Analysis of a Bianchi-like equation satisfied by the Mars-Simon tensor *

Florian Beyer$^1$ and Tim-Torben Paetz$^2$

$^1$Department of Mathematics and Statistics, University of Otago, P.O. Box 56, Dunedin 9054, New Zealand
$^2$Gravitational Physics, University of Vienna, Boltzmanngasse 5, 1090 Vienna, Austria

February 21, 2018

Abstract

The Mars-Simon tensor (MST), which e.g. plays a crucial role to provide gauge invariant characterizations of the Kerr-NUT-(A)(dS) family, satisfies a Bianchi-like equation. In this paper we analyze this equation in close analogy to the Bianchi equation, in particular it will be shown that the constraints are preserved supposing that a generalized Buchdahl condition holds. This permits the systematic construction of solutions to this equation in terms of a well-posed Cauchy problem. A particular emphasis lies on the asymptotic Cauchy problem, where data are prescribed on a spacelike $I^-$ (i.e. for $\Lambda > 0$). In contrast to the Bianchi equation, the MST equation is of Fuchsian type at $I^-$, for which existence and uniqueness results are derived.

1 Introduction

The Kerr-NUT-(A)(dS) family plays a distinguished role among solutions to Einstein’s field equations: Not merely is it an explicitly known family of vacuum solutions but it is also expected that e.g. the Kerr subfamily satisfies certain black hole uniqueness and stability results. In fact for the Kerr-de Sitter family stability has recently been established [20], while uniqueness is still open.

While the explicit form of the solutions employs coordinates which are adapted to the symmetries, it is equally important to have gauge invariant characterizations at hand. One approach to accomplish this is based on the so-called Mars-Simon tensor (MST) [21, 25, 26, 28, 37], which may be regarded as a generalization of the conformal Weyl tensor in spacetimes which admit a Killing vector: Given a vacuum solution of the Einstein equations with cosmological constant $\Lambda$, the vanishing of the Weyl tensor shows that the spacetime is locally Minkowski or (Anti-)de Sitter, respectively. Similarly, the vanishing of the MST implies that the spacetime is locally given by a certain explicitly known family of solutions, classified in [28], which contains in particular the Kerr-NUT-(A)dS family.

*Preprint UWThPh-2017-14.
It turns out that the MST satisfies an equation very similar to the Bianchi equation which is satisfied by the Weyl tensor. This “MST equation”, which has been derived in [21] for \( \Lambda = 0 \) and in [27] for arbitrary signs of \( \Lambda \), played an important role in the derivation of Kerr uniqueness results [1, 21], and in a characterization of \( \Lambda > 0 \)-vacuum spacetimes in terms of their asymptotic Cauchy data on \( \mathcal{I} \) [27]. In fact, the motivation for this paper comes from the latter one: In contrast to the Bianchi equation the MST equation (or rather its analog in an appropriately conformally rescaled spacetime) is not regular near a smooth spacelike \( \mathcal{I} \). Instead, the equation is shown there to be of Fuchsian type.

The MST equation can be split into a symmetric hyperbolic system of evolution equations, and a system of constraint equations [27]. As for the Bianchi equation, we will show that the constraints are preserved under evolution, supposing that a certain \textit{generalized Buchdahl condition} holds. Given a solution to the constraints on some Cauchy surface, the existence of a solution to the full system follows from standard results. When the initial hypersurface is a spacelike \( \mathcal{I} \), though, the Cauchy problem becomes more involved due to the Fuchsian character of the system at \( \mathcal{I} \). In [27] uniqueness of solutions which extend smoothly through \( \mathcal{I} \) with regular data at \( \mathcal{I} \) has been proved. It is a main goal of this paper to analyze in detail the behavior of solutions near \( \mathcal{I} \) and establish a well-posedness result for this \textit{singular initial value problem} (in contrast, it is easy to solve the constraints on \( \mathcal{I} \), whereas they are more involved on an ordinary Cauchy surface of the physical spacetime).

We tackle the singular initial value problem by means of the \textit{Fuchsian technique}. Earlier Fuchsian techniques by Rendall and co-authors, which were later applied to problems in general relativity [5, 12, 15, 19, 22, 23], were restricted to the real-analytic setting. The first attempts to overcome the analyticity restriction were made in [14, 35]. A series of papers [2–4, 7–10] led to a version of the Fuchsian technique which applies to a general class of quasilinear hyperbolic equations without the analyticity restriction.

We study the MST equation on a given background, and a solution thereof will generally not coincide with the MST of the background spacetime. The reasons why this is interesting are basically the same as for the Bianchi equation: A solution to the Bianchi equation is in one-to-one correspondence with a solution to the linearized Einstein equations (at least for \( \Lambda = 0 \)), and a similar result should be expected for a solution to the MST equation on a background with vanishing MST (we will only partially address this issue). Moreover, the importance of the Bianchi equation arises from its appearance as part of Friedrich’s conformal field equations [17]; one might think of a similar system of equations where the MST equation replaces the Bianchi equation in order to study certain classes of perturbations of spacetimes which admit a Killing vector field (e.g. since the (conformally rescaled) MST is much better behaved near the poles of a Kerr-de Sitter \( \mathcal{I} \) than the (conformally rescaled) Weyl tensor). Analyzing this equation on a given background then provides a toy model for this.

While the Bianchi equation is meaningful on any background spacetime, the MST equation can only be posed on a background which admits a Killing vector field. However, due to the Buchdahl condition well-posedness results for the Bianchi equation are only available on locally conformally flat backgrounds. Corresponding results for the MST equation permit a larger class of background spacetimes, namely those with vanishing MST such as e.g. the Kerr-NUT-(A)dS family.

The structure of the paper is as follows: In Section 2 we recall the definition of the MST. In Section 3 we review a couple of results on the Bianchi equation as a guideline for our analysis of the MST equation. Section 4 is devoted to the study of the MST equation. More precisely, in Section 4.1 we will prove that the linearized MST satisfies the MST equation on a background with vanishing MST. In the remainder of that section we will establish a generalized Buchdahl condition which ensures that the constraints are preserved under evolution.
In Section 5 we will consider the analog of the MST equation in Penrose’s conformally rescaled spacetimes. We then restrict attention to the case where the cosmological constant is positive, so that $\mathcal{I}$ is spacelike. The behavior of the constraint and evolution equations near $\mathcal{I}$ is unveiled in Section 5.2.

In Section 6, we solve the singular initial value problem of the MST equations with data on $\mathcal{I}^-$. In Section 6.1 we establish some basic notation, list our assumptions and formulate the main theorem, Theorem 6.1. The remainder of Section 6 is devoted to the proof of that theorem. Section 6.2 provides a detailed study of a particular matrix in the evolution equations which largely determines the leading singular behavior of solutions at $\mathcal{I}^-$. In Section 6.3 we investigate the evolution equation of the remainder of the singular initial value problem and establish certain decay conditions. The leading-order term for the singular initial value problem is derived in Section 6.4. By that stage we will have identified the singular data and established a local existence theory for the singular initial value problem of the evolution equations. In Section 6.5, we then find conditions on the singular data which guarantee that the asserted family of solutions of the evolution equations also satisfies the constraints. Finally, we show in Section 6.6 that the same conditions also imply that the solutions extend smoothly through $\mathcal{I}^-$ once an overall singular term expected for generic MSTs has been subtracted. Notice here that according to the general structure of the singular data, generic solutions of the evolution equations do not have this property.

1.1 Notation

For the convenience of the reader let us finish the introduction with some remarks concerning the notation. The “physical” spacetime, solution to Einstein’s vacuum field equations with cosmological constant $\Lambda$, will be denoted by $(M, g)$. Throughout the paper we assume $(M, g)$ to be a smooth 3 + 1-dimensional connected, oriented and time-oriented Lorentzian manifold with signature $(-, +, +, +)$. Its associated Levi-Civita covariant derivative, connection coefficients and volume form are denoted by $\nabla_\mu$, $\Gamma^\alpha_{\mu\nu}$ and $\eta_{\mu\nu\rho\sigma}$, respectively. Our conventions concerning Riemann tensor, Ricci tensor etc. follow those in [38].

The conformally rescaled counterpart of $(M, g)$ will be denoted by $(\tilde{M}, \tilde{g})$. Correspondingly, objects associated to $\tilde{g}$ are decorated with a tilde, $\tilde{\nabla}_\mu$, $\tilde{\Gamma}^\alpha_{\mu\nu}$ and $\tilde{\eta}_{\mu\nu\rho\sigma}$.

Spacetime indices are Greek. Coordinates in 3 + 1 splits are denoted by $\{x^\mu\} = \{t \equiv x^0, x^i\}$ with corresponding tensorial indices. Objects associated to the family $t \mapsto g_{ij}(t, x^k)$ of Riemannian metrics are marked with a slash, $\tilde{\nabla}_i$, $\tilde{\nabla}_i \tilde{v}_j$ and $\tilde{\eta}_{ijk}$. The action of $\tilde{\nabla}_i$ on spacetime tensors is defined as follows:

$$\tilde{\nabla}_i v_0 := \partial_i v_0 , \quad \tilde{\nabla}_i v_j := \partial_i v_j - \tilde{\nabla}_j v_i$$

and similarly for tensors of higher rank. Again, the corresponding objects associated to the family $t \mapsto \tilde{g}_{ij}(t, x^k)$ in the conformally rescaled spacetime are decorated with a tilde.

For $\Lambda > 0$ and in spacetimes with an appropriate fall-off behavior of the gravitational field, (past) null infinity is represented in $(\tilde{M}, \tilde{g})$ by a spacelike hypersurface $\mathcal{I}^-$. The Riemannian metric induced by $\tilde{g}_{\mu\nu}$ on $\mathcal{I}^-$ will be denoted by $h_{ij}$, its covariant derivative by $\mathcal{D}_i$ and the volume form by $\epsilon_{ijk}$ (the only exception will be Section 3.2, where $\mathcal{D}_i$ and $\epsilon_{ijk}$ denote covariant derivative and volume form of the induced metric on an arbitrary Cauchy surface).
\section{Mars-Simon tensor (MST)}

Let \((M, g)\) be a \(\Lambda\)-vacuum spacetime which admits a Killing vector field \(X\). The \textit{Mars-Simon tensor (MST)}, cf. \([21, 25, 26, 28, 37]\), is defined as follows:

\[
S_{\mu\nu\sigma\rho} := C_{\mu\nu\sigma\rho} + Q U_{\mu\nu\sigma\rho},
\]

where

\[
C_{\mu\nu\sigma\rho} := C_{\mu\nu\sigma\rho} + i C^*_{\mu\nu\sigma\rho},
\]

\[
U_{\mu\nu\sigma\rho} := F_{\mu\nu} F_{\sigma\rho} + \frac{1}{3} F^2 I_{\mu\nu\sigma\rho},
\]

\[
I_{\mu\nu\sigma\rho} := \frac{1}{4} (2 g_{\mu(\sigma} g_{\rho]\nu} + i \eta_{\mu\nu\sigma\rho}),
\]

\[
F_{\mu\nu} := F_{\mu\nu} + i F^*_{\mu\nu},
\]

\[
F^2 := F_{\mu\nu} F^\mu\nu,
\]

\[
F_{\mu\nu} := \nabla_\mu X_\nu.
\]

In these expressions, \(\eta_{\mu\nu\sigma\rho}\) is the volume form of \(g\) and \(*\) the corresponding Hodge dual. \(C_{\mu\nu\sigma\rho}\) and \(F_{\mu\nu}\) are the self-dual Weyl tensor and the self-dual Killing form. They satisfy \(C_{\mu\nu\sigma\rho} = -i C^*_{\mu\nu\sigma\rho}\) and \(F_{\mu\nu} = -i F^*_{\mu\nu}\). The symmetric double two-form \(I_{\mu\nu\sigma\rho}\) provides a metric in the space of self-dual two-forms in the sense that \(I_{\mu\nu\sigma\rho} W^{\sigma\rho} = W_{\mu\nu}\) for any self-dual two-form \(W_{\mu\nu}\). Some useful identities satisfied by self-dual tensors can be found e.g. in \([21, 24]\).

The MST is a Weyl field, i.e. it has all the algebraic symmetries of the Weyl tensor,

\[
S_{(\mu\nu)} = 0, \quad S_{\mu\nu\sigma\rho} = S_{\sigma\mu\nu}, \quad g^{\rho\sigma} S_{\mu\nu\sigma\rho} = 0, \quad S_{[\mu\nu\sigma\rho]} = 0.
\]

Moreover, its Lie derivative along the associated Killing vector \(X\) vanishes

\[
\mathcal{L}_X S_{\mu\nu\sigma\rho} = 0,
\]

supposing that \(Q\) satisfies \(\mathcal{L}_X Q = 0\) (as it is the case for the \(Q\) defined below).

In the literature different definitions of the function \(Q : M \to \mathbb{C}\) have proven to be advantageous in different contexts \([21, 27, 28, 32]\). All the different choices for \(Q\) are obtained by requiring a certain component of the MST (or a derivative thereof) to vanish, and are therefore equivalent in spacetimes where the MST vanishes (supposing that certain quantities are non-zero). Here we are interested in the Bianchi-like equations satisfied by the MST, and this requires a particular choice for the function \(Q\) \([28]\):

\[
Q := \frac{3J}{R} - \frac{\Lambda}{R^2},
\]

\[
R := -\frac{i}{2} \sqrt{F^2},
\]

\[
J := \frac{R + \sqrt{R^2 - \Lambda \sigma}}{\sigma},
\]

and where \(\sigma\) denotes the Ernst potential of the closed 1-form

\[
\sigma_\beta := 2 X^\alpha F_{\alpha\beta}.
\]

In general the Ernst potential exists only locally and is defined only up to an additive complex constant. In this paper we will eventually restrict attention to background spacetimes with
vanishing MST and in that case the Ernst 1-form is exact and the additive constant is determined by the requirement on the MST to vanish [28].

The definition of the complex square roots depends on the sign of the cosmological constant. For $\Lambda > 0$ (the case which we are primarily interested in) the square root is preferably chosen in such a way that the real part of $R$ approaches minus infinity at $I$, in agreement with the usual behavior for e.g. the Kerr-de Sitter family in Boyer-Lindquist type coordinates (cf. [27] for more details). With the signs chosen in (2.10)-(2.12) this is achieved by taking the positive branch, i.e. the one that takes positive real numbers and gives positive real values.

In [27] this choice for $Q$ was denoted by $Q_{ev}$. Since we assume $Q = Q_{ev}$ throughout the paper, no confusion arises and we will simply write $Q$ henceforth.

**Proposition 2.1** ([27], cf. [21] for the $\Lambda = 0$-case) Let $(M, g, X)$ be a $\Lambda$-vacuum spacetime with Killing vector field $X$ such that

$$F^2 \neq 0, \quad QF^2 + 8\Lambda \neq 0, \quad \sigma \neq 0.$$  \hspace{1cm} (2.14)

Then the MST (2.1) with $Q$ given by (2.10) satisfies the Bianchi-like equation

$$\nabla_\rho S_{\alpha\beta\mu} = J(S)_{\alpha\beta\mu},$$  \hspace{1cm} (2.15)

where

$$J(S)_{\alpha\beta\mu} = 4\Lambda \frac{5QF^2 + 4\Lambda}{QF^2 + 8\Lambda} F_{\alpha\beta\mu} F^{-4} X^\sigma F^\gamma S_{\gamma\delta\sigma} - Q X^\sigma F_{\mu\rho} S_{\alpha\sigma} X^\rho - \frac{2}{3} F^{-2} \left( QF^2 - 2\Lambda \frac{5QF^2 + 4\Lambda}{QF^2 + 8\Lambda} \right) L_{\alpha\beta\mu} X^\sigma F^\gamma S_{\gamma\delta\sigma}.$$  \hspace{1cm} (2.16)

**Remark 2.2** $J(S)_{\alpha\beta\mu}$ has the following properties [27]:

$$J(S)_{\alpha\beta\mu} = J(S)_{[\alpha\beta]\mu}, \quad J(S)_{[\alpha\beta\mu]} = 0, \quad J(S)^\mu_{\alpha\beta\mu} = 0.$$  

It is further self-dual in the first pair of anti-symmetric indices.

### 3 Bianchi equation

The MST has the same algebraic symmetries as the Weyl tensor, and fulfills an equation similar, though more complicated, to the Bianchi equation for the Weyl tensor. In fact, the MST was introduced as a generalization of the Weyl tensor in $\Lambda$-vacuum spacetimes which admit a Killing vector field: Its vanishing (supplemented by certain additional conditions) provides a local gauge-independent characterization of Kerr-(A)(dS) family in much the same way as the vanishing of the Weyl tensor provides a local characterization of Minkowski and (Anti-)de Sitter spacetime, respectively [25, 26, 28].

While the MST equation comes along with some additional features such as the Fuchsian behavior near a spacelike $I$, a couple of results which hold for the Bianchi equation can be derived for the MST equation, as well. Even more, the analysis of the Bianchi equation provides a guideline how to analyze the MST equation and what results should be expected. It is the aim of this section to review some crucial results on the Bianchi equation.

Consider a spacetime $(M, g)$ which satisfies the vacuum equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. Then the Weyl tensor of $g$ satisfies the Bianchi equation

$$\nabla_\mu C_{\mu\nu\sigma}^\rho = 0.$$  \hspace{1cm} (3.1)

Now, given a $\Lambda$-vacuum spacetime $(M, g)$, it is of interest to analyze this equation on that given background and construct solutions from appropriate initial surfaces:
(i) First of all the Bianchi equation is part of Friedrich’s conformal field equations (cf. e.g. [17]) which replace Einstein’s vacuum field equations in a setting where the metric is conformally rescaled. Analyzing (3.1) on a background therefore provides a kind of toy model for these equations.

(ii) On a conformally flat background, solutions to the linearized field equations provide solutions to the Bianchi equation, cf. Lemma 3.1 below. Even more, solutions to (3.1) turn out to be in one-to-one correspondence with solutions to the linearized field equations (to our knowledge, this result has only been established on a flat background so far, cf. Lemma 3.2 below, though an analog result should be expected on an (Anti-)de Sitter background, as well).

3.1 Linearized gravity

Let us elaborate somewhat more detailed on (ii). We consider a Lorentzian metric $\eta$ and denote by

$$g = \eta + h$$

(3.2)

a perturbation thereof. In the following we will decorate all fields related to the background metric $\eta$ with superscript $(0)$. The curvature tensor associated to the metric $g$ takes the form (cf. e.g. [13])

$$R_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho}^{(0)} + \nabla_{\rho}^{(0)} h_{\mu\nu,}\sigma - \nabla_{\sigma}^{(0)} h_{\mu\nu,}\rho - h_{[\mu}^{\kappa} R_{\nu]\sigma\kappa\rho}^{(0)} + O(h^2).$$

(3.3)

Two perturbations $h$ and $h'$ describe the same physical perturbation if and only if there exists a one-form $\xi$ such that $h'_\mu - h_\mu = \nabla_{(\mu} \xi_{\nu)}$. It is convenient to employ this gauge freedom

$$h_{\mu\nu} \mapsto h_{\mu\nu} + \nabla_{(\mu} \xi_{\nu)}$$

(3.4)

to impose the gauge condition

$$\nabla_{(0)}^{\alpha} h_{\alpha}^{\beta} = \frac{1}{2} \nabla_{(0)}^{\alpha} \text{tr}_g h.$$

(3.5)

(It is preserved under evolution of the linearized Einstein equations (3.11) below.) Assuming further that the background metric satisfies the $\Lambda$-vacuum equations, we obtain from (3.3) ($\Box_g := g^{\mu\nu} \nabla_{(0)}^{\mu} \nabla_{(0)}^{\nu}$)

$$R_{\mu\nu\sigma\rho} = \frac{2}{3} \Lambda \eta_{[\mu} h_{\nu]\sigma\rho} + C_{\mu\nu\sigma\rho}^{(0)} + \nabla_{\rho}^{(0)} h_{\mu\nu,}\sigma - \nabla_{\sigma}^{(0)} h_{\mu\nu,}\rho - h_{[\mu}^{\kappa} C_{\nu]\sigma\kappa\rho}^{(0)} + O(h^2),$$

where

$$R_{\mu\nu} = \Lambda \eta_{\mu\nu} - \frac{1}{2} \Box_g h_{\mu\nu} + \frac{4}{3} \Lambda h_{\mu\nu} - \frac{\Lambda}{3} \text{tr}_g h \eta_{\mu\nu} - C_{\mu\alpha\nu\beta}^{(0)} h_{\alpha\beta} + O(h^2),$$

(3.7)

and

$$R = 4\Lambda - \frac{1}{2} \Big( \Box_g + 2\Lambda \Big) \text{tr}_g h + O(h^2),$$

(3.8)
and one easily computes the Weyl tensor associated to $g$,

$$\begin{align*}
C_{\mu\nu\sigma\rho} &= R_{\mu\nu\sigma\rho} - g_{\sigma[\mu} R_{\nu]\rho] + g_{\rho[\mu} R_{\nu]\sigma] + \frac{1}{3} R g_{\sigma[\mu} g_{\nu]\rho]} \\
&= C^{(0)}_{\mu\nu\sigma\rho} + \nabla^{(0)} \eta_{[\mu} \nabla_{\nu]} h_{\sigma\rho]} - \nabla^{(0)} \eta_{[\mu} \nabla^{(0)} h_{\nu]\sigma]} \\
&\quad + \frac{1}{2} \eta_{[\mu} \eta_{\nu]} \nabla h_{\sigma\rho]} - \frac{1}{2} \eta_{\rho[\mu} \nabla h_{\nu]\sigma]} - \frac{1}{6} \eta_{\nu][\rho} \eta_{\nu]} h_{\sigma\rho]} \\
&\quad - 2 \Lambda (\eta_{[\mu} h_{\nu]\sigma]} - \eta_{\rho[\mu} h_{\nu]\sigma]} + \frac{1}{2} \eta_{\nu][\rho} \eta_{\nu]} h_{\sigma\rho]} \\
&\quad - h_{[\mu} \eta C^{(0)}_{\nu][\rho]} + \eta_{\mu} C^{(0)}_{[\sigma]} C_{\mu\nu\sigma\rho] + \eta_{[\rho} \eta C^{(0)}_{\nu]\sigma]} h_{\alpha\beta} - \eta_{\mu} C^{(0)}_{\nu][\sigma] h_{\alpha\beta} + O(h^2). \\
\end{align*}$$

(3.9)

It follows from (3.7) that the linearized Einstein equations on a vacuum background $(M, \eta)$ read

$$\Box g h_{\mu\nu} = -2 \Lambda (h_{\mu\nu} - \nabla h \eta_{\mu\nu}) - 2 C^{(0)}_{\mu\nu\rho\sigma} h^{\alpha\beta}.$$  

(3.11)

Let us now assume that the linearized field equations (3.11) are satisfied. Then the Weyl tensor (3.10) has the expansion

$$C_{\mu\nu\sigma\rho} = C^{(0)}_{\mu\nu\sigma\rho} + C^{(\text{lin})}_{\mu\nu\sigma\rho} + O(h^2),$$

(3.12)

where

$$C^{(\text{lin})}_{\mu\nu\sigma\rho} = \nabla^{(0)} \eta_{[\mu} \nabla_{\nu]} h_{\sigma\rho]} - \nabla^{(0)} \eta_{[\mu} \nabla^{(0)} h_{\nu]\sigma]} - \frac{1}{2} (\eta_{[\mu} h_{\nu]\sigma]} - \eta_{\rho[\mu} h_{\nu]\sigma]} - h_{[\mu} \eta C^{(0)}_{\nu][\rho]}.$$  

(3.13)

We compute the divergence. Using the linearized equations (3.11), the gauge condition (3.5), and the Bianchi equation for the background metric, we obtain

$$\nabla^{(0)} \nabla C_{\mu\nu\sigma\rho} = -2 h^{\alpha\beta} \nabla^{(0)} C^{(0)}_{\nu\alpha\sigma\beta} + C^{(0)}_{\alpha\sigma\beta}[\mu \nabla^{(0)} h^{\alpha\beta} - C^{(0)}_{\sigma[\beta} \alpha \nabla^{(0)} h_{\nu]} \beta} + C^{(0)}_{\mu\nu\sigma\rho]} h^{\alpha\beta} + O(h^2).$$

(3.14)

That yields the well-known (cf. e.g. [39])

**Lemma 3.1.** Let $(M, \eta)$ be a $\Lambda$-vacuum spacetime and let $h$ be a solution to the linearized vacuum equations. Then the linearized Weyl tensor $C^{(\text{lin})}_{\mu\nu\sigma\rho}$ associated to the perturbation $g = \eta + h$ satisfies the Bianchi equation, $\nabla^{(0)} C^{(\text{lin})}_{\mu\nu\sigma\rho} = 0$, supposing that the background metric $\eta$ is locally conformally flat.

