THE KONTSEVICH–ZORICH COCYCLE OVER VEECH–MCMULLEN FAMILY OF SYMMETRIC TRANSLATION SURFACES

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ABSTRACT. We describe the Kontsevich–Zorich cocycle over an affine invariant orbifold coming from a (cyclic) covering construction inspired by works of Veech and McMullen. In particular, using the terminology in a recent paper of Filip, we show that all cases of Kontsevich–Zorich monodromies of $SU(p,q)$ type are realized by appropriate covering constructions.

1. THE VEECH–MCMULLEN FAMILY OF SYMMETRIC TRANSLATION SURFACES

1.1. Definition and notations. Let $k$ be a positive integer, and let $\ell$ be an integer at least equal to 3. We denote by $R$ the rotation of $\mathbb{C}$ centered at 0 of angle $2\pi/\ell$, by $S$ the symmetry $z \rightarrow -\bar{z}$ with respect to the imaginary axis.

We write $\mathbb{Z}_m$ for the standard cyclic group with $m$ elements and $\mathbb{Z}_m$-homogeneous space of pairs of consecutive elements of $\mathbb{Z}_m$.

Let $Q_\text{even} \subset \mathbb{C}$ be the closed regular polygon whose vertices are the roots of unity of order $\ell$. Let $Q_\text{odd} := S(Q_\text{even}) = -Q_\text{even}$.

Consider $k$ copies of $Q_\text{even}$, indexed by the even elements of $\mathbb{Z}_2^k$, and $k$ copies of $Q_\text{odd}$, indexed by the odd elements of $\mathbb{Z}_2^k$.

The vertices of $Q_\text{even}$ and $Q_\text{odd}$ are indexed by $A(\text{even}, j) = R^{-j}(1)$, $A(\text{odd}, j) = -A(\text{even}, j)$, $\forall j \in \mathbb{Z}_\ell$.

For $j' = (j, j+1) \in \mathbb{Z}_\ell^2$, $\epsilon \in \{\text{even}, \text{odd}\}$, we denote by $M(\epsilon, j')$ the midpoint between $A(\epsilon, j)$ and $A(\epsilon, j+1)$, by $\mathbb{N}^+(\epsilon, j)$ the oriented segment from $A(\epsilon, j)$ to $M(\epsilon, j')$, and by $\mathbb{N}^-(\epsilon, j+1)$ the oriented segment from $A(\epsilon, j+1)$ to $M(\epsilon, j')$.

For $j \in \mathbb{Z}_\ell$ and $i \in \mathbb{Z}_2^k$, with parity $\epsilon$, we denote by $A(i, j)$, $M(i, (j, j+1))$, $\mathbb{N}^\pm(i, j)$ the copies in $Q_i$ of $A(\epsilon, j)$, $M(\epsilon, (j, j+1))$, $\mathbb{N}^\pm(\epsilon, j)$.

DEFINITION 1.1. The translation surface $\mathcal{M}_{k,\ell}$ is obtained from the disjoint union of the $Q_i$, $i \in \mathbb{Z}_2^k$ by identifying through the appropriate translation, for each $i \in \mathbb{Z}_2^k$, $j \in \mathbb{Z}_\ell$, the segment $\mathbb{N}^+(i, j)$ with the segment $\mathbb{N}^-(i+1, j+1)$. We denote by $\mathbb{N}((i, i+1), (j, j+1))$ the image of these segments in $\mathcal{M}_{k,\ell}$.
1.2. Basic properties and symmetries of the translation surface $\mathcal{M}_{k,\ell}$. We start by computing the ramification at the singular set, associated to the $M(i,(j,j+1))$ and $A(i,j)$.

For each $j' \in \hat{Z}_\ell$, the points $M(i,j'), i \in \mathbb{Z}_{2k}$ are identified into a single point $M(j')$ on $\mathcal{M}_{k,\ell}$ where the total angle is $2\pi k$.

On the other hand, when rotating counterclockwise around $A(i,j)$, a sector of angle $\frac{\pi(j'-2)}{\ell}$ in $Q_i$ is followed by a sector of the same angle at $A(i-1,j-1)$ in $Q_{i-1}$. The points of $\mathcal{M}_{k,\ell}$ corresponding to the $A(i,j)$ are therefore naturally indexed by the orbits of the transformation $(i,j) \rightarrow (i-1,j-1)$ on $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$. We denote by $A(\Delta)$ the point of $\mathcal{M}_{k,\ell}$ associated to an orbit $\Delta$. The number of such orbits is the greatest common divisor $\omega$ of $2k$ and $\ell$. The total angle at such a point $A(\Delta)$ is $\frac{2\pi k(\ell-2)}{\omega}$.

We denote by $\Sigma$ the set of marked points $M(j'), A(\Delta)$ of $\mathcal{M}_{k,\ell}$, of cardinality $\ell + \omega$.

The genus of $\mathcal{M}_{k,\ell}$ is thus given by

$$g = \ell k + 1 - k - \frac{1}{2}(\ell + \omega).$$

Observe that $\ell$ and $\omega$ have the same parity.

**Remark 1.2.** The translation surface $\mathcal{M}_{1,\ell}$ has been first studied by Veech [9]. He shows that it is a Veech surface and that the image of the Veech group in $PSL(2,\mathbb{R})$ is the lattice generated by $R$ and the parabolic element

$$\left( \begin{array}{cc} 1 & 0 \\ 2\cot \frac{\pi}{\ell} & 1 \end{array} \right).$$

It follows easily that the the subset $\Sigma_M \subset \mathcal{M}_{1,\ell}$ consisting of the $\ell$ points $M(j'), j' \in \hat{Z}_\ell$ is invariant under the group of affine homeomorphisms of $\mathcal{M}_{1,\ell}$. The same is true of the the subset $\Sigma_A$ consisting of the one (if $\ell$ is odd) or two (if $\ell$ is even) points $A(\Delta)$.

**Remark 1.3.** By the previous remark, the image in $\mathcal{M}_{1,\ell}$ of the ramification set of the natural projection $\mathcal{M}_{k,\ell} \twoheadrightarrow \mathcal{M}_{1,\ell}$ is invariant under the group of affine homeomorphisms of $\mathcal{M}_{1,\ell}$. It follows from a result of Gutkin–Judge [4] that $\mathcal{M}_{k,\ell}$ is a Veech surface.

1.3. Covers of hyperelliptic components of strata. Algebraically, the translation surface $\mathcal{M}_{1,\ell}$ corresponds to the Riemann surface $y^2 = (x-x_1)\ldots(x-x_\ell)$ together with the holomorphic one-form $cdx/y$ for appropriate choices of $\ell$ distinct points $x_1, \ldots, x_\ell \in \mathbb{C}$ and a constant $c \in \mathbb{C}^*$: see Veech [9]. More generally, $\mathcal{M}_{k,\ell}$ is the covering of $\mathcal{M}_{1,\ell}$ given by the Riemann surface $y^{2k} = (x-x_1)\ldots(x-x_\ell)$ and the holomorphic one-form $cdx/y^k$: see McMullen [7]. For this reason, we define the Veech–McMullen family $\mathcal{F}_{k,\ell}$ of translation surfaces the Riemann surfaces $y^{2k} = (x-x_1)\ldots(x-x_\ell)$ equipped with $cdx/y^k$ for arbitrary choices of $\ell$ distinct points $x_1, \ldots, x_\ell \in \mathbb{C}$ and constants $c \in \mathbb{C}^*$.
The Veech–McMullen family $\mathcal{F}_{k,\ell}$ for $\ell$ odd, resp. $\ell$ even, is the hyperelliptic component of the stratum $\mathcal{H}(\ell - 2 - \omega)/\omega, \ldots, (\ell - 2 - \omega)/\omega)$ of translation surfaces with $\omega$ conical singularities with total angle $\frac{2\pi k(\ell - 2)}{\omega}$; see [5].

In general, the Veech–McMullen family $\mathcal{F}_{k,\ell}$ is an affine suborbifold of the stratum $\mathcal{H}(k - 1, \ldots, k - 1, k\ell - 2 - \omega)/\omega, \ldots, k(\ell - 2 - \omega)/\omega)$ given by a covering construction. Therefore, the Kontsevich–Zorich cocycle over the Teichmüller geodesic flow on $\mathcal{F}_{k,\ell}$ is coded by the hyperelliptic Rauzy diagrams with arrows decorated by certain matrices (describing actions on the homology of canonical translation surfaces in $\mathcal{F}_{k,\ell}$).

In particular, following the discussion in our previous paper [1], one can associate a Rauzy–Veech group $RV(k, \ell)$ to $\mathcal{F}_{k,\ell}$. By definition, the matrices in $RV(k, \ell)$ preserve the natural symplectic intersection form on the absolute homology of the translation surfaces in $\mathcal{F}_{k,\ell}$.

**Remark 1.4.** In this notation, the main result from our previous paper [1] asserts that $RV(1, k)$ is naturally isomorphic to an explicit finite-index subgroup of the integral symplectic group $Sp(2g, \mathbb{Z})$ (where $g$ is the genus of $\mathcal{M}_{k,\ell}$).

**1.4. Statement of the main result.** In this paper, we study the structure of the Rauzy–Veech groups of $\mathcal{F}_{k,\ell}$.

**Theorem 1.5.** The real Hodge bundle over $\mathcal{F}_{k,\ell}$ decomposes into a direct sum $H_1 \oplus \cdots \oplus H_k$ of flat subbundles $H_r$ associated to the eigenspaces of the generator of the deck group of $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$ (cf. (2) and (3) below). The Rauzy–Veech group of $\mathcal{F}_{k,\ell}$ respects this decomposition. Moreover, if one denotes by $RV(k, \ell)|_{H_r}$ the group associated to the restrictions to $H_r$ of the matrices in $RV(k, \ell)$, then:

(a) $RV(k, \ell)|_{H_k}$ is naturally isomorphic to $RV(1, \ell)$; thus, $RV(k, \ell)|_{H_k}$ is isomorphic to a finite-index subgroup of $Sp(2g, \mathbb{Z})$;

(b) for each $0 < r < k$, the symplectic intersection form on $H_r$ induces a Hermitian form $Q_{r/2k}$ of signature $([\ell(r/2k) - 1], [\ell(1 - (r/2k)) - 1])$ which is preserved by $RV(k, \ell)|_{H_r}$; furthermore,

- if $\ell(r/2k) < 1$ and $r/2k \neq 1/6, 1/4, 1/3$, then $RV(k, \ell)|_{H_r} \cap SU(Q_{r/2k})$ is dense in $SU(Q_{r/2k})$ (for the usual topology);

- if $\ell(r/2k) \notin \mathbb{Z}$ and $r/2k \neq 1/6, 1/4, 1/3$, then $RV(k, \ell)|_{H_r} \cap SU(Q_{r/2k})$ is Zariski dense in $SU(Q_{r/2k})$.

**Remark 1.6.** Actually, our discussion in Section 4 provides a precise version of Theorem 1.5: in particular, we compute the Zariski closure of $RV(\ell, k)|_{H_r} \cap SU(Q_{r/2k})$ in the exceptional cases $r/2k = 1/6, 1/4, 1/3$. However, we have not included all possibilities in Theorem 1.5 in order to get a “cleaner” statement.

This result provides explicit examples showing that all cases of $SU(p, q)$ Kontsevich–Zorich monodromies discussed in Filip’s paper [3] actually occur.

A direct consequence of Theorem 1.5 and the simplicity criterion of Avila–Viana [2] (as stated in Subsection 2.5 of [6]) is the following.
**Corollary 1.7.** The Lyapunov exponents of the restriction of the Kontsevich–Zorich cocycle over $\mathcal{F}_{k,\ell}$ to $H_r$ are “simple” in the sense that:

- they have multiplicity one when $r = k$;
- for $0 < r < k$, $\ell(r/2k) \notin \mathbb{Z}$ and $r/2k \neq 1/6, 1/4, 1/3$, the multiplicity of all non-zero Lyapunov exponents is one and there are exactly $[\ell(1 - (r/2k)) - 1] - [\ell(r/2k) - 1]$ vanishing Lyapunov exponents.

**Proof.** As it is explained in [2, §7.3], the Kontsevich–Zorich cocycle over $\mathcal{F}_{k,\ell}$ is coded by a locally constant cocycle whose supporting monoid $B$ contains all matrices of the form $B_r BB_\gamma$, where $B_\gamma$ is a fixed Kontsevich–Zorich matrix and $B$ are the Kontsevich–Zorich matrices associated to all oriented loops based at a fixed vertex of the Rauzy diagram. Since the group generated by all such matrices $B$ is $RV(k, \ell)$, it follows from Theorem 1.5 that the restriction of the supporting monoid $B$ to each $H_r$ is pinching and twisting in the sense of [6, §2.5]. The desired result now follows from [6, Theorem 2.17].

**1.5. Organization of the paper.** In Section 2, we study a decomposition $H_1 \oplus \cdots \oplus H_k$ of the first absolute homology group of $\mathcal{M}_{k,\ell}$. In Section 3, we describe the restrictions $RV(k, \ell)|_{H_r}$ of the Rauzy–Veech group $R(k, \ell)$ to the summands of the decomposition $H_1(\mathcal{M}_{k,\ell}, \mathbb{R}) = H_1 \oplus \cdots \oplus H_k$ in terms of complex matrices on the vector space $\mathcal{C}_{\ell}$ equipped with adequate hermitian forms. In Section 3, we reduce the proof of Theorem 1.5 to the investigation of certain groups of complex matrices. Finally, we analyze in Section 4 the relevant groups of matrices in order to establish Theorem 1.5.

**Remark 1.8.** We hope that the arguments in this paper might be useful to study the question of non-continuity of the central Oseledets subspaces of the Kontsevich–Zorich cocycle.

2. Homology groups

2.1. Subgroups of affine diffeomorphisms. We now define a finite subgroup $G$ of order $4k\ell$ of the group of affine diffeomorphisms of $\mathcal{M}_{k,\ell}$.

The group $Z_{2k}$ acts by direct affine diffeomorphisms on $\mathcal{M}_{k,\ell}$: the element $r \in Z_{2k}$ sends $Q_i$ onto $Q_i, r$, with derivative $id$ if $r$ is even, $-id$ if $r$ is odd. It sends $A(i, j)$ to $A(i + r, j)$, $\mathbb{N}(i, i + 1) \cup (j, j + 1)$ to $\mathbb{N}(i + r, i + r + 1) \cup (j, j + 1)$ and fixes each $M(j')$.

The group $Z_{\ell}$ acts on $\mathcal{M}_{k,\ell}$ by direct affine diffeomorphisms, the derivative of the action of $s \in Z_{\ell}$ is the rotation $R_s$. This action preserves each $Q_i$, sending $A(i, j)$ to $A(i, j + s)$, $M((j, j + 1))$ to $M((j + s, j + s + 1))$, $\mathbb{N}(i, i + 1) \cup (j, j + 1)$ to $\mathbb{N}(i, i + 1) \cup (j + s, j + s + 1)$.

This two actions commute and they combine to define an action of the product group $Z_{2k} \times Z_{\ell}$ on $\mathcal{M}_{k,\ell}$.

We now define an affine involution $\sigma$ of $\mathcal{M}_{k,\ell}$ whose derivative is $S$. For $i \in Z_{2k}$, $\sigma$ sends $Q_i$ onto $Q_i, -1$, $M((j, j + 1))$ to $M((-j - 1, -j))$, $A(i, j)$ to $A(-i, -j)$, $\mathbb{N}(i, i + 1) \cup (j, j + 1)$ to $\mathbb{N}((-i, -i + 1) \cup (-j - 1, -j))$. 


