Continued fraction expansions and permutative representations of the Cuntz algebra $\mathcal{O}_\infty$

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Abstract

We show a correspondence between simple continued fraction expansions of irrational numbers and irreducible permutative representations of the Cuntz algebra $\mathcal{O}_\infty$. With respect to the correspondence, it is shown that the equivalence of real numbers with respect to modular transformations is equivalent to the unitary equivalence of representations. Furthermore, we show that quadratic irrationals are related to irreducible permutative representations of $\mathcal{O}_\infty$ with a cycle.

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1 Introduction

The purpose of this paper is to show a new relation between number theory and the representation theory of operator algebras. At the beginning, we show our motivation. Explicit mathematical statements will be given after §1.2. The main theorems will be shown in §1.4.

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1.1 Motivation

Continued fractions furnish important tools in number theory \([9, 11, 16]\), and continued fraction transformations induce typical dynamical systems \([3, 4, 10, 17]\). It is well known that the dynamical system of the simple continued fraction transformation on the set of irrational numbers in the interval \([0, 1]\) is conjugate with the one-sided full shift on the set \(N^\infty \equiv \{(n_i)_{i \geq 1} : n_i \in N \text{ for all } i\}\) \([15]\) by using continued fraction expansions where \(N \equiv \{1, 2, 3, \ldots\}\).

On the other hand, such dynamical systems induce representations of Cuntz algebras and their relations were studied \([12, 14]\). For example, the one-sided full shift on \(N^\infty\) induces the shift representation of \(\mathcal{O}_\infty\), which acts on the representation space \(l_2(N^\infty)\) \([2]\). The shift representation is decomposed into the direct sum of irreducibles unique up to unitary equivalence and the decomposition is multiplicity free. Every irreducible component in the decomposition is a permutative representation, and any irreducible permutative representation appears in the decomposition.

From these facts, we are interested in a relation between continued fraction expansions of irrationals and such representations of \(\mathcal{O}_\infty\) by the intermediary of the one-sided full shift on \(N^\infty\). We roughly illustrate relations among theories as follows:

- **One-sided full shift on** \(N^\infty\)
- **Expansion of continued fraction**
- **Shift representation**

\[ \text{Irrationals in } [0, 1] \quad \mapsto \quad ? \quad \mapsto \quad \text{Representations of } \mathcal{O}_\infty \]

The position in the above question mark is the content of this study.

1.2 Continued fraction expansion map and continued fraction transformation on the set of irrationals

We review the continued fraction expansion map and the continued fraction transformation according to \([9]\). Let \([0, 1]\) denote the closed interval from 0 to 1, and let \(\Omega\) denote the set of all irrationals in \([0, 1]\), that is,

\[ \Omega = [0, 1] \setminus \mathbb{Q}. \quad (1.1) \]

Remark that we consider only \(\Omega\) but not the whole of \([0, 1]\) in this paper. Any \(x \in \Omega\) has a unique infinite continued fraction expansion \([9, \text{Theorem}\].
that is, there exists a unique infinite sequence \((a_i(x))_{i \geq 1}\) of positive integers such that

\[
x = \frac{1}{a_1(x) + \frac{1}{a_2(x) + \frac{1}{a_3(x) + \ddots}}}.
\] (1.2)

From this, we define the map \(\text{CFE}\) from \(\Omega\) to \(\mathbb{N}^\infty\) by

\[
\text{CFE}(x) \equiv (a_i(x))_{i \geq 1}.
\] (1.3)

It is known that the map \(\text{CFE}\) is bijective ([9], Theorem 161, 166, 169). We call \(\text{CFE}\) the continued fraction expansion map. If \(x \in \Omega\) is a solution of a quadratic equation with integral coefficients, then we call \(x\) a quadratic irrational.

**Fact 1.1** ([9], Theorem 177) If \(x \in \Omega\) is a quadratic irrational, then there exists a finite sequence \((m_1, \ldots, m_l) \in \mathbb{N}^l \cup \{\emptyset\}\) and a nonperiodic finite sequence \((n_1, \ldots, n_k) \in \mathbb{N}^k\) such that

\[
\text{CFE}(x) = (m_1, \ldots, m_l, n_1, \ldots, n_k, n_1, \ldots, n_k, \ldots).
\] (1.4)

In Fact 1.1, we call \((n_1, \ldots, n_k)\) the repeating block of \(\text{CFE}(x)\) ([16], Chap. 3) where we suppose that there is no shorter such repeating block and that the initial block does not end with a copy of the repeating block. Hence the repeating block of \(\text{CFE}(x)\) is uniquely defined for each quadratic irrational \(x\) in \(\Omega\). The converse of Fact 1.1 is also true ([9], Theorem 176).

**1.3 Permutative representations of \(O_\infty\)**

We review permutative representations of the Cuntz algebra \(O_\infty\) in this subsection.

**1.3.1 \(O_\infty\)**

Let \(O_\infty\) denote the Cuntz algebra [6], that is, a \(C^*\)-algebra which is universally generated by \(\{s_i : i \in \mathbb{N}\}\) satisfying

\[
s_i^* s_j = \delta_{ij} I \quad (i, j \in \mathbb{N}), \quad \sum_{i=1}^{k} s_i s_i^* \leq I, \quad (k \in \mathbb{N}),
\] (1.5)
where $I$ denotes the unit of $O_\infty$.

