Kondo Quartet
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This article describes some recently obtained results on the low-energy properties of the "Kondo quartet" model of two spin-1/2 impurities interacting with two channels (flavours) of conduction electrons. We shall particularly emphasize the connections between conformal field-theory methods and bosonisation approaches, which are first illustrated on the example of the single-impurity, two-channel Kondo problem. This article is dedicated to the memory of Claude Itzykson, and will appear in the Proceedings of the Conference "Advanced Quantum Field Theory", held in La Londe Les Maures in Sept, 1996 (Nucl. Phys. B, Proc. Supp., V. Rittenberg, J. Frölich and A. Schwimmer eds.).

1. The two channel Kondo model

Two methods leading to a low-energy solution of the single impurity, two channel Kondo model have been recently put forward. One, due to Affleck and Ludwig, uses the general framework of (boundary) conformal field theory (CFT). The other, due to Emery and Kivelson, relies on abelian bosonisation and establishes an exact mapping onto a resonant-level model which reduces to free fermions for a special value of the Kondo coupling \( J_z \). This is analogous to the Toulouse limit of the single-channel case. The aim of this section is to provide the reader with an introduction to the two channel Kondo physics by reviewing partly these methods and, more importantly, by establishing the precise connection between the two approaches. This will prove to be very useful in the study of the corresponding two impurity problem.

1.1. Model and Symmetries

We formulate the model in terms of left-moving chiral fermions \( \psi_{i\alpha}(x) \) on the full axis \(-\infty < x < +\infty\), and of a spin-1/2 impurity spin \( \vec{S} \) placed at the origin. \( i = 1, 2 \) is a channel index, and \( \alpha = \uparrow, \downarrow \) is a spin index. The Hamiltonian is written as:

\[
H = iv_F \sum_{i\alpha} \int_{-\infty}^{+\infty} dx \psi_{i\alpha}^\dagger(x) \frac{\partial}{\partial x} \psi_{i\alpha}(x) + J_z S_z J_z(0) + J(S_x J_x(0) + S_y J_y(0))
\]

In these formulae, \( J(x) = \sum_{i=1}^{2} \sum_{\alpha\beta} \psi_{i\alpha}^\dagger(x) \frac{\sigma_{\alpha\beta}}{2} \psi_{i\beta}(x) \) denotes the total conduction electron spin current at position \( x \). A Kondo coupling which is anisotropic in spin space has been considered, for reasons which will become clear below. The anisotropy \( J_{-} - J_{+} \) is actually irrelevant at the non-trivial fixed point. As demonstrated by Nozières and Blandin and widely discussed since, both the free fermion \( J = 0 \) and the strong-coupling \( J = +\infty \) fixed points are unstable under renormalization. The low-energy physics is controlled by an intermediate coupling fixed point, which has non Fermi liquid properties \( \chi_{imp} \sim C/T \sim \ln 1/T \).

The interacting Hamiltonian has, in the isotropic case \( J_{-} = J_{+} \), a global symmetry \( SU(2)_{spin} \otimes SU(2)_{flavour} \otimes U(1)_{charge} \). This global symmetry group is smaller than the symmetry of the non-interacting theory with \( J_{z} = J = 0 \), which has a \( U(4) = SU(4) \otimes U(1) \) invariance mixing all four species \( i, \alpha \). This invariance is partially broken by the Kondo interaction. In fact, as explained in detail in [3], the non-
interacting theory has a full $SO(8)$ global symmetry including transformations which mix flavour and charge, and the interacting theory itself has a larger global symmetry than the spin-flavour-charge one, namely an $SU(2)_{\text{spin}} \otimes SO(5)_{c,f}$ invariance. We shall make use of this larger symmetry later in this paper.

The basic assumption of the CFT approach is that, at a fixed point, these global symmetries are promoted to local conformal symmetries. The symmetry algebra becomes the Kac-Moody algebra \cite{7}. The Kac-Moody symmetry puts strong constraints on the theory. In particular, it dictates the general form of the finite-size spectrum at any fixed point of the model as:

$$\frac{L \Delta E}{\pi v_F} = \frac{j(j+1)}{4} + m + \frac{j_f(j_f+1)}{4} + m_f + Q^2 + 8m_c(4)$$

(3)

Here, $SU_k(2)$ stands for the level-$k$ $SU(2)$ Kac-Moody algebra \cite{8}. The Kac-Moody symmetry puts strong constraints on the theory. In particular, it dictates the general form of the finite-size spectrum at any fixed point of the model as:

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In this expression, $\Delta E = E - E_{gs}$ are the excitation energies, $v_F$ is the Fermi velocity and $L$ the radial length of the system. The quantum number $j$ labels the representation of the $SU_2(2)_{\text{spin}}$ algebra and takes the possible values $j = 0, 1/2, 1$. Similarly, $j_f = 0, 1/2, 1$ labels the representation of $SU_2(2)_{\text{flavour}}$ and $Q$ is a (positive or negative) integer corresponding to the charge of $U(1)_c$. $m, m_f, m_c = 0, 1, 2, 3, ...$ are integers corresponding to towers of states built on these primary states. The degeneracies are also specified uniquely in terms of these quantum numbers.

The general form (3) dictates the possible energy levels, but does not specify which set of quantum numbers $j, j_f, Q$ actually appears in the spectrum at a given fixed point. The strategy followed by Affleck and Ludwig is first to identify the appropriate selection rules for the free fermion fixed point ($J = 0$). For this fixed point, all energy levels can be built following the Pauli principle. A major insight is then that the spectrum on the non-trivial infra-red stable fixed point can be obtained from a 'fusion principle'. Specifically, the spectrum is obtained by acting on the primary operator in the spin sector associated with a given state of the free-fermion fixed point with the operator ($j = 1/2$) of the $SU_2(2)_{\text{spin}}$ algebra (leaving $j_f$ and $Q$ unchanged). Each admissible values of $j = j_{FF}$ at the non-interacting fixed point then gives rise to at most two admissible values at the new fixed point, according to the fusion rule of CFT:

$$j = 0 \rightarrow j' = 1/2$$

$$j = 1/2 \rightarrow j' = 0 \text{ and } j' = 1$$

$$j = 1 \rightarrow j' = 1/2$$

(5)

Physically, this fusion principle reflects the (partial) screening of the impurity spin by the conduction electrons. The conformal towers appearing in the resulting spectrum are detailed in Refs.\cite{2,3}. This construction can be extended to find the operator content at the non-trivial fixed point (by a double fusion).

### 1.2. Bosonisation

The decomposition of a fermion $\psi_{ia}(x)$ into spin-flavour-charge degrees of freedom can be made explicit in two possible ways. The first is to make use of 'non-abelian bosonisation' which has the advantage of respecting all global symmetries explicitly. This route is the natural one within the CFT approach. The fermion field is written as:

$$\psi_{ia}(x)^+ = g_{ia}^+ h_i^+ e^{i\Phi_c/2}$$

(6)

In this expression, $g_{ia}^+, h_i^+$ are Wess-Zumino-Witten fields belonging to the fundamental representation of the $SU_2(2)_s$ and $SU_2(2)_f$ spin and flavour algebra, respectively, and $\Phi_c$ is a charge boson. Alternatively, one may take a more 'pedestrian' (and often more explicit) route by using abelian bosonisation, even though the global $SU(2)$ invariances are no longer manifest. This is the starting point of the approach due to Emery and Kivelson \cite{4}, who introduce four boson fields such that:

$$\psi_{ia}(x) = \frac{1}{\sqrt{2\pi a_0}} e^{-i\Phi_{ia}(x)}$$

(7)

where:

$$\Phi_{ia}(x) = \sqrt{\pi} \left\{ \int_{-\infty}^{x} dx' \Pi_{ia}(x') - \phi_{ia}(x) \right\}$$

(8)
with:
\[ \phi_{i\alpha}(x), \Pi_{j\beta}(x') = i\delta_{ij}\delta_{\alpha\beta}\delta(x - x') \] (9)

Our conventions for boson fields are such that an operator \( O = e^{i\Phi} \) has dimension \( \Delta = g^2/2 \) (i.e., its correlation function behaves as \( O(0)O(x) \sim 1/x^{2\Delta} \) for \( |x| \to \infty \)).

Linear combinations are then formed, corresponding to charge \( \phi_c \), spin \( \phi_s \), flavor \( \phi_f \) and spin-flavor \( \phi_{sf} \) degrees of freedom (similarly \( \Phi_c, \Phi_s, \Phi_f, \Phi_{sf} \)):
\[ \phi_c = \frac{1}{2}(\phi_{\uparrow\uparrow} + \phi_{\downarrow\downarrow} + \phi_{\uparrow\downarrow} + \phi_{\downarrow\uparrow}) \] (10)
\[ \phi_s = \frac{1}{2}(\phi_{\uparrow\uparrow} - \phi_{\downarrow\downarrow} + \phi_{\uparrow\downarrow} - \phi_{\downarrow\uparrow}) \] (11)
\[ \phi_f = \frac{1}{2}(\phi_{\uparrow\uparrow} + \phi_{\downarrow\downarrow} - \phi_{\uparrow\downarrow} - \phi_{\downarrow\uparrow}) \] (12)
\[ \phi_{sf} = \frac{1}{2}(\phi_{\uparrow\uparrow} - \phi_{\downarrow\downarrow} - \phi_{\uparrow\downarrow} + \phi_{\downarrow\uparrow}) \] (13)

In terms of these fields, the components of the total spin current read:
\[ J^a = \frac{1}{\pi a_0}\cos\Phi_s\cos\Phi_{sf}, \quad J_y = \frac{1}{\pi a_0}\sin\Phi_s\cos\Phi_{sf} \]
\[ J_z = -\frac{1}{2\pi}\frac{\partial\Phi}{\partial x} \] (14)

Only \( \Phi_s \) and \( \Phi_{sf} \) enter these expressions. This is expected since these fields correspond to the sum and the difference of the bosons associated with spin degrees of freedom for each channel, namely \( \Phi_{1\uparrow} - \Phi_{1\downarrow} \) and \( \Phi_{2\uparrow} - \Phi_{2\downarrow} \). Furthermore, note that among the four independent real combinations of these fields, only \textit{three} appear: \( \sin\Phi_{sf} \) is not involved in the total spin current. That three real fields enter the current reflects the fact that the \( \hat{SU}(2) \) Kac-Moody algebra has central charge \( c = 3/2 \). These fields have dimension 1/2 and can alternatively be considered as three real fermionic fields (Majorana fermions) \( \chi^a, \ a = x, y, z \):
\[ \chi^x = \sin\Phi_s, \ \chi^y = \cos\Phi_s, \ \chi^z = \cos\Phi_{sf} \] (15)
in terms of which the total spin current reads:
\[ \mathcal{J}^a = i\epsilon_{abc}\chi^b\chi^c \] (16)

This representation of the \( \hat{SU}(2) \) algebra in terms of three Majorana fermions is well known in CFT [3], and will prove to be most useful below in the solution of the two impurity problem.

