Exact Fermi-edge singularity exponent in a Luttinger liquid

Andrei Komnik and Reinhold Egger
Fakultät für Physik, Albert-Ludwigs-Universität, Hermann-Herder-Straße 3, D-79104 Freiburg, Germany

Alexander O. Gogolin
Department of Mathematics, Imperial College, 180 Queen’s Gate, London SW7 2BZ, United Kingdom

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I. INTRODUCTION

The singular response of conduction electrons in metals to a transient potential is a centerpiece of condensed-matter theory. It is related to basic phenomena such as Anderson’s orthogonality catastrophe,\(^1\) the Fermi-edge singularity in x-ray absorption spectra,\(^2\)\(^3\)\(^4\) and the Kondo effect.\(^5\) While the response of conduction electrons to a time-dependent perturbation is well understood by now for the case of uncorrelated conduction electrons, it is both of fundamental and practical interest to gain insight into the behavior of correlated electrons under transient potentials. The Luttinger liquid model for correlated one-dimensional (1D) electrons\(^6\) provides the perhaps simplest case study for such an investigation. The correlation degree is measured by a single dimensionless interaction strength \(g\); for repulsive Coulomb interactions, one has \(g < 1\). Due to recent fabrication advances, experimental applications for the Fermi-edge singularity in a Luttinger liquid have emerged in the 1D quantum wires in semiconductor heterostructures.\(^7\)\(^8\)

The Fermi-edge singularity reflects the instability of the electron sea to the local core-hole potential generated by an x-ray absorption process.\(^9\)\(^10\) Quantities of particular interest are, e.g., the overlap integral between the ground-state wavefunctions \(|f\rangle\) with and without the perturbation, or the core-hole Green’s function \(D(t) = \langle i | d(t) d(0) | i \rangle\), which is connected to the x-ray photoemission spectrum. Here, the operator \(d\) empties the core-hole level. As discussed below, the overlap integral and the core-hole Green’s functions are closely related quantities, and both are characterized by the same universal exponent \(\alpha\). This exponent enters the x-ray absorption rate for frequencies very close to the threshold frequency. The exponents of other response functions of interest\(^11\)\(^12\) can be related to \(\alpha\) and will not be discussed in what follows.

There have been several previous works related to the Fermi-edge singularity and the orthogonality catastrophe in a Luttinger liquid\(^13\)\(^14\)\(^15\)\(^16\). The forward-scattering contribution due to a transient potential was evaluated in Refs.\(^14\)\(^17\)\(^18\) and since the absorption rate factorizes into forward- and backward-scattering parts,\(^13\)\(^19\) we here focus on the more interesting backwardscattering contribution solely. Essentially based on the assumption of an open boundary fixed point\(^18\) the Fermi-edge singularity exponent \(\alpha = 1/8\) has been found for all \(g < 1\) by various techniques in Refs.\(^14\)\(^17\)\(^19\)\(^20\). The assumption of an open boundary fixed point\(^18\) has recently been questioned by Oreg and Finkel’stein\(^21\) who find the result \(\alpha = 0\) instead. This discrepancy has also been addressed in a recent numerical work.\(^22\) Here we decide the issue by presenting an exact derivation of the exponent \(\alpha = 1/8\).

Our calculation is performed at the special interaction strength \(g = 1/2\). This suffices to resolve the controversy about the open boundary fixed point. At the special point \(g = 1/2\), the bosonized Luttinger liquid Hamiltonian can be renormalized in a simple way by employing a suitable chiral fermion basis. This trick has been exploited previously for conductance calculations in the presence of a static impurity.\(^23\)\(^24\) Due to the transient nature of the core-hole potential, the situation is more intricate here, but nevertheless it allows for an exact solution by applying the Wiener-Hopf technique. We mention that the point \(g = 1/2\) can allow for exact results even when the quantity of interest cannot be written in terms of chiral fermions, see, e.g., a recent calculation of the Friedel oscillation around a static impurity.\(^25\)

In Ref.\(^17\) which addresses the Fermi-edge singularity problem from the boundary conformal field theory point of view, it was noted that even in conventional metals the final state of the system may have a non-Fermi-liquid character (albeit the initial state is a Fermi liquid). This can happen, for instance, when the core hole created in the absorption process has internal degrees of freedom like spin or orbital quantum numbers. For a final-state impurity leading to a multi-channel Kondo problem, the x-ray edge exponents were found in Ref.\(^17\). On the other hand, considerable progress in the theory of the \(k\)-channel Kondo problem has recently been achieved. In
In the framework of Abelian bosonization, Toulouse-limit type solutions were found for the \( k = 2 \) and \( k = 4 \) models. In the latter two cases, the Toulouse limit turns out to be equivalent to an impurity problem in a Luttinger liquid. Confusingly, the x-ray edge exponents of Ref. [3] for the Kondo problem are different from the \( \alpha = 1/8 \) result for the Luttinger liquid. This puzzle is addressed and resolved here.

The outline of this paper is as follows. In Sec. I, we briefly discuss the Luttinger liquid and fermionize the theory at \( g = 1/2 \). Sec. II presents the derivation of an integral equation for the core-hole Green’s function. The asymptotically exact solution of this integral equation is discussed in Sec. III. Thereby the Fermi-edge singularity exponent \( \alpha = 1/8 \) is found. In Sec. IV we re-derive the exponents for the \( k \)-channel Kondo problem and show that they are in fact consistent with the Luttinger liquid models.

