Research Article
Constraints Optimal Control Governing by Triple Nonlinear Hyperbolic Boundary Value Problem

Jamil A. Ali Al-Hawasy and Lamyaa H. Ali

Department of Mathematics, College of Science, Mustansiriyah University, Baghdad, Iraq

Correspondence should be addressed to Jamil A. Ali Al-Hawasy; jhawassy17@uomustansiriyah.edu.iq

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The focus of this work lies on proving the existence theorem of a unique state vector solution (Stvs) of the triple nonlinear hyperbolic boundary value problem (TNHBVP) when the classical continuous control vector (CCCVE) is fixed by using the Galerkin method (Galm), proving the existence theorem of a unique constraints classical continuous optimal control vector (CCCOCVE) with vector state constraints (equality EQVC and inequality INEQVC). Also, it consists of studying for the existence and uniqueness adjoint vector solution (Advs) of the triple adjoint vector equations (TAEqs) associated with the considered triple state equations (Tsteqs). The Fréchet Derivative (Frde.) of the Hamiltonian (HAM) is found. At the end, the theorems for the necessary conditions and the sufficient conditions of optimality (Necoop and Sucoop) are achieved.

1. Introduction

The subject of optimal control problem (OCP) plays a basic role in many real life problems in different branches of sciences; for example, in, medicine [1], engineering and social sciences [2], biology [3], ecology [4], electric power [5], aerospace [6], and many other branches.

This role encouraged many researchers to go deeply into studying the OCPS governed by differential equations (deqs). Such OCP problems are studied at the beginning for the systems which are controlled by nonlinear ordinary deqs (nodeqs) [7] or by linear deqs (lpdeqs) [8]. Later great interests have been made to study this subject but for systems which are controlled by pdeqs of elliptic type (ET) [9], or of hyperbolic type (HT) [10], or of parabolic type (PT) [11], or by couple of npdeqs of ET [12], or of PT [13], or of HT [14].

Recently, the attention in this subject is magnified to deal with more general types as the studying the CCCOPCVE controlling by TBVP of ET [15], and of PT [16]. All these studies motivated us to look deep inside the CCCOPCVE controlled by TNHBVP.

In this work and at first, we give a mathematical description for the CCCOPCVE, and then the TNHBVP is written in its weak form (wkf), and the existence and uniqueness theorem of the Stvs for the TNHBVP using the Galm with the Aubin compactness theorem is proved under appropriate hypotheses when the CCCVE is given. Under reasonable hypotheses, the objective function and the EQVC and INEQVC are proved continuous. The proof of the existence theorem of a CCCOPCVE governed by the TNHBVP is achieved. Under certain hypotheses, the study of the existence theorem for a unique Advs of the TAEqs associated with the considered Tsteqs is done. The Fréchet Derivative (Frde.) of the Hamiltonian (HAM) is found. At the end, the theorems for the necessary conditions and the sufficient conditions of optimality (Necoop and Sucoop) are achieved.

1.1. Problem Description. Let \( I = [0, T], \ T < \infty, \ E \) be a bounded and open region in \( \mathbb{R}^2 \) with Lipschitz (Lip.) boundary \( \partial E, \ \Pi = E \times I, \) and \( \partial \Pi = \partial E \times I. \) The considered CCCOPCVE consists of the Steq which is given by the TNLHPDEqs:

\[
\psi_{1tt} - \Delta \psi_1 + \psi_1 - \psi_2 - \psi_3 = k_1(x, t, \psi_1, \omega_1), \quad \text{in } \Pi, \quad (1)
\]

\[
\psi_{2tt} - \Delta \psi_2 + \psi_2 + \psi_3 + \psi_1 = k_2(x, t, \psi_2, \omega_2), \quad \text{in } \Pi, \quad (2)
\]

\[
\psi_{3tt} - \Delta \psi_3 + \psi_3 + \psi_1 - \psi_2 = k_3(x, t, \psi_3, \omega_3), \quad \text{in } \Pi, \quad (3)
\]

with the BCs and ICs.
\[ \psi_1(x, t) = 0, \psi_2(x, t) = 0, \psi_3(x, t) = 0, \quad \text{on } \partial \Pi, \]  
\[ \psi_1(x, 0) = \psi_1^0(x), \psi_2(x, 0) = \psi_2^0(x), \psi_3(x, 0) = \psi_3^0(x), \quad \text{on } E, \]  
\[ \psi_{1t}(x, 0) = \psi_{11}^1(x), \psi_{2t}(x, 0) = \psi_{11}^2(x), \psi_{3t}(x, 0) = \psi_{11}^3(x), \quad \text{on } E, \]  
where \( \psi = (\psi_1, \psi_2, \psi_3) \in (H^1(E))^3 \) is the Stvs, corresponding to the CCCVE \( \overline{w} = (\omega_1, \omega_2, \omega_3) \in (L^2(\Pi))^3 \) and \( (k_1, k_2, k_3) \in (L^2(\Pi))^3 \) is a function defined on \( (\Pi \times \mathcal{R} \times C_i) \times (\Pi \times \mathcal{R} \times C_i) \times (\Pi \times \mathcal{R} \times C_i) \) with \( C_i \subset R \), for \( i = 1, 2, 3 \).

The controls set are \( \overline{w} \in \overline{W}, \overline{W} \subset (L^2(\Pi))^3 \) with \( \overline{W} = \{ \overline{W} \in (L^2(\Pi))^3 \mid \overline{W} \in C, \text{ a.e. in } \Pi, \text{ with } C \subset R \}. \)

The cost function is
\[ M_0(\overline{w}) = \sum_{i=1}^{3} \int_{\Pi} m_{\bar{\omega}_i}(x, t, \psi, \omega_i) dx dt. \]  
(7)

The EQVC and INEQVC on the state vectors are
\[ M_r(\overline{w}) = \sum_{i=1}^{3} \int_{\Pi} m_{\bar{\omega}_i}(x, t, \psi, \omega_i) dx dt = 0, \quad 1 \leq r \leq p, \]  
(8)
\[ M_r(\overline{w}) = \sum_{i=1}^{3} \int_{\Pi} m_{\bar{\omega}_i}(x, t, \psi, \omega_i) dx dt \leq 0, \quad p + 1 \leq r \leq q. \]  
(9)

The set of admissible control vector is \( \overrightarrow{W}_A = \{ \overline{w} \in \overrightarrow{W} \mid M_0(\overline{w}) = 0, M_{r+p}(\overline{w}) \leq 0, 1 \leq r \leq p \}. \)

The continuous optimal control problem is to find \( \overline{w} \in \overrightarrow{W}_A \) such that
\[ M_0(\overline{w}) = \overrightarrow{W}_A \min M_0(\overline{w}). \]

Let
\[ Y = Y_1 \times Y_2 \times Y_3 = \{ \overline{v} \mid \overline{v} \in (H^1(\Omega))^3, \text{ with } v_1 = v_2 = v_3 = 0 \text{ on } \partial \Pi, \overline{v} = (v_1, v_2, v_3) \}. \]

We denote by \( (v, v), (v, v), \) and \( (\overline{v}, \overline{v}) \) the inner products in \( L^2(E), H^1(E), \) and \( Y \) respectively while the norm in these spaces is denoted by \( v_0, v_1, \) and \( \overline{v} = \sum_{i=1}^{3} v_{1i}, Y \) is denoted the dual of \( Y \). Also, the symbol \( \wedge \) will be used to indicate that the convergence of a sequence is weak, while the strong convergence of a sequence will be indicated by \( \rightarrow \).

The wft of problem (1)–(6) when \( \overrightarrow{w} \in (H^1_0(E))^3 \) is given almost everywhere on \( I \) for each \( v_1 \in Y_1, v_2 \in Y_2, \) and \( v_3 \in Y_3; \)

\[ \langle \psi_{1t}, v_1 \rangle + (\nabla \psi_1, \nabla v_1) + (\psi_1 - \psi_2 - \psi_3, v_1) = (k_1 \psi_1, v_1), \quad \psi_1 (x, t) \in Y_1, \]  
(10)
\[ \langle \psi_{11}, v_1 \rangle = (\psi_1(0), v_1), \]  
(11)
\[ \langle \psi_2, v_1 \rangle = (\psi_2(0), v_1), \]  
(12)
\[ \langle \psi_3, v_1 \rangle = (\psi_3(0), v_1), \]  
(13)
\[ \langle \psi_{11}, v_2 \rangle + (\nabla \psi_2, \nabla v_2) + (\psi_2 + \psi_3, v_2) = (k_2 \psi_2, v_2), \quad \psi_2 (x, t) \in Y_2, \]  
(14)
\[ \langle \psi_{11}, v_3 \rangle + (\nabla \psi_3, \nabla v_3) + (\psi_3 - \psi_2, v_3) = (k_3 \psi_3, v_3), \quad \psi_3 (x, t) \in Y_3, \]  
(15)

The following assumptions (Assums.) are needed to investigate the classical continuous optimal control problem (CCOPCP).

