ALEXANDER INVARIANTS AND TRANSVERSALITY

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Abstract. We show that some of the main results in Laurentiu Maxim’s paper [10] can be obtained (even in a slightly more general setting) using the theory of perverse sheaves of finite rank over \( \mathbb{Q} \) as described for instance in author’s recent book [3].

1. The main results

Let \( X \subset \mathbb{C}^{n+1} \) with \( n > 1 \) be a reduced hypersurface given by an equation \( f = 0 \). We say that \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is \( \infty \)-transversal if the projective closure \( V \) of \( X \) in \( \mathbb{P}^{n+1} \) is transversal in the stratified sense to the hyperplane at infinity \( H = \mathbb{P}^{n+1} \setminus \mathbb{C}^{n+1} \).

Consider the affine complement \( M_X = \mathbb{C}^{n+1} \setminus X \), and denote by \( M_X^\infty \) its infinite cyclic covering corresponding to the morphism

\[
f^\#: \pi_1(M_X) \to \pi_1(\mathbb{C}^*) = \mathbb{Z}.
\]

Then, for any positive integer \( m \), the homology group \( H_m(M_X^\infty, \mathbb{C}) \) of the hypersurface \( X \) is torsion for \( m < n + 1 \).

Since \( M_X^\infty \) is an \((n+1)\)-dimensional CW complex, one has \( H_m(M_X^\infty, \mathbb{C}) = 0 \) for \( m > n + 1 \), while \( H_{n+1}(M_X^\infty, \mathbb{C}) \) is free. In this sense, the above result is optimal.

The aim of this note is to give an alternative proof for generalizations of Corollary 3.8 and Theorem 4.1, as well as for a reformulation of Theorem 4.2 in [10]. With the above notation, the first two results in [10] can be stated as follows.

**Theorem 1.1.** Assume that \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is \( \infty \)-transversal. Then the Alexander modules \( H_m(M_X^\infty, \mathbb{C}) \) of the hypersurface \( X \) are torsion for \( m < n + 1 \).

Theorem 1.2. Assume that \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) is \( \infty \)-transversal. Let \( \lambda \in \mathbb{C}^* \) be such that \( \lambda^d \neq 1 \), where \( d \) is the degree of \( V \). Then \( \lambda \) is not a root of the Alexander polynomials \( \Delta_m(t) \) for \( m < n + 1 \).

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Recall that the construction of the Alexander modules and polynomials was generalized in the obvious way in \cite{5} to the case when $\mathbb{C}^{n+1}$ is replaced by a smooth affine variety $U$.

Let $W' = W'_0 \cup ... \cup W'_m$ be a hypersurface arrangement in $\mathbb{P}^N$ for $N > 1$. Denote by $M_{W'} = \mathbb{P}^N \setminus W'$ the corresponding complement. Let $d_j$ denote the degree of $W'_j$ and let $g_j = 0$ be a reduced defining equation for $W'_j$ in $\mathbb{P}^N$. Let $Z \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $n + 1 > 1$ and set $W_j = W'_j \cap Z$ for $j = 0, ..., m$ be the corresponding hypersurfaces in $Z$ considered as subscheme defined by the principal ideals generated by the $g_j$'s.

Let $W = W_0 \cup ... \cup W_m$ denote the corresponding hypersurface arrangement in $Z$. We assume throughout in this paper that the following hold.

\textbf{(H1)} All the hypersurfaces $W_j$ are distinct, reduced and irreducible; moreover $W_0$ is smooth.

\textbf{(H2)} The hypersurface $W_0$ is transverse in the stratified sense to $V = W_1 \cup ... \cup W_m$.

The complement $U = Z \setminus W_0$ is a smooth affine variety and we consider the hypersurface $X = U \cap V$ in $U$ and its complement $M_W = Z \setminus W = U \setminus X$.

\textbf{Theorem 1.3.} Assume that $d_0$ divides the sum $\sum_{j=1,m} d_j$, say $d_0 = \sum_{j=1,m} d_j$. Then one has the following.

