ERGODIC PROPERTIES OF ERDÖS MEASURE, THE ENTROPY OF THE GOLDSHIFT, AND RELATED PROBLEMS

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To the memory of Paul Erdős

Abstract. We define a two-sided analog of Erdős measure on the space of two-sided expansions with respect to the powers of the golden ratio, or, equivalently, the Erdős measure on the 2-torus. We construct the transformation (goldsnift) preserving both Erdős and Lebesgue measures on $T^2$ which is the induced automorphism with respect to the ordinary shift (or the corresponding Fibonacci toral automorphism) and proves to be Bernoulli with respect to both measures in question. This provides a direct way to obtain formulas for the entropy dimension of the Erdős measure on the interval, its entropy in the sense of Garsia-Alexander-Zagier and some other results. Besides, we study central measures on the Fibonacci graph, the dynamics of expansions and related questions.

0. Introduction

Among numerous connections between ergodic theory and the metric theory of numbers, the questions related to algebraic irrationals, expansions associated with them and ergodic properties of related dynamical systems, are of special interest. The simplest case, i.e. the golden ratio, the Fibonacci automorphism etc., has served as a deep source of problems and conjectures.

In 1939 P. Erdős [Er] proved in particular the singularity of the measure on the segment which is defined as the one corresponding to the distribution of the random variable $\sum_{1}^{\infty} \varepsilon_k \lambda^{-k}$, with $\lambda$ being the larger golden ratio and $\varepsilon_k$ independently taking the values 0 and 1 (or $\pm 1$) with probabilities $\frac{1}{2}$ each. We think it is natural to call this measure the Erdős measure. This work gave rise to many publications and numerous generalizations (see, e.g., [AlZa] and references therein). Nevertheless, little attention was paid to the dynamical properties of this natural measure. The

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aim of this paper is to begin studying dynamical properties of Erdös measure and its two-sided extension. We

(1) define the two-sided generalization of Erdös measure (Section 1);
(2) introduce a special automorphism (“goldenshift”) which preserves Erdös measure and which is a Bernoulli automorphism with a natural generator with respect to Erdös and Lebesgue measures (Section 2);
(3) compute the entropy of this automorphism and prove that it is related to the pointwise dimension of the Erdös measure defined in [Y] and Garsia’s entropy considered in [AlZa] (Section 3);
(4) discuss the connections with some properties of the Fibonacci graph, its central measures and the adic transformation on it (see Appendix A);
(5) define a new kind of expansions corresponding to the goldenshift (see Appendix B).

We will describe all this in more detail below.

Several years ago certain connections between symbolic dynamics of toral automorphisms and arithmetic expansions associated with their eigenvalues were established. The first step in this direction was also related to the golden ratio (see [Ver5]) and led to a natural description of the Markov partition in terms of the arithmetic of the 2-torus and homoclinic points of the Fibonacci automorphism. The main idea was to consider the natural extension of the shift in the sense of ergodic theory and the adic transformation on the space of one-sided arithmetic expansions and to identify the set of two-sided expansions with the 2-torus (see also [Ver6], [Ver3], [KenVer]).

In the present paper we use the same idea for a detailed study of the Erdös measure. Namely, we define the two-sided Erdös measure as a measure on the space of expansions infinite to both sides and identify it with a measure on the 2-torus; in the same way, Lebesgue measure on the 2-torus can be considered as a two-sided version of the Markov invariant measure on the corresponding Markov compactum – see below and [Ver5], [Ver6]. We study the properties of the ordinary shift and the goldenshift as a transformation on the space of expansions introduced by means of the notion of block. The goldenshift turns out to preserve both Lebesgue and Erdös measures, both being Bernoulli in the natural sense with respect to the goldenshift; this is one of the main results of the paper (Theorem 2.7). By the way, this immediately yields a proof of the Erdös theorem on the singularity of Erdös measure. Moreover, the two-sided goldenshift is an induced automorphism for the Fibonacci automorphism of the torus. Other important consequences of our approach follow from the fact that the entropy of the goldenshift is directly related to the entropy of Erdös measure in the sense of Garsia and Alexander-Zagier, i.e. to the entropy of the random walk with equal transition measures on the Fibonacci graph (Theorem 3.3).

In [AlZa] it was attempted to compute the entropy of Erdös measure as the infinite convolution of discrete measures, which was in fact introduced by A. Garsia [Ga] in a more general situation. Note that it proved to be the entropy of a random walk on the Fibonacci graph. In [LePo] the authors compute the dimension of the Erdös measure on the interval in the sense of L.-S. Young [Y] and relate a certain two-dimensional dynamics to it.
Finally, making use of a version of Shannon’s theorem for random walks (see [KaVer]) yields the value of the dimension of the Erdős measure in the sense of Young (Theorem 3.7).

Thus, the dynamical viewpoint for arithmetic expansions and for measures related to them provides new information and an essential simplification of computations of the invariants involved. One may expect that the methods of this paper will apply to more general algebraic irrationals and also to some nonstationary problems.

The contents of the present paper are as follows. In Section 1 we present auxiliary notions (canonical expansions and others) and give the main definitions (Erdős measure on the interval and the 2-torus, normalization, the Markov measure corresponding to Lebesgue measure etc.). In addition, we deduce some preliminary facts about the one-sided and two-sided Erdős measures. In Section 2 we study the combinatorics of blocks in terms of canonical expansions, introduce the notion of the goldenshift (both one-sided and two-sided) and prove its Bernoulli property with respect to Lebesgue and Erdős measures. Section 3 contains our main results on the entropy and dimension of the Erdős measure and their relationships to the random walk on the Fibonacci graph.

In the appendices we consider some related problems. Namely, in Appendix A the combinatorial and algebraic theory of the Fibonacci graph is presented. In particular, we describe the ergodic central measures on this graph and the action of the adic transformation, which is defined as the transfer to the immediate successor in the sense of the natural lexicographic order (in our case it is just the next expansion of a given real in the sense of the natural ordering of the expansions). We study the metric type of the adic transformation with respect to the ergodic central measures. In Appendix B we consider arithmetic block expansions of almost all points of the interval. The interest in them is due to the fact that the “digits” of the block expansions are independent with respect both to Erdős and Lebesgue measures. Note that there are some peculiarities caused by the difference between the one-sided and two-sided shifts. For instance, the one-sided Erdős measure is only quasi-invariant under the one-sided shift, while the two-sided measure is shift-invariant. In Appendix C the densities of the Erdős measure with respect to the shift and to the rotation by the golden ratio are computed by means of blocks. Finally, in Appendix D another proof of Alexander-Zagier’s formula for the entropy is given. It is worthwhile because of its connection with the geometry of the Fibonacci graph.

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1. Erdős Measure on the Interval and on the 2-Torus.

1.1. Canonical expansions. Let $\Sigma = \prod_1^{\infty} \{0,1\}$ endowed with the one-sided Bernoulli shift $\sigma$ and let $X \subset \Sigma$ be the stationary Markov compactum with the transition matrix $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, i.e. the set $X = \{(x_1x_2\ldots) \in \Sigma : x_i x_{i+1} = 0, \ i \geq 1\}$ endowed with the topology of pointwise convergence. Let next $\lambda = \frac{\sqrt{5}+1}{2}$ and $L : X \to [0,1]$ be the mapping acting by the formula

\begin{equation}
L(x_1x_2\ldots) := \sum_{k=1}^{\infty} x_k \lambda^{-k}.
\end{equation}
It is well known that $L$ is one-to-one, except for a countable number of sequences whose tail is of the form $0^\infty$ or $(01)^\infty$. The inverse mapping $L^{-1}$ is specified with the help of the greedy algorithm. Namely, let $Tx = \{\lambda x\}$ and

$$\varepsilon_k = [\lambda T^{k-1} x], \quad k \geq 1.$$

We call the constructed sequence $(\varepsilon_1(x) \varepsilon_2(x) \ldots)$ the canonical expansion of $x$. Note that usually the canonical expansions are called $\beta$-expansions (for $\beta = \lambda$). They were introduced in [Re] and [Ge] and thoroughly studied in [Pa].

Note that the mapping $L$ can be naturally extended to $\Sigma$, and let $L'$ stand for this extension. However, $L'(\Sigma) = [0, \lambda]$, that is why we define the projection $\pi : \Sigma \to [0, 1]$ by the formula $\pi(x) := \lambda^{-1} L'(x)$. It will be frequently used below. Note that $\pi$ is not one-to-one.

1.2. The Markov measure on $X$. The transformation $T : (0, 1) \to (0, 1)$ is transferred by $L$ to the Markov compactum $X$ and acts as the one-sided shift $\tau$:

$$\tau(\varepsilon_1 \varepsilon_2 \varepsilon_3 \ldots) := \varepsilon_2 \varepsilon_3 \ldots$$

Thus, we have $\tau = L^{-1} TL$. The transformation $T$ has been thoroughly studied, and it was shown that there exists the $T$-invariant measure $m'$ equivalent to Lebesgue measure $m_1$. Its density is given by the formula

$$\rho(x) = \frac{dm'}{dm_1} = \begin{cases} \lambda^2 / \sqrt{5}, & 0 < x \leq \lambda^{-1} \\ \lambda / \sqrt{5}, & \lambda^{-1} < x \leq 1 \end{cases}$$

(see, e.g., [Ge], [Pa]). The corresponding Markov measure $L^{-1} m'$ on $X$ is the one with the stationary initial distribution \( \left( \frac{\lambda^{-1}}{\sqrt{5}}, \frac{\lambda}{\sqrt{5}} \right) \) and the transition probability matrix \( \begin{pmatrix} \lambda^{-1} & \lambda^{-2} \\ 1 & 0 \end{pmatrix} \). The $L$-preimage of the Lebesgue measure $m$ on $X$ differs from this stationary Markov measure only by its initial distribution \( \begin{pmatrix} \lambda^{-1} \\ 1 \end{pmatrix} \). Note that for the adic transformation on $X$ (for the definition see [Ver2] or Appendix A) with the alternating ordering on the paths the latter measure is unique invariant, as this adic transformation turns into the rotation by the angle $\lambda^{-1}$ under the mapping $L$ (for more details see [VerSi]).

1.3. Erdös measure and normalization. Let us define the Erdös measure. By definition, the continuous Erdös measure $\mu$ on the unit interval is the infinite convolution $\vartheta_1 \ast \vartheta_2 \ast \ldots$, where $\supp \vartheta_n = \{0, \lambda^{-n-1}\}$, and $\vartheta_n(0) = \vartheta_n(\lambda^{-n-1}) = \frac{1}{2}$ (see [Er]).

We are going to specify this measure more explicitly. Let $p$ denote the product measure with the equal multipliers \( \left( \frac{1}{2}, \frac{1}{2} \right) \) on the compactum $\Sigma$. Then it is easy to see that $\mu = \pi(p)$.

We are also interested in the specification of the Erdös measure on the Markov compactum $X$. Of course, it is just $L^{-1} \mu$; however, it is worthwhile to introduce a direct mapping.
Definition. Let \( x \in \Sigma \), \( x = \{ x_k \}_{k=1}^{\infty} \); we define \([0, 1] \ni c(x) = \sum_{k=1}^{\infty} x_k \lambda^{-k-1} = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k} \), where \( \{ \varepsilon_k \} \) is the canonical expansion of \( c(x) \). We define
\[
  n(x) := \varepsilon = \{ \varepsilon_k \}_{k=1}^{\infty}.
\]
The mapping \( n : \Sigma \to X \) is called normalization.

We will also describe the mapping \( n \) directly avoiding the expansion of a number from \([0, 1]\). Namely, let \( x = (x_1, x_2, \ldots) \in \Sigma \); we put \( x_0 = 0 \) and look for the first occurrence of the triple 011, after which we replace it by 100. The next step is the same, i.e., we return to the zero coordinate and start from there until we meet again 011, etc. It is easy to see that the process leads to stabilization at a normalized sequence. Note that this algorithm is rather rough, as it is known that there exists a finite automaton carrying out the process of normalization faster (see, e.g., [Fr]). It is obvious that this definition is equivalent to the one given above.

Now we can define Erdős measure also on \( X \) as the image of the product measure \( p = \prod_{k=1}^{\infty} \left( \frac{1}{2}, \frac{1}{2} \right) \):
\[
  \mu = n(p)
\]
(we preserve the same notation for \( X \) as for \([0, 1]\)).

Below we will see that this definition of the Erdős measure is not very suitable for deducing its dynamical properties (the quasi-invariance under \( T \) and the rotation by the golden ratio, etc.); we will give another definition related to the two-sided theory.

We begin with its self-similar property which is typical for this measure and completely characterizes it.

Lemma 1.1. The Erdős measure \( \mu \) on the interval \([0, 1]\) satisfies the following self-similar relation:
\[
  \mu E = \begin{cases}
    \frac{1}{2} \mu(\lambda E), & E \subset [0, \lambda^{-2}) \\
    \frac{1}{2} \mu(\lambda E) + \mu(\lambda E - \lambda^{-1})), & E \subset [\lambda^{-2}, \lambda^{-1}) \\
    \frac{1}{2} \mu(\lambda E - \lambda^{-1}), & E \subset [\lambda^{-1}, 1]
  \end{cases}
\]
for any Borel set \( E \).

Proof. Let \( F_1 = 1, F_2 = 2, \ldots \) be the sequence of Fibonacci numbers. Let \( f_n(k) \) denote the number of representations of a nonnegative integer \( k \) as a sum of not more than \( n \) first Fibonacci numbers. We first show that for \( n \geq 3 \),
\[
  f_n(k) = \begin{cases}
    f_{n-1}(k), & 0 \leq k \leq F_n - 1 \\
    f_{n-1}(k) + f_{n-1}(k - F_n), & F_n \leq k \leq F_{n+1} - 2 \\
    f_{n-1}(k - F_n), & F_{n+1} - 1 \leq k \leq F_{n+2} - 2.
  \end{cases}
\]
(1.2)

To prove this, we represent \( f_n(k) \) as \( f_n(k) = f'_n(k) + f''_n(k) \) for each \( k < F_{n+2} - 1 \), where \( f'_n(k) \) is the number of representations with \( \varepsilon_n = 0 \), and \( f''_n(k) \) is the number with \( \varepsilon_n = 1 \). Obviously, if \( k \leq F_n - 1 \), then \( k = \sum_{j=1}^{n} \varepsilon_j F_j = \sum_{j=1}^{n-1} \varepsilon_j F_j \), whence \( f_n(k) = f'_n(k) \). If \( F_{n+1} - 1 \leq k \leq F_{n+2} - 2 \), then \( f_n(k) = f''_n(k) \). In the case \( F_n \leq k \leq F_{n+1} - 2 \), obviously, \( f'_n(k) > 0, f''_n(k) > 0 \). It remains to note that \( f'_n(k) = f_{n-1}(k) \) and \( f''_n(k) = f_{n-1}(k - F_n) \).

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Now from (1.2), and from the definition of the Erdős measure it follows that if, say, an interval $E \subset [0, \lambda^{-2})$, then

$$\mu E = \lim_{n \to \infty} \sum_{k:\tau_{k+2}^n \in E} \frac{f_n(k)}{2^n} = \frac{1}{2} \lim_{n \to \infty} \sum_{k:\tau_{k+1}^n \in \lambda E} \frac{f_{n-1}(k)}{2^{n-1}} = \frac{1}{2} \mu(\lambda E).$$

The other cases are studied in the same way. \(\Box\)

**Remark.** The Erdős measure $\mu$ (as a Borel measure) is completely determined by the above self-similar relation. Indeed, by induction one can determine its values for any interval $(a, b)$ with $a, b \in (\mathbb{Z} + \lambda \mathbb{Z}) \cap [0, 1]$.

**Corollary 1.2.** $\mu(0, \lambda^{-2}) = \mu(\lambda^{-2}, \lambda^{-1}) = \mu(\lambda^{-1}, 1) = \frac{1}{3}$.

The next step consists in introducing a two-sided analog of the Erdős measure, which will lead to the one-sided shift-invariant measure equivalent to $\mu$.

### 1.4. Two-sided theory.

Consider the two-sided space $\tilde{\Sigma} = \prod_{-\infty}^{\infty} \{0, 1\}$ and its subset the two-sided Markov compactum $\tilde{X} = \{(\varepsilon_k)_{k=-\infty}^{\infty} : \varepsilon_k \in \{0, 1\}, \varepsilon_k\varepsilon_{k+1} = 0, k \in \mathbb{Z}\}$.\(^1\) To define the two-sided Erdős measure on $\tilde{X}$, we are going to construct a two-sided analog of normalization. Furthermore, we will use an arithmetic mapping from $\tilde{X}$ onto $T^2$ which semiconjugates the two-sided Markov shift and the Fibonacci automorphism of the torus in order to specify Erdős measure on the 2-torus and to study its properties (see item 1.6).

Let $\tilde{\sigma}$ denote the two-sided shift on $\tilde{\Sigma}$, i.e. $(\tilde{\sigma}x)_k = x_{k+1}$, and let $\tilde{\tau}$ stand for the two-sided shift on the Markov compactum $\tilde{X}$. We denote by $\tilde{m}$ the stationary two-sided Markov measure on the compactum $\tilde{X}$ with the invariant initial distribution $\left(\frac{\lambda/\sqrt{5}}{\lambda^{-1}/\sqrt{5}}\right)$ and the transition probability matrix $\left(\begin{array}{cc} \lambda^{-1} & \lambda^{-2} \\ 1 & 0 \end{array}\right)$. As is well known, $\tilde{m}$ is the unique measure of maximal entropy for the shift $\tilde{\tau}$.

There is an important action of $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ on $\tilde{X}$. Namely, let $w_0, w_1$ be the generators, i.e. $\mathbb{Z}^2 = \{nw_0 + mw_1 \mid n, m \in \mathbb{Z}\}$. Let us describe the action $A_g : \tilde{X} \to \tilde{X}$, $g \in \mathbb{Z}^2$. First, $A_{w_1} = \tilde{\tau}^{-1} A_{w_0} \tilde{\tau}$, and $A_{w_0}$ is addition by 1 in the sense of the arithmetic of $\tilde{X}$. More precisely, considering $\tilde{X}$ as the set of formal series $\{\sum_{n=-\infty}^{\infty} \varepsilon_n w_n \mid \{\varepsilon_n\} \in \tilde{X}\}$, we define $w_n + w_{n+1} = w_{n+2}$, and $2w_n = w_{n-1} + w_{n+2}$ (implying the representation $w_n \leftrightarrow \lambda^{-n}$), whence the operation $\{\varepsilon_n\} \mapsto \{\varepsilon_n\} + w_0$ is well defined for $\tilde{m}$-a.e. $\{\varepsilon_n\} \in \tilde{X}$, as well as the sum of a.e. pair of sequences (see [Ver5], [Ver3]).

**Proposition.** (see [Ver5], [Ver3].) The measure $\tilde{m}$ is the unique Borel measure invariant under the action of $\mathbb{Z}^2$ described above.