Assumption A. \( k_i \) is of the Carathéodory type (Caraty.) on \( \Pi \times (\mathcal{R} \times C_i) \) and satisfies the following conditions for \( (x, t) \in \Pi \) and \( \forall i = 1, 2, 3; \)
\[ |k_i(x,t,\psi,\omega_i)| \leq F_i(x,t) + \beta_i |\psi_i|, \quad \text{where } \psi_i, \omega_i \in \mathcal{R}, \beta_i > 0, F_i \in L^2(\Omega), \]
\[ |k_i(x,t,\psi,\omega_i)| - k_i(x,t,\overline{\psi},\omega_i)| \leq L_i |\psi_i - \overline{\psi}|, \]

where \( \psi_i, \overline{\psi}, \omega_i \in \mathcal{R}, L_i > 0. \) \hfill (16)

1.2. The Solution of the State Equations. In this part, the existence theorem of a unique solution for triple nonlinear hyperbolic partial differential equations (TNLHPDEqs) under Assumption A is proved when the control vector is given, and the following proposition will be needed.

Proposition 1 (see [17]). Suppose \( D \subset \mathcal{R}^s (s = 2,3) \), \( k: D \times \mathcal{R}^n \rightarrow \mathcal{R}^m \) is of Caraty. It satisfies \( \|k(v,x)\| \leq a(v) + \beta(v)\|x\|^n \), for each \((v,x) \in D \times \mathcal{R}^n\), where \( x \in L^b(D,\mathcal{R}^n), a \in L^1(D,\mathcal{R}), \beta \in L^b(D,\mathcal{R}), a \in [0,b]\), if \( b \neq \infty \), \( a = 0 \) if \( b = \infty \). Then, the functional \( \mathcal{K}(x) = \int_k k(v,x)dv \) is cont.

Theorem 1. Existence and Uniqueness of the Stvs: with Assumption A, for any given \( \mathcal{D} \in (L^2(\Omega))^3 \), the wfk of \((10) - (15)\) has a unique solution \( \psi = (\psi_1, \psi_2, \psi_3) \) s.t., \( \overline{\psi} \in (L^2(I, Y))^3 \), \( \overline{\psi} = (\psi_{1\infty}, \psi_{2\infty}, \psi_{3\infty}) \in (L^2(I, Y))^3 \), and \( \overline{\psi}_{\infty} = (\psi_1, \psi_2, \psi_3) \in (L^2(I, Y))^3 \).

Proof. Let \( \overline{\mathcal{Y}}_n = Y_n \times Y_n \times Y_n \subset \mathcal{Y} \) (for each \( n \)) be the set of cont. and piecewise affine function in \( E \). \( \left\{ \overline{\mathcal{Y}}_n \right\}_{n=1}^{\infty} \) be a sequence of subspaces of \( \overline{\mathcal{Y}} \), such that \( \forall \overline{\psi} = (v_1, v_2, v_3) \in \overline{\mathcal{Y}}, \) there exists a sequence \( \{\overline{\psi}_n\} \) with \( \overline{\psi}_n = (v_{1n}, v_{2n}, v_{3n}) \in \overline{\mathcal{Y}}_n \) and \( \overline{\psi}_n \rightarrow \overline{\psi} \) in \( \overline{\mathcal{Y}} \) in \( L^2(E) \). Let \( \overline{\psi}_n = (\psi_{1n}, \psi_{2n}, \psi_{3n}) \) be the Galerkin approximate solution (Galaso) to the exact solution \( \overline{\psi} = (\psi_1, \psi_2, \psi_3) \) such that
\[ \overline{\psi}_n = \sum_{j=1}^{n} c_{ij}(t)v_{ij}(x), \] \hfill (17)
where \( c_{ij}(t) \) are unknown functions of \( t, \forall j = 1,2,\ldots,n \) and, \( \forall i = 1,2,3 \).

The wfk of \((10) - (15)\) is approximated with respect to \( x \) using the Galo, and substituting \( \overline{\psi}_{\infty} = \psi_{m}^{\infty}(i = 1,2,3) \) in the obtained equations, they become for each \( v_1, v_2, v_3 \in \mathcal{Y}_n \):
\[ \langle \zeta_{1\infty}, v_1 \rangle + (\nabla \psi_{1\infty}, \nabla v_1) + (\psi_{1\infty}, v_1) - (\psi_{2\infty}, v_1) - (\psi_{3\infty}, v_1) = (k_1(\psi_{1\infty}, \omega_1), v_1), \] \hfill (18)
\[ (\psi_{0\infty}, v_1) = (\psi_{0\infty}, v_1), \] \hfill (19)
\[ (\psi_{1\infty}, v_1) = (\psi_{1\infty}, v_1), \] \hfill (20)
\[ (\psi_{0\infty}, v_2) = (\psi_{0\infty}, v_2), \] \hfill (21)
\[ (\psi_{1\infty}, v_2) = (\psi_{1\infty}, v_2), \] \hfill (22)
\[ (\psi_{0\infty}, v_3) = (\psi_{0\infty}, v_3), \] \hfill (23)
\[ (\psi_{1\infty}, v_3) = (\psi_{1\infty}, v_3), \] \hfill (24)
\[ \psi_{m}^{\infty}(x) = \psi_{m}^{\infty}(x,0) \in \mathcal{Y}_n \] (respectively, \( \psi_{m}^{\infty} = \psi_{m}^{\infty}(x) = \psi_{m}^{\infty}(x,0) \in L^2(E) \)) be the projection of \( \psi_{m}^{\infty} \) onto \( V \) (be the projection of \( \psi_{m} \) onto \( L^2(E) \), \( \mathcal{V} \)), \( \forall i = 1,2,3, \) i.e.,
\[ \psi_{m}^{\infty} \rightarrow \psi_{m}^{\infty} \in V, \text{ with } \|\psi_{m}^{\infty}\|_0 \leq b_0 \text{ and } \|\psi_{m}^{\infty}\|_0 \leq b_0, \] \hfill (24)
\[ \psi_{m}^{\infty} \rightarrow \psi_{m}^{\infty} \in L^2(E) \text{ and } \|\psi_{m}^{\infty}\|_0 \leq b_1. \] \hfill (25)

Substituting (17) for each \( i = 1,2,3 \), respectively, in the pairs (18) and (19), (20) and (21), and in (22) and (23), setting \( v_1 = v_{1j}, v_2 = v_{2j}, v_3 = v_{3j} \), the obtained equations will be written as the following 1st order nodeqs with their ICs and have a unique solution \( \overline{\psi}_{n} \in C(I, \overline{\mathcal{Y}}) \), i.e., for each \( I = 1,2,3 \) and \( n = 0,1 \):
\[ A_1D_1'(t) + B_1C_1(t) - E_1C_2(t) - H_1C_3(t) = b_1 \left( \nabla'_1(x)C_1(t) \right), \]
\[ A_1C_1(0) = b_1^0, \]
\[ A_1D_1(0) = b_1^1, \]
\[ A_2D_2'(t) + B_2C_2(t) + E_2C_1(t) + H_2C_3(t) = b_2 \left( \nabla'_2(x)C_2(t) \right), \]
\[ A_2C_2(0) = b_2^0, \]
\[ A_2D_2(0) = b_2^1, \]
\[ A_3D_3'(t) + B_3C_3(t) + E_3C_1(t) - H_3C_3(t) = b_3 \left( \nabla'_3(x)C_3(t) \right), \]
\[ A_3C_3(0) = b_3^0, \]
\[ A_3D_3(0) = b_3^1, \]