The involution $\sigma$ conjugates the action of every element $(r, s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ to the action of $(-r, -s)$. Combining the action of $\sigma$ and the action of $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ defines an action of a group $G$ of order $4k\ell$. It is the usual dihedral group of order $4k\ell$ when $\omega = 1$. The group $G$ is the group of permutations of $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ of the form $(i', j') \rightarrow (r, s) \pm (i', j')$, for $r \in \mathbb{Z}_{2k}$, $s \in \mathbb{Z}_{\ell}$.

2.2. **Conjugacy classes in $G$.** Let $1_{2k}$, $1_{\ell}$ be the standard generators of the corresponding cyclic groups.

- Assume first that $\ell$ is odd. There are $k\ell + 1$ conjugacy classes of $G$ contained in $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$, more precisely 2 elements of order 2 and $k\ell - 1$ pairs of distinct elements inverse to each other. The other 2 conjugacy classes are those of $\sigma$ and $\sigma 1_{2k}$ and have size $k\ell$.
- Assume now that $\ell$ is even. There are $k\ell + 2$ conjugacy classes of $G$ contained in $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$, more precisely 4 elements of order 2 and $k\ell - 2$ pairs of distinct elements inverse to each other. The other 4 conjugacy classes are those of $\sigma$, $\sigma 1_{2k}$, $\sigma 1_{\ell}$, $\sigma 1_{2k} 1_{\ell}$ and have size $\frac{1}{2} k\ell$.

2.3. **Irreducible representations of $G$ over $\mathbb{R}$ or $\mathbb{C}$.** The irreducible representations of $G$ over $\mathbb{C}$ are all defined over $\mathbb{R}$ and have dimension 1 or 2.

When $\ell$ is odd, there are 4 nonequivalent 1-dimensional representations of $G$, sending $1_{\ell}$ to 1 and $1_{2k}$ and $\sigma$ to $\pm 1$ (independently).

When $\ell$ is even, there are 8 nonequivalent 1-dimensional representations of $G$, sending $1_{\ell}$, $1_{2k}$ and $\sigma$ to $\pm 1$ (independently).

To parametrize the 2-dimensional representations, we define $\mathcal{R}(2k, \ell)$ to be the quotient of $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ by $\{\pm 1\}$ and denote by $\mathcal{R}^*(2k, \ell)$ the subset of $\mathcal{R}(2k, \ell)$ associated to elements $(r, s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ of order $> 2$. The cardinality of $\mathcal{R}(2k, \ell)$ (resp. $\mathcal{R}^*(2k, \ell)$) is equal to $k\ell + 1$ (resp. $k\ell - 1$) if $\ell$ is odd, to $k\ell + 2$ (resp. $k\ell - 2$) if $\ell$ is even.

For $(r, s) \in \mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$, the representation $\pi_{r,s}$ sends $\sigma$ to $S$, $1_{2k}$ to a rotation of angle $\frac{2\pi}{k}$, and $1_{\ell}$ to a rotation of angle $2\pi \frac{j}{\ell}$.

The character $\chi_{r,s}$ of $\pi_{r,s}$ vanishes on $G - (\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell})$ and its values on $\mathbb{Z}_{2k} \times \mathbb{Z}_{\ell}$ are given by

$$\chi_{r,s}(i, j) = 2 \cos 2\pi \left( \frac{ri + sj}{2k} \right).$$

When $(r, s)$ has order $> 2$, the representation $\pi_{r,s}$ is irreducible and is conjugated by $S$ to $\pi_{-r,-s}$.

When $(r, s)$ has order 2, the representation $\pi_{r,s}$ split into two representations $\pi_{r,s}^+$ (with $\chi_{r,s}^+(\sigma) = 1$) and $\pi_{r,s}^-$ (with $\chi_{r,s}^-(\sigma) = -1$) of dimension 1. The isomorphism classes of irreducible representations are thus parametrized by $\mathcal{R}^*(2k, \ell)$.

2.4. **Irreducible representations of $G$ over $\mathbb{Q}$.** Let $\Pi := 2k\ell/\omega$ be the least common multiple of $2k$ and $\ell$, which is also the least common multiple of the elements of $G$. By a general result of Brauer (see [8, Theorem 24, Section 12.3]), which is easily checked in the case of $G$, all irreducible representations of $G$ over $\mathbb{C}$ are actually defined over the cyclotomic field $\mathbb{Q}(\Pi)$. To see how to group together the $\pi_{r,s}$ in order to obtain the irreducible representations of $G$ over $\mathbb{Q}$.
Q, consider the action of the Galois group \((\mathbb{Z}_n)^*\) of \(\mathbb{Q}(\mathbb{I})\) over \(\mathbb{Q}\) on \(\mathcal{R}(2k, \ell)\) defined by
\[
t.(r, s) = (tr, ts).
\]
The group \((\mathbb{Z}_n)^*\) also acts on \(G\) through \(t.g := g^t\). Observe that we have, for all \((r, s) \in \mathcal{R}(2k, \ell)\), \(g \in G\), \(t \in (\mathbb{Z}_n)^*\)
\[
t.*g \chi_{r,s}(g) = \chi_{r,s}(t.g) = \chi_{t.(r,s)}(g).
\]

The \(1\)-dimensional representations of \(G\) are all defined over \(\mathbb{Q}\). Similarly, each of the points in \(\mathcal{R}(2k, \ell) \sim \mathcal{R}(2k, \ell)_t\) is fixed by \((\mathbb{Z}_n)^*\).

By [8, Theorem 29, section 13.1], we conclude that the irreducible representations of \(G\) over \(\mathbb{Q}\) of dimension > 1 are parametrized by the orbits of the action of \((\mathbb{Z}_n)^*\) on \(\mathcal{R}(2k, \ell)_t\): for an orbit \(\mathcal{O}\), the associated representation is
\[
\pi_{\mathcal{O}} := \bigoplus_{(r, s) \in \mathcal{O}} \pi_{r,s}.
\]

**Remark 2.1.** To compute the orbits of the action of \((\mathbb{Z}_n)^*\) on \(\mathcal{R}(2k, \ell)\), it is sufficient to consider the action over \(\mathbb{Z}_{2k} \times \mathbb{Z}_\ell\), as \(-1 \in (\mathbb{Z}_n)^*\). Then one uses the Chinese remainder theorem to split \(\mathbb{I}\) into prime powers. Write \(\mathbb{I} = \prod_p \mathbb{P}^p\), \(2k = \prod_p \mathbb{P}^{p^a}\), \(\ell = \prod_p \mathbb{P}^{p^b}\), with \(C_p = \max(A_p, B_p)\) for each prime \(p\). The action of \((\mathbb{Z}_n)^*\) on \(\mathbb{Z}_{2k} \times \mathbb{Z}_\ell\) is the product of the actions of \((\mathbb{Z}_{p^c})^*\) over \(\mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}\) (with \(A = A_p, B = B_p, C = C_p\)). Let \((r, s) \in \mathbb{Z}_{p^a} \times \mathbb{Z}_{p^b}\). Let \(p^a\) (resp. \(p^b\)) be the order of \(r\) in \(\mathbb{Z}_{p^a}\) (resp. of \(s\) in \(\mathbb{Z}_{p^b}\)), with \(0 \leq a \leq A, 0 \leq b \leq B\). For any \((r', s')\) in the orbit of \((r, s)\), the order of \(r'\) (resp. \(s'\)) is also \(p^a\) (resp. \(p^b\)). When \(\min(a, b) = 0\), there is exactly one orbit with these orders. When \(\min(a, b) > 0\), the stabilizer of \((r, s)\) as above has cardinality equal to \(p^{C_{\max(a, b)}}\), hence the corresponding orbit has cardinality equal to \(p^{\max(a, b) - 1}(p - 1)\). As the number of pairs \((r, s)\) with orders \((p^a, p^b)\) is equal to \(p^{a+b-2}(p-1)^2\), the number of orbits associated to these values of \(a, b\) is equal to \(p^{\min(a, b) - 1}(p - 1)\).

2.5. **Decomposition of the first relative homology group.** The classes of the oriented segments \(\mathcal{N}(i', j')\) ( \(i' \in \hat{\mathbb{Z}}_{2k}, j' \in \hat{\mathbb{Z}}_\ell\) ) obviously span the first relative homology group \(H_1(\mathcal{M}_{k, \ell}, \Sigma, \mathbb{Q})\).

Going around the boundary of \(Q_i\) gives the relation
\[
\sum_{\mathcal{Z}_t} \mathcal{N}((i - 1, i), j') = \sum_{\mathcal{Z}_t} \mathcal{N}((i + 1, i), j'), \quad \forall i \in \mathbb{Z}_{2k},
\]
providing \(2k - 1\) independent relations between the \(\mathcal{N}(i', j')\). As
\[
2k\ell - (2k - 1) = 2g + (\#\Sigma - 1) = \dim H_1(\mathcal{M}_{k, \ell}, \Sigma, \mathbb{Q}),
\]
these are the only relations between the \(\mathcal{N}(i', j')\).

Denote by \((e_i)\), \((E_i')\) the canonical bases of \(\mathbb{Q}^{\mathbb{Z}_{2k}}, \mathbb{Q}^{\hat{\mathbb{Z}}_{2k} \times \hat{\mathbb{Z}}_{\ell}}\) respectively. We equip \(Q, \mathbb{Q}^{\mathbb{Z}_{2k}}, \mathbb{Q}^{\hat{\mathbb{Z}}_{2k} \times \hat{\mathbb{Z}}_{\ell}}\) with structures of \(G\)-module by defining
\[
1_k.x = 1_\ell.x = x, \quad 1_k.x = -x, \quad \forall x \in \mathbb{Q},
\]
\[
1_k.e_i = e_{i+1}, \quad 1_\ell.e_i = e_i, \quad \sigma.e_i = -e_{1-i}, \quad \forall i \in \mathbb{Z}_{2k},
\]
\[
1_k.E_i.j' = E_1 + i'.j', \quad 1_\ell.E_i.j' = E_i', \quad \sigma.E_i.j' = E_{1-i}.j', \quad \forall (i', j') \in \hat{\mathbb{Z}}_{2k} \times \hat{\mathbb{Z}}_\ell.
\]
We have then an exact sequence of $G$-modules.

\[(1) \quad 0 \to Q \to Q^Z_{2k} \to Q^{\mathbb{Z}_2 \times \mathbb{Z}_\ell} \to H_1(M_{k,\ell}, \Sigma, Q) \to 0.\]

The maps in this exact sequence are as follows. The map from $Q$ to $Q^Z_{2k}$ sends 1 to $\Sigma_i e_i$. The map from $Q^Z_{2k}$ to $Q^{\mathbb{Z}_2 \times \mathbb{Z}_\ell}$ sends $e_i$ to $\sum j'(E_{[i,i+1],j'} - E_{[i-1,i],j'})$. The map from $Q^{\mathbb{Z}_2 \times \mathbb{Z}_\ell}$ to $H_1(M_{k,\ell}, \Sigma, Q)$ sends $E_{i',j'}$ to the class of $N(i',j')$.

From (1), we deduce the character $\chi_{rel}$ of the $G$-module $H_1(M_{k,\ell}, \Sigma, Q)$. Firstly, for $(i, j) \in \mathbb{Z}_2k \times \mathbb{Z}_\ell$, we have

$$\chi_{rel}(i, j) = 1 - 2k\delta_{0i} + 2k\ell\delta_{0i}\delta_{0j}.$$

- When $\ell$ is odd, $\chi_{rel}$ is equal to 1 everywhere on $G - (\mathbb{Z}_2k \times \mathbb{Z}_\ell)$. One obtains
  $$\chi_{rel} = 1 + \sum_{(r,s) \in \mathbb{R}^+(2k,\ell), s \neq 0} \chi_{rs},$$

- When $\ell$ is even, the values of $\chi_{rel}$ on the conjugacy classes of $\sigma, \sigma 1_{2k}, \sigma 1_\ell, \sigma 1_{2k} 1_\ell$ are respectively $-1, +1, +3, +1$. One obtains
  $$\chi_{rel} = 1 + \chi_+ + \chi_- + \sum_{(r,s) \in \mathbb{R}^+(2k,\ell), s \neq 0} \chi_{rs},$$

where $\chi_+$ (resp. $\chi_-$) is the 1-dimensional character with value $-1$ at $\sigma$ and $1$ at $\ell$ and value $+1$ (resp. $-1$) at $1_{2k}$.

### 2.6. Decomposition of the first absolute homology group.

The exact sequence of $G$-modules for relative homology reads

$$0 \to H_1(M_{k,\ell}, Q) \to H_1(M_{k,\ell}, \Sigma, Q) \to H_0(\Sigma, Q) \to Q \to 0.$$

This gives

$$\chi_{ab} = \chi_{rel} - \chi_{\Sigma} + 1,$$

where $\chi_{ab}, \chi_{\Sigma}$ are the characters of $H_1(M_{k,\ell}, Q), H_0(\Sigma, Q)$ respectively.

Each of the subsets $\Sigma_M := \{ M_{j'} \mid j' \in \mathbb{Z}_\ell \}$ and $\Sigma_A := \{ A(\Delta) \mid \Delta \in \mathbb{Z}_0 \}$ is invariant under $G$, hence the character $\chi_{\Sigma}$ splits as $\chi_{\Sigma_M} + \chi_{\Sigma_A}$.

The character $\chi_{\Sigma_M}$ satisfies $\chi_{\Sigma_M}(i, j) = \ell\delta_{0j}$.

When $\ell$ is odd, it is equal to 1 on all of $G - (\mathbb{Z}_2k \times \mathbb{Z}_\ell)$. This gives in this case

$$\chi_{\Sigma_M} = 1 + \sum_{(0,s) \in \mathbb{R}^+(2k,\ell)} \chi_{0,s}.$$

When $\ell$ is even, $\chi_{\Sigma_M}$ takes the value 0 on the conjugacy classes of $\sigma$ and $\sigma 1_{2k}$, and the value 2 on the conjugacy classes of $\sigma 1_\ell$ and $\sigma 1_{2k} 1_\ell$. This gives in this case

$$\chi_{\Sigma_M} = 1 + \chi_+ + \sum_{(0,s) \in \mathbb{R}^+(2k,\ell)} \chi_{0,s}.$$

The character $\chi_{\Sigma_A}$ satisfies $\chi_{\Sigma_A}(i, j) = \delta$ if $i, j$ are congruent modulo $\delta$, to $\chi_{\Sigma_A}(i, j) = 0$ otherwise.

When $\ell$ is odd, it is equal to 1 on all of $G - (\mathbb{Z}_2k \times \mathbb{Z}_\ell)$. This gives

$$\chi_{\Sigma_A} = 1 + \sum_{(r,s) \in \mathbb{R}^+(2k,\ell), r+\frac{s}{2} \in \mathbb{Z}} \chi_{rs}.$$
When $\ell$ is even, $\chi_{\sigma A}$ takes the value 0 on the conjugacy classes of $\sigma$ and $\sigma 1_{2k} 1_\ell$, and the value 2 on the conjugacy classes of $\sigma 1_\ell$ and $\sigma 1_{2k}$. This gives

$$\chi_{\Sigma A} = 1 + \chi_\sigma + \sum_{(r,s) \in \mathfrak{R}^*(2k,\ell), 0 \neq r, s \neq 0, \frac{r}{\ell} + \frac{s}{2k} \in \mathbb{Z}} \chi_{r,s}.$$  

In the formula $\chi_{ab} = \chi_{rel} - \chi_{\Sigma A} + 1$ we have now computed all the terms in the right-hand side. We obtain

$$\chi_{ab} = \sum_{(r,s) \in \mathfrak{R}(2k,\ell), 0 \neq r, s \neq 0, \frac{r}{\ell} + \frac{s}{2k} \in \mathbb{Z}} \chi_{r,s}.$$  

**Remark 2.2.** With the conditions $r \neq 0, s \neq 0, \frac{r}{2k} + \frac{s}{\ell} \notin \mathbb{Z}$, there is no difference between $\mathfrak{R}(2k,\ell)$ and $\mathfrak{R}^*(2k,\ell)$.

