Weak gravitational lensing by Einstein-non-linear-Maxwell-Yukawa black hole

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In this article, we analyze the weak gravitational lensing in the context of Einstein-non-linear Maxwell-Yukawa black hole. To this desire, we derive the deflection angle of light by Einstein-non-linear Maxwell-Yukawa black hole using the Gibbons and Werner method. For this purpose, we obtain the Gaussian curvature and apply the Gauss-Bonnet theorem to find the deflection angle of Einstein-non-linear Maxwell-Yukawa black hole in weak field limits. Moreover, we derive the deflection angle of light in the influence of plasma medium. We also analyze the graphical behavior of deflection angle by Einstein-non-linear Maxwell-Yukawa black hole in the presence of plasma as well as non-plasma medium.

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I. INTRODUCTION

In our universe the black holes (BHs) are essential components and when stars are died, they become very dense objects which is the most important discoveries of astrophysics. To test the fundamental laws of universe black holes provide a golden opportunity. For example neutron star merges and the gravitational waves from black holes has been discover recently. Therefore, the gravitational lensing of black holes has attain incredible attention in the last few decade, essentially because of the solid proof of supermassive black holes at the focal point of galaxies [1, 2]. By utilizing the gravitational lensing we can simplified the study of black holes, which is a general investigative method for acquiring the time delays of the images, magnification and positions. Darwin [3] was the first who analyses the Schwarzschild geometry. After that in 1985 Herschel [4] published his similar article and after that many authors [5, 6], extended this geometry to general spherically symmetric black holes and Reissner-Nordstrom geometry. Kerr black holes [7]-[10] was also discussed for acquiring the time delays of the images, magnification and positions by using gravitational lensing. Many modification has been done through deflection of light and modification in the reference of non-linear electrodynamics (NLE) [11] has been studied through the various alternative gravity theories [12]. Classically, a black hole contains singularity and also horizon because in general theory of relativity the spacetime singularities create a dozen of problems which are physical as well as mathematical. Therefore many people use various methods to remove these singularities from the black holes like modified gravity, effect of quantum gravity and NLE. Bozza [13] discussed these topics in his recent article in details which also include observational prospectus and additional references [14]-[20].

The main focus of this article is to calculate the deflection angle of photon by using GBT. The aspect of deflection of light has been widely studied in different astrophysical system in the influence of strong as well as weak gravitational lensing [18]-[21]. Gibbons and Werner [22] made a very important role who contended about the significance of topological effect on the deflection of angle by utilizing the GBT and optical geometry. Additionally, they have calculated deflection angle for the schwarzschild black holes which is different from the other method by supposing light ray by taking domain outside, where the mass is closed in the given area on space is strongly related to the lensing effect. Recently, Gibbons and Werner (GW) [22] computed the deflection angle by applying GBT as follows

$$
\sigma = -\int \int_{D_{\infty}} K dS.
$$

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Here, $\mathcal{K}$ is Gaussian curvature and $dS$ represents the surface element. Afterwards, Werner [23] extend this method and obtained the deflection of light by Kerr-Newman BHs by applying the Nazim’s method for Randers-Finsler metric. Recently, deflection angle for an asymptotically flat and spherically symmetric spacetime by using the finite distance from an observer to a light source has been calculated by Ishihara et al. [24]-[26]. Moreover, Asada et al. [27] in stationary axisymmetric spacetime have calculated the weak gravitational lensing. In all of these methods by using GBT they have calculated the weak gravitational lensing which shows the global properties. The study of gravitational lensing in the presence of plasma medium discussed in number of cases. Initially, Bisnovatyi-kogan [28]-[29] show that the gravitational deflection in plasma is different from vacuum deflection angle due to presive properties of plasma. Recently, Gallo and Crisnejo [30] discussed the deflection angle of photon in a plasma medium.

In Einstein-Maxwell theory (EMT) for hairy BH in the context of Weak gravitational lensing with a non-minimally coupled dilaton has been examined by Javed et al. [31]. After that, there is a lot of literature [32]-[75] related to the investigation of weak gravitational lensing through the method of GBT on various black holes, cosmic strings and wormholes. In this paper, we study the weak gravitational lensing by Einstein-non-linear Maxwell-Yukawa black holes.

