Dynamic logic assigned to automata

Ivan Chajda · Jan Paseka

Abstract A dynamic logic $B$ can be assigned to every automaton $A$ without regard if $A$ is deterministic or nondeterministic. This logic enables us to formulate observations on $A$ in the form of composed propositions and, due to a transition functor $T$, it captures the dynamic behaviour of $A$. There are formulated conditions under which the automaton $A$ can be recovered by means of $B$ and $T$.

Keywords dynamic logic · automaton · state-transition relation · transition functor · modal functor

Mathematics Subject Classification (2000) 03B60 · 03D05 · 68S05

1 Introduction

The aim of the paper is to assign a certain logic to a given automaton without regard to whether it is deterministic or nondeterministic. This logic has to be dynamic in the sense to capture dynamicity of working automaton. We consider an automaton as $A = (X, S, R)$, where $X$ is a non-empty set of inputs, $S$ is a non-empty set of states and $R \subseteq X \times S \times S$ is the set of labelled transitions. In this case we say that $R$ is a state-transition relation and it is considered as a dynamics of $A$. Hence, the automaton $A$ can be visualized as a graph whose vertices are states and edges denote (possibly multiple) transitions $s \xrightarrow{x} t$ from one state $s$ to another state $t$ provided an input $x$ is coming; this is visualized by a label $x$ on the edge $(s, t)$. In particular, motivated by the above considerations and e.g. by the paper [1] where

Both authors acknowledge the support by a bilateral project New Perspectives on Residuated Posets financed by Austrian Science Fund (FWF): project I 1923-N25, and the Czech Science Foundation (GAČR): project 15-34697L.

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A denumerable set of vertices is used in studying quantum automata to recover the Weyl, Dirac and Maxwell dynamics in the relativistic limit we have to assume that the sets $X$ and $S$ can have arbitrarily large cardinality.

Any physical system can be in some sense considered as an automaton. Its states are then states of the automaton and the transition relation is a transition of a physical system from a given state to an admissible one. It should be noted that a quantum physical system is nondeterministic since particles can pass through a so-called superposition, i.e., they may randomly select a state from the set of admissible states.

On the other hand, we often formulate certain propositions on an automaton $A$ and deduce conclusions about the behaviour of $A$ in the present (i.e., a description) or in a (near) future (i.e., a forecast). It is apparent that for this aim we need a certain logic which is derived from a given automaton and which enables us to formulate propositions on $A$ and to deduce conclusions and consequences. Due to the mentioned dynamics of $A$, our logic $B$ should contain a tool for a certain dynamics. This tool will be called a transition functor. This transition functor will assign to every proposition $p \in B$ and input $x \in X$ another proposition $q$. In a certain case, this functor can be considered as a modal functor with one more input from $X$. The above mentioned approach has a sense if our logic $B$ with a transition functor $T$ enables us to reconstruct the dynamics of a given automaton $A$. One can compare our approach with the approach from [1] where an automaton can be represented by an operator over a Hilbert space or with the approach from [16] or [11] where the role of the transition functor is played by a map from $S$ to $(M^S)^X$ where $M$ is a bounded lattice of truth-values or by a map from $S$ to $(\{0, 1\}^S)^X$.

In what follows, we are going to involve a systematic approach how to reach such a transition functor and the logic $B$ such that the reconstruction of the state-transition relation $R$ is possible. Since the conditions of our approach are formulated in a pure algebraic way, we need to develop an algebraic background (see e.g. also in [2]). It is worth noticing that the transition functor will be constructed formally in a similar way as tense operators introduced by J. Burgess [3] for the classical logic and developed by the authors for several non-classical logics, see [4], [5] and [6], and also the monograph [7]. Because we are not interested in outputs of the automaton $A$, we will consider $A$ as the so-called acceptor only.

It is worth noticing that certain (temporal) logics assigned to automata were already investigated by several authors, see e.g. the seminal papers on temporal logics for programs by Vardi [14], [15], the papers [9, 12] and the monograph [10] for additional results and references. However, our approach is different. Namely, our logic assigned to an automaton is equipped with the so-called transition operator which makes the logic to be dynamic.

Besides of the previous, the observer or a user of an automaton can formulate propositions revealing our knowledge about it depending on the input. The truth-values of these propositions depend on states and inputs and let us assume that these propositions can acquire only two values, namely either TRUE of FALSE. For example, if we fix an input $x \in X$, the proposition $p/x$ can be true if the automaton $A$ is in the state $s$ but false if $A$ is not in the state $s$. Hence, for each state $s \in S$ we can evaluate the truth-value of $p/x$. As mentioned above, $p/x(s) \in \{0, 1\}$ where 0 indicates the truth-value FALSE and 1 indicates TRUE.

Denote by $B$ the set of propositions about the automaton $A$ formulated by the observer. We can introduce the order $\leq$ on $B$ as follows:

$$\text{for } p, q \in B, p \leq q \text{ if and only if } p(s) \leq q(s) \text{ for all } s \in S.$$  

One can immediately check that the contradiction, i.e., the proposition with constant truth-value 0, is the least element and the tautology, i.e., the proposition with the constant truth-
value 1 is the greatest element of the partially ordered set \( (B; \leq) \); this fact will be expressed by the notation \( B = (B; \leq, 0, 1) \) for the bounded partially ordered set of propositions about the automaton \( A \).

