GEOMETRY OF COMPLETE GRADIENT SHRINKING RICCI SOLITONS

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Dedicated to Professor S.-T. Yau on the occasion of his 60th birthday

The notion of Ricci solitons was introduced by Hamilton [24] in mid 1980s. They are natural generalizations of Einstein metrics. Ricci solitons also correspond to self-similar solutions of Hamilton’s Ricci flow [22], and often arise as limits of dilations of singularities in the Ricci flow. In this paper, we will focus our attention on complete gradient shrinking Ricci solitons and survey some of the recent progress, including the classifications in dimension three, and asymptotic behavior of potential functions as well as volume growths of geodesic balls in higher dimensions.

1. Gradient Shrinking Ricci Solitons

Recall that a Riemannian metric $g_{ij}$ is Einstein if its Ricci tensor $R_{ij}$ is a constant multiple of $g_{ij}$: $R_{ij} = \rho g_{ij}$ for some constant $\rho$. A smooth $n$-dimensional manifold $M^n$ with an Einstein metric $g_{ij}$ is an Einstein manifold. Ricci solitons, introduced by Hamilton [24], are natural generalizations of Einstein metrics.

Definition 1.1. A complete Riemannian metric $g_{ij}$ on a smooth manifold $M^n$ is called a gradient Ricci soliton if there exists a smooth function $f$ on $M^n$ such that the Ricci tensor $R_{ij}$ of the metric $g_{ij}$ satisfies the equation

$$R_{ij} + \nabla_i \nabla_j f = \rho g_{ij}$$

for some constant $\rho$. For $\rho = 0$ the Ricci soliton is steady, for $\rho > 0$ it is shrinking and for $\rho < 0$ expanding. The function $f$ is called a potential function of the Ricci soliton.

Note that when the potential function $f$ is a constant Ricci solitons are simply Einstein metrics. In this paper, we will focus our attention on complete gradient shrinking Ricci solitons, which are possible Type I singularity models in the Ricci flow. We refer the readers to [4, 5] and the references therein for a quick overview and more information on singularity formation of the Ricci flow on 3-manifolds and the role shrinking solitons played.

Throughout the paper we will assume our gradient shrinking solitons satisfy the equation

$$R_{ij} + \nabla_i \nabla_j f = \frac{1}{2} g_{ij}. \quad (1.1)$$

This can be achieved by a suitable scaling of any shrinking soliton metric $g_{ij}$.

To get a good feeling of how gradient shrinking solitons correspond to self-similar ancient solutions of the Ricci flow, we first observe that if $g_{ij}$ is an Einstein metric

\(^1\)Partially supported by NSF.
with positive scalar curvature $R = n/2$, then $g_{ij}$ corresponds to a homothetic shrinking solution $g_{ij}(t)$ to the Ricci flow

$$\frac{\partial g_{ij}(t)}{\partial t} = -2R_{ij}(t)$$

with $g_{ij}(0) = g_{ij}$. Indeed, if

$$R_{ij} = \frac{1}{2}g_{ij},$$

then

$$g_{ij}(t) = (1 - t)g_{ij}$$

is a solution to the Ricci flow which shrinks homothetically to a point as $t \to 1$. Note that $g_{ij}(t)$ exists on the ancient time interval $(-\infty, 1)$, hence an ancient solution, and the scalar curvature $R(t) \to \infty$ like $1/(1 - t)$ as $t \to 1$. Similarly, a complete gradient shrinking Ricci soliton satisfying equation (1.1) corresponds to a self-similar solution $\tilde{g}_{ij}(t)$ to the Ricci flow with

$$\tilde{g}_{ij}(t) := (1 - t)\varphi_t^*(g_{ij}), \quad -\infty < t < 1,$$

where $\varphi_t$ is the 1-parameter family of diffeomorphisms generated by $\nabla f / (1 - t)$.

A theorem of Perelman states that given any non-flat $\kappa$-noncollapsed ancient solution $g_{ij}(t)$ to the Ricci flow with bounded and nonnegative curvature operator, the limit of some suitable blow-back of the solution converges to a non-flat gradient shrinking soliton (see Proposition 11.2 in [35], or Theorem 6.2.1 in [8]). Thus knowing the geometry of gradient shrinking solitons helps us to understand the asymptotic behavior of ancient solutions.

For compact shrinking Ricci solitons in low dimensions, we have

**Proposition 1.1. (Hamilton [24] for $n = 2$, Ivey [26] for $n = 3$) In dimension $n \leq 3$, there are no compact shrinking Ricci solitons other than those of constant positive sectional curvature.**

However, when $n \geq 4$, besides positive Einstein manifolds and products of positive Einstein manifolds with Euclidean spaces, there do exist non-Einstein compact gradient shrinking solitons. Also, there exist complete noncompact non-flat shrinking solitons. We list below the main examples.

**Example 1.1. Quotients of round $n$-sphere $S^n/\Gamma$**

As we pointed out before, any positive Einstein manifold is a (compact) shrinking Ricci soliton. In particular, the round $n$-sphere $S^n$, or any of its metric quotient $S^n/\Gamma$ by a finite group $\Gamma$, is a compact shrinking soliton.

**Example 1.2. The Gaussian shrinker**

It is easy to check that the flat Euclidean space $(\mathbb{R}^n, \delta_{ij})$ is a gradient shrinker with potential function $f = |x|^2/4$:

$$\partial_i \partial_j f = \frac{1}{2}\delta_{ij}.$$ 

$(\mathbb{R}^n, \delta_{ij}, |x|^2/4)$ is called the Gaussian shrinking soliton.