A $\ast$-representation of $O_\infty$ is a pair $(H, \pi)$ such that $\pi$ is a $\ast$-homomorphism from $O_\infty$ to the $C^*$-algebra $L(H)$ of all bounded linear operators on a complex Hilbert space $H$ [1]. We call $\ast$-representation as representation for the simplicity of description. For two representations $(H_1, \pi_1)$ and $(H_2, \pi_2)$ of $O_\infty$, $(H_1, \pi_1)$ and $(H_2, \pi_2)$ are unitarily equivalent if there exists a unitary $u$ from $H_1$ onto $H_2$ such that $u\pi_1(x)u^* = \pi_2(x)$ for each $x \in O_\infty$. We state that a representation $(H, \pi)$ of $O_\infty$ is irreducible if there is no invariant closed subspace of $H$ except $\{0\}$ and $H$; $(H, \pi)$ is multiplicity free if any two subrepresentations of $(H, \pi)$ are not unitarily equivalent. A representation $(H, \pi)$ is irreducible if and only if the commutant $\{X \in L(H) : X\pi(A) = \pi(A)X \text{ for all } A \in O_\infty\}$ of $\pi(O_\infty)$ equals to $CI$.

Since $O_\infty$ is simple, that is, there is no nontrivial closed two-sided ideal, any representation of $O_\infty$ is injective. If $\{t_i : i \in \mathbb{N}\}$ are bounded operators on a Hilbert space $\mathcal{H}$ such that $\{t_i : i \in \mathbb{N}\}$ satisfy (1.5), then the correspondence $s_i \mapsto t_i$ for $i \in \mathbb{N}$ is uniquely extended to a unital $\ast$-representation of $O_\infty$ on $\mathcal{H}$ from the uniqueness of $O_\infty$. Therefore we call such a correspondence among generators by a representation of $O_\infty$ on $\mathcal{H}$. Assume that $\{s_i : i \in \mathbb{N}\}$ are realized as operators on a Hilbert space $\mathcal{H}$. According to (1.5), $\mathcal{H}$ is decomposed into orthogonal subspaces as $\oplus_{i \in \mathbb{N}}s_i\mathcal{H}$. Since $s_i$ is an isometry, $s_i\mathcal{H}$ has the same dimension as $\mathcal{H}$. From this, we see that there is no finite dimensional representation of $O_\infty$ which preserves the unit. The following illustration is helpful in understanding $\{s_i : i \in \mathbb{N}\}$:

The algebra $O_\infty$ appears in quantum field theory [13] and metrical number theory [13].

1.3.2 Permutative representations

We review permutative representations in this subsubsection.

**Definition 1.2** [2, 7, 8, 13] Let $\{s_i : i \in \mathbb{N}\}$ denote the canonical generators of $O_\infty$. 

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A representation \((\mathcal{H}, \pi)\) of \(O_\infty\) is permutative if there exists an orthonormal basis \(\mathcal{E}(\subset \mathcal{H})\) of \(\mathcal{H}\) such that \(\pi(s_i)\mathcal{E} \subset \mathcal{E}\) for each \(i \in \mathbb{N}\).

For \(J = (j_l)_{l=1}^k \in \mathbb{N}^k\) with \(1 \leq k < \infty\), let \(P(J)\) denote the class of representations \((\mathcal{H}, \pi)\) of \(O_\infty\) with a cyclic unit vector \(\nu \in \mathcal{H}\) such that \(\pi(s_J)\nu = \nu\) and \(\{\pi(s_{j_1} \cdots s_{j_l})\nu\}_{l=1}^k\) is an orthonormal family in \(\mathcal{H}\) where \(s_J \equiv s_{j_1} \cdots s_{j_k}\).

For \(J = (j_l)_{l \geq 1} \in \mathbb{N}^\infty\), let \(P(J)\) denote the class of representations \((\mathcal{H}, \pi)\) of \(O_\infty\) with a cyclic unit vector \(\nu \in \mathcal{H}\) such that \(\{\pi(s_{J(n)})^*\nu : n \in \mathbb{N}\}\) is an orthonormal family in \(\mathcal{H}\) where \(J(n) \equiv (j_1, \ldots, j_n)\).

The vector \(\nu\) in both (ii) and (iii) is called the GP vector of \((\mathcal{H}, \pi)\).

We recall properties of these classes as follows: For any \(J, P(J)\) in Definition 1.2(ii) and (iii) always exists and it is a class of permutative representations, which contains only one unitary equivalence class. From this, we can always identify \(P(J)\) with a representative of \(P(J)\) \([2, 7, 8]\).

1.4 Main theorems

We show our main theorems in this subsection. For this purpose, we construct two representations of \(O_\infty\) as follows. For a nonempty set \(A\), let \(l_2(A)\) denote the complex Hilbert space with an orthonormal basis \(\{e_a : a \in A\}\). We call \(\{e_a : a \in A\}\) the standard basis of \(l_2(A)\).

**Definition 1.3** Let \(\Omega\) be as in (1.1) and let \(\{s_i : i \in \mathbb{N}\}\) denote the canonical generators of \(O_\infty\).