The converse is also true (at least for vanishing cosmological constant). The proof can be found e.g. in [6].

**Lemma 3.2.** Let $(M, \eta)$ be a Minkowski background, and let $C_{\mu\nu\sigma\rho}$ be a solution to the Bianchi equation $\nabla^{(0)} C^{(0)}_{\mu\nu\sigma\rho} = 0$ with all the algebraic symmetries of the Weyl tensor. Then there exists a unique (up to gauge transformations (3.4)) perturbation $h$ which satisfies the linearized vacuum equations (3.11), such that $C_{\mu\nu\sigma\rho}$ is the linearized Weyl tensor of $g = \eta + h$.

### 3.2 Cauchy problem and Buchdahl condition

Having a motivation at hand that the Bianchi equation on a given background is of interest by itself, one would like to establish well-posedness of the associated Cauchy problem.
Let $\Sigma \subset M$ be a Cauchy surface in $(M, \eta)$, and introduce Gaussian coordinates $(t \equiv x^0, x^i)$ near $\Sigma$. Taking the algebraic symmetries of the Weyl tensor into account, it is easy to check that the Bianchi equation is equivalent to the system ($\ast$ denotes the Hodge dual)

\[
\nabla^0_\rho C^\rho_{0(ij)} = 0, \quad \nabla^0_\rho C^\ast_{0(ij)} = 0,
\]

(3.15)

\[
\nabla^0_\rho C^\rho_{000} = 0, \quad \nabla^0_\rho C^\ast_{000} = 0.
\]

(3.16)

One checks that (3.15) forms a symmetric hyperbolic system of evolution equations, for which standard well-posedness results are available. The system (3.16) does not contain transverse derivatives and therefore provides a system of constraint equations. It requires the initial data $C_{\mu\nu\sigma\rho}|_\Sigma$ to satisfy the following constraint equations

\[
\nabla^0 k C^0_{i00} k|_\Sigma = 0, \quad \nabla^0 k C^\ast_{i00} k|_\Sigma = 0.
\]

(3.17)

This is equivalent to the existence of a complex symmetric trace-free tensor $E_{ij}$:

\[
E_{ij} := C^0_{i00} j + i C^\ast_{i00} j|_\Sigma
\]

solves

\[
D_k E_{ik} = i \varepsilon^{ilm} K_{kl} E_{mk},
\]

(3.18)

where $D_i$ and $\varepsilon_{ijk}$ denote the covariant derivative and the volume form of the induced metric on $\Sigma$, while $K_{ij} = \frac{1}{2} \mathcal{L}_0 g_{ij}$ denotes the second fundamental form. To solve this constraint one may employ York’s splitting method: Here $E_{ij}$ is written in the form

\[
E_{ij} = B_{ij} + 2 D^i Y^j - \frac{2}{3} \eta_{ij} \mathcal{D}_k Y^k,
\]

(3.19)

where the symmetric, trace-free tensor $B_{ij}$ provides certain prescribed “seed” data. The constraint equation (3.18) then becomes a linear elliptic PDE-system for the complex vector field $Y$. In the time-symmetric case, for instance, solving the constraint equations simply amounts to the construction of symmetric trace-free and divergence-free tensors on the initial surface.

One then needs to make sure that (3.17) is preserved under evolution. It turns out that this is the case if and only if the Buchdahl condition \cite{11} holds,

\[
C^{0}_{[\alpha\beta\gamma} C^{0(0)*}_{i]0\alpha\beta\gamma} = 0, \quad C^{0}_{[\alpha\beta\gamma} C^{0(0)*}_{i]0\alpha\beta\gamma} = 0
\]

(3.20)

which is certainly the case whenever the background metric $\eta$ is locally conformally flat, or when $C_{\mu\nu\sigma\rho}$ is the Weyl tensor associated to $\eta$. In the latter case this is obvious for the first condition in (3.20), for the second we have

\[
C^{0}_{[0} \alpha\beta\gamma C^{0(0)*}_{i]0\alpha\beta\gamma} = \frac{i}{2} \eta^{\beta\gamma\delta\rho} C^{0(0)\alpha}_{[0} \beta\gamma C^{0(0)}_{i]0\alpha\delta\rho} = \frac{i}{2} \eta^{\beta\gamma\delta\rho} C^{0(0)\alpha}_{[0} \delta\rho C^{0(0)}_{i]0\alpha\beta\gamma} = -C^{0(0)\alpha\beta\gamma}_{[0} C^{0(0)*}_{i]0\alpha\beta\gamma}.
\]

To sum it up, the Bianchi equation, regarded as an equation on a given background $(M, \eta)$ is of particular relevance whenever $(M, \eta)$ is locally conformally flat. In that case the Buchdahl condition is satisfied and solutions are obtained from data on some Cauchy surface $\Sigma \subset M$ if and only if these data satisfy the constraint equations (3.17).

4 Bianchi-like equation for the MST

In this section we will carry out a similar analysis for the Bianchi-like equation (2.16) satisfied by the MST as we did for the Bianchi equation in the previous section.
4.1 Linearized MST

First of all we would like to derive an analog of Lemma 3.1. This is to explore the analogies between the Bianchi and the MST equation, but, even more important, we would like to provide a motivation to analyze the MST equation on a given background. It seems likely that there is an analog of Lemma 3.2 which we have not explored here.

Since the computations turn out to be fairly lengthy we will restrict attention to the case of a vanishing cosmological constant

\[ \Lambda = 0 \]  

(4.1)

Nonetheless, we expect a corresponding result to hold for \( \Lambda \neq 0 \) as well. This is to be analyzed elsewhere.

Let us assume we have been given a \( \Lambda = 0 \)-vacuum spacetime \((M, \eta)\) which admits a Killing vector field \(X(0)\). In this setting \((2.15)-(2.16)\) simplifies to

\[
\nabla^\rho S_{\alpha\beta\mu\rho} + Q(0) X^{(0)\sigma} f_{\mu\rho} S_{\alpha\beta\sigma\rho} - \frac{2}{3} Q(0) X^{(0)\sigma} f_{\mu\rho} f^{(0)\gamma\delta} S_{\gamma\delta\sigma\rho} = 0.
\]

(4.2)

We want to analyze under which conditions the linearized MST \(S^{(\text{lin})}_{\mu\nu\sigma\rho}\) of the perturbed metric \(g = \eta + h\) satisfies this equation, when \(h\) is some perturbation which satisfies the linearized vacuum equations

\[
\Box_h h_{\mu\nu} = -2 C_{\alpha\beta}^{(0)} h^{\alpha\beta}.
\]

(4.3)

As before we will impose the gauge condition \((3.5)\)

\[
\nabla_\beta h^{\alpha\beta} = \frac{1}{2} \nabla^{(0)} h^{\alpha\beta} h_{\alpha\beta}.
\]

(4.4)

To define the MST of the perturbed metric \(g\), it needs to admit a Killing vector field \(X\). In fact, to determine a linearization it is sufficient to have a vector field which satisfies the Killing equation up to and including the linear order in the perturbation \(h\). As before, we denote by \(X(0)\) the Killing vector of \((M, \eta)\). We write \(X\) as \(X = X(0) + X^{(\text{lin})} + O(h^2)\). Since

\[
\nabla_{(\mu} X_{\nu)} = \frac{1}{2} \mathcal{L}_{X(0)} h_{\mu\nu} + \nabla_{(\mu} X^{(\text{lin})}_{\nu)} + O(h^2)
\]

we need to assume that the perturbation \(\eta + h\) admits a vector field \(X^{(\text{lin})}\) such that

\[
\mathcal{L}_{X(0)} h_{\mu\nu} + 2 \nabla_{(\mu} X^{(\text{lin})}_{\nu)} = 0,
\]

(4.5)

which we assume henceforth.

A solution to (4.2) will be required to be self-dual w.r.t. the background \(\eta\), while the MST associated to \(g\) will be self-dual w.r.t. \(g\). In general these two notions only coincide in the leading order. In other words, the linearized MST of \(g\) will certainly be self-dual w.r.t. \(\eta\) if the MST associated to the background metric \(\eta\) vanishes (we have indicated in the formula w.r.t. which metric the Hodge dual is taken),

\[
S^{(0)} + S^{(\text{lin})} + O(h^2) = S = i S^* = \left(1 + \frac{1}{2} \operatorname{tr} h \right) i S^{(0)*\eta} + i S^{(\text{lin})*\eta} + O(h^2).
\]

For the remainder of this section we will thus assume

\[
S^{(0)}_{\mu\nu\sigma\rho} = 0.
\]

(4.7)

This is in accordance with the Bianchi equation where we needed to assume the vanishing of the Weyl tensor of the background metric in order to derive an analog result (note that the vanishing of the Weyl tensor is equivalent to the vanishing of the self-dual Weyl tensor).
Remark 4.1 \Lambda \text{-vacuum spacetimes with vanishing MST (4.7) have been classified in [28, 29]. This class includes in particular the Kerr-NUT-(\Lambda)(dS) family. So while, compared to the Bianchi equation, the class of admissible perturbations is restricted to those which preserve a symmetry, the class of admissible background spacetimes is much larger.}

Let us now determine the linearization of the MST

\[ S_{\mu\nu\sigma\rho} = C_{\mu\nu\sigma\rho} - Q \left( F_{\mu\nu} F_{\sigma\rho} - \frac{1}{3} F^2_{\mu\nu\sigma\rho} \right) . \]  

(4.8)

The linearized Weyl tensor has already been given in (3.13). We deduce that its self-dual counterpart satisfies (for \( \Lambda = 0 \))

\[ C_{\mu\nu\sigma\rho} = C_{\mu\nu\sigma\rho}^{(0)} - h_{[\mu} \eta^{(0)}_{\nu\sigma\rho]} - i(h_{\mu}^\gamma)_{(t)} \eta_{\nu\sigma\rho]}^{(0)} \kappa^{(0)}_{\mu\nu\gamma\delta} \]

\[ - 4 I_{\sigma\rho}^{(0)\gamma\delta} \nabla_{(0)}^{\gamma} \nabla_{(0)}^{(0)} h_{\mu\nu\delta} + O(h^2) , \]

(4.9)

where \( (\nu\mu)_{tt} \) denotes the \( \eta \)-trace-free part of the corresponding two-tensor. Moreover,

\[ \eta_{\mu\nu\sigma\rho} = \left( 1 + \frac{1}{2} (\nabla_{\rho} h) \right) \eta_{\mu\nu\sigma\rho}^{(0)} + O(h^2) , \]

(4.10)

\[ I_{\mu\nu\sigma\rho} = I_{\mu\nu\sigma\rho}^{(0)} + \frac{1}{2} h_{[\mu} h_{\nu\sigma\rho]} - \frac{1}{2} \eta_{[\mu} h_{\nu\sigma\rho]} + \frac{i}{8} \text{tr}_h \eta_{\mu\nu\sigma\rho}^{(0)} + O(h^2) , \]

(4.11)

and, employing (4.6),

\[ F_{\mu\nu} = F_{\mu\nu}^{(0)} - h_{[\mu} F_{\nu]\alpha}^{(0)} + X_{(0)\alpha}^{(0)} h_{[\nu] \alpha} + \nabla_{(0)\mu} (\nabla_{\nu} X)^{(0)} + O(h^2) , \]

(4.12)

\[ F_{\mu\nu} = F_{\mu\nu}^{(0)} - h_{[\mu} F_{\nu]\alpha}^{(0)} + 2 X_{(0)\alpha}^{(0)} h_{[\nu] \alpha} + G_{\mu\nu}^{(\text{lin})} + O(h^2) , \]

(4.13)

\[ F^2 = (F_{\mu\nu})^2 + 4 X_{(0)\alpha}^{(0)} h_{(\nu) \alpha} + 2 X_{(0)\mu}^{(0)} G_{\mu\nu}^{(\text{lin})} + O(h^2) , \]

(4.14)

where

\[ G_{\mu\nu}^{(\text{lin})} := (\nabla^{(0)}_{\mu} X)^{(0)} + i(\nabla^{(0)}_{\nu} X)^{(0)} + O(h) \]

(4.15)

denotes the self-dual two-form associated to \( X^{(\text{lin})} \).

It remains to determine the linearization of \( Q = Q^{(0)} + Q^{(\text{lin})} + O(h^2) \). Unfortunately, this term cannot be computed explicitly. It follows from the definition of \( Q \) (2.10) that

\[ \nabla_{\mu} Q = - \frac{1}{6} Q^2 \sigma_{\mu} = - \frac{1}{3} Q^2 X^\alpha F_{\alpha\mu} \]

(4.16)

\[ = - \frac{1}{6} (Q^{(0)})^2 \sigma_{\mu}^{(0)} - \frac{1}{3} (Q^{(0)})^2 \sigma^{(0)}_{\alpha\mu} + \frac{1}{3} (Q^{(0)})^2 X^{(0)\alpha} h_{\gamma[\alpha} F_{\beta]\gamma}^{(0)} \]

\[ - \frac{2}{3} (Q^{(0)})^2 X^{(0)\alpha} h_{[\gamma} F^{(0)}_{\alpha\beta]} X^{(0)\gamma} h_{\rho] \gamma} - \frac{1}{3} (Q^{(0)})^2 X^{(\text{lin})\alpha} F_{\alpha\mu}^{(0)} \]

\[ - \frac{1}{3} (Q^{(0)})^2 X^{(0)\alpha} G^{(\text{lin})}_{\alpha\mu} + O(h^2) , \]

(4.17)

whence \( Q^{(\text{lin})} \) satisfies the differential equation

\[ \nabla^{(0)}_{\mu} Q^{(\text{lin})} = - \frac{1}{3} (Q^{(0)})^2 Q^{(\text{lin})} \sigma_{\mu}^{(0)} + \frac{1}{3} (Q^{(0)})^2 X^{(0)\alpha} h_{\gamma[\alpha} F_{\beta]\gamma}^{(0)} \]

\[ - \frac{2}{3} (Q^{(0)})^2 X^{(0)\alpha} h_{[\gamma} F^{(0)}_{\alpha\beta]} X^{(0)\gamma} h_{\rho] \gamma} - \frac{1}{3} (Q^{(0)})^2 X^{(\text{lin})\alpha} F_{\alpha\mu}^{(0)} \]

\[ - \frac{1}{3} (Q^{(0)})^2 X^{(0)\alpha} G^{(\text{lin})}_{\alpha\mu} + O(h^2) . \]

(4.18)
From these expansions we determine the linearization of the MST. After some simplifications we find that

\[
S_{\mu\nu\sigma\rho} = -4T_{\mu\nu}^{(0)} T_{\sigma\rho}^{(0)} + Q_{(0)}^{(0)} U_{\sigma\rho\mu}^{(0)} \alpha h_{\nu}\alpha
\]

\[
-2Q_{(0)}^{(0)} F_{\mu\nu}^{(0)} T_{\sigma\rho}^{(0)} X^{(0)}_{\alpha} \nabla_{\alpha}^{(0)} h_{\beta\gamma} - 2Q_{(0)}^{(0)} F_{\mu\nu}^{(0)} T_{\sigma\rho}^{(0)} X_{\alpha}^{(0)} \nabla_{\alpha}^{(0)} h_{\beta\gamma}
\]

\[
+ \frac{4}{3} Q_{(0)}^{(0)} X_{(0)}^{(0)} \nabla_{\gamma}^{(0)} h_{\delta\gamma} T_{\mu\nu}^{(0)} + Q^{(0)} U_{\mu\nu\alpha}\rho 
\]

\[
- Q_{(0)}^{(0)} F_{\mu\nu}^{(0)} G_{\sigma\rho}^{(0)} - Q_{(0)}^{(0)} F_{\mu\nu}^{(0)} F_{\sigma\rho}^{(0)} + \frac{2}{3} Q_{(0)}^{(0)} F_{\alpha\beta}^{(0)} G^{(lin)\alpha\beta} T_{\mu\nu\alpha\beta}^{(0)} + O(h^2), \quad (4.19)
\]

and \(Q^{(0)}\) is given by (4.18). With regard to (4.2) we also compute the following contraction,

\[
F_{\sigma\rho}^{(0)} S_{\mu\nu\sigma\rho} = -4F_{\mu\nu}^{(0)} \nabla_{\mu}^{(0)} \nabla_{\nu}^{(0)} h_{\nu\lambda} - \frac{2}{3} Q_{(0)}^{(0)} F_{\mu\nu}^{(0)} h_{\nu\lambda}
\]

\[
-2Q_{(0)}^{(0)} (F_{\mu\nu}^{(0)})^2 T_{\mu\nu}^{(0)} X_{\alpha}^{(0)} \nabla_{\alpha}^{(0)} h_{\beta\gamma} - \frac{2}{3} Q^{(lin)} (F_{\mu\nu}^{(0)})^2 F_{\mu\nu}^{(0)}
\]

\[
- Q_{(0)}^{(0)} (F_{\mu\nu}^{(0)})^2 G_{\mu\nu}^{(lin)} - \frac{1}{3} Q_{(0)}^{(0)} F_{\alpha\beta}^{(0)} G^{(lin)\alpha\beta} F_{\mu\nu}^{(0)} + O(h^2), \quad (4.20)
\]

where we employed one more time the vanishing of the MST of the background metric.

Before we proceed, let us collect a couple of useful relations satisfied by \(F_{\mu\nu}^{(0)}\) in any \(\Lambda = 0\)-vacuum spacetime \((M, \eta, X^{(0)})\) with vanishing MST [28]

\[
\nabla_{\kappa}^{(0)} F_{\alpha\beta}^{(0)} = \frac{1}{2} Q_{(0)}^{(0)} \nabla_{\kappa}^{(0)} T_{\alpha\beta}^{(0)} + \frac{1}{3} Q_{(0)}^{(0)} (F_{\mu\nu}^{(0)})^2 X_{\alpha\beta}^{(0)} F_{\kappa\lambda}^{(0)}, \quad (4.21)
\]

\[
\nabla_{\eta}^{(0)} F_{\alpha}^{(0)} = 0, \quad (4.22)
\]

\[
\nabla_{\mu}^{(0)} (F_{\mu\nu}^{(0)})^2 = \frac{2}{3} Q_{(0)}^{(0)} (F_{\mu\nu}^{(0)})^2 \sigma_{\mu}^{(0)}, \quad (4.23)
\]

\[
\sigma_{\mu}^{(0)} F_{\mu\nu}^{(0)} = -\frac{1}{2} (F_{\mu\nu}^{(0)})^2 \eta_{\mu}^{\alpha\beta} \eta_{\nu}^{\gamma\delta}, \quad (4.24)
\]

\[
\nabla_{\kappa}^{(0)} Q_{\mu\nu}^{(0)} = -\frac{1}{6} (Q_{\mu\nu}^{(0)})^2 \eta_{\kappa}^{\alpha\beta}, \quad (4.25)
\]

Let \(V_{\alpha\beta}\) be an antisymmetric tensor, \(V_{\alpha\beta} = V_{\alpha\beta} + iV_{\alpha\beta}^*\) its self-dual counterpart, and \(W_{\alpha\beta}\) self-dual, then (using the properties of the Levi-Civita symbol)

\[
W_{\gamma\alpha} V_{\beta\gamma} = W_{\gamma\alpha} V_{\beta\gamma} + \frac{i}{2} W_{\gamma\alpha} \eta_{\beta\gamma} = W_{\gamma\alpha} V_{\beta\gamma} - \frac{1}{4} \eta_{\gamma\alpha} \eta_{\beta\gamma} W_{\alpha\beta} V_{\delta\rho} = 2W_{\gamma\alpha} V_{\beta\gamma}. \quad (4.26)
\]

From (4.6) we deduce

\[
0 = \nabla_{\alpha}^{(0)} (L_{X^{(0)} h_{\beta\gamma}}^{(0)} + \nabla_{\beta}^{(0)} X_{\rho}^{(lin)} + \nabla_{\rho}^{(0)} X^{(lin)})
\]

\[
= \nabla_{\alpha}^{(0)} X^{(0)} \nabla_{\alpha}^{\gamma} h_{\beta\gamma} + \nabla_{\beta}^{(0)} X^{(0)} \nabla_{\beta}^{\gamma} h_{\alpha\rho} - \nabla_{\rho}^{(0)} X^{(0)} \nabla_{\rho}^{\gamma} h_{\alpha\gamma} - \nabla_{\beta}^{(0)} X^{(0)} \nabla_{\beta}^{\gamma} h_{\rho\gamma}
\]

\[
+ X^{(0)} \nabla_{\alpha}^{(0)} h_{\beta\rho} + C^{(0)} \nabla_{\alpha}^{(0)} + \nabla_{\rho}^{(0)} X^{(0)} \nabla_{\alpha}^{(0)} X_{\beta\gamma}^{(lin)} + \nabla_{\beta}^{(0)} X^{(0)} \nabla_{\beta}^{(lin)}, \quad (4.27)
\]
whence

\[ \mathcal{I}_{\mu}^{(0)\alpha\beta}(\partial^{(0)}\sigma)\rho_{\rho',\gamma}X^{(0)\gamma}\gamma_{\alpha}|_{\beta}h_{\gamma\beta} = -\frac{1}{2}F^{(0)\alpha\beta}_{\gamma}(X^{(0)\gamma})\rho_{\alpha\beta} + \frac{1}{8}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})tr_{\gamma}h \]

\[ + \frac{1}{2}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} \]

(4.28)

\[ \mathcal{I}_{\alpha}^{(0)\alpha\beta}\rho_{\rho',\gamma}X^{(0)\gamma}\gamma_{\alpha}|_{\beta}h_{\gamma\beta} = -\frac{1}{2}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} + \frac{1}{8}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})tr_{\gamma}h \]

(4.29)

\[ \mathcal{I}_{\mu}^{(0)\alpha\beta}\rho_{\rho',\gamma}X^{(0)\gamma}\gamma_{\alpha}|_{\beta}h_{\gamma\beta} = -\frac{1}{2}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} - \frac{1}{2}F^{(0)\alpha\beta}_{\rho}(X^{(0)\rho})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} \]

(4.30)

