Exact double averages of twisted $L$-values

Kimball Martin

Received: 8 June 2022 / Accepted: 21 July 2022 / Published online: 6 September 2022
© The Author(s), under exclusive licence to Springer-Verlag GmbH Germany, part of Springer Nature 2022

Abstract
Consider central $L$-values of even weight elliptic or Hilbert modular forms $f$ twisted by ideal class characters $\chi$ of an imaginary quadratic extension $K$. Fixing $\chi$, and assuming $K$ is inert at each prime dividing the level, one knows simple exact formulas for averages over newforms $f$ of squarefree levels satisfying a parity condition on the number of prime factors. These averages stabilize when the level is large with respect to $K$ (the “stable range”). In weight 2, we obtain exact formulas for a simultaneous average over both $f$ and $\chi$. We allow for non-squarefree levels with any number of prime factors, and ramification or splitting of $K$ above the level. Under elementary conditions on the level, these double averages are “stable” in all ranges. Two consequences are generalizations of the aforementioned stable (single) averages and effective results on nonvanishing of central $L$-values.

1 Introduction

1.1 Prime power level

We first explain our results in the case of prime power levels over $\mathbb{Q}$. Denote by $S_2(N)$ the space of weight 2 elliptic modular forms of level $\Gamma_0(N)$. Let $\mathcal{F}_{\text{new}}(N)$ denote the set of level $N$ newforms in $S_2(N)$. Denote by $L(s, -)$ a completed $L$-function with center $s = \frac{1}{2}$, and by $L_{\text{fin}}(s, -)$ the nonarchimedean part of $L(s, -)$.

Suppose $N$ is a prime, $K/\mathbb{Q}$ an imaginary quadratic field of discriminant $D_K$ which is inert at $N$, and $\chi$ a character of $\text{Cl}(K)$. Assuming $D_K$ is odd, [19] proved an exact formula for weighted averages of the central values $L(\frac{1}{2}, f, \chi)$, where $f$ ranges over $\mathcal{F}_{\text{new}}(N)$. In the stable range $N > |D_K|$, these averages simplify to yield:

$$\frac{u_K^2 \sqrt{-D_K}}{8\pi^2} \sum_{f \in \mathcal{F}_{\text{new}}(N)} L_{\text{fin}}\left(\frac{1}{2}, f, \chi\right) \left(\frac{f}{f}\right) = h_K \left(u_K - \delta_{\chi, 1} \frac{12h_K}{N - 1}\right).$$

(1.1)

Here $\delta_{\chi, 1}$ is the Kronecker delta and $u_K = \# \mathcal{O}_K^2$. The work [19] also allows for weighting by Fourier coefficients of $f$ and arbitrary even weights. From the generalization in [7], (1.1) still holds when $D_K$ is even.

Kimball Martin
kimball.martin@ou.edu

1 Department of Mathematics, University of Oklahoma, Norman, OK 73019, USA
Our main theme is that the arithmetic of quaternion algebras allows us to provide exact formulas for any range of \((N, |D_K|)\) when we average over both \(f\) and \(\chi\). These double average value formulas are strikingly simple for “balanced” levels, and yield stable single average value formulas generalizing (1.1).

**Theorem 1.1** Suppose \(N = p^{2r+1} \geq 11\) for a prime \(p\) with \(N \neq 27\), and \(K/\mathbb{Q}\) is an imaginary quadratic field which is not split at \(p\). If \(r > 0\), assume \(K/\mathbb{Q}\) is inert at \(p\). Put

\[
C_K = \frac{1}{2} e_p(K/\mathbb{Q}) u_K^2 \sqrt{-D_K},
\]

where \(e_p(K/\mathbb{Q}) = 1\) if \(K/\mathbb{Q}\) is inert at \(p\) and \(e_p(K/\mathbb{Q}) = 2\) if \(K/\mathbb{Q}\) is ramified at \(p\).

(i) If either \(p \equiv 1\) mod 12 or \(u_K > 1\), we have the exact double average formula

\[
\frac{C_K}{4\pi^2} \sum_{\chi \in \widehat{\text{Cl}}(K)} \sum_{f \in \mathcal{F}_{\text{new}}(N)} L_{\text{fin}}\left(\frac{1}{2}, f, \chi\right) \frac{\left(L_{f}\left(\frac{1}{2}, f, \chi\right)\right)}{\left(L_{f}(1, f, \chi)\right)} = \begin{cases} 
\frac{h_{K}^2}{\eta \chi} \left(u_K - \frac{12}{p-1}\right) & r = 0, \\
\frac{h_{K}^2}{\eta \chi} \left(1 - \frac{1}{p^2}\right) & r \geq 1.
\end{cases}
\]

(ii) If \(p > |D_K|\), we have the stable (single) average formula

\[
\frac{C_K}{4\pi^2} \sum_{f \in \mathcal{F}_{\text{new}}(N)} L_{\text{fin}}\left(\frac{1}{2}, f, \chi\right) \frac{\left(L_{f}\left(\frac{1}{2}, f, \chi\right)\right)}{\left(L_{f}(1, f, \chi)\right)} = \begin{cases} 
\frac{h_{K}^2}{\eta \chi} \left(u_K - \delta_{\chi, 1} \frac{12h_{K}}{p-1}\right) & r = 0, \\
\frac{h_{K}^2}{\eta \chi} \left(1 - \frac{1}{p^2}\right) & r \geq 1.
\end{cases}
\]

Since the formula in (i) only depends on \(h_K\) and \(u_K\), by analogy with [19], we think of this double average as being “stable” in all ranges when \(p \equiv 1\) mod 12 (in which case we call \(N = p^{2r+1}\) balanced) or if \(u_K > 1\). If we do not assume that \(p \equiv 1\) mod 12, then one still has a double average value formula but it involves height pairings and is less elementary. One can also weight the \(L\)-values by Fourier coefficients of \(f\) as in [19], but then the formulas also involve certain representation numbers of quadratic forms. We also treat levels \(N = p^2\) when \(K\) is ramified at \(p\), and allow splitting of \(K\) above the level when \(N\) has multiple prime factors. The issue with levels \(N = p^{2r}\) with \(r > 1\) is explained below.

The most interesting, or at least classical, case of (ii) is when \(\chi = 1\) is trivial, so that our twisted \(L\)-function factors as \(L(s, f, 1) = L(s, f) L(s, f \otimes \eta_K)\). In this case if \(N = p^{2r+1}\) and \(K\) is split at \(p\), then we are forced to have \(L(1, f, 1_K) = 0\) because the root number is \(-1\). So for prime power level \(N\) and imaginary quadratic \(K\), the cases of most interest are when \(K\) is inert or ramified at \(p\), as in the theorem.

Using an extension of (i) to allow \(p \not\equiv 1\) mod 12, we still have elementary upper and lower bounds on the double average. We leverage this to get lower (and upper) bounds for the single average (1.3) when \(\chi = 1\) is trivial. These bounds are valid outside of the stable range \(p > |D_K|\) in (ii), and yield the following sharper effective non-vanishing result.

**Corollary 1.2** Fix an imaginary quadratic field \(K\). Then for any odd power of a prime \(N = p^{2r+1}\) such that \(p\) is inert in \(K\), there exists a newform \(f \in S_2(N)\) such that \(L(\frac{1}{2}, f) L(1, f \otimes \eta_K) \neq 0\), provided that (i) \(p > \frac{12h_{K}}{u_K} + 1\) if \(N = p\) is prime; or (ii) \(p^2 > \frac{3h_{K}}{u_K}\) if \(N > 27\) is not prime.

The prime level case is already in [19, Theorem 3]. See Proposition 5.9 for an analogue in level \(p^2\). We remark that effective non-vanishing results of this type in levels \(p^2\) and \(2p^2\) have been applied to generalized Fermat equations in [1, 6].

The prime level case of Theorem 1.1(i) (and its extension to general \(p\)) follows easily from Gross’s \(L\)-value formula [9]. (Gross’s formula was also the basis of the proof of the weight 2 case of [19].) However, to our knowledge, this had not been observed before.
main results of [19] have been extended to squarefree levels \( N \) which have an odd number of prime factors in [7] (which also treats totally real base fields). Note that [7] also requires \( K \) to be inert at each prime dividing the level. The approach there is via a relative trace formula, and is based on the central \( L \)-value formula in terms of periods on quaternion algebras from [17].

Our approach also uses the relative trace formula from [17], but instead of computing geometric orbital integrals as in [7], we directly compute averages of periods using the arithmetic of quaternion algebras. The main technical difficulties in this work arise from treating levels which are both not squarefree and which may have an even number of prime factors. A novel feature of our approach is that it allows \( K \) to be split or ramified at some primes dividing the level when \( \chi = 1_K \). Moreover, our approach to compute period averages is much simpler than the geometric trace formula calculations in [7]. On the other hand, we do not get an independent proof of the stable range formulas, nor is it clear how well our method could treat higher weights. (However, the case of weight 2 where our method readily applies is the precisely the case in which the geometric trace formula calculations are the most complicated.) That said, we expect that blending our methods with those of [7] should allow one to extend [7] to more general levels and ramification behavior of \( K \) in arbitrary even weight. (E.g., see Remark 1.6.)

1.2 A more general class of levels

In this paper, we treat a broad class of level structures for parallel weight 2 Hilbert modular forms over a totally real base field of class number 1. However, we will restrict to elliptic modular forms in the introduction, as well as for some of our more explicit results at the end.

Roughly, we can treat levels \( N \) such that if \( p^{2r} \) sharply divides \( N \) then \( r \leq 1 \). However, there is also a parity condition if there is no prime sharply dividing \( N \). The quadratic fields \( K \) can have arbitrary ramification at primes sharply dividing \( N \), but must be ramified at \( p \) when \( p^2 \) sharply divides \( N \) and inert at \( p \) when \( p^3 \mid N \). Here is the precise setup.

By a nice level type \((N_1, N_2, M)\) we mean pairwise coprime positive integers \( N_1, N_2 \) and \( M \) such that (i) each prime dividing \( N_1 \) occurs to an odd power; (ii) \( N_2 \) is the square of a squarefree number; (iii) \( \# \{p \mid |N_1 N_2| \} \) is odd; and (iv) \( M \) is squarefree. We say an imaginary quadratic field \( K \) is \((N_1, N_2, M)\)-admissible if \( K/\mathbb{Q} \) is non-split at each \( p \mid N_1 N_2 \), split at each \( p \mid M \), ramified at each \( p \mid N_2 \), and unramified at each \( p \) such that \( p^3 \mid N_1 \).

From now on, assume that \( N = N_1 N_2 M \) with \((N_1, N_2, M)\) a nice level type, and that \( K \) is an \((N_1, N_2, M)\)-admissible imaginary quadratic field. Our main results for elliptic modular forms extend and refine Theorem 1.1 such \( N \) and \( K \). For general nice level types, our formulas will involve a contribution from smaller levels, though when \( M = 1 \) we will be able to isolate a sum over newforms. We address the reasons for our restrictions in the outline of the proof below.

Let \( \mathcal{F}(N) \) be the set of normalized eigenforms \( f \in S_2(N) \) which are newforms of some level \( N' \mid N \) such that (i) \( \text{ord}_p(N') \) is odd for each \( p \mid N_1 \), and (ii) the local representation of \( \text{GL}_2(\mathbb{Q}_p) \) associated to \( f \) is discrete series for each \( p \mid N_2 \). For a character \( \chi \) of \( \text{Cl}(K) \), denote by \( \mathcal{F}(N; \chi) \) the subset of \( f \in \mathcal{F}(N) \) such that \( a_p(f) = \chi_p(\sqrt{-D_K}) \) for each \( p \mid N \) which ramifies in \( K \).

Put

\[
C(K; N) = 2^\#(p \mid \gcd(D_K, N)) - 1 u_K^2 \sqrt{-D_K} \prod_{p \mid N_2} \frac{1}{1 + p^{-1}}.
\] (1.4)
Note that since \( \{ p | N_2 \} = \{ p | N : \text{ord}_p(N) = 2 \} \), we see \( C(K; N) \) only depends on \( N \) and not the precise choice of triple \((N_1, N_2, M)\), which in general is not uniquely determined by \( N \). Set
\[
c(N_1, N_2, M) = \frac{12}{N} \prod_{p|N_{1,2}} \frac{1}{1 - p^{-1}} \prod_{p|N_2M} \frac{1}{1 + p^{-1}}.
\]

We also define a weight \( \Lambda_N(f, \chi) = \prod_{p|N} \Lambda_p(f, \chi) \). Here the local factor \( \Lambda_p(f, \chi) = 1 \) if \( p|N_1 \), is defined by (5.8) when \( p|N_2 \), and by (5.4) or equivalently (5.11) when \( p|M \).

When \( p|N_2 \), the factor \( \Lambda_p(f, \chi) \) does not actually depend on \( \chi \), only whether \( f \) is \( p \)-new and whether \( f \) is \( p \)-minimal.

Let \( N'_1 = \prod p^{\text{ord}_p(N_1)} \) where \( p \) runs over primes such that \( \text{ord}_p(N_1) > 1 \). Let \( \omega'(n) \) be the number of odd prime divisors of \( n \). Denote by \( \text{deg} T_m \) the degree of the Hecke operator on \( T_m \), which is 1 if \( m = 1 \) and \( p + 1 \) if \( m = p \) is a prime not dividing \( N \). (More precisely, it is the degree of \( T_m \) on a certain space of quaternionic modular forms—see Sect. 4.1 for our precise definition.)

For a new form \( f \), let \( N_f \) denote the (exact) level of \( f \), and \((f, f) = (f, f)_{N_f} \) be the usual Petersson norm of \( f \) and whether \( f \) is \( N \)-admissible implies that there is a special order (in the sense of Hijikata–Pizer–Shemanske [12]) \( \mathcal{O} \subset B \) of level type \((N_1, N_2, M)\), which in general is not uniquely determined by \( N \).

Our main result, Theorem 5.5, specialized to the case \( F = \mathbb{Q} \) says the following, with remaining notation defined below.

**Theorem 1.3** Let \( m \geq 1 \) be coprime to \( N'_1 N_2 \). Then
\[
\frac{C(K; N)}{4\pi^2} \sum_{\chi \in \text{Cl}(K)} \sum_{f \in \mathcal{F}(N; \chi)} a_m(f) \Lambda_N(f, \chi) \frac{N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right)}{N} \Lambda_N(f, \chi) \frac{N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right)}{N}
\]
\[
= h_K \left( \sum w_i h_i a_i(m) - \delta^+(N, m) \text{deg } T_m 2^{\omega'(N_f)} h_K c(N_1, N_2, M) \right).
\]

In particular, if \( m = 1 \), we have
\[
\frac{C(K; N)}{4\pi^2} \sum_{\chi \in \text{Cl}(K)} \sum_{f \in \mathcal{F}(N; \chi)} \Lambda_N(f, \chi) \frac{N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right)}{N} \Lambda_N(f, \chi) \frac{N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right)}{N}
\]
\[
= h_K \left( \sum w_i h_i - 2^{\omega'(N_f)} h_K c(N_1, N_2, M) \right).
\]

The weighting of our average \( L \)-values by \( \Lambda_N(f, \chi) \frac{N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right)}{N} \) is simply for two reasons: to account for oldforms at \( p|M \), and because we wrote the above average in terms of the Petersson norm \((f, f)\) rather than \( L(1, f, \text{Ad}) \). The factors \( \Lambda_p(f, \chi) \) are just 1 for \( p|M \) when \( f \) is \( p \)-new, and the factors \( \frac{N_f}{N} \) and \( \Lambda_p(f, \chi) \) for \( p|N_2 \) are not needed if one replaces \((f, f)\) with \( L(1, f, \text{Ad}) \) (see Theorem 5.5.)

We will now explain the remaining quantities in this theorem, but the main point is that the quantity \( \sum w_i h_i \) is approximately \( h_K \) (or exactly \( u_K h_K \) if \( u_K > 1 \)), and is exactly \( h_K \) under either elementary conditions on \((N_1, N_2, M)\) or when we are in a stable range.

Let \( B/\mathbb{Q} \) be the definite quaternion algebra with discriminant \( D_B = \prod_{p|N_1N_2} p \). Then the condition that \( K = (N_1, N_2, M) \)-admissible implies that there is a special order (in the sense of Hijikata–Pizer–Shemanske [12]) \( \mathcal{O} \subset B \) of level type \((N_1, N_2, M)\) in which \( \sigma_K \) embeds. In particular, if \( N \) is squarefree, we simply mean that \( \mathcal{O} \) is an Eichler order of level \( N \) in \( B \). See Sect. 3.1 for the general definition. We remark that \( c(N_1, N_2, M) \) is simply the reciprocal of the mass of \( \mathcal{O} \).

Let \( \text{Cl}(\mathcal{O}) = \{ \mathcal{I}_1, \ldots, \mathcal{I}_n \} \) denote the set of (invertible) right \( \mathcal{O} \)-ideal classes of \( B \). Let \( w_i \) be one half of the number of units in the left order of \( \mathcal{I}_i \). We usually have \( w_i = 1 \), and
if \( N \geq 5 \) then each \( w_i \leq 3 \) (see Lemma 5.7). Fix an embedding \( \sigma_K \subset \mathcal{O} \), which induces an ideal class map \( \text{Cl}(K) \to \text{Cl}(\mathcal{O}) \). Let \( h_i \) be the size of the preimage of \( \mathcal{I}_i \) under this map. Hence \( \sum h_i = h_K \), and we see that \( h_K \leq \sum w_i h_i \leq 3h_K \). We note that \( \sum w_i h_i \) can be interpreted in terms of heights of special cycles as in [9]. This explains all of the notation in the \( m = 1 \) case.

For \( m \geq 1 \), \( \delta^+(N, m) \) is either 0 or 1. It is always 1 if \( m = 1 \) or if \( N_2 = 1 \). See Proposition 4.1 for the full definition, where it is denoted \( \delta^+(\mathcal{O}, m) \). Finally, \( a_{ij}(m) \) denotes the \( i \)-th diagonal element of the \( m \)-th Brandt matrix associated to the given ordering of \( \text{Cl}(\mathcal{O}) \).

This can be expressed in terms of the number of ways a quadratic form associated to (the left order of) \( \mathcal{I}_i \) represents \( m \)—see Sect. 3.3.

We say a triple \((N_1, N_2, M)\) is balanced if it satisfies the conditions of Corollary 4.10. These are simple elementary conditions on the primes dividing \( N_1, N_2 \) and \( M \), that force \( \sum w_i h_i = h_K \) for any \((N_1, N_2, M)\)-admissible \( K \). For instance, the triple \((N_1, N_2, M)\) is balanced if \( N_2 > 9 \), or if there exists a \( p \equiv 1 \mod 12 \) dividing \( N_1 \), or if there exists a \( p \equiv 11 \mod 12 \) dividing \( M \). Hence for balanced level types, the above double average formula, at least for \( m = 1 \), is completely elementary.

To get from Theorem 1.3 to Theorem 1.1(i), we want to isolate the newform contribution at level \( N \). This is possible, at least for \( m = 1 \), when \( \sum w_i h_i = h_K \) in both level \( N \) as well as any relevant level \( N' \mid N \) for which we need to remove the contribution of oldforms. (This is the reason for the at-first-glance curious exclusion of \( N = 27 \)—cf. Example 6.3.)

