INVARIANTS OF MODULAR CURVES AND SHARIFI’S CONJECTURES
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Abstract. We compute some invariants attached to the Eisenstein quotient of an étale cohomology group of the modular curve $X_0(N)$ for $N \geq 5$. We also give a relation between one of the invariants and Sharifi’s conjecture on the modular curve $X_1(N)$.

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1. Introduction

A modular curve is an algebraic curve obtained from a quotient space $\Gamma \backslash \mathbb{H}$, where $\Gamma$ is a congruence subgroup of $\text{SL}_2(\mathbb{Z})$ and $\mathbb{H}$ is the Poincaré half-plane. More generally, it is a moduli space of elliptic curves with additional level structure. The theory of modular curves has been widely used in number theory. In Mazur’s seminal work [18], he studied the arithmetic properties of the modular curve $X_0(N)$ in detail. For example, Mazur gave a very nice description of the Eisenstein component $\mathfrak{h}$ of the $p$-adic cuspidal Hecke algebra of this modular curve. For a prime $p \geq 5$, he proved that there is an isomorphism $\mathbb{Z}_p/(N - 1)\mathbb{Z}_p \rightarrow \mathfrak{h}/I$, and the $p$-adic Hecke algebra $\mathfrak{h}$ is Gorenstein. Moreover, he gave a description of the Eisenstein kernel of the $p$-divisible group $J[p^\infty]$ attached to the modular Jacobian $J := \text{Jac}(X_0(N))$. That is, he proved that $J[I,p^\infty] \cong C \oplus \Sigma$, where $I \subset \mathfrak{h}$ is the Eisenstein ideal, $C$ is a constant group scheme generated by the image of divisor $(0) - (\infty)$, and $\Sigma$ is a multiplicative group scheme which is called the Shimura subgroup. Note that we have a Galois-equivariant perfect pairing

$$H \times J[p^\infty] \rightarrow \mathbb{Q}_p/\mathbb{Z}_p,$$

where $H := H^1_{\text{ét}}(X_0(N)/\mathbb{Q},\mathbb{Z}_p)$, and the Hecke operators are self-adjoint with respect to this pairing.

Thus, we get a corresponding $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module decomposition of $H/IH$

$$H/IH \cong H^+ / IH^+ \oplus H^- / IH^-.$$

Here, $\pm$ denotes the decomposition under complex conjugation. Via (1), $H^+ / IH^+ = \text{Hom}(C, \mathbb{Q}_p/\mathbb{Z}_p)$ and $H^- / IH^- = \text{Hom}(\Sigma, \mathbb{Q}_p/\mathbb{Z}_p)$. It is easy to see that $H^+ / IH^+$ is a trivial Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)-module, and the action of Gal($\overline{\mathbb{Q}}/\mathbb{Q}$) on $H^- / IH^-$ is via $\chi_p^{-1}$, where $\chi_p$ is the $p$-adic cyclotomic character.

One natural question is: what can we say about the $\mathfrak{h}[$Gal($\overline{\mathbb{Q}}/\mathbb{Q}$)$]$-module $H/I^2H$?

Let $K = \mathbb{Q}(\zeta_q)$, where $q$ is the highest power of $p$ dividing $N - 1$. Viewing $H/I^2H$ as representation of Gal($\overline{\mathbb{Q}}/K$), we have following four group homomorphisms:

1. $a : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Aut}_{\mathfrak{h}}(H^- / I^2H^-)$,
2. $b : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Hom}_{\mathfrak{h}}(H^+ / I^2H^+, H^- / I^2H^-)$,
3. $c : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Hom}_{\mathfrak{h}}(H^- / I^2H^-, H^+ / I^2H^+)$,
4. $d : \text{Gal}(\overline{\mathbb{Q}}/K) \rightarrow \text{Aut}_{\mathfrak{h}}(H^+ / I^2H^+)$.

One can prove that the homomorphisms $a$ and $d$ cut out a field extension $F_0$, the homomorphism $b$ cuts out a field $F_{-1}$, and the homomorphism $c$ cuts out a field $F_1$, where $F_i$ means that the group Gal($\mathbb{Q}(\zeta_q)/\mathbb{Q}$) acts on Gal($F_i/K$) via $\chi_p$. 


Let \( U = \mathbb{F}_p^N/(\mathbb{F}_p^N)^p \). Using global class field theory, we have the following canonical isomorphisms of abelian groups:

1. \( \text{Gal}(F_0/K) \cong U \),
2. \( \text{Gal}(F_1/K) \cong \mu_q \),
3. \( \text{Gal}(F_{-1}/K) \cong U^{\otimes 2} \otimes \mu_q^{-1} \),

where \( \mu_q^{-1} = \text{Hom}(\mu_q, \mathbb{Z}/q\mathbb{Z}) \). The \( F_i \) are class fields, and all of them are totally tamely ramified at \( N \) over \( K \).

In [18], Mazur uses modular symbols to construct a canonical winding isomorphism:

\[
e : I/I^2 \cong \text{Hom}(\mu_q, \mathbb{Z}/q\mathbb{Z}).
\]

In [18], Mazur also proved that \( H^+, H^- \) are both free \( \mathfrak{h} \)-modules of rank 1, which means that \( H^+/I^2H^+ \) and \( H^-/I^2H^- \) are free \( \mathfrak{h}/I^2\mathfrak{h} \)-modules of rank one. We know that the images of \( a, d \) are both in \( 1 + I/I^2 \). Since there is an canonical isomorphism between the multiplicative abelian group \( 1 + I/I^2 \) and the additive abelian group \( I/I^2 \), we may view \( a \) and \( d \) as homomorphisms:

1. \( a : U \to I/I^2 \),
2. \( d : U \to I/I^2 \).

By composing with the winding homomorphism \( \varpi \), we get maps \( I/I^2 \to I/I^2 \) that are multiplication by an integer in \( \mathbb{Z}/q\mathbb{Z} \). We use \( \bar{a} \) and \( \bar{d} \) to denote these two numbers. The invariants \( \bar{a} \) and \( \bar{d} \) have been already computed by B. Mazur, F. Calegari, R. Sharifi and W. Stein in the unpublished manuscript [19], and their results are \( \bar{a} = -1, \bar{d} = 1 \).

For the homomorphisms \( b \) and \( c \), we need a more careful analysis. We carefully choose a generator of \( H^+/IH^+ \):

\[
\text{see Remark 3.24}.
\]

For \( H^-/IH^- \), we can identify it with \( I/I^2 \otimes \mu_q^{-1} \) using \( \varpi \).

It is easy to see that the image of \( b \in \text{Hom}_\mathfrak{h}(H^+/IH^+, I^-/I^2H^-) \), the image of \( c \in \text{Hom}_\mathfrak{h}(H^-/IH^-, I^+/I^2H^+) \).

Since

1. \( \text{Hom}_\mathfrak{h}(H^+/IH^+, I^-/I^2H^-) \cong I^-/I^2H^- \cong I^2/I^3 \otimes \mu_q^{-1} \),
2. \( \text{Hom}_\mathfrak{h}(H^-/IH^-, I^+/I^2H^+) \cong \text{Hom}_\mathfrak{h}(I/I^2 \otimes \mu_q^{-1}, I/I^2) \cong \mu_q \),

we can rewrite \( b \) and \( c \) as follows:

1. \( b : \text{Gal}(F_{-1}/K) \cong U^{\otimes 2} \otimes \mu_q^{-1} \to I^2/I^3 \otimes \mu_q^{-1} \),
2. \( c : \text{Gal}(F_1/K) \cong \mu_q \to \mu_q \).

Hence the homomorphisms \( b \) and \( c \) give us two integers in \( \mathbb{Z}/q\mathbb{Z} \) which were constructed in [19]. Let’s use \( \bar{b} \) and \( \bar{c} \) to denote them. The integers \( \bar{b} \) and \( \bar{c} \) are units in \( \mathbb{Z}/q\mathbb{Z} \), which can be seen using [4], or [32]. What we want to understand are the exact values of \( b \) and \( c \) based on the canonical winding isomorphism \( I/I^2 \cong U \) in [3]. Our result is \( \bar{c} = 1 \). For the details, see Theorem 7.10.

The homomorphism \( b \) has a conjectural inverse which is closely related to Sharifi’s conjectures for the modular curve \( X_1(N) \). In [22], Sharifi formulated a series of remarkable conjectures which relate the arithmetic of cyclotomic fields to the homology of modular curves. Roughly speaking, he used cup products of cyclotomic units to define two maps:

\[
\varpi^0 : \tilde{H}_1^+ \to H^2(\mathbb{Z}[\zeta_{Np}, \zeta_p(2)], \mu_{q^2}), \quad \varpi : \tilde{H}_1^+ \to H^2(\mathbb{Z}[\zeta_{Np}, \zeta_p(2)]),
\]

where \( \tilde{H}_1 := H_1(X_1(N), \tilde{C}_\infty, \mathbb{Z}_p) \), \( \tilde{C}_\infty \) is the set of cusps of \( X_1(N) \) that lie above the \( \infty \)-cusp of \( X_0(N) \) and \( H_1 := H_1(X_1(N), \mathbb{Z}_p) \). The map \( \varpi^0 \) takes the adjusted Mainin symbol \([u, v]^* \) (Definition 6.2) to the cup product \( (1 - \zeta_N^2, 1 - \zeta_N) \).

**Conjecture 1.1** (Sharifi). Let \( \mathfrak{i}_\infty \) be the ideal of the modular Hecke algebra generated by \( T_l - 1 - (l)l \) for \( l \nmid N \), and \( T_N - 1 \), and let \( I_\infty \) be the image of \( \mathfrak{i}_\infty \) in the cuspidal Hecke algebra. Then \( \varpi^0, \varpi \) induce maps:

\[
\varpi^0 : \tilde{H}_1^+/I_\infty \tilde{H}_1^+ \to H^2(\mathbb{Z}[\zeta_{Np}, \zeta_p(2)], \mu_{q^2}), \quad \varpi : \tilde{H}_1^+/I_\infty \tilde{H}_1^+ \to H^2(\mathbb{Z}[\zeta_{Np}, \zeta_p(2)]).
\]

In [9], Fukaya and Kato proved these conjectures for the modular curve \( X_1(N) \) when \( p \mid N \). For modular curves with level not divisible by \( p \), the conjecture is still open.

By using the following theorem in [18], one can connect the homology of \( X_0(N) \) with the cohomology of \( X_1(N) \).

**Theorem 1.2** (Mazur). Let \( \pi : X_1(N) \to X_0(N) \) be the natural map between modular curves. Then we have an isomorphism

\[
\pi_* : H^+_1/(I_\infty + I_G)H^+_1 \cong IH^-/(1/I^2H^-)(1),
\]

where \( G = (\mathbb{Z}/N\mathbb{Z})^\times /\{ \pm 1 \} \).

**Conjecture 1.3.** Viewing \( b \) as an element in \( \text{Hom}(\text{Gal}(F_{-1}/K), H^+_1/(I_\infty + I_G)H^+_1) \) and assuming the Eisenstein quotient conjecture, we have

\[
b \circ \varpi_G = 1,
\]

where \( \varpi_G \) is the map induced on \( G \)-coinvariants by \( \varpi \).
Assuming Conjecture 1.1 and Conjecture 1.3 we use the Theorem \(\tilde{b} = 1\) to compute the invariant \(\tilde{b}\). Our result is \(\tilde{b} = 1\); for the details, see Theorem 6.30. In [9], Fukaya and Kato defined a map \( \infty : H^2(Y_1(N)_{\mathbb{Z}[1/Np]}, \mathbb{Z}_p(2)) \to H^2(\mathbb{Z}[\zeta_N], \mathbb{Z}_p(2))^\perp \), where \(Y_1(N)\) (Definition 2.1) is the moduli scheme over \(\mathbb{Z}[1/N]\) with \(\mathbb{C}\)-points \(Y_1(N)_{\mathbb{C}}\). And they proved that this map factors through the Eisenstein ideal when restricted to the submodule generated by the elements \(g_0, \frac{1}{\zeta} \cup g_0, \frac{1}{\zeta}\), where \(g_0, \frac{1}{\zeta}\) and \(g_0, \frac{1}{\zeta}\) are Siegel units. One can check directly that
\[
\infty (g_0, \frac{1}{\zeta} \cup g_0, \frac{1}{\zeta}) = (1 - \zeta_N^p, 1 - \zeta_N^p).
\]
If we can prove that the map
\[
\rho' : \tilde{H}_1 \to H^2(Y_1(N), \mathbb{Z}_p(2)),
[u, v]^* \mapsto g_u \cup g_v,
\]
is well-defined and Hecke-equivariant, then Conjecture 1.1 is true.

For \(p \mid N\), in Chapter 5, under some assumptions, we use the method in [9] to prove the well-defined property and the Hecke-equivariance of the map:
\[
\rho' \otimes \mathbb{Q}_p : \tilde{H}_1 \to H^2(Y_1(N), \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p.
\]

In Chapters 2 and 3, we review the general theory of modular curves and Mazur’s work on \(X_0(N)\). In Chapter 4, we compute the invariants assuming Sharifi’s conjectures. In Chapter 5, we list some partial results on Sharifi’s conjectures.

**Notation 1.4.** Let \(N \geq 5\) and \(p \geq 5\) be prime numbers such that \(N \neq p\). We fix an embedding \(\overline{\mathbb{Q}} \to \mathbb{C}\) and an embedding \(\mathbb{Q} \to \mathbb{Q}_p\). For \(n \geq 1\), let \(\zeta_n = e^{2\pi i/n} \in \overline{\mathbb{Q}} \subset \mathbb{C}\) and \(\zeta_n^{1/n} = \zeta_n + \zeta_n^{-1}\).

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### 2. Background

#### 2.1. Modular curves and modular forms.

**Definition 2.1.** Let \(Y_1(N)\) be the \(\mathbb{Z}[1/N]\)-scheme that represents the functor taking a \(\mathbb{Z}[1/N]\)-scheme \(S\) to the set of pairs \((E, \alpha)\), where \(E\) is a elliptic curve over \(S\) and \(\alpha\) is a closed immersion \(\mathbb{Z}/N\mathbb{Z} \to E\) of \(S\)-group schemes. Let \(Y_1(N) := Y_1(N) \otimes \mathbb{Q}\).

**Fact 2.2.** As a \(\mathbb{Z}[1/N]\)-scheme, \(Y_1(N)\) is a smooth and geometrically irreducible curve.

**Remark 2.3.** In [10], [30], they work with the curve \(Y_\mu(N)\), which classifies pairs \((E, \beta)\) where \(\beta : \mu_N \to E\) is a closed immersion. For the relation between \(Y_1(N)\) and \(Y_\mu(N)\), one can see [9] Section 1.4.

**Definition 2.4.** Let \(X_1(N)\) be the \(\mathbb{Z}[1/N]\)-scheme representing the functor taking a \(\mathbb{Z}[1/N]\)-scheme \(S\) to the set of pairs \((E, \alpha)\), where \(E\) is a generalized elliptic curve over \(S\) and \(\alpha\) is a closed immersion \(\mathbb{Z}/N\mathbb{Z} \to E^{reg}\) of \(S\)-group schemes. Here \(E^{reg}\) denotes the smooth locus of \(E/S\). Let \(X_1(N) := X_1(N) \otimes \mathbb{Q}\).

**Fact 2.5.** The moduli scheme \(X_1(N)\) is proper, smooth and geometrically irreducible curve over \(\mathbb{Z}[1/N]\).

**Definition 2.6.** Let \(Y_0(N)\) be the coarse moduli scheme over \(\mathbb{Z}^*[1/N]\) classifying the pairs \((E, C)\), where \(E\) is a elliptic curve over a \(\mathbb{Z}^*[1/N]\)-scheme \(S\) and \(C \subset E[S]\) is a finite flat group scheme of order \(N\) that is locally free and of rank \(N\). Let \(Y_0(N) := Y_0(N) \otimes \mathbb{Q}\).

**Definition 2.7.** Let \(X_0(N)\) be the coarse moduli scheme over \(\mathbb{Z}^*[1/N]\) classifying generalized elliptic curves together with a locally free subgroup scheme of order \(N\) in place of \(\alpha\) in Definition 2.1. Let \(X_0(N) := X_0(N) \otimes \mathbb{Q}\).

We give the definition of cusps. For the details, one can see [30] Section 1.1.

**Definition 2.8 (Cusps).** Let \(C_{/\mathbb{Z}[1/N]} := X_1(N) - Y_1(N)\). As a closed subscheme of \(X_1(N)\), it is finite étale over \(\mathbb{Z}[1/N]\).

For \(X_0(N)\), its subscheme of cusps is a disjoint union of two copies of \(\text{Spec} \mathbb{Z}[1/N]\), usually called \(0\) and \(\infty\) (cf. [6] Example 9.3.4 for their descriptions via Tate curves). Let \(C_{0/\mathbb{Z}[1/N]}\) and \(C_{\infty/\mathbb{Z}[1/N]}\) be their inverse images under the natural morphism

\[
X_1(N) \to X_0(N).
\]
One can prove that $C_{\infty/\mathbb{Z}[1/N]}$ is isomorphic to $\text{Spec} \mathbb{Z}[\zeta_N^+]\{1/N\}$, and $C_{0/\mathbb{Z}[1/N]}$ is isomorphic to the disjoint union of $\frac{\Delta}{\alpha}$ copies of $\text{Spec} \mathbb{Z}[1/N]$.

**Remark 2.9.** Let $Y_i(N) = \Gamma_i(N)\backslash \mathbb{H}$ and $X_i(N) = \Gamma_i(N)\backslash \mathbb{H}$ for $i = 0, 1$, where

$\mathbb{H} = \{ \tau \in \mathbb{C} | \text{Im}(\tau) > 0 \}$,

$\mathbb{H} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$,

$\Gamma_0(N) = \left\{ \alpha \in \text{SL}_2(\mathbb{Z}) | \alpha \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \text{ mod } N \right\}$,

$\Gamma_1(N) = \left\{ \alpha \in \text{SL}_2(\mathbb{Z}) | \alpha \equiv \begin{pmatrix} * & * \\ 0 & 1 \end{pmatrix} \text{ mod } N \right\}$.

One can prove that $Y_i(N)$ and $X_i(N)$ are all algebraic curves over $\mathbb{C}$, and $Y_i(N)$ and $X_i(N)$ are their $\mathbb{Z}[\frac{1}{N}]$-models. For the details, one can see [5, Section 8.2].

### 2.2. Hecke operators
In this subsection, we define Hecke correspondences on $X_1(N)$. For the details, one can see [30, Section 1.2], [10, Section 3]. Let $G := (\mathbb{Z}/N\mathbb{Z})^\times / \{ \pm 1 \}$. For each $a \in G$, we have an automorphism $\langle a \rangle$ of $X_1(N)$ over $X_0(N)$ which sends $(E, \alpha)$ to $(E, a\alpha)$. By pullback, this action induces automorphisms on singular cohomology, de Rham cohomology and étale cohomology. We will use $\langle a \rangle$ to denote these automorphisms.