(i) For \(i \in \mathbb{N}\), define the map \(\alpha_i\) from \(\Omega\) to \(\Omega\) by

\[
\alpha_i(x) \equiv \frac{1}{x + i} \quad (x \in \Omega).\tag{1.6}
\]

Define the representation \(\pi_\alpha\) of \(O_\infty\) on \(l_2(\Omega)\) by

\[
\pi_\alpha(s_i)e_x \equiv e_{\alpha_i(x)} \quad (x \in \Omega, i \in \mathbb{N}).\tag{1.7}
\]

(ii) For \(i \in \mathbb{N}\), define the map \(\beta_i\) from \(\mathbb{N}^\infty\) to \(\mathbb{N}^\infty\) by

\[
\beta_i(n_1, n_2, \ldots) \equiv (i, n_1, n_2, \ldots) \quad ((n_1, n_2, \ldots) \in \mathbb{N}^\infty, i \in \mathbb{N}).\tag{1.8}
\]

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Define the representation $\pi_\beta$ of $\mathcal{O}_\infty$ on $l_2(\mathbb{N}^\infty)$ by
\[ \pi_\beta(s_i)e_a \equiv e_{\beta_i(a)} \quad (a \in \mathbb{N}^\infty, i \in \mathbb{N}). \] (1.9)
The representation $(l_2(\mathbb{N}^\infty), \pi_\beta)$ is called the shift representation [2].

Then the following holds.

**Theorem 1.4** Two representations $(l_2(\Omega), \pi_\alpha)$ and $(l_2(\mathbb{N}^\infty), \pi_\beta)$ are unitarily equivalent.

From Theorem 1.4, we can compare irreducible components of $(l_2(\Omega), \pi_\alpha)$ with those of $(l_2(\mathbb{N}^\infty), \pi_\beta)$ and consider how an irreducible component in $(l_2(\Omega), \pi_\alpha)$ is realized. For this purpose, we introduce an equivalence relation of real numbers as follows.

**Definition 1.5** [5, 9, 16] If $x$ and $y$ are two real numbers such that
\[ x = \frac{ay + b}{cy + d} \quad (a, b, c, d \in \mathbb{Z}) \] (1.10)
where $ad - bc = \pm 1$, then $x$ is said to be equivalent to $y$. In this case, we write $x \sim y$. The transformation (1.10) is called a modular transformation in a broad sense.

Remark that the transformation (1.10) for $a, b, c, d$ with $ad - bc = 1$ is also called a modular transformation in a narrow sense.

According to the unitary equivalence in Theorem 1.4, the following holds.

**Theorem 1.6** For $x \in \Omega$, let $[x]$ denote the equivalence class of $x$ in $\Omega$ with respect to modular transformations, that is, $[x] = \{ y \in \Omega : y \sim x \}$. Let $(l_2(\Omega), \pi_\alpha)$ be as in Definition 1.3(i).

(i) The following irreducible decomposition holds:
\[ l_2(\Omega) = \bigoplus_{[x] \in \Omega/\sim} \mathcal{H}_{[x]} \] (1.11)
where $\mathcal{H}_{[x]}$ denotes the closed subspace of $l_2(\Omega)$ generated by the subset $\{ e_y : y \in [x] \}$.

(ii) For $x \in \Omega$, let $\eta_{[x]}$ denote the subrepresentation of $\pi_\alpha$ associated with the subspace $\mathcal{H}_{[x]}$ in (1.11), that is,
\[ \eta_{[x]} \equiv \pi_\alpha|_{\mathcal{H}_{[x]}}. \] (1.12)
Then $\eta_{[x]}$ and $\eta_{[y]}$ are unitarily equivalent if and only if $x \sim y$. Especially, (1.11) is multiplicity free.
(iii) Let CFE be as in (1.3) and let \( \Omega^{(2)} \) denote the set of all quadratic irrationals in \( \Omega \).

(a) If \( x \in \Omega \setminus \Omega^{(2)} \), then \( \eta_{[x]} \) is \( P(CFE(x)) \).

(b) If \( x \in \Omega^{(2)} \), then \( \eta_{[x]} \) is \( P(CFE_0(x)) \) where \( CFE_0(x) \) denotes the repeating block of \( CFE(x) \).

(iv) Any irreducible permutative representation of \( O_\infty \) is unitarily equivalent to \( \eta_{[x]} \) for some \( [x] \in \Omega/\sim \).

In consequence, Theorem 1.6 shows that the set \( \Omega/\sim \) of all equivalence classes of irrationals in \([0,1] \) is one-to-one correspondence in the set \( IPR(O_\infty)/\sim \) of all unitary equivalence classes of irreducible permutative representations of \( O_\infty \):

\[
\Omega/\sim \cong IPR(O_\infty)/\sim; \quad [x] \mapsto \eta_{[x]} \tag{1.13}
\]

where we identify \( \eta_{[x]} \) with the unitary equivalence class of \( \eta_{[x]} \) for convenience. Especially, the following equivalence holds as the restriction of \( \eta \) on the subset \( \Omega^{(2)}/\sim \) of \( \Omega/\sim \):

\[
\Omega^{(2)}/\sim \cong IPR_{cycle}(O_\infty)/\sim \tag{1.14}
\]

where \( IPR_{cycle}(O_\infty)/\sim \) denotes the set of all unitary equivalence classes of irreducible permutative representations of \( O_\infty \) with a cycle.

**Remark 1.7** The equivalence of numbers by modular transformations is well known in number theory, such that the discriminant of an irrational is invariant with respect to modular transformations [5] [18]. On the other hand, the unitary equivalence of representations is basic in the representation theory of \( * \)-algebras. Since these two equivalence relations are independently introduced in different mathematical areas, the equivalence of two equivalence relations in Theorem 1.6(ii) is nontrivial.

From Theorem 1.6, the following natural questions are thought up.

**Problem 1.8**

(i) Show the meaning of the discriminant for the representation associated with a quadratic irrational. What are the discriminant and the class number ([18], p. 59) in the representation theory of \( O_\infty \)?