II. LUTTINGER LIQUID

We treat the 1D interacting electron liquid in the framework of the Luttinger liquid model focusing on the spinless case. The most appealing theoretical description is offered by the bosonization treatment, which equivalently expresses the interacting fermionic system as a harmonic fluid composed of bosonic plasmon modes. In terms of right- and left-moving boson fields \( \varphi_{L,R} \) satisfying the algebra (\( p = R, L = \pm \))

\[
[\varphi_p(x), \varphi_{p'}(x')] = -i\pi p \delta_{p,p'} \text{sgn}(x - x') , \tag{2.1}
\]

the Hamiltonian takes the form (we put \( \hbar = 1 \) and the sound velocity \( v = 1 \))

\[
H_i = \frac{1}{8\pi} \int dx \left[ (\partial_x \varphi_R)^2 + (\partial_x \varphi_L)^2 \right] . \tag{2.2}
\]

This initial Hamiltonian describes the unperturbed Luttinger liquid prior to the absorption process at time \( t_i = 0 \), i.e., for \( t < t_i \). The local backscattering due to the core-hole potential acting at \( x = 0 \) and \( t > t_i \) leads to

\[
H_f = H_i + V \cos[\sqrt{g} (\varphi_R(0) - \varphi_L(0))] . \tag{2.3}
\]

The spinless Luttinger liquid is characterized by the dimensionless interaction strength parameter \( g \), where \( g = 1 \) is the non-interacting Fermi liquid value, and \( g < 1 \) corresponds to repulsive Coulomb interactions. The strength of the transient backscattering is denoted by \( V \).

In what follows, it is advantageous to introduce chiral right-moving boson fields, \( \phi_{\pm}(x) = [\varphi_R(x) \mp \varphi_L(-x)]/\sqrt{2} \), which from Eq. (2.1) obey the algebra

\[
[\phi_p(x), \phi_{p'}(x')] = -i\pi \delta_{p,p'} \text{sgn}(x - x') . \tag{2.4}
\]

Written in terms of \( \phi_{\pm}(x) \), the Hamiltonian decouples into a sum of odd and even fields,

\[
H_i = \frac{1}{8\pi} \int dx \left[ (\partial_x \phi_+)^2 + (\partial_x \phi_-)^2 \right] , \quad H_f = H_i + V \cos[\sqrt{2g} \phi_+(0)] . \tag{2.5}
\]

Because \( \phi_- \) does not couple to the impurity and at the same time commutes with \( \phi_+ \), see Eq. (2.4), we drop the \( (\partial_x \phi_-)^2 \) term in \( H_i \) in the following.

Let us now consider the special value \( g = 1/2 \), where the Hamiltonian \( H_f \) can be re-fermionized by using the chiral right-moving fermion field operator

\[
\psi(x) = (2\pi a)^{-1/2} \exp[i\phi_+(x)] , \tag{2.6}
\]

with a short-distance cutoff \( a \) (say, a lattice spacing). Absorbing the factor \( \sqrt{2\pi a} \) into the backscattering strength \( V \), we obtain the fermionic description

\[
H_i = -i \int dx \psi(x) \partial_x \psi(x) , \quad H_f = H_i + V \left[ \psi(0) + \psi^\dagger(0) \right] .
\]

To proceed further, it is convenient to use a trick introduced by Matveev and write

\[
\tilde{\psi}(x) = (c + c^\dagger) \psi(x) , \tag{2.7}
\]

with new fermion operators \( c \) and \( \tilde{\psi}(x) \). It is a simple matter to check that \( \tilde{\psi} \) is still fermionic in this representation. Since \( (c + c^\dagger)^2 = 1 \), the Hamiltonian takes the form

\[
H_i = -i \int dx \tilde{\psi}^\dagger(x) \partial_x \tilde{\psi}(x) , \quad H_f = H_i + V \left[ \psi(0) - \tilde{\psi}^\dagger(0) \right] .
\]

Finally, we switch to a representation in terms of real Majorana fermion operators

\[
c = (b + ia)/\sqrt{2} , \quad c^\dagger = (b - ia)/\sqrt{2} , \tag{2.8}
\]

\[
\tilde{\psi}(x) = (\eta(x) + i\xi(x))/\sqrt{2} , \quad \tilde{\psi}^\dagger(x) = (\eta(x) - i\xi(x))/\sqrt{2} .
\]

Here, we have the anticommutator algebra

\[
[\xi(x), \xi(x')]_+ = [\eta(x), \eta(x')]_+ = \delta(x - x') , \quad [\eta(x), \xi(x')]_+ = 0 ,
\]

and likewise for \( a, b \). Written in terms of these new fields,

\[
H_i = -(i/2) \int dx [\eta(x)\partial_x \eta(x) + \xi(x)\partial_x \xi(x)] , \quad H_f = H_i + iVb\xi(0) . \tag{2.9}
\]

Since the \( \eta \) field does not couple to the local backscattering potential, we drop it in the following. These expressions are the basis of the subsequent treatment.
III. INTEGRAL EQUATION FOR THE CORE-HOLE GREEN’S FUNCTION

One relevant dynamical quantity which allows extraction of the exponent \( \alpha \) is the core-hole Green’s function. It can be defined as

\[
D(t) = \langle e^{iH_i t} e^{-iH_i t} \rangle_0 ,
\]

where \( \langle \cdots \rangle_0 \) denotes a ground-state average using

\[
H_i = -(i/2) \int dx \xi(x) \partial_x \xi(x) .
\]

At long times, the core-hole Green’s function behaves like

\[
D(t) \sim e^{-i\Delta E t} t^{-\alpha} ,
\]

with the energy shift \( \Delta E \) of the core-hole level and the Fermi-edge singularity exponent \( \alpha \). The latter quantity is the aim of our calculation.

