Abstract

This article deals with the study of the following singular quasilinear equation:

\[
(P) \begin{cases} 
-\Delta_p u - \Delta_q u = f(x)u^{-\delta}, & u > 0 \text{ in } \Omega; \\
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary $\partial\Omega$, $1 < q < p < \infty$, $\delta > 0$ and $f \in L^{\infty}_{\text{loc}}(\Omega)$ is a non-negative function which behaves like dist($x, \partial\Omega$)$^{-\beta}$, $\beta \geq 0$ near the boundary of $\Omega$. We prove the existence of a weak solution in $W^{1,p}_{\text{loc}}(\Omega)$ and its behavior near the boundary for $\beta < p$. Consequently, we obtain optimal Sobolev regularity of weak solutions. By establishing the comparison principle, we prove the uniqueness of weak solution for the case $\beta < 2 - \frac{1}{p}$. For the case $\beta \geq p$, we prove the non-existence result. Moreover, we prove H"older regularity of the gradient of weak solution to a more general class of quasilinear equations involving singular nonlinearity (see (1.5)). This result is of independent interest. In addition to this, we prove H"older regularity of minimal weak solutions of $(P)$ for the case $\beta + \delta \geq 1$.

Key words: $(p,q)$-Singular equations, existence and uniqueness result, comparison principle, non-existence results, regularity results.

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1 Introduction

The purpose of this article is to study the existence and regularity of the weak solution to the following prototype singular problem:

\[
(P)\begin{cases}
-\Delta_p u - \Delta_q u = f(x)u^{-\delta}, & u > 0 \text{ in } \Omega \\
u = 0 & \text{ on } \partial\Omega,
\end{cases}
\]

where $\Omega$ is a bounded domain in $\mathbb{R}^N$ with $C^2$ boundary $\partial\Omega$, $1 < q < p < \infty$ and $\delta > 0$. $\Delta_p$ is the $p$-Laplace operator, defined as $\Delta_p u := \text{div}(|\nabla u|^{p-2}\nabla u)$. The operator $A_{p,q} := -\Delta_p - \Delta_q$ is known as $(p,q)$-Laplacian which arises while studying the stationary solutions of general reaction-diffusion equation

\[
u_t = \text{div}[A(u)\nabla u] + r(x,u),
\]

where $A(u) = |\nabla u|^{p-2} + |\nabla u|^{q-2}$. The problem (1.1) has applications in biophysics, plasma physics and chemical reactions, with double phase features, where the function $u$ corresponds to the concentration term, the first term on the right side represents diffusion with a diffusion coefficient $A(u)$ and the second term is the reaction which relates to sources and loss processes. For more details, readers are referred to [27] and its references.

The energy functional of equations driven by the $(p,q)$-Laplacian falls in the category of the so-called functionals with nonstandard growth conditions of $(p,q)$-type, according to Marcellini’s terminology [28]. These kinds of functionals involve integrals of the form

\[I(u) = \int_{\Omega} h(x,\nabla u(x)) \, dx,\]

where the energy density, $h$, satisfies

\[|\xi|^p \leq |h(x,\xi)| \leq |\xi|^q + 1, \quad 1 \leq p \leq q.\]

The physical significance of these models lie in the field of nonlinear elasticity, specifically in homogenisation theory. A particular form of the above class of functionals is the double phase functional given by

\[u \mapsto \int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q) \, dx, \quad 0 \leq a(x) \leq L, \quad 1 < p < q.\]

This functional was first introduced by Zhikov in [37], to model the Lavrentiev phenomenon on strongly anistropic materials. The study has been continued by Mingione et al. [2, 6] and Rădulescu et al. [29, 30, 33].

For the case $p = q$, problem $(P)$ involves the homogeneous $p$-Laplacian operator and the problem takes the following form

\[-\Delta_p u = f(x)u^{-\delta}, \quad u > 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial\Omega.
\]
This type of equation has applications in the physical world such as non-newtonian flows in porous media and heterogeneous catalysts. There has been an extensive study in this direction since the pioneering work of Crandall, Rabinowitz and Tartar [7], see for instance [1] [3] [9] [10] [17] [19] [20] [23] and references therein. In [7], authors studied (1.2) for \( p = 2 \) with \( f \) as a nonnegative bounded function and \( \delta > 0 \). In this work, they proved the existence and uniqueness of solution in \( C^2(\Omega) \cap C(\overline{\Omega}) \) and behavior of the solution near the boundary is also discussed when \( \delta = 1 \). Lazer and Mckenna in [23], considered (1.2) when \( p = 2 \) and \( f \in C^\alpha(\Omega) \) is positive. Authors proved the existence of unique solution in \( C^{2+\alpha}(\Omega) \cap C(\overline{\Omega}) \) for all \( \delta > 0 \). Moreover, they proved that the solution is not in \( C^1(\overline{\Omega}) \) if \( \delta > 1 \) and it is in \( H^1_0(\Omega) \) if and only if \( \delta < 3 \). Subsequently, Boccardo and Orsina considered (1.2) when the leading differential operator takes the form \(-\text{div}(A(x)\nabla u)\), where \( A \) is a bounded elliptic operator and \( f \) is either a nonnegative function belonging to some Lebesgue space or a nonnegative bounded radon measure. Here they proved the existence and some Sobolev regularity results. Concerning the case, when \( f \) has a singularity, Diaz, Hernández and Rakotoson [9] considered the case where \( f \) behaves as some negative power of the distance function. Here, regularity of \( \nabla u \) in Lorentz spaces is proved. Furthermore, for the case of \( \delta \in (0,1) \), Haitao in [17], and Hirano, Saccon and Shioji in [20] studied (1.2) with the critical growth perturbation with respect to the Sobolev embedding. Using Perron’s method Haitao proved global existence result while Hirano et al. used the Nehari manifold method to prove existence of at least two solutions. Adimurthi and Giacomoni [1] considered problem (1.2) for the case \( n = 2 \) and \( 0 < \delta < 3 \) with a perturbation of critical growth with respect to the Trudinger-Moser embeddings. For a thorough analysis of semilinear elliptic equations with singular nonlinearities we refer to the monograph by Ghergu and Rădulescu [11] and an overview article by Hernández, Mancebo and Vega [18].

For the quasilinear case, that is \( p \neq 2 \), Giacomoni and Sreenadh [15] studied (1.2) with \( p - 1 \) superlinear growth perturbation and \( f(x) = \lambda \), a real parameter. Authors proved that there exists a weak solution in \( W^{1,p}_0(\Omega) \cap C(\overline{\Omega}) \), for small \( \lambda > 0 \), if and only if \( \delta < 2 + 1/(p - 1) \). Subsequently, Giacomoni, Schindler and Takač [13] studied (1.2) with subcritical and critical perturbation with respect to Sobolev embedding for the case \( 0 < \delta < 1 \) and \( f(x) = \lambda \). Using variational methods, authors proved existence of multiple solutions in \( C^{1,\alpha}(\overline{\Omega}) \), for some \( \alpha \in (0,1) \). Here global multiplicity of solutions is also proved with respect to the parameter \( \lambda \). Thereafter, Canino, Sciunzi and Trombetta [5], and Bougherara, Giacomoni and Hernandez [4] studied problem (1.2) under different summability conditions on \( f \). Under the assumption that \( f \in L^1(\Omega) \), Canino et al. in [5], proved the existence result and with higher integrability assumptions, they obtained the uniqueness result. While in [4], authors considered \( f \in L^{\infty}_{\text{loc}}(\Omega) \), more general case, a nonnegative function which behaves like \( \text{dist}(x, \partial \Omega)^{-\beta} \) near the boundary \( \partial \Omega \), for \( \beta \geq 0 \). Exploiting the method of sub and supersolution authors proved the existence of a solution for all \( \delta > 1 - p \) and \( \beta \in [0,p) \). In this work, behavior near the
boundary and Sobolev regularity of the solution is also discussed. Moreover, authors proved the uniqueness result when \( 1 - p < \delta < 2 - \beta + \frac{1}{p - 1} \). For the case of \( p = 1 \), Cicco, Giachetti, Oliva and Petitta [8] studied (1.2) with the nonlinear term \( f(x)h(u) \), where \( h \) has a singularity at 0. Under certain assumptions on \( f \) and \( h \), authors proved the existence, uniqueness and regularity result.

As far as the equations with nonhomogeneous operators involving singular nonlinearity are concerned, we would like to draw the attention of readers towards the recent works Kumar, Rădulescu and Sreenadh [21] and Papageorgiou, Rădulescu and Repovš [31]. In [21], authors consider \((p,q)\)-Laplace equation of type (P) with critical growth perturbation with respect to the Sobolev embedding and \( f(x) = \lambda \). Splitting the Nehari manifold authors proved the existence of at least two positive solutions and the \( L^\infty \) estimates. Furthermore, they obtained the global existence result using Perron’s method. Authors in [31] considered the following equation

\[-\text{div}A(\nabla u) = \lambda \nu(u) + f(x,u) \quad u > 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{on } \partial \Omega,
\]

where \( A : \mathbb{R}^N \to \mathbb{R}^N \) satisfies certain structure condition, \( \nu \) behaves like \( x^{-\delta} \) for \( \delta \in (0,1) \) and \( f \) is a Carathéodory function with subcritical growth. Here authors proved the existence of \( \lambda_* > 0 \) such that the problem has at least two solutions if \( \lambda < \lambda_* \), at least one solution for \( \lambda = \lambda_* \) and no solution for \( \lambda > \lambda_* \).

Regarding the regularity results for solution to (P), we mention the work of Lieberman [25, 26] for solutions of quasilinear elliptic equation with nonsingular nonlinearity. Consider the following equation

\[-\text{div}A(x,u,z) = B(x,u,z) + g(x) \quad \text{in } \Omega. \quad (1.3)
\]

In [25], author proved that weak solutions of (1.3) are in \( C^{1,\alpha}(\Omega) \) when \( A \) and \( B \) satisfy the following structure condition \( \lambda(\kappa + |z|)^{p-2}|\xi|^2 \leq a^{ij}(x,u,z)\xi_i\xi_j \leq \Lambda(\kappa + |z|)^{p-2}|\xi|^2 \) and \( |B(x,u,z)| \leq \Lambda(\kappa + |z|)^p \), for \((x,u,z) \in \Omega \times \mathbb{R} \times \mathbb{R}^N\), with \( g = 0 \), where \( a^{ij} = \partial A^i/\partial z_j \), \( 0 < \lambda \leq \Lambda \) and \( \kappa \in [0,1] \). Subsequently, in [26] interior Hölder continuity result is established for the gradient of solution to (1.3) when \( A \) and \( B \) satisfies structure condition involving special kind of Orlicz functions and \( g = 0 \). Concerning the quasilinear equations with singular nonlinearity, Giacomoni, Schindler and Takáč [13, 14] obtained Hölder continuity results for weak solution to (1.3) when \( A(x,u,z) = A(x,z) \) and \( B = 0 \). In [13], following the approach of [25], authors proved that the weak solution, which behaves like the distance function near the boundary, is in \( C^{1,\alpha}(\Omega) \), for some \( \alpha \in (0,1) \), when \( 0 \leq g \leq C d(x)^{-\sigma} \) for \( \sigma < 1 \). In the latter work [14], authors proved that solution \( u \in W^{1,p}_0(\Omega) \), is in \( C^{0,\alpha}(\Omega) \) when \( 0 \leq u \leq C d(x)^{-\sigma} \) and \( 0 \leq g(x) \leq C d(x)^{-\sigma} \) with \( 0 < \sigma' < \sigma < \sigma' + 1 \).

Inspired from above discussion, in this work we consider singular problem driven by the nonhomogeneous \((p,q)\)-Laplace operator. We assume that \( f \in L^\infty_{loc}(\Omega) \) satisfies the following
condition
\[ c_1 d(x)^{-\beta} \leq f(x) \leq c_2 d(x)^{-\beta} \quad \text{in } \Omega_\theta, \]  
(1.4)

where \( d(x) := \text{dist}(x, \partial \Omega) \), \( c_1, c_2 \) are nonnegative constants, \( \beta \geq 0 \) and \( \Omega_\theta := \{ x \in \overline{\Omega} : d(x) < \theta \} \) for \( \theta > 0 \). To prove the existence of a weak solution, we perturb the problem \((P)\) by taking the nonlinear term as \( f_\varepsilon \leq f \) and replacing \( u - \delta \) by \( (u + \varepsilon) - \delta \). Thus standard Schauder fixed point theory and elliptic regularity theory can be applied to get the existence of a unique solution \( u_\varepsilon \in C^{1,\alpha}(\overline{\Omega}) \) (see Lemma 2.1). By establishing comparison of \( u_\varepsilon \) with some function of the distance function, we prove convergence of \( u_\varepsilon \) to \( u \), the minimal weak solution to problem \((P)\). Due to the nonhomogeneous nature of the leading operator, unlike the case of \( p\)-Laplace equation, we cannot use some scalar multiple of eigenfunctions of \(-\Delta_p\) to obtain the suitable sub and super solution involving the distance function. To overcome this difficulty, we exploit the \( C^2 \) regularity of the boundary \( \partial \Omega \), and that the distance function is \( C^2 \) in some neighborhood of the boundary. We use this in place of the first eigenfunction of \(-\Delta_p\) to construct a suitable sub and super solutions. The aforementioned behavior of \( u_\varepsilon \) near the boundary helps us to establish the optimal Sobolev regularity for the weak solution to \((P)\) obtained as the limit of \( u_\varepsilon \), that is, we prove \( u_\varepsilon \in W^{1,p}_0(\Omega) \) if and only if \( \rho > \rho_0 \). Another consequence of this boundary behavior is that by comparison with suitable \( u_\varepsilon \), we establish the non-existence result for the case of \( \beta \geq p \). This result is new even for the homogeneous quasilinear elliptic operators like \( p\)-Laplacian. Moreover, we prove a comparison principle for sub and super solution of \((P)\) in \( W^{1,p}_{\text{loc}}(\Omega) \) for the case of \( \beta < 2 - \frac{1}{p} \). Using suitably the Hardy inequality, this result improves former contribution even for the operators like \( p\)-Laplacian, by considering a new notion of solutions and a larger class of weight functions \( f \). In [14], authors obtained comparison principle when the solution is in the energy space, \( W^{1,p}_0(\Omega) \) while Canino et al. in [5] considered the case when \( f \) belongs to some Lebesgue space. A direct consequence is the uniqueness result for the case \( \beta < 2 - \frac{1}{p} \).

Since the boundary \( \partial \Omega \) is \( C^2 \), it follows from [16] Lemma 14.6, p. 355 that there exists \( \mu > 0 \) such that \( d \in C^2(\Omega_\mu) \). Without loss of generality, we may assume \( \theta \leq \min\{ \frac{\mu}{2}, 1 \} \), so that \( |\Delta d| \in L^\infty(\Omega_\theta) \) and \( (1.4) \) also holds. We define the notion of weak solution to \((P)\) as follows.

**Definition 1.1** A function \( u \in W^{1,p}_{\text{loc}}(\Omega) \) is said to be a weak sub-solution (resp. super-solution) of problem \((P)\) if the following holds

(i) for every \( K \subseteq \Omega \), there exists a constant \( C_K > 0 \) such that \( u \geq C_K \) in \( K \),

(ii) for all \( \phi \in C_c^\infty(\Omega) \), with \( \phi \geq 0 \) in \( \Omega \),

\[ \int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi + \int_\Omega |\nabla u|^{q-2} \nabla u \nabla \phi \leq (\text{resp.} \geq) \int_\Omega f(x) u^{-\delta} \phi, \]

(iii) there exists \( \gamma \geq 1 \) such that \( u^\gamma \in W^{1,p}_0(\Omega) \).
A function which is both sub and super solution of \( P \) is called a weak solution.