This paper is composed as; In Section 2, we concisely review about Einstein-non-linear Maxwell-Yukawa black hole. In Section 3, by using the Gauss-Bonnet theorem we compute the deflection angle of Einstein-non-linear Maxwell-Yukawa black hole. In Section 4, we work to investigate the deflection angle in the influence of plasma medium. We also demonstrate the graphical effect of deflection angle in the context of Einstein-non-linear Maxwell-Yukawa black hole for plasma and also for non-plasma medium. In Section 5, we present our result.

II. COMPUTATION OF WEAK LENSING BY EINSTEIN-NON-LINEAR MAXWELL-YUKAWA BLACK HOLE AND GAUSS-BONNET THEOREM

The Einstein-non-linear Maxwell-Yukawa BH in a static and spherically symmetric form is given as [76]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

The metric function $f(r)$ yields

$$f(r) \simeq 1 + \frac{2M}{r} - \frac{4qC_0}{r^2} + \frac{4qC_0\alpha}{3r} - qC_0\alpha^2 + O(\alpha^3). \quad (2)$$

Here, $M$ rendered as mass of BH, $C_0$ is an integration constant [77], $q$ represents charge of BH that is located at the origin and $\alpha$ is a positive constant and it can be chosen as $\alpha = 1$.

The optical spacetime simply written as

$$dt^2 = \frac{dr^2}{f(r)} + \frac{r^2d\phi^2}{f(r)} \quad (3)$$

From Eq. 1 we obtain non-zero Christopher symbols given below

$$\Gamma^0_{00} = -\frac{f'(r)}{f(r)}, \quad \Gamma^0_{11} = \frac{2f(r) - rf'(r)}{2rf(r)}, \quad \Gamma^1_0 = \frac{r^2f'(r) - f(r)2r}{2}. \quad (4)$$

Now, we calculate the Ricci Scalar corresponds to the optical metric by using the non-zero Christopher symbols that are stated as:

$$\mathcal{R} = \frac{-f'(r)^2}{2} + f''(r)f(r). \quad (5)$$

The Gaussian curvature that is computed as follows:

$$\mathcal{K} = \frac{\text{RicciScalar}}{2} \quad (6)$$

After simplifying, Gaussian optical curvature for Einstein-non-linear Maxwell-Yukawa black hole is stated as:

$$\mathcal{K} = -\frac{f'(r)^2}{4} + \frac{f(r)f''(r)}{2}, \quad (7)$$
where \( f(r) \) is given in Eq. 2 so that Gaussian optical curvature for Einstein-non-linear Maxwell-Yukawa black hole in leading order term is obtained as:

\[
K \approx -12 \frac{C_0 q}{r^4} + 4 \frac{C_0 q \alpha}{3r^3} + \left( 2r^{-3} - 24 \frac{C_0 q}{r^5} + 4 \frac{C_0 q \alpha}{r^4} \right) M.
\] (8)

III. DEFORMATION ANGLE OF EINSTEIN-NON-LINEAR MAXWELL-YUKAWA BLACK HOLES AND GAUSS-BONNET THEOREM

Now, by utilizing GBT, we will calculate the deflection angle of photon by Einstein-non-linear Maxwell-Yukawa BH. By using GBT in the region \( g_R \), given as

\[
\int \int_{g_R} KdS + \int_{\partial g_R} kdt + \sum_t \hat{\alpha} = 2\pi \mathcal{X}(g_R),
\] (9)

where, \( k \) represent the geodesic curvature, \( K \) denotes the Gaussian curvature and one can define \( k \) as \( k = \hat{g}(\nabla \hat{\alpha}, \hat{\alpha}) \) in that way \( \hat{g}(\hat{\alpha}, \hat{\alpha}) = 1 \), where \( \hat{\alpha} \) represent the unit acceleration vector and \( \alpha_t \) denotes the exterior angle at \( t^{th} \) vertex respectively. As \( R \to \infty \), we obtain the jump angles approximate to \( \pi/2 \). Thus \( \alpha_O + \alpha_S \to \pi \). Here, \( \mathcal{X}(g_R) = 1 \) is a Euler characteristic number and \( g_R \) denotes the non-singular domain. Therefore, we obtain

\[
\int \int_{g_R} KdS + \int_{\partial g_R} kdt + \hat{\alpha} = 2\pi \mathcal{X}(g_R).
\] (10)

where, the total jump angle is \( \hat{\alpha} = \pi \). When \( R \) approaching to infinity, then the remaining part is \( k(D_R) = | \nabla_{\partial g_R} \hat{D}_R | \).

