$N = 1, 2$ Super-NLS Hierarchies as Super-KP Coset Reductions.

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Abstract

We define consistent finite-superfields reductions of the $N = 1, 2$ super-KP hierarchies via the coset approach we already developed for reducing the bosonic KP-hierarchy (generating e.g. the NLS hierarchy from the $sl(2)/U(1) - \mathcal{K} \mathcal{M}$ coset). We work in a manifestly supersymmetric framework and illustrate our method by treating explicitly the $N = 1, 2$ super-NLS hierarchies. W.r.t. the bosonic case the ordinary covariant derivative is now replaced by a spinorial one containing a spin $\frac{1}{2}$ superfield. Each coset reduction is associated to a rational super-$\mathcal{W}$ algebra encoding a non-linear super-$\mathcal{W}_\infty$ algebra structure. In the $N = 2$ case two conjugate sets of superLax operators, equations of motion and infinite hamiltonians in involution are derived. Modified hierarchies are obtained from the original ones via free-fields mappings (just as a m-NLS equation arises by representing the $sl(2) - \mathcal{K} \mathcal{M}$ algebra through the classical Wakimoto free-fields).
1 Introduction

The hierarchy of integrable equations leading to a solitonic behaviour has been a widely studied subject since the fundamental work of Gardner, Green, Kruskal and Miura [1] concerning the KdV equation. In more recent time it has particularly deserved physicists’ attention especially in connection with matrix models, which are a sort of effective description for two-dimensional gravity. In such a context the partition functions of matrix models satisfy the so-called Virasoro-$\mathcal{W}$ constraints and can be expressed in terms of the $\tau$-functions of the hierarchies of classical integrable equations (for a review on that, see e.g. [2] and references therein).

Looking at a classification of all possible hierarchies is therefore a very attracting problem for both mathematical and physical reasons. Working in the formalism of pseudo-differential operators (PDO) such a problem can be formalized as follows: determining all possible algebraic constraints, consistent with the KP flows, on the infinite fields entering the KP operator (reduction procedure). Apart from the well-known solutions of Drinfeld-Sokolov type [3], which can be expressed in terms of purely differential Lax operators, in the literature other solutions, called multi-fields KP reductions, have been obtained [4, 5, 6]. Their basic features can be stated as follows: in the dispersionless limit they give rise to a Lax operator fractional in the momentum $p$. Moreover, the algebra of Virasoro-$\mathcal{W}$ constraints turns out to be a $\mathcal{W}_\infty$ algebra.

Inspired by the works [7] (see also [8]), in a previous paper [9] we have shown how such reductions can be derived via a coset construction, involving a factorization of a Kac-Moody subalgebra out of a given Kac-Moody or polynomial $\mathcal{W}$ algebra. In our framework we have immediately at disposal a Poisson-brackets structure providing a (multi)hamiltonian dynamics. Furthermore, the non-linear $\mathcal{W}_\infty$ algebra can be compactly interpreted as a finite rational $\mathcal{W}$ algebra (to our knowledge, the notion of rational $\mathcal{W}$ algebra has been first introduced in [10]; in [11] it has been shown that rational $\mathcal{W}$ algebras appear in the somehow different context of coset construction; a detailed analysis of classical rational $\mathcal{W}$ algebras and their quantum deformations has been given in [12]).

Even if we have not yet attempted to give a general formal proof the examples worked out so far strongly suggest that to each coset is associated a corresponding KP reduction; there exists indeed a well-defined procedure telling how to associate to a given coset a possible KP reduction. In the absence of a general theorem the consistency of the derived reduced operator with the KP flows should be explicitly checked, leading to lengthy but straightforward computations. No counterexample has been found so far.

A point should be made clear: in our framework we do not need to introduce Dirac’s brackets since we do not impose hamiltonian reductions; due to that we are able to derive modified hierarchies via free-field mappings provided by (the classical version of) the Wakimoto representation [13] and their generalizations.

In this paper we address the problem of extending the previous bosonic construction to the $N = 1, 2$ supersymmetric cases, leading to consistent coset reductions of the super-KP hierarchy. The fundamental reference we will follow concerning the definition of the KP hierarchy for odd-graded derivative is [14]. Supersymmetric integrable hierarchies have a vast literature, see among others e.g. [15] and for the $N = 2$ case [16].

We will work in a manifestly supersymmetric formalism and illustrate our procedure
by explicitly showing the coset derivation of the $N = 1, 2$ super-NLS equations. Due to the above considerations our framework can be straightforwardly applied to derive more complicated coset theories.

The basic difference with respect to the bosonic case lies in the fact that now the subalgebra we will factor out is generated by spin $\frac{1}{2}$ supercurrents which enter a spinorial, fermionic, covariant derivative.

The supercurrents algebra generating the $N = 1$ super-NLS theory involve two oppositely charged fermionic superfields of spin $\frac{1}{2}$. The coset construction can be performed as in the bosonic case leading to a non-linear super-$\mathcal{W}$ algebra involving an infinite number of primary superfields, one for each integral or half integral value of the spin $s \geq 1$. Such superalgebra can be regarded as a rational super-$\mathcal{W}$ algebra. It guarantees the existence of a consistent reduction of the super-KP hierarchy, which in its turn implies the integrability of the super-NLS equation. The totally new feature with respect to the bosonic case, i.e. that the subsector of fermionic superfields only appears in the reduced super-KP operator, will be fully discussed.

The first two fermionic superfields in the coset super-algebra are the first two (fermionic) hamiltonian densities of the super-NLS equation. Two compatible super-Poisson brackets are derived as in the bosonic case. A super-Wakimoto representation for the supercurrents algebra enables us to introduce the associated modified super-NLS hierarchy.

Our results concerning the $N = 1$ super-NLS equation should be compared with that of [17] (and [18]). The equation we derive coincide with the one analyzed in [17]. While the coefficients in [17] were suitably chosen in order to provide integrability, in our case they are automatically furnished by the coset construction. We remark here that the Lax operator given in [17] is of matricial type, while our super-KP reduced operator is of scalar type. More comments on the connection of the two approaches will be given in the text.

For what concerns the $N = 2$ case we will make use of the formalism, already developed in [19] for Toda theories, based on chiral and antichiral superfields. They are equivalent to $N = 1$ superfields, which allows us to reduce the $N = 2$ case to the previous one. Any object in the $N = 2$ theory (namely superfields, covariant derivatives, hamiltonians, Lax operators) has its chirally conjugated counterpart.

The scheme of the paper is the following: in the next two sections the bosonic construction is reviewed in detail and the basic structures which are used also in the super-case are discussed. We would like to point out that some of the results here presented are new. Next, the $N = 1$ formalism is introduced and the definition of super-algebra cosets is given. The $N = 1$ super-NLS hierarchy is analyzed. The last two sections are devoted to introduce the formalism and extend the results to the $N = 2$ case.

2 The coset reduction of the bosonic KP hierarchy.

In this section we summarize the basic results of [9] concerning the coset reduction of the bosonic KP hierarchy.

Let us state the problem first: the KP hierarchy (we follow [20] and the conventions
there introduced) is defined through the pseudodifferential Lax operator

\[ L = \partial + \sum_{i=0}^{\infty} U_i \partial^{-i} \]  

(2.1)

where the \( U_i \) are an infinite set of fields depending on the spatial coordinate \( x \) and the time parameters \( t_k \). Let us denote as \( L^k_+ \) the purely differential part of the \( k \)-th power of the \( L \) operator; an infinite set of differential equations, or flows, for the fields \( U_i \) is introduced via the equations

\[ \frac{\partial L}{\partial t_k} = [L^k_+, L] \]  

(2.2)

The quantities

\[ F_k = < L^k > \]  

(2.3)

are first integrals of motion for the flows (2.2). Here the symbol \( < A > \) denotes the integral of the residue \( (\mathrm{int}_w = \int \mathrm{dw} \mathrm{a}_w(w)) \) for the generic pseudodifferential operator \( A = \ldots + a_{-1} \partial^{-1} + \ldots \).

An infinite set of compatible Poisson brackets structures can be introduced, leading to a (multi)-hamiltonian structure for the flows (2.2). The first integrals of motion are hamiltonians in involution with respect to all Poisson brackets.

The flows (2.2) involve an infinite set of fields. The reduction procedure of the KP hierarchy consists in introducing algebraic constraints on such fields, so that only a finite number of them would be independent. Such constraints must be compatible with the flows (2.2). As a final result one gets a hierarchy of integrable differential equations involving a finite number of fields only.

The canonical way to perform a reduction consists in imposing the constraint

\[ L^n = L^n_+ \]  

(2.4)

which tells that the \( n \)-th power of \( L \) is a purely differential operator, for a given positive integer \( n = 2, 3, \ldots \). Such reductions lead to generalized KdV hierarchies: for \( n = 2 \) one gets the KdV equation, for \( n = 3 \) the Boussinesq one and so on. The hamiltonian structure for such reduced hierarchies is induced from the hamiltonian structure of the original unreduced KP. These hierarchies are the ones originally described by Drinfeld-Sokolov [3]. Under the Poisson brackets structure the fields entering \( L^n \) satisfy a classical finite non-linear \( \mathcal{W} \) algebra (of polynomial type).