We remark that the definition of weak solution considered above is a weaker notion of solution with respect to \([13, 14]\). Moreover, the condition (iii) in the above definition appears due to lack of the trace mapping in \( W^{1,p}_{\text{loc}}(\Omega) \) and this also implies the following definition of the boundary datum of \( u \).

**Definition 1.2** We say that \( u \leq 0 \) on \( \partial \Omega \), if \((u - \epsilon)^+ \in W^{1,p}_{0}(\Omega)\) for every \( \epsilon > 0 \) and \( u = 0 \) on \( \partial \Omega \) if \( \beta \geq 0 \) in \( \Omega \) and \( u \leq 0 \) on \( \partial \Omega \).

**Definition 1.3** We say a weak solution \( u \) of \( P \), is in the conical shell \( C_{\delta,\beta} \) if it is continuous and satisfies the following

\[
\eta d(x) \leq u(x) \leq \Gamma d(x) \quad \text{if} \quad \beta + \delta < 1,
\]

\[
\eta d(x) \log^{\frac{1}{\beta}} \left( \frac{L}{a(x)} \right) \leq u(x) \leq \Gamma d(x) \log^{\frac{1}{\beta}} \left( \frac{L}{a(x)} \right) \quad \text{if} \quad \beta + \delta = 1,
\]

\[
\eta d(x) \log^{\frac{1}{\beta}} \left( \frac{L}{a(x)} \right) \leq u(x) \leq \Gamma d(x) \log^{\frac{1}{\beta}} \left( \frac{L}{a(x)} \right) \quad \text{if} \quad \beta + \delta > 1,
\]

for some positive constants \( \eta, \Gamma > 0 \) and \( L > 0 \) is sufficiently large.

Now, we state our main existence result.

**Theorem 1.4** Let \( \beta \in [0,p) \), then problem \( P \) admits a weak minimal solution \( u \in W^{1,p}_{\text{loc}}(\Omega) \cap C_{\delta,\beta} \), in the sense of definition 1.2. Moreover, \( u^\rho \in W^{1,p}_{0}(\Omega) \) if and only if \( \rho > \rho_0 := \frac{1-\beta}{(p-1)(\beta+\delta-1)} > 0 \). Further, \( u \in W^{1,p}_{\text{loc}}(\Omega) \) if and only if \( \delta < 2 + \frac{1-\beta}{p-1} \).

To obtain the uniqueness result, we establish the following weak comparison principle.

**Theorem 1.5** Let \( \beta < 2 - \frac{1}{p} \) and \( u,v \in W^{1,p}_{\text{loc}}(\Omega) \) be sub and super solution of \( P \), respectively in the sense of definition 1.1. Then, \( u \leq v \) a.e. in \( \Omega \).

Next, we have a non-existence result for weak solution of \( P \).

**Theorem 1.6** Let \( \beta \geq p \) in (1.4). Then, there does not exist any weak solution of problem \( P \) in the sense of definition 1.1.

Theorem 1.6 shows that Theorem 1.4 is sharp. Regarding the Hölder regularity of solutions to problem \( P \), we study a more general quasilinear form of \( (P) \). Consider the following equation,

\[
-\text{div}A(x, Du) = B(x, u, \nabla u) + g(x) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega, \quad (1.5)
\]

where \( A : \overline{\Omega} \times \mathbb{R}^N \rightarrow \mathbb{R}^N \) is a continuous function. We assume the following conditions

(A1) \( |A(x,z)| + |\partial_z A(x,z)z| \leq \Lambda(|z|^{q-1} + |z|^{p-1}) \leq 2\Lambda(1 + |z|^{p-1}) \)

(A2) \( z.A(x,z) \geq \nu(|z|^p + |z|^q) \)

(A3) \( \sum_{i=1}^{N} |A^i(x, z) - A^i(y, z)| \leq \Lambda(1 + |z|^{p-1})|x - y|^\omega, \)
\[ |B(x, u, z)| \leq \Lambda (1 + |z|)^p \quad \text{for} \quad (x, u, z) \in \Omega \times [-M_0, M_0] \times \mathbb{R}^N, \]

where \( 0 < \nu \leq \Lambda \) are constants, \( M_0 > 0, 1 < q < p < \infty \) and \( \omega \in (0, 1) \). Furthermore, assume \( g \) satisfies the following

\[ 0 \leq g(x) \leq Cd(x)^{-\sigma}, \quad (1.6) \]

where \( \sigma \in [0, 1) \) and \( C > 0 \) is a constant. First we prove Hölder continuity result up to the boundary for the gradient of the weak solution to (1.5). Consequently, the weak solution to (P) is in \( C^{1, \alpha}(\Omega) \) for the case \( \beta + \delta < 1 \). The interior regularity follows from \cite{20} Theorem 1.7. Inspired from the ideas of \cite{13} and \cite{25}, we consider a perturbation of the problem (1.5) (see (4.2)). We estimate various quantities involving supremum and oscillation of the gradient of the solution to the perturbed problem by means of the weak Harnack inequality, local maximum principle and suitable barrier arguments. Using these estimates, we establish control over the Campanato norm of the solution \( u \) to problem (1.5), which helps us to finally obtain the Hölder regularity result. Here, we would like to mention that the Hölder regularity result is new even for the equations involving singular nonlinearity and the gradient terms, where the leading differential operator is a homogeneous quasilinear elliptic operator like \( p \)-Laplacian. Whereas for the non singular case, that is \( \beta = 0 = \delta \), we provide a complete proof of the Hölder continuity result for the gradient of weak solution which complements the interior regularity result of \cite{26} Theorem 1.7. We state our main regularity result in this case as follows.

**Theorem 1.7** Let \( u \in W^{1, p}_0(\Omega) \) be a weak solution of problem (1.5) such that \( 0 \leq u \leq M_0 \) in \( \Omega \). Let \( \sigma \in (0, 1) \) in (1.6) and suppose there exists \( C > 0 \) such that \( 0 \leq u(x) \leq Cd(x)^{-\sigma} \) a.e. in \( \Omega \). Then, there exists a constant \( \alpha \in (0, 1) \), depending only on \( N, p, \omega, \sigma, \nu, \Lambda \), such that \( u \in C^{1, \alpha}(\Omega) \) and

\[ |u|_{C^{1, \alpha}(\Omega)} \leq C(N, \nu, \Lambda, \omega, p, \sigma, M_0, \Omega). \]

Next, we prove Hölder continuity result for weak solution, \( u \in W^{1, p}_{loc}(\Omega) \) of problem (P), for the case \( \beta + \delta \geq 1 \). By taking into consideration \( u^\gamma \), for some suitable \( \gamma > 1 \), we transform the problem (P) to a new quasilinear equation involving a form of weighted \( (p, q) \)-Laplacian operator and lower order terms (see (4.19)). Using the behavior of \( u \) near the boundary, we choose \( \gamma \) appropriately so that the nonlinear term in the transformed equation belongs to \( L^\infty(\Omega) \). Then, we follow the idea of Ladyzhenskaya and Ural’tseva \cite{22} to obtain Morrey type estimates on \( u^\gamma \). This proves Hölder continuity of \( u^\gamma \), which in turns implies the continuity result in the sense of Hölder for \( u \). The main result in this regard is as given as below.

**Theorem 1.8** Let \( \beta + \delta \geq 1 \) and \( u \in W^{1, p}_{loc}(\Omega) \) be a bounded nonnegative weak solution of problem (P) in the sense of definition (1.4). Furthermore, suppose there exists \( \Gamma, \tilde{\sigma} > 0 \) such that \( 0 \leq u(x) \leq \Gamma d(x)^{\tilde{\sigma}} \) a.e. in \( \Omega \). Then, there exists \( \alpha \in (0, 1) \), depending only on \( N, p, \|u\|_{L^\infty(\Omega)}, \tilde{\sigma}, \Gamma, \beta \) and \( \delta \), such that \( u \in C^{0, \alpha}(\Omega) \).
We remark that the preceding theorem complements Theorem A.1 in [14] for equations involving $p$-Laplacian with singular nonlinearity, where the solutions are considered to be in the energy space $W^{1,p}_0(\Omega)$.

**Corollary 1.9** Let $u \in W^{1,p}_{loc}(\Omega)$ be either a unique solution or the minimal solution (i.e., obtained as a limit of solution to the approximated problem $(P_\epsilon)$) of problem $(P)$, then

1. $u \in C^{1,\alpha}(\Omega)$, for $\alpha \in (0, 1)$ given by theorem 1.7, in the case of $\beta + \delta < 1$.
2. $u \in C^{0,\alpha}(\Omega)$, for $\alpha \in (0, 1)$ given by theorem 1.8, in the case of $\beta + \delta \geq 1$.

**Remark 1.10** We remark that our results are true for more general class of quasilinear elliptic operators with slight modification in the proofs. Some examples of the differential operators are the following:

(i) The operator $-\Delta_p - a(x)\Delta_q$, for some non-negative function $a \in C(\Omega)$.

(ii) The operator $-\text{div}(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u)$, where $0 \leq a(x) \in W^{1,\infty}(\Omega) \cap C(\Omega)$ with $1 < q < p < \infty$.

Turning to the layout of the paper: In Section 2, we establish our main existence theorem and optimal Sobolev regularity, here we prove Theorem 1.4. In Section 3, we prove Theorem 1.5 and consequently, we obtain the uniqueness result. Here we provide proof of Theorem 1.6. In Section 4, we establish the Hölder regularity results, precisely we prove Theorems 1.7 and 1.8.

## 2 Existence results

First we consider the following perturbed problem

$$(P_\epsilon) \quad \left\{ \begin{array}{ll}
-\Delta_p u_\epsilon - \Delta_q u_\epsilon &= f_\epsilon(x)(u_\epsilon + \epsilon)^{-\delta}, \\
    u_\epsilon &= 0 \text{ on } \partial \Omega,
\end{array} \right.$$

where

$$f_\epsilon(x) := \begin{cases} 
(f(x) \frac{1}{\beta} + \epsilon \frac{p-1+\delta}{p-\beta})^{-\beta} & \text{if } f(x) > 0 \\
0 & \text{otherwise.}
\end{cases}$$

It is easy to observe that, for $\beta < p$, the function $f_\epsilon$ increases as $\epsilon \downarrow 0$ and $f_\epsilon \leq f$ for all $\epsilon > 0$.

**Lemma 2.1** For each $\epsilon > 0$, there exists a unique solution $u_\epsilon \in W^{1,p}_0(\Omega)$ of $(P_\epsilon)$. Furthermore, for $\beta < p$, the sequence $\{u_\epsilon\}$ is increasing as $\epsilon \downarrow 0$ and for each $\Omega' \subseteq \Omega$, there exists $C_{\Omega'} > 0$ such that for all $\epsilon > 0$,

$$u_\epsilon \geq C_{\Omega'} \text{ in } \Omega'.$$  \hfill (2.1)
Proof. For fixed \( \epsilon > 0 \) and for each \( v \in L^p(\Omega) \), we consider the following auxiliary problem

\[
-\Delta_p w - \Delta_q w = f_\epsilon(x)(|v| + \epsilon)^{-\delta}, \quad w > 0 \text{ in } \Omega; \quad w = 0 \text{ on } \partial \Omega. \tag{2.2}
\]

By standard minimization technique we can prove that there exists a unique solution \( w \in W^{1,p}_0(\Omega) \) of (2.2). Indeed, the corresponding energy functional \( J : W^{1,p}_0(\Omega) \to \mathbb{R} \) defined by

\[
J(w) := \frac{1}{p} \int_\Omega |\nabla w|^p dx + \frac{1}{q} \int_\Omega |\nabla w|^q dx - \int_\Omega f_\epsilon(x)(|v| + \epsilon)^{-\delta} w dx,
\]

is continuous, strictly convex and coercive. We define the operator \( S : L^p(\Omega) \to L^p(\Omega) \) as follows

\[
S(v) = w,
\]

where \( w \) is the unique solution to (2.2). By means of Poincare inequality, we observe that

\[
\|S(v)\|_{L^p(\Omega)}^p = \|w\|_{L^p(\Omega)}^p \leq C\|\nabla w\|_{L^p(\Omega)}^p \leq C\int_\Omega (|\nabla w|^p + |\nabla w|^q) = C\int_\Omega f_\epsilon(x)(|v| + \epsilon)^{-\delta} \leq C\epsilon^{-\delta - \beta/\tau} \int_\Omega |w| dx \leq C\epsilon^{-\delta - \beta/\tau} |\Omega| \cdot \frac{p}{p-\beta},
\]

where \( \tau = \frac{p-\beta}{p-1+\delta} \). Then, it is standard procedure to verify that \( S \) is continuous, compact and invariant on the ball of \( L^p(\Omega) \) with radius \( \left(C\epsilon^{-\delta - \beta/\tau} |\Omega| \cdot \frac{p}{p-\beta}\right)^{1/(p-1)} \). Therefore, by Schauder’s fixed point theorem, there exists \( u_\epsilon \in W^{1,p}_0(\Omega) \) such that \( u_\epsilon = S(u_\epsilon) \), that is, \( u_\epsilon \) is a solution of \( (P_\epsilon) \). Since \( f_\epsilon(x)(|v| + \epsilon) \geq 0 \), by standard elliptic regularity theory, we deduce that \( u_\epsilon \geq 0 \) and \( u_\epsilon \in L^\infty(\Omega) \). Consequently, regularity result of Theorem 1.7 with \( \sigma = 0 \) (or [26, Theorem 1.7]) gives us \( u_\epsilon \in C^{1,\alpha}(\Omega) \) and the strong maximum principle of [32, p. 111, 120] implies \( u_\epsilon > 0 \) in \( \Omega \).

