Decomposition of the vertex operator algebra $V_{\sqrt{2}A_3}$

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1 Introduction

A conformal vector with central charge $c$ in a vertex operator algebra is an element of weight two whose component operators satisfy the Virasoro algebra relation with central charge $c$. Then the vertex operator subalgebra generated by the vector is isomorphic to a highest weight module for the Virasoro algebra with central charge $c$ and highest weight 0 (cf. [M]).

Let $V_{\sqrt{2}A_l}$ be the vertex operator algebra associated with a lattice $\sqrt{2}A_l$, where $\sqrt{2}A_l$ denotes $\sqrt{2}$ times an ordinary root lattice of type $A_l$. Motivated by the problem of looking for maximal associative algebras of the Griess algebra [G], a class of conformal vectors in $V_{\sqrt{2}A_l}$ were studied and constructed in [DLMN]. It was shown in [DLMN] that the Virasoro element $\omega$ of $V_{\sqrt{2}A_l}$ is decomposed into a sum of $l+1$ mutually orthogonal conformal vectors $\omega^i; 1 \leq i \leq l+1$ with central charge $c_i = 1 - 6/(i+2)(i+3)$ for $1 \leq i \leq l$ and $c_{l+1} = 2l/(l+3)$. The vertex operator subalgebra generated by conformal vector $\omega^i$ is exactly the irreducible highest weight module $L(c_i, 0)$ for the Virasoro algebra. The vertex operator subalgebra $T = T_l$ generated by these conformal vectors is isomorphic to a tensor product $\otimes_{i=1}^{l+1} L(c_i, 0)$ of the Virasoro vertex operator algebras $L(c_i, 0)$ and $V_{\sqrt{2}A_l}$ is a direct sum of irreducible $T$-submodules.

In this paper we determine the decomposition of $V_{\sqrt{2}A_3}$ into the direct sum of irreducible $T$-modules completely. The direct summands have been determined [KMY] in the case $l = 2$. For general $l$ only the direct summands with minimal weights at most two are known [Y].

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The main idea for the decomposition in this paper is to embed $V_{\sqrt{2}A_3}$ into the vertex operator algebra $V_{(\sqrt{2}A_1)^{\otimes 3}}$ by considering the lattice $\sqrt{2}A_3$ as a sublattice of $(\sqrt{2}A_1)^{\otimes 3}$. It turns out that $V_{\sqrt{2}A_3}$ is isomorphic to the vertex operator subalgebra $V_{(\sqrt{2}A_1)^{\otimes 3}}^+$ which is the fixed points of the involution of $V_{(\sqrt{2}A_1)^{\otimes 3}}$ induced from the $-1$ isometry of $(\sqrt{2}A_1)^{\otimes 3}$. Moreover, $V_{(\sqrt{2}A_1)^{\otimes 3}}^+$ has a subalgebra isomorphic to $V_{\sqrt{2}A_2}^+ \otimes V_F^+$ where $F$ is a rank one lattice spanned by an element of square length 6 and $V_{(\sqrt{2}A_1)^{\otimes 3}}^+$ is a direct sum of 3 irreducible modules for $V_{\sqrt{2}A_2}^+ \otimes V_F^+$. Then using [DG] on the decomposition of lattice type vertex operator algebra of rank 1 into the direct sum of irreducible modules for the Virasoro algebra and results in [KMY], we can determine all the irreducible $T$-modules in $V_{\sqrt{2}A_3}$. We should also mention that the sum of certain irreducible modules for $T$ inside $V_{\sqrt{2}A_3}$ forms a rational vertex operator algebra from our picture.

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2 Some automorphisms of $V_{\mathbb{Z}A}$

Our notation for the vertex operator algebra $V_L = M(1) \otimes \mathbb{C}[L]$ associated with a positive definite even lattice $L$ is standard [FLM]. In particular, $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} L$ is an abelian Lie algebra and extend the the bilinear form to $\mathfrak{h}$ by $\mathbb{C}$-linearity, $\hat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ is the corresponding affine algebra, $M(1) = \mathbb{C}[\alpha(n) | \alpha \in \mathfrak{h}, n < 0]$, where $\alpha(n) = \alpha \otimes t^n$, is the unique irreducible $\mathfrak{h}$-module such that $\alpha(n)1 = 0$ for all $\alpha \in \mathfrak{h}$ and $n$ positive, and $K = 1$. The element in the group algebra $\mathbb{C}[L]$ of the additive group $L$ corresponding to $\beta \in L$ will be denoted by $e^\beta$. Note that the central extension $L$ of $L$ by the cyclic group of order 2 is split if the square length of any element in $L$ is a multiple of 4 (cf. [FLM]). For example, $\sqrt{2}A_1$ is a such lattice. The vacuum vector $1$ of $V_L$ is $1 \otimes e^0$ and the Virasoro element $\omega$ is $\frac{1}{2} \sum_{d=1}^{d=1} \beta_i(-1)^2$ where $\{\beta_1, ..., \beta_d\}$ is an orthonormal basis of $\mathfrak{h}$.

