SHARP DOUBLE INEQUALITY FOR COMPLETE ELLIPTIC INTEGRAL OF THE FIRST KIND

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Abstract. For \( r \in (0, 1) \), the function \( \mathcal{K}(r) = \int_0^{\pi/2} (1 - r^2 \sin^2 t)^{-1/2} \, dt \) is known as the complete elliptic integral of the first kind. In this paper, we prove the absolute monotonicity of two functions involving \( \mathcal{K}(r) \). As a consequence, we improve Alzer and Richards’ result.

1. Introduction

In the past few centuries, the complete elliptic integral of the first kind (cf. [1, 3, 6]) \( \mathcal{K}(r) \) defined on \((0, 1)\) by

\[
\mathcal{K} = \mathcal{K}(r) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - r^2 \sin^2 t}} \, dt = \frac{\pi}{2} F \left( \frac{1}{2}, \frac{1}{2}; 1; r^2 \right),
\]

where \( F \) denotes the classical Gaussian hypergeometric function (cf. [5, 7])

\[
F(a, b; c; x) = {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)n!} x^n, \quad |x| < 1,
\]

where \((a, n)\) is the Pochhammer symbol or shifted factorial defined as \((a, 0) = 1\) for \( a \neq 0\), and

\[
(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1) = \frac{\Gamma(n + a)}{\Gamma(a)}
\]

for \( n \in \mathbb{N} = \{1, 2, 3, \ldots\}\), where

\[
\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} \, dt \quad (x > 0)
\]

is the classical Euler Gamma function (cf. [1]).

It is well known that the complete elliptic integrals have many important applications in physics, engineering, geometric function theory, quasiconformal analysis, theory of mean values, number theory and other related fields (cf [2, 4, 8–12, 15–18, 21–23]).

Recently, the complete elliptic integrals have attracted the attention of numerous mathematicians. It is well known that complete elliptic integrals cannot be represented by the elementary transcendental functions. Therefore, there is a need for sharp computable bounds for the family of integrals. In particular, many remarkable properties and inequalities for the complete elliptic integrals can be found in the literature [12, 14, 18, 19, 24, 26].

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For example, in order to refine the following well-known asymptotic formula
\[
\lim_{r \to 1^-} \left[ \mathcal{K}(r) - \log \left( \frac{4}{\sqrt{1 - r^2}} \right) \right] = 0,
\] (1.5)
Anderson, Vamanamurthy and Vuorinen in [14] conjectured that the inequality
\[
\mathcal{K}(r) < \log \left( \frac{4}{\sqrt{1 - r^2}} \right) - \left( \log 5 - \frac{\pi}{2} \right) (1 - r)
\] (1.6)
holds for each \( r \in (0, 1) \). Later, the conjecture was proved by Qiu et al. in [12].
In 2020, Wang, Chu, Li and Chu in [13] improved (1.6), and showed that the inequality
\[
\log \left( 1 + \frac{4}{\sqrt{1 - r^2}} \right) - \left( \log 5 - \frac{\pi}{2} \right) + \frac{\pi}{8} - \frac{2}{5} r^2 + \alpha^* r^4 < K(r)
\] (1.7)
is valid for \( r \in (0, 1) \) with the best possible constants \( \alpha^* = 9\pi/128 - 11/50 = 0.000893 \cdots \) and \( \beta^* = 2/5 + \log 5 - 5\pi/8 = 0.0459 \cdots \).
Very recently, Alzer and Richards in [19] obtained the following upper bound for \( K(r) \), that is,
\[
K(r) < \frac{\pi}{2} \frac{16 - 5 \log(1 - r^2)}{16 + (5\pi - 16)r^2} \quad (0 < r < 1).
\] (1.8)
Observe that the upper bound of \( K(r) \) in (1.8) is concise and meaningful. Based on the known results such as those above mentioned, the following question is natural:

**Question 1.1.** Whether we can find an improved upper bound and a similar form of lower bound for (1.8)?

The main purpose of this paper is to give a positive answer to Question 1.1. Our results are following Theorems 1.2-1.3 and Corollary 1.4. For convenience, we let \( \mathbb{N} \) denotes the set of positive integers as usual, put
\[
\theta = \frac{\pi (17 - 5\pi)}{32} = 0.126845 \cdots, \tag{1.9}
\]
\[
\lambda = \frac{8}{5} - \log 4 = 0.213705 \cdots, \tag{1.10}
\]
\[
\alpha = \frac{85}{8} \pi - \frac{185}{32} \pi^2 + \frac{25}{32} \pi^3 = 0.544425 \cdots, \tag{1.11}
\]
\[
\beta = (8 - 10 \log 2) \pi - \frac{85}{32} \pi^2 + \frac{25}{32} \pi^3 = 1.364397 \cdots, \tag{1.12}
\]
\[
\delta = \frac{128}{5} - 32 \log 2 - \frac{17}{2} \pi + \frac{5}{2} \pi^2 = 1.389763 \cdots, \tag{1.13}
\]
\[
\zeta = \frac{128}{5} + 32 \log 2 + \left( \frac{47}{8} - 10 \log 2 \right) \pi + \frac{5}{8} \pi^2 = -0.569791 \cdots. \tag{1.14}
\]
**Theorem 1.2.** Let \( \theta, \alpha \) and \( \beta \) are given in (1.9), (1.11) and (1.12), respectively. Define the function \( f \) on \((0, 1)\) by
\[
f(r) = \frac{\pi}{2} \left[ 16 - 5 \log(1 - r^2) \right] - \left[ \theta r^2 + \mathcal{K}(r) \right] \left[ 16 + (5\pi - 16)r^2 \right].
\]
Then all coefficients are positive in the Maclaurin series for \( f_1 \equiv f/r^4 \) in powers of \( r^2 \) with range \((\alpha, \beta)\). In other words, \( f_1 \) is absolutely monotonic on \((0, 1)\).

**Theorem 1.3.** Let \( \lambda \) and \( \delta \) are given in (1.10) and (1.13), respectively. Define the function \( g \) on \((0, 1)\) by
\[
g(r) = \left[ \lambda r^2 + \mathcal{K}(r) \right] \left[ 16 + (5\pi - 16)r^2 \right] - \frac{\pi}{2} \left[ 16 - 5 \log(1 - r^2) \right].
\]
Then all coefficients are negative in the Maclaurin series for \( g_1 \equiv g/r^2 \) in powers of \( r^2 \) with range \((0, \delta)\). In other words, \(-g_1\) is absolutely monotonic on \((0, 1)\).

**Corollary 1.4.** According to Theorem 1.2-1.3, we can find better bounds for \( \mathcal{K}(r) \) than (1.8). For example, Theorem 1.2 (Theorem 1.3) implies that the function \( f_1 \) (\( g_1 \)) is strictly increasing and convex (decreasing and concave) from \((0, 1)\) onto \((\alpha, \beta)\) ((0, \(\delta\)), respectively). Consequently, the double inequality
\[
\max \left\{ \frac{\pi[16 - 5 \log(1 - r^2)] - 2[\alpha + (\beta - \alpha)r^4]}{32 + 2(5\pi - 16)r^2} - \theta r^2, \frac{\pi[16 - 5 \log(1 - r^2)] + 2\delta(1 - r^2)r^2}{32 + 2(5\pi - 16)r^2} - \lambda r^2 \right\} \leq \mathcal{K}(r) \leq \min \left\{ \frac{\pi[16 - 5 \log(1 - r^2)] - 2\alpha r^4}{32 + 2(5\pi - 16)r^2} - \theta r^2, \frac{\pi[16 - 5 \log(1 - r^2)] + 2\delta r^2}{32 + 2(5\pi - 16)r^2} - \lambda r^2 \right\},
\]
holds for all \( r \in (0, 1) \). The first (second) equality holds if and only if \( r \to 0 \) or \( r \to 1 \) \((r \to 0, \) respectively).

