THE ŁOJASIEWICZ EXPONENT IN NON-DEGENERATE DEFORMATIONS OF SURFACE SINGULARITIES

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Abstract. We prove the constancy of the Łojasiewicz exponent in non-degenerate \( \mu \)-constant deformations of surface singularities. This is a positive answer to a question posed by B. Teissier.

1. Introduction

Let \( f(z) = f(z_1, \ldots, z_n) \in \mathbb{C}\{z_1, \ldots, z_n\} =: \mathcal{O}_n \) be a convergent power series defining an isolated singularity at the origin \( 0 \in \mathbb{C}^n \), i.e., \( f(0) = 0 \) and the gradient of \( f \),

\[
\nabla f := \left( \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0),
\]

has an isolated zero at \( 0 \in \mathbb{C}^n \). The Łojasiewicz exponent \( \mathcal{L}_0(f) \) of \( f \) is the smallest (= infimum) \( \theta > 0 \) such that there exists a neighbourhood \( U \) of \( 0 \in \mathbb{C}^n \) and a constant \( C > 0 \) such that

\[
|\nabla f(z)| \geq C |z|^\theta \quad \text{for } z \in U.
\]

Remark 1. One can similarly define the Łojasiewicz exponent \( \mathcal{L}_0(F) \) of any holomorphic mapping \( F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0) \) having an isolated zero at \( 0 \in \mathbb{C}^n \) (replacing \( \nabla f \) by \( F \) in the above definition). Moreover, using the usual conventions for the infimum, one can also extend this definition to any holomorphic mapping \( F : (\mathbb{C}^n, 0) \to \mathbb{C}^p \); then \( \mathcal{L}_0(F) = 0 \) if \( F(0) \neq 0 \), and \( \mathcal{L}_0(F) = +\infty \) if \( F(0) = 0 \) but the zero is not isolated.

It is known \( \mathcal{L}_0(f) \) is a rational positive number and it is an analytic invariant of \( f \). Moreover, it depends only on the ideal \( (\partial f/\partial z_1, \ldots, \partial f/\partial z_n) \) in \( \mathcal{O}_n \) and can be calculated using analytic paths, i.e.,

\[
\mathcal{L}_0(f) = \sup_{\Phi} \frac{\text{ord} (\nabla f \circ \Phi)}{\text{ord} \Phi} = \max_{\Phi} \frac{\text{ord} (\nabla f \circ \Phi)}{\text{ord} \Phi},
\]

where \( 0 \neq \Phi = (\varphi_1, \ldots, \varphi_n) \in \mathbb{C}\{t\}^n, \Phi(0) = 0, \) and \( \text{ord} \Phi := \min_i \text{ord} \varphi_i \). It is also known that \( [\mathcal{L}_0(f)] + 1 \) is the degree of \( C^0 \)-sufficiency of \( f \); nevertheless, it is still an open and difficult problem whether the Łojasiewicz exponent is a topological invariant, i.e., whether \( \mathcal{L}_0(f) = \mathcal{L}_0(g) \) if isolated singularities \( f \) and \( g \) are topologically \( \mathcal{R} \)-equivalent (this is true for \( n = 2 \)). The behaviour of \( \mathcal{L}_0(f) \) in analytic families of singularities is also enigmatic. As much as the Milnor number \( \mu(f) = \dim_{\mathbb{C}} \mathcal{O}_n/(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n}) \) of \( f \) is semi-continuous from above in such families (and this is a quite natural property because the Milnor number is defined as the

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multiplicity of a mapping – exactly the gradient mapping), the Łojasiewicz exponent has no such property. Using, for instance, the formula for the Łojasiewicz exponent for functions of two variables given by A. Lenarcik [9] (see Remark 4), we easily verify

**Example 1.** 1. For the family \( f_t(x, y) = x^2 + ty^2 + y^3, \ t \in \mathbb{C}, \) we have \( L_0(f_0) = 2 \) and \( L_0(f_t) = 1 \) for \( t \neq 0. \)

2. For the family \( f_t(x, y) = xy^5 + tx^2 + x^8, \ t \in \mathbb{C}, \) we have \( L_0(f_0) = 7 \) and \( L_0(f_t) = 9 \) for \( t \neq 0. \)

This shows that in order to obtain positive results on the Łojasiewicz exponent in families of singularities, one has to impose some assumptions on the families.

