RICCATI EQUATIONS FOR LINEAR HAMILTONIAN SYSTEMS
WITHOUT CONTROLLABILITY CONDITION

PETER SEPITKA

Department of Mathematics and Statistics, Faculty of Science, Masaryk University
Kotlářská 2, CZ-61137 Brno, Czech Republic

Dedicated to the memory of Professor Russell A. Johnson.

(Communicated by Alberto Bressan)

ABSTRACT. In this paper we develop new theory of Riccati matrix differential equations for linear Hamiltonian systems, which do not require any controllability assumption. When the system is nonoscillatory, it is known from our previous work that conjoined bases of the system with eventually the same image form a special structure called a genus. We show that for every such a genus there is an associated Riccati equation. We study the properties of symmetric solutions of these Riccati equations and their connection with conjoined bases of the system. For a given genus, we pay a special attention to distinguished solutions at infinity of the associated Riccati equation and their relationship with the principal solutions at infinity of the system in the considered genus. We show the uniqueness of the distinguished solution at infinity of the Riccati equation corresponding to the minimal genus. This study essentially extends and completes the work of W. T. Reid (1964, 1972), W. A. Coppel (1971), P. Hartman (1964), W. Kratz (1995), and other authors who considered the Riccati equation and its distinguished solution at infinity for invertible conjoined bases, i.e., for the maximal genus in our setting.

1. Introduction. Riccati differential equations for self-adjoint linear differential systems play fundamental role in mathematical research as well as in applications. Specifically, if \( n \in \mathbb{N} \) is a given dimension and \( A, B, C : [a, \infty) \to \mathbb{R}^{n \times n} \) are given piecewise continuous matrix-valued functions such that \( B(t) \geq 0 \) for all \( t \in [a, \infty) \) (1.1), the Riccati matrix differential equation

\[
Q' + QA(t) + A^T(t)Q + QB(t)Q - C(t) = 0 \quad (R)
\]

is associated with the linear Hamiltonian system

\[
x' = A(t)x + B(t)u, \quad u' = C(t)x - A^T(t)u, \quad (H)
\]

see [7, 9, 17, 22, 23, 24]. It is known that under the Legendre condition

\[
B(t) \geq 0 \quad \text{for all } t \in [a, \infty) \quad (1.1)
\]

the Riccati equation (R) has many applications in various disciplines, for example in the oscillation and spectral theory [2, 7, 17, 22, 23, 24], filtering and prediction theory [16, 23], calculus of variations and optimal control theory [1, 3, 8, 12, 10, 2010 Mathematics Subject Classification. 34C10.

Key words and phrases. Linear Hamiltonian system, Riccati differential equation, genus of conjoined bases, distinguished solution at infinity, principal solution at infinity, controllability.

This research was supported by the Czech Science Foundation under grant GA16-00611S.
In [20], Reid showed that when the system \((H)\) is completely controllable and nonoscillatory, the Riccati equation \((R)\) has the so-called distinguished solution \(Q(t)\) at infinity, which is the smallest symmetric solution of \((R)\) existing on an interval \([\alpha, \infty)\) for some \(\alpha \geq a\). In the subsequent paper [21], Reid derived the minimality of the distinguished solution of \((R)\) at infinity also for a noncontrollable system \((H)\) by considering invertible principal solutions \((\hat{X}, \hat{U})\) of \((H)\) at infinity. Recently, the author and Šimon Hilscher developed the theory of principal solutions at infinity for a general nonoscillatory and possibly abnormal system \((H)\). We showed in [28, 29] the existence of principal solutions \((\hat{X}, \hat{U})\) at infinity with all ranks of \(\hat{X}(t)\) in a specific range depending on the maximal order of abnormality \(\alpha\) of \((H)\), their classification and limit properties with antiprincipal solutions at infinity [30], and the geometric structure of the set of all conjoined bases [31]. In particular, conjoined bases of \((H)\) with eventually the same image of the first component form a genus \(G\), which can be represented by an orthogonal projector \(R_G(t)\) satisfying the Riccati type matrix differential equation

\[
R'_G(t) - A(t) R_G(t) - R_G(t) A^T(t) + R_G(t) [A(t) + A^T(t)] R_G(t) = 0.
\] (1.2)

This leads under (1.1) to a complete description of the set \(\Gamma\) of all genera of conjoined bases of a nonoscillatory system \((H)\), being a complete lattice [31, Theorem 4.8]. This result was recently extended to a possibly oscillatory system \((H)\) in [27, Theorem 4.14].

In this paper we continue in the above study of linear Hamiltonian system \((H)\) by developing the corresponding theory of Riccati matrix differential equations. The presented approach and results are novel in three directions:

(i) we do not require any controllability assumption on system \((H)\),
(ii) for every genus \(G\) we associate a Riccati equation

\[
Q' + QA(t) + A^T(t) Q + QB(t) Q - C(t) = 0,
\] (R)

where the coefficients \(A(t), B(t),\) and \(C(t)\) are given by

\[
\begin{align*}
A(t) &:= A(t) R_G(t) - A^T(t) [I - R_G(t)], \\
B(t) &:= B(t), \\
C(t) &:= R_G(t) C(t) R_G(t),
\end{align*}
\] (1.3)

(iii) we show that every such a Riccati equation \((R)\) possesses a distinguished solution at infinity (defined in a suitable way), which corresponds to a principal solution of \((H)\) at infinity in the genus \(G\). We also prove (Theorem 7.16) that for every symmetric solution \(Q(t)\) of \((R)\) on \([\alpha, \infty)\) with (1.4) there exists a distinguished solution \(\hat{Q}(t)\)

More precisely, given a genus \(G\) of conjoined bases of \((H)\), we show (Theorems 4.18 and 4.21) a fundamental connection between the symmetric solutions \(Q(t)\) of \((R)\) on \([\alpha, \infty)\) with some \(\alpha \geq a\) satisfying

\[
\text{Im } Q(t) \subseteq \text{Im } R_G(t), \quad t \in [\alpha, \infty),
\] (1.4)

and the conjoined bases \((X, U)\) of \((H)\) with constant kernel on \([\alpha, \infty)\), which belong to \(G\). We define (Definition 7.1) a distinguished solution \(\hat{Q}(t)\) at infinity for each Riccati equation \((R)\), which corresponds to a principal solution \((\hat{X}, \hat{U})\) of \((H)\) at infinity in the genus \(G\). We also prove (Theorem 7.16) that for every symmetric solution \(Q(t)\) of \((R)\) on \([\alpha, \infty)\) with (1.4) there exists a distinguished solution \(\hat{Q}(t)\)
of \((\mathcal{R})\) satisfying the inequality
\[ Q(t) \geq \hat{Q}(t) \quad \text{on } [\alpha, \infty). \quad (1.5) \]

The above results are particularly important for the minimal genus \(G = G_{\text{min}}\), which is formed by the conjoined bases \((X, U)\) of \((H)\) with minimal possible rank of the matrix \(X(t)\), i.e., with rank \(X(t) = n - d_{\infty}\) on \([\alpha, \infty)\). In this case the associated distinguished solution \(Q_{\text{min}}(t)\) at infinity is unique and minimal among all symmetric solutions \(Q(t)\) of \((\mathcal{R})\) satisfying (1.4). This latter situation generalizes the classical controllable results of Reid and Coppel [7, 20, 22], since in this case \(d_{\infty} = 0\) and the orthogonal projector \(R_{G}(t) \equiv I\) on \([a, \infty)\), so that the Riccati equation \((\mathcal{R})\) reduces to \((\mathcal{R})\). We note that the original results by Reid [21, 23] for noncontrollable system \((H)\) and Riccati equation \((\mathcal{R})\) correspond in our new theory to the maximal genus \(G = G_{\text{max}}\) of conjoined bases \((X, U)\) with eventually invertible matrix \(X(t)\), i.e., to \(R_{G}(t) \equiv I\) on \([a, \infty)\). Therefore, the present study can be regarded as a generalization and completion of the theory of the Riccati equations \((\mathcal{R})\) for completely controllable systems \((H)\) using the minimal genus \(G = G_{\text{min}}\), as well as the noncontrollable systems \((H)\) using the maximal genus \(G = G_{\text{max}}\).

Among other new results in this paper (Theorem 6.3 and Corollary 6.4) we mention a connection of the symmetric solutions \(Q(t)\) of \((\mathcal{R})\) with the implicit Riccati equation
\[
R_{G}(t) [Q' + Q A(t) + A^T(t) Q + Q B(t) Q - C(t)] R_{G}(t) = 0. \quad (1.6)
\]

Such implicit Riccati equations occur in the study of nonnegative quadratic functional associated with system \((H)\), see [13, Section 6].

The study of the Riccati equations in the context of the present paper is also motivated by several situations in the literature, which are equivalent to using the Riccati matrix differential equation for an uncontrollable linear Hamiltonian system. For example, in [35, pg. 886], [1, pp. 621–622], [11, Sections 4 and 6], and [12, pp. 17–18] the authors use a cascade system of three differential equations for the investigation of calculus of variations or optimal control problems with variable endpoints – the Riccati equation \((\mathcal{R})\), a linear differential equation, and an integrator. These three differential equations are together equivalent to a Riccati equation in dimension \(2n\), which corresponds to an uncontrollable system \((H)\) in dimension \(4n\). This connection is discusses in details in [11, Remark 6.3].

The results of this paper open new directions in the theory of Riccati matrix differential equations associated with general uncontrollable linear Hamiltonian systems. They demonstrate that, as in the completely controllable case, distinguished solutions at infinity play a prominent role in the structure of the space of symmetric solutions of \((\mathcal{R})\). Moreover, the intimate connection with the principal solutions of \((H)\) at infinity points to effective applications of the distinguished solutions of \((\mathcal{R})\) at infinity in other fields of mathematics and engineering.

The paper is organized as follows. In Section 2 we display the notation and preliminary results about system \((H)\) and its solutions. In Section 3 we present properties of principal solutions of \((H)\) at infinity and recall the concept of a genus of conjoined bases of \((H)\). In Section 4 we develop the theory of Riccati differential equations for a given genus \(G\). In Section 5 we study inequalities for Riccati type quotients associated with the Riccati equation \((\mathcal{R})\). In Section 6 we analyze the relationship between the two Riccati equations \((\mathcal{R})\) and (1.6). In Section 7 we define
the notion of a distinguished solution of (R) at infinity and study its minimality properties. Finally, in Section 8 we provide examples illustrating our new theory.

2. Preliminaries about linear Hamiltonian systems. In this section we review some recent results about linear Hamiltonian systems (H) from [18, 33, 28, 29, 30, 31]. For a general theory of these systems we refer to [7, 17, 22]. By a matrix solution of (H) we mean a pair of functions (X,U) solution of (H). In order to shorten the notation and the calculations, we sometimes suppress the argument t in the solutions. For any two matrix solutions (X1,U1) and (X2,U2) of (H) their Wronskian X1TU2 − U1TX2 is constant on [a, ∞). A solution (X,U) of (H) is called a conjoined basis if rank (XT(t), UT(t))T = n and XT(t)U(t) is symmetric at some and hence at any t ∈ [a, ∞). The principal solution (Xα, Uα) at the point α ∈ [a, ∞) is defined by the initial conditions Xα(α) = 0 and Uα(α) = I. By [17, Corollary 3.3.9], a given conjoined basis (X,U) can be completed to a fundamental system of (H) by another conjoined basis (X,U). In addition, the conjoined basis (X,U) can be chosen so that (X,U) and (X,U) are normalized, i.e., we have

\[ X^T U - U^T X = I. \] (2.1)

The oscillation of conjoined bases of (H) is defined via the concept of proper focal points, see [36, Definition 1.1]. However, this concept will not be explicitly needed in this paper. By [33, Definition 2.1], a conjoined basis (X,U) of (H) is called nonoscillatory if there exists α ∈ [a, ∞) such that Ker X(t) is constant on [α, ∞). The main result of [33] then describes the nonoscillatory behavior of system (H), see Proposition 2.1 below. Based on this result we say that system (H) is nonoscillatory if one and hence all conjoined bases of (H) are nonoscillatory.

**Proposition 2.1.** Assume that the Legendre condition (1.1) holds. Then there exists a nonoscillatory conjoined basis of (H) if and only if every conjoined basis of (H) is nonoscillatory.

For a subspace V ⊆ Rn we denote by PV the orthogonal projector onto V. That is, PV is a symmetric and idempotent n × n matrix such that Im PV = V = Ker (I − PV) and Ker PV = V⊥ = Im (I − PV). Orthogonal projectors can be constructed by using the Moore–Penrose pseudoinverse. More precisely, for a given matrix M ∈ Rn×n and its pseudoinverse M† the matrix MM† is the orthogonal projector onto Im M, and the matrix M†M is the orthogonal projector onto Im M† = Im M†. Moreover, rank M = rank MM† = rank M†M and Ker (MN) = Ker (M†MN) for any matrices M, N ∈ Rn×n. For the theory of pseudoinverse matrices we refer to [4], [5, Chapter 6], and [6, Section 1.4]. In particular, we will need the following results on the differentiability of the Moore–Penrose pseudoinverse of a matrix-valued function M(t).

**Remark 2.2.** By [6, Theorems 10.5.1 and 10.5.3], for a differentiable matrix-valued function M(t) on an interval [α, ∞) its Moore–Penrose pseudoinverse M†(t) is differentiable on [α, ∞) if and only if rank M(t) is constant on [α, ∞). In this case (suppressing the argument t)

\[ (M^†)' = -M^†M'M^† + (I - M^†M) (M^†)^T M^†M^† + M^†M^†M^† (M^†)^T (I - M M^†) \] (2.2)
on $[\alpha, \infty)$, see also [28, Remark 2.3]. Moreover, when $\ker M(t)$ is constant on $[\alpha, \infty)$, then we have $\ker M(t) \subseteq \ker M'(t)$ on $[\alpha, \infty)$ and hence (2.2) reduces to

$$(M')'(t) = -M'(t) M'(t) M'(t) + M'(t) M'T(t) (M')^T(t) [I - M(t) M'(t)]$$

(2.3)

for every $t \in [\alpha, \infty)$. In particular, when the matrix $M(t)$ is symmetric and $\ker M(t)$ is constant on $[\alpha, \infty)$, then (2.3) yields the standard formula

$$(M')'(t) = -M'(t) M'(t) M'(t), \quad t \in [\alpha, \infty).$$

In the rest of this section (except of Theorem 2.9) we present known properties of conjoined bases of (H) with the corresponding references to the literature. Given a conjoined basis $(X, U)$ of (H), by the kernel, image, and rank of $(X, U)$ we mean the kernel, image, and rank of the component $X$. On the interval $[\alpha, \infty)$ we define the orthogonal projectors onto the subspaces $\text{Im} X(t)$ and $\text{Im} X(t)$ by

$$P(t) := \mathcal{P}_{\text{Im} X(t)} = X(t) X(t), \quad R(t) := \mathcal{P}_{\text{Im} X(t)} = X(t) X(t).$$

(2.4)

If $(X, U)$ has constant kernel on $[\alpha, \infty) \subseteq [\alpha, \infty)$, then by (2.4) the function $P(t)$ is constant on $[\alpha, \infty)$ and we set

$$P := P(t) \quad \text{on} \quad [\alpha, \infty).$$

(2.5)

In this case $(X, U)$ has constant rank $r$ on $[\alpha, \infty)$ with

$$r := \text{rank} X(t) = \text{rank} P = \text{rank} R(t) \quad \text{on} \quad [\alpha, \infty),$$

(2.6)

and hence it follows from Remark 2.2 that the function $X^\dagger \in C^1_p$ on $[\alpha, \infty)$. Consequently, the Riccati quotient

$$Q(t) := X(t) X^\dagger(t) U(t) X^\dagger(t) = R(t) U(t) X^\dagger(t), \quad t \in [\alpha, \infty),$$

(2.7)

is piecewise continuously differentiable on $[\alpha, \infty)$ as well. In addition, by [25, pg. 24] the matrix $Q(t)$ is symmetric and satisfies on $[\alpha, \infty)$ the properties (suppressing the argument $t$)

$$X^T Q X = X^T U, \quad \text{Im} Q \subseteq \text{Im} R, \quad Q X = RU.$$

(2.8)

The next statement is proven in [28, Theorem 4.2 and Equation (4.8)]. We observe that the Legendre condition (1.1) is not needed in this case.

**Proposition 2.3.** Let $(X, U)$ be a conjoined basis of (H) with constant kernel on the interval $[\alpha, \infty) \subseteq [\alpha, \infty)$ and let $P$, $R(t)$, and $Q(t)$ be the corresponding matrices in (2.5), (2.4), and (2.7). Then the equalities

$$\text{Im} [U(t) (I - P)] = \ker R(t), \quad B(t) = R(t) B(t) = B(t) R(t)$$

(2.9)

hold for all $t \in [\alpha, \infty)$. Moreover, the matrix $R(t)$ solves the Riccati equation (1.2) on $[\alpha, \infty)$, while $X^\dagger$ satisfies on $[\alpha, \infty)$ the formula

$$(X^\dagger)' = X^\dagger A^T (I - R) - X^\dagger A R - X^\dagger B Q.$$

(2.10)

Following [28, Section 4], with any conjoined basis $(X, U)$ of (H) with constant kernel on $[\alpha, \infty)$ we associate the $S$-matrix as the matrix-valued function

$$S_\alpha(t) := \int_\alpha^t X^\dagger(s) B(s) X^\dagger(s) \, ds, \quad t \in [\alpha, \infty).$$

(2.11)

Under (1.1), the matrix $S_\alpha(t)$ is symmetric, nonnegative definite, $S_\alpha \in C^1_p$ on $[\alpha, \infty)$, and by [28, Theorem 4.2] the set $\text{Im} S_\alpha(t)$ is nondecreasing on $[\alpha, \infty)$ and hence eventually constant with $\text{Im} S_\alpha(t) \subseteq \text{Im} P$. By the symmetry of $S_\alpha(t)$, the set $\ker S_\alpha(t)$ is nonincreasing on $[\alpha, \infty)$ and hence eventually constant with $\ker P \subseteq \ker S_\alpha(t)$. 

This implies that the orthogonal projector onto the set $\text{Im} \, S_\alpha(t)$ is eventually constant and we write

$$
P_{S_\alpha}(t) := \mathcal{P}_{\text{Im} \, S_\alpha(t)} = S_\alpha(t) S_\alpha^d(t) = S_\alpha^d(t) S_\alpha(t),
$$

$$
P_{S_\alpha, \infty} := P_{S_\alpha}(t) \quad \text{for} \ t \to \infty.
$$

In addition, on $[\alpha, \infty)$ we have the inclusions

$$
\text{Im} \, S_\alpha(t) = \text{Im} \, P_{S_\alpha}(t) \subseteq \text{Im} \, P_{S_\alpha, \infty} \subseteq \text{Im} \, P.
$$

The main properties of the function $S_\alpha(t)$ are summarized in the following statement, which follows from the definition of $S_\alpha(t)$ in (2.11), Remark 2.2, and (1.1), see also [28, Theorem 6.1].

**Proposition 2.4.** Assume (1.1). Let $(X, U)$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty)$ and let $S_\alpha(t)$ be the corresponding matrix defined in (2.11). Then the matrix function $S_\alpha(t)$ is nondecreasing on $[\alpha, \infty)$. Moreover, if $S_\alpha(t)$ has constant kernel on a subinterval $I \subseteq [\alpha, \infty)$, then $S_\alpha^1 \in C^1(I)$ and $S_\alpha^1(t)$ is nonincreasing on $I$. In particular, if $S_\alpha(t)$ has constant kernel on $I = [\beta, \infty)$, then the limit of $S_\alpha^1(t)$ as $t \to \infty$ exists.

**Remark 2.5.** Under (1.1), the results in Proposition 2.4 and the properties of the matrix function $S_\alpha(t)$ discussed above imply that for every conjoined basis $(X, U)$ of $(H)$ with constant kernel on an interval $[\alpha, \infty)$ the limit

$$
T_\alpha := \lim_{t \to \infty} S_\alpha^1(t)
$$

is well defined and it is referred to as the $T$-matrix corresponding to the conjoined basis $(X, U)$ on $[\alpha, \infty)$. Moreover, the matrix $T_\alpha$ is symmetric, nonnegative definite, and $\text{Im} \, T_\alpha \subseteq \text{Im} \, P_{S_\alpha, \infty}$ by (2.12) and $\text{Im} \, S_\alpha^1(t) = \text{Im} \, S_\alpha^1(t) = \text{Im} \, S_\alpha(t)$ on $[\alpha, \infty)$.

**Remark 2.6.** The matrix $S_\alpha(t)$ is intimately connected with a certain class of conjoined bases of $(H)$ which are normalized with $(X, U)$. As we showed in [28, Theorem 4.4], for a given conjoined basis $(X, U)$ with constant kernel on $[\alpha, \infty)$ there exists a conjoined basis $(\hat{X}, \hat{U})$ of $(H)$ such that $(X, U)$ and $(\hat{X}, \hat{U})$ are normalized, i.e., (2.1) holds, and

$$
X^\dagger(\alpha) \hat{X}(\alpha) = 0.
$$

The matrices $\hat{X}(t), \hat{X}(t) P,$ and $\hat{U}(t) P$ are uniquely determined by $(X, U)$ on the interval $[\alpha, \infty)$ and

$$
\hat{X}(t) P = X(t) S_\alpha(t), \quad \hat{U}(t) P = U(t) S_\alpha(t) + X^\dagger(t) + U(t) (I - P) \hat{X}^T(t) \hat{X}(t)
$$

for every $t \in [\alpha, \infty)$, where $P$ is given in (2.5), see [28, Remark 4.5.(ii)]. We also note that according to [28, Theorem 5.2] the matrix $S_\alpha(t)$ satisfies the identities

$$
\hat{X}_\alpha(t) = X(t) S_\alpha(t) X^T(\alpha), \quad X^\dagger(t) \hat{X}_\alpha(t) = S_\alpha(t) X^T(\alpha), \quad t \in [\alpha, \infty),
$$

where $(\hat{X}_\alpha, \hat{U}_\alpha)$ is the principal solution of $(H)$ at the point $\alpha$.

As it is common, see [23, Section 3] or [28, Section 5], we denote by $\Lambda[\alpha, \infty)$ the linear space of $n$-dimensional vector-valued functions $u \in C^n_0$, which satisfy the equations $u' = -A(t) u$ and $B(t) u = 0$ on $[\alpha, \infty)$. The functions $u \in \Lambda[\alpha, \infty)$ correspond to the vector solutions $(x \equiv 0, u)$ of system $(H)$ on $[\alpha, \infty)$. The space $\Lambda[\alpha, \infty)$ is finite-dimensional with $\delta(\alpha, \infty) := \dim \Lambda[\alpha, \infty) \leq n$. The number $\delta(\alpha, \infty)$ is called the order of abnormality of system $(H)$ on the interval $[\alpha, \infty)$. 
We remark that system (H) is said to be normal on \([\alpha, \infty)\) if \(d(\alpha, \infty) = 0\), while it is called identically normal (or completely controllable) on \([\alpha, \infty)\) if \(d(I) = 0\) for every nondegenerate subinterval \(I \subseteq [\alpha, \infty)\). Moreover, for a given \(t \in [\alpha, \infty)\) we denote by \(\Lambda_I(\alpha, \infty)\) the subspace in \(\mathbb{R}^n\) of values of functions \(u \in \Lambda(\alpha, \infty)\) at the point \(t\), i.e.,

\[
\Lambda_I(\alpha, \infty) := \{ c \in \mathbb{R}^n, \ u(t) = c \text{ for some } u \in \Lambda(\alpha, \infty) \}, \quad t \in [\alpha, \infty).
\] (2.17)

It is easy to see that \(\dim \Lambda_I(\alpha, \infty) = d(\alpha, \infty)\) for all \(t \in [\alpha, \infty)\). We note that the set \(\Lambda_I(\alpha, \infty)\) is nondecreasing in \(t\) on \([\alpha, \infty)\) and hence it is eventually constant. This means that the integer-valued function \(d(t, \infty)\) is nondecreasing, piecewise constant, and right-continuous on \([\alpha, \infty)\). In particular, there exists the limit

\[
d_\infty := \lim_{t \to \infty} d(t, \infty) = \max_{t \in [\alpha, \infty)} d(t, \infty), \quad 0 \leq d_\infty \leq n,
\] (2.18)

which is called the maximal order of abnormality of (H). The monotonicity of the function \(d(t, \infty)\) justifies the existence of the point \(\alpha_\infty \in [\alpha, \infty)\) such that

\[
\alpha_\infty := \min\{\alpha \in [\alpha, \infty), \ d(\alpha, \infty) = d_\infty\}.
\] (2.19)

From (2.18) and (2.19) we then obtain that the subspace \(\Lambda(\alpha_\infty, \infty)\) satisfies

\[
\begin{align*}
\Lambda(\alpha_\infty, \infty) &= \lim_{\alpha \to \infty} \Lambda(\alpha, \infty) = \max_{\alpha \in [\alpha, \infty)} \Lambda(\alpha, \infty), \\
\Lambda(\alpha, \infty) &\equiv \Lambda(\alpha_\infty, \alpha), \quad \alpha \in [\alpha_\infty, \infty).
\end{align*}
\] (2.20)

On the other hand, for any \(\alpha \in [\alpha, \infty)\) the subspace \(\Lambda(\alpha, t)\) is nonincreasing in \(t\) on \((\alpha, \infty)\) and hence it is eventually constant. In particular, the integer-valued function \(d(\alpha, t)\) is nonincreasing, piecewise constant, and left-continuous on \((\alpha, \infty)\), see also [28, Section 5]. Moreover, we get

\[
d(\alpha, \infty) = \lim_{t \to \infty} d(\alpha, t) = \min_{t \in [\alpha, \infty)} d(\alpha, t),
\] (2.21)

\[
\Lambda(\alpha, \infty) = \lim_{t \to \infty} \Lambda(\alpha, t) = \min_{t \in [\alpha, \infty)} \Lambda(\alpha, t)
\] (2.22)

for all \(\alpha \in [\alpha, \infty)\). For any such a point \(\alpha\) the relation in (2.21) and the above properties of the function \(d(\alpha, t)\) yield the existence of the point \(\tau(\alpha, \infty)\) in the interval \([\alpha, \infty)\) such that

\[
\tau(\alpha, \infty) := \inf\{t \in (\alpha, \infty), \ d(\alpha, t) = d(\alpha, \infty)\}.
\] (2.23)

**Remark 2.7.** We note that the subspace \(\Lambda(\alpha, t)\), resp. \(\Lambda(\alpha, \infty)\) is closely related with the matrix \(S_\alpha(t)\) in (2.11). More precisely, under (1.1) for every conjoined basis \((X, U)\) of (H) with constant kernel on \([\alpha, \infty)\) the corresponding matrices \(P_{S_\alpha}(t)\) and \(P_{S_\alpha, \infty}\) in (2.12) satisfy

\[
\text{Im} \ X(\alpha) \ P_{S_\alpha}(t) = (\Lambda_\alpha[\alpha, t])^\perp, \quad t \in (\alpha, \infty),
\]

\[
\text{Im} \ X(\alpha) \ P_{S_\alpha, \infty} = (\Lambda_\alpha[\alpha, \infty])^\perp.
\] (2.24)

The proof of the first formula in (2.24) is based on (2.16) and [28, Equation 5.6] in which we showed that \(\Lambda_\alpha[\alpha, t] = \text{Ker} X_\alpha(t)\) holds on \([\alpha, \infty)\), where \((X_\alpha, U_\alpha)\) is the principal solution at the point \(\alpha\). The second identity in (2.24) follows from the first one by using (2.12) and (2.22). Moreover, in [28, Theorem 5.2 and Remark 5.3] we proved the equalities

\[
\text{Im} \ S_\alpha(t) = \text{Im} \ P_{S_\alpha, \infty}, \quad \text{rank} \ S_\alpha(t) = \text{rank} \ P_{S_\alpha, \infty} = n - d(\alpha, \infty)
\] (2.25)

on \((\tau(\alpha, \infty), \infty)\) with \(\tau(\alpha, \infty)\) defined in (2.23).
Throughout this paper we will consider only the intervals $(\alpha, \infty)$ with the maximal order of abnormality $d_\infty$ defined in (2.18). The next remark shows how this condition reflects the properties of $S$-matrices corresponding to conjoined bases of $(H)$ with constant kernel.

