Ricci-like Solitons with Arbitrary Potential and Gradient Almost Ricci-like Solitons on Sasaki-like Almost Contact B-metric Manifolds

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Abstract. Ricci-like solitons with arbitrary potential are introduced and studied on Sasaki-like almost contact B-metric manifolds. A manifold of this type can be considered as an almost contact complex Riemannian manifold which complex cone is a holomorphic complex Riemannian manifold. The soliton under study is characterized and proved that its Ricci tensor is equal to the vertical component of both B-metrics multiplied by a constant. Thus, the scalar curvatures with respect to both B-metrics are equal and constant. In the 3-dimensional case, it is found that the special sectional curvatures with respect to the structure are constant. Gradient almost Ricci-like solitons on Sasaki-like almost contact B-metric manifolds have been proved to have constant soliton coefficients. Explicit examples are provided of Lie groups as manifolds of dimensions 3 and 5 equipped with the structures under study.

Mathematics Subject Classification. Primary 53C25, 53D15, 53C50; Secondary 53C44, 53D35, 70G45.

Keywords. Ricci-like soliton, \(\eta\)-Ricci soliton, einstein-like manifold, \(\eta\)-Einstein manifold, almost contact B-metric manifold, almost contact complex Riemannian manifold.

Introduction

Ricci soliton is a special self-similar solution of the Hamilton’s Ricci flow and it is a natural generalization of the notion of Einstein metric. According to [19], a (pseudo-)Riemannian manifold admits a Ricci soliton if the metric, its Ricci tensor and the Lie derivative of the metric along a vector field (called potential)
are linear dependent. If the coefficients of this dependence are functions then the soliton is called an *almost Ricci soliton* \[32\]. If a function exists so that the potential is its gradient, then the (almost) Ricci soliton is called a *gradient (almost) Ricci soliton* (see, e.g., \[14, 15, 35\]).

The topic became more popular after Perelman’s proof of the Poincaré conjecture, following Hamilton’s program to use the Ricci flow (see \[31\]). Ricci solitons have been explored by a number of authors (see, e.g., \[2, 3, 9–11, 17, 21, 30, 33\]).

Ricci solitons are also of interest to physicists, and in physical literature are called *quasi-Einstein* (see, e.g., \[12, 16\]).

The presence of the structure 1-form \(\eta\) on manifolds with almost contact or almost paracontact structure motivates the need to introduce so-called \(\eta\)-Ricci solitons. Then, \(\eta \otimes \eta\) is the restriction of the metric on the orthogonal complement to the (para)contact distribution, determined by the structure vector field \(\xi\). By adding a term proportional to \(\eta \otimes \eta\) into the defining equality of a Ricci soliton, it is defined the notion of \(\eta\)-Ricci soliton, introduced in \[13\]. Later, it has been studied on almost contact and almost paracontact manifolds by many authors (e.g., \[1, 4, 5, 8, 34\]). For almost \(\eta\)-Ricci solitons, see, for example \[6, 7\].

Our global goal is to study the differential geometry of almost contact B-metric manifolds investigated since 1993 \[18, 29\].

Unlike almost contact metric manifolds, almost contact B-metric manifolds have two metrics that are mutually associated with structural endomorphism. The restrictions of both B-metrics on the orthogonal distribution to the contact distribution is \(\eta \otimes \eta\). This is the reason for introducing in \[27\] a further generalization of the notions of a Ricci soliton and an \(\eta\)-Ricci soliton, the so-called *Ricci-like soliton*, using both B-metrics and \(\eta \otimes \eta\). There, we have explored these objects with potential Reeb vector field on some important kinds of manifolds under consideration: Einstein-like, Sasaki-like and having a torse-forming Reeb vector field. In \[28\], we continue to study Ricci-like solitons, whose potential is the Reeb vector field or pointwise collinear to it.

In the present paper, our goal is to investigate Ricci-like solitons with arbitrary potential on almost contact B-metric manifolds of Sasaki-like type, as well as gradient almost Ricci-like solitons on these manifolds.

The paper is organized as follows. In Sect. 1, we recall basic definitions and properties of almost contact B-metric manifolds of Sasaki-like type and obtain several immediate consequences. Section 2 includes some necessary results and a 5-dimensional example for a Ricci-like soliton with a potential Reeb vector field. In Sect. 3, we study Ricci-like solitons with an arbitrary potential. Then, we prove an identity for the soliton constants and a property of the potential, as well as that the Ricci tensor is a constant multiple of \(\eta \otimes \eta\). For the 3-dimensional case, we find the values of the sectional curvatures of the special 2-planes with respect to the structure and construct an explicit example. In
Sect. 4, we introduce gradient almost Ricci-like solitons on Sasaki-like manifolds and prove that their Ricci tensor has the same form as in the previous section. For the example in Sect. 3, we find a potential function to illustrate the obtained results.

1. Sasaki-like Almost Contact B-metric Manifolds

A differentiable manifold $M$ of dimension $(2n + 1)$, equipped with an almost contact structure $(\varphi, \xi, \eta)$ and a B-metric $g$ is called an almost contact B-metric manifold and it is denoted by $(M, \varphi, \xi, \eta, g)$. More concretely, $\varphi$ is an endomorphism of the tangent bundle $TM$, $\xi$ is a Reeb vector field, $\eta$ is its dual contact 1-form and $g$ is a pseudo-Riemannian metric of signature $(n + 1, n)$ satisfying the following conditions [18]

$$\varphi \xi = 0, \quad \varphi^2 = -1 + \eta \otimes \xi, \quad \eta \circ \varphi = 0, \quad \eta(\xi) = 1,$$

$$g(\varphi x, \varphi y) = -g(x, y) + \eta(x)\eta(y), \quad (1.1)$$

where $\iota$ stands for the identity transformation on $\Gamma(TM)$.

In the latter equality and further, $x, y, z, w$ will stand for arbitrary elements of $\Gamma(TM)$ or vectors in the tangent space $T_pM$ of $M$ at an arbitrary point $p$ in $M$.

The following equations are immediate consequences of (1.1)

$$g(\varphi x, y) = g(x, \varphi y), \quad g(x, \xi) = \eta(x), \quad g(\xi, \xi) = 1, \quad \eta(\nabla x \xi) = 0, \quad (1.2)$$

where $\nabla$ denotes the Levi-Civita connection of $g$.

The associated metric $\tilde{g}$ of $g$ on $M$ is also a B-metric and it is defined by

$$\tilde{g}(x, y) = g(x, \varphi y) + \eta(x)\eta(y).$$

In [18], almost contact B-metric manifolds (also known as almost contact complex Riemannian manifolds) are classified with respect to the $(0,3)$-tensor $F$ defined by

$$F(x, y, z) = g((\nabla_x \varphi) y, z).$$

It has the following basic properties:

$$F(x, y, z) = F(x, z, y) = F(x, \varphi y, \varphi z) + \eta(y)F(x, \xi, z) + \eta(z)F(x, y, \xi),$$

$$F(x, \varphi y, \xi) = (\nabla_x \eta)y = g(\nabla_x \xi, y).$$

This classification consists of eleven basic classes $\mathcal{F}_i$, $i \in \{1, 2, \ldots, 11\}$.