B. Teissier in [16] proved that, under the additional assumption of \( \mu \)-constancy of the family, the Łojasiewicz exponent is semi-continuous from below. This result has been generalized to mappings by A. Płoski in [15]. In the aforementioned paper B. Teissier posed the problem whether in this case (i.e. for a \( \mu \)-constant family) \( L_0(f_t) \) must also be constant. For \( n = 2, \) i.e., in the case of plane curve singularities, it is true because \( \mu \)-constancy in a family of plane curve singularities \( (f_t(x, y)) \) implies the curves \( \{ f_t(x, y) = 0 \} \) are pairwise topologically equivalent (by the Lê-Ramanujam theorem, for instance). This, in turn, gives the constancy of \( L_0(f_t) \) as the Łojasiewicz exponent is a topological invariant of such singularities (see [14, Cor. 1.5] for a formula for the Łojasiewicz exponent expressed in terms of the Puiseux characteristics of the branches of a curve as well as their intersection multiplicities).

In the article, we solve Teissier’s problem in the affirmative for surface singularities, in the particular case of non-degenerate deformations. More precisely, we prove that in \( \mu \)-constant non-degenerate families of surface singularities, the Łojasiewicz exponent is also constant. In a similar spirit, we remark that the constancy of the multiplicity (= the order of \( f_t \) at 0) in \( \mu \)-constant non-degenerate families of hypersurface singularities of any dimension, which is a particular case of the famous Zariski problem, has already been settled by Y. O. M. Abderrahmane in [1].

The proof of our result is based on [3], where we gave an explicit formula for the Łojasiewicz exponent of a non-degenerate surface singularity in terms of its Newton polyhedron (see formula (2.1) below), and on [4] which gives a characterization of \( \mu \)-constant non-degenerate families of surface singularities. We notice that the characterization from [4] has recently been extended (in an equivalent form) to any dimension by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky [10, Thm. 2.15]. Among other papers concerning the properties of \( \mu \)-constant non-degenerate families of hypersurface singularities, one can mention [11], [1].

### 2. The Newton Polyhedron of a Singularity

Let \( 0 \neq f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) be a holomorphic function defined by a convergent power series \( \sum_{\nu \in \mathbb{N}^n} a_\nu z^\nu, z = (z_1, \ldots, z_n). \) Let \( \mathbb{R}_+^n := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : x_i \geq 0, \ i = 1, \ldots, n \}. \) We define \( \text{supp} f := \{ \nu \in \mathbb{N}^n : a_\nu \neq 0 \} \subset \mathbb{R}_+^n. \) In the sequel, we will identify \( \nu = (\nu_1, \ldots, \nu_n) \in \text{supp} f \) with their associated monomials \( z^\nu = z_1^{\nu_1} \cdots z_n^{\nu_n}. \)

We define the Newton polyhedron \( \Gamma_+(f) \subset \mathbb{R}_+^n \) of \( f \) as the convex hull of \( \{ \nu + \mathbb{R}_+^n : \nu \in \text{supp} f \}. \) We say \( f \) is convenient if \( \Gamma_+(f) \) has non-empty intersection with each coordinate \( x_i \)-axis \( (i = 1, \ldots, n). \) Let \( \Gamma(f) \) be the set of compact boundary faces of any dimension of \( \Gamma_+(f) \) – the Newton boundary of \( f. \) Denote by \( \Gamma^k(f) \) the set of all
Remark 4. A. Lenarcik in [9] proved an alike formula for \( n = 2 \). Precisely,
\[
\mathcal{L}_0(f) = \begin{cases} 
\max_{S \in \Gamma^1(f) \setminus E(f)} m(S) - 1, & \text{if } \Gamma^1(f) \setminus E(f) \neq \emptyset \\
1, & \text{if } \Gamma^1(f) \setminus E(f) = \emptyset 
\end{cases}
\]

Remark 5. It is an open problem if a formula of the above type holds also in the \( n \)-dimensional case \( n > 3 \).