We need to know explicit expressions of the vertex operators $Y(u, z)$ for $u = h(-1)$ or $u = e^\beta$ for $h \in \mathfrak{h}$ and $\beta \in L$ in the next section to do certain calculations. We assume that the square length of any element in $L$ is a multiple of 4. The operator $Y(h(-1), z)$ is defined as

$$Y(h(-1), z) = \sum_{n \in \mathbb{Z}} h(n)z^{-n-1} = \sum_{n \in \mathbb{Z}} h(-1)_n z^{-n-1}$$  \hspace{1cm} (2.1)

where $h(n)$ acts on $M(1)$ if $n \neq 0$ and $h(0)$ acts on $\mathbb{C}[L]$ so that $h(0)e^\gamma = \langle h, \gamma \rangle e^\gamma$ for $\gamma \in L$. In order to define $Y(e^\beta, z)$ we need to define operators $e_\beta$ and $z^\beta$ acting on $V_L$ such that $e_\beta(u \otimes e^\gamma) = u \otimes e^{\beta+\gamma}$ and $z^\beta(u \otimes e^\gamma) = z^{(\beta, \gamma)} u \otimes e^\gamma$ for $u \in M(1)$ and $\gamma \in L$. Then

$$Y(e^\beta, z) = \sum_{n \in \mathbb{Z}} e_\beta^n z^{-n-1} = e^{\sum_{n > 0} \frac{\beta(n)}{n} z^{-n}} e^{\sum_{n > 0} \frac{\beta(n)}{n} z^{-n}} e_\beta z^\beta.$$  \hspace{1cm} (2.2)

We refer the reader to [FLM] for the definition of vertex operators $Y(u, z)$ for general $u \in V_L$. 

2
Let \( L^\circ = \{ \alpha \in h | (\alpha, L) \subset \mathbb{Z} \} \) be the dual lattice of \( L \). Then \( L^\circ / L \) is a finite group. For each \( \lambda \in L^\circ \) the corresponding untwisted Fock space \( V_{L+\lambda} = M(1) \otimes \mathbb{C}[L + \lambda] \) is an irreducible module for \( V_L \) \[FLM\]. Let \( L^\circ = \bigcup_{i \in L^\circ / L} (L + \lambda_i) \) be a coset decomposition. Then \( V_{L+\lambda_i} \) are all inequivalent irreducible \( V_L \)-modules \[FLM\].

Let \( V_{Z\alpha} \) be the vertex operator algebra associated with a rank one lattice \( Z\alpha \) such that \( (\alpha, \alpha) = 2 \). The homogeneous subspace \( g = (V_{Z\alpha})_1 \) of \( V_{Z\alpha} \) of weight one possesses a Lie algebra structure given by \( [u,v] = u_0v \) with a symmetric invariant form \( \langle \cdot, \cdot \rangle \) such that \( \langle u,v \rangle 1 = u_1v \) \( [FLM, \text{Section 8.9}] \). We have

\[
[e^\alpha, e^{-\alpha}] = \alpha(-1), \quad \langle e^\alpha, e^{-\alpha} \rangle = 1, \quad \langle e^\alpha, e^\pm \rangle = \pm 2e^\pm, \quad \langle \alpha(-1), \alpha(-1) \rangle = 2,
\]

and \( \langle u,v \rangle = 0 \) for the other pairs \( u,v \) in \( \{ \alpha(-1), e^\alpha, e^{-\alpha} \} \). In particular, \( \{ \alpha(-1), e^\alpha, e^{-\alpha} \} \) is a standard basis of \( g \cong \mathfrak{sl}_2(\mathbb{C}) \).

