On the Conformal change of a Douglas space of second kind with special \((\alpha, \beta)\)-metric

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Abstract

The notion of a Douglas space of second kind of a Finsler space with \((\alpha, \beta)\)-metric was introduced by I. Y. Lee \(^9\). Since then, so many geometers have studied this topic e. g., \(^{14}\). In this paper, we prove that a Douglas space of second kind with special \((\alpha, \beta)\)-metric \(\alpha + \epsilon \beta + k \frac{\beta^2}{\alpha}\) is conformally transformed to a Douglas space of second kind. Further, we obtain some results which prove that a Douglas space of second kind with certain \((\alpha, \beta)\)-metrics such as Randers metric, Kropina metric, first approximate Matsumoto metric and Finsler space with square metric is conformally transformed to a Douglas space of second kind.

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1 Introduction

The notion of Douglas space was introduced by S. Bacso and M. Matsumoto [3] as a generalization of Berwald space from viewpoint of geodesic equations. Also, they consider the notion of Landsberg space as a generalization of Berwald space. Recently, the notion of weakly-Berwald space as another generalization of Berwald space was introduced by S. Bacso and B. Szilagyi [4]. It is remarkable that a Finsler space is a Douglas space if and only if the Douglas tensor $D^i_{hjk}$ vanishes identically [5].

The theories of Finsler spaces with an $(\alpha, \beta)$-metric and Berwald spaces with an $(\alpha, \beta)$-metrics ([1], [11], [15]) have contributed a lot to the development of Finsler geometry [12]. The conformal theory of Finsler spaces was introduced by M. S. Knebman [7] in 1929 and this theory has been investigated in detail by M. Hashiguchi [6]. Later on Y.D. Lee [10] and B. N. Prasad [16] found conformally invariant tensors in the Finsler space with $(\alpha, \beta)$-metric under conformal $\beta$-change.

The purpose of the present paper is to prove that a Douglas space of second kind with a special $(\alpha, \beta)$-metric given by $L = \alpha + \epsilon \beta + k \beta^2$, where $\epsilon$ and $k$ are constants is conformally transformed to a Douglas space of second kind. Further, we obtain some results which prove that the Douglas space of second kind with certain $(\alpha, \beta)$-metrics such as Randers metric, Kropina metric, first approximate Matsumoto metric and Finsler space with square metric is conformally transformed to a Douglas space of second kind.

2 Preliminaries

A Finsler space $F^n = (M^n, L(\alpha, \beta))$ is said to be with an $(\alpha, \beta)$-metric, if $L(\alpha, \beta)$ is a positively homogenous function of $\alpha$ and $\beta$ of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ and $\beta = b_i(x)y^i$. The space $R^n = (M^n, \alpha)$ is called the Riemannian space associated with $F^n$. We shall use the following symbols [12]:

$$b^i = a^{ir}b_r, \quad b^2 = a^{rs}b_rb_s$$
$$2r_{ij} = b_{ij} + b_{ji}, \quad 2s_{ij} = b_{ij} - b_{ji},$$
$$s^i = a^{ir}s_{rj}, \quad s^i = a^{ir}s_{rj}, \quad s_j = b_{r}s^r_{,j}.$$
The Berwald connection $B \Gamma = \{G^i_{jk}, G^j_{ik}\}$ of $F^n$ plays an important role in the present paper. Denote by $B^i_{jk}$, the difference tensor of $G^i_{jk}$ from $\gamma^i_{jk}$:

$$G^i_{jk}(x, y) = \gamma^i_{jk}(x) + B^i_{jk}(x, y).$$  \hspace{0.5cm} (1)

With the subscript 0, transvecting by $y^i$, we have

$$G^i_j = \gamma^i_{0j} + B^i_j \quad \text{and} \quad 2G^i = \gamma^i_{00} + 2B^i,$$  \hspace{0.5cm} (2)

and then $B^i_j = \partial_j B^i$ and $B^i_{jk} = \partial_k B^i_j$.

