THE SEMI-CLASSICAL LIMIT WITH DELTA POTENTIALS

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Introduction

The semi-classical limit of Quantum Mechanics for Gaussian coherent states is a well established subject and, in the case of a regular ($C^2$) potential $V$, many comprehensive results are available. Here we are interested in the case where the potential has no regularity at all: $V = \delta_0$. 
**Introduction**

The semi-classical limit of Quantum Mechanics for Gaussian coherent states is a well established subject and, in the case of a regular ($C^2$) potential $V$, many comprehensive results are available. Here we are interested in the case where the potential has no regularity at all: $V = \delta_0$. We focus our attention on normalized Gaussian coherent states of the following form: given $\sigma_0 > 0$, we define, for any $\sigma \in \mathbb{C}$, $\xi \equiv (q, p) \in \mathbb{R}^2$,

$$\psi_{\sigma, \xi} : \mathbb{R} \rightarrow \mathbb{C},$$

by

$$\psi_{\sigma, \xi}^\hbar (x) := \frac{1}{(2\pi \hbar)^{1/4} \sqrt{\sigma}} \exp \left( -\frac{1}{4 \hbar \sigma_0 \sigma} (x - q)^2 + \frac{i}{\hbar} p(x - q) \right).$$
Notice that for the expectations on such states of the position $Q$ and momentum $P$ operators one has

\[
\langle Q \rangle_{\psi^\hbar_{\sigma_0,\xi}} := \langle \psi^\hbar_{\sigma_0,\xi}, Q \psi^\hbar_{\sigma_0,\xi} \rangle_{L^2(\mathbb{R})} = q,
\]

\[
\langle P \rangle_{\psi^\hbar_{\sigma_0,\xi}} := \langle \psi^\hbar_{\sigma_0,\xi}, Q \psi^\hbar_{\sigma_0,\xi} \rangle_{L^2(\mathbb{R})} = p
\]

and the $\psi^\hbar_{\sigma_0,\xi}$'s saturate the uncertainty relations, i.e.,

\[
\langle \Delta Q \rangle_{\psi^\hbar_{\sigma_0,\xi}} \langle \Delta P \rangle_{\psi^\hbar_{\sigma_0,\xi}} = \hbar / 2,
\]

\[
\langle \Delta Q \rangle_{\psi^\hbar_{\sigma_0,\xi}} := \sqrt{\langle Q^2 \rangle_{\psi^\hbar_{\sigma_0,\xi}} - \langle Q \rangle_{\psi^\hbar_{\sigma_0,\xi}}^2} ;
\]

\[
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\]
In the case the evolution is generated by the free Hamiltonian

\[ H_0 : H^2(\mathbb{R}) \subseteq L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad H_0 \psi := -\frac{\hbar^2}{2m} \psi'' , \]

the quantum-classical correspondence is exact:

\[ \left( e^{-i t \frac{\hbar}{\hbar} H_0} \psi_{\sigma_0, \xi} \right)(x) = e^{i A_t} \left( e^{i t L_0} \Phi_{\sigma_t, x}^{\hbar} \right)(\xi) . \]
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\[ \left( e^{-i\frac{t}{\hbar}H_0} \psi^h_{\sigma_0,\xi} \right)(x) = e^{i\frac{t}{\hbar}A_t} \left( e^{itL_0} \phi^h_{\sigma_t,\xi} \right)(\xi). \]