In [20], it is introduced the type of a Sasaki-like manifold among almost contact B-metric manifolds. The definition condition is its complex cone to be a Kähler-Norden manifold, i.e. with a parallel complex structure. A Sasaki-like manifold with almost contact B-metric structure is determined by the condition

$$(\nabla_x \varphi) y = -g(x, y)\xi - \eta(y) x + 2\eta(x)\eta(y)\xi, \quad (1.3)$$
which is equivalent to the following \((\nabla_x \varphi) y = g(\varphi x, \varphi y) \xi + \eta(y) \varphi^2 x\).

Obviously, Sasaki-like manifolds form a subclass of the class \(\mathcal{F}_4\). Moreover, the following identities are valid for it [20]

\[
\begin{align*}
\nabla_x \xi &= -\varphi x, & (\nabla_x \eta) (y) &= -g(x, \varphi y), \\
R(x, y; \xi) &= \eta(y) x - \eta(x) y, & \rho(x, \xi) &= 2n \eta(x), \\
R(\xi, y; z) &= g(y, z) \xi - \eta(z) y, & \rho(\xi, \xi) &= 2n,
\end{align*}
\]

(1.4)

where \(R\) and \(\rho\) stand for the curvature tensor and the Ricci tensor of \(\nabla\).

The corresponding curvature tensor of type \((0, 4)\) is determined as usually by \(R(x, y, z, w) = g(R(x, y)z, w)\).

Further, we use an arbitrary basis \(\{e_i\}, i \in \{1, 2, \ldots, 2n + 1\}\) of \(T_p M, p \in M\).

On an arbitrary almost contact B-metric manifold, there exists a \((0, 2)\)-tensor \(\rho^*\), which is associated with \(\rho\) regarding \(\varphi\). It is defined by \(\rho^*(y, z) = g^{ij}R(e_i, y, z, \varphi e_j)\) and due to the first equality in (1.2) \(\rho^*\) is symmetric.

The following relation between \(\rho^*\) and \(\rho\) is valid for a Sasaki-like manifold

\[\rho^*(y, z) = \rho(y, \varphi z) + (2n - 1)g(y, \varphi z).\]

(1.5)

It follows by taking the trace for \(x = e_i\) and \(w = e_j\) the following property of a Sasaki-like manifold [20]

\[
R(x, y, \varphi z, w) - R(x, y, z, \varphi w)
= \{g(y, z) - 2\eta(y)\eta(z)\} g(x, \varphi w) + \{g(y, w) - 2\eta(y)\eta(w)\} g(x, \varphi z)
- \{g(x, z) - 2\eta(x)\eta(z)\} g(y, \varphi w) - \{g(x, w) - 2\eta(x)\eta(w)\} g(y, \varphi z).
\]

(1.6)

As a corollary of (1.5) we have that \(\rho(y, \varphi z) = \rho(\varphi y, z)\), i.e. \(Q \circ \varphi = \varphi \circ Q\), where \(Q\) is the Ricci operator, i.e. \(\rho(y, z) = g(Qy, z)\).

The scalar curvature \(\tilde{\tau}\) of \(\tilde{g}\) is defined by \(\tilde{\tau} = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{R}(e_i, e_k, e_l, e_j)\), where \(\tilde{R}\) is the curvature tensor of \(\tilde{g}\) and \(\tilde{g}^{ij} = -\varphi_i \varphi_j + \xi^i \xi^j\) holds. In addition, the associated quantity \(\tau^*\) of \(\tau\) with respect to \(\varphi\) is determined by \(\tau^* = g^{ij}\rho(e_i, \varphi e_j)\). For them, using (1.6) for a Sasaki-like manifold, we infer the following relation

\[\tilde{\tau} = -\tau^* + 2n.\]

(1.7)

In [26], it is given the following relations between \(\tau\) and \(\tilde{\tau}\)

\[d\tau \circ \varphi = d\tilde{\tau} + 2(\tau - 2n)\eta, \quad d\tilde{\tau} \circ \varphi = -d\tau + 2(\tilde{\tau} - 2n)\eta.\]

As corollaries we have

\[
\begin{align*}
d\tau \circ \varphi^2 &= d\tilde{\tau} \circ \varphi, & d\tilde{\tau} \circ \varphi^2 &= -d\tau \circ \varphi, \\
d\tau(\xi) &= 2(\tilde{\tau} - 2n), & d\tilde{\tau}(\xi) &= -2(\tau - 2n).
\end{align*}
\]

(1.8) (1.9)
Proposition 1.1. On a Sasaki-like manifold \((M, \varphi, \xi, \eta, g)\) of dimension \(2n+1\), the following formulae for the Ricci operator \(Q\) are valid

\[
(\nabla_x Q)\xi = Q\varphi x - 2n\varphi x, \quad (1.10)
\]
\[
(\nabla_\xi Q)y = 2Q\varphi y. \quad (1.11)
\]

Proof. For a Sasaki-like manifold, according to (1.4), the equalities \(Q\xi = 2n\xi\) and \(\nabla_x \xi = -\varphi x\) holds. Using them, we obtain immediately the covariant derivative in (1.10).

Now, we apply \(\nabla_z\) to the expression of \(R(x, y)\xi\) in (1.4) and then, using the form of \(\nabla_\eta\) in (1.4), we get the following

\[
(\nabla_z R)(x, y)\xi = R(x, y)\varphi z - g(y, \varphi z)x + g(x, \varphi z)y.
\]

We take the trace of the above equality for \(z = e_i\) and \(x = e_j\) and use (1.5) to obtain

\[
g^{ij}(\nabla_{e_i} R)(e_j, y)\xi = -Q\varphi y - 2n\varphi y.
\]

By virtue of the following consequence the second Bianchi identity

\[
g^{ij}(\nabla_{e_i} R)(\xi, y)e_j = (\nabla_y Q)(\xi) - (\nabla_\xi Q)y,
\]

the symmetries of \(R\) and (1.10), we get (1.11). □

As consequences of (1.10) and (1.11) we obtain respectively

\[
\eta((\nabla_x Q)\xi) = 0, \quad \eta((\nabla_\xi Q)y) = 0. \quad (1.12)
\]

Let us recall [27], an almost contact B-metric manifold \((M, \varphi, \xi, \eta, g)\) is said to be Einstein-like if its Ricci tensor \(\rho\) satisfies

\[
\rho = a g + b \tilde{g} + c \eta \otimes \eta \quad (1.13)
\]

for some triplet of constants \((a, b, c)\). In particular, when \(b = 0\) and \(b = c = 0\), the manifold is called an \(\eta\)-Einstein manifold and an Einstein manifold, respectively.

If \(a, b, c\) are functions on \(M\), then the manifold is called almost Einstein-like, almost \(\eta\)-Einstein and almost Einstein, respectively.