Figure 1. Exceptional faces with respect to the \( x \)-axis in two and three dimensions

\( k \)-dimensional faces of \( \Gamma(f) \), \( k = 0, \ldots, n-1 \). Then \( \Gamma(f) = \bigcup_{k=0}^{n-1} \Gamma^k(f) \). For each face \( S \in \Gamma(f) \), we define the quasihomogeneous polynomial \( f_S := \sum_{\nu \in S} a_{\nu} z^\nu \). We say \( f \) is non-degenerate on \( S \) if the system of polynomial equations \( \{ \frac{\partial f}{\partial z_i} = \cdots = \frac{\partial f}{\partial z_j} = 0 \} \) has no solution in \( (\mathbb{C}^*)^n \); \( f \) is non-degenerate (in the Kushnirenko sense) if \( f \) is non-degenerate on each face \( S \in \Gamma(f) \).

For each \((n-1)\)-dimensional (compact) face \( S \in \Gamma^{n-1}(f) \) the unique affine hyperplane \( \Pi_S \) containing \( S \) intersects each coordinate \( x_i \)-axis in a point with a positive \( x_i \)-coordinate \( m(S)_{x_i} \). We define
\[
m(S) := \max m(S)_{x_i}.\]

It is interesting that for typical non-degenerate isolated singularities \( f \) in \( \mathbb{C}^2 \) and \( \mathbb{C}^3 \) the Lojasiewicz exponent \( \mathcal{L}_0(f) \) can be expressed in terms of \( m(S) \) where \( S \) run over some special \((n-1)\)-dimensional faces of \( \Gamma(f) \). We define them now.

We say \( S \in \Gamma^{n-1}(f) \) is exceptional with respect to the \( x_i \)-axis if one of the partial derivatives \( \frac{\partial f}{\partial z_j} \), \( j \neq i \), is a pure power of \( z_j \). Geometrically, this means \( S \) is an \((n-1)\)-dimensional pyramid with the base lying in one of the \((n-1)\)-dimensional coordinate hyperplanes containing the \( x_i \)-axis and with the apex lying at distance 1 from this axis (see Figure 1). We denote the set of exceptional faces of \( f \) with respect to \( x_i \)-axis by \( E_{x_i}(f) \).

A face \( S \in \Gamma^{n-1}(f) \) is exceptional if \( S \) is exceptional with respect to some axis. We denote the set of all exceptional faces of \( f \) by \( E(f) \). Then the formula for the Lojasiewicz exponent given in [3] reads

Theorem 2. If \( f : \mathbb{C}^3, 0 \to (\mathbb{C}, 0) \) is a non-degenerate isolated surface singularity possessing non-exceptional faces, i.e., \( \Gamma^2(f) \setminus E(f) \neq \emptyset \), then
\[
\mathcal{L}_0(f) = \max_{S \in \Gamma^2(f) \setminus E(f)} m(S) - 1.\]

Remark 3. The case \( \Gamma^2(f) \setminus E(f) = \emptyset \) is relatively simpler. The third-named author in [12, Prop. 3.4, Thm. 3.8] showed that in this case, if we denote the variables in \( \mathbb{C}^3 \) by \( x, y, z \), there is exactly one segment \( S \in \Gamma^1(f) \) joining monomials \( xy \) and \( z^k \), \( k \geq 2 \) (up to permutation of the variables), and then \( \mathcal{L}_0(f) = k - 1 \).
It turns out the set of non-exceptional faces in formula (2.1) could be narrowed – it suffices to allow only non-exceptional faces having “furthest intersections” with the axes. The definition capturing such faces for surface singularities is as follows.

We say \( S \in \Gamma^2(f) \) is proximate for the \( x_i \)-axis (\( i \in \{1, 2, 3\} \)) if \( S \) is a non-exceptional face with respect to the \( x_i \)-axis (\( S \notin E_{x_i}(f) \)), has a vertex either on \( x_i \)-axis or lying at distance 1 from this axis, and touches both coordinate planes containing this axis. Possible proximity faces are illustrated in Figure 2.

It is easy to prove (cf. Lemma 3.1 and Theorem 3.8 in [12] and Lemma 6 in [3]) the following properties of proximity faces.

**Proposition 6.**

1. If \( \Gamma^2(f) \setminus E(f) \neq \emptyset \), then each axis has a proximate face; moreover, all these faces are non-exceptional.

2. Once exists, a proximate face for a given axis is unique (see Figure 2(b)) if it is non-convenient with respect to this axis (and not necessarily unique in the opposite case; see Figure 2(a)).