For \( t \to \infty \), the function (3.1) gives the overlap of the initial and the final many-body wavefunction, which vanishes with the dimensionless system size \( N \) as

\[
D(\infty) \sim N^{-\alpha/2} .
\]

This is the famous Anderson orthogonality catastrophe.\[3\]

The factor 1/2 in the exponent compared with Eq. (3.3) is explained as follows. For the overlap (3.4), the backscattering potential is switched on at time \( t_i = 0 \) but never switched off. However, at finite times \( t \), for the core-hole Green’s function \( D(t) \), the potential is effectively switched off at time \( t \). Therefore there is one more switching processes which results in a doubling of the exponent compared to the overlap integral [3].

Using Eqs. (2.9) and (3.1), and switching to imaginary time \( \tau = it \), the core-hole Green’s function becomes

\[
D(\tau_i, \tau_f) = \left\langle T \exp \left[ -iV \int_{\tau_i}^{\tau_f} d\tau b(\tau) \xi(\tau) \right] \right\rangle_0 ,
\]

where \( \xi(\tau) \equiv \xi(x = 0, \tau) \) and \( T \) is the time-ordering operator. This can be written more suitably by first introducing the modified function

\[
D_{\lambda}(\tau_i, \tau_f) = \left\langle T \exp \left[ -i\lambda V \int_{\tau_i}^{\tau_f} d\tau b(\tau) \xi(\tau) \right] \right\rangle_0 ,
\]

which fulfills (time integrations always go from \( \tau_i \) to \( \tau_f \) in what follows)

\[
\frac{\partial}{\partial \lambda} D_{\lambda} = -i\lambda V \int d\tau Y_{\lambda}(\tau, \tau) D_{\lambda}
\]

with the auxiliary function

\[
Y_{\lambda}(\tau, \tau') = \langle T b(\tau) \xi(\tau') \rangle_{\lambda} .
\]

Here, \( \langle \cdots \rangle_{\lambda} \) refers to a ground-state average using

\[
H_{\lambda} = H_i + i\lambda V b(0) .
\]

Note that since the backscattering is only present for times \( \tau \) in the interval \( \tau_i < \tau < \tau_f \), the quantity (3.8) depends on \( \tau_i, \tau_f \). Substituting \( \lambda \to \lambda V \), we then obtain from Eq. (3.7)

\[
\ln D(\tau_i, \tau_f) = -i \int_0^V d\lambda \int d\tau Y_{\lambda}(\tau, \tau) .
\]

To exploit this equation, we will now formulate and solve an integral equation for the function \( Y_{\lambda} \).

For that purpose, we first need the free Green’s functions

\[
G_b(\tau - \tau') = (1/2) \text{sgn}(\tau - \tau') .
\]

They can easily be computed from their respective equation of motion. Since \( \partial_c b = [b, H_i] = 0 \) and \( b^2 = (e + c)^2/2 = 1/2 \) [see Eq. (2.8)], this reads

\[
\frac{\partial}{\partial \tau} G_b(\tau - \tau') = \delta(\tau - \tau') ,
\]

with the solution

\[
G_b(\tau - \tau') = (1/2) \text{sgn}(\tau - \tau') .
\]

For \( G_{\xi}(\tau - \tau') \), we first consider the correlation function \( \langle T \xi(x, \tau) \xi(x', \tau') \rangle_0 \). With \( \partial_c \xi(x) = [\xi(x), H_i] = i\partial_x \xi(x) \), its equation of motion reads

\[
\left( \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial x} \right) \langle T \xi(x, \tau) \xi(x', \tau') \rangle_0 = \delta(\tau - \tau') \delta(x - x') .
\]

Specializing the solution to \( x = x' = 0 \), we obtain

\[
G_{\xi}(\tau - \tau') = \frac{1}{2\pi(\tau - \tau')}.\]

The equation-of-motion method can similarly be employed to derive an integral equation for the function \( Y_{\lambda}(\tau, \tau') \) defined in Eq. (3.8). Under \( H_{\lambda} \) [note that the factor \( V \) in Eq. (2.9) has meanwhile been absorbed in \( \lambda \)], the time evolution of \( b \) and \( \xi(x) \) is governed by

\[
\frac{\partial}{\partial \tau} b = [b, H_{\lambda}] = i\lambda \xi(0)
\]

\[
\frac{\partial}{\partial \tau} \xi(x) = [\xi(x), H_{\lambda}] = i\frac{\partial}{\partial x} \xi(x) - i\lambda \delta(x) b .
\]

Applying a time derivative to Eq. (3.8), we have

\[
\frac{\partial}{\partial \tau} Y_{\lambda}(\tau, \tau') = i\lambda \langle T \xi(\tau) \xi(\tau') \rangle_{\lambda} .
\]

Similarly, applying a time derivative to the function \( \langle T \xi(x, \tau) \xi(\tau') \rangle_{\lambda} \) generates the equation of motion
\[
\left( \frac{\partial}{\partial \tau} - i \frac{\partial}{\partial x} \right) \langle T \xi(x,\tau) \xi(\tau') \rangle_\lambda = \delta(x)\delta(\tau - \tau') - i\lambda \delta(x)Y_\lambda(\tau,\tau').
\]

