Abstract: Does the time-space series of nature and society follow a unified probability distribution? We analyzed representative naturally occurring data series (time and space series) and data series for social activities to show a common shape of probability curves, for which we developed a new mathematical concept. We found that probability density functions (PDFs) first or higher order differences of the data are fat-tailed bell-shaped curves and that their cumulative distribution functions (CDFs) are S-shaped in relation to the straight line of the normal distribution CDF. Interestingly, we found that initial values and data perturbations have no influence on the PDF of a chaotic system. A new mathematical concept of probability stability is introduced by defining a map from the state space to the probability space. If the PDF of a dynamical system does not change with data perturbation, the system is probabilistically stable; otherwise, unstable. This definition is an extension of Lyapunov stability theory, and it has application in various areas such as global climate change, quantum mechanics, economics and biology.

1. INTRODUCTION

Origins of questions: We observed an interesting phenomenon during our research into detecting pipe bursts in a city water supply system, where flow fluctuates over time. The fluctuations form a stationary random process (Fig.1(a)). We calculated the cumulative distribution function (CDF) of water pressure and
the first difference of flow rate for different sets of water pressure and flow data; an example is shown in Fig.1. What we found interesting was that all the curves had characteristically similar shapes: the middle section of the CDF curve (i.e., close to the mean value) matches a normal distribution, but the other two sections deviate significantly from the straight line, forming an asymmetric S-shaped curve. This phenomenon was observed in the flow datasets for all the Chinese cities in which we investigated pipe bursts.

![Image](image1)

FIG. 1. **Statistical analysis of pressure and flow fluctuations in a water supply system.**

- a, water supply system of Guangzhou City, China. b, water pressure in Guangzhou. c, histogram of water pressure. d, CDF of water pressure first order difference; e, histogram of water pressure first order difference. f, flow rate of Guangzhou. g, histogram of flow rate. h, CDF of flow rate first order difference. i, histogram of flow rate first order difference.

We were interested by this common water flow pattern in various city water supply systems. We collected data for other dynamical systems from different natural or social systems, such as brain waves, seismic waves, sunspots, and stock markets. The CDFs of their first or higher order differences were all similar in shape to the water supply system CDFs. This observation caused us to ask the following three questions:

1. **Is this phenomenon caused by errors in instrumentation or in data processing?**
2. **Is this natural or social phenomenon occasional or universal?**
3. **A chaotic dynamical system is sensitive to its initial conditions and perturbations; can its state be statistically predictable, or is it unique in statistical sense?**
Question 1 is discussed in Section II. The phenomenon is not caused by data processing; time step size does not affect the PDF over a limited range. Question 2 is discussed in detail in the section III “natural and social cases of space-time series”. Despite differences in spatial or temporal scales and frequencies, the CDFs of first or higher order differences are all similar. We think that this statistical pattern may represent a universal natural and social phenomenon rather than an occasional occurrence.

Most of the cases we consider in this study are known to be chaotic and unstable dynamical systems in terms of Lyapunov stability. Initial value sensitivity has for a long time been problematic for mathematicians who research chaos. To address question 3, we begin by defining probabilistic stability in a probability space: if the PDF of a dynamical system does not change with perturbations, the system is probabilistically stable; otherwise, the system is probabilistically unstable. A Lyapunov stable dynamical system is necessarily probabilistically stable, but the converse does not hold. The definition of probabilistic stability allows us to understand that in terms of Lyapunov stability the world can be irregular and unpredictable, but that events in the world can be ordered and predictable if they are probabilistically stable in some probability space. The epistemological significance of our definition is that it can explain some problems of natural philosophy, such as Heisenberg’s uncertainty principle, probability waves and Schrödinger’s cat, and Laplacian determinism.
A. Definition of space-time series differences

Three different definitions of the first order difference of a time series are given by:

\[ \Delta \hat{f}(i) = \hat{f}(i) - \hat{f}(i-1) \]  
\[ \Delta \hat{f}_m(i) = \frac{\hat{f}(i) - \hat{f}(i-1)}{\sum_{i-K+1}^{i} \hat{f}(j)/K} \]  
\[ \Delta \ln \hat{f}(i) = \ln(\hat{f}(i)) - \ln(\hat{f}(i-1)) \]

where \( \hat{f} \) is a time series, \( \Delta \hat{f}(i) \) is the first order difference, \( \Delta \hat{f}_m(i) \) is the difference in the ratio of the time series to the mean value of the nearest \( K \) samples, and \( \Delta \ln \hat{f}(i) \) is the logarithmic first order difference.

These three forms all depend on short-term variation in the time series data, and each is an approximation of the first order difference. Equation (1) can be used when the sample does not greatly vary. Equation (2) is recommended if the time series (e.g., surface runoff) changes greatly or contains negative numbers because it makes the data dimensionless and thus enables the aggregation of different time series into one dataset.

Equation (3) can be used when a positive variable changes to a large degree (e.g., a stock index fluctuation rate is auto-correlated). Successive second or higher order differences can be calculated from Equations (1–3). Consistency of statistical characteristics can be ensured by the choice of suitable time steps. Note that any numerical errors in difference calculation from a lower-order to a higher-order difference will be amplified.

B. Bathtub effects of error between differential and difference for dynamical system

We address the first question we raised in this paper: are there instrumental or processing errors? The difference between real-life time series differences and the mathematical differential of a dynamical system
is that a time series difference includes some measurement errors which distort the calculations of higher order differences. The first order difference of a time series can be expressed by:

\[ \Delta \hat{y}(i) = f(i) \Delta t + \frac{\Delta t^n}{n!} \sum_{n=2}^{\infty} \frac{\Delta t^n}{n!} (\epsilon(i) - \epsilon(i-1)) \]  

(4)

Where \( \hat{y}(i) \) is measurement value, and \( \epsilon \) is the measurement error. As the time step \( \Delta t \) decreases, \( \Delta \hat{y}(i) \) becomes increasingly affected by any original observation error. As the time step \( \Delta t \) increases, the effect of the error on higher order differences becomes more significant. The deviation \( \sigma_{y(i-\Delta t)} / \sigma_{y(i)} \) between the first order difference and the first order differential is a bathtub curve (Fig. 2).

\[ \sigma_{y(i-\Delta t)} / \sigma_{y(i)} \]

FIG. 2. Bathtub curve showing discrepancy between data difference and data differential from the Lorentz system. Lorentz system with \( \sigma = 10, \beta = 8/3, \rho = 28 \); the right hand side of the equation contains white noise disturbance with mean value 0.002, added to measurement white noise with standard deviation 0.001 from sampling.