Next, for the case \( \beta < p \), we will prove that the sequence \( \{u_\epsilon\} \) is increasing as \( \epsilon \downarrow 0 \). Let \( u_\epsilon \) and \( u_\epsilon' \) be weak solutions of \( (P_\epsilon) \) and \( (P'_\epsilon) \), respectively with \( \epsilon' \leq \epsilon \). We observe that the term on the right in \( (P_\epsilon) \) is non-singular, therefore by density argument, we can take \( (u_\epsilon - u_\epsilon')^+ \) as a test function in the weak formulations. Thus, due to the fact \( 0 \leq f_\epsilon \leq f_\epsilon' \), we obtain

\[
\int_\Omega (|\nabla u_\epsilon|^{p-2}\nabla u_\epsilon - |\nabla u_\epsilon'|^{p-2}\nabla u_\epsilon') \nabla (u_\epsilon - u_\epsilon')^+
+ \int_\Omega (|\nabla u_\epsilon|^{q-2}\nabla u_\epsilon - |\nabla u_\epsilon'|^{q-2}\nabla u_\epsilon') \nabla (u_\epsilon - u_\epsilon')^+
= \int_\Omega \left( f_\epsilon(x)(u_\epsilon + \epsilon)^{-\delta} - f_\epsilon'(x)(u_\epsilon' + \epsilon')^{-\delta}\right)(u_\epsilon - u_\epsilon')^+
\leq \int_\Omega f_\epsilon'(x) \left((u_\epsilon + \epsilon)^{-\delta} - (u_\epsilon' + \epsilon')^{-\delta}\right)(u_\epsilon - u_\epsilon')^+
\leq 0.
\]
Using the inequality: for $p > 1$, there exists a constant $C_1 = C(p) > 0$ such that for all $\xi, \zeta \in \mathbb{R}^N$ with $|\xi| + |\zeta| > 0$, the following holds
\[
(|\xi|^{p-2}\xi - |\zeta|^{p-2}\zeta) \cdot (\xi - \zeta) \geq C_1 (|\xi| + |\zeta|)^{p-2}|\xi - \zeta|^2,
\] (2.3)
we deduce that
\[
\int_{\Omega} (|\nabla u_\epsilon| + |\nabla u_\epsilon'|)^{p-2} |\nabla u_\epsilon - \nabla u_\epsilon'|^2 \leq 0.
\]
This implies that $(u_\epsilon - u_\epsilon')^+ = 0$ a.e. in $\Omega$ and therefore, $u_\epsilon \leq u_\epsilon'$ in $\Omega$. Consequently, (2.1) holds for all relatively compact subsets of $\Omega$ on the account of $u_1 \in C^{1,\alpha}(\overline{\Omega})$ and $u_1 > 0$ in $\Omega$. For the case $\beta > p$, it is easy to see that $f_\epsilon$ decreases as $\epsilon \downarrow 0$ and proceeding similarly as above, we can prove that the sequence $\{u_\epsilon\}$ is decreasing as $\epsilon \downarrow 0$. The uniqueness of $u_\epsilon$ follows using similar assertions and arguments used to prove monotonocity of $u_\epsilon$ in $\epsilon$. This completes proof of the lemma. \hfill $\Box$

**Lemma 2.2** Let $\beta + \delta > 1$ and $\beta < p$. Suppose $u_\epsilon$ be the solution of $(P_\epsilon)$. Then, there exist constants $\eta, \Gamma > 0$, independent of $\epsilon$, such that the following holds for $x \in \Omega$,
\[
\eta \left( (d(x) + \epsilon^{\frac{p-\delta-\beta}{p-\beta}})^{\frac{p-\beta}{p-1}} - \epsilon \right) \leq u_\epsilon(x) \leq \Gamma \left( (d(x) + \epsilon^{\frac{p-\delta-\beta}{p-\beta}})^{\frac{p-\beta}{p-1}} - \epsilon \right).
\]

**Proof.** Set $\tau = \frac{p-\beta}{p-1+\delta} (\in (0, 1))$ and $v_\epsilon = \eta \left( (d(x) + \epsilon^{\frac{1}{1+\delta}})^r - \epsilon \right)$. Then
\[
\nabla v_\epsilon = \eta \tau (d(x) + \epsilon^{\frac{1}{1+\delta}})^{r-1} \nabla d.
\]
Since $\Delta d \in L^\infty(\Omega_\delta)$, there exists $M > 0$ such that $|\Delta d| \leq M$ in $\Omega_\delta$. Therefore, for $\psi \in C_c^\infty(\Omega_\delta)$ with $\psi \geq 0$ and noting the fact that $|\nabla d| = 1$, we deduce that
\[
\int_{\Omega_\delta} -\Delta_p v_\epsilon \psi = (\eta \tau)^{p-1} \int_{\Omega_\delta} (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)} \nabla d \nabla \psi
\]
\[
= (\eta \tau)^{p-1} \int_{\Omega_\delta} \left[ -\Delta d (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)} \psi + (p-1)(1-\tau)(d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)-1} \psi \right]
\]
\[
\leq (\eta \tau)^{p-1} \int_{\Omega_\delta} \left[ M (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)} + (p-1)(d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)-1} \psi \right]
\]
\[
\leq C (\eta \tau)^{p-1} \int_{\Omega_\delta} (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)-1} \psi
\]
where $C = 2 \max\{M, (p-1)\}$. Similar steps yield
\[
\int_{\Omega_\delta} -\Delta_q v_\epsilon \psi \leq C (\eta \tau)^{q-1} \int_{\Omega_\delta} (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(q-1)-1} \psi \leq C (\eta \tau)^{q-1} \int_{\Omega_\delta} (d(x) + \epsilon^{\frac{1}{1+\delta}})^{(r-1)(p-1)-1} \psi.
\]
Thus, using the definition of $\tau$, we have
\[
\int_{\Omega_\delta} \left( -\Delta_p v_\epsilon - \Delta_q v_\epsilon \right) \psi \leq C ((\eta \tau)^{p-1} + (\eta \tau)^{q-1}) \int_{\Omega_\delta} (d(x) + \epsilon^{\frac{1}{1+\delta}})^{-\delta \tau - \beta} \psi.
\]
Therefore, using (1.4), we deduce that
\[ \frac{1}{f_\epsilon(x)}(-\Delta_p v_\epsilon - \Delta q v_\epsilon) \leq C(\eta^{p-1} + \eta^{q-1})(d(x) + \epsilon^{\frac{1}{\tau}})^{-\delta \tau} \quad \text{in } \Omega_\theta. \] (2.4)

Next, we observe that \((v_\epsilon + \epsilon)^{-\delta} = (\eta(d + \epsilon^{1/\tau})^\tau + (1 - \eta)\epsilon)^{-\delta}\) and distinguish the following cases:

Case (i): \(\eta(d(x) + \epsilon^{1/\tau})^\tau \geq (1 - \eta)\epsilon\) for \(x \in \Omega\).

In this case, we have
\[ (v_\epsilon(x) + \epsilon)^{-\delta} \leq 2^{-\delta} \eta^{-\delta}(d(x) + \epsilon^{1/\tau})^{-\tau \delta}. \]

Therefore, from (2.4) for sufficiently small \(\eta > 0\), independent of \(\epsilon\), we obtain
\begin{align*}
\frac{1}{f_\epsilon(x)}(-\Delta_p v_\epsilon - \Delta q v_\epsilon) & \leq C(\eta^{p-1} + \eta^{q-1})(d(x) + \epsilon^{\frac{1}{\tau}})^{-\delta \tau} \leq 2^{-\delta} \eta^{-\delta}(d(x) + \epsilon^{1/\tau})^{-\tau \delta} \\
& \leq (v_\epsilon(x) + \epsilon)^{-\delta}.
\end{align*}

Case (ii): \(\eta(d(x) + \epsilon^{1/\tau})^\tau \leq (1 - \eta)\epsilon\) for \(x \in \Omega\).

We have
\[ (v_\epsilon(x) + \epsilon)^{-\delta} \geq 2^{-\delta}(1 - \eta)^{-\delta} \epsilon^{-\delta}. \]

Again, we can choose \(\eta > 0\) small enough and independent of \(\epsilon\) so that
\begin{align*}
\frac{1}{f_\epsilon(x)}(-\Delta_p v_\epsilon - \Delta q v_\epsilon) & \leq C(\eta^{p-1} + \eta^{q-1})(d(x) + \epsilon^{\frac{1}{\tau}})^{-\delta \tau} \leq (\eta^{p-1} + \eta^{q-1})\epsilon^{-\delta} \\
& \leq 2^{-\delta}(1 - \eta)^{-\delta} \epsilon^{-\delta} \\
& \leq (v_\epsilon(x) + \epsilon)^{-\delta}.
\end{align*}

Therefore, in either case, we can choose \(\eta > 0\) sufficiently small and independent of \(\epsilon\) such that
\[ -\Delta_p v_\epsilon - \Delta_q v_\epsilon \leq f_\epsilon(x)(v_\epsilon(x) + \epsilon)^{-\delta} \quad \text{in } \Omega_\theta. \]

On account of (2.4), we choose \(\eta > 0\) small enough, independent of \(\epsilon\), such that in addition to the preceding relations in cases (i) and (ii), the following holds
\[ v_\epsilon(x) \leq \eta \text{diam}(\Omega)^\tau \leq C_\theta \leq u_1(x) \leq u_\epsilon(x) \quad \text{in } \Omega \setminus \Omega_\theta. \]

Therefore, by comparison principle, we get \(v_\epsilon \leq u_\epsilon\) in \(\Omega_\theta\), that is,
\[ \eta((d + \epsilon^{1/\tau})^\tau - \epsilon) \leq u_\epsilon(x) \quad \text{in } \Omega. \]

Next, we will prove the upper bound for \(u_\epsilon\), for this consider \(u_\epsilon = \Gamma((d(x) + \epsilon^{\frac{1}{\tau}})^\tau - \epsilon)\), where \(\Gamma\) is a constant. Proceeding as above, for \(\psi \in C^\infty(\Omega)\) with \(\psi \geq 0\), we obtain
\begin{align*}
\int_{\Omega_\theta} -\Delta_p u_\epsilon \psi & = (\Gamma\tau)^{p-1} \int_{\Omega_\theta} [-\Delta d(d(x) + \epsilon^{\frac{1}{\tau}})^{(\tau-1)(p-1)}\psi + (p - 1)(1 - \tau)(d(x) + \epsilon^{\frac{1}{\tau}})^{(\tau-1)(p-1)-1}\psi] \\
& \geq (\Gamma\tau)^{p-1} \int_{\Omega_\theta} (-M(d(x) + \epsilon^{\frac{1}{\tau}})^{(\tau-1)(p-1)} + (p - 1)(1 - \tau)(d(x) + \epsilon^{\frac{1}{\tau}})^{(\tau-1)(p-1)-1}\psi).
\end{align*}
On a similar note, we have
\[ \int_{\Omega_\varepsilon} -\Delta_p w_\varepsilon \psi \geq (\Gamma\tau)^{q-1} \int_{\Omega_\varepsilon} \left( -M(d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(q-1)} + (q-1)(1-\tau)(d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(q-1)-1} \right) \psi. \]
(2.5)

Furthermore, if necessary by reducing \( p \) further, we may assume that there exists \( C_3 > 0 \) such that
\[(p-1)(1-\tau)(d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(p-1)-1} - M(d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(p-1)} \geq C_3(d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(p-1)-1} \text{ in } \Omega_\varepsilon,\]
and the right hand quantity in (2.5) is nonnegative (this is possible because \((\tau-1)(p-1) \leq 0\)). Therefore,
\[ \int_{\Omega_\varepsilon} -\Delta_p w_\varepsilon \psi - \Delta_q w_\varepsilon \psi \geq C_3(\Gamma\tau)^{p-1} \int_{\Omega_\varepsilon} (d(x) + \epsilon^{\frac{1}{p}})^{(r-1)(p-1)-1} \psi. \]

Taking into account (1.4), we obtain
\[ \frac{1}{f_\varepsilon(x)}(-\Delta_p w_\varepsilon - \Delta_q w_\varepsilon) \geq C_3(\Gamma\tau)^{p-1}(d(x) + \epsilon^{\frac{1}{p}})^{-\delta \tau} \text{ in } \Omega_\varepsilon. \]

By using the lower estimate of \( u_\varepsilon \) by \( v_\varepsilon \), for the right hand side of \((P_\varepsilon)\), we obtain
\[ f_\varepsilon(x)(u_\varepsilon + \epsilon)^{-\delta} \leq f(x)(v_\varepsilon + \epsilon)^{-\delta} \leq f(x)\eta^{-\delta}d^{-\tau \delta}. \]

Sine \( f \in L^\infty_{\text{loc}}(\Omega) \), we observe that \( f(x)\eta^{-\delta}d^{-\tau \delta} \in L^\infty_{\text{loc}}(\Omega) \). Therefore, by \( L^\infty \) estimate of \cite{22}, we get \( u_\varepsilon \in L^\infty_{\text{loc}}(\Omega) \) and the bound is independent of \( \epsilon \), say \( \|u_\varepsilon\|_{L^\infty(\Omega \setminus \Omega_\varepsilon)} \leq K \). Now, we choose \( \Gamma \) sufficiently large and independent of \( \epsilon \) satisfying last two inequalities in the following
\[ w_\varepsilon = \Gamma((d + \epsilon^{1/\tau})^\tau - \epsilon) \geq \Gamma(d^{\tau} - \frac{\theta^\tau}{2}) \geq \Gamma\frac{\theta^\tau}{2} \geq K \geq u_\varepsilon(x) \text{ in } \Omega \setminus \Omega_\varepsilon \]
for all \( \epsilon < \theta^\tau/2 \). Then, by comparison principle, we get \( u_\varepsilon \leq w_\varepsilon \) in \( \Omega \). This completes proof of the lemma.

**Lemma 2.3** Let \( \beta + \delta = 1 \) and \( u_\varepsilon \) be the solution of \((P_\varepsilon)\). Then, there exist constants \( \eta, \Gamma > 0 \), independent of \( \epsilon \), such that the following holds in \( \Omega \),
\[ (\eta d + \epsilon') \log^{1/\beta} \left( \frac{L}{\eta d + \epsilon'} \right) - \epsilon' \log^{1/\beta} \left( \frac{L}{\epsilon'} \right) \leq u \leq (\Gamma d + \epsilon') \log^{1/\beta} \left( \frac{L}{\Gamma d + \epsilon'} \right) - \epsilon' \log^{1/\beta} \left( \frac{L}{\epsilon'} \right), \]
where \( L > 0 \) is large enough and \( \epsilon = \epsilon' \log^{1/(p - \beta)} \left( \frac{L}{\epsilon'} \right) \).

**Proof.** Set \( w_\varepsilon = \eta \log^{1/(p - \beta)} \left( \frac{L}{\eta d + \epsilon'} \right) - \epsilon' \log^{1/(p - \beta)} \left( \frac{L}{\epsilon'} \right) \). Then,
\[ \nabla w_\varepsilon = \eta \log^{1/(p - \beta)} \left( \frac{L}{\eta d + \epsilon'} \right) \left[ \log \left( \frac{L}{\eta d + \epsilon'} \right) - \frac{1}{p - \beta} \right] \nabla d. \]
For $\psi \in C_c^\infty(\Omega_\delta)$ with $\psi \geq 0$, using the fact that $|\nabla d| = 1$, we get

$$
\int_\Omega -\Delta_p u \psi = \eta^{p-1} \int_\Omega \nabla d \nabla \psi \log \frac{L}{\eta \delta + \epsilon'} \left[ \log \left( \frac{L}{\eta \delta + \epsilon'} \right) - \frac{1}{p - \beta} \right]^{p-1}.
$$

A simple manipulation yields

$$-\Delta_p u = \eta^{p-1} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}} \left[ (-\Delta d)(\eta \delta + \epsilon') \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{2-p} \left( \log \frac{L}{\eta \delta + \epsilon'} - \frac{1}{p - \beta} \right)^{p-1}
+ \frac{\eta(1-p+\beta)(p-1)}{\eta \delta + \epsilon'} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{1-p} \left( \log \frac{L}{\eta \delta + \epsilon'} - \frac{1}{p - \beta} \right)^{p-1}
+ \eta(p-1) \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{2-p} \left( \log \frac{L}{\eta \delta + \epsilon'} - \frac{1}{p - \beta} \right)^{p-2} \right].$$

Since $| - \Delta d | \leq M$ in $\Omega_\delta$ and $\eta < 1$, we deduce that

$$-\Delta_p u \leq \eta^{p-1} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}} \left[ M(\eta \delta + \epsilon') \log \frac{L}{\eta \delta + \epsilon'} + \frac{(p-1-\beta)(p-1)}{p-\beta}
+ (p-1) \left( 1 - \frac{1}{(p-\beta) \log \left( \frac{L}{\eta \delta + \epsilon'} \right)} \right) \right]^{p-2}.$$

Choosing $L >> 1$ sufficiently large such that $\log \left( \frac{L}{\text{diam}(\Omega) + 1} \right) \geq 2/(p - \beta)$ and if necessary by reducing $\delta$ further, we get $(\eta \delta + \epsilon') \log \frac{L}{\eta \delta + \epsilon'} \leq C_1$ in $\Omega_\delta$. Therefore, the quantity in the bracket is bounded by a positive constant $C$, independent of $\epsilon$. Thus,

$$-\Delta_p u \leq C \eta^{p-1} (\eta \delta + \epsilon')^{-1} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}}.$$

