THE BOUNDED EULER CLASS AND QUASI-MORPHISMS ON GROUPS OF SYMPLECTOMORPHISMS OF THE DISK

SHUHEI MARUYAMA

ABSTRACT. On groups of symplectomorphisms of the disk, we construct and study two homogeneous quasi-morphisms which relate to the bounded Euler class of the group \( \text{Diff}_+(S^1) \) of orientation preserving diffeomorphisms of the circle.

1. Introduction

A quasi-morphism on a group \( \Gamma \) is a function \( \phi : \Gamma \to \mathbb{R} \) such that the value
\[
D(\phi) = \sup_{\gamma_1, \gamma_2 \in \Gamma} |\phi(\gamma_1 \gamma_2) - \phi(\gamma_1) - \phi(\gamma_2)|
\]
is finite. Denote by \( Q(\Gamma) \) the \( \mathbb{R} \)-vector space of quasi-morphisms on the group \( \Gamma \). A quasi-morphism \( \phi \) is called homogeneous if the equation \( \phi(\gamma^n) = n\phi(\gamma) \) holds for any \( \gamma \in \Gamma \) and \( n \in \mathbb{Z} \). We denote by \( Q^h(\Gamma) \) the subspace of \( Q(\Gamma) \) which consists of homogeneous quasi-morphisms on \( \Gamma \). For a quasi-morphism \( \phi \), we obtain the homogeneous quasi-morphism \( \overline{\phi} \) associated to \( \phi \) by
\[
\overline{\phi}(g) = \lim_{n \to \infty} \frac{\phi(g^n)}{n}.
\]
This \( \overline{\phi} \) is called the homogenization of \( \phi \).

Let \( D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\} \) be the unit disk in \( \mathbb{R}^2 \) and \( \omega = dx \wedge dy \) be the symplectic form on \( D \). Let \( \text{Symp}(D, \partial D) \) be the group of symplectomorphisms, which are the identity on a neighborhood of the boundary. Entov-Polterovich\[4\] and Gambaudo-Ghys\[6\] proved that the vector space \( Q^h(\text{Symp}(D, \partial D)) \) is infinite dimensional. Entov and Polterovich constructed uncountably many homogeneous quasi-morphisms on \( \text{Symp}(D, \partial D) \) as pullbacks of the Calabi quasi-morphism on the Hamiltonian diffeomorphism group \( \text{Ham}(S^2) \) of sphere by injections \( \phi_\epsilon : \text{Symp}(D, \partial D) \to \text{Ham}(S^2) \). This map \( \phi_\epsilon \) is defined by, for embedding \( h_\epsilon : D \to S^2 \), extending symplectomorphisms of \( D \) as identity at the outside of the embedded disk. After that, Gambaudo and Ghys constructed countably many homogeneous quasi-morphisms on \( \text{Symp}(D, \partial D) \) by averaging the value of the signature of the braid associated to a symplectomorphism in \( \text{Symp}(D, \partial D) \). This construction relies on the fact that the group \( \text{Symp}(D, \partial D) \) is contractible.

Let \( G = \text{Symp}(D) \) be the group of symplectomorphisms of \( D \) without any condition on the boundary \( \partial D \). Since \( G \) is not contractible and the injections \( \phi_\epsilon \) is not well-defined on \( G \), the above two constructions cannot be applied on \( G \).

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In this paper, we construct a homogeneous quasi-morphism on $G$. Let $\eta$ be a 1-form on $D$ satisfying $d\eta = \omega$. The map $\tau_\eta : G \to \mathbb{R}$ is defined by

$$\tau_\eta(g) = \int_D g^* \eta \wedge \eta.$$ 

The restriction of this map to the kernel of the surjection $p : G \to \text{Diff}^+(S^1)$ coincides with the Calabi invariant, where $\text{Diff}^+(S^1)$ denotes the group of orientation preserving diffeomorphisms of the circle. This map $\tau_\eta$ depends on the choice of $\eta$ and is not a homomorphism. However, this map $\tau_\eta$ gives rise to a quasi-morphism. Put $\tau$ the homogenization of $\tau_\eta$.