From this character formula, we get

$$H_1(\mathcal{M}_{k,\ell},\mathbb{R}) = \bigoplus_{(r,s) \in \mathfrak{R}(2k,\ell), 0 \neq r, s \neq 0, \frac{r}{\ell} + \frac{s}{2k} \in \mathbb{Z}} \pi_{r,s} = \bigoplus_{0 < r < k} H_r,$$

with

$$H_r := \bigoplus_{0 < s < \ell, \frac{r}{\ell} + \frac{s}{2k} \neq 1} \pi_{r,s}$$

for $0 < r < k$ and

$$H_k := \bigoplus_{0 < s < \ell/2} \pi_{k,s}.$$  

**Proposition 2.3.** The subrepresentation $H_r$ is defined over $\mathbb{Q}$ if and only if $r/2k$ is equal to 1/6, 1/4, 1/3, or 1/2.

**Proof.** Indeed, for $0 < r \leq k$, let $\mathfrak{R}_r$ be the image in $\mathfrak{R}^*(2k,\ell)$ of the set

$$\left\{ (r,s) \mid 0 < s < \ell, \frac{r}{2k} + \frac{s}{\ell} \neq 1 \right\}.$$  

By subsection 2.4, the subrepresentation $H_r$ is defined over $\mathbb{Q}$ iff $\mathfrak{R}_r$ is invariant under the action of $(\mathbb{Z}_H)^*$. As the subsets $\mathbb{Z}_{2k} \times \{0\}$ and $\left\{ (r,s) \mid \frac{r}{2k} + \frac{s}{\ell} \in \mathbb{Z} \right\}$ of $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$ are invariant under the action of $(\mathbb{Z}_H)^*$, $\mathfrak{R}_r$ is invariant under the action of $(\mathbb{Z}_H)^*$ iff the subset $\{r\}$ of $\mathbb{Z}_{2k}$ is invariant under the action of $(\mathbb{Z}_H)^*$, which has the same orbits in $\mathbb{Z}_{2k}$ that the action of $(\mathbb{Z}_{2k})^*$. But the only integers $n > 1$ such that $(\mathbb{Z}_n)^* = \{1\}$ are 2, 3, 4, 6. This proves the proposition. \qed

**Remark 2.4.** The subspace $H_k$ is identified with the $H_1(\mathcal{M}_{1,\ell},\mathbb{R})$ under the natural projection $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$. In general, the subspaces $H_r$ are given by the eigenspaces of the generator of the group of deck transformations of $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$. In particular, the decomposition $H_1(\mathcal{M}_{k,\ell},\mathbb{R}) = \bigoplus_{0 < r \leq k} H_r$ can be extended to all translation surfaces of the Veech–McMullen family $\mathcal{F}_{k,\ell}$.
2.7. Relation with the symplectic intersection form. Let $\chi_\sigma$ be the 1-dimen-
sional character on $G$ with kernel $\mathbb{Z}_{2k} \times \mathbb{Z}_\ell$. The intersection form $\omega$ on $H_1(M_{k,\ell}, \mathbb{Q})$ satisfies, for any $g \in G$, $v, w \in H_1(M_{k,\ell}, \mathbb{Q})$

$$
\omega(g, v, g, w) = \chi_\sigma(g) \omega(v, w).
$$

**Proposition 2.5.** The 2-dimensional summands in the decomposition of the $G$-module $H_1(M_{k,\ell}, \mathbb{Q})$ in the last subsection are mutually $\omega$-orthogonal.

**Proof.** Let $E, F$ be two such distinct summands. The intersection form defines an homomorphism $u$ from $E$ to the dual $F^*$ of $F$. We have to prove that $u = 0$. If $v \in E$ belongs to the kernel of $u$, then the same is true for $g, v$, for any $g \in G$. As $E$ is irreducible, this proves that $u$ is either invertible or equal to 0. If $u$ is invertible, the formula above shows that it is an isomorphism of $G$-modules from $E$ to the tensor product of the contragredient representation of $F$ by $\chi_\sigma$. But this tensor product is isomorphic as $G$-module to $F$ itself. As $E, F$ are distinct, we conclude that $u$ cannot be invertible, hence $u = 0$. \hfill $\square$

For $0 < r \leq k$, let $H_r$ be the sum of the summands in the decomposition of $H_1(M_{k,\ell}, \mathbb{Q})$ with characters $\chi_{r,s}$, $s$ varying according to the prescription above. Assume now that $0 < r < k$. We equip $H_r$ with the complex structure such that

$$1_{2k} \cdot v = \exp \left( i \pi \frac{r}{k} \right) v.
$$

The formula

$$
\langle v, v \rangle := \left( \sin \left( \pi \frac{r}{k} \right) \right)^{-1} \omega(1_{2k} \cdot v, v)
$$

defines an hermitian form on $H_r$ whose imaginary part is $\omega$.

The signature of this hermitian form is calculated in Subsection 3.6.

3. Hyperelliptic Rauzy diagrams and Rauzy–Veech groups $RV(k, \ell)$

3.1. Review of the description of hyperelliptic Rauzy diagrams. In this subsection, we recall the content of Subsection 2.1 of our previous paper [1].

Given an integer $d \geq 2$, denote by $A_d$ the arithmetic progression $d - 1, d - 3, \ldots, 1 - d$. Note that $A_d$ has a natural involution $v(k) = -k$.

The hyperelliptic Rauzy class $R_d$ over $A_d$ and the associated Rauzy diagram $\mathcal{D}_d$ are inductively defined as follows.

The Rauzy class $R_d$ contains the central vertex $\pi^* = \pi^*(d)$:

$$
\pi^*_i(k) = \frac{1}{2} (d + 1 + k), \quad \pi^*_b(k) = \frac{1}{2} (d + 1 - k).
$$

For $d = 2$, this is the sole vertex. For $d \geq 2$, $R_{d+1}$ is the disjoint union of $\pi^*(d + 1)$, $j_f(R_d)$ and $j_b(R_d)$, where $f, j_f, j_b$ are the following injective maps: for

---

1It comes from the fact that $H_r$ is associated to eigenspaces of the eigenvalues $\exp(\pm \frac{2\pi i r}{2k})$ for the action on complex cohomology of the generator $1_{2k}$ of the deck group of $\mathcal{M}_{k,\ell} \to \mathcal{M}_{1,\ell}$ (cf. Remark 2.4).

2In the literature, this hermitian form is called Hodge form.
\( \pi \in \mathcal{R}_d \), if we denote \( j_t(\pi) = t\pi \), \( j_b(\pi) = b\pi \), then

\[
\begin{align*}
t\pi_t(-d) &= 1, \\
t\pi_b(-d) &= \pi_b(d-3), \\
t\pi_t(k) &= 1 + \pi_t(k-1), \\
t\pi_b(k) &= \begin{cases} \pi_b(k-1) & \text{if } \pi_b(k-1) < \pi_b(d-3), \\ \pi_b(k-1) + 1 & \text{if } \pi_b(k-1) \geq \pi_b(d-3), \end{cases}
\end{align*}
\]

for \( 2 - d \leq k \leq d \), and

\[
\begin{align*}
b\pi_t(d) &= 1, \\
b\pi_b(d) &= \pi_t(3 - d), \\
b\pi_t(k) &= 1 + \pi_b(k + 1), \\
b\pi_b(k) &= \begin{cases} \pi_t(k + 1) & \text{if } \pi_t(k + 1) < \pi_t(3 - d), \\ \pi_t(k + 1) + 1 & \text{if } \pi_t(k + 1) \geq \pi_t(3 - d), \end{cases}
\end{align*}
\]

for \( -d \leq k \leq d - 2 \).

The arrows of \( \mathcal{D}_d \) are given by the following one-to-one maps \( R_t, R_b \) from \( \mathcal{R}_d \) to itself:

\[
\begin{align*}
\begin{cases}
R_t(\pi^* (d + 1)) &= j_t(\pi^* (d)), \\
R_b(\pi^* (d + 1)) &= j_b(\pi^* (d)), \\
\end{cases}
\begin{cases}
R_t \circ j_t \circ R_t^{-1} &= j_t, \\
R_b \circ j_t \circ R_b^{-1} &= j_t, \\
\end{cases}
\begin{cases}
R_t \circ j_t \circ R_t^{-1}(\pi) &= j_t(\pi), \\
R_b \circ j_b \circ R_b^{-1}(\pi) &= j_b(\pi), \\
\end{cases}
\begin{cases}
R_t(\pi^* (d + 1)) &= \pi^* (d + 1) = R_b \circ j_b \circ R_b^{-1}(\pi^* (d))).
\end{cases}
\end{align*}
\]

The elements of \( \mathcal{R}_d \) correspond bijectively to the words in \( t, b \) of length \( < d - 1 \) via the following map \( W_d \): let \( W_d(\pi^*(d)) \) be the empty word, \( W_d(j_t(\pi)) \) is the word \( tW_{d-1}(\pi) \) and \( W_d(j_b(\pi)) \) is the word \( bW_{d-1}(\pi) \).

One recovers from \( W_d(\pi) \) the winners of the arrows starting from \( \pi \) as follows: the winner of the arrow of top type starting from \( \pi \) is the letter \( d - 1 - 2w_b(\pi) \) of \( \mathcal{A}_d \), where \( w_b(\pi) \) is the number of occurrences of \( b \) in \( W_d(\pi) \); similarly, the winner of the arrow of bottom type starting from \( \pi \) is the letter \( 1 - d + 2w_t(\pi) \) of \( \mathcal{A}_d \). Observe that we have always

\[
d - 1 - 2w_b(\pi) > 1 - d + 2w_t(\pi).
\]

A vertex \( \pi \in \mathcal{R}_d \) is connected to the central vertex \( \pi^*(d) \) by a unique oriented simple path \( \gamma^*(\pi) \) in \( \mathcal{D}_d \) from \( \pi^*(d) \) to \( \pi \).

As it turns out, all non-trivial simple loops in \( \mathcal{R}_d \) are elementary; they consist of arrows of the same type. Any such loop \( \gamma \) contains a unique vertex \( \pi \) such that \( \gamma \) passes through \( \pi \) but \( \gamma^*(\pi) \) does not contain any arrow of \( \gamma \), and, furthermore, \( \pi \) is the vertex of \( \gamma \) such that \( |W_d(\pi)| \) is minimal, and its value is \( d - 1 - |\gamma| \). In the sequel, we denote by \( \gamma' \) the non-oriented loop at \( \pi^*(d) \) defined by

\[
\gamma' := \gamma^*(\pi) \ast \gamma \ast (\gamma^*(\pi))^{-1}.
\]
3.2. Mapping classes attached to the arrows of $\mathcal{D}_d$. In this subsection, we essentially review the content of Section 4 of our previous paper [1].

Given $\pi \in \mathcal{R}_d$, denote by $M_\pi$ the canonical translation surface with combinatorial data $\pi$ whose length data $\lambda^{can}$ and suspension data $\tau^{can}$ are:

$$\lambda^\text{can}_\alpha = 1, \quad \tau^\text{can}_\alpha = \pi_b(\alpha) - \pi_t(\alpha), \quad \forall \alpha \in A.$$

We obtain $M_\pi$ by identifying parallel sides of an appropriate polygon $P_\pi$. The set of marked points of $M_\pi$ is denoted by $\Sigma_\pi$, and the middle points of the sides of $P_\pi$ together with a base point $O_\pi$ in the interior of $P_\pi$ form a subset $\Sigma^*_\pi$ of $M_\pi$.

The Rauzy–Veech operation associated to each arrow $\gamma: \pi \to \pi'$ of $\mathcal{D}_d$ is encoded by the isometry class of a homeomorphism

$$H_\gamma: (M_\pi, \Sigma_\pi \cup \Sigma^*_\pi) \to (M_{\pi'}, \Sigma_{\pi'} \cup \Sigma^*_{\pi'})$$

constructed in Subsection 4.1 of [1]. The map $\gamma \mapsto [H_\gamma]$ induces (by functoriality\(^3\)) a morphism $\pi_1(\mathcal{D}_d, \pi^*) \to \text{Mod}(\pi^*)$ from the fundamental group $\pi_1(\mathcal{D}_d, \pi^*)$ of the non-oriented Rauzy diagram $\mathcal{D}_d$ associated to $\mathcal{D}_d$ based at the central vertex $\pi^*$ to the mapping class group $\text{Mod}(\pi^*)$ of $(M_{\pi^*}, \Sigma_{\pi^*})$.

In this context, given $\gamma$ a simple loop in $\mathcal{D}_d$, the action of $\gamma'$ as an isometry class on $M_{\pi'}$ was computed in Proposition 4.5 of [1]: it is a Dehn twist about the straight line joining the midpoints of the sides of $P_{\pi'}$ indexed by the letter of $A_d$ winning in the loop $\gamma$.

Note that the elements of $\text{Mod}(\pi^*)$ can be viewed also as mapping classes on the translation surface $M_{1,\ell}$ where $\ell = d + 1$, and, a fortiori, they can be lifted to $M_{k,\ell}$ via the natural projection $M_{k,\ell} \to M_{1,\ell}$.

**Definition 3.1.** The Rauzy–Veech group $RV(k, \ell)$ is the group generated by the actions on $H_1(M_{k,\ell}, \mathbb{R})$ of all $\gamma'$ associated to all elementary loops $\gamma$ in $\mathcal{R}_d$ (where $\ell = d + 1$).

The Rauzy–Veech group $RV(1, \ell)$ was computed in our previous paper [1]: it is isomorphic to an explicit finite-index subgroup of $Sp(2g, \mathbb{Z})$ (where $g$ is the genus of $M_{1,\ell}$).

**Remark 3.2.** In general, natural projection $M_{k,\ell} \to M_{1,\ell}$ takes $H_k$ to $H_1(M_{1,\ell}, \mathbb{R})$ in such a way that $RV(k, \ell)|_{H_k}$ is isomorphic to $RV(1, \ell)$, so that $RV(k, \ell)|_{H_k}$ is the explicit finite-index subgroup described in our previous paper [1, Theorem 2.9].

3.3. Lifting the action of the loops in $\mathcal{D}_d$: top case. Let $\gamma$, $\pi$, $\gamma'$ be as in the previous subsection. Let $k$ be an integer $\geq 2$. Let $M_{k,d+1}$ be the surface considered in the first section. We have a canonical projection $M_{k,d+1} \to M_{1,d+1}$. The action of $\gamma'$, as an isometry class of the translation surface $M_{1,d+1}$ with marked points at the $M(j)$ and the $A(\delta)$, was already discussed in the previous subsection. We describe now the lift of this action to $M_{k,d+1}$.