The Hamiltonian is easily written in terms of the boson fields. The free (kinetic energy) part reads:
\[ H_0 = \frac{v_F}{4\pi} \sum_{l=s,c,f,sf} \int dx (\frac{\partial \Phi_l}{\partial x})^2 \] (17)

while the Kondo interaction takes the form:
\[ H_{int} = -\frac{J}{2\pi} S^z \frac{\partial \Phi_s}{\partial x}(0) + \frac{J}{\pi a_0} [S^x \sin\Phi_s(0) + S^y \cos\Phi_s(0)] \] (19)

\[ H_{int} = -\frac{J}{2\pi} S^z \frac{\partial \Phi_s}{\partial x}(0) + \frac{J}{\pi a_0} S^x \cos\Phi_{sf}(0) \] (21)

Hence, for the special value of the coupling, \( J_z = 2\pi v_F \), \( \Phi_s \) no longer enters the Hamiltonian, which takes the very simple form:
\[ H_0 + \frac{J}{\pi a_0} S^x \cos\Phi_{sf}(0) \] (22)

This describes a resonant level model for the Majorana fermion \( \chi^z = \cos\Phi_{sf} \), with scattering at the Fermi level. So \( \chi^z \) (which was left unchanged by the rotation \( U \)) suffers a phase shift or, more appropriately for a real object, suffers a change of boundary condition from periodic \( \chi^z(0^+) = \chi^z(0^-) \) to antiperiodic \( \chi^z(0^+) = -\chi^z(0^-) \). The ‘twist’ operator \( \sigma_z \) which achieves this change of boundary conditions when acting on \( \chi^z \) has dimension \( \Delta(\sigma_z) = 1/16 \), corresponding to the shift in the ground-state energy under a change of boundary conditions of one free Majorana fermion: \( L\Delta E/\pi v_F = 1/16 \). Hence \( \Delta(\sigma_z) = 1/16 \).

In contrast, the \textit{transformed} fields \( \tilde{\chi}^x \) and \( \tilde{\chi}^y \) drop out from the Hamiltonian at the solvable
point, and hence remain free and unaffected. The relation of these transformed fields to the original ones is easily worked out by considering the Dirac fermion:

$$\tilde{\psi}_s \equiv \frac{1}{\sqrt{2\pi a_0}} e^{-i\Phi_s}$$

and observing that:

$$\tilde{\psi}_s(x) \equiv U\psi_s(x)U^\dagger = e^{iS^z\Phi_s(0)} \frac{e^{-i\Phi_s(x)}}{\sqrt{2\pi a_0}} e^{-iS^z\Phi_s(0)} = e^{-i\pi S^z \text{sign}(x)} \psi_s(x)$$

Since $S^z$ has eigenvalues $\pm 1/2$, one sees from this equation that $\psi_s(0^+) = e^{\pm i\pi} \psi_s(0^-)$ and hence that the original field $\psi_s$ also suffers a phase shift $\delta = \pi/2$ at the solvable point. In this case, the operator $U$ which achieves this has the simple boson representation (20) because it changes both components of a Dirac fermion. Its dimension is thus $1/8 = \delta^2/2\pi^2$, twice that of a twist operator for a single Majorana fermion.

Hence, the crucial feature of the Emery-Kivelson solvable point is that the three (original) Majorana fermions $\chi_{x,y,z}$ introduced in Eq. (13) in order to represent the total spin current suffer a change of boundary condition from periodic to antiperiodic. (Actually, as discussed below, we need to consider both the antiperiodic and periodic Majorana sectors to construct the free fermion spectrum of the four original Dirac fermions with e.g. antiperiodic boundary conditions. At the Kondo fixed point, the Majoranas which were originally periodic become antiperiodic, and the Majoranas which were originally antiperiodic formally become periodic, but in fact do not contribute to the spectrum at the Kondo fixed point.) This interpretation of the boundary condition associated with the non-trivial fixed point has also been discussed independently by Maldacena and Ludwig and more recently by Ye.

In the bosonic language, the new boundary condition associated with the non-trivial fixed point has also been discussed independently by Maldacena and Ludwig and more recently by Ye. In the bosonic language, the new boundary condition reads: $\Phi_s(0^+) = \pi + \Phi_s(0^-)$, $\Phi_{sf}(0^+) = \pi - \Phi_{sf}(0^-)$. The operator which achieves this change of boundary condition has been constructed in the present formulation (which is not explicitly $SU(2)$ symmetric) as the product $\sigma_z \cdot U$ of the transformation $U$ and of the twist operator for $\chi$. The total dimension of this operator is thus $1/16 + 1/8 = 3/16$. Within the CFT approach (which is manifestly $SU(2)$ invariant) this operator is identified as the primary operator corresponding to the fundamental representation $j = 1/2$ of the $SU(2)$ Kac-Moody algebra, i.e. to the WZW field $g_\alpha$. Note that its dimension is indeed $j(j+1)/4 = 3/16$. Thus, the explicit solution using abelian bosonisation at the Emery-Kivelson solvable point provides us with an explicit derivation of the 'fusion principle' of the CFT solution.

1.4. Finite-size spectrum

The whole finite-size spectrum can be simply recovered from the Majorana fermion approach (see also [8]). As in the CFT approach, the first task is to construct the spectrum at the free-fermion fixed point. Let us choose antiperiodic boundary conditions for the four original Dirac fermions $\psi_{1,2}$. The corresponding free-fermion partition function reads:

$$Z_{FF} = \prod_{m=0}^{\infty} (1 + q^{m+1/2})^8$$

where $q = e^{-4\pi L/\beta v_f}$. We must find how to construct this spectrum in terms of 8 free Majorana fermions, 3 of them ($\bar{\chi}$) being associated with the $SU(2)$ spin degrees of freedom, and 5 being associated with the $SO(5)$ charge-flavour degrees of freedom. It turns out that both the antiperiodic and the periodic sector must be considered, together with a projection onto states with an even total fermion number $F \equiv F_{spin} + F_{\bar{\chi}}$. The corresponding partition function in then obtained in the form (using formulas of Ref. [3]):

$$Z_{FF} = \text{tr}_A \frac{1+(-1)^F}{2} q^{L_0} + \text{tr}_F \frac{1+(-1)^F}{2} q^{L_0}$$

$$= \frac{1}{2} \prod_{m=0}^{\infty} (1 + q^{m+1/2})^8 + \frac{1}{2} \prod_{m=0}^{\infty} (1 - q^{m+1/2})^8 + 8q^{1/2} \prod_{m=1}^{\infty} (1 + q^m)^8$$

$$= q^{1/8} \left\{ \frac{1}{2} \sqrt{\frac{2q}{\pi}} \right\}^8 + \frac{1}{2} \left\{ \sqrt{\frac{2q}{\pi}} \right\}^8 + 8 \left\{ \sqrt{\frac{2q}{\pi}} \right\}^8$$

(26)

in which the Jacobi functions $\theta$’s and $\eta$ have argument $q$. This can be checked to coincide precisely with (25).

In order to obtain the spectrum at the Kondo fixed point from this expression, one must perform a modular transform $q \rightarrow w \equiv e^{-\pi \beta v_f/L}$, ...
and twist the boundary conditions of the three Majorana fermions associated with the spin sector, as explained above. This amounts to acting on the states with the operator $(-1)^F \psi_{nu}^\dagger$. It turns out that only the sector which was originally antiperiodic at the free-fermion fixed-point (and which has been twisted to periodic) contributes to the spectrum at the Kondo fixed point. The resulting partition function (with ground-state energy substracted) reads:

$$Z_{2\text{channel}} = \frac{1}{\sqrt{2\pi}} w^{-1/48} \left( \left( \frac{\sqrt{\eta}}{\pi} \right)^3 \left( \frac{\sqrt{\eta}}{\pi} \right)^5 \right) \left( \frac{\sqrt{\eta}}{\pi} \right)^{24} \left( \frac{\sqrt{\eta}}{\pi} \right)^{12}$$

$$= 2 + 4w^{1/8} + 10w^{1/2} + 12w^{5/8} + 26w + \cdots$$

in which the $\theta$'s and $\eta$ have argument $w$. From this, the lowest energy levels (and degeneracies) are read off as: $L\Delta E/\pi v_F = 0(2), 1/8(4), 1/2(10), 5/8(12), 1/26, \cdots$. This can be checked to coincide with the result obtained from the fusion principle in the CFT approach.