Proceeding similarly, we obtain

$$-\Delta_p u \leq \eta^{p-1} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}} \left[ M(\eta \delta + \epsilon') \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}}
+ \frac{(p-1-\beta)(q-1)}{(p-\beta) \log \left( \frac{L}{\eta \delta + \epsilon'} \right)}
+ (q-1) \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}} \left( 1 - \frac{1}{(p-\beta) \log \left( \frac{L}{\eta \delta + \epsilon'} \right)} \right) \right]^{q-2}.$$

Using the same assertions as in the estimate of $-\Delta_p u$, we get

$$-\Delta_p u \leq C \eta^{q-1} (\eta \delta + \epsilon')^{-1} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{\frac{p-1}{p-\beta}}.$$

Noting the fact that $(\tilde{u}_\epsilon + \epsilon)^{-\beta} = (\eta \delta + \epsilon')^{-\beta} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{(\beta-1)/(p-\beta)}$ and proceeding similar to lemma 2.2 for sufficiently small $\eta > 0$, independent of $\epsilon$, we get

$$\frac{1}{f_\epsilon(x)} \left( -\Delta_p u - \Delta_p u \right) \leq C \eta^{q-1} (\eta \delta + \epsilon')^{-\beta} \left( \log \frac{L}{\eta \delta + \epsilon'} \right)^{(\beta-1)/(p-\beta)} \leq u_\epsilon^{-\beta} \quad \text{in } \Omega_\delta.$$

Moreover, using (2.1), we obtain

$$u_\epsilon(x) \leq u(x) \quad \text{in } \Omega \setminus \Omega_\delta,$$
for sufficiently small $\eta > 0$ independent of $\epsilon$. Therefore, by comparison principle we deduce that $u_\epsilon \leq u_\epsilon$ in $\Omega$. This gives the lower bound for $u$. To obtain the upper bound, we set

$$\bar{u}_\epsilon = (\Gamma d + \epsilon') \log^{1/(p-\beta)} \left( \frac{L}{\Gamma d + \epsilon'} \right) - \epsilon' \log^{1/(p-\beta)} \left( \frac{L}{\epsilon'} \right).$$

Then, proceeding as in the previous case and after simplification, we get

$$-\Delta_p \bar{u}_\epsilon \geq \frac{\Gamma^{q-1}}{\Gamma d + \epsilon'} \left( \log \frac{L}{\Gamma d + \epsilon'} \right)^{\frac{q-1}{p-\beta}} \left[ -M(\Gamma d + \epsilon') \log \frac{L}{\Gamma d + \epsilon'} \left( 1 - \frac{1}{(p-\beta) \log \frac{L}{\Gamma d + \epsilon'}} \right)^{q-1} \right.$$\n
$$\left. + \frac{\Gamma(p-1-\beta)(q-1)}{(p-\beta)^2} \left( \log \frac{L}{\Gamma d + \epsilon'} \right)^{\frac{q-1}{p-\beta}} \left( 1 - \frac{1}{(p-\beta) \log \frac{L}{\Gamma d + \epsilon'}} \right)^{-1} \right].$$

And proceeding similarly,

$$-\Delta_q \bar{u}_\epsilon \geq \frac{\Gamma^{q-1}}{\Gamma d + \epsilon'} \left( \log \frac{L}{\Gamma d + \epsilon'} \right)^{\frac{q-1}{p-\beta}} \left[ -M(\Gamma d + \epsilon') \log \frac{L}{\Gamma d + \epsilon'} \left( 1 - \frac{1}{(p-\beta) \log \frac{L}{\Gamma d + \epsilon'}} \right)^{q-1} \right.$$\n
$$\left. + \frac{\Gamma(p-1-\beta)(q-1)}{(p-\beta)^2} \left( \log \frac{L}{\Gamma d + \epsilon'} \right)^{\frac{q-1}{p-\beta}} \left( 1 - \frac{1}{(p-\beta) \log \frac{L}{\Gamma d + \epsilon'}} \right)^{-1} \right].$$

We reduce $\varrho$ further so that $(\Gamma d + \epsilon') \log \frac{L}{\Gamma d + \epsilon'} \leq \frac{\Gamma(q-1)}{2M(p-\beta)}$ and $\log \frac{L}{\Gamma d + \epsilon'} \geq 2/(p-\beta)$ in $\Omega$, thus the quantity in the bracket is bounded from below by some positive constant $c$. Therefore,

$$-\Delta_p \bar{u}_\epsilon - \Delta_q \bar{u}_\epsilon \geq \frac{c\Gamma^{q-1}}{\Gamma d + \epsilon'} \left( \log \frac{L}{\Gamma d + \epsilon'} \right)^{\frac{q-1}{p-\beta}}.$$

Combining the approach of previous case with the assertions and arguments used in the case of supersolution in lemma 2.2, we obtain the required upper bound. This completes proof of the lemma.

**Lemma 2.4** Let $\beta + \delta \geq 1$ and $\beta \in [0, p)$, then the sequence $\{u_\epsilon^{(p+\delta-1)/(p-\beta)}\}$ is uniformly bounded in $W_0^{1,p}(\Omega)$. Moreover, $\{u_\epsilon\}$ is uniformly bounded in $W_{loc}^{1,p}(\Omega)$.

**Proof.** We first consider the case $\beta + \delta > 1$ and take $u_\gamma^\gamma$ as a test function in the weak formulation of (2.2) for some $\gamma > 0$. Therefore,

$$\int_\Omega |\nabla u_\gamma|^{p-2} \nabla u_\gamma \nabla u_\gamma^\gamma + \int_\Omega |\nabla u_\gamma|^{q-2} \nabla u_\gamma \nabla u_\gamma = \int_\Omega f_\epsilon(x) \frac{u_\gamma^\gamma}{(u_\epsilon + \epsilon)^\delta} \leq \int_\Omega f(x) u_\gamma^{\gamma - \delta}. \quad (2.6)$$

We first observe that

$$\int_\Omega |\nabla u_\gamma|^{p-2} \nabla u_\gamma \nabla u_\gamma^\gamma = \gamma \left( \frac{p}{p + \gamma - 1} \right)^p \int_\Omega |\nabla u_\epsilon^{(p+\gamma-1)/p}|^p$$
and similar result holds for the second term on the left of (2.6). Owing to (1.4) and behavior of $u_\varepsilon$ near the boundary proved in lemma 2.3 from [26], we infer that

$$\gamma \left( \frac{p}{p - \gamma - 1} \right)^p \int_\Omega |\nabla u_\varepsilon|^{(p-\gamma)/p} \leq C \int_\Omega d(x)^{-\beta + \frac{(\gamma - \delta)(p-\beta)}{p - \beta - 1}} dx,$$

the right side quantity is finite if and only if $\gamma > \delta + \frac{(\beta-1)(p-1+\delta)}{p-\beta}$. Thus, $u_\varepsilon^\rho \in W^{1,p}_0(\Omega)$ is uniformly bounded for all $\rho > \frac{1}{p} \left( p - 1 + \delta + \frac{(\beta-1)(p-1+\delta)}{p-\beta} \right) = \frac{(p-1)(p-1+\delta)}{p(p-\beta)}$. For the case $\beta + \delta = 1$, we take $u_\varepsilon^\delta$ as a test function in the weak formulation of $(P)$ and notice that the right hand side can be made independent of $u_\varepsilon$ and the function $d^{-\delta}$ is integrable, since $\beta < 1$. Proceeding similarly, we obtain $\{u_\varepsilon^{(\beta-1)/\beta}\}$ is uniformly bounded in $W^{1,p}_0(\Omega)$.

Next, we will prove the existence of unique weak solution to $(P)$ when $\beta + \delta < 1$. To construct a suitable subsolution for this case, we recall the following proposition proved by Papageorgiou et al. [31]. The main ingredient of the proof is strong maximum principle of Pucci and Serrin [32] and the strong comparison principle for general quasilinear elliptic equations. For this purpose, we define the following set

$$\text{int } C_+ := \{ u \in C^1(\overline{\Omega}) : u > 0 \text{ in } \Omega, u = 0 \text{ on } \partial \Omega, \frac{\partial u}{\partial \nu}|_{\partial \Omega} < 0 \}.$$

**Lemma 2.5 [31, Proposition 10]** For all $\rho > 0$, there exists a unique solution $\tilde{u}_\rho \in \text{int } C_+$ to the following problem

$$-\Delta_p u - \Delta_q u = \rho \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad (2.7)$$

Furthermore, the map $\rho \mapsto \tilde{u}_\rho$ is increasing from $(0, 1]$ to $C_0^1(\overline{\Omega})$ and $\tilde{u}_\rho \to 0$ in $C_0^1(\overline{\Omega})$ as $\rho \to 0^+$.

**Lemma 2.6** Let $\beta + \delta < 1$, then there exists a unique weak solution $u \in W^{1,p}_0(\Omega)$ of $(P)$.

**Proof.** We define the energy functional $I : W^{1,p}_0(\Omega) \to \mathbb{R}$ associated to $(P)$ as follows

$$I(u) := \frac{1}{p} \int_\Omega |\nabla u|^p dx + \frac{1}{q} \int_\Omega |\nabla u|^q dx - \frac{1}{1 - \delta} \int_\Omega f(x)|u|^{1-\delta} dx.$$

An easy consequence of Young and Hardy inequality, for any $\varepsilon > 0$, implies that

$$\frac{1}{1 - \delta} \int_\Omega f(x)|u|^{1-\delta} dx \leq \varepsilon \int_\Omega \left( \frac{|u|}{d} \right)^p + C(\varepsilon) \int_\Omega |u|^{\frac{p(1-\delta-\beta)}{p-\beta}} + C(\varepsilon) \int_\Omega |u|^{\frac{p(1-\delta-\beta)}{p-\beta}}.$$

Using the fact that $p(1 - \delta - \beta)/(p - \beta) < p$, we infer that $I$ is coercive and weakly lower semicontinuous in $W^{1,p}_0(\Omega)$. Moreover, $I$ is strictly convex on $W^{1,p}_0(\Omega)_+$, the positive cone of
there exists a unique global minimizer \( u \in W^{1,p}_0(\Omega) \) of \( I \) and without loss of generality we may assume \( u \geq 0 \) a.e. in \( \Omega \). Now, we will prove that \( u \) is in fact a solution of \((P)\). For fixed \( \rho > 0 \), let \( \tilde{u}_\rho \) be the unique solution of \((2.7)\) obtained in lemma \ref{lem:2.5}.

We observe that \( I \) is differentiable at \( \tilde{u}_\rho \), because \( \tilde{u}_\rho \in \text{int } C_+ \), and hence

\[
I'(\tilde{u}_\rho) = -\Delta \tilde{u}_\rho - \Delta_q \tilde{u}_\rho - f(x)\tilde{u}_\rho^{-\delta} = \rho - f(x)\tilde{u}_\rho^{-\delta} < 0,
\]

for \( \rho > 0 \) sufficiently small, since \( \tilde{u}_\rho \to 0 \) in \( C^1_0(\overline{\Omega}) \) as \( \rho \to 0^+ \). Set \( w = (\tilde{u}_\rho - u)^+ \) and \( \xi(t) = I(u + tw) \) for \( t > 0 \). Due to the fact \( u + tw \geq t\tilde{u}_\rho \) for \( t \in (0, 1] \) and Hardy inequality, we obtain that \( \xi \) is differentiable in \((0, 1]\). Since \( \xi \) is strictly convex, we have \( t \mapsto \xi'(t) \) is nonnegative and nondecreasing. Therefore,

\[
0 \leq \xi'(1) - \xi'(t) \leq \xi'(1) = I'(\tilde{u}_\rho) < 0,
\]

a contradiction if support of \( v \) has non zero measure. Thus, \( \tilde{u}_\rho \leq u \) in \( \Omega \) and since \( \tilde{u}_\rho \in \text{int } C_+ \), we get \( c_1 d(x) \leq u \). This implies that \( I \) is Gâteaux differentiable at \( u \), therefore \( u \) is a weak solution of \((P)\).

We prove behavior of the solution near the boundary, for this we first prove the following proposition.

**Proposition 2.7** Let \( u \in W^{1,p}_0(\Omega) \cap L^\infty_{\text{loc}}(\Omega) \) be a weak solution of the problem \((P)\) with \( \beta + \delta < 1 \). Then, there exists a constant \( C > 0 \) such that

\[
0 \leq u(x) \leq C \, d(x) \quad \text{in } \Omega.
\]

**Proof.** To prove the proposition, we will construct a suitable super solution to \((P)\). For this purpose, we recall the following observations from \cite[Lemma A.7]{13}: there exists a \( C^1 \) function \( \Theta_\alpha : [0, R_\alpha) \to [0, \infty) \) satisfying

\[
\begin{align*}
-\frac{d}{dr} \left( |\Theta'_\alpha(r)|^{p-2}\Theta'_\alpha(r) \right) &= \Theta_\alpha(r)^{-\delta-\beta}, \quad 0 < r < R_\alpha \\
\Theta_\alpha(0) &= 0, \quad \Theta'_\alpha(0) = \alpha > 0,
\end{align*}
\]

where \( R_\alpha > 0 \) is the supremum of all \( s \in (0, \infty) \) such that \( \Theta'_\alpha(s) > 0 \). We also observe that \( \Theta_\alpha \) is strictly increasing and \( \Theta'_\alpha \) is strictly decreasing in \([0, R_\alpha)\). By making the substitution

\[
\Theta_\alpha(r) = \frac{\alpha^{\frac{p}{\beta+\delta}}}{\Theta_1(\alpha^{\frac{p}{\beta+\delta}} r)}, \quad 0 \leq r \leq R_\alpha, \quad R_\alpha = \alpha^{\frac{p}{\beta+\delta}} R_1,
\]

we can choose \( R_\alpha > 0 \) such that \( R_\alpha > \text{diam}(\Omega) \). Here \( \Theta_1 \) and \( R_1 \) are given by \cite[(A.19)]{13} and \cite[(A.20)]{13}, respectively. An easy computation yields

\[
-\frac{d}{dr} \left( |\Theta'_1(r)|^{q-2}\Theta'_1(r) \right) \geq 0, \quad 0 < r < R_1
\]
and the same is true when $\Theta_1$ is replaced by $\Theta_\alpha$. Define $w = \Gamma \Theta_\alpha(d)$ in $\Omega$, where $\Gamma > 1$ (to be chosen later). Then,
\[ \nabla w = \Gamma \Theta_\alpha'(d) \nabla d. \]

Therefore, by observing the fact that $|\nabla d| = 1$, for $\psi \in C_c^\infty(\Omega)$ with $\psi \geq 0$, we deduce that
\[
\int_{\Omega_\psi} -\Delta_p w \psi = \Gamma^{p-1} \int_{\Omega} \Theta_\alpha'(d)^{p-1} \nabla d \nabla \psi = \Gamma^{p-1} \int_{\Omega_\psi} \left( -\left( \Theta_\alpha'(d)^{p-1} \right)' + \Theta_\alpha'(d)^{p-1}(-\Delta d) \right) \psi \\
\geq \Gamma^{p-1} \int_{\Omega_\psi} \left( -\left( \Theta_\alpha'(d)^{p-1} \right)' - M \Theta_\alpha'(d)^{p-1} \right) \psi.
\]
\[ \tag{2.10} \]