where \( C_1(t) = (c_{ij}(t))_{n \times 1}, \)
\( C_2(t) = (c_{ij}^2(t))_{n \times 1}, \)
\( D_1(t) = (d_{ij}(t))_{n \times 1}, \)
\( D_2(t) = \left( d_{ij}^2(t) \right)_{n \times 1}, \)
\( b_1 = (b_{ij}^1)_{n \times 1}, \)
\( b_2 = (b_{ij}^2)_{n \times 1}, \)
\( c_{ij}(t), \)
\( c_{ij}^2(t), \)
\( d_{ij}(t), \)
\( d_{ij}^2(t), \)
\( b_{ij}^1, \)
\( b_{ij}^2, \)
\( f_{1j} = (f_{1j})_{n \times 1}, \)
\( f_{2j} = (f_{2j})_{n \times 1}, \)
\( f_{3j} = (f_{3j})_{n \times 1}, \)
\( b_{ij}, \)
\( c_{ij}, \)
\( c_{ij}^2, \)
\( d_{ij}, \)
\( d_{ij}^2, \)
\( b_{ij}^1, \)
\( b_{ij}^2, \)
\( f_{1j}, \)
\( f_{2j}, \)
\( f_{3j}, \)
\( (\psi_{1n}^1, v_{1n}), \)
\( (\psi_{1n}^2, v_{1n}), \)
\( (\psi_{2n}^1, v_{2n}), \)
\( (\psi_{2n}^2, v_{2n}), \)
\( (\psi_{3n}^1, v_{3n}), \)
\( (\psi_{3n}^2, v_{3n}), \)
\( (\psi_{3n}^3, v_{3n}), \)
\( (\psi_{3n}^4, v_{3n}), \)
\( (\psi_{3n}^5, v_{3n}), \)
\( (\psi_{3n}^6, v_{3n}), \)

\[ H_2 = (f_{3j})_{n \times 1}, \]
\[ f_{3j} = (u_{1j}, u_{3j}), \]
\[ b_{ij} = (b_{ij})_{n \times 1}, \]
\[ b_{ij}^1 = [(\psi_{1n})_{n \times 1}, (\psi_{2n})_{n \times 1}, (\psi_{3n})_{n \times 1}], \]

Then, corresponding to the sequence \( \{ \psi_n \} \), there exists a sequence of the following approximation problems, i.e., for each \( \psi_n = (u_{1n}, u_{2n}, u_{3n}) \subset \nabla_n, \) and \( n = 1, 2, \ldots : \)

\[ \langle \psi_{1n}, \psi_{1n} \rangle + \langle \nabla \psi_{1n}, \nabla \psi_{1n} \rangle + (\psi_{1n} - \psi_{2n} - \psi_{3n}, \psi_{1n} \rangle = (k_1(\psi_{1n}, \omega_1), \psi_{1n} \rangle, \] (27)
\[ \langle \psi_{1n}, \psi_{1n} \rangle = (\psi_{1n}, \psi_{1n} \rangle, \] (28)
\[ \langle \psi_{2n}, \psi_{2n} \rangle + \langle \nabla \psi_{2n}, \nabla \psi_{2n} \rangle + (\psi_{2n} + \psi_{3n} + \psi_{1n}, \psi_{2n} \rangle = (k_2(\psi_{2n}, \omega_2), \psi_{2n} \rangle, \] (29)
\[ \langle \psi_{2n}, \psi_{2n} \rangle = (\psi_{2n}, \psi_{2n} \rangle, \] (30)
\[ \langle \psi_{3n}, \psi_{3n} \rangle + \langle \nabla \psi_{3n}, \nabla \psi_{3n} \rangle + (\psi_{3n} + \psi_{1n} - \psi_{2n}, \psi_{3n} \rangle = (k_3(\psi_{3n}, \omega_3), \psi_{3n} \rangle, \] (31)
\[ \langle \psi_{3n}, \psi_{3n} \rangle = (\psi_{3n}, \psi_{3n} \rangle, \] (32)

which has a sequence of unique solution \( \{ \psi_n \} \). Substituting \( \psi_n = \psi_{1n} \) for \( i = 1, 2, 3 \) in (25)–(27), respectively, adding the three obtained equations together, and employing Lemma 1.2 in [18] for the first term of the LHS, to get (33) which is given by

\[ \frac{d}{dt} \left( \| \psi_n(t) \|_0^2 + \| \psi_n \|_1^2 \right) = 2 \left( \| \psi_{2n} \psi_{1n} \|_0 + \| \psi_{3n} \psi_{1n} \|_0 - \| \psi_{1n} \psi_{2n} \|_0 - \| \psi_{3n} \psi_{2n} \|_0 - \| \psi_{1n} \psi_{3n} \|_0 + \| \psi_{2n} \psi_{3n} \|_0 \right) + (k_1(\psi_{1n}, \omega_1), \psi_{1n} \rangle + (k_2(\psi_{2n}, \omega_2), \psi_{2n} \rangle + (k_3(\psi_{3n}, \omega_3), \psi_{3n} \rangle). \] (33)

Or (33) can be rewritten as (34) which is

\[ \frac{d}{dt} \left( \| \psi_n(t) \|_0^2 + \| \psi_n \|_1^2 \right) = 2 \left( \| \psi_{2n} \psi_{1n} \|_0 + \| \psi_{3n} \psi_{1n} \|_0 + \| \psi_{1n} \psi_{2n} \|_0 + \| \psi_{3n} \psi_{2n} \|_0 + \| \psi_{1n} \psi_{3n} \|_0 + \| \psi_{2n} \psi_{3n} \|_0 \right) + (k_1(\psi_{1n}, \omega_1), \psi_{1n} \rangle + (k_2(\psi_{2n}, \omega_2), \psi_{2n} \rangle + (k_3(\psi_{3n}, \omega_3), \psi_{3n} \rangle). \] (34)
Using Assumption A for the RHS of (34), integrating both sides on \([0, T]\), using \(\|\psi_n\|_0 \leq \|\psi_n\|_1 \leq \|\psi_n\|_1\) and \(\|\psi_{in}\|_0 \leq \|\psi_{in}\|_1\), we get

\[
\int_0^T \frac{d}{dt} \left[ \|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \right] dt \\
\leq \int_0^T \left( \|\psi_{in}(t)\|_0^2 + \|\psi_{in}(t)\|_1^2 \right) dt + \int_0^T \sum_{i=1}^3 \|F_i\|_0^2 dt \\
+ \beta_4 \int_0^T \left( \|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \right) dt + \int_0^T \|\psi_n(t)\|_1^2 dt \\
\leq \beta_4 \int_0^T \left( \|\psi_{in}(t)\|_0^2 + \|\psi_{in}(t)\|_1^2 \right) dt + \beta_6 \int_0^T \left( \|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \right) dt \\
\leq \beta_4 + \beta_6 \int_0^T \left( \|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \right) dt,
\]

where \(\beta_4 = \sum_{i=1}^3 \beta_i\), \(\beta_5 = 1 + \beta_4\), \(\beta_6 = 2 + \beta_4\), \(\beta_7 = \max(\beta_5, \beta_6), \beta_8 = \sum_{i=1}^3 \beta_i\).

Since \(\|\psi_{in}\|_0 \leq b_1\), and \(\|\psi_{n}^0\|_0 \leq b_0\), with \(\beta_4 = b_0 + b_1 + \beta_7\), then (35) is reduced to

\[
\|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \leq \beta_4 + \beta_6 \int_0^T \left( \|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \right) dt.
\]

(36)

Applying the Belman–Gronwall (BGIn) inequality, the abovementioned inequality gives \(\forall t \in [0, T] \) :

\[
\|\psi_n(t)\|_0^2 + \|\psi_n(t)\|_1^2 \leq \beta_4 e^{\beta_6 T} = b^2(c)
\]

\[
\Rightarrow \|\psi_n(t)\|_0^2 \leq b^2(c), \quad \forall t \in [0, T].
\]

Easily, one can obtain that \(\|\psi_n(t)\|_0 \leq b_1(c)\) and \(\|\psi_n(t)\|_1 \leq b_2(c)\).