(ii) Find similar relations between the Cuntz algebra \( O_N \) with \( 2 \leq N < \infty \) [6] and real numbers.
(iii) From Theorem 1.6, we suspect that there exists a relation between the group of all modular transformations and \( O_\infty \). Make clear this relation.

In § 2, we prove Theorem 1.4 and 1.6. In § 3, we show examples of Theorem 1.6.

2 Proofs of theorems

In this section, we prove main theorems.

2.1 Continued fraction expansion and one-sided full shift

We review relations between continued fraction expansions and the one-sided full shift on \( \mathbb{N}^\infty \) in this subsection. Let \([a_1(x), a_2(x), \ldots]\) denote (1.2) for simplicity of description.

\[ x = [a_1, \ldots, a_m, c_1, c_2, \ldots], \quad y = [b_1, \ldots, b_n, c_1, c_2, \ldots]. \tag{2.1} \]

Define the simple continued fraction transformation (or the Gauss map) \( \tau \) from \( \Omega \) to \( \Omega \) by

\[ \tau(x) \equiv \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor \quad (x \in \Omega), \tag{2.2} \]

where \( \left\lfloor \cdot \right\rfloor \) denotes the floor (entire) function \([10, 17]\). Then we see that \( \tau([a_1, a_2, \ldots]) = [a_2, a_3, \ldots] \) for the continued fraction \([a_1, a_2, \ldots]\).

Define the map \( \sigma \) from \( \mathbb{N}^\infty \) to \( \mathbb{N}^\infty \) by

\[ \sigma(n_1, n_2, \ldots) \equiv (n_2, n_3, \ldots). \tag{2.3} \]

The dynamical system \( (\mathbb{N}^\infty, \sigma) \) is called the one-sided full shift on \( \mathbb{N}^\infty \) \([15]\, \S\, 7.2\). Then we see that CFE in (1.3) satisfies the following equation \([10, (1.1.2)]\):

\[ \text{CFE}(\tau(x)) = \sigma(\text{CFE}(x)) \quad (x \in \Omega). \tag{2.4} \]

From (2.4), two dynamical systems \( (\Omega, \tau) \) and \( (\mathbb{N}^\infty, \sigma) \) are conjugate.

**Definition 2.2** For \( a, b \in \mathbb{N}^\infty \), let \( a \sim b \) denote when there exist \( p, q \geq 1 \) such that \( \sigma^p(a) = \sigma^q(b) \).
We call \( \sim \) the tail equivalence in \( \mathbb{N}^\infty \) \( [2] \), Chap. 2). Then the following holds by definition.

**Fact 2.3** For \( x, y \in \Omega \), \( x \sim y \) if and only if \( \text{CFE}(x) \sim \text{CFE}(y) \).

For \( \{ \alpha_i : i \in \mathbb{N} \} \) and \( \{ \beta_i : i \in \mathbb{N} \} \) in Definition 1.3, the following holds:

\[
\text{CFE}(\alpha_i(x)) = \beta_i(\text{CFE}(x)) \quad (x \in \Omega, i \in \mathbb{N}).
\]

(2.5)

Remark that they are closely related to \( \tau \) and \( \sigma \) as follows:

\[
(\tau \circ \alpha_i)(x) = x, \quad (\sigma \circ \beta_i)(a) = a \quad (x \in \Omega, a \in \mathbb{N}^\infty, i \in \mathbb{N}).
\]

(2.6)

### 2.2 Properties of representations of \( \mathcal{O}_\infty \)

In this subsection, we recall properties of permutative representations and the shift representation of \( \mathcal{O}_\infty \). Define \( \mathbb{N}^* \equiv \bigcup_{1 \leq k < \infty} \mathbb{N}^k \). For \( J = (j_l)_{l=1}^m, K = (k_l)_{l=1}^{m'} \in \mathbb{N}^* \), we write \( J \sim K \) if \( m = m' \) and \( K = pJ \) where \( pJ = (j_{p(1)}, \ldots, j_{p(m)}) \) for any cyclic permutation \( p \in \mathbb{Z}_m \). For \( J \in \mathbb{N}^* \), we call \( J \) **nonperiodic** if \( J \not\sim J \) for any cyclic permutation \( p \neq id \). If \( J \in \mathbb{N}^* \) is nonperiodic, then we see that \( (\mathcal{H}, \pi) \) is \( P(J) \) if and only if there exists a cyclic vector \( v \in \mathcal{H} \) such that \( \pi(s_J)v = v \). For \( J \in \mathbb{N}^\infty \), we call \( J \) **nonperiodic** if \( J \) has no repeating block.

**Proposition 2.4** Let \( P(J) \) be as in Definition 1.3

(i) Any permutative representation of \( \mathcal{O}_\infty \) is decomposed into the direct sum of cyclic permutative representations uniquely up to unitary equivalence.

(ii) Any cyclic permutative representation is either one of the following two cases:

(a) \( P(J) \) for \( J \in \mathbb{N}^* \).

(b) \( P(J) \) for \( J \in \mathbb{N}^\infty \).

(iii) For two representations \( \pi_1 \) and \( \pi_2 \) of \( \mathcal{O}_\infty \), let \( \pi_1 \sim \pi_2 \) denote the unitary equivalence between \( \pi_1 \) and \( \pi_2 \). If \( J \in \mathbb{N}^* \) and \( K \in \mathbb{N}^\infty \), then \( P(J) \not\sim P(K) \).

