CORRESPONDENCE OF DONALDSON-THOMAS AND GOPAKUMAR-VAFA INVARIANTS ON LOCAL CALABI-YAU 4-FOLDS OVER $V_5$ AND $V_{22}$

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ABSTRACT. We compute Gromov-Witten (GW) and Donaldson-Thomas (DT) invariants (and also descendant invariants) for local CY 4-folds over Fano 3-folds, $V_5$ and $V_{22}$ up to degree 3. We use torus localization for GW invariants computation, and use classical results for Hilbert schemes on $V_5$ and $V_{22}$ for DT invariants computation. From these computations, one can check correspondence between DT and Gopakumar-Vafa (GV) invariants conjectured by Cao-Maulik-Toda in genus 0. Also we can compute genus 1 GV invariants via the conjecture of Cao-Toda, which turned out to be 0. These fit into the fact that there are no smooth elliptic curves in $V_5$ and $V_{22}$ up to degree 3.

1. INTRODUCTION

1.1. Motivation. Recently, Donaldson-Thomas (DT) invariants for Calabi-Yau 4-folds and the correspondence among related invariants, like stable pair (PT) and Gopakumar-Vafa invariants (GV or BPS) have been actively studied in [CL14, CMT21, CMT18, CK20, CK18, CKM21].

Virtual cycles of DT$_4$ moduli space has been developed by Borisov-Joyce [BJ17] and by Oh-Thomas [OT20]. For a CY 4-fold $X$ and a curve class $\beta \in H_2(X, \mathbb{Z})$, we denote $M_\beta(X)$ the moduli space of 1-dimensional stable sheaves $F$ on $X$, such that $[F] = \beta$ and $\chi(F) = 1$. Then DT$_4$ invariants are defined by integrations of cohomology insertions over virtual cycles $[M_\beta(X)]^{vir}$. Note that the virtual cycle is defined via some suitable choice of an orientation on the moduli space $M_\beta(X)$.

Insertions are defined by the following way. Consider the product space $M \times X$ and a (normalized)universal sheaf $F$ over it. For a cohomology element $\gamma \in H^*(X, \mathbb{Z})$, let

$$\tau_i(\gamma) := (\pi_M)_*(\pi_X^*\gamma \cup ch_{dim(X)+i-1}(F)) \in H^{*+2(i-1)}(M, \mathbb{Z})$$

where $M$ is an abbreviation of $M_\beta(X)$. DT$_4$ invariants and descendant invariants (with a single insertion) are defined by

$$\langle \tau_i(\gamma) \rangle_\beta := \int_{[M_\beta(X)]^{vir}} \tau_i(\gamma)$$

2010 Mathematics Subject Classification. 14N35, 14C05, 14C15, 14J45.

Key words and phrases. Gromov-Witten invariants, Gopakumar-Vafa invariants, Donaldson-Thomas invariants, Fano varieties.
via some suitable choice of orientation on the moduli space $M_\beta$. $\langle \tau_0(\gamma) \rangle_\beta$ is called DT$_4$ invariants, and we denote it by DT$_4(X)(\beta|\gamma)$. When $i \geq 1$, we call $\tau_i(\gamma)$ descendant insertions, and integrations of descendant insertions over the virtual class are called descendant invariants.

On the other hand, we can consider DT$_3$ invariants on Fano 3-folds. Note that DT$_3$ moduli space for a Fano 3-fold is isomorphic to the DT$_4$ moduli space of the corresponding local CY 4-fold [CL14, Theorem 6.5]. Moreover, DT$_3$ invariants for a Fano 3-fold is equal to the DT$_4$ invariant of the corresponding local CY 4-fold when we choose some suitable orientation in the DT$_4$ moduli space [Cao19a, Cao19b].

Now we review some famous conjectures between DT$_4$ and GV invariants. Let us denote $n_{0,\beta}(\gamma)$ (resp. $n_{1,\beta}$) be the genus 0 (resp. genus 1) Gopakumar-Vafa invariants on a CY 4-fold $X$ defined in [KP08].

**Conjecture 1.1.** [CMT18, Conjecture 0.2] Via some suitable choice of orientation on the moduli space, we have

\[(1) \quad n_{0,\beta}(\gamma) = \text{DT}_4(\beta|\gamma), \quad \text{GW}(\gamma)_{0,\beta}^X = \sum_{k|\beta} \frac{1}{k^2} \text{DT}_4(X)(\beta/k|\gamma) \]

for all $\gamma \in H^4(X, \mathbb{Z})$.

Note that two equations in (1) are equivalent via the definition of $n_{0,\beta}(\gamma)$ in [KP08]. In [CMT18], Conjecture 1.1 is proven for some cases: CY 4-folds with a elliptic structure and local curves/surfaces. Also, in [CT21], the following conjecture is proposed which relates DT$_4$ descendent invariants and genus one GV invariants. Let $m_{\beta_1,\beta_2}$ be the meeting invariants defined in Section 0.3 in [KP08] (for a detail, see Section 4).

**Conjecture 1.2.** [CT21, Conjecture 0.2] Via some suitable choice of orientation on the moduli space, we have

\[
\langle \tau_1(\gamma) \rangle_\beta = \frac{n_{0,\beta}(\gamma^2)}{2(\gamma \cdot \beta)} - \sum_{\beta_1 + \beta_2 = \beta} \frac{(\gamma \cdot \beta_1)(\gamma \cdot \beta_2)}{4(\gamma \cdot \beta)} m_{\beta_1,\beta_2} - \sum_{k \geq 1, k|\beta} \frac{(\gamma \cdot \beta)}{k} n_{1,\beta/k}
\]

for all $\gamma \in H^2(X, \mathbb{Z})$.

Here, the number $m_{\beta_1,\beta_2}$ is a virtual count of degree $\beta_1$ curves in $X$ which meets with degree $\beta_2$, called meeting invariants. We will compute these numbers in Section 4.

In [CT21], Conjecture 1.2 is proved in several cases: CY 4-folds with an elliptic fibration structure and local Fano 3-fold over $\mathbb{P}^3$. In this paper, as a continuation of the latter case, our study focus on the Fano 3-folds: $V_5$ and $V_{22}$. See Section 3.1.1 and 3.1.3 for definitions.
of $V_5$ and $V_{22}$. These Fano 3-folds arise as a minimal compactification of the complex 3-space $\mathbb{C}^3$. For detailed descriptions, see the beginning parts of Section 3.1.

1.2. Main results. In this paper, we compute some low degree DT$^3$ invariants for Fano 3-folds $V_5$ and $V_{22}$. For the cases $\beta = d \cdot [\text{line}], 1 \leq d \leq 3$ where [line] is a class of $\mathbb{P}^1$ in their projective embedding via the very ample generators, we review some classical results on description of DT$^3$ moduli space of $V_5$ and $V_{22}$ and universal sheaves on them. Using this, we will compute DT$^3$ invariants of these Fano 3-folds (equivalently DT$^4$ invariants of corresponding local CY 4-fold) in Section 3.1.

On the other hand, in Section 2 we compute twisted Gromov-Witten (GW) invariants on $V_5$ and $V_{22}$ for the cases $\beta = d \cdot [\text{line}], 1 \leq d \leq 3$, using quantum Lefschetz property [KKP03] and torus localization [GP99]. We will introduce some recipe for computing genus zero GW invariants on Grassmannian varieties, which is a direct analogue of the formula in [GP99, HTK+03] for genus zero GW invariants on projective spaces. From these calculations, we obtain the following main results of the paper.

**Theorem 1.3** (Theorem 3.18). Using the choice of orientation as in [CT21, (0.7)] on DT$^4$ moduli spaces on local CY 4-folds, the Conjecture 1.1 holds for $V_5$ and $V_{22}$, $\beta = d \cdot [\text{line}], 1 \leq d \leq 3$.

**Theorem 1.4** (Theorem 4.2). Using the choice of orientation as in [CT21, (0.7)] on DT$^4$ moduli spaces on local CY 4-folds, the Conjecture 1.2 holds for $V_5$ and $V_{22}$, $\beta = d \cdot [\text{line}], 1 \leq d \leq 3$ if and only if genus 1 GV invariants $n_{1, \beta} \equiv 0$.

