On Second-Moment Stability of Discrete-Time Linear Systems with General Stochastic Dynamics

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Abstract—This paper provides a new unified framework for second-moment stability of discrete-time linear systems with stochastic dynamics. Relations of notions of second-moment stability are studied for the systems with general stochastic dynamics, and associated Lyapunov inequalities are derived. Any type of stochastic process can be dealt with as a special case in our framework for determining system dynamics, and our results together with assumptions (i.e., restrictions) on the process immediately lead us to stability conditions for the corresponding special stochastic systems. As a demonstration of usefulness of such a framework, three selected applications are also provided.

Index Terms—Discrete-time linear systems, stochastic dynamics, stability analysis, Lyapunov inequalities, LMIs.

I. INTRODUCTION

Randomness is a common concept in many fields, with which various kinds of phenomenon are interpreted and evaluated. Examples can be readily found such as packet interarrival times in networks [1], failure occurrences in distributed systems [2] and chance of precipitation in weather forecast [3]. The systems having this kind of randomness (more precisely, the systems whose underlying randomness is regarded as essential) are called stochastic systems. This paper focuses on randomness of dynamics rather than that of input for discrete-time linear systems, for which internal stability and associated Lyapunov inequalities are discussed toward future control applications.

The systems with stochastic dynamics are called random dynamical systems in the field of analytical dynamics [4]. The arguments in this paper begin with the most general stochastic dynamics for discrete-time linear systems, whose system class is completely consistent with the discrete-time linear case of random dynamical systems. One of the motivations of our dealing with such systems is that the system class is considered to be compatible with sequential Monte Carlo methods such as the ensemble Kalman filter [5], [6] and the Gaussian mixture filter [7], [8]. Parameters and their distributions in systems can be sequentially estimated by those methods, and their mixture with theory for stochastic systems is expected to open a new frontier of control with sequential learning. This paper has the role of providing a way to give a guarantee on control performance (in stability) for the stochastic systems from the theoretical viewpoint. In particular, several notions of second-moment stability are introduced, and their relationships are discussed for the stochastic systems. Then, associated Lyapunov inequalities are derived, with which the convergence rate of the second moment of system state can be evaluated.

While systems with general stochastic dynamics are relatively complicated to deal with for control even in our discrete-time linear case and few results have been reported, some subclasses are well studied, each of which has formed an independent research field in the control society. One of the famous subclasses is Markov jump systems [9]. Although the dynamics of a standard Markov jump system is described with a finite-mode Markov chain, the earlier study [10] succeeded in alleviating this part of assumption so that a more general stationary Markov process can be dealt with in stability analysis. Another noteworthy subclass is the systems with white parameters [11], which we call the systems with dynamics determined by an independent and identically distributed (i.i.d.) process [12], [13]. This subclass further involves systems with state-multiplicative noise [14], [15].

Each of the above subclasses has an independent research history in the control society. At least in analysis of second-moment stability, however, our results turn out to deal with all the above systems in a unified framework, which facilitates the study of their relationships and generalizations drastically. For example, there is a difference between the above Markov and i.i.d. cases that the former (i.e., Markov case) assumed the essential boundedness of coefficient random matrices (depending on a Markov process) for derivation of a Lyapunov inequality in the earlier study while the latter did not; a random matrix depending on a standard Markov chain (i.e., the coefficients of a standard Markov jump system) obviously satisfies this assumption. Because of this difference, the results in the i.i.d. case were not covered by those in the Markov case, even though i.i.d. processes are a special case of Markov processes; indeed, the Lyapunov inequality in [10] does not readily reduce to that in [13] even when the processes are restricted to i.i.d. type. By using the results in this paper, we can easily clarify the essential reason of this difference. In addition to such an academic investigation, our results can be used also for generalization of earlier results as already stated, e.g., so that periodically stationary (i.e., periodically distributed) processes can be dealt with. Some associated applications will be provided later as a demonstration of powerfulness of our new framework.

The purposes of this paper are summarized as follows: (i) complete systematization of theory for second-moment stability of discrete-time linear systems with general stochastic dynamics, (ii) clarification of relationships among some subclasses of stochastic systems in stability analysis, and (iii)
generalization of selected earlier results. The purposes (i) and (ii) are related with academic significance, and the rest is with usefulness of the proposed framework.

This paper is organized as follows. Section II describes discrete-time linear systems with dynamics determined by a general stochastic process, and states the treatment of the initial condition for the systems associated with the underlying processes. Since the processes determining stochastic dynamics in this paper are general, we constantly use conditional expectations, with which some readers might be less familiar. Hence, the section also makes a brief preliminary for conditional expectations. After these preliminaries, five notions of second-moment stability are introduced, whose relations are discussed in Section III as a part of main results in this paper. The relations in the most general case of systems are first discussed, and then, further relations are discussed under an assumption on the systems to have a sort of ‘time-invariance’ property. Since exponential stability, which is one of the above five stability notions, is compatible with stability analysis based on Lyapunov inequalities, Section IV derives the Lyapunov inequalities characterizing it without loss of generality. In particular, we drives two types of Lyapunov inequalities, one of which is for the general systems and the other is for the systems having essentially bounded coefficient matrices. These results also become a key in clarifying the reason of the difference between the conventional results for Markov and i.i.d. cases stated above. As a demonstration of powerfulness of our results, Section V provides some selected applications, which not only clarify relationships of earlier and our results but also generalize the former in a very simple fashion; our results together with additional assumptions on the systems immediately lead us to the corresponding Lyapunov inequality conditions (including conventional ones). The framework proposed in this paper can unify all the results about second-moment stability of discrete-time linear systems having stochastic dynamics, and is expected to facilitate the studies in this field drastically.

We use the following notation in this paper. The set of real numbers, that of positive real numbers, that of integers and that of non-negative integers are denoted by \( \mathbb{R} \), \( \mathbb{R}^+ \), \( \mathbb{Z} \) and \( \mathbb{N}_0 \), respectively. Subsets of \( \mathbb{Z} \) are defined as \( \mathbb{Z}_+(t) := [t, \infty) \cap \mathbb{Z} \) and \( \mathbb{Z}_-(t) := (-\infty, t] \cap \mathbb{Z} \) for \( t \in \mathbb{Z} \). The set of \( n \times n \)-dimensional real column vectors and that of \( m \times n \) real matrices are denoted by \( \mathbb{R}^n \) and \( \mathbb{R}^{m \times n} \), respectively. The set of \( n \times n \) symmetric matrices and that of \( n \times n \) positive definite matrices are denoted by \( \mathbb{S}^{n \times n} \) and \( \mathbb{S}^{n \times n}_+ \), respectively. The Borel \( \sigma \)-algebra on the set \( \mathbb{C} \) is denoted by \( \mathcal{B}(\mathbb{C}) \). The maximum singular value is denoted by \( \sigma_{\text{max}}(\cdot) \). The Euclidean norm is denoted by \( ||(\cdot)|| \). For random variables \( s_1 \) and \( s_2 \), the expectation of \( s_1 \) and the conditional expectation of \( s_1 \) given \( s_2 \) are denoted by \( E[s_1] \) and \( E[s_1|s_2] \), respectively; this notation is used also for random matrices.

II. DISCRETE-TIME LINEAR SYSTEMS WITH GENERAL STOCHASTIC DYNAMICS AND SECOND-MOMENT STABILITY

Let \( (\Omega, \mathcal{F}, P) \) be a complete probability space, where \( \Omega \), \( \mathcal{F} \) and \( P \) are a sample space, a \( \sigma \)-algebra and a probability measure, respectively. All the random variables and processes in this paper will be defined on this common probability space. That is, for a set \( X \), an \( X \)-valued random variable \( X_0 \) is defined as a mapping \( X_0 : (\Omega, \mathcal{F}) \to (X, \mathcal{B}(X)) \); we describe this mapping also as \( X_0 : \Omega \to X \) for short. Similarly, an \( X \)-valued stochastic process \( X = (X_k)_{k \in \mathbb{T}} \) on the set \( \mathcal{T} \) of time instants is defined as a mapping \( X : (\Omega, \mathcal{F}) \to \{X^T, B(X^T)\} \), where \( X^T \) is the set of all the possible \( X \)-valued functions of \( k \in \mathcal{T} \) that map \( \mathcal{T} \) to \( X \).

For details of the terms about probability theory, see [16], [17] and other sophisticated books.

With the above probability space, this section first describes discrete-time linear systems with general stochastic dynamics. Then, several definitions for second-moment stability are given.

A. Discrete-Time Linear Systems with General Stochastic Dynamics

Let us consider the \( \mathbb{R}^{n \times n} \)-valued (i.e., matrix-valued) stochastic process \( \hat{A} = (\hat{A}_k)_{k \in \mathbb{Z}} : \Omega \to \{\mathbb{R}^{n \times n}\}^\mathbb{Z} \) defined on \( (\Omega, \mathcal{F}, P) \), and the associated discrete-time linear system

\[
x_{k+1} = \hat{A}_k x_k
\]

with the finite-dimensional state \( x_k \), where \( k \) is the discrete time (which is supposed to go forward). It is obvious that the above equation describes the most general discrete-time linear finite-dimensional (input-free) systems with stochastic dynamics, if no restrictions are imposed on \( (\Omega, \mathcal{F}, P) \) and \( \hat{A} \).

For convenience in discussing technical results, we next introduce an alternative representation of such systems without causing any loss of generality in the system description. To this end, we first consider the column expansion of \( \hat{A}_k \) for each \( k \), and denote it by \( \xi_k \in \mathbb{R}^{n \times n} \). Then, it is obvious that \( \hat{A}_k = A(\xi_k) \) by introducing the time-invariant mapping \( A(\cdot) \) in an obvious fashion. This observation immediately implies that confining the system description to

\[
x_{k+1} = A(\xi_k) x_k,
\]

where \( \xi_k \in \mathbb{R}^Z \) for each \( k \), does not lead to any loss of generality compared with (1), as long as the classes of the probability space \( (\Omega, \mathcal{F}, P) \), the integer \( Z \in \mathbb{N} \), the stochastic process \( \xi = (\xi_k)_{k \in \mathbb{Z}} : \Omega \to (\mathbb{R}^Z)^\mathbb{Z} \), and the Borel-measurable matrix-valued function \( A : \mathbb{R}^Z \to \mathbb{R}^{n \times n} \) are arbitrary. After discussing in the following subsection how the initial condition at a given initial time instant \( k \in \mathbb{Z} \) should be handled for these systems, this paper discusses for the first time the stability problems of the general stochastic system in the form (2).

