Random-field Solutions to Linear Hyperbolic Stochastic Partial Differential Equations with Variable Coefficients

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Abstract

In this article we show the existence and uniqueness of a random-field solution to linear stochastic partial differential equations whose partial differential operator is hyperbolic and has variable coefficients that may depend on the temporal and spatial argument. The main tools for this, pseudo-differential and Fourier integral operators, come from microlocal analysis. The equations that we treat are second-order and higher-order strictly hyperbolic, and second-order weakly hyperbolic with uniformly bounded coefficients in space. For the latter one we show that a stronger assumption on the correlation measure of the random noise is necessary. Moreover, we show that the well-known case of the stochastic wave equation can be embedded into the theory presented in this article.

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Keywords: stochastic partial differential equations, stochastic wave equation, hyperbolic partial differential equations, fundamental solution, variable coefficients, Fourier integral operators

1 Introduction

In the recent years there has been a huge progress in the solution theory to stochastic partial differential equations (SPDEs). A linear SPDE is given by the following equation

\[ Lu(t, x) = \gamma(t, x) + \sigma(t, x) \dot{F}(t, x), \quad (1.1) \]
where \( L \) is a partial differential operator, \( \gamma, \sigma : \mathbb{R}^{1+d} \to \mathbb{R} \) are functions, subject to certain regularity conditions and \( F \) is a random noise term that will be described in detail in Section 2.1. Due to the singularity of the random noise, the sample paths are in most situations not in the domain of the operator \( L \).

One way to make sense of this equation in the case of constant coefficients is the following: we define the solution to (1.1) as a sum of a deterministic term accounting for the initial conditions, a stochastic and a deterministic convolution of the terms on the right-hand side with the fundamental solution to the differential operator \( L \):

\[
\begin{align*}
  u(t, x) &= I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\sigma(s, y)M(ds, dy) \\
  &\quad + \int_0^t \int_{\mathbb{R}^d} \Lambda(t-s, x-y)\gamma(s, y)dyds,
\end{align*}
\]

where \( \Lambda \) is the fundamental solution to the associated partial differential equation (briefly, PDE in the following) \( Lu = 0 \), \( I_0 \) is a term that accounts for the initial conditions and \( M \) is the martingale measure derived from the random noise \( \dot{F} \), see Section 2.1. Solutions of this type are called mild solutions and were introduced in [32] and later generalized in [10, 7]. Note that the solution \( u \) is defined as a random variable for each \((t, x) \in [0, T] \times \mathbb{R}^d\), where \( T > 0 \) is the time horizon of the equation. Due to that striking feature, we call these solutions random-field solutions in contrast to function-valued solutions, which cannot be evaluated in the spatial argument, but only as a Hilbert- or Banach-space valued random element in the temporal argument, see [12] for that theory.

Many interesting properties of random-field solutions for SPDEs have been studied for the case when the partial differential operator \( L \) has constant coefficients, e.g. the regularity of the probability measure induced by the solution [22, 20, 27], large deviation principles [21, 16], Varadhan estimates [17, 26], support theorems [18, 13], path properties such as Hölder continuity [25, 9] and much more. See also the references in these works for a more detailed account. Due to the restriction on constant coefficients, the set of concrete examples for random-field solutions to SPDEs is essentially limited to the stochastic heat equation and the stochastic wave equation. Note furthermore, that in [28] the existence of a random-field solution to a class of parabolic equations with variable coefficients has been shown.

With the present article, we aim to study the case of hyperbolic equations with variable coefficients which has, to our knowledge, not been considered yet, producing a random-field solution similar to (1.2), and so enlarging the set of examples for random-field solutions to SPDEs which one may treat explicitly.

Let us briefly explain the contents of this article. We consider linear SPDEs whose partial differential operators have variable coefficients that may depend on space and time. For these equations we want to derive conditions on the coefficients such that there exist unique random-field solutions. The equations that we will consider are general second- and higher-order hyperbolic equations on the whole space \( \mathbb{R}^d \). The main result of this paper, Theorem 3.1, shows that if the coefficients of the partial differential operator

\[
L = \partial^2_t - \sum_{j,k=1}^d a_{j,k}(t,x)\partial_{x_j} \partial_{x_k} - \sum_{j=1}^d b_j(t,x)\partial_{x_j} - c(t,x)
\]

where \( L \) is a partial differential operator, \( \gamma, \sigma : \mathbb{R}^{1+d} \to \mathbb{R} \) are functions, subject to certain regularity conditions and \( F \) is a random noise term that will be described in detail in Section 2.1. Due to the singularity of the random noise, the sample paths are in most situations not in the domain of the operator \( L \). One way to make sense of this equation in the case of constant coefficients is the following: we define the solution to (1.1) as a sum of a deterministic term accounting for the initial conditions, a stochastic and a deterministic convolution of the terms on the right-hand side with the fundamental solution to the differential operator \( L \):

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\end{align*}
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where \( \Lambda \) is the fundamental solution to the associated partial differential equation (briefly, PDE in the following) \( Lu = 0 \), \( I_0 \) is a term that accounts for the initial conditions and \( M \) is the martingale measure derived from the random noise \( \dot{F} \), see Section 2.1. Solutions of this type are called mild solutions and were introduced in [32] and later generalized in [10, 7]. Note that the solution \( u \) is defined as a random variable for each \((t, x) \in [0, T] \times \mathbb{R}^d\), where \( T > 0 \) is the time horizon of the equation. Due to that striking feature, we call these solutions random-field solutions in contrast to function-valued solutions, which cannot be evaluated in the spatial argument, but only as a Hilbert- or Banach-space valued random element in the temporal argument, see [12] for that theory.

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\]
are smooth and bounded with bounded derivatives of all orders in $x$ and continuous in $t$ ($a_{j,k}$ have to be differentiable in $t$), and the operator $L$ is of strictly hyperbolic type, i.e. it satisfies:

$$
\sum_{j,k=1}^{d} a_{j,k}(t,x) \xi_j \xi_k \geq C|\xi|^2,
$$

(1.4)

for all $t \in [0,T]$ and all $x, \xi \in \mathbb{R}^d$, then the SPDE (1.1) admits a unique random-field solution. The main tools for achieving this objective, e.g. pseudo-differential operators and Fourier integral operators, come from microlocal analysis. To our knowledge, this is the first time that their full potential is applied rigorously within the theory of random-field solutions to SPDEs. Note however the case of SPDEs with a pseudo-differential operator in the framework of function-valued solutions, see [30].

The paper is organized as follows.

In the first part of Section 2 we review the notions of stochastic integration with respect to martingale measures and random-field solutions to SPDEs. Since, in contrast to the classic references [32, 10], we do not assume the partial differential operator to have constant coefficients, its fundamental solution is no longer stationary in time and space, i.e. it cannot be written as $\Lambda(t-s,x-y)$ as in (1.2), but rather as $\Lambda(t,s,x,y)$. This small difference will have some consequences for the conditions for the existence and uniqueness of solutions to SPDEs with such partial differential operators. The second part of that section is devoted to a quick introduction to microlocal analysis, where we gather all the tools necessary for constructing the fundamental solution to a second-order and higher-order hyperbolic equation. At this point we have to note that the concept of fundamental solutions to PDEs, which is used in the framework of random-field solutions to SPDEs (in Section 2.1), is different from the one that is used in microlocal analysis (in Section 2.2). The main idea which will allow us to relate both concepts to each other is given in Remark 2.24: the fundamental solution of Section 2.1 is the Schwartz kernel of the fundamental solution of Section 2.2. The main result of Section 2 is Proposition 2.18, where we calculate the Fourier transform with respect to the second variable of the Schwartz kernel of the fundamental solution.

Section 3 is devoted to our main result: we show that the operator in (1.3) under condition (1.4) has a fundamental solution that satisfies all the necessary assumptions for the existence and uniqueness of a random-field solution for (1.1). For the proof, we follow the same ideas as in [2, 3, 4], where fundamental solutions to deterministic hyperbolic PDEs have been obtained. We reduce the second-order equation to a first-order system, for which one can compute the fundamental solution explicitly, see [15]. From the fundamental solution of the system we compute the fundamental solution to the second-order equation, and show that the conditions on the fundamental solution for the well-definedness of the stochastic and deterministic convolutions and therefore the existence and uniqueness of a random-field solution to the SPDE are fulfilled.

In the subsequent two sections 4 and 5 we deal with generalizations of the second-order hyperbolic equations treated in Section 3.

In Section 4 we relax the assumption of the strict hyperbolicity on the partial differential operator and provide an example of an operator of the form (1.3) which does not satisfy (1.4). We show that, in this case, the existence
and uniqueness conditions have to be strengthened in order to deal with this degeneracy.

Section 5 is devoted to strictly hyperbolic equations of higher order \( n \in \mathbb{N}, n \geq 2 \); we show that the coefficients have to satisfy similar conditions as in the case of second-order strictly hyperbolic equations in order to obtain the existence and uniqueness conditions of a random-field solution to the SPDE. However, we will show that we can allow for more general spectral measures. In fact, the proof of Theorem 5.1 points out that the set of spectral measures \( \mu \) (see Section 2.1) that can be used to produce a random field solution to higher order strictly hyperbolic SPDEs enlarges as the order \( n \) of the equation increases, see formula (5.16) and the discussion below it. The strategy to show both extensions is similar to the one used in Section 3: reducing the equation to a suitable first-order system and using the fundamental solution of the system to derive properties of the fundamental solution of the original equation.

At the end of this article, in Section 6, we present the example of the stochastic wave equation, showing that it can be fit into the framework of this paper and that its treatment is consistent with the approach in [10].

2 Preliminaries

In this section we gather the basic concepts and notations. Throughout this article, let for all \( \xi \in \mathbb{R}^d \), \( |\xi| := (\sum_{j=1}^d \xi_j^2)^{1/2} \) and \( \langle \xi \rangle := (1 + |\xi|^2)^{1/2} \). Let moreover \( \alpha \) denote a multiindex with the usual arithmetic operations. We will denote partial derivatives with \( \partial \). Moreover, we set \( D = -i\partial \), \( i \) the imaginary unit, for the sake of Fourier transform. We will denote by \( C^m(X) \), \( C^m_b(X) \), \( C^m_0(X) \), \( S(X) \), \( D(X) \), \( S'(X) \) and \( D'(X) \) the \( m \)-times continuously differentiable functions, the \( m \)-times continuously differentiable functions with uniformly bounded derivatives of all orders \( \leq m \), the \( m \)-times continuously differentiable functions with compact support, the Schwartz functions, the test functions, the tempered distributions and the distributions on some finite or infinite-dimensional space \( X \) respectively. We shall denote \( H^r(\mathbb{R}^d) \) the Sobolev space of order \( r \geq 0 \) on \( L^2(\mathbb{R}^d) \). We will use the notation \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \) and \( \mathbb{R}^d_* := \mathbb{R}^d \setminus \{0\} \). Let furthermore \( C > 0 \) be a generic constant, whose value can change from line to line without further notice.

2.1 Stochastic integration with respect to martingale measures and spatially non-homogeneous SPDE

In this section we introduce the framework to treat mild solutions to SPDEs similarly as in [12]. We explain how stochastic integration with respect to martingale measures is defined, collect some conditions on the integrands and provide a theorem for the existence and uniqueness of solutions to SPDEs in the case of variable coefficients. The main novelty of this section compared with [10, 7] is that we do not make the assumption of spatial homogeneity. The price we pay is that we cannot treat semilinear SPDEs, see the comment at the end of this subsection. So let us consider the following solution to the SPDE in (1.1)

\[
\begin{align*}
  u(t, x) = I_0(t, x) + \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y) \sigma(s, y) M(ds, dy)
\end{align*}
\]  

(2.1)
\[
\int_0^t \int_{\mathbb{R}^d} A(t, s, x, y) \gamma(s, y) dy ds,
\]
and provide conditions to make sense of each term.

Let in the following \( \{ F(\phi) ; \phi \in C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d) \} \) be a Gaussian process with mean zero and covariance functional given by

\[
E[F(\phi)F(\psi)] = \int_0^\infty \int_{\mathbb{R}^d} (\phi(t) \ast \tilde{\psi}(t))(x) \Gamma(dx) dt,
\]

where \( \tilde{\psi}(t, x) := \psi(t, -x) \) and \( \Gamma \) is a nonnegative, nonnegative definite, tempered measure on \( \mathbb{R}^d \). Then [20] Chapter VII, Théorème XVIII implies that there exists a nonnegative tempered measure \( \mu \) on \( \mathbb{R}^d \) such that \( F \mu = \Gamma \), where \( F \) denotes the Fourier transform given for functions \( f \in L^1(\mathbb{R}^d) \) by

\[
(Ff)(\xi) := \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx,
\]

where \( x \cdot \xi \) denotes the inner product in \( \mathbb{R}^d \). We can then extend the Fourier transform to tempered distributions \( T \in S'(\mathbb{R}^d) \) by the relation

\[
\langle FT, \phi \rangle = \langle T, F\phi \rangle,
\]

for all \( \phi \in S(\mathbb{R}^d) \).

As explained in [11], by approximating indicator functions with \( C_0^\infty \)-functions, the process \( F \) can be extended to a worthy martingale measure \( M = (M_t(A); t \in \mathbb{R}_+, A \in \mathcal{B}_b(\mathbb{R}^d)) \) where \( \mathcal{B}_b(\mathbb{R}^d) \) denotes the bounded Borel subsets of \( \mathbb{R}^d \). The natural filtration generated by this martingale measure will be denoted in the sequel by \( (\mathcal{F}_t)_{t \geq 0} \).

In the following we shall use [32, 10, 7] as reference for an integration theory with respect to the martingale measure constructed above. Fix \( T > 0 \). For stochastic processes \( f \) and \( g \), indexed by \( (t, x) \in [0, T] \times \mathbb{R}^d \) and satisfying suitable conditions, we define the inner product

\[
(f, g)_0 = \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} (f(s) \ast \tilde{g}(s))(x) \Gamma(dx) ds \right],
\]

where the corresponding norm \( \| \cdot \|_0 \) is defined in the usual way. Moreover, we define the norm

\[
\| f \|^2 := \mathbb{E} \left[ \int_0^T \int_{\mathbb{R}^d} |f(s)| \Gamma(dx) ds \right].
\]

Let \( \mathcal{E} \) denote the set of simple processes \( g \), that is, stochastic processes of the form

\[
g(t; x; \omega) = \sum_{j=1}^m 1_{(a_j, b_j]}(t) 1_{A_j}(x) X_j(\omega),
\]
for some \( m \in \mathbb{N} \), where \( 0 \leq a_j < b_j \leq T \), \( A_j \in \mathcal{B}(\mathbb{R}^d) \) and \( X_j \) is a bounded and \( \mathcal{F}_{A_j} \)-measurable random variable for all \( 1 \leq j \leq n \). The stochastic integral of \( g \) with respect to the martingale measure \( M \), denoted by \( g \cdot M \) is given by

\[
(g \cdot M)_t := \sum_{j=1}^{m} (M_{t \wedge b_j}(A_j) - M_{t \wedge a_j}(A_j))X_j,
\]

where \( x \land y := \min\{x, y\} \). One can show by applying the definition that

\[
\mathbb{E}\left[ (g \cdot M)^2_t \right] = \|g\|_0^2,
\]

for all \( g \in \mathcal{E} \). Following [10], we denote by \( P_0 \) the completion of \( \mathcal{E} \) with respect to \( \langle \cdot, \cdot \rangle_0 \). Then \( P_0 \) is a Hilbert space consisting of predictable processes which may contain tempered distributions in the \( x \)-argument (whose Fourier transform are functions, \( \mathbb{P} \)-almost surely). The norm in this space is given by the \( \| \cdot \|_0 \)-norm defined above in (2.6) and for sufficiently smooth elements of \( P_0 \), this norm can be also written as in (2.5). Note that \( P_0 \) is not defined as the set of predictable processes \( g \) for which \( \|g\|_0 < \infty \). In fact, it can be shown that the latter space is not complete. So we have that \( P_0 \) is the space of all integrable (with respect to \( M \)) processes and the stochastic integrals are defined as an \( L^2(\Omega) \)-limit of simple processes via the isometry (2.7). In [27, Lemma 2.2], it was shown that \( P_0 = L^2_{\mathbb{P}}([0, T] \times \Omega, \mathcal{H}) \), where here \( L^2_{\mathbb{P}}(\text{dots}) \) stands for the predictable stochastic processes in \( L^2(\ldots) \) and \( \mathcal{H} \) is the Hilbert space which is obtained from completing the Schwartz functions with respect to the inner product \( \langle \cdot, \cdot \rangle_0 \).

On the other hand, we define \( P_+ \) to be the set of all predictable processes for which \( \|g\|_+ < \infty \). Then \( P_+ \) is a Banach space and the simple processes are dense in this space, see [32, Proposition 2.3]. Note that since \( \| \cdot \|_0 \leq \| \cdot \|_+ \), we have \( P_+ \subset P_0 \), and this inclusion is strict.

Now we describe the way how to integrate time- and space-dependent integrands of a special form \( \Lambda \sigma \) into the SPDE (1.1). Here, \( \Lambda \) is the fundamental solution to the associated PDE and \( \sigma \) is the coefficient on the right-hand side of the SPDE depending on the time and space parameter. That means we want to make sense of the stochastic integral

\[
\int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\sigma(s, y)M(ds, dy).
\]

Note that in this integral and throughout this article we write \( \Lambda(t, s, x, y) \) although this object will be a distribution in the last argument. This abuse of notation is for the sake of briefness.

With the help of (2.7) we calculate the second moment of (2.8), from where we can deduce sufficient conditions for its existence. We have to distinguish between the case when the coefficient \( \sigma \) depends on the spatial argument \( y \) and the case when it does not. In [10, Example 9] it has already been hinted how to treat the latter case. Indeed, we have writing \( \cdot \) for the temporal and \( * \) for the spatial argument

\[
\|\Lambda(t, \cdot, x, \cdot)\sigma(\cdot)\|_0^2 = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\Lambda(t, s, x, y - z)\sigma(s)\sigma(s)dy\Gamma(dz)ds
\]
If however the coefficient $\sigma$ depends on the spatial argument, we need a more elaborate analysis. We compute using (2.7), Fubini’s theorem and a well-known equality for the Fourier transform of products

$$
\|\Lambda(t,\cdot,\cdot,\cdot)\sigma(\cdot,\cdot,\cdot)\|_{0}^{2} \\
= \int_{0}^{t} \int_{\mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \Lambda(t,t,x,y)\Lambda(t,x,y-z)\sigma(s,y)\sigma(s,y-z) dy \Gamma(dz) ds
$$

$$
= \int_{0}^{t} \int_{\mathbb{R}^{d}} \left( \int_{\mathbb{R}^{d}} F\Lambda(t,x)(\xi + \eta_1)F\sigma(s)(-\eta_1)d\eta_1 \right) \times \left( \int_{\mathbb{R}^{d}} F\Lambda(t,x)(\xi - \eta_2)F\sigma(s)(\eta_2)d\eta_2 \right) \mu(d\xi) ds
$$

$$
\leq \int_{0}^{t} \left( \sup_{\eta \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} |F\Lambda(t,x)(\xi + \eta)|^{2}\mu(d\xi) \right) \left( \int_{\mathbb{R}^{d}} |F\sigma(s)(\eta)|d\eta \right)^{2} ds. \quad (2.10)
$$

We see that $F\sigma(s)$ has to be in $L^{1}(\mathbb{R}^{d})$ so that the previous term is finite. We will also assume that $\sigma(s) \in L^{\infty}(\mathbb{R}^{d})$ for every $s \in [0,T]$ in order to simplify an argument later. In fact, this condition is not very restrictive, since it is already implied if we assume $\sigma(s) \in L^{1}(\mathbb{R}^{d})$ for every $s \in [0,T]$. In this case, we get, by using a well-known continuity property of the Fourier transform

$$
\|\sigma(s)\|_{L^{\infty}(\mathbb{R}^{d})} = \|F(F^{-1}\sigma(s))\|_{L^{\infty}(\mathbb{R}^{d})} \leq C_{a}\|F^{-1}\sigma(s)\|_{L^{1}(\mathbb{R}^{d})}
$$

$$
= C_{a}\|F\sigma(s)\|_{L^{1}(\mathbb{R}^{d})}. \quad (2.11)
$$

This implies that for each $s \in [0,T]$, $\sigma(s)$ has to be (essentially) bounded. Let in the following $\Delta_{T}$ be the simplex given by $0 \leq t \leq T$ and $0 < s < t$. In order for the existence of the stochastic integral, we need to assume the following.

