DYNAMICS OF THE TEICHMÜLLER FLOW ON COMPACT INVARIANT SETS

URSULA HAMENSTÄDT

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ABSTRACT. Let $S$ be an oriented surface of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$. For a compact subset $K$ of the moduli space of area-one holomorphic quadratic differentials for $S$, let $\delta(K)$ be the asymptotic growth rate of the number of periodic orbits for the Teichmüller flow $\Phi^t$ which are contained in $K$. We relate $\delta(K)$ to the topological entropy of the restriction of $\Phi^t$ to $K$. Moreover, we show that $\sup_K \delta(K) = 6g - 6 + 2m$.

1. INTRODUCTION

An oriented surface $S$ is said to be of finite type if $S$ is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called punctures, have been deleted. We assume that $S$ is nonexceptional, i.e., that $3g - 3 + m \geq 2$. This means that $S$ is not a sphere with at most four punctures or a torus with at most one puncture. Since the Euler characteristic of $S$ is negative, the Teichmüller space $\mathcal{T}(S)$ of $S$ is the quotient of the space of all complete hyperbolic metrics on $S$ of finite volume under the action of the group of diffeomorphisms of $S$ which are isotopic to the identity.

The fiber bundle $\mathcal{Q}^1(S)$ over $\mathcal{T}(S)$ of all holomorphic quadratic differentials of area one can naturally be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the Teichmüller metric. The Teichmüller geodesic flow $\Phi^t$ on $\mathcal{Q}^1(S)$ commutes with the action of the mapping class group Mod(S) of all isotopy classes of orientation-preserving self-homeomorphisms of $S$. Thus this flow descends to a flow on the quotient $\mathcal{Q}^1(S)/\text{Mod}(S)$, again denoted by $\Phi^t$. This quotient is a noncompact orbifold.

In his seminal paper, Veech [17] showed that the asymptotic growth rate of the number of periodic orbits of the Teichmüller flow $\Phi^t$ on $\mathcal{Q}^1(S)/\text{Mod}(S)$ is at least $h = 6g - 6 + 2m$ (we use here a normalization for the Teichmüller flow which differs from the one used by Veech). Recently Eskin and Mirzakhani [6] obtained a sharp counting result: they show that as $r \to \infty$, the number of periodic orbits for $\Phi^t$ of period at most $r$ is asymptotic to $e^{hr}/hr$. An earlier partial result for the Teichmüller flow on the space of abelian differentials is due to Bufetov [4].

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In this note, we are interested in the dynamics of the restriction of the Teichmüller flow to a compact invariant set. For the formulation of our first result, a continuous flow $\Phi^t$ on a compact metric space $(X,d)$ is called expansive if there is a constant $\delta > 0$ with the following property. Let $x \in X$ and let $s: \mathbb{R} \to \mathbb{R}$ be any continuous function with $s(0) = 0$ and $d(\Phi^t(x), \Phi^{s(t)}(x)) < \delta$ for all $t$. If $y \in X$ is such that $d(\Phi^t(x), \Phi^{s(t)}(y)) < \delta$ for all $t$ then $y = \Phi^t(x)$ for some $t \in \mathbb{R}$ [11]. Note that this definition of expansiveness does not depend on the choice of the metric $d$ defining the topology on $X$ so it makes sense to talk about an expansive flow on a compact metrizable space.

Let $\Gamma < \text{Mod}(S)$ be a torsion-free normal subgroup of $\text{Mod}(S)$ of finite index. For example, the subgroup of all elements which act trivially on $\Gamma$ has this property. Define $\hat{\mathcal{D}}(S) = \mathcal{D}^1(S)/\Gamma$. We show

**Theorem 1.** The restriction of the Teichmüller flow to every compact invariant subset $K$ of $\hat{\mathcal{D}}(S)$ is expansive.

Periodic orbits of expansive flows on compact spaces are separated, and their asymptotic growth rate can be related to the topological entropy of the flow.

For a compact $\Phi^t$-invariant subset $K$ of $\hat{\mathcal{D}}(S)$ let $h_{\text{top}}(K)$ be the topological entropy of the restriction of the Teichmüller flow to $K$. For any subset $U$ of $\mathcal{D}(S)$ (or of $\mathcal{D}^1(S)/\text{Mod}(S)$) and for a number $r > 0$ define $n_U(r)$ (or $n_U^r(r)$) to be the cardinality of the set of all periodic orbits for $\Phi^t$ of period at most $r$ which are entirely contained in $U$ (or which intersect $U$). Clearly $n_U^r(r) \geq n_U(r)$ for all $r$. We show

**Theorem 2.** Let $K \subset \hat{\mathcal{D}}(S)$ be a compact $\Phi^t$-invariant topologically transitive set. Then for every open neighborhood $U$ of $K$ we have

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K(r) \leq h_{\text{top}}(K) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_U(r).$$

It is not hard to see that Theorem 1 and Theorem 2 are equally valid for compact invariant subsets of $\mathcal{D}^1(S)/\text{Mod}(S)$. However, we did not find an argument which avoids using some differential geometric properties for the action of the mapping class group on Teichmüller space which is not in the spirit of this paper, so we omit a proof.

By the variational principle, the topological entropy of a flow $\Phi^t$ on a compact space $K$ equals the supremum of the measure-theoretic entropies of all $\Phi^t$-invariant Borel probability measures on $K$. Bufetov and Gurevich [5] showed that the supremum of the topological entropies of the restriction of the Teichmüller flow to the moduli space of abelian differentials is just the measure-theoretic entropy of the invariant probability measure in the Lebesgue measure class, moreover this Lebesgue measure is the unique measure of maximal entropy.

The following counting result implies that the entropy $h$ of the $\Phi^t$-invariant Lebesgue measure on $\mathcal{D}^1(S)/\text{Mod}(S)$ equals the supremum of the topological entropies of the restrictions of the Teichmüller flow to compact invariant sets.
**Theorem 3.**

1. \( \limsup_{r \to \infty} \frac{1}{r} \log n_K^r(r) \leq h \) for every compact subset \( K \) of \( \mathcal{D}^1(S)/\text{Mod}(S) \).
2. For every \( \epsilon > 0 \), there is a compact subset \( K \subset \mathcal{D}^1(S)/\text{Mod}(S) \) (or \( K \subset \hat{\mathcal{D}}(S) \)) such that
   \[
   \liminf_{r \to \infty} \frac{1}{r} \log n_K^r(r) \geq h - \epsilon.
   \]

The first part of Theorem 3 is immediate from the results of Eskin and Mirzakhani [6]; however, the proof given here is short and easy. The lower bound for the growth of periodic orbits which remain in a fixed compact set is the technically most involved part of this work.

The main tool we use for the proofs of the above results is the \textit{curve graph} \( \mathcal{C}(S) \) of \( S \) and the relation between its geometry and the geometry of Teichmüller space. In Section 2 we introduce the curve graph, and we summarize some results from [10] in the form used in the later sections. In Section 3 we investigate the Teichmüller flow \( \Phi^t \) on \( \hat{\mathcal{D}}(S) \) and we show Theorem 1. In Section 4 we use the results from Section 3 to establish a version of the Anosov Closing Lemma for the restriction of the Teichmüller flow to compact invariant subsets of \( \hat{\mathcal{D}}(S) \) and show Theorem 2. The proof of Theorem 3 is contained in Section 5.

2. **The Curve Graph and Its Boundary**

Let \( S \) be an oriented surface of genus \( g \geq 0 \) with \( m \geq 0 \) punctures and \( 3g - 3 + m \geq 2 \). The \textit{curve graph} \( \mathcal{C}(S) \) of \( S \) is the graph whose vertices are the free homotopy classes of \textit{essential} simple closed curves on \( S \), i.e., simple closed curves which are neither contractible nor freely homotopic into a puncture. Two such curves are joined by an edge if and only if they can be realized disjointly. Since \( 3g - 3 + m \geq 2 \) by assumption, \( \mathcal{C}(S) \) is connected (see [16]). However, the curve graph is locally infinite. In the sequel we often do not distinguish between a simple closed curve on \( S \) and its free homotopy class. Also, if we write \( \alpha \in \mathcal{C}(S) \) then we always mean that \( \alpha \) is an essential simple closed curve, i.e., \( \alpha \) is a vertex in the curve graph \( \mathcal{C}(S) \).

Providing each edge in \( \mathcal{C}(S) \) with the standard Euclidean metric of diameter 1 equips the curve graph with the structure of a geodesic metric space. Since \( \mathcal{C}(S) \) is not locally finite, this metric space \( (\mathcal{C}(S), d) \) is not locally compact. Masur and Minsky [16] showed that nevertheless its geometry can be understood quite explicitly. Namely, \( \mathcal{C}(S) \) is hyperbolic of infinite diameter. The mapping class group \( \text{Mod}(S) \) naturally acts on \( \mathcal{C}(S) \) as a group of simplicial isometries. A mapping class \( \phi \in \text{Mod}(S) \) is \textit{pseudo-Anosov} if the cyclic subgroup of \( \text{Mod}(S) \) generated by \( \phi \) acts on the curve graph \( \mathcal{C}(S) \) with unbounded orbits.

A \textit{geodesic lamination} for a complete hyperbolic structure on \( S \) of finite volume is a \textit{compact} subset of \( S \) which is foliated into simple geodesics. A geodesic lamination \( \lambda \) on \( S \) is said to be \textit{minimal} if each of its half-leaves is dense in \( \lambda \). Thus a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and...
is said to be *minimal arational*. A geodesic lamination $\lambda$ is said to *fill up* $S$ if every simple closed geodesic on $S$ intersects $\lambda$ transversely. This is equivalent to stating that the complementary components of $\lambda$ are all topological disks or once punctured topological disks.

A *measured geodesic lamination* is a geodesic lamination $\lambda$ together with a translation-invariant transverse measure. Such a measure assigns a positive weight to each compact arc in $S$ which intersects $\lambda$ nontrivially and whose endpoints are contained in complementary regions of $\lambda$. The geodesic lamination $\lambda$ is called the *support* of the measured geodesic lamination; it consists of a disjoint union of minimal components. Vice versa, every minimal geodesic lamination is the support of a measured geodesic lamination.

The space $\mathcal{ML}$ of measured geodesic laminations on $S$ can be equipped with the weak$^*$-topology. Its projectivization $\mathcal{PML}$ is called the space of *projective measured geodesic laminations* and it is homeomorphic to the sphere $S^{6g-7+2m}$. There is a continuous symmetric pairing $\iota: \mathcal{ML} \times \mathcal{ML} \to (0, \infty)$, the so-called *intersection form*, which satisfies $\iota(a\xi, b\eta) = ab\iota(\xi, \eta)$ for all $a, b \geq 0$ and all $\xi, \eta \in \mathcal{ML}$. By the Hubbard–Masur Theorem (see [12]), for every $x \in \mathcal{T}(S)$ the space $\mathcal{PML}$ of projective measured geodesic laminations can naturally be identified with the projectivized cotangent space of $\mathcal{T}(S)$ at $x$.

Since $\mathcal{C}(S)$ is a hyperbolic geodesic metric space, it admits a *Gromov boundary* $\partial\mathcal{C}(S)$ which is a (noncompact) metrizable topological space equipped with an action of $\text{Mod}(S)$ by homeomorphisms (see [3] for the definition of the Gromov boundary of a hyperbolic geodesic metric space and for references). Following Klarreich [13] (see also [8]), this boundary can naturally be identified with the space of all (unmeasured) minimal geodesic laminations which fill up $S$, equipped with the topology which is induced from the weak$^*$-topology on $\mathcal{PML}$ via the measure-forgetting map.

Now let $\mathcal{FML} \subset \mathcal{PML}$ be the $\text{Mod}(S)$-invariant Borel subset of all projective measured geodesic laminations whose support is minimal and fills up $S$. The discussion in the previous paragraph shows that there is a continuous $\text{Mod}(S)$-equivariant surjection

$$F: \mathcal{FML} \to \partial\mathcal{C}(S)$$

that associates to a projective measured geodesic lamination in $\mathcal{FML}$ its support.

Since the curve graph is a hyperbolic geodesic metric space, for every $c \in \mathcal{C}(S)$ there is a *visual metric* $\delta_c$ of uniformly bounded diameter on the Gromov boundary $\partial\mathcal{C}(S)$ of $\mathcal{C}(S)$ (we refer to Chapter III.H of [3] for details of this construction and for references). These distances are related to the intrinsic geometry of $\mathcal{C}(S)$ as follows.

For a point $c \in \mathcal{C}(S)$, the *Gromov product* at $c$ associates to points $x, y \in \mathcal{C}(S)$ the value

$$(x \mid y)_c = \frac{1}{2}(d(x, c) + d(y, c) - d(x, y)).$$
The Gromov product can be extended to a Gromov product \( (\cdot | \cdot)_c \) for pairs of distinct points in \( \partial \mathcal{C}(S) \) by defining
\[
(\xi | \zeta)_c = \sup \left\{ \liminf_{i,j \to \infty} (x_i | y_j)_c \right\},
\]
where the supremum is taken over all sequences \( (x_i) \) and \( (y_j) \) in \( \mathcal{C}(S) \) such that \( \xi = \lim x_i \) and \( \zeta = \lim y_j \). There are numbers \( \beta > 0 \) and \( \nu \in (0,1) \) such that
\[
\nu e^{-\beta(|\xi|)} \leq \delta_c(\xi, \zeta) \leq e^{-\beta(|\xi|)}
\]
for all \( \xi, \zeta \in \partial \mathcal{C}(S) \) and
\[
\delta_c(\xi) \leq e^{\beta_d(c,a) \delta_a}
\]
for all \( c, a \in \mathcal{C}(S) \).

