The blackbody theory is revisited in the case of thermal electromagnetic fields inside uniaxial anisotropic media in thermal equilibrium with a heat bath. When these media are hyperbolic, we show that the spectral energy density of these fields radically differs from that predicted by Planck’s blackbody theory. We demonstrate that the maximum of their spectral energy density is shifted towards frequencies smaller than Wien’s frequency making these media apparently colder. Finally, we derive Stefan-Boltzmann’s law for hyperbolic media which becomes a quadratic function of the heat bath temperature.

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In 1901 Planck [1] derived the famous law describing the spectral distribution of energy of a blackbody (BB) by introducing the concept of quantum of light laying so the foundation of quantum physics. In his description of the problem, the electromagnetic field inside a cavity made with opaque walls which is set at a constant temperature is studied. In this formulation [2], the cavity is at thermal equilibrium and acts as a heat bath. The walls of the cavity emit and absorb electromagnetic waves so that the field itself becomes equilibrated. The internal energy density of the electromagnetic field in the cavity with volume $V$ for both principal polarization states (abbreviated by $s$ and $p$) is then given by

$$U_{BB}^{s/p} = \frac{1}{2} \int_0^\infty d\omega \frac{\omega^2}{\pi^2 c^3} \frac{h\omega}{e^\frac{\hbar\omega}{k_B T} - 1} = \Gamma(4)\zeta(4) (k_B T)^4 \frac{\omega^2}{2\pi^2} \frac{(\hbar c)^3}{k^3},$$

where $\hbar$, $k_B$ and $c$ denote Planck’s constant, Boltzmann’s constant and the velocity of light in vacuum, while $\Gamma$ and $\zeta$ are Riemann’s gamma and zeta functions. Here and in the following we neglect vacuum fluctuations.

In this Letter, we revisit this old problem when the cavity is filled with a uniaxial medium with a relative permittivity tensor of the form

$$\epsilon = \begin{pmatrix} \epsilon_\perp & 0 & 0 \\ 0 & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix}. \quad (2)$$

Here without loss of generality we assume that the optical axis points into the $z$-direction; $\epsilon_\parallel$ is the permittivity along the optical axis and $\epsilon_\perp$ is the permittivity perpendicular to the optical axis. For convenience, if not specified differently we neglect dispersion, dissipation and nonlocal effects in the following. Within such materials so-called ordinary modes (OMs) and extraordinary modes (EMs) exist and satisfy the dispersion relations

$$k_\perp^2 + k_\parallel^2 = \frac{\omega^2}{c^2}, \quad (OM) \quad (3)$$

$$k_\parallel^2 + k_\perp^2 = \frac{\omega^2}{c^2}, \quad (EM) \quad (4)$$

where $k_\perp$ ($k_\parallel$) is the wave number component perpendicular (parallel) to the optical axis. In usual dielectric uniaxial media the principal constants $\epsilon_\perp$ and $\epsilon_\parallel$ are both positive and the iso-frequency surfaces defined by relations (3) and (4) are spheres or ellipsoids, resp., as illustrated in Fig. 1(a). On the other hand, when $\epsilon_\parallel < 0$ and $\epsilon_\perp > 0$ or $\epsilon_\parallel > 0$ and $\epsilon_\perp < 0$ the iso-frequency surfaces of the EM are two- or one-sheeted hyperboloids [see Fig. 1(b)]. The first class of such uniaxial medium is called hyperbolic of type I while the second one hyperbolic of type II [4, 5]. Of course both $\epsilon_\perp$ and $\epsilon_\parallel$ can

Figure 1: (a) Cavity at temperature $T$ containing an isotropic medium of permittivity $\epsilon > 0$ or an anisotropic (uniaxial) medium with $\epsilon_\perp > 0$ and $\epsilon_\parallel > 0$. The particular case $\epsilon_\perp = \epsilon_\parallel = 1$ corresponds to the classical BB. (b) Cavity at temperature $T$ containing a hyperbolic medium of type I ($\epsilon_\perp > 0$ and $\epsilon_\parallel < 0$) and of type II ($\epsilon_\perp < 0$ and $\epsilon_\parallel > 0$). The isofrequency curves are plotted inside the cavities in the plane $(k_\perp, k_\parallel)$.
also be negative. In such uniaxial metallic-like materials no propagating modes exist and the upcoming quantities are all zero.

