A new (and optimal) result for boundedness of solution of a quasilinear chemotaxis–haptotaxis model (with logistic source)

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Abstract

This article deals with an initial-boundary value problem for the coupled chemotaxis-haptotaxis system with nonlinear diffusion

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, \; t > 0, \\
    \tau v_t &= \Delta v - v + u, \quad x \in \Omega, \; t > 0, \\
    w_t &= -vw, \quad x \in \Omega, \; t > 0,
\end{align*}
\]

(0.1)

under homogeneous Neumann boundary conditions in a smooth bounded domain \( \Omega \subset \mathbb{R}^N (N \geq 1) \), where \( \tau \in \{0, 1\} \) and \( \chi, \xi \) and \( \mu \) are given nonnegative parameters. The diffusivity \( D(u) \) is assumed to satisfy

\[ D(u) \geq C_D (u + 1)^{m-1} \] for all \( u \geq 0 \) and \( C_D > 0 \).

In the present work it is shown that if

\[ m \geq 2 - \frac{2}{N} \lambda \] with \( 0 < \mu < \tilde{\lambda}, \)

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\[ m > 2 - \frac{2}{N} \lambda \text{ with } \mu \geq \bar{\lambda}, \]

or

\[ m > 2 - \frac{2}{N} \text{ and } \mu = 0 \]

or

\[ m = 2 - \frac{2}{N} \text{ and } C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})^3}{4} (2 - \frac{2}{N})^2 \bar{\lambda}, \]

then for all reasonably regular initial data, a corresponding initial-boundary value problem for (0.1) possesses a unique global classical solution that is uniformly bounded in \( \Omega \times (0, \infty) \), where

\[
\lambda = \begin{cases} 
\max_{s \geq 1} \frac{1}{\lambda_0^{1+s}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) & \text{if } \tau = 1 \text{ and } \mu > 0, \\
\frac{\chi}{(\chi - \mu)_+} & \text{if } \tau = 0 \text{ and } \mu > 0
\end{cases}
\]

and

\[
\bar{\lambda} = \begin{cases} 
\max_{s \geq 1} \frac{1}{\lambda_0^{1+s}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) & \tau = 1, \\
\chi & \text{if } \tau = 0.
\end{cases}
\]

Here \( C_{GN} \) and \( \lambda_0 \) are the constants which are corresponding to the Gagliardo–Nirenberg inequality (see Lemma 2.1) and the maximal Sobolev regularity (see Lemma 2.2), respectively. As far as we know, this situation provides the first rigorous result which (precisely) gives the relationship between \( m, \xi, \chi \) and \( \mu \) that yields to the boundedness of the solutions. Moreover, these results thereby significantly extending results of previous results of several authors (see Remarks 1.1 and 1.2) and some optimal results are obtained.

**Key words:** Boundedness; Chemotaxis–haptotaxis; Nonlinear diffusion; Global existence

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1 Introduction

In order to describe the cancer cell invasion into surrounding healthy tissue, in 2005, Chaplain and Lolas ([4]) proposed a pioneering mathematical model which is called chemotaxis–haptotaxis model

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (D \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + u - v, \quad x \in \Omega, t > 0, \\
\frac{\partial w}{\partial t} &= -vw, \quad x \in \Omega, t > 0,
\end{align*}
\]

(1.1)

where \(D, \chi, \xi\) and \(\mu\) are the cancer cell random motility, the chemotactic coefficients, the haptotactic coefficients and the proliferation rate of the cells, respectively. Here \(\tau \in \{0, 1\}\), \(\Omega \subset \mathbb{R}^N (N \geq 1)\) is the physical domain which we assume to be bounded with smooth boundary, and the unknown quantities \(u, v\) and \(w\) represent the density of cancer cells, the concentration of matrix degrading enzymes (MDE) and the density of extracellular matrix (ECM), respectively.

As a subsystem, (1.1) contains the celebrated Keller–Segel ([16]) chemotaxis system (with logistic source, \(\mu \neq 0\))

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \nabla \cdot (D \nabla u) - \chi \nabla \cdot (u \nabla v) + \mu u(1 - u), \quad x \in \Omega, t > 0, \\
\frac{\partial v}{\partial t} &= \Delta v + u - v, \quad x \in \Omega, t > 0
\end{align*}
\]

(1.2)

by setting \(w \equiv 0\). Over the last four decades, there is a wide variety of patterns associated Keller–Segel system (1.2) have been studied extensively, and the main interest lies in whether the solution is global or blow up (see e.g., Cieślak [6], Cieślak and Winkler [5], Ishida et al. [12], Painter and Hillen [26], Winkler [46, 50, 50, 48], Li and Xiang [18], Tello and Winkler [41], Wang et al. [43], Zheng et al. [60]). In fact, if \(\mu = 0\), the two behaviors (boundedness and blow-up) of solutions strongly depend on the space dimension and the total mass of cells ([2, 9, 10, 47]). When \(\tau = 0\), Tello and Winkler ([41]) mainly proved that the global boundedness for model (1.1) exists under the condition \(\mu > \frac{(N-2)^+}{N} \chi\), moreover, they gave the weak solutions for arbitrary small \(\mu > 0\). Kang and Stevens [15] (see also [52, 11]) improve the results of Tello and Winkler ([41]) to the case \(\mu \geq \frac{(N-2)^+}{N} \chi\). While if \(\tau = 1\) and \(\mu > \frac{(N-2)^+}{N} \chi C_{\frac{N}{2}+1} \frac{1}{N+1}(\text{where } C_{\frac{N}{2}+1} \text{ is a positive constant})\), Zheng ([60]) proved that for any
sufficiently smooth initial data, the corresponding initial-boundary value problem for (1.2) possesses a globally defined bounded solution, which give the lower bound estimation for the logistic source, so that, improves the result of [46]. Furthermore, some recent studies have shown that the blow-up of solutions can be inhibited by the nonlinear diffusion (see Ishida et al. [12], Winkler et al. [11, 33, 54, 53, 45, 51] and nonlinear logistic term (see [53, 55]).

There have been large literature on the global existence and the large time behavior of solutions to the system (1.1). We refer to [3, 21, 32, 36, 37, 40, 59] and the references therein. In fact, when \( \tau = 0 \), MDEs diffuses much faster than cells (see [14, 37]), Tao and Wang [32] proved that model (1.1) possesses a unique global bounded classical solution for any \( \mu > 0 \) in two space dimensions, and for large \( \mu > 0 \) in three space dimensions. In [37], Tao and Winkler improved the condition on \( \mu (\mu > \frac{(N-2)^+}{N}) \), so that it coincides with the best one known for the parabolic-elliptic Keller-Segel system (1.2) (see Tello and Winkler [41]), moreover, in additional explicit smallness on \( w_0 \), they gave the exponential decay of \( w \) in the large time limit. However, this problem is left open for the critical case \( \mu = \frac{(N-2)^+}{N} \). While, if \( \tau = 1 \), refined approaches involving a more subtle analysis of (1.1), Tao (31) and Cao (3) obtained the boundedness of global solution for the 2D and 3D space respectively, especially, for the 3D space, similar to the chemotaxis-only system (60, 46), the global solution is obtained only for large \( \mu \), and it remains open for small \( \mu \).

The diffusion of cancer cell may depend nonlinearly on their densities (8, 29, 34), and so we are led to consider the cell motility \( D \) as a nonlinear function of the cancer cell density, \( D \equiv D(u) = C_D(u + 1)^{m-1}, m \in \mathbb{R}, C_D > 0 \). Introducing this into the model (1.1) leads to the following chemotaxis-haptotaxis system with nonlinear diffusion

\[
\begin{align*}
    u_t &= \nabla \cdot (D(u) \nabla u) - \chi \nabla \cdot (u \nabla v) - \xi \nabla \cdot (u \nabla w) + \mu u(1 - u - w), \quad x \in \Omega, t > 0, \\
    \tau v_t &= \Delta v + u - v, \quad x \in \Omega, t > 0, \\
    w_t &= -vw, \quad x \in \Omega, t > 0, \\
    \frac{\partial u}{\partial v} &= \frac{\partial v}{\partial v} = 0, \quad x \in \partial \Omega, t > 0, \\
    u(x, 0) &= u_0(x), \tau v(x, 0) = \tau v_0(x), w(x, 0) = w_0(x), \quad x \in \Omega,
\end{align*}
\]

(1.3)
where \( u, v \) and \( w \) are denoted as before, \( \mu \geq 0, \tau = 0 \) or \( 1 \), the diffusion function \( D(u) \) fulfills

\[
D \in C^2([0, \infty)) \tag{1.4}
\]

and there exist constants \( m \in \mathbb{R} \) and \( C_D \) such that

\[
D(u) \geq C_D(u + 1)^{m-1} \text{ for all } u \geq 0. \tag{1.5}
\]

This parabolic-parabolic-ODE system \( (\tau = 1 \text{ in } (1.3)) \) and its parabolic-elliptic-ODE simplifications \( (\tau = 0 \text{ in } (1.3)) \) have been objects of extensive studies in recent decades. In fact, in \( N = 2 \), Zheng et al. \([61]\) studied the global boundedness for model \((1.3)\) with \( D \) satisfies \((1.4)-(1.5)\) and \( m > 1 \), moreover, in additional explicit smallness on \( w_0 \), they gave the exponential decay of \( w \) in the large time limit. Moreover, if \( D \) satisfies \((1.4)-(1.5)\) with \( m > \max\{1, \tilde{m}\} \) and

\[
\tilde{m} := \begin{cases} 
\frac{2N^2+4N-4}{N(N+4)} & \text{if } N \leq 8, \\
\frac{2N^2+3N+2-\sqrt{8N(N+1)}}{N(N+20)} & \text{if } N \geq 9,
\end{cases}
\tag{1.6}
\]

Tao and Winkler \([34]\) proved that model \((1.3)\) possesses at least one nonnegative global classical solution, however, their boundedness is left as an open problem. Using the boundedness of \( \int_{\Omega} |\nabla v|^l (1 \leq l < \frac{N}{N-1}) \), Wang \([44]\) and Li, Lankeit \([19]\) proved that the global solvability and boundedness of classical solution (or weak solution) for any \( D \) satisfies \((1.4)-(1.5)\) and \( m > 2 - \frac{2}{N} \). Recently, Zheng \([57]\) and Jin \([13]\) extended these results to the case \( m > \frac{2N}{N+2} \) and \( m > 0 \) (as well as large \( \mu \)), respectively. But the cases \( m \leq 0 \) remain unknown. Other variants of the model that are commonly treated include the (nonlinear) logistic types and the re-establishment of ECM components, please refer to \([28, 25, 38, 56]\), etc, and references therein. Thus it is meaningful to analyze the following question:

\((Q)\): Which size of \( m, \chi, \xi \) and \( \mu \) are sufficient to ensure boundedness of solutions to \((1.3)\)?

It is our goal in this work to give answers to \((Q)\). To the best of our knowledge, this is the first result which gives a explicit condition between \( m, \chi, \xi \) and \( \mu \) that yields to the boundedness of the solution.

Motivated by the above works, the aim of the present paper is to study the quasilinear parabolic–elliptic–ODE \( (\tau = 0 \text{ in } (1.3)) \) and parabolic–parabolic–ODE \( (\tau = 1 \text{ in } (1.3)) \)
chemotaxis–haptotaxis model \((1.3)\) under the conditions \((1.4)–(1.5)\). Our main result is the following:

**Theorem 1.1.** Let \(\Omega \subset \mathbb{R}^N (N \geq 1)\) be a bounded domain with smooth boundary \(\partial \Omega\). Assume that \(D\) satisfy \((1.4)–(1.5)\) and the initial data \((u_0, w_0)\) fulfills

\[
\begin{align*}
&u_0 \in C(\bar{\Omega}) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\
&w_0 \in C^{2+\theta}(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega
\end{align*}
\]

with some \(\theta \in (0, 1)\).

If one of the following cases holds:

(i) \(m \geq 2 - \frac{2}{N} (\chi - \mu)_+\) with \(0 < \mu < \chi\);

(ii) \(m > 2 - \frac{2}{N} (\chi - \mu)_+\) with \(\mu \geq \chi\);

(iii) \(m > 2 - \frac{2}{N}\) with \(\mu = 0\);

(iv) \(m = 2 - \frac{2}{N}\) and \(C_D > \frac{C_{GN}(1+\|u_0\|_{L^1(\Omega)})^3}{4}(2 - \frac{2}{N})^2\chi\);

then there exists a triple \((u, v, w)\) \(\in (C_0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)))^3\) which solves \((1.3)\) in the classical sense. Here \(C_{GN}\) is a positive constant which is corresponding to the Gagliardo–Nirenberg inequality (see Lemma 2.1). Moreover, both \(u, v\) and \(w\) are bounded in \(\Omega \times (0, \infty)\).

Before we prove the theorems, there are a few remarks in order.

**Remark 1.1.** (i) Theorem 1.1 extends the results of Theorem 1.1 of Tao and Winkler ([37]) for the critical case \(\mu = \frac{(N-2)_+}{N}\chi\) and \(D(u) = 1\).

(ii) If \(\mu = 0\), in comparison to the result for the corresponding haptotaxis-free system ([51], \(w \equiv 0\)), it is easy to see that the restriction on \(m\) here is optimal.

(iii) Observing that if \(\mu \geq \frac{(N-2)_+}{N}\chi\) and \(w \equiv 0\), then \(2 - \frac{2}{N} (\chi - \mu)_+ < 1\), therefore, Theorem 1.1 also extends the results of Theorem 1.1 of Tello and Winkler ([41]).

(iv) Obviously, if \(\mu > \chi\), then \(2 - \frac{2}{N} (\chi - \mu)_+ < 1\), so that, Theorem 1.1 extends the results of Theorem 1.1 of Tao and Winkler ([36]).

(v) Obviously, if \(w \equiv 0\) and \(\mu > 0\), then \(2 - \frac{2}{N} (\chi - \mu)_+ < 2 - \frac{2}{N}\), so that, Theorem 1.1 also partly extends the results of Theorem 1.1 of Wang et al. ([43]).
(vi) Obviously, if \( w \equiv 0 \), \( \mu \geq \frac{(N-2)+\chi}{N} \), and \( D(u) \equiv 1 \), then \( 2 - \frac{2}{N} \frac{\chi}{(\lambda-\mu)_+} \leq 1 \), so that, Theorem 1.1 is consistent with the results of Kang and Stevens (15).