The distances \( \delta_c \) are equivariant with respect to the action of Mod(S) on \( \mathcal{C}(S) \) and on \( \partial \mathcal{C}(S) \). For \( c \in \mathcal{C}(S) \), \( \xi \in \partial \mathcal{C}(S) \) and \( r > 0 \), denote by \( D_c(\xi, r) \subset \partial \mathcal{C}(S) \) the closed ball of radius \( r \) about \( \xi \) with respect to the distance function \( \delta_c \).

We will need a more precise quantitative relation between the distance functions \( \delta_c \) for various \( c \in \mathcal{C}(S) \). Even though this property is well known, we did not find an explicit reference in the literature and we include a sketch of a proof.

For a formulation, for a number \( m > 1 \), an \( m \)-quasigeodesic in a metric space \( (X, d) \) is a map \( \gamma: J \to X \) such that
\[
|s - t|/m - m \leq d(\gamma(s), \gamma(t)) \leq m|s - t| + m \quad \text{for all } s, t \in J,
\]
where \( J \subset \mathbb{R} \) is a closed connected set. Since \( \mathcal{C}(S) \) is hyperbolic, every quasi-geodesic ray \( \gamma: [0, \infty) \to \mathcal{C}(S) \) converges as \( t \to \infty \) in \( \mathcal{C}(S) \cup \partial \mathcal{C}(S) \) to an endpoint \( \gamma(\infty) \in \partial \mathcal{C}(S) \).

**Lemma 2.1.** For every \( m > 1 \) there are constants \( a(m) > 1, b(m) > 0, a_0(m) > 0 \) with the following property. Let \( \gamma: [0, \infty) \to \mathcal{C}(S) \) be an \( m \)-quasigeodesic ray with endpoint \( \gamma(\infty) \in \partial \mathcal{C}(S) \). Then for all \( t \geq 0 \), we have
\[
\delta_{\gamma(t)} \leq a(m)e^{-b(m)t}\delta_{\gamma(0)} \quad \text{on } D_{\gamma(t)}(\gamma(\infty), a_0(m)).
\]

**Proof.** Since \( \mathcal{C}(S) \) is a hyperbolic geodesic metric space, there is a constant \( p > 1 \) depending on the hyperbolicity constant such that every point \( c \in \mathcal{C}(S) \) can be connected to every point \( \xi \in \partial \mathcal{C}(S) \) by a \( p \)-quasigeodesic (for the particular case of the curve graph see [8, 10, 13]). Similarly, any two points \( \xi \neq \zeta \in \partial \mathcal{C}(S) \) can be joined by a \( p \)-quasigeodesic.

Let \( m \geq p \). By hyperbolicity, there is a number \( r(m) > 0 \) and for every \( m \)-quasigeodesic triangle \( T \) in \( \mathcal{C}(S) \) with vertices \( c \in \mathcal{C}(S) \) and \( \xi \neq \zeta \in \partial \mathcal{C}(S) \), there is a point \( u \in \mathcal{C}(S) \) whose distance to each of the sides of \( T \) is at most \( r(m) \). The (nonunique) point \( u \) is called a center of \( T \). We claim that there is a number \( \chi(m) > 0 \) depending only on \( m \) and the hyperbolicity constant such that
\[
\delta_c(\xi, \zeta) \in \left[ \chi(m)e^{-\beta_d(c,u)}, e^{-\beta_d(c,u)}/\chi(m) \right].
\]
Namely, let \( \gamma_1, \gamma_2: [0, \infty) \to \mathcal{C}(S) \) be \( m \)-quasigeodesic rays which connect \( c \) to \( \xi, \zeta \). There is a universal constant \( b > 0 \) depending only on the hyperbolicity constant for \( \mathcal{C}(S) \) such that
\[
(\gamma_1(\infty) | \gamma_2(\infty))_c - b \leq \liminf_{s,t \to \infty}(\gamma_1(s) | \gamma_2(t))_c \leq (\gamma_1(\infty) | \gamma_2(\infty))_c
\]
By hyperbolicity, for sufficiently large $s$ and $t$, we have

$$|d(c, \gamma_1(s)) + d(c, \gamma_2(t)) - d(\gamma_1(s), \gamma_2(t)) - 2d(u, c)| \leq a,$$

where $a > 0$ is a constant depending only on $m$ and on the hyperbolicity constant for $\mathcal{C}(S)$. Together with (2), (3) and (5), this shows the estimate (4).

Now let $\gamma: [0, \infty) \to \mathcal{C}(S)$ be any $m$-quasigeodesic ray with endpoint $\gamma(\infty) = \xi \in \partial \mathcal{C}(S)$, let $\zeta \neq \xi \in \partial \mathcal{C}(S)$ and let $T$ be an $m$-quasigeodesic triangle with side $\gamma$ and vertices $\gamma(0), \xi, \zeta$. If $u \in \mathcal{C}(S)$ is a center for $T$ and if $\sigma \geq 0$ is such that $d(u, \gamma(\sigma)) \leq r(m)$, then for every $s \in [0, \sigma]$ the distance between $u$ and a center for any $m$-quasigeodesic triangle with vertices $\gamma(s), \xi, \zeta$ is bounded from above by a constant depending only on $m$ and the hyperbolicity constant for $\mathcal{C}(S)$. In particular, by the above discussion and the properties of an $m$-quasigeodesic, there are constants $a(m) > 0, b(m) > 0$ such that

$$\delta_{\gamma(0)}(\xi, \zeta) \leq a(m)e^{-b(m)s}\delta_{\gamma(s)}(\xi, \zeta) \quad \text{for every } s \in [0, \sigma].$$

From this the lemma follows. \hfill \Box

By a result of Bers [2], there is a constant $\chi_0 = \chi_0(S) > 0$ such that for every complete hyperbolic metric $x$ on $S$ of finite volume there is a pants decomposition for $S$ consisting of simple closed geodesics of $x$-length at most $\chi_0$. Define a map

$$\gamma_{\mathcal{F}}: \mathcal{F}(S) \to \mathcal{C}(S)$$

by associating to a marked hyperbolic metric $x \in \mathcal{F}(S)$ a simple closed curve $\gamma_{\mathcal{F}}(x)$ whose $x$-length $\ell_x(\gamma_{\mathcal{F}}(x))$ does not exceed $\chi_0$. There are choices involved in the definition of $\gamma_{\mathcal{F}}(x)$, but for any two such choices and any $x$ the distance between the images of $x$ is uniformly bounded.

Denote by $d_{\mathcal{F}}$ the distance on $\mathcal{F}(S)$ defined by the Teichmüller metric. Then, Lemma 2.2 of [10] shows that there is a number $L > 1$ such that

$$d(\gamma_{\mathcal{F}}(x), \gamma_{\mathcal{F}}(y)) \leq Ld_{\mathcal{F}}(x, y) + L \quad \text{for all } x, y \in \mathcal{F}(S).$$

Choose a smooth function $\sigma: [0, \infty) \to [0, 1]$ such that $\sigma([0, \chi_0]) = \{1\}$ and $\sigma([2\chi_0, \infty)) = \{0\}$. For every $x \in \mathcal{F}(S)$ we obtain a finite Borel measure $\mu_x$ on $\mathcal{C}(S)$ by defining

$$\mu_x = \sum_{\beta} \sigma(\ell_x(\beta))\Delta_{\beta},$$

where $\Delta_{\beta}$ denotes the Dirac mass at $\beta$. The total mass of $\mu_x$ is bounded from above and below by a universal positive constant, and the diameter of the support of $\mu_x$ in $\mathcal{C}(S)$ is uniformly bounded as well. The measures $\mu_x$ are equivariant with respect to the action of the mapping class group on $\mathcal{F}(S)$ and $\mathcal{C}(S)$, and they depend continuously on $x \in \mathcal{F}(S)$ in the weak* topology. This means that for every bounded function $f: \mathcal{C}(S) \to \mathbb{R}$ the mapping $x \mapsto \int f \, d\mu_x$ is continuous.

For $x \in \mathcal{F}(S)$ define a distance $\delta_x$ on $\partial \mathcal{C}(S)$ by

$$\delta_x(\xi, \zeta) = \int \delta(\xi, \zeta) \, d\mu_x(c).$$
Clearly the metrics $\delta_x$ are equivariant with respect to the action of $\text{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial \mathcal{E}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$\delta_x \leq e^{\kappa x_f(x,y)} \delta_y \quad \text{and} \quad \kappa^{-1} \delta_x \leq \delta_{\mathcal{Y}_{\mathcal{T}}(x)} \leq \kappa \delta_x$$

for all $x, y \in \mathcal{T}(S)$ [9, p. 230 and p. 231].

The main theorem of [10] relates the geometry of Teichmüller space to the geometry of the curve graph via the map $\mathcal{Y}_{\mathcal{T}}$. For its formulation, denote for $\epsilon > 0$ by $\mathcal{T}(S)_e$ the set of all hyperbolic metrics whose systole (i.e., the shortest length of a closed geodesic) is at least $\epsilon$. For sufficiently small $\epsilon$ the set $\mathcal{T}(S)_e$ is connected, and the mapping class group acts properly and cocompactly on $\mathcal{T}(S)_e$.

**Theorem 2.2.**

1. For every $L > 1$, there is a number $\epsilon = \epsilon(L) > 0$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set of length at least $1/\epsilon$ and let $\gamma: J \to \mathcal{T}(S)$ be an $L$-quasigeodesic. If $\mathcal{Y}_{\mathcal{T}} \circ \gamma$ is an $L$-quasigeodesic in $\mathcal{E}(S)$, then there is a Teichmüller geodesic $\xi: J' \to \mathcal{T}(S)_{e}$ such that the Hausdorff distance between $\gamma(J)$ and $\xi(J')$ is at most $1/\epsilon$.

2. For every $\epsilon > 0$, there is a number $L(\epsilon) > 1$ with the following property. Let $J \subset \mathbb{R}$ be a closed connected set and let $\gamma: J \to \mathcal{T}(S)$ be a $1/\epsilon$-quasigeodesic. If there is a Teichmüller geodesic arc $\xi: J' \to \mathcal{T}(S)_{e}$ such that the Hausdorff distance between $\gamma(J)$ and $\xi(J')$ is at most $1/\epsilon$, then $\mathcal{Y}_{\mathcal{T}} \circ \gamma$ is an $L(\epsilon)$-quasigeodesic in $\mathcal{E}(S)$.

Let $\mathcal{D}^1(S)$ be the bundle of area-one quadratic differentials over Teichmüller space $\mathcal{T}(S)$ for $S$. The mapping class group $\text{Mod}(S)$ acts properly discontinuously on $\mathcal{D}^1(S)$ as a group of bundle automorphisms. This action commutes with the action of the Teichmüller geodesic flow $\Phi^t$.

An area-one quadratic differential $q \in \mathcal{D}^1(S)$ is determined by a pair $(q_v, q_h) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations, the *vertical* and the *horizontal* measured geodesic lamination of $q$, respectively, with $(q_v, q_h) = 1$. For every $t \in \mathbb{R}$ the pair $(e^t q_v, e^{-t} q_h)$ corresponds to the quadratic differential $\Phi^t q$. The *strong stable manifold*

$$W^{ss}(q) \subset \mathcal{D}^1(S)$$

of $q$ is defined as the set of all quadratic differentials of area one whose vertical measured geodesic lamination coincides *precisely* with the vertical measured geodesic lamination of $q$. The *strong unstable manifold*

$$W^{su}(q) \subset \mathcal{D}^1(S)$$

is the set of all quadratic differentials of area one whose horizontal measured geodesic lamination coincides precisely with the horizontal measured geodesic lamination of $q$. Define the *stable manifold* $W^s(q)$ of $q$ and the *unstable manifold* $W^u(q)$ of $q$ by $W^s(q) = \bigcup_{t \in \mathbb{R}} \Phi^t W^{ss}(q)$ and $W^u(q) = \bigcup_{t \in \mathbb{R}} \Phi^t W^{su}(q)$. As $q$
varies through $\mathcal{D}^1(S)$, the manifolds $W^{ss}(q), W^{su}(q), W^s(q), W^u(q)$ define continuous foliations of $\mathcal{D}^1(S)$ which are called the strong stable, the strong unstable, the stable and the unstable foliation. These foliations are invariant under the action of Mod(S) and under the action of the Teichmüller geodesic flow $\Phi'$. For every $q \in \mathcal{D}^1(S)$ the map which associates to a quadratic differential its vertical measured geodesic lamination restricts to a homeomorphism of the unstable manifold $W^u(q)$ of $q$ onto an open dense subset of $\mathcal{ML}$. The space $\mathcal{ML}$ admits a natural Mod(S)-invariant measure in the Lebesgue measure class which lifts to a locally finite measure on $W^u(q)$ in the Lebesgue measure class. The induced family of conditional measures $\tilde{\lambda}_q (q \in \mathcal{D}^1(S))$ on strong unstable manifolds transform under the Teichmüller flow by $d\tilde{\lambda}_{\Phi'q} = e^{ht}d\tilde{\lambda}_q$ where as before, $h = 6g - 6 + 2m$. The measures $\tilde{\lambda}_q$ are equivariant under the action of the mapping class group. Moreover, they are conditional measures for a Mod(S)-invariant locally finite measure $\tilde{\lambda}$ on $\mathcal{D}^1(S)$ in the Lebesgue measure class. The measure $\tilde{\lambda}$ is the lift of a finite measure $\lambda$ on $\mathcal{D}^1(S)/\text{Mod}(S)$ which is $\Phi'$-invariant and mixing. The measure $\lambda$ gives full measure to the set of all quadratic differentials whose horizontal and vertical measured geodesic laminations are uniquely ergodic and fill up $S$ (see [15, 17] for details). Moreover, by the Poincaré Recurrence Theorem, $\lambda$-almost every $q \in \mathcal{D}^1(S)/\text{Mod}(S)$ is recurrent, i.e., it is contained in the $\omega$-limit set of its own orbit under the Teichmüller flow.

