Detailed proof of a theorem on coincidence of homological dimensions of Fréchet algebras of smooth functions on a manifold with the dimension of the manifold

O. S. Ogneva

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Given work is devoted to the proof of the following assertion.

**Theorem 1.** For the topological algebra \( C^\infty(M) \) of smooth functions on a smooth \( m \)-dimensional real manifold \( M \) the small global dimension \( (d_s C^\infty(M)) \), the global homological dimension \( (d_g C^\infty(M)) \) and the bidimension \( (d_b C^\infty(M)) \) are equal to \( m \) (all dimensions are understood in the sense of the homology of topological (locally convex) algebras [1]).

The bidimension of the topological algebra \( C^\infty(U) \), where \( U \) is an open set in \( R^m \), was computed by Taylor in the paper [2]. The proof of this fact, like the proof of purely algebraic Hilbert’s Syzygy Theorem [3, Chapter VII, §7] is substantially based upon the possibility of constructing a special free resolution of length \( m \), the so-called Koszul resolution. However in the general case of the topological algebra of smooth functions on an arbitrary smooth real manifold \( C^\infty(M) \) there is no natural system of commuting operators (of the type of the multiplication operators by an independent variable in the case of \( M \subset R^m \)), which would let us construct free Koszul resolutions. Nevertheless we will show, that the modules over the algebra \( C^\infty(M) \) always have projective (generally speaking, non-free) resolutions of length \( m \), however they are more complicated than the Koszul resolutions. In the proof the projectivity of some natural class of modules (see Section 2) is employed essentially. These modules are used for the construction of Koszul resolutions and of more complicated resolutions, received then by means of the smooth Čech cochain complex. Let us emphasise that the main result is related to the topological (‘locally convex’) homology and its proof uses the specific machinery of this theory.

1. Recall some basic notions and facts, which we will use. Let \( L \) be a locally convex space, let \( (T_1, T_2, \ldots, T_m) \) be a system of commuting continuous operators on \( L \). The complex

\[
0 \to L \otimes E_m \xrightarrow{d_m} L \otimes E_{m-1} \to \cdots \to L \otimes E_1 \xrightarrow{d_1} L \to 0,
\]
where $E_i = \bigwedge^i C^m$,

$$d_i(x \otimes e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{k=1}^i (-1)^{k-1} T_{jk} x \otimes e_{j_1} \wedge \cdots \wedge e_{j_{k-1}} \wedge e_{j_{k+1}} \wedge \cdots \wedge e_{j_i},$$  \hspace{1cm} (1)

is called the Koszul complex of the pair $(L; (T_1, \cdots, T_m))$ (we denote this complex by Kos$(L; (T_1, \cdots, T_m))$, and the complex

$$0 \to L \otimes E_m \overset{d_m}{\to} L \otimes E_{m-1} \to \cdots \to L \otimes E_1 \overset{d_1}{\to} L \overset{\varepsilon}{\to} E/D_m \to 0,$$  \hspace{1cm} (2)

where $D_m = \text{Im} T_1 + \cdots + \text{Im} T_m$, $\varepsilon$ is the natural projection, is called the augmented Koszul complex of the pair $(L; (T_1, \cdots, T_m))$.

For the pair $(L; (T_1, \cdots, T_m))$ we consider the spaces $D_i = \text{Im} T_1 + \cdots + \text{Im} T_i$, $i = 1, \cdots, m$, $D_0 = \{0\}$. Then the operator $T_{i+1}$, which leaves the space $D_i$ invariant, induces the operator $\tilde{T}_{i+1}$ on $L/D_i$. It is known that (see [1, Theorem V.1.3], [2, Proposition 4.1]) an augmented Koszul complex $\tilde{T}_i$ has a contracting homotopy in the category of locally convex spaces provided the operator $\tilde{T}_i$ has a left inverse continuous operator for all $i = 1, \cdots, m$.