**Theorem A.** The homogeneous quasi-morphism $\tau : G \to \mathbb{R}$ is surjective. Moreover, the bounded cohomology class $[\delta \tau] \in H^2_b(G; \mathbb{R})$ coincides with the pullback of the bounded Euler class of $\text{Diff}^+(S^1)$ by projection $p : G \to \text{Diff}^+(S^1)$ up to non-zero constant multiple.

In [9], Poincaré introduced the translation number and the rotation number. Although, in general, the translation number and the rotation number are defined on the group $\text{Homeo}^+(S^1)$ and its universal covering space, we consider them on the group $\text{Diff}^+(S^1)$ and its universal covering $\widetilde{\text{Diff}}^+(S^1)$. Note that, in this paper, we identify the circle $S^1$ with the quotient $\mathbb{R}/2\pi\mathbb{Z}$. The translation number is a quasi-morphism $\widetilde{\text{rot}} : \widetilde{\text{Diff}}^+(S^1) \to \mathbb{R}$ defined by

$$\widetilde{\text{rot}}(\widetilde{\varphi}) = \lim_{n \to \infty} \frac{\widetilde{\varphi}^n(0)}{2\pi n}.$$ 

It is known translation number $\widetilde{\text{rot}}$ descends to the map rot : $\text{Diff}^+(S^1) \to \mathbb{R}/\mathbb{Z}$ (see Frigerio[5]).

Let us consider the homogeneous quasi-morphism $\tau/\pi^2 : G \to \mathbb{R}$ modulo $\mathbb{Z}$ and denote it by $\mathcal{R} : G \to \mathbb{R}/\mathbb{Z}$. Also consider the homomorphism $\mathcal{R} : G \to \mathbb{R}/\mathbb{Z}$ defined in section 5.

**Theorem B.** The following holds:

$$\mathcal{R} + \mathcal{R} = \text{rot} : G \to \mathbb{R}/\mathbb{Z}.$$ 

Here the rot : $G \to \mathbb{R}/\mathbb{Z}$ is the pullback of the rotation number by $G \to \text{Diff}^+(S^1)$.

Let $G^0 = \{g \in G \mid g(0,0) = (0,0)\}$ be the subgroup of $G$ consisting of symplectomorphisms which preserve the origin $(0,0) \in D$. On the group $G^0$, we also construct a homogeneous quasi-morphism $\rho : G^0 \to \mathbb{R}$, which is a homogenization of the quasi-morphism $\rho_{\eta;\gamma} : G^0 \to \mathbb{R}$ defined by

$$\rho_{\eta;\gamma}(g) = \int_\gamma g^* \eta - \eta.$$ 

Here the symbol $\gamma$ is a path from the origin to a point on the boundary. Then we have
**Theorem C.** The homogeneous quasi-morphism \( \rho : G^0 \to \mathbb{R} \) is surjective. Moreover, the bounded cohomology class \([\delta \rho] \in H^2_b(G^0; \mathbb{R})\) coincides with the pullback of the bounded Euler class of \( \text{Diff}_+(S^1) \) by projection \( p : G \to \text{Diff}_+(S^1) \) up to non-zero constant multiple.

By comparing the two homogeneous quasi-morphisms \( \tau \) and \( \rho \), we obtain

**Theorem D.** The difference \( \tau - \pi \rho : G^0 \to \mathbb{R} \) is a surjective homomorphism. In particular, the group \( G^0 \) is not perfect.

The present paper is organized as follows. In section 2, we recall the bounded cohomology and quasi-morphisms on groups. In section 3, we construct the homogeneous quasi-morphism \( \tau : G \to \mathbb{R} \) and study its properties. In section 4, we show the relation between the homogeneous quasi-morphism \( \tau \), the homomorphism \( R : G \to \mathbb{R} \), and the rotation number. In section 5, we construct another homogeneous quasi-morphism \( \rho : G^0 \to \mathbb{R} \). In section 6, we study the relation between the homogeneous quasi-morphisms \( \tau \) and \( \rho \).

