An explicit model of a polarised K3 surface of genus 5 with a symplectic action of order 192

Stevell Muller

Abstract

We give a projective model of a complex polarised K3 surface via the knowledge of a finite group acting on it. This paper presents the theory used to develop an algorithm for this purpose. It relies in particular on the existence of a Gauss elimination theorem in the context of a semisimple group algebra over an algebraically closed field of characteristic zero.

Contents

1 Preliminaries on representation theory .............................................. 3
   1.1 Linear representations and group algebra modules ............................. 3
   1.2 Characters of representations .................................................. 4
   1.3 Group actions and projective representations .................................. 5

2 Parametrizing submodules of a given group algebra module .................... 6
   2.1 Isotypical modules and Gauss elimination ...................................... 6
   2.2 Moduli space of submodules .................................................... 8
   2.3 Determinantal character ....................................................... 8

3 Finding complete intersections with prescribed symmetry .................... 11
   3.1 From invariant ideals to group algebra modules ............................... 11
   3.2 Classification of projectively faithful representations ...................... 12
   3.3 Application to the case of K3 surfaces .................................. 13

Introduction

A K3 surface is a simply-connected, compact, complex manifold X admitting a nowhere vanishing holomorphic symplectic 2-form σ_X ∈ H^{2,0}(X), which is unique up to scaling. Any pair (X, L) consisting of a projective K3 surface X and a primitive ample line bundle L on X is called a primitively polarised K3 surface and we define the genus of (X, L) to be \( \frac{1}{2}(c_1(L)^2 + 2) > 1 \), where \( c_1(L) ∈ H^2(X, \mathbb{Z}) \) is the first Chern class of L. Polarized K3 surfaces and their projective models have been studied by A.L. Mayer [May72] and B. Saint-Donat [SD74] who provided some examples of projective models for low genera. For instance, any double cover of \( \mathbb{P}_\mathbb{C}^2 \) branched over a sextic defines a polarized K3 surface of genus 2, and any such polarized K3 surface can be obtained in this way.

For a K3 surface X, we call an automorphism \( f ∈ \text{Aut}(X) \) symplectic if \( f^*σ_X = σ_X \), where \( f^* \) denotes the map induced by f on \( H^2(X, \mathbb{C}) \). In [Muk88], S. Mukai shows that for a K3 surface X the normal subgroup of symplectic automorphisms \( G_ν ⊆ \text{Aut}(X) \) is a subgroup of one among 11 maximal ones. For each of the latter groups, he provides explicit projective models of polarized K3 surfaces admitting such a group of symplectic automorphisms. In the thesis [Smi07, Chapter 1], J. P. Smith constructs families of polarized K3 surfaces of genus 2 (resp. of genus 3) from the...
same 11 (maximal) groups classified by Mukai. These models are obtained by lifting linear actions of these groups on \( \mathbb{P}_3^2 \) to \( \mathbb{C}^3 \) (resp. \( \mathbb{P}_5^2 \) to \( \mathbb{C}^4 \)) and computing invariant polynomials of degree 6 (resp. degree 4). More recently, P. Comparin and R. Demelle prove in [CD22] that there are infinitely many polarized K3 surfaces admitting a symplectic action of the Mathieu group \( M_{20} \); they produce several new explicit models as intersection of quadrics in larger projective spaces via Veronese embeddings of already known models (see [BH21, §6.1]). They also compute a model for a polarized K3 surface of genus 7 using non-trivial finite central extensions of \( M_{20} \) by a cyclic group of order 4 in order to lift linear actions of \( M_{20} \) on \( \mathbb{P}_5^2 \) to \( \mathbb{C}^8 \).

The main goal of this paper is to develop an algorithm for computing projective models of polarized K3 surfaces given as smooth complete intersections, or s.c.i., of 4 hyperplanes of the same degree \( d \) in the projective space \( \mathbb{P}_d^5 \), with prescribed group of symmetries \( G \). This is the case for instance for general polarized K3 surfaces of genus 3 (resp. genus 5) since they admit a projective model given as a smooth quartic in \( \mathbb{P}_4^3 \) (resp. as the s.c.i. of 3 quadrics in \( \mathbb{P}_3^5 \)).

We use a similar approach as in [Smi07] and [CD22]. In Section 1 we review some results about linear representations of finite groups as well as their projective representations. The core of this paper is focused in Section 2 where we develop an algorithmic way of studying submodules of group algebra modules. We show in particular that one can parametrize, as for Grassmannian varieties, submodules of a given dimension and given character of a higher dimensional group algebra module. Finally, we show in Section 3 how one can use this systematic study of group algebra modules to compute defining ideals of complete intersections fixed under an action of a group on their ambient projective space. In particular, we prove the following:

**Theorem 0.1.** The polarized K3 surface \( (X, L) \) of genus 5 corresponding to the case 77b in [BH21] admits a projective model in \( \mathbb{P}_5^5 \) given by, for any \( \lambda \in \mathbb{C}^* \),

\[
S_\lambda: \begin{cases}
ix_0x_1 + x_0x_2 + x_1x_3 + ix_2x_3 + \lambda x_4^2 = 0 \\
ix_0x_1 - x_0x_2 - x_1x_3 + ix_2x_3 + \lambda x_5^2 = 0 \\
-x_0x_3 - x_1x_2 + \lambda x_4x_5 = 0
\end{cases}
\]

It admits a maximal symplectic action of \( T_{192} \) and is invariant under the linear action of \( G := T_{192} \rtimes \mu_2 \) on \( \mathbb{P}_5^5 \) given by

\[
\begin{align*}
\sigma_1 &= \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\omega^3 & 0 \\ 0 & 0 & 0 & 0 & 0 & \omega^3 \end{pmatrix} & \sigma_2 &= \begin{pmatrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & i+1 & i-1 \\ 0 & 0 & 0 & 0 & -i+1 & -i-1 \end{pmatrix} \\
\sigma_4 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} & \sigma_5 &= \begin{pmatrix} -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 \\ 1 & -i & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 \end{pmatrix}
\end{align*}
\]

where \( T_{192} \) is one of the maximal subgroups of symplectic automorphisms classified by Mukai in [Muk88].

All the algorithms written for the purpose of this paper have been implemented in the CAS Oscar [OSC22], running on Julia [BEKS17], and are accessible at [Mul22].

**Acknowledgements**

The author would like to thank Simon Brandhorst for suggesting him the subject, for the useful discussions and the help provided. The author would also like to thank Max Horn and Matthias Zach for helping to optimize the implementation of the algorithms used for the purpose of this paper. Finally, the author would like to thank Tommy Hofmann for the precious comments made on a review of this paper.
1 Preliminaries on representation theory

In this section we recall standard results about representation theory and character theory, and we fix notation for the rest of the paper. Throughout this paper, we will work over algebraically closed fields of characteristic zero, e.g. \( \mathbb{C} \), and all groups are supposed to be finite.

1.1 Linear representations and group algebra modules

The definitions and results of this subsection can be found in any classical book about representation theory of finite groups. We refer, for instance, to the book [EGH'11].

Let \( K \) be a algebraically closed field of characteristic zero and let \( E \) be a finite group. By Maschke’s theorem [EGH'11, Theorem 3.1], the group algebra \( KE \) is semisimple, that is, all of its modules are semisimple and therefore can be decomposed as the direct sum of simple submodules. Throughout this paper, we describe \( KE \)-modules as pairs \( (V, \rho) \), where \( V \) is a finite-dimensional \( K \)-vector space and \( \rho \) is a \( K \)-linear representation of \( E \) on \( V \), that is \( \rho \) is a homomorphism \( \rho: E \to \text{GL}(V) \).

Two linear representations \( \rho, \rho': E \to \text{GL}(V) \) are called equivalent if there exists an automorphism \( L: V \to V \) such that, for all \( e \in E \),

\[
L \circ \rho(e) = \rho'(e) \circ L.
\]

Moreover, given two \( KE \)-modules \( M = (V, \rho) \) and \( M' = (V', \rho') \), we say that \( M \) and \( M' \) are equivalent, and we write \( M \cong M' \), if there exists an invertible \( K \)-linear map \( L: V \to V' \) such that \( \rho'^L \) and \( \rho' \) are equivalent representations of \( E \) on \( V' \), where for all \( e \in E \),

\[
\rho'^L(e) := L \circ \rho(e) \circ L^{-1}.
\]

We say in this paper that \( M \) and \( M' \) are \( E \)-equivariant, and we write \( M \equiv E M' \), if moreover \( \rho'^L = \rho' \).