**Remark.** We define the following identification of some pairs of points in $\tilde{X}$ whose measure $\tilde{m} \times \tilde{m}$ is 0 in order to turn $\tilde{X}$ into the additive group. Let the equivalence relation $\sim$ be defined as follows:

$$(*1000000\ldots) \sim (*010101\ldots),$$
$$(*\ldots01010100*) \sim (*\ldots10101001*),$$

---

\(^1\)Henceforth the sign *tilde* will always stand for the two-sided objects.
where $*$ denotes an arbitrary (but the same for both sequences) tail starting at the same term. Besides, extending the equivalence relation $\sim$ by continuity, we get $(0.1)^{\infty} \sim (1.0)^{\infty} \sim 0^{\infty}$ (henceforth the point will mark the border between the negative and nonnegative coordinates of a sequence). Now one can easily check that the set $\tilde{X}' = \tilde{X}/\sim$ is a compact connected group in addition (see [Ver5], [Ver3]).

**Example.** Here is an example of subtraction in $\tilde{X}$: 

$-\lambda = \lambda^2 + \lambda^4 + \lambda^6 + \ldots = 1 + \lambda^3 + \lambda^5 + \lambda^7 + \ldots$, both sequences representing one and the same element of $\tilde{X}'$. Similarly, for any sequence $\varepsilon \in \tilde{X}$ finite to both sides, $-\varepsilon$ is a pair of sequences finite to the right and cofinite to the left, i.e. with the left tail $(01)^{\infty}$.

The purpose for the definition of the operation of addition will be explained in item 1.6, where the automorphism $(\tilde{X}, \tilde{m}, \tilde{\tau})$ will be related to the 2-torus.

Following the one-sided framework, we are going to define the two-sided generalization of the operation of one-sided normalization. Namely, we define the two-sided normalization $\tilde{n}$ as the mapping from $\tilde{\Sigma}$ to $\tilde{X}$. Consider the set $\tilde{\Sigma}' \subset \tilde{\Sigma}$ defined as follows:

$$\tilde{\Sigma}' = \left\{ x \in \tilde{\Sigma} : \#\{k < 0 : x_k = 0\} = \infty \right\}.$$ 

On the set $\tilde{\Sigma}'$ we will define two-sided normalization. Let $x \in \tilde{\Sigma}'$ and $0 \geq k_1 > k_2 > \ldots$, $x_{k_i} = 0$, $x_k \neq 0, k \neq k_i$ for all $i$. We set $x^{(0)} = x$. Let $n(\{x_i^{(0)}\}_{k_1+1}^{\infty}) = \{\varepsilon_i^{(1)}\}_{i=k_1}^{\infty}$. We define

$$x_i^{(1)} = \begin{cases} \varepsilon_i^{(1)} & i \geq k_1 \\ x_i & i < k_1. \end{cases}$$

By induction, let $n(\{x_i^{(n-1)}\}_{k_n+1}^{\infty}) = \{\varepsilon_i^{(n)}\}_{i=k_n}^{\infty}$. Then, by definition,

$$x_i^{(n)} = \begin{cases} \varepsilon_i^{(n)} & i \geq k_n \\ x_i & i < k_n. \end{cases}$$

Obviously, the process leads to the stabilization of $x_i^{(n)}$ in $n$.

**Definition.** The two-sided normalization $\tilde{n}$ at $x \in \tilde{\Sigma}'$ is defined as follows:

$$\tilde{n}(\{x_i\}_{-\infty}^{\infty}) = \lim_{n \to \infty} \{x_i^{(n)}\}.$$ 

**Definition.** The two-sided Erdős measure $\tilde{\nu}$ on the Markov compactum $\tilde{X}$ is the image under the mapping $\tilde{n}$ of the measure $\tilde{\rho}$, which is the product of infinite factors $(\frac{1}{2}, \frac{1}{2})$ on the full compactum $\tilde{\Sigma}$.

Since the set $\tilde{\Sigma}'$ has full measure $\tilde{\rho}$, we have the homomorphism of the measure spaces:

$$\tilde{n} : (\tilde{\Sigma}, \tilde{\rho}) \to (\tilde{X}, \tilde{\nu}).$$

Let now

$$\rho : \tilde{\Sigma} \to \Sigma, \rho(\{x_n\}) = (x_1, x_2, \ldots),$$

$$\rho' : \tilde{X} \to X, \rho'((\varepsilon_n)) = (\varepsilon, \varepsilon_0).$$
be the projections.

Let us write the following diagram:

\[
\begin{array}{ccccccc}
\Sigma & \xrightarrow{\sigma} & \Sigma & \xleftarrow{\rho} & \tilde{\Sigma} & \xleftarrow{\tilde{\sigma}} & \tilde{\Sigma} \\
\eta & \downarrow & \eta & \downarrow & \tilde{\eta} & \downarrow & \tilde{\eta} \\
X & \xleftarrow{\tau} & X & \xleftarrow{\rho'} & \tilde{X} & \xleftarrow{\tilde{\tau}} & \tilde{X}
\end{array}
\]

Note that \( \rho' \tilde{n} \neq n \rho, \tau n \neq n \sigma \).

This is the reason why the one-sided theory has some difficulties. The two-sided theory is more natural for this purpose (the right part of the diagram does commute).

**Proposition 1.3.** The two-sided Erdős measure \( \tilde{\nu} \) is invariant under the two-sided shift, i.e. \( \tilde{\tau} \tilde{\nu} = \tilde{\nu} \) (cf. the one-sided case, where it does not take place).

**Proof.** From the above specification of the mapping \( \tilde{n} \) it follows that

\[
(1.3) \quad \tilde{n} \tilde{\sigma} = \tilde{\tau} \tilde{n},
\]

hence \( \tilde{\nu}(\tilde{\tau}^{-1}E) = \tilde{\rho}(\tilde{n}^{-1}\tilde{\tau}^{-1}E) = \tilde{\rho}(\tilde{\sigma}^{-1}\tilde{n}^{-1}E) = (\tilde{\sigma} \tilde{\rho})(\tilde{n}^{-1}E) = \tilde{\rho}(\tilde{n}^{-1}E) = \tilde{\nu}(E) \) for any Borel set \( E \subset \tilde{X} \).

**Proposition 1.4.** For any cylinder \( \tilde{C} = (\varepsilon_1 = i_1, \ldots, \varepsilon_r = i_r) \subset \tilde{X} \), its measure \( \tilde{\nu} \) is strictly positive.

**Proof.** It follows from the direct specification of the two-sided normalization described above that for the cylinder \( C' = (\varepsilon_0 = 0, \varepsilon_1 = i_1, \ldots, \varepsilon_r = i_r, \varepsilon_{r+1} = 0, \varepsilon_{r+2} = 0) \subset \tilde{\Sigma} \), we have \( \tilde{n}^{-1}(\tilde{C}) \supset C' \), whence, by definition of the Erdős measure, \( \tilde{\nu}(\tilde{C}) \geq 2^{-r-3} \).

**Proposition 1.5.** The two-sided shift \( \tilde{\tau} \) on the Markov compactum \( \tilde{X} \) with the two-sided Erdős measure is a Bernoulli automorphism.

**Proof.** We observe that this dynamical system is a factor of the Bernoulli shift \( \tilde{\sigma} : \tilde{\Sigma} \to \tilde{\Sigma} \) with the product measure \( (\frac{1}{2}, \frac{1}{2}) \) (see relation (1.3)) and apply the theorem due to D. Ornstein [Or] on the Bernoullicity of all Bernoulli factors.

**Remark.** Note that the measure \( \tilde{\nu} \) is not Markov on the compactum \( \tilde{X} \). It would be interesting to prove that the two-sided Erdős measure is a Gibbs measure for a certain natural potential.

**1.5. Dynamical properties of Erdős measure.** Let the measure \( \nu \) on the one-sided compactum \( X \) be defined as the projection of the two-sided Erdős measure \( \tilde{\nu} \).

In other words, the dynamical system \( (\tilde{X}, \tilde{\nu}, \tilde{\tau}) \) is the natural extension of \( (X, \nu, \tau) \).

We recall that it means by definition that for any cylinder \( C = (\varepsilon_1 = i_1, \ldots, \varepsilon_k = i_k) \subset X \) its measure \( \nu \) equals \( \tilde{\nu}(\tilde{C}) \), where \( \tilde{C} = (\varepsilon_1 = i_1, \ldots, \varepsilon_k = i_k) \subset \tilde{X} \). Below we will denote the \( L \)-image of \( \nu \) on the interval \([0, 1]\) by the same letter.
Proposition 1.6. The measure $\nu$ is $\tau$-invariant and ergodic.

Proof. The $\tau$-invariance of $\nu$ is a consequence of the $\tilde{\tau}$-invariance of $\tilde{\nu}$. Furthermore, since the automorphism $(\tilde{X}, \tilde{\nu}, \tilde{\tau})$ is Bernoulli, it is ergodic, thus, the endomorphism $(X, \nu, \tau)$ is also ergodic.

We are going to establish a relation between the mappings $\rho'\tilde{n}$ and $n\rho$ in order to prove the equivalence of the measures $\mu$ and $\nu$ on the interval. Note first that in these terms $\mu = (n\rho)(\tilde{p})$, and $\nu = (\rho'\tilde{n})(\tilde{p})$. We also note that the fact that $\mu \prec \nu$ is shown easily, while in the opposite direction it is not straightforward. The following claims yield the same proofs for both sides.

Lemma 1.7. There exists a subset of $\tilde{\Sigma}$ of full measure $\tilde{\rho}$ and its countable partition into the sets $\{E_k\}_{k=1}^\infty$ and the corresponding set $\{f_k\}_{k=1}^\infty$ of finite sequences in $\tilde{\Sigma}$ such that

\begin{equation}
(\rho'\tilde{n})(x) = (n\rho)(x + f_k), \quad x \in E_k
\end{equation}

with group addition in the set $\tilde{\Sigma} = \prod_{-\infty}^\infty \mathbb{Z}/2$. Similarly, there exists a $\tilde{\rho}$-a.e. partition of $\tilde{\Sigma}$ into the sets $\{D_k\}_{k=1}^\infty$ and the corresponding set $\{g_k\}_{k=1}^\infty$ of finite sequences in $\tilde{\Sigma}$ such that

\begin{equation}
(n\rho)(x) = (\rho'\tilde{n})(x + g_k), \quad x \in D_k.
\end{equation}

Remark. The actions $x \mapsto x + f_k, \ x \mapsto x + g_k$ change finitely many coordinates of $x$. We need only this property.

Proof. Both assertions are proved in the same way. Let us prove the first one. The idea of the proof is based on the fact that a $\tilde{\rho}$-typical sequence from $\tilde{\Sigma}$ can be splitted into finite pieces so that its normalization splits into the concatenation of the corresponding normalizations.

We assume that $x$ has two successive zero coordinates with negative indices and four successive zero coordinates with positive indices. So, let $x = (A_000A_10000A_2)$, where $A_0$ and $A_2$ are infinite and $A_1$ is a finite fragment of $x$ containing $x_0$ and $x_1$. Then $\tilde{n}(x)$ is the concatenation of the normalizations of its pieces $A_00, 0A_1000$ and $0A_2$, whence

\begin{equation}
(\rho'\tilde{n})(x) = (C0n(0A_2))
\end{equation}

with a certain finite admissible word $C$ ending with two zeroes. We have a countable number of possibilities for $C$. Let $E_k := E_C$ be defined as the set of $x \in \tilde{\Sigma}'$ such that relation (1.6) holds with some $A_2$.

To construct $f_k$, we consider two cases. If $C$ begins with 0, i.e. if $C = 0C'$, then we set $x' := x + f_k = (A_00\ldots0 | C'00A_2)$, where "$|$" denotes the border between positive and nonpositive coordinates. For such an $x'$ relation (1.4) is satisfied. If $C$ begins with 1, i.e. if $C = (10)^j0C'$ for some $j \geq 1$ and admissible $C'$, then we set $x' := (A_00\ldots0 | 10(11)^{j-1}C'00A_2)$. The proof is complete.

Relations (1.4) and (1.5) together yield the main assertion.
Proposition 1.8. The measures \( \mu \) and \( \nu \) are equivalent.

Proof. Consider the group which acts on \( \tilde{\Sigma} \) by adding the finite sequences (= by changing a finite number of coordinates). Since the action of this group preserves the measure \( \tilde{\rho} \), we obtain from relation (1.4) \((\rho' \tilde{n})(\tilde{p}) \prec (n \rho)(\tilde{p})\), and from relation (1.5) we get \((n \rho)(\tilde{p}) \prec (\rho' \tilde{n})(\tilde{p})\), whence \( \nu \prec \mu \), and \( \mu \prec \nu \). \( \square \)

Remark. It is possible to show that there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \mu(E) \leq \nu(E) \leq -C_2 \mu(E) \log \mu(E)
\]
for any Borel \( E \). Note that the right estimate is attainable, for instance, at the sequence of sets \( E = E_n = (0, \lambda^{-n}) \), as by Lemma 1.1 and Corollary 1.2, \( \mu(0, \lambda^{-n}) = \frac{4}{3} \cdot 2^{-n} \), while \( \nu(0, \lambda^{-n}) \propto n2^{-n} \) (see the proof of Proposition 1.10 below). However, 0 is the only point of the interval \([0, 1]\), where the density \( \frac{d\nu}{d\mu} \) is unbounded (see Appendix C).

Let \( R \) denote the rotation of the circle \( \mathbb{R}/\mathbb{Z} \) by the angle \( \lambda^{-1} \).

Corollary 1.9. The Erdős measure \( \mu \) is quasi-invariant ergodic with respect to \( T \) and \( R \).

Proof. The first claim follows directly from the equivalence of the measures \( \mu \) and \( \nu \). To prove the second one, we note that by Lemma 1.1,
\[
(1.7) \quad \mu(T^{-1}E) = \frac{1}{2} (\mu E + \mu(E + \lambda^{-2} \text{ mod } 1)),
\]
whence follows the required assertion.

We recall the following well-known claim which is a corollary of the individual ergodic theorem. Namely, two Borel measures invariant and ergodic with respect to one and the same transformation of a metric space, either coincide or are mutually singular.

Now we can present a new (dynamical) proof of the Erdős theorem on the singularity of the measure \( \mu \).

Proposition 1.10. (Erdős theorem, see [Er]) The Erdős measure \( \mu \) is singular with respect to Lebesgue measure \( m \).

Proof. We proved that \( T \nu = \nu \), and above it was noted that \( Tm' = m' \) (see item 1.2), hence by the corollary of the ergodic theorem, either \( \nu \perp m' \) or \( \nu = m' \). Indeed, we can apply it, because \( \nu \) is ergodic by Proposition 1.6, and and the ergodicity of \( m' \) is a classical fact (see, e.g., [Ge], [Re]). To show that \( \nu \neq m' \), we observe that \( m'(0, \lambda^{-n}) \propto \lambda^{-n} \), while \( \nu(0, \lambda^{-n}) = \nu(\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 0) = O(n2^{-n}) \), because if for a sequence \( x = \{x_n\} \in \Sigma, \tilde{n}(x) \in (\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_n = 0) \), then either \( x_i = 0, 1 \leq i \leq n \), or \( x_i = 0, k + 1 \leq i \leq n \), and \( x_k = 1, x_{k-1} = 1, x_{k-2} = 0, x_{k-3} = 1, x_{k-4} = 0 \), etc. for some \( k \). So, \( \nu \perp m' \), hence, \( \mu \perp m_1 \), as \( m' \approx m_1 \), \( \mu \approx \nu \). \( \square \)

Remark 1. Note that the initial proof of Erdős followed the traditions of those times and was based on the study of the Fourier transform of \( \mu \).

Remark 2. The present proof fills a gap in the proof of this statement in the previous joint paper by the authors [VerSi]. Another dynamical proof of Erdős theorem is given in Corollary 2.8 (see below).
Remark 3. The problem of computing the densities \( \frac{d\nu}{d\mu}, \frac{d(R\mu)}{d\mu} \) and \( \frac{d(T\mu)}{d\mu} \) will be solved in Appendix C. Note that all these densities prove to be piecewise constant and unbounded.

We recall that \( \tilde{m} \) is the Markov measure on \( \tilde{X} \) with maximal entropy (see item 1.4).

**Proposition 1.11.** The two-sided Erdős measure \( \tilde{\nu} \) is singular with respect to the Markov measure \( \tilde{m} \).

**Proof.** We again apply the corollary of the ergodic theorem to the transformation \( \tilde{\tau} \) and the measures \( \tilde{m} \) and \( \tilde{\nu} \) on the two-sided Markov compactum. The distinction of the two measures is a consequence of the noncoincidence of their one-sided restrictions (see Proposition 1.10), therefore, \( \tilde{m} \perp \tilde{\nu} \).

At the end of the item we prove a claim which we will need in the next item. Recall that \( \tilde{X} \) has an additive structure (item 1.4).

**Proposition 1.12.** The Erdős measure \( \tilde{\nu} \) is invariant under the transformation \( i: \{\varepsilon_n\} \mapsto -\{\varepsilon_n\} \).

**Proof.** Note that \( i(\{\varepsilon_n\}) = \tilde{n}(\varepsilon'_n) \), where \( \varepsilon'_n \in \tilde{\Sigma} \), and \( \varepsilon'_n = 1 - \varepsilon_n \). Therefore, for any Borel \( E \subset \tilde{X} \) and any sequence \( \{x_n\} \) from the set \( \tilde{n}^{-1}(-E) \) there exists a unique sequence \( \{x'_n\} \in \tilde{n}^{-1}(E) \) such that \( x'_n = 1 - x_n \). Now the claim of the proposition follows from the definition of \( \tilde{\nu} \) and the symmetricity of the measure \( \tilde{\nu} \) on \( \tilde{\Sigma} \).

### 1.6. Erdős measure on the 2-torus.

There exists an important smooth interpretation of the two-sided theory. It is related to a general arithmetic approach to the coding of the hyperbolic automorphisms of the torus. Here we will give only some definitions and primary claims whose aim is to describe a two-sided analog of the Erdős measure. Some necessary bibliographic references will be given at the end of the item.

Consider the *Fibonacci automorphism* \( \tilde{T} \) of the 2-torus, i.e. the automorphism with the matrix \( \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). Later it will be clear that this automorphism can be considered as a natural extension of the endomorphism \( T x = \{\lambda x\} \) of the interval.

There exists a natural way to define a mapping semiconjugating the shift \( \tilde{\tau} \) on the Markov compactum \( \tilde{X} \) and the Fibonacci automorphism, namely the mapping which naturally generalizes the \( \beta \)-expansions with \( \beta = \lambda \) to the two-sided case. It is defined by the formula

\[
\tilde{l}(\{\varepsilon_k\}_{-\infty}^{\infty}) = \left( \sum_{k=-\infty}^{\infty} \varepsilon_k \lambda^{-k} \mod 1, \sum_{k=-\infty}^{\infty} \varepsilon_k \lambda^{-k-1} \mod 1 \right). 
\]

The convergence of the series involved follows from \( \lambda \) being a *PV number*. Indeed, as \(-\lambda^{-1}\) is the Galois conjugate of \( \lambda \), we have \( \|\lambda^n\| \leq \lambda^{-n} \) for any \( n \geq 1 \), where as usual, \( \|x\| := \min(\{|x\}, 1 - |x|) \). Let us explain the background of formula (1.8). Consider first \( x \geq 0 \) and its expansion \( x = \sum_{k=-\infty}^{\infty} \varepsilon_k \lambda^{-k} \) which is the canonical expansion natural extended to all nonnegative reals with \( \varepsilon_k \equiv 0 \) for \( k \leq K(x) \).