Now consider three automorphisms (cf. \[FLM\]) \( \theta_1, \theta_2, \sigma \) of \( (g, \langle \cdot, \cdot \rangle) \) of order two such that

\[
\theta_1 : \alpha(-1) \mapsto \alpha(-1), \quad e^\alpha \mapsto -e^\alpha, \quad e^{-\alpha} \mapsto -e^{-\alpha}, \\
\theta_2 : \alpha(-1) \mapsto -\alpha(-1), \quad e^\alpha \mapsto e^\alpha, \quad e^{-\alpha} \mapsto -e^{-\alpha}, \\
\sigma : \alpha(-1) \mapsto e^\alpha + e^{-\alpha}, \quad e^\alpha + e^{-\alpha} \mapsto \alpha(-1), \quad e^\alpha - e^{-\alpha} \mapsto -(e^\alpha - e^{-\alpha}).
\]

Clearly \( \sigma \theta_1 \sigma = \theta_2 \). These automorphisms of \( (g, \langle \cdot, \cdot \rangle) \) can be uniquely extended to automorphisms of the vertex operator algebra \( V_{Z\alpha} \). In order to see this we recall that the Verma module \( V(1,0) \) for the affine algebra \( A^{(1)}_1 = \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \oplus \mathbb{C}c \) is the quotient of \( U(A^{(1)}_1) \) modulo the left ideal generated by \( x \otimes t^n, c - 1 \) for \( x \in \mathfrak{sl}_2(\mathbb{C}) \) and \( n \geq 0 \). Note that the automorphism group \( \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \) of the Lie algebra \( \mathfrak{sl}_2(\mathbb{C}) \) acts on \( A^{(1)}_1 \) by acting on the first tensor factor of \( \mathfrak{sl}_2(\mathbb{C}) \otimes \mathbb{C}[t,t^{-1}] \) and trivially on \( \mathbb{C} \). As a result \( \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \) acts on \( U(A^{(1)}_1) \) as algebra automorphisms. Clearly this induces an action of \( \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \) on \( V(1,0) \). Note that \( V_{Z\alpha} \) is the irreducible quotient of \( V(1,0) \) modulo the maximal submodule for \( A^{(1)}_1 \). It is easy to see from this construction that \( \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \) acts on \( V_{Z\alpha} \). In fact, the subgroup of \( \text{Aut}(\mathfrak{sl}_2(\mathbb{C})) \) consisting of those preserving the invariant bilinear form can be regarded as a subgroup of the automorphisms of the vertex operator algebra \( V_{Z\alpha} \). (This observation works for any finite dimensional semisimple Lie algebra in the position of \( \mathfrak{sl}_2(\mathbb{C}). \)) Since \( \theta_1, \theta_2 \) and \( \sigma \) preserve the bilinear form on \( \mathfrak{sl}_2(\mathbb{C}) \) they act on \( V_{Z\alpha} \) as vertex operator algebra automorphisms.

We denote the corresponding automorphisms of \( V_{Z\alpha} \) by the same symbols \( \theta_1, \theta_2, \) and \( \sigma \). Then on \( V_{Z\alpha} \), we have \( \theta_1(u \otimes e^\beta) = (-1)^{(\alpha,\beta)/2}u \otimes e^\beta \) for \( u \in M(1) \) and \( \beta \in Z\alpha \) and \( \theta_2 \) is the automorphism induced from the isometry \( \beta \mapsto -\beta \) of \( Z\alpha \) \[FLM\]. We still have \( \sigma \theta_1 \sigma = \theta_2 \). We also have

\[
\sigma(\alpha(-1)^2) = \alpha(-1)^2 \quad \text{and} \quad \sigma(e^{\pm \alpha}) = \frac{1}{2}(\alpha(-1) \mp (e^\alpha - e^{-\alpha})).
\]

We should mention that \( \sigma(\alpha(-1)^2) = \alpha(-1)^2 \) is not obvious. Using the definitions of
\( Y(e^{\pm \alpha}, z) \) and \( \sigma \) we see that
\[
\sigma(\alpha(-1)^2) = \sigma(\alpha(-1))_1 \sigma(\alpha(-1)) = (e^\alpha + e^{-\alpha})_1 (e^\alpha + e^{-\alpha}) = \alpha(-1)^2.
\]