Similar calculation yields
\[
\int_{\Omega_\psi} -\Delta_q w \psi = \Gamma^{q-1} \int_{\Omega} \Theta_\alpha'(d)^{q-1} \nabla d \nabla \psi \geq \Gamma^{q-1} \int_{\Omega_\psi} \left( -\left( \Theta_\alpha'(d)^{q-1} \right)' - M \Theta_\alpha'(d)^{q-1} \right) \psi.
\]
\[ \tag{2.11} \]

Therefore, coupling (2.10) and (2.11) and using (2.8) together with (2.9), we get
\[
-\Delta_p w - \Delta_q w \geq \Gamma^{p-1} \left[ -\left( \Theta_\alpha'(d)^{p-1} \right)' - M \Theta_\alpha'(d)^{p-1} \right] + \Gamma^{q-1} \left[ -\left( \Theta_\alpha'(d)^{q-1} \right)' - M \Theta_\alpha'(d)^{q-1} \right] \\
\geq \Gamma^{p-1} \left[ \Theta_\alpha(d)^{-\beta - \delta} - M \Theta_\alpha'(d)^{p-1} \right] - \Gamma^{q-1} M \Theta_\alpha'(d)^{q-1},
\]
weakly in $\Omega_\psi$. Since $\Theta_\alpha$ is strictly increasing and $\Theta_\alpha'$ is strictly decreasing together with $\Theta_\alpha(0) = 0$ and $\Theta_\alpha'(0) = \alpha$, we obtain $\Theta_\alpha(d) \leq \alpha d$ and $\alpha^{p-1} \geq \Theta_\alpha'(d)^{p-1}$. Therefore, if necessary, we can further reduce $\varrho > 0$ such that the following holds
\[
\Theta_\alpha(d)^{-\beta - \delta} - M \Theta_\alpha'(d)^{p-1} - M \Theta_\alpha'(d)^{q-1} \geq c \Theta_\alpha(d)^{-\beta - \delta} \quad \text{in } \Omega_\varrho,
\]
for some positive constant $c$. Thus,
\[
-\Delta_p w - \Delta_q w \geq \Gamma^{p-1} \Theta_\alpha(d)^{-\beta - \delta} \geq c \alpha^{-\beta} d^{-\beta} \Gamma^{p-1} \Theta_\alpha(d)^{-\delta},
\]
where we used the relation $\Theta_\alpha(d) \leq \alpha d$. Choosing $\Gamma > 1$ large enough so that $c \alpha^{-\beta} \Gamma^{p-1} \geq c_2 \Gamma^{-\delta}$, we obtain
\[
-\Delta_p w - \Delta_q w \geq f(x) w^{-\delta} \quad \text{in } \Omega_\varrho.
\]

By the fact that $u \in L^\infty_{loc}(\Omega)$, we have
\[
\Gamma \Theta_\alpha(d) \geq \Gamma \Theta_\alpha(g) \geq \|u\|_{L^\infty(\Omega \setminus \Omega_\varrho)} \geq u(x) \quad \text{in } \Omega \setminus \Omega_\varrho,
\]
for sufficiently large $\Gamma$. Therefore, by comparison principle, we get
\[
u \leq w = \Gamma \Theta_\alpha(d) \leq \Gamma \alpha d \quad \text{in } \Omega.
\]
This completes proof of the proposition. \qed
Remark 2.8 We remark that the proof of Proposition 2.7 can be used to obtain similar bounds on the bounded weak solution to the following problem, for $\delta < 1$,
\[
\left\{-\Delta_p u - \Delta_q u = \lambda u^{-\delta} + u^{r-1}, \quad u > 0 \text{ in } \Omega; \quad u = 0 \text{ on } \partial \Omega,\right.\]
where $\lambda > 0$ and $r \leq p^* - 1 = \frac{np}{N-p} - 1$. Indeed, since $u \in L^\infty(\Omega)$, we have $u(x) \leq \|u\|_\infty$ and hence
\[
-\Delta_p u - \Delta_q u = \lambda u^{-\delta} + u^{r-1} \leq \lambda(1 + \lambda^{-1}\|u\|_{p^{-1}})u^{-\delta} := \hat{\lambda}u^{-\delta},
\]
where $\hat{\lambda} = \lambda(1 + \lambda^{-1}\|u\|_{p^{-1}})$. Then, rest of the proof follows similarly, by observing the fact that if we take $u$ as a subsolution of $(P)$ instead of weak solution the proof of Proposition 2.7 does not change.

Proof of Theorem 1.4 For the case $\beta + \delta < 1$, by means of lemma 2.6 we get the existence of a weak solution $u \in W^{1,p}_0(\Omega)$ satisfying $c_1d(x) \leq u(x)$ a.e. in $\Omega$. Next, since $\beta + \delta < 1$, following the procedure of [21] Lemma 3.2, we obtain $u \in L^\infty(\Omega)$. Then, applying Proposition 2.7 we obtain $u(x) \leq Cd(x)$ in $\Omega$.

For the case $\beta + \delta \geq 1$, due to lemma 2.4, the sequence $\{u_\epsilon^{(p+\delta-1)/(p-\beta)}\}$ is uniformly bounded in $W^{1,p}_0(\Omega)$. Therefore, we can extract a subsequence, still denoting by $u_\epsilon$, such that $u_\epsilon(x) \rightarrow u(x)$ a.e. in $\Omega$, for some $u \in W^{1,p}_0(\Omega)$. By the local H"older regularity result of Lieberman [26] Theorem 1.7, we obtain the sequence $u_\epsilon$ converges to $u$ in $C^1_{\text{loc}}(\Omega)$. Therefore, $u$ satisfies equation $(P)$ in the sense of distribution. Moreover, from lemmas 2.2 and 2.3 we deduce that
\[
\eta d \log \frac{u^\beta}{d} \left(\frac{A}{d}\right) \leq u \leq \Gamma d \log \frac{u^{\beta}}{d} \left(\frac{A}{d}\right) \quad \text{if } \beta + \delta = 1,
\]
\[
\eta d(x)^{\frac{p-\beta}{p-1+\delta}} \leq u(x) \leq \Gamma d(x)^{\frac{p-\beta}{p-1+\delta}} \quad \text{if } \beta + \delta > 1.
\]
Repeating the proof of lemma 2.4 and using above comparison estimates, we see that $u^{\frac{p+\delta-1}{p-\beta}} \in W^{1,p}_0(\Omega)$. Thus, $u$ is a weak solution to problem $(P)$ in the sense of definition 1.1.

On account of the fact that the minimal weak solution (thus obtained) exhibits aforementioned behavior near the boundary, taking $\lim_{x \to x_0 \in \partial \Omega} u(x)$, we get $u \in C_0(\Omega)$, thus $u \in C_{\beta,\delta}$. For the last part of the theorem, suppose $u^\rho \in W^{1,p}_0(\Omega)$, for some $\rho \geq 1$. Then from the weak formulation, it is clear that $\int_\Omega f(x)u^{\rho - \delta} < \infty$. Using behavior of $u$ near the boundary, we see that this is equivalent to $-\beta + (\rho - \delta)\frac{p-\beta}{p-1+\delta} > -1$, this gives us $\rho > \frac{(p-1)(\beta+\delta-1)}{p-\beta} := \rho_0$. Furthermore, we note that $u \in W^{1,p}_0(\Omega)$ if $\rho_0 < 1$, which yields $\delta < 2 + \frac{1-\beta p}{p-1}$. This completes proof of the theorem.

\[\Box\]

3 Comparison principle and non-existence result

In this section, we first establish a comparison principle for weak sub and super solution of $(P)$ and as a consequence of this, we obtain the uniqueness result. We remark that the proof
of weak comparison principle when \( u, v \in W^{1,p}_0(\Omega) \), is much simpler and it follows by taking \((u - v)^+\) as a test function in the weak formulation of \((P)\).

**Proof of Theorem 1.5.** For fixed \( m > 0 \), we define \( g_m : \mathbb{R} \rightarrow \mathbb{R}^+ \) as follows

\[
g_m(s) := \begin{cases} 
\min\{s^{-\delta}, m\} & \text{if } s > 0 \\
m & \text{otherwise.}
\end{cases}
\]

Let \( \Upsilon_m \) be the primitive of \( g_m \) such that \( \Upsilon_m(1) = 0 \). We define a functional \( \mathcal{I}_m : W^{1,p}_0(\Omega) \rightarrow \mathbb{R} \cup \{\infty\} \) as

\[
\mathcal{I}_m(\phi) := \frac{1}{p} \int_\Omega |\nabla \phi|^p \, dx + \frac{1}{q} \int_\Omega |\nabla \phi|^q \, dx - \int_\Omega f(x) \Upsilon_m(\phi) \, dx,
\]

for all \( \phi \in W^{1,p}_0(\Omega) \). Set

\[
\mathcal{M} := \{\phi \in W^{1,p}_0(\Omega) : 0 \leq \phi \leq v \text{ a.e. in } \Omega\},
\]

which is a closed and convex set. First we observe that for any bounded sequence \( \{u_n\} \in \mathcal{M} \) and \( \theta > 0 \), to be chosen later,

\[
\int_\Omega d^{-\beta} u_n \, dx \leq \left( \int_\Omega \left( \frac{u_n}{d}\right)^p \right)^{\frac{1}{p}} \left( \int_\Omega u_n^{\frac{q}{p}} \right)^{\frac{1}{q}} \left( \int_\Omega d^{(1-\beta-\theta)} \right)^{\frac{1}{p}} \leq C \left( \int_\Omega u_n^{\frac{q}{p}} \right)^{\frac{1}{q}} \left( \int_\Omega d^{(1-\beta-\theta)} \right)^{\frac{1}{p}},
\]

where \( r < p^* \), if \( p < N \), \( \frac{1-q}{p} + \frac{q}{p} + 1 = 1 \) and in the last inequality, we used Hardy inequality and boundedness of \( \{u_n\} \) in \( W^{1,p}_0(\Omega) \). This requires \( (1 - \beta - \theta)l > -1 \), which is equivalent to \( \theta < \frac{2pr-pr\beta-r}{pr-r+p} \). Due to the fact that \( \beta < 2 - 1/p \) and by above observation, it is easy to deduce that \( \mathcal{I}_m \) is weakly lower semicontinuous on \( \mathcal{M} \). Therefore, there exists a minimizer \( w \) of \( \mathcal{I}_m \) in \( \mathcal{M} \) and the following holds

\[
\int_\Omega(|\nabla w|^{p-2} + |\nabla w|^{q-2}) \nabla w \nabla (\phi - w) \, dx \geq \int_\Omega f(x) \Upsilon_m'(w)(\phi - w) \, dx \tag{3.1}
\]

for \( \phi \in w + \left( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \right) \) with \( 0 \leq \phi \leq v \) a.e. in \( \Omega \).

**Step I:** We claim that, for all \( \phi \in C^\infty(\Omega) \) with \( \phi \geq 0 \), there holds

\[
\int_\Omega(|\nabla w|^{p-2} + |\nabla w|^{q-2}) \nabla w \nabla (\phi(t) - w) \, dx \geq \int_\Omega f(x) \Upsilon_m'(w)(\phi(t) - w) \, dx. \tag{3.2}
\]

Let \( h \in C^\infty(\mathbb{R}) \) such that \( 0 \leq h \leq 1 \), \( h \equiv 1 \) in \([-1,1]\) and \( \text{supp}(h) \subset (-2,2) \). Now, for \( \phi \in C^\infty(\Omega) \) satisfying \( \phi \geq 0 \) in \( \Omega \), we define \( \phi_k := h(\frac{\phi}{k}) \phi \) and \( \phi_{k,t} := \min\{w + t \phi_k, v\} \), for \( k \geq 1 \) and \( t > 0 \). It is easy to observe that \( \phi_{k,t} \in w + \left( W^{1,p}_0(\Omega) \cap L^\infty(\Omega) \right) \) with \( 0 \leq \phi_{k,t} \leq v \) a.e. in \( \Omega \). From \( \text{3.1} \), we infer that

\[
\int_\Omega(|\nabla w|^{p-2} + |\nabla w|^{q-2}) \nabla w \nabla (\phi_{k,t} - w) \, dx \geq \int_\Omega f(x) \Upsilon_m'(w)(\phi_{k,t} - w) \, dx.
\]
Therefore, from (3.3), we obtain

\[ c \int_{\Omega} (|\nabla w| + |\nabla \phi_{k,t}|)^{p-2} |\nabla (\phi_{k,t} - w)|^2 \leq \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} \nabla \phi_{k,t} - |\nabla w|^{p-2} \nabla w) \nabla (\phi_{k,t} - w) dx \]
\[ + \int_{\Omega} (|\nabla \phi_{k,t}|^{q-2} \nabla \phi_{k,t} - |\nabla w|^{q-2} \nabla w) \nabla (\phi_{k,t} - w) dx \]
\[ \leq \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} + |\nabla \phi_{k,t}|^{q-2}) \nabla \phi_{k,t} \nabla (\phi_{k,t} - w) \]
\[ - \int_{\Omega} f(x) \Upsilon_m'(w)(\phi_{k,t} - w) dx. \]

This implies that

\[ c \int_{\Omega} (|\nabla w| + |\nabla \phi_{k,t}|)^{p-2} |\nabla (\phi_{k,t} - w)|^2 \leq \int_{\Omega} f(x) (\Upsilon_m'(\phi_{k,t}) - \Upsilon_m'(w))(\phi_{k,t} - w) \]
\[ \leq \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} + |\nabla \phi_{k,t}|^{q-2}) \nabla \phi_{k,t} \nabla (\phi_{k,t} - w - t \phi_k) \]
\[ - \int_{\Omega} f(x) \Upsilon_m'(\phi_{k,t})(\phi_{k,t} - w - t \phi_k) \]
\[ + t \left[ \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} + |\nabla \phi_{k,t}|^{q-2}) \nabla \phi_{k,t} \nabla \phi_k \right] \]
\[ - \int_{\Omega} f(x) \Upsilon_m'(\phi_{k,t}) \phi_k \right]. \]

Simplifying it further and using the observation that the first term on the left is nonnegative, we obtain

\[ - \int_{\Omega} f(x) (\Upsilon_m'(\phi_{k,t}) - \Upsilon_m'(w))(\phi_{k,t} - w) \leq \int_{\Omega} (|\nabla v|^{p-2} + |\nabla v|^{q-2}) \nabla v \nabla (v - w - t \phi_k) \]
\[ - \int_{\Omega} f(x) \Upsilon_m'(\phi_{k,t})(\phi_{k,t} - w - t \phi_k) \]
\[ + t \left[ \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} + |\nabla \phi_{k,t}|^{q-2}) \nabla \phi_{k,t} \nabla \phi_k \right]
\[ - \int_{\Omega} f(x) \Upsilon_m'(\phi_{k,t}) \phi_k \right]. \]

From the definition of \( \Upsilon_m \), it is clear that \( v \) is a super solution to the following equation

\[ - \Delta_p v - \Delta_q v = \Upsilon_m'(v). \]

Therefore, from (3.3), we obtain

\[ - \int_{\Omega} f(x) (\Upsilon_m'(\phi_{k,t}) - \Upsilon_m'(w))(\phi_{k,t} - w) \leq t \left[ \int_{\Omega} (|\nabla \phi_{k,t}|^{p-2} + |\nabla \phi_{k,t}|^{q-2}) \nabla \phi_{k,t} \nabla \phi_k \right]
\[ - \int_{\Omega} f(x) \Upsilon_m'(\phi_{k,t}) \phi_k \right]. \]

Since supports of \( \phi_{k,t} - w \) and \( \phi_k \) are compact, using dominated convergence theorem, we pass the limit \( t \to 0 \). Thus,

\[ \int_{\Omega} (|\nabla w|^{p-2} + |\nabla w|^{q-2}) \nabla w \nabla \phi_k \]
\[ - \int_{\Omega} f(x) \Upsilon_m'(w) \phi_k \geq 0. \]
Taking $k \to \infty$, we complete the proof of (3.2).

