ON ABEL MAPS OF STABLE CURVES

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Abstract. We construct Abel maps for a stable curve $X$. Namely, for each one-parameter deformation of $X$ to a smooth curve, having regular total space, and each integer $d \geq 1$, we construct by specialization a map $\alpha^d_X : \dot{X} \to P^d_X$, where $\dot{X} \subseteq X$ is the smooth locus, and $P^d_X$ is the moduli scheme for balanced line bundles on semistable curves over $X$. For $d = 1$, we show that $\alpha^1_X$ extends to a map $\alpha^1_X : X \to P^1_X$, and does not depend on the choice of the deformation. Finally, we give a precise description of when $\alpha^1_X$ is injective.

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1. Introduction

Let $C$ be a smooth projective curve, and Pic$^d C$ its degree-$d$ Picard variety parametrizing line bundles of degree $d$ on $C$. For each $d > 0$ there exists a remarkable morphism, often called the $d$-th Abel map:

$$C^d \longrightarrow \text{Pic}^d C \qquad (p_1, \ldots, p_d) \mapsto \mathcal{O}_C(\sum p_i).$$

Such a map has been extensively studied and used in the literature. For example if $d = 1$, after a base point on $C$ is chosen, it gives the Abel–Jacobi embedding $C \hookrightarrow \text{Pic}^1 C \cong \text{Pic}^0 C$ (unless $C \cong \mathbb{P}^1$). For an interesting historic survey see [K04] or [K05].

What about Abel maps for singular curves? Not much is known for reducible curves, whereas the case of integral curves is better understood, as we will explain further down. In the present paper we construct Abel maps for stable curves (including reducible ones, of course).

As we see it, Abel maps should satisfy the following natural properties. First, they should have a geometric meaning. More explicitly, recall that for a smooth curve $C$ the $d$-th Abel map is the moduli map of a natural line bundle on $C^d \times C$; see 2.5 We want a similar property to hold for singular curves as well.
Second, Abel maps should vary continuously in families. In particular, given a one-parameter family of smooth curves specializing to a singular one, we expect the $d$-th Abel maps of the smooth fibers to specialize to the $d$-th Abel map of the singular fiber.

Both requirements turn out to be nontrivial. In order to address the second one we view stable curves as limits of smooth ones. More precisely, let $X$ be a stable curve. Let $f : X \to B$ be a family of curves over a local one-dimensional regular base $B$, with regular total space $X$, smooth generic fiber and $X$ as closed fiber. We observe that there exists a canonical way to partially extend the $d$-th Abel map of the generic fiber of $f$, by using the Néron model $N_f^d$ of the degree-$d$ Picard scheme of that fiber. The Néron mapping property yields a close relative of the $d$-th Abel map of $X$, defined on the nonsingular locus $X^d \subseteq X^d$, which we call the $d$-th Abel–Néron map of $X$; see 2.6. The target of this map is the closed fiber of $N_f^d$, rather than the Picard scheme.

The great advantage of Abel–Néron maps is their naturality, obtained directly from the universal property of Néron models. However, they have two major drawbacks. First, they do not have any a-priori modular interpretation. Second, they are not defined on the whole $X^d$.

To attack these problems we consider the modular compactified Picard variety introduced in \[C94\] and further studied in \[C05\]. If $X$ is suitably general, more precisely “$d$-general” (Definition 3.6), there exists a natural proper $B$-scheme $P^d_f$ parametrizing equivalence classes of “balanced” line bundles on semistable curves; see 3.8. These are line bundles whose multidegree satisfies certain inequalities; see Definition 3.2. It is shown in \[C05\] that $P^d_f$ contains $N_f^d$ as a dense open subscheme, hence it gives a geometrically meaningful description and completion of $N_f^d$.

So, assume for now that $X$ is $d$-general. Let $P^d_X$ be the closed fiber of $P^d_f$; it does not depend on the choice of $f$. Viewing $N_f^d$ inside $P^d_f$, it is thus possible to exhibit the $d$-th Abel–Néron map as a map $\alpha^d_X : \hat{X}^d \to P^d_X$ (see Proposition 3.10) and thus give it a geometric interpretation. Moreover, since $P^d_X$ is complete, we are able to study whether $\alpha^d_X$ extends.

Although $\alpha^d_X$ is modular, an explicit description for it is hard to exhibit in full generality; we do that only for curves with two components (see Proposition 3.12). Without such a description it is difficult to study whether and how $\alpha^d_X$ extends.

The case $d = 1$ turns out to be easier. By means of Theorem 4.6 we give an explicit description of the line bundle defining $\alpha^1_X$. Using this description, we show in Corollary 4.11 that $\alpha^1_X$ does not depend on the choice of $f$, a remarkable property not to be expected in general for $d > 1$; see Remark 3.14.

Refining our modular description of $\alpha^1_X$, we construct a completion for it as a regular map $\overline{\alpha}^1_X : X \to \overline{P}^1_X$ (Theorem 5.5). Finally, we prove that $\overline{\alpha}^1_X$ is as close as it can be to an injection (see Proposition 5.9 for the precise statement).
Now suppose that $X$ is not 1-general, so that $X$ lies in a proper closed subset of $\overline{M}_g$ for $g$ even; see Proposition 3.15. Then $\overline{P}^1_f$ fails to contain $N^1_f$; nevertheless our existence results do extend, suitably modified (see 5.10), whereas uniqueness and injectivity results (like Proposition 5.9) may fail. In this case the setup is significantly more complicated for standard technical reasons (presence of non-GIT-stable points, or of nonfine moduli spaces.) This is why we chose to first work under the assumption of 1-generality, and to later indicate, in 5.10 and 5.13, how to modify proofs and statements to include the special case.

Abel maps were constructed for all integral curves in [AK], and further studied in [EGK00], [EGK02] and [EK05]. In [AK], it is shown that the first Abel map of an integral singular curve is an embedding into its compactified Picard scheme.

Constructing Abel maps for reducible curves presents further difficulties, due to the lack of natural, separated target spaces. The use of Néron models as target spaces is not new in the literature: in [E98] Abel–Jacobi maps for nodal curves were studied by means of the Néron mapping property, similarly to what we do here with our Abel–Néron maps. However, Néron models are seldom proper and thus we cannot expect Abel maps to Néron models to be defined everywhere. In this framework, our contribution is that of bringing compactified Picard schemes into the picture. This enables us to compactify Néron models and hence to obtain a target space into which complete Abel maps could be defined. In fact, we prove that $\alpha^1_X$ extends; it is still an open problem how to extend $\alpha^d_X$ for $d > 1$.

It is our intention to construct the first Abel map of any Gorenstein singular curve, using the modular compactified Picard schemes introduced in [E01]. For such curves the connection with Néron models is not available. Nonetheless, prompted by the results of the present paper, we found significant evidence for the existence of a modular interpretation similar to the present case.

Our paper is organized as follows. Section 2 is devoted to preliminaries of various types. In Section 3 we describe degree-$d$ Abel–Néron maps to the compactified Picard scheme. In Section 4 we establish the modular description of the Abel–Néron map in degree 1, and show that it is independent of the choice of the deformation. Finally, in Section 5 we construct the completed degree-1 Abel map, give it a modular description and study when it is injective. Finally, from 5.10 to the end of the paper, we explain how to handle the special case of non-1-general curves.

2. Néron models of Picard schemes

2.1. Setup. We work over a fixed algebraically closed field $k$. All schemes are assumed locally of finite type over $k$, unless stated otherwise.

For us, a curve is a reduced and connected projective scheme of dimension 1. Mostly, we will deal with nodal curves, that is, curves whose only singularities are nodes.
A regular pencil (of curves) is a flat projective morphism \( f : \mathcal{X} \to B \) between connected, regular, schemes such that \( \dim B = 1 \), every geometric fiber of \( f \) is a curve, and \( f \) is smooth over a dense open subscheme of \( B \).

We call a regular pencil \( f : \mathcal{X} \to B \) local if \( B = \text{Spec } R \), where \( R \) is a discrete valuation ring (having \( k \) as residue field). If \( X \) is the closed fiber, we will also say that \( f \) is a regular smoothing of \( X \).

For each regular pencil \( f : \mathcal{X} \to B \) we shall let \( K : = k(B) \), the field of rational functions of \( B \), and denote by \( \mathcal{X}_K \) the generic fiber of \( f \). Notice that \( \mathcal{X}_K \) is a smooth curve over \( K \).

Given any morphism \( f : \mathcal{X} \to B \), and any integer \( d \geq 1 \), let \( f^d : \mathcal{X}_B^d \to B \) denote the \( d \)-th fibered power of \( \mathcal{X} \) over \( B \). If \( f \) is a regular pencil, we denote the open subset of \( \mathcal{X}_B^d \) where \( f^d \) is smooth by \( \check{\mathcal{X}}_B^d \); so \( \check{\mathcal{X}}_B^d := \mathcal{X}_B^d \setminus \text{Sing}(f^d) \).

If \( f \) is a local regular pencil, let \( \check{\mathcal{X}}^d \) denote the closed fiber of \( \check{\mathcal{X}}_B^d \to B \); so \( \check{\mathcal{X}}^d = \{ (p_1, \ldots, p_d) : p_i \in X \setminus X_{\text{sing}} \} \).

Given any morphism \( f : \mathcal{X} \to B \) and any \( B \)-scheme \( T \), the base change of \( f \) to \( T \) is denoted \( f_T : \mathcal{X}_T \to T \).

2.2. The relative Picard scheme. Let \( f : \mathcal{X} \to B \) be a regular pencil, and \( d \) an integer. The closed fibers of \( f \) are geometric, by our general assumption, and the general fiber is smooth. Thus the irreducible components of the fibers of \( f \) are geometrically irreducible. By a theorem of Mumford’s, [BLR], Thm. 2, p. 210, the (relative) Picard scheme \( \text{Pic} f \) of \( f \) exists, and is locally of finite type over \( S \). Furthermore, \( \text{Pic} f \) is formally smooth over \( B \) by [BLR], Prop. 2, p. 232, whence smooth over \( B \) by [BLR], Prop. 6, p. 37.

Let \( \text{Pic}^d f \) be the degree-\( d \) Picard scheme of \( f \), the open subscheme of \( \text{Pic} f \) parametrizing line bundles of relative degree \( d \). Given any \( B \)-scheme \( T \) and any line bundle \( \mathcal{L} \) on \( \mathcal{X}_T \) of \( f_T \)-relative degree \( d \), there is a moduli map associated to \( \mathcal{L} \),

\[
\mu_{\mathcal{L}} : T \to \text{Pic}^d f \quad \quad t \mapsto L_t,
\]

where \( L_t \in \text{Pic}^d_x X_t \) is the restriction of \( \mathcal{L} \) to the fiber \( X_t := f_T^{-1}(t) \). The map \( \mu_{\mathcal{L}} \) determines \( \mathcal{L} \) up to tensoring with pullbacks of line bundles from \( T \). Notice that to a map \( T \to \text{Pic}^d f \) there does not necessarily correspond a line bundle on \( \mathcal{X}_T \), though the line bundle will exist, for instance, if \( f \) admits a section; see [BLR], Prop. 4, p. 204.

2.3. Néron models of Picard schemes. Let \( f : \mathcal{X} \to B \) be a regular pencil, and \( d \) an integer. Recall that a basic characteristic (and a drawback for various applications) of the Picard scheme \( \text{Pic}^d f \) is that it is not separated over \( B \), if \( f \) has reducible special fibers. One way to fix this is to introduce the Néron model:

\[
N^d_f := \text{N}(\text{Pic}^d_x \mathcal{X}_S).
\]

The Néron model is a smooth, separated (possibly not proper) scheme of finite type over \( B \) with generic fiber equal to \( \text{Pic}^d x \mathcal{X}_K \), which satisfies a fundamental mapping property that uniquely determines it. Namely, for
every smooth $B$-scheme $Z$ each map $Z_K \to \text{Pic}^d X_K$ extends uniquely to a map $Z \to N^d_f$; see [BLR], Def. 1, p. 12.

The existence of $N^d_f$ for any regular pencil $f$ is likely well known. Since this result is fundamental for our work, but we could not find the precise statement to refer to, we sketch a proof of it using results in [BLR]. First, assume that $f$ is local, that is, $B$ is the spectrum of a discrete valuation ring $R$. Then there is a Néron model of $\text{Pic}^d X_K$ over $B$, which is equal to $\text{Pic}^d_f$ if $f$ is smooth. Indeed, since $\text{Pic}^d X_K$ is a $\text{Pic}^0 X_K$-torsor, by descent theory we may assume that $R$ is a strictly Henselian ring; see [BLR], Cor. 3, p. 158. In this case, $f$ admits a section through its smooth locus, by [BLR], Prop. 5, p. 47. This section can be used to produce a $B$-isomorphism $\text{Pic}^d_f \to \text{Pic}^0_f$. We may thus assume that $d = 0$. In this case, there is a Néron model of $\text{Pic}^0 X_K$ over $B$ because $\text{Pic}^0 X_K$ is an Abelian variety over $K$; see [BLR], Cor. 2, p. 16. Furthermore, if $f$ is smooth, $\text{Pic}^0_f$ is an Abelian $B$-scheme, whence is the Néron model of $\text{Pic}^0 X_K$ over $B$ by [BLR], Prop. 8, p. 15.

Now, consider the general case. Let $U \subseteq B$ be the largest open subscheme over which $f$ is smooth, and set $h := f|_{f^{-1}(U)}$. As we saw above, $\text{Pic}^d_h$ restricts to the Néron model of $\text{Pic}^d X_K$ locally around each point of $U$. Then $\text{Pic}^d_h$ is the Néron model of $\text{Pic}^d X_K$ over $U$ by [BLR], Prop. 4, p. 13. Finally, the local existence of Néron models of $\text{Pic}^d X_K$ around each point of $B$, and the existence of the Néron model over the dense open subscheme $U \subseteq B$ imply the (global) existence of the Néron model over $B$, by [BLR], Prop. 1, p. 18.

Since $\text{Pic}^d_f$ is smooth over $B$, a first consequence of the mapping property of the Néron model $N^d_f$ is the existence of a canonical $B$-morphism

\begin{equation}
q_f : \text{Pic}^d_f \longrightarrow N^d_f
\end{equation}

which is the identity on the generic fiber.

Assume now that the geometric fibers of $f$ are nodal. Let $X$ be a closed fiber of $f$. In the description of the Néron model, and also in our paper, the following set $\text{Tw}_f X$ of (isomorphism classes of) distinguished line bundles plays an important role:

$$
\text{Tw}_f X := \left\{ \mathcal{O}_X(D) \mid D \in \text{Div} \mathcal{X} \text{ with Supp } D \subseteq X \right\} \subseteq \text{Pic}^0 X.
$$

The divisors $D$ appearing above are simply sums with integer coefficients of the components of $X$, which are Cartier divisors of $\mathcal{X}$ because $\mathcal{X}$ is regular. Line bundles in $\text{Tw}_f X$ are called twisters. Here is a useful observation:

\begin{equation}
\forall T, T' \in \text{Tw}_f X, \quad T = T' \iff \deg T = \deg T'.
\end{equation}

Since twisters are specializations of the trivial line bundle of the generic fiber, $\mathcal{O}_X$, all of them must be identified in any separated quotient of $\text{Pic}^0_f$. In particular, $q_f(T) = q_f(\mathcal{O}_X)$ for each $T \in \text{Tw}_f X$.