Here

\[ \phi^h_{\sigma,\xi} : \mathbb{R}^2 \rightarrow \mathbb{C}, \quad \phi^h_{\sigma,\xi} \in L^\infty(\mathbb{R}^2), \quad \phi^h_{\sigma,\xi}(\xi) := \psi^h_{\sigma,\xi}(x), \]
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Here
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and \[ e^{itL_0} f(q, p) = f(q + \frac{p}{m} t, p) , \]
i.e. \[ e^{itL_0} \] is the realization in \( L^\infty(\mathbb{R}^2) \) of the strongly continuous (in \( L^2(\mathbb{R}^2) \)) group of evolution generated by the self-adjoint operator
\[ L_0 := -i X_0 \cdot \nabla , \quad X_0(q, p) := \left( \frac{p}{m}, 0 \right) . \]
Now we consider the case in which $H_0$ is replaced by
\[
H_\alpha := H_0 + \alpha \delta_0 , \quad \alpha \in \mathbb{R} \cup \{\infty\},
\]
where the self-adjoint $H_\alpha$ is defined by means of the bounded from below, closed quadratic form
\[
\text{dom}(Q_\alpha) = H^1(\mathbb{R}) , \quad Q_\alpha(\psi) := \|\psi'\|_{L^2(\mathbb{R})}^2 + \alpha |\psi(0)|^2 ,
\]
\[
\text{dom}(Q_\infty) = H^1_0(\mathbb{R}_-) \oplus H^1_0(\mathbb{R}_+) , \quad Q_\infty(\psi) := \|\psi'\|_{L^2(\mathbb{R})}^2 .
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The suggestion coming from the free case is clear: since $H_\alpha$ is a self-adjoint extension of $H^0 = H_0|\mathcal{C}_c^\infty(\mathbb{R}\setminus\{0\})$, one is lead to look for an approximation of the kind

$$e^{-i\frac{t}{\hbar}H_\alpha} \psi_{\sigma_0,\xi}(x) \simeq e^{i\frac{t}{\hbar}A_t} (e^{it\Lambda_\beta} \phi_{\sigma_t,x}^\hbar)(\xi), \quad \hbar \ll 1,$$
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The suggestion coming from the free case is clear: since $H_\alpha$ is a self-adjoint extension of $H_0^0 = H_0|C^\infty_c(\mathbb{R}\setminus\{0\})$, one is lead to look for an approximation of the kind

$$e^{-i\frac{t}{\hbar}H_\alpha} \psi_0^\hbar,\xi(x) \simeq e^{i\hbar A_t} \left( e^{i t L_\beta} \phi_0^\hbar,\xi(x) \right)(\xi) , \quad \hbar \ll 1 ,$$

where $L_\beta$ is some self-adjoint extension, if any, of

$$L_0^\circ := L_0|C^\infty_c(M_0) , \quad M_0 := \mathbb{R}^2 \setminus \Sigma_0 , \quad \Sigma_0 := \{(0, p) | p \in \mathbb{R}\} ,$$

and $\phi_0^\hbar,\xi(x)$ is transformed using the realization in $L^\infty(\mathbb{R}^2)$, if any, of the unitary $e^{itL_\beta}$. 
Let \( h_0(q, p) = \frac{p^2}{2m} \) be the Hamiltonian function of a free particle in \( \mathbb{R} \) and let \( X_0(q, p) = \left( \frac{p}{m}, 0 \right) \) be the related Hamiltonian vector field.
Singular perturbations of the free classical dynamics

Let \( h_0(q, p) = \frac{p^2}{2m} \) be the Hamiltonian function of a free particle in \( \mathbb{R} \) and let \( X_0(q, p) = (\frac{p}{m}, 0) \) be the related Hamiltonian vector field. Then we consider the differential operator

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L f := -i X_0 \cdot \nabla f = -i \frac{p}{m} \frac{\partial}{\partial q}.
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Since the flow of \( X_0 \) in \( M_0 \) is obviously not complete, by Povzner’s theorem \( L|C_c^\infty(M_0) \) is not essentially self-adjoint in \( L^2(\mathbb{R}^2) \); it has infinitely many self-adjoint extensions and its defect indices are \((+\infty, +\infty)\).
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Even if it would be possible to find all such self-adjoint extensions, we are interested in a subset of extensions \( L_\beta \) parametrized by \( \beta \in \mathbb{R} \cup \{\infty\} \) such that the \( \beta = 0 \) case gives the free dynamics and the \( \beta = \infty \) case gives the dynamics corresponding to complete reflection.
Denoting by $L_0$, $\text{dom}(L_0) := \{ f \in L^2(\mathbb{R}^2) : Lf \in L^2(\mathbb{R}^2) \}$, the maximal realization of $L$ in $L^2(\mathbb{R}^2)$, $L_0$ is self-adjoint and its spectrum is purely absolutely continuous, $\sigma(L_0) = \sigma_{ac}(L_0) = \mathbb{R}$. We set $R_0^z := (L_0 - z)^{-1}$, $z \in \mathbb{C}\setminus\mathbb{R}$. 
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We set $R_0^z := (L_0 - z)^{-1}$, $z \in \mathbb{C}\setminus\mathbb{R}$. Given the trace map