Tracing (1.13) and using (1.7), the scalar curvatures \(\tau\) and \(\tilde{\tau}\) of an Einstein-like almost contact B-metric manifold have the form

\[
\tau = (2n + 1)a + b + c, \quad \tilde{\tau} = 2n(b + 1). \quad (1.14)
\]

For a Sasaki-like manifold \((M, \varphi, \xi, \eta, g)\) with \(\dim M = 2n + 1\) and a scalar curvature \(\tau\) regarding \(g\), which is Einstein-like with a triplet of constants \((a, b, c)\), the following equalities are given in [27]:

\[
a + b + c = 2n, \quad \tau = 2n(a + 1). \quad (1.15)
\]

Then, for \(\tilde{\tau}\) on an Einstein-like Sasaki-like manifold we obtain

\[
\tilde{\tau} = 2n(b + 1) \quad (1.16)
\]
and because (1.14)–(1.16), the expression (1.13) becomes
\[
\rho = \left( \frac{\tau}{2n} - 1 \right) g + \left( \frac{\tilde{\tau}}{2n} - 1 \right) \tilde{g} + \left( 2(n + 1) - \frac{\tau + \tilde{\tau}}{2n} \right) \eta \otimes \eta.
\]

**Proposition 1.2.** Let \((M, \varphi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional Sasaki-like manifold. If it is almost Einstein-like with functions \((a, b, c)\) then the scalar curvatures \(\tau\) and \(\tilde{\tau}\) of \(g\) and \(\tilde{g}\), respectively, are constants
\[
\tau = \text{const}, \quad \tilde{\tau} = 2n
\]
and \((M, \varphi, \xi, \eta, g)\) is \(\eta\)-Einstein with constants
\[
(a, b, c) = \left( \frac{\tau}{2n} - 1, 0, 2n + 1 - \frac{\tau}{2n} \right).
\]

**Proof.** If \((M, \varphi, \xi, \eta, g)\) is almost Einstein-like then \(\rho\) has the form in (1.13), where \((a, b, c)\) are a triad of functions. Then, according to (1.7), (1.13), (1.14) and the expression for \(\rho(\xi, \xi)\) on a Sasaki-like manifold, given in (1.4), we have the following
\[
a + b + c = 2n, \quad \tau = 2n(a + 1), \quad \tilde{\tau} = 2n(b + 1). \quad (1.17)
\]

Using (1.4), we can express \(R(x, y)\xi\) and \(R(x, \xi)y\) as follows
\[
R(x, y)\xi = \frac{1}{4n^2} \left\{ 2n [\eta(x)Qy - \eta(y)Qx + \rho(x, \xi)y - \rho(y, \xi)x] + (\tau - 2n) [\eta(y)x - \eta(x)y] + (\tilde{\tau} - 2n) [\eta(y)\varphi x - \eta(x)\varphi y] \right\},
\]
\[
R(x, \xi)y = \frac{1}{4n^2} \left\{ 2n [\rho(x, y)\xi + g(x, y)Q\xi - \rho(y, \xi)x - \eta(y)Qx] + (\tau - 2n) [\eta(y)x - g(x, y)\xi] + (\tilde{\tau} - 2n) [\eta(y)\varphi x - g(x, \varphi y)\xi] \right\}.
\]

Then, for \(y = \xi\) in either of the last two equalities, we have
\[
R(x, \xi)\xi = \eta(x)\xi - \frac{1}{2n} Qx - \frac{1}{4n^2} \left\{ [\tau - 2n(2n + 1)]\varphi^2 x - [\tilde{\tau} - 2n] \varphi x \right\}.
\]

After that, we compute the covariant derivative of \(R(x, \xi)\xi\) with respect to \(\nabla\). Since (1.3) and (1.4), we obtain
\[
(\nabla_{\xi} R) (x, \xi)\xi = -\frac{1}{2n} \{ (\nabla_{\xi} Q)x - \eta(x)Q\varphi z \} - \frac{1}{4n^2} \{ d\tau(z)\varphi^2 x - d\tilde{\tau}(z)\varphi x \\
- [\tau - 2n(2n + 1)]g(x, \varphi z)\xi + [\tilde{\tau} - 2n]g(\varphi x, \varphi z)\xi \} - \eta(x)\varphi z,
\]
which by taking the trace for \(z = e_i\) and \(x = e_j\) and (1.8) gives the following
\[
g^{ij} g((\nabla_{e_i} R)(e_j, \xi)\xi, y) = -\frac{1}{4n} d\tau(y) - \left\{ \frac{\tilde{\tau}}{2n} - 1 \right\} \eta(y). \quad (1.18)
\]

By virtue of the following consequence of the second Bianchi identity
\[
g^{ij} g((\nabla_{e_i} R)(y, \xi)\xi, e_j) = \eta((\nabla_y Q)\xi) - \eta((\nabla_\xi Q) y) \quad (1.19)
\]
and (1.12), we have that the trace in the left hand side of (1.19) vanishes. Then, (1.18) and (1.19) imply
\[ \text{d}\bar{r}(y) = -2\{\bar{r} - 2n\}\eta(y), \]
which comparing with (1.9) implies
\[ \text{d}\bar{r}(\xi) = 0, \quad \bar{r} = 2n. \]
The latter equalities together with (1.8) and (1.17) complete the proof. \(\Box\)

2. Ricci-like Solitons with Potential Reeb Vector Field on Sasaki-like Manifolds

In [27], by a condition for Ricci tensor, it is introduced the notion of a Ricci-like soliton with potential \(\xi\) on an almost contact B-metric manifold.

Now, we generalize this notion for a potential, which is an arbitrary vector field as follows. We say that \((M,\varphi,\xi,\eta,g)\) admits a \textit{Ricci-like soliton with potential vector field} \(v\) if the following condition is satisfied for a triplet of constants \((\lambda,\mu,\nu)\)
\[
\frac{1}{2}\mathcal{L}_vg + \rho + \lambda g + \mu \tilde{g} + \nu \eta \otimes \eta = 0, \quad (2.1)
\]
where \(\mathcal{L}\) denotes the Lie derivative.

If \(\mu = 0\) (respectively, \(\mu = \nu = 0\)), then (2.1) defines an \(\eta\)-\textit{Ricci soliton} (respectively, a \textit{Ricci soliton}) on \((M,\varphi,\xi,\eta,g)\).

If \(\lambda, \mu, \nu\) are functions on \(M\), then the soliton is called \textit{almost Ricci-like soliton}, \textit{almost \(\eta\)-Ricci soliton} and \textit{almost Ricci soliton}, respectively.