3 Decomposition of \( V_{\sqrt{2}A_3} \)

Let \( L \) be a lattice with basis \( \{\alpha_1, \alpha_2, \alpha_3\} \) such that \( \langle \alpha_i, \alpha_j \rangle = 2\delta_{ij} \). Then \( L = A_1 \oplus A_1 \oplus A_1 \) where \( A_1 \) is the root lattice of \( sl_2(\mathbb{C}) \). Set
\[
\beta_1 = (\alpha_1 + \alpha_2)/\sqrt{2}, \quad \beta_2 = (-\alpha_2 + \alpha_3)/\sqrt{2}, \quad \beta_3 = (-\alpha_1 + \alpha_2)/\sqrt{2}.
\]
Then \( \{\beta_1, \beta_2, \beta_3\} \) forms the set of simple roots of type \( A_3 \). Set \( \gamma = -\alpha_1 + \alpha_2 + \alpha_3 \). We consider two sublattices of \( L \):
\[
N = \sum_{i,j=1}^3 \mathbb{Z}(\alpha_i \pm \alpha_j), \quad D = E \oplus F,
\]
where \( E = \mathbb{Z}(\alpha_1 + \alpha_2) + \mathbb{Z}(-\alpha_2 + \alpha_3) = \mathbb{Z}\sqrt{2}\beta_1 + \mathbb{Z}\sqrt{2}\beta_2 \) and \( F = \mathbb{Z}\gamma \).

**Lemma 3.1** (1) We have that \( N = \{\beta \in L | \langle \alpha_1 + \alpha_2 + \alpha_3, \beta \rangle \equiv 0 \pmod{4}\} \), \( N \) is isometric to \( \sqrt{2}A_3 \), and \( [L : N] = 2 \).

(2) \( [L : D] = 3 \) and \( L = D \cup (D + \alpha_2) \cup (D - \alpha_2) \)

(3) \( \alpha_2 = \sqrt{2}(\beta_1 - \beta_2)/3 + \gamma/3 \) and each element of the coset \( D + \alpha_2 \) can be uniquely written as an orthogonal sum of an element in \( E + \sqrt{2}(\beta_1 - \beta_2)/3 \) and an element in \( F + \gamma/3 \).

(4) \( E \) is isometric to \( \sqrt{2}A_2 \).

**Proof** (1) The first assertion can be verified easily. Since \( N = \mathbb{Z}\sqrt{2}\beta_1 + \mathbb{Z}\sqrt{2}\beta_2 + \mathbb{Z}\sqrt{2}\beta_3 \), the second assertion holds. We have \( \alpha_i \not\in N \) and \( L = N \cup (N + \alpha_i) \) for any \( i \). In particular \( [L : N] = 2 \).

(2)–(4) are obvious. \( \Box \)

Since \( E \) and \( F \) are even lattices \( V_E \) and \( V_F \) are vertex operator subalgebras which can be regarded as vertex operator subalgebras of \( V_{\sqrt{2}A_3} \) (with different Virasoro algebras). Note that \( \langle E + \sqrt{2}(\beta_1 - \beta_2)/3, E \rangle \subset \mathbb{Z} \) and \( \langle F + \gamma/3, F \rangle \subset \mathbb{Z} \). Thus \( V_{E + \sqrt{2}(\beta_1 - \beta_2)/3} \) is an irreducible \( V_E \)-module and \( V_{F + \gamma/3} \) is an irreducible \( V_F \)-module (cf. [FLM]).

The lattice \( L \) is a direct sum of \( \mathbb{Z}\alpha_i; i = 1, 2, 3 \) and thus the vertex operator algebra \( V_L \) associated with \( L \) is a tensor product \( V_L = V_{\mathbb{Z}\alpha_1} \otimes V_{\mathbb{Z}\alpha_2} \otimes V_{\mathbb{Z}\alpha_3} \) (see [FHL] for the definition of tensor product vertex operator algebra). Define three automorphisms of \( V_L \) of order two by
\[
\psi_1 = \theta_1 \otimes \theta_1 \otimes \theta_1, \quad \psi_2 = \theta_2 \otimes \theta_2 \otimes \theta_2, \quad \tau = \sigma \otimes \sigma \otimes \sigma,
\]
where $\theta_1$, $\theta_2$, and $\sigma$ are the automorphisms of $V_{\mathbb{Q}^l}$ described in Section 2. Then

$$\psi_1(u \otimes e^\beta) = (-1)^{(\alpha_1 + \alpha_2 + \alpha_3, \beta)/2} u \otimes e^\beta$$

for $u \in M(1)$ and $\beta \in \mathcal{L}$, $\psi_2$ is the automorphism induced from the isometry $\beta \mapsto -\beta$ of $\mathcal{L}$, and $\tau \psi_1 \tau = \psi_2$.