(iv) For \( J, K \in \mathbb{N}^* \cup \mathbb{N}^\infty \), then \( P(J) \sim P(K) \) if and only if \( J \sim K \) where we define \( J \not\sim K \) when \( J \in \mathbb{N}^* \) and \( K \in \mathbb{N}^\infty \).

(v) For \( J \in \mathbb{N}^* \cup \mathbb{N}^\infty \), \( P(J) \) is irreducible if and only if \( J \) is nonperiodic.
Proof. See Appendix A.1

Next, we show properties of the shift representation of $O_\infty$ (see also [2], Chap. 6).

**Proposition 2.5** Let $(l_2(\mathbb{N}^\infty), \pi_\beta)$ be as in Definition 1.3(ii). Define $[a] \equiv \{b \in \mathbb{N}^\infty : b \sim a\}$ where $\sim$ is as in Definition 2.2.

(i) The following irreducible decomposition holds:

$$l_2(\mathbb{N}^\infty) = \bigoplus_{[a] \in \mathbb{N}^\infty/\sim} \mathcal{K}_a$$  \hspace{1cm} (2.7)

where $\mathcal{K}_a$ denotes the closed subspace of $l_2(\mathbb{N}^\infty)$ generated by the set $\{e_b : b \in [a]\}$.

(ii) For $a \in \mathbb{N}^\infty$, let $\theta_a$ denote the subrepresentation of $\pi_\beta$ associated with the subspace $\mathcal{K}_a$, that is,

$$\theta_a \equiv \pi_\beta|_{\mathcal{K}_a}.$$  \hspace{1cm} (2.8)

Then $\theta_a$ and $\theta_b$ are unitarily equivalent if and only if $a \sim b$. Especially, (2.7) is multiplicity free.

(iii) (a) If $a \in \mathbb{N}^\infty$ has no repeating block, then $\theta_a$ is $P(a)$.

(b) If $a \in \mathbb{N}^\infty$ has the repeating block $a'$, then $\theta_a$ is $P(a')$.

(iv) Any irreducible permutative representation of $O_\infty$ is unitarily equivalent to $\theta_a$ for some $[a] \in \mathbb{N}^\infty/\sim$.

Proof. See Appendix A.2

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2.3 Proofs of Theorem 1.4 and 1.6

We prove Theorem 1.4 and 1.6 in this subsection.

**Proof of Theorem 1.4** Let CFE be as in (1.3). Define the unitary $U$ from $l_2(\Omega)$ to $l_2(\mathbb{N}^\infty)$ by

$$U e_x \equiv e'_{\text{CFE}(x)} \quad (x \in \Omega)$$  \hspace{1cm} (2.9)

where we write $\{e_x : x \in \Omega\}$ and $\{e'_a : a \in \mathbb{N}^\infty\}$ as standard basis of $l_2(\Omega)$ and $l_2(\mathbb{N}^\infty)$, respectively. From (2.9), we can verify that $U \pi_\alpha(s_i) U^* = \ldots$
\[ \pi_\beta(s_i) \] for each \( i \in \mathbb{N} \). This implies the unitary equivalence between two representations \( (l_2(\Omega), \pi_\alpha) \) and \( (l_2(\mathbb{N}^\infty), \pi_\beta) \):

\[ U\pi_\alpha(A)U^* = \pi_\beta(A) \quad (A \in \mathcal{O}_\infty). \quad (2.10) \]

Hence the statement holds.

**Proof of Theorem 1.6.** From Fact 2.3, we see that

\[ \text{CFE}([x]) = [\text{CFE}(x)] \quad (x \in \Omega). \quad (2.11) \]

By this and (2.9),

\[ U\mathcal{H}_x = K_{[\text{CFE}(x)}} \quad (x \in \Omega). \quad (2.12) \]

From Proposition 2.5(i) and Theorem 1.4, the statement of (i) holds.

By (2.10) and (2.12), we obtain that

\[ U\eta_{[x]}(A)U^* = \theta_{[\text{CFE}(x)]}(A) \quad (x \in \Omega, A \in \mathcal{O}_\infty). \quad (2.13) \]

From Proposition 2.5(ii), (iii), (iv) and (2.13), the statements of (ii), (iii) and (iv) hold, respectively.

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### 3 Permutative representations of \( \mathcal{O}_\infty \) associated with quadratic irrationals

We show examples of Theorem 1.6(iii)-(b) in this section.

**Example 3.1** For \( k \in \mathbb{N} \), define \( x \in \Omega \) by

\[ x = \frac{\sqrt{k^2 + 4} - k}{2}. \quad (3.1) \]

Then \( x = [k, k, k, \ldots] \) as an infinite continued fraction. Therefore \( \eta_{[x]} \) is \( P(k) \). For example,

\[ \eta_{[\sqrt{5} - 1]} = P(1), \quad \eta_{[\sqrt{2} - 1]} = P(2), \quad \eta_{[\sqrt{13} - 3]} = P(3). \quad (3.2) \]

In [13], we showed that the restriction of \( P(1) \) on the algebra \( \mathcal{B} \) of bosons is unitarily equivalent to the Fock representation of \( \mathcal{B} \). Hence the first equation in (3.2) shows a relation between the golden ratio \((\sqrt{5} - 1)/2\) and the Fock representation of bosons.
Example 3.2 For $j, k \in \mathbb{N}$, define $x \in \Omega$ by

$$x = \frac{\sqrt{(jk)^2 + 4jk} - jk}{2j}.$$  \hspace{1cm} (3.3)

When $j \neq k$,

$$\eta_{[x]} = P(j, k).$$ \hspace{1cm} (3.4)

For example, $\eta_{[\sqrt{3} - 1]} = P(1, 2)$. If $j = k$, then (3.3) equals to (3.1). Therefore $\eta_{[x]}$ is $P(k)$.