The geodesics of an n-dimensional Finsler space $F^n = (M^n, L)$ are given by the system of differential equations [6]

$$\frac{d^2x^i}{dt^2} - \frac{d^2x^i}{dt^2} y^i + 2(G^i_j y^j - G^j_i y^i) = 0; \quad y^i = \frac{dx^i}{dt},$$  \hspace{0.5cm} (3)

in a parameter $t$. The spray function $G^i(x, y)$ is given by

$$2G^i(x, y) = g^{ij}(y^r \partial_j \partial_r F - \partial_j F) = \gamma^i_{jk} y^j y^k,$$  \hspace{0.5cm} (4)

where $\partial_i = \frac{\partial}{\partial x^i}$, $F = \frac{L^2}{2}$, $\gamma^i_{jk}$ are the Christoffel symbols constructed from $g_{ij}(x, y)$ with respect to $x^i$ and $g^{ij}(x, y)$ is the inverse of the metric tensor $g_{ij}(x, y)$.

It is well known [3] that a Finsler space $F^n$ becomes a Douglas space if and only if the Douglas tensor

$$D^h_{ijk} = G^h_{ijk} - \frac{1}{n+1} (G_{ijk} y^h + G_{ij} \delta^h_k + G_{jk} \delta^h_i + G_{ki} \delta^h_j),$$  \hspace{0.5cm} (5)

vanishes identically, where $G^h_{ijk} = \partial_k G^h_{ij}$ is the $\nu$-curvature tensor of the Berwald connection $B \Gamma$.

Further, a Finsler space $F^n$ is said to be a Douglas space [13], if

$$D^{ij} = G^i(x, y) y^j - G^j(x, y) y^i,$$  \hspace{0.5cm} (6)

are homogenous polynomials in $(y^i)$ of degree three.

Differentiating (3) by $y^m$ and contracting $m$ and $j$ in the obtained equation, we have

$$D^{im}_{m} = (n+1) G^i - G^m_{m} y^i.$$  \hspace{0.5cm} (7)

Thus a Finsler space $F^n$ becomes a Douglas space of the second kind if and only if (7) are homogenous polynomials in $(y^i)$ of degree two.
Definition 2.1. A Finsler space $F^n$ is said to be a Douglas space of second kind if $D_m^{im} = (n + 1)G^i - G^i_m y^i$ is a homogenous polynomial in $(y^i)$ of degree two.

On the other hand, a Finsler space with an $(\alpha, \beta)$-metric is said to be a Douglas space of second kind if and only if

$$B_m^i = (n + 1)B^i - B_m^i y^i,$$  

are homogenous polynomials in $(y^i)$ of degree two, where $B_m^i$ is given by [8]. Furthermore, differentiating the above with respect to $y^h$, $y^j$ and $y^k$, we get

$$B_{hjk}^i = B_{hjk}^i = 0.$$  

Definition 2.2. A Finsler space $F^n$ with $(\alpha, \beta)$-metric is said to be a Douglas space of second kind, if $B_m^i = (n + 1)B^i - B_m^i y^i$ is a homogenous polynomial in $(y^i)$ of degree two.

3 Douglas Space of second kind with $(\alpha, \beta)$-metric

In this section, we deal with the condition for a Finsler space with an $(\alpha, \beta)$-metric to be a Douglas space of second kind.

Let $G^i(x, y)$ be the spray function of a Finsler space $F^n$ with an $(\alpha, \beta)$-metric. According to [11], $G^i(x, y)$ is written in the form

$$2G^i = \gamma_{00} + 2B^i,$$

$$B^i = \frac{\alpha L^\beta}{L^\alpha} s^i_0 + C^* \left[ \frac{\beta L^\beta}{\alpha L^\alpha} y^i - \frac{\alpha L^\alpha}{\alpha L^\alpha} \left( \frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta} \right) \right],$$

where

$$C^* = \frac{\alpha (r_{00} L^\alpha - 2\alpha s_0 L^\beta)}{2(\beta^2 L^\alpha + \alpha \gamma^2 L_{\alpha\alpha})},$$

$$\gamma^2 = b^2 \alpha^2 - \beta^2.$$  

Since $\gamma_{00} = \gamma_{jk}(x)y^jy^k$ is hp(2), equation (10) yields

$$B_{ij} = \frac{\alpha L^\beta}{L^\alpha} (s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 L^\alpha}{\beta L^\alpha} C^* (b^i y^j - b^j y^i).$$

By means of (6) and (12), we have the following lemma [13]:

4
Lemma 3.1. A Finsler space $F^n$ with an $(\alpha, \beta)$-metric becomes a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are hp(3).