This idea leads to a generalization of Theorem 1.1(i) to where one restricts to forms which are new away from \( M \). (The factors \( \Lambda_N(f, \chi) \) prevent us from exactly isolating forms which are new also at \( M \), but we can at least approximate the contribution from forms new at \( M \)—e.g., Proposition 5.10.)

Let \( \mathcal{F}_0(N) \) be the set of normalized eigenforms \( f \in \mathcal{F}(N) \) which are newforms for some level \( N' \mid N \) such that \( \text{ord}_p(N') = \text{ord}_p(N) \) for \( p \mid N_1N_2 \), i.e., \( f \) is \( N_1N_2 \)-new. In particular, if \( M = 1 \) each \( f \in \mathcal{F}_0(N) \) is new of level \( N \) (though one does not get all newforms if \( N_2 > 1 \)). For a character \( \chi \) of \( \text{Cl}(K) \), put \( \mathcal{F}_0(N; \chi) = \mathcal{F}_0(N) \cap \mathcal{F}(N; \chi) \).

**Theorem 1.4** Asssume that one of the following holds:

1. the triple \((D_B, 1, M)\) is balanced;
2. we are in the stable range \( D_B > |D_K| \) with \( \gcd(D_B, D_K) = 1 \); or
3. \( u_K > 1 \) and \( N \geq 11 \) with \( N \neq 27 \).

Then

\[
\frac{C(K; N)}{4\pi^2} \sum_{\chi \in \text{Cl}(K)} \sum_{f \in \mathcal{F}_0(N; \chi)} \Lambda_N(f, \chi) N_f L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right) N \left( \frac{1}{2}, f, \chi \right) \frac{1}{N} = h_K^2 \left( u_K \prod_{p \mid N_1} \left( 1 - \frac{1}{p^2} \right) \cdot \prod_{p \mid N_2} \frac{p}{p+1} - \delta \frac{12}{N} \prod_{p \mid N_1N_2} \frac{1}{1 - p^{-1}} \prod_{p \mid N_2M} \frac{1}{1 + p^{-1}} \right),
\]

where \( \delta = 1 \) if \( N_1 \) is squarefree and \( N_2 \) is odd; otherwise \( \delta = 0 \).

Now one can ask what these results tell us about single averages over \( f \) as in (1.1). The above theorems clearly give upper bounds on averages for fixed \( \chi \) which are exact, though certainly suboptimal. More interesting are exact formulas and lower bounds. First we state a stable average value formula generalizing Theorem 1.1(ii).
Corollary 1.5 Suppose $N_2 = 1$. Fix a character $\chi$ of $\Cl(K)$. Then, in the stable range $D_B > |D_K|$ with $\gcd(D_B, D_K) = 1$, we have
\[
\frac{C(K; N)}{4\pi^2} \sum_{f \in \mathcal{F}_0(N; \chi)} \Lambda_M(f, \chi) \frac{N_f L_{\text{fin}}(\frac{1}{2}, f, \chi)}{N} \frac{1}{(f, f)} = h_K \left( u_K \prod_{p|N_1} \left( 1 - \frac{1}{p^2} \right) - \delta \frac{12h_K}{N} \prod_{p|N_1} \frac{1}{1 - p^{-1}} \prod_{p|\mathcal{M}} \frac{1}{1 + p^{-1}} \right),
\]
where $\delta = 1$ if both $N_1$ is squarefree and $\chi = 1$, and $\delta = 0$ otherwise.

The novel aspects of this formula are that $N$ is not required to be squarefree, $N$ may be divisible by an even number of primes, and $K$ may be split at primes dividing $N$. The restriction to $N_2 = 1$ is not due to a real obstruction in the method, but simply to make the second term on the right (which comes from Eisenstein series) easy to describe.

Our final results are effective lower bounds for average $L$-values $L_{\text{fin}}(\frac{1}{2}, f, 1_K)$, where $f$ runs over level $N$ newforms in $\mathcal{F}(N; 1_K)$. For simplicity, we just treat 2 cases:

1. $N = p^r N_0$, where $N_0$ is a squarefree product of an even number of primes, $r$ is odd or 2, and $K$ is inert or ramified at each prime dividing $N$ (necessarily ramified at $p$ if $r = 2$, and inert at $p$ if $r > 2$; here $M = 1$)

2. $N = N_1 p$ is a squarefree product of an even number of primes, and $K$ is inert or ramified at each prime dividing $N_1$ and $K$ is split at $p$ (here $M = p$)

In each of these situations, the lower bound implies a generalization of Corollary 1.2: when $p$ or $p^2$ is larger than an explicit multiple of $h_K$, there exists a newform $f \in S_2^{\text{new}}(N)$ such that $L(\frac{1}{2}, f)L(\frac{1}{2}, f \otimes \eta_K) \neq 0$. See Sect. 5.7 for precise statements.

As a check on our formulas, we note that our single stable average formulas match with those in [7, 19] when $N = N_1$ is squarefree and $K$ is inert at each $p|N$. We also present a few examples in Sect. 6 that provide a numerical check on our formulas in other situations.

1.3 Methods

The proof of the exact double average formula has two main steps. First, one computes averages of squares of periods for trivial weight quaternionic modular (i.e., automorphic) forms associated to an order $O$ in a definite quaternion algebra $B$. These quaternionic modular forms are $\mathbb{C}$-valued functions on the finite set $\Cl(O)$ which have a Hecke action, and we can think of a basis of quaternionic Hecke eigenforms as being analogous to the complex irreducible characters of a finite group $G$. The analogue of column orthogonality of characters of $G$ (or a generalization when one weights by Hecke eigenvalues) leads easily to an exact double average formula for periods. This much is carried out for general orders $O$ in definite quaternion algebras over arbitrary totally real fields in Sect. 4.

Second, one relates periods of quaternionic modular forms to central $L$-values with the relative trace formula. For this, one first needs to precisely relate the quaternionic modular forms for $O$ to classical modular forms, which is a refinement of the Jacquet–Langlands correspondence. We carried this out for special orders $O$ of level $N = N_1 N_2 M$ in [16], under the first 3 defining conditions (i)–(iii) for $(N_1, N_2, M)$ to be a nice level type. Here $M$ need not be squarefree. The condition that $N_2$ must be the square of a squarefree number arises due to complications in the precise description of the space of quaternionic oldforms when higher even powers of primes sharply divide $N_2$. 
To finish the second step, we need to precisely relate the quaternionic periods to \( L \)-values of modular forms. This is essentially carried out in [17], but the technical complication is that in general one also needs to relate periods of quaternionic oldforms to \( L \)-values. This requires an understanding of the local quaternionic oldforms at primes \( p | N_1 N_2 \) from [16] and a calculation of local spectral distributions for oldforms when \( p | M \). The latter is possibly quite complicated in general, and we only carry this out when \( M \) is squarefree and \( K \) is split at each \( p | M \). However, in principle, one could extend this to allow \( M \) to be non-squarefree and \( K \) to be split or ramified at \( p | M \).

The remaining conditions on the admissibility of \( K \) and the restriction of \( f \in \mathcal{F}(N; \chi) \) are what we need to see that the relevant periods appear when working with special orders of level \( N \) in \( B \). In general, provided that an \( L \)-function \( L(s, f, \chi) \) has root number \(+1\), the central value corresponds to a period on a unique quaternion algebra, which may or may not be \( B \).

This leads to Theorem 1.3. Then one derives Theorem 1.4 by isolating the \( N_1 N_2 \)-new contribution using the inclusion-exclusion principle.

The restriction to base fields of class number 1 in Theorem 5.5 for Hilbert modular forms is simply due to the fact that the relative trace formula of Jacquet used in the \( L \)-value formula of [17] was only established in for representations whose base change to \( K \) has trivial central character.

**Remark 1.6** Our calculation of local spectral distributions for \( p | M \) is based off of a similar calculation from [8] at a prime \( p \) sharply dividing \( M \) such that \( K \) is inert at \( p \) and \( \chi_p \) is ramified. If \( K \) is inert at \( p | M \) and \( \chi_p \) is unramified (as in the present paper), one does not see the relevant \( L \)-values with periods on \( B \), but on a quaternion algebra which is ramified at \( p \).

It should be possible to use our calculations in Sect. 5.3 to extend the average value formulas in [7, 8] to levels which are products of even numbers of primes, and allow for \( K \) to be split at primes dividing the level, with the caveat that now one needs to weight the \( L \)-values of oldforms by the factors \( \Lambda_N(f, \chi) \).

Now we briefly explain how to derive the lower bounds and stable formulas for single averages. Fixing \( \chi \), the average of \( L \)-values \( L(\frac{1}{2}, f, \chi) \) for \( f \in \mathcal{F}(N; \chi) \) plus an Eisenstein contribution is essentially the height \( \langle c_{K,\chi}, c_{K,\chi} \rangle \) of a special divisor \( c_{K,\chi} \). These heights are maximized when \( \chi = 1_K \) is the trivial character, hence the single average of \( L(\frac{1}{2}, f, 1_K) \) must be at least \( \frac{1}{\eta_K} \) times the double average over \( f \) and \( \chi \), minus the Eisenstein contribution. This leads to the lower bounds in Sect. 5.7, and then Corollary 1.2.

We now define \((N, K)\) to be in stable range if \( \langle c_{K,\chi}, c_{K,\chi} \rangle \) is dependent only on \( K \) (and our triple \((N_1, N_2, M)\)) but not \( \chi \). Then, in the stable range, the average over \( f \in \mathcal{F}(N; \chi) \) for any fixed \( \chi \) is given by \( \frac{1}{\eta_K} \) times the double average over \( f \) and \( \chi \), minus the Eisenstein contribution. The converse is also true, and so by comparing our double average formulas with the stable averages in [7, 19], we can conclude that their stable range is contained in our definition of stable range. In other words, we use [7, 19] to verify that \( D_B > |D_K| \) implies \((D_B, K)\) is in the stable range by our definition. Then we use this to deduce \((N, K)\) is also in the stable range. This leads to Theorem 1.1(ii) and Corollary 1.5.

**Remark 1.7** Since we make a variety of different hypotheses on our level types \((N_1, N_2, M)\) and the quadratic extension \( K \) for different results, and we also impose running assumptions at certain points, in Sect. 1 we present a table summarizing which hypotheses are used in which results for the convenience of the reader.
1.4 Related work

Here we briefly discuss some additional related work.

First, we know of two works on stable averages which allow non-squarefree level. In [21], Nelson established a quite general formula for Rankin–Selberg averages $L(\frac{1}{2}, f \times g)$ over $\mathbb{Q}$. Here $g$ is a fixed form and one averages over an orthogonal basis of cusp forms $f$ of some weight $k \geq 4$. The full formula is quite complicated, but it stabilizes in various situations, including large prime-power levels. However, Nelson does not get averages over newforms (or in weight 2).

Recently, Pi [23] proved a certain stable average formula similar to that of [7], which allows for squares and cubes dividing the level. More precisely, Pi averages over representations which are fixed depth 0 or simple supercuspidals $\pi_v$ at a given set of places. This restriction on the type of local representations forces the bound for the stable range to be quite large. Moreover, the character $\chi$ is prescribed according to the local representation type of $\pi_v$, so one cannot always take $\chi = 1_K$. Consequently, even for level $p^3$ it does not give a stable average result as in Theorem 1.1(ii).

We remark that there are also a number of asymptotic results on averages for a fixed $\chi$. See for instance [26] for weights $\geq 6$ when $\chi = 1_K$.

The main consequences of our double average value formula that we considered here are about single averages over $f$ for fixed $K$ and $\chi$. In particular, we thought of $K$ as being fixed and $N$ varying. However one nice feature of our double average formula, especially the stable double average in Theorem 1.4, is that we may simultaneously vary $N$ and $K$.

For a fixed form $f$ with $|D_K| \to \infty$, an asymptotic for averages over $\chi \in \hat{\text{Cl}}(K)$ was established in [20], which led to a quantitative lower bound on the number of nonvanishing twists $L(\frac{1}{2}, f, \chi)$. The authors assumed $f$ is weight 2 of prime level over $\mathbb{Q}$ for simplicity, though their method (using equidistribution of special cycles) clearly generalizes. This asymptotic average was extended in [13] where now one can vary both $f$ and $K$. Our double average value formula at least provides exact upper bounds on such averages in more general situations, though it is too crude to obtain averages for a fixed $f$ (unless $\#\mathcal{F}_0(N) = 1$).

Finally, we remark that many earlier average value results have yielded subconvexity results. One might wonder if it is possible to obtain some form of subconvexity from our double average value formula. The problem is that our double average involves too many $L$-values. Indeed, the upper bound one naively from gets Theorem 1.3 is that $L(\frac{1}{2}, f, \chi) \ll (f, f)|D_K|^{\frac{1}{2}+\varepsilon}$, which is no better than convexity.

2 Notation

Let $F$ be a number field with integer ring $\mathfrak{o}_F$ and discriminant $D_F$. We denote a place of $F$ by $v$. If $A$ is an $F$- or $\mathfrak{o}_F$-algebra, denote by $A_v$ the localization at $v$, and by $\hat{A} = \prod_{v<\infty} A_v$ the restricted direct product with respect to $\{R_v\}_{v<\infty}$ where $R \subset A$ is a maximal $\mathfrak{o}_F$-order. In particular, $\delta_F = \prod_{v<\infty} \mathfrak{o}_{F,v}$ and the adele ring is $\mathbb{A}_F = \prod_v A_v = \hat{F} \times F_\infty$. The ideal class group of $F$ is denoted by $\text{Cl}(F)$, which we identify with both $F^\times \backslash \hat{F}^\times / \hat{\mathbb{Q}}$ and $F^\times \backslash F_\infty^\times / \mathbb{Q}_v^\times$. Put $h_F = \#\text{Cl}(F)$ and let $\hat{\text{Cl}}(F)$ denote the character group of $\text{Cl}(F)$.

For a finite place $v$ of $F$, we use $p_v$ to denote both the associated prime ideal of $\mathfrak{o}_F$ and the maximal ideal of $\mathfrak{o}_{F,v}$—when the distinction matters, the context will make this notation clear. Throughout $\mathfrak{N}$ will be a nonzero ideal in $\mathfrak{o}_F$, and $\text{ord}_v(\mathfrak{N})$ denotes the $p_v$-adic valuation of $\mathfrak{N}$. Let $\mathfrak{m}_{F,v}$ denote a uniformizer in $\mathfrak{o}_{F,v}$ and $q_v$ be the size of the residue field of $\mathfrak{o}_{F,v}$. 

\(\text{Springer}\)
When we specialize to $F = \mathbb{Q}$, we typically use the roman form of the corresponding fraktur letters to denote the positive generator of an ideal, e.g., $N$ is the positive generator of a non-zero ideal $\mathfrak{m} \subset \mathbb{Z}$.

In what follows, $F$ is a totally real number field, and we often write $\mathcal{O} = \mathcal{O}_F$, $\mathcal{O}_v = \mathcal{O}_{F,v}$, etc. In Sect. 5, we further assume $h_F = 1$. Denote by $B$ a definite quaternion algebra over $F$, and $\mathcal{O}$ an $\mathfrak{o}$-order in $B$. Let $\text{Cl}(\mathcal{O})$ be the set of (invertible) right $\mathcal{O}$-ideal classes in $B$. As with number fields, we identify $\text{Cl}(\mathcal{O}) = B^\times \backslash B^\times (\mathbb{A}_F) / \hat{\mathcal{O}}^\times B^\times (F_\infty) = B^\times \backslash \hat{B}^\times / \hat{\mathcal{O}}^\times$. Write $n = h(\mathcal{O}) = \#\text{Cl}(\mathcal{O})$ for the class number of $\mathcal{O}$.

For a simple algebra $A/F$ (we have in mind a quaternion algebra $B/F$ or a quadratic field extension $K/F$), let $N_{A/F}$ denote the reduced norm map from $A$ to $F$. When understood or unimportant, we often omit the subscript, and simply write $N$ for the norm map. Further, denote by $\text{Ram}(A)$ the set of finite primes $v$ of $F$ at which $A$ ramifies, and $\mathcal{O}_A = \prod_{v \in \text{Ram}(A)} \mathfrak{p}_v$ the (reduced) relative discriminant of $A$ (over $F$).

For a quadratic extension $K/F$ of number fields, we denote by $\eta_K = \eta_{K/F}$ the associated quadratic idele class character of $F$, and analogously for quadratic extensions of local fields. Excluding the right regular representation, all representations are assumed to be irreducible. We often use 1 to denote a trivial character. However, in the case of a trivial local or global Hecke character over a field $k$, we sometimes write $1_k$ for clarity.

### 3 Quaternionic modular forms

In this section, we discuss quaternionic modular forms for arbitrary orders $\mathcal{O} \subset B$, but for some results we need to restrict to special orders.

#### 3.1 Special orders

Fix a definite quaternion algebra $B/F$. A class of quaternionic orders $\mathcal{O} \subset B$ generalizing Eichler orders was introduced was introduced in [12], which the authors termed special. First we recall the local notion of special orders.

Let $v$ be a finite place of $F$. If $B_v$ is split, a local special order $\mathcal{O}_v \subset B_v \simeq M_2(F_v)$ of level $r$ (or level $\mathfrak{p}_v^r$) is a local Eichler order of level $r$, i.e., an order conjugate to $\begin{pmatrix} \mathcal{O}_v & \mathcal{O}_v \\ \mathfrak{p}_v^r & \mathcal{O}_v \end{pmatrix}$.

Now suppose $B_v$ is division and $E_v/F_v$ is a quadratic extension of local fields. Let $\mathcal{O}_{B,v}$ denote the unique maximal order of $B_v$ and $\mathfrak{p}_v$ its unique maximal (2-sided) ideal. Set

$$\mathcal{O}_r(E_v) = \mathcal{O}_{E,v} + \mathfrak{p}_v^{r-1}, \quad r \geq 1.$$  

A local order $\mathcal{O}_v \subset B_v$ is special if $\mathcal{O}_v \simeq \mathcal{O}_r(E_v)$ for some $r$, $E_v/F_v$, in which case we say the level of $\mathcal{O}_v$ is $r$ (or $\mathfrak{p}_v^r$) if $r$ is minimal such that $\mathcal{O}_v \simeq \mathcal{O}_r(E_v)$. In particular, $\mathcal{O}_v$ is special if and only if it contains the integer ring of some quadratic subfield, and thus when $B_v$ is division special orders are the same as what are called basic orders in [29], where it is also shown that these are the same as Bass orders.

We recall some facts about local special orders.

**Lemma 3.1** [12] Suppose $B_v$ is division. Then:

1. Every special order in $B_v$ is of the form $\mathcal{O}_r(E_v)$ where either $r \geq 1$ is odd and $E_v/F_v$ is unramified, or $r \geq 1$ is arbitrary and $E_v/F_v$ is ramified.
2. $\mathcal{O}_1(E_v) = \mathcal{O}_{B,v}$ is the maximal order for every $E_v/F_v$.
(iii) Suppose $E_v/F_v$ and $E'_{v}/F_v$ are quadratic extensions and $r, r' \geq 1$. If $r \neq r'$, then $\mathcal{O}_v(E_v) \simeq \mathcal{O}_{r}(E'_{v})$ implies $E_v \simeq E'_{v}$ is unramified and $|r - r'| = 1$ with $\min\{r, r'\}$ odd.

(iv) If $E_v/F_v$ is unramified and $E'_{v}/F_v$ is ramified, then $\mathcal{O}_v(E_v) = \mathcal{O}_{r}(E'_{v})$ if and only if $r = r' = 1$.

