LINEAR CONVERGENCE OF DISTRIBUTED DYKSTRA’S ALGORITHM FOR SETS UNDER AN INTERSECTION PROPERTY

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Abstract. We show the linear convergence of a distributed Dykstra’s algorithm for sets intersecting in a manner slightly stronger than the usual constraint qualifications.

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1. Introduction

Let $G = (V, E)$ be an undirected graph. For all $i \in V$, let $C_i \subset \mathbb{R}^m$ be closed convex sets, and $\bar{x}_i \in \mathbb{R}^m$. For a closed convex set $C$, let $\delta_C(\cdot)$ be its indicator function. Consider the distributed optimization problem

$$\min_{x \in \mathbb{R}^m} \sum_{i \in V} \left[ \delta_{C_i}(x) + \frac{1}{2} \|x - \bar{x}_i\|^2 \right],$$

where communications between two vertices in $V$ occur only along edges in $E$. In Remark 2.3 we explain that we can assume that all $\bar{x}_i$ are equal to some $\bar{x}$ without losing any generality. The problem is therefore equivalent to projecting $\bar{x}$ onto $\bigcap_{i \in V} C_i$ in a distributed manner.

1.1. A review of the distributed Dykstra’s splitting. In our earlier paper [Pan18a], we considered the more general problem than (1.1) where $\delta_{C_i}(\cdot)$ can be general closed convex functions instead. We proposed a deterministic distributed asynchronous decentralized algorithm based on dual ascent for (1.1) that converges to the primal minimizer, and call it the distributed Dykstra’s algorithm. Our approach was motivated by work on Dykstra’s algorithm in [Dyk83, BDS85, GM89, HD97]. See also [Han88]. We also remark that the dual ascent idea had been discussed in [CDV11, CDV10, ACP+17]. We refer to the introduction in [Pan18a] for

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more historical summary of these methods. Part of the contribution in [Pan18a] was to point out that the dual ascent idea leads to a desirable distributed optimization algorithm. We give more details of the distributed Dykstra’s algorithm in Section 2.

1.2. Linear convergence of Dykstra’s algorithm. A well known algorithm for solving (1.1) is Dykstra’s algorithm. The primal problem and its corresponding (Fenchel) dual are typically written as

\[
\begin{align*}
\min_{x \in \mathbb{R}^m} & \frac{1}{2} \|x - \bar{x}\|^2 + \sum_{i \in V} \delta_{C_i}(x) \\
\max_{z_i \in \mathbb{R}^m, i \in V} & \frac{1}{2} \|\bar{x}\|^2 - \frac{1}{2} \left\| \bar{x} - \sum_{i \in V} z_i \right\|^2 - \sum_{i \in V} \delta^*_{C_i}(z_i)
\end{align*}
\]

respectively, and solved by block coordinate maximization on the dual problem. (See [BD85, Han88, GM89].) (Note that this dual is different from (2.4).) In the case when \(C_i\) are halfspaces, linear convergence of Dykstra’s algorithm was established in [Pi90], with refined rates given in [DH94]. We extended the linear rates to polyhedra in [Pan17].

A linear convergence rate of Dykstra’s algorithm assures that a high accuracy solution can be obtained in a reasonable amount of time. This would then allow the algorithm to be used as a subroutine of other optimization algorithms. For example, the distributed optimization algorithms [AH16, TSDS18] (and perhaps many others) make use of the averaged consensus algorithm as a subroutine. (The linear convergence rate of averaged consensus is used in the convergence proof of the main distributed optimization algorithm.) Since averaged consensus is a particular case of the distributed Dykstra’s algorithm with all \(C_i\) being \(\mathbb{R}^m\), it is plausible to make use of the distributed Dykstra’s algorithm to help solve constrained distributed problems.

1.3. Contributions of this paper. Even though we have observed linear convergence rates of the distributed Dykstra’s algorithm in [Pan18b] in our numerical experiments for the case when some of the terms are indicator functions of closed convex sets, it seems that there is no theoretical justification yet of linear convergence for both Dykstra’s original algorithm and for the distributed Dykstra’s algorithm beyond the polyhedral case. As is well-known, the intersection \(\cap_{i \in V} C_i\) can be sensitive to the perturbation of the sets \(C_i\) [Kru06], so additional constraint qualifications are needed for the linear convergence of the method of alternating projections (see for example [BB96]).

In this paper, we prove the asymptotic linear convergence of the distributed Dykstra’s algorithm when the functions are indicator functions of sets that are not necessarily polyhedral. We assume that the sets satisfy a property on systems of intersections of sets stronger than what is typically studied in the method of alternating projections. We also make assumptions that are closely related to conditions used to prove linear convergence in proximal algorithms.

1.4. Notation. Variables in bold, like \(\mathbf{x}\) and \(\mathbf{z}_i\), typically lie in the space \([\mathbb{R}^m]^{|V|}\), while variables not in bold, like \(x\) and \(y\), typically lie in \(\mathbb{R}^m\). All norms shall be the 2-norm. We often use \(\mathbf{e}^i\) to represent the unit vector in a given direction. For example, \(\hat{x}_i^0 = \frac{\mathbf{e}^i}{\|\mathbf{e}^i\|}\).
2. Preliminaries

In this section, we lay down the preliminaries of the paper. For each \( i \in V \), let \( f_i : [\mathbb{R}^m]^{\vert V \vert} \to \mathbb{R} \cup \{ \infty \} \) be defined by

\[
f_i(x) = \delta_{C_i}([x]_i). 
\]

For each \((i, j) \in E\), define the halfspaces \( H_{(i, j)} \) to be

\[
H_{(i, j)} := \{ x \in [\mathbb{R}^m]^{\vert V \vert} : x_i = x_j \}.
\]

Since the graph is connected, the intersection of all these halfspaces is the diagonal set defined by

\[
D := \cap_{e \in E} H_e = \{ x \in [\mathbb{R}^m]^{\vert V \vert} : x_1 = x_2 = \cdots = x_{\vert V \vert} \}. 
\]

For each \( e \in E \), define \( f_e : [\mathbb{R}^m]^{\vert V \vert} \to \mathbb{R} \) by \( f_e(x) = \delta_{H_e}(x) \). The setting for the distributed Dykstra’s algorithm that is easily seen to be equivalent to (1.1) is

\[
\min_{x \in [\mathbb{R}^m]^{\vert V \vert}} \frac{1}{2}\|x - \bar{x}\|^2 + \sum_{i \in V} f_i(x) + \sum_{e \in E} \delta_{H_e}(x), 
\]

where \( \bar{x} \in [\mathbb{R}^m]^{\vert V \vert} \) is such that each component of \([\bar{x}]_i\), where \( i \in V \), is equal to \( \bar{x} \).

Let the dual variables be \( z = \{ z_\alpha \}_{\alpha \in V \cup E} \), where each \( z_\alpha \in [\mathbb{R}^m]^{\vert V \vert} \). The (Fenchel) dual of (2.3) can be calculated to be

\[
\max_{z_\alpha \in [\mathbb{R}^m]^{\vert V \vert}, \alpha \in V \cup E} \frac{1}{2}\|x - \bar{x}\|^2 - \frac{1}{2} \|x - \sum_{\alpha \in V \cup E} z_\alpha \|^2 - \sum_{i \in V} \delta^*_C(z_i) - \sum_{e \in E} \delta^*_H(z_e). 
\]

Proposition 2.1. \( (Sparsity) \) If the value in (2.4) is finite, then

1. If \( i \in V \), then \( z_i \in [\mathbb{R}^m]^{\vert V \vert} \) is such that \([z_i]_j = 0 \) for all \( j \in V \setminus \{i\} \).
2. If \((i, j) \in E\), then \( z_{(i, j)} \in [\mathbb{R}^m]^{\vert V \vert} \) is such that \([z_{(i, j)}]_k = 0 \) for all \( k \in V \setminus \{i, j\} \), and \([z_{(i, j)}]_i + [z_{(i, j)}]_j = 0 \).