We summarize our description as follows:
- every automaton \( A \) will be identified with the triple \( (B, X, S) \), where \( B \) is the set of propositions about \( A \), \( X \) is the set of possible inputs and \( S \) is the set of states on \( A \);
- we are given a set of labelled transitions \( R \subseteq X \times S \times S \) such that, for an input \( x \in X \), \( A \) can go from \( s \) to \( t \) provided \( (x, s, t) \in R \);
- the set \( B \) is partially ordered by values of propositions as shown above.

If \( s \xrightarrow{x} t_1 \) and \( s \xrightarrow{x} t_2 \) yields \( t_1 = t_2 \) for all \( s, t_1, t_2 \in S \) and \( x \in X \) we say that \( A \) is a deterministic automaton. If \( A \) is not deterministic we say that it is nondeterministic.

To shed light on the previous concepts, let us present the following example.

**Example 1** At first, let us present a very simple automaton \( A \) describing a SkyLine Terminal Transfer Service at an airport between Terminals 1 and 2. The SkyLine train is housed, repaired and maintained in the engine shed and the only way how to get there is through Terminal 2.

The observer can distinguish three states as follows:
- \( s_1 \) means that the SkyLine train is in Terminal 1,
- \( s_2 \) means that the SkyLine train is in Terminal 2,
- \( s_3 \) means that the SkyLine train is in the engine shed.

There are two possible actions:
- \( x_1 \) means that the passengers entered the SkyLine train,
- \( x_2 \) means that the SkyLine train has to be moved to the engine shed.

If the SkyLine train is in Terminal 1 or in Terminal 2 then, after the passengers entered it, it moves to the other terminal. If the SkyLine train is in Terminal 2 then, after the request that the SkyLine train has to be moved to the engine shed is issued, it moves to the engine shed. If the SkyLine train is in the engine shed then, regardless of what action is requested, it stays there.

The set \( R \) of labelled transitions on the set \( S = \{s_1, s_2, s_3\} \) of states under actions from the set \( X = \{x_1, x_2\} \) is of the form

\[
R = \{(x_1, s_1, s_2), (x_1, s_2, s_1), (x_1, s_3, s_1), (x_2, s_2, s_3), (x_2, s_3, s_3)\}
\]

and it can be visualized as follows.

![Fig. 1 The transition graph of \( R \)](image)

The set \( B = \{0, p, q, r, p', q', r', 1\} \) of possible propositions \( B \) about the automaton \( A \) is as follows:
- 0 means that the SkyLine train is in no state of $S$,
- $p$ means that the SkyLine train is in Terminal 1,
- $q$ means that the SkyLine train is in Terminal 2,
- $r$ means that the SkyLine train is in the engine shed,
- 1 means that the SkyLine train is in at least one state of $S$.

Considering $B$ as a classical logic (represented by a Boolean algebra $(B; \lor, \land, \neg, 0, 1)$), we can apply logical connectives conjunction $\land$, disjunction $\lor$, negation $\neg$ and implication $\rightarrow$ to create new propositions about $\mathcal{A}$. In our case, we can get e.g. $p' = q \lor r$ which means that the SkyLine train is either in Terminal 2 or in the engine shed, etc. Altogether, we obtain eight propositions. We may identify $B$ with the Boolean algebra $\{0, 1\}^S$ as follows:

$$0 = (0,0,0), \quad p = (1,0,0), \quad q = (0,1,0), \quad r = (0,0,1),$$

$$p' = (0,1,1), \quad q' = (1,0,1), \quad r' = (1,1,0), \quad 1 = (1,1,1).$$

The interpretation of propositions from $B$ is as follows: for any $\alpha \in B$, $\alpha$ is true in the state $s_i$ of the automaton $\mathcal{A}$ if and only if $\alpha(s_i) = 1$.

### 2 Algebraic tools

For the above mentioned construction of a suitable logic with a transition functor and the reconverse of the given relation, we recall the following necessary algebraic tools and results in this section.

Let $S$ be a non-empty set. Every subset $R \subseteq S \times S$ is called a relation on $S$ and we say that the couple $(S,R)$ is a transition frame. The fact that $(s,t) \in R$ for $s,t \in S$ is expressed by the notation $s R t$.

Let $A$ be a non-empty set. A relation on $A$ is called a partial order if it is reflexive, antisymmetric and transitive. In what follows, partial order will be denoted by the symbol $\leq$ and the pair $A = (A; \leq)$ will be referred to as a partially ordered set (shortly a poset).

Let $(A; \leq)$ and $(B; \leq)$ be partially ordered sets, $f, g: A \rightarrow B$ mappings. We write $f \leq g$ if $f(a) \leq g(a)$, for all $a \in A$. A mapping $f$ is called order-preserving or monotone if $a,b \in A$ and $a \leq b$ together imply $f(a) \leq f(b)$ and order-reflecting if $a,b \in A$ and $f(a) \leq f(b)$ together imply $a \leq b$. A bijective order-preserving and order-reflecting mapping $f: A \rightarrow B$ is called an isomorphism and then we say that the partially ordered sets $(A; \leq)$ and $(B; \leq)$ are isomorphic.

Let $(A; \leq)$ and $(B; \leq)$ be partially ordered sets. A mapping $f: A \rightarrow B$ is called residuated if there exists a mapping $g: B \rightarrow A$ such that $f(a) \leq b$ if and only if $a \leq g(b)$ for all $a \in A$ and $b \in B$. In this situation, we say that $f$ and $g$ form a residuated pair or that the pair $(f, g)$ is a (monotone) Galois connection. The role of Galois connections is essential for our constructions.