---

\(^3\)I.e., the image of a loop under the morphism is the composition of the $[H_\gamma]$ attached to its arrows.
Assume first that \( \gamma \) is of top type. Let \( w := w_b(\pi) \in \{0, \ldots, d-2\} \). The winner \(^4\) of \( \pi \) is then \( p := d-1-2w \). We write \( L_p^t \) for the action of \( \gamma' \) on the homology of \( \mathcal{M}_{k,d+1} \).

For \( i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_{d+1} \), the image of the relative homology class \( \mathcal{N}((i, i+1), (j, j+1)) \) by \( L_p^t \) is equal to

- \( \mathcal{N}((i, i+1), (j, j+1)) \) if \( j \) is neither 0 nor \( w + 1 \);
- \( \mathcal{N}((i-1, i), (0, 1)) + \sum_1^w (\mathcal{N}((i-1, i), (m, m+1)) - \mathcal{N}((i, i+1), (m, m+1))) \) if \( j = w + 1 \);
- \( \mathcal{N}((i-1, i), (w+1, w+2)) + \sum_{w+2}^d (\mathcal{N}((i-1, i), (m, m+1)) - \mathcal{N}((i, i+1), (m, m+1))) \) if \( j = 0 \).

We define

\[
V_i(p) := \sum_0^w \mathcal{N}((i-1, i), (m, m+1)) - \sum_1^{w+1} \mathcal{N}((i, i+1), (m, m+1)),
\]

so that we have

\[
L_p^t(\mathcal{N}((i, i+1), (j, j+1))) = \mathcal{N}((i, i+1), (j, j+1)) + (\delta_{j,w+1} - \delta_{j,0})V_i(p).
\]

This gives

\[
L_p^t(V_i(p)) = -V_{i-1}(p).
\]

Let \( 0 < r < k, 0 < s < \ell := d+1 \) with \( \frac{r}{2k} + \frac{s}{\ell} \neq 1 \); write

\[
\begin{align*}
x_{i,j} = \cos 2\pi \left( \frac{r i}{2k} + \frac{s j}{\ell} \right), & \quad y_{i,j} = \sin 2\pi \left( \frac{r i}{2k} + \frac{s j}{\ell} \right).
\end{align*}
\]

Define

\[
X_{r,s}^t := \sum_{i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_{\ell}} x_{i,j} \mathcal{N}((i, i+1), (j, j+1)), \quad Y_{r,s}^t := \sum_{i \in \mathbb{Z}_{2k}, j \in \mathbb{Z}_{\ell}} y_{i,j} \mathcal{N}((i, i+1), (j, j+1)).
\]

We have

\[
L_p^t(X_{r,s}^t) = X_{r,s}^t + \sum_{i \in \mathbb{Z}_{2k}} (x_{i,w+1} - x_{i,0})V_i(p),
\]

\[
L_p^t(Y_{r,s}^t) = Y_{r,s}^t + \sum_{i \in \mathbb{Z}_{2k}} (y_{i,w+1} - y_{i,0})V_i(p).
\]

Set

\[
V_{\cos}(p, r) := \sum_{i \in \mathbb{Z}_{2k}} x_{i,0}V_i(p), \quad V_{\sin}(p, r) := \sum_{i \in \mathbb{Z}_{2k}} y_{i,0}V_i(p).
\]

Then we have

\[
L_p^t(V_{\cos}(p, r)) = \sum_{i \in \mathbb{Z}_{2k}} x_{i,0}L_p^t(V_i(p)) = -\sum_{i \in \mathbb{Z}_{2k}} x_{i,0}V_{i-1}(p) = -\sum_{i \in \mathbb{Z}_{2k}} x_{i+1,0}V_i(p)
\]

\[
= \cos \left( \pi \left(1 + \frac{r}{k}\right) \right) V_{\cos}(p, r) - \sin \left( \pi \left(1 + \frac{r}{k}\right) \right) V_{\sin}(p, r)
\]

and

\[
L_p^t(V_{\sin}(p, r)) = \sin \left( \pi \left(1 + \frac{r}{k}\right) \right) V_{\cos}(p, r) + \cos \left( \pi \left(1 + \frac{r}{k}\right) \right) V_{\sin}(p, r).
\]

\(^4\)In the sequel, it is useful to recall that an elementary loop of given (top or bottom) type is uniquely determined the parameter \( w \), or equivalently, the parameter \( p \).
We also compute the image of $V_{\cos}(p', r)$ and $V_{\sin}(p', r)$ for $p' \neq p$. We write $p' = d - 1 - 2w'$. Assume first that $p' < p$ (i.e., $w' > w$). One has

$$L_p^t(V_{\cos}(p', r))$$

$$= \sum_{i \in Z_{2k}} x_{i,0} \left( \sum_{i=0}^{w'} L_p^t(N((i-1, i), (m, m+1))) - \sum_{i=1}^{w'+1} L_p^t(N((i, i+1), (m, m+1))) \right)$$

$$= V_{\cos}(p', r) + \sum_{i \in Z_{2k}} x_{i,0} \left( \sum_{i=0}^{w'} (\delta_{m, w+1} - \delta_{m, 0}) V_{i-1}(p) - \sum_{i=1}^{w'+1} (\delta_{m, w+1} - \delta_{m, 0}) V_i(p) \right) - \sum_{i \in Z_{2k}} x_{i,0} V_i(p)$$

$$= V_{\cos}(p', r) - \sum_{i \in Z_{2k}} x_{i,0} V_i(p)$$

$$= V_{\cos}(p', r) - V_{\cos}(p, r).$$

Similarly

$$L_p^t(V_{\sin}(p', r)) = V_{\sin}(p', r) - V_{\sin}(p, r).$$

When $p' > p$ (i.e., $w' < w$), one obtains

$$L_p^t(V_{\cos}(p', r))$$

$$= V_{\cos}(p', r) + \sum_{i \in Z_{2k}} x_{i,0} \left( \sum_{i=0}^{w'} (\delta_{m, w+1} - \delta_{m, 0}) V_{i-1}(p) - \sum_{i=1}^{w'+1} (\delta_{m, w+1} - \delta_{m, 0}) V_i(p) \right)$$

$$= V_{\cos}(p', r) - \sum_{i \in Z_{2k}} x_{i,0} V_i(p)$$

$$= V_{\cos}(p', r) + L_p^t(V_{\cos}(p, r)),$$

and similarly

$$L_p^t(V_{\sin}(p', r)) = V_{\sin}(p', r) + L_p^t(V_{\sin}(p, r)).$$

In summary, for each $0 < r < k$, we deduce that

- The subspace $H_r$ is fixed by the lift of the action of $\gamma'$ (which commutes with the action of $Z_{2k}$); therefore, the Rauzy–Veech group $RV(k, \ell)$ gives rise to well-defined groups $RV(k, \ell)|H_r$ (obtained by restriction to $H_r$);

- $H_r$ is the direct sum of the 2-dimensional subspace generated by $V_{\cos}(p, r)$ and $V_{\sin}(p, r)$ on which $\gamma'$ acts by a rotation of $-\pi(1 + \frac{k}{r})$, and a subspace of codimension 2 on which $\gamma'$ acts by the identity.

3.4. Lifting the action of the loops in $D_q$: bottom case. In the same setting that in the last subsection, we now assume that $\gamma$ is of bottom type.

Let $w := w_1(\pi) \in \{0, \ldots, d - 2\}$. The winner of $\pi$ is then $p := -d + 1 + 2w$. We write $L_p^b$ for the action of $\gamma'$ on the homology of $X_{k,d+1}$.

For $i \in Z_{2k}$, $j \in Z_{d+1}$, the image of $N((i-1, i), (-j, -j+1))$ by $L_p^b$ is equal to

- $N((i-1, i), (-j, -j+1))$, if $j$ is neither 0 nor $w + 1$;

- $N((i, i+1), (0, 1)) + \sum_{i=1}^{w}(N((i, i+1), (-m, -m+1)) - N((i-1, i), (-m, -m+1)))$ if $j = w + 1$;

- $N((i, i+1), (-w - 1, -w)) + \sum_{i=0}^{w+2}(N((i, i+1), (-m, -m+1)) - N((i-1, i), (-m, -m+1)))$ if $j = 0$. 


We now have
\[
\sum_{0}^{w} \mathbb{N}((i, i+1), (-m, -m+1)) - \sum_{1}^{w+1} \mathbb{N}((i-1, i), (-m, -m+1)) = V_i(p)
\]

hence
\[
L_p^b(\mathbb{N}((i-1, i), (-j, -j+1))) = \mathbb{N}((i-1, i), (-j, -j+1)) + (\delta_j, w+1 - \delta_j, 0)V_i(p).
\]

This gives
\[
L_p^b(V_i(p)) = -V_{i+1}(p).
\]

Let 0 < r < k, 0 < s < \ell with \(\frac{s}{\ell} \neq \frac{r}{k}\); we have
\[
L_p^b(V_{\cos}(p, r)) = \sum_{i \in \mathbb{Z}_k} x_{i, 0} L_p^b(V_i(p)) = -\sum_{i \in \mathbb{Z}_k} x_{i, 0} V_{i+1}(p) = -\sum_{i \in \mathbb{Z}_k} x_{i-1, 0} V_i(p)
\]

and
\[
L_p^b(V_{\sin}(p, r)) = -\sin\left(\pi \left(1 + \frac{r}{k}\right)\right) V_{\cos}(p, r) + \cos\left(\pi \left(1 + \frac{r}{k}\right)\right) V_{\sin}(p, r).
\]

We also compute the image of \(V_{\cos}(p', r)\) and \(V_{\sin}(p', r)\) for \(p' \neq p\). We write \(p' = -d + 1 + 2w'\). Assume first that \(p' > p\) (i.e., \(w' > w\)). One has
\[
L_p^b(V_{\cos}(p', r)) = \sum_{i \in \mathbb{Z}_k} x_{i, 0} \left(\sum_{0}^{w'} L_p^b(\mathbb{N}((i, i+1), (-m, -m+1))) - \sum_{1}^{w'+1} L_p^b(\mathbb{N}((i-1, i), (m, m+1)))\right)
\]

\[
= V_{\cos}(p', r) + \sum_{i \in \mathbb{Z}_k} x_{i, 0} \left(\sum_{0}^{w'} (\delta_m, w+1 - \delta_m, 0)V_{i+1}(p) - \sum_{1}^{w'+1} (\delta_m, w+1 - \delta_m, 0)V_i(p)\right)
\]

\[
= V_{\cos}(p', r) - \sum_{i \in \mathbb{Z}_k} x_{i, 0} V_i(p)
\]

Similarly
\[
L_p^b(V_{\sin}(p', r)) = V_{\sin}(p', r) - V_{\sin}(p, r).
\]

When \(p' < p\) (i.e., \(w' < w\)), one obtains
\[
L_p^b(V_{\cos}(p', r)) = V_{\cos}(p', r) + \sum_{i \in \mathbb{Z}_k} x_{i, 0} \left(\sum_{0}^{w'} (\delta_m, w+1 - \delta_m, 0)V_{i+1}(p) - \sum_{1}^{w'+1} (\delta_m, w+1 - \delta_m, 0)V_i(p)\right)
\]

\[
= V_{\cos}(p', r) - \sum_{i \in \mathbb{Z}_k} x_{i, 0} V_i(p)
\]

\[
= V_{\cos}(p', r) + L_p^b(V_{\cos}(p, r)).
\]
and similarly
\[ L^b_p(\sin(p', r)) = \sin(p', r) + L^b_p(\sin(p, r)). \]
Thus we see that \( L_p^b \) is the inverse of \( L_p^b \).
In summary, the group \( RV(k, \ell)|_{H_\tau} \) is generated by the operators \( L_p^b|_{H_\tau} \).

3.5. **Formulas for \( L_p^f \) as a complex operator.** Let \( \rho := \exp(i \pi \frac{r}{k}) \). The complex structure on \( H_\tau \) is given by \( 1_{2k} \cdot v = \rho \cdot v \). We have thus

\[
L_p^f(\cos(p, r)) = \sum_{i \in \mathbb{Z}_{2k}} x_{i,0}L_p^f(V_i(p))
= -\sum_{i \in \mathbb{Z}_{2k}} x_{i,0}V_{i-1}(p)
= -\rho^{-1}V_{\cos}(p, r).
\]

For \( p' > p \),
\[
L_p^f(\cos(p', r)) = \cos(p', r) - \rho^{-1}V_{\cos}(p, r),
\]
and for \( p' < p \)
\[
L_p^f(\cos(p', r)) = \cos(p', r) - \rho^{-1}V_{\cos}(p, r).
\]

3.6. **Computation of the hermitian form on \( H_\tau \).** We fix \( 0 < r < k \) and abbreviate \( Z(p) := V_{\cos}(p, r) \), \( x_i := x_{i,0} \). We first compute the hermitian product
\[
\langle Z(p), Z(p) \rangle := \left( \sin \left( \frac{r}{k} \right) \right)^{-1} \omega(1_{2k} \cdot Z(p), Z(p)).
\]

**Lemma 3.3.** Let \((a_i)_{i \in \mathbb{Z}_{2k}} \) and \((b_i)_{i \in \mathbb{Z}_{2k}} \) be real numbers with \( \sum_i a_i = \sum_i b_i = 0 \). Then, we have
\[
\omega \left( \sum_i a_i V_i(p), \sum_i b_i V_i(p) \right) = \sum_{1 \leq i \leq i' < 2k} (a_i b_i' - a_i' b_i).
\]
For \( p > p' \), we have
\[
\omega \left( \sum_i a_i V_i(p), \sum_i b_i V_i(p') \right) = \sum_{1 \leq i \leq i' < 2k} a_i b_i'.
\]

**Proof.** Observe first that, although \( V_i(p) \) is only a relative homology class with nonzero boundary equal to \( M(0, 1) - M(w + 1, w + 2) \) (where \( p = d - 1 - 2w \)), the condition \( \sum_i a_i = 0 \) insures that \( \sum_i a_i V_i(p) \) is an absolute homology class so the intersection form \( \omega \) is well-defined in the formulas of the lemma.

In the second case, it is easy to represent the cycles \( V_i(p) \) (resp. \( V_i(p') \)) by paths from \( M(w + 1, w + 2) \) (resp. \( M(w' + 1, w' + 2) \)) to \( M(0, 1) \) in \( Q_i \) so that the intersection takes place only at \( M(0, 1) \). A direct inspection at this point gives the formula of the lemma.