---

2I.e. an algebraic integer greater than 1 whose Galois conjugates have the moduli less than 1, see, e.g., [Cas].
So, we identify the set of sequences finite to the left with \( \mathbb{R}_+ \). Consider now the inclusion \( \mathbb{R}_+ \leftrightarrow \mathcal{R} = \{(\{x\}, \{\lambda^{-1}x\}) \mid x \geq 0\} \subset \mathbb{T}^2 \). Since the set \( \mathcal{R} \) is the half-leaf of the unstable foliation for the Fibonacci automorphism (corresponding to its eigenvalue \( \lambda \)), we make sure that \((\tilde{l}\tau)(\{\varepsilon_n\}) = \tilde{Tl}(\{\varepsilon_n\})\), where \( \{\varepsilon_n\} \) is finite to the left.

As the set \( \mathcal{R} \) is dense in the 2-torus, as well as the set of sequences finite to the left is dense in \( \tilde{X} \), we can extend the relation above to the whole compactum \( \tilde{X} \), i.e.
\[
\tilde{l}\tau = \tilde{Tl}
\]
everywhere.

Besides, \( \tilde{l} \) is surjective and from the proposition cited in item 1.4 and the fact that Lebesgue measure \( m_2 \) is the only measure invariant under the translations by a dense set of points of the 2-torus, it follows that \( m_2 = \tilde{l}(\tilde{m}) \).

The important property of the mapping \( \tilde{l} \) is that it is not bijective a.e. Nevertheless, as is well known, the automorphisms \((\mathbb{T}^2, m_2, \tilde{T}) \) and \((\tilde{X}, \tilde{m}, \tau) \) are metrically isomorphic, see, e.g., [AdWe]. Below we will introduce a conjugating mapping for these dynamical systems which will be bijective a.e.

We recall that after a certain identification of pairs of sequences of measure zero, the Markov compactum \( \tilde{X} \) becomes a group in addition which we denoted by \( \tilde{X}' \) (see item 1.4). Note that the mapping \( \tilde{l} \) is well defined on \( \tilde{X}' \), i.e. it is constant on the equivalence classes.

**Lemma 1.13.** The mapping \( \tilde{l} : \tilde{X}' \to \mathbb{T}^2 \) is a group homomorphism.

**Proof.** It suffices to check that \( \tilde{l}(e_{n-1}) = \tilde{l}(e_n) + \tilde{l}(e_{n+1}) \), where \( e_k \) is a sequence having 1 at the \( k \)’th place and 0 at the other places. This follows directly from formula (1.8).

Now we are going to prove the following assertion.

**Proposition 1.14.** The mapping \( \tilde{l} \) is 5-to-1 a.e.

We are going to give two different proofs of this proposition.

**First (geometric) proof.** Consider an arbitrary sequence \( \varepsilon = \{\varepsilon_n\}_{n=-\infty}^{\infty} \in \tilde{X} \). We split it into two pieces \( \{\varepsilon_n\}_{n=-\infty}^{0} \) and \( \{\varepsilon_n\}_{n=0}^{\infty} \) and define \( x_1(\varepsilon) = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k} \), \( x_2 = \sum_{k=0}^{\infty} \varepsilon_{-k} (-\lambda)^{-k} \). It is a direct inspection that \( x_1 \in [0, 1] \), \( x_2 \in [-1, \lambda] \). Using the relation \( \{\lambda^n\} = \{(-1)^{n+1} \lambda^{-n}\} \), \( n \geq 0 \), we make sure that \( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n} = x_1 - x_2 \mod 1 \) and similarly, \( \sum_{n=-\infty}^{\infty} \varepsilon_n \lambda^{-n-1} = \lambda^{-1} x_1 + \lambda x_2 \mod 1 \). Thus, we have the sequence of mappings

\[
\tilde{X} \xrightarrow{\varphi} \mathbb{R}^2 \xrightarrow{b} \mathbb{R}^2,
\]

where \( \varphi(\varepsilon) = (x_1, x_2) \), and \( b(x_1, x_2) = (x_1 - x_2, \lambda^{-1} x_1 + \lambda x_2) \). Now
\[
\tilde{l}(\varepsilon) = (b\varphi)(\varepsilon) \mod \mathbb{Z}^2.
\]

Note that since \( (\varepsilon_0, \varepsilon_1) \neq (1, 1) \), the \( \varphi \)-image of \( \tilde{X} \) is in fact the difference of the rectangles \( \Pi = ([0, 1] \times [-1, \lambda]) \setminus ([\lambda^{-1}, 1] \times [\lambda^{-1}, \lambda]) \). As the area of \( \Pi \) is \( \lambda^2 - \lambda^{-2} = \sqrt{5} \), and the linear transformation \( b = \begin{pmatrix} 1 & -1 \\ \lambda & 1 \end{pmatrix} \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) has determinant \( \sqrt{5} \), the image \( (b_{\varphi})(\tilde{X}) \) has area 5. Since this mentioned image is also the difference of some.
rectangles whose vertices are \((1, -\lambda), (2, -1), (\lambda^{-2}, \lambda), (0, \lambda^{-1}\sqrt{5}), (-1, 3), (-\lambda, \lambda^2)\). One can immediately check that this set is a 5-tuple fundamental domain with respect to the lattice \(\mathbb{Z}^2\). \(\square\)

**Remark.** On the other hand, the final statement of the proof follows from Lemma 1.13, as from its claim it follows that the \(\widetilde{l}\)-preimage of a.e. point of the 2-torus has one and the same capacity.

**Second (algebraic) proof.** From Lemma 1.13 it follows that one needs only to describe the kernel of the homomorphism \(\widetilde{l}\). From the general considerations we conclude that the mapping \(\widetilde{l}\) is bounded-to-one, whence \(#\text{Ker} \widetilde{l} < +\infty\). Furthermore, this kernel is obviously invariant under the shift \(\widetilde{r}\), hence it consists of purely periodic sequences only.

Let \(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r)^\infty\) be such a sequence. From formula (1.8) it follows that

\[
\|\xi \lambda^n\| \to 0, \quad n \to +\infty,
\]

where \(\xi = \xi(\varepsilon) = (\sum_1^r \varepsilon_k \lambda^{-k})/(\lambda^r - 1) \in \mathbb{Q}(\lambda)\). Let \(\mathcal{G} := \{\xi : \|\xi \lambda^n\| \to 0, \ n \to \infty\}\). Obviously, \(\mathcal{G}\) is a group in addition. The following lemma answers the question on the structure of the group \(\mathcal{G}\) which is important itself and will be used below. Let \(\mathbb{Z}[\lambda]\) denote the additive group of the ring of all Laurent polynomials in powers of \(\lambda\), i.e. \(\{m + n\lambda \mid m, n \in \mathbb{Z}\}\).

**Lemma 1.15.** The group \(\mathcal{G}\) is isomorphic to \(\mathbb{Z}^2\). Its elements are described as follows:

\[
\mathcal{G} = \left\{ \frac{m + n\lambda}{5} : m, n \in \mathbb{Z}, \ 2n - m \equiv 0 \ (\text{mod} \ 5) \right\}.
\]

The factor group \(\mathcal{H} = \mathcal{G}/\mathbb{Z}[\lambda]\) is the cyclic group \(\{a^{\lambda+2}/5 \mid a \in \mathbb{Z}/5\mathbb{Z}\}\).

**Proof.** By the theorem of Pisot-Vijayaraghavan on the structure of the group \(\mathcal{G}\) (see [Cas]), \(\xi\) is necessarily algebraic and belongs to the field \(\mathbb{Q}(\lambda)\), and the necessary and sufficient condition for \(\xi\) to belong to \(\mathcal{G}\) is \(\text{Tr}(\xi) \in \mathbb{Z}, \text{Tr}(\lambda\xi) \in \mathbb{Z}\), where \(\text{Tr}\) denotes the trace of an algebraic number. Solving these equations for \(\xi = (m + n\lambda)/q\) with \(m, n \in \mathbb{Z}, \ q \in \mathbb{N}\) and at least one of the numbers \(m, n\) being coprime with \(q\), we come to the system of congruences

\[
\begin{cases}
2m + n \equiv 0 \pmod{q} \\
m + 3n \equiv 0 \pmod{q},
\end{cases}
\]

whence \(5m \equiv 0 \pmod{q}, \ 5n \equiv 0 \pmod{q}\), and thus, \(q = 1\) or \(5\). Besides, the system above with \(q = 5\) is equivalent to one congruence \(2n - m \equiv 0 \pmod{5}\). The second claim of the lemma is a direct computation.

Return to the second proof of the proposition. From the formula for \(\xi = \xi(\varepsilon)\) above it follows that \(\xi\) belongs to \(\mathbb{Z}[\lambda]\) if and only if its period \(r \leq 2\), i.e. when the sequence \(\varepsilon\) is equivalent to \(0^\infty\) in \(\tilde{X}'\) (see the definition of \(\tilde{X}'\) in item 1.4). Besides, if for two periodic sequences \(\varepsilon\) and \(\varepsilon'\), \(\xi(\varepsilon)\) and \(\xi(\varepsilon')\) belong to one and the same element of the factor group \(\mathcal{H}\), we see that \(\tilde{l}(\varepsilon) = \tilde{l}(\varepsilon')\).

Thus, \(\#\text{Ker} \tilde{l} \leq 5\). Let the sequence \(\varepsilon^{(0)} = (0.100)^\infty\); consider the set \(\mathcal{H}' = \{0, \varepsilon^{(j)} \mid 0 < j < 3\}\), where \(\varepsilon^{(j)} = \tilde{\tau}^j(\varepsilon^{(0)})\).
It is verified directly that $\mathcal{H}'$ is a subgroup of $\tilde{X}'$. We claim that $\mathcal{H}' = \text{Ker}\tilde{l}$. By the above, it suffices to prove the inclusion $\mathcal{H}' \subset \text{Ker}\tilde{l}$. From formula (1.8), $\tilde{l}(\varepsilon^{(j)}) = (a, a), \ 0 \leq j \leq 3$, where

$$a = \lim_{n \to \infty} \left\| \frac{\lambda^n}{\lambda^4 - 1} \right\| = \lim_{n \to \infty} \left\| \frac{\lambda^n}{\lambda^2 + 1} \right\| = \lim_{n \to \infty} \left\| \frac{\lambda^n}{\sqrt{5}} \right\| = 0,$$

as $\lambda^n = F_{n-1}\lambda + F_{n-2}, \ n \geq 3$, and we have $\lambda^n \sim F_{n-1}\sqrt{5}$, whence $\|\frac{1}{\sqrt{5}}\lambda^n\| \to 0$ as $n \to \infty$.

Remark. There exists a simple explanation of the origin of the sequences $\varepsilon^{(k)}$. Note first that if we identify a positive integer $n$ with the sequence equal to $n e_0$, we obtain a natural inclusion $\mathbb{N} \subset \tilde{X}$. For example, $2 = \lambda + \lambda^{-2}$, $3 = \lambda^2 + \lambda^{-2}$, etc. It is a direct computation that the Fibonacci numbers have the following representations in the compactum $\tilde{X}$:

$$F_k = \lambda^{k-1} + \lambda^{k-5} + \cdots + \lambda^{-k+3} + \lambda^{-k}, \quad k \text{ even},$$

$$F_k = \lambda^{k-1} + \lambda^{k-5} + \cdots + \lambda^{-k+5} + \lambda^{-k+1}, \quad k \text{ odd}.$$  

Thus, $\varepsilon^{(j)} = \lim_k (F_{4k+j} e_0), \ 0 \leq j \leq 3$ in the compactum $\tilde{X}$. Since $\lim_k \|\lambda F_k\| = 0$, we have $\tilde{l}(\varepsilon^{(j)}) = (0, 0)$.

Now we are going to present a simple modification of $\tilde{l}$ which proves to be a bijection.

Definition. Let the mapping $\tilde{L} : \tilde{X} \to \mathbb{T}^2$ be defined by the formula

$$\tilde{L} (\{\varepsilon_k\}_{-\infty}^{\infty}) = \left( \sum_{k=-\infty}^{\infty} \varepsilon_k \frac{\lambda^{-k}}{\sqrt{5}} \mod 1, \sum_{k=-\infty}^{\infty} \varepsilon_k \frac{\lambda^{-k-1}}{\sqrt{5}} \mod 1 \right).$$

By formula (1.9), a series $\sum_{k \in \mathbb{Z}} \varepsilon_k \frac{\lambda^{-k}}{\sqrt{5}}$ converges modulo 1 for any $\{\varepsilon_k\} \in \tilde{X}$.

Remark. In fact, we can treat formula (1.10) as formula (1.8) with the set of digits \{0, 1/\sqrt{5}\} instead of \{0, 1\} (see the remark about the references at the end of the item).

Furthermore, $\tilde{L}$ semiconjugates the shift and the Fibonacci automorphism, and by the same purposes as for $\tilde{l}$, we have $m_2 = \tilde{l}(\tilde{m})$.

Proposition 1.16. The mapping $\tilde{L} : \tilde{X} \to \mathbb{T}^2$ is bijective a.e.

Proof. Let the projection $P : \tilde{X} \to \tilde{X}$ be defined by the formula $P(e_n) = e_{n-1} + e_{n+1}$ and extended to the whole compactum $\tilde{X}$ by linearity (here $e_n$ is, as above, the sequence having the only 1 at the $n$th place). Then by the relation $\lambda^k = \lambda^{k-1}/\sqrt{5} + \lambda^{k+1}/\sqrt{5}$ and formulas (1.8) and (1.10), we have $\tilde{l} = \tilde{L} P$. Now, considering $A = \tilde{L} P(\tilde{L})^{-1} : \mathbb{T}^2 \to \mathbb{T}^2$, we see that $A = \tilde{T} + \tilde{T}^{-1} = \left( \begin{array}{cc} 1 & 2 \\ 2 & -1 \end{array} \right)$. Since $|\det(A)| = 5$, $\tilde{l}$ is 5-to-1 and $\tilde{l} = \tilde{L} A$, we complete the proof.

So, we proved the following theorem.
Theorem 1.17. The mapping $\tilde{L}$ is a metric isomorphism of the two-sided Markov shift $(\tilde{X}, \tilde{m}, \tilde{\tau})$ and the Fibonacci automorphism $(T^2, m_2, \tilde{T})$.

Remark 1. Furthermore, $\tilde{L}$ is a group isomorphism of the groups $\tilde{X}'$ and $T^2$.

Remark 2. Actually, the mappings $\tilde{l}$ and $\tilde{L}$ lead to different interpretations of the torus as the image of $\tilde{X}$. Namely, $T^2 \cong (\mathbb{Q}(\lambda) \otimes \mathbb{R})/(\mathbb{Z} + \lambda \mathbb{Z})$ for $\tilde{l}$, while for $\tilde{L}$ we have $T^2 \cong (\mathbb{Q}(\lambda) \otimes \mathbb{R})/\sqrt{5} \mathbb{Z}$.

Now we are ready to define the two-dimensional Erdős measure.

Definition. Let the measure $\tilde{\mu}$ on the 2-torus be defined as $\tilde{\mu} = \tilde{L}(\tilde{\nu})$. We call it the two-dimensional Erdős measure.

Let us show that in a sense the two-dimensional Erdős measure is defined canonically. Note that any mapping $\Psi$ from $\tilde{X}$ onto $T^2$ such that

1. $\Psi(\tilde{\tau} \epsilon) = \tilde{T} \Psi(\epsilon)$,
2. $\Psi(\epsilon + \epsilon') = \Psi(\epsilon) + \Psi(\epsilon')$ for a.e. $\epsilon, \epsilon' \in \tilde{X}$,
3. $\Psi$ is one-to-one a.e.

is of the form

$$
\Psi(\{\epsilon_k\}_{-\infty}^{\infty}) = \tilde{l}_\xi(\{\epsilon_k\}_{-\infty}^{\infty}) = \left(\sum_{k=-\infty}^{\infty} \epsilon_k \xi \lambda^{-k} \mod 1, \sum_{k=-\infty}^{\infty} \epsilon_k \xi \lambda^{-k-1} \mod 1\right),
$$

where $\xi$ is some real number.

Remark. Dealing with the leaf of the stable foliation for $\tilde{T}$ (see the explanation after formula (1.8) above), we come to a similar family of mappings:

$$
\tilde{\kappa}_\xi(\{\epsilon_k\}_{-\infty}^{\infty}) = \left(\sum_{k=-\infty}^{\infty} \epsilon_k \xi (-\lambda)^k \mod 1, \sum_{k=-\infty}^{\infty} \epsilon_k \xi (-\lambda)^{k+1} \mod 1\right).
$$

However, it does not yields anything new, since below we will show that $\xi$ should be a quadratic irrational, and it is easy to see that for such a $\xi$, $\tilde{\kappa}_\xi = \tilde{l}_\xi$, where $\xi$ denotes the algebraic conjugate of $\xi$.

We formulate the claim which answers the question on the possible values for $\xi$ and the bijectivity of $\tilde{l}_\xi$.

Proposition 1.18.

1. For the series in formula (1.11) to converge, $\xi$ must belong to the group $G$.
2. The mapping $\tilde{l}_\xi$ is one-to-one a.e. if and only if $\xi = \pm \frac{\lambda^k}{\sqrt{5}}$ for $k \in \mathbb{Z}$.

Proof. The first claim is a consequence of the condition $\|\xi \lambda^n\| \to 0$, $n \to \infty$, which defines the group $G$. Let us prove the second one. Using the same geometric arguments and the same notation as in the proof of Proposition 1.14 and also the fact that $\xi \lambda^n = \xi(-\lambda)^{-n}$, we obtain the following sequence of mappings

$$
\tilde{X} \xrightarrow{\varphi} \Pi \xrightarrow{b_\xi} \mathbb{R}^2,
$$
where
\[ b_\xi = \begin{pmatrix} \xi & -\xi \\ \lambda^{-1} & \lambda \end{pmatrix}, \]
and \((\tilde{\ell}_\xi)(\varepsilon) = (b_\xi \varphi)(\varepsilon) \mod \mathbb{Z}^2\). The area of the set \((b_\xi \varphi)(X)\) is \(5|\xi\tilde{\xi}| = 5|N(\xi)|\) (the algebraic norm of \(\xi\)). So, the condition of the bijectivity of \(\tilde{l}\) a.e. is
\[ N(\xi) = \pm \frac{1}{5}, \]
where \(\xi \in \mathcal{G}\). We are going to use Lemma 1.15; let \(\xi = \frac{m+n\lambda}{5}\) with the extra condition
\[ (\ast) \quad 2n - m \equiv 0 \pmod{5}. \]
Then \(N(\xi) = \frac{1}{25}(m^2 + mn - n^2)\), and we come to the Diophantine equation
\[ m^2 + mn - n^2 = \pm 5 \]
together with \(\ast\). Putting \(u = n\), \(v = \frac{2m+n}{5} \in \mathbb{Z}\), we get a classical Pell’s equation
\[ u^2 - 5v^2 = \pm 4. \]
Using the usual method (see, e.g., the monograph [Lev, vol. 1, Th. 8–7]), we obtain its general solution in the form \((u, v) = (u_k, v_k)\), where
\[ u_k + \sqrt{5}v_k = \pm 2\lambda^k, \quad k \in \mathbb{Z}. \]
Returning to the variables \(m = \frac{5v-u}{2}\) and \(n = u\), we have
\[ \frac{2n - m + (2m+n)\lambda}{5} = \pm \lambda^k, \quad k \in \mathbb{Z}, \]
whence
\[ \xi = \pm \frac{\lambda^k}{\sqrt{5}}, \quad k \in \mathbb{Z}. \]
\[ \square \]
Remark 1. Note that the condition \(\ast\) proved to be satisfied automatically.

