Abstract

Some scientific publications are under suspicion of fabrication of data. Since humans are bad random number generators, there might be some evidential value in favor of fabrication in the statistical results as presented in such papers. In line with Uri Simonsohn (2012, 2013) we study the evidential value of the results of an ANOVA study in favor of the hypothesis of a dependence structure in the underlying data.

1 Evidential Value in Forensic Statistics

At some crime scene a trace has been found that links a suspect to the crime. In the court case the prosecutor puts forward the hypothesis \( H_p \) that the suspect is the donor of the trace. The defendant claims the hypothesis \( H_d \) holds, which states that an unknown person, not the suspect, is the donor of the trace. The juror (judge, jury) has to decide in favor of \( H_p \) or \( H_d \). An important current scientific approach to such criminal court cases is via the so-called Bayesian Paradigm of Forensic Statistics.

Within this paradigm the juror has to construct a prior opinion about \( H_p \) and \( H_d \). This means that the juror has to decide beforehand, before seeing the evidence, how likely the hypothesis of the prosecutor is in comparison to the hypothesis of the defendant. This prior opinion might be based on e.g. the number of possible offenders, and it may be formulated in terms of the prior odds in favor of the hypothesis of the prosecutor, namely

\[ P(H_p) / P(H_d). \]
The evidence in such a court case consists of the trace found at the crime scene and characteristics of the suspect. Let us denote it by $E$. The forensic expert has to determine now the probability that a randomly chosen person would leave a trace like the one found, at the crime scene. This probability is denoted by $P(E \mid H_d)$. Likewise he has to determine $P(E \mid H_p)$, the probability that the suspect would leave a trace like the one found, at the crime scene. The ratio

$$\frac{P(E \mid H_p)}{P(E \mid H_d)}$$

is called the likelihood ratio. Multiplying the prior odds and the likelihood ratio the juror obtains the so-called posterior odds in favor of the hypothesis of the prosecutor

$$\frac{P(H_p \mid E)}{P(H_d \mid E)},$$

i.e., the odds in favor of $H_p$ after having seen the evidence. The juror has to base his decision on these posterior odds. In summary, the Bayesian Paradigm of Forensic Statistics reads as follows

$$\frac{P(H_p)}{P(H_d)} \cdot \frac{P(E \mid H_p)}{P(E \mid H_d)} = \frac{P(H_p \mid E)}{P(H_d \mid E)}.$$  \hspace{1cm} (1)

The validity of equation (1) may be checked straightforwardly by applying the definition of conditional probability, which is

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $A \cap B$ is the intersection of $A$ and $B$. Since the likelihood ratio in (1) may be interpreted as the weight that the evidence should have in the decision of the juror, it is often called the evidential value in forensic science.

The evidence $E$ is viewed here as a realization of a random mechanism, both under $H_d$ and $H_p$. In case this random mechanism produces outcomes via probability density functions $f(E \mid H_p)$ and $f(E \mid H_d)$, the probabilities in the likelihood ratio or evidential value are replaced by the corresponding probability density functions, resulting in

$$\frac{P(H_p)}{P(H_d)} \cdot \frac{f(E \mid H_p)}{f(E \mid H_d)} = \frac{P(H_p \mid E)}{P(H_d \mid E)}.$$  \hspace{1cm} (2)
2 Modelling Fabrication of Data Underlying an ANOVA Study

In Analysis of Variance the basic assumption is that all observations may be viewed as realizations of independent normally distributed random variables with the same variance $\sigma^2$ and with means that depend on the values of some categorical covariates. Let $I$ be the total number of cells that are defined via these categorical covariates, and let the number of observations per cell be the same, namely $n$. The random variables denoting the observations are then

$$X_{ij} = \mu_i + \varepsilon_{ij}, \quad i = 1, \ldots, I, \ j = 1, \ldots, n. \quad (3)$$

The cell means $\mu_i$ are unknown real numbers, and the measurement errors $\varepsilon_{ij}$ are independent, normally distributed random variables with mean 0 and variance $\sigma^2$.