In the first case, we choose two distinct representations for each \( V_i(p) \) as paths from \( M(w + 1, w + 2) \) to \( M(0, 1) \) so that the intersection takes place only at \( M(w + 1, w + 2) \) and \( M(0, 1) \). Again a direct inspection at these points gives the formula of the lemma. \( \square \)
Using the first part of the lemma, we get (as $\sum_i x_i = 0$)
\[
\omega(1_{2k}.Z(p), Z(p)) = \omega(\sum x_i 1_{2k}.V_i(p), \sum x_i V_i(p))
\]
\[
= \omega(\sum x_i V_{i+1}(p), \sum x_i V_i(p))
\]
\[
= \omega(\sum x_{i-1} V_i(p), \sum x_i V_i(p))
\]
\[
= \sum_{1 \leq i \leq i' < 2k} (x_{i-1} x_{i'} - x_i x_{i'-1})
\]
\[
= \sum_{1 \leq i \leq 2k-1} x_0 x_i + \sum_{1}^{2k-2} x_i x_{2k-i} - \sum_{1}^{2k-1} x_i x_{i-1} - \sum_{1}^{2k-2} x_i^2
\]
\[
= -x_0^2 - x_{-1} (x_0 + x_{-1}) - \sum_{1}^{2k-1} x_i x_{i-1} - \sum_{1}^{2k-2} x_i^2
\]
\[
= -\sum_{i \in \mathbb{Z}_{2k}} (x_i x_{i-1} + x_i^2)
\]
\[
= -\frac{1}{4} \sum_{i \in \mathbb{Z}_{2k}} (\rho^i + \rho^{-i}) (\rho^i + \rho^{-i} + \rho^{i-1} + \rho^{-i+1})
\]
\[
= -\frac{k}{2} (2 + \rho + \rho^{-1})
\]
\[
= -k \left(1 + \cos \frac{r}{k}\right).
\]
We have thus
\[
\langle Z(p), Z(p) \rangle := -k \frac{1 + \cos \pi \frac{r}{k}}{\sin(\pi \frac{r}{k})}.
\]
We now compute
\[
2\Re(\langle Z(p), Z(p') \rangle) = \langle Z(p) + Z(p'), Z(p) + Z(p') \rangle - \langle Z(p), Z(p) \rangle - \langle Z(p'), Z(p') \rangle
\]
for $p > p'$. We have (using the second part of Lemma 3.3)
\[
2 \sin \left(\frac{\pi}{k}\right) \Re(\langle Z(p), Z(p') \rangle)
\]
\[
= \omega(1_{2k}.Z(p), Z(p')) + \omega(1_{2k}.Z(p'), Z(p))
\]
\[
= \omega\left(\sum x_i V_i(p), \sum x_i V_i(p')\right) - \omega\left(\sum x_i V_i(p), \sum x_{i-1} V_i(p')\right)
\]
\[
= \sum_{1 \leq i \leq i' < 2k} (x_{i-1} x_{i'} - x_i x_{i'-1})
\]
\[
= -k \left(1 + \cos \frac{r}{k}\right).
\]
This gives
\[
\Re(\langle Z(p), Z(p') \rangle) = -\frac{k}{2} \frac{1 + \cos \pi \frac{r}{k}}{\sin(\pi \frac{r}{k})}.
\]
We finally compute
\[
2\Re(\rho \langle Z(p), Z(p') \rangle) = \langle \rho Z(p) + Z(p'), \rho Z(p) + Z(p') \rangle - \langle Z(p), Z(p) \rangle - \langle Z(p'), Z(p') \rangle
\]
for \( p > p' \). We have

\[
2 \sin\left( \frac{\pi r}{k} \right) \Re(\rho(Z(p), Z(p'))) = \omega\left( \frac{1}{2k} Z(p), Z(p') \right) + \omega\left( \frac{1}{2k}, Z(p), Z(p') \right)
\]

\[
= \omega\left( \frac{1}{2k}, Z(p), Z(p') \right) + \omega(\frac{1}{2k}, Z(p), Z(p') \right) + \omega(Z(p'), Z(p))
\]

\[
= \omega\left( \sum x_{i-1} V_i(p), \sum x_{i+1} V_i(p') \right) - \omega\left( \sum x_i V_i(p), \sum x_i V_i(p') \right)
\]

\[
= \sum_{1 \leq i \leq l < 2k} (x_{i-1} x_{i+1} - x_i x_i')
\]

\[
= \frac{2}{2k} \cos \pi \frac{r}{k}
\]

This gives

\[
\Re(\rho(Z(p), Z(p'))) = -k \frac{1 + \cos \pi \frac{r}{k}}{2 \sin \pi \frac{r}{k}}.
\]

We finally get, for \( p > p' \),

\[
\Im((Z(p), Z(p'))) = \Im((p))^{-1}(\Re(p) \Re((Z(p), Z(p'))) - \Re(p) \Re(Z(p), Z(p')))
\]

\[
= -\frac{k}{2} \frac{1 + \cos \pi \frac{r}{k}}{\sin \pi \frac{r}{k}} - \frac{1 + \cos \pi \frac{r}{k}}{\sin \pi \frac{r}{k}}
\]

\[
= \frac{k}{2}.
\]

It is probably nicer to scale the hermitian form by the factor \(-k \frac{1 + \cos \pi \frac{r}{k}}{\sin \pi \frac{r}{k}}\) in order to have \((Z(p), Z(p)) = 1\) for all \( p \). One has then, for \( p > p' \)

\[
\langle Z(p), Z(p') \rangle = \frac{1}{2} \left( 1 - i \tan \frac{\pi r}{2k} \right).
\]

Let \( u := \tan \frac{\pi r}{2k} \). Consider the hermitian form on \( C_{A_u} \) defined by

\[
A((w(p))_{p \in A_u}) := \langle \sum w(p) Z(p), \sum w(p) Z(p) \rangle = A_1 - u A_2,
\]

with

\[
A_1 = \frac{1}{2} \left( \sum_p |w(p)|^2 + \sum_p |w(p')|^2 \right)
\]

and

\[
A_2 = \sum_{p > p'} \Im(w(p') w(p)).
\]

Observe that \( A_1 \) is positive, so we can diagonalize simultaneously \( A_1 \) and \( A_2 \). Let indeed \( \xi := \exp \frac{2i \pi}{\ell} \) (recall that \( \ell = d + 1 \)). Define, for \( 0 < s < \ell \), \( p = d - 1 - 2w \)

\[
w_s(p) = \xi^{sw}.
\]
We have therefore, for $z_1, \ldots, z_d \in \mathbb{C}$

$$A_1(\sum z_s w_s) = \frac{1}{2} \left( \sum_{p} \left| \sum z_s \xi^{sw} \right|^2 + \left| \sum_{p} z_s \xi^{sw} \right|^2 \right)$$

$$= \frac{1}{2} \left( \sum_{p} \sum_{s} \sum_{s'} z_s \overline{z}_{s'} \xi^{(s-s')w} + \left| \sum z_s \xi^{sw} \right|^2 \right)$$

$$= \frac{\ell}{2} \sum |z_s|^2,$$

and

$$A_2(\sum z_s w_s) = \sum_{p > p'} \Im \left( \sum_{s'} \sum_{s} z_{s'} \overline{z}_s w_{s'}(p') \overline{w}_s(p) \right)$$

$$= \sum_{s'} \sum_{s} \Im \left( z_{s'} \overline{z}_s \sum_{0 < w < w' < \ell - 1} \xi^{sw'-sw} \right).$$

For $s \neq s'$, one has

$$\sum_{0 < w < w' < \ell - 1} \xi^{sw'-sw} = \sum_{0 < w' < \ell - 1} \frac{\xi^{sw'-sw} - 1}{\xi^{-s} - 1} = -1 - \xi^{-s-s'} + 1 + \xi^{-s'-1}$$

$$= \xi^{s-s'},$$

and

$$\Im \left( z_{s'} \overline{z}_s \xi^{s-s'} + z_s \overline{z}_{s'} \xi^{s-s'} \right) = 0.$$

On the other hand, for $0 < s < \ell$, we have

$$\sum_{0 < w < w' < \ell - 1} \xi^{sw'-sw} = \sum_{0 < w' < \ell - 1} \xi^{sw} \frac{\xi^{-sw'} - 1}{\xi^{-s} - 1} = \frac{\ell + \xi^{s} - 1}{\xi^{-s} - 1}$$

We have therefore

$$A_2(\sum z_s w_s) = \sum_s \Im \left( |z_s|^2 \sum_{0 < w < w' < \ell - 1} \xi^{sw'-sw} \right)$$

$$= \frac{\ell}{2} \sum_{s} |z_s|^2 \Im \frac{1}{\xi^{-s} - 1}$$

$$= \frac{\ell}{2} \sum_{s} \left( \tan \frac{\pi s \ell}{\ell} \right)^{-1} |z_s|^2.$$

In particular, this shows that the hermitian form on $H_{\ell}$ has the signature described in Theorem 1.5 (as expected from McMullen’s paper [7]).

3.7. Matricial description of $RV(k, \ell)_{H_{\ell}}$. Our discussion so far is summarized as follows. The group $RV(k, \ell)_{H_{\ell}}$ is generated by the operators $L_{p'}^t$, $p \in A_d$, given by

$$L_{p'}^t(V_{\cos}(p', r)) = \begin{cases} V_{\cos}(p', r) - \rho^{-1} V_{\cos}(p, r) & \text{if } p' > p \\ V_{\cos}(p', r) - V_{\cos}(p, r) & \text{if } p' < p \\ -\rho^{-1} V_{\cos}(p, r) & \text{if } p' = p \end{cases}$$
These operators preserve the hermitian form
\[
\langle V_{\cos}(p, r), V_{\cos}(p, r) \rangle = 1, \quad \langle V_{\cos}(p, r), V_{\cos}(p', r) \rangle = \frac{1}{2} \left( 1 - i \tan \frac{\pi r}{2k} \right) \quad \forall \ p > p'
\]

At this point, we reduced the proof of Theorem 1.5 to the analysis of the group generated by the matrices above.

4. Matrices

In this section, we complete the proof of Theorem 1.5 by studying the groups of matrices. Since we do not need anymore to make reference to the homology groups translation surfaces \( \mathcal{M}_{k, \ell} \), we are going to rewrite below the formulas from Subsection 3.7 using a slightly more abstract notation.

4.1. Setting.

- \( d \) is an integer \( \geq 2 \), \( \ell = d + 1 \).
- \( \rho \) is a complex number of modulus 1 with positive imaginary part, frequently a root of unity; \( \zeta := \rho^{-1} \). We write \( \rho = \exp 2\pi i \alpha, \ \alpha \in (0, \frac{1}{2}) \).
- \( p \) is an integer running from \( 1 - d \) to \( d - 1 \) with step 2, and thus taking \( d \) values. We denote by \( \mathcal{A}_d \) the set of values of \( p \).
- \((e_p)\) is the canonical basis of \( \mathbb{C}^{\mathcal{A}_d} \).

4.2. The operators. For \( p \in \mathcal{A}_d \), we define an operator \( L_p = L_p^t \) on \( \mathbb{C}^{\mathcal{A}_d} \) by
\[
L_p(e_q) = \begin{cases} 
  e_q - e_p & \text{if } q > p \\
  e_q - \zeta e_p & \text{if } q < p \\
  -\zeta e_p & \text{if } q = p.
\end{cases}
\]

Observe that the inverse \( L_p^{-1} = L_p^b \) is given by
\[
L_p^b(e_q) = \begin{cases} 
  e_q - e_p & \text{if } q < p \\
  e_q - \rho e_p & \text{if } q > p \\
  -\rho e_p & \text{if } q = p.
\end{cases}
\]
i.e., the same formula, changing \( \zeta \) to \( \rho \) and inverting the order on \( \mathcal{A}_d \).

4.3. The invariant hermitian form. Let \( Q_\alpha \) be the hermitian form on \( \mathbb{C}^{\mathcal{A}_d} \) such that
\[
Q_\alpha(e_p) = 1, \quad \forall p \in \mathcal{A}_d, \quad Q_\alpha(e_p, e_p') = (1 + \zeta)^{-1} = \frac{1}{2} (1 + i \tan \pi \alpha), \forall p > p'.
\]
4.4. The case \( d = 2 \).

4.4.1. A special element. We compute \( L_{-1}^b \circ L_1^f \) as

\[
\begin{align*}
\{ L_{-1}^b \circ L_1^f(e_{-1}) &= L_{-1}^b(e_{-1} - \zeta e_1) = -\rho e_{-1} - \zeta(e_1 - \rho e_{-1}) = (1 - \rho)e_{-1} - \zeta e_1, \\
L_{-1}^b \circ L_1^f(e_1) &= L_{-1}^b(-\zeta e_1) = -\zeta(e_1 - \rho e_{-1}) = e_{-1} - \zeta e_1.
\end{align*}
\]

We thus have \( \text{det}(L_{-1}^b \circ L_1^f) = 1 \) and \( \text{tr}(L_{-1}^b \circ L_1^f) = 1 - (\rho + \rho^{-1}) \).

**Lemma 4.1.** The operator \( L_{-1}^b \circ L_1^f \) has infinite order if \( \rho \) is a root of unity but \( \rho^4 \neq 1, \rho^6 \neq 1 \). It is hyperbolic if \( \rho + \rho^{-1} < -1 \).

**Proof.** The second assertion is clear. For the first, if \( \rho \) is a root of unity but \( \rho^4 \neq 1, \rho^6 \neq 1 \), there exists a Galois-conjugate \( \rho' \) of \( \rho \) such that \( \rho' + \rho'^{-1} < -1 \). Then the corresponding Galois-conjugate of the matrix of \( L_{-1}^b \circ L_1^f \) has infinite order. The same is true of the matrix of \( L_{-1}^b \circ L_1^f \).

4.4.2. The cases \( \alpha = \frac{1}{6}, \frac{1}{4} \). In these cases, the hermitian form has signature \((2,0)\). The coefficients of the matrices of the group generated by \( L_1 \) and \( L_{-1} \) belong to the ring of integers of the quadratic field \( \mathbb{Q}(\rho) \) and are bounded, hence the group generated by \( L_1 \) and \( L_{-1} \) is finite.

4.4.3. The case \( \alpha = \frac{1}{3} \). Let \( \rho = j := \exp \frac{2\pi i}{3}, e := e_{-1} + je_1, f := e_{-1} + j^2 e_1 \). We have \( L_{-1}(e) = L_1(e) = e \) and \( L_{-1}(f) = j^2 e - j^2 f, L_1(f) = -je - j^2 f \). The subgroup generated by \( L_{-1} \) and \( L_1 \) is therefore contained in

\[ \Gamma : = \{ L \in \text{GL}(2, \mathbb{C}) \mid L(e) = e, L(f) = \mu f + \omega e, \mu, \omega \in \mathbb{Z} \} \cdot \]

Observe that \( L_{-1}^b \circ L_1^f \) is parabolic, i.e. satisfies \( \mu = 1, \omega \neq 0 \).

4.4.4. The case \( 0 < \alpha < \frac{1}{3}, \alpha \neq \frac{1}{4}, \frac{1}{6} \). In this case, the hermitian form has signature \((2,0)\). Denote by \( U(Q_\alpha) \) and \( SU(Q_\alpha) \) the associated unitary and special unitary groups.

The operator \( L_{-1}^b \circ L_1^f \) belongs to \( SU(Q_\alpha) \) and has infinite order hence the closed (for the usual topology) subgroup generated by \( L_{-1}^b \circ L_1^f \) is a one-parameter group isomorphic to a circle, consisting of those transformations of \( SU(Q_\alpha) \) having the same eigenvectors than \( L_{-1}^b \circ L_1^f \). An infinitesimal generator of this one-parameter group is the element \( X \) of the Lie algebra \( su(Q_\alpha) \) which satisfies \( Xv_\pm = \pm i v_\pm \), where \( v_\pm \) are the eigenvectors of \( L_{-1}^b \circ L_1^f \).