To start, we focus our attention on the electromagnetic field inside a cavity filled with an arbitrary uniaxial medium. The spectral density of states (DOS) defined as the energy density $U$ normalized to the mean energy of a harmonic oscillator and associated to the thermal field can be calculated either by counting the modes in the wavevector space using expressions (3) and (4) or by means of the generalized trace formula [6] [7] [9]

$$D(\omega) = \frac{\omega}{c^2} \text{ImTr}[\epsilon \mathbf{G}^{\text{EE}}(\mathbf{r}, \mathbf{r}, \omega) + \mu \mathbf{G}^{\text{HH}}(\mathbf{r}, \mathbf{r}, \omega)]$$

(5)

where $\mathbf{G}^{\text{EE}}$ and $\mathbf{G}^{\text{HH}}$ are the electric and magnetic Green’s dyadics for the bulk material and $\mu$ is the relative permeability tensor. The result for a classical BB can be retrieved by using $\epsilon = \mu = 1$ with the unit dyad $\mathbf{1}$. Then the above expression reduces to the well-known expression $D_{\text{BB}}(\omega) = \frac{\omega}{c^2} \pi$. In the following, for the sake of clarity, we consider non-magnetic materials (i.e. $\mu = 1$) only. By inserting the general expression of dyadic Green’s tensors of uniaxial materials [8] in the trace formula [6] it is straightforward to derive the DOS for the three different classes of uniaxial media. Assuming that those media are lossless then in dielectric anisotropic media the DOS $D^0_\text{D}$ for the OMs and $D^\text{e}_\text{D}$ for the EMs are given by the following expressions [9] [10]

$$D^0_\text{D}(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{\epsilon_\perp \sqrt{\epsilon_\perp}}{2} \quad \text{and} \quad D^\text{e}_\text{D}(\omega) = \frac{\omega^2}{\pi^2 c^3} \frac{\epsilon_\parallel \sqrt{\epsilon_\parallel}}{2}.$$  

On the other hand, in the hyperbolic case we obtain

$$D^0_\text{I} = \frac{\omega^2}{\pi^2 c^3} \frac{\epsilon_\perp \sqrt{\epsilon_\perp}}{2},$$

$$D^\text{e}_\text{I} = \frac{\omega}{\pi^2 c^3} \left( \frac{k^2_{\perp, \max}}{\epsilon_\perp} \right) \frac{\epsilon_\perp}{\epsilon_\parallel} + \frac{\omega^2}{c^2} \epsilon_\perp - \frac{\omega}{c} \epsilon_\perp,$$  

(8)

and

$$D^0_\text{II} = 0,$$

$$D^\text{e}_\text{II} = \frac{\omega}{\pi^2 c^3} \frac{\epsilon_\parallel \sqrt{\epsilon_\parallel}}{2} \frac{k^2_{\perp, \max}}{\epsilon_\parallel} - \frac{\omega^2}{c^2} \epsilon_\perp,$$  

(10)

for the type I and type II media, respectively. Note that, we have introduced a cutoff wavenumber $k_{\perp, \max}$ which for dispersive media can be a function of the frequency and which is determined by the concrete (atomic or meta) structure of the medium. For an ideal hyperbolic material $k_{\perp, \max}$ is infinity so that the DOS diverges as was pointed out previously [11]. However, for any real structure $k_{\perp, \max}$ is a finite quantity [12]. For artificial hyperbolic structures it is mainly determined by the unit-cell size of the meta structure. Note further that the DOS of the EMs of type I and type II hyperbolic media coincides for $k_{\perp, \max} > \frac{\omega}{c} \sqrt{\epsilon_\parallel}$ and is given by

$$D^0_\text{I} \approx D^0_\text{II} \approx \frac{\omega}{\pi^2 c^3} \frac{\sqrt{\epsilon_\parallel \epsilon_\perp}}{2} k_{\perp, \max}.$$  

(11)

With the help of the DOS we can determine the thermodynamic potentials of the photon gas inside the uniaxial material. By definition, the internal and the free energy per unit volume are given by

$$U = \int_0^\infty d\omega \, D(\omega) U(\omega, T),$$

$$F = \int_0^\infty d\omega \, D(\omega) F(\omega, T)$$

(12)  

(13)

where

$$U(\omega, T) = \frac{\hbar \omega}{e^{\frac{\hbar \omega}{k_B T}} - 1},$$

$$F(\omega, T) = k_B T \ln(1 - e^{-\frac{\hbar \omega}{k_B T}}).$$

(14)  

(15)

Finally, from the internal and free energy we can also determine the entropy per unit volume by

$$S = \frac{U - F}{T}.$$  

(16)

Clearly, by means of these expressions we can derive any thermodynamic property of the photon gas inside the cavity as the pressure $P = U \theta^3$, the photonic heat capacity $C_V = \frac{\partial U}{\partial T}$, etc.