3. Let \( S \) be a proximate face for the \( x_i \)-axis. Then the supporting plane of \( S \) has the highest coordinate of intersection with the \( x_i \)-axis among all the \( x_i \)-non-exceptional faces.

The assumption \( \Gamma^2(f) \setminus E(f) \neq \emptyset \) in item 1. of the above proposition is essential.

**Example 2.** The surface singularity \( f(x, y, z) := xz + yz + y^3 \in \mathcal{O}_3 \) has a unique 2-dimensional face. The face is exceptional, \( x \)-proximate, \( y \)-proximate and not \( z \)-proximate.

The formula we will use in the proof of our main theorem is a stronger version of Theorem 2.

**Theorem 7** ([3, Cor. 2]). If \( f : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) is a non-degenerate isolated surface singularity possessing non-exceptional faces, i.e., \( \Gamma^2(f) \setminus E(f) \neq \emptyset \), and \( S_{x_i} \) is any proximate face for the \( x_i \)-axis (\( i = 1, 2, 3 \)), then

\[
L_0(f) = \max_i m(S_{x_i}) - 1.
\]

3. The Main Result

First, we recall some notions. Let \( f_0 \in \mathcal{O}_n \) be an isolated singularity. If \( f_0 \) is convenient, then \( \nu(f_0) \) is the Newton number of \( f_0 \) ([8]). A deformation of \( f_0 \) is a holomorphic function germ \( f(t, z) : (\mathbb{C} \times \mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) such that \( f(0, z) = f_0(z) \) and \( f(t, 0) = 0 \). Each deformation \( f(t, z) \) will also be treated as a family \((f_t)\) of germs at \( 0 \in \mathbb{C}^n \) by putting \( f_t(z) := f(t, z) \). Since \( f_0 \) has an isolated critical point.
Theorem 8. Let $f_t : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a non-degenerate isolated surface singularity and let $(f_t)$ be its non-degenerate $\mu$-constant deformation. Then $\mathcal{L}_0(f_t) = \mathcal{L}_0(f_0)$ for small $t$.

To prove this theorem, we give some auxiliary facts. The first one is a result by A. Ploski [13, Cor. 1.4].

Proposition 9. Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an isolated singularity. Then

$$\text{rank } \begin{bmatrix} \frac{\partial^2 f}{\partial x_i \partial x_j} (0) \end{bmatrix} \geq n - 1$$

if and only if $\mu(f) = \mathcal{L}_0(f)$.

The second fact is the following observation.

Proposition 10. Let $f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be an isolated surface singularity of order 2. If $(f_t)$ is a $\mu$-constant deformation of $f_0$, then $\mathcal{L}_0(f_t) = \mathcal{L}_0(f_0)$ for small $t$.

Proof. Let $H(f_t)$ denote the Hessian matrix of $f_t$ at 0. By the assumption, we have rank $H(f_0) > 0$. If rank $H(f_0) \geq 2$, then rank $H(f_t) \geq 2$ for small $t$ so by Proposition 9 we get $\mathcal{L}_0(f_t) = \mu(f_t) = \mu(f_0) = \mathcal{L}_0(f_0)$. Let, now, rank $H(f_0) = 1$, and let $x, y, z$ denote the variables in $\mathbb{C}^3$. Using splitting lemma (see [6, Thm. 2.47]), after a holomorphic change of coordinates, we may assume that $f_0 = x^2 + g_0(y, z)$, where $\text{ord } g_0 \geq 3$. Hence, $f_t = f_0 + \hat{h}_t(x, y, z)$, where $\text{ord } \hat{h}_t \geq 2$. Applying the procedure from the splitting lemma to the variable $x$, we easily find that there exists a holomorphic change of coordinates of the form $x \mapsto \Phi(t, x, y, z)$, $y \mapsto y$, $z \mapsto z$, $t \mapsto t$, where $\text{ord } \Phi(0, x, 0, 0) = 1$, bringing $f_t$ into the form $f_t = f_0 + \Phi(0, y, z)$, where $\text{ord } \Phi \geq 2$ for small $t$. Since both $\mu$ and $\mathcal{L}_0$ are invariants of the stable equivalence (see, e.g., [11, Thm. 21]), we may remove $x^2$ from $f_t$ and infer that $(g_t) := (g_0 + \hat{h}_t(y, z))$ is a $\mu$-constant deformation of the isolated plane curve singularity $g_0$ and $\mathcal{L}_0(g_t) = \mathcal{L}_0(f_t)$. As explained in the Introduction, this implies that $\mathcal{L}_0(f_t) = \mathcal{L}_0(g_0)$ for small $t$, hence also $\mathcal{L}_0(f_t) = \mathcal{L}_0(f_0)$. \hfill $\square$

