Adiabatic Spectra During Slowly Evolving

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In general, for single field, the scale invariant spectrum of curvature perturbation can be given by either its constant mode or its increasing mode. We show that during slowly expanding or contracting, the spectrum of curvature perturbation given by its increasing mode can be scale invariant. The perturbation mode can be naturally extended out of horizon, and the amplitude of perturbation is consistent with the observations. We briefly discuss the implement of this scenario.

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The nearly scale invariance of curvature perturbation is required by the observations. How obtaining it is still a significant issue, especially for single field. In general, the equation of motion of curvature perturbation $\zeta$ in $k$ space is given by

$$u_k'' + \left(\frac{k^2}{z} - \frac{z'}{z}\right) u_k = 0,$$

where $'$ is derivative for $\eta = \int dt/a$, $u_k = z\zeta_k$ and $z \simeq a \sqrt{|\epsilon|}$, in which $\epsilon = -\dot{H}/H^2$ and $M_P^2 = 1$ is set. When $k^2 \ll z''/z$, the solution of Eq. (1) is $\zeta_k \simeq C + D \int \frac{dz}{z}$, e.g. [3], where $C$ mode is the constant mode, and $D$ mode evolves with the time.

In general, the scale invariance of spectrum requires $z'' \sim (\eta_s - \eta)^2$. In principle, both $a$ and $|\epsilon|$ can be changed, and together contribute the change of $z$. When both are changed, the case is complicated for studying. The simplest case is that one of both is changed while another is hardly changed. When $|\epsilon|$ is nearly constant,

$$a \sim \frac{1}{\eta_s - \eta} \quad \text{or} \quad (\eta_s - \eta)^2$$

have to be satisfied, where initially $\eta \ll -1$ and $\eta_s$ is around the ending time of the corresponding evolution. The evolution with $a \sim 1/(\eta_s - \eta)$ is that of the inflation [4, 13, 14, 15], in which $\epsilon \simeq 0$. While another is that of the contraction with $\epsilon \simeq 1.5$ or $(\eta_s - \eta)^4$, both are dual [5].

The increasing or decaying of $D$ mode is determined by

$$\int \frac{d\eta}{z^2} \sim \int \frac{d\eta}{a^2|\epsilon|} \sim (\eta_s - \eta)^{1-p},$$

where $a^2|\epsilon| \sim (\eta_s - \eta)^p$ is applied. Thus it is decaying for $p < 1$ and is increasing for $p > 1$. In general, for inflation, $D$ mode is decaying, the spectrum is determined by the constant mode, since $p < 1$, while for the contraction with $\epsilon \simeq 1.5$, the spectrum is determined by the increasing $D$ mode, since $p > 1$.

Thus the scale invariance of $\zeta$ can be given by either its constant mode or its increasing mode, and the different modes implies different scenarios of early universe. In principle, when $|\epsilon|$ is nearly constant, the increasing mode of metric perturbation $\Phi$, which is scale invariant for $\epsilon \gg 1$ or $\epsilon \ll -1$, might dominate the curvature perturbation. The evolution with $\epsilon \gg 1$ corresponds to the slowly contracting, which is that of ekpyrotic universe [11], while $\epsilon \ll -1$ is the slowly expanding [12].

It is found with Eq. (3) that for $|\epsilon| \sim (\eta_s - \eta)^4$, the scale invariance of spectrum is determined by the increasing $D$ mode, while another is determined by the constant mode. Eq. (1) is only the exchange of $\sqrt{|\epsilon|}$ with $a$ in Eq. (2). However, the evolutions are completely different, which might be regarded as dual in certain sense.

When $\epsilon \sim (\eta_s - \eta)^4$ and $a$ is nearly constant, one of the solutions is $H \simeq \frac{1}{\alpha} + \Lambda_s$, in which $\alpha$ is constant and $\alpha \ll \Lambda_s$ since initially $t \ll -1$. Thus $H$ is also hardly changed for some times. This is adiabatic ekpyrosis given in [13], see [16] for the detailed discussion for the solutions and [17] for criticism. Here though $k^2 \ll z''/z$, the perturbation mode is actually still inside the Hubble horizon, since $k = aH$ and both $a$ and $H$ are hardly changed. Thus a period after it is required to extend the perturbation mode out of the Hubble horizon.