Finally, we need to determine the linearization of the divergence of the MST. First of all we deduce from the linearized Einstein equations (3.11) and the gauge condition (3.5)

\[ \nabla^{(0)}_{\rho}(\mathcal{I}_{\rho}^{(0)\alpha\beta}\rho_{\sigma\tau})(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} = -\frac{1}{2}h^{(0)\alpha\beta}\nabla^{(0)}_{\rho}(\mathcal{I}_{\rho}^{(0)\alpha\beta}\rho_{\sigma\tau})(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} + \frac{1}{8}h^{(0)\alpha\beta}\nabla^{(0)}_{\rho}(\mathcal{I}_{\rho}^{(0)\alpha\beta}\rho_{\sigma\tau})(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} \]

(4.31)

Then a rather lengthy calculation which employs (3.11), (3.5), the vanishing of the background MST (to express the self-dual-Weyl tensor in terms of \( F_{\alpha\beta} \)) as well as the relations (4.21)-(4.25) and (4.28)-(4.30) reveals that

\[ \nabla^{(0)}_{\rho}(\mathcal{I}_{\rho}^{(0)\alpha\beta}\rho_{\sigma\tau})(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} = \frac{1}{2}Q^{(0)}(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} + \frac{1}{8}Q^{(0)}(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} + \frac{1}{8}Q^{(0)}(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} + \frac{1}{8}Q^{(0)}(X^{(0)\gamma})\gamma_{\alpha}|_{\beta}h_{\gamma\beta} \]

(4.32)

Let us consider the term in brackets in the first two lines somewhat more carefully. The tensor

\[ \Xi_{\alpha\beta\mu\nu} := \nabla^{(0)}_{\mu}X^{(0)\gamma}|_{\beta}h_{\gamma\beta} = \frac{1}{2}h^{(0)\alpha\beta}\mathcal{C}_{\alpha\beta\mu\nu} \]

(4.33)
is manifestly antisymmetric in its first and last pair of indices. One checks that it is also trace-free,
\[ \Xi_{\alpha\beta\mu} = \frac{1}{4} \nabla^{(0)}_{\alpha} \nabla^{(0)}_{\mu} \text{tr}_{\h} h - \frac{1}{2} \nabla^{(0)}_{\alpha} \nabla^{(0)}_{[\mu} \eta_{\nu]} \rho' + \frac{1}{4} \Box_{\eta} h_{\alpha\mu} + \frac{1}{2} h^{\alpha\beta} C^{(0)}_{\alpha\kappa\beta} = 0 \]
by the linearized field equations (3.11) and the gauge condition (3.5). This implies that
\[ \eta^{(0)}_{\alpha\beta} \gamma^\delta \Xi_{\mu\nu\gamma\delta} = \eta_{\mu
u}^{(0)} \Xi_{\alpha\beta\gamma\delta}. \]
To see this, we observe that, applying \( \eta^{(0)}_{\mu
u\kappa\rho}, \) (4.35) is equivalent to \( \eta^{(0)}_{\mu\nu\kappa\rho} \gamma_{\alpha\beta\gamma\delta} = 4 \Xi_{\alpha\beta\mu\nu}, \)
in which it follows from the tracelessness of \( \Xi_{\alpha\beta\mu\nu}, \)
\[ \eta^{(0)}_{\mu
u\kappa\rho} \gamma_{\alpha\beta\gamma\delta} = 24 \delta_{\alpha\beta} \delta_{\gamma\delta} \Xi_{\mu\nu\gamma\delta} = 24 \delta_{\alpha\beta} \Xi_{\mu\nu\gamma\delta} = 4 \Xi_{\alpha\beta\kappa\rho}. \]
It follows that the term in brackets vanishes
\[
\eta_{\mu\nu}^{(0)} \Xi_{\alpha\beta\gamma\delta} = \gamma_{\alpha\beta} \Xi_{\mu\nu\gamma\delta} = 2 \delta_{\alpha\beta} \Xi_{\mu\nu\gamma\delta} = 2 \Xi_{\alpha\beta\mu\nu}. \]
Finally, from (4.19)-(4.20) we compute
\[
X^{(0)}_{\kappa} \left( \nabla^{(0)}_{\sigma} \eta^{(0)}_{\mu\nu\kappa\rho} - \frac{2}{3} \eta^{(0)}_{\mu\nu\kappa\rho} \Xi^{(0)}_{\sigma\gamma\delta} \right) S_{\gamma\delta\kappa\rho} = \frac{Q^{(lin)}}{6} \nabla^{(0)}_{\kappa} \left( \nabla^{(0)}_{\sigma} \eta^{(0)}_{\mu\nu\kappa\rho} + \frac{4}{9} (\nabla^{(0)}_{\sigma} \eta^{(0)}_{\mu\nu\kappa\rho})^{2} \nabla^{(0)}_{\tau\rho} \eta^{(0)}_{\kappa\sigma\rho\tau} - 4 X^{(0)}_{\kappa} \nabla^{(0)}_{\sigma} \eta^{(0)}_{\mu\nu\kappa\rho} \nabla^{(0)}_{\tau\rho} \eta^{(0)}_{\kappa\sigma\rho\tau} \right) \Xi^{(0)}_{\mu\nu\kappa\rho} \nabla^{(0)}_{\sigma} \eta^{(0)}_{\mu\nu\kappa\rho}
\end{equation}
and one readily checks that
\[ \nabla^{(0)}_{\rho} S_{\alpha\beta\mu\nu} + Q^{(0)} X^{(0)}_{\sigma} \left( \nabla^{(0)}_{\rho} \eta^{(0)}_{\mu\nu\kappa\rho} \nabla^{(0)}_{\tau\rho} \eta^{(0)}_{\kappa\sigma\rho\tau} \right) S_{\sigma\delta\rho\tau} = O(h^{2}). \]
We have proved:

**Proposition 4.2** Let \((M, \eta, X^{(0)})\) be a \( \Lambda = 0 \)-vacuum spacetime with a Killing vector \( X^{(0)} \) such that the associated MST vanishes. Let \( h \) be a vacuum perturbation of \( \eta \) such that \( \eta + h \) admits a Killing vector which is a perturbation of \( X^{(0)} \), i.e. which satisfies \( \mathcal{L}_{X^{(0)}} h_{\mu\nu} + 2 \nabla^{(0)}_{\rho} X^{(0)}_{\rho} = 0 \) for some perturbation \( X^{(0)} \) of \( X^{(0)} \). Then the linearized MST associated to \((M, \eta + h, X^{(0)} + X^{(0)})\) satisfies the MST equation (4.2) on the background \((M, \eta, X^{(0)})\).
Remark 4.3 As indicated above, one should expect this result to hold for any sign of the cosmological constant.

Remark 4.4 We have not attempted to derive an analog of Lemma 3.2. Nevertheless, we expect that a solution $S_{\alpha\beta\mu\nu}$ to the MST equation on a vacuum background $(M, \eta, X^{(0)})$ with vanishing MST defines a unique (up to gauge transformations (3.4)) perturbation $h$ of $\eta$ and $X^{(\text{lin})}$ of $X^{(0)}$ such that the linearized vacuum equations (3.11) and $\mathcal{L}_{X^{(0)}} h_{\mu\nu} + 2\nabla_{(\mu}^0 X_{\nu)}^{(\text{lin})} = 0$ are fulfilled, and such that $S_{\alpha\beta\mu\nu}$ is the linearized MST of $(M, \eta + h, X^{(0)} + X^{(\text{lin})})$. Note, though, that in Lemma 3.2 a vanishing cosmological constant has been assumed, so before considering this issue for the MST with $\Lambda \neq 0$, the corresponding issue for the Bianchi equation should be analyzed.

4.2 Constraint and evolution equations

Let $\Sigma \subset M$ be a Cauchy surface in some $\Lambda$-vacuum spacetime $(M, g)$ which admits a Killing vector $X$. Later on, when we work in a conformally rescaled spacetime $\tilde{g} = \Theta^2 g$ we will introduce Gaussian coordinates, at this stage, though, we merely assume coordinates $(t \equiv x^0, x^1)$ near $\Sigma$ where

$$g_{0i} = 0.$$  

Remark 4.5 In Section 3.1 we have denoted the background spacetime by $(M, \eta, X^{(0)})$, and the associated fields have been decorated with superscript $(0)$. Since from now on the background spacetime will be the only one we are working with, we will simply denote it by $(M, g, X)$ henceforth without any superscripts marking the associated fields. However, we will still denote any solution of the MST equation (2.15) by $S_{\alpha\beta\mu\nu}$. The MST associated to the spacetime $(M, g, X)$ will be denoted by $\mathcal{S}_{\alpha\beta\mu\nu}$.

It follows readily from the algebraic symmetries of $\mathcal{S}_{\alpha\beta\mu\nu}$ and its self-duality that the MST equation is equivalent to the system (cf. also Remark 2.2),

$$\nabla_\rho S_{\rho 0(12)} = J(S)_{0(12)},$$  

(4.39)

$$\nabla_\rho S_{000} = J(S)_{000}.$$  

(4.40)

As for the Bianchi equation one shows that (4.39) forms a regular symmetric hyperbolic system of evolution equations (supposing that there are no blow-ups in the denominator of (2.16)), for which standard well-posedness results are available, cf. [27]. Equation (4.40) does not contain transverse derivatives and therefore provides a set of constraint equations.

4.3 Generalized Buchdahl condition

Let us devote attention to the issue whether the constraint equations (4.40) are preserved under evolution. The main aim of this section is to derive an analog of the Buchdahl condition (3.20).

Employing the algebraic Weyl symmetries of the fields involved as well as self-duality we find
(the vacuum equations are not needed at this stage)

\[
\nabla_0 \nabla_\rho S_{000} = -g_{00} \nabla^j \nabla_0 S_{0ij} + R_{0j0} \nabla_i S_{000} - R_{0j} S_{000}^j
\]

\[
= -g_{00} \nabla^j \nabla_\rho S_{0ij} + g_{00} \nabla_j \nabla_i S_{000} + R_{0j0} \nabla_i S_{000} - R_{0j} S_{000}^j
\]

\[
- R_{0j} S_{000}^j
\]

\[
= -g_{00} \nabla^j \nabla_\rho S_{0ij} + g_{00} \nabla_i \nabla_\rho S_{000} + R_{0j0} \nabla_\rho S_{000} - R_{0j} S_{000}^j
\]

\[
+ \frac{1}{2} g_{00} R_{0j0} \nabla_i S_{000}^j + R_{0j0} \nabla_\rho S_{000} - R_{0j} S_{000}^j
\]

\[
= -g_{00} \nabla^j \nabla_\rho S_{0ij} + g_{00} \nabla_i \nabla_\rho S_{000} + R_{0j0} \nabla_\rho S_{000} - R_{0j} S_{000}^j
\]

\[
+ \frac{1}{2} g_{00} C_{jkl} \nabla_\rho S_{000}^j + \frac{1}{2} g_{00} C_{jkl} \nabla_\rho S_{000} - C_{0j0} \nabla_\rho S_{000} + C_{0j0} \nabla_\rho S_{000}^j
\]

\[
+ 2 C_{jkl} \nabla_\rho S_{000}^j.
\]

Here \(\nabla\) denotes the Levi-Civita connection associated to the Riemannian family \(t \mapsto g_{ij}(t, x^k)\).

If we assume that the evolution equations

\[
\nabla_\rho S_{0ij} = J(\mathcal{S})_{0ij}
\]

are fulfilled we derive from that formula an equation which is satisfied by the constraint violation operator

\[
\Xi_i := \nabla_\rho S_{000} - J(\mathcal{S})_{000}.
\]

We obtain

\[
\left(\delta^k_i (\partial_0 + \frac{3}{2} \Gamma^j_{00} - 2 \Gamma^j_{00}) - \frac{1}{2} \eta_{ilk} (\nabla_i j + 2 \Gamma^0_{ij}) - \frac{3}{2} \Gamma^h_{00}\right) \Xi_j = -g_{00} \nabla^j J(\mathcal{S})_{000} + 2 C_{jkl} \nabla_\rho S_{000}^j. \quad (4.43)
\]

Before we proceed let us compare this with the Bianchi case where \(J(\mathcal{S})\) vanishes. Then the symmetric hyperbolic system

\[
\left(\delta^k_i (\partial_0 + \frac{3}{2} \Gamma^j_{00} - 2 \Gamma^j_{00}) - \frac{1}{2} \eta_{ilk} (\nabla_i j + 2 \Gamma^0_{ij}) - \frac{3}{2} \Gamma^h_{00}\right) \nabla_\rho S_{000} = 2 C_{jkl} \nabla_\rho S_{000}^j.
\]

ensures that the constraints are preserved, i.e. zero data yield the zero-solution, if and only if

\[
C_{jkl} \nabla_\rho S_{000}^j = 0 \iff C_{0j0} \nabla_\rho S_{000} = 0.
\]

This recovers the Buchdahl condition (3.20) in self-dual language.

### 4.3.1 Divergence of \(J(\mathcal{S})\)

In order to analyze the general case, we need to determine \(\nabla^\mu J(\mathcal{S})_{0\mu}\) in (4.43). This requires some computational effort. It is convenient to collect some relations needed for this computation first.

Any self dual two form \(V_{\alpha\beta}\) satisfies

\[
V_{\mu\alpha} V_{\nu}^\alpha = \frac{1}{4} \gamma^2 g_{\mu\nu}.
\]

(4.45)
In any $\Lambda$-vacuum space-time a self-dual Killing form fulfills the following relations [28],

$$\nabla_{\mu} F_{\alpha \beta} = -X^\nu \left( C_{\mu \nu \alpha \beta} + \frac{4}{3} \Lambda L_{\mu \nu \alpha \beta} \right)$$ \hspace{1cm} (4.46)

$$= -X^\nu \left( \bar{S}_{\mu \nu \alpha \beta} + Q F_{\mu \nu} F_{\alpha \beta} - \frac{1}{3} (Q F^2 - 4\Lambda) L_{\mu \nu \alpha \beta} \right),$$ \hspace{1cm} (4.47)

$$\nabla_{\mu} F_{\alpha}{}^\mu = \Lambda X_\alpha,$$ \hspace{1cm} (4.48)

$$\nabla_{\mu} F^2 = -2X^\nu \left( F_{\alpha \beta} C_{\mu \nu \alpha \beta} + \frac{4}{3} \Lambda F_{\mu \nu} \right)$$ \hspace{1cm} (4.49)

$$= \frac{4}{3} (Q F^2 + 2\Lambda) X^\nu F_{\mu \nu} - 2X^\nu F_{\alpha \beta} \bar{S}_{\mu \nu \alpha \beta}.$$

(4.50)

We need to determine the gradient of $Q$. For this recall the definition (2.10)-(2.12) of $Q$. In particular it implies

$$R^2 = -\frac{1}{4} F^2,$$ \hspace{1cm} (4.51)

whence

$$\nabla_{\mu} (Q F^2) = -12 \nabla_{\mu} (JR)$$

$$= -6 \left( \frac{J}{R} + \frac{1}{\sigma} + \frac{R}{(J \sigma - R)} \right) \nabla_{\mu} R^2 + 6 \frac{R}{\sigma} \left( \frac{\Lambda}{J \sigma - R} + 2J \right) \nabla_{\mu} \sigma$$

$$= \frac{3 J^2 \sigma}{2 R (J \sigma - R)} \nabla_{\mu} F^2 + 12 \frac{J^2 R}{J \sigma - R} X^\alpha F_{\alpha \mu}$$

(4.50)

$$\equiv (Q F^2 - 4\Lambda) X^\alpha F_{\alpha \mu} - 2 \left( \frac{Q F^2 + 2\Lambda}{Q F^2 + 8\Lambda} \right) F_{\mu \nu} F^{\alpha \beta} \bar{S}_{\mu \nu \alpha \beta}. \hspace{1cm} (4.52)$$

Finally, using all these relations, we compute the divergence of $J(S)$. A somewhat lengthy calculation reveals that

$$\nabla^\mu J(S)_{\alpha \beta \mu} = -\frac{4}{3} \Lambda (Q F^2 - 4\Lambda) \frac{5Q F^2 + 4\Lambda}{Q F^2 + 8\Lambda} F^{-4} F_{\alpha \beta \mu \nu} X^\sigma X^\gamma \nabla_{\mu} S_{\rho \sigma \gamma \delta}$$

$$-4\Lambda \frac{5Q F^2 + 4\Lambda}{Q F^2 + 8\Lambda} F_{\alpha \beta \mu \nu} F^{-4} X^\gamma \nabla_{\mu} S_{\rho \sigma \gamma \delta}$$

$$+ \frac{2}{3} \frac{Q F^2 - 2\Lambda}{Q F^2 + 8\Lambda} \left( \frac{Q F^2 + 4\Lambda}{Q F^2 + 8\Lambda} \right) X^\sigma X^\gamma \nabla_{\mu} S_{\rho \sigma \gamma \delta}$$

$$+ \frac{2}{3} \frac{Q F^2 - 2\Lambda}{Q F^2 + 8\Lambda} \left( \frac{Q F^2 + 4\Lambda}{Q F^2 + 8\Lambda} \right) X^\sigma X^\gamma \nabla_{\mu} S_{\rho \sigma \gamma \delta}$$

$$+ \frac{2}{3} \frac{Q F^2 - 2\Lambda}{Q F^2 + 8\Lambda} \left( \frac{Q F^2 + 4\Lambda}{Q F^2 + 8\Lambda} \right) I_{\alpha \beta \mu \nu} X^\gamma \nabla_{\mu} S_{\rho \sigma \gamma \delta}$$

$$+ Q X^\sigma F_{\mu \rho} \nabla_{\mu} S_{\rho \sigma \gamma \delta} + f_{\alpha \beta \mu \nu} S_{\mu \nu \rho \sigma},$$

where the precise form of the generic tensor field $f_{\alpha \beta \mu \nu}$, which depends on the metric $g$, the Killing vector field $X$ and $S_{\alpha \beta \mu \nu}$ will be irrelevant for our purposes.

The right-hand side contains derivatives of $S_{\mu \nu \rho \sigma}$. We observe that they can be expressed as a linear combination of derivatives of the form $X^\alpha \nabla_{\alpha} S_{\alpha \beta \mu \nu}$ and $\nabla_{\mu} S_{\rho \sigma \gamma \delta}$. It follows from the algebraic symmetries of $S_{\mu \nu \rho \sigma}$ that

$$\nabla_{[\mu} S_{\rho] \sigma \alpha \beta} = \frac{i}{3} \eta_{\rho \sigma \gamma \delta} \nabla_{\gamma} S_{\alpha \beta \mu \nu}.$$ \hspace{1cm} (4.53)
Taking this into account we obtain
\[
\nabla^\mu J(S)_{\alpha\beta\mu} = \frac{4}{3} \Lambda (Q F^2 - 4 \Lambda) \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} \mathcal{I}_{\alpha\beta\mu} X^\nu X^\sigma F^{-\delta} F_\rho^\mu S_{\gamma\delta\rho} \\
+ 4A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{\alpha\beta} X^\sigma F_\sigma^\kappa F^{-\delta} \nabla \lambda S_{\gamma\delta\kappa} \\
+ \frac{2}{9} F^{-2} (Q F^2 + 8\Lambda) \left( Q F^2 - 2\Lambda \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} \right) X^\nu X^\sigma F^{-\delta} F_{\nu[\alpha} S_{\beta]\gamma\delta} \\
- \frac{2}{3} F^{-2} Q F^2 - 4A \left( \frac{Q F^2}{Q F^2 + 8\Lambda} - \Lambda \frac{Q F^2}{Q F^2 + 8\Lambda} \right) X^\nu X^\sigma F^{-\delta} F_{\nu[\alpha} S_{\beta]\gamma\delta} \\
+ \frac{2}{3} F^{-2} (Q F^2 - 2\Lambda \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F_{\alpha\beta} X^\kappa F^{-\delta} \nabla \lambda S_{\gamma\delta\kappa} \\
- Q X^\sigma F_\sigma^\kappa \nabla \lambda S_{\alpha\beta\kappa} + 2A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{\alpha\beta} F^\mu_\rho F^{-\delta} \nabla \lambda S_{\mu\rho\gamma\delta} \\
- \frac{1}{6} F^{-2} (Q F^2 + 4A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F^{-\delta} \nabla \lambda S_{\alpha\beta\gamma\delta} - \frac{1}{2} Q F^{-\delta} F_{\nu[\alpha} S_{\beta]\gamma\delta} + f_{\alpha\beta}^{\mu\nu\sigma\rho} \Xi_{\mu
u\sigma\rho}.
\]

Rewriting yields after another tedious computation
\[
\nabla^\mu J(S)_{\alpha\beta\mu} = 4\Lambda \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{\alpha\beta} X^\sigma F_\sigma^\kappa F^{-\delta} (\nabla \lambda S_{\gamma\delta\kappa} - J(S)_{\gamma\delta\kappa}) \\
+ \frac{2}{3} F^{-2} (Q F^2 - 2\Lambda \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F_{\alpha\beta} X^\kappa F^{-\delta} (\nabla \lambda S_{\gamma\delta\kappa} - J(S)_{\gamma\delta\kappa}) \\
- Q X^\sigma F_\sigma^\kappa (\nabla \lambda S_{\alpha\beta\kappa} - J(S)_{\alpha\beta\kappa}) + 2A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{\alpha\beta} F^\mu_\rho F^{-\delta} \nabla \lambda S_{\mu\rho\gamma\delta} \\
- \frac{1}{6} (Q F^2 + 4A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F^{-2} F^{-\delta} \nabla \lambda S_{\alpha\beta\gamma\delta} \\
+ \frac{1}{2} Q H_{\alpha\beta\sigma\gamma\delta} + f_{\alpha\beta}^{\mu\nu\sigma\rho} \Xi_{\mu
u\sigma\rho}.
\]

For this step we used the following relation, which follows from the self-duality of the fields involved,
\[
\mathcal{I}_{\alpha\beta\mu} X^\nu X^\sigma F_\rho^\mu F^{-\delta} S_{\gamma\delta\rho} = X^\nu X^\sigma F^{-\delta} F_{\nu[\alpha} S_{\beta]\gamma\delta}.
\]

Altogether we have shown that (4.43) can be written as
\[
\delta^k (\partial_0 + \frac{3}{2} \gamma^0 j_0 - 2\Gamma^0_{00}) - \frac{i}{2} \eta_0 j^k (\nabla j_0 + 2\Gamma_{0j}^0) - \frac{3}{2} \Gamma^k_0 k
\]
\[
= -g_0 \left[ 4A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{0i} X^\sigma F_\sigma^\kappa F^{-\delta} (\nabla \lambda S_{\gamma\delta\kappa} - J(S)_{\gamma\delta\kappa}) \\
+ \frac{2}{3} F^{-2} (Q F^2 - 2\Lambda \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F_{0i} X^\kappa F^{-\delta} (\nabla \lambda S_{\gamma\delta\kappa} - J(S)_{\gamma\delta\kappa}) \\
- Q X^\sigma F_\sigma^\kappa (\nabla \lambda S_{0i\kappa} - J(S)_{0i\kappa}) + 2A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} F^{-4} F_{0i} F^\mu_\rho F^{-\delta} \nabla \lambda S_{\mu\rho\gamma\delta} \\
- \frac{1}{6} (Q F^2 + 4A \frac{5Q F^2 + 4A}{Q F^2 + 8\Lambda} ) F^{-2} F^{-\delta} \nabla \lambda S_{0i\gamma\delta} + f_{0i}^{\mu\nu\sigma\rho} \Xi_{\mu
u\sigma\rho}.\]
Let us plug in the evolution equations $\nabla_\rho S_{\alpha(ij)}{^\rho} = \mathcal{J}(S)_{\alpha(ij)}$.