1.5. Physical properties at and near the fixed point.

Physical properties associated with the stable fixed point can also be analyzed in the bosonised language, and connections made with the CFT approach. It is convenient to fully refermionize the hamiltonian in terms of the Dirac fermion $\psi_s$ and its spin-flavour counterpart:

$$\psi_{sf} \equiv \frac{1}{\sqrt{2\pi u_0}} e^{-i\Phi_{sf}}$$

and to represent the impurity spin components in terms of three local Majorana fermions:

$$S^x = \frac{b}{\sqrt{2}}, \quad S^y = \frac{a}{\sqrt{2}}, \quad S^z = -ia b$$

The hamiltonian can be written as a sum of four terms $H = H_c + H_f + H_{sf} + H_s$, where $H_c$ and $H_f$ are the free charge and flavour parts, and the spin-flavour and spin parts read, in terms of these fermionic variables:

$$H_{sf} = iv_F \int_{-\infty}^{+\infty} dx \psi_{sf}^\dagger(x) \frac{\partial}{\partial x} \psi_{sf}(x)$$

$$-i\lambda a b \psi_{sf}^\dagger(0) \psi_{sf}(0)$$

where the coupling constant $\lambda \equiv J_z - 2\pi v F$ measures the deviation from the solvable point. The interpretation of $H_{sf}$ as a resonant level model is very clear in this form. One also sees that only the $b$ component of the impurity spin degrees of freedom hybridizes at the solvable point, in the same way that only the $\psi_{sf}^\dagger + \psi_{sf} \propto \chi z$ component of the conduction electron enters $H_{sf}$. $a$ remains a local fermion, with non-decaying correlations:

$$< Ta(0)a(\tau) > = \frac{1}{2} \text{sign}(\tau)$$

This gives rise to a residual entropy at the fixed point $S_{imp} = ln2/2$. It is a sense, only half of the impurity and conduction electron spin-flavour degrees of freedom hybridise. In contrast to $a, b$ hybridises with $\chi z$ and thus acquires dimension 1/2, so that impurity spin correlations at the fixed point decay as:

$$< S(0)S(\tau) > \propto b(0)b(\tau) \propto 1/\tau$$

Exactly at the solvable point (where one has a quadratic fermion hamiltonian), the impurity susceptibility $\chi_{imp}$ vanishes and the specific heat coefficient $C/T$ is non-singular. The non-Fermi liquid singularities in these quantities are controlled by the first non-trivial order of perturbation theory in the coupling $\lambda$ away from the solvable point. Indeed, the corresponding operator $ab\psi_{sf}^\dagger(0) \psi_{sf}(0)$, of dimension 3/2, precisely coincides with the CFT identification of the leading irrelevant operator at the non-trivial fixed point. To see this, we first note that $b$ can be traded for $\chi z$ in the expression of this operator since the two fields hybridize at the solvable point. Then, we note that $\psi_{sf}^\dagger \psi_{sf} \propto \chi z \chi z$, so that the operator can be written as:

$$iv_{abc}\chi^a \chi^b \chi^c \propto (\vec{\chi} \times \vec{\chi}) \cdot \vec{\chi}$$

The triplet of Majorana fermions $\vec{\chi}$, of dimension 1/2, is the $\phi_{j=1}$ primary operator of the Kac-Moody algebra $SU_2(2)$, so that (36) appears as the descendant operator $\vec{J}_{-1} \cdot \phi_{j=1}$, product of this operator and the spin current, which is precisely the CFT identification. Perturbation theory in this operator leads to the non-Fermi
liquid behaviour \( C/T \propto \chi_{imp} \propto \ln T \) with a universal Wilson ration \( R_W = 8/3 \).

It is important to note that the original impurity spin variables (before the transformation \( \mathcal{U} \) is performed) also have a simple representation in terms of the Majorana fermions \( \chi \) and \( a, b \). Indeed:

\[
S_+ = S^x + iS^y = \mathcal{U}^+ \tilde{S}^z \mathcal{U} = e^{iab \phi_c (0)} (b + ia)e^{-iab \phi_c (0)}
= (b + ia) \psi_x (0) / \sqrt{2}
\]

and \( S^z = \tilde{S}^z \). Hence, we obtain:

\[
S^x = a \chi^x (0) - b \chi^y (0), \quad S^y = a \chi^y (0) + b \chi^x (0) \quad S^z = iab \quad (37)
\]

The most relevant part of the impurity spin operator, of dimension 1/2 can thus be identified with \( a \chi (0) \) (where the fermion \( b \) has been traded for \( \chi^x (0) \), with which it hybridizes). This identification plays a crucial role in the solution of the two-impurity problem presented below.

2. The two-impurity, two-channel Kondo model

We now turn to the ‘Kondo quartet’ model, of two spin-1/2 impurities coupled to two channels of conduction electrons. This model was first investigated using numerical renormalization group methods \([10-12]\) Recently, an analytical solution of the low-energy universal properties of the model was found (13 and 1410).

In its original formulation, the problem involves two spin-1/2 impurities, \( \vec{S}_1, \vec{S}_2 \) located at positions \( \pm \vec{R}/2 \) and interacting with two-channel of free conduction electrons. The electron gas is described by operators \( c_{\vec{r} \alpha a} \), where \( \vec{k} \) is the momentum, \( \alpha = \uparrow, \downarrow \) the spin index. The Hamiltonian reads:

\[
H = \sum_{\vec{k} \alpha a} \epsilon (\vec{k}) c_{\vec{r} \alpha a}^+ c_{\vec{r} \alpha a} + J_K [\vec{S}_1 \cdot \vec{s} (+ \vec{R}/2) + \vec{S}_2 \cdot \vec{s} (- \vec{R}/2)] - I \vec{S}_1 \cdot \vec{S}_2 \quad (38)
\]

In these expressions, \( \vec{s}(\vec{r}) \) is the total conduction electron spin density at position \( \vec{r} \), \( J_K \) is the Kondo coupling (which we always take to be antiferromagnetic \( J_K > 0 \)) and the inter-impurity coupling \( I \) has been taken as an independent parameter. Of course, such a coupling is generated anyhow from the second-order process involving the conduction electron bath (i.e. the RKKY interaction, of order \( J_K^2 \) for small \( J_K \)). Standard renormalization-group arguments allow us to include this coupling by hand (see also 14).

The problem (38) can be considerably simplified by reducing it exactly (at low energy) to one-dimensional model involving 8 left-moving (chiral) fermionic fields \( \psi_{li \alpha}(x) \), where the extra-index \( l = 1, 2 \) originates from the presence of the two impurity sites. The procedure has been described in detail by other authors (see e.g. 19) and we shall only quote the final form of the Hamiltonian:

\[
H = iv_F \sum_{li \alpha} \int_{-\infty}^{+\infty} dx \psi_{li \alpha}^+(x) \frac{d}{dx} \psi_{li \alpha}^-(x)
+ J_1 (\vec{S}_1 + \vec{S}_2) \cdot (\vec{J}_1 (0) + \vec{J}_2 (0))
+ J_m (\vec{S}_1 - \vec{S}_2) \cdot (\vec{J}_1 (0) - \vec{J}_2 (0))
+ J_- (\vec{S}_1 + \vec{S}_2) \cdot \sum_{i, \alpha \beta} \psi_{li \alpha}^+ (0) \frac{\vec{\sigma} \cdot \vec{\sigma}}{2} \psi_{li \beta} (0)
+ \psi_{li \alpha} (0) \frac{\vec{\sigma} \cdot \vec{\sigma}}{2} \psi_{li \beta} (0))
- I \vec{S}_1 \cdot \vec{S}_2 \quad (39)
\]

In this formula \( \vec{J}_l (x) \equiv \sum_{i, \alpha \beta} \psi_{li \alpha}^+ (x) \frac{\vec{\sigma} \cdot \vec{\sigma}}{2} \psi_{li \beta} (x) \) denotes the spin-current at position \( x \) of conduction electrons corresponding to \( l = 1, 2 \). The reader is directed to Ref. [14] for the expression of the (bare values of) the coupling constants \( J_1, J_m, J_- \) in terms of \( J_K \) and \( \Gamma \). Alternatively, one could work in the even-odd basis:

\[
\psi_{eia} = (\psi_{1,ia} + \psi_{2,ia}) / \sqrt{2}, \quad \psi_{oia} = (\psi_{1,ia} - \psi_{2,ia}) / \sqrt{2} \quad (40)
\]

in terms of which the Kondo couplings in Eq. (38) read:

\[
(\vec{S}_1 + \vec{S}_2) \cdot (\Gamma_e \vec{J}_e (0) + \Gamma_o \vec{J}_o (0))
+ \Gamma_m (\vec{S}_1 - \vec{S}_2) \cdot \sum_{i, \alpha \beta} \psi_{eia}^+ (0) \frac{\vec{\sigma} \cdot \vec{\sigma}}{2} \psi_{oia} (0) + h.c
\]

Where the couplings \( \Gamma_e,o,m \) are the ones used in Ref. [14]:

\[
\Gamma_e = (J_+ + J_-)/2, \quad \Gamma_o = (J_+ - J_-)/2, \quad \Gamma_m = J_m \quad (41)
\]

The Hamiltonian (39) is invariant under a parity transformation, which exchanges the indices \( l = 1, 2 \) for both impurity spins and conduction electrons:

Parity : \( \psi_{1ia} \leftrightarrow \psi_{2ia}, \quad \vec{S}_1 \leftrightarrow \vec{S}_2 \quad (42) \)
Exchanging the even and odd combinations:

\[ \psi_{e,i\alpha} \leftrightarrow \psi_{o,i\alpha}; \psi_{1,i\alpha} \leftrightarrow -\psi_{2,i\alpha} \]

(43)

is a discrete symmetry of all terms in (39), except the \( J_{-} \propto \Gamma_{e} - \Gamma_{o} \) coupling.

When written in the simplified form (39), the hamiltonian is automatically invariant under a particle-hole transformation. Breaking particle-hole symmetry in the original hamiltonian of the model will induce potential scattering terms at low-energy, of the form:

\[ V_{e}\psi_{e}^{\dagger}\psi_{e} + V_{o}\psi_{o}^{\dagger}\psi_{o} \]

(44)

2.1. Overview of the results

We give here a brief overview of the results that will be derived in the following. Let us start by discussing the possible fixed points of the model in the space of the various coupling constants. There are several couplings \( (I, J_{\pm}, J_{m}) \) in the hamiltonian [9], but we shall show below that the whole phase diagram can actually be discussed in terms of only two parameters. These parameters (denoted \( x \) and \( y \) in the following) are phase-shifts which depend in a non-universal way on the original coupling constants, and can only be calculated analytically in some limiting regime. These two phase shifts are the natural variables in terms of which the problem is best described, and in terms of which numerical results should be interpreted. The precise phase diagram in terms of these two parameters will be found in analytical form in Sec.2.3.