For each prime number $l$, we denote by $X_1(N, l)$ the moduli scheme which classifies the triples $(E, \alpha, C)$, where $(E, \alpha)$ is as in Definition 2.4, and $C$ is a locally free subgroup scheme of $E$ of order $l$. Here we require that the group generated by the image of $\alpha$ and $C$ meets every geometric irreducible component of $E/S$, and moreover that image of $\alpha \cap C$ is trivial when $l = N$. We have two morphisms:

$$\pi : X_1(N, l) \to X_1(N), \quad \psi : X_1(N, l) \to X_1(N),$$

which are uniquely determined by the following rules for points of $Y_1(N)$ (cf. [10, Section 3]):

$$\pi(E, P, C) = (E, P), \quad \psi(E, P, C) = (E/C, P + C/C).$$

Then we define the following operators on the cohomology of modular curves:

$$T(l) = (\psi l)_* \pi^*, \quad T^*(l) = \pi_* \psi_1^*.$$

**Definition 2.10.** Let $T(1) = 1$. If $l \nmid N$, for $e \in \mathbb{Z}_{\geq 0}$, we define

$$T(l^{e+2}) = T(l)T(l^{e+1}) - T(l^e)(l)l,$$

$$T^*(l^{e+2}) = T^*(l)T^*(l^{e+1}) - T^*(l^e)(l)^{-1}l.$$

We also define

$$\begin{align*}
T(N^e) &= T(N)^e, \\
T^*(N^e) &= T^*(N)^e.
\end{align*}$$

**Proposition 2.11 (relative multiplicative property).** We have

$$\begin{align*}
T(n_1)(T(n_2)) &= T(n_2)T(n_1), \\
T^*(n_1)(T^*(n_2)) &= T^*(n_2)T^*(n_1), \\
T(n)(a) &= aT(n), \\
T^*(n)(\alpha) &= \alpha T^*(n), \\
T(n) &= T^*(n)(n) \text{ if } (n, N) = 1.
\end{align*}$$

**Fact 2.12.** The operators $T(n)$ and $T^*(n)$ are transposes of each other under Poincaré duality

$$H^1(X_1(N)(\mathbb{C}), \mathbb{Z}) \times H^1(X_1(N)(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}$$

and

$$H^1(Y_1(N)(\mathbb{C}), \mathbb{Z}) \times H^1_c(Y_1(N)(\mathbb{C}), \mathbb{Z}) \to \mathbb{Z}.$$

**Definition 2.13 (Atkin-Lehner involution).** Let $W_N$ denote the involution of $X_0(N)$ induced by $\tau \mapsto \frac{1}{N\tau}$ on the upper half-plane. For the details, one can see [10, IV], [13, Section 6].

For the modular scheme $X_1(N) \otimes \mathbb{Z}[1/N, \zeta_N]$, we have an involution $W_{\zeta_N}$. For the details, one can see [10, Section 6]. For an $\mathbb{Z}[1/N, \zeta_N]$-algebra $S$, the involution $W_{\zeta_N}$ sends an $S$-valued point $(E, \alpha)$ of $Y_1(N)$ to $(E', \alpha')$, where $E' = E/(\text{Image of } \alpha)$ and $\alpha'$ sends the identity of $(\mathbb{Z}/N\mathbb{Z})^\times$ to the image to $E'$ of an $N$-division point $s$ of $E$ such that the Weil-pairing of $\alpha(1)$ and $s$ equals $\zeta_N$. Via the natural projection map $X_1(N) \to X_0(N)$, the operator $W_{\zeta_N}$ maps to $W_N$.

**Definition 2.14.** For $i = 0, 1$, we define

$$S_2(\Gamma_i(N), \mathbb{Q}) := H^0(X_i(N), \Omega^1_{/\mathbb{Q}})$$
\[ M_2(\Gamma_1(N), \mathbb{Q}) := H^0(X_1(N), \Omega^1_{/\mathbb{Q}}(\log\{\text{cusps}\})) \]

where \( \Omega^1_{/\mathbb{Q}} \) is the sheaf of relative differential forms, and \( \Omega^1_{/\mathbb{Q}}(\log\{\text{cusps}\}) \) is the sheaf of differential forms which may have logarithmic poles along the cusps.

2.3. Modular symbols and Manin symbols. In this section, we introduce modular symbols and Manin symbols. The homology and cohomology groups in this section are all singular cohomology groups. For the details, one can see [33].

**Definition 2.15.** Let \( \mathbb{M}_2' \) be the free abelian group generated by \( \{\alpha, \beta\} \) for \( \alpha, \beta \in \mathbb{P}^1(\mathbb{Q}) \). Let \( \mathbb{M}_2 \) be the quotient of \( \mathbb{M}_2' \) by the relations
\[
\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0.
\]
We define the action \( \text{GL}_2(\mathbb{Q}) \) on \( \mathbb{M}_2 \) by
\[
g\{\alpha, \beta\} = \{g(\alpha), g(\beta)\} \quad \text{for all } g \in \text{GL}_2(\mathbb{Q}).
\]

**Definition 2.16.** We define
\[
\mathbb{M}_2(\Gamma_1(N)) = \left( \mathbb{M}_2 / \langle x - g(x) \mid x \in \mathbb{M}_2, g \in \Gamma_1(N) \rangle \right) / \text{torsion}.
\]

**Theorem 2.17 (Manin).** The homomorphism
\[
\phi : \mathbb{M}_2(\Gamma_1(N)) \rightarrow H_1(X_1(N), \text{cusps}, \mathbb{Z})
\]
that sends \( \{\alpha, \beta\} \) to the corresponding geodesic path in \( H_1(X_1(N), \text{cusps}, \mathbb{Z}) \) is an isomorphism.

*Proof.* For the proof, see [20].

**Proposition 2.18 (Manin).** Let \( N \) be a positive integer and \( \alpha_0, \ldots, \alpha_m \) a set of right coset representatives for \( \Gamma_1(N) \) in \( \text{SL}_2(\mathbb{Z}) \). Then \( \{\alpha_0 \{0, \infty\}, \ldots, \alpha_m \{0, \infty\}\} \) is a set of generators of \( \mathbb{M}_2(\Gamma_1(N)) \) as a \( \mathbb{Z} \)-module.

*Proof.* For the proof, see [33] Section 3.3.

We use the following proposition to give an expression for \( \alpha_i \).

**Proposition 2.19.** Let \( S \) be the following set
\[
\{[u, v] \in (\mathbb{Z}/N\mathbb{Z})^2 \mid (u, v) = 1 \in \mathbb{Z}/N\mathbb{Z}\}.
\]
We have the following bijection of sets
\[
S \xrightarrow{\psi} \Gamma_1(N) \backslash \text{SL}_2(\mathbb{Z}),
\]
\[
\psi([u, v]) = \Gamma_1(N) \left( \begin{array}{cc} a & b \\ c & d \end{array} \right),
\]
where \( c, d \in \mathbb{Z} \) are liftings of \( u, v \).

*Proof.* For the proof, see [33] Section 3.3.

**Notation 2.20.** From now on, we use \([u, v]\) to denote \( \psi([u, v])[0, \infty] \in H_1(X_1(N), \text{cusps}, \mathbb{Z}) \).

**Remark 2.21.** By Theorem 2.17, Proposition 2.18 and Proposition 2.19 we know that the pairs \([u, v]\) are generators of \( H_1(X_1(N), \text{cusps}, \mathbb{Z}) \), which are called Manin symbols.

It can be checked that the diamond operator \( (l) \) (\( l \mid N \)) acts on \([u, v]\) as follows:
\[
(l)[u, v] = [lu, lv].
\]

It is easy to check that Manin symbols satisfy the following relations:
\begin{enumerate}
  \item \([u, v] + [-v, u] = 0,\)
  \item \([u, v] = [u, u + v] + [u + v, v],\)
  \item \([-u, -v] = [u, v],\)
  \item \((j) [u, v] = [ju, jv]\) for \( j \in \mathbb{Z} \) prime to \( N \).
\end{enumerate}

For the details, one can see [20].

**Theorem 2.22 (Manin).** The above relations generate all the relations among the generators \([u, v]\), where \( u, v \in \mathbb{Z}/N\mathbb{Z} \) and \((u, v) = (1).\)

*Proof.* For the proof, see [33] Section 3.3

**Definition 2.23.** Define
\[
[u, v]^+ = \frac{1}{2}([u, v] + [u, -v]).
\]
Remark 2.24. We have an extra relation for symbols $[u, v]^+$:

$$[u, -v]^+ = [u, v]^+$$

The Manin symbols $[u, v]^+$ generate $H_1(X_1(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}[1/2])^+$ as a $\mathbb{Z}[1/2]$-module.

Set $X_i = X_i(N)(\mathbb{C}), Y_i = Y_i(N)(\mathbb{C})$, and $C_i = X_i - Y_i$ for $i = 0, 1$. We have the following exact sequence in homology

$$0 \to H_1(X_i, \mathbb{Z}) \to H_1(X_i, C_i, \mathbb{Z}) \to \tilde{H}_0(C_i, \mathbb{Z}) \to 0,$$

where $\tilde{H}_0$ is used to denote reduced homology.

The exact sequence (5) does not need to split as Hecke modules. However, it does if we tensor (5) by $\mathbb{Q}$.

Theorem 2.25 (Drinfeld-Manin splitting). The exact sequence

$$0 \to H_1(X_i, \mathbb{Q}) \to H_1(X_i, C_i, \mathbb{Q}) \to \tilde{H}_0(C_i, \mathbb{Q}) \to 0$$

splits canonically as a sequence of Hecke modules.

Proof. For the proof, one can see [14, Section 2, Chapter IV]. \hfill \square

Let $N, p$ be as in the beginning of Chapter 1. We are interested in the following two sequences:

$$0 \to H_1(X_0(N)(\mathbb{C}), \mathbb{Z}_p) \to H_1(X_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}_p) \to \tilde{H}_0(\text{cusps}, \mathbb{Z}_p) \to 0,$$

$$0 \to H_1(X_0(N)(\mathbb{C}), \mathbb{Q}_p) \to H_1(X_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Q}_p) \to \tilde{H}_0(\text{cusps}, \mathbb{Q}_p) \to 0.$$  

Remark 2.26. Since $N$ is prime, it is easy to see that $\tilde{H}_0(\text{cusps}, \mathbb{Z}_p) \cong \mathbb{Z}_p$, and $e = (\infty) - (0)$ is a generator.

The Drinfeld-Manin splitting provides a map

$$\pi : H_1(X_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Q}_p) \to H_1(X_0(N)(\mathbb{C}), \mathbb{Q}_p).$$

Notation 2.27 (Winding element). Let

$$\{0, \infty\}_{DM} := \pi(\{0, \infty\}) \in H_1(X_0(N)(\mathbb{C}), \mathbb{Q}_p).$$

where $\{0, \infty\} \in H_1(X_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}_p)$.

2.4. Siegel units.

Definition 2.28. Let $\mathcal{Y}(N)$ be the $\mathbb{Z}[1/N]$-scheme that represents the functor taking a $\mathbb{Z}[1/N]$-scheme $S$ to a set of triples $(E, P_1, P_2)$, where $E$ is an elliptic curve over $S$ and $P_1, P_2 \in E[N]$ are such that $\mathbb{Z}/N \mathbb{Z} \times \mathbb{Z}/N \mathbb{Z} \to E; (a, b) \mapsto aP_1 + bP_2$ is injective. For the representable property, one can see [25].

Definition 2.29. Let $(\mathcal{E}, e_1, e_2)$ be the triple corresponding to the identity morphism in $\mathcal{Y}(N)(\mathcal{Y}(N))$. $\mathcal{E}$ is called the universal elliptic curve.

In order to define Siegel units, we need to define theta functions on the universal elliptic curve first.

Theorem 2.30 (Kato). For any $c$ prime to 6, there is an unique element $c \theta_{\mathcal{E}} \in \mathcal{O}(\mathcal{E} \setminus \mathcal{E}[c]^x$ with divisor $c^2(0) - \mathcal{E}[c]$ which is invariant under the norm maps induced by the following maps

$$b : \mathcal{E} \setminus \mathcal{E}[bc] \to \mathcal{E} \setminus \mathcal{E}[c],$$

$$x \mapsto bx$$

where $b$ is an integer prime to $c$.

Proof. For the proof, one can see [25, Proposition 1.3]. \hfill \square

Definition 2.31 (Siegel units). Let $\mathcal{E}$ be the universal elliptic curve over $\mathcal{Y}(N)$, $N \geq 5$. For $c$ prime to $6N$, we define

$$c g_{\alpha, \beta} = \iota_{\alpha, \beta} \mathcal{E} \in \mathcal{O}(\mathcal{Y}(N))^x,$$

where $(\alpha, \beta) = \left(\frac{a}{N}, \frac{b}{N}\right) \in \left(\frac{\mathbb{Z}}{N \mathbb{Z}} \times \frac{\mathbb{Z}}{N \mathbb{Z}}\right) \setminus (0, 0)$, and $\iota_{\alpha, \beta} = ae_1 + be_2 : \mathcal{Y}(N) \to \mathcal{E} \setminus \mathcal{E}[c]$.

By taking $c$ such that $c \equiv 1 \mod N$ and $c \neq \pm 1$, let $g_{\alpha, \beta} = c g_{\alpha, \beta} \otimes (c^2 - 1)^{-1} \in \mathcal{O}(\mathcal{Y}(N))^x \otimes \mathbb{Q}$. It can be checked that $g_{\alpha, \beta}$ is independent of the choice of $c$.

Definition 2.32. For

$$\left(\begin{array}{c|c} s & u \\ \hline t & v \end{array}\right) \in \text{GL}_2(\mathbb{Z}/N \mathbb{Z}),$$

we define

$$\left(\begin{array}{c|c} s & u \\ \hline t & v \end{array}\right)^* c g_{\alpha, \beta} = c g_{\alpha', \beta'}, (\alpha', \beta') = (\alpha, \beta) \left(\begin{array}{c|c} s & u \\ \hline t & v \end{array}\right).$$
Remark 2.33. By the definition of \( \mathcal{Y}_1(N) \), we know that \( c_{g_0, \beta} \in \mu^*(\mathcal{O}(\mathcal{Y}_1(N))^\times) \) where \( \mu \) is the natural projection of \( \mathcal{Y}(N) \) to \( \mathcal{Y}_1(N) \) which sends \( (E, e_1, e_2) \) to \( (E, e_2) \). For a prime number \( l \nmid N, l \geq 5 \), we then have elements

\[
g_{0, \beta} \in \mathcal{O}(\mathcal{Y}_1(N))^\times \otimes \mathbb{Z}_l.
\]

**Proposition 2.34.** As a meromorphic modular function on \( \mathcal{Y}_1(N) \), \( g_{0, \beta} \) has the following \( q \)-expansion:

\[
g_{0, \beta}(\tau) = q^{\frac{1}{12}} \prod_{n \geq 0} \left( 1 - q^n \zeta_N^b \right) \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right),
\]

where \( \mathbb{E}_2(x) = x^2 - x + \frac{1}{6} \).

**Proof.** For the proof, see [25, 1.9]. \( \square \)

From Remark 2.33 we have the Siegel unit \( g_{0, \beta} \in \mathcal{O}(\mathcal{Y}_1(N))^\times \otimes \mathbb{Z}_p \). Let \( F \) be its norm to \( \mathcal{O}(\mathcal{Y}_0(N))^\times \otimes \mathbb{Z}_p \).

**Proposition 2.35.** The unit \( F \) has \( q \)-expansion:

\[
F = Nq^{\frac{N-1}{12}} \prod_{n > 0} \left( \frac{1 - q^n}{1 - q^n} \right)^2.
\]

**Proof.** Since

\[
g_{0, \beta} = q^{\frac{1}{12}} \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right) \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right),
\]

we have

\[
F = \prod_{i=1}^{N-1} g_{0, \beta} = q^{\frac{N-1}{12}} \prod_{i=1}^{N-1} \left( 1 - \zeta_N^b \right) \prod_{i=1}^{N-1} \left( \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right) \right) \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right).
\]

It is easy to see that

\[
(1) \quad \prod_{i=1}^{N-1} (1 - \zeta_N^b) = N,
\]

\[
(2) \quad \prod_{n > 0} \left( 1 - q^n \zeta_N^b \right) = \prod_{i=1}^{N-1} \left( 1 - q^n \zeta_N^b \right) = \frac{1 - q^N}{1 - q}.
\]

So we have

\[
\prod_{i=1}^{N-1} g_{0, \beta} = q^{\frac{N-1}{12}} N \prod_{n > 0} \left( \frac{1 - q^n}{1 - q^n} \right)^2.
\]

\( \square \)

**Definition 2.36.** We define \( g = \frac{N}{\frac{N-1}{12}} \).

**Definition 2.37.** Let \( \Delta(z) \in S_{12}(\text{SL}_2(\mathbb{Z}), \mathbb{C}) \) be the discriminant function. For the properties of discriminant function, see [7, Section 1.1]. Define

\[
f = \frac{\Delta(z)}{\Delta(Nz)}.
\]

It is a modular function of \( \mathcal{Y}_0(N) \). A similar function has already been considered by Ogg, for the details, see [27]. We have the following two propositions for the function \( f \). The proof is left as an exercise for the reader.

**Proposition 2.38.** We have

\[
\text{Div}(f) = (N - 1)((0) - (\infty)).
\]

**Proposition 2.39.** The leading coefficient of \( f(z) \) at \( \infty \) is \( 1 \), and the leading coefficient of \( f(z) \) at \( 0 \) is \( N^{12} \).

**Remark 2.40.** From the \( q \)-expansion, we know that \( f = g^{-12} \), and the leading coefficient of \( g \) at \( 0 \) is \( \frac{1}{N} \).

2.5. Mazur’s results on the homology of \( X_0(N) \). Let \( N, p \) be as Chapter 2, and assume \( p \nmid (N - 1) \). Let \( g \) be the highest \( p \)-power dividing \( N - 1 \). Let \( \mathfrak{b}_0(N) \) be the subring of \( \text{End}_{\mathbb{Z}_p}(H^1(\mathcal{Y}_0(N)(\mathbb{C}), \mathbb{Z}_p)) \) generated by \( T(n) (n \geq 1, n \nmid N) \) and \( W_N \), and let \( \mathfrak{b}_0(N) \) be its image in \( \text{End}_{\mathbb{Z}_p}(H^1(X_0(N)(\mathbb{C}), \mathbb{Z}_p)) \).

**Notation 2.41.** Let \( n \) be the numerator of \( \frac{N-1}{12} \).

**Theorem 2.42** (Mazur). There is a surjective algebra homomorphism:

\[
\phi : \mathfrak{b}_0(N) \rightarrow \mathbb{Z}_p/(n),
\]

\[
T(l) \mapsto 1 + l.
\]

The kernel of \( \phi \) is generated by \( \eta_l := 1 + l - T(l) \) and \( W_N + 1 \). We will use \( I \) (the Eisenstein ideal) to denote this kernel.
Proof. For the proof, see [18, Proposition 9.7]. □

Let $H := H_1(X_0(N)(\mathbb{C}), \mathbb{Z}_p)$. Let $H^+$ be the subspace of $H$ fixed by the complex conjugation $c$ and $H_+$ be the largest quotient of $H$ on which $c$ acts trivially.

**Remark 2.43.** Since $p$ is odd, we have a direct sum decomposition of $H$:

$$H = \frac{1+c}2 H \oplus \frac{1-c}2 H.$$

We identify $H^+$ and $H_+$ with $\frac{1+c}2 H$.

**Definition 2.44.** Let

$$e^+: I \to H^+$$

be the $h_0(N)$-morphism such that

$$x \mapsto x\{0, \infty\}_{DM}.$$

Let

$$\tilde{e}: I h_0(N)_I \to H^+_I$$

denote the induced map on $I$-adic completions. We call this map the winding homomorphism.