The Lorenz system 2 provides an example. The CDF of the third order difference is similar in shape to the CDFs found in this paper. The time step \( \Delta t \) is 0.002 for this numerical example. Measurement error and perturbations were added manually to the data. The measurement error was Gaussian white noise, with a 0.1% difference between maximum and minimum. Perturbations added to the equations were also Gaussian white noise, with a magnitude about 0.02% of the difference between maximum and minimum. Fig. 3a shows the CDFs of third-order differentials using sampling steps of \( \Delta t, 2\Delta t, 4\Delta t, 8\Delta t, 16\Delta t, 32\Delta t, \) and \( 64\Delta t \). As the time-step increases for third-order difference calculation, the CDF gradually becomes a Gaussian normal distribution. The results show that when the time step is too short or too long, the CDF of the
difference is close to that of a normal distribution. Use of the typhoon data shows similar results
(supertyphoon Lekima, International No 1909, Fig.3(b)). Both sets of results show that the CDFs of
differences change when the time step is within a certain range. Errors increase exponentially as the order of
the differences increases. Thus, care must be taken when empirically observed data are used. Generally, the
numerical resolution of differences for dynamical systems was adequately precise when the time steps were
within a certain range. Random noise pollution does not seriously affect the statistical analysis.

**FIG.3. CDF shape changes for differences in S2 with different time steps.**

- **a**, within a certain time step interval, CDF does not vary with
  perturbations ($\sigma = 10, \beta = 8 / 3, \rho = 28, \Delta t = 0.0025$): (a1) CDF of $x$ third order difference without perturbation; (a2–a8)
  CDFs with different time steps with perturbation error 0.004 showing that as the time step increases the CDF gradually approaches a
  Gaussian distribution.
- **b**, CDFs of first order differences with multiple time steps using wind speed data measured during supertyphoon
  Lekima (time step 0.1–102.4s).

Discrete differences approximate a continuous differential, as is usually assumed in computing science.

Data used in most of the cases we studied are publicly available. Measurement errors were within acceptable
ranges, and therefore did not significantly affect the accuracy of the statistical analysis we performed. Water
pressure data and flow rate data were obtained from different cities that used different instrumentation. The
sampling periods ranged from 1 to 15 minutes. The CDFs are similarly shaped. Measurement errors between
the Dow Jones stock index and the China stock index were negligible. The precision of temperature measurements was 0.1 ºC, and variation in temperature is about 3.1 ºC. The upper bound of the first difference error is <6.1% of the standard deviation; the second difference error is 12.2% of the standard deviation. It is believed that all cases’ error will not affect the final conclusion.

III. NATURAL AND SOCIAL CASES OF SPACE-TIME SERIES

A. Case information

The normal distribution is used in a standard solution of the linear diffusion equation. There is no evidence that there is a unique form of the probability distribution curve in natural or social systems. We investigated the CDFs of 21 cases in six categories (Table 1): (1) numerical solutions of chaotic systems; (2) time series data for various natural phenomena on Earth; (3) signals from artificial structures and experimental devices; (4) biological signals; (5) time series data of social behaviour; and (6) spatial differences. Most of the data samples were downloaded from public websites; their physical characteristics, sampling frequency and the measuring equipment used to obtain them are irrelevant.

| TABLE 1 Case information. |
|---------------------------|
| Case type                | Case name              | Sampling period/frequency | Time or space span | Physical background | Sample source                     |
| Classical chaotic dynamical equation | Lorenz system          | 0.002                     | >100 Cycle         | Meteorology         | Numerical simulation               |
|                           | Duffing oscillator     | 0.02                      | >100 Cycle         | Mechanical vibration| Numerical simulation               |
|                           | Chua circuit           | 0.002                     | >100 Cycle         | Electro circuit     | Numerical simulation               |
| Natural phenomena on Earth | Ambient wind speed     | 0.1s                      | 1 hour             | Ambient wind speed  | Measurement in Yueqing by authors  |
|                           | Tide current           | 10min                     | 30 days            | Tide                | Measurement in East China Sea      |
|                           | Land runoff            | 1 day                     | 135 years          | Hydrology           | Web site (UK_NRFA)                 |
|                           | Seismic signals        | 0.005s~0.01s              | 30~60 s            | Earthquake          | Collection by authors              |
|                           | Sunspot                | 1 day                     | 36 years           | Astronomy           | Web site (NASA )                   |
| Social behaviours         | Dow Jones stock index (DJIA) | 1 day                  | 113 years          | Economy             | Web site (Collected by authors)    |
|                           | Stock price market (China) | 5min                  | 5 days             | Economy             | Web site (sina)                    |
|                           |                        | 10min                    | 10 day             | Economy             | Web site (sina)                    |
### Biological signals

| Description                                      | Frequency (Hz) | Duration | Field   | Web Site                  |
|--------------------------------------------------|----------------|----------|---------|---------------------------|
| Human electrocardiogram (adfecgd)                | 1000           | 5 min    | Biology | Web site (PhysioBank ATM )|
| Human electrocardiogram (apean)                  | 1000           | 5 min    | Biology | Web site (PhysioBank ATM )|
| Dog electrocardiogram                             | 1000Hz         | 195 s    | Biology | Laboratory measurements, China |
| PCG (heart tone)                                  | 333 Hz         | 20 min   | Biology | Web site (PhysioBank ATM )|
| Brain waves (patient)                             | 2048 Hz        | 10 min   | Biology | Web site collection       |
| Brain waves (normal person)                       | 2048 Hz        | 10 min   | Biology | Web site collection       |
| Electroencephalogram (ltsrvp)                    | 2048 Hz        | 10 min   | Biology | Web site (PhysioBank ATM )|

### Artificial structures and experimental device signals

| Description                                      | Frequency (Hz) | Duration | Field               | Web Site                  |
|--------------------------------------------------|----------------|----------|---------------------|---------------------------|
| Ground vibration of high-speed train             | 1280Hz         | 5min     | Mechanical vibration | Field measurement         |
| Wind tunnel experiments                           | 0.01s          | 10 min   | Aerodynamics        | Laboratory measurements, Japan |