From the fact that the degree of the defining equation of $V_5$ and $V_{22}$ is $\leq 2$ and the varieties does not contain plane, one can check that the smooth elliptic curves of the degree $\leq 3$ in $V_5$ and $V_{22}$ does not exist. Hence the vanishing of GV invariants $n_{1, \beta}$ is desirable.

We have posted all source codes we have used during the computation of GW invariants and their outputs at the second author’s website:

https://sites.google.com/view/sanghyeon-lee/reference?authuser=0

Acknowledgements. The authors gratefully acknowledge the many helpful suggestions of Yalong Cao and Young-Hoon Kiem during the preparation of the paper. The second named author thanks to Jeongseok Oh and Hyeonjun Park for much advice on Donaldson-Thomas invariants.

2. Twisted GW invariants on Grassmannians

Let $Y$ be a smooth Fano 3-fold embedded in the Grassmannian variety $G = \text{Gr}(r, d)$, which is a zero section of a bundle over $G$. In this subsection, we present an algorithm of the computation of GW invariants of $|K_Y|$, a total space of a canonical line bundle of $Y$. For any smooth projective variety $X$, we denote the moduli space of stable maps,
with $k$-marked points with degree $\beta \in \mathbb{H}^2(X, \mathbb{Z})$ by $M_{0,k}(X, \beta)$. When $X = G$, we will denote $M_{0,k}(G, d)$ the stable map space with degree $d \cdot |\text{line}|$. We will usually abbreviate $M_{0,k}(X, \beta)$ by $M(X)$ in the following. In this section, we will introduce classical methods to compute GW invariants, quantum Lefschetz principle and torus localization when the target variety is $|K_Y|$. If you are not interested in this part, you may skip the details and just see computational results in Section 2.4.

Consider the forgetful map $p : M = M_{0,k}(G, d) \to \mathcal{M}_{0,k}$ where $\mathcal{M}_{0,k}$ is the moduli space of prestable genus 0 curves with $k$ marked points. We usually will abbreviate $\mathcal{M}_{0,k}$ by $\mathcal{M}$. There is a usual (relative) perfect obstruction theory [Beh96] $E_{M(X)/\mathcal{M}} \to L_{M(X)/\mathcal{M}}$ of $M(X) \to \mathcal{M}$, where

\begin{equation}
E_{M/\mathcal{M}} := [R\pi_* f^* T_G]^\vee \in D^b(M(X)).
\end{equation}

Also, as in [BF97] the virtual cycle correspond to this perfect obstruction theory is given by:

\begin{equation}
[M(X)]^{\text{vir}} := 0^!_{h^1/h^0(E_{M/\mathcal{M}})}(C_{M(X)/\mathcal{M}}) \in A_{\text{vdim}}(M(X))
\end{equation}

where $h^1/h^0(E_{M/\mathcal{M}})$ is a vector bundle stack correspond to $E_{M/\mathcal{M}}$, $C_{M(X)/\mathcal{M}}$ is the (relative) intrinsic normal cone defined in [BF97], and $\text{vdim}$ is given by

$$\text{vdim} = (1 - g)(\dim(X) - 3) + k - \int c_1(K_X)$$

which is called virtual dimension.

2.1. Quantum Lefschetz principle. We will briefly review some aspect of famous quantum Lefschetz principle [KKP03] in this section. We consider a negative vector bundle $E$ on $G$, so that $H^0(\mathcal{C}, f^* E) = 0$ for any non-constant morphism $f : \mathbb{P}^1 \to G$. Let $|E|$ be the total space of $E$ and $p : |E| \to G$ be the projection. Consider a stable map $f : C \to |E|$. Then $p \circ f : C \to G$ is a stable map and $f$ induces an element of $H^0(C, (p \circ f)^* E)$. But since $E$ is a direct sum of negative degree line bundles, $H^0(C, (p \circ f)^* E) = 0$. Hence we have a natural isomorphism of moduli spaces of stable maps:

$$M_{0,k}(|E|, d) \cong M_{0,k}(G, d).$$

Note that $M_{0,k}(G, d)$ is well known to be smooth. Also, we have the short exact sequence:

$$0 \to p^* T_G \to T_{|E|} \to p^* E \to 0.$$  

From the condition that $E$ is negative and the fact that $R^1\pi_*(f^* p^* T_G) = 0$, we obtain the following by taking the higher direct image functor $R\pi_* f^*$ to the above short exact sequence.

$$E_{M(|E|)/\mathcal{M}}^\vee = R\pi_* T_{|E|} = \left[ \pi_*(f^* T_G) \xrightarrow{0} R^1\pi_*(f^* p^* E) \right]$$
Note that $\pi_*(f^*p^*T_G) \cong T_M(G)$. We can easily check that the intrinsic normal cone $\mathcal{C}_{M(G)/\mathfrak{M}}$ is $[M(G)/T_M(G)/\mathfrak{M}]$ and $h^1/h^0(E^\vee) = [R^1\pi_*(f^*p^*E)/T_M(G)/\mathfrak{M}]$. From the definition of the virtual cycle in (3), we have

$$[M(G)]^{\text{vir}} = 0^!_{[R^1\pi_*(f^*p^*E)/T_M(G)/\mathfrak{M}]}[M(G)/T_M(G)/\mathfrak{M}] = 0^!_{R^1\pi_*(f^*p^*E)}[M(G)] = e(R^1\pi_*(f^*p^*E)) \cap [M(G)]$$

where the second identity comes from properties of Gysin pull-backs via bundle stacks in [Kre99]. We call this phenomenon **B-twist**. Note that this phenomenon also arises when we replace $G$ by any projective variety. Using this, we define a **twisted GW invariants** of a Fano variety $X$ by:

**Definition 2.1 (Twisted GW invariant).** The twisted Gromov-Witten invariant a Fano variety $X$ is defined by the integration

$$\text{GW}_{0,\beta}^{\text{twist}}(X)(\gamma) := \int_{[M_{0,1}(X,\beta)]^{\text{vir}}} e(R^1\pi_*f^*K_X) \cup \text{ev}^*(\gamma) \in \mathbb{Q}$$

where $\text{ev} : M_{0,1}(X, \beta) \rightarrow X$ be the evaluation map.

Note that twisted GW invariants are usually considered as a definition of GW invariants of the total space $|K_X|$, because we cannot define it directly since $|K_X|$ is not compact.

Next we consider a complete intersection in $G = G(r,n)$. Let $E$ be the vector bundle on $G$ and let $Y \subset G$ be a zero section of the generic section $s : \mathcal{O}_G \rightarrow E$. Assume that $E$ is convex, so that $H^1(G, \varphi^*E) = 0$ for any non-constant morphism $\varphi : \mathbb{P}^1 \rightarrow G$. For example, the direct sum $\bigoplus_i \mathcal{O}(a_i)$, $a_i > 0$ of line bundles on $G$ is a convex vector bundle. From the convexity of $E$, $\pi_*f^*E$ is locally free, where $\pi : \mathcal{C} \rightarrow M$ is the universal curve and $f : \mathcal{C} \rightarrow G$ is the universal morphism. Let $M(Y)$ denote the stable map space $M_{0,k}(Y,d)$. Then we have

$$E_{Y}^{\vee}_{M(Y)/\mathfrak{M}} = R\pi_*f^*T_Y = [\pi_*f^*T_G \rightarrow \pi_*f^*E] = [T_M \rightarrow \pi_*f^*E].$$

Consider the section $s : \mathcal{O}_G \rightarrow E$ and the induced section $\tilde{s} : M \rightarrow \pi_*f^*E$ defined by $\tilde{s}([C,f]) = (C, f^*s) \in H^0(C, f^*E).$ Then we have $Z(\tilde{s}) = M(Y) \subset M$. We can easily check that $\mathcal{C}_{M/\mathfrak{M}} = [C_{M(Y)}/M/T_M/\mathfrak{M}]$ and $h^1/h^0(E^\vee) = [\pi_*f^*E/T_M/\mathfrak{M}]$. Then by the definition of the virtual cycle, we have

$$[M(Y)]^{\text{vir}} = 0^!_{[\pi_*f^*E/T_M]}[C_{M(Y)/M}/T_M/\mathfrak{M}] = 0^!_{\pi_*f^*E}[C_{M(Y)/M}] \in A_*(M(Y)).$$