**Remark 2.8.** (i) Assume (1.1) and let $(X, U)$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty)$. In [26, Theorem 4.1.12] we proved that the condition $d(\alpha, \infty) = d_\infty$ holds if and only if the matrix $S(\alpha)(t)$ in (2.11) associated with $(X, U)$ satisfies the equalities

$$\text{Im} \left[ P_{S(\alpha)\infty} - S(\alpha)(t) T_\alpha \right] = \text{Im} P_{S(\alpha)\infty} = \text{Im} \left[ P_{S(\alpha)\infty} - S(\alpha)(t) T_\alpha \right]^T \quad (2.26)$$

for every $t \in [\alpha, \infty)$, where $P_{S(\alpha)\infty}$ and $T_\alpha$ are corresponding matrices in (2.12) and (2.13). We note that the identities in (2.26) can be equivalently replaced by

$$\text{rank} \left[ P_{S(\alpha)\infty} - S(\alpha)(t) T_\alpha \right] = n - d(\alpha, \infty) \quad \text{on} \ [\alpha, \infty),$$

see [28, Theorem 6.9]. In addition, by [28, Equation 5.13] the conjoined basis $(X, U)$ satisfies the conditions

$$n - d_\infty = n - d(\alpha, \infty) \leq \text{rank} X(t) \leq n \quad \text{for all} \ t \in [\alpha, \infty). \quad (2.27)$$

(ii) Let $T_\beta$ be the $T$-matrix in (2.13), which is associated with $(X, U)$ on the interval $[\beta, \infty)$ for $\beta \in [\alpha, \infty)$. In [26, Theorem 4.3.1(ii)] we showed that the set $\text{Im} T_\beta$ is constant in $\beta$ on $[\alpha, \infty)$ if and only if the condition $d(\alpha, \infty) = d_\infty$ holds.

The following theorem is an extension of the result presented in Remark 2.8(i).

**Theorem 2.9.** Assume (1.1) and let $(X, U)$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty)$. Moreover, let $P$, $S(\alpha)(t)$, and $T_\alpha$ be its corresponding matrices in (2.5), (2.11), and (2.13), respectively. Then the condition $d(\alpha, \infty) = d_\infty$ is equivalent with the formulas

$$\text{Im} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{Im} P = \text{Im} \left[ P - S(\alpha)(t) T_\alpha \right]^T, \quad t \in [\alpha, \infty). \quad (2.28)$$

**Proof.** First we remark that by $PT_\alpha = T_\alpha$ and $PS(\alpha)(t) = S(\alpha)(t)$ on $[\alpha, \infty)$ we always have the inclusions

$$\text{Im} \left[ P - S(\alpha)(t) T_\alpha \right] \subseteq \text{Im} P, \quad \text{Im} \left[ P - S(\alpha)(t) T_\alpha \right]^T = \text{Im} \left[ P - T_\alpha S(\alpha)(t) \right] \subseteq \text{Im} P \quad (2.29)$$

for every $t \in [\alpha, \infty)$. And since $\text{rank} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{rank} \left[ P - S(\alpha)(t) T_\alpha \right]^T$ for all $t \in [\alpha, \infty)$, it is sufficient to show the equivalence of $d(\alpha, \infty) = d_\infty$ and the equality $\text{Im} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{Im} P$ on $[\alpha, \infty)$. Assume that $d(\alpha, \infty) = d_\infty$ holds. Fix $t \in [\alpha, \infty)$ and let $v \in \text{Ker} \left[ P - S(\alpha)(t) T_\alpha \right]^T = \text{Ker} \left[ P - T_\alpha S(\alpha)(t) \right]$, that is, $\left[ P - T_\alpha S(\alpha)(t) \right] v = 0$. Using the latter equation and the identities $P_{S(\alpha)\infty} P = P_{S(\alpha)\infty}$ and $P_{S(\alpha)\infty} T_\alpha = T_\alpha$, yields $\left[ P_{S(\alpha)\infty} - T_\alpha S(\alpha)(t) \right] v = P_{S(\alpha)\infty} \left[ P - T_\alpha S(\alpha)(t) \right] v = 0$ and hence, the vector $v \in \text{Ker} \left[ P_{S(\alpha)\infty} - T_\alpha S(\alpha)(t) \right] = \text{Ker} \left[ P_{S(\alpha)\infty} - S(\alpha)(t) T_\alpha \right]^T = \text{Ker} P_{S(\alpha)\infty}$, by the first equality in (2.26). Moreover, with the aid of identity $S(\alpha)(t) = S(\alpha)(t) P_{S(\alpha)\infty}$ we then get

$$P v = \left[ P - T_\alpha S(\alpha)(t) \right] v + T_\alpha S(\alpha)(t) v = \left[ P - T_\alpha S(\alpha)(t) \right] v + T_\alpha S(\alpha)(t) P_{S(\alpha)\infty} v = 0,$$

which shows that $v \in \text{Ker} P$. Therefore, the inclusion $\text{Ker} \left[ P - S(\alpha)(t) T_\alpha \right]^T \subseteq \text{Ker} P$, or equivalently, the inclusion $\text{Im} P \subseteq \text{Im} \left[ P - S(\alpha)(t) T_\alpha \right]$ holds for every $t \in [\alpha, \infty)$. Combining the latter relation with the first property in (2.29) gives the equality $\text{Im} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{Im} P$ on $[\alpha, \infty)$. Conversely, if $\text{Im} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{Im} P$ is satisfied for all $t \in [\alpha, \infty)$, then

$$\text{Im} \left[ P_{S(\alpha)\infty} - S(\alpha)(t) T_\alpha \right] = \text{Im} P_{S(\alpha)\infty} \left[ P - S(\alpha)(t) T_\alpha \right] = \text{Im} P_{S(\alpha)\infty} P = \text{Im} P_{S(\alpha)\infty}.$$
on $[\alpha, \infty)$, showing the first identity in (2.26). Finally, the condition $d(\alpha, \infty) = d_\infty$ holds by Remark 2.8(i), which completes the proof. 

The next statement is a combination of [28, Theorem 4.6] and [26, Theorem 2.3.3]. We again note that the Legendre condition (1.1) is in this case not needed.

**Proposition 2.10.** Let $(X, U)$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty)$ and let $P$ be its corresponding orthogonal projector in (2.5). Moreover, let $(\tilde{X}, \tilde{U})$ be a solution of $(H)$, which is expressed in terms of $(X, U)$ via matrices $M, N \in \mathbb{R}^{n \times n}$, that is,

$$
\begin{pmatrix}
\tilde{X} \\
\tilde{U}
\end{pmatrix} = 
\begin{pmatrix}
X & \tilde{X} \\
U & \tilde{U}
\end{pmatrix} 
\begin{pmatrix}
M \\
N
\end{pmatrix} \quad \text{on } [\alpha, \infty),
$$

(2.30)

where $(\tilde{X}, \tilde{U})$ is a conjoined basis of $(H)$ satisfying (2.1) and (2.14) with regard to $(X, U)$. Then the inclusion $\text{Im} \tilde{X}(\alpha) \subseteq \text{Im} X(\alpha)$ holds if and only if $\text{Im} N \subseteq \text{Im} P$.

In this case the matrices $M$ and $N$ do not depend on the particular choice of $(\tilde{X}, \tilde{U})$.

In addition, if $(\tilde{X}, \tilde{U})$ is a conjoined basis with constant kernel on $[\alpha, \infty)$ and the equality $\text{Im} \tilde{X}(\alpha) = \text{Im} X(\alpha)$ holds, then

$$
M \text{ is nonsingular, } M^T N = N^T M, \quad \text{Im } N^T \subseteq \text{Im } \tilde{P},
$$

(2.31)

where $\tilde{P}$ is the matrix in (2.5) associated with $(\tilde{X}, \tilde{U})$.

**Remark 2.11.** (i) Combining (2.14) with formulas (2.15) and (2.30) at $t = \alpha$ we obtain that the solutions $(X, U)$ and $(\tilde{X}, \tilde{U})$ in Proposition 2.10 satisfy, see also [26, Equation (2.52)],

$$
\tilde{X}(\alpha) = X(\alpha) M, \quad \tilde{U}(\alpha) = U(\alpha) M + X^T(\alpha) N, \quad \text{Im } N \subseteq \text{Im } P.
$$

The first equality in (2.15) allows to rewrite the expression for the matrix $\tilde{X}(t)$ in (2.30) into the form

$$
\tilde{X}(t) = X(t) [M + S_\alpha(t) N] = X(t) [PM + S_\alpha(t) N] \quad \text{on } [\alpha, \infty),
$$

(2.32)

where $S_\alpha(t)$ is the $S$-matrix in (2.11) associated with $(X, U)$. In particular, this shows that the inclusion $\text{Im} \tilde{X}(t) \subseteq \text{Im} X(t)$ holds for every $t \in [\alpha, \infty)$, see [26, Theorem 2.3.3]. We also note that the matrix $N$ is the (constant) Wronskian of $(X, U)$ and $(\tilde{X}, \tilde{U})$.

(ii) Let $(\tilde{X}, \tilde{U})$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty)$ such that $\text{Im} \tilde{X}(\alpha) = \text{Im} X(\alpha)$ holds. Then $\text{Im} \tilde{X}(t) = \text{Im} X(t)$ on $[\alpha, \infty)$, as we proved in [28, Theorem 4.10]. If $\tilde{S}_\alpha(t)$ is the $S$-matrix which corresponds to $(\tilde{X}, \tilde{U})$ on $[\alpha, \infty)$, then we have identities

$$
[PM + S_\alpha(t) N]^\dagger = \tilde{P} M^{-1} - \tilde{S}_\alpha(t) N^T,
$$

(2.33)

$$
\text{Im} [PM + S_\alpha(t) N] = \text{Im } P, \quad \text{Im} [PM + S_\alpha(t) N]^T = \text{Im } \tilde{P},
$$

(2.34)

$$
\tilde{S}_\alpha(t) = [PM + S_\alpha(t) N] \tilde{S}_\alpha(t) M^{-1} \tilde{P}, \quad \text{Im } \tilde{S}_\alpha(t) = \text{Im } \tilde{P} M^{-1} S_\alpha(t)
$$

(2.35)

for every $t \in [\alpha, \infty)$, see [28, Remark 4.7, Theorem 4.10]. In particular, since $S_\alpha(\alpha) = 0 = \tilde{S}_\alpha(\alpha)$ by (2.11), formula (2.33) at $t = \alpha$ and the inclusion in (2.31) give the equalities

$$
(PM)^\dagger = \tilde{P} M^{-1}, \quad N(PM)^\dagger = N\tilde{P} M^{-1} = NM^{-1}.
$$

(2.36)
Moreover, the identities in (2.32) and (2.34) yield $X(t) = \tilde{X}(t) [PM + S_\alpha(t) N]^\dagger$ and the formulas for the pseudoinverses

\[
\begin{align*}
\tilde{X}(t) &= [PM + S_\alpha(t) N]^\dagger X(t), \\
X(t) &= [PM + S_\alpha(t) N] \tilde{X}(t),
\end{align*}
\] on $[\alpha, \infty)$.

(2.37)

3. **Principal solutions at infinity.** Following [29, Definition 7.1], we say that a conjoined basis $(\tilde{X}, \tilde{U})$ of $(H)$ is a principal solution at infinity if $(\tilde{X}, \tilde{U})$ has constant kernel on $[\alpha, \infty)$ and its corresponding matrix $\tilde{S}_\alpha(t)$ defined in (2.11) through $X(t)$ satisfies $\tilde{S}_\alpha(t) \to 0$ as $t \to \infty$, that is, $\tilde{T}_\alpha = 0$ in (2.13). In this case we will say that $(\tilde{X}, \tilde{U})$ is a principal solution of $(H)$ at infinity with respect to the interval $[\alpha, \infty)$. By (2.27), the principal solutions of $(H)$ can be classified according to the rank of $\tilde{X}(t)$ on $[\alpha, \infty)$. In particular, the minimal principal solution $(\tilde{X}_{\min}, \tilde{U}_{\min})$ of $(H)$ at infinity satisfies $\tilde{X}(t)_{\min} = n - d_\infty$, while the maximal principal solution $(\tilde{X}_{\max}, \tilde{U}_{\max})$ of $(H)$ at infinity is determined by rank $\tilde{X}(t)_{\max} = n$, hence $\tilde{X}(t)_{\max}$ is invertible on $[\alpha, \infty)$, see [29, Remark 7.2].

In the next proposition we recall from [29, Theorem 7.6] and [28, Theorems 7.6] the characterization of the nonoscillation of system $(H)$ by the existence of a principal solution of $(H)$ at infinity with any possible rank, as well as the uniqueness of the minimal principal solution.

**Proposition 3.1.** Assume that (1.1) holds. Then the following statements are equivalent.

(i) System $(H)$ is nonoscillatory.

(ii) There exists a principal solution of $(H)$ at infinity.

(iii) For any integer $r$ satisfying $n - d_\infty \leq r \leq n$ there exists a principal solution of $(H)$ at infinity with rank equal to $r$.

In particular, system $(H)$ is nonoscillatory if and only if there exists a minimal principal solution of $(H)$ at infinity. In this case the minimal principal solution is unique up to a right nonsingular constant multiple.

In [29, Equation 7.4] we defined for a nonoscillatory system $(H)$ the point $\hat{\alpha}_{\min} \in [\alpha, \infty)$ by

\[
\hat{\alpha}_{\min} := \inf \{ \alpha \in [\alpha, \infty), (\tilde{X}_{\min}, \tilde{U}_{\min}) has constant kernel on [\alpha, \infty) \},
\]

(3.1)

where $(\tilde{X}_{\min}, \tilde{U}_{\min})$ is the minimal principal solution of $(H)$ at infinity. We note that the equality $d(\alpha, \infty) = d_\infty$ holds for every $\alpha > \hat{\alpha}_{\min}$, see [29, Theorem 7.9]. In turn, combining this fact with formula (2.19) we obtain that

\[
d(\hat{\alpha}_{\min}, \infty) = d_\infty, \quad i.e., \quad \hat{\alpha}_{\min} \geq \alpha_\infty.
\]

The next results are based on [28, Theorem 7.5] and [29, Lemma 7.5 and Remark 7.11].

**Proposition 3.2.** Assume that (1.1) holds and system $(H)$ is nonoscillatory with $\hat{\alpha}_{\min}$ defined in (3.1). Then the following statements hold.

(i) If $(\tilde{X}, \tilde{U})$ is a principal solution of $(H)$ at infinity with respect to the interval $[\alpha, \infty)$, then $d(\alpha, \infty) = d_\infty$. Moreover, the pair $(\tilde{X}, \tilde{U})$ is a principal solution of $(H)$ at infinity also with respect to the interval $[\beta, \infty)$ for every $\beta \geq \alpha$. 


(ii) Every principal solution \((X, U)\) of (H) at infinity is a principal solution with respect to \([α, ∞)\) for every \(α \in (\hat{α}_{\text{min}}, ∞)\). In other words, the conjoined basis \((X, U)\) has constant kernel on the open interval \((\hat{α}_{\text{min}}, ∞)\) and its corresponding matrix \(\hat{S}_a(t)\) in (2.11) satisfies \(\hat{S}_a'(t) \to 0\) as \(t \to ∞\) for every \(α > \hat{α}_{\text{min}}\).

**Remark 3.3.** We note that the orthogonal projector \(P_{S_∞}\) in (2.12) associated with the principal solution \((X, U)\) through the matrix \(\hat{S}_a(t)\) is the same for all initial points \(α \in (\hat{α}_{\text{min}}, ∞)\), see [29, Remark 7.11]. Therefore, we will use the notation

\[
P_{S_∞} := P_{S_∞} \quad \text{for } α \in (\hat{α}_{\text{min}}, ∞).
\]

Given a principal solution \((X, U)\) of (H) at infinity, we define the point \(\hat{α} \in [α, ∞)\) associated with \((X, U)\) by

\[
\hat{α} := \inf \{α \in [α, ∞), (X, U) \text{ is a principal solution of (H) with respect to } [α, ∞) \}\.
\]

From Proposition 3.2 it immediately follows that the point \(\hat{α}\) in (3.3) satisfies the inequalities \(α_∞ ≤ \hat{α} ≤ \hat{α}_{\text{min}}\) with \(α_∞\) defined in (2.19). We also note that the set \((\hat{α}, ∞)\) is the maximal open interval with the property that \((X, U)\) is a principal solution of (H) with respect to \([α, ∞)\) for every \(α \in (\hat{α}, ∞)\). Therefore, we will often say that \((X, U)\) is a principal solution of (H) at infinity with respect to the maximal interval \((\hat{α}, ∞)\). In particular, the conjoined basis \((X, U)\) has constant kernel on the open interval \((\hat{α}, ∞)\) and the \(S\)-matrix \(\hat{S}_a(t)\) in (2.11) associated with \((X, U)\) satisfies \(\hat{S}_a'(t) \to 0\) as \(t \to ∞\) for every \(α > \hat{α}\). In the next theorem we derive an exact relation between the points \(\hat{α}\) and \(\hat{α}_{\text{min}}\).

**Theorem 3.4.** Assume that (1.1) holds and system (H) is nonoscillatory with \(\hat{α}_{\text{min}}\) defined in (3.1). Let \((X, U)\) be a principal solution of (H) at infinity and let \(\hat{α}\) be its corresponding point in (3.3). Then the equality \(\hat{α} = \hat{α}_{\text{min}}\) holds.

**Proof.** Let \((X, U), \hat{α}, \text{ and } \hat{α}_{\text{min}}\) be as in the proposition and suppose that \(\hat{α} < \hat{α}_{\text{min}}\). According to (3.3) there exists a point \(β \in (\hat{α}, \hat{α}_{\text{min}})\) such that \((X, U)\) is a principal solution of (H) at infinity with respect to the interval \([β, ∞)\). By Proposition 3.2(i) with \(α := β\) we know that \(d(β, ∞) = d_∞\). Let \((\hat{X}_{\text{min}}, \hat{U}_{\text{min}})\) be the minimal principal solution of (H) at infinity. By [29, Theorem 7.3] it follows that the pair \((\hat{X}_{\text{min}}, \hat{U}_{\text{min}})\) is a minimal principal solution at infinity with respect to the interval \([β, ∞)\). For this we note that \((\hat{X}_{\text{min}}, \hat{U}_{\text{min}})\) is contained in \((X, U)\) on \([β, ∞)\) according to the properties of the relation “being contained” in [28, Section 5]. The uniqueness of the minimal principal solution and the definition of \(\hat{α}_{\text{min}}\) in (3.1) then yield that \(β ≥ \hat{α}_{\text{min}}\), which is a contradiction. Therefore, the equality \(\hat{α} = \hat{α}_{\text{min}}\) holds and the proof is complete.

In the following result we present a construction of a principal solution of (H) at infinity from a conjoined basis of (H) with constant kernel on \([α, ∞)\). It is a generalization of [32, Equation (10)], where only the minimal principal solution of (H) was considered. This result will be utilized for the construction of a distinguished solution of (R) at infinity in Theorem 7.16.

**Theorem 3.5.** Assume that condition (1.1) holds and system (H) is nonoscillatory. Let \(α \in [α, ∞)\) be such that \(d(α, ∞) = d_∞\) and let there exists a conjoined basis of
(H) with constant kernel on \([\alpha, \infty)\). Then a solution \((\hat{X}, \hat{U})\) of (H) is a principal solution at infinity with respect to the interval \([\alpha, \infty)\) if and only if
\[
\begin{pmatrix}
\hat{X} \\
\hat{U}
\end{pmatrix} := \begin{pmatrix} X & \dot{X} \\ U & \dot{U} \end{pmatrix} \begin{pmatrix} I \\ -T_\alpha \end{pmatrix} \text{ on } [\alpha, \infty),
\]
for some conjoined basis \((X, U)\) of (H) with constant kernel on \([\alpha, \infty)\). Here the conjoined basis \((\hat{X}, \hat{U})\) and the matrix \(T_\alpha\) are associated with \((X, U)\) through Remark 2.6 and (2.13).

**Proof.** If \((\hat{X}, \hat{U})\) is a principal solution of (H) at infinity with respect to \([\alpha, \infty)\), then \((\hat{X}, \hat{U})\) has constant kernel on \([\alpha, \infty)\) and the associated matrix \(T_\alpha\) in (2.13) satisfies \(T_\alpha = 0\). Formula (3.4) then holds trivially with \((X, U) := (\hat{X}, \hat{U})\). Conversely, let \((X, U)\) be a conjoined basis of (H) with constant kernel on \([\alpha, \infty)\) and let \(P, S_\alpha(t), P_{S_\alpha}, \) and \(T_\alpha\) be the matrices in (2.5), (2.11), (2.12), and (2.13) corresponding to \((X, U)\) on \([\alpha, \infty)\). Consider the matrix solution \((\hat{X}, \hat{U})\) of (H) in (3.4). Since \(PT_\alpha = T_\alpha\), it follows from Proposition 2.10 with \(M := I, N := -T_\alpha\) and \((\hat{X}, \hat{U}) := (\hat{X}, \hat{U})\) that \((\hat{X}, \hat{U})\) is a conjoined basis of (H), which in turn by (2.32) yields that \(X(t) = X(t) [P - S_\alpha(t) T_\alpha]\) on \([\alpha, \infty)\). Then, by \(P S_\alpha(t) = S_\alpha(t)\) and using (2.28) from Theorem 2.9, we get
\[
\text{Ker} \hat{X}(t) = \text{Ker} \begin{pmatrix} X(t) & \dot{X}(t) \end{pmatrix} [P - S_\alpha(t) T_\alpha] = \text{Ker} P [P - S_\alpha(t) T_\alpha]
\]
\[
\text{Ker} [P - S_\alpha(t) T_\alpha] \overset{\text{(2.28)}}{=} \left(\text{Im } P\right)^\perp = \text{Ker } P
\]
on \([\alpha, \infty)\). This shows that \((\hat{X}, \hat{U})\) has constant kernel on \([\alpha, \infty)\) as well. Moreover, if \(\hat{P}, \hat{S}_\alpha(t)\) and \(P_{\hat{S}_\alpha}\) are the matrices in (2.5), (2.11), and (2.12) corresponding to \((\hat{X}, \hat{U})\) on \([\alpha, \infty)\), then by using the first equations in (2.36) and (2.35), respectively, we have that \(\hat{P} = P\) and
\[
\hat{S}_\alpha(t) = [P - S_\alpha(t) T_\alpha]^\dagger S_\alpha(t) P = [P - S_\alpha(t) T_\alpha]^\dagger S_\alpha(t) \quad (3.5)
\]
for all \(t \in [\alpha, \infty)\). On the other hand, applying the second identity in (2.35) yields the equalities \(\text{Im } \hat{S}_\alpha(t) = \text{Im } \hat{P} S_\alpha(t) = \text{Im } PS_\alpha(t) = \text{Im } S_\alpha(t)\) on \([\alpha, \infty)\), which in particular by (2.12) imply that \(P_{\hat{S}_\alpha} = P_{S_\alpha}\). Let \(\tau_\alpha, \infty\) be defined in (2.23). Then \(S_\alpha(t) S_\alpha(t) S_\alpha(t) = P_{S_\alpha} \hat{S}_\alpha = S_\alpha(t) \hat{S}_\alpha = P_{S_\alpha} \hat{S}_\alpha\) as well. Moreover, \(\text{Ker } T_\alpha P_{S_\alpha} = T_\alpha = P_{S_\alpha} T_\alpha\) we obtain that
\[
\begin{align*}
\hat{S}_\alpha(t) &= P_{S_\alpha} \hat{S}_\alpha(t) \\
&= P_{S_\alpha} S_\alpha(t) \hat{S}_\alpha(t) S_\alpha(t) S_\alpha(t) \hat{S}_\alpha(t) = S_\alpha(t) P S_\alpha(t) \hat{S}_\alpha(t) \\
&= S_\alpha(t) [P - S_\alpha(t) T_\alpha] \hat{S}_\alpha(t) S_\alpha(t) \hat{S}_\alpha(t) = S_\alpha(t) [P - S_\alpha(t) T_\alpha] P_{S_\alpha} \\
&= S_\alpha(t) [P - S_\alpha(t) T_\alpha] P_{S_\alpha} = S_\alpha(t) - T_\alpha \quad (3.6)
\end{align*}
\]
for every \(t \in (\tau_\alpha, \infty)\). Finally, formula (3.6) implies that \(\hat{S}_\alpha(t) \to 0\) as \(t \to \infty\). This shows that the conjoined basis \((\hat{X}, \hat{U})\) is a principal solution of (H) at infinity with respect to the interval \([\alpha, \infty)\).

**Remark 3.6.** It follows from Proposition 2.10 and Remark 2.11(i) that the principal solution \((\hat{X}, \hat{U})\) constructed in (3.4) satisfies the equality \(\text{Im } \hat{X}(\alpha) = \text{Im } X(\alpha)\).
Moreover, as we noted in Remark 2.11(ii), this condition is valid on the whole interval \([\alpha, \infty]\), i.e., \(\text{Im}\, X(t) = \text{Im}\, \tilde{X}(t)\) holds for all \(t \in [\alpha, \infty)\). In particular, the last equality means that the conjoined bases \((X, U)\) and \((\tilde{X}, \tilde{U})\) belong to the same genus of conjoined bases of \((H)\) as we define below, see also Remark 3.13.