Employing the free Green’s function (3.12), the solution of this equation (for \( x = 0 \)) is

\[
\langle T \xi(\tau) \xi(\tau') \rangle_\lambda = G_\xi(\tau - \tau') - i\lambda \int d\tau_1 G_\xi(\tau - \tau_1) Y_\lambda(\tau_1,\tau').
\]

(3.14)

On the other hand, from Eqs. (3.11) and (3.13), we have

\[
Y_\lambda(\tau,\tau') = i\lambda \int d\tau_1 G_b(\tau - \tau_1) \langle T \xi(\tau_1) \xi(\tau') \rangle_\lambda.
\]

(3.15)

Inserting Eq. (3.14) into Eq. (3.12) results in the integral equation

\[
Y_\lambda(\tau,\tau') = -i\lambda f(\tau,\tau') - \lambda^2 \int d\tau_1 f(\tau,\tau_1) Y_\lambda(\tau_1,\tau').
\]

(3.16)

The singular kernel \( f(\tau,\tau') \) is defined by

\[
f(\tau,\tau') = \int d\tau_1 G_b(\tau - \tau_1) G_\xi(\tau' - \tau_1),
\]

\[
= \frac{1}{4\pi} \int d\tau_1 \frac{\text{sgn}(\tau - \tau_1)}{\tau' - \tau_1}.
\]

(3.17)

Let us conclude this section by summarizing and slightly simplifying the main result. Redefining \( Y_\lambda \rightarrow Y_\lambda/\lambda \), we have derived the integral equation

\[
Y_\lambda(\tau,\tau') = -i f(\tau,\tau') - \lambda^2 \int d\tau_1 f(\tau,\tau_1) Y_\lambda(\tau_1,\tau')
\]

(3.18)

with the kernel \( f(\tau,\tau') \) given in Eq. (3.17). Once this equation is solved, the core-hole Green’s function and the overlap integral can be determined from

\[
\ln D(\tau_1,\tau_f) = -i \int_0^\tau_f d\lambda \int_{\tau_i}^{\tau_f} d\tau Y_\lambda(\tau,\tau),
\]

(3.19)

see Eq. (3.10). The solution of the integral equation (3.18) for the overlap integral, i.e., for \( \tau_i = 0 \) and \( \tau_f \rightarrow \infty \), is discussed in the next section.

As a simple consistency check for these expressions, the short-time behavior of the core-hole Green’s function can be calculated by perturbation theory in \( \lambda \) from Eqs. (3.18) and (3.13). The result coincides with the result from Ref. 14 obtained by straightforward perturbation theory in \( V \) using the original bosonized picture.

### IV. WIENER-HOPF SOLUTION FOR THE OVERLAP INTEGRAL

We now turn to the calculation of the overlap integral \( (\tau_i = 0, \tau_f \rightarrow \infty) \). Then the integral equation (3.18) can be solved using the Wiener-Hopf method. For \( \tau, \tau' > 0 \), the singular kernel (3.17) takes the form

\[
f(\tau - \tau') = f_1(\tau - \tau') + f_2(\tau'),
\]

(4.1)

where

\[
f_1(\tau - \tau') = -\frac{1}{2\pi} \ln|\tau - \tau'|
\]

\[
f_2(\tau') = \frac{1}{4\pi} [\ln(\tau_f - \tau') + \ln(\tau')] \,
\]

The function \( f_2(\tau') \) requires a finite cutoff \( \tau_f \). However, it will turn out that terms related to \( f_2 \) do not contribute to the overlap integral, and therefore this formal divergence is not detrimental to us.

We next switch to (partial) Fourier transforms,

\[
Y_\lambda(\tau,\tau') = \int \frac{d\omega}{2\pi} e^{-i\omega\tau} Y_\lambda^0(\omega,\tau')
\]

\[
f_1(\tau - \tau') = \int \frac{d\omega}{2\pi} e^{-i\omega(\tau - \tau')} \hat{f}_1(\omega),
\]

where

\[
\hat{f}_1(\omega) = \frac{1}{2|\omega|}.
\]

(4.2)

For application of the Wiener-Hopf method, it is necessary to introduce the functions \( (\epsilon \rightarrow 0) \)

\[
\tilde{Y}_\lambda^{(\pm)}(\omega,\tau') = \mp \int \frac{d\omega'}{2\pi i} \frac{\tilde{Y}_\lambda^0(\omega',\tau')}{\omega' + i\epsilon},
\]

(4.3)

which are analytic functions in the upper/lower \( \omega \)-plane \( \Pi^{(\pm)} \), respectively. Naturally, the sum of both reproduces the function itself, \( \tilde{Y}_\lambda = \tilde{Y}_\lambda^{(+)} + \tilde{Y}_\lambda^{(-)} \).