Also we have

$$\iota_*[M(Y)]^{\text{vir}} = e(\pi_*f^*E) \cap [M] \in A_*(M)$$

where $\iota : M(Y) \rightarrow M$ is the inclusion. See [KKP03] for the proof of the general case. We call this phenomenon **A-twist**.
2.2. Torus localization. We will briefly review some aspect of torus localization [GP99] in this section. Again we consider the Grassmannian variety $G = \text{Gr}(r, d)$ and the stable map space $M = M_{0,k}(G, d)$. We give a $\mathbb{C}^*$-action on $\mathbb{C}^n$ with weights $-\alpha_1, \ldots, -\alpha_n$, which induces the $\mathbb{C}^*$-action on $\text{Gr}(r, n)$ and $M$. Let $M^F \subset M$ be the fixed locus of the action and let $M^F = \bigcup_i M_i^F$ be the irreducible decomposition. Note that $E^\vee_{M/\mathbb{R}}$ has an induced $\mathbb{C}^*$-action and we have a decomposition

$$E^\vee_{M/\mathbb{R}}|_{M_i^F} \cong N^\text{fix}_i \oplus N^\text{vir}_i$$

where $N^\text{fix}_i$ has weight 0 under the $\mathbb{C}^*$-action and $N^\text{vir}_i$ is a direct sum of vector bundles with non-zero weights.

Let $(A^T)_*(M)$ (resp. $(A^T)^*(M)$) be the equivariant Chow group (resp. equivariant Chow cohomology group) of $M$ and $e^T(E)$ be the equivariant Euler class of a locally free sheaf $E$. If $E_M$ has locally free resolution $[E_0 \to E_1]$, we define the (equivariant) Euler class by

$$e^T(N^\text{vir}) = \frac{e^T(E^0_m)}{e^T(E^m_0)} \in (A^T)^*(M) \otimes \mathbb{Q}[t, 1/t].$$

By the virtual localization theorem in [GP99], we have

$$[M]^\text{vir} = \sum_i \frac{[M_i^F]^\text{vir}}{e^T(N^\text{vir})} \in A^T_*(M) \cong (A^T)_*(M) \otimes \mathbb{Q}[t, 1/t].$$

2.3. Computation of the virtual normal bundle. For $M = M_{0,k}(G, d)$, we can do more specific computation. In a similar manner as in [GP99] and [HTK+03], which dealt with the case $G = \mathbb{P}^n$, fixed loci $M_i^F$ are indexed by decorated graphs $\Gamma$. We denote $M_i^F$ by $M_\Gamma$ for the corresponding decorated graph $\Gamma$. Note that $M_\Gamma$ is smooth and thus $[M_\Gamma]^\text{vir} = [M_\Gamma]$. Also $N^\text{vir}$ denotes the virtual normal bundle defined in [BF97].

For a stable map $[(C, x_1, \ldots, x_k, f)] \in M_\Gamma$, the fiber of the (K-theoretic) virtual normal bundle is given by the moving part of $\text{Ext}^1(\Omega_C(x_1 + \ldots, x_k), \mathcal{O}_C) = \text{Ext}^0(\Omega_C(x_1 + \ldots, x_k), \mathcal{O}_C) + (H^0 - H^1)(f^*T_G)$. By [GP99] and [HTK+03], we have

$$e^T(\text{Ext}^0(\Omega_C(x_1 + \ldots, x_k), \mathcal{O}_C)) = \prod_{v \in \text{Vertices}} \left( \omega_F - \text{val}(v) \right)$$

and

$$e^T(\text{Ext}^1(\Omega_C(x_1 + \ldots, x_k), \mathcal{O}_C)) = \prod_{F \in \text{Flags}} (\omega_F - e_F),$$

where $\omega_F := \frac{\alpha_{\text{fix}}(F) - \alpha_{\text{vir}}(F)}{\text{val}(i(F))} \geq 3$ and $e_F$ is the $\psi$-class correspond to the flag $F$. Also, by [GP99] and [HTK+03], we have the following by using the normalization sequence of nodal curves.

$$H^0 - H^1(f^*T_G) = \bigoplus_{v \in \text{Vertices}} T_{p_v}G + \bigoplus_{e \in \text{Edges}} H^0(C_e, f^*T_G)$$

$$- \bigoplus_{F \in \text{Flags}} T_{p_i(F)}G - \bigoplus_{v \in \text{Vertices}} H^1(C_v, f^*T_G)$$

(4)
Then the equivariant Euler classes of their moving part are given by the followings. For the first term of (4), if \( p_v = x_{u_1, \ldots, u_r} = (e_{u_1}, \ldots, e_{u_r}) \in \text{Gr}(r, n) \), then

\[
e^T(T_{p_v}(G)) = \prod_{1 \leq j \leq r} \prod_{k \in [n] \setminus \{u_1, \ldots, u_r\}} (\alpha_{u_j} - \alpha_k).
\]

For the third term of (4), if \( p_i(F) = x_{u_1, \ldots, u_r} = (e_{u_1}, \ldots, e_{u_r}) \in \text{Gr}(r, n) \), then we have the same formula as above:

\[
T_{p_i(F)} = \prod_{1 \leq j \leq r} \prod_{k \in [n] \setminus \{u_1, \ldots, u_r\}} (\alpha_{u_j} - \alpha_k).
\]

For the fourth term of (4), by [HTK+03], we have the following. If valency(\( v \)) = 2 and there is no marking on \( v \), then

\[
e^T(H^1(C_v, f^*T_G)) = \omega_{F_{v,1}} + \omega_{F_{v,2}}.
\]

Otherwise, we have

\[
e^T(H^1(C_v, f^*T_G)) = 1.
\]

For the second term of (4), consider the Euler sequence:

\[
0 \to S' \otimes S \to S' \otimes O_G^{\oplus n} \to S' \otimes Q = T_G \to 0
\]

where \( S, Q \) are tautological bundle and universal quotient bundle on \( G \). Take pull-back via the map \( f : C_e \to G \). Let \( e = \{v_1, v_2\} \), \( p_{v_1} = x_{u_1, \ldots, u_{r-1}, u_a} \) and \( p_{v_2} = x_{u_1, \ldots, u_{r-1}, u_b} \). Note that \( u_a \neq u_b \). Then we have \( f^*(S') \cong O(\alpha_{u_1}) \oplus \ldots \oplus O(\alpha_{u_i}) \oplus O(d_e) \) where \( O(\alpha_i) \) is an equivariant trivial bundle where \((\mathbb{C}^*)^n\) acts on it with a weight \( \alpha_i \). Note that \( H^0(C_e, O(d_e)) \) has weights \( \frac{\alpha_a + \alpha_b}{d_e} \) for \( \alpha_a + \alpha_b = d_e \). Let \( f|_{C_e} = f_e \). By taking \( f_e^* \) and the cohomology in (5), we have the following exact sequence:

\[
0 \to H^0(f_e^*(S' \otimes S)) \to H^0((f_e^*S' \otimes O_G^{\oplus n})) \to H^0(f_e^*T_G) \to H^1((f_e^*S' \otimes S)) \to 0.
\]

By a direct calculation, we have

\[
e^T((H^1 - H^0)(f_e^*(S' \otimes S)))
= (-1)^{d_e-1} \prod_{1 \leq i \leq r-1} (\alpha_{u_i} - \alpha_{u_1}) (\alpha_{u_i} - \alpha_{u_r}) \prod_{1 \leq i < j \leq r-1} (\alpha_{u_i} - \alpha_{u_j})^2.
\]