Differentiating (12) with respect to $y^h, y^k, y^p$ and $y^q$, we have $D^{ij}_{hkpq} = 0$, which are equivalent of $D^{im}_{hkpm} = (n + 1)D^{i}_{hk} = 0$. Thus, if a Finsler space $F^n$ satisfies the condition $D^{ij}_{hkpq} = 0$, we call it Douglas space. Further differentiating (12) by $y^m$ and contracting m and j in the obtained equation, we obtain

$$B^{im}_m = \frac{(n + 1)\alpha L_\beta s^i_0}{L_\alpha} + \frac{\alpha\{(n + 1)\alpha^2\Omega L_{aa}b^i + \beta\gamma^2 Ay^i\}r_{00}}{2\Omega^2} - \frac{\alpha^2\{(n + 1)\alpha^2\Omega L_\beta L_{aa}b^i + By^i\}s_0}{L_\alpha\Omega^2} - \frac{\alpha^3 L_{aa}y^i r_0}{\Omega}, \quad (13)$$

where

$$\Omega = (\beta^2 L_\alpha + \alpha \gamma^2 L_{aa}), \text{ provided that } \Omega \neq 0,$$

$$A = \alpha L_\alpha L_{aaa} + 3L_\alpha L_{aa} - 3\alpha(L_{aa})^2,$$

$$B = \alpha\beta \gamma^2 L_\alpha L_\beta L_{aa} + \beta\{(3\gamma^2 - \beta^2)L_\alpha - 4\alpha \gamma^2 L_{aa}\}L_\beta L_{aa} + \Omega LL_{aa}. \quad (14)$$

We use the following result [9]:

Theorem 3.1. The necessary and sufficient condition for a Finsler space $F^n$ with an $(\alpha, \beta)$-metric to be a Douglas space of second kind is that, $B^{im}_m$ are homogenous polynomials in $(y^m)$ of degree two, where $B^{im}_m$ is given by (13) and (14), provided that $\Omega \neq 0$.

4 Conformal change of Douglas space of second kind with $(\alpha, \beta)$-metric.

In the present section, we find the condition on conformal change, so that a Douglas space of second kind with $(\alpha, \beta)$-metric is conformally transformed to a Douglas space of second kind.

Let $F^n = (M^n, L)$ and $\tilde{F}^n = (M^n, \tilde{L})$ be two Finslers spaces on the same underlying manifold $M^n$. If we have a function $\sigma(x)$ in each coordinate neighbourhoods of $M^n$ such that $\tilde{L}(x,y) = e^\sigma L(x,y)$, then $F^n$ is called conformal to $\tilde{F}^n$ and the change $L \to \tilde{L}$ of metric is called a conformal change.
A conformal change of \((\alpha, \beta)\)-metric is given as \((\alpha, \beta) \rightarrow (\bar{\alpha}, \bar{\beta})\), where 
\[
\bar{\alpha} = e^{\sigma} \alpha \quad \text{and} \quad \bar{\beta} = e^{\sigma} \beta.
\]
Therefore, we have
\[
\bar{a}_{ij} = e^{2\sigma} a_{ij}, \quad \bar{b}_i = e^{\sigma} b_i \quad \text{(15)}
\]
and
\[
b^2 = a^{ij} b_i b_j = \bar{a}^{ij} \bar{b}_i \bar{b}_j. \quad \text{Thus, we state the following theorem for further use:}
\]

**Theorem 4.1.** A Finsler space with \((\alpha, \beta)\)-metric with the length \(b\) of \(b_i\) with respect to the Riemannian metric \(\alpha\) is invariant under any conformal change of \((\alpha, \beta)\)-metric.

From (16), it follows that the conformal change of Christoffel symbols is given by [6]:
\[
\bar{\gamma}^i_{jk} = \gamma^i_{jk} + \delta_j^i \sigma_k + \delta_k^i \sigma_j - \sigma^i a_{jk}, \quad \text{(17)}
\]
where \(\sigma_j = \partial_j \sigma\) and \(\sigma^i = a^{ij} \sigma_j\).

