Decidability of the extension problem for maps into odd-dimensional spheres

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Abstract

In a recent paper [3], it was shown that the problem of existence of a continuous map $X \to Y$ extending a given map $A \to Y$ defined on a subspace $A \subseteq X$ is undecidable, even for $Y$ an even-dimensional sphere. In the present paper, we prove that the same problem for $Y$ an odd-dimensional sphere is decidable. More generally, the same holds for any $d$-connected target space $Y$ whose homotopy groups $\pi_k Y$ are finite for $k > 2d$.

1. Introduction

The main object of study of the present paper is the extension problem. Given spaces $X$, $Y$ and a map $f: A \to Y$ defined on a subspace $A \subseteq X$, it questions the existence of a continuous extension

\[
\begin{array}{ccc}
A & \xrightarrow{f} & Y \\
\downarrow & & \downarrow \\
X & \xrightarrow{g} & 
\end{array}
\]

If $Y$ is allowed non-simply connected, this problem is undecidable by a simple reduction to the word problem in groups. Thus, we restrict ourselves to the situation of a simply connected $Y$.

In [8], Steenrod expressed a hope that the extendability problem would be algorithmically solvable. It was proved in [11] that this is indeed the case if one restricts to a suitably “stable” situation, i.e. if $\dim X \leq 2 \text{conn} Y + 1$. The algorithm of that paper depended on computations with abelian groups of homotopy classes of maps that are not available unstably. Later, the authors showed in [3] that the previous positive result was very much the best possible: the extension problem with $\dim X > 2 \text{conn} Y + 1$ is undecidable, even for such a simple target space as $S^{d+1}$ with $d + 1$ even. This undecidability result has implications to other problems, namely, [5] shows the undecidability of the problem of existence of a robust zero of a given PL-map $K \to \mathbb{R}^{d+2}$, again for $d$ even.

It may thus come as a bit of a surprise that the last two problems with $d + 1$ odd are decidable – this is the content of Theorem 1 below. It applies to $Y = S^{d+1}$, $d + 1$ odd, since in this case, $\pi_n S^{d+1}$ is finite for $n > d + 1$. Again, [5] implies the decidability of the problem of existence of a robust zero of a given PL-map $K \to \mathbb{R}^{d+2}$, $d$ odd.

Theorem 1. There exists an algorithm that, given a pair of finite simplicial sets $(X, A)$, a finite $d$-connected simplicial set $Y$, $d \geq 1$, with homotopy groups $\pi_n Y$ finite for all $2d < n < \dim X$ and a simplicial map $f: A \to Y$, decides the existence of a continuous extension $g: X \to Y$ of $f$. 

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We do not have any bounds on the running time of such an algorithm. In the light of the \#P-hardness of the computation of the homotopy group \( \pi_k Y \) when \( k \) is a part of the input (in unary), see [3], one should not expect that this algorithm is polynomial-time when the dimension of \( X \) is not fixed. However, even if \( \dim X \) is bounded, it seems that our algorithm will not have polynomial running time. Nevertheless, the contrast with the undecidability for even-dimensional \( X \) is huge.

In Section 5 we briefly discuss an extension of Theorem 4 to the fibrewise equivariant situation of [4]. In the special case \( A = \emptyset \), such an extension implies the decidability of the problem of existence of a \( \mathbb{Z}/2 \)-equivariant map \( X \to S^{d+1} \) when \( d+1 \) is odd. The index of \( X \), denoted \( \text{ind} X \), is the smallest \( d+1 \) for which such an equivariant map \( X \to S^{d+1} \) exists; it has many applications in geometry and combinatorics. Thus, with the equivariant version of Theorem 4 it is possible to narrow \( \text{ind} X \) down to two possible values.

2. Sets with an action and mappings to abelian groups

Let \( S \) and \( T \) be sets with a binary operation \(+ : S \times T \to S\) that has a right-sided zero \( 0 \in T \), i.e. such that \( x + 0 = x \). We use the bracketing convention \( x + y + z = (x + y) + z \). We define a “derived” action of \( T \) on \( S \) by

\[
  x + \theta y = x + y + \cdots + y.
\]

Again, it has a right-sided zero \( 0 \). The following lemma will be our main technical tool.

