Low-frequency dynamics of disordered \( XY \) spin chains and pinned density waves: from localized spin waves to soliton tunneling

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A long-standing problem of the low-energy dynamics of a disordered \( XY \) spin chain is re-examined. The case of a rigid chain is studied where the quantum effects can be treated quasiclassically. It is shown that as the frequency decreases, the relevant excitations change from localized spin waves to two-level systems to soliton-antisoliton pairs. The linear-response correlation functions are calculated. The results apply to other periodic glassy systems such as pinned density waves, planar vortex lattices, stripes, and disordered Luttinger liquids.

In this Letter I revisit the problem of the low-frequency response of a periodic elastic string pinned by a weak random potential. This problem has been repeatedly studied in connection with dynamics of density waves, Luttinger liquids, glasses, and random-field \( XY \) spin chains. It is also germane for statistical mechanics of periodic planar systems (Josephson junctions, stripes) with columnar disorder. The approach presented below is customized to one dimension (1D) but the results should apply to all dimensions \( d < 4 \) after a few minor modifications. The difficulty of the problem stems from strong nonperturbative disorder effects at low enough frequencies and large enough length scales. Such effects are termed pinning [1] or localization depending on a context. Rigorous results regarding the pinned regime are restricted to one particular value of the chain elastic modulus where the problem maps onto noninteracting fermions in a random potential, whose dynamics is governed by the famous Mott–Halperin–Berezinskii (MHB) law [2, 3]. In this special case the chain is extremely soft and behaves as a quantum object. On the other hand, real charge density waves (CDW) and low-density electron gases are much more rigid and consequently quasiclassical. Despite many previous attempts [4–6] to calculate the response of the latter type of systems, no fully satisfactory solution has emerged. The existing controversy between different authors is clouded by an uncontrolled nature of approximations they use. The purpose of this Letter is to demonstrate a feasibility of a quantitative analysis of the problem and to elucidate the physical picture of the low-frequency excitations in the glassy phase of a rigid (strongly interacting) system. Most of the discussion is focused on the classical dynamics; however, at the end I also consider quantum effects and establish the connection with the MHB law. In plain words, I offer a definitive answer the following basic question: If we shake a pinned elastic chain, how does it respond?

Model. — Consider an \( XY \) spin chain with spin stiffness \( c \), spin wave velocity \( v \), lattice constant \( a \), which is described by the Lagrangean

\[
\mathcal{L} = \sum_j \left\{ \frac{c}{2} \left[ \frac{a^2}{v^2} \left( \partial_t \phi_j - \phi_j \right)^2 - \frac{v^2}{a} \left( \phi(x_{j+1}) - \phi(x_j) \right)^2 \right] \right. \\
- \hbar_j e^{i\phi(x_j)} - \hbar_j^* e^{-i\phi(x_j)} \right\}, 
\]

(1)

where \( x_j = aj \) and \( h_j \) are quenched Gaussian random variables with zero mean and variance \( \langle |h_j|^2 \rangle = \Delta \). Depending on the context, \( \phi \) can equally well represent the phase of a CDW, the bosonized density field of a Luttinger liquid or a transverse displacement of a flexible line object. The quantity of interest is the propagator \( D(\omega, q) \) of the field \( \phi \), or equivalently, the quantity

\[
\sigma(\omega, q) = -\frac{i e^{2\omega}}{4\pi^2\hbar^2} D(\omega, q),
\]

(2)

which has the meaning of the ac conductivity. Of particular interest is \( \sigma(\omega, q = 0) \equiv \sigma(\omega) \).

Classical glass. — A classical chain respond to an external perturbation by elastic vibrations around the ground state \( \phi^0(x) \). It is therefore instructive to recall the basic properties of \( \phi^0 \), described by the collective pinning theory [1, 4]. The spatial structure of \( \phi^0 \) is determined by the competition between disorder and elasticity: on short scales the elasticity prevents large phase distortions but on long scales the disorder eventually wins and the distortions grow without a bound: \( \langle |\phi^0(x) - \phi^0(0)|^2 \rangle \to \infty \) as \( x \to \infty \). There is a characteristic scale \( R_c \) where a \( 2\pi \)-distortion is accumulated. At this scale the typical elastic and disorder energies of a chain segment are of the same order, \( c/R_c \sim \sqrt{R_c \Delta/a} \). It is easy to see that \( R_c \sim (\varepsilon^2 a/\Delta)^{1/3} \). A crude but useful picture [4] is to imagine that the chain consists of domains of size \( R_c \) individually pinned by collective potential wells generated by random fields \( h_j \).