**Assumption 2.1.** For $(t,s,x) \in \Delta_{T} \times \mathbb{R}^{d}$, let $\Lambda(t,s,x)$ be a deterministic function with values in $S'_{r}(\mathbb{R}^{d})$ (the set of tempered distributions with rapid decrease), and let $\sigma$ be a function in $L^{2}([0,T], L^{\infty}(\mathbb{R}^{d}))$ such that:

**(A1)** for all $(t,s,x), \xi \mapsto F\Lambda(t,s,x)(\xi)$ is a function, the mapping $(t,s,x,\xi) \mapsto F\Lambda(t,s,x)(\xi)$ is measurable, $s \mapsto F\sigma(s) \in L^{2}([0,T], L^{1}(\mathbb{R}^{d}))$, and moreover

$$
\int_{0}^{T} \left( \sup_{\eta \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} |F\Lambda(t,s,x)(\xi + \eta)|^{2}\mu(d\xi) \right) \left( \int_{\mathbb{R}^{d}} |F\sigma(s)(\eta)|d\eta \right)^{2} ds < \infty.
$$

(2.12)

**(A2)** $\Lambda$ and $\sigma$ are as in (A1) and

$$
\lim_{b \downarrow 0} \int_{0}^{T} \left( \sup_{\eta \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} \sup_{r \in (s,s+b)} |F(\Lambda(t,s,x) - \Lambda(t,r,x))(\xi + \eta)|^{2}\mu(d\xi) \right) \times \left( \int_{\mathbb{R}^{d}} |F\sigma(s)(\eta)|d\eta \right)^{2} ds = 0.
$$
In the case where the coefficient does not depend on the spatial argument, we rephrase the assumptions.

**Assumption 2.2.** For \((t, s, x) \in \Delta_T \times \mathbb{R}^d\), let \(\Lambda(t, s, x)\) be a deterministic function with values in \(S'(\mathbb{R}^d)\), and let \(\sigma \in L^2([0, T])\) such that:

\[(A1')\]

for all \((t, s, x)\), \(\xi \mapsto F \Lambda(t, s, x)(\xi)\) is a function, the mapping \((t, s, x, \xi) \mapsto F \Lambda(t, s, x)(\xi)\) is measurable, and moreover

\[
\int_0^T \sigma(s)^2 \int_{\mathbb{R}^d} |F \Lambda(t, s, x)(\xi)|^2 \mu(d\xi) ds < \infty.
\]

\[(A2')\]

\(\Lambda\) and \(\sigma\) are as in \((A1')\) and

\[
\lim_{h \downarrow 0} \int_0^T \sigma(s)^2 \left( \int_{\mathbb{R}^d} \sup_{r \in (s, s+h)} |F(\Lambda(t, s, x) - \Lambda(t, r, x))(\xi)|^2 \mu(d\xi) \right) ds = 0.
\]

The reason for the assumption that \(\Lambda(t) \in S'(\mathbb{R}^d)\) is that in this case the Fourier transform in the second spatial argument is a Schwartz function and the convolution of such a distribution with any other distribution is well-defined, see [29, Chapter VII, §5]. A necessary and sufficient condition for \(T \in S'(\mathbb{R}^d)\) is that each regularization of \(T\) with a \(C_0^\infty\)-function is a Schwartz function. This will be true in our case due to Proposition 2.18 and the fact that the Fourier transform is a bijection on the Schwartz functions, see Lemma 2.19.

In the spatially homogeneous case investigated in [10, 7], when \(\Lambda\) is the solution of the heat or the wave equation, [24, Lemma 6.1] shows that the corresponding condition (2.12) is equivalent to

\[
\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty.
\]

(2.13)

So one of our aims in the subsequent sections is to find a similar estimate for (2.12), which reads

\[
\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1 + |\xi + \eta|^2} \mu(d\xi) < \infty,
\]

(2.14)

for some \(\nu \in (0, 1]\). Note that if the correlation measure \(\Gamma\) is absolutely continuous, then condition (2.14) is equivalent to (2.13), see [23].

In contrast to the methods used in the proof of [24, Lemma 6.1], (2.14) will follow easily from a quick investigation of the order of the symbol associated to the fundamental solution using the tools presented in Section 2.2.

We can now prove, similarly to [4, Theorem 3.1] that under the two assumptions above, the stochastic integral is well-defined.

**Theorem 2.3.** Assume \((A1)\) and \((A2)\). Then \(\Lambda \sigma \in \mathcal{P}_0\). In particular, the stochastic integral \((\Lambda(\cdot, \cdot, \cdot) \sigma(\cdot, \cdot)) \cdot M\) is well-defined and

\[
\mathbb{E} \left[ \left( (\Lambda(\cdot, \cdot, \cdot) \sigma(\cdot, \cdot)) \cdot M \right)^2 \right]
\leq \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |F \Lambda(t, s, x)(\xi + \eta)|^2 \mu(d\xi) \right) \left( \int_{\mathbb{R}^d} |F \sigma(s)(\eta)| d\eta \right)^2 ds.
\]
Proof. Fix throughout this proof \( (t, x) \in [0, T] \times \mathbb{R}^d \) and \( s \in [0, t) \). Take \( \psi \in C^\infty_c(\mathbb{R}^d) \) such that \( \text{supp} \psi \subseteq B_d(0, 1) \) (the unit ball in \( \mathbb{R}^d \)). Then set for all \( n \in \mathbb{N} \),
\[
\psi_n(y) := n^d \psi(ny) \quad \text{and} \quad \Lambda_n(t, s, x) := \Lambda(t, s, x) * \psi_n.
\]
Then we have \( |\mathcal{F} \psi_n(\xi)| \leq 1 \), \( |\mathcal{F} \Lambda(\xi)| \to 1 \) pointwise, and \( |\mathcal{F} \Lambda_n(t, s, x)(\xi)| = |\mathcal{F} \Lambda(t, s, x)(\xi)| |\mathcal{F} \psi_n(\xi)| \). If we have that \( \Lambda_n(t, \cdot, x) \sigma(\cdot, s) \in \mathcal{P}_0 \) for all \( n \in \mathbb{N} \), then performing the same steps as in (2.11) yields
\[
\begin{align*}
&\| (\Lambda(t, \cdot, x, *) - \Lambda_n(t, \cdot, x, *)) \sigma(\cdot) \|_0^2 \\
&\quad \leq \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}(\Lambda(t, s, x) - \Lambda_n(t, s, x))(\xi + \eta)|^2 \mu(d\xi) \right) ds \\
&\quad \times \left( \int_{\mathbb{R}^d} |\mathcal{F} \sigma(s)(\eta)|^2 d\eta \right)^2 \\
&\quad = \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}(\Lambda(t, s, x)(\xi + \eta)^2 |1 - \mathcal{F} \psi_n(\xi + \eta)|^2 \mu(d\xi) \right) ds \\
&\quad \times \left( \int_{\mathbb{R}^d} |\mathcal{F} \sigma(s)(\eta)|^2 d\eta \right)^2 ds,
\end{align*}
\]
and \( |1 - \mathcal{F} \psi_n(\xi + \eta)|^2 \leq 4 \) and the bounded convergence theorem imply that the latter term goes to zero, which in turn implies the assertion. In order to show that \( \Lambda_n(t, \cdot, x, *) \sigma(\cdot, s) \in \mathcal{P}_0 \), we define
\[
\Lambda_{n,m}(t, s, x, y) := \sum_{j=0}^{2^m-1} \Lambda_n(t, t_{2^m}^j, x, y)1_{[t_{2^m}^j, t_{2^m}^{j+1})}(s),
\]
for all \( m \in \mathbb{N} \), where \( t_{2^m}^j = jT2^{-m} \). Then \( \Lambda_{n,m}(t, s, x, *) \in \mathcal{S}(\mathbb{R}^d) \) and
\[
\begin{align*}
&\| \Lambda_{n,m}(t, \cdot, x, *) \sigma(\cdot, s) \|_+^2 \\
&\quad = \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |\Lambda_{n,m}(t, s, x, y)||\Lambda_{n,m}(t, s, x, y - z)||\sigma(s, y)||\sigma(s, y - z)|dy \Gamma(dz)ds \\
&\quad \leq \int_0^t \| \sigma(s) \|_{L^\infty(\mathbb{R}^d)} \left( \int_{\mathbb{R}^d} |\Lambda_{n,m}(t, s, x, y)|dy \Gamma(dz) \right) ds.
\end{align*}
\]
This is the (only) place, where we have used that \( \sigma(s) \in L^\infty(\mathbb{R}^d) \) for almost all \( s \in [0, T) \). Now Leibniz’ formula [31] Exercise 26.4 implies that for each \( s \in [0, t) \), the term in the brackets in the last line of the previous inequality is finite. Moreover, since \( \Lambda_{n,m}(t, \cdot, x, *) \) was a step function in \( s \), it is also uniformly bounded and (2.11) together with the assumption on \( \sigma \) implies the finiteness of this term. Therefore \( \Lambda_{n,m} \sigma \in \mathcal{P}_+ \), which implies that there exists a sequence of step functions approximating this object.

The last step in this proof is to show that \( \Lambda_{n,m}(t, \cdot, x, *) \sigma(\cdot, s) \) converges to \( \Lambda_n(t, \cdot, x, *) \sigma(\cdot, s) \) in \( \mathcal{P}_0 \) for all \( (t, x) \in [0, T] \times \mathbb{R}^d \). We compute using (2.11)
\[
\begin{align*}
&\| (\Lambda_{n,m}(t, \cdot, x, *) - \Lambda_n(t, \cdot, x, *)) \sigma(\cdot, s) \|_0^2 \\
&\quad \leq \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}(\Lambda_n(t, s, x) - \Lambda_{n,m}(t, s, x))(\xi + \eta)|^2 \mu(d\xi) \right) ds \\
&\quad \times \left( \int_{\mathbb{R}^d} |\mathcal{F} \sigma(s)(\eta)|^2 d\eta \right)^2 ds.
\end{align*}
\]
\[
\leq \int_0^t \sup_{\eta \in \mathbb{R}^d} \sup_{r \in (s,s+T^2-m)} |\mathcal{F}(\Lambda_n(t,s,x) - \Lambda_n(t,r,x))(\xi + \eta)|^2 \mu(d\xi)
\times \left( \int_{\mathbb{R}^d} |\mathcal{F}\sigma(s)(\eta)| d\eta \right)^2 ds,
\]
which goes to zero by (A2) which ends the proof. \(\square\)

A similar, but much easier proof can be formulated if we assume (A1') and (A2').

Now we treat the pathwise integral in (2.1). Similar to the stochastic integral we first compute an estimate for its second moment from which we can deduce suitable sufficient conditions for its existence. Again we have to distinguish between the case when the coefficient \(b\) depends on the spatial argument and the case when it does not. We start with the former and compute

\[
\left( \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds \right)^2
\leq T \int_0^t \left( \mathcal{F}(\Lambda(t,s,x,\ast)\gamma(s, \ast))(0) \right)^2 ds
\leq C \int_0^t \left( \int_{\mathbb{R}^d} \mathcal{F}\Lambda(t,s,x,-\eta)\mathcal{F}\gamma(s)(\eta)d\eta \right)^2 ds
\leq C \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(t,s,x)(\eta)| \right) \left( \int_{\mathbb{R}^d} |\mathcal{F}\gamma(s)(\eta)| d\eta \right)^2 ds. \tag{2.15}
\]

The case where \(\gamma\) does not depend on the spatial argument can be treated as in (2.9) and we get

\[
\left( \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)\gamma(s,y)dyds \right)^2 \leq T \int_0^t \gamma(s)^2 \left( \mathcal{F}(\Lambda(t,s,x,\ast))(0) \right)^2 ds.
\]

In order to give a rigorous meaning to the pathwise integral, some additional assumptions are needed. These are the following.

**Assumption 2.4.** For \((t,s,x) \in \Delta_T \times \mathbb{R}^d\), let \(\Lambda(t,s,x)\) be a deterministic function with values in \(S'_r(\mathbb{R}^d)\) and let \(\gamma \in L^2([0,T],L^\infty(\mathbb{R}^d))\) such that

(A3) for all \((t,s,x), \xi \mapsto \mathcal{F}\Lambda(t,s,x)(\xi)\) is a function, the mapping \((t,s,x,\xi) \mapsto \mathcal{F}\Lambda(t,s,x)(\xi)\) is measurable, \(s \mapsto \mathcal{F}\gamma(s) \in L^2([0,T],L^1(\mathbb{R}^d))\), and moreover

\[
\int_0^t \left( \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}\Lambda(t,s,x)(\eta)| \right)^2 \left( \int_{\mathbb{R}^d} |\mathcal{F}\gamma(s)(\eta)| d\eta \right)^2 ds < \infty. \tag{2.16}
\]

(A4) Let \(\Lambda\) and \(\gamma\) be as in (A3)

\[
\lim_{h \downarrow 0} \int_0^t \left( \sup_{\eta \in \mathbb{R}^d} \sup_{r \in (s,s+h)} |\mathcal{F}(\Lambda(t,s,x) - \Lambda(t,r,x))(\eta)|^2 \right) \times \left( \int_{\mathbb{R}^d} |\mathcal{F}\gamma(s)(\eta)| d\eta \right)^2 ds = 0.
\]
Similar to \((A1')\) and \((A2')\) we reformulate the assumptions \((A3)\) and \((A4)\) when the coefficient does not depend on the spatial argument.

**Assumption 2.5.** For \((t, s, x) \in \Delta_T \times \mathbb{R}^d\), let \(\Lambda(t, s, x)\) be a deterministic function with values in \(S'_r(\mathbb{R}^d)\) and \(\gamma \in L^2([0, T])\) such that

\[(A3')\quad \text{for all } (t, s, x), \xi \mapsto \mathcal{F}(\Lambda(t, s, x)(\xi)) \text{ is a function, the mapping } (t, s, x, \xi) \mapsto \mathcal{F}(\Lambda(t, s, x)(\xi)) \text{ is measurable, and moreover}
\]
\[\int_0^t \gamma(s)^2 |\mathcal{F}(\Lambda(t, s, x)(0))|^2 ds < \infty.\]

\[(A4')\quad \Lambda \text{ and } \gamma \text{ are as in } (A3')\text{ and } \lim_{h \downarrow 0} \int_0^T \gamma(s)^2 \left( \sup_{r \in (s, s+h)} |\mathcal{F}(\Lambda(t, s, x) - \Lambda(t, r, x))(0)|^2 \right) ds = 0.\]

Note that the two conditions \((A3)\) and \((A4)\) coincide with \((A1)\) and \((A2)\) respectively if \(\mu = \delta_0\). Assuming \((A3)\) and \((A4)\) (or \((A3')\) and \((A4')\)), one can show using the very same arguments as in the proof of Theorem 2.3 replacing \(\mu\) by \(\delta_0\), that the pathwise integral is well-defined and \((2.15)\) holds.

We now make a last assumption on the first term \(I_0\) in \((2.1)\), that accounts for the initial conditions.

**Assumption 2.6.** \((A5)\) For every \((t, x) \in [0, T] \times \mathbb{R}^d\), \(I_0(t, x)\) is finite.

With all these preparations we can now state the existence and uniqueness theorem for stochastic partial differential equations which are nonhomogeneous in space.

**Theorem 2.7.** Under the assumptions \((A1)\) \((A2)\) \((A3)\) \((A4)\) (or \((A1')\) \((A2')\) \((A3')\) \((A4')\)) and \((A5)\) the SPDE \((1.1)\) has a unique random-field solution, which is given by the mild solution of \((2.1)\).

**Proof.** We calculate the second moment of \(u(t, x)\) in \((2.1)\) for any fixed \((t, x) \in [0, T] \times \mathbb{R}^d\) and obtain

\[\mathbb{E}[u(t, x)^2] \leq C \left[ I_0(t, x)^2 + \mathbb{E} \left[ \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\sigma(y)\mathcal{M}(ds, dy) \right]^2 \right] + \left( \int_0^t \int_{\mathbb{R}^d} \Lambda(t, s, x, y)\gamma(s, y)dyds \right)^2 \]

\[\leq C \left( I_0(t, x)^2 + \int_0^t \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}(\Lambda(t, s, x)(\xi + \eta)|^2 \mu(d\xi) \left( \int_{\mathbb{R}^d} |\mathcal{F}(\sigma)(\eta)|^2 d\eta \right)^2 ds \right.\]

\[+ \left. \int_0^t \sup_{\eta \in \mathbb{R}^d} |\mathcal{F}(\Lambda(t, s, x)(\eta)|^2 \left( \int_{\mathbb{R}^d} |\mathcal{F}(\gamma)(\eta)|^2 d\eta \right)^2 ds, \right)\]

which is finite by assumption, so that \(u(t, x)\) is well-defined as a random variable in \(L^2(\Omega)\) for every \((t, x) \in [0, T] \times \mathbb{R}^d\). \(\square\)
Note that if in the previous inequality all the terms on the right-hand side can be uniformly bounded in $t$ and $x$, then we have for the solution that
\[ \sup_{(t,x)\in[0,T] \times \mathbb{R}^d} \mathbb{E}[u(t,x)^2] < \infty. \]
This condition is essential in order to treat semilinear SPDEs. In fact, in this situation one can reproduce the results from [20] that if the fundamental solution is a function or a nonnegative distribution, one can incorporate coefficients $\sigma$ and $\gamma$ which depend on the solution $u$. However, if the fundamental solution is only a general distribution as in [7], one also needs a stationarity condition which is not satisfied in the case when the partial differential operator has variable coefficients. Therefore we keep our attention to the linear case, because we cannot tell from the methods presented in Section 2.2 whether the fundamental distribution is a function, a nonnegative distribution or a distribution.