If \((M,\varphi,\xi,\eta,g)\) is Sasaki-like, we have
\[
(\mathcal{L}_\xi g)(x,y) = g(\nabla_x \xi, y) + g(x, \nabla_y \xi) = -2g(x, \varphi y),
\]
i.e. \(\frac{1}{2}\mathcal{L}_\xi g = -\tilde{g} + \eta \otimes \eta\). Then, because of (2.1), \(\rho\) takes the form
\[
\rho = -\lambda g + (1 - \mu)\tilde{g} - (1 + \nu)\eta \otimes \eta.
\]

**Theorem 2.1** ([27]). Let \((M,\varphi,\xi,\eta,g)\) be a \((2n + 1)\)-dimensional Sasaki-like manifold and let \(a, b, c, \lambda, \mu, \nu\) be constants that satisfy the following equalities:
\[
a + \lambda = 0, \quad b + \mu - 1 = 0, \quad c + \nu + 1 = 0.
\]
Then, the manifold admits a Ricci-like soliton with potential \(\xi\) and constants \((\lambda, \mu, \nu)\), where \(\lambda + \mu + \nu = -2n\), if and only if it is Einstein-like with constants \((a, b, c)\), where \(a + b + c = 2n\).

In particular, we get:

(i) The manifold admits an \(\eta\)-Ricci soliton with potential \(\xi\) and constants \((\lambda, 0, -2n - \lambda)\) if and only if the manifold is Einstein-like with constants \((-\lambda, 1, \lambda + 2n - 1)\).
(ii) The manifold admits a shrinking Ricci soliton with potential $\xi$ and constant $-2n$ if and only if the manifold is Einstein-like with constants $(2n, 1, -1)$.

(iii) The manifold is $\eta$-Einstein with constants $(a, 0, 2n - a)$ if and only if it admits a Ricci-like soliton with potential $\xi$ and constants $(-a, 1, a - 2n - 1)$.

(iv) The manifold is Einstein with constant $2n$ if and only if it admits a Ricci-like soliton with potential $\xi$ and constants $(-2n, 1, -1)$.

2.1. Example 1

In Example 2 of [20], it is given a Lie group $G$ of dimension 5 (i.e. $n = 2$) with a basis of left-invariant vector fields $\{e_0, \ldots, e_4\}$ and the corresponding Lie algebra is defined as follows

\[
\begin{align*}
[e_0, e_1] &= pe_2 + e_3 + qe_4, \quad [e_0, e_2] = -pe_1 - qe_3 + e_4, \\
[e_0, e_3] &= -e_1 - qe_2 + pe_4, \quad [e_0, e_4] = qe_1 - e_2 - pe_3,
\end{align*}
\]

$p, q \in \mathbb{R}$.

After that $G$ is equipped with an almost contact B-metric structure defined by

\[
\begin{align*}
g(e_0, e_0) &= g(e_1, e_1) = g(e_2, e_2) = -g(e_3, e_3) = -g(e_4, e_4) = 1, \\
g(e_i, e_j) &= 0, \quad i, j \in \{0, 1, \ldots, 4\}, \quad i \neq j,
\end{align*}
\]

$\xi = e_0, \quad \varphi e_1 = e_3, \quad \varphi e_2 = e_4, \quad \varphi e_3 = -e_1, \quad \varphi e_4 = -e_2$.

It is verified that the constructed almost contact B-metric manifold $(G, \varphi, \xi, \eta, g)$ is Sasaki-like.

In [27], it is proved that $(G, \varphi, \xi, \eta, g)$ is $\eta$-Einstein with constants

\[ (a, b, c) = (0, 0, 4). \quad (2.2) \]

Moreover, it is clear that $\tau = \tilde{\tau} = 4$.

It is also found there that $(G, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential $\xi$ and constants

\[ (\lambda, \mu, \nu) = (0, 1, -5). \quad (2.3) \]

Therefore, this example is in unison with Theorem 2.1 (iii) for $a = 0$.

3. Ricci-like Solitons with Arbitrary Potential on Sasaki-like Manifolds

Theorem 3.1. Let $(M, \varphi, \xi, \eta, g)$ be a $(2n + 1)$-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with arbitrary potential vector field $v$ and constants $(\lambda, \mu, \nu)$ then it is valid the following identities

\[
\begin{align*}
\lambda + \mu + \nu &= -2n, \\
\nabla_v \xi &= -\varphi v.
\end{align*}
\]

(3.1)
Proof. According to (2.1), a Ricci-like soliton with arbitrary potential vector field \( v \) is defined by \((L_v g)(y, z) = -2\rho(y, z) - 2\lambda g(y, z) - 2\mu \tilde{g}(y, z) - 2\nu \eta(y)\eta(z)\). Then, bearing in mind (1.4), the covariant derivative with respect to \( \nabla_x \) has the form
\[
(\nabla_x L_v g)(y, z) = -2(\nabla_x \rho)(y, z) - 2\mu \{g(\varphi_x, \varphi_y)\eta(z) + g(\varphi_x, \varphi_z)\eta(y)\}
\]
\[
+ 2(\mu + \nu)\{g(x, \varphi_y)\eta(z) + g(x, \varphi_z)\eta(y)\}.
\]
(3.2)

We use the following formula from [36] for a metric connection \( \nabla \)
\[
(\nabla_x L_v g)(y, z) = g((L_v \nabla)(x, y), z) + g((L_v \nabla)(x, z), y),
\]
which due to symmetry of \( L_v \nabla \) can read as
\[
2g((L_v \nabla)(x, y), z) = (\nabla_x L_v g)(y, z) + (\nabla_y L_v g)(z, x) - (\nabla_z L_v g)(x, y).
\]
(3.3)

Applying (3.3) to (3.2), we obtain
\[
g((L_v \nabla)(x, y), z) = -(\nabla_x \rho)(y, z) - (\nabla_y \rho)(z, x) + (\nabla_z \rho)(x, y)
\]
\[
-2\mu g(\varphi_x, \varphi_y)\eta(z) + 2(\mu + \nu)g(x, \varphi_y)\eta(z).
\]
(3.4)

Setting \( y = \xi \) in the equality above and using (1.10) and (1.11), we get
\[
(L_v \nabla)(x, \xi) = -2Q \varphi_x.
\]
(3.5)

The covariant derivative of the above equation by using of (1.4) has the form
\[
(\nabla_y L_v \nabla)(x, \xi) = (L_v \nabla)(x, \varphi_y) - 2(\nabla_y Q)\varphi x + 2\eta(x)Q y
\]
\[
-4n g(x, y) - 2(2n + 1)\eta(x)\eta(y)\xi.
\]
(3.6)

We apply the latter equality to the following formula from [36]
\[
(L_v R)(x, y)z = (\nabla_x L_v \nabla)(y, z) - (\nabla_y L_v \nabla)(x, z)
\]
(3.7)

and owing to symmetry of \( L_v \nabla \), we obtain the following consequence of (3.5)–(3.7)
\[
g((L_v R)(x, y)\xi, z) = -(\nabla_x \rho)(\varphi_y, z) + (\nabla_y \rho)(x, \varphi_z) - (\nabla_z \rho)(x, \varphi_y)
\]
\[
+ (\nabla_y \rho)(\varphi_x, z) - (\nabla_\varphi \rho)(y, z) + (\nabla_z \rho)(\varphi x, y)
\]
\[
- 2\eta(\rho y, z) + 2\eta(y)\rho(x, z).
\]
(3.8)