**Remark 1.1.** By the Krull-Schmidt theorem [EGH'11, Theorem 2.19], if a \( KE \)-module is semisimple then its decomposition into a direct sum of simple submodules is unique up to equivalence and order of the summands. Moreover, according to [Isa76, Corollary (2.5)], the equivalence classes of simple \( KE \)-modules correspond bijectively to the conjugacy classes of \( E \).

If \( M = (V, \rho) \) is a \( KE \)-module, then by Maschke’s theorem, one can write

\[
M = \bigoplus_{i=1}^{t} M_i
\]

where we assume that there exists \( 1 = i_0 < i_1 < \ldots < i_{t-1} < i_t = t + 1 \) such that for all \( 0 \leq j \leq t - 1 \)

\[
W_j := \bigoplus_{k=i_j}^{i_{j+1}-1} M_k \cong M_{i_j}^{\oplus i_{j+1}-i_j}
\]

is the complete sum of all simple submodules of \( M \) which are equivalent to \( M_{i_j} \). For \( 0 \leq j \leq t - 1 \), we call \( W_j \) an isotypical component of \( M \) and we say that

\[
M = \bigoplus_{j=0}^{t-1} W_j
\]

is an isotypical decomposition of \( M \) (which is unique up to equivalence and order of the summands). For all \( 0 \leq j \leq t - 1 \), \( W_j \) is itself a \( KE \)-module which we say to be isotypical of weight \( \text{wgt}(W_j) := \dim_K(M_{i_j}) \) (to be understood, the \( K \)-dimension of the underlying vector space).

We conclude this subsection by stating one of the key results we use several times in this paper.
Theorem 1.2 (Schur’s lemma; [EGH’11], Proposition 1.16, Corollary 1.17). Let \( M \cong W^{\otimes t} \) and \( M' \cong W^{\otimes t'} \) be two isotypical \( KE \)-modules, where \( W \) and \( W' \) are simple. Then, under the assumption that \( K \) is algebraically closed, one has

\[
\text{Hom}_{KE}(M, M') = \begin{cases} 
M_{t,t'}(K) & \text{if } W \cong W' \\
0 & \text{else}
\end{cases}
\]

where \( M_{t,t'}(K) \) denotes the set of \( t \times t' \) matrices with entries in \( K \). In particular, the \( KE \)-automorphism group of a simple \( KE \)-module can be identified with \( K^* \).

1.2 Characters of representations

To any \( K \)-linear representation of \( E \) on a vector space \( V \), one can associate a so-called character. These characters encode most of the information one needs to study \( KE \)-modules. To read more about characters we refer to [Isa76].

Again, let \( K \) be an algebraically closed field of characteristic 0, let \( E \) be a finite group, and let \( M = (V, \rho) \) be a \( KE \)-module. We define the \( K \)-character \( \chi_M \) of \( M \) to be the mapping

\[ \chi_M : E \to K, \ e \mapsto \text{Tr}(\rho(e)). \]

We say that \( M \) affords \( \chi_M \) and that \( \chi_M \) is afforded by \( M \). One notes that \( \chi_\rho(1_E) = \dim_K(V) \) and \( \chi_\rho \) is constant on each conjugacy class of \( E \). More generally, \( K \)-characters of \( E \) are a special case of what we call class functions on \( E \), and they are all of the form \( \chi_M \) for some \( KE \)-module \( M \).

We define sum and product of \( K \)-characters of \( E \) as pointwise sum and product of their respective images in \( K \). So for instance, if \( \chi \) and \( \chi' \) are two \( K \)-characters of \( E \) afforded by \( M \) and \( M' \) respectively, then \( \chi + \chi' \) is afforded by \( M \oplus M' \) and vice-versa. A \( K \)-character \( \chi \) of \( E \) is said to be simple, or irreducible, if \( \chi \) cannot be written non-trivially as sum of other \( K \)-characters of \( E \).

Proposition 1.3 ([Isa76], Corollary (2.5)). The number of simple \( K \)-characters of \( E \) is equal to the number of conjugacy classes of \( E \) (recall that \( E \) is a finite group here). In particular, simple \( K \)-characters of \( E \) are afforded by simple \( KE \)-modules.

Proposition 1.4 ([Isa76], Corollary (2.9)). Two \( KE \)-modules \( M \) and \( M' \) are equivalent if and only if they afford the same \( K \)-character of \( E \).

We define the degree of a \( K \)-character \( \chi \) of \( E \) as \( \chi(1_E) \). For all \( n \geq 1 \), we denote \( \text{Irr}_K^n(E) \), the set of all simple \( K \)-characters of \( E \) of degree \( n \) and let \( \text{Irr}_K(E) := \bigcup_{n \geq 1} \text{Irr}_K^n(E) \). According to [Isa76, Theorem (2.8)], any \( K \)-character \( \chi \) of \( E \) admits a unique decomposition

\[ \chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu \]

where \( e_\mu \in \mathbb{Z}_{\geq 0} \) is called the multiplicity of the irreducible character \( \mu \) in \( \chi \). Given two \( K \)-characters \( \chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu \) and \( \chi' = \sum_{\mu \in \text{Irr}_K(E)} e'_\mu \mu \) of \( E \), we define their scalar product

\[ \langle \chi, \chi' \rangle := \sum_{\mu \in \text{Irr}_K(E)} e_\mu e'_\mu. \]

In particular, for \( \mu \in \text{Irr}_K(E) \), \( \langle \mu, \mu \rangle = 1 \) and \( \langle \chi, \mu \rangle \) is equal to the multiplicity of \( \mu \) in \( \chi \). If \( \chi = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu \) and \( \chi' = \sum_{\mu \in \text{Irr}_K(E)} e'_\mu \mu \) are two \( K \)-characters of \( E \) such that \( 0 \leq e_\mu \leq e'_\mu \) for all \( \mu \in \text{Irr}_K(E) \), then we say that \( \chi \) is a constituent of \( \chi' \).

We see that the decomposition of the \( K \)-character afforded by a \( KE \)-module depends only on its isotypical decomposition. In particular, we say that a \( K \)-character \( \chi \) of \( E \) is isotypical if \( \chi \) is afforded by an isotypical \( KE \)-module, i.e. it is a positive multiple of an irreducible \( K \)-character of \( E \).
1.3 Group actions and projective representations

In this subsection we state some relevant results about projective representations of finite groups which we use throughout this paper. We refer to [Isa76, Chapter 11] for the readers who are not familiar with the notion of projective representations. The key point is that one can relate linear actions of a group $G$ on $\mathbb{P}^n(\mathbb{C})$ to linear actions of a possibly larger group, called a Schur cover, on $\mathbb{C}^{n+1}$.

Let $K$ be algebraically closed of characteristic zero. Given a finite group $G$ and a finite dimensional $K$-vector space $V$, we call a projective representation of $G$ on $V$ any homomorphism $\rho \colon G \to \text{PGL}(V)$.

Such a representation is called faithful if it is injective. For any group $G$, there exists a finite abelian group $M(G)$ called the Schur multiplier of $G$ (see [Isa76, Definition (11.12)]), which can be identified with $H^2(G, K^*)$, the second cohomology group of $G$ with coefficients in $K^*$. In [Sch04], I. Schur proved that for any finite group $G$, there exists a group $E$ and an exact sequence

$$1 \to H \xrightarrow{i} E \xrightarrow{\rho} G \to 1$$

such that $H \cong M(G)$, $i(H) \subseteq E'$ (the derived subgroup of $E$) and such that for any projective representation $\rho \colon G \to \text{PGL}(V)$ of $G$ on a finite vector space $V$, there exists a linear representation $\rho' \colon E \to \text{GL}(V)$ making the following diagram with exact rows commute

$$
\begin{array}{cccccc}
1 & \rightarrow & H & \xrightarrow{i} & E & \xrightarrow{\rho} & G & \rightarrow & 1 \\
& \downarrow{\beta} & \downarrow{\bar{\rho}} & \downarrow{\rho} & \downarrow{\pi} & & \downarrow{\pi} \\
1 & \rightarrow & K^* & \xrightarrow{id_{V'}} & \text{GL}(V) & \xrightarrow{\pi} & \text{PGL}(V) & \rightarrow & 1.
\end{array}
$$

Here $\beta$ is induced by the restriction of $\rho$ to $i(H)$. This result is known as Schur’s theorem (see [Isa76, Theorem (11.17)]) and the group $E$ is referred to as a Schur cover of $G$. We moreover refer to $\rho$ as a lift of $\rho$ and the latter as the reduction of the former. Schur’s theorem allows us to use the results from the theory of linear representations of finite groups to work with projective representations of finite groups. In particular, given a finite group $G$, a Schur cover $E$ of $G$ and a finite dimensional complex vector space $V$, one can relate a classification of projective representations of $G$ on $V$ to a classification of linear representations of $E$ on $V$.