### Spatial differential

| Description                                      | Radius (m)     | Area (km²) | Field   | Web Site                  |
|--------------------------------------------------|----------------|------------|---------|---------------------------|
| Cosmic background radiation                      | 5 arc min      | Entire universe | Cosmology | Web site(ESA)            |
| Earth surface elevation                          | 150 m          | 108 km x 108 km | Geography | Web site (USAG)         |

**B. Examples of natural and social dynamical systems**

**Examples of chaotic dynamical systems:** Chaotic systems are neither periodic nor convergent due to their sensitivity to initial values. The Lorenz system is a set of differential equations for air flow; the Duffing equation is a nonlinear vibration model; the Chua circuit is a nonlinear electronic circuit. All of them behave chaotically (APPENDIX A). Numerical models using these equations (Fig.4) conform to our hypothesis in that at some order of difference they show a bell-shaped histogram that gives an S-shaped CDF. When the order of the difference increases (to an order higher than that of current variables), CDFs for these variables have similar shapes (Fig.2). The shape of the PDF is independent of the initial values and perturbations (see the section IV).
FIG. 4. Chaotic dynamical systems. a, Lorenz system: (a1) state space in x–z plane, (a2) histogram of x third order difference, (a3) CDF of x third order difference. b, Duffing equation: (b1) state space in x–z plane, (b2) histogram of x second order difference, (b3) CDF of x second order difference. c, Chua circuit: (c1) 3D graph of Chua circuit, (c2) histogram of x third order difference, (c3) CDF of x third order difference.

Natural phenomena and artificial structures: Phenomena such as air temperature, wind speed, tidal currents, and runoff usually vary at different temporal or spatial scales. The first case we examine is the daily temperature in Nanjing recorded at 11:00 or 11:25 from 2001-01-01 to 2018-05-25. The histogram of the temperature data contains multiple peaks (Fig.5(a2)). In the first order difference, a single peak remains and the histogram shifts slightly to the left (cooling occurs quickly, but heating is slow). The second order difference curve becomes symmetric (Figs.5(a4), 3(a6)). Wind speed changes randomly at the atmospheric boundary layer. The wind speed data were recorded at Yueqing, China. The wind speed PDF is asymmetric, but the first order difference PDF is symmetric, and the corresponding CDF deviates from the central straight line at both ends (Figs.5(b2), 3(b3)). Tides are due to the gravitational attraction of Earth, moon and sun; they are periodic. Tide data were recorded for the East China Sea (Fig.5(c)). Daily surface runoff data for the River Thames in the United Kingdom were recorded from 1883-01-01 to 2017-09-20 (Fig.5; original data can be downloaded from NRFA 5). A large change in surface runoff data can be observed. Runoff values in flood periods can be hundreds of times those in a normal period. We used Equation (2) to
investigate the small fluctuations in surface runoff data. Fig.5(d) shows daily runoff and daily fluctuation rate. We note that the range of daily fluctuation rate has greatly increased in the past 30–40 years.

**FIG.5. Representations of data for natural geophysical Earth phenomena.** a, characteristic statistics of noontime air temperature at Nanjing, China (unit: °C): (a1) location of Nanjing, (a2) air temperature histogram, (a3) air temperature first order difference histogram, (a4) CDF of air temperature first order difference, (a5) air temperature second order difference histogram, (a6) CDF of air temperature second order difference; b, characteristic statistics of wind speed (sampling period 0.1s, unit: m/s): (b1) ultrasonic anemometer at the top of a corner column in Yueqing Stadium, (b2) histogram of wind speed, histogram of wind speed first-order difference, and CDF of wind speed first order difference in Yueqing stadium from 11:00 to 12:00 on 2013-07-14 (wind speed was in the range 1–2 m/s), (b3) histogram and CDF of wind speed first-order difference in 3 directions, (b4) path of supertyphoon Lekima and monitoring position from 2019-09-09 14:00 to 2019-09-10 10:00 (maximum wind speed 25 m/s); c, tidal current velocity in the East China Sea (30.995281°N, 122.807595°E) from 2014-05-31 to 2014-06-30 (sampling interval 10 minutes, unit: m/s): (c1) tidal current monitoring position, (c2) tidal current speed, (c3) histogram of current speed, (c4) histogram of current speed first order difference, (c5) CDF of current speed first order difference. d, characteristic statistics of daily runoff into the River Thames over 135 years (unit: m³/d): (d1) monitoring position, (d2) photo of monitoring position, (d3) histogram of daily runoff, (d4) daily runoff first order difference (the red line is the standard deviation for daily flow over two years; it has increased by 30%–50% in the past 30–40 years), (d5) CDF of daily runoff first order difference.
**Seismic waves:** a seismic wave is a short-term time series. The three sets of seismic waves used in this study have different power spectra but they have similar first order difference CDFs (Fig.6). Note that the seismic wave data for the first order difference of the ground velocity were measured by the accelerometer sensors (Fig.6(d)).

**FIG. 6.** Seismic waves (principal earthquake wave). a. seismic wave propagation and monitoring. b, locations of seismic wave sources: Hanshin-Awaji and Chūetsu earthquakes in Japan, and Chi-Chi earthquake in Taiwan (the sampling period of Hanshin and Chūetsu is
0.01 s and of Chi-Chi is 0.005 s). c, spectra of earthquake waves (there is no relationship between them; even in the same earthquake, the horizontal and vertical power spectra are quite different). d, histogram of earthquake acceleration and CDF (the seismic accelerometer had calculated the first order difference of ground speed). e, histogram and CDF of earthquake acceleration first order difference.

**Sunspots:** sunspots are comparatively dark areas on the surface of the sun where magnetic fields gather. The main cycle of sunspot intensity is about 11 years. The time series is chaotic. Individual sunspots or groups of sunspots may occur anywhere on the surface of the sun and remain for a period of a few days to a few months. Sunspots eventually decay and disappear. Sunspot data observed from 1977-01-01 to 2013-12-31 were downloaded from NASA. The CDF of the data is S-shaped, as shown in Extended Data Fig.3.

**Artificial structures and experimental devices:** fluctuations in water pressure and flow rate in a water supply system can be due to changes in water consumption, changes in water supply controls, and measurement errors from monitoring instruments. We selected two dynamical systems that include artificial structures (a high-speed railway in China and a wind tunnel) which are unrelated to human behaviour. The
data for the acceleration of ground vibrations along the high-speed railway and wind pressure data for an atmospheric boundary layer from a wind tunnel simulation have similarly shaped CDFs. The ground vibration data had a sampling frequency of 1280 Hz. The 2004 wind pressure experiment was conducted at the Wind Engineering Research Center of Tokyo Polytechnic University (Fig. 8).