From (16) and (17), we have the following identities:
\[
\nabla_j \bar{b}_i = e^{\sigma} (\nabla_j b_i + \rho a_{ij} - \sigma_i b_j),
\]
\[
\bar{r}_{ij} = e^{\sigma} [r_{ij} + \rho a_{ij} - \frac{1}{2} (b_i \sigma_j + b_j \sigma_i)],
\]
\[
\bar{s}_{ij} = e^{\sigma} [s_{ij} + \frac{1}{2} (b_i \sigma_j - b_j \sigma_i)],
\]
\[
\bar{s}_j^i = e^{-\sigma} [s_j^i + \frac{1}{2} (b^i \sigma_j - b_j \sigma^i)],
\]
\[
\bar{s}_j = s_j + \frac{1}{2} (b^2 \sigma_j - \rho b_j), \quad \text{(18)}
\]
where \(\rho = \sigma_i b^i\).

From (17) and (18), we can easily obtain the following:
\[
\bar{\gamma}^{0i} = \gamma^{0i} + 2 \sigma_0 y^i - \alpha^2 \sigma_j, \quad \text{(19)}
\]
\[
\bar{r}_{00} = e^{\sigma} (r_{00} + \rho a^2 - \sigma_0 \beta), \quad \text{(20)}
\]
\[
\bar{s}_0^i = e^{-\sigma} [s_0^i + \frac{1}{2} (\sigma s_0 b^i - \beta \sigma^i)], \quad \text{(21)}
\]
\[
\bar{s}_0 = s_0 + \frac{1}{2} (\sigma_0 b^i - \rho \beta). \quad \text{(22)}
\]
Next, we find the conformal change of $B_{ij}$ given in (12). We have $ar{L}(\alpha, \beta) = e^\sigma L(\alpha, \beta)$, and

$$
\bar{L}_\alpha = L_\alpha, \quad \bar{L}_\alpha = e^{-\sigma} L_{\alpha\alpha}, \quad \bar{L}_\beta = L_\beta, \quad \bar{\gamma}^2 = e^{2\sigma} \gamma^2.
$$

(23)

By using (11), (22), (23) and theorem (3.1), we obtain

$$
C^* = e^\sigma (C^* + D^*),
$$

(24)

where

$$
D^* = \frac{\alpha\beta[(\beta\alpha^2 - \sigma_0\beta) L_\alpha - \alpha(b^2\sigma_0 - \rho\beta) L_\beta]}{2(\beta^2 L_\alpha + \alpha\gamma^2 L_{\alpha\alpha})}.
$$

(25)

Hence, under the conformal change $B_{ij}$ can be written as:

$$
B_{ij} = \frac{\alpha L_\beta (\sigma^0_i y^j - \sigma^0_j y^i)}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha} C^* (b^j y^i - b^i y^j)}{L_\alpha}
$$

$$
+ \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha} D^*}{L_\alpha}\right) (b^i y^j - b^j y^i) - \frac{\alpha^2 L_\beta (\sigma^i y^j - \sigma^j y^i)}{2L_\alpha},
$$

where

$$
C^{ij} = \left(\frac{\alpha\sigma_0 L_\beta}{L_\alpha} + \frac{\alpha^2 L_{\alpha\alpha} D^*}{L_\alpha}\right) (b^i y^j - b^j y^i) - \frac{\alpha^2 L_\beta (\sigma^i y^j - \sigma^j y^i)}{2L_\alpha}.
$$

From (14), we have

$$
\bar{\Omega} = e^{2\sigma} \Omega, \quad \bar{A} = e^{-\sigma} A, \quad \bar{B} = e^{2\sigma} B.
$$

(26)

Now, we apply conformation transformation to $B_{im}^{im}$, and obtain

$$
B_{im}^{im} = B_{im}^{im} + K_{im}^{im},
$$

(27)

where

$$
2K_{im}^{im} = \frac{(n + 1)\alpha L_\beta (\sigma_0 b^i - \beta \sigma^i)}{L_\alpha} + \alpha \left\{ \frac{(n + 1)\alpha^2 \Omega L_{\alpha\alpha} b^i + \beta \gamma^2 A y^i}{\Omega^2} \right\} (\rho \alpha^2 - \sigma_0 \beta)
$$

$$
- \left[ \alpha^2 \frac{(n + 1)\alpha^2 \Omega L_{\alpha\alpha} b^i + B y^i}{L_\alpha \Omega^2} - \frac{\alpha^3 L_{\alpha\alpha} y^i}{\Omega} \right] (b^2 \sigma_0 - \rho \beta).
$$