Also we have,

\[
e^T((H^1 - H^0)(f_e^*S' \otimes O_G^{\oplus n})) = \prod_{1 \leq j \leq r} \prod_{k \in [n] \setminus \{u_1, \ldots, u_r\}} (\alpha_{u_j} - \alpha_k)
\times \prod_{k \in [n] \setminus \{u_1, \ldots, u_r\}} \prod_{c_1, c_2 \neq 0} \prod_{c_1 + c_2 = d_e} \prod_{(c_1, k) \neq (0, u_0), (d_e, u_a)} \left( \frac{c_1 \alpha_{u_a} + c_2 \alpha_{u_b}}{d_e} - \alpha_k \right).
\]
2.4. Computation of GW invariants on Fano 3-folds. Combining above arguments, we have expression of $e^T(N^{vir}|_{M_r})$. Next we represent A-twist and B-twist in section 2.1 in equivariant cohomology. Let $F_1 := \pi_* f^* \left( \bigoplus_i \mathcal{O}(a_i) \right)$ and $F_2 := R^1 \pi_* f^* \mathcal{O}(-b)$ where $b$ is the Fano index of $X$, $\pi : \mathcal{C} \rightarrow M(X)$ is the universal curve and $f : \mathcal{C} \rightarrow X$ is the universal morphism. We have

\[
\iota_* [M(X)]^{vir} = \sum_{\Gamma} \frac{e^T(F_1|_{M_{r}\Gamma}) \cup e^T(F_2|_{M_{r}\Gamma})}{e^T(N^{vir})} [M_{r\Gamma}] \in (A^T)_*(M) \otimes \mathbb{Q}[t, 1/t] \tag{6}
\]

where $A_{r\Gamma}$ is the order of the automorphism group of a generic element in $M_{r\Gamma}$. We have $|A_{r\Gamma}| = |\text{Aut}(\Gamma)| \cdot \prod_{e \in \text{Edges}} d_e$ where $\text{Aut}(\Gamma)$ is the automorphism group of the decorated graph $\Gamma$. One can check more detail on the group $A_{r\Gamma}$ in [GP99]. Note that we can find specific expressions of $e^T(F_1|_{M_{r\Gamma}})$ and $e^T(F_2|_{M_{r\Gamma}})$ using the normalization sequence:

\[
0 \rightarrow \mathcal{O}_C \rightarrow \bigoplus_{v \in \text{Vertices}} \mathcal{O}_{C_v} \oplus \bigoplus_{e \in \text{Edges}} \mathcal{O}_{C_e} \rightarrow \bigoplus_{F \in \text{Flags}} \mathbb{C}_{x_F} \rightarrow 0. \tag{7}
\]

Combining arguments in section 2.2, 2.3, 2.1 and Hodge integrals computed in [FP00], we can express the right hand side terms of (6) by formal weights of $\mathbb{C}^*$-action. Therefore, we can compute genus 0 Gromov-Witten invariants of the total space of the canonical line bundle over the Fano 3-fold, which is a zero section of an equivariant vector bundle over a Grassmannian variety.

In this paper, we consider two cases: (a) $Y = V_5$ and (b) $Y = V_{22}$. In case (a), $F_1 = \pi_* f^* \mathcal{O}(1)^{\otimes 3}$ and $F_2 = R^1 \pi_* f^* \mathcal{O}(-2)$. In case (b), $F_1 = \pi_* f^* (\wedge^2 S^c)$ and $F_2 = R^1 \pi_* f^* \mathcal{O}(-1)$. The actual computation has been done by a computer program. Firstly we make a dataset of all possible decorated graphs $\Gamma$ and their information. Secondly, using this dataset, we make a code computing the right hand side terms in (6) for each localization graph $\Gamma$ and adding up them. As a result, we obtain the following table are twisted GW invariants.

**Proposition 2.2** (Twisted GW invariants). The twisted GW invariants for $Y = V_5$ and $V_{22}$ are given by the numbers of the following table.

| $d$ | $GW^{\text{twist}}_{0,d}(V_5)(h_2)$ | $GW^{\text{twist}}_{0,d}(V_{22})(h_2)$ |
|-----|---------------------------------|----------------------------------|
| 1   | $-5$                            | $2$                              |
| 2   | $-36\frac{1}{4}$               | $-6\frac{1}{2}$                 |
| 3   | $-490\frac{5}{9}$             | $28\frac{2}{9}$                 |

Here $h_2$ is the generator of $H^4(Y, \mathbb{Z}) \cong \mathbb{Z}$.

**Remark 2.3.** The degree 4 twisted GW invariant on $V_5$ is given by $GW^{\text{twist}}_{0,4}(V_5)(\gamma_2) = -8829\frac{1}{16}$. In principle we can compute GW invariants for higher degrees, but the time taken for the calculation super-exponentially increases.
In this section, we compute DT invariants and descendant invariants for some local Fano 3-folds $|K_{V_1}|$ and $|K_{V_2}|$ for degree $1 \leq d \cdot [\text{line}] \leq 3$. We will abbreviate $d \cdot [\text{line}]$ by $d$.

In these cases, $M_\beta$ naturally isomorphic to moduli space of stable sheaves on $|K_Y|$.

**Definition 3.1** (Twisted DT invariant). Let $M_\beta = M_\beta(Y)$ be the moduli space of stable sheaves $F$ on $Y$ with $[F] = \beta \in H_2(Y, \mathbb{Z})$ and $\chi(F) = 1$. Let

$$\tau_0 : H^4(Y, \mathbb{Z}) \to H^2(M_\beta, \mathbb{Z}), \quad \tau_0(\gamma) = \tau_{\bar{X}}(\pi_X^* \gamma \cup \text{ch}_2(F))$$

be the primary insertion of $\gamma \in H^4(Y, \mathbb{Z})$. Here $\mathcal{F}$ is the universal sheaf and the maps $\tau_{\bar{X}}, \tau_Y$ are the canonical projection maps. The twisted genus zero DT invariant is defined by

$$DT^\text{twist}_3(Y)(\beta|\gamma) := (-1)^{c_1(Y) \cdot \beta - 1} \int_{[M_\beta(Y)]^\text{vir}} \tau_0(\gamma) \in \mathbb{Z}.$$  

where $[M_\beta(Y)]^\text{vir}$ is the virtual class defined in [Tho00, Corollary 3.39].

Since $DT_4$ invariant of $|K_Y|$ is equal to $DT_3$ invariant of $Y$, it is enough to compute twisted $DT_3$ invariant.

Note that if the moduli space $M_\beta(Y)$ is smooth, the virtual cycle is the Poincaré dual of the top Chern class of the obstruction bundle. Combining computation of GW invariants in Section 2 and DT invariant computation in this section, we will check Conjecture 1.1 for $1 \leq d \leq 3$, which can be rewritten by

**Conjecture 3.2.** For the cohomology class $\gamma \in H^4(Y, \mathbb{Z})$,

$$n_{0,\beta}(\gamma) = DT_4(|K_Y|)(d|\gamma)$$

and

$$GW_{0,\beta}(Y)^\text{twist}(\gamma) = \sum_{k|\beta} \frac{1}{k^2} \cdot DT_4(|K_Y|)(d/k|\gamma).$$

On the other hand, Cao and Toda suggest genus one GV-type invariant on CY 4-fold $X$ by using the descendent insertion in [CT21].

**Definition 3.3** (Descendent insertion). For an integral class $\gamma \in H^2(X, \mathbb{Z})$, let us define the descendent insertion as

$$\tau_1 : H^2(X, \mathbb{Z}) \to H^2(M_\beta, \mathbb{Z}), \quad \tau_1(\gamma) = \pi_M^*(\pi_X^* \gamma \cup \text{ch}_4(\mathcal{F}_\text{norm})).$$

where $\mathcal{F}_\text{norm} := \mathcal{F} \otimes \pi_M^*(\pi_M\det(\mathcal{F}))^{-1}$ is the normalized universal sheaf of the universal sheaf $\mathcal{F} \in \text{Coh}(M_\beta \times X)$ and the maps $\pi_M$ and $\pi_X$ are the canonical projection maps from $M_\beta \times X$ into $M_\beta$ and $X$ respectively.
Descendant invariants from the *descendent insertions* are defined by

\[
\langle \tau_1(\gamma) \rangle_\beta := \int_{[M_\beta]_{\text{vir}}} \tau_1(\gamma).
\]

**Remark 3.4.** For \(\gamma \in H^{4-2i}(X, \mathbb{Z})\), the insertion becomes

\[
\tau_i(\gamma) = \pi_{\text{vir}}^* (\pi_Y^* \gamma \cup \{\text{ch}(\mathcal{F}_{\text{norm}}) \cdot \text{td}(K_Y)^{-1}\})_{2+i}
\]

by Grothendieck-Riemann-Roch theorem.