Plugging \( y = z = \xi \) in (3.8) and using (3.5), we obtain
\[
(L_v R)(x, \xi)\xi = 0.
\]
(3.9)

On the other hand, applying \( L_v \) to the expression of \( R(x, \xi)\xi \) from (1.4) and using (2.1), as well as the formulae for \( R(x, y)\xi \) and \( R(\xi, y)z \) from the same referent equalities, we get
\[
(L_v R)(x, \xi)\xi = (L_v \eta)(x)\xi + g(x, L_v \xi)\xi - 2\eta(L_v \xi)x
\]
or in an equivalent form
\[
(L_v R)(x, \xi)\xi = \{(L_v \eta)(x) + g(x, L_v \xi) - 2\eta(L_v \xi)\eta(x)\}\xi + 2\eta(L_v \xi)\varphi^2 x.
\]
(3.10)
Comparing (3.9) and (3.10), we obtain the following system of equations
\[ (\mathcal{L}_v \eta)(x) + g(x, \mathcal{L}_v \xi) - 2\eta(\mathcal{L}_v \xi) \eta(x) = 0, \quad \eta(\mathcal{L}_v \xi) = 0, \]
i.e.
\[ (\mathcal{L}_v \eta)(x) + g(x, \mathcal{L}_v \xi) = 0, \quad \eta(\mathcal{L}_v \xi) = 0. \tag{3.11} \]
According to (2.1) and \( \rho(x, \xi) = 2n\eta(x) \) from (1.4), we have for a Sasaki-
like manifold
\[ (\mathcal{L}_v g)(x, \xi) = -2(\lambda + \mu + \nu + 2n)\eta(x) \tag{3.12} \]
and as a consequence for \( x = \xi \) the following
\[ (\mathcal{L}_v g)(\xi, \xi) = -2(\lambda + \mu + \nu + 2n). \tag{3.13} \]
The Lie derivative of \( g(x, \xi) = \eta(x) \) with respect to \( v \) gives
\[ (\mathcal{L}_v g)(x, \xi) = (\mathcal{L}_v \eta)(x) - g(x, \mathcal{L}_v \xi), \tag{3.14} \]
which for \( x = \xi \) leads to
\[ (\mathcal{L}_v g)(\xi, \xi) = -2\eta(\mathcal{L}_v \xi). \tag{3.15} \]
From (3.13) and (3.15) we obtain
\[ \eta(\mathcal{L}_v \xi) = \lambda + \mu + \nu + 2n. \]
The latter equality implies (3.1), by virtue of the second equality in (3.11).
Substituting (3.1) in (3.12) gives the vanishing of \( (\mathcal{L}_v g)(x, \xi) \) and because
of (3.14) we have \( (\mathcal{L}_v \eta)(x) = g(x, \mathcal{L}_v \xi) \). Hence, bearing in mind the first
equality in (3.11), we get
\[ \mathcal{L}_v \xi = 0, \]
which together with \( \nabla \xi = -\varphi \) from (1.4) completes the proof. \( \square \)

**Proposition 3.2.** Let \((M, \varphi, \xi, \eta, g)\) be a \((2n + 1)\)-dimensional Sasaki-like man-
ifold. If it admits a Ricci-like soliton with arbitrary potential \( v \) then the Ricci
tensor \( \rho \) of \( g \) and the scalar curvatures \( \tau \) and \( \tilde{\tau} \) of \( g \) and \( \tilde{g} \), respectively, satisfy
the following equalities
\[ (\mathcal{L}_v \rho)(x, \xi) = 0, \quad \tau = 2n, \quad \tilde{\tau} = \text{const.} \]

**Proof.** By (3.7) we find the following
\[ g((\mathcal{L}_v R)(x, y)\xi, z) = -g((\mathcal{L}_v \nabla)(x, \varphi y), z) + g((\mathcal{L}_v \nabla)(\varphi x, y), z) \]
\[ -2(\nabla_x \rho)(\varphi y, z) + 2(\nabla_y \rho)(\varphi x, z) \]
\[ -2\eta(x)\rho(y, z) + 2\eta(y)\rho(x, z). \]
Taking the trace of the last equality for \( x = e_i \) and \( z = e_j \) and using (3.4) and
(1.7), \( d\tau = 2 \text{ div } \rho \), we obtain successively
\[ g^{ij} g((\mathcal{L}_v \nabla)(e_i, \varphi y), e_j) = -d\tau(\varphi y), \]
\[ g^{ij} g((\mathcal{L}_v \nabla)(\varphi e_i, y), e_j) = d\tilde{\tau}(y), \]
\[ g^{ij} g((\mathcal{L}_v R)(e_i, y)\xi, e_j) = (\mathcal{L}_v \rho)(y, \xi) \tag{3.16} \]
and therefore the following formula is valid
\[(\mathcal{L}_v\rho)(y,\xi) = -d\tilde{\tau}(y) + 2(\tau - 2n)\eta(y),\] (3.17)
which for \(y = \xi\) implies
\[(\mathcal{L}_v\rho)(\xi,\xi) = -d\tilde{\tau}(\xi) + 2(\tau - 2n).\] (3.18)

On the other hand, according to (3.9) and (3.16), \((\mathcal{L}_v\rho)(\xi,\xi)\) vanishes and therefore (3.17) and (3.18) imply
\[(\mathcal{L}_v\rho)(x,\xi) = d\tilde{\tau}(\varphi^2x), \quad d\tilde{\tau}(\xi) = 2\tau - 4n.\] (3.19)
The latter equalities, due to (1.8) and (1.9), imply consequently \(d\tilde{\tau}(\xi) = 0\) and
\[\tau = 2n, \quad \tilde{\tau} = \text{const}.\]
In conclusion, because of (3.19), we infer the assertion. □

**Theorem 3.3.** Let \((M,\varphi,\xi,\eta,\rho)\) be a \((2n+1)\)-dimensional Einstein-like Sasaki-like manifold. If it admits a Ricci-like soliton with potential \(v\) then the Ricci tensor is \(\rho = 2n\eta \otimes \eta\) and the scalar curvatures are \(\tau = \tilde{\tau} = 2n\).

**Proof.** The assertion follows from Theorem 3.1, Propositions 1.2 and 3.2. □

**Corollary 3.4.** Let \((M,\varphi,\xi,\eta,\rho)\), \(\dim M = 2n + 1\), be an Einstein-like Sasaki-like manifold. Then it is \(\eta\)-Einstein with constants \((0, 0, 2n)\), which is equivalent to the existence on \(M\) of a Ricci-like soliton with potential \(\xi\) and constants \((0, 1, -2n - 1)\).