As a corollary, we prove a particular case of the main theorem (in fact, an even stronger assertion since we do not assume non-degeneracy of $f_t$) when the set of 2-dimensional faces $\Gamma^2(f_0)$ is empty or consists of exceptional faces only.

Corollary 11. Let $f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be an isolated surface singularity such that $\Gamma^2(f_0) \setminus E(f_0) = \emptyset$. If $(f_t)$ is a $\mu$-constant deformation of $f_0$, then $\mathcal{L}_0(f_t) = \mathcal{L}_0(f_0)$ for small $t$.

Proof. The assumption $\Gamma^2(f_0) \setminus E(f_0) = \emptyset$ implies by Theorem 1.8 in [12] that $\Gamma^1(f_0)$ has the unique edge joining vertices $z^k$ and $xy$, $k \geq 2$ (up to permutation of variables $x, y, z$). Since $xy$ is a vertex of $\Gamma^+(f_0)$, Proposition 10 delivers the assertion. \hfill $\square$

Now we may prove the main theorem.
Proof of the main theorem. Let $f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ be a non-degenerate isolated singularity and let $(f_t)$ be its non-degenerate $\mu$-constant deformation. If $\Gamma^2(f) \setminus E(f) = \emptyset$, then the assertion follows from Corollary 11. Assume now $\Gamma^2(f) \setminus E(f) \neq \emptyset$. Then by Proposition 6, item 1 each coordinate axis has a proximate face and these faces are all non-exceptional. Observe that it is enough to prove the theorem for convenient $f_0$ and $f_t$. In fact, if $f_0$ is non-convenient with respect to the $x_i$-axis, $i \in \{1, 2, 3\}$, then we put $\tilde{f}_0 := f_0 + z_i^k$, $\tilde{f}_t := f_t + z_i^k$, where $k > \mu(f_0) = \mu(f_t)$.

Possibly increasing $k$, we get that $\tilde{f}_t$ are also non-degenerate (see for instance [5, Lemma 3.7]), $\mu(f_0) = \mu(f_0) = \mu(f_t)$ and, by the Ploski theorem [13], $L_0(f_t) = L_0(f_t)$. Hence, in the sequel we may assume $f_0$ and $f_t$ are convenient. As $f_t$ are non-degenerate, by the Kushnirenko theorem we have $\nu(f_t) = \mu(f_t) = \text{const.}$

By the monotonicity of the Newton number with respect to Newton polyhedra (see e.g. [7], [10, Cor. 2.3]), it is enough to prove the theorem for monomial deformations, i.e., deformations of the form $f_t = f_0 + t\alpha^\gamma$, where $\alpha \in \Gamma^+(f_0)$ (the case $\alpha \in \Gamma^+(f_0)$ follows from Theorem 2). We may also assume $\Gamma^+(f_t) = \text{const}$ for sufficiently small $t \neq 0$.

Denote the variables in $\mathbb{C}^3$ by $x, y, z$. Then $f_t(x, y, z) = f_0(x, y, z) + t\alpha^\gamma z^\gamma$, $(\alpha_1, \alpha_2, \alpha_3) \notin \Gamma^+(f_0)$. Since $\nu(f_t)$ is constant and $\Gamma^+(f_0) \subset \Gamma^+(f_t)$, using [4, Theorem 1] we infer that the vertex $C := x^{\alpha_1}y^{\alpha_2}z^{\alpha_3}$ of $\Gamma^+(f_t)$ lies in one of the coordinate planes $H$, say the $xy$-plane, and $P := \Gamma^+(f_t) \setminus \Gamma^+(f_0)$ is a pyramid with the base $\Gamma^+(f_t) \setminus \Gamma^+(f_0) \cap H$ whose apex $D \in \Gamma^0(f_0)$ has its $z$-coordinate equal to one (see Figure 3). Since ord $f_0 \geq 2$, $D$ does not belong to the $z$-axis.