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1. It was observed by Z.-H. Zhang [13] that $V = \nabla f$ is a complete vector field if the soliton metric $g_{ij}$ is complete.
2. See alternative proofs in (Proposition 5.21, [14]) or (Proposition 5.1.10, [8]), and [13].
3. See [17] for alternative proofs.
Example 1.3. The round cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$

The product of a positive Einstein manifold with the Euclidean Gaussian shrinker is a (noncompact) shrinking soliton. In particular, the round cylinder $\mathbb{S}^k \times \mathbb{R}^{n-k}$ is a noncompact shrinking soliton.

Example 1.4. Compact rotationally symmetric Kähler shrinkers

For real dimension 4, the first example of a compact shrinking soliton was constructed in early 90’s by Koiso [27] and independently by the author [24] on compact complex surface $\mathbb{C}P^2\#(-\mathbb{C}P^2)$, the blow-up of the complex projective plane at one point. Here $(-\mathbb{C}P^2)$ denotes the complex projective space with the opposite orientation. This is a gradient shrinking Kähler-Ricci soliton, which has $U(2)$ symmetry and positive Ricci curvature. More generally, Koiso and the author found $U(n)$-invariant Kähler-Ricci solitons on certain twisted projective line bundle over $\mathbb{C}P^{n-1}$ for $n \geq 2$.

Example 1.5. Compact toric Kähler shrinkers

In [39], Wang-Zhu found a gradient Kähler-Ricci soliton on $\mathbb{C}P^2\#2(-\mathbb{C}P^2)$ which has $U(1) \times U(1)$ symmetry. More generally, they proved the existence of gradient shrinking Kähler solitons on all Fano toric varieties of complex dimension $n \geq 2$ with non-vanishing Futaki invariant.

Example 1.6. Noncompact gradient Kähler shrinkers

Feldman-Ilmanen-Knopf [19] found the first complete noncompact $U(n)$-invariant gradient shrinking Kähler solitons on certain twisted complex line bundles over $\mathbb{C}P^{n-1}$ ($n \geq 2$) which are cone-like at infinity and have quadratic curvature decay. Moreover, the shrinker has positive scalar curvature but the Ricci curvature changes sign. The simplest such example is defined on the tautological line bundle $\mathcal{O}(-1)$ of $\mathbb{C}P^n$, the blow-up of $\mathbb{C}^2$ at the origin.

Remark 1.1. Recently, Dancer-Wang [15] produced new examples of rotationally symmetric compact and complete noncompact gradient Kähler shrinkers on bundles over the product of positive Kähler-Einstein manifolds, generalizing those in Example 1.4 and Example 1.6.

Next we describe some useful results about gradient shrinking solitons.

Proposition 1.2. (Hamilton [25]) Let $(M^n, g_{ij}, f)$ be a complete gradient shrinking soliton satisfying Eq. (1.1). Then we have

$$\nabla_i R = 2 R_{ij} \nabla_j f,$$

and

$$R + |\nabla f|^2 - f = C_0$$

for some constant $C$. Here $R$ denotes the scalar curvature.

Proof. Let $(M^n, g_{ij}, f)$ be a complete gradient Ricci soliton satisfying equation (1.1). Taking the covariant derivatives and using the commutating formula for covariant derivatives, we obtain

$$\nabla_i R_{jk} - \nabla_j R_{ik} + R_{ijkl} \nabla_l f = 0.$$
Taking the trace on \( j \) and \( k \), and using the contracted second Bianchi, we get
\[
\nabla_i R = 2R_{ij} \nabla_j f.
\]
Thus
\[
\nabla_i (R + |\nabla f|^2 - f) = 2(R_{ij} + \nabla_i \nabla_j f - \frac{1}{2} g_{ij}) \nabla_j f = 0.
\]
Therefore
\[
R + |\nabla f|^2 - f = C_0
\]
for some constant \( C_0 \).

Note that if we normalize \( f \) by adding the constant \( C_0 \) to it, then (1.5) becomes
\[
R + |\nabla f|^2 - f = 0.
\]

**Proposition 1.3. (Perelman)\(^{36}\)** Let \((M^n, g_{ij}, f)\) be a complete noncompact gradient shrinking Ricci soliton with bounded Ricci curvature which satisfies (1.1) and the normalization (1.6). Let \( r(x) = d(x, x_0) \) denote the distance from \( x \) to a fixed point \( x_0 \in M \). Then there exist positive constants \( C_1, C_2, c_1, c_2 \) such that, for \( r(x) \) sufficiently large, \( f \) satisfies the following estimates:
\[
\begin{align*}
\frac{1}{4} (r(x) - c_1)^2 &\leq f(x) \leq C_1 (r(x) + c_2)^2; \\
|\nabla f|(x) &\leq C_2 (r(x) + 1).
\end{align*}
\]

**Proof.** By assumption, we know the Ricci curvature is bounded by \(|Rc| \leq C\) for some positive constant \( C > 0 \). From soliton equation (1.1) and the Ricci lower bound \( Rc \geq -nC \), we immediately have the upper estimate on Hessian of \( f \):
\[
\nabla_i \nabla_j f \leq (C + 1/2) g_{ij},
\]
from which the upper estimate on \( f \) in (1.7) follows. Now (1.8) follows from this upper estimate \( f(x) \), the scalar curvature lower bound \( R \geq -nC \), and (1.5) in Proposition 1.2.