**Theorem 1.2.** Let \( \Omega \subset \mathbb{R}^N (N \geq 1) \) be a bounded domain with smooth boundary \( \partial \Omega \). Assume that \( D \) satisfy (1.4)-(1.5) and the initial data \((u_0, v_0, w_0)\) fulfills

\[
\begin{align*}
&u_0 \in W^{1,\infty}(\Omega) \text{ with } u_0 \geq 0 \text{ in } \Omega \text{ and } u_0 \not\equiv 0, \\
v_0 \in W^{1,\infty}(\Omega) \text{ with } v_0 \geq 0 \text{ in } \Omega \text{ and } \frac{\partial v_0}{\partial \nu} = 0 \text{ on } \partial \Omega, \\
w_0 \in C^{2+\theta}(\bar{\Omega}) \text{ with } w_0 \geq 0 \text{ in } \bar{\Omega} \text{ and } \frac{\partial w_0}{\partial \nu} = 0 \text{ on } \partial \Omega
\end{align*}
\]

with some \( \theta \in (0, 1) \). If one of the following cases holds:

(i) \( m \geq 2 - \frac{2}{N} \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \) with \( 0 < \mu < \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \);

(ii) \( m > 2 - \frac{2}{N} \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \) with \( \mu \geq \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \);

(iii) \( m = 2 - \frac{2}{N} \) and \( C_D > \frac{C_{GN}(1+\|u_0\|_{L^1(\Omega)})^3}{4}(2 - \frac{2}{N})^2 \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \);

then there exists a pair \((u, v, w) \in (C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))\)^2 which solves (1.3) in the classical sense, where \( C_{GN} \) and \( \lambda_0 := \lambda_0(\gamma) \) are the constants which are corresponding to the Gagliardo-Nirenberg inequality (see Lemma 2.1) and the maximal Sobolev regularity (see Lemma 2.2), respectively. Moreover, both \( u \) and \( v \) are bounded in \( \Omega \times (0, \infty) \).

**Remark 1.2.** (i) Obviously, if \( \mu > \frac{(N-2)+\chi}{N} \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) \), then, Theorem 1.2 extends the results of Ke and Zheng (59) and partly extends the result of Liu et al (20).

(ii) Obviously, if \( \mu > 0 \), \( 2 - \frac{2}{N} \max_{s \geq 1} \frac{1}{2} \left( \frac{\lambda_0^{1+\gamma} (\chi + \|u_0\|_{L^\infty(\Omega)})}{1 + \gamma} \right) < 2 - \frac{2}{N} \), hence Theorem 1.2 extends the results of Wang (44) and Li and Lankeit (19).

(iii) Theorem 1.2 extends the results of Zheng et al. (58) for the critical case \( \mu \geq \frac{(N-2)+\chi}{N} \)

as well as \( w \equiv 0 \) and \( D(u) = 1 \).

(iv) In comparison to the result for the corresponding haptotaxis-free system (35, 45), it is easy to see that the restriction on \( m \) here is optimal.
If \( N = 2 \), then
\[
2 - \frac{2}{N} \left[ \max_{s \geq 1} \frac{1}{\lambda_0^{\frac{1}{s+1}}} \left( \chi + \xi \| w_0 \|_{L^\infty(\Omega)} \right) \right] = 2 - \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}) - \mu} < 1,
\]
therefore, Theorem 1.1 extends the results of Wang et al. ([61]), who proved the possibility of global and bounded, in the cases, \( D \) satisfies (1.4)–(1.5) with \( m > 1 \).

The main novelty and difficulty of the paper is how to control the chemotaxis term \( \chi \nabla \cdot (u \nabla v) \), haptotaxis term \( \xi \nabla \cdot (u \nabla w) \) and strong degeneracies caused by system (1.3). To overcome this difficulty, the purpose of the present paper is to demonstrate how far an adequate combination of maximal Sobolev regularity theory and develop new \( L^p \)-estimate techniques (see Lemmas 3.3–3.10) can be used to obtain the global existence and boundedness of solutions to (1.3).

The rest of the paper is organized in the following way. Section 2 will be concerned with preliminaries, including some basic facts and a local existence result. In section 3, by careful analysis, this paper develops some \( L^p \)-estimate techniques to raise the a priori estimate of a solution from \( L^1(\Omega) \to L^{70-\epsilon}(\Omega) \to L^{70}(\Omega) \to L^{70+\epsilon}(\Omega) \), where
\[
\gamma_0 = \begin{cases} 
\frac{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}) - \mu} & \text{if } \tau = 1 \text{ and } 0 < \mu < \max_{s \geq 1} \lambda_0^{\frac{1}{s+1}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}), \\
\frac{\chi}{(\chi - \mu)} & \text{if } \tau = 0 \text{ and } 0 < \mu < \chi.
\end{cases}
\]
To this end, by using the maximal Sobolev regularity and the standard estimate for the solution, we may derive entropy-like inequalities (see (3.11) and (3.30)). Then in order to estimate the right term \( \int_\Omega u^{k+1} \) and \( \int_\Omega u^k \) on the rightmost of (3.31) and (3.30), we need to deal with for two steps from \( \| u(\cdot, t) \|_{L^{70-\epsilon}(\Omega)} \to \| u(\cdot, t) \|_{L^{70}(\Omega)} \) (see the proof of Lemmas 3.3 and 3.4), which plays a key rule in obtaining the main results. Then employing a bootstrap argument (see (3.47) and (3.48)), one could derive the boundedness of \( \| u(\cdot, t) \|_{L^{70+\epsilon}(\Omega)} \) (see Lemma 3.5). Relying on this, we develop new \( L^p \)-estimate techniques to raise the a priori estimate of solutions from \( L^{70+\epsilon}(\Omega) \to L^p(\Omega) \) (for all \( p > 1 \)) (see Lemmas 3.6–3.10). Finally, applying the standard Alikakos–Moser iteration, we prove the main results of this paper in the last part.
2 Preliminaries

In this section, we will recall some lemmas and elementary inequalities which will be used frequently later.

To begin with, let us collect some basic solution properties which essentially have already been used in [17].

**Lemma 2.1.** ([7]) Let \( \theta \in (0, p) \). There exists a positive constant \( C_{GN} \) such that for all \( u \in W^{1,2}(\Omega) \cap L^\theta(\Omega) \),

\[
\|u\|_{L^p(\Omega)} \leq C_{GN}(\|\nabla u\|_{L^2(\Omega)}^{1-a} \|u\|_{L^\theta(\Omega)}^{1-a} + \|u\|_{L^\theta(\Omega)})
\]

is valid with \( a = \frac{Np - N}{1 - \frac{N}{2} + \frac{N}{\theta} } \in (0, 1) \).

**Lemma 2.2.** ([17]) Suppose that \( \gamma \in (1, +\infty) \) and \( g \in L^\gamma((0, T); L^\gamma(\Omega)) \). Consider the following evolution equation

\[
\begin{cases}
v_t - \Delta v + v = g, & (x, t) \in \Omega \times (0, T), \\
\frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, T), \\
v(x, 0) = v_0(x), & (x, t) \in \Omega.
\end{cases}
\]

For each \( v_0 \in W^{2,\gamma}(\Omega) \) such that \( \frac{\partial v_0}{\partial \nu} = 0 \) and any \( g \in L^\gamma((0, T); L^\gamma(\Omega)) \), there exists a unique solution \( v \in W^{1,\gamma}((0, T); L^\gamma(\Omega)) \cap L^\gamma((0, T); W^{2,\gamma}(\Omega)) \). In addition, if \( s_0 \in [0, T) \), \( v(\cdot, s_0) \in W^{2,\gamma}(\Omega)(\gamma > N) \) with \( \frac{\partial v(\cdot, s_0)}{\partial \nu} = 0 \), then there exists a positive constant \( \lambda_0 := \lambda_0(\Omega, \gamma, N) \) such that

\[
\int_{s_0}^T e^{\gamma s} \|v(\cdot, t)\|_{W^{2,\gamma}(\Omega)}^\gamma ds \leq \lambda_0 \left( \int_{s_0}^T e^{\gamma s} \|g(\cdot, s)\|_{L^\gamma(\Omega)}^\gamma ds + e^{\gamma s_0}(\|v_0(\cdot, s_0)\|_{W^{2,\gamma}(\Omega)}) \right).
\]

The local-in-time existence of classical solutions to the chemotaxis–haptotaxis model (1.3) is quite standard; see similar discussions in [34, 20]. Therefore we omit it.

**Lemma 2.3.** Assume that the nonnegative functions \( u_0, v_0, \) and \( w_0 \) satisfies (1.8) (or (1.7), if \( \tau = 0 \)) for some \( \vartheta \in (0, 1) \), \( D \) satisfies (1.4) and (1.3). Then there exists a maximal
existence time $T_{\text{max}} \in (0, \infty]$ and a triple of nonnegative functions

$$u \in C^0(\bar{\Omega} \times [0,T_{\text{max}}])) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}})),
\quad v \in C^0(\bar{\Omega} \times [0,T_{\text{max}}])) \cap C^{2,1}(\bar{\Omega} \times (0,T_{\text{max}})),
\quad w \in C^{2,1}(\bar{\Omega} \times [0,T_{\text{max}}]))$$

which solves \((1.3)\) classically and satisfies $w \leq \|w_0\|_{L^\infty(\Omega)}$ in $\Omega \times (0,T_{\text{max}})$. Moreover, if $T_{\text{max}} < +\infty$, then

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} \to \infty \quad \text{as} \quad t \to T_{\text{max}}. \quad (2.1)$$

Employing the same arguments as in the proof of Lemma 2.3 in [40] (see also [31]), we derive the following Lemma:

**Lemma 2.4.** Let $(u, v, w)$ solve \((1.3)\) in $\Omega \times (0,T_{\text{max}})$. Then

$$-\Delta w(x,t) \leq \tau \|w_0\|_{L^\infty(\Omega)} \cdot v(x,t) + \kappa \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad t \in (0,T_{\text{max}}), \quad (2.2)$$

where

$$\kappa := \|\Delta w_0\|_{L^\infty(\Omega)} + 4\|\nabla \sqrt{w_0}\|_{L^\infty(\Omega)}^2 + \frac{\|w_0\|_{L^\infty(\Omega)}}{e}. \quad (2.3)$$

### 3 A priori estimates

In this section, we are going to establish an iteration step to develop the main ingredient of our result. The iteration depends on a series of a priori estimates. Firstly, the following two lemmas provide some elementary material that will be essential to our bootstrap procedure.

**Lemma 3.1.** Let $\mu = 0$, then the solution $(u, v, w)$ of \((1.3)\) satisfies

$$\|u(\cdot,t)\|_{L^1(\Omega)} = \|u_0\|_{L^1(\Omega)} \quad \text{for all} \quad t \in (0,T_{\text{max}}). \quad (3.1)$$

In contrast to the situation without source terms ($\mu = 0$ in \((1.3)\)), we cannot hope for mass conservation in the first component. Nevertheless, the following inequality still holds:

**Lemma 3.2.** (see e.g. [59]) Assume that $\mu > 0$. There exists a positive constant $K_0$ such that the solution $(u, v, w)$ of \((1.3)\) satisfies

$$\|u(\cdot,t)\|_{L^1(\Omega)} \leq K_0 \quad \text{for all} \quad t \in (0,T_{\text{max}}) \quad (3.2)$$
and
\[
\int_t^{t+\tau} \int_\Omega u^2 \leq K_0 \quad \text{for all } t \in (0, T_{\max} - \tau),
\]  
(3.3)
where
\[
\tau := \min\{1, \frac{1}{6} T_{\max}\}.
\]  
(3.4)

Now, we now proceed to derive a uniform upper bound for \(u\), which turns out to be the key to obtain all the higher order estimates and thus to extend the classical solution globally. To do this, employing the maximal Sobolev regularity, in light of Lemma 3.1, as a first conclusion towards global existence of the classical solutions is the following a priori estimate which asserts that, in sharp contrast to the case \(\mu = 0\) (see also [46]) is a priori uniformly bounded in \(L^k(\Omega)\) for some \(k\) larger than one. In order to deal with the critical case \((k = \lambda)\), the novelty of paper, we first obtain the bounded of \(\|u(\cdot, t)\|_{L^{k_0}(\Omega)}\) where \(k_0 \in (\max\{1, \lambda - \frac{N}{\chi}\}, \lambda)\). And then by some careful analysis, one can finally derive the bounded of the critical case, which are the following Lemmas:

**Lemma 3.3.** Let \((u, v, w)\) be a solution to (1.3) on \((0, T_{\max})\). If \(\tau = 0\) and \(\mu > 0\), then for any
\[
k \in \begin{cases} 
(1, \frac{\chi}{(\chi - \mu)^+}), & \text{if } 0 < \mu < \chi \text{ and } m \geq 2 - \frac{2}{N} \frac{\chi}{(\chi - \mu)^+}, \\
(1, \frac{\chi}{(\chi - \mu)^+}), & \text{if } \mu \geq \chi,
\end{cases}
\]  
(3.5)
one can find a positive constant \(C\) such that
\[
\|u(\cdot, t)\|_{L^k(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\max})
\]  
(3.6)
holds.