With these conventions, the integral equation (3.18) is equivalently expressed in the form

\[
\tilde{Y}_\lambda(\omega,\tau') = -i\epsilon e^{i\omega\tau'} \hat{f}_1(\omega) - 2\pi i \delta(\omega) f_2(\tau')
\]

\[
- \lambda^2 \hat{f}_1(\omega) \tilde{Y}_\lambda^{(+)}(\omega,\tau') - 2\pi \lambda^2 \delta(\omega) \int d\tau_1 f_2(\tau_1) Y_\lambda(\tau_1,\tau').
\]

(4.4)

This integral equation is of Wiener-Hopf form. That is made apparent by introducing the quantities \( X_\lambda^{(\pm)}(\omega) \) fulfilling

\[
X_\lambda^{(\pm)}(\omega) X_\lambda^{(-)}(\omega) = \frac{1}{1 + \lambda^2 f_1(\omega)},
\]

(4.5)

with the explicit form
\[ \ln X^{(\pm)}_\lambda(\omega) = \pm \int \frac{d\omega'}{2\pi i} \ln \left(1 + \frac{\lambda^2 \tilde{f}_1(\omega')}{\omega' - \omega + i\epsilon} \right). \] (4.6)

With these definitions, Eq. (4.4) becomes

\[ \frac{\tilde{Y}^{(+)}_\lambda(\omega, \tau')}{X^{(+)}_\lambda(\omega)} + \tilde{Y}^{(-)}_\lambda(\omega, \tau')X^{(-)}_\lambda(\omega) = -ie^{i\omega\tau'}\tilde{f}_1(\omega)X^{(-)}_\lambda(\omega) - 2\pi i\delta(\omega)f_2(\tau')X^{(-)}_\lambda(\omega) - 2\pi \lambda^2 \delta(\omega)X^{(-)}_\lambda(\omega) \int d\tau f_2(\tau)f_1(\tau_1, \tau'). \] (4.7)

As mentioned above, all terms related to \( f_2(\tau') \) do not contribute to the solution of Eq. (4.7). This can be seen by inspecting the \( \omega \to 0 \) limit of \( X^{(-)}_\lambda(\omega) \). From Eqs. (4.2) and (4.4), we have

\[ \ln X^{(-)}_\lambda(\omega) = \int \frac{d\omega'}{2\pi i} \ln \left(1 + \frac{\lambda^2 \tilde{f}_1(\omega')}{\omega' + i\epsilon} \right) \]

\[ = -\int_0^\infty d\omega' \frac{\epsilon/\pi}{\omega'^2 + \epsilon^2} \ln \left(1 + \frac{\lambda^2}{2\omega'} \right). \]

As \( \epsilon \to 0 \), this expression approaches \(-\infty \) such that

\[ \lim_{\omega \to 0} X^{(-)}_\lambda(\omega) = 0. \]

Therefore, when integrating over \( \omega \) in Eq. (4.7), all terms containing \( f_2(\tau') \) drop out due to the \( \delta(\omega) \) factors.

In the end, only the translationally invariant part \( f_1 \) of the kernel matters,

\[ \tilde{Y}^{(+)}_\lambda(\omega, \tau') = \frac{-i\tilde{f}_1(\omega)e^{i\omega\tau'}}{1 + \lambda^2 \tilde{f}_1(\omega)} - X^{(+)}_\lambda(\omega)X^{(-)}_\lambda(\omega)\tilde{Y}^{(-)}_\lambda(\omega, \tau') \].

(4.8)

Since Eq. (4.19) can be written in the form

\[ \ln D(\tau_f) = -i \int_0^\tau d\tau_0 \int d\lambda \int_0^\infty d\tau e^{-i\omega\tau'} \tilde{Y}^{(+)}_\lambda(\omega, \tau) \]

we obtain two contributions to \( D(\tau_f) \) from Eq. (4.18).

The first term generates the energy shift \( \Delta E \) of the core-hole level in Eq. (3.3),

\[ \ln D^{(1)}(\tau_f) = -i\Delta E(-i\tau_f) \]

where

\[ \Delta E = \int_0^\infty d\lambda \int_0^\infty d\omega \frac{\tilde{f}_1(\omega)}{2\pi 1 + \lambda^2 \tilde{f}_1(\omega)}. \] (4.10)

After analytic continuation to real time, \(-i\tau_f \to t\), this gives exactly the \( \exp(-i\Delta E t) \) factor in Eq. (1.13). The value of \( \Delta E \) can be computed from Eq. (4.10), but it is not of immediate interest to us in the following.

The second term in Eq. (1.8) is responsible for the orthogonality catastrophe. For the overlap integral, this gives

\[ \ln D(\infty) = i \int_0^V d\lambda \int_0^\infty d\tau e^{-\epsilon\tau} \int \frac{d\omega}{2\pi} \frac{e^{-i\omega\tau}}{\omega' - \omega + i\epsilon} \] (4.11)

\[ \times X^{(+)}_\lambda(\omega)X^{(-)}_\lambda(\omega)\tilde{Y}^{(-)}_\lambda(\omega, \tau). \]

To get \( \tilde{Y}^{(-)}_\lambda(\omega, \tau) \), we now apply the Wiener-Hopf trick to Eq. (4.7) [note that the \( f_2 \) terms can be omitted]; The r.h.s. of Eq. (4.7) can be separated into two functions which are analytic in \( \Pi^{(\pm)} \), respectively. This yields

\[ \tilde{Y}^{(-)}_\lambda(\omega, \tau) = \frac{i}{X^{(-)}_\lambda(\omega)} \int \frac{d\omega'}{2\pi i} \frac{e^{i\omega'\tau} \tilde{f}_1(\omega')X^{(-)}_\lambda(\omega')}{\omega' - \omega + i\epsilon}, \] (4.12)

which is now inserted into Eq. (4.11) and gives

\[ \ln D(\infty) = -i \int_0^V d\lambda \int_0^\infty d\tau e^{-\epsilon\tau} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi i} \int \frac{d\omega'}{2\pi i} 
\times e^{-i\omega\tau} X^{(+)}_\lambda(\omega) \left( \frac{e^{i\omega'\tau} \tilde{f}_1(\omega')X^{(-)}_\lambda(\omega')}{\omega' - \omega + i\epsilon} \right). \]