Let us first have a look at the expressions for the ordinary uniaxial material. In this case we obtain

$$U^0_\text{D} = U^0_{\text{BB}} \epsilon_\perp \sqrt{\epsilon_\perp} \quad \text{and} \quad U^\text{e}_\text{D} = U^p_{\text{BB}} \epsilon_\parallel \sqrt{\epsilon_\perp}.$$  

(17)

Therefore, when $\epsilon_\perp = \epsilon_\parallel = 1$ we recover the classical blackbody result. And the relation between the internal energy, the free energy and the entropy have the familiar forms

$$F^{0/e}_\text{D} = -\frac{1}{3} U^{0/e}_\text{D} \quad \text{and} \quad S^{0/e}_\text{D} = 4 \frac{U^{0/e}_\text{D}}{3}.$$  

(18)

Note that these relations are the same as for a usual BB because the DOS of the field inside a dielectric uniaxial medium is proportional to $\omega^2$.

On the contrary, in type I and type II hyperbolic media we have seen that the DOS of the EMs is linear in $\omega$ as in a 2D photon gas in vacuum. It follows that the relations between the thermodynamic properties of the photon gas are radically different in that case. Indeed, we obtain

$$U^{0}_I = \sqrt{\epsilon_\parallel \epsilon_\perp} \frac{1}{2} k_{\perp, \max} \frac{1}{\pi^2 c^3} \frac{1}{\hbar^2} \Gamma(3) \zeta(3) (k_B T)^3$$  

(19)

and

$$F^{0}_I = -\frac{1}{2} U^{0}_I \quad \text{and} \quad S^{0}_I = \frac{3}{2} U^{0}_I.$$

(20)
Hence $U$, $F$ and $S$ are proportional to $T^3$ and not anymore to $T^4$. This result is a direct consequence of the linear dependence of the electromagnetic DOS inside hyperbolic media with respect to $\omega$. Naturally, for the OMs we find

$$U_{\parallel}^0 = U_{\perp}^0 \text{ and } U_{\parallel}^\perp = 0. \quad (21)$$

Note that for the type II hyperbolic material the internal energy of the OMs is zero, since there are no OMs in such a material. The internal energy of the OMs in a type I hyperbolic materials is just the same as in a dielectric uniaxial medium. Hence, the relations between the thermodynamic potentials are the same as in a dielectric uniaxial medium. However, in typical hyperbolic (meta)materials the maximal wave vector is much larger than the vacuum wave vector $k_{\perp,\text{max}} \gg \frac{c}{\omega}$, making the material properties dominated by the EMs.

Another consequence of the linearity of the DOS with respect to $\omega$ inside a hyperbolic medium is the spectral shift of Wien’s frequency $\omega_{\text{max}}$ (resp. wavelength $\lambda_{\text{max}}$) at which the energy distribution function has its maximum. For both type I and type II hyperbolic media we find after a straightforward calculation from relations (9) and (11) ($k_{\perp,\text{max}} \gg \frac{c}{\omega}$) that this maximum is reached when

$$\frac{\hbar \omega_{\text{max}}}{k_B T} = 1.59 \text{ or } \frac{2\pi l_c}{\lambda_{\text{max}}} = 3.92 \quad (22)$$

whereas for a usual BB $\frac{\hbar \omega_{\text{max}}}{k_B T} = 2.82$ and $2\pi l_c/\lambda_{\text{max}} = 4.965$. Here we have introduced the thermal coherence length $l_c \equiv \frac{\hbar c}{k_B T}[6]$. Hence we see that Wien’s frequency is shifted toward smaller values (i.e. the medium appears colder than a classical BB) and the maximum vacuum wavelength to larger values (see Fig. 2).

It is now interesting to compare the internal energy of the EMs in a hyperbolic material with that of a classical BB. From expressions [7] and [19] we immediately get

$$\frac{U_{\parallel}^0}{U_{\parallel}^{\text{BB}}} \approx \sqrt{|\epsilon_{\parallel} \epsilon_{\perp}|} (k_{\perp,\text{max}} l_c) \frac{\Gamma(3) \zeta(3)}{\Gamma(4) \zeta(4)}. \quad (23)$$

If $\Lambda$ denotes the unit-cell size of our hyperbolic material then $k_{\perp,\text{max}} = (2\pi)/\Lambda$, so that

$$\frac{U_{\parallel}^0}{U_{\parallel}^{\text{BB}}} \propto \frac{l_c}{\Lambda}. \quad (24)$$

At a temperature of 300 K the coherence length is $l_c = 7.6 \mu m$. The period of realistic hyperbolic metamaterials is typically larger than $\Lambda \approx 10 nm$. In natural hyperbolic materials the unit cell size reduces to the interatomic spacing, i.e. $\Lambda \approx 1 \AA$. Hence, the internal energy of thermal radiation inside a hyperbolic cavity can be 3 to 5 orders of magnitude larger than that of a perfect BB. The same is of course also true for the free energy and the entropy. This result suggests that the radiative heat flux inside a hyperbolic material is dramatically enhanced compared to that of a classical BB.