If authors are fiddling around with data and are fabricating and falsifying data, they tend to underestimate the variation that the data should show due to the randomness within the model. Within the framework of the above ANOVA case, we model this by introducing dependence between the normal random variables $\varepsilon_{ij}$, which represent the measurement errors. Actually, we assume that the measurement errors in any cell have correlation coefficient $\rho$ with respect to the corresponding measurement errors in the other cells. More precisely formulated, we assume that the correlations between the random variables $\varepsilon_{ij}$ no longer all vanish, but satisfy

$$\rho(\varepsilon_{ij}, \varepsilon_{hj}) = \rho, \quad j = 1, \ldots, n, \ 0 \leq i \neq h \leq I, \quad (4)$$

with all other correlations still being equal to 0. In the sequel we restrict attention to nonnegative values of $\rho$ and we exclude $\rho = 1$ for technical reasons, so $0 \leq \rho < 1$. We note that under the standard assumptions of ANOVA $\rho = 0$ holds. Furthermore, we note that within cells observations may be renumbered in order to get the structure (4). Nevertheless, we still assume (3) to hold and the measurement errors to be normally distributed with mean 0 and variance $\sigma^2$.

A way in which fabrication of measurement errors may take place is by copying some of them. This might be modelled as follows. Let $U_j, \ j = 1, \ldots, n$, and $V_{ij}, \ i = 1, \ldots, I, \ j = 1, \ldots, n$, be independent and identically distributed normal random variables with mean 0 and variance $\sigma^2$. Independent of these, let the random indicators $\Delta_{ij}, \ i = 1, \ldots, I, \ j = 1, \ldots, n$, be independent and identically distributed Bernoulli random variables with

$$P(\Delta_{ij} = 1) = \sqrt{\rho} \quad \text{and} \quad P(\Delta_{ij} = 0) = 1 - \sqrt{\rho}. \quad \text{Then}$$

$$\varepsilon_{ij} = \Delta_{ij}U_j + (1 - \Delta_{ij})V_{ij}, \quad i = 1, \ldots, I, \ j = 1, \ldots, n, \quad (5)$$

satisfy (4) and (3). Note that for $0 \leq i \neq h \leq I$ we have $\varepsilon_{ij} = \varepsilon_{hj} = U_j$ with probability $\sqrt{\rho^2} = \rho$ then, and the measurement errors satisfy (4).
Finally, we note that (4) is just one possible way to model dependence, and that the actual way in which fabrication has been implemented, might lead to quite different dependence structures. However, this model will come close to some types of fabrication and falsification.

3 Evidential Value for Fabrication of Data Underlying an ANOVA Study

Consider a study in a scientific research paper. The data in this study are analyzed by ANOVA and presented via the sample averages of the cells and the values of some F-statistics. The underlying data themselves are not published and are not available. The conclusion of this study is that the $I$ cells can be grouped into $K$ groups of $I_k$ cells ($\sum_{k=1}^{K} I_k = I$), such that (possibly after renumbering of the cells) group $k$ consists of cells $i = L_{k-1} + 1, \ldots, L_k$, $k = 1, \ldots, K$, with $0 = L_0 < L_1 < \cdots < L_K = I$, $L_k - L_{k-1} = I_k$, and such that for each group the population cell means are the same, i.e.,

$$\mu_i = \nu_k, \quad i = L_{k-1} + 1, \ldots, L_k, \quad k = 1, \ldots, K,$$

(6)

for some values $\nu_k$, $k = 1, \ldots, K$.

There are two hypotheses to be formulated about the data underlying the ANOVA study. The hypothesis $H_p$ of fabrication of the data underlying the results presented in the paper, is $0 < \rho < 1$. The other hypothesis $H_d$ represents the situation that data have been collected according to (3) with independent $X_{ij}$, i.e., $\rho = 0$. We want to determine the evidential value of the ANOVA study, i.e., of the sample means of the cells and the published F-statistics, with respect to these hypotheses $H_p$ and $H_d$.

To this end we first note that the sample averages in the cells,

$$X_{i.} = \frac{1}{n} \sum_{j=1}^{n} X_{ij}, \quad i = 1, \ldots, I,$$

(7)

have a joint $I$-dimensional multivariate normal distribution. Actually, the dependence structure (4) implies

$$\begin{pmatrix} X_{1.} \\ \vdots \\ X_{I.} \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_I \end{pmatrix}, \sigma^2 n^{-1} \begin{pmatrix} 1 & \rho & \rho & \cdots & \rho \\ \rho & 1 & \rho & \cdots & \rho \\ \rho & \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \rho & \cdots & 1 \end{pmatrix} \right),$$

(8)

In stead of assuming (4), we could have started right away from (8).