**Remark 2.45.** These maps are well defined since

$$I H_1(Y_0(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}) \subset H_1(X_0(N)(\mathbb{C}), \mathbb{Z}).$$

**Definition 2.46 (Shimura subgroup).** The Shimura covering is the maximal unramified subcover intermediate to $X_1(N) \to X_0(N)$ whose covering group is the unique quotient group of $(\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$ which has order $n$. From the natural map $X_1(N) \to X_0(N)$, we obtain a Picard morphism $\text{Jac}(X_0(N)) \to \text{Jac}(X_1(N))$. We define the Shimura subgroup $\Sigma$ to be the kernel of this map on $\mathbb{Q}$-points.

**Remark 2.47.** Mazur proved that the Shimura subgroup is killed by the Eisenstein ideal. For the proof, see [18, Proposition 11.7].

**Lemma 2.48 (Mazur).** The group $H^+/IH^+$ is cyclic of order $q$. There is a canonical isomorphism

$$\phi: (\mathbb{Z}/N\mathbb{Z})^\times/(\mathbb{Z}/N\mathbb{Z})^\times \to H^+/IH^+$$

which identifies $H^+/IH^+$ with the Galois group of $q$-subcover of the Shimura covering.

**Proof.** For the proof, see [18, Lemma 18.7]. □

**Proposition 2.49 (Mazur).** Let $a$ and $b$ be coprime integers with $b$ relatively prime to $N$. Let $\tilde{b}$ denote the image of $b$ in $(\mathbb{Z}/N\mathbb{Z})^\times$. Let $\Phi(a/b) \in H^+/IH^+$ denote the image of the modular symbol $\{a/b\}$ in $H^+/IH^+$. Then

$$\Phi(a/b) = \phi(\tilde{b}^{-1}).$$

**Proof.** For the proof, see [18, Proposition 18.8] □

**Example 2.50.** By Proposition 2.49, we have

$$\Phi([x,1]^+) = \phi(x^{-1}),$$

where $x \in (\mathbb{Z}/N\mathbb{Z})^\times$.

**Proposition 2.51 (Congruence formula for the winding homomorphism).** Let $\eta_l = 1 + l - T(l)$ and $\tilde{e}^+: I/I^2 \to H^+/IH^+$ be the homomorphism induced from $\tilde{e}: I \to H^+_I$. Then:

$$\tilde{e}^+(\eta_l) = (l-1)\phi(l),$$

where $l$ is any prime number different from $N$.

**Proof.** For the proof, see [18, Theorem 18.10]. □

**Definition 2.52.** Let $\mathfrak{P}$ be the ideal $(p, I)$ of $h_0(N)$. By Theorem 2.42 it is a maximal ideal of $h_0(N)$, and $h_0(N)/\mathfrak{P} \cong \mathbb{F}_p$.

**Definition 2.53 (good prime).** Suppose that $w$ is a prime number dividing $n$. Let $l$ be a prime different from $N$. We say that $l$ is good for $(w, N)$ if either

1. one of $l$ or $w$ is odd, $l$ is not a $w$-th power modulo $N$, and $\frac{l-1}2 \not\equiv 0 \mod w$.
2. $l = w = 2$ and $-4$ is not an 8-th power modulo $N$. 


Theorem 2.54 (Mazur). Let $p$ be a odd prime number dividing $n$. Let $l$ be a prime number different from $N$. Then $\eta_l$ is a generator of the ideal $I_\ell = H_0(N)_{\ell} \subset h_0(N)_{\ell}$ if and only if $l$ is a good prime number for $(p,N)$. Moreover, the winding homomorphism $\hat{e}^+: I_\ell \to H^1_\ell$ is an isomorphism of Hecke modules.

Corollary 2.55. The Hecke algebra $h_0(N)_{\ell}$ is Gorenstein.

3. Structure of homology modulo Eisenstein ideal

Notation 3.1. For a scheme $C$ over $\mathbb{Q}$, we will use $H^1_{et}(C)$ to denote $H^1_{et}(C \otimes \overline{\mathbb{Q}}, \mathbb{Z}_p)$.

Via the comparison theorem between étale cohomology and singular cohomology, we may view $H^1(X_0(N)(C), \mathbb{Z}_p)$ as a Galois module. We list some well-known properties of this Galois module.

1. As a $h_0(N) \otimes \mathbb{Q}_p$-module, $H^1(X_0(N)(C), \mathbb{Q}_p) \cong H^1_{et}(X_0(N)) \otimes \mathbb{Q}_p$ is free of rank 2.
2. For a prime number $l \mid pN$, the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is unramified at $l$, and we have
   \[ \det(1 - T_l^{-1}) = 1 - T(l)u + l(l)u^2 = 1 - (l)T^*(l)u + l(l)u^2. \]
3. The determinant of $\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is $\kappa(\sigma)^{-1}(\sigma)^{-1}$, where $\kappa : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{Z}_p^\times$ is the cyclotomic character, and $(\sigma)$ denotes $(a)$ for $a \in (\mathbb{Z}/N\mathbb{Z})^\times$ such that $\sigma(\zeta_N) = \zeta_N^a$.

For the $h_0(N)_{\ell} \otimes \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$-module structure of $H^1_{et}(X_0(N))_{\ell}$, we have the following filtration:

\[ 0 \to H^1_{et}(X_0(N))_{\ell, \text{sub}} \to H^1_{et}(X_0(N))_{\ell} \to H^1_{et}(X_0(N))_{\ell, \text{quo}} \to 0. \]

Remark 3.2. In fact, $H^1_{et}(X_0(N))_{\ell, \text{sub}}$ is a free $h_0(N)_{\ell}$-module, and $H^1_{et}(X_0(N))_{\ell, \text{ord}}$ is a dualizing $h_0(N)_{\ell}$-module. Moreover, we have the following.

1. The action of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on $H^1_{et}(X_0(N))_{\ell, \text{quo}}(1)$ is unramified.
2. The action of $\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ on $H^1_{et}(X_0(N))_{\ell, \text{sub}}$ is unramified.
3. Let $\alpha_p$ be the unit root of the equation $X^2 - T_pX + p = 0$ in the Hecke algebra $h_0(N)_{\ell}$. Then $\text{Fr}_{p,\ell}$ acts on $H^1_{et}(X_0(N))_{\ell, \text{quo}}(1)$ by multiplication by $\alpha_p$.

Remark 3.3. For the proof of the above statements, one can see [35, Proposition 3.3.4, Lemma 3.3.6].

Notation 3.4. Let $H^1(X) = H^1_{et}(X_0(N))$ and $H^1(Y) = H^1_{et}(Y_0(N))$.

Remark 3.5. We have an isomorphism from Poincaré duality

\[ H^1(X)(1) \cong H_1(X_0(N), \mathbb{Z}_p), \]

which respects complex conjugation. We use these isomorphisms to transfer from homology groups to cohomology groups.

In this section, following the method in [9, Section 6.3] and [32, Section 4], we construct a global exact sequence which gives the $h_0(N)_{\ell} \otimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-module structure of $H^1(X)/1H^1(X)$. Let $\xi = \frac{N}{12}$. By (1), we have the following exact sequence of $h_0(N)\otimes \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-modules:

\[ 0 \to H^1(X)(1) \to H^1(Y)(1) \to \mathbb{Z}_p\xi \to 0, \]

where $e = (\infty) - (0)$.

Remark 3.6. Via duality, the sequence (1) is isomorphic to the following sequence of homology groups

\[ 0 \to H_1(X_0(N), \mathbb{Z}_p) \to H_1(X_0(N), \text{cusps}, \mathbb{Z}_p) \to \mathbb{Z}_p\xi \to 0. \]

For the details, see [32, Proposition 3.5].

Lemma 3.7. We have

\[ H^1(Y)_{DM,\ell, \ell} / H^1(X)_{\ell} \cong h_0(N)/I \cong \mathbb{Z}_p/\xi. \]

Proof. The isomorphism

\[ h_0(N)/I \cong \mathbb{Z}_p/\xi \]

is directly deduced from Theorem 2.52.

It is easy to see that $H^1(Y)_{DM,\ell}/H^1(X)_{\ell}$ is generated by the image of $\{0, \infty\}_{DM,\ell}$. The isomorphism

\[ H^1(Y)_{DM,\ell}/H^1(X)_{\ell} \cong h_0(N)/I \]

follows from the fact that $(0) - (\infty)$ has order $n$.

Lemma 3.8. In $H^1(Y)_{DM,\ell}$, the element $\xi\{0, \infty\}_{DM}$ of $H^1(X)_{\ell}$ is a part of a $\mathbb{Z}_p$-basis of $H_{\ell}$. 


Proof. The modular symbol \(\{0, \infty\}_{DM, \mathfrak{p}}\) is a generator of \(H^1(Y)_{DM, \mathfrak{p}}/H^1(X)_{\mathfrak{p}}\). From the isomorphism \(H^1(Y)_{DM, \mathfrak{p}}/H^1(X)_{\mathfrak{p}} \cong \mathbb{Z}_p/\xi\), we know that \(\{0, \infty\}_{DM, \mathfrak{p}}\) is part of a \(\mathbb{Z}_p\)-basis of \(H_{\mathfrak{p}}\).

Definition 3.9. Using the Kummer sequence:
\[
0 \to \mathbb{Z}/p^n \mathbb{Z}(1) \to G_m \to G_m \to 0
\]
on the étale site of \(\mathcal{Y}_0(N)_{\mathcal{Z}[1/Np]}\), we get a map:
\[
\mathcal{O}(\mathcal{Y}_0(N))_{\mathcal{K}} \otimes \mathbb{Z}/p^n \to H^1(\mathcal{Y}_0(N) \otimes \mathbb{Z}[1/p^m], \mathbb{Z}_p(1)) \to H^1(\mathcal{Y}(1)).
\]

Let \(\bar{g}\) be the image of modular unit \(g\) in \(H^1(Y)(1)\) via this map.

Lemma 3.10. Via the map \(H^1(Y)(1) \to H^1(Y)(1)_{\mathfrak{p}}/H^1(X)(1)_{\mathfrak{p}} \cong \mathbb{Z}_p/\xi\), the element \(\bar{g}\) maps to \(\xi e\).

Proof. This is from the \(q\)-expansion of \(g\). See Proposition 2.38.

Lemma 3.11. The kernel of \(H^1(Y)(1)_{\mathfrak{p}} \to H^1(Y)(1)_{DM, \mathfrak{p}}\) is generated by \(\bar{g}\).

Proof. Let \(U \subset H^1(Y)(1)_{\mathfrak{p}}\) be the image of \(\mathcal{O}(\mathcal{Y}_0(N))_{\mathcal{K}} \otimes \mathbb{Z}_p\) under the Kummer map. Then \(U\) is a \(\mathfrak{h}_0(N)\)-submodule generated by \(\bar{g}\). As \(\mathbb{Z}_p\)-modules, \(U \cong \mathbb{Z}_p\), so \(U\) maps injectively into \(H^1(Y)(1)_{\mathfrak{p}}/H^1(X)(1)_{\mathfrak{p}}\). It means that \(U \cap H^1(X)(\mathfrak{p})(1) = 0\), so \(U\) is contained in the kernel of \(H^1(Y)(1)_{\mathfrak{p}} \to H^1(Y)(1)_{DM, \mathfrak{p}}\). By the definition of splitting, this kernel maps injectively into \(H^1(Y)(1)/H^1(X)(1) \cong \mathbb{Z}_p\), and by Lemma 3.11 and the definition of congruence module (cf. [29, Lemma 1.1.4]), the image of this kernel in \(H^1(Y)(1)/H^1(X)(1) \cong \mathbb{Z}_p\) is \(\xi\mathbb{Z}_p\). By Lemma 3.10, \(\bar{g}\) maps to \(\xi\) in \(H^1(Y)(1)/H^1(X)(1)\). So \(\bar{g}\) generates the kernel.

Notation 3.12. Let \(P = H^1(X)^{-}/IH^1(X)^{-}\), \(Q = H^1(X)^{+}/IH^1(X)^{+}\), \(R = H^1(Y)_{DM}/H^1(X)\), where \(\pm\) is with respect to the action of complex conjugation on \(X_0(N)(\mathbb{C})\).

Remark 3.13. The group \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) acts on \(R\) via \(\kappa^{-1}\). This is because \(R(1)\) is generated by the cusps of \(X_0(N)\), which are both rational over \(\mathbb{Q}\), so it has trivial Galois action.

Lemma 3.14. We have a Galois-equivariant perfect pairing \((, )\) from the cup product of étale cohomology:
\[
H^1(X) \times H^1(X) \to \mathbb{Z}_p(-1),
\]
such that
\[
(tx, y) = (x, ty), x, y \in H^1(X), t \in \mathfrak{h}_0(N).
\]

Proof. This follows from Poincaré duality of étale cohomology.

Definition 3.15. We define a \(\mathbb{Z}_p\)-linear map from \(H^1(X)/IH^1(X)\) to \(\mathbb{Z}_p/(\xi)\) as follows. Since \(\{0, \infty\}_{DM, \mathfrak{p}} \in H^1(X)_{\mathfrak{p}}\), we get a map:
\[
\begin{align*}
H^1(X)_{\mathfrak{p}} & \to \mathbb{Z}_p/\xi \mathbb{Z}_p \\
(x) & \mapsto (x, W_N \xi \{0, \infty\}_{DM, \mathfrak{p}}).
\end{align*}
\]

Since \(\mathfrak{h}/I \cong \mathbb{Z}_p/\xi \mathbb{Z}_p\), the above map factors through \(H^1(X)/IH^1(X)\). So we get a map from \(H^1(X)/IH^1(X)\) to \(\mathbb{Z}_p/\xi \mathbb{Z}_p\).

Lemma 3.16. The map \((- , W_N \xi \{0, \infty\}_{DM})\) is surjective.

Proof. By Lemma 3.17, \(\{0, \infty\}_{DM}\) is part of a \(\mathbb{Z}_p\)-basis of \(H^1(X)_{\mathfrak{p}}\). Since the Poincaré duality pairing is perfect, the map that we are considering is surjective.

Notation 3.17. In order to simplify the notation, let \(T = H^1(X)/IH^1(X), P' = \ker(T \to \mathbb{Z}_p/\xi),\) and \(Q' = T/P'\).

Proposition 3.18. The group \(P'\) is a Galois submodule of \(T\), and the Galois action on \(Q'\) is trivial.

Remark 3.19. We have an exact sequence of \(\mathfrak{h}_0(N)[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]\)-modules:
\[
0 \to P' \to T \to Q' \to 0.
\]

Proof. For any element \(x \in P'\), we have \((x, W_N \xi \{0, \infty\}_{DM}) \in \mathbb{Z}_p\). For \(\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), we need to prove that \((\sigma x, W_N \xi \{0, \infty\}_{DM}) \in \xi \mathbb{Z}_p\). Note that \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) acts on \(R\) via \(\kappa^{-1}\). Hence, \(\mathbb{Z}_p\{0, \infty\}_{DM} = \kappa(\sigma)\{0, \infty\}_{DM} + y\) for some \(y \in H^1(X)_{\mathfrak{p}}\). Note that
\[
(\sigma x, W_N \xi \{0, \infty\}_{DM}) = (\sigma x, \sigma^{-1} W_N \xi \{0, \infty\}_{DM}).
\]
By the properties of Poincaré pairing, we know that
\[(\sigma x, \sigma^*W_N\xi\{0,\infty\}_{DM}) = \kappa(\sigma)^{-1}(x, W_N\xi(\kappa(\sigma)\{0,\infty\}_{DM} + y)) \]
\[= (x, W_N\xi\{0,\infty\}_{DM}) + \kappa(\sigma)^{-1}(x, W_N\xi y).\]
Since \((x, W_N\xi\{0,\infty\}_{DM}) \in \xi\mathbb{Z}_p\) and \(\kappa(\sigma)^{-1}(x, W_N\xi y) \in \xi\mathbb{Z}_p\), we know that \((\sigma x, W_N\xi\{0,\infty\}_{DM}) \in \xi\mathbb{Z}_p\). Similarly, one can prove that the Galois action on \(Q\) is trivial.

Using the local filtration \(S\), we try to find a local splitting of \(\mathcal{I}\).

Let \(\tau\) be an element of the inertia subgroup of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}_p)\) whose image in \(\text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)\) is a generator. Let \(f = \kappa(\tau)^{-1}\). Let \(S = \{x \in H^1(X) | \tau x = fx\}\).

**Proposition 3.20.** There is a direct sum decomposition of Hecke-modules,
\[H^1(X)_{\mathfrak{P}} = H^1(X)_{\mathfrak{P},\text{sub}} \oplus S_{\mathfrak{P}}.\]

**Proof.** The action of \(\tau\) on \(H^1(X)_{\mathfrak{P},\text{sub}}\) is trivial, and the action of \(\tau\) on \(S\) is multiplication by \(f\). It is clear that \(f \neq 1 \mod \mathfrak{P}\).

**Lemma 3.21.** The composition
\[\phi : H^1(X)_{\mathfrak{P},\text{sub}}/IH^1(X)_{\mathfrak{P},\text{sub}} \rightarrow H^1(X)/IH^1(X) \rightarrow Q'\]
is an isomorphism. The canonical map \(P' \rightarrow H^1(X)_{\mathfrak{P},\text{quo}}/IH^1(X)_{\mathfrak{P},\text{quo}}\) is an isomorphism, and the action of \(\tau\) on \(P'\) is multiplication by \(f\).

**Proof.** Since we have a surjection from \(H^1(X)_{\mathfrak{P},\text{quo}}/IH^1(X)_{\mathfrak{P},\text{quo}}\) to \(\text{Coker}(\phi)\), and the action of \(\tau\) on \(H^1(X)_{\mathfrak{P},\text{quo}}\) is \(f\), it follows that the action of \(\tau\) on \(\text{Coker}(\phi)\) is also by \(f\). But we know that the action of \(\tau\) on \(Q'\) is trivial, so \(\phi\) has to be a surjection. Note that \(Q' \cong h_0(N)_{\mathfrak{P}}/I\) and \(H^1_{\mathfrak{P},\text{sub}}/IH^1_{\mathfrak{P},\text{sub}} \cong h_0(N)_{\mathfrak{P}}/I\), so \(\phi\) is an isomorphism. It follows that \(P' \rightarrow H^1(X)_{\mathfrak{P},\text{quo}}/IH^1(X)_{\mathfrak{P},\text{quo}}\) is also an isomorphism, and \(\tau\) acts on \(P'\) by \(f\), and \(P' = S/IS\).

**Lemma 3.22.** The action of \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\) on \(P'\) is given by \(\kappa^{-1}\).

**Proof.** Since \(H^1(X)_{\mathfrak{P}} = H^1(X)_{\mathfrak{P},\text{sub}} \oplus S_{\mathfrak{P}}\), we can write down the following components of \(\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\),
1. \(a(\sigma) \in \text{Hom}_{h_0(N)_{\mathfrak{P}}}(H^1(X)_{\mathfrak{P},\text{sub}}, H^1(X)_{\mathfrak{P},\text{sub}})\),
2. \(b(\sigma) \in \text{Hom}_{h_0(N)_{\mathfrak{P}}}(H^1(X)_{\mathfrak{P},\text{sub}}, S_{\mathfrak{P}})\),
3. \(c(\sigma) \in \text{Hom}_{h_0(N)_{\mathfrak{P}}}(S_{\mathfrak{P}}, H^1(X)_{\mathfrak{P},\text{sub}})\),
4. \(d(\sigma) \in \text{Hom}_{h_0(N)_{\mathfrak{P}}}(S_{\mathfrak{P}}, S_{\mathfrak{P}})\).

