BOWERS-STEPHENSON’S CONJECTURE ON THE CONVERGENCE OF
INVERSIVE DISTANCE CIRCLE PACKINGS TO THE RIEMANN MAPPING

YUXIANG CHEN, YANWEN LUO, XU XU, SIQI ZHANG

ABSTRACT. Bowers and Stephenson [3] introduced the notion of inversive distance circle packings as a natural generalization of Thurston’s circle packings [28]. They conjectured that discrete conformal maps induced by inversive distance circle packings converge to the Riemann mapping. Motivated by the recent work of Luo-Sun-Wu [21], we prove Bowers-Stephenson’s conjecture for Jordan domains by establishing a maximal principle, an infinite rigidity theorem and a solvability theorem of certain prescribing combinatorial curvature problems for inversive distance circle packings.

CONTENTS

1. Introduction 1
2. Inversive distance circle packings and weighted Delaunay triangulations 4
  2.1. Basic properties of inversive distance circle packings 4
  2.2. Weighted Delaunay triangulations 7
3. A maximal principle, a ring lemma and spiral hexagonal triangulations 10
  3.1. A maximal principle 10
  3.2. A ring lemma 13
  3.3. Spiral hexagonal triangulations and linear discrete conformal factors 14
4. Rigidity of infinite inversive distance circle packings 17
5. The convergence of inversive distance circle packings 21
  5.1. Proof of the main theorem 21
  5.2. Inversive distance circle packings along flows 24
  5.3. Standard subdivisions of an equilateral triangle 26
  5.4. Proof of Theorem 5.2 30
References 32

1. INTRODUCTION

In [29], Thurston proposed a constructive approach to the Riemann mapping theorem by approximating conformal mappings in simply connected domains using circle packings. Thurston conjectured that the discrete conformal maps induced by circle packings converge to the Riemann mapping. Thurston’s conjecture has been proved elegantly by Rodin-Sullivan [25]. Since then, there have been lots of important works on the convergence of discrete conformal maps to the Riemann mapping. See [4, 12, 15, 16, 21, 30] and others.

Key words and phrases. Inversive distance circle packings, maximal principles, infinite rigidity, convergence.
MSC (2020): 52C25, 52C26.
Motivated by Thurston’s circle packings [28], Bowers-Stephenson [3] introduced the notion of inverse distance circle packings and conjectured that the Riemann mapping could be approximated by inverse distance circle packings. In this paper, we prove Bowers-Stephenson’s conjecture for Jordan domains as a counterpart of Thurston’s conjecture [29] in the setting of circle packings. The main idea comes from the recent work of Luo-Sun-Wu [21].

Suppose $S$ is a topological surface possibly with boundary and $\mathcal{T}$ is a triangulation of $S$. We use $V = V(\mathcal{T})$, $E = E(\mathcal{T})$ and $F = F(\mathcal{T})$ to denote the set of vertices, edges, and faces of $\mathcal{T}$ respectively. A piecewise linear metric $d$ (PL metric for simplicity) on $(S, \mathcal{T})$ is a flat cone metric on $S$ such that each face in $F$ in the metric is isometric to a non-degenerate Euclidean triangle. In this case, one can represent the PL metric on $(S, \mathcal{T})$ as a length function $l : E \to \mathbb{R}_{>0}$, which satisfies the strict triangle inequality for any face in $F$. Conversely, given a function $l : E \to \mathbb{R}_{>0}$ satisfying the strict triangle inequality, one can construct a PL metric on $(S, \mathcal{T})$ by isometrically gluing Euclidean triangles along edges in pairs. Hence, we also refer to a PL metric on $(S, \mathcal{T})$ as a function $l : E \to \mathbb{R}_{>0}$ satisfying the strict triangle inequality for any face in $F$.

For a PL metric $l : E \to \mathbb{R}_{>0}$ on $(S, \mathcal{T})$, the combinatorial curvature is a map $K : V \to (-\infty, 2\pi)$ sending an interior vertex $v \in V$ to $2\pi$ minus the sum of angles at $v$ and a boundary vertex $v \in V$ to $\pi$ minus the sum of angles at $v$. The combinatorial curvature $K$ for a PL metric on $(S, \mathcal{T})$ satisfies the discrete Gauss-Bonnet formula

$$
\sum_{v \in V} K(v) = 2\pi \chi(S),
$$

where $\chi(S)$ is the Euler characteristic of the surface. A vertex $v$ is flat in a PL metric if $K(v) = 0$. A PL metric is flat if all interior vertices are flat.

**Definition 1.1 ([3]).** Suppose $(S, \mathcal{T})$ is a triangulated surface with a weight $I : E \to (-1, +\infty)$. A PL metric $l : E \to \mathbb{R}_{>0}$ on $(S, \mathcal{T})$ is an inverse distance circle packing metric on the weighted triangulated surface $(S, \mathcal{T}, I)$ if there exists a function $u : V \to \mathbb{R}$ such that for any edge $e \in E$ with vertices $v$ and $v'$, the length $l(e)$ is given by

$$
l(e) = \sqrt{e^{2u(v)} + e^{2u(v')} + 2I(e)e^{u(v)} + e^{u(v')}}.
$$

The function $u : V \to \mathbb{R}$ is called a label on $(S, \mathcal{T}, I)$. Two inverse distance circle packing metrics $(S, \mathcal{T}, I, l)$ and $(S, \mathcal{T}, \bar{I}, \bar{l})$ are conformally equivalent if $I = \bar{I}$. In this case, we set $w = \bar{u} - u$ and denote this relation as $l^* = \bar{l} = w * l$. The function $w$ is called a discrete conformal factor on $(S, \mathcal{T}, I, l)$.

If we set $r(v) = e^{u(v)}$ for $v \in V$, then the weight $I(e)$ in (2) is the inverse distance of the two circles centered at $v$ and $v'$ with radii $r(v)$ and $r(v')$ respectively. The map $r : V \to (0, +\infty)$ is referred as an inverse distance circle packing on the weighted triangulated surface $(S, \mathcal{T}, I)$. Thurston’s circle packing [28] is a special type of inverse distance circle packing with $I \in [0, 1]$ in (2). An excellent source for the comprehensive theory of circle packings is [27].

The main focus of this paper is to provide an affirmative answer to Bowers-Stephenson’s conjecture on the convergence of discrete conformal maps induced by inverse distance circle packings to the Riemann mapping for Jordan domains. Specifically, let $\Omega$ be a Jordan domain in the plane with three distinct boundary points $p, q, r$ specified. By the Riemann mapping theorem, there exists a conformal map from $\Omega$ to the interior of an equilateral Euclidean triangle.
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

△ABC with unit edge length, which could be uniquely extended to be a homeomorphism \( g \) from \( \Omega \) to \( \triangle ABC \) with \( p, q, r \) sent to \( A, B, C \) respectively by Caratheodory’s extension theorem \([24]\). The map \( g \) and \( g^{-1} \) are referred as the Riemann mapping for \( (\Omega, (p, q, r)) \). Let \((D, T, I)\) be an oriented weighted polygonal disk in the plane with three distinct boundary vertices \( p, q, r \) and \( l \) be a flat inversive distance circle packing metric on \((D, T, I)\). Suppose that there exists a function \( w : V \to \mathbb{R} \) such that \( l^* = w \ast l \) is an inversive distance circle packing metric on \((D, T, I)\) with total area \( \sqrt{\frac{\pi}{4}} \) combinatorial curvature \( \frac{2\pi}{3} \) at \( p, q, r \), and flat at other vertices. Then \((D, T, l^*)\) is isometric to a triangulated unit equilateral triangle \((\triangle ABC, \mathcal{T}')\) with some triangulation \( \mathcal{T}' \) and the standard flat metric. Let \( f \) be the orientation-preserving piecewise linear map induced by the map sending the vertices of \( T \) to the corresponding vertices of \( \mathcal{T}' \) such that \( f(A) = p, f(B) = q \) and \( f(C) = r \). The map \( f \) is called the discrete conformal map associated to \((D, T, I, l, \{p, q, r\})\). We prove the following theorem on the convergence of discrete conformal maps induced by a specific sequence of inversive distance circle packings on \( \Omega \).

**Theorem 1.2.** Let \( \Omega \) be a Jordan domain in the complex plane with three distinct boundary points \( p, q, r \) specified. Let \( f \) be the Riemann mapping from the equilateral triangle \( \triangle ABC \) to \( \overline{\Omega} \) such that \( f(A) = p, f(B) = q, f(C) = r \). Then there exists a sequence of weighted triangulated polygonal disks \( (\Omega_n, T_n, I_n, (p_n, q_n, r_n)) \) with inversive distance circle packing metrics \( l_n \), where \( T_n \) is a triangulation of \( \Omega_n \), \( I_n : E_n \to (1, +\infty) \) is a weight defined on \( E_n = E(T_n) \) and \( p_n, q_n, r_n \) are three distinct boundary vertices of \( T_n \), such that

(a) \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \) with \( \Omega_n \subset \Omega_{n+1} \), and \( \lim_n p_n = p, \lim_n q_n = q, \lim_n r_n = r \).

(b) discrete conformal maps \( f_n \) from \( \triangle ABC \) to \((\Omega_n, T_n, I_n, l_n)\) with \( f_n(A) = p_n, f_n(B) = q_n, f_n(C) = r_n \) exist.

(c) discrete conformal maps \( f_n \) converge uniformly to the Riemann mapping \( f \).

In comparison with Rodin-Sullivan’s convergence theorem for circle packings in \([25]\), which allows the approximating triangulated polygonal disks to be arbitrarily selected, Theorem 1.2 requires that the approximating weighted triangulated polygonal disks should be carefully selected. The key difference is that the discrete conformal map does not exist for general inversive distance circle packings on weighted triangulated polygonal disks with inversive distance \( I : E \to (1, +\infty) \), while Koebe-Andreev-Thurston theorem ensures the existence of discrete conformal maps for any circle packings on triangulated polygonal disks. In the rest of this paper, we assume that \( I : E \to (1, +\infty) \) unless otherwise stated. This condition corresponds to the “S-packings” introduced by Bowers-Stephenson \([3]\).

The paper is organized as follows. In Section 2 we give some preliminaries on inversive distance circle packings and weighted Delaunay triangulations. In Section 3 we derive a maximal principle and a ring lemma for inversive distance circle packings. We also study the properties of inversive distance circle packings on spiral hexagonal triangulations in this section. In Section 4 we prove the rigidity of infinite inversive distance circle packings on the hexagonal triangulated plane. In Section 5 we solve some prescribing combinatorial curvature problem for inversive distance circle packings and prove Theorem 1.2.

**Acknowledgement.** The research of Xu Xu is supported by the Fundamental Research Funds for the Central Universities under Grant No. 2042020k0199.
2. INVERSIVE DISTANCE CIRCLE PACKINGS AND WEIGHTED DELAUNAY TRIANGULATIONS

In this section, we collect some basic properties of inversive distance circle packings and weighted Delaunay triangulations. We first describe the admissible space of inversive distance circle packings on a triangle and the variation of inner angles in this space. Then we discuss a notion of generalized weighted Delaunay triangulations and their relationships with inversive distance circle packings.

2.1. Basic properties of inversive distance circle packings. Let \( (S, \mathcal{T}, I) \) be a weighted triangulated surface. We use \( v_i \) to denote a vertex in \( V \), \( e_{ij} = v_iv_j \) to denote an edge in \( E \) and \( \triangle v_iv_jv_k \) to denote a face in \( F \). We will denote \( f_i = f(v_i) \) if \( f \) is a function defined on \( V \), \( f_{ij} = f(v_iv_j) = f(e_{ij}) \) if \( f \) is a function defined on \( E \), and \( f_{ijk} = f(\triangle v_iv_jv_k) \) if \( f \) is a function defined on \( F \).

For any function \( u : V \rightarrow \mathbb{R} \), the formula (2) produces a positive function \( l \) on \( E \). However, for a face \( \triangle v_iv_jv_k \) in \( (S, \mathcal{T}, I) \), the positive numbers \( l_{ij}, l_{ik}, l_{jk} \) may not satisfy the strict triangle inequality

\[
\forall r, s, t \in \{i, j, k\}: \quad l_{rs} < l_{rt} + l_{st},
\]

(3)

The label \( u : V \rightarrow \mathbb{R} \) is said to be admissible if the function \( l : E \rightarrow (0, +\infty) \) determined by \( u : V \rightarrow \mathbb{R} \) via the formula (2) satisfies the strict triangle inequality (3) for every face in \( (S, \mathcal{T}, I) \). We also say that the corresponding inversive distance circle packing \( r : V \rightarrow \mathbb{R}_{>0} \) on \( (S, \mathcal{T}, I) \) with \( r_i = e^{u_i} \) is admissible, if it causes no confusion in the context. The admissible space of inversive distance circle packings on \( (S, \mathcal{T}, I) \) consists of all the admissible inversive distance circle packings on \( (S, \mathcal{T}, I) \). For an admissible inversive distance circle packing \( r \) on \( (S, \mathcal{T}, I) \), every face in \( (S, \mathcal{T}, I) \) is isometric to a non-degenerate Euclidean triangle with edge lengths given by (2). We also say that \( r : V \rightarrow \mathbb{R}_{>0} \) generates a PL metric on \( (S, \mathcal{T}, I) \) for simplicity in this case.

If three positive numbers \( l_{ij}, l_{ik}, l_{jk} \) satisfy the triangle inequality

\[
\forall r, s, t \in \{i, j, k\}: \quad l_{rs} \leq l_{rt} + l_{st},
\]

(4)

then \( l_{ij}, l_{ik}, l_{jk} \) generate a generalized Euclidean triangle \( \triangle v_iv_jv_k \). If \( l_{ij} = l_{ik} + l_{jk} \), the generalized triangle \( \triangle v_iv_jv_k \) is flat at \( v_k \), and the inner angle at \( v_k \) is defined to be \( \pi \). In this case, the generalized triangle is referred as a degenerate triangle. A function \( l : E \rightarrow \mathbb{R}_{>0} \) is called a generalized PL metric on \( (S, \mathcal{T}) \) if the triangle inequality (4) is satisfied for every face in \( (S, \mathcal{T}) \). A PL metric is a special type of generalized PL metric with the strict triangle inequality (3) for every face in \( (S, \mathcal{T}) \). The combinatorial curvature of generalized PL metrics is defined the same as that of PL metrics and still satisfies the discrete Gauss-Bonnet formula (1).

A generalized PL metric \( l : E \rightarrow \mathbb{R}_{>0} \) is called a generalized inversive distance circle packing metric on a weighted triangulated surface \( (S, \mathcal{T}, I) \) if there exists a map \( u : V \rightarrow \mathbb{R} \) such that \( l \) is determined by \( u \) via the formula (2). In this case, the map \( r : V \rightarrow \mathbb{R}_{>0} \) with \( r_i = e^{u_i} \) is said to be a generalized inversive distance circle packing on \( (S, \mathcal{T}, I) \). We will denote it as \( (S, \mathcal{T}, I, l) \), \( (S, \mathcal{T}, I, u) \), or \( (S, \mathcal{T}, I, r) \) interchangeably.

We have a characterization of the admissible space of inversive distance circle packings on a weighted triangle and an extension of inner angles for generalized triangles generated by generalized inversive distance circle packings.
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

**Lemma 2.1** ([13] [31] [32]). Let $\triangle v_1 v_2 v_3$ be a face in $(S, \mathcal{T})$ with three weights $I_1, I_2, I_3 \in (1, +\infty)$ defined on edges opposite to the vertices $v_1, v_2, v_3$ respectively. Let $u : \{v_1, v_2, v_3\} \to \mathbb{R}$ be a function defined on the vertices, inducing edge lengths by

$$l_{ij} = \sqrt{e^{2u_i} + e^{2u_j} + 2e^{u_i+u_j}I_k} = \sqrt{r_i^2 + r_j^2 + 2r_i r_j I_k},$$

where $r_i = e^{u_i}$, $\{i, j, k\} = \{1, 2, 3\}$.

(a) $l_{12}, l_{13}, l_{23}$ generate a non-degenerate Euclidean triangle if and only if

(b) The admissible space $\Omega_{123}$ of inversive distance circle packings $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$ on $\triangle v_1 v_2 v_3$ is

$$\Omega_{123} = \mathbb{R}^3_{>0} \setminus \bigcup^3_{i=1} V_i,$$

where $\bigcup^3_{i=1} V_i$ is a disjoint union of

$$V_i = \left\{ (r_1, r_2, r_3) \in \mathbb{R}^3_{>0} | \kappa_i \geq \frac{-B_i + \sqrt{\Delta_i}}{2A_i} \right\}$$

with

$$A_i = I_i^2 - 1, \quad B_i = -2(\kappa_j \gamma_k + \kappa_k \gamma_j), \quad \Delta_i = 4(I_1^2 + I_2^2 + I_3^2 + 2I_1 I_2 I_3 - 1)(\kappa_j^2 + \kappa_k^2 + 2\kappa_j \kappa_k \gamma_i).$$

Let $\theta_i$ be the inner angle of $\triangle v_1 v_2 v_3$ at $v_i$, then the inner angles of $\triangle v_1 v_2 v_3$ could be uniquely continuously extended by constants as follows

$$\tilde{\theta}_i(r_1, r_2, r_3) = \begin{cases} \theta_i, & \text{if } (r_1, r_2, r_3) \in \Omega_{123}; \\ \pi, & \text{if } (r_1, r_2, r_3) \in V_i; \\ 0, & \text{otherwise}. \end{cases}$$

**Corollary 2.2** ([13] [31] [32]). If $v_i$ is the flat vertex of the degenerate triangle $\triangle v_1 v_2 v_3$ generated by $(r_1, r_2, r_3) \in \mathbb{R}^3_{>0}$, then $(r_1, r_2, r_3) \in \partial V_i$, i.e. $\kappa_i = \frac{B_i + \sqrt{\Delta_i}}{2A_i}$.

The following lemma describes the change of inner angles along PL metrics generated by smooth families of labels on $(S, \mathcal{T}, I)$.

