Completeness of shifted dilates in invariant Banach spaces of tempered distributions

Hans G. Feichtinger∗ Anupam Gumber †
August 17, 2020

Abstract

We show that well-established methods from the theory of Banach modules and time-frequency analysis allow to derive completeness results for the collection of shifted and dilated version of a given (test) function in a quite general setting. While the basic ideas show strong similarity to the arguments used in a recent paper by V. Katsnelson we extend his results in several directions, both relaxing the assumptions and widening the range of applications. There is no need for the Banach spaces considered to be embedded into \((L^2(\mathbb{R})), \|\cdot\|_2)\), nor is the Hilbert space structure relevant. We choose to present the results in the setting of the Euclidean spaces, because then the Schwartz space \(S'(\mathbb{R}^d)(d \geq 1)\) of tempered distributions provides a well-established environment for mathematical analysis. We also establish connections to modulation spaces and Shubin classes \((Q_s(\mathbb{R}^d), \|\cdot\|_{Q_s})\), showing that they are special cases of Katsnelson’s setting (only) for \(s \geq 0\).

Keywords: Beurling algebra, Shubin spaces, modulation spaces, approximation by translations, Banach spaces of tempered distributions, Banach modules, compactness

2010 Mathematics Subject Classification. Primary 43A15, 41A30, 43A10, 41A65, 46F05, 46B50; Secondary 43A25, 46H25, 46A40

1 Introduction

The motivation for the present paper lies in the study of [24], which shows that the set of all shifted, dilated Gaussians is total in certain translation and modulation invariant Hilbert spaces of functions which are continuously embedded into \((L^2(\mathbb{R})), \|\cdot\|_2)\).

By working in a more general setting we show that the results presented in that paper can be extended in various directions. During the studies it turned out that the setting used in [4] appears to be most appropriate, although it is mainly the existence of a double module structure (namely with respect to convolution and pointwise multiplication) which makes the key arguments work. Such Banach spaces have been discussed already long ago by the first author, under the name of standard spaces in order to study compactness in function spaces of distributions, or in order to derive a number of module theoretical properties of such spaces, as given in [3]. The setting which we choose is also closely related to Triebel’s systematic work concerning the theory of function spaces.

The paper is organized as follows. After providing basic notations we will describe the setting of invariant function spaces of tempered distributions. Since we are addressing the question of totality of a set of shifted and dilated version of a given test function we have to restrict our attention to Banach spaces which contain \(S(\mathbb{R}^d)\) as a dense subspace. On the other hand it is convenient and still very general to work within the realm of tempered distributions. While the methods employed are valid in a more general context this setting should make the reading of the paper easier for the majority of readers.

In preparation of the main result we then go on to prove some technical results concerning the approximation of convolution products in Beurling algebras. Subsequently we will derive our main result. The remaining sections will be devoted to an exploration of the wide range of applicability of the result presented. In the final section we will explain, why our results contain the key results of Katsnelson’s paper and in which sense we are going (far) beyond the setting described in his paper.

∗NuHAG, Faculty of Mathematics, University of Vienna, Oskar-Morgenstern-Platz 1,1090 Vienna, AUSTRIA, email: hans.feichtinger@univie.ac.at
†Department of Mathematics, Indian Institute of Science,560022 Bangalore, INDIA, email: anupamgumber@iisc.ac.in
2 Notations and Conventions

For a bounded linear operator $T$ on a Banach space $(B, \| \cdot \|_B)$ we write $T \in \mathcal{L}(B)$, and use the symbol $\|T\|_B$ in order to describe its operator norm

$$\|T\|_B = \sup_{\|f\|_B \leq 1} \|T(f)\|_B.$$

We will make use of standard facts concerning tempered distributions. Recall that $\mathcal{D}(\mathbb{R}^d)$, the subspace of smooth function with compact support is a dense subspace of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ with the standard topology.

The Fourier invariance of $\mathcal{S}(\mathbb{R}^d)$ allows to extend the classical Fourier transform in a unique (weak-$\ast$-continuous) form by the rule $\hat{\sigma}(f) = \sigma(\hat{f})$, $f \in \mathcal{S}(\mathbb{R}^d)$, for any $\sigma \in \mathcal{S}(\mathbb{R}^d)$. This extended Fourier transform provides an automorphism of $\mathcal{S}^\prime(\mathbb{R}^d)$, and hence for any Banach space $(B, \| \cdot \|_B)$ continuously embedded into $\mathcal{S}^\prime(\mathbb{R}^d)$, we use the symbol $(B, \| \cdot \|_B) \hookrightarrow \mathcal{S}^\prime(\mathbb{R}^d)$, i.e. satisfying

$$\|f_n - f\|_B \to 0 \text{ in } (B, \| \cdot \|_B) \text{ for } n \to \infty \Rightarrow f_n(g) \to f(g), \quad \forall g \in \mathcal{S}(\mathbb{R}^d)$$

also $\mathcal{F}B$ is a well defined Banach space of distributions with the natural norm $\|\hat{f}\|_{\mathcal{F}B} = \|f\|_B, \ f \in B$.

In addition to the usual function spaces such as $(L^p(\mathbb{R}^d), \| \cdot \|_p)$, with $1 \leq p \leq \infty$ we also need their weighted versions. Given as strictly positive weight $w(x) > 0$ we obtain Banach spaces

$$L^p_w(\mathbb{R}^d) := \{f \mid f \in L^p(\mathbb{R}^d)\}, \text{ resp. } B_w = \{f \mid f w \in B\}$$

with the norm $\|f\|_{L^p_w} = \|f\|_{L^p(\mathbb{R}^d)} = \|f w\|_{L^p(\mathbb{R}^d)}$.

We will be only interested in translation invariant function spaces of this form, hence we restrict our attention to moderate weight functions, which (without loss of generality) can be assumed to be strictly positive and continuous.

On such spaces every translation operator, defined via $T_s f(x) = f(x-s)$ is bounded on $(L^p_w(\mathbb{R}^d), \| \cdot \|_{p, w})$ and the operator norm $w(s) := \|T_s\|_B$ is a so-called Beurling weight function, i.e. a strictly positive submultiplicative function satisfying

$$w(x+y) \leq w(x)w(y), \quad x, y \in \mathbb{R}^d. \quad (2)$$

Such weight functions generate weighted $L^1$-spaces which are Banach algebras with respect to convolution (so-called Beurling algebras), due to the pointwise (a.e.) estimate

$$|f \ast g| w \leq |f| w * |g| w, \quad f, g \in L^1_w(\mathbb{R}^d), \quad (3)$$

which imply the norm estimate

$$\|f \ast g\|_{1, w} \leq \|f\|_{1, w} \|g\|_{1, w}, \quad f, g \in L^1_w(\mathbb{R}^d). \quad (4)$$

Details concerning these so-called Beurling algebras are found in Reiter’s book [27] (or [28], Ch.2.6.3. Among others they are translation invariant, with the property

$$\|T_s f\|_{1, w} \leq w(x) \|f\|_{1, w}, \quad \forall x \in \mathbb{R}^d, f \in L^1_w(\mathbb{R}^d),$$

and that shifts depend continuously on $x$, meaning that $x \to T_x f$ is a continuous mapping from $\mathbb{R}^d$ to $(L^1_w(\mathbb{R}^d), \| \cdot \|_{1, w})$, or equivalently

$$\lim_{x \to 0} \|T_x f - f\|_{1, w} = 0, \quad \forall f \in L^1_w(\mathbb{R}^d). \quad (6)$$

Occasionally we will work with the Banach convolution algebra $(M^1_w(\mathbb{R}^d), \| \cdot \|_{M^1_w(\mathbb{R}^d)})$, the space of Radon measures $\mu$ such that $\mu w$ is a bounded measure, or equivalently, the dual space of $(C^{0,0}_1(\mathbb{R}^d), \| \cdot \|_{C^{0,0}_1(\mathbb{R}^d)})$. It contains $(L^1_w(\mathbb{R}^d), \| \cdot \|_{1, w})$ as a closed ideal.