If a partially ordered set $A$ has both a bottom and a top element, it will be called bounded; the appropriate notation for a bounded partially ordered set is $(A; \leq, 0, 1)$. Let $(A; \leq, 0, 1)$ and $(B; \leq, 0, 1)$ be bounded partially ordered sets. A morphism $f: A \rightarrow B$ of bounded partially ordered sets is an order, top element and bottom element preserving map.

We can take the following useful result from [4] Observation 1.

**Observation 1** (H) Let $A$ and $M$ be bounded partially ordered sets, $S$ a non-empty set, and $h_s: A \rightarrow M, s \in S$, morphisms of bounded partially ordered sets. The following conditions are equivalent:
(i) \((\forall s \in S) h_s(a) \leq h_s(b) \) \implies a \leq b for any elements \(a, b \in A\);

(ii) The map \(i_A^s : A \to M^s\) defined by \(i_A^s(a) = (h_s(a))_{s \in S}\) for all \(a \in A\) is order reflecting.

We then say that \(\{h_s : A \to M; s \in S\}\) is a full set of order-preserving maps with respect to \(M\).

Note that we may in this case identify \(A\) with a bounded subposet of \(M^s\) since \(i_A^s\) is an order reflecting morphism alias embedding of bounded partially ordered sets. For any \(s \in S\) and any \(p = (p_i)_{i \in S} \in M^s\) we denote by \(p(s)\) the \(s\)-th projection \(p_i\). Note that \(i_A^s(a)(s) = h_s(a)\) for all \(a \in A\) and all \(s \in S\).

3 Transition frames and transition operators

The aim of this section is to recall a construction of two operators on partially ordered sets derived by means of a given relation and a construction of relations induced by these operators. For more details see the paper \([8]\).

In what follows, let \(M = (M; \leq, 0, 1)\) be a bounded partially ordered set and the bounded subposets \(A = (A; \leq, 0, 1)\) and \(B = (B; \leq, 0, 1)\) of \(M^s\) will play the role of possibly different logics of propositions pertaining to our automaton \(A\), a corresponding set of states \(S\), and a state-transition relation \(R \subseteq S\). The operator \(T_k : B \to M^s\) will prescribe to a proposition \(b \in B\) about \(A\) a new proposition \(T_k(b) \in M^s\) such that the truth value of \(T_k(b)\) in state \(s \in S\) is the greatest truth value that is smaller or equal than the corresponding truth values of \(b\) in all states that can be reached from \(s\). If there is no such state the truth value of \(T_k(b)\) in state \(s\) will be 1. Similarly, the operator \(P_k : A \to M^s\) will prescribe to a proposition \(a \in A\) about \(A\) a new proposition \(P_k(a) \in M^s\) such that the truth value of \(P_k(a)\) in state \(t \in S\) is the smallest truth value that is greater or equal than the corresponding truth values of \(b\) in all states such that \(t\) can be reached from them. If there is no such state the truth value of \(P_k(a)\) in state \(t\) will be 0.

Specifically, if \(M = \{0, 1\}\) then \(T_k(b)\) is true in state \(s\) if and only if there is no state \(t \in S\) that can be reached from \(s\) and \(b\) is false in \(t\), and \(P_k(a)\) is false in state \(t\) if and only if there is no state \(s \in S\) such that \(t\) can be reached from \(s\) and \(b\) is true in \(s\).

Consider a complete lattice \(M = (M; \leq, 0, 1)\) and let \(A = (A; \leq, 0, 1)\) and \(B = (B; \leq, 0, 1)\) be bounded partially ordered sets with a full set \(S\) of morphisms of bounded partially ordered sets into a non-trivial complete lattice \(M\). We may assume that \(A\) and \(B\) are bounded subposets of \(M^s\). Further, let \((S, R)\) be a transition frame.

Define mappings \(P_k : A \to M^s\) and \(T_k : B \to M^s\) as follows: For all \(b \in B\) and all \(s \in S\),

\[
T_k(b)(s) = \forall_{M^s} \{b(t) \mid sRt\}
\]

and, for all \(a \in A\) and all \(t \in S\),

\[
P_k(a)(t) = \exists_{M^s} \{a(s) \mid sRt\}
\]

Then we say that \(T_k\) (or \(P_k\)) is an upper transition functor (lower transition functor) constructed by means of the transition frame \((S, R)\), respectively. We have that \(T_k\) is an order-preserving map such that \(T_k(1) = 1\) and similarly, \(P_k\) is an order-preserving map such that \(P_k(0) = 0\).

As an illustration of our approach we present the following example.

Example 2 Consider the automaton \(A\) and the set of propositions \(B\) of Example\([1]\). Then \(R = \{x_1\} \times R_{x_1} \cup \{x_2\} \times R_{x_2}\) where \(R_{x_1} = \{(x_1, x_2), (x_2, x_1), (x_3, x_1)\}\) and \(R_{x_2} = \{(x_2, x_3), (x_3, x_3)\}\).
Using our formulas (\*) and (**), we can compute the upper transition functors \( T_{R_{\downarrow}} : B \to 2^S \) and the lower transition functors \( P_{R_{\downarrow}}, P_{R_{\uparrow}} : B \to 2^S \) as follows:

\[
\begin{align*}
T_{R_{\downarrow}}(0) &= 0, & T_{R_{\downarrow}}(1) &= 1, & T_{R_{\downarrow}}(0) &= p, & T_{R_{\downarrow}}(1) &= 1, \\
T_{R_{\downarrow}}(p) &= q, & T_{R_{\downarrow}}(p') &= q', & T_{R_{\downarrow}}(p) &= p, & T_{R_{\downarrow}}(p') &= 1, \\
T_{R_{\downarrow}}(q) &= p, & T_{R_{\downarrow}}(q') &= p', & T_{R_{\downarrow}}(q) &= p, & T_{R_{\downarrow}}(q') &= 1, \\
T_{R_{\downarrow}}(r) &= r, & T_{R_{\downarrow}}(r') &= r', & T_{R_{\downarrow}}(r) &= 1, & T_{R_{\downarrow}}(r') &= p, \\
P_{R_{\downarrow}}(0) &= 0, & P_{R_{\downarrow}}(1) &= 1, & P_{R_{\downarrow}}(0) &= 0, & P_{R_{\downarrow}}(1) &= r, \\
P_{R_{\downarrow}}(p) &= q, & P_{R_{\downarrow}}(p') &= q', & P_{R_{\downarrow}}(p) &= 0, & P_{R_{\downarrow}}(p') &= r, \\
P_{R_{\downarrow}}(q) &= p, & P_{R_{\downarrow}}(q') &= p', & P_{R_{\downarrow}}(q) &= r, & P_{R_{\downarrow}}(q') &= r, \\
P_{R_{\downarrow}}(r) &= r, & P_{R_{\downarrow}}(r') &= r', & P_{R_{\downarrow}}(r) &= r, & P_{R_{\downarrow}}(r') &= r.
\end{align*}
\]

E.g., \( T_{R_{\downarrow}}(q) = p \) means that if the Skyline train is in Terminal 1 then, after any possible transition under the action that the passengers entered the Skyline train, it will change to Terminal 2, and \( T_{R_{\downarrow}}(q') = p' \) means that if the Skyline train is in Terminal 2 or in the engine shed then, after any possible transition under the action that the passengers entered the Skyline train, it will be in Terminal 1 or in the engine shed. Similarly, \( T_{R_{\downarrow}}(1) = 1 \) means that if the Skyline train is in at least one state of \( S \) then, after any possible transition under the action that the Skyline train has to be moved to the engine shed, it will be in at least one state of \( S \), and \( T_{R_{\downarrow}}(p) = p \) means that if the Skyline train is in Terminal 1 then, after any possible transition under the action that the Skyline train has to be moved to the engine shed (which can be done only at Terminal 2 or at the engine shed), it will stay in Terminal 1.

Let \( P : A \to B \) and \( T : B \to A \) be morphisms of partially ordered sets, \((A; \leq)\) and \((B; \leq)\) subposets of \( M \). Let us define the relations

\[
R_T = \{(s, t) \in S \times S \mid (\forall b \in B)(T(b)(s) \leq b(t))\} \quad (\dagger)
\]

and

\[
R^P = \{(s, t) \in S \times S \mid (\forall a \in A)(a(s) \leq P(a)(t))\}. \quad (\ddagger)
\]

The relations \( R_T \) and \( R^P \) on \( S \) will be called the upper \( T \)-induced relation by \( M \) (shortly \( T \)-induced relation by \( M \)) and lower \( P \)-induced relation by \( M \) (shortly \( P \)-induced relation by \( M \)), respectively.

**Example 3** Consider the automaton \( \mathcal{A} \) of Example 1. Let \( P \) be a restriction of the operator \( P_{R_{\downarrow}} \) of Example 2 and let \( T \) be a restriction of the operator \( T_{R_{\downarrow}} \) of the same example. Let us compute \( R_T \) and \( R^P \). We have \( R_T = R^P = \{(s_2, s_1), (s_3, s_3)\} \). Hence the transition relation \( R_T \) of Example 2 coincides with our induced transitions relations \( R_T \) and \( R^P \). We can see from above that the operator \( T_{R_{\downarrow}} \) bears the maximal amount of information about the transition relation \( R_T \) on the subposet of all fixpoints of \( P_{R_{\downarrow}} \circ T_{R_{\downarrow}} \). The same conclusion holds for the operator \( P_{R_{\downarrow}} \).

Now, let \((S, R)\) be a transition frame and \( T_R, P_R \) functors constructed by means of the transition frame \((S, R)\). We can ask under what conditions the relation \( R \) coincides with the relation \( R_T \) constructed as in (\dagger) or with the relation \( R^P \) constructed as in (\ddagger). If this is the case we say that \( R \) is recoverable from \( T_R \) or that \( R \) is recoverable from \( P_R \). We say that \( R \) is recoverable if it is recoverable both from \( T_R \) and \( P_R \).
Example 4 Consider the automaton \( \mathcal{A} \) of Example 1. Let us put \( A = B = \{0,1\}^3 \). Let \( P: \{0,1\}^3 \to \{0,1\}^3 \) and \( T: \{0,1\}^3 \to \{0,1\}^3 \) be morphisms of partially ordered sets given as follows:

\[
\begin{align*}
T(0) &= 0, T(p) = q, T(q) = p, T(r) = r, T(p') = q', T(q') = p', T(r') = r', T(1) = 1, \\
P(0) &= 0, P(p) = q, P(q) = p, P(r) = r, P(p') = q', P(q') = p', P(r') = r', P(1) = 1.
\end{align*}
\]

Note that \( P \) coincides with the operator \( P_{R_0} \) of Example 2 and \( T \) coincides with the operator \( T_{R_0} \) of the same example. We have \( R_T = R^p = \{(s_1,s_2),(s_2,s_1),(s_3,s_3)\} \). The transition relation \( R_{T_1} \) of Example 1 coincides with our induced transitions relations \( R_T \) and \( R^p \).

