamma, \sigma} := \{(X + P^\gamma(\alpha), \alpha - i_X \sigma) \mid X \in \Gamma(H^\gamma), \alpha \in \Gamma((H^\gamma)^0)\}. \tag{40}$$

It is clear that $D^{\gamma, \sigma}$ is a regular distribution whose rank is just equal to $\dim M$. By straightforward computations, one can show that $D^{\gamma, \sigma}$ is an isotropic distribution relative to the symmetric form. Moreover, we have the following fact.

**Proposition 7.1.** For every admissible Poisson connection $\gamma \in \text{Conn}_H(M, F, P)$ and a Hamiltonian form $\sigma$ of $\text{Curv}^\gamma$ satisfying (29), the associated distribution $D^{\gamma, \sigma}$ in (40) is a Dirac structure on $M$.

**Proof.** We only need to prove that $D^{\gamma, \sigma}$ is closed under the Courant bracket. Taking into account that

$$D^{\gamma, \sigma} = \text{Graph}(P)|_{\mathbb{H}^\epsilon} \oplus \text{Graph}(\sigma)|_{\mathbb{H}^\epsilon},$$

we fix a set of (local) generators of $D$, defined by the elements of the form $e_\alpha = (P^\gamma(\alpha), \alpha)$ and $e_X = (X, -i_X \sigma)$, with $X \in \Gamma_H(H^\gamma)$ and $\alpha \in \Gamma((H^\gamma)^0)$. Since the bivector $P$ and the connection $\gamma$ are Poisson, we have

$$[e_\alpha, e_\beta] = (P^\gamma(L_{P\gamma}(\beta) - i_{P\gamma}(\beta) d\alpha), L_{P\gamma}(\alpha) - i_{P\gamma}(\alpha) d\alpha) \in D^{\gamma, \sigma}.$$

In addition, if we consider curvature identity (27), we obtain

$$[e_X, e_\alpha] = (P^\gamma(L_X \alpha), L_X \alpha + i_{P\gamma}(\alpha) d\alpha) \in D^{\gamma, \sigma}.$$

Finally, the admissibility of $\gamma$ and the curvature identity imply that

$$[e_X, e_Y] = ([X, Y], -L_X i_Y \sigma + i_Y d\alpha \sigma),$$

$$= ([X, Y]_{1,0}, -i_{[X, Y]} \sigma) + (\text{Curv}^\gamma(X, Y), -i_Y i_X d\sigma) \in D^{\gamma, \sigma}.$$

\[\square\]

**Remark 7.1.** In fact, $D^{\gamma, \sigma}$ is a coupling Dirac structure on the foliated manifold $(M, F)$ associated with the geometric data $(P, \gamma, \sigma)$, [19, 20].

**Invariant Dirac Structures.** Now we suppose that on the Poisson foliation $(M, F, P)$, we are given a canonical action of a connected and compact Lie group $G$ which is contangential respect to $P$ with pre-momentum map $\mu$, that is, $\xi_M = P^\mu \mu^\xi$ for all $\xi \in \mathfrak{g}$.

**Lemma 7.2.** Under the hypothesis above, the $G$-action is cotangential on the Dirac structure $D^{\gamma, \sigma}$.

**Proof.** We set $\tilde{\mu} := \gamma^* \circ \mu$. By straightforward computations, we prove that $\tilde{\mu}$ makes the $G$-action cotangential. For every $\xi \in \mathfrak{g}$, we have

$$P^\xi(\tilde{\mu}^\xi) = P^\xi((\mu^\xi)_{0,1}) = P^\xi(\mu^\xi) = \xi_M,$$

$$\tilde{\mu}^\xi(X) = \mu^\xi(\gamma(X)) = 0, \quad \forall \ X \in \Gamma((H^\gamma)^0).$$

Thus, $(\xi_M, \tilde{\mu}^\xi) \in D^{\gamma, \sigma}$. \[\square\]
Lemma 7.3. Let $\gamma \in \text{Conn}_H(M,F,P)$ be an arbitrary connection and let $\sigma$ be a Hamiltonian form of the curvature of $\gamma$. Set $\bar{\gamma} := (\gamma)^G$ and $\bar{\sigma}$ the 2-form given by (32). Then, the invariance of the distribution $D^{\bar{\gamma},\bar{\sigma}}$ under the $G$-action is equivalent to the $G$-invariance of the 2-form $\bar{\sigma}$, $L_\xi \bar{\sigma} = 0$ for all $\xi \in \mathfrak{g}$.