By computing descendant invariants, we can obtain genus 1 GV invariants \(n_{1,\beta}\) via Conjecture 1.2.

### 3.1. Computations on Fano 3-folds.

It is very well-known that the following list of smooth Fano 3-folds have the same Betti numbers of that of \(\mathbb{P}^3\):

\[\mathbb{P}^3, \ Q_2 \subset \mathbb{P}^4, \ V_5 \subset \mathbb{P}^6, \ V_{22} \subset \mathbb{P}^{13}.\]

In special, the odd cohomology of these varieties vanish. All of varieties are rigid except \(V_{22}\). The moduli of \(V_{22}\)'s is six-dimensional. The first two varieties in the list are homogeneous and the others are not. In this section, we compute the primary and descent invariants for non-homogeneous cases.

#### 3.1.1. The case \(V_5\).

The Fano threefold \(V_5\) is defined by the linear intersection \(V_5 = \text{Gr}(2, \mathbb{C}^5) \cap H_1 \cap H_2 \cap H_3\) where \(H_i\) are the general hyperplane in \(\mathbb{P}(\wedge^2 \mathbb{C}^5) = \mathbb{P}^9\).

- \(\text{Pic}_\mathbb{Z}(V_5) \cong \mathbb{Z}\langle H \rangle, \deg(V_5) = 5, \ K_{V_5} = -2H.\)
- The cohomology ring of \(Y\) over \(\mathbb{Z}\) is isomorphic to
  \[H^*(V_5, \mathbb{Z}) \cong \mathbb{Z}[h_1, h_2, h_3]/\langle h_1^2 - 5h_2, h_1^3 - 5h_3, h_1h_2 - h_3, h_2^2 \rangle\]
  where \(\deg(h_i) = 2i, 1 \leq i \leq 3.\) Moreover, \(h_1 = c_1(O_Y(1))\) and \(h_i\) is the Poincaré dual of the linear space of dimension \(3 - i\) for \(i = 2, 3.\)
- Let \(S_Y\) and \(Q_Y\) be the restriction of the universal bundles \(S\) and \(Q\) on \(\text{Gr}(2, 5)\). The Chern classes are
  1. \(c(S_Y) = 1 - h_1 + 2h_2,\)
  2. \(c(Q_Y) = 1 + h_1 + 3h_2 + h_3,\)
  3. \(c(Y) = 1 + 2h_1 + 12h_2 + 4h_3.\)

Unless otherwise stated, we omit the subscription \(V_5\) in the universal bundles.

Let us denote by \(M_d := M_{\beta}(V_5)\) for \(\beta = d[\text{line}] \in H_2(V_5, \mathbb{Z}).\) For \(1 \leq d \leq 3,\) one can easily see that \(M_d(Y)\) is isomorphic to the Hilbert scheme \(H_d(V_5)\) of curves with Hilbert polynomial \(dm + 1\) (cf. [Chu19, Proposition 3.1]). Thus one can borrow the description of Hilbert scheme of rational curves in \(V_5.\)

**Proposition 3.5** ([Fae05, FN89, Ili94, San14]). \(M_1 \cong \mathbb{P}^2, \ M_2 \cong \mathbb{P}^4, \ M_3 \cong \text{Gr}(2, 5).\)
Remark 3.6. By taking an explicit choice of the hyperplanes $H_i$, the results for $d = 1$ and 2 of Proposition 3.5 has been reproved by the birational-geometric method (For the detail, see [CHL18, Section 7]).

The universal sheaves over $M_d$ were explicitly presented in Proposition 2.20, Proposition 2.32, and Proposition 2.46 of [San14]. Let $i : C_d \hookrightarrow M_d \times V_5$ be the universal curve in $M_d \times V_5$. The free resolutions of $i_*\mathcal{O}_{C_d}$ on $M_d \times V_5$ are

1. $(d = 1) 0 \to \mathcal{O}_{\mathbb{P}^2}(-3) \boxtimes S \to \mathcal{O}_{\mathbb{P}^2}(-2) \boxtimes \mathcal{O}^* \to \mathcal{O}_{\mathbb{P}^2 \times V_5} \to i_*\mathcal{O}_{C_1} \to 0$,
2. $(d = 2) 0 \to \mathcal{O}_{\mathbb{P}^4}(-2) \boxtimes \mathcal{O}_{V_5}(-1) \to \mathcal{O}_{\mathbb{P}^4}(-1) \boxtimes S \to \mathcal{O}_{\mathbb{P}^4 \times V_5} \to i_*\mathcal{O}_{C_2} \to 0$,
3. $(d = 3) 0 \to S(-1) \boxtimes \mathcal{O}_{V_5}(-1) \to \mathcal{O}_{\Gr(2,5)}(-1) \boxtimes \mathcal{O}(-1) \to \mathcal{O}_{\Gr(2,5) \times V_5} \to i_*\mathcal{O}_{C_3} \to 0$.

Note that each of the moduli spaces is smooth, and we can check that the virtual class $[M_d]_{\text{vir}}$ is the Euler class of a K-theoretic obstruction bundle $\text{ob}_{M_d}$. One can compute this K-theoretic obstruction bundle by the formula

$$[\text{ob}_{M_d}] = [T_{M_d}] + [R_{\pi_{M_d}} \mathcal{R}Hom_{M_d \times V_5}(i_*\mathcal{O}_{C_d}, i_*\mathcal{O}_{C_d})[1]] - [\mathcal{O}]$$

which appears in the proof of [CT21, Proposition 2.13].

| $d$ | $R_{\pi_{M_d}} \mathcal{R}Hom_{M_d \times V_5}(i_*\mathcal{O}_{C_d}, i_*\mathcal{O}_{C_d})[1]$ | $[\text{ob}_{M_d}]$ |
|-----|-------------------------------------------------|-----------------|
| 1   | $-3[\mathcal{O}_{\mathbb{P}^2}] + 3[\mathcal{O}_{\mathbb{P}^2}(1)] + 5[\mathcal{O}_{\mathbb{P}^2}(2)] - 5[\mathcal{O}_{\mathbb{P}^2}(3)]$ | $[\mathcal{O}_{\mathbb{P}^2}] - 5[\mathcal{O}_{\mathbb{P}^2}(2)] + 5[\mathcal{O}_{\mathbb{P}^2}(3)]$ |
| 2   | $-3[\mathcal{O}_{\mathbb{P}^4}] + 10[\mathcal{O}_{\mathbb{P}^4}(1)] - 7[\mathcal{O}_{\mathbb{P}^4}(2)]$ | $[\mathcal{O}_{\mathbb{P}^4}] - 5[\mathcal{O}_{\mathbb{P}^4}(1)] + 7[\mathcal{O}_{\mathbb{P}^4}(2)]$ |
| 3   | $-2[\mathcal{O}_{\Gr(2,5)}] + 10[\mathcal{O}_{\Gr(2,5)}(1)] - 7[S^*(1)] + 5[S^*] - [S^* \otimes S]$ | $[\mathcal{O}_{\Gr(2,5)}] - 10[\mathcal{O}_{\Gr(2,5)}(1)] + 7[S^*(1)] - 5[S^*] + [S^* \otimes S] + [S^* \otimes \mathcal{O}]$ |

Here, the bundles $S$ and $Q$ in the fourth row are the universal sub-bundle and quotient bundle of $M_3 = \Gr(2,5)$. By using the computer algebra system, Macaulay2 ([GS]), we have

**Proposition 3.7.** The invariants $\langle \tau_i(h_{2-i}) \rangle_d$ are given by the numbers of the following table.

| $d$ | $i = 0$ | $i = 1$ |
|-----|--------|--------|
| 1   | 5      | $\frac{35}{2}$ |
| 2   | 35     | $\frac{35}{2}$ |
| 3   | 490    | $-\frac{490}{2}$ |

**Proof.** Let us present the computation of the invariants for the degree $d = 3$ case. The other cases are more simple and thus we omit it. The cohomology ring structure of $\Gr(2,5)$ is very well-known as follow. Let $m_i := c_i(S)$ be the $i$-th Chern class of the universal sub-bundle $S$ of Grassmannian $\Gr(2,5)$. The cohomology ring of $\Gr(2,5)$ is given by ([EH16, Theorem 5.26])

$$H^*(\Gr(2,5), \mathbb{Z}) = \mathbb{Z}[m_1, m_2]/(-m_1^5 + 4m_1^3m_2 - 3m_1m_2^2, m_1^4 - 3m_1^2m_2 + m_2^2).$$
Note that the dual of the point class is \([\text{point}]^* = m_2^3\). Then the Chern class of the obstruction bundle \(\text{ob}_{M_d}\) is
\[
c(\text{ob}_{M_d}) = 1 - 11m_1 + (48m_1^2 + 7m_2) + (-102m_1^3 - 56m_1m_2 + 451m_1^2m_2) - 78m_2^2 - 490m_1m_2^2.
\]
Thus the virtual class of \(M_3\) is \([M_3]^{\text{vir}} = -490m_1m_2^2\).