\[
\left[\delta^p_j(\partial_0 + \frac{3}{2} \Gamma^j_0) - 2 \Gamma^0_{00} + \frac{i}{2} \eta_{ij} X^j + 2 \Gamma^0_{0j} - \frac{3}{2} \Gamma^0_{0j}
\right]
\n+ Q \left( X^j F_{00} \delta^p_j + \frac{i}{2} \eta_{ij} X^j F_{00} \right)
\n- 8 \Lambda \eta_{00} F^{-4} \left[ 5 Q F^2 + 4 \Lambda \right] F_{00} \left( 2 X^j F_{0j} F_{00} + i \eta_{00} X^p F_{0p} F_{00} \right)
\n- \frac{4}{3} \eta_{00} F^{-2} \left( Q F^2 - 2 \Lambda \right) \left( 2 \partial_0 \partial_j X^j F_{00} + i \eta_{00} F_{0j} X^p F_{00} \right)
\n= - g_{00} \left[ 32 \Lambda \frac{5 Q F^2 + 4 \Lambda}{Q F^2 + 8 \Lambda} F^{-2} F_{00} F_{00} + \frac{2}{3} \left( Q F^2 + 4 \Lambda \frac{5 Q F^2 + 4 \Lambda}{Q F^2 + 8 \Lambda} \right) \delta^k \right] F^{-2} F_{0j} \mathcal{L}_X S_{0j0k}
\n+ f_{0\mu} \eta_{\mu\sigma} \mathcal{E}_{\mu\nu\sigma}.
\]

Employing one more time the self-duality of the fields involved this can be written as

\[
\left[\delta^p_j(\partial_0 + \frac{3}{2} \Gamma^j_0) - 2 \Gamma^0_{00} + \frac{i}{2} \eta_{ij} X^j + 2 \Gamma^0_{0j} - \frac{3}{2} \Gamma^0_{0j}\right]
\n+ Q \left( X^j F_{00} \delta^p_j + \frac{i}{2} \eta_{ij} X^j F_{00} \right)
\n- 8 \Lambda \eta_{00} F^{-4} \left[ 5 Q F^2 + 4 \Lambda \right] F_{00} \left( 2 X^j F_{0j} F_{00} + i \eta_{00} X^p F_{0p} F_{00} \right)
\n- \frac{4}{3} \eta_{00} F^{-2} \left( Q F^2 - 2 \Lambda \right) \left( 2 \partial_0 \partial_j X^j F_{00} + i \eta_{00} F_{0j} X^p F_{00} \right)
\n= - g_{00} \left[ 32 \Lambda \frac{5 Q F^2 + 4 \Lambda}{Q F^2 + 8 \Lambda} F^{-2} F_{00} F_{00} + \frac{2}{3} \left( Q F^2 + 4 \Lambda \frac{5 Q F^2 + 4 \Lambda}{Q F^2 + 8 \Lambda} \right) \delta^k \right] F^{-2} F_{0j} \mathcal{L}_X S_{0j0k}
\n+ f_{0\mu} \eta_{\mu\sigma} \mathcal{E}_{\mu\nu\sigma}.
\]

### 4.3.2 Generalized Buchdahl condition and its realization

We deduce from (4.54) that the analog of the Buchdahl condition for the MST equation (2.15), which is necessary for the preservation of the constraints under evolution, adopts the form

\[
f_{0j} F_{0k} \mathcal{L}_X S_{0j0k} + f_{0\mu} \eta_{\mu\sigma} \mathcal{E}_{\mu\nu\sigma} = 0.
\]

Since (4.54) is symmetric hyperbolic – its principal part has the same structure as the principal part of the system discussed in the specific case in Section 6.5 – the validity of the Buchdahl condition is also sufficient for the preservation of the constraints. Similar to how one proceeds to construct solutions to the Bianchi equation, let us assume that the MST of the background metric vanishes,

\[
\mathcal{E}_{\mu\nu\sigma} = 0.
\]

(4.56)

(It also follows from the considerations in Section 4.1 and the relevance of spacetimes with vanishing MST that this is a reasonable ansatz to realize (4.55).)

In contrast to the Bianchi equation, though, this is not sufficient to satisfy the generalized Buchdahl condition (4.55), whence we assume, in addition to (4.56), that

\[
\mathcal{L}_X S_{\alpha\beta\mu\nu} = 0.
\]

(4.57)

Recall that the Lie derivative of the MST always vanishes, $\mathcal{L}_X S_{\alpha\beta\mu\nu} = 0$. Having the expectation in mind (cf. Remark 4.4) that a solution to the MST equation on a given background with vanishing MST can be interpreted as the linearized MST of some perturbed metric, (4.57) seems to be a very natural condition.
In contrast to (4.56), though, condition (4.57) involves the solution and not just the background spacetime. Fortunately, this spacetime condition can be realized by an appropriate choice of the initial data as will be discussed next.

Let us consider again (2.15). Note that it follows from (4.46) and (4.52) that
\[
\mathcal{L}_X F_{\alpha\beta} = 0, \quad \mathcal{L}_X (QF^2) = 0,
\]
whence the Lie derivative w.r.t. the Killing vector $X$ of the coefficients in (2.15) vanishes. That implies that once we have solved the evolution equations, the relation
\[
\nabla_\rho \mathcal{L}_X S_{0(1j)}{}^\rho = \mathcal{L}_X \nabla_\rho S_{0(1j)}{}^\rho = J(\mathcal{L}_X S)_{0(1j)}
\]
holds automatically, i.e. the Lie derivative satisfies an identical system of equations. The condition (4.57) can therefore be realized by an appropriate choice of the initial data, namely
\[
\mathcal{L}_X S_{\alpha\beta\mu\nu}|_\Sigma = 0.
\]

Remark 4.6 In fact, in order to fulfill the Buchdahl condition it suffices if, in addition to (4.56), certain components of the Lie derivative vanish, namely
\[
F^k_0 L_X S_{0j0k} = 0 \iff \mathcal{L}_X (F^{\mu\nu} S_{\alpha\beta\mu\nu}) = 0
\]
holds. However, one would need to make sure that (4.61) follows from the evolution equations for appropriately chosen initial data sets, and it does not seem to be possible to derive a homogeneous system of equations for $\mathcal{L}_X (F^{\mu\nu} S_{\alpha\beta\mu\nu})$.

Let us consider a Cauchy problem for (2.15) in a $\Lambda$-vacuum spacetime $(M, g, X)$ with vanishing MST and which satisfies (2.14) (at least in some neighborhood of the Cauchy surface). Then the evolution equations (4.39) form a regular symmetric hyperbolic system for which well-posedness results are available.

The Cauchy problem is therefore reduced to the issue to construct initial data sets which satisfy both the constraint equations and our requirement that the Lie derivative of $S$ vanishes,
\[
\mathcal{L}_X S_{00|j}|_\Sigma = 0, \quad \nabla_j S_{00j} - J(S)_{00|j}|_\Sigma = 0.
\]

In this paper we are particularly interested in the construction solutions from a spacelike $\mathcal{I}$, so we will not attempt to solve this system here. Instead we will consider its analog on a spacelike $\mathcal{I}$ in more detail, where it is in fact simpler since e.g. (4.62) is always an inner equation on $\mathcal{I}$ (otherwise one would have to eliminate the transverse derivative via the evolution equations).

5 Bianchi-like equation in a conformally rescaled spacetime

5.1 Conformally rescaled spacetime

In view of Penrose’s notion of a smooth conformal structure at infinity, let us conformally rescale the spacetime $(M, g, X)$,
\[
g \mapsto \tilde{g} = \Theta^2 g, \quad M \mapsto \tilde{M}, \quad \Theta|_\varphi(M) > 0.
\]
In this “unphysical” spacetime Einstein’s vacuum field equations are most conveniently replaced by Friedrich’s conformal field equations [17]. Since we will rarely need these equations, we will
not discuss them here and refer to the literature. Nevertheless, two of these equations will be needed later: Set \( \tilde{s} := \frac{1}{4}\Box g\Theta + \frac{1}{24}R\Theta \) and denote by \( \tilde{L}_{\mu\nu} \) the Schouten tensor of \( \tilde{g} \). Then the following equations hold in any spacetime \((\tilde{M}, \tilde{g}, \Theta)\) which arises from a \( \Lambda \)-vacuum spacetime \((M, g)\),

\[
\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Theta = -\Theta \tilde{L}_{\mu\nu} + \tilde{s} \tilde{g}_{\mu\nu},
\]

(5.2)

\[
\tilde{\nabla}_\mu \Theta \tilde{\nabla}^\mu \Theta = 2\Theta \tilde{s} - \frac{\Lambda}{3}.
\]

(5.3)

A Killing vector field \( X \) in \((M, g)\) is mapped to a conformal Killing vector field \( \tilde{X} \) in \((\tilde{M}, \tilde{g}, \Theta)\) (which we identify with \( X \)) which, in addition, satisfies \( \tilde{\nabla}_\mu \tilde{\nabla}^\mu \Theta = \Theta \tilde{L}_{\mu\nu} \Theta \tilde{g}^{\mu\nu} \),

\[
\tilde{\nabla}_\mu \tilde{\nabla}_\nu \Theta = \Theta \tilde{L}_{\mu\nu} \Theta \tilde{g}^{\mu\nu} + \tilde{s} \tilde{g}^{\mu\nu}.
\]

(5.4)

Here and henceforth objects associated with the conformally rescaled metric \( \tilde{g} \) will be decorated with a \( \tilde{\ } \). The indices of those objects will be raised and lowered with \( \tilde{g} \).

Due to the simple behavior of \((2.15)\) under conformal rescalings of the metric, one easily shows \(27\) that in \((\tilde{M}, \tilde{g}, \Theta, \tilde{X})\) the conformally rescaled MST

\[
\tilde{T}_{\mu\nu\sigma} := \Theta^{-1} S_{\mu\nu\sigma}
\]

(5.5)

satisfies the equation

\[
\tilde{\nabla}_\rho \tilde{T}_{\alpha\beta\mu\nu} = J(\tilde{T})_{\alpha\beta\mu\nu} \equiv -4\Lambda \frac{5QF^2 + 4\Lambda}{QF^2 + 8\Lambda} \tilde{U}_{\alpha\beta\mu\nu} F^{-4} X^\gamma g^{\delta\kappa} F_{\kappa\mu\nu} \tilde{T}^\gamma^\delta^\sigma^\rho + QX^\alpha \left( \frac{2}{3} \tilde{U}_{\alpha\beta\mu\nu} g^{\gamma\kappa} \tilde{F}_{\kappa\mu\nu} \tilde{T}^\gamma^\delta^\sigma^\rho - F_{\mu\nu} \tilde{U}_{\alpha\beta\mu\nu} \right),
\]

(5.6)

where the fields on the right-hand side need to be expressed in terms of the “unphysical” fields,

\[
F_{\mu\nu} = \Theta^{-3} \tilde{H}_{\mu\nu} + \Theta^{-2} \tilde{F}_{\mu\nu},
\]

(5.7)

\[
F^2 = \Theta^{-2} \tilde{H}^2 + 2\Theta^{-1} \tilde{F}_{\alpha\beta} \tilde{H}^{\alpha\beta} + \tilde{F}^2,
\]

(5.8)

\[
\tilde{U}_{\alpha\beta\mu\nu} = -\Theta^{-6} \left( \tilde{H}_{\alpha\beta} \tilde{H}_{\mu\nu} - \frac{1}{3} \tilde{H}^2 \tilde{U}_{\alpha\beta\mu\nu} \right),
\]

(5.9)

\[
\tilde{U}_{\alpha\beta\mu\nu} = -\Theta^{-6} \left( \tilde{F}_{\alpha\beta} \tilde{H}_{\mu\nu} + \tilde{H}_{\alpha\beta} \tilde{F}_{\mu\nu} - \frac{2}{3} \tilde{F}_{\kappa\mu\nu} \tilde{H}^{\kappa\sigma} \tilde{U}_{\alpha\beta\mu\nu} \right),
\]

(5.10)

As in Section 4.2 we introduce Gaussian normal coordinates (near some Cauchy surface say) where

\[
\tilde{g}_{00} = -1, \quad \tilde{g}_{0i} = 0,
\]

(5.11)

such that the system splits into a symmetric hyperbolic system of evolution equations \( \tilde{\nabla}_\rho \tilde{T}_{0i(j)} = J(\tilde{T})_{0i(j)} \) and a system of constraint equations \( \tilde{\nabla}_\rho \tilde{T}_{00} = J(\tilde{T})_{00} \).
5.2 Behavior of the MST equation near a spacelike $\mathcal{I}$

5.2.1 Expansions near $\mathcal{I}$

Let us consider spacetimes which admit a smooth conformal completion at infinity à la Penrose \cite{33, 34}. By this it is meant that $(M, g)$ admits a conformal rescaling \cite{5.1} such that $(\bar{M}, \bar{g}, \Theta)$ admits a representation of null infinity

$$\mathcal{I} = \{ \Theta = 0, \ d\Theta \neq 0 \} \cap \partial \phi(M) \quad (5.12)$$

through which $\bar{g}$ and $\Theta$ can be smoothly extended. $\mathcal{I}$ is a smooth hypersurface consisting of two subsets $\mathcal{I}^-$ and $\mathcal{I}^+$, distinguished by the absence of endpoints of future and past causal curves in $(M, g)$, respectively.

The causal character of $\mathcal{I}$ is determined by the sign of the cosmological constant (cf. \cite{5.3}). A positive cosmological constant

$$\Lambda > 0 \quad (5.13)$$

yields a spacelike $\mathcal{I}$. From now on we will assume that \cite{5.13} holds. We further take $\mathcal{I}^-$ to be a connected component of past null infinity and restrict, if necessary, $(\bar{M}, \bar{g}, \Theta)$ to the domain of dependence of $\mathcal{I}^-$. Our aim is to construct solutions to \cite{5.6} by prescribing appropriate data on the Cauchy surface $\mathcal{I}^-$.

In \cite{27} the leading order behavior of the coefficients in \cite{5.6} at $\mathcal{I}^-$ has been computed, and it has been shown that it is actually a Fuchsian system. It is one of the main purposes of this work to analyze this system. It turns out that for this one needs to know the next-to-leading order terms, which have not been determined in \cite{27}.

In Gaussian coordinates we have (as a consequence of \cite{5.3})

$$\partial_0 \Theta\big|_{\mathcal{I}^-} = \sqrt{\frac{\Lambda}{3}}. \quad (5.14)$$

The gauge freedom which arises from the artificially introduced conformal factor $\Theta$ can be employed \cite{16, 31} to achieve that

$$\tilde{s}\big|_{\mathcal{I}^-} = 0. \quad (5.15)$$

Then we have by \cite{5.2}

$$\partial_0 \tilde{g}_{ij}\big|_{\mathcal{I}^-} = 0, \quad \partial_0 \partial_0 \Theta\big|_{\mathcal{I}^-} = 0. \quad (5.16)$$

In particular,

$$\Theta = \sqrt{\frac{\Lambda}{3}} t + O(t^4). \quad (5.17)$$

In addition, there remains the gauge freedom to conformally rescale the initial 3-manifold which we shall employ later.

We denote the induced 3-metric on $\mathcal{I}^-$ by $h$, the volume form of $h$ by $\epsilon_{ijk}$, its covariant derivative by $\mathcal{D}$. It follows from \cite{5.4} that the conformal Killing vector $\tilde{X}$ has no transverse component on $\mathcal{I}$. It induces a conformal Killing vector on $(\mathcal{I}^-, h)$ which we denote by $Y$. Moreover, let $Y$ and $N$ denote divergence and curl of $Y$,

$$f := \mathcal{D}_i Y^i, \quad N_i := \epsilon_{ijk} \mathcal{D}^j Y^k. \quad (5.18)$$
In the gauge (5.11) and (5.15) we find the following expansions,

\[
\begin{align*}
\tilde{X}^i &= Y^i + O(\Theta^2), \\
\tilde{X}^0 &= \frac{1}{3} \sqrt{\frac{3}{A}} Y + O(\Theta^2), \\
\tilde{H}_{0i} &= -\sqrt{\frac{A}{3}} Y_i + O(\Theta^2), \\
\tilde{H}_{ij} &= \frac{i}{3} Y_{ij} k + O(\Theta^2), \\
\tilde{\eta}^2 &= -4 \frac{A}{3} |Y|^2 + O(\Theta^2), \\
\tilde{\bar{F}}_{\alpha\beta} \tilde{F}^{\alpha\beta} &= 2 \sqrt{\frac{A}{3}} Y_k N^k + O(\Theta), \\
\tilde{F}_{0i} &= \frac{i}{2} N_i + O(\Theta), \\
\tilde{F}_{ij} &= \frac{i}{2} \epsilon_{ijk} N^k + O(\Theta).
\end{align*}
\]  

and [27]  

\[ Q = O(\Theta^4). \]  

Moreover, we have the useful relations,

\[
\begin{align*}
\tilde{H}^{\gamma\delta} \tilde{F}_{\gamma\delta\rho} &= 4 \tilde{X}^{\gamma} \tilde{\nabla}^{\delta} \tilde{\Theta} \tilde{F}_{\gamma\delta\rho} = 4 \sqrt{\frac{A}{3}} Y^k \tilde{\bar{F}}_{0k\rho} + (O(\Theta^2) \tilde{F})_{\sigma}^{\rho}, \\
\tilde{\bar{F}}^{\gamma\delta} \tilde{F}_{\gamma\delta\rho} &= 2 \tilde{\nabla}^{\gamma} \tilde{X}^{\delta} \tilde{F}_{\gamma\delta\rho} = -2i N^k \tilde{\bar{F}}_{0k\rho} + (O(\Theta) \tilde{F})_{\sigma}^{\rho}.
\end{align*}
\]  

### 5.2.2 Evolution equations and remaining gauge freedom

We analyze the behavior of the evolutionary part of the system (5.6) at \( \mathcal{F}^- \). In Gaussian coordinates it is given by the \( \alpha\beta\mu = 0(ij) \)-components. We denote by \( \tilde{\nabla} \) the Levi-Civita connection associated to the family \( t \mapsto \tilde{g}_{ij}(t, x^k) \) of \textit{conformally rescaled} Riemannian metrics, its volume form is denoted by \( \tilde{\eta}_{ijk} \).

For the left-hand side of (5.6) we find

\[ \tilde{\nabla}_{\rho} \tilde{F}_{0(ij)} \equiv \partial_{\rho} \tilde{F}_{0(ij)} - i \tilde{\eta}_{0k} \tilde{\nabla}_{[k} \tilde{F}_{0(ij)]l} + (O(\Theta) \tilde{F})_{0(ij)}, \]

while the right-hand side satisfies

\[
\begin{align*}
\mathcal{F}(\tilde{F})_{0(ij)} &= \sqrt{\frac{A}{3}} |Y|^{-2} \Theta^{-1} \left( \frac{3}{2} Y_{(i} Y^k \tilde{F}_{0(j)0k} - 3 |Y|^{-2} Y_{i} Y_{j} Y^{k} Y^{l} \tilde{F}_{0kl0} + \frac{1}{2} b_{ij} Y^{k} Y^{l} \tilde{F}_{0kl0} \right) \\
&\quad + 2i |Y|^{-4} Y_{0} Y_{p} \left( \frac{9}{8} Y_{(i} Y^k \tilde{F}_{0(j)0k} - 3 |Y|^{-2} Y_{i} Y_{j} Y^{k} Y^{l} \tilde{F}_{0kl0} + \frac{1}{4} b_{ij} Y^{k} Y^{l} \tilde{F}_{0kl0} \right) \\
&\quad \frac{3}{4} |Y|^{-2} \left( Y_{(i} N^k + N_{i} Y^k \tilde{F}_{0(j)0k} - 3 |Y|^{-2} Y_{i} Y_{j} N^k + N_{i} Y_{j} Y^k \tilde{F}_{0kl0} \right) \\
&\quad + \frac{1}{3} b_{ij} Y^{k} \tilde{F}_{0kl0} - \frac{2}{3} f |Y|^{-2} Y_{(i} \tilde{\eta}_{j)p} Y^{k} Y^{l} \tilde{F}_{0kl0} + (O(\Theta) \tilde{F})_{0(ij)}.
\end{align*}
\]
Then, taking further into account that 

\[ \text{observe that } \mathcal{E}_{ij} \text{ contains all independent components of the MST}, \]

\[
\partial_t \mathcal{E}_{ij} - i \bar{\eta}_{ij}(\langle \mathcal{D} \rangle) = \sqrt{\frac{\Lambda}{3}} Y^{-2} \Theta^{-1} \left( \frac{3}{2} Y_{(i} Y^k \mathcal{E}_{j)k} - 3|Y|^{-2} Y_{i} Y^k Y^l \mathcal{E}_{kl} + \frac{1}{2} h_{ij} Y^k Y^l \mathcal{E}_{kl} \right) + 2i|Y|^{-4} Y_{p} N_{p} \left( \frac{9}{8} Y_{(i} Y^k \mathcal{E}_{j)k} - 3|Y|^{-2} Y_{i} Y^k Y^l \mathcal{E}_{kl} + \frac{1}{4} h_{ij} Y^k Y^l \mathcal{E}_{kl} \right) \\
\frac{3}{4} i |Y|^{-2} \left( Y_{(i} N^k + N_{(i} Y^k \right) \mathcal{E}_{j)k} - 3|Y|^{-2} Y_{i} Y_j N^k + N_{(i} Y^k Y^l \mathcal{E}_{kl} \\
+ \frac{1}{3} h_{ij} Y^k N^l \mathcal{E}_{kl} - \frac{2}{3} f |Y|^{-2} (\epsilon_{ij}^p k Y^p Y^l \mathcal{E}_{kl}) + (O(\Theta)) \mathcal{E}_{ij} \right) .
\]

(5.31)

To simplify the analysis of (5.31) we exploit the above mentioned remaining gauge freedom, which is to conformally rescale the initial 3-manifold \( (\mathcal{F}, h) \).