B. Treatment of the Initial Condition

This paper basically assumes that we are given the initial time instant \( k_0 \in \mathbb{Z} \), and is interested in studying the behavior of the state \( x_{k_0} \) of (1) or (2) for \( k \in \mathbb{Z}_+(k_0) \), even though \( k_0 \) is eventually assumed to be arbitrary so that stability of these systems can be defined appropriately and then studied thoroughly. Hence, each time \( k_0 \) is fixed, we assume that the initial state \( x_{k_0} \in \mathbb{R}^n \) is given, and we regard it as a deterministic vector. This initial state alone, however, is
not enough as the initial condition of these systems when we aim at discussing their behavior for \( k \in \mathbb{Z}_+ (k_0) \). This is because these systems are associated with the stochastic process \( A \), and the behavior of \( x_k \) for \( k \in \mathbb{Z}_+ (k_0) \) depends on the (conditional) distribution of \( \{A_{k_0} \in \mathbb{Z}_+ (k_0) \} \) given all the information available at time \( k_0 \). This implies that the initial condition of these systems consists not only of the initial state \( x_{k_0} \) but also of the initial condition of the stochastic process \( A \) at \( k_0 \). For example, when \( A \) is a Markov process in \([1]\), its initial condition is nothing but the conditional distribution of \( \hat{A}_{k_0} \) given \( \hat{A}_{k_0-1} \), where \( \hat{A}_{k_0-1} \) is viewed as a deterministic matrix. Similarly, when \( \xi \) is a Markov process in \([2]\), its initial condition is nothing but the conditional distribution of \( \xi_{k_0} \) given \( \xi_{k_0-1} \), where \( \xi_{k_0-1} \) is viewed as a deterministic vector.

For adequately dealing with the initial condition at \( k_0 \) of the general stochastic process \( \xi \) in \([2]\), let us introduce its subsequences \( \xi^{k+} = (\xi_t)_t \in \mathbb{Z}_+ (k) : \Omega \rightarrow (\mathbb{R}^3 \mathbb{Z}_+ (k)) \) and \( \xi^{-k} = (\xi_t)_t \in \mathbb{Z}_- (k) : \Omega \rightarrow (\mathbb{R}^2 \mathbb{Z}_+ (k)) \) for each \( k \in \mathbb{Z} \). The intuitive interpretation of such partitioning of the stochastic process \( \xi \) is that \( \xi^{(k-1)-} \) for each \( k \in \mathbb{Z} \) can be regarded as the (possibly redundant) information that is sufficient for determining the distribution of \( \xi^{k+} \) (more precisely, its conditional distribution available at time \( k \)). In particular, when \( k \) equals the initial time \( k_0 \) of the system \([2]\), the associated \( \xi^{(k-1)-} \) can be regarded as determining the initial condition of the (future) stochastic process \( \xi_{k_0-} \), which together with the initial state \( x_{k_0} \) determines the behavior of \( x_k \) for \( k \in \mathbb{Z}_+ (k_0) \). Hence, this paper assumes that \( \xi^{(k-1)-} \) viewed as a (past) deterministic vector series is given as information determining the initial condition of the stochastic process \( \xi \) at \( k_0 \), together with the initial state \( x_{k_0} \).

In our later discussions on stability, the initial state vector \( x_{k_0} \) will be treated as being arbitrary in \( \mathbb{R}^n \). Similarly, the initial condition of \( \xi^{(k-1)-} \) at \( k_0 \) will be treated as being arbitrary by regarding \( \xi^{(k-1)-} \) as being arbitrary (past) vector series in its support, which we denote by \( \mathbb{E}_0 \).

\section*{C. Preliminaries about Conditional Expectation}

In this paper, we deal with several notions of second-moment stability \([18]\). Roughly speaking, second-moment stability is the concept about the convergence of (or the existence of a certain uniform bound for) the second moment of \( \|x_k\| \) with respect to \( k \in \mathbb{Z}_+ (k_0) \). As is clear from our treatment of the initial condition for \([2]\), the second moment of \( \|x_k\| \) should precisely refer to the conditional expectation of \( \|x_k\|^2 \) given \( \xi^{(k-1)-} \). When we wish to be clear that \( \xi^{(k-1)-} \) is viewed as a deterministic series in this context, we could instead introduce the notation \( \hat{\xi}^{(k-1)-} \) to denote the path of the (past) stochastic process \( \xi^{(k-1)-} \) up to time \( k_0 - 1 \); in this context, the initial condition of \( \xi^{k_0+} \) at \( k_0 \) could more precisely be written as the equation \( \xi^{(k-1)-} = \hat{\xi}^{(k-1)-} \). Under this notational standpoint, the second moment can be written as the conditional expectation \( E[\|x_k\|^2 | \xi^{(k-1)-}] = \hat{\xi}^{(k-1)-} \). Since the conditional expectations of other quantities are also handled repeatedly, we introduce the shorthand notation

\[ E_0(\cdot) := E(\cdot | \xi^{(k-1)-}) = \hat{\xi}^{(k-1)-} \].

\section*{D. Stability Notions}

To define second-moment stability, we introduce the following assumption on system \([2]\).

\begin{assumption}
For each \( k_0 \in \mathbb{Z} \), there exists \( M_1 = M_1 (k_0) \in \mathbb{R}_+ \) such that

\[ E_0[A_{ij}(\xi_{k_0})^2] \leq M_1 (k_0) \]

\((\forall i, j = 1, \ldots, n; \forall \xi^{(k-1)-} \in \mathbb{E}_0)\),

\end{assumption}

where \( A_{ij}(\xi_{k_0}) \) denotes the \((i, j)\)-entry of \( A(\xi_{k_0}) \).

Since

\[ \sigma_{\max}(A(\xi_{k_0}))^2 \leq \sum_{i=1}^n \sum_{j=1}^n A_{ij}(\xi_{k_0})^2 \]

regardless of \( k_0 \), this assumption implies the existence of a \( k_0 \)-dependent upper bound of \( E_0[\sigma_{\max}(A(\xi_{k_0}))^2]\). The assumption is a minimal requirement for dealing with the second moment \( E_0[\|x_k\|^2] \) for each initial time instant \( k_0 \) and every \( k \in \mathbb{Z}_+ (k_0) \), as can be confirmed in the following lemma.
Lemma 1: For system (2), the following two conditions are equivalent.

1) Assumption \( I \) is satisfied.
2) For each \( k_0 \in \mathbb{Z} \) and every \( k \in \mathbb{Z}^+(k_0) \), there exists \( M_2 = M_2(k, k_0) \in \mathbb{R}_+ \) such that
\[
E_0 \left[ \|x_k\|^2 \right] \leq M_2(k, k_0) \|x_{k_0}\|^2
\]
\[
\forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0 .
\] (10)

That is, the second moment \( E_0 \left[ \|x_k\|^2 \right] \) is bounded for each \( k_0 \) and every \( k \).

Proof: 1\( \Rightarrow \)2: Fix \( k_0 \) and \( \xi^{(k_0-1)-} \), and take an arbitrary \( k \in \mathbb{Z}^+(k_0) \). We proceed with the proof by tentatively relating this \( k \) with \( k_0 \) in (8) so that we could evaluate the conditional expectation of \( A_{ij}(\xi_k)^2 \) or \( \sigma_{\max}(A(\xi_k))^2 \) in an appropriate sense (while maintaining the overall standpoint that \( k_0 \) always refers to the one we fixed at the beginning of the proof except for the specific tentative treatment here). More precisely, we give a restatement (in terms of \( k \in \mathbb{Z}^+(k_0) \) rather than \( k_0 \) that we fixed) of what (8) implies with respect to \( A_{ij}(\xi_k) \) when Assumption \( I \) is satisfied. First, the assumption obviously implies that the conditional expectation of \( E_0[A_{ij}(\xi_k)^2|F_{k-1}] \) given \( \xi^{(k_0-1)-} \) is well-defined for \( i, j = 1, \ldots, n \) regardless of \( \xi_{k_0}, \xi_{k_0+1}, \ldots, \xi_{k-1} \) such that the vector series \( \xi^{(k_0-1)-} \) belongs to the support of \( \xi^{(k_0-1)-} \) (where \( E_0[\cdot] \) is with respect to \( k_0 \) and \( \xi^{(k_0-1)-} \) that we fixed at the beginning of the proof). Hence, by (9), \( \alpha(k) = n^2M_1(k) \) satisfies
\[
E_0 \left[ \sigma_{\max}(A(\xi_k))^2|F_{k-1} \right] \leq \alpha(k) \text{ a.s.} \] (11)
for each \( k \in \mathbb{Z}^+(k_0) \). This, together with (5) and (6), leads us to
\[
E_0 \left[ \|x_{k+1}\|^2 \right] = E_0 \left[ \|A(\xi_k)x_k\|^2 \right] \leq E_0 \left[ \sigma_{\max}(A(\xi_k))^2 \|x_k\|^2 \right] = E_0 \left[ \sigma_{\max}(A(\xi_k))^2 \|x_k\|^2 \|F_{k-1} \right] \leq \alpha(k)E_0 \left[ \|x_k\|^2 \right] \] (\( \forall k \in \mathbb{Z}^+(k_0); \forall x_{k_0} \in \mathbb{R}^n \)). (12)
A recursive use of this inequality leads to (10) with
\[
M_2(k, k_0) := \alpha(k_0)\alpha(k_0 + 1) \cdots \alpha(k - 1) .
\] (13)
This completes the proof.