An explicit algorithm for finding \( \phi^0(x) \) was originally given by Feigelman [5]. Consider the segment of the chain with \( j \) leftmost spins, which satisfies the boundary conditions at the left and has the angular coordinate of the \( j \)th spin fixed at a given value \( \phi_j \). Let \( E^-(x_j, \phi_j) \) be the minimal energy of such a segment optimized with respect to
all spin coordinates in the interior of the segment. Function \( E^- \) satisfies the recurrence relation [5]

\[
E^-(x_{j+1}, \phi) = \min_{\phi'} \left\{ E^-(x_j, \phi') + \frac{c}{2a} (\phi - \phi')^2 \right\} + aU(x_{j+1}, \phi),
\]

\[
U(x_{j+1}, \phi) = a^{-1} (h_{j+1} e^{i\phi} + h_{j+1} e^{-i\phi}).
\]

Let \( E^+(x_j, \phi) \) be a similar function for the right end of the chain, then the desired \( \delta \phi \) is the value of \( \phi \) that minimizes the sum \( E(x_j, \phi) = E^-(x_j, \phi) + E^+(x_j, \phi) \). For \( R_c \gg a, \delta \phi \) varies little from \( j \) to \( j+1 \), and so Eq. (3) possesses a meaningful continuum limit,

\[
E^{-}_x = - (1/2c)(E^{-}_x)^2 + U(x, \phi),
\]

which is the Kardar-Parisi-Zhang (KPZ) equation [5, 7].

Due to the \( 2\pi \)-periodicity of \( U \) the solutions of Eqs. (3) and (5) become \( 2\pi \)-periodic at large \( x \) irrespectively of the boundary conditions at the ends. A typical behavior of \( E(\phi) \) is illustrated in Fig. 1. \( E(\phi) \) gives direct information about the rigidity of the system. The minimal work needed to twist a given spin of the chain to the angular coordinate \( \phi \) is equal to \( E(\phi) - E(\phi^0) \). For small \( \delta \phi = \phi - \phi^0 \), it is quadratic in \( \delta \phi \). The elastic distortion caused by the twist is localized predominantly within a domain of length \( R_c \) around the chosen spin. When \( |\delta \phi| \) exceeds a certain value, the chain suddenly snaps into a conformation corresponding to a competing metastable state. At such \( \phi \), function \( E(\phi) \) exhibits upward cusps (typically, one per period — see Fig. 1), extensively studied in the context of the KPZ equation and Burgers turbulence [7, 8].

Of interest to us here are the low-frequency excitations of the chain. They can be visualized as localized mechanical oscillations of essentially rigid segments of size \( R_c \). This concept can be succinctly expressed by means of the following low-frequency local effective action

\[
\mathcal{L}_{\text{eff}} = \frac{1}{2} M (\partial_t \phi)^2 - E(\phi),
\]

which describes the motion of a particle ("domain") of mass \( M \sim R_c \sqrt{c/v^2} \) in the potential \( E(\phi) \). To justify this action, I will follow Ref. [5] but with important modifications, leading to very different end results.

To calculate the linear response we expand \( L \) [Eq. (1)] in \( \delta \phi \) and keep only the second order terms. After that we perform the usual spectral decomposition to obtain

\[
D(\omega; x, x') = \frac{1}{a} \sum_n \frac{\psi_n(x) \psi_n(x')}{\varepsilon_n - \varepsilon(\omega) + i0}, \quad \varepsilon(\omega) \equiv \frac{c\omega^2}{v^2}.
\]