Let

$$\pi: \mathcal{D}^1(S) \to \mathcal{PM}\mathcal{L}$$

be the map which associates to a quadratic differential its vertical projective measured geodesic lamination. For every $q \in \mathcal{D}^1(S)$ the restriction of the projection $\pi$ to $W^{z}(q)$ is a homeomorphism of $W^{z}(q)$ onto the open subset of $\mathcal{PM}\mathcal{L}$ of all projective measured geodesic laminations $\mu$ which together with $\pi(-q)$ jointly fill up $S$, i.e., are such that for every measured geodesic lamination $\eta \in \mathcal{ML}$ we have $\iota(\mu, \eta) + \iota(\pi(-q), \eta) \neq 0$ (note that this makes sense even though the intersection form $\iota$ is defined on $\mathcal{ML}$ rather than on $\mathcal{PM}\mathcal{L}$). The measure class of the push-forward under $\pi$ of the measure $\tilde{\lambda}_q$ on $W^{z}(q)$ does not depend on $q$ and defines a Mod(S)-invariant ergodic measure class on $\mathcal{PM}\mathcal{L}$.

### 3. Compact invariant sets are expansive

Let again $\mathcal{D}^1(S)$ be the bundle of area-one quadratic differentials over Teichmüller space $\mathcal{T}(S)$ of an oriented surface $S$ of genus $g \geq 0$ with $m \geq 0$ punctures and where $3g - 3 + m \geq 2$. The mapping class group $\text{Mod}(S)$ acts on $\mathcal{D}^1(S)$, but this action is not free and the quotient space $\mathcal{D}^1(S)/\text{Mod}(S)$ is a noncompact orbifold rather than a manifold.

To overcome this (mainly technical) difficulty we choose a torsion-free normal subgroup $\Gamma$ of $\text{Mod}(S)$ of finite index. For example, the group of all elements which act trivially on $H_1(S, \mathbb{Z}/3\mathbb{Z})$ has this property. Define $\mathcal{D}(S) = \ldots$
The goal of this section is to show the following theorem.

**Theorem 3.1.** The restriction of the Teichmüller flow to every compact invariant subset $K$ of $\hat{Q}(S)$ is expansive.

We begin with the construction of a convenient metric on $Q^1(S)$ and $\hat{Q}(S)$ inducing the usual topology. To this end, call a distance $d$ on a space $X$ a length metric if the distance between any two points is the infimum of the lengths of all paths connecting these points. In the sequel denote by

$$\Pi : Q^1(S) \to \hat{Q}(S)$$

be the canonical projection. Since the action of $\Gamma$ on $Q^1(S)$ is free, the map $\Pi$ is a covering. The Teichmüller flow $\Phi^t$ acts on $\hat{Q}(S)$.

**Lemma 3.2.** There is a complete $\text{Mod}(S)$-invariant length metric $d$ on $Q^1(S)$ with the following properties.

1. The metric $d$ induces the usual topology.
2. The canonical projection $P : (Q^1(S), d) \to \mathcal{T}(S)$ is distance-nondecreasing where $\mathcal{T}(S)$ is equipped with the Teichmüller metric.
3. Every orbit of the Teichmüller flow with its natural parametrization is a minimal geodesic for $d$ parametrized by arc length.

**Proof.** By the Hubbard–Masur Theorem (see [12] for a presentation of this celebrated and by now classical result), the restriction of the canonical projection $P : Q^1(S) \to \mathcal{T}(S)$ to every unstable (or stable) manifold in $Q^1(S)$ is a homeomorphism onto $\mathcal{T}(S)$. Thus the Teichmüller metric on $\mathcal{T}(S)$ lifts to a length metric on the leaves of the stable and of the unstable foliation.

Call a path $\rho : [0, 1] \to Q^1(S)$ admissible if there is a finite partition $0 = t_0 < \cdots < t_k = 1$ such that the restriction of $\rho$ to each interval $[t_{i-1}, t_i]$ is entirely contained in a stable or in an unstable manifold. For each such admissible path $\rho$ we can define its length to be the sum of the lengths with respect to the lifts of the Teichmüller metric of the subsegments of $\rho$ entirely contained in a stable or an unstable manifold. For $q_0, q_1 \in Q^1(S)$ define $d(q_0, q_1)$ to be the infimum of the lengths of all admissible paths connecting $q_0$ to $q_1$. Then $d$ is a (a priori not finite) distance function on $Q^1(S)$ which satisfies the second requirement in the lemma. The third property holds since the projection to $\mathcal{T}(S)$ of an orbit of the Teichmüller flow is a Teichmüller geodesic of the same length and hence realizes the distance between its endpoints. By the second property for $d$ and the definition, the $d$-length of every path in $Q^1(S)$ which is entirely contained in a stable or an unstable manifold coincides with the length of its projection to $\mathcal{T}(S)$. As a consequence, the metric $d$ is a length metric.

We are left with showing that $d$ induces the usual topology on $Q^1(S)$. For this let $q \in Q^1(S)$ and let $\varepsilon > 0$. We have to show that the $\varepsilon$-ball about $q$ for
the distance \( d \) contains a neighborhood of \( q \) in \( \mathcal{Q}^1(S) \). Namely, denote for \( z \in \mathcal{Q}^1(S) \) and \( r > 0 \) by \( B^i(q, r) \) the closed \( r \)-ball about \( q \) in \( W^i(q) \) with respect to the lift of the Teichmüller metric \((i = s, u)\). For each \( z \in B^i(q, \varepsilon/2) \), the closed ball \( B^u(z, \varepsilon/2) \) of radius \( \varepsilon/2 \) about \( z \) in \( W^u(z) \) is a compact neighborhood of \( z \) in \( W^u(z) \) which depends continuously on \( z \) in the Hausdorff topology for compact subsets of \( \mathcal{Q}^1(S) \) by the Hubbard–Masur Theorem. Then \( U = \bigcup_{z \in B^i(q, \varepsilon/2)} B^u(z, \varepsilon/2) \) is a neighborhood of \( q \) in \( \mathcal{Q}^1(S) \). Moreover by construction, \( U \) is contained in the \( \varepsilon \)-ball about \( q \) with respect to the distance function \( d \). This completes the proof of the lemma.

The distance \( d \) on \( \mathcal{Q}^1(S) \) constructed in Lemma 3.2 induces a metric on \( \hat{\mathcal{Q}}(S) \), again denoted by \( d \), via

\[
d(x, y) = \inf \{d(\hat{x}, \hat{y}) \mid \Pi(\hat{x}) = \Pi(\hat{y})\}
\]

where as before, \( \Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S) \) is the canonical projection. In the sequel we always use these distance functions on \( \mathcal{Q}^1(S) \) and \( \hat{\mathcal{Q}}(S) \) without further mentioning. With these preparations, we can prove Theorem 3.1.

**Proof of Theorem 3.1.** Let \( K \subset \hat{\mathcal{Q}}(S) \) be a compact \( \Phi^t \)-invariant subset and let \( \hat{K} \subset \mathcal{Q}^1(S) \) be the preimage of \( K \) under the canonical projection \( \Pi : \mathcal{Q}^1(S) \rightarrow \hat{\mathcal{Q}}(S) \). Let as before \( P : \mathcal{Q}^1(S) \rightarrow \mathcal{F}(S) \) be the canonical projection. By the second part of Theorem 2.2, there is a constant \( p > 0 \) depending only on \( K \) such that for every \( q \in \hat{K} \) the curve \( t \rightarrow \Upsilon_F(P\Phi^t q) \) is a \( p \)-quasigeodesic in \( \mathcal{C}(S) \), i.e., we have for all \( s, t \in \mathbb{R} \)

\[
|t - s|/p - p \leq d(\Upsilon_F(P\Phi^t q), \Upsilon_F(P\Phi^s q)) \leq p|t - s| + p.
\]

Since, by inequality (7), the map \( \Upsilon_F \) is coarsely Lipschitz, this shows that lifts of orbits of \( \Phi^t|K \) which are contained in the same unstable manifold diverge linearly in forward direction. We therefore just have to relate distances in the curve graph to distances in \( \mathcal{Q}^1(S) \) for the metric \( d \) in a quantitative way. The remainder of the argument establishes such a control.

Let \( \mathcal{F} : \mathcal{Q}^1(S) \rightarrow \mathcal{Q}^1(S) \) be the flip \( q \rightarrow \mathcal{F}(q) = -q \). This flip is equivariant with respect to the action of the mapping class group and hence it descends to a continuous involution of \( \hat{\mathcal{Q}}(S) \) which we denote again by \( \mathcal{F} \). Recall that the Gromov boundary \( \partial \mathcal{C}(S) \) of \( \mathcal{C}(S) \) can be identified with the set of all (unmeasured) minimal geodesic laminations on \( S \) which fill up \( S \). Let again \( \pi : \mathcal{Q}^1(S) \rightarrow \mathcal{PML} \) be the canonical projection. If \( q \in \hat{K} \cup \mathcal{F}(\hat{K}) \) then the support \( F(\pi q) \) of the vertical measured geodesic lamination of \( q \) is uniquely ergodic, which means that \( F(\pi q) \) admits a unique transverse measure up to scale, and \( F(\pi q) \) is minimal and fills up \( S \), [15]. Moreover, the \( p \)-quasigeodesic \( t \rightarrow \Upsilon_F(P\Phi^t q) \) converges in \( \mathcal{C}(S) \cup \partial \mathcal{C}(S) \) to \( F(\pi q) \) (the latter statement follows from the explicit identification of \( \partial \mathcal{C}(S) \) with the set of minimal geodesic laminations which fill up \( S \) established in [8, 13], see also [10]).

Write \( A = \pi(\hat{K} \cup \mathcal{F}(\hat{K})) \). Then \( A \) is a \( \Gamma \)-invariant Borel subset of \( \mathcal{PML} \). By the consideration in the previous paragraph, the restriction to \( A \) of the map \( F : \mathcal{F}ML \rightarrow \partial \mathcal{C}(S) \) introduced in (1) of Section 2 is a \( \Gamma \)-equivariant continuous
injection $F_A : A \to \partial \mathcal{C}(S)$ which associates to a projective measured geodesic lamination contained in $A$ its support. In other words, we can identify the set $A$ with a subset of $\partial \mathcal{C}(S)$.

Recall from equation (8) the definition of the distances $\delta_x (x \in \mathcal{T}(S))$ on the Gromov boundary $\partial \mathcal{C}(S)$ of $\mathcal{C}(S)$. Since the map $F_A : A \to \partial \mathcal{C}(S)$ is injective, for every $q \in \mathcal{L}(S)$ the function $(\xi, \zeta) \in A \times A \to \delta_{pq}(F_A \xi, F_A \zeta)$ is a distance on $A$. For simplicity of notation, we denote this distance again by $\delta_{pq}$. The topology on $A$ defined by this distance is just the subspace topology of $A$ as a subset of $\mathcal{P}\mathcal{M}\mathcal{L}$ (this is the result of [13]).