**Lemma 2.3** ([13] [31] [32]). Let $\triangle v_1 v_2 v_3$ be a face in $(S, \mathcal{T}, I)$ given by Lemma 2.1

(a) Suppose that the label $u \in \mathbb{R}^3$ induces a non-degenerate Euclidean triangle $\triangle v_1 v_2 v_3$, then

$$\frac{\partial \theta_i}{\partial u_j} = \frac{\partial \theta_j}{\partial u_i} = \frac{h_{i,j,k}}{l_{ij}}, \quad \frac{\partial \theta_i}{\partial u_k} = -\frac{\partial \theta_i}{\partial u_j} - \frac{\partial \theta_i}{\partial u_k} < 0,$$

where

$$h_{i,j,k} = \frac{y_1^2 y_2^2 y_3^2}{A_{123} l_{ij}} \left[ \kappa_i^2(1 - I_k^2) + \kappa_j \kappa_k \gamma_i + \kappa_k \kappa_j \gamma_j \right] = \frac{r_i^2 r_j^2 r_k^2 \kappa_i \kappa_j \kappa_k \gamma_i}{A_{123} l_{ij}} h_k$$

with $A_{123} = l_{12} l_{13} \sin \theta_1$ and

$$h_k = \kappa_k(1 - I_k^2) + \kappa_i \gamma_j + \kappa_j \gamma_i.$$
(b) If \( u = (u_1, u_2, u_3) \in \mathbb{R}^3 \) is not admissible, then one of \( h_1, h_2, h_3 \) is negative and the other two are positive. Specially, if \( u \in \mathbb{R}^3 \) generates a degenerate triangle \( \triangle v_1v_2v_3 \) having \( v_3 \) as the flat vertex, then \( h_1 > 0, h_2 > 0, h_3 < 0 \) at \( u \). Moreover, in this case,

\[
h_{12,3} \to -\infty, \quad h_{13,2} \to +\infty, \quad h_{23,1} \to +\infty
\]
as \((\tilde{r}_1, \tilde{r}_2, \tilde{r}_3) \in \Omega_{123} \) tends to \((r_1, r_2, r_3) = (e^{u_1}, e^{u_2}, e^{u_3}) \in \partial \Omega_{123}\).

Note that \( h_{ij,k} \) is only defined for non-degenerate inversive distance circle packings \((r_1, r_2, r_3) \in \Omega_{123} \subseteq \mathbb{R}_{>0}^3\), while \( h_i \) is defined for any \((r_1, r_2, r_3) \in \mathbb{R}_3^3\).

For a non-degenerate inversive distance circle packing metric \( l \) on \((S, \mathcal{T}, I)\), set \( \eta^{k}_{ij} = h_{ij,k}/l_{ij} \) and define the conductance \( \eta : E \to \mathbb{R} \) for \((S, \mathcal{T}, I, l)\) by

\[
\eta_{ij} = \begin{cases} 
\eta^{k}_{ij} + \eta^{m}_{ij}, & v_i v_j \text{ is an interior edge contained in } \triangle v_i v_j v_k \text{ and } \triangle v_i v_j v_m; \\
\eta^{k}_{ij}, & v_i v_j \text{ is a boundary edge contained in } \triangle v_i v_j v_k.
\end{cases}
\]

As a direct corollary of formula (8), we have the following variation of combinatorial curvatures.

**Corollary 2.4** \((13, 31, 32)\). Suppose \( w(t) \ast l \) is a family of inversive distance circle packing metrics on \((S, \mathcal{T}, I)\) induced by a smooth family of discrete conformal factor \( w(t) \in \mathbb{R}^V \). Let \( K(t) \) and \( \eta(t) \) be the combinatorial curvature and the conductance of \((S, \mathcal{T}, I, w(t) \ast l)\). Then

\[
\frac{dK_i(t)}{dt} = \sum_{j \sim i} \eta_{ij}(t) \left( \frac{dw_i}{dt} - \frac{dw_j}{dt} \right).
\]

We prove the following results on inversive distance circle packings.

**Proposition 2.5.** Let \( \triangle v_1v_2v_3 \) be a face in \((S, \mathcal{T}, I)\) given by Lemma [2.7]

(a) For any fixed \( r_i, r_j \in (0, +\infty) \), the set of \( r_k \in (0, +\infty) \) such that \((r_i, r_j, r_k)\) is an admissible inversive distance circle packing on \( \triangle v_1v_2v_3 \) is an open interval. As a result, if \((r_i, r_j, \tilde{r}_k)\) and \((r_i, r_j, \tilde{r}_k)\) are two generalized inversive distance circle packings on \( \triangle v_1v_2v_3 \) with \( \tilde{r}_k < \tilde{r}_k \), then for any \( r_k \in (\tilde{r}_k, \tilde{r}_k) \), \((r_i, r_j, r_k)\) generates a non-degenerate triangle \( \triangle v_1v_2v_3 \).

(b) If \( \triangle v_1v_2v_3 \) generated by \((r_1, r_2, r_3) \in \mathbb{R}_3^3\) is a degenerate triangle having \( v_3 \) as the flat vertex, then there exists \( \varepsilon > 0 \) such that \((r_1, r_2, r_3 + t) \in \Omega_{123} \) and

\[
\frac{\partial h_{12,3}}{\partial r_3}(r_1, r_2, r_3 + t) > 0
\]

for \( t \in (0, \varepsilon) \).

**Proof.** To prove part (a), without loss of generality, set \( \{i, j\} = \{2, 3\} \), \( k = 1 \) and

\[
f(\kappa_1) = (1 - I_1^2)\kappa_1^2 + 2\kappa_1(\kappa_2 \gamma_3 + \kappa_3 \gamma_2) + \kappa_2^2(1 - I_2^2) + \kappa_3^2(1 - I_3^2) + 2\kappa_2 \kappa_3 \gamma_1.
\]

By Lemma [2.7](a), we need to show that the solution of \( f(\kappa_1) > 0 \) with \( \kappa_1 \in (0, +\infty) \) is an open interval. The inequality \( f(\kappa_1) > 0 \) is equivalent to the following quadratic inequality

\[
(\kappa_1^2 - 1)\kappa_1^2 - 2\kappa_1(\kappa_2 \gamma_3 + \kappa_3 \gamma_2) - \kappa_2^2(1 - I_2^2) - \kappa_3^2(1 - I_3^2) - 2\kappa_2 \kappa_3 \gamma_1 < 0.
\]

By \( I > 1 \), we have

\[
-\frac{b}{2a} = \frac{\kappa_2 \gamma_3 + \kappa_3 \gamma_2}{I_2^2 - 1} > 0.
\]
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

and the discriminant of the quadratic polynomial defined in (7) satisfies
\[ \Delta = 4(I_1^2 + I_2^2 + I_3^2 + 2I_1I_2I_3 - 1)(\kappa_2^2 + \kappa_3^2 + 2\kappa_2\kappa_3 I_1) > 0. \]

This implies that the solution of \( f(\kappa_1) > 0 \) with \( \kappa_1 > 0 \) is an open interval in \((0, +\infty)\).

To prove part (b), recall that the triangle \( \triangle v_1v_2v_3 \) is degenerate if and only if \( Q = 0 \) by Lemma 2.1 (a), where \( Q \) is defined by (6). By direct calculations, we have \( \frac{\partial Q}{\partial r_3^3} = 2h_3 < 0 \) at \((r_1, r_2, r_3)\) by Lemma 2.3 (b), which implies that \( \frac{\partial Q}{\partial r_3^3} = -\frac{\partial Q}{\partial r_3^3} = -\frac{1}{r_3^2} \frac{\partial Q}{\partial r_3} > 0 \) around \((r_1, r_2, r_3)\). Therefore, for small \( t > 0 \), \( Q(r_1, r_2, r_3 + t) > 0 \) and \((r_1, r_2, r_3 + t)\) generates a non-degenerate triangle.

Using the identities \( Q = \kappa_1 h_1 + \kappa_2 h_2 + \kappa_3 h_3 \) and \( A_{123}^2 = r_1^2 r_2^2 r_3^2 Q \), we can deduce from the definition (9) of \( h_{123} \) that
\[ \frac{\partial h_{123}}{\partial \kappa_3} = \frac{r_1^2 r_2^2 r_3^2}{A_{123}^3} [r_1^2 r_2^2 r_3^2 (\kappa_1 h_1 + \kappa_2 h_2) h_3 - A_{123}^2 (\kappa_1 \gamma_2 + \kappa_2 \gamma_1)]. \]

Note that \( v_3 \) is the flat vertex of the degenerate triangle \( \triangle v_1v_2v_3 \) generated by \((r_1, r_2, r_3)\), then \( A_{123} = 0 \) and \( h_1 > 0, h_2 > 0, h_3 < 0 \) at \((r_1, r_2, r_3)\) by Lemma 2.3 (b), which implies that \( \frac{\partial h_{123}}{\partial \kappa_3} < 0 \) around \((r_1, r_2, r_3)\) in the admissible space \( \Omega_{123} \) by (13). Note that \( \frac{\partial h_{123}}{\partial r_3} = \frac{\partial h_{123}}{\partial \kappa_3} = -\frac{1}{r_3^2} \frac{\partial h_{123}}{\partial r_3} \). Therefore, there exists \( \epsilon > 0 \) such that \( \frac{\partial h_{123}}{\partial r_3} (r_1, r_2, r_3 + t) > 0 \) for \( t \in (0, \epsilon) \).

Q.E.D.

2.2. Weighted Delaunay triangulations. Weighted Delaunay triangulations are natural generalizations of the classical Delaunay triangulations, where the sites generating the corresponding Voronoi decomposition are disks instead of points. It has wide applications in computational geometry. See [2, 5] and others. In this subsection, we propose an alternative characterization of weighted Delaunay triangulations for inversive distance circle packing metrics and generalize weighted Delaunay triangulations for non-degenerate inversive distance circle packing metrics to generalized inversive distance circle packing metrics.

Assume \( r : V \to (0, +\infty) \) is a non-degenerate inversive distance circle packing on a weighted triangulated surface \((S, T, I)\). Let \( \triangle v_1v_2v_3 \) be a Euclidean triangle in the plane isometric to a face in \((S, T, I, r)\). Then there exists a unique geometric center \( C_{123} \) such that its power distances to \( v_i \), defined by \( |C_{123} - v_i|^2 - r_i^2 \), are equal for \( i = 1, 2, 3 \). Projections of the geometric center \( C_{123} \) to the lines \( v_1v_2, v_1v_3, v_2v_3 \) give rise to the geometric centers of these edges, which are denoted by \( C_{12}, C_{13}, C_{23} \) respectively. Please refer to Figure 1. One can refer to [6, 7, 8, 9] for more information on the geometric center generated by discrete conformal structures on manifolds.

Denote \( d_{ij} \) as the signed distance of \( C_{ij} \) to the vertex \( v_i \) and \( h_{ij,k} \) as the signed distance of \( C_{123} \) to the edge \( v_iv_j \). Glickenstein [7] obtained the following identities
\[ d_{ij} = \frac{r_i^2 + r_j^2 I_{ij}}{l_{ij}}, \quad h_{ij,k} = \frac{d_{ik} - d_{ij} \cos \theta_i}{\sin \theta_i}. \]

Note that \( d_{ij} \in \mathbb{R}_{>0} \) could be defined by (14) independent of the existence of the geometric center \( C_{ijk} \), and \( h_{ij,k} \) is symmetric in the indices \( i \) and \( j \), while \( d_{ij} \) is not.

For a weighted triangulated surface with a non-degenerate inversive distance circle packing \((S, T, I, r)\), an interior edge \( v_iv_j \) is weighted Delaunay if \( h_{ij,k} + h_{ij,l} \geq 0 \), where \( \triangle v_iv_jv_k \) and \( \triangle v_iv_jv_l \) are two triangles in \( F \) sharing the common edge \( v_iv_j \). And \((S, T, I, r)\) is weighted
Delaunay if all the interior edges are weighted Delaunay. Note that weighted Delaunay triangulations are only defined for non-degenerate inversive distance circle packings. We need to introduce the definition of weighted Delaunay triangulations for generalized inversive distance circle packing metrics. To this end, we introduce the following notion.

**Definition 2.6.** Let \( r \in \mathbb{R}_V^+ \) be a generalized inversive distance circle packing on a weighted triangulated surface \((S, T, I)\). Suppose that \( \triangle v_1 v_2 v_3 \) is a generalized triangle in \((S, T, I, r)\). If \( \triangle v_1 v_2 v_3 \) is non-degenerate, define

\[
\theta_{ij,k} = \arctan \frac{h_{ij,k}}{d_{ij}}.
\]

If \( \triangle v_1 v_2 v_3 \) is degenerate, define \( \theta_{ij,k} \) as

\[
\theta_{ij,k} = \begin{cases} 
\frac{\pi}{2}, & \text{if } v_i \text{ or } v_j \text{ is the flat vertex,} \\
-\frac{\pi}{2}, & \text{if } v_k \text{ is the flat vertex.}
\end{cases}
\]

Note that for a non-degenerate triangle \( \triangle v_1 v_2 v_3 \) in \((S, T, I, r)\), \( \theta_{ij,k} \) is in fact the signed angle \( \angle_{\mathcal{C}ijk} \), which is negative if \( h_{ij,k} < 0 \) and non-negative otherwise. Please refer to Figure 2 for this.

**Figure 1.** Sign distances of the geometric center.

**Figure 2.** The angle \( \theta_{ij,k} \) when \( h_{ij,k} < 0 \) (left) and \( h_{ij,k} > 0 \) (right).

For non-degenerate inversive distance circle packings on a weighted triangle \( \triangle v_1 v_2 v_3 \), \( \theta_{ij,k} \) is a continuous function of \((r_1, r_2, r_3) \in \Omega_{123}\) and satisfies \( \theta_{ij,k} + \theta_{ik,j} = \theta_i \). We further have
the following property on $\theta_{ij,k}$ for generalized inversive distance circle packings on a weighted triangle.  

**Lemma 2.7.** Suppose $\Delta v_1v_2v_3$ is a face in a weighted triangulated surface $(S, T, I)$. Then $\theta_{ij,k}(r_1, r_2, r_3)$ is a continuous function defined on $\Omega_{123}$ and satisfies 

\begin{equation}
\theta_{ij,k} + \theta_{ik,j} = \theta_i. 
\end{equation}

**Proof.** We just need to prove that $\theta_{ij,k}(r_1, r_2, r_3) \rightarrow \theta_{ij,k}(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ as $(r_1, r_2, r_3) \in \Omega_{123}$ tends to a point $(\bar{r}_1, \bar{r}_2, \bar{r}_3) \in \partial \Omega_{123}$. If $v_k$ is the flat vertex of the degenerate triangle $\Delta v_1v_2v_3$ generated by $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$, then $h_{ij,k}(r_1, r_2, r_3) \rightarrow -\infty$ as $(r_1, r_2, r_3) \rightarrow (\bar{r}_1, \bar{r}_2, \bar{r}_3)$ by Lemma 2.3. As a result, we have $\theta_{ij,k}(r_1, r_2, r_3) = \arctan \frac{h_{ij,k}}{d_{ij}} \rightarrow -\pi = \theta_{ij,k}(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ by Definition 2.6. If $v_i$ is the flat vertex of the degenerate triangle $\Delta v_1v_2v_3$ generated by $(\bar{r}_1, \bar{r}_2, \bar{r}_3)$, then $h_{ij,k}(r_1, r_2, r_3) \rightarrow +\infty$ as $(r_1, r_2, r_3) \rightarrow (\bar{r}_1, \bar{r}_2, \bar{r}_3)$. As a result, we have $\theta_{ij,k}(r_1, r_2, r_3) \rightarrow \pi = \theta_{ij,k}(\bar{r}_1, \bar{r}_2, \bar{r}_3)$ as $(r_1, r_2, r_3) \rightarrow (\bar{r}_1, \bar{r}_2, \bar{r}_3)$ by Definition 2.6. The same argument applies to the case that $v_j$ is the flat vertex. Q.E.D.

Weighted Delaunay triangulations for non-degenerate inversive distance circle packings have a simple characterization using $\theta_{ij,k}$. 

**Corollary 2.8.** Suppose $v \in \mathbb{R}_0^V$ is a non-degenerate inversive distance circle packing on a weighted triangulated surface $(S, T, I)$. An edge $v_iv_j \in E$ is shared by two adjacent non-degenerate triangles $\Delta v_iv_jv_k$ and $\Delta v_iv_jv_l$ in $(S, T, I, r)$. Then the edge $v_iv_j$ is weighted Delaunay in the inversive distance circle packing $r$ if and only if 

$$\theta_{ij,k} + \theta_{ij,l} \geq 0.$$ 

**Proof.** Since $\theta_{ij,k} = \arctan \frac{h_{ij,k}}{d_{ij}} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\theta_{ij,l} = \arctan \frac{h_{ij,l}}{d_{ij}} \in (-\frac{\pi}{2}, \frac{\pi}{2})$ by Definition 2.6 we have 

$$\frac{h_{ij,k} + h_{ij,l}}{d_{ij}} = \tan \theta_{ij,k} + \tan \theta_{ij,l} = \frac{\sin(\theta_{ij,k} + \theta_{ij,l})}{\cos \theta_{ij,k} \cos \theta_{ij,l}},$$

which implies $h_{ij,k} + h_{ij,l} \geq 0$ is equivalent to $\theta_{ij,k} + \theta_{ij,l} \geq 0$ by $d_{ij} > 0$. Q.E.D.

**Remark 2.9.** Under the conditions in Corollary 2.8 we further have that $h_{ij,k} + h_{ij,l} > 0$ is equivalent to $\theta_{ij,k} + \theta_{ij,l} > 0$ for non-degenerate inversive distance circle packings.

Note that $h_{ij,k}$ is only defined for non-degenerate inversive distance circle packings, while $\theta_{ij,k}$ could be defined for generalized inversive distance circle packings. We introduce the following definition of weighed Delaunay triangulations for generalized inversive distance circle packings, which generalizes the classical definition of weighed Delaunay triangulations for non-degenerate inversive distance circle packings.

**Definition 2.10.** Suppose $r : V \rightarrow (0, +\infty)$ is a generalized inversive distance circle packing on a weighted triangulated surface $(S, T, I)$. Let $v_iv_j \in E$ be an edge shared by two adjacent triangles $\Delta v_iv_jv_k$ and $\Delta v_iv_jv_l$ in $T$. An interior edge $v_iv_j \in E$ is weighted Delaunay in the generalized inversive distance circle packing $r$ if 

$$\theta_{ij,k} + \theta_{ij,l} \geq 0.$$ 

The triangulation $T$ is weighted Delaunay in the generalized inversive distance circle packing $r$ if every interior edge is weighed Delaunay.
For simplicity, we also say that $r$ is a generalized weighted Delaunay inverse distance circle packing on $(S, T, I)$ if $T$ is weighted Delaunay in $r$.

We further have the following monotonicity for the angle $\theta_{ij,k}$ in Definition 2.6.