**Definition 1.5** ([Isa76], Definition(1.18); [Isa76], Page 177). Let $G$ be a finite group and let $V$ be a finite-dimensional $K$-vector space. Two projective representations $\bar{\rho}, \bar{\rho}' \colon G \to \text{PGL}(V)$ are called similar if there exists an automorphism $\mathcal{L} \colon V \to V$ such that, for all $g \in G$,

$$\mathcal{L} \circ \bar{\rho}(g) = \bar{\rho}'(g) \circ \mathcal{L}$$

where $\mathcal{L} \colon \mathbb{P}(V) \to \mathbb{P}(V)$ is induced by $\mathcal{L}$.

**Lemma 1.6** ([Isa76], Page 178). Let $G$ be a finite group, $E$ a Schur cover of $G$ and $V$ a finite-dimensional $K$-vector space. Assume that there are two projective representations $\bar{\rho}, \bar{\rho}' \colon G \to \text{PGL}(V)$ lifting respectively to $\rho, \rho' \colon E \to \text{GL}(V)$ as in Diagram 1. Then $\bar{\rho}$ and $\bar{\rho}'$ are similar if and only if there exists a homomorphism $\epsilon \colon E \to K^*$ such that $\rho$ and $\epsilon \rho'$ are equivalent.

**Proof.** Suppose that $\bar{\rho}$ and $\bar{\rho}'$ are similar and let $\mathcal{L} \in \text{GL}(V)$ such that, for all $g \in G$,

$$\mathcal{L} \circ \bar{\rho}(g) \circ \mathcal{L}^{-1} = \bar{\rho}'(g).$$

Now by commutativity of Diagram 1, for all $e \in E$, one obtains that

$$\pi(\mathcal{L} \circ \rho(e) \circ \mathcal{L}^{-1}) = \mathcal{L} \circ (\pi(\rho(e))) \circ \mathcal{L}^{-1} = \bar{\rho}'(p(e)) = \pi(\rho'(e)).$$
Hence, there exists a homomorphism \( \epsilon: E \to K^\times \) such that \( \rho \) and \( \epsilon \rho' \) are equivalent. Now, suppose there exists a map \( \epsilon: E \to K^\times \) and \( L \in \text{GL}(V) \) such that, for all \( e \in E \),

\[
L \circ \rho(e) \circ L^{-1} = \epsilon(e) \rho'(e).
\]

One deduce that \( \overline{\rho} \) and \( \overline{\rho}' \) are similar by commutativity of Diagram 1 and surjectivity of \( p \).

Note that, in the context of Lemma 1.6, given a linear representation \( \rho \) of \( E \) on \( V \), one can always define a projective representation of \( G \) on \( V \) which makes Diagram 1 commute, by setting \( \overline{\rho} := \pi \circ \rho \circ s \) where \( s \) is any section of \( p \) that maps \( 1_G \) to \( 1_E \) (it can be easily shown that this definition does not depend on the choice of \( s \)). In other words, any linear representation of \( E \) always admits a (unique) reduction to \( G \).

## 2 Parametrizing submodules of a given group algebra module

Let \( K \) be an algebraically closed field of characteristic zero, \( E \) a finite group and \( M = (V, \rho) \) a \( KE \)-module. In this section we show that the moduli space parametrizing the \( KE \)-submodules of \( M \) is algebraic, and its irreducible components are actually rational.

### 2.1 Isotypical modules and Gauss elimination

The goal of this subsection is to bring a constructive approach to the proof of the existence of a Gauss elimination theorem for isotypical \( KE \)-modules.

**Lemma 2.1** (Key lemma). Let \((V, \rho)\) and \((V', \rho')\) be two equivalent simple \( KE \)-modules. Then, up to scalar multiplication, there exists a unique \( L' \in \text{GL}(V') \) such that

\[
(L'V', \rho'\rho) \equiv_E (V, \rho).
\]

**Remark 2.2.** In other words, Lemma 2.1 tells us that if two simple \( KE \)-modules \( M \) and \( M' \) are equivalent, one can always perform a base change on the underlying space of \( M' \), for instance, in such a way that the action of \( E \) on the respective \( K \)-bases of both \( M \) and \( M' \) is the same.

**Proof.** Since \((V, \rho)\) and \((V', \rho')\) are equivalent, there exists a \( K \)-linear isomorphism \( L: V \to V' \) such that \( \rho' \) and \( \rho' \circ L \) are equivalent. Therefore, there exists a base change \( L' \in \text{GL}(V') \) such that for all \( e \in E \)

\[
L' \circ \rho'(e) \circ L'^{-1} = L \circ \rho(e) \circ L^{-1}.
\]

Thus, \( L \) induces an \( E \)-equivariance between \((L'V', \rho'\rho)\) and \((V, \rho)\).

Now, if there exists another \( L'' \in \text{GL}(V') \) satisfying the same property, then for all \( e \in E \)

\[
\rho'^{-1}(L''e) = \rho'(L'e)
\]

and therefore \( \rho'^{-1}(L') = \rho' \). This means that \( L'^{-1}L' \) is a \( K \)-linear automorphism of \( V' \) which commutes with the action of \( E \) given by \( \rho' \). By Schur’s lemma (Theorem 1.2), since \((V', \rho')\) is simple, one has that \( L'^{-1}L' \in K^\times \text{Id}_V \).

**Example 1.** We show here that the assumption on \( K \) to be algebraically closed is crucial for the unicity up to scalar condition in Lemma 2.1 (which fails if we can’t use Schur’s lemma). Let \( K = \mathbb{R} \) and let \( E = \mathbb{Z}/4 \) with generator \( e \). We define

\[
\rho: E \to \text{GL}_2(\mathbb{R}), \ e \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}
\]

and

\[
V := \text{Span}_\mathbb{R}(e_1, e_2) \text{ with } e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}.
\]

We have that \((V, \rho)\) is a simple 2-dimensional \( \mathbb{R}E \)-module and the respective actions of \( E \) on the two distinct bases \( \{e_1, e_2\} \) and \( \{e_1 - e_2, e_1 + e_2\} \) of \( V \) are the same, given by \( \rho \). However, the matrix of base change between these two bases is not a multiple of the identity.
Let $M = (V, \rho)$ be a $K E$-module of $K$-dimension $n$. Then for $L \in \text{GL}(V)$, we denote

$$L \cdot (V, \rho) := (LV, \rho^L).$$

We can actually extend the result from Lemma 2.1 to any isotypical $KE$-module.

**Theorem 2.3.** Let $M = \bigoplus_{i=1}^{t} (V_i, \rho_i)$ be an isotypical $KE$-module of weight $n$, i.e. all the $(V_i, \rho_i)$’s are simple and equivalent to a fixed $n$-dimensional simple $KE$-module $M' = (W, \mu)$. Then for all $1 \leq j \leq t$ there exists an unique, up to scaling, $L_j \in \text{GL}(V_j)$ such that $L_j \cdot M_j \cong M'$ and

$$M = \bigoplus_{j=1}^{t} L_j \cdot M_j.$$

Moreover, this decomposition is unique up to the action of $GL_n(K)$ on the summands of $M$.

**Proof.** It follows directly from Lemma 2.1. The unicity up to the action of $GL_n(K)$ follows also from Schur’s lemma (Theorem 1.2) which tells us that $\text{Aut}(M) \cong GL_n(K)$. \qed

Let $M = (V, \rho)$ be isotypical of weight $n$ and $K$-dimension $tn$ such that $M \cong (W, \mu)^{\otimes t}$ with $(W, \mu)$ simple, and let’s fix bases $B_V$ and $B_W$ of $V$ and $W$ respectively. Let $M_{\rho, \mu}(B_V, B_W)$ be the matrix space of $n$-by-$tn$ matrices $B$ such that, for all $e \in E$

$$B \rho(e) = \mu(e) B,$$

where we identify $\text{GL}(V)$ and $\text{GL}(W)$ with the respective group of invertible matrices using the fixed bases of $V$ and $W$. This space contains the zero matrix, it is closed under addition and multiplication by $K$-scalars: it is a $K$-vector space. Since $(W, \mu)$ is simple, any non-zero matrix in $M_{\rho, \mu}(B_V, B_W)$ is of full rank $n$. Moreover, Theorem 2.3 tells us in particular that $M_{\rho, \mu}(B_V, B_W)$ is finite-dimensional of $K$-dimension $t$. A basis of $M_{\rho, \mu}(B_V, B_W)$ is algorithmically computable (see for instance [CIK97, Theorem 2] for $t = 1$; the cases for $t > 1$ can be done similarly). The rows of any matrix in $M_{\rho, \mu}(B_V, B_W)$ define the coordinates in $B_V$ of a basis of a copy of $(W, \mu)$ in $M$.