We next apply the relation

\[ \lambda^2 \tilde{f}_1(\omega')X^{(-)}_\lambda(\omega') = \frac{1}{X^{(+)}_\lambda(\omega') - X^{(-)}_\lambda(\omega')}, \]

which follows from Eq. (4.3). This gives from Eq. (4.13) the expression

\[ \ln D(\infty) = -i \int_0^V d\lambda \int_0^\infty d\tau e^{-\epsilon\tau} \int \frac{d\omega}{2\pi} \frac{d\omega'}{2\pi i} \int \frac{d\omega'}{2\pi i} \int \frac{d\omega}{2\pi} 
\times \left( \frac{1}{X^{(+)}_\lambda(\omega') - X^{(-)}_\lambda(\omega')} \right) X^{(+)}_\lambda(\omega) \frac{X^{(-)}_\lambda(\omega)}{\omega' - \omega + i\epsilon}. \]

The 1/X^{(+)}(\omega') term does not contribute, such that we obtain with the variables \( \omega_1 = \omega'/\lambda^2, \omega_2 = \omega/\lambda^2, \gamma = \lambda^2 \), and \( \epsilon \to \epsilon/\lambda^2 \)

\[ \ln D(\infty) = \frac{1}{2} \int_0^V d\gamma \int_0^\infty d\tau e^{-\epsilon\tau} \int \frac{d\omega_1}{2\pi} \int \frac{d\omega_2}{2\pi i} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} 
\times e^{\gamma\tau}(\omega_1 - \omega_2) X^{(+)}_1(\omega_2)X^{(-)}_1(\omega_1) \]

\[ \frac{1}{\omega_1 - \omega_2 + i\epsilon}. \]

Remarkably, the integrand is symmetric under the exchange of \( \tau \) and \( \gamma \), since only their product appears. In Eq. (1.14) we have also exploited the fact

\[ X^{(\pm)}_\lambda(\omega) = X^{(\pm)}_1(\omega/\lambda^2). \]

which is apparent from the definition (4.6).

Naturally, Eq. (1.14) is an infra-red divergent expression since the overlap vanishes for the infinite system.
For a finite system of dimensionless length \( N \), the shortest possible timescales are of the order \( \sim 1/N \), and the \( \tau \) integration in Eq. (1.14) should be cut off from below at a scale \( \tau_0/N \) (where \( \tau_0 \) is a timescale of the infinite system). Because of the formal symmetry of the integrand of Eq. (1.14), under exchange of \( \gamma \) and \( \tau \), we can alternatively cut off the \( \gamma \) integration from below. The \( \tau \) integration can then be carried out and gives a factor \( i/\left[\gamma(\omega_1 - \omega_2 + i\epsilon)\right] \). In the end, the result becomes

\[
\ln D(\infty) = -\frac{1}{2} \int_{\tau_0/N}^{V^2} \frac{d\gamma}{\gamma} \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} \frac{X_1^{(+)}(\omega_2)X_1^{(-)}(\omega_1)}{(\omega_1 - \omega_2 + i\epsilon)^2}.
\]

(4.15)

For \( N \to \infty \), this yields indeed the power law (3.3) for the overlap integral, and we can read off the exponent \( \alpha \) by comparing Eqs. (3.4) and (4.15),

\[
\alpha = \int \frac{d\omega_1}{2\pi i} \int \frac{d\omega_2}{2\pi i} \frac{X_1^{(+)}(\omega_2)X_1^{(-)}(\omega_1)}{(\omega_1 - \omega_2 + i\epsilon)^2}.
\]

(4.16)

The calculation of the Fermi-edge singularity exponent is now straightforward. Carrying out the \( \omega_2 \) integration in Eq. (4.16), one obtains

\[
\alpha = \int \frac{d\omega}{2\pi i} \frac{1}{\frac{1}{2\omega}} \frac{\partial}{\partial \omega} \ln X_1^{(+)}(\omega) \int d\omega' \frac{\ln(1 + \frac{1}{2\omega'})}{2\pi i} \omega' - \omega - i\epsilon,
\]

where we have used Eq. (4.6) in the second line. After a partial integration and the rescaling \( \omega_1 = 2\omega, \omega_2 = 2\omega' \), this takes the form

\[
\alpha = -\frac{1}{\pi^2} \int_0^\infty d\omega \int_0^\infty \frac{d\omega_2}{\omega_2(\omega^2_2 - \omega^2_1)(\omega_1 + 1)(\omega_2 + 1)}
\]

\[
= \frac{1}{2\pi^2} (I_+ + I_-).
\]

The quantities \( I_\pm \) are

\[
I_\pm = \int_0^\infty d\omega \int_0^\infty \frac{1}{\omega_1 \pm \omega_2}.
\]

with the result \( I_+ = \pi^2/4 \) and \( I_- = 0 \). In the end, we arrive at the central result of this calculation, \( \alpha = 1/8 \). This is the exact Fermi-edge singularity exponent for a Luttinger liquid.