Proof. The proof is elementary and exactly the same as that in [Pan18a]. (Part (1) makes use of the fact that \( f_i(\cdot) \) depends on only the \( i \)-th coordinate of the input, while part (2) makes use of the fact that \( \delta^*_H(z_{(i, j)}(\cdot)) = \delta_{H_{(i, j)}}(\cdot) \), and \( \delta^*_H(z_{(i, j)})) < \infty \) if and only if the conditions in (2) hold.) \( \Box \)

In view of Proposition 2.1, the vector \( z_i \) for all \( i \in V \) are such that \([z_i]_j = 0 \) if \( j \neq i \). Letting \( z_i := [z_i]_i \), we let the dual function \( F : ([\mathbb{R}^m]^{\vert V \vert})^{V \cup E} \to \mathbb{R} \) be

\[
F(z) := \sum_{i \in V} \delta^*_C(z_i) + \sum_{e \in E} \delta^*_H(z_e) + \frac{1}{2} \left\|x - \sum_{\alpha \in V \cup E} z_\alpha \right\|^2_\infty. 
\]

It is clear to see that \( F(z) \) differs from (2.4) by a sign and a constant. It is known that strong duality between (2.3) and (2.4) holds (even though a dual minimizer may not exist). Minimizing \( F(\cdot) \) allows one to find the optimal value to (2.4), and also the optimal solution to (2.3). It turns out that the only variables that need to be tracked are \( z_i \in \mathbb{R}^m \) for all \( i \in V \) and \( x \in [\mathbb{R}^m]^{\vert V \vert} \) as marked above. We shall prove that \( x \) converges linearly to the optimal primal solution under some additional assumptions. We refer to the \( i \)-th coordinate of \( x \) as \( x_i \). Also, if \( x^* \), the projection of \( \bar{x} \) onto \( \cap_{i \in V} C_i \), were to be zero, then \( F(x) \) takes the minimum of zero when \( x \) is the primal optimal solution and \( \{ z_i \}_{i \in V} \) are optimal multipliers.
Here are the first set of assumptions we need to prove our linear convergence result.

**Assumption 2.2.** Suppose that the following assumptions hold:

1. Let \( x^* \in \mathbb{R}^m \) be the optimal solution to (1.1). We assume that \( x^* = 0 \).
2. The \( \bar{x}_i \) are all equal for all \( i \in V \).
3. (Existence of dual minimizers) There exists \( \{z_i\}_{i \in V} \) such that \( z_i \in N_{C_i}(x^*) \) and \( \sum_{i \in V} z_i = |V| \bar{x} \).
4. (Regularity of the sets \( C_i \)) The sets satisfy the semismoothness property of order 2.
5. (Graph connectedness) The (undirected) graph \( G = (V, E) \) is connected.
6. (Regularity of the sets \( C_i \)) The sets satisfy a nondegeneracy constraint qualification: There is a neighborhood \( U \) of \( x^* \) and parameters \( M_{\max} > 1 \) and \( M_{\min} > 0 \) such that if the multipliers \( \{z_i\}_{i \in V} \) and points \( \{x_i\}_{i \in V} \) are such that \( x_i \in U \) and \( z_i \in N_{C_i}(x_i) \) for all \( i \in V \) and \( [\bar{x} - \sum_{a \in V \cup E} z_a]_j \in U \) for all \( j \in V \), then
   \[
   M_{\min} \leq ||z_i|| \leq M_{\max} \quad \text{for all} \quad i \in V.
   \]
   Let \( H_i \) be the hyperplane \( \{x : x_i^T(x - x_i) = 0\} \) for all \( i \in V \). Assume that for all \( x \in \mathbb{R}^m \), there is some constant \( \kappa_1 > 0 \) such that \( d(x, \cap_{i \in V} H_i) \leq \kappa_1 \max_{i \in V} d(x, H_i) \).
7. (First order property on normals) There is a neighborhood \( U \) of \( x^* \) and \( \kappa_3 > 0 \) such that for all \( i \in V \), if \( x \in U \cap C_i \), and \( z \in N_{C_i}(x) \setminus \{0\} \), then there is a \( z^* \in N_{C_i}(x^*) \setminus \{0\} \) such that
   \[
   ||\frac{\bar{x}_i^T}{x_i^T} - \frac{z^*}{z^*_i^T}|| \leq \kappa_3 \|x - x^*\|.
   \]
8. (A linear regularity property on the normal cones) Define the set \( M \subset [\mathbb{R}^m]^{\left| V \right|} \) of optimal multipliers to be \( M := M_1 \cap M_2 \), where
   \[
   M_1 := N_{C_1}(x^*) \times \cdots \times N_{C_{\left| V \right|}}(x^*)
   \]
   and
   \[
   M_2 := \{z \in [\mathbb{R}^m]^{\left| V \right|} : \sum_{i \in V} z_i = |V| \bar{x}\}.
   \]
   Assume there is a \( \kappa_4 > 0 \) such that
   \[
   d(z, M_1 \cap M_2) \leq \kappa_4 d(z, M_2) \quad \text{for all} \quad z \in M_1.
   \]

We remark about Assumption 2.2\( ^{[5]} \). The linear regularity property is usually stated as \( d(z, M_1 \cap M_2) \leq \kappa_4 \max\{d(z, M_1), d(z, M_2)\} \) for all \( z \), but we state a weaker version of it in Assumption 2.2\( ^{[5]} \) because that is what our proof needs. The stronger linear regularity is satisfied whenever the normal cones \( N_{C_i}(x^*) \) are polyhedral (see for example [BB96, Corollary 5.26]), so this assumption is quite reasonable.

Assumption 2.2\( ^{[3]} \) is stronger than the usual transversality condition typically studied in the method of alternating projections. Now that we are working with
an optimization problem (1.1) rather than a feasibility problem, it may be more appropriate to compare to the Robinson constraint qualification. We seek to study this assumption further in future work.

We make the following remark.

\textbf{Remark 2.3.} (On Assumption 2.2(2)) We now show that Assumption 2.2(2) does not lose any generality. Suppose that the $\bar{x}_i$ are not all necessarily the same. Note that $\sum_{i \in V} \frac{1}{2} \| x - \bar{x}_i \|^2 = \sum_{i \in V} \left( \frac{1}{2} \| x - a \|^2 + \frac{1}{2} \| \bar{x}_i \|^2 - \frac{1}{2} \| a \|^2 \right)$, where $a = \frac{1}{|V|} \sum_{i \in V} \bar{x}_i$. Thus all the $\bar{x}_i$ can be replaced by $a$. Note that this does not mean that the primal iterate $x$ needs to be such that all its coordinates are $a$ at the start.

We now state Algorithm 2.4 which minimizes $F(\cdot)$ by block coordinate minimization.