**Theorem 2.4** (Gauss elimination). Let $M \cong (W, \mu)^{\otimes t}$ be an isotypical $KE$-module of weight $n$ and $K$-dimension $tn$. Let $N$ be a $KE$-submodule of $M$. Then

1. $N$ is isotypical of weight $n$ and $K$-dimension $rn$ for some $1 \leq r \leq t$, and $r = 1$ if and only if $N$ is simple.
2. For $1 \leq r \leq t$, the space of $KE$-submodules of $M$ of dimension $rn$ is isomorphic to the Grassmannian $\text{Gr}(r, t)$.

**Proof.**

1. Since $N$ is a $KE$-submodule of $M$, its character is a constituent of the character of $M$. Therefore $N$ is isotypical of weight $\dim_K(W) = n$ and dimension $rn$ with $1 \leq r \leq t$. Moreover, $r = 1$ if and only if $N$ affords the same character as $(W, \mu)$, i.e. $N$ is simple.
2. Let us fix a basis $B_V$ of the underlying $K$-vector space $V$ of $M$, and a basis $B_W$ of $W$. Let $N$ be a submodule of $M$. Then by Item 1, there exists $1 \leq r \leq t$ such that $N$ is isotypical of weight $n$ and $K$-dimension $rn$, so in particular $N \cong (W, \mu)^\otimes r$. Therefore, the coordinates in $B_V$ of a basis of any copy of $(W, \mu)$ in $N$ are given by the rows of an element of $M_{\rho, \mu}(B_V, B_W)$. In particular, if $B_1, \ldots, B_r$ are $r$ matrices whose respective rows define basis vectors for each of $r$ distinct copies of $(W, \mu)$ in $N$, then $B_1, \ldots, B_r$ are $K$-linearly independent. Therefore, $B_1, \ldots, B_r$ define a $r$-space inside the $t$-dimensional $K$-vector space $M_{\rho, \mu}(B_V, B_W)$. Conversely, given any such $r$-space $A$, for $1 \leq r \leq t$, one can define a submodule of $M$ of $K$-dimension $rn$ by taking the $K$-linear span of the vectors of $V$ whose coordinates in $B_V$ are given by the rows of the matrices in a basis of $A$. It is clear that this construction does not depend on the choice of the basis of $A$. Hence, choosing a $KE$-submodule of $M$ of dimension $rn$ is equivalent to find a $r$-space in $M_{\rho, \mu}(B_V, B_W)$, which is of $K$-dimension $t$, so we can conclude. \qed
2.2 Moduli space of submodules

In this subsection we show that, using Gauss elimination for isotypical KE-modules, the space of KE-submodules with a given character of any KE-module is rational.

Recall that \( M = (V, \rho) \) is a KE-module, and let \( n := \dim_K(V) \). For all \( 1 \leq t \leq n \), we define \( \mathcal{M}(M, t) \) to be the moduli space of \( t \)-dimensional KE-submodule of \( M \). Using iteratively an argument from [MWY20, Theorem 5.11], we have that \( \mathcal{M}(M, t) \) is a closed subvariety of \( \text{Gr}(t, V) \subseteq \mathbb{P}^{n-1}_K \). In general, \( \mathcal{M}(M, t) \) is not irreducible, and we give two ways to decompose it: we use the first one computationally to parametrize all \( t \)-dimensional submodules of \( M \).

Let \( \chi \) be the \( K \)-character of \( E \) afforded by \( M \). For all \( 1 \leq t \leq \chi(1_E) \), we define \( ch_\chi(t) \) to be the set all of \( t \)-dimensional \( K \)-characters of \( E \) that are consistent with \( \chi \). For \( 1 \leq t \leq \chi(1_E) \), each \( \eta \in ch_\chi(t) \) defines an equivalence class of \( t \)-dimensional submodules of \( M \). We denote by \( \mathcal{M}(M, \eta) \) the moduli space of \( t \)-dimensional submodules of \( M \) affording \( \eta \). One has that

\[
\mathcal{M}(M, t) = \bigsqcup_{\eta \in ch_\chi(t)} \mathcal{M}(M, \eta).
\]

The following holds:

**Theorem 2.5.** Let \( 1 \leq t \leq \chi(1_E) \). For all \( \eta \in ch_\chi(t) \), \( \mathcal{M}(M, \eta) \) is a rational closed subvariety of \( \mathcal{M}(M, t) \) of dimension \( \eta, \chi - \eta \). In particular, \( \{ \mathcal{M}(M, \eta) \}_{\eta \in ch_\chi(t)} \) is the set of irreducible components of \( \mathcal{M}(M, t) \).

**Proof.** Let \( \eta \in ch_\chi(t) \) and let \( \eta = \sum_{\mu \in \text{Irr}_K(E)} e_\mu \mu \) be the isotypical decomposition of \( \eta \). Then, for a KE-module \( N \in \mathcal{M}(M, \eta) \), \( N \) is a direct sum of isotypical KE-submodules of \( M \) affording respectively \( e_\mu \mu \) for \( \mu \in \text{Irr}_K(E) \) such that \( e_\mu \mu \neq 0 \). Let

\[
M = \bigoplus_{\mu \in \text{Irr}_K(E)} W^\oplus_{\mu} f_{\mu}
\]

be an isotypical decomposition of \( M \) (where \( W_\mu \) affords \( \mu \) and some of the \( f_{\mu}'s \) can be zero). For all \( \mu \in \text{Irr}_K(E) \), one has \( 0 \leq e_\mu \mu \leq f_{\mu} \), and the isotypical component \( N^e_{\mu} \) of \( N \) affording \( e_\mu \mu \) is a KE-submodule of \( W^e_{\mu} \), i.e.

\[
N^e_{\mu} \in \mathcal{M}(W^e_{\mu}, e_\mu \mu).
\]

By Theorem 2.4, \( \mathcal{M}(W^e_{\mu}, e_\mu \mu) \) is isomorphic to the Grassmannian variety \( \text{Gr}(e_\mu, f_{\mu}) \), which is of dimension \( e_\mu(f_{\mu} - e_\mu) = (e_\mu \mu, f_{\mu} - e_\mu \mu) \). Therefore, noticing that

\[
\mathcal{M}(M, \eta) = \prod_{\mu \in \text{Irr}_K(E), e_\mu \neq 0} \mathcal{M}(W^e_{\mu}, e_\mu \mu)
\]

one deduces that \( \mathcal{M}(M, \eta) \) is rational by the rationality of Grassmannian varieties, which is preserved under products. \( \mathcal{M}(M, \eta) \) is therefore an irreducible closed subvariety of \( \mathcal{M}(M, t) \) of dimension

\[
\sum_{\mu \in \text{Irr}_K(E)} (e_\mu \mu, f_{\mu} - e_\mu \mu) = \langle \eta, \chi - \eta \rangle.
\]

**Theorem 2.5** offers a feasible way to parametrising \( t \)-dimensional submodules of a given KE-module \( M \). Indeed, for all \( t \)-dimensional constituent \( \eta \) of the character \( \chi \) of \( M \), one may use Theorem 2.4 and Theorem 2.5 to construct a concrete parametrisation of \( \mathcal{M}(M, \eta) \).

2.3 Determinantal character

In this subsection, we give another decomposition of \( \mathcal{M}(M, t) \) from Section 2.2, by looking at the determinantal characters of the \( t \)-dimensional KE-submodules of \( M \).