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2. Bounded Cohomology

Let \( \Gamma \) be a group. A \( p \)-cochain of \( \Gamma \) is a map from \( \Gamma^p \) to \( \mathbb{R} \). Denote by \( C^p(\Gamma; \mathbb{R}) \) the set of the \( p \)-cochains of \( \Gamma \). Define the coboundary operator \( \delta : C^p(\Gamma; \mathbb{R}) \to C^{p+1}(\Gamma; \mathbb{R}) \) by

\[
\delta c(\gamma_1, \ldots, \gamma_{p+1}) = c(\gamma_2, \ldots, \gamma_{p+1}) + \sum_{i=1}^{p} (-1)^i c(\gamma_1, \ldots, \gamma_i \gamma_{i+1}, \ldots, \gamma_{p+1}) + (-1)^{p+1} c(\gamma_1, \ldots, \gamma_p).
\]

Then the cohomology of the cochain complex \( (C^\bullet(\Gamma; \mathbb{R}), \delta) \) is called the group cohomology of \( \Gamma \) and denoted by \( H^\bullet(\Gamma; \mathbb{R}) \). Let us consider the subcomplex \( C^p_b(\Gamma; \mathbb{R}) \) of \( C^p(\Gamma; \mathbb{R}) \) consisting of all bounded function \( c : \Gamma^p \to \mathbb{R} \). The cohomology of this subcomplex \( (C^\bullet_b(\Gamma; \mathbb{R}), \delta) \) is called the bounded cohomology of \( \Gamma \) and denoted by \( H^\bullet_b(\Gamma; \mathbb{R}) \). The inclusion \( C^p(\Gamma; \mathbb{R}) \to C^p_b(\Gamma; \mathbb{R}) \) induces the map \( \iota : H^p(\Gamma; \mathbb{R}) \to H^p_b(\Gamma; \mathbb{R}) \) called the **comparison map**. The kernel of comparison map is denote by \( EH^p_b(\Gamma; \mathbb{R}) \) and called the **exact bounded cohomology**.

The second bounded cohomology is closely related to quasi-morphisms. Denote by \( Q^b(\Gamma) \) the vector space consisting of all homogeneous quasi-morphisms on \( \Gamma \). It is known that the map \( Q^b(\Gamma) \to EH^2_b(\Gamma; \mathbb{R}); \phi \mapsto [\delta \phi] \) induces an isomorphism

\[
Q^b(\Gamma) / \text{Hom}(\Gamma, \mathbb{R}) \cong EH^2_b(\Gamma; \mathbb{R}),
\]

where \( \text{Hom}(\Gamma, \mathbb{R}) \) consists of the homomorphisms from \( \Gamma \) to \( \mathbb{R} \) (see [5]). This implies that, if a homogeneous quasi-morphism is not a homomorphism, then the corresponding bounded cohomology class is non-trivial.
3. CALABI INVARIANT AND THE QUASI-MORPHISM $\tau$

Let $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$ be the unit disk with a symplectic form $\omega = dx \wedge dy$. Denote by $G = \text{Symp}(D)$ the symplectomorphism group of $D$ and by $\text{Diff}_+(S^1)$ the orientation preserving diffeomorphism group of the unit circle $S^1 = \partial D$. It is known that the restriction to the boundary $p : G \to \text{Diff}_+(S^1)$ is a surjective (see Tsuboi[10]). Thus we have an exact sequence

$$1 \longrightarrow G_{\text{rel}} \longrightarrow G \xrightarrow{p} \text{Diff}_+(S^1) \longrightarrow 1,$$

where the group $G_{\text{rel}}$ is the kernel of the map $G \to \text{Diff}_+(S^1)$.

The Calabi invariant $\text{Cal} : G_{\text{rel}} \to \mathbb{R}$ is defined by

$$(3.1) \quad \text{Cal}(h) = \int_D h^* \eta \wedge \eta$$

where $\eta$ is a 1-form satisfying $d\eta = \omega$. This map $\text{Cal}$ is a surjective homomorphism and is independent of the choice of $\eta$ (see Moriyoshi[8]). The map $\tau_\eta : G \to \mathbb{R}$ is defined as the same way in (3.1):

$$(3.2) \quad \tau_\eta(g) = \int_D g^* \eta \wedge \eta.$$

Note that the map $\tau_\eta$ is not a homomorphism and does depend on the choice of $\eta$. In [8], Moriyoshi proved the following transgression formula:

$$(3.3) \quad -\delta \tau_\eta(g, h) = \pi^2 \chi(p(g), p(h)) + \pi^2 / 2 \quad (g, h \in G).$$