Observe that $\Gamma^2(f_t)$, $t \neq 0$, differ from $\Gamma^2(f_0)$ on some 2-dimensional polytopes (precisely triangles). Write $\Gamma^2(f_t) = (\Gamma^2(f_0) \setminus \{S_1, \ldots, S_l\}) \cup \{N_1, \ldots, N_k\}$, where $S_1, \ldots, S_l, l \geq 1$, are the sides that get removed from $\Gamma^2(f_0)$ and $N_1, \ldots, N_k, k \geq 1$, the new ones showing in $\Gamma^2(f_t)$. (As an illustration, in Figure 3(a) the three sides of $\Gamma^2(f_0)$ get replaced by two new ones, marked with dashed contours, taken from $\Gamma^2(f_t)$; in Figure 3(b) the three sides of $\Gamma^2(f_0)$ are also replaced by two new ones but in this case one of them, namely $CDE$, is an extension of the one removed, i.e., $DEF$. This also shows that $N_i, S_j$ need not constitute sides of $P$.)

Since $N_i, S_j$ are triangles with the bases contained in the $xy$-plane and sharing the common vertex $D$ whose $z$-coordinate is equal to 1, we notice that if any of the faces $N_i, S_j$ is proximate for the $z$-axis, then either $D = (1, 0, 1)$ or $D = (0, 1, 1)$ (in the non-convenient case this is because any proximity face of an axis has a vertex lying at distance 1 from this axis). But if this happens, we get the assertion by applying Proposition 10.
Thus, we may suppose none of the faces \( N_i, S_j \) is proximate for the \( z \)-axis. This also means that, in order to finish the proof, it is enough to show that the intersections of (the supporting planes of) the \( x \)- and \( y \)-proximate faces of \( \Gamma_+(f_t) \) with the corresponding axes are the same as those of \( \Gamma_+(f_0) \); indeed, the formula in Theorem 7 then implies that \( L_0(f_t) = L_0(f_0) \) for small \( t \). To this end, we consider the following cases:

a) The vertex \( D \notin (xz \text{-plane} \cup yz \text{-plane}) \). Then \( C \) does not belong to the \( x \)-axis and to the \( y \)-axis.

b) The vertex \( D \in (xz \text{-plane} \cup yz \text{-plane}) \). Without loss of generality, we may assume that \( D \) belongs to the \( xz \)-plane (see Figure 4(b)), then \( DV \) is the common edge of some \( x \)-proximate face of \( \Gamma_+(f_0) \) and some \( x \)-proximate face of \( \Gamma_+(f_t) \). This means that the proximity faces for the \( x \)-axis of both \( \Gamma_+(f_0) \) and \( \Gamma_+(f_t) \) intersect this axis in the same coordinate, viz., \( V_x \).

If none of the faces \( N_1, \ldots, N_k, S_1, \ldots, S_l \) touches the \( x \)- or \( y \)-axis, then none of these faces touches the \( xz \)- or \( yz \)-plane, see Figure 4(a). Hence none of these faces is proximate for the \( x \)- or \( y \)-axis. Then the Newton polyhedron \( \Gamma_+(f_t) \) of \( f_t \), for \( t \neq 0 \), has the same proximity faces, for all three axes, as those of \( \Gamma_+(f_0) \). Consequently, also the intersections of (the supporting planes of) the proximity faces of \( \Gamma_+(f_t) \) with the axes are the same as those of \( \Gamma_+(f_0) \).

If some \( S_i \) or \( N_j \) touches the \( x \)-axis (for the \( y \)-axis we could reason analogously) in the vertex \( V \) (see Figure 4(b)), then \( DV \) is the common edge of some \( x \)-proximate face of \( \Gamma_+(f_0) \) and some \( x \)-proximate face of \( \Gamma_+(f_t) \). This means that the proximity faces for the \( x \)-axis of both \( \Gamma_+(f_0) \) and \( \Gamma_+(f_t) \) intersect this axis in the same coordinate, viz., \( V_x \).