\textbf{Algorithm 2.4.} (Distributed Dykstra’s algorithm) Our distributed Dykstra’s algorithm is as follows:

\begin{itemize}
  \item 01 Let
  \begin{itemize}
    \item $z_{i,0}^{1} \in [\mathbb{R}^m]^{|V|}$ be a starting dual vector for $f_i(\cdot)$ for each $i \in V$ so that $[z_{i,0}^{1}]_j = 0$ for all $j \in V \setminus \{i\}$.
    \item $z_{(i,j),1}^{1} \in [\mathbb{R}^m]^{|V|}$ be a starting dual vector for each edge $(i, j)$ so that $[z_{(i,j),1}^{1}]_i + [z_{(i,j),1}^{1}]_{j'} = 0$ for all $j' \in V \setminus \{i, j\}$.
  \end{itemize}
  \item 02 For $n = 1, 2, \ldots$
  \item 03 For $w = 1, 2, \ldots, \tilde{w}$
  \item 04 Choose a set $S_{n,w} \subset E \cup V$ such that $S_{n,w} \neq \emptyset$.
  \item 05 Define $\{z_{\alpha,n,w}^{\alpha,0}\}_{\alpha \in S_{n,w}}$ by
  \begin{equation}
  \{z_{\alpha,n,w}^{\alpha,0}\}_{\alpha \in S_{n,w}} = \arg \min_{z_{\alpha,n,w}^{\alpha,0}} \left\{ \frac{1}{2} \| x - \sum_{\alpha \notin S_{n,w}} z_{\alpha,n,w}^{\alpha,w-1} - \sum_{\alpha \in S_{n,w}} z_{\alpha,n,w}^{\alpha,w} \|_2^2 + \sum_{\alpha \in S_{n,w}} f_{\alpha}^{\alpha}(z_{\alpha,n,w}^{\alpha,0}) \right\}.
  \end{equation}
  \item 06 Set $z_{\alpha,n,w}^{\alpha,w} := z_{\alpha,n,w}^{\alpha,w-1}$ for all $\alpha \notin S_{n,w}$.
  \item 07 End For
  \item 08 Let $z_{\alpha,n,w}^{\alpha,w+1,0} := z_{\alpha,n,w}^{\alpha,w}$ for all $\alpha \in V \cup E$.
  \item 09 End For
\end{itemize}

To provide some intuition to Algorithm 2.4 we mention that minimizing only one $z_i$ at a time for some $i \in V$ (i.e., $S_{n,k} = \{i\}$) reduces (2.11) to a standard proximal problem. Minimizing only one $z_{(i,j)}$ for some $(i, j) \in E$ (i.e., $S_{n,k} = \{(i, j)\}$) has the natural interpretation of averaging the $i$-th and $j$-th components of $x$.

Let the function $f^\ast : [\mathbb{R}^m]^{|V|} \rightarrow \mathbb{R} \cup \{\infty\}$ to be defined to be $f^\ast(\cdot) = \delta_{H^\ast}(\cdot)$. Let $x^\ast$ be the optimal solution of (2.3). Before we prove the result, we note that using a technique in [GAMS9], the duality gap between the primal and dual pair (2.3) and
Theorem 3.1. (Linear convergence of dual value) Suppose Assumptions 2.2 and 2.5 hold. For Algorithm 2.4, there is a constant \( r \in (0, 1) \) such that \( F(z^{n+1,0}) \leq r F(z^{n-1,0}) \). Together with (2.12), this implies that the distance \( \{ \| x^0_n - x^* \| \}_{n \geq 1} \) converges linearly to zero for all \( i \in V \).

We need positive parameters \( \tilde{c}, \theta_D \) and \( \theta_Z \) to be small enough so that they satisfy \( \tilde{c}|V|(2\kappa_2 M_{\text{max}} + 1) \leq \frac{1}{1}, c_2(\theta_D, \theta_D, 0) > 0 \) and \( (3.23) \), where \( c(\cdot) \) and \( c_2(\cdot) \) are defined in (3.20) and (3.29), and the other constants are described in Assumption 2.2 and in the course of the proof. It is easy to see that the parameters \( \tilde{c}, \theta_D \) and \( \theta_Z \) can be chosen to satisfy these conditions.

The first two cases of the proof of Theorem 3.1 are easier than the third case. To simplify notation, we let \( z^n_{\alpha}^w \) to be written simply as \( z^n_{\alpha} \) for all \( w \in \{0, 1, \ldots, \tilde{w} \} \) and \( \alpha \in V \cup E \), and the dropping of “\( n \)” appears in all other variables as well. Let \( x_i^w \) be the \( i \)-th coordinate of \( x^w \), and let \( z_i^w \) be \( [z^w_i] \), the \( i \)-th coordinate of \( z^w \). If \( S_{n,k} = \{ i \} \) for some \( i \in V \), then \( x_i^k \) and \( z_i^k \) are the solutions to the primal dual pair.
We have 2 cases.

Also, we can assume that at index 

The proof is split into 3 cases:

Proof of cases 1 and 2 of Theorem 3.1. We have

Case 1a:

Case 1b:

In this case,

Note that

Case 1:

(3.2)

We can assume that at index 

We have 2 cases.

Case 1a: \( \|x^k\|^2 \leq \frac{1}{\kappa_2 M_{\text{max}} + 1} \delta_{\delta_C}(z^0) \)

In this case,

\[
\delta_{\delta_C}(z^0) \leq \frac{1}{2} \|x^k\|^2 \leq \frac{1}{2} \delta_{\delta_C}(z^0) \leq \frac{1}{2} \delta_{\delta_C}(z^0) + \frac{1}{2} \|s^{k-1}\|^2.
\]

Recall that \( z^0 = z^{k-1} \). Since \( S_{n,k} = \{i^*\} \), we also have \( z_i = z^{k-1} \) for all \( i \neq i^* \) and \( x_i = x_i^{k-1} \) for all \( i \neq i^* \). We have

\[
F(z^k) = \sum_{i \in V} \left( \delta_{\delta_C}(z^0) + \frac{1}{2} \|x_i\|^2 \right) 
\]

\[
\leq \sum_{i \neq i^*} \left( \delta_{\delta_C}(z^0) + \frac{1}{2} \|x_i\|^2 \right) + \frac{1}{2} \delta_{\delta_C}(z^0) + \frac{1}{2} \|s^{k-1}\|^2 
\]

\[
\leq F(z^{k-1}) - \frac{1}{2} \delta_{\delta_C}(z^0) \leq F(z^{k-1}) - \frac{1}{2} \delta_{\delta_C}(z^0) \leq \left( 1 - \frac{1}{|V| + 1} \right) F(z^0) 
\]

Case 1b: \( \|x_i^0\|^2 \geq \frac{1}{\kappa_2 M_{\text{max}} + 1} \delta_{\delta_C}(z^0) \)

Note that \( \frac{1}{2} \|x_i^0 - x_i^0\|^2 \) is an estimate of the decrease of the dual objective value.

We choose \( \varepsilon > 0 \) so that \( \varepsilon |V| (2\kappa_2 M_{\text{max}} + 1) \leq \frac{1}{2} \). We have

\[
\|x_i^0\|^2 \leq \varepsilon |V| \delta_{\delta_C}(z^0) \leq \varepsilon |V| (2\kappa_2 M_{\text{max}} + 1) \|x_i^0\|^2 \leq \frac{1}{2} \|x_i^0\|^2. 
\]
We then have

\[
\|x_i^k - x_i^{k-1}\|^2 \geq (\|x_i^k\| - \|x_i^{k-1}\|)^2 \geq \frac{1}{4}\|x_i^k\|^2 \geq \frac{1}{4(2\kappa_2 M_{\max} + 1)} \delta_{C_{i^*}}(z_{i^*}^0).
\]  

(3.8)

Then

\[
\sum_{k'=1}^k \|x_i^{k'} - x_i^{k'-1}\|^2 \geq \frac{1}{k} \left( \sum_{k'=1}^k \|x_i^{k'} - x_i^{k'-1}\| \right)^2 \geq \frac{1}{k}\|x_i^0 - x_i^{k'}\|^2.
\]  

(3.9)