We shall now identify multidegrees that differ by multidegrees of twisters. Let $\gamma$ be the number of irreducible components of $X$, and set

$$
\Lambda_X := \{\deg T : T \in \text{Tw}_f X\} \subseteq \mathbb{Z}^\gamma.
$$
Define now an equivalence relation “≡” on multidegrees by setting
\[ d \equiv d' \iff d - d' \in \Lambda_X. \]

The set of multidegree classes \( d + \Lambda_X \) with fixed total degree \( |d| \) equal to \( d \) is denoted by \( \Delta^d_X \). Thus
\[
\Delta^d_X := \{ d \in \mathbb{Z}^\gamma : |d| = d \} \equiv.
\]

It is well known that \( \Delta^0_X \) is a finite group, a purely combinatorial invariant of \( X \) often called the degree class group of \( X \), or the group of connected components of \( N^d \). In addition, for each \( d \) there is a (nonunique) bijection \( \Delta^0_X \rightarrow \Delta^d_X \), obtained by summing with any multidegree \( d \) with \( |d| = d \).

For each \( \delta \in \Delta^d_X \), let \( d \) be any multidegree representing \( \delta \), and set
\[
\text{Pic}_f^d := \{ L \in \text{Pic}^d_X : \text{deg} \, L|_X = d \} \subset \text{Pic}^d_X.
\]

This definition is easily seen not to depend on the choice of the representative \( d \) (cf. [C05], 3.9).

Assume now that \( f \) is a regular smoothing of \( X \). At this point we are able to describe the Néron model of \( \text{Pic}^d_X \):
\[
N^d_f \equiv \coprod_{\delta \in \Delta^d_X} \text{Pic}^d_f \sim_K,
\]

where \( \sim_K \) denotes the gluing along the generic fiber, equal to \( \text{Pic}^d X \); see [C05], Lemma 3.10.

Let \( N^d_X \) denote the closed fiber of \( N^d_f \). Observe that \( N^d_X \) is a disjoint union of finitely many copies of the generalized Jacobian of \( X \): picking a representative \( d \) for each class \( \delta \in \Delta^d_X \), we have
\[
N^d_X \equiv \coprod_{\delta \in \Delta^d_X} \text{Pic}^d \, X.
\]

Although the above isomorphism is not canonical, we see that the scheme structure of \( N^d_X \) does not depend on \( f \). The closed points of \( N^d_X \) are in 1-1 correspondence with the degree-\( d \) line bundles on \( X \) modulo twisters. In particular, for \( d = 0 \), we have \( q^{-1}_f(q_f(O_X)) = \text{Tw}_f X \).

### 2.4. Néron maps

Let \( f : \mathcal{X} \rightarrow B \) be a regular pencil. Let \( T \) be a \( B \)-scheme, and \( \mathcal{L} \) a line bundle on \( \mathcal{X}_T \) of relative degree \( d \) over \( T \). Let \( \mu_\mathcal{L} : T \rightarrow \text{Pic}^d_f \) be the moduli map of \( \mathcal{L} \), defined in 2.2. Consider the composition:
\[
\overline{\mu}_\mathcal{L} : T \xrightarrow{\mu_\mathcal{L}} \text{Pic}^d_f \xrightarrow{q_f} N^d_f.
\]

We call \( \overline{\mu}_\mathcal{L} \) the Néron map of \( \mathcal{L} \). Notice that \( \mathcal{L} \) is certainly not determined by its Néron map, not even modulo pullbacks of line bundles on \( T \). In fact, if \( D \subset \mathcal{X} \) is a Cartier divisor entirely supported on a closed fiber of \( f \), then \( \mathcal{L} \otimes \mathcal{O}_{\mathcal{X}_T}(D_T) \) has the same Néron map as \( \mathcal{L} \), because \( N^d_f \rightarrow B \) is separated.

### 2.5. Abel–Néron maps

Let us recall the precise definition of the Abel map of a smooth curve, using the same set up of [GIT], Section 6 p.118, 119. Let \( h : \mathcal{C} \rightarrow S \) be a smooth curve over a scheme \( S \), so that \( h \) is a smooth
morphism whose fibers are curves. For each integer \( d \geq 1 \), let \( C^d_S \) be the \( d \)-th fibered power of \( C \) over \( S \). There is a canonical \( S \)-morphism

\[
C^d_S \equiv (p_1, \ldots, p_d) \mapsto \mathcal{O}_{C^d}(p_1 + \cdots + p_d),
\]

defined over each \( s \in S \) by taking a \( d \)-tuple of points of the fiber \( C_s \) to the line bundle associated to their sum, which we shall call the \( d \)-th Abel map of \( h \). Recall that the above map is the moduli map of a natural line bundle on \( C^d_S \times_S C \), namely the one associated to the Cartier divisor \( \sum_i S_i \), where each \( S_i \) is the image of the \( i \)-th natural section \( \sigma_i \) of the first projection \( C^d_S \times_S C \to C^d_S \), given by

\[
\sigma_i(p_1, \ldots, p_d) = ((p_1, \ldots, p_d), p_i).
\]

We may apply this construction to a regular pencil \( f : \mathcal{X} \to B \). First of all, since \( \mathcal{X}_K \) is smooth over \( K \), we may consider the \( d \)-th Abel map of \( \mathcal{X}_K \):

\[
\alpha^d_K : \mathcal{X}^d_K \to \text{Pic}^d \mathcal{X}_K.
\]

The above map extends to a map \( \mathcal{X}^d_B \to \text{Pic}^d_f \). Indeed, the extension is the moduli map of the line bundle associated to the Cartier divisor \( E^d \) of \( \mathcal{X}^d_B \times_B \mathcal{X} \), where \( E^d \) is the sum of the images of the \( d \) natural sections \( \sigma_1, \ldots, \sigma_d \) given by \( \mathcal{X}^d_B \times_B \mathcal{X} \to \mathcal{X}^d_B \). Composing with the map \( q_f \) of \( \mathcal{X}^d \), we obtain the Néron map of \( \mathcal{O}(\mathcal{X}^d_B \times_B \mathcal{X}(E^d)) \) (cf. [2.4]), which is also an extension of \( \alpha^d_K \).

The first simple but crucial observation is the following (well-known) fact:

**Lemma - Definition 2.6.** Let \( f : \mathcal{X} \to B \) be a regular pencil. For each integer \( d \geq 1 \) there exists a unique morphism, which we call the \( d \)-th Abel–Néron map of \( f \),

\[
N(\alpha^d_K) : \mathcal{X}^d_B \to \text{N}(\text{Pic}^d \mathcal{X}_K) = N^d_f
\]

whose restriction to the generic fiber is \( \alpha^d_K \). The map \( N(\alpha^d_K) \) is the Néron map of \( \mathcal{O}(\mathcal{X}^d_B \times_B \mathcal{X}(E^d + D)) \) for every Cartier divisor \( D \) of \( \mathcal{X}^d_B \times_B \mathcal{X} \) supported on any finite number of closed fibers of \( \mathcal{X}^d_B \times_B \mathcal{X} \to B \).

**Proof.** The existence of an extension of \( \alpha^d_K \) is a straightforward consequence of the Néron mapping property, [BLR], Def. 1, p. 12: since \( \mathcal{X}^d_B \) is smooth over \( B \) and has generic fiber \( \mathcal{X}^d_K \), the Abel map \( \alpha^d_K \) admits a unique extension \( N(\alpha^d_K) : \mathcal{X}^d_B \to \text{N}(\text{Pic}^d \mathcal{X}_K) \).

Since also the Néron map of \( \mathcal{O}(\mathcal{X}^d_B \times_B \mathcal{X}(E^d + D)) \), for any \( D \) as described in the lemma, extends \( \alpha^d_K \), the last statement follows from the fact that \( N^d_f \) is separated over \( B \). \( \blacksquare \)

### 3. Abel maps to balanced Picard schemes

We want to give a modular interpretation of the Abel–Néron maps and, at the same time, study the problem of completing them. To do this we shall use some results of [CT5], where Néron models are glued together...
over the moduli space of stable curves and are thereby endowed of a ge-
ometrically meaningful completion. Our moduli problem is centered around
Definition 3.2 below. First, we recall a few concepts.

3.1. Let $X$ be a nodal curve of arithmetic genus $g \geq 2$. Denote by $\omega_X$ its
canonical, or dualizing bundle. For each proper subcurve $Z \subsetneq X$, always
assumed to be complete, let $Z' := X \setminus Z$ and $k_Z := \#Z \cap Z'$. Also, let
$w_Z := \deg_Z \omega_X$. If $Z$ is connected, denote by $g_Z$ its arithmetic genus, and
recall that
\[(10) \quad w_Z = 2g_Z - 2 + k_Z,\]
a well-known identity that can be proved using adjunction.

We call $X$ semistable (resp. stable) if $k_Z \geq 2$ (resp. $k_Z \geq 3$) for each
smooth rational component $Z$ of $X$. Those $Z$ for which $k_Z = 2$ are called
exceptional. A semistable curve is called quasistable if two exceptional com-
ponents never meet each other. If $X$ is semistable, it follows from (10) that
$w_Z \geq 0$ for each subcurve $Z \subsetneq X$, with equality if and only if $Z$ is a union
of exceptional components.

A family of semistable (resp. stable, resp. quasistable) curves is a flat,
projective map $f : \mathcal{X} \to B$ whose geometric fibers are semistable (resp.
stable, resp. quasistable) curves. A line bundle of degree $d$ on such a family
$f : \mathcal{X} \to B$ is a line bundle on $\mathcal{X}$ whose restriction to each fiber has degree $d$.

Definition 3.2. Let $X$ be a semistable curve of arithmetic genus $g \geq 2,$
and let $L \in \text{Pic}^d X$.

(i) We say that $L$ and its multidegree $\deg L$ are semibalanced if for each
connected proper subcurve $Z \subsetneq X$ the Basic Inequality below holds:
\[(11) \quad m_Z(d) \leq \deg_Z L \leq M_Z(d),\]
where
\[M_Z(d) := \frac{dw_Z}{2g-2} + \frac{k_Z}{2},\]
and
\[m_Z(d) := \begin{cases} M_Z(d) - k_Z & \text{if } Z \text{ is not an exceptional component}, \\ 0 & \text{if } Z \text{ is an exceptional component}. \end{cases}\]

(ii) We call $L$ and $\deg L$ balanced if they are semibalanced and if for each
exceptional component $E \subset X$ we have
\[\deg_E L = 1.\]

(iii) We call $L$ and $\deg L$ stably balanced if they are balanced and if for
each connected proper subcurve $Z \subsetneq X$ such that $\deg_Z L = m_Z(d)$ the
complement $Z'$ is a union of exceptional components.

(iv) We denote by $\tilde{B}_X^d$ the set of balanced multidegrees on $X$, and by $\tilde{B}_X^d$
the subset of stably balanced ones.\(^1\)

(v) A line bundle (of degree $d$) on a family of semistable curves is called
semibalanced (balanced or stably balanced) if its restriction to each
geometric fiber of the family is.

\(^1\)In \cite{CO05} $\tilde{B}_X^d$ denotes the semibalanced multidegrees.
Remark 3.3. Definition 3.2 is 4.6 of [C05], which originates from [C94], Section 3.1 and from [CCC04], 5.1.1. There the Basic Inequality is
\[ |\deg_Z L - \frac{dw_Z}{2g-2}| \leq \frac{k_Z}{2}, \]
and the extra condition for when \( Z \) is an exceptional component is imposed separately. For convenience, we presented a set of inequalities, (11), that includes the exceptional cases. Abusing the terminology, we still call (11) the Basic Inequality.

We mention some simple but useful consequences of the definition.

(A) If \( X \) is stable, then (i) and (ii) coincide, i.e. a semibalanced line bundle is always balanced.

(B) There can only exist a balanced line bundle on a semistable curve \( X \) if \( X \) is quasistable. Indeed, let \( Z \subset X \) be a connected chain of exceptional components. If \( L \) is a semibalanced line bundle on \( X \), then \( L \) has degree 0 on every component of \( Z \) but possibly one, where \( L \) may have degree 1. However, if \( L \) is balanced, \( L \) cannot have degree 0 on any component of \( Z \). Then \( Z \sim P^1 \) and \( \deg_Z L = 1 \).

(C) To check whether \( L \) is semibalanced it is enough to check whether \( \deg_Z L \geq m_Z(d) \) for each connected proper subcurve \( Z \subset X \). Indeed, let \( Z \) be such a subcurve, and let \( Y_1, \ldots, Y_n \) denote the connected components of \( Z' \). By hypothesis, \( \deg_{Y_i} L \geq m_{Y_i}(d) \geq \frac{dw_{Y_i}}{2g-2} - \frac{k_{Y_i}}{2} \) for each \( i \). Since \( k_{Y_1} + \cdots + k_{Y_n} = k_{Z'} = k_Z \), summing up the above inequalities we get
\[ \deg_{Z'} L \geq \frac{dw_{Z'}}{2g-2} - \frac{k_Z}{2}. \]

Now, since \( \deg_Z L + \deg_{Z'} L = d \) and \( w_Z + w_{Z'} = 2g - 2 \), we get
\[ \deg_Z L = d - \deg_{Z'} L \leq d - \frac{dw_{Z'}}{2g-2} - \frac{k_Z}{2} = \frac{dw_Z}{2g-2} + \frac{k_Z}{2}. \]

3.4. In [C05], Lemma 4.4, it is proved that each multidegree class has a semibalanced representative. More precisely, fix an integer \( d \), and let \( X \) be a stable curve. Recall the notation in (4) and 3.2 (iv). Then Lemma 4.4 of [C05] implies that the natural map below is surjective (square brackets denoting classes):
\[
(12) \quad [\ldots] : B_X^d \rightarrow \Delta_X^d \ni [d] \rightarrow \Delta_X^d
\]

We shall say that \( X \) is “\( d \)-general” if the map (12) is bijective; see Definition 3.6 below

3.5. The moduli problem for balanced line bundles was introduced and studied in [C94] to compactify the universal Picard scheme over \( M_g \). That compactification was constructed as a GIT-quotient. We do not need to recall the details of the construction here, only a few facts. There are morphisms
\[ H_d \xrightarrow{\pi_d} H_d/G \xrightarrow{\phi_d} M_g \]
where \( H_d \) is an open subscheme of a suitable Hilbert scheme, acted upon by an algebraic group \( G \), the map \( \pi_d \) is a GIT-quotient map, and \( \phi_d \) is a surjective, projective morphism. The quotient scheme \( H_d/G \), denoted by \( \overline{T}_{d,g} \), is integral and projective. The fiber of \( \phi_d \) over a general smooth curve \( X \) is exactly \( \text{Pic}^d X \); see [C94], Thm. 6.1.