$$ (\gamma \phi)(p) := \frac{1}{2} \left( \phi(0^+, p) + \phi(0^-, p) \right), $$

we define the bounded operators $G_z : L^2(\mathbb{R}, |p|^{-1} dp) \to L^2(\mathbb{R}^2)$,

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($\theta$ denotes the Heaviside step function)
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($\theta$ denotes the Heaviside step function) and

$$\Lambda^\beta_z := \left( \frac{1}{\beta} - \gamma G_z \right)^{-1} : L^2(\mathbb{R}, |p| dp) \rightarrow L^2(\mathbb{R}, |p|^{-1} dp),$$
Denoting by $L_0$, $\text{dom}(L_0) := \{f \in L^2(\mathbb{R}^2) : Lf \in L^2(\mathbb{R}^2)\}$, the maximal realization of $L$ in $L^2(\mathbb{R}^2)$, $L_0$ is self-adjoint and its spectrum is purely absolutely continuous, $\sigma(L_0) = \sigma_{ac}(L_0) = \mathbb{R}$. We set $R_z^0 := (L_0 - z)^{-1}$, $z \in \mathbb{C} \setminus \mathbb{R}$. Given the trace map

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$$(\Lambda_z^\beta \phi)(p) = \frac{\phi(p)}{\frac{1}{\beta} - \text{sgn}(\text{Im}(z)) \frac{i m}{2|p|}}.$$
**Theorem.** For any $\beta \in (\mathbb{R}\setminus\{0\}) \cup \{\infty\}$, the family of linear bounded operators

$$R^\beta_z := R^0_z + G_z \Pi \Lambda^\beta_z \Pi G^*_z,$$

$z \in \mathbb{C}\setminus\mathbb{R},$

is the resolvent of a self-adjoint extension of the densely defined symmetric operator $L|_{C_c^\infty(\mathcal{M}_0)}$. Here $\Pi$ denotes the projection onto even functions $(\Pi \phi)(p) := \frac{1}{2} (\phi(p) + \phi(-p))$. 


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is the resolvent of a self-adjoint extension of the densely defined symmetric operator $L|C_c^\infty(\mathcal{M}_0)$. Here $\Pi$ denotes the projection onto even functions \((\Pi \phi)(p) := \frac{1}{2} (\phi(p) + \phi(-p))\). Such an extension $L_\beta$ acts on its domain

$$
dom(L_\beta) := \{ f \in L^2(\mathbb{R}^2) : f = f_z + G_z \phi, f_z \in dom(L_0), \phi \in L^2(\mathbb{R}, |p|^{-1} dp), \Pi \gamma f = \frac{1}{\beta} \phi \}
$$

by \((L_\beta - z)f = (L_0 - z)f_z\), equivalently

$$
L_\beta f = Lf - \phi \delta_{\Sigma_0} \equiv Lf - \beta \Pi \gamma f \delta_{\Sigma_0}.
$$

Moreover, $\sigma(L_\beta) = \sigma_{ac}(L_\beta) = \mathbb{R}$. 
The action of the unitary group $e^{itL_\beta}$ can be calculated by the inverse Laplace transform of the resolvent $R_z^\beta$:
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**Proposition.** Let $\beta \in (\mathbb{R}\backslash\{0\}) \cup \{\infty\}$ and $f \in L^2(\mathbb{R}^2)$. Then, for all $t \in \mathbb{R}$, one has

$$
(e^{itL_\beta} f)(q, p) = (e^{itL_0} f)(q, p) - \frac{\theta(-t q p) \theta\left(\frac{|pt|}{m} - |q|\right)}{1 - i \text{sgn}(t) \frac{2|p|}{m^2}} (e^{-itL_0} f_{ev})(q, p),
$$

where

$$(e^{itL_0} f)(q, p) = f\left(q + \frac{pt}{m}, p\right), \quad f_{ev}(q, p) := f(q, p) + f(-q, -p).$$
The action of the unitary group $e^{itL_\beta}$ can be calculated by the inverse Laplace transform of the resolvent $R^\beta_z$:

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(this results also shows that $e^{itL_\beta}$ has a realization as a group of evolution in $L^\infty(\mathbb{R}^2)$)
Notice that while the free operator \( e^{itL_0} \) maps real-valued functions into real-valued functions the same does not hold true for \( e^{itL_\beta} \), unless \( \beta = \infty \) (corresponding to the case of Dirichlet boundary conditions, for which there is complete reflection);
Notice that while the free operator $e^{itL_0}$ maps real-valued functions into real-valued functions the same does not hold true for $e^{itL_\beta}$, unless $\beta = \infty$ (corresponding to the case of Dirichlet boundary conditions, for which there is complete reflection); in this particular case one has

$$(e^{itL_\infty} f)(q, p)$$

$$= \left(1 - \theta(-t q) \theta\left(\frac{|pt|}{m} - |q|\right)\right) f\left(q + \frac{pt}{m}, p\right)$$

$$- \theta(-t q) \theta\left(\frac{|pt|}{m} - |q|\right) f\left(-q - \frac{pt}{m}, -p\right).$$
Notice that while the free operator $e^{itL_0}$ maps real-valued functions into real-valued functions the same does not hold true for $e^{itL_β}$, unless $β = \infty$ (corresponding to the case of Dirichlet boundary conditions, for which there is complete reflection); in this particular case one has

\[
(e^{itL_∞} f)(q, p) = \left(1 - \theta(-t q p) \theta(\frac{|p t|}{m} - |q|)\right) f\left(q + \frac{pt}{m}, p\right) - \theta(-t q p) \theta\left(\frac{|p t|}{m} - |q|\right) f\left(-q - \frac{pt}{m}, -p\right).
\]

This also justifies our introduction of the projector on even functions $Π$: it leads to a family of self-adjoint extensions containing the generator of the dynamics corresponding to complete reflection.
Now we provide explicit formulae for the classical wave operators defined by
\[ W_\beta^{\pm} f := \lim_{t \to \pm \infty} e^{itL_0} e^{-itL_\beta} f \]
and for the corresponding classical scattering operator
\[ S^{cl}_\beta := (W_\beta^+)^* W^-_\beta. \]
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Notice that the classical wave operators for a couple of flows in the phase space \( \mathbb{R}^2 \), \( \varphi^0_t \) (the free one) and \( \varphi_t \) (the interacting one), are defined pointwise by

\[ w^\pm := \lim_{t \to \pm \infty} \varphi_{-t} \circ \varphi^0_t \]
Now we provide explicit formulae for the classical wave operators defined by

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These induces the wave operators acting on functions defined by

\[ W^\pm f := \lim_{t \to \pm \infty} f(\phi_{-t} \circ \phi^0_t) = \lim_{t \to \pm \infty} e^{itL_0} f(\phi_{-t}). \]
Proposition. The above limits exist pointwise in $\mathbb{R}^2$ and in $L^2(\mathbb{R}^2)$ for any $f \in L^2(\mathbb{R}^2)$; it results

$$(W^\pm_\beta f)(q,p) = f(q,p) - \frac{\theta(\mp q p)}{1 \pm i \frac{2|p|}{m_\beta}} f_{ev}(q,p),$$
Proposition. The above limits exist pointwise in $\mathbb{R}^2$ and in $L^2(\mathbb{R}^2)$ for any $f \in L^2(\mathbb{R}^2)$; it results

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(W_{\beta}^{\pm} f)(q, p) = f(q, p) - \frac{\theta(\mp q \cdot p)}{1 \pm i \frac{2|p|}{m_\beta}} f_{\text{ev}}(q, p),
$$