Here we follow a route which is physically more transparent, and display a schematic version of the phase diagram in the space of two variables, in terms of which the phase diagram has been investigated numerically by Ingersent and Jones [11,12]. These two parameters are the ratio \( I/T_{K} \) of the RKKY interaction to the Kondo temperature of the single impurity model and the ratio \( (\Gamma_{e} - \Gamma_{o})/(\Gamma_{e} + \Gamma_{o}) \propto J_{-} \) measuring the relative asymmetry between the odd and even Kondo couplings. In order to reduce oneself to this parameter space, one possibility is to assume that the coupling \( \delta J \equiv J_{+} - J_{m} \) is taken to be zero (we shall see that \( \delta J \) actually corresponds to an irrelevant operator around the decoupled impurity fixed point).

In this restricted two-parameter space, the schematic phase diagram of the model is depicted in Fig.1. The figure is not meant to be quantitatively precise (a numerical determination using the Wilson RG has been given by Ingersent and Jones in [11,12]), but merely indicates the various possible fixed points and directions of the RG flow. Some of the fixed points in Fig.1 correspond to simple limits which can be easily discussed qualitatively [10]:

- The fixed point at \( I = \delta J = 0, J_{-} = 0 \) \((\Gamma_{e} = \Gamma_{o})\) corresponds to two decoupled impurities interacting with two channels of conduction electrons. The Nozieres-Blandin single impurity physics is found at this fixed point.

- The fixed point at \( I = -\infty, J_{-} = 0 \) \((\Gamma_{e} = \Gamma_{o})\) obviously corresponds to free electrons. Indeed, for very strong antiferromagnetic RKKY coupling, the impurity spins bind into a singlet state, leaving a local Fermi liquid with no residual scat-
tering for both the even and the odd combinations ($\delta_e = \delta_o = 0$).

- The fixed point at $I = +\infty$, $J_- = 0$ ($\Gamma_e = \Gamma_o$) corresponds to binding the impurity spins into an $S = 1$ triplet state, while maintaining perfect symmetry between the odd and even parity combinations. As conjectured in $[10]$, it is natural to expect that the physics at this fixed point is identical to that of a single $S = 1$ impurity with twice two channels of conduction electrons, i.e. four channels. Since $4 > 2 \times 1$, this is an overscreened non-Fermi liquid fixed point, with $\chi_{imp} \sim C/T \sim T^{-1/3}$. We shall prove this conjecture in the following.

- Finally, the fixed point at $I = 0$, $J_- \to +\infty$ ($\Gamma_e >> \Gamma_o$) also has a simple physical interpretation. At this fixed point, the odd electrons decouple and the Kondo effect takes place for each impurity with the even combination only. Hence, the scattering is characterized by $\delta_e = \pi/2, \delta_o = 0$). Again, one has a local Fermi-liquid. When the RKKY coupling is ferromagnetic ($I > 0$), we find that an arbitrary small odd-even asymmetry drives the system towards this fixed point.

In view of the above interpretation of the strong ferromagnetic fixed point, we see that this even Kondo fixed point must have identical physics to a single-impurity, spin-1, 2-channel fixed point, which is indeed an exactly screened situation and hence a Fermi-liquid. On the antiferromagnetic side, this fixed-point controls only part of the parameter space, namely large enough $J_-$ and small enough $I$ (Fig.1).

Apart from these fixed points that can be found from simple arguments, the model also displays a continuous family of non-Fermi liquid fixed points (in dark on Fig.4). As indicated on Fig.4, this continuous family is made up of a marginal line restricted to the $J_- = 0$ ($\Gamma_e = \Gamma_o$) axis when $I > 0$ is antiferromagnetic, and of a two-dimensional domain when $I < 0$ is antiferromagnetic. This results from the finding that the RKKY coupling is an exactly marginal perturbation around the decoupled impurity fixed point $I = J_- = 0$, while the even-odd asymmetry is relevant for $I > 0$, and again exactly marginal for $I < 0$. Within this marginal domain, universal low-energy properties (i.e. the low-temperature critical behaviour of the various physical quantities and the form of the finite-size spectrum) depend continuously on two parameters, according to formulas derived below. As an example, for ferromagnetic RKKY and $J_- = 0$, a power-law behaviour $\chi_{imp} \sim C/T \sim 1/T^\theta$ is found, with an exponent varying continuously with $I/T_K$ between the limits $\theta(I = 0) = 0$ (corresponding to the Nozieres-Blandin logarithmic behaviour) and $\theta = 1/3$ (corresponding to the $S = 1$, 4-channel model). A separatrix exists in the phase-diagram, which separates the attraction basin of the free-electron (RKKY) fixed point and that of the even-parity Kondo fixed point. This separatrix hits the boundaries of the marginal domain at a multicritical point (Fig.1).

Hence, the main physical message is that interimpurity effects do not destroy non-Fermi liquid behaviour in this model (in the particle-hole symmetric case), but do modify it as compared to the single-impurity case and generate a rich variety of new behaviour. The competition between the Kondo screening and the RKKY ordering is highly involved in this model. This is to be contrasted with the two-impurity single-channel model (again at particle-hole symmetry). In this case, a RKKY (free electrons) fixed point controls the physics for sufficiently antiferromagnetic $I/T_K \leq (I/T_K)_c$, while a phase with separate Kondo screening of the impurities is found for $I/T_K > (I/T_K)_c$. Both of these fixed point are Fermi liquids and only the (unstable) multicritical point at $(I/T_K)_c$ displays non-Fermi liquid properties.

2.2. The odd-even symmetric case

2.2.1. Symmetries

We concentrate in this section on the case $J_- = 0$ ($\Gamma_e = \Gamma_o$) corresponding to a hamiltonian invariant under odd-even exchange. When the coupling between impurities is turned off ($I = 0, J_+ = J_m$), the model has a global invariance $(SU(2)_{\text{spin}} \otimes SO(5)_{fc})^2$. At the decoupled impurity fixed point, the symmetry algebra consists in two copies of a product of Kac-Moody algebra for spin, channel and charge: $(SU(2)_{\text{spin}} \otimes SO(5)_{fc})^2$. When coupling the two impurities ($I \neq 0, J_+ \neq J_m$) while keeping $J_- = 0$, the
independent charge and flavour symmetries are left unaffected (this is no longer true for \( J \neq 0 \)), but the spin symmetry is reduced to the ‘diagonal’ \( SU(2) \) for which \( \psi_1 \) and \( \psi_2 \) are transformed identically. This reduces the conformal symmetry of the spin sector at a fixed point to a \( SU(2) \) algebra. The generators of this algebra are the sum of the generators of the two \( SU(2)_s \) for each impurity, that is the sum of the spin currents \( \tilde{J}_1(x) + \tilde{J}_2(x) \). Hence we must understand how the product \( SU(2)_s \otimes SU(2)_s \) can be decomposed into \( SU(4)_s \), plus some residual degrees of freedom. This procedure is known in CFT as a ‘coset construction’ \[20\]. The residual degrees of freedom define an algebra \( A(2,2) \) such that:

\[
SU(2)_s \otimes SU(2)_s = SU(4)_s \otimes A(2,2) \tag{45}
\]

The conservation of the total number of degrees of freedom between the two sides of (45) is expressed by the conservation of the total central charge. Since the central charge of \( SU(2)_s \) is \( c = 3/2 \) and that of \( SU(4)_s \) is \( c = 2 \), we deduce that the central charge of the \( A(2,2) \) algebra is \( c = 3 + 2/3 = 1 \). This algebra is actually known from CFT \[21\] to be a \( N = 1 \) superconformal unitary model corresponding to the \( m = 4 \) member of the discrete series with central charges \( c = \frac{2}{m(m+2)} \).

### 2.2.2. Bosonisation

We shall give a schematic derivation of the operator content of the algebra \( A(2,2) \) using the Majorana fermion representation and abelian bosonisation. We introduce two triplets of Majorana fermions \( \chi^a \) such that the \( SU(2)_s \) spin currents \( J^a_i(x) (i = 1, 2; a = x, y, z) \) read:

\[
J^a_i(x) = \tilde{\psi}_a \chi^a_i.
\]

These six Majorana fermions can be combined into three Dirac fermions, and bosonised as:

\[
\chi^a_i(x) + i\chi^a_i(x) = \frac{1}{\sqrt{2\pi a_0}} e^{-i\Phi_a(x)}; \quad a = x, y, z \tag{46}
\]

In terms of these fields, the total spin current corresponding to the diagonal \( SU(4)_s \) algebra reads:

\[
\mathcal{J}^x \equiv \mathcal{J}^x_1 + \mathcal{J}^x_2 = \cos(\Phi_y - \Phi_z), \quad \mathcal{J}^y = \cos(\Phi_x - \Phi_z)
\]

\[
\mathcal{J}^z = \cos(\Phi_x - \Phi_y)
\]

It is convenient to introduce three linear combinations of boson fields as follows:

\[
\Phi = \frac{1}{\sqrt{3}}(\Phi_x + \Phi_y + \Phi_z), \quad \mu = \frac{1}{\sqrt{2}}(\Phi_x - \Phi_y)
\]

\[
\nu = \frac{1}{\sqrt{6}}(\Phi_x + \Phi_y - 2\Phi_z) \tag{47}
\]

Let us also note for further use the inverse relations:

\[
\Phi_x = \frac{1}{\sqrt{3}}\Phi + \frac{1}{\sqrt{2}}\mu + \frac{1}{\sqrt{6}}\nu, \quad \Phi_y = \frac{1}{\sqrt{3}}\Phi - \frac{1}{\sqrt{2}}\mu + \frac{1}{\sqrt{6}}\nu, \quad \Phi_z = \frac{1}{\sqrt{3}}\Phi - \frac{2}{\sqrt{6}}\nu \tag{48}
\]

In term of these combinations, the components of the total spin current read:

\[
\mathcal{J}^x = \cos(\frac{\Phi_y - \Phi_z}{\sqrt{2}}), \quad \mathcal{J}^y = \cos(\frac{\Phi_y - \Phi_z}{\sqrt{6}})
\]

\[
\mathcal{J}^z = \cos(\sqrt{2} \mu) \tag{49}
\]

Note that \( \Phi \) does not enter these expressions. The two bosons \( \mu, \nu \) are associated with the \( SU(4)_s \) algebra (which has \( c = 2 \)), while \( \Phi \) corresponds to the residual \( A(2,2) \) degree of freedom \((c = 1)\). This boson is compact, with a radius \( R = \sqrt{3} \), which means that \( \Phi \) and \( \Phi + 2\pi R \) are identified. Hence, \( A(2,2) \) contains all operators of the form \( e^{i(n\sqrt{3} + m/2\sqrt{6})\Phi} \) and \( \partial^a \Phi \) (with \( n, m \) integers). This yields primary operators of dimensions: \((0),(1/24),(1/6),(3/8),(1/6 + 1/2),(1)\), as summarized in Table I. In addition, it can be shown that \( \Phi \) and \( -\Phi \) must be identified, so that \( A(2,2) \) is in fact an orbifold theory. This implies that the algebra contains, in addition to the operators above, two twist operators of dimension \( 1/16 \) and two twist operators of dimension \( 9/16 \). The full set of primary operators of the \( A(2,2) \) algebra and their boson representation (when it exists) is given in Table I.