In the limit of dispersionless Lax equation (which is taken by assuming the fields not depending on the spatial coordinate \( x \)) and in the Fourier-transformed basis (the operator \( \partial \equiv p \), \( p \) the momentum) the Lax operator \( L^n \) is just given by a polynomial in \( p \) of order \( n \).

The set of reductions given by the constraint (2.4) does not exhaust the set of all possible reductions compatible with the flows of KP. Indeed in the literature other consistent reductions have been discussed (see e.g. [4],[5]). They are called multi-fields reductions of KP. In the language of [3] they are labelled by two positive integers \( p, q \) and called
generalized \((p, q)\) KdV hierarchies. For this new class of reductions there exists no integer \(n\) such that the constraint (2.4) holds. As a basic feature of this new class, a non-linear \(W_{\infty}\) algebra is associated to each reduction, instead of just the polynomial \(W\) algebra associated to the standard Drinfeld-Sokolov reductions.

In [9] we have shown, working out explicitly some examples, that this new set of reductions can be derived from factoring a Kac-Moody subalgebra out of a given Kac-Moody or polynomial \(W\) algebra (coset construction). Furthermore, the structure of non-linear \(W_{\infty}\) algebra associated to such a coset is encoded in an underlining structure of finite rational \(W\) algebra (since the notion of rational \(W\) algebra has been fully explained in [11, 9], we will not discuss it here). Even if we do not dispose of a formal proof telling that any coset factorization determines its corresponding KP reduction, we believe this statement to be true. Indeed, for any example of coset worked out so far we were able to find its associated KP-reduced hierarchy.

Before going ahead, let us constraint \(U_0 \equiv 0\) in (2.2) and let us discuss the first two flows for \(k = 1, 2\). We get respectively

\[
\frac{\partial}{\partial t_1} U_j = U'_j
\]

\[
\frac{\partial}{\partial t_2} U_j = U''_j + 2U'_{j+1} - 2\sum_{r=1}^{j-1} (-1)^r \left( \begin{array}{c} j-1 \\ r \end{array} \right) U_{j-r} \partial^r U_1
\]

(from now on we use the standard convention of denoting the spatial derivative with a prime and the time derivative with a dot if no confusion concerning the flow arises) for any \(j = 1, 2, \ldots\).

The first flow is trivial, while the second one provides a set of non-linear equations.

For later convenience (and in order to derive the KP reduction we are going to discuss from an underlining coset algebra which provides the hamiltonian structure) let us introduce at this point a covariant derivative \(D\) (whose precise definition will be given later), acting on covariant fields with definite charge. An important point is that the covariant derivative satisfies the same rules, in particular the Leibniz rule, as the ordinary derivative and coincides with the latter one when acting upon chargeless fields. At a formal level, the formulas giving the action of covariant derivatives on covariant fields look the same as those involving ordinary derivatives. An example of that is the following important commutation rule

\[
D^{-k} f = f D^{-k} + \sum_{r=1}^{\infty} (-1)^r \left( \begin{array}{c} k+r-1 \\ r \end{array} \right) f^{(r)} D^{-k-r}
\]

(here \(f^{(r)} \equiv D^r f\) and \(k\) is a positive integer).

A consistent reduced version of the KP hierarchy can be expressed as the Lax operator

\[
L = D + J_+ \cdot D^{-1} J_+ \equiv \partial + J_+ \cdot D^{-1} J_+
\]

(2.7)

Let us introduce the composite fields \(V_n = J_+ \cdot D^n J_+\). The reduction (2.7) implies the identification

\[
U_n = (-1)^{n-1} V_{n-1}, \quad n = 1, 2, \ldots
\]

1the following discussion will be limited to covariant derivatives defined for an abelian \(U(1) - K.M\) algebra, even if the non-abelian case can be considered on the same foot as well.
where the $U_i$’s are the fields appearing in (2.1). It can be easily checked that the above position is indeed a reduction, namely that is consistent with the flows (2.2); this statement is proved as follows: at first one should notice that, due to the properties of the covariant derivative, an algebraic relation holds

$$V_{p+1} \cdot V_0 = V_0 \cdot \partial V_p + (V_1 - \partial V_0) V_p$$

(2.9)

which allows to algebraically express the fields $V_p$, for $p \geq 2$, in terms of the fundamental ones $V_0$ and $V_1$. Due to standard properties of the Newton binomial, the equations for $j > 2$ in the flows (2.5, b) are compatible with the algebraic relation (2.9) after taking into account the substitutions (2.8).

So far we have discussed the reduction of the KP hierarchy at a purely algebraic level, without mentioning any hamiltonian structure. Up to now the introduction of a covariant derivative was not effective since, as we have already remarked, covariant and ordinary derivatives play the same role if only algebra is concerned. The introduction of a covariant derivative is at least a very convenient tool to make contact with the hamiltonian dynamics and it proves to be crucial for regarding the (2.7) reduction as a coset construction.

Let us assume the fields $J_\pm(x), J_0(x)$ to satisfy the $sl(2)$ Kac-Moody algebra

$$\{J_+(z), J_-(w)\} = \partial_w \delta(z - w) - 2J_0(w)\delta(z - w) \equiv \mathcal{D}(w)\delta(z - w) \quad \{J_0(z), J_\pm(w)\} = \pm J_\pm(w)\delta(z - w) \quad \{J_0(z), J_0(w)\} = -\frac{1}{2}\partial_w \delta(z - w) \quad \{J_\pm(z), J_\pm(w)\} = 0$$

(2.10)

the covariant derivative $\mathcal{D}$ is defined acting on covariant fields $\Phi_q$ of definite charge $q$ as

$$\mathcal{D} = (\partial + 2qJ_0)\Phi_q$$

(2.11)

The property of covariance for the field $\Phi_q$ being defined through the relation

$$\{J_0(z), \Phi_q(w)\} = q\Phi_q(w)\delta(z - w)$$

(2.12)

As its name suggests, the covariant derivative maps covariant fields of charge $q$ into new covariant fields having the same charge. In particular $J_\pm$ are covariant fields with respect to $J_0$ and have charge $\pm 1$ respectively, so that

$$\mathcal{D}J_\pm = \partial J_\pm \pm 2J_0 \cdot J_\pm$$

(2.13)

The algebraic relations (2.10) of the $sl(2)$-Kac-Moody can be seen as a first Poisson bracket structure (denoted as $\{\cdot, \cdot\}_1$) for the reduced (2.7) hierarchy. It is a trivial check indeed to show that the first two integrals of motion $F_{1,2}$ (2.3) are proportional to $H_{1,2}$:

$$H_1 = \int (J_- \cdot J_+)$$

$$H_2 = -\int (J_- \cdot \mathcal{D}J_+)$$

(2.14)
which are hamiltonians in involution with respect to the (2.10) Poisson brackets: $H_{1,2}$ reproduce respectively the first and the second flow of (2.5) under the substitution (2.8):

$$\frac{\partial}{\partial t_1} V_n = \{H_1, V_n\} = V'_n$$
$$\frac{\partial}{\partial t_2} V_n = \{H_2, V_n\} = V''_n - 2V'_{n+1} - 2 \sum_{r=1}^{n} \binom{n}{r} V_{n-r} \partial^r V_0$$

(2.15)

for $n = 0, 1, \ldots$.

Our framework allows us to accomodate a second compatible Poisson brackets structure which is given by

$$\{J_-(z), J_-(w)\} = 0$$
$$\{J_+(z), J_+(w)\} = -\delta(z-w)(J_+)^2(w)$$
$$\{J_+(z), J_-(w)\} = \mathcal{D}_w^2 \delta(z-w) + \delta(z-w)J_+(w)J_-(w)$$

(2.16)

To understand the above relations, one should notice that they are obtained from the corresponding relations for the first Poisson brackets structure (2.10) after taking into account the substitutions

$$J_- \mapsto J_-$$
$$J_+ \mapsto -\mathcal{D}J_+$$

(2.17)

The compatibility of first and second Poisson brackets simply means the following equality being satisfied

$$\dot{f} = \{H_1, f\} = \{H_2, f\}$$

(2.18)

The composite fields $V_n$ entering (2.10) are by construction chargeless, i.e. they have vanishing Poisson brackets with respect to $J_0$

$$\{J_0(z), V_n(w)\} = 0$$

(2.19)

They constitute a linearly independent basis for the composite chargeless bilinear fields (bilinear invariants); namely any such field can be obtained as a linear combination of the $V_n$ fields and ordinary derivatives acting on them. Under the first Poisson brackets structure the fields $V_n$ form a closed non-linear algebra. The only finite subset which is closed with respect to this algebra is given by $V_0$ itself: as soon as every other field is added, one needs the whole infinite set of fields to close the algebra. These bilinear invariants therefore provide the reduction (2.7) with the structure of a non-linear $W_\infty$ algebra. Since however the $V_n$ fields, even if linearly independent, are not algebraically independent due to relations like (2.9), the non-linear $W_\infty$ algebra structure can be regarded as encoded in the more compact structure of finite rational $W$ algebra. For more details and for the explicit expression of such rational algebra see [9].