The definitions of topological (locally convex) algebras (in particular, Fréchet algebras), of modules over them, and of their homological characteristics are given in [1], [4]. For a locally convex algebra $A$ we denote by $A dh X$ the homological dimension of a left $A$-module $X$ (that is the minimal length of the projective resolution of $X$); by $ds A$, $dg A$, $db A$ we denote, respectively, the left small homological dimension, the left global dimension, and the bidimension of the algebra $A$. These values are defined as follows: $ds A = \sup\{A dh X : X$ is a left $A$-module, $\dim X < \infty\}$, $dg A = \sup\{A dh X : X$ is a left $A$-module\}, $db A = A^e dh A$, where $A^e = A \otimes A^{\text{op}}$ is the enveloping algebra of the algebra $A$ (here $\otimes$ is the complete projective tensor product, and $A^{\text{op}}$ is the algebra with the opposite multiplication).

We will consider the commutative Fréchet algebra $C^\infty(M)$ of smooth functions on a smooth $m$-dimensional real manifold $M$. The topology on the space $C^\infty(M)$ is given by the system of seminorms

$$\|f\|_{(W_i, \omega_i), K, n_1, \cdots, n_m} = \max_{\omega_i(K)} \left| \frac{\partial^{n_1+\cdots+n_m}}{\partial x_1^{n_1} \cdots \partial x_m^{n_m}} f(\omega_i^{-1}(x)) \right|,$$

where $(W_i, \omega_i)$, $i = 1, 2, \cdots$ are charts, i.e., open sets $W_i \subset M$ along with the fixed homeomorphisms $\omega_i$ onto open subsets in $R^m$, $K \subset W_i$ is a compact set, $n_1, \cdots, n_m$ are non-negative integers, $x = (x_1, \cdots, x_m) = \omega_i(s) \in R^m$, $s \in W_i$. We recall that by Grothendieck’s Theorem [3, Chapitre II, §3] for any smooth manifolds $M_1$ and $M_2$ the spaces $C^\infty(M_1 \times M_2) = C^\infty(M_1 \times C^\infty(M_2))$ and $C^\infty(M_1) \otimes C^\infty(M_2)$ are topologically isomorphic. Moreover it is evident that the isomorphism $C^\infty(M)^e = C^\infty(M) \otimes C^\infty(M) \simeq C^\infty(M \times M)$ is an isomorphism of algebras.

Since (see [1]) for arbitrary locally convex algebra

$$\text{ds } A \leq \text{dg } A \leq \text{db } A,$$

$$2$$
in order to prove Theorem it is sufficient to find a finite-dimensional \( C^{\infty}(\mathcal{M}) \)-module with the homological dimension not less than \( m \) (Section 3 is devoted to this) and to establish that \( C^{\infty}(\mathcal{M}) \cdot dh C(\mathcal{M}) \leq m \) (see Section 5).

2. For an open set \( U \subset \mathcal{M} \) the space \( C^{\infty}(U) \) of smooth functions is a Fréchet module over the algebra \( C^{\infty}(\mathcal{M}) \) with respect to the pointwise outer multiplication \( f \cdot g(s) = f(s)g(s) \), where \( f \in C^{\infty}(\mathcal{M}) \), \( g \in C^{\infty}(U) \), \( s \in U \).

**Theorem 2.** Let \( U \) be an open set in \( \mathcal{M} \), which is contained in a chart. Then \( C^{\infty}(U) \) is a projective \( C^{\infty}(\mathcal{M}) \)-module.

\(<\) It is sufficient to prove that \( C^{\infty}(U) \) is a retract of the free \( C^{\infty}(\mathcal{M}) \)-module \( C^{\infty}(\mathcal{M}) \otimes C^{\infty}(U) \), i.e., that the canonical projection

\[ \pi_U : C^{\infty}(\mathcal{M} \times U) \to C^{\infty}(U), \]

\[ \pi_U(f) = f(s,s), \ s \in U \] has a left inverse morphism \( \rho \) \cite[Theorem III.1.30]{[H]}.

Let \( U \subset W \), \( (W, \omega) \) be a chart. We consider the positive continuous function

\[ \psi(x) = \min(1, \text{dist}(x, \partial\omega(U))), \]

where \( \text{dist} \) is the Euclidean distance from the point \( x \in \omega(U) \) to the boundary.