To prove the lower estimate on \( f \) in (1.7), consider any minimizing normal geodesic \( \gamma(s) \), \( 0 \leq s \leq s_0 \) for some arbitrary large \( s_0 > 0 \) starting from \( \gamma(0) = x_0 \). Denote by \( X(s) = \dot{\gamma}(s) \) the unit tangent vector along \( \gamma \). Then, by the second variation of arc length, we have
\[
\int_0^{s_0} \phi^2 Rc(X, X) ds \leq (n-1) \int_0^{s_0} |\phi(s)|^2 ds
\]
for every nonnegative function \( \phi(s) \) defined on the interval \([0, s_0]\). Now, following Hamilton \([25]\), we choose \( \phi(s) \) by
\[
\phi(s) = \begin{cases} 
  s, & s \in [0, 1], \\
  1, & s \in [1, s_0 - 1], \\
  s_0 - s, & s \in [s_0 - 1, s_0].
\end{cases}
\]

\(^5\)This result was essentially sketched by Perelman (see p.3 in \([30]\)), and a detailed argument was presented in \([8]\) (see p.385-386 in \([8]\)).
Then
\[
\int_0^{s_0} Rc(X, X) ds = \int_0^{s_0} \phi^2 Rc(X, X) ds + \int_0^{s_0} (1 - \phi^2) Rc(X, X) ds \\
\leq (n - 1) \int_0^{s_0} |\phi'(s)|^2 ds + \int_0^{s_0} (1 - \phi^2) Rc(X, X) ds \\
\leq 2(n - 1) + \max_{B_{r_0}(1)} |Rc| + \max_{B_{r_0}(1)} |Rc|.
\]

On the other hand, by (1.1), we have
\[
\nabla_X f = \nabla_X \nabla_X f = \frac{1}{2} - Rc(X, X).
\]

Integrating (1.10) along \( \gamma \) from 0 to \( s_0 \), we get
\[
\dot{f}(\gamma(s_0)) - \dot{f}(\gamma(0)) = \frac{1}{2} s_0 - \int_0^{s_0} Rc(X, X) ds.
\]

Since by assumption \( g_{ij} \) has bounded Ricci curvature \( |Rc| \leq C \), it follows that
\[
\dot{f}(\gamma(s_0)) \geq \frac{s_0}{2} + \dot{f}(\gamma(0)) - 2(n - 1) - \max_{B_{r_0}(1)} |Rc| - \max_{B_{r_0}(1)} |Rc| \\
\geq \frac{1}{2} s_0 - \dot{f}(\gamma(0)) - 2(n - 1) - 2C = \frac{1}{2} (s_0 - c),
\]
and
\[
f(\gamma(s_0)) \geq \frac{1}{4} (s_0 - c)^2 - f(x_0) - \frac{c^2}{4}.
\]

This completes the proof of Proposition 1.6. \( \square \)

Remark 1.2. When \( Rc \geq 0 \), (1.11) also implies that
\[
f(x) \leq \frac{1}{4} (r(x) + c_2)^2.
\]

On the other hand, replacing \( |Rc| \leq C \) by \( Rc \geq 0 \), Ni \( 32 \) showed that
\[
f(x) \geq \frac{1}{8} r^2(x) - C_1.
\]

Remark 1.3. A precise asymptotic estimate on \( f \), without any curvature bound assumption, has been obtained more recently by Cao-Zhou \( 7 \) (see also Theorem 3.2 in this paper).

2. Classification of 3-dimensional Gradient Shrinking Solitons

In this section, we present the classification results on 3-dimensional complete shrinking Ricci solitons by Perelman \( 36 \), Ni-Wallach \( 33 \), and Cao-Chen-Zhu \( 6 \).

In \( 36 \), Perelman obtained the following classification result of 3-dimensional complete shrinking solitons which is an improvement of a result of Hamilton (Theorem 26.5, \( 25 \)).

**Theorem 2.1. (Perelman)** Let \((M^3, g_{ij}, f)\) be a complete nonflat 3-dimensional gradient shrinking Ricci soliton. Suppose \( g_{ij} \) has bounded and nonnegative sectional curvature \( 0 \leq Rm \leq C \) and is \( \kappa \)-noncollapsed on all scales for some \( \kappa > 0 \). Then \((M^3, g_{ij})\) is one of the following:

(i) the round three-sphere \( S^3 \), or one of its metric quotients;

(ii) the round infinite cylinder \( S^2 \times \mathbb{R} \), or one of its \( \mathbb{Z}_2 \) quotients.
We remark that, by the strong maximum principle of Hamilton [23], a complete shrinking Ricci soliton with nonnegative curvature operator either has strictly positive curvature operator everywhere or its universal cover splits as a product $N \times \mathbb{R}^k$, where $k \geq 1$ and $N$ is a shrinking soliton with positive curvature operator. On the other hand, we know that compact shrinking solitons with positive curvature operator are isometric to finite quotients of round spheres, thanks to the works of Hamilton [22, 23] (for $n = 3, 4$) and Böhm-Wilking [1] (for $n \geq 5$).

In view of the proceeding remark, to prove Theorem 2.1 it suffices to rule out the existence of a complete noncompact nonflat $\kappa$-noncollapsed 3-dimensional gradient shrinking Ricci soliton with bounded and positive sectional curvature.

**Proposition 2.1. (Perelman [35])** There is no three-dimensional complete noncompact $\kappa$-noncollapsed gradient shrinking soliton with bounded and positive sectional curvature.

**Sketch of the Proof.** By contradiction, suppose there is a 3-dimensional complete noncompact $\kappa$-noncollapsed gradient shrinking soliton $(M^3, g_{ij}, f)$ with bounded and positive sectional curvature. Consider any minimizing normal geodesic $\gamma(s)$ starting from $\gamma(0) = x_0$. As we saw in the proof of Proposition 1.3, for $s$ sufficiently large we have

$$\left| \nabla f \cdot \dot{\gamma}(s) - \frac{s}{2} \right| \leq C,$$

and

$$\left| f(\gamma(s)) - \frac{s^2}{4} \right| \leq C \cdot (s + 1).$$

In particular, $f$ has no critical points outside some large geodesic ball $B_{x_0}(s_0)$.