At this stage, we can compare the coset construction made above with that encountered in the 2-impurity, single channel Kondo model \[13\]. There, the symmetry algebra at the decoupled impurity fixed point is \( SU(2) \otimes SU(2) \). It is broken down to \( SU(2)_s \) when coupling the two impurities. The coset algebra for the residual degrees of freedom thus have central charge \( c = 1 + 1 - 3/2 = 1/2 \) and identifies with the Ising model:

\[
SU(2)_s \otimes SU(2)_s = SU(2)_s \otimes (I_{sing}) \tag{50}
\]
The Ising algebra contains three primary operators, of dimension 0 (identity), 1/16 (order parameter), and 1/2 (energy density). Affleck and Ludwig [13] have shown that the properties of the non-Fermi liquid unstable critical point [17] encountered for this model are obtained from the decoupled impurity fixed point, by a fusion involving the order-parameter operator of the Ising sector. In the present case, we shall see that no direct fusion principle with an operator of the RKKY term, we make use of the two coset operators of the algebra, except at special points in the phase diagram.

### 2.2.3. The RKKY interaction as an exactly marginal perturbation

We consider the effect of turning on the couplings \( I \) and \( \delta J \equiv J_++J_- \) away from the decoupled impurity fixed point (but keeping \( J_- = 0 \)). At the decoupled point, the impurity spin has dimension 1/2, and the conduction electron spin current has dimension 1. Thus, the RKKY coupling has dimension 1 and is marginal at lowest order, while \( \delta J \) has dimension 3/2 and is irrelevant. We shall see that \( I \) is actually exactly marginal to all orders, while the dimension of \( \delta J \) is actually continuously modified as one departs from the decoupled fixed point. In order to represent the RKKY term, we make use of the two sets of Majorana fermions \( \chi_1, \chi_2 \) above and introduce, as in Sec. I, two sets of Majorana fermions \( a_1, b_l \) \((l = 1, 2)\) in order to represent the impurity spins \( \vec{S}_I \) as in Eq. (1). A direct fusion principle with an operator of dimension 1/2 \((\partial \Phi)\) around the decoupled point. Note that the perturbing term in Eq. (52) is perfectly compatible with the orbifold nature of the \( A(2, 2) \) algebra, provided the transformation \( \Phi \to -\Phi \) is always made simultaneously with \( d \to d^+ \) on the local degree of freedom. Since \( d \) remains a local fermion (with non-decaying correlations), it is clear from Eq. (52) that the RKKY coupling is associated with a dimension 1 operator and is thus an exactly marginal perturbation. This results in the line of fixed points on the \( \Gamma_e = \Gamma_o \) axis of Fig. I.

The Hamiltonian (52) is very similar to the X-ray edge problem in the bosonised form (22). The interaction term on the boundary can be ab-

**Table 1**

Primary operators of the \( A(2, 2) \) algebra. \((\Delta)\) labels an operator of dimension \( \Delta \). The second line displays the bosonic representation of the operator (when it exists). \( NS \) and \( R \) stand for the Neveu-Schwarz and Ramond sectors of the algebra.

| \( \langle 0 \rangle \) | \( \langle \frac{1}{2} \rangle \) | \( \langle \frac{1}{16} \rangle \) | \( \langle \frac{1}{4} \rangle \) | \( \langle \frac{3}{8} \rangle \) | \( \langle \frac{3}{16} + \frac{1}{2} \rangle \) | \( \langle \frac{9}{16} \rangle \) | \( \langle \frac{1}{6} + \frac{1}{2} \rangle \) | \( \langle 1 \rangle \) |
|---|---|---|---|---|---|---|---|---|
| \( NS \) | \( R \) | \( NS \) | \( R \) | \( NS \) | \( R \) | \( NS \) | \( R \) | \( NS \) |

\( e^{\pm i\frac{\pi}{2\sqrt{3}}} \) \n
**end of section**

At the decoupled point, the impurity spin has dimension 1 and is marginal at lowest order. Thus, the RKKY coupling term in the \( A(2, 2) \) sector of the Hamiltonian:

\[
H_{A(2, 2)} = v_F \int dx (\frac{\partial \Phi}{\partial x})^2 + \tilde{I}(d^+ d - \frac{1}{2}) \frac{\partial \Phi}{\partial x}
\]

where we have set:

\[
d^+ \equiv (a_1 + ia_2)/\sqrt{2}
\]

with a normalisation \( a_1^2 = a_2^2 = 1/2 \), so that \( \{d, d^+\} = 1 \). Here, \( \tilde{I} \) is some (non-universal) function of \( I \) and \( \delta J \equiv J_+ - J_- \), which has a perturbative expansion \( \tilde{I} = I + O(I^2, \delta J^2) \) around the decoupled point. Note that perturbative expansion (52) is perfectly compatible with the orbifold nature of the \( A(2, 2) \) algebra, provided the transformation \( \Phi \to -\Phi \) is always made simultaneously with \( d \to d^+ \) on the local degree of freedom. Since \( d \) remains a local fermion (with non-decaying correlations), it is clear from Eq. (52) that the RKKY coupling is associated with a dimension 1 operator and is thus an exactly marginal perturbation. This results in the line of fixed points on the \( \Gamma_e = \Gamma_o \) axis of Fig. I.

The Hamiltonian (52) is very similar to the X-ray edge problem in the bosonised form (22). The interaction term on the boundary can be ab-
sorbed into a redefinition of the field:
\[
\Phi(x) \rightarrow \Phi(x) + i \frac{2 \pi}{v_F} (d^+ d - \frac{1}{2}) \text{sign}(x) \tag{54}
\]
This changes the boundary condition on the field \( \Phi \) by a phase-shift \( \delta \), such that \( \Phi(0^+) - \Phi(0^-) = \pm 2\delta \) when \( d^+ d - 1/2 = \pm 1 \), respectively. The weak-coupling expansion of this phase shift is:
\[
\delta = \frac{I}{2v_F} = \frac{I}{2v_F} + O(I^2, \delta J^2) \tag{55}
\]
As observed by Schotte and Schotte \cite{22} and also recently discussed in \cite{23}, this means that the interacting hamiltonian is related to the non-interacting one by a canonical transformation:
\[
H_{A(2,2)} = U_\delta H_{I=0} U_\delta, \quad U_\delta \equiv e^{i\frac{2\pi}{\delta} (d^+ d - \frac{1}{2})} \tag{56}
\]
This transformation changes the dimension of the operators \( e^{i\Phi} \) from \( \frac{2\pi}{\delta} \) to \( \frac{1}{2}(g \pm \frac{2\pi}{\delta})^2 \). Note that these dimensions do not correspond in general to dimensions of existing operators in the \( A(2,2) \) algebra. This reflects the fact that \( U_\delta \) is itself not a primary operator of this algebra in general.

Since the available dimensions in this algebra are given by \( g = n \sqrt{3} + m/2\sqrt{3} \), the maximal possible scattering (corresponding to a shift of unity of the integer \( m \)) is reached for \( \frac{\delta_{max}}{\delta} = \frac{1}{2\sqrt{3}} \).

Each fixed-point along the marginal line in fig. \( \ref{fig:fig1} \) corresponds to a specific value of the phase-shift \( -\delta_{max} \leq \delta \leq \delta_{max} \), which specifies completely all universal properties at the fixed point. It is convenient to make use of the normalised parameter:
\[
x = \frac{\delta}{\delta_{max}} = \frac{2\sqrt{3} \delta}{\pi} \tag{57}
\]
Note that \( x > 0 \) (resp. \( x < 0 \)) corresponds to a ferromagnetic (resp. antiferromagnetic) RKKY coupling. As will be shown below, the marginal lines terminates at infinitely strong RKKY coupling \( I = +\infty \) on the ferromagnetic side, corresponding to maximal scattering \( x = +1 \), while the marginal behaviour only persists for \( x > x = x_{min} = -(3 - \sqrt{6}) \simeq -0.55 \) on the antiferromagnetic side.

### 2.2.4. Physical properties

We can investigate the operator content and low-temperature critical behaviour of physical quantities for non-zero \( I \), \( \delta J \) (corresponding to a specific value of \( x \)), using the fact that a given operator \( \mathcal{O} \) is changed to \( U_\delta \mathcal{O} U_\delta^+ \) for a non-zero \( x \). We shall deal first with impurity spin correlation functions. As already noted, the most relevant operator (of dimension 1/2) corresponding to each impurity spin at the decoupled impurity fixed point reads \( S_I \), and we thus have the identification:
\[
S_i^a + S_i^a \propto d e^{i\Phi} + d^+ e^{-i\Phi} \tag{58}
\]
We act on these operators with the transformation \( U_\delta \), using \( U_\delta d U_\delta^+ = \exp (-i x\Phi(0)/\sqrt{3}) \) \( d \).