In order to evaluate the flux radiated by a cavity filled with a hyperbolic medium into a hyperbolic medium and to derive Stefan Boltzmann’s law we calculate now the Poynting vector in the cavity in the direction of the principal optical axis by assuming, for convenience, that the cavity opening (see Fig. 2) is along this axis. Using the framework of fluctuational electrodynamics theory the ensemble average of the Poynting vector (for any dispersive and dissipative anisotropic medium) reads [19] (Einstein’s convention)

$$\langle S_\gamma \rangle = \zeta_{\alpha\beta\gamma} 2 \Re \int_0^{\infty} \frac{d\omega}{2\pi} \frac{2\omega^3 \mu_0}{c^2} \int V \frac{\Gamma(3) \zeta(3)}{\Gamma(4) \zeta(4)}.$$

Here we have introduced the Levi-Civita tensor $\zeta_{\alpha\beta\gamma}$ and the permeability of vacuum $\mu_0$. Note that $G^{\text{HE}}(\mathbf{r}, r') = \frac{1}{16 \pi \mu_0 c^2} \nabla \times G^{\text{EE}}(\mathbf{r}, r')$ and that $\dagger$ denote the conjugate transpose. In order to determine the heat flux, we assume that the cavity is infinitely large so that we can replace it by a uniaxial halfspace at a given temperature $T$. Inserting the Green’s dyadic [8] and integrating over this halfspace with volume $V$ we find after a lengthy calculation (see Ref. [9]) in the lossless limit, the relatively simple ex-
Hence, in the non-dispersive case, where we find \( \Phi_o \) for dielectric anisotropic material, whereas as a consequence we find change. Before seeing this, let us first consider the OMs. The BB value, which is a well-known fact [14].

In this case, we see that the heat flux is proportional to \( \frac{\omega^2}{c^2} \epsilon \) and \( \frac{\omega^2}{c^2} \epsilon \) for non-dispersive materials this simplifies to \( \Phi_o/\epsilon = \frac{\epsilon}{c^2} \). On the other hand, inside the BB value, which is a well-known fact [14].

When \( \epsilon \neq \epsilon \neq 1 \) we find again the usual BB result, i.e. Stefan-Boltzmann’s law. On the other hand, inside a uniaxial material (as inside an isotropic material) with \( \epsilon > 1 \) and \( \epsilon > 1 \) the radiative heat flux is larger than the BB value, which is a well-known fact [14].

In the case of hyperbolic media these results radically change. Before seeing this, let us first consider the OMs. For \( \Phi_o/\epsilon \) we find of course the same relation as for the dielectric anisotropic material, whereas as a consequence that there do not exist any OMs in a type II hyperbolic material we find \( \Phi_o = 0 \). On the contrary, for the EMs we find

\[
\Phi_o = \int_0^\infty d\omega U(\omega, T) \frac{\omega^2}{\pi^2 c^4} \frac{1}{2} \left\{ \epsilon \right\}
\]

For non-dispersive materials this simplifies to

\[
\Phi_o = \frac{c}{4} T^{s/p} \left\{ \epsilon \right\}
\]

When \( \epsilon = \epsilon \neq 1 \) we find again the usual BB result, i.e. Stefan-Boltzmann’s law. On the other hand, inside a uniaxial material (as inside an isotropic material) with \( \epsilon > 1 \) and \( \epsilon > 1 \) the radiative heat flux is larger than the BB value, which is a well-known fact [14].

The type I hyperbolic BB behaves like a perfect BB and the type II hyperbolic BB behaves like a perfect metal or a “white” body. To summarize, we have extended the BB theory to arbitrary uniaxial materials. For dielectric anisotropic media we have seen that the thermodynamic properties of the photon gas inside such media are very similar to that of a classical BB. On the other hand, when these media are hyperbolic, the spectral energy distribution
of radiation is shifted towards frequencies smaller than Wien’s frequency making these media apparently colder. We have also shown that in contrast to Stefan Boltzmann’s law, the heat flux radiated by these media depends quadratically on their temperature. Nevertheless, the magnitude of heat flux carried by these media can be several orders of magnitude larger than the flux radiated by a classical BB and may even exceed the heat flux carried by conduction in superlattices. Detailed derivations of the above relations and the underlying assumptions as well as more detailed discussions will be given elsewhere [9].

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