Here the cocycle $\chi \in C^2(\text{Diff}_+(S^1); \mathbb{R})$ is defined by

$$\chi(\gamma_1, \gamma_2) = \frac{1}{4\pi^2} \int_0^{2\pi} (\tilde{\gamma}_1 \circ \tilde{\gamma}_2(x) - \tilde{\gamma}_1(x) - \tilde{\gamma}_2(x)) dx,$$

where $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ in $\text{Diff}_+(S^1)$ are lifts of $\gamma_1$ and $\gamma_2$ respectively. This cocycle $\chi$ represents the Euler class $e_{\mathbb{R}} \in H^2(\text{Diff}_+(S^1); \mathbb{R})$ of $\text{Diff}_+(S^1)$. Since the cocycle $\chi$ is a bounded 2-cocycle, the map $\tau_\eta : G \to \mathbb{R}$ is a quasi-morphism. Let us denote by $\tau$ the homogenization of $\tau_\eta$.

**Proposition 3.1.** The homogenization $\tau$ is independent of the choice of $\eta$.

**Proof.** Let $\eta$ and $\lambda$ be 1-forms such that their exterior derivatives are equal to $\omega$. Put $\varphi = \tau_\eta - \tau_\lambda$, then the map $\varphi$ is a quasi-morphism on $G$. The difference of homogenizations of $\tau_\eta$ and $\tau_\lambda$ is equal to the homogenization $\varphi$ of $\varphi$. Therefore it is enough to show that the map $\varphi$ is equal to 0. For any $g \in G$ and $h \in G_{\text{rel}}$, the following hold (see [8]):

$$\tau_\eta(gh) = \tau_\eta(hg) = \tau_\eta(g) + \text{Cal}(h)$$

$$\tau_\lambda(gh) = \tau_\lambda(hg) = \tau_\lambda(g) + \text{Cal}(h).$$
From this equalities, we have
\begin{align*}
\varphi(gh) &= \tau_\eta(gh) - \tau_\lambda(gh) = \tau_\eta(g) + \text{Cal}(h) - \tau_\lambda(g) - \text{Cal}(h) = \varphi(g) \\
\varphi(hg) &= \tau_\eta(hg) - \tau_\lambda(hg) = \tau_\eta(g) + \text{Cal}(h) - \tau_\lambda(g) - \text{Cal}(h) = \varphi(g)
\end{align*}
for any \( g \in G \) and \( h \in G_{\text{rel}} \). This implies that the map \( \varphi \) defines a map from \( \text{Diff}_+(S^1) \), that is, there exists a map \( \tilde{\psi} : \text{Diff}_+(S^1) \to \mathbb{R} \) such that \( \varphi(g) = \tilde{\psi}(p(g)) \) for any \( g \in G \). Since \( p : G \to \text{Diff}_+(S^1) \) is surjective and \( \varphi \) is quasi-morphism, the map \( \tilde{\psi} : \text{Diff}_+(S^1) \to \mathbb{R} \) becomes a quasi-morphism. So we have \( \tilde{\tau}(g) = \tilde{\psi}(p(g)) \) where \( \tilde{\psi} \) is the homogenization of \( \psi \). Since the group \( \text{Diff}_+(S^1) \) is uniformly perfect, there is no non-trivial homogeneous quasi-morphism. Therefore we have \( \tilde{\psi} = 0 \). □

**Proposition 3.2.** The homogenization \( \tau : G \to \mathbb{R} \) is a surjective homogeneous quasi-morphism.

**Proof.** For \( h \in G_{\text{rel}} \), we have
\[
\tau(h) = \lim_{n \to \infty} \frac{\tau_\eta(h^n)}{n} = \lim_{n \to \infty} \frac{\text{Cal}(h^n)}{n} = \lim_{n \to \infty} \frac{n \text{Cal}(h)}{n} = \text{Cal}(h).
\]
Since the Calabi invariant is surjective, the homogenization \( \tau \) is also surjective. □