Example 3.3 For $i, j, k \in \mathbb{N}$, define $x \in \Omega$ by

$$x = \frac{-ijk + i + k - j + \sqrt{D}}{2(ij + 1)},$$ \hspace{1cm} (3.5)

$$D = (ijk + i + j + k)^2 + 4.$$ \hspace{1cm} (3.6)

If $(i, j, k)$ is nonperiodic, then $\eta_{[x]} = P(i, j, k)$. If $i = j = k$, then (3.5) equals to (3.1).

Example 3.4 For $i, j, k, l \in \mathbb{N}$, define $x \in \Omega$ by

$$x = \frac{-ijkl + ij + kl + li - jk + \sqrt{D}}{2(ijk + i + k)},$$ \hspace{1cm} (3.7)

$$D = (ijkl + ij + jk + kl + li)(ijkl + ij + jk + kl + li + 4).$$ \hspace{1cm} (3.8)

If $(i, j, k, l)$ is nonperiodic, then $\eta_{[x]} = P(i, j, k, l)$.

Remark that the symbol $D$ in neither (3.6) nor (3.8) always means the discriminant of $x$. For example, when $(i, j, k) = (1, 2, 3)$, $D$ in (3.6) is 148. On the other hand, $x$ in (3.5) is $\frac{-5 + \sqrt{37}}{2}$ with the discriminant 37.

Problem 3.5 For a given $J \in \mathbb{N}$ for $n \geq 5$, compute $x \in \Omega$ such that $\eta_{[x]} = P(J)$.

Appendix

A Proofs of propositions

We prove Proposition 2.4 and 2.5 in this section.
A.1 Proof of Proposition 2.4

Let \((\mathcal{H}, \pi)\) be a permutative representation of \(\mathcal{O}_\infty\). By assumption, we see that there exists a family \(\{f_i : i \in \mathbb{N}\}\) of maps on a set \(\Lambda\) and an orthonormal basis \(\{e_n : n \in \Lambda\}\) such that

\[
\pi(s_i) e_n = e_{f_i(n)} \quad (i \in \mathbb{N}, n \in \Lambda).
\]

(A.1)

From this and (1.5), we see that \(#\Lambda = \infty\), the map \(f_i: \Lambda \to \Lambda\) is injective for each \(i\), \(f_i(\Lambda) \cap f_j(\Lambda) = \emptyset\) when \(i \neq j\) and \(\Lambda = \bigcup_{i \in \mathbb{N}} f_i(\Lambda)\). The family \(\{f_i : i \in \mathbb{N}\}\) is called a branching function system (2, Definition 2.1). On the other hand, if a branching function system is given, then we can construct a permutative representation of \(\mathcal{O}_\infty\) as (A.1). Hence we write \(\pi\) in (A.1) as \(\pi_f\). For a given branching function system \(\{f_i : i \in \mathbb{N}\}\), define the coding map \(F\) of \(\{f_i : i \in \mathbb{N}\}\) by the map from \(\Lambda\) to \(\Lambda\) as \(F(n) \equiv f_i^{-1}(n)\) when \(n \in f_i(\Lambda)\).

Proof of Proposition 2.4. (i) Let \((\mathcal{H}, \pi)\) be a permutative representation of \(\mathcal{O}_\infty\). Then there exists a branching function system \(f = \{f_i : i \in \mathbb{N}\}\) on a set \(X\) such that \((\mathcal{H}, \pi)\) is unitarily equivalent to \((l_2(X), \pi_f)\) by definition. For \(x, y \in X\), define the equivalence relation \(\sim\) as \(x \sim y\) if and only if there exist \(i, j \in \mathbb{N}\) such that \(F^i(x) = F^j(y)\) where \(F\) denotes the coding map of \(f\). Let \(\{X_\lambda : \lambda \in \Xi\}\) denote the set of all equivalence classes in \(X\) with respect to \(\sim\). Then we see that \(X\) is decomposed into the disjoint union of subsets \(\{X_\lambda : \lambda \in \Xi\}\) such that \(f_i(X_\lambda) \subset X_\lambda\) for each \(i \in \mathbb{N}\) and \(\lambda \in \Xi\). Therefore \(f\) is decomposed into the direct sum of branching function systems \(f^{(\lambda)} \equiv \{f_i|_{X_\lambda} : i \in \mathbb{N}\}\) for \(\lambda \in \Xi\). By the definition of \(X_\lambda\), \((l_2(X_\lambda), \pi_{f^{(\lambda)}})\) is cyclic and \(\pi_{f^{(\lambda)}} = \pi_{f|_{l_2(X_\lambda)}}\). This implies that \(\mathcal{H}\) is decomposed into the direct sum of cyclic subspaces \(\{l_2(X_\lambda) : \lambda \in \Xi\}\). Hence the statement of decomposition holds.