2\( \Rightarrow \)1: By taking \( k = k_0 + 1 \), the inequality in (10) leads to
\[
x_k^TE_0[A(\xi_k)^T A(\xi_k)]x_{k_0} = E_0 \left[ \|A(\xi_k)x_k\|^2 \right] \leq M_2(k_0 + 1, k_0) \|x_{k_0}\|^2 .
\] (14)
Since this inequality holds for all \( x_{k_0} \in \mathbb{R}^n \), there exists \( M_1 = M_1(k_0) \) satisfying \( 0 \leq M_1(k_0) \leq M_2(k_0 + 1, k_0) \). This completes the proof.

For system (2) satisfying Assumption \( I \) we first define the most basic notion of second-moment stability as follows.

Definition 1 (Stability): The system (2) (satisfying Assumption \( I \)) is said to be stable in the second moment if for each \( \epsilon \in \mathbb{R}_+ \) and every \( k_0 \in \mathbb{Z} \), there exists \( \delta = \delta(\epsilon, k_0) \) such that
\[
\|x_{k_0}\|^2 \leq \delta(\epsilon, k_0) \Rightarrow \|x_k\|^2 \leq \epsilon \]
\[
\forall k \in \mathbb{Z}^+(k_0); \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0 .
\] (15)
Note that the inequality (10) resulting from Assumption \( I \) ensures that it is indeed meaningful to refer to \( E_0[\|x_k\|^2] \) in the above definition (but (10) itself does not immediately imply the second uniform inequality in \( k \) in the above definition).

Uniform stability is further defined as follows.

Definition 2 (Uniform Stability): The system (2) is said to be uniformly stable in the second moment if for each \( \epsilon \in \mathbb{R}_+ \), there exists \( \delta = \delta(\epsilon) \) such that
\[
\|x_{k_0}\|^2 \leq \delta(\epsilon) \Rightarrow \|x_k\|^2 \leq \epsilon \]
\[
\forall k \in \mathbb{Z}^+(k_0); \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0; \forall k_0 \in \mathbb{Z} .
\] (16)
The above two stability notions are about the existence of a certain (i.e., \( k \)-independent) upper bound uniform in \( k \) for the second moment \( E_0[\|x_k\|^2] \). The following three notions are about the convergence of the second moment to 0.

Definition 3 (Asymptotic Stability): The system (2) is said to be asymptotically stable in the second moment if it is stable in the second moment and for each \( k_0 \in \mathbb{Z} \),
\[
E_0[\|x_k\|^2] \to 0 \text{ as } k \to \infty \quad (\forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0). \] (17)

Definition 4 (Uniform Asymptotic Stability): The system (2) is said to be uniformly asymptotically stable in the second moment if it is uniformly stable in the second moment and
\[
E_0[\|x_k\|^2] \to 0 \text{ as } k \to \infty \quad (\forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0), \]
uniformly in \( k_0 \in \mathbb{Z} \). (18)

What the latter condition (18) precisely means is that for each \( \epsilon \in \mathbb{R}_+ \), there exists \( k_0 \)-independent \( K = K(\epsilon) \in \mathbb{N}_0 \) such that
\[
E_0[\|x_k\|^2] \leq \epsilon \|x_{k_0}\|^2 \]
\[
(\forall k \in \mathbb{Z}^+(k_0 + K(\epsilon)); \forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0; \forall k_0 \in \mathbb{Z}). \] (19)

Definition 5 (Exponential Stability): The system (2) is said to be exponentially stable in the second moment if there exist \( a \in \mathbb{R}_+ \) and \( \lambda \in (0, 1) \) such that
\[
E_0[\|x_k\|^2] \leq a \|x_{k_0}\|^2 \lambda^{2(k-k_0)} \]
\[
(\forall k \in \mathbb{Z}^+(k_0); \forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)-} \in \hat{\mathbb{S}}_0; \forall k_0 \in \mathbb{Z}). \] (20)

One of the main purposes of this paper is to discuss the relationships among these stability notions for system (2).

Remark 1: Second-moment stability is also called mean square stability (18) in some literature. Hence, for instance, the stability notion defined in Definition 3 corresponds to asymptotic mean square stability, which might be more familiar to some readers. Such paraphrases could facilitate understanding of the relationship between our study and other earlier results.
III. RELATIONS OF STABILITY NOTIONS

A. Relations for general stochastic systems

This section first discusses the relations of the five stability notions introduced above for the most general case of discrete-time linear systems with stochastic dynamics (i.e., only under Assumption 1). To state the conclusion in advance, the relations shown with solid arrows in Fig. 1 hold for the system (2) under Assumption 1 where all the arrows are in the direction from a strong stability notion to a weak stability notion. If the two notions are linked with two arrows with different directions, the notions are equivalent under the corresponding assumptions (some assumptions are additionally introduced later). By definitions of stability notions, the relations described with solid arrows are trivial except the equivalence between uniform asymptotic stability and exponential stability (the relations about attractivity and quadratic stability will be discussed separately after introducing additional assumptions).

Hence, we here focus only on the non-trivial equivalence, and show the following theorem for the general stochastic systems.

Theorem 1: Suppose the system (2) satisfies Assumption 1.

The following two conditions are equivalent.

1) The system is uniformly asymptotically stable in the second moment.

2) The system is exponentially stable in the second moment.

Proof:

2⇒1: This assertion is almost obvious; it follows from (20) and 0 < λ < 1 that the system is uniformly stable, and for each ε ∈ R+, there exists K = K(ε) ∈ N₀ satisfying (19).

1⇒2: Linearity of the system (2) frequently used in this part of the proof is not explicitly referred to so as not to make the arguments verbose. The initial step for the proof of this assertion is similar to Theorem 1 in [13]. That is, we first introduce the decomposition

\[ x_{k0} = \beta_{k0} \sum_{i=1}^{n} a_{k0i} \sigma_{k0i} e^{(i)} \]  \tag{21}

with the non-negative scalars \( \beta_{k0}, a_{k0i} (i = 1, \ldots, n) \) satisfying \( \sum_{i=1}^{n} a_{k0i} = 1 \), the integers \( \sigma_{k0i} \in \{-1, 1\} (i = 1, \ldots, n) \) and the standard basis vectors \( e^{(i)} (i = 1, \ldots, n) \) for the n-dimensional Euclidean space. By definition, we have

\[ \|x_{k0}\|^2 = \beta_{k0}^2 (a_{k01}^2 + \ldots + a_{k0n}^2) \geq \beta_{k0}^2 / n. \]  \tag{22}

Associated with this decomposition of \( x_{k0} \), we can also decompose the corresponding \( x_k \) as

\[ x_k = \beta_{k0} \sum_{i=1}^{n} a_{k0i} \sigma_{k0i} x_k^{(i)}, \]  \tag{23}

where \( x_k^{(i)} \) is the state at \( k \) for the initial state \( x_{k0} = e^{(i)} \). It follows from (19) that there exists \( K \in N_0 \) such that

\[ E_0 \|x_k^{(i)}\|^2 \leq 1/(2n^2) \]

\[ (i = 1, \ldots, n; \forall k \in \mathbb{Z}_+(k_0 + K); \forall \xi^{(k_0-1)} \in \hat{\Xi}_0; \forall k_0 \in \mathbb{Z}). \]  \tag{24}

Then, we have

\[ E_0 \|x_k^2\| \leq \beta_{k0}^2 E_0 \left[ \left( \sum_{i=1}^{n} a_{k0i} \sigma_{k0i} x_k^{(i)} \right)^2 \right] \]

\[ \leq \beta_{k0}^2 E_0 \sum_{i=1}^{n} a_{k0i} \|\sigma_{k0i} x_k^{(i)}\|^2 \]

\[ = \beta_{k0}^2 \sum_{i=1}^{n} a_{k0i} E_0 \|x_k^{(i)}\|^2 \]

\[ \leq \beta_{k0}^2 / (2n^2) \]

\[ (\forall k \in \mathbb{Z}_+(k_0 + K); \forall x_{k0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)} \in \hat{\Xi}_0; \forall k_0 \in \mathbb{Z}). \]  \tag{25}

where the first inequality follows from Jensen’s inequality. Hence, it follows from (22) and (25) that

\[ E_0 \|x_{K+k_0}\|^2 \leq \|x_{k0}\|^2 / 2 \]

\[ (\forall x_{k0} \in \mathbb{R}^n; \forall \xi^{(k_0-1)} \in \hat{\Xi}_0; \forall k_0 \in \mathbb{Z}) \]  \tag{26}
for the same $K$. Since this inequality holds for each $k_0 \in \mathbb{Z}$ and every $\xi^{(k_0-1)}$, belonging to the support $\mathfrak{E}_0$ of $\xi^{(k_0-1)}$, (recall the arguments about the treatment of $k_0$ at the beginning of the proof of Lemma 1), it follows for each $k \in \mathbb{Z}_+^{(k_0)}$ and every sample of the series $\xi_{k_0}, \ldots, \xi_{k-1}$ determining $x_k$ that

$$E\|x_{k+K}\|^2 \xi^{(k_0-1)} = \xi^{(k_0-1)}, \xi_{k_0}, \ldots, \xi_{k-1} \leq \|x_k\|^2 / 2$$

(∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z),

(27)

which implies

$$E_0[\|x_{k+K}\|^2 | \mathcal{F}_{k-1}] \leq \|x_k\|^2 / 2 \ a.s.$$  

(∀k ∈ Z_+(k_0); ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(28)

This together with (5) further implies

$$E_0[\|x_{k+K}\|^2 \leq E_0[\|x_k\|^2] / 2$$  

(∀k ∈ Z_+(k_0); ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(29)

For each $k \in Z_+(k_0)$, take $c, j \in N_0$ such that $k = c + jK + k_0$ ($0 \leq c < K$). Then, a recursive use of (29) leads to

$$E_0[\|x_{k}\|^2] = E_0[\|x_{c+jK+k_0}\|^2] \leq E_0[\|x_{c+k_0}\|^2] / 2$$

$$= \varphi^{c/K} E_0[\|x_{c+k_0}\|^2] / (2^{1/K})^{(k-k_0)} \leq 2E_0[\|x_{c+k_0}\|^2] / (2^{1/K})^{(k-k_0)}$$  

(∀k ∈ Z_+(k_0); ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(30)

Here, $x_{c+k_0}$ can be decomposed as (23), and thus, satisfies

$$E_0[\|x_{c+k_0}\|^2] \leq \beta_0^2 \sum_{i=1}^n a_{ki} E_0[\|e^{(i)}(1)\|^2]$$

(∀c ∈ [0, K]; ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(31)

Since the system is uniformly stable (and $\|x_{c+k_0}\|^2 = \|e^{(1)}\|^2 = 1$), there exists $\epsilon'$ such that

$$E_0[\|e^{(1)}(1)\|^2] \leq \epsilon'$$  

(i = 1, ..., n; ∀c ∈ [0, K]; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z),  

(32)

which together with (22) and (31) implies

$$E_0[\|x_{c+k_0}\|^2] \leq n\epsilon' \|x_{c+k_0}\|^2$$

(∀c ∈ [0, K]; ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(33)

Hence, from (30) and (33), we obtain

$$E_0[\|x_k\|^2] \leq 2n\epsilon' \|x_{c+k_0}\|^2 / (2^{1/K})^{(k-k_0)}$$

(∀k ∈ Z_+(k_0); ∀x_k ∈ R^n; ∀ξ^{(k_0-1)} ∈ $\mathfrak{E}_0$; ∀k_0 ∈ Z).  