### 2.2 Microlocal Analysis

In this section we collect some results which will be useful to us when we construct the fundamental solution to hyperbolic equations in the sections below. Our main tool will be the Fourier transform $F$, which was defined in (2.3) for all functions $f \in L^1(\mathbb{R}^d)$ and extended to Schwartz distributions in (2.4). The inverse of the Fourier transform can be given by
\[ (F^{-1}f)(x) := (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot \xi} f(\xi) d\xi = (2\pi)^{-d} (Ff)(-x), \]
for all $f \in L^1(\mathbb{R}^d)$, and extended to Schwartz distributions, so that for all $T \in \mathcal{S}'(\mathbb{R}^d)$ we have $F^{-1} FT = T$.

Using the Fourier transform we can now define Fourier integral operators, that is the operators we will need in order to construct the fundamental solution. For their definition we need three ingredients, symbols, phase functions and oscillatory integrals.

**Definition 2.8 (Symbols).**

(i) Let $m \in \mathbb{R}$. A $C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$-function $p$ is called a *symbol* of class $S^m$ if for every $R > 0$ and for all $\alpha, \beta \in \mathbb{N}_0^d$ there exists a constant $C_{R,\alpha,\beta} > 0$ such that
\[ |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq C_{R,\alpha,\beta} |\xi|^{-m-|\alpha|}, \]
for all $x, \xi \in \mathbb{R}^d$ with $|\xi| \geq R$. We say that $p$ is a symbol of order $m$.

(ii) We define $S^{-\infty} := \cap_{m \in \mathbb{R}} S^m$ and $S^{+\infty} := \cup_{m \in \mathbb{R}} S^m$, and we trivially have that for every $m_1 \leq m_2$ it holds $S^{-\infty} \subset S^{m_1} \subset S^{m_2} \subset S^{+\infty}$.

(iii) For $p \in S^m$ we define the seminorms $|p|^{(m)}_{l,R}$ for all $l \in \mathbb{N}_0$, $R > 0$ by
\[ |p|^{(m)}_{l,R} := \max_{|\alpha + \beta| \leq l, \xi \in \mathbb{R}^d, |\xi| \geq R} \sup_{x \in \mathbb{R}^d} |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| |\xi|^{-m-|\alpha|}. \]

The space $S^m$ endowed with the family of seminorms $(|\cdot|^{(m)}_{l,R} : l \in \mathbb{N}_0, R > 0)$ becomes a Fréchet space and for any $p \in S^m$, we have by definition
\[ |\partial_\xi^\alpha \partial_x^\beta p(x, \xi)| \leq |p|^{(m)}_{|\alpha + \beta|, R} |\xi|^{-m-|\alpha|}, \quad (2.17) \]
for all $x \in \mathbb{R}^d$, $|\xi| \geq R$, where $|p|_{l+\beta,R}^{(m)}$ is the smallest constant assuring (2.17).

In this paper we denote the special case $|p|_{l,1}^{(m)}$ by $|p|^{(m)}_l$.

**Definition 2.9** (Asymptotic expansion). Let $(p_j)_{j \in \mathbb{N}}$ be a sequence of symbols $p_j \in S^{m_j}$, where $(m_j)_{j \in \mathbb{N}}$ is a nonincreasing sequence with $m_j \to -\infty$ as $j \to \infty$. Then we say that a symbol $p \in S^m$ has the asymptotic expansion

$$p \sim \sum_{j=1}^{\infty} p_j,$$

(2.18)

if for any integer $n \in \mathbb{N}$

$$p - \sum_{j=1}^{n-1} p_j \in S^{m_n},$$

(2.19)

Note that this concept does not imply the convergence of the sum $\sum_{j=1}^{\infty} p_j$ in (2.18) in any sense, although the order of the difference in (2.19) goes to $-\infty$. An object of the form $\sum_{j=1}^{\infty} p_j$ in (2.18) is called a formal series. It is possible to show, see [5], that every symbol $p \in S^m$ is uniquely determined (modulo an element of $S^{-\infty}$) by its asymptotic expansion.

The next step is to define phase functions.

**Definition 2.10** (Phase functions). A phase function is a $C^\infty$-function $\varphi : \mathbb{R}^d_0 \times \mathbb{R}^d_1 \to \mathbb{R}$ that is homogeneous of degree one in the second argument, i.e. $\varphi(x, t\xi) = t \varphi(x, \xi)$ for all $t > 0$, and $\nabla_x, \xi \varphi(x, \xi) \neq 0$ in $\mathbb{R}^d_0 \times \mathbb{R}^d_1$.

**Example 2.11.** The most simple example of a phase function is given by $\varphi(x, \xi) = x \cdot \xi$ for $x, \xi \in \mathbb{R}^d$.

We consider now the double integral

$$\int_{\mathbb{R}^d_0 \times \mathbb{R}^d_1} e^{i\varphi(x, \xi)} p(x, \xi)v(x)dx d\xi,$$

where $\varphi$ is a phase function, $p \in S^m$, $v \in \mathcal{D}(\mathbb{R}^d_0)$ and $d\xi := (2\pi)^{-d_2} d\xi$. This integral may be not absolutely convergent. In order to give a rigorous meaning to it, we introduce the concept of oscillatory integral, following [14]. If $\varphi$ is a phase function, then it is possible to show the existence of a partial differential operator of first order $L$ such that $L(e^{i\varphi(x, \xi)}) = e^{i\varphi(x, \xi)}$. Then, integrating by parts $l$ times, where $l > m + d_2$, we have

$$\int_{\mathbb{R}^d_0 \times \mathbb{R}^d_1} e^{i\varphi(x, \xi)} p(x, \xi)v(x)dx d\xi = \int_{\mathbb{R}^d_0 \times \mathbb{R}^d_1} L^l(e^{i\varphi(x, \xi)}) p(x, \xi)v(x)dx d\xi$$

$$= \int_{\mathbb{R}^d_0 \times \mathbb{R}^d_1} e^{i\varphi(x, \xi)} (LT)^l(p(x, \xi)v(x))dx d\xi,$$

where the integrand on the right-hand side behaves like $(\xi)^{m-l} 1_{\text{supp } v}(x)$, where $\text{supp } v$ is the support of $v$, so the integral is finite. Now the integral on the right-hand side is a Lebesgue integral and we can compute it with Fubini’s theorem integrating first over $x$ (using the compact support of $v$) and then over $\xi$. As a consequence, we can define the following object.
Definition 2.12 (Oscillatory integral distribution). Let \( \varphi \) be a phase function and \( p \in S^m \). Then the oscillatory integral distribution of \( e^{i\varphi(x,\cdot)}p(x,\cdot) \) is the distribution defined for all test functions \( v \in \mathcal{D}(\mathbb{R}^d) \) by

\[
\langle O_S - \int_{\mathbb{R}^d} e^{i\varphi(\cdot,\xi)}p(\cdot,\xi)d\xi, v \rangle = \int_{\mathbb{R}^d \times \mathbb{R}^n} e^{i\varphi(x,\xi)}p(x,\xi)v(x)d\xi.
\]

(2.20)

Note that there are other (equivalent) ways of defining an oscillatory integral. If, however, for some \( x \in \mathbb{R}^d \), \( e^{i\varphi(x,\cdot)}p(x,\cdot) \in L^1(\mathbb{R}^d) \), then the oscillatory integral can be evaluated in this \( x \) and its value equals the Lebesgue integral of \( e^{i\varphi(x,\cdot)}p(x,\cdot) \) over \( \xi \). The concept of oscillatory integrals allows us to make rigorous some arguments that involve diverging integrals.

Example 2.13. (i) Since the Fourier transform of the Dirac delta distribution \( \delta_0 \) is equal to 1, the inverse Fourier transform of 1 should be \( \delta_0 \). However, \( \xi \mapsto e^{ix\cdot\xi} \) is not integrable for any \( x \in \mathbb{R}^d \) with respect to the Lebesgue measure. But for all \( v \in \mathcal{D}(\mathbb{R}^d) \)

\[
\langle O_S - \int_{\mathbb{R}^d} e^{ix\cdot\xi}d\xi, v \rangle = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{ix\cdot\xi}v(x)d\xi = \int_{\mathbb{R}^d} Fv(-\xi)d\xi = v(0).
\]

This implies that \( O_S - \int_{\mathbb{R}^d} e^{ix\cdot\xi}d\xi = \delta_0 \).

(ii) The previous example can be generalized. Let \( d \in \mathbb{N} \), \( n \in \mathbb{N}_0 \) such that \( n \leq d \) and \( G : \mathbb{R}^d \to \mathbb{R}^n \). Then a similar calculation as in the previous example yields that \( O_S - \int_{\mathbb{R}^n} e^{i\xi\cdot G(x)}d\xi \) is the \( \delta \)-distribution supported on the submanifold \( \{G = 0\} \). In particular, we will need the case where \( G(x) = x' - c \), where \( x' := \pi_n(x) \) is the projection of \( x \) onto its first \( n \) coordinates and \( c \in \mathbb{R}^n \). Then \( G(x) = 0 \) if and only if \( x' = c \) and the distribution \( O_S - \int_{\mathbb{R}^n} e^{i\xi\cdot(x'-c)}d\xi \) is given for any test function \( v \in \mathcal{D}(\mathbb{R}^d) \) by

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^n} e^{i\xi\cdot(x'-c)}v(x)d\xi = \int_{\mathbb{R}^{d-n}} v(c, x'')dx'',
\]

where \( x'' \) denotes the projection of \( x \) onto its last \( d-n \) coordinates.

With all this we can now define the so-called Fourier integral operators and the subclass of pseudo-differential operators.

Definition 2.14 (Fourier integral operators). Let \( \phi \) be a \( C^\infty \)-function on \( \mathbb{R}^d \times \mathbb{R}^d \), homogeneous of degree one with respect to \( \xi \) and \( p \in S^m \). A Fourier integral operator \( P_\phi : S(\mathbb{R}^d) \to S(\mathbb{R}^d) \) with phase function \( \phi \) and symbol \( p \) is defined by

\[
(P_\phi v)(x) = \int_{\mathbb{R}^d} e^{i\phi(x,\xi)} p(x,\xi) Fv(\xi)d\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi(x,\xi) - iy\cdot\xi} p(x,\xi)v(y)dyd\xi,
\]

(2.21)

for all \( v \in S(\mathbb{R}^d) \). We will write \( P_\phi = P_\phi(x, D_x) = p_\phi(x, D_x) \) to denote a Fourier integral operator with phase function \( \phi \) and symbol \( p \).

Note that strictly speaking, \( \phi \) in the above definition is not a phase function in the sense of Definition 2.10 but the function \( \varphi(x, y, \xi) := \phi(x, \xi) - y \cdot \xi \) is indeed a phase function both with respect to the three arguments \( (x, y, \xi) \).
and with respect to the two arguments \((y, \xi)\). Therefore the oscillatory integral converges as explained above and we will refer in the following to \(\phi\) as a “phase function”.

Now we provide a few examples for Fourier integral operators, which will be used throughout this article.

**Example 2.15.** (i) With the special choice of \(\phi(x, \xi) = x \cdot \xi\), we have

\[
(P\phi)u(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi} p(x, \xi) F u(\xi) d\xi = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y) \cdot \xi} p(x, \xi) u(y) dy d\xi.
\]

Operators of the form (2.22) are called pseudo-differential operators; given a symbol \(p \in S^m\) we denote by \(P_\phi(x, D_x) = p(x, D_x)\) a pseudo-differential operator with symbol \(p\), omitting in the notation the dependence on the phase \(x \cdot \xi\).

(ii) A partial differential operator of second order given by

\[
P = \sum_{j,k=1}^d a_{j,k}(x) \partial_{x_j} \partial_{x_k}
\]

with \(C^\infty_b(\mathbb{R}^d)\)-coefficients \(a_{j,k}\) can be reinterpreted as a pseudo-differential operator with symbol \(p(x, \xi) = -\sum_{j,k=1}^d a_{j,k}(x) \xi_j \xi_k \in S^2\).

(iii) The operators \((D_x)^2\) and \(|D_x|^2\) which are defined for all \(f \in S(\mathbb{R}^d)\) by

\[
(D_x)^2 f := 1 + \sum_{j=1}^d \partial_{x_j}^2 f \quad \text{and} \quad |D_x|^2 f := \sum_{j=1}^d \partial_{x_j}^2 f,
\]

are the pseudo-differential operators with the symbols \(p(x, \xi) = \langle \xi \rangle\) and \(p(x, \xi) = |\xi|\) respectively. Both are of order 1.

(iv) For all \(\tau \in \mathbb{R}\), the function \(\phi(x, \xi) = x \cdot \xi + \tau |\xi|\) is a phase function, and the operator \(I_\phi\) defined by

\[
(I_\phi v)(x) = \int_{\mathbb{R}^d} e^{ix \cdot \xi + i\tau |\xi|} F v(\xi) d\xi,
\]

for all \(v \in S(\mathbb{R}^d)\) is the Fourier integral operator with phase \(\phi\) and symbol 1, which will be used in Section 6.

Throughout all the article we are going to write, for the sake of brevity, FIO instead of Fourier integral operator, and PDO instead of pseudo-differential operator.

**Definition 2.16.** Given a FIO \(P_\phi = p_\phi(x, D_x)\), we can define the adjoint \(P_\phi^*\) of \(P\) by \(\langle P_\phi u, v \rangle := \langle u, P_\phi^* v \rangle\) for all \(u, v \in S(\mathbb{R}^d)\), and one can show that the adjoint \(P_\phi^*\) has phase function \(-\phi\) and symbol with the following asymptotic expansion:

\[
p^*(x, \xi) = \sum_{\alpha \in \mathbb{N}^d} \frac{(-1)^{|\alpha|}}{\alpha!} \partial_{\xi}^\alpha \partial_x^\alpha p(x, \xi).
\]

(2.23)
This allows us to generalize FIOs to a larger domain.

**Definition 2.17.** Let \( P_\phi = p_\phi(x, D_x) \) be a FIO on \( S(\mathbb{R}^d) \). We can extend \( P_\phi \) to the tempered distributions \( T \in S'(\mathbb{R}^d) \) by

\[
\langle P_\phi T, v \rangle := \langle T, P_\phi^* v \rangle,
\]

for all \( v \in S(\mathbb{R}^d) \).

Recall that the Schwartz kernel of a linear operator \( A : D(\mathbb{R}^d) \to D'(\mathbb{R}^d) \) is the distribution \( K_A \in D'(\mathbb{R}^d \times \mathbb{R}^d) \) given by

\[
\langle K_A, u \otimes v \rangle = \langle Av, u \rangle,
\]

for all \( u, v \in D(\mathbb{R}^d) \).

We see directly from the definition of FIO and Fubini’s theorem w.r.t. \( dx \) and \( dyd\xi \) above that for all \( u, v \in S(\mathbb{R}^d) \)

\[
\langle P_\phi v, u \rangle = \int_{\mathbb{R}^d} (P_\phi v)(x)u(x)dx
\]

\[
= \int_{\mathbb{R}^d} \left( \int_{\mathbb{R}^d \times \mathbb{R}^d} e^{i\phi(x, \xi) - iy \xi} p(x, \xi)v(y)dyd\xi \right) u(x)dx
\]

\[
= \int_{\mathbb{R}^d \times \mathbb{R}^d} \int_{\mathbb{R}^d} e^{i\phi(x, \xi) - iy \xi} p(x, \xi)u(x)v(y)dxdy\xi
\]

\[
= \left( O_S - \int_{\mathbb{R}^d} e^{i\phi(x, \xi)} - iy \xi p(x, \xi)\xi, u \otimes v \right).
\]

This implies that the Schwartz kernel of a FIO \( P_\phi \) is given by

\[
K_{P_\phi}(x, y) = O_S - \int_{\mathbb{R}^d} e^{i\phi(x, \xi) - iy \xi} p(x, \xi)\xi.
\]

Furthermore we see that for every \( x \in \mathbb{R}^d \), \( K_{P_\phi}(x, \cdot) \in S'(\mathbb{R}^d) \). This observation allows us to compute the Fourier transform in the second argument of the Schwartz kernel of a FIO for every \( x \in \mathbb{R}^d \) fixed.

**Proposition 2.18.** Let \( P_\phi \) be a FIO with symbol \( p \) and let \( K_{P_\phi} = (K_{P_\phi}(x, \cdot); x \in \mathbb{R}^d) \) denote its Schwartz kernel. Then the Fourier transform in the second argument of its Schwartz kernel, \( \mathcal{F}_{y \mapsto \eta} K_{P_\phi}(x, \cdot) \), is given by

\[
(\mathcal{F}_{y \mapsto \eta} K_{P_\phi}(x, \cdot))(\eta) = e^{i\phi(x, -\eta)} p(x, -\eta).
\] (2.24)

**Proof.** Let \( u, v \in S(\mathbb{R}^d) \). First we note that due to (2.4), the Fourier transform of \( K_{P_\phi}(x, \cdot) \) is defined for all fixed \( x \in \mathbb{R}^d \) by

\[
(\mathcal{F}_{y \mapsto \eta} K_{P_\phi}(x, \cdot))v = K_{P_\phi}(x, \cdot)(\mathcal{F}_{y \mapsto \eta} v).
\]

We compute using the second representation of the Schwartz kernel in (2.21)

\[
\langle \mathcal{F}_{y \mapsto \eta} K_{P_\phi} v, u \rangle = \int_{\mathbb{R}^d} \left( (\mathcal{F}_{y \mapsto \eta} K_{P_\phi}(x, \cdot)) v \right) u(x)dx
\]

\[
= \int_{\mathbb{R}^d} (K_{P_\phi}(x, \cdot)(\mathcal{F}_{y \mapsto \eta} v)) u(x)dx
\]
and Proposition 2.18, we conclude that the proof.

The function of the right-hand side of the previous equality is obviously in η growth. We show now that $K_{P_\phi}$ has a pointwise interpretation in $x$, and in the last line we have used the change of variable $\xi \mapsto -\eta$.

Applying this proposition, we can show an assumption on the fundamental solution for the existence and uniqueness of a random-field solution in Lemma 2.19.

**Lemma 2.19.** Let $P_\phi$ be a FIO with symbol $p$, and let $K_{P_\phi}$ denote its Schwartz kernel. Then, for every $x \in \mathbb{R}^d$, $K_{P_\phi}(x, \cdot)$ is a tempered distribution with rapid decrease.

**Proof.** Fix $x \in \mathbb{R}^d$ and $\psi \in \mathcal{D}(\mathbb{R}^d)$. We know by [29] p. 244/245 that the regularization of $K_{P_\phi}(x, \cdot) \ast \psi$ is an infinitely differentiable function of slow growth. We show now that $K_{P_\phi}(x, \cdot) \ast \psi$ is indeed a Schwartz function; this implies the assertion, again by [29] p. 244/245. For this we take the Fourier transform of $K_{P_\phi}(x, \cdot) \ast \psi$ (in the sense of distributions) and using [15] Theorem 1.5.3(2) and Proposition 2.18, we conclude that

$$
F_{y \mapsto \eta}(K_{P_\phi}(x, \cdot) \ast \psi)(\eta) = F_{y \mapsto \eta}K_{P_\phi}(x, \eta)F\psi(\eta) = e^{i\phi(x, -\eta)}p(x, -\eta)F\psi(\eta).
$$

The function of the right-hand side of the previous equality is obviously in $C^\infty(\mathbb{R}^d)$ with respect to $\eta$. The fact that $\phi$ is of order $1$ in $\eta$, $p$ is of finite order in $\eta$ and $F\psi$ is a Schwartz function imply that the function $\eta \mapsto F_{y \mapsto \eta}(T(x) \ast \psi)(\eta)$ is a Schwartz function, and hence its inverse Fourier transform too. This finishes the proof.