In the second part of this section we recall basic concepts from the theory of genera of conjoined bases of \((H)\) from our recent work [27, Section 4]. We wish to point out that in this context the Legendre condition \((1.1)\) is not assumed and/or system \((H)\) is allowed to be oscillatory. Define the orthogonal projector

\[ R_{\Lambda}(t) := \mathcal{P}_{U_t^\perp}, \quad \text{where} \quad U_t := \Lambda_x(\alpha, \infty), \quad t \in [\alpha, \infty), \quad (3.7) \]

where the point \(\alpha_\infty\) is determined in \((2.19)\) and the subspace \(\Lambda_x(\alpha_\infty, \infty)\) is defined in \((2.17)\). From the second identity in \((2.20)\) it follows that for any \(\alpha \geq \alpha_\infty\) the matrix \(R_{\Lambda}(t)\) defined in \((3.7)\) is the orthogonal projector onto the set \((\Lambda_x(\alpha))^\perp\) on \([\alpha, \infty)\), i.e.,

\[ R_{\Lambda}(t) = \mathcal{P}_{U_t^\perp}, \quad \text{where} \quad U_t := \Lambda_x(\alpha, \infty), \quad t \in [\alpha, \infty). \quad (3.8) \]

**Remark 3.7.** Assume the Legendre condition \((1.1)\). Let \((X, U)\) be a conjoined basis of \((H)\) with constant kernel on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) and let \(P\) and \(P_{S_{\alpha_\infty}}\) be the corresponding orthogonal projectors in \((2.4)\) and \((2.12)\), respectively. Combining \((2.24)\) and \((3.8)\) then yields the identity

\[ \text{Im}\, X(\alpha)\, P_{S_{\alpha_\infty}} = \text{Im}\, R_{\Lambda}(\alpha). \quad (3.9) \]

Moreover, since \(\text{Im}\, [X(\alpha)\, P_{S_{\alpha_\infty}}]^T = P_{S_{\alpha_\infty}},\ P P_{S_{\alpha_\infty}} = P_{S_{\alpha_\infty}},\) and \(X^\dagger(\alpha)\, X(\alpha) = P\), we have that

\[ [X(\alpha)\, P_{S_{\alpha_\infty}}]^\dagger = P P_{S_{\alpha_\infty}} [X(\alpha)\, P_{S_{\alpha_\infty}}]^\dagger = X^\dagger(\alpha)\, X(\alpha)\, P_{S_{\alpha_\infty}} [X(\alpha)\, P_{S_{\alpha_\infty}}]^\dagger \]

\[ = X^\dagger(\alpha)\, [X(\alpha)\, P_{S_{\alpha_\infty}}] [X(\alpha)\, P_{S_{\alpha_\infty}}]^\dagger \]

\[ \overset{(3.9)}{=} X^\dagger(\alpha)\, R_{\Lambda}(\alpha). \quad (3.10) \]

The orthogonal projector \(R_{\Lambda}(t)\) defined in \((3.7)\) plays a crucial role in the following notion. According to [27, Definition 4.3] we say that two conjoined bases \((X_1, U_1)\) and \((X_2, U_2)\) of \((H)\) have the same genus (or they belong to the same genus) if there exists \(\alpha \in [\alpha_\infty, \infty)\) such that

\[ \text{Im}\, X_1(t) + \text{Im}\, R_{\Lambda_1}(t) = \text{Im}\, X_2(t) + \text{Im}\, R_{\Lambda_2}(t), \quad t \in [\alpha, \infty). \]

From this definition it follows that the relation “having (or belonging to) the same genus” is an equivalence on the set of all conjoined bases of \((H)\). Therefore, there exists a partition of this set into disjoint classes of conjoined bases of \((H)\) with the same genus. This allows to interpret each such an equivalence class \(G\) as a genus itself. The following result is proven in [27, Theorem 4.5].

**Proposition 3.8.** Let \((X_1, U_1)\) and \((X_2, U_2)\) be conjoined bases of \((H)\). Then the following statements are equivalent.

(i) The conjoined bases \((X_1, U_1)\) and \((X_2, U_2)\) belong to the same genus.

(ii) The equality \(\text{Im}\, X_1(t) + \text{Im}\, R_{\Lambda_1}(t) = \text{Im}\, X_2(t) + \text{Im}\, R_{\Lambda_2}(t)\) is satisfied for every \(t \in [\alpha_\infty, \infty)\).

(iii) The equality \(\text{Im}\, X_1(t) + \text{Im}\, R_{\Lambda_1}(t) = \text{Im}\, X_2(t) + \text{Im}\, R_{\Lambda_2}(t)\) is satisfied for some \(t \in [\alpha_\infty, \infty)\).

Let \(G\) be a genus of conjoined bases of \((H)\) and let \((X, U)\) be a conjoined basis belonging to \(G\). The results in Proposition 3.8 imply that for all \(t \in [\alpha_\infty, \infty)\) the
subspace $\text{Im} \, X(t) + \text{Im} \, R_{\Lambda}(t)$ does not depend on the particular choice of $(X,U)$ in $\mathcal{G}$. Therefore, the orthogonal projector onto $\text{Im} \, X(t) + \text{Im} \, R_{\Lambda}(t)$, i.e., the matrix

$$R_\varphi(t) := P_{\mathcal{V}_t}, \quad \text{where} \quad \mathcal{V}_t := \text{Im} \, X(t) + \text{Im} \, R_{\Lambda}(t), \quad t \in [\alpha, \infty), \quad (3.11)$$

is uniquely determined for each genus $\mathcal{G}$. The next statement is from [27, Theorem 4.7].

**Proposition 3.9.** Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ and let $R_\varphi(t)$ be the orthogonal projector defined in (3.11). Then the matrix $R_\varphi(t)$ is a solution of the Riccati equation (1.2) on $[\alpha, \infty)$ and the inclusion $\text{Im} \, R_{\Lambda}(t) \subseteq \text{Im} \, R_\varphi(t)$ holds for every $t \in [\alpha, \infty)$.

**Remark 3.10.** If the orthogonal projector $R_\varphi(t)$ satisfies $R_\varphi(t) = R_{\Lambda}(t)$ on $[\alpha, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\min}$ is called minimal, while if $R_\varphi(t) \equiv I$ on $[\alpha, \infty)$, then the genus $\mathcal{G} = \mathcal{G}_{\max}$ is called maximal.

The next result describes important properties of nonoscillatory conjoined bases from a given genus $\mathcal{G}$. These properties will be utilized in Section 4 to show their connection with symmetric solutions of the Riccati equation (R) associated with the genus $\mathcal{G}$, see Theorem 4.18.

**Proposition 3.11.** Let $\mathcal{G}$ be a genus of conjoined basis of $(H)$ with the corresponding orthogonal projector $R_\varphi(t)$ in (3.11). Furthermore, let $(X,U)$ be a conjoined basis of $(H)$ with constant kernel on $[\alpha, \infty) \subseteq [\alpha, \infty)$ such that $(X,U)$ belongs to $\mathcal{G}$ and let $R(t)$ and $Q(t)$ be the matrices in (2.4) and (2.7). Then the equality $R_\varphi(t) = R(t)$ holds for all $t \in [\alpha, \infty)$. Moreover, the matrices $X(t)$, $X^\dagger(t)$, and $U(t)$ satisfy on $[\alpha, \infty)$ the equations

$$X' = (A + B \, Q) \, X, \quad (X^\dagger)' = -X^\dagger (A + B \, Q), \quad (3.12)$$

$$U' = AU + [C - (A + A^T) \, Q] \, X, \quad (3.13)$$

where the matrices $A(t)$ and $B(t)$ are defined in (1.3).

**Proof.** For the proof of $R_\varphi(t) = R(t)$ we refer to [27, Proposition 4.16]. We will prove that (3.12) and (3.13) hold. From the definition of the matrix $A(t)$ in (1.3) it follows that $A(t) \, R_\varphi(t) = A(t) \, R_\varphi(t)$ on $[\alpha, \infty)$. Moreover, using (1.3), (2.8), (2.9), and the identity $R_\varphi(t) \, X(t) = X(t)$ on $[\alpha, \infty)$ yields the formula

$$X' = (1.9) \, AR_\varphi X + BRU = (2.8) \, AR_\varphi X + BQX = (1.3) \, (A + B \, Q)X$$

on $[\alpha, \infty)$. Since the function $X^\dagger \in C_0^1$, equation (2.10) in Proposition 2.3 becomes

$$(X^\dagger)' = -X^\dagger [AR_\varphi - A^T (I - R_\varphi)] - X^\dagger BQ \quad ((1.9) \Rightarrow -X^\dagger (A + B \, Q)$$

on $[\alpha, \infty)$. Finally, by using $R_\varphi(t) \, U(t) = Q(t) \, X(t)$ for every $t \in [\alpha, \infty)$ we get

$$U' - AU = (1.3) \, CX - A^T U - [AR_\varphi - A^T (I - R_\varphi)]U = CX - (A + A^T) \, R_\varphi U$$

$$= (2.8) \, CX - (A + A^T) \, QX = [C - (A + A^T) \, Q] \, X$$

on $[\alpha, \infty)$. Thus, the matrix $U(t)$ solves (3.13) on $[\alpha, \infty)$. The proof is complete. \hfill \square

**Remark 3.12.** Let $\Phi_\alpha(t)$ be the fundamental matrix of the system $Y' = [A(t) + B(t) \, Q(t)] \, Y'$ for $t \in [\alpha, \infty)$ satisfying $\Phi_\alpha(\alpha) = I$. It is well-known that $\Phi_{\alpha^{-1}}^T(t)$ is the fundamental matrix of the adjoint system $Y' = -[A(t) + B(t) \, Q(t)]^T \, Y$ for $t \in [\alpha, \infty)$. From (3.12) we then obtain by the uniqueness of solutions that

$$X(t) = \Phi_\alpha(t) \, X(\alpha), \quad X^\dagger(t) = X^\dagger(\alpha) \, \Phi_{\alpha^{-1}}(t), \quad t \in [\alpha, \infty). \quad (3.14)$$
Remark 3.13. In [29, Theorem 7.12] we proved that every genus $G$ of conjoined bases of nonoscillatory system (H) contains a principal solution of (H) at infinity. Moreover, in Theorem 3.5 we described the construction of any such principal solution in terms of conjoined bases from the genus $G$, see also Remark 3.6.

In the next proposition we recall from [29, Theorem 7.13] a complete classification of all principal solutions of (H) at infinity within the genus $G$.

Proposition 3.14. Assume that (1.1) holds and system (H) is nonoscillatory with $\alpha_{\min}$ defined in (3.1). Let $(X, U)$ be a principal solution of (H) at infinity, which belongs to a genus $G$. Moreover, let $\hat{P}$ and $P_{S\infty}$ be the orthogonal projectors defined through the function $\hat{X}(t)$ on $(\alpha_{\min}, \infty)$ in (2.5), (2.12), and 3.2. Then a solution $(X, U)$ of (H) is a principal solution belonging to $G$ if and only if for some (and hence for every) $\alpha \in (\alpha_{\min}, \infty)$ there exist matrices $\hat{M}, \hat{N} \in \mathbb{R}^{nxn}$ such that

$$X(\alpha) = \hat{X}(\alpha) \hat{M}, \quad U(\alpha) = \hat{U}(\alpha) \hat{M} + \hat{X}^T(\alpha) \hat{N},$$

$\hat{M}$ is nonsingular, $\hat{M}^T \hat{N} = \hat{N}^T \hat{M}$, $\text{Im} \hat{N} \subseteq \text{Im} \hat{P}$, $P_{S\infty} \hat{N}^T \hat{M}^{-1} P_{S\infty} = 0$.

4. Riccati matrix differential equation for given genus. In this section we present a new theory extending the results by Reid in [22, 23] about Riccati matrix differential equation (R) to general possibly uncontrollable systems (H). Namely, for every genus $G$ of conjoined bases of (H) we consider the Riccati matrix differential equation (R). In Lemma 4.1, Theorem 4.3, and Corollary 4.5 we first derive properties of solutions of (R) in the relation with the associated projector $R_\psi(t)$ in (3.11). In (4.9) and (4.18) we introduce an auxiliary linear differential system and the so-called $F$-matrix for a solution of this system, which serve as main tools for the formulation of the results in this section. In particular, in Theorem 4.16 we present additional properties of solutions of (R) obtained through the above mentioned $F$-matrix. The main results concerning the correspondence between the solutions of the Riccati equation (R) and conjoined bases of (H) from the genus $G$ are contained in Theorems 4.18 and 4.21.

First we derive some auxiliary properties of the projector $R_\psi(t)$, being a solution of the Riccati equation (1.2), and the coefficient $A(t)$ in (1.3). In particular, we represent $R_\psi(t)$ as a solution of a linear differential system

$$R_\psi' = [A(t), R_\psi] = A(t) R_\psi - R_\psi A(t).$$

We note that since $R_\psi(t)$ is symmetric, then it solves also the system

$$R_\psi' = [R_\psi, A^T(t)] = R_\psi A^T(t) - A^T(t) R_\psi.$$

being the transpose of the system in (4.1). For $M, N \in \mathbb{R}^{nxn}$ the notation $[M, N]$ used in (4.1) and (4.2) means their commutator, i.e., $[M, N] := MN - NM$.

Lemma 4.1. Let $G$ be a genus of conjoined bases of (H) and let $R_\psi(t)$ and $A(t)$ be the corresponding matrices in (3.11) and (1.3). Then $[R_\psi(t), A(t) + A^T(t)] = 0$ for every $t \in [\alpha, \infty)$, i.e., the matrices $R_\psi(t)$ and $A(t) + A^T(t)$ commute on $[\alpha, \infty)$. Moreover, the orthogonal projector $R_\psi(t)$ satisfies on $[\alpha, \infty)$ system (4.1).

Proof. First we note that from the definition of the matrix $A(t)$ in (1.3) we have on $[\alpha, \infty)$ the formulas

$$AR_\psi = AR_\psi, \quad R_\psi A^T = R_\psi A^T,$$

$$R_\psi A = R_\psi AR_\psi - R_\psi A^T (I - R_\psi) = R_\psi (A + A^T) R_\psi - R_\psi A^T.$$


By combining equality (4.4) with the second identity in (4.3) we obtain that
\[ R_\varphi (A + A^T) = R_\varphi A + R_\varphi A^T = [R_\varphi (A + A^T) R_\varphi - R_\varphi A^T] + R_\varphi A^T \]
\[ = R_\varphi (A + A^T) R_\varphi \] (4.5)
on \([a_\infty, \infty)\). From (4.5) it then follows that the matrix \(R_\varphi(t) [A(t) + A^T(t)]\) is symmetric for every \(t \in [a_\infty, \infty)\), which in turn implies the equality
\[ R_\varphi(t) [A(t) + A^T(t)] = [A(t) + A^T(t)] R_\varphi(t) \]
for all \(t \in [a_\infty, \infty)\). In particular, this means that the matrices \(R_\varphi(t)\) and \(A(t) + A^T(t)\) commute on \([a_\infty, \infty)\), i.e., the commutator \([R_\varphi(t), A(t) + A^T(t)] = 0\) for every \(t \in [a_\infty, \infty)\), showing the first part of the lemma. For the proof of the second part we note that the orthogonal projector \(R_\varphi(t)\) solves the Riccati equation (1.2) on \([a_\infty, \infty)\), by Proposition 3.9. Moreover, by using formula (4.4) and the first identity in (4.3) equation (1.2) reads on \([a_\infty, \infty)\) as
\[ R_\varphi(t) = AR_\varphi - [R_\varphi (A + A^T) R_\varphi - R_\varphi A^T] (4.4) \equiv AR_\varphi - R_\varphi A = [A, R_\varphi]. \]
Thus, the matrix \(R_\varphi(t)\) solves system (4.1) on \([a_\infty, \infty)\). \(\square\)

**Remark 4.2.** We remark that the formulas in (4.1) are equivalent with
\[ (I - R_\varphi)' = [A(t), I - R_\varphi] = [I - R_\varphi, A^T(t)], \quad t \in [a_\infty, \infty), \] (4.6)
as one can easily check. The matrix \(R_\varphi(t)\) satisfies on \([a_\infty, \infty)\) also the relations
\[ R_\varphi(t) = \begin{aligned} \mathcal{A}(t) R_\varphi(t) + R_\varphi(t) A^T(t) - R_\varphi(t) [A(t) + A^T(t)] R_\varphi(t), \\ \text{Im} \mathcal{B}(t) \subseteq \text{Im} R_\varphi(t). \end{aligned} \] (4.7)
Note that the first equation in (4.7) is the same as (1.2) with \(A(t)\) instead of \(A(t)\).

Next we derive properties of the solutions of (\(\mathcal{R}\)), which are based on the projector \(R_\varphi(t)\) and Lemma 4.1.

**Theorem 4.3.** Let \(\mathcal{G}\) be a genus of conjoined bases of (H) with the corresponding matrix \(R_\varphi(t)\) in (3.11) and let \(Q(t)\) be a solution of the Riccati equation (\(\mathcal{R}\)) on \([a, \infty) \subseteq [a_\infty, \infty)\). Then the matrices \(R_\varphi(t) Q(t), Q(t) R_\varphi(t), \) and \(R_\varphi(t) Q(t) R_\varphi(t)\) also solve (\(\mathcal{R}\)) on \([a, \infty)\).

**Proof.** Let \(R_\varphi(t)\) and \(Q(t)\) be as in the theorem. By using (\(\mathcal{R}\)), the second formula in (4.1), and the identities \(R_\varphi(t) C(t) = C(t)\) and \(\mathcal{B}(t) R_\varphi(t) = \mathcal{B}(t)\) for every \(t \in [a, \infty)\) we obtain that
\[ \begin{aligned} (R_\varphi Q)' &= R_\varphi Q + R_\varphi Q'^{(4.1)} = (\mathcal{R}) \quad [R_\varphi, A^T] Q + R_\varphi (C - Q A - A^T Q - Q B Q) \\ &= R_\varphi A^T Q - A^T R_\varphi Q + C - R_\varphi A - R_\varphi A^T Q - R_\varphi B R_\varphi Q \\ &= -A^T (R_\varphi Q) - (R_\varphi Q) A - (R_\varphi Q) B (R_\varphi Q) + C \end{aligned} \]
on \([a, \infty)\). Thus, the matrix \(R_\varphi(t) Q(t)\) solves (\(\mathcal{R}\)) on \([a, \infty)\). Similarly, by the first formula in (4.1) and the identities \(C(t) R_\varphi(t) = C(t)\) and \(R_\varphi(t) \mathcal{B}(t) = \mathcal{B}(t)\) for all \(t \in [a, \infty)\) we get
\[ \begin{aligned} (Q R_\varphi)' &= Q' R_\varphi + Q R_\varphi' = (\mathcal{R}) \quad (C - Q A - A^T Q - Q B Q) R_\varphi + Q [A, R_\varphi] \\ &= C - Q AR_\varphi - A^T QR_\varphi - QR_\varphi B QR_\varphi + Q AR_\varphi - QR_\varphi A \\ &= -A^T (QR_\varphi) - (QR_\varphi) A - (QR_\varphi) B (QR_\varphi) + C \]
on \([\alpha, \infty)\), showing that the matrix \(Q(t)R_c(t)\) solves \((R)\) on \([\alpha, \infty)\). Finally, by combining these results we get that also \(R_c(t)[Q(t)R_c(t)] = [R_c(t)Q(t)]R_c(t) = R_c(t)Q(t)R_c(t)\) is a solution of \((R)\) on \([\alpha, \infty)\) and the proof is complete. \(\square\)

**Remark 4.4.** The symmetry of equation \((R)\) implies that the matrix \(Q^T(t)\) solves equation \((R)\) on \([\alpha, \infty)\). By applying Theorem 4.3 for \(Q := Q^T\) we then obtain that also the matrices \(R_c(t)Q^T(t), Q^T(t)R_c(t),\) and \(R_c(t)Q^T(t)R_c(t)\) are solutions of \((R)\) on \([\alpha, \infty)\).

**Corollary 4.5.** With the assumptions and notations of Theorem 4.3, the matrix \(Q(t)\) satisfies the inclusion \(\text{Im} Q(t) \subseteq \text{Im} R_c(t), \text{resp. the inclusion} \text{Im} Q^T(t) \subseteq \text{Im} R_c(t), \text{for all} t \in [\alpha, \infty) \text{ if and only if the inclusion} \text{Im} Q(t_0) \subseteq \text{Im} R_c(t_0), \text{resp. the inclusion} \text{Im} Q^T(t_0) \subseteq \text{Im} R_c(t_0), \text{holds for some} t_0 \in [\alpha, \infty).\)

**Proof.** From Theorem 4.3 and Remark 4.4 we know that the matrices \(Q_*(t) := R_c(t)Q(t)\) and \(Q_{**}(t) := R_c(t)Q^T(t)\) solve equation \((R)\) on \([\alpha, \infty)\). Fix \(t_0 \in [\alpha, \infty).\) If \(\text{Im} Q(t_0) \subseteq \text{Im} R_c(t_0),\) then the matrix \(Q_*(t_0) = Q(t_0)\) and by the uniqueness of solutions of \((R)\) we obtain the equality \(Q_*(t) = Q(t)\) on \([\alpha, \infty).\) The latter identity means that the inclusion \(\text{Im} Q(t) \subseteq \text{Im} R_c(t)\) holds for every \(t \in [\alpha, \infty).\) By using the similar arguments the relation \(\text{Im} Q^T(t_0) \subseteq \text{Im} R_c(t_0)\) implies that \(Q_{**}(t_0) = Q(t_0)\) and consequently, we have the equality \(Q_{**}(t) = Q(t)\) on \([\alpha, \infty).\) Hence, the inclusion \(\text{Im} Q^T(t) \subseteq \text{Im} R_c(t)\) holds for every \(t \in [\alpha, \infty).\) The proof of opposite implications is trivial. \(\square\)

For our reference we now present an auxiliary result from linear algebra about orthogonal projectors.

**Lemma 4.6.** Let \(Z \in \mathbb{R}^{n \times n}\) be an orthogonal projector. Then \(K, L \in \mathbb{R}^{n \times n}\) satisfy

\[
\text{Im} K \subseteq \text{Im} Z \quad \text{and} \quad \text{Im} L \subseteq \text{Ker} Z
\]

(4.8) if and only if \(K = ZE\) and \(L = (I - Z) E\) for some matrix \(E \in \mathbb{R}^{n \times n}.\) In this case the equality \(\text{Ker} K \cap \text{Ker} L = \text{Ker} E\) holds.

**Proof.** Let \(Z\) be as in the lemma. If the matrices \(K\) and \(L\) satisfy (4.8), then for \(E := K + L\) we have that

\[
ZE = ZK + ZL = K, \quad (I - Z) E = (I - Z) K + (I - Z) L = L.
\]

The opposite implication is trivial. Finally, it is easy to see that in this case we have the equality \(\text{Ker} K \cap \text{Ker} L = \text{Ker} E,\) which completes the proof. \(\square\)

**Remark 4.7.** Let \(Z\) be an orthogonal projector and let \(K\) and \(L\) be matrices satisfying (4.8). The results in Lemma 4.6 then show that \(\text{Ker} K \cap \text{Ker} L = \{0\}\) if and only if the matrix \(E = K + L\) is nonsingular. In particular, in this case the inclusions in (4.8) are implemented as equalities, i.e., the identities \(\text{Im} K = \text{Im} Z\) and \(\text{Im} L = \text{Ker} Z\) hold. We also note that the condition \(\text{Ker} K \cap \text{Ker} L = \{0\}\) is equivalent with \(\text{rank}(K^T, L^T)^T = n.\)

Let \(G\) be a genus of conjoined bases of \((H)\) and let \(R_c(t)\) be its representing orthogonal projector in (3.11). For a given solution \(Q(t)\) of the Riccati equation \((R)\) on a subinterval \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) we consider the following system of first order linear differential equations

\[
\begin{align*}
\Theta' &= [A(t) + B(t) Q(t)] \Theta, \\
\Omega' &= A(t) \Omega + [I - R_c(t)][C(t) - [A(t) + A^T(t)] Q(t)] \Theta.
\end{align*}
\]

(4.9)
on \([\alpha, \infty)\) together with the initial conditions
\[
\Theta(\alpha) = K, \quad \Omega(\alpha) = L,
\] (4.10)
where the matrices \(K, L \in \mathbb{R}^{n \times n}\) satisfy
\[
\text{Im } K \subseteq \text{Im } R_\phi(\alpha), \quad \text{Im } L \subseteq \text{Ker } R_\phi(\alpha), \quad \text{rank}(K^T, L^T)^T = n.
\] (4.11)
We will study the properties of solutions of system (4.9), which will serve for the formulation and proofs of the main results of this section. The first equation in (4.9) is motivated by the approach in [23, Chapter 2, Lemma 2.1], which is adopted here to the setting of uncontrollable systems (H).

**Remark 4.8.** Given a solution \(Q(t)\) of \((R)\) on \([\alpha, \infty)\) we note that for any matrices \(K\) and \(L\) satisfying (4.11) there exist unique matrices \(\Theta(t)\) and \(\Omega(t)\), which solve the equations in (4.9) on \([\alpha, \infty)\) with \(\Theta(\alpha) = K\) and \(\Omega(\alpha) = L\). Moreover, in this case we have from Lemma 4.6 and Remark 4.7 with \(Z := R_\phi(\alpha)\) and \(\Theta(\alpha) = R_\phi(\alpha) E\) and \(\Omega(\alpha) = [I - R_\phi(\alpha)] E\) for some nonsingular matrix \(E\). These observations then imply that the initial value problem (4.9)–(4.10) with (4.11) has the solution \((\Theta, \Omega)\), which is unique up to a right nonsingular multiple. More precisely, if \((\Theta_0, \Omega_0)\) is another solution of (4.9)–(4.11), then there exists a constant nonsingular matrix \(M \in \mathbb{R}^{n \times n}\) such that \(\Theta_0(t) = \Theta(t) M\) and \(\Omega_0(t) = \Omega(t) M\) for all \(t \in [\alpha, \infty)\).