The corresponding scattering operator is given by

$$
(S_{\beta}^{\text{cl}} f)(q, p) = f(q, p) - \frac{f_{\text{ev}}(q, p)}{1 - i \frac{2|p|}{m_\beta}}.
$$
The quantum dynamics with a delta potential

Let us recall that for $\alpha > 0$ the Hamiltonian $H_\alpha$ has purely absolutely continuous spectrum $\sigma(H_\alpha) \equiv \sigma_{ac}(H_\alpha) = [0, \infty)$; in this case, two sets of generalized eigenfunctions for $H_\alpha$ are given by

$$\varphi_k^\pm(x) := \frac{e^{ikx}}{\sqrt{2\pi}} + R_\pm(k) \frac{e^{-i|k||x|}}{\sqrt{2\pi}}, \quad R_\pm(k) = -\frac{1}{1 \pm i \frac{\hbar^2|k|}{m\alpha}} \quad (k \in \mathbb{R}).$$
The quantum dynamics with a delta potential

Let us recall that for $\alpha > 0$ the Hamiltonian $H_\alpha$ has purely absolutely continuous spectrum $\sigma(H_\alpha) \equiv \sigma_{ac}(H_\alpha) = [0, \infty)$; in this case, two sets of generalized eigenfunctions for $H_\alpha$ are given by

$$\varphi_k^\pm(x) := \frac{e^{ikx}}{\sqrt{2\pi}} + R_\pm(k) \frac{e^{-|k|x}}{\sqrt{2\pi}}, \quad R_\pm(k) = -\frac{1}{1 \pm i \frac{\hbar^2|k|}{m\alpha}} \quad (k \in \mathbb{R}).$$

For $\alpha < 0$, the absolutely continuous spectrum of $H_\alpha$ remains $\sigma_{ac}(H_\alpha) = [0, \infty)$; again, this part of the spectrum is related to the set of generalized eigenfunctions $\varphi_k^\pm$. In addition, $H_\alpha$ possesses a proper eigenvalue $\lambda_\alpha < 0$ with corresponding (normalized) eigenfunction $\varphi_\alpha$:

$$\lambda_\alpha := -\frac{m\alpha^2}{2\hbar^2}, \quad \varphi_\alpha(x) := \sqrt{\frac{m|\alpha|}{\hbar}} e^{-\frac{m|\alpha|}{\hbar^2}|x|}.$$
For any $\alpha \in \mathbb{R}$, one introduces the bounded operators

$$F_{\pm} : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (F_{\pm} \psi)(k) := \int_\mathbb{R} dx \varphi_{k}^{\pm}(x) \psi(x)$$

and the orthogonal projector onto the eigenspace generated by $\varphi_{\alpha}$

$$P_\alpha : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (P_\alpha \psi)(x) := \theta(-\alpha) \varphi_{\alpha}(x) \int_\mathbb{R} dy \varphi_{\alpha}(y) \psi(y).$$
For any $\alpha \in \mathbb{R}$, one introduces the bounded operators

$$\mathcal{F}_\pm : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R}), \quad (\mathcal{F}_\pm \psi)(k) := \int_{\mathbb{R}} dx \, \varphi_k^\pm(x) \psi(x)$$

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Then, the time evolution of any state $\psi \in L^2(\mathbb{R})$ induced by the unitary group $e^{-i \frac{t}{\hbar} H_\alpha}$ can be written as

$$(e^{-i \frac{t}{\hbar} H_\alpha} \psi)(x) = \int_{\mathbb{R}} dk \, e^{-i \frac{t}{\hbar} \frac{\hbar^2 k^2}{2m}} \varphi_k^\pm(x) (\mathcal{F}_\pm \psi)(k) + e^{-i \frac{t}{\hbar} \lambda_\alpha} (P_\alpha \psi)(x).$$
For any $\alpha \in \mathbb{R}$, one introduces the bounded operators