(34)

This implies the existence of $a = 2n\epsilon'$ and $\lambda = 2^{-1/K}$ such that $a \in R_+$, $\lambda \in (0, 1)$ and (20) hold. This completes the proof.

The essential differences between this proof and that for Theorem 1 in [13] are the treatment of the conditional expectation from (26) through (29) and the use of uniform stability in taking $k_0$-independent $\epsilon'$ in (32) so that $a$ in (26) is $k_0$-independent. The latter will be related with the discussions on the time-invariance property of stochastic systems in the following subsection.

### B. Time-invariance property of stochastic systems

The relations described by the solid arrows in Fig. 1 are known to hold also for linear time-varying deterministic systems [19]. In the case of deterministic systems, it is also known that stability implies uniform stability if the system is time-invariant. A similar result can be obtained for our stochastic systems by using the following assumption.

**Assumption 2:** The stochastic process $\xi$ is stationary (in the strict sense); i.e., none of the characteristics of $\xi_k$ changes with time $k$.

This assumption leads to the following lemma (see the dashed arrows in Fig. 1), which is obvious from the definition of each stability notion.

**Lemma 2:** Suppose the system (2) satisfies Assumptions 1 and 2. The system is uniformly stable in the second moment if and only if the system is stable in the second moment. Similarly, the system is uniformly asymptotically stable in the second moment if and only if the system is asymptotically stable in the second moment.

In addition, Assumption 2 and Theorem 1 lead us to the following theorem.

**Theorem 2:** Suppose the system (2) satisfies Assumptions 1 and 2. The system is uniformly asymptotically stable in the second moment if and only if (15) holds.

**Proof:** Since the necessity assertion is obvious, we here only prove the sufficiency assertion, i.e., uniform asymptotic stability being implied only with (15) (under Assumptions 1 and 2). Since uniform asymptotic stability is implied by exponential stability under Assumption 1 by Theorem 1, it suffices to show that (15) implies exponential stability under Assumption 2. The key point for showing this claim is to note where we had to use uniform stability (i.e., (16)) in showing exponential stability in the part 1 ⇒ 2 of the proof of Theorem 1 if we can show it with additional Assumption 2 instead of the uniform stability assumption, then the proof is completed. Regarding the above key point, we readily see that uniform stability was used in showing the existence of the $k_0$-independent constant $\epsilon'$ in (32). Such a $k_0$-independent constant always exists if the system satisfies not only Assumption 1 but Assumption 2 indeed, since

$$\sup_{i \in \{1, ..., n\}, c \in \{0, ..., K-1\}} E_0[\|x_{c+k_0}\|^2]$$

is bounded by Assumption 1 (recall Lemma 1) and $k_0$-independent by Assumption 2 and taking it as $\epsilon'$ in (32) is sufficient. This completes the proof.}

The property described by (18) is called (uniform) attractivity [19]. It is known in the case of deterministic systems that asymptotic stability can be ensured only with attractivity if the system is linear and time-invariant. Hence, Theorem 2 corresponds to a stochastic counterpart of such a result for deterministic systems. Although a similar result was already shown in our earlier study [13] for stochastic systems as Corollary 1, it was assumed throughout the study that $\xi_k$ is independent and identically distributed (i.i.d.) with respect to $k \in N_0$ (i.e., $\xi$ is not only stationary but also temporally-independent). The present Theorem 2 was derived only with
Assumption 2 (in addition to Assumption 1), and hence, is more general than the corollary in the earlier study.

According to the above arguments, Assumption 2 seems to let the stochastic system have a sort of “time-invariance” property. However, it will be clearer in the next section that exponential stability is not equivalent to quadratic stability (dealt with, e.g., in [20], [21] for deterministic systems and in [13] for a special case of stochastic systems) only by Assumption 2 because exponential stability cannot be characterized by the Lyapunov inequality with a constant (i.e., time-invariant deterministic) Lyapunov matrix in that case; it is shown in [13] that if is i.i.d. with respect to then exponential stability becomes equivalent to quadratic stability, as is the case with linear time-invariant deterministic systems. Hence, the complete “time-invariance” property does not follow only with Assumption 2 and other additional assumptions (e.g., temporal independence of ) are needed depending on the required level of the property. This is a difference between the case of deterministic systems and that of stochastic systems.

IV. LyaPunov Inequalities

Among the five stability notions introduced in Section II, exponential stability is the most compatible with the approach of stability analysis using Lyapunov inequalities. Hence, we first derive in this section Lyapunov inequalities giving a necessary and sufficient condition for exponential stability of system 2 only under Assumption 1 (i.e., the most general case); it is obvious from Theorem I that such inequalities can be used also for uniform asymptotic stability. The Lyapunov matrix in the inequalities will be revealed soon to be independent of the necessity of the condition. If we restrict the Lyapunov matrix to a -independent constant matrix, the necessity does not hold in general. Since such a special case of the condition is closely related with so called quadratic stability, we also give some associated comments. Then, we further derive another type of Lyapunov inequality condition that can be used for stochastic systems having essentially bounded coefficient matrices.

A. Lyapunov inequalities for general stochastic systems

To show the Lyapunov inequality characterizing exponential stability of system 2, let us introduce the time shift operator for processes such that is defined by with . . . ; . Since is nothing but . With this operator, we can show the following theorem giving a necessary and sufficient inequality condition for exponential stability in the most general case.

Theorem 3: Suppose the system 2 satisfies Assumption 1. The following two conditions are equivalent.

1) The system is exponentially stable in the second moment.

2) There exist , , such that

Theorem 3: Suppose the system 2 satisfies Assumption 1. The following two conditions are equivalent.

1) The system is exponentially stable in the second moment.

2) There exist , , such that

Proof:

2⇒1: It follows from the inequality in (38) that

Since this inequality holds for each , every , we have

which implies (by 35)

A recursive use of this inequality leads to

For the left-hand side of this inequality, (46) leads to

while for the right-hand side, (47) leads to

Hence, we have (20) with and which means by definition that the system is exponentially stable in the second moment.

1⇒2: For , take such that and define

for Markov jump systems has the same stance as the latter study.
for \(k_1, k_2 \in \mathbb{Z}\) such that \(k_2 \geq k_1\). Then, (20) can be rewritten as
\[
x^T_{k_0} E_0 \Gamma_k (S_{k_0} \xi^{(k_0)} +) T \Gamma_k (S_{k_0} \xi^{(k_0)} +) x_{k_0}
\leq x^T_{k_0} (\alpha (\lambda / \lambda_1)^{2(k-k_0)} I) x_{k_0}
\]
for \((\forall k \in \mathbb{Z}_+(k_0))\); \(\forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0)-} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}\),

which implies
\[
E_0 [\Gamma_k (S_{k_0} \xi^{(k_0)} +) T \Gamma_k (S_{k_0} \xi^{(k_0)} +)] \leq \alpha (\lambda / \lambda_1)^{2(k-k_0)} I
\]
\((\forall k \in \mathbb{Z}_+(k_0)); \forall \xi^{(k_0)-} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}\).

We next define
\[
P_k (S_k \xi^{(k+)} := \lambda_1^{-2} \sum_{t=k}^K \Gamma_t (S_t \xi^{(k+)} T \Gamma_t (S_t \xi^{(k+)}
\]

for \(k \geq k\). Then, it satisfies
\[
\lambda_1^2 P_k (S_k \xi^{(k+)} - A(\xi_k)^T P_k (S_k \xi^{(k+)} + A(\xi_k) = I
\]

for \(k \geq k_0 + 1\), and thus,
\[
\lambda_1^2 E_0 [P_k (S_k \xi^{(k+)} + ]
\geq E_0 [A(\xi_k)^T E_0 [P_k (S_k \xi^{(k+)} + ] [F_{k_0}] A(\xi_k) ]
\]
by (5) and (6). On the other hand, (48) also implies that the sequence of
\[
E_0 [P_k (S_k \xi^{(k+)} ] = \lambda_1^{-2} \sum_{t=k}^K E_0 [\Gamma_t (S_t \xi^{(k+)} T \Gamma_t (S_t \xi^{(k+)}
\]

with respect to \(K\) for each fixed \(k\) is monotonically non-decreasing under the semi-order relation based on positive semidefiniteness, i.e.,
\[
E_0 [P_k (S_k \xi^{(k+)} ] \leq E_0 [P_{k+1} (S_{k+1} \xi^{(k+)} ]
\]

In addition, it follows from (47) that
\[
E_0 [P_k (S_k \xi^{(k+)} ] \leq \lambda_1^{-2} \alpha \left( \sum_{t=k_0}^K (\lambda / \lambda_1)^{2(t-k_0)} \right) I
\]
whose right-hand side converges to a constant matrix as \(K \to \infty\). Hence, the conditional expectation of \(P_\infty (S_k \xi^{(k+)} ) = P_k (S_k \xi^{(k+)} ) \to \infty\) given \(\xi^{(k_0)-} \) is bounded (for each \(k_0 \in \mathbb{Z}\), every \(\xi^{(k_0)-} \in \mathbb{E}_0\) and every \(k \in \mathbb{Z}_+(k_0)\). We take \(P(\cdot) = P_\infty(\cdot)\), which itself is independent of \(k_0\) and \(k\) because \(A(\cdot)\) is. By definition, this \(P\) satisfies (35) (by (45) and (48)) and (37) (by (53)) with appropriate \(\xi_1, \tau_1 \in \mathbb{R}_+.\) In addition, letting \(K \to \infty\) in (50) leads to (35). This completes the proof. \[\square\]

The inequality (35) is a Lyapunov inequality for the system (2) satisfying Assumption 1 which is a generalization of the usual Lyapunov inequality for discrete-time linear deterministic systems.