**Proposition 4.9.** Let \(G\) be a genus of conjoined bases of \((H)\) with the corresponding matrix \(R_\phi(t)\) in (3.11). Moreover, let \(Q(t)\) be a solution of equation (\(R\)) on \([\alpha, \infty)\) and let \((\Theta, \Omega)\) be a solution of the associated system in (4.9) on \([\alpha, \infty)\). Then the matrix \(\Theta(t)\) has a constant kernel on \([\alpha, \infty)\) and the matrices \(V(t) := [I - R_\phi(t)] \Theta(t)\) and \(W(t) := R_\phi(t) \Omega(t)\) solve on \([\alpha, \infty)\) the linear differential equation
\[
Y' = \mathcal{A}(t) Y.
\] In addition, if the matrices \(K := \Theta(\alpha)\) and \(L := \Omega(\alpha)\) satisfy the conditions in (4.11), then for all \(t \in [\alpha, \infty)\) we have
\[
\text{Im } \Theta(t) = \text{Im } R_\phi(t), \quad \text{Im } \Omega(t) = \text{Ker } R_\phi(t), \quad \text{rank}(\Theta^T(t), \Omega^T(t))^T = n.
\] (4.12)

**Proof.** Let \(R_\phi(t), Q(t), \Theta(t), \Omega(t), K, \) and \(L\) be as in the proposition. By the uniqueness of solutions of the first equation in (4.9) we have that \(\Theta(t) = \Phi_\alpha(t) \Theta(\alpha)\), where \(\Phi_\alpha(t)\) is the associated fundamental matrix normalized at the point \(\alpha\), i.e.,
\[
\Phi_\alpha = [\mathcal{A}(t) + \mathcal{B}(t) Q(t)] \Phi_\alpha, \quad t \in [\alpha, \infty), \quad \Phi_\alpha(\alpha) = I.
\] (4.13)
This implies that \(\text{Ker } \Theta(t) = \text{Ker } \Theta(\alpha)\) for every \(t \in [\alpha, \infty)\), i.e., the matrix \(\Theta(t)\) has constant kernel on \([\alpha, \infty)\). Next we show that the matrices \(V(t)\) and \(W(t)\) satisfy on \([\alpha, \infty)\) the equation \(Y' = \mathcal{A}(t) Y\). Indeed, by using (4.1), (4.6), (4.9), and the inclusion in (4.7) we obtain on \([\alpha, \infty)\) that
\[
V' = (I - R_\phi) \mathcal{A} (I - R_\phi) \Theta + (I - R_\phi) \mathcal{A} (I - R_\phi) \mathcal{B} Q \Theta = \mathcal{A} (I - R_\phi) \Theta + (I - R_\phi) \mathcal{A} \Theta = \mathcal{A} V,
\] (4.14)
\[
W' = R_\phi \mathcal{A} R_\phi \mathcal{A} (I - R_\phi) \mathcal{A} (I - R_\phi) \mathcal{B} Q \Theta = \mathcal{A} R_\phi \mathcal{A} \mathcal{A} (I - R_\phi) R_\phi \mathcal{A} \mathcal{A} \mathcal{A} = \mathcal{A} W.
\] (4.15)
Moreover, suppose that the matrices \(K\) and \(L\) satisfy (4.11). Then \(V(\alpha) = 0 = W(\alpha)\), which in turn, by uniqueness of solutions of (4.14) and (4.15), implies that \(V(t) = 0 = W(t)\) for all \(t \in [\alpha, \infty)\). Therefore, we have \(\text{Im } \Theta(t) \subseteq \text{Im } R_\phi(t)\) and \(\text{Im } \Omega(t) \subseteq \text{Ker } R_\phi(t)\) on \([\alpha, \infty)\). And since the matrices \(\Theta(t)\) and \(R_\phi(t)\) have constant ranks on \([\alpha, \infty)\) and the equality rank \(\Theta(\alpha) = \text{rank } R_\phi(\alpha)\) holds by Remark 4.8, we
obtain that even \( \text{Im} \Theta(t) = \text{Im} R_\gamma(t) \) for every \( t \in [\alpha, \infty) \). Now we shall prove the last condition in (4.12), which is clearly equivalent with the identity \( \text{Ker } \Theta(t) \cap \text{Ker } \Omega(t) = \{0\} \) on \( [\alpha, \infty) \). Fix \( \beta \in [\alpha, \infty) \) and let \( v \in \text{Ker } \Theta(\beta) \cap \text{Ker } \Omega(\beta) \). From the fact that \( \text{Ker } \Theta(t) \) is constant on \( [\alpha, \infty) \) it then follows that \( \Theta(t) v = 0 \) for all \( t \in [\alpha, \infty) \). In particular, this means that the function \( w(t) := \Omega(t) v \) satisfies on \( [\alpha, \infty) \) the identity \( w'(t) = \mathcal{A}(t) w(t) \), by (4.9). But \( w(\beta) = 0 \) and hence, by the uniqueness of solutions of the equation \( y' = \mathcal{A}(t) y \) we get that \( w(t) = 0 \) for every \( t \in [\alpha, \infty) \). Therefore, the vector \( v \in \text{Ker } \Theta(\alpha) \cap \text{Ker } \Omega(\alpha) = \text{Ker } K \cap \text{Ker } L \), which in turn implies that \( v = 0 \), by (4.11). Thus, the subspace \( \text{Ker } \Theta(t) \cap \text{Ker } \Omega(t) = \{0\} \) for all \( t \in [\alpha, \infty) \). Finally, with the aid of Remark 4.7 we conclude that \( \text{Im } \Omega(t) = \text{Ker } R_\gamma(t) \) on \( [\alpha, \infty) \), showing the second condition in (4.12).

**Remark 4.10.** The results in Proposition 4.9 and Remark 2.2 imply that for any solution \((\Theta, \Omega)\) of (4.9)–(4.11) the matrix \( \Theta^\dagger \in \mathbb{C}_p^1 \) and satisfies the equation

\[
[\Theta^\dagger(t)]' = -\Theta^\dagger(t) [\mathcal{A}(t) + \mathcal{B}(t) Q(t) R_\gamma(t)], \quad t \in [\alpha, \infty).
\]

Moreover, from Remark 4.8 it follows that for a given solution \( Q(t) \) of the Riccati equation (\( \mathcal{R} \)) on \( [\alpha, \infty) \subseteq [\alpha, \infty, \infty) \) there exists the unique such a pair \((\Theta, \Omega)\) satisfying \( \Theta(\alpha) = R_\gamma(\alpha) \) and \( \Omega(\alpha) = I - R_\gamma(\alpha) \). Obviously, in this case we have the equalities \( \Theta(t) = \Phi_\alpha(t) R_\gamma(t) \) and \( \text{Im } \Theta^\dagger(t) = \text{Im } \Theta^\dagger(t) = \text{Im } R_\gamma(t) \) on \( [\alpha, \infty) \), where \( \Phi_\alpha(t) \) is the fundamental matrix in (4.13). In particular, the matrix \( \Theta^\dagger(t) \) then satisfies for every \( t \in [\alpha, \infty) \) the formula

\[
\Theta^\dagger(t) = R_\gamma(t) \Theta^\dagger(t) = \Phi_\alpha^{-1}(t) \Phi_\alpha(t) R_\gamma(t) \Theta^\dagger(t)
\]

\[
= \Phi_\alpha^{-1}(t) \Theta(t) \Theta^\dagger(t) \quad \text{(4.12)}
\]

On the other hand, by the aid of (4.12) we obtain the equality

\[
\text{Im } \Phi_\alpha(t) R_\gamma(t) = \text{Im } \Theta(t) \quad \text{(4.12)} \equiv \text{Im } R_\gamma(t), \quad t \in [\alpha, \infty).
\]

This equality is an important property of the matrix function \( \Phi_\alpha(t) \), which will be utilized in the proof of Theorem 4.16 below.

In the following remark we introduce an important matrix (called the \( F \)-matrix) in terms of a solution \( \Theta(t) \) of (4.9). For an invertible \( \Theta(t) \) this matrix was considered in [23, Section 2.2]. Here we allow \( \Theta(t) \) to be singular.

**Remark 4.11.** (i) The properties of the matrix \( \Theta(t) \) allow to define the function

\[
F_\alpha(t) := \int_{\alpha}^{t} \Theta^\dagger(s) \mathcal{B}(s) \Theta^\dagger T(s) ds, \quad t \in [\alpha, \infty),
\]

which will be referred to as the \( F \)-matrix corresponding to the solution \( Q(t) \) with respect to the genus \( \mathcal{G} \). From (4.18) it immediately follows that \( F_\alpha(t) \) is symmetric and the inclusion \( \text{Im } F_\alpha(t) \subseteq \text{Im } R_\gamma(t) \) holds for every \( t \in [\alpha, \infty) \) and \( F_\alpha \in \mathbb{C}_p^1 \). Moreover, under (1.1) the matrix \( F_\alpha(t) \) is nonnegative definite and nondecreasing with \( F_\alpha(0) = 0 \) on \( [\alpha, \infty) \). Therefore, the subspace \( \text{Ker } F_\alpha(t) \) is nonincreasing on \( [\alpha, \infty) \), and hence eventually constant. Consequently, the properties of Moore–Penrose pseudoinverse displayed in Remark 2.2 imply that \( F^\dagger_\alpha \in \mathbb{C}_p^1 \)

\[
(F^\dagger_\alpha)'(t) = -F^\dagger_\alpha(t) F_\alpha(t) F^\dagger_\alpha(t) = -F^\dagger_\alpha(t) \Theta^\dagger(t) \mathcal{B}(t) \Theta^\dagger T(t) F^\dagger_\alpha(t) \leq 0 \text{ for large } t.
\]

Thus, the matrix \( F^\dagger_\alpha(t) \) is nonincreasing for large \( t \). And since \( F^\dagger_\alpha(t) \) is nonnegative definite on \( [\alpha, \infty) \), it follows that the limit of \( F^\dagger_\alpha(t) \) exists as \( t \to \infty \), i.e.,

\[
D_\alpha := \lim_{t \to \infty} F^\dagger_\alpha(t).
\]
Clearly, the matrix $D_\alpha$ defined in (4.19) is symmetric and nonnegative definite and the inclusion $\text{Im} D_\alpha \subseteq \text{Im} R_\alpha(r)$ holds. In addition, we note that with the aid of (4.16) and the identity $R_\alpha(t) B(t) R_\alpha(t) = B(t)$ on $[\alpha, \infty)$ the matrix $F_\alpha(t)$ in (4.18) can be also represented as
\[
F_\alpha(t) = \int_\alpha^t \Phi^{-1}_\alpha(s) R_\alpha(s) B(s) R_\alpha(s) \Phi^{-1}_\alpha(s) \, ds
\]
for all $t \in [\alpha, \infty]$ with $\Phi_\alpha(t)$ defined in (4.13).

(ii) There is another important property of the $F$-matrix introduced in (4.18). For a given genus $G$ of conjoined bases of (H) with $R_\alpha(t)$ in (3.11) let $Q(t)$ be a solution of (R) on $[\alpha, \infty) \subseteq [\alpha, \infty)$. By Theorem 4.3 we know that also the matrix $\tilde{Q}(t) := R_\alpha(t) Q(t) R_\alpha(t)$ solves (R) on $[\alpha, \infty)$. Moreover, let $\Theta(t)$ and $\tilde{\Theta}(t)$ be the corresponding matrices from Remark 4.10, that is,
\[
\Theta'(t) = [A(t) + B(t) Q(t)] \Theta, \quad \tilde{\Theta}'(t) = [A(t) + B(t) \tilde{Q}(t)] \tilde{\Theta}, \quad t \in [\alpha, \infty).
\]
(4.21)

With the aid of the identities $R_\alpha(t) \tilde{\Theta}(t) = \tilde{\Theta}(t)$ and $B(t) R_\alpha(t) = B(t)$ on $[\alpha, \infty)$ we obtain the equality
\[
\tilde{\Theta}'(t) = [A(t) + B(t) \tilde{Q}(t)] \tilde{\Theta}(t) = A(t) \tilde{\Theta}(t) + B(t) R_\alpha(t) Q(t) R_\alpha(t) \tilde{\Theta}(t) = A(t) \tilde{\Theta}(t) + B(t) Q(t) \tilde{\Theta}(t) = [A(t) + B(t) Q(t)] \tilde{\Theta}(t)
\]
for every $t \in [\alpha, \infty)$. Therefore, the matrices $\Theta(t)$ and $\tilde{\Theta}(t)$ solve the same equation on $[\alpha, \infty)$ and hence, $\Theta(t) = \tilde{\Theta}(t)$ for all $t \in [\alpha, \infty)$ by the last condition in (4.21). Consequently, the matrices $F_\alpha(t)$ and $\tilde{F}_\alpha(t)$ in (4.18) associated with the solutions $Q(t)$ and $\tilde{Q}(t)$, respectively, satisfy the equality $F_\alpha(t) = \tilde{F}_\alpha(t)$ on $[\alpha, \infty)$.

The representation of the matrix $F_\alpha(t)$ in (4.20) in terms of the fundamental matrix $\Phi_\alpha(t)$ of (4.13) allows to apply the original result in [23, Lemma 2.1, pg. 12] for symmetric solutions $Q(t)$ of (R). This yields the following statement, which will be utilized in our further analysis.

**Proposition 4.12.** Let $G$ be a genus of conjoined bases of (H) with the matrix $R_\alpha(t)$ in (3.11) and let $Q(t)$ be a symmetric solution of the Riccati equation (R) on $[\alpha, \infty) \subseteq [\alpha, \infty)$. Moreover, let $\Phi_\alpha(t)$ and $F_\alpha(t)$ be the corresponding matrices in (4.13) and (4.20), respectively. Then an $n \times n$ matrix-valued function $\tilde{Q}(t)$ solves (R) on $[\alpha, \infty)$ if and only if the constant matrix $G := \tilde{Q}(\alpha) - Q(\alpha)$ is such that the matrix $I + F_\alpha(t) G$ is nonsingular on $[\alpha, \infty)$ and
\[
\tilde{Q}(t) = Q(t) + \Phi^{-1}_\alpha(t) G [I + F_\alpha(t) G]^{-1} \Phi^{-1}_\alpha(t) \quad \text{for every } t \in [\alpha, \infty).
\]
(4.22)

**Remark 4.13.** Let $K$ be a given $n \times n$ matrix and let $\tilde{Q}(t)$ be the solution of the Riccati equation (R) satisfying $\tilde{Q}(\alpha) = K$. From Proposition 4.12 it then follows that the matrix $\tilde{Q}(t)$ as a solution of (R) can be extended to the whole interval $[\alpha, \infty)$ if and only if the matrix $G := K - Q(\alpha)$ is such that the matrix $I + F_\alpha(t) G$ is nonsingular for all $t \in [\alpha, \infty)$. In this case, the solution $\tilde{Q}(t)$ has the representation in (4.22).
Formula (4.22) allows to derive inequalities between symmetric solutions of the Riccati equation (R). We note that the statement about the constant rank of \( \dot{Q}(t) - Q(t) \) corresponds to [23, Corollary 2, pg. 13].

**Corollary 4.14.** Let \( \mathcal{G} \) be a genus of conjoined bases of (H) with the corresponding matrix \( R_c(t) \) in (3.11) and let \( Q(t) \) and \( \dot{Q}(t) \) be symmetric solutions of the Riccati equation (R) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\). Then the quantities

\[
\text{rank}[\dot{Q}(t) - Q(t)] \quad \text{and} \quad \text{ind}[\dot{Q}(t) - Q(t)] \quad \text{are constant on} \quad [\alpha, \infty). \tag{4.23}
\]

In particular, the inequality \( \dot{Q}(t) \geq Q(t) \) holds on \([\alpha, \infty)\) if and only if \( \dot{Q}(\alpha) \geq Q(\alpha) \), and the inequality \( \dot{Q}(t) > Q(t) \) holds on \([\alpha, \infty)\) if and only if \( \dot{Q}(\alpha) > Q(\alpha) \).

**Proof.** Let \( Q(t) \) and \( \dot{Q}(t) \) be as in the corollary and set \( G := \dot{Q}(\alpha) - Q(\alpha) \). With the aid of formula (4.22) we then obtain that \( \text{rank}[\dot{Q}(t) - Q(t)] = \text{rank} G = \text{rank} [\dot{Q}(\alpha) - Q(\alpha)] \) on \([\alpha, \infty)\). Moreover, the continuity of the matrices \( Q(t) \) and \( \dot{Q}(t) \) implies that also the quantity \( \text{ind}[\dot{Q}(t) - Q(t)] \) is constant on \([\alpha, \infty)\), completing the proof of the statements in (4.23). Finally, the assertions in the second part of the corollary follow immediately from (4.23). \( \square \)

The next statement extends to an arbitrary genus \( \mathcal{G} \) the result in [7, Corollary (iv), pp. 52–53], in which we consider one system (H).

**Corollary 4.15.** Assume (1.1) and let \( \mathcal{G} \) be a genus of conjoined bases of (H) with the corresponding matrix \( R_c(t) \) in (3.11). Let \( Q(t) \) be a symmetric solution of the Riccati equation (R) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) and let \( \dot{Q}(t) \) be a symmetric solution of the Riccati equation (R) satisfying the condition \( \dot{Q}(\alpha) \geq Q(\alpha) \). Then the matrix \( \dot{Q}(t) \) solves (R) on the whole interval \([\alpha, \infty)\) such that the inequality \( \dot{Q}(t) \geq Q(t) \) holds for all \( t \in [\alpha, \infty)\).

**Proof.** Let \( F_\alpha(t) \) be the matrix in (4.18) associated with \( Q(t) \) on \([\alpha, \infty)\) and set \( G := \dot{Q}(\alpha) - Q(\alpha) \). We will show that the matrix \( I + F_\alpha(t)G \) is nonsingular on \([\alpha, \infty)\). Fix \( t \in [\alpha, \infty)\) and let \( v \in \text{Ker}[I + F_\alpha(t)G] \), i.e., the equality \( v = -F_\alpha(t)Gv \) holds. Since the matrix \( G \) is symmetric and satisfies \( G \geq 0 \) and from Remark 4.11 we know that under the Legendre condition (1.1) the matrix \( F_\alpha(t) \) is nonsingular definite, we have that \( 0 \leq v^T G v = -v^T G^T F_\alpha(t) G v \leq 0 \). Thus, \( v^T G v = 0 \) and consequently, \( Gv = 0 \). Therefore, the vector \( v = -F_\alpha(t)Gv = 0 \) and the matrix \( I + F_\alpha(t)G \) is nonsingular. Finally, according to Remark 4.13 and Corollary 4.14 this then means that the matrix \( \dot{Q}(t) \) solves the Riccati equation (R) on the whole interval \([\alpha, \infty)\) and satisfies the inequality \( \dot{Q}(t) \geq Q(t) \) for every \( t \in [\alpha, \infty) \). \( \square \)

In the next result we show further properties of the solutions of the Riccati equation (R). Namely, we characterize a certain class of the values \( K \) of the initial conditions at some point \( \beta \), which guarantee that the corresponding solution \( \dot{Q}(t) \) of (R) with \( \dot{Q}(\beta) = K \) exists on the whole interval \([\beta, \infty)\). Another interpretation of the following statement is that any symmetric solution \( Q(t) \) of (R) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) satisfying inclusion (1.4) can be decomposed into the product \( R_c(t) \dot{Q}(t) R_c(t) \) for a suitable, in general nonsymmetric, solution \( \dot{Q}(t) \) of (R). This result can be regarded as a partial converse to Theorem 4.3 and it will be utilized for the classification of solutions of (R) in Remark 4.20 below.

**Theorem 4.16.** Let \( \mathcal{G} \) be a genus of conjoined bases of (H) and let \( R_c(t) \) be its corresponding matrix in (3.11). Moreover, let \( Q(t) \) be a symmetric solution of (R) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) satisfying condition (1.4). Let \( \beta \in [\alpha, \infty) \) and \( K \in \mathbb{R}^{n \times n} \) be
given and consider the solution \( \tilde{Q}(t) \) of (\( R \)) with \( \tilde{Q}(\beta) = K \). Then the following
statements are equivalent.

(i) The matrix \( Q(t) \) solves the Riccati equation (\( R \)) on the whole interval \([\alpha, \infty)\) such that \( R_\psi(t) \tilde{Q}(t) R_\psi(t) = Q(t) \) holds for every \( t \in [\alpha, \infty) \).

(ii) The matrix \( K \) satisfies the equality \( R_\psi(\beta) K R_\psi(\beta) = Q(\beta) \).

Proof. First we note that assertion (i) implies (ii) trivially. Therefore, suppose that (ii) holds, i.e., the matrix \( K \) satisfies \( R_\psi(\beta) K R_\psi(\beta) = Q(\beta) \). Let \( \Phi_\alpha(t) \) and \( F_\alpha(t) \) be the matrices in (4.13) and (4.18) associated with \( Q(t) \) and put

\[
E := I - \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta). \tag{4.24}
\]

We observe that from Remark 4.11(i), inclusion (1.4) with symmetric \( Q(t) \), and (4.17) at \( t = \beta \) we have

\[
F_\alpha(\beta) = R_\psi(\alpha) F_\alpha(\beta), \quad R_\psi(\beta) Q(\beta) \overset{(1.4)}{=} Q(\beta) \overset{(1.4)}{=} Q(\beta) R_\psi(\beta), \quad R_\psi(\beta) \Phi_\alpha(\beta) R_\psi(\alpha) \overset{(4.17)}{=} \Phi_\alpha(\beta) R_\psi(\alpha).
\]

Then the matrix \( E \) in (4.24) satisfies

\[
R_\psi(\alpha) E \overset{(4.24)}{=} R_\psi(\alpha) - R_\psi(\alpha) \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) F_\alpha(\beta) = R_\psi(\alpha) - R_\psi(\alpha) \Phi_\alpha^T(\beta) [R_\psi(\beta) K R_\psi(\beta) - Q(\beta)] \Phi_\alpha(\beta) R_\psi(\alpha) F_\alpha(\beta) = R_\psi(\alpha). \tag{4.25}
\]

We will show that the matrix \( E \) is nonsingular. Let \( v \in \text{Ker} E \). This means according to (4.24) that

\[
v = \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) F_\alpha(\beta) v. \tag{4.27}
\]

Moreover, from (4.26) it follows that \( R_\psi(\alpha) v = R_\psi(\alpha) Ev = 0 \). Combining the latter equality together with (4.27) and with the first identity in (4.25) yields

\[
v \overset{(4.27)}{=} \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) F_\alpha(\beta) v = \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) F_\alpha(\beta) R_\psi(\alpha) v = 0,
\]

which proves the nonsingularity of \( E \). In particular, formula (4.26) is then equivalent with the equality \( R_\psi(\alpha) E^{-1} = R_\psi(\alpha) \). Now set

\[
G := E^{-1} \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) \tag{4.28}
\]

and consider the solution \( Q_\star(t) \) of (\( R \)) satisfying the initial condition \( Q_\star(\alpha) = Q(\alpha) + G \). We shall prove that the solution \( Q_\star(t) \) is defined on the whole interval \([\alpha, \infty)\) such that \( Q_\star(\beta) = K \). First observe that with the aid of (4.24) and (4.28) we get the identity

\[
\Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) \overset{(4.28)}{=} E G \overset{(4.24)}{=} G - \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) F_\alpha(\beta) G,
\]

which implies immediately the formula

\[
G = \Phi_\alpha^T(\beta) [K - Q(\beta)] \Phi_\alpha(\beta) [I + F_\alpha(\beta) G]. \tag{4.29}
\]
Furthermore, by using the equality $R_{c}(\alpha) E^{-1} = R_{c}(\alpha)$ and (4.25) we obtain that

\[
R_{c}(\alpha) G R_{c}(\alpha) (4.28) = R_{c}(\alpha) E^{-1} \Phi_{\alpha}^{T}(\beta) [K - Q(\beta)] \Phi_{\alpha}(\beta) R_{c}(\alpha)
\]

\[
= R_{c}(\alpha) \Phi_{\alpha}^{T}(\beta) [K - Q(\beta)] \Phi_{\alpha}(\beta) R_{c}(\alpha)
\]

\[
(4.25) = R_{c}(\alpha) \Phi_{\alpha}^{T}(\beta) [R_{c}(\beta) KR_{c}(\beta) - Q(\beta)] \Phi_{\alpha}(\beta) R_{c}(\alpha)
\]

\[
= 0.
\]

Fix $t \in [\alpha, \infty)$ and let $v \in \text{Ker} [I + F_{\alpha}(t) G]$, i.e., the equality $v = -F_{\alpha}(t) G v$ holds. In particular, the vector $v \in \text{Im} F_{\alpha}(t) \subseteq \text{Im} R_{c}(\alpha)$, by Remark 4.11(i). This means that $v = R_{c}(\alpha) v$, which in turn together with the equality $F_{\alpha}(t) R_{c}(\alpha) = F_{\alpha}(t)$ and equation (4.30) yields $v = -F_{\alpha}(t) G v = -F_{\alpha}(t) R_{c}(\alpha) G R_{c}(\alpha) v = 0$. Thus, the matrix $I + F_{\alpha}(t) G$ is nonsingular on $[\alpha, \infty)$ and by Remark 4.13 the solution $Q_{c}(t)$ exists on the whole interval $[\alpha, \infty)$ such that

\[
Q_{c}(t) = Q(t) + \Phi_{\alpha}^{T}(\alpha) [I + F_{\alpha}(t) G]^{-1} G \Phi_{\alpha}^{-1}(\alpha) + \Phi_{\alpha}^{T}(\beta) [R_{c}(\beta) K R_{c}(\beta) - Q(\beta)] \Phi_{\alpha}(\beta) R_{c}(\alpha)
\]

\[
= 0.
\]

Therefore, by the uniqueness of solutions of (R) the matrix $\tilde{Q}(t)$ solves (R) on the whole interval $[\alpha, \infty)$ with $\tilde{Q}(t) = Q_{c}(t)$ for every $t \in [\alpha, \infty)$. In turn, the matrix $R_{c}(t) \tilde{Q}(t) R_{c}(t)$ is also a solution of (R) on $[\alpha, \infty)$, by Theorem 4.3. Finally, since $R_{c}(\beta) \tilde{Q}(\beta) R_{c}(\beta) = R_{c}(\beta) K R_{c}(\beta) = Q(\beta)$, we conclude by using the uniqueness of solutions of (R) once more that $R_{c}(t) \tilde{Q}(t) R_{c}(t) = Q(t)$ for all $t \in [\alpha, \infty)$. This shows (i) and the proof is complete.

\[\square\]

Remark 4.17. For the completeness we note that the matrix $\tilde{Q}(t)$ in Theorem 4.16 satisfies the formula

\[
\tilde{Q}(t) = Q(t) + \Phi_{\alpha}^{T}(\alpha) [I + F_{\alpha}(t) G]^{-1} \Phi_{\alpha}^{-1}(\alpha), \quad t \in [\alpha, \infty).
\]

This follows directly from the equality $\tilde{Q}(t) = Q_{c}(t)$ on $[\alpha, \infty)$ and the representation of the matrix $Q_{c}(t)$ in (4.31).