**Lemma 2.11.** Suppose $\triangle v_1v_2v_3$ is a face in a weighted triangulated surface $(S, T, I)$. Let $(r_1, r_2, \tilde{r}_3)$ and $(r_1, r_2, r_3)$ be two generalized inverse distance circle packings on $\triangle v_1v_2v_3$ with $\tilde{r}_3 < r_3$. If $r_1$ and $r_2$ are fixed, then $\theta_{123}$ is strictly increasing in $r_3 \in [\tilde{r}_3, \bar{r}_3]$.

**Proof.** By Proposition 2.5 (a), $(r_1, r_2, r_3)$ generates a non-degenerate triangle $\triangle v_1v_2v_3$ for $r_3 \in (\tilde{r}_3, \bar{r}_3)$. For $r_3 \in (\tilde{r}_3, \bar{r}_3)$, $h_{123}$ and $\theta_{123}$ are smooth functions of $r_3$. By the definition of $h_i$ and $\gamma_i$, we can deduce that

$$\frac{\partial h_{123}}{\partial \kappa_3} = \frac{r_1^2r_2^2r_3^2}{A_{123}^3 l_{12}} \left[ \frac{1}{r_2^2} \left( r_3^2 (\kappa_1 h_1 + \kappa_2 h_2) h_3 - A_{123}^2 (\kappa_2 \gamma_1 + \kappa_1 \gamma_2) \right) \right]$$

$$= \frac{r_1^4r_2^4r_3^4}{A_{123}^3 l_{12}} \left( 1 - I_{12}^2 - I_{13}^2 - I_{23}^2 - 2I_{12}I_{13}I_{23} \right) (\kappa_1^2 + \kappa_2^2 + 2\kappa_1\kappa_2) < 0.$$ 

This implies

$$\frac{\partial \theta_{123}}{\partial r_3} = -\frac{d_{12} \kappa_3}{d_{12}^2 + (h_{123})^2} \cdot \frac{\partial h_{123}}{\partial \kappa_3} > 0, \quad \forall r_3 \in (\tilde{r}_3, \bar{r}_3)$$

by the definition of $\theta_{123}$. Note that $\theta_{123}$ is a continuous function of $r_3 \in [\tilde{r}_3, \bar{r}_3]$ by Lemma 2.7, then $\theta_{123}$ is strictly increasing in $r_3 \in [\tilde{r}_3, \bar{r}_3]$.

Q.E.D.

3. A MAXIMAL PRINCIPLE, A RING LEMMA AND SPIRAL HEXAGONAL TRIANGULATIONS

3.1. A maximal principle. Let $P_n$ be a star-shaped $n$-sided polygon in the plane with boundary vertices $v_1, \ldots, v_n$ ordered cyclically ($v_{n+i} = v_i$). Assume $v_0$ is an interior point of $P_n$ and it induces a triangulation $T$ of $P_n$ with triangles $\triangle v_0v_iv_{i+1}$. Then an assignment of radii $r : V(T) \rightarrow \mathbb{R}_{>0}$ is a vector in $\mathbb{R}^{n+1}$. For any two vectors $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ in $\mathbb{R}^{n+1}$, we use $x \geq y$ to denote $x_i \geq y_i$ for all $i \in \{0, \ldots, n\}$.

![Figure 3. A star triangulation of a polygon.](image)

We have the following maximal principle for inverse distance circle packings.

**Theorem 3.1.** Let $T$ be a star triangulation of $P_n$ with boundary vertices $v_1, \ldots, v_n$ and a unique interior vertex $v_0$. Let $I : E \rightarrow (1, +\infty)$ be a weight. Suppose $\bar{T}$ and $r$ are two generalized inverse distance circle packings on $(P_n, T, I)$ such that
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

\[(a) \text{ and } r \text{ are generalized weighted Delaunay inversive distance circle packings,}
(b) \text{ the combinatorial curvatures } K_0(r) \text{ and } K_0(\bar{r}) \text{ at the vertex } v_0 \text{ satisfy } K_0(r) \leq K_0(\bar{r}),
(c) \max\{\frac{\theta_j}{\tau_j} | i = 1, 2, \cdots, n\} \leq \frac{\theta_0}{\tau_0}.
\]

Then there exists a constant \(c > 0\) such that \(r = c\bar{r}\).

We use the following notations to prove Theorem 3.1. For \(i \in \{1, \cdots, n\}\), we denote \(I_i\) as \(I_i\) for simplicity. For two adjacent triangles \(\triangle v_0v_jv_{j+1}\) in \(\mathcal{T}\), set \(\theta_{j,j+1}\) to be the inner angle at \(v_0\) in the triangle \(\triangle v_0v_jv_{j+1}\). Moreover, set

\[h^-_j = h_{0j,j-1}, h^+_j = h_{0j,j+1}, \theta^-_j = \theta_{0j,j-1}, \theta^+_j = \theta_{0j,j+1}.\]

The proof of the maximal principle is based on the following key lemma.

**Lemma 3.2.** If \(r, \bar{r} : \{v_0, v_1, \cdots, v_n\} \to \mathbb{R}_{>0}\) satisfy \((a), (b), (c)\) in Theorem 3.1 and there exists \(j \in \{1, 2, \cdots, n\}\) such that \(\frac{r_j}{\tau_j} < \frac{\theta_j}{\theta_0}\), then there exists \(\bar{r} \in \mathbb{R}_{>0}\) such that

\[(a) \bar{r}_i \geq r_i \text{ for } i \in \{1, \cdots, n\},
(b) \frac{\bar{r}_j}{\tau_j} \leq \frac{\theta_j}{\theta_0} \text{ for all } i = 1, 2, \cdots, n,
(c) \bar{r} \text{ is a generalized weighted Delaunay inversive distance circle packing on } (P_n, \mathcal{T}, I),
(d) \text{ if } \alpha(r) \text{ is the cone angle of the inversive distance circle packing } r \text{ at } v_0, \text{ then}
\]

\[(16) \alpha(\bar{r}) > \alpha(r).\]

**Proof.** Up to a scaling, we may assume that \(r_0 = \bar{r}_0\). Then the condition \((c)\) in Theorem 3.1 is equivalent to \(r_i \leq \bar{r}_i\) for all \(i \in \{1, 2, \cdots, n\}\). Set

\[J = \{j \in \{1, 2, \cdots, n\} | r_j < \bar{r}_j\},
K = \{k \in \{1, 2, \cdots, n\} | \tau_k = \bar{r}_k\},
\gamma(r) = \sum_{j \in J} (\theta_{0j,j+1} + \theta_{0j,j-1}) = \sum_{j \in J} (\theta^+_j + \theta^-_j),
\beta(r) = \sum_{k \in K} (\theta_{0k,k+1} + \theta_{0k,k-1}) = \sum_{k \in K} (\theta^+_k + \theta^-_k).\]

Then \(J \neq \emptyset\) by assumption. By \((15)\), we have \(\alpha(r) = \beta(r) + \gamma(r), \alpha(\bar{r}) = \beta(\bar{r}) + \gamma(\bar{r})\), which further implies

\[(17) \beta(\bar{r}) + \gamma(\bar{r}) \leq \beta(r) + \gamma(r)\]

by the condition \(K_0(r) \leq K_0(\bar{r})\).

**Claim 1:** For any \(j \in J, \theta^0_{j-1,j}(r) < \pi \text{ and } \theta^0_{j,j+1}(r) < \pi\).

We will prove that for any \(j \in J, v_0\) is not the flat vertex if the triangle \(\triangle v_0v_jv_{j+1}\) is degenerate. Otherwise, suppose that for some \(j \in J, v_0\) is the flat vertex of the degenerate triangle \(\triangle v_0v_jv_{j+1}\) generated by \(r\). By Corollary 2.2, \(r\) satisfies \(\kappa_0 = f(\kappa_{j-1}, \kappa_j)\), where

\[f(\kappa_{j-1}, \kappa_j) = \frac{1}{I^{j-1}_{j} - 1}[(\kappa_{j} \gamma_{j-1} + \kappa_{j-1} \gamma_{j})
+ (I^2 + I^2_{j-1} + I^2_{j,j-1} + 2I_{j-1}I_{j,j-1} - 1)^{1/2}(\kappa^2_{j} + \kappa^2_{j-1} + 2I_{j,j-1} \kappa_{j} \kappa_{j-1})^{1/2}].\]

Note that \(\kappa_j > \kappa_{j-1}\) and \(\kappa_{j-1} \geq \kappa_{j-1}\). Then we have

\[\bar{\kappa}_0 = \kappa_0 = f(\kappa_{j-1}, \kappa_j) < f(\kappa_{j-1}, \kappa_{j}).\]
This implies that \((\tau_0, \tau_j, \tau_{j-1})\) is in the complement of the space of generalized inversive distance circle packings on \(\triangle v_0v_jv_{j-1}\) in \(\mathbb{R}^3_{>0}\) by Lemma 2.11(b), which contradicts the assumption that \(\tau\) is a generalized inversive distance circle packing on \((P_n, \mathcal{T}, I)\).

**Claim 2:** There exists \(j \in J\) such that \(\theta^+_{j}(r) + \theta^-_{j}(r) > 0\).

We will consider the two cases \(K \neq \emptyset\) and \(K = \emptyset\).

**Case 1:** \(K \neq \emptyset\).

By Lemma 2.11, for any \(i \in K\), \(\theta^-\) and \(\theta^+\) are strictly increasing in \(r_{i-1}\) and \(r_{i+1}\) respectively, which implies that \(\beta(r) \leq \beta(\hat{r})\). As \(J \neq \emptyset\), there exists \(i \in K\) such that \(i-1\) or \(i+1\) is in \(J\). Say \(i-1 \in J\), then \(r_{i-1} < \hat{r}_{i-1}\) and then \(\theta^-_{i}(r) < \theta^-_{i}(\tau)\) by Lemma 2.11. Thus, \(\beta(r) < \beta(\hat{r})\), which implies 0 \(\leq \gamma(\hat{r}) < \gamma(r)\) by (17). Therefore, there exists \(j \in J\) such that \(\theta^+_{j}(r) + \theta^-_{j}(r) > 0\) by the definition of \(\gamma(r)\).

**Case 2:** \(K = \emptyset\).

If \(K = \emptyset\), we have \(J = \{1, \ldots, n\}\),

\[
\gamma(r) = \sum_{j \in J} (\theta^+_{j}(r) + \theta^-_{j}(r)) = \alpha(r) \geq 0.
\]

If \(\alpha(r) > 0\), there exists \(j \in J\) such that \(\theta^+_{j}(r) + \theta^-_{j}(r) > 0\). If \(\alpha(r) = 0\), for any triangle \(\triangle v_0v_jv_{j-1}\), \(j = 1, \ldots, n\), the inner angle at \(v_0\) is equal to zero. Thus all triangles are degenerate. For any triangle \(\triangle v_0v_jv_{j-1}\), the flat vertex is \(v_j\) or \(v_{j-1}\) by Claim 1. Then \(\{\theta^-_{j}(r), \theta^+_{j}(r)\} = \{\frac{\pi}{2}, -\frac{\pi}{2}\}, \forall j \in \{1, \ldots, n\}\). Without loss of generality, we may assume \(v_1\) is the flat vertex of \(\triangle v_0v_1v_2\). Then \(\theta^+_{1}(r) = \frac{\pi}{2}, \theta^-_{1}(r) = -\frac{\pi}{2}\) by Definition 2,6 and \(l_{02}(r) = l_{01}(r) + l_{12}(r) > l_{01}(r)\). By the weighted Delaunay condition (a) in Theorem 3,1, \(\theta^-_{j}(r) = \frac{\pi}{2}\), which implies \(\theta^-_{j}(r) = \frac{\pi}{2}\) and \(l_{03}(r) = l_{02}(r) + l_{23}(r) > l_{02}(r)\). By induction, we have a contradiction

\[
l_{01}(r) < l_{02}(r) < \cdots < l_{0n}(r) < l_{01}(r).
\]

This completes the proof of Claim 2.

Now we fix \(j \in J\) in Claim 2. Then we have

(18)

\[
\theta^+_{j}(r) + \theta^-_{j}(r) > 0.
\]

In the following, we show that there exists \(\epsilon > 0\) such that \(\hat{r} = (r_0, \ldots, r_j + t, \ldots, r_n)\) satisfies Lemma 3.2 for \(t \in (0, \epsilon)\). It is easy to check that for \(t \in (0, \tau_j - r_j)\), \(\hat{r}\) satisfies Lemma 3.2(a) and (b).

To see part (c) of Lemma 3.2 we first show that there exists \(\epsilon > 0\) such that \(\hat{r}\) is a generalized inversive distance circle packing on \((P_n, \mathcal{T}, I)\) for \(t \in (0, \epsilon)\). Furthermore, we will show that the triangles \(\triangle v_0v_jv_{j\pm 1}\) generated by \(\hat{r}\) are non-degenerate.

The triangle \(\triangle v_0v_jv_{j-1}\) generated by \(r\) is non-degenerate or degenerate with \(v_j\) or \(v_{j-1}\) as the flat vertex by Claim 1. By Proposition 2.5 we just need to prove that \(v_{j-1}\) is not the flat vertex of the triangle \(\triangle v_0v_jv_{j-1}\) generated by \(r\) if it is degenerate. Otherwise, we have \(\theta^-_{j}(r) = -\frac{\pi}{2}\) by Definition 2.6, which implies \(\theta^+_{j}(r) > \frac{\pi}{2}\) by (18). However, this is impossible since \(\theta^+_{j}(r) \in [-\frac{\pi}{2}, \frac{\pi}{2}]\) by Definition 2.6. Therefore, \(v_{j-1}\) can never be the flat vertex of the triangle \(\triangle v_0v_jv_{j-1}\) if it is degenerate. Similar arguments applying to the triangle \(\triangle v_0v_jv_{j+1}\) show that \(v_{j+1}\) can never be the flat vertex of the triangle \(\triangle v_0v_jv_{j+1}\) if it is degenerate. Therefore, by Proposition 2.5(b), there exists \(\epsilon > 0\) such that for \(t \in (0, \epsilon)\), \(\hat{r}\) is a generalized inversive distance circle packing on \((P_n, \mathcal{T}, I)\) and the triangles \(\triangle v_0v_jv_{j\pm 1}\) generated by \(\hat{r}\) are non-degenerate.

Next, we show that \(\hat{r}\) satisfies the weighted Delaunay condition. As \(\hat{r}\) differs from \(r\) only at the \(j\)-th position, we just need to consider the edges \(v_0v_j\) and \(v_0v_{j\pm 1}\). For the edge \(v_0v_j\), since
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

\[ \theta_j^+(r) + \theta_j^-(r) > 0, \text{ we have } \theta_j^+(\hat{r}) + \theta_j^-(\hat{r}) > 0 \text{ for small } t > 0 \text{ by the continuity of } \theta_j^\pm \text{ in Lemma 2.7. For the edge } v_0v_{j-1}, \theta_{j-1}(r) = \theta_{j-1}(\hat{r}). \text{ We further have } \theta_{j-1}^+(r) < \theta_{j-1}^-(r) \text{ for } t \in (0, \pi_j - r_j) \text{ by Lemma 2.11, which implies } \theta_{j-1}^+(\hat{r}) + \theta_{j-1}^-(\hat{r}) > \theta_{j-1}^+(r) + \theta_{j-1}^-(r) \geq 0. \text{ This implies that the edge } v_0v_{j-1} \text{ satisfies the weighted Delaunay condition for } \hat{r}. \text{ The same arguments apply to the edge } v_0v_{j+1}. \]

To see part (d) of Lemma 3.2 by the arguments for part (c), there exists \( \epsilon > 0 \) such that the triangles \( \Delta v_0v_jv_{j+1} \) are non-degenerate in \( \hat{r} \) and \( \theta_j^+(\hat{r}) + \theta_j^-(\hat{r}) > 0 \) for \( t \in (0, \epsilon) \), which implies \( h_j^+(\hat{r}) + h_j^-(\hat{r}) > 0 \) for \( t \in (0, \epsilon) \) by Remark 2.9. Note that \( \alpha(\hat{r}) \) is continuous for \( t \in [0, \epsilon] \), smooth for \( t \in (0, \epsilon) \), and

\[
\frac{\partial \alpha}{\partial t}(\hat{r}) = \frac{h_j^+(\hat{r}) + h_j^-(\hat{r})}{l_{0j}} > 0, \quad t \in (0, \epsilon)
\]

by Lemma 2.3(a), we have \( \alpha(\hat{r}) > \alpha(r) \) for \( t \in (0, \epsilon) \).

Q.E.D.

Now we can prove Theorem 3.1.

**Proof for Theorem 3.1.** Without loss of generality, we assume \( r_0 = \bar{r}_0 \) and \( r_i \leq \bar{r}_i \) for all \( i = 1, 2, \ldots, n \). We prove the theorem by contradiction. Suppose that there exists a weighted Delaunay inversive distance circle packing \( P_n, T, I \) such that \( r_0 = \bar{r}_0 \), \( r_i \leq \bar{r}_i \) for all \( i = 1, 2, \ldots, n \) with one \( r_{i_0} < \bar{r}_{i_0} \) and \( \alpha(\bar{r}) \leq \alpha(r) \). By Lemma 3.2, we may assume that

\[
\alpha(\bar{r}) < \alpha(r).
\]

On the other hand, consider the set

\[ X := \{ x \in \mathbb{R}^{n+1} | r \leq x \leq \bar{r}, \text{ and } x \text{ is a generalized weighted Delaunay inversive distance circle packing on } (P_n, T, I) \}. \]

Obviously, \( r \in X \) and \( X \) is bounded. By Lemma 2.7, \( X \) is a closed subset of \( \mathbb{R}^{n+1} \). Therefore, \( X \) is a nonempty compact set and \( \alpha(x) \) has a maximum point on \( X \). Let \( t \in X \) be a maximum point of the continuous function \( f(x) = \alpha(x) \) on \( X \). If \( t \neq \bar{r} \), then by Lemma 3.2, we can find a weighted Delaunay inversive distance circle packing \( \hat{t} \) on \( (P_n, T, I) \) such that \( \hat{t} \geq t, \hat{t}_0 = \bar{r}_0, \hat{t}_i \leq \bar{r}_i \) and \( \alpha(\hat{t}) > \alpha(t) \), which implies that \( t \) is not a maximum point of \( f(x) = \alpha(x) \) on \( X \). So, \( t = \bar{r} \) and then

\[
\alpha(\bar{r}) = \alpha(t) \geq \alpha(r) > \alpha(\bar{r}),
\]

where the last inequality comes from (19). This is a contradiction. Q.E.D.