**FIG. 8. Artificial structures and experiments.** a, acceleration signal of ground vibration along a high-speed railway (unit: mv): (a1) high speed railway data were obtained for the up line of the Shanghai–Nanjing high-speed railway in 2016, at a speed of 200–300 km/h; (a2) serial acceleration data serial with sampling frequency 1280 Hz (the vibration sensor uses a DC differential wiring method to filter out low frequency components below 0.7 Hz); (a3) histogram of acceleration; (a4) CDF of acceleration data; (a5) histogram of acceleration first order difference; (a6) CDF of data first-order difference. b, wind pressure measurement in a wind tunnel (unit: Pa): (b1) wind tunnel test in the wind tunnel (which has an operational section 15 m long, 2.2 m wide and 1.8 m high) of the Wind Engineering Research Center of Tokyo Polytechnic University; a thickness $Z_0 = 100$ cm of two boundary layers with terrain types having power laws 0.195 and 0.346 was generated using a set of spikes and a number of roughness blocks on the wind tunnel floor using a sampling interval of 15 s; (b2) histograms of wind pressure and CDFs of wind pressure first order difference.

**Bioelectrical signals:** Electrocardiograms (ECGs) and electroencephalograms (EEGs) represent bioelectrical signals from the heart and the brain. An ECG measures change in the potential of cardiomyocytes from the endocardial–pericardial sequence. Human ECG data was downloaded from
PhysioNet\textsuperscript{15}, and an EEG of a dog (\textit{Canis lupus familiaris}) was provided by the School of Pharmaceutical Sciences, Zhejiang University. CDFs of the ECG first-order difference deviate from the straight line (normal distribution) outside the range of ±2–3 times the variance. The CDFs have the same shape as those for other natural time series. The EEGs were downloaded from EPFL\textsuperscript{16} (one sample is a patient EEG, the other is a healthy human EEG\textsuperscript{17}) and PhysioNet\textsuperscript{15}. Although EEGs are complex and noisy, CDFs of their first-order and higher order differences are similar in shape (Fig.10(b)). A phonocardiogram (PCG) represents cardiac sounds caused by the movement of blood in the heart and blood vessels. PCG data were downloaded from PhysioNet\textsuperscript{18,19}. The CDF of PCG first-order difference had a similar shape to the other CDFs discussed in this paragraph.

FIG.9. Bioelectrical signals (ECG, EEG and PCG, some images from internet). a, electrocardiogram, (a2) electrocardiogram from Apnea–ECG Database, (a3, a4) CDF of human ECG first order difference, (a5) CDF of dog ECG first order difference. b, brain wave diagram, (b2, b3) raw time series data, CDFs and histograms of first and second order differences, (b4) ltrsvp EEG at PO8, including raw signals, first and second order difference data, and CDFs. c, cardiac sound diagram, (c2) PCG time series source, (c3) CDF of PCG first order difference.
**Social phenomena:** stock trading is a social behaviour that can easily be accurately digitized. The Dow Jones Industrial Average index (DJIA) is the first weighted average price index to have been created, in 1884. Since then it has increased by a factor of 500. Equation (3) was used to process the DJIA data. The history of the Chinese stock market is much shorter than the lifespan of the DJIA. Each stock price on the Chinese market is a different time series. Equation (2) converts the time series of different individual stocks into dimensionless data which we then aggregated into one dataset. Figs. 10(b1) and 10(b2) are the dimensionless daily stock price data, with fluctuations, in China. Figs. 10(b3) and 10(b4) are 5-minute and 10-minute price data of 900 stocks (the parameter $K = 5$). Both the New York and Chinese stock markets, having different time scales, have stock price CDFs that deviate from the normal distribution at both ends (Figs. 10(a4), 110(b2)–5(b4)). The CDF for copper spot prices published by the Chilean government has a similar shape (Fig. 11).

**FIG.10. Stock market fluctuations.** a, Dow Jones Industrial Average index: (a1) New York Stock Exchange (image from internet), (a2) logarithmic DJIA histogram, (a3) DJIA histogram and CDF of logarithmic first order difference. b, China stock market prices: (b1) histogram of daily prices in the Chinese stock market using 934,842 data records from 2015-01-06 to 2018-01-29, (b2) CDF of daily prices in China stock market, (b3) CDF of 5-minute prices in China stock market from 2019-04-22 to 2019-04-26, (b4) CDF of 10-minute prices in China stock market.
Spatial differences: the data series considered so far were time series. Spatial data series contain data points that vary over multidimensional space rather than time. We collected two examples of spatial data series: one is terrestrial elevation, the other is the cosmic microwave background (CMB). Ground elevation data for 38°N, 102°W was downloaded from the American Geographic Society. The CDFs for both north-south and east-west directions have the same shape as those for the time series datasets (Figs.13(a1)–13(a4)). CMB is electromagnetic radiation that is a remnant from an early stage of the universe. The cosmic background radiation temperature map was generated using data collected by the European Space Agency (ESA) Planck satellite. Four products (SMICA, NILC, SEVEM and COMMANDER-Ruler) can be downloaded from ESA. The CDFs (Fig.13(b3)) were generated from the SMICA.
product, with a resolution of 2048. The CDFs from the calculated first order differences in both directions (Fig.13(b2)) gradually deviate from the normal distribution curve at points close to 1% or 99%, and turn sharply near 0.3% and 99.7%.

**FIG. 12. Spatial distribution of cosmic background radiation and terrestrial elevation.**

- **a.** terrestrial elevation: (a1) ground elevation, (a2) CDF of ground elevation, (a3, a4) CDF of elevation first-order difference in west-east and north-south directions.
- **b.** cosmic background radiation: (b1) sky map of CMB radiation temperature \( T_{CMB} \), (b2) data of CMB radiation temperature (SMICA), (b3) CDF of CMB radiation temperature and CMB radiation temperature first-order difference along two directions.

C. **Statistical curve shape of differences for time space-time data series**

our analysis of the preceding cases provided three consistent characteristics of CDFs for first (or higher) order differences:
(1) The mean value of the difference is zero or close to zero; the ratio of mean value to standard deviation $\frac{\mu}{\sigma}$ can be used as an index (except for DJIA, which has been increasing over time).