**Proof.** Multiplying the first equation in (1.3) by \((u + 1)^{k-1}\), and integrating in space and using \(w \geq 0\), we get
\[
\begin{align*}
\frac{1}{k} \frac{d}{dt} \|u + 1\|_{L^k(\Omega)}^k + (k - 1) \int_\Omega (u + 1)^{k-2}D(u)\left|\nabla u\right|^2 dx & \leq -\chi \int_\Omega \nabla \cdot (u\nabla v)(u + 1)^{k-1} dx - \xi \int_\Omega \nabla \cdot (u\nabla w)(u + 1)^{k-1} dx + \mu \int_\Omega (u + 1)^{k-1} u(1 - u - w) \\
& \leq -\chi \int_\Omega \nabla \cdot (u\nabla v)(u + 1)^{k-1} dx - \xi \int_\Omega \nabla \cdot (u\nabla w)(u + 1)^{k-1} dx + \mu \int_\Omega (u + 1)^{k-1} u(1 - u).
\end{align*}
\]  
(3.7)
Integrating by parts to the first term on the right hand side of (3.7) and from (1.3) we obtain
\[-\chi \int_{\Omega} \nabla \cdot (u \nabla v)(u + 1)^{k-1}
= (k - 1)\chi \int_{\Omega} u(u + 1)^{k-2} \nabla u \cdot \nabla v
= (k - 1)\chi \int_{\Omega} \nabla \int_{0}^{u} \tau(\tau + 1)^{k-2} d\tau \cdot \nabla v
\leq (k - 1)\chi \int_{\Omega} \int_{0}^{u} \tau(\tau + 1)^{k-2} d\tau u
\leq \frac{k-1}{k} \chi \int_{\Omega} (u + 1)^{k+1},
\]
where we have used the fact that \(v \geq 0\). Summing up (2.2) and (2.3) yields to
\[-\xi \int_{\Omega} \nabla \cdot (u \nabla w)(u + 1)^{k-1}
= -(k - 1)\xi \int_{\Omega} \int_{0}^{u} \tau(\tau + 1)^{k-2} d\tau \Delta w
\leq \kappa(k - 1)\xi \int_{\Omega} \int_{0}^{u} \tau(\tau + 1)^{k-2} d\tau
\leq \kappa\frac{(k - 1)\xi}{k} \int_{\Omega} (u + 1)^{k}
\leq \kappa\xi \int_{\Omega} (u + 1)^{k},
\]
where \(\kappa\) is given by (2.3) and we have used the fact that \(\tau = 0\) in (2.2).

Here, by some basic calculation, we deduce that
\[\mu \int_{\Omega} (u + 1)^{k-1} u(1 - u)
= -\mu \int_{\Omega} (u + 1)^{k-1} [(u + 1)^{2} - 3u - 1]
\leq -\mu \int_{\Omega} (u + 1)^{k-1} [(u + 1)^{2} - 3u - 3]
\leq -\mu \int_{\Omega} (u + 1)^{k+1} + 3\mu \int_{\Omega} (u + 1)^{k}.
\]
Therefore, combined with (3.9), (3.10), and (3.7) and (1.5), we have
\[\frac{1}{k} \frac{d}{dt} \|u + 1\|_{L^k(\Omega)}^{k} + \frac{4C_{D}(k - 1)}{(m + k - 1)^{2}} \|\nabla (u + 1)^{\frac{m+1}{2}}\|_{L^2(\Omega)}^{2}
\leq (-\mu + \frac{k-1}{k} \chi) \int_{\Omega} (u + 1)^{k+1} + C_{1} \int_{\Omega} (u + 1)^{k},
\]
with \(C_{1} = 3\mu + \kappa\xi\).

**Case \(\mu < \chi\):**

**Step 1.** The boundedness of \(\|u(\cdot, t)\|_{L^{k_0}(\Omega)}\) for all \(t \in (0, T_{max})\) and \(k_0 \in (\max\{1, \frac{\chi}{(\mu - \chi)^{\frac{1}{2}}}, \frac{\chi}{(\chi - \mu)^{\frac{1}{2}}}, \frac{\chi}{(\chi - \mu)^{\frac{1}{2}}}, \}.
\)
To this end, for any $\varepsilon > 0$, pick $k = \frac{\chi}{(\chi - \mu)_+} - \varepsilon$ in (3.11), then, $-\mu + \frac{k-1}{k}\chi < 0$ (by $0 < \mu < \chi$), so that, (3.11) implies that

$$
\frac{1}{k} \frac{d}{dt} \|u + 1\|^k_{L^k(\Omega)} + \frac{4C_D(k-1)}{(m + k - 1)^2} \|\nabla (u + 1)\|^2_{L^2(\Omega)} + \frac{1}{2} (\mu - \frac{k-1}{k}\chi) \int_\Omega (u + 1)^{k+1} \leq C_2
$$

(3.12)

by using the Young inequality. Applying the Gronwall lemma to (3.12), we derive

$$
\|u(\cdot, t)\|_{L^\left(\frac{\chi}{(\chi - \mu)_+} - \varepsilon\right)(\Omega)} \leq C_3 \quad \text{for all} \quad t \in (0, T_{\max}),
$$

(3.13)

which combined with the arbitrariness of $\varepsilon$ and the Hölder inequality yields to for any $k_0 \in (\max\{1, \frac{\chi}{(\chi - \mu)_+} - \frac{N}{2}\}, \frac{\chi}{(\chi - \mu)_+})$,

$$
\|u(\cdot, t)\|_{L^{k_0}(0, T_{\max})} \leq C_4 \quad \text{for all} \quad t \in (0, T_{\max}).
$$

(3.14)

**Step 2. The boundedness of $\|u(\cdot, t)\|_{L^k(\Omega)}$ for all $t \in (0, T_{\max})$ and $k \in (1, \frac{\chi}{(\chi - \mu)_+}]$.**

To achieve this, we pick $k = \frac{\chi}{(\chi - \mu)_+}$ in (3.11), then, $-\mu + \frac{k-1}{k}\chi = 0$ (by $0 < \mu < \chi$), so that, (3.11) implies that

$$
\frac{1}{k} \frac{d}{dt} \|u + 1\|^k_{L^k(\Omega)} + \frac{4C_D(k-1)}{(m + k - 1)^2} \|\nabla (u + 1)\|^2_{L^2(\Omega)} \leq C_1 \int_\Omega (u + 1)^k
$$

(3.15)

with $C_1 = 3\mu + \kappa \xi$. Now, observe that $m \geq 2 - \frac{2}{N} \frac{\chi}{(\chi - \mu)_+}$ and $k_0 > \max\{1, \frac{\chi}{(\chi - \mu)_+} - \frac{N}{2}\}$ implies that

$$
m + k - 1 + \frac{2}{N} \times k_0 > k,
$$

therefore, in view of (3.14), a use of the Gagliardo-Nirenberg inequality to (3.15) implies that there exist positive constants $C_5$ and $C_6$ such that

$$
C_1 \int_\Omega (u + 1)^k
= \|(u + 1)^{\frac{m + k - 1}{2}}\|^\frac{2k}{m + k - 1}_{L^\frac{2k}{m + k - 1}(\Omega)} \|\nabla (u + 1)\|^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}_{L^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}(\Omega)}
\leq C_5(\|\nabla (u + 1)\|^\frac{m + k - 1}{2}_{L^\frac{m + k - 1}{2}(\Omega)} + \|(u + 1)^{\frac{m + k - 1}{2}}\|^\frac{2k}{m + k - 1}_{L^\frac{2k}{m + k - 1}(\Omega)} \|\nabla (u + 1)\|^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}_{L^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}(\Omega)}
\leq C_6(\|\nabla (u + 1)\|^\frac{m + k - 1}{2}_{L^\frac{m + k - 1}{2}(\Omega)} + \|(u + 1)^{\frac{m + k - 1}{2}}\|^\frac{2k}{m + k - 1}_{L^\frac{2k}{m + k - 1}(\Omega)} \|\nabla (u + 1)\|^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}_{L^\frac{N(k - k_0)}{N(m + k - 1) + (2 - N)k_0}(\Omega)} + 1),
$$

(3.16)
where combined with \( \frac{2N(k-k_0)}{N(m+k-1)+(2-N)k_0} < 2 \) (by \( m + k - 1 + \frac{2}{N} \times k_0 > k \)) implies that
\[
C_1 \int_\Omega (u + 1)^k \leq \frac{2C_D(k-1)}{(m + k - 1)^2} \| \nabla (u + 1)^{\frac{m+k-1}{2}} \|_{L^2(\Omega)}^2 + C_7.
\]
Substituting the above inequality into (3.12), one can easily deduce that
\[
\frac{1}{k} \frac{d}{dt} \| u + 1 \|_{L^k(\Omega)}^k + C_1 \int_\Omega (u + 1)^k \leq C_8,
\]
which, upon a use of the Gronwall inequality, yields that (3.6) holds.

Case \( \mu \geq \chi \): In view of \( 1 < k < \frac{\chi}{(\chi-\mu)_+} \),
\[
-\mu + \frac{k-1}{k} \chi < 0,
\]
so that, (3.11) and the Young inequality yields to
\[
\frac{1}{k} \frac{d}{dt} \| u + 1 \|_{L^k(\Omega)}^k \leq \frac{1}{2} (-\mu + \frac{k-1}{k} \chi) \int_\Omega (u + 1)^{k+1} + C_9.
\]
Solving this the Gronwall inequality, we deduce from the Holder inequality that \( \| u(\cdot, t) \|_{L^k(\Omega)} \)
is bounded for all \( k \in (1, \frac{\chi}{(\chi-\mu)_+}) \).

**Lemma 3.4.** Let \( (u, v, w) \) be a solution to (1.3) on \( (0, T_{\max}) \) and \( \theta_0 = \max_{s \geq 1} \frac{1}{\lambda_0^s} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}) \). If \( \tau = 1 \) and \( \mu > 0 \), then for any
\[
k \in \begin{cases}
(1, \theta_0], & \text{if } 0 < \mu < \max_{s \geq 1} \frac{1}{\lambda_0^s} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}) \text{ and } m \geq 2 - \frac{2}{N} \theta_0, \\
(1, \theta_0), & \text{if } \mu \geq \max_{s \geq 1} \frac{1}{\lambda_0^s} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}),
\end{cases}
\]
one can find a positive constant \( C \) such that
\[
\| u(\cdot, t) \|_{L^k(\Omega)} \leq C \text{ for all } t \in (0, T_{\max})
\]
holds.

**Proof.** Multiplying (1.3) by \( u^{k-1} \), integrating over \( \Omega \) and using \( w \geq 0 \), we get
\[
\frac{1}{k} \frac{d}{dt} \| u \|_{L^k(\Omega)}^k + (k - 1) \int_\Omega u^{k-2} D(u) |\nabla u|^2 dx \leq -\chi \int_\Omega \nabla \cdot (u \nabla v) u^{k-1} dx - \xi \int_\Omega \nabla \cdot (u \nabla w) u^{k-1} dx + \mu \int_\Omega u^{k-1} (1 - u - w) \]
\[
\leq -\chi \int_\Omega \nabla \cdot (u \nabla v) u^{k-1} dx - \xi \int_\Omega \nabla \cdot (u \nabla w) u^{k-1} + \mu \int_\Omega u^{k-1} (1 - u - w).
\]
We now estimate the right hand side of (3.20) terms by terms. To this end, integrating by parts to the first term on the right hand side of (3.20) and from (1.3) we obtain
\[
-\chi \int_{\Omega} \nabla \cdot (u \nabla) u^k \nabla v = -\frac{(k-1)\chi}{k} \int_{\Omega} u^k \Delta v \leq \frac{(k-1)\chi}{k} \int_{\Omega} u^k |\nabla v| \leq \varepsilon_1 \int_{\Omega} u^{k+1} + \gamma_1 \varepsilon_1^{-k} \int_{\Omega} |\Delta v|^{k+1},
\]
where
\[
\gamma_1 = \frac{1}{k+1} \left( \frac{k+1}{k} \right)^{-k} \left( \frac{(k-1)\chi}{k} \right)^{k+1}.
\]

Due to (2.2) and (2.3), it follows that
\[
-\xi \int_{\Omega} \nabla \cdot (u \nabla) u^k \nabla w = -\frac{(k-1)\xi}{k} \int_{\Omega} u^k \Delta w \leq \kappa \frac{(k-1)\xi}{k} \int_{\Omega} (\xi \|w_0\|_{L^\infty(\Omega)}) u^k v \leq \kappa \xi \int_{\Omega} u^k + \gamma_2 \varepsilon_2^{-k} \int_{\Omega} u^{k+1},
\]
where
\[
\gamma_2 := \frac{1}{k+1} \left( \frac{k+1}{k} \right)^{-k} \left( \frac{(k-1)\xi\|w_0\|_{L^\infty(\Omega)}}{k} \right)^{k+1}.
\]

Here \(\kappa\) is give by (2.3) and we have used the fact that \(\tau = 1\) in (2.2).

On the other hand, in view of the Young inequality, we also derive that
\[
\mu \int_{\Omega} u^{k-1} u(1-u) = -\mu \int_{\Omega} u^{k+1} + (\mu + \frac{k+1}{k}) \int_{\Omega} u^k - \frac{k+1}{k} \int_{\Omega} u^k \leq -\mu \int_{\Omega} u^{k+1} + (\mu + 2) \int_{\Omega} u^k - \frac{k+1}{k} \int_{\Omega} u^k
\]
by using \(k > 1\). Therefore, combined with (3.21), (3.23), (5.20), and (3.25) and (1.5), we have
\[
\frac{1}{k} \frac{d}{dt} \|u^k\|_{L^k(\Omega)} + C_D(k-1) \int_{\Omega} u^{m+k-3} |\nabla u|^2 + \frac{k+1}{k} \int_{\Omega} u^k \\
\leq (-\mu + \varepsilon_1 + \varepsilon_2) \int_{\Omega} u^{k+1} + \gamma_1 \int_{\Omega} |\Delta v|^{k+1} + \gamma_2 \int_{\Omega} v^{k+1} + C_1 \int_{\Omega} u^k
\]
(3.26)
with \( C_1 = \kappa \xi + \mu + 2 \). For any \( t \in (s_0, T_{\text{max}}) \), applying the Gronwall Lemma to the above inequality shows that

\[
\frac{1}{k} \| u(\cdot, t) \|_{L^k(\Omega)}^k + C_D(k - 1) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{m+k-3} |\nabla u|^2 ds\,dt \\
\leq \frac{1}{k} e^{-(k+1)(t-s_0)} \| u(\cdot, s_0) \|_{L^k(\Omega)}^k + (\varepsilon_1 + \varepsilon_2 - \mu) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} dx\,ds \\
+ \gamma_1 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} |\Delta v|^{k+1} dx\,ds + C_1 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^k dx\,ds \\
+ \gamma_2 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} v^{k+1} dx\,ds
\]

(3.27)

where

\[
C_2 := C_2(k) = \frac{1}{k} \| u(\cdot, s_0) \|_{L^k(\Omega)}^k.
\]

Next, a use of Lemma 2.2 leads to

\[
\gamma_1 \varepsilon_1^{-k} \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} |\Delta v|^{k+1} dx\,ds \\
= \gamma_1 \varepsilon_1^{-k} e^{-(k+1)t} \int_{s_0}^t e^{(k+1)s} \int_{\Omega} |\Delta v|^{k+1} dx\,ds
\]