**Remark 3.3.** For simplicity, we only present the maximal principle for the inversive distance \( I : E \to (1, +\infty) \), which is enough for the application in the convergence of inversive distance circle packings in this paper. For a much more general version of the maximal principle for inversive distance circle packings with \( I : E \to (-1, +\infty) \) and generic discrete conformal structures on surfaces [9, 33], please refer to [22].

### 3.2. A ring lemma.

**Lemma 3.4.** Let \( T \) be a star triangulation of an \( n \)-sided polygon \( P_n \) with boundary vertices \( v_1, \ldots, v_n \) and a unique interior vertex \( v_0 \). Let \( I : E \to (1, +\infty) \) be a weight and \( r \) be a flat generalized inversive distance circle packing on \( (P_n, T, I) \). Then there exists a constant \( C = C(I, n) > 0 \) such that \( r_0 \leq Cr_i \) for all \( i \in \{1, 2, \ldots, n\} \).
developing map $\phi$ with any isometrically embedding in $C$. Suppose $w(n)$ connected triangulated surface by isometrically embedding $\phi$ lent to $\lim_{r \to 0}$. Spiral hexagonal triangulations and linear discrete conformal factors. Let $w(I)$ the definition of developing maps in [21]. Let $v(I) \Delta \phi \varepsilon \gamma \lambda$. The same arguments applying to the triangles $w(I)$ such that $\lim_{m \to \infty} k_1^{(m)} = +\infty$ for some $i \in \{1, 2, \cdots, n\}$. Without loss of generality, we can assume $i = 1$. For the triangle $\Delta v_0 v_1 v_2$, we set $I_0 = I_{12}, I_1 = I_{02}$ and $I_2 = I_{01}$ for simplicity. As $r^{(m)}$ is a generalized inversive distance circle packing on $(P_n, T, I)$, we have

$$(I_2^2 - 1)(k_2^{(m)})^2 + (I_1^2 - 1)(k_1^{(m)})^2 + (I_0^2 - 1) - 2k_1^{(m)} \gamma_2 - 2k_2^{(m)} \gamma_1 - 2k_1^{(m)} k_2^{(m)} \gamma_0 \leq 0$$

by Lemma [21](a), which implies

$$k_2^{(m)} \geq \frac{1}{I_2 - 1} [k_1^{(m)} \gamma_0 + \gamma_1 - \sqrt{(I_0^2 + I_1^2 + I_2^2 + 2I_0 I_1 I_2 - 1)((k_1^{(m)})^2 + 2I_2 k_1^{(m)} + 1)]}$$

$$= \frac{(I_2^2 - 1)(k_1^{(m)})^2 - 2I_2 k_1^{(m)} + I_0^2 - 1}{k_1^{(m)} \gamma_0 + \gamma_1 + \sqrt{(I_0^2 + I_1^2 + I_2^2 + 2I_0 I_1 I_2 - 1)((k_1^{(m)})^2 + 2I_2 k_1^{(m)} + 1)}}$$

Note that $\lim_{m \to \infty} k_1^{(m)} = +\infty$. We have $\lim_{m \to \infty} k_2^{(m)} = +\infty$ by (20), which is equivalent to $\lim_{m \to \infty} r_2^{(m)} = 0$. Combining $r_0^{(m)} = 1$ with $\lim_{m \to \infty} r_1^{(m)} = \lim_{m \to \infty} r_2^{(m)} = 0$, we have $\lim_{m \to \infty} (\theta_{12}^{(m)}(m) \to 0$, where $\theta_{12}^{(m)}$ is the inner angle of the triangle $\Delta v_0 v_1 v_2$ at $v_0$. The same arguments applying to the triangles $\Delta v_0 v_i v_{i+1}, i = 2, 3, \cdots, n$ subsequently give $\lim_{m \to \infty} (\theta_{i,i+1}^{(m)}(m) \to 0$ for all $i = 1, 2, \cdots, n$, which implies $\lim_{m \to \infty} K_0^{(m)} = 2\pi$. This contradicts the assumption that $v_0$ is a flat interior vertex of $(P_n, T, I, r^{(m)})$. Q.E.D.

3.3. Spiral hexagonal triangulations and linear discrete conformal factors. We first recall the definition of developing maps in [21]. Let $l$ be a flat polyhedral metric on a simply connected triangulated surface $(S, T)$. Then $S$ is homeomorphic to its universal covering, so the developing map $\phi : (S, T, l) \to \mathbb{C}$ for this polyhedral metric can be constructed by starting with any isometrically embedding in $\mathbb{C}$ of a Euclidean triangle $t \in F$. This defines an initial map $\phi|_t : (t, l) \to \mathbb{C}$, which can be extended to any adjacent triangle $s \in E$ with $e = t \cap s \in E$ by isometrically embedding $s$ in $\mathbb{C}$ such that $\phi(e) = \phi(s) \cap \phi(t)$. Since $S$ is simply connected, we can repeat this extension for all triangles, which induces a well-defined developing map up to isometries of $(S, T, l)$.

Proposition 3.5. Let $T_{st}$ be the standard hexagonal triangulation of the plane. Let $l$ be a weighted Delaunay inversive distance circle packing metric determined by a label $u : V \to \mathbb{R}$ on $(\mathbb{C}, T_{st}, I)$ such that the vertex set is a lattice $V = L = \{m\vec{v}_1 + n\vec{v}_2\}$, where $\{\vec{v}_1, \vec{v}_2\}$ is a geometric basis of the lattice $L$, and $I$ is invariant under the translations generated by $\{\vec{v}_1, \vec{v}_2\}$. Suppose $w : V \to \mathbb{R}$ is a non-constant linear function defined by two positive numbers $\lambda$ and $\mu$ via

$$w(m\vec{v}_1 + n\vec{v}_2) = m \log \lambda + n \log \mu$$

and $w \ast l$ is a generalized weighted Delaunay inversive distance circle packing metric on $(\mathbb{C}, T_{st}, I)$. Then the following statements hold.

(a) $(\mathbb{C}, T_{st}, I, w \ast l)$ is flat.
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

(b) Let \( \phi \) be the developing map for \((\mathbb{C}, T_{st}, I, w \ast l)\). If there exists a non-degenerate triangle in \((\mathbb{C}, T_{st}, I, w \ast l)\), then there are two different non-degenerate triangles \( t_1 \) and \( t_2 \) in \((\mathbb{C}, T_{st}, I, w \ast l)\) such that \( \phi(\text{int}(t_1)) \cap \phi(\text{int}(t_2)) \neq \emptyset \). In other words, \( \phi \) does not produce an embedding of \((\mathbb{C}, T_{st}, I, w \ast l)\) in the plane.

(c) If all the triangles in \((\mathbb{C}, T_{st}, I, w \ast l)\) are degenerate, then there exists an automorphism \( \psi \) of the triangulation \( T_{st} \) and two positive constants \( \bar{\lambda} = \bar{\lambda}(I, u) \) and \( \bar{\mu} = \bar{\mu}(I, u) \) such that \( w(\psi(m\vec{v}_1 + n\vec{v}_2)) = m \ln \bar{\lambda} + n \ln \bar{\mu} \).

**Proof.** The proof is a modification of the proof for Proposition 3.4 in \([21]\). For completeness, we include the proof here.

![Diagram of angles of spiral triangulations.](image)

**Figure 4.** Angles of spiral triangulations.

To prove part (a), consider two translations \( \tau_1 \) and \( \tau_2 \) of the triangulation \( T_{st} \) defined by

\[
\tau_1(v) = v + \vec{v}_1, \quad \tau_2(v) = v + \vec{v}_2, \quad v \in V.
\]

The lattice \( L \) is isometric to the abelian subgroup of the automorphism group of \( T_{st} \) generated by \( \tau_1 \) and \( \tau_2 \). Up to the action to this subgroup, all the triangles in \((\mathbb{C}, T_{st}, I, w \ast l)\) belong to two equivalent classes, one of which is equivalent to \( t_1 \) with vertices \( 0, \vec{v}_1, \vec{v}_2 \) and the other of which is equivalent to \( t_2 \) with vertices \( 0, \vec{v}_2 - \vec{v}_1, \vec{v}_1 \). Please refer to Figure 4. Since \( I \) is invariant under the translations generated by \( \{\vec{v}_1, \vec{v}_2\} \), \( w \ast l(\tau_1(e)) = \lambda w \ast l(e) \) and \( w \ast l(\tau_2(e)) = \mu w \ast l(e) \) for any edge \( e \in E \) by \([21]\) and \((\ref{eq:2})\). It is straightforward to check that the generalized triangle \( \tau_1(t) \) (\( \tau_2(t) \) respectively) is a scaling of the generalized triangle \( t \) by a factor \( \lambda \) (\( \mu \) respectively).

As a result, the triangles in the same equivalence class are similar to each other. Assume that the inner angles in \( t_i \) are \( \alpha_i, \beta_i \), and \( \gamma_i \) for \( i = 1, 2 \). Then it is clear from Figure 4 that the curvature \( K(v) = 0 \) for all \( v \in V \) by \( \alpha_i + \beta_i + \gamma_i = \pi \). Therefore, \((\mathbb{C}, T_{st}, I, w \ast l)\) is flat.

To prove part (b), note that \( (\lambda, \mu) \neq (1, 1) \) by the assumption that \( w \) is not a constant. Without loss of generality, assume \( \lambda \neq 1 \). By the similarity of triangles in \((\mathbb{C}, T_{st}, I, w \ast l)\) under the action of \( \tau_1 \) and \( \tau_2 \), the two translations \( \tau_1 \) and \( \tau_2 \) induce two affine transformations \( \eta \) and \( \zeta \) of the plane when composed with the developing map \( \phi \). Taking \( \tau_1 \) for example, by the similarities of triangles in \((\mathbb{C}, T_{st}, I, w \ast l)\) under the action of \( \tau_1 \) and \( \tau_2 \), there exists \( \theta \in [0, 2\pi) \) such that \( \phi(\tau_1(v)) - \phi(v) = \lambda e^{i\theta}[\phi(v) - \phi(\tau_1^{-1}(v))] \) for any \( v \in V \), which implies \( \phi(\tau_1(v)) - \lambda e^{i\theta}\phi(v) = \phi(v) - \lambda e^{i\theta}\phi(\tau_1^{-1}(v)) \). Therefore, \( \phi(\tau_1(v)) - \lambda e^{i\theta}\phi(v) \) is a constant for any \( v \in V \), denoted by \( c \in \mathbb{C} \), which implies \( \phi(\tau_1(v)) = \lambda e^{i\theta}\phi(v) + c \). Similar arguments
apply to $\tau_2$. Therefore, there exists an affine transformation $\eta(z) = \lambda^* z + c$ with $|\lambda^*| = \lambda \neq 1$ such that $\phi \circ \tau_2 = \eta \circ \phi$. Since $\lambda \neq 1$, $\eta$ has a unique fixed point $p \in \mathbb{C}$. By $\tau_1 \tau_2 = \tau_2 \tau_1$, we have $\eta \zeta = \zeta \eta$, which implies $\zeta(p) = p$ by the uniqueness of the fixed point of $\eta$. Set $\tilde{\phi}(v) = \phi(v) - p$, then $\tilde{\phi}$ is still a developing map. For simplicity, we still denote $\tilde{\phi}$ by $\phi$. Then $\phi(\tau_1(v)) = \eta \phi(v) = \lambda^* \phi(v)$ and $\phi(\tau_2(v)) = \zeta \phi(v) = \mu^* \phi(v)$ for some linear transformations $\eta(z) = \lambda^* z$ and $\zeta(z) = \mu^* z$. Notice that the group $G = \langle \eta, \zeta \rangle$ generated by $\eta$ and $\zeta$ is either a non-trivial cyclic group or an abelian group isomorphic to $\mathbb{Z}^2$.

By the assumption of part (b), $U = \phi(\mathbb{C})$ has non-empty interior containing the interior of a triangle $t_1$. If $G = \langle \eta, \zeta \rangle$ is a cyclic group, then there exists $(k, j) \neq (0, 0)$ such that $\eta^k \zeta^j$ is the identity map. Set $t_2 = \tau_1^k \tau_2^j(t_1)$. Then $\phi(t_2) = \phi(\tau_1^k \tau_2^j(t_1)) = \eta^k \zeta^j \phi(t_1) = \phi(t_1)$, which implies $\phi(t_1) \cap \phi(t_2) \neq \emptyset$. If $G = \langle \eta, \zeta \rangle$ is isometric to $\mathbb{Z}^2$, then $\eta^k \zeta^j$ is never identity and their action in the plane is not discontinuous. Specifically, for $W := \phi(int(t_1))$, there exists $(k, j) \neq (0, 0)$ so that $\eta^k \zeta^j(W) \cap W \neq \emptyset$. Set $t_2 = \tau_1^k \tau_2^j(t_1)$, then $\phi(t_1) \cap \phi(t_2) = \eta^k \zeta^j(W) \cap W \neq \emptyset$.

To see part (c), since all the triangles in $(\mathbb{C}, \mathcal{T}_s, I, w \ast l)$ are degenerate, the inner angles of the triangles $t_1$ and $t_2$ are 0 or $\pi$. Composing with an automorphism of the triangulation $\mathcal{T}_s$, we may assume $\alpha_1 = \gamma_2 = \pi$, where the angles are marked in Figure 4. For the degenerate triangle with vertices $0, -\vec{v}_1$ and $-\vec{v}_2$, it is flat at $-\vec{v}_2$ by assumption. By Corollary 2.2 if we use $\kappa^*$ to denote the reciprocal of radii in the metric $w \ast l$, then

$$k^*(-\vec{v}_2) = \frac{1}{I^2_{0,-\vec{v}_1} - 1} \left\{ \gamma_{-\vec{v}_1, -\vec{v}_2, 0}^* \kappa^*(0) + \gamma_{0,-\vec{v}_1,-\vec{v}_2} \kappa^*(-\vec{v}_1) \right\}$$

$$+ \sqrt{\Delta_{0,-\vec{v}_1,-\vec{v}_2} \left[ (\kappa^*(0))^2 + (\kappa^*(-\vec{v}_1))^2 + 2I_{0,-\vec{v}_1} \kappa^*(0) \kappa^*(-\vec{v}_1) \right]}$$

(22)

where $\gamma_{v_1,v_2,v_k} = I_{v_1v_2} + I_{v_2v_k} I_{v_1v_k}$ and

$$\Delta_{0,-\vec{v}_1,-\vec{v}_2} = I^2_{0,-\vec{v}_1} + I^2_{0,-\vec{v}_2} + I^2_{-\vec{v}_1,-\vec{v}_2} + 2I_{0,-\vec{v}_1} I_{0,-\vec{v}_2} I_{-\vec{v}_1,-\vec{v}_2} - 1.$$ 

Note that $\kappa^*(0) = \kappa(0), \kappa^*(-\vec{v}_1) = \kappa(-\vec{v}_1) \lambda$ and $\kappa^*(-\vec{v}_2) = \kappa(-\vec{v}_2) \mu$, we have

$$k^*(-\vec{v}_2) \mu = \frac{1}{I^2_{0,-\vec{v}_1} - 1} \left\{ \gamma_{-\vec{v}_1, -\vec{v}_2, 0} \kappa(0) + \gamma_{0,-\vec{v}_1,-\vec{v}_2} \kappa(-\vec{v}_1) \lambda \right\} + \sqrt{\Delta_{0,-\vec{v}_1,-\vec{v}_2} \left[ (\kappa(0))^2 + 2(\kappa(-\vec{v}_1))^2 \lambda^2 + 2I_{0,-\vec{v}_1} \kappa(0) \kappa(-\vec{v}_1) \lambda \right]}$$

by (22). Denote the right hand side of the equation (23) as $f_1(\lambda)$. Then $f_1(\lambda)$ is a strictly increasing function of $\lambda$. Furthermore, we have $\lim_{\lambda \to 0^+} f_1(\lambda) = C_1 > 0$ and $\lim_{\lambda \to +\infty} f_1(\lambda) = +\infty$. Dividing both sides of (23) by $\lambda$ gives

$$k^*(-\vec{v}_2) \frac{\mu}{\lambda} = \frac{1}{I^2_{0,-\vec{v}_1} - 1} \left\{ \gamma_{-\vec{v}_1, -\vec{v}_2, 0} \kappa(0) \lambda^{-1} + \gamma_{0,-\vec{v}_1,-\vec{v}_2} \kappa(-\vec{v}_1) \right\} + \sqrt{\Delta_{0,-\vec{v}_1,-\vec{v}_2} \left[ (\kappa(0))^2 \lambda^{-2} + 2(\kappa(-\vec{v}_1))^2 \lambda^{-2} + 2I_{0,-\vec{v}_1} \kappa(0) \kappa(-\vec{v}_1) \lambda^{-1} \right]}$$

(24)

which implies that $\frac{\mu}{\lambda}$ is a strictly decreasing function of $\lambda$ with $\lim_{\lambda \to 0^+} \frac{\mu}{\lambda} = +\infty$ and $\lim_{\lambda \to +\infty} \frac{\mu}{\lambda} = C_2 > 0$. 