Since the inverse of the covariance matrix in (8) equals

$$\frac{n}{\sigma^2 (1-\rho)(1+(I-1)\rho)} \begin{pmatrix} 1 + (I - 2)\rho & -\rho & \cdots & -\rho \\ -\rho & 1 + (I - 2)\rho & \cdots & -\rho \\ \vdots & \vdots & \ddots & \vdots \\ -\rho & -\rho & \cdots & 1 + (I - 2)\rho \end{pmatrix},$$

(9)
and the determinant of \( n\sigma^{-2} \) times this covariance matrix equals

\[
\begin{vmatrix}
1 & \rho & \rho \\
\rho & 1 & \rho \\
\rho & \rho & 1
\end{vmatrix} = \begin{vmatrix}
1 & \rho & \rho \\
\rho - 1 & 1 - \rho & 0 \\
\rho - 1 & 0 & 1 - \rho
\end{vmatrix} = \begin{vmatrix}
1 + (I - 1)\rho & \rho & \rho \\
0 & 1 - \rho & 0 \\
0 & 0 & 1 - \rho
\end{vmatrix} = (1 + (I - 1)\rho)(1 - \rho)^{I-1},
\]

(10)

(8) and (6) entail that the joint density of \( X_1, \ldots, X_I \) equals

\[
\left( \frac{n}{2\pi\sigma^2} \right)^{I/2} \left[ (1 + (I - 1)\rho)(1 - \rho)^{I-1} \right]^{-1/2} \exp \left( -\frac{n}{2\sigma^2} \left[ \frac{1}{1 - \rho} \sum_{k=1}^{K} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \nu_k)^2 \\
- \frac{\rho}{(1 + (I - 1)\rho)(1 - \rho)} \left( \sum_{k=1}^{K} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \nu_k) \right)^2 \right] \right)
\]

(11)

This density depends on the parameters \( \rho, \sigma^2, \nu_1, \ldots, \nu_K \). If the underlying data would be available their mean square error

\[
\frac{1}{I(n - 1)} \sum_{i=1}^{I} \sum_{j=1}^{n} (X_{ij} - X_i)^2
\]

(12)

would be the proper unbiased estimator of \( \sigma^2 \). The distribution of this estimator depends on \( \rho \), but its mean does not. Furthermore, standard ANOVA theory shows that this estimator is independent of the exponent in (11). Since the underlying data are not available, the value of the parameter \( \sigma^2 \) should be retrieved from the values of the F-statistics given. For a method to do this that does not depend on \( \rho \), see the next section. Let us call the resulting estimate \( \hat{\sigma}_n^2 \), and let us denote the density from (11) with \( \sigma \) replaced by \( \hat{\sigma}_n \) by \( f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho) \).

The hypothesis \( H_p \) of fabrication of the data corresponds to the parameter values \( 0 < \rho < 1 \) and \( \nu_1, \ldots, \nu_K \) arbitrary, and the hypothesis \( H_d \) of proper data corresponds to the parameter values \( \rho = 0 \) and \( \nu_1, \ldots, \nu_K \) arbitrary. The evidential value

\[
\frac{f(E \mid H_p)}{f(E \mid H_d)}
\]
from (2) in favor of $H_p$ versus $H_d$ becomes in this case (cf. Zhang (2009), Bickel (2012))

$$V = \frac{\sup_{0<\rho<1, \nu_1, \ldots, \nu_K \in \mathbb{R}} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho)}{\sup_{\nu_1, \ldots, \nu_K \in \mathbb{R}} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, 0)}. \quad (13)$$

Straightforward computation shows that for any $\rho$

$$\sup_{\nu_1, \ldots, \nu_K \in \mathbb{R}} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho) \quad (14)$$

is attained at

$$\nu_k = \bar{X}_k = \frac{1}{L_k - L_{k-1}} \sum_{i=L_{k-1}+1}^{L_k} X_i, \quad k = 1, \ldots, K.$$

This implies that the evidential value from (13) reduces to

$$V = \sup_{0<\rho<1} \chi_n(\rho) \quad (15)$$

with

$$\chi_n(\rho) = \left[ (1 + (I - 1)\rho)(1 - \rho)^{I-1} \right]^{-1/2} \exp \left( -\frac{n\rho}{2\hat{\sigma}^2_n(1 - \rho)} \sum_{k=1}^{K} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \bar{X}_k)^2 \right). \quad (16)$$

We need the additional notation

$$S_n = \frac{n}{I\hat{\sigma}_n^2} \sum_{k=1}^{K} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \bar{X}_k)^2, \quad (17)$$

$$\hat{\rho}_n = \frac{1}{2}(1 - S_n) \left[ 1 + \sqrt{1 - \frac{4S_n}{(I-1)(1 - S_n)^2}} \right]. \quad (18)$$

In Proposition A.1 of the Appendix the following is shown.

- If

$$S_n \geq \frac{\sqrt{I} - 1}{\sqrt{I} + 1}$$

holds, then the evidential value from (13) and (15) reduces to $V = 1$.