Let \( 1 \leq t \leq \chi(1_E) \), where we recall that \( \chi \) is the character afforded by a fixed KE-module \( M = (V, \rho) \) of dimension \( n \). Any element of the \( t \)-th exterior power \( \Lambda^t V \) of \( V \) is called a \( t \)-tensor
of $V$ and those of the form $v_1 \wedge \ldots \wedge v_t$ are called completely decomposable or pure. Any element of $\Lambda^t V$ can be written as a finite sum of pure $t$-tensors. There is moreover an induced action of $E$ on $\Lambda^t V$ given by, for all $e \in E$ and for any pure $t$-tensor $v_1 \wedge \ldots \wedge v_t$ of $V$,

$$ e \cdot (v_1 \wedge \ldots \wedge v_t) := (\rho(e)v_1) \wedge \ldots \wedge (\rho(e)v_t). $$

We denote by $\Lambda^t \rho$ the previous representation of $E$ on $\Lambda^t V$ and $\Lambda^t M := (\Lambda^t V, \Lambda^t \rho)$ the corresponding $KE$-module. We call it the $t$-antisymmetric part of $M$.

**Proposition 2.6.** Let $M$ be a $KE$-module of finite $K$-dimension $n$. Then $M(M,t)$ is non-empty if and only if $\Lambda^t M$ admits a 1-dimensional $KE$-submodule whose underlying $K$-vector space is spanned by a pure $t$-tensor.

**Proof.** This follows from Eq. (2) and from the fact that a pure tensor $v_1 \wedge \ldots \wedge v_m$ is non zero if and only if its components $v_1, \ldots, v_m$ are linearly independent in $V$. \qed

Let $\Lambda^t M = \bigoplus_{\mu \in \text{Irr}_K(E)} U^\otimes \mu$ be an isotypical decomposition of $\Lambda^t M$. Therefore, $\Lambda^t M$ has a 1-dimensional $KE$-submodule if and only there exists $\mu \in \text{Irr}_K(E)$ such that $g_\mu \neq 0$. For a linear (i.e. 1-dimensional) $K$-character $\mu$ of $E$ with $g_\mu \neq 0$, the action of $E$ on each summand of $U^\otimes \mu$ is given by $\mu$. Therefore, since $\mu$ is a homomorphism

$$ U^\otimes \mu = \bigcap_{e \in E} \text{Eigenspace} \left( \left( \Lambda^t \rho \right)(e), \mu(e) \right). $$

This proves the following.

**Theorem 2.7 (Ser77, Theorem 4).** With the same notation as in Proposition 2.6 and Eq. (3), $M(M,t)$ is non empty if and only if there exists $\mu \in \text{Irr}_K^1(E)$ such that $g_\mu \neq 0$ and $U^\otimes \mu_\mu$ contains a pure tensor.

Denote by $\chi$ the $K$-character of $E$ afforded by $M$. For any $\eta \in ch_\chi(t)$, we call the determinantal character of $\eta$, denoted $\det(\eta)$, the character afforded by the $t$-antisymmetric part of any $KE$-module affording $\eta$. This is a 1-dimensional character, constituent of the character $\Lambda^t \chi$ afforded by $\Lambda^t M$. Note that two distinct constituents $\eta, \eta' \in ch_\chi(t)$ of $\chi$ can have the same determinantal character. For any linear $K$-character $\mu$ of $E$, we denote $M(M,t,\mu)$ the moduli space of $t$-dimensional $KE$-submodules of $M$ having determinantal character equal to $\mu$. Then, we have the decompositions

$$ M(M,t) = \bigsqcup_{\mu \in \text{Irr}_K^1(E)} M(M,t,\mu) $$

and for all $\mu \in \text{Irr}_K^1(E)$

$$ M(M,t,\mu) = \bigsqcup_{\eta \in ch_\chi(t),\det(\eta)=\mu} M(M,\eta). $$

In the rest of this section, we show that for all $\mu \in \text{Irr}_K^1(E)$ such that the set $\{ \eta \in ch_\chi(t) \mid \det(\eta) = \mu \}$ is non empty, $M(M,t,\mu)$ is an algebraic subvariety of $M(M,t)$ by explaining how one can compute its defining ideal. For this, we make explicit how to find pure tensors in an isotypical component of weight 1 of $\Lambda^t M$.

**Proposition 2.8 (Har92, Page 64).** Let $V$ be a $K$-vector space of dimension $n$ and let $1 \leq t \leq n-1$. Then for any non-zero $w \in \Lambda^t V$, $w$ is a pure $t$-tensor if and only if the linear map

$$ \varphi: V \to \Lambda^{t+1} V, \quad v \mapsto v \wedge w $$

has kernel of dimension $t$.

**Proof.** The result follows from the observation that for a non-zero pure tensor $w = v_1 \wedge \ldots \wedge v_t$, $v_1, \ldots, v_t$ are $K$-linearly independent and any $v$ such that $v \wedge w = 0$ must lie in the $t$-dimensional $K$-vector space spanned by the $v_i$’s. In particular, for a general non-zero $t$-tensor $w \in \Lambda^t V$, the kernel of $\varphi$ is of dimension at most $t$, with equality if and only if $w$ is pure. \qed
Remark 2.9. We chose here to state Proposition 2.8 regarding the dimension of the kernel of $\varphi$, but one can state a similar result regarding the rank of $\varphi$, using the Rank-nullity theorem. Indeed, we see with the last comment of the proof that the dimension of the kernel of $\varphi$, for any tensor in $\bigwedge^t V$, will never exceed $t$. Therefore, for any non-zero tensor in $\bigwedge^t V$, $\varphi$ has rank at least $n-t$, with equality if the tensor is pure. In particular, one only needs to check whether the rank is at most $n-t$.

We state also a corollary to Proposition 2.8. For this, we consider the following: fix a volume form given by the natural pairing
\[
\bigwedge^t V \times \bigwedge^{n-t} V \rightarrow \bigwedge^n V \xrightarrow{\text{vol}} K
\]
\[
(w, w') \mapsto w \wedge w' \xrightarrow{\text{vol}} w'' \rightarrow \text{vol}(w'').
\]

To define vol, fix a basis $\{v_1, \ldots, v_n\}$ of $V$ and for $w'' \in \bigwedge^n V$ non zero, via permutation of the factors, one can write uniquely $w'' = \text{vol}(w'')v_1 \wedge \ldots \wedge v_n$ (and $\text{vol}(0) = 0$). Eq. (4) provides an isomorphism between $\bigwedge^t V$ and $(\bigwedge^{n-t} V)^*$. Moreover, there exists an isomorphism between $(\bigwedge^{n-t} V)^*$ and $\bigwedge^{n-t} V^*$, given on the elements of the basis by $(v_1 \wedge \ldots \wedge v_{n-t})^* \equiv v_1^* \wedge \ldots \wedge v_{n-t}^*$ for pairwise distinct indices $i_j$'s. Therefore, to each non-zero $w \in \bigwedge^t V$ one can associate a non-zero element $v^* \in \bigwedge^{n-t} V^*$ (which depends on the choice of the volume form). Via those descriptions, one sees that $w$ is a pure $t$-tensor if and only if $w^*$ is a pure $(n-t)$-tensor.

Corollary 2.10. Let $V$ be a $K$-vector space of dimension $n$ and let $1 \leq t \leq n-1$. Then for any non-zero $w \in \bigwedge^t V$, $w$ is a pure tensor if and only if the linear map
\[
V^* \rightarrow \bigwedge^{n-t+1} V^*
\]
\[
v^* \rightarrow w^* \wedge v^*
\]
has rank at most $t$ (since it is at least $t$, by duality with Remark 2.9). Here $w^*$ is obtained by fixing a volume form on $\bigwedge^n V$ (see Eq. (4)).

It is good to keep in mind both Proposition 2.8 and its Corollary 2.10 for computational aspects. Indeed, one has to choose to work either with $V$ or its dual depending on $t$ and $n$.