Remark 2. Thus, any value of a parameter \(\xi\) yielding the bijectivity of \(\tilde{l}_\xi\) is of the form \(\frac{1}{\sqrt{5}}\) times an arbitrary unit of the field \(\mathbb{Q}(\lambda)\).

Corollary 1.19. For any bijection \(\tilde{l}_\xi\) from \(\tilde{X}\) onto \(\mathbb{T}^2\) having the form (1.11), the image of the Erdős measure \(\tilde{\nu}\) under it will coincide with \(\tilde{\mu}\).

Proof. It suffices to use the form of any mapping of such a kind deduced in the previous proposition and the invariance of \(\tilde{\nu}\) under the shift \(\tilde{\tau}\) (Proposition 1.3) and the operation \(\varepsilon \mapsto -\varepsilon\) (Proposition 1.12).

So, the two-dimensional Erdős measure is defined canonically, and we show that the result analogous to the Erdős theorem holds for the two-dimensional Erdős measure.
Theorem 1.20. The two-dimensional Erdős measure $\tilde{\mu}$ on $\mathbb{T}^2$ is singular with respect to Lebesgue measure.

Proof. It follows from the Proposition 1.11 that $\tilde{m}$ is singular with respect to $\tilde{\nu}$, and by the bijectivity of $\tilde{L}$ a.e. with respect to both measures, we deduce the mutual singularity of their images.

Remark. Let us formulate a number of open questions about the properties of the two-dimensional Erdős measure.

(1) Is the measure $\tilde{\mu}$ Gibbs with respect to the Fibonacci automorphism for some natural potential?

(2) Is $\tilde{\mu}$ invariant under the endomorphism $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ (if it was so, the mapping $\tilde{L}(\tilde{\nu})$ would coincide with $\tilde{\mu}$)?

(3) Both questions can be reformulated in terms of the compactum $\tilde{X}$ and remain valid for it.

We conclude this item by mentioning some necessary references. The first precise symbolic coding of the hyperbolic automorphisms of the 2-torus had been proposed in [AdWe] and was developing then by a number of authors (see the references in [Ver6], [KenVer]). In [Ber] in connection with the arithmetic of PV numbers were considered two-sided expansions and the corresponding mapping semiconjugating the two-sided (in general, sofic) shift and the endomorphism of the torus with a companion matrix. Note that for the case of the golden ratio it coincides with the mapping $\tilde{L}$.

In the works [Ver3], [Ver5], [Ver6], [KenVer] an arithmetic approach to the coding of hyperbolic automorphisms of the torus has been developing. In one of the versions of such an approach which generalizes adic transformation to the two-sided case, it leads to the two-sided $\beta$-expansions, and another one being applicable to a more general algebraic numbers (not only PV) leads to a scheme of $\tilde{L}$; it uses the digits from the field of an irrationality being not always integers (see [Ver6], [KenVer]). Later this approach was developed and detailed in the dissertation [Leb].

The group $H'$ and its action on $\tilde{X}$ (see above) were considered in the recent work [FrSa] in connection with the study of certain finite automata.

More detailed analysis of bijective arithmetic codings for hyperbolic automorphisms of the 2-torus was recently given in [SiVer].

1.7. Numerical properties of the measure $\nu$. We are going first to give now an explicit formula for $\nu$.

Proposition 1.21. The following relation holds:

\[
\nu E = \begin{cases} 
\frac{2}{3} \mu E + \frac{1}{3} \mu (E + \lambda^{-2}) + \frac{1}{3} \mu (E + \lambda^{-1}), & E \subset [0, \lambda^{-2}) \\
\frac{2}{3} \mu E + \frac{1}{3} \mu (E + \lambda^{-2}), & E \subset [\lambda^{-2}, \lambda^{-1}) \\
\frac{1}{2} \mu E + \frac{1}{3} \mu (E - \lambda^{-1}), & E \subset [\lambda^{-1}, 1]. 
\end{cases}
\]

Proof. Let the measure $\nu'$ be defined by formula (1.12). We need to show that $\nu' = \nu$. Note first that the $T$-invariance of $\nu'$ is checked directly using Lemma 1.1.
Let, say, $E \subset (0, \lambda^{-2})$. Then $T^{-1}E = \lambda^{-1}E \cup (\lambda^{-1}E + \lambda^{-1})$. Hence

$$
\nu'(T^{-1}E) = \nu'(\lambda^{-1}E) + \nu'(\lambda^{-1}E + \lambda^{-1})
$$

$$
= \frac{2}{3}\mu(\lambda^{-1}E) + \frac{1}{3}\mu(\lambda^{-1}E + \lambda^{-2}) + \frac{1}{6}\mu(\lambda^{-1}E + \lambda^{-1}) + \frac{1}{2}\mu(\lambda^{-1}E + \lambda^{-1})
$$

$$
+ \frac{1}{3}\mu(\lambda^{-1}E)
$$

$$
= \frac{1}{2}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1}) + \frac{1}{6}\mu E
$$

(by Lemma 1.1)

$$
= \frac{2}{3}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1})
$$

$$
= \nu'(E).
$$

The cases $E \subset (\lambda^{-2}, \lambda^{-1})$ and $E \subset (\lambda^{-1}, 1)$ are studied in the same way.

To prove now that $\nu' = \nu$, we observe that since $\mu$ is quasi-invariant with respect to $T$ and the rotations by $\lambda^{-1}$ and $\lambda^{-2}$, the measure $\nu'$ is also ergodic with respect to $T$ (see (1.7) and (1.12)). Since $\mu \approx \nu$ and $\mu < \nu'$, we have $\nu < \nu'$, and by the corollary of the ergodic theorem, $\nu' = \nu$. □

**Corollary 1.22.** $\nu(0, \lambda^{-2}) = \frac{4}{9}$, $\nu(\lambda^{-2}, \lambda^{-1}) = \nu(\lambda^{-1}, 1) = \frac{5}{18}$.

**Proposition 1.23.** The following relation holds:

$$
\nu = \lim_n T^n \mu.
$$

**Proof.** The sketch of the proof is as follows. Let, for example, $E \subset (0, \lambda^{-2})$. Considering successively the sets $T^{-n}E$, $n \geq 1$ and using Lemma 1.1, we deduce similarly to relation (1.7) that $\mu(T^{-2}E) = \frac{2}{3}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{4}\mu(E + \lambda^{-1})$, $\mu(T^{-3}E) = \frac{2}{3}\mu E + \frac{2}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1})$, etc., whence by induction,

$$
\mu(T^{-n}E) = \left(\frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n}{2^n}\right)\mu E + \left(\frac{1}{2} - \frac{1}{4} + \cdots + \frac{(-1)^{n+1}}{2^n}\right)\mu(E + \lambda^{-2}) + \left(\frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n}{2^n}\right)\mu(E + \lambda^{-1}) = \frac{2}{3}\mu E + \frac{1}{3}\mu(E + \lambda^{-2}) + \frac{1}{6}\mu(E + \lambda^{-1}) + O(2^{-n}) = \nu E + O(2^{-n})
$$

by formula (1.12). The cases $E \subset (\lambda^{-2}, \lambda^{-1})$ and $E \subset (\lambda^{-1}, 1)$ are considered in the same way.

**Remark.** It is appropriate, following the well-known framework of the baker’s transformation which serves as a model for the full two-sided shift on $\Sigma$, to represent the two-sided shift on $\tilde{X}$ as the Fibonacci-baker’s transformation.

Namely, we split a sequence $\{\varepsilon_k\} \in \tilde{X}$ into the two one-sided sequences, i.e. into $\varepsilon_1\varepsilon_2\ldots \in X$ and $\varepsilon_0\varepsilon_{-1}\ldots \in X$ with regard to the fact that $\varepsilon_0\varepsilon_{-1} = 0$. This last condition leads to the space $Y = ([0,1] \setminus [0,1]) \cup ([\lambda^{-1},1] \times [\lambda^{-1},1])$ similar to the set $\Pi$ described in the proof of Proposition 1.14.
Fig. 1. The natural domain for the Fibonacci-baker’s transformation

Thus, the shift $\tilde{\tau}$ on the two-sided Markov compactum $\tilde{X}$ is isomorphic to the transformation $F$ on the space $Y$ with

$$F(x, y) = \begin{cases} 
(\lambda x, \lambda^{-1} y), & x \in [0, \lambda^{-1}] \\
(\lambda x - 1, \lambda^{-1} y + \lambda^{-1}), & x \in (\lambda^{-1}, 1].
\end{cases}$$

We call $F$ the Fibonacci-baker’s transformation on the set $Y$ (see Fig. 1). For more general models this transformation was considered in [DaKrSo].

2. Symbolic dynamics of expansions

In this section we will study in detail the combinatorics of all possible representations of a real $x$ of the form (1.1) with $\varepsilon_k \in \{0, 1\}$ for all $k$.

2.1. Blocks. Let us give an important technical definition.

**Definition.** A finite 0-1 sequence without pairs of adjacent 1’s starting from 1 and ending by an even number of zeroes will be called a block if it does not contain any piece “1(00)$^l$1” with $l \geq 1$.

Let us make some remarks. Note first that each block has odd length; the simplest example of a block is “100”. Next, there are exactly $2^{n-1}$ blocks of length $2n + 1$. This assertion follows from the fact that a block $B$ can be represented in the form $1(00)^{a_1}(01)^{a_2}(00)^{a_3} \ldots (01)^{a_{t-1}}(00)^{a_t}$ for $t$ odd or $1(01)^{a_1}(00)^{a_2} \ldots (01)^{a_{t-1}}(00)^{a_t}$ for $t$ even. Thus, any block $B$ is naturally parametrized by means of a finite sequence of positive integers $a_1, \ldots, a_t$, and we will write $B = B(a_1, \ldots, a_t)$.

Let $I_0 := [\lambda^{-1}, 1)$, i.e. the interval corresponding to the cylinder $(\varepsilon_1 = 1) \subset X$.

**Definition.** Let $x$ lie in the interval $I_0$, and let the canonical expansion of $x$ have infinitely many pieces “1(00)$^l$1” with $l \geq 1$. We call such a point $x$ regular.

Almost every point $x$ in $I_0$ with respect to Lebesgue measure is regular. Now we split the canonical expansion of a regular $x$ into blocks as follows. Since $x \in I_0$, its canonical expansion starts with 1. It is just the beginning of the first block $B_1 = B_1(x)$. The first block ends, when an even number of zeroes followed by 1 appears for the first time. This 1 begins the second block $B_2 = B_2(x)$ of the canonical expansion of $x$, etc. We define thus a one-to-one mapping $\Psi$ acting from the set of all regular points of $(\lambda^{-1}, 1)$ into the space of block sequences.
Definition. The sequence \((B_1(x), B_2(x), \ldots) = \Psi(x)\) will be called the block expansion of a regular \(x\).

2.2. The cardinality of a 0-1 sequence and its properties. We are going to define an equivalence relation on the set of all finite 0-1 sequences.

Definition. Two 0-1 sequences (finite or not) \((x_1 x_2 \ldots)\) and \((x'_1 x'_2 \ldots)\) are called equivalent if \(\sum_k x_k \lambda^{-k} = \sum_k x'_k \lambda^{-k}\) (or, equivalently, if their normalizations coincide — see Section 1). Let for a finite 0-1 sequence \(x\), \(\mathcal{E}(x)\) denote the set of all 0-1 sequences equivalent to \(x\); this set is always finite. Let \(f(x) := \#\mathcal{E}(x)\). We call \(f\) the cardinality of a finite sequence (or the cardinality of an equivalence class).

Note that this function (of positive integers) was considered in [Car], [AlZa] and recently in [DuSiTh] and [Pu].

The assertions below answer the question about the cardinality of a block and explain the purpose of the introduction of blocks as natural structural units in this theory.

Lemma 2.1. Let a finite word \(x\) be a block, i.e. \(x = 1(00)^{a_1}(01)^{a_2}(00)^{a_3} \ldots (01)^{a_{t-1}}(00)^{a_t}\) or \(1(01)^{a_1}(00)^{a_2} \ldots (01)^{a_{t-1}}(00)^{a_t}\). Let \(p/q = [a_1, \ldots, a_t]\) be a finite continued fraction. Then

\[f(x) = p + q.\]

Proof. Note first that \(f(100) = 2 = p + q\). The desired relations for the blocks 10000 and 10100 follow by direct inspection. Next, let \(\varkappa = \varkappa(a_1, \ldots, a_k) = f(x)\). We need to show that similarly to the numerators and denominators of the convergents, \(\varkappa_t = \varkappa_t \varkappa_{t-1} + \varkappa_{t-2}\), whence the required assertion will follow.

Let, say, \(t\) be odd and \(B = 1(00)^{a_1} \ldots (00)^{a_{t-2}}(01)^{a_{t-1}}(00)^{a_t}\). We first consider all 0-1 sequences equivalent to \(B\) and ending by \((00)^{a_t}\). Their number is obviously \(\varkappa_{t-2}\), as they in fact should end by \((01)^{a_{t-1}}(00)^{a_t}\). Now we consider 0-1 sequences equivalent to \(B\) but not ending by \((00)^{a_t}\) and will show that their number is \(a_t \varkappa_{t-1}\).

Namely, let

\[
\begin{align*}
B^{(1)} &= 1(00)^{a_1} \ldots (00)^{a_{t-2}}(01)^{a_{t-1}}10011(00)^{a_t-1}, \\
B^{(2)} &= 1(00)^{a_1} \ldots (00)^{a_{t-2}}(01)^{a_{t-1}}0121(00)^{a_t-2}, \\
&\quad \vdots \\
B^{(a_t)} &= 1(00)^{a_1} \ldots (00)^{a_{t-2}}(01)^{a_{t-1}}10(01)^{a_t}.
\end{align*}
\]

It is clear that any \(B^{(j)}\) is equivalent to \(B\); now we observe that the number of 0-1 words equivalent to \(B\) and ending by \((00)^{a_t-j}\), \(1 \leq j \leq a_t\), is exactly \(\varkappa_{t-1}\), as the replaceable part of \(B^{(j)}\) with the fixed end \((00)^{a_t-j}\) is in fact \(1(00)^{a_1} \ldots (00)^{a_{t-j}}(01)^{a_{t-1}}100\), hence \(\varkappa_t = a_t \varkappa_{t-1} + \varkappa_{t-2}\), as \([a_1, \ldots, a_{t-1} - 1, 1] = [a_1, \ldots, a_{t-1}]\). The case of even \(t\) is treated in the same way. \(\Box\)

Remark. To any rational \(r \in (0, 1)\) exactly two blocks correspond, namely with \(r = [a_1, \ldots, a_t] = [a_1, \ldots, a_{t-1}, a_t - 1, 1]\), and the unique block “100” corresponds to \(r = 1\).

If \(E_1\) and \(E_2\) are two sets of sequences, then henceforward \(E = E_1 E_2\) is the concatenation of these two sets, i.e. any sequence in \(E\) begins with a word from \(E_1\) and ends with a word from \(E_2\).
Lemma 2.2. For any blocks \( B_1, \ldots, B_k \),

1. \( \mathcal{E}(B_1 \ldots B_k) = \mathcal{E}(B_1) \ldots \mathcal{E}(B_k) \).
2. The cardinality is blockwise multiplicative, i.e.

\[
f(B_1 \ldots B_k) = \prod_{i=1}^{k} f(B_i).
\]

Proof. It suffices to prove item (1). Let us restrict ourselves to the case \( k = 2 \) (the general case is studied in the same way). We will see that there is no sequence in \( \mathcal{E}(B_1 B_2) \) containing a triple 011 or 100 which crosses the “border” between the first \( |B_1| \) digits and the last \( |B_2| \). In other words, we will show that any sequence equivalent to \( B_1 B_2 \) can be constructed as the concatenation of a sequence equivalent to \( B_1 \) and a sequence equivalent to \( B_2 \).

Let “|” below denote the border in question. First, any sequence from \( \mathcal{E}(B_2) \) must begin either with 10 or with 01, hence, the situation \( (1|00) \) or \( (0|11) \) is impossible. Next, a sequence in \( \mathcal{E}(B_1) \) ends in either 00 or 11 (see the proof of the previous lemma), neither leading to \( (10|0) \) or \( (01|1) \).

This simple result shows that the space of all equivalent infinite 0-1 sequences for a given regular \( x \) splits into the direct infinite product of spaces, the \( k \)’th space consisting of all finite sequences equivalent to the block \( B_k(x) \). So, we see that the notion of block, initially arising in terms of the canonical expansion, can be naturally extended to all representations. Below we will explain the geometric sense of a block in terms of the Fibonacci graph (see p. 3.1).

Remark. Note that this block partition appeared for the first time in [Pu] in other terms and for algebraic and combinatorial purposes. Namely, let the partial ordering on a space \( \mathcal{E}(x) \) for some finite word \( x \) be defined as follows. We set \( x \prec x' \) if there exists \( k \geq 2 \) such that \( x_{k-1} = 0, x_k = 1, x_{k+1} = 1, x'_{k-1} = 1, x'_k = 0, x'_{k+1} = 0, \) and \( x_j = x'_j, |k-j| \geq 2 \). Next, one extends this ordering by transitivity. It was shown in [Pu] that any equivalence class has the structure of a distributive lattice in the sense of this order.

2.3. Goldenshift. We are going to give one of the central definitions of the present paper.

Definition. The transformation \( S \) acting from the set of regular points of the interval \((\lambda^{-1}, 1)\) into itself by the formula

\[
Sx = \tau^{n(x)} x, \quad x \text{ is regular},
\]

where \( n(x) \) is the length of the first block of the block expansion of \( x \), is called the goldenshift.

Remark 1. The transformation \( S \) is piecewise linear. More precisely, if \((\lambda^{-1}, 1) = \bigcup_r \Delta_r \) is the partition of \((\lambda^{-1}, 1) \mod 0 \) into intervals corresponding to that of \( \mathcal{B} \) into the states of the first block, then \( S \) is linear inside \( \Delta_r =: [\alpha_r, \beta_r] \), and \( S(\alpha_r) = \lambda^{-1}, S(\beta_r) = 1 \).

Remark 2. The transformation \( S \) is a generalized power of \( \tau \) in the sense of Dye (see, e.g., [Bel]). In other words, the goldenshift is a random power of the ordinary shift.
as the number of shifted coordinates depends on the length of the first block. Note that the goldenshift is not an induced endomorphism for \( \tau \) but for the two-sided case it is (see Proposition 2.3 and Theorem 2.12 below).