We then have

\[
F(z^k) \geq \sum_{i \in V} \delta_{C_i}(z_i^k) + \frac{1}{2}\|x^k\|^2 \\
\geq \sum_{i \in V} \delta_{C_i}(z_i^0) + \frac{1}{2}\|x^0\|^2 - \sum_{k'=1}^k \frac{1}{2}\|x_i^{k'} - x_i^{k'-1}\|^2 \\
\geq F(z^0) - \frac{1}{8w(2\kappa_2 M_{\max} + 1)} \delta_{C_{i^*}}(z_{i^*}^0) \\
\geq \left( 1 - \frac{1}{4w(2\kappa_2 M_{\max} + 1)(\bar{\tau}+\epsilon)} \right) F(z^0).
\]  

(3.10)

**Case 2:** \(\|x^0\|^2 \geq \epsilon \sum_{i \in V} \delta_{C_i}(z_i^0),\) and \(\|P_{D^*} x^0\|^2 \geq \theta_D \|x^0\|^2.\)

In this case, note that \(d(x^0, D) = \|P_{D^*} x^0\| \geq \sqrt{\theta_D} \|x^0\|.\) Since \(D = \bigcap_{e \in E} H_e\) and \(H_e\) are hyperplanes, there is some \(\kappa_D > 0\) such that \(\max_{e \in E} d(x^0, H_e) \geq \frac{1}{\kappa_D} d(x^0, D).\) Let \(e^*\) be such that \(d(x^0, H_{e^*}) = \max_{e \in E} d(x^0, H_e),\) and let \(k\) be such that \(x^k \in H_{e^*},\) which exists by Assumption (2.5) (1). We then have

\[
\|x^0 - x^k\| \geq d(x^0, H_{e^*}) \geq \frac{1}{\kappa_D} d(x^0, D) \geq \frac{2}{\kappa_D} \sqrt{\theta_D} \|x^0\|.
\]  

(3.11)

Now,

\[
\left( \frac{1}{2} + \frac{1}{2} \right) \|x^0\|^2 \geq \frac{1}{2}\|x^0\|^2 + \sum_{i \in V} \delta_{C_i}(z_i^0) \geq F(z^0).
\]  

(3.12)

We have \(\frac{1}{2} \sum_{i=1}^k \|x^i - x^{i-1}\|^2 \geq \frac{1}{2w} \left( \sum_{i=1}^k \|x^i - x^{i-1}\| \right)^2 \geq \frac{1}{2w}\|x^0 - x^k\|^2,\) so

\[
F(z^k) \leq F(z^0) - \frac{1}{2w} \sum_{i=1}^k \|x^i - x^{i-1}\|^2 \leq F(z^0) - \frac{1}{2w}\|x^0 - x^k\|^2 \leq F(z^0) - \frac{1}{2w} \theta_D \|x^0\|^2 \leq \left( 1 - \frac{1}{w\theta_D} \right) F(z^0).
\]  

(3.13)

Hence we are done.

This leaves us with Case 3, i.e.,

**Case 3:** \(\|x^0\|^2 \geq \epsilon \sum_{i \in V} \delta_{C_i}(z_i^0),\) and \(\|P_{D^*} x^0\|^2 \leq \theta_D \|x^0\|^2.\)
By the definition of $D$ in (2.2), all $|V|$ components of $P_Dx^0$ are equal to some value, which we call $a$. Then we have the inequalities

$$
\|P_Dx^0\|^2 = \|x^0\|^2 - \|P_Dx^0\|^2 \quad \text{Case 3} \quad \sum_{i \in V} \|x^0_i - a\|^2 = \|P_Dx^0\|^2 \quad \frac{\theta_D}{1 - \theta_D} \|P_Dx^0\|^2 \leq \frac{\theta_D}{1 - \theta_D} \|a\|^2, \quad (3.14)
$$

$$
|V|\|a\|^2 = \|P_Dx^0\|^2 \leq \|x^0\|. \quad (3.15)
$$

We have

$$
\|x^0_i\| \leq \|a\| + \|x^0_i - a\| \leq (1 + \sqrt{\frac{\theta_D}{1 - \theta_D}}|V|)\|a\|. \quad (3.17a)
$$

and

$$
\|x^0_i\| \geq \|a\| - \|x^0_i - a\| \geq (1 - \sqrt{\frac{\theta_D}{1 - \theta_D}}|V|)\|a\|. \quad (3.17b)
$$

and

$$
\|a - x^0_i\| \leq \sqrt{\frac{\theta_D}{1 - \theta_D}|V|\|a\|} \leq \left(1 - \sqrt{\frac{\theta_D}{1 - \theta_D}}|V|\right)^{-1} \sqrt{\frac{\theta_D}{1 - \theta_D}|V||x^0_i|^2}. \quad (3.18)
$$

We now show that there is a constant $\tilde{\kappa}_3 > 0$ such that $\|\tilde{z}^0_i - \tilde{z}_i\| \leq \tilde{\kappa}_3\|x^0_i\|$. We have $\|\tilde{z}^0_i - \tilde{z}_i\| \leq \kappa_3\|\tilde{x}_i\|$, where $\tilde{x}_i := x^{n-1,p(n-1,i)}$, and $p(n-1,i)$ is the index such that $i \notin S_{n-1,k}$ for all $k$ such that $p(n-1,i) < k \leq \bar{u}$. If we have $\|\tilde{z}^0_i - \tilde{z}_i\| > \tilde{\kappa}_3\|x^0_i\|$, then

$$
F(z^{n-1,p(n-1,i)}) \geq \frac{1}{2}\|\tilde{x}_i\|^2 \geq \frac{1}{2\kappa_3}\|\tilde{z}^0_i - \tilde{z}_i\|^2 \geq \frac{\tilde{\kappa}^2}{2\kappa_3}\|x^0_i\|^2 \quad (3.19)
$$

This would then give us $F(z^{n-1,0}) \geq \frac{\tilde{\kappa}^2}{\kappa_3} \left(1 - \sqrt{\frac{\theta_D}{1 - \theta_D}}\right)^2 \frac{1 - \theta_D}{|V| + 2\tau} F(z^{n+1,0})$. The parameter $\tilde{\kappa}_3$ can be chosen large enough so that the coefficient of $F(z^{n+1,0})$ is greater than 1, which once again leads to the conclusion in Theorem 3.1. Therefore, we shall assume

$$
\|\tilde{z}^0_i - \tilde{z}_i\| \leq \tilde{\kappa}_3\|x^0_i\| \quad (3.19)
$$

throughout. We now assume Assumption (2.5,[2]), and let $x_i^+$ and $z_i^+$ be $x_i^1$ and $z_i^1$ respectively.

**Proof of case 3 of Theorem 3.1** We consider $\{z^0_i\}_{i \in V}$ and $\{z_i^+\}_{i \in V}$, where $z_i^+ = P_{N_G}(x^+)(z^0_i)$. Recall $M \subset \mathbb{R}^m$ defined as the set of optimal multipliers defined in Assumption (2.2).[8] Let $(z^0_p, \ldots, z^0_{|V|}) \in \mathbb{R}^m$ be

$$
(z^0_p, \ldots, z^0_{|V|}) = P_M((z^+_p, \ldots, z^+_{|V|})), \quad (3.20)
$$

where $(z^+_p, \ldots, z^+_{|V|}) \in \mathbb{R}^m$. Let $Z$ be span$\{(z^+_p)_{i \in V}\}$. Let $d$ be the direction $P_Za$. There are two subcases to consider.