Yuxiang Chen, Yanwen Luo, Xu Xu, Siqi Zhang
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

On the other hand, for the triangle with vertices \(0, -\vec{v}_2\) and \(\vec{v}_1 - \vec{v}_2\), it is flat at \(-\vec{v}_2\) by assumption. Applying Corollary 2.2 to this triangle gives

\[
\kappa^*(\vec{v}_2) = \frac{1}{I^2_{0,\vec{v}_1-\vec{v}_2} - I_0}\left\{ \gamma_{0,-\vec{v}_2,\vec{v}_1-\vec{v}_2} \kappa^*(\vec{v}_1 - \vec{v}_2) + \gamma_{\vec{v}_1-\vec{v}_2,0,-\vec{v}_2} \kappa^*(0) \right. \\
+ \sqrt{\Delta_{0,-\vec{v}_2,\vec{v}_1-\vec{v}_2}((\kappa^*(0))^2 + (\kappa^*(\vec{v}_1 - \vec{v}_2))^2 + 2I_{0,-\vec{v}_1} \kappa^*(0)\kappa^*(\vec{v}_1 - \vec{v}_2))}. \tag{25}
\]

Note that \(\kappa^*(0) = \kappa'(0)\), \(\kappa^*(-\vec{v}_2) = \kappa(-\vec{v}_2)\mu\) and \(\kappa^*(\vec{v}_1 - \vec{v}_2) = \kappa(\vec{v}_1 - \vec{v}_2)\frac{\mu}{\lambda}\), we have

\[
\kappa(-\vec{v}_2)\mu = \frac{1}{I^2_{0,\vec{v}_1-\vec{v}_2} - I_0}\left\{ \gamma_{0,-\vec{v}_2,\vec{v}_1-\vec{v}_2} \kappa(\vec{v}_1 - \vec{v}_2)\frac{\mu}{\lambda} + \gamma_{\vec{v}_1-\vec{v}_2,0,-\vec{v}_2} \kappa(0) \right. \\
+ \sqrt{\Delta_{0,-\vec{v}_2,\vec{v}_1-\vec{v}_2}((\kappa(0))^2 + \kappa^2(\vec{v}_1 - \vec{v}_2)\frac{\mu^2}{\lambda^2} + 2I_{0,-\vec{v}_1} \kappa^*(0)\kappa(\vec{v}_1 - \vec{v}_2)\frac{\mu}{\lambda})}. \tag{26}
\]

by (25). Denote the right hand side of the equation (26) as \(f_2(\lambda)\). Then \(f_2(\lambda)\) is a strictly decreasing function of \(\lambda\) by the fact that \(\kappa^*\) is a strictly decreasing function of \(\lambda\). Furthermore, \(\lim_{\lambda \to 0+} f_2(\lambda) = +\infty\) and \(\lim_{\lambda \to +\infty} f_2(\lambda) = C_3 > 0\). Set \(f(\lambda) = f_1(\lambda) - f_2(\lambda)\), then \(f(\lambda)\) is a strictly increasing continuous function of \(\lambda \in \mathbb{R}_{>0}\) with \(\lim_{\lambda \to 0+} f(\lambda) = -\infty\) and \(\lim_{\lambda \to +\infty} f(\lambda) = +\infty\), which implies that there exists a unique number \(\lambda = \lambda(I, u) \in \mathbb{R}_{>0}\) such that \(f_1(\lambda) = f_2(\lambda)\). As a result, the system (23) and (26) has a unique solution \(\lambda = \lambda(I, u)\) and \(\mu = \mu(I, u)\) in \(\mathbb{R}_{>0}\). This completes the proof for part (c). \(\text{Q.E.D.}\)

We call the weight \(I\) in Proposition 3.5 translation invariant on \(T_{st}\) since \(I(e) = I(e + \delta)\) for any \(\delta \in V = L = \{m\vec{v}_1 + n\vec{v}_2\}\) and \(e \in E\). It is in fact determined by the three weights on the edges of any triangle in \(T_{st}\).

4. Rigidity of Infinite Inversive Distance Circle Packings

The conformality of the limit of discrete conformal maps \(f_n\) in Theorem 1.2 is a consequence of the rigidity of infinite inversive distance circle packings in the plane, which is also conjectured by Bowers-Stephenson [3]. The main result of this section confirms this conjecture.

**Theorem 4.1.** Let \((\mathbb{C}, T_{st}, I)\) be a weighted hexagonal triangulated plane such that the weight \(I : E \to (1, +\infty)\) is translation invariant. Assume \(I\) is a weighted Delaunay inversive distance circle packing metric on \((\mathbb{C}, T_{st}, I)\) induced by a constant label. If \((\mathbb{C}, T_{st}, I, w*l)\) is a weighted Delaunay triangulated surface isometric to an open set in the plane, then \(w\) is a constant function.

The rigidity of inversive distance circle packings with prescribed combinatorial curvatures on weighted triangulated compact surfaces has been proved in [13, 20, 31, 32] based on variational principles. Theorem 4.1 provides a result on the rigidity of infinite inversive distance circle packings in the non-compact plane. In the case of Thurston’s circle packings, the rigidity of infinite circle packings in the plane has been explored in [14, 25, 26].

To prove Theorem 4.1, recall the following definition and properties of embeddable flat polyhedral surfaces in [21].

**Definition 4.2 ([21], Definition 4.1).** Suppose \((S, T)\) is a simply connected triangulated surface with a generalized PL metric \(l\) and \(\phi\) is a developing map for \((S, T, l)\). Then \((S, T, l, \phi)\) is said to be embeddable into \(\mathbb{C}\) if for every simply connected finite subcomplex \(P\) of \(T\), there...
exist a sequence of flat PL metrics on $P$ whose developing maps $\phi_n : P \to \mathbb{C}$ are topological embeddings and converge uniformly to $\phi|_P$.

**Lemma 4.3** ([21], Lemma 4.2). Let $(S, \mathcal{T}, l)$ be a flat polyhedral metric on a simply connected surface with a developing map $\phi$.

1. Suppose $\phi$ is embeddable. If two simplices $s_1, s_2$ represent two distinct non-degenerate triangles or two distinct edges in $\mathcal{T}$, then $\phi(\text{int}(s_1)) \cap \phi(\text{int}(s_2)) = \emptyset$.

2. If $\phi$ is the pointwise convergent limit $\lim_{n \to \infty} \psi_n$ of the developing maps $\psi_n$ of embeddable flat polyhedral metrics $(X, \mathcal{T}, l_n)$, then $(X, \mathcal{T}, l)$ is embeddable.

The standard hexagonal geodesic triangulations of open sets in $\mathbb{C}$ are embeddable. On the other hand, the generic Doyle spirals produce circle packings with overlapping disks, so the corresponding polyhedral metrics are not embeddable.

**Lemma 4.4.** Let $(S, \mathcal{T}_{st}, I)$ be a weighted hexagonal triangulated plane with the weight $I$ translation invariant, and $l_0$ be a weighted Delaunay inverse distance circle packing metric on $(S, \mathcal{T}_{st}, I)$ generated by a label $w_0 : V \to \mathbb{R}$ such that the vertex set is a lattice $L = V$. Suppose $(w - w_0) * l_0$ is a flat generalized weighted Delaunay inverse distance circle packing metric on the plane $(S, \mathcal{T}_{st}, I)$. For any $\delta \in V$, set $u(v) = w(v + \delta) - w(v)$. Then $u * ((w - w_0) * l_0) = (u + w - w_0) * l_0$ is a flat generalized weighted Delaunay inverse distance circle packing metric on $(S, \mathcal{T}_{st})$. Furthermore, if $u(v_0) = \max_{v \in V} u(v)$, then $u$ is a constant.

**Proof.** Suppose $e \in E$ is an edge with vertices $v$ and $v'$. By Definition[1.1] we have

$$
u * ((w - w_0) * l_0)(e) = [e^{2w(v) + 2w(v')} + e^{2w(v') + 2w(v)} + 2I(e)e^{w(v) + w(v')}]^{1/2} = [e^{2w(v+\delta)} + e^{2w(v'+\delta)} + 2I(e + \delta)e^{w(v+\delta) + w(v'+\delta)}]^{1/2} = (w - w_0) * l_0(e + \delta),$$

where $I(e) = I(e + \delta)$ is used in the third line. By the condition that $(w - w_0) * l_0$ is a flat generalized weighted Delaunay inverse distance circle packing metric on $(S, \mathcal{T}_{st}, I)$, we have $u * ((w - w_0) * l_0)$ is a flat generalized weighted Delaunay inverse distance circle packing metric on $(S, \mathcal{T}_{st}, I)$. The rest of the proof is an application of the discrete maximal principle, i.e. Theorem[3.1]. Q.E.D.

The function $u$ in Lemma[4.4] could be taken as a discrete version of the directional derivative of $w$.

**Lemma 4.5.** Let $(S, \mathcal{T}_{st}, I)$ be a weighted hexagonal triangulated plane with the weight $I$ translation invariant, and $l_0$ be a weighted Delaunay inverse distance circle packing metric on $(S, \mathcal{T}_{st}, I)$ generated by a constant label $w_0 : V \to \mathbb{R}$ such that the vertex set is a lattice $L = V = \{mu_1 + nu_2 | m, n \in \mathbb{Z}\}$. Suppose $w * l_0$ is a flat generalized weighted Delaunay inverse distance circle packing metric on the plane $(S, \mathcal{T}_{st}, I)$. Then for any $\delta \in \{\pm u_1, \pm u_2, \pm (u_1 - u_2)\}$, there exists a sequence $\{v_n\} \subset V$ such that

$$w_n(v) := w(v + v_n) - w(v_n)$$

satisfies

(a) for all $v \in V$, the limit $w_\infty(v) = \lim_{n \to \infty} w_n(v)$ exists.

(b) $w_n * l_0$ and $w_\infty * l_0$ are flat generalized weighted Delaunay inverse distance circle packing metrics on $(S, \mathcal{T}_{st}, I)$.

18
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

(c) \( w_\infty(v + \delta) - w_\infty(v) = a := \sup\{w(v + \delta) - w(v)|v \in V\} \) for all \( v \in V\).

(d) the normalized developing maps \( \phi_{w_n*l_0} \) of \( w_n * l_0 \) converges uniformly on compact subcomplex of \( (S, T_{st}) \) to the normalized developing maps \( \phi_{w_\infty*l_0} \) of \( w_\infty * l_0 \). As a result, if \( (S, T_{st}, I, w * l_0) \) is embeddable, then \( (S, T_{st}, I, w_\infty * l_0) \) is embeddable.

Proof. Since \( w_0 \) is a constant function, without loss of generality, we can assume that \( w_0 = 0 \), otherwise applying a scaling to \( l_0 \). To see part (a), notice that Lemma 3.4 implies that there exists a positive constant \( M \) such that

\[
M = M(V, I) = \sup\{|w(v + \delta) - w(v)||v \in V, \delta \in \{\pm u_1, \pm u_2, \pm (u_1 - u_2)\}\}
\]

Then for fixed \( \delta \in \{\pm u_1, \pm u_2, \pm (u_1 - u_2)\} \), we have

\[
a := \sup\{w(v + \delta) - w(v)|v \in V\} \leq M.
\]

Therefore, there exist a sequence \( \{v_n\} \) in \( V \) such that

\[
(28) \quad a - \frac{1}{n} \leq w_n(\delta) = w(v_n + \delta) - w(v_n) \leq a.
\]

Furthermore, we have \( w_n(0) = 0 \) and

\[
(29) \quad w_n(v + \delta) - w_n(v) = w(v + \delta + v_n) - w(v + v_n) \leq a, \forall v \in V
\]

by the definition of \( w_n \) and \( a \). By Lemma 3.4, if \( v \in V \) is of combinatorial distance \( m \) to 0, then

\[
|w_n(v)| = |w_n(v) - w_n(0)|
\]

\[
\leq \sum_{i=1}^{m} |w_n(v_i) - w_n(v_{i-1})| = \sum_{i=1}^{m} |w(v_i + v_n) - w(v_{i-1} + v_n)|
\]

\[
= \sum_{i=1}^{m} |w(v_i + v_n) - w(v_{i-1} + v_n)| \leq mM,
\]

where \( v_m = v \), \( v_0 = 0 \) and \( v_0 \sim v_1 \sim \cdots \sim v_m \) is a path of combinatorial distance \( m \) between 0 and \( v \). By the diagonal argument, there exists a subsequence of \( \{v_n\} \), still denoted by \( \{v_n\} \) for simplicity, such that \( w_\infty(v) := \lim_{n \to \infty} w_n(v) \) exists for all \( v \in V \).

To see part (b), for any fixed \( n \in \mathbb{N} \) and any edge \( e \in E \), we have

\[
(30) \quad w_n * l_0(e) = e^{-w(v_n)} w * l_0(e + v_n)
\]

by the translation invariance \( I(e) = I(e + \delta) \) for the weight \( I \). This implies that \( w_n * l_0 \) is a flat generalized weighted Delaunay inversive distance circle packing metric on \( (S, T_{st}, I) \) by the assumption that \( w * l_0 \) is a flat generalized weighted Delaunay inversive distance circle packing metric on \( (S, T_{st}, I) \). By \( w_\infty(v) = \lim_{n \to \infty} w_n(v) \) and continuity, we have \( w_\infty * l_0 \) is a flat generalized weighted Delaunay inversive distance circle packing metric on \( (S, T_{st}, I) \).

Moreover, we have \( w_\infty(v + \delta) - w_\infty(v) \leq a \) for any \( v \in V \) by \( (29) \), which implies

\[
(31) \quad \sup\{w_\infty(v + \delta) - w_\infty(v)|v \in V\} \leq a.
\]

To see part (c), by \( w_n(0) = 0 \), \( (28) \) and \( (31) \), we have \( w_\infty(0) = 0 \) and

\[
(32) \quad w_\infty(\delta) - w_\infty(0) = w_\infty(\delta) = a \geq \sup\{w_\infty(v + \delta) - w_\infty(v)|v \in V\},
\]

which implies that \( w_\infty(v + \delta) - w_\infty(v) \) attains the maximal value \( \sup\{w_\infty(v + \delta) - w_\infty(v)|v \in V\} \) at \( v = 0 \). Note that for a fixed \( \delta \) and \( u(v) := w_\infty(v + \delta) - w_\infty(v) \), \( u * (w_\infty * l_0) \) is a flat generalized weighted Delaunay inversive distance circle packing metric on \( (S, T_{st}, I) \) by Lemma 4.4. By Theorem 3.1, we have \( w_\infty(v + \delta) - w_\infty(v) = a \) for any \( v \in V \).
If \((S, T_{st}, I, w \ast l_0)\) is embeddable, then \((S, T_{st}, I, w_n \ast l_0)\) is embeddable by [30]. The rest of the proof is an application of Lemma 4.3. Q.E.D.

Theorem 4.1 follows from the following more general result.

**Theorem 4.6.** Let \((S, T_{st}, I)\) be a weighted hexagonal triangulated plane with the weight \(I\) translation invariant, and \(l_0\) be a weighted Delaunay inverse distance circle packing metric on \((S, T_{st}, I)\) generated by a constant label \(w_0 : V \rightarrow \mathbb{R}\) such that the vertex set is a lattice \(L = V = \{mu_1 + nu_2|m, n \in \mathbb{Z}\}\). Suppose \(w \ast l_0\) is a flat generalized weighted Delaunay inverse distance circle packing metric on the plane \((S, T_{st}, I)\) and \((S, T_{st}, I, w \ast l_0)\) is embeddable into \(\mathbb{C}\). Then \(w\) is a constant function.

**Proof.** The idea of proof can be summarized as follows. Assume \(w\) is not a constant, we will construct a sequence of discrete conformal factor \(w_n\) by extracting “directional derivatives” of \(w\) at different base points. This construction relies heavily on the symmetric structure of the lattice \(V(T_{st}) = L\) generated by \(I\) and \(w_0\), which implies that the limit of this sequence produce a linear discrete conformal factor \(w_\infty\). By Lemma 4.5, \((S, T_{st}, I, w_\infty \ast l_0)\) is embeddable. However, by Proposition 3.5 if \(w_\infty\) is not a constant, \((S, T_{st}, I, w_\infty \ast l_0)\) contains overlapping triangles under the developing maps. This leads to a contradiction.

**Step 1:** construct a linear discrete conformal factor \(\hat{w}\). Since \(w\) is assumed to be different from a constant function, then there exists a \(\delta_1 \in \{ \pm u_1, \pm u_2, \pm (u_1 - u_2) \}\) such that \(a_1 = \sup \{w(v + \delta_1) - w(v)|v \in V\} > 0\). By Lemma 3.4, \(a_1 \in (0, \infty)\). Applying Lemma 4.5 to \(w \ast l_0\) in the direction \(\delta_1\), we can find a function \(w_\infty : V \rightarrow \mathbb{R}\) such that \((S, T_{st}, I, w_\infty \ast l_0)\) is embeddable and
\[
\hat{w}(v + \delta_1) - \hat{w}(v) = a_1, \forall v \in V.
\]

Further applying Lemma 4.5 to \(w_\infty \ast l_0\) in the direction \(\delta_2 \in \{ \pm u_1, \pm u_2, \pm (u_1 - u_2)\} - \{ \pm \delta_1 \}\) gives rise to a function \(\hat{w} = (w_\infty)_\infty : V \rightarrow \mathbb{R}\) such that \((S, T_{st}, I, \hat{w} \ast l_0)\) is embeddable. Moreover,
\[
\hat{w}(v + \delta_1) - \hat{w}(v) = a_1, \hat{w}(v + \delta_2) - \hat{w}(v) = a_2, \forall v \in V,
\]

which shows that \(\hat{w}(v)\) is an affine function of the form \(\hat{w}(n\delta_1 + m\delta_2) = na_1 + ma_2 + a_3\) with \(a_1 \in (0, +\infty), a_2, a_3 \in \mathbb{R}\). Without loss of generality, we can assume \(\hat{w}(n\delta_1 + m\delta_2) = na_1 + ma_2\), because the weighted Delaunay property of generalized inverse distance circle packing metrics is invariant under scaling. Furthermore, \((S, T_{st}, I, \hat{w} \ast l_0)\) is embeddable.

**Step 2:** Overlapping of \((S, T_{st}, I, \hat{w} \ast l_0)\).

By Step 1, there are two positive numbers \(\bar{\lambda} \in (1, +\infty)\) and \(\bar{\mu} \in (0, +\infty)\) so that
\[
\hat{w}(m\delta_1 + n\delta_2) = m\log \bar{\lambda} + n\log \bar{\mu}
\]

\[20\]

**Figure 5.** Three cases of degenerate triangulations.
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

and \((S, T_{st}, I, \hat{w} \ast l_0)\) is embeddable. Then there is no non-degenerate triangle in the image of the developing map \(\hat{\phi}\) for \((S, T_{st}, I, \hat{w} \ast l_0)\). Otherwise by Proposition 3.5 there are two triangles with overlapping interiors. Therefore, all the triangles in the image of \((S, T_{st}, I, \hat{w})\) under \(\hat{\phi}\) are degenerate. All the inner angles are either 0 and \(\pi\). Up to obvious automorphisms of \(T_{st}\), there are three cases in Figure 5 showing triangles in the star of the origin. Case 1 and Case 2 are differed by a rotation \(\gamma\), and Case 1 and Case 3 are differed by an orientation-reversing automorphism \(\rho\) of \(T_{st}\) such that \(\rho(0) = 0\), \(\rho(\vec{v}_1) = \vec{v}_2\), and \(\rho(\vec{v}_2) = \vec{v}_2 - \vec{v}_1\). Therefore, we only need to consider Case 1. By Proposition 3.5 (c), the constants \(\bar{\lambda}\) and \(\bar{\mu}\) depend only on \(I\) and \(w_0\).