We would like to achieve that the conformal Killing vector \( Y \) and its curl \( N \) are parallel, i.e. that the cross product vanishes,

\[
0 \equiv (Y \times N)_{i} = \epsilon_{ijk} Y^{j} N^{k} = 2Y^{j} \partial_{i} Y_{j} ,
\]

(5.32)

For this purpose let us consider a conformal rescaling of the induced Riemannian metric \( h \),

\[
h \mapsto \hat{h} := \omega^{2} h .
\]

(5.33)

Then the following relations hold on the rescaled Riemannian manifold \( (\mathcal{F}, \hat{h}) \) (we denote the associated Levi-Civita covariant derivative by \( \hat{\mathcal{D}} \)),

\[
|\hat{Y}|^{2} = \omega^{2} |Y|^{2} ,
\]

(5.34)

\[
\hat{f} = f + \frac{3}{2} \omega^{-2} Y^j \partial_j \omega^2 ,
\]

(5.35)

\[
2\hat{Y}^j \hat{\mathcal{D}}_{(i} \hat{Y}_{j)} = \partial_i |\hat{Y}|^{2} - \frac{2}{3} \hat{f} \hat{Y}_i \\
= \partial_i (\omega^2 |Y|^2) - \frac{2}{3} \omega^2 \left( f + \frac{3}{2} \omega^{-2} Y^j \partial_j \omega^2 \right) Y_i \\
= \partial_i (\omega^2 |Y|^2) - \frac{2}{3} \omega^2 f Y_i - Y^j \partial_j \omega^2 .
\]

(5.36)

Let us now make the gauge choice

\[
\omega^{2} = |Y|^{-2} .
\]

(5.37)

Then, taking further into account that \( Y \) is a conformal Killing vector, (5.34)-(5.36) simply become

\[
|\hat{Y}|^{2} = 1 ,
\]

(5.38)

\[
\hat{f} = f - \frac{3}{2} |Y|^{-2} Y^i \partial_i |Y|^2 = f - 3|Y|^{-2} Y^i Y^k \mathcal{D}_{(i} Y_{j)k} = 0 ,
\]

(5.39)

\[
2\hat{Y}^j \hat{\mathcal{D}}_{(i} \hat{Y}_{j)} = -\frac{2}{3} |Y|^{-2} Y_i \left( f - \frac{3}{2} |Y|^{-2} Y^j \partial_j |Y|^2 \right) = 0 ,
\]

(5.40)

so in particular (5.32) holds.
We therefore can and will impose the gauge condition
\[ |Y|^2 = 1, \quad f = 0, \quad N = \lambda Y \] (5.41)
for some real function \( \lambda = \lambda(x^i) \). In particular, \( Y \) is a proper Killing vector in this gauge. It simplifies the evaluation of (5.31) significantly which now reads (using also (5.17)),
\[
\partial_t \mathcal{E}_{ij} - i \tilde{g}^{kl} \partial_k [\mathcal{E}_{ij}]_t = \left( \frac{1}{t} + \frac{i}{2} \lambda \right) \left( \frac{3}{2} Y (\partial_t \mathcal{E}_{ij})^k - 3 Y_i Y^k + \frac{1}{2} h_{ij} Y^k \right) Y^l \mathcal{E}_{kl} + (O(t) \mathcal{E})_{ij}, \quad (5.42)
\]

### 5.2.3 Constraint equations

Next, we compute the asymptotic behavior of the constraint equations, i.e. of the \( \alpha \beta \mu = 0 \) components of (5.6) at \( \mathcal{I}^- \). In a gauge where (5.11) and (5.15) hold we obtain for the right-hand side
\[
\mathcal{J} (\tilde{T})_{00} = -\frac{1}{2} \sqrt{\frac{\Lambda}{3}} \Theta^{-1} |Y|^{-2} \epsilon_{ij} Y^j Y^k \tilde{T}_{0k0l} + \frac{1}{2} f |Y|^{-4} (Y_i Y_j)_{kl} Y^k \tilde{T}_{0k0l}^j - \frac{1}{4} Y |^{-2} \epsilon_{ij} Y^j N^k \tilde{T}_{0k0l}^i + \frac{3}{4} |Y|^{-4} \epsilon_{ij} Y^j N^k Y^m \tilde{T}_{0k0l}^j + \frac{1}{2} |Y|^{-4} \epsilon_{ij} Y_m N^m Y^l \tilde{T}_{0k0l}^j + (O(\Theta) \tilde{T})_{00},
\]
(5.43)
where we have used the various expansions derived in Section 5.2.1. For the left-hand side of (5.6) we find
\[
\tilde{\nabla}_\rho \tilde{T}_{00}^\rho = \mathcal{D}_j \tilde{T}_{00}^j + (O(\Theta) \tilde{T})_{00},
\]
(5.44)
and combined, again with \( \mathcal{E}_{ij} \equiv \tilde{T}_{00} \)
\[
\mathcal{D}_j \mathcal{E}^j = -\frac{1}{2} \sqrt{\frac{\Lambda}{3}} \Theta^{-1} |Y|^{-2} \epsilon_{ij} Y^j Y^k \mathcal{E}_{kl} + \frac{1}{2} f |Y|^{-4} (Y_i Y_j)_{kl} Y^k \mathcal{E}_{kl}^j - \frac{1}{4} |Y|^{-2} \epsilon_{ij} Y^j N^k \mathcal{E}_{kl} + \frac{3}{4} |Y|^{-4} \epsilon_{ij} Y^j N^k Y^m \mathcal{E}_{kl}^j + \frac{1}{2} |Y|^{-4} \epsilon_{ij} Y_m N^m Y^l \mathcal{E}_{kl} + (O(\Theta) \mathcal{E})_{l}.
\]
(5.45)

Imposing, in addition, the gauge condition (5.41), this becomes
\[
\mathcal{D}_j \mathcal{E}^j + i \left( \frac{1}{t} + \frac{i}{2} \lambda \right) \epsilon_{ij} Y^j Y^k \mathcal{E}_{kl} + (O(t) \mathcal{E})_{l} = 0.
\]
(5.46)

We sum up the result of Section 5.2 by the following

**Lemma 5.1** Let \((\tilde{M}, \tilde{g}, \Theta, \tilde{X})\) be a solution to the conformal field equations with cosmological constant \( \Lambda > 0 \) which admits a conformal Killing vector field \( \tilde{X} \) which satisfies (5.4), and which admits a smooth \( \mathcal{I} \) where \( |\tilde{X}|^2_{\mathcal{I}} > 0 \). Assume further that the gauge conditions (5.11), (5.15) and (5.41) hold. Then the evolution and constraint part of (5.6) show a Fuchsian behavior near \( \mathcal{I} \) as given by (5.42) and (5.46), respectively.

**Remark 5.2** The inequalities (2.14) hold near \( \mathcal{I} \) as long as \( \tilde{X} \) has no zeros on \( \mathcal{I} \). This follows from the expansions computed in [27].
5.3 Admissible data sets on a spacelike $I$

5.3.1 Preservation of the constraints from $I$

For the time being let us consider again the physical spacetime $(M, g, X)$. Let us assume that $S_{\mu\nu\rho\sigma}$ fulfills the evolution equations (4.39) and that (4.56)-(4.57) hold. Then (4.54) becomes

$$
\left[ \left( \delta^i_j \partial_0 + \frac{3}{2} \Gamma^j_{0j} - 2 \Gamma^0_{00} \right) \eta_{0i} + \frac{3}{2} \Gamma^0_{0i} \right] 
+ Q \left( \frac{2}{3} X^j F_{0j} \delta_p^p - \frac{5}{6} X_0 F_{0i} - \frac{1}{6} X^p F_{0i} - \frac{1}{6} X_i F_0^p \right) 
- \frac{2}{3} \Lambda F^p + 4 \Lambda \left( 36 \theta_{00} X^j F_{0j} F_{0i} - 2 F^2 X^p F_{0i} - 2 F^2 X_i F_0^p \right) 
- F^2 X_0 F_{0i} - F^2 \delta_k^p X^j F_{0j} \right] \Xi_p = 0.
$$

To transform this equation into the unphysical spacetime, note that the algebraic properties of $S_{\mu\nu\rho\sigma}$ imply that

$$
\nabla_\lambda S_{0\rho0}^\lambda - J(S)_{0\rho0} = \Theta \left( \tilde{\nabla}_\lambda \tilde{T}_{0\rho0}^\lambda - J(\tilde{T})_{0\rho0} \right) =: \Theta \tilde{\Xi}_p.
$$

We further employ the relations (5.7)-(5.10) and (5.19)-(5.27) and take the behavior of the Levi-Civita connection under conformal transformations into account to obtain, in Gaussian normal coordinates (5.11), a homogeneous equations for $\tilde{\Xi}_i$,

$$
\left[ \left( \delta^k_i \partial_0 - \frac{i}{2} \eta^k_{ij} \tilde{\nabla}_j \right) + \frac{1}{4} \left( \delta^k_i + 13 |Y|^{-2} Y_i Y^k \right) \right] \tilde{\Xi}_i = 0.
$$

5.3.2 Realization of the Buchdahl condition at $I$

We need to make sure that the generalized Buchdahl condition (4.55) holds, so let us translate it into the unphysical spacetime $(\tilde{M}, \tilde{g}, \Theta, \tilde{X})$.

Under conformal rescaling we have (set $\tilde{F} := \frac{1}{4} \tilde{\nabla}_n \tilde{X}^n$

$$
\mathcal{L}_X S_{0\beta\mu}^{\nu} = \Theta \left( \tilde{\mathcal{L}}_\tilde{X} \tilde{T}_{0\beta\mu}^{\nu} + \tilde{\Xi}_i \tilde{\nabla}_i \tilde{T}_{0\beta\mu}^{\nu} \right) = \Theta(\mathcal{L}_X \tilde{F}) \tilde{T}_{0\beta\mu}^{\nu},
$$

as follows from (5.4). In the conformally rescaled spacetime the Buchdahl condition (4.55) therefore adopts the form

$$
\tilde{f}^{(1)}_{\mu\nu\rho}(\mathcal{L}_X \tilde{F}) \tilde{T}_{\mu\nu\rho} + \tilde{f}^{(2)}_{\mu\nu\rho} \tilde{\Xi}_{\mu\nu\rho} = 0,
$$

where $\tilde{\Xi}_{\mu\nu\rho}$ denotes the rescaled MST of the background spacetime $(\tilde{M}, \tilde{g}, \Theta, \tilde{X})$.

We assume that the background spacetime $(\tilde{M}, \tilde{g}, \Theta, \tilde{X})$ has a vanishing (rescaled) MST, and in order to make sure that (5.50) is fulfilled we will further make sure that

$$
(\mathcal{L}_X \tilde{F}) \tilde{T}_{\mu\nu\rho} = 0.
$$
The analog of (4.59) in the conformally rescaled spacetime reads

\[
\tilde{\nabla}_\rho((\mathcal{L}_\tilde{X} + \tilde{F})\tilde{T}_{0(ij)}^\rho) = \tilde{\nabla}_\rho(\Theta^{-1}\mathcal{L}_X S_{0(ij)}^\rho) = \Theta^{-1}\tilde{\nabla}_\rho \mathcal{L}_X S_{0(ij)}^\rho - \Theta^{-2}\tilde{\nabla}_\rho \Theta \mathcal{L}_X S_{0(ij)}^\rho = \mathcal{L}_X \mathcal{J}(\tilde{T})_{0(ij)} + \Theta^{-1} \mathcal{L}_X \tilde{\Theta} \mathcal{J}(\tilde{T})_{0(ij)} = \mathcal{L}_X(\Theta^{-1} \mathcal{J}(S)_{0(ij)}) + \Theta^{-1} \mathcal{L}_X \tilde{\Theta} \mathcal{J}(\tilde{T})_{0(ij)} = \Theta^{-1} \mathcal{J}(\mathcal{L}_X S)_{0(ij)} = \mathcal{J}((\mathcal{L}_\tilde{X} + \tilde{F})\tilde{T})_{0(ij)},
\]

so \((\mathcal{L}_\tilde{X} + \tilde{F})\tilde{T}_{\mu\nu\sigma\rho}\) satisfies the same equation as \(\tilde{T}_{\mu\nu\sigma\rho}\).

Let us determine the expansion of \((\mathcal{L}_\tilde{X} + \tilde{F})\tilde{T}_{\alpha\beta\mu\nu}\) near \(\mathcal{I}\). Assuming that the evolution equations hold, we find

\[
(\mathcal{L}_\tilde{X} + \tilde{F})\tilde{T}_{0000} = \tilde{X}^0\tilde{\nabla}_0 \tilde{T}_{0000} + \tilde{X}^k \tilde{\nabla}_k \tilde{T}_{0000} + \tilde{F} \tilde{T}_{0000} + 2\tilde{T}_{0000} \tilde{\nabla}_j \tilde{X}^j + 2\tilde{T}_{0000} \tilde{\nabla}_0 \tilde{X}^0 + 2\tilde{T}_{0000} \tilde{\nabla}_j \tilde{X}^j + 2\tilde{T}_{0000} \tilde{\nabla}_0 \tilde{X}^0 = \tilde{X}^0 \mathcal{J}(\tilde{T})_{0(ij)} - \tilde{X}^0 \tilde{\nabla}_k \tilde{T}_{0(ij)}^k + \tilde{X}^k \tilde{\nabla}_k \tilde{T}_{0000} + \tilde{F} \tilde{T}_{0000} + 2\tilde{T}_{0000} \tilde{\nabla}_j \tilde{X}^j + 2\tilde{T}_{0000} \tilde{\nabla}_0 \tilde{X}^0 + 2\tilde{T}_{0000} \tilde{\nabla}_j \tilde{X}^j + 2\tilde{T}_{0000} \tilde{\nabla}_0 \tilde{X}^0.
\]

In Gaussian normal coordinates we have the expansions (5.19), (5.20) and

\[
\tilde{F} = \frac{1}{3} f + O(\Theta),
\]

whence

\[
(\mathcal{L}_\tilde{X} + \tilde{F})\mathcal{E}_{ij} = f[Y]^{-2}\left(\frac{1}{2} Y_i^l Y_j^k \mathcal{E}_{lk} + (\frac{1}{2} Y_i^l Y^k Y^l \mathcal{E}_{kl}) + (\mathcal{L}_Y + f)\mathcal{E}_{ij} - (O(\Theta) \mathcal{E}_{ij} + O(\Theta) \mathcal{E}_{ij}).
\]

In a gauge where (5.41) holds this becomes

\[
\mathcal{L}_\tilde{X} \mathcal{E}_{ij} = \mathcal{L}_Y \mathcal{E}_{ij} + (O(\Theta) \mathcal{E}_{ij} + (O(\Theta) \mathcal{E}_{ij}).
\]

In Section 6.5 it will be analyzed how initial data need to be chosen in order to satisfy the constraint equations and the Buchdahl condition, and for this the equations (5.48) and (5.52) will be relevant.

6 Fuchsian analysis near \(\mathcal{I}^−\)

6.1 Preliminaries and the main result

For the purpose of this whole section, we shall introduce certain small variations of previous conventions and notations. Pick \(\delta > 0\) and an orientable 3-dimensional differentiable manifold \(\Sigma\). We refer to \(\tilde{M} = (\tilde{\delta}, \delta) \times \Sigma\) as the conformal (or unphysical) spacetime and to \(M = (0, \delta) \times \Sigma\) as the physical spacetime; we always identify the physical spacetime with this subset of the conformal spacetime explicitly. As before, both manifolds are equipped with conformally related
Lorentzian metrics; in this section we only deal with the conformal (unphysical) metric \( \tilde{g}_{\mu\nu} \) on \( \tilde{M} \). Let \( t \) be the parameter on the \((-\delta, \delta)\)-factor of \( \tilde{M} \). We refer to the \( t = \text{const} \)-hypersurface for any \( t \in (-\delta, \delta) \) as \( \Sigma_t \), i.e.,

\[
\Sigma_t = \{ t \} \times \Sigma
\]

which is clearly a subset of \( \tilde{M} \) diffeomorphic to \( \Sigma \). We assume that all these hypersurfaces are spacelike Cauchy surfaces of \( \tilde{M} \). Notice that \( \Sigma_0 \) agrees with \( \mathcal{I}^- \). We assume the existence of a conformal Killing vector field \( \tilde{X}^\mu \) of \( \tilde{g}_{\mu\nu} \) on \( \tilde{M} \).

Let \( \tilde{N}^\mu \) be the future-pointing unit normal of the surfaces \( \Sigma_t \) in \( \tilde{M} \) with respect to \( \tilde{g}_{\mu\nu} \). It will be convenient for the following discussion to adopt a slightly different index convention than before: Tensor indices \( \mu, \nu, \ldots \) are still considered as abstract spacetime indices, while \( i, j, \ldots \) shall now denote abstract indices which have been projected orthogonally into the hypersurfaces \( \Sigma_t \) with respect to \( \tilde{N}^\mu \). Correspondingly, the index 0 denotes projections onto \( \tilde{N}^\mu \). Interpreting all indices in this coordinate invariant manner has several advantages for the following discussion. It is only a slight shift of the viewpoint and it is in full consistency with the conventions in previous sections when tensors are expressed in terms of Gaussian coordinate frames. Most of the tensor fields we are dealing with in this section are completely intrinsic to \( \Sigma_t \), i.e., fully spatial, and will henceforth carry indices \( i, j, \ldots \) exclusively. A particular important example is the tensor \( \tilde{\epsilon}_{ij} \) derived from the MST.

The metric induced on \( \Sigma_t \) from the conformal metric \( \tilde{g}_{\mu\nu} \) is therefore denoted by \( \tilde{g}_{ij} \) (we shall sometimes write \( \tilde{g}_{ij}(t) \) when the particular value of \( t \) is relevant or when we want to emphasize the fact that this metric is time dependent). On any \( \Sigma_t \), this Riemannian metric \( \tilde{g}_{ij} \) determines a volume form \( \tilde{h}_{ijk} \) and a Levi-Civita covariant derivative \( \tilde{\nabla}_k \). The consistent use of our abstract index conventions above makes it unnecessary to introduce a special symbol for the Levi-Civita connection of the Riemannian metric \( \tilde{g}_{ij} \).

The metric induced on \( \mathcal{I}^- = \Sigma_0 \) will play a particular role in the following and is denoted by \( \tilde{h}_{ij} = \tilde{g}_{ij}(0) \) as before. Via Lie transport along \( \tilde{N}^\mu \), this metric \( \tilde{h}_{ij} \) can be dragged to any surface \( \Sigma_t \); the resulting field on \( \tilde{M} \) shall be referred to as \( \tilde{h}_{ij} \) as well for simplicity. We shall do the same for all quantities derived from \( \tilde{h}_{ij} \), in particular, for the volume form \( \epsilon_{ijk} \) and the covariant derivative \( \partial_k \) associated with \( h_{ij} \). Since such fields are therefore by definition invariant under Lie transport along \( \tilde{N}^\mu \) we say that they are time-independent. Any field on \( \tilde{M} \) (or \( M \)) that is invariant in this way shall be referred to as time-independent; otherwise we call it time-dependent.

According to the discussion in Section 5 we shall now make certain assumptions about the behavior of various quantities at \( t = 0 \); recall that most of these conditions constitute no loss of generality because they can always be achieved by an appropriate choice of gauge. First we assume that

\[
\tilde{g}_{ij}(t) = \tilde{h}_{ij} + \mathcal{O}(t^2), \quad \tilde{K}_{ij}(t) = \mathcal{O}(t^2), \quad (6.1)
\]

where, in this coordinate invariant sense, the \( \mathcal{O} \)-symbol is defined with respect to any time-independent Riemannian reference metric uniformly on \( \Sigma \) in all of what follows. Second we assume that the conformal Killing vector field \( \tilde{X}^\mu \) can be written as

\[
\tilde{X}^0(t) = \mathcal{O}(t^2), \quad \tilde{X}^i(t) = Y^i + \mathcal{O}(t^2) \quad (6.2)
\]

according to Eq. (5.19) where \( Y^i \) is a conformal Killing vector field of \( \tilde{h}_{ij} \) without zeros on \( \Sigma_0 \). We shall interpret \( Y^i \) as a time-independent field on \( \tilde{M} \). In agreement with Eqs. (5.41) and (5.18) we also assume

\[
\tilde{h}_{ij} Y^i Y^j = 1, \quad \epsilon_{ijk} \partial_k Y^l = \lambda Y^i, \quad \partial_l Y^l = 0, \quad (6.3)
\]

where the quantity \( \lambda \) is some, in principle known, smooth time-independent function.

27
Following the earlier discussion, the MST is represented by a time-dependent purely spatial complex symmetric \( \tilde{g}_{ij} \)-trace-free \((0,2)\)-tensor field \( \mathcal{E}_{ij} \) as follows. According to Eq. (5.42), the MST evolution equations are

\[
\tilde{\nabla}_0 \mathcal{E}_{ij} - i \tilde{\eta}_{i,ij} \tilde{g}^{ik} \tilde{g}^{jl} \tilde{\nabla}_l \mathcal{E}_{ij} = T^{kl}_{\ i j} \mathcal{E}_{kl}.
\] (6.4)

They are formally singular at \( t = 0 \) and hence only make sense for \( t > 0 \), i.e., on the subset \( M \) of \( \tilde{M} \). Near \( t = 0 \), the smooth time-dependent field \( T^{kl}_{\ i j} \) is given as

\[
T^{kl}_{\ i j}(t) = T^{kl}_{\ i j(0)} + \frac{i\lambda}{2} T^{kl}_{\ i j(0)} + t^2 T^{kl}_{\ i j(1)}(t)
\] (6.5)

with

\[
T^{kl}_{\ i j(0)} = \frac{3}{2} \left( Y_i^k Y_j^l - 3 Y_i Y_j Y^k Y^l + \frac{1}{2} h_{ij} Y^k Y^l + \frac{1}{2} h^{kl} Y_i Y_j - \frac{1}{6} h_{ij} h^{kl} \right),
\] (6.6)

which is hence time-independent, and with some known smooth time-dependent field \( T^{kl}_{\ i j(1)} \).

According to Eq. (5.46), the constraint equations take the form

\[
0 = \partial_t \mathcal{E}_{ij} + \frac{i}{2} \left( l + \frac{1}{2} \right) \epsilon_{ij}^{\ kl} Y^k Y^l \mathcal{E}_{kl} - t \mathcal{L}^{kl}_{\ i j}(t) \mathcal{E}_{kl} = : \mathcal{N}_{ij},
\] (6.7)

on \( M \), where \( \mathcal{L}^{kl}_{\ i j} \) is some smooth time-dependent tensor field on \( M \) which is also known. All index operations in Eqs. (6.1) – (6.7) are performed with the metric \( h_{ij} \).

Before we continue, a few remarks are in place. Recall that Eqs. (6.4) – (6.7) had been derived in previous sections assuming Gauss coordinates and further particular gauge choices. Once these equations have been derived, however, we can forget about this and consider them as fully invariant tensorial equations. Indeed, for large parts of the analysis in the following section we only need to impose Eqs. (6.1) – (6.3). Only for specific steps of our discussion, we explicitly need to introduce Gauss coordinates. One obtains a Gauss coordinate system within the general setup above by imposing the following restrictions on the foliation \( \Sigma_t \) with respect to the time function \( t \). We pick spatial coordinates \( x^i \) on each leaf \( \Sigma_t \) and make the additional assumption that \( \tilde{N}^\mu = \partial_t^\mu \) and \( \tilde{N}_\mu = dt^\mu \) for the corresponding spacetime coordinates \( (t, x^i) \).

The reader will notice that (6.6) looks significantly different from the corresponding terms in Eq. (5.42) (inside the second pair of brackets there). As one can easily check, however, both expressions are equivalent if \( \mathcal{E}_{ij} \) is symmetric and \( \tilde{g}_{ij} \)-trace-free. Since in some of the intermediate steps of our arguments below we will allow \( \mathcal{E}_{ij} \) to be non-symmetric and non-trace-free, the expression in Eq. (6.6) is more suitable than the one in Eq. (5.42). The field in Eq. (6.6) is by construction explicitly symmetric and \( h_{ij} \)-trace-free with respect to both pairs of indices. The field in Eq. (6.5) is identically symmetric and \( \tilde{g}_{ij} \)-trace-free with respect to both pairs of indices.

The overall goal is to solve an initial value problem for Eq. (6.4) with data prescribed at \( t = 0 \) such that Eq. (6.7) is satisfied identically on \( M \). As noticed before, Eq. (6.4) is formally singular at \( t = 0 \) and hence the initial value problem in the standard sense does not make sense. Instead we consider a singular initial value problem. The Fuchsian method shall allow us to prove the main result of this section, Theorem 6.1, below. Before we can state this theorem, however, we must introduce some further notation and terminology.