Now by (1.4) in Proposition 1.2 and the assumption of $Rc > 0$, we have

$$\nabla R \cdot \nabla f = 2Rc(\nabla f, \nabla f) > 0 \quad (2.1)$$

at all points $x$ with $d(x, x_0) \geq s_0$. So outside $B_{x_0}(s_0)$, the scalar curvature $R$ is strictly increasing along the gradient curves of $f$. Hence

$$\bar{R} = \limsup_{d(x, x_0) \to +\infty} R(x) > 0. \quad (2.2)$$

**Claim:** $\bar{R} = 1$.

First of all, if we choose a sequence of points $\{x_k\}$ so that $R(x_k) \to \bar{R}$, then it follows from the noncollapsing assumption and the compactness theorem that a subsequence of $(M^3, g_{ij}, x_k)$ converges to some limit $(\bar{M}^3, \bar{g}_{ij}, \bar{x})$, which splits off a line and hence is a round cylinder with scalar curvature $\bar{R}$, corresponding to a shrinking soliton defined (at least) on the ancient time interval $(-\infty, 1)$. Thus we conclude that $\bar{R} \leq 1$. In particular, the scalar curvature of our original gradient shrinking soliton satisfies the upper bound

$$R(x) < 1 \quad (2.3)$$

outside $B_{x_0}(s_0)$.

To see that in fact $\bar{R} = 1$, we consider the level surfaces of $f$

$$\Sigma_s = \{ x \in M : f(x) = s \}.$$
Note that the second fundamental form of level surface $\Sigma_s$ is given by
\[ h_{ij} = \nabla_i \nabla_j f / |\nabla f|, \quad i, j = 1, 2, \]
where \(\{e_1, e_2\}\) is any orthonormal basis tangent to $\Sigma_s$. Now, for 3-manifolds, the positivity of sectional curvature is equivalent to $Rg_{ij} \geq 2R_{ij}$. If we choose an orthonormal basis $\{e_1, e_2\}$ such that $Rc(e_1, e_2) = 0$, then by (1.1) we have
\[ \nabla_{e_i} \nabla_{e_j} f = \frac{1}{2} \delta_{ij} - Rc(e_i, e_j) \geq \left( \frac{1}{2} - \frac{R}{2} \right) \delta_{ij}, \quad i = 1, 2. \] (2.4)

In particular, $\Sigma_s$ is convex and it follows that
\[
\frac{d}{ds} \text{Area} (\Sigma_s) = \int_{\Sigma_s} \frac{1}{|\nabla f|} \text{div}(\frac{\nabla f}{|\nabla f|}) \\
\geq \int_{\Sigma_s} \frac{1}{|\nabla f|} (1 - R) \\
> \frac{1 - \bar{R}}{2s} \text{Area} (\Sigma_s) \geq 0
\]
for $s \geq s_0$. This implies that $\text{Area} (\Sigma_s)$ is strictly increasing as $s$ increases, and
\[
\log \text{Area} (\Sigma_s) > (1 - \bar{R}) \log \sqrt{s} - C
\]
for some constant $C$, and $s \geq s_0$. But $\text{Area} (\Sigma_s)$ is uniformly bounded from above by the area of the round sphere with scalar curvature one. Thus we conclude that $\bar{R} = 1$, and
\[
\text{Area} (\Sigma_s) < 8\pi
\] (2.5)
for $s$ large enough. This proves the Claim.

Now, denote by $\nu$ the unit normal to the level surface. By using the Gauss equation and the soliton equation (1.1), the intrinsic curvature $K$ of the level surface $\Sigma_s$ can be computed as
\[
K = R_{1212} + \det(h_{ij})
\]
\[
= \frac{R}{2} - Rc(\nu, \nu) + \frac{\det(\nabla^2 f)}{|\nabla f|^2}
\]
\[
\leq \frac{R}{2} - Rc(\nu, \nu) + \frac{(1 - \bar{R} + Rc(\nu, \nu))^2}{4|\nabla f|^2} < \frac{1}{2}
\]
for $s$ sufficiently large. But, this together with (2.5) lead to a contradiction to the Gauss-Bonnet formula.

**Corollary 2.1.** The only three-dimensional complete noncompact $\kappa$-noncollapsed gradient shrinking soliton with bounded and nonnegative sectional curvature are either $\mathbb{R}^3$ or quotients of $S^2 \times \mathbb{R}$.

Note that Perelman’s proof of Proposition 2.1 relies upon the Gauss-Bonnet formula, which cannot be used in higher dimensions. In the past a few years, there have been considerable efforts to improve or generalize the above results of Perelman. Ni-Wallach [33] and Naber [31] were able to drop the assumption on $\kappa$-noncollapsing condition and replace nonnegative sectional curvature assumption by that of nonnegative Ricci curvature. In addition, Ni-Wallach [33] allows the curvature $|Rm|$ to grow as fast as $e^{ar(x)}$, where $r(x)$ is the distance function to some origin $x_0$ and $a > 0$ is a suitable small positive constant.
Theorem 2.2. (Ni-Wallach [33]) Any 3-dimensional complete noncompact non-flat gradient shrinking soliton with nonnegative Ricci curvature $Rc \geq 0$ and curvature bound $|Rm|(x) \leq Ce^{ar(x)}$ is a quotient of the round sphere $S^3$ or round cylinder $S^2 \times \mathbb{R}$.