Since \(S/IS\) is stable under the action of Galois, we have \(c(\sigma) \equiv 0 \mod I\). Since \(H^1(X)_{\mathfrak{P},\text{sub}}\) is free of rank 1 over \(h_0(N)_{\mathfrak{P}}\), we have \(a(\sigma) \in h_0(N)_{\mathfrak{P}}\). Since \(S_{\mathfrak{P}}\) is a dualizing module, \(d(\sigma) \in h_0(N)_{\mathfrak{P}}\). Since
\[\det(\sigma) = \kappa^{-1}(\sigma) = a(\sigma)d(\sigma) - b(\sigma)c(\sigma),\]
we know that \(a(\sigma)d(\sigma) \equiv \kappa^{-1}(\sigma) \mod I\). Note that \(a(\sigma) \equiv 1 \mod I\) because \(H^1_{\mathfrak{P},\text{sub}}/IH^1_{\mathfrak{P},\text{sub}} \cong Q'\). Hence \(d(\sigma) \equiv \kappa^{-1}(\sigma) \mod I\).

**Remark 3.23.** Since the action of complex conjugation on \(P'\) is multiplication by \(-1\), this proves that \(P' = P\).

**Remark 3.24.** Since we have canonical isomorphism
\[Q = H^1(X)^+/IH^1(X)^+ \cong Q' \cong \mathbb{Z}_p/\xi\mathbb{Z}_p,\]
we call the element of \(H^1(X)^+/IH^1(X)^+\) corresponding to \(1 \in \mathbb{Z}_p/\xi\mathbb{Z}_p\) the canonical generator.

From the construction, we know that the sequence \(\mathcal{I}\) splits locally at \(p\). In fact, Mazur proved a stronger result.

**Theorem 3.25** (Mazur). The exact sequence
\[(11) \quad 0 \rightarrow P \rightarrow T \rightarrow Q \rightarrow 0\]
of Galois modules splits. The map
\[P \rightarrow \text{Coker}(H^1(X_1(N)) \rightarrow T)\]
is an isomorphism, which gives a canonical splitting of \(\mathcal{I}\).

**Proof.** See [18 Section 16].

**Remark 3.26.** In fact, Mazur proved that there is a decomposition of the \(I\)-torsion on the Jacobian:
\[\text{Jac}(X_0(N))_{\mathfrak{P}}(I) = \Sigma_p \oplus C_p,\]
where \(\Sigma_p\) is the \(p\)-part of the Shimura subgroup \(\Sigma\) and \(C_p\) is the \(p\)-part of the cuspidal subgroup.
Remark 3.27. Via the duality between the étale cohomology group and the Picard group, we have
\[ P \cong \text{Hom}(\Sigma, \mathbb{Q}_p/\mathbb{Z}_p), \]
\[ Q \cong \text{Hom}(C, \mathbb{Q}_p/\mathbb{Z}_p), \]
which gives the splitting of the sequence in Theorem 3.25.

4. Extension classes

In this section, we construct the extension classes we are interested in and compute the corresponding invariants.

The construction of such extension classes has been considered in \[19\]; we revisit it in this section.

Let \( q = p^f \) be the highest \( p \) power dividing \( N - 1 \), and let \( K = \mathbb{Q}(\zeta_q) \). Let \( \Delta = \text{Gal}(K/\mathbb{Q}) \) and \( U = \mathbb{F}_N^\times/(\mathbb{F}_N^\times)^q \).

We want to consider the following exact sequence:

\[ 0 \to H^1(X)/IH^1(X) \otimes I/I^2 \to H^1(X)/I^2 \to H^1(X)/I^2 \to H^1(X)/I^2 \to 0. \]

By pushout and pullback, we get the following exact sequences:

1. \( 0 \to Q \otimes I/I^2 \to \to Q \to 0, \)
2. \( 0 \to P \otimes I/I^2 \to \to P \to 0, \)
3. \( 0 \to Q \otimes I/I^2 \to \to P \to 0, \)
4. \( 0 \to P \otimes I/I^2 \to \to P \to 0. \)

Notation 4.1. Let \( a \in H^1(\mathbb{Z}[1/\mathbb{N}_p], I/I^2) \) correspond to the exact sequence \( A \), \( b \in H^1(\mathbb{Z}[1/\mathbb{N}_p], P \otimes I/I^2) \) correspond to the exact sequence \( B \), \( c \in H^1(\mathbb{Z}[1/\mathbb{N}_p], P^{-1} \otimes I/I^2) \) correspond to the exact sequence \( C \), and \( d \in H^1(\mathbb{Z}[1/\mathbb{N}_p], I/I^2) \) correspond to the exact sequence \( D \).

Definition 4.2. Let

1. \( \chi_a \in H^1(\mathbb{Z}[1/\mathbb{N}_p, \zeta_q], I/I^2)^\Delta \) be the image of \( a \),
2. \( \chi_b \in H^1(\mathbb{Z}[1/\mathbb{N}_p, \zeta_q], P \otimes I/I^2)^\Delta \) be the image of \( b \),
3. \( \chi_c \in H^1(\mathbb{Z}[1/\mathbb{N}_p, \zeta_q], P^{-1} \otimes I/I^2)^\Delta \) be the image of \( c \),
4. \( \chi_d \in H^1(\mathbb{Z}[1/\mathbb{N}_p, \zeta_q], I/I^2)^\Delta \) be the image of \( d \).

Proposition 4.3. For \( k = 1, -1, 0 \), there exists a unique field extension \( F_k/K \) that is Galois over \( \mathbb{Q} \), satisfying the following conditions:

1. \( \Delta \) acts on \( \text{Gal}(F_k/K) \) via \( \kappa^k \), where \( \kappa \) is the mod \( q \) cyclotomic character,
2. \( \text{Gal}(F_k/K) \cong \mathbb{Z}/q\mathbb{Z} \),
3. \( F_k/K \) is unramified outside \( p \) and \( N \),
4. \( F_k/K \) is totally tamely ramified at \( N \),
5. \( F_0 \) is the unique \( q \)-subextension of \( K(\zeta_N)/K \),
6. \( F_1 \) is the unique \( q \)-ramification at \( p \) over \( K \) and equals \( K(N^{1/q}) \),
7. \( F_{-1}/K \) splits completely at \( p \).

Proof. See \[4\] Lemma 3.9, Proposition 5.4.

Notation 4.4. We use \( G_k \) to denote \( \text{Gal}(F_k/K) \).

By Proposition 4.3, we know that \( \chi_a, \chi_d \in \text{Hom}(G_0, I/I^2) \), \( \chi_b \in \text{Hom}(G_{-1}, P \otimes I/I^2) \), \( \chi_c \in \text{Hom}(G_1, P^{-1} \otimes I/I^2) \).

Lemma 4.5. There are canonical isomorphisms:

1. \( G_0 \cong U \),
2. \( G_1 \cong \mu_q \),
3. \( G_{-1} \cong U^\otimes 2 \otimes \mu^{-1} \).

Proof. (1) We have \( \text{Gal}(K(\zeta_N)/K) \cong \mathbb{F}_N^\times \), so \( G_0 \cong \mathbb{F}_N^\times/(\mathbb{F}_N^\times)^q = U \).

(2) For \( G_1 \), we have the Kummer map
\[ G_1 \to \mu_q, \quad \sigma \mapsto \sigma^{(N^{1/q})}/N^{1/q}, \]
which gives the canonical isomorphism \( G_1 \cong \mu_q \).

(3) For the isomorphism \( G_{-1} \cong U^\otimes 2 \otimes \mu^{-1} \),
see Remark 4.3.

\( \square \)
Definition 4.6. By the winding isomorphism (Proposition [2.51]), \(I/I^2\) is identified with \(U\). By Lemma [2.48] and Poincaré duality, \(P\) is identified with \(I/I^2 \otimes \mu_q^{-1}\). Hence by Lemma [4.9], \(\chi_a, \chi_d\) are characters from \(U\) to \(U\), \(\chi_b\) is a character from \(U^\otimes q \otimes \mu_q^{-1}\) to \(U^\otimes q \otimes \mu_q^{-1}\), \(\chi_c\) is a character from \(\mu_q\) to \(\mu_q\). So, we have four integers attached to \(\chi_a, \chi_b, \chi_c, \chi_d\), and we use \(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}\) to denote the invariants.

The invariants \(\tilde{a}\) and \(\tilde{d}\) have been already computed in [19]. We review their argument here.

Fixing a basis \(e_+, e_\) of \(H^1(X)^+_{\mathbb{Q}}/I^2 \oplus H^1(X)_{\mathbb{Q}}^\times/I^2\), one can write the representation as follows:

\[
\rho(\sigma) = \begin{pmatrix} 1 + A(\sigma) & B(\sigma) \\ C(\sigma) & \kappa^{-1}(\sigma)(1 + D(\sigma)) \end{pmatrix}
\]

for \(\sigma \in \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\), where \(A, B, C, D : \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to I/I^2\) are homomorphisms, and \(\kappa\) is the \(p\)-adic cyclotomic character.

Proposition 4.7. For a prime \(l \not\equiv 1 \pmod{q}, l \not\equiv N\), we have

\[
1 + A(\text{Frob}_l^{-1}) = 1 + \frac{\eta_l}{l - 1}
\]

where \(\eta_l = 1 + l - T(l)\).

Proof. From the representation \(\rho\), we know that \(1 + A(\text{Frob}_l^{-1})\) is the root of \(t^2 - T(l)t + l = 0 \mod I^2\) which equal to \(1 \mod I\). We claim that

\[
1 + A(\text{Frob}_l^{-1}) \equiv 1 + \frac{\eta_l}{l - 1} \mod I^2.
\]

This is seen via the following string of equalities:

\[
\left(1 + \frac{\eta_l}{l - 1}\right)^2 - (1 + l - \eta_l)(1 + \frac{\eta_l}{l - 1}) + l = \left(1 + \frac{\eta_l}{l - 1}\right)(\frac{\eta_l}{l - 1} - \frac{\eta_l - l}{l - 1}) + l
\]

\[
= \left(1 + \frac{\eta_l}{l - 1}\right)(\frac{l}{l - 1} - \frac{\eta_l - l}{l - 1}) + l
\]

\[
= l\left(\frac{\eta_l}{l - 1} + 1\right)(\frac{\eta_l - l}{l - 1} + 1) + l
\]

\[
= 0 \mod I^2.
\]

Theorem 4.8 (Calegari-Mazur-Sharifi-Stein). We have \(\tilde{a} = -\tilde{d} = -1\).

Proof. Since

\[
1 + A(\text{Frob}_l^{-1}) + \kappa^{-1}(\text{Frob}_l^{-1})(1 + D(\text{Frob}_l^{-1})) = T(l) = 1 + l - \eta_l,
\]

and

\[
1 + A(\text{Frob}_l^{-1}) \equiv 1 + \frac{\eta_l}{l - 1} \mod I^2,
\]

we know that

\[
1 + D(\text{Frob}_l^{-1}) = 1 - \frac{\eta_l}{l - 1}.
\]

The image of \(\text{Frob}_l \in U\) is \(l\), so \(\chi_d\) maps \(l^{-1} \in U\) to \(\frac{1}{l - 1} \eta_l \in I/I^2\). Via the winding isomorphism, \(\eta_l\) maps to \(l^{-1} \in U\). Thus, the composite maps \(l^{-1} \to l\), which means that \(\tilde{a} = -1\).

As for \(\chi_d\), it maps \(l^{-1} \in U\) to \(-\frac{1}{l - 1} \eta_l \in I/I^2\). Via the winding isomorphism, \(-\frac{1}{l - 1} \eta_l\) maps to \(l^{-1} \in U\). The composite then maps \(l^{-1}\) to \(l^{-1}\), which means that \(\tilde{d} = 1\).

5. The invariant \(\tilde{c}\)

In this section, we use the method developed by Fukaya and Kato in [9 Section 9.6.3] to compute the invariant \(\tilde{c}\). Recall that the invariant \(\tilde{c}\) (resp. cocycle \(c\)) is from the following extension class:

\[
0 \to \frac{I}{I^2} \otimes H^1(X)^+_{\mathbb{Q}} \to \frac{IH^1(X)^+_{\mathbb{Q}}}{I^2H^1(X)^+} \oplus \frac{IH^1(X)^-_{\mathbb{Q}}}{I^2H^1(X)^-} \to \frac{H^1(X)^-}{I^2H^1(X)^-} \to 0
\]

In this section, we will relate the sequence \([13]\) to another sequence that arises from the cuspidal extension.

Let \(E = H^1(Y)_{DM, \mathbb{Q}}/\text{Ker}(H^1(X)_{\mathbb{Q}} \to Q)\). Then we have the following exact sequence:

\[
0 \to Q \to E \to R \to 0.
\]
Since 
\[ H^1(Y)_{DM_\mathfrak{p}}/H^1(X)_{\mathfrak{p}} \cong H^1(Y)_{DM_\mathfrak{p}}/H^1(X)_{\mathfrak{p}}, \quad H^1(Y)_{DM_\mathfrak{p}} = H^1(X)_{\mathfrak{p}}, \]
we may rewrite (14) as:
\[ \begin{align*}
0 \to & \frac{H^1(X)^+}{IH^1(X)^+} \to \frac{H^1(Y)_{DM}}{IH^1(Y)_{DM}} \oplus \frac{H^1(X)^+}{IH^1(X)^+} \to \frac{H^1(Y)^-}{IH^1(Y)^-}_{DM} \to 0 
\end{align*} \] (15)

Note that the direct sum is with respect to the Hecke module structure.

Since there is a canonical generator \( \{0, \infty\}_{DM} \) of \( R(1) \), the sequence (15) gives an element \( c' \) in \( H^1(\mathbb{Z}[1/N], Q(1)) \cong H^1(\mathbb{Z}[1/N], \mu_\eta) \).

**Proposition 5.1.** We have \( c = c' \) in \( H^1(\mathbb{Z}[1/N], \mu_\eta) \).

**Proof.** We can rewrite sequence (15) as follows:
\[ \begin{align*}
0 \to & I/I^2 \otimes \frac{H^1(X)^+}{IH^1(X)^+} \to \frac{IH^1(Y)_{DM}}{I^2H^1(Y)_{DM}} \oplus \frac{IH^1(X)^+}{I^2H^1(X)^+} \to \frac{IH^1(Y)^-}{I^2H^1(Y)^-}_{DM} \to 0. 
\end{align*} \]

Since \( I \) is a principal ideal in the Eisenstein component and \( H^1(Y)_{DM_\mathfrak{p}}/H^1(X)_{\mathfrak{p}} \cong h/I \cong I/I^2 \), choosing a generator \( \eta \in I/I^2 \), we get the isomorphisms \( \frac{H^1(X)^+}{IH^1(X)^+} \cong \frac{I}{I^2} \otimes \frac{H^1(X)^+}{IH^1(X)^+} \) and \( \frac{IH^1(Y)^-}{I^2H^1(Y)^-}_{DM} \cong \frac{IH^1(Y)^-}{I^2H^1(Y)^-}_{DM} \). By doing this, we identify two exact sequences. Although the identification depends on the choice of the generator, the extension class does not change.

So, in order to compute \( c \) or \( c' \), it suffices to compute the extension class obtained from (14). We need to make some preparations.

Let \( J \) be the Jacobian variety of \( X_0(N) \), and let \( GJ \) be the generalized Jacobian variety of \( X_0(N) \) with respect to the cusps of \( X_0(N) \). For the general properties of Jacobians and generalized Jacobians of algebraic curves, one can see [28, Section 3]. We have the following exact sequence:
\[ \begin{align*}
0 \to & T \to GJ \to J \to 0, 
\end{align*} \] (16)

where \( T := \ker(GJ \to J) \), and we have a canonical isomorphism \( T \cong \text{Coker}(\mathbb{G}_m \to \mathbb{G}_m \times \mathbb{G}_m) \).

Note that we have the following duality between étale cohomology groups and generalized Jacobians
\[ H^1(Y) \cong GJ[p]\wedge, \quad H^1(X) \cong J[p]\wedge. \]
where \( \wedge \) is \( \mathbb{Q}/\mathbb{Z} \)-dual. By this duality, the sequence
\[ \begin{align*}
0 \to & T \to GJ \to J \to 0 
\end{align*} \]
is dual to
\[ \begin{align*}
0 \to & H^1(X) \to H^1(Y) \to \mathbb{Z}_p \to 0. 
\end{align*} \]

Taking the \( q \)-kernel of the first of the above two sequences, we have the following exact sequence:
\[ \begin{align*}
0 \to & T[q] \to GJ[q] \to J[q] \to 0 
\end{align*} \] (17)

The exactness of above sequence is from the exactness of the following sequence:
\[ \begin{align*}
0 \to & H^1(X)/q \to H^1(Y)/q \to \mathbb{Z}_p e/q \to 0, 
\end{align*} \] (18)

where \( e = (\infty) - (0) \in H^1(Y)/H^1(X) \).

**Notation 5.2.** Let \( \xi = \frac{a}{m} \) and \( n = vq \), where \( (n,m) = 1 \). Note that \( \frac{\xi}{q} \) and \( \frac{1}{v} \) can be viewed as elements in \((\mathbb{Z}/q\mathbb{Z})\wedge\). To avoid ambiguity, we choose \( \bar{r}, \bar{v} \in \mathbb{Z} \) such that \( \bar{r} \equiv \frac{\xi}{q} \in \mathbb{Z}/q\mathbb{Z}, \bar{v} \equiv \frac{1}{v} \in \mathbb{Z}/q\mathbb{Z} \).

Let \( D = v(0) - v(\infty) \in J[p\wedge] \). The following lemma is from [9, section 9.6].

**Lemma 5.3.** Let
\[ \begin{align*}
\frac{H^1(Y)(1)_{DM}}{H^1(X)(1)} & \to H^1(X)(1) \otimes \frac{\mathbb{Q}_p}{\mathbb{Z}_p}, 
\end{align*} \]
be the map induced by \( H^1(Y)_{DM}(1) \to H^1(X)(1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \). Then \( \iota \) maps the image of \( v\{0, \infty\}_{DM} \) to the image of \( D \) in \( H^1(X)(1) \otimes \mathbb{Q}_p/\mathbb{Z}_p \).