Denote by \( U \subseteq \overline{T}_{d,g} \) the nonempty open subscheme over which \( \pi_d \) restricts to a geometric GIT-quotient, i.e. all fibers are orbits, and all stabilizers are finite and reduced.

**Definition 3.6.** Let \( X \) be a stable curve of arithmetic genus \( g \geq 2 \). We say that \( X \) is \( d \)-general if any of the following equivalent conditions hold:

(i) \( \phi_d^{-1}(X) \subset U \).
(ii) The class map (12) of 3.3 is bijective.
(iii) Every balanced line bundle on \( X \) is stably balanced, i.e. \( \tilde{B}_X^d = B_X^d \).

The equivalence of these three conditions follows from [C94] Lemma 6.1.

It is known that every stable curve of genus \( g \) is \( d \)-general if and only if the integers \( d \) and \( g \) satisfy \( (d - g + 1, 2g - 2) = 1 \); see [C94], Prop 6.2.

3.7. We need to recall when two semibalanced line bundles are defined to be equivalent. Let \( X \) be a stable curve, and \( X_1 \) and \( X_2 \) two semistable curves having \( X \) as stable model. For each \( i = 1, 2 \) let \( L_i \) be a semistable line bundle on \( X_i \). Let \( Y_i \) be the semistable curve obtained by contracting all exceptional components of \( X_i \) where \( L_i \) has degree 0. Then there is a unique line bundle \( M_i \) on \( Y_i \) whose pullback to \( X_i \) is \( L_i \). Since \( L_i \) is semistable, \( M_i \) is balanced and \( Y_i \) is quasistable. Let \( F_i \subseteq Y_i \) be the union of all the exceptional components of \( Y_i \), and let \( \tilde{Y}_i := F_i' \), the complementary subcurve of \( F_i \) in \( Y_i \). Then \( L_1 \) and \( L_2 \) are equivalent if \( Y_1 = Y_2 \) and \( M_1|_{\tilde{Y}_1} \cong M_2|_{\tilde{Y}_2} \).

Notice that \( M_i \) is equivalent to \( L_i \) for \( i = 1, 2 \).

Thus, every equivalence class includes always a balanced line bundle \( N \) on a quasistable curve \( Y \). The quasistable curve \( Y \) is unique, but \( N \) is not. What is unique is the restriction of \( N \) to \( \tilde{Y} \), the complementary subcurve of the union \( F \) of all the exceptional components of \( Y \). The quasistable curve \( Y \) and \( N|_Y \) determine the equivalence class. The restriction \( N|_F \) is also unique, since a balanced line bundle must have degree 1 on every exceptional component. So, our equivalence relation disregards the gluing data of the bundles over the points in \( \tilde{Y} \cap F \).

If \( X \to B \) is a family of semistable curves, then two semistable line bundles \( L_1 \) and \( L_2 \) on \( X \to B \) are called equivalent if and only if their restrictions to every geometric fiber of \( X \to B \) are equivalent in the sense explained above.

3.8. Let \( d \) and \( g \) be integers, with \( g \geq 2 \). Assume first that \( d - g + 1 \) and \( 2g - 2 \) are coprime, so that every stable curve of arithmetic genus \( g \) is \( d \)-general. Then the construction summarized in 3.5 can be improved, by considering stacks. More precisely, there exist two (modular) Deligne–Mumford stacks \( \overline{T}_{d,g} \) and \( \overline{P}_{d,g} \), each one equipped with a natural, strongly representable morphism to \( \overline{M}_g \). (To tie in with 3.3 notice that \( \overline{P}_{d,g} \) is the quotient stack \([H_d/G]\).) The following properties hold; see [C05], Section 5 for details:
(A) For each (d-general) stable curve \(X\), denote by \(P^d_X\) and \(\overline{P}^d_X\) the fibers of \(\mathcal{P}_{d,g}\) and \(\overline{\mathcal{P}}_{d,g}\) over \(X\). Since \(\mathcal{P}_{d,g}\) and \(\overline{\mathcal{P}}_{d,g}\) are strongly representable over \(\overline{\mathcal{M}}_g\), both \(P^d_X\) and \(\overline{P}^d_X\) are quasiprojective schemes. The first, \(P^d_X\), is the fine moduli scheme of degree-\(d\) balanced line bundles on \(X\). The second, \(\overline{P}^d_X\), is the coarse moduli scheme of equivalence classes of degree-\(d\) semibalanced line bundles on semistable curves having \(X\) as stable model; see 3.7. Actually, \(\overline{P}^d_X\) is not far from being a fine moduli scheme; see (C) below. \(P^d_X\) lies naturally inside \(\overline{P}^d_X\) as an open and dense subscheme.

(B) Let \(f : \mathcal{X} \to B\) be any family of (d-general) stable curves of genus \(g\), and consider the schemes
\[
P^d_f := B \times_{\overline{\mathcal{M}}_g} \mathcal{P}_{d,g} \quad \text{and} \quad \overline{P}^d_f := B \times_{\overline{\mathcal{M}}_g} \overline{\mathcal{P}}_{d,g}.
\]
(That these are indeed schemes follows, again, from the fact that the maps \(\mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g\) and \(\overline{\mathcal{P}}_{d,g} \to \overline{\mathcal{M}}_g\) are strongly representable.) We have a natural inclusion \(P^d_f \subset \overline{P}^d_f\). An explicit description of \(P^d_f\), for when \(f\) is a local regular pencil, is (13). As for \(\overline{P}^d_f\), the following fact holds: to each triple \((T, Y \to T, L)\) where \(T\) is a \(B\)-scheme, \(Y \to T\) is a family of semistable curves with stable model \(f_T : \mathcal{X}_T \to T\), and \(L\) is a semibalanced line bundle of degree \(d\) on \(Y \to T\), there corresponds a moduli map \(\hat{\mu}_L : T \to \overline{P}^d_f\), taking each geometric point \(t\) of \(T\) to the equivalence class of the restriction of \(L\) to the (geometric) fiber of \(Y\) over \(t\). We call \(\hat{\mu}_L\) the moduli map of \(L\).

The image of \(\hat{\mu}_L\) is contained in \(P^d_f\) if and only if \(L\) has degree 0 on every exceptional component of every geometric fiber of \(Y \to T\).

(C) The scheme \(P^d_f\) is a fine moduli scheme; see [C05], Cor. 5.14 and Rmk. 5.15. Also, \(\overline{P}^d_f\) is not far from being a fine moduli scheme. In fact, it is endowed with a quasiversal pair \((Z \to \overline{P}^d_f, N)\), where \(Z \to \overline{P}^d_f\) is a family of quasiprojective curves having \(f_{\overline{P}^d_f} : \mathcal{X}_{\overline{P}^d_f} \to \overline{P}^d_f\) as stable model, and \(N\) is a balanced line bundle of degree \(d\) on \(Z \to \overline{P}^d_f\) that has a role similar to that of a Poincaré bundle. Indeed, for each triple \((T, \mathcal{Y} \to T, \mathcal{L})\) where \(T\) is a \(B\)-scheme, \(\mathcal{Y} \to T\) is a family of semistable curves with model \(f_T : \mathcal{X}_T \to T\), and \(\mathcal{L}\) is a semibalanced line bundle of degree \(d\) on \(\mathcal{Y} \to T\), there is a map \(\mathcal{Y} \to Z\) such that the diagram of maps below is commutative,

\[
\begin{array}{ccc}
\mathcal{Y} & \to & Z \\
\downarrow & & \downarrow \\
T & \xrightarrow{\hat{\mu}_L} & \overline{P}^d_f,
\end{array}
\]
and such that \(\mathcal{L}\) is equivalent to the pullback of \(N\) to \(\mathcal{Y}\); see 3.7. (The map \(\mathcal{Y} \to Z\) is certainly not uniquely determined, which is why we
call the pair \((Z \to \overline{P}_f^d, \mathcal{N})\) quasiuniversal.) Furthermore, if \(\mathcal{Y} \to T\) is a family of quasistable curves, and \(\mathcal{L}\) is balanced, then the map \(\mathcal{Y} \to Z\) can be chosen such that the above diagram is a fibered product diagram.

**Remark 3.9.** If \((d - g + 1, 2g - 2) \neq 1\), almost everything in \([5,8]\) works over the open subset of \(\mathcal{M}_g\) parametrizing \(d\)-general stable curves. For a proof, it suffices to argue exactly as for Theorem 5.9 in \([23,8]\), after replacing \(\mathcal{M}_g\) with the substack of \(d\)-general curves, and the stacks \(\mathcal{P}_{d,g}\) and \(\overline{\mathcal{P}}_{d,g}\) with the corresponding substacks (over \(d\)-general curves).

The only assertion in \([5,8]\) that does not hold is the existence of a “Poincaré” line bundle, in \([C]\), which will never be used in this paper.

We are ready to go back to the study of Abel maps.

**Proposition 3.10.** Let \(f : X \to B\) be a regular pencil of \(d\)-general stable curves. Then there exists a canonical map

\[
\alpha_f^d : \check{X}^d \to P_f^d
\]

which restricts to the \(d\)-th Abel map on the generic fiber.

**Proof.** We may glue local extensions of the \(d\)-th Abel map of \(X_K\) because they are unique. Thus we may assume \(f\) is local; let \(X\) be the closed fiber of \(f\). In this case, the explicit description of \(P_f^d\) is (see \([23,8]\), Cor. 5.14)

\[
P_f^d = \bigsqcup_{d \in B^d_X} \text{Pic}_{\check{X}^d} \sim_K^d,
\]

where, as in \([23,8]\) “\(\sim_K^d\)” means gluing over the generic fiber.

As we know from \([23,8]\) \(N_f^d\) is described in a very similar way. Indeed, by \([23,8]\), Thm. 6.1, we have a canonical isomorphism

\[
\epsilon^d_f : P_f^d \xrightarrow{\sim} N_f^d.
\]

(14)

To describe it precisely is straightforward: for each \(d \in B^d_X\) the restriction of \(\epsilon^d_f\) to \(\text{Pic}^d_f\) is the natural isomorphism

\[
\text{Pic}^d_f \xrightarrow{\sim} \text{Pic}^d_{\check{X}} \subset N_f^d,
\]

restricting to the identity on the generic fibers. The isomorphism \(\epsilon^d_f\) is completely described because, since \(X\) is \(d\)-general, the class map \(B^d_X \to \Delta^d_X\) is bijective, by Definition 3.6. To conclude, use Lemma 2.6 and (14) to define \(\alpha_f^d := (\epsilon^d_f)^{-1} \circ N(\alpha_K^d)\).

We call \(\alpha_f^d\) the \(d\)-th Abel map of \(f\). The natural problem now is to describe \(\alpha_f^d\) as the moduli map of a balanced line bundle on \(\pi : \check{X}^d \times_B X \to \check{X}^d\). Since \(P_f^d\) is a fine moduli scheme, this should be possible. In fact, the proof of \([23,8]\), Prop. 4.1, p. 621, can be used to produce an algorithm for determining the necessary twisters we need to tensor \(\mathcal{O}_{\check{X}^d \times_B X}(E^d)\) with to get the balanced line bundle.

However the explicit description of this line bundle turns out to be difficult to find in general. In Section \([11]\) we will find it for \(d = 1\). In the next subsection, \([3,11]\) we will do that for every \(d\) in a special case.
3.11. Two-component curves. Let \( X \) be a stable curve with only two irreducible components, \( C_1 \) and \( C_2 \). Let \( g \) be the arithmetic genus of \( C \) and \( \delta := \#C_1 \cap C_2 \). Set (cf. 3.3)

\[
m := \left\lfloor m_{C_1}(d) \right\rfloor = \left\lfloor \frac{dw_{C_1}}{2g - 2} - \frac{\delta}{2} \right\rfloor.
\]

The set \( B_X^d \) of balanced multidegrees of \( X \) satisfies

\[
B_X^d \supseteq \{(m, d - m), (m + 1, d - m - 1), \ldots, (m + \delta - 1, d - m - \delta + 1)\},
\]

with equality if and only if \( m_{C_1}(d) \) is not integer, if and only if \( X \) is \( d \)-general.

For each integer \( a \) such that \( 0 \leq a \leq d \) define \( r(a) \) to be the integer determined by the following two conditions:

\[
0 \leq r(a) < \delta \quad \text{and} \quad a - m \equiv r(a) \mod \delta.
\]

Using this notation we have:

**Proposition 3.12.** Let \( X \) be a stable curve with exactly two irreducible components, \( C_1 \) and \( C_2 \). Let \( \delta := \#C_1 \cap C_2 \). For each regular smoothing \( f : X \to B \) of \( X \), let

\[
\mathcal{L}^{(d)} := \mathcal{O}_{\hat{X}^d \times B X} \left( E^d + \sum_{a=0}^{d} \frac{a - m - r(a)}{\delta} C_1^a \times C_2^{d-a} \times C_1 \right)
\]

where, abusing notation, we view

\[
C_1^a \times C_2^{d-a} \times C_1 \subset X^d \times X \subset \hat{X}^d \times_b X
\]

as a Cartier divisor of \( \hat{X}^d \times_b X \), by restriction. Then \( \mathcal{L}^{(d)} \) is balanced on \( f_{\hat{X}^d : \hat{X}^d \times_b X \to \hat{X}^d} \). Furthermore, if \( X \) is \( d \)-general, then the \( d \)-th Abel map \( \alpha_f^d : \hat{X}^d \to P_f^d \) is the moduli map of \( \mathcal{L}^{(d)} \).

**Proof.** Since each \( C_1^a \times C_2^{d-a} \times C_1 \) is supported over the closed point of \( B \), the line bundle \( \mathcal{L}^{(d)} \) coincides with \( \mathcal{O}_{\hat{X}^d \times_b X} (E^d) \) over the generic point of \( B \). This implies that \( \mathcal{L}^{(d)} \) and \( \mathcal{O}_{\hat{X}^d \times_b X} (E^d) \) have the same Néron map.

If \( X \) is \( d \)-general, \( \alpha_f^d \) is defined and coincides with the Néron map of \( \mathcal{O}_{\hat{X}^d \times_b X} (E^d) \) on \( \hat{X}^d \). On the other hand, if the line bundle \( \mathcal{L}^{(d)} \) is balanced on \( \pi : \hat{X}^d \times_b X \to \hat{X}^d \), there is an associated moduli map \( \hat{\mu}_{\mathcal{L}^{(d)}} : \hat{X}^d \to P_f^d \) by 3.8 11. Since \( \hat{\mu}_{\mathcal{L}^{(d)}} \) coincides with the Néron map of \( \mathcal{L}^{(d)} \) on \( \hat{X}^d \), it follows that \( \hat{\mu}_{\mathcal{L}^{(d)}} = \alpha_f^d \). Thus, it suffices to prove that \( \mathcal{L}^{(d)} \) is balanced.