The frequency \( \varepsilon_n \) of \( n \)th eigenmode and its wavefunction \( \psi_n \) satisfy the discrete Schrödinger equation

\[
-\frac{c}{2} \nabla^2 \psi_n(x_j) + U(x_j, \phi_j) \psi_n(x_j) = \varepsilon_n \psi_n,
\]

where \( \nabla \) is the lattice derivative and \( \varepsilon_n = \varepsilon(\omega_n) \) is the "energy" of the mode. From the scaling of the "kinetic" and the disorder energies with distance, we quickly deduce that the low-energy modes are necessarily trapped in the potential wells of the random potential \( U \). To determine their typical localization length \( L_{\text{loc}} \) we can compare the kinetic energy \( c/L_{\text{loc}}^2 \) of a wavepacket of size \( L_{\text{loc}} \) with the depth \( \sqrt{\Delta U} L_{\text{loc}} \alpha \) of a typical potential well, which yields \( L_{\text{loc}} \sim R_c \). Thus, \( R_c \) is the unique characteristic length of the classical glass regime, which shows up both in statics and dynamics [9]. The crossover to this regime occurs around the pinning frequency \( \omega_p = v/R_c \). Note that the depths of individual potential wells have a broad distribution around the typical value \( \sqrt{\Delta U} R_c \sim \varepsilon(\omega_p) \), which gives rise to an inhomogeneously broadened spectrum extending down to \( \varepsilon = 0 \). Modes with \( \varepsilon_n \ll \varepsilon(\omega_p) \) stem from near cancellations between the kinetic and potential terms.

Let \( x \) be a point near which one of such modes, \( \psi_n \), is localized, then for all \( \omega < \omega_p \), Eq. (7) implies that \( D(\omega; x, x) = A/\left(1 + \varepsilon_n - \varepsilon + i0\right) + \) smaller terms, with \( A \equiv \psi_n^2(x)/a \sim R_c^{-1} \). On the other hand, \( D(0; x, x) = E_{\phi}(\phi^0) = \alpha(x) \) [5], and so \( D(\omega; x, x) \sim \omega \sim \alpha - M \omega^2 + o(\alpha) \). This indicates that \( L_{\text{eff}} \) indeed has the correct form for small \( \delta \phi \). (This is all we need to start using \( L_{\text{eff}} \) for calculating the classical linear response). Equation (6) is also correct for adiabatically slow motions, \( \partial_t \phi \to 0 \). Thus, it cannot be badly wrong for all \( \delta \phi \) smaller than the distance from \( \phi_0 \) to the nearest cusp of \( E(\phi) \). Indeed, local smooth distortions should appear to the rest of the system as adiabatically slow ones provided their characteristic frequencies are sufficiently small, \( \omega \ll \omega_p \).

From Eq. (6) we see that the disorder-averaged low-\( \omega \) response of the chain is encoded in the small-\( \alpha \) behavior of the distribution function \( P(\alpha) \) of the local ground-state rigidity \( \alpha(x) \), e.g., the spin wave density of states \( \rho(\varepsilon) = (N_s \alpha)^{-1} \sum_n \delta(\varepsilon - \varepsilon_n) \) is given by

\[
\rho(\varepsilon) \sim P(\alpha = \varepsilon R_c), \quad \varepsilon < \varepsilon(\omega_p).
\]

Here \( N_s \) is the number of spins in the chain. Similarly, starting from the spectral representation

\[
\Re \sigma(\omega) = \frac{e^2 \omega}{4\pi N_s} \left\langle \sum_n \delta(\varepsilon - \varepsilon_n) \right\rangle,
\]
where \( d_n \equiv \sum \psi_n(x_j) \), we obtain \( d_n \sim (R_c/a)^{1/2} \) and

\[
\Re \sigma(\omega) \sim e^2 \omega R_c \rho(\epsilon(\omega)), \quad \omega < \omega_p.
\]  

(11)

Now I intend to show that at small \( \alpha \)

\[
P(\alpha) \propto \alpha^s,
\]

(12)

where \( s = 3/2 \). Combined with Eqs. (9) and (11), this entails [10]

\[
\Re \sigma(\omega) \sim (e^2 v R_c/c)(\epsilon/\omega_p)^4, \quad \omega \ll \omega_p.
\]  

(13)