For any $\psi_2$-invariant subspace $U$ of $V_L$ we shall write $U^\pm = \{v \in V_L \mid \psi_2(v) = \pm v\}$.

**Lemma 3.2** We have $\tau(V_N) = V_L^+$.

**Proof** From the action of $\psi_1$ on $V_L$ and Lemma 3.1 (1) it follows that $V_N = \{v \in V_L \mid \psi_1(v) = v\}$. Since $\psi_2 \tau = \tau \psi_1$, the assertion holds. □

**Lemma 3.3** (1) $V_L^+ \cong D_D^+ \oplus V_{D+\alpha_2}$ as $V_D^+$-modules.

(2) $V_D^+ = (V_E^+ \otimes V_F^+) \oplus (V_E^- \otimes V_F^-)$.

(3) $V_{D+\alpha_2} = V_E \sqrt{2(\beta_1 - \beta_2)} \otimes V_F^{\gamma/3}$.

**Proof** Lemma [3.1] (2) implies $V_L = V_D \oplus V_{D+\alpha_2} \oplus V_{D-\alpha_2}$. Since $\psi_2$ leaves $V_D$ invariant and interchanges $V_{D+\alpha_2}$ and $V_{D-\alpha_2}$, we have

$$V_L^+ = V_D^+ \oplus (V_{D+\alpha_2} \oplus V_{D-\alpha_2})^+$$

$$\cong V_D^+ \oplus V_{D+\alpha_2}$$

as $V_D^+$-modules. Now $D = E \oplus F$ and both of $E$ and $F$ are $\psi_2$-invariant. Hence (2) holds.

(3) follows from Lemma [3.1] (3). □

Similarly we have the following result which is not used in this paper to study the decomposition of $V_N$ into the sum of irreducible $T$-modules.

**Lemma 3.4** (1) $V_L^- \cong D_D^- \oplus V_{D+\alpha_2}$ as $V_D^+$-modules.

(2) $V_D^- = (V_E^+ \otimes V_F^+) \oplus (V_E^- \otimes V_F^-)$.

Let us recall the conformal vectors introduced in [DLMM]. For any positive integer $l > 0$ let $\Phi_l$ be the root system of type $A_l$ generated by simple roots $\beta_1, \ldots, \beta_l$ and $\Phi_l^+$ be the set of positive roots. We assume that the square length of a root is 2. We also assume that $\Phi_l$ is a sub-system of $\Phi_l$ with simple roots $\beta_1, \ldots, \beta_i$ for $i \leq l$. Let $\sqrt{2}A_i$ be the positive definite even lattice spanned by $\sqrt{2} \Phi_i$. Then $\sqrt{2}A_i$ is a sublattice of $\sqrt{2}A_l$ and $V_{\sqrt{2}A_i}$ is a vertex operator subalgebra of $V_{\sqrt{2}A_i}$.

For $\beta \in \Phi_l$ set

$$w_{\beta}^\pm = \beta(-1)^2 \pm 2(e^{\sqrt{2} \beta} + e^{-\sqrt{2} \beta})$$

and

$$s_i = \frac{1}{2(i+3)} \sum_{\beta \in \Phi_i^+} w_{\beta}, \quad \omega = \frac{1}{2(l+1)} \sum_{\beta \in \Phi_l^+} \beta(-1)^2.$$
Then the conformal vectors of $V_{\sqrt{2}A_l}$ defined in [DLMN] are

$$\omega^1 = s^1, \quad \omega^{i+1} = s^{i+1} - s^i, \quad i = 1, ..., l - 1, \quad \omega^{l+1} = \omega - s^l.$$  (3.1)

Note that $\omega$ is the Virasoro element.

By Lemma 3.2, $V_N$ is isomorphic to $V_L^+$ as a vertex operator algebra. We next calculate the images of the conformal vectors $\omega^1, \omega^2, \omega^3,$ and $\omega^4$ in $V_N$ under the automorphism $\tau$. Note that $\Phi^+_2 = \{\beta_1, \beta_2, \beta_1 + \beta_2\}$ and $\Phi^+_3 = \Phi^+_2 \cup \{\beta_3, \beta_2 + \beta_3, \beta_1 + \beta_2 + \beta_3\}$.