**Case 3a:** $\|P_Za\|^2 \leq \theta_Z\|a\|^2$
Since \( d = P_{Z^*} a \in Z^* \), we have
\[
d \perp z^p_i \quad \text{for all } i \in V. \tag{3.21}\]

We would be projecting \( x^0_i + z^0_i \) onto \( C_i \) for all \( i \in V \). Let an outer approximate of \( C_i \) be
\[
P_i := \{ x : (z^0_i)^T x \leq \epsilon_i, (z^p_i)^T x \leq 0 \}, \quad \text{where} \quad \epsilon_i := \delta^*_C(z^0_i). \tag{3.22}\]

Since \( C_i \subset P_i \), we have \( \delta_{P_i}(\cdot) \leq \delta_{C_i}(\cdot) \), and so \( \delta_{P_i}(z^0_i) = \delta_{C_i}(z^0_i) \). Since \( d \in Z^* \) and \( \hat{x} \in Z \), we have \( d^T \hat{x} = 0 \). Proposition 2.10 implies that \( \sum_{i \in V} \sum_{a \in E} [z^0_{a_i}] = 0 \). So we have
\[
d^T \sum_{i \in V} (x^0_i + z^0_i) \quad \text{Prop 2.12} \quad \sum_{i \in V} (x^0_i + [z^0_{a_i}]) \quad \sum_{a \in E} [z^0_{a_i}] = \quad \sum_{i \in V} \hat{x} = 0. \tag{3.23}\]

Hence there is some \( i \) such that
\[
d^T (x^0_i + z^0_i) \quad \tag{3.24}\]

Then we move ahead with this \( i \) (without labeling it as \( i^* \) to save notation).

Since \( d = P_{Z^*} a \), we have \( d^T a = a^T P_{Z^*} a = a^T P_{Z^*} a = \|P_{Z^*} a\|^2 \). Note that
\[
\|d\|^2 = \|P_{Z^*} a\|^2 = \|a\|^2 - \|P_{Z^*} a\|^2 \tag{3.25a} \\
\text{Case 3a} \quad \text{and} \quad \|d - a\|^2 = \|P_{Z^*} a\|^2 \tag{3.25b}
\]

so
\[
d^T x^0_i = d^T a + d^T (x^0_i - a) \geq \|P_{Z^*} a\|^2 - \|d\| \|x^0_i - a\| \tag{3.26}
\]

Let \( c(\theta_Z, \theta_D) \) be the formula marked above. Let \( \hat{d} = d/\|d\| \). We have
\[
\hat{d}^T \hat{z}^p_{i^*} = \frac{1}{\|z^0_{i^*}\|} \left( \hat{d}^T (x^0_{i^*} + z^0_{i^*}) - \hat{d}^T x^0_{i^*} \right) \leq \frac{-c(\theta_Z, \theta_D)}{\max_{i^*}} \|x^0_{i^*}\|. \tag{3.27}\]

We then project \( x^0_i + z^0_i \) onto \( P_i \). Suppose \( \hat{z}^p_i \) is close enough to \( \hat{z}^p_i \) so that \( \|\hat{z}^p_i - \hat{z}^p_i\| \leq \frac{1}{2} \). Then
\[
(\hat{z}^p_i)^T z^0_i = (\hat{z}^p_i)^T z^0_i + (\hat{z}^p_i - \hat{z}^p_i)^T z^0_i \geq \|z^0_i\| (1 - \|\hat{z}^p_i - \hat{z}^p_i\|) \geq \frac{1}{2} M_{\min}. \tag{3.28}\]

If we assume that \( x^0_i \) is close enough to \( x^* \) so that \( \|x^0_i\| \leq \frac{1}{3} M_{\min} \), then \( (\hat{z}^p_i)^T x^0_i \geq -\|z^0_i\| \|x^0_i\| = -\|x^0_i\| \geq -\frac{1}{3} M_{\min} \), and so
\[
(\hat{z}^p_i)^T (x^0_i + z^0_i) \geq -\frac{1}{3} M_{\min} + \frac{1}{2} M_{\min} = \frac{1}{6} M_{\min} > 0. \tag{3.29}\]
This means that \( x_0^0 + z_0^0 \) does not satisfy the second inequality in the definition of \( P_i \) in (3.22), so at least one of the inequalities there must be active at \( P_i(x_0^0 + z_0^0) \). We let the point \( P_i(x_0^0 + z_0^0) \) be \( \bar{x}_i^+ \).

**Claim.** Recall that \( \lim_{(\theta_Z, \theta_D) \to (0, 0)} c(\theta_Z, \theta_D) = 1 \). Let \( \kappa_5 \) be \( \frac{3\kappa_4|V|\max_{\kappa_3+1}}{\min_{\kappa_{\min}}} + \tilde{\kappa}_3 \), which is checked to be greater than 1. Suppose \( \theta_Z, \theta_D > 0 \) are chosen small enough so that the following conditions hold:

\[
\begin{align*}
\frac{2\kappa_4|V|\max_{\kappa_3+1}}{\min_{\kappa_{\min}}} + \tilde{\kappa}_3 & \leq \kappa_5, \quad (3.29a) \\
\left(1 - \frac{\theta_D|V|}{1 - \theta_D}\right)^{-1} \left(1 + \frac{\theta_D|V|}{1 - \theta_D}\right)^{-1} & \leq \frac{1}{2\max_{\kappa_5}}, \quad (3.29b) \\
c(\theta_Z, \theta_D) & \geq \frac{1}{2\max_{\kappa_5}}. \quad (3.29c)
\end{align*}
\]

Then \( \|x_0^0 - \bar{x}_i^+\| \geq \frac{1}{2\max_{\kappa_5}}\|x_0^0\| \).

We now prove the claim. For \( \bar{x}_i^+ = P_i(x_0^0 + z_0^0) \), there are three different cases.

**Case 3a-1:** Only the constraint \( (\hat{z}_i^p)^T \bar{x} \leq 0 \) in (3.22) is active at \( \bar{x}_i^+ \).

If that active constraint is \( (\hat{z}_i^p)^T x \leq 0 \), then by the KKT conditions, \( \bar{x}_i^+ \) would be of the form \( \bar{x}_i^+ = x_i^0 + z_i^0 - \lambda \hat{z}_i^p \), and hence

\[
d^T \bar{x}_i^+ = d^T(x_i^0 + z_i^0) - \lambda d^T \hat{z}_i^p \leq 0. \quad (3.30)
\]

Then

\[
\|x_0^0 - \bar{x}_i^+\| \geq \tilde{d}^T(x_0^0 - \bar{x}_i^+) \geq c(\theta_Z, \theta_D)\|x_0^0\| \geq \frac{1}{2\max_{\kappa_5}}\|x_0^0\|. \quad (3.29b, 3.30)
\]

**Case 3a-2:** Both constraints in (3.22) are active at \( \bar{x}_i^+ \).

**Step 1:** Bounding \( \|z_i^0 - \hat{z}_i^p\| \).

For all \( i \in V \), we have

\[
\|z_i^0 - \hat{z}_i^p\| \leq \|(z_1^0, \ldots, z_{|V|}^0) - (z_1^0, \ldots, z_{|V|}^0)\| \leq \kappa_4d((z_1^0, \ldots, z_{|V|}^0), M_2). \quad (3.29b, 3.30)
\]

The projection of \( (z_1^0, \ldots, z_{|V|}^0) \) onto \( M_2 \) is \( (z_1^0 - \delta, z_2^0 - \delta, \ldots, z_{|V|}^0 - \delta) \), where \( M_2 \) is as defined in (2.0a) and \( \delta = \frac{1}{|V|} (\sum_{i \in V} z_i^0 - |V| \bar{x}) \). This means that

\[
d((z_1^0, \ldots, z_{|V|}^0), M_2) = \sqrt{|V|\|\delta\|} = \frac{\|z_i^0 - |V| \bar{x}\|}{\sqrt{|V|}}. \quad (3.32)
\]