We say that an open subset \( \Omega \) of \( M \) with compact closure in \( \tilde{M} \) is a lens-shaped region (cf. Section 3.1 in [18]) if its boundary is the union of two smooth spacelike hypersurfaces in \( \tilde{M} \) with respect to the conformal metric \( \tilde{g}_{\mu\nu} \) and if the boundaries of these two hypersurfaces coincide and are smooth. When one of these two spacelike hypersurfaces is called \( S \) in \( \tilde{M} \), then we say
that $\Omega$ is a lens-shaped region with respect to $S$. For any $t \in (-\delta, \delta)$, we define $\Omega_t = \Omega \cap \Sigma_t$; we allow $\Omega_t$ to be empty. We shall also write $\Omega(t_0, t)$ to denote the intersection of $\Omega$ with the spacetime slab $(t_0, t) \times \Sigma$ with $0 < t_0 < t < \delta$.

In anticipation of the results in Section 6.2, we define the following time-independent fields (index operations are performed with respect to $h_{ij}$)

$$E_{kl(1)} = \frac{1}{\sqrt{6}} (3Y_kY_l - h_{kl}),$$

$$E_{kl(2)} = \frac{1}{\sqrt{2}} (e_{[1]}[e_{[2]}] + c_{[1]}e_{[2]}),$$

$$E_{kl(3)} = \frac{1}{\sqrt{2}} (e_{[1]}k - e_{[2]}k),$$

$$E_{kl(4)} = \frac{1}{\sqrt{2}} (Y_ke_{[1]} + Y_de_{[2]}),$$

$$E_{kl(5)} = \frac{1}{\sqrt{2}} (Y_ke_{[2]} + Y_de_{[2]}),$$

where $(e_{[1]}, e_{[2]}, Y^i)$ is any time-independent, $Y^i$-invariant $h_{ij}$-orthonormal frame.

**Theorem 6.1** (Singular initial value problem of the MST equations) Pick $\delta > 0$ and an orientable 3-dimensional differentiable manifold $\Sigma$. Equip $\tilde{M} = (-\delta, \delta) \times \Sigma$ with a smooth Lorentzian metric $\tilde{g}_{\mu\nu}$ with a conformal Killing vector field $\tilde{X}^\mu$ without zeros on $\tilde{M}^- = \Sigma_0$ and Gaussian coordinates $(t, x^i)$ as before. Suppose that the MST of $(M, \tilde{g}_{\mu\nu}, \tilde{X}^\mu)$ vanishes and that Eqs. (6.1) - (6.3) hold with respect to our coordinate system. Pick any non-empty open subset $S_0$ of $\Sigma$ with compact closure and non-empty smooth boundary, and, any non-empty lens-shaped region $\Omega$ with respect to the subset $\{t = 0\} \times S_0$ of $\partial M \subset \tilde{M}$ where $M = (0, \delta) \times \Sigma \subset \tilde{M}$.

Then, for any smooth complex time-independent functions $c_1$, $c_2$, and $c_3$ on $\Omega$ with the property

$$\mathcal{L}_Y c_1 = \mathcal{L}_Y c_2 = \mathcal{L}_Y c_3 = 0,$$

where $Y^i$ is defined by $\tilde{X}^\mu$ through Eq. (6.2), there is a smooth solution $\tilde{E}_{ij}$ of Eqs. (6.4) - (6.7), i.e., of the full Mars-Simon equation (5.6), of the form

$$\tilde{E}_{ij}(t, x) = c_1(x)E_{ij(1)}(x) + i\delta_{ij}c_1(x)E_{ij(2)}(x) + c_2(x)E_{ij(2)}(x) + c_3(x)E_{ij(3)}(x)$$

$$- \frac{2i}{\sqrt{3}} \phi_{[1]} c_1(x)E_{ij(4)}(x) + \frac{2i}{\sqrt{3}} \phi_{[1]} c_1(x)E_{ij(5)}(x) + E_{ij}(t, x),$$

for every $(t, x) \in \Omega$ provided $\delta > 0$ is sufficiently small. Here, $E_{ij}$ is some smooth complex symmetric $\tilde{g}_{ij}$-trace-free field which can be extended smoothly through $t = 0$ and $\lim_{t \to 0} E_{ij}(t, x) = 0$ for all points $x$. Any two smooth solutions $\tilde{E}_{ij}$ and $\tilde{E}_{ij}$ of this form given by the same data $c_1$, $c_2$ and $c_3$ are identical on $\Omega$.

The remainder of this section is devoted to the proof of this theorem.

### 6.2 Spectral analysis of the principal part matrix

An essential first step in the proof of Theorem 6.1 is a detailed analysis of the field $T^{kl}_{ij(0)}$ defined in Eq. (6.6). This will lead naturally to the quantities $E_{ij(1)}, \ldots, E_{ij(5)}$ above and to the structure of the leading-order term in Eq. (6.8).

The whole discussion in this subsection only involves time-independent fields. It can therefore be carried out on the abstract Cauchy surface $\Sigma$ without any reference to time $t$. For this whole subsection, complex conjugates of complex fields are denoted with a bar. All index operations are performed with the metric $h_{ij}$. We pick an arbitrary point $x \in \Sigma$ and refrain from writing $x$ in the following formulas for simplicity. We consider the 9-dimensional complex vector space of
(0, 2)-tensors of $T_x \Sigma$; we do not impose any symmetry or trace-free conditions at this stage yet. On this vector space we have an inner product

$$\left( E_{ij}, \tilde{E}_{ij} \right) \mapsto E^{ij} \tilde{E}_{ij}.$$

(6.9)

The quantity $T_{ij}^{kl}(0)$ defined in Eq. (6.6) can be considered as an endomorphism of this vector space which we find to be self-adjoint

$$T_{ij}^{kl}(0) = T^{kl}_{ij}(0).$$

It is thus diagonalizable, the eigenvalues are real and the respective eigenspaces are mutually orthogonal. Using any $\mathfrak{h}_{ij}$-orthonormal basis $(e_{[1]}^i, e_{[2]}^i, Y^i)$ of $T_x \Sigma$ (recall Eq. (6.3)) we can show:

**Eigenvalue** $-1$: This eigenspace is spanned by the tensor

$$E_{kl(1)} = \frac{1}{\sqrt{6}} (3Y_k Y_l - \mathfrak{h}_{kl}),$$

(6.10)

and is therefore a 1-dimensional subspace.

**Eigenvalue** $0$: This eigenspace is spanned by all antisymmetric $(0, 2)$-tensors (3-dimensional), all pure $\mathfrak{h}_{ij}$-trace $(0, 2)$-tensors (1-dimensional), and, all symmetric $\mathfrak{h}_{ij}$-trace-free $(0, 2)$-tensors $\epsilon_{kl}$ with the property $\epsilon_{kl}Y^l = 0$ (2-dimensional). The latter subspace is spanned by

$$E_{kl(2)} = \frac{1}{\sqrt{2}} (\epsilon_{[1]k} e_{[2]l} + \epsilon_{[1]l} e_{[2]k}), \quad E_{kl(3)} = \frac{1}{\sqrt{2}} (\epsilon_{[1]k} e_{[1]l} - \epsilon_{[2]k} e_{[2]l}).$$

(6.11)

In total this eigenspace is therefore 6-dimensional.

**Eigenvalue** $3/4$: This eigenspace is spanned by

$$E_{kl(4)} = \frac{1}{\sqrt{2}} (Y_k e_{[1]l} + Y_l e_{[1]k}), \quad E_{kl(5)} = \frac{1}{\sqrt{2}} (Y_k e_{[2]l} + Y_l e_{[2]k}),$$

(6.12)

and is therefore 2-dimensional.

Now, when we restrict the map $T_{ij}^{kl}(0)$ to the 5-dimensional subspace of symmetric, $\mathfrak{h}_{ij}$-trace-free $(0, 2)$-tensors, it is still a self-adjoint endomorphism and the analogue eigenspace decomposition can be carried out. The only difference is that the eigenspace of the eigenvalue 0 is now reduced to the two dimensional subspace spanned by Eq. (6.11). The self-adjoint property implies that all eigenspaces are mutually orthogonal with respect to the scalar product Eq. (6.9). Thanks to the normalizations chosen above we find

$$E_{ij(p)} \tilde{E}_{ij(q)}^{ij} = \delta_{pq}, \quad \text{for all } p, q = 1, \ldots, 5.$$

(6.13)

The collection of fields $E_{ij(1)}$ to $E_{ij(5)}$ therefore constitutes an orthonormal basis of the complex vector space of symmetric $\mathfrak{h}_{ij}$-trace-free $(0, 2)$-tensors at $x \in T_x \Sigma$.

Since Theorem 6.1 only refers to open proper subsets $S_0$ of $\Sigma$ with compact closure we can use without loss of generality that $\Sigma$ is a compact orientable 3-dimensional manifold and therefore parallelizable. We may therefore assume that $(e_{[1]}^i, e_{[2]}^i, Y^i)$ is a global orthonormal frame on $\Sigma$ and then construct $E_{ij(1)}$ to $E_{ij(5)}$ at each $x \in \Sigma$ as above. In this way we may interpret these as smooth fields on $\Sigma$ which satisfy all the properties above at each $x \in \Sigma$. We
may even consider $E_{ij(1)}$ to $E_{ij(5)}$ as smooth time-independent fields on $\tilde{M}$ which satisfy all the properties above at each $(t, x) \in \tilde{M}$. Without loss of generality we can assume additionally that the orthonormal frame $(e^i_{(1)}, e^i_{(2)}, Y^i)$ is invariant under Lie transport along $Y^i$ globally on $\Sigma$. This implies that

$$\mathcal{L}_Y E_{ij(p)} = 0$$

(6.14)
everywhere on $\tilde{M}$.

The remainder of this subsection is now devoted to some further technical properties of the fields $E_{ij(p)}$, which turn out to be useful later. Given (6.10), the following quantity related to the curl of $E_{ij(1)}$ can be written as

$$\epsilon_{(i}^{kl}\mathcal{D}[k]E_{ji)(1)} = \sqrt{3} \epsilon_{(i}^{kl}(\mathcal{D}[k]Y_{j})_{i} Y_{l} + \sqrt{3} \epsilon_{(i}^{kl}Y_{j})_{l} Y_{l}$$

$$= \sqrt{3} \epsilon_{(i}^{kl}(\mathcal{D}[k]Y_{j})_{i} Y_{l} + \sqrt{3} \lambda Y_{j}$$

where we have used the second condition of Eq. (6.11). The third condition there implies that $\mathcal{D}[k]Y_{j}$ is $h_{ij}$-trace-free. Since $Y^i$ is a conformal Killing vector field of $h_{ij}$, it therefore follows that

$$\mathcal{D}[k]Y_{j} = \mathcal{D}[k]Y_{j} = \frac{1}{2} \epsilon_{nkj}e^{ml} \mathcal{D}_m Y_{l} = \frac{\lambda}{2} \epsilon_{nkj} Y^n.$$  

(6.15)

We plug this into the expression above:

$$\epsilon_{(i}^{kl}\mathcal{D}[k]E_{ji)(1)} = \frac{1}{2} \sqrt{\frac{3}{2}} \epsilon_{i}^{kl}(\mathcal{D}[k]Y_{j})_{i} Y_{l} + \frac{1}{2} \sqrt{\frac{3}{2}} \epsilon_{j}^{kl}(\mathcal{D}[k]Y_{i})_{j} Y_{l} + \sqrt{\frac{3}{2}} \lambda Y_{j}$$

$$= \frac{1}{2} \sqrt{\frac{3}{2}} \epsilon_{i}^{kl} \left( \frac{\lambda}{2} \epsilon_{nkj} Y^n \right) Y_{l} + \frac{1}{2} \sqrt{\frac{3}{2}} \epsilon_{j}^{kl} \left( \frac{\lambda}{2} \epsilon_{nkj} Y^n \right) Y_{l} + \sqrt{\frac{3}{2}} \lambda Y_{j}.$$  

The first term is

$$\frac{\lambda}{4} \sqrt{\frac{3}{2}} \delta^{mk} \epsilon_{nkj} Y^n Y_{j} = \frac{\lambda}{4} \sqrt{3} \delta^{mk} \left( \delta^{nl} \delta^{j}_{i} - \delta^{n} \delta^{l}_{i} \right) Y^n Y_{j} = \frac{\lambda}{4} \sqrt{3} \left( Y_{j} Y_{j} - h_{ij} \right).$$

Since this is symmetric in $i$ and $j$, it follows

$$\epsilon_{(i}^{kl}\mathcal{D}[k]E_{ji)(1)} = \frac{\lambda}{4} \sqrt{3} \left( Y_{j} Y_{j} - h_{ij} \right) + \sqrt{\frac{3}{2}} \lambda Y_{j} = \frac{3}{2} \lambda E_{ij(1)},$$

(6.16)

using Eq. (6.10).

Another useful identity can be derived for the divergence of $E_{ij(1)}$. Using again Eq. (6.10), we find

$$\mathcal{D}^k E_{ik(1)} = \sqrt{\frac{3}{2}} \mathcal{D}^k (Y_{j} Y_{i} = \sqrt{3} \mathcal{D}^k (Y_{j} Y_{i},$$

which follows from the third relation in Eq. (6.3). Eq. (6.15) yields

$$\mathcal{D}^k E_{ik(1)} = \sqrt{\frac{3}{2}} Y_{j} \frac{\lambda}{2} \epsilon_{njl} Y^n = 0.$$  

(6.17)

Finally, we discuss the following

$$\epsilon_{(i}^{kl}E_{ji)(1)} = \frac{1}{\sqrt{6}} \epsilon_{(i}^{kl} (3 Y_{j} Y_{l} - h_{ijl}) = \frac{1}{\sqrt{6}} (3 \epsilon_{(i}^{kl} Y_{j} Y_{l} - \epsilon_{(i}^{kl} Y_{j} Y_{l})) = \sqrt{\frac{3}{2}} \epsilon_{(i}^{kl} Y_{j} Y_{l},$$

(6.18)
using Eq. (6.10). If now we assume in addition to the above that the orthonormal frame \((e_1^i, e_2^i, Y^i)\) is oriented according to \(\epsilon_{kij} e_1^j e_2^j Y^i = 1\), then Eqs. (6.10) – (6.12) imply

\[
\epsilon_{ij}^{kl} E_{j(i)} E_{(p)}^{ij} = \begin{cases} 
\frac{\sqrt{\eta}}{2} e_2^{k ij}, & p = 1, 2, 3, \\
\frac{\sqrt{\eta}}{2} e_2^{k ij}, & p = 4, \\
-\frac{\sqrt{\eta}}{2} e_1^{k ij}, & p = 5,
\end{cases}
\]  

(6.18)

and,

\[
\epsilon_{ij}^{kl} Y^j Y^i E_{ij(p)} = \begin{cases} 
0, & p = 1, 2, 3, \\
\frac{\sqrt{\eta}}{2} e_2^{j ij}, & p = 4, \\
-\frac{\sqrt{\eta}}{2} e_1^{j ij}, & p = 5.
\end{cases}
\]  

(6.19)

6.3 The singular initial value problem of the evolution equations

Next we are going to discuss the singular initial value problem of the evolution equations Eq. (6.4) with "data" prescribed at \(t = 0\). This is an application of the Fuchsian method developed in [2, 3] for general quasilinear symmetric hyperbolic systems. The main first step of the analysis of any singular initial value problem is to split the unknown into two parts. The first part is some explicitly known function – in most cases a generalized power series expansion about the initial time, and the free Cauchy data can be identified with some of its coefficients.

The leading-order term corresponds to a (truncated) Taylor series of the solution about the initial time, and the free Cauchy data can be identified with some of its coefficients.

There the leading-order term corresponds to a (truncated) Taylor series of the solution about the singular time \(t = 0\). The second part, i.e., the difference of the original unknown and this leading-order term, is called the remainder. It is considered as the unknown of the singular initial value problem. The aim is then to show that given the leading-order term, there exists a remainder which is uniquely determined by the equation and which decays with some sufficiently high order at \(t = 0\) relative to the leading-order term in some suitable sense.

We remark that this decomposition also applies to the standard regular initial value problem. There the leading-order term corresponds to a (truncated) Taylor series of the solution about the initial time, and the free Cauchy data can be identified with some of its coefficients.

In the case of Eq. (6.4), the main tasks are therefore, (i), to derive appropriate leading-order terms (see Section 6.4), and, (ii), to analyze the equation for the remainder. This subsection now is devoted to (ii). In a first step, we do not yet explicitly specify a leading-order term (and hence the free data). Instead we work with the following equation

\[
t \tilde{\nabla}_j E_{ij} - \eta \tilde{g}_{m(l} (\delta_k^{ij})^{k)} \tilde{\nabla}_m E_{kl} = T^{kl}_{ij} E_{kl} + E_{ij}
\]  

(6.20)

where the unknown is a complex time-dependent purely spatial \((0, 2)\)-tensor field \(E_{ij}\). The time-dependent source term field \(E_{ij}\) is considered as given in most of the following discussion. It will eventually describe the contribution from the leading-order term of the singular initial value problem to the original equation (6.4), and the unknown \(E_{ij}\) will be identified with the remainder of the singular initial value problem; see the end of Section 6.4. In this subsection it is convenient to perform index operations with the metric \(\tilde{g}_{ij}\). Notice that the second term in Eq. (6.20) differs from the corresponding term in Eq. (6.4); however, both terms are the same in the eventual case of interest when \(E_{ij}\) is symmetric and \(\tilde{g}_{ij}\)-trace-free. For the time being we allow \(E_{ij}\) and \(E_{ij}\) to be any smooth time-dependent purely spatial \((0, 2)\)-tensor fields. The purpose of the following now is to derive precise conditions on the source term field which guarantee the existence of a uniquely determined solution \(E_{ij}\) with a sufficient decay at \(t = 0\).

Under all the conditions before, Eq. (6.20) is a linear symmetric hyperbolic system with smooth coefficients (the regularity of the source term field \(E_{ij}\) has not been fixed yet). We shall therefore follow the analysis of this class of equations in [2, 3]. Notice however that in contrast to
these references, the spatial domain $\Sigma$ here may not be a torus. In fact, the techniques presented below shall have a particular emphasis on localization and "domain of dependence" arguments; the torus will still play an intermediate role as we will see.

We derive now the fundamental energy estimates for solutions of Eq. (6.20). In contrast to the estimate in [2, 3], we shall here take the tensorial character of the unknown $E_{ij}$ explicitly into account and, in addition, localize the estimate. To this end, we suppose that the field $E_{ij}$ in Eq. (6.20) is defined on some open subset $\Omega$ of $M$ (not necessarily a lens-shaped region) and is sufficiently smooth there (we specify the minimal regularity later). Suppose also that $E_{ij}$ is a sufficiently smooth solution of Eq. (6.20) defined on $\Omega$. We find

$$t\nabla_0 E_{ij} + i t \partial_m E_{ij} \bar{\nabla}_m = T_{kl} E_{ij} + \bar{E}_{ij},$$

where, as before, complex conjugates are denoted with a bar. Hence, for any $\mu \in \mathbb{R}$, we have

$$t \tilde{\nabla}_0 \left( t^{-2} \bar{E}_{ij} E_{ij} \right) = -2 \mu t^{-2} \bar{E}_{ij} E_{ij}$$

$$+ i t^{-2} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m + T_{kl} \bar{E}_{ij} + \bar{E}_{ij}$$

$$+ i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m + T_{kl} \bar{E}_{ij} + \bar{E}_{ij}$$

$$= -2 \mu t^{-2} \bar{E}_{ij} E_{ij}$$

$$+ i t^{-2} \left( E_{ij} T_{kl} \bar{E}_{ij} + \bar{E}_{ij} T_{kl} \bar{E}_{ij} \right)$$

$$+ i t^{-2} \mu \left( E_{ij} \bar{E}_{ij} + \bar{E}_{ij} E_{ij} \right) - i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m + i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m + \bar{\nabla}_m + \bar{\nabla}_m \left( i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m \right).$$

The last two terms can be written as follows

$$- i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m + i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m$$

$$= - i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) V_m + i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m$$

$$= - i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) V_m + i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m$$

$$= \tilde{\nabla}_m \left( i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m \right) = \tilde{\nabla}_m \left( i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m \right).$$

We have therefore found the following identity

$$t \tilde{\nabla}_0 \left( t^{-2} \bar{E}_{ij} E_{ij} \right) = - i t^{-2} \bar{E}_{ij} \left( 2 \mu t^{-1} \bar{E}_{ij} - T_{ij} \bar{E}_{ij} \right) E_{kl}$$

$$= M_{kl} E_{ij}$$

$$+ i t^{-2} \mu \left( \bar{E}_{ij} E_{ij} + \bar{E}_{ij} E_{ij} \right) + \tilde{\nabla}_m \left( i t^{-2} \mu t^{-1} \partial_m \left( t \partial_m \bar{E}_{ij} \right) \bar{\nabla}_m \right).$$

If $\delta > 0$ is now sufficiently small, the linear map $M_{ij}^{kl}$ on the complex vector space of $(0,2)$-tensors at any given $(t, x) \in \Omega$ is positive definite with respect to the scalar product

$$(E_{ij}, \bar{E}_{ij}) \rightarrow E_{ij} \bar{E}_{ij},$$

provided the linear map

$$M_{ij}^{kl} := 2 \mu \delta^{i} \bar{\delta}^{j} - T_{ij}^{kl} - T_{ij}^{kl}.$$

(6.23)
is positive definite at \( t = 0 \). Recall that index operations are performed with the metric \( \tilde{g}_{ij} \) in this subsection and hence the inner product in \((6.22)\) is distinct from the one in Eq. \((6.9)\).

Using that \( \tilde{g}_{ij} \) equals the metric \( h_{ij} \) at \( t = 0 \) according to Eq. \((6.1)\), we conclude that the endomorphism \( M^{kl}_{ij(0)} \) is self-adjoint, and it is therefore positive definite at each \( (0, x) \in \Sigma_0 \) provided that \( \mu \) is strictly larger than the largest eigenvalue of the endomorphism \( T^{kl}_{ij(0)} \).

According to the results in Section 6.2, Eq. \((6.21)\) therefore implies the inequality

\[
t \tilde{\nabla}_0 \left( t^{-2\mu} E^{ij} E_{ij} \right) \leq t^{-2\mu} \left( \tilde{E}^{ij} E_{ij} + \tilde{E}^{ij} F_{ij} \right) + \tilde{\nabla}_m \left( t^{-2\mu+1} \tilde{E}^{ij} \eta^{-m(l)} (\delta^i j^k) E_{kl} \right),
\]

for any \((t, x) \in \Omega\) provided that \( \mu \) is any constant larger than \( 3/4 \) and \( \delta \) is any sufficiently small positive constant.

Let us now, for the time being, pick \( \Omega = M \) and \( \Sigma = T^3 \). Given this, let us, first, replace the spatial covariant derivative \( \nabla_m \) (defined with respect to \( \tilde{g}_{ij} \)) by \( \mathcal{D}_m \) (defined with respect to \( h_{ij} \)) in Eq. \((6.21)\); this yields an additional (tensorial) contribution of \( O(t) \) to \( M^{kl}_{ij} \) in Eq. \((6.21)\).