(2) The PDF of first or higher order difference is basically a bell-shaped symmetric curve.

(3) The central sections of the CDF curves coincide with the normal distribution CDF, but both ends deviate significantly. The CDF curves are S-shaped and are similar to a $t$-distribution or a Cauchy distribution.

The first characteristic is easy to understand. In a finite system, over a long period, \( \lim_{n \to \infty} \frac{1}{n} \sum \Delta y(i) = 0 \). The second and third characteristics have not yet been mathematically explained. Extended Data Table 2 shows that curve fitting results match a $t$-distribution. All the correlation coefficients are $>0.9$, which indicates that a $t$-distribution is a good approximation.

![Normalized cumulative probability functions.](image)

**FIG.13. Normalized cumulative probability functions.**

**TABLE 2 $t$-distribution fittings**

| Case                | $\frac{\mu}{\sigma}$ | $\nu$    | Correlation coefficient |
|---------------------|------------------------|----------|-------------------------|
| Chaotic equations   |                        |          |                         |
| Lorentz system      | $-1.93E-03$            | 2.491    | 0.9819                  |
| Duffing equation    | $-7.77E-06$            | 2.658    | 0.9788                  |
| Chua circuit        | $2.10E-06$             | 1.114    | 0.9413                  |
| Natural phenomenon  |                        |          |                         |
| Temperature         | $-8.73E-05$            | 5.925    | 0.9994                  |
| Wind speed          | $1.96E-04$             | 3.760    | 0.9980                  |
|                                | Tide current | Surface runoff | Sunspot area | Earth signals |
|--------------------------------|--------------|----------------|-------------|---------------|
| 3.11E04                        | 3.317        | 3.154          | 0.9988      |               |

|                                | Dow Jones index | Daily stock price in China | 5-minute stock price in China | 10-minute stock price in China |
|--------------------------------|-----------------|---------------------------|-----------------------------|-------------------------------|
| 1.61E−02                      | 2.995           | 0.0118                    | 2.872                       | 0.9600                        |

|                                | Social time series | Biological signal | Artificial structures | Earth surface elevation |
|--------------------------------|---------------------|-------------------|-----------------------|-------------------------|
|                                |                    | Dog electrocardiogram | Railway vibration | East-west direction |
|                                |                    | 1.80E−05           | 1.51E−05            | 0.00341                  |
|                                |                    | 2.594              | 3.350                | 4.769                     |
|                                |                    | Brain waves        | Wind tunnel pressure | North-south direction |
|                                |                    | −5.92E−04          | −8.52E−04           | −0.01598                  |
|                                |                    | 13.153             | 6.220                | 4.732                     |

|                                | 0.9386            | 0.9923            | 0.9931               | 0.9938                   |

|                                | Earth signals     | Biological signal | Artificial structures | Earth surface elevation |
|--------------------------------|-------------------|-------------------|-----------------------|-------------------------|
|                                | 1.21E−04          | −4.41E−04         | Railway vibration     | East-west direction     |
|                                | 2.561             | 2.805             | 1.61E−05              | 0.00341                 |
|                                | 0.9908            | 0.996             | 3.154                 | 4.769                    |
|                                |                   | 0.9908            | 2.872                 | 4.769                    |
|                                |                   |                   | 0.9923                | 4.732                    |
|                                |                   |                   | 0.9931                | 0.9938                   |

|                                | Earth signals     | Biological signal | Artificial structures | Earth surface elevation |
|--------------------------------|-------------------|-------------------|-----------------------|-------------------------|
|                                | 1.21E−04          | −4.41E−04         | Railway vibration     | East-west direction     |
|                                | 2.561             | 2.805             | 1.61E−05              | 0.00341                 |
|                                | 0.9908            | 0.996             | 2.872                 | 4.769                    |
|                                |                   | 0.9908            | 2.872                 | 4.732                    |
|                                |                   |                   | 0.9923                | 4.732                    |
|                                |                   |                   | 0.9931                | 0.9938                   |

|                                | Earth signals     | Biological signal | Artificial structures | Earth surface elevation |
|--------------------------------|-------------------|-------------------|-----------------------|-------------------------|
|                                | 1.21E−04          | −4.41E−04         | Railway vibration     | East-west direction     |
|                                | 2.561             | 2.805             | 1.61E−05              | 0.00341                 |
|                                | 0.9908            | 0.996             | 2.872                 | 4.769                    |
|                                |                   | 0.9908            | 2.872                 | 4.732                    |
|                                |                   |                   | 0.9923                | 4.732                    |
|                                |                   |                   | 0.9931                | 0.9938                   |

### IV. PROBABILISTIC STABILITY OF DYNAMICAL SYSTEMS

The Lyapunov exponents of a chaotic dynamical system are not all negative, which indicates that perturbation increases gradually. Because of its sensitivity to initial values, a dynamical system is unpredictable in the long term. This unpredictability leads scientists to make use of statistical tools to study natural phenomena. For example, for climate change, both numerical modeling and observed data are presented statistically. In the statistical analysis that we performed, we asked: Was there sensitivity to initial or boundary values or was there parameter stability? To answer this question, we mapped the phase space to a probability space and created a statistical definition of dynamical stability. We illustrate the probabilistic stability of dynamical systems through a case study. More information can be found in APPENDIX B and C.

**Traditional definition of dynamical system stability:** Let $\bar{x}(t)$ be an equilibrium special solution to the differential equation $\dot{x} = f(t, x), t > 0$ where $x \in G \subseteq \mathbb{R}^n$. If, for all $\varepsilon > 0$ and all $t > \tau$, $\tau \in \mathbb{R}^+$, and all initial values $x_0$, there exists $\delta > 0$ such that $\|x_0 - \bar{x}(\tau)\| < \delta$ and $\|x(t, \tau) - \bar{x}(\tau)\| < \varepsilon$, then $x(t)$ is called stable, otherwise $x(t)$ is called unstable.