We are now ready to formulate the main results of this section (Theorems 4.18 and 4.21), in which we connect the solutions $Q(t)$ of the Riccati equation (R) on $[\alpha, \infty)$ with conjoined bases $(X, U)$ with constant kernel on $[\alpha, \infty)$ from the genus $\mathcal{G}$. These results extend the well known correspondence between the symmetric solutions $Q(t)$ of (R) on $[\alpha, \infty)$ and conjoined bases $(X, U)$ of (H) with $X(t)$ invertible on $[\alpha, \infty)$, i.e.,

\[
Q(t) = U(t) X^{-1}(t), \quad t \in [\alpha, \infty)
\]

(4.32) to the case of possibly noninvertible $X(t)$ on $[\alpha, \infty)$.

Theorem 4.18. Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{c}(t)$ in (3.11). Moreover, let $(X, U)$ be a conjoined basis of (H) belonging to $\mathcal{G}$ such that $(X, U)$ has constant kernel on a subinterval $[\alpha, \infty) \subseteq [\alpha, \infty)$ and let $Q(t)$ be the corresponding Riccati quotient in (2.7). Then the matrix $Q(t)$ is a symmetric solution of the Riccati equation (R) on $[\alpha, \infty)$ such that the condition in (1.4) holds and the matrices $\Theta(t)$ and $\Omega(t)$ defined by

\[
\Theta(t) := X(t), \quad \Omega(t) := U(t) - Q(t) X(t), \quad t \in [\alpha, \infty),
\]

(4.33)
solve the initial value problem (4.9)–(4.10) on $[\alpha, \infty)$ with (4.11).

**Proof.** Let $R(t)$ be the orthogonal projector in (2.4) associated with $(X, U)$. From Proposition 3.11 we know that $R(t) = R_\alpha(t)$ on $[\alpha, \infty)$. By (2.7) the matrix $Q(t)$ then satisfies the equality $Q(t) = R_\alpha(t)U(t)X(t)'$ for every $t \in [\alpha, \infty)$. Moreover, using the identities in (1.3), (3.12), and (4.2) we obtain on $[\alpha, \infty)$ that

$$Q' = R_\alpha' UX' + R_\alpha U X' = (4.33)$$

$$= \left[ R_\alpha, A^T \right] UX' + R_\alpha (CX - A^T U) X' + R_\alpha U X' \quad (4.1)$$

$$= R_\alpha A^T UX' - A^T R_\alpha UX' + R_\alpha CR_\alpha - R_\alpha A^T UX' - R_\alpha UX'(A + BQ) \quad (4.2)$$

$$= -Q A - A^T Q - BQ + C. \quad (4.3)$$

Thus, the matrix $Q(t)$ solves the Riccati equation (R) on $[\alpha, \infty)$ and condition (1.4) holds. Furthermore, according to (3.12) in Proposition 3.11 the matrix $\Theta(t) = X(t)$ satisfies the first equation in (4.11) on $[\alpha, \infty)$ while applying (3.13) and (R) yields for the matrix $\Omega(t)$ the equality

$$\Omega' - A \Omega = (U - QX)' - A (U - QX) \quad (\text{R})$$

$$\Omega' - A \Omega = [C - (A + A^T) Q] X - (Q' X - QX') + A Q X \quad (\text{R})$$

$$= [C - (A + A^T) Q] X + (A + A^T) Q X - C X \quad (4.33)$$

on $[\alpha, \infty)$. Moreover, by using (4.5), (1.3), and the equalities $R_\alpha(t) Q(t) = Q(t)$ and $R_\alpha(t) X(t) = R_\alpha(t)$ for every $t \in [\alpha, \infty)$ the last two terms in (4.33) become

$$= (A + A^T) Q X - C X \quad (1.3)$$

$$= (A + A^T) R_\alpha Q X - R_\alpha CR_\alpha X \quad (4.5)$$

$$= R_\alpha (A + A^T) R_\alpha Q X - R_\alpha CX \quad (4.7)$$

$$= R_\alpha (A + A^T) Q X - R_\alpha CX \quad (4.35)$$

$$= -R_\alpha [C - (A + A^T) Q] X \quad (4.33)$$

on $[\alpha, \infty)$. By combining formulas (4.33) and (4.35) we obtain that

$$\Omega' - A \Omega = (I - R_\alpha) [C - (A + A^T) Q] X = (I - R_\alpha) [C - (A + A^T) Q] \Theta$$

on $[\alpha, \infty)$, showing the second equation in (4.9). Finally, from the first identity in (3.33) we have that $\text{Im} \Theta(\alpha) = \text{Im} X(\alpha) = \text{Im} R_\alpha(\alpha)$, while the second one together with the last formula in (2.8) give

$$R_\alpha(\alpha) \Omega(\alpha) = R_\alpha(\alpha) U(\alpha) - R_\alpha(\alpha) Q(\alpha) X(\alpha) = R_\alpha(\alpha) U(\alpha) - Q(\alpha) X(\alpha) = 0.$$ 

Hence, the inclusion $\text{Im} \Omega(\alpha) \subseteq \text{Ker} R_\alpha(\alpha)$ holds. On the other hand, with the aid of (3.33) and the fact that $(X, U)$ is a conjoined basis one can easily check that $\text{Ker} \Theta(\alpha) \cap \text{Ker} \Omega(\alpha) = \text{Ker} X(\alpha) \cap \text{Ker} U(\alpha) = \{0\}$, which is equivalent with the equality $\text{rank} (\Theta^T(\alpha), \Omega^T(\alpha))^T = n$. Therefore, the matrices $\Theta(\alpha)$ and $\Omega(\alpha)$ satisfy the conditions in (4.11) and the proof is complete. \[\square\]

**Remark 4.19.** Let $S_\alpha(t)$ be the matrix in (2.11) associated with the conjoined basis $(X, U)$ on $[\alpha, \infty)$ and let $F_\alpha(t)$ be the matrix in (4.18), which corresponds to
the Riccati quotient $Q(t)$ on $[\alpha, \infty)$. The identities in (3.14) and (4.20) then give
\[
S_\alpha(t) = \int_\alpha^t X^1(s) B(s) X^1T(s) \, ds \overset{(3.14)}{=} \int_\alpha^t X^1(\alpha) \Phi^{-1}_\alpha(s) B(s) \Phi^{-1}_\alpha(s) X^1T(\alpha) \, ds
\]
\[
\overset{(1.2)}{=} X^1(\alpha) \int_\alpha^t \Phi^{-1}_\alpha(s) B(s) \Phi^{-1}_\alpha(s) \, ds X^1T(\alpha)
\]
\[
\overset{(4.20)}{=} X^1(\alpha) F_\alpha(t) X^1T(\alpha)
\]
for all $t \in [\alpha, \infty)$, where $\Phi_\alpha(t)$ is the fundamental matrix in (4.13). On the other hand, with the aid of the equalities $X(\alpha)X^1(\alpha) = R_\alpha(t) = X^1T(\alpha)X^T(\alpha)$ and $F_\alpha(t) = R_\alpha(t)F_\alpha(t)R_\alpha(\alpha)$ on $[\alpha, \infty)$ expression (4.36) yields the formula
\[
F_\alpha(t) = R_\alpha(t)F_\alpha(t)R_\alpha(\alpha) = X(\alpha)X^1(\alpha)F_\alpha(t)X^1T(\alpha)X^T(\alpha)
\]
\[
\overset{(4.36)}{=} X(\alpha)S_\alpha(t)X^T(\alpha).
\]
(4.37)

Furthermore, let $P, P_{S_\alpha}(t)$, and $P_{S_{\alpha\infty}}$ be the matrices in (2.4) and (2.12) associated with $(X, U)$. By combining (4.37) with the identities $X^T(\alpha)X^1(\alpha) = P$ and $S_\alpha(t)P = S_\alpha(t)$ we get $F_\alpha(t)X^1T(\alpha) = X(\alpha)S_\alpha(t)P = X(\alpha)S_\alpha(t)$, which in turn through (2.24) implies that
\[
\text{Im } F_\alpha(t) = \text{Im } X(\alpha)S_\alpha(t) \overset{(2.12)}{=} \text{Im } X(\alpha)P_{S_{\alpha}(t)} \overset{(2.24)}{=} \left(\Lambda_\alpha[\alpha, \infty]\right)_1,
\]
where the subspace $\Lambda_\alpha[\alpha, \infty]$ is defined in Section 2. Note that equality (4.38) is in a full agreement with the monotonicity of the subspace Ker $F_\alpha(t)$ in Remark 4.11(i).

Moreover, by using the relation in (2.22) we obtain
\[
\text{Im } F_\alpha(t) \overset{(4.38)}{=} \left(\Lambda_\alpha[\alpha, \infty]\right)_1 \overset{(2.22)}{=} \left(\Lambda_\alpha[\alpha, \infty]\right)_1 \text{ on } (\tau_\alpha, \infty),
\]
(4.39)

where the point $\tau_\alpha$ is defined in (2.23).

Remark 4.20. Based on Theorem 4.18, the result in Theorem 4.16 enables to determine all the solutions $Q(t)$ of the Riccati equation $(R)$ on $[\alpha, \infty)$, for which the matrix $R_\alpha(t)Q(t)R_\alpha(t)$ is the (symmetric) Riccati quotient associated with the conjoined basis $(X, U)$ on $[\alpha, \infty)$. More precisely, if $\beta \in [\alpha, \infty)$ is a given point and $Q(t)$ is a solution of $(R)$ defined on a neighborhood of $\beta$, then the matrix $Q(t)$ solves $(R)$ on the whole interval $[\alpha, \infty)$ and the matrix $R_\alpha(t)Q(t)R_\alpha(t)$ is the Riccati quotient associated with $(X, U)$ for every $t \in [\alpha, \infty)$ if and only if the matrix $R_\alpha(\beta)Q(\beta)R_\alpha(\beta)$ is the Riccati quotient for $(X, U)$ at the point $\beta$. In addition, from Remarks 4.11(ii) and 4.19 it follows that the matrix $F_\alpha(t)$ in (4.18), which corresponds to every such a solution $Q(t)$, satisfies formulas (4.36)–(4.39).

Theorem 4.21. Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ with the orthogonal projector $R_\alpha(t)$ in (3.11) and let $Q(t)$ be a solution of the Riccati equation $(R)$ on $[\alpha, \infty]$ such that the matrix $R_\alpha(t)Q(t)R_\alpha(t)$ is symmetric on $[\alpha, \infty)$. Moreover, let $(\Theta, \Omega)$ be a solution of (4.9)–(4.11) on $[\alpha, \infty)$ and define the matrices
\[
X(t) := \Theta(t), \quad U(t) := Q(t)\Theta(t) + \Omega(t), \quad t \in [\alpha, \infty).
\]
(4.40)

Then the following statements hold.

(i) The pair $(X, U)$ is a conjoined basis of $(H)$ such that $(X, U)$ has a constant kernel on $[\alpha, \infty)$ and belongs to the genus $\mathcal{G}$.

(ii) The matrix $R_\alpha(t)Q(t)R_\alpha(t)$ is the Riccati quotient in (2.7) associated with $(X, U)$ on $[\alpha, \infty)$, i.e., the equality $R_\alpha(t)Q(t)R_\alpha(t) = R(t)U(t)X^1(t)$ holds for all $t \in [\alpha, \infty)$, where $R(t)$ is the corresponding projector in (2.4).
Proposition 4.9 we know that the matrix \( \Theta(\cdot) \) has constant kernel on \([\alpha, \infty)\) and \( \operatorname{Im} \Theta(t) = \operatorname{Im} R_\varphi(t) \) for every \( t \in [\alpha, \infty) \). Thus, the matrix \( X(t) \) has constant kernel on \([\alpha, \infty)\) and \( \operatorname{Im} X(t) = R_\varphi(t) \) for all \( t \in [\alpha, \infty) \). And since \( \Theta(t) \) solves the first equation in (4.9) on \([\alpha, \infty)\), we have that

\[
X'(t) = [A(t) + B(t) Q(t)] X(t), \quad t \in [\alpha, \infty).
\]

(4.41)

Moreover, by using (1.3), (4.40), and the inclusion in (4.7) we obtain the formula

\[
B(t) U(t) = (1.3), (4.40), (4.47) \quad B(t) Q(t) X(t) + B(t) R_\varphi(t) \Omega(t) = B(t) Q(t) X(t)
\]

(4.42)

for every \( t \in [\alpha, \infty) \). Combining identities (4.41) and (4.42) with (4.3) and the equality \( R_\varphi(t) X(t) = X(t) \) for every \( t \in [\alpha, \infty) \) then yields on \([\alpha, \infty)\) that

\[
X' = A X + B Q X \quad (4.42) = A R_\varphi X + B U \quad (4.3) = A R_\varphi X + BU = AX + BU.
\]

(4.43)

Next we derive some additional properties of the matrices \( \Theta(t) \) and \( \Omega(t) \), which will simplify our calculations. In particular, by \((R)\) and the first equation in (4.9) we have on \([\alpha, \infty)\) that

\[
(Q \Theta)' (R) = (4.9) \quad (C - Q A - A^T Q - Q B Q) \Theta + Q (A + B Q) \Theta = (C - A^T Q) \Theta.
\]

(4.44)

On the other hand, by (1.3) and the identities \( R_\varphi(t) \Omega(t) = 0 \) and \( R_\varphi(t) \Theta(t) = \Theta(t) \) for all \( t \in [\alpha, \infty) \), the second equation in (4.9) reads on \([\alpha, \infty)\) as

\[
\Omega \quad (4.9), (1.3) \quad [AR_\varphi - A^T (I - R_\varphi)] \Omega + [C - (A + A^T) Q] \Theta - R_\varphi [C - (A + A^T) Q] \Theta
\]

\[
= -A^T \Omega + [C - A^T Q] \Theta - A Q \Theta - R_\varphi C R_\varphi \Theta + R_\varphi (A + A^T) Q \Theta
\]

\[
(1.3) \quad -A^T \Omega + [C - A^T Q] \Theta - C \Theta + [R_\varphi (A + A^T) - A] Q \Theta
\]

\[
(1.3) \quad -A^T \Omega + [C - A^T Q] \Theta - [C - A^T Q] \Theta.
\]

(4.45)

Now by using (4.44) and (4.45) we obtain that the matrix \( U(t) \) in (4.40) satisfies

\[
U' = (Q \Theta)' + \Omega' = (C - A^T Q) \Theta - A^T \Omega + [C - A^T Q] \Theta - [C - A^T Q] \Theta
\]

\[
= C \Theta - A^T (Q \Theta + \Omega) \quad (4.40) = CX - A^T U \quad \text{on} \quad [\alpha, \infty).
\]

(4.46)

Hence, equalities (4.43) and (4.46) show that the pair \((X, U)\) solves system \((H)\) on \([\alpha, \infty)\). Moreover, the matrix

\[
X^T(t) U(t) = \Theta^T(t) Q(t) \Theta(t) + \Theta^T(t) \Omega(t) = \Theta^T(t) R_\varphi(t) Q(t) R_\varphi(t) \Theta(t)
\]

is symmetric and the subspace \( \operatorname{Ker} X(t) \cap \operatorname{Ker} U(t) = \operatorname{Ker} \Theta(t) \cap \operatorname{Ker} \Omega(t) = \{0\} \) for every \( t \in [\alpha, \infty) \), both by Proposition 4.9. Therefore, the solution \((X, U)\) is a conjoined basis with constant kernel on \([\alpha, \infty)\). And since the equality \( \operatorname{Im} X(t) = \operatorname{Im} R_\varphi(t) \) holds for every \( t \in [\alpha, \infty) \), we conclude that \((X, U)\) belongs to the genus \( \mathcal{G} \). For the proof of part (ii) we note that the orthogonal projector \( R(t) \) in (2.4) associated with \((X, U)\) satisfies \( R(t) = R_\varphi(t) \) for all \( t \in [\alpha, \infty) \). In particular, by using the identities \( \Theta(t) \Theta^T(t) = R_\varphi(t) \) and \( R_\varphi(t) \Omega(t) = 0 \) on \([\alpha, \infty)\) we obtain that

\[
R(t) U(t) X'(t) \quad (4.40) = R_\varphi(t) Q(t) \Theta(t) \Theta^T(t) + R_\varphi(t) \Omega(t) \Theta^T(t) = R_\varphi(t) Q(t) R_\varphi(t)
\]

for every \( t \in [\alpha, \infty) \). Thus, the matrix \( R_\varphi(t) Q(t) R_\varphi(t) \) is the Riccati quotient in (2.7) associated with \((X, U)\) on \([\alpha, \infty)\) and the proof is complete. \( \square \)
Remark 4.22. We note that the matrices $F_\alpha(t)$ and $S_\alpha(t)$ in (4.18) and (2.11) associated with the matrix $Q(t)$ and the conjoined basis $(X, U)$ in Theorem 4.21, respectively, satisfy the identities in (4.36) and (4.37). This follows directly from Theorem 4.18 and Remark 4.11(ii).

Remark 4.23. Let $\mathcal{G}$ be a genus of conjoined basis of $(H)$ with the associated matrix $R_\xi(t)$ in (3.11) and let $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ be a given interval. The results in Theorems 4.18 and 4.21 provide a correspondence between the set of all symmetric solutions $Q(t)$ of the Riccati equation $(R)$ on $[\alpha, \infty)$ satisfying condition (1.4). More precisely, for every such a conjoined basis $(X, U)$ of $(H)$ with constant kernel on $[\alpha, \infty)$, which belong to the genus $\mathcal{G}$, its Riccati quotient $Q(t)$ in (2.7) is a symmetric solution of $(R)$ on $[\alpha, \infty)$ with $\text{Im} Q(t) \subseteq \text{Im} R_\xi(t)$ for every $t \in [\alpha, \infty)$, as we claim in Theorem 4.18. Conversely, if $Q(t)$ is a symmetric solution of $(R)$ on $[\alpha, \infty)$ such that $\text{Im} Q(t) \subseteq \text{Im} R_\xi(t)$ for every $t \in [\alpha, \infty)$, then there exists a conjoined basis $(X, U)$ of $(H)$ from the genus $\mathcal{G}$ with constant kernel on $[\alpha, \infty)$ such that the matrix $Q(t)$ is its corresponding Riccati quotient in (2.7), by Theorem 4.21, and the equality $R_\xi(t)Q(t)R_\xi(t) = Q(t)$ on $[\alpha, \infty)$. In addition, every such a conjoined basis $(X, U)$ has the form of (4.40) for some solution $(\Theta, \Omega)$ of (4.9)–(4.11) on $[\alpha, \infty)$. Finally, the observations in Remark 4.8 then imply that the conjoined basis $(X, U)$ is uniquely determined up to a right nonsingular multiple by the genus $\mathcal{G}$ and the matrix $Q(t)$.

Remark 4.24. The representation of the solution $Q(t)$ of $(R)$ in (4.32) corresponds to the results in Theorems 4.18 and 4.21 with the maximal genus $\mathcal{G} = G_{\max}$. In this case $R_\xi(t) \equiv I$, and (1.3) yields that the Riccati equations $(R)$ and $(\bar{R})$ coincide.

5. Inequalities for Riccati quotients in given genus. In this section we derive a mutual representation of the Riccati quotients corresponding to conjoined bases of $(H)$ from a given genus $\mathcal{G}$ (Theorem 5.3). This representation is then utilized for obtaining inequalities between two Riccati quotients (Corollary 5.5). The results presented in this section essentially generalize the discussion in [7, pg. 54] to possibly uncontrollable systems $(H)$.

First we prove an auxiliary property of the image of the matrix $F_\alpha(t)$ in (4.18).

Lemma 5.1. Assume (1.1). Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ with the matrix $R_\xi(t)$ in (3.11) and let $Q(t)$ be a solution of the Riccati equation $(R)$ on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that the matrix $R_\xi(t)Q(t)R_\xi(t)$ is symmetric on $[\alpha, \infty)$. Moreover, let $F_\alpha(t)$ be the matrix in (4.18), which corresponds to $Q(t)$, and $R_{\alpha,\infty}(t)$ be the orthogonal projector defined in (3.7). Then

$$\text{Im } F_\alpha(t) \subseteq \text{Im } R_{\alpha,\infty}(\alpha) \quad \text{for every } t \in [\alpha, \infty).$$

(5.1)

Proof. Let $(X, U)$ be a conjoined basis of $(H)$ from the genus $\mathcal{G}$, which corresponds to the matrix $Q(t)$ through Theorem 4.21. It follows that $(X, U)$ has constant kernel on $[\alpha, \infty)$ and the matrix $X(t)$ satisfies the equality $\text{Im } X(t) = \text{Im } R_\xi(t)$ for every $t \in [\alpha, \infty)$. Moreover, the matrix $R_\xi(t)Q(t)R_\xi(t)$ is the Riccati quotient in (2.7) associated with $(X, U)$ on $[\alpha, \infty)$, by Theorem 4.21(iii). Let $S_\alpha(t)$ be the $S$-matrix in (2.11) corresponding to the conjoined basis $(X, U)$ on $[\alpha, \infty)$. From Remark 4.11(ii) we know that $F_\alpha(t)$ is the $F$-matrix in (4.18) associated with $R_\xi(t)Q(t)R_\xi(t)$. Thus, by combining (3.8) and (4.39) we obtain the identity

$$\text{Im } F_\alpha(t) \overset{(4.39)}{=} (\Lambda_\alpha[\alpha, \infty])^{-1} \overset{(3.8)}{=} \text{Im } R_{\alpha,\infty}(\alpha) \quad \text{on } (\tau_{\alpha, \infty}, \infty),$$

(5.2)
where the point $\tau_{\alpha, \infty}$ is defined in (2.23). And since by Remark 4.11(i) the subspace $\text{Im} F_\alpha(t)$ is nondecreasing on $[\alpha, \infty)$, the inclusion in (5.1) now immediately follows from (5.2). The proof is complete.

Remark 5.2. (i) Let $S_\alpha(t)$ and $P_{S_\alpha, \infty}$ be the matrices in (2.11) and (2.12) which correspond to the conjoined basis $(X, U)$ on $[\alpha, \infty) \subseteq [\alpha_0, \infty)$. From (2.25) and (5.2) it follows that

$$\text{Im} S_\alpha(t) = \text{Im} P_{S_\alpha, \infty}, \quad \text{Im} F_\alpha(t) = \text{Im} R_{\lambda, \infty}(\alpha),$$

and

$$\text{rank} S_\alpha(t) = \text{rank} P_{S_\alpha, \infty} = n - d(\alpha, \infty) = \text{rank} R_{\lambda, \infty}(\alpha) = \text{rank} F_\alpha(t)$$

on $(\tau_{\alpha, \infty}, \infty)$. Moreover, the matrices $S_\alpha(t)$ and $F_\alpha(t)$ satisfies the identities

$$\begin{align*}
S_\alpha^\dagger(t) &= P_{S_\alpha, \infty} X^T(\alpha) F_\alpha^\dagger(t) X(\alpha) P_{S_\alpha, \infty}, \\
F_\alpha^\dagger(t) &= R_{\lambda, \infty}(\alpha) X^T(\alpha) S_\alpha^\dagger(t) X(\alpha) R_{\lambda, \infty}(\alpha)
\end{align*}$$

for every $t \in (\tau_{\alpha, \infty}, \infty)$, which can verify by direct computation with the aid of (3.9)–(3.10), (4.36), (4.37), and (5.3).

(ii) Furthermore, upon taking $t \to \infty$ in (5.5) we obtain the formulas

$$\begin{align*}
T_\alpha &= P_{S_\alpha, \infty} X^T(\alpha) D_\alpha X(\alpha) P_{S_\alpha, \infty}, \\
D_\alpha &= R_{\lambda, \infty}(\alpha) X^T(\alpha) T_\alpha X(\alpha) R_{\lambda, \infty}(\alpha)
\end{align*}$$

where the matrices $T_\alpha$ and $D_\alpha$ are defined in (2.13) and (4.19), respectively. The equalities in (5.6) then yield that $\text{Im} D_\alpha \subseteq \text{Im} R_{\lambda, \infty}(\alpha)$ and $\text{rank} T_\alpha = \text{rank} D_\alpha$.

In particular, combining the last formula with Remark 2.8(ii) and the fact that $d(\alpha, \infty) = d_\infty$ implies that $\text{rank} D_\beta$ is constant with respect to $\beta \in [\alpha, \infty)$.

In the following main result of this section we present a representation of two Riccati quotients to two conjoined bases of $(H)$ from a genus $\mathcal{G}$. This result will be utilized in the classification of all distinguished solutions of $(\mathcal{R})$ at infinity in Section 7. When $\mathcal{G} = \mathcal{G}_{\text{max}}$ is the maximal genus (in particular, when system $(H)$ is controllable), this representation coincides with the statement in Proposition 4.12. We note that for a given genus $\mathcal{G}$ we now compare those solutions $Q(t)$ and $\tilde{Q}(t)$ of $(\mathcal{R})$, which are Riccati quotients according to their definition in (2.7). However, the Riccati equation $(\mathcal{R})$ may also have other solutions, which are not of this particular form.