Hence the dimensions \( \Delta_x^{spin} \) of \( S_i^a \pm S_i^a \) read, for arbitrary \( x \):
\[
\Delta^{spin} = \frac{1}{2} \left( \frac{1-x}{\sqrt{3}} \right)^2 + \frac{1}{3}, \quad \Delta^{spin} = \frac{1}{2} \left( \frac{1+x}{\sqrt{3}} \right)^2 + \frac{1}{3} \tag{59}
\]
From this, we find the low-temperature behaviour of the impurity spin susceptibility:
\[
\chi_{imp} \propto \int^{1/T} \frac{dT}{t^{2\Delta^{spin}}} \sim \frac{1}{T^{\theta(x)}} \tag{60}
\]
while \( \chi_{imp} \) remains finite on the antiferromagnetic side \( x < 0 \). Note that the critical behaviour \cite{20} interpolates continuously between the known limits \cite{3}: \( \chi_{imp} \sim ln(T) \) at the decoupled impurity fixed point, and \( \chi_{imp} \sim 1/T^{1/3} \) of the \( S = 1 \), four channel model at the strong coupling ferromagnetic fixed point. Similarly, we find that the staggered susceptibility \( \chi_{st} \) (defined as the response to a field coupled to \( S_1^+ - S_2^+ \)) is finite on the ferromagnetic side \( x > 0 \), and diverges on the antiferromagnetic side as:
\[
\chi_{st} \sim \frac{1}{T^{\theta(-x)}} \quad \theta(-x) = 1 - 2\Delta^{spin} = \frac{x(2-x)}{3} \quad (x \geq 0) \tag{61}
\]
This continuous dependance of the critical exponents on \( x \) establish the existence of a line of fixed points extending on both sides of the decoupled impurity fixed point. In order to find the precise extension of this line, and the low-temperature behaviour of the specific heat along it, we look for the leading irrelevant perturbation(s) compatible with parity and the even/odd symmetry. At
the decoupled impurity fixed point, there are two such operators, of dimension 3/2, corresponding to the couplings $J_+$ and $J_m$:

$$
O_+ \equiv (\vec{S}_1 + \vec{S}_2), (\vec{J}_1(0) + \vec{J}_2(0))
$$

$$
O_m \equiv (\vec{S}_1 - \vec{S}_2), (\vec{J}_1(0) - \vec{J}_2(0)) \tag{62}
$$

We can find the bosonised form of $O_+$, using the representation of $\vec{S}_1 + \vec{S}_2$ given above, and the expression \cite{49} of the total spin current:

$$
O_+ \propto d e^{i(\frac{1}{3}\Phi b + \frac{1}{\sqrt{3}}\nu)} + d e^{i(\frac{1}{3}\Phi b + \sqrt{2}\mu - \frac{2}{\sqrt{3}}\nu)} + h.c \tag{63}
$$

Acting with $U_\delta$ on this expression, we find that both terms give rise for arbitrary $x$ to an operator of dimension:

$$
\Delta_+ = \frac{4}{3} + \frac{1}{2} \left( 1 - \frac{x}{\sqrt{3}} \right)^2 = 1 + \Delta_{pin}^+ \tag{64}
$$

Similarly, for $O_m$, we used the bosonised expressions: $\vec{J}_1^2 - \vec{J}_2^2 \propto \cos(\Phi_b + \Phi_z)$ (and permutations) to obtain:

$$
O_m \propto d e^{-i\sqrt{3}\Phi} + d e^{i(\frac{1}{3}b - \sqrt{2}\mu - \frac{2}{\sqrt{3}}\nu)} + h.c \tag{65}
$$

Upon acting with $U_\delta$, the first operator becomes of dimension $(\sqrt{3} + x/\sqrt{3})^2/2$, while the second has the same dimension than $O_+$. Hence the operator of lowest dimension which is generated has:

$$
\Delta_m = \min \left\{ \frac{1}{2}(\sqrt{3} + \frac{x}{\sqrt{3}})^2, \Delta_+ \right\} \tag{66}
$$

The low-temperature behaviour of the specific heat is related to the dimension $\Delta$ of the irrelevant operator of lowest dimension by:

$$
\frac{C}{T} \propto \frac{\partial^2 F}{\partial T^2}, \quad F \propto \int_{-1/2}^{1/2} \frac{dt}{t^{2\Delta - 1}} \sim T^{2\Delta - 1} \tag{67}
$$

Hence:

$$
\frac{C}{T} \sim T^{-\alpha}, \quad \alpha = 3 - 2\Delta \tag{68}
$$

For $x > 0$ (ferromagnetic side), $O_+$ has the lowest dimension, and we find $\alpha = 3 - 2\Delta_+ = 1 - 2\Delta_{pin}^+ = \theta(x)$, so that:

$$
x > 0 : \quad \frac{C}{T} \sim \chi_{imp} \sim T^{-\theta(x)} \tag{69}
$$

and we expect a universal ($x$-dependent) Wilson ratio, which can be calculated using perturbation theory along the lines of Ref.\cite{5}. For $x < 0$ however (antiferromagnetic side), $O_m$ has the lowest dimension, and we find:

$$
x < 0 : \quad \frac{C}{T} \sim T^{-\alpha(x)}, \quad \alpha(x) = 3 - 2\Delta_m = |x|\frac{6 + x}{3} \tag{70}
$$

Note that $\chi_{st}$ behaves with a different exponent. The limiting value of $x = x_{min}$, associated with the termination of the marginal line for antiferromagnetic coupling, is reached when $O_m$ becomes relevant, $\alpha = \Delta_m = 1$. This yields:

$$
x_{min} = -3 - \sqrt{3} \approx -0.55 \tag{71}
$$

For $x < x_{min}$, the system flows to the ‘strong antiferromagnetic’ (or ‘RKKY’) fixed point, and the two impurities bind into a singlet state. Obviously, this fixed point has the properties of a local Fermi-liquid. On the ferromagnetic side however, the marginal line extends all the way up to the infinitely strong RKKY coupling $I = +\infty$ (corresponding to maximal scattering $x = +1$), since $O_+$ is still irrelevant at this point. Note that the critical properties derived above coincide, for $x = +1$, with that of the spin-1, 4-channel Kondo problem (with spin dimension $1/3$ and leading irrelevant operator of dimension $4/3$), in agreement with the conjecture made in Ref.\cite{10} and with the physical picture that the two impurities bind into a $S = 1$ triplet state with effectively 4 channel of conduction electrons (because of even-odd symmetry).

### 2.2.5. Finite-size spectrum

We now investigate the finite-spectrum of the model at a given fixed point along the marginal line, as a function of $x$. In order to derive the spectrum, one first classifies the states of the decoupled impurity fixed point according to the $SU_3(2)_s \otimes A(2, 2) \otimes (SO_1(5)_{fc})^2$ decomposition, and then act on each state with the transformation $U_\delta$. This modifies the contribution of the $A(2, 2)$ sector to the total energy of the state. The dimension $1/16$ of the twist operators can be shown to be unchanged by the action of $U_\delta$. Under multiplication by $U_\delta$, the dimension of an operator $e^{ik\Phi}$ is changed to $\frac{1}{3}(k \pm \frac{1}{2})^2$, for $d^+d - 1/2 = \pm 1/2$, respectively. Hence, we also
need to associate to each state an eigenvalue of $d^2 d - 1/2 = \pm 1/2$ to decide which of the two possible new dimensions is produced. This can be done, when constructing the spectrum at the decoupled point, by keeping track of the relative sign between the impurity spin and the total spin of the state. More precisely, the impurity spin ($\hat{S}$) is proportional to the adjoint operator of $SU_2(2)$ ($\chi$) up to a sign which depends on the state. The product of these two signs for $l = 1, 2$ yields the eigenvalue of $2d^2 d - 1$. In particular, this ‘selection rule’ is essential to insure that the spectrum of the $S = 1$ four-channel model is indeed obtained at $x = +1$. This observation is also a way to recover this selection rule in an empirical manner.

The resulting spectrum is displayed in Table 2. The normalised excitation energy of a given state, $L \Delta E / \pi v_F$ (with $L$ the radial length of the bulk system and $v_F$ the Fermi velocity), is obtained from the total dimension $\Delta_{tot}$ given in the table, by the formula:

$$\frac{L \Delta E}{\pi v_F} = \Delta_{tot} - \frac{1}{3} \left(1 - \frac{1}{x}\right)^2, \quad (\text{for } x > 0) \quad (72)$$

$$\frac{L \Delta E}{\pi v_F} = \Delta_{tot} - \frac{3}{8} \left(1 + \frac{1}{x}\right)^2, \quad (\text{for } x < 0) \quad (73)$$

The ground-state is the triplet of lowest energy for $x > 0$ (ferromagnetic coupling), and the singlet of lowest energy for $x < 0$ (antiferromagnetic coupling). Comparison with the numerical renormalization group results of K. Ingersent and B. Jones [10] [12] reveals that the numerical spectra are excellently fitted by formula (73) depending on the single parameter $x$ (the agreement with the quantum numbers and degeneracies of each state has also been checked). Note that, despite the different degeneracies of the finite-size ground-state for $x > 0$ and $x < 0$, the residual entropy defined by taking the infinite-volume limit first is constant along the marginal line:

$$S_{imp} \equiv \lim_{T \to 0} \lim_{L \to \infty} \Delta S = \ln 2 \quad (74)$$

while $S_{imp} = 0$ at the strong antiferromagnetic fixed point. This is expected from the boundary version of the ‘c-theorem’.