For the parameters \( (z_1^0, \ldots, z_{|V|}^0) \), we note from Proposition 2.1 that \( z_i^0 = |\bar{x}|_i \), \( |\bar{x}|_i = 0 \) for all \( j \neq i \) and \( \sum_{j \in V} |\sum_{\alpha \in E} z_{\alpha j}^0| = 0 \), which gives

\[
\sum_{i \in V} z_i^0 - |V| \bar{x} = \sum_{i \in V} (z_i^0 - z_i^0) + \sum_{j \in V} \left[ \sum_{\alpha \in E} z_{\alpha j}^0 + \sum_{i \in V} z_i^0 - \bar{x}\right]_j = \sum_{i \in V} (z_i^0 - z_i^0) - \sum_{i \in V} x_i^0. \quad (3.33)
\]
Recall that $z^*_i = P_{NC_i(x^*)}(z^0_i)$, and Assumption 2.2[7]. This gives $\|z^*_i - z^0_i\| \leq \tilde{\kappa}_3\|x^0_i\|$ and

$$\|z^*_i - z^0_i\| = d(z^0_i, NC(x^*)) \leq \|z^0_i - \tilde{z}^*_i\|\|z^0_i\| = \|z^0_i\|\|z^*_i - z^0_i\| \leq M\max\tilde{\kappa}_3\|x^0_i\|.$$  
(3.34)

So

$$\left\| \sum_{i \in V} z^*_i - |V|\bar{x} \right\| \leq \sum_{i \in V} \|z^*_i - z^0_i\| + \sum_{i \in V} \|x^0_i\| \leq \sum_{i \in V} (M\max\tilde{\kappa}_3 + 1)\|x^0_i\| \leq |V|(M\max\tilde{\kappa}_3 + 1)\|x^0\|.$$  
(3.35)

Hence, for all $i \in V$, we have

$$\|z^*_i - z^0_i\| \leq \kappa_4\sqrt{|V|(M\max\tilde{\kappa}_3 + 1)}\|x^0\|.$$  
(3.36)

Also,

$$\|z^*_i - \tilde{z}^*_i\| = \left| \|z^*_i\| - \|z^0_i\| \right| \leq \|z^*_i - z^0_i\| \|z^0_i\| \leq \kappa_4\sqrt{\|z^0_i\|\}}.$$  
(3.37)

Since $z^*_i = P_{NC_i(x^*)}(z^0_i)$, Assumption 2.2[7] shows us that $\|z^0_i - \tilde{z}^*_i\| \leq \kappa_4\|x^0_i\|$. Note that for any $i \in V$,

$$\|x^0_i\|^2 \leq \|x^0\|^2 \leq |V| \left( 1 + \sqrt{\frac{\|z^0_i\|}{1 - \|z^0_i\|}} \right)^2 \|x^0\|^2 \leq |V| \left( 1 + \sqrt{\frac{\|z^0_i\|}{1 - \|z^0_i\|}} \right)^2 \|x^0\|^2.$$  
(3.38)

We thus have

$$\|z^0_i - \tilde{z}^*_i\| \leq \kappa_5\|x^0_i\|.$$  
(3.39)

**Step 2:** Showing $\|x^0_i - \tilde{x}^*_i\|$ is large enough.

Since both constraints in $P_i$ (see 3.2[22]) are tight at $\tilde{x}^*_i$, the projection of $x^0_i + z^0_i$ onto $P_i$ is equivalent to the projection of $P_{(\tilde{z}^*_i)^T}(x^0_i + z^0_i)$ onto $\{x : (P_{(\tilde{z}^*_i)^T}(z^0_i))^T x = \epsilon_i\}$. We have

$$\|P_{(\tilde{z}^*_i)^T}(z^0_i)\| = \|P_{(\tilde{z}^*_i)^T}(\tilde{z}^0_i - \tilde{z}^0_i)\| \leq \|\tilde{z}^0_i - \tilde{z}^0_i\| \leq \kappa_5\|x^0_i - x^*\|.$$  
(3.40)

Note that by the KKT conditions, $P_{(\tilde{z}^*_i)^T}(z^0_i) = z^0_i - \lambda\tilde{z}^0_i$ for some $\lambda \in \mathbb{R}$. So

$$\tilde{d}^T(P_{(\tilde{z}^*_i)^T}(z^0_i)) = \tilde{d}^T(z^0_i - \lambda\tilde{z}^0_i) \leq \frac{c(\theta_2, \theta_3)}{M\max\|x^0_i - x^*\|} \leq 0.$$  
(3.41)

Then we have

$$\frac{\tilde{d}^T(P_{(\tilde{z}^*_i)^T}(z^0_i))}{\|d\|_{P_{(\tilde{z}^*_i)^T}(z^0_i)}} \leq -\frac{c(\theta_2, \theta_3)}{M\max\tilde{\kappa}_5}.$$  
(3.42)
Also,
\[
\begin{align*}
\left( x_i^0 - d \right)^T P_{\mathcal{P}_i} (z_i^0) & \geq \left( x_i^0 - d \right)^T x_i^0 + d^T P_{\mathcal{P}_i} (z_i^0) \\
\| x_i^0 - a \| & \leq \frac{\| x_i^0 - a \|}{\| x_i^0 - a \|} + d^T P_{\mathcal{P}_i} (z_i^0) \\
& \geq \frac{\| x_i^0 - a \|}{\| x_i^0 - a \|} + d^T P_{\mathcal{P}_i} (z_i^0) \\
& \geq \frac{\| x_i^0 - a \|}{\| x_i^0 - a \|} + d^T P_{\mathcal{P}_i} (z_i^0)
\end{align*}
\]
(3.43)

Note that \( \bar{x}_i^+ \) is the deflection of \( x_i^0 \) along the normal \( P_{\mathcal{P}_i} (z_i^0) \), i.e., \( \bar{x}_i^+ = x_i^0 + \lambda P_{\mathcal{P}_i} (z_i^0) \) for some \( \lambda \geq 0 \). Moreover, we have
\[
\begin{align*}
P_{\mathcal{P}_i} (z_i^0) & = \left( P_{\mathcal{P}_i} (z_i^0) \right)^T \\
\bar{x}_i^+ & = \frac{x_i^0}{\| x_i^0 - a \|}
\end{align*}
\]
since the two constraints in the definition of \( P_i \) in (3.22) are tight. The distance of \( \bar{x}_i^+ \) must be at least
\[
\| \bar{x}_i^+ - x_i^0 \| \geq \left( \frac{\| x_i^0 - a \|}{\| x_i^0 - a \|} + d^T P_{\mathcal{P}_i} (z_i^0) \right) (\bar{x}_i^+ - x_i^0) \\
\geq \frac{1}{\| x_i^0 - a \|} - \left( \frac{1}{\| x_i^0 - a \|} \right) \| x_i^0 - a \| \| \bar{x}_i^+ - x_i^0 \|,
\]
which concludes the proof for this case.