**Step II:** In this step we will show that $u \leq w + \epsilon$ in $\Omega$ for all $\epsilon > 0$.

Since $w \in W^{1,p}_0(\Omega)$, the function $(u - w - \epsilon)^+$ is in $W^{1,p}_0(\Omega)$. By density argument and Fatou lemma, we see that (3.2) holds if $T_k((u - w - \epsilon)^+)$ is taken as a test function, that is,

$$
\int_\Omega (|\nabla w|^{p-2} + |\nabla w|^{q-2}) \nabla w \nabla T_k((u - w - \epsilon)^+) \geq \int_\Omega f(x) \Gamma'_m(w) T_k((u - w - \epsilon)^+),
$$

where $T_k(s) = \min\{s, k\}$. Let $\tilde{\phi}_n \in C^\infty(\Omega)$ be such that $\tilde{\phi}_n \to (u - w - \epsilon)^+$ in $W^{1,p}_0(\Omega)$. Set $\phi_{n,k} := T_k(\min\{(u - w - \epsilon)^+, \tilde{\phi}_n^+\})$. It is easy to observe that $\phi_{n,k} \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega)$, therefore by density argument, we obtain

$$
\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \nabla \phi_{n,k} \leq \int_\Omega f(x) u^{-\delta} \phi_{n,k}.
$$

Consequently, using dominated convergence theorem, we get

$$
\int_\Omega (|\nabla u|^{p-2} + |\nabla u|^{q-2}) \nabla u \nabla T_k((u - w - \epsilon)^+) \leq \int_\Omega f(x) u^{-\delta} T_k((u - w - \epsilon)^+). \tag{3.5}
$$

For $m > \epsilon^{-\delta}$, proceeding similar to lemma 2.4, subtracting (3.5) from (3.4), we deduce that

$$
c \int_\Omega (|\nabla u| + |\nabla w|)^{p-2} \nabla T_k((u - w - \epsilon)^+) \leq \int_\Omega f(x)(u^{-\delta} - \Gamma'_m(w)) T_k((u - w - \epsilon)^+)
$$

$$
\leq \int_\Omega f(x)(\Gamma'_m(u) - \Gamma'_m(w)) T_k((u - w - \epsilon)^+)
$$

$$
\leq 0,
$$

where the last inequality holds in the support of $(u - w - \epsilon)^+$. This implies that $T_k((u - w - \epsilon)^+) = 0$ a.e. in $\Omega$ and since it is true for every $k > 0$, we get $u \leq w + \epsilon$ in $\Omega$. By the arbitrariness of $\epsilon$ and the fact that $w \leq v$, we obtain the required result of the lemma. \(\square\)

**Corollary 3.1** Let $\beta < 2 - 1/p$, then there exists a unique weak solution of \((P)\).

**Proof.** Suppose there exist two weak solutions $u$ and $v$ of problem \((P)\) in $W^{1,p}_{loc}(\Omega)$. Then, we can treat $u$ as a subsolution and $v$ as a supersolution to \((P)\). Consequently, the comparison principle implies $u \leq v$ a.e. in $\Omega$. Reversing the role, we get $u = v$ a.e. in $\Omega$. \(\square\)

**Proof of Theorem 1.6** On the contrary suppose there exists a solution $u_0 \in W^{1,p}_{loc}(\Omega)$ of \((P)\) and $\gamma_0 \geq 1$ such that $u_0^\gamma_0 \in W^{1,p}_{loc}(\Omega)$. From (1.4), we have

$$
c_1 d(x)^{-\beta} \leq f(x) \leq c_2 d(x)^{-\beta} \quad \text{in } \Omega_\delta.
$$

For $\tilde{\beta} < p$, which we will specify later, we choose $f_{\tilde{\beta}} \in L^\infty_{loc}(\Omega)$ such that $mf_{\tilde{\beta}}(x) \leq f(x)$ a.e. in $\Omega$, for some constant $m \in (0,1)$ independent of $\tilde{\beta}$ (this can be achieved by choosing $m < c_1/c_2$), and

$$
c_1 d(x)^{-\tilde{\beta}} \leq mf_{\tilde{\beta}}(x) \leq c_2 d(x)^{-\tilde{\beta}} \quad \text{in } \Omega_\delta.
$$
Now we will construct a suitable subsolution near the boundary \( \partial \Omega \), to arrive at some contradiction. For \( \epsilon > 0 \), let \( w_\epsilon \in W^{1,p}_0(\Omega) \) be the unique solution to the following problem

\[
-\Delta_p w_\epsilon - \Delta w_\epsilon = m f_{\beta, \epsilon}(x) (w_\epsilon + \epsilon)^{-\delta}, \tag{3.6}
\]

where \( f_{\beta, \epsilon}(x) := (f_{\beta}(x) \epsilon^\delta + \epsilon^{\frac{p-2+1}{\delta}})^{-\beta} \) if \( f_{\beta}(x) > 0 \) and 0 otherwise.

Next, we will prove that \( w_\epsilon \leq u_0 \) in \( \Omega \). We observe that \( w_\epsilon \in C^{1,\alpha}(\Omega) \) and \( w_\epsilon = 0 \) on \( \partial \Omega \). Therefore, for given \( \sigma > 0 \), there exists \( \rho > 0 \) such that \( w_\epsilon \leq \sigma/2 \) in \( \Omega_\rho \). Moreover, \( w_\epsilon - u_0 - \sigma \leq -\sigma/2 < 0 \) in \( \Omega_\rho \), because \( u_0 \geq 0 \), we have

\[\text{supp}(w_\epsilon - u_0 - \sigma)^+ \subset \Omega \setminus \Omega_\rho \in \Omega.\]

Therefore, \( (w_\epsilon - u_0 - \sigma)^+ \in W^{1,p}_0(\Omega) \) and from the weak formulation of (3.6), we obtain

\[
\int_\Omega (|\nabla w_\epsilon|^p - 2 |\nabla w_\epsilon|^{q-2}) \nabla w_\epsilon \nabla k \left( (w_\epsilon - u_0 - \sigma)^+ \right) = \int_\Omega \frac{m f_{\beta, \epsilon}(x)}{(w_\epsilon + \epsilon)^\delta} T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right), \tag{3.7}
\]

where \( T_k(s) := \min\{s, k\} \) for \( k > 0 \) and \( s \geq 0 \). Furthermore, since \( u_0 \in W^{1,p}_0(\Omega) \) is a weak solution to \((P)\), for all \( \psi \in C_c^\infty(\Omega) \), we have

\[
\int_\Omega (|\nabla u_0|^p - 2 |\nabla u_0|^{q-2}) \nabla u_0 \nabla \psi = \int_\Omega f(x) u_0^\delta \psi. \tag{3.8}
\]

Let \( \psi_n \in C_c^\infty(\Omega) \) be such that \( \psi_n \rightarrow (w_\epsilon - u_0 - \sigma)^+ \) in \( W^{1,p}_0(\Omega) \). Set \( \tilde{\psi}_{n,k} := T_k \left( \min\{(w_\epsilon - u_0 - \sigma)^+, \psi_n^+\} \right). \) Then, \( \tilde{\psi}_{n,k} \in W^{1,p}_0(\Omega) \cap L_c^\infty(\Omega) \), therefore from (3.8), we infer that

\[
\int_\Omega (|\nabla u_0|^p - 2 |\nabla u_0|^{q-2}) \nabla u_0 \nabla \tilde{\psi}_{n,k} = \int_\Omega f(x) u_0^\delta \tilde{\psi}_{n,k}.
\]

Using the fact that \( \text{supp}(w_\epsilon - u_0 - \sigma)^+ \in \Omega \) and Fatou lemma, we obtain

\[
\int_\Omega (|\nabla u_0|^p - 2 |\nabla u_0|^{q-2}) \nabla u_0 \nabla T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right) \geq \int_\Omega f(x) u_0^\delta T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right) \geq \int_\Omega m f_{\beta, \epsilon}(x) u_0^\delta T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right). \tag{3.9}
\]

Taking into account (3.7) and (3.9), we deduce that

\[
\int_\Omega (|\nabla w_\epsilon|^p - 2 |\nabla w_\epsilon|^{q-2}) \nabla w_\epsilon \nabla k \left( (w_\epsilon - u_0 - \sigma)^+ \right) dx + \int_\Omega (|\nabla w_\epsilon|^p - 2 |\nabla w_\epsilon|^{q-2}) \nabla w_\epsilon \nabla T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right) dx \\
\leq \int_\Omega m f_{\beta, \epsilon}(x) \left( (w_\epsilon + \epsilon)^{-\delta} - u_0^{-\delta} \right) T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right) dx \\
\leq \int_\Omega m f_{\beta, \epsilon}(x) \left( w_\epsilon^{-\delta} - u_0^{-\delta} \right) T_k \left( (w_\epsilon - u_0 - \sigma)^+ \right) dx \leq 0.
\]
To estimate the quantities on the left side, we use the inequality (2.3), therefore
\[ C \int_{\Omega} (|\nabla w_\epsilon| + |\nabla u_0|)^{p-2} |\nabla T_k((w_\epsilon - u_0 - \sigma)^+)|^2 \leq 0, \]
this implies that \( T_k((w_\epsilon - u_0 - \sigma)^+) = 0 \) a.e. in \( \Omega \) and since it is true for every \( k > 0 \), we get \( w_\epsilon \leq u_0 + \sigma \) in \( \Omega \). Moreover, the arbitrariness of \( \sigma \) proves \( w_\epsilon \leq u_0 \) in \( \Omega \). Owing to the estimates of \( w_\epsilon \) given by lemma 2.2, we have
\[ \eta((d(x) + \epsilon (p-\beta)/(p+\beta) - \epsilon) \leq w_\epsilon(x)\leq u_0(x) \text{ in } \Omega. \]
Since \( u_0^{0p} \in W^{1,p}_0(\Omega) \), by Hardy inequality, we obtain
\[ \eta^{0p} \int_{\Omega} ((d(x) + \epsilon (p-\beta)/(p+\beta) - \epsilon)^{0p} \leq C \int_{\Omega} u_0^{0p} < \infty, \]
choosing \( \tilde{\beta} < p \) sufficiently close to \( p \) and taking \( \epsilon \downarrow 0 \), we obtain the quantity on the left side is not finite, which yields a contraction. This completes proof of the theorem. \( \square \)

4 Hölder regularity

In this section, first we study Hölder regularity results for weak solution of equation (1.5), which is general form of (P). The interior Hölder regularity for solutions of (1.5) follows from [26] Theorem 1.7, p.320. To prove regularity results up to the boundary, we flatten the boundary \( \partial \Omega \) by a \( C^2 \) diffeomorphism \( \Phi \) such that \( d(\Phi(x)) = (\Phi(x))_N \) for all \( x \in \mathbb{R}^N \), where \( d \) denotes the distance from an open ball centered at the origin with \( \Phi(x)_N \geq 0 \). For \( r > 0 \), we fix the following notation
\[ B^+_r(y) = \{ x \in \mathbb{R}^N : |x-y| < r, x_N - y_N > 0 \}, \quad B^0_r(y) = \{ x \in \mathbb{R}^N : |x-y| < r, x_N - y_N = 0 \}. \]
As a consequence of the above transformation, problem in (1.5) takes the following form
\[ -\text{div}A(x, \nabla u) = B(x, u, \nabla u) + g(x) \text{ in } B^+_1(0), \quad u = 0 \text{ on } B^+_0(0). \quad (4.1) \]
Conditions in Theorem 1.7 implies
(B1) \( 0 \leq g(x) \leq Cx_N^{-\sigma} \) for a.e. \( B^+_1(0) \).
(B2) \( 0 \leq u(x) \leq Cx_N \) for a.e. \( B^+_1(0) \).

For any \( x_0 \in B^+_{1/2}(0) \) and \( 0 < R < 1/2 \), we consider the following perturbed problem
\[ -\text{div}A(x_0, \nabla v) = 0 \text{ in } B^+_R(x_0), \quad v = u \text{ on } \partial B^+_R(x_0). \quad (4.2) \]
To estimate the various quantities involving the solution \( v \), we consider the normalized form of problem (4.2) with \( x_0 = 0 \in \mathbb{R}^N \), that is,
\[ -\text{div}A(0, \nabla v) = 0 \text{ in } B^+_R(0), \quad v = u \text{ on } \partial B^+_R(0). \quad (4.3) \]
In what follows, we denote \( B^+_r(0) \) as \( B^+_r \) for all \( r > 0 \).
Lemma 4.1 There exists a unique solution \( v \in W^{1,p}(B^+_R) \) of (4.3). Furthermore, the following hold

(i) \( \sup_{B^+_{r/2}} |\nabla v| \leq C \left( R^{-\frac{N}{p}} \|\nabla v\|_{L^p(B^+_R)} + \chi \right) \);

(ii) \( \text{osc}_{B^+_{r/2}} \nabla v \leq C \left( \frac{r}{R} \right)^\zeta \left( \sup_{B^+_{r/2}} |\nabla v| + \chi \right) \) for \( 0 < r < R/7 \);

(iii) \( \int_{B^+_R} |\nabla v|^p \leq C \int_{B^+_R} (1 + |\nabla u|)^p \ dx \);

(iv) \( \sup_{B^+_R} |u - v| \leq \text{osc}_{B^+_R} u \leq \sup_{B^+_R} u \leq CR \).

Here \( \chi > 0 \) is a constant which depends only on \( \Lambda, \nu \) and \( p \), and the constants \( C, \zeta > 0 \) depend only on \( \Lambda, \nu, p, N \) and \( \omega \).