Then, by applying Alaoglu’s theorem (Algth), \(\{\psi_n\}_{n \in N}\) has a subsequence; it is not loss of generality to say \(\{\psi_n\}_{n \in N}\) such that \(\psi_n \rightharpoonup \psi\) in \((L^2(\Omega))^3\) and \(\psi_n \rightarrow \psi\) in \((L^2(I,Y))^3\), and

\[
(L^2(\mathcal{R},Y))^3 \subset \left((L^2(\mathcal{R},E))^3\right) \subset \left(L^2(\mathcal{R},Y^*)ight)^3,
\]

(38)

Then, the Aubin compactness theorem [18] can be applied here to get that \(\psi_n \rightharpoonup \psi\) in \((L^2(\Omega))^3\). Now, multiplying both sides of (27) and (29), and (31) by \(\chi_i(t) \in C^3[0,T],\) such that \(\chi_i(T) = \chi_i(0) = 0, \chi_i(0) \neq 0, \chi_i^{(3)}(0) \neq 0, \forall i = 1,2\), integrating on, finally integrating by parts twice the 1\textsuperscript{st} term of each one of the obtained three equations, led to

\[
-\int_0^T \frac{d}{dt} \left( \psi_1 n \psi_1 n \chi_1 (t) \right) dt + \int_0^T \left[ (\psi_1 n \psi_1 n \chi_1 (t)) + ((\psi_1 n \psi_1 n) \beta_4) + (\psi_2 n \psi_1 n) \chi_1 (t) \right] dt
\]

(39)

\[
= \int_0^T \left( k_1 (\psi_1 n, \chi_1, \psi_1 n) \chi_1 (t) \right) dt + (\psi_1 n, \chi_1, \psi_1 n) \chi_1 (0),
\]

(40)

\[
\int_0^T \left( (\psi_1 n \psi_1 n) \chi_1 (t) \right) dt + \int_0^T \left[ (\psi_1 n \psi_1 n) \chi_1 (t) + ((\psi_1 n \psi_1 n) \beta_4) + (\psi_2 n \psi_1 n) \chi_1 (t) \right] dt
\]

(41)

\[
= \int_0^T \left( k_1 (\psi_1 n, \chi_1, \psi_1 n) \chi_1 (t) \right) dt + (\psi_1 n, \chi_1, \psi_1 n) \chi_1 (0) + \int_0^T \left( k_2 (\psi_1 n, \chi_1, \psi_1 n) \chi_1 (t) \right) dt + (\psi_2 n, \psi_1 n) \chi_1 (t) dt
\]

(42)

\[
\int_0^T \left( (\psi_2 n \psi_2 n) \chi_2 n (t) \right) dt + \int_0^T \left[ (\psi_2 n \psi_2 n) \chi_2 n (t) + ((\psi_2 n \psi_2 n) \beta_4) + (\psi_3 n \psi_2 n) \chi_2 n (t) \right] dt
\]

(43)
\[
\begin{align*}
\int_0^T (\psi_{3n}, v_{3n}) (t)dt + \int_0^T \left[ (\nabla \psi_{3n}, \nabla v_{3n})_X (t) + \left( (\psi_{3n}, v_{3n}) + (\psi_{1n}, v_{3n}) - (\psi_{2n}, v_{3n}) \right)_X (t)dt \right] \\
= \int_0^T (k_3 (\psi_{3n}, \omega_3), v_{3n})_X (t)dt + \left( (\psi_{3n}, v_{3n})_X (0) + (\psi_{3n}, v_{3n})_X (0) \right).
\end{align*}
\]

Now, for each \(i = 1, 2, 3\), we have the following convergences:

First, since
\[
\begin{align*}
v_{in} \to v_i \text{ in } Y \quad &\Rightarrow \quad \begin{cases} v_{in} \to v_i \text{ in } L^2 (I, Y) \\
v_{in} \to v_i \text{ in } L^2 (E) \end{cases} \\
v_{in} \to v_i \text{ in } L^2 (E) \quad &\Rightarrow \quad \begin{cases} v_{in} \to v_i \text{ in } L^2 (I, Y) \\
v_{in} \to v_i \text{ in } L^2 (E) \end{cases}
\end{align*}
\]

On the other hand, since
\[
\begin{align*}
v_{in} \to v_i \text{ in } L^2 (E) \quad &\Rightarrow \quad \begin{cases} v_{in} \to v_i \text{ in } L^2 (I, Y) \\
v_{in} \to v_i \text{ in } L^2 (E) \end{cases} \\
v_{in} \to v_i \text{ in } L^2 (E) \quad &\Rightarrow \quad \begin{cases} v_{in} \to v_i \text{ in } L^2 (I, Y) \\
v_{in} \to v_i \text{ in } L^2 (E) \end{cases}
\end{align*}
\]

Second, we have
\[
\begin{align*}
\psi_{in} \to \psi_i \quad &\Rightarrow \quad \begin{cases} \psi_{in} \to \psi_i \text{ in } L^2 (I, Y) \\
\psi_{in} \to \psi_i \text{ in } L^2 (E) \end{cases}
\end{align*}
\]

Third, let \(w_{in} = v_{in}^X_i\) and \(w_i = v_{X_i}\), then \(w_{in} \to w_i\) in \(L^2 (E)\) and \(w_{in}\) is measurable in \(E\), so from Assumption (A-1) and Proposition 1, the integral
\[
\int_0^T k_i (x, t, \psi_{in}, \omega_i) w_{in} dx dt \text{ is cont. with respect to } \psi_{in}, \omega_i, w_{in}, \text{ then}
\]
\[
\int_0^T (k_i (\psi_i, \omega_i), v_{im})_X (t)dt \to \int_0^T (k_i (\psi_i, \omega_i), v_{i})_X (t)dt.
\]

Now, from these convergences and (24) and (25), for \((i = 1)\), we can pass to the limits in (39) and (40) to get

\[
\begin{align*}
\int_0^T (\psi_{1t}, v_1)_X (t)dt + \int_0^T (\nabla \psi_1, \nabla v_1) + (\psi_1 - \psi_2 - \psi_3, v_1)X_1 (t)dt \\
= \int_0^T (k_1 (\psi_1, \omega_1), v_1)_X (t)dt + (\psi_1, v_1)_X (0) \\
= \int_0^T (k_1 (\psi_1, \omega_1), v_1)_X (t)dt + (\psi_1, v_1)_X (0) + (\psi_0, v_1)_X (0).
\end{align*}
\]

Case 1. Choose \(\chi_i \in C^2 [0, T]\), s.t. \(\chi_i (0) = \chi_i (T) = \chi_i'' (T) = 0\). Using these values in (47) for \((i = 1)\), using integration by parts twice for the first terms in the LHS of the obtained equation, yields to

\[
\int_0^T (\psi_{1tt}, v_1)_X (t)dt + \int_0^T (\nabla \psi_1, \nabla v_1) + (\psi_1 - \psi_2 - \psi_3, v_1))X_1 (t)dt \\
= \int_0^T (k_1 (\psi_1, \omega_1), v_1)_X (t)dt,
\]

which give that \(\psi_1\) is a solution of (10) (a.e. on \(I\)).

Similarly can be used for \((i = 2, 3)\), with (41)–(44) respectively to get that \(\psi_2\) and \(\psi_3\) are solutions of (12) and (14) respectively (a.e. on \(l\)).

Case 2. Choose \(\chi_i \in C^2 [0, T]\), such that \(\chi_i (0) \neq 0\) and \(\chi_i (T) \neq 0\). For \((i = 1)\), integrating both sides of (10) on \([0, T]\) after multiplying it by \(\chi_i (t)\), using integrating by parts for the first term in the LHS of the obtained equation, then subtracting the obtained equation from (41), we get
\[
(\psi_1, v_1)_X (0) = (\psi_0, v_1)_X (0).
\]

Also, similar way can be used but for \((i = 2, 3)\) with the pairs (12) and (47) and (14) and (48), respectively, to get the same result.

Case 3. Choose \(\chi_i \in C^2 [0, T]\), such that \(\chi_i (0) = \chi_i (T) = \chi_i'' (T) = 0, \chi_i (0) \neq 0\). For \((i = 1)\), integrating both sides of (10) on \([0, T]\) after multiplying it by \(\chi_i (t)\), using integrating by parts for the first term in the LHS of the obtained equation, then subtracting this obtained equation from (47), we get
\[
(\psi_0, v_1)_X (0) = (\psi_0, v_1)_X (0), \text{ also, for } i = 2, 3 \text{ and by using (12) and (14), we can use a similar way to get the same result.}
\]

From the last two cases, easily we can get the ICs (11) and (13), and (15).