V. ORTHOGONALITY EXPONENTS FOR THE MULTICHANNEL KONDO CASE

In the following, we discuss the relationship between the orthogonality exponent \( \alpha = 1/8 \) in a Luttinger liquid and the respective exponent for a Kondo impurity interacting with \( k \) uncorrelated conduction electron channels. If the core hole has internal degrees of freedom, \( d \to d_{\sigma} \), the final-state Hamiltonian takes the form

\[
H_f = H_i + \sum_{\sigma} \vec{s} \vec{J}(0) ,
\]

(5.1)

where \( H_i \) describes the uncorrelated conduction electrons prior to the x-ray absorption process. The following consideration is restricted to \( \sigma \) denoting a (pseudo)spin-1/2 index, such that the impurity spin operator is \( \vec{s} = d_{\sigma}^\dagger \vec{\tau}_{\sigma\sigma'} d_{\sigma'} \) with the spin-1/2 matrices \( \vec{\tau} \). The conduction electron spin density operators are defined by

\[
\vec{J}(x) = \sum_{i=1}^{k} \psi_{i\sigma}^\dagger(x) \vec{\tau}_{\sigma\sigma'} \psi_{i\sigma'}(x)
\]

for \( k \) conduction electron channels. Without loss of generality, \( x \) can be understood as a 1D coordinate As found in Ref. [17] the x-ray edge exponent for the problem (5.1) is

\[
\alpha_k = \frac{3}{2(2 + k)}.
\]

(5.2)

We bosonize the electron field operators in the standard way, see Eq. (2.4),

\[
\psi_{i\sigma}(x) = \frac{1}{\sqrt{2\pi a}} \exp[i\phi_{i\sigma}(x)],
\]

where \( \phi_{i\sigma}(x) \) is a chiral (right-moving) boson field. Here only spin degrees of freedom are relevant, i.e., only the combinations

\[
\phi_i(x) = \frac{1}{\sqrt{2}} [\phi_{i\uparrow}(x) - \phi_{i\downarrow}(x)]
\]

enter the bosonized form of the spin density operators,

\[
J^- = \sum_{i=1}^{k} \psi_{i\downarrow}(x) \psi_{i\uparrow}(x) = \frac{1}{2\pi a} \sum_{i=1}^{k} \exp[i\phi_i(x)],
\]

(5.3)

\[
J^z = \sum_{i=1}^{k} \frac{1}{\sqrt{2}} \partial_x \phi_i(x) ,
\]

where the total spin field \( \phi_s \) is defined as

\[
\phi_s(x) = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} \phi_i(x).
\]

The spin density operators \( J^a(x) (a = x, y, z) \) are also referred to as spin currents since they satisfy the commutation relations of the \( SU(2) \) level-k Kac-Moody current algebra

\[
[J^a(x), J^b(y)] = i\epsilon^{abc} \delta(x - y) J^c(x) + \frac{iK}{4\pi} \delta'(x - y) \delta^{ab}.
\]

(5.4)

In order to investigate how the result (5.2) fits into the Abelian treatment of the Kondo problem [24] and
whether it is consistent with the Luttinger liquid exponent $\alpha = 1/8$, we next exploit the following observation. Assume that we have constructed a unitary transformation $U$ relating the initial and the final Hamiltonian,

$$U^\dagger H_f U = H_i.$$ 

Then the core-hole Green’s function $D(t)$ can be represented as

$$D(t) = \langle i | U^\dagger(t) U(0) | i \rangle ,$$

where the time evolution of the unitary operator $U(t)$ is governed by the initial Hamiltonian, $U(t) = e^{iH_t U e^{-iH_t t}}$. In general, the Toulouse limit solution of the Kondo model amounts to finding a special unitary operator, $U$, which transforms the Kondo Hamiltonian $H_K(= H_f)$ into the “Toulouse limit Hamiltonian” $H_T$,

$$U^\dagger H_f U = H_T ,$$

(5.5)

where $H_T$ is (in some sense) exactly solvable. However, one may have neglected irrelevant operators in performing the transformation (5.5), see below. The exactly solvable Toulouse limit Hamiltonian can in turn be mapped onto a free fermion Hamiltonian by yet another canonical transformation $U_f$,

$$U_f^\dagger H_f U_f = H_0 ,$$

so that the canonical transformation $U$ relevant for the x-ray edge problem is $U = U_f U_T$. The $U_f$ transformation is known for the cases $k = 2$ and $k = 4$. We shall briefly review these two derivations and refer the reader to the original papers for a more detailed discussion.

Two-channel Kondo problem. The relevant part of the bosonized Kondo Hamiltonian can be written as

$$H_K = H_0[\phi_s] + H_0[\phi_{sf}]$$

$$+ \frac{I_s}{2\pi a} \left[ s_x e^{i\phi_s(0)} \cos \phi_{sf}(0) + \text{H.c.} \right] + \frac{I_s}{2\pi} s_x \partial_x \phi_s(0) ,$$

where the “spin-flavor” boson field is defined by

$$\phi_{sf}(x) = \frac{1}{\sqrt{2}} [\phi_1(x) - \phi_2(x)] .$$

Emery and Kivelson noticed that after the unitary transformation $U_S = \exp[is_z \phi_s(0)]$, the Hamiltonian (5.6) takes the form

$$\tilde{H}_K = H_0[\phi_s] + H_0[\phi_{sf}] + \frac{I_s}{\pi a} s_x \cos \phi_{sf}(0) + \frac{\lambda}{2\pi} s_z \partial_x \phi_s(0)$$