The connection between relations induced by means of transition functors \( T \) and \( P \) is shown in the following lemma and theorem.

Lemma 1 \([8]\) Let \( M \) be a non-trivial complete lattice and \( S \) a non-empty set such that \( A \) and \( B \) are bounded subposets of \( M^S \). Let \( P: A \to M^S \) and \( T: B \to M^S \) be morphisms of partially ordered sets such that, for all \( a \in A \) and all \( b \in B \),

\[
P(a) \leq b \iff a \leq T(b).
\]

(a) If \( P(A) \subseteq B \) then \( R_T \subseteq R^p \).
(b) If \( T(B) \subseteq A \) then \( R^p \subseteq R_T \).
(c) If \( P(A) \subseteq B \) and \( T(B) \subseteq A \) then \( R_T = R^p \).

Among other things, the following theorem shows that if a given transition relation \( R \) can be recovered by the upper transition functor then, under natural conditions, it can be recovered by the lower transition functor and vice versa.

Theorem 2 \([8]\) Let \( M \) be a non-trivial complete lattice and \( (S,R) \) a transition frame. Let \( A \) and \( B \) be bounded subposets of \( M^S \). Let \( P_T: A \to M^S \) and \( T_R: B \to M^S \) be functors constructed by means of the transition frame \((S,R)\). Then, for all \( a \in A \) and all \( b \in B \),

\[
P_T(a) \leq b \iff a \leq T_R(b).
\]

Moreover, the following holds.

(a) Let for all \( t \in S \) exist an element \( b' \in B \) such that, for all \( s \in S \), \((s,t) \notin R \), we have \( \bigwedge_{M} \{u_s(b') \mid sRu\} \leq t(b') \neq 1 \). Then \( R = R_{T_R} \).
(b) Let for all \( s \in S \) exist an element \( a' \in A \) such that, for all \( t \in S \), \((s,t) \notin R \), we have \( \bigvee_{M} \{u_s(a') \mid uRt\} \geq s(a') \neq 0 \). Then \( R = R^p_{P_T} \).
(c) If \( R = R_{T_R} \) and \( T_R(b) \subseteq A \) then \( R = R_{T_R} = R^p_{P_T} \).
(d) If \( R = R^p_{P_T} \) and \( P_T(A) \subseteq B \) then \( R = R_{T_R} = R^p_{P_T} \).

The following corollary of Theorem 2 shows that if the set \( B \) of propositions on the system \((B,S)\) is large enough, i.e., if it contains the full set \( \{0,1\}^S \) then the transition relation \( R \) can be recovered by each of the transition functors.

Corollary 1 \([8]\) Let \( M \) be a non-trivial complete lattice and \( (S,R) \) a transition frame. Let \( B \) be a bounded subposet of \( M^S \) such that \( \{0,1\}^S \subseteq B \). Let \( P_T: B \to M^S \) and \( T_R: B \to M^S \) be functors constructed by means of the transition frame \((S,R)\). Then \( R = R^p_{P_T} = R_{T_R} \).
4 The labelled transition functor characterizing the automaton

The aim of this section is to derive the logic $\mathbf{B}$ with transition functors corresponding to a given automaton $\mathcal{A} = (X, S, R)$. This logic $\mathbf{B}$ will be represented via the partially ordered set of its propositions. In the rest of the paper, truth-values of our logic $\mathbf{B}$ will be considered to be from the complete lattice $\mathbf{M}$. Thus $\mathbf{B}$ will be a bounded subposet of $\mathbf{M}^\mathcal{T}$ for the complete lattice $\mathbf{M}$ of truth-values.

Let us consider an automaton $\mathcal{A} = (X, S, R)$. Clearly, $R$ can be written in the following form

$$R = \bigcup_{x \in X} \{x\} \times R_x$$

where $R_x \subseteq S \times S$ for all $x \in X$. Hence, for all $x \in X$, using our formulas $(\ast)$ and $(\ast\ast)$, we obtain the upper transition functor $T_R : B \to \mathbf{M}^S$ and the lower transition functor $P_R : B \to \mathbf{M}^S$. It follows that we have functors $T_R = (T_R)_x \subseteq X : B \to (\mathbf{M}^S)^x$ and $P_R = (P_R)_x \subseteq X : B \to (\mathbf{M}^S)^x$. We say that $T_R$ is the labelled upper transition functor constructed by means of $\mathcal{A}$ and $P_R$ is the labelled lower transition functor constructed by means of $\mathcal{A}$. Note that any mapping $T : B \to (\mathbf{M}^S)^X$ corresponds uniquely to a mapping $\tilde{T} : X \times B \to \mathbf{M}^S$ such that, for all $x \in X, T = (\tilde{T}(x,-))_{x \in X}$. Hence, $T_R$ and $P_R$ will play the role of our transition functor.

Now, let $P = (P_x)_{x \in X} : B \to (\mathbf{M}^S)^x$ and $T = (T_x)_{x \in X} : B \to (\mathbf{M}^S)^x$ be morphisms of partially ordered sets. For all $x \in X$, let $R^P$ be the lower $P_x$-induced relation by $\mathbf{M}$ and $R_T$ be the upper $T_x$-induced relation by $\mathbf{M}$. Then $R^P = \bigcup_{x \in X} \{x\} \times R^P_x$ is called the lower $P$-induced state-transition relation and $R_T = \bigcup_{x \in X} \{x\} \times R_T^x$ is called the upper $T$-induced state-transition relation. The automaton $\mathcal{A}^P = (X, S, R^P)$ is said to be the lower $P$-induced automaton and the automaton $\mathcal{A}_T = (X, S, R_T)$ is said to be the upper $T$-induced automaton.