(i) The function $f : U \to \mathbb{C}$ given by

$$f(x) = \frac{g_1(x)...g_m(x)}{g_0(x)^d}$$

is a well-defined regular function on $U$ whose generic fiber $F$ is connected.

(ii) The above Theorems 1.1 and 1.2 hold for the Alexander modules and the Alexander polynomials associated to the hypersurface $X$ in $U$.

Note that we need the connectedness of $F$ since this is one of the general assumptions made in \cite{5}. The next result says roughly that an $\infty$-transversal polynomial behaves as a homogeneous polynomial up-to (co)homology of degree $n - 1$. In these degrees, the determination of the Alexander polynomial of $X$ in $U$ is reduced to the simpler problem of computing a monodromy operator.

\textbf{Corollary 1.4.} With the assumption in Theorem 1.3, the following hold.

(i) Let $\iota : \mathbb{C}^* \to \mathbb{C}$ be the inclusion. Then, for each $m < n$ there is a local system $\mathcal{L}_m$ on $\mathbb{C}^*$ such that

$$R^m f_* \mathcal{C}_U = \iota_! \mathcal{L}_m.$$ 

In particular, for each $m < n$, the monodromy operators of $f$ at the origin $T^m_0$ and at infinity $T^m_\infty$ acting on $H^m(F, \mathbb{C})$ coincide and the above local system $\mathcal{L}_m$ is precisely the local system corresponding to this automorphism of $H^m(F, \mathbb{C})$. 
(ii) For \( m < n \), there is an isomorphism \( H^m(F, \mathbb{C}) \to H^m(M_c W, \mathbb{C}) \) which is compatible with the obvious actions. In particular, the associated characteristic polynomial
\[
\det(t\text{Id} - T_0^m) = \det(t\text{Id} - T_\infty^m)
\]
coincides to the \( m \)-th Alexander polynomial of \( X \) in \( U \) for \( m < n \).

The next result can be regarded as similar to some results in [1], [8] and [6].

**Theorem 1.5.** Let \( g = g_0 \ldots g_m = 0 \) be the equation of the hypersurface arrangement \( W \) in \( Z \) and let \( F(g) \) be the corresponding global Milnor fiber given by \( g = 1 \) in the cone \( CZ \) over \( Z \). Then
\[
H^j(F(g), \mathbb{C}) = H^j(M_W, \mathbb{C})
\]
for all \( j < n + 1 \). In other words, the action of the monodromy on \( H^j(F(g), \mathbb{C}) \) is trivial for all \( j < n + 1 \).

Now we turn to the following reformulation of Theorem 4.2 in [10] in our more general setting described above.

**Theorem 1.6.** Assume that \( d_0 \) divides the sum \( \sum_{j=1}^{m} d_j \), say \( dd_0 = \sum_{j=1}^{m} d_j \). Let \( \lambda \in \mathbb{C}^* \) be such that \( \lambda^d = 1 \) and let \( \sigma \) be a non negative integer. Assume that \( \lambda \) is not a root of the \( q \)-th local Alexander polynomial \( \Delta_q(t)_x \) of the hypersurface singularity \((V, x)\) for any \( q < n + 1 - \sigma \) and any point \( x \in W_1 \), where \( W_1 \) is an irreducible component of \( W \) different from \( W_0 \).

Then \( \lambda \) is not a root of the global Alexander polynomials \( \Delta_q(t) \) associated to \( X \) for any \( q < n + 1 - \sigma \).

The proofs we propose below are based on Theorem 4.2 in [5] (which relates Alexander modules to the cohomology of a class of rank one local systems on the complement \( M_W \)) and on a general idea of getting vanishing results via perverse sheaves (based on Artin’s vanishing Theorem) introduced in [11] and developed in [3], Chapter 6. We use mainly the notation from [5].

**1.7. Open problem.** It would be interesting to compare the following properties for a polynomial function \( f : \mathbb{C}^{n+1} \to \mathbb{C} \).

(i) \( f \) is \( \infty \)-transversal;

(ii) \( f \) is \( h \)-good as defined in [4];

(iii) \( f \) is \( M_0 \)-tame as defined in Theorem 2.1 in [4].