(3.28)

\[
\leq \gamma_1 \varepsilon_1^{-k} e^{-(k+1)t} \lambda_0 \left( \int_{s_0}^t \int_{\Omega} e^{(k+1)s} u^{k+1} dx\,ds + \int_{s_0}^t \int_{\Omega} e^{(k+1)s} v^{k+1} dx\,ds \right)
\]

and

\[
\gamma_2 \varepsilon_2^{-k} \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} v^{k+1} dx\,ds \\
= \gamma_2 \varepsilon_2^{-k} \int_{s_0}^t e^{(k+1)s} \int_{\Omega} v^{k+1} dx\,ds
\]

(3.29)

\[
\leq \gamma_2 \varepsilon_2^{-k} e^{-(k+1)t} \lambda_0 \left( \int_{s_0}^t \int_{\Omega} e^{(k+1)s} u^{k+1} dx\,ds + \int_{s_0}^t \int_{\Omega} e^{(k+1)s} v^{k+1} dx\,ds \right)
\]

for all \( t \in (s_0, T_{\text{max}}) \). On the other hand, choosing \( \varepsilon_1 = \frac{(k-1)\chi}{k+1} \lambda_0^\frac{1}{k+1} \), and \( \varepsilon_2 = \frac{(k-1)\xi}{k+1} \lambda_0^\frac{1}{k+1} \xi \| w_0 \|_{L^\infty(\Omega)} \), with the help of (3.22) and (3.24), a simple calculation shows that

\[
\varepsilon_1 + \gamma_1 \lambda_0 \varepsilon_1^{-k} = \frac{(k-1)\chi}{k} \lambda_0^\frac{1}{k+1} \chi
\]

and

\[
\varepsilon_2 + \gamma_2 \lambda_0 \varepsilon_2^{-k} = \frac{(k-1)\xi}{k} \lambda_0^\frac{1}{k+1} \xi \| w_0 \|_{L^\infty(\Omega)},
\]

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so that, substituting (3.28)–(3.29) into (3.27) implies that

\[
\frac{1}{k} \|u(\cdot, t)\|_{L_k(\Omega)}^k + C_D(k - 1) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{m+k-3} |\nabla u|^2 ds.
\]

\[
\leq (\varepsilon_1 + \gamma_1 \lambda_0 \varepsilon_1 - k + \varepsilon_2 + \gamma_2 \lambda_0 \varepsilon_2 - k - \mu) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} ds + C_1 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k} dx ds + C_2
\]

\[
+ (\gamma_1 \varepsilon_2 - k + \gamma_2 \varepsilon_2 - k) e^{-(k+1)(t-s_0)} \lambda_0 \|v(\cdot, s_0)\|_{W_{2, k+1}}^{k+1} + C_1 \int_{s_0}^t \int_{\Omega} u^{k} dx ds + C_2
\]

\[
= (k-1) \frac{k}{k} \lambda_0^{\frac{1}{k+1}} \chi + \frac{(k-1)}{k} \lambda_0^{\frac{1}{k+1}} \xi \|w_0\|_{L^\infty(\Omega)} - \mu) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} dx ds
\]

\[
+ (\gamma_1 \varepsilon_2 - k + \gamma_2 \varepsilon_2 - k) e^{-(k+1)(t-s_0)} \lambda_0 \|v(\cdot, s_0)\|_{W_{2, k+1}}^{k+1} + C_1 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k} dx ds + C_2
\]

\[
\leq \left[ \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu) \right] \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} dx ds
\]

\[
+ C_1 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k} dx ds + C_3
\]

(3.30)

with

\[
C_3 = (\gamma_1 \varepsilon_2 - k + \gamma_2 \varepsilon_2 - k) e^{-(k+1)(t-s_0)} \lambda_0 \|v(\cdot, s_0)\|_{W_{2, k+1}}^{k+1} + C_2.
\]

In the sequel, we wish to bound the terms on the right-hand side of (3.30) in terms of the dissipation term on its left-hand side. Case \( \mu < \max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) \): **Step 1.**

**The boundedness of** \( \|u(\cdot, t)\|_{L_{k_0}(\Omega)} \) **for all** \( t \in (0, T_{max}) \) **and** \( k_0 \in (\max\{1, \theta_0 - \frac{N}{2}\}, \theta_0) \)

**with** \( \theta_0 = \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)}{\max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} \).

To this end, for any \( \varepsilon > 0 \), pick \( k = \theta_0 - \varepsilon \) in (3.30), then,

\[-\mu + \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) < 0 \quad \text{by} \quad 0 < \mu < \max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}),
\]

so that, (3.30) implies that

\[
\frac{1}{k} \|u(\cdot, t)\|_{L_k(\Omega)}^k \leq 2 \left[ \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) - \mu \right] \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} dx ds + C_4.
\]

(3.31)

by using the Young inequality. This combined with the arbitrariness of \( \varepsilon \) and the Hölder inequality yields to for any \( k_0 \in (\max\{1, \theta_0 - \frac{N}{2}\}, \theta_0) \),

\[
\|u(\cdot, t)\|_{L_{k_0}(\Omega)} \leq C_4 \quad \text{for all} \quad t \in (0, T_{max}).
\]

(3.32)

**Step 2.** **The boundedness of** \( \|u(\cdot, t)\|_{L_k(\Omega)} \) **for all** \( t \in (0, T_{max}) \) **and** \( k \in (1, \theta_0] \),

**where** \( \theta_0 = \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{k+1}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)} - \mu)} \).
To achieve this, we pick \( k = \theta_0 \) in (3.30), then,

\[ -\mu + \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{k}} (\chi + \xi \| w_0 \|_{L^\infty(\Omega)}) = 0, \]

so that, by (3.30), we have

\[
\frac{1}{k} \| u(\cdot, t) \|_{L^k(\Omega)}^{k} + C_D(k-1) \int_{0}^{t} e^{-(k+1)(t-s)} \int_{\Omega} u^{m+k-3} |\nabla u|^2 \, dx \, ds \leq C_1 \int_{0}^{t} e^{-(k+1)(t-s)} \int_{\Omega} u^k \, dx \, ds + C_3
\]

with

\[
C_3 = (\gamma_1 \varepsilon_2^{-k} + \gamma_2 \varepsilon_2^{-k}) e^{-(k+1)(t-s_0)} \lambda_0 \| v(\cdot, s_0) \|_{W^{2,1}_2}^{k+1} + C_2.
\]

In the following, we shall apply the Gagliardo-Nirenberg interpolation inequality to control the second integral on the right-hand side of (3.33). To this end, in view of \( m \geq 2 - \frac{2}{N} \theta_0 \) and \( k_0 > \max\{1, \theta_0 - \frac{N}{2}\} \) implies that

\[
m + k - 1 + \frac{2}{N} \times k_0 > k,
\]

therefore, in view of (3.32), we deduce from the Gagliardo–Nirenberg inequality that there exist positive constants \( C_5 \) and \( C_6 \) such that

\[
C_1 \int_{\Omega} u^k \leq C_5 \left( \| \nabla u \|^\frac{m+k-1}{2} \| u \|^\frac{m+k-1}{2} \right)_{L^{2}(\Omega)} + C_6 \left( \| \nabla u \|^\frac{m+k-1}{2} \| u \|^\frac{m+k-1}{2} \right)_{L^{2}(\Omega)}
\]

which, together with the fact

\[
\frac{2N(k-k_0)}{N(m+k-1)+(2-N)k_0} < 2 \quad \text{(by } m + k - 1 + \frac{2}{N} \times k_0 > k),
\]

immediately gives that

\[
C_1 \int_{\Omega} u^k \leq \frac{2C_D(k-1)}{(m+k-1)^2} \| \nabla u \|^\frac{m+k-1}{2} \| u \|_{L^2(\Omega)}^2 + C_7.
\]

Substituting the above inequality into (3.33), we can get (3.19).
Case $\mu \geq \max_{s \geq 1} \lambda_0^{+\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})$: By (3.26), we obtain for any $\delta_1 > 0$, 
\[
\frac{1}{k} \frac{d}{dt} \|u(t)\|_{L^k(\Omega)}^k + \frac{k+1}{k} \int_{\Omega} u^k \leq \left(-\mu + \varepsilon_1 + \varepsilon_2 + \delta_1\right) \int_{\Omega} u^{k+1} + \gamma_1 \int_{\Omega} |\Delta v|^{k+1} + \gamma_2 \int_{\Omega} v^{k+1} + C_8
\] 
(3.35)
by using the Young inequality. For any $t \in (s_0, T_{\text{max}})$, again, from the Gronwall lemma, we derive that
\[
\frac{1}{k} \|u(\cdot, t)\|_{L^k(\Omega)}^k \leq \frac{1}{k} e^{-(k+1)(t-s_0)} \|u(\cdot, s_0)\|_{L^k(\Omega)}^k + \left(\delta_1 + \varepsilon_1 + \varepsilon_2 - \mu\right) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} \, dx \, ds 
+ \gamma_1 \int_{s_0}^t e^{-(k+1)(t-s)} \left(\int_{\Omega} |\Delta v|^{k+1} + C_8 \int_{s_0}^t e^{-(k+1)(t-s)} \, ds\right) \int_{\Omega} v^{k+1} \, dx \, ds 
+ \gamma_2 \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} v^{k+1} \, dx \, ds 
\leq \left(\delta_1 + \varepsilon_1 + \varepsilon_2 - \mu\right) \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} \, dx \, ds 
+ \gamma_1 \int_{s_0}^t e^{-(k+1)(t-s)} \left(\int_{\Omega} |\Delta v|^{k+1} + \gamma_2 \int_{s_0}^t e^{-(k+1)(t-s)} \, ds\right) \int_{\Omega} v^{k+1} \, dx \, ds + C_9,
\] 
(3.36)
where
\[
C_9 := C_9(k) = \frac{1}{k} \|u(\cdot, s_0)\|_{L^k(\Omega)}^k + C_8 \int_{s_0}^t e^{-(k+1)(t-s)} \, ds.
\]
The same argument as in the derivation of (3.28)–(3.29) into then shows that
\[
\frac{1}{k} \|u(\cdot, t)\|_{L^k(\Omega)}^k \leq \left[\delta_1 + \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) - \mu\right] \int_{s_0}^t e^{-(k+1)(t-s)} \int_{\Omega} u^{k+1} 
+ \left(\gamma_1 \varepsilon_{1e}^k + \gamma_2 \varepsilon_{2e}^k\right) e^{-(k+1)(t-s_0)} \max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) \int_{\Omega} v^{k+1} \, dx \, ds + C_{10}.
\] 
(3.37)
For any $\varepsilon > 0$, we choose $k = \frac{\max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})}{\max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) - \mu} - \varepsilon$. Then
\[
\frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)}) < \mu,
\]
so that, by picking $\delta_1$ appropriately small (e.g. $\delta_1 < \frac{1}{2} \mu - \frac{(k-1)}{k} \max_{s \geq 1} \lambda_0^{\frac{1}{(k+1)}} (\chi + \xi \|w_0\|_{L^\infty(\Omega)})$) in (3.37), we derive that there exists a positive constant $C_{11}$ such that
\[
\int_{\Omega} u^k(x, t) \, dx \leq C_{11} \quad \text{for all} \quad t \in (0, T_{\text{max}}).
\] 
(3.38)
Thereupon, combined with the arbitrariness of $\varepsilon$ and the Hölder inequality, we can derive (3.19). The proof Lemma 3.4 is complete. \qed

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Lemma 3.5. Let \((u, v, w)\) be a solution to (3.3) on \((0, T_{\text{max}})\). If \(0 < \mu < \kappa_0\) and \(m \geq 2 - \frac{2}{N} \lambda\), then there exist positive constants \(\alpha_0 > \lambda\) and \(C\)

\[
\|u(\cdot, t)\|_{L^{\alpha_0}(\Omega)} \leq C \quad \text{for all} \quad t \in (0, T_{\text{max}})
\]

(3.39)

holds, where

\[
\lambda = \begin{cases} \frac{\kappa_0}{(\kappa_0 - \mu)_+} & \text{if } \tau = 1, \\ \frac{\kappa_0}{(\kappa_0 - \mu)_+} & \text{if } \tau = 0 \end{cases}
\]

(3.40)

and

\[
\kappa_0 = \begin{cases} \max_{s \geq 1} \frac{1}{s + 1} (\lambda + \xi) \|w_0\|_{L^\infty(\Omega)} & \text{if } \tau = 1, \\ \chi & \text{if } \tau = 0. \end{cases}
\]

(3.41)

Proof. Case \(\tau = 0\): Therefore, \(\kappa_0 = \chi\) and \(\lambda = \frac{\chi}{(\lambda - \mu)_+}\) by (3.40) and (3.41), so that in view of Lemma 3.3, we deduce that

\[
\|u(\cdot, t)\|_{L^{\frac{\chi}{(\lambda - \mu)_+}}(\Omega)} \leq C_1 \quad \text{for all} \quad t \in (0, T_{\text{max}})
\]

(3.42)

and some positive constant \(C_1\). On the other hand, by (3.11), we derive that for any \(\delta_1 > 0\),

\[
\begin{align*}
\frac{1}{k} \frac{d}{dt} \|u + 1\|_{L^k(\Omega)}^k &+ \frac{4C_D(k - 1)}{(m + k - 1)^2} \|\nabla (u + 1)^{\frac{k+1}{2}}\|_{L^2(\Omega)}^2 + \int_{\Omega} (u + 1)^k \\
&\leq (-\mu + \frac{k - 1}{k} \chi) \int_{\Omega} (u + 1)^{k+1} + (C_1 + 1) \int_{\Omega} (u + 1)^k \\
&\leq (-\mu + \frac{k - 1}{k} \chi + \delta_1) \int_{\Omega} (u + 1)^{k+1} + C_2
\end{align*}
\]