Proof. The invariance property for the averaged connection $\bar{\gamma}$ implies that the corresponding splittings (2) and (3) are also invariant under the $G$-action. The $G$-invariance condition for $D^{\bar{\gamma},\bar{\sigma}}$ means that for any sections $X \in \Gamma(H^\gamma)$, $\alpha \in \Gamma((H^\gamma)^\theta)$ and $g \in G$, we have

\begin{align}
\Phi_g^*X + \Phi_g^*P^\delta(\alpha) &= \bar{X} + P^\delta(\bar{\alpha}), \quad (41) \\
\Phi_g^*\alpha - i_{\Phi_g^*X}\Phi_g^*\sigma &= \bar{\alpha} - i_{\bar{X}}\sigma \quad (42)
\end{align}

for some $\bar{X} \in \Gamma(H^\bar{\gamma})$ and $\bar{\alpha} \in \Gamma((H^\bar{\gamma})^\theta)$. Taking into account that the action preserves the vertical $V$ and horizontal $H^\bar{\gamma}$ distributions, from (41) we conclude that $\Phi_g^*X \in \Gamma(H^\bar{\gamma})$, $\Phi_g^*P^\delta(\alpha) \in \Gamma(V)$ and hence $\bar{X} = \Phi_g^*X$. Moreover, it follows from (42) that $\bar{\alpha} = \Phi_g^*\alpha$ and $i_{\bar{X}}\sigma = i_{\bar{X}}\Phi_g^*\sigma$ for all $\bar{X} \in \Gamma(H^\bar{\gamma})$. This implies that $\Phi_g^*\sigma = \sigma$.

Now, we formulate a generalized version of the averaging theorem for Dirac structures [20].

Theorem 7.4. Let $\gamma \in \text{Conn}_H(M,F,P)$ be an admissible Poisson connection and $\sigma$ a Hamiltonian form of $\text{Curv}^\gamma$ satisfying (29). Then, the averaged Poisson connection $\bar{\gamma} := (\gamma)^G$ induces a $G$-invariant Dirac structure $D^{\bar{\gamma},\bar{\sigma}+C}$, where $\bar{\sigma}$ is given by (32), and $C \in C^2$ is an arbitrary $d^\gamma$-cocycle, $d^\gamma C = 0$. Moreover, the $G$-action on $D^{\bar{\gamma},\bar{\sigma}+C}$ is canonical and cotangent with pre-momentum map $\bar{\gamma}^* \circ \mu$.

Proof. By Theorem 5.4 and Proposition 7.1 the distribution $D^{\bar{\gamma},\bar{\sigma}+C}$ defines a Dirac structure. By Lemma 7.2, the $G$-action is cotangential of $D^{\bar{\gamma},\bar{\sigma}+C}$ with pre-momentum map $\bar{\gamma}^* \circ \mu$. In order to prove the $G$-invariance of $D^{\bar{\gamma},\bar{\sigma}+C}$, let us consider the presymplectic foliations $(S,\omega)$ and $(S,\bar{\omega})$, associated to $D^{\gamma,\sigma}$ and $D^{\bar{\gamma},\bar{\sigma}}$ respectively. By (40), the characteristic distribution of $D^{\gamma,\sigma}$ is

$$TS = H^\perp \oplus P^\delta(T^*M),$$

with presymplectic form $\omega_S = \sigma \oplus \tau_S$, where $\tau$ is the leafwise symplectic form of $P$. On the other hand, the characteristic distribution of $D^{\bar{\gamma},\bar{\sigma}}$ is

$$TS = H^\perp \oplus P^\delta(T^*M),$$

with presymplectic form $\bar{\omega}_S = \bar{\sigma} \oplus \tau_S$. A generating family of vector fields for $TS$ is

$$\left\{ \bar{X} = X + P^\delta d(Q(X)) \quad \text{and} \quad P^\delta(df) \mid X \in \Gamma(H^\gamma), \ f \in C^\infty(M) \right\}.$$