It is a general fact that a positive, continuous weight function $m$ is moderate (with respect to $w$) if and only if it satisfies

$$m(x+y) \leq m(x)w(y), \quad x, y \in \mathbb{R}^d \quad (7)$$

f for some submultiplicative weight function $w$ ( [10][22]). In this note we will concentrate on polynomially moderated weights, i.e. weights for which one can use

$$w(x) = w_s(x) := \langle x \rangle^s = (1 + |x|^2)^{1/2} \approx (1 + |x|)^s, \quad s \geq 0. \quad (8)$$

Any function $\langle x \rangle^{s}, s \in \mathbb{R}$ is a moderate weight, with respect to $\langle x \rangle^{|s|}$. Consequently the weighted, translation invariant spaces $L^p_w(\mathbb{R}^d)$ will be continuously embedded into $\mathcal{S}^\prime(\mathbb{R}^d)$. 

Beurling algebras share several important properties with \((L^1(\mathbb{R}^d), \| \cdot \|_1)\): Compactly supported functions are dense, and there are bounded approximate identities in \((L^1_{0w}(\mathbb{R}^d), \| \cdot \|_{1,w})\), i.e. bounded nets \((e_\alpha)_{\alpha \in I}\) such that for each \(h \in L^1_{0w}(\mathbb{R}^d)\):

\[
|e_\alpha * h - h|_{1,w} \to 0 \quad \text{for} \quad \alpha \to \infty.
\]

The boundedness of such a family also allows to extend this property to relatively compact sets, hence one has: For every relatively compact set \(M \subset (B, \| \cdot \|_B)\) and \(\varepsilon > 0\) one can find some index \(\alpha_0\) such that \(\alpha \geq \alpha_0\) implies:

\[
|e_\alpha * h - h|_{1,w} \leq \varepsilon, \quad \forall h \in M.
\]

For us, approximate identities obtained by compression of a given function \(\hat{g}(0) \neq 0\) will be important. Without loss of generality let us assume that \(\hat{g}(0) = \int_{\mathbb{R}^d} g(x)dx = 1\). We use the \(L^1\)-isometric compression

\[
g_\rho(x) = St_\rho g(x) = \rho^{-d} g(x/\rho), \quad \rho > 0,
\]

with \(\text{supp}(St_\rho g) = \rho \text{supp}(g)\) and \(\|g_\rho\|_{L^1} = \|St_\rho g\|_{L^1} = \|g\|_{L^1}\) for \(g \in (L^1(\mathbb{R}^d), \| \cdot \|_1)\).

For any radial symmetric, increasing weight \(w\) satisfying \(w(y) \leq w(x)\) if \(|y| \leq |x|\), one has

\[
\|g_\rho\|_{1,w} = \|St_\rho g\|_{1,w} \leq \|g\|_{1,w} \quad \text{for all} \quad \rho \in (0, 1).
\]

Without loss of generality we will make this assumption concerning \(w\), satisfied by all the usual examples. In the general case it is possible to replace a given weight function by another one satisfying this condition. With this extra condition we obtain bounded approximate identities by compression:

**Lemma 1.** For any \(f \in L^1_{0w}(\mathbb{R}^d)\) and \(g \in L^1_{0w}(\mathbb{R}^d)\) with \(\hat{g}(0) = 1\) one has

\[
\lim_{\rho \to 0} \|g_\rho * f - f\|_{1,w} = \lim_{\rho \to 0} \|St_\rho g * f - f\|_{1,w} = 0.
\]

Polynomial weights \(w_s\) satisfy the so-called *Beurling-Domar condition* (see [28]) and hence

\[
\{f \mid f \in L^1_{0w}(\mathbb{R}^d), \text{spec}(f) = \text{supp}(\hat{f}) \text{ is compact}\}
\]

the subset of all *band-limited* elements is a dense subspace of \((L^1_{0w}(\mathbb{R}^d), \| \cdot \|_{1,w})\).

For two topological vector spaces \(B^1\) and \(B^2\) we will write \(B^1 \hookrightarrow B^2\) if the embedding is continuous. If both of them are normed spaces this means that there exists some constant \(C > 0\) such that \(\|f\|_{B^2} \leq C\|f\|_{B^1}\), for all \(f \in B^1 \subset B^2\). For Banach spaces continuously embedded into \(\mathcal{S}'(\mathbb{R}^d)\) the boundedness of an inclusion mapping follows from the simple inclusion \(B^1 \subseteq B^2\), via the Closed Graph Theorem.

For the rest of this paper we will work with the following *standard assumptions*, similar to the setting chosen in [3]:

**Definition 1.** A Banach space \((B, \| \cdot \|_B)\) is called a *minimal tempered standard space* (abbreviated as MINTSTA) if the following conditions are valid:

1. One has the following sandwiching property:

\[
\mathcal{S}(\mathbb{R}^d) \hookrightarrow (B, \| \cdot \|_B) \hookrightarrow \mathcal{S}'(\mathbb{R}^d);
\]

2. \(\mathcal{S}(\mathbb{R}^d)\) is dense in \((B, \| \cdot \|_B)\) (minimality);

3. \((B, \| \cdot \|_B)\) is translation invariant, and for some \(n_1 \in \mathbb{N}\) and \(C_1 > 0\) one has

\[
\|T_x f\|_B \leq C_1 \langle x \rangle^{n_1} \|f\|_B \quad \forall x \in \mathbb{R}^d;
\]

4. \((B, \| \cdot \|_B)\) is modulation invariant, and for some \(n_2 \in \mathbb{N}\) and \(C_2 > 0\) one has

\[
\|M_y f\|_B \leq C_2 \langle y \rangle^{n_2} \|f\|_B \quad \forall y \in \mathbb{R}^d.
\]

**Remark 1.** The notion of MINTSTAs relates the approach to the use of “standard spaces” in the work of the first author, starting in the 70th, specifically the use of double module properties in [3].

The formal definition provided here is inspired by the work [4], where such spaces are called TMIBs. The density of \(\mathcal{S}(\mathbb{R}^d)\) is a part of their definition. The few interesting spaces which do not satisfy this extra condition are typically DTMBs in their terminology (dual translation, modulation invariant Banach spaces). See also [5] for a more general setting.