To verify this, we must compute the multidegree of the restriction of \( \mathcal{L}^{(d)} \) to every singular fiber of \( \pi : \hat{X}^d \times X \to \hat{X}^d \). Of course, all singular fibers lie over \( \hat{X}^d \). So, let \( p \in \hat{X}^d \), and denote by \( X_p \) the fiber \( \pi^{-1}(p) \). The point \( p \) determines a unique pair of nonnegative integers \((a_0, b_0)\) such that \( a_0 + b_0 = d \) and \( p \in C_1^{a_0} \times C_2^{b_0} \). Then

\[
(E^d \cdot C_1, E^d \cdot C_2) = (a_0, b_0) = (a_0, d - a_0).
\]

To compute the intersection degrees with \( C_1 \) and \( C_2 \) of the remaining summands defining \( \mathcal{L}^{(d)} \), notice first that, since \( p \in C_1^{a_0} \times C_2^{b_0} \), the only nonzero degrees come from the summand indexed by \( a = a_0 \). Now, using

\[
(C_1^{a_0} \times C_2^{b_0} \times C_1) \cdot (C_1, C_2) = (-\delta, \delta)
\]
and (10), we get
\[ \deg_{x_p} L^{(d)} = (m + r(a_0), d - m - r(a_0)), \]
which is balanced because \( 0 \leq r(a_0) < \delta; \) see (15).

**Example 3.13.** Let \( X \) be a “split” curve of arithmetic genus \( g \), that is, \( X = C_1 \cup C_2 \) with \( C_i \cong \mathbb{P}^1 \) and \( \#C_1 \cap C_2 = g + 1 \). Then, for each \( d = 1, \ldots, g \) and any regular smoothing \( f : X \to B \) of \( X \), the map \( \alpha_f^d \) is the moduli map of \( \mathcal{O}_{\tilde{X}^d \times_B X}(E^d) \).

In particular, for any \( p_1, \ldots, p_d \in \tilde{X} \) we have, independently of \( f \),
\[ \alpha_f^d(p_1, \ldots, p_d) = \mathcal{O}_X(p_1 + \cdots + p_d). \]

A split curve is \( d \)-general if and only if \( d \equiv g \mod 2 \). Actually, the conclusion of Example 3.13 is valid regardless of \( X \) being \( d \)-general, because at any rate \( \mathcal{O}_{\tilde{X}^d \times_B X}(E^d) \) is stably balanced on \( f_{\tilde{X}^d} \).

**Remark 3.14.** The case of split curves is in a sense special. In general, we should expect the restriction \( \alpha_f^d|_X \) of the \( d \)-th Abel map of Proposition 3.12 to depend on the choice of smoothing \( f \). For a simple concrete example of this dependence, consider the case \( d = 2 \) and \( \delta = 2 \). Then \( X \) is stable and \( 2 \)-general if \( C_1 \) and \( C_2 \) have distinct positive arithmetic genera. Suppose \( C_1 \) has smaller genus. Then \( m = 0 \), and thus \( r(0) = 0 \), \( r(1) = 1 \), but \( r(2) = 0 \).

Let \( p, q \in C_1 \setminus C_1 \cap C_2 \). Then
\[ \alpha_f^2(p, q) = \mathcal{O}_X(p + q) \otimes \mathcal{O}_X(C_1)|_X. \]

Now, as \( f \) varies through all smoothings of \( X \), the restriction \( \mathcal{O}_X(C_1)|_X \) varies through all line bundles restricting to \( \mathcal{O}_{C_1}(-r_1 - r_2) \) and \( \mathcal{O}_{C_2}(r_1 + r_2) \), where \( r_1 \) and \( r_2 \) are the nodes of \( X \). So \( \alpha_f^2(p, q) \) depends on the choice of \( f \).

Denote by \( \Sigma_g^d \) the locus in \( \overline{M}_g \) of curves that are not \( d \)-general. Then \( \Sigma_g^d \) is a proper closed subset of \( \overline{M}_g \); in fact, with the notation of 3.5 \( \Sigma_g^d \) is the image via \( \phi_d \) of the complement of \( U \). As we mentioned in 3.5 \( \Sigma_g^d \) is empty if and only if \( (d - g + 1, 2g - 2) = 1 \). Since the rest of our paper is devoted to Abel maps for \( d = 1 \), we conclude this section by describing the locus of curves that are not 1-general.

**Proposition 3.15.** Let \( g \geq 2 \).

(i) If \( g \) is odd then \( \Sigma^1_g \) is empty.

(ii) If \( g \) is even then \( \Sigma^1_g \) is the closure in \( \overline{M}_g \) of the locus of curves \( X \) such that \( X = C_1 \cup C_2 \), with \( C_1 \) and \( C_2 \) smooth of the same genus and \( \#C_1 \cap C_2 \) odd.

**Proof.** If \( g \) is odd then
\[ (1 - g + 1, 2g - 2) = (g - 2, g - 1) = 1. \]

So Part (i) follows; see 3.6.

For Part (ii), let \( X \) be a stable curve. Suppose first that \( X \) has the description given in 11. Then
\[ \left( \frac{1 - k}{2}, \frac{1 + k}{2} \right) \in B^1_X \prec \tilde{B}^1_X, \]
where $k := k_{C_1} = k_{C_2}$. So $X$ is in $\Sigma^1_g$. Since $\Sigma^1_g$ is closed in $\overline{M}_g$, it contains the closure of the locus defined in (11).

Suppose now that $X$ is in $\Sigma^1_g$, i.e. there is a line bundle $L$ on $X$ such that $\deg L \in B^1_X \setminus \overline{B}^1_X$. Then there is a connected, proper subcurve $Z \subseteq X$ such that either $m_Z(1)$ or $M_Z(1)$ is equal to $\deg_Z L$. Then both $m_Z(1)$ and $M_Z(1)$ are integers. Thus

\begin{equation}
\frac{w_Z}{w} + \frac{k_Z}{2} \in \mathbb{Z}
\end{equation}

where $w := \deg \omega_X = 2g - 2$. Now, since $X$ is stable,

\begin{equation}
0 < \frac{w_Z}{w} < 1.
\end{equation}

In particular, $\frac{w_Z}{w}$ is never an integer, and thus (17) implies that $k_Z$ is odd. Since $k_Z$ is odd, (17) and (18) immediately yield $\frac{w_Z}{w} = \frac{1}{2}$.

If $Z'$ is connected, then, since $\frac{w_{Z'}}{w} = \frac{1}{2}$ as well, we have $g_Z = g_{Z'}$. Since both $Z$ and $Z'$ are limits of smooth curves, $X$ lies in the closure of the locus described in (11).

Thus it remains to show that $Z'$ is also connected. Let $Z'_1, \ldots, Z'_m$ be the connected components of $Z'$. Notice that

\begin{equation}
k_{Z'_1} + \cdots + k_{Z'_m} = k_{Z'} = k_Z.
\end{equation}

Suppose by contradiction that $m > 1$. Then

\begin{equation}
0 < \frac{w_{Z'_i}}{w} < \frac{1}{2}
\end{equation}

for each $i$. Since $\deg L \in B^1_X$, we have

\begin{equation}
\frac{w_{Z'_i}}{w} + \frac{k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{w_{Z'_i}}{w} + \frac{k_{Z'_i}}{2}
\end{equation}

for each $i$. Using (20), we get

\begin{equation}
\frac{1 - k_{Z'_i}}{2} \leq \deg_{Z'_i} L \leq \frac{k_{Z'_i}}{2}
\end{equation}

for each $i$. Summing up, and using (19), we get

\begin{equation}
\frac{m - k_Z}{2} \leq \deg_{Z'} L \leq \frac{k_Z}{2}.
\end{equation}

Now, since $\deg_L = 1$, we must have

\begin{equation}
\frac{2 - k_Z}{2} \leq \deg_{Z} L \leq \frac{2 - m + k_Z}{2}.
\end{equation}

Suppose first that $\deg_{Z} L = M_Z(1)$, that is, $\deg_{Z} L = (1 + k_Z)/2$. Then (21) implies $1 + k_Z \leq 2 - m + k_Z$, and hence $m \leq 1$, a contradiction.

Finally, suppose $\deg_{Z} L = m_Z(1)$, that is, $\deg_{Z} L = (1 - k_Z)/2$. Then (21) implies $2 - k_Z \leq 1 - k_Z$, a contradiction as well.  

\hfill \square
4. Modular interpretation of the first Abel map

The following diagram represents the families we shall deal with in this section, starting from a regular pencil of stable curves \( f : \mathcal{X} \to B \):

\[
\begin{array}{c}
\hat{\mathcal{X}} \times_B \mathcal{X} \hookrightarrow \mathcal{X}_B^2 \longrightarrow \mathcal{X} \\
\downarrow \hat{\pi} \quad \downarrow \pi \quad \downarrow f \\
\hat{\mathcal{X}} \quad \hookrightarrow \quad \mathcal{X} \quad \longrightarrow \quad B
\end{array}
\]

(22)

where \( \pi \) is the projection onto the first factor.

We denote by \( \Delta \subset \mathcal{X}_B^2 \) the diagonal. Its restriction to \( \hat{\mathcal{X}} \times_B \mathcal{X} \) is a Cartier divisor. Denote by \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \) the associated line bundle. We may view \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \) as a family of degree-1 line bundles on the fibers of \( \hat{\pi} \). Recall that the first Abel map of the generic fiber of \( f \) is the moduli map of the restriction of \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \); see 2.5. We want to interpret the first Abel map \( \alpha_f^1 \), defined in Proposition 3.10, as the moduli map of a balanced line bundle on \( \hat{\pi} \), which will necessarily be a (possibly trivial) twist of \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \).

In fact, we shall see that \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \) fails to be balanced over points of a singular fiber \( X \) of \( f \) only if \( X \) has a separating node. To fix this, we will tensor \( O_{\hat{\mathcal{X}} \times_B \mathcal{X}}(\Delta) \) by twisters supported on so-called “tails”.

We need a few preliminary results which hold for any curve \( X \), possibly having singularities other than nodes. For the sake of future applications of the techniques developed in this paper, from now until 4.5 and in 4.8, 4.11 and 4.13, we shall be in this more general situation, i.e. \( X \) will be any (reduced, connected and projective) curve over an algebraically closed field.

Let \( r \) be a node of \( X \) and \( X'_r \to X \) be the normalization of \( X \) at \( r \) only. If \( X'_r \) is not connected, \( r \) is called a separating node of \( X \). For every tail \( Q \subset X \) we have that \( g = g_Q + g_{Q'} \). So, at least one of the two tails attached to a separating node has arithmetic genus at most \( g/2 \). If the curve \( X \) is stable and 1-general, it follows from Proposition 3.15 that no tail of \( X \) can have genus equal to \( g/2 \), or in other words that every tail of \( X \) is either small or large.

**Definition 4.1.** Let \( X \) be a curve of arithmetic genus \( g \). A proper subcurve \( Q \subset X \) will be called a tail of \( X \) if \( Q \) intersects the complementary subcurve \( Q' \) in a separating node \( r \) of \( X \). We say that \( Q \) is attached to \( r \) or that \( r \) generates \( Q \). A tail \( Q \) of \( X \) will be called small if \( g_Q < g/2 \) and large if \( g_Q > g/2 \). Let

\[
Q(X) := \{ Q \subset X : Q \text{ is a small tail of } X \}.
\]

If \( X \) has no separating node, for instance if \( X \) is smooth, then \( Q(X) = \emptyset \).

If \( r \) is a separating node of \( X \), then \( X'_r \) has two connected components, isomorphic to the two tails generated by \( r \); hence every tail is connected.

For every tail \( Q \subset X \) we have that \( g = g_Q + g_{Q'} \). So, at least one of the two tails attached to a separating node has arithmetic genus at most \( g/2 \). If the curve \( X \) is stable and 1-general, it follows from Proposition 3.15 that no tail of \( X \) can have genus equal to \( g/2 \), or in other words that every tail of \( X \) is either small or large.

**Remark 4.2.** Let \( r \) be a separating node of \( X \) generating the tails \( Q \) and \( Q' \). If \( Z \subset X \) is a connected subcurve not containing \( r \), then \( Z \) is entirely contained in either \( Q \) or \( Q' \).

**Lemma 4.3.** Let \( X \) be a curve and \( Q_1 \) and \( Q_2 \) two tails of \( X \). Then

\[
Q_1 \cup Q_2 = X \quad \text{or} \quad Q_1 \cap Q_2 = \emptyset \quad \text{or} \quad Q_1 \subseteq Q_2 \quad \text{or} \quad Q_2 \subseteq Q_1.
\]
Proof. For each $i = 1, 2$ let $r_i$ be the separating node of $X$ generating $Q_i$. If $r_1 = r_2$ then either $Q_1 = Q_2$, and hence $Q_1 \cup Q_2 = X$, or $Q_1 = Q_2$. So we may assume that $r_1 \neq r_2$.

Thus $r_1 \notin Q_2$ or $r_1 \notin Q_2'$. Suppose first that $r_1 \notin Q_2$. Since $Q_2$ is connected, either $Q_2 \subset Q_1$ or $Q_2 \subset Q_1'$ by Remark 4.2. If $Q_2 \subset Q_1'$ then $Q_1 \cap Q_2 \subset Q_1 \cap Q_1' = \{r_1\}$, and hence $Q_1 \cap Q_2 = \emptyset$ because $r_1 \notin Q_2$. So either $Q_2 \subset Q_1$ or $Q_1 \cap Q_2 = \emptyset$.

The case where $r_1 \notin Q_2'$ is treated similarly. In this case, either $Q_2' \subset Q_1'$, and hence $Q_1 \subset Q_2$, or $Q_2' \cap Q_2' = \emptyset$, and hence $Q_1 \cup Q_2 = X$. ■

Lemma 4.4. Let $X$ be a curve, and $Q$ a tail of $X$. Then, for any two line bundles $L_1$ on $Q$ and $L_2$ on $Q'$, there is, up to isomorphism, a unique line bundle $L$ on $X$ such that $L|_Q \cong L_1$ and $L|_{Q'} \cong L_2$.

Proof. Let $r$ be the separating node of $X$ to which $Q$ is attached. For each isomorphism $\mu : L_1|_{\{r\}} \to L_2|_{\{r\}}$, let $L$ be the kernel of the composition

$$
\phi_\mu : L_1 \oplus L_2 \to L_1|_{\{r\}} \oplus L_2|_{\{r\}} \xrightarrow{\hat{\mu}} L_2|_{\{r\}}
$$

where $\hat{\mu} := (-\mu, 1)$. Since $\phi_\mu$ is surjective, $L \neq L_1 \oplus L_2$. Also, since $\mu$ is an isomorphism, $L \neq L_1(-r) \oplus L_2$ and $L \neq L_1 \oplus L_2(-r)$. Since $r$ is a node of $X$, it follows that $L$ is a line bundle, and $L|_Q \cong L_1$ and $L|_{Q'} \cong L_2$.

Conversely, if $N$ is a line bundle on $X$ for which there are isomorphisms $\lambda_1 : N|_{Q} \to L_1$ and $\lambda_2 : N|_{Q'} \to L_2$, then $N$ is the kernel of $\phi_\mu$, where $\mu := \lambda_2|_{\{r\}} \circ \lambda_1^{-1}|_{\{r\}}$.