**Proof.** Using Theorem 3.3, we obtain the following expression \(\mathcal{L}_v\rho = -2\lambda g - 2\mu \tilde{g} + 2(\lambda + \mu)\eta \otimes \eta\), which holds for \(\lambda = 0, \mu = 1\) in the case \(v = \xi\). Therefore Theorem 2.1 is restricted to its case (iii) and \(a = 0\). □

Let us recall, every non-degenerate 2-plane (or section) \(\beta\) with a basis \(\{x, y\}\) with respect to \(g\) in \(T_p M, p \in M\), has the following sectional curvature
\[k(\beta; p) = \frac{R(x, y, y, x)}{g(x, x)g(y, y) - [g(x, y)]^2}.\] (3.20)
A section \(\beta\) is said to be \(\varphi\)-holomorphic if the condition \(\beta = \varphi\beta\) holds. Every \(\varphi\)-holomorphic section has a basis of the form \(\{\varphi x, \varphi^2x\}\). A section \(\beta\) is called a \(\xi\)-section if it has a basis of the form \(\{x, \xi\}\).

**Theorem 3.5.** Let \((M,\varphi,\xi,\eta,\rho)\) be a 3-dimensional Sasaki-like manifold. If it admits a Ricci-like soliton with potential \(v\) then:
(i) the sectional curvatures of its \(\varphi\)-holomorphic sections are equal to \(-1\);
(ii) the sectional curvatures of its \(\xi\)-sections are equal to \(1\).
Proof. It is well known that the curvature tensor of a 3-dimensional manifold has the form
\begin{equation}
R(x, y)z = g(y, z)Qx - g(x, z)Qy + \rho(y, z)x - \rho(x, z)y \\
- \frac{\tau}{2} \{g(y, z)x - g(x, z)y\}.
\end{equation}
(3.21)

Then, substituting \( y = z = \xi \) and recalling (1.4), we have
\[ \rho = \frac{1}{2} \left\{ (\tau - 2)g - (\tau - 6)\eta \otimes \eta \right\}, \]
which means that the manifold is \( \eta \)-Einstein. Therefore, because of Theorem 3.3, we have
\[ \rho = 2 \eta \otimes \eta, \quad \tau = \tilde{\tau} = 2. \]

Substituting the latter two equalities for \( \tau \) and \( \rho \) in (3.21), we get
\begin{align*}
R(x, y)z &= - \left[ g(y, z) - 2 \eta(y)\eta(z) \right] x + 2 g(y, z) \eta(x) \xi \\
&\quad + \left[ g(x, z) - 2 \eta(x)\eta(z) \right] y - 2 g(x, z) \eta(y) \xi.
\end{align*}

Using a basis \( \{ \varphi x, \varphi^2 x \} \) of an arbitrary \( \varphi \)-holomorphic section, we calculate its sectional curvature by (3.20), replacing \( x \) and \( y \) by \( \varphi x \) and \( \varphi^2 x \), respectively. Then, bearing in mind (1.1) and (1.2), we obtain
\[ k(\varphi x, \varphi^2 x) = -1. \]

Similarly, for a \( \xi \)-section with a basis \( \{ x, \xi \} \), we get \( k(x, \xi) = 1 \), which completes the proof. \( \square \)

Remark 3.6. Examples of 3-dimensional Sasaki-like manifolds as a Lie group from type \( Bia(VII)0(1) \), a matrix Lie group, an \( S^1 \)-solvable extension on a Kähler-Norden 2-manifold, and their geometrical properties are studied in [22–25], respectively.

Remark 3.7. The constructed 5-dimensional example in Sect. 3.1 of a Sasaki-like manifold with the results in (2.2) and (2.3) supports also Theorem 3.1, Proposition 3.2, Theorem 3.3 and Corollary 3.4 for the case of \( v = \xi \) and \( n = 2 \).

3.1. Example 2

Let us consider \( M \) as a set of points in \( \mathbb{R}^3 \) with coordinates \( (x^1, x^2, x^3) \) and let \( M \) be equipped with an almost contact B-metric structure defined by
\begin{align*}
g(\partial_1, \partial_1) &= -g(\partial_2, \partial_2) = \cos 2x^3, \quad g(\partial_1, \partial_2) = \sin 2x^3, \\
g(\partial_1, \partial_3) &= g(\partial_2, \partial_3) = 0, \quad g(\partial_3, \partial_3) = 1, \\
\varphi \partial_1 &= \partial_2, \quad \varphi \partial_2 = -\partial_1, \quad \xi = \partial_3,
\end{align*}
where \( \partial_1, \partial_2, \partial_3 \) denote briefly \( \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3} \), respectively. Then, the vectors determined by
\begin{align}
e_1 &= \cos x^3 \partial_1 + \sin x^3 \partial_2, \quad e_2 = -\sin x^3 \partial_1 + \cos x^3 \partial_2, \quad e_3 = \partial_3 \quad (3.22)
\end{align}
form an orthonormal $\varphi$-basis of $T_p M$, $p \in M$, i.e.
\begin{align*}
g(e_1, e_1) &= -g(e_2, e_2) = g(e_3, e_3) = 1 \\
g(e_i, e_j) &= 0, \quad i, j \in \{1, 2, 3\}, \ i \neq j, \quad (3.23) \\
\varphi e_1 &= e_2, \quad \varphi e_2 = -e_1, \quad \xi = e_3.
\end{align*}

Immediately from (3.22) we obtain the commutators of $e_i$ as follows
\begin{align*}
[e_0, e_1] &= e_2, \\
[e_0, e_2] &= -e_1, \\
[e_1, e_2] &= 0.
\end{align*}

Then, according to Example 1 in [20] for $n = 1$, the solvable Lie group of dimension 3 with a basis of left-invariant vector fields $\{e_1, e_2, e_3\}$ defined by (3.24) and equipped with the $(\varphi, \xi, \eta, g)$-structure from (3.23) is a Sasaki-like almost contact B-metric manifold.

In the well-known way, we calculate the components of the Levi-Civita connection $\nabla$ for $g$ and from there the corresponding components $R_{ijkl} = R(e_i, e_j, e_k, e_l)$ and $\rho_{ij} = \rho(e_i, e_j)$ of the curvature tensor $R$ and the Ricci tensor $\rho$, respectively. The non-zero ones of them are the following (keep in mind the symmetries of $R$)
\begin{align*}
\nabla e_1 e_2 &= \nabla e_2 e_1 = -e_3, \\
\nabla e_1 e_3 &= -e_2, \\
\nabla e_2 e_3 &= e_1; \\
R_{1221} &= R_{1331} = -R_{2332} = 1, \\
\rho_{33} &= 2.
\end{align*}

The latter equality means that the Ricci tensor has the following form
\begin{equation}
\rho = 2\eta \otimes \eta, \quad (3.27)
\end{equation}
i.e. the manifold is Einstein-like with constants $(a, b, c) = (0, 0, 2)$. Therefore, the scalar curvatures with respect to $g$ and $\tilde{g}$ are $\tau = \tilde{\tau} = 2$.

The values of $R_{ijkl}$ in (3.26) imply the sectional curvatures
\begin{align*}
k_{12} &= -k_{13} = -k_{23} = -1,
\end{align*}
which supports Theorem 3.5.