Sketch of the Proof. It suffices to show that if $(M^3, g_{ij}, f)$ is a complete gradient shrinking soliton with positive Ricci curvature $Rc > 0$ and curvature bound $|Rm|(x) \leq Ce^{ar(x)}$, then $(M^3, g_{ij})$ is a finite quotient of $S^3$. The basic ingredient of Ni-Wallach’s proof is to use the identity

$$
\Delta \left( \frac{|Rc|^2}{R^2} \right) = \nabla \left( \frac{|Rc|^2}{R^2} \right) \cdot \nabla f + \frac{2}{R^4} |R\nabla Rc - \nabla RRc|^2 - \frac{2}{R} \nabla \left( \frac{|Rc|^2}{R^2} \right) \cdot \nabla R + \frac{4P}{R^3} \tag{2.7}
$$

satisfied by the soliton metric, where

$$
P = \frac{1}{2}(\lambda + \mu - \nu)^2(\lambda - \mu)^2 + (\mu + \nu - \lambda)^2(\mu - \nu)^2 + (\nu + \lambda - \mu)^2(\nu - \lambda)^2, \tag{2.8}
$$

and $\lambda \geq \mu \geq \nu$ are the eigenvalues of $Rc$. We remark that Hamilton [22] showed that for any solution $g_{ij}(t)$ to the Ricci flow on 3-manifolds there holds the evolution equation

$$
\frac{\partial}{\partial t} \left( \frac{|Rc|^2}{R^2} \right) = \Delta \left( \frac{|Rc|^2}{R^2} \right) - \frac{2}{R^4} |R\nabla Rc - \nabla RRc|^2 + \frac{2}{R} \nabla \left( \frac{|Rc|^2}{R^2} \right) \cdot \nabla R - \frac{4P}{R^3},
$$

of which (2.7) is simply a special case when $g_{ij}(t)$ is given by the self-similar solution (1.3) corresponding to the shrinking soliton metric $g_{ij}$.

Now, by multiplying $|Rc|^2e^{-f}$ to (2.7) and integration by parts, Ni-Wallach [33] deduced that

$$
0 = \int_M \left( |\nabla \left( \frac{|Rc|^2}{R^2} \right)|^2 R^2 + \frac{2|\nabla|^2}{R^4} |R\nabla Rc - \nabla RRc|^2 + \frac{4P}{R^3} |Rc|^2 \right) e^{-f}.
$$

Thus: (i) $\frac{|Rc|^2}{R^2} = \text{constant}$; (ii) $R\nabla Rc - \nabla RRc = 0$; and (iii) $P = 0$, provided the integration by parts is legitimate. Moreover, it is clear from (2.8) that $Rc > 0$ and $P = 0$ imply $\lambda = \mu = \nu$. Thus $R_{ij} = \frac{R}{2} g_{ij}$, which in turn implies that $R$ is a (positive) constant and $(M^3, g_{ij})$ is a space form.

Finally, based on the quadratic growth of $f$ in (1.13), Ni-Wallach argued that the integration by parts can be justified when the curvature bound $|Rm|(x) \leq Ce^{ar(x)}$ is satisfied.

Subsequently, B.-L. Chen, X.-P. Zhu and the author [6] observed that one can actually remove all the curvature bound assumptions in Theorem 2.2.

Theorem 2.3. (Cao-Chen-Zhu [6]) Let $(M^3, g_{ij}, f)$ be a 3-dimensional complete non-flat gradient shrinking soliton. Then $(M^3, g_{ij})$ is a quotient of the round sphere $S^3$ or round cylinder $S^2 \times \mathbb{R}$.

Sketch of the Proof. It was shown by B.-L. Chen [12] that every complete 3-dimensional ancient solution to the Ricci flow, without assuming bounded curvature, has nonnegative sectional curvature

$$
Rm \geq 0, \quad (2.9)
$$

In particular, this means every complete noncompact 3-dimensional gradient shrinking soliton has $Rm \geq 0$. Since we now have $R \geq 0$, it follows from (1.6) that

$$
0 \leq |\nabla f|^2 \leq f
$$
Hence,\[ |\nabla \sqrt{f}| \leq \frac{1}{2} \] whenever \( f > 0 \). Thus \( \sqrt{f} \) is a Lipschitz function, and it follows that\[ \sqrt{f(x)} \leq \frac{1}{2} r(x) + \sqrt{f(x_0)}, \] where \( x_0 \in M^n \) is a fixed base point and \( r(x) = d(x, x_0) \). Thus,\[ f(x) \leq \frac{1}{4} (r(x) + 2 \sqrt{f(x_0)})^2, \] (2.11)\[ |\nabla f|(x) \leq \frac{1}{2} r(x) + \sqrt{f(x_0)}, \] (2.12)\[ R(x) \leq \frac{1}{4} (r(x) + 2 \sqrt{f(x_0)})^2. \] (2.13) Consequently,\[ |Rm|(x) \leq C(r^2(x) + 1). \] (2.14) Therefore, Theorem 2.3 follows from (2.9), (2.14) and Theorem 2.2.

Remark 2.1. For \( n = 4 \), Ni and Wallach \[34\] showed that any 4-dimensional complete gradient shrinking soliton with nonnegative curvature operator and positive isotropic curvature, satisfying certain additional assumptions, is a quotient of either \( S^4 \) or \( S^3 \times \mathbb{R} \). Based in part on this result of Ni-Wallach, Naber \[31\] proved

**Theorem 2.4. (Naber \[31\])** A 4-dimensional non-flat complete noncompact shrinking Ricci soliton with bounded and nonnegative curvature operator is isometric to a finite quotient of \( S^3 \times \mathbb{R} \) or \( S^2 \times \mathbb{R}^2 \).