We, in this paper, will consider the evolution with $|\epsilon| \sim (\eta_s - \eta)^4$ and constant $a$. In this case, the spectrum of the curvature perturbation is induced by its increasing mode, while in [13, 20], the spectrum is induced by the
constant mode. This difference is significant. We will see that, different from that in [19, [20], here the perturbation mode can naturally leave the horizon, and also there is not the problem pointed out in [21].

We begin with $|e| \sim (t_* - t)^4$, since $a$ is nearly constant which brings $\eta \sim t$. Thus by integral for it, we have

$$ H \sim \pm \frac{1}{\Lambda^4(t_* - t)^5}, \quad (5) $$

where initially $t \ll -1$ and $|t| \gg |t_*|$, and $\Lambda \sim 1/|t_*|$ is constant, which is regarded as the exit scale of $H$. The positive solution, i.e. the expansion solution, is for $\epsilon < 0$. The minus, i.e. the contraction solution, is for $\epsilon > 0$. $a$ can be obtained by $\ln a = \int H dt$, which is

$$ \ln a \sim \pm \frac{1}{\Lambda^4(t_* - t)^4}. \quad (6) $$

When initially $t \ll -1$, $a \simeq 1$ and $|H|$ is highly small and negligible, see Eq.(5), however, $|e| \sim \Lambda^4(t_* - t)^4 \gg 1$ is not constant and will gradually decrease with the evolution. When $t \approx O(1)t_*$, this phase of background evolution ends. In this epoch, $a \approx e$ implying $a$ is slowly expanding during this phase, or $1/e$ implying $a$ is slowly contracting, $|H| \sim |H_*| \sim \Lambda_*$ which is large, and $|e| \sim 1$. In principle, it can be expected that after this slowly expanding or contracting phase, the evolution of standard cosmology begins. This will bring completely distinct scenarios of early universe. We list Tab.1, which is a brief of above discussions.

The slowly evolving of $a$ and the rapidly increasing of $H$ lead that the perturbation modes can be naturally extended out of Hubble horizon during this phase. The efolding number for the primordial perturbation generated during this phase is

$$ N_c \approx \ln \left( \frac{H_*}{H} \right), \quad (7) $$

since $k = a|H|$ and $a$ is nearly constant. Thus in principle, the enough efolding number can be obtained. The details of the model and the evolution of perturbation mode are visualized in Fig.1, in which the initial time $|t| \approx 10|t_*|$ is set for simplicity.

When $k^2 \gg z''/z$, $u_k \rightarrow \frac{1}{2k^2} e^{ik\eta}$, which gives the initial condition of mode evolution. $z''/z$ is increased with the time. When $k^2 \ll z''/z$, the solution of Eq.11 brings

$$ \mathcal{P}_\zeta^{1/2} \sim \sqrt{k^2} \frac{|u_k|}{z} \sim \frac{1}{|t_*|^3 \Lambda_*^2} \sim |H_*|, \quad (8) $$

since $|H_*| \sim 1/|t_*| \sim \Lambda_*$ and $a$ is constant. In certain sense, $\mathcal{P}^{1/2}$ is determined by $|H_*|$ around the exiting time is the reflection that on superhorizon scale $\zeta$ is increased, since when it leaves the horizon $|H|$ is quite small. The observations requires $|H_*| \sim \Lambda_* \sim 10^{-5}$, since $\mathcal{P}^{1/2} \sim 10^{-5}$, which implies that the scale around the exiting is about $10^{10}$Gev. This is not any finetuning.

| Nearly scale invariance of $\zeta$ (single field) |
|-----------------------------------------------|
| $|e|$ is slowly changed | $a$ is slowly changed |
| $a \sim t^n (n > 1)$ | $|e| \sim \frac{1}{(t_* - t)^{4\epsilon}}$ (initially $|e| \simeq 0$) |
| $D$ | $a \sim (t_* - t)^4$ | $|e| \sim (t_* - t)^4$ (initially $|e| \gg 1$) |