**Probabilistic stability of a continuous dynamical system:** Consider a continuous dynamical system $x = f(t, x, c)$, with $x(0) = x_0, t \geq 0$, where $x(t) \in G \subseteq \mathbb{R}^n$ and $c \in D \subseteq \mathbb{R}^m$ are respectively the system state vector
and the parameter vector, and \( f : G \rightarrow \mathbb{R}^n \) is continuous on \( G \). Let \( P(x_0, c) \) be the PDF of \( x_0 \in G \) in the state space of \( f \) with \( c \in D \) and let \( P(c) \) be the PDF of \( c \) in the probability space of \( f \) for all \( x_0 \in G \) and \( c \in D \). Then either \( P(x_0, c) \) or \( P(c) \) is the PDF of the solution of \( f \). \( E^s(x_0, c) \) and \( E^s(c) \) are \( q \)th order moments. Then:

1. If, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| c - c' \| < \delta \), then \( \| P(x_0, c) - P(x_0, c') \| < \varepsilon \), then the continuous dynamical equation \( f \) is called \textit{probabilistically stable} for parameter \( c \) at \( x_0 \in G \). If, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| c - c' \| < \delta \), then \( \| E^s(x_0, c) - E^s(x_0, c') \| < \varepsilon \), then \( f \) is called \textit{probabilistically weakly stable} for parameter \( c \) at \( x_0 \in G \).

2. If, for all \( \varepsilon > 0 \), there exists \( \| c - c' \| < \delta \) satisfying \( \| P(c) - P(c') \| < \varepsilon \), then the continuous dynamical equation \( f \) is called \textit{probabilistically stable} for parameter \( c \). If, for all \( \varepsilon > 0 \), there exists \( \| c - c' \| < \delta \) satisfying \( \| E^s(c) - E^s(c') \| < \varepsilon \), then \( f \) is called \textit{probabilistically weakly stable} for parameter \( c \).

**Probabilistic stability of a discrete dynamical system:** Consider a discrete dynamical system

\[ x_{k+1} = g(k, x_k, c), \quad k \geq 0, x(0) = x_0, \text{ where } x(t) \in \mathbb{R}^n \text{ and } c \in D \subseteq \mathbb{R}^m \text{ are respectively the system state vector and the parameter vector. Let } P(x_0, c) \text{ be the PDF of the state space of } g \text{ at } x_0 \in G \text{ with } c \in D \text{ and let } P(c) \text{ be the probability distribution function of } g \text{ for some initial } x_0 \in G. \text{ } E^s(x_0, c) \text{ and } E^s(c) \text{ are the } q \text{th order moments. Then:}

1. If, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| c - c' \| < \delta \), then \( \| P(x_0, c) - P(x_0, c') \| < \varepsilon \), then the state equation \( g \) is called \textit{probabilistically stable} for parameter \( c \) at \( x_0 \in G \). If, for all \( \varepsilon > 0 \), there exists \( \delta > 0 \) such that if \( \| c - c' \| < \delta \), then \( \| E^s(x_0, c) - E^s(x_0, c') \| < \varepsilon \), then the state equation \( g \) is called \textit{probabilistically weakly stable} for parameter \( c \) at \( x_0 \).

2. If, for all \( \varepsilon > 0 \), there exists \( \| c - c' \| < \delta \) satisfying \( \| P(c) - P(c') \| < \varepsilon \), the state equation \( g \) is called \textit{probabilistically stable} for parameter \( c \). If, for all \( \varepsilon > 0 \), there exists \( \| c - c' \| < \delta \) satisfying \( \| E^s(c) - E^s(c') \| < \varepsilon \), then the state equation \( c \) is called \textit{probabilistically weakly stable} for parameter \( c \).
Fig. 14(a2) shows that a slight deviation in the Lorenz system causes unpredictable behaviour. Fig. 14(a3) shows that even if there is a great discrepancy between the initial values, the deviation in the statistical value gradually converges to 0. Figs. 14(a4) and 14(b4) show that as long as the parameters are determined, the shapes of the statistical distributions of numerical solutions are similar regardless of the initial values. For both continuous and discrete models (such as the Lorentz system, the Rössler equation, the logistic map, the quadratic map, the Hénon map, and the Ikeda map), when they have been assigned fixed parameter values, their PDFs are consistent even if their initial states are different. More details are presented in APPENDIX C.

Our results indicate that chaos can be investigated and understood using the concept of probabilistic stability in the probability space to overcome the problem of sensitivity to initial values. Consequently, many practical cases, such as stocks or brain waves, can be analyzed and understood from a new perspective that has solid mathematical support. It is often difficult to find exact solutions in complex nonlinear systems, such as those of fluid mechanics. The results given by computational fluid dynamics (CFD) are often questioned by engineers because of numerical errors caused by truncation. If a truncation error is considered to be some kind of perturbation, its statistical outcome, with a suitable simulation mesh size, could be accepted without question by engineers because of probabilistic stability. By analyzing more interesting things, such as the Kirkwood gaps in the asteroid belt or the Cassini Division in Saturn’s rings, we can find similar phenomena in these celestial chaotic systems. For example, when the logistic map parameter is 3.6 or 3.75 and the initial value is 0.7, after several iterations the value is no longer around 0.7. Thus, there is a gap in the logistic map (Fig. 14(b3)).
FIG. 14. Probabilistic stability of dynamical systems. a, probabilistic stability of the Lorenz system: (a1) Lorenz curve, (a2) even though there is only a small deviation from the initial value, the discrepancy increases quickly and unpredictably, (a3) different initial values for which the discrepancy appears with probabilities 25% and 75%, gradually converging to zero, (a4) different initial values, with the same parameters, for which the probability distribution does not change, showing that the parameters are important factors that affect the probability distribution. b, the gap in the probability distribution for the logistic map is independent of initial values: (b1) Kirkwood gaps of asteroids near Jupiter, (b2) Cassini Division of Saturn rings, (b3) gaps in the logistic map (initial value 0.7, after 1000 iterations), (b4) probability distribution of the logistic map is unrelated to initial values and depends only on parameters.

V DISCUSSION

We collected some typical independent time series and space series data that occur in nature, in social activity, and in artificial structures and analyzed them. Some datasets were chaotic and unstable in the conventional sense. Despite the diversity of their natural properties and their different frequency spectra, the shapes of the CDFs of first or higher order differences were similar in that the mean value was close to zero. They were also similar in that the PDF had a single peak and was a symmetric bell-shaped curve that approximated a t-distribution. Our statistical analysis showed that the CDF shape is universally similar.
We also discovered some very interesting phenomena. For example, the fluctuation in runoff into the River Thames has increased over the last few decades. We would like to know whether this increase is due to measurement error, human activity, or climate change. Our results show that the first order differences of a normal person ECG or a patient ECG are not white noise. These illustrative observations indicate that our analysis of first and higher order differences can establish previously unrecognized regularities in chaotic physical or social systems. There is historical precedent for this claim: for example, Newton discovered the law of momentum by examining difference in velocities; Faraday discovered the laws of electromagnetism by analyzing the rate of change of magnetic flux. Further research into the nature of first and higher order difference may uncover more physical or social regularities and lead to the recognition of underlying law.