---

1 In contrast to [2] we do not assume density of the embedding whenever we use this symbol. We rather prefer to put this as an explicit extra assumption.
Remark 2. The term Banach spaces in standard situation has been used in a number of papers of the first author, e.g. in order to prove results about compactness in such spaces (13), in order to introduce Wiener amalgam spaces (12), or in order to study spaces with a double module structure (3). In each of these cases it is important that it is meaningful for the objects under consideration (functions, measures or distributions) to allow pointwise products with suitable test functions (leading to a localization), and make use of this fact that this is definitely possible for elements in the dual of a space $A_c = C_c(\mathbb{R}^d) \cap A$, of compactly supported test functions, where $A$ is a suitable pointwise Banach algebra of test functions which is also translation invariant.

Remark 3. The situation described in Definition 1 are special cases of this more general notion of standard spaces, with the main restriction (more or less made for the convenience of the reader, and in accordance with [4]) that we assume $(B, \| \cdot \|_B) \hookrightarrow S'(\mathbb{R}^d)$.

One should also observe that for a non-trivial Banach space of tempered distributions satisfying the invariance properties (under translation and modulation) one always has the continuous embedding $S(\mathbb{R}^d) \hookrightarrow (B, \| \cdot \|_B)$. In fact, given properties 3. and 4. in Definition 1 one can show that there is a minimal space in the corresponding family of spaces, namely $W(F\mathcal{E}_{v_2}^1, \ell^1_{v_1})$ according to [14], and hence the following chain of inclusions is valid:

$$S(\mathbb{R}^d) \hookrightarrow W(F\mathcal{E}_{v_2}^1, \ell^1_{v_1}) \hookrightarrow (B, \| \cdot \|_B).$$

(17)

The minimality condition ensures that these embeddings are dense embeddings.

The following result is stated for later reference. The proof is left to the reader as an exercise.

Proposition 1. (i) For any MINTSTA $(B, \| \cdot \|_B)$ also its Fourier version $\mathcal{F}B = \{ \hat{f} \mid f \in B \}$ is a MINTSTA with respect to the natural norm

$$\| \hat{f} \|_{\mathcal{F}B} = \| f \|_B, \quad f \in B.$$  

(18)

(ii) Given two MINTSTAs $(B^1, \| \cdot \|^{(1)}_B)$ and $(B^2, \| \cdot \|^{(2)}_B)$, also their intersection (or their sum) is a MINTSTA, with the corresponding natural norms, e.g.

$$\| f \|_{B^1 \cap B^2} := \| f \|_{B^1} + \| f \|_{B^2}, \quad f \in B^1 \cap B^2.$$

2.1 Equivalent Assumptions

Let us first discuss a few alternative assumptions which lead to the same family of spaces.

Lemma 2. i) Assume that $(B, \| \cdot \|_B)$ is a Banach space satisfying conditions 1. and 2. of Definition 1. Then 3. and 4. together are equivalent to the claim that the space $(B, \| \cdot \|_B)$ is invariant under TF-shifts $\pi(z) = M_y T_z$, with $z = (x, y) \in \mathbb{R}^{2d}$, and that for some constant $C_3 > 0$ and $s \geq 0$ one has:

$$\| \pi(z)g \|_B \leq C_3(z)^s, \quad \forall z \in \mathbb{R}^{2d}$$

or equivalently described:

$$\| \pi(z)f \|_B \leq C_3(z)^s \| f \|_B, \quad \forall z \in \mathbb{R}^{2d}, \forall f \in B.$$  

(19)

(20)

ii) For any minimal tempered standard space one has: for any $g \in B$

$$z \mapsto \pi(z)g$$

is continuous from $\mathbb{R}^{2d}$ to $(B, \| \cdot \|_B)$.

(21)

iii) Conversely, assuming that a Banach space $(B, \| \cdot \|_B)$ continuously embedded into $S'(\mathbb{R}^d)$ satisfies (20) and (21). Then (13) and (16) are valid, for $n_1 = s = n_2$. Moreover, $S(\mathbb{R}^d)$ is embedded into $(B, \| \cdot \|_B)$ as a dense subspace.\footnote{This fact justifies the use of the word minimality.}

Proof. i) It is clear that the estimates (15) and (16) are just special cases of (19), e.g. for $n_1 = n = n_2$ being any integer $n$ with $s \leq n$.

Conversely assume that the two estimates (15) and (16) are valid, and that we have to estimate the norm of $\pi(z)g$ in $(B, \| \cdot \|_B)$. Clearly

$$\| \pi(z)g \|_B = \| M_y T_z g \|_B \leq \| M_y \|_B \| T_z \|_B \| g \|_B \leq C_1 C_2(y)^s \langle z \rangle^{n_1} \| g \|_B.$$  

(22)

By choosing $s = n_1 + n_2$ we obtain for $C_3 = C_1 C_2$:

$$\| \pi(z)g \|_B \leq C_1 C_2(z)^s \langle z \rangle^{n_1} \| g \|_B \leq C_3(z)^s.$$  

(23)

ii) The continuous shift property is clear for $g \in S(\mathbb{R}^d)$ in the Schwartz topology. Due to the continuous embedding of $S(\mathbb{R}^d)$ into $(B, \| \cdot \|_B)$ condition (21) is valid for $g \in S(\mathbb{R}^d)$. Using the (uniform) boundedness of TF-shifts with say $|z| \leq 1$ it follows easily that (21) is valid for any $g \in B$ by the usual approximation argument.
iii) The continuity of translation implies that every element \( g \in (B, \| \cdot \|_B) \subset S'(\mathbb{R}^d) \) can be regularized, i.e. it can be approximated by functions in the Schwartz space, because the usual regularization procedures of the form \( \sigma \mapsto S_{\mu, \alpha} \ast (D_{\mu, \alpha} \ast g) \) map \( g \in S(\mathbb{R}^d) \), but also approximate \( g \in (B, \| \cdot \|_B) \). Details are found in [3], where the closure of \( S(\mathbb{R}^d) \) is characterized as \( B_{AC} = B_{CA} \).

When comparing with the setting of [4] we have the following connection:

**Lemma 3.** A Banach space \( (B, \| \cdot \|_B) \) is a minimal tempered Fourier standard space if and only if \( (B, \| \cdot \|_B) \) as well as its Fourier image \( FB \), with the norm \( \| \hat{f} \|_{FB} = \| f \|_B \) are translation invariant Banach spaces of distributions in the sense of [3], containing \( D(\mathbb{R}) \) as a dense subspace.

**Proof.** According to Theorem 1 of [4] a translation invariant Banach space of tempered distributions satisfies conditions 1. to 3. of our definition. Being sandwiched between \( S(\mathbb{R}^d) \) and \( S'(\mathbb{R}^d) \) its Fourier transform \( FB \) is a well-defined Banach space, which is itself again a Banach space in sandwich position.

The fact that translation on the Fourier transform side corresponds to modulation on the time-side (combined with the corresponding) implies immediately that the validity of 4. in Definition [3] is equivalent to a polynomial estimate of the translation operator for \((FB, \| \cdot \|_{FB})\).