By taking \( X \) to be a hyperplane arrangement with at least two parallel hyperplanes, we see that (iii) does not imply (i). By taking \( X \) to be a central hyperplane arrangement, we see that (i) does not imply (ii). This example also shows that for an \( \infty \)-transversal polynomial \( f \), the closure of the nearby fibers \( X_t = f^{-1}(t) \) are not
necessarily transverse to the hyperplane at infinity, i.e. $f - t$ is not necessarily an $\infty$-transversal polynomial for $t \neq 0$.

On the positive side, note that an $\infty$-transversal polynomial $f$ satisfies a partial rational version of the conditions of being $h$-good as defined in [5]. More precisely, the necessary condition $H_q(T_b; F; \mathbb{Z}) = 0$ for all bifurcation points $b$ of $f$ and all $q < n + 1$ in the case of an $h$-good mapping is replaced by the condition $H_q(T_b; F; \mathbb{Q}) = 0$ for all non-zero bifurcation points $b$ of $f$ and all $q < n + 1$ (which follows from Theorem 2.10.v in [5] and Theorem 1.3 (ii) above.

A positive answer to the implication (i) implies (iii) would give another proof for Corollary 3.8 in view of Theorem 2.10 v in [5].

2. The proofs

We start with the following easy remark, which proves the first claim of Theorem 1.3 in the case considered by Maxim, i.e. $W_0 = H$ is the hyperplane at infinity (for this reason we use here the notation from the beginning of our paper.

Lemma 2.1. With the above notation, if the closure $V$ of $X$ in $\mathbb{P}^{n+1}$ has a positive dimension singular locus, i.e. $\dim V_{\text{sing}} > 0$, and $H$ is transversal to $V$ except at finitely many points, then

$$\dim V_{\text{sing}} = \dim (V_{\text{sing}} \cap H) + 1.$$ 

In particular, the singular locus $V_{\text{sing}}$ cannot be contained in $H$.

Let $f = 0$ be a reduced equation for the affine hypersurface $X$. It follows from the above result that $f$ is a primitive polynomial, i.e. its generic fiber $F$ is connected and hence the results in [5] apply to this situation.

Now we start the proof of Theorem 1.3. In order to establish the first claim, note that the closure $\mathcal{F}$ of $F$ is a general member of the pencil

$$g_1(x)\ldots g_m(x) - tg_0(x)^d = 0.$$ 

As such, it is smooth outside the base locus given by $g_1(x)\ldots g_m(x) = g_0(x) = 0$. A closer look shows that a singular point is located either at a point where at least two of the polynomials $g_j$ for $1 \leq j \leq n$ vanish, or at a singular point on one of the hypersurfaces $W_j$ for $1 \leq j \leq n$. It follows that $\text{codimSing}(\mathcal{F}) \geq 3$, hence $\mathcal{F}$ is irreducible. This implies that $F$ is connected.

According to Theorem 4.2 in [5], to prove the second claim in Theorem 1.3 it is enough to prove the following.

Proposition 2.2. Let $\lambda \in \mathbb{C}^*$ be such that $\lambda^d \neq 1$, where $d$ is the quotient of $\sum_{j=1}^{m} d_j$ by $d_0$. If $\mathcal{L}_\lambda$ denotes the corresponding local system on $M_W$, then $H_q(M_W, \mathcal{L}_\lambda) = 0$ for all $q \neq n + 1$. 

**Proof.** First we shall recall the construction of the rank one local system $L_\lambda$. Any such local system on $M_W$ is given by a homomorphism from $\pi_1(M_W)$ to $\mathbb{C}^*$. To define our local system consider the composition

$$\pi_1(M_W) \to \pi_1(M'_W) \to H_1(M'_W) = \mathbb{Z}^{m+1}/(d_0, \ldots, d_m) \to \mathbb{C}^*$$

where the first morphism is induced by the inclusion, the second is the passage to the abelianization and the third one is given by sending the classes $e_0, \ldots, e_m$ corresponding to the canonical basis of $\mathbb{Z}^{m+1}$ to $\lambda^{-d}, \lambda, \ldots, \lambda$ respectively. For the isomorphism in the middle, see for instance [2], p. 102.