On the other hand, the insertion classes on \(H^*(M_3 \times V_3)\) are
\[
\tau_0(h_2) = m_1^2h_2 - m_2h_2 - m_1h_3,
\]
\[
\tau_1(h_1) = m_1^3h_1 - \frac{3}{2}m_1m_2h_1 - \frac{5}{2}m_1^2h_2 + \frac{1}{2}m_1h_3
\]
From these one, we have
\[
\int_{[M_3]^{\text{vir}}} \tau_0(h_2) = 490m_2^3h_3, \quad \int_{[M_3]^{\text{vir}}} \tau_1(h_1) = -245m_2^3h_3.
\]
\[\square\]

**Remark 3.8.** From the description of the universal curve \(C_1\) in [FN89, Lemma 2.1, 2.2], one can easily check that the obstruction bundle is isomorphic to \(\text{ob}_{M_1} \cong O_{M_1}(5)\) and thus its cohomology matches with our computation.

**Remark 3.9.** The universal curve \(C_2\) is a regular section of the vector bundle \(O_{\mathbb{P}^4}(1) \boxtimes S^*\) ([San14, Proposition 2.32]). Hence the Chern character is given by
\[
\text{ch}(i_*O_{C_2}) = c_2(O_{\mathbb{P}^4}(1) \boxtimes S^*) \cdot \text{td}(O_{\mathbb{P}^4}(1) \boxtimes S^*)^{-1},
\]
and thus its cohomology class matches with our computation. In the following subsection, we find the fundamental class of \(C_3\) by using Porteous’ formula.

### 3.1.2. The universal cubic curves \(C_3\) via degeneracy loci.

Recall that the space \(M_3\) is isomorphic to \(\text{Gr}(2,V)\) such that \(\dim V = 5\). In this subsection, we describe the universal family \(C_3\) of cubic curves in a geometric way which confirms the calculation of previous subsection. Let us recall the isomorphism \(M_3 \cong \text{Gr}(2,V)\). Consider the Schubert variety
\[
\sigma_{2,0}(l) := \{[l'] \in \text{Gr}(2,V) | l \cap l' \neq \emptyset\}
\]
which is a degree 3 and 4-dimensional subvariety of \(\text{Gr}(2,V)\). By taking the hyperplane sections \(H_1 \cap H_2 \cap H_3\) with this \(\sigma_{2,0}(l)\), we obtain a twisted cubic curve
\[
C_l := \sigma_{2,0}(l) \cap H_1 \cap H_2 \cap H_3 \subset \text{Gr}(2,5) \cap H_1 \cap H_2 \cap H_3 = V_5,
\]
that is, \(\pi_{V_5}(\pi_{M_3}^{-1}([l])) = C_l\). Conversely, for a point \(p = [L] \in Y \subset \text{Gr}(2,V)\), the inverse image \(\pi_{V_5}^{-1}(p)\) consists of the twisted cubic curves \([C_l]\) such that \(l \cap L \neq \emptyset\). This implies that \(\pi_{M_3}(\pi_{V_5}^{-1}(p)) = \sigma_{2,0}(L) \cap \text{Gr}(2,V) = M_3\). Note that the Schubert variety \(\sigma_{2,0}\) is a cone of rational normal scroll \(\mathbb{P}(O_{\mathbb{P}^4}(1) \boxtimes \mathbb{P}^3) \cong \mathbb{P}^2 \times \mathbb{P}^1\) in \(\mathbb{P}^6\). Thus the universal cubic \(C_3\) is an irreducible variety of dimension 7. Also, it is well-known that the Schubert variety \(\sigma_{2,0}\) can be defined by a degeneracy loci of vector bundles over Grassmannian.
Example 3.10. Let \( i: W \subset V \) be one-dimensional subvector space of \( V \). For a line \( l = \mathbb{P}(W) \subset \mathbb{P}(V) \), let
\[
\phi: U \to (V/W) \otimes \mathcal{O}_{\Gr(2,V)}
\]
be the canonical morphism induced by the injection \( i \). Then one can check that the degeneracy locus \( D_{\leq 1}(\phi) \subset \Gr(2, V) \) of the map \( \phi \) whose rank is \( \leq 1 \) has the support \( \sigma_{2,0}(l) \). Also it has the expected dimension \( \dim \Gr(2,V) - (2 - 1) \cdot (3 - 1) = 4 \). Thus, by Porteous’ formula, the fundamental class of \( D_{\leq 1}(\phi) \) is given by
\[
[D_{\leq 1}(\phi)] = c_2([(V/W) \otimes \mathcal{O}_{\Gr(2,V)}] - U).
\]

In our case, by relativizing over \( V_5 \), we can find the fundamental form \([C_3]\) over \( M_3 \times V_5 \).

**Proposition 3.11.** Let \( S_{M_3} \) be the universal subbundle of \( M_3 \) and \( Q_{V_5} \) be the restriction of the universal quotient bundle \( V_5 \subset \Gr(2, V) \). Then the fundamental class of the universal cubic curves is given by
\[
[C_3] = c_2(Q_{V_5} - S_{M_3}) \in H^2(M_3 \times Y).
\]

**Proof.** Since \( V_5 \subset \Gr(2, V) \), we have an universal sequence
\[
0 \to S_{V_5} \to V \otimes \mathcal{O}_{V_5} \to Q_{V_5} \to 0.
\]

Let us consider the relative Grassmannian bundle \( \Gr(2, V \otimes \mathcal{O}_{V_5}) \to V_5 \) with the structure morphism \( \pi_{V_5} : \Gr(2, V \otimes \mathcal{O}_{V_5}) \to V_5 \). Here we denote the same notation with the projection map because \( \Gr(2, V \otimes \mathcal{O}_{V_5}) \cong \Gr(2, V) \times V_5 \). From the universal sequence,
\[
0 \to S_G \to \pi_Y^*(V \otimes \mathcal{O}_{V_5}) \to Q_G \to 0
\]
over \( G := \Gr(2, V \otimes \mathcal{O}_{V_5}) \) and the pull-back of the sequence (8), we obtain a bundle morphism
\[
\phi_G : S_G \to \pi_{V_5}^*Q_{V_5}
\]
over \( G \). Note that \( S_G = \pi_{M_3}^*S_{M_3} \) by its definition. The space \( C_3 \) is reduced because it is a generically reduced and Cohen–Macaulay space. Thus the degeneracy locus \( D_{\leq 1}(\phi_G) \) of the map \( \phi_G \) is \( C_3 \). By Porteous’ formula,
\[
[D_{\leq 1}(\phi_G)] = c_2(Q_{V_5} - S_{M_3}).
\]

\( \square \)

**Remark 3.12.** The Poincaré dual of the fundamental class of the universal cubic curves is
\[
[C_3] = m_1^2 - m_2 - m_1 h_1 + 3h_2 \in H^2(M_3 \times V_5),
\]
which matches our computation of Subsection 3.1.1.
The case $V_{22}$. Let us recall the definition of the variety $V_{22}$. Let $S$ and $Q$ be the universal bundles of $Gr(3,7)$. Then $V_{22}$ is defined as a zero section of $\wedge^2 (S^*)^{\oplus 3}$. Alternatively, $V_{22}$ can be regraded as a subvariety of the net of quadrics $N(4;2,3)$.