Finally, if $\mu' : L_1|_{\{r\}} \to L_2|_{\{r\}}$ is another isomorphism, the kernel of $\phi_\mu$ is carried isomorphically to the kernel of $\phi_{\mu'}$ by the automorphism

$$(a, 1) : L_1 \oplus L_2 \to L_1 \oplus L_2,$$

where $a$ is the unique scalar such that $\mu = a\mu'$.

4.5. Twisters on tails. Let $X$ be a curve. By Lemma 4.4 for each tail $Q$ of $X$ there is a unique, up to isomorphism, line bundle on $X$ whose restrictions to $Q$ and $Q'$ are $\mathcal{O}_Q(-r)$ and $\mathcal{O}_{Q'}(r)$, where $r$ is the separating node of $X$ generating $Q$. Denote this bundle by $\mathcal{O}_X(Q)$.

For each formal sum $\sum a_Q Q$ of tails $Q$ with coefficients $a_Q \in \mathbb{Z}$, set

$$\mathcal{O}_X(\sum a_Q Q) := \bigotimes \mathcal{O}_X(Q)^{\otimes a_Q}.$$ 

If $X$ is a nodal curve, and a closed fiber of a regular pencil $f : X \to B$, then

$$\mathcal{O}_X(\sum a_Q Q)|_X \cong \mathcal{O}_X(\sum a_Q Q).$$

So twisters supported on tails do not depend on the chosen regular pencil. To check (23) it is enough to observe that, for each tail $Q$ of $X$, since

$$\mathcal{O}_X(Q)|_Q \cong \mathcal{O}_Q(-r) \text{ and } \mathcal{O}_X(Q)|_{Q'} \cong \mathcal{O}_Q(r),$$

Lemma 4.4 implies that $\mathcal{O}_X(Q)|_X \cong \mathcal{O}_X(Q)$.

To state the main result of this section we need some notation, similar to the one used in Proposition 3.12. Let $f : X \to B$ be a regular pencil. Let $Z$ be a subcurve of $X$, where $X \subset X$ is a singular fiber of $f$. Then $Z$ is a
divisor of $\mathcal{X}$. Now, the restriction $\pi_Z$ of the first projection $\pi : \mathcal{X}^2_B \to \mathcal{X}$ over $Z$ is the trivial family

$$\mathcal{X}^2_B \ni Z \times X \xrightarrow{\pi_Z} Z.$$  

Thus, for any other subcurve $Z_1 \subseteq X$, the product $Z \times Z_1$ can be viewed as a Weil divisor of $\mathcal{X}^2_B$. Now, since the open subscheme $\mathcal{X} \times_B \mathcal{X} \subset \mathcal{X}^2_B$ is regular, the restriction of $Z \times Z_1$ to it is a Cartier divisor. Let $O_{\mathcal{X} \times_B \mathcal{X}}(Z \times Z_1)$ denote the associated line bundle. Using this notation, we have an explicit description of the map $\alpha_f^1 : \mathcal{X} \to \mathbb{P}^1_f$ defined in Proposition 3.10.

**Theorem 4.6.** Let $f : \mathcal{X} \to B$ be a regular pencil of stable curves. Then the line bundle

$$L^{(1)} := O_{\mathcal{X} \times_B \mathcal{X}}\left(\Delta + \sum_{Q \in \mathcal{Q}(X)} Q \times Q\right).$$

is balanced on $\hat{\pi} : \hat{\mathcal{X}} \times_B \mathcal{X} \to \hat{\mathcal{X}}$. Furthermore, assume that the fibers of $f$ are 1-general. Then the following two statements hold.

(i) The morphism $\alpha_f^1 : \hat{\mathcal{X}} \to P^1_f$ is the moduli map of $L^{(1)}$.

(ii) If $\mathcal{M}$ is a balanced line bundle on $\hat{\pi} : \hat{\mathcal{X}} \times_B \mathcal{X} \to \hat{\mathcal{X}}$ having $\alpha_f^1$ as moduli map, then $\mathcal{M} \cong L^{(1)}$, up to pullbacks from $\hat{\mathcal{X}}$.

**Remark 4.7.** But for Part (ii), the theorem generalizes to curves that are not 1-general. See 5.10 and Remark 5.13.

**Proof.** The divisor $\sum Q \times Q$ is entirely supported on a union of closed fibers of $\mathcal{X} \times_B \mathcal{X} \to B$. By Lemma 2.6, the Néron maps of $L^{(1)}$ and $O(\Delta)$ are equal. Now, if the fibers of $f$ are 1-general, $\alpha_f^1$ agrees with the Néron map of $O(\Delta)$ on $X_K$. On the other hand, if $L^{(1)}$ is balanced on $\hat{\pi} : \hat{\mathcal{X}} \times_B \mathcal{X} \to \hat{\mathcal{X}}$, there is an associated moduli map $\hat{\mu}_{L^{(1)}} : \hat{\mathcal{X}} \to P^1_f$ by 3.8 (B). Since $\hat{\mu}_{L^{(1)}}$ coincides with the Néron map of $L^{(1)}$ on $X_K$, it follows that $\hat{\mu}_{L^{(1)}} = \alpha_f^1$. Thus, to prove Part (i) it is enough to prove that $L^{(1)}$ is balanced on $\hat{\pi}$.

To prove Part (ii), it is also enough to show that $L^{(1)}$ is balanced, since $P^1_f$ is a fine moduli scheme; see 3.8 (C).

Let us now prove that $L^{(1)}$ is indeed balanced. We need only check this on each singular fiber of $f$, whence we may assume $f$ is local. Let $X$ be the closed fiber. It suffices to consider the singular fibers of the first projection

$$\hat{\pi} : \hat{\mathcal{X}} \times_B \mathcal{X} \to \hat{\mathcal{X}},$$

which are all isomorphic to $X$. Let $p \in X$ be a nonsingular point, and set $L_p^{(1)} := L^{(1)}|_{\hat{\pi}^{-1}(p)}$. Then

$$L_p^{(1)} \cong O_X(p) \otimes O_X(\sum_{Q \in \mathcal{Q}(X)} Q).$$

We conclude by Lemma 13.4 (ii), observing that, since $X$ is stable, a semibalanced line bundle on $X$ is necessarily balanced; see 3.8 (A).  

The next two lemmas are needed to finish the proof of Theorem 4.6.
Lemma 4.8. Let $X$ be a curve, and $Z$ a connected, proper subcurve. Let
\begin{equation}
Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q_n
\end{equation}
be a chain of tails of $X$, and let $r_i$ be the separating node of $X$ generating $Q_i$ for each $i = 1, \ldots, n$. Then
\[-1 \leq \deg_Z \mathcal{O}_X(\sum Q_i) \leq 1.\]
Furthermore, the extremes are attained if and only if there is a unique $j$ such that $r_j \in Z \cap Z'$. In this case, the lower bound is attained if $Z \subseteq Q_j$, and the upper bound is attained if $Z \subseteq Q'_j$.

Proof. If $r_\ell \not\in Z$ for any $\ell = 1, \ldots, n$, then $\deg_Z \mathcal{O}_X(\sum Q_i) = 0$. Suppose now that $Z$ contains at least one $r_\ell$. Let $i$ and $j$ be the smallest and greatest integers such that $r_i \in Z$ and $r_j \in Z$, respectively. Since $Z$ is connected, $Z$ contains as well all the irreducible components of $X$ containing $r_{i+1}, \ldots, r_{j-1}$. In particular, $r_\ell \not\in Z \cap Z'$ for any $\ell = i+1, \ldots, j-1$.

If $Z \cap Z'$ contains both $r_i$ and $r_j$ or neither of them, $\deg_Z \mathcal{O}_X(\sum Q_i) = 0$. If $Z \cap Z'$ contains $r_i$ but not $r_j$, then $\deg_Z \mathcal{O}_X(\sum Q_i) = 1$ and $Z \subseteq Q'_j$. At last, if $Z \cap Z'$ contains $r_j$ but not $r_i$, then $\deg_Z \mathcal{O}_X(\sum Q_i) = -1$ and $Z \subseteq Q_j$. □

Lemma 4.9. Let $X$ be a semistable curve, and $p$ a nonsingular point. Then the following two statements hold.

(i) The line bundle $\mathcal{O}_X(p)$ is semibalanced if and only if $p$ does not belong to any small tail of $X$.

(ii) The line bundle
\[ L_p^{(1)} := \mathcal{O}_X(p) \otimes \mathcal{O}_X(\sum_{Q \subseteq Q(X)} Q) \]

is semibalanced.

Proof. The “if part” of (i) is a consequence of (ii), as the sum of tails in (ii) is zero when $p$ does not belong to any small tail. As for the “only-if part”, recall from 3.1 that
\begin{equation}
\forall Z \subseteq X \quad w_Z = 2g_Z - 2 + k_Z
\end{equation}
for every connected proper subcurve $Z \subseteq X$. In particular,
\begin{equation}
\forall Q \subseteq X \quad w_Q < g - 1
\end{equation}
for every small tail $Q$ of $X$. So, if $p$ is contained in a small tail $Q$, then
\[ M_Q(1) < 1 = \deg_Q \mathcal{O}_X(p). \]
Hence the Basic Inequality (i) is not satisfied for $Q$. So $\mathcal{O}_X(p)$ is not semibalanced.

We need only prove (ii) now. First, since $X$ is semistable, $w_Z \geq 0$ for every subcurve $Z \subseteq X$; see 3.1. As a consequence,
\begin{equation}
\forall Z_1, Z_2 \subseteq X \quad w_{Z_1} \leq w_{Z_2}
\end{equation}
for all subcurves $Z_1$ and $Z_2$ of $X$ with $Z_1 \subseteq Z_2$. 

Let $Q_1, \ldots, Q_n$ be the small tails of $X$ containing $p$, and $r_1, \ldots, r_n$ their generating nodes. Since $w_{Q_i} + w_{Q_j} < 2g - 2$ by (27), we have $Q_i \cup Q_j \neq X$ for each $i$ and $j$. By Lemma 4.3 up to reordering, we may assume that

$$Q_1 \subset Q_2 \subset \cdots \subset Q_{n-1} \subset Q_n.$$ 

Let $N := O_X(\sum_1^n Q_i)$; so $L_p^{(1)} = O_X(p) \otimes N$. Let $Z$ be any connected, proper subcurve of $X$. Then $\deg Z N \geq -1$ by Lemma 4.8 and hence $\deg Z L_p^{(1)} \geq -1$. As pointed out in Remark 3.3 (C), we need only show that $\deg Z L_p^{(1)} \geq m_Z(1)$.

First, suppose $\deg Z N = -1$. By Lemma 4.8 there is $j$ such that $r_j \in Z$ and $Z \subseteq Q_j$. By (27) and (28),

$$(29) \quad w_Z \leq w_{Q_j} < g - 1,$$

and hence $m_Z(1) \leq 0$. Thus, if $p \in Z$,

$$\deg Z L_p^{(1)} = 0 \geq m_Z(1).$$

Second, suppose $\deg Z N \geq 0$. Then $\deg Z L_p^{(1)} \geq 0$. By (28) we have that $w_Z \leq w_X = 2g - 2$. So, if $Z$ is not a large tail of $X$, then $m_Z(1) \leq 0$, and hence

$$\deg Z L_p^{(1)} \geq 0 \geq m_Z(1).$$

On the other hand, suppose that $Z$ is a large tail. At any rate, $m_Z(1) \leq 1/2$. Thus, if $p \in Z$,

$$(30) \quad \deg Z L_p^{(1)} \geq 1 \geq 1/2 \geq m_Z(1).$$

Finally, suppose $p \notin Z$. Then $p$ lies on $Z'$, which is a small tail of $X$. Thus $Z' = Q_j$ for some $j$, and hence $Z = Q_j'$. It follows that $\deg Z N = 1$, and hence (30) holds as well.

Let $X$ be a 1-general stable curve. Let $\dot{X} := X \setminus X_{\text{sing}}$. For any regular smoothing $\dot{f}$ of $X$, let

$$\alpha^1_X := \alpha^1_f |_{\dot{X}} : \dot{X} \to P^1_{\dot{X}}.$$ 

The notation is not ambiguous by the following consequence of Theorem 4.6.

**Corollary 4.10.** Let $X$ be a 1-general stable curve. Then $\alpha^1_X$ does not depend on $f$. In fact, for each nonsingular point $p \in X$ we have

$$\alpha^1_X(p) = O_X(p) \otimes O_X(\sum_{Q \in \varphi(X)} Q).$$

**Proof.** The expression of $\alpha^1_X(p)$ follows from (24) in the proof of Theorem 4.6. By 4.5 the map $\alpha^1_X$ does not depend on $f$. □
If \( X \) is free from separating nodes then \( \alpha_X^1 \) is injective. This follows immediately from Lemma 4.13 below. The same lemma will be used in the proof of Proposition 5.9, a more general and precise statement. For the lemma and the proposition, the definition below is used.

**Definition 4.11.** Let \( X \) be a curve. A rational, smooth component \( C \) of \( X \) is called a *separating line* if \( C \) intersects \( X \smallsetminus C \) in separating nodes of \( X \). More generally, a connected subcurve \( Z \subseteq X \) of arithmetic genus 0 is called a *separating tree of lines* if \( Z \) intersects \( X \smallsetminus Z \) in separating nodes of \( X \).

4.12. Let \( X \) be a curve, and \( Z \subseteq X \) a proper connected subcurve such that \( Z \) intersects \( X \smallsetminus Z \) in separating nodes of \( X \). Then the connected components of \( X \smallsetminus Z \) are tails of \( Z \). In addition, if \( r \) is a separating node of \( Z \), then \( r \) is a separating node of \( X \).

A curve of arithmetic genus 0 is a curve of compact type, i.e. a nodal curve with every node separating, whose irreducible components are smooth and rational. So, if \( Z \) is a separating tree of lines, every node of \( Z \) is a separating node of \( Z \), and hence of \( X \). It follows that every connected subcurve of \( Z \) is also a separating tree of lines. In particular, every irreducible component of \( Z \) is a separating line.

We shall later need the following lemma.

**Lemma 4.13.** Let \( X \) be a curve, and \( p \) and \( q \) distinct nonsingular points of \( X \). Let \( C \subseteq X \) be the irreducible component containing \( p \). Then there is an isomorphism \( \mathcal{O}_X(p) \cong \mathcal{O}_X(q) \) if and only if \( C \) contains \( q \) and is a separating line of \( X \).

**Proof.** Assume first that \( C \) contains \( q \) and is a separating line of \( X \). Since \( C \) is smooth and rational, \( \mathcal{O}_C(p) \cong \mathcal{O}_C(q) \). We may thus assume \( C \neq X \).

Since \( C \) meets \( C' := X \smallsetminus C \) in separating nodes, applying Lemma 4.4 a few times, we can show that a line bundle on \( X \) is uniquely determined by its restrictions to \( C \) and to \( C' \). Since \( \mathcal{O}_X(p) \) and \( \mathcal{O}_X(q) \) restrict to isomorphic line bundles on \( C \) and to the trivial line bundle on \( C' \), it follows that \( \mathcal{O}_X(p) \cong \mathcal{O}_X(q) \).