(28)

Thus, we have the following result:

**Theorem 4.2.** The necessary and sufficient condition for a conformal change of Douglas space of the second kind to be a Douglas space of second kind, is that $K_{im}^{im}(x)$ are homogenous polynomial in $(y^i)$ of degree two.
5 Conformal change of Douglas space of second kind with special \((\alpha, \beta)\)-metric \(L = \alpha + \epsilon \beta + k \frac{\beta^2}{\alpha}\)

Let us consider a Finsler space with special \((\alpha, \beta)\)-metric

\[
L = \alpha + \epsilon \beta + k \frac{\beta^2}{\alpha},
\]

where \(\epsilon\) and \(k\) are constants.

Then, from (29), we can easily find

\[
\begin{align*}
L_\alpha &= 1 - \frac{k \beta^2}{\alpha^2}, \\
L_\beta &= \epsilon + \frac{2k \beta}{\alpha}, \\
L_{\alpha\alpha} &= 2k \frac{\beta^2}{\alpha^3}, \\
L_{\alpha\alpha\alpha} &= -6k \frac{\beta^2}{\alpha^4}.
\end{align*}
\]

Hence, from (14), we have

\[
\begin{align*}
\Omega &= -3k \frac{\beta^4}{\alpha^2} + (1 + 2k b^2) \frac{\alpha^2 \beta^2}{\alpha^2}, \\
A &= \frac{-12k^2 \beta^4}{\alpha^5}, \\
B &= \frac{2k}{\alpha^6} \left\{ (1 + 2k b^2) \alpha^4 \beta^4 - 6\epsilon k b^2 \alpha^3 \beta^5 - 2k (2 + 7k b^2) \alpha^2 \beta^6 \\
&\quad + 6\epsilon k \alpha \beta^7 + 15k^2 \beta^8 \right\}.
\end{align*}
\]

Thus, \(K_m^{\text{inv}}\) in (28), reduces to

\[
2K_m^{\text{inv}} = \frac{(n + 1)(\epsilon \alpha^3 + 2k \alpha \beta)(\sigma_0 b^i - \beta \sigma^i)}{(\alpha^2 - k \beta^2)} + p_1 + p_2 + p_3 + p_4,
\]

(32)
where

\[ p_1 = \frac{(n+1)b^i}{\{k\beta^4 - (1 - 2kb^2)\alpha^2\beta^2\}^2} \left\{ 2k\rho(1 - 2kb^2)\alpha^6 \beta^4 - 2k^2\rho\alpha^4 \beta^6 - 12k^2\rho b^2\alpha^2 \beta^7 y^i \right. \\
- \left. (1 - 2kb^2)2k\sigma_0\alpha^4 \beta^5 + (\sigma_0 - 6\rho y^i)2k^2\alpha^2 \beta^7 + 12k^2\sigma_0 b^2\alpha^2 \beta^6 y^i - 12k^2\sigma_0 \beta^8 y^i \right\}, \]

\[ p_2 = \frac{-12k^2\beta^5\gamma^2(\rho\alpha^2 - \sigma_0\beta)y^i}{\{1 + 2kb^2\alpha^2\beta^2 - 3k\beta^4\}^2}, \]

\[ p_3 = \frac{2k(b^2\sigma_0\alpha^2 y^i - \rho\alpha^2 \beta y^i)}{(\alpha^2 - k\beta^2)\{(1 + 2kb^2)\alpha^2\beta^2 - 3k\beta^4\}^2} \left\{ (1 + 2kb^2)\alpha^4 \beta^4 - 6\epsilon kb^2\alpha^3 \beta^5 \\
- 2k(2 + 7kb^2)\alpha^2 \beta^6 + 6\epsilon k\alpha \beta^7 + 15k^2\beta^8 \right\}, \]

\[ p_4 = \frac{2kb^2\alpha^2 y^i\sigma_0 - 2k\rho \alpha^2 \beta y^i}{(1 + 2kb^2)\alpha^2 - 3k\beta^2}. \]

which shows that \( K_{im}^{km} \) is a homogenous polynomial in \((y^i)\) of degree two. Hence, we have the following theorem:

**Theorem 5.1.** A Douglas space of second kind with special \((\alpha, \beta)\)-metric \(L = \alpha + \epsilon \beta + k\beta^2\alpha\), where \(\epsilon\) and \(k\) are constants, is conformally transformed to a Douglas space of second kind.