- $\text{Pic}_Z(V_{22}) \cong Z\langle H \rangle$, $\text{deg}(V_{22}) = 22$, $K_{V_{22}} = -H$.
- The cohomology ring of $V_{22}$ over $\mathbb{Z}$ is isomorphic to

$$H^*(V_{22}, \mathbb{Z}) \cong Z[h_1, h_2, h_3]/\langle h_1^2 - 22 h_2, h_1^3 - 22 h_3, h_1 h_2 - h_3, h_2^3\rangle$$

where $\text{deg}(h_i) = i$, $1 \leq i \leq 3$. Moreover, $h_1 = c_1(O_{V_{22}}(1))$ and $h_i$ is the Poincaré dual of the linear space of dimension $3 - i$ for $i = 2, 3$.
- The Chern classes of tautological bundles on $V_{22}$ are
  (1) $c(S_{V_{22}}) = 1 - h_1 + 10 h_2 - 2 h_3$,
  (2) $c(Q_{V_{22}}) = 1 + h_1 + 12 h_2 + 4 h_3$,
  (3) $c(V_{22}) = 1 + h_1 + 24 h_2 + 4 h_3$

where $S_{V_{22}}$ and $Q_{V_{22}}$ are the restriction of the universal bundles $S$ and $Q$ on $Gr(3,7)$. Unless otherwise stated, we omit the subscription $V_{22}$ of the universal bundles.

By the same reason as in the case of $V_5$, the moduli space $M_d(V_{22})$ is isomorphic to the Hilbert scheme $H_d(V_{22})$ of curves with Hilbert polynomial $dm + 1$ for $1 \leq d \leq 3$. The later space $H_d(V_{22})$ has been studied by many authors.

**Proposition 3.13** ([KPS18, Fae14, KS04]). $M_1 \cong Q$, $M_2 \cong \mathbb{P}^2$, $M_3 \cong \mathbb{P}^3$, where $Q$ is a singular planar quartic curve.

Let us compute the degree $d = 1$ case by the result of Pirola ([Pir85]). By the Chern class computation, the virtual dimension of $M_1$ is $\text{virt. dim} M_1 = 1$ and the virtual fundamental class is given by the following.

**Lemma 3.14.**

$$[M_1]_{\text{vir}} = [M_1]$$

**Proof.** By the deformation invariance of DT invariants, we may assume that $V_{22}$ is not Mukai-Umemura 3-folds. Then $M_1 \cong Q \subset \mathbb{P}^2$ is a regular embedding, hence the intrinsic normal cone $C_{M_1}$ of $M_1$ is given by the bundle stack

$$C_{M_1} = [N_Q/\mathbb{P}^2/T_{\mathbb{P}^2}|_Q].$$

Let $E \rightarrow L_{M_1}$ be the usual perfect obstruction theory. Then by [BF97], we have the closed embedding

$$C_{M_1} \hookrightarrow h^1/h^0(E^\vee).$$

Since they are both bundle stacks with dimension 0 (as Artin stacks), we have $C_{M_1} = h^1/h^0(E^\vee)$. Hence, by the definition of the virtual cycle in [BF97], we have

$$[M_1]_{\text{vir}} = 0_{h^1/h^0(E^\vee)}[C_{M_1}] = 0_{h^1/h^0(E^\vee)}[h^1/h^0(E^\vee)] = [M_1].$$
Proposition 3.15. The degree $d = 1$ DT invariant and descendent invariant on $V_{22}$ are given by

$$\langle \tau_0(h_2) \rangle_1 = 2 \text{ and } \langle \tau_1(h_1) \rangle_1 = 22$$

Proof. Let $C_1$ be the universal curve over $M_1 (= Q)$. We compute the invariants by using the degeneracy loci method. In [AF06, Lemma 3.1], the authors describe how to obtain lines in $V_{22}$. We relativize their construction. Let $K$ be the vector bundle on $V_{22}$ with data $\text{rk}(K) = 5$, $c_1(K) = -2$, $c_2(K) = 40$, $c_3(K) = -20$, and $\dim \text{Hom}(K, S) = 3$. Let $B^* = \text{Hom}(K, S)$. Note that $Q \subset \mathbb{P}(B^*)$. The universal curve $C_1 \subset \mathbb{P}(B^*) \times V_{22}$ is the degeneracy loci of the canonical homomorphism

$$\Phi : S^* \to K^* \otimes \mathcal{O}_{\mathbb{P}(B^*)}(1).$$

In fact, the map $\Phi$ is the dual of the composition of the pull-back of the evaluation map $\text{ev} : K \otimes \text{Hom}(K, S) \to S$ on $V_{22}$ and the tautological map $\mathcal{O}_{\mathbb{P}(B^*)}(-1) \to B \otimes \mathcal{O}_{\mathbb{P}(B^*)}$ on $\mathbb{P}(B^*)$.

Let $H^*(\mathbb{P}(B^*), \mathbb{Z}) = \mathbb{Z}[m_1]/(m_1^3)$ with $\text{deg}(m_1) = 1$. By Proposition 3.14 in [Pir85], the Chern character of the structure sheaf $\mathcal{O}_{C_1}$ over $\mathbb{P}(B^*) \times V_{22}$ is given by

$$\text{ch}(\mathcal{O}_{C_1}) = (2m_1^2 h_1 + 4m_1 h_2) + (-8m_1^2 h_2 + 2m_1 h_3) + (\text{the terms of higher degree}).$$

Thus, by the Grothendieck-Riemann-Roch theorem, the invariants are

$$\int_{[M_1]} \tau_0(h_2) = 2m_1^2 h_3 = 2[pt], \quad \int_{[M_1]} \tau_1(h_1) = 22m_1^2 h_3 = 22[pt].$$

□

Remark 3.16. By Porteous’ formula, the dual class of the fundamental class $[C_1]$ is

$$[C_1] = 2m_1^2 h_1 + 4m_1 h_2.$$ 

The intersection number of $[C_1]$ with the line class $h_2$ in $V_{22}$ is $[C_1] \cdot h_2 = 2m_1^2 h_3$. This matches with the fact that the degree of the surface $S$ sweeping out by lines in $V_{22}$ is $\text{deg}(S) = 2$ ([Ame98, Section 3]).

For the degree $d = 2$ and $3$ cases, the universal curves over $M_d$ have been studied in [Fae14, Lemma 4.1] and [KS04, Theorem 2.4]. Let $i : C_d \to M_d \times V_{22}$ be the universal curve in $M_d \times V_{22}$. The free resolutions of $i_* \mathcal{O}_{C_d}$ on $M_d \times V_{22}$ are

1. $(d = 2) \quad 0 \to S \boxtimes \mathcal{O}_{\mathbb{P}^2}(-4) \to Q^* \boxtimes \mathcal{O}_{\mathbb{P}^2}(-3) \to \mathcal{O}_{V_{22} \times \mathbb{P}^2} \to i_* \mathcal{O}_{C_2} \to 0,$
2. $(d = 3) \quad 0 \to E \boxtimes \mathcal{O}_{\mathbb{P}^3}(-3) \to S \boxtimes \mathcal{O}_{\mathbb{P}^3}(-2) \to \mathcal{O}_{V_{22} \times \mathbb{P}^3} \to i_* \mathcal{O}_{C_3} \to 0,$

where $\text{rk}(E) = 2$, $c_1(E) = -1$, $c_2(E) = 7$. By the same method for the case $V_5$, one can find the (virtual) fundamental class $[M_d]$. 

where the last equality comes from properties of Gysin pull-back via bundle stacks [Kre99].