For \( n \geq 4 \), various results on classification of gradient shrinking solitons with vanishing Weyl curvature tensor have been obtained. In the compact case, see Eminenti-La Nave-Mantegazza \[17\], while in the noncompact case, see Ni-Wallach \[33\], Petersen-Wylie \[37\], X. Cao-Wang \[10\], and Z.-H. Zhang \[44\]. In particular, we have

**Proposition 2.2. (Ni-Wallach \[33\])** Let \( (M^n, g) \) be a complete, locally conformally flat gradient shrinking soliton with nonnegative Ricci curvature. Assume that\[ |Rm|(x) \leq e^{\alpha(r(x) + 1)} \] for some constant \( \alpha > 0 \), where \( r(x) \) is the distance function to some origin. Then its universal cover is \( \mathbb{R}^n \), \( S^n \) or \( S^{n-1} \times \mathbb{R} \).

**Proposition 2.3. (Petersen-Wylie \[37\])** Let \( (M^n, g) \) be a complete gradient shrinking Ricci soliton with potential function \( f \). Assume the Weyl tensor \( W = 0 \) and\[ \int_M |Rc|^2 e^{-f} < \infty, \] then \( (M^n, g) \) is a finite quotient of \( \mathbb{R}^n \), \( S^n \) or \( S^{n-1} \times \mathbb{R} \). In \[44\], Z.-H. Zhang showed that an \( n \)-dimensional \((n \geq 4) \) complete gradient shrinking soliton with vanishing Weyl tensor necessarily has nonnegative curvature operator. Thus, combining with Proposition 2.2, we have\[ \text{The estimate here is more precise than that given in \[6\], see also Lemma 2.3 in \[7\].} \]
Proposition 2.4. (Z.-H. Zhang [44]) Any n-dimensional \((n \geq 4)\) complete non-flat gradient shrinking soliton with vanishing Weyl tensor must be a finite quotient of \(S^n\) or \(S^{n-1} \times \mathbb{R}\).

Remark 2.2. Most recently, we have learned that Munteanu-Sesum [29] have shown that \(\int_M |\nabla| e^{-f} < \infty\) for any complete gradient shrinking soliton. Thus, in view of Proposition 2.3, this gives an alternative proof of Proposition 2.4. We refer the readers to [29] for more information.

3. Geometry of complete gradient solitons

In the previous section, we described the classification results on complete gradient shrinking solitons in dimensions \(n = 3\) and \(n = 4\), as well as in the locally conformal flat case when \(n \geq 4\). In all three cases, the proofs depend on the crucial fact that the gradient shrinking soliton under the consideration has non-negative curvature operator. We also saw the condition of nonnegative curvature operator is automatically satisfied when \(n = 3\) or when the shrinker is locally conformally flat. However, when \(n \geq 4\) one cannot expect that in general a complete gradient shrinking soliton has nonnegative sectional curvature, or even nonnegative Ricci curvature. For example, the noncompact gradient \(\mathbb{K}\)-shrinkers of Feldman-Ilmanen-Knopf [19] does not have nonnegative Ricci curvature. While it is certainly of considerable interest to classify complete gradient shrinking solitons with nonnegative curvature operator or nonnegative Ricci curvature in dimension \(n \geq 5\), it is also important to understand as much as possible the geometry of a general complete noncompact shrinker \((M^n, g_{ij}, f)\) for \(n \geq 4\). In this section, we report some recent progress in this direction. The results we are going to describe are mainly concerned with the asymptotic behavior of potential functions, and volume growth rates of geodesic balls.

First of all, we shall need the logarithmic Sobolev inequality proved recently by Carrillo-Ni [11] for complete noncompact gradient shrinking solitons. To state their result, instead of normalizing the potential function \(f\) by (1.6), we will use a different normalization by requiring

\[
\int_M e^{-f} = (4\pi)^{\frac{n}{2}}. \tag{3.1}
\]

Remark 3.1. According to [30, 40] or [7], \(\int_M e^{-f}\) is finite on a complete gradient shrinking soliton \((M^n, g_{ij}, f)\). Hence normalization (3.1) is always possible.

Now under the normalization (3.1) on \(f\), we have

\[
R + |\nabla f|^2 - f = \mu_0 \tag{3.2}
\]

for some (not necessarily positive) constant \(\mu_0\).

Proposition 3.1. (Carrillo-Ni [11]) Let \((M^n, g_{ij}, f)\) be a complete noncompact gradient shrinking soliton satisfying (1.1). Assume that the scalar curvature is bounded below by a negative constant. Then, for any function \(u\in C_0^\infty(M)\) with \(\int_M u^2 = (4\pi)^{n/2}\), one has

\[
(4\pi)^{-\frac{n}{2}} \int_M (Ru^2 + 4|\nabla u|^2 - u^2 \log u^2 - nu^2) \geq \mu_0, \tag{3.3}
\]

where \(\mu_0\) is the same constant as given in (3.2).
We shall also need the following very useful fact due to B.-L. Chen \[12\].

**Proposition 3.2.** (B.-L. Chen \[12\]) Let \((M^n, g_{ij}, f)\) be a complete shrinking Ricci soliton. Then the scalar curvature \(R\) of \(g_{ij}\) is nonnegative:
\[ R \geq 0. \]

As a consequence of Propositions 3.1 and 3.2, one can deduce the following Perelman-type non-collapsing result, which is a slight improvement of Corollary 4.1 in [11], for gradient shrinking solitons by using a similar argument as in the proof of Theorem 3.3.3 in [8].

**Corollary 3.1.** Let \((M^n, g_{ij}, f)\) be a complete noncompact gradient shrinking soliton satisfying (1.1). Then there exists a positive constant \(\kappa = \kappa(\mu_0) > 0\) such that whenever \(r \leq 1\) and \(R \leq \frac{C}{r^2}\) on a geodesic ball \(B_r \subset M\), one has \(\text{Vol}(B_r) \geq \kappa r^n\).