For any sufficiently small \( \delta > 0 \), this new map \( M^{kl}_{ij} \) is therefore positive definite under the same conditions as above. Second, noting that \( \tilde{\nabla}_0 \) acts on a scalar function in Eq. \((6.21)\), it can therefore be interpreted as the directional derivative along \( \tilde{N}^\mu \). When we now introduce arbitrary coordinates \((t, x')\) on \( M \), for example (but not necessarily) Gauss coordinates, for the same time function \( t \) as above and hence obtain that \( \tilde{N}^\mu = \alpha \partial^\mu + \beta^\mu \) for a strictly positive function \( \alpha \) and a purely spatial vector field \( \beta^\mu \), the spatial derivative associated with the \( \beta^\mu \)-term yields an additional contribution to the last and to the first term on the right side of Eq. \((6.21)\).

The contribution to the first term is \( O(t) \) and does therefore not change the positivity criterion of the new map \( M^{kl}_{ij} \). Third, we integrate over the spatial domain \( \Sigma_t = T^3 \) with respect to the volume element of \( h_{ij} \) at any \( t \in (0, \delta) \). The last term in Eq. \((6.21)\) (including the additional contribution obtained in the second step above) then disappears as a consequence of Stokes’ theorem. If \((e^i_a) \) \((a, b, \ldots = 1, 2, 3)\) is any smooth \( \tilde{g}_{ij} \)-orthonormal frame intrinsic to each surface \( \Sigma_t \) (for example, consider the frame in Section 6.2) and we denote by \( \hat{E}(t, x) \) the 9-dimensional complex vector of frame components of \( E_{ij}(t, x) \) and by \( \hat{F}(t, x) \) the corresponding vector for \( F_{ij}(t, x) \), then

\[
\frac{d}{dt} \left\| t^{-\mu} E(t, \cdot) \right\|_{L^2(T^3, h_{ij})}^2 \leq 2C \text{Re} \left\langle t^{-\mu} E(t, \cdot), t^{-\mu} \hat{F}(t, \cdot) \right\rangle_{L^2(T^3, h_{ij})},
\]

for some uniform constant \( C > 0 \) (determined by the function \( \alpha \) above). In this notation, the metric \( h_{ij} \) determines the volume element with respect to which the norm and scalar product are defined. An application of the Cauchy-Schwarz inequality and integration with respect to \( t \) over any interval \((t_0, t)\) with \( 0 < t_0 < t < \delta \) yield

\[
\left\| t^{-\mu} E(t, \cdot) \right\|_{L^2(T^3, h_{ij})} \leq C \left( \left\| t^{-\mu} E(t_0, \cdot) \right\|_{L^2(T^3, h_{ij})} + \int_{t_0}^{t} \left\| s^{-\mu} \hat{F}(s, \cdot) \right\|_{L^2(T^3, h_{ij})} s^{-1} ds \right); \quad (6.26)
\]

see Section 7.2 in [36] for some of the basic technical steps going from \((6.25)\) to \((6.26)\). By adapting the constant \( C \) we can conclude that

\[
\left\| t^{-\mu} E(t, \cdot) \right\|_{L^2(T^3)} \leq C \left( \left\| t^{-\mu} E(t_0, \cdot) \right\|_{L^2(T^3)} + \int_{t_0}^{t} \left\| s^{-\mu} \hat{F}(s, \cdot) \right\|_{L^2(T^3)} s^{-1} ds \right),
\]

for any \( t_0 > 0 \) and \( t \in [t_0, \delta) \). These norms are now defined with respect to the Euclidean flat metric on \( T^3 \). Thanks to this estimate, we are now in the position to apply the methods in [2, 3] (see Props. 2.10 and 2.12 in [2], and, Prop. 3.5 in [3]), to establish the following fundamental existence result.
Proposition 6.2 ($T^3$-existence and uniqueness: Evolution equations) Pick any sufficiently small $\delta > 0$, any integer $q > 3/2 + 1$ and set $\Sigma = T^3$. Equip $M = (-\delta, \delta) \times \Sigma$ with a smooth Lorentzian metric $\tilde{g}_{\mu\nu}$ with a conformal Killing vector field $\tilde{X}^\mu$ without zeros on $\mathcal{I}^+ = \Sigma_0$. Suppose that Eqs. (6.1) - (6.3) hold. Consider any complex field $\tilde{E}_{ij}$ on $M = (0, \delta) \times \Sigma$ whose orthonormal frame components $\tilde{E}_{ab}$ are in $X_{\delta,\nu,q}(T^3)$ for some constant $\nu > 3/4$. Then Eq. (6.20) has a classical solution $E_{ij}$ defined on $M$ whose orthonormal frame components $E_{ab}$ are in $X_{\delta,\nu,q}(T^3)$ for any constant $\mu < \nu$. Moreover, all functions $t \tilde{N}(E_{ab})$ are in $X_{\delta,\mu,q-1}$ for the same $\mu$. This solution $E_{ij}$ is uniquely determined within the class of all fields $\tilde{E}_{ij}$ defined on $M$ whose orthonormal frame components $\tilde{E}_{ab}$ are continuously differentiable and which satisfy

$$\lim_{t \to 0} \|t^{-\eta} \tilde{E}(t)\|_{L^2(T^3)} = 0$$

for some constant $\eta > 3/4$.

The Banach spaces $X_{\delta,\mu,q}(T^3)$ are defined [2, 3] as the completion of the set of smooth functions $f$ on $(0, \delta) \times T^3$ with respect to the norm

$$\|f\|_{\delta,\mu,q} := \sup_{t \in (0, \delta)} \|t^{-\mu}f(t, \cdot)\|_{H^q(T^3)},$$

where $H^q(T^3)$ is the standard $L^2$-Sobolev space on $T^3$ (with respect to the flat metric) of differentiability order $q$. Notice that in general the exponent $\mu$ is allowed to be any smooth function on $T^3$; for the purpose of our studies here it is sufficient to work with constant exponents $\mu$ exclusively. The interested reader can find the details in the references above and in [4, 10].

Proposition 6.2 is our fundamental existence and uniqueness result. However, several of its aspects are not fully satisfying. The first issue we address now is the restriction $\Sigma = T^3$. We thereby obtain a localized existence and uniqueness result.

To this end, we shall first determine the characteristics of the hyperbolic system Eq. (6.20). The characteristic surfaces are expected to bound the domain of dependence of solutions of the singular initial value problem. By definition, the normal to any characteristic surface at $(t, x)$ in $M$ is a spacetime covector $(\xi_0, \xi_m)$ for which the determinant of the principal symbol of Eq. (6.20)

$$\xi_0 \delta^{\ell}_{\xi_0} - i \xi_m \tilde{m}^{(\ell \delta^{\ell}_{\xi_0})}_{(i \delta_{ij})}$$

vanishes; this is interpreted as an endomorphism on the complex vector space of $(0, 2)$-tensors at $(t, x)$. For any choice of spatial covector $\xi_m$, this is the case if and only if $\xi_0$ is an eigenvalue of the linear map

$$i \xi_m \tilde{m}^{(\ell \delta^{\ell}_{\xi_0})}_{(i \delta_{ij})}.$$  (6.27)

By straightforward calculations we show that this map is diagonalizable on our 9-dimensional complex vector space. The eigenspace of the eigenvalue 0 is 5-dimensional, and the eigenspace of each of the four eigenvalues $\pm |\xi|/2$ and $\pm |\xi|$ is 1-dimensional where $|\xi| = (\tilde{g}^{ij} \xi_i \xi_j)^{1/2}$. Solutions to Eq. (6.20) therefore propagate at either speed 0, half of the speed of light or the full speed of light.

Consider now any non-empty open subset $S_0$ of $\Sigma_0$ with compact closure and smooth non-empty boundary and pick a lens-shaped region $\Omega$ with respect to $S_0$. Suppose first now that $\tilde{E}_{ij}$ vanishes identically on $\Omega$, and that $E_{ij}$ is a solution of Eq. (6.20) defined on $\Omega$ whose orthonormal frame components (chosen as in Proposition 6.2) are $C^1$-functions on $\Omega$ and, for any $t$ and $t_0$ with $0 < t_0 < t < \delta$, extend as $C^1$-functions to the boundary of $\Omega(t_0, t)$. Integrating (6.24) over
$\Omega_{(t_0,t)}$ with respect to the volume element of the metric $\tilde{g}_{\mu\nu}$ assuming $\mu > 3/4$ and applying Stokes’ theorem, we get

$$\int_{\Omega_t} t^{-2\mu} E^{ij} E_{ij}(t) \text{Vol}_{\tilde{g}_{ij}(t)} \leq \int_{\Omega_{t_0}} t^{-2\mu} E^{ij} E_{ij}(t_0) \text{Vol}_{\tilde{g}_{ij}(t_0)}.$$ \hspace{1cm} (6.28)

Recall here that the boundary of the lens-shaped region is spacelike everywhere and hence all remaining boundary integrals implied by Stokes' theorem have a “good” sign thanks to the properties of the characteristics established above. If now we assume that

$$\lim_{t \searrow 0} \int_{\Omega_t} t^{-2\mu} E^{ij}(t,x) E_{ij}(t,x) \text{Vol}_{\tilde{g}_{ij}(t)} = 0,$$

and we take the limit $t_0 \searrow 0$ in Eq. (6.28), we conclude that $E_{ij}(t,x) = 0$ for all $(t,x) \in \Omega$. Because $\tilde{g}_{ij}$ is a smooth Riemannian metric near $t = 0$, this is the case if and only if

$$\lim_{t \searrow 0} \int_{\Omega_t} t^{-2\mu} E^{ij}(t,x) E_{ij}(t,x) dx = 0,$$

where $dx$ denotes the flat volume element. This completes the proof of the following lemma.

**Lemma 6.3** Pick any sufficiently small $\delta > 0$ and orientable 3-dimensional differentiable manifold $\Sigma$. Equip $M = (-\delta, \delta) \times \Sigma$ with a smooth Lorentzian metric $\tilde{g}_{\mu\nu}$ with a conformal Killing vector field $\tilde{X}^\mu$ without zeros on $\mathcal{J}^- = \Sigma_0$. Suppose that Eqs. (6.1) - (6.3) hold. Pick any non-empty open subset $S_0$ of $\Sigma$ with compact closure and non-empty smooth boundary, and, any lens-shaped region $\Omega$ with respect to the subset $\{t = 0\} \times S_0$ of $\partial M \subset M$ where $M = (0, \delta) \times \Sigma \subset M$. Suppose that $E_{ij}$ vanishes identically on $\Omega$, and that $E_{ij}$ is a solution of Eq. (6.20) defined on $\Omega$ whose orthonormal frame components are $C^1$-functions on $\Omega$, and, for any $t$ and $t_0$ with $0 < t_0 < t < \delta$, extend as $C^1$-functions to the boundary of $\Omega_{(t_0,t)}$. If in addition

$$\lim_{t \searrow 0} \int_{\Omega_t} t^{-2\mu} E^{ij}(t,x) E_{ij}(t,x) dx = 0$$ \hspace{1cm} (6.29)

for some $\mu > 3/4$, then $E_{ij}(t,x) = 0$ for all $(t,x) \in \Omega$.

Due to the linearity of the equations, this lemma yields conditions under which the values of the solution $E_{ij}$ on $\Omega$ are guaranteed to be independent of the values of the source term $E_{ij}$ outside of $\Omega$.

We remark that this local uniqueness statement here is more general than Lemma 4.14 in [27] because only the class of solutions of the MST evolution equations which extend as $C^1$-functions through $t = 0$ is considered there. We will see below that the evolution equations admit solutions which violate this property. Only towards the end of this whole section we will see how to restrict the class of solutions appropriately in order to establish smoothness through $t = 0$; see Section 6.6.

In consistency with the localized character of Theorem 6.1 (and therefore Lemma 6.3) we shall henceforth ignore the dynamics outside of $\Omega$ now. In particular, since $\Omega$ is a proper subset of $M$, it is no loss of generality to pick $\Sigma = T^3$ as in Proposition 6.2 now in all of what follows. Suppose that $E_{ij}$ is any smooth field on $\Omega$ which can be extended smoothly to $M = (0, \delta) \times T^3$ such that the vector of its frame components satisfies

$$\lim_{t \searrow 0} \|t^{-\mu} F(t,\cdot)\|_{H^4(T^3)} = 0$$
for some $\nu > 3/4$ and for all positive integers $q$. This is equivalent to the condition that this extended field is in $X_{\delta,\nu,q}(T^3)$ for some $\nu > 3/4$ for all positive integers $q$. Proposition 6.2 therefore implies the existence of a classical (in fact smooth) solution $E_{ij}$ on $M$. Since the vector $E$ of its orthonormal frame components is in $X_{\delta,\nu,q}$ for any $\mu < \nu$, it is therefore possible to satisfy Eq. (6.29) for some $\mu \in (3/4, \nu)$. Lemma 6.3 thus implies that the restriction of $E_{ij}$ to $\Omega$ is unaffected by the extension of $\tilde{E}_{ij}$ to $M = (0, \delta) \times T^3$.

**Proposition 6.4** (Local existence and uniqueness: Evolution equations) Pick any sufficiently small $\delta > 0$ and orientable 3-dimensional differentiable manifold $\Sigma$. Equip $\tilde{M} = (-\delta, \delta) \times \Sigma$ with a smooth Lorentzian metric $\tilde{g}_{\mu\nu}$ with a conformal Killing vector field $\tilde{X}^\mu$ without zeros on $\mathcal{I}^- = \Sigma_0$. Suppose that Eqs. (6.1) - (6.3) hold. Pick any non-empty open subset $S_0$ of $\Sigma$ with compact closure and non-empty smooth boundary, and, any lens-shaped region $\Omega$ with respect to the subset $\{t = 0\} \times S_0$ of $\partial M \subset M$ where $M = (0, \delta) \times \Sigma \subset M$. Consider any smooth complex field $\tilde{E}_{ij}$ on $\Omega$ which can be extended smoothly to $(0, \delta) \times T^3$ (considering $S_0$ as a subset of $T^3$) such that the orthonormal frame components of the extended field are in $X_{\delta,\nu,q}(T^3)$ for some constant $\nu > 3/4$ and every positive integer $q$. Then Eq. (6.20) has a smooth solution $\bar{E}_{ij}$ defined on $\Omega$ that satisfies

$$\lim_{t \to 0} \int_{\Omega_t} t^{-2\mu} \bar{E}^{ij}(t, x) \bar{E}_{ij}(t, x) dx = 0$$

(6.30)

for any $\mu < \nu$. The same estimate holds for any spatial derivative of any order of the frame components of $E_{ij}$. If there is any other solution $\bar{E}_{ij}$ on $\Omega$ whose orthonormal frame components are $C^1$-functions on $\Omega$, and, for any $t$ and $t_0$ with $0 < t_0 < t < \delta$, extend as $C^1$-functions to the boundary of $\Omega_{(t_0, t)}$, and, has the property

$$\lim_{t \to 0} \int_{\Omega_t} t^{-2\mu} \tilde{E}^{ij}(t, x) \bar{E}_{ij}(t, x) dx = 0$$

for some $\mu > 3/4$, then

$$\bar{E}_{ij}(t, x) = E_{ij}(t, x)$$

for every $(t, x) \in \Omega$.

The following is a direct consequence of the symmetry and trace-free-ness of all terms in Eqs. (6.4) and (6.5).

**Corollary 6.5** In addition to the hypothesis of Proposition 6.4, suppose that the field $E_{ij}$ is $\tilde{g}_{ij}$-traceless and symmetric on $\Omega$. Then the solution $E_{ij}$ asserted by Proposition 6.4 is $\tilde{g}_{ij}$-traceless and symmetric.

More specifically, this corollary is proved by first decomposing the unknown into its symmetric trace-free part, its antisymmetric part and its pure trace part. Due to the symmetry and trace-free-ness of all terms in the equations, the first part can be handled as above, while the equations for the second and third parts become trivial. All solutions compatible with condition (6.30) therefore have vanishing antisymmetric and pure trace parts.

### 6.4 The leading-order term

According to Section 6.2 there are real time-independent symmetric, $h_{ij}$-trace-free purely spatial tensor fields $E_{ij}(1), \ldots, E_{ij}(5)$ which form an orthonormal basis of the complex vector space of symmetric and $h_{ij}$-trace-free purely spatial tensors at any $(t, x)$ in $\tilde{M}$. Any complex symmetric
Hence $E$ is necessary here because the fields $\eta_{ij}$, for complex functions $f_1, \ldots, f_6$ (whose regularity shall be specified later). The $O(t^2)$-correction is necessary here because the fields $E_{ij(p)}$ are $h_{ij}$- and therefore not $\eta_{ij}$-trace-free.

Now we assume Gaussian coordinates $(t, x')$ where $t$ is the time function introduced earlier. Hence $\tilde{N}^\mu = \partial_t^\mu$. We plug (6.31) into (6.4) with (6.5), noticing that due to Eqs. (5.11) and (6.1) the additional terms picked up when $\tilde{\nabla}_0$ is expressed in terms of $\partial_t^\mu$ contribute smoothly to the last term of (6.5) only. Then the properties of the fields $E_{ij(p)}$ established before imply the system (index operations are performed with the metric $h_{ij}$ for this subsection)

$$t\partial_t f_1 + f_1 = 2i\sum_{p=1}^{5} \tilde{\eta}_{mn} g^{mk} g^{nl} \tilde{\nabla}_k (f_p E_{jl(p)}) E^{ij}_{(1)} - \frac{i \lambda}{2} f_1 + t^2 \sum_{p=1}^{5} T_p^t f_p,$$  

$$t\partial_t f_2 = i 2 \sum_{p=1}^{5} \tilde{\eta}_{mn} g^{mk} g^{nl} \tilde{\nabla}_k (f_p E_{jl(p)}) E^{ij}_{(2)} + t^2 \sum_{p=1}^{5} T_p^t f_p,$$  

$$t\partial_t f_3 = i 2 \sum_{p=1}^{5} \tilde{\eta}_{mn} g^{mk} g^{nl} \tilde{\nabla}_k (f_p E_{jl(p)}) E^{ij}_{(3)} + t^2 \sum_{p=1}^{5} T_p^t f_p,$$  

$$t\partial_t f_4 = \frac{3}{4} f_4 = i 2 \sum_{p=1}^{5} \tilde{\eta}_{mn} g^{mk} g^{nl} \tilde{\nabla}_k (f_p E_{jl(p)}) E^{ij}_{(4)} + \frac{3}{4} \frac{i \lambda}{2} f_4 + t^2 \sum_{p=1}^{5} T_p^t f_p,$$  

$$t\partial_t f_5 = \frac{3}{4} f_5 = i 2 \sum_{p=1}^{5} \tilde{\eta}_{mn} g^{mk} g^{nl} \tilde{\nabla}_k (f_p E_{jl(p)}) E^{ij}_{(5)} + \frac{3}{4} \frac{i \lambda}{2} f_5 + t^2 \sum_{p=1}^{5} T_p^t f_p,$$

for smooth, in principle known functions $T^p_q(t, x)$ defined for all $p, q = 1, \ldots, 5$. The aim is now to derive expansions of the solutions of this system which shall determine the leading-order term of our singular initial value problem eventually.

It is ok to use slightly loose and imprecise language in a first step now because we will eventually justify the resulting expressions fully rigorously. With this in mind we suppose now that $f_1 = O(t^{-1})$, and $f_2, f_3, f_4, f_5 = O(1)$ at $t = 0$, possibly, with additional log $t$ factors which we control formally by incorporating an arbitrarily small positive constant $\eta > 0$ into the following arguments. Under this assumption, Eq. (6.32) can be simplified as follows

$$t\partial_t f_1 + f_1 = i \epsilon_{ij} \partial_k (f_1 E_{jk(1)}) E^{ij}_{(1)} - \frac{i \lambda}{2} f_1 + O(t^{1-\eta}),$$

where we recall that $\epsilon_{ijk}$ and $\partial_k$ are the volume form and covariant derivative associated with $h_{ij}$. The family of functions

$$f_1(t, x) = c_1(x) t^{-1} + i \epsilon_{ij} \partial_k (c_1(x) E_{jk(1)}(x)) E^{ij}_{(1)} + \frac{i \lambda}{2} \tilde{c}_1(x)$$

given by an arbitrary (time independent) complex function $c_1$ satisfies this equation, i.e., it represents the leading terms of a formal expansion of Eq. (6.32). Given this, (6.33) and (6.34) become (we refrain from writing the arguments $x$ and $t$ now)

$$t\partial_t f_{2,3} = i \epsilon_{ij} \partial_k (c_1 E_{jk(1)}) E^{ij}_{(2,3)} + O(t^{1-\eta}),$$

38
for which we find
\[ f_{2,3} = i e^{ki} \mathcal{H}(c_1 E_{ij(1)}) E_{ij(2,3)}^{(2)} \log t + c_{2,3}, \]
for arbitrary complex functions \( c_2(x) \) and \( c_3(x) \). Finally, (6.35) and (6.36) are written as
\[ t \partial_t f_{4,5} - \frac{3}{4} f_{4,5} = i e^{ki} \mathcal{H}(c_1 E_{ij(1)}) E_{ij(4,5)}^{(1)} + O(t^{1-\eta}), \]
which leads to
\[ f_{4,5} = -\frac{4i}{3} e^{ki} \mathcal{H}(c_1 E_{ij(1)}) E_{ij(4,5)}^{(1)} + c_{4,5} t^{3/4}, \]
for arbitrary complex functions \( c_4(x) \) and \( c_5(x) \).

These formal leading-order expressions for the functions \( f_1, \ldots, f_5 \) can be simplified significantly using
\[ i e^{ki} \mathcal{H}(c_1 E_{ij(1)}) E_{ij(p)}^{(1)} = i e^{ki} \mathcal{H}(c_1 E_{ij(1)}) E_{ij(p)}^{(1)} \mathcal{H} c_1 + \frac{3i}{2} c_1 \delta_{p,1}, \tag{6.37} \]
which holds for every \( p = 1, \ldots, 5 \) as a consequence of Eqs. (6.16) and (6.13), and using Eq. (6.18). Observe in particular that the logarithmic terms drop out thanks to Eq. (6.18). Given all this and Eq. (6.31), we set
\[
\mathcal{E}_{kl,s}(t, x) := (c_1(x) t^{-1} + i c_1(x)) E_{kl(1)}(x) + c_2(x) E_{kl(2)}(x) + c_3(x) E_{kl(3)}(x) \\
+ \left( c_4(x) t^{3/4} - \frac{2i}{\sqrt{3}} \mathcal{H}_{ij} c_1(x) \right) E_{kl(4)}(x) \\
+ \left( c_5(x) t^{3/4} + \frac{2i}{\sqrt{3}} \mathcal{H}_{ij} c_1(x) \right) E_{kl(5)}(x) \\
= c_1(x) t^{-1} E_{kl(1)}(x) \\
+ i c_1(x) E_{kl(1)}(x) + c_2(x) E_{kl(2)}(x) + c_3(x) E_{kl(3)}(x) \\
- \frac{2i}{\sqrt{3}} \mathcal{H}_{ij} c_1(x) E_{kl(4)}(x) + \frac{2i}{\sqrt{3}} \mathcal{H}_{ij} c_1(x) E_{kl(5)}(x) \\
+ t^{3/4} \left( c_4(x) E_{kl(4)}(x) + c_5(x) t^{3/4} E_{kl(5)}(x) \right),
\tag{6.38} \]
which is determined by arbitrary time-independent complex functions \( c_1, \ldots, c_5 \). Anticipating the following results we shall refer to this field as the **leading-order term** of our singular initial value problem, and the functions \( c_1, \ldots, c_5 \) as the (singular) data.