For $q \in K \cup \mathcal{F}(K)$ denote by $D_q(\pi(q), r) \subset A$ the closed ball of radius $r$ in $A$ about $\pi(q)$ with respect to this distance. By continuity of the projection $\pi$ and the map $F_A$ and by the relation (9) between the distances $\delta_x$ and $\delta_y$ for $x, y \in \mathcal{T}(S)$, the ball $D_q(\pi(q), r)$ depends continuously on $q$ in the following sense. For every $q \in K \cup \mathcal{F}(K)$, every $r > 0$ and every $\epsilon \in (0, r)$ there is a neighborhood $U$ of $q$ in $K \cup \mathcal{F}(K)$ such that for every $u \in U$ we have

$$D_u(\pi(u), r - \epsilon) \subset D_q(\pi(q), r) \subset D_u(\pi(u), r + \epsilon).$$

By inequalities (9) and (14) and by Lemma 2.1, there are numbers $\alpha > 0$, $a > 1$, and $b > 0$ such that for every $q \in K$ and for all $t > 0$ we have

$$\delta_{\rho \Phi^t \cdot q} \leq ae^{-bt} \delta_{pq} \quad \text{on } D_q(\pi(q), 2\alpha)$$

and

$$\delta_{\rho \Phi^t \cdot q} \leq ae^{-bt} \delta_{pq} \quad \text{on } D_{\mathcal{F}(q)}(\pi(\mathcal{F} q), 2\alpha).$$

For $q \in \mathcal{L}(S)$ and $\beta > 0$ denote by $B(q, \beta)$ the closed ball of radius $\beta$ about $q$ in $\mathcal{L}(S)$ with respect to the length metric $d$ defined in Lemma 3.2. Since the projection $\pi$ is continuous, for every $q \in K \cup \mathcal{F}(K)$ there is a number $\epsilon(q) > 0$ such that $\pi(B(q, \epsilon(q))) \cap (K \cup \mathcal{F}(K)) \subset D_q(\pi(q), \alpha)$ where $\alpha > 0$ is as in the inequalities (16). By the continuity properties (15) of the balls $D_u(\pi(u), r)$ for $u \in K \cup \mathcal{F}(K)$ and $r > 0$, by invariance under the action of the group $\Gamma < \text{Mod}(S)$ and cocompactness, we can find a universal number $\beta_0 > 0$ such that

$$\pi(B(q, \beta_0) \cap (K \cup \mathcal{F}(K))) \subset D_q(\pi(q), \alpha) \quad \text{for all } q \in K \cup \mathcal{F}(K).$$

Denote again by $d$ the distance on $\hat{\mathcal{L}}(S)$ induced in equation (13) from the distance on $\mathcal{L}(S)$. Since $K \cup \mathcal{F}(K)$ is compact and $\Pi : \mathcal{L}(S) \to \mathcal{L}(S)$ is a covering, there is a number $\beta < \beta_0$ such that for every $q \in K \cup \mathcal{F}(K)$ and every lift $\tilde{q}$ of $q$ to $\mathcal{L}(S)$ the ball $B(q, \beta)$ in $\hat{\mathcal{L}}(S)$ of radius $\beta$ about $q$ is the homeomorphic image under $\Pi$ of the ball $B(\tilde{q}, \beta)$. Now the orbits of $\Phi^t$ are geodesics for the distance $d$. Hence if $x \in K$, if $s : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $s(0) = 0$ and $d(\Phi^t q, \Phi^{s(t)} q) < \beta/2$ for all $t$ then by the choice of $\beta$ we have $|t - s(t)| < \beta/2$ for all $t$.

This implies that for this function $s$, if $u \in K$ is such that $d(\Phi^t q, \Phi^{s(t)} u) < \beta/2$ for all $t$ then $d(\Phi^t u, \Phi^{s(t)} u) < \beta/2$ (since $|t - s(t)| < \beta/2$ and orbits of the Teichmüller flow are geodesics) and hence $d(\Phi^t u, \Phi^t q) < \beta$ for all $t$ by the triangle inequality. In particular, for a lift $\tilde{q} \in \mathcal{L}(S)$ of $q$ and a lift $\tilde{u}$ of $u$ with $d(\tilde{q}, \tilde{u}) < \beta$ we have $d(\Phi^t \tilde{q}, \Phi^t \tilde{u}) < \beta$ for all $t \in \mathbb{R}$.
Let $W^s_{\text{loc}}(q)$ be the connected component containing $q$ of the intersection of $B(q, \beta)$ with the stable manifold $W^s(q)$ of $q$. We claim that $u \in W^s_{\text{loc}}(q)$. For this assume otherwise. Let again $\tilde{q}$ be a preimage of $q$ in $\mathcal{S}^1(S)$ and let $\tilde{u} \in \mathcal{S}^1(S)$ be the preimage of $u$ with $d(\tilde{q}, \tilde{u}) < \beta$. If $\pi(\tilde{q}) \neq \pi(\tilde{u})$ then $\delta P_{\tilde{q}}(\pi(\tilde{q}), \pi(\tilde{u})) > 0$ and by continuity, our choice of $\alpha$ and the estimates (16), there is a number $t > 0$ such that $\delta P_{\tilde{q}}(\pi(\tilde{q}), \pi(\tilde{u})) = \alpha$. On the other hand, for every $s \in [0, t]$ the distance in $\mathcal{S}^1(S)$ between $\Phi^s \tilde{q}, \Phi^s \tilde{u}$ is smaller than $\beta$ which contradicts the choice of $\beta < \beta_0$ and the relation (17).

In the same way we conclude that $u$ is contained in the intersection of $B(q, \beta)$ with the local unstable manifold of $q$. Now the intersection of the local stable manifold $W^s_{\text{loc}}(q)$ with the local unstable manifold $W^u_{\text{loc}}(q)$ is contained in the orbit of $q$ under the Teichmüller flow $\Phi^t$ which completes the proof of Theorem 3.1.

For a compact $\Phi^t$-invariant subset $K$ of $\hat{\mathcal{S}}(S)$ let $h_{\text{top}}(K)$ be the topological entropy of the restriction of $\Phi^t$ to $K$. For $r > 0$ let moreover $n_K(r)$ be the number of periodic orbits of $\Phi^t$ of period at most $r$ which are contained in $K$. Since by Theorem 3.1 the restriction of the flow $\Phi^t$ to $K$ is expansive, by [11, Proposition 3.2.14] we have

**Corollary 3.3.** Let $K \subset \hat{\mathcal{S}}(S)$ be a compact $\Phi^t$-invariant set. Then,

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K(r) \leq h_{\text{top}}(K).$$

4. **An Anosov Closing Lemma**

The goal of this section is to establish a version of the Anosov Closing Lemma for the restriction of the Teichmüller flow to a compact invariant set $K \subset \hat{\mathcal{S}}(S)$. The classical Anosov Closing Lemma roughly states that for a hyperbolic flow on a closed Riemannian manifold, a closed curve consisting of sufficiently long orbit segments which are connected at the endpoints by sufficiently short arcs is closely fellow-traveled by a periodic orbit.

We continue to use the assumptions and notations from Sections 2 and 3. In particular, we always use the distances on $\mathcal{S}^1(S)$ and $\hat{\mathcal{S}}(S)$ defined in Lemma 3.2 and in equation (13). For a precise formulation of our version of an Anosov Closing Lemma for the Teichmüller flow, using the assumptions and notations from Sections 2 and 3, we define the following.

**Definition 4.1.** For $n > 0, \epsilon > 0$, an $(n, \epsilon)$-pseudoorbit for the Teichmüller flow $\Phi^t$ on $\hat{\mathcal{S}}(S)$ consists of a sequence of points $q_0, q_1, \ldots, q_k \in \hat{\mathcal{S}}(S)$ and a sequence of numbers $t_0, \ldots, t_{k-1} \in [n, \infty)$ with the following properties.

1. For every $j \leq k$ the $2\epsilon$-neighborhood of $q_j$ is contained in a contractible subset of $\hat{\mathcal{S}}(S)$.
2. For every $j < k$ we have $d(\Phi^{t_j}q_j, q_{j+1}) \leq \epsilon$.

The pseudoorbit is contained in a compact set $K$ if for all $i$ and all $t \in [0, t_i]$ we have $\Phi^t q_i \in K$. The pseudoorbit is called closed if $q_0 = q_k$. 

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An \((n, \epsilon)\)-pseudoorbit \(q_0, \ldots, q_k\) determines an essentially unique arc connecting \(q_0\) to \(q_k\) which we call a **characteristic arc**. Namely, by assumption, for each \(j < k\) the \(2\varepsilon\)-neighborhood of \(q_{j+1}\) is contained in a contractible subset of \(\hat{\mathcal{D}}(S)\). Hence the homotopy class with fixed endpoints in \(\hat{\mathcal{D}}(S)\) of an arc of length smaller than \(2\varepsilon\) connecting \(\Phi^t q_j\) to \(q_{j+1}\) is unique. We define a characteristic arc of the \((n, \epsilon)\)-pseudoorbit to be an arc connecting \(q_0\) to \(q_k\) which is obtained by successively joining the endpoint of the orbit segment \([\Phi^t q_j \mid 0 \leq t \leq t_j]\) to \(q_{j+1}\) with an arc of length smaller than \(2\varepsilon\) which is parametrized on the unit interval \((j = 0, \ldots, k-1)\). The points \(q_i\) \((1 \leq i \leq k)\) are called the **break points** of the characteristic arc of the pseudoorbit. The **characteristic homotopy class** of the pseudoorbit is the homotopy class with fixed endpoints of a characteristic arc connecting \(q_0\) to \(q_k\). Note that this is independent of the choice of a characteristic arc. If the pseudoorbit is closed then it determines a closed characteristic curve and hence a free homotopy class of closed curves in \(\hat{\mathcal{D}}(S)\) which we call the **characteristic free homotopy class** of the closed pseudoorbit.

By abuse of notation, denote again by \(P: \hat{\mathcal{D}}(S) \to \mathcal{T}(S)/\Gamma\) the canonical projection.

**Definition 4.2.** An \((n, \epsilon)\)-pseudoorbit \(q_0, \ldots, q_k\) as in Definition 4.1 is said to be **\(\delta\)-shadowed** by an orbit segment \(\zeta = [\Phi^t q \mid t \in [0, \tau]]\) for some \(q \in \hat{\mathcal{D}}(S)\) and some \(\tau > 0\) if the following holds.

1. There is a number \(\alpha \leq \delta\) such that the \(\alpha\)-neighborhoods of \(Pq_0, Pq_k\) in \(\mathcal{T}(S)/\Gamma\) with respect to the projection of the Teichmüller metric are contractible and contain \(Pq, P\Phi^\tau q\).
2. There is a lift \(\tilde{\zeta}\) to \(\mathcal{D}^1(S)\) of the orbit segment \(\zeta\) and a lift \(\tilde{\gamma}\) to \(\mathcal{D}^1(S)\) of a characteristic arc \(\gamma\) for the pseudoorbit with the following properties. The distance between the endpoints of \(P\tilde{\gamma}, P\tilde{\zeta}\) is at most \(\alpha\), and the Hausdorff distance between \(\tilde{\gamma}\) and \(\tilde{\zeta}\) is at most \(\delta\).

A closed pseudoorbit in \(\hat{\mathcal{D}}(S)\) is said to be \(\delta\)-shadowed by a periodic orbit if in addition to the above requirements the orbit \([\Phi^t q \mid t \in [0, \tau]]\) is closed.

Note that every point in the orbifold \(\mathcal{D}^1(S)/\text{Mod}(S)\) admits a contractible neighborhood, so we can use the above definition is the same way for the Teichmüller flow on the orbifold \(\mathcal{D}^1(S)/\text{Mod}(S)\).

Using Definition 4.1 and Definition 4.2, we can now formulate a version of the Anosov Closing Lemma for the Teichmüller flow, which is the main result of this section.

**Theorem 4.3.** For every compact \(\Phi^t\)-invariant set \(K \subset \hat{\mathcal{D}}(S)\), there are numbers \(\epsilon_1 = \epsilon_1(K) > 0\), \(n = n(K) > 0\), and \(b = b(K) > 0\) such that every \((n, \epsilon_1)\)-pseudoorbit contained in \(K\) is \(b\)-shadowed by an orbit. Moreover, for every \(\delta > 0\) there is a number \(\epsilon_2 = \epsilon_2(K, \delta) < \epsilon_1\) such that a closed \((n, \epsilon_2)\)-pseudoorbit contained in \(K\) is \(\delta\)-shadowed by a periodic orbit which is contained in the characteristic free homotopy class of the closed pseudoorbit.

For the proof of Theorem 4.3 we need the following technical preparation, which is also used in Section 5. To this end, recall that a point \(q \in \mathcal{D}^1(S)/\text{Mod}(S)\)
is recurrent if $q$ is contained in the $\omega$-limit set of its own orbit under the Teichmüller flow. For some $m > 1$, an unparametrized $m$-quasigeodesic in $\mathcal{C}(S)$ is an arc $\gamma : J \to \mathcal{C}(S)$ with the property that there is an orientation-preserving homeomorphism $\phi : I \to J$ such that $\gamma \circ \phi : I \to \mathcal{C}(S)$ is an $m$-quasigeodesic. We show

**Lemma 4.4.**

1. For every compact $\Phi^t$-invariant set $K \subset \hat{\mathcal{D}}(S)$ there are numbers $\epsilon_0 = \epsilon_0(K) > 0$, $n_0 = n_0(K) > 0$, and $\ell_0 = \ell_0(K) > 1$ depending on $K$ with the following property. Let $q_0, \ldots, q_k \in \hat{\mathcal{D}}(S)$ be an $(n_0, \epsilon_0)$-pseudoorbit contained in $K$ and let $\bar{\gamma}$ be a lift to $\mathcal{D}^1(S)$ of a characteristic arc connecting $q_0$ to $q_k$. Then the arc $t \to \mathcal{Y}_\mathcal{T}(P\bar{\gamma}(t))$ is an $\ell_0$-quasigeodesic in $\mathcal{C}(S)$.