**Lemma 2.** Let \( f : S \to G \) be an arbitrary mapping of \( S \) into an abelian group \( G \). Then, for each prime power \( q = p^m \) and \( \ell > 0 \), there exists \( \ell' \geq \ell_0 \), a function \( D_{q,\ell} f : S \times T^{\ell} \to G \) such that \( D_{q,\ell} f(x; y_1, \ldots, y_\ell) = 0 \) whenever \( y_i = 0 \) for some \( i \), and \( \theta > 0 \) such that

\[
  f(x + \theta y) \equiv f(x) + D_{q,\ell} f(x; y, \ldots, y) \quad (\text{mod } q). \tag{mod q}
\]

In fact, \( D_{q,\ell} f \) is a formal expression in terms of \( f \), the action of \( T \) on \( S \) and the group structure on \( G \) and works universally for all \( f : S \to G \). Moreover, this expression is computable.

We will make a heavy use of higher-order differences

\[
  \Delta_\ell f(x; y_1, \ldots, y_\ell) = \sum_{0 \leq k \leq \ell} \sum_{1 \leq i_1 < \cdots < i_k \leq \ell} (-1)^{\ell-k} f(x + y_{i_1} + \cdots + y_{i_k}).
\]

Clearly, \( \Delta_\ell f(x; y_1, \ldots, y_\ell) = 0 \) whenever \( y_i = 0 \) for some \( i \).

For any formal expression written in terms of the action of the group \( T \) on \( S \), we will use a superscript \((-)^{\theta}\) to denote the expression obtained by replacing each \( x + y \) by \( x + \theta y \). In this way, we yield \( \Delta_\ell^{(\theta)} f \). The function \( D_{q,\ell} f \) will be an integral combination of the \( \Delta_\ell^{(\theta)} f \).

**Proof.** We let \( \ell = p^m \) be any power of \( p \) for which \( \ell > \ell_0 \) and \( \theta = p^{n+m-1} \). The proof is executed by induction with respect to \( m \). By definition, \( f(x + p^n m - 1 y) \) equals

\[
  f(x + p^n m - 1 y) = \Delta_\ell^{(p^{n-1})} f(x; y, \ldots, y) - \sum_{j=0}^{p^n-1} (-1)^{p^n-j} \binom{p^n}{j} f(x + j p^{m-1} y).
\]

For \( j > 0 \), write \( j = p^{n'} j' \) where \( j' \) is prime to \( p \) and observe that

\[
  j \binom{p^n}{j} = p^n \binom{p^n-1}{j-1}
\]

is divisible by \( p^n \), so that \( p^{n-n'} \mid \binom{p^n}{j} \). Setting \( n' + m = n + m' \), we have either \( m' \leq 0 \), in which case \( n - n' \geq m \) and the binomial coefficient is divisible by \( q = p^m \), or we obtain for \( q' = p^{n'} \) by induction

\[
  f(x + j p^{m-1} y) = f(x + p^{n+m-1} j' y) \equiv f(x) + D_{q',\ell} f(x; y, \ldots, y) \quad (\text{mod } q').
\]
(this holds even for \( j = 0 \) when the last term is interpreted as 0). Upon multiplication by \( \binom{p^n}{j} \), that is divisible by \( p^{n-n'} = q/q' \), we obtain even

\[
\left( \binom{p^n}{j} \right) f(x + jy^{p^{n-1}}y) \equiv \left( \binom{p^n}{j} \right) f(x) + \left( \binom{p^n}{j} \right) D_{q', \ell} (x; y, \ldots, y). \tag{mod q}
\]

Since \( \sum_{j=0}^{p^n-1} (-1)^{n-j} \binom{p^n}{j} = -1 \), substituting the previous equation into the first yields

\[
f(x + p^{n+m-1}y) \equiv f(x) + \Delta_4^{(p^{m-1})} f(x; y, \ldots, y) - \sum_{j=0}^{p^n-1} (-1)^{n-j} \left( \binom{p^n}{j} \right) D_{q', \ell} (x; y, \ldots, y)
\]

where we set \( D_{q', \ell} = \Delta_4^{(p^{m-1})} - \sum_{j=0}^{p^n-1} (-1)^{n-j} \binom{p^n}{j} D_{q', \ell} \).