**Proof.** The proof can be found in [9, Lemma 9.6.5]. For the convenience of the reader, we review the proof here. Viewing \( v\{0, \infty\} \) as an element in \( H^1(Y)(1) \), we know that the image of \( v\{0, \infty\} \) in \( H^1(X)(1) \) is \(-D\). Since the order of \( D \) is \( q \), we have \( qD = \text{Div}(h) \) for some \( h \in \mathcal{O}(Y) \wedge \). This means that \(-qv\{0, \infty\} \equiv [h] \text{ mod } H^1(X)(1) \), where \([h] \) is
the Kummer class of $h$. We can write
\begin{equation}
- q\nu(0, \infty) = [h] + x
\end{equation}
for some $x \in H^1(X)(1)$. We write $x$ as a projective system $(x_1, x_2, \ldots)$, where $x_i \in \frac{H^1(X)(1)}{\rho(H^1(X)(1))}$ for $i \in \mathbb{Z}_{\geq 0}$. Modulo $q$, we have
\[
[h] = -x_f \in \frac{H^1(Y)(1)}{qH^1(Y)(1)}.
\]
From the exact sequence
\[
0 \to H^1(X)(1) \to H^1(Y)(1) \to \mathbb{Z}_p \to 0,
\]
we know that the map $\iota : \frac{H^1(Y)(1)}{qH^1(Y)(1)} \to \frac{H^1(Y)(1)}{qH^1(Y)(1)}$ is injective. By [9, Lemma 9.6.4], we know that $\iota(D) = [h]$ and $\iota(D + x_f) = x_f + [h] = 0$. Hence,
\begin{equation}
D = -x_f \in \frac{H^1(X)(1)}{qH^1(X)(1)}.
\end{equation}
Taking the Drinfeld-Manin splitting map for $\nu(19)$, since $[h]$ maps to 0 and $x$ maps to $x$, we have
\begin{equation}
\nu(0, \infty)^{DM} = -x \in H^1(Y)(1)_{DM}.
\end{equation}
Note that we have a canonical injection
\[
\frac{H^1(Y)(1)^{DM}}{H^1(X)(1)} \hookrightarrow H^1(X)(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p.
\]
By (21), we have $\nu(0, \infty)^{DM} = (-x) \otimes \frac{1}{q}$. By (20), we have $(-x) \otimes \frac{1}{q} = D \in H^1(X)(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$. \hfill $\Box$

Remark 5.4. The map $\nu$ maps $\{0, \infty\}^{DM}$ to $\nu D \in J[q]$.

**Lemma 5.5.** The map
\[
(\cdot, W_N \xi : \{0, \infty\}^{DM}) : H^1(X) \to \mathbb{Z}_p/\xi\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow \mathbb{Q}_p
\]
corresponds to the element $-\nu D$ in $J[p^\infty]$.

**Proof.** We identify $\frac{H^1(Y)(1)^{DM}}{H^1(X)(1)}$ with $J[p^\infty]$ using the following canonical isomorphism
\[
\frac{H^1(Y)(1)^{DM}}{H^1(X)(1)} \cong H^1(X)(1) \otimes_{\mathbb{Q}_p} \mathbb{Q}_p = J[p^\infty].
\]
From the pairing
\[
H^1(X) \times J[p^\infty] \to \mathbb{Q}_p/\mathbb{Z}_p,
\]
we have the following pairing
\[
H^1(X) \times \frac{H^1(Y)(1)^{DM}}{H^1(X)(1)} \to \mathbb{Q}_p/\mathbb{Z}_p.
\]
Via the isomorphism
\[
\frac{H^1(Y)(1)^{DM}}{H^1(X)(1)} \to J[p^\infty],
\]
\[
\{0, \infty\}^{DM} \text{ maps to } \nu D \in J[q] \text{ (Remark 5.4).} \]
Note that the image of $W_N$ in $h_0(N)_p$ is $-1$. Therefore, the map $(\cdot, W_N \xi : \{0, \infty\}^{DM})$ can be restated as
\[
H^1(X) \to \frac{\mathbb{Q}_p}{\mathbb{Z}_p}[q] \cong \mathbb{Q}_p/\mathbb{Z}_p,
\]
where the map $s$ corresponds to $-\nu D \in \text{Hom}(H^1(X), \mathbb{Q}_p/\mathbb{Z}_p[q])$. Then the lemma follows. \hfill $\Box$

**Corollary 5.6.** Taking the $\mathbb{Q}_p$-dual of the map $H^1(X) \to Q = \mathbb{Z}/q\mathbb{Z}$, we get a map
\[
Q^\vee \to J[q].
\]
Then the canonical generator of $Q^\vee$ maps to $-r\nu D \in J[q]$.

**Proof.** For $\mathbb{Q}_p[q]$, we have a canonical generator $\frac{1}{q}$. So we have a canonical isomorphism
\[
Q^\vee \cong \mathbb{Z}/q\mathbb{Z}.
\]
Note that in Lemma 5.5 we choose a generator $\frac{1}{q} \in \mathbb{Q}_p[q]$. So the canonical generator of $Q^\vee$ maps to $-r\nu D$. \hfill $\Box$
Definition 5.7. In the sequence \([17]\), pulling back by the map \(\mathbb{Z}/q\mathbb{Z} \to J[q]\) which takes 1 to \(-r\vec{v}D\), and pushing out by the map \(T[q] \to \mu_q\) corresponding to \(e\), we get an extension class:

\[0 \to \mu_q \to \mathbb{Z}/q\mathbb{Z} \to 0.\]

Remark 5.8. The extension class \([14]\) is obtained from the sequence \([18]\) via pullback by \(e\), and pushout by the map \(H^1(X) \to Q\). If we take the Pontryagin dual, it is the sequence that is obtained from the sequence \([18]\) via pullback by the map of \(\mathbb{Z}/q\mathbb{Z} \to J[q]\) which maps 1 to \(-r\vec{v}D\) and pushout by \(e\). So the extension class \([22]\) is negative of the extension class given by \([14]\). We will compute the extension class given by \([22]\).

For the computation of the extension class given by \([22]\), we have the following general proposition which is from \([9\) Section 9.6]:

Proposition 5.9. Let \(C\) be a smooth projective curve over a field \(k\) with characteristic 0. Let \(\Sigma\) be a finite set of \(k\)-rational points of \(C\). Fix two degree 0 divisors \(D_1\) and \(D_2\) such that both of them are supported on \(\Sigma\), and \(D_1\) is order \(q\). Starting from the following exact sequence

\[0 \to T[q](\bar{k}) \to GJ[q](\bar{k}) \to J[q](\bar{k}) \to 0,\]

we may pull back by the map \(\mathbb{Z}/q\mathbb{Z} \to J[q]; 1 \mapsto D_1\) and push out by the map \(D_2 : T[q] \to \mu_q\) to obtain an exact sequence

\[0 \to \mu_q \to ? \to \mathbb{Z}/q\mathbb{Z} \to 0.\]

Let \(F\) be a rational function on \(C\) such that \(qD_1 = (F)\), and let \(h\) be a rational function on \(C\) such that \(\text{div}(h) - D_1\) is supported away from \(\Sigma\). Then the extension class of \([22]\) coincides with the Kummer class of

\[\left(\frac{F}{h}\right)(-D_2) := \prod_{x \in \Sigma} \frac{F}{h^q}(x)^{-m(x)},\]

where \(D_2 = \sum_{x \in \Sigma} m(x) \cdot x\).

Proof. We will use the following description of \(GJ(\bar{k})\):

\[GJ(\bar{k}) = \text{Div}^0_{\Sigma^q}(\bar{C})/\{\text{div}(f) : f \in K(\bar{C}), f \equiv 1 \mod \Sigma\},\]

where \(\text{Div}^0_{\Sigma^q}(\bar{C})\) is the group of degree 0 divisors away from \(\Sigma\). Choose a rational function \(\alpha \in K(\bar{C})\) such that \(\frac{F}{h^q\alpha} \equiv 1 \mod \Sigma\), and let \(A = \text{div}(\alpha)\). From the assumption, we know that

\[qD_1 - q\text{div}(h) - qA = 0 \in GJ(\bar{k}).\]

So we have \(D_1 - \text{div}(h) - A \in GJ[q](\bar{k})\). Note that \(D_1 - \text{div}(h) - A = D_1 \in J[q](\bar{k})\), since \(\text{div}(h)\) and \(A\) are principal divisors. Hence the extension class is given by

\[\sigma \mapsto \sigma(D_1 - \text{div}(h) - A) - (D_1 - \text{div}(h) - A).\]

Note that \(D_1 - \text{div}(h)\) is \(k\)-rational, so the extension class is the class of

\[\sigma \mapsto A - \sigma A,\]

which is the Kummer class of \(\alpha^{-q}\). Since \(\frac{F}{h^q\alpha} \equiv 1 \mod \Sigma\), we have

\[\frac{F}{h^q}(x) = \alpha^q(x)\text{ for all }x \in \Sigma.\]

Hence, after pushout by \(D_2\), the extension class is given by the Kummer class of \((\frac{F}{h})(-D_2)\). \(\square\)

Theorem 5.10. The extension class \(c\) is the Kummer class of \(N\), and the invariant \(\tilde{c}\) equals 1.

Proof. We apply Proposition 5.9 to the case that the curve is the modular curve \(X_0(N)\), and \(D_1 = -r\vec{v}D = -r\vec{v}((0) - (\infty))\) and \(D_2 = (\infty) - (0)\). We need to find a function \(F\) such that \(\text{div}(F) = -qr\vec{v}D\). By Remark 2.40, we know that \(\text{div}(g^{-1}) = \xi(((0) - (\infty))\). Hence \(\text{div}(g^{\frac{\bar{v}}{v}r\vec{v}}) = -qr\vec{v}((0) - (\infty))\). It is easy to see that \(\frac{m}{v}r\vec{v} \equiv 1 \mod q\). Again by Remark 2.40 we have

\[\frac{g^{\frac{\bar{v}}{v}r\vec{v}}((0))}{g^{\frac{\bar{v}}{v}r\vec{v}}((\infty))} = N^a,\]

where \(a \equiv -1 \mod q\), which means that the extension class \([22]\) is given by the Kummer class of \(N^{-1}\). Therefore, the extension class \([14]\) is given by the Kummer class of \(N\). By the isomorphism of Lemma 4.5 (2), the invariant \(\tilde{c}\) equals 1. \(\square\)
6. The cocycle $\chi_b$

Recall the cocycle $\chi_b \in \text{Hom}(G_{-1}, P \otimes I/I^2) \cong \text{Hom}(G_{-1}, I^2/I^3 \otimes \mu_q^{-1})$. In this section, we try to give a conjectural inverse map to $\chi_b$.

6.1. $K$-theory of integer rings and Galois cohomology. We will use the following theorem to identify $K$-groups with étale cohomology groups:

**Theorem 6.1.** For a number field $F$, let $S$ be a finite set of places containing the places above $p$. Then there is a \( \text{étale Chern character inducing an isomorphism:} \)

$$K_2(\mathfrak{a}_{F,S}) \otimes \mathbb{Z}_p \rightarrow H^2_{\text{ét}}(\mathfrak{a}_{F,S}, \mathbb{Z}_p(2)).$$

For the proof, one can see [24].

**Remark 6.2.** We may identify Galois cohomology groups with étale cohomology groups: for the details, see [17].

**Lemma 6.3.** We have the following exact sequence:

$$0 \rightarrow K_2(\mathbb{Z}[\zeta_N]) \otimes \mathbb{Z}_p \rightarrow K_2(\mathbb{Z}[\zeta_N, \frac{1}{Np}]) \otimes \mathbb{Z}_p \rightarrow K_1(F) \otimes \mathbb{Z}_p \rightarrow 0.$$

**Proof.** This is from the localization sequence of $K$-groups and the facts that

1. $K_2(F) = 0$.
2. $K_1(\mathbb{Z}[\zeta_N]) \otimes \mathbb{Z}_p \hookrightarrow K_1(\mathbb{Z}[\zeta_N, \frac{1}{Np}]) \otimes \mathbb{Z}_p$.

$\square$

**Corollary 6.4.** Using the Chern class maps of Theorem 6.1, we have the following exact sequence in étale cohomology:

$$0 \rightarrow H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mathbb{Z}_p(2)) \rightarrow H^2(\mathbb{Z}[\zeta_N, \frac{1}{Np}], \mathbb{Z}_p(2)) \rightarrow \mathbb{F}_N^\times \otimes \mathbb{Z}_p \rightarrow 0.$$

Let $G = \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. We have the following lemma:

**Lemma 6.5.** Via the injection of Corollary 6.4, we have

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mathbb{Z}_p(2)) = I_GH^2(\mathbb{Z}[\zeta_N, \frac{1}{Np}], \mathbb{Z}_p(2)).$$

**Proof.** It suffices to show that the map induced on coinvariants

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{Np}], \mathbb{Z}_p(2)) \cong \mathbb{F}_N^\times \otimes \mathbb{Z}_p$$

from the exact sequence of Corollary 6.4 is an isomorphism. But

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{Np}], \mathbb{Z}_p(2)) \cong H^2(\mathbb{Z}[\frac{1}{Np}], \mathbb{Z}_p(2))$$

by [24] Proposition 3.3.11] and

$$H^2(\mathbb{Z}[\frac{1}{Np}], \mathbb{Z}_p(2)) \cong K_2(\mathbb{Z}[\frac{1}{Np}]) \otimes \mathbb{Z}_p = \mathbb{F}_N^\times \otimes \mathbb{Z}_p,$$

so the surjection is an isomorphism by equality of orders. $\square$

**Corollary 6.6.** Fix the isomorphism

$$I_G/I_G^2 \rightarrow G,$$

$$g - 1 \mapsto g.$$

We have canonical isomorphisms:

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mathbb{Z}_p(2)) \cong I_GH^2(\mathbb{Z}[\zeta_N, \frac{1}{Np}], \mathbb{Z}_p(2)) \cong I_G/I_G^2 \otimes \mathbb{F}_N^\times \otimes \mathbb{Z}_p \cong U \otimes^G.$$

**Proposition 6.7.** We have a canonical isomorphism:

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mathbb{Z}_p(2)) \cong G_{-1} \otimes \mu_q.$$

**Proof.** Let $\Delta = \text{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$. Since $H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mathbb{Z}_p(2)) \cong U \otimes^G \cong \mathbb{Z}/q\mathbb{Z}$ and Spec $\mathbb{Z}[\zeta_N, \frac{1}{p}]$ has $p$-cohomological dimension 2 (see [24] Proposition 8.3.18)), we know that

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mu_q^{\otimes 2}) \cong H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mu_q^{\otimes 2}) \Delta.$$

Since

$$H^2(\mathbb{Z}[\zeta_N, \frac{1}{p}], \mu_q^{\otimes 2}) \cong H^2(\mathbb{Z}[\zeta_Nq, \frac{1}{p}], \mu_q^{\otimes 2}) \Delta,$$

we have the desired isomorphism.
by [24 Proposition 3.3.11], we only need to understand $H^2(\mathbb{Z}[\zeta_{Nq}, \frac{1}{p}], \mu_q^{\otimes 2}) \circ \Delta$. By [24 Proposition 8.3.11], we have the following sequence

$$0 \to A_{Q(\zeta_{Nq})} \otimes \mathbb{Z}/q\mathbb{Z} \to H^2(\mathbb{Z}[\frac{1}{p}, \zeta_{Nq}], \mu_q) \to \bigoplus_{v|p} H^2(Q(\zeta_{Nq})_v, \mu_q) \to \mathbb{Z}/q\mathbb{Z} \to 0,$$

where $A_{Q(\zeta_{Nq})} \otimes \mathbb{Z}/q\mathbb{Z}$ is ideal class group of $Q(\zeta_{Nq})$ modulo the ideal classes $[p]$, for $p|p$. Taking the $\kappa^{-1}$-eigenquotient, where $\kappa$ is the modulo $q$ cyclotomic character, since $H^2(Q(\zeta_{Nq})_v, \mu_q^{\otimes 2}) \cong \mu_q$ and $\Delta$ does not permute the $v | p$, we have $\bigoplus_{v|p} H^2(Q(\zeta_{Nq})_v, \mu_q^{\otimes 2})_\Delta = 0$. Thus, we have the following:

$$(A_{Q(\zeta_{Nq})} \otimes \mu_q) \cong H^2(\mathbb{Z}[\zeta_{Nq}, \frac{1}{p}], \mu_q^{\otimes 2})_\Delta.$$

Hence

$$H^2(\mathbb{Z}[\zeta_{Nq}, \frac{1}{p}], \mu_q^{\otimes 2}) \circ \Delta \cong (A_{Q(\zeta_{Nq})} \otimes \mu_q).$$

Now let $F/Q(\zeta_{Nq})$ be the field extension that corresponds to $(A_{Q(\zeta_{Nq})} \otimes \mu_q)\kappa^{-1}$. Note that by Proposition [4.3] we have

$$G_{-1} \cong \text{Gal}(F_{-1}/K),$$

where $F_{-1}$ is tamely ramified at $N$ and split completely at the prime dividing $p$. Hence, $F_{-1}Q(\zeta_{Nq})$ is unramified over $Q(\zeta_{Nq})$ with $\kappa^{-1}$-action and split completely at the primes dividing $p$. Then, we have $F_{-1}Q(\zeta_{Nq}) \subseteq F$, which means that $G_{-1}$ is a quotient of $(A_{Q(\zeta_{Nq})} \otimes \mu_q)\kappa^{-1}$, but they have the same size, so we have

$$(A_{Q(\zeta_{Nq})} \otimes \mu_q) \kappa^{-1} \cong G_{-1}.$$

\[\square\]

**Remark 6.8.** By Corollary [0.6] and Proposition [0.7] we have a canonical isomorphism:

$$G_{-1} \cong U^{\otimes 2} \otimes \mu_q^{-1}.$$

6.2. The map from homology to Galois cohomology.

**Definition 6.9** (Adjusted Manin symbol). For $u, v \in \mathbb{Z}/N\mathbb{Z}$ with $(u, v) = 1$, we let

$$[u, v]' = W_N[u, v] = \left\{ \frac{a \cdot b \cdot c}{bN} \right\},$$

where $a, b, c, d \in \mathbb{Z}$ satisfy $ad - bc = 1$, $u = c \mod N$, $v = d \mod N$.

Let $[u, v]^* := W_N[u, v]^* = \frac{1}{2}([u, v]' + [u, -v]')$

**Lemma 6.10.** The diamond operator $(j)$ $(j \in \mathbb{F}_N^\times)$ acts on $[u, v]'$ as follows:

$$(j)[u, v]' = [j^{-1}u, j^{-1}v'].$$

**Proof.** We have

$$(j)[u, v]' = (j)W_N[u, v] = W_N(j)^*[u, v] = [j^{-1}u, j^{-1}v'].$$

\[\square\]

**Theorem 6.11** (Manin). The relative homology group $H_1(X_1(N), \text{cusps}, \mathbb{Z}_p)^+$ has a representation as a $\mathbb{Z}_p[\mathbb{F}_N^\times]$-module with generators $[u, v]^*$ for $u, v \in \mathbb{F}_N$ and $(u, v) = 1$, subject to the following relations:

1. $[u, v]^* + [-v, u]^* = 0$,
2. $[u, v]^* = [u, u + v]^* + [u + v, v]^*$,
3. $[-u, -v]^* = [u, v]^* = [u, -v]^*$,
4. $(j)[u, v]^* = [j^{-1}u, j^{-1}v]^*$ for $j \in \mathbb{F}_N^\times$.

**Remark 6.12.** We have the following isomorphisms via Poincaré duality

$$H^1(X_1(N)(\mathbb{C}), \mathbb{Z}_p) \cong H_1(X_1(N)(\mathbb{C}), \mathbb{Z}_p),$$

$$H^1(X_1(N)(\mathbb{C}), \mathbb{Z}_p) \cong H_1(X_1(N)(\mathbb{C}), \text{cusps}, \mathbb{Z}_p).$$

Note that these isomorphisms are not Hecke compatible, they transfer from the $T(l)^*$-action to the $T(l)$-action. For the details, one can see [32 Proposition 3.5].