(v) Suppose $E_v, E'_{v}$ are two non-isomorphic ramified quadratic extensions of $F_v$. If $r \leq 2$, then $\mathcal{O}_v(E_v) \simeq \mathcal{O}_{r}(E'_{v})$; the converse also holds if $v$ is odd.

See [12, Theorem 3.10] for the converse of (v) when $v$ is dyadic.

Following [12], we say a global order $\mathcal{O} \subset B$ is special of level $\mathfrak{N}$ if $\mathcal{O}_v$ is special of level $\text{ord}_v(\mathfrak{N})$ for each $v < \infty$. However, $\mathfrak{N}$ alone does not determine the local isomorphism type of $\mathcal{O}$. We will always place the following additional assumption on a global special order $\mathcal{O}$ of level $\mathfrak{N}$: for $v < \infty$ such that $B_v$ is division and $r_v := \text{ord}_v(\mathfrak{N})$ is odd, $\mathcal{O}_v \simeq \mathcal{O}_{r_v}(E_v)$ where $E_v/F_v$ is the unramified quadratic extension. Further, when we write $\mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{M}$, it will always mean that $\mathfrak{N}_1, \mathfrak{N}_2$ and $\mathfrak{M}$ are pairwise coprime ideals such that (i) for finite $v|\mathfrak{N}_1 \mathfrak{N}_2$ if and only if $B_v$ is division, and (ii) for $v|\mathfrak{N}_1 \mathfrak{N}_2$, $v|\mathfrak{N}_1$ if and only if $\text{ord}_v(\mathfrak{N}_1)$ is odd. Thus $\mathfrak{N}_1$ (resp. $\mathfrak{N}_2$) is divisible exactly by the finite primes $v$ ramifying in $B$ such that $\mathcal{O}_v \simeq \mathcal{O}_{r_v}(E_v)$ for some $r_v$ where $E_v$ is an unramified (resp. ramified) quadratic extension of $F_v$, and $\mathfrak{M}$ is divisible exactly by the finite primes $v$ splitting $B$ where $\mathcal{O}_v$ is a non-maximal local Eichler order.

Sometimes to emphasize the above conventions, we will say $\mathcal{O}$ is a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. Note that by Lemma 3.1, if $\text{ord}_v(\mathfrak{N}_2) = 2$ for all $v|\mathfrak{N}_2$, then any two special orders of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ are locally isomorphic, i.e., in the same genus.

If $\mathcal{O}$ is special of level type $(\mathfrak{N}_1, \mathfrak{o}, \mathfrak{M})$, we say $\mathcal{O}$ is of unramified quadratic type. This means that $\mathcal{O}_v$ contains the maximal order of an unramified or split quadratic extension of $F_v$ for each $v < \infty$. Such orders are nice to work with because then one has that $N(\hat{\mathcal{O}}^\times) = \hat{o}^\times$.

Note that an order $\mathcal{O}$ is Eichler if and only if $\mathcal{O}$ is of unramified quadratic type with $\mathfrak{N}_1$ squarefree, i.e., $\mathfrak{N}_1 = \mathfrak{D}_B$ and $\mathfrak{N}_2 = \mathfrak{o}$.

### 3.2 Automorphic forms

Fix an arbitrary $v_B$-order $\mathcal{O} \subset B$. We define the space of quaternionic modular forms (or automorphic forms) of level $\mathcal{O}$ on $B$ with trivial weight to be $M(\mathcal{O}) = \{\varphi : \text{Cl}(\mathcal{O}) \to \mathbb{C}\}$. Set $n = h(\mathcal{O}) = \#\text{Cl}(\mathcal{O})$, and write $\text{Cl}(\mathcal{O}) = \langle x_1, \ldots, x_n \rangle$. By abuse of notation, we use also use $x_i$ to denote an element of $\hat{B}^\times$ representing the corresponding element of $\text{Cl}(\mathcal{O}) = B^\times \setminus \hat{B}^\times / \hat{\mathcal{O}}^\times$. Let $\mathcal{I}_i$ denote a classical right $\mathcal{O}$-ideal representing $x_i$. Let $\mathcal{O}_\ell(\mathcal{I}_i) = \{\alpha \in B : \alpha \mathcal{I}_i \subset \mathcal{I}_i\} = x_i \mathcal{O}_\ell x_i^{-1} \cap B$ be the left order of $\mathcal{I}_i$. Put $w_i = [\mathcal{O}_\ell(\mathcal{I}_i)^\times : \mathfrak{o}^\times]$, which is always finite. We define an inner product on $M(\mathcal{O})$ by

$$
(\varphi, \varphi') = \sum_{i=1}^{n} \frac{1}{w_i} \varphi(x_i) \varphi'(x_i). \quad (3.1)
$$

This inner product can also be defined by integrating $\varphi$ against $\varphi'$ over $\hat{F}^\times B^\times \setminus \hat{B}^\times$ with an appropriate normalization of measure (e.g., see [14]).

Let $\omega : F^\times \setminus H_F^\times \to \mathbb{C}^\times$ be an idele class character of $F$ which is trivial at each infinite place. Let $L^2(\hat{B}^\times \setminus \hat{B}^\times, \omega)$ be the space of $L^2$-functions $\varphi : \hat{B}^\times \setminus \hat{B}^\times \to \mathbb{C}$ such that $\varphi(\bar{z}x) = \omega(z)\varphi(x)$ for all $z \in \hat{F}^\times, x \in \hat{B}^\times$. Setting $M(\mathcal{O}, \omega) = M(\mathcal{O}) \cap L^2(\hat{B}^\times \setminus \hat{B}^\times, \omega)$, we see that

$$
M(\mathcal{O}) = \bigoplus M(\mathcal{O}, \omega), \quad (3.2)
$$

\(\bigcirc\) Springer
where $\omega$ ranges over idele class characters which are trivial on $\sigma_F^\times \times F_F^\times = \mathbb{A}_F^\times \cap (\hat{\mathcal{O}}^\times \times B_\infty^\times)$, i.e., $\omega$ runs over the group $\hat{\text{Cl}}(F)$ of ideal class characters of $F$.

By an automorphic representation $\pi$ of $B^\times$ with trivial weight and (central) character $\omega$, we mean an irreducible unitary subrepresentation of the right regular representation of $\hat{B}_\infty^\times$ on $L^2(B^\times \backslash \hat{B}_\infty^\times, \omega)$. Call $\pi$ cuspidal if $\pi$ is not 1-dimensional. We say that $\pi$ occurs in $M(O)$ if $\pi \cap M(O) \neq 0$, which is equivalent to $\pi_{\hat{\mathcal{O}}^\times} \neq 0$, and speak similarly for $M(O, \omega)$. Accordingly we get decompositions
\[
M(O) = \bigoplus \pi \hat{\mathcal{O}}^\times, \quad M(O, \omega) = \bigoplus \pi \hat{\mathcal{O}}^\times, \tag{3.3}
\]
where $\pi$ respectively runs over automorphic representations occurring in $M(O)$ and $M(O, \omega)$.

We define the Eisenstein subspace of $M(O)$ to be $\text{Eis}(O) = \bigoplus \pi \hat{\mathcal{O}}^\times$, where $\pi$ runs over 1-dimensional automorphic representations occurring in $M(O)$. All such $\pi$ must be of the form $\mu \circ N_{B/F}$ for some character $\mu : F^\times \hat{\mathcal{A}}_F^\times \to \mathbb{C}^\times$. We may view $N = N_{B/F}$ as a map from $B^\times \backslash B^\times (\hat{\mathcal{A}}_F) / \hat{\mathcal{O}}^\times B_\infty^\times$ to
\[
\text{Cl}^+(N(\hat{\mathcal{O}})) := F^\times \hat{\mathcal{A}}_F^\times / N(\hat{\mathcal{O}}^\times) F_\infty^+, \]
where $F_\infty^+$ denotes the totally positive elements of $F_\infty$. Thus we may take $\mu$ to be a character of $\text{Cl}^+(N(\hat{\mathcal{O}}))$. If $O$ is special of unramified quadratic type, then $\text{Cl}^+(N(\hat{\mathcal{O}}))$ is just the narrow class group $\text{Cl}^+(F)$.

Put $\text{Eis}(O, \omega) = \text{Eis}(O) \cap M(O, \omega)$. Note $\text{Eis}(O, 1)$ always contains the constant function $1$. In general, a basis of $\text{Eis}(O, \omega)$ is given by the characters $\mu \circ N$, where $\mu$ runs over the characters of $\text{Cl}^+(N(\hat{\mathcal{O}}))$ such that $\mu^2 = \omega$. Note that for any character $\mu \circ N \in \text{Eis}(O)$, its norm $(\mu \circ N, \mu \circ N)$ equals the mass of $O$,
\[
m(O) := \sum_{i=1}^n \frac{1}{w_i}.
\]

**Lemma 3.2** Let $O$ be a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. Then
\[
m(O) = m(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) := 2^{-\frac{1}{2}}|F:Q| h_F |\xi_F(-1)| N(\mathfrak{M}) \prod_{v | \mathfrak{N}_1 \mathfrak{N}_2} (1 - q_v^{-1}) \prod_{v | \mathfrak{N}_2 \mathfrak{M}} (1 + q_v^{-1}).
\]

**Proof** Let $O'$ be a special order of level type $(\mathfrak{N}_1', \mathfrak{M})$ containing $O$ where $\mathfrak{N}_1' = \mathfrak{N}_1 \prod_{v | \mathfrak{N}_2} p_v$. A formula for $m(O')$ is given in [14, (1.6)]. Since one can interpret masses as orders of volumes in $\hat{B}_\infty^\times$ with respect to suitable Haar measures, one has that
\[
m(O) = m(O') \prod_{v | \mathfrak{N}_2 \mathfrak{M}} [(O')^\times : O_v^\times].
\]
From [12, Proposition 2.6], one knows $[(O')^\times : O_v^\times] = (q_v + 1)q_v^{e_v - 2}$ where $e_v = \text{ord}_v(\mathfrak{M}_2)$. These two calculations combine to give the lemma. (The $F = \mathbb{Q}$ case is already stated in [12, Theorem 6.8].) \hfill $\square$

**Lemma 3.3** Let $O$ be a special order of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ such that $\text{ord}_v(\mathfrak{M}_2) = 2$ for each $v | \mathfrak{N}_2$. Then there is a basis of $\text{Eis}(O, 1)$ consisting of $1$ and the characters $\eta_{E/F} \circ N$, where $E/F$ runs over quadratic extensions such that each finite $v$ ramified in $E/F$ is odd and divides $\mathfrak{N}_2$.

**Proof** Consider a character $\mu \circ N$ occurring in $M(O, 1)$ and let $F_v^{(2)}$ denote the subgroup of squares of $F_v^\times$. Then $\mu^2 = 1$ and $\mu_v$ factors through $F_v^\times / N(O_v^\times) F_v^{(2)}$ for $v < \infty$. For $v | \infty$, the only requirement on $\mu_v$ is that it factors through $\mathbb{R}^\times / \mathbb{R}_{>0} \simeq \{ \pm 1 \}$. For a finite $v \mid \mathfrak{N}_2$, \hfill $\blacksquare$ Springer
we have \( N(O_v^\times) = \sigma_v^\times \), so \( F_v^\times / N(O_v^\times)F_v^{(2)} \simeq (\sigma_v) / (\sigma_v^2) \) and we see \( \mu_v \) can be trivial or the unramified quadratic character of \( F_v^\times \).

Now suppose \( v | \mathfrak{M}_2 \). To elucidate the assumption on \( \mathfrak{M}_2 \), first just suppose \( \text{ord}_v(\mathfrak{M}_2) = 2r \) with \( r \geq 1 \), and write \( O_v = O_{2r}(L_v) \) where \( L_v / F_v \) is a ramified quadratic extension. Then \( O_v^\times = \sigma_v^\times (1 + \mathfrak{M}_v^{2r-1}) \). One has \( N(1 + \mathfrak{M}_v^{2r-1}) = 1 + p_v^r \), and thus \( F_v^\times / N(O_v^\times)F_v^{(2)} \simeq (\sigma_v^\times / N(\sigma_v^\times)) (1 + p_v^r) \times (\sigma_v) / (\sigma_v^2) \). Thus \( \mu_v \) can be unramified or any ramified quadratic character \( \eta_{E_v/F_v} \) such that \( N(\sigma_v^\times) \subset N(\sigma_v^\times)(1 + p_v^r) \). Now assume \( r = 1 \). It is well known that \( N(\sigma_v^\times) \subset 1 + p_v \) if and only if \( v \) is odd, and thus if \( v \) is even \( \mu_v \) must be unramified. If \( v \) is odd, then \( O_2(L_v) \simeq O_2(E_v) \), so we may take \( L_v \simeq E_v \) to see that \( \mu_v \) can be any quadratic character.

\[ \square \]

Remark 3.4 Let \( O \) be as in Lemma 3.3. Since the 1-dimensional automorphic representations occurring in \( M(O) \) form a group, the above lemma yields a complete description of \( \text{Eis}(O) \). Namely, if \( \omega = \mu^2 \) for some \( \mu \in \hat{\mathbb{C}}^+ \) \( (N(\hat{\mathbb{O}})) \), then \( \text{Eis}(O, \omega) = \{(\mu \circ N)\varphi \mid \varphi \in \text{Eis}(O, 1)\} \). Otherwise \( \text{Eis}(O, \omega) = 0 \).

We define the cuspidal subspaces \( S(O) \) and \( S(O, \omega) \) of \( M(O) \) and \( M(O, \omega) \) to be the orthogonal complements of the Eisenstein subspaces. Hence we have decompositions

\[
S(O) = \bigoplus \pi \hat{\mathbb{O}}^\times, \quad S(O, \omega) = \bigoplus \pi \hat{\mathbb{O}}^\times,
\]

where \( \pi \) respectively runs over cuspidal representations occurring in \( M(O) \) and \( M(O, \omega) \).

3.3 Brandt matrices and representation numbers

We may realize \( M(O) \) as \( \mathbb{C}^n \) via \( \varphi \mapsto (\varphi(x_1), \ldots, \varphi(x_n)) \). Let \([\varphi] \) denote the column vector \( ^t(\varphi(x_1) \ldots \varphi(x_n)) \). Then we can define a Hecke action in terms of Brandt matrices as in [5, 16].

For a nonzero integral ideal \( m \), we define the \( n \times n \) Brandt matrix \( A_m = (a_{ij}(m)) \) via

\[
a_{ij}(m) = \# \left\{ \gamma \in I_iI_j^{-1} \mid N(\gamma)\alpha = mN(I_iI_j^{-1}) \right\} / (O_l(I_j)^\times) \tag{3.4}
\]

Define the Hecke operator \( T_m : M(O) \rightarrow M(O) \) via matrix multiplication: \([T_m \varphi] = A_m[\varphi] \).

We can also express

\[
(T_m \varphi)(x) = \sum \varphi(x\beta),
\]

where \( \beta \) runs over the integral right \( O \)-ideals of norm \( m \). Note \( T_\emptyset \) is the identity operator. The collection of \( T_m \)’s is a commuting family of real self-adjoint operators on \( M(O) \).

Under a class number 1 assumption, the Brandt matrix entries can be expressed as classical representation numbers of quadratic forms as follows. For a (full) \( \sigma_F \)-lattice \( \Lambda \subset B \), let \( O_F(\Lambda) = \{ \alpha \in B : \Lambda \alpha \subset \Lambda \} \) and \( O_F(\Lambda)^1 \) be the subset of norm 1 elements in \( O_F(\Lambda) \). Let \( \sigma_+ \) be the subset of totally positive elements of \( \sigma \), and \( \sigma_+^\times \) be the totally positive units. For \( y \in \sigma_+ \), define the representation number

\[
r_\Lambda(y) = \# \{ \lambda \in \Lambda : N(\lambda) = y \}. \tag{3.5}
\]

Now suppose \( h_F^+ = 1 \). Since \( h_F = h_F^+ \), every element of \( \sigma_+^\times \) is a square, and thus \( N_B/F(\sigma_+^\times) = \sigma_+^\times \). Fix \( y \in \sigma_+ \). Then \( N(\lambda)\sigma = y\sigma \) is equivalent to \( N(\lambda) \in y\sigma_+^\times \). Since \( O_F(\Lambda)^\times = \sigma^\times O_F(\Lambda)^1 \), we see there is a bijection of sets,

\[
\{ \lambda \in \Lambda : N(\lambda) \in y\sigma_+^\times \}/O_F(\Lambda)^\times \simeq \{ \lambda \in \Lambda : N(\lambda) = y \}/O_F(\Lambda)^1.
\]
Put \( \Lambda_{ij} = I_i I_j^{-1} \) and let \( \alpha_{ij} \) be a totally positive generator of \( N(I_i I_j^{-1}) \). Note \( O_\ell(\Lambda_{ij}) = O_\ell(I_j) \). Also \( o^\times \cap O_\ell(\Lambda_{ij}) = \{ \pm 1 \} \), which implies \( \# O_\ell(\Lambda_{ij}) = 2w_j \). Hence, assuming \( h_\ell^+ = 1 \) and \( y \in o_+ \), we can rewrite our Brandt matrix entries as

\[
a_{ij}(y\alpha) = \frac{1}{2w_j} r_{\Lambda_{ij}}(y\alpha_{ij}). \tag{3.6}
\]

Note that when \( i = j \), we get \( \Lambda_{ii} = O_\ell(I_i) \), so the diagonal Brandt matrix entries are simply

\[
a_{ii}(y\alpha) = \frac{1}{2w_i} r_{O_\ell(I_i)}(y). \tag{3.7}
\]

### 3.4 Jacquet–Langlands correspondence

The Jacquet–Langlands correspondence is a dictionary between automorphic representations of \( B^\times \) and \( GL(2) \). Here we present a refinement of it at the level of modular forms from [16].

Let \( O \subset B \) be a special order of level type \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}) \) and write \( \mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{M} \). Let \( S_2(\mathfrak{N}, \omega) \) denote the space of adelic holomorphic Hilbert cusp forms of level \( \mathfrak{N} \), parallel weight 2, and central character \( \omega \). (By level \( \mathfrak{N} \), we mean “level \( W(\mathfrak{N}) \),” which is Shimura’s analogue of \( \Gamma_1(N) \)—see, e.g., [16, Section 5.1].) For relatively prime integral ideals \( a, b, c \) of \( o \) such that \( a|\mathfrak{N}_1 \) and \( bc|\mathfrak{N}_2 \), let \( S_2^{[a,b,c]}(abc\mathfrak{M}, \omega) \) be the subspace of \( S_2(\mathfrak{N}, \omega) \) generated by eigenforms which are \( p \)-new for each \( p|abc \) and whose corresponding local representations of \( GL_2(F_p) \) are: (i) local discrete series for \( \nu|abc \), (ii) supercuspidal for \( \nu|b \), and (iii) special (i.e., twisted Steinberg) for \( \nu|c \).

For an integral ideal \( \mathfrak{A} \subset o \), let \( \mathcal{H}^{\mathfrak{A}} \) be the Hecke algebra generated by the \( T_p \) for \( p \nmid \mathfrak{A} \). (For our purposes, it does not matter if we work with Hecke algebras over \( \mathbb{Z} \) or \( \mathbb{Q} \) or \( \mathbb{C} \).)