Consider the lengths of edges \(e_1 = v_0v_3\), \(e_2 = v_0v_6\), \(e_3 = v_6v_7\) and their respective lengths \(l_1, l_2, l_3\) in \(\hat{w} \ast l_0\) in Figure 6. Notice that \(l_3 = (\bar{\lambda}/\bar{\mu})l_2\) and \(l_1 = (\bar{\mu}/\bar{\lambda})l_2\), then

\[l_1 + l_3 \geq 2l_2 > l_2.\]

Since \((S, T_{st}, I, \hat{w} \ast l_0)\) with a developing map \(\hat{\phi}\) is embeddable, there exist a sequence of flat polyhedral metrics with developing maps \(\phi_n\) which are embeddings such that \(\phi_n\) converge to \(\hat{\phi}\) uniformly on compact sets. Then for \(n\) large enough, the images of \(e_1\) and \(e_3\) under \(\phi_n\) intersect by the inequality above. This is because the angle condition at \(v_0\) forces \(e_3\) to rotate clockwise and the angle condition at \(v_0\) forces \(e_1\) to rotate counterclockwise. Then the intersection contradicts the fact that \((S, T_{st}, I, \hat{w} \ast l_0)\) is embeddable. Q.E.D.

5. THE CONVERGENCE OF INVERSIVE DISTANCE CIRCLE PACKINGS

5.1. Proof of the main theorem. Recall the main theorem of this paper.

**Theorem 5.1.** Let \(\Omega\) be a Jordan domain in the complex plane bounded by a Jordan curve \(\partial \Omega\) with three distinct points \(p, q, r \in \partial \Omega\). Let \(f\) be the Riemann mapping from the unit equilateral triangle \(\triangle ABC\) to \(\overline{\Omega}\) such that \(f(A) = p, f(B) = q, f(C) = r\). There exists a sequence of weighted triangulated polygonal disks \((\Omega_n, T_n, I_n, (p_n, q_n, r_n))\) with inversive distance circle
packing metrics \( l_n \), where \( \mathcal{T}_n \) is an equilateral triangulation of \( \Omega_n \), \( I_n : E_n \to (1, +\infty) \) is a weight defined on \( E_n = \mathcal{E}(\mathcal{T}_n) \) and \( p_n, q_n, r_n \) are three boundary vertices of \( \mathcal{T}_n \), such that

1. \( \Omega = \bigcup_{n=1}^{\infty} \Omega_n \) with \( \Omega_n \subset \Omega_{n+1} \), and \( \lim_n p_n = p \), \( \lim_n q_n = q \) and \( \lim_n r_n = r \),
2. the discrete conformal maps \( f_n \) from \( \triangle ABC \) to \( (\Omega_n, \mathcal{T}_n, I_n, l_n) \) with \( f_n(A) = p_n, f_n(B) = q_n, f_n(C) = r_n \) exist,
3. the discrete conformal maps \( f_n \) converge uniformly to the Riemann mapping \( f \).

To prove Theorem 5.1 we need to establish the existence of discrete conformal maps induced by inverse distance circle packings from a flat polyhedral disk to an equilateral triangle. In the case of Thurston’s circle packings, the Koebe-Andreev-Thurston theorem guarantees the existence of a circle packing of the unit disk with any given triangulation of a disk as the nerve of the packing. In the case of vertex scaling [19], the discrete uniformization theorem [10, 11] gives the existence of a discrete conformal fact or with prescribed combinatorial curvature on marked closed surfaces. Unfortunately, there is no known existence theorem for inverse distance circle packings on arbitrary triangulations. Theorem 5.2 below establishes such an existence theorem for inverse distance circle packings when the triangulations of flat polyhedral disks are subdivided in a scheme as follows.

Let \( (\mathcal{P}, \mathcal{T}, l) \) be a flat polyhedral disk with an equilateral triangulation, in which all triangles are equilateral. Then the length function \( l \) is a constant function on \( E \). Given an equilateral Euclidean triangle \( \triangle \) in the plane, the \( n \)-th standard subdivision of \( \triangle \) is the equilateral triangulation of \( \triangle \) by \( n^2 \) equilateral triangles. Applying this subdivision to each triangle in an equilateral triangulation of a flat polyhedral disk \( (\mathcal{P}, \mathcal{T}, l) \), we obtain its \( n \)-th standard subdivision \( (\mathcal{P}, \mathcal{T}_n, l_n) \). Furthermore, if \( l \) is an inverse distance circle packing metric induced by a constant label \( u \) and a constant weight \( I : E \to (1, +\infty) \), we require that \( l_n \) is also an inverse distance circle packing metric induced by a constant label \( u_n \) and a constant weight \( I_n : E_n \to (1, +\infty) \) taking the same value as \( I : E \to (1, +\infty) \).

**Theorem 5.2.** Suppose \( (\mathcal{P}, \mathcal{T}, l) \) is a flat polyhedral disk with an equilateral triangulation \( \mathcal{T} \) such that exactly three boundary vertices \( p, q, r \) have curvature \( \frac{2\pi}{3} \), and the metric \( l \) is an inverse distance circle packing metric induced by a constant label \( u \) and a constant weight \( I : E \to (1, +\infty) \). Then for sufficiently large \( n \), there is a discrete conformal factor \( w : V_n \to \mathbb{R} \) for the \( n \)-th standard subdivision \( (\mathcal{P}, \mathcal{T}_n, l_n) \) such that

1. \( K_i(w \ast l_n) = 0 \) for all \( v_i \in V_n - \{p, q, r\} \),
2. \( K_i(w \ast l_n) = \frac{2\pi}{3} \) for all \( v_i \in \{p, q, r\} \),
3. there is a constant \( \theta_0 = \theta_0(I) > 0 \) independent of \( n \) such that all inner angles of triangles in \( (\mathcal{T}_n, w \ast l_n) \) are in the interval \( \left[ \theta_0, \frac{\pi}{2} + \theta_0 \right] \).

Note that the underlying metric space of \( (\mathcal{P}, \mathcal{T}_n, l_n) \) is an equilateral triangle, and \( (\mathcal{P}, \mathcal{T}_n, l_n) \) is weighted Delaunay for each \( n \).

Assuming Theorem 5.2, the proof of Theorem 5.1 is a standard argument using the properties of quasiconformal maps in the plane. To achieve this aim, we first recall the following three theorems on the extension and convergence of quasiconformal maps.

**Theorem 5.3** ([11, Corollary in Page 30]). If \( f : \mathbb{D} \to \Omega \) is a \( K \)-quasiconformal map from the open unit disk \( \mathbb{D} \) onto a Jordan domain \( \Omega \), then \( f \) extends continuously to a homeomorphism \( \overline{f} : \overline{\mathbb{D}} \to \overline{\Omega} \).

The following theorem is a simple consequence of Lemma 2.1 and Theorem 2.2 in [17].
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

**Theorem 5.4.** If \( f_n : \mathbb{D} \to \Omega_n \) is a sequence of \( K \)-quasiconformal maps such that \( \Omega_n \) is uniformly bounded, then every subsequence of \( f_n \) contains a subsequence that converge locally uniformly. Moreover, the limit of this subsequence is a \( K \)-quasiconformal map or a constant map.

A sequence of Jordan curves \( J_n \) in \( \mathbb{C} \) converge uniformly to a Jordan curve \( J \) in \( \mathbb{C} \) if there exist homeomorphisms \( \phi_n : \mathbb{S}^1 \to J_n \) and \( \phi : \mathbb{S}^1 \to J \) such that \( \phi_n \) converge uniformly to \( \phi \).

**Theorem 5.5** ([23], Corollary 1). Assume that \( \Omega_n \) is a sequence of Jordan domains such that \( \partial \Omega_n \) converge uniformly to \( \partial \Omega \). If \( f_n : \mathbb{D} \to \Omega_n \) is a \( K \)-quasiconformal map for each \( n \), and the sequence \( \{f_n\} \) converge to a \( K \)-quasiconformal map \( f : \mathbb{D} \to \Omega \) uniformly on compact sets of \( \mathbb{D} \), then \( f_n \) converge to \( f \) uniformly on \( \mathbb{D} \).

**Proof of Theorem 5.7** By taking the intersection of scalings of the standard hexagonal triangulation in the plane with \( \Omega \), we can construct a sequence of nested polygonal disks \( \Omega_n \) such that \( \partial \Omega_n \) converge uniformly to \( \partial \Omega \) and there are three boundary vertices \( p_n, q_n, r_n \subset \partial \Omega_n \) such that \( \lim_n p_n = p, \lim_n q_n = q \) and \( \lim_n r_n = r \). By adding or subtracting boundary vertices if necessary, we can assume that the curvatures at \( p_n, q_n, r_n \in \partial \Omega_n \) are \( \frac{2\pi}{3} \) and the curvatures at all other boundary vertices of \( \Omega_n \) are not \( \frac{2\pi}{3} \).

By Theorem 5.2 we produce some standard subdivision \( T_n \) of \( \Omega_n \) and some discrete conformal factors \( w_n \) such that \( (\Omega_n, T_n, w_n * l_{st}) \) is isometric to the unit equilateral triangle \((\triangle ABC, T_n)\), where \( A, B, C \) correspond to \( p_n, q_n, r_n \) respectively. Let \( f_n : (\triangle ABC, T_n, (A, B, C)) \to (\Omega_n, T_n, (p_n, q_n, r_n)) \) be the discrete conformal map induced by the correspondence of triangulations. Let \( f \) be the Riemann mapping from \( \triangle ABC \) to \( \Omega \) sending \( A, B, C \) to \( p, q, r \) respectively. We claim that \( f_n \) converges uniformly to \( f \) on \( \triangle ABC \).

By Theorem 5.2 all angles of triangles in \((\triangle ABC, T_n, w * l_{st})\) are at least \( \epsilon_0 > 0 \). Then the discrete conformal maps \( f_n \) are \( K \)-quasiconformal from \( \text{int}(\triangle ABC) \) to \( \text{int}(\Omega_n) \) for some constant \( K \) independent of \( n \) and continuous from \( \triangle ABC \) to \( \Omega_n \). Let \( f_n \) be the restriction of \( f_n \) in \( \text{int}(\triangle ABC) \). Theorem 5.4 implies that every convergent subsequence of \( \{f_n\} \) converge to a \( K \)-quasiconformal map \( \tilde{g} \) from \( \text{int}(\triangle ABC) \) to \( \text{int}(\Omega) \). Since \( \Omega = \bigcup_n \Omega_n \), \( \tilde{g} \) is onto \( \text{int}(\Omega) \). Theorem 5.3 implies that \( \tilde{g} \) extends to a homeomorphism \( g : \triangle ABC \to \Omega \). Theorem 5.5 implies that \( f_n \) converge uniformly to \( g \) on \( \triangle ABC \). It is straightforward to check that \( g(A) = p, g(B) = q \), and \( g(C) = r \).

Notice that the Riemann mapping \( f \) is the only continuous extension of a conformal map from \( \text{int}(\triangle ABC) \) to \( \Omega \) with \( f(A) = p, f(B) = q \), and \( f(C) = r \). This means that if we can show \( g \) is conformal, then \( g = f \) and all limits of convergent subsequences of \( \{f_n\} \) are \( f \). This will complete the proof of \( f_n \to f \) uniformly on \( \triangle ABC \).

The conformality of \( g \) follows from Theorem 4.6 by the same argument as the Hexagonal Packing Lemma in [25]. We briefly repeat the arguments here for completion. For a vertex \( v_0 \in T_{st} \), let \( B_n \) be the \( n \)-ring neighborhood of \( v_0 \) in \( T_{st} \). Then \( B_n \) is a finite simplicial complex whose underlying space is a topological disk \( \mathbb{D} \). Assume that \( l_n \) is a flat inverse distance circle packing on \( B_n \) with the constant weight \( I \). Let \( s_n \) be the maximal ratio of radii of two adjacent circles in \( l_n \) of \( B_n \). Lemma 3.4 implies that \( s_n \) is uniformly bounded by some constant \( C(I) \). As \( n \to \infty \), we can pick a convergent subsequence of \( (\mathbb{D}, B_n, I, l_n) \), still indexed with \( n \), such that all circles converge geometrically. We claim that \( \lim_n s_n = 0 \). Otherwise, as \( n \to \infty \), the limit produces an inverse distance circle packing on \( T_{st} \) such that circles have different sizes. This contradicts the fact that \( w \) is a constant in Theorem 4.6.
As \( n \to \infty \), the arguments above show that \( s_n \) of \( \mathcal{T}_n \) goes to zero. Equilateral triangles in \( \mathcal{T}_n \) contained in a compact subset of \( \Omega \) are mapped by \( f_n^{-1} \) to triangles in \( (\triangle ABC, \mathcal{T}_n) \) which are close to be equilateral. Then \( f_n \) restricted in each triangle converge to a similarity map. The dilatations \( K_n \) of \( f_n \) converge to 1. Therefore, \( g \) is 1-conformal, which is equivalent to be conformal. Q.E.D.

The rest of this paper is devoted to prove Theorem 5.2. To find the discrete conformal factors in Theorem 5.2, we will construct a system of ordinary differential equations proposed in [12] to deform the discrete conformal factors via discrete curvatures at vertices. We first consider such a flow on a standard subdivision of an equilateral triangle in Theorem 5.10, then use the flow to construct the discrete conformal factor required in Theorem 5.2.

In the rest of this section, we assume that any initial polyhedral metric \( l \) on \((\mathcal{P}, \mathcal{T})\) is an inversive distance circle packing metric induced by a constant weight \( I > 1 \) and a constant label, which is weighted Delaunay. For any discrete conformal factor \( w \) on \( V \), denote the angle at \( v_k \) of a triangle \( \triangle v_i v_j v_k \) in the metric \( w * l \) as \( \theta^l_{ij}(w) \). Similarly, the conductance of \( w * l \) defined by the formula (11) is denoted as \( \eta(w) \), and the curvature of \( w * l \) is denoted as \( K(w) \).

The notation \( a = O(b) \) refers to the fact that \( |a| \leq C|b| \) for some constant \( C = C(I) > 0 \).

### 5.2. Inversive distance circle packings along flows

In this subsection, we will solve the following prescribing curvature problem: assume \( V_0 \subset V \) and the initial curvature of \((\mathcal{P}, \mathcal{T}, l)\) is \( K^0 \). Given a prescribed curvature \( K^* \) on \( V - V_0 \), find a discrete conformal factor \( w \) such that \( w|_{V_0} = 0 \) and \( K(w) = K^* \).

Consider a smooth family of discrete conformal factors \( w(t) \) satisfying

\[
\begin{align*}
K_i(w(t)) &= (1 - t)K^0_i + tK^*_i, & v_i \in V - V_0, \\
w_i(t) &= 0, & v_i \in V_0,
\end{align*}
\]

and \( w(0) = 0 \). This family of \( w(t) \), if it exists in the interval \([0, 1]\), provides a linear interpolation between the initial curvature \( K^0 \) and the prescribed curvature \( K^* \). Therefore, \( w(1) * l \) has curvature \( K^* \). Hence, we need to show that this flow exists on \([0, 1]\) for some standard subdivisions of \( \mathcal{T} \) on \( V - V_0 \). To this end, we recall some basic notions of analysis on graphs.

Given a graph \((V, E)\), the set of all oriented edges in \((V, E)\) is denoted by \( \vec{E} \). If \( \vec{v}_i \vec{v}_j \) is an edge in \( E \), we denote it as \( i \sim j \). A conductance on \( G \) is a function \( \eta : \vec{E} \to \mathbb{R}_{\geq 0} \) so that \( \eta_{ij} = \eta_{ji} \). The following definitions and results are well-known. See [18] for details.

**Definition 5.6.** Given a finite graph \((V, E)\) with a conductance \( \eta \), the gradient \( \nabla : \mathbb{R}^V \to \mathbb{R}^E \) is defined by

\[
(\nabla f)_{ij} = \eta_{ij}(f_i - f_j),
\]

the Laplace operator associated to \( \eta \) is the linear map \( \Delta : \mathbb{R}^V \to \mathbb{R}^V \) defined by

\[
(\Delta f)_i = \sum_{j \sim i} \eta_{ij}(f_i - f_j).
\]

Given a set \( V_0 \subset V \) and a function \( g : V_0 \to \mathbb{R} \), the solution to the Dirichlet problem is a function \( f : V \to \mathbb{R} \) satisfying

\[
(\Delta f)_i = 0, \forall v_i \in V - V_0, \text{ and } f|_{V_0} = g.
\]

**Proposition 5.7.** Suppose \((V, E)\) is a finite connected graph with a conductance \( \eta(e) > 0 \) for any edge \( e \in E \). Given a nonempty \( V_0 \subset V \) and \( g : V_0 \to \mathbb{R} \), the solution \( f \) to the Dirichlet problem exists. Moreover,
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

(a) (Maximum principle) \(\max_{v_i \in V} f_i = \max_{v_i \in V_0} f_i\).

(b) (Strong maximum principle) If \(V - V_0\) is connected and \(\max_{v_i \in V - V_0} f_i = \max_{v_i \in V_0} f_i\), then \(f|_{V - V_0}\) is a constant function.

Recall that the formula (11) defines a conductance \(\eta\) for any inversive distance circle packing on \((S, T, I)\). If it is weighted Delaunay, then \(\eta_{ij} \geq 0\). In the rest of this paper, we assume that the Laplace operator \(\Delta\) is induced from this conductance \(\eta\) for an inversive distance circle packing.

By the variation formula of curvatures in (12), we have the following system of ODEs by taking derivative of equation (32) with respect to \(\theta\).

\[
(\Delta w')_i = \sum_{j \sim i} \eta_{ij} (w'_i - w'_j) = K_i^* - K_i^0, \quad v_i \in V - V_0,
\]

with the initial value \(w(0) = 0\) and \(w'_i = \frac{dw_i}{dt}\). We will show that the solution to the system (33) exists for all \(t \in [0, 1]\) if \((P, T, l)\) is chosen carefully. Prior to the existence, we first characterize the maximal interval for the existence of the solution to (33).

Given a weighted triangulated surface \((S, T, I)\) with an inversive distance circle packing metric \(l\), consider the set of discrete conformal factors \(W \subset \mathbb{R}^V\) defined by

\[
W = \{w \in \mathbb{R}^V | w \ast l \text{ is an inversive distance circle packing metric} \}
\]

on \((S, T, I)\) such that \(\eta_{ij} > 0\) for all edges.

**Lemma 5.8.** Let \((P, T, I)\) be a weighted triangulated surface with an inversive distance circle packing metric \(l\) generated by a label \(u\). The initial valued problem (33) defined on \(W\) has a unique solution in a maximum interval \([0, t_0]\) with \(t_0 > 0\) if \(V_0 \neq \emptyset\) and \(0 \in W\). Moreover, if \(t_0 < \infty\), then either \(\liminf_{t \to t_0} \theta_{ij}^t(w(t)) = 0\) for some angle \(\theta_{ij}^t\) or \(\liminf_{t \to t_0} \eta_{ij}(w(t)) = 0\) for some edge \(v_i v_j\).