Let \(\lambda_{\text{min}}\) and \(\lambda_{\text{min}}\) be respectively the infimum of \(\lambda\) such that there exists \(\alpha \in \mathbb{R}_+\) satisfying (20) and that of \(\lambda_1\) such that there exist \(\xi_1, \tau_1 \in \mathbb{R}_+\) and \(P : (\mathbb{R}^2)^{N_0} \to \mathbb{S}^{n \times n}\) satisfying (36)-(38). Then, since \(\lambda_1\) in the part 1 \(\Rightarrow 2\) of the above proof can be taken arbitrarily close to \(\lambda\), and since \(\lambda = \lambda_1\) in the proof of the opposite direction, we have the following equality.
\[
\lambda_{\text{min}} = \lambda_{1\text{min}}
\]

This implies that we can characterize the convergence rate of the sequence \((\sqrt{E_0 [\|x_k\|^2]} )_{k \in \mathbb{Z}_+(k_0)}\) by the inequality condition (36)-(38) without loss of generality.

However, if we are interested only in whether the system is stable and not in the convergence rate, the Lyapunov inequality without \(\lambda_1\) shown in the following lemma would be sufficient.

Lemma 3: Suppose the system (2) satisfies Assumption 1. The following two conditions are equivalent.

1) There exist \(\xi_1, \tau_1 \in \mathbb{R}_+, \lambda_1 \in (0, 1)\) and \(P : (\mathbb{R}^2)^{N_0} \to \mathbb{S}^{n \times n}\) satisfying (36)-(38).

2) There exist \(\xi_1, \tau_1, \xi_1 \in \mathbb{R}_+\) and \(P : (\mathbb{R}^2)^{N_0} \to \mathbb{S}^{n \times n}\) satisfying (36), (37) and
\[
E_0 [P(S_k \xi^{(k_0)+}) - A(\xi_k)^T E_0 [P(S_k \xi^{(k_0)+}) ] [F_{k_0}] A(\xi_k) ]
\]
\geq \xi_1 (\forall k \in \mathbb{Z}_+(k_0); \forall \xi^{(k_0)-} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}].
\]

Proof: Adding \(E_0 [(1 - \lambda_1^2) P(S_k \xi^{(k_0)+}) ]\) to (38) and using (36) lead to (55) with \(\xi_1 = (1 - \lambda_1^2) \lambda_1 > 0\). The opposite assertion is obvious from (37). \[\square\]

As already stated, the inequality condition (36), (37) and (55) (and thus, (36)-(38)) is necessary and sufficient not only for exponential stability but also for uniform asymptotic stability (recall Theorem 1 and Fig. 1 under Assumption 1).

B. Quadratic stability

The Lyapunov inequalities (38) and (55) involve the \(\xi\)-dependent Lyapunov matrix, and hence, the direct use of them for numerical analysis is considered not so easy. In the case of deterministic linear time-varying (or parameter-varying) systems, we sometimes consider restricting the Lyapunov matrix to a constant matrix for ease of numerical analysis as in (24), (25), and a similar idea might be useful for our Lyapunov inequalities. If we introduce the restriction \(P(\cdot) = P_0\) for \(P_0 \in \mathbb{S}^{n \times n}\), the inequality condition in Theorem 2 reduces to the following form: there exist \(\lambda_1 \in (0, 1)\) and \(P_0 \in \mathbb{S}^{n \times n}\) such that
\[
E_0 [\lambda^2 P_0 - A(\xi_0)^T P_0 A(\xi_0) ]
\geq \xi_1 (\forall k \in \mathbb{Z}_+(k_0); \forall \xi^{(k_0)-} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}].
\]

While this condition is only sufficient for exponential stability, it is also necessary for quadratic stability defined in the following.

Definition 6 (Quadratic Stability): The system (2) is said to be quadratically stable if there exist \(P_0 \in \mathbb{S}^{n \times n}\) and \(\lambda \in (0, 1)\) such that
\[
E_0 [x^T_{k+1} P_0 x_{k+1}] \leq \lambda^2 E_0 [x^T_k P_0 x_k]
\]
\((\forall k \in \mathbb{Z}_+(k_0); \forall x_{k_0} \in \mathbb{R}^n; \forall \xi^{(k_0)-} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}].
\]

We can show that (57) implies (56) (i.e., the above necessity assertion) through taking \(k = 0\) and \(\lambda_1 = \lambda\); the opposite
direction is obvious from the proof of Theorem 3 (see the arguments around (41)).

It is obvious from the introduced restriction that the equivalence between exponential stability and quadratic stability does not hold in general (see Fig. 1). This is true even when Assumption 2 is additionally satisfied. However, in the case of i.i.d. processes (13), the equivalence is known to hold. We will revisit this special case as one of the selected applications later.

C. Lyapunov inequalities under essential boundedness assumption

In this section, we derived new Lyapunov inequalities (38) and (55) for system (2) only with Assumption 1 which was a minimal requirement for defining the notions of second-moment stability. In this subsection, we consider another assumption on system (2) that is stronger than Assumption 1 and derive different Lyapunov inequalities. The assumption we use here is the following.

Assumption 3: There exists $M_3 \in \mathbb{R}_+$ such that

\[
|A_{ij}(\xi_{k0})| < M_3 \quad \text{a.s.}
\]

(\forall i, j = 1, \ldots, n; \forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}),
\]

(58)

where $| \cdot |$ denotes the absolute value.

By this assumption, the square entries of $A(\xi_k)$ become essentially bounded. Hence, $A(\xi_k)$ satisfying this assumption also satisfies Assumption 1.

With Assumption 3, we can show the following theorem.

Theorem 4: Suppose the system (2) satisfies Assumption 3.

The following two conditions are equivalent.

1) There exist $\xi_1, \tau_1 \in \mathbb{R}_+, \lambda_1 \in (0, 1)$ and $P : (\mathbb{R}^Z)^{N_0} \to \mathbb{S}^{n \times n}$ satisfying (36)–(38).

2) There exist $\xi_2, \tau_2, \xi_2 \in \mathbb{R}_+, \lambda_2 \in (0, 1)$ and $R : (\mathbb{R}^Z)^{N_0} \to \mathbb{S}^{n \times n}$ such that

\[
E_0|R(S_{k0} \xi^{(k0)} + |F_{k0}|) \geq \xi_2 I \quad \text{a.s.}
\]

(59)

\[
E_0|R(S_{k0} \xi^{(k0)} + |F_{k0}|) \leq \tau_2 I \quad \text{a.s.}
\]

(60)

\[
\lambda_2^2 E_0|R(S_{k0} \xi^{(k0)} + |F_{k0}|) \geq A(\xi_k)^T E_0[R(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq \xi_2 I \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(61)

The intuitive interpretation of the above theorem is that $A(\xi_k)$ in (38) can be taken out from the conditional expectation as in (61) through shifting time for conditions of conditional expectations so that $E_0[\cdot]$ is replaced by $E_0[|F_{k0}|]$, when the entries of $A(\xi_k)$ are essentially bounded (otherwise, (61) does not make sense). For example, uniformly distributed entries of $A(\xi_k)$ can be dealt with in the inequality condition (59)–(61) (while normally distributed entries cannot). The proof of the theorem is as follows.

Proof: 2⇒1: Taking the conditional expectations $E_0[\cdot]$ for (59)–(61) leads to (36)–(38) with $\xi_1 = \xi_2$, $\tau_1 = \tau_2$ and $P = R$.

1⇒2: It follows from (38) that

\[
E_0|^2 P(S_{k0} + \xi^{(k0)} + |F_{k0}|) - A(\xi_k)^T E_0[P(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq 0 \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(62)

Take $\lambda_2$ satisfying $\lambda_1 < \lambda_2 < 1$ and define

\[
\xi_1 : = (\lambda_2^2 - \lambda_1^2) \xi_2.
\]

(63)

Then, (62) together with (36) leads us to

\[
E_0|^2 P(S_{k0} + \xi^{(k0)} + |F_{k0}|) - A(\xi_k)^T E_0[P(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq (\lambda_2^2 - \lambda_1^2) E_0[P(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq \xi_1 I \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(64)

Take

\[
R(S_{k} \xi^{(k+1)}) = A(\xi_k)^T P(S_{k+1} \xi^{(k+1)}) + \xi_1 I.
\]

(65)

Then, since

\[
E_0[R(S_{k0} \xi^{(k0)} + |F_{k0}|)] = A(\xi_k)^T E_0[P(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k) + \xi_1 I
\]

holds, (36) leads to (59) with $\xi_2 = \xi_1 > 0$, while (37) and Assumption 3 lead to (60) with appropriate $\tau_2$. By using this $R$, (64) can be rewritten as

\[
\lambda_2^2 E_0[R(S_{k0} + \xi^{(k0)} + |F_{k0}|)] - E_0[R(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq 0 \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(67)

By post- and pre-multiplying $A(\xi_k)$ and its transpose respectively on this inequality, we further obtain

\[
\lambda_2^2 E_0[A(\xi_k)^T P(S_{k0} + \xi^{(k0)} + |F_{k0}|) A(\xi_k)] - A(\xi_k)^T E_0[R(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq 0 \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(68)

which implies (61) with $\epsilon_2 = \lambda_2^2 \epsilon_1 \in \mathbb{R}_+$. This completes the proof.