Conversely, suppose \( \mathcal{O}_X(p) \cong \mathcal{O}_X(q) \). Since \( \mathcal{O}_X(p) \) has degree 1 on \( C \), so has \( \mathcal{O}_X(q) \), and hence \( q \in C \) as well. Now, since \( \mathcal{O}_X(p) \cong \mathcal{O}_X(q) \), in particular \( \mathcal{O}_C(p) \cong \mathcal{O}_C(q) \). Since \( p \neq q \), it follows from \([AK]\), Thm. 8.8, p. 108 that \( C \cong \mathbb{P}^1 \).

If \( C = X \) we are done. Suppose thus that \( C \neq X \), and let \( C' := X \smallsetminus C \). Also, suppose by contradiction that \( C \cap C' \) is not made of separating nodes of \( X \). Then there is a connected subcurve \( Z \subseteq C' \) such that \( C \cap Z \) is a scheme of length at least 2.

Since \( X \) is connected, the restriction \( \tau : H^0(X, \mathcal{O}_X(p)) \to H^0(C, \mathcal{O}_C(p)) \) is injective. But it is not surjective. Indeed, if a nonconstant \( \sigma \in H^0(C, \mathcal{O}_C(p)) \) could be extended to \( \tilde{\sigma} \in H^0(X, \mathcal{O}_X(p)) \), then \( \tilde{\sigma} \) would have to be constant on \( Z \) and hence \( \sigma \) would be constant on \( C \cap Z \). Since \( C \cong \mathbb{P}^1 \), this is impossible, \( \sigma \) being a nonconstant section of \( \mathcal{O}_{\mathbb{P}^1}(1) \). So \( \tau \) is injective, but not surjective, and hence \( h^0(X, \mathcal{O}_X(p)) = 1 \). Since \( \mathcal{O}_X(p) \cong \mathcal{O}_X(q) \), it follows that \( p = q \), an absurd.

\[ \blacksquare \]
5. Completing the first Abel map

The main result of this section is Theorem 5.5. We shall prove it first in a simpler case in Proposition 5.2, where we have a neater statement concerning the modularity, see 5.3.

As in Section 4, certain basic results of this section hold in more generality for curves having singularities other than nodes. Apart from the notation set in 5.1 below, these results are concentrated in 5.4.

5.1. Let $X$ be a curve. For each node $r \in X$, let $X_r \rightarrow X$ denote the partial normalization of $X$ at $r$, and let $\hat{X}_r$ be the curve obtained by adding to $X_r$ a smooth rational curve $E_r$ connecting the two branches over $r$. Thus

$\hat{X}_r = X_r \cup E_r$,

and there is a natural surjection $\sigma_r : \hat{X}_r \rightarrow X$ such that $\sigma_r(E_r) = \{r\}$, and such that $\sigma_r$ is an isomorphism away from $E_r$.

Assume now that $X$ is a 1-general stable curve, and let $r$ be a nonseparating node of $X$. Let $r \in E_r \subset \hat{X}_r$ such that $r$ is a nonsingular point of $\hat{X}_r$. Then the line bundle $O_{\hat{X}_r}(r) \in \text{Pic}^1 \hat{X}_r$ is balanced by Lemma 4.9, and hence determines a point of $\mathbb{P}^1_X \setminus \mathbb{P}^1_X$; see 3.8 (B). This point does not depend on the choice of $r$; see 3.7. Thus we shall denote it by $\ell_r$.

Proposition 5.2. Let $f : X \rightarrow B$ be a regular pencil of 1-general stable curves free from separating nodes. Then $\alpha_1^f : \hat{X} \rightarrow \mathbb{P}^1_f$ extends to an injection

$\overline{\alpha}_f : X \rightarrow \mathbb{P}^1_f$

such that $\overline{\alpha}_f(r) = \ell_r \in \mathbb{P}^1_X$ for each node $r$ of each closed fiber $X$ of $f$.

Remark 5.3. More precisely, the proof will show that $\overline{\alpha}_f$ is the moduli map of the line bundle $\mathcal{O}_Y(\tilde{\Delta})$, where $Y \rightarrow X^2_B$ is a partial resolution of singularities, and $\Delta$ is the proper transform in $Y$ of the diagonal $\Delta$; see 5.4.

Proof. Denote by $\rho : Y \rightarrow X^2_B$ the partial resolution of singularities described in 5.4 from where we take some of the properties mentioned below. The map $\rho$ is an isomorphism away from the points $(r, r)$ for $r \in X \setminus \hat{X}$. On the other hand, if $r \in X \setminus \hat{X}$, then $\rho^{-1}(r, r)$ is a copy of $\mathbb{P}^1$. In addition, composing $\rho$ with the first projection $\pi$,

$Y \overset{\rho}{\rightarrow} X^2_B \overset{\pi}{\rightarrow} X$,

we obtain a family of quasistable curves $Y \rightarrow X$ having $\pi : X^2_B \rightarrow X$ as stable model.

For each closed fiber $X$ of $f$, and each $r \in X_{\text{sing}} \subset X$, let $Y_r$ be the fiber of $\pi \circ \rho$ over $r$. Then $Y_r = \hat{X}_r$, where $\hat{X}_r$ is as defined in 5.1. On the other hand, each fiber of $\pi \circ \rho$ over $\hat{X} \setminus \hat{X}$ is the same as the corresponding fiber of $\pi$.

Let $\tilde{\Delta} \subset Y$ be the proper transform of $\Delta$. By Property 5.4 [13], the map $\rho$ restricts to an isomorphism between $\tilde{\Delta}$ and $\Delta$. Also, $\tilde{\Delta}$ meets each fiber $Y_r$ of $\pi \circ \rho$ over $X \setminus \hat{X}$ transversally at a nonsingular point $\mathcal{U}$ contained in the exceptional component $E_r$.

To prove that $\overline{\alpha}_f$ extends, we prove two claims: first, that $\mathcal{O}_Y(\tilde{\Delta})$ is balanced on $\pi \circ \rho : Y \rightarrow X$, so it induces a morphism $\overline{\alpha}_f : X \rightarrow \mathbb{P}^1_f$; its
moduli map; and second, to show that $\overline{\alpha^1_f}$ extends $\alpha^1_f$, that the restriction of $O_Y(\hat{\Delta})$ to the each fiber of $\pi \circ \rho$ over $X$ is isomorphic to the corresponding restriction of $L^{(1)}$, whose moduli map is $\alpha^1_f$ by Theorem 4.6.

We may now assume that $f$ is local. Let $X$ be its closed fiber. For each $r \in X_{\text{sing}}$, since $\hat{\Delta}$ intersects $Y_r$ transversally at $r$, we have

$$O_Y(\hat{\Delta})|_{Y_r} \cong O_{\hat{X}_r}(\mathcal{F}),$$

which is balanced by Lemma 4.9. In addition, for each nonsingular point $p \in X$ we have

$$O_Y(\hat{\Delta})|_{Y_p} \cong O_X(p),$$

which is balanced and isomorphic to the corresponding restriction of $L^{(1)}$, also by Lemma 4.9. Therefore $O_Y(\hat{\Delta})$ induces a moduli map

$$\overline{\alpha^1_f} : \mathcal{X} \to \mathbb{P}^1$$

which extends $\alpha^1_f$. Notice that (31) also shows that $\overline{\alpha^1_f}(r) = \ell_r$ for each $r \in X_{\text{sing}}$.

To show that $\overline{\alpha^1_f}$ is injective it suffices to consider singular points of $X$, by Corollary 4.10 and by the fact that $\overline{\alpha^1_f}(r) \in \mathbb{P}^1 \setminus \mathbb{P}^1$ for every node $r \in X$. Now, if $r \in X_{\text{sing}}$, then $\overline{\alpha^1_f}(r)$ represents a balanced line bundle on $\hat{X}_r$. Hence, two different nodes $r$ and $r'$ of $X$ are mapped to two points of $\mathbb{P}^1$ corresponding to balanced line bundles on different quasistable curves, namely $\hat{X}_r$ and $\hat{X}_{r'}$. Thus $\overline{\alpha^1_f}(r) \neq \overline{\alpha^1_f}(r')$; see 3.7.

5.4. Resolution of singularities. In the proof of Proposition 5.2 we used a partial resolution of singularities of $X_B^2$ which we are now going to describe in detail, and in more generality.

Let $f : \mathcal{X} \to B$ be a regular pencil. The threefold $X_B^2$ is singular at the points $(r_1, r_2)$, where $r_1$ and $r_2$ are (not necessarily distinct) singular points of the same closed fiber of $f$.

Let $X$ be a closed fiber of $f$, and $r_1$ and $r_2$ nodes of $X$. Since $f$ is regular, locally around $r_i$ the surface $\mathcal{X}$ is formally equivalent to the surface in $\mathbb{A}^3$ given by the equation $x_i y_i = t$, where $t$ denotes a local parameter of $B$ at the closed point covered by $X$. Pulling back these local equations to $X_B^2$ under the two projection maps $X_B^2 \to \mathcal{X}$, and abusing of the same notation, we get that $X_B^2$ is formally equivalent, locally around $(r_1, r_2)$, to the threefold in $\mathbb{A}^5$ given the equations

$$\begin{cases} x_1 y_1 = t, \\ x_2 y_2 = t. \end{cases}$$

If $r_1 = r_2$, then the diagonal $\Delta \subset X_B^2$ contains $(r_1, r_2)$, and we may assume that it is given locally around $(r_1, r_2)$ by

$$\begin{cases} x_1 y_1 = t, \\ x_2 = x_1, \\ y_2 = y_1. \end{cases}$$

Locally around $(r_1, r_2)$ we may eliminate $t$, and view $X_B^2$ as the cone $C \subset \mathbb{A}^4$ over the smooth quadric in $\mathbb{P}^3$ given by $x_1 y_1 = x_2 y_2$. Also, if $r_1 = r_2$, we may view $\Delta$ as the plane $D \subset \mathbb{A}^4$ given by $x_2 = x_1$ and $y_2 = y_1$. 


Notice that $C$ is singular only at the origin. To resolve this singularity we need only blow up a plane in $C$ containing the origin. Any plane will do, but let us blow up the plane given by $x_1 = x_2 = 0$. The blowup is the nonsingular threefold $\tilde{C} \subset \mathbb{P}^1 \times \mathbb{A}^4$ given by the equations

$$\left\{ \begin{array}{l}
\xi_2 x_1 = \xi_1 x_2, \\
\xi_1 y_1 = \xi_2 y_2,
\end{array} \right.$$ 

where $\xi_1, \xi_2$ are homogeneous coordinates of $\mathbb{P}^1$. The blowup $\gamma : \tilde{C} \to C$ is isomorphic to $C$ away from the origin. In addition, the fiber $F$ over the origin is given by $x_1 = x_2 = y_1 = y_2 = 0$, and hence is isomorphic to $\mathbb{P}^1$.

The exceptional divisor $E$ of the blow up $\tilde{C}$ is given by $x_2 = 0$ where $\xi_2 \neq 0$, and $x_1 = 0$ where $\xi_1 \neq 0$. In particular, $F \subset E$. Now, since $\xi_2 x_1 = \xi_1 x_2$, summing the divisor given by $\xi_1 = 0$ to $E$ we get the principal divisor given by $x_1 = 0$. Thus $E \cdot F = -1$.

Suppose $r_1 = r_2$. Then $\gamma^{-1}(D)$ is given by $x_1(\xi_1 - \xi_2) = y_2(\xi_1 - \xi_2) = 0$ where $\xi_1 \neq 0$, and by $x_2(\xi_1 - \xi_2) = y_1(\xi_1 - \xi_2) = 0$ where $\xi_2 \neq 0$. Thus $\gamma^{-1}(D)$ is the union of the Cartier divisor given by $\xi_1 = \xi_2$ and the fiber $F$. The strict transform $\tilde{D}$ of $D$ is thus a Cartier divisor intersecting $F$ transversally at a point.

For $i = 1, 2$, let $\phi_i : \tilde{C} \to \mathbb{A}^2$ be the composition of $\gamma$ with the projection onto the plane with coordinates $x_i, y_i$. Its fiber over the origin is given by $x_1 = y_1 = \xi_1 x_2 = \xi_2 y_2 = 0$. It is the union of $F$ and the affine lines $N_1$, given by $x_1 = y_1 = \xi_1 = y_2 = 0$, and $N_2$, given by $x_1 = y_1 = \xi_2 = x_2 = 0$. The lines $N_1$ and $N_2$ do not meet, and $F$ intersects each $N_i$ transversally at a single point. Also, $\phi_2$ maps $N_1$ and $N_2$ isomorphically onto the lines $y_2 = 0$ and $x_2 = 0$, respectively.

The exceptional divisor $E$ contains $N_2$, and intersects $N_1$ transversally. Since $\xi_1 \neq 0$ on $N_2$, we have $E \cdot N_2 = 0$. If $r_1 = r_2$, the strict transform $\tilde{D}$ does not meet either $N_1$ or $N_2$, and intersects $F$ transversally.

(An analogous description holds if we reverse the roles of $\phi_1$ and $\phi_2$.)

We will now consider the global picture. Let $\mathcal{I}_\Delta$ denote the ideal sheaf of the diagonal $\Delta \subset X_B^2$, and let $\tilde{\mathcal{I}}_\Delta$ denote the dual sheaf, i.e.

$$\tilde{\mathcal{I}}_\Delta := \text{Hom}(\mathcal{I}_\Delta, \mathcal{O}_{X_B^2}).$$

Since $\mathcal{I}_\Delta$ is a sheaf of ideals, $\tilde{\mathcal{I}}_\Delta$ is a sheaf of fractional ideals of $X_B^2$. A piece of notation: for each open subscheme $U \subseteq X_B^2$ and each sheaf of fractional ideals $\mathcal{M}$ of $U$, consider its powers $\mathcal{M}^n$, and form the associated sheaf of Rees algebras:

$$\mathcal{R}(\mathcal{M}) := \mathcal{O}_U \oplus \mathcal{M} \oplus \mathcal{M}^2 \oplus \cdots \oplus \mathcal{M}^n \oplus \cdots.$$ 

Set $\mathcal{Y} := \text{Proj}_{X_B^2}(\mathcal{R}(\tilde{\mathcal{I}}_\Delta))$, and let $\rho : \mathcal{Y} \to X_B^2$ be the structure map.