Assume that \(g = \{g_i : i \in \mathbb{N}\}\) is another branching function system on a set \(Y\) associated with \(\pi\). Since dimensions of the representation space of both \(\pi_f\) and \(\pi_g\) are same, we can assume that \(Y = X\). Define the map \(\phi\) from \(X\) to \(X\) by \(\phi(x) \equiv (g_i \circ f_i^{-1})(x)\) when \(x \in f_i(X)\). Then \(\phi\) is bijective and induces the same decomposition up to conjugacy. Define the unitary \(U\) on \(\mathcal{H}\) by \(U e_x \equiv e_{\phi(x)}\) for the standard basis \(\{e_x : x \in X\}\). Then we see that \(U\pi_f(\cdot)U^* = \pi_g\). Hence the statement of uniqueness holds.

(ii) Let \((\mathcal{H}, \pi)\) be a cyclic permutative representation of \(\mathcal{O}_\infty\). From the proof of (i), we can assume that there exists a branching function system \(f = \{f_i : i \in \mathbb{N}\}\) on a set \(X\) such that \((\mathcal{H}, \pi)\) is unitarily equivalent to \((l_2(X), \pi_f)\) with a cyclic vector \(e_{x_0}\) for a point \(x_0 \in X\). For the coding map
If a unit vector \( v \) satisfies \( s = w \langle \omega \rangle \), we can always assume \( J, K \) \( J, K \neq 0 \).

Therefore \( v \in H \), space \( H \).

(b) Assume \( v \neq w \). Let \( J, K \) \( J, K \neq 0 \) for some \( n, m \). Assume that \( J, K \) \( J, K \neq 0 \) for some \( n, m \).

(iii) If they are equivalent, then there exists an action of \( O_\infty \) on a Hilbert space \( \mathcal{H} \) with two cyclic unit vectors \( v \) and \( v' \) such that \( v \) and \( v' \) are GP vectors of \( P(J) \) and \( P(K) \), respectively. Then \( \langle v|v' \rangle = \langle v|s_Jv' \rangle = \cdots = \langle v|(s_J^n)v' \rangle \to 0 \) \( n \to \infty \) because of the assumption of \( P(K) \). From this, \( \langle v|v' \rangle = 0 \). This implies that \( \langle s_{LV}|v' \rangle = 0 \) for each \( L \in \mathbb{N}^* \). Therefore \( v = 0 \). This is a contradiction.

(iv) From (iii), it is sufficient to show the following two cases.

(a) Assume \( J, K \in \mathbb{N}^* \). We can assume that \( O_\infty \) acts on two Hilbert spaces \( \mathcal{H} \) and \( \mathcal{H}' \) with cyclic unit vectors \( v \) and \( v' \) which are GP vectors of \( P(J) \) and \( P(K) \), respectively. Then \( \langle v|v' \rangle = \langle v|(s_J^n)v_{K} \rangle \) for each \( n, m \geq 1 \). Since \( J \neq K \), we see that \( (s_J^n)v_{K} = 0 \) for some \( n, m \). Therefore \( \langle v|v' \rangle = 0 \). This implies that \( \langle s_{LV}|v' \rangle = 0 \) for each \( L \in \mathbb{N}^* \). Therefore \( v = 0 \). This is a contradiction.

(b) Assume \( J, K \in \mathbb{N}^\infty \). Then \( P(J) \sim P(K) \) if and only if \( J \sim K \) from the analogy of the case (a).

(v) We consider the following two cases.

(a) Assume \( J \in \mathbb{N}^* \). Assume that \( J \) is nonperiodic and \( O_\infty \) acts on a Hilbert space \( \mathcal{H} \) with a cyclic unit vector \( v \) which is the GP vector of \( P(J) \). Let \( w \in \mathcal{H} \). Since \( \mathcal{H} \) is generated by the set \( \{s_Kv : K \in \mathbb{N}^* \} \), we can write \( w = \sum_L a_Ls_{LV} \) such that the set \( \{s_{LV} \} \) is an orthonormal family in \( \mathcal{H} \). If \( w \neq 0 \), there exists \( L_0 \in \mathbb{N}^* \) such that \( a_{L_0} \neq 0 \). By replacing \( w \) by \( a_{L_0}^{-1}s_{L_0}w \), we can always assume \( \langle w|v \rangle = 1 \). On the other hand, \( s_{K}^*s_{LV} = 0 \) when \( K \neq J^n \) for some \( n \geq 0 \). Hence \( (s_J^n)^* w \to v \) when \( n \to \infty \). From this, we can obtain \( v \) from a given \( w \in \mathcal{H} \), \( w \neq 0 \). This implies that \( \mathcal{H} \) is irreducible.

Assume that \( J \) is not nonperiodic (=periodic). Then there exists \( J_0 \in \mathbb{N}^* \) such that \( J \) is the \( n \)-times concatenation of \( J_0 \) for some \( n \geq 2 \). Assume that a unit vector \( v \) satisfies \( s_Jv = v \). Define two vectors \( w_1 \) and \( w_2 \) by

\[
 w_1 \equiv v + s_{J_0}v + \cdots + s_{J_0}^{n-1}v, \quad (A.2)
\]

\[
 w_2 \equiv v + \zeta s_{J_0}v + \cdots + \zeta^{n-1}s_{J_0}^{n-1}v \quad (A.3)
\]
where \( \zeta \equiv e^{2\pi \sqrt{-1}/n} \). Since \( s_{\beta_0}w_1 = w_1 \) and \( s_{\beta_0}w_2 = \zeta^{-1}w_2 \), \( \langle w_1|w_2 \rangle = 0 \). Define \( V_i \equiv \overline{O_\infty w_i} \) for \( i = 1, 2 \). Then we can verify that \( V_1 \) and \( V_2 \) are orthogonal. Since \( \{0\} \neq V_i \subset V_1 \oplus V_2 \subset \mathcal{H} \) for \( i = 1, 2 \), \( \mathcal{H} \) is not irreducible.