\[\square\]

Example 3. In this example, we have \( q = p^m = 4 \) and \( \ell = 4 \). Then

\[
f(x + 8y) = \Delta_4^{(2)} f(x; y, y, y, y) + 4f(x + 6y) - 6f(x + 4y) + 4f(x + 2y) - f(x)
\]

and we continue in a similar way with the third term,

\[
f(x + 4y) = \Delta_4 f(x; y, y, y, y) + 4f(x + 3y) - 6f(x + 2y) + 4f(x + y) - f(x).
\]

Substituting into the first equation, we get

\[
f(x + 8y) \equiv f(x) + \Delta_4^{(2)} f(x; y, y, y, y) + 2\Delta_4 f(x; y, y, y, y) \tag{mod 4}
\]

and \( D_{4,4} f = \Delta_4^{(2)} f + 2\Delta_4 f \).

3. Postnikov tower

We assume that \( Y \) is \( d \)-connected simplicial set and has all homotopy groups \( \pi_n Y \) finite for \( 2d < n < \dim X = D \). In the following theorem, \( K(\pi, n+1) \) is the Eilenberg-MacLane space and \( E(\pi, n) \) its path space; more precisely, we use the canonical minimal models with both simplicial sets minimal and the projection \( \delta: E(\pi, n) \to K(\pi, n+1) \) a minimal fibration, see [7].

Theorem 4. For each simply connected simplicial set \( Y \), it is possible to construct simplicial sets \( P_n \) for \( n < D \), and a sequence of simplicial maps

\[
Y \xrightarrow{\varphi_n} P_n
\]

such that \( \varphi_n: \pi_i(Y) \to \pi_i(P_n) \) is an isomorphism for \( i \leq n \) and \( \pi_i(P_n) = 0 \) for \( i > n \).

Further, for \( 2d < n < D \), it is possible to construct simplicial sets \( P_{n,i} \) that fit into a pullback square

\[
P_{n,i} \xrightarrow{k} K(\mathbb{Z}/q, n+1)
\]

with \( q = p^m \) a prime power (depending on \( n \) and \( i \); the same applies to \( k \)) and \( P_{n-1} = P_{n,0} \), \( P_n = P_{n,r} \), where \( r \) is some integer that depends on \( n \). The composition of the projections \( P_{n,i} \to P_{n,i-1} \) for \( i = 1, \ldots, r \) is a map \( p_n: P_n \to P_{n-1} \) for which \( p_n \varphi_n = \varphi_{n-1} \).
Proof. The paper [2] gives the simplicial sets $P_n$. To obtain their refinements $P_{n,i}$, we compute a decomposition

$$\pi_n \cong \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_r$$

of the $n$-th homotopy group into a sum of cyclic groups of prime power orders. Then we define $\pi_{n,i} = \mathbb{Z}/q_i \oplus \cdots \oplus \mathbb{Z}/q_i$ with obvious projections $pr: \pi_n \to \pi_{n,i}$; $P_{n,i}$ is the following pullback

$$
\begin{array}{ccc}
P_{n,i} & \xrightarrow{\delta} & E(\pi_{n,i}, n) \\
\downarrow & & \downarrow \\
P_{n-1} & \xrightarrow{k_n} & K(\pi_{n+1}, n + 1)
\end{array}
$$

\[\text{Theorem 5.} \text{ It is possible to construct an action } x + \Theta y, \Theta \gg 0, \text{ of } P_{2d} \text{ on each } P_{n,i}, \text{ for } 2d \leq n < D, \text{ that has a right-sided zero } 0 \in P_{2d}. \text{ The projections } P_{n,i} \to P_{n,i-1} \text{ respect this action.}\]

Proof. We will construct, by induction with respect to $n$ and $i$, positive integers $\Theta_{n,i}$ and an action $x + \Theta_{n,i} y$ of $P_{2d}$ on $P_{n,i}$. The action $x + \Theta y$ from the statement is then obtained by setting $\Theta = \Theta_{D-1,i}$ and deriving the action $\Theta_{n,i}$: this is possible since $\Theta_{n,i} \mid \Theta$ by construction. Starting with $n = 2d$, the paper [1] constructs an abelian H-group structure on $P_{2d}$, i.e. an action of $P_{2d}$ on itself; we set $\Theta_{2d+1,0} = 1$.