The pullback of the real Euler class \( e_\mathbb{R} \in H^2(\text{Diff}_+(S^1); \mathbb{R}) \) to \( H^2(G; \mathbb{R}) \) is equal to 0 since the transgression formula \([3, 2]\) holds. However, if we work on the bounded cohomology, the pullback of bounded Euler class remains to be a non-trivial cohomology class. Indeed, it is known that, for a surjective homomorphism \( p : A \to B \), the induced map \( p^* : H^2_b(B; \mathbb{R}) \to H^2_b(A; \mathbb{R}) \) is injective (see [5]). Therefore the pullback \( p^* e_b \) is non-trivial and belongs to the exact bounded cohomology \( EH^2_b(G; \mathbb{R}) \). Then the homogeneous quasi-morphism \( \tau \) relates to the bounded Euler class as follows:

**Theorem 3.3.** The bounded cohomology class \([\delta \tau] \in H^2_b(G; \mathbb{R})\) is equal to \(-\pi^2\) times the pullback \( p^* e_b \) of the bounded Euler class \( e_b \).

**Proof.** Recall that the difference between a quasi-morphism and its homogenization is a bounded function. Thus we have \( \delta \tau_\eta - \delta \tau = \delta b \) where \( b = \tau_\eta - \tau \) is a bounded function. This implies that the bounded cohomology class \([\delta \tau_\eta] \) coincides with \([\delta \tau] \). Moreover, the class \([\delta \tau_\eta] \) is equal to the pullback \( p^* e_b \) up to non-zero constant multiple because of the transgression formula \([3, 2]\). □

4. THE QUASI-MORPHISM \( \tau \) AND THE POINCARÉ’S ROTATION NUMBER

In this section, we show the relation between the homogeneous quasi-morphism \( \tau \) and the classical rotation number introduced by Poincaré [9].

At first, we recall the homomorphism \( R : \tilde{G} \to \mathbb{R} \) introduced by Banyaga [1], where the group \( \tilde{G} \) is the universal covering of \( G \) with respect to the \( C^\infty \)-topology (see also Tsuboi [10]). Let \( \mathcal{L}_\omega(D) \) be the set of divergence free vector fields which are tangent to the boundary. For any vector field \( X \) in \( \mathcal{L}_\omega(D) \), there is a unique function \( f_X : D \to \mathbb{R} \) such that \( i_X \omega = df_X \) and \( f_X|_{\partial D} = 0 \). For any path \( g_t \) in \( G \), we define the time-dependent vector field \( X_t \) by \( X_t = (\partial g_t/\partial t) \circ g_t^{-1} \). Since \( g_t \) are
symplectomorphisms, the vector fields $X_t$ are in $L^\omega(D)$. Then we define the map $R : \tilde{G} \to \mathbb{R}$ by

$$R([g_t]) = \int_0^1 \left( \int_D f_{X_t} \omega \right) dt.$$  

This map $R$ is a well-defined homomorphism (see Banyaga[1]).

**Lemma 4.1.** Let $g_t$ be a path in $G$ such that $g_0 = \text{id}$ and $X_t$ be the time-dependent vector field defined by $X_t = (\partial g_t / \partial t) \circ g_t^{-1}$, then

$$\tau_\eta(g_1) + 2R([g_t]) = \int_{\partial D} \left( \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \eta$$

**Proof.** Note that the following identities hold (see Banyaga[1]):

$$g_t^*\eta - \eta = d \left( \int_0^1 g_t^* f_{X_t} dt + \int_0^1 g_t^*(i_{X_t}\eta) dt \right)$$

$$(i_{X_t}\eta)\omega = \eta \wedge i_{X_t}\omega = \eta \wedge df_{X_t} = f_{X_t} \omega - d(f_{X_t}\eta).$$

Therefore we have

$$\tau_\eta(g_1) = \int_D (g_1^*\eta - \eta) \wedge \eta$$

$$= \int_D d \left( \int_0^1 g_t^* f_{X_t} dt + \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \wedge \eta$$

$$= \int_{\partial D} \left( \int_0^1 g_t^* f_{X_t} dt + \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \eta$$

$$- \int_D \left( \int_0^1 g_t^* f_{X_t} dt + \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \omega$$

$$= \int_{\partial D} \left( \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \eta - \int_0^1 \left( \int_D f_{X_t} \omega \right) dt - \int_0^1 \left( \int_D (i_{X_t}\eta) \omega \right) dt$$

$$= \int_{\partial D} \left( \int_0^1 g_t^*(i_{X_t}\eta) dt \right) \eta - 2R([g_t]).$$

**Corollary 4.2.** [1] Let $h_t$ be a path in $G_{rel}$ such that $h_0 = \text{id}$, then $\text{Cal}(h_1) = -2R([h_t])$.