Our work also has application in the field of numerical modeling and simulation. The normal distribution is widely used in science and engineering. However, the solution of a linear diffusion equation may not be the best way to explain a nonlinear phenomenon: a fat-tailed PDF may be a more realistic representation than the normal distribution; a Monte Carlo simulation with a $t$-distribution hypothesis may be more accurate than a normal distribution.

The widely-used Fourier transform maps the time domain to the frequency domain. We developed a transformation for a dynamical system to map the state space to the probability space. In state space, a chaotic system is sensitive to initial values and thus generally unpredictable; however, the PDF of the system does not diverge as perturbations occur. Note that initial-value sensitivity in chaotic systems, even though adumbrated, has been problematic in dynamical system research since the three-body problem was recognized in physics and Newtonian mechanics. In probability space, the PDF of a chaotic data (time or space) series can be stable. However, it is generally unstable when transitioning from one state space to another state space. Thus, the concept of probabilistic stability can improve numerical solutions to chaotic system problems, such as those of predicting future global climate, by establishing a solid mathematical foundation for analyzing and understanding them. We think that Lyapunov stability is a mathematic
description of causal determinism, and that the determinism and indeterminism of dynamical systems may become compatible in the mapping from state space to probability space. Future work will focus on *transforming* state equations to probability equations, which remains a major challenge in dynamical systems research.

**DATA AVAILABILITY**

The authors declare that all relevant data supporting the finding of this study are available within the paper and its APPENDIX files. Additional data are available from the corresponding author upon request. Most of the data comes from public websites. We strongly recommend that readers download it directly from the websites marked within references.

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**COMPETING FINANCIAL INTERESTS**

The authors declare no competing financial interests.

**APPENDIX**

**APPENDIX A: CHAOTIC DYNAMICAL SYSTEM EQUATIONS**

Lorenz developed a simplified mathematical model for atmospheric convection, the Lorenz system, in 1963.

The model is a system of three ordinary differential equations known as the Lorenz equations:

\[
\begin{align*}
\frac{dx}{dt} &= \sigma(y - x) \\
\frac{dy}{dt} &= \rho x - xz - y \\
\frac{dz}{dt} &= xy - \beta z
\end{align*}
\]  

(A1)
These equations describe the properties of a two-dimensional fluid layer that is uniformly warmed from below and cooled from above. The equations describe the rate of change of three quantities with respect to time: $x$, $y$ and $z$ are proportional to the convection rate, horizontal change in temperature, and vertical change in temperature. The constants $\sigma$, $\rho$, and $\beta$ are system parameters proportional to the Prandtl number, the Rayleigh number, and certain physical dimensions of the fluid layer. The Lorenz system is used in simplified models of lasers, dynamos, thermosiphons, brushless DC motors, electric circuits, chemical reactions, and forward osmosis in oil sand membranes.

The Duffing equation (or Duffing oscillator) is a nonlinear second order differential equation used to model certain damped and driven oscillators. It can be expressed as:

$$\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \alpha x + \beta x^3 = \gamma \cos(\omega t)$$

where $x$ is displacement at time $t$ and $\alpha$, $\gamma$, $\beta$, and $\omega$ are given constants. The Duffing equation represents a dynamical system that exhibits chaotic behaviour. In some cases, a Duffing system will show a jump in the frequency domain, which is a type of frequency hysteresis.

A Chua circuit (Equation 3) is a simple electronic circuit that exhibits typical chaotic behaviour. It is a nonperiodic oscillator which produces an oscillating wave. Unlike an ordinary electronic oscillator, it never repeats. It is an example of an ubiquitous real-world chaotic system and is known as a paradigm for chaos.

$$\frac{dx}{dt} = \alpha \left[ y - x - f(x) \right]$$
$$RC \frac{dy}{dt} = x - y - Rz$$
$$\frac{dz}{dt} = -\beta y$$

(A3)
APPENDIX B: PROBABILISTIC FUNCTION OF DYNAMIC SYSTEM

If a dynamical system has a solution, whether or not it is Lyapunov stable, a map from state space to probability space can be defined. The factors that influence the probability distribution function of the system are primarily the initial values and the parameters. Probability functions of dynamical systems can be divided into three categories (Fig.3(a)) depending whether the initial values alone, the parameters alone, or both initial values and parameters affect the PDF. The three categories are shown in Fig.B1.

Figs.3(b)–3(g) illustrate this categorization. Figs.3(b)–3(c) give examples of category 1.1. The range of the solution is affected by the initial velocity. The PDF is determined by the initial values. When $v_0 < \sqrt{2gl}$, its PDF is stable. The initial value $v_0 = \sqrt{2gl}$ is the inflexion point of the PDF. In Fig.3(c), the dynamic system is unstable at its highest point and its solution depends on the direction of initial motion. Thus its PDF also depends on the initial value. Fig.3(d) is a simple harmonic vibration, belonging to categories 2.1 and 3.1. Its PDF depends on vibration amplitude. Before the logistic map becomes chaotic $(a < a_c)$, its solution is determinate, but it is not continuous at the bifurcation points. For example, in Fig.3(f), at the first bifurcation point, the probability function on the left side is

$$p(a < a_c) = \begin{cases} 1 & x = x_i \\ 0 & x \neq x_i \end{cases}; \text{ the probability function on the right side is }$$

$$p(a > a_c) = \begin{cases} 0.5 & x = x, x_2 \text{ or } x_3 \\ 0 & x \neq x, x_2 \text{ or } x_3 \end{cases}.$$ 

However, the mathematical expectation is continuous

$$\lim_{a \to a_c} E(x) = \lim_{a \to a_c} E(x).$$

When $a < a_c$, the logistic map has stable solutions, and the PDF is independent of the initial values.