We take an arbitrary smooth function \( \varphi(x), x \in \omega(U) \) such that

\[ 0 < \varphi(x) < \psi(x) \]

and, using this function, we define on \( \omega(U) \times \omega(U) \) the smooth function \( \theta \) by

\[ \theta(x,y) = \begin{cases} \exp \frac{|x-y|^2}{|x-y|^{\varphi(y)}} & \text{if } |x-y| \leq \varphi(y), \\ 0, & \text{if } |x-y| > \varphi(y). \end{cases} \]

For \( (s,t) \in \mathcal{M} \times U \) we set

\[ F(s,t) = \begin{cases} \theta(\omega(s),\omega(t)), & \text{if } s \in U, \\ 0, & \text{if } s \notin U. \end{cases} \]

Since it is evident that the support of \( F(s,t) \) belongs to \( U \times U \) then \( F(s,t) \in C^{\infty}(\mathcal{M} \times U) \), moreover \( F(s,s) = 1 \). Then we can define the required map \( \rho : C^{\infty}(U) \to C^{\infty}(\mathcal{M} \times U) \) by

\[ (\rho f)(s,t) = \begin{cases} f(s)F(s,t), & \text{if } s \in U, \\ 0, & \text{if } s \notin U. \end{cases} \]

It is clear that \( \rho \) is well-defined, it is a morphism, and \( \pi_U \rho = 1_{C^{\infty}(U)} \).

Thus the projectivity of the module \( C^{\infty}(U) \) is proved. \(\triangleright\)

3. For any open set \( U \subset \mathcal{M} \), lying entirely in a chart, e.g., \( (W, \omega) \), we define the multiplication operators by the \( k \)-th coordinate function \( T^{m}_k(U) : C^{\infty}(U) \to C^{\infty}(U) \). Namely for \( \omega(t) = (\omega^1(t), \ldots, \omega^m(t)) \in \mathbb{R}^m \), \( t \in U \) we set

\[ T^{m}_k(U)f = \omega^k(t)f(t), \ k = 1, \ldots, m. \]

It is evident that \( T^{m}_k(U) \) are \( C^{\infty}(\mathcal{M}) \)-module morphisms.
We take an arbitrary point $s_0 \in U$ and an open set $U_0$, containing this point and belonging to a chart. Without loss of generality one can suppose that $U_0$ is homeomorphic to $R^m (\omega(U_0) = R^m)$ and $\omega(s_0) = 0$.

We denote by $C_0$ the one-dimensional $C^\infty(M)$-module $C$ with the outer multiplication $f \cdot \lambda = f(s_0)\lambda$, where $\lambda \in C$, $f \in C^\infty(M)$.

**Theorem 3.** The complex

$$\text{Kos}(C^\infty(U_0); (T^m_1(U_0), \cdots, T^m_m(U_0)) \xrightarrow{\pi_0} C_0, \quad (3)$$

over $C_0$, where $\pi_0 : C^\infty(U_0) \to C_0$, $\pi_0(f) = f(s_0)$, is a projective resolution of the $C^\infty(M)$-module $C_0$.

\(<\text{ Since all modules } C^\infty(U_0) \otimes E_i \text{ are projective as projective summands of projective modules and the maps } d_i, \pi_0 \text{ are morphisms of } C^\infty(M) \text{-modules it is sufficient to prove that the complex } (3) \text{ is admissible, i.e., it has a contracting homotopy in the category of Fréchet spaces.}\)

It is clear that the condition $\omega(U_0) = R^m$ implies the isomorphism of the Fréchet spaces of the complex in question and that of the complex

$$0 \to C^\infty(R^m) \otimes E_m \xrightarrow{d_m} C^\infty(R^m) \otimes E_{m-1} \to \cdots \to C^\infty(R^m) \otimes E_1 \xrightarrow{d_1} C^\infty(R^m) \xrightarrow{\pi_0} C_0 \to 0 \quad (4)$$

in the category of complexes, that is the isomorphism between $(3)$ and the complex

$$\text{Kos}(C^\infty(R^m); (T_1, \cdots, T_m)) \xrightarrow{\pi'_0} C_0,$$

where $T_k(f) = x_k f(x), \ x = (x_1, \cdots, x_m) \in R^m$, $\pi'_0 : C^\infty(R^m) \to C_0$, $\pi'_0(f) = f(0)$.