- If

$$S_n < \frac{\sqrt{I} - 1}{\sqrt{I} + 1}$$

holds, then $\hat{\rho}_n$ is well-defined and the evidential value from (13) and (15) reduces to

$$V = \max \{ \chi_n(\hat{\rho}_n), 1 \}. \quad (19)$$
4 Estimating $\sigma^2$ from F-Statistics

Table 1 in Stapel, Koomen and Van der Pligt (1996) presents the sample means in a three-way layout ANOVA study with a $3 \times 2 \times 2$ design.

| Prime type          | Positive | Negative | Irrelevant |
|---------------------|----------|----------|------------|
| Impersonal / Memory | 2.3      | 3.5      | 2.9        |
| Impersonal / Impression | 3.4  | 2.5      | 2.9        |
| Personal / Memory   | 3.3      | 2.3      | 2.9        |
| Personal / Impression | 3.5    | 2.5      | 3.0        |

The estimate of the error variance $\sigma^2$ is not given. It should be possible to retrieve this estimate from the value of any F-statistic. On page 441 of ibid. the value of the F-statistic for testing three-way interactions is given, namely, $F(2, 326) = 3.21$. We assume that the 338 observations are approximately uniformly distributed over the 12 cells. This yields an average of 28.17 observations per cell. Applying e.g. Table 4.5.2 (Analysis of Variance of the Three-Way Layout with $M$ Observations per Cell) of Scheffé (1959) we obtain by some computation that the mean square error for interaction equals 7.769. Dividing this by the value 3.21 of the F-statistic we get 2.420 as the mean square for error, i.e., the estimate for $\sigma^2$. However, this is not the value that we would have gotten, would we have used the underlying observations, since the cell means, which are used in the above computation, are given in very low precision.

In an ANOVA of the upper half of Table 1 in ibid. the two way interaction terms are tested by an F-statistic with value $F(2, 164) = 14.28$. By Table 4.3.1 of Scheffé (1959) a similar computation as above yields 1.095 as the value of the mean square for error, based on 170 observations.

Note that a value like 2.3 for a cell mean implies that the actual value of the cell mean lies in the interval $[2.25, 2.35]$. Using this rounding off property we may conclude that the first three F-values given on page 442 of ibid., which have 1 and 164 degrees of freedom, imply that the value of the mean square for error, based on 170 observations, lies in the interval $[0.918, 1.218]$. Note that 1.095 belongs to this interval. Averaging the values of the mean square for error that we get from the last four F-values, we obtain 1.047 as our estimate.

The F-values presented on page 442 of ibid. that are based on the second half of Table 1 of ibid., namely $F(2, 162) = 11.49$ and $F(1, 162) = 23.00$, yield 1.223 and 1.217 as value of the mean square for error, based on the remaining 168 observations. Averaging yields 1.220. Pooling 1.047 and 1.220 we obtain

$$\hat{\sigma}^2_n = 1.134$$

as our final estimate for $\sigma^2$. Note that this deviates considerably from the value 2.42, which has been obtained from the F-value 3.21 for three-way interactions.
interaction. Let us presume here that this is a misprint and that this F-value should have been something like 6.9.

In order to take care of the rounding off of the values of the cell means, we have adapted Table 1 in a direction that increases the double sum in (16) as much as possible and that should decrease the evidential value. The resulting table is given below.

| Prime type           | Positive | Negative | Irrelevant |
|----------------------|----------|----------|------------|
| Impersonal / Memory  | 2.25     | 3.55     | 2.85       |
| Impersonal / Impression | 3.35   | 2.55     | 2.85       |
| Personal / Memory    | 3.25     | 2.25     | 2.85       |
| Personal / Impression | 3.55   | 2.55     | 3.05       |

Analyzing the same F-statistics as above and performing the same computations we see that the F-statistics for interactions yield exactly the same values for the mean square for error. Only the three F-statistics of the type $F(1, 164)$ yield different values. Averaging the four values for the mean square for error that we get out of the four F-values related to the upper half of the table, we arrive at 1.117. The F-statistics for the second half of the table yield the same estimate 1.220. Pooling 1.117 and 1.220 we obtain

$$\hat{\sigma}_n^2 = 1.168$$

as our final estimate for $\sigma^2$ based on our version of Table 1 of ibid.