The fact that the fields $V_n$ have vanishing Poisson brackets with respect to the $U(1) - \mathcal{KM}$ subalgebra of the Kac-Moody $sl(2)$ means that we have found the explicit link between our KP-reduction (2.7) and the coset factorization.
In [9] another such reduction was considered in full detail; it was associated to the Lax operator
\[
\tilde{L} = D^2 + T + W_+ \cdot D^{-1} W_-
\] (2.20)

Such operator has not the form of a KP operator, but it is however possible to introduce the uniquely defined “square root” \( \tilde{L}^{1/2} \) of \( \tilde{L} = \tilde{L}^{1/2} \cdot \tilde{L}^{1/2} \) which is of KP-type (\( \tilde{L}^{1/2} = D + \ldots \)).

The fields \( T, W_\pm \) entering (2.20), are respectively a chargeless stress-energy tensor and two (opposite charged) bosonic spin \( \frac{3}{2} \) fields; the charge being defined with respect to an \( U(1) - \mathcal{KM} \) current \( J \) entering the covariant derivative. The fields \( J, T, W_\pm \) form a closed algebra which is nothing else that the non-linear Polyakov-Bershadski \( \mathcal{W} \) algebra [21]. It plays the same role of first Poisson brackets structure leading to a hamiltonian dynamics for the flow associated to the (2.20) Lax operator, just like the \( sl(2) - \mathcal{KM} \) algebra in the previous case. The same steps done before can be repeated in this case too.

In general, starting from a given coset algebra, it is quite an easy Ansatz to find out the form of the reduced KP Lax operator; the following steps should be performed: at first the Kac-Moody currents of the factorized subalgebra should be accommodated into a single covariant derivative, then with the help of dimensional considerations one should identify the \( U_n \) fields of (2.1) with invariants constructed out of covariant fields, the original ones in the algebra as well as the covariant derivatives applied on them. The only difficulty left consists in explicitly checking the consistency of such reduction with the KP flow, as well as its link with the hamiltonian dynamics provided by the algebra itself.

We close this section with a remark: in the limit of dispersionless Lax equation, and taking into account that \( J_0 \) is a constant (\( \equiv \alpha \)) with respect to any flow due to the relations (2.19), the reduced operators (2.7) and (2.20) are respectively given by
\[
\begin{align*}
L & \rightarrow p + \frac{\lambda}{p + \alpha} \\
\tilde{L} & \rightarrow p^2 + t + \frac{\lambda}{p + \alpha}
\end{align*}
\] (2.21)

with \( \alpha, \lambda \) and \( t \) constants.

It is remarkable that the reductions associated to rational \( \mathcal{W} \) algebras lead, in the dispersionless limit, to Lax operators fractional in \( p \) (this is always true in any case of coset construction), while the Drinfeld-Sokolov reductions associated to polynomial \( \mathcal{W} \) algebras lead to Lax operators polynomial in \( p \).

3 From NLS to a modified NLS hierarchy via Wakimoto representation of the \( sl(2) - \mathcal{KM} \) algebra.

In this section we study more closely the hierarchy associated to the reduced KP operator (2.7). We show that it coincides with the two-components formulation of the NLS hierarchy. In terms of the second hamiltonian \( H_2 = -\int (J_-D J_+) \) we get indeed the following equations
\[
\dot{J}_\pm = \{J_\pm, H_2\}_1 = \pm D^2 J_\pm \pm 2 (J_+ J_-) J_\pm
\] (3.22)
This is the coupled system associated to the NLS equation. Due to the results mentioned in the previous section it is consistent to set \( J_0 \equiv 0 \), which further implies \( D^2 J_\pm = J_\pm'' \). Next, the standard NLS equation is recovered by letting the time being imaginary. Such “Wick rotation” allows making the identification

\[
J_-^* = J_+ = u
\]  

(3.23)

We obtain finally

\[
i\dot{u} = u'' + 2u|u|^2
\]  

(3.24)

which is the NLS equation in its standard form \[22\].

At this point we should recall that equivalent integrable equations can arise in two different ways: either because they are associated to different hamiltonians belonging to the same hierarchy of hamiltonians in involution, or because there exists a mapping between them. This is the case concerning the relation between KdV and m-KdV equations, the latter being the equation involving the free field \( \varphi \), which is related via Miura transformation to the \( v \) field satisfying the KdV equation; for the KdV Lax operator this reads as follows

\[
\partial^2 + v = (\partial - \varphi)(\partial + \varphi) = \partial^2 + \varphi' - \varphi^2
\]  

(3.25)

Generalizations of this construction hold for any hierarchy of Drinfeld-Sokolov type.

The framework we developed in the previous section is particularly useful for describing the analogue free-fields mappings in the case of coset reductions. There exists indeed a standard free field representation of the \( sl(2) - K\mathcal{M} \) algebra which is given by the (classical) Wakimoto representation \[13\]. It is realized in terms of the weight 1 field \( \nu \) and the bosonic \( \beta - \gamma \) system of weight \((1, 0)\), satisfying the algebra

\[
\{\beta(z), \gamma(w)\} = -\{\gamma(z), \beta(w)\} = \delta(z - w)
\]

\[
\{\nu(z), \nu(w)\} = \partial_w \delta(z - w)
\]  

(3.26)

(any other Poisson bracket is vanishing).

The \( sl(2) - K\mathcal{M} \) algebra given in \[29\] is reproduced through the identifications

\[
J_+ = \beta
\]

\[
J_0 = -\beta\gamma + \frac{i}{\sqrt{2}}\nu
\]

\[
J_- = \beta\gamma^2 - i\sqrt{2}\gamma\nu + \partial\nu
\]  

(3.27)

Representing the hamiltonian \( H_2 \) in terms of the Wakimoto fields, one can derive the coupled system

\[
\dot{\beta} = \{\beta, H_2\}_1 = \beta'' + 2\beta^2\gamma' - 2\beta^3\gamma^2
\]

\[
\dot{\gamma} = \{\gamma, H_2\}_1 = \gamma'' - 2\gamma^2\beta' - 2\gamma^3\beta^2
\]  

(3.28)

\(^2\) in the standard notation for the quantum Wakimoto representation \( \nu \equiv \partial\phi \), \( \phi \) is the fundamental field satisfying the OPE \( \phi(z)\phi(w) \sim log(z - w) \).
(we used here the consistent constraint $J_0 = 0$ to get rid of the field $\nu$ in the above equations). The fields $\beta, \gamma$ enter in a symmetric way in the above system and we can forget about the different weights between them we originally introduced to define the Wakimoto representation. If we let indeed the spatial coordinate being imaginary ($\partial_x \mapsto i\partial_x$), it is consistent to set

$$\gamma^* = \beta = \lambda$$

so that the final result is

$$\dot{\lambda} = -\lambda'' + 2\lambda(i\lambda \partial \lambda^* - |\lambda|^4)$$

It is a remarkable fact that the modified NLS system (3.28) should be regarded as some sort of dual version of the original NLS system (3.22). In the latter case the reduction to the single component NLS equation is done by assuming the time being imaginary, while in the m-NLS case this is provided by assuming the space being imaginary.

The construction here discussed can be trivially extended to any coset arising from generic Kac-Moody algebra. The free-fields analogue of the Wakimoto representation is in this case provided by (the classical version of) the results of ref. [23].

Let us finally stress the point that in our approach to the KP-coset reduction the connection with the free-fields representation is particularly explicit, since we did not need to introduce any Dirac brackets arising from the constraint $J_0 \equiv 0$: in our framework all computations are performed using the original Poisson brackets structure.

4 The coset derivation of the $N = 1$ super-NLS equation.

In this section we will set up a manifestly supersymmetric framework to derive via coset construction $N = 1$ supersymmetric integrable hierarchies. There are two basic motivations for doing that. The first concerns of course the construction of superintegrable hierarchies, which are interesting by their own, and have been widely studied in the literature (see e.g. [15, 16]). The second motivation lies in better understanding the coset construction itself. Before any attempt of classifying the cosets and before giving general formal proofs of their link with the hierarchies, it is interesting to investigate how they look in the case of superalgebras.

It should be kept in mind that even if our discussion will concern the super-NLS hierarchy only, in no respect this example is crucial. The same approach here discussed can be straightforwardly applied to derive other supersymmetric coset hierarchies. It is enough for that to apply the machinery here developed to any given coset algebra. The advantage of discussing the super-NLS case lies in its technical simplicity.