Theorem 5.3. Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ and let $(X, U)$ and $(\tilde{X}, \tilde{U})$ be two conjoined bases of $(H)$ with constant kernel on $[\alpha, \infty) \subseteq [\alpha_0, \infty)$ belonging to $\mathcal{G}$. Moreover, let $P$ and $S_\alpha(t)$ be the matrices in (2.5) and (2.11) associated with $(X, U)$. Suppose that $(\tilde{X}, \tilde{U})$ is expressed in terms of $(X, U)$ via matrices $M$ and $N$ as in Proposition 2.10. Then the Riccati quotients $Q(t)$ and $\tilde{Q}(t)$ in (2.7) corresponding to $(X, U)$ and $(\tilde{X}, \tilde{U})$, respectively, satisfy

$$\tilde{Q}(t) = Q(t) + X^T(t) N [P M + S_\alpha(t) N] X^\dagger(t), \quad t \in [\alpha, \infty).$$

Proof. Let $R_\gamma(t)$ be the orthogonal projector in (3.11) and let $R(t)$ and $\tilde{R}(t)$ be the matrices in (2.4), which correspond to the conjoined bases $(X, U)$ and $(\tilde{X}, \tilde{U})$, respectively. According to Proposition 3.11 and the second identity in (2.4) we then have the equalities

$$X(t) X^\dagger(t) \overset{(2.4)}{=} R(t) = R_\gamma(t) = \tilde{R}(t) \overset{(2.4)}{=} \tilde{X}(t) \tilde{X}^\dagger(t)$$

and

$$Q(t) \overset{(2.11)}{=} X^\dagger(t) S_\alpha(t) X(t) \overset{(2.5)}{=} P S_\alpha(t)$$

and

$$\tilde{Q}(t) \overset{(2.11)}{=} \tilde{X}(t) \tilde{X}^\dagger(t) \tilde{S}_\alpha(t) \tilde{X}(t) \overset{(2.5)}{=} \tilde{P} \tilde{S}_\alpha(t).$$

Since $R(t)$ and $\tilde{R}(t)$ coincide, we obtain the desired formula.
for all \( t \in [\alpha, \infty) \). Moreover, the symmetry of the matrices \( Q(t), \dot{Q}(t), R(t), \) and \( \dot{R}(t) \) on \([\alpha, \infty)\), the fact that the matrix \( N \) is the Wronskian of \((X, U)\) and \((\dot{X}, \dot{U})\) by Remark 2.11(i), and the equations in (5.8) imply that (we omit the argument \( t \))

\[
\dot{Q} - Q = 2(\dot{R} \dot{U} \dot{X}^\dagger - RUX^\dagger) + 5.9 \Rightarrow R \dot{U} \dot{X}^\dagger - X^\dagger U^T \dot{R}
\]

\( = X^\dagger (X^T \dot{U} - U^T \dot{X}) \dot{X}^\dagger = X^\dagger N \dot{X}^\dagger \) (5.9)

on \([\alpha, \infty)\). Finally, inserting the expression for the matrix \( \dot{X}^\dagger(t) \) in (2.37) into the equality in (5.9) yields formula (5.7) on \([\alpha, \infty)\) and the proof is complete. \( \square \)

**Remark 5.4.** By substituting the matrix \( X^\dagger(t) \) instead of \( \dot{X}^\dagger(t) \) in (5.9) we get another formula for the difference \( Q(t) - \dot{Q}(t) \). Namely, inserting the second identity in (2.37) into (5.9) and using the equality \( PN = N \) and the symmetry of \( S_\alpha(t) \) on \([\alpha, \infty)\) yields the formula

\[
\dot{Q}(t) - Q(t) = \dot{X}^\dagger(t) [PM + S_\alpha(t) N]^T N \dot{X}^\dagger(t)
\]

\( = \dot{X}^\dagger(t) [M^T N + N^T S_\alpha(t) N] \dot{X}^\dagger(t) \) (5.10) for every \( t \in [\alpha, \infty) \). In addition, if \( \dot{P} \) is the projector in (2.5) associated with the conjoined basis \((\dot{X}, \dot{U})\), then by the identities \( X^T \dot{X}^\dagger = \dot{P} = \dot{X}^\dagger \dot{X} \) on \([\alpha, \infty)\), \( N \dot{P} = N \), and \( M^T N = N^T M \) formula (5.10) implies (suppressing the argument \( t \))

\[
\dot{X}^T [\dot{Q} - Q] \dot{X} = \dot{X}^T \dot{X}^\dagger [M^T N + N^T S_\alpha(t) N] \dot{X}^\dagger \dot{X}
\]

\( = \dot{P} [M^T N + N^T S_\alpha(t) N] \dot{P} = M^T N + N^T S_\alpha(t) N \) (5.11) on \([\alpha, \infty)\). Moreover, from (5.10) and (5.11) it immediately follows that

\[
\begin{align*}
\text{rank} \left[ \dot{Q}(t) - Q(t) \right] &= \text{rank} \left[ M^T N + N^T S_\alpha(t) N \right], \\
\text{ind} \left[ \dot{Q}(t) - Q(t) \right] &= \text{ind} \left[ M^T N + N^T S_\alpha(t) N \right]
\end{align*}
\]

(5.12) for every \( t \in [\alpha, \infty) \). In particular, since \( S_\alpha(\alpha) = 0 \), by evaluating (5.12) at \( t = \alpha \) we obtain the equalities

\[
\text{rank} \left[ \dot{Q}(\alpha) - Q(\alpha) \right] = \text{rank} M^T N, \quad \text{ind} \left[ \dot{Q}(\alpha) - Q(\alpha) \right] = \text{ind} M^T N.
\]

Formula (5.7) in Theorem 5.3 yields the following inequalities between two Riccati quotients associated with two conjoined bases from the genus \( \mathcal{G} \).

**Corollary 5.5.** With the assumptions and notation of Theorem 5.3, the Riccati quotients \( Q(t) \) and \( \dot{Q}(t) \) satisfy the formulas

\[
\text{rank} \left[ \dot{Q}(t) - Q(t) \right] \equiv \text{rank} N, \quad \text{ind} \left[ \dot{Q}(t) - Q(t) \right] \equiv \text{ind} NM^{-1}
\]

(5.14) on \([\alpha, \infty)\). Moreover, the following statements hold.

(i) The inequality \( \dot{Q}(t) \geq Q(t) \) holds for all \( t \in [\alpha, \infty) \) if and only if \( NM^{-1} \geq 0 \).

(ii) The inequality \( \dot{Q}(t) \leq Q(t) \) holds for all \( t \in [\alpha, \infty) \) if and only if \( NM^{-1} \leq 0 \).

(iii) The inequality \( \dot{Q}(t) > Q(t) \), resp. \( \dot{Q}(t) < Q(t) \), holds on the subspace \( \text{Im} R_c(t) \) for all \( t \in [\alpha, \infty) \) if and only if the inequality \( NM^{-1} > 0 \), resp. \( NM^{-1} < 0 \), holds on \( \text{Im} P \).
Proof. From Theorem 4.18 we know that the matrices \( Q(t) \) and \( \dot{Q}(t) \) are symmetric solutions of \( (R) \) on \([\alpha, \infty)\). Therefore, the quantities \( \text{rank} [\dot{Q}(t) - Q(t)] \) and \( \text{ind} [\dot{Q}(t) - Q(t)] \) are constant on \([\alpha, \infty)\), by Corollary 4.14. According to (2.31) the matrix \( M \) is nonsingular and the matrix \( M^T N \) is symmetric. Thus, the matrix \( M^T - M^T N M^{-1} = NM^{-1} \) is symmetric and
\[
\text{rank } M^T N = \text{rank } NM^{-1} = \text{rank } N, \quad \text{ind } M^T N = \text{ind } NM^{-1}.
\]
(5.15)
The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)

The formulas in (5.14) now follow from (5.13) and (5.15). Furthermore, assertions (i) and (ii) are direct consequences of the equalities in (5.14). For the proof of statement (iii) we note that the matrix \( \tilde{R} \) and (ii) are direct consequences of the equalities in (5.14). For the proof of

\[
\begin{align*}
\text{rank } M^T N &= \text{rank } NM^{-1} = \text{rank } N, \\
\text{ind } M^T N &= \text{ind } NM^{-1}.
\end{align*}
\]

(5.15)
and define the functions (we omit the argument $t$)
\[
\mathcal{E}_1 := [R_{\alpha}^+ Q' + QA + A^T Q + QBQ - C] R_{\alpha}, \\
\mathcal{E}_2 := [R_{\alpha}^+ Q' + QA + A^T Q + QBQ - C] R_{\alpha},
\]
(6.2) on $[\alpha, \infty)$. By using (1.3) and (4.3) together with the identity $[R_{\alpha}(t)]^2 = R_{\alpha}(t)$ for every $t \in [\alpha, \infty)$ we then obtain (suppressing the argument $t$)
\[
\mathcal{E}_1 \stackrel{(6.2)}{=} R_{\alpha} Q'R_{\alpha} + R_{\alpha} Q AR_{\alpha} + R_{\alpha} A^T Q R_{\alpha} + R_{\alpha} QBQ R_{\alpha} - R_{\alpha} CR_{\alpha}
\]
\[
\stackrel{(1.3),(4.3)}{=} R_{\alpha} Q'R_{\alpha} + R_{\alpha} Q AR_{\alpha} + R_{\alpha} A^T Q R_{\alpha} + R_{\alpha} QBQ R_{\alpha} - R_{\alpha} CR_{\alpha} \stackrel{(6.2)}{=} \mathcal{E}_2
\]
on $[\alpha, \infty)$, which proves directly the statement of the lemma. 

**Remark 6.2.** It is easy to see that for a given orthogonal projector $R_{\alpha}(t)$ in (3.11) any matrix $Q(t)$, which solves the Riccati equation ($\mathcal{R}$) on some subinterval $[\alpha, \infty) \subseteq [\alpha, \infty)$, satisfies also the implicit Riccati equation (6.1) on $[\alpha, \infty)$.

Following the above remark, we now establish the opposite relation between the solutions of the implicit Riccati equation (6.1) and the Riccati equation ($\mathcal{R}$).

**Theorem 6.3.** Let $\mathcal{G}$ be a genus of conjoined bases of (H) with the orthogonal projector $R_{\alpha}(t)$ in (3.11). Let $Q(t)$ be a solution of the implicit Riccati equation (6.1) on $[\alpha, \infty) \subseteq [\alpha, \infty)$. Then the matrix $R_{\alpha}(t) Q(t) R_{\alpha}(t)$ solves ($\mathcal{R}$) on $[\alpha, \infty)$.

**Proof.** Let $R_{\alpha}(t)$ and $Q(t)$ be as in the theorem. With the aid of (4.1), (4.2), (6.1), and the equalities (suppressing the argument $t$) $R_{\alpha} CR_{\alpha} = C$ and $B = R_{\alpha} BR_{\alpha}$ on $[\alpha, \infty)$ we get
\[
(R_{\alpha} Q R_{\alpha})' = R_{\alpha} Q R_{\alpha} + R_{\alpha} Q' R_{\alpha} + R_{\alpha} QR_{\alpha}'
\]
\[
\stackrel{(4.1),(4.2)}{=} [R_{\alpha}, A^T] Q R_{\alpha} + R_{\alpha} Q' R_{\alpha} + R_{\alpha} Q [A, R_{\alpha}]
\]
\[
\stackrel{(6.1)}{=} C - (R_{\alpha} Q R_{\alpha}) A - A^T (R_{\alpha} Q R_{\alpha}) - (R_{\alpha} Q R_{\alpha}) B (R_{\alpha} Q R_{\alpha})
\]
on $[\alpha, \infty)$. Hence, the matrix $R_{\alpha}(t) Q(t) R_{\alpha}(t)$ solves ($\mathcal{R}$) on $[\alpha, \infty)$. 

The results in Theorem 6.3 and Lemma 6.1 yield the following.

**Corollary 6.4.** Let $\mathcal{G}$ be a genus of conjoined bases of (H) and let $R_{\alpha}(t)$ be the corresponding orthogonal projector in (3.11). Moreover, let $Q(t)$ be a symmetric matrix defined on $[\alpha, \infty) \subseteq [\alpha, \infty)$ such that condition (1.4) holds. Then the following statements are equivalent,

(i) The matrix $Q(t)$ solves the Riccati equation ($\mathcal{R}$) on $[\alpha, \infty)$.

(ii) The matrix $Q(t)$ solves the implicit Riccati equation (6.1) on $[\alpha, \infty)$.

(iii) The matrix $Q(t)$ solves the implicit Riccati equation (1.6) on $[\alpha, \infty)$.

**Proof.** The implication (i) $\Rightarrow$ (ii) follows by Remark 6.2. The equivalence of the assertions in (ii) and (iii) is a direct consequence of Lemma 6.1. Now assume (ii), i.e., suppose that the matrix $Q(t)$ is a solution of (6.1) on $[\alpha, \infty)$. The result of Theorem 6.3 and the identities
\[
R_{\alpha}(t) Q(t) R_{\alpha}(t) \stackrel{(1.4)}{=} Q(t) R_{\alpha}(t) = Q(t) R_{\alpha}(t) = [R_{\alpha}(t) Q(t)]^T \stackrel{(1.4)}{=} Q(t)
\]
for $t \in [\alpha, \infty)$ then imply that $Q(t)$ solves ($\mathcal{R}$) on $[\alpha, \infty)$, showing (i).
7. Distinguished solutions at infinity. In this section we study, for a given genus $\mathcal{G}$, symmetric solutions of the Riccati equation ($\mathcal{R}$), which correspond to principal solutions of $(H)$ at infinity belonging to the genus $\mathcal{G}$. This correspondence is based on the results in Theorems 4.18 and 4.21 and in Remark 4.20. We introduce the notion of a distinguished solution of ($\mathcal{R}$) at infinity (Definition 7.1) and prove its main properties. In particular, we establish the results about distinguished solutions of ($\mathcal{R}$) at infinity regarding their relationship to principal solutions at infinity (Theorems 7.4 and 7.5) and to the nonoscillation of system $(H)$ at infinity (Theorem 7.8), their interval of existence (Theorem 7.13), their mutual classification within the genus $\mathcal{G}$ (Theorem 7.15), and their minimality in a suitable sense (Theorems 7.16 and 7.18).

It may be surprising that these results comply with the known theory of distinguished solutions of the Riccati equation ($\mathcal{R}$) for a controllable system $(H)$ only partially. In many aspects the presented theory for general uncontrollable system $(H)$ is substantially different. This is related to the nature of the problem, since for each genus $\mathcal{G}$ of conjoined bases of $(H)$ there is a different Riccati equation ($\mathcal{R}$), but even within one genus $\mathcal{G}$ there may be many distinguished solutions of ($\mathcal{R}$) at infinity. We discuss these issues in Remark 7.25 at the end of this section. We note that the true uniqueness and minimality of the distinguished solution of ($\mathcal{R}$) at infinity is satisfied only in the minimal genus $\mathcal{G}_{\text{min}}$ (see Theorem 7.23).

The following definition extends the notion of a distinguished solution (also called a principal solution) of ($\mathcal{R}$) at infinity for a controllable system $(H)$ in [7, pg. 53].

**Definition 7.1.** Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ with the orthogonal projector $R_{\mathcal{G}}(t)$ in (3.11). A symmetric solution $\hat{Q}(t)$ of the Riccati equation ($\mathcal{R}$) is said to be a distinguished solution at infinity if the matrix $\hat{Q}(t)$ is defined on an interval $[\alpha, \infty) \subseteq [\alpha_{\infty}, \infty)$ and its corresponding matrix $\hat{F}_\alpha(t)$ in (4.18) satisfies $\hat{F}_\alpha^\dagger(t) \to 0$ as $t \to \infty$.

The notion in Definition 7.1 also extends the distinguished solution of ($\mathcal{R}$) introduced by W. T. Reid in [21, Section IV] and [23, Section 2.7], which in our context corresponds to the maximal genus $\mathcal{G} = \mathcal{G}_{\text{max}}$ (for which $R_{\mathcal{G}}(t) \equiv I$).

**Remark 7.2.** When it is clear from the context, we will often drop the term “at infinity” in the terminology in Definition 7.1. We also remark that a distinguished solution of the Riccati equation ($\mathcal{R}$) associated with the genus $\mathcal{G}$ is also defined by the property $\hat{D}_\alpha = 0$ with the matrix $\hat{D}_\alpha$ in (4.19) corresponding to $\hat{F}_\alpha(t)$.

In the next auxiliary statement we show that the property of being a distinguished solution of ($\mathcal{R}$) is invariant under the multiplication by the orthogonal projector $R_{\mathcal{G}}(t)$. This property will be utilized in the proofs of the subsequent main results.

**Lemma 7.3.** Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ with the matrix $R_{\mathcal{G}}(t)$ in (3.11). Let $Q(t)$ be a symmetric solution of the Riccati equation ($\mathcal{R}$) on the interval $[\alpha, \infty) \subseteq [\alpha_{\infty}, \infty)$. Then $Q(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$ if and only if the matrix $R_{\mathcal{G}}(t)Q(t)R_{\mathcal{G}}(t)$ is a distinguished solution of ($\mathcal{R}$) at infinity with respect to $[\alpha, \infty)$.

**Proof.** From Theorem 4.3 we know that the matrix $R_{\mathcal{G}}(t)Q(t)R_{\mathcal{G}}(t)$ solves ($\mathcal{R}$) on $[\alpha, \infty)$. And since by Remark 4.11(ii) the matrices $Q(t)$ and $R_{\mathcal{G}}(t)Q(t)R_{\mathcal{G}}(t)$ have the same $F$-matrices in (4.18) with respect to the interval $[\alpha, \infty)$, the statement follows directly from Definition 7.1. □
The following two results show that in the context of Theorems 4.18 and 4.21 the distinguished solutions of (R) correspond to the principal solutions of (H) at infinity from the genus \( G \).

**Theorem 7.4.** Let \( G \) be a genus of conjoined bases of (H) and \( R_\alpha(t) \) be the projector in (3.11). Moreover, let \( \hat{Q}(t) \) be a distinguished solution of (R) at infinity with respect to the interval \([\alpha, \infty) \subseteq [\alpha, \infty)\). Then every conjoined basis \((\hat{X}, \hat{U})\) of (H), which is associated with \( \hat{Q}(t) \) on \([\alpha, \infty)\) via Theorem 4.21, is a principal solution of (H) at infinity with respect to \([\alpha, \infty)\) belonging to the genus \( G \).

**Proof.** Let \( R_\phi(t) \) and \( \hat{Q}(t) \) be as in the theorem. According to Remark 7.2 the matrix \( \hat{D}_\alpha \) in (4.19) corresponding to \( \hat{Q}(t) \) satisfies \( \hat{D}_\alpha = 0 \). Let \((\hat{X}, \hat{U})\) be a conjoined basis of (H), which is associated with the matrix \( \hat{Q}(t) \) on \([\alpha, \infty)\) via Theorem 4.21. Then \((\hat{X}, \hat{U})\) belongs to the genus \( G \) such that \((\hat{X}, \hat{U})\) has constant kernel on \([\alpha, \infty)\). Moreover, if \( \hat{T}_\alpha \) is the \( T \)-matrix in (2.13) associated with \((\hat{X}, \hat{U})\) through the matrix \( \hat{S}_\alpha \) in (2.11), then we have \( \text{rank} \hat{T}_\alpha = \text{rank} \hat{D}_\alpha = 0 \), by Remark 5.2(ii). Hence, \( \hat{T}_\alpha = 0 \) and \((\hat{X}, \hat{U})\) is a principal solution at infinity.

**Theorem 7.5.** Let \((\hat{X}, \hat{U})\) be a principal solution of (H) at infinity with respect to the interval \([\alpha, \infty)\), which belongs to a genus \( G \). Moreover, let \( \hat{Q}(t) \) be the Riccati quotient in (2.7) associated with \((\hat{X}, \hat{U})\) on \([\alpha, \infty)\). Then \( \hat{Q}(t) \) is a distinguished solution of the Riccati equation (R) at infinity with respect to \([\alpha, \infty)\).

**Proof.** By using Proposition 3.2(i) we have the equality \( d(\alpha, \infty) = d_\infty \), which means that \([\alpha, \infty) \subseteq [\alpha, \infty)\), by (2.19). Moreover, the matrix \( \hat{T}_\alpha \) in (2.13) associated with \((\hat{X}, \hat{U})\) satisfies \( \hat{T}_\alpha = 0 \). From Theorem 4.18 it follows that the matrix \( \hat{Q}(t) \) is a symmetric solution of the Riccati equation (R) on \([\alpha, \infty)\). Finally, if \( \hat{D}_\alpha \) is the matrix in (4.19), which corresponds to \( \hat{Q} \) through its \( F \)-matrix \( \hat{F}_\alpha(t) \) in (4.18), then \( \text{rank} \hat{D}_\alpha = \text{rank} \hat{T}_\alpha = 0 \), by Remark 5.2(ii). Thus, \( \hat{D}_\alpha = 0 \) and \( \hat{Q}(t) \) is a distinguished solution at infinity, by Remark 7.2.

**Remark 7.6.** We note that according to Theorem 4.18 the distinguished solution \( \hat{Q}(t) \) at infinity in Theorem 7.5 satisfies the additional property (1.4), i.e., the inclusion \( \text{Im} \hat{Q}(t) \subseteq R_\phi(t) \) for all \( t \in [\alpha, \infty) \). In particular, the latter relation together with the symmetry of the matrix \( \hat{Q}(t) \) on \([\alpha, \infty)\) yields the identity \( \hat{Q}(t) = R_\phi(t) \hat{Q}(t) R_\phi(t) \) for every \( t \in [\alpha, \infty) \). Moreover, from Lemma 7.3 it follows that every symmetric solution \( Q(t) \) of (R) on \([\alpha, \infty)\), for which the matrix \( R_\phi(t) Q(t) R_\phi(t) \) is the Riccati quotient in (2.7) associated with \((\hat{X}, \hat{U})\), is also a distinguished solution at infinity with respect to \([\alpha, \infty)\). In general, however, such a matrix \( Q(t) \) does not need to satisfy the inclusion in (1.4).

From Theorems 7.4 and 7.5 it follows that the property of the existence of a principal solution of (H) at infinity in the genus \( G \), as stated in [29, Theorem 7.12], transfers naturally to the existence of a distinguished solution at infinity of the associated Riccati equation (R).

**Corollary 7.7.** Let \( G \) be a genus of conjoined bases of (H) with the orthogonal projector \( R_\phi(t) \) in (3.11). Then there exists a principal solution of (H) at infinity belonging to the genus \( G \) if and only if there exists a distinguished solution of the Riccati equation (R) at infinity. In this case, the set of all Riccati quotients in (2.7), which correspond to the principal solutions \((\hat{X}, \hat{U})\) of (H) at infinity from the
genus $\mathcal{G}$, coincides with the set of all matrices $R_{G} \hat{Q} R_{G}$, where $\hat{Q}$ is a distinguished solution of $(R)$ at infinity.

Proof. The statement follows directly from Theorems 7.4 and 7.5 and from Remark 7.6.

In the following result we characterize the nonoscillation of system (H) in terms of the existence of a distinguished solution of the Riccati equation $(R)$ in a given (or every) genus $\mathcal{G}$. This corresponds to [29, Theorems 7.6 and 7.12] regarding the principal solutions of (H) at infinity.

**Theorem 7.8.** Assume (1.1). Then the following statements are equivalent.

(i) System (H) is nonoscillatory.

(ii) There exists a distinguished solution of equation $(R)$ for some genus $\mathcal{G}$.

(iii) There exists a distinguished solution of equation $(R)$ for every genus $\mathcal{G}$.

The proof of Theorem 7.8 is displayed below after the following two remarks.

**Remark 7.9.** The result in Theorem 7.8 justifies the development of the theory of genera of conjoined bases for possibly oscillatory system (H). Of course, assuming that system (H) is nonoscillatory, then it is sufficient to use the theory of genera of conjoined bases from [29, Section 6] and [31, Section 4] for the construction of distinguished solutions of the Riccati equation $(R)$ for a genus $\mathcal{G}$. It is the converse to this implication, which requires a more general approach, since in this case we need to define the coefficients of equation $(R)$ without the assumption of nonoscillation of system (H). This natural requirement was the initial motivation for the study presented in [27].

**Remark 7.10.** We note that the result in Theorem 7.8 remains valid also with the additional condition (1.4) for solutions $Q(t)$ of $(R)$ in parts (ii) and (iii). More precisely, system (H) is nonoscillatory if and only if there exists a distinguished solution of $(R)$ at infinity for some (and hence for every) genus $\mathcal{G}$, which satisfies condition (1.4) for all sufficiently large $t \in [\alpha_{\infty}, \infty)$. This observation follows directly from Lemma 7.3 and Theorem 7.8.

**Proof of Theorem 7.8.** If (H) is nonoscillatory, then by Remark 3.13 for any genus $\mathcal{G}$ of conjoined bases of (H) there exists a principal solution of (H) at infinity belonging to $\mathcal{G}$. In turn, there exists a distinguished solution of the Riccati equation $(R)$ at infinity for every genus $\mathcal{G}$, by Corollary 7.7. Moreover, assertion (iii) implies (ii) trivially. Finally, by using Corollary 7.7 once more, assertion (ii), that is, the existence of a distinguished solution of $(R)$ at infinity for some genus $\mathcal{G}$, means that there exists a principal solution of (H) at infinity, which belongs to $\mathcal{G}$. Since every principal solution is a nonoscillatory conjoined basis, system (H) is nonoscillatory, by Proposition 2.1. This shows the validity of (i) and completes the proof.

The next two results deal with the interval of existence of distinguished solutions of $(R)$. In particular, we determine the maximal interval of existence for each particular distinguished solution of $(R)$. Moreover, we show that this maximal interval is the same for all distinguished solutions of $(R)$ as well as for all genera $\mathcal{G}$.

**Theorem 7.11.** Assume (1.1) and let $\mathcal{G}$ be a genus of conjoined bases of (H) with the matrix $R_{G}(t)$ in (3.11). Moreover, let $Q(t)$ be a distinguished solution of the Riccati equation $(R)$ at infinity with respect to the interval $[\alpha, \infty) \subseteq [\alpha_{\infty}, \infty)$.
Then the matrix $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity also with respect to the interval $[\beta, \infty)$ for every $\beta \geq \alpha$.

Proof. Let $(\hat{X}, \hat{U})$ be a conjoined basis of $(H)$ corresponding to $\hat{Q}(t)$ on $[\alpha, \infty)$ via Theorem 4.21. In particular, the matrix $R_\alpha(t) \hat{Q}(t) R_\alpha(t)$ is the Riccati quotient in (2.7) associated with $(\hat{X}, \hat{U})$ on $[\alpha, \infty)$. Moreover, from Theorem 7.4 we know that $(\hat{X}, \hat{U})$ is a principal solution of $(H)$ at infinity with respect to $[\alpha, \infty)$, which belongs to the genus $\mathcal{G}$. Fix now $\beta \geq \alpha$. Then $(\hat{X}, \hat{U})$ is a principal solution of $(H)$ at infinity with respect to $[\beta, \infty)$, by Proposition 3.2(i). Consequently, the matrix $R_\alpha(t) \hat{Q}(t) R_\alpha(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to $[\beta, \infty)$, by Theorem 7.5. Finally, by using Lemma 7.3 we conclude that also the matrix $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to the interval $[\beta, \infty)$.

Remark 7.12. For a given distinguished solution $\hat{Q}(t)$ of the Riccati equation $(\mathcal{R})$ at infinity we define the point $\alpha_\hat{Q} \in [\alpha_\infty, \infty)$ by

$$
\alpha_\hat{Q} := \inf \{ \alpha \in [\alpha, \infty), \, \hat{Q}(t) \text{ is a distinguished solution of } (\mathcal{R}) \text{ with respect to } [\alpha, \infty) \}.
$$

The result in Theorem 7.11 then implies that $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ with respect to $[\alpha, \infty)$ for every $\alpha \in (\alpha_\hat{Q}, \infty)$. In fact, the set $(\alpha_\hat{Q}, \infty)$ is the maximal open interval on which the matrix $\hat{Q}(t)$ exists as a solution of $(\mathcal{R})$. Indeed, if the matrix $\hat{Q}(t)$ solves the equation $(\mathcal{R})$ on the interval $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$, then according to Remark 5.2(ii) the corresponding matrix $D_\alpha$ in (4.19) satisfies $D_\alpha = 0$, because $d(\alpha, \infty) = d_\infty$, by (2.19). Thus, $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ with respect to the interval $[\alpha, \infty)$, by Remark 7.2.