**Theorem 3.5** [16] Assume \( \ord_p(\mathfrak{N}_1) = 2 \) for each \( p|\mathfrak{N}_2 \). Let \( \mathfrak{N}' = \mathfrak{N}^2 \mathfrak{O}_B^{-1} \). We have the following isomorphism of \( \mathcal{H}^{\mathfrak{N}'} \)-modules:

\[
S(O, \omega) \simeq \bigoplus 2^{\#[p|b]} S_2^{[a,b,c]}(abc\mathfrak{M}, \omega),
\]

where (i) \( a \) runs over divisors of \( \mathfrak{N}_1 \) such that \( v_p(a) \) is odd for all \( p|\mathfrak{N}_1 \), and (ii) \( b, c \) run over relatively prime divisors of \( \mathfrak{N}_2 \) such that \( p|bc \) for each \( p|\mathfrak{N}_2 \).

**Proof** The result [16, Corollary 5.5] shows the stated isomorphism holds for \( \mathcal{H}^{\mathfrak{N}'} \)-modules. Note \( \mathcal{H}^{\mathfrak{N}_1} \) is the algebra generated by \( \mathcal{H}^{\mathfrak{N}_1} \) and the \( T_p \) for \( p|\mathfrak{N}_1 \) such that \( \ord_p(\mathfrak{N}_1) = 1 \). Since any eigenform occurring in either side of the isomorphism is \( p \)-new for \( p \) sharply dividing \( \mathfrak{N}_1 \), the above isomorphism also preserves the action of each such \( T_p \) (e.g., see [15] or [16, Remark 5.2]).

We remark that [16] also establishes an analogous Hecke isomorphism for **newspaces** without assuming \( \ord_p(\mathfrak{N}_2) = 2 \) when \( p|\mathfrak{N}_2 \). The difficulty in removing this assumption in the above theorem boils down to an issue in the local determination of oldspaces.

### 3.5 Average values of quaternionic modular forms

The following simple linear-algebraic relation forms the basis of our calculation of period averages in the next section.
Proposition 3.6 Let $O$ be an arbitrary order in $B$, and $T$ a real self-adjoint operator on $M(O)$. Let $(a_{ij})$ be the matrix of $T$ with respect to the basis $\{e_i\}$, where $e_i \in M(O)$ is the indicator function of $x_i \in \text{Cl}(O)$. Let $\Phi$ be an orthogonal basis for $M(O)$. Then, for $1 \leq i, j \leq n$, we have

$$\sum_{\varphi \in \Phi} \frac{(T \varphi)(x_i)\overline{\varphi(x_j)}}{\langle \varphi, \varphi \rangle} = w_j a_{ij}.$$

Proof The left hand side of the equation above equals

$$\sum_{\varphi \in \Phi} \frac{(T \varphi, w_i e_i)(\varphi, w_j e_j)}{\langle \varphi, \varphi \rangle} = \sum_{\varphi \in \Phi} \frac{(\varphi, T w_i e_i)(\varphi, w_j e_j)}{\langle \varphi, \varphi \rangle} = (T w_i e_i, w_j e_j) = w_i a_{ji} = w_j a_{ij}.$$  

Remark 3.7 In the special case $T = T_\sigma$ is the identity, we get

$$\sum_{\varphi \in \Phi} \frac{\varphi(x_i)\overline{\varphi(x_j)}}{\langle \varphi, \varphi \rangle} = \delta_{ij} w_i. \quad (3.8)$$

One can think of quaternionic eigenforms, which are functions on finite class sets associated to irreducible representations, as a kind of analogue of characters of finite groups. With this point of view, we can think of $(3.8)$ as an analogue of column orthogonality for characters of finite groups. Here is a proof of $(3.8)$ which is perhaps more parallel to a typical proof of column orthogonality:

Write $\Phi = \{\varphi_1, \ldots, \varphi_n\}$. Let $A = (a_{ij})$ be the $n \times n$ matrix with $a_{ij} = \frac{\psi(x_i)}{\sqrt{\psi(x_i)\overline{\psi(x_i)}}}$. Then $A' \hat{A} = I$, and $(3.8)$ follows from orthogonality of the columns of $A$.

4 Periods and embeddings

In this section, we study periods of quaternionic modular forms and their averages. As in the previous section, we will often work with arbitrary orders $O \subset B$, but in certain cases restrict to special orders.

4.1 Periods and averages

Let $O$ be a $\sigma$-order of $B$, and $K/F$ a quadratic subfield of $B$ such that $\sigma_K \subset O$. Define the ideal class map $t \mapsto x(t)$ from $\text{Cl}(K)$ to $\text{Cl}(O)$ by $K^* t \hat{\sigma}_K^* \mapsto B^* t \hat{\sigma}_O^*$. This map depends on the specific embedding of $K$ into $B$. Let $h_K$ be the class number of $K$.

Let $\text{Pic}_C(\text{Cl}(O))$ be the set of formal $C$-linear combinations of elements of $\text{Cl}(O)$. For $c = \sum c_i x_i, d = \sum d_i x_i \in \text{Pic}_C(O)$, define the height pairing

$$\langle c, d \rangle = \sum_{i=1}^n w_i c_i \overline{d_i},$$

and let $\tilde{c} \in M(O)$ be the dual element given by $\tilde{c}(x_i) = w_i c_i$ for $1 \leq i \leq n$. Then $c \mapsto \tilde{c}$ defines an isometry of $\text{Pic}_C(\text{Cl}(O))$ with $M(O)$.
For $\chi \in \hat{\text{Cl}}(K)$, consider the complex divisor $c_{K,\chi} = \sum \chi(t)x(t) \in \text{Pic}_C(\text{Cl}(O))$, which we also denote by $c_K$ when $\chi = 1$. For $\varphi \in M(O)$, we define the period along $K$ against $\chi$ to be

$$P_{K,\chi}(\varphi) = (\varphi, \hat{c}_{K,\chi}) = \sum_{t \in \text{Cl}(K)} \varphi(x(t))\chi^{-1}(t).$$

(4.1)

We also denote this by $P_K(\varphi)$ when $\chi = 1$. Define the ideal class embedding numbers $h_i = h_{K,i} = \#\{t \in \text{Cl}(K) : x(t) = x_i\}$ for $1 \leq i \leq n$. Thus $\sum h_i = h_K$, and $P_K(\varphi) = \sum h_i\varphi(x_i)$.

We consider averages of absolute squares of periods in two directions. First, note that orthogonality of characters of $\text{Cl}(K)$ implies that

$$\sum_{\chi \in \hat{\text{Cl}}(K)} |P_{K,\chi}(\varphi)|^2 = h_K \sum_{t \in \text{Cl}(K)} |\varphi(x(t))|^2 = h_K \sum_{i=1}^n h_i|\varphi(x_i)|^2.$$ 

(4.2)

This expression was used in [20] to study such averages asymptotically (for simplicity, restricted to prime level over $\mathbb{Q}$).

Second, for $T = (a_{ij})$ a real self-adjoint operator on $M(O)$ and $\Phi$ an orthogonal basis for $M(O)$, we have

$$\sum_{\varphi \in \Phi} \frac{P_{K,\chi}(T\varphi)P_{K,\chi}(\varphi)}{(\varphi, \varphi)} = \sum_{\varphi \in \Phi} \frac{(\varphi, T\hat{c}_{K,\chi})(\hat{c}_{K,\chi}, \varphi)}{(\varphi, \varphi)} = (T\hat{c}_{K,\chi}, \hat{c}_{K,\chi}).$$

(4.3)

This expression (in terms of the height pairing) was used for the exact average formula in [19].

Let $\mathfrak{N}' = \mathfrak{N}(\mathfrak{B}^{-1})$, where $\mathfrak{N}$ is the (reduced) discriminant of $O$. Thus $p|\mathfrak{N}'$ if and only if $O_p$ is not a maximal order. Then for a nonzero integral ideal $m \subset \mathfrak{o}$ which is coprime to $\mathfrak{N}'$, and for any $\pi$ occurring in $M(O)$, $T_m$ acts on $\pi^{\mathfrak{O}_m^\times}$ by a scalar $\lambda_m(\pi)$ since the local space $\pi^{\mathfrak{O}_m^\times}$ of invariants is 1-dimensional for $v|m$. For $\varphi \in \pi^{\mathfrak{O}_m^\times}$, set $\lambda_m(\varphi) = \lambda_m(\pi)$.

**Proposition 4.1** Let $\Phi$ be an orthogonal basis of eigenforms for $M(O)$. For a nonzero integral ideal $m \subset \mathfrak{o}$ coprime to $\mathfrak{N}'$, consider the divisor $a(m) = \sum_{i=1}^n a_{ii}(m)x_i$ associated to the diagonal of the Brandt matrix $A_m$. Then we have

$$\sum_{\varphi \in \Phi} \sum_{\chi \in \hat{\text{Cl}}(K)} \frac{\lambda_m(\varphi)|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K \langle c_K, a(m) \rangle = h_K \sum_{i=1}^n w_i h_i a_{ii}(m).$$

(4.4)

**Proof** This follows from applying (4.2) and then Proposition 3.6 with $i = j$ to the left hand side.

For $m$ as above, define the degree of $\deg_T m$ on $M(O)$ to be the number of integral right $O$-ideals of norm $m$. Note each row of the Brandt matrix $A_m$ sums to $\deg_T m$. For a prime $p \not| \mathfrak{N}'$, note $\deg_T p = q_p + 1$, where $q_p = \#(\mathfrak{o}/p)$.

**Corollary 4.2** Let $\Phi_0$ be an orthogonal basis of eigenforms for $S(\mathfrak{O})$. If $m$ is coprime to $\mathfrak{N}'$, then

$$\sum_{\varphi \in \Phi_0} \sum_{\chi \in \hat{\text{Cl}}(K)} \frac{\lambda_m(\varphi)|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_K \langle c_K, a(m) \rangle - \delta^+(O, m) \deg_T m \frac{h_K^2|C^+(\mathfrak{N}(\mathfrak{O}))|}{m(O)},$$

where $\delta^+(O, m)$ is 1 if the class of $m$ is trivial in $C^+(\mathfrak{N}(\mathfrak{O}))$ and 0 otherwise.
Proof  Recall that a basis of \( \text{Eis}(\mathcal{O}) \) is given by \( \{ \mu \circ N \} \), where \( \mu \) runs over the characters of \( \text{Cl}^+(N(\mathcal{O})) \). For \( \varphi = \mu \circ N \), we see that \( P_{K,\chi}(\varphi) = \sum_{t \in \text{Cl}(K)} \mu(N_{K/F}(t)) \chi^{-1}(t) \) is \( h_K \) if \( (\mu \circ N)|_{\text{Cl}(K)} = \chi \) and 0 otherwise. Since each such \( \varphi \) has norm \( (\varphi, \varphi) = m(\mathcal{O}) \) and \( \lambda_m(\mu \circ N) = \mu(m)\lambda_m(1) = \mu(m) \deg T_m \), we see that the Eisenstein contribution to \( (4.4) \) is

\[
\sum_\mu \lambda_m(\mu \circ N) \frac{h_k^2}{m(\mathcal{O})} = \frac{h_k^2}{m(\mathcal{O})} \sum_\mu \deg T_m \mu(m),
\]

where \( \mu \) runs over the characters of \( \text{Cl}^+(N(\mathcal{O})) \). \( \square \)

This double average formula simplifies in various situations. We explicate two now. We call \( \mathcal{O} \) balanced if \( w_i = 1 \) for all \( 1 \leq i \leq n \), i.e., if \( n = m(\mathcal{O}) \). (Balanced implies that each ideal class in \( \text{Cl}(\mathcal{O}) \) has the same volume with respect to a Haar measure, though it is not exactly equivalent.) In Sect. 4.4, we give elementary criteria for orders to be balanced.

Corollary 4.3  Let \( \Phi_0 \) be an orthogonal basis of eigenforms for \( S(\mathcal{O}) \). Then:

(i) If \( \mathcal{O} \) is balanced, then

\[
\sum_{\varphi \in \Phi_0} \sum_{\chi \in \text{Cl}(K)} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_k^2 \left( 1 - \frac{|\text{Cl}^+(N(\mathcal{O}))|}{m(\mathcal{O})} \right).
\]

(ii) If \( h_k^2 = 1 \), \( \mathcal{O} \) is special of unramified quadratic type, and \( m \) is coprime to \( \mathfrak{N} \), then

\[
\sum_{\varphi \in \Phi_0} \sum_{\chi \in \text{Cl}(K)} \lambda_m(\varphi) \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = h_k \sum_{i=1}^n w_i h_i a_i(m) - \deg T_m \frac{h_k^2}{m(\mathcal{O})}.
\]

4.2 Exact averages and stability

Here we will use the double average formula in the previous section to obtain some precise results about the single averages (4.3) for fixed \( \chi \). We continue the notation of the previous section. In particular, \( \mathfrak{o}_K \subset \mathcal{O} \), where \( \mathcal{O} \) is an arbitrary order in \( B \).

Lemma 4.4  We have \( \langle c_{K,\chi}, c_{K,\chi} \rangle \leq \langle c_{K,\chi}, c_{K,\chi} \rangle \) for all \( \chi \in \text{Cl}(K) \). Further, \( \langle c_{K,\chi}, c_{K,\chi} \rangle = \langle c_{K,\chi}, c_{K,\chi} \rangle \) if and only if only if \( h_i \leq 1 \) for all \( 1 \leq i \leq n \).

Proof  Note that

\[
\langle c_{K,\chi}, c_{K,\chi} \rangle = \sum_{i=1}^n w_i \left( \sum_{x(t) = x_i} \chi(t) \right)^2,
\]

where in the inner sum \( t \) runs over the elements of \( \text{Cl}(K) \) which map to \( x_i \) via the ideal class map. Clearly this is maximized for \( \chi \) being the trivial character, and we have equality among all \( \chi \) when \( h_i = \# \{ t \in \text{Cl}(K) : x(t) = x_i \} \leq 1 \) for all \( i \). Conversely, if \( h_i \) is large for some \( i \), then there exist distinct \( t, t' \in \text{Cl}(K) \) such that \( x(t) = x(t') = x_i \). Now \( \chi(t) \neq \chi(t') \) for some \( \chi \in \text{Cl}(K) \), and we obtain a strict inequality for such \( \chi \). \( \square \)

We say that the pair \( (\mathcal{O}, K) \) lies in the semistable range if \( h_i \leq 1 \) for all \( 1 \leq i \leq n \), and in the stable range if in addition \( h_i = 1 \) implies \( w_i = u_K := [\mathfrak{o}_K^\times : \mathfrak{o}_K^\times] \). Note that if \( h_i > 0 \), then \( \mathfrak{o}_K \subset \mathcal{O}(\mathfrak{I}_i) \), and thus \( u_K \mid w_i \). If \( \mathcal{O} \) is balanced, then the stable range is the same as the semistable range.
Corollary 4.5  Let $\Phi_0$ be an orthogonal basis of eigenforms for $S(\mathcal{O})$ and $\chi \in \hat{\text{Cl}}(K)$. Then we have the bounds
\[
\sum_{i=1}^{n} w_i h_i - m^+(\mathcal{O}, 1) \frac{h_K^2}{m(\mathcal{O})} \leq \sum_{\varphi \in \Phi_0} |P_K(\varphi)|^2 \leq h_K \sum_{i=1}^{n} w_i h_i - m^+(\mathcal{O}, 1) \frac{h_K^2}{m(\mathcal{O})},
\]
where $m^+(\mathcal{O}, \chi)$ is the number of characters $\mu$ of $\hat{\text{Cl}}^+(N(\mathcal{O}))$ such that $\mu \circ N_{K/F} = \chi$. The lower bound is an equality if and only if $(\mathcal{O}, K)$ is in the semistable range, in which case we have
\[
\sum_{\varphi \in \Phi_0} |P_{K, \chi}(\varphi)|^2 = \sum_{i=1}^{n} w_i h_i - m^+(\mathcal{O}, \chi) \frac{h_K^2}{m(\mathcal{O})}
\]
for all $\chi \in \hat{\text{Cl}}(K)$.

Proof  We extend $\Phi_0$ to an orthogonal basis $\Phi$ of $M(\mathcal{O})$ by adjoining $\mu \circ N$ for each $\mu \in \hat{\text{Cl}}^+(N(\mathcal{O}))$. Then comparing Proposition 4.1 (with $m = \circ$), (4.3) (for $T = T_0$ the identity) and Lemma 4.4, we see that
\[
\sum_{i=1}^{n} w_i h_i \leq \sum_{\varphi \in \Phi} |P_K(\varphi)|^2 \leq h_K \sum_{i=1}^{n} w_i h_i,
\]
with the lower bound being equality if and only if we are in the semistable range, in which case (4.3) for $T = T_0$ is independent of $\chi$. Now we simply subtract of the Eisenstein contribution as in Corollary 4.2.

Note that if $h_F^+ = 1$ and $\mathcal{O}$ is a special order of unramified quadratic type, $m^+(\mathcal{O}, \chi)$ is simply 1 if $\chi = 1$ and 0 otherwise. If in addition we are in the stable range, then we simply get
\[
\sum_{\varphi \in \Phi_0} |P_{K, \chi}(\varphi)|^2 = h_K \left( u_K - \delta_{\chi, 1} \frac{h_K}{m(\mathcal{O})} \right),
\]
where $\delta_{\chi, \chi'}$ is the Kronecker delta. This is always nonnegative because being in the stable range implies $u_K m(\mathcal{O}) \geq h_K$.

In general, we can make the above bounds more elementary by observing that
\[
h_K \leq \sum_{i=1}^{n} w_i h_i \leq w_B h_K,
\]
where $w_B = \max[\mathcal{O}_B^\times : \sigma^\times]$, where $\mathcal{O}_B$ runs over maximal orders of $B$. If $F = \mathbb{Q}$ and $D_B > 3$, then $w_B \leq 3$ and can be determined by congruence conditions.

Finally, we remark the following inheritance properties for suborders.

Lemma 4.6  Let $\mathcal{O} \subset B$ be an order, and $\mathcal{O}'$ be a suborder. If $\mathcal{O}$ is balanced, then so is $\mathcal{O}'$. Similarly if $\sigma_K \subset \mathcal{O}'$, and $(\mathcal{O}, K)$ is in the (semi-)stable range then so is $(\mathcal{O}', K)$.

Proof  We can write $\text{Cl}(\mathcal{O}') = \{y_{ij}\}$, where as double cosets in $\hat{B}^\times$ we have $x_i = \bigsqcup_j y_{ij}$ for all $1 \leq i \leq n$. Then each $\mathcal{O}_x(y_{ij}) \subset \mathcal{O}_x(x_i)$, so each $[\mathcal{O}_x(y_{ij})^\times : \sigma^\times] \leq w_i$. In addition if $t \mapsto y(t)$ denotes the corresponding class map $\text{Cl}(K) \to \text{Cl}(\mathcal{O}')$, then we see $\sum_j h_{ij}' = h_i$, where $h_{ij}' = \# \{ t \in \text{Cl}(K) : y(t) = y_{ij}\}$. \qed
4.3 Integral embeddings

Recall that a quadratic field extension $K/F$ embeds in $B$ if and only if $K_v/F_v$ is non-split at each $v \in \text{Ram}(B)$ and each $v|\infty$. Here we recall some facts about embedding $\sigma_K$ into a quaternionic order $O$.