For the construction of the fundamental solution, we need to know how to multiply PDOs with FIOs. We introduce the following notation: for a phase function $\phi$ we define the mean value of the gradient in the convex hull of $x, x'$ and $\xi, \xi'$ as

$$
\hat{\nabla}_x \phi(x, x'; \xi) = \int_0^1 \nabla_x \phi(x' + \theta(x - x'), \xi) d\theta,
$$

$$
\hat{\nabla}_x \phi(x; \xi', \xi) = \int_0^1 \nabla_x \phi(x, \xi + \theta(\xi' - \xi)) d\theta,
$$

and then we have the following result.

**Proposition 2.20** ([15], Theorems 10.2.1 and 10.2.2). Let $P_\phi$ be a FIO with symbol $p \in S^{m_1}$, and let $Q$ be a PDO with symbol $q \in S^{m_2}$. Then $P_\phi Q$ and $QP_\phi$ are FIOs with phase function $\phi$ and symbols $r_1$ and $r_2$ (of order $m_1 + m_2$) respectively, where $r_1$ and $r_2$ have asymptotic expansions

$$
r_1(x, \xi) \sim \sum_{\alpha \in N_0^d} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial^\alpha} \left( p(x, \xi) D_\xi^\alpha q(\hat{\nabla}_x \phi(x; \xi, \xi'), \xi') \right) \bigg|_{\xi' = \xi}
$$

$$
r_2(x, \xi) \sim \sum_{\alpha \in N_0^d} \frac{1}{\alpha!} \frac{\partial^\alpha}{\partial^\alpha} \left( q(x, \xi) D_\xi^\alpha p(\hat{\nabla}_x \phi(x; \xi, \xi'), \xi') \right) \bigg|_{\xi' = \xi}
$$
Proposition 2.23. Let \( P \) and \( Q \) be PDOs with symbols \( p(x,\xi) \in S^{m_1} \) and \( q(x,\xi) \in S^{m_2} \). Then \( PQ \) is a PDO with symbol \( r(x,\xi) \in S^{m_1+m_2} \) having the asymptotic expansion

\[
\sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_{x'}^\alpha p(x,\xi) D_{x'}^\alpha q(x,\xi) = r_2(x,\xi) \sim \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_{x'}^\alpha p(x,\xi) D_{x'}^\alpha q(x,\xi) \bigg|_{x' = x}.
\]

The asymptotic expansions of the symbols to the second order are given by

\[
r_1(x,\xi) = p(x,\xi)q(\nabla_\xi \phi(x,\xi),\xi) + \sum_{j=1}^d \partial_{\xi_j} p(x,\xi) D_{\xi_j} q(\nabla_\xi \phi(x,\xi),\xi) + \frac{i}{2} p(x,\xi) D_{\xi_j} D_{\xi_k} q(\nabla_\xi \phi(x,\xi),\xi) \frac{\partial^2 \phi}{\partial_{\xi_j} \partial_{\xi_k}}(x,\xi) + r_1^*(x,\xi),
\]

and

\[
r_2(x,\xi) = p(x,\nabla_\xi \phi(x,\xi))q(x,\xi) + \sum_{j=1}^d \partial_{\xi_j} p(x,\nabla_\xi \phi(x,\xi)) D_{\xi_j} q(x,\xi) + \frac{i}{2} \left( \sum_{j,k=1}^d \partial_{\xi_j} \partial_{\xi_k} p(x,\nabla_\xi \phi(x,\xi)) \frac{\partial^2 \phi}{\partial_{\xi_j} \partial_{\xi_k}}(x,\xi) \right) q(x,\xi) + r_2^*(x,\xi),
\]

where \( r_1^*, r_2^* \in S^{m_1+m_2-2} \).

The following corollary immediately follows from Proposition 2.20.

**Corollary 2.21.** Let \( P \) and \( Q \) be PDOs with symbols \( p(x,\xi) \in S^{m_1} \) and \( q(x,\xi) \in S^{m_2} \). Then \( PQ \) is a PDO with symbol \( r(x,\xi) \in S^{m_1+m_2} \) having the asymptotic expansion

\[
r(x,\xi) = \sum_{\alpha \in \mathbb{N}_0^n} \frac{1}{\alpha!} \partial_{x'}^\alpha p(x,\xi) D_{x'}^\alpha q(x,\xi).
\]

A consequence of Corollary 2.21 is that the commutator \([P,Q] := PQ - QP\) between two PDOs \( P,Q \) with symbols \( p \in S^{m_1} \) and \( q \in S^{m_2} \) respectively is of order \( m_1 + m_2 - 1 \), since the leading term of the asymptotic expansion of the symbols of both products \( PQ \) and \( QP \) is \( p(x,\xi)q(x,\xi) \).

The following Proposition, a generalization of the Calderón-Vaillancourt Theorem, states the boundedness of FIOs acting on Sobolev spaces.

**Proposition 2.22.** ([15], Theorem 10.2.3). Let \( P_\phi = p_\phi(x,D_x) \) and \( r \in \mathbb{R} \). The operator \( P_\phi \) defines a continuous map \( H^{r+m} \to H^r \), and there exists a constant \( C = C_{r,m} > 0 \) and an integer \( \ell \geq 0 \) such that for every \( u \in H^{r+m} \) it holds

\[
\| P_\phi u \|_r \leq C |p|^{(m)}_\ell \| u \|_{r+m}.
\]

Finally, we give a Proposition concerning the composition of \( n \) FIOs, which is a simplified version of Theorem 10.6.8 in [15], referring to [15] for the details.

**Proposition 2.23.** For \( 1 \leq j \leq n \), let \( P_{j,\phi_j} \) be FIOs with phase functions \( \phi_j \) and symbols \( p_j \in S^{m_j} \). There exist a symbol \( p \) of order \( m = m_1 + \ldots + m_n \) and
a phase function $\phi$ such that $P_1,\phi_1 \cdots P_n,\phi_n(x,D_x) = p_0(x,D_x)$, and moreover for every integer $\ell \geq 0$ there exists a constant $C_{\ell} > 0$ and an integer $\ell' \geq 0$ such that
\begin{equation}
|p_{\ell}(m)| \leq C_{\ell}^{m-1} \prod_{j=1}^{n} |p_j(m_j)|. \tag{2.25}
\end{equation}

The phase function of the composition $P_1,\phi_1 \cdots P_n,\phi_n$ can be explicitly computed, see Section 4.5 in [15], especially formulas (5.4) and (5.5). In the statement of Proposition 2.23 we focus only on formula (2.25) which will be crucial in the proof of the main theorem of this paper, without being precise about the phase function $\phi$, which will not be used in our computations.

We conclude this section with a remark that comments on the two concepts of fundamental solutions to PDEs that we deal with in this article.

Remark 2.24. In this paper we are going to construct the fundamental solution to an initial value problem (in the sense of [15, Section 10.7]) of the form
\begin{equation}
\begin{cases}
L(t,x,D_x)U(t,x) = G(t,x), & (t,x) \in [0,T] \times \mathbb{R}^d \\
U(0,x) = U_0(x),
\end{cases}
\end{equation}
where $L = \partial_t \text{id} - iD(t,x,D_x) + R(t,x,D_x)$ is a square matrix of PDOs with symbols of first order, $D$ is the diagonal principal part and $R$ is some PDO of order less than 1, that satisfies some conditions. That is, we are going to construct a family of FIOs $E(t,s)$, indexed by two time parameters $(t,s) \in \Delta_T$, where $0 < \bar{T} \leq T$ is the (modified) time horizon of the PDE, such that
\begin{equation}
\begin{cases}
LE(t,s) = 0, & (t,s) \in \Delta_T \\
E(s,s) = \text{id} & s \in [0,T].
\end{cases}
\end{equation}
Then, we can compute the solution of Problem (2.26) using Duhamel’s formula
\begin{equation}
U(t,x) = (E(t,0)U_0)(x) + \int_0^t (E(t,s)G(s))(x)ds. \tag{2.27}
\end{equation}
This equality rewritten, using the definition of Schwartz kernels, with $\Lambda$ denoting the Schwartz kernel of $E$, is given by
\begin{equation*}
U(t,x) = \langle \Lambda(t,0,x,\cdot),U_0 \rangle + \int_0^t \langle \Lambda(t,s,x,\cdot),G(s,\cdot) \rangle ds,
\end{equation*}
or, using the abuse of notation from Section 2.1
\begin{equation*}
U(t,x) = \int_{\mathbb{R}^d} \Lambda(t,0,x,y)U_0(y)dy + \int_0^t \int_{\mathbb{R}^d} \Lambda(t,s,x,y)G(s,y)dyds.
\end{equation*}
In the case of constant coefficients, $\Lambda$ can be shown to be the solution to the abstract Cauchy problem
\begin{equation*}
\begin{cases}
L\Lambda(t,x) = \delta_{0,0}, & (t,x) \in [0,T] \times \mathbb{R}^d, \\
\Lambda(0,x) = 0, & x \in \mathbb{R}^d,
\end{cases}
\end{equation*}
where $\delta_{0,0}$ is the space-time Dirac distribution in $(0,0)$. This concept of fundamental solution, which is fairly common in PDE theory, is the one that we referred to in the Introduction and in Section 2.1.

From now on we will refer to both concepts as fundamental solution when there is no risk of confusion. We will however make the distinction when applying Proposition 2.18.
3 Second order hyperbolic equations - the case of strict hyperbolicity

In this section we consider the case where the partial differential operator $L$ in (1.1) is given by (1.3). More specifically, this section is devoted to the proof of the following theorem.

**Theorem 3.1.** Let us consider an SPDE (1.1) where the partial differential operator $L$ is given by

$$L = \partial^2_t - \sum_{j,k=1}^d a_{j,k}(t,x)\partial_{x_j}\partial_{x_k} - \sum_{j=1}^d b_j(t,x)\partial_{x_j} - c(t,x), \quad (3.1)$$

where for the coefficients we assume $a_{j,k} \in C^1([0,T];C^\infty_0(\mathbb{R}^d))$ for $1 \leq j,k \leq d$, $b_j \in C([0,T];C^\infty_0(\mathbb{R}^d))$ for $1 \leq j \leq d$ and $c \in C([0,T];C^\infty_0(\mathbb{R}^d))$. Suppose that $L$ is a strictly hyperbolic operator, i.e. there exists a constant $C > 0$ such that

$$\sum_{j,k=1}^d a_{j,k}(t,x)\xi_j\xi_k \geq C|\xi|^2, \quad (3.2)$$

for all $(x,\xi) \in \mathbb{R}^d \times \mathbb{R}^d$. Assume for the initial conditions that $u_0 \in H^r(\mathbb{R}^d)$ and $u_1 \in H^{r-1}(\mathbb{R}^d)$, where $2r > d$. Furthermore, assume for the spectral measure that (2.14) with $\nu = 1$ holds, and that $\sigma$ and $\gamma$ are as in (A1) of Assumption 2.1 and (A3) of Assumption 2.4 respectively.

Then, for some time horizon $0 < T \leq \bar{T}$, the conditions (A1) (A2) (A3) (A4) and (A5) hold, and therefore there exists a unique solution to the SPDE (1.1) with partial differential operator given by (1.3).

Note that if $\sigma$ or $\gamma$ do not depend on the spatial argument, then one has to assume that they are as in (A1’) or (A3’) and then the assertion is that (A1’) or (A3’) hold respectively.

The proof of this theorem is divided in three steps, following [2, 3, 4]. First we reduce the second order hyperbolic equation to a first order system keeping track of the transformations on the right-hand side and the initial conditions. Then we compute the fundamental solution of the resulting first order system and solve it using Duhamel’s formula. From this we obtain a representation formula for the fundamental solution of the second order equation and for its Fourier transform which we will use to show conditions (A1)(A5). We will now for the sake of briefness assume that the coefficients do not depend on space and check (A1) - (A5). The case where the coefficients do not depend on space can be easily checked by the same methods.

**Proof of Theorem 3.1.** We focus on the Cauchy problem

$$\begin{cases}
L(t,x,\partial_t,\nabla_x)u(t,x) = f(t,x), & (t,x) \in (0,T] \times \mathbb{R}^d, \\
u(0,x) = u_0(x), & x \in \mathbb{R}^d, \\
\partial_t u(0,x) = u_1(x), & x \in \mathbb{R}^d, 
\end{cases} \quad (3.3)$$

for the equation (1.1), where $L$ is the partial differential operator in (3.1), and for the right-hand side we choose an arbitrary function $f$, which satisfies some
regularity conditions. In the last step, we choose

\[ f(t, x) := \gamma(t, x) + \sigma(t, x)\dot{F}(t, x). \]

This is of course an abuse of notation, but it is to some extent justified by the fact that the only operations we perform with it are composing it with FIOs, which is justified by Definition 2.17 and integration, where then formally

\[ \int_0^t \int_{\mathbb{R}^d} \sigma(s, y) \dot{F}(s, y) dy ds := \int_0^t \int_{\mathbb{R}^d} \sigma(s, y) M(ds, dy), \]

where \( M \) is the martingale measure associated to the stochastic noise \( F \) as defined in Section 2.1. Another way to make explicitly sense of the right-hand side is to consider an absolutely continous approximation of \( \dot{F} \), denoted by \((\dot{F}_n)_{n \in \mathbb{N}}\) such that in some sense \( F_n \to F \) as \( n \to \infty \). Then one does all the arguments below on the level of approximations and passes to the limit as \( n \to \infty \) at the end.

**First step: Reduction to a first-order system.** Using the relation \( D = -i\partial \) we restate (3.3) as

\[
\begin{aligned}
P(t, x, D_t, D_x) u(t, x) &= -f(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, \\
u(0, x) &= u_0(x), \quad x \in \mathbb{R}^d, \\
u_t(0, x) &= u_1(x), \quad x \in \mathbb{R}^d,
\end{aligned}
\]  

(3.4)

with

\[
P = D_t^2 - \sum_{j, k=1}^d a_{j, k}(t, x) D_{x_j} D_{x_k} + i \sum_{j=1}^d b_j(t, x) D_{x_j} + c(t, x).
\]

The symbol of the differential operator \( P \) is given by

\[
p(t, x, \tau, \xi) = \tau^2 - \sum_{j, k=1}^d a_{j, k}(t, x) \xi_j \xi_k + i \sum_{j=1}^d b_j(t, x) \xi_j + c(t, x),
\]

and its principal part is \( p_2(t, x, \tau, \xi) = \tau^2 - \sum_{j, k=1}^d a_{j, k}(t, x) \xi_j \xi_k \). Its characteristic roots, i.e., the solutions to the equation \( p_2(t, x, \tau, \xi) = 0 \) with respect to \( \tau \), are real-valued, and they are given by

\[
\tau = \pm \left( \sum_{j, k=1}^d a_{j, k}(t, x) \xi_j \xi_k \right)^{1/2} =: \pm \lambda(t, x, \xi),
\]

(3.5)

which are real-valued thanks to the strict hyperbolicity condition (3.2). Moreover, \( \pm \lambda \in C^1([0, T], S^1) \). Now we denote \( \pm \lambda(t, x, D_x) \) the PDOs with symbols \( \pm \lambda(t, x, \xi) \) and we set

\[
\begin{aligned}
v_1(t, x) &:= (D_x) u(t, x) \\
v_2(t, x) &:= (D_t + \lambda(t, x, D_x)) u(t, x).
\end{aligned}
\]

(3.6)

With these definitions we compute, at operator’s level

\[
(D_t + \lambda(t, x, D_x)) v_1(t, x) = (D_t + \lambda(t, x, D_x)) (D_x) u(t, x)
\]

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where \( R_0 := -\lambda \langle D_x \rangle [D_x]^{-1} \) is a PDO with symbol of order zero, and

\[
(D_t - \lambda(t, x, D_x))v_2(t, x) = (D_t - \lambda(t, x, D_x))(D_t + \lambda(t, x, D_x))u(t, x) = \left(D_t^2 + D_t \lambda(t, x, D_x) - \lambda(t, x, D_x)D_t - \lambda^2(t, x, D_x)\right)u(t, x).
\]

Using Corollary 2.21 we can develop the symbol of the operator \( \lambda^2 = \lambda \lambda \) and we get

\[
\lambda^2(t, x, D_x) = \sum_{j,k=1}^d a_{j,k}(t, x) D_x^j D_{x_k} + R_1(t, x, D_x),
\]

where \( R_1 \) is PDO with symbol of order (at most) 1. Therefore (omitting the notation \((t, x, D_x)\))

\[
(D_t - \lambda)v_2 = P u - \left(i \sum_{j=1}^d b_j D_{x_j} + c - D_t \lambda + \lambda D_t + R_1\right) u
\]

\[
= -f - \left(i \sum_{j=1}^d b_j D_{x_j} + c + \frac{i}{\partial t} \lambda(t, x, \xi) + R_1(t, x, \xi)\right) \langle D_x \rangle^{-1} v_1
\]

\[
= -f - T_0 v_1,
\]

where \( \frac{\partial \lambda}{\partial t} = \frac{\partial \lambda}{\partial t}(t, x, D_x) \) stands for a PDO with symbol \( \frac{\partial \lambda}{\partial t}(t, x, \xi) \in C([0, T]; \mathbb{S}^1) \) by (3.5), and \( T_0 = T_0(t, x, D_x) \) is a PDO with symbol

\[
T_0(t, x, \xi) := \left(i \sum_{j=1}^d b_j \xi_j + c + \frac{i}{\partial t} \lambda(t, x, \xi) + R_1(t, x, \xi)\right) \langle \xi \rangle^{-1},
\]

which is in \( C([0, T]; \mathbb{S}^0) \). For the initial conditions we have

\[
v_1(0) = \langle D_x \rangle v_1(0, \cdot) = \langle D_x \rangle u_0,
\]

\[
v_2(0) = -i \frac{\partial \lambda}{\partial t} u_0 + \lambda(0, x, D_x) u_0(0, \cdot) = -i u_1 + \lambda(0, x, D_x) u_0.
\]