From theorem 5.1, we can easily prove that a Douglas space of second kind for certain Finsler spaces with \((\alpha, \beta)\)-metric is conformally transformed to a Douglas space of second kind. We have the following cases.

**Case (i).** If \(\epsilon = 1\) and \(k = 0\), the special \((\alpha, \beta)\)-metric reduces to \(L = \alpha + \beta\), which is the well known Randers metric. In this case, \(2K_{im}^{km}\) reduces to

\[ 2K_{im}^{km} = (n+1)\alpha(\sigma_0 b^i - \beta \sigma^i), \quad (33) \]

which shows that \( K_{im}^{km} \) is a homogenous polynomial in \((y^i)\) of degree two.

Hence, we have the following:

**Corollary 5.1.** A Douglas space of second kind with Randers metric \(L = \alpha + \beta\), is conformally transformed to a Douglas space of second kind.
Case (ii). If $\epsilon = 0$ and $k = 1$, the special $(\alpha, \beta)$-metric reduces to $L = \alpha + \frac{\beta^2}{\alpha}$. In this case, $2K^m$ reduces to

$$2K^m = \frac{(n + 1)(2\alpha \beta)(\sigma_0 b^i - \beta \sigma^i)}{(\alpha^2 - \beta^2)} + q_1 + q_2 + q_3 + q_4, \quad (34)$$

where

$$q_1 = \frac{(n + 1)b^i}{\{\beta^4 - (1 - 2b^2)\alpha^2 \beta^2 \}^2} \left\{ 2\rho(1 - 2b^2)\alpha^6 \beta^4 - 2\rho \alpha^4 \beta^6 - 12\rho b^2 \alpha^2 \beta^7 y^i \\
- (1 - 2b^2)2\sigma_0 \alpha^4 \beta^3 + (\sigma_0 - 6\rho y^i)2\alpha^2 \beta^7 + 12\sigma_0 b^2 \alpha^2 \beta^6 y^i - 12\sigma_0 \beta^5 y^i \right\},$$

$$q_2 = \frac{-12\beta^5 \gamma^2 (\rho \alpha^2 - \sigma_0 \beta) y^i}{\{(1 + 2b^2)\alpha^2 \beta^2 - 3\beta^4 \}^2},$$

$$q_3 = \frac{2(b^2 \sigma_0 \alpha^2 y^i - \rho \beta \gamma^2 y^i)}{(\alpha^2 - \beta^2)\{(1 + 2b^2)\alpha^2 \beta^2 - 3\beta^4 \}^2} \left\{ (1 + 2b^2)\alpha^4 \beta^4 - 2(2 + 7b^2)\alpha^2 \beta^6 \\
+ 15\beta^8 \right\},$$

$$q_4 = \frac{2b^2 \alpha^2 y^i \sigma_0 - 2\rho \beta \gamma^2 y^i}{(1 + 2b^2)\alpha^2 - 3\beta^2}.$$

This shows that $K^m$ is a homogenous polynomial in $(y^i)$ of degree two. Hence, we have the following:

**Corollary 5.2.** A Douglas space of second kind with special $(\alpha, \beta)$-metric $L = \alpha + \frac{\beta^2}{\alpha}$ is conformally transformed to a Douglas space of second kind.

Case (iii). If $\epsilon = 1$ and $k = 1$, the special $(\alpha, \beta)$-metric reduces to the form $L = \alpha + \beta + \frac{\beta^2}{\alpha}$, which is the first approximate Matsumoto metric. In this case, $2K^m$ reduces to

$$2K^m = \frac{(n + 1)(\alpha^3 + 2\alpha \beta)(\sigma_0 b^i - \beta \sigma^i)}{(\alpha^2 - \beta^2)} + r_1 + r_2 + r_3 + r_4, \quad (35)$$
where