The super-NLS case is however not an academical exercise and it is interesting to compare our results with that of [18, 17]. In [18] two distinct supersymmetrizations, one of these involving a free parameter, of the NLS equation have been proposed. It is stated that both lead to an integrable hierarchy. In [17] manifestly supersymmetric NLS equations have been investigated. It has been shown that applying on such equations
conventional tests of integrability only the supersymmetric system without any free parameter is selected. Moreover there exists a discrepancy in the coefficients with respect to [18]. The coset construction we are going to discuss will automatically provide the super-NLS integrable system of ref. [17] with the same coefficients (therefore supporting the statement of [17] that a misprint occurs in [18]). Our coset construction implies that associated to such system there exists a non-linear super-\(\mathcal{W}_\infty\) algebra involving an infinite series of primary bosonic (of integral dimension \(h = 1, 2, \ldots\)) and fermionic (of half-integral dimension \(h = \frac{3}{2}, \frac{5}{2}, \ldots\)) \(N = 1\) superfields. Such super-\(\mathcal{W}_\infty\) algebra can be regarded as a rational super-\(\mathcal{W}\) algebra. The existence of this non-linear super-\(\mathcal{W}_\infty\) algebra is already an indication of the integrability properties of our super-NLS system. This statement is made precise by associating to the coset a consistent reduction of the super-KP hierarchy. Our Lax operator is different from the one discussed in [17].

Let us fix now our conventions concerning the superspace. We denote with capital letters the \(N = 1\) supercoordinates \((X \equiv x, \theta, \text{ with } x \text{ and } \theta \text{ real, respectively bosonic and grassmann, variables})\). The supersymmetric spinor derivative is given by

\[
D \equiv D_X = \frac{\partial}{\partial \theta} + \theta \frac{\partial}{\partial x}
\]

With the above definition \(D_X X = \frac{\partial}{\partial x}\).

The supersymmetric delta-function \(\Delta(X,Y)\) is a fermionic object

\[
\Delta(X,Y) = \delta(x - y)(\theta - \eta)
\]

It satisfies the relations

\[
\Delta(X,Y) = -\Delta(Y,X) \quad D_X \Delta(X,Y) - D_Y \Delta(X,Y)
\]

Our convention for the integration over the grassmann variable is

\[
\int d\theta \cdot \theta = -1
\]

For any given superfield \(F(X)\) we get then

\[
\int dY \Delta(X,Y) F(Y) = F(X)
\]

As in the bosonic case, the (super)-line integral over a total derivative gives a vanishing result. The canonical dimensions \(d\) are respectively given by

\[
d(D) = d(\Delta) = -d(\theta) = -2d(x) = \frac{1}{2}
\]

The role which in the bosonic case is played by the ordinary derivative is now played by the spinor derivative of dimension \(d = \frac{1}{2}\). It makes plausible that now covariant spinor derivatives should be constructed in terms of spin \(\frac{1}{2}\) fermionic superfields. An example of supersymmetric rational \(\mathcal{W}\) algebra involving such kind of derivatives has indeed been given in [24].
The $N = 1$ counterpart of the $U(1) - \mathcal{KM}$ current $J_0(z)$ should be expressed by the fermionic superfield $\Psi_0(X) = \psi_0(x) + \theta J_0(x)$, satisfying the super-Poisson brackets relation
\begin{equation}
\{\Psi_0(X), \Psi_0(Y)\} = D_Y \Delta(X,Y) \tag{4.37}
\end{equation}
which implies, at the level of components
\begin{align*}
\{\psi_0(x), \psi_0(y)\} &= -\delta(x - y) \\
\{J_0(x), J_0(y)\} &= -\partial_y \delta(x - y) \tag{4.38}
\end{align*}
Super-covariant fields and the supercovariant derivative can now be introduced through
\begin{align*}
\{\Psi_0(X), \Phi_q(Y)\} &= q\Delta(X,Y) \Phi_q(Y) \\
\mathcal{D} \Phi_q &= D \Phi_q + q \psi_0 \Phi_q \tag{4.39}
\end{align*}
$\Phi_q$ is a covariant superfield (either bosonic or fermionic).

We are now in the position to discuss the algebra providing the first (super)-Poisson brackets structure for the super-NLS equation. As suggested in [17], the component fields should be accommodated in two fermionic spin $\frac{1}{2}$ superfields $\Psi_\pm = \psi_\pm + \theta J_\pm$. With the above choice one can identify the bosonic components $J_\pm$ with the analogue fields we already encountered in the bosonic case. The relevant algebra can therefore be simply guessed to be the supersymmetric analogue of the $sl(2) - \mathcal{KM}$ algebra, introduced through the relations
\begin{align*}
\{\Psi_0(X), \Psi_\pm(Y)\} &= \pm \Delta(X,Y) \Psi_\pm(Y) \\
\{\Psi_+(X), \Psi_-(Y)\} &= \mathcal{D}_Y \Delta(X,Y) = D_Y \Delta(X,Y) + \Delta(X,Y) \Psi_0(Y) \tag{4.40}
\end{align*}
One indeed recover the (2.10) algebra by setting all the component spin $\frac{1}{2}$ fermionic fields equal to 0.

We can define, just like in the bosonic case, the composite superfields $V_n(X)$, where
\begin{equation}
V_n = \Psi_- \mathcal{D}^n \Psi_+ \quad n = 0, 1, 2, ... \tag{4.41}
\end{equation}
By construction they have vanishing Poisson brackets with respect to $\Psi_0$:
\begin{equation}
\{\Psi_0(X), V_n(Y)\} = 0 \tag{4.42}
\end{equation}
The superfields $V_n$ are respectively bosonic for even values of $n$ and fermionic for odd values. They play the same role as the corresponding fields in the purely bosonic case: they constitute a basis of linearly independent superfields for the chargeless composite superfields. The super Poisson brackets (4.37, 4.40) provide such basis of fields with the structure of non-linear super-$\mathcal{W}_\infty$ algebra that will be discussed later in more detail.

In order to associate to the coset algebra a hamiltonian dynamics like we did in the bosonic case we proceed as follows: we recall that the superfields $V_n$ have positive
\footnote{we recall that super-Poisson brackets are symmetric when taken between odd elements, antisymmetric otherwise.}
dimensions $d(V_n) = \frac{n+2}{2}$; then we look for all possible hamiltonian densities of a given
dimension that one can algebraically construct out of the superfields $V_n$ and the covariant
derivative (of dimension $\frac{1}{2}$) acting upon them. For any given dimension only a finite
number of such combinations are allowed. Since now we are working in a manifestly
supersymmetric framework and the (super)-line integral is fermionic, so the hamiltonian
densities must be fermionic of half-integral dimension. The first two possible hamiltonian
densities at dimension $3/2$ and $5/2$ respectively are just given by $V_1$ and $V_3$. The latter is
indeed the unique, up to a total derivative, chargeless $d = \frac{5}{2}$ object.

It can easily checked now that $H_{1,2}$ given by

$$
H_1 = \int dXV_1(X) = \int dX(\Psi_- \cdot D\Psi_+)
$$
$$
H_2 = \int dXV_3(X) = \int dX(\Psi_- \cdot D^3\Psi_+)
$$

have vanishing Poisson brackets among themselves with respect to $[4.37, 4.40]$ and can
therefore been regarded as hamiltonians in involution. Two compatible flows are defined
through

$$
\frac{\partial}{\partial t_1} \Psi_+ = \{H_1, \Psi_+\} = D^2\Psi_+
$$
$$
\frac{\partial}{\partial t_2} \Psi_+ = \{H_2, \Psi_+\} = \pm D^4\Psi_+ \mp \Psi_+ D(\Psi_+ D\Psi_+)
$$

(4.44)

The latter equation is the $N = 1$ supersymmetric version of the two-components NLS. As
in the bosonic case, if we let the time $t_2$ be imaginary we can consistently set

$$
\Psi_+ = \Psi_-^* = \Psi
$$

to get the super-NLS equation

$$
i\dot{\Psi} = \Psi^{(4)} - D\Psi D(\Psi^* \Psi^{(1)})
$$

(4.45)

(in order to simplify the notation from now on the symbol $A^{(n)} \equiv D^n A$ will be used).

Since $\Psi_0 = 0$ makes consistent to set $\Psi_0 = 0$, the above equation leads to the following
system in component fields ($\Psi = \phi + \theta q$, $\phi$ fermionic and $q$ bosonic):

$$
i\dot{\phi} = \phi_{xx} + \phi(\phi^* \phi_x - q^* q)$$
$$
i\dot{q} = q_{xx} - (qq^*) q + (\phi_x^* - \phi^* \phi_x) q + (\phi \phi^*) q_x
$$

(4.46)

As already stated, this equation coincides with the integrable super-NLS equation of ref. [17].