Now we are ready to prove the following result obtained by X.-P. Zhu and the author [9].

**Theorem 3.1.** (Cao-Zhu \[9\]) Let \((M^n, g_{ij}, f)\) be a complete noncompact gradient shrinking Ricci soliton. Then \((M^n, g_{ij})\) has infinite volume:
\[ \text{Vol}(M^n, g_{ij}) = \infty. \]

More precisely, there exists some positive constant \(C_3 > 0\) such that
\[ \text{Vol}(B_{x_0}(r)) \geq C_3 \ln \ln r \]
for \(r > 0\) sufficiently large.

**Proof.** We are going to show that if \(\text{Vol}(M^n, g_{ij}) < \infty\), then we shall get a contradiction to the logarithmic Sobolev inequality (3.3). The argument is similar in spirit to that of the proof of Theorem 3.3.3 in [8], and is an adoption of the Perelman’s proof on the uniform diameter estimate for the normalized Kähler Ricci flow on Fano manifolds.

Let \((M^n, g_{ij}, f)\) be any complete noncompact gradient shrinking soliton satisfying equation (1.1) and the normalization condition (3.1). Then, by Proposition 3.2, we have \(R \geq 0\). Thus the log Sobolev inequality (3.3) of Carrillo-Ni is valid for \((M^n, g_{ij}, f)\).

Pick a base point \(x_0 \in M\) and denote by \(r(x) = d(x, x_0)\) the distance from \(x\) to \(x_0\). Also denote by \(A(k_1, k_2)\) the annulus region defined by
\[ A(k_1, k_2) = \{ x \in M : 2^{k_1} \leq d(x, x_0) \leq 2^{k_2} \}, \]
and
\[ V(k_1, k_2) = \text{Vol}(A(k_1, k_2)), \]
the volume of \(A(k_1, k_2)\).

Note that for each annulus \(A(k, k + 1)\), the scalar curvature is bounded above by \(R \leq C2^{2k}\) for some uniform constant \(C > 0\). Since \(A(k, k + 1)\) contains at least \(2^{2k-1}\) balls \(B_r\) of radius \(r = 2^{-k}\) and each of these \(B_r\) has \(\text{Vol}(B_r) \geq \kappa(2^{-k})^n\) by Corollary 3.1,
\[ V(k, k + 1) \geq \kappa 2^{2k-1} 2^{-kn}. \]

Now, suppose \(\text{Vol}(M^n, g_{ij}) < \infty\).

---

\[ ^7\text{This is a special case of Proposition 5.5 stated in [4] which was essentially proved in [12].} \]
Then, for each $\epsilon > 0$, there exists a large constant $k_0 > 0$ such that if $k_2 > k_1 > k_0$, then
\[ V(k_1, k_2) \leq \epsilon. \]  
(3.5)

We claim that we can choose $k_1$ and $k_2$ in such a way that $V(k_1, k_2)$ also satisfies the following volume doubling property:
\[ V(k_1, k_2) \leq 2^{4n}V(k_1 + 2, k_2 - 2). \]  
(3.6)

Indeed, we can choose a very large number $K > 0$ and pick $k_1 \approx K/2$ and $k_2 \approx 3K/2$.

Then we consider whether or not
\[ V(k_1 + 2, k_2 - 2) \leq 2^{4n}V(k_1 + 4, k_2 - 4). \]

If yes, then we are done. Otherwise we repeat the process. After $j$ steps, we have
\[ V(k_1, k_2) > 2^{4nj}V(k_1 + 2j, k_2 - 2j). \]

However, when $j \approx K/4$ and using (3.4), this implies that
\[ \text{Vol}(M^n) > V(k_1, k_2) \geq 2^{nK}V(K, K + 1) \geq \kappa 2^{2K-1}. \]

But $\text{Vol}(M^n)$ is supposed to be finite, so after finitely many steps (3.6) must hold for a pair of large numbers $k_2 > k_1$. Thus we can choose $k_1 = k_1(\epsilon)$ and $k_2 \approx 3k_1$ so that both (3.5) and (3.6) are valid.

Next by using the Co-Area formula and the fundamental theorem of calculus, we can choose $r_1 \in [2^{k_1}, 2^{k_1+1}]$ and $r_2 \in [2^{k_2}, 2^{k_2+1}]$ such that
\[ \text{Vol}(S_{r_1}) \leq \frac{V(k_1, k_2)}{2^{k_1}} \quad \text{and} \quad \text{Vol}(S_{r_2}) \leq \frac{V(k_1, k_2)}{2^{k_2}}, \]
where $S_r$ denotes the geodesic sphere of radius $r$ centered at $x_0$.

Then, by integration by parts and using (2.12),
\[
| \int_{A(r_1,r_2)} \Delta f | \leq \int_{S_{r_1}} |\nabla f| + \int_{S_{r_2}} |\nabla f| \leq \frac{V(k_1, k_2)}{2^{k_1}} C 2^{k_1+1} + \frac{V(k_1, k_2)}{2^{k_2}} C 2^{k_2+1} \leq CV(k_1, k_2).
\]

Therefore, since $R + \Delta f = n/2$, it follows that
\[ \int_{A(r_1,r_2)} R \leq CV(k_1, k_2), \]  
(3.7)
for some universal positive constant $C > 0$.