For the complex $(4)$ the space $D_k = \text{Im} T_1 + \cdots + \text{Im} T_k$ coincides with the space $A_k = \{ f \in C^\infty(R^m) : f(0, \cdots, 0, x_{k+1}, \cdots, x_m) = 0 \}$: the inclusion $D_k \subseteq A_k$ is evident, and from Hadamard’s Lemma it follows that functions $f \in A_k$ are representable in the form $f(x) = \sum_{i=1}^k x_i f_i(x)$, where $f_i \in C^\infty(R^m)$, i.e., they belong to $D_k$. Therefore $C^\infty(R^m)/D_k \simeq C^\infty(R^{m-k})$ (in particular, $C^\infty(R^m)/D_m \simeq C_0$) and the natural projection $\varepsilon : C^\infty(R^m) \to C^\infty(R^m)/D_m$ is identified with the morphism $\pi'_0$. Thus we have showed that up to isomorphism the complex $(3)$ is the augmented Koszul complex of the pair $(C^\infty(R^m); (T_1, \cdots, T_m))$, besides, the established isomorphisms let to consider the operator $T_{k+1}$ to be acting on the space $C^\infty(R^{m-k})$:

$$T_{k+1}(g(x_{k+1}, \cdots, x_m)) = x_{k+1} g(x_{k+1}, \cdots, x_m).$$

Then the continuous operator

$$S_{k+1} : C^\infty(R^{m-k}) \to C^\infty(R^{m-k}),$$
Corollary. \( S_k(g(x_{k+1}, \ldots, x_m)) = (g(x_{k+1}, \ldots, x_m) - g(0, x_{k+2}, \ldots, x_m))/x_{k+1} \) is a left inverse to \( T_{k+1} \). Thus the sufficient conditions from Section 1 for the existence of a contracting homotopy for the augmented Koszul complex \( \mathcal{H} \) in the category of Fréchet spaces are fulfilled. Therefore the complex \( \mathcal{H} \) and the complex \( \mathcal{K} \), isomorphic to it, are admissible. \( \triangleright \)

Proposition 1. \( C^{\infty}(\mathcal{M}) \) dh \( C_0 = m \).

\( \triangleright \) In view of the fact that \( [1] \) Theorem III.5.4] for an arbitrary locally convex algebra \( A \) and for an \( A \)-module \( X \)

\[ A \text{ dh } X = \sup \{ k : A \text{ Ext}^{k+n}(X, Y) = 0 \text{ for any } n > 0 \text{ and there is a module } Y, \text{ such that }_A \text{ Ext}^k(X, Y) \neq 0 \} \]

it is sufficient to prove that \( C^{\infty}(\mathcal{M}) \text{ Ext}^m(C_0, C_0) \neq 0 \). For this we use the projective resolution \( \mathcal{K} \) of the module \( C_0 \). Up to an isomorphism the left end of the resolution has the form

\[ 0 \to C^{\infty}(U_0) \xrightarrow{d_m} \underbrace{C^{\infty}(U_0) \oplus \cdots \oplus C^{\infty}(U_0)}_{m} \to \cdots, \]

where \( d_m(f) = (\omega^1(t)f(t), \ldots, \omega^m(t)f(t)) \), \( t \in U_0 \). Since for any function \( f \in C^{\infty}(U_0) \) there exists a sequence \( \{ f_n \} \), \( f_n \in C^{\infty}(\mathcal{M}) \), such that \( f = \lim_{n \to \infty} f_n|_{U_0} \) in \( C^{\infty}(U_0) \), then for any \( C^{\infty}(\mathcal{M}) \)-module morphism \( \alpha : C^{\infty}(U_0) \to C_0 \) we have

\[ \alpha(f) = \alpha(\lim_{n \to \infty} f_n|_{U_0}) = \lim_{n \to \infty} \alpha(f_n|_{U_0}) = \lim_{n \to \infty} (f_n \cdot \alpha(1)) = \lim_{n \to \infty} f_n(s_0)\alpha(1) = f(s_0)\alpha(1). \]