**Case 3a-3:** Only the constraint \( (z_i^0)^T x \leq \epsilon_i \) in (3.22) is active at \( \bar{x}_i^+ \).

We now show that this case is impossible by showing that \( (z_i^0)^T x \geq \epsilon_i > 0 \) and \( (z_i^0)^T \bar{x}_i^+ \leq 0 \) cannot hold at the same time. We have
\[
\| x_i^0 - a \| \leq \sqrt{\frac{\theta_i |V_i|}{1 - \theta_i D}} \| a \|.
\]
By the nonexpansiveness of the projection operation, we have
\[
\begin{align*}
\| P_{\mathcal{P}_i} (x_i^0) - d \| & \leq \| x_i^0 - d \| \leq \| x_i^0 - a \| + \| d - a \| \\
& \leq \sqrt{\frac{\theta_i |V_i|}{1 - \theta_i D}} \| a \| \leq \frac{1}{\sqrt{1 - \theta_i D}} \left( \sqrt{\frac{\theta_i |V_i|}{1 - \theta_i D}} \| d \| \right).
\end{align*}
\]
(3.44)

Define \( x'_i \) to be the point such that \( x'_i = x_i^0 + \lambda \bar{z}_i^p \) and \( (z_i^0)^T x'_i = 0 \). Note that \( P_{\mathcal{P}_i} (x_i^0) \) is of the form \( x_i^0 + \lambda \bar{z}_i^p \) with \( (\bar{z}_i^p)^T P_{\mathcal{P}_i} (x_i^0) = 0 \). Further arithmetic gives us
\[
x'_i = x_i^0 - \frac{(\bar{z}_i^p)^T x_i^0}{(\bar{z}_i^p)^T x_i^0} z_i^0 + P_{\mathcal{P}_i} (x_i^0) = x_i^0 - [(\bar{z}_i^p)^T x_i^0] z_i^0.
\]
(3.45)

Now
\[
(\bar{z}_i^p)^T x_i^0 = \frac{1}{\| z_i^0 \|} z_i^0 - \frac{1}{\| z_i^0 \|} \| z_i^0 \| = \frac{1}{\| z_i^0 \|} \| z_i^0 \| = \frac{1}{\| z_i^0 \|} \| z_i^0 \| = \frac{1}{\| z_i^0 \|} \| z_i^0 \| = \frac{1}{\| z_i^0 \|} \| z_i^0 \|.
\]
(3.46)

Also,
\[
\begin{align*}
\| \bar{z}_i^p \| & \leq \| z_i^0 - \bar{z}_i^p \| + \left( \frac{1}{\| z_i^0 \|} \right)^2 \| z_i^0 \| \| z_i^0 \| \| z_i^0 \|, \quad \text{since} \ |z_i^0 - \bar{z}_i^p| \text{ can be made arbitrarily small by (3.19) and (3.20a)}
\end{align*}
\]
we can assume
that there is an $\gamma_1$ such that $\|\frac{1}{M_1} z_i^0 - \hat{z}_i^0\| \leq \gamma_1$ throughout. So

$$\|x'_i - P(z_i^0) + (x_i^0)\| \leq \|\frac{1}{M_1} z_i^0 - \hat{z}_i^0\| \leq \gamma_1$$

(3.48)

By the KKT conditions, the point $\hat{x}_i^+$ has the form $\hat{x}_i^+ = x_i^0 + \lambda z_i^0$ for some $\lambda \geq 0$. We show that points of the form $x_i^0 + \lambda z_i^0$, where $\lambda \in \mathbb{R}$, cannot satisfy both $(z_i^0)^T(x_i^0 + \lambda z_i^0) \geq 0$ and $(z_i^0)^T(x_i^0 + \lambda z_i^0) \leq 0$ at the same time. Since $x_i^0 - x_i'$ is a multiple of $z_i^0$, we can prove our results for points of the form $x_i^0 + \lambda z_i^0$. Now,

$$\begin{align*}
(z_i^0)^T(x_i') &= (P_{(\theta_D \theta_Z \theta_D)} z_i^0 + P_{(\theta_D)} z_i^0)^T x_i' \\
&= (P_{(\theta_D \theta_Z \theta_D)} z_i^0 + P_{(\theta_D)} z_i^0)^T x_i' + (P_{(\theta_D \theta_Z \theta_D)} z_i^0 + P_{(\theta_D)} z_i^0)^T (x_i' - d) + (P_{(\theta_D \theta_Z \theta_D)} z_i^0 + P_{(\theta_D)} z_i^0)^T (d) \\
&\leq 0 + ||P_{(\theta_D \theta_Z \theta_D)} z_i^0|| |x_i' - d| - \frac{c(\theta_D \theta_Z \theta_D)}{\|\theta_D \theta_Z \theta_D\|} ||P_{(\theta_D \theta_Z \theta_D)} z_i^0|| |d|. \\
\end{align*}$$

(3.49)

In view of $\|x_i' - d\| \leq \|x_i' - P_{(\theta_D \theta_Z \theta_D)} x_i^0\| + ||P_{(\theta_D)} z_i^0 - d||$, (3.44) and (3.47), and the fact that $\lim_{(\theta_D \theta_Z \theta_D) \to (0,0)} c(\theta_Z, \theta_D) = 1$, we can choose $\theta_D, \theta_Z > 0$ small enough so that $(z_i^0)^T(x_i') \leq 0$. So if $(z_i^0)^T(x_i') \geq 0$, then $\lambda \geq 0$, which implies that $(z_i^0)^T(x_i^0 + \lambda z_i^0) = \lambda(z_i^0)^Tz_i^0 > 0$. This completes the proof of the claim. \ △

Let the minimizer of $\frac{1}{2} ||(x_i^0 + z_i^0) - \cdot||^2 + \delta_{C_i}(\cdot)$ be $z_i^+$. It is standard to obtain $\hat{x}_i^+ + \hat{z}_i^+ = x_i^0 + z_i^0$. We have

$$\begin{align*}
\hat{z}_i^+ \text{ minimizer} &\geq \frac{1}{2} ||(x_i^0 + z_i^0) - z_i^+||^2 + \delta_{C_i}(z_i^0) \\
&\geq \frac{1}{2} ||(x_i^0 + z_i^0) - z_i^+||^2 + \delta_{C_i}(\hat{z}_i^+) + \frac{1}{2} \|z_i^0 - \hat{z}_i^+\|^2 \\
\delta_{C_i}(\cdot) &\geq \frac{1}{2} ||(x_i^0 + z_i^0) - z_i^+||^2 + \delta_{C_i}(\hat{z}_i^+) + \frac{1}{2} \|x_i^0 - \hat{x}_i^+\|^2. \\
\end{align*}$$

(3.50)

Note that $\|z_i^0\| \geq (1 - \sqrt{\frac{\theta_D V}{1-\theta_D}}) \|a\| \geq (1 - \sqrt{\frac{\theta_D V}{1-\theta_D}}) \sqrt{\frac{1-\theta_D}{\theta_D} \|a\|}$. Also,

$$\frac{\theta_D}{2 \theta_D} \|x_i^0\|^2 \geq F(z_i^0).$$

(3.49)

$$\begin{align*}
F(z_i^0) &\leq \sum_{i \in V} (\delta_{C_i}(z_i^0) + \frac{1}{2} \|x_i^0\|^2) \\
&\leq \sum_{i \in V} (\delta_{C_i}(z_i^0) + \frac{1}{2} \|x_i^0\|^2) - \frac{1}{2} \|x_i^0 - \hat{x}_i^+\|^2 \leq F(z_i^0) - \frac{1}{2} \|x_i^0 - \hat{x}_i^+\|^2 \\
\end{align*}$$

(3.51)

This once again leads to linear convergence.