**Proof.** Existence of the unique solution of (4.3) is standard. We first prove (i) and (iii). To prove (i), we will apply the local maximum principle proved by Trudinger in [36] for the case of cubes, however the same proof can be adopted to get similar results for the case of balls, see for instance remark at the end of [26, Theorem 1.2, p. 316]. Applying the aforementioned result on the nonnegative weak solution \( v \) of (4.3), for \( \rho \in (0,1) \), we have

\[
\sup_{B^+_{\rho R}} v \leq \frac{C}{(1 - \rho)^N} \left( R^{-\frac{N}{p}} \|v\|_{L^p(B^+_R)} + \chi R \right),
\]

where \( C, \chi > 0 \) are constants depending only on \( N, \Lambda, \nu, p, \|v\|_{L^\infty(B^+_R)} \). By means of Poincaré inequality (note that \( v = 0 \) on \( B^+_R \)), we obtain

\[
\sup_{B^+_{\rho R}} \frac{v}{R} \leq \frac{C}{(1 - \rho)^N} \left( R^{-\frac{N}{p}} \left( \int_{B^+_R} |\nabla v|^p \ dx \right)^{1/p} + \chi \right). \quad (4.4)
\]

To estimate the term on the left, we will use barrier argument as in [25, Section 2]. Let us fix \( r \in (0, R), \ x_0 \in B^+_{r/4} \) and set

\[
w(x) = 16r^{-2} \sup_{B^+_r} v \left( |x - (x_0, 0)|^2 + \frac{N \Lambda}{\nu} (r x_N - x^2_N) \right) \quad \text{for} \ x \in B^+_r.
\]

It is easy to observe that \( w \geq v \) on \( \partial B^+_{r/2} \) and by direct computation, we get

\[-\text{div} A(0, \nabla w) \geq 0 \quad \text{in} \ B^+_{r/2}.
\]

That is, \( w \) is a classical super solution, therefore employing the maximum principle, we obtain \( w \geq v \) in \( B^+_{r/2} \). Evaluating these at \( x = x_0 \), we obtain

\[\frac{v(x_0)}{x_0 N} \leq C r^{-1} \sup_{B^+_r} v,\]
thus, by arbitrariness of $x_0$, we get
\[
\sup_{B^+_r/4} \frac{v(x)}{x_N} \leq C \sup_{B^+_r} \frac{v(x)}{r}.
\]

Next, we estimate $\sup |\nabla v|$ by virtue of (4.5) in $B^+_{r/16}$. Let $x_0 \in B^+_{r/16} \setminus B^0_{R}$ and $x_1$ be the projection of $x_0$ on $B^0_{R}$. Then, we have $d := d(x_0, B^0_{R}) \leq r$. By the interior gradient estimate of [26, (5.3a)], we have
\[
G(|\nabla v(x_0)|) \leq C d^{-N} \int_{B^+_{d/4}(x_0)} G(|\nabla v|) dx,
\]
where $G(t) := t^{p} + t^{q}$ for $t \geq 0$. Using Caccioppoli inequality for interior balls (see [6, (2.12)]), we deduce that
\[
G(|\nabla v(x_0)|) \leq C \sup_{B^+_{d/2}(x_0)} G\left(\frac{v}{d}\right) \leq C \sup_{B^+_{d/4}(x_0)} G\left(\frac{v}{d}\right).
\]

Since $x_N/2 \leq d$ in $B_{d/2}(x_0)$ and $B_{d/2}(x_0) \subset B^+_{2d}(x_1) \subset B^+_{r/8}(x_1) \subset B^+_{r/4}$, from above inequality and (4.5), we obtain
\[
G(|\nabla v(x_0)|) \leq C \sup_{B^+_{d/2}(x_0)} G\left(\frac{v}{d}\right) \leq C \sup_{B^+_{r/4}(0)} G\left(\frac{v}{x_N}\right) \leq C \sup_{B^+_{r}} G\left(\frac{v}{r}\right).
\]

Using the invertibility of $G$ and arbitrariness of $x_0 \in B^+_{r/16}$, we get
\[
\sup_{B^+_{r/16}} |\nabla v| \leq C \sup_{B^+_{r}} \frac{v}{r}.
\]

On account of interior gradient estimate [26, (5.3a)] and Caccioppoli inequality in the interior, we have similar bound for interior balls too. Therefore, by covering argument, for suitable $r > 0$, we obtain
\[
\sup_{B^+_{R/2}} |\nabla v| \leq C \sup_{B^+_{R}} \frac{v}{R}.
\]

Now, coupling (4.4) and (4.6), we get the required result in (i). Proof of (iii) follows exactly on the same lines of proof of [25, Lemma 5, (3.3)].

To complete the proof of lemma 4.1 we need the following two lemmas. For $\rho, R > 0$, define the following sets
\[
G(\rho, R) := \{ x \in \mathbb{R}^N : |x'| < R, \ 0 < x_N < \rho R \} \quad \text{and}
\]
\[
G'(\rho, R) := \{ x \in \mathbb{R}^N : |x'| < R, \ \rho R < x_N < \frac{3}{2} \rho R \}.
\]
Lemma 4.2 Let $L$ be an elliptic operator of the form $Lu = a^{ij}D_{ij}u$ with $$\nu(|z|^{p-2} + |z|^{q-2})|\xi|^2 \leq \alpha^2(z)|z|^{p-2} + |z|^{q-2})|\xi|^2$$ for $x \in \mathbb{B}_1^+$, $z, \xi \in \mathbb{R}^N$, where $\Lambda, \nu$ are positive constants with $\nu \leq \Lambda$. Furthermore, let us assume $u \in C^2(B_1^+)$ be such that $0 \leq u \leq H x_N$ in $B_1^+$ and $Lu(x) = 0$. Then, for $\rho = \rho(N, \Lambda, \nu)$, small enough and $R < 1$, there exist positive constants $C$ and $\varsigma$, depending only on $N, \Lambda, \nu$, such that the following holds $$\text{osc}_{G(\rho, r)} \frac{u}{x_N} \leq C \left( \frac{r}{R} \right)^{\varsigma} \left( \text{osc}_{G(\rho, R)} \frac{u}{x_N} + \chi \right),$$ for $r \in (0, R)$.

Proof. For $r > 0$ and $i \in \{1, 2, 3, 4\}$, set the following $$m_i = \inf_{G(\rho, ir)} \frac{u}{x_N} \text{ and } M_i = \sup_{G(\rho, ir)} \frac{u}{x_N}.$$ Proceeding similar to [24, Lemma 5.2], we will apply the weak Harnack inequality [26] to $u - m_2x_N$ in $G(\rho, 2r)$. Noting $|Lu| \leq \chi x_N^{p-1}$, for some $\sigma \in (0, 1)$, and since $x_N/r$ is trapped between $\rho$ and $2\rho$, there exists $s > 0$ such that $$\left( |G'(\rho, 2r)|^{-1} \int_{G'(\rho, 2r)} (u - m_2x_N)^s \right)^{1/s} \leq C \left( \inf_{G'(\rho, 2r)} (u - m_2x_N) + \chi r^{1+\sigma} \right) \leq C r \left( \inf_{G'(\rho, 2r)} \frac{u}{x_N} - m_2 + \chi r^\sigma \right),$$ where $C > 0$ is a constant. Following the proof of [24, Lemma 5.1] with $F_1 = 0$, we have $$\inf_{G'(\rho, 2R)} \frac{u}{x_N} \leq 4 \inf_{G(\rho, R)} \frac{u}{x_N}.$$ Therefore, we obtain $$\left( |G'(\rho, 2r)|^{-1} \int_{G'(\rho, 2r)} (u - m_2x_N)^s \right)^{1/s} \leq C \left( m_1 - m_2 + \chi r^\sigma \right).$$ Similar estimate holds for $M_2x_N - u$, that is, $$\left( |G'(\rho, 2r)|^{-1} \int_{G'(\rho, 2r)} (M_2x_N - u)^s \right)^{1/s} \leq C \left( M_2 - M_1 + \chi r^\sigma \right).$$ Adding these inequalities, we get $$(M_2 - m_2)r \leq C \left( M_2 - m_2 - M_1 + m_1 + \chi r^\sigma \right).$$ Writing $\omega(ir) = M_i - m_i$, we have $$\omega(r) \leq \frac{C - 1}{C} \omega(2r) + \chi r^\sigma,$$ then rest of the proof can be completed as in [24, Lemma 5.2].
Lemma 4.3 Let \( v \in W^{1,p}(B_R^+) \) be the unique solution of (4.3) and assume \( v \in C^2(B_R^+) \). Then, there exists a positive constant \( C = C(N, \omega, \Lambda, p) \) such that

\[
\text{osc}_{B_r^+} \nabla v \leq C \left( \frac{r}{R} \right)^\chi \left( \sup_{B_{r/2}^+} |\nabla v| + \chi \right),
\]

for \( r \in (0, R/7) \).

**Proof.** Due to the structure of the differential operator in (4.3), we see that Lemma 4.2 can be applied to \( v \), consequently \( \partial_{x_N} v(0) := D_Nv(0) \) exists. Therefore, we define

\[
w(x) = v(x) - D_Nv(0)x_N.
\]

It is easy to observe that \( \text{osc}_S \nabla w = \text{osc}_S \nabla v \) for any \( S \). Therefore, for any \( y \in B_r^+ \), by the interior regularity estimate of [26, Theorem 1.7], we get \( w \in C^{1,\alpha}_{\text{loc}}(\Omega) \) and by the equivalence of Campanato and Hölder norms, we have

\[
\text{osc}_{B_{\rho}(y)} \nabla w \leq C \left( \frac{\rho}{r} \right)^\alpha \left( \text{osc}_{B_{r/2}^+} \nabla w + \chi r^\sigma \right), \tag{4.7}
\]

for \( \sigma > 0 \), if \( 0 < \rho < r \). Now, we estimate \( |\nabla w(y)| \) in terms of \( |w| \). For this purpose, we set

\[
d_{x_N} := \{ x \in D : \text{dist}(x, B_1^0) > \epsilon \}, \quad |u|^{(1)}_{\infty, D} = \sup_{\epsilon > 0} (\epsilon \sup_{D_\epsilon} |u|),
\]

\[
[u]_\alpha,D := \sup \{ |u(x) - u(y)||x-y|^{-\alpha} : x, y \in D, x \neq y \}, \text{ and } [u]_{\alpha+1,D}^* := \sup_{\epsilon > 0} \epsilon^{\alpha+1} |\nabla u|_{\alpha,D_\epsilon}.
\]

Let \( D = B_{yN/2}(y) \) and \( \epsilon \in (0, y_N) \). For \( x, y \in D_\epsilon \), if \( |x - y| < \epsilon \), we take \( \rho = |x - y| \) and \( r = \epsilon \) in (4.7), thus

\[
\epsilon^\alpha \frac{|\nabla w(x) - \nabla w(y)|}{|x-y|^{\alpha}} \leq C \left( \sup_{D_\epsilon} |\nabla w| + \chi \epsilon^\sigma \right).
\]

The above inequality is trivially true if \( |x - y| \geq \epsilon \). Therefore, multiplying by \( \epsilon \) and taking supremum over \( \epsilon \in (0, y_N) \), we obtain

\[
[w]_{\alpha+1,D}^* \leq C \left( |\nabla w|^{(1)}_{\infty,D} + \chi y_N^{\sigma+1} \right). \tag{4.8}
\]

Using the standard interpolation identity

\[
|\nabla w|^{(1)}_{\infty,D} \leq 2 \mu^{-1} \|w\|_{L^\infty(D)} + 2^{1+\alpha} \mu^\alpha [w]_{\alpha+1,D}^* \forall \mu \in (0, 1/2],
\]

for suitable \( \mu \) and (4.8), we obtain

\[
y_N |\nabla w(y)| \leq |\nabla w|^{(1)}_{\infty,D} \leq C \left( \sup_{B_{yN/2}(y)} |w| + \chi y_N^{\sigma+1} \right). \tag{4.9}
\]
For \( x \in B_{y_N}(y) \), we have \( y_N/2 \leq x \leq 3y_N/2 \) and since \( y \in B_{r}^{+} \), we get \( B_{y_N/2}(y) \subset B_{3r/2}^{+} \). Then, for \( x \in B_{y_N}(y) \), using the definition of \( w \) and Lemma 4.2 for \( r \in (0, R/7) \), we deduce that

\[
|w(x)| \leq C y_N \operatorname{osc}_{B_{3r/2}^{+}} v \leq C y_N \left( \frac{r}{R} \right)^{\gamma} \left( \operatorname{osc}_{B_{3r/2}^{+}} \frac{v}{x_N} + \chi \right)
\leq C y_N \left( \frac{r}{R} \right)^{\gamma} \left( \sup_{B_{3r/2}^{+}} \frac{v}{x_N} + \chi \right)
\leq C y_N \left( \frac{r}{R} \right)^{\gamma} \left( \sup_{B_{R/4}^{+}} \frac{v}{R} + \chi \right),
\]

where in the last inequality we have used (4.5). Now from [6, (2.11)], we obtain

\[
|w(x)| \leq C y_N \left( \frac{r}{R} \right)^{\gamma} G^{-1} \left[ \left| B_{R/2}^{+} \right|^{-1} \int_{B_{R/2}^{+}} G(|\nabla v|) \, dx \right] \leq C y_N \left( \frac{r}{R} \right)^{\gamma} \left( \sup_{B_{R/2}^{+}} |\nabla v| + \chi \right). \tag{4.10}
\]

Coupling (4.9) and (4.10), we get

\[
\sup_{B_{r}^{+}} |\nabla w| \leq C \left( \frac{r}{R} \right)^{\gamma} \left( \sup_{B_{R/2}^{+}} |\nabla v| + \chi \right) \quad \text{for } 0 < r < R/7.
\]

Then the required result of the lemma follows from the observation \( \operatorname{osc}_{B_{r}^{+}} \nabla v = \operatorname{osc}_{B_{r}^{+}} \nabla w \leq 2 \sup_{B_{r}^{+}} |\nabla w| \). \( \square \)

**Proof of Lemma 4.1 continued:** To prove (ii) and (iv) of the lemma, we follow the approximation argument as in [26, Lemma 5.2], with \( g(t) = t^{p-1} + t^{q-1} \). Proceeding similarly, we can construct a sequence of operators \( A_{1/j}(\cdot) \), with sufficiently smooth coefficient, converging uniformly to \( A(0, \cdot) \). Here the coefficient of \( A_{1/j} \) exhibit similar bounds depending only on \( \Lambda, \nu, p, \omega \) and \( N \). For each \( j \) large enough, by standard existence theorem, we get a \( C^2 \) solution to the problem

\[-\operatorname{div} A_{1/j}(\nabla v_j) = 0 \quad \text{in } B_{R}^{+}(0), \quad v_j = u \quad \text{on } \partial B_{R}^{+}(0).\]

Then, proof of (ii) follows from Lemma 4.3 and noting the fact that constant \( C \) depends only on \( N, \beta, \Lambda, p \) together with the convergence of \( v_j \) to \( v \). Moreover, a consequence of maximum principle implies that \( v_j \) attains its maximum and minimum on the boundary, and thus the same will be true for \( v \) also. Since \( u \geq 0 \), we get \( \operatorname{osc} u \leq \sup u \) and the fact \( u \leq C x_N \) completes proof of (iv) of the lemma. \( \square \)

**Proof of Theorem 1.7:** We take \( u - v \) as a test function in the weak formulations of (4.1)
and (4.3), on account of Lemma 4.1(iv), (A3) and (A4), we deduce that

\[
\int_{B_R^+} (A(0, \nabla u) - A(0, \nabla v)) \nabla (u - v) = \int_{B_R^+} (A(x, \nabla u) - A(0, \nabla v)) \nabla (u - v)
\]

\[
+ \int_{B_R^+} (A(0, \nabla u) - A(x, \nabla u)) \nabla (u - v)
\]

\[
\leq \int_{B_R^+} (|B(x, u, \nabla u)| + g(x)) |u - v|
\]

\[
+ \Lambda \int_{B_R^+} |x|^\omega (1 + |\nabla u|)^{p-1} |\nabla u - \nabla v|
\]

\[
\leq C \left[ \Lambda R \int_{B_R^+} (1 + |\nabla u|)^p + R \int_{B_R^+} x^{-\sigma} \, dx
\]

\[
+ R^\omega \int_{B_R^+} (1 + |\nabla u|)^{p-1} |\nabla u - \nabla v| \right].
\]

Noting the fact that \( \sigma, \omega, R < 1 \) and using Young inequality, for \( \varepsilon > 0 \), we obtain

\[
\int_{B_R^+} [A(0, \nabla u) - A(0, \nabla v)] \nabla (u - v) \leq c R^{N+1-\sigma} + \varepsilon R^\omega \int_{B_R^+} |\nabla (u - v)|^p
\]

\[
+ (C_\varepsilon + C) R^\omega \int_{B_R^+} [1 + |\nabla u|]^p
\]