The eigenvalues \( \lambda_\pm \) of \( L_{-1}^b \circ L_1^f \) are solutions of \( \lambda^2 - (1 - \zeta - \rho)\lambda + 1 = 0 \), and the corresponding eigenvectors are

\[ v_\pm = a_\pm e_{-1} + e_1, \quad a_\pm = -1 - \rho \lambda_\pm \]

The coefficients of \( X \) are given by

\[ X_{-1} = \frac{i(a_+ + a_-)}{a_+ - a_-} = -X_{11}, \quad X_{1} = \frac{2i}{a_+ - a_-}, \quad X_{-1} = \frac{2i a_+ a_-}{a_+ - a_-}. \]
For \( n \in \mathbb{Z} \), write \( \text{ad}(L_{-1}^n)X =: X(n) \), which is the infinitesimal generator of the previous one-parameter group conjugated by \( L_{-1}^n \). Setting also \( L_{-1}^n v_\pm =: v_\pm(n) =: a_\pm(n)e_1 + e_1 \), we have

\[
X_{-1}^{-1}(n) = \frac{i(a_+(n) + a_-(n))}{a_+(n) - a_-(n)} = -X_{11}(n), \quad X_{-1}(n) = \frac{2i}{a_+(n) - a_-(n)},
\]

with the recurrence relation \( a(n+1) = -\zeta a(n) - 1 \).

We now show that the vector fields \( X = X(0), X(1), X(2) \) are linearly independent (over \( \mathbb{C} \)). The sequences \( s(n) := a_+(n) + a_-(n) \) and \( p(n) := a_+(n)a_-(n) \) satisfy the recurrence relations

\[
s(n+1) = -\zeta s(n) - 2, \quad p(n+1) = \zeta^2 p(n) + \zeta s(n) + 1
\]

with initial conditions \( s(0) = -1 - \rho + \rho^2, \quad p(0) = \rho \). We have thus

\[
s(1) = \zeta - 1 - \rho, \quad s(2) = -\zeta^2 + \zeta - 1, \quad p(1) = \rho, \quad p(2) = \zeta^2.
\]

As we have

\[
\det \begin{pmatrix} 1 & p(0) & s(0) \\ 1 & p(1) & s(1) \\ 1 & p(2) & s(2) \end{pmatrix} = (1 - \zeta^2)(1 - \rho^2) \neq 0,
\]

the vector fields \( X = X(0), X(1), X(2) \) are indeed linearly independent. The Lie algebra \( su(Q_a) \) has dimension 3, hence is generated by \( X(0), X(1), X(2) \).

We conclude that the intersection of \( SU(Q_a) \) with the group generated by \( L_1 \) and \( L_{-1} \) is dense (for the usual topology) in \( SU(Q_a) \).

4.4.5. The case \( \frac{1}{2} < \alpha < \frac{1}{3} \). In this case, the hermitian form has signature (1,1). Denote by \( U(Q_a) \) and \( SU(Q_a) \) the associated unitary and special unitary groups.

The operator \( L_{-1}^b \circ L_1^b \) belongs to \( SU(Q_a) \) and has eigenvalues \( \lambda_+ > 1 \) and \( \lambda_- = \lambda_+^{-1} \). Let \( v_\pm \) be the associated eigenvectors. The Zariski closure of the group generated by \( L_{-1}^b \circ L_1^b \) is the one-parameter group having for infinitesimal generator a vectorfield \( X \) satisfying \( Xv_\pm = \pm v_\pm \). The same calculation as in the previous case shows that \( X, \text{ad}(L_{-1})X \) and \( \text{ad}(L_1^2)X \) span \( su(Q_a) \). We conclude that the intersection of \( SU(Q_a) \) with the group generated by \( L_1 \) and \( L_{-1} \) is dense (for the Zariski topology) in \( SU(Q_a) \).

4.5. The induction step in the generic case.

4.5.1. Restrictions. For \( d \geq 3, p \in A_d \), let \( t_p \) be the embedding of \( A_{d-1} \) into \( A_d \) sending \( q \) to \( q - 1 \) if \( q < p \) and to \( q + 1 \) if \( q > p \) (observe that \( p \notin A_d \). The image of \( t_p \) is \( A_d - \{ p \} \). We also denote by \( t_p \) the embedding of \( C^{A_{d-1}} \) into \( C^{A_d} \) such that \( t_p(e_q) = e_{t_p(q)} \), and by \( H_p \) the hyperplane of \( C^{A_d} \) which is the image of this embedding.

From the defining formulas, for all \( p \in A_d, q \in A_{d-1} \), the hyperplane \( H_p \) is invariant under \( L_q \) and we have

\[
t_p \circ L_q = L_{t_p(q)} \circ t_p.
\]
In order to avoid confusion, we denote by $Q'_a$ the hermitian form on $\mathbb{C}^{A_d}$ denoted by $Q_a$ previously, and keep the notation $Q_a$ for the hermitian form on $\mathbb{C}^{A_d}$.

**Lemma 4.2.** For any $p \in A_d$, the restriction to $H_p$ of the form $Q_a$ on $\mathbb{C}^{A_d}$ is equal to the image of $Q'_a$ under $\iota_p$.

*Proof.* This is clear from Subsection 4.2.

When $\rho^{d+1} \neq 1$, we denote by $H'_p$ the 1-dimensional subspace which is the $Q_a$-orthogonal of $H_p$. As $Q_a$ is non-degenerate and $\bigcap_{p \in A_d} H_p = \{0\}$, $\mathbb{C}^{A_d}$ is the direct sum of the $H'_p$.

When moreover $\rho^d \neq 1$, the restriction of $Q_a$ to each $H_p$ is non-degenerate by the lemma. Therefore $\mathbb{C}^{A_d}$ is the direct sum of $H_p$ and $H'_p$. From the formulas for $L_q, q \neq p$, we see that the line $H'_p$ is point wise fixed under all $L_q, q \neq p$.

**4.5.2. A result on stabilizers.**

**Proposition 4.3.** Let $d \geq 3$, $Q$ a non-degenerate hermitian form on $\mathbb{C}^d$, $u_1, u_2 \in \mathbb{C}^d$ linearly independent vectors such that $Q(u_1) = Q(u_2) \neq 0$. Let $SU(Q)$ be the special unitary group of $Q$, and, for $j = 1, 2$, let $G_j$ be the stabilizer of $u_j$ in $SU(Q)$. Then the smallest closed subgroup containing $G_1 \cup G_2$ is $SU(Q)$.

*Proof.* Let $G$ be the smallest closed subgroup of $SU(Q)$ containing $G_1 \cup G_2$. Let $c$ be the common value of the $Q(u_j)$. It is sufficient to prove that $G$ acts transitively on $\{Q(u) = c\}$. Indeed, assume this is true, and let $h$ be an element of $SU(Q)$; there exists $g \in G$ such that $g(u_1) = h(u_1)$; then $g^{-1} h \in G_1$ and $h \in G$. To prove that $G$ acts transitively on $\{Q(u) = c\}$, we observe that, as $\{Q(u) = c\}$ is connected\footnote{Since we are assuming that $c = Q(u_1) = Q(u_2) \neq 0$, the desired connectedness follows from Witt’s theorem (see also the claim in the proof of Lemma 4.4 below).}, it is sufficient to show that the orbits of $G$ have non-empty interior (and so are open) in $\{Q(u) = c\}$.

Let $u_0$ be a vector such that $Q(u_0) = c$.

**Lemma 4.4.** If either $Q(Gu_0, u_1)$ or $Q(Gu_0, u_2)$ have non-empty interior in $\mathbb{C}$, then $Gu_0$ has non-empty interior in $\{Q(u) = c\}$.

*Proof.* Recall the following fact: for $p + q \geq 2$ and $b \neq 0$, $SU(p, q)$ acts transitively on $\{\sum_1^p |z_i|^2 - \sum_{p+1}^q |z_i|^2 = b\} \subset \mathbb{C}^{p+q}$.

Assume for instance that $Q(Gu_0, u_1)$ has non-empty interior in $\mathbb{C}$. Let $W$ be a non-empty open set which is contained in $Q(Gu_0, u_1)$ and is disjoint from the circle $\{|z| = Q(u_1)\}$. For any $w \in W$, the intersection $\{Q(u) = Q(u_1)\} \cap \{Q(u, u_1) = w\}$ consists of vectors of the form $u = \alpha u_1 + \nu$, with $\alpha = \frac{w}{Q(u_1)}$, $Q(\nu, u_1) = 0$, and $Q(\nu) = (1 - |\alpha|^2) Q(u_1)$. As $|\alpha| \neq 1$, it follows from the fact recalled above that the intersection of $\{Q(u) = Q(u_1)\}$ with $\{Q(u, u_1) = w\}$ is contained in $Gu_0$.

**Lemma 4.5.** Let $i \in \{1, 2\}$. If $Q(Gu_0, u_1)$ is not contained in the circle $|z| = c$, then $Q(Gu_0, u_{3-i})$ has non-empty interior in $\mathbb{C}$.
Proof. We may assume for instance that \( u_0 = \alpha u_1 + v_0 \) with \( |\alpha| \neq 1 \), \( Q(v_0, u_1) = 0 \), \( Q(v_0) = (1 - |\alpha|^2)Q(u_1) \neq 0 \). Write \( u_2 = \beta u_1 + v_2 \) with \( Q(v_2, u_1) = 0 \), \( v_2 \neq 0 \). For \( g \in G_1 \), we have
\[
Q(g, u_0, u_2) = a\bar{\beta}Q(u_1) + Q(g, v_0, v_2).
\]
From the fact recalled above, the set \( G_1 v_0 \) consists of the vectors \( v \) orthogonal to \( u_1 \) satisfying \( Q(v) = (1 - |\alpha|^2)Q(u_1) \). Any linear projection of this set on \( C \) has nonempty interior, which proves the assertion of the lemma.

We can now end the proof of the proposition. In view of the two lemmas above, we know that \( Gu_0 \) has non-empty interior in \( \{Q(u) = c\} \) except perhaps if both \( Q(Gu_0, u_1) \) and \( Q(Gu_0, u_2) \) are contained in the circle \( \{|z| = c\} \). We will now prove that this exceptional case is impossible. Write as before \( u_0 = \alpha u_1 + v_0 \), \( u_2 = \beta u_1 + v_2 \), with \( Q(v_0, u_1) = Q(v_2, u_1) = 0 \), \( v_2 \neq 0 \). Exchanging \( u_1, u_2 \) if necessary, we may assume that \( v_0 \neq 0 \). We may also assume that \( |\alpha| = 1 \), \( Q(v_0) = 0 \). Choose a 2-dimensional subspace \( E \) of the hyperplane \( H \) orthogonal to \( u_1 \) with the following properties:

- The subspace \( E \) contains \( v_0 \).
- The restriction of \( Q \) to \( E \) is non-degenerate.
- The orthogonal projection of \( v_2 \) on \( E \) is \( \neq 0 \).

Such a choice is possible because the last two conditions are open and dense amongst 2-dimensional subspaces of \( H \) containing \( v_0 \). Choose a basis \( e, f \) of \( E \) such that \( v_0 = e + f \) and \( Q(xe + yf) = |x|^2 - |y|^2 \). For any \( a, b \in \mathbb{C} \) with \( |a|^2 - |b|^2 = 1 \), we can find \( g \in G_1 \) such that
\[
g.e = ae + bf, \quad g.f = \bar{a}e + \bar{b}f.
\]
Therefore any vector of the form \( ze + \bar{z}f \), with \( z \in \mathbb{C}^* \) belongs to \( G_1 v_0 \).

Let \( se + tf \neq 0 \) be the orthogonal projection of \( v_2 \) on \( E \). Then \( z\bar{s} - \bar{z}t \) belongs to \( Q(G_1 v_0, v_2) \) for any \( z \in \mathbb{C}^* \). As the set \( \{z\bar{s} - \bar{z}t \mid z \in \mathbb{C}^*\} \) contains a straight segment in \( \mathbb{C} \), the set \( Q(G_1 u_0, u_2) = a\bar{\beta}c + Q(G_1 v_0, v_2) \) is not contained in the circle \( \{|z| = c\} \).

This concludes the proof of the proposition.

4.5.3. Application.

**Proposition 4.6.** Assume that \( d \geq 3 \), \( \rho^d, \rho^{d+1} \neq 1 \). Assume also that the intersection of the group generated by the operators \( L_q, q \in A_{d-1} \), on \( C^{A_{d-1}} \) with the special unitary group \( SU(Q^d_a) \) is dense (resp. Zariski dense) in \( SU(Q^d_a) \). Then the intersection of the group generated by the operators \( L_p, p \in A_d \), on \( C^{A_d} \) with the special unitary group \( SU(Q_a) \) is dense (resp. Zariski dense) in \( SU(Q_a) \).

**Proof.** Let \( p_1, p_2 \) be two distinct elements of \( A_d \). Denote by \( u_1, u_2 \) generators of \( H_{p_1}', H_{p_2}' \) respectively, satisfying \( Q_a(u_1) = Q_a(u_2) \neq 0 \). Such a choice is possible because the restrictions of \( Q_a \) to \( H_{p_1}, H_{p_2} \) have the same signature by lemma 4.2.

By the assumption of the proposition, for \( i \in \{1, 2\} \), the intersection of the group generated by the operators \( L_p, p \in A_d \), \( p \neq p_i \), with the special unitary group \( SU(Q_a) \) is dense (resp. Zariski dense) in the stabilizer \( G_i \) of \( u_i \) in \( SU(Q_a) \).
Proposition 4.3, the smallest closed group containing $G_1 \cup G_2$ is $SU(Q_a)$. We thus obtain the conclusion of the proposition.

4.6. **The induction step in the non-exceptional degenerate case.**

4.6.1. *The setting.* We assume in this subsection that $d \geq 4$ and that $(d + 1)\alpha$ is an integer. Therefore the hermitian form $Q_a$ on $\mathbb{C}^{A_d}$ is degenerate. As $0 < \alpha < \frac{1}{2}$, $d\alpha$ is not an integer and the hermitian form $Q'_a$ on $\mathbb{C}^{A_{d-1}}$ is non-degenerate. As the restriction of $Q_a$ to each hyperplane $H_p$ is isomorphic to $Q'_a$ (Lemma 4.2), the kernel of $Q_a$ has dimension 1.

**Remark 4.7.** In the case $d = 3$, we must have $\alpha = \frac{1}{4}$, then the induction hypothesis (see below) is not satisfied.

As $Q_a$ is invariant under each $L_p$ the kernel of $Q_a$ is invariant under the $L_p$. But the eigenvalues of $L_p$ are $1$ with multiplicity $(d - 1)$ and $-\zeta$ with multiplicity $1$, and the eigenvector associated to the eigenvalue $-\zeta$ is $e_p$, which is not an eigenvector of $L_q$ for $q \neq p$. We conclude that the kernel of $Q_a$ is pointwise fixed by each $L_p$.

We make the following **induction hypothesis**: on $\mathbb{C}^{A_{d-1}}$, the intersection of the subgroup generated by the $L_p$, $p \in A_{d-1}$ with the special unitary group $SU(Q'_a)$ is dense for the ordinary topology (resp. Zariski dense) in $SU(Q'_a)$.

4.6.2. *The induction step.* As the restriction of $Q_a$ to each $H_p$ is non-degenerate, the kernel $\mathbb{C}e$ of $Q_a$ is not contained in any $H_p$.

Let us denote by $SU^*(Q_a)$ the subgroup of $GL(\mathbb{C}^{A_d})$ formed by linear automorphisms which preserve $Q_a$, fix $e$ (and not simply the line $\mathbb{C}e$) and have determinant 1. If one writes these automorphisms in the basis $(e, e_{3-d}, \ldots, e_{d-1})$ (using that $\mathbb{C}^{A_d} = \mathbb{C}e \oplus H_{1-d}$), the matrix takes a block triangular form

$$M = \begin{pmatrix} 1 & v \\ 0 & g \end{pmatrix},$$

with $g \in SU(Q'_a)$. Let $D$ be the subgroup of $SU^*(Q_a)$ formed of automorphisms whose matrix in the selected basis satisfies $v = 0$.