Uniqueness of the solution: let \(\psi = (\psi_1, \psi_2, \psi_3)\) and \(\overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3)\) be two solutions of the wkf (10)–(15), i.e., \(\psi_i, \overline{\psi}_i, \text{ for each } i = 1, 2, 3\) are satisfied the wkf (10)–(15), subtracting each equality from the other and letting \(v_i = \psi_i - \overline{\psi}_i\), yields to
\[ \langle (\psi_i - \psi_j)_{t=0}, \psi_i - \psi_j \rangle + \|\psi_i - \psi_j\|^2_1 = (k_i (\psi_i, \omega_i) - k_i (\psi_j, \omega_j), \psi_i - \psi_j), \]
\[ (\langle \psi_i - \psi_j \rangle(0), \psi_i - \psi_j(0) \rangle = 0, \]
\[ ((\psi_i - \psi_j)(0), (\psi_i - \psi_j)(0)) = 0. \]  

Adding these three equations, using Lemma 1.2 in ref. [18] on the first term in LHS of the obtained equation which will be positive, integrating both sides from 0 to \( t \), using the initial conditions, the Lipschitz property on the RHS, and lastly applying the B–G inequality, to get

\[ \int_0^t \left[ \frac{d}{dt} (\psi - \psi_j)(t)\right]^2 + 2\|\psi - \psi_j\|^2_1 \right] dt \leq 2L \int_0^t \left[ \|\psi_i - \psi_j\|^2_1 + \|\psi - \psi_j\|^2_1 \right] dt \]
\[ \|\psi_i - \psi_j\|^2_1 + \|\psi - \psi_j\|^2_1 \leq \|\psi - \psi_j\|^2_1 = 0, \quad \forall t \in I \Rightarrow \]
\[ \|\psi_i - \psi_j\|^2(t)_{L^2(I,Y)} = 0 \Rightarrow \quad \text{the solution is unique.} \]

**Lemma 1.** In addition to Assumption A, if the functions \( k_i \) (for each \( i = 1, 2, 3 \)) is Lip. with respect to \( y_i \) and \( \omega_i \), and if the control vector is bounded, then the operator \( \omega \rightarrow \psi \) from \((L^2(\Pi))^3 \) into \((L^\infty(I,L^2(E)))^3 \) or in to \((L^2(I,Y)) \) or in to \((L^2(\Pi))^3 \) is cont.

**Proof.** Let \( \overrightarrow{\omega} = (\omega_1, \omega_2, \omega_3), \overrightarrow{\psi} = (\psi_1, \psi_2, \psi_3) \) are their corresponding states’ solutions which satisfy the wkf of (10)–(15), setting \( \Delta \psi = (\Delta \psi_1, \Delta \psi_2, \Delta \psi_3) = (\psi - \psi_j), \) then

\[ \langle \Delta \psi_{1, t}, v_1 \rangle + (\nabla \Delta \psi, \nabla v_1) + (\Delta \psi_1 - \Delta \psi_2 - \Delta \psi_3, v_1) = (k_1 (\psi_1 + \Delta \psi_1, \omega_1 + \Delta \omega_1) - k_1 (\psi_1, \omega_1), v_1), \]
\[ \Delta \psi_1 (x, 0) = 0, \]
\[ \Delta \psi_{1, t} (x, 0) = 0. \]  

Substituting \( v_i = \Delta \psi_i \) for each \( i = 1, 2, 3 \) in (51), (53), and (55), respectively, adding the obtained three equations together, and using the same way that we used to get (32), a similar equation can be obtained but with \( \Delta \psi \) in state of \( \psi \) and then integration of both sides on \([0, t]\), using the Lip. property on \( k_i, i = 1, 2, 3, \) with respect to each dependent variable, yields

\[ \int_0^t \frac{d}{dt} \left[ \|\Delta \psi_i(t)\|^2 + \|\Delta \psi_j\|^2 \right] dt \leq 2 \int_0^t \left[ \|\Delta \psi_2\| + \|\Delta \psi_3\| \right] \|\Delta \psi_{1, t}\| + \left[ \|\Delta \psi_1\| + \|\Delta \psi_3\| \right] \|\Delta \psi_{2, t}\| dt \]
\[ + 2 \int_0^t \left[ \|\Delta \psi_{1, t}\| + \|\Delta \psi_3\| \right] \|\Delta \psi_{3, t}\| dt + \left( T_2 \|\Delta \psi_1\| + T_2 \|\Delta \psi_3\| \right) \|\Delta \psi_{3, t}\| dt. \]
Using the definitions of the norms and the relations between them, we get

\[
\|\Delta \psi(t)\|_2^2 + \|\Delta \psi\|_1^2 \leq 2 \int_0^T \left( \|\Delta \psi_0\|_2^2 + \|\Delta \psi_1\|_1^2 \right) dt + \bar{L}_1 \int_0^T \left( \|\Delta \psi_0\|_2^2 + \|\Delta \psi_1\|_1^2 \right) dt \\
+ T^2 \int_0^T \|\Delta \omega\| dt + T^2 \int_0^T \|\Delta \psi\|_1^2 dt \\
\leq T^2 \|\Delta \omega(t)\|_1 + L_1 \int_0^T \left( \|\Delta \psi_0\|_2^2 + \|\Delta \psi_1\|_1^2 \right) dt,
\]

where \( \bar{L}_1 = \max(T_1, T_2, T_3) \), \( T^2 = \max(T_1, T_2, T_3) \), and \( L_1 = \max(2 + \bar{L}_1, 2 + \bar{L}_1 + \bar{T}^2) \).

Applying the BGin, with \( L^2 = T^2 e^t \), we get

\[
\|\Delta \psi(t)\|_2^2 + \|\Delta \psi\|_1^2 \leq L^2 \|\Delta \omega(t)\|_1^2, \quad \forall t \in T \Rightarrow \\
\|\Delta \psi(t)\|_2^2 \leq L^2 \|\Delta \omega(t)\|_1^2, \quad \forall t \in T \Rightarrow \\
\|\Delta \psi\|_{L^2([t, t+1]; R^2)} \leq L \|\Delta \omega\|_1, \\
\|\Delta \psi\|_{L^2([t, t+1]; R^2)} \leq L \|\Delta \omega\|_1 \text{ and } \|\Delta \psi\|_1 \leq L \|\Delta \omega\|_1.
\]

Hence, the following assumption and lemmas will be needed.

**Assumption B.** Consider \( m_{1r} \) (for \( r = 0, \ldots, q \) and \( i = 1, 2, 3 \)) is of Caraty. on \( \Pi \times (\mathcal{R} \times \mathbb{C}) \) and satisfies \( \psi_i \in \mathcal{R} \) and \( \omega_i \in \mathbb{C} \):

\[
|m_{1r}(x, t, \psi_i, \omega_i)| \leq M_{1r}(x, t) + c_{i3}|\psi_i|^2, \quad \text{where } M_{1r} \in L^1(\Pi), \forall i = 1, 2, 3, \forall r = 0, \ldots, q.
\]

In addition \( m_{1r} \) is independent of \( \omega_i \) (\( i = 1, 2, 3 \), and \( r = 1, \ldots, p \)), \( m_{1r}(x, i = 1, 2, 3, \text{ and } r = 1, \ldots, p) \) is cox. with respect to \( \omega_i \) for fixed \( x, t, \psi_i \), there exists a CCOPCV.

**Proof.** From the Assumption on \( \mathbb{C} \subset \mathcal{R} \forall i = 1, 2, 3 \) and Egorov's theorem, one obtains that \( \bar{W}_1 \times \bar{W}_2 \times \bar{W}_3 = \bar{W} \) is weakly com. Since \( \bar{W}_A \neq \emptyset \), hence there is \( \bar{\omega} \in \bar{W}_A \) s.t. \( r(\bar{\omega}) = 0, 1 \leq r \leq p, M_{k} (\bar{\omega}) \leq 0 \), for \( p + 1 \leq r \leq q \) and there is a minimizing sequence \( \{\bar{\omega}_p\} \) s.t. \( \bar{\omega}_p \in \bar{W}_A, \forall p \), which satisfies \( p \lim_{\rho \to \infty} M_{k} (\bar{\omega}_p) = \bar{\omega} \in \bar{W}_A M_{k}(\bar{\omega}) \). Since \( \bar{\omega}_p \in \bar{W}_A, \forall p \) and \( \bar{W} \) is weakly com., then \( \{\bar{\omega}_p\} \) has a subsequence say again \( \{\bar{\omega}_p\} \) which converges weakly to some \( \bar{\omega} \in \bar{W} \), i.e., \( \bar{\omega}_p \to \omega \) in \( (L^2(\Pi))^3 \) and \( \|\bar{\omega}_p\|_1 \leq c, \forall p \). From Theorem 1, for any given control \( \bar{\omega}_p \), then \( \bar{\psi}_p = \bar{\psi}_{\bar{\omega}_p} \) is a unique solution for the Tsteqs, and \( \|\bar{\psi}_p\|_{L^2(\Pi)} \).
Then, by applying the Aubin compactness theorem [18], the sequence \( \{ \overrightarrow{\psi}_p \} \) has a strongly converging subsequence, say for simplicity, \( \{ \overrightarrow{\psi}_p \} \) such that \( \overrightarrow{\psi}_p \rightharpoonup \overrightarrow{\psi} \) in \( (L^2(\Pi))^3 \).