**Remark 4.** It is noteworthy to mention that the sandwiching properties 1. and 2. above follow often from 3. and 4., e.g. if \( (B, \| \cdot \|_B) \) is a solid BF-space containing

\[ C_c(\mathbb{R}^d) = \{ k | \text{ k continuous, complex valued on } \mathbb{R}^d, \text{ with supp}(k) \text{ compact} \} \]

as a dense subspace and satisfying 3. (because 4. above is trivial for solid spaces, see [3,10]).

By a slight adaptation of the terminology of Y. Katznelson (see [25]) we call a Banach space \( (B, \| \cdot \|_B) \) in “sandwich position” a **homogeneous Banach space of tempered distributions** if

1. Translations are isometric on \( (B, \| \cdot \|_B) \): \( \| T_x f \|_B = \| f \|_B, \forall f \in B \);
2. translation is continuous, i.e. \( \lim_{x \to 0} \| T_x f - f \|_B = 0, \forall f \in B \).

### 3 Discretization of convolution in Beurling algebras

For the rest we assume that the weight function \( w \) is not only a continuous and submultiplicative function on \( \mathbb{R}^d \), but in addition that it is radial symmetric, with increasing profile, i.e. \( w(y) \leq w(x) \) whenever \( |y| \leq |x|, x, y \in \mathbb{R}^d \).

This is no loss of generality, because any general weight function (of polynomial growth) is dominated by another submultiplicative function with this extra property. The main advantage of this assumption is the fact that it implies that the dilation operator \( g \mapsto S_{\mu, \alpha} g \) is non-expansive on \( (L^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{1, \rho}) \) as well as on \((M^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{M^1_{\rho, \alpha}(\mathbb{R}^d)})\) for \( \rho \in (0, 1) \).

In short, based on a variant of the key result of [18], \( (B, \| \cdot \|_B) \) is a Banach module over \( (L^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{1, \rho}) \), and in fact over \( (M^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{M^1_{\rho, \alpha}(\mathbb{R}^d)}) \). Hence we have

\[ \| \mu * f \|_B \leq \| \mu \|_{M^1_{\rho, \alpha}} \| f \|_B, \forall \mu \in M^1_{\rho, \alpha}(\mathbb{R}^d), f \in B. \] (24)

Let us not forget to mention the validity of the innocent looking associative law:

\[ (\mu_1 * \mu_2) * f = \mu_1 * (\mu_2 * f), \mu_1, \mu_2 \in M^1_{\rho, \alpha}(\mathbb{R}^d), f \in B. \] (25)

This result concerning “integrated group representations” can be considered a folklore result, see for example [2], Chap.8, working for general group representations on a Banach space. Similar results are given in [3] and [4], for example.

A minor modification of the results in [6] gives the following characteristic:

**Lemma 4.** Let \( (B, \| \cdot \|_B) \hookrightarrow S'(\mathbb{R}^d) \) be a Banach convolution module over some weighted measure algebra \( (M^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{M^1_{\rho, \alpha}(\mathbb{R}^d)}) \). Then, viewed as a Banach module over the corresponding Beurling algebra \( (L^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{1, \rho}) \) it is an essential Banach module if and only if translation is continuous in \( (B, \| \cdot \|_B) \), i.e.

\[ \| T_x f - f \|_B = 0 \quad \text{for } x \to 0, \forall f \in B \] (26)

or equivalently

\[ \lim_{x \to 0} \| e_{\alpha} * f - f \|_B = 0 \quad \forall f \in B \] (27)

for any bounded approximate identity \( (e_{\alpha})_{\alpha \in I} \) in \( (L^1_{\rho, \alpha}(\mathbb{R}^d), \| \cdot \|_{1, \rho}) \).
Proof. Since \( \delta_x \in M_1^1(\mathbb{R}^d) \) for any \( x \in \mathbb{R}^d \) it is clear that \( B \) is translation invariant and:
\[
\|T_x f\|_B = \|\delta_x * f\|_B \leq \|\delta_x\|_{M_1^1} \|f\|_B = w(x)\|f\|_B \quad \forall f \in B, x \in \mathbb{R}^d.
\]  
(28)
If translation is continuous in \( (B, \|\cdot\|_B) \) (see \([29]\)) the usual approximate units (convolution with \( L^1 \)-normalized bump functions with small support) act as expected, i.e. for \( \varepsilon > 0 \) there exists some \( h \in L^1_w(\mathbb{R}^d) \) such that
\[
\|h * f - f\|_B < \varepsilon.
\]
Hence obviously \( (B, \|\cdot\|_B) \) is an essential Banach module over \((L^1_w(\mathbb{R}^d), \|\cdot\|_{1,w})\), i.e. (by the definition) that the linear span of \( L^1_w(\mathbb{R}^d) \ast B \) is dense in \((B, \|\cdot\|_B)\).

The equivalence to both stated properties (namely \([20]\) and \([27]\)) follows therefrom. \( \square \)

As a special case which will be used frequently in the sequel we have

**Corollary 1.** \((L^1_w(\mathbb{R}^d), \|\cdot\|_{1,w})\) is a closed ideal in \((M^1_1(\mathbb{R}^d), \|\cdot\|_{M^1_1(\mathbb{R}^d)})\), i.e. one has
\[
\|\mu * f\|_{L^1_w} \leq \|\mu\|_{M_1^1} \|f\|_{L^1_w}, \quad \forall \mu \in M_1^1(\mathbb{R}^d), \forall f \in L^1_w(\mathbb{R}^d).
\]  
(29)
\(L^1_w(\mathbb{R}^d)\) consists exactly of those elements in \( M_1^1(\mathbb{R}^d) \) which have the continuous shift property.

Moreover, any bounded linear operator \( T \) on \( L^1_w(\mathbb{R}^d) \) which commutes with all translations is of the form \( T(f) = \mu * f \), for a uniquely determined \( \mu \in M^1_1(\mathbb{R}^d) \).

**Proof.** Since \( L^1_w(\mathbb{R}^d) \) is an \( M^1_1(\mathbb{R}^d) \)-module it is clear that we have the first two statements. The isometric embedding of \((L^1_w(\mathbb{R}^d), \|\cdot\|_{1,w})\) into \((M^1_1(\mathbb{R}^d), \|\cdot\|_{M^1_1(\mathbb{R}^d)})\) is a routine task.
The subspace of measures in \( M^1_1(\mathbb{R}^d) \) with continuous shift form an essential \( L^1_w(\mathbb{R}^d) \)-module. Hence these elements can be approximated by elements of the form \( c_{\alpha} * \mu \in L^1_w(\mathbb{R}^d) \ast M^1_1(\mathbb{R}^d) \subset L^1_w(\mathbb{R}^d) \) (cf. \([6]\)).
The additional statement about multipliers is just a reformulation of the main result of Gaudry \([20]\), provided a complementary perspective. This correspondence is in fact an isometric one.

For the technical part of our proof we need the following joint estimate on the discretization operators \( D_\psi \), showing their uniform boundedness over the family \((\Psi, \|\Psi\| \leq 1)\).