The strong-coupling ferromagnetic fixed point found for $x = +1$ deserves special comments. There, the impurity spins bind into an $S = 1$ triplet state. Thus, it should be possible to describe this point, in the spirit of the CFT approach [9], by a direct fusion in the spin $SU_2(2)$ sector with the triplet operator $j = 1$ (of dimension $j(j+1)/6 = 1/3$) applied on the free fermion fixed point. One can check that performing this fusion gives precisely the spectrum of a single-impurity, four channel Kondo model. Furthermore, one can also check that exactly the same spectrum is obtained when acting on the spectrum of the decoupled impurity fixed point with the operator $\exp(i(d^2 d - 1/2)\Phi(0)/\sqrt{3})$ (corresponding to $U_3$ for $x = +1$), taking into account the selection rule discussed above. Note that this operator is then a member of the $A(2, 2)$ algebra, of dimension $1/24$. Accordingly, the $A(2, 2)$ operator labelling any eigenstate at $x = 0$ is changed at $x = +1$ into another operator of the $A(2, 2)$ algebra. This strong-coupling assignment is also displayed in Table 2. It is compatible with the operator product expansion of the $(1/24)$ operator with any other operator of the $A(2, 2)$ algebra:

$$(\frac{1}{24}) \times (\frac{1}{24}) \to (0) + (1) + (\frac{1}{6}) \quad (75)$$

$$(\frac{1}{24}) \times (\frac{1}{6})_{NS} \to (\frac{1}{6})_{R} + (\frac{1}{6})_{L} \quad (76)$$

$$(\frac{1}{24}) \times (\frac{1}{6})_{R} \to (\frac{1}{6})_{NS} + (\frac{1}{6} + \frac{1}{2})_{NS} \quad (77)$$

$$(\frac{1}{24}) \times (\frac{1}{6}) \to (\frac{3}{8}) + (\frac{2}{24}) \quad (78)$$

$$(\frac{1}{24}) \times (\frac{3}{8}) \to (\frac{1}{3} + \frac{1}{2}) \quad (79)$$

$$(\frac{1}{24}) \times (\frac{3}{8})_{R} \to (\frac{1}{6})_{NS} + (\frac{1}{6} + \frac{1}{2})_{NS} \quad (80)$$

$$(\frac{1}{24}) \times (\frac{1}{6} + \frac{1}{2})_{NS} \to (\frac{1}{6})_{R} + (\frac{1}{6})_{L} \quad (81)$$

The selection rule discussed above dictates, for each state, which of the possible operators appearing in the r.h.s of these rules are actually generated when going from the decoupled impurity spectrum to the strong ferromagnetic one. For example, the singlet state in Table 2 and the multiplet of degeneracy 10 are both associated with the operator $(\frac{1}{2})$ for $x = 0$, but are assigned different operators $(\frac{1}{6} + \frac{3}{8})$ for the first one, $(\frac{1}{6})$ for the second at the strong ferromagnetic point.

### 2.3. The effect of odd-even asymmetry

We now consider the effect of a non-zero value of the coupling $J_-$, breaking the symmetry between even and odd combinations ($\Gamma_e \neq \Gamma_o$). In the presence of this coupling, the independent
Table 2

Finite-size spectrum of the first 76 low-lying states in the even-odd symmetric case \( J_- = 0 \) (\( \Gamma_e = \Gamma_o \)). \( j \) is the total spin quantum number associated with \( SU_2(4)^{\text{spin}} \). The second column gives the charge-flavour representation in the \( \widetilde{SO}_1(5)^{\text{even}} \otimes \widetilde{SO}_1(5)^{\text{odd}} \) decomposition. \( \sigma \) is the sign of \( d^x \cdot d - 1/2 = \pm 1/2 \). The third column displays the \( A(2, 2) \) operator associated with each eigenstate at the decoupled impurities fixed point \( (I = 0, \text{i.e.} \ x = 0) \), whereas the fifth column displays the corresponding operator at the strong ferromagnetic fixed point \( (I = +\infty, \text{i.e.} \ x = +1) \). The degeneracy of each state is displayed in the last column, while \( \Delta_{\text{tot}} \) is the total conformal dimension at arbitrary \( x \). The normalised excitation energies are given by \( L \Delta E/\pi v_F = \Delta_{\text{tot}} - \Delta_{\text{tot}}^0 \).

| \( j \) | \( SO(5)^{\text{even}} \otimes SO(5)^{\text{odd}} \) | \( A(2, 2) \) deg | \( \sigma \) | \( \Delta_{\text{tot}} \) | \( A(2, 2) \) strong ferro. | Deg. |
|---|---|---|---|---|---|---|
| 0 | (1, 1) | \( (\frac{3}{2}) \) | + | \( \frac{1}{8}(1 + \frac{3}{3})^2 \) | \( (\frac{1}{6} + \frac{1}{2}) \) | 1 |
| 1 | (1, 1) | \( (\frac{3}{2}) \) | − | \( \frac{1}{8} + (1 - x)^2 \) | \( (0) \) | 3 |
| \( \frac{1}{2} \) | \( (1, 4) \oplus (4, 1) \) | \( (\frac{1}{16}) \) | + | \( \frac{1}{16} \) | \( (\frac{1}{16}) \) | 16 |
| 0 | \( (4, 4) \) | \( (0) \) | − | \( \frac{8}{8} + \frac{2}{24} \) | \( (\frac{1}{24}) \) | 16 |
| 0 | \( (1, 5) \oplus (5, 1) \) | \( (\frac{1}{5}) \) | − | \( \frac{1}{2} + \frac{4}{16} \) | \( (\frac{1}{6}) \) | 10 |
| 1 | \( (1, 5) \oplus (5, 1) \) | \( (\frac{1}{24}) \) | + | \( \frac{8}{8} + (1 - x)^2 \) | \( (\frac{1}{6}) \) | 30 |

flavour and charge global symmetry \((SO(5))^2\) no longer holds: only identical transformations for both impurities are allowed. The symmetry group in the flavour and charge sector is thus broken down to the \( SO(5) \) diagonal subgroup, the sum of the flavour and charge generators. Correspondingly, at a fixed point, the model has a \( SU_2(5)^{\text{ferro}} \otimes SO_2(5)^{\text{ferro}} \) Kac-Moody symmetry, and we must consider the following ‘coset construction’ in the flavour-charge sector (in addition to the one above in the spin sector):

\[
SO_1(5) \otimes \widetilde{SO}_1(5) = \widetilde{SO}_2(5) \otimes K
\]

It is crucial in this case to take into account the full \( SO(5) \) symmetry of the charge-flavour sector, and not only its \( SU(2)_f \otimes U(1)_c \) subgroup. Indeed, this excludes some marginal perturbations that would be naively admissible in the independent flavour/charge language and explains how multiplets of states at \( J_- = 0 \) are broken into multiplets of smaller degeneracies when \( J_- \neq 0 \).

The current algebra of \( \widetilde{SO}_1(5) \) is essentially a straightforward generalization of that of \( SU_2(2) \) to the case of five Majorana fermions (it has central charge \( c = 5/2 \)). Again, five Dirac fermions are built out of the two sets of Majorana’s, and converted into five bosonic fields \( \Phi_i \). The diagonal combination \( \Phi = \sum_i \Phi_i/\sqrt{5} \) is associated with the coset algebra \( K \), which has central charge \( c = 1 \). The four other combinations correspond to \( SO_2(5) \) (with \( c = 4 \)). The coset \( K \) is a subalgebra of the product of cosets \( A(2, 2)_f \otimes U(1), \) which have both \( c = 1 \). Accordingly, the boson \( \Phi_i \) is a specific linear combinations of the two bosons associated with the flavour and charge cosets, namely:

\[
\Phi = \sqrt{\frac{3}{5}} \Phi_f + \sqrt{\frac{2}{5}} \Phi_5 - \Phi_\xi
\]

We now ask whether turning on a small coupling \( J_- \) is a relevant perturbation at a given fixed point on the marginal line of the previous section. To answer this, we consider the corresponding operator:

\[
O_- = \sum_{i,\alpha} \langle (\tilde{S}_i + \tilde{S}_{2}) \rangle \cdot \langle \psi_{1,\alpha}\rangle(0) \frac{\delta_{\alpha\alpha}}{\sqrt{2}} \langle \psi_{2,\alpha}\rangle(0)
\]
in bosonised form:

\[
\psi_{2\alpha}(0)\frac{\sigma^z}{2}\psi_{1\beta}(0)
\]

A naive counting of dimension would suggest that this operator has dimension 1/2 + 1 = 3/2 at the decoupled impurity fixed point. This is incorrect, however, since \(\mathcal{O}_-\) is a product of an impurity and fermion parts which are not independent. Instead, we shall see that \(\mathcal{O}_-\) is, to lowest order in \(J_-\), a marginal operator (of dimension 1) at the decoupled impurity fixed point. It is convenient to find the bosonised form of \(\mathcal{O}_-\) at this fixed point. \(\bar{S}_1+\bar{S}_2\) is represented as in eq.\((38)\), and we need to concentrate on \(\psi_{1\alpha}^\dagger(0)\frac{\sigma^z}{2}\psi_{2\beta}(0)+h.c.\)

This operator is a spin triplet, and a flavour and charge singlet. Using the spin-flavour-charge decomposition \(SU_2(2)_s \otimes SU_2(2)_f \otimes U(1)_c\) for each impurity index \(l=1,2\), we have to build a triplet combination out of two \(j=1/2\) operators of the \(SU_2(2)_s\) algebra (each of dimension \(j(j+1)/4=3/16\), so that the spin contribution to the overall dimension is 3/16 + 3/16 = 3/8). Similarly, we build a singlet operator out of two \(j=1/2\) operators of the \(SU_2(2)_f\) algebra (also of dimension 3/8), and a singlet charge combination (of dimension 1/8 + 1/8 = 1/4).