**Case 3b:** $\|P_2 a\|^2 \geq \theta_D \|a\|^2$.

For each $i \in V$, define the hyperplanes $H_i, H_{i_0}$ and $H_{i_0}^T$ by

$$H_i := \{ x : (\hat{z}_i^+)^T x = \delta_{C_i}(\hat{z}_i^+) \}, \quad H_{i_0} := \{ x : (\hat{z}_i^+)^T x = 0 \} \quad \text{and} \quad H_{i_0}^T := \{ x : (\hat{z}_i^+)^T x = 0 \}.$$  

Recall that by Assumption 2.24, $z_i^0$ are big enough so that $x_i^0 + z_i^0$ is always outside $C_i$, so that $P_{C_i}(x_i^0 + z_i^0)$ is onto the boundary of $C_i$ (and not in the interior of $C_i$). Recall that the dual vectors after the projection are $\{z_i^+\}_{i \in V}$. The term
\( \delta^*_{C_i}(z^0) \) in the definition of \( H_i \) implies that \( H_i \) is a supporting hyperplane of \( C_i \) at \( x_i^+ \) with normal vector \( z_i^+ \). Due to the fact that the dual function is decreasing, we have \( \delta^*_{C_i}(z_i^+) + \frac{1}{2}\|x_i^+\|^2 \leq \delta^*_{C_i}(z_i^0) + \frac{1}{2}\|x_i^0\|^2 \) for all \( i' \in V \), so

\[
\|x_i^+\|^2 \leq 2\delta^*_{C_i}(z_i^+) + \|x_i^+\|^2 \leq 2\sum_{j \in V} \delta^*_{C_j}(z_j^0) + \|\mathbf{x}^0\|^2 \leq (1 + \frac{2}{p})\|\mathbf{x}^0\|^2. \tag{3.50}
\]

If a point \( x_i^+ \) is on \( C_i' \), then the distance of the supporting hyperplane of \( C_i \) at \( x_i^+ \) to the origin is \( o(\|x_i^+\|) = o(\|\mathbf{x}^0\|) \) by Assumption 2.2(3). (We actually have \( O(\|\mathbf{x}^0\|^2) \), but \( o(\|\mathbf{x}^0\|) \) is enough for this part of the proof.) So we have 

\[
d(0, H_i) = o(\|x_i^0\|). \tag{3.51}
\]

Since \( \|x_i^0\| \in \Theta(|a|) \), the term \( \delta^*_{C_i}(z_i^+) \) is \( o(|a|) \), for any \( \varepsilon > 0 \), we have \( \varepsilon|a| \geq d(0, H_i) \) for all \( i \in V \) if \( x^0 \) is close enough to \( x^* \), which gives

\[
d(a, H_i) = d(a, H_i, 0) - \varepsilon|a| \text{ for all } i \in V. \tag{3.52}
\]

We have 

\[
d(a, \bigcap_{i \in V} H_{P_i}) = d(a, Z^+) = \|P_Z a\| \geq \sqrt{\theta_Z} |a|. \tag{3.53}
\]

Next, by Assumption 2.2(1), we have

\[
|(\delta^*_i)^T a| \geq d(a, H_{P_i}) \geq \frac{1}{\kappa_1} d(a, \bigcap_{j \in V} H_{P_j}) \geq \frac{\sqrt{\theta_Z}}{\kappa_1} |a|. \tag{3.54}
\]

We have 

\[
d(a, H_i) \geq d(a, H_{P_i}) - 2\varepsilon_1 |a| \geq \left( \frac{\sqrt{\theta_Z}}{\kappa_1} - 2\varepsilon_1 \right) |a|. \tag{3.55}
\]

We have 

\[
\|x_i^0 - x_i^+\| \geq d(x_i^0, H_i) \geq d(a, H_i) - \|x_i^0 - a\| \geq \left( \frac{\sqrt{\theta_Z}}{\kappa_1} - 2\varepsilon_1 - \sqrt{\theta_D |V| \over 1 - \theta_D} \right) |a|. \tag{3.56}
\]

Since \( \theta_Z, \theta_D, \varepsilon_1 > 0 \) are chosen so that \( c_2(\theta_Z, \theta_D, \varepsilon_1) > 0 \), we have 

\[
\|x_i^0 - x_i^+\|^2 \geq c_2(\theta_Z, \theta_D, \varepsilon_1)^2 \|\mathbf{x}^0\|^2 \geq (1 + \varepsilon)^{-1} c_2(\theta_Z, \theta_D, \varepsilon_1)^2 F(\mathbf{z}^0). \tag{3.57}
\]

This leads to linear convergence like in the last three lines of (3.49). \( \Box \)
4. Lifting Assumption [2.5]  

In this section, we show how to adjust the proof of the main result in Section 3 so that Assumption [2.5] can be lifted. We let \( z_i^0 \) and \( x_i^0 \) be what they were in the proof of Theorem 3.1 in Section 3. We shall treat case 3a first, and then explain the similarities in case 3b.

We can assume that there is an index \( k \) such that \( i \notin S_{n,k} \) for all \( k' \in \{1, \ldots, k-1\} \) (which implies \( z_i^0 = z_i^{k-1} \)) and \( S_{n,k} = \{i\} \). Let the operator \( T: \mathbb{R}^n \to \mathbb{R}^n \) be \( T(x') = \arg \min_x \frac{1}{2} \|x' - x\|^2 + \delta_{P_i}(x) \). Define \( \hat{x}_i^k \) as

\[
\hat{x}_i^k := T(x_i^{k-1} + z_i^{k-1}) = T(x_i^{k-1} + z_i^0).
\]

Note also that \( \hat{x}_i^+ = T(x_i^0 + z_i^0) \). Since \( \partial\delta_{P_i}(\cdot) \) is a monotone operator, the operator \( T(\cdot) \) is nonexpansive (see for example the textbook [BC11]), which gives \( \|\hat{x}_i^k - \hat{x}_i^+\| \leq \|x_i^{k-1} - x_i^0\| \). We have

\[
\|x_i^0 - \hat{x}_i^+\| \leq \|x_i^0 - x_i^{k-1}\| + \|x_i^{k-1} - \hat{x}_i^k\| + \|\hat{x}_i^k - \hat{x}_i^+\| \leq 2\|x_i^0 - x_i^{k-1}\| + \|x_i^{k-1} - \hat{x}_i^k\|.
\]

Then

\[
\begin{align*}
\sum_{k'=1}^{k-1} \|x_{i}^{k'} - x_{i}^{k'-1}\|^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 &\geq \sum_{k'=1}^{k-1} \|x_{i}^{k'} - x_{i}^{k'-1}\|^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 \\
&\geq \frac{1}{k-1} \left( \sum_{k'=1}^{k-1} \|x_{i}^{k'} - x_{i}^{k'-1}\|^2 \right)^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 \\
&\geq \frac{1}{w} \left( \|x_{i}^{k-1} - x_{i}^0\|^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 \right) \\
&\geq \frac{1}{8w} \|x_{i}^{k-1} - x_{i}^0\|^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 \geq \frac{1}{8w} \|x_{i}^0 - \hat{x}_{i}^+\|^2.
\end{align*}
\]

The same steps as (3.48) leads us to

\[
\frac{1}{2} \|x_{i}^{k-1} + z_{i}^{k-1}\|^2 - \delta_{C_i}(z_{i}^{k-1}) \geq \frac{1}{2} \|x_{i}^{k-1} + z_{i}^{k-1}\|^2 + \sum_{k'=1}^{k-1} \|x_{i}^{k'} - x_{i}^{k'-1}\|^2 + \|x_{i}^{k-1} - \hat{x}_{i}^k\|^2 \geq \frac{1}{8w} \|x_{i}^0 - \hat{x}_{i}^+\|^2.
\]

Once again, the steps similar to (3.49) gives

\[
\begin{align*}
F(z^k) &\leq F(z^{k-1}) - \frac{1}{2} \|x_i^{k-1} - \hat{x}_i^k\|^2 \\
&\leq F(z^0) - \frac{1}{16w} \|x_i^0 - \hat{x}_i^+\|^2 \\
&\leq \frac{1}{16w} \|x_i^0 - \hat{x}_i^+\|^2 \\
&\leq \frac{1}{32w(M_{\text{max}} + \epsilon)} \left( 1 - \frac{\theta_P}{1 - \theta_P} \frac{|V|}{V_2} \right)^2 \left( 1 - \frac{\theta_P}{1 - \theta_P} \frac{\hat{r}}{2} \right) F(z^0).
\end{align*}
\]

The adjustments for case 3b is similar, except that the set \( P_i \) is set to be \( C_i \), and \( \hat{x}_i^+ \) and \( \hat{x}_i^k \) can be replaced by \( x_i^+ \) and \( x_i^k \) respectively.

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