2. There is a number $m > 1$ and for every recurrent point $q \in \mathcal{D}^1(S)/\text{Mod}(S)$ there are numbers $\epsilon_0(q) > 0$ and $n_0(q) > 0$ with the following properties. Let $q_0, \ldots, q_k$ be an $(n_0(q), \epsilon_0(q))$-pseudoorbit with $d(q_i, q) \leq \epsilon_0(q)$ for all $i$, and let $\bar{\gamma}$ be a lift to $\mathcal{D}^1(S)$ of a characteristic arc $\gamma$ connecting $q_0$ to $q_k$. Then the arc $t \to \mathcal{Y}_\mathcal{T}(P\bar{\gamma}(t))$ is an unparametrized $m$-quasigeodesic in $\mathcal{C}(S)$.

**Proof.** Recall from (3) and (8) of Section 2 the definition of the distance functions $\delta_c$ for $c \in \mathcal{C}(S)$ and $\delta_x$ for $x \in \mathcal{T}(S)$ on the Gromov boundary $\partial \mathcal{C}(S)$ of $\mathcal{C}(S)$. By hyperbolicity, every quasigeodesic ray $\zeta : [0, \infty) \to \mathcal{C}(S)$ converges as $t \to \infty$ to a point $R(\zeta) \in \partial \mathcal{C}(S)$. Moreover, for every $p > 1$, there are numbers $m(p) > 0$, $\alpha(p) > 0$, and $\beta(p) > 1$ with the following property.

Let $k > 0$ and let $\zeta_0, \ldots, \zeta_{k-1} : \mathbb{R} \to \mathcal{C}(S)$ be infinite $p$-quasigeodesics. Let $L > 1$ be as in inequality (7). Suppose that for every $j \leq k - 2$ there is a number $T_j > m(p)$ such that $d(\zeta_j(T_j), \zeta_{j+1}(0)) \leq 2L$. For each $j \leq k - 1$, let $\rho_j : [0, 1] \to \mathcal{C}(S)$ be any map whose image is contained in the $2L$-neighborhood of $\zeta_j(T_j)$ and let $\tilde{\zeta}_j$ be the composition of $\zeta_j[0, T_j]$ with $\rho_j$ parametrized in the natural way on $[0, T_j + 1]$. If $\delta_{\sum_i T_i + k}(R(\zeta_j), R(\zeta_{j+1})) < \alpha(p)$ for all $j$, then the curve $\zeta : [0, \sum_i T_i + k] \to \mathcal{C}(S)$ defined by

$$
\zeta(t) = \tilde{\zeta}_j \left( t - \sum_{i=0}^{j-1} T_i - j \right) \quad \text{for} \quad t \in \left[ \sum_{i=0}^{j-1} T_i + j, \sum_{i=0}^{j} T_i + j + 1 \right]
$$

is a $\beta(p)$-quasigeodesic.

Let $K \subset \hat{\mathcal{D}}(S)$ be a compact $\Phi^t$-invariant set and let $\bar{K} \subset \mathcal{D}^1(S)$ be the preimage of $K$ under the natural projection. By the second part of Theorem 2.2 there is a number $p > 1$ such that for every $q \in \bar{K}$ the assignment $t \to \mathcal{Y}_\mathcal{T}(P\Phi^t q)$ ($t \in \mathbb{R}$) is a $p$-quasigeodesic.

For $q \in \bar{K}$, we have $F(\pi(q)) = R(t \to \mathcal{Y}_\mathcal{T}(P\Phi^t q)) \in \partial \mathcal{C}(S)$. Let $k > 0$ be as in (9) of Section 2. By continuity, for every $q \in \bar{K}$ there is a number $\epsilon(q) > 0$ such that for every point $\tilde{q} \in \bar{K}$ which is contained in the $2\epsilon(q)$-neighborhood of $q$ the $\delta_{pq}$-distance between $F(\pi(q))$ and $F(\pi(\tilde{q}))$ is smaller than $\alpha(p)/k$ where $\alpha(p) > 0$ is as in the first paragraph of this proof. By continuity, invariance under the action of the mapping class group on $\mathcal{D}^1(S)$ and $\partial \mathcal{C}(S)$ and cocompactness of
the action of \( \Gamma \) on \( \tilde{K} \), there is a number \( \epsilon_0 \in (0, 1/2) \) which has this property for all \( q \in \tilde{K} \) (compare the proof of Theorem 3.1 for a similar statement).

Let \( m(p) > 0 \) be as in the first paragraph of this proof. Let \( \tilde{\gamma} \) be the lift to \( \mathcal{D}^1(S) \) of a characteristic arc of an \((m(p), \epsilon_0)\)-pseudoorbit \( q_0, q_1, \ldots, q_k \) contained in \( K \subset \mathcal{D}^1(S) \). Then \( \tilde{\gamma} \) is a concatenation of curves \( \tilde{\gamma}_i \) (\( i = 1, \ldots, k \)), where the curve \( \tilde{\gamma}_i \) is a lift to \( \mathcal{D}^1(S) \) of an orbit segment for the Teichmüller flow of length at least \( m(p) \) beginning at \( x_{i-1} \) and an arc of length at most \( 2\epsilon_0 < 1 \) parametrized on \([0, 1]\), with endpoint \( x_i \). By the choice of \( m(p) \), of \( \epsilon_0 > 0 \) and by the construction of the curves \( \tilde{\gamma}_i \), the curve \( \Upsilon_\gamma(P\tilde{\gamma}) \) is of the form described in the second paragraph of this proof and hence it is a \( \beta(p) \)-quasigeodesic in \( \mathcal{C}(S) \). This shows the first part of the lemma.

The second part of the lemma follows in the same way. By [16] there is a number \( \ell > 0 \) such that for every \( \tilde{q} \in \mathcal{D}^1(S) \) the mapping \( t \mapsto \Upsilon_{\tilde{\gamma}}(P\Phi^t \tilde{q}) \) is an unparametrized \( \ell \)-quasigeodesic (this number \( \ell > 0 \) does not coincide with the number \( p > 1 \) above, see also [10]). Let \( m(\ell) > 0 \), \( a(\ell) > 0 \), and \( \rho(\ell) > 1 \) be as in the first paragraph of this proof. Let \( q \in \mathcal{D}^1(S)/\text{Mod}(S) \) be a recurrent point and let \( \tilde{q} \in \mathcal{D}^1(S) \) be a lift of \( q \). Then the vertical measured geodesic lamination of \( \tilde{q} \) is uniquely ergodic and fills up \( S \) [15], and the unparametrized \( \ell \)-quasigeodesic \( t \mapsto \Upsilon_{\tilde{\gamma}}(P\Phi^t \tilde{q}) \) is of infinite diameter. By continuity and by Lemma 2.4 of [10] there is a neighborhood \( V \) of \( \tilde{q} \) in \( \mathcal{D}^1(S) \) and a number \( T(q) > 0 \) such that

\[
(19) \quad d(\Upsilon_{\tilde{\gamma}}(P\Phi^t u), \Upsilon_{\tilde{\gamma}}(Pu)) \geq \ell m(\ell) + \ell \quad \text{for all } u \in V \text{ and } t \geq T(q).
\]

In particular, if \( u \in V \), \( a \in (0, \infty) \) and \( \rho_u : [0, a) \to [0, \infty) \) is a homeomorphism with the property that the mapping \( t \mapsto \Upsilon_{\tilde{\gamma}}(P\Phi^{\rho_u(t)} u) \) is a parametrized \( \ell \)-quasigeodesic in \( \mathcal{C}(S) \), then \( \rho_u(m(\ell)) \leq T(q) \).

The subset \( \mathcal{U} \) of \( \mathcal{D}^1(S) \) of all points with uniquely ergodic vertical measured geodesic lamination which fills up \( S \) is dense [15]. Each \( u \in \mathcal{U} \) defines a point \( F\pi(u) \in \partial \mathcal{C}(S) \) which is just the endpoint of the infinite unparametrized \( \ell \)-quasigeodesic \( t \mapsto \Upsilon_{\tilde{\gamma}}(P\Phi^t u) \). The map \( u \in \mathcal{U} \to F\pi(u) \in \partial \mathcal{C}(S) \) is continuous. Thus we can find a number \( \epsilon_0(q) \in (0, 1/2) \) which is small enough that the \( 2\epsilon_0(q) \)-neighborhood \( W \) of \( \tilde{q} \) in \( \mathcal{D}^1(S) \) is contained in \( V \) and that for every point \( u \in W \cap \mathcal{U} \) the \( \delta_{P\tilde{\gamma}} \)-distance between \( F(\pi(q)) \) and \( F(\pi(u)) \) is smaller than \( a(\ell)/\kappa \) where as before, \( \kappa > 0 \) is as in (9) in Section 2. The second part of the lemma holds for the numbers \( m = m(\ell) > 0 \), \( \epsilon_0(q) > 0 \), and \( n_0(q) = T(q) > 0 \).}
Let $\tilde{\gamma}$ be a lift of $\gamma$ to $\mathcal{D}^1(S)$. By Lemma 4.4, the assignment $t \rightarrow Y_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an $\ell_0$-quasigeodesic in $\mathcal{E}(S)$ for a number $\ell_0 > 1$ depending only on $K$. The map $t \rightarrow P\tilde{\gamma}(t) \in (\mathcal{T}(S), d_\mathcal{T})$ is one-Lipschitz. Since, by inequality (7), there is a number $L > 0$ such that $d_\mathcal{T}(Pq, Pz) \geq d(Y_{\mathcal{T}}(Pq), Y_{\mathcal{T}}(Pz))/L - L$ for all $q, z \in \mathcal{D}^1(S)$, we conclude that the curve $t \rightarrow P\tilde{\gamma}(t)$ is a uniform quasigeodesic in $\mathcal{T}(S)$. By the first part of Theorem 2.2, this implies that there is a Teichmüller geodesic whose Hausdorff distance to $P\tilde{\gamma}$ is bounded from above by a universal constant. As a consequence, the Hausdorff distance between $\tilde{\gamma}$ and the tangent line of this geodesic is bounded from above by a universal constant $b > 0$.

A mapping class $g \in \text{Mod}(S)$ is pseudo-Anosov if the cyclic subgroup of $\text{Mod}(S)$ generated by $g$ acts on the curve graph $\mathcal{E}(S)$ with unbounded orbits. In this case the conjugacy class of $g$ can be represented by a closed orbit for the Teichmüller flow $\Phi^t$ on $\mathcal{D}^1(S)/\text{Mod}(S)$, and it can be represented by a closed orbit for the Teichmüller flow on $\mathcal{D}(S)$ if the conjugacy class of $g$ is contained in the normal subgroup $\Gamma$ of $\text{Mod}(S)$. Assume now that the $(n_0, \epsilon_0)$-pseudoorbit $q_0, \ldots, q_c$ contained in $K$ is closed. Let $\tilde{\gamma}$ be a lift to $\mathcal{D}^1(S)$ of a closed characteristic arc $\gamma$ for the pseudoorbit. By the first part of Lemma 4.4, the curve $t \rightarrow Y_{\mathcal{T}}(P\tilde{\gamma}(t))$ is an infinite $\ell_0$-quasigeodesic in $\mathcal{E}(S)$ which is invariant under an element $g \in \Gamma < \text{Mod}(S)$ of the mapping class group. The mapping class $g$ acts on this quasigeodesic as a translation and hence it is pseudo-Anosov. As a consequence, there is a unique $g$-invariant Teichmüller geodesic in $\mathcal{T}(S)$ whose cotangent line in $\mathcal{D}^1(S)$ projects to a periodic orbit of $\Phi^t$ in $\mathcal{D}(S)$ which defines the free homotopy class of $\gamma$. In other words, there is a closed orbit for $\Phi^t$ in $\mathcal{D}(S)$ which is freely homotopic to $\gamma$.

By the first part of Theorem 2.2, applied to the bi-infinite quasigeodesic $P\tilde{\gamma}$ in $\mathcal{T}(S)$, this orbit is contained in a compact subset $C_0 \supset K$ of $\mathcal{D}(S)$ not depending on the pseudoorbit. Moreover, it $b$-shadows the pseudoorbit for a number $b > 0$ depending only on $K, n_0, \epsilon_0$. This shows the first part of Theorem 4.3.

Let $C \subset C_0$ be the $\Phi^t$-invariant subset of $C_0$ of all points whose $\Phi^t$-orbit is entirely contained in $C_0$. The periodic orbit defined by the conjugacy class of the pseudo-Anosov element $g$ is contained in $C$. Let $\tilde{C}$ be the preimage of $C$ in $\mathcal{D}^1(S)$. Then every lift to $\mathcal{D}^1(S)$ of a periodic orbit in $\mathcal{D}(S)$ determined as above by a closed $(n_0, \epsilon_0)$-pseudoorbit contained in $K$ is contained in $\tilde{C}$.

Let again $\pi: \mathcal{D}^1(S) \rightarrow \mathcal{PM}\mathcal{L}$ be the canonical projection. Write $A = \pi(\tilde{C} \cup \mathcal{F}(\tilde{C}))$ where $\mathcal{F}: \mathcal{D}^1(S) \rightarrow \mathcal{D}^1(S)$ is the flip $q \rightarrow \mathcal{F}(q) = -q$. As in Section 3, let $F_A = F|A: A \rightarrow \partial\mathcal{E}(S)$ be the measure-forgetting injection. For $q \in \tilde{C} \cup \mathcal{F}(\tilde{C})$ let $\delta_{pq}$ be the distance on $\partial\mathcal{E}(S)$ defined in equation (8) and denote by $D_q(\pi(q), r)$ the closed ball of radius $r$ about $\pi(q)$ in $A$ with respect to the distance $(x, y) \in A \times A \rightarrow \delta_{pq}(F_A x, F_A y) \in [0, \infty)$ which we denote again by $\delta_{pq}$ (compare the proof of Theorem 3.1).