**Remark 6.13.** If we identify $H^1(X_1(N)(\mathbb{C}), \mathbb{Z}_p)$ (resp. $H^1(Y_1(N)(\mathbb{C}), \mathbb{Z}_p)$) with $H^1_{\text{et}}(X_1(N))$ (resp. $H^1_{\text{et}}(Y_1(N))$), Poincaré duality also changes the Galois action. In fact we have an isomorphism

$$H^1_{\text{et}}(X_1(N))(1) \cong H_1(X_1(N)(\mathbb{C}), \mathbb{Z}_p),$$

which respects complex conjugation. For the details, one can see [32 Section 3.5].
Lemma 6.14. Let $\tilde{C}_0$ denote the cusps of $X_1(N)(\C)$ that lie above the 0 cusp in $X_0(N)$, and let $\tilde{C}_\infty$ denote the cusps of $X_1(N)(\C)$ that lie above the $\infty$ cusp in $X_0(N)$. The Manin symbols $[u,v]$ ($u \neq 0, v \neq 0$) generate the relative homology group $H_1(X_1(N), \tilde{C}_0, \Z_p)$, and Manin symbols $[u,v]'$ generate the relative homology group $H_1(X_1(N), \tilde{C}_\infty, \Z_p)$.

Proposition 6.15 (Sharifi). There exists a homomorphism

$$\varpi^0 : H_1(X_1(N), \tilde{C}_\infty, \Z_p)^+ \to H^2(\Z[\zeta_N, \frac{1}{N}], \Z_p(2))^+. $$

$[u,v]^* \mapsto (1 - \zeta_N^u, 1 - \zeta_N^v)^+.$

Definition 6.16. Since $H_1(X_1(N), \Z_p)^+ \subset H_1(X_1(N), \tilde{C}_\infty, \Z_p)^+$, via restriction, we have the following homomorphism:

$$\varpi : H_1(X_1(N), \Z_p)^+ \to H^2(\Z[\zeta_N, \frac{1}{N}], \Z_p(2))^+. $$

Definition 6.17. (Eisenstein ideal for $X_1(N)$) Let $\mathcal{I}_0$ be the ideal of $\text{End}_{\Z_p}(H_1(X_1(N), \tilde{C}_\infty, \Z_p))$ generated by $T(l) - l \cdot \langle l, \rangle$, for $l \neq N$, and $T(N) - N$, and let $I_0$ be the image of $\mathcal{I}_0$ in $\text{End}_{\Z_p}(H_1(X_1(N), \Z_p))$. Let $\mathcal{I}_\infty$ be the ideal of $\text{End}_{\Z_p}(H_1(X_1(N), \tilde{C}_\infty, \Z_p))$ generated by $T(l) - 1 - l \cdot \langle l, \rangle$, for $l \neq N$, and $T(N) - 1$, and let $I_\infty$ be the image of $\mathcal{I}_\infty$ in $\text{End}_{\Z_p}(H_1(X_1(N), \Z_p))$.

Lemma 6.18. We have $\mathcal{I}_0 H_1(X_1(N), \tilde{C}_\infty, \Z_p) \subset H_1(X_1(N), \Z_p)$.

Proof. One can check that $\mathcal{I}_0$ kills $\tilde{C}_\infty$. For the Hecke action on cusps, one can see [30, Section 1.2].

6.3. Topological boundary and arithmetic boundary. In this section, we compare the following three exact sequences:

(24) $0 \to H^2(\Z[\frac{1}{N}, \zeta_N^+], \Z_p(2)) \to H^2(\Z[\frac{1}{N}, \zeta_N^+], \Z_p(2)) \xrightarrow{\partial} H^2(\Q(\zeta_N^+), \Z_p(2)) \to 0,$

where $\Q(\zeta_N^+)$ is the completion of $\Q(\zeta_N^+)$ at the place dividing $N$,

(25) $0 \to K_2(\Z[\zeta_N^+]) \otimes \Z_p \to K_2(\Z[\frac{1}{N}, \zeta_N^+]) \otimes \Z_p \xrightarrow{\partial} K_1(\Q(\zeta_N^+)) \otimes \Z_p \to 0,$

(26) $0 \to H_1(X_1(N), \Z_p)^+ \to H_1(X_1(N), \tilde{C}_\infty, \Z_p)^+ \xrightarrow{\partial'} \bigoplus_{\tilde{C}_\infty}^0 \Z_p \to 0,$

where $\bigoplus^0$ denotes the elements which sum to zero.

Remark 6.19. We identify sequence (24) with the sequence (25) via the étale chern class map.

Proposition 6.20. In sequence (24), the residue map $\partial$ satisfies

$$\partial(1 - \zeta_N^u, 1 - \zeta_N^v)^+ = \frac{u}{v} \in \Q(\zeta_N^+) \otimes \Z_p.$$ 

Proof. By the definition of tame symbol, we have

$$\partial(1 - \zeta_N^u, 1 - \zeta_N^v) = (-1)\frac{1 - \zeta_N^u}{1 - \zeta_N^v} \equiv -\frac{u}{v} \in \Q(\zeta_N^+) \otimes \Z_p.$$ 

Note that $\partial(1 - \zeta_N^u, 1 - \zeta_N^v)$ is trivial in $\Q(\zeta_N^+) \otimes \Z_p$.

Proposition 6.21. In sequence (26), the boundary map $\partial'$ satisfies

$$\partial'(a, b) = \left(\frac{c}{Na} - \frac{d}{Nb}\right),$$

where $a, b, c, d \in \Z$ with $ad - bc = 1$, and $(a, b) \equiv (c, d) \mod N$.

Proof. It is almost immediate from the definition.

Definition 6.22. Let $s = \frac{c}{a}$ be an element of $\Q \cup \{\infty\}$ with $\gcd(a, c) = 1$ and $N | a$. We define a map $t$ as follows:

$$t : \Z_p(\tilde{C}_\infty) \to \Q(\zeta_N^+) \otimes \Z_p,$$

$$t(s) = c \in \Q(\zeta_N^+) \otimes \Z_p.$$ 

Remark 6.23. The map $t$ is well defined. One can check this by using [7, Proposition 3.8.3].
Proposition 6.24. The following diagram commutes:

\[
\begin{array}{c}
H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p) \xrightarrow{\partial} \bigoplus_{C_\infty}^{0} \mathbb{Z}_p \\
\downarrow \varpi^0 \quad \quad \quad \quad \quad \downarrow \quad \\
K_2(\mathbb{Z}[\zeta_N^{+}, \frac{1}{p}]) \otimes \mathbb{Z}_p \xrightarrow{\partial'} \mathbb{F}_N^x \otimes \mathbb{Z}_p.
\end{array}
\]

(27)

Proof. Using Proposition [6.21], we know that
\[
t(\partial([u, v]^*)) = \frac{u}{v}.
\]
And using Proposition [6.20] and the definition of \(\varpi\), we know that
\[
\partial' \varpi^0([u, v]^*) = \frac{c}{d} = \frac{u}{v} \in \mathbb{F}_N^x \otimes \mathbb{Z}_p.
\]

So we have the following proposition.

Proposition 6.25. The image of \(\varpi\) is in \(H^2(\mathbb{Z}[\zeta_N^{+}, \frac{1}{p}], \mathbb{Z}_p(2))^+\) or \(K_2(\mathbb{Z}[\zeta_N^{+}]) \otimes \mathbb{Z}_p\).

Proof. This is the map on kernels of the horizontal maps in the commutative diagram [27].

Remark 6.26. By Remark [6.13], we may also view \(\varpi\) as a map:
\[
\varpi : H^1(X_1(N), \mathbb{Z}_p)^- (1) \rightarrow H^2(\mathbb{Z}[\zeta_N^{+}]) \otimes \mathbb{Z}_p(2))^+.
\]

6.4. Sharifi’s conjecture. In this section, we list several conjectures made by Sharifi.

Conjecture 6.27. The map \(\varpi^0\) satisfies
\[
\varpi^0(\eta x) = 0
\]
for all \(\eta \in \mathcal{J}_\infty\) and \(x \in H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p)\).

Lemma 6.28 (Busuioc, Sharifi). We have
\[
\varpi^0 \circ (T(2) - 1 - 2(2)) = 0,
\]
\[
\varpi^0 \circ (T(3) - 1 - 3(3)) = 0.
\]

Proof. For the proof, one can see [3] Theorem 1.2.

Remark 6.29. Conjecture 6.27 has been proved under the assumption \(p \mid N\): see [9] Theorem 5.2.3.

Let \(\tilde{I} = I_{\infty} + I_G\). Then we have the following map of exact sequences which gives the relationship between the cohomology of \(X_1(N)\) and \(X_0(N)\):

\[
\begin{array}{cccccc}
0 & \xrightarrow{\pi} & H^1(X_1(N), \mathbb{Z}_p)^- & \xrightarrow{\partial} & H^1(X_1(N), \mathbb{Z}_p)^+ & \xrightarrow{\partial'} & \mathbb{F}_N^x \otimes \mathbb{Z}_p \\
0 & \xrightarrow{1/2} & H^1(X_0(N), \mathbb{Z}_p)^- & \xrightarrow{1/2} & H^1(X_0(N), \mathbb{Z}_p)^+ & \xrightarrow{1/2} & \mathbb{F}_N^x \otimes \mathbb{Z}_p
\end{array}
\]

Remark 6.30. The surjectivity of the vertical maps is from Theorem [3.25] The injectivity of the vertical maps is from [16] Proposition 4.5.

Remark 6.31. Note that the sequence
\[
0 \rightarrow \frac{H^1(X_1(N), \mathbb{Z}_p)^-}{\text{IH}^1(X_1(N), \mathbb{Z}_p)} \rightarrow \frac{H^1(X_1(N), \mathbb{Z}_p)^+}{\text{IH}^1(X_1(N), \mathbb{Z}_p)} \rightarrow \frac{H^1(X_1(N), \mathbb{Z}_p)^+}{\text{IH}^1(X_1(N), \mathbb{Z}_p)} \rightarrow 0
\]
also gives the same extension class as \(b\) in \(H^1(\mathbb{Z}[\zeta_N^{+}], \mathbb{Z}_p(1))\). It gives a character
\[
\chi_b : G_{\infty} \rightarrow \frac{H^1(X_1(N), \mathbb{Z}_p)^-}{\text{IH}^1(X_1(N), \mathbb{Z}_p)} \cong \frac{\text{IH}^1(X_0(N), \mathbb{Z}_p)^-}{\mathbb{F}_N^x} \cong \frac{\mathbb{F}_N^x}{\mathbb{F}_N} \otimes \mathbb{Z}_p.
\]

In [32], Sharifi made a general conjecture about the relationship between homology of modular curves and Galois cohomology of number fields. We just express the related form that we need here.
Conjecture 6.32 (Sharifi).  

(1) The map $\varpi$ induces an isomorphism:
\[
\frac{H^1(X_1(N), \mathbb{Z}_p)^+}{I_\infty H^1(X_1(N), \mathbb{Z}_p)^+} \cong \frac{H_1(X_1(N), \mathbb{Z}_p)^+}{I_\infty H_1(X_1(N), \mathbb{Z}_p)^+} \cong H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)).
\]

(2) Let $\varpi_G$ be the map induced by $\varpi$ on $G$-coinvariants. It is a map
\[
\frac{IH^1(X_0(N), \mathbb{Z}_p)^+}{I^2 H_1(X_0(N), \mathbb{Z}_p)^+} \to H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \cong G \otimes \mu_q.
\]

Twisting the coefficients, we have a map
\[
\varpi_G \otimes \mu_q^{-1} : \frac{IH^1(X_0(N), \mathbb{Z}_p)^+}{I^2 H_1(X_0(N), \mathbb{Z}_p)^+} \to H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \cong G \otimes \mu_q^{-1} \cong G_{-1}.
\]

Then
\[
\chi_b \circ (\varpi_G \otimes \mu_q^{-1}) = (\varpi_G \otimes \mu_q^{-1}) \circ \chi_b = 1.
\]

Remark 6.33. The analogous conjecture for modular curves of level divisible by $p$ has been proved by Takako Fukaya and Kazuya Kato when certain hypotheses hold. For details, one can see [9].

Lemma 6.34. In the following diagram
\[
\begin{array}{ccc}
H_1(X_1(N), \mathbb{Z}_p)^+ & \to & H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \\
\downarrow \pi & & \downarrow \\
H_1(X_0(N), \mathbb{Z}_p)^+ & \to & H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)),
\end{array}
\]
the vertical maps are both surjective. The map $\varpi_G^0$ factors through the quotient by $IH_1(X_0(N), \mathbb{Z}_p)^+$. 

Proof. The surjectivity of vertical maps is from the definitions and the fact that there are only two cusps on $X_0(N)$. Note that $\pi([u,v]^*) = -\pi((v)[\frac{u}{v}, 1]) = -[\frac{u}{v}, 1] = -[0, \frac{1}{v}]$. By Proposition 6.20, $[u,v]^*$ maps to $\frac{1}{v} \in H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2))$. So $\varpi_G^0$ maps $\{0, \frac{1}{v}\}$ to $\frac{1}{v}$. From this, we know that $\varpi_G^0 = \phi^{-1}$, where $\phi$ is the map defined in Proposition 2.49. By Proposition 2.49, we know that $\varpi_G^0$ factors through $IH_1(X_0(N), \mathbb{Z}_p)^+$. 

Remark 6.35. We also have the following commutative diagram
\[
\begin{array}{ccc}
H_1(X_1(N), \mathbb{Z}_p)^+ & \to & H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \\
\downarrow \pi & & \downarrow \\
IH_1(X_0(N), \mathbb{Z}_p)^+ & \to & H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \cong G \otimes \mu_q.
\end{array}
\]

However, we cannot prove that $\varpi_G$ factors through the quotient by $I^2 H_1(X_0(N), \mathbb{Z}_p)^+$. If we know that 2 or 3 are good prime numbers, we can conclude that it does by Theorem 2.54.

6.5. Computation of $b$. In this section, assuming Conjecture 6.32 and Conjecture 6.32, we compute the invariant $b$. Recall that in Lemma 2.48, we have a canonical isomorphism:
\[
\phi : U \to H_1(X_0(N), \mathbb{Z}_p)^+/IH_1(X_0(N), \mathbb{Z}_p)^+.
\]

We identify the two groups by $\phi$.

Lemma 6.36. For $u, v \in \mathbb{F}_N^*$, the image of $(l + l - T(l))[u, v]^*$ in $IH_1(X_0(N), \mathbb{Z}_p)^+$ is $\eta_l \otimes \frac{1}{v}$, where we view $\frac{u}{v}$ as an element in $U$.

Proof. By definition, we have $\pi(l + l - T(l))[u, v]^* = \eta_l \pi([u, v]^*)$. Now, let us compute $\pi([u, v]^*) = \pi(W_N([u, v])) = W_N \pi([u, v])$. Using Hensel’s lemma, we have that the image of $W_N$ in $h_0(N)$ is $\mathfrak{P}$ as in Definition 6.32 is -1. Hence, in $H_1(X_0(N), \mathbb{Z}_p)^+$, we have $\pi([u, v]^*) = -\pi((v)[\frac{u}{v}, 1]) = -[\frac{u}{v}, 1]$. Let $x \in \mathbb{Z}$ be a lifting of $\frac{u}{v}$. By computation, we have $[\frac{u}{v}, 1] = \{0, \frac{1}{v}\}$. Using Proposition 2.49, we have that the element $-\{0, \frac{1}{v}\}$ corresponds to $\frac{u}{v} \in U$. 

Lemma 6.37. Via $\partial \circ \varpi^0$, $[u, v]^*$ maps to $\frac{u}{v}$ in $H^2(\mathbb{Q}_N(\zeta_N), \mathbb{Z}_p(2)) \cong U$. 

Proof. The image of $\varpi^0([u, v]^*)$ in $H^2(\mathbb{Q}_N(\zeta_N), \mathbb{Z}_p(2))$ is $(1 - \zeta_N, 1 - \zeta_N^*)$ by definition. Via the map $H^2(\mathbb{Q}_N(\zeta_N), \mathbb{Z}_p(2)) \cong U$, the image of $(1 - \zeta_N, 1 - \zeta_N^*)$ is exactly $\frac{u}{v}$ by Proposition 6.20. 

Since $(l + l - T(l))[u, v]^* \in H_1(X_1(N), \mathbb{Z}_p)$, via $\varpi$, it maps into
\[
I_G/I_G^2 \otimes H^2(\mathbb{Z}[\zeta_N^+, \frac{1}{p}], \mathbb{Z}_p(2)) \cong G \otimes U.
\]
Lemma 6.38. Via \( \varpi \), the element \( (l + \langle l \rangle - T(l))[u, v]^* \) maps to \( l^{l-1} \otimes \frac{u}{v} \).

Proof. Note that by definition, we have
\[
\varpi((l + \langle l \rangle - T(l))[u, v]^*) = \varpi^0((l + \langle l \rangle - T(l))[u, v]^*).
\]
If we assume the Eisenstein quotient conjecture for \( \varpi^0 \) and note that
\[
(l + \langle l \rangle - T(l))(l - 1)(1 - \langle l \rangle) \mod I_\infty.
\]
We have
\[
\varpi^0((l + \langle l \rangle - T(l))[u, v]^*) = \varpi^0((l - 1)(1 - \langle l \rangle))[u, v]^*) = (l - 1)(1 - \sigma^{l-1})[u, v]^*.
\]
By the isomorphism in Corollary 6.4, \( l - 1)(1 - \sigma^{l-1}) \) maps to \( l^{l-1} \in G \otimes \mathbb{Z}_p \). Since \( \partial \varpi^0([u, v]^*) = \frac{u}{v} \in U \), we have proved that \( \varpi((l + \langle l \rangle - T(l))[u, v]^*) = l^{l-1} \otimes \frac{u}{v} \).

\[\square\]

Theorem 6.39. Suppose that Conjecture 6.27 and Conjecture 6.32 hold. Then the invariant \( \tilde{b} \) equals 1.

Proof. By Lemma 6.36 and Lemma 6.38, we know that
\[
\varpi_G(u \otimes \frac{u}{v}) = l^{l-1} \otimes \frac{u}{v}.
\]
Assuming Conjecture 6.32, we know that
\[
\chi_b(l^{l-1} \otimes \frac{u}{v} \otimes \zeta_q) = u \otimes \frac{u}{v} \otimes \zeta_q.
\]
Since the winding isomorphism identifies \( l^{l-1} \) with \( \eta_b \) (Proposition 2.51), we know that \( \tilde{b} = 1 \).

\[\square\]

7. Eisenstein Quotient Conjecture

7.1. Goncharov and Brunault’s map and the \( \infty \)-map. In \[2\], Brunault constructed a map from the first homology group of a modular curve to the second \( K \)-group of a modular variety tensored with \( \mathbb{Q} \). In \[11\], Goncharov constructed a similar map to the second \( K \)-group of a modular variety without tensoring with \( \mathbb{Q} \).

Theorem 7.1 (Brunault, Goncharov). There is a well-defined map \( \rho' \) \( \rho' : H_1(X_1(N), \text{cusp}, \mathbb{Z}) \rightarrow K_2(Y_1(N)) \otimes \mathbb{Z}[\frac{1}{N}] \),
\[
\rho'([u, v]) = g_0, \frac{u}{v} \cup g_0, \frac{v}{u}.
\]

Corollary 7.2. Since \( p \nmid N \), we have a map
\[
H_1(X_1(N), \text{cusp}, \mathbb{Z}_p) \rightarrow K_2(Y_1(N)) \otimes \mathbb{Z}_p.
\]

Remark 7.3. Composing \( \rho' \) with the \( \text{étale} \) chern class map, we get a map from a homology group to \( H^2_{\text{ét}}(Y_1(N))(2) \). We still use \( \rho' \) to denote this map.