So setting \( V := (v_1, v_2)^T, \ G := (0, -f)^T, \)

\[
P := \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \begin{pmatrix} \lambda(t, x, D_x) & -\langle D_x \rangle \\ 0 & -\lambda(t, x, D_x) \end{pmatrix} + \begin{pmatrix} R_0 & 0 \\ T_0 & 0 \end{pmatrix},
\]

and

\[
V_0 = (\langle D_x \rangle u_0, -i u_1 + \lambda(0, x, D_x) u_0)^T,
\]

we have reduced the Cauchy problem (3.3) to the following equivalent system of two first-order equations:

\[
\begin{cases}
PV(t, x) = G(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, \\
V(0, x) = V_0(x), \quad x \in \mathbb{R}^d.
\end{cases}
\]
Now we want to diagonalize the principal part of the operator matrix in (3.9). To this end, we start working at the level of symbols and we look for a matrix of the form
\[
M(t, x, \xi) := \begin{pmatrix} 1 & m(t, x, \xi) \\ 0 & 1 \end{pmatrix},
\]
with a good choice for the element \( m \) such that
\[
M^{-1}(t, x, \xi) \begin{pmatrix} \lambda(t, x, \xi) & -\langle \xi \rangle \\ 0 & -\lambda(t, x, \xi) \end{pmatrix} M(t, x, \xi) = \begin{pmatrix} \lambda(t, x, \xi) & 0 \\ 0 & -\lambda(t, x, \xi) \end{pmatrix}.
\]
We have
\[
\begin{pmatrix} 1 & -m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda & -\langle \xi \rangle \\ 0 & -\lambda \end{pmatrix} \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \lambda & 2\lambda m - \langle \xi \rangle \\ 0 & -\lambda \end{pmatrix}.
\]
So the matrix on the right-hand side is diagonal if and only if
\[
m(t, x, \xi) = \frac{\langle \xi \rangle}{2\lambda(t, x, \xi)}.
\]
Since we assume in this section that the partial differential operator (1.3) is strictly hyperbolic, we have that \(|\lambda(t, x, \xi)| > 0\) if \(|\xi| > 0\) so that the symbol \( m \) is well-defined. Coming now to the level of operators, we define the operator matrix
\[
M(t, x, D_x) := \begin{pmatrix} 1 & m(t, x, D_x) \\ 0 & 1 \end{pmatrix},
\]
(3.12)
where \( m(t, x, D_x) \) is the PDO with symbol \( m(t, x, \xi) \). We define the PDO
\[
Q_0(t, x, D_x) := \lambda(t, x, D_x)m(t, x, D_x) + m(t, x, D_x)\lambda(t, x, D_x) - \langle D_x \rangle,
\]
and developing its symbol as in Corollary 2.21 we can conclude that \( Q_0(t, x, \xi) \in C([0, T], S^0) \) because of the choice of \( m(t, x, \xi) \). Therefore, we can diagonalize the principal part of the operator in (3.9) obtaining
\[
M^{-1}(t, x, D_x) \begin{pmatrix} \lambda(t, x, D_x) & -\langle D_x \rangle \\ 0 & -\lambda(t, x, D_x) \end{pmatrix} M(t, x, D_x) = \begin{pmatrix} \lambda(t, x, D_x) & 0 \\ 0 & -\lambda(t, x, D_x) \end{pmatrix} + \begin{pmatrix} 0 & Q_0(t, x, D_x) \\ 0 & 0 \end{pmatrix}.
\]
Performing the change of variable
\[
W := M^{-1}V, \quad W = (w_1, w_2)^T,
\]
we have shown that system (3.11) is equivalent to the following first-order system
\[
\begin{cases}
\dot{P}W(t, x) = \tilde{G}(t, x), & (t, x) \in (0, T] \times \mathbb{R}^d, \\
W(0) = W_0, & x \in \mathbb{R}^d,
\end{cases}
\]
(3.13)
where the operator \( \dot{P} \) is given by (dropping the arguments \( t, x, D_x \) in the second equality)
\[
\dot{P}(t, x, D_x) := M^{-1}(t, x, D_x)P(t, x, D_x)M(t, x, D_x)
\]
23
where 

\[
\begin{pmatrix}
D_t & 0 \\
0 & D_t
\end{pmatrix} + \begin{pmatrix}
0 & D_t m - mD_t \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
0 & 0 \\
0 & -\lambda
\end{pmatrix} + \begin{pmatrix}
0 & Q_0 \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
R_0 - mT_0 \\
T_0 m
\end{pmatrix} \]

that is

\[
\tilde{P}(t, x, D_x) = \begin{pmatrix}
D_t & 0 \\
0 & D_t
\end{pmatrix} + \begin{pmatrix}
0 & D_t m - mD_t \\
0 & 0
\end{pmatrix} + \begin{pmatrix}
\lambda(t, x, D_x) & 0 \\
0 & -\lambda(t, x, D_x)
\end{pmatrix} + \mathcal{R}(t, x, D_x),
\]

where

\[
\mathcal{R} := \begin{pmatrix}
R_0 - mT_0 & D_t m - mD_t + Q_0 + R_0 m - mT_0 m \\
T_0 m
\end{pmatrix}
\]

is a PDO with symbol \(r(t, x, \xi) \in C([0, T], S^0)\), the right hand side is

\[
\tilde{G} := M^{-1}(t, x, D_x) \begin{pmatrix}
0 \\
f
\end{pmatrix} = \begin{pmatrix}
m(t, x, D_x)f \\
f
\end{pmatrix},
\]

and the initial conditions in (3.10) become \(W_0 = (w_1(0), w_2(0))^T\), where

\[
W_0 := M^{-1}(0, x, D_x) V_0 = \begin{pmatrix}
\langle D_x \rangle u_0 + im(0, x, D_x) u_1 - m(0, x, D_x) \lambda(0, x, D_x) u_0 \\
- iu_1 + \lambda(0, x, D_x) u_0
\end{pmatrix}.
\]

Second step: Computing the (fundamental) solution of system (3.13). The system in (3.13) is in the form of [15, Section 10.7] and we can apply [15, Theorem 10.7.2] to obtain its solution. To this end, let \(\phi_{\pm} = \phi_{\pm}(t, s, x, \xi)\) be two phase functions, where \(x, \xi \in \mathbb{R}^d\) and \((t, s) \in \Delta_T\), where \(0 < T \leq T\) is sufficiently small \((T\) will be made precise later). The exact form of the phase functions is to be determined by the following arguments. We define the operator matrix

\[
I_{\phi}(t, s) := I_{\phi}(t, s, x, D_x) := \begin{pmatrix}
I_{\phi_{+}}(t, s, x, D_x) & 0 \\
0 & I_{\phi_{-}}(t, s, x, D_x)
\end{pmatrix},
\]

where \(I_{\phi_{\pm}}\) are the FIOs with phase function \(\phi_{\pm}\) and symbol 1. From this definition together with Proposition [2.20] we see that

\[
D_t I_{\phi_{\pm}} \pm \lambda(t, x, D_x) I_{\phi_{\pm}} = \int_{\mathbb{R}^d} e^{i\phi_{\pm}(t, s, x, \xi)} \frac{\partial \phi_{\pm}}{\partial t}(t, s, x, \xi) \xi \xi \pm \int_{\mathbb{R}^d} e^{i\phi_{\pm}(t, s, x, \xi)} \lambda(t, x, \nabla_x \phi_{\pm}(t, s, x, \xi)) \xi \
+ \int_{\mathbb{R}^d} e^{i\phi_{\pm}(t, s, x, \xi)} b_{0, \pm}(t, s, x, \xi) \xi \
\]

where \(b_{0, \pm}(t, s) \in S^0\). The first two integral terms on the right-hand side of (3.13) cancel if we choose \(\phi_{\pm}\) to be the solutions of the so-called eikonal equations. These equations are given by

\[
\begin{cases}
\partial_t \phi_{\pm}(t, s, x, \xi) \pm \lambda(t, x, \nabla_x \phi_{\pm}(t, s, x, \xi)) = 0, & (t, s, x, \xi) \in \Delta_T \times \mathbb{R}^d \times \mathbb{R}^d, \\
\phi_{\pm}(s, s, x, \xi) = x \cdot \xi, & s \in [0, T].
\end{cases}
\]

(3.19)
Indeed, [15, Theorem 10.4.1] states that for a sufficiently small $0 < T < T$ there exists a unique solution to the two eikonal equations. In the following we choose $\phi_\pm$ as the solutions to \eqref{3.19}. Denoting by $B_{0, \pm}(t, s, x, \xi)$ the PDOs with symbols $b_{0, \pm}(t, s, x, \xi)$ in \eqref{3.18}, we define the family $(W_i(t, s) ; (t, s) \in \Delta_T)$ of FIOs by

$$W_i(t, s, x, D_x) := -i \left( \begin{pmatrix} B_{0, +}(t, s, x, D_x) & 0 \\ 0 & B_{0, -}(t, s, x, D_x) \end{pmatrix} + \mathcal{R}(t, x, D_x) \right) I_\phi(t, s, x, D_x).$$

From \eqref{3.14}, \eqref{3.18}, \eqref{3.19} and \eqref{3.20} we obtain that

$$\hat{\mathcal{P}}(t, x, D_x) I_\phi(t, s, x, D_x) = iW_i(t, s, x, D_x),$$

that is $iW_i$ is the residual of system \eqref{3.14} for $I_\phi$. We define then by induction the sequence of $2 \times 2$-matrices of FIOs, denoted by $(W_n(t, s) ; (t, s) \in \Delta_T)_{n \in \mathbb{N}}$, by

$$W_{n+1}(t, s, x, D_x) = \int_0^\ell W_1(t, \theta, x, D_x) W_n(\theta, s, x, D_x) d\theta.$$  \hfill(3.22)

We are now going to prove that the operator norms of $W_n$, seen as operators from the Sobolev space $H^r$ for any fixed $r \in \mathbb{R}_+$ into itself, can be estimated from above by

$$\|W_n(t, s)\| \leq \frac{C_r^{n-1} |t-s|^{n-1}}{(n-1)!} \leq \frac{C_r^{n-1} \bar{\sigma}^{n-1}}{(n-1)!},$$  \hfill(3.23)

for all $(t, s) \in \Delta_T$ and $n \in \mathbb{N}$, where $C_r$ is a constant which only depends on the index of the Sobolev space.

To deal with the operator norms in \eqref{3.23}, we need to explicitly write the matrices $W_n$; an induction in \eqref{3.22} easily shows that

$$W_n(t, s) = \int_0^\ell \int_0^{\theta_1} \cdots \int_0^{\theta_{n-2}} W_1(t, \theta_1) \cdots W_1(\theta_{n-2}, \theta_{n-1}) d\theta_{n-1} \cdots d\theta_1.$$  \hfill(3.24)

The integrand is a product of $n - 1$ $2 \times 2$-matrices of FIOs, therefore it is an operator matrix whose entries consist of $2^{n-2}$ summands of products of $n - 1$ FIOs. Denoting by $Q_1 \ldots Q_{n-1}$ one of these products, where each of the $Q_j$ is one of the four entries of the $2 \times 2$-matrix of FIOs $W_1$, we have from Proposition \ref{2.25} that $Q_1 \ldots Q_{n-1}$ is again a FIO with symbol $\sigma_{n-1}$ of order zero, and for all $\ell \in \mathbb{N}$ there exists $C_\ell > 0$ and $\ell' \in \mathbb{N}_0$ such that

$$|\sigma_{n-1}(t, \theta_1, \ldots, \theta_{n-1})|^{(0)}_{\ell} \leq C_\ell^{n-2} |q_1(t, \theta_1)|^{(0)}_{\ell'} \cdots |q_{n-1}(\theta_{n-2}, \theta_{n-1})|^{(0)}_{\ell'},$$

where for $j = 1, \ldots, n-1$, $q_j(t, s)$ denotes the symbol of the FIO $Q_j(t, s)$, $(t, s) \in \Delta_T$. Now we set

$$\bar{\sigma} := \sup_{j=1, \ldots, n-1} \sup_{(t, s) \in \Delta_T} |q_j(t, s)|^{(0)}_{\ell'} < \infty,$$

so that

$$|\sigma_{n-1}(t, \theta_1, \ldots, \theta_{n-1})|^{(0)}_{\ell} \leq C_\ell^{n-2} \bar{\sigma}^{n-1}.$$
By Proposition 2.22 applied to products of the type $Q_1 \ldots Q_{n-1}$ and from the previous inequality, for every $r \geq 0$ there exist constants $C_r > 0$ (depending only on the index of the Sobolev space) and $\ell_r \in \mathbb{N}_0$ such that for all $u \in H^r$

$$\|Q_1(t, \theta_1) \ldots Q_{n-1}(\theta_{n-2}, \theta_{n-1})u\|_r \leq C_r \sigma_{n-1}(t, \theta_1, \ldots, \theta_{n-1})I_{(0)}^{(0)}\|u\|_r$$

$$\leq C_r C_{n-1}^{n-2} \sigma_{n-1} \|u\|_r. \quad (3.25)$$

Therefore, in the operator matrix $W_1(t, \theta_1) \ldots W_1(\theta_{n-2}, \theta_{n-1})$, the operator norm of each entry can be bounded from above by $2^{n-2}C_r C_{n-1}^{n-2} \sigma_{n-1}$, since there are $2^{n-2}$ products of $n-1$ FIOs. Now by (3.24) and (3.25) we deduce that

$$\|W_n(t, s)\| \leq \int_s^t \int_s^{\theta_1} \ldots \int_s^{\theta_{n-2}} \|W_1(t, \theta_1) \ldots W_1(\theta_{n-2}, \theta_{n-1})\| d\theta_{n-1} \ldots d\theta_1$$

$$\leq 2^{n-2}C_r C_{n-1}^{n-2} \sigma_{n-1} \int_s^t \int_s^{\theta_1} \ldots \int_s^{\theta_{n-2}} d\theta_{n-1} \ldots d\theta_1$$

$$\leq \frac{2^{n-2}C_r C_{n-1}^{n-2} \sigma_{n-1} |t-s|^{n-1}}{(n-1)!} = \tilde{C}_r |t-s|^{n-1} (n-1)! \quad (3.26)$$

for a new constant $\tilde{C}_r$ depending only on $r$, which yields the claim (3.23).

Now, using the estimate (3.23) we can show that the sequence of FIOs defined for all $(t, s) \in \Delta_T$ and all $N \in \mathbb{N}$ by

$$E_N(t, s) = I_\phi(t, s) + \int_s^t I_\phi(t, \theta) \sum_{n=1}^N W_n(\theta, s) d\theta \quad (3.27)$$

is a well-defined FIO on $H^r$ for every $r \geq 0$ and converges to the well-defined FIO

$$E(t, s) = I_\phi(t, s) + \int_s^t I_\phi(t, \theta) \sum_{n=1}^\infty W_n(\theta, s) d\theta, \quad (3.28)$$

which is the fundamental solution to the system (3.13) in the sense that it satisfies

$$\begin{cases} \tilde{P}E(t, s) = 0 & (t, s) \in \Delta_T, \\ E(s, s) = \text{id} & s \in [0, T]. \end{cases} \quad (3.29)$$

We now check that (3.29) holds. At symbols level, for every $l \in \mathbb{N}$ and multi-indices $\alpha, \beta$ such that $|\alpha + \beta| \leq \ell$, we have

$$|\partial^\alpha_x \partial^\beta_x \sigma (E_N(t, s)) (x, \xi)|$$

$$\leq \int_s^t \sum_{n=1}^N |\partial^\alpha_x \partial^\beta_x \sigma (W_n(\theta, s)) (x, \xi)| d\theta$$

$$\leq \sum_{n=1}^N \int_s^t \int_s^{\theta_1} \ldots \int_s^{\theta_{n-2}} |\partial^\alpha_x \partial^\beta_x \sigma (W_1(t, \theta_1) \ldots W_1(\theta_{n-2}, \theta_{n-1})) (x, \xi)| d\theta_{n-1} \ldots d\theta_1 d\theta$$

$$\leq \sum_{n=1}^N \int_s^t \ldots \int_s^{\theta_{n-2}} |\sigma (W_1(t, \theta_1) \ldots W_1(\theta_{n-2}, \theta_{n-1}))|^{(0)} (\xi)^{-\alpha} d\theta_{n-1} \ldots d\theta$$
\[ \leq \langle \xi \rangle^{-\alpha} \sum_{n=1}^{N} \frac{2^{n-2} C_{\ell}^{n-2} \varphi_{n-1} |t-s|^{n-1}}{(n-1)!}. \]  

(3.30)

This equality means that for all \( \ell \in \mathbb{N} \)

\[ |\sigma(E_{N}(t,s))|^{(0)} \leq \sum_{n=0}^{N-1} \frac{(C_{\ell} |t-s|)^{n}}{n!}, \]

for a new constant \( C_{\ell, r} > 0 \). As \( N \to \infty \) we have

\[ |\sigma(E(t,s))|^{(0)} \leq \exp(C_{\ell, r}(t-s)) < \infty. \]

Thus the FIO (3.28) has a well-defined symbol. At the level of operators, by definitions (3.27) and (3.14) we have

\[ \hat{P} E_{N} = \hat{P} I_{\phi} + \hat{P} \int_{s}^{t} I_{\phi}(t, \theta) \sum_{n=1}^{N} W_{n}(\theta, s) d\theta \]

\[ = \hat{P} I_{\phi} - i \sum_{n=1}^{N} W_{n}(t, s) + \int_{s}^{t} \hat{P} I_{\phi}(t, \theta) \sum_{n=1}^{N} W_{n}(\theta, s) d\theta. \]

An induction shows that

\[ \sum_{n=1}^{N} W_{n}(t, s) = -i(\hat{P} I_{\phi})(t, s) - i \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) \sum_{n=1}^{N-1} W_{n}(\theta, s) d\theta. \]  

(3.31)

Indeed, for \( N = 2 \) we have by (3.21)

\[ W_{1}(t, s) + W_{2}(t, s) = -i(\hat{P} I_{\phi})(t, s) - i \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) W_{1}(\theta, s) d\theta. \]

The induction step \( N \mapsto N + 1 \) is shown as follows:

\[ \sum_{n=1}^{N+1} W_{n}(t, s) = W_{N+1}(t, s) + \sum_{n=1}^{N} (t, s) W_{n}(t, s) \]

\[ = -i \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) W_{N}(\theta, s) d\theta - i(\hat{P} I_{\phi})(t, s) \]

\[ - i \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) \sum_{n=1}^{N-1} W_{n}(\theta, s) d\theta \]

\[ = -i(\hat{P} I_{\phi})(t, s) - i \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) \sum_{n=1}^{N} W_{n}(\theta, s) d\theta, \]

which yields (3.31) for \( N + 1 \). Therefore

\[ (\hat{P} E_{N})(t, s) = \int_{s}^{t} (\hat{P} I_{\phi})(t, \theta) W_{N}(\theta, s) d\theta. \]

Now, for \( N \to \infty \), \( \|W_{N}(t, s)\| \to 0 \) in face of (3.20). Thus

\[ \hat{P} E_{N} \to \hat{P} E = 0. \]
Moreover, $E(s, s) = \text{id}$, which is easily verified. So we have constructed the fundamental solution of $\vec{\mathcal{P}}$, i.e. a family of FIOs $(E(t, s); (t, s) \in \Delta_T)$ satisfying the system (3.20). Notice that $(t, s) \mapsto E(t, s) \in \mathcal{C}(\Delta_T)$, with values in the space of the FIOs with some phase function $\phi$ and a symbol of order 0. This can be seen from (3.28), because $E$ is obtained by continuous operations of operators which are continuous in $t, s$. By Duhamel’s formula, the solution to system (3.13) is given by

$$W(t) = E(t, 0)W_0 + i \int_0^t E(t, \theta)G(\theta) d\theta.$$ 

Therefore, from (3.17), (3.16) we get

$$W(t) = E(t, 0) \left( \begin{align*}
(\langle D_x \rangle - m(0, \cdot, D_x)\lambda(0, \cdot, D_x))u_0 + im(0, \cdot, D_x)u_1 \\
\lambda(0, \cdot, D_x)u_0 - iu_1
\end{align*} \right) + i \int_0^t E(t, \theta) \left( \begin{align*}
(mf)(\theta) \\
-f(\theta)
\end{align*} \right) d\theta. \tag{3.32}
$$

Third step: Computing the fundamental solution of the equation (3.3). From the solution of the first-order system we can then go back to the solution of the original equation (3.3). For this we reverse the transformations (3.6) from $V$ to $V$, then from $V$ to $W$ and get

$$u(t) = \langle D_x \rangle^{-1}v_1(t) = \langle D_x \rangle^{-1}(w_1(t) + m(t)w_2(t)). \tag{3.33}$$

In the sequel, we denote by $(e_{jk}(t, s))_{j, k=1, 2}$ the entries in the operator matrix $E(t, s)$. Combining (3.33) and (3.32), we obtain the following representation for the solution $u$ of (3.3)

$$u(t) = T_1(t)u_0 + T_2(t)u_1 + \int_0^t T_3(t, s)f(s)ds, \tag{3.34}$$

where

$$T_1(t) := \langle D_x \rangle^{-1} \left[ (e_{1,1}(t, 0) + m(t)e_{2,1}(t, 0)) \cdot \langle D_x \rangle - m(0)\lambda(0) \right] + (e_{1,2}(t, 0) + m(t)e_{2,2}(t, 0))\lambda(0),$$

$$T_2(t) := i\langle D_x \rangle^{-1} \left[ (e_{1,1}(t, 0) - e_{1,2}(t, 0) + m(t)e_{2,1}(t, 0))m(0) - m(t)e_{2,2}(t, 0) \right],$$

$$T_3(t, s) := i\langle D_x \rangle^{-1} \left[ (e_{1,1}(t, s) + m(t)e_{2,1}(t, s))m(s) - e_{1,2}(t, s) + m(t)e_{2,2}(t, s) \right].$$

We see that $T_1(t) = T_1(t, x, D_x)$ is a FIO with symbol of order 0 and $T_2(t) = T_2(t, x, D_x)$ is a FIO with symbol of order $-1$. So if $u_0 \in H^r$ and $u_1 \in H^{r-1}$, then $g(t) := T_1(t)u_0 + T_2(t)u_1 \in H^r$. Due to the assumption that $2r > d$, we conclude by Sobolev’s Embedding Theorem that $g(t) \in \mathcal{C}(\mathbb{R}^d)$. Moreover, since $t \mapsto g(t)$ is continuous, we have that at every point $(t, x) \in [0, \bar{T}] \times \mathbb{R}^d$, $g(t, x)$ is well-defined, which implies (A5).