By the second part of Theorem 2.2, applied to the projection into $\mathcal{T}(S)$ of the preimage $\tilde{C}$ of the compact $\Phi^t$-invariant set $C \subset \mathcal{D}(S)$, by Lemma 2.1 and by inequality (9) of Section 2, there are numbers $a_0 < 1/2$, $a > 1$, and $b > 0$ such
that for every \( q \in \mathcal{C} \) and all \( t > 0 \) we have
\[
(20) \quad \delta_p \Phi^t \cdot q \leq a e^{-bt} \delta_p q \quad \text{on} \quad D_q(\pi(q), 4a_0).
\]
Moreover, for every \( 0 < \alpha < a_0 \) there is a number \( \beta = \beta(\alpha) < 1 \) such that for every \( q \in \mathcal{C} \) we have \( A \cap B(q, \beta) \subset D_q(\pi(q), \alpha) \) where \( B(q, \beta) \) is the closed ball of radius \( \beta \) about \( q \) in \( \mathcal{W}_1(S) \) (compare the proof of Theorem 3.1).

Let \( n = \max(a_0, \log(4a)/b) \), let \( \alpha < a_0 \) and let \( \sigma = \min\{\epsilon_0, \beta(\alpha), \kappa^{-1} \log 2\} \) where \( \kappa > 0 \) is as in inequality (9). We claim that for a lift \( \tilde{\gamma} : [0, T] \to \mathcal{W}_1(S) \) of a characteristic arc \( \gamma \) of any \((n, \sigma)\)-pseudoorbit contained in \( K \) we have
\[
\delta_{\tilde{\gamma}(0)}(\pi(\tilde{\gamma}(0)), \pi(\tilde{\gamma}(T))) \leq \alpha.
\]
To see this we proceed by induction on the number of break points of the pseudoorbit. The case without break point is trivial, so assume that the claim is known whenever the number of break points of the pseudoorbit is at most \( k - 1 \geq 0 \). Let \( \tilde{\gamma} \) be a lift to \( \mathcal{W}_1(S) \) of a characteristic arc \( \gamma \) of an \((n, \sigma)\)-pseudoorbit contained in \( K \) with \( k \) break points. Let \( t_0 \geq n + 1 \) be such that \( \gamma(t_0) \) is the first break point of \( \gamma \). By assumption and the choice of the parametrization of a characteristic arc we have \( d(\tilde{\gamma}(t_0), \tilde{\gamma}(t_0 - 1)) \leq \sigma \). Since \( \tilde{\gamma}(t_0) \in \mathcal{C}, \tilde{\gamma}(t_0 - 1) \in \mathcal{C} \), by the choice of \( \sigma \) we have
\[
(21) \quad \delta_{\tilde{\gamma}(t_0)}(\pi(\tilde{\gamma}(t_0)), \pi(\tilde{\gamma}(t_0 - 1))) \leq \alpha.
\]
Moreover, the distances \( \delta_{\tilde{\gamma}(t_0)}, \delta_{\tilde{\gamma}(t_0 - 1)} \) are 2-bi-Lipschitz equivalent (recall that the projection \( P : \mathcal{W}_1(S) \to \mathcal{F}(S) \) is distance-nonincreasing).

Now \( \pi(\tilde{\gamma}(T)) \in D_{\tilde{\gamma}(t_0)}(\pi(\tilde{\gamma}(t_0)), \alpha) \) by the induction hypothesis and therefore
\[
(22) \quad \delta_{\tilde{\gamma}(t_0)}(\pi(\tilde{\gamma}(t_0 - 1)), \pi(\tilde{\gamma}(T))) \leq 4\alpha.
\]
On the other hand, since \( \alpha \leq a_0 \), since \( n \geq \log(4a)/b \) and since \( \pi(\tilde{\gamma}(t_0 - 1)) = \pi(\tilde{\gamma}(0)) \), we infer from the estimate (20) that
\[
(23) \quad \delta_{\tilde{\gamma}(t_0 - 1)}(\pi(\tilde{\gamma}(t_0 - 1)), \pi(\tilde{\gamma}(T))) \geq 4\delta_{\tilde{\gamma}(t_0)}(\pi(\tilde{\gamma}(0)), \tilde{\gamma}(T)).
\]
Inequalities (22) and (23) imply the claim.

Note that the argument in the previous paragraph together with the estimate (20) also shows that
\[
(24) \quad \delta_{\tilde{\gamma}(t)}(\pi(\tilde{\gamma}(t)), \pi(\tilde{\gamma}(T))) \leq 4\alpha
\]
for all \( t \in [0, T] \) with the additional property that \( \tilde{\gamma}(t) \in \mathcal{C} \) (namely this holds for every \( t \) such that \( \tilde{\gamma}(t) \) projects to an orbit segment defining the pseudoorbit).

Let again \( \tilde{\gamma} \) be a bi-infinite lift to \( \mathcal{W}_1(S) \) of a closed characteristic curve \( \gamma \) for an \((n, \sigma)\)-pseudoorbit contained in \( K \). The curve \( \mathcal{Y}_\mathcal{F}(P\tilde{\gamma}) \) is a uniform quasigeodesic in \( \mathcal{C}(S) \) invariant under a pseudo-Anosov element \( g \in \Gamma < \text{Mod}(S) \) on \( \mathcal{D} \mathcal{C}(S) \). The conjugacy class of \( \gamma \) defines the free homotopy class of \( \gamma \). The oriented cotangent line of the axis of \( g \) is contained in \( \mathcal{C} \). If \( z \in \mathcal{C} \) is a point in this cotangent line then \( \pi(z) \in A \) is a fixed point for the action of \( g \) on \( \mathcal{P} \mathcal{M} \mathcal{L} \). An inductive application to longer and longer subsegments of \( \tilde{\gamma} \) of the argument which led to the estimate (24) shows that for every \( t \in \mathbb{R} \) such that \( \tilde{\gamma}(t) \in \mathcal{C} \) the fixed point \( \pi(z) \in A \) of \( g \) is contained in the ball \( D_{\mathcal{Y}(t)}(\pi(\tilde{\gamma}(t)), 4\alpha a) \). The
same argument also shows that the fixed point $\pi(-z)$ for the action of $g$ is contained in $D_{-v(t)}(\pi(-v(t)), 4a\alpha)$. The periodic orbit on $\tilde{\mathcal{O}}(S)$ defined by $g$ is contained in the compact $\Phi^t$-invariant subset $C \supset K$ of $\tilde{\mathcal{O}}(S)$ determined above.

By the considerations in Theorem 3.1 and its proof, applied to the compact $\Phi^t$-invariant subset $C$ of $\tilde{\mathcal{O}}(S)$, this means that for every $\delta > 0$ there is a constant $\beta > 0$ depending only on $K$ with the following property: Let $q_0, \ldots, q_k$ be a closed $(n, \beta)$-pseudoorbit contained in $K$. Then there is a closed orbit for $\Phi^t$ contained in $C$ whose Hausdorff distance to a closed characteristic curve defined by the pseudoorbit is at most $\delta$. From this Theorem 4.3 follows. \hfill \square

The Anosov Closing Lemma implies the existence of many periodic orbits near any nonwandering point of a compact $\Phi^t$-invariant subset $K$ of $\tilde{\mathcal{O}}(S)$. However, as for compact invariant hyperbolic sets in the usual sense of smooth dynamical systems (see [11]), these periodic orbits are in general not contained in $K$. The next corollary is an immediate adaptation of Corollary 6.4.19 of [11] and shows that the periodic orbits can be chosen to be contained in an arbitrarily small neighborhood of $K$.

**Corollary 4.5.** Let $K$ be a compact $\Phi^t$-invariant subset of $\tilde{\mathcal{O}}(S)$ and let $U$ be an open neighborhood of $K$. Then every nonwandering point $q \in K$ is an accumulation point of periodic points of $\Phi^t$ whose orbits are entirely contained in $U$.

In the case of a topologically transitive compact invariant set $K \subset \tilde{\mathcal{O}}(S)$ we can say more.

**Lemma 4.6.** Let $K$ be a compact $\Phi^t$-invariant topologically transitive subset of $\tilde{\mathcal{O}}(S)$. Then for every $\sigma > 0$, there is a periodic orbit for $\Phi^t$ whose Hausdorff distance to $K$, as subsets of $\tilde{\mathcal{O}}(S)$, is at most $\sigma$.

**Proof.** Let $K \subset \tilde{\mathcal{O}}(S)$ be a compact $\Phi^t$-invariant topologically transitive set and let $\sigma > 0$. Let $n = n(K) > 0$, $\varepsilon_2 = \varepsilon_2(K, \sigma/2) < \sigma/2$ be as in Theorem 4.3. Since $K$ is topologically transitive by assumption, there is some $q \in K$ and there is some $T > n$ such that $d(q, \Phi^T q) < \varepsilon_2$ and that moreover the Hausdorff distance between the set $K$ and its subset $B = \{\Phi^t q \mid 0 \leq t \leq T\}$ is at most $\sigma/2$.

By Theorem 4.3, applied to the closed $(n, \varepsilon_2)$-pseudoorbit defined by the orbit segment $\{\Phi^t q \mid 0 \leq t \leq T\}$, there is a periodic orbit for $\Phi^t$ whose Hausdorff distance to $B$ is at most $\sigma/2$. This means that the Hausdorff distance between this orbit and the set $K$ is at most $\sigma$ and shows the lemma. \hfill \square

For a compact $\Phi^t$-invariant subset $K \subset \tilde{\mathcal{O}}(S)$ denote by $h_{top}(K)$ the topological entropy of the restriction of $\Phi^t$ to $K$. For an arbitrary subset $U \subset \tilde{\mathcal{O}}(S)$ and a number $r > 0$ let $n_{top}(r)$ be the number of all periodic orbits of $\Phi^t$ of period at most $r$ which are contained in $U$. The following corollary is another fairly immediate consequence of Theorem 4.3. Together with Corollary 3.3 it shows Theorem 2 from the introduction.
Corollary 4.7. Let \( K \subset \hat{\mathcal{S}}(S) \) be a compact \( \Phi^t \)-invariant topologically transitive set. Then for every open neighborhood \( U \) of \( K \) we have

\[
h_{\text{top}}(K) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_U(r).
\]

Proof. Let \( K \subset \hat{\mathcal{S}}(S) \) be a topologically transitive compact \( \Phi^t \)-invariant set and let \( U \) be an open neighborhood of \( K \). Then there is a number \( \beta > 0 \) such that \( U \) contains the \( \beta \)-neighborhood of \( K \).

Let \( \delta < \beta \) be so small that the \( \delta \)-neighborhood of every point in \( K \) is contained in a contractible subset of \( \hat{\mathcal{S}}(S) \). Let \( n = n(K) > 0, \varepsilon_2 = \varepsilon_2(K, \delta/8) < 1 \) be as in Theorem 4.3. Since the Teichmüller flow on \( K \) is topologically transitive by assumption, by compactness of \( K \times K \) there is a number \( N > n \) with the following property. Let \( q, q' \in K \); then there is some \( u \in K \) and some \( T \in [n, N] \) with \( d(u, q') < \varepsilon_2 \) and \( d(\Phi^Tu, q) < \varepsilon_2 \).

A subset \( E \) of \( K \) is said to be \((m, \delta)\)-separated for some \( m \geq 0 \) if for any two points \( q \neq u \in E \) we have

\[
d(\Phi^t q, \Phi^t u) \geq \delta \quad \text{for some} \quad t \in [0, m].
\]

Let \( m > n \) and let \( E_m \subset K \) be any \((m, \delta)\)-separated set. Let \( q \in E_m \). By the choice of \( N > n \) there is some \( u \in K \) and some \( T \in [n, N] \) such that \( d(u, \Phi^m q) < \varepsilon_2 \) and \( d(\Phi^Tu, q) < \varepsilon_2 \). By Theorem 4.3, the closed \((n, \varepsilon_2)\)-pseudoorbit \( q, u, q \) is \( \delta/8 \)-shadowed by a periodic orbit which defines the characteristic free homotopy class of the pseudoorbit. Since periodic orbits for \( \Phi^t \) in \( \hat{\mathcal{S}}(S) \) minimize the length in their free homotopy class, the length of the periodic orbit does not exceed \( m + N + 2\varepsilon_2 \). Moreover, by the choice of \( \delta \) this periodic orbit is contained in \( U \). There is a point \( \zeta(q) \) on the orbit with \( d(q, \zeta(q)) \leq \delta/8 \). In other words, there is a map \( \zeta \) which associates to every point \( q \in E_m \) a point \( \zeta(q) \in U \) whose orbit under \( \Phi^t \) is entirely contained in \( U \) and is periodic of period at most \( m + N + 2\varepsilon_2 \).