**Proposition 4.8.** A subgroup of $SU^*(Q_a)$ which contains $D$ is equal to $D$ or to $SU^*(Q_a)$.

**Proof.** Let $D'$ be such a subgroup. We identify $H_{1-d}$ to $\mathbb{C}^{A_{d-1}}$ through $t_{1-d}$. Let

$$V(D') := \left\{ v \in (\mathbb{C}^{A_{d-1}})^* \left| \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \in D' \right. \right\}.$$

Clearly $V(D')$ is an additive subgroup of $(\mathbb{C}^{A_{d-1}})^*$. Moreover, as

$$\begin{pmatrix} 1 & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} 1 & v.g \\ 0 & 1 \end{pmatrix},$$

$V(D')$ is invariant under the natural action of $SU(Q'_a)$ on $(\mathbb{C}^{A_{d-1}})^*$. In view of the lemma below, $V(D')$ must be equal to $\{0\}$ or $(\mathbb{C}^{A_{d-1}})^*$, which corresponds to $D' = D$ and $D' = SU^*(Q_a)^*$. \qed
**Lemma 4.9.** Let $Q$ be a non-degenerate hermitian form on $\mathbb{C}^N$, $N \geq 3$. The only additive subgroups of $\mathbb{C}^N$ which are invariant under $SU(Q)$ are $\{0\}$ and $\mathbb{C}^N$.

**Proof.** We first observe that, as $N \geq 3$, the orbit $SU(Q).v_0$ of a vector $v_0$ is equal to
- $\{0\}$ if $v_0 = 0$;
- $\{v \neq 0 \mid Q(v) = 0\}$ if $v_0 \neq 0$, $Q(v_0) = 0$;
- $\{Q(v) = c\}$ if $Q(v_0) = c \neq 0$.

Indeed, Witt's theorem implies that the orbit $U(Q).v_0$ is as stated. As $N \geq 3$, there exists a vector $v_1$ in this orbit with at least one coordinate vanishing (in an orthogonal basis for $Q$). But then the image of the stabilizer of $v_1$ (in $U(Q)$) by the determinant map is the full unit circle. This means that the orbits $U(Q).v_1$ and $SU(Q).v_1$ are equal.

Let $V$ be an additive subgroup of $\mathbb{C}^N$ which is also $SU(Q)$-invariant. It is sufficient to show that, if $V$ contains a non-zero vector, then $V$ has non-empty interior. Let $v_0 \in V$, $v_0 \neq 0$. Then $V$ contains $V_0 := \{v \neq 0 \mid Q(v) = Q(v_0)\}$. The set $V_0 - v_0$ is also contained in $V$. The map

$$v \to Q(v - v_0) = 2Q(v_0) - 2\Re Q(v, v_0)$$

is not constant in a neighborhood of $v_0$ in $V_0$, hence its image contains a non-trivial interval. This implies that $V$ has non-empty interior. \hfill \square

**Corollary 4.10.** Under the induction hypothesis stated above, the intersection of the subgroup generated by the $L_p$ with $SU(Q_a)$ is dense (resp. Zariski dense) in $SU^*(Q_a)$.

**Proof.** Let $G$ be the closure (resp. the Zariski closure) of the intersection of the subgroup generated by the $L_p$, $1 - d \leq p \leq d - 1$ with $SL(n, \mathbb{C})$. We have seen earlier that $G$ is contained in $SU^*(Q_a)$.

Let $G'$ be the closure (resp. the Zariski closure) of the intersection of the subgroup generated by the $L_p$, $3 - d \leq p \leq d - 1$ with $SL(n, \mathbb{C})$. We have seen earlier that $G' \subset D$. It follows from the induction hypothesis that $G' = D$. As $L_{1-d} \circ L_{d-1}^{-1}$ has determinant 1 but does not preserve $H_{1-d}$, $G$ is not equal to $D$. It follows from the proposition that $G = SU^*(Q_a)$. \hfill \square

4.7. **From the degenerate case to the non-degenerate case.**

4.7.1. **The setting.** We assume in this subsection that $d \geq 5$ and that $d \alpha$ is an integer. Therefore the hermitian form $Q_a$ is non-degenerate, but the restrictions of $Q_a$ to each hyperplane $H_p$, which are isomorphic to $Q_a$, are degenerate. It means that the $Q_a$-orthogonal of $H_p$ is a line $\mathbb{C} w_p$ contained in $H_p$.

We make the following **induction hypothesis**: The intersection of the subgroup generated by the $L_p$, $p \in A_{d-1}$ with $SL(\mathbb{C}^d, \mathbb{C})$ is dense for the ordinary topology (resp. Zariski dense) in the subgroup $SU^*(Q_a)$ defined in the previous subsection.
4.7.2. Stabilizers. Let $Q$ be a non-degenerate non-definite hermitian form on $\mathbb{C}^N$. Let $w$ be a nonzero vector such that $Q(w) = 0$. Let $H$ be the hyperplane $Q$-orthogonal to $\mathbb{C}w$. It contains $\mathbb{C}w$. Denote by $Q'$ the restriction of $Q$ to $H$, which is degenerate. Choose a vector $w'$ such that $Q(w, w') = 1$. Denote by $H'$ the orthogonal of the plane $\mathbb{C}w \oplus \mathbb{C}w'$, and by $Q''$ the restriction of $Q$ to $H'$, which is non-degenerate. We have direct sums

$\mathbb{C}^N = H \oplus \mathbb{C}w', \quad H = \mathbb{C}w \oplus H'$.

Write $\text{Stab}(w)$ for the stabilizer of $w$ in $SU(Q)$. The matrix of an element of $\text{Stab}(w)$ in the decomposition $\mathbb{C}w \oplus H' \oplus \mathbb{C}w'$ of $\mathbb{C}^N$ takes a block-triangular form

$M = \begin{pmatrix}
1 & v & t \\
0 & g & h \\
0 & 0 & \omega
\end{pmatrix} \quad (4)$

However, not all such matrices $M$ are associated to elements of $SU(Q)$. Define a one-dimensional subgroup

$K := \left\{ \begin{pmatrix}
1 & 0 & t \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \Re t = 0 \right\}$.

In the next proposition, the group $SU^*(Q')$ was defined in the last subsection.

**Proposition 4.11.** We have an exact sequence

$1 \rightarrow K \rightarrow \text{Stab}(w) \rightarrow SU^*(Q') \rightarrow 1$

The homomorphism $\theta$ from $\text{Stab}(w)$ to $SU^*(Q')$ is induced by restriction to $H$. The exact sequence is not split.

**Proof.** If a matrix of the form (4) is unitary, we must have $\omega = 1$ because the scalar product $Q(w, w')$ is preserved.

As $\omega \equiv 1$, the homomorphism $\theta$ takes values in $SU^*(Q')$. It is onto by Witt’s theorem.

An elementary computation shows that the kernel of $\theta$ is equal to $K$ (see also below).

It remains to show that $\theta$ has no section. Assume by contradiction that such a section $\sigma$ exists. Consider

$u(v) := \sigma \begin{pmatrix}
1 & v \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & v & t(v) \\
0 & 1 & h(v) \\
0 & 0 & 1
\end{pmatrix}$.

As $u(v)$ is unitary, we have, for all $v \in (H')^*, x \in H'$

$Q(u(v)x, u(v)w') = Q(v(x)w + x, w' + h(v) + t(v)w) = v(x) + Q(x, h(v)) = 0$.

This determines $h$ as a semi-linear isomorphism from $(H')^*$ to $H'$. On the other hand, as $\sigma$ is a homomorphism, we have

$t(v + v') = t(v) + t(v') + v(h(v'))$,
for all \( v, v' \in (H')^* \). Therefore \( v(h(v')) \) is a symmetric function of \( v, v' \). As \( h \) is \( \mathbb{C} \)-antilinear, we should have \( v(h(v')) = 0 \). This is not true because \( h \) is an isomorphism. \( \square \)

We now obtain in our particular setting:

**Corollary 4.12.** Let \( p \in A_d \). The intersection with \( SU(Q_a) \) of the subgroup generated by the \( L_q, q \neq p \), is Zariski dense in the stabilizer \( Stab(w_p) \).

**Remark 4.13.** Probably one doesn’t get the density in the usual topology, even if we started from this form of the induction hypothesis.

**Proof.** We already know that \( L_q(w_p) = w_p \) for \( q \neq p \). Therefore the Zariski closure \( G_p \) of the intersection with \( SU(Q_a) \) of the subgroup generated by the \( L_q, q \neq p \) is a Zariski closed subgroup contained in \( Stab(w_p) \). On the other hand, the induction hypothesis (applied to the restriction of \( Q_a \) to \( H_p \), which is isomorphic to \( Q_a' \)) implies that restriction to \( H_p \) induces an homomorphism of \( G_p \) onto \( SU^*(Q_a') \).

The kernel of the homomorphism from \( G_p \) onto \( SU^*(Q_a) \) is a Zariski closed subgroup of \( K \) hence it is either equal to \( K \) (in which case \( G_p = Stab(w_p) \)) or to \( \{1\} \). (If this subgroup is only closed for the usual topology, it could be an infinite discrete subgroup of \( K \)). But the second case is impossible since the exact sequence of the proposition is not split. \( \square \)

4.7.3. More stabilizers. Let \( Q \) be a non-degenerate non-definite hermitian form on \( \mathbb{C}^N \). Let \( w \) be a nonzero vector such that \( Q(w) = 0 \). Let \( Stab(w) \) be the stabilizer of \( w \) in \( SU(Q) \). From Witt’s theorem, one gets

**Lemma 4.14.** \((N \geq 3) \) For any \( c \in \mathbb{C}, c \neq 0 \), the group \( Stab(w) \) acts transitively on \( N(w,c) := \{ u \in \mathbb{C}^N \ | \ Q(u) = 0, Q(u,w) = c \} \).

**Proof.** Let \( u \in N(w,c) \). It is sufficient to see that the determinant of an element \( g \in SU(Q) \) which stabilizes \( w \) and \( u \) can have any determinant of modulus one. This is clear since the restriction of \( Q \) to the orthogonal of \( \langle u, w \rangle \) is non-degenerate and the restriction of \( g \) to this subspace is any unitary matrix. \( \square \)

**Lemma 4.15.** Let \( F \) be a nontrivial linear subspace of \( \mathbb{C}^N \). Assume that the restriction of \( Q \) to \( F \) is non-degenerate.

1. If the restriction of \( Q \) to \( F \) is indefinite, any translate \( u + F \) intersects \( \{Q = 0\} \).
2. Assume that the restriction of \( Q \) to \( F \) is positive definite (resp. negative definite). Then a translate \( u + F, u \in F^\perp \), intersects \( \{Q = 0\} \) iff \( Q(u) \leq 0 \) (resp. \( Q(u) \geq 0 \)).

**Proof.**

1. The real-valued function \( f \rightarrow Q(u + f) = Q(u) + Q(f) + 2\Re Q(u,f) \) takes on \( F \) arbitrarily large positive and negative values, hence must vanish somewhere.
2. One has now \( Q(u + f) = Q(u) + Q(f) \geq Q(u) \); the conclusion follows. \( \square \)
Lemma 4.16. Assume $N = 2$. Let $w, w'$ be a basis of $C^2$, such that $Q(w) = 0$. Then $z \to Q(w + zw')$ takes positive and negative values in any neighborhood of $0$.

Proof. Indeed, one has $Q(w + zw') = |z|^2 Q(w') + 2\Re(zQ(w', w))$ with $Q(w', w) \neq 0$ as $Q$ is non-degenerate and $Q(w) = 0$.

Proposition 4.17. Let $(w_1, \ldots, w_N)$ be a basis of $C^N$ with $Q(w_i) = 0$ for $1 \leq i \leq N$, $\langle w_i, w_j \rangle \neq 0$ for $1 \leq i \neq j \leq N$. Let $G_i$ be the stabilizer of $w_i$ in $SU(Q)$. Then the smallest closed subgroup containing $G_1, \ldots, G_N$ is $SU(Q)$.

Proof. Let $G$ be the smallest subgroup containing $G_1, \ldots, G_N$. It is sufficient to show that $G$ acts transitively on $\Omega := \{ Q(u) = 0 \mid u \neq 0 \}$. As this last set is connected, it is sufficient to show that any orbit of $G$ in $\Omega$ has non-empty interior.

Lemma 4.18. Let $u_0 \in \Omega$. There exists an index $1 \leq i \leq N$ such that the image of $G.u_0$ by the map $u \to Q(u, w_i)$ has non-empty interior in $C$.

From this lemma, we may assume $c_i := Q(u_0, w_i)$ is different from 0 and that a small neighborhood of $c_i$ in $C$ is contained in the image of $G.u_0$ by the map $u \to Q(u, w_i)$. Let $u \in \Omega$ be close to $u_0$. There exists $u_1 \in G.u_0$ such that $Q(u_1, w_i) = Q(u, w_i) \neq 0$. By Lemma 4.14, one has $u \in G_1, u_1 \in G.u_0$.

Proof of Lemma 4.18. We first claim that there exist distinct indices $i, j$ and $u_1 \in G.u_0$ such that $Q(u_1, w_i) \neq 0, Q(u_1, w_j) \neq 0$.

Let $i$ be an index such that $c_i := Q(u_0, w_i)$ is different from 0. Let $j \neq i$. As $Q(w_i, w_j) \neq 0$, there exists $\lambda \in C$ such that $\Re(\lambda c_i) = 0$ and $Q(u_0 + \lambda w_i, w_j) \neq 0$. Take $u_1 := u_0 + \lambda w_i$. One has $Q(u_1) = 0, Q(u_1, w_j) =: c_j \neq 0, Q(u_1, w_i) = c_i$ and $u_1 \in G_i, u_0$ by Lemma 4.14. This proves the claim.

Let $F$ be the codimension 2 subspace of $C^N$ orthogonal to $w_i, w_j$. Since $Q(w_j, w_j) \neq 0$, the restriction of $Q$ to $F$ is nondegenerate.

• If the restriction of $Q$ to $F$ is positive definite, Lemma 4.15 implies that for any $c_i'$ close to $c_i$ there exist $u_2 \in \Omega$ such that $Q(u_2, w_i) = c_i', Q(u_2, w_j) = c_j$. By Lemma 4.14, one has $u_2 \in G_j, u_1$. This proves the assertion of the lemma in this case.

• Assume that the restriction of $Q$ to $F$ is positive definite (the negative case is symmetric). Then the restriction of $Q$ to $F^\perp$ is indefinite. Identify $F^\perp$ to $C^2$ through $u \to (Q(u, w_i), Q(u, w_j))$. We have $Q(c_i, c_j) \leq 0$ by Lemma 4.15. If $Q(c_i, c_j) < 0$, we proceed as in the first case to get the conclusion of the lemma (using again Lemma 4.15). If $Q(c_i, c_j) = 0$, by Lemma 4.16 there exists $c_i'$ close to $c_j$ such that $Q(c_i', c_j') < 0$. Then there exists $u_i' \in \Omega$ such that $Q(u_i', w_j) = c_i, Q(u_i', w_j) = c_j$. One has $u_i' \in G_i, u_1$ from Lemma 4.14. Then the end of the argument is the same than for $Q(c_i, c_j) < 0$.

Putting together Proposition 4.17 and Corollary 4.12, we obtain

Corollary 4.19. The intersection with $SU(Q_a)$ of the subgroup generated by the $L_p$ is Zariski dense in $SU(Q_a)$. 