$$
\mathcal{F}_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R}), \quad (\mathcal{F}_\pm \psi)(k) := \int_{\mathbb{R}} dx \; \varphi_k^{\pm}(x) \psi(x)
$$

and the orthogonal projector onto the eigenspace generated by $\varphi_\alpha$

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(e^{-i\frac{t}{\hbar}H_\alpha} \psi)(x) = \int_{\mathbb{R}} dk \; e^{-i\frac{t}{\hbar}k^2/2m} \; \varphi_k^{\pm}(x) \; (\mathcal{F}_\pm \psi)(k) + e^{-i\frac{t}{\hbar}\lambda_\alpha} (P_\alpha \psi)(x).
$$

Existence and asymptotic completeness of the wave operators

$$
\Omega_\alpha^{\pm} := s- \lim_{t \to \pm \infty} e^{i\frac{t}{\hbar}H_\alpha} e^{-i\frac{t}{\hbar}H_0}
$$

is well-known and they have an explicit expression in terms of $\mathcal{F}_\pm$ and of the Fourier transform $\hat{F}$:

$$
\Omega_\alpha^{\pm} = \mathcal{F}_\pm^* \mathcal{F}.
$$
The semiclassical limits with a delta potential

Proposition.

\[
\left( e^{-i \frac{t}{\hbar} H_{\alpha}} \psi_{\xi,\sigma_0}^{h} \right)(x) = \left( e^{-i \frac{t}{\hbar} H_{0}} \psi_{\xi,\sigma_0}^{h} \right)(x)
+ \theta(qp) F_{+,t}^{h} (- \text{sgn}(q)|x|) + \theta(-qp) F_{-,t}^{h} (- \text{sgn}(q)|x|)
+ E_{1,t}^{h}(x) + E_{2,t}^{h}(x) + E_{\alpha,t}^{h}(x),
\]

where we set

\[
F_{\pm,t}^{h}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \ e^{-i \frac{\hbar t}{2m} k^2} e^{ikx} R_{\pm}(k) \hat{\psi}_{\xi,\sigma_0}^{h}(k),
\]
\[
E_{1, t}^{\hbar}(x) := \frac{1}{2\pi} \int_{\mathbb{R}} dk \ e^{-i \frac{\hbar t}{2m} k^2} \left( e^{ikx} R_-(k) + e^{-i |k||x|} |R_+(k)|^2 \right) \times \\
\times \int_{\mathbb{R}} dy \ (e^{i |k||y|} - e^{i \text{sgn}(q)|k||y|}) \psi_{\xi, \sigma_0}(y),
\]

\[
E_{2, t}^{\hbar}(x) := \text{sgn}(qp) \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} dk \ e^{-i \frac{\hbar t}{2m} k^2} \ e^{i \text{sgn}(qp) k|x|} \left( R_-(k) - R_+(k) \right) \hat{\psi}_{\xi, \sigma_0}(\text{sgn}(p)k),
\]

and

\[
E_{\alpha, t}^{\hbar}(x) := e^{-i \frac{t}{\hbar} \lambda_\alpha} (P_\alpha \psi_{\xi, \sigma_0}^{\hbar})(x).
\]
The only relevant terms are $F_{\pm,t}^h$:

**Lemma.** There exists a constant $C > 0$ such that, for all $t \in \mathbb{R}$, for all $\lambda > 0$, there holds

$$
\| E_{1,t}^h \|_{L^2(\mathbb{R})} \leq C e^{-\frac{q^2}{4h\sigma_0^2}},
$$

$$
\| E_{2,t}^h \|_{L^2(\mathbb{R})} \leq C e^{-\frac{p^2\sigma_0^2}{h}} \left( \left( \frac{\hbar}{(m\alpha\sigma_0)^{2/3}} \right)^{\frac{3}2 - \frac{3\lambda}{2}} + e^{-\frac{1}{2} \left( \frac{1}{\hbar} (m\alpha\sigma_0)^{2/3} \right)^{2\lambda}} \right),
$$