**Remark 3.** Let \( \mathfrak{X} \) denote the space of block sequences. The goldenshift may be treated as the one-sided shift in the space \( \mathfrak{X} \), i.e. \( S(B_1B_2B_3\ldots) = (B_2B_3\ldots) \).

Now we are going to define the two-sided goldenshift. Let, as above, \( \tilde{\tau} \) denote the two-sided shift on \( \tilde{\mathfrak{X}} \). In order to define the two-sided goldenshift on the Markov compactum \( \tilde{\mathfrak{X}} \), we give the following definition.

**Definition.** The \( \tilde{\tau} \)-invariant set \( \tilde{\mathfrak{X}}_{\text{reg}} \subset \tilde{\mathfrak{X}} \) is defined as the set consisting of all sequences containing pieces "10\( l \)1" with \( l \geq 1 \) infinitely many times both to the left and to the right with respect to the first coordinate.

Obviously, \( \tilde{\nu}(\tilde{\mathfrak{X}}_{\text{reg}}) = 1 \), as by Proposition 1.4, the measure \( \tilde{\nu} \) of any cylinder in \( \tilde{\mathfrak{X}} \) is positive, and by Proposition 1.5, the automorphism \( (\tilde{\mathfrak{X}}, \tilde{\nu}, \tilde{\tau}) \) is ergodic, so, it suffices to apply the ergodic theorem. Let \( \tilde{X}_0 := \bigcup_{k=1}^{\infty} (x_{-2k} = 1, x_{-2k+1} = \cdots = x_0 = 0, x_1 = 1) \), and let \( \tilde{\mathfrak{X}}_{\text{reg}} = \tilde{\mathfrak{X}}_{\text{reg}} \cap \tilde{X}_0 \).

**Definition.** The two-sided goldenshift \( \tilde{S} : \tilde{\mathfrak{X}}_{\text{reg}} \to \tilde{\mathfrak{X}}_{\text{reg}} \) on the Markov compactum \( \tilde{\mathfrak{X}} \) is, by definition, the shift by the length of \( B_1 \).

**Remark.** Considering the space of two-sided block sequences \( \mathfrak{x} = \prod_{-\infty}^{\infty} \mathfrak{B} \) and implying that \( B_1 \) begins with the first coordinate of \( \tilde{\mathfrak{X}} \) (i.e. with \( \varepsilon_1 = 1 \)), we see that the two-sided goldenshift \( \tilde{S} \) is the shift in the space \( \tilde{\mathfrak{X}} \).

**Proposition 2.3.** The two-sided shift \( \tilde{\tau} \) on the set \( \tilde{\mathfrak{X}}_{\text{reg}} \) is a special automorphism over the goldenshift \( \tilde{S} \) on the set \( \tilde{\mathfrak{X}}_{\text{reg}} \). The number of steps over a sequence \( (\varepsilon_k) \in \tilde{\mathfrak{X}}_{\text{reg}} \) is equal to the length of the block beginning with \( \varepsilon_1 = 1 \).

**Proof.** It suffices to present the steps of the corresponding tower. Let, by definition, \( \tilde{\mathfrak{X}}_{\text{reg}} = \tilde{\tau} \tilde{X}_0 \), and \( \tilde{\mathfrak{X}}_{\text{reg}} = \tilde{\tau} \tilde{X}_{2j-1} \), \( \tilde{\mathfrak{X}}_{\text{reg}} = \tilde{\tau} \tilde{X}_{2j} \setminus (\varepsilon_1 = 1) \), \( j \geq 1 \). This completes the proof, as \( \tilde{\mathfrak{X}}_{\text{reg}} = \bigcup_{j=0}^{\infty} \tilde{X}_{\text{reg}}^j \), the union being disjoint.

![Fig. 2. The steps of the special automorphism \( \tilde{\tau} \)](image)

Below the corresponding result will be established for the metric case with the two-sided Erdös measure.
2.4. Bernoullicity of the goldenshift. In this subsection we will show that the goldenshift (one-sided or two-sided) is a Bernoulli shift in the space $\mathcal{X}$ (respectively $\tilde{\mathcal{X}}$) with respect both to Lebesgue and Erdős measures and compute their one-dimensional distributions.

We recall that above we denoted the set of blocks by $\mathcal{B}$; let the same letter stand for the totality of cylinder sets $\{B_1 = B\} \subset \mathcal{X}$ for all blocks $B$. Let $\mathcal{B}_n$ denote all cylinder sets $\{B_1 = B\} \subset \mathcal{X}$ such that the length of $B$ is $2n + 1$. Thus, $\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{B}_n$.

Recall that the mapping $\Psi$ assigns to each regular $x$ its block expansion, i.e. a certain sequence in the space $\mathcal{X}$. We denote by $m$ the normalized Lebesgue measure on the interval $(\lambda^{-1}, 1)$; let $m_\mathcal{X}$ stand for the measure $\Psi(m)$ in the space $\mathcal{X}$, and, similarly, let $\mu_\mathcal{X}$ denote $\Psi(\mu_0)$ (this measure is well defined, as $\mu$-a.e. point $x \in (\lambda^{-1}, 1)$ is also regular).

We recall that to any block we associated the interval $\Delta_r$ defined as the image of the cylinder “$B(r)1$" in $X$ by the mapping (1.1) (see Remark 1 after the definition of the goldenshift).

We are going to show that the measures $m_\mathcal{X}$ and $\mu_\mathcal{X}$ are Bernoulli in the space $\mathcal{X}$ and to compute their one-dimensional distributions.

**Theorem 2.4.** The measure $m_\mathcal{X}$ in the space $\mathcal{X}$ is a product measure with equal multipliers, i.e. a Bernoulli measure.

**Proof.** By the linearity of $S$ on each interval $\Delta_r$ and the fact that $S(\Delta_r) = (\lambda^{-1}, 1)$, we have for any Borel set $E \subset (\lambda^{-1}, 1)$,

$$m(S^{-1}E \cap \Delta_r) = mE \cdot m\Delta_r,$$

whence the required assertion immediately follows by virtue of the obvious $S$-invariance of $m$, and by setting $E = \Delta_{r'}$ in relation (2.1) for any $r'$, which yields the $m_\mathcal{X}$-independence of the first and the second blocks.

So, it remains to compute the one-dimensional distribution of $m_\mathcal{X}$.

**Corollary 2.5.** For any cylinder set $\{B_1 = B\} \subset \mathcal{B}_n$ its measure $m_\mathcal{X}$ equals $\lambda^{-2n-1}$. The measure $m_\mathcal{X}$ of $\mathcal{B}_n$ is equal to $\frac{1}{2^n} (\frac{2}{\lambda})^n$.

**Proposition 2.6.** The measure $\mu_\mathcal{X}$ is a product measure on $\mathcal{X}$ with equal multipliers.

**Proof.** It suffices to establish a relation similar to (2.1) for the measure $\mu_\mathcal{X}$ and for any finite block sequence $E = B_1 \ldots B_k$. Note first that by virtue of Lemma 2.2, $\mathcal{E}(B_1 \ldots B_k) = \mathcal{E}(B_1) \ldots \mathcal{E}(B_k)$ for any blocks $B_1, \ldots, B_k$. Next,

$$n^{-1}(B_1 \ldots B_k1) = \mathcal{E}(B_1) \ldots \mathcal{E}(B_k)n^{-1}(\varepsilon_1 = 1).$$

We are going to show that

$$\mu_\mathcal{X}\{B_1 \ldots B_k\} = \mu_\mathcal{X}\{B_1\} \ldots \mu_\mathcal{X}\{B_k\} = \frac{f(B_1)}{2^{|B_1|}} \cdots \frac{f(B_k)}{2^{|B_k|}}, \ k \geq 1.$$
To do this, we use previous remarks and the definition of the Erdős measure on \( X \) by means of the normalization (see Section 1). We have \( \mu_X \{ B_1 \ldots B_k \} = \mu(B_1 \ldots B_k) / \mu(\varepsilon_1 = 1) \), and

\[
\mu(B_1 \ldots B_k) = p(n^{-1}(B_1 \ldots B_k)) = p(\mathcal{E}(B_1) \ldots \mathcal{E}(B_k)n^{-1}(\varepsilon_1 = 1))
\]

\[
= \frac{f(B_1)}{2^{|B_1|}} \cdot \cdots \cdot \frac{f(B_k)}{2^{|B_k|}} \cdot \mu(\varepsilon_1 = 1)
\]

(by Lemma 2.2), whence the required assertion follows.

Thus, we have proved one of the main results of the present paper.

**Theorem 2.7.** The goldenshift \( S \) is a Bernoulli automorphism with respect to Lebesgue and Erdős measures.

Now we are ready to give the second proof of Erdős theorem (see Section 1).

**Corollary 2.8.** (a new proof of Erdős theorem) The Erdős measure is singular with respect to Lebesgue measure.

**Proof.** In fact, by the corollary of the ergodic theorem, the measures involved are mutually singular on the interval \((\lambda^{-1}, 1)\) (recall that \( \mu(0, \lambda^{-2}) = \frac{1}{3} \), whence the measure \( \mu \) differs from the Lebesgue measure). This yields the assertion of the corollary, as any infinite convolution of discrete measures, if it is not stochastically constant, is known to be either singular or absolutely continuous with respect to Lebesgue measure (the “Law of Pure Types”, see [JeWi]).

The one-dimensional distribution of \( \mu_X \) is a bit more sophisticated than for \( m_X \). It is described as follows (see formula (2.2)):

**Proposition 2.9.** For a block \( B = B(a_1, \ldots, a_t) \) of length \( 2n + 1 \),

\[
\mu_X \{ B_1 = B \} = \frac{f(B)}{2^{|B|}} = \frac{p + q}{2^{2n+1}},
\]

where, as usual, \( p/q = [a_1, \ldots, a_t] \). The measure \( \mu_X \) of the set \( \mathcal{B}_n \) equals \( \frac{1}{2} \cdot (\frac{3}{2})^n \).

**2.5. Concluding remarks on the Erdős measure.** We conclude the study of ergodic properties of Erdős measures (one-sided and two-sided) and the transformations of shift and goldenshift.

Recall that the Erdős measure \( \mu \) is quasi-invariant under the one-sided shift \( \tau \), and the equivalent measure \( \nu \) is \( \tau \)-invariant. It is worthwhile to know the behavior of \( \nu \) with respect to the goldenshift \( S \).

Let \( \nu_X \) be defined in the same way as \( \mu_X \). We formulate the following claim (for more details see Appendix C).

**Proposition 2.10.** The measure \( \nu_X \) on the space \( X \) of block sequences is quasi-invariant under the goldenshift \( S \). More precisely, any two cylinders \( \{ B_j = B'_j \} \) and \( \{ B_i = B'_i \} \) with \( i \neq j \) are \( \nu_X \)-independent, and for \( B = B(a_1, \ldots, a_t) \),

\[
\nu_X \{ B_k = B \} = \mu_X \{ B_k = B \} = \frac{f(B)}{2^{|B|}} = \frac{p + q}{2 \cdot 4^{a_1 + \cdots + a_t}}, \quad k \geq 2,
\]

\[
\nu_X \{ B_1 = B \} = \left\{ \begin{array}{ll}
\frac{p + q}{2 \cdot 4^{a_1 + \cdots + a_t}}, & B = 100 \ldots \\
\frac{p + q}{2 \cdot 4^{a_1 + \cdots + a_t}}, & B = 101 \ldots
\end{array} \right.
\]

Now we will prove a numerical claim useful for the next section.
Corollary 2.11. $\tilde{\nu}\tilde{X}_0 = \frac{1}{9}$.

Proof. We have by the definition of the set $\tilde{X}_0$, Proposition 2.10 and the fact that $\nu(\varepsilon_1 = 1) = \frac{5}{18}$ (see Corollary 1.22),

$$\tilde{\nu}\tilde{X}_0 = \sum_{k=1}^{\infty} \tilde{\nu}(1(00)^k1) = \frac{5}{18} \sum_{k=1}^{\infty} \frac{4 + \frac{6}{k}}{1 + k} \cdot \frac{k + 1}{2^{2k+1}} = \frac{1}{9}.$$  

Finally, we prove a metric version of Proposition 2.3.

Theorem 2.12. The two-sided shift $\tilde{\tau}$ with the measure $\tilde{\nu}$ is a special automorphism over the goldenshift $\tilde{S}$ with the measure $\tilde{\mu}$ on the space $\tilde{X}_0$. The step function is defined as the length of the block beginning with the first coordinate.

Proof. It suffices to show that the lifting measure for $\tilde{\mu}$ coincides with $\tilde{\nu}$. This in turn is implied again by the corollary of the ergodic theorem applied to $\tilde{\tau}$ that preserves both measures which are ergodic. Since they are clearly equivalent, we are done.

3. The entropy of the goldenshift and applications

In this section we will establish a relationship between the entropy of the Erdős measure in the sense of A. Garsia and the entropy of the goldenshift with respect to $\mu$, i.e. between two different entropies. As an application, we will reprove the formula for Garsia’s entropy proved in [AlZa]. Further, we use random walk theory to compute the dimension of the Erdős measure on the interval.

3.1. Fibonacci graph, random walk on it and Garsia’s entropy.

The combinatorics of equivalent 0-1 sequences may be expressed graphically, namely by means of the Fibonacci graph introduced in [AlZa]. Let, as in Section 1, $\Sigma = \prod_{1}^{\infty} \{0; 1\}$, and let the mapping $\pi : \Sigma \to [0, 1]$ be defined as

$$\pi(\varepsilon_1, \varepsilon_2, \ldots) = \sum_{k=1}^{\infty} \varepsilon_k \lambda^{-k-1}.$$  

Since the $\varepsilon_n$ assume the values 0 and 1 without any restrictions, a typical $x$ will have a continuum number of representations, and they all may be illustrated with the help of the Fibonacci graph depicted in Fig. 3. This figure appeared for the first time in the work due to J. C. Alexander and D. Zagier [AlZa]. Let us give the precise definition.

Definition. The Fibonacci graph $\Phi$ is a binary graph with the edges labeled with 0 if to the left and 1 if to the right. Any vertex at the $n$’th level corresponds to a certain $x$, for which some representation (3.1) is finite with the length $n$ (obviously, in this case $x = \{N\lambda\}$ for some $N \in \mathbb{Z}$). The paths are 0-1 sequences treated as representations of the form (3.1).³

³The term “Fibonacci graph” is overloaded, as the authors know several different graphs also called “Fibonacci”. Nevertheless, we hope that there will be no confusion with any of them.
Remark. The vertices of the $n$'th level of the graph $\Phi$ can be treated as the nonnegative integers from 0 to $F_{n+2} - 2$. Namely, if a path $(\varepsilon_1 \varepsilon_2 \ldots \varepsilon_n)$ goes to a vertex $k$, then, by definition, $k = \sum_1^n \varepsilon_j F_{n-j}$ (obviously, this sum does not depend on the choice of a path).

Let $Y(\Phi)$ denote the set of paths in the graph $\Phi$. Obviously, $Y(\Phi)$ is naturally isomorphic to $\Sigma$, and sometimes we will not make a distinction between them. Let $(\varepsilon_1 \varepsilon_2 \ldots)$ be a path, and let the projection from $Y(\Phi)$ onto $[0,1]$ be also denoted by $\pi$ (see formula (3.1)).

Let, as above, $f_n(k)$ denote the number of representations of a nonnegative integer $k$ as a sum of not more than $n$ first Fibonacci numbers. It is easy to see that $f_n(k)$ is also the frequency of the vertex $k$ on the $n$'th level of the graph $\Phi$. Let $D_n = \{k : k = \sum_{k=1}^n \varepsilon_k F_{n-k}, \varepsilon_k \in \{0,1\}\}$ (or, equivalently, the $n$'th level of the Fibonacci graph), and $D'_n = \{w : w = \sum_{k=1}^n \varepsilon_k \lambda^{-k-1}, \varepsilon_k \in \{0,1\}\}$. These sets are clearly isomorphic ($w \leftrightarrow k$), and $\#D_n = \#D'_n = F_{n+2} - 1$. The use of $D_n$ instead of $D'_n$ is due only to technical reasons. We recall that the sequence of distributions $(2^{-n}f_n(w))_{n=1}^\infty$ tends to the distribution of Erdős measure (see Section 1).

We have proved in the previous section that block is an object defined on the space of almost all 0-1 sequences (not only admissible), i.e. on the Fibonacci graph. Let us now explain the geometric treatment of the block expansion.

Note first that an odd level $2n + 1$ contains $2^{n-1}$ specific vertices which we will call, following [AlZa], the Euclidean vertices. They are defined recursively. The central vertex on the third level is Euclidean; then, any Euclidean vertex generates exactly two new Euclidean vertices on the next odd level by means of the arcs 00 and 11. We call the set of all Euclidean vertices the Euclidean tree.

It is evident that for a given path in the Fibonacci graph its first block can end only at some Euclidean vertex. To find out, if it does end at a vertex $v$, we consider the induced Fibonacci graph with $v$ as the top and also the induced Euclidean tree. The criterion in question is that a given path should go through some induced

\footnote{Thus, relation (1.2) completely determines the whole graph $\Phi$.}
Euclidean vertex earlier than through any initial one. Next, considering the induced Fibonacci graph, we can find the second block of the block expansion, etc.

We denote the entropy of the discrete distribution on $D'_n$ described above, by $H^{(n)}$. Thus,

$$H^{(n)} = - \sum_{k=0}^{F_{n+2} - 2} \frac{f_n(k)}{2^n} \log \frac{f_n(k)}{2^n}.$$ 

Then, by definition,

$$H_\mu := \lim_{n \to \infty} \frac{H^{(n)}}{n \log \lambda}$$

(this limit is known to exist and is obviously independent of the choice of the base of logarithms, see [Ga]). Now we choose once and for all $\lambda$ as the base of logarithms.

The quantity $H_\mu$ can be considered as the entropy of the random walk on the Fibonacci graph with the probabilities $(\frac{1}{2}, \frac{1}{2})$. In the next item it will be shown that in fact $H_\mu$ is proportional to the entropy of the goldenshift. The Erdős measure is the projection of the Markov measure $(\frac{1}{2}, \frac{1}{2})$ on the graph $\Phi$ under the mapping $\pi$. We consider the random walk on the Fibonacci graph with the equal transition measures.

**Definition.** The *Fibonacci semigroup* (resp. *group*) is by definition, the semigroup (resp. group) with the generators $a, b$ and the relation $ab^2 = ba^2$.

The following claims are straightforward.

**Proposition 3.1.** The Fibonacci graph is the Cayley graph of the Fibonacci semigroup.

In [Av] (see also [KaVer]) was introduced the notion of the entropy of a random walk on a finitely generated group (or semigroup). The following claim establishes a relation between the two notions of entropy. Note that the Fibonacci semigroup can be naturally embedded into the Fibonacci group, that is why we can use the theory of random walks on groups.

**Proposition 3.2.** The entropy $H_\mu$ is equal to the entropy of the random walk on the Fibonacci semigroup with the probabilities $(\frac{1}{2}, \frac{1}{2})$.