If the parameters $a$ are not equal, the PDFs are not equal, and thus the system is weakly stable. When the logistic map becomes chaotic $(a > a_c)$, its solution is sensitive to the initial value. If the solution space is continuous, the definition of the probability function might be used in the stability problem. If the solution space is discontinuous, it is recommended to use mathematical expectation to calculate the function. Fig.3(g) is 200 000 simulation results of the Lorentz system (the initial value is a random uniform distribution given by $[12 + 2 \times \text{random}, 2 + 2 \times \text{random}, 9 + 2 \times \text{random}]$; total simulation time $t$ is 500; time step is 0.005). The 200 000 solutions are projected onto the $x$–$z$ plane; the shape is similar to that of a single equation in a time series. The PDFs have the same shape. Even though chaotic systems are sensitive to the initial values, the results show that the PDF has nothing to do with the initial value. This feature has potential theoretical values in
the research of chaotic dynamic systems, and it helps us to understand why the world looks orderly no matter with perturbations.

FIG. B1. Different categories of probabilistic stability for a dynamic system. a, categories of the dynamic system probability function. b, a small ball oscillates in the trough. c, a small ball on the peak is unstable. d, PDF of simple harmonic vibration. e, logistic map. f, bifurcation point’s probabilistic function of logistic map. g, x-z projection and PDF of Lorenz equation solution with 1,000,000 uniform random distribution initial values.
APPENDIX C: PROBABILISTIC STABILITY OF CHAOTIC SYSTEMS

The Lorenz system (Equation (1)), the Rössler attractor equations, the logistic map, the quadratic map, the Hénon map, and the Ikeda map are well-known chaotic dynamical systems. Their system characteristics have been widely studied. We explored them statistically. These unrelated dynamical systems all show that if their parameters are determinate, their PDFs will be insensitive to initial values. The results of using these equations are shown in the following figures.

Rössler equations

\[
\begin{align*}
\frac{dx}{dt} &= -y - z \\
\frac{dy}{dt} &= x + ay \\
\frac{dz}{dt} &= b + z(x - c)
\end{align*}
\]

(C1)

Logistic map

\[x(n + 1) = a[1 - x(n)]\]  

(C2)

Quadratic map

\[
\frac{dx}{dt} = c - x^2
\]  

(C3)

Hénon map

\[
\begin{align*}
\frac{dx}{dt} &= 1 - ax^2 + y \\
\frac{dy}{dt} &= bx
\end{align*}
\]

(C4)

Ikeda map

\[
\begin{align*}
\frac{dx}{dt} &= 1 + v[x \cos(t) - y \sin(t)] \\
\frac{dy}{dt} &= v[x \sin(t) - y \cos(t)]
\end{align*}
\]

(C5)
FIG.C1. Probabilistic stability of Lorenz system. a, the calculated results of the Lorenz system are not sensitive to initial values but vary depending on parameter $\rho$ ($\sigma=10, \beta=8/3, \rho=28-37$). b, the calculated results of the Lorenz system are sensitive to initial values, ($\sigma=10, \beta=8/3, \rho=28, (x, y, z)_0^1=(6,2,9), (x, y, z)_0^2=(6+10^{-6},2,9)$, integration step size 0.01). c, depending on the initial values, there is discrepancy between states corresponding to probabilities 25% and 75%, increasing gradually to zero ($\sigma=10, \beta=8/3, \rho=28, (x, y, z)_0^1=(6,2,9), (x, y, z)_0^2=(3,6,5)$, integration step size 0.01).
FIG.C2. Probabilistic stability of the Rössler equation. 

a. the calculated results of the Rössler equation are not sensitive to the initial values 

\[(a=0.2, b=0.2, c=5.7 \sim 20.7)\] ; b. the calculated results of the Rössler equation are sensitive to the initial values 

\[(a=0.2, b=0.2, c=5.7 \sim 20.7), (x, y, z)_{k}=(12,2,9), (x, y, z)_{k}'=(12+10^{k},2,9), \Delta t=0.01\]. 

c. depending on the different initial values, there is discrepancy between states corresponding to probabilities 25% and 75%, converging gradually to zero 

\[(a=0.2, b=0.2, c=5.7, (x, y, z)_{0}=(12,2,9), (x, y, z)_{0}'=(12,5,9), \Delta t=0.01)\].
FIG. C3. Probabilistic stability of logistic map. 

a, calculated results of the logistic map are not sensitive to initial values $a = 3.0 \sim 3.75$. b, calculated results of the logistic map are sensitive to the initial values $a = 3.8, x_0 = 0.9, x_0^2 = 0.9 + 10^{-6}$. c, depending on the different initial values, there is discrepancy between states corresponding to probability 25% and 75%, converging gradually to zero ($a = 3.8, x_0 = 0.9, x_0^2 = 0.8$).
FIG. C4. Probabilistic stability of quadratic map. a, calculated results of the quadratic map are not sensitive to initial values ($c = 0.5 \sim 1.4$). b, calculated results of the quadratic map are sensitive to initial values ($c = 1.5, x_0^1 = 0.6, x_0^2 = 0.6 + 10^{-4}$). c, depending on the different initial values, there is discrepancy between states corresponding to probability 25% and 75%, converging gradually to zero ($c = 1.5, x_0^1 = 0.6, x_0^2 = 0.3$).
FIG. C5. Probabilistic stability of Hénon map. (a) calculated results of Hénon equation are not sensitive to initial values ($c = 1.1 \sim 1.35, b = 0.3$); (b) calculated results of Hénon map are sensitive to initial values ($a = 1.2, b = 0.3, (x_0, y_0)^1 = (0.3, 0.4), (x_0, y_0)^2 = (0.3 + 10^{-4}, 0.4)$). (c) depending on the different initial values, there is discrepancy between states corresponding to probability 25% and 75%, converging gradually to zero ($a = 1.2, b = 0.3, (x_0, y_0)^1 = (0.3, 0.4), (x_0, y_0)^2 = (0.5, 0.6)$).
Probabilistic stability of Ikeda map. a, calculated results of Ikeda equation are not sensitive to initial values ($v = 0.15 \sim 0.9$, $(x_0, y_0) = (0.2, 0.2) \sim (0.65, 0.65)$). b, calculated results of Ikeda equation are sensitive to initial values

$\left( v = 0.9, (x_0, y_0) = (0.3, 0.4), (x_0, y_0) = (0.3 + 10^{-6}, 0.4) \right)$ and $\left( v = 0.9, (x_0, y_0) = (0.3, 0.4), (x_0, y_0) = (0.5, 0.6) \right)$; (c) depending on the different initial values, there is discrepancy between states corresponding to probability 25% and 75%, converging gradually to zero.

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