**Lemma 5.** Given a Bearing weight \( w \) on \( \mathbb{R}^d \), there is a uniformly estimate for the family of discretization operators with respect to BUPUs of size \( |\Psi| \leq 1 \). For some \( C_1 > 0 \) one has:
\[
\sum_{i \in I} |\mu(\psi)| w(x_i) = \|D_\psi \mu\|_{1,w} \leq C_1 \|\mu\|_{M^1_1(\mathbb{R}^d)}, \quad \forall \mu \in M_1^1(\mathbb{R}^d).
\]  
(30)

**Proof.** We will use that \((M^1_1(\mathbb{R}^d), \|\cdot\|_{M^1_1(\mathbb{R}^d)})\) is the dual space of \((C^0_{1,w}, \|\cdot\|_{C^0_{1,w}})\), with the natural norm \( \|f/w\|_\infty \). Using the density of \( C_c(\mathbb{R}^d) \) in \((C^0_{1,w}, \|\cdot\|_{C^0_{1,w}})\) we first verify the adjointness relation
\[
[D_\psi \mu](f) = \mu(Sp_\psi f), \quad \forall f \in C_c(\mathbb{R}^d),
\]  
(31)
justified by
\[
Sp_\psi^* \mu(f) = \mu(Sp_\psi f) = \mu \left( \sum_{i \in I} f(\psi_i) \psi_i \right) = \left( \sum_{i \in I} \mu(\psi_i) \delta_{\psi_i} \right) (f) = D_\psi \mu(f),
\]  
(32)
and look for an estimate of \( f \mapsto Sp_\psi f \) on \((C^0_{1,w}(\mathbb{R}^d), \|\cdot\|_{C^0_{1,w}(\mathbb{R}^d)})\).

Given the (continuous) weight function \( w \) we set \( C_1 = \max_{|x| \leq 1} w(x) \). Then for any BUPU \( \Psi \) with \( |\Psi| \leq 1 \) we have, using \( \text{supp}(\psi_i) \subseteq B_1(\xi_i) \) for each \( i \in I\)
\[
1/w(x) \leq w(\xi - x)/w(\xi) \leq C_1/w(\xi), \quad x \in \text{supp}(\psi_i),
\]  
(33)
and consequently the following pointwise estimate for any \( f \in C^0_{1,w}(\mathbb{R}^d)\):
\[
|Sp_\psi f(x)|/w(x) \leq \sum_{i \in I} |f(x)\psi_i(x)w(x)| \leq C_1 \sum_{i \in I} |f(\xi)|/w(\xi)|\psi_i(x)| \leq C_1 |f/w|_\infty
\]  
(34)
or in terms of the norm on \( C^0_{1,w}(\mathbb{R}^d)\):
\[
\|Sp_\psi f\|_{C^0_{1,w}} \leq C_1 \|f\|_{C^0_{1,w}}, \quad f \in C^0_{1,w}(\mathbb{R}^d),
\]  
(35)
respectively expressed by operator norms:
\[
\|D_\psi\|_{M^1_1(\mathbb{R}^d)} = \|Sp_\psi\|_{C^0_{1,w}(\mathbb{R}^d)} \leq C_1, \quad \forall |\Psi| \leq 1.
\]  
(36)
\( \square \)
Next we show that a convolution product within a Beurling algebra \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \) can be discretized, i.e., a convolution product can be approximated by a finite linear combination of shifted version of either convolution factor. This result is inspired by Chap.1.2 of Reiter’s book [27] and can be viewed as a variant of Theorem 2.2 in [8].

**Theorem 1.** Given two functions \( g, f \) in some Beurling algebra \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \) and \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that one has for any \( \Psi = (\psi_i)_{i \in I} \) with \( |\Psi| \leq \delta \):

\[
\| g * f - g * D \Psi f \|_{L_w^\infty(\mathbb{R}^d)} < \varepsilon. \tag{37}
\]

**Remark 5.** This result is closely related to the compactness criteria for function spaces [11] and [13]. It is clear that by the tightness and boundedness in \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \) - of the family \( D \Psi f, |\Psi| \leq 1 \), also \( g * D \Psi f \) is a bounded and tight family. It is also clear that it is equicontinuous in \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \), since we can control the shift error as follows:

\[
\| g * D \Psi f - T_s(g * D \Psi f)\|_{L_w^1(\mathbb{R}^d)} \leq \| g - T_s g \|_{L_w^1(\mathbb{R}^d)} \cdot \| D \Psi f \|_{M_w^\infty} \leq \| g - T_s g \|_{L_w^1(\mathbb{R}^d)} \cdot C_1 \| f \|_{L_w^1(\mathbb{R}^d)} ,
\]

which tends to zero for \( s \to 0 \), since translation is continuous in any Beurling algebra \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \) (see [27], Chap.1.6.3.).

According to [11] this implies that this set is relatively compact in \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \), and hence there is a subsequence which converges in the norm. However, we want to prove actual convergence of the net, for \( |\Psi| \to 0 \), not only for a subsequence.

**Proof.** We start by fixing \( \varepsilon > 0 \) and assume that \( f, g \in L_w^1(\mathbb{R}^d) \) are given. For simplicity we assume without loss of generality that both \( g, f \) are normalized in \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \), in order to make the presentation more straightforward.

We will prove the estimate by reduction to the dense subspace \( C_c(\mathbb{R}^d) \) of \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \). First, we choose \( k_g, k_f \in C_c(\mathbb{R}^d) \) such that \( \| g - k_g \|_{L_w^1(\mathbb{R}^d)} < \eta \) and \( \| f - k_f \|_{L_w^1(\mathbb{R}^d)} < \eta \) for some \( \eta \in (0, \varepsilon/(12C_1)) \). Since convolution is a continuous, bilinear operation in the Banach convolution algebra \( (L_w^1(\mathbb{R}^d), \| \cdot \|_{1,w}) \) we can choose \( \eta > 0 \) such that in addition

\[
\| g * f - k_g * k_f \|_{L_w^1(\mathbb{R}^d)} < \varepsilon/4. \tag{39}
\]

As now \( k_g, k_f \in C_c(\mathbb{R}^d) \) the convolution product \( k_g * k_f \) also has compact support, but also all the functions \( k_g * D \Psi k_f \) have joint compact support \( Q_2 \), for any \( |\Psi| \leq 1 \).