Now, for each \( \rho \), substitute the solution \( (\psi_{1\rho}, \psi_{2\rho}, \psi_{3\rho}) \) in the wkf of (18), (20), and (22), then multiply both sides of each one by \( \chi_i(t) \) (with \( \chi_i \in C^2[0,T] \)), such that \( \chi_i(T) = \chi_i'(T) = 0 \), \( \chi_i(0) \neq 0 \), \( \chi_i'(0) \neq 0 \), for \( i = 1, 2, 3 \). After rewriting the first terms in the LHS in each one of them, integrating both sides on \([0,T]\), and then by applying integration by parts for these first terms, we get

\[
\int_0^T \frac{d}{dt} (\psi_{1\rho}, v_1) \cdot \chi_1(t) dt + \int_0^T \left[ (\nabla \psi_{1\rho}, \nabla v_1) + (\psi_{1\rho}, v_1) - (\psi_{2\rho}, v_1) - (\psi_{3\rho}, v_1) \right] \chi_1(t) dt \\
= \int_0^T \left( k_{11}(x,t,\psi_{1\rho}) + k_{12}(x,t)\omega_{1\rho}, v_1 \right) \chi_1(t) dt, \\
\int_0^T \frac{d}{dt} (\psi_{2\rho}, v_2) \cdot \chi_2(t) dt + \int_0^T \left[ (\nabla \psi_{2\rho}, \nabla v_2) + (\psi_{2\rho}, v_2) + (\psi_{4\rho}, v_2) + (\psi_{5\rho}, v_2) \right] \chi_2(t) dt \\
= \int_0^T \left( k_{21}(x,t,\psi_{2\rho}) + k_{22}(x,t)\omega_{2\rho}, v_2 \right) \chi_2(t) dt, \\
\int_0^T \frac{d}{dt} (\psi_{3\rho}, v_3) \cdot \chi_3(t) dt + \int_0^T \left[ (\nabla \psi_{3\rho}, \nabla v_3) + (\psi_{3\rho}, v_3) + (\psi_{4\rho}, v_3) - (\psi_{2\rho}, v_3) \right] \chi_3(t) dt \\
= \int_0^T \left( k_{31}(x,t,\psi_{3\rho}) + k_{32}(x,t)\omega_{3\rho}, v_3 \right) \chi_3(t) dt.
\]

One can pass the limits in the LHS of (50)–(52) by applying the same manner which is applied in the proof of Theorem 1 to pass the limits in RHS of these equations; we suppose (\( \forall i = 1, 2, 3 \)), \( v_i \in C[\overline{\Omega}] \), \( \omega_i = v_i \chi_i(t) \), then \( w_i \in C[\overline{\Pi}] \subseteq L^\infty(\overline{I}, V) \subseteq L^2(\Pi) \), set \( \overrightarrow{k_{1i}}(\psi_{ip}) = k_{1i}(\psi_{ip})w_i \), then \( \overrightarrow{k_{1i}}: \overline{\Pi} \times \mathbb{R} \rightarrow \mathbb{R} \) is of Carat., using Proposition 1, to get the integral \( \int_\Pi \overrightarrow{k_{1i}}(\psi_{ip})w_idx dt \) is cont. with respect to \( \psi_{ip} \), but \( \psi_{ip} \longrightarrow \psi_i \) in \( L^2(\Pi) \) and \( \omega_i \longrightarrow \omega_i \) in \( L^2(\Pi) \), then

\[
\int_\Pi k_{1i}(\psi_{ip})w_idx dt \longrightarrow \int_\Pi k_{1i}(\psi_i)w_idx dt, \quad \forall w_i \in C[\overline{\Pi}] \text{ for } i = 1, 2, 3.
\]

\[
\int_\Pi k_{1i}(x,t)\omega_ip_idx dt \longrightarrow \int_\Pi k_{1i}(x,t)\omega_ip_idx dt, \quad \forall w_i \in C[\overline{\Pi}] \text{ for } i = 1, 2, 3.
\]

Thus, (66) and (67) are hold for every \( v_i \in Y \), since \( C(\overline{\Pi}) \) is dense in \( Y \); hence, we get the wkf (10), (12), (14). Also, the same manner which is applied in the proof of Theorem 2 can be used here to pass the limits in the ICs. Hence, \( (\psi_1, \psi_2, \psi_3) \) is a solution of the wkf of (10)–(15).

On the other hand, since \( m_{ri}(\cdot, \cdot, \psi_{ip}) \) for \( i = 1, 2, 3 \) and \( r = 1, 2, \ldots, p \) is independent of \( \omega \) and cont. with respect to \( \psi_{ip} \), by Lemma 2, \( \int_\Pi m_{ri}(x,t,\psi_{ip})dx dt \) is cont. with respect to \( \psi_{ip} \), \( \psi_{ip} \longrightarrow \psi_i \in (L^2(\Pi))^3 \), then

\[
\int_\Pi m_{ri}(x,t,\psi_{ip})dx dt \longrightarrow \int_\Pi m_{ri}(x,t,\psi_{ip})dx dt.
\]

Hence, \( M_r(\overrightarrow{\omega}) = \rho \lim_{\rho \to 0} \co M_r(\overrightarrow{\omega}_p) = 0. \)
then $M_r(\overrightarrow{\omega}) \leq 0$ (r = p + 1, ..., q) since $\overrightarrow{\omega}_p \in \overrightarrow{W}_A$; we get that
\[
M_0(\overrightarrow{\omega}) \leq \rho \lim_{\rho \to \infty} \inf M_0(\overrightarrow{\omega}_p) = \lim_{\rho \to \infty} M_0(\overrightarrow{\omega}_p) = \overrightarrow{\omega} \in \overrightarrow{W}_A M_0(\overrightarrow{\omega}) \tag{71}
\]
Thus, $\overrightarrow{\omega}$ is a CCOPCV.

Assumption C. Assume for r = 0, ..., q and i = 1, 2, 3, the functions $k_i$, $k_{i\psi}$, $k_{i\omega}$, $m_{i\psi}$, and $m_{i\omega}$ are defined and are of

\[
\xi_{1st} - \Delta \xi_1 + \xi_1 + \xi_2 + \xi_3 = \xi_1 k_{1\psi}, (x, t, \psi, \omega_1) + m_{1\psi}, (x, t, \psi, \omega_1), \quad \text{on } \Pi,
\]
\[
\xi_1 = 0, \quad \text{on } \Sigma,
\]
\[
\xi_{2st} - \Delta \xi_2 + \xi_2 - \xi_1 - \xi_3 = \xi_2 k_{2\psi}, (x, t, \psi, \omega_2) + m_{2\psi}, (x, t, \psi, \omega_2), \quad \text{on } \Pi,
\]
\[
\xi_2 = 0, \quad \text{on } \Sigma,
\]
\[
\xi_{3st} - \Delta \xi_3 + \xi_3 - \xi_1 + \xi_2 = \xi_3 k_{3\psi}, (x, t, \psi, \omega_3) + m_{3\psi}, (x, t, \psi, \omega_3), \quad \text{on } \Pi,
\]
\[
\xi_3 = 0, \quad \text{on } \Sigma
\]
and the Ham is given by $\mathcal{H}(x, t, \psi, \overrightarrow{\omega}, \overrightarrow{\xi}) = \sum_{i=1}^{3} \xi_i (x, t, \psi, \omega_i)$, where $\xi_i = \xi_i k_i (x, t, \psi, \omega_i) + m_i (x, t, \psi, \omega_i)$, for each i = 1, 2, 3.