Put $\eta = (r^2 d\theta)/2$ where $(r, \theta) \in D$ is the polar coordinates. A path $g_t$ in $G$ defines a path $\varphi_t$ in $\text{Diff}^+(S^1)$. Let $\xi_t$ be a time-dependent vector field defined by $\xi_t = (\partial \varphi_t / \partial t) \circ \varphi_t^{-1}$. Let $\tilde{\varphi}_t \in \text{Diff}^+(S^1)$ be a lift of $\varphi_t$ such that $\tilde{\varphi}_0 = \text{id}$. Then the
right-hand side of the equation (4.1) can be written as
\[
\int_{\partial D} \left( \int_0^1 g^*_t(i_X, \eta) dt \right) \eta = \frac{1}{4} \int_{S^1} \left( \int_0^1 \varphi^*_t(i_{\xi}, d\theta) dt \right) d\theta \\
= \frac{1}{4} \int_0^{2\pi} \left( \int_0^1 \frac{\partial \varphi_t}{\partial t} dt \right) dx \\
= \frac{1}{4} \int_0^{2\pi} (\varphi_1(x) - x) dx.
\]

Let us define a map \( \sigma : \tilde{\text{Diff}}_+(S^1) \to \mathbb{R} \) by \( \sigma(\tilde{\varphi}) = \frac{1}{4\pi^2} \int_0^{2\pi} (\tilde{\varphi}(x) - x) dx \). Then we have the following formula:
\[
(4.2) \quad \tau_\eta(g_1) + 2R([g_1]) = \pi^2 \sigma(\tilde{\varphi}_1).
\]
Note that, for any \( \tilde{\varphi}, \tilde{\psi} \) in \( \tilde{\text{Diff}}_+(S^1) \), the inequality
\[
|\tilde{\varphi}\tilde{\psi}(x) - \tilde{\varphi}(x) - \tilde{\psi}(x)| < 4\pi
\]
hold. Thus we have
\[
|\sigma(\tilde{\varphi}\tilde{\psi}) - \sigma(\tilde{\varphi}) - \sigma(\tilde{\psi})| < \frac{1}{4\pi^2} \int_0^{2\pi} \left( |\tilde{\varphi}\tilde{\psi}(x) - \tilde{\varphi}(x) - \tilde{\psi}(x)| + x \right) dx \\
< 2 + \frac{1}{2}
\]
and this implies that the map \( \sigma \) is a quasi-morphism. Let \( \overline{\sigma} : \text{Diff}_+(S^1) \to \mathbb{R} \) be the homogenization of \( \sigma \). By taking the homogenization of the both sides of the equation (4.2), we have
\[
(4.3) \quad \tau(g_1) + 2R([g_1]) = \pi^2 \overline{\sigma}(\varphi_1).
\]
The translation number \( \tilde{\text{rot}} : \text{Homeo}_+(S^1) \) introduced by Poincaré defines also a homogeneous quasi-morphism on \( \text{Diff}_+(S^1) \).

**Proposition 4.3.** The homogeneous quasi-morphism \( \overline{\sigma} : \text{Diff}_+(S^1) \to \mathbb{R} \) coincides with the translation number.

**Proof.** The translation number \( \tilde{\text{rot}} : \text{Diff}_+(S^1) \to \mathbb{R} \) is a homogeneous quasi-morphism and satisfies \( \tilde{\text{rot}}(T_k) = k \), where \( T_k \) is the translation of \( \mathbb{R} \) by \( 2\pi k \).

The homogeneous quasi-morphism \( \overline{\sigma} : \text{Diff}_+(S^1) \to \mathbb{R} \) also satisfies the equation \( \overline{\sigma}(T_k) = k \). Thus the map \( \tilde{\text{rot}} - \sigma \) is a homogeneous quasi-morphism which is equal to 0 on the kernel of the projection \( \text{Diff}_+(S^1) \to \text{Diff}_+(S^1) \). This implies that the map \( \tilde{\text{rot}} \) coincides with \( \overline{\sigma} \) on the whole group \( \text{Diff}_+(S^1) \) (see Ben Simon-Salamon Remark 4).