For the induction step, we apply Lemma 2 to the Postnikov invariant $k: P_{n,i-1} \to K(\mathbb{Z}/q, n + 1)$ – its target is a simplicial abelian group, i.e. an abelian group in each dimension. The function

$$D_{\mathbb{Z}/q, \ell}^{(\Theta_{n,i-1})} k: P_{n,i-1} \times P_{2d} \times \cdots \times P_{2d} \to K(\mathbb{Z}/q, n + 1)$$

(formally, it is not derived from $D_{\mathbb{Z}/q, k}$ since $x + y$ is not defined, but we want to emphasize that it is with respect to the action $x + \Theta_{n,i-1} y$) is zero whenever at least one of the components in $P_{2d}$ is zero and thus we have a diagram

$$
\begin{array}{ccc}
P_{n,i-1} \times \{\text{fat wedge}\} & \xrightarrow{0} & E(\mathbb{Z}/q, n) \\
\downarrow & & \downarrow \\
P_{n,i-1} \times P_{2d} \times \cdots \times P_{2d} & \xrightarrow{M'} & K(\mathbb{Z}/q, n + 1)
\end{array}
$$

(the fat wedge consists of those $\ell$-tuples $(y_1, \ldots, y_\ell) \in P_{2d} \times \cdots \times P_{2d}$ with at least one $y_i$ equal to the basepoint 0). The cofibre of the map on the left is $(P_{n,i-1})_+ \wedge P_{2d} \wedge \cdots \wedge P_{2d}$ and is $(\ell(d + 1) - 1)$-connected. Therefore, when $\ell \gg 0$, a diagonal $M'$ exists; it can be computed as in [1]. We define $M(x, y) = M'(x, y, \ldots, y)$, so that

$$\delta M(x, y) = D_{\mathbb{Z}/q, \ell}^{(\Theta_{n,i-1})} k(x, y, \ldots, y) = k(x + \theta \Theta_{n,i-1} y) - k(x),$$

where $\theta$ is the output of Lemma 2. Denoting $\Theta_{n,i} = \theta \Theta_{n,i-1}$, this allows us to define a new action on $P_{n,i} \subseteq P_{n,i-1} \times E(\mathbb{Z}/q, n)$ by the formula

$$(x, c) + \Theta_{n,i} y = (x + \Theta_{n,i} y, c + M(x, y))$$

(the compatibility holds since $\delta (c + M(x, y)) = \delta c + \delta M(x, y) = k(x) + (k(x + \Theta_{n,i} y) - k(x)) = k(x + \Theta_{n,i} y))$. 

After the following simple observation, we will be ready to prove Theorem 1.

\[\text{Lemma 6.} \text{ For each } g': X \to P_{2d} \text{ and } 2d < n < D, \text{ it is possible to compute the finite set of homotopy classes of all lifts } g: X \to P_n.\]

Proof. This follows from the fact that each $\pi_n$ is finite for $2d < n < D$. Namely, since $\pi_{2d+1}$ is finite, the number of all lifts of $g'$ to a map $X \to P_{2d+1}$ is finite. Thus, it is possible to go through all these partial lifts and compute all their lifts to $P_n$ by recursion.
4. Proof of Theorem