(3.43)

with \(C_1 = 3\mu + \kappa_\xi\). Next, in view of (3.42), we conclude from the Gagliardo–Nirenberg inequality that there exist positive constants \(C_3 = C_3(k)\) and \(C_4 = C_4(k)\) such that

\[
\begin{align*}
\int_{\Omega} (u + 1)^{k+1} &\leq \|(u + 1)^{\frac{m+k-1}{2}}\|_{L^{\frac{m+k-1}{2}}(\Omega)}^{\frac{2(k+1)}{m+k-1}} \|\nabla (u + 1)^{\frac{k+1}{2}}\|_{L^2(\Omega)}^{\frac{k+1-\gamma_0}{m+k-1}} \|(u + 1)^{\frac{m+k-1}{2}}\|_{L^{\frac{2\gamma_0}{2N(1+\gamma_0)}(\Omega)}^{\frac{2(c+1)}{2N(1+\gamma_0)}(\Omega)}}^{\frac{2(k+1)}{m+k-1}} \\
&\leq C_3(\|\nabla (u + 1)^{\frac{k+1}{2}}\|_{L^2(\Omega)}^{\frac{k+1-\gamma_0}{m+k-1}} \|(u + 1)^{\frac{m+k-1}{2}}\|_{L^{\frac{2\gamma_0}{2N(1+\gamma_0)}(\Omega)}^{\frac{2(c+1)}{2N(1+\gamma_0)}(\Omega)}}^{\frac{2(k+1)}{m+k-1}} \\
&\quad + \|(u + 1)^{\frac{m+k-1}{2}}\|_{L^{\frac{2\gamma_0}{2N(1+\gamma_0)}(\Omega)}^{\frac{2(c+1)}{2N(1+\gamma_0)}(\Omega)}}^{\frac{2(k+1)}{m+k-1}}) + 1),
\end{align*}
\]

(3.44)
where $\gamma_0 = \frac{\chi}{(\chi - \mu)_{+}}$. Since $m \geq 2 - \frac{2}{N} \lambda$, one can easily see that
\[ \frac{2N(k + 1 - \gamma_0)}{\gamma_0(2 - N) + N(m + k - 1)} \leq 2, \]
so that, (3.44) yields to
\[ \int_\Omega (u + 1)^{k+1} \leq C_5(\|\nabla (u + 1)^{\frac{m+k-1}{2}}\|_{L^2(\Omega)}^2 + 1) \] (3.45)
for some positive constant $C_5(k) > 0$. Substituting (3.45) into (3.43), we obtain that
\[ \frac{1}{k} \frac{d}{dt} \|u + 1\|_{L^k(\Omega)}^k + \left[ \frac{4C_D(k - 1)}{(m + k - 1)^2} - (\mu + \frac{k - 1}{k} \chi)C_5 + \delta_1 C_5 \right] \|\nabla (u + 1)^{\frac{m+k-1}{2}}\|_{L^2(\Omega)}^2 \]
\[ + \int_\Omega (u + 1)^k \leq C_6. \] (3.46)

Let $k > \frac{\chi}{(\chi - \mu)_{+}}$. Then by some basic calculation, we derive that
\[ \lim_{k \to (\chi - \mu)_{+}} \left[ \frac{4C_D(k - 1)}{(m + k - 1)^2} - (\mu + \frac{k - 1}{k} \chi)C_5 \right] = \frac{4C_D}{(m + \frac{\mu}{(\chi - \mu)_{+}})^2} > 0 \] (3.47)
and
\[ \frac{4C_D(k - 1)}{(m + k - 1)^2} - \mu + \frac{k - 1}{k} \chi > 0. \] (3.48)
Collecting (3.45)–(3.48), we may choose $k > \frac{\chi}{(\chi - \mu)_{+}}$ which is close to $\frac{\chi}{(\chi - \mu)_{+}}$ such that
\[ \lim_{k \to (\chi - \mu)_{+}} \left[ \frac{4C_D(k - 1)}{(m + k - 1)^2} - (\mu + \frac{k - 1}{k} \chi)C_5 \right] = \frac{2C_D}{(m + \frac{\mu}{(\chi - \mu)_{+}})^2} \] (3.49)
Next, substitute (3.49) into (3.46) and choose $\delta_1$ suitably small (e.g. $\delta_1 < \frac{2C_D}{C_5(m + \frac{\mu}{(\chi - \mu)_{+}})^2}$), then we have
\[ \frac{1}{k} \frac{d}{dt} \|u + 1\|_{L^k(\Omega)}^k + \int_\Omega (u + 1)^k \leq C_6. \] (3.50)

The Gronwall inequality implies assertion (3.39).

Case $\tau = 1$ can be proved very similarly. Therefore, we omit it.

Along with the basic estimate from Lemmas 3.3–3.5, this immediately implies the following Lemma:
Lemma 3.6. Assume that \( \mu > 0 \). If

\[
m \geq 2 - \frac{2}{N} \lambda \quad \text{with} \quad \mu < \kappa_0,
\]

then for \( p > \max\{N+1, N(m+1)\} \), there exists a positive constant \( C = C(p, |\Omega|, \mu, \lambda_0, \xi, \chi, m, C_D) \) such that the solution of (1.3) from Lemma 2.3 satisfies

\[
\int_{\Omega} u^p(x,t) dx \leq C \quad \text{for all} \quad t \in (0, T_{\max}),
\]

where \( \lambda \) and \( \kappa_0 \) are given by (3.40) and (3.41), respectively.

\textbf{Proof.} Firstly, due to Lemma 3.5, we derive that there exists a positive constant \( C_1 \) such that

\[
\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C_1 \quad \text{for all} \quad t \in (0, T_{\max}).
\]

Next, we multiply the first equation of (1.3) by \( u^{p-1} \) and integrate the resulting equation to discover

\[
\frac{1}{p} \frac{d}{dt} \|u\|^p_{L^p(\Omega)} + C_D(p-1) \int_{\Omega} u^{m+p-3} |\nabla u|^2 \leq -\chi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla v) - \xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) + \int_{\Omega} u^{p-1} \mu u - \mu u^2
\]

\[
= \chi (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v - \xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) + \int_{\Omega} u^{p-1} \mu u - \mu u^2
\]

\[
\leq \chi (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v - \xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) + \mu \int_{\Omega} u^p \quad \text{for all} \quad t \in (0, T_{\max}),
\]

which leads to

\[
\frac{1}{p} \frac{d}{dt} \|u\|^p_{L^p(\Omega)} + C_D(p-1) \int_{\Omega} u^{m+p-3} |\nabla u|^2 + \frac{p+1}{p} \int_{\Omega} u^p \leq -\chi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla v) - \xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w) + \mu \int_{\Omega} u^p
\]

for all \( t \in (0, T_{\max}) \). In the following, we will estimate the terms on the right hand side of (3.55) one by one. To this end, firstly, the Young inequality guarantees that

\[
\frac{p+1}{p} \int_{\Omega} u^p \leq \int_{\Omega} u^{p+1} + C_1,
\]

where

\[
C_1(p) = \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} \left( \frac{p+1}{p} + \mu \right)^{p+1} |\Omega|.
\]

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Once more integrating by parts, we also find that
\[
\begin{align*}
\chi(p - 1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v
&= \chi(p - 1) \int_\Omega \nabla u^p \cdot \nabla v \\
&\leq \chi \frac{(p - 1)}{p} \int_\Omega u^p |\Delta v|.
\end{align*}
\]
(3.57)

Here we use the Young inequality to estimate the integrals on the right of (3.57) according to
\[
\begin{align*}
\chi \frac{(p - 1)}{p} \int_\Omega u^p |\Delta v|
&\leq \int_\Omega u^{p+1} + \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \chi \frac{(p - 1)}{p} \right)^{p+1} \int_\Omega |\Delta v|^{p+1} \\
&= \int_\Omega u^{p+1} + C_2 \int_\Omega |\Delta v|^{p+1},
\end{align*}
\]
(3.58)

where
\[
C_2 := \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \left( \chi \frac{(p - 1)}{p} \right)^{p+1}.
\]

Recalling that (3.23) and (3.9), the last term in (3.55) can be estimated by
\[
\begin{align*}
-\xi \int_\Omega \nabla \cdot (u \nabla w) u^{p-1}
&\leq \kappa \xi \int_\Omega u^p + \xi \| w_0 \|_{L^\infty(\Omega)} \int_\Omega u^p v \\
&\leq 2 \int_\Omega u^p + C_4 \int_\Omega v^{p+1} + C_3,
\end{align*}
\]
(3.59)

where
\[
C_3 = \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} (\kappa \xi)^{p+1} |\Omega|
\]
and
\[
C_4 := \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} (\xi \| w_0 \|_{L^\infty(\Omega)})^{p+1}.
\]

While (3.55), (3.56), (3.58) and (3.59) imply that
\[
\begin{align*}
\frac{1}{p} \frac{d}{dt} \| u \|_{L^p(\Omega)}^p + \frac{4C_D(p - 1)}{(m + p - 1)^2} \| \nabla u^{\frac{m+p-1}{2}} \|_{L^2(\Omega)}^2
&\leq 4 \int_\Omega u^{p+1} - \frac{p+1}{p} \int_\Omega u^p + C_5 \int_\Omega (|\Delta v|^{p+1} + v^{p+1}) + C_1 \quad \text{for all } t \in (0, T_{max})
\end{align*}
\]
(3.60)
with $C_5 = C_2 + C_4$. Employing the variation-of-constants formula to (3.60), we obtain

$$
\frac{1}{p}\|u(\cdot, t)\|_{L^p(\Omega)}^p \leq \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|_{L^p(\Omega)}^p - \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u_{m+p-1}^\alpha\|_{L^2(\Omega)}^2 ds \\
+ 4 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds \\
+C_5 \int_0^t e^{-(p+1)(t-s)} \int_\Omega (|\Delta v|^{p+1} + u^{p+1}) ds + C_1 \int_0^t e^{-(p+1)(t-s)} ds
$$

(3.61)

for all $t \in (0, T_{max})$, where $C_7 = C_5 e^{-(p+1)t} \lambda_0 \|v_0\|_{W^{2,p+1}_{2,p+1}(\Omega)}^{p+1}$. Inserting (3.62) into (3.61), we deduce that

$$
\frac{1}{p}\|u(\cdot, t)\|_{L^p(\Omega)}^p \leq \left(4 + C_5 \lambda_0\right) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds \\
- \frac{4C_D(p - 1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u_{m+p-1}^\alpha\|_{L^2(\Omega)}^2 ds + C_6 + C_7.
$$

(3.63)

for all $t \in (0, T_{max})$. For any $p > \max\{N+1, N(m+1), \alpha_0 - 1, 1, 1 - m + \frac{N-2}{N} \alpha_0\}$, $m \geq 2 - \frac{2}{N} \lambda$ together with $\alpha_0 > \lambda$ yields to

$$
p + 1 \leq m + p - 1 + \frac{2}{N} \lambda < m + p - 1 + \frac{2}{N} \alpha_0.
$$

so that, in particular, according to by the Gagliardo–Nirenberg inequality and (3.53), one
can get there exist positive constants $C_8$ and $C_9$ such that
\[(4 + C_5 \lambda_0) \int_\Omega u^{p+1} \leq (4 + C_5 \lambda_0) \| u \|^{m+p-1}_{\frac{m+p-1}{2}} \| u \|^{2(p+1)}_{\frac{2(p+1)}{m+p-1}(\Omega)} \leq C_8(\| \nabla u \|^{m+p-1}_{\frac{m+p-1}{2}} \| u \|^{m+p-1}_{\frac{m+p-1}{2}} L^{\frac{m+p-1}{2}(\Omega)} + \| u \|^{m+p-1}_{\frac{m+p-1}{2}} L^{\frac{m+p-1}{2}(\Omega)})^{\frac{2(p+1)}{m+p-1}} \leq C_9(\| \nabla u \|^{m+p-1}_{\frac{m+p-1}{2}} L^{\frac{m+p-1}{2}(\Omega)} + 1).\]

In view of $m \geq 2 - \frac{2}{N} \lambda$ and $\alpha_0 > \lambda$, by some basic calculation, we derive that
\[
\frac{N(p+1) - N\alpha_0}{(2-N)\alpha_0 + N(m+p-1)} < 1,
\]
so that, which returns, using again the Young inequality, for any $\delta_1 > 0$,
\[(4 + C_5 \lambda_0) \int_\Omega u^{p+1} \leq \delta_1 \| \nabla u \|^{m+p-1}_{\frac{1}{2}} L^{\frac{1}{2}(\Omega)} + C_{10}.\] (3.65)

Combining the above three estimates and choosing $\delta_1$ appropriately small, we arrive at
\[
\int_\Omega u^p(x, t) dx \leq C_{11} \text{ for all } t \in (0, T_{max}),
\] (3.66)
from which we readily infer (3.52). Case $\tau = 0$ can be proved very similarly. Thus we omit it. The proof of Lemma 3.6 is completed. \(\square\)

**Lemma 3.7.** Assume that $\mu > 0$. If
\[m > 2 - \frac{2}{N} \lambda \text{ and } \mu \geq \kappa_0,\] (3.67)
then for all $p > \max\{N+1, N(m+1)\}$, there exists a positive constant $C = C(p, |\Omega|, \mu, \lambda_0, \xi, \chi, m, C_D)$ such that the solution of (1.3) from Lemma 2.3 satisfies
\[
\int_\Omega u^p(x, t) dx \leq C \text{ for all } t \in (0, T_{max}),
\] (3.68)
where $\lambda$ and $\kappa_0$ are given by (3.40) and (3.41), respectively.

**Proof.** In the following, we will only prove the case $\tau = 1$, since, $\tau = 0$ can be proved very similarly and easily. To this end, we begin with (3.63). Firstly, in view of Lemma 3.4, there exists a positive constant $C_1$ such that
\[
\int_\Omega u^b(x, t) dx \leq C_1 \text{ for all } t \in (0, T_{max}),
\] (3.69)
where $l_0 = \lambda - \varepsilon$ with $\varepsilon = \frac{1}{3}N(m - 2 + \frac{2}{N}\lambda)$. On the other hand, since $m > 2 - \frac{2}{N}\lambda$, yields to $p + 1 < m + p - 1 + \frac{2}{N}l_0$, so that, in particular, according to by the Gagliardo–Nirenberg inequality and (3.69), one can get there exist positive constants $k_1$ and $k_2$ such that

$$
(4 + C_5\lambda_0) \int_\Omega u^{p+1} \leq (4 + C_5\lambda_0) \left\| u \frac{m+p-1}{2} \right\|_{L^2(\Omega)}^{\frac{2(p+1)}{m+p+1}} + \left\| u \frac{m+p-1}{2} \right\|_{L^{\frac{m+p+1}{m+p}}(\Omega)}^{\frac{2(p+1)}{m+p+1}} (3.70)
$$

where $C_5$ is the same as (3.63). This together with $\frac{N(p+1)-Nl_0}{(2-N)l_0+N(m+p-1)} < 1$ (by $m > 2 - \frac{2}{N}\lambda$) and the Young inequality implies that for any $\delta_1 > 0$, 

$$
(4 + C_5\lambda_0) \int_\Omega u^{p+1} \leq \delta_1 \left\| \nabla u \frac{m+p-1}{2} \right\|_{L^2(\Omega)}^2 + k_3. (3.71)
$$

which combined with (3.63) implies that

$$
\int_\Omega u^p(x,t)dx \leq C_{11} \quad \text{for all } t \in (0, T_{\text{max}}) (3.72)
$$

by picking $\delta_1$ appropriately small in (3.71). Finally, using the Hölder inequality, we can get (3.68). The proof of Lemma 3.7 is completed.