Remark 7.4. It is easy to see that \( g_0, \frac{u}{v} \cup g_0, \frac{v}{u} \) lies in \( H^2_{\text{ét}}(Y_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \). And the map \( H^2_{\text{ét}}(Y_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \rightarrow H^3_{\text{ét}}(Y_1(N))(2) \) is injective, which can be seen from the commutative diagram
\[
\begin{array}{cccccc}
0 & \rightarrow & H^2(\mathbb{Z}[\frac{1}{N}], \mathbb{Z}_p(2)) & \rightarrow & H^2_{\text{ét}}(Y_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) & \rightarrow & H^1(\mathbb{Z}[\frac{1}{N}], H^3_{\text{ét}}(Y_1(N))(2)) & \rightarrow & 0 \\
(1) & & \downarrow & & \downarrow & & \downarrow & & (2) \\
0 & \rightarrow & H^2(\mathbb{Q}, \mathbb{Z}_p(2)) & \rightarrow & H^2_{\text{ét}}(Y_1(N), \mathbb{Z}_p(2)) & \rightarrow & H^1(\mathbb{Z}[\frac{1}{N}], H^3_{\text{ét}}(Y_1(N))(2)) & \rightarrow & 0,
\end{array}
\]
where the rows of the diagram are from the Hochschild-Serre spectral sequence and the maps (1), (2) are both injective. We can view \( \rho' \) as a map from a homology group to \( H^2_{\text{ét}}(Y_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \).

Remark 7.5. The element \( g_0, \frac{u}{v} \cup g_0, \frac{v}{u} \) is called a Beilinson-Kato element. See \[11\], \[19\], \[23\].

In \[9\], Fukaya and Kato defined a map \( \infty(0, 1) \) and they proved in \[9\] the following theorem.

Theorem 7.6 (Fukaya-Kato). We have
\[
\varpi([u, v]^*) = \infty(0, 1)(g_0, \frac{u}{v} \cup g_0, \frac{v}{u}).
\]
The key property of \( \infty(0, 1) \) is found in the following proposition.

Proposition 7.7 (Fukaya-Kato). Restricted to the image of the Beilinson-Kato elements, the map \( \infty(0, 1) \) is killed by the Eisenstein ideal \( J_\infty \).
Proof: For the proof, see [9] Proposition 5.1.5, Proposition 5.1.7, Theorem 5.1.9. □

**Conjecture 7.8.** The map \( \rho \) is Hecke-equivariant.

**Corollary 7.9.** If Conjecture 7.8 is true, then Conjecture 6.2 is true.

Note that we have the following exact sequence from the Hochschild-Serre spectral sequence:

\[
0 \to H^2(\mathbb{Z}[\frac{1}{Np}], \mathbb{Z}_p(2)) \to H^2_{et}(\mathcal{Y}_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \xrightarrow{\partial} H^1(\mathbb{Z}[\frac{1}{N}], H^1_{et}(\mathcal{Y}_1(N))(2)) \to 0.
\]

We define

\[
\rho : H_1(X_1(N), \tilde{C}_\infty, \mathbb{Z}_p) \to H^2_{et}(\mathcal{Y}_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p \cong H^1(\mathbb{Z}[\frac{1}{N}], H^1_{et}(\mathcal{Y}_1(N))(2)).
\]

In this section, we prove that the map

\[
\rho : H_1(X_1(N), \tilde{C}_\infty, \mathbb{Z}_p) \to H^2_{et}(\mathcal{Y}_1(N) \otimes \mathbb{Z}[1/p], \mathbb{Z}_p(2)) \otimes \mathbb{Q}_p
\]

is well-defined and Hecke-equivariant.

### 7.2. Beilinson-Kato elements on \( Y(M, N) \)

Let \( M, N \in \mathbb{Z}_+ \), \( M + N \geq 5 \). Let \( \mathcal{Y}(M, N) \) be the \( \mathbb{Z}[\frac{1}{MN}] \)-scheme that represents the functor of triples \((E, e_1, e_2)\), where \( e_1 \) has order \( M \), \( e_2 \) has order \( N \), and \( \mathbb{Z}/M \times \mathbb{Z}/N \to E : (a, b) \mapsto ae_1 + be_2 \) is injective. Similarly to Siegel units on \( \mathcal{Y}(N) \), we can define Siegel units on \( \mathcal{Y}(M, N) \).

Let \( E \) be the universal elliptic curve over \( \mathcal{Y}(M, N) \).

**Definition 7.10.** For \((\alpha, \beta) = (\frac{a}{M}, \frac{b}{N}) \in (\frac{1}{M} \mathbb{Z} \times \frac{1}{N} \mathbb{Z}) \backslash (0, 0)\) and for an integer \( c > 1 \) prime to \( 6MN \), we define the Siegel unit \( c\gamma_{a, b} \) as

\[
c\gamma_{a, b} = \tau_{a, b}(c \theta_E) \in O(\mathcal{Y}(M, N))^\times,
\]

where \( \tau_{a, b} = a_1 + be_2 : \mathcal{Y}(M, N) \to E \backslash \mathcal{E} \). We define

\[
g_{a, b} = c\gamma_{a, b} \otimes (c^2 - 1)^{-1} \in O(\mathcal{Y}(M, N))^\times \otimes \mathbb{Q}.
\]

In [9], the following objects are defined.

**Definition 7.11** (Fukaya-Kato). For a matrix \( R = (\frac{a}{s}, \frac{b}{t}) \in M_2(\mathbb{Z}) \) such that \((s, u) \neq (0, 0)\) and \((t, v) \neq (0, 0)\), define:

\[
c_dz_{m, M}(R) := c\gamma_{a, b} \cup a\gamma_{a, b} \in K_2(\mathcal{Y}(m, M)).
\]

Let \( z_m(R) = g_{\frac{a}{M}, \frac{b}{N}} \cup g_{\frac{s}{L}, \frac{t}{M}} \in K_2(\mathcal{Y}(m, M)) \otimes \mathbb{Q} \).

### 7.3. Beilinson-Kato elements on \( Y_1(M) \otimes \mathbb{Q}(\zeta_m) \) and zeta values.

In this section, we assume that \( M \geq 4 \) and \( m \geq 1 \). We have the following two propositions that are from [9].

**Proposition 7.13.** Let \( N, M \geq 1 \), and assume \( N + M \geq 5 \). Assume that the prime divisors of \( N \) are prime divisors of \( M \). Let \( L \geq 1 \). Let \( R \in GL_2(\mathbb{Z}[1/L]) \). Let \( c, d \in \mathbb{Z} \) and assume that \((cd, 6NML) = 1\). Then the norm map \( K_2(\mathcal{Y}(NL, ML)) \to K_2(\mathcal{Y}(N, M)_{/\mathbb{Z}[1/L]}) \) sends \( c_dz_{NL, ML}(R) \) to

\[
\left( \prod_{l \mid L, l \mid N} P_l \right) c_dz_{NL, ML}(R),
\]

where

1. \( P_l = 1 - T^*(l) \begin{pmatrix} \frac{1}{l} & 0 \\ 0 & 1 \end{pmatrix} \) if \( l \nmid M \),
2. \( P_l = 1 - T^*(l) \begin{pmatrix} 0 & 0 \\ \frac{1}{l} & 1 \end{pmatrix} \) if \( l \mid M \).

**Proposition 7.14.** Let \( L \geq 1 \), and let \( m, M \) be as in the beginning of this section. Then the norm map \( K_2(Y_1(M) \otimes \mathbb{Z}[\frac{1}{mL}, \zeta_m]) \to K_2(Y_1(M) \otimes \mathbb{Z}[\frac{1}{mL}, \zeta_m]) \) sends \( c_dz_{1, M, M}(u, v) \) to

\[
\prod_{l \in C'} (1 - \sigma_l^{-1} \otimes T^*(l) + \sigma_l^{-2} \otimes (l^{-1}l)l) \prod_{l \in C} (1 - \sigma_l^{-1} \otimes T^*(l)) c_dz_{1, M, M}(u, v),
\]

where \( C' \) denotes the set of all prime numbers which divide \( L \) but do not divide \( mM \), and \( C \) denotes the set of all primes which divide \( LM \) but do not divide \( m \).
Definition 7.15 (Fukaya-Kato). Let $Z_{1,M,m}(s) = \prod_{l} P_l(l^s) 1$ be the operator-valued zeta function acting on $\mathbb{Z}[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes \mathbb{Z}[[\pm 1]] H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$, where

$$P_l(t) = 1 - \sigma_1^{-1} \otimes T^*(l)t + \sigma_1^{-2} \otimes (l)^{-1}t^2$$

if $l \nmid M$,

$$P_l(t) = 1 - \sigma_1^{-1} \otimes T^*(l)t$$

if $l \mid M$,

$$P_l(t) = 1$$

if $l \mid m$.

Remark 7.16. The zeta function $Z_{1,M,m}(s)$ converges when $\text{Re}(s) > 2$, and has meromorphic continuation to the whole complex plane and is holomorphic except at $s = 2$.

Definition 7.17. Let $c,d[u,v] \in \mathbb{Z}[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ be

$$c,d[u,v] = c^2d^2 \otimes [u,v] - c^2d \otimes [u, dv] - d^2c \otimes [cu,v] + \sigma_{cd} \otimes [cu, dv],$$

where $c, d$ are prime to $6Mm$.

Let $M \geq 5$. For $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, let $\{\alpha, \beta\} \in H_1(X_1(M)(\mathbb{C}), \text{cusps}, \mathbb{Z})$ be the class of the geodesic line on the upper half plane from $\alpha$ to $\beta$. Via the canonical isomorphism of Poincare duality, we regard it as an element in $H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})(1)$.

Definition 7.18. Let $M_2(M)/\mathbb{Q}$ be the space of all modular forms of weight 2 and level $M$ defined over $\mathbb{Q}$. We have the period map

$$\text{per}': M_2(M)/\mathbb{Q} \rightarrow H^1(Y_1(M)(\mathbb{C}), \mathbb{C}),$$

$$f \mapsto \phi_f : \{\alpha, \beta\} \mapsto \frac{1}{2\pi i} \int_{\alpha}^{\beta} f dz.$$

Definition 7.19. Define the equivariant period map,

$$\text{per} : M_2(M)/\mathbb{Q} \otimes \mathbb{Q}(\zeta_m) \rightarrow \mathbb{Z}[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes H^1(X_1(M)(\mathbb{C}), \mathbb{C}),$$

$$x \otimes y \mapsto \sum_{g \in \text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})} g \otimes g^{-1}(y) \text{per}'(x).$$

Definition 7.20. Consider the following composition:

$$\lim_{n \to \infty} K_2(Y_1(Mp^n)) \xrightarrow{(1)} \lim_{n \to \infty} H^2(Y_1(Mp^n), \mathbb{Z}_p(2)) \xrightarrow{(2)} H^2(Y_1(Mp^n), \mathbb{Z}_p(1)) \xrightarrow{(3)} H^{2}(Y_1(M)_{\overline{p}\mathbb{Q}}, \mathbb{Z}_p(1)) \xrightarrow{(4)} H^1(\mathcal{O}_p, H^1_{\text{et}}(Y_1(M)(1))) \xrightarrow{(5)} M_2(M)/\mathbb{Q}_p$$

Remark 7.21. In [25, 9.3] the map (1) is the étale Chern class map, the map (2) is defined by $\otimes (\mathbb{Q}_p^{\otimes (1)})_n$, the map (3) is the map corresponding the natural projection $Y_1(Mp^n) \rightarrow Y_1(M)$, the map (4) is the map from the spectral sequence, and the map (5) is the restriction map. The map $\exp^*$ is the dual exponential map, for the precise definition, one can see [23, 9.3].

Theorem 7.22 (Fukaya-Kato). Assume $p \mid m$. Then the map $\exp^*$ in Definition 7.20 sends $(c,dz_{1,M,m,p^n}(u,v))_{n \geq 0}$ to an element of $M_2(M)/\mathbb{Q} \otimes \mathbb{Q}(\zeta_m)$ whose period image in $\mathbb{Z}[\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})] \otimes H^1(Y_1(M)(\mathbb{C}), \mathbb{C})$ coincides with $-Z_{1,M,m}(1)c,d[u,v]^*$.

Remark 7.23. This is Theorem 2.4.9 in [9], but we drop the assumption $p \mid M$, since the condition $p \mid M$ is not used.

7.4. The map $z_{1,N,p^\infty}$. In this section, we focus on the modular curve $Y_1(N)$. Let $p \nmid N$ and let

$$A := \mathbb{Z}_p[[\mathbb{Z}_p^\times]] \cong \mathbb{Z}_p[[\text{Gal}(\mathbb{Q}(\zeta_{p^\infty})/\mathbb{Q})]].$$

Let

$$\tilde{A} := \lim_{n \to \infty} H^1(\mathbb{Z}[\frac{1}{p^n}], \mathbb{Q}_p(1)).$$

Lemma 7.24. The kernel of

$$\exp^* : \tilde{A} \rightarrow \lim_{n \to \infty} M_2(N) \otimes \mathbb{Q}(\zeta_{p^n}) \otimes \mathbb{Q}_p$$

coincides with the $A$-torsion of $\tilde{A}$.

Proof. For the proof, see [9, Lemma 3.1.4].
Theorem 7.25 (Fukaya-Kato). There exists a unique Hecke-equivariant $\Lambda$-module homomorphism

$$z_{1,N,p^\infty} : \Lambda \otimes H^1(Y_1(M)(\mathbb{C}), \mathbb{Z}) \to \mathfrak{f} \otimes Q(\Lambda)$$

satisfying the following conditions:

1. For any $\gamma \in \Lambda \otimes H^1(Y_1(M)(\mathbb{C}), \mathbb{Z})$ and for any $c,d$ such that $(cd, 6Np) = 1$ and $c \equiv d \equiv 1 \mod N$, $(\sigma_c - c)(\sigma_d - d)z_{1,N,p^\infty}(\gamma)$ belongs to the image of $\mathfrak{f}$ in $\mathfrak{f} \otimes Q(\Lambda)$.

2. For $n \geq 1$, consider the dual exponential map

$$\exp^*_n : H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(1)) \otimes Q_p \to M_2(N) \otimes Q(\zeta_{p^n}) \otimes Q_p$$

and the period map

$$\text{per}_{p^n} : M_2(M) \otimes Q(\zeta_{p^n}) \to \mathbb{Z}[\text{Gal}(Q(\zeta_{p^n})/Q)] \otimes H^1(Y_1(N)(\mathbb{C}), \mathbb{C}).$$

Then for any $\gamma \in H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})$, the image of $z'_{1,N,p^\infty}(1 \otimes \gamma)$ in $M_2(N) \otimes Q(\zeta_{p^n}) \otimes Q_p$ under $\exp^*$ is an element of $M_2(M)Q \otimes Q(\zeta_{p^n})$ whose image in $\mathbb{Z}[\text{Gal}(Q(\zeta_{p^n})/Q)] \otimes H^1(Y_1(N)(\mathbb{C}), \mathbb{C})$ under $\text{per}_{p^n}$ coincides with $Z_{1,N,p^\infty}(1) \cdot \gamma$.

Proof. The uniqueness is from Lemma 7.24 and the injectivity of the period map.

Let $\gamma \in H^1(Y_1(N)(\mathbb{C}), \mathbb{Z}_p)$. Suppose $\gamma = \sum a_i[u_i, v_i]^*$ where $a_i \in \mathbb{B}_N$ and the $[u_i, v_i]^*$ are adjusted Manin symbols. Take $c, d \in \mathbb{Z}\{\pm 1\}$ such that $(cd, 6M_m) = 1$. Then $c - \sigma_c$ and $d - \sigma_d$ are non-zero divisors in $\Lambda$. Let

$$z'_{1,N,p^\infty}(\gamma) = (c - \sigma_c)^{-1}(d - \sigma_d)^{-1} \sum_i a_i tw_{-1}(c, d z_{1,N,p^\infty}(u_i, v_i)) \in \mathfrak{f} \otimes Q(\Lambda).$$

Here $tw_{-1}$ is the twist by $\mathbb{Z}_p(-1)$. From the definition, we know that if this map is well defined, then it is Hecke-equivariant. Suppose we have two different expressions for $\gamma$:

$$\gamma = \sum_i a_i[u_i, v_i]^* = \sum_j b_j[u_j, v_j]^*.$$ 

Let

$$A = (c - \sigma_c)^{-1}(d - \sigma_d)^{-1} \sum_i a_i tw_{-1}(c, d z_{1,N,p^\infty}(u_i, v_i)),$$

and

$$B = (c' - \sigma_{c'})^{-1}(d' - \sigma_{d'})^{-1} \sum_j b_j tw_{-1}(c, d z_{1,N,p^\infty}(u_j, v_j)).$$

By Theorem 7.22, $\text{per} \circ \exp^*$ maps both $A$ and $B$ to $Z_{1,N,p^\infty}(1) \cdot \gamma$. Then by Lemma 7.24 and the injectivity of period map, we know that $z'_{1,N,p^\infty}(\gamma)$ is well-defined. Then it is easy to check that $z'_{1,N,p^\infty}(\gamma)$ satisfies the properties in Theorem 7.25. \hfill \Box

Remark 7.26. This theorem is exactly the same as Theorem 3.1.5 in [9]. Here we don’t require that $p \mid N$, but the proof is exactly the same.

By twisting coefficients, we have the following corollary.

Corollary 7.27. We have a Hecke-equivariant $\Lambda$-homomorphism

$$z_{1,N,p^\infty} : \Lambda \otimes H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})(1) \to \mathfrak{f}(1) \otimes Q(\Lambda)$$

induced by $z'_{1,N,p^\infty}$ such that

1. $z_{1,N,p^\infty}$ sends $c,d[u,u]^*$ to $(c,d z_{1,N,p^\infty}(u,u))^*$, and
2. the image of $[u,v]^*$ in $H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(2)) \otimes Q_p$ is $-z_{1,N,p^\infty}(u,v)$.

So in the first layer of this tower, we have a Hecke-equivariant map:

$$\mathbb{Z}_p[\text{Gal}(Q(\zeta_{p^n})/Q)] \otimes H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})(1) \to H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(2)) \otimes Q_p,$$

which induces a map

$$Z : H^1(Y_1(N)(\mathbb{C}), \mathbb{Z})(1) \to H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(2)) \otimes Q_p$$

which sends $1 \otimes [u,v]^*$ to $-z_{1,N,p^\infty}(u,v)$.

Proposition 7.28. The norm map

$$H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(2)) \otimes Q_p \to H^1(Z[\frac{1}{Np}, \zeta_{p^n}], H^1_{et}(Y_1(N))(2)) \otimes Q_p,$$

takes $z_{1,N,p}(u,v)$ to $(1 - T^*(p) + p(p)^{-1})z_{1,N,1}(u,v)$.
Proof. It is immediate from Proposition 7.14. □

Remark 7.29. We have a Hecke-equivariant map

\[ Z': H_1(X_1(N)(\mathbb{C}), \tilde{C}_\infty, \mathbb{Z}_p) \to H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(Y_1(N))(2)) \otimes \mathbb{Q}_p \]

which sends \([u, v]^*\) to \((1 - T^*(p) + p(p)^{-1})z_{1, N, 1}(u, v)\).