Combining Proposition 4.3 with the equation (4.3), we have the following:
Theorem 4.4. Let \( p : \tilde{G} \to G \) be the projection. Then, we have
\[
p^* \tau + 2R = \pi^2 \tilde{\text{rot}} : \tilde{G} \to \mathbb{R}.
\]
Here the map \( \tilde{\text{rot}} : \tilde{G} \to \mathbb{R} \) is the pullback of the translation number by the surjection \( \tilde{G} \to \text{Diff}_+(S^1) \).

The translation number descends to the map \( \text{rot} : \text{Diff}_+(S^1) \to \mathbb{R}/\mathbb{Z} \) and this is called the Poincaré’s rotation number. The homomorphism \( 2R/\pi^2 : \tilde{G} \to \mathbb{R} \) also descends to the homomorphism \( \tilde{\text{rot}} : \tilde{G} \to \mathbb{R}/\mathbb{Z} \). To see this, let \( g_t, h_t \) be paths in \( G \) such that \( g_1 = h_1 \) and \( \varphi_t, \psi_t \) be paths in \( \text{Diff}_+(S^1) \) corresponding to \( g_t, h_t \) respectively. Since \( \varphi_1 = \psi_1 \), there exists some integer \( k \) such that \( \tilde{\varphi}_1 = \tilde{\psi}_1 + 2\pi k \). Thus we have
\[
2\pi^2 R([g_t]) = \sigma(\tilde{\varphi}_1) - \frac{1}{\pi^2} \tau_\eta(g_1) = \sigma(\tilde{\psi}_1 + 2\pi k) - \frac{1}{\pi^2} \tau_\eta(h_1) \nonumber
= \sigma(\tilde{\psi}_1) + k - \frac{1}{\pi^2} \tau_\eta(h_1) = 2\pi^2 R([h_t]) + k.
\]
So the homomorphism \( 2R/\pi^2 : \tilde{G} \to \mathbb{R} \) descends to the homomorphism \( \tilde{\text{rot}} : \tilde{G} \to \mathbb{R}/2\pi\mathbb{Z} \).

Thus we have the following theorem.

Theorem 4.5. Let \( \tau : G \to \mathbb{R}/\mathbb{Z} \) be the composition of the homogeneous quasi-morphism \( \tau/\pi^2 : G \to \mathbb{R} \) and the projection \( \mathbb{R} \to \mathbb{R}/\mathbb{Z} \), then
\[
\tau + R = \text{rot}.
\]
Here the \( \text{rot} : G \to \mathbb{R}/\mathbb{Z} \) is the pullback of the rotation number by the projection \( G \to \text{Diff}_+(S^1) \).

5. The Flux Homomorphism and the Quasi-Morphism \( \rho \)

Let us consider the subgroup \( G^0 \) of \( G \):
\[
G^0 = \{ g \in G \mid g(0,0) = (0,0) \in D \}.
\]
Put \( G^0_{\text{rel}} = G_{\text{rel}} \cap G^0 \). Then the following sequence of groups is exact:
\[
1 \to G^0_{\text{rel}} \to G^0 \to p^* \to \text{Diff}_+(S^1) \to 1.
\]
On the group \( G^0_{\text{rel}} \), the Calabi invariant is defined as the restriction \( \text{Cal}|_{G^0_{\text{rel}}} : G^0_{\text{rel}} \to \mathbb{R} \). In \([7]\), the author defined another homomorphism \( \text{Flux}_\mathbb{R} : G^0_{\text{rel}} \to \mathbb{R} \), which we call flux homomorphism. This flux homomorphism \( \text{Flux}_\mathbb{R} \) is defined by
\[
\text{Flux}_\mathbb{R}(h) = \int_\gamma h^* \eta - \eta
\]
where \( \gamma \) is a path from the origin \( (0,0) \) to a point on the boundary \( \partial D \). Note that the flux homomorphism is a surjective homomorphism and is independent of the choice of \( \eta \) and \( \gamma \).
As in the case of Calabi invariant, the flux homomorphism can be extended to the group \( G^0 \), that is, we define the map \( \rho_{\eta,\gamma} : G^0 \to \mathbb{R} \) by

\[
\rho_{\eta,\gamma}(g) = \int_\gamma g^* \eta - \eta.
\]