Let us consider a vector field, determined by the following
\begin{align*}
v &= v^1 e_1 + v^2 e_2 + v^3 e_3, \\
v^1 &= -\{c_1 \cos x^3 + c_2 \sin x^3\} x^1 + \{c_2 \cos x^3 - c_1 \sin x^3\} x^2 + \sin x^3, \\
v^2 &= -\{c_2 \cos x^3 - c_1 \sin x^3\} x^1 - \{c_1 \cos x^3 + c_2 \sin x^3\} x^2 + \cos x^3, \\
v^3 &= c_3, \quad (3.28)
\end{align*}
and $c_1, c_2, c_3$ are arbitrary constants.

Using (3.22), (3.23), (3.25) and (3.28), we obtain the following
\begin{align*}
\nabla_{e_1} v &= -c_1 e_1 - (c_2 + c_3)e_2 - v^2 e_3, \\
\nabla_{e_2} v &= (c_2 + c_3)e_1 - c_1 e_2 - v^1 e_3, \\
\nabla_{e_3} v &= v^2 e_1 - v^1 e_2, \quad (3.29)
\end{align*}
that allow us to calculate the components \((\mathcal{L}_v g)_{ij} \) of the Lie derivative \(\mathcal{L}_v g\). Then, we get the following nonzero ones
\[
(\mathcal{L}_v g)_{11} = -(\mathcal{L}_v g)_{22} = -2c_1, \quad (\mathcal{L}_v g)_{12} = 2(c_2 + c_3),
\]
which implies that this tensor has the following expression
\[
\mathcal{L}_v g = -2c_1 g - 2(c_2 + c_3)\bar{g} + 2(c_1 + c_2 + c_3)\eta \otimes \eta. \tag{3.30}
\]
Substituting the latter equality and (3.27) in (2.1), we obtain that \((M, \varphi, \xi, \eta, g)\) admits a Ricci-like soliton with potential \(v\) determined by (3.28) and the potential constants are
\[
\lambda = c_1, \quad \mu = c_2 + c_3, \quad \nu = -c_1 - c_2 - c_3 - 2.
\]
These results are in accordance with Theorems 3.1 and 3.3. The conclusion in Proposition 3.2 follows from (3.27) and the subsequent formula
\[
(\mathcal{L}_\rho)(x, \xi) = -2g(\varphi x, v) + 2\eta(\nabla_x v),
\]
Together with the equalities in (3.28) and (3.29).

4. Gradient Almost Ricci-like Solitons

Let us consider a Ricci-like soliton, defined by (2.1) with the condition \(\lambda, \mu, \nu\) to be functions on \(M\). If its potential \(v\) is a gradient of a differentiable function \(f\), i.e. \(v = \text{grad} f\), then the soliton is called a gradient almost Ricci-like soliton of \((M, \varphi, \xi, \eta, g)\). In this case (2.1) is reduced to the following condition
\[
\text{Hess} f + \rho + \lambda g + \mu \bar{g} + \nu \eta \otimes \eta = 0, \tag{4.1}
\]
where Hess denotes the Hessian operator with respect to \(g\), i.e. \(\text{Hess} f\) is defined by
\[
(\text{Hess} f)(x, y) := (\nabla_x df)(y) = g(\nabla_x \text{grad} f, y). \tag{4.2}
\]
Taking the trace of (4.1), we obtain
\[
\Delta f + \tau + (2n + 1)\lambda + \mu + \nu = 0,
\]
where \(\Delta := \text{tr} \circ \text{Hess}\) is the Laplacian operator of \(g\). Also for the Laplacian of \(f\), the formula \(\Delta f = \text{div}(\text{grad} f)\) is valid, where div stands for the divergence operator.

The gradient Ricci-like soliton is said to be trivial when \(f\) is constant. Further, we consider only non-trivial gradient Ricci-like solitons.

Equality (4.1) with the recall of (4.2) provides the following
\[
\nabla_x v = -Qx - \lambda x - \mu \varphi x - (\mu + \nu)\eta(x)\xi, \tag{4.3}
\]
where \(Q\) is the Ricci operator and \(v = \text{grad} f\).

**Theorem 4.1.** Let \((M, \varphi, \xi, \eta, g)\) be a Sasaki-like almost contact B-metric manifold of dimension \(2n + 1\). If it admits a gradient almost Ricci-like soliton with functions \((\lambda, \mu, \nu)\) and a potential function \(f\), then \((M, \varphi, \xi, \eta, g)\) has constant scalar curvatures \(\tau = \bar{\tau} = 2n\) for both B-metrics \(g\) and \(\bar{g}\), respectively, and its Ricci tensor is \(\rho = 2n \eta \otimes \eta\).
Proof. Using (4.3), we compute the following curvature tensor

\[ R(x, y)v = - (\nabla_x Q) y + (\nabla_y Q) x \]
\[ + \{d\lambda(y) + \mu\eta(y)\} x - \{d\lambda(x) + \mu\eta(x)\} y \]
\[ + \{d\mu(y) + (\mu + \nu)\eta(y)\} x - \{d\mu(x) + (\mu + \nu)\eta(x)\} y \]
\[ + d(\mu + \nu)(y)\eta(x) - d(\mu + \nu)(x)\eta(y)\eta(x). \hspace{1cm} (4.4) \]

The latter expression implies the following equality

\[ R(\xi, y)v = - (\nabla_\xi Q) y + (\nabla_y Q) \xi + \{d\lambda(\xi) + \mu\} \varphi^2 y - \{d\mu(\xi) + \mu + \nu\} \varphi y \]
\[ + d(\lambda + \mu + \nu)(y)\xi - d(\lambda + \mu + \nu)(\xi)\eta(y)\xi, \]

where we apply (1.10) and (1.11) and get

\[ R(\xi, y)v = - Q\varphi y + \{d\lambda(\xi) + \mu\} \varphi^2 y - \{d\mu(\xi) + \mu + \nu + 2n\} \varphi y \]
\[ + d(\lambda + \mu + \nu)(y)\xi - d(\lambda + \mu + \nu)(\xi)\eta(y)\xi. \hspace{1cm} (4.5) \]

We put \( z = v \) in the equality for \( R(\xi, y)z \) in (1.4) and obtain the following expression

\[ R(\xi, y)v = df(y)\xi - df(\xi)y. \hspace{1cm} (4.6) \]

Combining (4.5) and (4.6), we find the following formula

\[ Q\varphi y = \{d(\lambda - f)(\xi) + \mu\} \varphi^2 y - \{d\mu(\xi) + \mu + \nu + 2n\} \varphi y \]
\[ + d(\lambda + \mu + \nu - f)(y)\xi - d(\lambda + \mu + \nu - f)(\xi)\eta(y)\xi. \hspace{1cm} (4.7) \]