$$
\| E_{\alpha,t}^h \|_{L^2(\mathbb{R})} \leq C \left( e^{-\frac{m|\alpha||q|}{8\hbar^2}} + e^{-\frac{q^2}{8h\sigma_0^2}} \right).
$$
Proposition. Let $\beta = \frac{2\alpha}{\hbar}$, $\xi \equiv (q, p) \in \mathbb{R}^2$. Then,

$$
eq \frac{i}{\hbar} A_t \left( e^{itL_{\beta}} \Phi_{\sigma_0, x}^h \right) (\xi) = \left( e^{-i\frac{t}{\hbar} H_0 \psi_{\sigma_0, x}^h} \right) (x) + \theta(qp) F_{+, t}^c (x) + \theta(-qp) F_{-, t}^c (x),$$

where we set

$$F_{\pm, t}^c (x) := \theta \left( \pm t \pm \frac{mq}{p} \right) R_{\pm} (p/\hbar) \times$$

$$\times \left( \left( e^{-i\frac{t}{\hbar} H_0 \psi_{\sigma_0, x}^h} \right) (x) + \left( e^{-i\frac{t}{\hbar} H_0 \psi_{\sigma_0, x}^h} \right) (-x) \right).$$
By taking into account the fact that the Gaussian state $\psi_{\sigma_0, \xi}(k)$ is concentrated in a neighborhood of $k = p$, one gets the estimate

**Lemma.**

$$
\| F_{\pm, t}^\hbar - R_{\pm} (p/\hbar) \, e^{-i t/\hbar H_0} \psi_{\sigma_0, \xi}^\hbar \|_{L^2(\mathbb{R})} \\
\leq C \left( \left( \frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}} \right)^{3/2-\lambda} + e^{-\frac{1}{2} \left( \frac{(m|\alpha|\sigma_0)^{2/3}}{\hbar} \right)^{2\lambda}} \right)
$$
Summing up, one gets, whenever

\[ \beta = \frac{2\alpha}{\hbar}, \]

**Theorem.** There exists a constant \( C > 0 \) such that, for any \( \lambda_1, \lambda_2 > 0 \), for any \( t \in \mathbb{R} \) and for any \( \xi \equiv (q, p) \in \mathbb{R}^2 \), \( qp \neq 0 \),

\[
\left\| e^{-i \frac{t}{\hbar} H_\alpha} \psi_{\sigma_0, \xi}^h - e^{i \frac{t}{\hbar} A_t \left( e^{itL_\beta} \phi_{\sigma_t, (\cdot)}^h \right)}(\xi) \right\|_{L^2(\mathbb{R})} \leq C \left( \left( \frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}} \right)^{\frac{3}{2} - \lambda_1} + e^{-\frac{1}{2} \left( \frac{(m|\alpha|\sigma_0)^{2/3}}{\hbar} \right)^{2\lambda_1}} \right. \\
+ e^{-\frac{\sigma_0 p_2^2}{\hbar}} \left( \left( \frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}} \right)^{\frac{3}{2} - \frac{3}{2} \lambda_2} + e^{-\frac{1}{2} \left( \frac{(m|\alpha|\sigma_0)^{2/3}}{\hbar} \right)^{2\lambda_2}} \right) \right. \\
+ e^{-\frac{q^2}{8\hbar^2 \sigma_0^2}} + e^{-\frac{m|\alpha| |q|}{8\hbar^2}} + e^{-\frac{(q+pt/m)^2}{4\hbar|\sigma_t|^2}} \right).
Thus, whenever $t$ is not too close to the collision time

$$t_{\text{coll}}(\xi) := -\frac{mq}{p}, \quad \xi \equiv (q, p),$$

Corollary. Let $\beta = \frac{2\alpha}{\hbar}$. Then, for any $0 < \lambda < 3/2$, there exist constants $c_0, C_*, h_*> 0$ such that

$$h := \max \left\{ \frac{\hbar \sigma_0^2}{q^2}, \frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}} \right\} < h_*$$