Consider an $\sigma$-order $O \subset B$. Let $\text{Emb}(\sigma_K, O)$ denote the set of embeddings of $\sigma_K$ into $O$ up to conjugation by $O \times$. Similarly, define $\text{Emb}(\sigma_{K,v}, O_v)$ for $v < \infty$. We have the following formula (see [3] or [29, Theorem 30.7.3]):

$$\sum_{i=1}^{n} \# \text{Emb}(\sigma_K, O_{\ell}(I_i)) = h_K \prod_{v|\infty} \# \text{Emb}(\sigma_{K,v}, O_v).\quad(4.7)$$

On the left, $O_{\ell}(I_i)$ ranges over all isomorphism classes of orders in the genus of $O$ (with multiplicity).

Now suppose $O$ is special of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$. In particular, we see that $\sigma_K$ embeds into a special order in the genus of $O$ if and only if $\sigma_{K,v}$ locally embeds into $O_v$ for all $v|\mathfrak{N} = \mathfrak{N}_1\mathfrak{N}_2\mathfrak{M}$ and $K/F$ is totally imaginary.

Local embedding numbers have been computed for Eichler orders in [11] and for special orders in [12] (see also [29, Chapter 30]). The precise description is complicated in general, and we will only state it under the following assumptions:

$$\text{ord}_v(\mathfrak{N}_2) = 2$$

for all $v|\mathfrak{N}_2$ and $\mathfrak{M}$ is squarefree. \quad(4.8)

We recall this implies that all special orders of level type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ lie in the same genus.

**Lemma 4.7** Assume $O$ is a special order type $(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})$ such that (4.8) holds, and that $K/F$ is a quadratic field which embeds in $B$.

1. For $v|\mathfrak{N}_1$,

$$\# \text{Emb}(\sigma_{K,v}, O_v) = \begin{cases} 2 & K_v/F_v \text{ unramified}, \\ 1 & K_v/F_v \text{ ramified, } \text{ord}_v(\mathfrak{N}_1) = 1, \\ 0 & K_v/F_v \text{ ramified, } \text{ord}_v(\mathfrak{N}_1) > 1. \end{cases}$$

2. For $v|\mathfrak{N}_2$,

$$\# \text{Emb}(\sigma_{K,v}, O_v) = \begin{cases} 0 & K_v/F_v \text{ unramified}, \\ q_v + 1 & K_v/F_v \text{ ramified}. \end{cases}$$

3. For $v|\mathfrak{M}$,

$$\# \text{Emb}(\sigma_{K,v}, O_v) = \begin{cases} 0 & K_v/F_v \text{ unramified}, \\ 1 & K_v/F_v \text{ ramified}, \\ 2 & K_v/F_v \text{ split}. \end{cases}$$

**Proof** The first two parts follow from [12, Theorems 5.12 and 5.19] and Lemma 3.1(v). The third part follows from [11, Section 2] (see also [29, Lemma 30.6.16]). \hfill \Box
Now (4.7) and Lemma 4.7 immediately yield the following.

**Corollary 4.8** Let \( \mathfrak{N} = \mathfrak{N}_1 \mathfrak{N}_2 \mathfrak{M} \) be a nonzero ideal in \( \mathfrak{o} \) such that \( \mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M} \) are pairwise coprime, \( \#\{v|\mathfrak{N}_1 \mathfrak{N}_2\} + [F : \mathbb{Q}] \) is even, and \( \text{ord}_v(\mathfrak{N}_1) \) is odd for \( v|\mathfrak{N}_1 \). Further assume (4.8).

Let \( K/F \) be a quadratic field extension. Then \( \mathfrak{o}_K \) embeds in some special order \( \mathcal{O} \subset B \) of level type \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \) if and only if the following conditions hold:

(i) \( K/F \) is totally imaginary;
(ii) \( K_v/F_v \) is non-split for each \( v|\mathfrak{N}_1 \mathfrak{N}_2 \);
(iii) \( K_v/F_v \) is unramified for each \( v|\mathfrak{N}_1 \) such that \( \text{ord}_v(\mathfrak{N}_1) > 1 \);
(iv) \( K_v/F_v \) is ramified for each \( v|\mathfrak{N}_2 \);
(v) \( K_v/F_v \) is either ramified or split for each \( v|\mathfrak{M} \).

When \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \) satisfies the hypotheses in the first 2 sentences of Corollary 4.8, we will say \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \) is a nice level type. When all conditions in this corollary hold (i.e., \( \mathfrak{o}_K \) embeds in a special order of nice level type \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \)) and the further condition that \((v')\) \( K_v/F_v \) is split for each \( v|\mathfrak{M} \) also holds, we will say \( K/F \) is \((\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M})\)-admissible. This extra condition \((v')\) will be needed in Sect. 5.4.

### 4.4 Balanced orders

We can rephrase the notion of balanced orders in terms of the existence of certain embeddings.

**Lemma 4.9** Let \( \mathfrak{e}_1, \ldots, \mathfrak{e}_r \) be a set of generators for \( \mathfrak{o}_+^\times \). Then an arbitrary order \( \mathcal{O} \) is balanced if and only if no ring of the form \( \mathfrak{o}[u] \) embeds the genus of \( \mathcal{O} \), where either \( u \) is a root of unity of order \( \geq 3 \) or \( u = \sqrt{-\mathfrak{e}} \) where \( \mathfrak{e} = \prod S \mathfrak{e}_i \) and \( \emptyset \neq S \subset \{1, \ldots, r\} \).

**Proof** Clearly \( \mathcal{O} \) is unbalanced if and only if there exist an order \( \mathcal{O}' = \mathcal{O}_1(\mathcal{I}_2) \) in the genus of \( \mathcal{O} \) such that \( (\mathcal{O}')^\times \) contains a unit \( u \notin \mathfrak{o}_+^\times \). Suppose this is the case. Then \( K = F(u) \) is a CM extension in \( B/F \). Let \( \mu_K \) denote the roots of unity of \( K \). Hasse’s unit index \( Q_{K/F} = [\mathfrak{o}_+^\times : \mu_K \mathfrak{o}_+^\times] \) is either 1 or 2. If \( Q_{K/F} = 1 \), then \( u \in \mu_K \). Assume \( Q_{K/F} = 2 \). Then \( u^2 \in \zeta \mathfrak{o}_+^\times \) for some \( \zeta \in \mu_K \). Hence \( \mathfrak{o}([\zeta]) \) embeds in \( \mathcal{O}' \), and we may restrict to the case \( \zeta = \pm 1 \). Then \( u^2 = -\mathfrak{e}^+ \) for some \( \mathfrak{e}^+ \in \mathfrak{o}_+^\times \). Write \( \mathfrak{e}^+ = \prod S \mathfrak{e}_i^d_i \). Then, for an appropriate choice of \( \eta \in \mathfrak{o}_+^\times \), one sees \( \mathfrak{e} := \mathfrak{e}^+ \eta^2 \) is of the form \( \prod S \mathfrak{e}_i \) for some \( \emptyset \neq S \subset \{1, \ldots, r\} \).

By replacing \( u \) with \( u' = \sqrt{-\mathfrak{e}} \), we see that \( \mathfrak{o}[u] = \mathfrak{o}[u'] = \mathfrak{o}([\sqrt{-\mathfrak{e}}]) \). This proves the “if” direction; the other direction is clear. \( \square \)

Lemma 4.9 implies that we can guarantee an order \( \mathcal{O} \) is balanced by checking a finite number of local conditions, which depend upon \( F \). Namely, there are finitely many CM extensions \( F(u) \) of \( F \), with \( u \) as in Lemma 4.9: either \( u = \zeta_m \) is a primitive \( m \)-th root of unity for some \( m \geq 3 \) such that the totally real subfield of \( \mathbb{Q}(\zeta_m) \) is contained in \( F \), or \( u = \sqrt{-\mathfrak{e}} \) for one of the finitely many \( \mathfrak{e} \in \mathfrak{o}_+^\times \) of the form above. For each such \( u \), \( \mathfrak{o}[u] \) does not embed in the genus of \( \mathcal{O} \) if either (i) \( B/F \) is ramified at some prime where \( F(u)/F \) splits, or (ii) if \( \mathfrak{o}[u] = \mathfrak{o}_K \) where \( K = F(u) \) and \( \mathcal{O} \) is a special order of nice level type \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \) but \( K \) does not satisfy the conditions of Corollary 4.8. (When \( \mathfrak{o}[u] \neq \mathfrak{o}_K \), one can also use results about optimal embeddings to get similar conditions.) In particular, if (i) is satisfied for all such \( u \), then every order \( \mathcal{O} \subset B \) is balanced.

We explicate our criteria in the case of \( F = \mathbb{Q} \).

**Corollary 4.10** Suppose \( F = \mathbb{Q} \) and \( \mathcal{O} \subset B \) is a special order of nice level type \( (\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{M}) \). Then \( \mathcal{O} \) is balanced if and only if both (1) and (2) below hold.
(1) One of the following holds:
   (a) there exists a prime \( p \equiv 1 \mod 4 \) such that \( p | N_1 \); or
   (b) there exists a prime \( p \equiv 3 \mod 4 \) such that \( p | M \); or
   (c) \( N_2 \notin \{1, 4\} \); or
   (d) 8\( | N_1 \).

(2) And one of the following holds:
   (a) there exists a prime \( p \equiv 1 \mod 3 \) such that \( p | N_1 \); or
   (b) there exists a prime \( p \equiv 2 \mod 3 \) such that \( p | M \); or
   (c) \( N_2 \notin \{1, 9\} \); or
   (d) 27\( | N_1 \).

**Proof** By Lemma 4.9, \( \mathcal{O} \) is balanced if and only if neither \( \mathbb{Z}[i] \) nor \( \mathbb{Z}[\zeta_3] \) embed in the genus of \( \mathcal{O} \). Now apply Corollary 4.8.

In particular, in the setting of this corollary, we see that \( \mathcal{O} \) is balanced if \( N_2 > 9 \), or if there exists a \( p \equiv 1 \mod 12 \) dividing \( N_1 \), or if there exists a \( p \equiv 11 \mod 12 \) dividing \( M \).

Alternatively, one can treat special orders over \( \mathbb{Q} \) by comparing the mass formula with the class number formula from [12, Theorem 8.6].

Now we briefly discuss some sufficient conditions for balanced orders \( \mathcal{O} \) when \( F = \mathbb{Q}(\sqrt{d}) \) is real quadratic. Assume \( d > 1 \) is squarefree and let \( \varepsilon \) be a generator for the rank 1 group \( \mathcal{O}^\times_+ \). Then, assuming \( \mathcal{O}[\varepsilon] \) does not embed into the genus of \( \mathcal{O} \), Lemma 4.9 tells us that \( \mathcal{O} \) is balanced if and only if no cyclotomic ring \( \mathbb{Z}[\zeta_m] \) with \( m \geq 3 \) embeds in the genus of \( \mathcal{O} \).

If \( \mathbb{Z}[\zeta_m] \) embeds in the genus of \( \mathcal{O} \), then \( K = K_m := F(\zeta_m) \) embeds in \( \mathbb{B} \). The only possibilities are \( K = F \mathbb{Q}(i) \), \( K = F \mathbb{Q}(\zeta_3) \) or that \( K = \mathbb{Q}(\zeta_m) \) is cyclotomic of degree 4 containing \( F \). The latter possibility implies that \( m = 8 \) and \( d = 2, m = 12 \) and \( d = 3 \), or \( m = d = 5 \). In these 3 cases, \( \mathbb{Z}[\zeta_m] \) embeds in the genus of \( \mathcal{O} \) if and only if if \( \mathcal{O}[\varepsilon] \) does, so we can apply Corollary 4.8. When \( m = 3, 4, \) i.e., \( K = F \mathbb{Q}(i) \) or \( K = F \mathbb{Q}(\zeta_3) \), it is not necessarily true that \( \mathcal{O}_K = \mathcal{O}_F[\zeta_m] \). However, if \( d > 5 \) and \( d = 1 \mod 4 \), then \( \mathbb{Q}(i), \mathbb{Q}(\zeta_3) \) and \( F \) have pairwise coprime discriminants, implying that they are pairwise linearly independent, and thus \( \mathcal{O}_K = \mathcal{O}_F[\zeta_m] \) for \( m = 3, 4 \). In this situation, we can again use Corollary 4.8.

**Example 4.11** Let \( F = \mathbb{Q}(\sqrt{5}) \). Then \( \varepsilon = \frac{\sqrt{5} - 1}{2} \) generates \( \mathcal{O}^\times_+ \). Let \( p_{11} = (1 + \frac{3 + \sqrt{5}}{2}) \) be a prime of \( F \) above 11. Both \( F(\varepsilon) \) and \( \mathbb{Q}(\zeta_5) \) split over \( p_{11} \), whereas both \( F \mathbb{Q}(i) \) and \( F \mathbb{Q}(\zeta_3) \) split over the prime ideal \( (7) \). Hence if \( B \) is any definite quaternion algebra ramified at both \( (2) \) and \( p_{11} \), then any order \( \mathcal{O} \subset B \) is balanced.

Alternatively, \( F(\varepsilon) \) and \( F \mathbb{Q}(i) \) have rings of integers of the form \( \mathcal{O}[u] \) where \( u = \varepsilon \) or \( u = i \), and both are ramified above \( (2) \). Also \( F \mathbb{Q}(\zeta_3) / F \) is split above \( (2) \). Hence if \( B \) is any definite quaternion algebra ramified at \( (2) \) and also at some prime \( p \) lying above a prime \( p \) of \( \mathbb{Q} \) that completely splits \( \mathbb{Q}(\zeta_5) \) (i.e., \( p \equiv 1 \mod 5 \)), then any special order \( \mathcal{O} \subset B \) of level type \( (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}) \) is balanced if \( 8 \mid \mathfrak{m}_1 \).

## 5 Twisted \( L \)-values

From now on, we assume that \( h_F = 1, K / F \) is quadratic, \( \mathfrak{m} = \mathfrak{m}_1 \mathfrak{m}_2 \mathfrak{m} \) is a nonzero integral \( \mathcal{O} \)-ideal, and that \( \mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m} \) and \( K / F \) satisfy the conditions in Corollary 4.8. In particular, \( (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}) \) is a nice level type. From Sect. 5.4 onwards, we will further assume that \( K_v/F_v \) is split for each \( v \mid \mathfrak{m} \), i.e., that \( K / F \) is \( (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}) \)-admissible.
Let $B/F$ be the definite quaternion algebra with discriminant $\mathcal{D}_B = \prod_{p | \mathfrak{M}_1, \mathfrak{M}_2} p$. Then there exists a special order $\mathcal{O} \subset B$ of level type $(\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M})$ and an embedding of $K$ into $B$ such that $\mathfrak{o}_K \subset \mathcal{O}$. We fix such an $\mathcal{O}$ and such an embedding, which defines a class map $\text{Cl}(K) \to \text{Cl}(\mathcal{O})$, periods $P_{K,\chi}$, divisors $c_{K,\chi}$, and ideal class embedding numbers $h_1, \ldots, h_n$ as in the previous section.

For an (irreducible) cuspidal representation $\pi$ of $B^\times(\mathbb{A}_F)$ or $\text{GL}_2(\mathbb{A}_F)$, denote by $c(\pi)$ the conductor of $\pi$. This means that $c(\pi)$ is the ideal in $\mathfrak{o}_F$ corresponding to the exact level of a newform $\varphi \in \pi$. In particular, $c(\pi)$ is the conductor of the Jacquet–Langlands transfer of $\pi$ to $\text{GL}_2(\mathbb{A}_F)$ in the case of $B^\times(\mathbb{A}_F)$. For $v < \infty$, the local conductor is $c(\pi_v) := \text{ord}_v(c(\pi))$.

### 5.1 Local vanishing criteria

Let $\varphi \in M(\mathcal{O})$ be a eigenform and $\pi$ the associated representation of $B^\times(\mathbb{A}_F)$ (necessarily of trivial central character since $h_F = 1$). Let $\chi$ be a character of $\text{Cl}(K)$. It is easy to see that $P_{K,\chi}(\varphi) = 0$ unless the restriction of $\chi$ to $\mathbb{A}_K^\times$ is trivial, so let us assume this.

Our definition (4.1) of the period $P_{K,\chi}$ can also be expressed as an integral

$$P_{K,\chi}(\varphi) = \int_{K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times} \varphi(t)\chi^{-1}(t) \, dt,$$

for a suitable choice of Haar measure on $\mathbb{A}_K^\times$. Via this integral, $P_{K,\chi}$ extends to a $\chi$-equivariant linear functional on $\pi$. In other words, $P_{K,\chi}$ is an element of $\text{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$. Thus the following gives sufficient criteria for the vanishing of $P_{K,\chi}(\varphi)$, where $\varphi \in \pi$.

Denote by $\text{St}_v$ the Steinberg representation of $\text{GL}_2(F_v)$ for $v < \infty$.

**Proposition 5.1** Let $\pi$ be a cuspidal representation of $B^\times(\mathbb{A}_F)$ occurring in $S(\mathcal{O})$. Let $\chi$ be a character of $\text{Cl}(K)$ which is trivial on $\mathbb{A}_F^\times$. The global Hom space $\text{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$ is nonzero if and only if both of the following hold for all $v | \mathfrak{M}$ such that $K_v/F_v$ is ramified:

1. if $v | \mathfrak{M}_1, \mathfrak{M}_2$ and $\pi_v$ is 1-dimensional, then $\pi_v(\varpi_K, v) = \chi_v(\varpi_K, v)$;
2. if $v | \mathfrak{M}$ and $\pi_v \simeq \text{St}_v \otimes \mu_v$, then $\mu_v(\varpi_K, v) = -\chi_v(\varpi_K, v)$.

**Proof** The global Hom space $\text{Hom}_{\mathbb{A}_K^\times}(\pi, \chi)$ is nonzero if and only if each local Hom space $\text{Hom}_{\mathbb{A}_K^\times}(\pi_v, \chi_v)$ is. For $v | \infty$, $\pi_v$ and $\chi_v$ are both trivial, so this local Hom space is clearly nonzero. Assume from now on $v$ is finite. Necessary and sufficient criteria for each local Hom space to be nonzero were given by Tunnell [28] and Saito [25]. In particular, if either $\pi_v$ is an (unramified or ramified) principal series or $K_v/F_v$ is split, then necessarily $B_v$ is split and $\text{Hom}_{\mathbb{A}_K^\times}(\pi_v, \chi_v) \neq 0$ for all $\chi_v : K_v^\times / F_v^\times \to \mathbb{C}^\times$.

In all other cases, $v | \mathfrak{M}$ and either $\pi_v$ is a representation of the unit group $D_v^\times$ of the local quaternion division algebra or $\pi_v$ is a discrete series representation of $\text{GL}_2(F_v)$. In either case, let $\pi'_v$ be the Jacquet–Langlands correspondent on the other group. Furthermore, $K_v/F_v$ is a field extension which embeds into $D_v$, and we have the dichotomy relation:

$$\dim \text{Hom}_{\mathbb{A}_K^\times}(\pi_v, \chi_v) + \dim \text{Hom}_{\mathbb{A}_K^\times}(\pi'_v, \chi_v) = 1.$$

Let $\pi_{D,v}$ be the element of $\{\pi_v, \pi'_v\}$ which is a representation of $D_v^\times$. Then $\text{Hom}_{\mathbb{A}_K^\times}(\pi_{D,v}, \chi_v) \neq 0$ if and only if $\pi_{D,v}|_{\mathbb{A}_K^\times}$ contains $\chi_v$ as a subrepresentation. We note that the Tunnell–Saito criterion says that this is the case if and only if $\varepsilon(\frac{1}{2}, \pi_{K,v} \otimes \chi_v) = -1$, though we do not directly use this in our proof.