Let $\lambda_{\min}^1$ be the infimum of $\lambda_2$ such that there exist $\xi_2, \tau_2, \epsilon_2 \in \mathbb{R}_+$ and $R : (\mathbb{R}^Z)^{N_0} \to \mathbb{S}^{n \times n}$ satisfying (59)–(61). Then, it follows from the above proof that $\lambda_{\min}^1 = \lambda_{\min}^2$, and thus,

\[
\lambda_{\min}^1 = \lambda_{\min}^2.
\]

In addition, we can also show the following lemma.

Lemma 4: Suppose the system (2) satisfies Assumption 3.

The following two conditions are equivalent.

1) There exist $\xi_2, \tau_2, \epsilon_2 \in \mathbb{R}_+$, $\lambda_2 \in (0, 1)$ and $R : (\mathbb{R}^Z)^{N_0} \to \mathbb{S}^{n \times n}$ satisfying (59)–(61).

2) There exist $\xi_2, \tau_2, \epsilon_2 \in \mathbb{R}_+$, and $R : (\mathbb{R}^Z)^{N_0} \to \mathbb{S}^{n \times n}$ satisfying (59)–(61) and

\[
E_0[R(S_{k0} \xi^{(k0)} + |F_{k0}|)] - A(\xi_k)^T E_0[R(S_{k0} + \xi^{(k0)} + |F_{k0}|)] A(\xi_k)
\]

\[
\geq \epsilon_2 I \quad \text{a.s.} \quad (\forall \xi_{(k-1)} \in \mathbb{E}_0; \forall k_0 \in \mathbb{Z}).
\]

(70)
Lyapunov inequalities (38) and (55) correspond to a generalization of the earlier result in [13] for systems with dynamics determined by an i.i.d. process while another type of Lyapunov inequalities (61) and (70) correspond to that in [10] for systems with dynamics determined by a Markov process, both of which will be revisited in the following section. As already stated, the results in the two earlier frameworks do not cover each other, and we had a question of what is the essential reason for this difference. According to Theorem 3 the two types of Lyapunov inequalities can be shown to be equivalent under Assumption 5. This implies that the assumptions on the process \( \xi \) to be i.i.d. or Markov are not essential, and only it is important in the selection of the type of Lyapunov inequalities whether Assumption 3 is satisfied or not (note a Lyapunov inequality of the same type as (38) and (55) can be always derived under Assumption 1). This is the answer to the question, which was led to by our new unified framework for second-moment stability of systems with stochastic dynamics.

Since Assumption 3 is stronger than Assumption 1, the relations in Fig. 1 automatically hold even under Assumption 3. In addition, the above arguments in turn imply that we can derive a Lyapunov inequality in the type of (61) and (70) even for the i.i.d. case under Assumption 4 and that in the type of (38) and (55) even for the Markov case under Assumption 1. The associated results will be also shown in the following section.

V. SELECTED APPLICATIONS

In this section, we demonstrate usefulness of the results obtained in the preceding sections through providing their applications. Our results, together with additional assumptions on \( \xi \), readily lead us to Lyapunov inequalities for the corresponding special cases of systems. We here deal with temporally-independent processes, Markov processes and polytopic martingales for determining system dynamics as selected examples, and discuss associated stability conditions. These arguments not only reveal the connections between earlier and our results but also lead to generalizations of the former.

A. Temporally-Independent Process Case

We first consider the following assumption on \( \xi \).

**Assumption 4**: For \( \xi = (\xi_k)_{k \in \mathbb{Z}} \), the random vectors \( \xi_k (k \in \mathbb{Z}) \) are independently distributed.

We call the process \( \xi \) satisfying this assumption a temporally-independent process. With this assumption, the Lyapunov matrix in (36)–(38) becomes independent of \( \xi_{(k_0-1)} \) for each \( k_0 \). Hence, the associated conditional expectations can be replaced by the standard expectations. Let

\[
P_{k_0} = E[P(S_{k_0} \xi_{k_0}^+)].
\]

Then, this implies that (38) reduce to

\[
P_{k_0} \geq \lambda_1 I,
\]

\[
P_{k_0} \leq \tau_1 I,
\]

\[
E[\lambda_1^2 P_{k_0} - A(\xi_{k_0})^T P_{k_0+1} A(\xi_{k_0})] \geq 0 \quad (\forall k_0 \in \mathbb{Z})
\]

under Assumption 4 respectively. In this manner, Assumption 4 can lead us to Lyapunov inequalities involving a deterministic (time-varying) Lyapunov matrix.

We further consider the situation where Assumptions 2 and 4 are both satisfied. This corresponds to the assumption that \( \xi \) is an i.i.d. process, which is used in [13]. Then, the above Lyapunov matrix \( P_{k_0} \) becomes time-invariant, since

\[
E[P(S_{k_0} \xi_{k_0}^+)] = E[P(S_{k_0+1} \xi_{(k_0+1)}^+)]
\]

for each \( k_0 \in \mathbb{Z} \). This implies that the corresponding Lyapunov inequality can be described with a deterministic time-invariant Lyapunov matrix. In particular, for \( P_0 \in \mathbb{S}_+^{n \times n} \),

\[
E[\lambda_1^2 P_0 - A(\xi_{k_0})^T P_0 A(\xi_{k_0})] \geq 0 \quad (\forall k_0 \in \mathbb{Z})
\]

if and only if

\[
E[\lambda_1^2 P_0 - A(\xi_0)^T P_0 A(\xi_0)] \geq 0
\]

in this situation. Hence, the following corollary holds.

**Corollary 1**: Suppose the system (2) satisfies Assumptions 2 and 4. The system is exponentially stable in the second moment if and only if there exist \( \lambda_1 \in (0, 1) \) and \( P_0 \in \mathbb{S}_+^{n \times n} \) satisfying (76).

The Lyapunov inequality (76) is nothing but that in [13] (see also the dotted arrow in Fig. 1). As was shown above, the Lyapunov inequalities derived in the present paper for systems with general stochastic dynamics can easily lead us to those for special cases of systems (the Lyapunov inequality (73) can be seen as an extension of the earlier result (76)).

Essentially the same techniques can be applied to (59)–(61). However, since \( R(S_{k_0} \xi_{k_0}^+) \) is not independent of \( \xi_{k_0} \) even under Assumption 4, the conditional expectation \( E_0[ R(S_{k_0} \xi_{k_0}^+) | F_{k_0} ] \) cannot be replaced by the standard expectation; it becomes a random matrix depending on \( \xi_{k_0} \). We denote such a random matrix by \( R_{k_0} (\xi_{k_0} \xi_{k_0}) \). Then, taking account of this difference and Theorems 3 and 4 lead us to the following corollary.

**Corollary 2**: Suppose the system (2) satisfies Assumptions 2, 3 and 4. The system is exponentially stable in the second moment if and only if there exist \( \lambda_2, \tau_2, \epsilon_2 \in \mathbb{R}_+ \), \( \lambda_2 \in (0, 1) \) and \( R_0 : \mathbb{R}^n \rightarrow \mathbb{S}_+^{n \times n} \) such that

\[
R_0(\xi_0) \geq \lambda_2 I \quad a.s.,
\]

\[
R_0(\xi_{k_0}) \leq \tau_2 I \quad a.s.,
\]

\[
\lambda_2^2 R_0(\xi_0) - A(\xi_0)^T E[R_0(\xi_0)] A(\xi_0) \geq \epsilon_2 I \quad a.s.
\]

If we introduce the support \( \Xi_0 \) of \( \xi_0 \), the inequality condition in this corollary can be rewritten as

\[
R_0(\xi_0) \geq \lambda_2 I,
\]

\[
R_0(\xi_{k_0}) \leq \tau_2 I,
\]

\[
\lambda_2^2 R_0(\xi_0) - A(\xi_0)^T E[R_0(\xi_0)] A(\xi_0) \geq \epsilon_2 I \quad (\forall \xi_0 \in \Xi_0),
\]

which might be easier to interpret.

Compared to Corollary 1 the inequality condition in Corollary 2 does not require us to deal with the expectation of square entries of \( A(\xi_0) \), which might enable us to extend the inequality conditions toward other control problems by using linear matrix inequality (LMI) optimization techniques [14], [26] more easily. Instead, however, we have to directly deal with a random matrix (i.e., \( R_0(\xi_0) \)) as a decision variable in
Corollary 2 which deteriorates the tractability of the condition in numerical analysis. Since the inequality condition in Corollary 1 can be extended (at least) to that for stabilization synthesis [13], [27], and since Assumption 5 is stronger than Assumption 1 the superiority of Corollary 2 over Corollary 1 might be limited at this moment.

In the above, we discussed the case of stationary processes. The associated assumption (i.e., Assumption 2), however, can be easily alleviated in our framework; this is obvious because we already obtained (73) without Assumption 2. For example, let us consider the following assumption about $N$-periodically stationary processes.

**Assumption 5:** The stochastic process $\xi$ is $N$-periodically stationary; i.e., for each $i = 0, 1, \ldots, N-1$, none of the characteristics of $\xi_{\kappa+N\tau}$ changes with $\kappa \in \mathbb{Z}$.

Then, the following periodic version of Corollary 1 can also be extended (at least) to that for stabilization in numerical analysis. Since the inequality condition in Corollary 4 in numerical analysis would not be much different from that of the condition in Corollary 5 in the Markov process case. The difference between the two conditions is related with whether $A(\xi_k)$ is essentially bounded or not (i.e., with or without Assumption 3). In other words, $A(\xi_0)$ should be contained in the (conditional) expectation as in (92) of Corollary 4 unless Assumption 3 is satisfied.