**Proof.** Follows from the definitions and Proposition 3.1.

**3.2. Main theorem.** We prove an assertion that is one of the central points of the present paper. Note that in [AlZa] Garsia’s entropy was computed by means of generating functions. We will see that $H_\mu$ is closely connected with the entropy of the goldenshift, which gives a new and simplified proof of their relation and relates it to the dynamics of the Erdős measure.

**Theorem 3.3.** The following relation holds:

$$h_\mu(S) = 9H_\mu.$$ 

**Proof.** We are going to apply Abramov’s formula for the entropy of the special automorphism (see [Ab]) to the dynamical systems $(\tilde{X}, \tilde{\nu}, \tilde{\tau})$ and $(\tilde{X}_0, \tilde{\mu}, \tilde{S})$. By Abramov’s formula and Theorem 2.12,

$$h_\mu(\tilde{S}) = \frac{1}{9} h_{\tilde{\nu}}(\tilde{\tau}).$$
From this relation we will deduce the required one.
1. Since the dynamical system \((\tilde{X}, \tilde{\mu}, \tilde{S})\) is the natural extension of \((X, \mu, S)\), we have \(h_{\tilde{\mu}}(\tilde{S}) = h_\mu(S)\) and by the same reason, \(h_{\tilde{\nu}}(\tilde{\tau}) = h_\nu(\tau)\).
2. By Corollary 2.11, \(\tilde{\nu}\tilde{X}_0 = \frac{1}{\theta}\).
3. The rest of the proof is devoted to establishing the validity of the relation

\[
h_\nu(\tau) = H_\mu.
\]

Let \(\eta\) denote the partition of \(X\) into the cylinders \((\varepsilon_1 = 0)\) and \((\varepsilon_1 = 1)\). Since \(\eta\) is a generating partition for \(\tilde{\tau}\), we have \(h_\nu(\tau) = h_\nu(\tau, \eta)\) by Kolmogorov’s theorem. By definition,

\[
h_\nu(\tau, \eta) = \lim_{n \to \infty} \frac{1}{n} H_\nu(\eta^{(n)}),
\]

where \(\eta^{(n)}\) is the partition of \(X\) into \(F_{n+1}\) admissible cylinders of the form \((\varepsilon_1 = i_1, \ldots, \varepsilon_n = i_n)\). We need to prove that

\[
H_\nu(\eta^{(n)}) \sim H^{(n)}.
\]

By virtue of the equivalence of the measures \(\mu\) and \(\nu\) it suffices to show this for \(H_\mu(\eta^{(n)})\) instead of \(H_\nu(\eta^{(n)})\). Let \(\theta_n(k) = 2^{-n} f_n(k)\). Then, by relation (1.2), for \(n \geq 3\),

\[
\theta_n(k) = \begin{cases} 
\frac{1}{2} \theta_{n-1}(k), & 0 \leq k \leq F_n - 1 \\
\frac{1}{2} (\theta_{n-1}(k) + \theta_{n-1}(k - F_n)), & F_n \leq k \leq F_{n+1} - 2 \\
\frac{1}{2} \theta_{n-1}(k - F_n), & F_{n+1} - 1 \leq k \leq F_{n+2} - 2. 
\end{cases}
\]

We are going to obtain almost the same recurrence relation for the distribution \(\eta^{(n)}\). To do this, we return to the interval \([0, 1]\) and denote by \(\eta^{(n)}\) the partition into \(F_{n+1}\) intervals which is the image of the corresponding partition of \(X\) with the help of the canonical expansion. So, let \(\eta^{(n+1)} = (J_n(k))_{k=0}^{F_{n+2} - 1}\) with the ordered intervals \(J_n(k)\). Finally, let \(\mu_n(k) := \mu J_n(k)\). Then by Lemma 1.1, for \(n \geq 3\),

\[
\mu_n(k) = \begin{cases} 
\frac{1}{2} \mu_{n-1}(k), & 0 \leq k \leq F_n - 1 \\
\frac{1}{2} (\mu_{n-1}(k) + \mu_{n-1}(k - F_n)), & F_n \leq k \leq F_{n+1} - 1 \\
\frac{1}{2} \mu_{n-1}(k - F_n), & F_{n+1} \leq k \leq F_{n+2} - 3. 
\end{cases}
\]

Also, \(\mu_n(F_{n+2} - 2) = O(2^{-n})\), \(\mu_n(F_{n+2} - 1) = O(2^{-n})\). Thus, by induction on \(n\), there exists \(C > 0\) such that

\[
\frac{1}{C} \leq \frac{\theta_n(k)}{\mu_n(k)} \leq C, \quad 0 \leq k \leq F_{n+2} - 2,
\]

Since the measure \(\nu\) is \(\tau\)-invariant and ergodic, we can apply the Shannon-McMillan-Breiman theorem and deduce that the entropies of the distributions \(\mu_n(k)\) and \(\theta_n(k)\) are equivalent. \(\square\)

### 3.3. **Alexander-Zagier’s theorem.** We first compute the entropy of the gold-enrichment with respect to the Erdös and Lehman measures.
Proposition 3.4. The metric entropies of the goldenshift $S$ with respect to the two measures in question are computed as follows:

$$h_m(S) = -\sum_{B \in \mathcal{B}} m_X(B) \log m_X(B) = 4\lambda + 3 = 9.4721356\ldots,$$

$$h_\mu(S) = -\sum_{B \in \mathcal{B}} \mu_X(B) \log \mu_X(B) = -\sum_{n=1}^{\infty} \sum_{B : |B| = 2n+1} \frac{p + q}{2^{2n+1}} \log \frac{p + q}{2^{2n+1}} = 8.961417\ldots$$

Proof. This is a direct computation using the Bernoullicity of $S$ with respect to both measures and Corollaries 2.5 and 2.9.

Corollary 3.5. Let $\Lambda := \log \lambda 2$, and

$$k_n = \sum_{t \geq 1} (a_1, \ldots, a_t) \in \mathbb{N}^t \quad \text{subject to} \quad a_1 + \cdots + a_t = n \quad \text{and} \quad p/q = [a_1, \ldots, a_t]$$

Then

$$(3.2) \quad h_\mu(S) = 9 \left( \Lambda - \frac{1}{18} \sum_{n=1}^{\infty} \frac{k_n}{4^n} \right).$$

Proof. We have

$$h_\mu(S) = -\sum_{n=1}^{\infty} \sum_{B : |B| = 2n+1} \frac{p + q}{2^{2n+1}} \log \frac{p + q}{2^{2n+1}}$$

$$= -\frac{1}{2} \sum_{n=1}^{\infty} 4^{-n} \left( k_n - \Lambda (2n + 1) \sum_{a_1 + \cdots + a_t = n} (p + q) \right).$$

It suffices now to recall that

$$\sum_{t \geq 1} (p + q) = 2 \cdot 3^{n-1}$$

and to compute the value of the corresponding series. □

Note that the quantity $k_n$ appeared for the first time in [AlZa] in somewhat different notation. Namely, let $k$ and $i$ be positive integers, and let $e(k, i)$ denote the length of the simple Euclidean algorithm for $k$ and $i$ (formally: $e(i, i) = 0$, $e(i + k, i) = e(i + k, k) = e(i, k) + 1$). Then obviously

$$k_n = \sum_{k > i > 0} k \log \lambda k.$$
Proposition 3.6. (Alexander-Zagier, 1991). The following relation holds:

\[ H_\mu = \Lambda - \frac{1}{18} \sum_{n=1}^{\infty} \frac{k_n}{4^n} = 0.995713 \ldots \]

Proof. An application of Theorem 3.3 and of relation (3.2).

3.4. The dimension of the Erdős measure. As an application of the treatment of the Erdős measure as the projection of the measure of the uniform random walk on the Fibonacci graph, we will compute the dimension of \( \mu \) in the sense of L.-S. Young.

We first give a number of necessary definitions (see [Y]).

Definition. Let \( \nu \) be a Borel probability measure on a compact space \( Y \). The quantities

\[ \dim_H \nu = \inf \{ \dim_H A : A \subset Y, \nu A = 1 \} \]
\[ \overline{C}(\nu) = \limsup_{\delta \to 0} \inf \{ \overline{C}(A) : A \subset Y, \nu A \geq 1 - \delta \} \]
\[ \underline{C}(\nu) = \liminf_{\delta \to 0} \inf \{ \underline{C}(A) : A \subset Y, \nu A \geq 1 - \delta \} \]

(where \( \overline{C}(A) \) and \( \underline{C}(A) \) are respectively the upper and lower capacities of \( A \)) are called the Hausdorff dimension of a measure \( \nu \) and the upper and lower capacities of \( \nu \), respectively.

Let next \( N(\varepsilon, \delta) \) denote the minimal number of balls of radius \( \varepsilon > 0 \) which are necessary to cover a set of \( \nu \)-measure \( \geq 1 - \delta \).

Definition. The quantities

\[ \underline{C}_L(\nu) = \limsup_{\delta \to 0} \liminf_{\varepsilon \to 0} \frac{\log N(\varepsilon, \delta)}{\log(1/\varepsilon)} \]
\[ \overline{C}_L(\nu) = \limsup_{\delta \to 0} \limsup_{\varepsilon \to 0} \frac{\log N(\varepsilon, \delta)}{\log(1/\varepsilon)} \]

are called the lower and upper Ledrappier capacities of \( \nu \).

Definition. Let \( H_\nu(\varepsilon) = \inf \{ H_\nu(\xi) : \text{diam} \xi \leq \varepsilon \} \), where \( H_\nu(\xi) \) is the entropy of a finite partition \( \xi \). The quantities

\[ \overline{R}(\nu) = \limsup_{\varepsilon \to 0} \frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)} \]
\[ \underline{R}(\nu) = \liminf_{\varepsilon \to 0} \frac{H_\nu(\varepsilon)}{\log(1/\varepsilon)} \]

are called respectively the upper and lower informational dimensions of \( \nu \) (= Rényi dimensions).
Theorem. (L.-S. Young [Y], 1982). Let \( \nu \) be a Borel probability measure on a metric space \( Y \), and let \( B(x, r) \) denote the ball with the center at \( x \) of radius \( r \). If

\[
\alpha(x) := \lim_{r \to 0} \frac{\log \nu B(x, r)}{\log r} \equiv \alpha
\]

for \( \nu \)-a.e. point \( x \in Y \), then

\[
\dim_H \nu = \overline{C}(\nu) = C_L(\nu) = C_L(\nu) = \underbar{R}(\nu) = \overline{R}(\nu) = \alpha.
\]

Definition. If the condition of Young’s theorem is satisfied, then this \( \alpha \) is called the pointwise dimension of a measure \( \nu \) and is denoted by \( \dim(\nu) \). This notion was also proposed in [Y].

Theorem 3.7. For the Erdős measure \( \mu \),

\[
\dim(\mu) = H_\mu.
\]

Proof. Fix a path \( \varepsilon = (\varepsilon_1\varepsilon_2\ldots) \in Y(\Phi) \), and let \( x = \pi(\varepsilon) \) and \( Y_n = Y_n(\varepsilon) \) be the interval whose every point \( x' \) has a path \( \varepsilon' \in Y(\Phi) \) such that \( \varepsilon_i \equiv \varepsilon'_i \), \( 1 \leq i \leq n \). Clearly, \( Y_n(\varepsilon) = [\sum_1^n \varepsilon_k \lambda^{-k-1}, \sum_1^n \varepsilon_k \lambda^{-k-1} + \lambda^{-n}] \), We recall that \( \mu = \vartheta_1 * \vartheta_2 * \ldots \) (see Section 1). Let \( \mu^{(n)} := \vartheta_1 * \cdots * \vartheta_n \). Then by Shannon’s theorem for the random walks (see Theorem 2.1 in [KaVer] and also [De]) and because \( H_\mu \) is the entropy of the random walk on the Fibonacci semigroup (see Proposition 3.2),

\[
\lim_{n \to \infty} \frac{\log \lambda \mu^{(n)}(Y_n(\varepsilon))}{n} = -H_\mu
\]

for \( \mu \)-a.e. \( x \in [0, 1] \). We set \( Y'_n(\varepsilon) = [\sum_1^n \varepsilon_k \lambda^{-k-1}, \sum_1^n \varepsilon_k \lambda^{-k-1} + \lambda^{-n+2}] \supset Y_n(\varepsilon) \).

Obviously, \( \mu^{(n)}(Y'_n(\varepsilon)) \sim \mu^{(n)}(Y_n(\varepsilon)) \). Given \( h > 0 \), we choose \( n = n(h) \) such that \( Y_{n+1}(\varepsilon) \subset (x, x + h) \subset Y'_n(\varepsilon) \) for any \( \varepsilon \in \pi^{-1}\{x\} \). Hence it follows that for \( \mu \)-a.e. \( x \),

\[
\lim_{h \to 0} \frac{\log \lambda \mu(x, x + h)}{\log \lambda h} = -\lim_{n \to \infty} \frac{\log \lambda \mu Y_n(\varepsilon)}{n} = H_\mu,
\]

as \( h \asymp \lambda^{-n} \). \( \square \)

Remark. When the present paper was in preparation, the authors were told that the claim of Theorem 3.7 can be obtained as a corollary of several results including the new one due to F. Ledrappier and A. Porzio. More precisely, it was shown in [AIYo] that \( H_\mu = \overline{R}(\mu) = \overline{R}(\mu) \), and in [LePo] it was proved that the limit in Young’s theorem does exist for the Erdős measure. This proves Theorem 3.7.

Our proof is straightforward and, what is more important, is a direct corollary of a Shannon-like theorem, so far it leads to new connections between geometric and dynamical properties of the Erdős measure.
Corollary 3.8.

\[ \dim_H \mu = \overline{C}(\mu) = C(\mu) = \overline{C}_L(\mu) = C_L(\mu) = \overline{R}(\mu) = R(\mu) = H_\mu = 0.995713 \ldots \]

Remark 1. Another proof of Theorem 3.7 can be obtained by using the Bernoulli structure of the measure \( \mu \). More precisely, for a regular \( x \in (\lambda^{-1}, 1) \) having a normal block expansion with respect to the measure \( \mu_x \), as is easy to compute, the limit in the definition of the dimension equals \( \frac{1}{9} h_\mu(S) \). This yields also another proof of Theorem 3.3. The details are left to the interested reader.

Remark 2. In fact, we have computed the \( \mu \)-typical Lipschitz exponent of the distribution function of the Erdős measure. Note that in [Si] it was proved that the best possible Lipschitz exponent of this function for all \( x \) is \( \Lambda - \frac{1}{2} = 0.9404 \ldots \).

Remark 3. As a conjecture, we claim that for the two-dimensional Erdős measure \( \tilde{\mu} \) (see Section 1),

\[ \dim(\tilde{\mu}) = 2 \dim(\mu) = 1.991426 \ldots \]

The proof could apparently follow from the theorem due to L.-S. Young [Y] relating the pointwise dimension to the entropy of an automorphism and Lyapunov exponents of an ergodic measure. Besides, we think that the measure \( \tilde{\mu} \) has a local structure of direct product, which would also explain this relation.

3.5. Fibonacci exponents. We recall that in Section 2 we defined the function \( f \) acting on the set of finite 0-1 words and counting the number of words equivalent to an argument. Let \( x \in (0, 1) \) and \((\varepsilon_1 \varepsilon_2 \ldots)\) be its canonical representation. Let now the finite word \( x_n = (\varepsilon_1 \ldots \varepsilon_n) \).

Definition. The limit

\[ E(x) = \lim_{n \to \infty} \frac{\log_\lambda f(x_n)}{n} \]

(if exists) will be called the Fibonacci exponent of a point \( x \).

We will show that for a.e. \( x \) with respect to Erdős measure the Fibonacci exponent exists and is the same, as well as for Lebesgue measure. Besides, we reprove one theorem due to S. Lalley (see [Lal] and references therein). The proof in [Lal] can be applied to any PV number \( \lambda \) but for the golden ratio our proof is more direct.

Proposition 3.9. ([Lal, Th. 2]). For \( \mu \)-a.e. \( x \),

\[ E(x) = E_\mu := \Lambda - H_\mu = 0.44469 \ldots \]

Proof. It suffices to consider a regular \( x \in (\lambda^{-1}, 1) \). Let \( B_1 B_2 \ldots \) be its block expansion. Then

\[ E(x) = \lim_{k \to \infty} \frac{\log_\lambda f(B_1 \ldots B_k)}{|B_1| + \cdots + |B_k|} = \frac{E_\mu \log_\lambda f(B_1)}{E_\mu |B_1|} \]

(where \( E_\mu \) denotes mathematical expectation with respect to Erdős measure) by the ergodic theorem applied to the goldenshift and Erdős measure. Now it suffices to observe that \( E_\mu |B_1| = 9 \), and \( E_\mu \log_\lambda f(B_1) = \frac{1}{2} \sum_{n \geq 1} \frac{k_n}{4^n} \) and apply Proposition 3.6. \( \Box \)

In the same way we obtain...
Proposition 3.10. For a.e. \( x \) with respect to Lebesgue measure,

\[
E(x) := E_m = \frac{E_m \log \lambda f(B_1)}{E_m |B_1|} = \frac{\sum_{n=1}^{\infty} \ell_n \lambda^{-2n-1}}{4\lambda + 3},
\]

where

\[
\ell_n = \sum_{i \geq 1} \log \lambda (p + q).
\]

Remark 1. We established in Proposition 3.9 a relation between the pointwise dimension of the Erdős measure and its typical Fibonacci exponent. This gives us an occasion to state without proof a similar claim for Lebesgue measure:

\[
\lim_{h \to 0} \frac{\log \mu(x, x+h)}{\log h} = \Lambda - E_m
\]

for a.e. \( x \) with respect to the Lebesgue measure. The proof is the same as the one mentioned in Remark 1 after Corollary 3.8. Apparently, this dimensional characteristic has not been considered yet.

Remark 2. Let \( f(n) \) be the number of representations of a positive integer \( n \) as a sum of distinct Fibonacci numbers. If \( n = \sum_{j=1}^{k} \varepsilon_j F_j \) is such a representation with \( \varepsilon_j \in \{0, 1\} \), \( \varepsilon_j \varepsilon_{j+1} = 0 \), \( 1 \leq j \leq k-1 \), and \( \varepsilon_k = 1 \), then evidently \( f(n) = f(\varepsilon_k \ldots \varepsilon_1) \) in the usual sense, which explains the choice of the notation. It is known that \( f(n) = O(\sqrt{n}) \) is an attainable estimate (see [Pu]) and that the average behavior of \( f \) in the sense of its summation function is \( \sum_{n < N} f(n) \asymp N^A \) (for more precise results see [DuSiTh]). It is worth asking the question about the “typical exponent” of \( f(n) \) with respect to density, i.e.

\[
E_d = \lim_{n \to \infty} \frac{\log f(n)}{\log n},
\]

where \( J \subset \mathbb{N} \) is a subsequence of density 1. We conjecture that this exponent exists, and \( E_d = E_m \), i.e. density 1 corresponds to full Lebesgue measure.

**Appendix A. The ergodic central measures and the adic transformation on the Fibonacci graph**

In this appendix we will study in detail some properties of the space of paths \( Y(\Phi) \) of the Fibonacci graph introduced in Section 3. We first give a necessary definition which is close to the definition of canonical expansions but reflects the fact that 0 and 1 have the same rights in the graph \( \Phi \).