For $n = D - 1$, let $f: A \to P_n$ also denote the composition $f: A \xrightarrow{i} Y \xrightarrow{\pi_n} P_n$. By the usual obstruction theory, it is enough to check whether an extension to $g: X \to P_n$ exists − the higher obstructions are all zero. Thus, we consider the Postnikov stage $P_n$ with an action $x + \Theta g$ by the stage $P_{2d}$. Consider the commutative square (the $R$ and $R'$ are the restriction maps while $\Pi_X$ and $\Pi_A$ are post-compositions with the projection $P_n \to P_{2d}$)

$$[g] \in [X, P_n] \xrightarrow{\Pi_X} [X, P_{2d}]$$

$$[f] \in [A, P_n] \xrightarrow{\Pi_A} [A, P_{2d}] \triangleright [f']$$

with $[f'] = \Pi_A[f]$. We compute the groups on the right explicitly as in [1] and consider the subset $H = (R')^{-1}([f'])$ of all possible extensions of $f'$ to a map $X \to P_{2d}$. There is a finite set $H_0 \subseteq H$ such that $H = H_0 + \Theta \ker R'$; namely, if $[h_0] \in H$ and we identify $\ker R' \cong \mathbb{Z}/q_1 \oplus \cdots \oplus \mathbb{Z}/q_r$ (possibly with some $q_i = 0$ giving $\mathbb{Z}/0 \cong \mathbb{Z}$), we may take for $H_0$ all $r$-tuples of the form $[h_0] + (z_1, \ldots, z_r) \in H$ with each $|z_i| \leq \Theta/2$.

Suppose first that $g$ is any extension of $f$ and express its image in $[X, P_{2d}]$ as $\Pi_X[g] = [h] - \Theta[k]$ with $[h] \in H_0$ and $[k] \in \ker R'$. Then $[\hat{g}] = [g] + \Theta[k] \in \Pi_{X}^2(H_0)$ also gives an extension of $f$ since

$$R[\hat{g}] = R([g] + \Theta[k]) = [f] + \Theta R'[k] = [f],$$

(the operations in homotopy classes are natural and $[k] \in \ker R'$). Thus, we see that an extension $g$ exists if and only if $[f] \in R\Pi_{X}^{-1}(H_0)$. This set is finite and its representatives can be computed using Lemma [3]. For each $[\hat{f}] \in R\Pi_{X}^{-1}(H_0)$, we may then test whether $[\hat{f}] = [f]$ by the main theorem of [3].

5. A fibrewise equivariant version

The same argument could be repeated in the fibrewise equivariant setup of [11], though actions with a strict right-sided zero have to be replaced by ones with a weak zero. Denoting $I = \Delta^1$, this structure is a map

$$(1 \times P_{n,i} \times_B P_{2d}) \cup (I \times P_{n,i} \times_B B) \to P_{n,i}$$

consisting of an action and a homotopy $x \sim x + 0$.

The most significant difference lies in the proof of Theorem[5] The space $P_{n,i-1} \times P_{2d} \times \cdots \times P_{2d}$ has to be replaced by the following subspace of $I^\ell \times (P_{n,i-1} \times_B P_{2d} \times_B \cdots \times_B P_{2d})$: 

$$\bigcup_{1 \leq t_1 < \cdots < t_{\ell}} (d_{t_1}^+ \cdots d_{t_{\ell}}^+ I^\ell) \times (P_{n,i-1} \times_B \bigvee_{k=1}^{\ell} P_{2d}),$$

(1)

where $d_{t_1}^+ \cdots d_{t_{\ell}}^+ I^\ell \subseteq I^\ell$ consists of those $\ell$-tuples $(t_1, \ldots, t_{\ell})$ with $t_{i_1} = \cdots = t_{i_k} = 1$ and where $\bigvee_{k=1}^{\ell} P_{2d} \subseteq P_{2d} \times_B \cdots \times_B P_{2d}$ is formed by those $\ell$-tuples $(y_1, \ldots, y_{\ell})$ whose components $y_j$ with $j \notin \{i_1, \ldots, i_k\}$ lie on the zero section $B$. In particular, $\bigvee_B P_{2d} = B \times B \cdots \times B$ and $\bigvee_B P_{2d} = P_{2d} \times_B \cdots \times_B P_{2d}$.

The subspace $P_{n,i-1} \times \{\text{fat wedge}\}$ is replaced by the subspace of [11] formed by those elements whose component in $I^\ell$ has at least one component equal to 0. By the methods of [4], it is then easy to equip this pair with effective homology, compute the variation of the map $M'$ from the proof of Theorem[5] and use it to define a new weak action of $P_{2d}$ on $P_{n,i}$.
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