For the next step we recall that the convolution between a measure \( \mu \) with a test function \( k \) can be determined pointwise by \( \mu * k(x) = \mu(T_x k') \), with \( k' (x) = k(-x) \). Thus

\[
(k_g * k_f - k_g * D \Psi k_f)(x) = (k_f - D \Psi k_f)(T_x k_g')
\]

in the pointwise sense. In fact, it is valid uniformly over compact sets, but outside of \( Q_2 \) all the functions are zero anyway, hence we have uniform convergence and joint compact support. Since the weight \( w \) is bounded over \( Q_2 \) it is then clear that one has

\[
\lim_{|\Psi| \to 0} \| k_g * k_f - k_g * D \Psi k_f \|_{L_w^1(\mathbb{R}^d)} = 0.
\]

In other words, one can find \( \delta > 0 \) such that for \( |\Psi| \leq \delta \) (\( \leq 1 \)) one has

\[
\| k_g * k_f - k_g * D \Psi k_f \|_{L_w^1(\mathbb{R}^d)} < \varepsilon/4. \tag{40}
\]

We also have to control the transition to the discretized form:

\[
\| k_g * D \Psi k_f - g * D \Psi f \|_{L_w^1(\mathbb{R}^d)} \leq \| k_g * D \Psi k_f - k_g * D \Psi f \|_{L_w^1(\mathbb{R}^d)} + \| k_g * D \Psi f - g * D \Psi f \|_{L_w^1(\mathbb{R}^d)},
\]

which thanks to (29) can be continued by the estimate

\[
\leq \| k_g * D \Psi (k_f - f) \|_{L_w^1(\mathbb{R}^d)} + \|(k_g - g) * D \Psi f \|_{L_w^1(\mathbb{R}^d)} \leq \| k_g \|_{L_w^1(\mathbb{R}^d)} \cdot \| D \Psi (k_f - f) \|_{M_w^\infty} + \|(k_g - g) \|_{L_w^1(\mathbb{R}^d)} \cdot \| D \Psi f \|_{M_w^\infty},
\]

and finally, using the normalization assumption \( \| g \|_{L_w^1(\mathbb{R}^d)} = 1 = \| f \|_{L_w^1(\mathbb{R}^d)} \), we have:

\[
\leq 2\| g \|_{L_w^1(\mathbb{R}^d)} \cdot C_1 \| k_f - f \|_{L_w^1(\mathbb{R}^d)} + \| k_g - g \|_{L_w^1(\mathbb{R}^d)} \cdot C_1 \| f \|_{L_w^1(\mathbb{R}^d)} \leq 3C_1 \eta.
\]

By the choice of \( \eta \) we get:

\[
\| k_g * D \Psi k_f - g * D \Psi f \|_{L_w^1(\mathbb{R}^d)} < \varepsilon/4. \tag{41}
\]

Combining the estimates (39), (40) and (41) the claim, i.e. formula (37) in the theorem is verified. \qed
Next we will use the Cohen-Hewitt factorization theorem to show that a similar result is true for the action on a Banach module. Since we are dealing with a commutative situation and functions and measures over \( \mathbb{R}^d \) we keep the order and write convolution from the right.

**Theorem 2.** Any minimal TMIB Banach space of tempered distributions \((B, \| \cdot \|_B)\) is an essential Banach module over some Beurling algebra \((L^1_0(\mathbb{R}^d), \| \cdot \|_{1,0})\). Moreover, one has for any \( g \in B \) and \( k \in L^1_0(\mathbb{R}^d) \):

\[
\| g \ast k - g \ast D_\psi k \|_B \to 0 \quad \text{for } |\psi| \to 0.
\]

**Proof.** Since \((B, \| \cdot \|_B)\) is an essential Banach module over the Banach convolution algebra \((L^1_0(\mathbb{R}^d), \| \cdot \|_{1,0})\) we can apply the Cohen-Hewitt factorization Theorem, \((23), \text{Chap.32})\), i.e. any \( g \in B \) can be written as \( g = g_1 \ast h \), with \( h \in L^1_0(\mathbb{R}^d) \). Using the associativity law for Banach modules we obtain therefrom:

\[
\| g \ast k - g \ast D_\psi k \|_B = \| g_1 \ast h \ast k - g_1 \ast h \ast D_\psi k \|_B \to 0
\]
as \( |\psi| \to 0 \), according to Theorem 1. \(\square\)

## 4 The Main Result

We are now ready to formulate our main result:

**Theorem 3.** Given a minimal tempered standard space \((B, \| \cdot \|_B)\) on \( \mathbb{R}^d \), and any \( g \in \mathcal{S}(\mathbb{R}^d) \) with \( \int_{\mathbb{R}^d} g(x) \, dx = \widehat{g}(0) \neq 0 \), the set

\[
S(g) := \{ T_{s,t}g \mid x \in \mathbb{R}^d, \rho \in (0,1] \}
\]
is total in \((B, \| \cdot \|_B)\), i.e. the finite linear combinations are dense.

**Proof.** The claim requires to find, for any given \( f \in B \) and \( \varepsilon > 0 \), some finite linear combination \( h \) of elements from \( S(g) \) such that

\[
\| f - h \|_B < \varepsilon.
\]

We will verify something slightly stronger: Given \( f, \varepsilon > 0 \) there exists \( \rho_0 < 1 \) such that for any (fixed) \( \rho \in (0,\rho_0] \) one can find a finite set \((x_i)_{i \in F}\) and coefficients \((c_i)_{i \in F}\) such that \( h = \sum_{i \in F} c_i T_{x_i}g_\rho \) satisfies (44).

This approximation will be achieved in four steps:

1. By the density of \( \mathcal{S}(\mathbb{R}^d) \) in \((B, \| \cdot \|_B)\) and the density of compactly supported functions in \( \mathcal{S}(\mathbb{R}^d) \) (in the Schwartz topology), we can find some \( k \in \mathcal{D}(\mathbb{R}^d) \subset B \cap L^1_0(\mathbb{R}^d) \) with

\[
\| f - k \|_B < \varepsilon/4.
\]

2. In the next step we apply Lemma 2 for the specific approximate unit \((g_\rho)_{\rho \to 0}\), according to Lemma 1. Hence there exists some \( \rho_0 \) such that for any \( \rho \in (0,\rho_0] \) one has

\[
\| g_\rho \ast k - k \|_B < \varepsilon/4.
\]

Let us fix one such parameter \( \rho \) for the rest.

3. The final step is the discretization of the convolution \( g_\rho \ast k \), by replacing \( k \) by some finite, discrete measure in \( M^*_0(\mathbb{R}^d) \), by applying Theorem 2 with \( g = g_\rho \) and \( k \in \mathcal{D}(\mathbb{R}^d) \). By choosing \( \delta_0 > 0 \) properly we can guarantee that \( |\psi| \leq \delta_0 \) implies

\[
\| k \ast g_\rho - (D_\psi k) \ast g_\rho \|_B < \varepsilon/4.
\]

Note that

\[
 h = (D_\psi k) \ast g_\rho = \sum_{i \in I} c_i \delta_{x_i} \ast g_\rho = \sum_{i \in F} c_i T_{x_i}g_\rho
\]

has the required form, because \( F = \{ i \in I \mid \supp(k) \cap \supp(\psi_i) \neq \emptyset \} \) is a finite set, due to the compactness of \( \supp(k) \). It depends only on \( \supp(k) \) and \( \Psi \).