**TABLE I:** The possibilities of nearly scale invariance of $\zeta$ for single field. The $C$ or $D$ denotes that $\zeta$ is dominated by its constant mode or increasing mode. When $|e|$ is slowly changed, $a \sim t^n (n > 1)$ is that of inflation [4, 5, 6, 7], while $a \sim (t_* - t)^4$ is that of the contraction with $\epsilon \simeq 1.5$ or $\omega \simeq 0$. When $a$ is slowly changed, $|e| \sim \frac{1}{(t_* - t)^{4\epsilon}}$ implies that $|e| \sim 0$ initially and $|e| \gg 1$ around the ending since $t < 0$, which is that of adiabatic ekpyrosis given in [18, 19], while $|e| \sim (t_* - t)^4$ implies that $|e| \gg 1$ initially and $|e| \approx 1$ around the ending, which is that given in this paper. These listed here, for constant $c_0$, might be the simplest possibilities obtaining scale invariant $\zeta$. In principle, both $a$ and $|e|$ can be changed, however, the case is slightly intractable.

![FIG. 1: The black lines are the evolutions of $a$, and the solid line and dashed line correspond to the slowly expanding and slowly contracting, respectively. The dark yellow line is that of $1/|H|$. The lines are plotted with Eqs.(5) and (6), in which for simplicity $\Lambda = |t_*| = 1$ is set and the initial time is $t \approx 10t_*$. We can see that the change of $a$ is not negligible is only around $t \approx 2t_*$, at this epoch the exiting is assumed to occur. During this phase, due to the rapidly change of $H$, the perturbation mode initially inside the horizon, i.e. $\lambda \sim a \ll 1/|H|$, will naturally leave the horizon, i.e. $\lambda \gg 1/|H|$.](image)

We will calculate the perturbation of energy density, following [21]. The equation of metric perturbation is same as Eq.(11), however, here $u$ is defined as $H\sqrt|e| u_k \simeq \Phi_k$ and $z$ is replaced with $\theta = 1/z$. Thus the solution is $u_k \sim \theta \int \frac{H}{\sqrt|e|} \frac{dz}{z} \approx \Phi_k$ for $k^2 \ll \theta'/\theta$. That of increasing mode is $u \sim \theta$, since $a$ is hardly changed and $z \sim \sqrt{|e|} \sim \Lambda^5(t_* - t)^2$. Thus we have

$$ \Phi_k \simeq \frac{H\sqrt|e|}{z} \sim H. \quad (9) $$

The amplitude of the energy density perturbation on
large scale is given by \((\delta \rho/\rho)_k \simeq \Phi_k + \dot{\Phi}_k/H \sim \dot{\Phi}_k/H\), since here \(\Phi_k \ll \dot{\Phi}_k/H\) for \(\Lambda^4(t_* - t)^4 \ll 1\). Thus when the slow expanding phase ends, \(|H_*| \simeq 1/t_* \simeq \Lambda_*\), we approximately have

\[
\left(\frac{\delta \rho}{\rho}\right)_k \sim \frac{1}{(t_* - t)} \simeq |H_*|, \tag{10}
\]

which is consistent with [8]. In [21], it has been pointed that the perturbations of the energy density in [18] can be too large invalidating the use of the perturbation theory. The reason is that the amplitude of the perturbation after leaving the horizon is constant, which is too small to be responsible for the observations, thus an astronomically large value of parameter \(c\) is required to uplift this amplitude [18]. However, here since the amplitude of the perturbation is increasing, at the end time of slow expansion a suitable amplitude of perturbation can be naturally obtained.

The equation of motion of the tensor perturbation \(h_k\) is that in Eq. (11) with the replacements of \(u_k \simeq a h_k\) and \(z = a\), in which \(h_k\) is the \(k\) mode of the tensor perturbation \(h_{ij}\), which satisfies \(\delta^{ij} h_{ij} = 0\) and \(\partial_i \partial_j h_{ij} = 0\), here \(i\) and \(j\) are the spatial index. When the expansion or contraction is slow, the dominated term of \(a''/a\) is

\[
a'' \sim \frac{1}{\Lambda^4(t_* - t)^6} \sim \mathcal{O}(0) \sim \frac{\mathcal{O}(0)}{(n_* - n)^2} \tag{11}
\]

for \(|t| \gg |t_*|\). Thus \(\mathcal{P}^{1/2}_T \sim k\) is quite blue, which implies that the amplitude of tensor perturbation is negligible on large scale. This is the universal character of the slowly evolving background, e.g. [12]. Thus the detection of tensor perturbation is significant for falsifying the slowly expanding or contracting model.