Then, the Fré. of $G$ is defined by
\[
M'(\overrightarrow{\omega}) \Delta \overrightarrow{\omega} = \int_{\Pi} \mathcal{H}(x, t, \psi, \overrightarrow{\omega}, \overrightarrow{\xi}) \Delta \overrightarrow{\omega} \, dx \, dt, \tag{74}
\]

### Proof.
At first, let the wkf of the TAEqs are given $\forall v_1, v_2, v_3 \in Y$, by
\[
\langle \xi_{1st}, v_1 \rangle + (\nabla \xi_1, \nabla v_1) + (\xi_1 + \xi_2 + \xi_3, v_1) = (\xi_1 k_{1\psi} + m_{1\psi}, v_1), \quad \forall v_1 \in Y \text{ a.e. on } I,
\]
\[
(\xi_1(T), v_1) = (\xi_{1T}(T), v_1) = 0, \tag{75}
\]
\[
\langle \xi_{2st}, v_2 \rangle + (\nabla \xi_2, \nabla v_2) + (\xi_2 - \xi_1 - \xi_3, v_2) = (\xi_2 k_{2\psi} + m_{2\psi}, v_2), \quad \forall v_2 \in Y \text{ a.e. on } I,
\]
\[
(\xi_2(T), v_2) = (\xi_{2T}(T), v_2) = 0 \tag{76}
\]
\[
\langle \xi_{3st}, v_3 \rangle + (\nabla \xi_3, \nabla v_3) + (\xi_3 - \xi_1 + \xi_2, v_3) = (\xi_3 k_{3\psi} + m_{3\psi}, v_3), \quad \forall v_2 \in Y \text{ a.e. on } I,
\]
\[
(\xi_3(T), v_3) = (\xi_{3T}(T), v_3) = 0 \tag{77}
\]
From the assumptions and using the same manner which is applied in the proof of Theorem 1, once can prove that the wksf (75)–(80) has a unique solution
\[ \overrightarrow{\xi} = (\xi_1, \xi_2, \xi_3) \in (L^2(\Pi))^3. \]

Substituting \( u_i = \Delta \psi_i \) for each \( i = 1, 2, 3 \) in (75), (77), and (79) and integrating both sides on \([0, T]\), we get
\[
\int_0^T \left[ \langle \Delta \psi_1, \xi_{1t} \rangle + (\nabla \psi_1, \nabla \psi_1) + (\xi_1 + \xi_3 + \xi_2, \Delta \psi_1) \right] dt = \int_0^T \left( \xi_1 k_{1\psi_1} + g_{1\psi_1}, \Delta \psi_1 \right) dt, \tag{81}
\]
\[
\int_0^T \left[ \langle \Delta \psi_2, \xi_{2t} \rangle + (\nabla \psi_2, \nabla \psi_2) + (\xi_2 - \xi_1 - \xi_3, \Delta \psi_2) \right] dt = \int_0^T \left( \xi_2 k_{2\psi_2} + g_{2\psi_2}, \Delta \psi_2 \right) dt, \tag{82}
\]
\[
\int_0^T \left[ \langle \Delta \psi_3, \xi_{3t} \rangle + (\nabla \psi_3, \nabla \psi_3) + (\xi_3 - \xi_1 + \xi_2, \Delta \psi_3) \right] dt = \int_0^T \left( \xi_3 k_{3\psi_3} + g_{3\psi_3}, \Delta \psi_3 \right) dt. \tag{83}
\]

Now, let \( \overrightarrow{\omega}, \delta \overrightarrow{\omega} \in (L^2(\Pi))^3 \), \( \Delta \overrightarrow{\omega} = \overrightarrow{\omega} - \delta \overrightarrow{\omega} \in (L^2(\Pi))^3 \), and then by Theorem 1, \( \overrightarrow{\psi} = \overrightarrow{\omega} \rightarrow \overrightarrow{\psi} \) and \( \overrightarrow{\psi} = \overrightarrow{\omega} \rightarrow \overrightarrow{\psi} \) are their corresponding solutions. Let \( \Delta \overrightarrow{\psi} = (\Delta \psi_1, \Delta \psi_2, \Delta \psi_3) = \overrightarrow{\psi} - \overrightarrow{\psi}_0 \), substitute \( u_i = \xi_i \) for each \( i = 1, 2, 3 \) in (51), (53), and (55), integrate both sides on \([0, T]\), and then integrate by parts twice the first term in the LHS of each equation. Finding for each \( i = 1, 2, 3 \) the Frde of \( k_i \) in the RHS of each equation which are exist from the Assumption C, then by Lemma 1, and the inequality of Minkowski, one has
\[
\int_0^T \sum_{i=1}^{3} k_{i\omega_i} \Delta \omega_i, \xi_i \right] dt + O_4 \left( \Delta \overrightarrow{\omega} \right) = \int_0^T \sum_{i=1}^{3} (m_{i\psi_i} \Delta \psi_i) dt, \tag{84}
\]
where \( O_4 \left( \Delta \overrightarrow{\omega} \right) \rightarrow 0, \) as \( \Delta \overrightarrow{\omega} \rightarrow 0. \)

Now, by substituting (87) in (88), one has
\[
M \left( \overrightarrow{\omega} + \Delta \overrightarrow{\omega} \right) - M \left( \overrightarrow{\omega} \right) = \int_0^T \sum_{i=1}^{3} \xi_{i\omega_i} (x, t, \psi_i, \omega_i) \Delta \omega_i dx dt + O_4 \left( \Delta \overrightarrow{\omega} \right) \Delta \omega_{11}, \tag{89}
\]
where \( O_4 \left( \Delta \overrightarrow{\omega} \right) \rightarrow 0, \) as \( \Delta \overrightarrow{\omega} \rightarrow 0. \)

On the other hand, from the Assumption B on \( m_i \) (for \( i = 1, 2, 3 \)), the Frde definition, Lemma 1, and by applying the inequality of Minkowski, we obtain
\[
M \left( \overrightarrow{\omega} + \Delta \overrightarrow{\omega} \right) - M \left( \overrightarrow{\omega} \right) = \int_0^T \sum_{i=1}^{3} \left( m_{i\psi_i} \Delta \psi_i + m_{i\omega_i} \Delta \omega_i \right) dx dt + O_5 \left( \Delta \overrightarrow{\omega} \right) \Delta \omega_{11}. \tag{88}
\]

1.4. Necessary and Sufficient Conditions for Optimality.
This section deals with the theorems for the Necoop necessary under certain hypotheses which are proved as follows:
Theorem 4. Necoop (multipliers theorem):

(a) with Assumptions A, B, and C, if \( \overrightarrow{W} \) is cox. and the \( \overrightarrow{\omega} \in \overrightarrow{W}_A \) is optimal, then there are multipliers \( \kappa_r \in \mathbb{R}, r = 0, 1, \ldots, p, p + 1, \ldots, q \) with \( \kappa_r \geq 0 \), for

\[
\int_{\Pi} \mathcal{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\zeta}, \overrightarrow{\omega}) \Delta \overrightarrow{\omega} \, dx \, dt \geq 0, \quad \forall \overrightarrow{\omega} \in \overrightarrow{W}, \Delta \overrightarrow{\omega} = \overrightarrow{\omega} - \overrightarrow{\omega},
\]

(91)

\[
\kappa_rM_r(\overrightarrow{\omega}) = 0, \quad \text{for } r = p + 1, \ldots, q.
\]

(b) Inequality (91) is equivalent to the (weak) piecewise minimum principle

\[
\mathcal{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\zeta}, \overrightarrow{\omega}) \cdot \overrightarrow{\omega}(t) = \min_{\overrightarrow{\omega} \to \overrightarrow{\zeta}} \mathcal{H}_{\overrightarrow{\omega}}(x, t, \overrightarrow{\psi}, \overrightarrow{\zeta}, \overrightarrow{\omega}) \cdot \overrightarrow{\omega}(t), \quad \text{a.e. on } \Pi.
\]

(93)

Proof. (a) From Lemma 2, the functional \( M_r(\overrightarrow{\omega}) \) (for \( r = 0, 1, \ldots, q \)) is cont. and from Theorem 3, the functional \( M'_r(\overrightarrow{\omega}) \) (for \( r = 0, 1, \ldots, q \)) is cont. with respect to \( \overrightarrow{\omega} - \overrightarrow{\omega} \) and linear in \( \overrightarrow{\omega} - \overrightarrow{\omega} \), then \( M'_r \) is L– differential for every \( L \) and then applying the K. T. L. theorem [5], there are multipliers \( \kappa_r \in \mathbb{R}, r = 0, 1, \ldots, q \) with \( \kappa_r \geq 0 \), for \( r = 0, p + 1, \ldots, q \), \( \sum_{r=0}^q |\kappa_r| = 1 \), such that (91)–(93) are satisfied, by using Theorem 3, then (91) becomes

\[
\int_{\Pi} \overrightarrow{\zeta} \cdot (\overrightarrow{\omega}_k - \overrightarrow{\omega}) \, dx \, dt \geq 0, \quad \forall \overrightarrow{\omega} \in \overrightarrow{W} \text{ with } \overrightarrow{\zeta} \geq 0.
\]

(94)

(b) Let \( \overrightarrow{\omega}_i \) be a dense sequence (dse) in \( \overrightarrow{W} \), \( m \) denotes the Lebesgue measure on \( \Pi \), and \( \Gamma \subset \Pi \) be a measurable subset with property

\[
\overrightarrow{\omega}(x, t) = \begin{cases} 
\overrightarrow{\omega}_i(x, t), \text{ if } (x, t) \in \Gamma \\
\overrightarrow{\omega}(x, t), \text{ if } (x, t) \notin \Gamma
\end{cases}
\]