implies

$$\left\| e^{-it\frac{\hbar}{\hbar}H_\alpha} \psi^\hbar_{\sigma_0, \xi} - e^{itL_\beta} \Phi^\hbar_{\sigma_t, (\cdot)}(\xi) \right\|_{L^2(\mathbb{R})} \leq C_* h^{3/2 - \lambda}$$

uniformly for any $(t, \xi) \in \mathbb{R} \times \mathbb{R}^2$, such that

$$|t - t_{\text{coll}}(\xi)| \geq c_0 |t_{\text{coll}}(\xi)| \sqrt{\left( \frac{3}{2} - \lambda \right) \hbar |\ln \hbar|}.$$
Notice that, if one approximates the quantum dynamics with the classical dynamics corresponding to the infinite barrier case (i.e., using the operator $L_\infty$), which would seem the most natural choice, the estimates cannot be better than $O(\hbar)$:

\[
\| e^{-it\bar{\hbar}H_\alpha} \psi \bar{\hbar} \sigma_0, \xi - e^{it\bar{\hbar}A}(e^{itL_\infty} \varphi \bar{\hbar} \sigma_t, \cdot) (\xi) \|_{L^2(\mathbb{R})} \geq C^* \bar{\hbar} |p|^{\alpha}.
\]

Also notice that for generic regular potentials the estimates are $O(\bar{\hbar}^{1/2-\lambda})$, $0 < \lambda < 1/2$. 
Notice that, if one approximates the quantum dynamics with the classical dynamics corresponding to the infinite barrier case (i.e., using the operator $L_\infty$), which would seem the most natural choice, the estimates cannot be better than $O(\hbar)$:

**Lemma.** For any $\alpha \neq \infty$,

$$\left\| e^{-i\frac{t}{\hbar}H_\alpha} \psi_{\sigma_0,\xi}^\hbar - e^{i\frac{t}{\hbar}A_t} \left( e^{itL_\infty} \Phi_{\sigma_t,(\cdot)}^\hbar \right)(\xi) \right\|_{L^2(\mathbb{R})} \geq C_* \frac{\hbar|p|}{m|\alpha|}.$$

Notice that, if one approximates the quantum dynamics with the classical dynamics corresponding to the infinite barrier case (i.e., using the operator $L_\infty$), which would seem the most natural choice, the estimates cannot be better than $O(\hbar)$:

Lemma. For any $\alpha \neq \infty$,

$$\left\| e^{-i\frac{t}{\hbar}H_\alpha} \psi_{\sigma_0,\xi} - e^{i\hbar A_t \left( e^{itL_\infty \phi_{\sigma_t,\xi}} \right)}(\xi) \right\|_{L^2(\mathbb{R})} \geq C_* \frac{\hbar|p|}{m|\alpha|}.$$

Also notice that for generic regular potentials the estimates are $O(\hbar^{\frac{1}{2}-\lambda})$, $0 < \lambda < \frac{1}{2}$. 
Of course, the constraint $t \neq t_{coll}$ does not affect the semi-classical approximation for large times:

**Theorem.** Let $\beta = \frac{2\alpha}{\hbar}$. Then, for any $0 < \lambda < 3/2$ there exist constants $C_*, h_* > 0$ such that

$$\hbar := \max \left\{ \frac{\hbar \sigma_0^2}{q^2}, \frac{\hbar}{(m|\alpha|\sigma_0)^{2/3}}, \frac{\hbar}{\sigma_0^2 p^2} \right\} < h_*$$

implies

$$\left\| \Omega_\alpha^{\pm} \psi^{\hbar}_{\sigma_0,\xi} - \left( W^{\pm}_\beta \phi^{\hbar}_{\sigma_0,(\cdot)} \right)(\xi) \right\|_{L^2(\mathbb{R})} \leq C_* h_*^{3/2-\lambda},$$

and

$$\left\| S_\alpha \psi^{\hbar}_{\sigma_0,\xi} - \left( S^{cl}_\beta \phi^{\hbar}_{\sigma_0,(\cdot)} \right)(\xi) \right\|_{L^2(\mathbb{R})} \leq C_* h_*^{3/2-\lambda}.$$
Thank you for your attention!