**Definition.** The generalized canonical expansion of a point in \( (0, 1) \) is defined as follows. We construct the sequence \( (\varepsilon_1 \varepsilon_2 \ldots) \) such that relation (3.1) holds, and either \( (\varepsilon_1 \varepsilon_2 \ldots) \in X \), or \( \varepsilon_1 = \cdots = \varepsilon_m = 1 \) for some \( m \in \mathbb{N} \), and the tail is in \( X \). The algorithm is a clear modification of the greedy algorithm.

**Remark.** In fact, the generalized canonical expansions lead to a normal form in the semigroup corresponding to the group \( G \) (see Section 3).

The tail partition \( \tau(\Phi) \) of \( Y(\Phi) \) is defined as follows:
Definition. Paths $(\varepsilon_n)$ and $(\varepsilon'_n)$, by definition, belong to one and the same element of $\eta(\Phi)$ iff

(i) $\pi(\varepsilon_1\varepsilon_2\ldots) = \pi(\varepsilon'_1\varepsilon'_2\ldots)$, and
(ii) there exists $N \in \mathbb{N}$ such that $\varepsilon_n \equiv \varepsilon'_n$, $n > N$.

The partial lexicographic ordering on $Y(\Phi)$ is defined for paths belonging to one and the same element of the tail partition $\eta(\Phi)$.

Definition. Let two paths $\varepsilon = (\varepsilon_1\varepsilon_2\ldots)$ and $\varepsilon' = (\varepsilon'_1\varepsilon'_2\ldots)$ belong to one and the same element of $\eta(\Phi)$. If $\varepsilon_{k-1} = 0$, $\varepsilon_k = 1$, $\varepsilon_{k+1} = 1$ and $\varepsilon'_{k-1} = 1$, $\varepsilon'_k = 0$, $\varepsilon'_{k+1} = 0$ for some $k \geq 2$, and $\varepsilon_j \equiv \varepsilon'_j$ for $k - j \geq 2$, then, by definition, $\varepsilon \prec \varepsilon'$. Next, by transitivity, $\varepsilon \prec \varepsilon'$, $\varepsilon' \prec \varepsilon''$ implies $\varepsilon \prec \varepsilon''$.

Remark. This definition is consistent, because any element of $\eta(\Phi)$ is isomorphic to a finite number of finite paths, and they all can be transferred into one another with the help of replacements $011 \leftrightarrow 100$. Note also that this linear ordering on each element of $\eta(\Phi)$ is stronger than the partial ordering introduced in [Pu] (see the end of item 2.2). For example, $(100011*) < (011100*)$ in the above sense but in the sense of the partial order they are noncomparable.

Definition. The adic transformation $T_\Phi$ assigns (if possible) to a path $\varepsilon \in Y(\Phi)$ the path $\varepsilon'$ such that $\varepsilon'$ belongs to the same element of the tail partition as $\varepsilon$ and is the immediate successor of $\varepsilon$ in the sense of the lexicographic order.

It is clear that the adic transformation $T_\Phi$ is not everywhere well defined. More precisely, it is well defined on the paths $\varepsilon$ containing at least one triple $\varepsilon_k = 0$, $\varepsilon_{k+1} = 1$, $\varepsilon_{k+2} = 1$. Let us describe its action in more detail. Let $(\varepsilon_1\varepsilon_2\ldots) \in Y(\Phi)$ be as described. After finding the first triple $\varepsilon_k = 0$, $\varepsilon_{k+1} = 1$, $\varepsilon_{k+2} = 1$, we

1) replace it by $\varepsilon_k = 1$, $\varepsilon_{k+1} = 0$, $\varepsilon_{k+2} = 0$,
2) leave the tail $(\varepsilon_{k+3}\varepsilon_{k+4}\ldots)$ without changes,
3) find the minimal possible $(\varepsilon'_1\ldots\varepsilon'_{k-1})$ equivalent to $(\varepsilon_1\ldots\varepsilon_{k-1})$ in the sense of Section 2.

To carry out 3), we may use the algorithm of “anti-normalization”, i.e. the process analogous to the ordinary normalization but changing “100” to “011” (cf. Section 1).

So, the generalized canonical expansions are just the maximal paths, i.e. the ones where $T_\Phi$ is not well defined; thus, the set of maximal paths is naturally isomorphic mod 0 to the interval $[0, 1]$. Geometrically the generalized canonical expansion corresponds to the right most possible path descending to a given vertex. Similarly, the minimal paths (i.e. the ones, on which $T_\Phi^{-1}$ is not well defined) are just the so-called lazy expansions (for the definition see, e.g., [ErJoKo]).

For more general definitions of adic transformations and investigation of their properties see [Ver1], [Ver2], [LivVer] and [VerSi] and [StVo] and [VerKe] for more details). We recall that topologically the space $Y(\Phi)$ is a nonstationary Markov compactum (see [Ver2] for definition).

Definition. A measure $\xi$ on $\Phi$ with the distribution $\xi_n$ on its $n$’th level is called Markov if the sequence $(D_n, \xi_n)$ of random variables is a (nonhomogenous) Markov chain.

Now we can define for a Markov measure the notion of conditional measures.
Definition. A Markov measure $\xi$ on the graph $\Phi$ is called central if any of the following equivalent conditions is satisfied:

1. For any vertex in this graph the conditional measure on the set of all paths descending to this vertex, is uniform.
2. $\xi$ is $T_\Phi$-invariant.
3. $\xi$ is invariant with respect to the tail partition $\eta(\Phi)$.

Definition. A central measure $\xi$ on $\Phi$ is called ergodic if either of the following two equivalent conditions is satisfied:

1. The adic transformation $T_\Phi$ is ergodic with respect to it.
2. The tail partition is $\xi$-trivial, i.e. contains only sets whose $\xi$-measure is either 0 or 1.

The aim of this section is to describe
1) all ergodic central measures on $\Phi$.
2) the action of the adic transformation $T_\Phi$ on $\Phi$.

In the following theorem we will describe the ergodic central measures and the corresponding components of the action of $T_\Phi$. As was noted above, $T_\Phi$ interchanges representations of one and the same $x$. We will see that the regularity or irregularity of the generalized canonical expansion of a given $x$ leads to three types of possible ergodic components of the action of $T_\Phi$, namely, to a “full” odometer, an irrational rotation of the circle or a special automorphism over a rotation.

Theorem A.1. 1. The ergodic central measures on $\Phi$ are naturally parametrized by the points of the interval $[0,1]$. We denote by $\mu_x$ the measure corresponding to $x$.

2. The measure $\mu_x$ is continuous if and only if $x \neq \{N\lambda\}$ for any $N \in \mathbb{Z}$ (or, equivalently, if $x$ has infinite canonical expansion).

3. The action of the adic transformation $T_\Phi$ is not transitive, and its trajectories are described as follows. Let $x$ be as in the previous item, and let $\varphi_x$ denote the space of paths in $\Phi$ such that $\varphi_x = \text{supp} \mu_x$. The set $\varphi_x$ is invariant under $T_\Phi$ and we have the following alternatives.

a. If the generalized canonical expansion of $x$ contains infinitely many pieces “1(00)$^l$1” with $l \geq 1$ (let us call such a piece even), then $T_\Phi|_{\varphi_x}$ is strictly ergodic and metrically isomorphic to the shift by 1 on the group of certain $a$-adic integers with $a = (a_1, a_2, \ldots)$ being a sequence of positive integers, generally speaking, nonstationary. Thus, $T_\Phi|_{\varphi_x}$ has a purely discrete rational spectrum.

b. If the generalized canonical expansion of $x$ does not contain even pieces at all, then $T_\Phi|_{\varphi_x}$ is also strictly ergodic and metrically isomorphic to a certain irrational rotation of the circle.

c. Finally, if the generalized canonical expansion of $x$ contains a finite number of even pieces, then $T_\Phi|_{\varphi_x}$ is metrically isomorphic to some special automorphism over a rotation of the circle, i.e. to a shift on the space $S^1 \ltimes \mathbb{Z}/k$ for some $k \in \mathbb{N}$. 
Proof. (1) Let $\varphi_x$ be the set of all paths projecting to $x \in [0, 1]$ (a $\pi$-fiber over $x$). Obviously, the set $\varphi_x$ is invariant under $T_\Phi$ for any $x$. Thus, $T_\Phi$ is not transitive, and its action splits into components, each acting on a certain $\varphi_x$ (below we will see that for all $x$, except for some countable set, the action of $T_\Phi|_{\varphi_x}$ is strictly ergodic).

Henceforward in this proof we assume that $x$ has infinite canonical expansion. Let $x = \sum_{j=1}^{\infty} \varepsilon_j \lambda^{-j-1}$ be the generalized continued fraction of $x$. We first split it in the following way: $(\varepsilon_1 \varepsilon_2 \ldots) = B^{(0)} B^{(1)}$, where $B^{(0)}$ is either $0^s$ or $1^s$ for some $s \geq 0$ (if $s = 0$, then $B^{(0)} = \emptyset$), and $B^{(1)}$ begins with “$1^0$”. Such a splitting is caused by the trivial reason that the action of $T_\Phi$ does not touch at all the set $B^{(0)}$, as $T_\Phi$ only interchanges certain triples “$100$” and “$011$.” So, we have the following cases (they correspond to those enumerated in the theorem).

a. If $B^{(1)}$ contains infinitely many even pieces, then $x$ is regular (see Section 2), hence,

$$B^{(1)} = B_1 B_2 B_3 \ldots,$$

where

$$B_j = 1(00)^{a_1(j)} (01)^{a_2(j)} (00)^{a_3(j)} \ldots (00)^{a_{t_j(j)}}$$

or

$$B_j = 1(01)^{a_1(j)} (00)^{a_2(j)} (01)^{a_3(j)} \ldots (00)^{a_{t_j(j)}}$$

with $a_i(j) \in \mathbb{N}$ and $t_j < \infty$.

b. If $B^{(1)}$ does not contain any even piece, then obviously

$$B^{(1)} = 1(00)^{a_1} (01)^{a_2} (00)^{a_3} \ldots$$

or

$$B^{(1)} = 1(01)^{a_1} (00)^{a_2} (01)^{a_3} \ldots$$

with $a_j \in \mathbb{N}$ for any $j \geq 1$.

c. Finally, if the number of even pieces is finite (but nonzero), then

$$B^{(1)} = B_1 B_2 \ldots B_m \tilde{B}^{(1)},$$

where $B_1, \ldots, B_m$ have the form described in the previous item, and $\tilde{B}^{(1)}$ is an infinite block of the form described in item b.

(3) Consider items a, b, c from the viewpoint of the action of $T_x := T_\Phi|_{\varphi_x}$.

a. The idea of the study of $T_x$ in this case is based on two assertions of the previous section, namely on Lemmas 2.1 and 2.2. In particular, from Lemma 2.2 it follows that blocks $B_i$ and $B_{i+1}$ for all $i \in \mathbb{N}$ are replaced by any equivalent sequences independently, and hence it is clear that for such a point $x$ the transformation $T_x$ is the shift by 1 in the group of $\alpha$-adic integers with $\alpha = (p_1 + q_1, p_2 + q_2, \ldots)$. This transformation $T_x$ is known to be strictly ergodic, i.e. there is a unique (product) measure $\mu_x$ invariant under it.

b. We recall that in this case

$$B^{(1)} = 1(00)^{a_1} (01)^{a_2} (00)^{a_3} \ldots$$

or

$$B^{(1)} = 1(01)^{a_1} (00)^{a_2} (01)^{a_3} \ldots$$

Let $\alpha = [1, a_1, a_2, \ldots]$ denote a (regular) continued fraction. We claim that in this case the transformation $T_x$ is strictly ergodic and metrically isomorphic to the rotation through $\alpha$. The idea of the proof lies in recoding the space $\varphi_x$ into the space of continued fractions $[1, a_1, a_2, \ldots]$.
second model of the adic realization of the rotation from [VerSi] (see Section 3 of the cited work and Example 3 below). The unique invariant measure can be described with the help of Theorem 2.3 from the cited work.

c. This case in a sense is a “mixture” of the previous ones. One can easily see that if \( \tilde{B}^{(1)} \) is parametrized by the infinite sequence \((a_1, a_2, \ldots)\) in the sense of the previous item, and if \( B_j = B_j(a_1^{(j)}, \ldots, a_t^{(j)}) \), then \( T_x \) acts on \( \varphi_x \) as the special automorphism over the rotation through \( \alpha = [1, a_1, a_2, \ldots] \) with the constant step function (\( \equiv 1 \)) and the number of upper steps equal to \( \prod_1^{m}(p_j + q_j) - 1 \). So, \( T_x \) is again strictly ergodic. The proof of the theorem is complete.

Remarks. 1. It is known (see, e.g., [VerKe]) that any ergodic central measure on the Pascal graph is also parametrized by a real in \([0, 1]\) but in a completely different manner, namely by means of the first transition measure. It is appropriate to compare that situation with the Fibonacci graph. We see that in the graph \( \Phi \), for any \( \alpha \in [0, 1] \) there exists a central ergodic measure \( \mu \) such that \( \mu(\varepsilon_1 = 0) = \alpha \). If \( \alpha \) is irrational, then this measure is unique, namely \( \mu = \mu_x \) for \( x = \sum_j \varepsilon_j \lambda^{-j-1} \) with \( \alpha = [1, a_1, a_2, \ldots] \) and \((\varepsilon_1\varepsilon_2\ldots) = 1(00)^{a_1}(01)^{a_2} \ldots \) for \( \alpha > \frac{1}{2} \), and \( 1 - \alpha = [1, a_1, a_2, \ldots] \) and \((\varepsilon_1\varepsilon_2\ldots) = 1(01)^{a_1}(00)^{a_2} \ldots \) otherwise. If \( \alpha \) is rational, then there exists a whole interval of \( x \) in \([0, 1]\) such that \( \mu_x(\varepsilon_1 = 0) = \alpha \).

2. A typical \( x \) from the viewpoint of Lebesgue measure, of course, corresponds to the case a. of the theorem.

Examples. We illustrate the possible situations in the previous theorem with four examples. For a better illustration we will use the following convention:

1. \( x = \lambda^{-2} \sim (1000\ldots) \). Here \( \varphi_x \) is countable and isomorphic to the stationary Markov compactum with the matrix \( \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). This compactum consists of the sequence \( 0^\infty \) and the sequences \( 0^k1^\infty \) for \( k \geq 0 \). For a central measure \( \mu_x, \mu_x(0^k1^\infty) = \mu_x(0^l1^\infty) \) for any \( k, l \), hence, this measure is concentrated on the path \( 0^\infty \) corresponding to the path \((010101\ldots)\) in the initial compactum (see Fig. 5 below).
2. \( x = \frac{1}{2} \sim (100)^\infty \). Here \( B^{(0)} = \emptyset, B^{(1)} = (1(00)^1)^\infty \). We have \( \varphi_x = \prod_{1}^{\infty} \{011, 100\} \), and thus \( T_x \) is isomorphic to the 2-adic shift, i.e. the shift by 1 in the group of dyadic integers. Therefore, \( T_x \) has the binary rational purely discrete spectrum. We depict the way of recoding the paths in \( \varphi_x \) into the full dyadic compactum by the rule “011 \sim 0, 100 \sim 1” (see Fig. 6).

3. \( x \sim (1(0001)^\infty) \). Here \( \alpha = [1, 1, 1, \ldots] = \lambda^{-1} \), and \( T_x \) acts as the rotation by the golden ratio. Fig. 8 shows the way of recoding the paths in \( \varphi_x \) into the usual model for this rotation (“Fibonacci compactum”). Note that the natural ordering of these paths is alternating (see Fig. 7, 8).
4. $x \sim (1001(0001)\infty)$. For this $x$, the transformation $T_x$ acts as the special automorphism over the rotation by $\lambda^{-1}$ with a single step equal to the base (see Fig. 9).
Appendix B. Arithmetic expression for block expansions

We recall that in Section 2 we have defined the mapping $\Psi$ assigning to a regular $x \in (\lambda^{-1}, 1)$ the sequence of blocks $B_1(x), B_2(x), \ldots$. In this Appendix we are going to specify the mapping $\Psi$ in an arithmetic way. To this end, we gather the canonical expansion of a given regular $x$ blockwise.

Recall that similarly to the canonical expansion (1.1) of reals, there exists the corresponding representation of positive integers. Namely, each $N \in \mathbb{N}$ has a unique representation in the form

$$N = \sum_{i=1}^{k} \varepsilon_i F_i,$$

where $\varepsilon_i \in \{0, 1\}$, $\varepsilon_i \varepsilon_{i+1} = 0$, $\varepsilon_k = 1$ for some $k \in \mathbb{N}$. It is usually called the Zeckendorf decomposition. We denote by $\mathcal{F}$ the class of positive integers whose Zeckendorf decomposition has $\varepsilon_1 = 1$ and $\varepsilon_i \equiv 0$ for all even $i$. Obviously, $\mathcal{F}$ as a subset of $\mathbb{N}$ has zero density. Let the height of $e$ with a finite canonical expansion of the form $e = \sum_j \varepsilon_j \lambda^{-j}$ be, by definition, the positive integer $h(e) := \max\{j : \varepsilon_j = 1\}$.

**Proposition B.1.** Each regular $x \in (\lambda^{-1}, 1)$ has a unique representation of the form

$$x = \sum_{j=1}^{\infty} e_j(x) \lambda^{-\sum_{i=1}^{j} n_i(x)},$$

where

(i) $e_j(x) = m\lambda - n$, $n \in \mathcal{F}$, $n = [m\lambda]$.
(ii) $n_j$ is odd, $n_j \geq 3$ for $j \geq 1$.
(iii) $n_j \geq 2h(e_j) + 1$ for all $j$. 

Fig. 9. Recoding the paths for case c
Proof. Let $\Psi(x) = B_1B_2\ldots$ be the block expansion of $x$. Suppose $B_j = B_j(a^{(j)}_1,\ldots,a^{(j)}_{t_j})$. We set $n_j(x) := 2\sum_{i=1}^{t_j}a^{(j)}_i + 1$, $j \geq 1$, i.e. $n_j$ is the length of the $j$’th block. Let $m_0^{(j)} = 0$, $m_k^{(j)} = \sum_{i=1}^{k}a^{(j)}_i$, and let $e_j$ be the “value” of $B_j$ in the sense of formula (1.1) as if it were the first block, i.e.

$$
e_j(x) = \lambda^{-1} + \lambda^{-1} \sum_{k=1}^{t_j} \sum_{\nu=1}^{a^{(j)}_k} \lambda^{-2(m_{2k-1}^{(j)}+\nu)}$$

$$= \lambda^{-1} + \lambda^{-2} \sum_{k=1}^{t_j} \left(\lambda^{-2m_{2k-1}^{(j)}} - \lambda^{-2m_{2k-1}^{(j)}}\right), \ t_j \ odd, \nonumber$$

$$e_j(x) = \lambda^{-1} + \lambda^{-1} \sum_{k=1}^{t_j} \sum_{\nu=1}^{a^{(j)}_{2k+t_j}} \lambda^{-2(m_{2k+2}^{(j)}+\nu)}$$

$$= \lambda^{-1} + \lambda^{-2} \sum_{k=1}^{t_j} \left(\lambda^{-2m_{2k+2}^{(j)}} - \lambda^{-2m_{2k+1}^{(j)}}\right), \ t_j \ even. \nonumber$$

The uniqueness of expansion (B.1) follows from the condition (iii) and from the uniqueness of expansion (1.1) for any finite sequence (and, therefore, for any block). The fact that $n \in F$ follows from the definition of block.