5 Computing Evidential Value

Let us group the cells of the tables in the preceding section into three groups, namely the groups corresponding to the covariate Prime type with the first two cells in the row Impersonal / Memory interchanged; $I = 12, K = 3, I_1 = I_2 = I_3 = 4$. According to the social psychology theory as put forward in Stapel, Koomen and Van der Pligt (1996), the participants within these groups should have similar mean scores. By (15) through (19) we may compute the evidential value $V$ in favor of the hypothesis $H_p$ that these data have been fabricated in some way resulting in (8) with $0 < \rho < 1$. For the first table from the preceding section, i.e., Table 1 from ibid., this yields

$$V = 56.88$$

and for the second, adapted table from the preceding section this yields

$$V = 1.92.$$
6 Interpreting Evidential Value

With the evidential value $V$ defined as in (15) through (19) the Bayesian paradigm for criminal court cases (2) becomes

$$\frac{P(H_p)}{P(H_d)} \overset{\text{evidential value}}{\underset{\text{prior odds}}{\approx}} \frac{P(H_p|E)}{P(H_d|E)} \overset{\text{posterior odds}}{=} \frac{P(H_p|E)}{P(H_d|E)}.$$  (20)

An important principle in criminal court cases is ‘in dubio pro reo’, which means that in case of doubt the accused is favored. In science one might argue that the leading principle should be ‘in dubio pro scientia’, which should mean that in case of doubt a publication should be withdrawn. Within the framework of this paper this would imply that if the posterior odds in favor of hypothesis $H_p$ of fabrication equal at least 1, then the conclusion should be that $H_p$ is true. So an ANOVA study for which

$$\frac{P(H_p)}{P(H_d)} \overset{\text{evidential value}}{\underset{\text{prior odds}}{\approx}} \frac{P(H_p|E)}{P(H_d|E)} \overset{\text{posterior odds}}{=} \frac{P(H_p|E)}{P(H_d|E)} > 1.$$  (21)

holds, should be rejected and disqualified scientifically.

We conclude with some notes.

- ANOVA studies are based on the assumption of normality. Often this assumption is not satisfied, but the technique is still applied. This is the case in Stapel et al. (1996), since in Table 1 of ibid. the measurements are averages of two 7 point Likert scales, which hardly behave like normal random variables. However, in view of the central limit theorem cell means like in our basic model (8) behave approximately like (jointly multivariate) normal random variables.

- Note that (19) implies $V \geq 1$. Consequently, within this framework there does not exist exculpatory evidence. This is reasonable since bad science cannot be compensated by very good science. It should be very good anyway.

- When a paper contains more than one study based on independent data, then the evidential values of both studies can and may be combined into an overall evidential value by multiplication in order to determine the validity of the whole paper; see the preceding item.

- One may wonder if the way in which the mean square error (12) is retrieved from the values of F-statistics, interferes with the randomness
As mentioned in Section 3, standard ANOVA theory shows that this estimator is independent of the exponent in (11) and hence (13), provided the underlying data have a normal distribution; see also item 1.

7 Evidential Value for Fabrication of Data Underlying an ANOVA Study Based on an Alternative Dependence Structure

In this Section we present an analysis as in Sections 2 and 3 but under a different dependence structure. Given the group structure of the cells as presented in the first paragraph of Section 3 we assume the existence of \( \rho_1, \ldots, \rho_K \in [0, 1] \) such that

\[
\rho(\varepsilon_{ij}, \varepsilon_{hj}) = \rho_k, \quad j = 1, \ldots, n, \quad L_{k-1} + 1 \leq i \neq h \leq L_k, \quad k = 1, \ldots, K,
\]

(22)

hold with all other correlations being equal to 0. This implies independence between different groups of cells. We note that (22) is just another possible way to model dependence, and we note again that the actual way in which fabrication has been implemented, might lead to quite different dependence structures.

We reconsider the ANOVA study presented via the sample averages of the cells and the values of some F-statistics. Again the underlying data themselves are not published and are not available, and the conclusion of this study is given by (6). There are two hypotheses to be formulated about the data underlying the ANOVA study. The hypothesis \( H_p \) of fabrication of the data underlying the results, is that at least one of the \( \rho_k \)s is positive. The other hypothesis \( H_d \) represents the situation that data have been collected according to (3) with independent \( X_{ij} \), i.e., \( \rho_1 = \cdots = \rho_K = 0 \). We want to determine the evidential value of the ANOVA study, i.e., of the sample means of the cells and the published F-statistics, with respect to these hypotheses \( H_p \) and \( H_d \). Here the evidential value is defined analogously to (13) with the supremum taken over \( 0 < \rho_k < 1, \ k = 1, \ldots, K \).