Theorem 7.13. Assume that (1.1) holds and system $(H)$ is nonoscillatory with $\alpha_\min$ defined in (3.1). Let $\mathcal{G}$ be a genus of conjoined bases of $(H)$ and let $R_\alpha(t)$ be its corresponding orthogonal projector in (3.11). Moreover, let $\hat{Q}(t)$ be a distinguished solution of the Riccati equation $(\mathcal{R})$ at infinity with $\alpha_\hat{Q}$ defined in (7.1). Then the equality $\alpha_\hat{Q} = \alpha_\min$ holds.

Proof. Let $\alpha \in [\alpha_\infty, \infty)$ be such that the matrix $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ with respect to the interval $[\alpha, \infty)$. Let $(\hat{X}, \hat{U})$ be a conjoined basis of $(H)$ at infinity with respect to $[\alpha, \infty)$, which is associated with $\hat{Q}(t)$ via Theorem 4.21. Then $(\hat{X}, \hat{U})$ is a principal solution of $(H)$ at infinity with respect to the interval $[\alpha, \infty)$, by Theorem 7.4. Moreover, from Theorem 3.4 we know that $(\hat{X}, \hat{U})$ is a principal solution with respect to the maximal open interval $(\alpha_\min, \infty)$. Thus, we have the inequality $\alpha_\min \leq \alpha$. And since $\alpha \in [\alpha_\infty, \infty)$ was chosen arbitrarily with regard to $Q(t)$, we obtain that $\alpha_\hat{Q} \leq \alpha_\min$, by (7.1). Now we show that the last inequality is implemented as the equality. Suppose that $\alpha_\hat{Q} < \alpha_\min$. According to (7.1) there exists $\beta \in (\alpha_\hat{Q}, \alpha_\min)$ such that $\hat{Q}(t)$ is a distinguished solution of $(\mathcal{R})$ at infinity with respect to the interval $[\beta, \infty)$. In turn, the conjoined basis $(X, U)$ in Theorem 4.21 applied to $Q(t) := \hat{Q}(t)$ on $[\beta, \infty)$ is a principal solution of $(H)$ at infinity with respect to $[\beta, \infty)$, by Theorem 7.4. Applying formula (3.3) and Theorem 3.4 with $(\hat{X}, \hat{U}) := (X, U)$ then yields the inequality $\beta \geq \alpha_\min$, which is a contradiction. Therefore, $\alpha_\hat{Q} = \alpha_\min$ holds and the proof is complete.
Remark 7.14. Given a genus $G$ of conjoined bases of (H) with the matrix $R_\alpha(t)$ in (3.11), from Theorem 7.13 it follows that any distinguished solution $\hat{Q}(t)$ of (R) is defined on the maximal interval $(\alpha, \infty)$ and the corresponding matrix $F_\alpha(t)$ in (4.18) satisfies $F_\alpha^\top(t) \to 0$ as $t \to \infty$ for every $\alpha > \hat{\alpha}_\min$.

In the following result we present a mutual classification of all distinguished solutions of the Riccati equation (R). This classification is formulated in terms of the initial values of the involved distinguished solutions at some point $\alpha$ from the maximal interval $(\hat{\alpha}_\min, \infty)$.

Theorem 7.15. Assume that (1.1) holds and system (H) is nonoscillatory with $\hat{\alpha}_\min$ and $R_\lambda(t)$ defined in (3.1) and (3.7), respectively. Let $G$ be a genus of conjoined bases of (H) and let $R_\alpha(t)$ be the matrix in (3.11). Moreover, let $Q(t)$ be a distinguished solution of the Riccati equation (R) at infinity. Then a symmetric solution $Q(t)$ of (R) defined on a neighborhood of some point $\alpha \in (\hat{\alpha}_\min, \infty)$ is a distinguished solution at infinity if and only if

$$R_\lambda(t)Q(\alpha)R_\lambda(t) = R_\lambda(\alpha)\hat{Q}(\alpha)R_\lambda(\alpha).$$

(7.2)

Proof. Fix $\alpha \in (\hat{\alpha}_\min, \infty)$ and let $\hat{Q}(t)$ be as in the theorem. From Definition 7.1 and Remark 7.14 we know that $Q(t)$ is a distinguished solution of (R) at infinity with respect to the interval $[\alpha, \infty)$. Moreover, let $(\hat{X}, \hat{U})$ be a conjoined basis of (H), which corresponds to $\hat{Q}(t)$ on $[\alpha, \infty)$ via Theorem 4.21. Then $(\hat{X}, \hat{U})$ is a principal solution of (H) at infinity with respect to $[\alpha, \infty)$ belonging to the genus $G$, by Theorem 7.4. Let $\hat{S}_\alpha(t)$ be the S-matrix in (2.11) associated with $(\hat{X}, \hat{U})$. Suppose that $Q(t)$ is a distinguished solution of (R) at infinity. Thus, $Q(t)$ is a distinguished solution of (R) with respect to $[\alpha, \infty)$ and the corresponding conjoined basis $(X, U)$ of (H) in Theorem 4.21 is a principal solution with respect to $[\alpha, \infty)$, which belongs to the genus $G$. From Proposition 3.14 it then follows that there exist matrices $\hat{M}, \hat{N} \in \mathbb{R}^{n \times n}$ such that

$$X(\alpha) = \hat{X}(\alpha)\hat{M}, \quad U(\alpha) = \hat{U}(\alpha)\hat{M} + \hat{X}^\top(\alpha)\hat{N},$$

(7.3)

$$\hat{M} \text{ is nonsingular, } \hat{M}^T \hat{N} = \hat{N}^T \hat{M}, \quad \text{Im} \hat{N} \subseteq \text{Im} \hat{P}, \quad P_{\hat{S}_\alpha} \hat{N} \hat{M}^{-1} P_{\hat{S}_\alpha} = 0,$$

(7.4)

where $\hat{P}$ and $P_{\hat{S}_\alpha}$ are the matrices in (2.5), (2.12), and (3.2) associated with the functions $\hat{X}(t)$ and $\hat{S}_\alpha$ on $(\hat{\alpha}_\min, \infty)$, respectively. In particular, the matrices $\hat{M}$ and $\hat{N}$ in (7.3)–(7.4) represent the conjoined basis $(X, U)$ in terms of $(\hat{X}, \hat{U})$ on $[\alpha, \infty)$ in Proposition 2.10 and Remark 2.11. Moreover, the matrices $R_\alpha(t)Q(t)R_\alpha(t)$ and $R_\alpha(t)\hat{Q}(t)R_\alpha(t)$ are the Riccati quotients in (2.7) associated with $(X, U)$ on $[\alpha, \infty)$, by Theorem 4.21(ii). Consequently, according to (5.7) in Theorem 5.3 with $(X, U) := (\hat{X}, \hat{U}), \quad (\hat{X}, \hat{U}) := (X, U), \quad Q := R_\alpha \hat{Q} R_\alpha, \quad \hat{Q} := R_\alpha \hat{Q} R_\alpha, \quad S_\alpha := \hat{S}_\alpha$,

$$M := \hat{M}, \quad N := \hat{N}, \quad \text{and by using (2.36) and } \hat{S}_\alpha(\alpha) = 0 \text{ we obtain the identity}$$

$$R_\alpha(t)Q(\alpha)R_\alpha(t) = R_\alpha(\alpha)\hat{Q}(\alpha)R_\alpha(\alpha) + \hat{X}^\top(\alpha)\hat{N} \hat{M}^{-1} \hat{X}^\top(\alpha).$$

(7.5)

In order to simplify the notation we set

$$Z := R_\lambda(\alpha)Q(\alpha)R_\lambda(\alpha), \quad \hat{Z} := R_\lambda(\alpha)\hat{Q}(\alpha)R_\lambda(\alpha).$$

(7.6)
Since we have $R_{\alpha,\infty}(\alpha) R_{\theta}(\alpha) = R_{\alpha,\infty}(\alpha)$, we get $X(t) = X(t)$ with $t \in [\alpha, \infty)$. Hence, from Proposition 2.10 and Remark 2.11(i) with $(X, U) := (X, \hat{U})$ it then follows that there exists matrices $\hat{M}, \hat{N} \in \mathbb{R}^{n \times n}$ such that the formulas in (7.3) and the first three conditions in (7.4) hold. Similarly as in the first part of the proof, the matrix $R_{\alpha}(t) Q(t) R_{\alpha}(t)$ is the Riccati quotient in (7.7) associated with $(X, U)$ on $[\alpha, \infty)$ and the equality in (7.5) holds. Moreover, multiplying (7.5) by $R_{\alpha}(t)$, we get $v = -\hat{F}_t(t) \hat{G} v = -\hat{F}_t(t) R_{\alpha,\infty}(\alpha) \hat{G} R_{\alpha,\infty}(\alpha) v = 0$. Therefore, the matrix $I + \hat{F}_t(t) \hat{G}$ is nonsingular for every $t \in [\alpha, \infty)$. Consequently, according to Remark 4.13 with $Q := \hat{Q}$, $Q := \hat{Q}$, and $G := \hat{G}$, we conclude that the symmetric matrix $Q(t)$ solve equation (7.7) on the whole interval $[\alpha, \infty)$. Let $(X, U)$ be a conjoined basis of $(H)$ associated with $Q(t)$ on $[\alpha, \infty)$ via Theorem 4.21. Then $(X, U)$ has a conjoined matrix $R_{\alpha,\infty}(\alpha)$ from the both sides and using the identities $R_{\alpha,\infty}(\alpha) R_{\theta}(\alpha) = R_{\alpha,\infty}(\alpha) = R_{\alpha,\infty}(\alpha)$ yield

$$Z = \hat{Z} + R_{\alpha,\infty}(\alpha) \hat{X}^{\dagger} T(\alpha) \hat{N} \hat{M}^{-1} \hat{X}^{\dagger}(\alpha) R_{\alpha,\infty}(\alpha).$$

(7.7)

In turn, by using (7.6) and (7.2) we have $Z = \hat{Z}$ and hence formula (7.7) becomes

$$R_{\alpha,\infty}(\alpha) \hat{X}^{\dagger} T(\alpha) \hat{N} \hat{M}^{-1} \hat{X}^{\dagger}(\alpha) R_{\alpha,\infty}(\alpha) = 0.$$  

(7.8)

From Remark 3.7 it follows that $\hat{X}^{\dagger}(\alpha) R_{\alpha,\infty}(\alpha) K = P_{S_{\alpha\infty}}$, which means that we have $\hat{X}^{\dagger}(\alpha) R_{\alpha,\infty}(\alpha) K = P_{S_{\alpha\infty}}$ for some invertible matrix $K$. By using (7.8) we then obtain that

$$P_{S_{\alpha\infty}} \hat{N} \hat{M}^{-1} P_{S_{\alpha\infty}} = K T R_{\alpha,\infty}(\alpha) \hat{X}^{\dagger} T(\alpha) \hat{N} \hat{M}^{-1} \hat{X}^{\dagger}(\alpha) R_{\alpha,\infty}(\alpha) K = 0,$$

which is the last condition in (7.4). Thus, according to Proposition 3.14 the conjoined basis $(X, U)$ is a principal solution of $(H)$ at infinity with respect to $[\alpha, \infty)$. Finally, with the aid of Remark 7.6 the matrix $Q(t)$ is then a distinguished solution of $(R)$ at infinity with respect to $[\alpha, \infty)$. The proof is complete.

In the next three results we study the minimality of distinguished solutions of $(R)$. This minimality property needs to be understood in the following sense. For
every symmetric solution $Q(t)$ of $(R)$ there exists a distinguished solution of $(R)$, which exists on the same interval and is at the same time smaller than $Q(t)$ on this interval (Theorems 7.16 and 7.18). On the other hand, any symmetric solution of $(H)$, which is smaller than a distinguished solution of $(H)$ on some interval, is a distinguished solution itself with respect to this interval (Theorem 7.20). However, in general there is no universal “smallest” distinguished solution of the Riccati equation $(R)$, see Remark 7.21. We also note that in the first result we consider the case when the solutions satisfy condition (1.4), while in the second and third result this assumption is removed.

**Theorem 7.16.** Assume (1.1). Let $G$ be a genus of conjoined bases of $(H)$ with the matrix $R_G(t)$ in (3.11) and let $Q(t)$ be a symmetric solution of the Riccati equation $(R)$ on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$ such that inclusion (1.4) holds. Then there exists a distinguished solution $\hat{Q}(t)$ of $(R)$ at infinity with respect to $[\alpha, \infty)$ satisfying (1.4) such that $Q(t) \geq \hat{Q}(t)$ for every $t \in [\alpha, \infty)$.

**Proof.** Let $Q(t)$ be as in the theorem and let $(X,U)$ be its associated conjoined basis of $(H)$ in Theorem 4.21 or Remark 4.23. Then $(X,U)$ has constant kernel on $[\alpha, \infty)$ and belongs to the genus $G$. Moreover, through (1.4) the matrix $Q(t) = R_G(t) Q(t) R_G(t)$ is the Riccati quotient in (2.7) corresponding to $(X,U)$ on $[\alpha, \infty)$. Let $T_\alpha$ be the $T$-matrix in (2.13) associated with $(X,U)$ on $[\alpha, \infty)$ and consider the solution $(\hat{X},\hat{U})$ of $(H)$ in (3.4). From Theorem 3.5 and Remark 3.6 we know that $(\hat{X},\hat{U})$ is a principal solution of $(H)$ at infinity belonging to the genus $G$. Let $\hat{Q}(t)$ be its corresponding Riccati quotient in (2.7) on $[\alpha, \infty)$. According to Theorem 7.5 and Remark 7.6 the matrix $\hat{Q}(t)$ is a distinguished solution of $(R)$ at infinity with respect to $[\alpha, \infty)$ satisfying condition (1.4). Moreover, since the matrix $T_\alpha$ is nonnegative definite, by Remark 2.5, with the aid of Corollary 5.5(ii) with $\hat{Q} := \hat{Q}, N := -T_\alpha$, and $M := I$ we then have that $Q(\alpha) \geq \hat{Q}(\alpha)$. Finally, this inequality implies through Corollary 4.15 that $Q(t) \geq \hat{Q}(t)$ for all $t \in [\alpha, \infty)$, which completes the proof. \hfill $\square$

**Remark 7.17.** (i) By applying (5.7) in Theorem 5.3 we obtain an exact relation between the Riccati quotients $Q(t)$ and $\hat{Q}(t)$ on $[\alpha, \infty)$. Namely, the formula

$$\hat{Q}(t) = Q(t) - X^T(t) T_\alpha [P - S_\alpha(t) T_\alpha^\dagger X^\dagger(t)$$

holds for every $t \in [\alpha, \infty)$. In particular, for $t = \alpha$ the equality in (7.9) becomes

$$\hat{Q}(\alpha) = Q(\alpha) - X^T(\alpha) T_\alpha X^\dagger(\alpha).$$

(ii) According to Remark 7.14, the point $\alpha \in [\alpha_\infty, \infty)$ in Theorem 7.16 satisfies $\alpha > \hat{\alpha}_{\min}$. Moreover, from Theorems 4.3 and 7.16 it follows that the last inequality holds even when condition (1.4) regarding the matrix $Q(t)$ is dropped. Hence, we conclude that for any genus $G$ the open interval $(\hat{\alpha}_{\min}, \infty)$ is the maximal set such that there exists a symmetric solution of the Riccati equation $(R)$ on $(\hat{\alpha}_{\min}, \infty)$.

**Theorem 7.18.** Assume (1.1). Let $G$ be a genus of conjoined bases of $(H)$ with the orthogonal projector $R_G(t)$ in (3.11). Let $Q(t)$ be a symmetric solution of the Riccati equation $(R)$ on $[\alpha, \infty) \subseteq [\alpha_\infty, \infty)$. Then there exists a distinguished solution $\hat{Q}(t)$ of $(R)$ with respect to $[\alpha, \infty)$ such that $Q(t) \geq \hat{Q}(t)$ holds for every $t \in [\alpha, \infty)$.

**Proof.** We proceed similarly as in the proof of Theorem 7.16. Let $(X,U)$ be a conjoined basis of $(H)$ from the genus $G$, which corresponds to $Q(t)$ on $[\alpha, \infty)$ through
Theorem 4.21. In particular, \((X,U)\) has constant kernel on \([\alpha, \infty)\) and the symmetric matrix \(Q_\alpha(t) := R_\alpha(t) Q(t) R_\alpha(t)\) is the associated Riccati quotient in (2.7) for every \(t \in [\alpha, \infty)\). Moreover, according to formula (7.10) in Remark 7.17 with \(Q := Q\) the solution \(\hat{Q}(t)\) of \((R)\) satisfying the condition
\[
\hat{Q}_\alpha(\alpha) = Q_\alpha(\alpha) - X^\dagger T_\alpha X^\dagger(\alpha) = R_\alpha(\alpha) Q(\alpha) R_\alpha(\alpha) - X^\dagger T_\alpha X^\dagger(\alpha) \quad (\text{7.11})
\]
is a distinguished solution of \((R)\) at infinity with respect to \([\alpha, \infty)\). Here \(T_\alpha\) is the \(T\)-matrix in (2.13) associated with \((X,U)\). Furthermore, let \(D_\alpha\) be the matrix in (4.19), which corresponds to \(Q(t)\) through the \(F\)-matrix \(F_\alpha(t)\) in (4.18) on \([\alpha, \infty)\), and consider the symmetric solution \(\hat{Q}(t)\) of \((R)\) given by initial condition \(\hat{Q}(\alpha) := Q(\alpha) - D_\alpha\). We will show that \(\hat{Q}(t)\) is a distinguished solution of \((R)\) at infinity with respect to \([\alpha, \infty)\). Let \(R_{\alpha, \infty}(t)\) be the orthogonal projector defined in (3.7). Similarly, as in the proof of Theorem 7.15 we will use the notation
\[
\hat{Z} := R_{\alpha, \infty}(\alpha) \hat{Q}(\alpha) R_{\alpha, \infty}(\alpha), \quad \hat{Z}_\alpha := R_{\alpha, \infty}(\alpha) \hat{Q}_\alpha(\alpha) R_{\alpha, \infty}(\alpha). \quad (\text{7.12})
\]
With the aid of (3.11) and Remark 5.2(ii) together with the symmetry of the matrices \(R_{\alpha, \infty}(t), R_\alpha(t),\) and \(D_\alpha\) we have the identities \(R_{\alpha, \infty}(\alpha) R_\alpha(\alpha) = R_{\alpha, \infty}(\alpha) = R_\alpha(\alpha) R_{\alpha, \infty}(\alpha)\) and \(R_{\alpha, \infty}(\alpha) D_\alpha = D_\alpha = D_\alpha R_{\alpha, \infty}(\alpha)\). By combining these properties with (7.11) and the second equality in (5.6) we obtain that
\[
\hat{Z} \overset{\text{(7.12)}}{=} R_{\alpha, \infty}(\alpha) \hat{Q}(\alpha) R_{\alpha, \infty}(\alpha) = R_{\alpha, \infty}(\alpha) [Q(\alpha) - D_\alpha] R_{\alpha, \infty}(\alpha) = R_{\alpha, \infty}(\alpha) R_\alpha(\alpha) R_{\alpha, \infty}(\alpha) - D_\alpha \overset{\text{(7.11)}}{=} R_{\alpha, \infty}(\alpha) \hat{Q}_\alpha(\alpha) + X^\dagger T_\alpha X^\dagger(\alpha) R_{\alpha, \infty}(\alpha) - D_\alpha \overset{\text{(7.12)}}{=} \hat{Z}_\alpha + R_{\alpha, \infty}(\alpha) X^\dagger T_\alpha X^\dagger(\alpha) R_{\alpha, \infty}(\alpha) - D_\alpha \overset{\text{(5.6)}}{=} \hat{Z}. \quad (\text{7.13})
\]
Finally, since the point \(\alpha > \hat{\alpha}_{\text{min}}\) by Remark 7.17(ii), from (7.12), the equation in (7.13), and Theorem 7.15 it follows immediately that the solution \(\hat{Q}(t)\) is a distinguished solution of \((R)\) at infinity with respect to \([\alpha, \infty)\). In particular, the matrix \(\hat{Q}(t)\) solves equation \((R)\) on the whole interval \([\alpha, \infty)\). And since \(Q(\alpha) - \hat{Q}(\alpha) = D_\alpha \geq 0\), we conclude by Corollary 4.14 that the inequality \(\hat{Q}(t) \geq Q(t)\) holds for every \(t \in [\alpha, \infty)\), which completes the proof. \(\square\)

Remark 7.19. We note that the converse to Theorem 7.18 also holds. More precisely, if \(\hat{Q}(t)\) is a distinguished solution of \((R)\) at infinity with respect to the interval \([\alpha, \infty)\), then every symmetric solution \(Q(t)\) of \((R)\), which satisfies the condition \(Q(\alpha) \geq \hat{Q}(\alpha)\), exists on the whole interval \([\alpha, \infty)\) and the inequality \(\hat{Q}(t) \geq Q(t)\) holds on \([\alpha, \infty)\). This observation is a direct application of Corollary 4.15 with the choice \(Q := \hat{Q}\) and \(\hat{Q} := Q\).

Theorem 7.20. Assume (1.1) and let \(G\) be a genus of conjoined bases of \((H)\) with the matrix \(R_\alpha(t)\) in (3.11). Let \(\hat{Q}(t)\) be a distinguished solution of the Riccati equation \((R)\) with respect to the interval \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\). Moreover, let \(Q(t)\) be a symmetric solution of \((R)\) on \([\alpha, \infty)\) satisfying the initial condition \(Q(\alpha) \geq Q_\alpha(\alpha)\). Then \(Q(t)\) is a distinguished solution of \((R)\) at infinity with respect to \([\alpha, \infty)\) and the inequality \(\hat{Q}(t) \geq Q(t)\) holds for all \(t \in [\alpha, \infty)\).

Proof. Let \(\hat{Q}(t)\) and \(Q(t)\) be as in the theorem. By using Corollary 4.14 we obtain the inequality \(\hat{Q}(t) \geq Q(t)\) on \([\alpha, \infty)\). On the other hand, according to Theorem 7.18 there exists a distinguished solution \(\hat{Q}(t)\) of \((R)\) at infinity such that \(Q(t) \geq \hat{Q}(t)\).
for every \( t \in [\alpha, \infty) \). Hence, for \( t = \alpha \) we have the relations \( \hat{Q}(\alpha) \geq Q(\alpha) \geq \hat{Q}(\alpha) \). Consequently, by multiplying the last inequalities by the matrix \( R_{\infty}(\alpha) \) defined in (3.7) from the both sides we obtain that

\[
R_{\infty}(\alpha) \hat{Q}(\alpha) R_{\infty}(\alpha) \geq R_{\infty}(\alpha) Q(\alpha) R_{\infty}(\alpha) \geq R_{\infty}(\alpha) \hat{Q}(\alpha) R_{\infty}(\alpha). \tag{7.14}
\]

But Theorem 7.15 and the fact that both the solutions \( \hat{Q}(t) \) and \( \hat{Q}(t) \) are distinguished with respect to \([\alpha, \infty)\) yield \( R_{\infty}(\alpha) \hat{Q}(\alpha) R_{\infty}(\alpha) R_{\infty}(\alpha) \hat{Q}(\alpha) R_{\infty}(\alpha) \). Therefore, the inequalities in (7.14) are implemented as the equalities. In turn, applying Theorem 7.15 once more then implies that \( Q(t) \) is a distinguished solution of \((R)\) at infinity with respect to \([\alpha, \infty)\) as well. The proof is complete.

\[\square\]

Remark 7.21. Given a genus \( \mathcal{G} \) of conjoined bases of \((H)\) with the matrix \( R_\infty(t) \) defined in (3.11), let \( \hat{Q}(t) \) be a distinguished solution of \((R)\) at infinity with respect to the interval \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\). Then there exist distinguished solutions \( \hat{Q}_*(t) \) and \( \hat{Q}_*(t) \) of \((R)\) satisfying

\[
\hat{Q}_*(t) \leq \hat{Q}(t) \leq \hat{Q}_*(t), \quad t \in [\alpha, \infty). \tag{7.15}
\]

The solutions \( \hat{Q}_*(t) \) and \( \hat{Q}_*(t) \) are given, for example, by the initial conditions

\[
\hat{Q}_*(\alpha) = \hat{Q}(\alpha) - I + R_{\infty}(\alpha) \quad \text{and} \quad \hat{Q}_*(\alpha) = \hat{Q}(\alpha) + I - R_{\infty}(\alpha), \tag{7.16}
\]

where \( R_{\infty}(t) \) is the orthogonal projector defined in (3.7). Indeed, by using (7.16) and the basic properties of orthogonal projectors from Section 2 and by utilizing the notation in (7.6) we get

\[
R_{\infty}(\alpha) \hat{Q}_*(\alpha) R_{\infty}(\alpha) = R_{\infty}(\alpha) \hat{Q}(\alpha) - I + R_{\infty}(\alpha) \] \tag{7.16}

\[
R_{\infty}(\alpha) \hat{Q}_*(\alpha) R_{\infty}(\alpha) = R_{\infty}(\alpha) \hat{Q}(\alpha) + I - R_{\infty}(\alpha) \] \tag{7.15}

From Theorem 7.13 we know that \( \alpha > \alpha_{\min} \). Hence, by applying Theorem 7.15 we obtain immediately that \( \hat{Q}_*(t) \) and \( \hat{Q}_*(t) \) are distinguished solutions of \((R)\) at infinity with respect to \([\alpha, \infty)\). In addition, since \( \hat{Q}(\alpha) \geq \hat{Q}(\alpha) \) = \( Q_*(\alpha) \hat{Q}(\alpha) \) by (7.16) and \( I - R_{\infty}(\alpha) \geq 0 \), we have the inequalities \( \hat{Q}(\alpha) \leq \hat{Q}(\alpha) \leq \hat{Q}(\alpha) \). In turn, according to Corollary 4.14 the inequalities in (7.15) hold. This observation shows that for the case of a general (not necessarily controllable) system \((H)\) the partially ordered set of all distinguished solutions of \((R)\) has neither a minimal element nor a maximal element.

The considerations in Theorems 7.15 and 7.16 show that for the minimal genus \( \mathcal{G}_{\min} \), i.e., for \( R_\infty(t) = R_{\infty}(t) \), there exists a uniquely determined distinguished solution of the Riccati equation \((R)\) with

\[
A(t) := A(t) R_{\infty}(t) - A(t) [I - R_{\infty}(t)], \quad B(t) := B(t), \quad C(t) := R_{\infty}(t) C(t) R_{\infty}(t) \tag{7.17}
\]

which represents the smallest element in the set of all symmetric solutions \( Q(t) \) of equation \((R)\) satisfying (1.4).