We say that the automaton $\mathcal{A}$ is recoverable from $T_R (R_T)$ if, for all $x \in X$, $R_x$ is recoverable from $T_R (R_T)$, i.e., if $\mathcal{A} = \mathcal{A}_T \bowtie \mathcal{A} (\mathcal{A} = \mathcal{A}^P)$. The following results follow immediately from Lemma [1] Theorem [2] and Corollary [1].

**Theorem 3** Let $\mathbf{M}$ be a non-trivial complete lattice and $S, X$ non-empty sets such that $\mathbf{B}$ is a bounded subposet of $\mathbf{M}^S$. Let $P : B \to (\mathbf{M}^S)^X$ and $T : B \to (\mathbf{M}^S)^X$ be morphisms of partially ordered sets such that, for all $a, b \in B$ and all $x \in X$,

$$P_x(a) \leq b \iff a \leq T_x(b).$$

(a) If $P(B) \subseteq B^S$ then $R_T \subseteq R^P$.
(b) If $T(B) \subseteq B^S$ then $R^P \subseteq R_T$.
(c) If $P(B) \subseteq B^S$ and $T(B) \subseteq B^S$ then $R_T = R^P$ and $\mathcal{A}_T = \mathcal{A}^P$.

Hence, using Theorem [3], we can ask whether the functors computed by $(\ast)$ and $(\ast\ast)$ can recover a given relation $R$ on the set of states. The answer is in the following theorem.

**Theorem 4** Let $\mathbf{M}$ be a non-trivial complete lattice and $S, X$ non-empty sets equipped with a set of labelled transitions $R \subseteq X \times S \times S$. Let $\mathbf{B}$ be a bounded subposet of $\mathbf{M}^S$. Let $P_R : B \to (\mathbf{M}^S)^X$ and $T_R : B \to (\mathbf{M}^S)^X$ be labelled transition functors constructed by means of $R$. Then, for all $a, b \in B$ and all $x \in X$,

$$P_R(a) \leq b \iff a \leq T_R(b).$$

Moreover, the following holds.

(a) If $R = R_T$ and $T_R(B) \subseteq B^S$ then $R = R_T = R^P$. 

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(b) If \( R = R^S_k \) and \( P_R(B) \subseteq B^S \) then \( R = R_{T_k} = R^S_k \).

The following corollary illustrates the situation in the case when our partially ordered set \( B \) of propositions is large enough, i.e., the case when \( \{0, 1\}^3 \subseteq B \).

**Corollary 2** Let \( M \) be a non-trivial complete lattice and \( A = (X, S, R) \) an automaton. Let \( B \) be a bounded subposet of \( M^S \) such that \( \{0, 1\}^3 \subseteq B \). Then the automaton \( A \) is recoverable both from \( P_R \) and \( T_k \).

We can illustrate previous results in the following example.

**Example 5** Consider the automaton \( A \), the set of propositions \( B \) and the state-transition relation \( R \) of Example 1. From Example 2 we know the labelled upper transition functor \( T_k = (T_{R_{x_1}}, T_{R_{x_2}}) \) and the labelled lower transition functor \( P_R = (P_{R_{x_1}}, P_{R_{x_2}}) \) from \( B \) to \( (2^S)^X \). Since \( B = 2^S \) we have \( T_{R_{x_1}}(B) \cup T_{R_{x_2}}(B) \subseteq B \) and \( P_{R_{x_1}}(B) \cup P_{R_{x_2}}(B) \subseteq B \).

Now, we use \( T_k \) for computing the transition relations \( R_{T_{x_1}} \) and \( R_{T_{x_2}} \) (by the formula \( \dagger \) and Example 3) and \( P_R \) for computing the transition relations \( R^{P_{R_{x_1}}} \) and \( R^{P_{R_{x_2}}} \) (by the formula \( \dagger \dagger \) and Example 4). We obtain by Corollary 1 that \( R_{T_{x_1}} = R^{P_{R_{x_1}}} = R_{x_1} \) and \( R_{T_{x_2}} = R^{P_{R_{x_2}}} = R_{x_2} \). It follows that \( R_{T_k} = R^S_k = \{x_1\} \times R_{T_{x_1}} \cup \{x_2\} \times R_{T_{x_2}} = R \), i.e., our given state-transition relation \( R \) is simultaneously recoverable by the transition functors \( T_k \) and \( P_R \). Hence these functors are characteristics of the triple \((B, X, S)\).

5 Constructions of automata

By a synthesis in Theory of Systems is usually meant the task to construct an automaton \( A \) which realizes a dynamic process at least partially known to the user. Hence, we are given a description of this dynamic process and we know the set \( X \) of inputs. Our task is to set up the set \( S \) of states and a relation \( R \) on \( S \) labelled by elements from \( X \) such that the constructed automaton \((X, S, R)\) induces the logic, i.e., the partially ordered set of propositions, which corresponds to the original description.

The algebraic tools collected in previous sections enable us to solve the mentioned task. In what follows we involve a construction of \( S \) and \( R \) provided our logic with the transition functor representing the dynamics of our system is given. As in the previous section, our logic \( B \) will be considered to be a bounded subposet \( B \) of a power \( M^S \) where \( M \) is a complete lattice of truth-values. Our logic \( B \) is equipped with a transition functor \( T : B \to (M^S)^X \) where \( X \) is a set of possible inputs. We ask that either \( T = T_k \) or \( T = P_R \). Depending on the respective type of our considered logic and of the properties of \( T \) we will present some partial solutions to this task.