The supersymmetric character of the above equations is guaranteed by the invariance
of the hamiltonians $H_{1,2}$ under the transformations

$$
\delta \Psi_\pm = \pm \varepsilon D\Psi_\pm
$$

(4.47)

where $\varepsilon$ is a grassmann parameter.
The existence of a bihamiltonian structure is derived as in the bosonic case. The second super-Poisson brackets structure is given by

\[
\{\Psi_-(X), \Psi_-(Y)\}_2 = 0 \\
\{\Psi_-(X), \Psi_+(Y)\}_2 = \Delta^{(3)} - \Delta^{(1)} \Psi_- \Psi_+ + \Delta \Psi_{-}^{(1)} \Psi_+ \\
\{\Psi_+(X), \Psi_+(Y)\}_2 = \Delta^{(2)} \Psi_+ \Psi_{+}^{(1)} - \Delta^{(1)} \Psi_+ \Psi_+^{(2)} + \Delta \Psi_{+}^{(1)} \Psi_+^{(2)}
\]  

(4.48)

(The superfields on the right hand side are evaluated in \( Y \) and \( \Delta^{(n)} = D^n Y \Delta(X,Y) \)).

This second Poisson brackets structure is derived from the first one after the substitutions

\[
\Psi_- \mapsto \Psi_- \\
\Psi_+ \mapsto D^2 \Psi_+
\]  

(4.49)

are taken into account.

The compatibility of the two Poisson brackets structure is ensured, like in the bosonic case, by the relation

\[
\frac{dF}{dt} = \{H_1, F\}_2 = \{H_2, F\}_1
\]  

(4.50)

Precisely like the bosonic case, the two hamiltonians \( H_{1,2} \) are the first two of an infinite series of hamiltonians mutually in involution. This statement will be justified later when we show how to associate to the system \( \{4.44\} \) a reduction of the super-KP hierarchy.

A comment is in order. The algebra \( \{4.37, 4.40\} \) is the simplest possible algebra realized in terms of supercurrents and allowing a Kac-Moody coset construction. There is another very simple supercurrent algebra, which is realized by just coupling to the \( \Psi_0 \) superfield two bosonic superfields \( \Phi_{\pm} \) (instead of two fermionic ones) of dimension \( \frac{1}{2} \). The expression of this algebra looks like \( \{4.40\} \) but now one has to take into account the antisymmetric property when exchanging \( \Phi_{\pm} \) in the super-Poisson brackets. If we define the charges being the super-line integral over the supercurrents (as \( H = \int dX \Psi_0, E_{\pm} = \int dX \Psi_{\pm} \)), then the algebra \( \{4.37, 4.40\} \) generates a global \( sl(2) \) algebra for the charges, while the algebra determined by \( \Psi_0 \) and the bosonic super currents is promoted to the global superalgebra \( osp(1|2) \) (with generators \( H, F_{\pm} = \int dX \Phi_{\pm} \)). In \( \{17\} \) a zero-curvature formulation for the system \( \{4.44\} \) was found; it is based on the \( sl(2) \) algebra. They claimed being unable to derive an analogue formulation starting from \( osp(1|2) \). The reason is simply because this is associated with a radically different system, the dynamics being in this case defined for the bosonic \( \Phi_{\pm} \) superfields. The fact that the dynamics differs from the fermionic case can be immediately seen using the following argument: an invariant composite superfield \( W_0 \cdot W_1 \) \((W_n = \Phi_- D^n \Phi_+)\) is allowed entering in the second hamiltonian density of dimension \( \frac{5}{2} \), while the corresponding composite superfield \( V_0 \cdot V_1 \) is vanishing in the fermionic case for the antisymmetry of \( \Psi_{\pm} \). Our coset construction can be performed for this bosonic case as well, leading to an interesting superintegrable system, which of course has nothing to do with the supersymmetrization of the NLS equation, since the component bosonic fields have spin \( \frac{1}{2} \) and not 1. It is likely that for such a system the zero-curvature formulation would be based on the superalgebra \( osp(1|2) \). We leave a detailed discussion of it for a further publication.
5 Comments on the non-linear super-$\mathcal{W}_\infty$ coset algebra.

Let us make some more comments here concerning the non-linear super-$\mathcal{W}_\infty$ algebra structure of the coset algebra. Its linear generators are the superfields $V_n$, $n$ non-negative integer, defined in (4.41). The superfields are bosonic for even values of $n$, fermionic for odd values. The set $\{V_0, V_1\}$ constitutes a finite super-algebra, given by the Poisson brackets

$$
\{V_0(X), V_0(Y)\} = -\Delta(X,Y)(DV_0 + 2V_1)(Y)
$$
$$
\{V_0(X), V_1(Y)\} = \Delta^{(2)}(X,Y)V_0(Y) + \Delta^{(1)}(X,Y)V_1(Y) - \Delta(X,Y)DV_1(Y)
$$
$$
\{V_1(X), V_1(Y)\} = -2\Delta^{(2)}(X,Y)V_1(Y) - \Delta(X,Y)D^2V_1(Y)
$$

(5.51)

In terms of component fields it is given by two bosons of spin 1 and 2 respectively, and two spin $\frac{3}{2}$ fermions. It is the maximal finite subalgebra of the coset superalgebra: as soon as any other superfield is added to $V_0, V_1$, the whole set of fields $V_n$ is needed to close the algebra, giving to the coset the structure of a super-$\mathcal{W}_\infty$ algebra. Moreover such algebra closes in non-linear way.

Using the techniques developed in [11] it is possible to show the existence of an equivalent basis for expressing our super-$\mathcal{W}_\infty$ algebra, given by the infinite set of superfields $W_h(X)$, which are primary with conformal dimension $h$ with respect to the stress-energy tensor (having vanishing central charge) $T(X) \equiv W_3(X)$. To any integral value of $h$ ($h = 1, 2, ..$) is associated a bosonic primary superfield; to any half-integral value ($h = \frac{3}{2}, \frac{5}{2}, ...$) a fermionic one.

The condition of being primary means that the superfields $W_h$ satisfy the relation

$$
\{T(X), W_h(Y)\} = -h\Delta^{(2)}(X,Y)W_h(Y) + \frac{1}{2}\Delta^{(1)}(X,Y)DW_h(Y) - \Delta(X,Y)D^2W_h(Y)
$$

(5.52)

We have at the lowest orders

$$
W_1 = V_0 = \Psi_-\Psi_+
$$
$$
T = V_1 + \frac{1}{2}DV_0 = \frac{1}{4}D\Psi_- \cdot \Psi_+ + \frac{1}{2}\Psi_- \cdot D\Psi_+
$$
$$
W_2 = 3V_2 + DT - \frac{3}{2}DV_0 = \Psi_- \cdot D^2\Psi_+ + D\Psi_- \cdot D\Psi_+ - D^2\Psi_- \cdot \Psi_+
$$

(5.53)

We wish finally to make some comments on the rational character of the above defined super-$\mathcal{W}_\infty$ algebra: the whole set of algebraic relations can be expressed just in terms of closed rational super-$\mathcal{W}$ algebra involving 4 superfields as the following reasoning shows: let us introduce the superfields

$$
\Lambda_p = def \quad D\Psi_- \cdot D^{(p+1)}\Psi_+
$$
then

\[ \Lambda_p = DV_{p+1} - V_{p+2} \]

Due to standard properties of the covariant derivative we can write down for the superfields \( \Lambda_p \) the analogue of the relation (2.9) of the bosonic case:

\[ \Lambda_0 \Lambda_{p+1} = \Lambda_0 D \Lambda_p + (\Lambda_1 - D \Lambda_0) \Lambda_p \quad (5.54) \]

which implies that \( \Lambda_p \) are rational functions of \( \Lambda_{0,1} \), which in their turns are determined by \( V_i, i = 0, 1, 2, 3 \).

Inverting the relation (5.54) we can express any higher field \( V_{p+1} \) in terms of \( V_p, \Lambda_{p-1} \). As a consequence of this we have the (rational) closure of the superalgebra on the superfields \( V_0, V_1, V_2, V_3 \).

We remark that now is not possible, like in the bosonic case, to determine higher order superfields \( V_p \) from the formula (5.54) by simply inserting \( V_0, V_p \) in place of \( \Lambda_0, \Lambda_p \): this is due to the fact that any product \( V_0 \cdot V_{p+1} \) identically vanishes since it is proportional to a squared fermion \( (\Psi_+^2 = 0) \). That is the reason why four superfields are necessary to produce a finite rational algebra and not just two as one would have naively expected.

6 The \( N = 1 \) superWakimoto representation and the modified super-NLS equation.

In this very short section we will repeat the construction of section 3, furnishing the \( N = 1 \) super-Wakimoto representation of the \((4.37, 4.40)\) algebra and associating to the super-NLS equation (4.44) its modified version.

The classical super-Wakimoto representation is realized in terms of three free superfields, denoted as \( B, C, N \):

\[
\begin{align*}
B(X) &= b(x) + \theta \beta(x) \\
C(X) &= \gamma(x) + \theta c(x) \\
N(X) &= \mu(x) + \theta \nu(x)
\end{align*}
\]

(6.55)

\( B, N \) are assumed to be fermionic of dimension \( \frac{1}{2} \), while \( C \) is assumed to be a 0-dimensional bosonic superfield coupled to \( B \). At the level of components we have in particular the already encountered bosonic \( \beta - \gamma \) system of weight \((1, 0)\), plus now a fermionic \( b - c \) system of weight \((\frac{1}{2}, \frac{1}{2})\).