**Corollary 6:** Suppose the system (2) satisfies Assumptions 1, 2, and 6. The system is exponentially stable in the second moment if and only if there exist $\lambda_1 \in (0, 1)$ and a time-invariant function $P_0 : \mathbb{R}^Z \rightarrow S_{++}^{n \times n}$ such that

\[
P_0(\xi_0) \geq \xi_1 I,
\]

\[
P_0(\xi_0) \leq \tau_1 I,
\]

\[
E[\lambda_1^2 P_0(\xi_0) - A(\xi_0)^T P_{01}(\xi_0) A(\xi_0)|\xi_{-1} = \xi_0] \geq 0
\]

(90)

(91)

(92)

In a similar fashion, Theorem 4 also leads us to the following corollary.

**Corollary 5:** Suppose the system (2) satisfies Assumptions 1, 2, and 5. The system is exponentially stable in the second moment if and only if there exist $\lambda_1, \tau_2, \epsilon_2 \in \mathbb{R}_+$, $\lambda_2 \in (0, 1)$ and $R_0 : \mathbb{R}^Z \rightarrow S_{++}^{n \times n}$ such that [80], [81] and

\[
\lambda_2^2 R_0(\xi_0) - A(\xi_0)^T E[R_0(\xi_{-1})][\xi_{-1} = \xi_0] A(\xi_0) \geq \epsilon_2 I
\]

(93)

The Lyapunov inequality in Corollary 5 is essentially the same as that in [10]. In contrast to Corollary 1 (i.e., the temporally-independent process case), the Lyapunov matrix in Corollary 5 does not become a constant matrix even when Assumption 2 is used. Hence, the tractability of the condition in Corollary 4 in numerical analysis would not be much different from that of the condition in Corollary 5 under $N = 1$.

**B. Markov Process Case**

We next consider the following assumption on $\xi$ about Markov processes.

**Assumption 6:** The stochastic process $\xi$ has the Markov property; i.e., for each subset $\Xi \subseteq \mathbb{R}^Z$ and every $i, j \in \mathbb{Z}$ such that $i < j$,

\[
\Pr(\xi_j \in \Xi|\xi_i, \xi_{i-1}, \ldots) = \Pr(\xi_j \in \Xi|\xi_i),
\]

(84)

where $\Pr(\cdot|\cdot)$ denotes the conditional probability.

Under this assumption, the conditional expectation $E_0$ can be simplified as

\[
E_0[|] = E[|\xi_{k-1} = \xi_j],
\]

\[
E_0[|F_k] = E[|\xi_k] \quad (k \geq k_0)
\]

for $\xi_j$ belonging to the support of $\xi_{k-1}$ denoted by $\Xi_{k-1}$. Hence, $E_0[|P(S_{\xi_k}^T \eta_{k+1})] = E_0[|P(S_{\xi_{k+1}}^T \eta_{k+2})] = E_0[|P(S_{\xi_k}^T \eta_{k+1})]$ in (86)–(88) can be respectively simplified as $P_{k_0}(\xi_0)$ and $P_{k_0+1}(\xi_{k_0})$ with an appropriate time-varying function $P_{k_0} : \mathbb{R}^Z \rightarrow S_{++}^{n \times n}$. That is, (86)–(88) reduce to

\[
P_{k_0}(\xi_0) \geq \xi_1 I,
\]

\[
P_{k_0}(\xi_0) \leq \tau_1 I,
\]

\[
E[\lambda_1^2 P_{k_0}(\xi_0) - A(\xi_0)^T P_{k_0+1}(\xi_{k_0}) A(\xi_0)|\xi_{k-1} = \xi_0] \geq 0
\]

(87)

(88)

(89)

This, together with Assumption 2 about stationary $\xi$ and Theorem 3 leads us to the following corollary.

**Corollary 4:** Suppose the system (2) satisfies Assumptions 1, 2 and 6. The system is exponentially stable in the second moment if and only if there exist $\lambda_1 \in (0, 1)$ and a time-invariant function $P_0 : \mathbb{R}^Z \rightarrow S_{++}^{n \times n}$ such that

\[
P_0(\xi_0) \geq \xi_1 I,
\]

\[
P_0(\xi_0) \leq \tau_1 I,
\]

\[
E[\lambda_1^2 P_0(\xi_0) - A(\xi_0)^T P_{01}(\xi_0) A(\xi_0)|\xi_{-1} = \xi_0] \geq 0
\]

\[
(\forall \xi_0 \in \Xi_{-1})
\]

(90)

(91)

(92)
C. Polytopic Martingale Case

The preceding two subsections exemplified the generality of our results through clarifying their connections with some earlier studies. In particular, the arguments implied that the Lyapunov inequalities individually derived in the earlier studies can be seen as special cases of our results, which were led to in this section just by introducing associated assumptions. In this subsection, we further show potentials of our results by introducing an assumption uncommon in the sense that the corresponding stochastic systems have not been dealt with as the target for analysis and synthesis in the field of control, to the best knowledge of the authors. Interestingly, the associated results will also provide us with a new insight into a well-known robust stability condition for uncertain deterministic systems.

The following is the assumption we use in this subsection.

**Assumption 7:** For each \( k_0 \in \mathbb{Z} \) and every \( \xi^{(k_0 - 1)} - \in \mathbb{E}_0 \), the stochastic process \( \xi \) satisfies the following conditions.

1a) For each \( k \in \mathbb{Z}_+(k_0) \), \( \xi_k \) is \( \mathcal{F}_k \)-measurable (this is automatically satisfied by the present definition of \( \mathcal{F}_k \)).

1b) For each \( k \in \mathbb{Z}_+(k_0) \), \( E_0[\|\xi_k\|] < \infty \) (this is automatically satisfied by the following condition 2).

1c) For each \( k \in \mathbb{Z}_+(k_0) \),

\[
E_0[\xi_{k+1} | \mathcal{F}_k] = \xi_k \quad \text{a.s.}\]

(100)

2) The support of \( \xi_{k_0} \) belongs to (or given by)

\[
E^Z := \left\{ \theta \in \mathbb{R}^Z \left| \theta_i \geq 0 \ (i = 1, \ldots, Z), \sum_{i=1}^Z \theta_i = 1 \right. \right\},

(101)

where \( \theta_i \) is the \( i \)-th entry of \( \theta \).

Condition 1 in this assumption corresponds to the definition of martingales [17] (so \( \xi \) satisfying this assumption is a martingale). In addition to it, we also consider condition 2 for restricting \( A(\xi_k) \) in (2) to a polytopically-random matrix later. Obviously, \( \xi \) becomes neither temporally-independent nor Markovian only with this assumption. Hence, the system (2) with such \( \xi \) cannot be dealt with in the framework of the earlier studies referred to in the preceding subsections.

For given deterministic constant matrices \( A^{(i)} \) (\( i = 1, \ldots, Z \)), let system (2) further satisfy the following assumption.

**Assumption 8:** The function \( A : \mathbb{R}^Z \rightarrow \mathbb{R}^{n \times n} \) is given by

\[
A(\theta) = \sum_{i=1}^Z \theta_i A^{(i)}

(102)

for \( A^{(i)} \in \mathbb{R}^{n \times n} \) and \( \theta \in \mathbb{R}^Z \).

This assumption, together with Assumption [7] implies that \( A(\xi_k) \) takes values only in the polytope defined with the vertices \( A^{(i)} \) (\( i = 1, \ldots, Z \)); in particular, the sequence of such \( A(\xi_k) \) with respect to \( k \) is a martingale because

\[
E[A(\xi_{k+1}) | \mathcal{F}_k] = \sum_{i=1}^Z E[\xi_{i(k+1)} | \mathcal{F}_k] A^{(i)} = A(\xi_k) \quad \text{a.s.}

(103)

for \( \xi_k = [\xi_{1k}, \ldots, \xi_{zk}]^T \).

Since Assumptions [7] and [8] let Assumption [8] be automatically satisfied, we consider deriving a stability condition based on Theorem [4] rather than Theorem [5]. To this end, let us consider the Lyapunov matrix given by

\[
R(S_k \xi^{k+1}) = R_0(\xi_k) = \sum_{i=1}^Z \xi_{ik} R^{(i)}_0

(104)

for \( R^{(i)}_0 \in \mathbb{S}^n_{+} \) (\( i = 1, \ldots, Z \)), associated with Assumption [8]. Because of this restriction, the corresponding stability condition becomes conservative. However, it has the advantage that \( R(\xi_k) \in \mathbb{Z} \) also becomes a martingale, i.e.,

\[
E[R_0(\xi_{k+1}) | \mathcal{F}_k] = R_0(\xi_k) \quad \text{a.s.},

(105)

which will be a key in deriving a tractable stability condition with Theorem [4].

With Assumptions [7] and [8] and (104), the inequality condition (61) in Theorem [4] reduces to

\[
\lambda_2^2 R_0(\xi_{k_0}) - A(\xi_{k_0})^T E[R_0(\xi_{k_0+1}) | \mathcal{F}_k] A(\xi_{k_0}) \geq \epsilon_2 I \quad \text{a.s.}

\]

(106)

(note \( E[R_0(\xi_{k_0}) | \mathcal{F}_k] = R_0(\xi_{k_0}) \)); the other inequality conditions in the theorem are automatically satisfied under Assumption [7] and (104). Then, (106) further reduces to

\[
\lambda_2^2 R_0(\xi_{k_0}) - A(\xi_{k_0})^T R_0(\xi_{k_0}) A(\xi_{k_0}) \geq \epsilon_2 I \quad \text{a.s.}

\]

(107)

by (105). Hence, by using the \( S \)-variable (i.e., auxiliary variable) technique [26], we can show that there exists \( \epsilon_2 \in \mathbb{R}_+ \) satisfying (107) (i.e., (61)) if there exist \( \epsilon_2 \in \mathbb{R}_+ \) and \( S \in \mathbb{R}^{2n \times n} \) such that

\[
\begin{bmatrix}
\lambda_2^2 R_0(\xi_{k_0}) & 0 \\
0 & -R_0(\xi_{k_0})
\end{bmatrix}
+ \text{He} \left( S \begin{bmatrix} A(\xi_{k_0}) \ I \end{bmatrix} \right) \geq \epsilon_2 I \quad \text{a.s.}

(108)

where \( \text{He}(\cdot) := (\cdot)^T + (\cdot) \) for the square matrix \( (\cdot) \). Noting that the present \( A \) and \( R_0 \) have the polytopic structures (102) and (104), this immediately leads us to the following theorem.