Consequently, for any morphism \( \gamma : \underbrace{C^{\infty}(U_0) \oplus \cdots \oplus C^{\infty}(U_0)}_{m} \to C_0 \) we have

\[ \gamma d_m(f) = \gamma(\omega^1(t)f(t), \ldots, \omega^m(t)f(t)) = \gamma(\omega^1(t)f(t)) + \cdots + \gamma(\omega^m(t)f(t)) = 0 \]

and then \( C^{\infty}(\mathcal{M}) \text{ Ext}(C_0, C_0) = c^{\infty}(\mathcal{M}) \text{ hom}(C^{\infty}(U_0), C_0) \).

It remains to observe that the later space is not zero because it contains the non-trivial morphism \( \pi_0 \), defined by \( \pi_0(f) = f(s_0). \triangleright \)

From Proposition \( \mathcal{H} \) we immediately get a lower estimate for homological dimensions.

Corollary. \( dsC^{\infty}(\mathcal{M}) \geq m \).

4. The aim of the further exposition is to get an upper estimate for homological dimensions. At the beginning we make it ‘locally’, namely for the \( C^{\infty}(\mathcal{M})^e \)-module \( C^{\infty}(U) \) with the outer multiplication \( f \cdot g = f(s, s)g(s), f \in C^{\infty}(\mathcal{M})^e, g \in C^{\infty}(U), s \in U \), we establish that \( C^{\infty}(\mathcal{M}) \text{ dh } C^{\infty}(U) \leq m \).

As before we suppose that \( U \) is contained in a chart.
Theorem 4. The complex

\[
\text{Kos}(C^\infty(U \times U); (T_{2m-1}^1(U \times U) - T_{m+1}^{2m}(U \times U)), \ldots, T_{2m}^m(U \times U) - T_{2m}^{2m}(U \times U))) \xrightarrow{\pi} C^\infty(U),
\]
(5)

where \(\pi(f) = f(s,s), s \in U\) is a projective resolution of \(C^\infty(M)^e\)-module \(C^\infty(U)\).

\(~\)\footnote{By Theorem 2 the \(C^\infty(M)^e\)-modules \(C^\infty(U \times U) \otimes E_i\) are projective and consequently, as in Theorem 3 it is sufficient to establish the admissibility of the complex (5). But the complex (5) is isomorphic to the complex

\[
0 \to C^\infty(\omega(U) \times \omega(U)) \otimes E_m \xrightarrow{d_m} C^\infty(\omega(U) \times \omega(U)) \otimes E_{m-1} \to \cdots \\
\to C^\infty(\omega(U) \times \omega(U)) \xrightarrow{\pi} C^\infty(\omega(U)) 0,
\]
(6)

\((\omega(U) \subset R^m)\), which is known to be admissible [2, Proposition 4.4]. ~\)

\section*{Corollary.}
\(C^\infty(M)^e \text{ dh } C^\infty(U) \leq m.\)

We take an arbitrary set \(\{U_i\}, i = 1, 2, \cdots\) of open subsets of \(M\), each of which lies in a chart. Then the space \(\prod_{i=1}^{\infty} C^\infty(U_i)\) being a countable Cartesian product of Fréchet spaces and at the same time being \(C^\infty(M)^e\)-modules is a Fréchet \(C^\infty(M)^e\)-module with respect to the outer multiplication, defined componentwise.

Proposition 2. \(C^\infty(M)^e \text{ dh } \prod_{i=1}^{\infty} C^\infty(U_i) \leq m.\)

\(~\)\footnote{We consider the Cartesian product of the complexes (5) for \(C^\infty(M)^e\)-modules \(C^\infty(U_i)\):

\[
0 \to \prod_{i=1}^{\infty} C^\infty(U_i \times U_i) \otimes E_m \xrightarrow{d_m} \prod_{i=1}^{\infty} C^\infty(U_i \times U_i) \otimes E_{m-1} \xrightarrow{d_{m-1}} \cdots \\
\to \prod_{i=1}^{\infty} C^\infty(U_i \times U_i) \xrightarrow{\pi} \prod_{i=1}^{\infty} C^\infty(U_i) 0.
\]
(7)

We prove that the complex (7) is a projective resolution for \(\prod_{i=1}^{\infty} C^\infty(U_i)\).