We may view $\rho$ as a blowup. In fact, for any open subscheme $U \subseteq X_B^2$ over which there is an embedding $\iota : \tilde{\mathcal{I}}_\Delta|_U \to \mathcal{L}$ into an invertible sheaf $\mathcal{L}$, we may view $\rho : \rho^{-1}(U) \to U$ as the blowup of $U$ along the closed subscheme $V \subseteq U$ whose sheaf of ideals $\mathcal{I}_V|_U$ satisfies $\iota(\tilde{\mathcal{I}}_\Delta|_U) = \mathcal{I}_V|_U \mathcal{L}$. In other words, $\iota$ induces an isomorphism over $U$:

$$\text{Proj}_U(\mathcal{R}(\mathcal{I}_V|_U)) \longrightarrow \text{Proj}_U(\mathcal{R}(\tilde{\mathcal{I}}_\Delta)|_U).$$
In the same vein, for each invertible sheaf of ideals \( J \subseteq O_U \) we have that 
\[ \text{Hom}(I_\Delta|_U, J) = \mathcal{I}_\Delta|_U J, \]
and hence we obtain a canonical isomorphism over \( U \):
\[ \text{Proj}_U(\mathcal{R}(\mathcal{I}_\Delta)|_U) \longrightarrow \text{Proj}_U(\mathcal{R}(\text{Hom}(I_\Delta|_U, J))). \]

Since \( I_\Delta \) is invertible away from the points \((r, r)\) for \( r \in X \setminus \check{X} \), it follows from the above description that \( \rho \) is an isomorphism away from these same points. In addition, around the points \((r, r)\), where \( r \) is a node of a closed fiber of \( f \), the map \( \rho \) is formally equivalent to the blowup described above, because
\[ \text{Hom}_A((x_1 - x_2, y_1 - y_2), (x_1 - x_2)) = (x_1, x_2), \]
where \( A := \mathbb{C}[[x_1, x_2, y_1, y_2]]/(x_1y_1 - x_2y_2) \).

Then all of the properties above, verified locally, yield global properties of \( \rho \).

Indeed, assume that the fibers of \( f \) are nodal. (It would actually be enough to assume that the fibers are Gorenstein.) Then, recalling that \( \pi : X_B^2 \to X \) denotes the first projection, the following statements hold:

(A) The composition
\[ \pi \circ \rho : \mathcal{Y} \longrightarrow X \]
is a family of curves whose fiber \( Y_r \) over a point \( r \) of a closed fiber \( X \) of \( f \) is \( X \), if \( r \) is nonsingular, and \( \check{X}_r \), described in 5.1, if \( r \) is a node.

(B) Let \( \check{\Delta} \subset Y \) denote the proper transform of \( \Delta \). For each node \( r \) of each closed fiber \( X \) of \( f \), the transform \( \check{\Delta} \) intersects the fiber \( Y_r \) transversally at a point lying in the exceptional component \( E_r = \rho^{-1}(r, r) \).

(C) Let \( Q \) be a tail of a closed fiber \( X \) of \( f \), and \( r \) the node of \( X \) generating \( Q \). Let
\[ \check{Q}^2 := \rho^{-1}(Q \times Q). \]
Then \( \check{Q}^2 \) is a Cartier divisor of \( Y \) containing \( E_r \). Furthermore,
\[ \check{Q}^2 \cdot E_r = -1, \quad \check{Q}^2 \cdot \check{Q} = -1, \quad \text{and} \quad \check{Q}^2 \cdot \check{Q}' = 1, \]
where, using the notation in 5.1, \( \check{Q} := \sigma_r^{-1}(Q) \) and \( \check{Q}' := X_r \setminus \check{Q} \), i.e. \( \check{Q} \) is the tail of \( \check{X}_r \) mapping to \( Q \) and containing \( E_r \), and \( \check{Q}' \) is the complementary tail.

We may now generalize Proposition 5.2.

**Theorem 5.5.** Let \( f : X \to B \) be a regular pencil of 1-general stable curves. Then there exists a morphism
\[ \overline{\alpha}_f^1 : X \longrightarrow P^1_f \]
extending \( \alpha_1^f : \hat{X} \to P^1_f \). If \( r \) is a node of a closed fiber \( X \) of \( f \), then \( \overline{\alpha}_f^1(r) \in P^1_X \) if and only if \( r \) is a separating node of \( X \).

**Remark 5.6.** The result extends to curves that are not 1-general. See 5.10 and 5.13.

**Proof.** As in the proof of Proposition 5.10, we may work locally around each singular fiber. So, assume \( f \) is local, and let \( X \) denote its closed fiber.

The new difficulty with respect to Proposition 5.2 is that, if \( X \) has separating nodes, \( \alpha_1^f \) is the moduli map of a nontrivial “twist” of the diagonal...
by Theorem 4.6 and thus the same must hold for its completion. Fortunately, however, the divisors we need for the “twist” are already present in the partial resolution of singularities \( \rho : Y \to \mathcal{X}_B^2 \) described in 5.4.

Namely, let \( Q_1, \ldots, Q_m \) be all the small tails of \( X \). Let \( \hat{\Delta} \subset Y \) be the strict transform of \( \Delta \), and set \( \tilde{Q}_i^2 := \rho^{-1}(Q_i \times Q_i) \) for \( i = 1, \ldots, m \). As seen in 5.4 all the \( \tilde{Q}_i^2 \) and \( \hat{\Delta} \) are Cartier divisors. Define the line bundle

\[
\mathcal{M} := \mathcal{O}_Y\left( \hat{\Delta} + \tilde{Q}_1^2 + \cdots + \tilde{Q}_m^2 \right).
\]

We claim that \( \mathcal{M} \) is semibalanced on the composition \( \pi \circ \rho : Y \to \mathcal{X} \) of \( \rho \) with the first projection \( \pi : \mathcal{X}_B^2 \to \mathcal{X} \). Once the claim is proved, we may let \( \alpha_f^1 : \mathcal{X} \to \mathcal{P}_f^1 \) be the moduli map of \( \mathcal{M} \); see 3.8 [13].

To prove the claim, first observe that \( \rho \) is an isomorphism over \( \tilde{X} \times_B \mathcal{X} \), whence

\[
\mathcal{M}|_{\tilde{X} \times_B \mathcal{X}} \cong \mathcal{O}_{\tilde{X} \times_B \mathcal{X}}(\Delta + Q_1 \times Q_1 + \cdots + Q_m \times Q_m),
\]

which is balanced, by Theorem 4.6 and defines \( \alpha_f^1 \). Thus, once \( \mathcal{M} \) is shown to be semibalanced, we have that \( \alpha_f^1|_{\tilde{X}} = \alpha_f^1 \).

Now, let \( r \in X_{\text{sing}} \). The fiber \( Y_r := (\pi \circ \rho)^{-1}(r) \) is equal to \( \tilde{X}_r \), by Property 5.4 (A). Also, \( \hat{\Delta} \) intersects \( Y_r \) transversally at a point \( \tau \) of the exceptional component \( E_r = \rho^{-1}(r, r) \), by Property 5.4 (A).

For each \( i = 1, \ldots, m \), let \( r_i \) be the separating node of \( X \) generating \( Q_i \). Let

\[
\tilde{Q}_i := \sigma_r^{-1}(Q_i) \subset \tilde{X}_r,
\]

and let \( \tilde{Q}_i' \) be its complement in \( \tilde{X}_r \). Then \( \tilde{Q}_i \) is a small tail of \( \tilde{X}_r \) dominating \( Q_i \), and containing \( E_r \) if and only if \( r \in Q_i \). If \( r = r_i \) then also \( \tilde{Q}_i \setminus E_r \) is a small tail of \( \tilde{X}_r \). These are all the small tails of \( \tilde{X}_r \): the subcurves \( \tilde{Q}_1, \ldots, \tilde{Q}_m \), together with \( \tilde{Q}_i \setminus E_r \) in case \( r = r_i \).

For each \( i = 1, \ldots, m \), the subscheme \( Q_i \times Q_i \subset \mathcal{X}_B^2 \) is a Cartier divisor away from \( (r_i, r_i) \). Identifying \( Y_r \) with \( \tilde{X}_r \), we claim that

\[
\mathcal{O}_Y(\tilde{Q}_i^2)|_{Y_r} \cong \begin{cases} 
\mathcal{O}_{\tilde{X}_r} & \text{if } r \notin Q_i, \\
\mathcal{O}_{\mathcal{X}_r}(\tilde{Q}_i) & \text{if } r \in Q_i.
\end{cases}
\]

In fact, if \( r \notin Q_i \), then \( \tilde{Q}_i^2 \) does not meet \( Y_r \), and hence \( \mathcal{O}_Y(\tilde{Q}_i^2)|_{Y_r} \cong \mathcal{O}_{\tilde{X}_r} \) holds. Suppose now that \( r \in Q_i \). Recall that \( \tilde{Q}_i \) is a tail of \( \tilde{X}_r \). Let \( s_i \) denote its generating node. If \( r \neq r_i \) then, since \( Q_i \times Q_i \) is a Cartier divisor of \( \mathcal{X}_B^2 \) at \( (r, r_i) \), we have

\[
\mathcal{O}_Y(\tilde{Q}_i^2)|_{\tilde{Q}_i} \cong \mathcal{O}_{\tilde{Q}_i}(-s_i) \quad \text{and} \quad \mathcal{O}_Y(\tilde{Q}_i^2)|_{\tilde{Q}_i'} \cong \mathcal{O}_{\tilde{Q}_i'}(s_i).
\]

The same restrictions are achieved with \( \mathcal{O}_{\tilde{X}_r}(\tilde{Q}_i) \). Thus \( \mathcal{O}_Y(\tilde{Q}_i^2)|_{Y_r} \) follows from Lemma 4.4. Finally, if \( r = r_i \) then \( \tilde{Q}_i \setminus E_r \) still holds, by Property 5.4 (C), and hence \( \mathcal{O}_Y(\tilde{Q}_i^2)|_{Y_r} \) follows in the same way. The proof of \( \mathcal{O}_Y(\tilde{Q}_i^2)|_{Y_r} \) is complete.

Now, notice that \( Q_i \) contains \( r \) if and only if \( \tilde{Q}_i \) contains \( \tau \). In addition, if \( r = r_i \) then \( \tau \notin \tilde{Q}_i \setminus E_r \). Since \( \tilde{Q}_1, \ldots, \tilde{Q}_m \), and \( \tilde{Q}_i \setminus E_r \) if \( r = r_i \), are all
the small tails of $\tilde{X}_r$, it follows from \cite{33} that
\[ \mathcal{M}|_{Y_r} \cong \mathcal{O}_{X_r}(\mathcal{F}) \otimes \mathcal{O}_{X_r}(\sum_{Q \in \mathcal{Q}(X_r)} Q), \]
which is semibalanced by Lemma \cite{1.9}. Our claim is proved, and thus we finish the proof of the existence of $\alpha_1^j$.

To prove the second statement of the theorem, it suffices to prove that for any node $r \in X$ we have
\[ \deg_{E_r} \mathcal{M} = \begin{cases} 1 & \text{if } r \text{ is not separating}, \\ 0 & \text{otherwise}. \end{cases} \]

To prove this, notice that, if $r \neq r_i$ then $Q^2_r \cdot E_r = 0$, whereas if $r = r_i$ then $Q^2_r \cdot E_r = -1$ by Property \cite{5.3} (C). Since at any rate $\tilde{\Delta} \cdot E_r = 1$, the degree of $\mathcal{M}|_{E_r}$ is 1, unless $r = r_i$ for some $i$, in which case the degree is 0.

Since $X$ is 1-general, each separating node of $X$ generates a small tail, and hence is equal to $r_i$ for some $i$. So $\alpha_1^j(r) \in P_X^1$ if and only if $r$ is a separating node.

\[ \boxdot \]

**Example 5.7.** Let $X$ be a curve of compact type with two components, $C_1$ and $C_2$. Then $C_1$ and $C_2$ are smooth, and $C_1 \cap C_2 = \{ r \}$, where $r$ is the unique node of $X$. Assume $g_{C_1} < g_{C_2}$. Then $X_r = C_1 \cup E \cup C_2$ and $Q(X_r) = \{ C_1, C_1 \cup E \}$, where $E = \mathbb{P}^1$. The line bundle $\mathcal{M}$ in the proof of Theorem \cite{5.3} whose moduli map is $\mathcal{M}$, satisfies
\[ \mathcal{M} = \mathcal{O}_Y(\tilde{\Delta} + \tilde{C}^2_1). \]

In this case, it is easy to describe the completed Abel map. First notice that there is a canonical isomorphism $P_X^1 \cong \text{Pic}^0 C_1 \times \text{Pic}^1 C_2$, essentially by Lemma \cite{1.4}\text. Hence, a point $\ell \in P_X^1$ is represented by a pair $(L_1, L_2)$ with $L_i \in \text{Pic} C_i$. For $i = 1, 2$ let $q_i \in C_i$ lying above $r$. Then
\[ \alpha_1^j(p) = \begin{cases} (\mathcal{O}_{C_1}(p - q_1), \mathcal{O}_{C_2}(q_2)) & \text{if } p \in C_1, \\ (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(p)) & \text{if } p \in C_2. \end{cases} \]

In particular, $\alpha_1^j(r) = (\mathcal{O}_{C_1}, \mathcal{O}_{C_2}(q_2))$. Thus, composing $\alpha_1^j|_{C_1}$ with the projection $\text{Pic}^0 C_1 \times \text{Pic}^1 C_2 \to \text{Pic}^0 C_1$ we obtain the classical Abel–Jacobi map of $C_1$ with base point $q_1$, i.e.
\[ C_1 \to \text{Pic}^0 C_1, \quad p \mapsto \mathcal{O}_{C_1}(p - q_1). \]

The analogous composition for $C_2$ gives the first Abel map $C_2 \to \text{Pic}^1 C_2$.

Let $X$ be a 1-general stable curve, and $f$ a regular smoothing of $X$. The restriction $\alpha_1^j|_X$ coincides with $\alpha_1^j|_X$, whence does not depend on $f$ by Corollary \cite{4.10}. So $\alpha_1^j|_X$ does not depend on $f$ either, and we may set $\alpha_X := \alpha_1^j|_X$. 
5.8. Explicit description of the complete Abel map. Let $X$ be a 1-general stable curve and $p \in X$. We shall now explicitly describe $\overline{\alpha_X}(p)$, in formulas (35) and (36) below, following the proof of Theorem 5.5.

First some notation. Let $P_1, \ldots, P_m$ be all the small tails of $X$ containing $p$. (The unusual naming of the tails using “$P$” rather than “$Q$” is to match the notation of the proof of Proposition 5.9.) By Lemma 4.3 we can write

$$P_m \subset P_{m-1} \subset \cdots \subset P_2 \subset P_1.$$ 

Set $Z_i := P_i - P_{i+1}$ for each $i = 1, \ldots, m - 1$ and $Z_m := P_m$, so that $P_1 = \bigcup_i^m Z_i$. Also, put $Q := P'_m$, a large tail of $X$. Hence

$$X = P_1 \cup Q = \bigcup_i^m Z_i \cup Q.$$ 

Let $r_1, \ldots, r_m$ be the separating nodes of $X$ generating $P_1, \ldots, P_m$. Notice that $Z_i \cap Z'_i = \{r_i, r_{i+1}\}$ if $i = 1, \ldots, m - 1$, and $Z_m \cap Z'_m = \{r_m\}$. Therefore each of the $Z_i$ and $Q$ meets the complementary curve in separating nodes of $X$. Hence, by iterated use of Lemma 4.4 to give a line bundle on $X$ it suffices to give its restrictions to all the $Z_i$ and to $Q$.