**Theorem 5:** Suppose the system (2) satisfies Assumptions [7] and [8]. The system is exponentially stable in the second moment if there exist \( \lambda_2 \in (0, 1) \), \( R^{(i)}_0 \in \mathbb{S}^n_{+} \) (\( i = 1, \ldots, Z \)) and \( S \in \mathbb{R}^{2n \times n} \) such that

\[
\begin{bmatrix}
\lambda_2^2 R^{(i)}_0 & 0 \\
0 & -R^{(i)}_0
\end{bmatrix}
+ \text{He} \left( S \begin{bmatrix} A^{(i)} \ I \end{bmatrix} \right) > 0 \quad (i = 1, \ldots, Z).

(109)

Let \( \Xi \) denote the set of processes \( \xi \) satisfying Assumption [7] then, the above theorem actually implies that the system is stable for each \( \xi \in \Xi \) if a solution of (109) exists. Hence, the above theorem gives a robust stability condition of the system with respect to \( \xi \in \Xi \).

Here, recall that \( \xi \) given with \( \xi_k = \theta \ (\forall k \in \mathbb{Z}) \) for the deterministic vector \( \theta \in \mathbb{E}^Z \) (i.e., the deterministic process taking only \( \theta \) satisfies Assumption [7] and hence, belongs to \( \Xi \). This special case corresponds to nothing but the deterministic...
system with the polytopic uncertain parameter $\theta \in \mathbb{E}^Z$. The inequality condition (109) in Theorem 5 is conventionally used as a sufficient condition for robust stability of such a special case of systems. However, according to our theorem, the inequality condition actually ensures robust stability of the systems not only with such deterministic time-invariant $\xi$ but also with stochastic time-varying $\xi$ satisfying Assumption 7. This would not have been known in the field of robust control, and in turn demonstrates potentials of the results in this paper. Since the form of the inequality condition in (109) is consistent with that for deterministic uncertain systems, it can be readily extended, e.g., toward synthesis of robustly stabilizing state feedback, as is the case with deterministic systems.

Remark 2: In Theorem 5 if we confine $S$ to $[0, G]T$ for $G \in \mathbb{R}^{n \times n}$, then the inequality (109) reduces to

$$\begin{bmatrix} \lambda_2^2R_0^{(i)} & A^{(i)T}G^T \\ GA^{(i)} & G + G^T - R_0^{(i)} \end{bmatrix} > 0 \quad (i = 1, \ldots, Z), \quad (110)$$

which is essentially the same as the inequality condition in Theorem 2 in [20] for uncertain deterministic time-invariant systems. One might be more familiar with this special case, to which a comment similar to (109) applies; that is, it follows from our results that the inequality condition ensures robust stability of the systems not only for deterministic polytopic uncertainties but also for polytopic martingale uncertainties.

Remark 3: In the case of deterministic systems, parameter-dependent Lyapunov inequalities for deterministic time-varying uncertainties have been also studied, e.g., in [21], and one might be also interested in the relationship of Theorem 5 with this approach. Since $A(\xi_k)$ satisfying Assumptions 7 and 8 takes a value only in a $(k$-independent) polytope at each $k$, robust stability for that polytope in the deterministic sense leads us to robust second-moment stability, which is dealt with in Theorem 5 deterministic stability is stronger than stochastic stability, in general. However, such deterministic robust stability cannot be ensured only with the $Z$ inequalities in (109), and a more severe condition consisting of $Z^2$ inequalities is required in association with the parameter-dependent Lyapunov matrix (for details, see [21]). Hence, Theorem 5 is not covered by this approach.

VI. CONCLUSION

In this paper, we studied second-moment stability of discrete-time linear systems with general stochastic dynamics. We first showed relations of several notions of second-moment stability, discussed the time-invariance property of the systems, and then derived two types of Lyapunov inequalities characterizing second-moment exponential stability. One of them was derived for the systems with the most general stochastic dynamics (i.e., only under Assumption 1 which is a minimal requirement for defining second-moment stability), and the other was derived for the systems with essentially bounded random coefficient matrices (i.e., under Assumption 3). By introducing additional assumptions on the systems, our results can readily lead us to stability conditions for the corresponding special cases. As a demonstration of their usefulness, we provided three applications in which temporally-independent processes, Markov processes and polytopic martingales were dealt with for determining system dynamics. Discrete-time linear systems with any type of stochastic dynamics can be dealt with in our framework as in those applications. That is, our framework can unify all the results about second-moment stability of discrete-time linear systems with stochastic dynamics. Providing such a framework is expected to facilitate the studies on analysis and control of stochastic systems drastically. Although only stability conditions were discussed in this paper, the techniques used there are considered to be useful also for discussing control performance such as $H_2$ and $H_\infty$ norms (provided that those are appropriately defined in the stochastic sense).

REFERENCES

[1] V. Paxson and S. Floyd, “Wide area traffic: The failure of Poisson modeling,” IEEE/ACM Transactions on Networking, vol. 3, no. 3, pp. 226–244, 1995.
[2] M. Finkelstein, Failure Rate Modelling for Reliability and Risk. London, UK: Springer-Verlag, 2008.
[3] E. Kalnay, Atmospheric modeling, data assimilation and predictability. Cambridge, UK: Cambridge University Press, 2003.
[4] L. Arnold, Random Dynamical Systems. Berlin Heidelberg, Germany: Springer-Verlag, 1998.
[5] G. Evensen, “The ensemble Kalman filter: Theoretical formulation and practical implementation,” Ocean Dynamics, vol. 53, no. 4, pp. 343–367, 2003.
[6] G. Evensen, Data Assimilation: The Ensemble Kalman Filter, 2nd ed. Berlin Heidelberg, Germany: Springer-Verlag, 2009.
[7] J. H. Kotecha and P. M. Djuric, “Gaussian sum particle filtering,” IEEE Transactions on signal processing, vol. 51, no. 10, pp. 2602–2612, 2003.
[8] A. H. Alen, H. A. Karlsen, G. Nævdal, H. J. Skaug, and B. Vallels, “Bridging the ensemble Kalman filter and particle filters: The adaptive Gaussian mixture filter,” Computational Geosciences, vol. 15, no. 2, pp. 293–305, 2011.
[9] O. L. V. Costa, M. D. Fragoso, and R. P. Marques, Discrete-Time Markov Jump Linear Systems. London, UK: Springer-Verlag, 2005.
[10] O. L. V. Costa and D. Z. Figueiredo, “Stochastic stability of jump discrete-time linear systems with Markov chain in a general Borel space,” IEEE Transactions on Automatic Control, vol. 59, no. 1, pp. 223–227, 2014.
[11] W. L. De Koning, “Compensatabilaty and optimal compensation of systems with white parameters,” IEEE Transactions on Automatic Control, vol. 37, no. 5, pp. 574–579, 1992.
[12] Y. Hosoe, T. Hagiwara, and D. Peaucelle, “Robust stability analysis and state feedback synthesis for discrete-time systems characterized by random polytopes,” IEEE Transactions on Automatic Control, vol. 63, no. 2, pp. 556–562, 2018.
[13] Y. Hosoe and T. Hagiwara, “Equivalent stability notions, Lyapunov inequality, and its application in discrete-time linear systems with stochastic dynamics determined by an i.i.d. process,” IEEE Transactions on Automatic Control, to appear (DOI: 10.1109/TAC.2019.2905216).
[14] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory. Philadelphia, PA, USA: SIAM, 1994.
[15] E. Gershon and U. Shaked, “$H_\infty$ output-feedback control of discrete-time systems with state-multiplicative noise,” Automatica, vol. 44, no. 2, pp. 574–579, 2008.
[16] O. Knill, Probability and Stochastic Processes with Applications. New Delhi, India: Overseas Press, 2009.
[17] A. Klenke, Probability Theory: A Comprehensive Course, 2nd ed. London, UK: Springer-Verlag, 2014.
[18] S. Koskin, “A survey of stability of stochastic systems,” Automatica, vol. 5, no. 1, pp. 95–112, 1969.
[19] M. Vidyasagar, Nonlinear Systems Analysis, 2nd ed. Philadelphia, PA, USA: SIAM, 2002.
[20] M. C. de Oliveira, J. Bernussou, and J. C. Geromel, “A new discrete-time robust stability condition,” Systems & Control Letters, vol. 37, no. 4, pp. 261–265, 1999.
[21] J. Daafouz and J. Bernussou, “Parameter dependent Lyapunov functions for discrete time systems with time varying parametric uncertainties,” Systems & Control Letters, vol. 43, no. 5, pp. 355–359, 2001.
[22] E.-K. Boukas and P. Shi, “Stochastic stability and guaranteed cost control of discrete-time uncertain systems with Markovian jumping parameters,” *International Journal of Robust and Nonlinear Control*, vol. 8, no. 13, pp. 1155–1167, 1998.

[23] J. C. Geromel and P. Colaneri, “Stability and stabilization of discrete time switched systems,” *International Journal of Control*, vol. 79, no. 7, pp. 719–728, 2006.

[24] C. E. De Souza, M. Fu, and L. Xie, “$H_{\infty}$ analysis and synthesis of discrete-time systems with time-varying uncertainty,” *IEEE Transactions on Automatic Control*, vol. 38, no. 3, pp. 459–462, 1993.

[25] S. Xu, J. Lam, and C. Yang, “Quadratic stability and stabilization of uncertain linear discrete-time systems with state delay,” *Systems & control letters*, vol. 43, no. 2, pp. 77–84, 2001.

[26] Y. Ebihara, D. Peaucelle, and D. Arzelier, *S-Variable Approach to LMI-Based Robust Control*. London, UK: Springer-Verlag, 2015.

[27] Y. Hosoe and D. Peaucelle, “Static output feedback stabilization of discrete-time linear systems with stochastic dynamics determined by an independent identically distributed process,” *IEEE Control Systems Letters*, vol. 3, no. 3, pp. 673–678, 2019.