Actually, it is evident that the maps \(d_i, \pi\) are \(C^\infty(M)^e\)-module morphisms. The contracting homotopy maps for the complex (7) are obtained as Cartesian.
products of respective contracting homotopy maps for $C^\infty(M)^s$-module complexes for $C^\infty(U_i)$, therefore the admissibility of the complex (7) follows from the admissibility of the complex (5).

It remains to show that the modules $\prod_{i=1}^\infty C^\infty(U_i \times U_i)$ are projective. As in Theorem 2 for the canonical morphism

$$\pi_\infty : C^\infty(M \times M) \otimes \prod_{i=1}^\infty C^\infty(U_i \times U_i) \to \prod_{i=1}^\infty C^\infty(U_i \times U_i),$$

$$\pi_\infty(f \otimes \{g_i\}) = \{\pi_i(f \otimes g_i)\}$$

(here $\pi_i$ are canonical $C^\infty(M)^s$-module morphism for the module $C^\infty(U_i \times U_i)$ we suggest a morphism $\rho_\infty$ such that $\pi_\infty \rho_\infty = 1$. For the constructing the morphism $\rho$ we use the Fréchet space isomorphism [5, Chapter 1, §1, Proposition 1]

$$C^\infty(M \times M) \otimes \prod_{i=1}^\infty C^\infty(U_i \times U_i) \simeq \prod_{i=1}^\infty C^\infty(M \times M) \otimes C^\infty(U_i \times U_i)$$

and we define the morphism $\rho_\infty$ in the following manner

$$\rho_\infty : \prod_{i=1}^\infty C^\infty(U_i \times U_i) \to \prod_{i=1}^\infty C^\infty(M \times M) \otimes C^\infty(U_i \times U_i),$$

$$\rho_\infty(\{g_i\}) = \{\rho_i g_i\}$$

($\rho_i$ is a left inverse $C^\infty(M)^s$-module morphism to the morphism $\pi_i$, which exists because of projectivity of the module $C^\infty(U_i \times U_i)$.

Thus the length of the admissible resolution (7) for the $C^\infty(M)^s$-module $\prod_{i=1}^\infty C^\infty(U_i)$ does not exceed $m$ and consequently $C^\infty(M)^s \cdot dh \prod_{i=1}^\infty C^\infty(U_i) \leq m$. >

5. The final phase of the proof is to establish the estimate $C^\infty(M)^s \cdot dh C^\infty(M) \leq m$ by ‘pasting together’ the local upper estimates by the means of the Čech complex.

Theorem 5. $C^\infty(M)^s \cdot dh C^\infty(M) \leq m$

\(<\)We take a covering of the manifold $M$ with countable set of charts $\{(W_i, \omega_i)\}$. The covering dimension of $m$-dimensional smooth real manifold does not exceed $m$ (see [6, Theorem 2.15]), therefore one can inscribe a countable covering $\mathcal{U} = \{U_i\}$ of the multiplicity less at most $m$ in the covering $\{W_i\}$. In other words any point of the manifold is contained in the $m + 1$ sets of the system $\mathcal{U}$ at most. We construct the augmented smooth cochains complex corresponding to the covering $\mathcal{U}$:

$$0 \to C^\infty(M) \xrightarrow{\eta} \mathcal{C}^0(\mathcal{U}, C^\infty(M)) \xrightarrow{\partial^0} \mathcal{C}^1(\mathcal{U}, C^\infty(M)) \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^{m-1}} \mathcal{C}^m(\mathcal{U}, C^\infty(M)) \to 0,$$
where \( \tilde{C}^i(\mathcal{U}, C^\infty(\mathcal{M})) = \prod_i C^\infty(|\sigma_i|) \),

\begin{equation}
(\partial^{i-1} f)(\sigma) = \sum_{k=0}^{i} (-1)^k f(U_{j_0}, \ldots, U_{j_{k-1}}, U_{j_{k+1}}, \ldots, U_{j_i})|_{\sigma_i},
\end{equation}

(\text{the product is taken by the sets } \sigma_i = (U_{j_0}, \ldots, U_{j_i}), j_0 < \cdots < j_i \text{ such that } |\sigma_i| = U_{j_0} \cap \cdots \cap U_{j_i} \neq \emptyset)

We note that \( \tilde{C}^i(\mathcal{U}, C^\infty(\mathcal{M})) = 0 \) for \( i > m \) because \( |\sigma_i| = \emptyset \).