Let $L(c, h)$ be the irreducible highest weight module for the Virasoro algebra with central charge $c$ and highest weight $h$. Then $L(c, 0)$ is a vertex operator algebra if $c \neq 0$ (cf. [FZ]). The conformal vectors $\omega^1, \omega^2, \omega^3,$ and $\omega^4$ are mutually orthogonal and their central charges are $1/2, 7/10, 4/5,$ and $1$. Hence the subalgebra $T$ of $V_N$ generated by these conformal vectors is isomorphic to

$$L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{4}{5}, 0\right) \otimes L(1, 0).$$

Moreover, $V_N$ is a completely reducible $T$-module. The main purpose in this paper is to determine the irreducible $T$-modules in $V_N$ or equivalently in $V_L^+$. In order to achieve this we need to determine the images of $s^i$ under $\tau$.

**Lemma 3.5** We have

$$\tau(s^1) = \frac{1}{8} w^+_{\beta_3},$$
$$\tau(s^2) = \frac{1}{10} (w^+_{\beta_3} + w^-_{\beta_2} + w^+_{\beta_2 + \beta_3}),$$
$$\tau(s^3) = \frac{1}{12} (w^+_{\beta_3} + w^-_{\beta_2} + w^+_{\beta_2 + \beta_3} + w^-_{\beta_3} + w^-_{\beta_2 + \beta_3} + w^+_{\beta_2})$$
$$= \frac{1}{6} (\beta_2 (-1)^2 + \beta_3 (-1)^2 + (\beta_2 + \beta_3) (-1)^2),$$
$$\tau(\omega) = \omega.$$

**Proof** The proof is a straightforward computation. We compute $\tau(s^1)$ here and leave the others to the reader. Note that

$$s^1 = \frac{1}{8} \left(\beta_1 (-1)^2 - 2(e^{\sqrt{2} \beta_1} + e^{-\sqrt{2} \beta_1})\right)$$
$$= \frac{1}{16} (\alpha_1 (-1) + \alpha_2 (-1))^2 - \frac{1}{4} (e^{\alpha_1} e^{\alpha_2} + e^{-\alpha_1} e^{-\alpha_2}),$$

where $e^{\pm \alpha_1} e^{\pm \alpha_2}$ is understood as tensor product of $e^{\pm \alpha_1}$ with $e^{\pm \alpha_2}$ in $V_{\sqrt{2}\alpha_1} \otimes V_{\sqrt{2}\alpha_2}$. Recall the definitions of vertex operators $Y(h(-1), z)$ and $Y(e^{\beta}, z)$ from (2.1) and (2.2). Then
from the definition of $\tau$ we obtain

\[
\tau(s^1) = \frac{1}{16}(e^{\alpha_1 + e^{-\alpha_1} + e^{\alpha_2} + e^{-\alpha_2}} - 1)(e^{\alpha_1} + e^{-\alpha_1} + e^{\alpha_2} + e^{-\alpha_2})
\]

\[
- \frac{1}{16}(\alpha_1(-1) - (e^{\alpha_1} - e^{-\alpha_1}))(\alpha_2(-1) - (e^{\alpha_2} - e^{-\alpha_2}))
\]

\[
- \frac{1}{16}(\alpha_1(-1) + (e^{\alpha_1} - e^{-\alpha_1}))(\alpha_2(-1) + (e^{\alpha_2} - e^{-\alpha_2}))
\]

\[
= \frac{1}{16}(\alpha_1(-1)^2 + \alpha_2(-1)^2) + \frac{1}{8}(e^{\alpha_1+\alpha_2} + e^{\alpha_1-\alpha_2} + e^{-\alpha_1+\alpha_2} + e^{-\alpha_1-\alpha_2})
\]

\[
- \frac{1}{8}\alpha_1(-1)\alpha_2(-1) - \frac{1}{8}(e^{\alpha_1+\alpha_2} - e^{\alpha_1-\alpha_2} - e^{-\alpha_1+\alpha_2} + e^{-\alpha_1-\alpha_2})
\]

\[
= \frac{1}{8}(\beta_3(-1)^2 + 2(e^{\sqrt{2}\beta_3} + e^{-\sqrt{2}\beta_3}))
\]

\[
= \frac{1}{8}w^+_{\beta_3}.
\]

Clearly, $\tau(s^1)$, $\tau(s^2 - s^1)$, $\tau(s^3 - s^2)$ are not the conformal vectors associated to the lattice $\sqrt{2}A_2$ defined in [DLMM]. We shall compose $\tau$ with another automorphism of $V_L$ so that the resulting conformal vectors are those in [DLMM] and we can apply the decomposition result given in [KMY] for $V_{\sqrt{2}A_2}$.