Therefore, (94) becomes

\[
\int_{\Gamma} \overrightarrow{\zeta} \cdot (\overrightarrow{\omega}_i - \overrightarrow{\omega}) \, dx \, dt \geq 0, \text{ which implies to } \overrightarrow{\zeta} \cdot (\overrightarrow{\omega}_i - \overrightarrow{\omega}) \geq 0, \text{ a.e. on } \Pi.
\]

(95)

This means the inequality is satisfied on the whole region \( Q \) except in a subset \( \Pi_i \) such that \( m(\Pi_i) = 0, \forall i \), where \( m \) represents the Lebesgue measure; thus, the inequality holds on \( \Pi \) except in the union \( \cup_i \Pi_i \) with \( m(\cup_i \Pi_i) = 0 \), but \( \overrightarrow{\omega}_i \) is a dse in \( \overrightarrow{W} \), then there is \( \overrightarrow{\omega} \in \overrightarrow{W} \) such that

\[
\int_{\Pi} \overrightarrow{\zeta} \cdot (\overrightarrow{\omega}_i - \overrightarrow{\omega}) \geq 0, \text{ a.e. on } \Pi, \forall \overrightarrow{\omega} \in \overrightarrow{W}.
\]

That is, (91) gives (94). The converse is clear.

Theorem 5. Sucoop: besides the Assumptions A, B, and C, suppose \( \overrightarrow{W} \) is cox., with \( \overrightarrow{\zeta} \) cox., \( k_i \) and \( m_l \) (\( \forall r = 1, 2, \ldots, p \)) are affine with respect to \( (\psi_r, \omega) \) for each \( (x, t) \), \( M_r(\overrightarrow{\omega}) \) (\( \forall r = 0, p + 1, \ldots, q \)) are cox. with respect to \( (\psi_r, \omega) \) \( \forall (x, t) \), for \( i = 1, 2, 3 \). Then, the Necoop for Theorem 4 with \( \kappa_0 > 0 \) are sufficient.

Proof. Assume \( \overrightarrow{\omega} \in \overrightarrow{W}_A \) satisfies the condition (91) and (92). Let \( M(\overrightarrow{\omega}) = \sum_{r=0}^q \kappa_rM_r(\overrightarrow{\omega}) \), then using Theorem 3, we get

\[
M'_{\overrightarrow{\omega}}(\overrightarrow{\omega}) \Delta \overrightarrow{\omega} = \sum_{r=0}^q \kappa_rM'_r(\overrightarrow{\omega}) \Delta \overrightarrow{\omega} = \sum_{r=0}^q \kappa_r \sum_{i=1}^3 \xi_{\omega, i} \cdot \delta \omega, \text{dx dt} \geq 0.
\]

(96)

Since

\[
k_i(x, t, \psi_r, \omega) = k_{i1}(x, t)\psi_1 + k_{i2}(x, t)\omega_1 + k_{i3}(x, t)\omega_3, i = 1, 2, 3,
\]

Let \( \omega = (\omega_1, \omega_2, \omega_3) \) and \( \overrightarrow{\omega} = (\overrightarrow{\omega}_1, \overrightarrow{\omega}_2, \overrightarrow{\omega}_3) \) be two given control vectors, then \( \overrightarrow{\psi} = (\psi_{\omega_1}, \psi_{\omega_2}, \psi_{\omega_3}) = (\psi_1, \psi_2, \psi_3) \) and \( \overrightarrow{\psi} = (\overrightarrow{\psi}_{\omega_1}, \overrightarrow{\psi}_{\omega_2}, \overrightarrow{\psi}_{\omega_3}) = (\overrightarrow{\psi}_1, \overrightarrow{\psi}_2, \overrightarrow{\psi}_3) \) represent the corresponding state solutions. Substituting \( (\overrightarrow{\omega}, \overrightarrow{\psi}) \) in (1)–(6), multiplying all the obtained equalities by \( a \in [0, 1] \) once again, and lastly adding each pair from the corresponding equalities together, we obtain
\[
(\alpha \psi_1 + (1 - \alpha) \overline{\Psi}_1)_\omega - \Delta (\alpha \psi_1 + (1 - \alpha) \overline{\Psi}_1) + \alpha (\psi_1 - \psi_2 - \psi_3) + (1 - \alpha) (\overline{\Psi}_1 - \overline{\Psi}_2 - \overline{\Psi}_3) = 0,
\]

\[
(\alpha \psi_1 + (1 - \alpha) \overline{\Psi}_2)_\omega - \Delta (\alpha \psi_2 + (1 - \alpha) \overline{\Psi}_2) + \alpha (\psi_2 + \psi_3) + (1 - \alpha) (\overline{\Psi}_2 + \overline{\Psi}_1 - \overline{\Psi}_3) = 0,
\]

\[
(\alpha \psi_2 + (1 - \alpha) \overline{\Psi}_3)_\omega - \Delta (\alpha \psi_3 + (1 - \alpha) \overline{\Psi}_3) + \alpha (\psi_3 - \psi_2) + (1 - \alpha) (\overline{\Psi}_3 + \overline{\Psi}_2 - \overline{\Psi}_1) = 0.
\]

Equations (97)–(105) show that if the control vector is \( \overline{\omega} = (\overline{\omega}_1, \overline{\omega}_2, \overline{\omega}_3) \) with \( \overline{\omega} = \overline{\omega}_1 + (1 - \alpha) \overline{\omega}_2 \) then its corresponding state vector is \( \overline{\psi} = (\overline{\psi}_1, \overline{\psi}_2, \overline{\psi}_3) \) with \( \overline{\psi}_i = \psi_{i\omega_1 + (1 - \alpha)\overline{\omega}_1} + \alpha \psi_{i\overline{\omega}_1 + (1 - \alpha)\overline{\omega}_2}, \forall i = 1, 2, 3. \) This gives the operator \( \overline{\omega} \mapsto \overline{\psi} \) is convex–linear with respect to \( \overline{\psi} \) for any \( (\psi, \omega) \) with \( (r = 1, \ldots, p) \) with respect to \( (\psi, \omega) \) and \( \forall (x, t) \in \Pi \) is affine and the operator \( \overline{\omega} \mapsto \overline{\psi} \) is convex-linear.

The functions \( M_r(\overline{\omega}) \), \( \forall r = 0, 1, \ldots, q \) are cox. with respect to \( (\overline{\psi}, \overline{\omega}) \), \( \forall (x, t) \in \Pi \) (from the assumptions on the functions \( m_{r_1} \) and \( m_{r_2} \), \( \forall r = 0, 1, \ldots, q \)). Hence, \( M(\overline{\omega}) \) is cox. with respect to \( (\overline{\psi}, \overline{\omega}) \), \( \forall (x, t) \in \Pi \) in the cox. set \( \overline{\omega} \), and has a cont. Fréd satisfies \( M'(\overline{\omega}) \cdot (\overline{\omega} - \overline{\omega}') = 0 \Rightarrow M(\overline{\omega}) \) has a minimum at \( \overline{\omega} \Rightarrow M(\overline{\omega}) \leq M(\overline{\omega}) \), \( \forall \overline{\omega} \in \overline{\omega} \Rightarrow \overline{\omega} \Rightarrow \overline{\omega}.

Let \( \overline{\omega} \in \overline{\omega}_A \), with \( k_r \geq 0 \) (for \( r = p + 1, \ldots, q \)), then from (8), (9), and (77), the abovementioned inequality becomes

\[
\sum_{r=0}^{p} \kappa_r M_r(\overline{\omega}) + \sum_{r=p+1}^{q} \kappa_r M_r(\overline{\omega}) \leq \sum_{r=0}^{p} \kappa_r M_r(\overline{\omega}) + \sum_{r=p+1}^{q} \kappa_r M_r(\overline{\omega}), \quad \forall \overline{\omega} \in \overline{\omega}.
\]

2. Conclusions

The Gal'm with the Aubin compactness theorem are applied successfully to prove the existence of a unique “continuous state vector” solution for the TNLHPDEqs for a given cont. CCOPCVE. The existence theorem of governing by the considered the TNLHPDEqs with EQVC and INEQVC is proved. The existence of a unique solution of ATDEqs associated with the considered Tsteqs is studied. The Fréder. of the Ham is derived. The theorems of the
Necoop and the Sucoop of the constrained problem are proved.

Data Availability
No data were used to support this study.

Conflicts of Interest
The authors declare that they have no conflicts of interest.

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