5.1 Automata via partially ordered sets

Recall that (see e.g. [13]), for any bounded partially ordered set \( B = (B; \leq, 0, 1) \), we have a full set \( S_B \) of morphisms of bounded partially ordered set into the two-element Boolean algebra considered as a bounded partially ordered set \( 2 = ((0, 1); \leq, (0, 1)) \). The elements \( h_B : B \to (0, 1) \) of \( S_B \) (indexed by proper down-sets \( D \) of \( B \)) are morphisms of bounded partially ordered sets defined by the prescription \( h_B(a) = 0 \) iff \( a \in D \).

In other words, every bounded partially ordered set \( B \) can be embedded into a Boolean algebra \( 2^S \) for a certain set \( S \) via the mapping \( h_B \).
Hence, it looks hopeful to use the bounded partially ordered set $2 = (\{0, 1\}; \leq, 0, 1)$ for the construction of our state-transition relation $R_T \subseteq X \times S_B \times S_B$.

As mentioned in the beginning of this section, we are interested in a construction of an automaton $A = (X, S, R)$ for a given set $X$ of inputs and determined by a certain partially ordered set of propositions. We cannot assume that this set of propositions is necessarily a Boolean algebra. In the previous part we supposed that this logic $B$ is a bounded partially ordered set $B = (B; \leq, 0, 1)$. Now, we are going to solve the situation when it is only a subset $C$ of $B$.

**Theorem 5** Let $B = (B; \leq, 0, 1)$ be a bounded partially ordered set such that $B$ is a bounded subposet of $2^B$. Let $(C; \leq, 1)$ be a subposet of $B$ containing 1, and $X$ a non-empty set. Let $T = (T_x)_{x \in X}$ where $T_x : C \to 2^\alpha$ are morphisms of partially ordered sets such that $T_x(1) = 1$ for all $x \in X$. Let $R_T$ be the upper $T$-induced state-transition relation and $T_{R_T} : B \to (2^B)^X$ be the labelled upper transition functor constructed by means of the upper $T$-induced automaton $A_T = (X, S_B, R_T)$. Then, for all $b \in C$,

$$T(b) = T_{R_T}(b).$$

**Proof** Clearly, $T_{R_T} = (T_{R_T})_{x \in X}$, where $(T_{R_T}) : B \to 2^B$ are morphisms of partially ordered sets for all $x \in X$. We write $R_T = \bigcup_{x \in X} \{x\} \times R_{T x}$, where $R_{T x}, x \in X$ are the upper $T$-induced relation by 2.

Let us choose $b \in C$ and $x \in X$ arbitrarily, but fixed. We have to check that $T_x(b) = (T_{R_T})_x(b).$ Assume that $s \in S_B$. It is enough to verify that $T_x(b)(s) = \bigwedge \{b(t) \mid sT_x t\}$.

Evidently, for all $t \in S_B$ such that $sT_x t$, $T_x(b)(s) \leq b(t)$. Hence $T_x(b)(s) \leq \bigwedge \{b(t) \mid sT_x t\}$. To get the other inequality assume that $T_x(b)(s) < \bigwedge \{b(t) \mid sT_x t\}$. Then $T_x(b)(s) = 0$ and $\bigwedge \{b(t) \mid sT_x t\} = 1$. Put $V_x = \{z \in B \mid (\exists y \in C) (T_x y)(s) = 1 \text{ and } y \leq z\}$. It follows that $b \notin V_x$ and $V_x$ is an upper set of $B$ such that $1 \in V_x$ (since $T_x(1)(s) = 1(s) = 1$). Let $W_x$ be a maximal proper upper set of $B$ including $V_x$ such that $b \notin W_x$. Put $U_x = B \setminus W_x$. Then $U_x$ is a proper down-set, $0 \notin U_x$, $h_{U_x}(b) = 0$ and $h_{U_x}(z) = 1$ for all $z \in V_x$, i.e., $h_{U_x} \in S_B$ such that $T_x(a)(s) \leq a(h_{U_x})$ for all $a \in C$. But this yields that $sT_x h_{U_x}$, i.e., $1 = \bigwedge \{b(t) \mid sT_x t\} \leq b(h_{U_x}) = h_{U_x}(b) = 0$, a contradiction.

Using the relation $R_T$ instead of $R_T$, we can obtain a statement dual to Theorem 5.

5.2 Automata via Boolean algebras

As for bounded partially ordered sets we have that, for any Boolean algebra $B = (B; \lor, \land, \prime, 0, 1)$, there is a full set $S_B^{\text{bool}}$ of morphisms of Boolean algebras into the two-element Boolean algebra $2 = (\{0, 1\}; \lor, \land, \prime, 0, 1)$.

In what follows, we will modify our Theorem 5 for the more special case when the considered subposet $C$ is closed under finite infima.

We are now ready to show under which conditions our transition functor can be recovered.