Proposition 7.30. The operator

\[ 1 - T^*(p) + p(p)^{-1} \in \text{End}_{\mathbb{Q}_p} \left( H^1(\mathbb{Z}[\frac{1}{Np}], H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2)) \otimes \mathbb{Q}_p \right) \]

is injective.

Proof. For simplicity of notation, let \(\eta_p = 1 - T^*(p) + p(p)^{-1}\) and \(V = H^1_{\text{ét}}(X_1(N))(1) \otimes \mathbb{Q}_p\). We have \(V \cong \bigoplus f_i T_p(A_{f_i}) \otimes \mathbb{Q}_p\), where the \(f_i\) are weight 2 newforms of \(\Gamma_1(N)\). Note that Hecke action changes in the above isomorphism, so \(\eta_p\) acts on \(T_p(A_{f_i}) \otimes \mathbb{Q}_p\) by multiplying by \(1 - a_p(f) + p \chi_i(p)\), where \(\chi_i\) is the character of \(f_i\).

It is easy to see that \(1 - a_p(f) + p \chi_i(p) \neq 0\). That is because the root of the polynomial \(x^2 - a_p(f)x + p \chi_i(p)\) has absolute value \(p^{1/2}\) (cf. [25, 14.10.5]). We know that \(\eta_p\) acts injectively on \(V(1)\). Since \(V(1)\) is a finite dimensional vector space, \(\eta_p\) is an isomorphism of \(V(1)\). We can conclude that \(\eta_p\) is injective on \(H^1(\mathbb{Z}[\frac{1}{Np}], V(1))\). □

Proposition 7.31. We have a Hecke-equivariant map

\[ \rho: H_1(X_1(N)(\mathbb{C}), \tilde{C}_\infty, \mathbb{Z}_p) \to H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(Y_1(N))(2)) \otimes \mathbb{Q}_p \]

which sends \([u, v]^*\) to \(z_{1, N, 1}(u, v)\).

Proof. We have the following exact sequence

\[ 0 \to H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{Q}_p \to H^1_{\text{ét}}(Y_1(N))(2) \otimes \mathbb{Q}_p \to C(1) \to 0, \]

where \(C = \bigoplus_{\text{cusp}}^0 \mathbb{Q}_p\). Note that \(H^0(\mathbb{Z}[\frac{1}{Np}], C(1)) = 0\) since \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))\) acts trivially on \(C\). Taking Galois cohomology, we have the following exact sequence

\[ (33) \quad 0 \to H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{Q}_p) \to H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(Y_1(N))(2) \otimes \mathbb{Q}_p) \]

\[ \quad \to H^1(\mathbb{Z}[\frac{1}{Np}], C(1)). \]

Let \(\gamma \in H_1(X_1(N)(\mathbb{C}), \tilde{C}_\infty, \mathbb{Z}_p)\). Suppose we have two different expressions for \(\gamma\):

\[ \gamma = \sum_i a_i[u_i, v_i]^* = \sum_j b_j[u_j, v_j]^*, \]

where \(a_i, b_j\) are elements in the Hecke algebra. Let \(A = \sum_i a_i z_{1, N, 1}(u_i, v_i)\) and \(B = \sum_j b_j z_{1, N, 1}(u_j, v_j)\). By [21] Theorem 3.9.3 (iii), we have \(t(A - B) = 0\). By (33), we have \(A - B \in H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{Q}_p)\). By Remark 7.29, we know that \(\eta_p(A - B) = 0\). From Proposition 7.30, we know that \(\eta_p\) is injective on \(H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{Q}_p)\). So, we can conclude that \(A = B\). □

Proposition 7.32. The group \(H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(Y_1(N))(2))\) is \(\mathbb{Z}_p\)-torsion-free.

Proof. Consider the exact sequence

\[ 0 \to H^1_{\text{ét}}(X_1(N)) \to H^1_{\text{ét}}(Y_1(N)) \to C \to 0, \]

where \(C\) is the cokernel of \(i\). In order to prove that \(H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(Y_1(N))(2))\) is \(\mathbb{Z}_p\)-torsion-free, it suffices to prove that both \(H^1(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2))\) and \(H^1(\mathbb{Z}[\frac{1}{Np}], C(2))\) are \(\mathbb{Z}_p\)-torsion-free. From the sequence

\[ 0 \to \mathbb{Z}_p(2) \to \mathbb{Z}_p(2) \to \mathbb{Z}_p(2)^{p^2} \to 0, \]

it suffices to prove that

\[ (1) \quad H^0(\mathbb{Z}[\frac{1}{Np}], H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{F}_p) = 0, \]

\[ (2) \quad H^0(\mathbb{Z}[\frac{1}{Np}], C(2) \otimes \mathbb{F}_p) = 0. \]

Note that the \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-representation \(H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{Q}_p\) is crystalline. Then (1) is deduced from the statement that \(H^0(\mathbb{Q}_p, H^1_{\text{ét}}(X_1(N))(2) \otimes \mathbb{F}_p) = 0\). For the details, see [38] Section 3.2. For (2), it is true because \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_N))\) acts trivially on \(C(1)\). □

Corollary 7.33. If \(p \nmid N\phi(N)\), then the Eisenstein quotient conjecture is true.
Proof. If \( p \nmid N\phi(N) \), it is easy to see that \( H^2(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(2)) = 0 \). From [29], we have
\[
H^2_{et}(\mathcal{Y}_1(N) \otimes \mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(2)) \xrightarrow{\sim} H^1(\mathbb{Z}[\frac{1}{p}], H^1_{et}(\mathcal{Y}_1(N))(2)).
\]
Then the corollary follows from Proposition 7.31 and Proposition 7.32. \( \square \)

8. Sharifi's conjecture for the modular curve \( X_1(N)^{(p)} \)

In this section, enlightened by Lecouturier's work [16], we consider Sharifi's conjecture for \( X_1(N)^{(p)} \). We try to give some evidence for this conjecture.

Notation 8.1.

1. Let \( \tilde{G} := (\mathbb{Z}/N\mathbb{Z})^\times /\{\pm 1\} = G_p \times G' \), where \( G_p \) is the Sylow \( p \)-subgroup of \( G \) and \( G' \) is the maximal subgroup which has order not divisible by \( p \). Let \( \Lambda = \mathbb{Z}_p[G] \) and \( \Lambda_p = \mathbb{Z}_p[G_p] \).

2. Let \( \tilde{\xi} := \frac{N}{2} \sum_{i=1}^{N-1} \mathbb{Z}_2\left( \frac{i}{N} \right) \in \Lambda \), \( \tau = \sum_{\sigma \in G_p} \sigma \).

3. Let \( X_1(N)^{(p)} \) (resp., \( Y_1(N)^{(p)} \)) be the compact modular curve (resp., modular curve) corresponding to the congruence subgroup
\[
\Gamma_1(N)^{(p)} := \left\{ \begin{pmatrix} a & b \\
 & d \end{pmatrix} \in \Gamma_0(N) \mid a \in G' \mod N, d \in G' \mod N \right\}.
\]

We still use \( \hat{C}_\infty \) to denote the cusps that lie above the \( \infty \) cusp of \( X_0(N) \). Let \( \mathfrak{h} \) be the subring of \( \text{End}_{\mathbb{Z}_p}(H^1(X_1(N)^{(p)}(\mathbb{C}), \hat{C}_\infty, \mathbb{Z}_p)) \) generated by \( T^*(n) \) \((n \geq 1)\) and \( \langle n \rangle \) \((n, N) = 1\).

Let \( \mathfrak{h} \) be the subring of \( \text{End}_{\mathbb{Z}_p}(H^1(X_1(N)^{(p)}(\mathbb{C}), \mathbb{Z}_p)) \) generated by \( T^*(n) \) \((n \geq 1)\) and \( \langle n \rangle \) \((n, N) = 1\).

4. Let \( \mathbb{Q}((\zeta_N^{(p)})) \) be the maximal \( p \)-power subextension of \( \mathbb{Q}((\zeta_N)) / \mathbb{Q} \), and let \( \mathbb{Z}((\zeta_N^{(p)})) \) be its integer ring. Note that this field is totally real.

Lemma 8.2. The natural maps of homology groups give the following isomorphisms:
\[
H_1(X_1(N)^{(p)}, \mathbb{Z}_p) \cong H_1(X_1(N), \mathbb{Z}_p)_{G'},
\]
\[
H_1(X_1(N)^{(p)}, \hat{C}_\infty, \mathbb{Z}_p) \cong H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p)_{G'}.
\]

Proof. By [16] Lemma 4.3], we know that

1. \( I_{G'} H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p) \) is the kernel of the map
\[
H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p) \to H_1(X_1(N)^{(p)}, \hat{C}_\infty, \mathbb{Z}_p).
\]

2. \( I_{G'} H_1(X_1(N), \mathbb{Z}_p) \) is the kernel of the map
\[
H_1(X_1(N), \mathbb{Z}_p) \to H_1(X_1(N)^{(p)}, \mathbb{Z}_p).
\]

From [16] Corollary 4.2], we know that
\[
H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p) \to H_1(X_1(N)^{(p)}, \hat{C}_\infty, \mathbb{Z}_p)
\]
is surjective.
\[
H_1(X_1(N)^{(p)}, \hat{C}_\infty, \mathbb{Z}_p) \cong H_1(X_1(N), \hat{C}_\infty, \mathbb{Z}_p)_{G'}
\]
Since the order of \( G' \) is prime to \( p \), we know that the map
\[
H_1(X_1(N), \mathbb{Z}_p) \to H_1(X_1(N)^{(p)}, \mathbb{Z}_p)
\]
is also surjective.
\[
H_1(X_1(N)^{(p)}, \mathbb{Z}_p) \cong H_1(X_1(N), \mathbb{Z}_p)_{G'}.
\]

\( \square \)

Notation 8.3. Let \( H_p = H_1(X_1(N)^{(p)}, \mathbb{Z}_p) \) and \( \tilde{H}_p = H_1(X_1(N)^{(p)}, \hat{C}_\infty, \mathbb{Z}_p) \).

Theorem 8.4 (Lecouturier). The map \( \Lambda_p \to \mathfrak{h} \) given by \([d] \to \langle d \rangle^{-1}\) gives isomorphisms of \( \Lambda_p \)-modules
\[
\Lambda_p/\tilde{\xi} \cong \mathfrak{h} / \mathfrak{I}_\infty \cong \tilde{H}_p^+ / \mathfrak{I}_\infty \tilde{H}_p^+,
\]
\[
\Lambda_p/(\tilde{\xi}, \tau) \cong \mathfrak{h} / I_\infty \cong H_p^+ / I_\infty H_p^+.
\]

Proof. For the proof, see [16] Theorem 4.6, Theorem 4.9]. \( \square \)
Lemma 8.5. We have the following exact sequence of $\Lambda_p$-modules

\[ 0 \to H_p^+ / I_\infty H_p^+ \to \tilde{H}_p^+ / \mathcal{J}_\infty \tilde{H}_p^+ \to I_{G_p} H_0^+ \cdot (\infty) \to 0. \]

Proof. From the definition, we have the following exact sequence

\[ 0 \to H_p^+ \to \tilde{H}_p^+ \to I_{G_p} \cdot (\infty) \to 0. \]

Tensoring with $\mathfrak{H} / \mathcal{J}_\infty$, we have the following exact sequence

\[ H_p^+ / I_\infty H_p^+ \to \tilde{H}_p^+ / \mathcal{J}_\infty \tilde{H}_p^+ \to \mathfrak{H} / \mathcal{J}_\infty \otimes_B I_{G_p} \cdot (\infty) \to 0. \]

It is easy to see that $\mathfrak{H} / \mathcal{J}_\infty \otimes_B I_{G_p} \cdot (\infty) \cong I_{G_p} H_0^+ \cdot (\infty)$. From Theorem 8.4, we know that

\[ |H_p^+ / I_\infty H_p^+| \cdot q = |\tilde{H}_p^+ / \mathcal{J}_\infty \tilde{H}_p^+|. \]

So, the map $j$ has to be an injection. \qed

Similar to the case of the modular curve $X_1(N)$, we can define maps

\[ \varpi_{G'} : H_p^+ \to H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)), \]
\[ \varpi_{G'}^0 : \tilde{H}_p^+ \to H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)). \]

Lemma 8.6. We have the following commutative diagrams:

\begin{align*}
\begin{array}{ccc}
H_1(X_1(N), \tilde{C}_\infty, \mathbb{Z}_p)^+ & \xrightarrow{\varpi_{G'}^0} & H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \\
\downarrow & & \downarrow \\
\tilde{H}_p^+ & \xrightarrow{\varpi_{G'}^0} & H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)), \\
H_1(X_1(N), \mathbb{Z}_p)^+ & \xrightarrow{\varpi} & H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \\
\downarrow & & \downarrow \\
H_p^+ & \xrightarrow{\varpi_{G'}} & H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)).
\end{array}
\end{align*}

Proof. The diagram of $\varpi_{G'}^0$ commutes since $\varpi^0$ is $G$-equivariant, using the facts that $H_1(X_1(N), \tilde{C}_\infty, \mathbb{Z}_p)^+ \cong \tilde{H}_p^+$ and $H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \cong H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2))$. Via restriction, we get the diagram for $\varpi_{G'}$. \qed

Similar to the case of the modular curve $X_1(N)$, we have the following conjecture.

Conjecture 8.7.  
(1) The map $\varpi_{G'}^0$ factors through the Eisenstein ideal.
(2) The map $\varpi_{G'}$ induces an isomorphism from $H_p^+ / I_\infty H_p^+$ to $H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2))$.

Proposition 8.8. The map $\varpi_{G'}^0$ is surjective.

Proof. Note that $\varpi_{G'}^0$ takes values in $H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \cong H^2(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(2)) \cong U$. The map $(\varpi_{G'}^0)_{G'} = \varpi_{G'}^0$ is surjective since $\varpi_{G'}^0([u, v]^*) = \frac{u}{v} \in U$. Then the proposition follows from Nakayama’s lemma. \qed

Remark 8.9. Without using the Eisenstein quotient conjecture, we cannot prove that $\varpi_{G'}$ is surjective. If we assume the Eisenstein quotient conjecture, we can even prove that $\varpi_{G'}$ is an isomorphism. See Proposition 8.14.

Proposition 8.10. As $\Lambda_p$-modules, we have an isomorphism

\[ H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \cong \Lambda_p / (\tilde{\xi}). \]

Proof. Since

\[ H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \cong H^2(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}_p(2)) \cong \mathbb{F}_N \otimes \mathbb{Z}_p, \]
\[ H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \] is a cyclic $\Lambda_p$-module. We only need to prove that the Fitting ideal of $H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2))$ is $\tilde{\xi}$. This follows directly from the Coates-Sinnott conjecture for $\mathbb{Q}(\zeta_N) / \mathbb{Q}$ proven in [12] Theorem 6.19. \qed

For the convenience of the reader, we review the Coates-Sinnott conjecture. Let $K/k$ be abelian extension of number fields of Galois group $G$, and let $S$ be a finite set of primes in $k$ that contains all the ramified primes and all the primes over $\infty$. 

\[ H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2)) \]

is a cyclic $\Lambda_p$-module. We only need to prove that the Fitting ideal of $H^2(\mathbb{Z}[\zeta_N^+ \cdot \frac{1}{p}], \mathbb{Z}_p(2))$ is $\tilde{\xi}$. This follows directly from the Coates-Sinnott conjecture for $\mathbb{Q}(\zeta_N) / \mathbb{Q}$ proven in [12] Theorem 6.19. \qed
Definition 8.11. 
(1) Set $e_n(K/k) := \prod_{v \in \mathcal{S}_\infty} \left( \frac{\mathrm{N}^{1+(-1)^n\sigma_v}}{2} \right)$ if $k$ is totally real. Here, $\sigma_v$ is complex conjugation.
(2) Set $e_n(K/k) := 0$ if $k$ is not totally real.

In [12], the following theorem is proved.

Theorem 8.12 (Greither, Popescu). If $k$ is totally real, $p > 2$ and the classical Iwasawa $\mu$-invariant associated to the maximal CM-subfield of $K(\mu_p)$ is zero, then we have

(\begin{equation}
\text{Ann}_{\mathbb{Z}_p}[0] H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(n)) \Theta_{S,K/k}(1 - n) = e_n(K/k) \text{Fitt}(H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(n)), \mathbb{Z}_p(n)),
\end{equation})

where $\Theta_{S,K/k}(1 - n) = \frac{1}{n} N^{n-1} \sum \mathbb{B}_n(\frac{\mathcal{N}}{\mathfrak{n}})[i].$

Now let $k = \mathbb{Q}, K = \mathbb{Q}(\zeta_p^{(p)}, n = 2, S = \{N, \infty\}$. Since $K$ is abelian, the $\mu$-invariant vanishes. As proven in [12], one can show that

$H^1(\mathcal{O}_{K,S}, \mathbb{Z}_p(2))_{\text{tors}} = (\mathbb{Q}_p/\mathbb{Z}_p(2))^{G_K} = 0.$

Since $K = \mathbb{Q}(\zeta_p^{(p)})$ is already totally real, $e_2(K/k)$ acts on the Fitting ideal trivially, so we get

$\text{Fitt}(H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) = \Theta_{S,K/k}(-1)$

In the case we are interested in, we have

$\Theta_{S,K/k}(1 - 2) = \frac{1}{2} N \sum \mathbb{B}_2(\frac{\mathcal{N}}{\mathfrak{n}})[i] = \tilde{\xi}.$

By Theorem 8.12 we have

$\text{Fitt}(H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) = \tilde{\xi}.$

Corollary 8.13. As $\Lambda_p$-modules, we have

$H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) \cong (I_{G_p} + \tilde{\xi})/\tilde{\xi} \cong \Lambda_p/(\tilde{\xi} + \tau).$

Proof. From the localization sequence

$0 \to H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) \to H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) \to F^N \otimes \mathbb{Z}_p \to 0,$

we know that

$H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) = I_{G_p} H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2))) = (I_{G_p} + \tilde{\xi})/\tilde{\xi}.$

Proposition 8.14. Assuming Conjecture 5.7 (1), the map $\varpi^0_{G'}$ induces an isomorphism

$H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(2)) = \varpi^0_{G'},$

and $\varpi_{G'}$ induces an isomorphism

$H^2(\mathcal{O}_{K,S}, \mathbb{Z}_p(2)) = \varpi_{G'}.$

Proof. From Theorem 8.4 and Proposition 8.10 we know that as $\Lambda_p$-modules, $\tilde{H}_p^2/\mathcal{O}_{K,S} H_p^2$ and $H^2(\mathbb{Z}([\zeta_p^{(p)}, \frac{1}{Np}], \mathbb{Z}_p(2)))$ are both isomorphic to $\varpi_{G'}$. It suffices to prove that $\varpi^0_{G'}$ is a surjection, which is from Proposition 8.8. The statement on $\varpi^0_{G'}$ follows directly from the statement of $\varpi^0_{G'}$, and the exact sequence (14).

Remark 8.15. A similar result has been obtained in [16] Proposition 4.34 assuming [15] Conjecture 1.10 and Eisenstein quotient conjecture for $\varpi^0_{G'}$. By using the theorem of Greither-Popescu, we can get rid of the assumption on Conjecture 1.10 in [15].

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