Recall that \( \alpha \) is defined to be the second derivative of \( E(\phi) \) at the point of its global minimum \( \phi_0 \). Let us first demonstrate that at arbitrary local minima \( \text{Prob}(\alpha) \equiv P_0(\alpha) \propto |\alpha| \) at small \( \alpha \). Indeed, local extrema are the points where \( E_\phi(\phi) = 0 \); hence,

\[
P_0(\alpha) = (N_{-1}^{-1} f_{-1}'^{2\pi} d\phi (E_{\phi\phi}(\phi) - \alpha\delta(E_\phi)|E_{\phi\phi}|)\),
\]

where \( N_{-1} = \int_{-\pi}^{\pi} d\phi (E_{\phi\phi}(\phi))\). The term \( E_{\phi\phi} \) in the integrands is the Jacobian. In the limit \( \alpha \to 0 \) we obtain

\[
P_0(\alpha) = |\alpha| \left( N_{-1}^{-1} \int_{-\pi}^{\pi} d\phi (E_{\phi\phi}(\phi))\right)^{-1} \approx C_0 |\alpha|.
\]  

(14)

The main contribution to the integral that determines the constant \( C_0 \sim R_c^2/c^3 \) is supplied by configurations where \( E_{\phi\phi}^- \) and \( E_{\phi\phi}^+ \) have their typical values of the order of \( c/R_c \) but are opposite in sign and almost cancel each other. Thus, soft modes arise from the frustrations in the ground state, e.g., when \( \phi = \phi_0 \) gives a rather low energy to the left half of the chain but corresponds to a high energy state of the right half. Such frustrations can always happen because \( E^- \) and \( E^+ \) are uncorrelated.

Let us now consider the global minimum. In its vicinity

\[
E(\phi) = E(\phi^0) + \frac{\alpha}{2} \delta \phi^2 + \beta \delta \phi^3 + \gamma \delta \phi^4,
\]  

(15)

which leads to the double-minimum structure depicted in Fig. 1. It is easy to see that \( \phi^0 \) is the lower of the two minima only if \( |\beta| < \sqrt{2\alpha \gamma} \). Via a straightforward extension of the argument leading to Eq. (14), one can obtain \( P_0(\alpha, \beta) \propto 1 \) for the joint distribution function of small \( \alpha \) and \( \beta \). The restriction on allowed \( \beta \) at the global minimum leads to \( P(\alpha) \propto \sqrt{\alpha} P_0(\alpha, \beta) \propto \alpha^{3/2} \) as I claimed above. Here I ignore competing minima away from the immediate vicinity of \( \phi^0 \) because their number within a \( 2\pi \)-period is \( N \sim 1 \). They cannot yield any additional powers of the small parameter \( \alpha \).

Now I present my numerical results. The numerical solution of Eq. (3) does not pose major difficulties. This has to be done for a number of disorder realizations \( \{h_j\} \), followed by solving the eigenvalue problem (8), calculating the desired response functions, and their statistical averaging. Such a procedure produces the rigidity distribution \( P(\alpha) \) shown in Fig. 2. The power-law behavior predicted by Eq. (12) is apparent. Function \( \rho(\epsilon) \), plotted in the same figure, can be well fitted to the power law with the same exponent, which shows that Eq. (9) is also well satisfied. The actual value of the exponent from those fits, \( s = 1.7 \), deviates slightly from my analytical prediction \( s = 3/2 \). Further numerical work is needed to resolve this small discrepancy but there is enough accuracy to rule out \( s = 1/2 \) [4, 6] or \( s = 1 \) [5]. As for the functional shape of \( E(\phi) \), I observed the double-minimum structure sketched in Fig. 1 quite often, roughly in 20% of the cases.

Two-level systems.— For a typical double-well potential \( E(\phi) \) with a given oscillator frequency \( \omega \), the distance in \( \phi \) and the energy barrier between the two minima are of the order of \( \omega/\omega_p \) and \( (c/R_c)(\omega/\omega_p)^4 \), respectively. Both decrease with \( \omega \), and it can be verified that at \( \omega \sim \omega_q = \sqrt{1/3} \omega_p \) where \( K \equiv \hbar v/2\pi c \ll 1 \), the matrix element \( I \) for the quantum tunneling between the two minima becomes of the order of \( \hbar \omega \). This implies that at frequencies below \( \omega_q \) the response of the chain is dominated by quantum tunneling. It contributes to \( \Re \sigma(\omega) \) whenever the levels localized in the two minima are split in energy by exactly \( \hbar \omega \). A straightforward analysis [11] of the model defined by Eqs. (6), (14) and (15) leads to

\[
\Re \sigma(\omega) \sim \frac{e^2}{\hbar} R_c K^2 \frac{\omega}{\omega_p}, \quad \omega_q \ll \omega \ll \omega_q.
\]  