The sample averages in the cells, \( X_i \), from (7), have a joint \( I \)-dimensional multivariate normal distribution with

\[
\begin{pmatrix}
X_{L_{k-1}+1} \\
\vdots \\
X_{L_k}
\end{pmatrix}
\sim \mathcal{N}
\begin{pmatrix}
\nu_k \\
\vdots \\
\nu_k
\end{pmatrix}
, \sigma^2 n^{-1}
\begin{pmatrix}
1 & \rho_k & \cdots & \rho_k \\
\rho_k & 1 & \cdots & \rho_k \\
\vdots & \vdots & \ddots & \vdots \\
\rho_k & \rho_k & \cdots & 1
\end{pmatrix}
\]

(23)

for each \( k = 1, \ldots, K \) and with independence between groups with different
This entails that the joint density of $X_1, \ldots, X_I$ equals

$$
\left( \frac{n}{2\pi \sigma^2} \right)^{I/2} \prod_{k=1}^K \left[ (1 + (I_k - 1)\rho_k)(1 - \rho_k)^{I_k - 1} \right]^{-1/2}
$$

$$
\exp \left( -\frac{n}{2\sigma^2} \sum_{k=1}^K \frac{1}{1 - \rho_k} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \nu_k)^2 \right.
$$

$$
- \frac{\rho_k}{(1 + (I_k - 1)\rho_k)(1 - \rho_k)} \left( \sum_{i=L_{k-1}+1}^{L_k} (X_i - \nu_k) \right)^2 \right) \right).
$$

This density depends on the parameters $\rho_1, \ldots, \rho_K, \sigma^2, \nu_1, \ldots, \nu_K$. Again, we write $\hat{\sigma}_n^2$, for the estimate of $\sigma^2$.

With the notation

$$
\chi_{n,k}(\rho) = \left[ (1 + (I_k - 1)\rho)(1 - \rho)^{I_k - 1} \right]^{-1/2}
$$

$$
\exp \left( -\frac{n\rho}{2\hat{\sigma}_n^2(1 - \rho)} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \bar{X}_k)^2 \right),
$$

$$
S_{n,k} = \frac{n}{I_k \hat{\sigma}_n^2} \sum_{i=L_{k-1}+1}^{L_k} (X_i - \bar{X}_k)^2,
$$

$$
\hat{\rho}_{n,k} = \frac{1}{2} (1 - S_{n,k}) \left[ 1 + \sqrt{1 - \frac{4S_{n,k}}{(I_k - 1)(1 - S_{n,k})^2}} \right] 1_{[S_{n,k} < (\sqrt{I_k - 1})/(\sqrt{I_k} + 1)]}
$$

Proposition A.1 of the Appendix shows

$$
\mathbb{V} = \prod_{k=1}^K \chi_{n,k}(\hat{\rho}_{n,k}).
$$

Computing this evidential value for Table 1 in Stapel, Koomen and Van der Pligt (1996), i.e., for the first table of Section 4, we obtain

$$
\mathbb{V} = 14.49.
$$

The adapted table, namely the second table of Section 4, yields

$$
\mathbb{V} = 1.28.
$$

Here and in [13] we have defined the evidential value in the presence of the nuisance parameters $\nu_1, \ldots, \nu_K$ by replacing these parameters by their maximum likelihood estimators. An alternative approach is to compute the evidential value keeping these parameters fixed, and to subsequently
minimize the resulting evidential value over these nuisance parameters; in
formula
\[ \tilde{V} = \inf_{\nu_1, \ldots, \nu_K \in \mathbb{R}} \sup_{0 < \rho_k < 1, k = 1, \ldots, K} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho_1, \ldots, \rho_K) \]
where \( f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho_1, \ldots, \rho_K) \) is the density as given in (24) with \( \sigma \) replaced by \( \hat{\sigma}_n \). In fact, both definitions of evidential value yield the same value in the situation of this Section 7, as is shown in Theorem B.1.

A Appendix: Analysis \( \chi \) Function

Here we present a proof of the main result of Section 3.

Proposition A.1. In the notation (10) and (17) and for \( I \geq 2 \)
\[ \sup_{0 < \rho < 1} \chi_n(\rho) = 1_{\{S_n \geq (\sqrt{I-1})/\sqrt{I+1}\}} + \max \{\chi_n(\hat{\rho}_n), 1\} 1_{\{S_n < (\sqrt{I-1})/\sqrt{I+1}\}} \] (28)
holds.