**Theorem 6** Let $B = (B; \lor, \land, \prime, 0, 1)$ be a Boolean algebra such that $B$ is a sub-Boolean algebra of $2^B$. Let $C = (C; \leq, 1)$ be a subposet of $B$ containing 1 such that $x, y \in C$ implies $x \land y \leq C$, and $X$ a non-empty set. Let $T = (T_x)_{x \in X}$ where $T_x : C \to 2^B$ are mappings preserving finite meets such that $T_x(1) = 1$ for all $x \in X$. Let $R_T$ be the upper $T$-induced state-transition relation and $T_{R_T} : B \to (2^B)^X$ be the labelled upper transition functor constructed by means of the upper $T$-induced automaton $A_T = (X, S_B^{\text{bool}}, R_T)$. Then, for all $b \in C$,

$$T(b) = T_{R_T}(b).$$
Proof} Let us choose \( b \in C \) and \( x \in X \) arbitrarily, but fixed. Assume that \( s \in S^\text{bool} \). As in Theorem 5 it is enough to verify that \( T_i(b)(s) = \bigwedge \{b(t) | sR_T(t) \} \).

By the same considerations as in the proof of Theorem 5 we have \( T_i(b)(s) = \bigwedge \{b(t) | sR_T(t) \} \). To get the other inequality assume that \( T_i(b)(s) < \bigwedge \{b(t) | sR_T(t) \} \). Then \( T_i(b)(s) = 0 \) and \( \bigwedge \{b(t) | sR_T(t) \} = 1 \). Put \( V_i = \{ z \in B | (\exists y \in C)(T_i(y)(s) = 1 \text{ and } y \leq z) \} \). It follows that \( b \notin V_i \) and \( V_i \) is a filter of \( B \) such that \( 1 \in V_i \) (since \( y,z \in V_i \cap C \) implies \( T_i(y \wedge z)(s) = T_i(y)(s) \wedge T_i(z)(s) = 1 \amp 1 = 1 \) and \( T_i(1)(s) = 1 \)). Let \( W_i \) be a maximal proper filter of \( B \) including \( V_i \) such that \( b \notin W_i \). Then \( W_i \) is an ultrafilter of \( B \). The ultrafilter \( W_i \) determines a map \( g_{W_i} : S^\text{bool} \rightarrow B \) such that \( g_{W_i}(b) = 0 \) and \( g_{W_i}(c) = 1 \) for all \( c \in V_i \), i.e., \( g_{W_i} \in S^\text{bool} \) is such that \( T_i(a)(s) \leq g_{W_i}(a) \) for all \( a \in C \). This yields that \( sR_T,g_{W_i} \), i.e., \( 1 = \bigwedge \{b(t) | sR_T(t) \} \leq b(g_{W_i}) = g_{W_i}(b) = 0 \), a contradiction.

The example below shows an application of Theorem 6.

**Example 6** Consider again the set \( S = \{s_1,s_2,s_3\} \) of states, the set \( X = \{x_1,x_2\} \), and the set of propositions \( B = 2^X \) of Example 1. Recall that in this case \( S = S^\text{bool} \), and \( C = \{0,r,p',q',1\} \subseteq B \) from the logic \( B \) of Example 1.

Assume further that our partially known transition operator \( T \) from \( C \) to \( (2^S)^X \) is given as follows:

\[
T_i(0) = 0, \quad T_i(1) = 1, \quad T_i(0) = p, \quad T_i(1) = 1, \\
T_i(r) = r, \quad T_i(p') = q', \quad T_i(r) = 1, \quad T_i(p') = 1, \\
T_i(q') = p', \quad T_i(q') = 1.
\]

Note that \( T \) was chosen as a restriction of the operator \( T_b \) from Example 2 on the set \( C \).

Then, by an easy computation, we obtain from (7) that \( R_T = \{x_1\} \times R_{T_1} \cup \{x_2\} \times R_{T_2} \) where

\[
R_{T_1} = \{(s_1,s_2),(s_2,s_1),(s_3,s_3)\} \quad \text{and} \quad R_{T_2} = \{(s_2,s_3),(s_3,s_3)\}.
\]

From Theorem 6 we have that \( T \) is a restriction of the operator \( T_{bk} \) on the set \( C \).

Moreover, we can see that our state-transition relation \( R \) from Example 1 coincides with the induced state-transition relation \( R_T \), i.e., our partially known transition operator \( T \) gives us a full information about the automaton \( A \) from Example 1.

### 6 Conclusion

We have shown in our paper that to every automaton considered as an acceptor a certain dynamic logic can be assigned. The dynamic nature of an automaton is expressed via its transition relation labelled by inputs. The logic consists of propositions on the given automaton and its dynamic nature is expressed by means of the so-called transition functor. However, this logic enables us to derive again a certain relation on the set of states which is labelled by inputs. The main task is whether the relation derived from the logic and the transition functor is faithful, i.e., whether it coincides with the original transition relation of the automaton.

In fact, we have shown that if our set of propositions is large enough this recovering of the transition relation is possible. Several examples are included.

Conversely, having a set \( B \) of propositions that describe behaviour of our intended automaton and the transition functor which express the dynamicity of this process together with the set \( X \) of inputs (going from environment), we presented a construction of a set of

states $S$ and of a state-transition relation $R$ on $S$ such the constructed automaton $(X, S, R)$ realizes the description given by the propositions. It is shown that for every large enough set of states the induced transition functor coincides with the original one.

We believe that this theory enables us to consider automata from a different point of view which is more close to logical treatment and which enables us to make estimations and forecasts of the behaviour of automaton particularly in a nondeterministic mode. The next task will be to testify which type of automaton is determined by a suitable sort of logic.

Acknowledgement

This is a pre-print of an article published in International Journal of Theoretical Physics. The final authenticated version of the article is available online at: https://link.springer.com/article/10.1007/s10773-017-3311-0.

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