If we represent each fermion \(\psi_{1\alpha}^\dagger\) using non-abelian bosonisation (Eq.\((\ref{eq:non-abelian-bosonisation})\)), as \(g_{1\alpha}^\dagger h_{1\gamma}^\dagger e^{i\Phi^f_{\gamma}/2}\), this corresponds to forming the combination \(g_{1\alpha}^\dagger \sigma_{\alpha\beta} g_{2\beta}^\dagger h_{1\gamma}^\dagger h_{2\gamma}^\dagger e^{i(\Phi_1^f - \Phi_2^f)}/2\). The flavour singlet combination \(h_{1\gamma}^\dagger h_{2\gamma}^\dagger\) is built out of the three operators (of dimension 1/8) \(e^{i\Phi^f_{\gamma}/2}, a=x,y,z\) (which give a \(\pi/2\) phase shift to one of the three flavour Dirac fermions), as \(\exp(i\Phi^f_1 + \Phi^f_2 + \Phi^f_3)/2 = \exp i\sqrt{3}\Phi^f/2\). Similarly, we build the spin triplet \(g_{1\alpha}^\dagger \sigma_{\alpha\beta} g_{2\beta}^\dagger\) out of the \(e^{i\phi_{\gamma}/2}\) operators as: \(\exp i(-\Phi_x + \Phi_y + \Phi_z)/2\) (and permutations), i.e. as: \(\exp i(\sqrt{3}\Phi^f_2 - \Phi^f_1), a=x,y,z\). Multiplying by \(\bar{S}_1+\bar{S}_2\) given by Eq.\((38)\), we find that the operator of lowest dimension contained in \(\mathcal{O}_-\) reads in bosonised form:

\[
d\exp[i(\frac{x}{\sqrt{2}}\Phi(0) + \frac{y}{\sqrt{2}}\Phi^f(0)) + \frac{1}{\sqrt{2}}\frac{\Phi_1^f(0) - \Phi_2^f(0)}{\sqrt{2}})] + h.c.
\]

Hence, this operator is of dimension 1 at the decoupled impurity fixed point, as announced. Under the action of \(U_\delta\), the spin part is transformed into \(\exp i(\sqrt{3}/2 - x/\sqrt{3})\Phi(0)\) and hence the dimension is changed to: \(\Delta_-(x) = \frac{1}{2}(\frac{x}{\sqrt{3}} - \frac{y}{\sqrt{8}})^2 + \frac{5}{8}\)

Since \(\Delta_-(x) < 1\) for \(x > 0\), we conclude that the coupling \(J_-\) is a relevant perturbation on the ferromagnetic side, and that the system flows away from the marginal line. For e.g \(\Gamma_e > \Gamma_o\), the flow must be towards a fixed point where the odd combination eventually decouples, and thus corresponds \(\Phi\) to an even-parity Kondo effect with \(S=1\) and two channel of conduction electrons, which is a Fermi liquid fixed point (cf. fig.\((??)\)).

On the antiferromagnetic side of the line \((x < 0)\) however, we find that \(\Delta_-(x) > 1\) and thus that a small \(J_-\) is an irrelevant perturbation. Hence, the marginal behaviour must be preserved for small enough \(J_-\) on this side.

In order to find the new marginal operators generated by the coupling \(J_-\) for \(x < 0\), we consider the perturbation expansion in the operator \(\Phi\). This operator is irrelevant, but its operator product expansion with itself will generate the marginal operator \((d^+d - 1/2)\partial_x\Phi(0)\) (inducing simply a renormalisation of \(I\), proportional to \(J^2\) for small \(J_-\)), but also the marginal operator \((d^+d - 1/2)\partial_x\Phi(0)\), where \(\Phi\) is precisely the linear combination of flavour and charge bosons \(\Phi\) corresponding to the coset \(K\). This operator was forbidden for \(J_- = 0\). The fact that precisely this combination appears is due to the \(SO(5)\) symmetry of the theory. Had we used the separate flavour-charge decompositions of, we would have concluded incorrectly that \(two\) new marginal perturbations \(\partial_x\Phi^f(0)\) and \(\partial_x(\Phi^f_1 - \Phi^f_2)\) were \textit{a priori} allowed, corresponding to the two \(c = 1\) cosets \(A(2,2)_f\) and \(U(1)_{c_1 - c_2}\). Hence, we find that, in addition to the induced boundary term \(S\) in the spin sector, we have a single other boundary term induced in the flavour-charge sector, which reads:

\[
(d^+d - \frac{1}{2}) \frac{\partial^2\Phi}{\partial x^2}(0)
\]

The problem is again solved by a canonical transformation which now involves a single additional phase shift \(\delta'\) associated with \(\Phi\), depending on the couplings in a non-universal manner with \(\delta' \propto J^2\) for small \(J_-\). In the following, we make use of a parameter \(y\) such that \(\delta'/\pi = \sqrt{5}y/2\).
The overall canonical transformation reads:

$$U_{\delta\delta'} = \exp\{i(d^+d - \frac{1}{2})(\frac{x}{2\sqrt{3}}\Phi(0) + \frac{\sqrt{3}y}{2}\tilde{\Phi}(0))\}$$

(80)

For antiferromagnetic RKKY coupling \((x < 0)\), the marginal line extends into a two-dimensional marginal domain parametrized by the two parameters \(x\) and \(y\). In order to find its precise extension in the \((x,y)\) plane, we discuss the operator content at a given fixed point of the marginal domain, concentrating on the irrelevant operator(s) of lowest dimension. This analysis will also yield the low-temperature behaviour of physical quantities. The boundaries of the domain will be characterized by one of these operators becoming relevant. It turns out that we have to discuss only \(O_m\) and \(O_-\), all other operators (including \(O_+\)) being of higher dimension in this region of the phase diagram. We use the bosonised form (78) of \(O_-\) to find the dimension of the transformed operator \(U_{\delta\delta'}O_-U_{\delta\delta'}^\dagger\), which reads:

$$\Delta_-(x,y) = \frac{1}{2} \left(\frac{3}{2} - x\right)^2 + \frac{5}{2} \left(\frac{1}{2} - y\right)^2$$

(81)

Similarly, using eq. (65) for \(O_m\):

$$\Delta_m(x,y) = \frac{3}{2}(1 + \frac{x}{3})^2 + \frac{5}{2}y^2$$

(82)

Note that these expressions coincide with those established above in the limit \(y \leq 0\). Thus, we find that these operators are irrelevant \((\Delta > 1)\) outside the ellipses of equations: \(\Delta_-(x,y) = 1\) (for \(O_-\)) and \(\Delta_m(x,y) = 1\) (for \(O_m\)). The location of these ellipses in the \((x,y)\) plane is depicted on fig. 2. Remarkably, they are tangent at a single point \((x_c,y_c) = (-3/4,1/4)\). The shaded area in fig. 2 delimited by these two ellipses and the two coordinate axis, together with the whole half-axis \(y = 0, x \geq 0\) corresponds to the marginal domain. When crossing the boundary to the right of the point \((x_c,y_c)\) (`ferromagnetic side'), \(O_-\) becomes relevant and the system flows to the even-parity (or odd-parity) Fermi-liquid fixed point (except when \(y\) is strictly zero). When crossing the boundary to the left of \((x_c,y_c)\) (`antiferromagnetic side'), \(O_m\) becomes relevant and the system flows to the strong antiferromagnetic Fermi-liquid fixed point \(I = -\infty\). Note that \(y\) is even in \(J_+\) and thus that the phase diagram in the \(y > 0\) half-plane yields a symmetric phase diagram in the coupling \(J_-\), with odd and even coupling exchanged. Thus, we identify \((x_c,y_c)\) as the multicritical point separating the attraction basins of two different fixed points when the boundary of the marginal domain is crossed. It is worth noticing that at this point the dimension of the operator \(U_{\delta\delta'}\) becomes \(x_c^2/24 + 5y_c^2/8 = 1/16\), hence suggesting that a direct interpretation of this point as a fusion from the decoupled fixed point with some twist operator is probably possible. This is reminiscent of the multicritical point of the single-channel, two impurity model [18].

![Figure 2. The marginal domain in the plane of the phase shift parameters \((x,y)\) is made up of the shaded area and the half-axis \(x \geq 0, y = 0\). The (tangent) ellipses inside which the operators \(O_-\) and \(O_m\) become relevant are indicated: see text for a detailed explanation.](image-url)
Hence, the impurity susceptibility is found to be finite within the whole antiferromagnetic part $x < 0$ of the marginal domain, while the staggered susceptibility (defined as above) diverges as:

$$\chi_{st} \sim T^{-\theta}, \quad \theta = \frac{|x|(2 + x)}{3} - 5y^2 \quad (83)$$

In order to find the behaviour of the specific heat inside the marginal domain, we have to find which of the two irrelevant operators $\mathcal{O}_m$ and $\mathcal{O}_-\chi$ has lowest dimension. Using Eqs.(81,82), we find that $\Delta_\chi < \Delta_m$ to the right of the straight line $3x + 5y = 1$ (also depicted as a dashed line on fig.2), while $\Delta_m < \Delta_\chi$ to its left. Hence, the specific-heat behaves as: $C/T \sim T^{-\alpha}$ with:

$$\alpha = \begin{cases} 3 - 2\Delta_\chi(x,y) & \text{for } 3x + 5y < 1, \quad (y \neq 0) \\ 3 - 2\Delta_m(x,y) & \text{for } 3x + 5y > 1 \end{cases} \quad (84)$$

The straight line $3x + 5y = 1$ contains $(x_c, y_c)$ (where $\Delta_\chi = \Delta_m = 1$) and crosses the $y = 0$ axis at $x = -1/3 > x_{min}$. On this axis however, the operator $\mathcal{O}_\chi$ is not allowed because of the odd-even symmetry, and the low-temperature behaviour of $C/T$ was found to be controlled by $\mathcal{O}_m$. Hence our results lead to the somewhat surprising conclusion that the exponent of the specific heat should change discontinuously for $-1/3 < x < 0$ when $y$ is turned on.

The finite-size spectrum of the model in the marginal surface can be obtained in a similar manner than for the even-odd symmetric case, by acting, for each state, on the operator in the coset $A(2,2) \otimes K$ with the transformation $U_{\delta y'}$. The resulting spectrum is also in good agreement with available numerical data, and will be reported in another article [141].

3. Conclusion

We have shown that the competition between inter-impurity (RKKY) interactions and the (partial) screening associated with the Kondo effect, lead to a continuous two-parameter family of new non-Fermi liquid fixed points in the two-impurity, two-channel Kondo model at particle-hole symmetry. An analytical solution of the universal low-energy properties of this model has been obtained, using a combination of bosonisation and conformal field theory techniques.

Possible physical realizations of this model include magnetic impurities in coupled Heisenberg spin chains [1414] and coupled quantum dot de
dices [2425]. In those cases, one often faces an RKKY interaction which is not isotropic in spin space. This can be solved by bosonisation along the same lines as the isotropic case considered in this paper. The perturbation then acts not only in the coset algebra, but also in the $S\hat{U}_3(2)$ spin sector, and additional marginal operators (hence additional phase shifts) are induced.

An outstanding question that has not been dealt with above is the effect of particle-hole symmetry breaking on this family of fixed points. We have shown [16] that it is a relevant perturbation near the strong ferromagnetic fixed-point (and hence, by continuity, along the whole marginal line for $x > 0$). The situation on the antiferromagnetic side is currently under investigation.

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