We apply \( \eta \) of equality (4.7) and since \( Q \circ \varphi = \varphi \circ Q \) for a Sasaki-like manifold, we obtain the following

\[ d(\lambda + \mu + \nu - f)(y) = d(\lambda + \mu + \nu - f)(\xi)\eta(y), \hspace{1cm} (4.8) \]

which changes (4.7) and (4.5) as follows

\[ Q\varphi y = \{d(\lambda - f)(\xi) + \mu\} \varphi^2 y - \{d\mu(\xi) + \mu + \nu + 2n\} \varphi y, \]
\[ R(\xi, y)v = - Q\varphi y + \{d\lambda(\xi) + \mu\} \varphi^2 y - \{d\mu(\xi) + \mu + \nu + 2n\} \varphi y \]
\[ + \{df(y) - df(\xi)\eta(y)\} \xi \]

and therefore we have

\[ R(\xi, y, v, z) = - \rho(y, \varphi z) + \{d\lambda(\xi) + \mu\} g(\varphi y, \varphi z) \]
\[ - \{d\mu(\xi) + \mu + \nu + 2n\} g(y, \varphi z) \]
\[ + \{df(y) - df(\xi)\eta(y)\} \eta(z). \hspace{1cm} (4.9) \]

On the other hand, the expression of \( R(x, y)\xi \) from (1.4) and equality (4.4) imply respectively the following two equalities

\[ R(x, y, \xi, v) = df(x)\eta(y) - df(y)\eta(x), \]
\[ R(x, y, v, \xi) = - \eta((\nabla_x Q)y - (\nabla_y Q)x) + d(\lambda + \mu + \nu)(y)\eta(x) \]
\[ - d(\lambda + \mu + \nu)(x)\eta(y). \]
By summation of the latter two equalities, we find the following formula
\[
\eta((\nabla_x Q)y - (\nabla_y Q)x) = -d(\lambda + \mu + \nu - f)(x)\eta(y) \\
+ d(\lambda + \mu + \nu - f)(y)\eta(x),
\]
which because of (4.8) is simplified to the following form
\[
\eta((\nabla_x Q)y - (\nabla_y Q)x) = 0.
\]
On the other hand, the expression of \( R(\xi, y, z, v) \) from (1.4) yield
\[
R(\xi, y, z, v) = -df(\xi)g(\varphi y, \varphi z) - \{ df(y) - df(\xi)\eta(y) \} \eta(z),
\]
which together with (4.9) and the form of \( \rho(x, \xi) \) from (1.4) implies
\[
\rho(y, z) = \{ d\mu(\xi) + \mu + \nu + 2n \} g(\varphi y, \varphi z) + \{ d(\lambda - f)(\xi) + \mu \} g(y, \varphi z) \\
+ 2n\eta(y)\eta(z).
\]
The latter equality can be rewritten in the form
\[
\rho = -\{ d\mu(\xi) + \mu + \nu + 2n \} g + \{ d(\lambda - f)(\xi) + \mu \} \tilde{g} \\
+ \{ 4n + \nu - d(\lambda - \mu - f)(\xi) \} \eta \otimes \eta,
\]
which means that the manifold is almost Einstein-like with coefficient functions
\[
a = -d\mu(\xi) - \mu - \nu - 2n, \quad b = d(\lambda - f)(\xi) + \mu, \\
c = -d(\lambda - \mu - f)(\xi) + \nu + 4n.
\]
(4.10)

Then, using (1.14), we obtain
\[
\tau = -2n\{ d\mu(\xi) + \mu + \nu + 2n - 1 \}, \quad \tilde{\tau} = 2n\{ d(\lambda - f)(\xi) + \mu + 1 \}.
\]
(4.11)

Contracting (4.4) with respect to \( x \), we obtain
\[
\rho(y, v) = \frac12 d\tau(y) + 2n\, d\lambda(y) + d(\mu + \nu)(y) - d\mu(\varphi y) - \{ d(\mu + \nu)(\xi) - 2n\mu \} \eta(y)
\]
and consequently for \( y = \xi \) we have
\[
\rho(\xi, v) = \frac12 d\tau(\xi) + 2n\, d\lambda(\xi) + 2n\mu.
\]
(4.12)

We compute the left side of (4.12) by the formula \( \rho(x, \xi) = 2n\, \eta(x) \) from (1.4) and then from (4.12) and (1.9), we obtain
\[
\tilde{\tau} = -2n\{ d(\lambda - f)(\xi) + \mu - 1 \}.
\]
Comparing the latter equality with (4.11), we have
\[
d(\lambda - f)(\xi) = -\mu, \quad \tilde{\tau} = 2n.
\]
The former equality implies \( b = 0 \) in (4.10) and therefore the manifold is almost \( \eta \)-Einstein and the latter one means that \( d\tilde{\tau} = 0 \) and using (1.9), we obtain for \( \tau \) the following
\[
\tau = 2n.
Then, substituting the value of $\tau$ in (4.11), we obtain
\[ d\mu(\xi) = -\mu - \nu - 2n, \]
which implies $a = 0$ in (4.10) and finally we get $(a, b, c) = (0, 0, 2n)$. \qed

4.1. Example 3

Let $(M, \varphi, \xi, \eta, g)$ be the 3-dimensional Sasaki-like manifold, given in Example 2 of sect. 3.1. Now, let $f$ be a differentiable function on $M$, defined by
\[ f = -\frac{1}{2} s \{(x^1)^2 + (x^2)^2\} + x^2 + tx^3 \]
for arbitrary constants $s$ and $t$. Then, the gradient of $f$ with respect to the B-metric $g$ is the following
\[ \text{grad } f = -\{sx^1 \cos x^3 + (sx^2 - 1) \sin x^3\} e_1 + \{sx^1 \sin x^3 - (sx^2 - 1) \cos x^3\} e_2 + te_3. \] (4.13)

Using (3.22), we compute the components of $\mathcal{L}_{\text{grad } f} g$ as follows
\[ (\mathcal{L}_{\text{grad } f} g)_{11} = -(\mathcal{L}_{\text{grad } f} g)_{22} = -2s, \quad (\mathcal{L}_{\text{grad } f} g)_{12} = 2t, \]
which give us the following expression
\[ (\mathcal{L}_{\text{grad } f} g) = -2s g - 2t \tilde{g} + 2(s + t) \eta \otimes \eta. \]

The latter equality coincides with (3.30) for $s = c_1$, $t = c_2 + c_3$. Therefore, $(M, \varphi, \xi, \eta, g)$ admits a Ricci-like soliton with potential $v = \text{grad } f$ determined by (4.13) and the potential constants are
\[ \lambda = s, \quad \mu = t, \quad \nu = -s - t - 2. \]

In conclusion, the constructed 3-dimensional example of a Sasaki-like manifold with $\tau = \tilde{\tau} = 2$ and gradient Ricci-like soliton supports also Theorem 4.1.

Author contributions Not applicable.

Funding The author was supported by projects MU21-FMI-008 and FP21-FMI-002 of the Scientific Research Fund, University of Plovdiv Paisii Hilendarski, Bulgaria.

Data Availibility Data is contained within the article.

Declarations Conflicts of interest The author declares no conflict of interest. The funders had no role in the design of the study; in the collection, analyses, or interpretation of data; in the writing of the manuscript, or in the decision to publish the results.
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Received: August 13, 2020.
Accepted: June 8, 2022.

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