(5.7)

with $\lambda = I_z - 2\pi v_F$. Here $H_0$ denotes a free fermion Hamiltonian with Fermi velocity $v_F$. The Toulouse limit corresponds to $\lambda = 0$, leading to $H_T = \tilde{H}_K|_{\lambda=0}$. The $\lambda$ operator is irrelevant and therefore does not contribute to the long-time asymptotics of the $U_S$ correlation function. Therefore Eq. (5.7) is essentially the Hamiltonian (2.5) for a $g = 1/2$ Luttinger liquid containing an impurity. This can be seen by noticing that $s_x$ commutes with $H_T$ and hence can be set to $s_x = \pm 1/2$. Alternatively, one can reformalize Eq. (5.3) to obtain a Majorana resonant-level model.

Four-channel Kondo problem. In this case the representation (5.3) for the spin currents is not appropriate. A convenient representation which satisfies the $SU(2)$ level 4 commutation relations (5.3) is

$$J^-(x) = \frac{\sqrt{2}}{\pi a} \exp \left[ \frac{i}{\sqrt{2}} \phi_s(x) \right] \cos \sqrt{3} \phi_{sf}(x) ,$$

(5.8)

$$J_z(x) = \frac{1}{\sqrt{2\pi}} \partial_x \phi_s(x) .$$

Using the representation (5.8) in the Kondo Hamiltonian and employing the transformation $U_S = \exp[is_z \phi_s(0)/\sqrt{2}]$, one finds

$$\tilde{H}_T = H_0[\phi_s] + H_0[\phi_{sf}]$$

$$+ \frac{I_s}{\pi a} s_x \cos \sqrt{3} \phi_{sf}(0) + \frac{\lambda}{2\pi} s_z \partial_x \phi_s(0)$$

(5.9)

with $\lambda = \sqrt{2}(I_z - \pi v_F)$. The $\lambda$ operator is again irrelevant, and the Toulouse limit Hamiltonian $H_T = \tilde{H}_K|_{\lambda=0}$ is exactly the Hamiltonian describing a $g = 3/4$ Luttinger liquid with an impurity, see Eq. (2.3).

Therefore, the canonical transformation involved in the Toulouse limit solution is always a “spin rotation” $U_S = \exp[is_z \phi_s(0)]$ with $\kappa = 1$ for $k = 2$ and $\kappa = 1/\sqrt{2}$ for $k = 4$. The rotation angle is determined by the total spin field $\phi_s(0)$. Provided possible irrelevant operators can be neglected, the total spin field commutes with the impurity part of the Toulouse limit Hamiltonians. Consequently, the correlation function of $U_S$ must be computed with respect to the ground state of $H_0[\phi_s]$ which coincides with the ground state of $H_T$ as far as the field $\phi_s$ is concerned. Noticing that it does not matter in which spin state $s_x = \pm 1/2$ the $U_S$ correlation function is computed, and that the transformation $U_T$ is related to the spin-flavor field $\phi_{sf}$ (which commutes with $\phi_s$), we obtain the factorized form at long times,

$$D(t) = \langle i | U^\dagger(t) U(0) | i \rangle = \langle U^\dagger_0(t) U_S(0) \rangle \langle U^\dagger_T(t) U_T(0) \rangle ,$$

(5.10)

where $\langle U^\dagger_0(t) U_S(0) \rangle \sim t^{-\kappa^2/4}$. The second factor in Eq. (5.10) is easily obtained since the correlation function of $U_T$ determines the orthogonality exponent in a Luttinger liquid. Therefore, we obtain for arbitrary $k$ the correlator $\langle U^\dagger_0(t) U_T(0) \rangle \sim t^{-1/8}$.

In the end, we arrive at the following expression for the orthogonality exponent of the Kondo problem,
\[ \alpha_k = \frac{k^2}{4} + \frac{1}{8}. \]  

This equation establishes the relationship between the orthogonality exponents of the multi-channel Kondo impurity problem and the Luttinger liquid. In the two-channel case we have

\[ \alpha_{k=2} = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}, \]

while for the four-channel case

\[ \alpha_{k=4} = \frac{1}{8} + \frac{1}{8} = \frac{1}{4}. \]

This is in agreement with Eq. (5.2) obtained in Ref. 17. Unfortunately, so far Toulouse-limit type solutions have not been found for other values of \( k \).

VI. CONCLUSIONS

In this work, we have carried out an exact calculation of the Fermi-edge singularity exponent in a Luttinger liquid. The result \( \alpha = 1/8 \) determines both the scaling of the overlap \( \langle f| i \rangle \sim N^{-\alpha/2} \) of the wavefunctions with and without the local perturbation, and the asymptotic long-time behavior of the core-hole Green’s function, \( D(t) \sim \exp(-i\Delta E t) t^{-\alpha} \). Our result is in notable contrast to the calculation by Oreg and Finkel’stein 18, but verifies the result obtained previously from the open boundary fixed point assumption. 19 The latter assumption therefore holds both for static and transient local perturbations. In addition, we have extended the previously found relations between the multi-channel Kondo problem and the problem of an impurity in the Luttinger liquid to the case when these impurities are dynamic.

Note: After completion of this manuscript, we became aware of the simultaneous and independent work of Furusaki 32. Using essentially the same method, he has also studied the orthogonality exponent in a \( g = 1/2 \) Luttinger liquid and obtained the same result.

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