Now we deal with the third term in (3.34). We see that $T_3$ (or its Schwartz kernel) is the fundamental solution of the second order equation (3.3) with null initial conditions. By Proposition 2.20 it is easy to see that it is a sum of four
FIOs, each one of order $-1$. Moreover, $T_{3}(t,s)$ has a symbol in $C(\Delta, S^{-1})$ since it depends continuously on $E$ with symbol in $C(\Delta, S^{0})$ and $m(t,x,D_{x})$ with symbol $m(t,x,\xi) \in C([0,T], S^{0})$ ($m$ is continuous in time because it depends continuously on the characteristic roots $\pm \lambda$, and by definition \(\Delta\) the roots inherit the regularity with respect to time of the coefficients of the PDE).

With this we can finally show the conditions (A1)-(A4). In order to show (A1) and (A3) with $\Lambda(t,s)$ set to be the Schwartz kernel of $T_{3}(t,s)$, for each $(t,s) \in \Delta$ we invoke Proposition 2.18 together with (2.17) to see that

$$|F_{y}(t,s,\cdot)(\xi)|^2 = |\sigma(T_{3}(t,s))(x,-\xi)|^2 \leq C_{t,s}(\xi)^{-2}.$$  

Therefore the conditions (A1) and (A3) become

$$\int_{0}^{t} \sup_{\xi \in \mathbb{R}^{d}} \left| \int_{\mathbb{R}^{d}} |F_{y}(t,s,x,\cdot)(\eta + \xi)|^2 \mu(d\eta) \right| ds \leq \int_{0}^{t} C_{t,s} ds \sup_{\xi \in \mathbb{R}^{d}} \frac{1}{1 + |\xi|^2} \mu(d\eta),$$

$$\int_{0}^{t} \sup_{\xi \in \mathbb{R}^{d}} \left| \int_{\mathbb{R}^{d}} |F_{y}(t,s,x,\cdot)(\xi)|^2 \right| ds \leq \int_{0}^{t} C_{t,s} ds \sup_{\xi \in \mathbb{R}^{d}} \frac{1}{1 + |\xi|^2}.$$  

Since the constants $C_{t,s}$ can be chosen in such a way that they are continuous in $s$ and $t$, we have that (A1) holds as long as (2.14) holds and (A3) is always satisfied.

To check the two continuity conditions (A2) and (A4), by Lebesgue Theorem it is sufficient to check that the Schwartz kernel $\Lambda$ of the fundamental solution $T_{3}$ is continuous in $s$ and $t$ and there exists a bounded function $k : \mathbb{R}^{d} \to \mathbb{R}$ such that

$$|\mathcal{F}(t,s,\cdot)(\xi + \eta)| \leq k(\xi + \eta), \quad \text{and} \quad \sup_{\eta \in \mathbb{R}^{d}} \int_{\mathbb{R}^{d}} k(\xi + \eta)^2 \mu(d\xi) < \infty.$$  

But this immediately follows taking $k(x) := C(x)^{-1}$ for some constant $C > 0$, given by the maximum of $C_{t,s}$ over the simplex $\Delta$. The proof is complete.

**Remark 3.2.** Theorem 3.1 holds also if we substitute the assumption “$\gamma$ as in (A3)” with the (presumably) more general assumption $\gamma \in C([0,T], H^{r-1})$. In the latter case, the term $\int_{0}^{T} T_{3}(t,s) \gamma(s) ds$ in (3.32) can be put into the deterministic function $g(t) \in H^{r}$, and it is pointwise well defined. In the statement of Theorem 3.1 we choose to ask $\gamma$ as in (A3) that is to consider it in a similar fashion as the stochastic integral term. The reason for this is that if the problem of nonstationary nonlinear SPDEs (where $\sigma$ and $\gamma$ may depend on the solution $u$) with a general distribution as fundamental solution is solved, then the extension of the results in Theorem 3.1 to the nonlinear case is possible.

**Remark 3.3.** The proof of Theorem 3.1 points out that we only need to assume that the coefficients $a, b, c$ are $C_{\ell}^{\ell}$-functions in the spatial argument for a sufficiently large $\ell \in \mathbb{N}$. Such an $\ell$ has to be large enough such that for every $t \in [0,T]$ all the products $Q_{1} \ldots Q_{n-1}$ in the computation of $E$ map continuously Sobolev spaces into Sobolev spaces, by Proposition 2.22. This $\ell$ cannot be computed explicitly in the general case, but for a (simple enough) example it is possible to provide its precise value.
4 Second-order hyperbolic equations - the case of weak hyperbolicity

In this section we show that the assumption of strict hyperbolicity (3.2) on equation (3.1) is an important one. In fact, if this assumption does not hold, the associated PDE might not be well posed, neither in $C^\infty$ (or in usual Sobolev spaces), see [6], nor in weighted Sobolev spaces, see [19]. However, results of ”well-posedness with loss of derivatives” in Sobolev spaces can be obtained under suitable assumptions in the case of weakly hyperbolic equations, i.e. equations having real characteristic roots which are not necessarily distinct and separate at every time, see for example [2] and the references therein. Here ”well-posedness with loss of derivatives” means that in this case the fundamental solution $E(t,s)$ results to be a Fourier integral operator of order $\delta > 0$, and so by Duhamel’s formula (2.27) one obtains a solution $U$ which is less regular than the data, i.e. if the right-hand side $G$ and the initial data are in $H^r$ for some $r \geq 0$, then the solution $U$ is in $H^{r-\delta}$, as proved for instance in [3, 4].

This leads to the fact that without the assumption in (3.2) the Fourier transform of the fundamental solution might not behave as $|\xi|^{-1}$ as shown in the previous section, but only as $|\xi|^{-\nu}$ with $\nu \in (0, 1)$.

In this section we give an example of a weakly hyperbolic equation with fundamental solution that satisfies (2.14) with some $\nu \in [0, 1)$.

Let us so consider the following SPDE in spatial dimension $d = 1$:

$$
\begin{align*}
\left\{ \begin{array}{l}
(\partial^2_t - t^k \partial^2_x + ct^{i\rho} \partial_x) \ u(t,x) = \dot{F}(t,x), \\
n(0,x) = 0, \\
\partial_t u(0,x) = 0,
\end{array} \right. \\
(4.1)
\end{align*}
$$

where $(t,x) \in [0,1] \times \mathbb{R}$, $k \in \mathbb{N}$ with $k \geq 2$, $\rho = \frac{1}{2} - \frac{1}{k}$ and $c > 0$ is a constant that we will set later. Then we know by [3, Theorem 1.1] that the associated PDE to problem (4.1) is well-posed in Sobolev spaces with loss of derivatives, and the fundamental solution $\Lambda$ exists. Therefore, the solution to problem (4.1) is given by

$$
\begin{align*}
u(t,x) = \int_0^t \int_\mathbb{R} \Lambda(t,s,x-y) M(ds,dy),
\end{align*}
$$

since the coefficients and the right-hand side of the equation in (4.1) do not depend on the spatial argument. In this section we will construct the fundamental solution $\Lambda$ and derive an upper estimate as in (2.14).

Equation (4.1) can be reformulated using $D = -i\partial$:

$$
\begin{align*}
\left\{ \begin{array}{l}
(D^2_t - t^k D^2_x - cit^{i\rho} D_x) \ u(t,x) = -f(t,x), \\
n(0,x) = 0, \\
D_t u(0,x) = 0,
\end{array} \right. \\
(4.1)
\end{align*}
$$

where as in Section 3 the right-hand side is given by $f = \dot{F}$.

Now we proceed as in Section 3 computing the characteristic roots of the partial differential operator (4.1). Its principal symbol is given by $\tau^2 - t^k \xi^2$, so its characteristic roots are $\lambda(t,\xi) = \pm t^{k/2} |\xi|$ for all $(t,x) \in [0,1] \times \mathbb{R}$. Since these roots coincide (and vanish) at $t = 0$ for every fixed $\xi \neq 0$, we define the
approximated characteristic roots

\[ \tilde{\lambda}(t, \xi) := \pm \sqrt{t^k + \langle \xi \rangle^{-2}} \cdot |\xi| = \pm \sqrt{1 + t^k \langle \xi \rangle^2} \cdot (\langle \xi \rangle^{-1}) |\xi|. \]

Moreover, we set

\[ \zeta(t, \xi) := \sqrt{1 + t^k \langle \xi \rangle^2}, \]

and easily see that \( \zeta \in C([0, 1], S^1) \).

Now we define similarly as in (3.0), but using the approximated characteristic roots

\[
\begin{align*}
v_1(t, x) &:= \zeta(t, D_x)u(t, x) \\
v_2(t, x) &:= (D_t + \tilde{\lambda}(t, D_x))u(t, x),
\end{align*}
\]

and by performing similar calculations as in Section 3 we obtain

\[ (D_t + \tilde{\lambda}(t, D_x))v_1 = \zeta(t, D_x)(D_t + \tilde{\lambda}(t, D_x))u - \frac{i\partial \zeta}{\partial t}(t, D_x)u = \zeta(t, D_x)v_2 - R_0v_1, \]

where \( R_0 = R_0(t, D_x) \) is the PDO with symbol

\[ r_0(t, \xi) = \frac{i k t^{k-1}\langle \xi \rangle^2}{2(1 + t^k \langle \xi \rangle^2)} = \frac{i k t^{k-1}}{2(t^k + \langle \xi \rangle^{-2})}. \] (4.2)

Applying Corollary 2.21 we see that \( \tilde{\lambda} \) is a PDO with symbol \((t^k + \langle \xi \rangle^{-2})\xi^2\); then we obtain

\[ (D_t - \tilde{\lambda}(t, D_x))v_2 = (D_t - \tilde{\lambda}(t, D_x))(D_t + \tilde{\lambda}(t, D_x))u \]

\[ = \left( D_t^2 - \frac{i k t^{k-1}}{\langle \xi \rangle^2} D_x - (t^k + \langle D_x \rangle^{-2})D_x^2 \right)u \]

\[ = -f - \left( - \frac{c k t^{k-1}}{\sqrt{t^k + \langle \xi \rangle^{-2}}} |D_x| + \langle D_x \rangle^{-2} D_x^2 \right)u \]

\[ = -f(t, x) - T_0(t, D_x)v_1, \]

where \( T_0(t, D_x) \) is a PDO with symbol

\[ n_0(t, \xi) = -\frac{c k t^{k-1}}{\langle \xi \rangle^2 \sqrt{t^k + \langle \xi \rangle^{-2}}} + \frac{i k t^{k-1} |\xi|}{2(t^k + \langle \xi \rangle^{-2}) \langle \xi \rangle} + \frac{\xi^2}{\langle \xi \rangle^2 \zeta(t, \xi)}. \] (4.3)

So we obtain the equivalent first order system

\[
\begin{align*}
P V(t) &= G, \quad \text{in } [0, 1], \\
P V(0) &= 0,
\end{align*}
\] (4.4)

where \( V = (v_1, v_2), G = (0, -f), \) and

\[ P = D_t + \begin{pmatrix} \tilde{\lambda}(t, D_x) & -\zeta(t, D_x) \\ 0 & -\tilde{\lambda}(t, D_x) \end{pmatrix} + \begin{pmatrix} R_0(t, D_x) & 0 \\ T_0(t, D_x) & 0 \end{pmatrix}. \]

Now, before diagonalizing the system, we investigate the order of the two symbols in (4.2) and (4.3). For this we need three lemmas, the first one being an important integral inequality.
Lemma 4.1. For all $\alpha, \beta, \delta > 0$ we have

$$
\int_0^1 \frac{t^\alpha}{(t^\delta + \langle \xi \rangle^{-2})^\beta} dt \leq \begin{cases} 
C_{\alpha, \beta, \delta} \frac{1}{\alpha+1} + \log \left( \frac{\langle \xi \rangle^{2\beta/(\alpha+1)}}{\langle \xi \rangle^{2(\beta-\alpha-1)/\delta}} \right) & \text{for } \alpha - \beta \delta > -1, \\
\frac{1}{\alpha+1} & \text{for } \alpha - \beta \delta = -1, \\
C_{\alpha, \beta, \delta} \langle \xi \rangle^{2(\beta-\alpha-1)/\delta} & \text{for } \alpha - \beta \delta < -1.
\end{cases}
$$

Proof. By separating the domain of integration into $[0, h]$ and $[h, 1]$, where $h := \langle \xi \rangle^{-2\beta/(\alpha+1)}$ we obtain

$$
\int_0^1 \frac{t^\alpha}{(t^\delta + \langle \xi \rangle^{-2})^\beta} dt \leq \langle \xi \rangle^{2\beta} \int_0^h t^\alpha dt + \int_h^1 t^{\alpha-\beta \delta} dt \leq \frac{1}{\alpha+1} + \frac{1}{\alpha - \beta \delta + 1}.
$$

if $\alpha - \beta \delta > -1$; if $\alpha - \beta \delta = -1$, then

$$
\int_0^1 \frac{t^\alpha}{(t^\delta + \langle \xi \rangle^{-2})^\beta} dt \leq \frac{1}{\alpha+1} + \log (h^{-1}).
$$

In the case when $\alpha - \beta \delta < -1$, we obtain by the change of variable $t^\delta + \langle \xi \rangle^{-2} \mapsto s$

$$
\int_0^1 \frac{t^\alpha}{(t^\delta + \langle \xi \rangle^{-2})^\beta} dt = \frac{1}{\delta} \int_{\langle \xi \rangle^{-2}}^{1+\langle \xi \rangle^{-2}} \frac{(s - \langle \xi \rangle^{-2})^{\alpha+1/\delta-1}}{s^{\alpha+1-\beta \delta}} ds
\leq \frac{1}{\delta} \int_{\langle \xi \rangle^{-2}}^{1+\langle \xi \rangle^{-2}} s^{(\alpha+1-\beta \delta)/\delta} ds
= \frac{1}{\alpha + 1 - \beta \delta} \left( (1 + \langle \xi \rangle^{-2})^{(\alpha+1-\beta \delta)/\delta} - (\langle \xi \rangle^{-2})^{(\alpha+1-\beta \delta)/\delta} \right)
\leq \frac{1}{\beta \delta - \alpha - 1} \langle \xi \rangle^{2(\beta-\alpha-1)/\delta}.
$$

The other two lemma give bounds on the derivatives of some of the terms in the symbols of $R_0$ and $T_0$.

Lemma 4.2. For all $l \in \mathbb{N}_0$, and $j \in \{0, 1\}$

$$
\int_0^1 \left| \partial^l_{\xi} \frac{t^{k-j}}{t^k + \langle \xi \rangle^{-2}} \right| dt \leq C_l \langle \xi \rangle^{-l},
$$

where $C_l \leq l \cdot 2^{l+2}$.

Proof. An induction shows that the partial derivatives can be bounded form above by

$$
\partial^l_{\xi} \frac{t^{k-j}}{t^k + \langle \xi \rangle^{-2}} \leq \tilde{C}_l \sum_{m=1}^{l} \langle \xi \rangle^{-l-2m} \frac{t^{k-j}}{(t^k + \langle \xi \rangle^{-2})^{m+1}},
$$

where $\tilde{C}_l \leq 2^{l+2}$. Then integrating this over $[0, 1]$ and using Lemma 4.1 with $\alpha = k - j$, $\beta = m + 1$ and $\delta = k$, we get

$$
\int_0^1 \left| \partial^l_{\xi} \frac{1}{t^k + \langle \xi \rangle^{-2}} \right| dt \leq \tilde{C}_l \sum_{m=1}^{l} \langle \xi \rangle^{-l-2m} \int_0^1 \frac{t^{k-j}}{(t^k + \langle \xi \rangle^{-2})^{m+1}} dt
\leq \tilde{C}_l \langle \xi \rangle^{-l} \sum_{m=1}^{l} \frac{1}{nk + j - 1}
\leq l \cdot \tilde{C}_l \langle \xi \rangle^{-l}.
$$

\[\square\]
This lemma implies that \( r_0 \) in (1.2) is of order zero. Moreover, we immediately see that the third term on the right-hand side of (3.6) is of order \(-1\) since \( \zeta \) is of order 1. The second symbol is of order zero, being the product of the two symbols of order zero \( k t^{k-1}/(t^k + \langle \xi \rangle^{-2}) \) and \( \langle \xi \rangle/\langle \xi \rangle \).

The following lemma is needed to investigate the first term on the right-hand side of (3.3).

**Lemma 4.3.** For all \( l \in \mathbb{N}_0 \),

\[
\int_0^1 \left| \frac{\partial_l^k}{\sqrt{1 + t^k \langle \xi \rangle^2}} \right| dt \leq C_l \langle \xi \rangle^{-l} \log(1 + \langle \xi \rangle), \tag{4.5}
\]

for some constant \( C_l < 2l \cdot 1! \cdot (2l - 1)! \).