(16)

The origin of the frequency scale \( \omega_q = \omega_p \exp(-K^{-1}) \) is as follows. The energy splitting of our two-level system (2LS) is bounded from below by \( I \). Therefore, as \( \omega \) decreases, the tunneling barrier and the tunneling distance have to increase. At \( \omega < \omega_q \) a typical 2LS cannot possess such a small \( I \) and Eq. (16) ceases to be valid. As \( \omega \) continues to decrease, the dissipation is initially determined by rare 2LS with unusually large barriers, until at \( \omega_s = \omega_p \exp(-K^{-3/2}) \) the soliton mechanism becomes more prominent.
Soliton tunneling.— In the soliton mechanism a large tunneling action (small $I$) is due to a large tunneling mass. The object that is a segment of the chain of length $l_{tun}(\omega) \gg R_c$. The optimal way to accomplish such a tunneling is to send a virtual soliton (a $2\pi$-kink in $\phi$) over the distance $l_{tun}$. Another route to the concept of solitons is appealing to universality of the low-frequency physics of the glass phase (throughout the range $K < 3/2$). If it holds, the conductivity must be calculable by generalizing the Mott–Halperin argument [2]. This argument devised for $K = 1$ focuses on the tunneling of single electrons. But they are precisely the $2\pi$-solitons in the bosonized formulation (1), so replacing the word “electron” by “soliton” everywhere in the argument seems entirely natural.

The 1D MHB law can be written in the form [14]
\[
\Re \sigma(\omega) \sim Q^2 l_{tun}^2 \rho_s^2(\hbar \omega),
\]
where $\rho_s(E)$ is the density of states, $l_{tun} \sim l_s \ln(l_0/\hbar \omega)$ is the typical tunneling distance, and $l_s$ is the localization length of the tunneling charge-$Q$ objects ($Q = e$ for $2\pi$-solitons). A typical soliton has the size $R_c$ and the creation energy $E_s = e/R_c$, which implies $l_s \sim \hbar \omega/E_s \sim K R_c$ [15]. Under the standard assumption [16] that $\rho_s(E) \to \text{const} \sim (R_c E_s)^{-1}$ for $E \to 0$, we get
\[
\Re \sigma(\omega) \sim \frac{e^2}{\hbar} R_c K^5 \omega^2 \frac{\ln^2 \omega}{\omega_p^2}, \quad \omega \ll \omega_s.
\]

The frequency dependence of the conductance is summarized in Fig. 3. My results support the notion that the infrared behavior of nondissipative classical [11, 12] and quantum [13] glasses are universally controlled by the $\omega^4$-dependence (13) and the MHB law (18), respectively. In the model considered, as $K$ increases, the system becomes more quantum. The quasiclassical regimes (2LS and $\omega^4$) shrink and eventually get eliminated at $K \approx 1$, where the soliton tunneling crosses over directly to the Drude behavior, in agreement with Ref. [3].

Relation to experiments.— The obtained results have implications for a wide variety of physical systems. Perhaps, the most studied of them are the CDWs. Evidence for the 2LS in CDW compounds has been seen in the specific-heat [17]. Transport data typically fit the $\sigma(\omega) \propto T \omega^b$ dependence with $b \approx 1$ [18]. It may or may not be due to the 2LS physics, but in any case, it means the true zero-temperature limit I studied here has not been realized in the experiment. To verify Eqs. (13) and (16) new ac transport experiments at considerably lower $T$ are needed. As for Eq. (18), strong modifications due to Coulomb effects [14] are expected in real CDWs. Another class of materials to test would be the high-spin spin-chain compounds whose dynamics can be studied by electron spin resonance. Finally, some of the ideas behind Eq. (13) may also be relevant for understanding the $\omega^2$-broadening of phonons in glassy liquids [19].

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