**Lemma 3.8.** Let

$$
C_D > \frac{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})^3}{4} \left(2 - \frac{2}{N}\right)^2 \tilde{\lambda} (3.73)
$$

and

$$
h(p) := \frac{4C_D}{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})^3} - \frac{(1 - \frac{2}{N} + p)^2}{p} \tilde{\lambda},
$$

where

$$
\tilde{\lambda} = \begin{cases} 
\max_{s \geq 1} \lambda_0^{\frac{1}{\tau}} (\xi \|w_0\|_{L^\infty(\Omega)} + \chi), & \tau = 1, \\
\chi, & \text{if } \tau = 0,
\end{cases} (3.74)
$$

$p \geq 1, C_D, C_{GN}, \chi, \lambda_0$ and $\chi$ are positive constants. Then there exists a positive constant $\tilde{p}_0 > 1$ such that

$$
h(p) > 0 \quad \text{for all } p \in (1, \tilde{p}_0], (3.75)
$$
Proof. The idea comes from [58]. Indeed, due to (3.73), it is not difficult to verify that
\[ h(1) \geq 4C_D \frac{4C_G N(1 + \|u_0\|_{L^1(\Omega)})^3}{4} - (2 - \frac{2}{N})^2 \tilde{\lambda} > 0. \]
Next, by basic calculation, we derive that for any \( p \geq 1 \),
\[ h'(p) = \frac{(1 - \frac{2}{N}) + p}{p^2} (p + \frac{2}{N} - 1) \tilde{\lambda} < 0. \]
Therefore, from the monotonicity of \( h \), there exists a positive constant \( \tilde{p}_0 > 1 \) such that (3.75) holds.

Lemma 3.9. Assume that \( \mu = 0 \). If
\[ m > 2 - \frac{2}{N} \] (3.76)

or
\[ m = 2 - \frac{2}{N} \] and \[ C_D > \frac{C_G N(1 + \|u_0\|_{L^1(\Omega)})^3}{4} (2 - \frac{2}{N})^2 \tilde{\lambda}, \] (3.77)
then there exists a positive constant \( p_0 > 1 \) such that the solution of (1.3) from Lemma 2.3 satisfies
\[ \int_{\Omega} u^{p_0}(x, t) dx \leq C \quad \text{for all} \quad t \in (0, T_{max}), \] (3.78)
where \( \tilde{\lambda} \) is the same as (3.74).

Proof. Without loss of generality, we may assume that
\[ m = 2 - \frac{2}{N} \] and \[ C_D > \frac{C_G N(1 + \|u_0\|_{L^1(\Omega)})^3}{4} (2 - \frac{2}{N})^2 \tilde{\lambda}, \]
since, \( m > 2 - \frac{2}{N} \) can be proved similarly and easily.

We now consider two situations. Case \( \tau = 1 \): Then by (3.74) derive that \( \tilde{\lambda} = \max_{s \geq 1} \lambda_0^\frac{1}{s+1} \).
Assume that \( \tilde{p}_0 \) is the same as lemma 3.8 and let \( 1 < p \leq \min\{2, \tilde{p}_0\} \). Multiplying the first equation of (1.3) by \( u^{p-1} \) and using \( \mu = 0 \), we derive that
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + (p - 1) \int_{\Omega} D(u) u^{p-2} |\nabla u|^2
\]
\[
= -\chi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla v) - \xi \int_{\Omega} u^{p-1} \nabla \cdot (u \nabla w)
\]
\[
= \chi (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v \quad \text{for all} \quad t \in (0, T_{max}),
\] (3.79)
which combined with (1.5) yields to
\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + C_D (p - 1) \int_{\Omega} u^{p-2} |\nabla u|^2 + \frac{p+1}{p} \int_{\Omega} u^p
\]
\[
\leq \chi (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla v + \xi (p-1) \int_{\Omega} u^{p-1} \nabla u \cdot \nabla w + \frac{p+1}{p} \int_{\Omega} u^p
\] (3.80)

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for all $t \in (0, T_{\max})$. Now, we will estimate the terms on the right hand side of (3.80). In fact, employing the same arguments as in the proof of (3.23)-(3.25), we deduce the estimate

$$
\chi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla v + \xi(p-1) \int_\Omega u^{p-1} \nabla u \cdot \nabla w + \frac{p+1}{p} \int_\Omega u^p
$$

$$
= -\chi \frac{(p-1)}{p} \int_\Omega u^p \Delta v - \xi \frac{(p-1)}{p} \int_\Omega u^p \Delta w + \frac{p+1}{p} \int_\Omega u^p
$$

$$
\leq (\kappa \frac{(p-1)\xi}{p} + \frac{p+1}{p}) \int_\Omega u^p + (\frac{(p-1)\xi}{p} \|w_0\|_{L^\infty(\Omega)}) \int_\Omega u^p + \chi \frac{(p-1)}{p} \int_\Omega u^p |\Delta v| \quad (3.81)
$$

$$
\leq (2 + \kappa \xi) \int_\Omega u^p + (\varepsilon_1 + \varepsilon_2) \int_\Omega u^{p+1} + \tilde{\gamma}_2 \varepsilon_2^{-p} \int_\Omega u^{p+1} + \tilde{\gamma}_1 \varepsilon_1^{-p} \int_\Omega |\Delta v|^{p+1}
$$

$$
\leq \delta_1 \int_\Omega u^{p+1} + (\varepsilon_1 + \varepsilon_2) \int_\Omega u^{p+1} + \tilde{\gamma}_2 \varepsilon_2^{-p} \int_\Omega u^{p+1} + \tilde{\gamma}_1 \varepsilon_1^{-p} \int_\Omega |\Delta v|^{p+1} + C_1,
$$

where

$$
\varepsilon_1 = \frac{(p-1)\chi}{p+1} \lambda_0^{\frac{1}{p+1}} \quad \text{and} \quad \varepsilon_2 = \frac{(p-1)\xi \|w_0\|_{L^\infty(\Omega)}}{p+1} \lambda_0^{\frac{1}{p+1}} \quad (3.82)
$$

as well as

$$
C_1 = \frac{1}{p+1} (\delta_1 \frac{p+1}{p})^{-p} (2 + \kappa \xi)^{p+1} |\Omega| \quad (3.83)
$$

and

$$
\tilde{\gamma}_1 := \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} \left( \frac{(p-1)\chi}{p} \right)^{p+1} \quad \text{and} \quad \tilde{\gamma}_2 := \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} \left( \frac{(p-1)\xi \|w_0\|_{L^\infty(\Omega)}}{p} \right)^{p+1}, \quad (3.84)
$$

Here $\lambda_0$ and $\kappa$ are give by (2.3) and Lemma 2.2, respectively.

Hence (3.80), (3.81)-(3.84) results in

$$
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + \frac{4C_D(p-1)}{1 - \frac{2}{N} + \frac{2}{p}} \|\nabla u \chi^{\frac{1}{p+1}}\|_{L^2(\Omega)}^2 + \frac{p+1}{p} \int_\Omega u^p
$$

$$
\leq (\varepsilon_1 + \varepsilon_2) \int_\Omega u^{p+1} + \tilde{\gamma}_2 \varepsilon_2^{-p} \int_\Omega u^{p+1} + \tilde{\gamma}_1 \varepsilon_1^{-p} \int_\Omega |\Delta v|^{p+1} + C_1 \quad \text{for all} \ t \in (0, T_{\max}). \quad (3.85)
$$

Here, in order to estimate the rightmost term appropriately, we employ the Gagliardo-Nirenberg inequality to obtain $C_{GN} > 0$ such that

$$
\int_\Omega u^{p+1}
$$

$$
= \|u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^2 + \|u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)}^2
$$

$$
\leq C_{GN} (\|\nabla u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^2(\Omega)} \|u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)} + \|u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^{\frac{2(p+1)}{p}}(\Omega)})^2
$$

$$
\leq C_{GN} (1 + \|u_0\|_{L^1(\Omega)})^2 \|\nabla u^{\frac{1-\frac{2}{p+1}}{2}}\|_{L^2(\Omega)}^2 + 1, \quad (3.86)
$$
by using (3.1), where in the last inequality we have used $p \leq 2$ and $C_{GN}$ is the same as Lemma 2.1. In combination with (3.85) and (3.86), this shows that

$$\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p \leq (\delta_1 + \varepsilon_1 + \varepsilon_2 - \frac{4C_D(p - 1)}{1 - \frac{2N}{p} + p^2} \frac{1}{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})^3}) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds + \tilde{\gamma}_2 \varepsilon_2^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega v^{p+1} ds + \tilde{\gamma}_1 \varepsilon_1^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} ds + C_2$$

for all $t \in (0, T_{max})$,

$$\text{(3.87)}$$

where $C_2 = C_1 + \frac{4C_D(p-1)}{1 - \frac{2N}{p} + p^2}$. Employing the variation-of-constants formula to (3.87), we obtain

$$\frac{1}{p} \|u(\cdot, t)\|_{L^p(\Omega)}^p$$

\[ \leq (\delta_1 + \varepsilon_1 + \varepsilon_2 - \frac{4C_D(p - 1)}{1 - \frac{2N}{p} + p^2} \frac{1}{C_{GN}(1 + \|u_0\|_{L^1(\Omega)})^3}) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds + \tilde{\gamma}_2 \varepsilon_2^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega v^{p+1} ds + \tilde{\gamma}_1 \varepsilon_1^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} ds + C_2 \]

with

$$C_3 := C_3(\varepsilon_1, p) = \frac{1}{p} e^{-(p+1)t} \|u_0\|_{L^p(\Omega)}^p + C_2 \int_0^t e^{-(p+1)(t-s)} ds.$$ 

Now, due to Lemma 2.2 and the second equation of (1.3), we have

$$\tilde{\gamma}_2 \varepsilon_2^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega v^{p+1} ds$$

\[ = \tilde{\gamma}_2 \varepsilon_2^p e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_\Omega v^{p+1} ds \]

\[ \leq \tilde{\gamma}_2 \varepsilon_2^p e^{-(p+1)t} \lambda_0 \left[ \int_0^t \int_\Omega e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1}(\Omega)}^{p+1} \right] \]

\[ \leq \tilde{\gamma}_2 \varepsilon_2^p e^{-(p+1)t} \lambda_0 \int_0^t e^{(p+1)s} u^{p+1} ds + C_4 \]

and

$$\tilde{\gamma}_1 \varepsilon_1^p \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} ds$$

\[ = \tilde{\gamma}_1 \varepsilon_1^p e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_\Omega |\Delta v|^{p+1} ds \]

\[ \leq \tilde{\gamma}_1 \varepsilon_1^p e^{-(p+1)t} \lambda_0 \left[ \int_0^t \int_\Omega e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1}(\Omega)}^{p+1} \right] \]

\[ \leq \tilde{\gamma}_1 \varepsilon_1^p e^{-(p+1)t} \lambda_0 \int_0^t e^{(p+1)s} u^{p+1} ds + C_5 \]

(3.89)
for all \( t \in (0, T_{\text{max}}) \), where \( C_4 = \tilde{\gamma}_2 \varepsilon_2^{-p} e^{-(p+1)t} \lambda_0 \| v_0 \|_{W^{2,p+1}(\Omega)}^{p+1} \) and \( C_5 = \tilde{\gamma}_1 \varepsilon_1^{-p} e^{-(p+1)t} \lambda_0 \| v_0 \|_{W^{2,p+1}(\Omega)}^{p+1} \).

Recalling (3.88) and applying (3.89)–(3.90) now ensures that

\[
\frac{1}{p} \| u(\cdot, t) \|_{L^p(\Omega)}^p \leq \delta_1 + \varepsilon_1 \lambda_0 \varepsilon_1^{-p} - \frac{4C_D(p-1)}{(1 - \frac{2}{N} + p)^2 C_{GN}(1 + \| u_0 \|_{L^1(\Omega)})^3} \int_0^t e^{-(p+1)(t-s)} \int_\Omega u_{p+1}^p ds + \tilde{\gamma}_1 \varepsilon_1^{-p} \lambda_0 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u_{p+1}^p ds + C_6
\]

\[
+ C_6
\]

with \( C_6 = C_4 + C_5 + C_3 \). Now, by (3.81) and (3.83), some basic calculation yields to

\[
\varepsilon_1 + \gamma_1 \lambda_0 \varepsilon_1^{-p} = \frac{(p-1)}{p} \lambda_0^{\frac{1}{p+1}} \chi \leq \frac{(p-1)}{p} \max_{s \geq 1} \lambda_0^{\frac{1}{p+1}} \chi
\]

and

\[
\varepsilon_2 + \gamma_2 \lambda_0 \varepsilon_2^{-p} = \frac{(p-1)}{p} \lambda_0^{\frac{1}{p+1}} \xi \| w_0 \|_{L^\infty(\Omega)} \leq \frac{(p-1)}{p} \max_{s \geq 1} \lambda_0^{\frac{1}{p+1}} \xi \| w_0 \|_{L^\infty(\Omega)},
\]

so that, thus, by (3.77), we can choose \( \delta_1 \) small enough in (3.91), using Lemma 3.8 we derive that there exists a positive constant \( p_0 \geq 1 \) such that

\[
\int_\Omega u_{p_0}^p(x, t) dx \leq C_6 \quad \text{for all } t \in (0, T_{\text{max}}).
\]

(3.92)

Case \( \tau = 0 \) can be proved very similarly, therefore, we omit it. The proof of Lemma 3.9 is completed.

Our next goal is to make sure that Lemma 3.9 is sufficient to enforce boundedness of \( \| u(\cdot, t) \|_{L^p(\Omega)} \) for all \( t \in (0, T_{\text{max}}) \) and \( p > 1 \), which plays a key step in the derivation of our main results.