For geodesic curvature the radial component is described as:

\[
(\nabla_{\partial g_R} \hat{D}_R)^\rho = \hat{D}_R^\phi \partial_\phi \hat{D}_R^\rho + \Gamma^\rho_{\phi\phi}(\hat{D}_R^\phi)^2.
\] (11)

At \( R \) very high , \( D_R := r(\phi) = R = \text{const.} \) Thus, the first term of Eq. 11 vanishes and \( (\hat{D}_R^\phi)^2 = \frac{1}{f(r)^2} \). Recalling \( \Gamma^\nu_{\phi\phi} = \frac{2f(r) - rf'(r)}{2rf(r)} \), we get

\[
(\nabla_{\partial g_R} \hat{D}_R)^\rho \to \frac{-1}{R},
\] (12)

and \( k(D_R) \to R^{-1} \), so we write \( dt = Rd\phi \). Thus;

\[
k(D_R)dt = d\phi.
\] (13)

From the pervious results, we get

\[
\int \int_{g_R} Kds + \int_{\partial g_R} kdt \to R \to \infty \int \int_{S_{\infty}} KdS + \int_0^{\pi + \sigma} d\phi.
\] (14)

At \( 0^{th} \) order weak field deflection limit of the light is defined as \( r(t) = b/\sin \phi \). Hence, the deflection angle given as:

\[
\sigma = -\int_0^\pi \int_{b/\sin \phi} K \sqrt{detg} dud\phi,
\] (15)

here we put the first terms of Eq. 8 into above Eq. 15, so we get the deflection angle upto leading order term is computed as:

\[
\sigma \approx \frac{4M}{b} + \frac{C_0 M \alpha q \pi}{b^2} + 8 \frac{C_0 q \alpha}{3b} - 3 \frac{C_0 q \pi}{b^2} - 32 \frac{C_0 M q}{3 b^3}.
\] (16)

Note that the solution (16) with \( \alpha = q = 0 \) reduces to deflection angle of Schwarzschild BH in the leading order terms. Moreover, it can be seen that the \( \alpha \) parameter increases the deflection angle.
IV. WEAK LENSING BY EINSTEIN-NON-LINEAR MAXWELL-YUKAWA BLACK HOLE IN A PLASMA MEDIUM

This section is based on the calculation of weak gravitational lensing by Einstein-non-linear Maxwell-Yukawa black hole in plasma medium. For Einstein non-linear Maxwell-Yukawa black hole the refractive index \( n(r) \), is obtain as,

\[
n(r) = \sqrt{1 - \frac{\omega_e^2}{\omega_\infty^2 f(r),}} \tag{17}
\]

where electron plasma frequency is \( \omega_e \) and photon frequency measured by an observer at infinity is \( \omega_\infty \) are and respectively, then the corresponding optical metric illustrated as

\[
dt^2 = g_{ij}^{opt} dx^i dx^j = n^2(r) \left[ -f(r) dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2) \right]. \tag{18}
\]

The metric function \( f(r) \) is defined in Eq. 2.4 and the non-zero christopher symbols corresponding to the optical metric are computed as:

\[
\begin{align*}
\Gamma^0_{00} &= \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2} \right) \left[ -\frac{\omega_e^2 f'(r)}{2\omega_\infty^2} \right] - \frac{f'(r)}{f(r),} \\
\Gamma^1_{01} &= \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2} \right) \left(\frac{\omega_e^2 f'(r)}{2\omega_\infty^2} \right) + \frac{2f(r) - rf'(r)}{2rf(r)}; \\
\Gamma^0_{11} &= \left(1 + \frac{\omega_e^2 f(r)}{\omega_\infty^2} \right) \left(\frac{\omega_e^2 f(r)f'(r)r^2}{2\omega_\infty^2} \right) + \frac{r^2f'(r) - f(r)2r}{2}.
\end{align*}
\tag{19}
\]