Let $\varphi$ be the automorphism of $V_L$ defined by

\[
\varphi : u \otimes e^{\beta} \mapsto (-1)^{(-\alpha_2 + \alpha_3, \beta)/2} u \otimes e^{\beta}
\]

for $u \in M(1)$ and $\beta \in L$ and set

\[
\rho = (\theta_2 \otimes 1 \otimes 1)\varphi \tau,
\]

where $\theta_2 \otimes 1 \otimes 1$ is the automorphism which acts as $\theta_2$ on $V_{\alpha_2}$ and acts as the identity on $V_{2\alpha_2} \otimes V_{\alpha_3}$. Let $\tilde{\omega} = \rho(\omega^i)$.

**Lemma 3.6** (1) We have

\[
\rho(s^1) = \frac{1}{8}w^-_{\beta_1},
\]

\[
\rho(s^2) = \frac{1}{10} \sum_{\beta \in \Phi_+^2} w^-_{\beta},
\]

\[
\rho(s^3) = \frac{1}{6}(\beta_1(-1)^2 + \beta_2(-1)^2 + (\beta_1 + \beta_2)(-1)^2), \quad \rho(\omega) = \omega.
\]

(2) The $\tilde{\omega}^1$, $\tilde{\omega}^2$, and $\tilde{\omega}^3$ are the mutually orthogonal conformal vectors of $V_E \cong V_{\sqrt{2}A_2}$ defined in [DLMM] and

\[
\tilde{\omega}^1 = \rho(\omega) - \rho(s^3) = \frac{1}{12}\gamma(-1)^2
\]

is the Virasoro element of $V_E$ with central charge 1.
Proof (2) follows from (1) and the definition of the conformal vectors in $V_{\sqrt{\varphi A_2}}$ given in (3.3).

(1) follows from Lemma 3.3 and the definitions of all automorphisms involved. For example,

$$
\rho(s^1) = \frac{1}{8}((\theta_2 \otimes 1 \otimes 1)\varphi)w_{\beta_3}^+
= \frac{1}{16}((\theta_2 \otimes 1 \otimes 1)\varphi)((\alpha_1(-1) - \alpha_2(-1))^2 + 4(e^{\alpha_1}e^{-\alpha_2} + e^{-\alpha_1}e^{\alpha_2}))
= \frac{1}{16}(\theta_2 \otimes 1 \otimes 1)((\alpha_1(-1) - \alpha_2(-1))^2 - 4(e^{\alpha_1}e^{-\alpha_2} + e^{-\alpha_1}e^{\alpha_2}))
= \frac{1}{16}((\alpha_1(-1) + \alpha_2(-1))^2 - 4(e^{\alpha_1}e^{-\alpha_2} + e^{-\alpha_1}e^{\alpha_2}))
= \frac{1}{8}w_{\bar{\beta}_1}^+.
$$

□

Let $\tilde{T}'$ be the subalgebra generated by $\tilde{\omega}^1$, $\tilde{\omega}^2$, and $\tilde{\omega}^3$, and $\tilde{T}''$ the subalgebra generated by $\tilde{\omega}^4$. Then $\tilde{T}' \subset V_{E}^+$ and $\tilde{T}'' \subset V_{E}^+$. Moreover,

$$
\tilde{T}' \cong L(1, 0) \otimes L(7_{10}, 0) \otimes L(4_{5}, 0), \quad \tilde{T}'' \cong L(1, 0).
$$

As a $\tilde{T}'$-module $V_{E}^+$ decomposes into a direct sum of irreducible $\tilde{T}'$-submodules of the form $L(1, h_1) \otimes L(7_{10}, h_2) \otimes L(4_{5}, h_3)$. It follows from [KMY, Lemma 4.1] that $V_{E}^+$ is a direct sum of four irreducible submodules, which are isomorphic to

$$
L(1, 0) \otimes L(7_{10}, 0) \otimes L(4_{5}, 0), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 7),
L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 0), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 0),
$$

and $V_{E}^+$ is a direct sum of four irreducible submodules, which are isomorphic to

$$
L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 7), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 3),
L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 3), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 3).
$$