**Lemma 3.10.** Suppose that the conditions of Lemma 3.9 hold. Then for any \( p > 1 \), there exists a positive constant \( C := C(p, |\Omega|, C_D, C_{GN}, \lambda_0, m, \chi) \) such that

\[
\| u(\cdot, t) \|_{L^p(\Omega)} \leq C \quad \text{for all } t \in (0, T_{\text{max}}).
\]

(3.93)
Proof. Firstly, let \( p > \max\{N+1, N(m+1), p_0 - 1, 1, 1 - m + \frac{N-2}{N}\} \), where \( p_0 > 1 \) is the same as Lemma 3.9. Testing the first equation of (1.3) against \( u^{p-1} \), using \( \mu = 0 \), Lemma 2.4 and the Young inequality yields

\[
\frac{1}{p} \frac{d}{dt} \|u\|_{L^p(\Omega)}^p + C_D(p-1) \int_\Omega (u+1)^{m+p-3} |\nabla u|^2 + \frac{p+1}{p} \int_\Omega u^p \\
\leq -\chi \int_\Omega u^{p-1} \nabla \cdot (u \nabla v) - \xi \int_\Omega u^{p-1} \nabla \cdot (u \nabla w) + \frac{p+1}{p} \int_\Omega u^p \\
\leq (\kappa (p-1) \xi + \frac{p+1}{p}) \int_\Omega u^p + \tau (p-1) \xi \|w_0\|_{L^\infty(\Omega)} \int_\Omega u^p \chi \frac{(p-1)}{p} \int_\Omega u^p |\Delta v| \\
\leq (\kappa (p-1) \xi + \frac{p+1}{p}) \int_\Omega u^p + (p-1) \xi \|w_0\|_{L^\infty(\Omega)} \int_\Omega u^p \chi \frac{(p-1)}{p} \int_\Omega u^p |\Delta v| \\
\leq (2 + \kappa \xi) \int_\Omega u^p + \xi \|w_0\|_{L^\infty(\Omega)} \int_\Omega u^p v + \chi \int_\Omega u^p |\Delta v| \\
\leq 3 \int_\Omega u^{p+1} + C_1 \int_\Omega |\Delta v|^{p+1} + C_3 \int_\Omega v^{p+1} + C_2 \text{ for all } t \in (0, T_{\max}),
\]

where

\[
C_1 = \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} \chi^{p+1} \text{ as well as } C_2 = \frac{1}{p+1} \left( \frac{p+1}{p} \right)^{-p} (2 + \kappa \xi)^{p+1} |\Omega|
\]

and

\[
C_3 = \frac{1}{p+1} \left[ \frac{p+1}{p} \right]^{-p} (\xi \|w_0\|_{L^\infty(\Omega)})^{p+1}.
\]

Here we have used that \( \tau \in \{0, 1\} \). In order to control the second and third integral on the right of (3.94), we make use of the variation-of-constants formula to (3.94), so that, we derive

\[
\frac{1}{p} \|u(t)\|_{L^p(\Omega)}^p \\
\leq \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|_{L^p(\Omega)}^p - 4C_D(p-1) \int_0^t e^{-(p+1)(t-s)} \|\nabla u\|_{L^2(\Omega)}^{\frac{m+p-1}{2}} ds \\
+ 3 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds \\
+ C_1 \int_0^t e^{-(p+1)(t-s)} \int_\Omega |\Delta v|^{p+1} ds + C_3 \int_0^t e^{-(p+1)(t-s)} \int_\Omega v^{p+1} ds + C_2 \int_0^t e^{-(p+1)(t-s)} ds \\
\leq 3 \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds - 4C_D(p-1) \int_0^t e^{-(p+1)(t-s)} \|\nabla u\|_{L^2(\Omega)}^{\frac{m+p-1}{2}} ds \\
+ C_5 \int_0^t e^{-(p+1)(t-s)} \int_\Omega (|\Delta v|^{p+1} + v^{p+1}) ds + C_4
\]

with

\[
C_4 := \frac{1}{p} e^{-(p+1)t} \|u_0(\cdot)\|_{L^p(\Omega)}^p + C_2 \int_0^t e^{-(p+1)(t-s)} ds \text{ and } C_5 = C_1 + C_3.
\]
Now, we use Lemma 2.2, the second equation of (1.3) and the Hölder inequality to find

\[ C_5 \int_0^t e^{-(p+1)(t-s)} \int_\Omega (|\Delta v|^{p+1} + v^{p+1}) dx ds \]

\[ = C_5 e^{-(p+1)t} \int_0^t e^{(p+1)s} \int_\Omega (|\Delta v|^{p+1} + v^{p+1}) dx ds \]

\[ \leq C_5 e^{-(p+1)t} \lambda_0 \left[ \int_0^t \int_\Omega e^{(p+1)s} u^{p+1} ds + \|v_0\|_{W^{2,p+1} (\Omega)}^{p+1} \right] \]

\[ \leq C_5 e^{-(p+1)t} \lambda_0 \int_0^t e^{(p+1)s} u^{p+1} ds + C_6 \]

for all \( t \in (0, T_{max}) \), where \( C_6 = C_5 e^{-(p+1)t} \lambda_0 \|v_0\|_{W^{2,p+1} (\Omega)}^{p+1} \). Hence (3.95) and (3.96) results in

\[ \frac{1}{p} \|u(\cdot, t)\|^p_{L^p (\Omega)} \leq (3 + C_5 \lambda_0) \int_0^t e^{-(p+1)(t-s)} \int_\Omega u^{p+1} ds \]

\[ - \frac{4C_D(p-1)}{(m + p - 1)^2} \int_0^t e^{-(p+1)(t-s)} \|\nabla u\|^{m+p-1}_2 \|u\|^{2(p+1)}_{L^2(\Omega)} ds + C_4 + C_6. \]

for all \( t \in (0, T_{max}) \). Therefore, observe that \( m \geq 2 - \frac{2}{N} \) and \( p_0 > 1 \) yields to \( p + 1 < m + p - 1 + \frac{2}{N}p_0 \), so that, in view of the Gagliardo–Nirenberg inequality, (3.78) and using the Young inequality, one can get there exist positive constants \( C_7, C_8 \) and \( C_9 \) such that for any \( \delta_1 > 0 \)

\[ (3 + C_5 \lambda_0) \int_\Omega u^{p+1} \]

\[ = (3 + C_5 \lambda_0) \|u\|^{p+1}_{\frac{m+p-1}{2}} \|u\|^{\frac{2(p+1)}{m+p-1}}_{\frac{1}{m+p-1}} \left[ \frac{m+p-1}{2} \|u\|^{\frac{m+p-1}{p_0}-1}_{\frac{2}{m+p-1}} \|u\|^{p_0}_{L^{m+p-1}(\Omega)} \right] \]

\[ \leq C_7 (\|\nabla u\|^{\frac{m+p-1}{2}}_{L^{2}(\Omega)} \|u\|^{\frac{m+p-1}{2}}_{L^{2}(\Omega)} \|u\|^{\frac{2p_0}{m+p-1}}_{L^{m+p-1}(\Omega)} \|u\|^{\frac{2p_0}{m+p-1}}_{L^{m+p-1}(\Omega)})^{\frac{2(p+1)}{m+p-1}} \]

\[ \leq C_8 (\|\nabla u\|^{\frac{m+p-1}{2}}_{L^{2}(\Omega)} \|u\|^{\frac{2p_0}{m+p-1}}_{L^{2}(\Omega)} \frac{N(p+1)-Np_0}{(2-N)p_0+N(m+p-1)}) + 1) \]

\[ \leq \delta_1 \|\nabla u\|^{\frac{m+p-1}{2}}_{L^{2}(\Omega)} + C_9, \]

where we have used \( \frac{N(p+1)-Np_0}{(2-N)p_0+N(m+p-1)} < 1 \) together with \( m \geq 2 - \frac{2}{N} \) and \( p_0 > 1 \). Inserting (3.98) into (3.97), choosing \( \delta_1 \) appropriately small and using the Hölder inequality, we can get (3.93).

\[ \square \]

4 The proof of main results

In this section, we are going to prove our main result. To this end, we will proceed in two steps. Firstly, applying the standard regularity theory of partial differential equation, we turn the bounds from Lemma 3.10 into a higher order bound for \( \nabla v \).
Lemma 4.1. Suppose that the conditions of Theorem 1.1 (or Theorem 1.2) hold. Let $T \in (0, T_{\text{max}})$ and $(u, v, w)$ be the solution of (1.3). Then there exists a constant $C > 0$ independent of $T$ such that the component $v$ of $(u, v, w)$ satisfies

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \leq C \text{ for all } t \in (0, T).$$

(4.1)

Proof. Due to $\|u(\cdot, t)\|_{L^p(\Omega)}$ is bounded for any large $p$ (see Lemma 3.10), we infer from the standard regularity theory of parabolic equation (or elliptic equation, $\tau = 0$) that (4.1) holds.

The previous lemmas at hand, we can now pass to the proof of our main result. Its proof is based on a Moser-type iteration (see e.g. [35] and [17]).

Lemma 4.2. Under the assumptions of Theorem 1.1 (or Theorem 1.2), one can find a positive constant such that for every $T \in (0, T_{\text{max}})$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq C \text{ for all } t \in (0, T).$$

(4.2)

Proof. Throughout the proof of Lemma 4.2 we use $C_i$ ($i \in \mathbb{N}$) to denote the different positive constants independent of $p$ and $k$ ($k \in \mathbb{N}$).

Case $m \geq 1$: For any $p > 1$, multiplying both sides of the first equation in (1.3) by $(u+1)^{p-1}$, integrating over $\Omega$, integrating by parts and using the Young inequality and (4.1)
and (2.2), we derive that

\[
\frac{1}{p} \frac{d}{dt} \|u + 1\|_{L^p(\Omega)}^p + CD(p - 1) \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 \\
\leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v)(u + 1)^{p-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w)(u + 1)^{p-1} + \int_{\Omega} (u + 1)^{p-1}(\mu u - \mu u^2) \\
\leq \chi(p - 1) \int_{\Omega} (u + 1)^{p-2} |\nabla u| |\nabla v| - \xi(p - 1) \int_{\Omega} \nabla u(\nabla v) + (u + 1)^{p-1}(\mu u - \mu u^2) \\
\leq \chi(p - 1) C_1 \int_{\Omega} (u + 1)^{p-1} |\nabla u| + C_2 \int_{\Omega} (u + 1)^{p-1} + \int_{\Omega} (u + 1)^{p-1}(\mu u - \mu u^2) \\
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + \xi(p - 1)C_2^2 \int_{\Omega} (u + 1)^{p+1-m} + (C_2 + \mu) \int_{\Omega} (u + 1)^p \\
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + \chi^2(p - 1)C_2^2 \int_{\Omega} (u + 1)^{p+1} + (C_2 + \mu) \int_{\Omega} (u + 1)^p \\
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + C_3p \int_{\Omega} (u + 1)^p - \int_{\Omega} (u + 1)^p \text{ for all } t \in (0, T) \\
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + C_3p \int_{\Omega} (u + 1)^p - \int_{\Omega} (u + 1)^p \text{ for all } t \in (0, T).
\]

(4.3)

with \( C_1 = \sup_{t \in (0, T)} \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \), \( C_2 = \xi \|w_0\|_{L^\infty(\Omega)} C_1 + \kappa \), \( C_3 = \chi C_2^2 + C_2 + \mu + 1 \), where in the last inequality we have used the fact that \( \int_{\Omega} \mu u(u + 1)^{p-1} \leq \int_{\Omega} \mu u(u + 1)^p \) and \( u \geq 0 \).

Due to (4.3), we deduce that

\[
\frac{d}{dt} \|u + 1\|_{L^p(\Omega)}^p + \int_{\Omega} (u + 1)^{p} + C_3 \int_{\Omega} |\nabla (u + 1)^{\frac{m+p-1}{2}}|^2 \leq C_2p^2 \int_{\Omega} (u + 1)^p \text{ for all } t \in (0, T). \\
\]

(4.4)

Now, we let \( l_0 > \max\{1, m - 1\} \), \( p := p_k = 2^k(l_0 + 1 - m) + m - 1 \) and

\[
M_k = \max\{1, \sup_{t \in (0, T)} \int_{\Omega} (u + 1)^{pk}\} \text{ for } k \in \mathbb{N}. \\n\]

(4.5)

We now invoke the Gagliardo–Nirenberg inequality ensures that

\[
\int_{\Omega} (u + 1)^{pk} \\
\leq C_2p_k^2 \int_{\Omega} (u + 1)^{pk} \\
= C_2p_k^2 \|u + 1\|_{L^{pk}(\Omega)}^{\frac{2pk}{m+pk-1}} \|u + 1\|_{L^{pk}(\Omega)}^{\frac{2pk}{m+pk-1}} \\
\leq C_3p_k^2 \|\nabla (u + 1)^{\frac{m+pk-1}{2}}\|_{L^2(\Omega)} \leq \|u + 1\|_{L^1(\Omega)} + \|u + 1\|_{L^1(\Omega)} \leq \|u + 1\|_{L^1(\Omega)} + \|u + 1\|_{L^1(\Omega)} \\
(4.6)
\]

where

\[
\frac{2pk}{m+pk-1} = \frac{2pk}{m+pk-1} \left(1 - \frac{N(m+pk-1)}{2}\right) = \frac{2N(p_k + 1 - m)}{(N+2)(m+pk-1)} < 2
\]

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with some \( \rho > m \) and
\[
2 \frac{p_k}{m + p_k - 1} (1 - \varsigma_1) = \frac{2p_k}{m + p_k - 1} \left( 1 - \frac{N - \frac{N(m + p_k - 1)}{2p_k}}{1 - \frac{N}{2} + N} \right) = 2 \frac{2p_k + N(m - 1)}{(N + 2)(m + p_k - 1)}.
\]

Therefore, an application of the Young inequality yields
\[
C_2 p_k^2 \int \Omega (u + 1)^{pk} \leq C_4 \| \nabla (u + 1) \|_{L^2(\Omega)}^{\frac{m + p_k - 1}{2}} + C_5 p_k^{(N + 2)(m + p_k - 1)} \| (u + 1) \|_{L^1(\Omega)}^{\frac{2p_k + N(m - 1)}{m + p_k - 1}}\]
\[
+ C_6 p_k^2 \| (u + 1) \|_{L^1(\Omega)}^{\frac{m + p_k - 1}{2}} \| (u + 1) \|_{L^1(\Omega)}^{\frac{2p_k + N(m - 1)}{m + p_k - 1}}.
\]