Now, by using the above non-zero christopher symbols the value of Gaussian optical curvature is found as:

\[
\mathcal{K} \approx 3 \frac{\omega_e^2 M}{r^3 \omega_\infty^2} + 2 \frac{M}{r^3} - 104 \frac{C_0 qM \omega_e^2}{\omega_\infty^2 r^3} - 24 \frac{C_0 qM}{r^5} - 20 \frac{C_0 q \omega_e^2}{\omega_\infty^2 r^4} - 12 \frac{C_0 q \omega_e^2 M}{\omega_\infty^2 r^4} + 4 \frac{q \alpha C_0 M}{r^4} + 2 \frac{q \alpha C_0 \omega_e^2}{r^3 \omega_\infty^2} + 4/3 \frac{q \alpha C_0}{r^3}. \tag{20}
\]

For this, we use GBT to compute the deflection angle in order to relate it with non-plasma. As follows, for calculating angle in the weak field limits, as the light beams become a straight line. Therefore, the condition at zero order is \( r = \frac{4M}{b} \). Then the GBT reduces to this simple form for calculating the deflection angle \( \sigma \):

\[
\sigma = -\lim_{R \to \alpha} \int_{\theta}^{\pi} \int_{\frac{\pi}{3} \theta}^{R} \mathcal{K}dS \tag{21}
\]

So, the deflection angle of Einstein-non-linear Maxwell-Yukawa BH in the presence of plasma medium is obtained as follows:

\[
\sigma \approx \frac{4M}{b} + 4 \frac{q \alpha C_0 M \omega_e^2 \pi}{b^2 \omega_\infty^2} + \frac{C_0 qM \alpha \pi}{b^2} + 4 \frac{C_0 q \alpha \omega_e^2}{b^2 \omega_\infty^2} + 8/3 \frac{qC_0 \alpha}{b} - 5 \frac{C_0 q \omega_e^2 \pi}{b^2 \omega_\infty^2} - 3 \frac{qC_0 \pi}{b^2} - \frac{416 qC_0 M \omega_e^2}{9 b^3 \omega_\infty^2} - \frac{32 C_0 qM}{3 b^3} + \frac{6 M \omega_e^2}{b \omega_\infty^2}. \tag{22}
\]

The above result tells us that photon rays are moving into medium of homogeneous plasma. One can see that the plasma effect can be removed by neglecting \( \frac{q \alpha}{b} \) term from Eq. 22 and it reduced into Eq. 16.

V. GRAPHICAL INFLUENCE OF DEFLECTION ANGLE ON EINSTEIN-NON-LINEAR MAXWELL-YUKAWA BLACK HOLE

This section of the paper comprises the graphical influence of deflection angle of Einstein-non-linear Maxwell-Yukawa BH. We examine the impact of different parameters on deflection angle. Here for simplicity we take \( C_0 = c \).
• Figure 1 Left Plot indicates the behavior of deflection angle w.r.t $b$ by fixing the value of $M,c,\alpha$ and varying $q$. It is to be observed that for small constant value of $M$ and $q \geq 0$ the behavior of deflection angle gradually decreasing with respect to impact parameter $b$. For increasing value of $q$, deflection angle increases.

• Figure 1 Right Plot represents the behavior of deflection angle w.r.t $b$ by varying the mass $M$ and taking $q,c,\alpha$ fixed. We noticed that for values of $M \geq 0$ the behavior of deflection angle gradually positively decreasing with respect to $b$. But for the increasing values of $M$, the behavior of the deflection angle is increasing.

• Figure 1(a) Left Plot indicates the behavior of deflection angle w.r.t $b$ by fixing the value of $\alpha$ and varying $q,c,M$. It is to be observed that for decreasing the value of $\alpha < 0$ the deflection angle gradually decreasing. On the other hand, the deflection angle is increasing gradually with respect to $b$ for fixed $\alpha$.

• Figure 1(a) Right Plot represents the behavior of deflection angle w.r.t $b$ by varying the $c$ and taking $q,\alpha,M$ fixed. We noticed that for values of $b > 5$ and $c \geq 0$ the behavior of deflection angle gradually positively decreasing. For the increasing the value of $c$, the behavior of deflection angle is negatively decreasing.

VI. GRAPHICAL ANALYSIS FOR PLASMA MEDIUM

This section gives us graphical analysis of deflection angle $\alpha$ in the presence of plasma medium. For simplicity we take $C_0 = c$ in the Eq. 22.
• **Figure 2 Left Plot**, shows the behavior of deflection angle w.r.t \( b \) by taking the fixed values of \( M, q, c, \alpha \) and by varying \( \beta \). It is to analyzed that initially the behavior of deflection angle is positively decreasing for \( \beta \geq 0 \). Furthermore, for the increasing value of \( \beta \) the deflection angle positively increases for fixed \( b \).