(b) Assume \( J \in \mathbf{N}^\infty \). If \( J \) is nonperiodic, then we can prove the irreducibility of \( P(J) \) by the analogy of the case of (a).

Assume that \( J \) is not nonperiodic. We can assume that there exists \( J_0 \in \mathbf{N}^* \) such that \( J \) is purely periodic with the repeating block \( J_0 \). Assume that \( O_\infty \) acts on a Hilbert space \( \mathcal{K} \) with a cyclic unit vector \( v_0 \) such that \( s_{\beta_0}v_0 = v_0 \). Let \( \mathcal{L} \) denote the Hilbert space of all \( \mathcal{K} \)-valued functions \( \phi \) on \( U(1) \equiv \{ z \in \mathbf{C} : |z| = 1 \} \) such that \( \int_{U(1)} \| \phi(z) \|^2 d\mu(z) < \infty \) where \( \mu \) denotes the probabilistic Haar measure of the unitary group \( U(1) \). Define the action of \( O_\infty \) on \( \mathcal{L} \) by

\[
(s_i \phi)(z) \equiv z s_i \phi(z) \quad (z \in U(1), \phi \in \mathcal{L}, i \in \mathbf{N}).
\]

Let \( \tilde{v}_0 \) denote the constant function in \( \mathcal{L} \) such that \( \tilde{v}_0(z) \equiv v_0 \) for each \( z \in U(1) \). Then \( (s_{\beta_0} \tilde{v}_0)(z) = z \tilde{v}_0(z) \) for each \( z \in U(1) \). From this, we see that \( \{ (s_{J_0})(z) \} \) is an orthonormal family in \( \mathcal{L} \). Let \( \mathcal{L}_1 \) denote the cyclic subspace of \( \mathcal{L}_1 \) generated by \( \tilde{v}_0 \) with respect to the action of \( O_\infty \). Then \( \mathcal{L}_1 \) is \( P(J) \). Define the operator \( T \) on \( \mathcal{L} \) by \( (T \phi)(z) \equiv z \phi(z) \) for \( \phi \in \mathcal{L} \) and \( z \in U(1) \). Then \( T \mathcal{L}_1 \subset \mathcal{L}_1 \) and \( Ts_i = s_i T \) and \( Ts_i^* = s_i^* T \) for each \( i \in \mathbf{N} \) on \( \mathcal{L}_1 \). Hence \( T \) is the nontrivial element of the commutant of \( O_\infty \) on \( \mathcal{L}_1 \). Therefore neither \( \mathcal{L}_1 \) nor \( \mathcal{L} \) is irreducible. Hence \( P(J) \) is not irreducible. \( \blacksquare \)

### A.2 Proof of Proposition 2.5

(i) For \( [a] \in \mathbf{N}^\infty /\sim \), we see that \( \beta_i([a]) \subset [a] \) and \( \beta_i^{-1}([a]) \subset [a] \) for each \( i \). This implies that \( \pi_\beta(O_\infty) \mathcal{K}_[a] \subset \mathcal{K}_[a] \). By definition, \( l_2(\mathbf{N}^\infty) \) is decomposed into the direct sum of the family \( \{ \mathcal{K}_[a] : [a] \in \mathbf{N}^\infty /\sim \} \) of closed subspaces as a Hilbert space, the statement of decomposition holds. It is sufficient to show the irreducibility of \( \mathcal{K}_[a] \) for each \( [a] \). We consider the following two cases.

(a) Assume that \( a \) has the repeating block \( a' \). Let \( \langle a' \rangle \) denote the purely periodic sequence with the repeating block \( a' \). Then \( [a] = \{ \beta_J(a') : J \in \mathbf{N}^k, k \geq 0 \} \) and \( \beta_a^{-1}([a]) = ([a]) \in \mathcal{K}_[a] \). This implies that \( (\mathcal{K}_[a], \pi_\beta|_{\mathcal{K}_[a]}) \) is \( P(a') \) with the GP vector \( e_{\langle a' \rangle} \). From Proposition 2.4(v) and the nonperiodicity of \( a', \mathcal{K}_[a] \) is irreducible.

(b) Assume that \( a \) has no repeating block. By the analogy of the case of (a), we see that \( \mathcal{K}_[a] \) is \( P(a) \) with the GP vector \( e_a \). From Proposition 2.4(v) and the nonperiodicity of \( a, \mathcal{K}_[a] \) is irreducible.
(ii) From the proof of (i), \( \theta_{[a]} \) is \( P(J) \) for a nonperiodic element \( J \) in \( \mathbb{N}^* \cup \mathbb{N}^\infty \). From this and Proposition 2.4(iii) and (iv), the statement holds.

(iii) The proof has been already given in the proof of (i).

(iv) Let \((\mathcal{H}, \pi)\) be an irreducible permutative representation of \( \mathcal{O}_\infty \). From Proposition 2.4(ii), \((\mathcal{H}, \pi)\) is \( P(J) \) for some nonperiodic element \( J \) in \( \mathbb{N}^* \cup \mathbb{N}^\infty \). From (iii), if \( J \in \mathbb{N}^* \), then \( \theta_{[a]} \) is \( P(J) \) for \( a \equiv J^\infty \in \mathbb{N}^\infty \), and if \( J \in \mathbb{N}^\infty \), then \( \theta_{[J]} \) is \( P(J) \). Hence the statement holds.

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