The free superfields super-Poisson brackets are given by

\[
\begin{align*}
\{B(X), C(Y)\} &= \{C(X), B(Y)\} = \Delta(X, Y) \\
\{N(X), N(Y)\} &= D_Y \Delta(X, Y)
\end{align*}
\]

(6.56)

The \((1.37, 1.40)\) superalgebra is reproduced in terms of the superfields \( B, C, N \) through the identifications

\[
\begin{align*}
\Psi_+ &= B \\
\Psi_0 &= -BC + N \\
\Psi_- &= -\frac{1}{2} BC^2 + CN - DC
\end{align*}
\]

(6.57)
Representing $H_2$ in (4.43) via the above system we get an evolution equation for $B, C$. As in the bosonic case the $N$ superfield can be expressed through $B, C$ by setting $\Psi_0 = 0$. Finally, by letting the space being imaginary it is consistent to further set

$$(\mathcal{D}B)^* = C$$

which implies

$$\beta(x) = \gamma(x); \quad b'(x) = c(x)$$

(6.59)

At the end we arrive at the supersymmetric generalization of eq. (3.30), which is given by

$$\dot{B} = -D^4 B + B(D(C^*DC) - \frac{1}{2}|C|^4)$$

(6.60)

## 7 Integrable properties of the $N = 1$ super-NLS equation: the super-KP reduction.

We have already discussed the indications of integrability associated to the super-NLS equation arising from its bihamiltonian structure. Moreover we are aware of the results of [17] concerning the integrability. In this section we will show that the equation (4.44) deserves the name of super-NLS hierarchy by explicitly associating to it a reduction of the super-KP operator. Before doing that let us spend some words on the supersymmetric (with graded derivative) version of the KP hierarchy. The standard reference we follow in this case is [14].

The super-KP operator is given by

$$L = D + \sum_{i=0}^{\infty} U_i(X) D^{-i}$$

(7.61)

where now $D$ is the fermionic derivative and the $U_i$’s are superfields. For even values of $i$ they are fermionic, for odd values bosonic. In the following we will be interested only to the flows associated to even (bosonic) time. For a discussion concerning odd-time flows see e.g. [25]. The even-time flows are defined through

$$\frac{\partial L}{\partial t_k} = [L^{2k+}, L]$$

(7.62)

where $L^r_+$ denotes the purely differential part of $L^r$. The above flows provide a set of equations for the infinite series of superfields $U_i$. To derive such equations we recall that $D^{-1} = D\partial^{-1}$ and the commutation rule (2.6) can be employed.

If we set the constraint

$$DU_0 + 2U_1 = 0$$

(7.63)

then $L^2_+ = D^2 = \partial$ and the first flow is trivial. With the above constraint we get

$$L^4_+ = D^4 + FD + B$$

(7.64)
where
\[
F = 2DU_1 \\
B = 4U_3 + 2DU_2 - 6U_1U_1
\] (7.65)

The second flow \((k = 2)\) is non-trivial and provides the following set of equations
\[
\frac{\partial U_{2n}}{\partial t_2} = U_{2n}^{(4)} + 2U_{2n+2}^{(2)} + FU_{2n}^{(1)} + 2FU_{2n+1} - U_{2n-1}B^{(1)} + \\
\sum_{r=1}^{n-1} (-1)^{r+1} \binom{n-1}{r} (U_{2n-2r}B^{(2r)} + U_{2n-2r-1}B^{(2r+1)}) + \\
\sum_{r=1}^{n} (-1)^{r} \binom{n}{r} U_{2n-2r+1}F^{(2r)}
\] (7.66)
for the fermionic superfields, and
\[
\frac{\partial U_{2n-1}}{\partial t_2} = U_{2n-1}^{(4)} + 2U_{2n+1}^{(2)} + FU_{2n-1}^{(1)} - F^{(1)}U_{2n-1} + \\
\sum_{r=1}^{n-1} (-1)^{r+1} \binom{n-1}{r} (U_{2n-2r-1}B^{(2r)} + U_{2n-2r-1}F^{(2r+1)} + U_{2n-2r}F^{(2r)})
\] (7.67)
for the bosonic ones.

In order to define the reduced super-KP operator we compare this flows with the set of equations
\[
\dot{V}_n = \{V_n, H_2\}
\] (7.68)
for the superfields \(V_n = \Psi - \mathcal{D}^n\Psi_+\) introduced in (4.41), provided by the hamiltonian \(H_2\) given in (4.43), with respect to the (1.37)(4.40) Poisson brackets structure.

We get the following equations, for respectively fermionic and bosonic superfields
\[
\frac{\partial V_{2n+1}}{\partial t_2} = \partial^2 V_{2n+1} - 2\partial V_{2n+3} - V_{2n+1}\partial V_0 - V_{2n}\partial V_1 + \\
\sum_{k=0}^{n-1} \binom{n}{k} (V_{2k+1}\partial^{n-k}DV_1 - V_{2k}\partial^{n-k+1}V_1)
\] (7.69)
and
\[
\frac{\partial V_{2n}}{\partial t_2} = \partial^2 V_{2n} - 2\partial V_{2n+2} - V_{2n}\partial V_0 + \\
\sum_{k=0}^{n-1} \binom{n}{k} V_{2k}\partial^{n-k}DV_1
\] (7.70)

In order to produce a consistent super-KP reduction we must be able to fit the above equations in the corresponding equations for the \(U_i\) superfields. This can not be done,
or at least we were unable to do that, for the whole set of $V_n$ superfields. However the following considerations can be made: we remark that the equations of motion for bosonic superfields (labelled by an even integer) involve on the right hand side bosonic superfields only. It is therefore consistent with the dynamics to set all the bosonic superfields $V_{2n} \equiv 0$. We argue that this constraint should be imposed in order to proceed to the right supersymmetrization of the NLS hierarchy: indeed the corresponding generators of the coset algebra in the bosonic case are given by the $J_- D^n J_+$ fields, which implies having a single bosonic field for each integral value of the spin ($n + 2$). In the supersymmetric theory one expects that the fermionic counterparts should be associated to such fields: for each half-integral value of the spin one should have a single fermionic field. The set of superfields $V_n$, $n = 0, 1, 2, \ldots$ is in this respect highly redundant: it provides two bosons and two fermions respectively for each integer and half-integer spin value $s \geq \frac{3}{2}$, plus a single spin 1 bosonic field arising from $V_0$ which plays no role in the NLS hierarchy. To get rid of this redundancy, a constraint which kills the extra degrees of freedom should be imposed. A constraint which allows doing that is just provided by setting

$$V_{2n} = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(7.71)

It is remarkable the consistency of this constraint with the dynamics, as we have just pointed out.

After taking into account of (7.71), the equation for the fermionic superfields $V_{2n-1}$ is reduced to

$$\dot{V}_{2n-1} = \partial^2 V_{2n-1} - 2\partial V_{2n+1} + \sum_{k=0}^{n-1} \binom{n-1}{k} V_{2k-1} \partial^{n-k} D V_1$$

(7.72)

It is immediately checked at this point that a consistent reduction of the super-KP hierarchy is recovered by setting

$$U_{2n-1} = 0$$

$$U_{2n} = \frac{1}{2} (-1)^n V_{2n-1} \quad \text{for} \quad n = 1, 2, \ldots$$

(7.73)

The corresponding reduced super-KP operator can be compactly written as

$$L = D + \frac{1}{2} \Psi_+ D^{-2} \Psi_+^{(1)}$$

(7.74)

with $\Psi_+^{(1)} = D \Psi_+$.

The integrability properties of the super-NLS hierarchy are established due to the existence of such Lax operator.

## 8 The $N = 2$ formalism.

Let us introduce here the framework and conventions for working in a manifestly supersymmetric $N = 2$ formalism.

The $N = 2$ superspace is parametrized by the bosonic $x$ coordinate and two grassmann variables $\theta, \bar{\theta}$. A generic superfield is then expanded as

$$\Phi(X) = \phi(x) + \theta f(x) + \bar{\theta} \bar{f}(x) + \theta \bar{\theta} g(x)$$

(8.75)
The $N = 1$ case is recovered when letting $\theta = \bar{\theta}$.

Two spinor derivatives $\tilde{D}$, $\overline{D}$ are defined as

$$
\tilde{D} = \frac{\partial}{\partial \theta} + \bar{\theta} \partial_x \\
\overline{D} = \frac{\partial}{\partial \bar{\theta}} + \theta \partial_x
$$

(8.76)

They satisfy the relations

$$
\tilde{D}^2 = \overline{D}^2 = 0 \\
\{ \tilde{D}, \overline{D} \} = 2 \partial_x
$$

(8.77)

It is convenient (we come later on this point) to describe the $N = 2$ theory in terms of constrained superfields, namely the chiral ($\Psi$) and antichiral ($\overline{\Psi}$) superfields, defined respectively by

$$
\overline{D} \Psi = 0 \\
\tilde{D} \overline{\Psi} = 0
$$

(8.78)

Due to the above relation the derivated superfields $\tilde{D} \Phi$ and $\overline{D} \Phi$ are respectively antichiral and chiral superfields.