**Proof.** First we see what happens at the level of \( l = 0 \). Then the integrand in (4.4) can be written as

\[
\frac{t^{k \rho} \xi}{\langle \xi \rangle \sqrt{t^k + \langle \xi \rangle^2}} = \frac{\xi}{\langle \xi \rangle} \left( \frac{t^k}{t^k + \langle \xi \rangle^2} \right)^{\rho} \frac{1}{(t^k + \langle \xi \rangle^{-2})^{1/2}}, \tag{4.6}
\]

where, with the help of Lemma 4.2 the first two terms can be seen to be symbols of order zero. The third term is a symbol in \( S^{2/k} \subseteq S^1 \), and using Lemma 4.1 with \( \alpha = 0, \beta = k^{-1} \) and \( \delta = k \), we get

\[
\int_0^1 \frac{1}{(t^k + \langle \xi \rangle^{-2})^{1/k}} dt \leq 1 + \log(\langle \xi \rangle^{2/k}) \leq 1 + \log(\langle \xi \rangle). \tag{4.7}
\]

The next step is to investigate the derivatives of the term in (4.6). We define

\[
\theta(t, \xi) := \frac{t^{k \rho}}{\sqrt{1 + t^k \langle \xi \rangle^2}} = \frac{t^{k \rho}}{\langle \xi \rangle \sqrt{\langle \xi \rangle^{-2} + t^k}} = \left( \frac{t^k}{\langle \xi \rangle^{-2} + t^k} \right)^{\rho} \frac{1}{(\langle \xi \rangle^{-2} + t^k)^{1/k}} \langle \xi \rangle,
\]

and for all \( l \in \mathbb{N}_0 \)

\[
\Theta_l(t, \xi) := (-1)^l(2l - 1)!! \frac{t^{k(\rho+1)}}{(1 + t^k \langle \xi \rangle^2)(2l+1)/2} = (-1)^l(2l - 1)!! \frac{t^{k(\rho+1)}}{\langle \xi \rangle^{2l+1} + t^k(2l+1)/2}
\]

\[
= (-1)^l(2l - 1)!! \left( \frac{t^k}{\langle \xi \rangle^{-2} + t^k} \right)^{\rho+l} \frac{1}{(\langle \xi \rangle^{-2} + t^k)^{1/k}} \langle \xi \rangle^{2l+1},
\]

where \( n!! \) denotes the odd factorial of an odd number, i.e. \( n!! = n(n - 2) \ldots 1 \), and \( (-1)!! := 1 \). We have that \( \Theta_0(t, \xi) = \theta(t, \xi) \) and

\[
\partial_l \Theta_l(t, \xi) = \Theta_{l+1}(t, \xi) \partial_l \langle \xi \rangle = \Theta_{l+1}(t, \xi) \xi.
\]

Set furthermore

\[
\hat{\Theta}_l(t, \xi) := (-1)^l(2l - 1)!! \left( \frac{t^k}{\langle \xi \rangle^{-2} + t^k} \right)^{\rho+l} \frac{1}{(\langle \xi \rangle^{-2} + t^k)^{1/k}}, \tag{4.8}
\]
\[
\hat{\Theta}_l(t, \xi) = \Theta_l(t, \xi)(\xi)^{2l+1}.
\]

Note that the term in the brackets in (4.8) is bounded by 1.

In the sequel, we will deal with symbols of the form \( p_{a_1, a_2}(\xi) := \xi^{a_1}(\xi)^{a_2} \in S^0_{a_1 + a_2} \), where \( a_1, -a_2 \in \mathbb{N}_0 \). Note that the symbols \( \hat{\Theta}_l(t, \xi) \) and \( p_{a_1, a_2} \) are bounded by \((2l)!\) and 1 respectively. With this preparation, we can evaluate the derivatives of the function on the left-hand side of (4.6), which is equal to bounded by (2l)!! and 1 respectively. With this preparation, we can evaluate the derivatives of the function on the left-hand side of (4.6), which is equal to \( \theta(t, \xi) \xi \). Its derivatives are given by the following formula. Set \( l^* := l/2 + 1 \) if \( l \) is even and \( l^* := (l + 1)/2 + 1 \) if \( l \) is odd, i.e. \( l^* = [l/2 + 1] \). Then

\[
\sigma_{l,j} = \frac{\xi(t, \xi)}{2\lambda(t, \xi)} = \frac{\sqrt{1 + t^k(\xi)^2}}{2\sqrt{(\xi)^{-2} + t^k|\xi|}} = \frac{\langle \xi \rangle}{2|\xi|}.
\]

Note that using Corollary 2.21 the operator \( Q_0 := \hat{\lambda} m + \lambda \hat{\lambda} - \zeta \) can be computed to be the zero operator because we have for its symbol

\[
\sigma(Q_0) = \sqrt{\sqrt{k} + (\xi)^{-2}|\xi|\langle \xi \rangle^2\frac{2|\xi|}{2|\xi|} + \langle \xi \rangle\sqrt{\sqrt{k} + (\xi)^{-2}|\xi|} - \sqrt{1 + t^k(\xi)^2}} = 0.
\]

Then, with the change of variable \( W := M^{-1}V \) the system becomes

\[
\begin{aligned}
\dot{W}(t, x) &= \tilde{G}(t, x),\ (t, x) \in (0, 1] \times \mathbb{R}, \\
W(0, x) &= 0, \quad x \in \mathbb{R},
\end{aligned}
\]

with \( \tilde{G} = (mf, -f)^T \) and

\[
\dot{\tilde{P}}(t, x, D_x) = \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \begin{pmatrix} \tilde{\lambda}(t, x, D_x) & 0 \\ 0 & -\tilde{\lambda}(t, x, D_x) \end{pmatrix} + R(t, x, D_x),
\]

with

\[
R = \begin{pmatrix} R_0 - mT_0 & R_0m - mT_0m \\ T_0m \end{pmatrix}.
\]
Note that here $\mathcal{R}$ is a matrix of PDOs of order $2/k$, not of order zero as it was in Section 3, cfr. formula (3.15). It’s important to remark here that in the proof of Theorem 3.1 we constructed a FIO $W_1$ in (3.20) with the same order as $\mathcal{R}$; the behavior of $W_1$ was used to obtain the well definedness of the symbol of $E_N$ and its order, see (3.30). More precisely, to obtain formula (3.30) it was crucial to have a uniform in time estimate of $\int_1^T W_1(t,\theta)\,d\theta$.

Here we want to follow the same ideas, so now we derive an integral estimate for the symbols of the four operators in the matrix (4.11). It can be easily checked that the symbols of the operators $mT_0$, $T_0m$ and $mT_0m$ satisfy the same integral estimate as in (4.9) with the same constants $C_i$, which was derived in Lemma 4.3. For the latter estimate we consider as in Definition 2.8 the symbol only for $\xi$ outside the ball with radius $R > 1$. The symbols of the other two operators $R_0$ and $R_0m$ satisfy the same integral inequality as in Lemma 4.2 with the constant $C_l \cdot k$. The symbols of the four PDOs in $\mathcal{R}$, denoted by $r_{i,j}$ for $i, j \in \{1, 2\}$, satisfy therefore

$$\int_0^1 |\partial^\xi r_{i,j}(t,\xi)|\,dt \leq (cC_l^i \log(1 + \langle \xi \rangle) + C_l \cdot k)\langle \xi \rangle^{-1} \leq C_{k,i} (1 + c\log(1 + \langle \xi \rangle))\langle \xi \rangle^{-1},$$

where $C_{k,i} \leq C_l^i + C_l \cdot k$. Then for $\xi$ outside a sufficiently large ball, whose radius may depend on $c$, $c\log(1 + \langle \xi \rangle)$ dominates the constant $1$.

Now, to construct the fundamental solution $E$ to the system (4.10) we have to substitute the approximate characteristic roots by the true ones, rewriting the operator $P$ in the form

$$P(t, x, D_x) = \begin{pmatrix} D_t & 0 \\ 0 & D_x \end{pmatrix} + \begin{pmatrix} \lambda(t, x) & 0 \\ 0 & -\lambda(t, D_x) \end{pmatrix} + \mathcal{R}(t, x, D_x),$$

with

$$\mathcal{R} = \mathcal{R} + (\lambda - \lambda) \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$  

We notice that

$$(\lambda - \lambda)(t,\xi) = \left(\sqrt{t^k + \langle \xi \rangle^{-2} - t^{k/2}}\right)|\xi| = \frac{\langle \xi \rangle^{-2}|\xi|}{\sqrt{t^k + \langle \xi \rangle^{-2} + t^{k/2}}}$$

is a symbol of order zero, and by Lemma 4.4 with $\alpha = 0$, $\beta = 1/2$, $\delta = k$ we have (since $k > 2$)

$$\int_0^1 |(\lambda - \lambda)(t,\xi)|\,dt \leq \frac{2}{k - 2} \langle \xi \rangle^{1 - (2k)^{-1}} \langle \xi \rangle^{-2} |\xi| \leq \frac{2}{k - 2};$$

similarly, we can compute

$$\int_0^1 \left|\partial^\xi (\lambda(t,\xi) - \lambda(t,\xi))\right|\,dt \leq \frac{2}{k - 2} C_l \langle \xi \rangle^{-1},$$

with $C_l$ computed in Lemma 4.2. This means that $\mathcal{R}$ is again a matrix of PDOs of order $2/k$ and the symbols of its entries satisfy

$$\int_0^1 |\partial^\xi r_{i,j}(t,\xi)|\,dt \leq \left(2cC_{k,i}^i \log(1 + \langle \xi \rangle) + C_l \frac{2}{k - 2}\right)\langle \xi \rangle^{-1}.$$
\[
\leq C_{k, l} \log(1 + (\xi))(\xi)^{-t},
\]
(4.12)
with \(C_{k, l} \leq 2C_{k, l} + 2C_{l}(k - 2)\).

We define now \(I_{0}, \phi_{k}, W_{1}, W_{n}, E_{N}, E\) as in the proof of Theorem 3.1 with the difference that here we have \(E\) satisfying (4.12) instead of \(E\) of order zero in (3.20). In the estimate of the derivatives of the symbol of \(E_{N}\), cfr. formula (3.30), by Proposition 2.22 we have that for every given \(r > 0\) there exist a constant \(\bar{C}_{r} > 0\) and an integer \(l_{0} \in \mathbb{N}_{0}\) sufficiently large such that when setting

\[
\delta_{0} := \sup_{l \leq l_{0}} C_{k, l} = C_{k, l_{0}} \leq 2l_{0} \cdot l_{0}! \cdot (2l_{0} - 1)!!
\]

we get for every \(\ell \leq l_{0}\)

\[
|\partial^{\ell}_{x}(E_{N}(t, s))(x, \xi)| \leq \sum_{n=1}^{N} \int_{0}^{t} \int_{0}^{\theta_{1}} \cdots \int_{0}^{\theta_{n-2}} |\partial^{\ell}_{x}(W_{1}(t, \theta_{1}) \cdots W_{1}(\theta_{n-2}, \theta_{n-1}))(x, \xi)| d\theta_{n-1} \cdots d\theta_{1} d\theta
\]

\[
\leq \langle \xi \rangle^{-\ell} (2C_{\ell, r})^{-1} \sum_{n=1}^{N} \frac{(2C_{\ell, r} C_{r} \delta_{0} (1 + \log(\langle \xi \rangle)))^{n-1}}{(n - 1)!} \leq C_{\ell, r} \langle \xi \rangle^{\delta - \ell},
\]
(4.13)
where

\[
\delta := 2C_{\ell, r} \bar{C}_{r} \delta_{0},
\]
with a new constant \(C_{\ell, r} > 0\). Thus the FIO \(E\) defined as in (3.28) has a well-defined symbol of order \(\delta > 0\). The same computations as in Section 3 now give that \(E\) is the fundamental solution to the system

\[
\begin{cases}
\hat{P}E(t, s) = 0 & (t, s) \in \Delta_{\bar{T}} \\
E(s, s) = \text{id} & s \in [0, \bar{T}]
\end{cases}
\]

for some \(\bar{T} < 1\). For more precise computations (in a general setting) we refer to [3] Theorem 3.1.

Now if we set \(0 < c < (2C_{\ell, r} \bar{C}_{r} \delta_{0})^{-1}\), then \(0 < \delta < 1\). Next we compute the order of the fundamental solution to (4.11) using a backwards transformation similar to (3.33), i.e.

\[
u(t) = \zeta(t, D_{x})^{-1} (w_{1}(t) + mw_{2}(t)).
\]

So substituting the above expressions into this equation we obtain that the fundamental solution to (4.11) is given by

\[
T_{3}(t, s) = i\zeta(t)^{-1}(e_{1,1}(t, s)m - e_{1,2}(t, s) + me_{2,1}(t, s)m - me_{2,2}(t, s)).
\]

Since \(\zeta\) is an operator of order 1 and \(m\) is an operator of order 0, we have the fundamental solution is an operator with symbol of order \(\delta - 1 \in (-1, 0)\). Therefore, the condition (A1) on the spectral measure for the existence and uniqueness of a random-field solution in this case becomes

\[
\sup_{\eta \in \mathbb{R}} \int_{\mathbb{R}^{d}} \frac{1}{(1 + |\xi + \eta|^{2})^{1 - \delta}} \mu(d\xi) < \infty.
\]

The other conditions for the existence and uniqueness can be shown with the similar arguments as at the end of the proof of Theorem 5.1.
5 Higher-order hyperbolic equations

In this section we give another generalization of the solution theory presented in Section 3. We treat higher order equations of the form

$$P(t, x, D_t, D_x)u(t, x) = \gamma(t, x) + \sigma(t, x)\hat{F}(t, x),$$  \hspace{1cm} (5.1)

where for \(n \in \mathbb{N}, n \geq 2,\)

$$P(t, x, D_t, D_x) = D^n_t + \sum_{j=0}^{n-1} \sum_{|\alpha| = n-j} a_{\alpha, j}(t, x)D_\alpha x D^j_t,$$  \hspace{1cm} (5.2)

with some suitable coefficients \(a_{\alpha, j},\) see Theorem 5.1 below. As in Section 3, we assume \(P\) to be strictly hyperbolic, which means here that the symbol of the principal part given by

$$p_n(t, x, \tau, \xi) = \tau^n + \sum_{j=0}^{n-1} \sum_{|\alpha| = n-j} a_{\alpha, j}(t, x)\xi^\alpha \tau^j,$$  \hspace{1cm} (5.3)

factorizes as

$$p_n(t, x, \tau, \xi) = \prod_{j=1}^{n}(\tau + \lambda_j(t, x, \xi)),$$  \hspace{1cm} (5.4)

where the \(n\) characteristic roots \(\lambda_j\) of \(p_n\) are such that \(\lambda_j(t, x, \xi) \in \mathbb{R}\) for all \(1 \leq j \leq n,\) and

$$|\lambda_j(t, x, \xi) - \lambda_k(t, x, \xi)| \geq c|\xi|,$$  \hspace{1cm} (5.4)

for some \(c > 0\) and for all \(\xi \neq 0, j \neq k.\) Note that in general one cannot compute the roots \(\lambda_j\) explicitly, as in the case of second order equations, because of the lack of a general resolution formula for higher order polynomial equations. The result of this section is the following.

**Theorem 5.1.** Let us consider an SPDE (5.1) where the partial differential operator \(P\) is of the form (5.2) with coefficients \(a_{\alpha, j} \in C^{n-1}([0, T]; C^\infty_b(\mathbb{R}^d))\) for \(|\alpha| = n-j, a_{\alpha, j} \in C([0, T]; C^\infty_b(\mathbb{R}^d))\) for \(|\alpha| < n-j, 0 \leq j \leq n-1.\) Suppose that \(P\) is a strictly hyperbolic operator, i.e. (5.3) and (5.4) hold. Assume for the initial conditions that \(D^j_t u(0) =: u_j \in H^{r-j}(\mathbb{R}^d) 0 \leq j \leq n-1,\) where \(2r > d.\) Furthermore, assume that \(\gamma\) and \(\sigma\) are as in Theorem 3.1 and that

$$\sup_{\nu \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{(1 + |\xi + \eta|^2)^{n-1}} \mu(d\xi) < \infty.$$  \hspace{1cm} (5.5)

Then, for some time horizon \(0 < \bar{T} \leq T,\) the SPDE (5.1) with partial differential operator (5.2) has a unique random-field solution.

To prove Theorem 5.1 we reduce the Cauchy problem for the operator \(P\) to the Cauchy problem for a first order system \(PV = G,\) with \(P\) as in (5.4), but with \(n \times n\) instead of \(2 \times 2\) matrices. Then, following the proof of Theorem 3.1 and only changing the dimensions of the matrices, the steps of the diagonalization, the computation of the fundamental solution, and the return to the original equation follow by the same ideas, but more technicism.
Proof. Before starting with the proof, we need to point out that the factorization \(5.3\) of the principal symbol of \(P\) can be brought to the level of operators producing the following factorization for the principal part of the operator \(P\):

\[
\begin{align*}
P(t,x,D_t,D_x) &= \prod_{j=1}^{n} (D_t + \lambda_j(t,x,D_x)) + \sum_{j=0}^{n-1} S_j(t,x,D_x)D_t^j, \quad (5.6)
\end{align*}
\]

where \(S_j\) are PDOs with symbols \(S_j(t,x,\xi) \in C([0,T]; S^{n-j-1})\). This can be shown by induction, where the case \(n = 2\) has already been done in the proof of Theorem 3.1, see formulas (3.7) and (3.8). For a detailed proof of (5.6) we refer to [1, Proposition 3.2, \(p = 1, n = 0\)].