First suppose $K_v/F_v$ is unramified. By our embedding conditions, this means $v | \mathfrak{M}_1$, so in particular $\pi_v = \pi_{D,v}$. Then $K_v^\times = F_v^\times \mathcal{O}_{K,v}^\times$, so $\chi_v$ occurs in $\pi_v$ if and only if $\pi_v$ contains an
$\pi_{K,v}^\times$-fixed vector. In fact $\pi_v^{\Omega_v}$ contains an $\pi_{K,v}^\times$-fixed vector since $\sigma_{K,v} \subset O_v$. Thus in this case we always have $\dim \Hom_{K_v^\times}^{\times}(\pi_v, \chi_v) \neq 0$.

Now assume $K_v/F_v$ is ramified. By our embedding criteria, either $\pi_{D,v}$ is 1-dimensional or $v|\mathfrak{M}_1$ and $\pi_v = \pi_{D,v}$ has dimension $> 1$. In addition, there are 2 possibilities for $\chi_v$ according to $\chi_v(\sigma_{K,v}) = \pm 1$, since $K_v^\times/O_{K,v}^\times F_v^\times \simeq (\sigma_{K,v})/(q\sigma_{K,v})$.

If $v|\mathfrak{M}_1$ and $\pi_v = \pi_{D,v}$ has dimension $> 1$, then $\pi_v^\times$ is minimal supercuspidal representation of even conductor (in fact depth 0), so by [16, Theorem 3.6] $\pi_v|K_v^\times$ contains both unramified characters of $K_v^\times/F_v^\times$. In particular, $\Hom_{K_v^\times}(\pi_v, \chi_v) \neq 0$.

Finally we are reduced to the case that $\pi_{D,v} = \mu_v \circ N$ is 1-dimensional, so $\pi_v^\times \simeq St_v \otimes \mu_v$. It is clear that $\Hom_{K_v^\times}(\pi_{D,v}, \chi_v) \neq 0$ if and only if $\mu_v(\sigma_v) = \chi_v(\sigma_{K,v})$. The proposition now follows from the dichotomy relation by observing that $\pi_v = \pi_{D,v}$ if $v|\mathfrak{M}_1$ and $\pi_v = \pi_{D,v}$ if $v|\mathfrak{M}$.

\[ \square \]

### 5.2 Local $L$-factors

In this section, we describe local $L$-factors in cases that are relevant for us. Here $v$ denotes a finite place of $F$, $K_v/F_v$ is a quadratic field extension, $\pi_v$ is a representation of $\text{PGL}_2(F_v)$, and $\chi_v$ an unramified character of $K_v^\times/F_v^\times$. We say a supercuspidal $\pi_v$ is unramified dihedral if it is induced from a regular character of the unramified quadratic extension of $F_v$. A supercuspidal $\pi_v$ is unramified dihedral if and only if $\pi_v$ is minimal of even conductor [27, Proposition 3.5].

#### Lemma 5.2

We have the following local $L$-factor ratios.

1. Suppose $\pi_v$ is supercuspidal. Then

\[
\frac{L \left( \frac{1}{2}, \pi_{K,v} \otimes \chi_v \right)}{L(1, \pi_v, \text{Ad})} = \begin{cases} 
1 + q_v^{-1} & \text{if } \pi_v \text{ unramified dihedral} \\
1 & \text{if } \pi_v \text{ else.}
\end{cases}
\]

2. Suppose $K_v/F_v$ is ramified and $\pi_v \simeq St_v \otimes \mu_v$ is a twisted Steinberg representation, where $\mu_v$ is a quadratic character of $K_v^\times$ such that $\mu_v(\sigma_F) = \chi_v(\sigma_{K,v})$. Then

\[
\frac{L \left( \frac{1}{2}, \pi_{K,v} \otimes \chi_v \right)}{L(1, \pi_v, \text{Ad})} = 1 + q_v^{-1}.
\]

#### Proof

First suppose $\pi_v$ is supercuspidal. Then $\pi_v$ corresponds to an irreducible 2-dimensional representation $\sigma_v$ of the Weil group $W_{F_v}$ by the local Langlands correspondence, and the base change $\pi_{K,v}$ corresponds to the restriction of $\sigma_{K,v}$ to $W_{K,v}$, where $w$ is a place of $K$ above $v$. If $\sigma_{K,v}$ is irreducible, then $\pi_{K,v}$ is supercuspidal. Otherwise, $\sigma_{K,v} \simeq \tau_w \oplus \tau_w^{-1}$ for some character $\tau_w$ of $W_{K,v}$. By inductivity of $L$-functions, $L(s, \tau_w) = L(s, \sigma_v) = L(s, \pi_v) = 1$, so $L(s, \pi_{K,v}) = L(s, \tau_w)L(s, \tau_w^{-1}) = 1$ and $\pi_{K,v}$ must be a ramified principal series.

In either case, $\pi_{K,v}$ has conductor at least 2, so $L(s, \pi_{K,v} \otimes \chi_v) = 1$ since $\chi_v$ is unramified. From the calculations in [22], $L(s, \pi_{K,v}, \text{Ad})$ is either $(1 + q_v^{-s})^{-1}$ or 1 according to whether $\pi_v$ is unramified dihedral or not. This yields (1).

Next assume the hypotheses of (2). Via the local Langlands correspondence, it is easy to see that $\pi_{K,v} \simeq St_{K,v} \otimes (\mu_v \circ N)$. Thus $L(s, \pi_{K,v} \otimes \chi_v) = L(s, (\mu_v \circ N)\chi_v) = L(s, 1_{F_v})$, and $L(s, \pi_v, \text{Ad}) = (1 - q_v^{-s})^{-1} - 1$. This yields (2).
5.3 Local spectral distributions

Here we will compute relevant local spectral distributions at a nonarchimedean place $v$ which splits $K/F$. To ease notation, for this section only, all objects are local and we drop the subscripts $v$. In particular, $F$ is now a $p$-adic field, and $K = F \times F$.

Let $\pi = \pi(\mu, \mu^{-1})$ be an unramified principal series representation of $GL_2(F)$ with trivial central character. Consider a character $\chi$ of $K^\times = F^\times \times F^\times$ given by $\chi(a, b) = \chi_1(a)\chi_2(b)$, where $\chi_1, \chi_2$ are 2 unramified characters of $F^\times$ such that $\chi_2 = \chi_1^{-1}$. Let $\psi$ be an additive character of order 0, and $W$ be the $\psi$-Whittaker model for $\pi$. For $W \in \mathcal{W}$, we abbreviate $W\left(\begin{smallmatrix} a \\ 1 \end{smallmatrix}\right)$ by $W(a)$. (Alternatively, one can think of $W$ as an element of the Kirillov model.) Let $d^\times a$ denote the Haar measure on $F^\times$ which gives $\mathcal{O}_F^\times$ volume 1, i.e., $d^\times a$ is self-dual with respect to $\psi$.

Put

$$\ell(W) = \int_{F^\times} W(a)\chi_1^{-1}(a)\,d^\times a.$$ 

As in [17, Section 2.1], we define the local distribution

$$\tilde{J}_\pi(f) = \sum_W \ell(\pi(f)W)\overline{\ell(W)}, \quad (5.1)$$

where $f \in C_c^\infty(GL_2(F))$ and $W$ runs over an orthonormal basis for $\mathcal{W}$.

If $f_0$ is the characteristic function of $GL_2(\mathcal{o})$ divided by its volume, then (see loc. cit.)

$$\tilde{J}_\pi(f) = \frac{L(2, 1_F)}{L(1, 1_F)} \cdot \frac{L\left(\frac{1}{2}, \pi \otimes \chi_1\right)L\left(\frac{1}{2}, \pi \otimes \chi_2\right)}{L(1, \pi, \text{Ad})}. \quad (5.2)$$

Lemma 5.3 Consider the Eichler order $R = \left(\begin{smallmatrix} 0 & p \\ 0 & 0 \end{smallmatrix}\right)$ of level $p$ in $M_2(\mathcal{o})$. When $f = \text{vol}(R^\times)^{-1}1_{R^\times}$, we have

$$\tilde{J}_\pi(f) = \frac{L(2, 1_F)}{L(1, 1_F)} \left(\frac{L\left(\frac{1}{2}, \pi \otimes \chi_1\right)L\left(\frac{1}{2}, \pi \otimes \chi_2\right)}{L(1, \pi, \text{Ad})} + \frac{1}{L(1, 1_F)}\right).$$

Proof Let $W_0$ be the Whittaker new vector in $\mathcal{W}$ of norm 1 which is invariant under $GL_2(\mathcal{o})$. Then $\pi(f)$ acts as orthogonal projection onto the subspace $V$ of $\mathcal{W}$ spanned by $W_0$ and $W_1 = \pi(\begin{smallmatrix} \sigma & 0 \\ 0 & 1 \end{smallmatrix}) W_0$. From [8, Section 8B], an orthonormal basis for $V$ is given by $W_1$ and $W_2$, where

$$W_2 = \left(\frac{L(1, \pi, \text{Ad})L(1, 1_F)}{L(2, 1_F)^2}\right)^{\frac{1}{2}} \left(W_0 - \frac{\mu(\sigma) + \mu(\sigma)^{-1}}{q^{\frac{1}{2}}(1 + q^{-1})}W_1\right).$$

Hence $\tilde{J}_\pi(f) = |\ell(W_1)|^2 + |\ell(W_2)|^2$. It is well known that

$$\ell(W_0) = \left(\frac{L(2, 1_F)}{L(1, \pi, \text{Ad})L(1, 1_F)}\right)^{\frac{1}{2}} L\left(\frac{1}{2}, \pi \otimes \chi_1\right),$$

$$\ell(W_2) = \left(\frac{L(2, 1_F)}{L(1, \pi, \text{Ad})L(1, 1_F)}\right)^{\frac{1}{2}} L\left(\frac{1}{2}, \pi \otimes \chi_2\right).$$
and we see that \( \ell(W_1) = \chi_1(\varpi)\ell(W_0) \) and
\[
|\ell(W_2)|^2 = |1 - \chi_1(\varpi)(\mu(\varpi) + \mu(\varpi)^{-1})q^{-1/2} + q^{-1}|^2 \frac{L(1, \pi, \text{Ad})}{L(1, 1_F)} |\ell(W_0)|^2.
\]
This yields the lemma. \( \square \)

### 5.4 \( L \)-values and periods

For the rest of this section, assume that \( K_v/F_v \) is split for each \( v | \mathfrak{M} \), i.e., \( K/F \) is \((\mathfrak{M}_1, \mathfrak{M}_2, \mathfrak{M})\)-admissible.

Suppose \( \pi \) is a cuspidal representation of \( B^\times(\mathbb{A}_F) \) or \( \text{GL}_2(\mathbb{A}_F) \). Denote by \( S(\pi) \) the set of finite places \( v \) at which \( \pi \) is ramified and \( S(\pi, K) \) the subset of \( v \in S(\pi) \) such that \( K_v/F_v \) is ramified. Denote by \( \eta = \eta_{K/F} \) the quadratic character of \( \mathbb{A}_F^\times \) associated to \( K/F \) by class field theory. Recall \( u_K = [\sigma_K^\times : \sigma_F^\times] \). Put
\[
C(K; \mathfrak{M}) = 2^{#S(\pi, K)-1} h_F D_F^{-2} u_K^2 |D_K|^1/2 \prod_{v|\mathfrak{M}} \frac{1}{1 + q_v^{-1}}.
\] (5.3)

By our assumption that \( K_v/F_v \) is split at all \( v | \mathfrak{M} \), we see that \( S(\pi, K) \) and thus \( C(K; \mathfrak{M}) \) does not actually depend upon \( \pi \).

We also set \( \Lambda_{\mathfrak{M}}(\pi, \chi) = \prod_{v|\mathfrak{M}} \Lambda_v(\pi, \chi) \), where
\[
\Lambda_v(\pi, \chi) = \begin{cases} 
\frac{1}{1 + q_v^{-1}} & (1 + (1 - q_v^{-1}) \frac{L(1, \pi_v, \text{Ad})}{L(1, \pi_v, \text{Ad})}) \quad v|\mathfrak{M}, v \notin S(\pi) \\
1 & v|\mathfrak{M}, v \in S(\pi).
\end{cases}
\] (5.4)

**Proposition 5.4** Let \( \pi \) be a cuspidal representation of \( B^\times(\mathbb{A}_F) \) occurring in \( S(\mathcal{O}) \), and \( \Phi_{\pi} \) be an orthogonal basis for \( \pi \hat{\otimes} \mathcal{O}^\times \subset S(\mathcal{O}) \). Let \( \chi \in \hat{\mathcal{O}}(K) \) be such that \( \text{Hom}_{\mathbb{A}_K^\times}(\pi, \chi) \neq 0 \). Then
\[
N(\mathfrak{M}) \sum_{\varphi \in \Phi_{\pi}} |P_{K, \chi}(\varphi)|^2 = C(K; \mathfrak{M}) \Lambda_{\mathfrak{M}}(\pi, \chi) \frac{L\left(\frac{1}{2}, \pi_K \otimes \chi\right)}{L(1, \pi, \text{Ad})}. \] (5.5)

**Proof** Let \( \mathfrak{M}' = c(\pi) \), and let \( \mathcal{O}' \) be a special order of level type \((\mathfrak{M}_1', \mathfrak{M}_2', \mathfrak{M}')\) such that \( \mathfrak{M}' = \mathfrak{M}_1' \mathfrak{M}_2' \mathfrak{M}' \) and \( \mathcal{O}' \supset \mathcal{O} \).

Take for measures on \( B^\times(\mathbb{A}_F), \mathbb{A}_K^\times \) and \( \mathbb{A}_F^\times \) the product of the local Tamagawa measures. Consider a test function \( f = \prod f_v \in C^\infty_c(B^\times(\mathbb{A}_F)) \) where \( f_v \) is the characteristic function of \((\mathcal{O}_v')^\times\) divided by its volume for \( v < \infty \). For \( v|\infty \), choose \( f_v \) so that \( \int_{B^\times(F_v)} f_v(g) dg = 1 \). Then \( \pi(f) \) acts as orthogonal projection onto \( S(\mathcal{O}') \cap \pi \). For \( \varphi, \varphi' \in S(\mathcal{O}) \), put
\[
(\varphi, \varphi')_{\text{Tam}} = \int_{\mathbb{A}_F^\times \setminus B^\times(\mathbb{A}_F)} \varphi(x) \overline{\varphi'(x)} dx.
\]
Since \( \mathbb{A}_F^\times \mathbb{A}_K^\times \mathbb{A}_F^\times(\mathbb{A}_F) \) has Tamagawa volume 2, we see \( (\varphi, \varphi')_{\text{Tam}} = \frac{2}{m(\mathcal{O})} (\varphi, \varphi') \).

Let \( S_0(\pi) = S(\pi, K) \cup \{ v \in S(\pi) : c(\pi_v) \geq 2 \} \). Then by the proof of [17, Theorem 4.1] (see [8, Theorem 1.1] for a formulation consistent with our present choice of measures), we have the formula
\[
J_\pi(f) = \frac{2^{#S(\pi, K)-1}}{\pi[F:Q]} \sqrt{|D_F| |D_K|} L(\pi)(1, \eta) L(S(\pi)(2, 1_F) \frac{L(S_0(\pi))\left(\frac{1}{2}, \pi_K \otimes \chi\right)}{L(S_0(\pi))(1, \pi, \text{Ad})}.
\]
where \( J_\pi (f) \) is a certain spectral distribution, which for our choice of test function \( f \) is given by,

\[
J_\pi (f) = \frac{m(\mathcal{O})}{2} \left( \frac{\text{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} \right)^2 \sum_{\varphi} |P_{K,\chi}(\varphi)|^2, \tag{5.6}
\]

and \( \varphi \) runs over an orthogonal basis \( \Phi' \) for \( \pi (\hat{\mathcal{O}})^\times \). Note

\[
\frac{\text{vol}(K^\times \mathbb{A}_F^\times \backslash \mathbb{A}_K^\times)}{h_K} = 2L(1, \eta) = 2^{[F:Q]} \sqrt{D_F} h_K \sqrt{|D_K^\times|}.
\]

Putting everything together and using the identity

\[
\frac{L(2, 1_F)}{|\zeta_F(-1)|} = \frac{(2\pi)^{[F:Q]}}{D_F^{3/2}}
\]

shows that \( N(\mathfrak{M}) \sum_{\varphi \in \Phi'} |P_{K,\chi}(\varphi)|^2 \), equals

\[
C(K; \mathfrak{M}) \cdot \prod_{v \in S(\pi, K)} (1 + q_v^{-1}) \cdot \prod_{v \mid \mathfrak{M}(\mathfrak{M}')^{-1}} \frac{1}{1 + q_v^{-1}} \cdot \frac{L_{\mathfrak{S}_0}(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \text{Ad})}. \tag{5.7}
\]

By Lemma 5.2, we may rewrite this as

\[
N(\mathfrak{M}) \sum_{\varphi \in \Phi'} |P_{K,\chi}(\varphi)|^2 = C(K; \mathfrak{M}) \cdot \prod_{v \mid \mathfrak{M}(\mathfrak{M}')^{-1}} \frac{1}{1 + q_v^{-1}} \cdot \frac{L \left( \frac{1}{2}, \pi_K \otimes \chi \right)}{L(1, \pi, \text{Ad})}. \tag{5.7}
\]

Now we need to relate the sum of square periods over \( \Phi' \) to one over \( \Phi_\pi \).

By [16], we have \( \pi (\hat{\mathcal{O}})^\times = \pi (\hat{\mathcal{O}}')^\times \), where \( \mathcal{O}' \) is a special order of level type \( (\mathfrak{M}_1', \mathfrak{M}_2', \mathfrak{M}) \) lying between \( \mathcal{O}' \) and \( \mathcal{O} \). Now we take a test function \( f' = \prod f_v' \in C_c^\infty (B^\times (\mathbb{A}_F)) \) such that \( f_v' = f_v \) for \( v \mid \mathfrak{M}(\mathfrak{M}')^{-1} \) and \( f_v' \) is the characteristic function of \( (\mathcal{O}_v')^\times = \mathcal{O}_v^\times \) divided by its volume for \( v \mid \mathfrak{M}(\mathfrak{M}')^{-1} \). Then, as in [8, Section 8B], we can write

\[
J_\pi (f') = J_\pi (f) \cdot \prod_{v \mid \mathfrak{M}(\mathfrak{M}')^{-1}} \frac{\tilde{j}_{\pi,v}(f'_v)}{\tilde{j}_{\pi,v}(f_v)},
\]

where \( J_\pi (f') \) is given by (5.6) but now with \( \varphi \) running over \( \Phi_\pi \). Now comparing (5.2) with Lemma 5.3 shows that for \( v \mid \mathfrak{M}(\mathfrak{M}')^{-1} \),

\[
\frac{\tilde{j}_{\pi,v}(f'_v)}{\tilde{j}_{\pi,v}(f_v)} = 1 + (1 - q_v^{-1}) \cdot \frac{L(1, \pi_v, \text{Ad})}{L \left( \frac{1}{2}, \pi_{K_v} \otimes \chi \right)}.
\]

Combining this with (5.7) gives the proposition. \( \square \)

### 5.5 Average L-values

Let \( \mathcal{F}(\mathfrak{M}) \) denote the set of holomorphic parallel weight 2 cuspidal automorphic representations \( \pi \) of \( \text{GL}_2 (\mathbb{A}_F) \) with trivial central character such that (i) \( c(\pi) \mid \mathfrak{M} \); (ii) \( c(\pi_v) \) is odd for \( v \mid \mathfrak{M}_1 \); and (iii) \( \pi_v \) is a discrete series representation for \( v \mid \mathfrak{M}_2 \). For \( \pi \in \mathcal{F}(\mathfrak{M}) \), let \( \pi_B \) be the corresponding automorphic representation of \( B^\times (\mathbb{A}_F) \). For a character \( \chi \) of \( \text{Cl}(K) \), denote by \( \mathcal{F}(\mathfrak{M}; \chi) \) the subset of \( \pi \in \mathcal{F}(\mathfrak{M}) \) such that \( \text{Hom}_{\mathbb{A}_F} (\pi_B, \chi) \neq 0 \). Thus by Proposition 5.1, \( \square \)
\( \pi \in \mathcal{F}(\mathfrak{N}; \chi) \) if and only if, for any place \( v \mid c(\pi) \) such that \( K_v/F_v \) is ramified (so \( v|\mathfrak{N}_1\mathfrak{N}_2 \) and \( \pi_v \simeq \text{St}_v \otimes \mu_v \)), we have \( \nu_v(\sigma_{F,v}) = \chi_v(\sigma_K,v) \).