Now, if $W = (U, \mu^W)$ is an isotypical component of weight 1 of $\bigwedge^t M = (\bigwedge^t V, \bigwedge^t \rho)$, identifying $\mathbb{P}(\bigwedge^t V)$ with $\mathbb{P}_{K}^{t-1}$ as finite dimensional $K$-vector spaces, we have that the set of pure tensors $U'$ in $U$ is actually the same as
\[
U' = \mathbb{P}(U) \cap \text{Gr}(t, V).
\]

Here we implicitly identify the Grassmannian variety $\text{Gr}(t, V)$ with its image via
\[
\text{Gr}(t, V) \xrightarrow{\iota} \mathbb{P}_{K}^{t-1} \xrightarrow{\phi} \mathbb{P}(\bigwedge^t V)
\]
where $\iota$ is the usual Plucker embedding. Using Proposition 2.8, we can describe the ideal defining the projective variety $U'$ in $\mathbb{P}_{K}^{t-1}$. In fact, denoting $(u_1, \ldots, u_l)$ a basis of $U$, for any non-zero $u \in U$, there exist scalars $y_1, \ldots, y_l \in K$, not all zero, such that
\[
u = \sum_{i=1}^{l} y_i u_i.
\]

Therefore, $U'$ is defined as the set of tuples $(y_i)_{1 \leq i \leq l} \in K^l \setminus \{0\}$ for which the map
\[
\varphi(y_1, \ldots, y_l): V \rightarrow \bigwedge^{t+1} V
\]
associated to $u = \sum_{i=1}^{l} y_i u_i$ has rank $n-t$. Such a set can be obtained by constructing the polynomial matrix $P$ with entries in $K[y_1, \ldots, y_l]$ corresponding to the linear map
\[
\varphi(y_1, \ldots, y_l): V \rightarrow (\bigwedge^{t+1} V)[y_1, \ldots, y_l], \ v \mapsto v \wedge (\sum_{i=1}^{l} y_i u_i)
\]
and computing the ideal $I$ generated by all $(n-t+1)$-minors of $P$: in this situation, one can show that $U' = V(I)$. 

10
3 Finding complete intersections with prescribed symmetry

In this section, we show how one can use the method explained in the previous sections to find defining ideal of complete intersections that are fixed by a linear action of a finite group on their ambient complex projective space.

Let $X$ be a complex projective variety in $\mathbb{P}^n_C$ given as a complete intersection of $t$ hyperplanes of the same degree $d$. Let $G$ be a finite group and suppose that $G$ acts linearly on $\mathbb{P}^n_C$ while fixing $X$. We see that $X$ is therefore given by a regular section of $\mathcal{O}_{\mathbb{P}^n_C}(d)^{\otimes t}$ whose vanishing locus is invariant under the linear action of $G$ on $\mathbb{P}^n_C$. Let $E$ be a Schur cover of $G$ (see Section 1.3). We define $\overline{\rho}: G \to \text{PGL}_{n+1}(\mathbb{C})$ to be the projective representation of $G$ associated to its linear action on $\mathbb{P}^n_C$, and we let $\rho: E \to \text{GL}_{n+1}(\mathbb{C})$ be a lift of $\overline{\rho}$ making the following commutative diagram with exact rows commute

$$
\begin{array}{cccccc}
1 & \to & H & \to & E & \to & G & \to & 1 \\
\beta \downarrow & & \rho \downarrow & & \overline{\rho} \downarrow & & & \\
1 & \to & \mathbb{C}^\times & \to & \text{GL}(\mathbb{C}^{n+1}) & \to & \text{PGL}(\mathbb{C}^{n+1}) & \to & 1
\end{array}
$$

We write $X = V(s)$ where $s$ is a regular section of $\mathcal{O}_{\mathbb{P}^n_C}(d)^{\otimes t}$ $(n, d, t \in \mathbb{Z}_{\geq 0})$ with $G$-invariant vanishing locus: we say that $s$ is a (regular) $(G, n, d, t)$-section.

3.1 From invariant ideals to group algebra modules

In this subsection we transform the problem of finding $(G, n, d, t)$-sections to finding $t$-dimensional $CE$-submodules of $R_d$, the $d$-homogeneous part of the polynomial algebra associated to $\mathbb{C}^{n+1}$.

Let us fix a $(G, n, d, t)$-section $s$ and a projective representation $\overline{\rho}: G \to \text{PGL}(\mathbb{C}^{n+1})$: we are in the context of Diagram 5. The ideal $I$ defining the vanishing locus $V(s)$ of $s$ is homogeneous and generated by $t$ homogeneous polynomials $f_1, \ldots, f_t \in \mathbb{C}[x_0, \ldots, x_n]$ of common degree $d$. We denote by $R_\ast := \bigoplus_{h \geq 0} \mathbb{C}[x_0, \ldots, x_n]_h$ the $\mathbb{Z}$-graded $\mathbb{C}$-algebra of polynomials in $n + 1$ variables. Considering Diagram 5, the action of $E$ on $\mathbb{C}^{n+1}$ defined by $\rho: E \to \text{GL}(\mathbb{C}^{n+1})$ naturally induces, for all $h \geq 0$, a linear action on $R_h$. It is given as follows: for any $h \geq 0$, $P \in R_h$, $e \in E$ and $x \in \mathbb{C}^{n+1}$,

$$(e \cdot P)(x) := P(\rho(e)^{-1}(x)).$$

It is a well-defined action, because the action of $E$ on $\mathbb{C}^{n+1}$ is linear, which we denote by $\rho_h$. Collecting these actions for all $h \geq 0$ gives $(R_\ast, \rho_h)$ the structure of a $CE$-algebra: $R_\ast$ is a $\mathbb{Z}$-graded $\mathbb{C}$-algebra and all of its homogeneous components $R_h$ $(h \geq 0)$, equipped with the action $\rho_h$, are $CE$-modules.

**Proposition 3.1.** Let $K$ be a field, let $E$ be a group and let $(R_\ast, \rho_h)$ be a $\mathbb{Z}$-graded $KE$-algebra. Let $I$ be a homogeneous ideal of $R_\ast$ being finitely generated by $t$ homogeneous elements $r_1, \ldots, r_t \in R_\ast$ of respective degrees $d_1, \ldots, d_t$ (possibly non pairwise distinct). We denote by $I_h := I \cap R_h$ the $h$-homogeneous part of $I$. Then, $I$ is invariant for the given action of $E$ on $R_\ast$ if and only if $(I_{d_i}, \rho_{d_i})$ is a $KE$-submodule of $(R_{d_i}, \rho_{d_i})$ for all $i = 1, \ldots, t$ (here we use the same notation for the restriction of $\rho_h$ to $I_h$, $h \geq 0$).

**Proof.** First, remark that $I = \bigoplus_{h \in \mathbb{Z}} I_h = \bigoplus_{i = 1}^t I_{d_i}$ as $R_0$-modules since $I$ is generated by the $t$ homogeneous elements $r_1, \ldots, r_t$. Therefore, we see that if $E \cdot I = I$ (i.e. $I$ is $E$-invariant) then for all $i = 1, \ldots, t$, $E \cdot I_h = E \cdot (I \cap R_h) \subseteq I_h$, because $R_h$ is fixed under the action of $E$. Therefore, $(I_{d_i}, \rho_{d_i})$ is a $CE$-submodule of $(R_{d_i}, \rho_{d_i})$, for all $i = 1, \ldots, t$.

Now suppose that for all $i = 1, \ldots, t$, $I_{d_i}$ is $E$-invariant. Since $I$ is generated by $\bigcup_{i = 1}^t I_{d_i}$ as a $R_\ast$-module and $(R_{d_i}, \rho_{d_i})$ is a $KE$-module, then $I$ is $E$-invariant. \qed

Recall that $X = V(s)$ is the complete intersection in $\mathbb{P}^n_C$ defined by $s$. Considering Diagram 5 with $R_d$ instead of $\mathbb{C}^{n+1}$, we know that $\rho_d$ reduces to a unique projective representation of $G$ on $R_d$. By commutativity of Diagram 5, one sees that $(I_{d}, \rho_{d})$ is a $CE$-submodule of $(R_{d}, \rho_{d})$ if and
only if \( P(I_d) \) is invariant under the induced action of \( G \) on \( P(R_d) \). Moreover, the fact that \( X \) is fixed under the action of \( G \) on \( P(V) \) is equivalent to have \( P(I_d) \) invariant under the induced action of \( G \) on \( P(R_d) \). Therefore, according to Proposition 3.1, having \( X \) invariant under the action of \( G \) on \( P(V) \) is equivalent to have \( I \) invariant under the induced action of \( E \) on \( R \). This also means that if \( V \) is the \( \mathbb{C} \)-span of \( f_1, \ldots, f_t \) in \( R \), then \( I \) is the ideal of \( R \) generated by \( V \) and \( X \) being \( G \)-invariant is equivalent to have \( (V, \rho_d \mid V) \) being a \( CE \)-submodule of \( (R_d, \rho_d) \).

Therefore, in order to explicitly determine the \( f_i \)'s, one can equivalently search for a linear representation \( \rho \) of \( E \) on \( \mathbb{C}^{n+1} \) that reduces to a faithful projective representation of \( G \) (see Subsection 3.2), and a \( CE \)-submodule \( W \) of \( (R_d, \rho_d) \) whose underlying \( \mathbb{C} \)-vector space is spanned by \( t \) homogeneous polynomials of common total degree \( d \) (see Section 2).