The condition of chirality implies the following expansions in component fields

$$
\tilde{A} = a(x) + \theta \alpha(x) + \bar{\theta} \alpha(x)' \\
\overline{B} = b(x) - \bar{\theta} \beta(x) - \theta \beta(x)'
$$

(8.79)

and the derivated superfields are

$$
\tilde{D} \tilde{A} = \alpha(x) + 2\overline{\theta} \alpha(x)' - \theta \overline{\theta} \alpha(x)'' \\
\overline{D} \overline{B} = \beta(x) + 2\theta \beta(x)' + \overline{\theta} \theta \beta(x)''
$$

(8.80)

It is remarkable that chiral and antichiral superfields can be expressed as $N = 1$ superfields in relation with the superspaces

$$
\check{X} = (\hat{x} = x + \theta \bar{\theta}, \theta) \\
\check{\check{X}} = (\hat{x} = x - \theta \bar{\theta}, \bar{\theta})
$$

(8.81)

respectively.

Moreover if we introduce the $N = 1$ spinor derivative $D$ as

$$
D = D_x = \frac{\partial}{\partial \theta} + 2\theta \frac{\partial}{\partial x}
$$

(8.82)

(allowing a factor 2 difference with respect to the convention used in the previous sections), then we can write the derivated superfields as

$$
\tilde{D} \tilde{A} \equiv D \tilde{A}|_{\check{X}} \\
\overline{D} \overline{B} \equiv D \overline{B}|_{\check{X}}
$$

(8.83)
The existence of the $N = 1$ superfield representation for chiral and antichiral superfields is particularly useful for our purposes because it allows defining the $N = 2$ supersymmetric theory in terms of the $N = 1$ superfield formalism developed in the previous sections. In particular we can define super-Poisson brackets structures as done before: they will depend on the $N = 1$ supersymmetric delta-function already encountered (and the derivative acting on it).

The supersymmetric line integral for chiral and antichiral superfields are given respectively by

\[
\begin{align*}
    d\hat{X} &\equiv dX X = (x, \theta) \\
    d\check{X} &\equiv d\overline{X} \overline{X} = (x, \overline{\theta})
\end{align*}
\]

The two equivalence relations are due to the fact that the term proportional to $\theta\overline{\theta}$ is a total derivative for both chiral and antichiral superfields.

Let us spend now some more words about using (anti)-chiral superfields to describe $N = 2$ theories. The dynamics of a real superfield $\Psi$ can always be recovered from the dynamics of two conjugated chiral and antichiral superfields $\widetilde{\Psi}, \overline{\Psi}$ (we recall that the $g(x)$ field appearing in (8.75) is just an auxiliary field, dynamically determined in terms of the component fields $\phi, f, \bar{f}$).

It turns out that the $N = 2$ dynamics can be expressed by using two conjugated sets of equations of motion for chiral and antichiral superfields. Such equations of motion are defined in terms of conjugated (anti-)chiral hamiltonians whose combination gives a single real hamiltonian. For integrable systems the dynamics can also be expressed through two chirally conjugated Lax operators whose combination provide a single real Lax operator. Further details concerning such construction can be found in (20). In the following we will define the $N = 2$ super-NLS equation in terms of these two conjugated sets of chiral superfields.

### 9 The $N = 2$ super-NLS hierarchy.

Let us introduce now the $N = 2$ super-NLS hierarchy, extending to this case the procedure already worked out for the bosonic and $N = 1$ NLS theories.

According to the discussion developed in the previous section, it is clear that now we should define our $N = 2$ hierarchy by “doubling” the number of superfields of the $N = 1$ case: we should look for two (chiral and antichiral) covariant derivatives defined in terms of the spin $\frac{1}{2}$ superfields $\widetilde{\Psi}_0, \overline{\Psi}_0$. Moreover we should have two sets of opposite charged (anti-)chiral superfields $\widetilde{\Psi}_\pm, \overline{\Psi}_\pm$. These two sets of superfields should be seen as chirally conjugated.

Let us define now the two conjugated covariant derivatives: we introduce first the conjugate spin $\frac{1}{2}$ superfields $\widetilde{\Psi}_0, \overline{\Psi}_0$. There is a freedom in choosing the normalization condition for their super-Poisson brackets algebra. Let us fix it by assuming

\[
\begin{align*}
    \{\widetilde{\Psi}_0(X), \overline{\Psi}_0(Y)\} &= \{\overline{\Psi}_0(X), \overline{\Psi}_0(Y)\} = D_Y \Delta(X, Y) \\
    \{\widetilde{\Psi}_0(X), \overline{\Psi}_0(Y)\} &= 0
\end{align*}
\]

(9.85)
with $D_Y$ given by (8.82).

Next the notion of covariant superfield can be introduced: $V$ is said covariant with charges $(\tilde{q}, \bar{q})$ if it satisfies the relations

$$\{\tilde{\Psi}_0(X), V(Y)\} = \tilde{q}\Delta(X, Y)V(Y) \quad (9.86)$$

and an analogous one with $\tilde{\cdot} \mapsto \cdot$.

The covariant derivative $D$, mapping covariant superfields of charges $(\tilde{q}, \bar{q})$ into superfields of the same charge, is in this case given by

$$DV = (D + \tilde{q}\tilde{\Psi}_0 + \bar{q}\bar{\Psi}_0)V \quad (9.87)$$

At this point we have all the ingredients to define the complete super currents algebra involving $\tilde{\Psi}_0, \tilde{\Psi}_0, \bar{\Psi}_0, \bar{\Psi}_0$ which allows us to define the $N = 2$ super-NLS theory. After a little inspection one can realize that our game can be played by simply postulating such algebra as given by two separated copies of the $N = 1$ (4.37, 4.40) algebra. A fundamental point is that now, in order to recover the non-trivial equations of motion which involve together chiral and antichiral superfields, the two $N = 1$ supercurrents algebras should mix chiral and antichiral superfields.

We can assume the two copies being given by $(\tilde{\Psi}_-, \tilde{\Psi}_0, \bar{\Psi}_+)$ and $(\tilde{\Psi}_-, \tilde{\Psi}_0, \bar{\Psi}_+)$, with the following charges for the $\tilde{\Psi}_\pm, \bar{\Psi}_\pm$ superfields

$$\tilde{\Psi}_- \equiv (0, -1)$$
$$\tilde{\Psi}_+ \equiv (0, 1)$$
$$\bar{\Psi}_- \equiv (-1, 0)$$
$$\bar{\Psi}_+ \equiv (1, 0) \quad (9.88)$$

The complete algebra is given by

$$\{\tilde{\Psi}_0(X), \bar{\Psi}_+(Y)\} = \Delta(X, Y)\bar{\Psi}_+(Y)$$
$$\{\tilde{\Psi}_0(X), \tilde{\Psi}_-(Y)\} = -\Delta(X, Y)\tilde{\Psi}_-(Y)$$
$$\{\bar{\Psi}_+(X), \tilde{\Psi}_-(Y)\} = (D_Y - \tilde{\Psi}_0(Y))\Delta(X, Y) = D_Y\Delta(X, Y)$$
$$\{\tilde{\Psi}_0(X), \bar{\Psi}_+(Y)\} = \Delta(X, Y)\bar{\Psi}_+(Y)$$
$$\{\tilde{\Psi}_0(X), \tilde{\Psi}_+(Y)\} = -\Delta(X, Y)\tilde{\Psi}_+(Y)$$
$$\{\bar{\Psi}_+(X), \tilde{\Psi}_-(Y)\} = (D_Y - \tilde{\Psi}_0(Y))\Delta(X, Y) = D_Y\Delta(X, Y) \quad (9.89)$$

Together with (9.83). All other super-Poisson brackets are vanishing.

There exists of course a superWakimoto representation, provided by two sets of chirally conjugated superfields: the bosonic superfields $\hat{C}, \hat{\bar{C}}$ of weight 0, and the fermionic ones $\hat{B}, \hat{\bar{B}}, \hat{\bar{N}}, \hat{\bar{N}}$ of weight $\frac{1}{2}$. The $B$’s and $C$’s superfields generate two coupled systems.

The superalgebra of the free Wakimoto superfields is just provided by

$$\{\hat{B}(X), \hat{C}(Y)\} = \Delta(X, Y)$$
$$\{\hat{C}(X), \hat{B}(Y)\} = \Delta(X, Y)$$
$$\{\hat{\bar{N}}(X), \hat{\bar{N}}(Y)\} = D_Y\Delta(X, Y) \quad (9.90)$$
and an equivalent relation with $\hat{\cdot} \mapsto \check{\cdot}$. The superfields identifications are the same as in (5.57):

$$
\begin{align*}
\overline{\Psi}_+ &= \hat{B} \\
\overline{\Psi}_0 &= -\hat{B}\hat{C} + \hat{B} \\
\tilde{\Psi}_- &= -\frac{1}{2}\hat{B}\hat{C}^2 + \hat{C}\hat{N} - D\hat{C}
\end{align*}
$$

(9.91)

and the analogous relations involving the second set of superfields.