Let us now assume as in Proposition 6.4 that \( \Sigma \) is any 3-dimensional orientable manifold, \( \delta > 0 \) is sufficiently small, \( S_0 \) is a non-empty open subset of \( \Sigma_0 \) with compact closure and smooth non-empty boundary, and \( \Omega \) is a lens-shaped region with respect to \( S_0 \). Let us further suppose that \( c_1, \ldots, c_5 \) are smooth complex functions on \( S_0 \). Since \( \Omega \) is a proper subset of \( M \), we can assume without loss of generality that \( \Sigma = T^3 \) and that \( c_1, \ldots, c_5 \) have been extended as smooth functions to \( T^3 \) (where as before we consider \( S_0 \) as a subset of \( T^3 \)). The field \( \mathcal{E}_{ij, s} \) defined in Eq. (6.38) can then be interpreted as a smooth field for all \((t, x) \in (0, \delta) \times T^3 \). When we now write \( \mathcal{E}_{ij} = \mathcal{E}_{ij, s} + \mathcal{E}_{ij} \) and plug this into Eq. (6.4), we obtain Eq. (6.20) with
\[
\mathcal{E}_{ij} := -t \partial_t \mathcal{E}_{ij, s} + i \eta^{m(t)}_{\delta_j} e_m^{kl} \mathcal{E}_{kl, s} + T^{kl}_{ij} \mathcal{E}_{kl, s},
\tag{6.39} \]
which is therefore also a smooth field on \((0, \delta] \times T^3 \). We check easily that the frame components satisfy
\[ \mathcal{E}_{ab} \in X_{\delta,1,\infty}(T^3). \]

The following proposition is then a consequence of Proposition 6.4 and Corollary 6.5.
**Proposition 6.6** Consider the hypothesis of Proposition 6.4, introduce Gauss coordinates and let \(E_{ij}\) be given by Eqs. (6.38) and (6.39) on \(\Omega\) for arbitrary smooth time-independent complex functions \(c_1, \ldots, c_5\). The solution \(E_{ij}\) asserted by Proposition 6.4 for \(\nu = 1\) gives rise to a smooth solution \(\tilde{E}_{ij} = E_{ij,\ast} + E_{ij}\) of the Mars-Simon evolution equations (6.4) on \(\Omega\). In particular, the field \(E_{ij}\) is symmetric and \(\tilde{g}_{ij}\)-trace-free.

### 6.5 The constraints

We have now constructed smooth solutions \(E_{ij}\) of the singular initial value problem of the evolution equations (6.4) on certain subsets \(\Omega\) of \(M\). Through the relation \(\hat{T}_{ij} = \hat{\tau}_{00ij}\), this determines the field \(\hat{T}_{\mu\nu\sigma\rho}\) on \(\Omega\) which is supposed to be a MST. The next question we need to address is whether, given any such solution, the constraint violation quantities \(\tilde{Z}_i\) (see Eq. (6.7)) vanish identically on \(\Omega\). Only if this is the case, this field \(\hat{T}_{\mu\nu\sigma\rho}\) is a solution of the full Mars-Simon equations (5.6) on \(\Omega\) and therefore a MST.

First we need to make sure that the Buchdahl condition, see Sections 4.3.2 and 5.3.2, holds. To this end, we shall now assume that the background spacetime has a vanishing MST. The Buchdahl condition then reduces to Eq. (5.51). This means that we shall from now on only accept those solutions asserted by Proposition 6.6 for which the associated tensor field \(r_{\mu\nu\sigma\rho} = (\hat{\mathcal{L}}_g + \hat{F})\hat{T}_{\mu\nu\sigma\rho}\) vanishes identically on \(\Omega\). According to the discussion in Section 5.3.2, this tensor field satisfies the same evolution equations as \(\hat{T}_{\mu\nu\sigma\rho}\) and has the same algebraic properties as \(\hat{T}_{\mu\nu\sigma\rho}\). When Lemma 6.3 is applied to the tensor field \(\hat{r}_{ij} = \hat{r}_{00ij}\) instead of \(E_{ij}\), it follows that \(\hat{r}_{ij}\) and thereby \(r_{\mu\nu\sigma\rho}\), vanish identically on \(\Omega\) if the vector \(\hat{r}\) of its orthonormal frame components satisfies

\[
\lim_{t \searrow 0} \| t^{-\mu} \hat{r}(t, \cdot) \|_{L^2(\Omega)} = 0
\]

for some \(\mu > 3/4\). We conclude from Eqs. (5.54), (5.53) and (5.41) that this is the case if the orthonormal frame components of \(\mathcal{L}_Y \tilde{E}_{ij}\) vanish in this same sense in the limit \(t \searrow 0\). Because of Eq. (6.14), this is the case for the class of solutions given by Proposition 6.6, if the Lie derivatives of the data \(c_1, \ldots, c_5\) with respect to \(Y^i\) vanish.

**Lemma 6.7** Consider the hypothesis of Proposition 6.6. Suppose in addition that the MST of the background spacetime vanishes identically on \(\Omega\) and that

\[
\mathcal{L}_Y c_1 = \ldots = \mathcal{L}_Y c_5 = 0.
\]

Then the solution \(E_{ij}\) asserted by Proposition 6.6 of the evolution equations (6.4) satisfies the Buchdahl condition Eq. (5.50) on \(\Omega\).

Recall Section 4.3 where we established the constraint propagation system and discussed the role of the Buchdahl condition. Close to \(t = 0\), this system takes the form

\[
t\hat{\nabla}_i\tilde{Z}_j = \frac{i}{2}t\hat{U}_j^k\hat{U}_j\tilde{Z}_k + \frac{1}{4} (\delta^k {\cal k} + 13 Y^k) \tilde{Z}_k + O(t)\tilde{Z}_i = 0,
\]

where index operations are performed with the metric \(\tilde{g}_{ij}\).

Let us now make the same assumptions as in Lemma 6.7 where \(E_{ij} = E_{ij,\ast} + E_{ij}\) is the smooth solution asserted by Proposition 6.6. The fields \(\tilde{Z}_i\) are determined by Eq. (6.7) from \(E_{ij}\), which are therefore also smooth fields on \(\Omega\). With similar arguments as in Section 6.3, we establish
from Eq. (6.41) that

\[ t\nabla_0(t^{-2\mu}\tilde{\Xi}^i) = -\nabla_j\left(\frac{1}{2}t^{-2\mu+1}e^{ij}_{\ k}\tilde{\Xi}^k\right) \]

\[-\frac{1}{2}t^{-2\mu}\tilde{\Xi}^i\left((1+4\mu)\delta^k + 13Y^k\right)\tilde{\Xi}_k + O(t^{-2\mu}\tilde{\Xi}^i).\]

By choosing a time-dependent orthonormal frame with respect to \( \tilde{g}_{ij} \) which is smooth through \( t = 0 \) and which has the property that one of the frame vector fields agrees with \( Y^i \) at \( t = 0 \), we see that the map \( (1+4\mu)\delta^k + 13Y^k \) on the complex vector space of \((0,1)\)-tensors at any \((t,x)\) in \( \Omega \) is positive definite if \( \mu > -1/4 \). Similar to our discussion in Section 6.3, we then perform an integration over any spacetime slab \( \Omega(t_0,t) \subset M \) with \( 0 < t_0 \leq t < \delta \) with respect to the spacetime metric \( \tilde{g}_{\mu\nu} \). As a consequence of a characteristic analysis of Eq. (6.41), which establishes that all characteristic eigenvalues are 0 or \( \pm|\xi|/2 \), all additional boundary terms which arise from the application of Stokes’ theorem over the lens-shaped region have a “good” sign if \( \delta > 0 \) is sufficiently small. We conclude that the constraint violation quantities \( \tilde{\Xi} \) vanish identically on \( \Omega \) provided \( \lim_{t\to 0} ||t^{-\mu}\tilde{\Xi}(t,\cdot)||_{L^2(\Omega)} = 0 \) for some \( \mu > -1/4 \) where \( \tilde{\Xi} \) is the vector of orthonormal frame components.

When we now calculate the fields \( \tilde{\Xi}_i \) using Eqs. (6.7) for any solution \( \mathcal{E}_{ij} = \mathcal{E}_{ij,*} + \mathcal{E}_{ij} \) asserted by Proposition 6.6 with the additional hypothesis of Lemma 6.7, we find that \( \lim_{t\to 0} ||t^{-\mu}\tilde{\Xi}(t,\cdot)||_{L^2(\Omega)} = 0 \) for some \( \mu > -1/4 \) provided

\[ \mathcal{L}_Y c_1 = 0.\]

This is therefore always the case under the hypothesis of Lemma 6.7. Using Eq. (6.19) we then show that \( \lim_{t\to 0} ||t^{-\mu}\tilde{\Xi}(t,\cdot)||_{L^2(\Omega)} = 0 \) for some \( \mu > -1/4 \) provided

\[ c_4 = c_5 = 0.\]

We have therefore established the following result.

**Proposition 6.8** Consider the hypothesis of Proposition 6.6. Suppose in addition that the MST of the background spacetime vanishes identically on \( \Omega \) and that

\[ \mathcal{L}_Y c_1 = \mathcal{L}_Y c_2 = \mathcal{L}_Y c_3 = 0, \quad c_4 = c_5 = 0.\]

Then the solution \( \mathcal{E}_{ij} \) asserted by Proposition 6.6 of the evolution equations (6.4) on \( \Omega \) is a solution of the full Mars-Simon equations on \( \Omega \).

### 6.6 Smooth extendibility through \( \mathcal{I}^- \)

Any of the solutions of the Mars-Simon equations constructed in Proposition 6.8 is smooth on the lens-shaped region \( \Omega \). If the datum \( c_1 \) does not vanish, however, it is singular in the limit \( t \searrow 0 \). In fact, it has been observed in [27] that such a divergent behavior of the rescaled Mars-Simon tensor at \( \mathcal{I}^- \) needs to be expected and is in fact generic (unless certain components of the radiations field and the Cotton-York tensor of the induced metric on \( \mathcal{I}^- \) vanish). We now finally wish to show that if we subtract the leading singular term \( c_1 t^{-1}E_{kl(1)} \) from the solution \( \mathcal{E}_{ij} \), see Eq. (6.38), the resulting field extends smoothly through \( t = 0 \). Observe that the \( t^{3/4} \)-terms in Eq. (6.38) are not present as a consequence of the hypothesis of Proposition 6.8.

In fact, we attempt to show now that the remainder field \( \mathcal{E}_{ij} \) can be written as a truncated Taylor series of arbitrary order \( L > 0 \) for all \((t,x)\) ∈ \( \Omega \), i.e., that

\[ E_{ij}(t,x) = \sum_{\ell=1}^{L} E_{ij,\ell}(x)t^{\ell} + \tilde{E}_{ij}(t,x) \]

\[ (6.42) \]
for some new remainder $\tilde{E}_{ij}(t, x)$ with the property that $\lim_{\nu \to 0} \| t^{-\nu} \tilde{E}(t, \cdot) \|_{H^s(\Omega_t)} = 0$ for any $\mu < L + 1$ and any positive integer $q$, formed from the orthonormal frame components of $\tilde{E}_{ij}$.

To this end let us consider any solution $E_{ij}$ asserted by Proposition 6.6 under the additional conditions of Proposition 6.8. Then set $E_{ij}(t, x) = E_{ij} - E_{ij,*}$. Recall that $E_{ij}(t, x)$ satisfies Eq. (6.20) with $E_{ij}$ given by Eq. (6.39). Plugging Eq. (6.42) into (6.20), we find that $\tilde{E}_{ij}$ satisfies the same equation (6.20), just with a different source term

$$\tilde{E}_{ij}(t, x) := E_{ij} - \ell \delta_{ij} \sum_{\ell=1}^{L} E_{ij, \ell}(x) t^{\ell} + i t^{\ell-1} \eta_m \delta_{ij}^k \tilde{E}_{m} + \sum_{\ell=1}^{L} E_{ij, \ell}(x) t^{\ell} + T_{ij}^{kl} \sum_{\ell=1}^{L} E_{kl, \ell}(x) t^{\ell}. \quad (6.43)$$

We know that $E_{ij}$ is a smooth field through $t = 0$ and that it can hence be written as

$$E_{ij} = \sum_{\ell=1}^{L} E_{ij, \ell}(x) t^{\ell} + t^{L+1} \tilde{E}_{ij}(t, x)$$

for some smooth fields $E_{ij, \ell}(x)$ and $\tilde{E}_{ij}(t, x)$ through $t = 0$. According to Eq. (6.5) we can also write

$$T_{ij}^{kl}(t, x) = T_{ij}^{kl}(0)(x) + t \tilde{T}_{ij}^{kl}(t, x)$$

for another smooth field $\tilde{T}_{ij}^{kl}(t, x)$ through $t = 0$. All this yields that

$$\tilde{E}_{ij} = \sum_{\ell=1}^{L} \left[ E_{ij, \ell}(x) + \ell t \delta_{ij} \sum_{\ell=1}^{L} E_{ij, \ell}(x) t^{\ell} + i t^{\ell-1} \eta_m \delta_{ij}^k \tilde{E}_{m} + \sum_{\ell=1}^{L} E_{ij, \ell}(x) t^{\ell} + T_{ij}^{kl} \sum_{\ell=1}^{L} E_{kl, \ell}(x) t^{\ell} + t^{\ell+1} \tilde{T}_{ij}^{kl} \tilde{E}_{kl, \ell} \right]$$

$$+ t^{L+1} \tilde{E}_{ij}$$

$$= \sum_{\ell=1}^{L} \left[ E_{ij, \ell} + \ell t \delta_{ij} \sum_{\ell=1}^{L} E_{ij, \ell} t^{\ell} + i t^{\ell-1} \eta_m \delta_{ij}^k \tilde{E}_{m} + \sum_{\ell=1}^{L} E_{ij, \ell} t^{\ell} + T_{ij}^{kl} \sum_{\ell=1}^{L} E_{kl, \ell} t^{\ell} + t^{\ell+1} \tilde{T}_{ij}^{kl} \tilde{E}_{kl, \ell} \right]$$

$$+ t^{L+1} \left( \tilde{E}_{ij} + \ell t \delta_{ij} \sum_{\ell=1}^{L} E_{ij, \ell} t^{\ell} + i t^{\ell-1} \eta_m \delta_{ij}^k \tilde{E}_{m} + \sum_{\ell=1}^{L} E_{ij, \ell} t^{\ell} + T_{ij}^{kl} \sum_{\ell=1}^{L} E_{kl, \ell} t^{\ell} + t^{\ell+1} \tilde{T}_{ij}^{kl} \tilde{E}_{kl, \ell} \right).$$

Hence, $\lim_{\nu \to 0} \| t^{-\nu} \tilde{F}(t, \cdot) \|_{H^s(\Omega_t)} = 0$ for $\nu = L + 1$ and all $q$, formed from the frame components of $\tilde{F}_{ij}$ – and hence Proposition 6.4 applied to the equation for $\tilde{F}_{ij}$ discussed above establishes that the uniquely determined solution $\tilde{E}_{ij}$ on $\Omega$ has the required properties – provided the following finite hierarchy of linear algebraic equations has a solution

$$T_{ij}^{kl}(0) - \delta_{ij} \delta_{ij}^k \delta_{ij} \delta_{ij} E_{kl, 1} = -E_{ij, 1},$$

$$T_{ij}^{kl}(0) - \delta_{ij} \delta_{ij}^k \delta_{ij} \delta_{ij} E_{kl, -1} = -E_{ij, -1} + i \eta_m \eta_{m} \delta_{ij}^k \tilde{E}_{m} + T_{ij}^{kl} \sum_{\ell=1}^{L} E_{kl, \ell} t^{\ell} + t^{\ell+1} \tilde{T}_{ij}^{kl} \tilde{E}_{kl, \ell}$$

for all $\ell = 2, \ldots, L$. Indeed it does. The first equation has a unique solution $E_{kl, 1}$ since 1 is not an eigenvalue of $T_{ij}^{kl}(0)$, see Section 6.2. Given this solution $E_{kl, 1}$, the second equation for $\ell = 2$ has a unique solution $E_{kl, 2}$ since 2 is not an eigenvalue of $T_{ij}^{kl}(0)$. If $\ell$ is any integer $2 \ldots, L$, given any solution $E_{kl, \ell-1}$, the second equation for $\ell$ has a unique solution $E_{kl, \ell}$ since $\ell$ is not an eigenvalue of $T_{ij}^{kl}(0)$.

We have therefore now fully established Theorem 6.1.
Acknowledgments

TTP acknowledges financial support by the Austrian Science Fund (FWF): P 28495-N27. TTP wishes to thank the Department of Mathematics and Statistics, University of Otago, for hospitality and support during work on this paper.

References

[1] S. Alexakis, A.D. Ionescu, S. Klainerman: Uniqueness of smooth stationary black holes in vacuum: Small perturbations of the Kerr spaces, Comm. Math. Phys. 299 (2010) 89–127.

[2] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear Hyperbolic Fuchsian Systems and AVTD Behavior in $T^2$-Symmetric Vacuum Spacetimes. Ann. Henri Poincaré, 14(6):1445–1523, 2013.

[3] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. Quasilinear Symmetric Hyperbolic Fuchsian Systems in Several Space Dimensions. In M. Agranovsky, M. Ben-Artzi, G. J. Galloway, L. Karp, V. Maz’ya, S. Reich, D. Shoikhet, G. Weinstein, and L. Zalcman, editors, Complex Analysis and Dynamical Systems V. American Mathematical Society, Providence, Rhode Island, 2013.

[4] E. Ames, F. Beyer, J. Isenberg, and P. G. LeFloch. A class of solutions to the Einstein equations with AVTD behavior in generalized wave gauges. J. Geom. Phys., 121:42–71, 2017.

[5] L. Andersson and A. D. Rendall. Quiescent cosmological singularities. Commun. Math. Phys., 218(3):479–511, 2001.

[6] R. Beig, P.T. Chruściel: Shielding linearised gravity, Phys. Rev. D 95 (2017) 064063.

[7] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems and applications. Class. Quantum Grav., 27(24):245012, 2010.

[8] F. Beyer and P. G. LeFloch. Second-order hyperbolic Fuchsian systems: Asymptotic behavior of geodesics in Gowdy spacetimes. Phys. Rev. D, 84(8):084036, 2011.

[9] F. Beyer and J. Hennig. Smooth Gowdy-symmetric generalized Taub-NUT solutions, Class. Quantum Grav. 29, 245017 (2012)

[10] F. Beyer and P. G. LeFloch. Self-gravitating fluid flows with Gowdy symmetry near cosmological singularities. Commun. Part. Diff. Eq., 42(8):1199–1248, 2017.

[11] H.A. Buchdahl: On the compatibility of relativistic wave equations for particles of higher spin in the presence of a gravitational field, Nuovo Cimento 10 (1958) 96–103.

[12] Y. Choquet-Bruhat and J. Isenberg. Half polarized $U(1)$-symmetric vacuum spacetimes with AVTD behavior. J. Geom. Phys., 56(8):1199–1214, 2006.

[13] P.T. Chruściel: The geometry of black holes, lecture notes (2015), http://homepage.univie.ac.at/piotr.chrusciel/teaching/BlackHoles/BlackHolesViennaJanuary2015.pdf.

[14] C. M. Claudel and K. P. Newman. The Cauchy problem for quasi-linear hyperbolic evolution problems with a singularity in the time. Proc. R. Soc. London A, 454(1972):1073–1107, 1998.
[15] T. Damour, M. Henneaux, A. D. Rendall, and M. Weaver. Kasner-like behaviour for subcritical Einstein-matter systems. *Ann. Henri Poincaré*, 3(6):1049–1111, 2002.

[16] H. Friedrich: *Existence and structure of past asymptotically simple solutions of Einstein’s field equations with positive cosmological constant*, J. Geom. Phys. *3* (1986) 101–117.

[17] H. Friedrich: *Conformal Einstein evolution*, in: *The conformal structure of space-time – Geometry, analysis, numerics*, J. Frauendiener, H. Friedrich (eds.), Berlin, Heidelberg: Springer, 2002, 1–50.

[18] H. Friedrich and A. D. Rendall. The Cauchy Problem for the Einstein Equations. In *Einstein’s Field Equations and Their Physical Implications*, pages 127–223. Springer Berlin Heidelberg, Berlin, Heidelberg, 2000.

[19] M. Henneaux, D. Persson, and P. Spindel. Spacelike singularities and hidden symmetries of gravity. *Living Rev. Relativity*, 11(1), 2008.

[20] P. Hintz, A. Vasy: *The global non-linear stability of the Kerr-de Sitter family of black holes*, (2016), arXiv:1606.04014 [math.DG].

[21] A.D. Ionescu, S. Klainerman: *On the uniqueness of smooth, stationary black holes in vacuum*, Invent. Math. *175* (2009) 35–102.

[22] J. Isenberg and S. Kichenassamy. Asymptotic behavior in polarized $T^2$-symmetric vacuum space–times. *J. Math. Phys.*, 40(1):340, 1999.

[23] J. Isenberg and V. Moncrief. Asymptotic behaviour in polarized and half-polarized $U(1)$ symmetric vacuum spacetimes. *Class. Quantum Grav.*, 19(21):5361, 2002.

[24] W. Israel: *Differential forms in general relativity*, Commun. of the Dublin Institute for Advanced Studies, Series A, *19* (1970) 1–100.

[25] M. Mars: *A spacetime characterization of the Kerr metric*, Class. Quantum Grav. *16* (1999) 2507–2523.

[26] M. Mars: *Uniqueness properties of the Kerr metric*, Class. Quantum Grav. *17* (2000) 3353–3373.

[27] M. Mars, T.-T. Paetz, J.M.M. Senovilla, W. Simon: *Characterization of (asymptotically) Kerr-de Sitter-like spacetimes at null infinity*, Class. Quantum Grav. *33* (2016) 155001.

[28] M. Mars, J.M.M. Senovilla: *A spacetime characterization of the Kerr-NUT-(A)de Sitter and related metrics*, Ann. Henri Poincaré *16* (2015) 1509–1550.

[29] M. Mars, J.M.M. Senovilla: *Spacetime characterizations of $\Lambda$-vacuum metrics with a null Killing 2-form*, Class. Quantum Grav. *33* (2016) 195004.

[30] T.-T. Paetz: *KIDs prefer special cones*, Class. Quantum Grav. *31* (2014) 085007.

[31] T.-T. Paetz: *Killing Initial Data on space-like conformal boundaries*, (2014), J. Geom. Phys. *106* (2016) 51–69.

[32] T.-T. Paetz: *Algorithmic characterization results for the Kerr-NUT-(A)dS space-time. I. A space-time approach*, J. Math. Phys. *58* (2017) 042501.
[33] R. Penrose: *Asymptotic properties of fields and space-time*, Phys. Rev. Lett. 10 (1963) 66–68.

[34] R. Penrose: *Zero rest-mass fields including gravitation: Asymptotic behavior*, Proc. R. Soc. Lond. A 284 (1965) 159–203.

[35] A. D. Rendall. Fuchsian analysis of singularities in Gowdy spacetimes beyond analyticity. *Class. Quantum Grav.*, 17(16):3305–3316, 2000.

[36] H. Ringström. *The Cauchy problem in General Relativity*. ESI Lectures in Mathematics and Physics. European Mathematical Society, Zürich, Switzerland, 2009.

[37] W. Simon: *Characterizations of the Kerr metric*, Gen. Rel. Grav. 16 (1984) 465–476.

[38] R.M. Wald: *General relativity*, Chicago and London: The University of Chicago Press, 1984.

[39] R.M. Wald: *Spin-two fields and general covariance*, Phys. Rev. D 33 (1986) 3613–3625.