Here we have used the fact that \( \frac{2p_k + N(m - 1)}{N(m - 1) + m + p_k - 1} \leq \frac{2p_k}{m + p_k - 1} \) and \( \frac{(N + 2)(m + p_k - 1)}{p_k + (N + 1)(m - 1)} \geq 2 \) (by \( p_k > m - 1 \)). Thus, in light of \( m \geq 1 \), by means of (4.5)–(4.7),
\[
\frac{d}{dt} \| u + 1 \|_{L^{pk}(\Omega)}^2 + \int \Omega (u + 1)^{pk} \leq C_7 p_k^{(N + 2)(m + p_k - 1)} \| (u + 1) \|_{L^1(\Omega)}^{\frac{2p_k}{m + p_k - 1}} \]
\[
\leq \rho^k M_{k-1}^{2p_k} \leq \rho^k M_{k-1}^2 \quad \text{for all} \quad t \in (0, T)
\]
with some \( \rho > 1 \). Here we have used the fact that
\[
\frac{(N + 2)(m + p_k - 1)}{p_k + (N + 1)(m - 1)} = 2^{k(l_0 + 1 - m)} (N + 2) + 2(N + 2)(m - 1) \leq N + 2
\]
and
\[
\frac{2p_k}{m + p_k - 1} \leq 2(p_k + m - 1) \quad \text{and} \quad m + p_k - 1 = 2.
\]

Integrating (4.8) over \( (0, t) \) with \( t \in (0, T) \), we derive
\[
\int \Omega (u + 1)^{pk}(x, t) \leq \max \left\{ \int \Omega (u_0 + 1)^{pk}, \rho^k M_{k-1}^2 \right\} \quad \text{for all} \quad t \in (0, T). \quad (4.9)
\]
If \( \int \Omega (u + 1)^{pk}(x, t) \leq \int \Omega (u_0 + 1)^{pk} \) for any large \( k \in \mathbb{N} \), then we obtain (4.2) directly. Otherwise, by a straightforward induction, we have
\[
\int \Omega (u + 1)^{pk} \leq \rho^k (\rho^{k-1} M_{k-2}^2)^2
\]
\[
= \rho^{k+2(k-1)} M_{k-2}^2
\]
\[
\leq \rho^{k+\sum_{j=2}^{k}(j-1)} M_0^{2k}.
\]

In light of \( \ln(1 + z) \leq z \) for all \( z \geq 0 \), so that, taking \( p_k \)-th roots on both sides of (4.10), we can easily get (4.2).
Case $m < 1$: Due to Lemmas 3.51 and 3.10 we may choose
\[\tilde{p}_0 := \max\{6N(1-m), 5(1-m), (3N+3)(1-m), 1\}\] (4.11)
such that
\[\int_{\Omega} (u + 1)^{\tilde{p}_0}(x, t) \leq C_9 \quad \text{for all} \quad t \in (0, T_{\max}).\] (4.12)

Next, testing the first equation in (1.3) by $(u + 1)^{p-1}$, integrating over $\Omega$, integrating by parts and applying the Young inequality and (4.1), we derive that
\[
\frac{1}{p} \frac{d}{dt} \|u + 1\|_{L^p(\Omega)}^p + C_D(p - 1) \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2
\leq -\chi \int_{\Omega} \nabla \cdot (u \nabla v)(u + 1)^{p-1} - \xi \int_{\Omega} \nabla \cdot (u \nabla w)(u + 1)^{p-1} + \int_{\Omega} (u + 1)^{p-1}(\mu u - \mu u^2)
\leq \chi(p - 1) \int_{\Omega} u(u + 1)^{p-2} |\nabla u| |\nabla v| - \xi(p - 1) \int_{\Omega} \int_0^t \tau(\tau + 1)^{p-2} d\tau \Delta w + \int_{\Omega} (u + 1)^{p-1}(\mu u - \mu u^2)
\leq \chi(p - 1) \int_{\Omega} u(u + 1)^{p-2} |\nabla u| |\nabla v|
+ \frac{\xi(p - 1)}{4} \int_{\Omega} (u + 1)^{p-1}(\tau) |w_0|_{L^\infty(\Omega)} \cdot v(x, t) + \kappa + \int_{\Omega} (u + 1)^{p-1}(\mu u - \mu u^2)
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + \chi^2(p - 1)C_1 \int_{\Omega} (u + 1)^{p+1-m} + (C_2 + \mu) \int_{\Omega} (u + 1)^{p}
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + \chi^2(p - 1)C_1 \int_{\Omega} (u + 1)^{p+1-m} + (C_2 + \mu) \int_{\Omega} (u + 1)^{p+1-m}
\leq \frac{(p - 1)}{4} \int_{\Omega} (u + 1)^{m+p-3} |\nabla u|^2 + C_{10} p \int_{\Omega} (u + 1)^{p+1-m} - \int_{\Omega} (u + 1)^{p} \quad \text{for all} \quad t \in (0, T),\] (4.13)

with $C_1 = \sup_{t \in (0, T)} \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}, C_2 = \xi \|w_0\|_{L^\infty(\Omega)} C_1 + \kappa, C_{10} = \chi^2 C_1^2 + C_2 + \mu + 1$.

Here we have used the fact that $\int_{\Omega} u(u + 1)^{p-1} \leq \int_{\Omega} (u + 1)^p \leq \int_{\Omega} (u + 1)^{p+1-m}$ and $u \geq 0$.

Therefore, (4.13) yields to
\[
\frac{d}{dt} \|u + 1\|_{L^p(\Omega)}^p + \int_{\Omega} (u + 1)^p + C_{11} \int_{\Omega} |\nabla(u + 1)^\frac{m+p-1}{2}|^2 \leq C_{12} p^2 \int_{\Omega} (u + 1)^{p+1-m} \quad \text{for all} \quad t \in (0, T),\] (4.14)

Let $p := \tilde{p}_k = 2^k(\tilde{p}_0 + 1 - m) + m - 1$ and
\[\tilde{M}_k = \max\{1, \sup_{t \in (0, T)} \int_{\Omega} (u + 1)^\tilde{p}_k\} \quad \text{for} \quad k \in \mathbb{N},\] (4.15)
where \( \tilde{p}_0 \) is given by (4.11). As moreover by the Gagliardo-CNirenberg inequality, we have

\[
C_{12}\tilde{p}_k^2 \int_\Omega (u + 1)^{\tilde{p}_k + 1 - m} \\
= C_{12}\tilde{p}_k^2 \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^2(\Omega)} \left( \frac{2(\tilde{p}_k + 1 - m)}{m + \tilde{p}_k - 1} \right)^{2(\tilde{p}_k + 1 - m)} \left\| L^1(\Omega) \right\|_{m + \tilde{p}_k - 1} \left( u + 1 \right)^{\frac{m + \tilde{p}_k - 1}{2}} \left\| L^1(\Omega) \right\|_{m + \tilde{p}_k - 1}^{(1 - \varsigma_2)} + \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^1(\Omega)}^{2(\tilde{p}_k + 1 - m)},
\]

where

\[
\frac{2(\tilde{p}_k + 1 - m)}{m + \tilde{p}_k - 1} \varsigma_2 = \frac{2(\tilde{p}_k + 1 - m) N - \frac{N(m + \tilde{p}_k - 1)}{2(\tilde{p}_k + 1 - m)}}{1 - \frac{N}{2} + \frac{N}{2}} = \frac{2N\tilde{p}_k + 6N(1 - m)}{(m + \tilde{p}_k - 1)(N + 2)} < 2 \text{ if } \tilde{p}_k > (2N + 1)(1 - m)
\]

and

\[
\frac{2(\tilde{p}_k + 1 - m)}{m + \tilde{p}_k - 1} (1 - \varsigma_2) = \frac{2(\tilde{p}_k + 1 - m)}{m + \tilde{p}_k - 1} \left( 1 - \frac{N - \frac{N(m + \tilde{p}_k - 1)}{2(\tilde{p}_k + 1 - m)}}{1 - \frac{N}{2} + \frac{N}{2}} \right) = \frac{(m + \tilde{p}_k - 1)(N - 1)}{(m + \tilde{p}_k - 1)(N + 2)}.
\]

Therefore, in light of the Young inequality, we conclude that

\[
C_{12}\tilde{p}_k^2 \int_\Omega (u + 1)^{\tilde{p}_k + 1 - m} \\
\leq C_{11} \left\| \nabla (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^2(\Omega)}^2 + C_{14}\tilde{p}_k^{\frac{(m + \tilde{p}_k - 1)(N + 2)}{2(m + \tilde{p}_k - 1)(2N + 1)}} \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^1(\Omega)}^{2(\tilde{p}_k + (N - 1)(m - 1))} \\
+ C_{15}\tilde{p}_k \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^1(\Omega)}^{2(\tilde{p}_k + 1 - m)} \\
\leq C_{11} \left\| \nabla (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^2(\Omega)}^2 + C_{16}\tilde{p}_k^{\frac{(m + \tilde{p}_k - 1)(N + 2)}{2(m + \tilde{p}_k - 1)(2N + 1)}} \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^1(\Omega)}^{2(\tilde{p}_k + 1 - m)},
\]

where we have utilized the following facts

\[
\frac{2[\tilde{p}_k + (N - 1)(m - 1)]}{\tilde{p}_k + (2N + 1)(m - 1)} \geq \frac{2(\tilde{p}_k + 1 - m)}{m + \tilde{p}_k - 1} \quad \text{and} \quad \frac{(m + \tilde{p}_k - 1)(N + 2)}{\tilde{p}_k + (m - 1)(2N + 1)} \geq 2.
\]

The fact \( \tilde{p}_0 > (1 - m)(4N + 1) \) then ensures

\[
\frac{(m + \tilde{p}_k - 1)(N + 2)}{\tilde{p}_k + (m - 1)(2N + 1)} = \frac{(N + 2)(2k(\tilde{p}_0 + 1 - m) + 2(m - 1))}{2k(\tilde{p}_0 + 1 - m) + 2(m - 1)(2N + 1)} \leq \frac{(N + 2)(\tilde{p}_0 + 1 - m) + 2(m - 1)}{\tilde{p}_0 + 1 - m + (2N + 2)(m - 1)} \leq 2(N + 2),
\]

so that, in light of (4.11), (4.15)–(4.17),

\[
\frac{d}{dt} \left\| u + 1 \right\|_{L^p(\Omega)}^2 + \int_\Omega (u + 1)^{\tilde{p}_k} \\
\leq C_{16}\tilde{p}_k \left\| (u + 1)^{\frac{m + \tilde{p}_k - 1}{2}} \right\|_{L^1(\Omega)}^{2(\tilde{p}_k + (N - 1)(m - 1))} \\
\leq \rho^k \tilde{M}_{k-1} \quad \text{for all } t \in (0, T)
\]
with some $\tilde{\rho} > 1$, where
\[
\frac{2[\tilde{p}_k + (N - 1)(m - 1)]}{\tilde{p}_k + (2N + 1)(m - 1)} = 2 \frac{2^k(\tilde{\rho}_0 + 1 - m) + N(m - 1)}{2^k(\tilde{\rho}_0 + 1 - m) + (2N + 2)(m - 1)} = 2(1 + \frac{(N+2)(1-m)}{2^k(\tilde{\rho}_0 + 1 - m) + (2N + 2)(m - 1)}) := \kappa_k.
\]
Here we note that $\kappa_k = 2(1 + \varepsilon_k)$ for $k \geq 1$, where $\varepsilon_k$ satisfies $\varepsilon_k \leq \frac{C_{17}}{2^k}$ for all $k$ with some $C_{17} > 0$. Next, we integrate (4.18) over $(0, t)$ with $t \in (0, T)$, then yields to
\[
\int_{\Omega} (u + 1)\tilde{p}_k(x, t) \leq \max\{\int_{\Omega} (u_0 + 1)\tilde{p}_k, \tilde{\rho}_k \tilde{p}_k + \sum_{j=2}^{k} (j-1) \cdot \prod_{i=j}^{k} \kappa_i M_{\tilde{p}_k + 2(N+1)(m-1)} \} \text{ for all } t \in (0, T).
\] (4.19)
If $\int_{\Omega} (u + 1)\tilde{p}_k(x, t) \leq \int_{\Omega} (u_0 + 1)\tilde{p}_k$ for any large $k \in \mathbb{N}$, then we derive (4.2) holds. Otherwise, by a straightforward induction, we have
\[
\int_{\Omega} (u + 1)\tilde{p}_k \leq \tilde{\rho}_k \sum_{j=2}^{k} (j-1) \cdot \prod_{i=j}^{k} \kappa_i M_{\tilde{p}_k + 2(N+1)(m-1)} \text{ for all } k \geq 1.
\] (4.20)
On the other hand, due to the fact that $\ln(1 + x) \leq x$ (for all $x \geq 0$),
\[
\prod_{i=j}^{k} \kappa_i = 2^{k+1-j} e^{\sum_{i=j}^{k} \ln(1+\varepsilon_i)} \leq 2^{k+1-j} e^{\sum_{i=j}^{k} \varepsilon_i} \leq 2^{k+1-j} e^{C_{17}} \text{ for all } k \geq 1 \text{ and } j \{1, \ldots, k\}.
\]
In light of the above inequality, with the help of (4.20), we conclude that
\[
\left(\int_{\Omega} (u + 1)\tilde{p}_k\right)_{\tilde{p}_k} \leq \tilde{\rho}_k \sum_{j=2}^{k} (j-1) \cdot \prod_{i=j}^{k} \kappa_i M_{\tilde{p}_k + 2(N+1)(m-1)} \text{ for all } k \geq 1,
\] (4.21)
which after taking $k \to \infty$ readily implies that (4.2) holds.

The previous lemmas at hand, we can conclude main results in a straightforward manner.

The proof of main results Theorem 1.2 (and Theorem 1.1) will be proved if we can show $T_{\text{max}} = \infty$. Suppose on contrary that $T_{\text{max}} < \infty$. In view of (4.2), we apply Lemma 2.3 to reach a contradiction. Hence the classical solution $(u, v, w)$ of (1.3) is global in time and bounded.

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