We are now ready to describe $\overline{\alpha_X}(p)$ if $p$ is a nonsingular point or a separating node of $X$ (in which case of course $p = r_m$). Recall that by Theorem 5.5 $\overline{\alpha_X}(p)$ corresponds to a line bundle on $X$. We have

$$\overline{\alpha_X}(p) = \{O_Q(r_1), O_{Z_1}(r_2 - r_1), \ldots, O_{Z_{m-1}}(r_m - r_{m-1}), O_{Z_m}(p - r_m)\}.$$ 

Now, suppose that $p$ is a nonseparating node of $X$. Then we know that $\overline{\alpha_X}(p)$ corresponds to a line bundle on $\hat{X}_p$. Let $E \subset \hat{X}_p$ be the exceptional component of $\hat{X}_p$, and let $\hat{Z}_m$ denote the normalization of $Z_m$ at $p$ only. Keeping the above notation we have

$$\hat{X}_p = Q \cup Z_1 \cup \cdots \cup Z_{m-1} \cup \hat{Z}_m \cup E.$$ 

Now, recall from 3.7 that $\overline{\alpha_X}(p)$ is uniquely determined by a line bundle $L$, of degree 0, on the complementary curve of $E$; that is, arguing as above, by the string of the restrictions of $L$ to $Q, Z_1, \ldots, Z_{m-1}, \hat{Z}_m$. We have

$$\overline{\alpha_X}(p) = \{O_Q(r_1), O_{Z_1}(r_2 - r_1), \ldots, O_{Z_{m-1}}(r_m - r_{m-1}), O_{\hat{Z}_m}(-r_m)\}.$$ 

**Proposition 5.9.** Let $X$ be a 1-general stable curve. Let $p$ and $q$ be distinct points of $X$. Then $\overline{\alpha_X}(p) = \overline{\alpha_X}(q)$ if and only if $p$ and $q$ belong to the same separating tree of lines of $X$.

A similar result for the Abel–Jacobi map to the (degree-0) Jacobian is proved by B.Edixhoven in [B95], Prop. 9.5. His statement (necessarily) excludes the case where $p$ or $q$ is a nonseparating node, since there the target space of the map is a noncompactified Néron model.

**Proof.** Suppose first that $p$ and $q$ belong to a separating tree of lines of $X$, call it $Z$. Since $Z$ is connected, to prove that $\overline{\alpha_X}(p) = \overline{\alpha_X}(q)$ it is enough to consider the case where $p$ and $q$ are nonsingular points of $X$ in the same irreducible component $C$ of $Z$. Now, $C$ is a separating line of $X$; see 4.12. Thus $O_X(p) \cong O_X(q)$ by Lemma 4.13. Since $p$ and $q$ lie on the same component, it follows that $\alpha_X^1(p) = \alpha_X^1(q)$, and hence $\overline{\alpha_X}(p) = \overline{\alpha_X}(q)$.
Conversely, suppose \(\alpha_X(p) = \alpha_X(q)\). We claim that \(p\) and \(q\) are either nonisomorphic points or separating nodes of \(X\). Indeed, suppose by contradiction, and without loss of generality, that \(p\) is a nonseparating node of \(X\). Then \(\alpha_X(p) = \alpha_X(q)\) by Theorem 5.3. Since \(\alpha_X(p) = \alpha_X(q)\), it follows from Theorem 5.3 as well that \(q\) is also a nonseparating node of \(X\). However, \(\alpha_X(p)\) and \(\alpha_X(q)\) correspond to balanced line bundles on different quasistable curves, \(\hat{X}_p\) and \(\hat{X}_q\). So \(\alpha_X(p) \neq \alpha_X(q)\); see 5.7. The contradiction proves the claim.

Since \(p\) and \(q\) are nonisomorphic or separating nodes of \(X\), both \(\alpha_X(p)\) and \(\alpha_X(q)\) are line bundles on \(X\). Let \(P_1, \ldots, P_m\) be the small tails containing \(p\) and \(Q_1, \ldots, Q_n\) the small tails containing \(q\). (We may have \(m = 0\) or \(n = 0\).) It follows from Lemma 4.3 as in the proof of Theorem 4.6 that, up to reordering the tails,

\[
P_m \subset P_{m-1} \subset \cdots \subset P_2 \subset P_1 \quad \text{and} \quad Q_n \subset Q_{n-1} \subset \cdots \subset Q_2 \subset Q_1.
\]

Set \(P_0 := Q_0 := X\). Let \(r_1, \ldots, r_m\) be the separating nodes of \(X\) generating \(P_1, \ldots, P_m\), and \(s_1, \ldots, s_n\) those generating \(Q_1, \ldots, Q_n\). In addition, set \(P_{m+1} := Q_{n+1} := \emptyset\), and put \(r_{m+1} := p\) and \(s_{n+1} := q\).

We may assume \(m \leq n\), without loss of generality. Let \(i\) be the largest nonnegative integer such that \(i \leq m\) and \(P_i = Q_j\) for \(j = 0, 1, \ldots, i\). Then also \(r_j = s_j\) for \(j = 0, 1, \ldots, i\). We claim that \(P_{i+1} \cap Q_{i+1} = \emptyset\). Indeed, if \(i = m\) then \(P_{i+1}\) is already empty. Suppose \(i < m\). If \(P_{i+1} \subseteq Q_{i+1}\), then

\[
P_{i+1} \subseteq Q_{i+1} \subset Q_i = P_i,
\]

and hence \(P_{i+1} = Q_{i+1}\), contradicting the maximality of \(i\). In a similar way, \(Q_{i+1} \not\subseteq P_{i+1}\). Since \(P_{i+1} \cup Q_{i+1} \neq X\), because \(P_{i+1}\) and \(Q_{i+1}\) are small tails, it follows from Lemma 4.3 that \(P_{i+1} \cap Q_{i+1} = \emptyset\), proving our claim. In particular, \(r_{i+1} \neq s_{i+1}\).

As \(P_i = Q_i\), we may consider \(Y := \overline{P_i} \setminus (P_{i+1} \cup Q_{i+1})\). As \(P_{i+1}\) and \(Q_{i+1}\) do not meet, their union cannot be \(P_i\), a connected subcurve of \(X\). Thus \(Y\) is a subcurve of \(X\). It is also connected, being either equal to, or a tail of, \(\overline{P_i} \setminus P_{i+1}\), which in turn is either equal to, or a tail of, \(P_i\), a tail of \(X\).

Since \(Y \subseteq \overline{P_i} \setminus P_{i+1}\), the restriction of \(\alpha_X(p)\) to \(Y\) is \(\overline{O_Y(r_{i+1} - r_i)}\). Analogously, \(\alpha_X(q)\) restricts to \(O_Y(s_{i+1} - s_i)\). Since \(\alpha_X(p) = \alpha_X(q)\) and \(r_i = s_i\), it follows that \(O_Y(r_{i+1}) \cong O_Y(s_{i+1})\). Since \(r_{i+1} \neq s_{i+1}\), by Lemma 4.13 applied to the curve \(Y\), we see that \(r_{i+1}\) and \(s_{i+1}\) are contained in a separating line \(C\) of \(Y\) and \(Y \cap Y'\) is made of separating nodes of \(X\), so is \(C \cap C'\); see 4.13. In other words, \(C\) is a separating line of \(X\).

For \(\ell = 1, \ldots, m - i\) let \(Y_\ell := P_{i+\ell} - P_{i+\ell+1}\). Then \(\alpha_X(p)\) restricts to \(O_{Y_\ell}(r_{i+\ell} - r_{i+\ell+1})\). On the other hand, since \(Y_\ell \subseteq Q_i \setminus Q_{i+1}\) for each \(\ell\), but neither \(s_i \in Y_\ell\) nor \(s_{i+1} \in Y_\ell\), we have that \(\alpha_X(q)\) restricts to the trivial bundle \(O_{Y_\ell}\). Applying Lemma 4.13 to the curve \(Y_\ell\), we get that \(r_{i+\ell}\) and \(s_{i+\ell+1}\) are contained in a separating line \(C^p_\ell\) of \(Y_\ell\). As before, \(C^p_\ell\) is also a separating line of \(X\).

Similarly, for each \(\ell = 1, \ldots, n - i\) the points \(s_{i+\ell}\) and \(s_{i+\ell+1}\) are contained in a separating line \(C^q_\ell\) of \(X\). The union of all the separating lines, namely,

\[
\bigcup_{\ell=1}^{\infty} \bigcup_{i=0}^{\ell} C^p_{m-i}, C^p_1, C^q_1, C^q_2, \ldots, C^q_n,
\]
is a separating tree of lines containing \( p \) and \( q \).

5.10. **Curves that are not 1-general.** We conclude by discussing the case of stable curves which are not 1-general. Recall that such curves form a proper closed subset of \( \overline{M}_g \), nonempty if and only if \( g \) is even, and their combinatorial structure is described in Proposition 3.15.

What kind of complications occur for curves that are not 1-general, or not \( d \)-general?

The stack \( \overline{P}_{d,g} \) introduced in 3.8 in the case \((d - g + 1, 2g - 2) = 1\) is constructed as a quotient stack, i.e. \( \overline{P}_{d,g} := [H_d/G] \); notation as in 3.5. The same definition can be given for each \( d \), obtaining in this way a quotient stack \([H_d/G]\). However, when non-\( d \)-general curves appear, this stack presents some pathologies.

More precisely, recall from 3.5 that the scheme-theoretic quotient \( H_d/G \) is endowed with a natural surjective morphism \( \phi_d : H_d/G \to \overline{M}_g \). The open subset of \( \overline{M}_g \) over which the quotient map \( \pi_d : H_d \to H_d/G \) is a geometric quotient is exactly the locus of \( d \)-general curves. The problem is that, as soon as \( \pi_d : H_d \to H_d/G \) fails to be a geometric quotient, the following pathologies occur:

(i) \([H_d/G]\) fails to be a Deligne–Mumford stack.
(ii) The natural map of stacks \([H_d/G] \to \overline{M}_g\) fails to be representable.
(iii) Néron models are not parametrized by \([H_d/G]\).

However, when studying Abel maps, we can still obtain some results. Since the stack \( \overline{P}_{d,g} \) behaves badly, let us consider the scheme \( \overline{P}_{d,g} := H_d/G \) introduced in 3.8. As we mentioned above, there is always a surjective morphism \( \phi_d : \overline{P}_{d,g} \to \overline{M}_g \). By [C94], Thm. 6.1, p. 641, \( \overline{P}_{d,g} \) is an integral projective scheme. It is also normal, being a GIT-quotient of \( H_d \), which is nonsingular by [C94], Lemma 2.2, p. 609.

Although \( \overline{P}_{d,g} \) is not a coarse moduli space, not even away from curves with nontrivial automorphisms, \( \overline{P}_{d,g} \) does satisfy useful functorial properties.

Thus, let \( f : X \to B \) be a regular pencil of stable curves. Let \( B \to \overline{M}_g \) be the associated map, and define

\[
\overline{P}_f := B \times_{\overline{M}_g} \overline{P}_{d,g}.
\]

If \( X \) is a closed fiber of \( f \), denote by \( \overline{P}_X \) the corresponding fiber of \( \overline{P}_f \) over \( B \).

As mentioned above, \( \overline{P}_f \) may fail to contain the Néron model \( N_f \). However, a functorial property holds: the moduli property given in 3.8 (B) holds exactly as stated. More precisely, to any semibalanced line bundle \( \mathcal{L} \) on a family of semistable curves \( Y \to T \) having \( X_T \to T \) as stable model, where \( T \) is any \( B \)-scheme, we can associate a canonical moduli map \( \hat{\mu}_\mathcal{L} : T \to \overline{P}_f \); see [C94], Prop. 8.1, p. 653.

The main weakness, when nongeneral curves are present, is that different balanced line bundles on the same quasistable, or even stable, curve may be mapped to the same point in \( \overline{P}_f \). Let us give an example of this behavior with regard to Abel maps.
Example 5.11. Let $X = C_1 \cup C_2$ be a curve of compact type as in Example 5.7. However, assume now that $C_1$ and $C_2$ have the same genus. Then $X$ is not 1-general. As before, let $r$ be the unique node of $X$, and let $q_i$ be the point of $C_i$ lying over $r$ for $i = 1, 2$.

We shall now exhibit three nonequivalent balanced line bundles that correspond to the same point of $\tilde{P}_X^1$. Notice that, by Lemma 4.4, to give a line bundle on a curve of compact type is equivalent to give a line bundle on each irreducible component of the curve.

Let $p \in C_1 \setminus \{q_1\}$. Our first line bundle is $L_1 \in \text{Pic} X$ corresponding to the pair $(O_{C_1}(p), O_{C_2})$.

Let $Y := \hat{X}_r$; so $Y = C_1 \cup E \cup C_2$, where $E$ is the exceptional component. Our second line bundle is $L_2 \in \text{Pic} Y$ corresponding to the triple $(O_{C_1}(p - q_1), O_{E}(q_E), O_{C_2})$, where $q_E$ is the point of $E$ glued to $q_1$ in $Y$. Finally, our third line bundle is $L_3 \in \text{Pic} X$ corresponding to the pair $(O_{C_1}(p - q_1), O_{C_2}(q_2))$.

We leave out all proofs, referring to [94, 7.2, Example 2, p. 645 for more details.

The above example shows that, if $f : \mathcal{X} \to B$ is a regular pencil with a non-1-general fiber $X$, then $\tilde{P}_f^1$ and $\tilde{P}_X^1$ are not coarse moduli schemes for balanced line bundles.

However, we can still get a map $\alpha_f : \mathcal{X} \to \tilde{P}_f^1$ restricting to the classical Abel map of $\mathcal{X}_K$, by using our modular interpretation of the Abel map. In fact, essentially the same line bundle $\mathcal{M}$ given in (32) can be used to produce a moduli map $\alpha_f : \mathcal{X} \to \tilde{P}_f^1$. Most of the results in Sections 4 and 5 hold, provided we change one definition, as explained in Remark 5.12 below.

5.12. Small tails, again. Let $X$ be a stable curve of arithmetic genus $g$. Taking into account the case where $X$ is not 1-general, we need to adjust the definition of the set $\mathcal{Q}(X)$ in 4.1. Suppose that $X$ has a separating node that generates two tails $Q$ and $Q'$ of equal genus. It is easy to see that, if such a node exists, then it is unique. (A 1-general curve will never admit such a node by Proposition 3.15.) We must add to the set $\mathcal{Q}(X)$ of small tails of $X$ either $Q$ or $Q'$, thus making an arbitrary choice between $Q$ and $Q'$, which nonetheless turns out to be completely irrelevant.

So, $\mathcal{Q}(X)$ is defined as the set of all small tails of $X$ together with one tail of genus $g/2$, if any such tail exists.

Remark 5.13. The following results of the paper hold with essentially the same proof, as long as we use the modified definition of $\mathcal{Q}(X)$ for stable curves $X$ of 5.12:

(i) Theorem 4.6, excluding Part (ii).

(ii) Corollary 4.10.

(iii) Theorem 5.5.
What will certainly fail is the possibility to interpret the Abel map in a unique way. In other words, if $f: X \to B$ is a regular pencil, an extension $\alpha_f^1: X \to P_f^1$ of the Abel map of $X_K$ is obtained as the moduli map of a semibalanced line bundle, however the line bundle is not uniquely determined.

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