### 3.2 Classification of projectively faithful representations

In Definition 1.5 we define an equivalence relation on the linear representations of \( E \). In this subsection, we define another equivalence relation, coarser than the previous one. More precisely, we classify linear representations of \( E \) having faithful reduction, up to similarity of their respective reductions to \( G \).

Let \( E \) and \( G \) as before.

**Definition 3.2.** Let \( V \) be a finite-dimensional \( \mathbb{C} \)-vector space. A linear representation \( \rho: E \to \text{GL}(V) \) is said to be projectively faithful if \( \text{im}(\pi \circ \rho) \) is isomorphic to \( G \).

By commutativity of Diagram 5 and by surjectivity of \( \rho \), we see that any representation \( \rho \) of \( E \) on \( V \) is projectively faithful if and only if its reduction \( \pi: G \to \text{PGL}(\mathbb{C}^{n+1}) \) is faithful. Therefore, in order to find a \( CE \)-module \( W \) whose underlying vector space generates the defining ideal \( I \) of \( X \) (see Section 3.1), we start by classifying the projectively faithful representations of \( E \) on \( \mathbb{C}^{n+1} \).

**Definition 3.3 ([Isa76], Definition (2.26)).** Let \( \chi \) be the \( \mathbb{C} \)-character of \( E \) afforded by a \( CE \)-module \((V, \rho)\). Then, the center of the character \( \chi \) is defined to be

\[
Z(\chi) := \left\{ e \in E \mid \frac{\chi(e)}{\chi(1)} \text{ is a root of unity} \right\}.
\]

**Proposition 3.4.** With the notations of Definition 3.3 and Diagram 5, \( Z(\chi) = \ker(\pi \circ \rho) \).

*Proof.* According to [Isa76, Lemma (2.27)],

\[
Z(\chi) = \{ e \in E \mid \rho(e) \in \mathbb{C}^* \text{Id}_V \}
\]

so the first inclusion \( Z(\chi) \subseteq \ker(\pi \circ \rho) \) holds. Now, let \( e \in E \) such that \( \pi(\rho(e)) = 1_{\text{PGL}(V)} \). In particular, \( \rho(e) \in \ker(\pi) = \mathbb{C}^* \text{Id}_V \). Therefore, according to Eq. (6), \( e \in Z(\chi) \).

**Corollary 3.5.** With the notations of Definition 3.3 and Diagram 5, \( \pi \) is faithful if and only if \( E/Z(\chi) \cong G \).

Using Corollary 3.5, we are now able to decide whether a linear representation of \( E \) is projectively faithful or not. In order to classify them, we first consider the following: if \( \pi \) and \( \pi' \) are two similar projective representations of \( G \) on \( V \), we denote by \( \mathcal{L} \in \text{Aut}(V) \) an automorphism of \( V \) such that, for all \( g \in G \),

\[
\mathcal{L} \circ \pi(g) \circ \mathcal{L}^{-1} = \pi'(g).
\]

Then, a \( \mathbb{C} \)-subvector space \( W \subseteq V \) satisfies that \( P(W) \) is invariant under the action of \( G \) on \( \mathbb{P}(V) \) given by \( \pi \) if and only if \( P(\mathcal{L}W) \) is invariant under the action of \( G \) on \( \mathbb{P}(V) \) given by \( \pi' \). This means that the projective varieties whose defining ideals are respectively generated by \( W \) and \( \mathcal{L}W \) are \( G \)-equivariantly isomorphic. Using Lemma 1.6, we can thus classify projectively faithful representations of \( E \) on \( \mathbb{C}^{n+1} \) by equivalence modulo \( \text{Irr}_G^E(E) \). This concludes the classification: among all the linear representations of \( E \) on \( \mathbb{C}^{n+1} \), we choose only a finite number of them corresponding to representatives of classes in

\[
\{ \text{projectively faithful representations of } E \text{ on } \mathbb{C}^{n+1} \} / \{ \rho \sim \rho' \text{ if and only if } \exists \epsilon \in \text{Irr}_G^E(E) \text{ s.t. } \chi_\rho = \epsilon \chi_\rho' \}
\]
We ensure that there are only finitely many such classes since the number of equivalence classes of linear representations of $E$ on $\mathbb{C}^{n+1}$ is actually finite (see [Isa76, Corollary (2.5)]).

### 3.3 Application to the case of K3 surfaces

In this subsection, we apply the previous theory to compute projective models of K3 surfaces given as complete intersections of hyperplanes of the same degree.

In the paper [BH21], S. Brandhorst and K. Hashimoto study pairs $(S, G)$ consisting of a complex polarised K3 surface $S$ and a (maximal) finite subgroup $G \leq \text{Aut}(S)$ of automorphisms of $S$ such that the proper subgroup $G_s$ of symplectic automorphisms is among the 11 maximal symplectic subgroups classified by Mukai ([Muk88]). Such pairs $(S, G)$ come with a canonical polarization $L$. They gather such pairs into 42 different isomorphism classes and they exhibit 25 cases for which an explicit projective model is known. In particular, all of the pairs $(S, G)$ for $S$ of genus $\leq 6$ have been treated except one of genus 5 (case 77b). Following the notation in [BH21] (except the polarization that we denote "L" here and not "l"), we display in Table 1 some information about this isomorphism class of K3 surfaces.

| case | $G_s$ | $\Lambda^{\nu_s}_{K3}$ | SO($\Lambda^{\nu_s}_{K3}$) | $G/G_s$ | $c_1(L)^2$ | $G$ |
|------|-----|-----------------|-----------------|--------|----------|-----|
| 77b  | $T_{192}$ | $\begin{pmatrix} 4 & 0 & 0 \\ 0 & 8 & 4 \\ 0 & 4 & 8 \end{pmatrix}$ | $D_6$ | $\mu_2$ | 8 | $T_{192} \rtimes \mu_2$ GAP Id [384, 5602] |

Table 1: Specification for the triple $(S, G, G_s)$ considered in this paper

where $T_{192}$ is one of the maximal subgroups of symplectic automorphism classified by Mukai [Muk88].

According to a remark in [SD74, Page 615], either the polarization $L$ is hyperelliptic and $\vert L \vert$ defines a degree 2 map onto a curve of degree 4 in $\mathbb{P}_C^5$, or it is not hyperelliptic and $S$ is isomorphic to a smooth complete intersection (s.c.i.) of 3 quadrics in $\mathbb{P}_C^5$ (see [May72, Page 9]). Once such a polarization is known, it is algorithmically possible to check whether it is hyperelliptic or not:

**Theorem 3.6** ([SD74, Theorem 5.2.]). Let $\vert L \vert$ being a complete linear system on a K3 surface $S$ without fixed components and such that $c_1(L)^2 \geq 4$. Then $L$ is hyperelliptic only if one of the following holds

- There exists an irreducible curve $E \in \text{NS}(S)$ of genus 1 such that $c_1(L) \cdot E = 2$;
- There exists an irreducible curve $B \in \text{NS}(S)$ of genus 2 such that $c_1(L) = 2B$.

According to the recent database of S. Brandhorst and T. Hofmann [BH22], this pair $(S, G)$ corresponds to the case "77.2.1.3", and using in complement an algorithm of I. Shimada (see [Shi15, Algorithm 2.2]) one may show that the surface $S$ has a polarization $L$ with $c_1(L)^2 = 8$ and which does not satisfy any of the two conditions in **Theorem 3.6**. Therefore, we may take $L$ to be not hyperelliptic and $\vert L \vert$ defines an isomorphism $\varphi_{\vert L \vert}$ from $S$ to a s.c.i. of type $(2, 2, 2)$ in $\mathbb{P}_C^5$. Since the group $G$ acts faithfully on $S$ and it preserves $L$, $G$ acts faithfully on $\mathbb{P}_C^5 \cong \vert L \vert$ and the image of $\varphi_{\vert L \vert}$. This image can be seen as the vanishing locus of a smooth section $s$ of the vector bundle $\mathcal{O}_{\mathbb{P}_C^5}(2)\otimes$, i.e. a smooth $(G, 5, 2, 3)$-section.