In [KMY, Lemma 4.2] the irreducible $\tilde{T}'$-submodules with minimal weight $2/3$ of $V_{E+\sqrt{\varphi (\beta_1-\beta_2)}/2}$, which is denoted by $V^2$ in [KMY], are determined. By using fusion rules ([DMZ, W]) we see that as a $\tilde{T}'$-module $V_{E+\sqrt{\varphi (\beta_1-\beta_2)}/2}$ is a direct sum of four irreducible submodules, which are isomorphic to

$$
L(1, 0) \otimes L(7_{10}, 0) \otimes L(4_{5}, 2), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 3),
L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 2), \quad L(1, 0) \otimes L(7_{10}, 3) \otimes L(4_{5}, 3).
$$
The decompositions of $V^+_F$ and $V^+_{F+\gamma/3}$ as $\tilde{T}'\otimes\tilde{T}''$-modules can be found in [DG]; that is,

\begin{align*}
V^+_F &\cong (\oplus_{m\geq 0} L(1, 4m^2)) \oplus (\oplus_{m\geq 1} L(1, 3m^2)), \\
V^-_F &\cong (\oplus_{m\geq 0} L(1, 2m+1^2)) \oplus (\oplus_{m\geq 1} L(1, 3m^2)), \\
V^+_{F+\gamma/3} &\cong \oplus_{m\in\mathbb{Z}} L(1, (3m+1)^2/3).
\end{align*}

From these decompositions and Lemma 3.3 we know all irreducible direct summands of $V^+_L$ as a $\tilde{T}'\otimes\tilde{T}''$-module.

Finally, note that the automorphisms $(\theta_2 \otimes 1 \otimes 1)\varphi$ and $\psi_2$ of $V_L$ commute. Thus $\rho \psi_1 = \psi_2 \rho$ and $\rho(V_N) = V^+_L$. Since $\rho(T) = \tilde{T}' \otimes \tilde{T}''$, the decomposition of $V_N$ as a $T$-module and the decomposition of $V^+_L$ as a $\tilde{T}' \otimes \tilde{T}''$-module are the same. Using (3.2), (3.3), (3.4), (3.5), and Lemma 3.3 we conclude:

**Theorem 3.7** The decomposition of $V_N$ into a direct sum of irreducible $T$-submodules is as follows:

\[
V_{\sqrt{2}}A_3 = \left( L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{4}{5}, 0\right) \bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \otimes L\left(\frac{4}{5}, \frac{7}{5}\right) \\
\bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, 1\right) \otimes L\left(\frac{7}{10}, \frac{1}{2}\right) \otimes L\left(\frac{4}{5}, \frac{7}{5}\right) \bigoplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{4}{5}, 0\right) \bigotimes \left( (\oplus_{m\geq 0} L(1, 4m^2)) \bigoplus (\oplus_{m\geq 1} L(1, 3m^2)) \right) \\
\bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \otimes L\left(\frac{4}{5}, \frac{2}{5}\right) \bigoplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{1}{10}\right) \otimes L\left(\frac{4}{5}, \frac{2}{5}\right) \\
\bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{4}{5}\right) \otimes L\left(\frac{3}{5}, 3\right) \bigoplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{4}{5}, 3\right) \bigotimes \left( (\oplus_{m\geq 0} L(1, (2m+1)^2)) \bigoplus (\oplus_{m\geq 1} L(1, 3m^2)) \right) \\
\bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, 0\right) \otimes L\left(\frac{4}{5}, \frac{2}{3}\right) \bigoplus L\left(\frac{1}{2}, 0\right) \otimes L\left(\frac{7}{10}, \frac{3}{5}\right) \otimes L\left(\frac{4}{5}, \frac{1}{15}\right) \\
\bigoplus \bigoplus \bigoplus \bigoplus L\left(\frac{1}{2}, \frac{1}{2}\right) \otimes L\left(\frac{7}{10}, \frac{1}{2}\right) \otimes L\left(\frac{4}{5}, \frac{15}{15}\right) \bigoplus L\left(\frac{1}{2}, 1\right) \otimes L\left(\frac{7}{10}, \frac{3}{2}\right) \otimes L\left(\frac{4}{5}, \frac{2}{3}\right) \bigotimes (\oplus_{m\in\mathbb{Z}} L(1, (3m+1)^2/3)).
\]

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