4. Combining the estimates (45), (46) and (47), we have for the given choice

\[
\| f - h \|_B \leq \| f - k \|_B + \| k - g_\rho \ast k \|_B + \| k \ast g_\rho - h \|_B \leq 3 \varepsilon/4,
\]
i.e. we have obtained the desired estimate:

\[
\| f - \sum_{i \in F} c_i T_{x_i}g_\rho \|_B = \| f - h \|_B < \varepsilon,
\]

and the proof is complete.
Remark 6. There is some freedom for the choice of the points $x_i$. Their density depends on the translation behaviour of $g_0$ within $(B, \| \cdot \|_B)$. It is not obvious to find the optimal choice, requiring minimal density of these points, combined with a good robustness of the approximation. If $\rho$ is close to zero, then one expects that the finite family has to be chosen very densely within $\text{supp}(k)$. On the other hand, working with relatively large $\rho$, which appears to be better in this respect, the error $\|k - g_0 * k\|_B$ will become larger.

5 Application to concrete cases

This section contains essentially three parts. In the first part we collect a few basic facts about weighted spaces. These will be used in the sequel to convince the reader that the current setting includes all the cases which are covered by the paper [24], but in fact many more. This will be explained in the second part of this section. A short subsection is devoted to the case of Shubin classes $Q_s(\mathbb{R}^d)$. Further indication of the richness of examples is given in the final subsection.

5.1 Weighted spaces, basic properties

First let us summarize a few facts concerning function spaces, in particular weighted $L^p$-spaces over $\mathbb{R}^d$, which are the prototypical examples of MINTSTAs (resp. TMIBs).

Going back to the classical papers [7], [20], [10], and [22], let us recall a few general facts about translation invariant function spaces. Recall that two weights $m_1$ and $m_2$ are called equivalent (we write $m_1 \approx m_2$) if for some $C > 0$

$$C^{-1} m_1(x) \leq m_2(x) \leq C m_1(x), \; \forall x \in \mathbb{R}^d. \quad (50)$$

Lemma 6. Let $p \in [1, \infty)$ be given.

1. A weighted $L^p$-space $(L^p_m, \| \cdot \|_{L^p_m})$ is translation invariant if and only $m$ is moderate;
2. For any $k \in C_c(G)$ the function $x \mapsto \|k\|_{m,p} = \|km\|_{L^p}$ is equivalent to the weight function $m$;
3. Any moderate weight function is equivalent to a continuous one;
4. Two spaces $L^p_{m_1}$ and $L^p_{m_2}$ are equivalent if and only if $p_1 = p_2$ and $m_1 \approx m_2$.

The following lemma is a consequence of the main results of [15], choosing $p = 2$ there.

Lemma 7. The extended Fourier transform maps $L^2_{m_1} \cap \mathcal{F}L^2_{m_2}$ onto $L^2_{m_1} \cap \mathcal{F}L^2_{m_2}$. In particular, one has Fourier invariant spaces of the form $L^2_{m_1} \cap \mathcal{F}L^2_{m_2}$ if and only if $m_1 \approx m_2$.

5.2 Deducing Katsnelson’s results

The spaces considered by Katsnelson in [24] are of the form $B = L^p_{m_1} \cap \mathcal{F}L^p_{m_2}$, with their natural norm. We do not have to repeat the technical conditions made in the paper [24], but rather summarize the relevant consequences of the setting described in that paper which allow us to demonstrate that the setting chosen for the current manuscript covers the cases described in Katsnelson’s paper:

1. $(B, \| \cdot \|_B)$ is continuously embedded into $(L^2(\mathbb{R}^d), \| \cdot \|_2)$;
2. $(B, \| \cdot \|_B)$ is a Banach space, in fact even a Hilbert space;
3. The spaces are invariant under translation and modulations.

The argument to be used next is taken from Lemma 2.2. of [10].

Lemma 8. Given a space of the form $L^2_{m_1} \cap \mathcal{F}L^2_{m_2}$ with two continuous, moderate weights $m_1$ and $m_2$ one has a continuous embedding into $(L^2(\mathbb{R}^d), \| \cdot \|_2)$ if and only if both $m_1$ and $m_2$ are bounded away from zero, which in turn is equivalent to the assumption that both

$$L^2_{m_1} \hookrightarrow L^2(\mathbb{R}^d) \quad \text{and} \quad L^2_{m_2} \hookrightarrow L^2(\mathbb{R}^d).$$

In conclusion we have the following observation:

The setting described in the paper [24] is exactly equivalent to the assumptions made in Lemma 8. Obviously these spaces are then Banach spaces of tempered distributions in the sense of our Definition 1 and hence all the results of the current paper or facts in [7] apply to that situation (e.g. Prop. 3.4).

The interested reader is referred to [15] for details in this direction.
**Proposition 2.** For $m_1(x) = \langle x \rangle^s = m_2(x)$, $s \in \mathbb{R}$ the corresponding spaces $L^2_{m_1} \cap FL^2_{m_2}$ are Fourier invariant, as the intersection of a Sobolev space with the corresponding weighted $L^2$-space.

They can also be identified with the so-called Shubin classes $(Q_\sigma(\mathbb{R}^d), \| \cdot \|_{Q_\sigma})$, characterized as the Banach spaces of all tempered distributions with a short-time Fourier transform in $L^2_{\sigma}(\mathbb{R}^d \times \mathbb{R}^d)$.

**Remark 7.** For $d = 1$ the spaces $Q_\sigma(\mathbb{R})$ coincides with a space of tempered distributions having Hermite coefficients in a weighted $\ell^2$-space (with polynomial weight of the order $s/2$).

In order to show that the Shubin classes $(Q_\sigma(\mathbb{R}^d), \| \cdot \|_{Q_\sigma})$ (for $s \geq 0$) are covered even in the setting of Katsnelson’s paper we have to shortly recall the concept of modulation spaces (see [16,17]).

A meanwhile widely used variant of modulation spaces (skipping many technical details) are the space $(M^p,q_m(\mathbb{R}^d), \| \cdot \|_{M^p,q_m(\mathbb{R}^d)})$, which are those tempered distributions which have an STFT (Short-Time Fourier Transform) belonging to a (moderately) weighted mixed-norm space (with two independent parameters $p,q$). This STFT of $\sigma \in \mathcal{S}(\mathbb{R}^d)$ can be defined for any (say real-valued) Schwartz window $g \in \mathcal{S}(\mathbb{R}^d)$ by:

$$V_g(\sigma)(x,y) = \sigma(M_yT_xg), \quad (x,y) \in \mathbb{R}^d \times \mathbb{R}^d.$$  

(51)

The choice $m(x,y) = (y)^s$ then gives the classical modulation space $(M^p,q_m(\mathbb{R}^d), \| \cdot \|_{M^p,q_m(\mathbb{R}^d)})$. For $p = q$ and radial symmetric weights of the form $m(x,y) = v_s(z) = (1 + x^2 + y^2)^{s/2}$ one has the (Fourier invariant) modulation spaces $M^p,q_m(\mathbb{R}^d)$ (see [21]). For more information on modulation spaces see [1], [16], and [17].

In order to verify that the Shubin classes (see also [20]) are special cases of Katsnelson’s paper we need the following simple observations:

**Lemma 9.** (i) Given two modulation spaces $M^p,q_{m_1}$ and $M^p,q_{m_2}$, one has $M^p,q_{m_1} \cap M^p,q_{m_2} = M^p,q_m$, with $m = \max(m_1, m_2)$, and equivalence of the corresponding natural norms.