\[ r_1 = \frac{(n + 1)b^i}{\{3^4 - (1 - 2b^2)\alpha^2\beta^2\}^2} \left\{ 2\rho(1 - 2b^2)\alpha^6\beta^4 - 2\rho\alpha^4\beta^6 - 12\rho b^2\alpha^2\beta^7 y^i \right. \]

\[ \left. -(1 - 2b^2)2\sigma_0\alpha^4\beta^5 + (\sigma_0 - 6\rho y^i)2\alpha^2\beta^7 + 12\sigma_0 b^2\alpha^2\beta^6 y^i - 12\sigma_0\beta^8 y^i \right\}, \]

\[ r_2 = \frac{-12\beta^5\gamma^2(\rho\alpha^2 - \sigma_0\beta) y^i}{\{(1 + 2b^2)\alpha^2\beta^2 - 3\beta^4\}^2}, \]

\[ r_3 = \frac{2(b^2\sigma_0\alpha^2 y^i - \rho\alpha^2\beta y^i)}{\{(1 + 2b^2)\alpha^2\beta^2 - 3\beta^4\}^2} \left\{ (1 + 2b^2)\alpha^4\beta^4 - 6b^2\alpha^3\beta^5 \right. \]

\[ \left. -2(2 + 7b^2)\alpha^2\beta^6 + 6\alpha\beta^7 + 15\beta^8 \right\}, \]

\[ r_4 = \frac{2b^2\alpha^2 y^i\sigma_0 - 2\rho\alpha^2\beta y^i}{(1 + 2b^2)\alpha^2 - 3\beta^2}. \]

This shows that \( K_m^{im} \) is a homogenous polynomial in \( (y^i) \) of degree two. Hence, we have the following:

**Corollary 5.3.** A Douglas space of second kind with first approximate Matsumoto metric \( L = \alpha + \beta + \frac{\beta^2}{\alpha} \) is conformally transformed to a Douglas space of second kind.

**Case (iv).** If \( \epsilon = 2 \) and \( k = 1 \), the special \( (\alpha, \beta) \)-metric reduces to the form \( L = \frac{1}{\alpha^2} \), which is known as square metric. In this case, \( 2K_m^{im} \) reduces to

\[ 2K_m^{im} = \frac{(n + 1)(2\alpha^3 + 2\alpha\beta)(\sigma_0 b^i - \beta^i)}{(\alpha^2 - \beta^2)} + s_1 + s_2 + s_3 + s_4, \quad (36) \]

where

\[ s_1 = \frac{(n + 1)b^i}{\{3^4 - (1 - 2b^2)\alpha^2\beta^2\}^2} \left\{ 2\rho(1 - 2b^2)\alpha^6\beta^4 - 2\rho\alpha^4\beta^6 - 12\rho b^2\alpha^2\beta^7 y^i \right. \]

\[ \left. -(1 - 2b^2)2\sigma_0\alpha^4\beta^5 + (\sigma_0 - 6\rho y^i)2\alpha^2\beta^7 + 12\sigma_0 b^2\alpha^2\beta^6 y^i - 12\sigma_0\beta^8 y^i \right\}, \]

\[ s_2 = \frac{-12\beta^5\gamma^2(\rho\alpha^2 - \sigma_0\beta) y^i}{\{(1 + 2b^2)\alpha^2\beta^2 - 3\beta^4\}^2}, \]

\[ s_3 = \frac{2(b^2\sigma_0\alpha^2 y^i - \rho\alpha^2\beta y^i)}{\{(1 + 2b^2)\alpha^2\beta^2 - 3\beta^4\}^2} \left\{ (1 + 2b^2)\alpha^4\beta^4 - 6b^2\alpha^3\beta^5 \right. \]

\[ \left. -2(2 + 7b^2)\alpha^2\beta^6 + 6\alpha\beta^7 + 15\beta^8 \right\}, \]

\[ s_4 = \frac{2b^2\alpha^2 y^i\sigma_0 - 2\rho\alpha^2\beta y^i}{(1 + 2b^2)\alpha^2 - 3\beta^2}, \]
which shows that $K^i_m$ is a homogenous polynomial in $(y^i)$ of degree two.
Hence, we have the following:

**Corollary 5.4.** A Douglas space of second kind with square metric $L = (\frac{(\alpha+\beta)^2}{\alpha})$ is conformally transformed to a Douglas space of second kind.

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