Proof
Write \( \psi_n(\rho) = \log(\chi_n(\rho)) \), \( 0 \leq \rho < 1 \), and \( \psi'_n(\rho) \) for its derivative. Some computation shows that
\[ \psi_n(0) = 0, \quad \psi'_n(0) = -\frac{1}{2} IS_n, \] (29)
hold and that \( \psi'_n(\rho) \) is nonnegative on the interval \([0, 1)\) if and only if both
\[ S_n \leq (\sqrt{I-1})/\sqrt{I+1} \]
and
\[ \frac{1}{2}(1 - S_n) \left[ 1 - \sqrt{1 - \frac{4S_n}{(I-1)(1-S_n)^2}} \right] \leq \rho \leq \frac{1}{2}(1 - S_n) \left[ 1 + \sqrt{1 - \frac{4S_n}{(I-1)(1-S_n)^2}} \right] = \hat{\rho}_n \] (30)
hold. Consequently, \( \psi_n(\rho) \) and \( \chi_n(\rho) \) have (local) maxima at \( \rho = 0 \) and \( \rho = \hat{\rho}_n \) on \([0, 1)\). This implies (28). \( \square \)
Appendix: Alternative Definition of Evidential Value

The alternative definition (27) of evidential value yields the same value as (13) for the alternative dependence model as given in Section 7.

Theorem B.1. In the situation of Section 7 the evidential values as defined by (13) and (27) satisfy

\[ \hat{V} = V. \]  

Proof

First we note

\[ \hat{V} = \inf_{\nu_1, \ldots, \nu_K \in \mathbb{R}} \sup_{0 < \rho < 1} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho) \]  

\[ \leq \inf_{\nu_1, \ldots, \nu_K \in \mathbb{R}} \sup_{0 < \rho < 1, \nu_1', \ldots, \nu_K' \in \mathbb{R}} f_n(X_1, \ldots, X_I; \nu_1', \ldots, \nu_K', \rho) \]  

\[ = \sup_{0 < \rho < 1} \sup_{\nu_1, \ldots, \nu_K \in \mathbb{R}} f_n(X_1, \ldots, X_I; \nu_1, \ldots, \nu_K, \rho) = V. \]  

(32)

Subsequently, we note that by the product structure of (24) it suffices to consider the case \( K = 1 \) in proving \( \hat{V} \geq V \). Furthermore, by (24) with \( K = 1 \) we have

\[ \sup_{0 < \rho < 1} \frac{f_n(X_1, \ldots, X_I; \nu, \rho)}{f_n(X_1, \ldots, X_I; \nu, 0)} = \sup_{0 < \rho < 1} \frac{1 + (I - 1)\rho}{1 + (I - 1)\rho} \left( \sum_{i=1}^{I} (X_i - \nu)^2 \right)^{-1/2} \]

\[ \leq \sup_{0 < \rho < 1} \frac{1 + (I - 1)\rho}{1 + (I - 1)\rho} \left( \sum_{i=1}^{I} (X_i - \nu)^2 \right)^{-1/2} \]

\[ \leq \sup_{0 < \rho < 1} \frac{1 + (I - 1)\rho}{1 + (I - 1)\rho} \left( \sum_{i=1}^{I} (X_i - \nu)^2 \right)^{-1/2} = \sup_{0 < \rho < 1} \frac{1 + (I - 1)\rho}{1 + (I - 1)\rho} \left( \sum_{i=1}^{I} (X_i - \nu)^2 \right)^{-1/2} \]

(33)

With the notation \( \bar{X} = I^{-1} \sum_{i=1}^{I} X_i \), we obtain

\[ \sum_{i=1}^{I} (X_i - \nu)^2 - \frac{1}{1 + (I - 1)\rho} \left( \sum_{i=1}^{I} (X_i - \nu)^2 \right)^{2} \]

\[ = \sum_{i=1}^{I} (X_i - \bar{X})^2 + \left( I - \frac{I^2}{1 + (I - 1)\rho} \right) (\bar{X} - \nu)^2 \]

\[ \leq \sum_{i=1}^{I} (X_i - \bar{X})^2 \]

(34)
in view of \( \rho < 1 \). Together with (33) this inequality yields
\[
\tilde{V} \geq \inf_{\nu} \sup_{0<\rho<1} \left[ (1 + (I - 1)\rho)(1 - \rho)^{I-1} \right]^{-1/2} \exp \left( -\frac{n\rho}{2\sigma^2_n(1 - \rho)} \sum_{i=1}^{I} (X_i - \bar{X})^2 \right).
\]  
(35)

Since the infimum over \( \nu \) may be removed from (35), equations (15) and (16) with \( K = 1 \) imply \( \tilde{V} \geq V \), which completes the proof.

\[ \blacksquare \]

C Appendix: \( t \)-Statistic under Dependence

If one would be interested in the distribution of the exponent in (16) or of \( S_n \) from (17), the following lemma would come in handy.