**Proof.** The ODE system (33) can be written as

\[
\begin{cases}
A(w) \cdot w'(t) = b, \\
w(0) = 0,
\end{cases}
\]

where \(A(w)\) is a square matrix valued smooth function of \(w\), \(b\) is a column vector determined by curvature, and \(w'(t)\) is a column vector. Then \(A(w)\) is an invertible matrix for a fixed \(w \in W\). Indeed, consider the following system of linear equations for a fixed \(w\)

\[
A(w) \cdot f = 0.
\]

From (33) we know that equation (35) is equivalent to

\[
\begin{cases}
(\Delta f)_i = 0, \quad v_i \in V - V_0, \\
f_i = 0, \quad v_i \in V_0,
\end{cases}
\]

where \(\eta_{ij} > 0\) for all edges since \(w \in W\). The maximal principle in Proposition 5.7 implies that \(f = 0\). Therefore, \(A(w)\) is invertible.

As a result, (33) can be written as \(w'(t) = A(w)^{-1} b\). Picard’s existence theorem for solutions to the ODE systems implies that there exists an interval \([0, t_0]\) on which (33) has a solution.

If \(t_0 < \infty\) and \(t \nearrow t_0\), then \(w(t)\) leaves every compact set in \(W\). Consider subsets \(W_\delta = \{w \in W | \theta_{ij}^t \geq \delta, |w_i| \leq \frac{1}{\delta}, \eta_{ij} \geq \delta\}\). It is straightforward to check that \(W_\delta\) is compact. Since \(w(t)\) leaves every \(W_\delta\) for each \(\delta > 0\), one of the following three cases occurs:

25
Yuxiang Chen, Yanwen Luo, Xu Xu, Siqi Zhang

(1) \( \lim \inf_{t \to t_0} \theta^i_{jk}(w(t)) = 0 \) for some \( \theta^i_{jk} \), or
(2) \( \lim \inf_{t \to t_0} n_{ij}(w(t)) = 0 \) for some edge , or
(3) \( \lim \sup_{t \to t_0} |w_i(t)| = +\infty \) for some \( v_i \in V \).

We claim that the case (3) implies the case (1). Otherwise, there exists \( \delta > 0 \) such that \( \lim \inf_{t \to t_0} \theta^i_{jk}(w(t)) > \delta \) for all \( \theta^i_{jk} \). Since \( w'_i(t) = 0 \) for \( v_i \in V_0 \) along the flow (33), the radius \( r_i \equiv e^{w_i+\nu} \) does not change along the flow. Then for any triangle \( \triangle v_i v_j v_k \) with \( v_i \) as a vertex, the sine law implies that

\[
\frac{r_{ij}^2}{r_{ik}^2} \leq \frac{1}{\sin^2 \delta}, \quad \frac{r_{ik}^2}{r_{ij}^2} \leq \frac{1}{\sin^2 \delta},
\]

which further implies that \( r_j \leq \frac{\sqrt{T}}{\sin \delta} (r_i + r_k) \) and \( r_k \leq \frac{\sqrt{T}}{\sin \delta} (r_i + r_j) \) by \( I > 1 \) and (2). Therefore, \( r_j \) and \( r_k \) are of the same order. Specially, \( r_j \to +\infty \) if and only if \( r_k \to +\infty \). If \( w_k \) and \( w_j \) go to infinity, then

\[
\cos \theta^i_{jk} = \frac{r_{ij}^2 + r_{ik}^2 - r_{jk}^2}{2r_{ij}r_{ik}} = \frac{r_r^2 + r_i \kappa r + r_i \kappa r - r_j r_k I - r_j r_k I}{l_{ij} l_{ik}} \to -I < -1,
\]

which is impossible. Since the 1-skeleton of \( \mathcal{T} \) is a finite connected graph, we can show inductively that \( w_i \) is bounded for all \( v_i \in V \), which contradicts the assumption in case (3). This completes the proof for the claim. Q.E.D.

5.3. Standard subdivisions of an equilateral triangle. In this subsection, we consider the ODE system (33) when the polyhedral surface is an equilateral triangulation of an equilateral triangle. We will prescribe special curvatures at the boundary vertices such that the discrete conformal maps approximate the power functions in complex analysis. To apply the estimates in network theory, we need to bound the conductance of a weighted Delaunay triangulation as follows.

Lemma 5.9. Let \( \triangle v_1 v_2 v_3 \) be a weighted triangle generated by an inversive distance circle packing \((r_1, r_2, r_3)\) and the weight \( I > 1 \) is a constant. There exists a constant \( \theta_0 = \theta_0(I) \in (0, \frac{\pi}{6}) \) such that if the three inner angles of the triangle are bounded in \([\pi/6 - \theta_0, \pi/2 + \theta_0]\), then

(a) \( r_j / r_i \leq 20 \) for any two radii \( r_i \) and \( r_j \),
(b) \( C \leq \eta^k_{ij} \leq M \) for some constants \( C = C(I) > 0 \) and \( M = M(I) > 0 \).

Proof. Set

\[
\theta_0 = \min\{\frac{\pi}{1000}, \arcsin \frac{1}{10(20 + I)}\}.
\]

To prove part (a), by the angle bound and the sine law,

\[
\frac{l_{ij}}{l_{ik}} \leq 1/\sin(\pi/6 - \pi/1000) < \sqrt{5}
\]

for any two edges in the triangle. Without loss of generality, assume that \( r_i = 1 \). We will prove that \( r_j \leq 20 \) by contradiction in the following two cases.

If \( r_j > 20 \) and \( r_k / r_j \leq 1/5 \), then

\[
\frac{r_{ij}^2}{r_{ik}^2} \geq \frac{r_j^2/5 + 2Ir_j + 1 + 4r_j^2/5}{(r_j/5)^2 + 2Ir_j/5 + 1} \geq 5,
\]

which contradicts (36).
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

If \( r_j > 20 \) and \( r_k/r_j > 1/5 \), then \( r_k > 4 \) and the inner angle \( \theta^i_{jk} \) at \( v_i \) is the largest inner angle in \( \triangle v_i v_j v_k \). Note that in this case, we have \( I(r_k + r_j - r_k r_j) + 1 < 0 \), \( l_{ij} < \sqrt{I(r_j + 1)} \) and \( l_{ik} < \sqrt{I(r_k + 1)} \). As a result, by the cosine law, we have

\[
\cos \theta^i_{jk} = \frac{l_{ij}^2 + l_{ik}^2 - l_{jk}^2}{2l_{ij}l_{ik}} = \frac{I(r_k + r_j - r_k r_j) + 1}{l_{ij}l_{ik}} < \frac{-I(r_k + r_j + r_k r_j + 1) + 2I(r_k + r_j) + I + 1}{I(r_k + r_j + r_k r_j + 1)} \leq \frac{2I(r_k + r_j) + 2I}{I(r_k + r_j + r_k r_j + 1)} \leq \frac{-11}{21}.
\]

This contradicts that the angle bound is \([\pi/6 - \theta_0, \pi/2 + \theta_0]\).

To prove part (b), the definition of \( \eta^k_{ij} \) and the formula (9) implies \( \eta^k_{ij} = \frac{h_{ij,k}}{l_{ij}} = \frac{r^2 r_k h_k}{A l^2_{ij}} \), where \( A = l_{ij} l_{ik} \sin \theta^i_{jk} \). The sign of \( \eta^k_{ij} \) is determined by \( h_k \). We will show

\[
(37) \quad r_k h_k = (1 + I)(1 + I(r_k/r_j - 1)) \geq \frac{1 + I}{4} > 0.
\]

We just need to check the case that \( r_k/r_i \leq 1 \) and \( r_k/r_j \leq 1 \). If both \( r_k/r_i \geq 1/2 \) and \( r_k/r_j \geq 1/2 \), then \( r_k h_k \geq 1 + I \). Hence, we only need to consider the situation that \( r_k/r_i < 1/2 \) or \( r_k/r_j < 1/2 \). By the angle bound and cosine law, we have

\[
-\frac{1}{5(20 + I)} l_{ij}^2 \leq l_{jk}^2 + l_{ik}^2 - l_{ij}^2.
\]

This is equivalent to

\[
I(r_i r_k + r_j - r_i r_j) \geq -r_k^2 - \frac{1}{10(20 + I)}(r_i^2 + r_j^2 + 2I r_i r_j),
\]

which implies

\[
1 + I(r_k/r_i + r_k/r_j - 1) \geq 1 - \frac{r_k^2}{r_i r_j} - \frac{1}{10(20 + I)} \left( \frac{r_i}{r_j} + \frac{r_j}{r_i} + 2I \right) \geq \frac{4}{5} - \frac{r_k^2}{r_i r_j},
\]

where the results in part (a) of Lemma 5.9 is used in the last inequality. Then by the formula (10) of \( h_k \), we have

\[
r_k h_k = (1 + I)(1 + I(r_k/r_i + r_k/r_j - 1)) \geq (1 + I)(\frac{4}{5} - \frac{r_k^2}{r_i r_j}).
\]

Therefore, under the assumption that \( r_k/r_i \leq 1 \) and \( r_k/r_j \leq 1 \), we have \( r_k h_k \geq 3(1 + I)/10 > (1 + I)/4 \) when \( r_k/r_i < 1/2 \) or \( r_k/r_j < 1/2 \).

The sine law implies that \( l_{ij}^2/50 \leq A \leq 5l_{ij}^2 \). Combining with part (a) of Lemma 5.9, we can find two constants \( M = M(I) \) and \( C = C(I) \) such that

\[
M(I) \geq \frac{r_i^2 r_j^2 (1 + I)(1 + 40I)}{A l_{ij}^2} \geq \frac{r_i^2 r_j^2 r_k h_k}{A l_{ij}^2} \geq \frac{r_i^2 r_j^2 (1 + I)}{4 A l_{ij}^2} \geq C(I) > 0.
\]
Theorem 5.10. Let $\mathcal{P} = \triangle ABC$ be an equilateral triangle, $\mathcal{T}_{(n)}$ be the $n$-th standard subdivision of $\mathcal{P}$, $l$ be an inversive distance circle packing metric on $(\mathcal{P}, \mathcal{T}_{(n)})$ induced by a constant weight $1$ and a constant label. Set

$$V_0 = \{ v \in V | v \text{ is in the edge BC of the triangle } \triangle ABC \}.$$ 

Given any $\alpha \in [\frac{\pi}{6}, \frac{\pi}{2}]$, there exists a smooth family of discrete conformal factors $w(t) \in \mathbb{R}^V$ for $t \in [0, 1]$ such that $w(0) = 0$ and $w(t) * l$ is an inversive distance circle packing metric on $\mathcal{T}_{(n)}$ with curvature $K(t) = K(w(t) * l)$ satisfying

1. $K_A(t) = -t\alpha + (2 + t)\pi$, 
2. $K_i(t) = 0$ for all $v_i \in V - \{ A \} \cup V_0$, 
3. $w_i(t) = 0$ for all $v_i \in V_0$, 
4. all inner angles $\theta_{jk}(t)$ in the metric $w(t) * l$ are in the interval 
   $$[\frac{\pi}{3} - |\alpha - \frac{\pi}{3}|, \frac{\pi}{3} + |\alpha - \frac{\pi}{3}|] \subseteq [\frac{\pi}{6}, \frac{\pi}{2}]$$
5. for $v_i \neq A$, 
   $$|K_i(t) - K_i(0)| = O\left(\frac{1}{\sqrt{\ln(n)}}\right).$$

Moreover, 

$$\sum_{v_i \in V_0} |K_i(t) - K_i(0)| \leq \frac{\pi}{6}.$$ 

Notice that the angle at the vertex $A$ is $t\alpha + (1-t)\pi/3$ along $w(t)$, and curvatures of vertices stay zero except vertices in $BC$ and the vertex $A$. The piecewise linear map from $(\mathcal{P}, \mathcal{T}_{(n)}, l)$ to $(\mathcal{P}, \mathcal{T}_{(n)}, w * l)$ determined by Theorem 5.10 is a discrete analogue of the analytic function $f(z) = z^{3\alpha/\pi}$. This construction works for any $n$-th subdivision of equilateral triangulations.

To prove Theorem 5.10, we need the following two estimates for solutions to the Dirichlet problem when the graph is an equilateral triangulation of a polygonal disk.

Lemma 5.11 (Lemma 5.8). Assume $\triangle ABC, n, \mathcal{T}, V_0$ are as given in Theorem 5.10 Let $\tau : \mathcal{T} \rightarrow \mathcal{T}$ be the involution induced by the reflection of $\triangle ABC$ about the angle bisector of $\angle BAC$ and $\eta : E \rightarrow \mathbb{R}_{\geq 0}$ be a conductance so that $\eta \tau = \eta$ and $\eta_{ij} = \eta_{ji}$. Let $\Delta : \mathbb{R}^V \rightarrow \mathbb{R}^V$ be the Laplace operator defined by $(\Delta f)_i = \sum_{j \sim i} \eta_{ij} (f_i - f_j)$. If $f \in \mathbb{R}^V$ satisfies $(\Delta f)_i = 0$ for $v_i \in V - \{ A \} \cup V_0$ and $f|_{V_0} = 0$, then for all edges $v_i v_j$, the gradient $(\nabla f)_{ij} = \eta_{ij} (f_i - f_j)$ satisfies

$$|\nabla f| \leq \frac{1}{2} |\Delta f|.$$ 

Lemma 5.12 (Lemma 5.9). Assume $\triangle ABC, n, \mathcal{T}, V_0$ are as given in Theorem 5.10 Let $\eta : E(\mathcal{T}) \rightarrow [\frac{1}{M}, M]$ be a conductance function for some $M > 0$ and $\Delta$ be the Laplace operator on $\mathbb{R}^V$ associated to $\eta$. If $f : V \rightarrow \mathbb{R}$ solves the Dirichlet problem $(\Delta f)_i = 0, \forall v_i \in V - \{ A \} \cup V_0, f|_{V_0} = 0$ and $(\Delta f)_A = 1$, then for all $u \in V_0$, $|\nabla f| \leq \frac{2M}{\sqrt{\ln(n)}}$.

Proof of Theorem 5.10 We will prove Theorem 5.10 by considering the ODE system (33) for $(\triangle ABC, \mathcal{T}_{(n)}, l)$ when $n$ is sufficiently large. The prescribed curvature $K^*$ is 

$$K_A^* = \pi - \alpha, K_i^* = 0, v_i \in V - V_0 \cup \{ A \}.$$
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

The initial curvature $K^0$ is

$$K_i^0 = K_i(0) = \frac{2\pi}{3}, v_i \in \{A, B, C\}, K_i^0 = K_i(t) = 0, v_i \in V - \{A, B, C\}.$$ 

Then the ODE system (33) could be written as

$$\begin{cases} 
K'_A(t) = (\Delta w')_A = \frac{\pi}{3} - \alpha, \\
K'_i(t) = (\Delta w')_i = K_i^* - K_i^0 = 0, v_i \in V_0 \cup \{A\}, \\
w'_i(t) = 0, v_i \in V_0,
\end{cases}$$

with the initial value $w_i(0) = 0$. It is straightforward to check that $w(0) \in W$, where $W$ is defined by (34).

Then there exists a maximum $t_0 > 0$ such that a solution $w(t)$ to (40) exists. Moreover, Lemma 5.8 implies that there exists a maximal time $s_0$ such that $w(t) \in W$ and the statement (4) holds for $t \in [0, s_0)$. We will prove $s_0 \geq 1$. Moreover, $w(1)$ exists and $w(1) * l$ is an inversive distance circle packing metric satisfying (1)-(4) in Theorem 5.10. Without loss of generality, we assume that $s_0 < \infty$.

**Claim:** For any inner angle $\theta^i_{jk}$ and $t \in [0, s_0)$, we have

$$|\theta^i_{jk}(t) - \frac{\pi}{3}| \leq t|\alpha - \frac{\pi}{3}|.$$  

To prove this claim, notice that $\alpha \in [\pi/6, \pi/2]$ implies $\theta^i_{jk}(t) \in [\pi/6, \pi/2]$ for $t \in [0, s_0)$ by the statement (4). By Lemma 5.9, $\eta^k_{ij}(t) > C(I) > 0$ for any triangle $\Delta v_iv_jv_k$. Then

$$|(\nabla w')_{ij}| = \eta_{ij}|w'_i - w'_j| \geq \eta^k_{ij}|w'_i - w'_j|.$$  

By Lemma 5.11 and $\frac{dK_i}{dt} = (\Delta w')_i$, we obtain

$$|(\nabla w')_{ij}| \leq \frac{1}{2}|(\Delta w')_A| = \frac{1}{2} \left| \frac{dK_A}{dt} \right| = \frac{1}{2} \left| \alpha - \frac{\pi}{3} \right|.$$  

By the formula (8), we have

$$\left| \frac{d\theta^i_{jk}}{dt} \right| \leq \eta^k_{ik}|w'_i - w'_k| + \eta^k_{jk}|w'_i - w'_j| \leq |(\nabla w')_{ik}| + |(\nabla w')_{ij}| \leq |\alpha - \frac{\pi}{3}|.$$  

Then for all $t \in [0, s_0)$,

$$|\theta^i_{jk}(t) - \frac{\pi}{3}| = |\theta^i_{jk}(t) - \theta^i_{jk}(0)| \leq \int_0^t \left| \frac{d\theta^i_{jk}(t)}{dt} \right| \, dt \leq t|\alpha - \frac{\pi}{3}|.$$  

Now it is not hard to show that $s_0 \geq 1$ from the claim. Notice that $\liminf_{t \to s_0^-} \theta^i_{jk}(w(t)) \geq \pi/6$ and $\liminf_{t \to s_0^-} \eta_{ij}(w(t)) > 0$ for all edges by Lemma 5.9. Therefore, as $t \to s_0^-$, for some $\theta^i_{jk}$,

$$\limsup_{t \to s_0^-} |\theta^i_{jk}(w(t)) - \frac{\pi}{3}| = |\alpha - \frac{\pi}{3}|$$

by the definition of $s_0$ and Lemma 5.8. If $s_0 < 1$, then for all the inner angles, we have

$$\limsup_{t \to s_0^-} |\theta^i_{jk}(t) - \frac{\pi}{3}| \leq s_0|\alpha - \frac{\pi}{3}| < |\alpha - \frac{\pi}{3}|$$

by (41), which contradicts (42). Therefore, $s_0 \geq 1$.