Since the points in the set \( E_m \) are \((m, \delta)\)-separated by assumption and the orbits of \( \Phi^t \) are geodesics parametrized by arc length, the orbit segments \( c(q) = \bigcup_{t \in (-\delta/8, \delta/8]} \Phi^t \zeta(q) \) as \( q \) varies through \( E_m \) are pairwise disjoint. Thus for a fixed periodic orbit \( \gamma \) for \( \Phi^t \) of length at most \( m + N + 2\varepsilon_2 \) there are at most \( 4(m + N + 2)/\delta \) distinct points \( q \in E_m \) with \( \zeta(q) \in \gamma \). As a consequence, there are at least \( \delta \text{card}(E_m)/4(m + N + 2) \) distinct periodic orbits of period at most \( m + N + 2 \) in \( U \). This shows that the asymptotic growth as \( m \to \infty \) of the maximal cardinality of an \((m, \delta)\)-separated subset of \( K \) does not exceed the asymptotic growth of the numbers \( n_U(r) \) as \( r \to \infty \). The corollary is now an immediate consequence of the definition of the topological entropy of a continuous flow on a compact space (recall also from Theorem 3.1 that the Teichmüller flow on \( K \) is expansive and hence for all sufficiently small \( \delta > 0 \) its topological entropy is just the asymptotic growth rate of maximal \((m, \delta)\)-separated sets as \( m \to \infty \)).
5. LOWER BOUNDS FOR THE NUMBER OF PERIODIC ORBITS

In this section we complete the proof of Theorem 3 from the introduction. For this we continue to use the assumptions and notations from Sections 2 and 3. In particular, we always denote by $d_T$ the Teichmüller metric on Teichmüller space $\mathcal{T}(S)$ for $S$.

We begin with establishing the first part of Theorem 3 which is immediate from the work of Eskin and Mirzakhani [6]. Since the proof is short and easy, we include it for completeness.

The Poincaré series with exponent $\alpha > 0$ at a point $x \in \mathcal{T}(S)$ is defined to be the series

$$\sum_{g \in \text{Mod}(S)} e^{-\alpha d(x,gx)}.$$  

The critical exponent of $\text{Mod}(S)$ is the infimum of all numbers $\alpha > 0$ such that the Poincaré series with exponent $\alpha$ converges. Note that this critical exponent does not depend on the choice of $x$. Athreya, Bufetov, Eskin and Mirzakhani [1] showed that the critical exponent of the Poincaré series equals $h = 6g - 6 + 2m$ and that the Poincaré series diverges at the critical exponent.

For $r > 0$ and for a compact set $K \subset \mathcal{Q}^1(S)/\text{Mod}(S)$ let $n_K^r(r)$ be the number of all periodic orbits for the Teichmüller flow of period at most $r$ which intersect $K$. The next lemma is the first part of Theorem 3.

**Lemma 5.1.** For every compact subset $K$ of $\mathcal{Q}^1(S)/\text{Mod}(S)$, we have

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K^r(r) \leq 6g - 6 + 2m.$$  

**Proof.** Let $\hat{K}$ be any compact subset of the moduli space $\mathcal{M}(S) = \mathcal{T}(S)/\text{Mod}(S)$ which is the closure of an open set. Let $K_1 \subset \mathcal{T}(S)$ be a relatively compact fundamental domain for the action of $\text{Mod}(S)$ on the preimage $\hat{K}$ of $\hat{K}$ in $\mathcal{T}(S)$. Let $D$ be the diameter of $K_1$ and let $x \in K_1$ be any point. Let $g \in \text{Mod}(S)$ be a pseudo-Anosov element whose axis (i.e., the unique $g$-invariant Teichmüller geodesic on which $g$ acts as a translation) projects to a closed geodesic $\gamma$ in moduli space which intersects $\hat{K}$. Then there is a point $\hat{x} \in K_1$ which lies on the axis of a conjugate of $g$ which we denote again by $g$ for simplicity. By the properties of an axis, the length $\ell(\gamma)$ of the closed geodesic $\gamma$ equals $d_T(\hat{x}, g\hat{x})$. On the other hand, we have

$$d_T(x, gx) \leq d_T(\hat{x}, g\hat{x}) + 2d_T(x, \hat{x}) \leq \ell(\gamma) + 2D$$

by the definition of $D$, the choice of $\hat{x}$ and invariance of the Teichmüller metric under the action of $\text{Mod}(S)$. Therefore, if we denote by $K \subset \mathcal{Q}^1(S)/\text{Mod}(S)$ the preimage of $\hat{K} \subset \mathcal{M}(S)$ under the natural projection and if we define $N(r)$ for $r > 0$ to be the number of all $g \in \text{Mod}(S)$ with $d(x, gx) \leq r$, then we have

$$\limsup_{r \to \infty} \frac{1}{r} \log n_K^r(r) \leq \limsup_{r \to \infty} \frac{1}{r} \log N(r).$$
Since the critical exponent of the Poincaré series equals \(6g - 6 + 2m\), for every \(\epsilon > 0\) the Poincaré series converges at the exponent \(a = 6g - 6 + 2m + \epsilon\). Let \(c(\alpha) > 0\) be its value. Then for every \(r > 0\), the cardinality of the set \(|g \in \text{Mod}(S) | d(x, gx) \leq r\) does not exceed \(c(\alpha)e^{ar}\) (note that the term in the Poincaré series corresponding to such an element of \(\text{Mod}(S)\) is not smaller than \(e^{-ar}\)). This shows that \(\operatorname{limsup} \frac{1}{r} \log N(r) \leq 6g - 6 + 2m + \epsilon\). Since \(\epsilon > 0\) and the compact set \(\hat{K} \subset \mathcal{M}(S)\) were arbitrarily chosen, the lemma follows.

As an immediate consequence we obtain the following result.

**Corollary 5.2.** Let \(K \subset \mathcal{D}^1(S)/\text{Mod}(S)\) be a compact \(\Phi^t\)-invariant topologically transitive set. Then \(h_{\text{top}}(K) \leq 6g - 6 + 2m\).

**Proof.** Let \(K \subset \mathcal{D}^1(S)/\text{Mod}(S)\) be a compact \(\Phi^t\)-invariant topologically transitive set, let \(q \in K\) be a point whose orbit under \(\Phi^t\) is dense in \(K\) and let \(\hat{q} \in \hat{\mathcal{D}}(S)\) be a preimage of \(q\) under the natural projection \(\Theta: \hat{\mathcal{D}}(S) \to \mathcal{D}^1(S)/\text{Mod}(S)\). Let \(\hat{K}\) be the closure of the orbit of \(\hat{q}\); then \(\hat{K}\) is a compact \(\Phi^t\)-invariant topologically transitive set. By equivariance of the Teichmüller flow under the projection \(\Theta\), this set is mapped by \(\Theta\) onto \(K\). Moreover, by Corollary 4.7, for every open relatively compact neighborhood \(U\) of \(\hat{K}\) we have

\[
h_{\text{top}}(K) \leq h_{\text{top}}(\hat{K}) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_U(r).
\]

Now the projection \(\Theta\) maps periodic orbits for \(\Phi^t\) in \(U\) of period at most \(r\) to periodic orbits for \(\Phi^t\) of period at most \(r\) which are contained in the relatively compact set \(\Theta(U) \subset \mathcal{D}^1(S)/\text{Mod}(S)\). If the periodic orbits \(\gamma_1 \neq \gamma_2\) in \(U\) are mapped to the same periodic orbit in \(\Theta(U)\) then there is some element \(g\) from the factor group \(G = \text{Mod}(S)/\Gamma\) which maps \(\gamma_1\) to \(\gamma_2\). Since \(G\) is finite, the number of distinct periodic orbits in \(U\) which are mapped to the single orbit in \(\Theta(U)\) is uniformly bounded. Therefore by Lemma 5.1 we have

\[
\liminf_{r \to \infty} \frac{1}{r} \log n_U(r) \leq \liminf_{r \to \infty} \frac{1}{r} \log n_{\Theta(U)}(r) \leq 6g - 6 + 2m.
\]

This shows the corollary.

Now we are ready for the proof of the second part of Theorem 3 from the introduction.

**Proposition 5.3.** For every \(\epsilon > 0\) there is a compact \(\Phi^t\)-invariant subset \(K\) of \(\mathcal{D}^1(S)/\text{Mod}(S)\) with

\[
\liminf_{r \to \infty} \frac{1}{r} \log n_K(r) \geq 6g - 6 + 2m - \epsilon.
\]

**Proof.** As in Section 2, let \(\mathcal{F}\mathcal{M}\mathcal{L} \subset \mathcal{P}\mathcal{M}\mathcal{L}\) be the \(\text{Mod}(S)\)-invariant Borel subset of all projective measured geodesic laminations whose support is minimal and fills up \(S\) and let \(F: \mathcal{F}\mathcal{M}\mathcal{L} \to \mathcal{D}\mathcal{E}(S)\) be the continuous \(\text{Mod}(S)\)-equivariant surjection which associates to a projective measured geodesic lamination its
support. Let $\pi : \mathcal{Q}^1(S) \to \mathcal{PML}$ be the natural projection as defined in (10) and define

$$\mathcal{A} := \pi^{-1}\mathcal{FML} \subset \mathcal{Q}^1(S).$$

Let $\lambda$ be the $\Phi^t$-invariant probability measure on $\mathcal{Q}^1(S)/\text{Mod}(S)$ in the Lebesgue measure class constructed in [15, 17]. This measure is ergodic and mixing under the Teichmüller flow, with full support. In particular, the $\Phi^t$-orbit of $\lambda$-almost every point $q \in \mathcal{Q}^1(S)/\text{Mod}(S)$ returns to every neighborhood of $q$ for arbitrarily large times. The measure $\lambda$ lifts to a $\text{Mod}(S)$-invariant $\Phi^t$-invariant Radon measure $\check{\lambda}$ on $\mathcal{Q}^1(S)$ of full support which gives full measure to the $\text{Mod}(S)$-invariant Borel set $\mathcal{A}$ [15].

The Lebesgue measure $\check{\lambda}$ on $\mathcal{Q}^1(S)$ is absolutely continuous with respect to the strong unstable foliation. More precisely, for every $q \in \mathcal{Q}^1(S)$ there is a natural conditional measure $\lambda_q$ for $\check{\lambda}$ on the strong unstable manifold $W^{su}(q)$, and these conditional measures transform under the Teichmüller flow via $d\check{\lambda}_{\Phi_t q} \circ \Phi^t = e^{ht}d\check{\lambda}_q$ where $h = 6g - 6 + 2m$ as before. The image under the projection $\pi$ of the measure $\check{\lambda}_q$ on $W^{su}(q)$ is a locally finite Borel measure $\lambda_q$ on the open dense subset of $\mathcal{PML}$ of all projective measured geodesic laminations which together with $\pi(-q)$ jointly fill up $S$. The measures $\lambda_q$ are all absolutely continuous, and they depend continuously on $q \in \mathcal{Q}^1(S)$ in the weak$^*$-topology. Moreover, for each $q$ the measure $\lambda_q$ gives full measure to the set $\mathcal{FML}$ and hence it can be mapped via the surjection $F$ to a measure on $\partial\mathcal{E}(S)$ which we denote again by $\lambda_q$.

Recall from (8) and (9) the definition of the distances $\delta_x \{x \in \mathcal{T}(S)\}$ on $\partial\mathcal{E}(S)$ and their properties. For $q \in \mathcal{A}$ and $\chi > 0$ define

$$D(q, \chi) \subset \partial\mathcal{E}(S)$$

to be the closed $\delta_{pq}$-ball of radius $\chi$ about $F\pi(q) \in \partial\mathcal{E}(S)$. Note that $D(q, \chi)$ contains the image under the map $F$ of the ball $D_q(\pi(q), \chi)$ used in Section 4. Moreover, $D(q, \chi)$ depends on $q$ and not only on its center $F\pi(q)$ and the radius $\chi$ since in general the distances $\delta_{pq}$ and $\delta_{\Phi_t q}$ do not coincide for $t \neq 0$.

Let $q_0 \in \mathcal{Q}^1(S)/\text{Mod}(S)$ be a typical point for the Lebesgue measure $\lambda$ (so the Birkhoff Ergodic Theorem holds for $q_0$) and let $q_1$ be a lift of $q_0$ to $\mathcal{Q}^1(S)$. Assume without loss of generality that $Pq_1$ is not fixed by any element of $\text{Mod}(S)$. This is possible since the set of points in $\mathcal{T}(S)$ which are stabilized by a nontrivial element of $\text{Mod}(S)$ is closed and nowhere dense and since the Lebesgue measure is of full support.

Let $m > 1$ be as in the second part of Lemma 4.4. We may assume that the image under the map $Y_\mathcal{T}$ of every Teichmüller geodesic in $\mathcal{T}(S)$ is an unparametrized $m$-quasigeodesic in $\mathcal{E}(S)$. Since $q_0$ is a typical point for Lebesgue measure, the unparametrized quasigeodesic $t \mapsto Y_\mathcal{T}(P\Phi^t q_1)$ is of infinite diameter.