Lemma C.1. Let the correlated standard normal random variables \( Z_1, \ldots, Z_d \) have a joint multivariate normal distribution, namely
\[
Z = \begin{pmatrix} Z_1 \\ \vdots \\ Z_d \end{pmatrix} \sim \mathcal{N} \left( \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & \ddots & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix} \right)
\]  
(36)

with \( 0 \leq \rho < 1 \). Let
\[
\bar{Z}_d = \frac{1}{d} \sum_{i=1}^{d} Z_i, \quad S_d^2 = \frac{1}{d-1} \sum_{i=1}^{d} (Z_i - \bar{Z}_d)^2
\]  
(37)

be their sample mean and sample variance, respectively.

Then \( \bar{Z}_d \) and \( S_d^2 \) are independent, \( \bar{Z}_d \) has a normal distribution with mean 0 and variance \( (1 + (d - 1)\rho)/d \), and \((d - 1)S_d^2/(1 - \rho)\) has a chi squared distribution with \( d - 1 \) degrees of freedom.

Proof

The following classical trick for the case \( \rho = 0 \) also works for positive \( \rho \). Let \( A^T \) be an orthogonal (orthonormal) matrix, the first row of which is the row vector \((d^{-1/2}, \ldots, d^{-1/2})\). Define the column \( d \)-vector \( Y \) by \( Y = A^T Z \), and note
\[
Y_1 = d^{1/2} \bar{Z}_d,
\]
\[
(d - 1)S_d^2 = Z^T Z - d\bar{Z}_d^2 = Y^T A^T A Y - Y_1^2
\]
\[
= Y^T Y - Y_1^2 = \sum_{i=2}^{d} Y_i^2,
\]  
(38)

\( EY = 0 \),
and

\[
\begin{align*}
AT \begin{pmatrix}
1 & \rho & \cdots & \rho \\
\rho & 1 & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & 1 \\
\end{pmatrix} A & = AT \begin{pmatrix}
1 & 0 & & 0 \\
0 & 1 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 1 \\
\end{pmatrix} + \begin{pmatrix}
\rho & \rho & \cdots & \rho \\
\rho & \rho & \cdots & \rho \\
\vdots & \vdots & \ddots & \vdots \\
\rho & \rho & \cdots & \rho \\
\end{pmatrix} A \\
& = (1 - \rho) \begin{pmatrix}
1 & 0 & & 0 \\
0 & 1 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 1 \\
\end{pmatrix} + \begin{pmatrix}
\sqrt{d\rho} & \sqrt{d\rho} & \cdots & \sqrt{d\rho} \\
\sqrt{d\rho} & \sqrt{d\rho} & \cdots & \sqrt{d\rho} \\
\vdots & \vdots & \ddots & \vdots \\
\sqrt{d\rho} & \sqrt{d\rho} & \cdots & \sqrt{d\rho} \\
\end{pmatrix} A \\
& = (1 - \rho) \begin{pmatrix}
1 & 0 & & 0 \\
0 & 1 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 1 \\
\end{pmatrix} + \begin{pmatrix}
d\rho & 0 & & 0 \\
0 & 0 & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 0 \\
\end{pmatrix} A \\
& = \begin{pmatrix}
1 + (d - 1)\rho & 0 & & 0 \\
0 & 1 - \rho & & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & & 1 - \rho \\
\end{pmatrix}, \quad (39)
\end{align*}
\]

where the matrix equalities hold because \(A^T A\) equals the identity matrix and because all row vectors of \(A^T\) are orthogonal to its first row vector \((d^{-1/2}, \ldots, d^{-1/2})\), and hence to all multiples of \((1, \ldots, 1)\). Since \(A\) is the covariance matrix of the multivariate normally distributed vector \(Y\), it follows that \(Y_1, \ldots, Y_d\) are independent, and consequently, that \(\bar{Z}_d\) and \(S_d^2\) are. Finally, (38) and (39) imply that \(Y_2, \ldots, Y_d\) are independent identically distributed with a normal distribution with mean 0 and variance \(1 - \rho\), which yields that \((1 - \rho)^{-1} \sum_{i=2}^{d} Y_i^2\) has a chi squared distribution with \(d - 1\) degrees of freedom. \(\square\)

We note that as a consequence the statistic

\[
\sqrt{\frac{d(1-\rho)}{1 + (d-1)\rho} \frac{\bar{Z}_d}{S_d}}
\]

has a \(t\)-distribution with \(d - 1\) degrees of freedom.

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