□
Therefore we have

**Proposition 3.17.** The invariants \( \langle \tau_i (h_{2-i}) \rangle_d \) are given by the numbers of the following table.

| \( d \) | \( i = 0 \) | \( i = 1 \) |
|-------|-------|-------|
| 1     | 2     | 22    |
| 2     | 7     | 28    |
| 3     | 28    | 28    |

By combining Proposition 2.2, Proposition 3.7 and Proposition 3.17, we have

**Theorem 3.18.** Conjecture 1.1 (which is equivalent to Conjecture 3.2) is true when \( X = V_5 \) and \( V_{22} \) up to the degree 3.

### 4. Proof of Conjecture 1.2

Let us recall the definition of meeting invariants \( m_{\beta_1, \beta_2} \in \mathbb{Z} \) for \( \beta_1, \beta_2 \in H_2(X, \mathbb{Z}) \) ([KP08, Section 0.3]). It is given by the following rules:

1. \( m_{\beta_1, \beta_2} = m_{\beta_2, \beta_1} \).
2. If either \( \deg(\beta_1) \leq 0 \) or \( \deg(\beta_2) \leq 0 \), then \( m_{\beta_1, \beta_2} = 0 \).

Let \( \{S_1, \cdots, S_k\} \) be the basis of the torsion free part of \( H^4(X, \mathbb{Z}) \). Let \( (g_{ij}) \) be the inverse matrix of the intersection matrix \( (g_{ij}) \), \( g_{ij} = \langle S_i, S_j \rangle \).
3. If \( \beta_1 \neq \beta_2 \),

\[
m_{\beta_1, \beta_2} = \sum_{i,j} n_{0, \beta_1}(S_i)g_{ij}n_{0, \beta_2}(S_j) + m_{\beta_1, \beta_2-\beta_1} + m_{\beta_1-\beta_2, \beta_2}.
\]

4. If \( \beta_1 = \beta_2 = \beta \), then

\[
m_{\beta, \beta} = n_{0, \beta}(c_2(X)) + \sum_{i,j} n_{0, \beta}(S_i)g_{ij}n_{0, \beta}(S_j) - \sum_{\beta_1+\beta_2=\beta, k \geq 1} m_{\beta_1, \beta_2}.\]

We recall Conjecture 1.2 here. For each \( \gamma \in H^2(X, \mathbb{Z}) \),

\[
\langle \tau_1 (\gamma) \rangle_{\beta} = \frac{n_{0, \gamma}(\gamma^2)}{2(\gamma \cdot \beta)} - \sum_{\beta_1+\beta_2=\beta} \frac{(\gamma \cdot \beta_1)(\gamma \cdot \beta_2)}{4(\gamma \cdot \beta)} m_{\beta_1, \beta_2} - \sum_{k \geq 1, k | \beta} \frac{(\gamma \cdot \beta)}{k} n_{1, \beta/k}.
\]

**Remark 4.1.** It is believed that the invariants \( n_{1,d} \) come from a space of elliptic curves in \( Y \) (cf. [KP08, Section 3 and 5]). The defining equations of \( V_5 \) and \( V_{22} \) can be generated by quadric equations and they does not contain planes. Thus the space of elliptic curves of degree \( d \leq 3 \) should be empty. This implies that the invariants are \( n_{1,d} = 0 \).
Therefore the suitable choice of sign for the conjecture should be $-1$.

Theorem 4.2. Conjecture 1.2 holds for $Y = V_5$ and $V_{22}$ when we assume that $n_{1,d} = 0$ for $1 \leq d \leq 3$.

The remaining two subsections is devoted to the proof of Theorem 4.2.

4.1. The case $V_5$. Let $X = |K_Y|$, $\bar{X} = \mathbb{P}(K_Y \oplus O_Y)$ and $\pi : \bar{X} \to Y$ be the canonical projection map. By construction, we have

$$H^4(\bar{X}, \mathbb{Z}) = \langle T_1, T_2 \rangle,$$

where $T_1 = \text{PD}(H \cap Y)$, $T_2 = \pi^*(\text{PD}(L_1))$ for linear spaces $L_i$ of dimension $i$. Since the normal bundle of $H \cap Y$ in $X$ is $N_{H \cap Y,X} = \pi^*(O_Y(1) \oplus O_Y(-2))$, $T_1 \cdot T_1 = -2H^3 \cap Y = -10$. Hence the intersection matrix is given by

$$\begin{pmatrix}
T_1 & T_2 \\
T_1 & -10 & 1 \\
T_2 & 1 & 0
\end{pmatrix},
\quad
\begin{pmatrix}
g_{ij}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
$$

From $T_X = \pi^*(T_Y \oplus O_Y(-2))$, we have $c(X) = \pi^*(c(Y) \cdot (1 - 2h_1))$. Hence $c_2(X) = -8T_2$. Also we have $T_1|_Y = (T_1 \cdot T_1)T_2|_Y = -10T_2|_Y$. Using the formula of meeting invariants, we have

$$m_{1,1} = n_{0,1}(-8T_2) + 2n_{0,1}(T_1)n_{0,1}(T_2) + 10n_{0,1}(T_2)^2
= 40 - 20 \times 5^2 + 10 \times 5^2 = -210,$n_{1,2} = n_{0,1}(T_1)n_{0,2}(T_2) + n_{0,1}(T_2)n_{0,2}(T_1) + 10n_{0,1}(T_2)n_{0,2}(T_2) + m_{1,1}
= -10n_{0,1}(T_2)n_{0,2}(T_2) - 210 = -1960.$$

Motivated from the fact that the spaces of genus one curves on $Y$ with degree $d = 1, 2, 3$ are empty sets, let us assume that $n_{1,1} = n_{1,2} = n_{1,3} = 0$. Then Conjecture 1.2 has been written as

$$\langle \tau_1(H) \rangle_d = \frac{n_{0,d}(H^2)}{2d} - \sum_{d_1 + d_2 = d} \frac{d_1 \cdot d_2}{4d} m_{d_1,d_2}.$$

Note that we need to choose some suitable orientation of the moduli space. Here we use the orientation in [CT21, (0.7)]. By a direction calculation, one can check the identity as follows.

$$-\langle \tau_1(H) \rangle_1 = \frac{5n_{0,1}(T_2)}{2},
-\langle \tau_1(H) \rangle_2 = \frac{5n_{0,2}(T_2)}{4} - \frac{1}{8}m_{1,1},
-\langle \tau_1(H) \rangle_3 = \frac{5n_{0,3}(T_2)}{6} - \frac{1}{3}m_{1,2}.$$

Therefore the suitable choice of sign for the conjecture should be $-1$. 
4.2. The case $V_{22}$. Let $Y = V_{22}$, $X = |K_{V_{22}}|$, and $\tilde{X} = \mathbb{P}(K_Y \oplus \mathcal{O}_Y)$ and $\pi : \tilde{X} \to Y$ be the canonical projection map. Let $T_1$ and $T_2$ be generators of the cohomology group $H^2(X, \mathbb{Z})$, defined by

$$T_1 := PD(H \cap Y), T_2 := \pi^*(PD(L_1))$$

where $L_1$ is a class of line, equal to $(H^2/22) \cap Y$. Then the intersection matrix is computed by:

$$\begin{pmatrix}
  g_{ij} \\
\end{pmatrix} = \begin{pmatrix}
  T_1 & T_2 \\
  T_1 & -22 & 1 \\
  T_2 & 1 & 0 \\
\end{pmatrix}, \quad g^{ij} = (g_{ij})^{-1} = \begin{pmatrix}
  0 & 1 \\
  1 & 22 \\
\end{pmatrix}$$

Note that we can check $c_2(X) = 2T_2$ by direct calculation. Also, by the same manner as did in the case $Y = Y_5$, the meeting invariants are

$$m_{1,1} = -84, m_{1,2} = 224.$$ 

Under the assumption $n_{1,1} = n_{1,2} = n_{1,3} = 0$, we have the identities:

$$\langle \tau_1(H) \rangle_1 = \frac{22n_{0,1}(T_2)}{2}, -\langle \tau_1(H) \rangle_2 = \frac{22n_{0,2}(T_2)}{4} - \frac{1}{8}m_{1,1},$$

$$\langle \tau_1(H) \rangle_3 = \frac{22n_{0,3}(T_2)}{6} - \frac{1}{3}m_{1,2},$$

which confirms Theorem 4.2.

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