Let $G$ be the group with Id [384, 5602] (in the Small Group Library [BEO22]). Using GAP [GAP21], one can show that this group has Schur multiplier $M(G)$ isomorphic to $C_2^3$, and therefore any Schur cover of $G$ has order 3072. We compute such a Schur cover $E$, using for instance the GAP method SchurCover: the following steps may differ depending on the choice of the Schur cover, yet the final result shall remain true. $E$ has 12 classes of projectively faithful representations on $K^6$, where $F := \mathbb{Q}[x]/(x^{24} - 1) = \mathbb{Q}(z)$ with 24 being the exponent of $E$. Here we choose $F$ instead of $C$ for computational reasons: according to [Isa76, Corollary (9.15)], $F$ is a splitting field for $E$, so we are allowed to restrict to $F$ (the results remain true over $C$). In what follows, we denote $i := z^6$ and $\omega := z^3$. 

13
Let $M$ be the $FE$-module $(F^6, \rho)$, where $\rho$ is given by the $\sigma_i$’s in Theorem 0.1. The representation $\rho$ is projectively faithful, and $S^2M^\vee := (R_2, \rho_2)$ is a 21-dimensional $FE$-module (where $(R_*, \rho_*)$ is defined as in Section 3.1). Let $\chi$ be the $F$-character of $E$ afforded by $S^2M^\vee$. One has that $\text{ch}_\chi(3) = \{\mu\}$ where $\mu \in \text{Irr}_F^E(E)$ and $(\chi, \mu) = 2$. Therefore, we have that $\mathcal{M}(S^2M^\vee, 3)$ is irreducible of dimension 1, equal to $\mathcal{M}(S^2M^\vee, \mu)$. Let $W$ be the isotypical component of $S^2M^\vee$ affording $2\mu$. $W$ consists of the sum of two equivalent simple modules, affording $\mu$, whose respective $F$-bases are given by

$$
\begin{align*}
    w_1 := \left(\begin{array}{c}
i x_0 x_1 + x_0 x_2 + x_1 x_3 + i x_2 x_3 \\
i x_0 x_1 - x_0 x_2 - x_1 x_3 + i x_2 x_3 \\
-x_0 x_3 - x_1 x_2
\end{array}\right), \\
w_2 := \left(\begin{array}{c}x^2_1 \\
x^2_2 \\
-x_4 x_5
\end{array}\right)
\end{align*}
$$

where $(x_0, \ldots, x_5)$ is a basis for the dual space of $F^6$. Note that these bases are chosen in such a way that the actions of $E$ on each of them are given by the same representation $E \to \text{GL}(R_2)$, which is possible according to Theorem 2.3. We know that any 3-dimensional submodule of $W$ is then generated by a linear combination of $w_1$ and $w_2$ (Theorem 2.4). However, it is easy to see that the ideals respectively generated by $w_1$ and $w_2$ do not define smooth varieties. Now let $\lambda \in F^\times$. Then, the ideal generated by $w_1 + \lambda w_2$ defines the variety $S_\lambda$ given in Theorem 0.1. For all $\lambda \in F^\times$, $S_\lambda$ is by construction a K3 surface, and for distinct non-zero $\lambda_1 \neq \lambda_2$, it is clear that $S_{\lambda_1}$ and $S_{\lambda_2}$ are $G$-equivarantly isomorphic. In this case, we say that $(S_\lambda)_{\lambda \in F^\times}$ is a 1-dimensional $G$-isotrivial family. Finally, to ensure that $S_5$ ($\lambda = F^\times$) corresponds to the case 77b in [BH21], we need that the subgroup $G_S$ of automorphisms of $G$ acting symplectically on $S_\lambda$ is isomorphic to the group $T_{192}$. We use the following Lemma:

**Lemma 3.7** ([Muk88], Lemma (2.1)). Let $S$ be a smooth complete intersection of hypersurfaces $H_i = V(f_i)$ $(1 \leq i \leq k)$ in $\mathbb{P}^N$. Assume that $\sum_{i=1}^k \deg(f_i) = N + 1$ and let $\sigma \in \text{GL}_{N+1}(\mathbb{C})$ be a linear transformation preserving $V_S := \text{Vect}_\mathbb{C}((f_i)_{i=1,\ldots,k})$. $\sigma$ induces an automorphism of $S$ which we denote $\varphi$. Let $\omega \in H^2(S, O_S)$ be no-where vanishing.

1. If, for all $1 \leq i \leq k$, there exists $a_i \in \mathbb{C}^*$ such that $f_i^* = a_i f_i$, then

$$
\varphi^* \omega = \frac{\det(\sigma)}{\prod_{i=1}^k a_i} \omega.
$$

2. If $\sigma$ is of finite order and induces a linear transformation on $\tilde{\sigma}$ on $V_S$, then

$$
\varphi^* \omega = \frac{\det(\sigma)}{\det(\tilde{\sigma})} \omega.
$$

Lemma 3.7 offers a practical, and computationally feasible, way to compute $G_S$. Here, we can apply it to the group generated by the $\sigma_i$’s on $S_\lambda$. One finds that $G_S \cong T_{192}$ with, in particular, $\sigma_i$ acting symplectically on $S_5$ for $i = 1, 2, 3, 4$ and $\sigma_5$ being a non-symplectic involution. This concludes the proof of Theorem 0.1.

**Remark 3.8.** The methods explained in this paper have been used to compute projective models of 17 other (isotrivial families of) K3 surfaces from the database in [BH22]. They are available at [Mul22] in the format of an OSCAR-readable database.

**References**

[BEKS17] Jeff Bezanson, Alan Edelman, Stefan Karpinski, and Viral B. Shah. *Julia: A fresh approach to numerical computing*. SIAM Review, 59(1):65–98, 2017.

[BEO22] Hans U. Besche, Bettina Eick, and Eamonn O’Brien. *SmallGrp, The GAP Small Groups Library, Version 1.5*. https://gap-packages.github.io/smallgrp/, Apr 2022. Refereed GAP package.

[BH21] Simon Brandhorst and Kenji Hashimoto. *Extensions of maximal symplectic actions on K3 surfaces*. Ann. H. Lebesgue, 4:785–809, 2021.
[BH22] Simon Brandhorst and Tommy Hofmann. *Finite subgroups of automorphisms of K3 surfaces*, 2022. arXiv:2112.07715.

[CD22] Paola Comparin and Romain Demelle. *K3 surfaces with action of the group M_{20}* 2022. arXiv:2201.02150.

[CIK97] Alexander Chistov, Gábor Ivanyos, and Marek Karpinski. *Polynomial Time Algorithms for Modules over Finite Dimensional Algebras*. In *Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation*, ISSAC ’97, page 68–74. ACM, 1997.

[EGH’11] Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, and Elena Yudovina. *Introduction to representation theory*, volume 59 of Stud. Math. Libr. AMS, 2011.

[GAP21] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.11.1*, 2021. GAP.

[Har92] Joe Harris. *Algebraic Geometry: A First Course*, volume 133 of Graduate Texts of Mathematics. Springer, 1992.

[Isa76] Irvin M. Isaacs. *Character theory of finite groups*. Pure and applied mathematics, a series of monographs and textbooks. Academic Press New York, 1976.

[May72] Alan L. Mayer. *Families of K-3 surfaces*. Nagoya Math. J., 48:1–17, 1972.

[Muk88] Shigeru Mukai. *Finite groups of automorphisms of K3 surfaces and the Mathieu group*. Invent. Math. 94, 1:183–221, 1988.

[Mul22] Stevell Muller. *ProjModK3 - OSCAR methods*, 2022. GitHub.

[MWY20] Dinakar Muthiah, Alex Weekes, and Oded Yacobi. *The Equations Defining Affine Grassmannians in Type A and a Conjecture of Kreiman, Lakshmibai, Magyar, and Weyman*. Int. Math. Res. Not., 2022(3):1922–1972, 2020.

[OSC22] The OSCAR Team. *Oscar – Open Source Computer Algebra Research, Version 0.8.3-DEV*, 2022. OSCAR.

[Sch04] Issai Schur. *Über die Darstellung der endlichen Gruppen durch gebrochen lineare Substitutionen*. J. für Reine Angew. Math., 127:20–50, 1904.

[SD74] Bernard Saint-Donat. *Projective Models of K - 3 Surfaces*. Am. J. Math., 96:602–639, 1974.

[Ser77] Jean-Pierre Serre. *Linear representations of finite groups*, volume 42 of Graduate Texts in Mathematics. Springer, 1977.

[Shi15] Ichiro Shimada. *An algorithm to compute automorphism groups of K3 surfaces and an application to singular K3 surfaces*, 2015. arXiv:1304.7427v6.

[Smi07] James P. Smith. *Picard-Fuchs Differential Equations for Families of K3 Surfaces*, 2007. arXiv:0705.3658.