(ii) For polynomial weights of the form $m_1(x,y) = \langle x \rangle^s$ and $m_2(x,y) = \langle y \rangle^s$ for some $s \in \mathbb{R}$, one has:

$$\max(m_1, m_2)(x,y) \sim v_s(z) = v_s(x,y).$$

**Proof.** For the second statement one just has to observe that for $z = (x,y)$ one has $\max((x)^s, (y)^s) \sim (z)^s$, using the fact that $\max(|x|, |y|) \geq |z|$, since obviously

$$\max(|x|, |y|) \leq |z| \leq |x| + |y| \leq 2\max(|x|, |y|), \quad z = (x,y).$$

**Remark 8.** As a final remark let us observe that $Q_\sigma(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$ if and only if $s \geq 0$. Thus the results of [24] apply only for these spaces, while our general results allow the full range $s \in \mathbb{R}$, including their dual spaces, since $Q_\sigma(\mathbb{R}^d)^* = Q_{-s}(\mathbb{R}^d)$.

**6 Summary**

In conclusion this paper provides extensions of the main result of [24] in the following directions:

1. The Gauss function can be replaced by any Schwartz function with non-zero integral;
2. The results are valid for $\mathbb{R}^d$, for any $d \geq 1$;
3. We abolish the assumption that $(B, \| \cdot \|_{B})$ is a Hilbert space, as well as the rather restrictive property that it should be contained in $(L^2(\mathbb{R}^d), \| \cdot \|_{L^2})$;
4. Our main result applies to an abundance of function spaces for which such completeness statements can be shown; we just list particular examples;
5. As a benefit we establish a connection to the so-called Shubin classes and show that the completeness statement is also valid for their dual spaces.

In order to avoid the use of ultra-distributions and a generalized Fourier transform in the sense of such ultra-distributions we decided to work with the well-known setting of tempered distributions in the sense of Schwartz, i.e. with $\mathcal{S}(\mathbb{R}^d)$, the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions. Extension to the setting of ultra-distributions are no problem but will be discussed elsewhere. Also the setting of [8] (Theorem 2.2) should be helpful in this respect.
7 Acknowledgement

This paper was prepared during the visit of the second author to the NuHAG workgroup at the University of Vienna, supported by an Ernst Mach Grant-Worldwide Fellowship (ICM-2019-13302) from the OeAD-GmbH, Austria. The second author is very grateful to Professor Hans G. Feichtinger for his guidance, for hosting and arranging excellent research facilities at the University of Vienna. The second author is also grateful to the NBHM-DAE (0204/19/2019R&D-II/10472) and Indian Institute of Science, Bangalore for allowing the academic leave.

References

[1] A. Beyni and K. A. Okoudjou. Modulation Spaces. Springer (Birkhäuser), New York, 2020.
[2] N. Bourbaki. Integration. II. Chapters 7–9. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 2004.
[3] W. Braun and H. G. Feichtinger. Banach spaces of distributions having two module structures. J. Funct. Anal., 51:174–212, 1983.
[4] P. Dimovski, S. Pilipovic, and J. Vindas. New distribution spaces associated to translation-invariant Banach spaces. Monatsh. Math., 177(4):495–515, 2015.
[5] P. Dimovski, S. Pilipovic, B. Prangoski, and J. Vindas. Translation–modulation invariant Banach spaces of ultradistributions. J. Fourier Anal. Appl., 25(3):819–841, 2019.
[6] D. H. Dunford. Segal algebras and left normed ideals. J. Lond. Math. Soc. (2), 8:514–516, 1974.
[7] R. E. Edwards. The stability of weighted Lebesgue spaces. Trans. Amer. Math. Soc., 93:369–394, 1959.
[8] H. G. Feichtinger. Multipliers from $L^1(G)$ to a homogeneous Banach space. J. Math. Anal. Appl., 61:341–356, 1977.
[9] H. G. Feichtinger. Results on Banach ideals and spaces of multipliers. Math. Scand., 41(2):315–324, 1977.
[10] H. G. Feichtinger. Gewichtsfunktionen auf lokalkompakten Gruppen. Sitzber. d. österr. Akad. Wiss., 188:451–471, 1979.
[11] H. G. Feichtinger. A compactness criterion for translation invariant Banach spaces of functions. Analysis Mathematica, 8:165–172, 1982.
[12] H. G. Feichtinger. Banach convolution algebras of Wiener type. In Proc. Conf. on Functions, Series, Operators, Budapest 1980, volume 35 of Colloq. Math. Soc. Janos Bolyai, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.
[13] H. G. Feichtinger. Compactness in translation invariant Banach spaces of distributions and compact multipliers. J. Math. Anal. Appl., 102:289–327, 1984.
[14] H. G. Feichtinger. Minimal Banach spaces and atomic representations. Publ. Math. Debrecen, 34(3-4):231–240, 1987.
[15] H. G. Feichtinger. Generalized amalgams, with applications to Fourier transform. Canad. J. Math., 42(3):395–409, 1990.
[16] H. G. Feichtinger. Modulation spaces on locally compact Abelian groups. In R. Radha, M. Krishna, and S. Thangavelu, editors, Proc. Internat. Conf. on Wavelets and Applications, pages 1–56, Chennai, January 2002, 2003. New Delhi Allied Publishers.
[17] H. G. Feichtinger. Modulation Spaces: Looking Back and Ahead. Sampl. Theory Signal Image Process., 5(2):109–140, 2006.
[18] H. G. Feichtinger. A novel mathematical approach to the theory of translation invariant linear systems. In Peter J. Bentley and I. Pesenson, editors, Novel Methods in Harmonic Analysis with Applications to Numerical Analysis and Data Processing, pages 1–32, 2016.
[19] H. G. Feichtinger and A. T. Gürkanli. On a family of weighted convolution algebras. Int. J. Math. Math. Sci., 13(3):517–526, 1990.
[20] G. I. Gaudry. Multipliers of weighted Lebesgue and measure spaces. Proc. Lond. Math. Soc., III. Ser., 19:327–340, 1969.
[21] K. Gröchenig. Foundations of Time-Frequency Analysis. Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.
[22] K. Gröchenig. Weight functions in time-frequency analysis. In L. Rodino and et al., editors, *Pseudodifferential Operators: Partial Differential Equations and Time-Frequency Analysis*, volume 52 of *Fields Inst. Commun.*, pages 343–366. Amer. Math. Soc., Providence, RI, 2007.

[23] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups*. Springer, Berlin, Heidelberg, New York, 1970.

[24] V. Katsnelson. On the completeness of Gaussians in a Hilbert functional space. *Complex Anal. Oper. Theory*, 13(3):637–658, 2019.

[25] Y. Katznelson. *An Introduction to Harmonic Analysis. 2nd corr. ed.* Dover Publications Inc., New York, 1976.

[26] F. Luef and Z. Rahbani. On pseudodifferential operators with symbols in generalized Shubin classes and an application to Landau-Weyl operators. *Banach J. Math. Anal.*, 5(2):59–72, 2011.

[27] H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Clarendon Press, Oxford, 1968.

[28] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.