Definition. We call the expansion of $x \in (\lambda^{-1},1)$ of the form (B.1) satisfying the conditions (i)–(iii) the arithmetic block expansion.

Remark 1. $n_j$ and $e_j$ depend on $B_j$ only.

Remark 2. In fact, series (B.1) is nothing but series (1.1) rewritten in a different notation. However, we will see that it has its own dynamical sense (see relation (B.2) below).

Remark 3. By Item (iii), the quantities $e_j$ and $n_j$ are not completely independent. Let $l_j := h(e_j)$. Taking into consideration new quantities $s_j := n_j - 2l_j$ and representing $n_j$ as the sum $s_j$ and $2l_j$ in formula (B.1), we come to independent multipliers, but this new form of the block representation does not seem to be natural.

Remark 4. In terms of arithmetic block expansions the goldenshift acts as

$$S(x) = \sum_{j=2}^{\infty} e_j(x)\lambda^{-\sum_{i=2}^{j} n_i(x)}.$$ 

APPENDIX C. COMPUTATION OF DENSITIES AND THE POLYMORPHISM $\Pi$

We return to the subject of the first section. Recall that we have already denoted the transformation $x \mapsto \{\lambda x\}$ by $T$, and $R$ stands for the rotation of the circle $\mathbb{R}/\mathbb{Z}$ by the angle $\lambda^{-1}$.

C.1. Computation of densities.
**Proposition C.1.** The densities \( \frac{d(R\mu)}{d\mu}, \frac{d(\tau\mu)}{d\mu} \) and \( \frac{d\nu}{d\mu} \) are unbounded and piecewise constant with a countable number of steps.

**Proof.** By virtue of the results of Section 1, it suffices to prove the proposition only for \( \frac{d(R\mu)}{d\mu} \). Let \( E \) be a Borel subset of \((0,1)\). The idea of the study lies in the fact that \( R \) does not change any block beginning with the second. As usual, we consider three cases.

**I.** \( E \subseteq (0,\lambda^{-2}) \). If \( E \subseteq (\lambda^{-2k},\lambda^{-2k+1}) \), \( k \geq 1 \), then each point \( x \) of the set \( E \) has the canonical expansion (1.1) of the form \( 02^{k-1}10^* \). Hence the canonical expansion of \( x+\lambda^{-1} \) is \( 1(00)^{k-1}10^* \), and

\[
\frac{\mu(E+\lambda^{-1})}{\mu E} \equiv k,
\]
as \( f(1(00)^{k-1}) = k \). If, on the contrary, \( E \subseteq (\lambda^{-2k-1},\lambda^{-2k}) \), \( k \geq 1 \), then the situation is as follows. This interval in terms of the canonical expansion is \( \bigcup_{a} 02^k B \) mod 0, where the union runs over all closed blocks \( B \).\(^5\) We have two subcases.

**Ia.** Let in terms of the canonical expansion, \( E \subseteq 0^2 1(00)^{a_1}(01)^{a_2} \ldots (00)^{a_t} 1 \). Here \( E+\lambda^{-1} \subseteq 1(00)^{k-1}(01)(00)^{a_1}(01)^{a_2} \ldots (00)^{a_t} 1 \), hence \( \frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{f(B')}{f(B)} \), where \( B' \) is the closed block defined as \( B' = B'(k-1,1,a_1,a_2,\ldots,a_t) \). So, we conclude from Lemma 2.1 that

\[
\frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{kp+(k+1)q}{p+q},
\]
where, as usual, \( \frac{p}{q} = [a_1,a_2,\ldots,a_t] \).

**Ib.** In the same terms, suppose \( E \subseteq 0^2 1(01)^{a_1}(00)^{a_2} \ldots (00)^{a_t} 1 \). Similarly to the above,

\[
\frac{\mu(E+\lambda^{-1})}{\mu E} = \frac{(k+1)p+kq}{p+q}.
\]

**II.** Let \( E \subseteq (\lambda^{-2},\lambda^{-1}) \). This case is analogous to Case I. If \( E \subseteq (\lambda^{-2}+\lambda^{-2k-3},\lambda^{-2}+\lambda^{-2k-2}) \), \( k \geq 1 \), then

\[
\frac{\mu(E-\lambda^{-2})}{\mu E} \equiv 1 \frac{1}{k}.
\]
If \( E \subseteq (\lambda^{-2}+\lambda^{-2k-2},\lambda^{-2}+\lambda^{-2k-1}) \), \( k \geq 1 \), then

\[
\frac{\mu(E-\lambda^{-2})}{\mu E} \equiv \begin{cases} \frac{p+q}{kp+(k+1)q}, & E \subseteq 01(00)^{k-1}01(00)^{a_1}(01)^{a_2} \ldots (00)^{a_t} 1 \\ \frac{p+q}{(k+1)p+kq}, & E \subseteq 01(00)^{k-1}01(01)^{a_1}(00)^{a_2} \ldots (00)^{a_t} 1. \end{cases}
\]

**III.** Let \( E \subseteq (\lambda^{-1},1) \). If \( E \subseteq (\lambda^{-1},\lambda^{-1}+\lambda^{-4}) \), then \( \mu(E-\lambda^{-2}) = \mu E \). If \( E \subseteq (\lambda^{-1}+\lambda^{-4},1) \), then \( E-\lambda^{-2} \subseteq (\lambda^{-2},\lambda^{-1}) \), hence \( E-\lambda^{-2} \subseteq 010^* \).

**IIIa.** Let \( E-\lambda^{-2} \subseteq 1(00)^{a_1}(01)^{a_2} \ldots (00)^{a_t} 1 \), then

\[
\frac{\mu(E-\lambda^{-2})}{\mu E} = 1 + \frac{p}{q}.
\]

\(^5\)We say that a 0-1 word is a closed block if it has the form \( B1 \) for some block \( B \).
IIIb. Let \( E - \lambda^{-2} \subset 1(01)^{a_1}(00)^{a_2} \ldots (00)^{a_1}1 \). Here
\[
\frac{\mu(E - \lambda^{-2})}{\mu E} = 1 + \frac{q}{p}.
\]
The proof is complete.

*Remark 1.* Let \( d = \frac{d(R\mu)}{d\mu} \). Then by relation (1.7), \( \frac{d(\tau \mu)}{d\mu} = \frac{1}{2}(d + 1) \), and by formula (1.12),
\[
\frac{d\nu}{d\mu}(x) = \begin{cases}
\frac{2}{3} + \frac{1}{3}d(x) + \frac{1}{6}d^{-1}(x + \lambda^{-1}), & x \in [0, \lambda^{-2}) \\
\frac{2}{3} + \frac{1}{3}d(x), & x \in [\lambda^{-2}, \lambda^{-1}) \\
\frac{1}{2} + \frac{1}{3}d(x), & x \in [\lambda^{-1}, 1].
\end{cases}
\]

*Remark 2.* From this relation follows Proposition 2.10.

**C.2. The polymorphism II.** Let as above \( \sigma : \Sigma \to \Sigma \) be the one-sided shift. Let us ask the natural question: what is the image of \( \sigma \) on the interval \([0, 1]\) under the mapping \( \pi \) defined by the formula (3.1)?

Note first that the partition into \( \pi \)-preimages of singletons is not invariant under \( \sigma \). Indeed, if, say, \( x = 0110^\infty \), then \( \sigma \sigma(x) = 0^\infty \), while \( n \sigma(x) = 110^\infty \). Thus, \( \Pi := \pi \sigma \pi^{-1} : [0, 1] \to [0, 1] \) is a multivalued mapping, i.e. a *polymorphism* by the terminology of [Ver4].

Recall that a *measure-preserving polymorphism* of a measure space \((X, \mathcal{A}, \mu)\) is the diagram
\[
(X, \mu) \xleftarrow{\pi_1} (Y, \nu) \xrightarrow{\pi_2} (X, \mu),
\]
where \( \pi_1, \pi_2 \) are homomorphisms of measure spaces such that \( \pi_i \nu = \mu, \ i = 1, 2 \). Instead of an arbitrary \( Y \) it suffices to consider \( X \times X \) with the coordinate projections and a certain “bistochastic” measure \( \nu \) (i.e. a measure on the sigma-algebra \( \mathcal{A} \times \mathcal{A} \) with given marginal measures). Such a polymorphism is called *reduced*.

If \( T \) is an automorphism of the space \( Y \) with an invariant measure \( \gamma \) and \( \zeta \) is a measurable partition, one can define the polymorphism \( T_\zeta : (Y_\zeta, \gamma_\zeta) \) into itself as follows. Consider two partitions of \( Y \), namely, \( \zeta \) and \( T^{-1}\zeta \). Identifying \( Y_\zeta \) and \( Y_{T^{-1}\zeta} \) in the natural way, we obtain the diagram
\[
(Y_\zeta, \nu_\zeta) \leftarrow (Y, \nu) \rightarrow (Y_\zeta, \nu_\zeta).
\]
Let \( \xi = \zeta \vee T^{-1}\zeta \). Then \( Y_\xi \subset Y_\zeta \times Y_{T^{-1}\zeta} = Y_\zeta \times Y_\xi \), and the reduced automorphism
\[
(Y_\zeta, \nu_\zeta) \leftarrow (Y_\xi, \nu_\xi) \rightarrow (Y_\zeta, \nu_\zeta)
\]
can be easily interpreted: it is a Markov (multivalued) mapping of the factor space \( Y_\zeta \) with the invariant measure \( \nu_\zeta \). If \( \zeta \) is a \( T \)-invariant partition (i.e. \( T^{-1}\zeta \prec \zeta \)), then the polymorphism is the factor endomorphism \( T_\zeta : Y_\zeta \to Y_\zeta \). That is why the polymorphism in this context is a generalization of the notion of endomorphism (see [Ver4]).
We are going to make use of these notions in our case. We define the polymorphism $\Pi$ as the subset of $[0, 1] \times [0, 1]$ defined as

$$\Pi(x) = \begin{cases} 
\lambda x, & 0 \leq x < \lambda^{-2} \\
\lambda x \cup \lambda x - \lambda^{-1}, & \lambda^{-2} \leq x < \lambda^{-1} \\
\lambda x - \lambda^{-1}, & \lambda^{-1} \leq x \leq 1 
\end{cases}$$

and provided with the measure $\mu$ which is the image of the product measure $p$. By definition of the polymorphism and the Erdős measure, $\Pi \mu = \Pi^{-1} \mu = \mu$, and $\mu$ being the projection of $\mu$ to both axes.

This polymorphism was considered in [VerSi]. Note that for a Borel set $E$, $\Pi^{-1} E = \pi \sigma^{-1} \pi^{-1} E = \lambda^{-1} E \cup \lambda^{-1} E + \lambda^{-2}$. Let $\gamma = (\gamma_1, \gamma_2)$ be the corresponding partition of $\Pi$. It is possible to show that there exists a countable partition of $[0, 1]$ into the intervals $\{G_k\}_{k=1}^\infty$, such that for any $k$ and any $G \subseteq G_k$ the ratio $\mu/(\Pi^{-1} G \cap \gamma_1)/\mu/(\Pi^{-1} G \cap \gamma_2)$ is constant. In particular, if $G \subseteq (\lambda^{-2n}, \lambda^{-2n+1})$ for $n \geq 1$, then this ratio is equal to $n$ (specifically, for $G \subseteq (\lambda^{-1}, \lambda^{-2})$, it equals 1). The method of the proof is the same as in Proposition C.1.

**Appendix D. An Independent Proof of Alexander-Zagier’s Formula**

In this appendix we will present the second proof of formula (3.3) which reveals some new relations between certain structures of the Fibonacci graph $\Phi$.

We first recall that the quantity $f_n(k)$ is nothing but the frequency of the $k$'th vertex on the $n$'th level of the Fibonacci graph which was denoted by $D_n$ (see the beginning of Section 3). We have $\#D_n = F_{n+2} - 1$.

Consider level $n$ of the Fibonacci graph for $n = 2N + 1$. We denote the middle part of $D_n$, i.e. the segment from $F_n$ to $F_{n+1} - 1$, by $D'_n$. The Erdős measure of $D'_n$ obviously equals $\frac{1}{3} + O(\lambda^{-n})$, and we will introduce the partition of $D'_n$ into $2^{N-1} - 1$ intervals of vertices in the following way.

Recall that a Euclidean vertex is one, where the first block can end (they are marked in Fig. 3 for $D_3$ and $D_5$) and that these vertices form the Euclidean binary tree (see Section 3). There are $2^{N-1}$ such vertices at level $2N + 1$, and all of them lie in $D'_n$. Let $V^{(N)}_k$ denote the $k$'th Euclidean vertex from the left on level $2N + 1$. 

![Fig. 10. The polymorphism $\Pi$](image-url)
**Definition.** An open interval of vertices $\Omega_k^{(N)} := (V_k^{(N)}, V_{k+1}^{(N)})$ will be called a Euclidean interval.

So, we have divided the set of vertices $D'_n$ into $2^{N-1}$ Euclidean vertices and $2^{N-1} - 1$ open intervals $\Omega_k^{(N)}$. Now we introduce the subgraph $\Gamma_V$ associated with each Euclidean vertex $V$. It is defined as the one containing all the successors of $V$ in the sense of the Fibonacci graph, except any other Euclidean vertices (see Fig. 11).

![Fig. 11. The graph $\Gamma_V$](image)

We state a straightforward lemma.

**Lemma D.1.** For any Euclidean interval $\Omega_k^{(N)}$ there is a unique Euclidean vertex $V_i^{(j)}$, $j < N$, such that $\Omega_k^{(N)} \subset \Gamma_{V_i^{(j)}}$.

So, any Euclidean interval is determined by a certain Euclidean vertex on one of the preceding odd levels of the Fibonacci graph. Moreover, in the notation of the above lemma, the entropy of $\Omega_k^{(N)}$ may be computed in terms of the frequency of $V_i^{(j)}$ and the entropy of $\tilde{D}_{2N+1-j}$. Namely, let $H_n := \sum_{k=F_n}^{F_{n+1}-1} f_n(k) \log \lambda f_n(k)$, and let next $H_{2N+1} = \sum_{j=1}^{N} H_{2N+1}^{(j)}$, where $H_{2N+1}^{(j)}$ denotes the sum over the vertices $V \in \Gamma_{V_i^{(j)}}$ for all $i \leq 2^{j-1}$. So, $H_{2N+1}^{(j)}$ corresponds to all Euclidean vertices of level $2j + 1$.

Let next $\varphi_i^{(j)}$ denote the frequency of $V_i^{(j)}$. For instance, for $j = 3$, $\varphi_1^{(3)} = \varphi_4^{(3)} = 4$, $\varphi_2^{(3)} = \varphi_3^{(3)} = 5$. In this notation $k_j = \sum_{i=1}^{2^{j-1}} \varphi_i^{(j)} \log \lambda \varphi_i^{(j)}$.

For any $v \in (V_k^{(N)}, V_{k+1}^{(N)})$ its frequency equals $f(V_i^{(j)})$ times the frequency of the corresponding vertex of the central part of level $2N + 1 - j$. So, we established an essential relationship between the central part of level $2N + 1$ of the Fibonacci graph and all Euclidean vertices $V_i^{(j)}$, $1 \leq j \leq N$, $1 \leq i \leq 2^{j-1}$. 
Lemma D.2. The following recurrence relation holds:

\[
H_{2N+1} = \frac{2}{3} \sum_{j=1}^{N-1} 3^j H_{2N-2j} + \frac{1}{3} \cdot 4^N \cdot \sum_{j=1}^{N} \frac{k_j}{4^j} + O \left( \sum_{j=1}^{N} k_j \right), \quad N \to \infty.
\]

Proof. By the above considerations,

\[
H_{2N+1}^{(j)} = \sum_{i=1}^{2^{j-1} - 1} F_{2N-2j+1} \left( \sum_{k=F_{2N-2j}}^{2^{j-1} - 1} \varphi_i^{(j)} f_{2N-2j}(k) \log_\lambda \left( \varphi_i^{(j)} f_{2N-2j}(k) \right) \right)
\]

\[
= \sum_{i=1}^{2^{j-1} - 1} \varphi_i^{(j)} \left( H_{2N-2j} + \frac{1}{3} \log_\lambda \varphi_i^{(j)} \cdot (4^{N-j} + O(1)) \right)
\]

\[
= 2H_{2N-2j} \cdot 3^{j-1} + \frac{1}{3} \cdot \frac{k_j}{4^j} \cdot 4^N + O(k_j)
\]

(we used the fact that \(\sum_{i=1}^{2^{j-1} - 1} \varphi_i^{(j)} = 2 \cdot 3^{j-1}\) easily obtained from Proposition 2.9). Hence relation (D.1) follows.

Remark. Formula (D.1) shows that the entropy of the \(n\)'th level with \(n\) odd can be computed by means of the entropies of the previous even levels and the entropy of the Euclidean tree.

Now we are ready to complete the second proof of formula (3.3). We have

\[
nH_\mu \sim \sum_{k=0}^{F_{n+2} - 2} \frac{f_n(k)}{2^n} \log_\lambda \frac{2^n}{f_n(k)}.
\]

whence

\[
nH_\mu \sim 3 \sum_{k=F_n}^{F_{n+1} - 1} \frac{f_n(k)}{2^n} \log_\lambda \frac{2^n}{f_n(k)},
\]

and

\[
(H_n = \frac{1}{3} (\Lambda - H_\mu) n 2^n).
\]

From relation (D.2) it follows that in the sum \(\sum_{j=1}^{N-1} 3^j H_{2N-2j}\) the first terms are more valuable than the last. Thus, from formulas (D.1) and (D.2) and from the fact that \(k_N = \sum_{i=1}^{2^{N-1}} \varphi_i^{(N)} \log_\lambda \varphi_i^{(N)} < 2(N-1)3^{N-1}\) it follows that

\[
\frac{1}{3} (\Lambda - H_\mu) \cdot (2N + 1) 2^{2N+1} \sim \frac{2}{3} \sum_{j=1}^{N} 3^j \cdot \frac{1}{3} (\Lambda - H_\mu) (2N - 2j) 4^{N-j}
\]

\[
+ \frac{1}{3} \cdot 4^N \sum_{j=1}^{N} \frac{k_j}{4^j},
\]
whence, after straightforward computations,

\[ 18(\Lambda - H_\mu) \sim \sum_{j=1}^{N} \frac{k_j}{4^j}, \quad N \to \infty. \]

**Remark.** The Euclidean tree, being symmetric, naturally splits into two binary subtrees (left and right) being symmetric. If we label each vertex of the left subtree with the corresponding rational \( p/q \), then this left subtree turns out to coincide with the **Farey** tree introduced and studied in detail in [Lag].

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