(4.11)

where \( c, C, C_\varepsilon > 0 \) are constants. We fix the following

\[
J(w; R) = \int_{B_R^+} |\nabla w|^p \, dx \quad \text{and} \quad I(w; R) = \int_{B_R^+} |\nabla w - (\nabla w)_R|^p,
\]

with \((w)_R = \frac{1}{|B_R^+|} \int_{B_R^+} w(x) \, dx\). Now proceeding similar to [13] Page 150, for \( p \geq 2 \), we have

\[
(A(0, \nabla u) - A(0, \nabla v)) \nabla (u - v) \geq \nu \kappa_p |\nabla (u - v)|^p,
\]

where \( \kappa_p > 0 \) is a constant. Therefore, using this in (4.11), for suitable \( \varepsilon > 0 \), we obtain

\[
J(u - v; R) = \int_{B_R^+} |\nabla (u - v)|^p \leq C \left( R^{N+1-\sigma} + R^\omega \int_{B_R^+} [1 + |\nabla u|]^p \right).
\]

(4.12)

For the case \( 1 < p < 2 \), we recall the following result of [35] Lemma 1]

\[
\nu \int_{B_R^+} (1 + |\nabla u| + |\nabla v|)^{p-2} |\nabla (u - v)|^2 \leq C \int_{B_R^+} [A(0, \nabla u) - A(0, \nabla v)] \nabla (u - v).
\]

On the account of Lemma 4.1(iii) and by repeated application of H"older and Young inequality, for \( \gamma > 0 \), we deduce that

\[
J(u - v; R) \leq \left[ \int_{B_R^+} (1 + |\nabla u| + |\nabla v|)^{p-2} |\nabla (u - v)|^2 \right]^{\frac{p}{2}} \left[ \int_{B_R^+} (1 + |\nabla u|)^p \right]^{\frac{2-p}{2}}
\]

\[
\leq C \left[ R^{-\gamma} \int_{B_R^+} [A(0, \nabla u) - A(0, \nabla v)] \nabla (u - v) + R^{\gamma-\gamma} \int_{B_R^+} (1 + |\nabla u|)^p \right].
\]
Using (4.11) in the above expression, we get
\[ J(u - v; R) \leq CR^{-\frac{2\alpha}{p}} \left[ cR^{N+1-\sigma} + \varepsilon R^\omega \int_{B_R^+} |\nabla (u - v)|^p + (C_z + C)R^\omega \int_{B_R^+} [1 + |\nabla u|^p] \right. \]
\[ \left. + CR^{\frac{\omega}{p}} \int_{B_R^+} (1 + |\nabla u|)^p. \right] \]

We choose \( \gamma > 0 \) sufficiently small such that \( 1 - \sigma - 2\gamma/p > 0 \) and \( \omega - 2\gamma/p > 0 \), and we set \( \gamma_0 = \min\{1 - \sigma - 2\gamma/p, \omega - 2\gamma/p, 2\gamma/(2 - p)\} > 0 \). Therefore, for sufficiently small \( \varepsilon > 0 \), we infer that
\[ J(u - v; R) \leq CR^{\gamma_0} \left[ R^N + \int_{B_R^+} (1 + |\nabla u|)^p \right]. \tag{4.13} \]

Combining (4.12) and (4.13), we obtain
\[ J(u - v; R) = \int_{B_R^+} |\nabla (u - v)|^p \leq C \left[ R^{N+\gamma_0} + R^{\gamma_0} (R^N + J(u; R)) \right], \quad \text{for all } p > 1. \tag{4.14} \]

Furthermore, using Lemma 4.1(i) and (iii), we observe that
\[ J(v; r) = \int_{B_R^+} |\nabla v|^p \leq Cr^N \left( \sup_{B_{R/2}} |\nabla v| \right)^p \leq Cr^N \left[ R^{-N} J(v; R) + \chi \right] \]
\[ \leq C \left[ R^N + \left( \frac{r}{R} \right)^N J(u; R) \right]. \tag{4.15} \]

Taking into account (4.14) and (4.15) together with the estimate of \( J(u; 1) \), due to [34, Lemma 3], a standard procedure yields
\[ J(u; R) \leq CR^{N-\tau} \text{ for } 0 < R < 1 \text{ and any } \tau > 0. \tag{4.16} \]

Thus, (4.14) reduce to
\[ J(u - v; R) \leq C \left( R^{N+\gamma_0} + R^{N-\tau+\gamma_0} \right), \quad \text{for all } p > 1. \tag{4.17} \]

Now, consider
\[ I(u; r) = \int_{B_r^+} |\nabla u - (\nabla u)_v|^p \leq C \left[ \int_{B_r^+} |\nabla (u - v)|^p + \int_{B_r^+} |\nabla v - (\nabla v)_v|^p \right] \]
\[ \leq C \left[ J(u - v; R) + r^N (\text{osc}_{B_r^+} \nabla v)^p \right]. \]

Then, using (4.17) and Lemma 4.1(i)-(iii) in the above expression, we deduce that
\[ I(u; r) \leq C \left[ R^{N-\tau+\gamma_0} + r^N \left( \frac{r}{R} \right)^{\gamma_0} \right] \left[ R^{-N} J(v; R) + \chi \right] \leq C \left[ R^{N-\tau+\gamma_0} + r^{N+p} R^{-\gamma_0 - \tau} \right], \]
where in the last inequality we have used (4.16) to estimate \( J(u; R) \) and the fact that \( 0 < R < 1 \). Setting \( R = r^\theta \), appropriate choice of \( \tau \) and \( \theta \) yields
\[ I(u; r) \leq C r^{N+\alpha p}, \]
for some positive constant \( \alpha \). It is clear that the constant \( \alpha \) depends only on \( N, \omega, \sigma, \nu, \Lambda \) and \( p \). By the equivalence of Campanato and Hölder norms, and covering argument, we get \( u \in C^{1,\alpha}(\Omega) \).
Proof of Theorem 1.8. Since $u$ is a weak solution to (P), for all $\phi \in C_c^\infty(\Omega)$, the following holds

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi + \int_\Omega |\nabla u|^{q-2}\nabla u \nabla \phi = \int_\Omega f(x) u^{-\delta} \phi. \quad (4.18)$$

For $\gamma > 1$, to be chosen later, from (4.18), we infer that

$$\gamma^{1-p} \int_\Omega u^{(1-\gamma)(p-1)} |\nabla u|^p \nabla \phi + \gamma^{1-q} \int_\Omega u^{(1-\gamma)(q-1)} |\nabla u|^q \nabla \phi = \int_\Omega f(x) u^{-\delta} \phi,$$

that is,

$$\int_\Omega |\nabla u|^p \nabla \phi + (\gamma - 1)(p - 1) \int_\Omega u^{(1-\gamma)(p-1)} |\nabla u|^p \nabla \phi + \gamma^{p-q} |\nabla u|^q \nabla \phi = \gamma^{p-1} \int_\Omega f(x) u^{-\delta} \phi.$$

This implies that,

$$-u^{(1-\gamma)(p-1)} \Delta_p u^\gamma - \gamma^{p-q} u^{(1-\gamma)(q-1)} \Delta_q u^\gamma + (\gamma - 1)(p - 1) u^{(1-\gamma)(p-1)} |\nabla u|^p \nabla \phi + \gamma^{p-q} u^{(1-\gamma)(q-1)} |\nabla u|^q \nabla \phi = \gamma^{p-1} f(x) u^{-\delta},$$

equivalently,

$$-\Delta_p u^\gamma - \gamma^{p-q} u^{(1-\gamma)(q-1)} \Delta_q u^\gamma + (\gamma - 1)(p - 1) \frac{|\nabla u|^p}{u^\gamma} + \gamma^{p-q} (\gamma - 1)(q - 1) \frac{|\nabla u|^q}{u^\gamma + (\gamma - 1)(q - p)} = \gamma^{p-1} f(x) u^{-\delta}, \quad (4.19)$$

On the account of assumption that $u \in \Gamma d^p$, we see that the right hand side of (4.19) can be bounded from above by $C(\gamma, d, c_2) d^{-\beta + \delta - (\gamma - 1)(p - 1)}$. Therefore, we choose $\gamma > 1$ such that $u^\gamma \in W_0^{1,p}(\Omega)$ and $-\beta + \delta - (\gamma - 1)(p - 1) > 0$ so that the right hand side of (4.19) is in $L^\infty(\Omega)$. For convenience, we denote $u^\gamma = v \in W_0^{1,p}(\Omega)$, thus (4.19) takes the following form

$$-\Delta_p v - \gamma^{p-q} v^{(1-\gamma)(q-1)} \Delta_q v + (\gamma - 1)(p - 1) \frac{|\nabla v|^p}{v^\gamma} + \gamma^{p-q} (\gamma - 1)(q - 1) \frac{|\nabla v|^q}{v^{1+q} + (\gamma)(q-1)(q-p)/\gamma} = \gamma^{p-1} f(x) v^{-\delta}, \quad (4.20)$$

Now, we will prove that $v \in C^{0,\alpha_1}(\overline{\Omega})$, for some $\alpha_1 \in (0, 1)$. Let us assume that $0 \in \Omega$ and $\rho > 0$ be such that $B_\rho := B_\rho(0) \Subset \Omega$. Then, **Claim (i):** there exists a constant $C > 0$, depending on $N, p$ and $\|v\|_{L^\infty(\Omega)}$, such that

$$\int_{B_\rho} |\nabla v|^p < C \rho^{-N-p}.$$

We note that the similar result holds if $B_\rho$ is replaced by $\Omega \cap B_\rho$, when $B_\rho$ is centered on the
boundary of $\Omega$. Let $\zeta \in C^\infty_c(\Omega)$ such that $0 \leq \zeta \leq 1$, $\zeta = 1$ in $B_\rho$ and supp($\zeta$) $\subset B_{2\rho}$. We take $\phi = e^{\tau v}\zeta^p$, for some $\tau > 0$, in the weak formulation of (1.20), thus
\[
\int_\Omega |\nabla v|^{p-2}\nabla v \phi + \gamma^{p-q}|\nabla v|^{q-2}\nabla v \nabla (v^{(\gamma-1)(p-q)/\gamma}) \phi + (\gamma - 1)(p - 1)|\nabla v|^p \phi + \gamma^{p-q}(\gamma - 1)(q - 1)|\nabla v|^q \phi - \gamma^{p-1}f(x)v^{\delta + (\gamma - 1)(p-1)/\gamma} \phi \right] dx = 0.
\]
Since $\phi \geq 0$ and $v \geq 0$ in $\Omega$, we observe that the third and fourth term of the integrand is nonnegative. Therefore,
\[
\int_\Omega e^{\tau v} \left[ |\nabla v|^{p-2} |\nabla v|^{p} \zeta^p \nabla v (\tau \zeta^p \nabla v + p \zeta^{p-1} \nabla \zeta) + v^{(\gamma-1)(p-q)/\gamma} |\nabla v|^{q-2} \nabla v (\gamma \zeta^p \nabla v + p \zeta^{p-1} \nabla \zeta) + (\gamma - 1)(p - q)v^{(\gamma-1)(p-q)/\gamma} |\nabla v|^q \gamma \zeta^p - \gamma^{p-1}f(x)v^{\delta + (\gamma - 1)(p-1)/\gamma} \zeta^p \right] dx \leq 0,
\]
which again due to the nonnegativity of $\zeta$ and $v$, and for $\gamma > 1$ with $(\gamma - 1)(p - q) > \gamma$, implies
\[
\tau \int_\Omega e^{\tau v} |\nabla v|^p \zeta^p dx \leq C \int_\Omega e^{\tau v} \left[ |\nabla v|^{p-1} p \zeta^{p-1} |\nabla \zeta| + \|v\|_{L^\infty(\Omega)} |\nabla v|^{q-1} p \zeta^{p-1} |\nabla \zeta| + \gamma^{p-1}d^{-\beta + \delta(\gamma - 1)(p-1)/\gamma} \right] dx.
\]
Then, proof of the claim (i) can be completed similar to [22, Lemma 1.1, p. 247]. Employing [22, Theorem 1.1, p.251], we conclude that $v = u^\gamma \in C^{0,\alpha_1}(\bar{\Omega})$, here $\alpha_1 \in (0, 1)$ is a constant depending only on $\|u\|_{L^\infty(\Omega)}$, $N, p$. Therefore, we infer that $u \in C^{0,\alpha}(\bar{\Omega})$, where $\alpha = \alpha_1/\gamma$. This completes proof of the theorem. □

**Remark 4.4** For the case $\beta < 2 - 1/p$ and $\Omega = B_R(0)$, by standard procedure, we get the existence of a unique radial solution $u$ of (P). Moreover, similar arguments as in the proof of Theorem 1.7 imply $u^\gamma \in C^1(\bar{\Omega})$. In this manner the regularity results of Theorem 1.8 and Corollary 1.9 are improved.

**Proof of Corollary 1.9** We note that either by the uniqueness result or due to the minimality of the solution, on account of Theorem 1.4, we get $u \in C_{d_3, \delta}$. Thus, for the case of $\beta + \delta < 1$, Theorem 1.1 ensures that $u \in C^{1,\alpha}(\bar{\Omega})$, whereas for the case $\beta + \delta \geq 1$, Theorem 1.8 implies $u \in C^{0,\alpha}(\bar{\Omega})$. This completes proof of the corollary. □

**Remark 4.5** We remark that our Hölder continuity result of Theorem 1.7 for the gradient of weak solution, in the case of $\beta + \delta < 1$, can be used to obtain multiplicity results for the following quasilinear elliptic equation involving singular nonlinearity

\[
(S) \begin{cases}
-\Delta_p u - \Delta_q u = f(x)u^{-\delta} + g(x, u), \quad u > 0 \text{ in } \Omega, \\
u = 0 \text{ on } \partial \Omega,
\end{cases}
\]
where $g(x, t)$ is a Carathéodory function satisfying
\((g_1)\) \(g(x,t) \geq 0\) for all \((x,t) \in \overline{\Omega} \times \mathbb{R}^+\) with \(g(x,0) = 0\).

\((g_2)\) There exists \(r > p - 1\) with \(r \leq p^* - 1 := \frac{Np}{N-p} - 1\), if \(p < N\), otherwise \(r < \infty\) such that 
\[ g(x,t) \leq C(1 + t)^r \]
for all \((x,t) \in \Omega \times \mathbb{R}^+\), for some constant \(C > 0\).

We define the associated energy functional \(I : W^{1,p}_0(\Omega) \to \mathbb{R}\) as
\[
I(u) := \frac{1}{p} \int_{\Omega} |\nabla u|^p \, dx + \frac{1}{q} \int_{\Omega} |\nabla u|^q \, dx - \frac{1}{1-\delta} \int_{\Omega} f(x)|u|^{1-\delta} \, dx - \int_{\Omega} G(x,u) \, dx,
\]
where \(G(x,u) = \int_0^u g(x,t) \, dt\). Following the approach in [12], we can prove the following Sobolev versus Hölder minimizer result.

**Theorem 4.6** Let \(u_0 \in C^1(\overline{\Omega})\) satisfying \(u_0 \geq \eta d(x)\) in \(\Omega\), for some \(\eta > 0\), be a local minimizer of \(I\) in the \(C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})\) topology. Then, \(u_0\) is a local minimizer of \(I\) in \(W^{1,p}_0(\Omega)\) topology also.

Using the above theorem and strong comparison principle for singular problems we can prove the existence of one positive solution \(u_1\) to \((S)\). Consequently, using critical point analysis similar to [21, Section 5], we obtain the existence of second solution.

**Remark 4.7** We remark that Theorem 1.8 can be used to obtain the existence results for quasilinear elliptic systems driven by the nonhomogeneous \(p-q\) Laplace operator and involving singular nonlinearities by using Schauder fixed point theorem similar to [14].

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