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we have seen in Subsection 4.4 that the group generated by \(L\) is contained in this stabilizer. As

\[\text{tion with } SL(2,\mathbb{C})\]

For \(d = 4.9.\) Exceptional case II: \(\alpha = 1/4.\) At this stage, we can conclude that the Zariski closure of the intersection with \(SU(Q_\alpha)\) of the subgroup generated by the \(L_p\) contains the stabilizer of each \(w_p.\) Therefore it is equal to \(SU(Q_\alpha).\) \(\square\)

4.8. Exceptional case I: \(\alpha = 1/3.\) At this stage, we can conclude that the Zariski closure of the intersection with \(SU(Q_\alpha)\) of the subgroup generated by the \(L_p\) is equal to \(SU(Q_\alpha)\) when \(\alpha \neq \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\) and \((d + 1)\alpha\) is not an integer (so that \(Q_\alpha\) is non-degenerate). We may even replace Zariski closure by closure for the usual topology when the form is definite, i.e \((d + 1)\alpha < 1.\) Indeed it is sufficient to proceed by induction on the dimension \(d,\) starting with the results of Subsection 4.4, and applying successively either Proposition 4.6, Corollary 4.10, or Corollary 4.19.

In the two exceptional cases \(\alpha = \frac{1}{6}, \frac{1}{4}, \frac{1}{3}\), the group generated by \(L_1, L_{-1}\) in dimension 2 is a finite group so it is not a basis for a successful induction. These two cases will be dealt with in later subsections. However, the case \(\alpha = \frac{1}{3}\) does not need any supplemental work.

Remember that \(Q_1\) is degenerate for \(d = 2.\) The kernel of \(Q_\alpha\) is generated by \(e := e_{-1} + fe_1\) and fixed by \(L_{-1}\) and \(L_1.\) It was observed in Subsection 4.4 that the intersection of the group generated by \(L_{-1}\) and \(L_1\) with \(SL(2,\mathbb{C})\) contains a parabolic matrix. The determinant of \(L_{-1}\) and \(L_1\) is \(-\zeta,\) a sixth root of unity. This is sufficient to show that the Zariski closure \(G\) of the intersection of the group generated by \(L_{-1}\) and \(L_1\) with \(SL(2,\mathbb{C})\) is the stabilizer of \(e\) in \(SL(2,\mathbb{C}).\) Indeed, \(G\) is contained in this stabilizer. As \(R\)-Lie groups, the stabilizer of \(e\) in \(SL(2,\mathbb{C})\) has only three types of Zariski closed subgroups: the two trivial subgroups\(^6\), and (given any vector \(f\) independent of \(e\)) the subgroup \(G_f\) of the stabilizer formed of elements \(g\) such that \(g.f - f\) is a real multiple of \(e\) (there is a one-parameter family, parametrized by the 1-dimensional real projective space, of such subgroups). Here, the existence of a parabolic element guarantees that \(G\) is not reduced to the identity. It cannot be of the intermediate form, because conjugating by powers of \(L_1\) an element \(g\) such that \(g.f - f = e,\) we get elements \(g'\) such that \(g'.f - f = \omega e\) for any sixth root of unity. Therefore \(G\) is equal to the full stabilizer.

Thus the argument from Subsection 4.7 can be used to conclude the desired result (even though the dimension \(d\) is not as high as assumed there).

4.9. Exceptional case II: \(\alpha = 1/4.\) We assume in this subsection that \(\alpha = \frac{1}{4}.\) Then, we have seen in Subsection 4.4 that the group generated by \(L_{-1}\) and \(L_1\) is finite. For \(d = 3,\) the form \(Q_1\) is degenerate, but the Zariski closure \(G\) of the intersection with \(SL(3,\mathbb{C})\) of the group generated by \(L_{-2}, L_0\) and \(L_2\) is strictly smaller

\(^6\)I.e., \(\{Id\}\) and the full stabilizer itself.
than the group $SU^*(Q_{\frac{3}{4}})$ described in Subsection 4.6 (see below). It is only from dimension 4 that we get a “big” group generated by the $L_p$.

Consider first the case $d = 2$. It can be checked that the group $\Gamma$ generated by $L_{-1}$ and $L_1$ has order 96. The property of $\Gamma$ that will be useful in the sequel is

**Lemma 4.20.** The representation of $\Gamma$ on $\mathbb{C}^2 \simeq \mathbb{R}^4$ induced by the inclusion $\Gamma \subset U(Q_{\frac{3}{4}})$ is irreducible over $\mathbb{R}$.

**Proof.** It is clear that $L_1$ and $L_{-1}$ do not have a common eigenvector, therefore the representation is irreducible over $\mathbb{C}$. But $\Gamma$ contains the scalar multiplications by the fourth roots of unity, hence any $\mathbb{R}$-subspace invariant under $\Gamma$ has to be complex. \hfill \Box

We can now determine what is the Zariski closure $\Gamma_3$ of the group generated by the $L_p$ for $d = 3$. As in Subsection 4.6, let us denote by $e$ a generator of the kernel of $Q_{\frac{3}{4}}$, and write the operators in the basis $(e, e_0, e_2)$ (using that all coordinates of $e$ are non-zero). All $L_p$ fix $e$ so the matrices take the block-form

$$
\begin{pmatrix}
1 & v \\
0 & \gamma
\end{pmatrix}.
$$

Here, the $2 \times 2$ block $\gamma$ will vary exactly in the finite group $\Gamma$ mentioned above (hence $G$ will not contain $SU^*(Q_{\frac{3}{4}})$). Therefore we have an exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma_3 \longrightarrow \Gamma \longrightarrow 1,$$

where the kernel $K$ is the Zariski closed subgroup of $(H_{-2})^*$ formed of those $v$ such that $\begin{pmatrix} 1 & v \end{pmatrix}$ belongs to $\Gamma_3$. Note that $K$ is not only a Zariski closed subgroup of $(H_{-2})^*$, it is also invariant under the representation of $\Gamma$ dual to that induced by the inclusion of $\Gamma$ in $U(Q_{\frac{3}{4}})$: indeed, conjugating an element of the kernel by an element of $\Gamma_3$ with diagonal block $\gamma$ changes $v$ into $v, \gamma^{-1}$. By Lemma 4.20 this representation is irreducible over $\mathbb{R}$. We conclude that $K$ must be equal to either $\{0\}$ or $(H_{-2})^*$.

**Lemma 4.21.** The kernel $K$ is equal to $(H_{-2})^*$.

**Proof.** Otherwise, $K$ would be trivial and $\Gamma_3$ would be finite. As any representation of a finite group is semi-simple, there would exist a $\Gamma_3$-invariant complex 2-plane supplemented by $Ce$. But then the transposed matrices $^{t}L_{-2}, ^{t}L_0, ^{t}L_2$ would have a common eigenvector. This is clearly not the case. \hfill \Box

Having described $\Gamma_3$, we now go to the case $d = 4$. To represent the operators $L_p$, we chose a basis $w_{-3}, f_{-1}, f_1, w_3$ with the following properties:

- $w_{-3}$ is a generator of the orthogonal of the hyperplane $H_{-3}$;
- $w_3$ is a generator of the orthogonal of the hyperplane $H_3$;
- $Q_{\frac{3}{4}}(w_{-3}, w_3) = 1$
- the subspace $F$ generated by $f_{-1}, f_1$ is the orthogonal of the subspace $W$ generated by $w_{-3}, w_3$; note that the restriction of $Q_{\frac{3}{4}}$ to $W$ has signature
(1, 1), hence $W$ and $F$ are indeed transverse and the restriction of $Q_1$ to $F$ has signature $(2, 0)$.

- $f_{-1}, f_1$ form an orthonormal basis of $F$.

Observe that the first three vectors form a basis of $H_{-3}$, and the last three form a basis of $H_3$.

Consider the Zariski closure of the subgroup generated by $L_{-1}, L_1, L_3$. An element in this group preserves the hyperplane $H_{-3}$ and its restriction to $H_3$ is constrained exactly by the case $d = 3$: we can write it in $1 + 2 + 1$ block form as

$$
\begin{pmatrix}
1 & v & s \\
0 & \gamma & v' \\
0 & 0 & \omega
\end{pmatrix}
$$

with $\gamma \in \Gamma$. We restrict to the subgroup $G_{-3}$ of finite index such that $\gamma = 1$. As in Subsection 4.7, writing that the form $Q_1$ is preserved gives (when $\gamma = 1$)

$$
\omega = 1, \quad v' = -\bar{v}, \quad \Re s = -\frac{1}{2}||v||^2.
$$

**Lemma 4.22.** Conversely, any matrix of the prescribed form satisfying these relations belongs to $G_{-3}$.

**Proof.** Essentially the same as in Proposition 4.11 and Corollary 4.12. \hfill \Box

The 5-dimensional Lie algebra $g_{-3}$ of $G_{-3}$ is therefore the set of matrices of the form

$$
A = \begin{pmatrix}
0 & v_{-1} & v_1 & is \\
0 & 0 & 0 & -\bar{v}_{-1} \\
0 & 0 & 0 & -\bar{v}_1 \\
0 & 0 & 0 & 0
\end{pmatrix}
$$

with $v_{-1}, v_1 \in \mathbb{C}$ and $s \in \mathbb{R}$.

Similarly, from the action of $L_{-3}, L_{-1}, L_1$, one obtains a Zariski closed group $G_3$ whose Lie algebra $g_3$ is the set of matrices of the form

$$
B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-\bar{u}_{-1} & 0 & 0 & 0 \\
i r & u_{-1} & u_1 & 0
\end{pmatrix}
$$

with $u_{-1}, u_1 \in \mathbb{C}$ and $r \in \mathbb{R}$.

**Lemma 4.23.** The smallest Lie algebra containing $g_{-3}$ and $g_3$ is the Lie algebra $su(Q_4)$.

**Proof.** With $A, B$ as above, we have

$$
AB - BA = \begin{pmatrix}
-\bar{r} & -v_{-1} & isu_{-1} & isu_1 & 0 & isu_{-1} & isu_1 \\
-iv_{-1} & -\bar{v}_{-1} & v_{-1}u_{-1} - u_{-1}\bar{v}_{-1} & v_1 u_{-1} - u_1\bar{v}_{-1} & 0 & isu_{-1} & isu_1 \\
-iv_1 & -\bar{v}_1 & v_{-1}u_{-1} - u_{-1}\bar{v}_1 & v_1 u_{-1} - u_1\bar{v}_1 & 0 & isu_{-1} & isu_1 \\
i r & -iv_{-1} & -iv_1 & -iv_1 & r \bar{s} + u_{-1}\bar{v}_{-1} + u_1\bar{v}_1 & isu_{-1} & isu_1
\end{pmatrix}.
$$
After adding appropriate elements \( A', B' \) of \( g_{-3}, g_3 \) respectively, the matrix \( AB - BA + A' + B' \) is equal to

\[
C = \begin{pmatrix}
-rs - v_{-1} \bar{u}_{-1} - v_i \bar{u}_1 & 0 & 0 & 0 \\
0 & v_{-1} \bar{u}_{-1} - u_{-1} \bar{v}_{-1} & v_1 \bar{u}_{-1} - u_1 \bar{v}_{-1} & 0 \\
0 & 0 & v_{-1} \bar{u}_{-1} - u_{-1} \bar{v}_1 & v_1 \bar{u}_{-1} - u_1 \bar{v}_1 \\
0 & 0 & 0 & r s + u_{-1} \bar{v}_{-1} + u_1 \bar{v}_1
\end{pmatrix}.
\]

For \( u_{-1} = u_1 = v_{-1} = v_1 = 0, r = s = 1 \), we get

\[
C = \begin{pmatrix}
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

For \( r = s = v_{-1} = u_1 = 0, v_1 = u_{-1} = 1 \), we get

\[
C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

For \( r = s = v_{-1} = u_1 = 0, v_1 = i, u_{-1} = 1 \), we get

\[
C = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & i & 0 \\
0 & i & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

For \( r = s = v_{-1} = u_{-1} = 0, v_1 = 1, u_1 = i \), we get

\[
C = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}.
\]

For \( r = s = v_1 = u_1 = 0, v_{-1} = 1, u_{-1} = i \), we get

\[
C = \begin{pmatrix}
i & 0 & 0 & 0 \\
0 & 0 & -2i & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & i
\end{pmatrix}.
\]

These five matrices, together with \( g_{-3} \) and \( g_3 \), span a 15-dimensional vector space. As \( su(Q_{1/4}) \), being isomorphic to \( su(3, 1) \), has also dimension 15, the lemma is proved.

\[\square\]

**Corollary 4.24.** The intersection of the subgroup generated by \( L_{-3}, L_{-1}, L_1 \) and \( L_3 \) with \( SL(C^{\mathbb{A}_1}) \) is Zariski dense in \( SU(Q_{1/4}) \).

This provides an appropriate starting point for the induction for \( \alpha = 1/4 \). The results in higher dimension follow.
4.10. Exceptional case III: $\alpha = 1/6$. The proof is identical to the previous case $\alpha = 1/4$ due to the following fact, an improvement (in generality) on Lemma 4.20.

**Proposition 4.25.** For any $\alpha \in (0, 1/2)$ and any $d \geq 2$ such that $(d + 1)\alpha$ is not an integer, there is no non-trivial $\mathbb{R}$-subspace of $\mathbb{C}^{A_d}$ which is invariant under every $L_p$, $p \in A_d$.

*Proof.* The kernel of $L_p - \text{id}$ is the hyperplane

$$\mathcal{H}_p = \left\{ \zeta \sum_{q \leq p} x_q + \sum_{q > p} x_q = 0 \right\}.$$  

The other eigenvalue of $L_p$ is equal to $-\zeta$ and is simple. The associated eigenspace is $\mathbb{C} e_p$. We claim that the intersection $\bigcap_{p \in A_d} \mathcal{H}_p$ is trivial (if $\zeta^{d+1} \neq 1$). Indeed, the intersection $\mathcal{H}_p \cap \mathcal{H}_{p+2}$ is contained in $\{ x_p = \zeta x_{p+2} \}$ for $p < d - 1$, and the intersection $\mathcal{H}_{d-1} \cap \mathcal{H}_{1-d}$ is contained in $\{ \zeta x_{1-d} = \zeta^{-1} x_{d-1} \}$.

Let $p \in A_d$, and let $W$ be an $\mathbb{R}$-subspace of $\mathbb{C}^{A_d}$ that is invariant under $L_p$. If $W$ is not contained in $\mathcal{H}_p$, it contains $\mathbb{C} e_p$: indeed, if $w \in W$ has the form $h + t e_p$ with $h \in \mathcal{H}_p$ and $t \neq 0$, then $w - L_p(w) = (1 + \zeta) t e_p$ belongs to $W$ and $\mathbb{C} e_p \subset W$ as $\zeta$ is not real.

Assume now that $W$ is an $\mathbb{R}$-subspace of $\mathbb{C}^{A_d}$ that is invariant under all $L_p$, $p \in A_d$. If $W$ is contained in $\mathcal{H}_p$ for every $p \in A_d$, it is equal to $\{0\}$. Otherwise, there exists $p \in A_d$ such that $\mathbb{C} e_p \subset W$. For $q \neq p$, $e_p - L_q(e_p)$ is equal to $e_q$ or $\zeta e_q$, therefore we have also $\mathbb{C} e_q \subset W$. Then $W = \mathbb{C}^{A_d}$. 

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