For the flux homomorphism and \( \rho_{\eta,\gamma} \), the following equalities hold (see [7]):

\[
\rho_{\eta,\gamma}(gh) = \rho_{\eta,\gamma}(hg) = \rho_{\eta,\gamma}(g) + \text{Flux}_R(h)
\]

for \( g \in G^0 \) and \( h \in G^0_{\text{rel}} \). Moreover, the following transgression formula, which is similar to the Calabi invariant, holds:

\[
\text{Flux}_R(h) = \rho_{\eta,\gamma}(h) (h \in G^0_{\text{rel}})
\]

(5.1)

where \( \xi \in C^2(\text{Diff}_+(S^1); \mathbb{R}) \) is an Euler cocycle. Unlike the flux homomorphism, the map \( \tau_{\eta,\gamma} \) is not a homomorphism and depends on the choice of \( \eta \) and \( \gamma \). However, by the similar arguments in Proposition 3.1, we have:

**Proposition 5.1.** The homogenization \( \rho : G^0 \to \mathbb{R} \) of \( \rho_{\eta,\gamma} \) is independent of the choice of \( \eta \) and \( \gamma \).

Since the restriction of \( \rho \) to the group \( G^0_{\text{rel}} \) is equal to the flux homomorphism \( \text{Flux}_R \), we have:

**Proposition 5.2.** The homogenization \( \rho : G^0 \to \mathbb{R} \) is a surjective homogeneous quasi-morphism.

Furthermore, by the transgression formula (5.1), the following holds:

**Theorem 5.3.** The bounded cohomology class \( [\delta \rho] \) is equal to \(-\pi \) times the class \( p^*e_b \).

6. RELATION BETWEEN \( \tau \) AND \( \rho \)

The restriction \( \text{Cal} |_{G^0_{\text{rel}}} : G^0_{\text{rel}} \to \mathbb{R} \) of the Calabi invariant remains surjective. So the restriction \( \rho : G^0 \to \mathbb{R} \) is also surjective homogeneous quasi-morphism. Therefore we have two non-trivial homogeneous quasi-morphisms \( \tau, \rho \in Q^h(G^0) \). By Theorem 3.3 and Theorem 5.3, the class \( [\delta \rho] \) coincides with \( \pi[\delta \rho] \) in \( EH^2_h(G^0; \mathbb{R}) \). From the isomorphism (2.3), the difference \( \tau - \pi \rho \) is a homomorphism on \( G^0 \). Put \( h = \tau - \pi \rho \), then we have:

**Proposition 6.1.** The homomorphism \( h : G^0 \to \mathbb{R} \) is surjective.

**Proof.** On the group \( G^0_{\text{rel}} \), the homomorphism \( h \) is equal to \( \text{Cal} - \pi \text{Flux}_R \). Put the non-increasing \( C^\infty \)-function \( f : [0,1] \to \mathbb{R} \) which is equal to 1 near \( r = 0 \) and \( f(1) = 0 \). Then, for \( s \in \mathbb{R} \), we define diffeomorphisms \( g_s \) in \( G^0_{\text{rel}} \) by

\[
g_s(r, \theta) = (r, \theta + sf(r))
\]
where \((r, \theta) \in D\) is the polar coordinates. Since the Calabi invariant \(\text{Cal}\) and the flux homomorphism \(\text{Flux}_R\) is independent of the choice of \(\eta\) and \(\gamma\), we fix one of \(\eta\) and \(\gamma\) as

\[\eta = (xdy - ydx)/2 = (r^2 d\theta)/2, \quad \gamma(r) = (r, 0) \in D.\]

Then we have

\[
\text{Cal}(g_s) = \frac{s\pi}{2} \int_0^1 r^4 \frac{\partial f}{\partial r} dr,
\]

\[
\pi \text{Flux}_R(g_s) = \frac{s\pi}{2} \int_0^1 r^2 \frac{\partial f}{\partial r} dr.
\]

This implies that the difference \(h = \tau - \pi \rho\) is surjective on \(G^0_{\text{rel}}\), and so is on \(G^0\). □

**Corollary 6.2.** The following hold:

i) The homogeneous quasi-morphism \(\tau\) and \(\rho\) are linealy independent in \(Q^h(G^0)\).

ii) The group \(G^0\) is not perfect.

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