Evaluating $\bar{\omega}_S$ on the generating elements, we conclude that $\bar{\omega}_S = \omega_S - dQ|_S$. Since the $G$-action admits a pre-momentum map, the average of $\omega_S$ can be written as $\langle \omega_S \rangle^G = \omega_S - i_{\bar{\omega}}^\delta dQ$, where $i_{\bar{\omega}}^\delta: S \rightarrow M$ is the canonical injection, (see [20]). Hence, $\bar{\omega}_S = \langle \omega_S \rangle^G$ and the $G$-invariance of $\bar{\sigma}$ follows from here and the $G$-invariance of $\tau$. The $G$-invariance of $D^{\bar{\gamma},\bar{\sigma}+C}$ is a consequence of Lemma 7.3. Therefore, the $G$-action on $D^{\bar{\gamma},\bar{\sigma}+C}$ is canonical.

Corollary 7.5. The Dirac structures $D^{\gamma,\sigma}$ and $D^{\bar{\gamma},\bar{\sigma}}$ are related by gauge transformation associated with the exact 2-form $dQ$.

Remark 7.2. An alternative way to prove the $G$-invariance of the Dirac structure $D^{\gamma,\sigma}+C$ in Theorem 7.4 is to apply Proposition 5.3 with Lemma 7.3.
Combining Theorem 7.4 with the fact that a Dirac structure $D$ on $M$ around an embedded pre-symplectic leaf $S$ is realized as a coupling Dirac structure $[19]$, one can apply the averaging method relative to a cotangential and canonical $G$-action on $(M, D)$ in order to construct a $G$-invariant Dirac structure in a neighborhood of $S$.

**Example 7.6.** Let $(M, \mathcal{F})$ be a foliated manifold equipped with a leaf-tangent $G$-action. Let $D$ be a Dirac structure on $M$ and $S$ an embedded pre-symplectic leaf of $D$ such that $T_S M = TS \oplus T_S F$. Assume that the $G$-action is canonical and cotangential relative to $D$ with pre-momentum map $\mu : \mathfrak{g} \to \Omega^1(M)$. If $S$ is invariant with respect to the $G$-action then there exists a $G$-invariant neighborhood $U$ of $S$ in $M$ with the following properties: (a) the restriction $D_U = D|_U$ is a coupling Dirac structure on $(U, F_U)$, $D_U = D^\gamma \sigma$, where $\gamma, \sigma$ and the vertical Poisson tensor $P$ in (40) are uniquely defined by $D_U [19]$; (b) the $G$-action on $U$ is canonical and cotangential relative to $P$, $\xi_M|_U = P^j(\mu^i|_U)$. By Theorem 7.4, $\bar{D}^\gamma \sigma$ is a $G$-invariant Dirac structure on the neighborhood $U$ which is gauge equivalent to the original one.

**The Poisson Case.** Under the above assumptions, suppose that the Lie group $G$ acts in Hamiltonian fashion with momentum map $\mathcal{J} : \mathfrak{g} \to C^\infty(M)$, $\mu^\xi = d\mathcal{J} \xi$. Suppose that the Dirac structure $D$ is the graph of a bivector field $\bar{\Pi}$. Then, $\bar{\Pi}$ is a Poisson tensor on $M$ which is $G$-invariant. If the cohomology class of the $\bar{d}^G$-cocycle $\langle d^G \gamma, 0 \rangle \mathcal{J}$ is trivial, then the $G$-action is canonical with momentum map $\mathcal{J}$ relative to $\bar{\Pi}$ (see, also [1]).

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