Inspired by the $N = 1$ results we can define at this point our dynamics as determined by the two conjugated sets of (anti-)chiral hamiltonians in involution. The first two ($\tilde{H}_{1,2}$ and the conjugates $\overline{H}_{1,2}$) are given by

$$
\begin{align*}
\tilde{H}_1 &= \int dX \tilde{H}_1 = \int dX (\tilde{\Psi}_- D\overline{\Psi}_+) \\
\tilde{H}_2 &= \int dX \tilde{H}_2 = \int dX (\tilde{\Psi}_- D^3\overline{\Psi}_+)
\end{align*}
$$

(9.92)

and

$$
\begin{align*}
\overline{H}_1 &= \int dX \overline{H}_1 = \int dX (\overline{\Psi}_- D\tilde{\Psi}_+) \\
\overline{H}_2 &= \int dX \overline{H}_2 = \int dX (\overline{\Psi}_- D^3\tilde{\Psi}_+)
\end{align*}
$$

(9.93)

The real hamiltonians are given by

$$
H_{1,2} = \tilde{H}_{1,2} + \overline{H}_{1,2}
$$

(9.94)

They are invariant under the $N = 2$ supersymmetry transformations

$$
\begin{align*}
\delta \tilde{\Psi}_\pm &= \pm \epsilon D\overline{\Psi}_\pm \\
\delta \overline{\Psi}_\pm &= \pm \epsilon D\tilde{\Psi}_\pm
\end{align*}
$$

(9.95)

Moreover the hamiltonian densities $\tilde{H}_j, \overline{H}_j$ have by construction vanishing Poisson brackets with respect to the subalgebra generators $\tilde{\Psi}_0, \overline{\Psi}_0$, namely they are in the commutant.

The equations of motion are introduced through the following equations

$$
\frac{\partial}{\partial t_j} F = \{H_j, F\}
$$

(9.96)

After using the algebraic relations (9.85,9.89), and taking into account that we can consistently set

$$
\tilde{\Psi}_0 = \overline{\Psi}_0 = 0
$$

(9.97)

we get the flows:

$$
\begin{align*}
\frac{\partial}{\partial t_1} \tilde{\Psi}_\pm &= \tilde{\Psi}_\pm' \\
\frac{\partial}{\partial t_1} \overline{\Psi}_\pm &= \overline{\Psi}_\pm
\end{align*}
$$

(9.98)
and
\[
\frac{\partial}{\partial t_2} \tilde{\Psi}_\pm = \pm \tilde{\Psi}''_\pm \mp \tilde{\Psi}_\pm D(\tilde{\Psi}_\mp D\tilde{\Psi}_\pm)
\]
\[
\frac{\partial}{\partial t_2} \Psi_\pm = \pm \Psi''_\mp \mp \Psi_\pm \bar{D}(\tilde{\Psi}_\mp \bar{D}\Psi_\pm)
\]
(9.99)

The second flow provides the two-components \(N = 2\) super-NLS equation.

Notice that the chirality condition is respected by the equations of motion as it should be.

On the right hand side chiral and antichiral superfields are coupled together in the non-linear term. This ensures the theory having the genuine feature of a non-trivial \(N = 2\) supersymmetry. The \(N = 1\) equation is recovered by assuming \(\theta = \bar{\theta}\) which implies \(\tilde{\Psi}_\pm = \bar{\Psi}_\pm\).

It is clear that one can straightforwardly repeat the same steps as done in the \(N = 1\) construction. The same structures appear in this case as well. Let us recall them briefly:

i) existence of a compatible bihamiltonian structure relating the first two Hamiltonians.

ii) \(N = 2\) generalization of the modified super-NLS equation arising by the super-Wakimoto representation for the algebra \((9.83, 9.89)\).

iii) existence of the (coset) \(N = 2\) non-linear super-\(W_\infty\) algebra, promoted to be a finite rational super-\(W\) algebra. It is linearly generated by the chargeless superfields

\[
V_{2n} = \tilde{\Psi}_- D^{2n} \bar{\Psi}_+
\]
\[
V_{2n+1} = \tilde{\Psi}_- D^{2n+1} \bar{\Psi}_+
\]
\[
W_{2n} = \bar{\Psi}_- D^{2n} \tilde{\Psi}_+
\]
\[
W_{2n+1} = \bar{\Psi}_- D^{2n+1} \tilde{\Psi}_+
\]
(9.100)

The fermionic superfields \(V_{2n+1}, W_{2n+1}\) have half-integral spin \(\frac{2n+3}{2}\). When evaluated at \(\tilde{\Psi}_0 = \bar{\Psi}_0 = 0\) they are respectively chiral and antichiral, and can be expressed as

\[
V_{2n+1} = \tilde{\Psi}_- (2\partial)^n \bar{D}\Psi_+
\]
\[
W_{2n+1} = \bar{\Psi}_- (2\partial)^n \tilde{D}\tilde{\Psi}_+
\]
(9.101)

The bosonic superfields \(V_{2n}, W_{2n}\) of spin \(n + 1\) have not a definite chirality. Notice that, as it should be, our \(N = 2\) super-\(W\) algebra admits a “doubled” number of superfields with respect to the \(N = 1\) case.

iv) existence of a dynamically consistent constraint, which allows setting the bosonic superfields \(V_{2n}, W_{2n}\) equal to zero. This implies in its turn a “reduced dynamics” involving only the chiral and antichiral fermionic superfields; such a dynamics is particularly important because it gives rise to a consistent reduction of the \(N = 2\) super-KP hierarchy provided by the two conjugate Lax operators of definite chirality.

These two conjugate Lax operators are given by

\[
\bar{L} = \bar{D} + \bar{\Psi}_- D^{-2} \Psi_+^{(1)}
\]
\[
\bar{L} = \bar{D} + \bar{\Psi}_- D^{-2} \tilde{\Psi}_+^{(1)}
\]
(9.102)
where

\[ \Psi_+^{(1)} = D\Psi_+ \]
\[ \bar{\Psi}_+^{(1)} = D\bar{\Psi}_+ \]  

(9.103)

\( \bar{L} \) is chiral, \( \mathcal{T} \) antichiral.

Once expanded, they are expressed in terms of the \( V_{2n+1}, W_{2n+1} \) superfields respectively, which are invariants under the \( N = 2 \) Kac-Moody superalgebra (9.85):

\[ \bar{L} = \bar{D} + \sum_{k=0}^{\infty} (-1)^k V_{2k+1} \partial^{-k} \]
\[ \mathcal{T} = \mathcal{D} + \sum_{k=0}^{\infty} (-1)^k W_{2k+1} \partial^{-k} \]  

(9.104)

(we have replaced the covariant derivative with the standard one, which is allowed when \( \bar{L}, \mathcal{T} \) act on chargeless superfields).

The dynamics for the \( V_{2n+1}, W_{2n+1} \) superfields derived in terms of flows of the super-KP reduced operator (9.104) coincides with the just mentioned “reduced dynamics” of \( V_{2n+1}, W_{2n+1} \) arising from the hamiltonian formulation.

**Conclusions**

In this paper we have furnished a method to derive what we can call (in analogy to the bosonic case) multi-superfields reductions of the super-KP hierarchy, which are a further generalization of the commonly studied generalized super-KdV hierarchies.

In the particular example here considered we obtained some new results concerning the form of the super-Lax operator, the connection with a super\( \mathcal{W}_\infty \) algebra, the link with the modified super-NLS equation, etc.

According to our “coset method” the multifields reductions are obtained from cosets of (in this case super) Kac-Moody algebras.

We would like to spend some words about the coset method and why it deserves being further investigated: it allows having a nice algebraic interpretation for the Poisson brackets structures of the theories involved; more than that, it could provide an algebraic classification of the multi-fields (super) KP reductions if the attracting hypothesis that they are all associated to cosets proves to be correct. Since our method makes use of covariant derivatives and is not based on a hamiltonian reduction (and consequently on Dirac’s brackets) it implies a nice free-fields interpretation and mapping to modified hierarchies as explained in the paper. This could prove useful when discussing quantization (it is tempting indeed to repeat our procedure for let’s say the \( q \)-deformed affine \( sl(2) \) algebra).

In order to attack the most important point, concerning the classification of the (super) KP reductions some preliminary results will be needed: we can mention for instance understanding the coset method in the light of the AKS scheme, expliciting the connection between the (unreduced) KP hierarchy Poisson brackets structure and those coming from the cosets, computing the associated \( r \)-matrices with methods like those developed...
Such results are needed for a formal proof of the statement that any coset gives rise to a KP-reduction. We will address all these points in forthcoming papers.

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