**Theorem 5.5** Let \( \mathfrak{m} \) be a nonzero integral ideal of \( \mathfrak{o}_F \) which is coprime to \( \mathfrak{D}_B^{-1} \). Then

\[
C(K; \mathfrak{N}) \sum_{\chi \in \text{Cl}(K)} \sum_{\pi \in \mathcal{F}(\mathfrak{N}; \chi)} \lambda_{\mathfrak{m}}(\pi) \Lambda_{\mathfrak{N}}(\pi, \chi) \frac{L(\frac{1}{2}, \pi_K \otimes \chi)}{L(1, \pi, \text{Ad})} = N(\mathfrak{N})h_K \left( (c_K, a(m)) - \delta^+(\mathfrak{O}, m) \deg T_m \frac{h_K|\text{Cl}^+(N(\mathfrak{O}))|}{m(\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N})} \right).
\]

**Proof** Since the periods \( P_{K,\chi} \) automatically vanish on \( \pi_B \) if \( \text{Hom}_{\mathfrak{h}_K^*}(\pi_B, \chi) = 0 \), this theorem follows directly from Proposition 5.4, Corollary 4.2 and Theorem 3.5.

**Remark 5.6** By the \( \varepsilon \)-factor criteria of Tunnell and Saito discussed earlier, note that \( \pi \in \mathcal{F}(\mathfrak{N}; \chi) \) implies that \( \pi_K \otimes \chi \) has root number \( +1 \). (The converse is not typically true.) Thus the restriction to such \( \pi \) precludes one trivial reason for \( L(\frac{1}{2}, \pi_K \otimes \chi) \) to vanish, the root number over \( K \) being \( -1 \). There is also a trivial way for this central value to vanish if the root number of \( K \) is \( +1 \)—if \( L(s, \pi_K \otimes \chi) = L(s, \tau_1)L(s, \tau_2) \) factors as two degree 2 L-functions over \( F \) with root number \( -1 \). For instance, if \( \chi \) is trivial, then \( L(s, \pi_K) = L(s, \pi)L(s, \pi \otimes \eta_K/F) \) so \( L(\frac{1}{2}, \pi_K) \) is forced to vanish if \( \pi \) has root number \( -1 \). See also [18, Remark 6.10] regarding a factorization when \( \chi \) is quadratic.

As in Sect. 4, this double average formula simplifies in a variety of situations, can be used to obtain bounds not involving \( \mathfrak{O} \), and yields exact bounds on single averages. Under suitable conditions, we can also pick out the average restricted to newforms. For simplicity, we only carry this out over \( \mathbb{Q} \) below.

### 5.6 Consequences over \( \mathbb{Q} \)

We now assume \( F = \mathbb{Q} \), and explain how to deduce the double average value formula Theorem 1.3 from Theorem 5.5. Write \( N, N_s, M, D_B \) in place of \( \mathfrak{N}, \mathfrak{N}_s, \mathfrak{N}, \mathfrak{D}_B \), where the roman types now represent positive generators of their fraktur counterparts. If \( \pi \) is an automorphic representation corresponding to a weight 2 newform \( f \), we also write the factors \( \Lambda_*(\pi, \chi) \) as \( \Lambda_*(f, \chi) \).

**Proof of Theorem 1.3** Let \( f \in \mathcal{F}(N; \chi) \) be a newform level \( N_f | N \) and \( \pi \) be the associated cuspidal representation of \( \text{GL}_2(A_F) \). To translate our automorphic L-values to classical quantities first note that by [10, Theorem 5.1], we have

\[
L_{\text{fin}}^{S_2}(1, \pi, \text{Ad}) = \frac{8\pi^3}{N_f} \Delta(f, f),
\]

where \( (f, f) = (f, f)_{N_f} = \int_{\Gamma_0(N_f)\backslash \mathbb{H}} |f(x + iy)|^2 \, dx \, dy \) denotes the usual Petersson norm on \( X_0(N_f) \), and \( S_2(\pi) \) denotes the set of finite primes \( p \) where \( c(\pi_p) \geq 2 \). From the calculations in the proof of Lemma 5.2, we have

\[
L_{\text{fin}}(1, \pi, \text{Ad}) = \Lambda_{N_2}(f) L_{\text{fin}}^{S_2}(1, \pi, \text{Ad}),
\]
where $\Lambda_{N_2}(f) = \prod_{p \mid N_2} \Lambda_p(f)$ and

$$\Lambda_p(f) = \Lambda_p(f, \chi) = \begin{cases} 1 + p^{-1} & p \mid N_2, \pi_p \text{ supercuspidal}, \\ 1 - p^{-2} & p \mid N_2, \pi_p \text{ ramified twist of } St_p, \\ 1 & p \mid N_2, \pi_p \text{ unramified twist of } St_p. \end{cases} \quad (5.8)$$

Also note that for $v \mid \infty$,

$$\frac{L\left(\frac{1}{2}, \pi_K, \chi \otimes \chi_v \right)}{L(1, \pi_v, \text{Ad})} = 2\pi.$$  

Thus

$$\frac{L\left(1, \pi_K, \chi \otimes \chi \right)}{L(1, \pi, \text{Ad})} = \frac{N_f}{4\pi^2} \Lambda_{N_2}(f) \frac{L_{\text{fin}}\left(\frac{1}{2}, f, \chi \right)}{(f, f)}. \quad (5.9)$$

Now Theorem 1.3 follows from Theorem 5.5 together with Lemma 3.3. \qed

In the above proof, (5.9) means we can rewrite (5.5) as

$$\sum_{\varphi \in \Phi_\pi} \frac{|P_{K,\chi}(\varphi)|^2}{(\varphi, \varphi)} = \frac{C(K; N)}{4\pi^2} \Lambda_N(f, \chi) \frac{N_f}{N} \frac{L_{\text{fin}}\left(\frac{1}{2}, f, \chi \right)}{(f, f)}. \quad (5.10)$$

For some of our remaining results on $L$-value averages, we will use this together with the results of Sect. 4.

For explicit calculations for later use, we note that we can express the factors $\Lambda_p(f, \chi)$ for $p \mid M$, $p \nmid N_f$ from (5.4) in terms of Fourier coefficients $a_p(f)$ as

$$\Lambda_p(f, \chi) = \frac{1}{1 + p^{-1}} \left(1 + \frac{|1 + \chi(p)(\chi(p) - a_p(f))p^{-1}|^2}{2 + 2p^{-1} + (1 - a_p(f))^2} \right) \quad (for \ p \mid M, \ p \nmid N_f),$$

where $p$ denotes a prime of $K$ above $p$.

**Lemma 5.7** Let $O \subset B$ be a special order of nice level type $(N_1, N_2, M)$, and set $N = N_1N_2M$. Then $\#O^\times \leq 6$ unless $N \leq 4$.

**Proof** Suppose $\#O^\times > 6$. Note that if $O' \supset O$, then $\#(O')^\times \geq \#O^\times$. Moreover $\#O_B^\times \leq 6$ for every maximal order $O_B \subset B$ unless $D_B \in \{2, 3\}$, in which case $O_B$ is unique up to isomorphism. When $D_B = 2$, $\#O_B^\times = 24$, and when $D_B = 3$, $\#O_B^\times = 12$. In both cases, the $\mathbb{Z}$-order in $B$ generated by $O_B^\times$ is all of $O_B$. Hence the only possibility with $N > 3$ is that $D_B = 2$ and $O$ is a non-maximal order in $B$. Assume this.

Then $O^\times$ is a proper subgroup of $O_B^\times \simeq \text{SL}_2(\mathbb{F}_3)$ of order greater than 6, and the only possibility is $\#O^\times = 8$. Necessarily $\mathbb{Z}[i]$ embeds in the genus of $O$, so by Lemma 4.7 we must have that $N_1N_2$ is 2 or 4. If $N = 4$, then there is a unique order up to isomorphism and $\#O^\times = 8$ does occur.

Suppose $N > 4$. Then $M > 1$, and by enlarging $O$ if necessary we may suppose $M = p$ and $N = 2p$. Again by Lemma 4.7, we must have $p \equiv 1 \text{ mod } 4$. From the mass formula (Lemma 3.2), $m(O) = \frac{p+1}{12}$. By the class number formula, $h(O) = m(O) + \frac{7}{6}$ or $h(O) = m(O) + \frac{1}{2}$, depending on whether $p \equiv 1 \text{ mod } 3$ or $p \equiv 2 \text{ mod } 3$. Correspondingly $\sum(1 - \frac{1}{w_i})$ is either $\frac{7}{6}$ or $\frac{1}{2}$. In the latter case, there must be exactly one $w_i > 1$, and it must be 2, which implies $\#O^\times \leq 4$. So consider the former case. Then there must be exactly two $w_i$'s such that $w_i > 1$, and their values must be 2 and 3. In particular, $w_i = 4$ is impossible. \qed
Lemma 5.8 Suppose \( D_B > |D_K| \) with \( \gcd(D_B, D_K) = 1 \). Then for any order \( \mathcal{O} \subset B \) containing \( \mathfrak{o}_K \), the pair \((\mathcal{O}, K)\) lies in the stable range.

Proof The stable average value formula [7, Theorem 6.10] together with Corollary 4.5 implies that \((O_B, K)\) is in the stable range for any maximal order \( O_B \) containing \( \mathfrak{o}_K \). Now apply Lemma 4.6. \( \square \)

Proof of Theorem 1.4 We want to relate double averages over \( \mathcal{F}(N) \) to double averages of \( \mathcal{F}_0(N) \). We do this by subtracting away the contribution of the \( N_1N_2 \)-oldforms using inclusion-exclusion.

For a subset \( \Sigma \) of primes dividing \( D_B^{-1}N_1N_2 \) at which we want to lower the level, define \( N^\Sigma = N_{1}^\Sigma N_{2}^\Sigma M \) as follows. First put \( N_2^\Sigma = \prod p^{2} \), where \( p \) runs over primes dividing \( N_2 \) such that \( p \notin \Sigma \). Then set \( N_1^\Sigma = \prod p^{\text{ord}_p(N_1)} \prod q^{\text{ord}_q(N_1)-2} \prod r \), where \( p \) runs over the primes dividing \( N_1 \) not in \( \Sigma \), and \( q \) (resp. \( r \)) runs over the primes dividing \( N_1 \) (resp. \( N_2 \)) which are in \( \Sigma \).

Let

\[
A_0(N_1, N_2) = \frac{C(K; N)}{4\pi^2} \sum_{\chi \in \text{Cl}(K)} \sum_{f \in \mathcal{F}_0(N; \chi)} A_f(N_f, \chi) \Lambda_f \frac{L_{f \in \text{fin}}(s, f, \chi)}{(f, f)},
\]

and \( A(N_1, N_2) \) denote the corresponding expression where the sum over \( f \in \mathcal{F}_0(N; \chi) \) is replaced by the sum over \( f \in \mathcal{F}(N; \chi) \). Then

\[
A_0(N_1, N_2) = \sum_{\Sigma} (-1)^{\#\Sigma} \left( \prod_{p|N_2, p \notin \Sigma} \frac{1}{1 + p^{-1}} \right) A(N_1^\Sigma, N_2^\Sigma),
\]

where \( \Sigma \) runs over all subsets of \( \{ p : p|D_B^{-1}N_1N_2 \} \). It follows from Lemmas 4.6, 5.7 and 5.8, that any of the conditions (1)–(3) in the initial assumption of Theorem 1.4 imply the equality \( \sum h_i = u_K h_K \) for any order \( \mathcal{O} \supset \mathfrak{o}_K \) of nice level type \((N_1^\Sigma, N_2^\Sigma, M)\). Now from Theorem 1.3, we have

\[
A(N_1^\Sigma, N_2^\Sigma) = N^\Sigma h_K^2 (u - 2 \omega(N_2^\Sigma) c(N_1^\Sigma, N_2^\Sigma, M)).
\]

Then the theorem follows from the facts that

\[
\sum_{\Sigma} (-1)^{\#\Sigma} N^\Sigma \left( \prod_{p|N_2, p \notin \Sigma} \frac{1}{1 + p^{-1}} \right) = N \prod_{p|N_1} \left( 1 - \frac{1}{p^2} \right) \prod_{p|N_2} \frac{p}{p + 1},
\]

and

\[
\sum_{\Sigma} (-1)^{\#\Sigma} N^\Sigma \left( \prod_{p|N_2, p \notin \Sigma} \frac{1}{1 + p^{-1}} \right) 2^{\omega(N_2^\Sigma)} c(N_1^\Sigma, N_2^\Sigma, M)
\]

is 0 unless \( N_1' = 1 \) and \( N_2 \) is odd, in which case it is

\[
N c(N_1, N_2, M) = 12 \prod_{p|N_1N_2} \frac{1}{1 - p^{-1}} \prod_{p|N_2M} \frac{1}{1 + p^{-1}}.
\]

\( \square \)
Proof of Corollary 1.5 Keeping the notation of the previous proof, define \( A^x(N_1) \) and \( A^x_0(N_1) \) to be the “\( \chi \)-parts” of \( A(N_1, 1) \) and \( A_0(N_1, 1) \)—i.e., the analogous expressions with only a single sum over \( f \) in \( \mathcal{F}(N; \chi) \) or \( \mathcal{F}_0(N; \chi) \) for a fixed \( \chi \). Then the formula follows by the calculations from the previous proof. 

By (5.10) and Corollary 4.5, in the stable range we have

\[
A^x(N_1) = \sum_{\Sigma} (-1)^{\# \Sigma} A^x_1(N_1^{\Sigma})
\]

Then the formula follows by the calculations from the previous proof. □

5.7 Lower bounds and effective nonvanishing

We keep to the situation of \( F = \mathbb{Q} \) and obtain exact lower bounds for single averages of \( L \)-values in special situations. In particular, this will give the effective nonvanishing results stated in the introduction.

We first remark that, when \( N = N_1 > 4 \) is squarefree, (5.10) and Corollary 4.5 imply

\[
\frac{C(K; N)}{4\pi^2} \sum_{f \in \mathcal{F}_{\text{new}}(N)} \frac{L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right)}{(f, f)} = \sum_{i=1}^{n} w_i h_i - h_K^2 c(N_1, 1, 1) 
\geq h_K (u_K - \frac{12h_K}{\varphi(N)}).
\] (5.12)

Here \( \varphi(N) \) is the Euler totient function. In particular, we get that \( L \left( \frac{1}{2}, f \right) L \left( \frac{1}{2}, f \otimes \eta_K \right) \neq 0 \) for some \( f \in \mathcal{S}_{\text{new}}(N) \) as soon as \( \varphi(N) > \frac{12h_K}{u_K} \).

Proposition 5.9 Fix a prime \( p \) and let \( r > 1 \). Let \( N = N_0 p^r \geq 11 \), where \( N_0 \) is a squarefree product of an even number of primes (possibly \( N_0 = 1 \) coprime to \( p \)).

(i) If \( r \) is odd, \( N \neq 27 \) and \( K/\mathbb{Q} \) is \( (N, 1, 1) \)-admissible, we have

\[
\frac{C(K; N)}{4\pi^2} \sum_{f \in \mathcal{F}_{\text{new}}(N)} \frac{L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right)}{(f, f)} 
\geq h_K \left( u_K - \frac{3h_K}{p^2} \right).
\]

(ii) If \( r = 2 \) and \( K/\mathbb{Q} \) is \( (N_0, p^2, 1) \)-admissible, we have

\[
\frac{C(K; N)}{4\pi^2} \sum_{f \in \mathcal{F}_{\text{new}}(N)} \Lambda_{\mathcal{P}}(f) \frac{L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right)}{(f, f)} 
\geq h_K \left( u_K - \frac{3h_K}{p+1} \left( 1 + \frac{4}{(p-1)\varphi(N_0)} \right) \right).
\]

Proof We realize \( N = N_1 N_2 \) with our running conventions, so \( M = 1 \) and \( D_B = N_0 \). Let

\[
A_0(N) = \frac{C(K; N)}{4\pi^2} \sum_{f \in \mathcal{F}_0(N; 1K)} \Lambda_{N_2}(f) N f \frac{L_{\text{fin}} \left( \frac{1}{2}, f, 1K \right)}{(f, f)}.
\]

and \( A(N) \) denote the corresponding sum over \( f \in \mathcal{F}(N; 1K) \). (These are the \( \chi = 1K \) parts of the sums denoted \( A_0(N_1, N_2) \) and \( A(N_1, N_2) \) in the previous section.) Note that \( \mathcal{F}_0(N; 1K) \subset \mathcal{F}_{\text{new}}(N) \), so by non-negativity of central \( L \)-values, the left hand sides above are at least \( \frac{A_0(N)}{N} \). Hence it suffices to show the right hand sides are lower bounds for \( \frac{A_0(N)}{N} \).
We have $A_0(N) = A(N) - \frac{1}{1+p^{-1}} A(N/p)$ or $A_0(N) = A(N) - A(N/p^2)$, according to whether $r = 2$ or $r$ is odd. As in the proof of Theorem 1.4, our restrictions on $N$ imply that, for a special order of level $N/p$ or $N/p^2$, the quantity $\sum w_i h_i \leq 3h_K$. Now the asserted upper bound on $\frac{A_0(N)}{N}$ follows from using Corollary 4.5 for a lower bound on $A(N)$ and upper bounds on $A(N/p)$ and $A(N/p^2)$.

The $N_0 = 1$ case of this proposition, combined with (5.12), yields Corollary 1.2, and an analogue for levels $N = p^2$. E.g., if $K$ is ramified at $p \geq 7$, then one gets a non-vanishing result in level $p^2$ provided $p + 1 > \frac{5h_K}{u_K}$.

**Proposition 5.10** Suppose $N_1 > 3$ is squarefree, $N_2 = 1$ and $M = p > 3$. Assume $K/\mathbb{Q}$ is $(N_1, 1, p)$-admissible. Then

$$C(K; N) = \frac{4\pi}{4\pi^2} \sum_{\mathcal{F}(N; N)} \Lambda_p(f, 1_K) L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right) \geq h_K \left( u_K - \frac{3h_K \Xi(p)}{p} \right),$$

where $\Xi(p) := 2 \left( \frac{1+p^{-1/2}}{1-p^{-1/2}} \right)^2$.