The complex \( \tilde{\mathcal{C}} \) is exact [7]. Moreover, one can remark that in the proof of this fact the necessary contracting homotopy in the category of Fréchet spaces is constructed [7, Lemma VI.1.3]. Therefore the complex \( \tilde{\mathcal{C}} \) is admissible. We represent the complex in the form of Yoneda product of \( C^\infty(\mathcal{M})^e \)-modules and morphisms:

\begin{equation}
0 \to \text{Ker } \partial^{m-1} \to \tilde{C}^{m-1}(\mathcal{U}, C^\infty(\mathcal{M})) \overset{\partial^{m-1}}{\longrightarrow} \tilde{C}^m(\mathcal{U}, C^\infty(\mathcal{M})) \to 0, \tag{9}
\end{equation}

\begin{equation}
0 \to \text{Ker } \partial^i \to \tilde{C}^i(\mathcal{U}, C^\infty(\mathcal{M})) \overset{\partial^i}{\longrightarrow} \text{Ker } \partial^{i-1} \to 0, \tag{10}
\end{equation}

\begin{equation}
0 \to C^\infty(\mathcal{M}) \overset{\partial^0}{\longrightarrow} \tilde{C}^0(\mathcal{U}, C^\infty(\mathcal{M})) \overset{\partial^0}{\longrightarrow} \text{Ker } \partial^1 \to 0. \tag{11}
\end{equation}

For an arbitrary fixed \( C^\infty(\mathcal{M})^e \)-module \( Y \) each of the sequences [10]–[11] defines a long exact sequence [1, Theorm III.4.4]. For instance, for the sequence (11) the long exact sequence has the form:

\[ \cdots \to C^\infty(\mathcal{M})^e \overset{\text{Ext}^k(\partial^{m-1}, Y)}{\longrightarrow} \text{Ker } \partial^{m-1} \overset{\text{Ext}^k(\partial^{m-1}, Y)}{\longrightarrow} C^\infty(\mathcal{M})^e \overset{\text{Ext}^{k+1}(\partial^{m}, Y)}{\longrightarrow} \cdots. \]

Since \( C^\infty(\mathcal{M})^e \overset{\text{dh } \tilde{C}^{m-1}}{\leq} m \) and \( C^\infty(\mathcal{M})^e \overset{\text{dh } \tilde{C}^{m}}{\leq} m \), then

\[ C^\infty(\mathcal{M})^e \overset{\text{Ext}^k(\text{Ker } \partial^{m-1}, Y)}{=} 0 \]

when \( k > m \). Applying further the exact sequence for the functor \( C^\infty(\mathcal{M})^e \overset{\text{Ext}(\cdot, Y)}{\longrightarrow} \) to the sequence (10) when \( i = m - 2, \cdots, 1 \), and then to the sequence (11), and, taking into account Proposition 2, we get that \( C^\infty(\mathcal{M})^e \overset{\text{Ext}^k(\text{Ker } \partial^i, Y)}{=} 0 \) when \( k > m \), and finally, \( C^\infty(\mathcal{M})^e \overset{\text{Ext}^k(C^\infty(\mathcal{M}), Y)}{=} 0 \) when \( k > m \). Consequently, \( C^\infty(\mathcal{M})^e \overset{\text{dh } C^\infty(\mathcal{M})}{\leq} m \).

Joining the just obtained inequality and Corollary from Theorem 3 we get that

\[ \text{ds } C^\infty(\mathcal{M}) = \text{dg } C^\infty(\mathcal{M}) = \text{db } C^\infty(\mathcal{M}) = m. \]

The proof of Theorem 4 is finished.
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