2. **Proof of Theorem 1.2-1.3**

In order to prove our main results in this section, we need next lemma.

**Lemma 2.1.** For \( n \in \mathbb{N} \), the sequence
\[
Q_n = \frac{5\pi n}{5\pi n^2 - 16n + 4} - \left[ \frac{\Gamma(n - 1/2)}{\Gamma(n)} \right]^2
\]
is positive.

**Proof.** In [20, equation (1.3)], Kershaw proved that
\[
\left( x + \frac{s}{2} \right)^{-s} < \frac{\Gamma(x + 1)}{\Gamma(x + s)} < \left[ x - \frac{1}{2} + \left( \frac{1}{4} + s \right)^{1/2} \right]^{-s} \tag{2.1}
\]
holds for $x > 0$ and $0 < s < 1$. Hence by first inequality sign of (2.1) and by using the substitution $x = n - 1$, take $s = 1/2$, we have
\[
\left[ \frac{\Gamma(n - 1/2)}{\Gamma(n)} \right]^2 < \frac{4}{4n - 3}
\] (2.2)
is valid for $n \in \mathbb{N}$. It follows from (2.2) that
\[
Q_n > \frac{5\pi n}{5\pi n^2 - 16n + 4} - \frac{4}{4n - 3} = P_n.
\] (2.3)
It is enough to prove $P_n > 0$ for $n \in \mathbb{N}$, as a matter of fact,
\[
P_n > 0 \iff 5\pi n(4n - 3) - 4(5\pi n^2 - 16n + 4) > 0
\]
holds for $n \in \mathbb{N}$. Therefore, together with (2.3), yields the sequence \{Q_n\} is positive for $n \in \mathbb{N}$. $\square$

**Proof of Theorem 1.2**

By (1.2), expanding in power series yields
\[
f(r) = \frac{\pi}{2} \left( 16 + 5 \sum_{n=1}^{\infty} \frac{1}{n} r^{2n} \right) - [16 + (5\pi - 16)r^2] \left( \theta r^2 + \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} \right)
\]
\[
= 8\pi - 16\theta r^2 - \theta(5\pi - 16)r^4
\]
\[
+ \frac{5\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} r^{2n} - \frac{(5\pi - 16)\pi}{2} \sum_{n=1}^{\infty} \frac{(1/2, n - 1)^2}{[(n - 1)!]^2} r^{2n} - 8\pi \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n}
\]
\[
= \left( \frac{85}{8\pi} - \frac{185}{32\pi^2} + \frac{25}{32}\pi^3 \right) r^4
\]
\[
+ \sum_{n=3}^{\infty} \left( \frac{5\pi}{2n} - \frac{(5\pi n^2 - 16n + 4)\Gamma(n - 1/2)^2}{2\Gamma(n + 1)^2} \right) r^{2n}
\]
\[
= \alpha r^4 + \sum_{n=3}^{\infty} \frac{(5\pi n^2 - 16n + 4)Q_n r^{2n}}{2n^2},
\]

where $Q_n$ is given in lemma 2.1. Hence, $f_1 = f/r^4$ has following series expansion
\[
f_1(r) = \alpha + \sum_{n=3}^{\infty} \frac{(5\pi n^2 - 16n + 4)Q_n r^{2n}}{2n^2}. \quad (2.4)
\]
It is easy to verify that $5\pi n^2 - 16n + 4 > 0$ for $n \in \mathbb{N}$, hence it follows from Lemma 2.1 that all coefficients are positive in the Taylor series for $f_1$ in powers of $r^2$. By (2.4), we clearly see that
\[
f_1(0^+) = \lim_{r \to 0} f_1(r) = \alpha. \quad (2.5)
\]
From (1.5) it is easy to obtain the following asymptotic formula
\[
\mathcal{K}(r) \sim \log 4 - \frac{\log(1 - r^2)}{2}, \quad \text{as } r \to 1. \quad (2.6)
\]
Hence we obtain
\[
f(1^-) = \lim_{r \to 1} f_1(r)
\]
\[
= \lim_{r \to 1} \left\{ \frac{5\pi}{2} \log(1 - r^2) - \left( \frac{\log(4 - \log(1 - r^2))}{2} \right) \left[ 16 + (5\pi - 16)r^2 \right] \right\}
\]
\[
= (8 - 10\log 2)\pi - \frac{85\pi}{32} + \frac{25\pi}{32} = \beta. \quad (2.7)
\]