**Proof** Define quantities $A_0(N)$ and $A(N)$ as in the previous proof. By Deligne’s bound $a_p(f) \leq 2\sqrt{p}$ and (5.11), we have

$$\frac{1}{1+p^{-1}} \leq \Lambda_p(f, 1_K) \leq \Xi(p)$$

for $f \in \mathcal{F}(N_1; 1_K)$. Then use $A_0(N) \geq A(N) - \Xi(p)A(N/p)$ and proceed as before. □

Note $\Xi(p)$ is a decreasing function in $p$. In particular, $\Xi(p) < 4$ if $p \geq 13$ and $\Xi(p) < 3$ if $p \geq 31$. Hence one of the $L$-values appearing on the left of Proposition 5.10 must be nonzero if $p \geq \max\{\frac{12h_K}{u_K}, 13\}$ or if $p \geq \max\{\frac{9h_K}{u_K}, 31\}$.

### 6 Examples

In this section, we illustrate our formulas with a few examples when $F = \mathbb{Q}$. In each of examples, we numerically computed the relevant $L$-values to serve as a quality check for our formulas. We performed our calculations using a combination of Magma [2] and Sage [24]. We also used Dan Collins’ Sage code to compute Petersson norms (see [4]).

**Example 6.1** Let $N = D_B = 11$, so $O$ is a maximal order. Then $n = 2$ and there are 2 maximal orders up to isomorphism, one with 4 units and one with 6 units. Order the ideal classes $I_1, I_2$ so that $w_1 = 2$ and $w_2 = 3$. We may write $S(O) = \langle \varphi \rangle$, where $\varphi(x_1) = 2, \varphi(x_2) = -3$. Then $\varphi$ corresponds to the unique newform $f \in S_2(11)$.

Quadratic forms associated to $I_1$ and $I_2$ are $Q_1(x, y, z, w) = \frac{(2x+z)^2+(2y+w)^2+11z^2+11w^2}{4}$ and $Q_2(x, y, z, w) = \frac{(4x+z+2w)^2+(4y+2z-w)^2+11z^2+11w^2}{4}$. By (3.7), the diagonal Brandt matrix entries are $a_{11}(n) = \frac{r_{Q_1}(n)}{4}$ and $a_{22}(n) = \frac{r_{Q_1}(n)}{6}$. Eichler’s work on the basis problem shows $a_n(f) = 3a_{22}(n) - 2a_{11}(n) = \frac{r_{Q_1}(n)-r_{Q_1}(n)}{2}$ for all $n$. Comparing this with the trace of the $p$-th Brandt matrix shows, for $p \neq 11$, we have $a_p(f) = a_{11}(p) + a_{22}(p) - p - 1 = \frac{5r_{Q_1}(p)-12p-12}{8}$.
First suppose $K = \mathbb{Q}(i)$. Then necessarily $h_1 = 1$ and $h_2 = 0$ so $\frac{|P_K(\varphi)|^2}{(\varphi, \varphi)} = \frac{4}{5}$. From (5.10) or Theorem 1.3, we see

$$L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right) = \frac{4\pi^2}{5}. $$

Consider a prime $p \neq 11$. Then by Theorem 1.3 with $m = p$, we have

$$a_p(f) = \frac{5}{4\pi^2} a_p(f) L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right) = \frac{5r_{Q_1}(p) - 12p - 12}{8}. $$

This independently recovers the above formula for $a_p(f)$ which arose from linear relations of theta series.

Next suppose $K = \mathbb{Q}(\sqrt{-15})$ which also has class number 1. One can check that $\varphi_K$ embeds in $O_\ell(I_1)$, so $h_1 = 1$ and $h_2 = 0$. Again we have $\frac{|P_K(\varphi)|^2}{(\varphi, \varphi)} = \frac{4}{5}$. Then (5.10) or Theorem 1.3 implies that

$$C(\mathbb{Q}(\sqrt{-15}), 11)L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_{\mathbb{Q}(\sqrt{-15})} \right) = C(\mathbb{Q}(i), 11)L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_{\mathbb{Q}(i)} \right).$$

Hence

$$\frac{L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_{\mathbb{Q}(\sqrt{-15})} \right)}{L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_{\mathbb{Q}(i)} \right)} = \frac{4}{\sqrt{15}},$$

which agrees with numerical calculations. This provides a check on our local factors when $K$ is ramified at a prime $p|N_1$.

**Example 6.2** Let $D_B = 11$ and $N = 22$. Then $n = 3$ and we may order the ideal classes so that $w_1 = 2$, $w_2 = w_3 = 1$. We can take an orthogonal basis of $S(\mathcal{O})$ to be $\{\varphi_1, \varphi_2\}$ where $\varphi_1(x_1) = \varphi_1(x_2) = 2$, $\varphi_1(x_3) = -3$ and $\varphi_2(x_1) = 2$, $\varphi_2(x_2) = -1$, $\varphi_2(x_3) = 0$. (There is a maximal order $\mathcal{O}_B \supsetneq \mathcal{O}$ such that as double cosets $\text{Cl}(\mathcal{O}_B) = \{x_1 \cup x_2, x_3\}$ and $\varphi_1$ the old eigenform in $S(\mathcal{O}_B)$ denoted $\varphi$ in the previous example.) Here $S(\mathcal{O})$ is the $\hat{\mathcal{O}}^\times$-invariant subspace of the cuspidal representation $\pi$ of $B^\times(\mathbb{A}_\mathbb{Q})$ corresponding to the newform $f \in S_2(11)$.

Suppose $K = \mathbb{Q}(\sqrt{-15})$ so $h_K = 2$. We compute $h_1 = 0$ and $h_2 = h_3 = 1$. Hence

$$\frac{|P_K(\varphi_1)|^2}{(\varphi_1, \varphi_1)} + \frac{|P_K(\varphi_2)|^2}{(\varphi_2, \varphi_2)} = \frac{1}{15} + \frac{1}{3} = \frac{2}{5}. $$

Since $a_2(f) = -2$, we see $A_2(f, 1_K) = 4$ from (5.11). Then (5.10) tells us that

$$L_{\text{fin}} \left( \frac{1}{2}, f \right) L_{\text{fin}} \left( \frac{1}{2}, f \otimes \eta_K \right) = \frac{8\pi^2}{5\sqrt{15}}. $$

If $\chi$ is the nontrivial character of $\text{Cl}(K)$, we see

$$\frac{|P_{K, \chi}(\varphi_1)|^2}{(\varphi_1, \varphi_1)} + \frac{|P_{K, \chi}(\varphi_2)|^2}{(\varphi_2, \varphi_2)} = \frac{25}{15} + \frac{1}{3} = 2. $$

Since the Hilbert class field $H_K$ of $K$ is unramified at a prime $p_2$ of $K$ lying above 2, we see $\chi(p_2) = -1$, and get $A_2(f, \chi) = \frac{4}{5}$. Thus (5.10) tells us that

$$L_{\text{fin}} \left( \frac{1}{2}, f, \chi \right) = \frac{8\sqrt{15}\pi^2}{3}. $$
We numerically verified both of these \( L \)-values in Magma in terms of elliptic curve base change \( L \)-values. Namely, if \( E \) is an elliptic curve of conductor 11 associated to \( f \), then
\[
L\left(\frac{1}{2}, f, 1\right) = L\left(\frac{1}{2}, E_K\right), \text{ and } L_{\text{fin}}\left(\frac{1}{2}, f, \chi\right) = L\left(\frac{1}{2}, E_{H_K}\right)/L\left(\frac{1}{2}, E_K\right).
\]

While we could have obtained these \( L \)-values using the maximal order \( O_B \), this example provides a check on our formulas (5.4), (5.11) for the local factors \( \Lambda_p(f, \chi) \) when \( p|\mathcal{M} \).

**Example 6.3** Let \( N = 27 \), so \( DB = 3 \). Since \( \dim S_2(27) = 1 \), we have \( n = 2 \). We also have \( m(\mathcal{O}) = \frac{3}{2} \), so we can order \( \text{Cl}(\mathcal{O}) \) such that \( w_1 = 2, w_2 = 1 \). Thus \( S(\mathcal{O}) \) is spanned by the function \( \varphi \) given by \( \varphi(x_1) = 2, \varphi(x_2) = -1 \), and \( \varphi \) corresponds to the unique newform \( f \in S_2(27) \).

Suppose \( K = \mathbb{Q}(i) \). Then \( h_1 = 1 \) and \( h_2 = 0 \), so \( \frac{1}{2} |_{\mathcal{O}(\varphi, \varphi)}^2 = \frac{4}{3} \). Thus by Theorem 5.5 (or (5.10)) we have
\[
\frac{L_{\text{fin}}\left(\frac{1}{2}, f\right)}{(f, f)} \frac{L_{\text{fin}}\left(\frac{1}{2}, f \otimes \eta_K\right)}{(f, f)} = \frac{4\pi^2}{3},
\]
which matches numerical computations. Note \( N = 27 \) is excluded from Theorem 1.1 and this example shows (1.2) does not hold when \( N = 27 \).

**Example 6.4** Let \( N = 75 \) with \( DB = 5 \). This corresponds to the triple \((1, 25, 3)\), which is balanced so each \( w_i = 1 \). Here \( n = 8 \) with \( \dim \text{Eis}(\mathcal{O}) = 2 \). Let \( g \) be the unique newform in \( S_2(15) \). Note \( S_2^{\text{new}}(75) \) is generated by 3 newforms, say \( f_1, f_2, f_3 \). One of them, say \( f_1 \), is a quadratic twist of \( g \), and thus locally a ramified twist of Steinberg at 5. The other two, \( f_2 \) and \( f_3 \), are quadratic twists of each other and minimal at 5, whence locally supercuspidal at 5. We remark that Theorem 3.5 tells us that \( S(\mathcal{O}) \cong \mathbb{C}g \oplus \mathbb{C}f_1 \oplus 2\mathbb{C}f_2 \oplus 2\mathbb{C}f_3 \).

Let \( K = \mathbb{Q}(\sqrt{-5}) \), so \( h_K = 2 \) and \( K \) is split at 3 and ramified at 5. Then Theorem 1.3 says
\[
\frac{5\sqrt{5}}{12\pi^2} \sum_{\chi \in \text{Cl}(K)} \left( \sum_{i=1}^3 \Lambda_5(f_i) \frac{L_{\text{fin}}\left(\frac{1}{2}, f_i, \chi\right)}{(f_i, f_i)} + \frac{1}{5} \frac{L_{\text{fin}}\left(\frac{1}{2}, g, \chi\right)}{(g, g)} \right) = 3.
\]

From (5.8), we see that \( \Lambda_5(f_1) = \frac{24}{25} \) and \( \Lambda_5(f_2) = \Lambda_5(f_3) = \frac{6}{5} \). This agrees with numerical calculations of base change elliptic curve \( L \)-values.

**Acknowledgements** We thank Dan Collins for discussions about computations of Petersson norms. We thank the referee for a careful reading and detailed, thoughtful comments which led to several improvements in the exposition. This work was supported in part by a grant from the Simons Foundation (512927, KM).

**Appendix: Guide to hypotheses**

For the convenience of the reader, we present in Table 1 a summary of our primary assumptions on \( F \), the level type \((\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3)\) or the level \( \mathcal{O} \), and the CM extension \( K/F \) for various global results in this paper. Some “exceptional” hypotheses (e.g., \( N \geq 11 \) and \( N \not= 27 \) in Theorem 1.1, or coprimality of \( DB \) and \( DK \) in Corollary 1.5) are excluded for brevity.

We refer to Sect. 2 for general remarks on notation, with Sect. 3.1 explaining our standing conventions on \((\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3)\). The definitions of nice level types and admissibility of \( K \) are given in Sect. 1.2 when \( F = \mathbb{Q} \), and in Sect. 4.3 in general. The notation \( (P) \) in the bottom entries of the table refers to a parity condition on the number of primes dividing \( N_0 \) or \( N_1 \).
### Table 1 Table of hypotheses

| Theorem/Corollary | Condition | Description |
|-------------------|-----------|-------------|
| Theorem 1.1        | $\mathbb{Q}$ | $(p^{2r+1}, 1, 1)$ Non-split at $p$; Inert at $p$ if $r > 0$ |
| Corollary 1.2      | $\mathbb{Q}$ | $(p^{2r+1}, 1, 1)$ Inert at $p$ |
| Theorem 1.3        | $\mathbb{Q}$ | Nice level type Admissible |
| Theorem 1.4(1)     | $\mathbb{Q}$ | Nice level type, balanced Admissible |
| Theorem 1.4(2)     | $\mathbb{Q}$ | Nice level type Admissible, stable range |
| Theorem 1.4(3)     | $\mathbb{Q}$ | Nice level type Admissible, $u_K > 1$ |
| Corollary 1.5      | $\mathbb{Q}$ | Nice level type with $N_2 = 1$ Admissible, stable range |
| Theorem 3.5        | $\mathbb{Q}$ | $\mathfrak{O}_2$ cubefree |
| Proposition 3.6    | $\mathbb{Q}$ | $\sigma_K \subset \mathfrak{O}$ |
| Proposition 4.1    | $\mathbb{Q}$ | $\mathfrak{O}$ balanced |
| Corollary 4.2      | $\mathbb{Q}$ | $\mathfrak{O}$ unram. quad. |
| Corollary 4.3(i)   | $\mathbb{Q}$ | $h_F^+ = 1$ |
| Corollary 4.3(ii)  | $\mathbb{Q}$ | $h_F = 1$ |
| Proposition 5.1    | $\mathbb{Q}$ | $h_F = 1$ |
| Proposition 5.4    | $\mathbb{Q}$ | $h_F = 1$ |
| Theorem 5.5        | $\mathbb{Q}$ | $N = N_1$ sqfree, $(P)$ (5.12) |
| Proposition 5.9(i) | $\mathbb{Q}$ | $N = p^{2r+1}N_0$, $pN_0$ sqfree, $(P)$ |
| Proposition 5.9(ii)| $\mathbb{Q}$ | $N = p^2N_0$, $pN_0$ sqfree, $(P)$ |
| Proposition 5.10   | $\mathbb{Q}$ | $N = pN_1$ sqfree, $(P)$ |

### References

1. Bennett, M.A., Ellenberg, J.S., Ng, N.C.: The Diophantine equation $A^4 + 2^3B^2 = C^n$. Int. J. Number Theory 6(2), 311–338 (2010)
2. Bosma, W., Cannon, J., Playoust, C.: The Magma algebra system. I. The user language, Computational algebra and number theory (London, 1993). J. Symbolic Comput. 24(3–4), 235–265 (1997)
3. Brzezinski, J.: A combinatorial class number formula. J. Reine Angew. Math. 402, 199–210 (1989)
4. Collins, D.J.: Numerical computation of Petersson inner products and q-expansions. arXiv:1802.09740v1 (Preprint)
5. Dembélé, L., Voight, J.: Explicit methods for Hilbert modular forms, Elliptic curves, Hilbert modular forms and Galois deformations, Adv. Courses Math. CRM Barcelona, pp. 135–198. Birkhäuser, Basel (2013)
6. Ellenberg, J.S.: Galois representations attached to $\mathbb{Q}$-curves and the generalized Fermat equation $A^4 + B^2 = C^n$. Am. J. Math. 126(4), 763–787 (2004)
7. Feigon, B., Whitehouse, D.: Averages of central $L$-values of Hilbert modular forms with an application to subconvexity. Duke Math. J. 149(2), 347–410 (2009)
8. File, D., Martin, K., Pitale, A.: Test vectors and central $L$-values for GL(2). Algebra Number Theory 11(2), 253–318 (2017)
9. Gross, B.H.: Heights and the special values of $L$-series, Number theory, Montreal, Que., (1985), CMS Conf. Proc., vol. 7, Amer. Math. Soc., Providence, RI, pp. 115–187 (1987)
10. Hida, H.: Congruence of cusp forms and special values of their zeta functions. Invent. Math. 63(2), 225–261 (1981)
11. Hijikata, H.: Explicit formula of the traces of Hecke operators for $\Gamma_0(N)$. J. Math. Soc. Jpn. 26, 56–82 (1974)
12. Hijikata, H., Pizer, A., Shemanske, T.: Orders in quaternion algebras. J. Reine Angew. Math. 394, 59–106 (1989)
13. Liu, S.-C., Masri, R., Young, M.P.: Rankin–Selberg $L$-functions and the reduction of CM elliptic curves. Res. Math. Sci. 2, 23 (2015)
14. Martin, K.: The Jacquet–Langlands correspondence, Eisenstein congruences, and integral $L$-values in weight 2. Math. Res. Lett. 24(6), 1775–1795 (2017). Corrected version available at arXiv:1601.03284v4
15. Martin, K.: Congruences for modular forms mod 2 and quaternionic $S$-ideal classes. Can. J. Math. 70(5), 1076–1095 (2018)
16. Martin, K.: The basis problem revisited. Trans. Am. Math. Soc. 373(7), 4523–4559 (2020). https://doi.org/10.1090/tran/8077
17. Kimball, M., David, W.: Central $L$-values and toric periods for $GL(2)$. Int. Math. Res. Not. (1) Art. ID rnm 127, 141–191 (2009)
18. Martin, K., Wiebe, J.: Zeroes of quaternionic modular forms and central $L$-values. J. Number Theory 217, 460–494 (2020)
19. Michel, P., Ramakrishnan, D.: Consequences of the Gross–Zagier Formulae: Stability of Average $L$-Values, Subconvexity, and Non-vanishing mod $p$, Number Theory, Analysis and Geometry, pp. 437–459. Springer, New York (2012)
20. Philippe, M., Akshay, V.: Heegner Points and Non-vanishing of Rankin/Selberg $L$-Functions, Analytic Number Theory, Clay Math. Proc., vol. 7, pp. 169–183. American Mathematical Society, Providence (2007)
21. Nelson, P.D.: Stable averages of central values of Rankin-Selberg $L$-functions: some new variants. J. Number Theory 133(8), 2588–2615 (2013)
22. Nelson, P.D., Pitale, A., Saha, A.: Bounds for Rankin–Selberg integrals and quantum unique ergodicity for powerful levels. J. Am. Math. Soc. 27(1), 147–191 (2014)
23. Pi, Q.: Central values of twisted base change $L$-functions associated to Hilbert modular forms. J. Number Theory 191, 213–245 (2018)
24. The Sage Developers: Sagemath, the Sage Mathematics Software System (Version 8.9) (2019). https://www.sagemath.org
25. Saito, H.: On Tunnell’s formula for characters of $GL(2)$. Compos. Math. 85(1), 99–108 (1993)
26. Sugiyama, S., Tsuzuki, M.: Relative trace formulas and subconvexity estimates for $L$-functions of Hilbert modular forms. Acta Arith. 176(1), 1–63 (2016)
27. Tunnell, J.B.: On the local Langlands conjecture for $GL(2)$. Invent. Math. 46(2), 179–200 (1978)
28. Tunnell, J.B.: Local $\epsilon$-factors and characters of $GL(2)$. Am. J. Math. 105(6), 1277–1307 (1983)
29. John, V.: Quaternion Algebras, Graduate Texts in Mathematics, vol. 288. Springer, Cham (2021)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Springer Nature or its licensor holds exclusive rights to this article under a publishing agreement with the author(s) or other rightsholder(s); author self-archiving of the accepted manuscript version of this article is solely governed by the terms of such publishing agreement and applicable law.