Notice that $\theta^i_{jk}(t) \in [\pi/6, \pi/2]$ for all inner angles and $\eta_{ij}(t) \geq C(I) > 0$ by Lemma 5.9 when $t \in [0, s_0)$. This means that $w(t) * l$ is non-degenerate and strictly weighted Delaunay for...
Lemma 5.13 (\cite{21}, Proposition 5.10) \[ \text{We need the following lemma in the second step.} \]

In the second step, we eliminate these small curvatures of boundary vertices of nonzero curvatures. This step will diffuse the curvature of boundary vertices of polyhedral disk \( \mathcal{P} \) to interior vertices such that curvatures are small if the subdivision is sufficiently dense. In the second step, we eliminate these small curvatures using a flow similar to (33). We need the following lemma in the second step.

Lemma 5.13 (\cite{21}, Proposition 5.10). Suppose \((\mathcal{P}, \mathcal{T}, l)\) is polygonal disk with an equilateral triangulation and \( \mathcal{T} \) is the \( n \)-th standard subdivision of the triangulation \( \mathcal{T}' \) with \( n \geq e^{10^6} \). Let \( \eta : E = E(\mathcal{T}) \to [1, M] \) be a conductance function with \( M > 0 \) and \( \Delta : \mathbb{R}^V \to \mathbb{R}^V \) be the associated Laplace operator. Let \( V_0 \subset V(\mathcal{T}) \) be a thin subset such that for all \( v \in V \cap V_0 \)

\[ t \in [0, s_0). \] Therefore, Lemma 5.8 implies that \( 1 \leq s_0 < t_0 \). Then \( w(1) \in W \). By continuity, the metric \( w(1) \cdot l \) is a non-degenerated inversive distance circle packing metric satisfies (1)-(4) in Theorem 5.10.

Finally, we use Lemma 5.12 to prove the last statement in Theorem 5.10. Notice that by Lemma 5.9, \( 0 < C \leq \eta_{ij} \leq M \), where \( C = C(I), M = M(I) \). Applying Lemma 5.12 to the function \( f = \frac{dw(t)}{dt}/(\alpha - \pi/3) \), we obtain

\[ \left| \frac{dK_i(t)}{dt} \right| = |(\Delta w')_i| = |\alpha - \pi/3| \cdot |(\Delta f)_i| = O\left( \frac{1}{\sqrt{\ln(n)}} \right), \quad v_i \in V_0. \]

Then for \( v_i \neq A \), we have for \( t \in [0, 1] \),

\[ |K_i(t) - K_i(0)| \leq \int_0^t \left| \frac{dK_i(t)}{dt} \right| dt = O\left( \frac{1}{\sqrt{\ln(n)}} \right). \]

Moreover, if \( \alpha = \pi/3 \), then (38) is automatically true since the flow would be a constant flow by Proposition 5.7. Hence we assume \( \alpha \neq \pi/3 \). We claim that \( w_A'(t) \neq 0 \) for \( t \in [0, t_0) \). Otherwise, \( w_A'(s) = 0 \) for some \( s \in [0, t_0) \). Applying the maximum principle, i.e. Proposition 5.7 to the following Dirichlet problem

\[
\begin{cases}
(\Delta w')(s)_i = 0, & v_i \in V - \{A\} \cup V_0, \\
w'(s)_i = 0, & v_i \in \{A\} \cup V_0,
\end{cases}
\]

we obtain \( w'(s) = 0 \) for all \( v_i \in V \), which implies \( (\Delta w')_A(s) = 0 \). This contradicts \( (\Delta w')_A(s) = \alpha - \pi/3 \neq 0 \). This completes the proof of the claim. Furthermore, applying the maximal principal, i.e. Proposition 5.7 to (40) again shows that \( w_A'(t) \) and \( w'_i(t) \) have the same sign. Note that for \( v_i \in V_0, w'_i(t) = 0 \). We have

\[
w'_A(t)K'_i(t) = w'_A(t) \sum_{i \sim j} \eta_{ij}(w'_i - w'_j) = -\sum_{i \sim j} \eta_{ij}w'_j(t)w'_A(t) \leq 0,
\]

which implies \((K_i(t) - K_i(0))w'_A(t) \leq 0\). By the discrete Gauss-Bonnet formula (1), we have

\[ K_A(t) + \sum_{v_i \in V_0} K_i(t) = K_A(0) + \sum_{v_i \in V} K_i(0) = 2\pi. \]

Since \( K_i(t) - K_i(0) \) have the same sign for all \( v_i \in V_0 \), we conclude that for \( t \in [0, 1] \),

\[ \sum_{v_i \in V_0} |K_i(t) - K_i(0)| = |\sum_{v_i \in V_0} (K_i(t) - K_i(0))| = |K_A(t) - K_A(0)| = |t(\alpha - \pi/3)| \leq \pi/6. \]

Q.E.D.

5.4. Proof of Theorem 5.2. There are two steps to find the discrete conformal factor required in Theorem 5.2. In the first step, we construct a discrete conformal factor by Theorem 5.10 where the triangles contain boundary vertices of nonzero curvatures. This step will diffuse the curvature of boundary vertices of polyhedral disk \( \mathcal{P} \) to interior vertices such that curvatures are small if the subdivision is sufficiently dense. In the second step, we eliminate these small curvatures using a flow similar to (33). We need the following lemma in the second step.

Lemma 5.13 (\cite{21}, Proposition 5.10). Suppose \((\mathcal{P}, \mathcal{T}, l)\) is polygonal disk with an equilateral triangulation and \( \mathcal{T} \) is the \( n \)-th standard subdivision of the triangulation \( \mathcal{T}' \) with \( n \geq e^{10^6} \). Let \( \eta : E = E(\mathcal{T}) \to [1, M] \) be a conductance function with \( M > 0 \) and \( \Delta : \mathbb{R}^V \to \mathbb{R}^V \) be the associated Laplace operator. Let \( V_0 \subset V(\mathcal{T}) \) be a thin subset such that for all \( v \in V \cap V_0 \)}
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

and \(m \leq n/2, |B_m(v) \cap V_0| \leq 10m\). If \(f : V \to \mathbb{R}\) satisfies \((\Delta f)_i = 0\) for \(v_i \in V - V_0\), \(|(\Delta f)_i| \leq \frac{M}{\sqrt{\ln(n)}}\) for \(v_i \in V_0\) and \(\sum_{v_i \in V_0} |(\Delta f)_i| \leq M\), then for all edges \(v_j v_k\) in \(\mathcal{T}\),

\[|f_j - f_k| \leq \frac{200M^3}{\sqrt{\ln(\ln(n))}}\]

Proof of Theorem 5.2. We call each boundary vertex of \(\mathcal{P}\) other than \(p, q, r\) corner if it has nonzero curvature. Denote the set of corners as \(V\). Notice that by the assumption on \(\mathcal{P}\), each vertex in \(V\) has degree \(m = 3, 5\) or 6. Moreover, the standard subdivision of each triangle of \(\mathcal{P}\) does not introduce new corners. Thus, the cardinality \(|V_c|\) of \(V_c\) is independent of the subdivision \(\mathcal{T}(n)\) of \(\mathcal{T}\).

Let \(B_{[n/3]}(v)\) be the combinatorial ball in \(\mathcal{T}(n)\) centered at \(v \in V_c\) with radius \([n/3]\) where \([x]\) is the integer part of a real number \(x\). Notice that these ball are disjoint in \(\mathcal{T}(n)\). Each \(B_{[n/3]}(v)\) consists of \(m - 1\) copies of \([n/3]\)-th subdivision of equilateral triangles \(\Delta^v_1, \ldots, \Delta^v_{m-1}\).

Step 1: For every \(v \in V_c\), we will deform its curvature to zero. In particular, we apply Theorem 5.10 to \(\Delta^v_1, \ldots, \Delta^v_{m-1}\) with \(\alpha = \pi/(m-1) \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]\). It produces a discrete conformal factor \(w_i\) on \(\Delta^v_i\) for each \(i = 1, \ldots, m - 1\). Notice that the discrete conformal factor on \(\mathcal{T}(n)\) in Theorem 5.10 depends only on \(\alpha\). Then discrete conformal factor \(w_i\) are identical on each \(\Delta_i\). By symmetry, we can glue them together to form a discrete conformal factor \(\bar{w}\) on \(\mathcal{T}(n)\). Specifically, the value of \(\bar{w}\) on \(B_{[n/3]}(v)\) for \(v \in V_c\) is determined by Theorem 5.10 and the values of \(\bar{w}\) on other vertices are zero.

Let \(\bar{K}\) be the curvature of inversive distance circle packing \(\bar{l} = \bar{w} \ast l\). Let \(K\) be the curvature of the target equilateral triangle with \(K_i = 0\) for all \(v_i \in V(n) - \{p, q, r\}\) and \(K_i = \frac{2\pi}{3}\) for all \(v_i \in \{p, q, r\}\). Then Theorem 5.10 implies that

1. \(\bar{K}_i = K_i\) for all vertices \(v_i\) in the set of \(V_k := \{v_i|d_c(v_i, v) \neq \lceil n/3 \rceil, v \in V_c\}\),
2. \(\bar{w}_i = 0\) for all \(v_i \notin \cup_{v \in V_c} B_{[n/3]}(v)\),
3. all inner angles at \(v \in V\) satisfy \(\theta_{ij}^v \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right]\),
4. for all vertices \(v_i \notin V_k\), \(|\bar{K}_i - K_i| = O\left(\frac{1}{\sqrt{\ln(n)}}\right)\),
5. \(\sum_{v_i \in V} |\bar{K}_i - K_i| \leq \frac{2\pi N}{3}\), where \(N\) denote the number of corners.

Notice that the set \(V_k\) is the union of the sets \(V_0\) given by Theorem 5.10 for each \(v \in V_c\). Statement (1) and (2) are immediate by the construction. Statement (3), (4), and (5) are immediate by Theorem 5.10.

Step 2: we construct a flow to deform the curvatures of vertices in \(V_k\) to be zero when the subdivision is sufficiently dense. Specifically, consider the following ODE system on \(\mathcal{T}(n)\)

\[
\begin{cases}
\frac{dK_i(w(s))}{dt} = K_i - \bar{K}_i, & v_i \in V - \{p, q, r\}, \\
\frac{dw_i(s)}{dt} = 0, & v_i \in \{p, q, r\},
\end{cases}
\]

with initial value \(w(0) = 0\). The idea is the same as that of (33). Namely, we linearly interpolate the initial curvature \(\bar{K}\) and the target curvature \(K\). By Lemma 5.8 there exists a maximal \(s_0 > 0\) such that the solution \(w(s)\) to (43) exists and \(w(s) \ast \bar{l}\) satisfies that on \([0, s_0]\), all inner angles at \(v \in V\), \(\theta_{ij}^v \in [\frac{\pi}{6} - \theta_0, \frac{\pi}{2} + \theta_0]\), where \(\theta_0\) is the parameter given by Lemma 5.9.

Now we apply Lemma 5.13 to estimate the angle deformation along the flow (43). Set \(V_B = V(n) \setminus V_k\). First notice that \(V_B\) is a thin set in \(V(n)\) of \(\mathcal{T}(n)\). In particular, \(|B_r(v_i) \cap V_B| \leq 10r\)
for any $v_i \in V(n)$ and any $r \leq n/3$. Moreover, by (4) and (5) in Step 1, we obtain
\[
\sum_{v_i \in V_B} |K_i'| = \sum_{v_i \in V_B} |(\Delta w')_i| \leq \sum_{i \in V} |\bar{K}_i - K_i| \leq \frac{2\pi N}{3},
\]
and
\[
|K_i'| = |(\Delta w')_i| \leq |\bar{K}_i - K_i| = O\left(\frac{1}{\sqrt{\ln(n)}}\right), v_i \in V_B.
\]
Lemma 5.9 implies that $f = w'$ along the flow (43) satisfies the conditions in Lemma 5.13. Therefore, we obtain that if $i \sim j$, then
\[
|w'_i(s) - w'_j(s)| = O\left(\frac{1}{\sqrt{\ln(\ln(n))}}\right).
\]
As a result,
\[
\left| \frac{d\theta^k_{ij}}{ds} \right| \leq |n^i_k(w'_j - w'_k)| + |n^j_k(w'_i - w'_k)| = O\left(\frac{1}{\sqrt{\ln(\ln(n))}}\right),
\]
where Lemma 5.9 is used in the last equality. For all $s \in [0, s_0)$ and sufficiently large $n$,
\[
|\theta^k_{ij}(w(s)) - \theta^k_{ij}(0)| \leq \int_0^s \left| \frac{d\theta^k_{ij}(w(s))}{ds} \right| ds = O\left(\frac{1}{\sqrt{\ln(\ln(n))}}\right) \leq \theta_0 s_0.
\]
We claim that $s_0 > 1$. Otherwise, we can extend the solution to (43) to $[0, s_0 + \epsilon)$ for some small $\epsilon > 0$, which contradicts the maximality of $s_0$. Set $w^* = w(1)$ and $w = \bar{w} + w^*$. Then the curvature of the inversive distance circle packing metric $w*l$ is
\[
K(0) + \int_0^1 K'(s) ds = \bar{K} + (K - \bar{K}) = K.
\]
This implies that the discrete conformal factor $w = \bar{w} + w^*$ produces the discrete conformal map from $(\mathcal{P}, \mathcal{T}(n), I(n))$ to the equilateral triangle. Q.E.D.

REFERENCES

[1] L. Ahlfors, Lectures on quasiconformal mappings. Vol. 38. American Mathematical Soc, 2006.
[2] F. Aurenhammer, R. Klein, Voronoi Diagrams. Handbook of Computational Geometry, pp. 201-290. North-Holland, Amsterdam (2000)
[3] P. L. Bowers, K. Stephenson, Uniformizing dessins and Belyi maps via circle packing. Mem. Amer. Math. Soc. 170 (2004), no. 805.
[4] U. Bücking, Approximation of conformal mappings by circle patterns. Geom. Dedicata 137 (2008), 163-197.
[5] H. Edelsbrunner, Geometry and topology for mesh generation. Cambridge University Press, 2001.
[6] D. Glickenstein, A monotonicity property for weighted Delaunay triangulations. Discrete Comput. Geom. 38 (2007), no. 4, 651-664.
[7] D. Glickenstein, Discrete conformal variations and scalar curvature on piecewise flat two and three dimensional manifolds, J. Differential Geom. 87 (2011), no. 2, 201-237.
[8] D. Glickenstein, Geometric triangulations and discrete Laplacians on manifolds, arXiv:math/0508188 [math.MG].
[9] D. Glickenstein, J. Thomas, Duality structures and discrete conformal variations of piecewise constant curvature surfaces. Adv. Math. 320 (2017), 250-278.
[10] X. D. Gu, R. Guo, F. Luo, J. Sun, T. Wu, A discrete uniformization theorem for polyhedral surfaces II, J. Differential Geom. 109 (2018), no. 3, 431-466.
[11] X. D. Gu, F. Luo, J. Sun, T. Wu, A discrete uniformization theorem for polyhedral surfaces, J. Differential Geom. 109 (2018), no. 2, 223-256.
Bowers-Stephenson’s conjecture on convergence of inversive distance circle packings

[12] X. D. Gu, F. Luo, T. Wu, Convergence of discrete conformal geometry and computation of uniformization maps. Asian J. Math. 23 (2019), no. 1, 21-34.

[13] R. Guo, Local rigidity of inversive distance circle packing, Trans. Amer. Math. Soc. 363 (2011) 4757-4776.

[14] Z.-X. He, Rigidity of infinite disk patterns. Ann. of Math. (2) 149 (1999), no. 1, 1-33.

[15] Z.-X. He, O. Schramm, On the convergence of circle packings to the Riemann map. Invent. Math., 125 (1996), 285-305.

[16] Z.-X. He, O. Schramm, The $C^\infty$-convergence of hexagonal disk packings to the Riemann map, Acta Math. 180 (1998) 219-245.

[17] O. Lehto, Univalent functions and Teichmüller spaces. Vol. 109. Springer Science & Business Media, 2012.

[18] L. Lovász, Graphs and Geometry, vol. 65, American Mathematical Soc, 2019.

[19] F. Luo, Combinatorial Yamabe flow on surfaces. Commun. Contemp. Math. 6 (2004), no. 5, 765-780.

[20] F. Luo, Rigidity of polyhedral surfaces, III, Geom. Topol. 15 (2011), 2299-2319.

[21] F. Luo, J. Sun, T. Wu, Discrete conformal geometry of polyhedral surfaces and its convergence, Geom. Topol. 26 (2022), no. 3, 937-987.

[22] Y. Luo, X. Xu, S. Zhang, private communications.

[23] B. Palka, Fréchet distance and the uniform convergence of quasiconformal mappings. Duke Math. J. 39 (1972), 289-296.

[24] C. Pommerenke, Univalent functions. Studia MathematicaMathematische Lehrbücher, Band XXV. Vandenhoeck & Ruprecht, Göttingen, 1975. 376 pp.

[25] B. Rodin, D. Sullivan, The convergence of circle packings to the Riemann mapping. J. Differential Geom. 26 (1987) 349-360.

[26] O. Schramm, Rigidity of infinite (circle) packings. J. Amer. Math. Soc. 4 (1991), no. 1, 127-149.

[27] K. Stephenson, Introduction to circle packing. The theory of discrete analytic functions. Cambridge University Press, Cambridge, 2005.

[28] W. Thurston, Geometry and topology of 3-manifolds. Princeton lecture notes 1976, http://www.msri.org/publications/books/gt3m

[29] W. Thurston, The finite Riemann mapping theorem. An International Symposium at Purdue University on the Occasion of the Proof of the Bieberbach Conjecture, 1985.

[30] T. Wu, X. Zhu, The convergence of discrete uniformizations for closed surfaces, to appear in J. Differential Geom. [arXiv:2008.06744v2 [math.GT]]

[31] X. Xu, Rigidity of inversive distance circle packings revisited, Adv. Math. 332 (2018), 476-509.

[32] X. Xu, A new proof of Bowers-Stephenson conjecture, Math. Res. Lett. 28 (2021), no. 4, 1283-1306.

[33] X. Xu, Rigidity and deformation of discrete conformal structures on polyhedral surfaces. [arXiv:2103.05272v2 [math.DG]].

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R.China
Email address: chenyuxiang@whu.edu.cn

Department of Mathematics, Rutgers University, New Brunswick NJ, 08817
Email address: y11594@rutgers.edu

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R.China
Email address: xuxu2@whu.edu.cn

School of Mathematics and Statistics, Wuhan University, Wuhan, 430072, P.R.China
Email address: 2014301000156@whu.edu.cn