We will briefly conceive how the required background evolution might be obtained. We begin with the action with the negative potential \(\Lambda^4 \phi^3/M^3\), which will be expected to bring the evolution of the slowly contracting, and the action with negative kinetic energy and positive potential \(\Lambda^4 \phi^3/M^3\), which will be expected to bring that of the slowly expanding, in which \(M\) and \(\Lambda\) are the constants of the mass dimensions. The slowly change of \(a\) requires that initially \(|\epsilon| \gg 1\), which implies \(\rho \ll |P|\) is negligible. \(\rho \simeq 0\) brings \(\dot{\phi}^2 \simeq \Lambda^4 (\phi/M)^3\). Thus \(\phi \simeq \frac{4M^3}{\Lambda^4 (t_* - t)^2}\) is obtained. Thus

\[
\dot{\phi}^2 \simeq \Lambda^4 \left(\frac{\phi}{M}\right)^3 \sim \frac{64M^6}{\Lambda^8 (t_* - t)^6}, \tag{12}
\]

which is increased since initially \(t \ll -1\). We have

\[
\dot{H} \sim \frac{M^6}{\Lambda^6 (t_* - t)^6}, \tag{13}
\]

since \(\dot{H} \simeq -P\), which by the integral will induce Eq. (5), in which \(\Lambda_*^4 \sim M^6/\Lambda^8\). This result implies that, for the slowly contracting or expanding, we have to add a term like \((\partial \phi)^{2/3} \Box \phi\) in the action, which will assure

\[
\rho \sim H^2 \sim \frac{1}{(t_* - t)^{10}} \tag{14}
\]

is negligibly small but increased, since in this case \(\rho \simeq H \dot{\phi}^{1/3} \sim \frac{1}{t_*^{1/3}}\). Thus, \(\phi\) might be a Galileon [22], see its nontrivial generalization, e.g. kinematic braiding [22] or [24]. \(|\epsilon| \simeq \frac{t^6 (t_* - t)^4}{\Lambda^4} \gg 1\) is gradually decrease. When \(\epsilon \simeq \mathcal{O}(1) t_*\), \(|\epsilon| \simeq 1\), and \(\rho\) has become not negligible. Thus [13] is not any more right around this epoch, which signals the end of the slowly contracting or expanding phase. Thus in principle, we can design the required background evolution.

In certain sense, that of slow contraction might be only a simple change of the adiabatic ekpyrotic scenario, in which a different time dependence of \(\epsilon\) is selected. Its jointing with the standard cosmology requires a bounce mechanism, like in ekpyrotic scenario [11], or [22, 26], or quintom bounce [27]. However, that of slow expansion is completely different, in which the bounce is not required, and the jointing of the slowly expanding with standard cosmology is only simply reheating, like in phantom inflation [28], since the universe expands all along. However, since \(\epsilon < 0\), there is ghost instability for \(\zeta\). However, it can be thought that the evolution with \(\epsilon < 0\) might be only the approximative simulation of a fundamental theory below certain physical cutoff during certain period, which is generally not Lorentz invariant [30], and the full action should be ghost free. We will provide the details of the model building of slowly expanding scenario in the coming work [29], in which the evolution of background satisfies the required conditions and \(c_s^2\) is nearly constant, and there is not the ghost instability.

In conclusion, we have brought a possibility generating the scale invariant spectrum, by which a viable scenario of early universe might be implemented. In general, for single field, the scale invariant spectrum of curvature perturbation can be given by either its constant mode or its increasing mode. When \(|\epsilon|\) is rapidly changed while \(a\) is slowly expanding or contracting, the scale invariant spectrum of curvature perturbation can be induced by its increasing mode. The perturbation mode during this slowly evolving can be naturally extended out of horizon, which is distinguished with that of adiabatic ekpyrosis [18], in which the spectrum of curvature perturbation is given by its constant mode. Here \(c_s\) is constant. This work is supported in part by NSFC under Grant No:10775180, 11075205, in part by the Scientific Research Fund of GU-CAS(NO:055101BM03), in part by National Basic Research Program of China, No:2010CB832804.
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