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If none of the faces \( N_1, \ldots, N_k, S_1, \ldots, S_l \) touches the \( x \)- or \( y \)-axis, then none of these faces touches the \( xz \)- or \( yz \)-plane, see Figure 4(a). Hence none of these faces is proximate for the \( x \)- or \( y \)-axis. Then the Newton polyhedron \( \Gamma_+(f_t) \) of \( f_t \), for \( t \neq 0 \), has the same proximity faces, for all three axes, as those of \( \Gamma_+(f_0) \). Consequently, also the intersections of (the supporting planes of) the proximity faces of \( \Gamma_+(f_t) \) with the axes are the same as those of \( \Gamma_+(f_0) \).

If some \( S_i \) or \( N_j \) touches the \( x \)-axis (for the \( y \)-axis we could reason analogously) in the vertex \( V \) (see Figure 4(b)), then \( DV \) is the common edge of some \( x \)-proximate face of \( \Gamma_+(f_0) \) and some \( x \)-proximate face of \( \Gamma_+(f_t) \). This means that the proximity faces for the \( x \)-axis of both \( \Gamma_+(f_0) \) and \( \Gamma_+(f_t) \) intersect this axis in the same coordinate, viz., \( V_x \).

If none of the faces \( N_1, \ldots, N_k, S_1, \ldots, S_l \) touches the \( x \)- or \( y \)-axis, then none of these faces touches the \( xz \)- or \( yz \)-plane, see Figure 4(a). Hence none of these faces is proximate for the \( x \)- or \( y \)-axis. Then the Newton polyhedron \( \Gamma_+(f_t) \) of \( f_t \), for \( t \neq 0 \), has the same proximity faces, for all three axes, as those of \( \Gamma_+(f_0) \). Consequently, also the intersections of (the supporting planes of) the proximity faces of \( \Gamma_+(f_t) \) with the axes are the same as those of \( \Gamma_+(f_0) \).

If some \( S_i \) or \( N_j \) touches the \( x \)-axis (for the \( y \)-axis we could reason analogously) in the vertex \( V \) (see Figure 4(b)), then \( DV \) is the common edge of some \( x \)-proximate face of \( \Gamma_+(f_0) \) and some \( x \)-proximate face of \( \Gamma_+(f_t) \). This means that the proximity faces for the \( x \)-axis of both \( \Gamma_+(f_0) \) and \( \Gamma_+(f_t) \) intersect this axis in the same coordinate, viz., \( V_x \).

b) The vertex \( D \in (xz \text{-plane} \cup yz \text{-plane}) \). Without loss of generality, we may assume that \( D \) belongs to the \( xz \)-plane (see Figure 5). Since its \( z \)-coordinate equals one, we infer that all the faces \( N_1, \ldots, N_k, S_1, \ldots, S_l \) are exceptional with respect to the \( x \)-axis. Combining this observation with the assumption \( \Gamma^2(f) \setminus E(f) \neq \emptyset \), we conclude that \( \Gamma_+(f_t) \) and \( \Gamma_+(f_0) \) have the same proximity faces, for all the three axes, and so \( L_0(f_t) = L_0(f_0) \) for small \( t \).

This ends the proof. \( \square \)
4. Concluding remarks

Using the main theorem and some known facts, we may obtain additional information on families of surface singularities.

Corollary 12. Let \( f_0 : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \) be a non-degenerate isolated surface singularity and let \( (f_t) \) be its non-degenerate deformation. If \( f_t \) are pairwise topologically equivalent, then \( L_0(f_t) = L_0(f_0) \) for small \( t \).

Proof. Since \( f_t \) are pairwise topologically equivalent, they have the same Milnor numbers. So, \( (f_t) \) is \( \mu \)-constant deformation of \( f_0 \). By the main theorem, we get \( L_0(f_t) = L_0(f_0) \) for small \( t \). \( \square \)

We predict the main theorem is true also in the \( n \)-dimensional case. In the proof of our theorem we use two facts on surface singularities: the first one – a formula for the Lojasiewicz exponent of non-degenerate surface singularities (Theorem 7), and the second one – the characterization of these Newton polyhedra of surface singularities that have the same Newton numbers [4, Thm. 1]. The second fact has been recently generalized by M. Leyton-Álvarez, H. Mourtada and M. Spivakovsky in [10, Thm. 2.15] to the \( n \)-dimensional case. Since the Lojasiewicz exponent is one and the same for isolated non-degenerate singularities with a given Newton polyhedron ([2]), there “only” remains the question about a formula for the exponent, a formula expressed in terms of the Newton polyhedron (cf. Remark 5).

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