• **Figure 2 Right Plot**, illustrate the behavior of deflection angle w.r.t \( b \) by varying BH charge \( q \) and for fixed \( M, \alpha, c \) and \( \beta \). We examined that for positive values of \( q \) the deflection angle gradually positively decreasing. While, for increasing values of \( q \), the behavior of deflection angle is gradually increasing for fixed \( b \).

• **Figure 2(a) Left Plot**, examines the behavior of deflection angle w.r.t \( b \) by taking the fixed values of \( \beta, q, c, M \) and by varying \( \alpha \). We noticed that for small values of \( \alpha \) and \( \beta = 1, q = 1 \) then the deflection angle cannot define the behavior. While, for large values of \( \alpha \) the behavior of deflection angle is positively decreasing. For values of \( \alpha < 0 \) the behavior of deflection angle gradually increasing.

• **Figure 2(a) Right Plot**, we analyzed the behavior of deflection angle w.r.t \( b \) by varying \( c \) and for fixed \( M, \alpha, q \) and \( \beta \). we see that for \( c \geq 0 \) the deflection angle positively decreasing. For increasing values of \( c \), deflection angle is increasing for fixed \( b \).
\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3}
\caption{$\sigma$ versus $b$.}
\end{figure}

- Figure 3, manifest the influence of deflection angle w.r.t $b$ for constant values of $q, \beta, c, \alpha$ and varying $M$. It has been noted that for small range of $b < 15$ the behavior of deflection angle not defined. On the other hand for large $b \geq 15$ and for $M \geq 0$ the deflection angle positively decreasing and the behavior of deflection angle is increasing for increasing values of $M$ at fixed $b$.

VII. CONCLUSION

The present paper is about the investigation of deflection angle by Einstein-non-linear Maxwell-Yukawa BH in non-plasma as well as plasma medium. In this regard, we study the weak gravitational lensing by using GBT and obtain the deflection angle of photon for Einstein-non-linear Maxwell-Yukawa BH. The obtained deflection angle is given in Eq. 16 as follows:

$$\sigma \simeq \frac{4M}{b} + \frac{C_0 M \alpha q \pi}{b^2} + \frac{8}{3} \frac{C_0 q \alpha}{b} - \frac{3}{2} \frac{C_0 q \pi}{b^2} - \frac{32 C_0 M q}{3 b^3} + O(q^2, M^2, C_0^2).$$

We examine that by the reduction of some parameters the obtained deflection angle converted into the Schwarzschild deflection angle up to the first order terms. We also discuss the graphical effect of different parameters on deflection angle by Einstein-non-linear Maxwell-Yukawa BH. We also observed that deflection angle in the presence of plasma medium given by Eq. 22 which is;

$$\sigma \simeq \frac{4M}{b} + \frac{4 q \alpha C_0 M \omega_e^2 \pi}{b^2 b \omega_\infty^2} + \frac{C_0 q M \alpha \pi}{b^2} + \frac{4 q \alpha C_0 \omega_e^2}{b^2 b \omega_\infty^2} + \frac{8}{3} \frac{qC_0 \alpha}{b} - \frac{5 C_0 q \omega_e^2 \pi}{b^2 b \omega_\infty^2} - \frac{3 qC_0 \pi}{b^2} - \frac{416 q C_0 M \omega_e^2}{9 b^3 b \omega_\infty^2} - \frac{32 C_0 qM}{3 b^3} + \frac{6 \omega_e^2}{b \omega_\infty^2} + O(q^2, M^2, C_0^2, \omega_e^3).$$

When the $\omega_\infty$ approaches to zero, the plasma effect are removed. Furthermore, we scrutinized the graphical impact of deflection angle on Einstein-non-linear Maxwell-Yukawa BH in plasma medium versus some parameters. One can conclude that the deflection of photon is found outside of the lensing area which points that the gravitational lensing effect is a global and even topological effect. Hence, it is our belief that studying on the Virbhadra-Ellis lens equation, can provide us to improve the computation of the image positions, Einstein ring radii, magnification factors and the magnification ratio. This is going to be our next problem in the near future. Moreover, it will be interesting to investigate the deflection angle of black holes in MAXWELL $f(R)$ gravity theories and in fourth order gravity theories using the GBT in future [78–80].

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