Now we can derive a contradiction to the log Sobolev inequality (3.3). Pick a smooth cut-off function $0 < \zeta(t) \leq 1$ defined on the real line such that
\[
\zeta(t) = \begin{cases} 
1, & 2^{k_1+2} \leq t \leq 2^{k_2-2}, \\
0, & \text{outside } [r_1, r_2], 
\end{cases}
\]
and $|\zeta'| \leq 1$ everywhere. Define
\[ u = e^L \zeta(d(x, x_0)), \]
where the constant $L$ is chosen so that

$$(4\pi)^{n/2} = \int_M u^2 = e^{2L} \int_{A(r_1,r_2)} \zeta^2. \tag{3.8}$$

Then, by the log Sobolev inequality (3.3), we have

$$\mu_0 \leq 2\log L - 2n - C,$$

for some universal positive constant $C > 0$. But, this is a contradiction, since by (3.5) and (3.8) we can make $L$ arbitrary large by choosing $\epsilon > 0$ arbitrary small.

Moreover, if one examines the above proof carefully, one can see that in fact the geodesic balls $B_r(x_0)$ necessarily have at least $\ln \ln r$ growth. This completes the proof of Theorem 3.1. \hfill $\square$

More recently, D. Zhou and the author [7] obtained a rather precise estimate on asymptotic behavior of the potential function of a general complete gradient shrinking soliton.

**Theorem 3.2. (Cao-Zhou [7])** Let $(M^n, g_{ij}, f)$ be a complete noncompact gradient shrinking Ricci soliton satisfying (1.1) and the normalization (1.6). Then, the potential function $f$ satisfies the estimates

$$\frac{1}{4}(r(x) - c_3)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_4)^2. \tag{3.9}$$

Here $r(x) = d(x_0, x)$ is the distance function from some fixed point $x_0 \in M$, $c_3$ and $c_4$ are positive constants depending only on $n$ and the geometry of $g_{ij}$ on the unit ball $B_{x_0}(1)$. 

Remark 3.2. In view of the Gaussian shrinker in Example 1.2, whose potential function is $|x|^2/4$, the leading term $1/4 r^2(x)$ in (3.9) is optimal.

Remark 3.3. By Theorem 3.2, we can use $f$ to define a distance-like function on $M^n$ by

$$\rho(x) = 2\sqrt{f(x)}$$

so that

$$r(x) - c \leq \rho(x) \leq r(x) + c$$

with $c = \max\{c_3, c_4\} > 0$. Moreover, by (1.6) and Proposition 3.2, we have $|\nabla f|^2 \leq f$. Hence, whenever $f > 0$,

$$|\nabla \rho| = \frac{|\nabla f|}{\sqrt{f}} \leq 1.$$

(3.12)

Note also that (1.6) and the upper bound on $f$ in (3.9) imply that

$$R(x) \leq \frac{1}{4}(r(x) + 2\sqrt{f(x_0)})^2.$$

(3.13)

Using Theorem 3.2 and the soliton equation (1.1), and working with $\rho(x)$ as a distance function, an upper estimate on the volume growth was derived in [7].

Theorem 3.3. (Cao-Zhou [7]) Let $(M^n, g_{ij}, f)$ be a complete noncompact gradient shrinking Ricci soliton. Then, there exists some positive constant $C_4 > 0$ such that

$$\text{Vol}(B_{x_0}(r)) \leq C_4 r^n$$

for $r > 0$ sufficiently large.

Remark 3.4. The noncompact Kähler shrinker of Feldman-Ilmanen-Knopf [19] in Example 1.6 has Euclidean volume growth, with $Rc$ changing signs and $R$ decaying to zero. This shows that the volume growth rate in Theorem 3.3 is optimal.

Remark 3.5. Carrillo-Ni [11] proved that any non-flat gradient shrinking soliton with nonnegative Ricci curvature $Rc \geq 0$ must have zero asymptotic volume ratio, i.e., $\lim_{r \to \infty} \text{Vol}(B_{x_0}(r))/r^n = 0$.

In addition, the following result was also shown by Cao-Zhou [7].

Proposition 3.3. (Cao-Zhou [7]) Let $(M^n, g_{ij}, f)$ be a complete noncompact gradient shrinking Ricci soliton. Suppose the average scalar curvature satisfies the upper bound

$$\frac{1}{V(r)} \int_{D(r)} R \leq \delta$$

for some constant $0 < \delta < n/2$, and all sufficiently large $r$. Then, there exists some positive constant $C_5 > 0$ such that

$$\text{Vol}(B_{x_0}(r)) \geq C_5 r^{n-2\delta}$$

for $r$ sufficiently large. Here $D(r) = \{x \in M^n : \rho(x) < r\}$, with $\rho(x)$ given by (3.10), and $V(r) = \text{Vol}(D(r))$.

Remark 3.6. On a complete noncompact Riemannian manifold $X^n$ with nonnegative Ricci curvature, a theorem of Yau and Calabi (see [12]) asserts that the geodesic balls of $X$ have at least linear growth, while the classical Bishop volume comparison theorem says (cf. [38]) the geodesic balls of $X$ have at most Euclidean growth. Theorem 3.1 and Theorem 3.3 were motivated by these two well-known theorems.
respectively. It remains an interesting problem to see if every complete noncompact gradient shrinking soliton has at least linear volume growth.

**Remark 3.7.** On the topological side, Wylie [41] showed that a complete shrinking Ricci soliton has finite fundamental group. In the compact case, this result was proved by Derdzinski [16], and Fernández-López and García-Río [20] (see also a different proof by Eminenti-La Nave-Mantegazza [17]). Moreover, Fang-Man-Zhang [18] proved that a complete gradient shrinking Ricci soliton with bounded scalar curvature has finite topological type.

**Acknowledgments.** I would like to dedicate this paper to my teacher Professor S.-T. Yau, who taught me so much and guided me for over a quarter of century. I would also like to thank X.-P. Zhu for very helpful discussions.

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