Therefore, Theorem (1.2) directly follows from (2.5) and (2.7) together with (2.4). This completes the proof.

**Proof of Theorem (1.3)**

Similarly, it follows from (1.2) that
\[
g(r) = \left[ 16 + (5\pi - 16)r^2 \right] \left( \lambda r^2 + \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n} \right) - \frac{\pi}{2} \left( 16 + 5 \sum_{n=1}^{\infty} \frac{1}{n^2} r^{2n} \right)
\]
\[
= -8\pi + 16\lambda r^2 + \lambda(5\pi - 16)r^4
\]
\[
- \frac{5\pi}{2} \sum_{n=1}^{\infty} \frac{1}{n} r^{2n} + \frac{(5\pi - 16)\pi}{2} \sum_{n=1}^{\infty} \frac{(1/2, n - 1)^2}{[(n - 1)!]^2} r^{2n} + 8\pi \sum_{n=0}^{\infty} \frac{(1/2, n)^2}{(n!)^2} r^{2n}
\]
\[
= \left( \frac{128}{5} - 32\log 2 - \frac{17}{2} \pi + \frac{5}{2} \pi^2 \right) r^2
\]
\[
+ \left[ -\frac{128}{5} + 32\log 2 + \left( \frac{47}{8} - 10\log 2 \right) \pi + \frac{5}{8} \pi^2 \right] r^4
\]
\[
- \sum_{n=3}^{\infty} \frac{5\pi}{2n} - \frac{(5\pi n^2 - 16n + 4)\Gamma(n - 1/2)^2}{2\Gamma(n + 1)^2} \right] r^{2n}
\]
\[
= \delta r^2 + \zeta r^4 - \sum_{n=3}^{\infty} \frac{(5\pi n^2 - 16n + 4)Q_n}{2n^2} r^{2n-2},
\]

where \(Q_n\) is given in Lemma (2.1). Hence, \(g_1 = g/r^2\) has following series expansion
\[
g_1(r) = \delta + \zeta r^2 - \sum_{n=3}^{\infty} \frac{(5\pi n^2 - 16n + 4)Q_n}{2n^2} r^{2n-2}. \quad (2.8)
\]

Therefore, it follows from Lemma (2.1) that all coefficients are negative in the Taylor series for \(g_1\) in powers of \(r^2\). By (2.8), we clearly see that
\[
g_1(0^+) = \lim_{r \to 0} g_1(r) = \delta. \quad (2.9)
\]

Again using asymptotic formula (2.6), we clearly see that
\[
g_1(1^-) = \lim_{r \to 1} g_1(r)
\]
\[
\lim_{r \to 1} \left\{ -8\pi + \lambda \left[ 16 + (5\pi - 16)r^2 \right] r^2 \\
+ \frac{5\pi}{2} \log(1 - r^2) + \left[ 16 + (5\pi - 16)r^2 \right] \left( \log 4 - \frac{\log(1 - r^2)}{2} \right) \right\} = 0. \tag{2.10}
\]

Therefore, Theorem 1.3 directly follows from (2.8)-(2.10). This completes the proof. \qed

**Remark 2.2.** Clearly, the upper bound of inequality (1.15) is better than (1.8). Moreover, computer simulation and experiments of Maple 2016 show that (1.7) and (1.15) have their own merits.

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