First step: Reduction to a first-order system. Let us denote, as in the proof of second-order equations, a right-hand side by \(f_{t,x,D}\) arguments (\(t,x,\gamma\)). In the following, we set \(f(t,x) := \gamma(t,x) + \sigma(t,x)\dot{F}(t,x)\), and define the vector \(V := (v_1, \ldots, v_n)\) as follows:

\[
\begin{align*}
\begin{cases}
v_1 := (D_x)^{n-1}u, \\
v_j := (D_x)^{n-j}(D_t + \lambda_{j-1}) \cdots (D_t + \lambda_1)u, & j = 2, \ldots, n.
\end{cases} \quad (5.7)
\end{align*}
\]

With computations similar to the ones leading to (3.11) (and dropping the arguments \(t,x,D_x\)), we obtain that for all \(j = 1, \ldots, n-1\)

\[
\begin{align*}
(D_t + \lambda_j)v_j &= (D_t + \lambda_j)(D_x)^{n-j}(D_t + \lambda_{j-1}) \cdots (D_t + \lambda_1)u \\
&= (D_x)^{n-j}(D_t + \lambda_j)(D_t + \lambda_{j-1}) \cdots (D_t + \lambda_1)u \\
&\quad + [\lambda_j, (D_x)^{n-j}](D_t + \lambda_{j-1}) \cdots (D_t + \lambda_1)u \\
&= (D_x)v_{j+1} + [\lambda_j, (D_x)^{n-j}](D_x)^{-n(j)}v_j,
\end{align*}
\]

and for \(j = n\)

\[
(D_t + \lambda_n)v_n = \prod_{j=1}^{n} (D_t + \lambda_j(t,x,D_x))u = f - \sum_{j=0}^{n-1} S_j(t,x,D_x)D_t^j u.
\]

By the reduction (5.7), working again by induction (for a proof, see [1] formula (4.8), \(p = 1\)), we get

\[
D_t^j u = (D_x)^{-(n-j-1)} \sum_{\ell=1}^{j+1} S_{\ell}^{(0)}(t,x,D_x) v_{\ell},
\]

where \(S_{\ell}^{(0)}\) are PDOs of order zero, \(1 \leq \ell \leq j + 1\), and so

\[
(D_t + \lambda_n)v_n = f - \sum_{j=1}^{n} R_j(t,x,D_x)v_j,
\]

for some PDOs \(R_j\) of order zero. Summing up, equation (5.11) is equivalent to the first-order system

\[
\begin{align*}
\begin{cases}
P(t,x,D_t,D_x)V = G & \text{on } (0,T] \times \mathbb{R}^d, \\
V(0) = V_0, & \text{on } \mathbb{R}^d.
\end{cases}
\end{align*}
\]

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where $V = (v_1, \ldots, v_n)^T$, $P = D_t + K + \mathcal{R}$,

$$
\begin{pmatrix}
\lambda_1 - \langle D_x \rangle & 0 & 0 & \cdots & 0 & 0 \\
0 & \lambda_2 - \langle D_x \rangle & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\
0 & \cdots & \cdots & \lambda_{n-1} - \langle D_x \rangle & 0 & 0 \\
0 & \cdots & \cdots & 0 & \lambda_n & 0
\end{pmatrix}, \quad (5.8)
$$

$K$ is a matrix of PDOs of order zero, $G = (0, \ldots, f)^T$ and

$$
V_0 = (v_{0,i})_{1 \leq i \leq n} := \left( \sum_{\ell=0}^{j-1} S^{(n-\ell-1)} \right)_{1 \leq j \leq n}, \quad (5.9)
$$

with $S^{(n-\ell-1)}$ PDOs with symbols $S^{(n-\ell-1)}(x, \xi) \in \mathcal{S}^{n-\ell-1}$, $0 \leq \ell \leq n - 1$, and $v_{0,i}$ the Cauchy data of the original equation $(5.1)$.

Now we diagonalize the operator matrix in $(5.8)$. We first compute on the level of symbols, that a diagonalizer of $K$, which is bidiagonal, is given by

$$
M = (m_{ij})_{i,j=1,\ldots,n}, \quad m_{i,i} = 1, \quad m_{i,j} = 0 \quad \text{for} \quad i > j \quad \text{and} \quad m_{i,j}(t,x,\xi) = (-1)^{j-i} \left( \frac{\xi^j \cdot \lambda_i(t,x,\xi)}{\prod_{k=1}^{j-1} (\lambda_j(t,x,\xi) - \lambda_k(t,x,\xi))} \right), \quad (5.10)
$$

for $i < j$. Note that these symbols are in $C([0,T],\mathcal{S}^0)$, and that the matrix $M$ is invertible thanks to its special structure and to condition $(5.4)$, and the inverse $M^{-1}$ is a matrix of symbols of order zero. Then we set $W := M^{-1}V$, $\tilde{P} := M^{-1}PM$, $W_0 := M^{-1}V_0$, $\tilde{G} := M^{-1}G$ so that we obtain the system of first-order equations

$$
\begin{cases}
\tilde{P}W = \tilde{G}, \text{ for } (0,T], \\
W(0) = W_0,
\end{cases} \quad (5.11)
$$

where

$$
\tilde{P} = D_t + K_1 + \tilde{\mathcal{R}} \quad (5.12)
$$

$K_1$ is a diagonal operator matrix with the roots $\lambda_j$ as entries, and $\tilde{\mathcal{R}}$ is an $n \times n$-operator matrix with elements in $C([0,T],\mathcal{S}^0)$.

**Second step: Computing the (fundamental) solution to the first-order system.**

We are in the same situation as in the second step of the proof of Theorem 3.1 only with different dimensions. In the present case, we define the matrix of FIOs $I_\phi(t,s)$ as

$$
I_\phi(t,s) = \begin{pmatrix}
I_{\phi_1}(t,s) & 0 \\
& \ddots \\
0 & I_{\phi_n}(t,s)
\end{pmatrix},
$$

where $I_{\phi_j}(t,s)$ are the FIOs with symbol 1 and phase functions given by the solutions to the $n$ eikonal equations

$$
\partial_t \phi_j(t,s,x,\xi) + \lambda_j(t,x,\nabla_x \phi_j(t,s,x,\xi)) = 0, \quad \text{on } t \in [s, \bar{T}],
$$

$$
\phi_j(s,s,x,\xi) = x \cdot \xi, \quad \text{for } 0 \leq s \leq \bar{T},
$$

with $\phi_j(t,s,x,\xi) \in \mathcal{S}^{n-1}$, $0 \leq j \leq n$, and $\lambda_j(t,x,\xi) \in \mathcal{S}^0$. Then, the solution to the first-order system $(5.11)$ is given by

$$
W(t,s) = I_\phi(t,s)W_0,
$$

where $I_\phi(t,s)$ are the fundamental solutions to the eikonal equations with $\xi = x \cdot \xi$. Thus, the solution to the original equation $(5.1)$ is

$$
\phi(t,s,x,\xi) = \phi(t,s,x,\xi), \quad \text{for } t \in [s, \bar{T}].
$$
for $j = 1, \ldots, n$. Then we define $W_1$ and $W_{r+1}$ as in (5.20) and (3.22) respectively, and with the same arguments as in the proof of Theorem 3.1 we come to an $n \times n$ matrix of FIOs of order zero $E(t, s)$, which is the fundamental solution for the operator $\tilde{P}$ in (5.12) in the sense that it satisfies (3.29). Duhamel’s formula now gives the solution of (5.1):

$$W(t) = E(t, 0)W_0 + i \int_0^t E(t, \theta)\tilde{G}(\theta)d\theta = E(t, 0)M^{-1}(0, x, D_x)W_0 + i \int_0^t E(t, \theta)M^{-1}(\theta, x, D_x)G(\theta)d\theta.$$ 

The entries of the vector $W(t)$ are given for $1 \leq h \leq n$ by

$$w_k(t) = \sum_{n=1}^n \sum_{j=1}^n \left[ e_{k,h}(t, 0)m_{h,j}(0)v_{0,j} + i \int_0^t e_{k,h}(t, \theta)m_{h,j}(\theta)g_j(\theta)d\theta \right].$$

where $m_{ik}(t, x, D_x)$ stands for a PDO with symbol $m_{ik}(t, x, \xi)$ as in (5.10).

Third step: Computing the fundamental solution of equation (5.1). In order to go back to the solution of the original equation (5.1), we reverse all the transformations from $u$ to $V$, then from $V$ to $W$ so we obtain

$$u(t) = \langle D_x \rangle^{-(n-1)}v_1(t) = \langle D_x \rangle^{-(n-1)} \sum_{k=1}^n m_{i,k}(t, x, D_x)w_k(t),$$

Combining this with (5.13) and looking at (5.9) together with the definition of $G$, we obtain the following representation for the solution $u$ of (5.1):

$$u(t) = \sum_{k=1}^n \sum_{h=1}^n \sum_{j=1}^n \sum_{\ell=0}^{j-1} \langle D_x \rangle^{-(n-1)}m_{i,k}(t)\xi^{-1}m_{h,j}(0)E^{(n-\ell-1)}u_{\ell}$$

$$+ i \sum_{k=1}^n \sum_{h=1}^n \int_0^t \langle D_x \rangle^{-(n-1)}m_{i,k}(t)\xi^{-1}m_{h,a}(\theta)f(\theta)d\theta$$

$$= \sum_{\ell=0}^{n-1} T_\ell(t)u_{\ell} + \int_0^t T_n(t, \theta)f(\theta)d\theta$$

where $T_\ell(t) = T_\ell(t, x, D_x)$ are FIOs with symbols of order $-\ell$ for all $0 \leq \ell \leq n-1$, and $T_n(t, s) = T_n(t, s, x, D_x)$ a FIO with symbol of order $-(n-1)$. The term $g(t) = \sum_{\ell=0}^{n-1} T_\ell(t)u_{\ell} \in H^\tau$ since $u_{\ell} \in H^{r-\ell}$, so this term can be treated as the corresponding one in the proof of Theorem 3.1. As for the term $T_n(t, s)$, it can be handled exactly as $T_3(t, s)$ of Theorem 3.1 with the only difference that $T_n(t, s)$ has order $-(n-1)$; letting $\Lambda(t, s)$ be the Schwartz kernel of $T_n(t, s)$, we come to

$$|F_{y \rightarrow y} \Lambda(t, s, x, \xi)(\xi)|^2 = |\sigma(T_n(t, s))(x, -\xi)|^2 \leq C_{t, \tau}(\xi)^{-2(n-1)},$$

and then the existence and uniqueness of a random field solution follow as in the proof of Theorem 3.1.

The condition (5.5) has already been seen in (8) when dealing with higher-order beam equations.
6 An example - The stochastic wave equation

In this section we show how the concept of fundamental solution used in [10, 7], see also Section 2.1, can be fit into the theory presented in Section 3. We illustrate this using the wave equation in the whole space $\mathbb{R}^d$ for any spatial dimension $d \in \mathbb{N}$ as example. This equation is given by

$$ \begin{cases} \left( \partial_t^2 - \sum_{j=1}^{d} \partial^2_{x_j} \right) u(t, x) = \gamma(t, x) + \sigma(t, x)\bar{F}(t, x), & \text{in } (0, T] \times \mathbb{R}^d, \\ u(0, x) = u_0, & \text{on } \mathbb{R}^d, \\ \partial_t u(0, x) = u_1, & \text{on } \mathbb{R}^d. \end{cases} \tag{6.1} $$

In the above-mentioned articles, the fundamental solution to the wave equation was defined as the (distribution-valued) function $\Lambda : [0, T] \to \mathcal{S}'(\mathbb{R}^d)$ satisfying (6.1) with $u_0 = u_1 = 0$ and the right-hand side equal to $\delta_{0,0}$, where $\delta_{0,0}$ is the space-time Dirac delta distribution in $(0, 0)$. Applying the Fourier transform in the spatial argument to this equation, one can easily calculate the fundamental solution $\Lambda$ to be the inverse spatial Fourier transform of $\sin(t|\xi|)/|\xi|$. However, in the following we construct the fundamental solution $\Lambda$ of this equation in the way of Section 3.

Using again the notation $D = -i\partial$, the partial differential operator in (6.1) can be rewritten as $-D_t^2 + \sum_{j=1}^{d} D_{x_j}^2$. Then its symbol is $-\tau^2 + |\xi|^2$, and so the characteristic roots are given by $\tau = \pm |\xi|$. Note that they do not depend on $t$ and $x$, so they commute with $D_t$, $D_x$ and functions of these operators. Setting as in Section 3

$$ \begin{align*} v_1 &= \langle D_x \rangle \Lambda, \\ v_2 &= (D_t + |D_x|)\Lambda, \end{align*} $$

where the operator $|D_x|$ has been defined in Example 2.15(iii), we can compute

$$(D_t + |D_x|)v_1 = (D_t + |D_x|) \langle D_x \rangle \Lambda = \langle D_x \rangle (D_t + |D_x|)\Lambda = \langle D_x \rangle v_2,$$

and

$$(D_t - |D_x|)v_2 = (D_t - |D_x|) (D_t + |D_x|)\Lambda = (D_t^2 - |D_x|^2)\Lambda = -f.$$  

We let $f$ be a right-hand side and set in the final step $f = b + \sigma\bar{F}$. So the equivalent first-order system becomes

$$ \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} \begin{pmatrix} |D_x| & -\langle D_x \rangle \\ 0 & -|D_x| \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}, $$

with initial conditions $v_1(0) = \langle D_x \rangle u_0$ and $v_2(0) = -iu_1 + |D_x|u_0$. Note that the residual terms $R_0$ and $T_0$ in 3.11 are not present. Now we diagonalize this system using the same matrix as in (3.12). Here $m$ is the PDO with symbol $m(\xi) = \langle \xi \rangle / (2|\xi|)$. Note that due to the multiplication formula for PDOs in Corollary 2.21, we have that $|D_x|m(D_x) = m(D_x)|D_x|$ because both symbols do not depend on the spatial coordinate $x$. So the diagonalized system becomes with the notation $W = M^{-1}V$

$$ \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \begin{pmatrix} |D_x| & 0 \\ 0 & -|D_x| \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} mf \\ -f \end{pmatrix}, \tag{6.2} $$

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with initial conditions \( w_1(0) = (D_x)u_0 + imu_1 - m|D_x|u_0 \) and \( w_2(0) = -iu_1 + |D_x|u_0 \).

Now we set as in Section 3 \( \phi_{\pm} \) to be the solutions to the eikonal equations which have the form

\[
\partial_t \phi_{\pm}(t, s, x, \xi) = \mp|\nabla_x \phi_{\pm}(t, s, x, \xi)|,
\]

with the initial condition \( \phi_{\pm}(s, s, x, \xi) = x \cdot \xi \). One can solve these PDEs explicitly to obtain the solutions \( \phi_{\pm}(t, s, x, \xi) = x \cdot \xi \mp (t - s)|\xi| \) for all \( 0 \leq s \leq t \leq T \) and all \( x, \xi \in \mathbb{R}^d \).

Moreover, \( T = T \). Now we set

\[
I_\phi := \begin{pmatrix} I_{\phi_+} & 0 \\ 0 & I_{\phi_-} \end{pmatrix},
\]

where \( I_{\phi_{\pm}} \) is the FIO having phase function \( \phi_{\pm} \) and symbol 1. One can compute for all \( 0 \leq s \leq t \leq T \) that

\[
\left( \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} + \begin{pmatrix} |D_x| & 0 \\ 0 & -|D_x| \end{pmatrix} \right) I_{\phi}(t, s) = 0,
\]

\[
I_{\phi}(s, s) = \text{id},
\]

so that \( W_1 \) in (3.20) is identical to zero, which means that in (3.21) there is no residual \( R \) and \( I_{\phi} \) is the fundamental solution to this first-order system. Now Duhamel’s formula implies that the solution to the system (3.2) is

\[
\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = \begin{pmatrix} I_{\phi_+}(t, 0)((D_x)u_0 + imu_1 - m|D_x|u_0) \\ I_{\phi_-}(t, 0)(-imu_1 + |D_x|u_0) \end{pmatrix} + i \int_0^t \begin{pmatrix} I_{\phi_+}(t, \theta)(mf)(\theta) \\ -I_{\phi_-}(t, \theta)(f)(\theta) \end{pmatrix} d\theta.
\]

Now the solution \( u \) to the wave equation (6.1) with initial conditions \( u_0 \) and \( u_1 \) and a right-hand side \( f \) can be represented by using (3.34) in the following way

\[
u(t) = T_1(t)u_0 + T_2(t)u_1 + \int_0^t T_3(t, s)f(s)ds,
\]

where

\[
T_1(t) = (D_x)^{-1} \left[ I_{\phi_+}(t, 0)((D_x) - m\lambda) + mI_{\phi_-}(t, 0)\lambda \right],
\]

\[
T_2(t) = i(D_x)^{-1} \left[ I_{\phi_+}(t, 0)m - mI_{\phi_-}(t, 0) \right],
\]

\[
T_3(t, s) = i(D_x)^{-1} \left[ I_{\phi_+}(t, s)m - mI_{\phi_-}(t, s) \right].
\]

Due to the multiplication formulas in Proposition 2.20 we can compute

\[
T_3(t, s) = i \int_{\mathbb{R}^d} e^{ix \cdot \xi} \frac{1}{2i|\xi|} \frac{\sin((t-s)|\xi|)}{|\xi|} d\xi,
\]

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and $T_2(t) = T_3(t, 0)$. We can see that this is a PDO with symbol $\sin(t|\xi|)/|\xi|$. On the other hand, $T_1(t)$ becomes

$$T_1(t) = \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} \left( \frac{1}{\langle \xi \rangle} \left( \frac{\langle \xi \rangle}{2} |\xi| \right) \right) e^{it|\xi|} \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} \left( \frac{1}{\langle \xi \rangle} \frac{1}{2} |\xi| \right) e^{it|\xi|} d\xi,$$

$$= \frac{1}{2} \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} e^{it|\xi|} d\xi + \frac{1}{2} \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} \xi d\xi,$$

$$= \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} + \frac{1}{2} \xi d\xi,$$

$$= \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} \cos(t|\xi|) d\xi,$$

$$= \partial_t \int_{\mathbb{R}^d} e^{ix \cdot t|\xi|} \sin(t|\xi|) d\xi.$$

We can see that in this case all the three operators involve the inverse Fourier transform of $\sin(t|\xi|)/|\xi|$ or its derivative.

In order to show that this term is indeed the fundamental solution to (6.1), we set $u_0 = u_1 = 0$ and the right-hand side equal to $\delta_{0,0}$. Therefore, the solution to the system (6.2) is given by

$$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} = i \int_0^t \begin{pmatrix} I_{\phi_+}(t, \theta)(m\delta_{0,0})(\theta) \\ -I_{\phi_-}(t, \theta)(\delta_{0,0})(\theta) \end{pmatrix} d\theta = i \begin{pmatrix} I_{\phi_+}(t, 0)(m\delta_{0,0}) \\ -I_{\phi_-}(t, 0)(\delta_{0,0}) \end{pmatrix},$$

where $\delta_0$ is the Dirac distribution in space. In order to apply $T_3(t, s)$ to this Dirac delta distribution, we use Definition 2.16. Note that by 2.23 the symbol of the adjoint PDO $T_3(t, s)^*$ is the same, i.e. $p^*(x, \xi) = p(x, \xi) = \sin(t|\xi|)/|\xi|$. We can compute $\Lambda(t)$ to be

$$\langle T_3(t, s)\delta_0, v \rangle = \langle \delta_0, T_3(t, s)^* v \rangle$$

$$= \int_{\mathbb{R}^d} \frac{\sin((t-s)|\xi|)}{|\xi|} Fv(\xi) d\xi,$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-iy \cdot \xi} \frac{\sin((t-s)|\xi|)}{|\xi|} \xi v(y) dy d\xi,$$

$$= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{-iy \cdot \xi} \frac{\sin((t-s)|\xi|)}{|\xi|} \xi v(y) dy d\xi,$$  (6.3)

for any $v \in \mathcal{S}(\mathbb{R}^d)$, where in the last line we have used the change of variable $\xi \mapsto -\xi$. Therefore $\Lambda(t)$ is the inverse Fourier transform of $\sin(t|\xi|)/|\xi|$, which is the same as what we have obtained at the beginning of this section.

From these calculations we can draw some conclusions. We see that in the case of the wave equation, the solution formula that involves $T_1, T_2, T_3$, which was derived using FIOs, coincides with the PDE concept of fundamental solutions, Fourier transforming the PDE directly. Moreover, the representation formula of the solution using the fundamental solution and its time derivatives coincides with 3.34. We furthermore see in 6.3 that the Fourier transform of the Schwartz kernel of the fundamental solution operator is indeed the symbol with a negative second argument, i.e. $p(x, -\xi)$, which is what we have calculated in Proposition 2.18. All this implies that the PDE concept of fundamental solutions, which was used in [10], can be put into the framework of [15].

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