Definition 7.22. Let \( \mathcal{G}_{\min} \) be the minimal genus of conjoined bases of system \((H)\) with the minimal orthogonal projector \( R_{\infty}(t) \) in (3.7). A symmetric solution \( \hat{Q}(t) \) of the Riccati equation \((R)\) with the coefficients in (7.17) is said to be a minimal distinguished solution at infinity if the matrix \( \hat{Q}(t) \) is defined in some interval
These solutions are minimal distinguished solutions of the Riccati equation \((R)\) and are nonoscillatory as well. The minimal principal solution \(\hat{F}_\alpha(t)\) in (4.18) satisfies \(\hat{F}_\alpha'(t) \to 0\) as \(t \to \infty\).

The following result shows the existence and uniqueness of the minimal distinguished solution of the Riccati equation \((R)\) for the minimal genus \(G_{\min}\), as well as its minimality property.

**Theorem 7.23.** Assume (1.1). Then system \((H)\) is nonoscillatory if and only if there exists a minimal distinguished solution \(\hat{Q}(t)\) of the Riccati equation \((R)\) with the coefficients in (7.17). In this case, the minimal distinguished solution \(\hat{Q}(t)\) is determined uniquely and any symmetric solution \(Q(t)\) of \((R)\) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) with (7.18) satisfies \(Q(t) \geq \hat{Q}(t)\) on \([\alpha, \infty)\).

**Proof.** The first part of the theorem coincides with the statement of Remark 7.10 for the genus \(G = G_{\min}\) and for the corresponding orthogonal projector \(R_\alpha(t)\) in (3.11) equal to the minimal orthogonal projector \(R_\alpha(t)\) defined in (3.7). The uniqueness of the distinguished solution \(\hat{Q}(t)\) follows from Theorem 7.15 with \(G := G_{\min}\). More precisely, let \(Q(t)\) and \(Q_\ast(t)\) be two distinguished solutions of equation \((R)\) for the minimal genus \(G_{\min}\), which satisfy condition (1.4) (with \(R_\alpha(t) = R_\alpha(t)\)) on \([\alpha, \infty)\) for some \(\alpha \geq \alpha_\infty\). From Theorem 7.13 it follows that both matrices \(Q(t)\) and \(Q_\ast(t)\) solve \((R)\) on the maximal open interval \((\hat{\alpha}_{\min}, \infty)\) and hence, the point \(\alpha \in (\hat{\alpha}_{\min}, \infty)\). According to Corollary 4.5 we then obtain the inclusions \(\text{Im} \hat{Q}(t) \subseteq \text{Im} R_\alpha(t)\) and \(\text{Im} Q_\ast(t) \subseteq \text{Im} R_\alpha(t)\) for every \(t \in (\hat{\alpha}_{\min}, \infty)\). Consequently, combining these facts with the symmetry of \(\hat{Q}(t)\) and \(Q_\ast(t)\) yields

\[
R_\alpha(t) \hat{Q}(t) R_\alpha(t) = \hat{Q}(t), \quad R_\alpha(t) \hat{Q}_\ast(t) R_\alpha(t) = Q_\ast(t)
\]

for all \(t \in (\hat{\alpha}_{\min}, \infty)\). Finally, by using formula (7.2) in Theorem 7.15 with \(G := G_{\min}\) and \(Q := Q_\ast\) together with (7.19) we obtain on \((\hat{\alpha}_{\min}, \infty)\) that

\[
\hat{Q}(t) \overset{(7.19)}{=} R_\alpha(t) \hat{Q}(t) R_\alpha(t) \overset{(7.2)}{=} R_\alpha(t) \hat{Q}_\ast(t) R_\alpha(t) \overset{(7.19)}{=} \hat{Q}_\ast(t).
\]

Thus, the distinguished solutions \(\hat{Q}(t)\) and \(Q_\ast(t)\) coincide. Finally, the minimality property of the minimal distinguished solution of \((R)\) at infinity follows from Theorem 7.16. Namely, by using the latter reference for any symmetric solution \(Q(t)\) of \((R)\) on \([\alpha, \infty) \subseteq [\alpha_\infty, \infty)\) with (7.18) there exists a distinguished solution \(\hat{Q}(t)\) of \((R)\) at infinity with respect to \([\alpha, \infty)\) satisfying (7.18) such that \(Q(t) \geq \hat{Q}(t)\) for every \(t \in [\alpha, \infty)\). In fact, the matrix \(\hat{Q}(t)\) is the minimal distinguished solution of \((R)\) at infinity, by Definition 7.22. The proof is complete.

**Remark 7.24.** The minimal distinguished solution of \((R)\) at infinity in Theorem 7.23 will be denoted by \(\hat{Q}_{\min}\). The minimal distinguished solution \(\hat{Q}_{\min}\) plays for the theory of the Riccati differential equations \((R)\) or \((R)\) a similar role as the minimal principal solution \((\hat{X}_{\min}, \hat{U}_{\min})\) of system \((H)\) at infinity for the theory of principal solutions at infinity.

**Remark 7.25.** When system \((H)\) is completely controllable, the main results of this section give the classical statements about the distinguished solutions at infinity of the Riccati equation \((R)\). More precisely, the following holds.
• The results in Corollary 7.7 and Theorem 7.15 yield the correspondence between the unique principal solution of \((H)\) at infinity and the unique distinguished solution of \((R)\) at infinity, see [7, pg. 53] or [23, pp. 45–46].

• The result in Theorem 7.8 provides a characterization of the nonoscillation of system \((H)\) in terms of the existence of the unique distinguished solution of \((R)\) at infinity, see the necessary condition in [22, Theorem VII.3.3]. Note that the nonoscillation of \((H)\) is defined in [22, Section VII.3] in terms of disconjugacy of \((H)\), i.e., in terms of the nonexistence of mutually conjugate points, which is a stronger concept than the nonoscillation of \((H)\) as we define in Section 2. We note also that the sufficiency part of Theorem 7.8 is new also in the completely controllable case.

• The results in Theorems 7.18 and 7.20 yield the minimality property of the unique distinguished solution of \((R)\) at infinity, see [7, Theorem 8, pg. 54] or [23, Theorem IV.4.2].

Indeed, in this case \(d_\infty = 0\) and there is only one minimal/maximal genus of conjoined bases of \((H)\). This implies that \(\alpha_\infty = a\) and the orthogonal projector \(R_{\Lambda_\infty}(t)\) in (3.7) satisfies \(R_{\Lambda_\infty}(t) \equiv I\) on \([a, \infty)\). Therefore, the unique Riccati equation \((R)\) associated with the minimal/maximal genus coincides with the classical Riccati equation \((R)\). Moreover, under the Legendre condition (1.1) the nonoscillation of system \((H)\) is then equivalent with the existence of a unique (minimal) distinguished solution \(\hat{Q}\) of \((R)\) at infinity. In addition, the matrix \(\hat{Q}\) constitutes the smallest symmetric solution of the Riccati equation \((R)\), that is, every symmetric solution \(Q\) of \((R)\) on \([\alpha, \infty) \subseteq [a, \infty)\) satisfies inequality (1.5).

**Remark 7.26.** We note that the results commented on in Remark 7.25 hold under a weaker assumption than the complete controllability of \((H)\). More precisely, under (1.1) and the nonoscillation of \((H)\) the existence of a unique distinguished solution at infinity of a (unique) Riccati equation \((R)\) is equivalent with the fact that the maximal order of abnormality \(d_\infty = 0\).

8. **Examples.** In this section we provide several examples which illustrate the presented theory of Riccati equations for abnormal system \((H)\).

**Example 8.1.** In the first example we explore a controllable linear Hamiltonian system. For \(n = 1\), \(a = 0\) we consider system \((H)\) with \(A(t) = 0\), \(B(t) = 1 + t^2\), and \(C(t) = -2/(1 + t^2)^2\). This system comes from the second order Sturm–Liouville equation \([y'/(1 + t^2)]' + 2y/(1 + t^2)^2 = 0\). The matrix \(B(t) > 0\) on \([0, \infty)\), which implies that system \((H)\) is completely controllable on \([0, \infty)\) with \(d(0, \infty) = d_\infty = 0\) and \(\alpha_\infty = 0\), by (2.19). Thus, there exists only one (minimal/maximal) genus \(G\) of conjoined bases with the corresponding orthogonal projector \(R_G(t) \equiv 1\) on \([0, \infty)\) and consequently, the unique Riccati equation

\[
Q' + (1 + t^2)Q^2 + 2/(1 + t^2)^2 = 0, \quad t \in [0, \infty).
\]

(8.1)

In [30, Example 7.1] we showed that system \((H)\) is nonoscillatory and that the principal solutions at infinity are nonzero multiples of

\[
(\hat{X}(t), \hat{U}(t)) = (t, 1/(1 + t^2)),
\]

with \(\hat{\alpha}_{\min} = 0\), by (3.1). Therefore, by Theorem 7.5 and Remark 7.25 the unique (minimal) distinguished solution \(\hat{Q}\) of (8.1) at infinity satisfies

\[
\hat{Q}(t) = 1/[t(1 + t^2)], \quad t \in (0, \infty).
\]
Moreover, by using Proposition 4.12 with \( Q := \tilde{Q} \) the general solution \( Q(t, \alpha, p) \) of the Riccati equation (8.1) defined on an interval \([\alpha, \infty) \subseteq (0, \infty)\) has the form

\[
Q(t, \alpha, p) = \tilde{Q}(t) + p/[t^2 + pt(t - \alpha)(t + 1/\alpha)], \quad t \in [\alpha, \infty),
\]

with \( p \in [0, \infty) \cup \{\infty\} \), where

\[
Q(t, \alpha, \infty) := \lim_{p \to \infty} Q(t, \alpha, p) \quad (8.2) \quad \tilde{Q}(t) + 1/[t(t - \alpha)(t + 1/\alpha)], \quad t \in [\alpha, \infty). \quad (8.3)
\]

From formulas (8.2)–(8.3) it then follows that for any point \( \alpha > 0 \) and parameter \( p \in [0, \infty) \cup \{\infty\} \) we have the inequality \( Q(t, \alpha, p) \geq \tilde{Q}(t) \) on \([\alpha, \infty)\), as we claim in Theorem 7.18 or Remark 7.25.

**Example 8.2.** In this example we consider the so-called zero system \((H)\) with \( n \times n \) coefficient matrices \( A(t) = B(t) = C(t) \equiv 0 \) on \([a, \infty)\). This system is nonoscillatory and extremely abnormal, that is, \( d[a, \infty) = d_{\infty} = n \) and hence, \( \alpha_{\infty} = a \) and \( R_{\infty}(t) \equiv 0 \) on \([a, \infty)\). In [29, Example 8.2] and [31, Example 5.7] we showed that every conjoined basis of \((H)\) is a constant principal solution at infinity with respect to \([a, \infty)\) and that the set of all genera of conjoined bases of \((H)\) is isomorphic to the complete lattice of all subspaces in \( \mathbb{R}^n \). Namely, for every constant orthogonal projector \( R \in \mathbb{R}^{n \times n} \) there exists a unique genus \( G \) of conjoined bases of \((H)\) such that \( R_c(t) \equiv R \) on \([a, \infty)\). The associated Riccati equation \((\mathcal{R})\) reduces to \( Q' = 0 \). In this case, every symmetric solution of \((\mathcal{R})\) is a constant distinguished solution at infinity with respect to the interval \([a, \infty)\), so that \( \alpha_{\min} = a \). In particular, for any constant symmetric matrix \( M \in \mathbb{R}^{n \times n} \) the pair \((X, U) := (R, MR + I - R)\) constitutes a constant principal solution of \((H)\) at infinity belonging to \( G \), which corresponds to the distinguished solution \( \tilde{Q}(t) \equiv M \) of \((\mathcal{R})\) on \([a, \infty)\) via Theorems 4.21 and 7.4. Moreover, the minimal distinguished solution at infinity satisfies \( \tilde{Q}_{\min}(t) \equiv 0 \) on \([a, \infty)\).

In the previous two examples we studied the situation when system \((H)\) possessed only one Riccati equation \((\mathcal{R})\). However, this will not be the case of the system presented in the last example.

**Example 8.3.** For \( n = 3 \) and \( a = 0 \) we consider system \((H)\) with the coefficients \( A(t) = \text{diag}\{0, 0, 1\}, B(t) = \text{diag}\{1 + t^2, 0, 0\}, \) and \( C(t) = \text{diag}\{-2/(1 + t^2)^2, 0, 0\} \) on \([0, \infty)\). In this case we have \( d_{\infty} = 2, \alpha_{\infty} = 0, \) and \( R_{\infty}(t) \equiv \text{diag}\{1, 0, 0\} \) on \([0, \infty)\). Moreover, in [31, Example 5.8] we examined the set of all genera \( G \) of conjoined bases of \((H)\) and found a principal solution at infinity in each genus \( G \). We will continue in this study by illustrating the concept of distinguished solutions at infinity of the associated Riccati equations \((\mathcal{R})\). For the minimal genus \( G = G_{\min} \) represented by the orthogonal projector \( R_c(t) = R_{\Lambda}(t) \) on \([0, \infty)\) we have the Riccati equation \((\mathcal{R})\) with the coefficients in (7.17), i.e.,

\[
A_{\min}(t) = -A(t), \quad B_{\min}(t) = B(t), \quad C_{\min}(t) = C(t), \quad t \in [0, \infty).
\]

This Riccati equation possesses on \((0, \infty)\) the minimal distinguished solution

\[
\tilde{Q}_{\min}(t) = \text{diag}\{1/[t(1 + t^2)], 0, 0\}
\]

constructed from the minimal principal solution

\[
(\tilde{X}_{\min}(t), \tilde{U}_{\min}(t)) = \left( \text{diag}\{t, 0, 0\}, \text{diag}\{1/(1 + t^2), 1, e^{-t}\} \right)
\]

of \((H)\) at infinity via Theorem 4.18, as well as the distinguished solution

\[
\tilde{Q}_0(t) = \text{diag}\{1/[t(1 + t^2)], 1, -e^{2t}\},
\]

as presented in the last example.
which does not satisfy condition (7.18). In particular, the distinguished solutions in (8.4) and (8.5) are mutually incomparable on the interval \((0, \infty)\). Similarly, with the maximal genus \(G = G_{\text{max}}\) represented by the orthogonal projector \(R_G(t) \equiv I\) on \([0, \infty)\) there is associated the Riccati equation (\(R\)) with the pair of incomparable distinguished solutions at infinity

\[
\hat{Q}(t) = \text{diag}\{1/[t(1 + t^2)], -1, e^{-2t}\}, \quad \hat{Q}_*(t) = \text{diag}\{1/[t(1 + t^2)], 1, -e^{-2t}\} \quad (8.6)
\]

for \(t \in (0, \infty)\). We note that \(\hat{Q}(t)\) and \(\hat{Q}_*(t)\) in (8.6) are both the Riccati quotients in (2.7), which correspond to the maximal principal solutions

\[
(\bar{X}(t), \bar{U}(t)) = (\text{diag}\{t, 1, e^t\}, \text{diag}\{1/(1 + t^2), -1, e^{-t}\}),
\]

\[
(\hat{X}_*(t), \hat{U}_*(t)) = (\text{diag}\{t, 1, e^t\}, \text{diag}\{1/(1 + t^2), 1, -e^{-t}\})
\]

of \((H)\) at infinity, respectively. In the remaining part of this example we analyze for three different genera with rank equal to \(r = 2\) the corresponding Riccati equations (\(R\)) and their distinguished solutions. More precisely, according to [31, Example 5.8] we consider the genera \(G_1\), \(G_2\), and \(G_3\) given by

\[
R_{G_1}(t) \equiv \text{diag}\{1, 0, 1\}, \quad R_{G_2}(t) \equiv \text{diag}\{1, 1, 0\},
\]

\[
R_{G_3}(t) = \frac{1}{e^{2t} + 1} \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & e^t \\
0 & e^t & e^{2t}
\end{pmatrix}
\]

on \([0, \infty)\). With the genus \(G_1\) we associate the Riccati equation (\(R\)) with

\[
A_1(t) = A(t), \quad B_1(t) = B(t), \quad C_1(t) = C(t), \quad t \in [0, \infty),
\]

possessing the pair of incomparable distinguished solutions at infinity

\[
\hat{Q}_1(t) = \text{diag}\{1/[t(1 + t^2)], 0, -e^{-2t}\}, \quad \hat{Q}_{1*}(t) = \text{diag}\{1/[t(1 + t^2)], -1, 0\}
\]

for \((0, \infty)\). The matrix \(\hat{Q}_1(t)\) is the Riccati quotient in (2.7), which corresponds to the principal solution

\[
(\hat{X}_1(t), \hat{U}_1(t)) = (\text{diag}\{t, 0, e^t\}, \text{diag}\{1/(1 + t^2), 1, -e^{-t}\})
\]

of \((H)\) at infinity belonging to \(G_1\), while the distinguished solution \(\hat{Q}_{1*}(t)\) does not satisfy (1.4). Similarly, for the genus \(G_2\) we have the Riccati equation (\(R\)) with

\[
A_2(t) = -A(t), \quad B_2(t) = B(t), \quad C_2(t) = C(t), \quad t \in [0, \infty),
\]

which has the pair of incomparable distinguished solutions at infinity

\[
\hat{Q}_2(t) = \text{diag}\{1/[t(1 + t^2)], 1, 0\}, \quad \hat{Q}_{2*}(t) = \text{diag}\{1/[t(1 + t^2)], 0, e^{2t}\}
\]

for \((0, \infty)\). The matrix \(\hat{Q}_2(t)\) is the Riccati quotient in (2.7), which corresponds to the principal solution

\[
(\hat{X}_2(t), \hat{U}_2(t)) = (\text{diag}\{t, 1, 0\}, \text{diag}\{1/(1 + t^2), 1, e^{-t}\})
\]

of \((H)\) at infinity from the genus \(G_2\) and the distinguished solution \(\hat{Q}_{2*}(t)\) does not satisfy (1.4). Finally, for the genus \(G_3\) we obtain the Riccati equation (\(R\)) with

\[
A_3(t) = \frac{1}{e^{2t} + 1} \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 2e^t & e^{2t} - 1
\end{pmatrix}, \quad B_3(t) = B(t), \quad C_3(t) = C(t), \quad t \in [0, \infty),
\]
having on $(0, \infty)$ the pair of incomparable distinguished solutions at infinity

\[
\hat{Q}_3(t) = \frac{1}{(e^{2t} + 1)^2} \left( \begin{array}{ccc}
(e^{2t} + 1)^2/[t(1 + t^2)] & 0 & 0 \\
0 & 1 & e^t \\
0 & e^t & e^{2t}
\end{array} \right),
\]

\[
\hat{Q}_3^*(t) = \frac{1}{(e^{2t} + 1)^2} \left( \begin{array}{ccc}
(e^{2t} + 1)^2/[t(1 + t^2)] & 0 & 0 \\
0 & 2e^{2t} + 3 & e^{3t} + 2e^t \\
0 & e^{3t} + 2e^t & e^{2t}
\end{array} \right).
\]

In particular, the matrix $\hat{Q}_3(t)$ constitutes the Riccati quotient in (2.7) associated with the principal solution at infinity $(\hat{X}_3(t), \hat{U}_3(t)) = \left( \begin{array}{ccc}
t & 0 & 0 \\
0 & 1 & -1 \\
0 & e^t & -e^t
\end{array} \right), \left( \begin{array}{ccc}
1/(1 + t^2) & 0 & 0 \\
0 & -1 & 1 \\
0 & 2e^{-t} & -2e^{-t}
\end{array} \right)$

and the distinguished solution $\hat{Q}_3^*(t)$ does not satisfy condition (1.4).

**Acknowledgments.** The author is grateful to Professor Roman Šimon Hilscher for consultations regarding the subject of this paper.

**REFERENCES**

[1] J. Allwright and R. Vinter, Second order conditions for periodic optimal control problems, *Control Cybernet.*, 34 (2005), 617–643.

[2] F. V. Atkinson, *Discrete and Continuous Boundary Problems*, Academic Press, New York - London, 1964.

[3] R. Bellman, *Introduction to the Mathematical Theory of Control Processes. Vol. I: Linear Equations and Quadratic Criteria*, Mathematics in Science and Engineering, Vol. 40, Academic Press, New York, NY, 1967.

[4] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Second Edition, Springer-Verlag, New York, NY, 2003.

[5] D. S. Bernstein, *Matrix Mathematics. Theory, Facts, and Formulas with Application to Linear Systems Theory*, Princeton University Press, Princeton, 2005.

[6] S. L. Campbell and C. D. Meyer, *Generalized Inverses of Linear Transformations*, Reprint of the 1991 corrected reprint of the 1979 original, Classics in Applied Mathematics, Vol. 56, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.

[7] W. A. Coppel, *Disconjugacy*, Lecture Notes in Mathematics, Vol. 220, Springer-Verlag, Berlin - New York, 1971.

[8] Z. Došlá and O. Došlý, *Quadratic functionals with general boundary conditions*, *Appl. Math. Optim.*, 36 (1997), 243–262.

[9] P. Hartman, *Ordinary Differential Equations*, John Wiley, New York, 1964.

[10] M. R. Hestenes, *Quadratic control problems*, J. Optim. Theory Appl., 17 (1975), 1–42.

[11] R. Hilscher and V. Zeidan, Applications of time scale symplectic systems without normality, *J. Math. Anal. Appl.*, 340 (2008), 451–465.

[12] R. Hilscher and V. Zeidan, Time scale embedding theorem and coercivity of quadratic functionals, *Analysis (Munich)*, 28 (2008), 1–28.

[13] R. Hilscher and V. Zeidan, Riccati equations for abnormal time scale quadratic functionals, *J. Differential Equations*, 244 (2008), 1410–1447.

[14] R. Johnson, C. Núñez and R. Obaya, Dynamical methods for linear Hamiltonian systems with applications to control processes, *J. Dynam. Differential Equations*, 25 (2013), 679–713.

[15] R. Johnson, R. Obaya, S. Novo, C. Núñez and R. Fabbri, *Nonautonomous Linear Hamiltonian Systems: Oscillation, Spectral Theory and Control*, Developments in Mathematics, Vol. 36, Springer, Cham, 2016.

[16] R. E. Kalman and R. S. Bucy, New results in linear filtering and prediction theory, *Trans. ASME Ser. D J. Basic Engry.*, 83 (1961), 95–108.
[17] W. Kratz, Quadratic Functionals in Variational Analysis and Control Theory, Akademie Verlag, Berlin, 1995.

[18] W. Kratz, Definiteness of quadratic functionals, Analysis, 23 (2003), 163–183.

[19] B. P. Molinari, Nonnegativity of a quadratic functional, SIAM J. Control, 13 (1975), 792–806.

[20] W. T. Reid, Riccati matrix differential equations and non-oscillation criteria for associated linear differential systems, Pacific J. Math., 13 (1963), 665–685.

[21] W. T. Reid, Principal solutions of nonoscillatory linear differential systems, J. Math. Anal. Appl., 9 (1964), 397–423.

[22] W. T. Reid, Ordinary Differential Equations, John Wiley & Sons, Inc., New York – London – Sydney, 1971.

[23] W. T. Reid, Riccati Differential Equations, Academic Press, New York – London, 1972.

[24] W. T. Reid, Sturmian Theory for Ordinary Differential Equations, Springer-Verlag, New York – Berlin – Heidelberg, 1980.

[25] V. Ružičková, Discrete Symplectic Systems and Definiteness of Quadratic Functionals, Ph.D thesis, Masaryk University, Brno, 2006. Available from https://is.muni.cz/th/p9iz7/?lang=en.

[26] P. Šepitka, Theory of Principal Solutions at Infinity for Linear Hamiltonian Systems, Ph.D thesis, Masaryk University, Brno, 2014. Available from https://is.muni.cz/th/vqad7/?lang=en.

[27] P. Šepitka, Genera of conjoined bases for (non)oscillatory linear Hamiltonian systems: extended theory, submitted, 2017.

[28] P. Šepitka and R. Šimon Hilscher, Minimal principal solution at infinity for nonoscillatory linear Hamiltonian systems, J. Dynam. Differential Equations, 26 (2014), 57–91.

[29] P. Šepitka and R. Šimon Hilscher, Principal solutions at infinity of given ranks for nonoscillatory linear Hamiltonian systems, J. Dynam. Differential Equations, 27 (2015), 137–175.

[30] P. Šepitka and R. Šimon Hilscher, Principal and antiprincipal solutions at infinity of linear Hamiltonian systems, J. Differential Equations, 259 (2015), 4651–4682.

[31] P. Šepitka and R. Šimon Hilscher, Genera of conjoined bases of linear Hamiltonian systems and limit characterization of principal solutions at infinity, J. Differential Equations, 260 (2016), 6581–6603.

[32] P. Šepitka and R. Šimon Hilscher, Reid’s construction of minimal principal solution at infinity for linear Hamiltonian systems, in Differential and Difference Equations with Applications (eds. S. Pinelas, Z. Došlá, O. Došlá and P. E. Kloeden), Proceedings of the International Conference on Differential & Difference Equations and Applications (Amadora, 2015), Springer Proceedings in Mathematics & Statistics, Vol. 164, Springer, (2016), 359–369.

[33] R. Šimon Hilscher, On general Sturmian theory for abnormal linear Hamiltonian systems, in Dynamical Systems, Differential Equations and Applications (eds. W. Feng, Z. Feng, M. Grasselli, A. Ibragimov, X. Lu, S. Siegmund and J. Voigt), Proceedings of the 8th AIMS Conference on Dynamical Systems, Differential Equations and Applications (Dresden, 2010), Discrete Contin. Dyn. Syst., Suppl. 2011, American Institute of Mathematical Sciences (AIMS), (2011), 684–691.

[34] J. L. Speyer and D. H. Jacobson, Primer on Optimal Control Theory, Advances in Design and Control, Vol. 20, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2010.

[35] G. Stefani and P. Zezza, Constrained regular LQ-control problems, SIAM J. Control Optim., 35 (1997), 876–900.

[36] M. Wahrheit, Eigenvalue problems and oscillation of linear Hamiltonian systems, Int. J. Difference Equ., 2 (2007), 221–244.

[37] V. Zeidan, Sufficiency criteria via focal points and via coupled points, SIAM J. Control Optim., 30 (1992), 82–98.

[38] V. Zeidan, The Riccati equation for optimal control problems with mixed state-control constraints: necessity and sufficiency, SIAM J. Control Optim., 32 (1994), 1297–1321.

[39] V. Zeidan, New second-order optimality conditions for variational problems with $C^2$-Hamiltonians, SIAM J. Control Optim., 40 (2001), 577–609.

Received August 2017; revised July 2018.

E-mail address: sepitkap@math.muni.cz