Let $\kappa > 0$ be as in inequality (9). By Lemma 2.1 and inequality (9), there is a number $\alpha > 0$ depending on $m$ and there is a neighborhood $V$ of $q_1$ in $\mathcal{Q}^1(S)$
with diameter at most $\log 2/\kappa$ and a number $T_0 > 0$ such that
\begin{equation}
\delta_{p\Phi^t u} \geq 16\delta_{p u} \quad \text{on } D(\Phi^t u, a),
\end{equation}
for $t \geq T_0$ and $u \in V \cap \mathcal{A}$. Since $q_0$ is recurrent and hence the vertical measured geodesic lamination of $q_1$ is uniquely ergodic and fills up $S$, by the second part of Lemma 3.2 of [9] there is a number $\chi \leq a/4$ such that
\begin{equation}
F\pi(V \cap \mathcal{A} \cap W^{su}(q_1)) \supseteq D(q_1, \chi).
\end{equation}

Let $\epsilon_0 = \epsilon_0(q_0) > 0$ be as in the second part of Lemma 4.4. We may assume that the $\epsilon_0$-neighborhood of $q_0$ is contained in a contractible subset of $\mathcal{D}^1(S)/\text{Mod}(S)$. By continuity, there is a compact neighborhood $K \subset V$ of $q_1$ with the following properties:
\begin{enumerate}
\item the diameter of $K$ does not exceed max{$\epsilon_0, (\log 2)/\kappa$};
\item $F \circ \pi(K \cap \mathcal{A}) \subset D(q_1, \chi/4)$.
\end{enumerate}
By the second requirement for $K$, if $q, u \in K \cap \mathcal{A}$ then $\delta_{p q_t}(F\pi(u), F\pi(q)) \leq \chi/2$. The first property of $K$ together with the relation (9) for the distances $\delta_x$ for $x \in \mathcal{F}(S)$ then implies that $\delta_{p q_t}(F\pi(q), F\pi(u)) \leq \chi$ and $D(q, \chi) \subset D(u, 4\chi)$.

Following [7], a Borel covering relation for a Borel subset $C$ of a topological space $X$ is a family $V$ of pairs $(x, V)$ where $V \subset X$ is a Borel set, $x \in V$ and
\begin{equation}
C \subset \bigcup \{V \mid (z, V) \in V \text{ for some } z \in C\}.
\end{equation}
For $\chi > 0$ and the neighborhood $K \subset \mathcal{D}^1(S)$ of $q_1$ as above define
\begin{equation}
\mathcal{V}_{q_0, \chi, K} := \left\{(F\pi(q), gD(q_1, \chi)) \in W^{su}(q_1) \cap \mathcal{A}, g \in \text{Mod}(S), gK \cap \bigcup_{t > 0} \Phi^t q \neq \emptyset\right\}.
\end{equation}
By Proposition 3.5 of [9], via possibly decreasing the size of $\chi$ and $K$ we may assume that the covering relation $\mathcal{V}_{q_0, \chi, K}$ is a Vitali relation for the measure $\lambda_{q_1}$ on $\partial\mathcal{E}(S)$. In our context, this means that for every $T > 0$ there is a covering of $\lambda_{q_1}$-almost all of $D(q_1, \chi/4)$ by pairwise disjoint sets from the relation of the form
\begin{equation}
V(g, t) = (F\pi(u), gD(q_1, \chi)),
\end{equation}
where $u \in W^{su}(q_1) \cap \mathcal{A}$, $F\pi(u) \in D(q_1, \chi/4)$, $g \in \text{Mod}(S)$ and where $t \geq T$ is such that $\Phi^t u \in gK$ (we refer to Section 3 of [9] for a detailed discussion).

Since the measures $\lambda_u$ and the distances $\delta_{p u}$ on $\partial\mathcal{E}(S)$ depend continuously on $u \in \mathcal{D}^1(S)$, there is a number $a \leq \lambda_{q_1} D(q_1, \chi/4)$ such that $\lambda_q D(u, \chi) \in [a, a^{-1}]$ for all $q \in K, u \in K \cap \mathcal{A}$. By the transition properties for the measures $\lambda_u$ and invariance under the action of the mapping class group, if $g \in \text{Mod}(S)$, if $u \in W^{su}(q_1)$ and if $t > 0$ are such that $\Phi^t u \in gK$ for some $t > 0$ then $\lambda_{q_1}(gD(q_1, \chi)) \in [ae^{-ht}, e^{-ht}/a]$ (compare the discussion in [9]).

Let $n_0 = n_0(q_0) > 0$ be as in the second part of Lemma 4.4. Let $\epsilon > 0$ and let $T(\epsilon) = \max\{|T_0, n_0| + 2$ be sufficiently large that
\begin{equation}
\int_{T(\epsilon)}^{\infty} e^{-\epsilon s} \, ds \leq e^{-2h} a^2.
\end{equation}
Now choose a covering of $\lambda_{q_1}$-almost all of $D(q_1, \chi/4)$ by pairwise disjoint sets $V(g, t) = (F\pi(u), gD(q_1, \chi))$ from the relation $\mathcal{V}_{q_0, \chi, K}$, where $u \in W^{su}(q_1) \cap \mathcal{A}$,
\[ F \pi(u) \in D(q_1, \chi/4), \; g \in \text{Mod}(S) \text{ and where } t \geq T(e) \text{ is such that } \Phi^t u \in gK. \] By the inclusion (31), we have \( u \in \mathcal{V} \) and therefore from the assumption \( T(e) \geq T_0 \) and the estimate (30) we deduce that \( D(\Phi^t u, 4\chi) \subset D(u, \chi/4) \subset D(q_1, \chi) \). On the other hand, we have \( \Phi^t u \in gK \) and hence
\[
(33) \qquad gD(q_1, \chi) \subset D(\Phi^t u, 4\chi) \subset D(q_1, \chi).
\]
The total \( \lambda_{q_1} \)-mass of the balls from the covering is at least \( \lambda_{q_1}D(q_1, \chi/4) \geq a \).
Therefore there is a number \( T > T(e) + 2 \) such that the total volume of those balls \( \mathcal{V}(g, t) \) from the covering which correspond to a parameter \( t \in [T-2, T-1] \) is at least \( e^{-cT} e^{2h}/a \). Now the \( \lambda_{q_1} \)-volume of each such ball is at most \( e^{-h(T-2)}/a \) and hence the number of these balls is at least \( e^{(h-c)T} \).

Let \( \{g_1, \ldots, g_k\} \subset \text{Mod}(S) \) be the subset of \( \text{Mod}(S) \) defining these balls. By (33) above, we have for any \( i \) and \( j \),
\[ g_j g_i D(q_1, \chi) \subset g_j D(q_1, \chi). \]
Therefore, the sets \( g_j g_i D(q_1, \chi) \) (\( i, j = 1, \ldots, k \)) are pairwise disjoint. Namely, the sets \( g_j D(q_1, \chi) \) (\( j = 1, \ldots, k \)) are pairwise disjoint, and for each \( j \) the sets \( g_j g_i D(q_1, \chi) \subset g_i D(q_1, \chi) \) (\( i = 1, \ldots, k \)) are pairwise disjoint as well. By induction, we conclude that for two distinct words \( w_1 = g_{i_1} \cdots g_{i_r} \) and \( w_2 = g_{j_1} \cdots g_{j_m} \) in the letters \( g_1, \ldots, g_k \), viewed as elements of \( \text{Mod}(S) \), the images of \( D(q_1, \chi) \) under \( w_1, w_2 \) are either disjoint or properly contained in each other. This shows that the elements \( g_1, \ldots, g_k \) generate a free semi-subgroup \( \Lambda \) of \( \text{Mod}(S) \).

Since \( T(e) \geq n_0 \), each word \( w \) of length \( \ell \geq 1 \) in the letters \( g_1, \ldots, g_k \) defines a closed \((n_0, \epsilon_0)\)-pseudoorbit \( u_0, \ldots, u_{\ell} \) in \( \mathcal{H}^1(S)/\text{Mod}(S) \) with \( d(u_i, u_0) < \epsilon_0 \). This pseudoorbit consists of the successive projections to \( \mathcal{H}^1(S)/\text{Mod}(S) \) of flow lines \( \{\Phi^t u \mid t \in [0, \tau]\} \) where \( u \in \mathcal{V} \cap \mathcal{W}^{su}(q_1) \cap \mathcal{A} \) and \( \tau \in [T-2, T-1] \) are such that \( F \pi(u) \in g_j D(q_1, \chi/4) \) for some \( j \leq k \) and \( \Phi^t u \in g_j K \). Thus by the second part of Lemma 4.4, if \( \tilde{y} \) is a lift to \( \mathcal{H}^1(S) \) of a characteristic arc of such a pseudoorbit then \( \mathcal{Y}_{\mathcal{F}}(\tilde{y}) \) is a bi-infinite unparametrized \( m \)-quasigeodesic in \( \mathcal{C}(S) \) which is invariant under the element of \( \Lambda \subset \text{Mod}(S) \) defined by \( w \). In particular, this element is pseudo-Anosov, and its conjugacy class defines the characteristic free homotopy class of the closed pseudoorbit.

The length of the periodic orbit of \( \Phi^t \) determined by \( w \) does not exceed the length of a characteristic closed curve for the pseudoorbit and hence it is no bigger than \( T \ell \). Moreover, since by the choice of \( n_0 \) for any \( s < t \) with the property that \( \tilde{y}(s), \tilde{y}(t) \) project to distinct break points of \( y \) the distance between \( Y_{\mathcal{F}}(\tilde{y}(s)), Y_{\mathcal{F}}(\tilde{y}(t)) \) is at least \( 2c(m) \), it follows from [10, Lemma 2.4] that the unparametrized \( m \)-quasigeodesic \( Y_{\mathcal{F}}(\tilde{y}) \) is in fact a parametrized \( p \)-quasigeodesic for some \( p > m \). Using once more the first part of Theorem 2.2, this implies that the axis of the element of \( \Lambda \subset \text{Mod}(S) \) defined by \( w \) passes through a fixed compact neighborhood \( B \) of \( Pq_1 \) in \( \mathcal{F}(S) \), and the projection of its unit tangent line to \( \mathcal{H}^1(S)/\text{Mod}(S) \) is a periodic orbit for \( \Phi^t \) which is contained in a compact subset \( C_0 \) of \( \mathcal{H}^1(S)/\text{Mod}(S) \) not depending on \( w \). If we denote by \( C \) the closed subset of \( C_0 \) of all points \( z \in C_0 \) whose orbit under \( \Phi^t \) is entirely contained in \( C_0 \) then each of these orbits is contained in \( C \).
The above argument does not immediately imply that the asymptotic growth rate of the number of periodic orbits in $C$ is at least $h - \epsilon$. Namely, periodic orbits of the Teichmüller flow on $\mathcal{Q}(S)/\text{Mod}(S)$ correspond to conjugacy classes of pseudo-Anosov elements in $\text{Mod}(S)$. Thus if we want to count periodic orbits for $\Phi^t$ in $\mathcal{Q}(S)/\text{Mod}(S)$ using the semi-subgroup $\Lambda$ of $\text{Mod}(S)$ constructed above, then we must identify those elements of $\Lambda$ that are conjugate in $\text{Mod}(S)$.

For this recall that the axis of each element of the semi-subgroup $\Lambda$ of $\text{Mod}(S)$ passes through the fixed compact neighborhood $B$ of $Pq_1$. Thus if $\gamma, \zeta$ is the axis of $v, w \in \Lambda$ and if $v, w$ are conjugate in $\text{Mod}(S)$ then there is some $b \in \text{Mod}(S)$ with $w = b^{-1}vb$ and the following additional property. Let $\gamma[0, \tau]$ be a fundamental domain for the action of $v$ on $\gamma$ and such that $\gamma(0) \in B$. Such a fundamental domain always exists, perhaps after a reparametrization of $\gamma$. Then there is some $t \in [0, \tau]$ such that $b^{-1}\gamma(t) \in B$.

As a consequence, the number of all elements $w \in \Lambda$ which are conjugate to a fixed element $v \in \Lambda$ is bounded from above by the number of elements $b \in \text{Mod}(S)$ with $bB \cap \gamma[0, \tau] \neq \emptyset$. In particular, if $D$ is the diameter of $B$ then this number does not exceed the cardinality of the set

$$\{b \in \text{Mod}(S) \mid d_T(bPq_1, \gamma[0, \tau]) \leq D\}.$$  

However, this cardinality is bounded from above by a universal multiple of $\tau$. Therefore there is a constant $c > 0$ such that for all sufficiently large $r > 0$ the number of periodic orbits of $\Phi^t$ contained in $C$ of length at most $r$ is not smaller than $e^{(h-\epsilon)r}/cr$. This completes the proof of the proposition.

Remarks. 1. Proposition 5.3 is equally valid, with identical proof, for the Teichmüller flow $\Phi^t$ on $\hat{\mathcal{Q}}(S)$. Together with Theorem 2 it implies that the measure-theoretic entropy $h$ of the unique $\Phi^t$-invariant Lebesgue measure on $\hat{\mathcal{Q}}(S)$ in the Lebesgue measure class equals the supremum of the topological entropies of the restriction of $\Phi^t$ to compact invariant subsets of $\hat{\mathcal{Q}}(S)$. In [5], this fact was established for the Teichmüller flow on the moduli space of abelian differentials using symbolic dynamics.

2. The abundance of orbits of the Teichmüller flow which entirely remain in some compact set (depending on the orbit) was earlier established by Kleinbock and Weiss [14]. They show that this set is of full Hausdorff dimension.

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Ursula Hamenstädt <ursula@math.uni-bonn.de>: Mathematisches Institut der Rheinischen Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, D-53115 Bonn, Germany