It is of course enough to show the vanishing in cohomology, i.e. $H^q(M_W, L_\lambda) = 0$ for all $q \neq n + 1$. Let $i : M_W \to U$ and $j : U \to Z$ be the two inclusions. Then one clearly has $L_\lambda[n+1] \in \text{Perv}(M_W)$ and hence $\mathcal{F} = Ri_*(L_\lambda[n+1]) \in \text{Perv}(U)$, since the inclusion $i$ is a quasi-finite affine morphism. See for this and the following p. 214 in [3] for a similar argument.

Our vanishing result will follow from a study of the natural morphism

$$Rj_!\mathcal{F} \to Rj_*\mathcal{F}.$$ 

Extend it to a distinguished triangle

$$Rj_!\mathcal{F} \to Rj_*\mathcal{F} \to \mathcal{G} \to .$$

Using the long exact sequence of hypercohomology coming from the above triangle, we see exactly as on p.214 in [3] that all we have to show is that $\mathbb{H}^k(Z, \mathcal{G}) = 0$ for all $k < 0$. This vanishing obviously holds if we show that $\mathcal{G} = 0$.

This in turn is equivalent to the vanishing of all the local cohomology groups of $Rj_*\mathcal{F}$, namely $H^m(M_x, L_x) = 0$ for all $m \in \mathbb{Z}$ and for all points $x \in W_0$. Here $M_x = M_W \cap B_x$ for $B_x$ a small open ball at $x$ in $Z$ and $L_x$ is the restriction of the local system $L_\lambda$ to $M_x$.

The key observation is that, as already stated above, the action of an oriented elementary loop about the hypersurface $W_0$ in the local systems $L_\lambda$ and $L_x$ corresponds to multiplication by $\nu = \lambda^{-d} \neq 1$.

There are two cases to consider.

Case 1. If $x \in W_0 \setminus V$, then $M_x$ is homotopy equivalent to $\mathbb{C}^*$ and the corresponding local system $L_\nu$ on $\mathbb{C}^*$ is defined by multiplication by $\nu$, hence the claimed vanishings are obvious.

Case 2. If $x \in W_0 \cap V$, then due to the local product structure of stratified sets cut by a transversal, $M_x$ is homotopy equivalent to a product $(B' \setminus (V \cap B')) \times \mathbb{C}^*$, with $B'$ a small open ball centered at $x$ in $W_0$, and the corresponding local system is an external tensor product, the second factor being exactly $L_\nu$. The claimed vanishings follow then from the Künneth Theorem, see 4.3.14 [3].
This ends the proof of Proposition 2.2 and of Theorem 1.3.

Now we turn to the proof of Corollary 1.4. The first claim follows from Proposition 6.3.6 and Exercise 4.2.13 in [3] in conjunction to Theorem 2.10 v in [5]. In fact, to get the vanishing of \((R^m f_*\mathcal{C}U)_0\) one has just to write the exact sequence of the triple \((U, T_0, F)\) and to use the fact that \(H^m(U, \mathbb{C}) = 0\) for \(m < n + 1\). This vanishing comes from identifying \(U\) to a finite cyclic quotient of the Milnor fiber \(F(g_0)\) of the homogeneous isolated complete intersection singularity \(CW_0\) defined by \(g_0 : (CZ, 0) \to (\mathbb{C}, 0)\).

For the second claim, one has to use Theorem 2.10.i and Proposition 2.18 in [5].

The proof of Theorem 1.5 is completely similar to the second part of the proof above and we can safely leave it to the reader after the following remark. Let \(D = \sum_{j=0}^{m} d_j\) and let \(\alpha\) be a \(D\)-root of unity, \(\alpha \neq 1\). All we have to show is that \(H^q(M_W, \mathcal{L}_\alpha) = 0\) for all \(q \neq n + 1\), see for instance 6.4.6 in [3].

The action of an oriented elementary loop about the hypersurface \(W_0\) in the local systems \(\mathcal{L}_\alpha\) and in its restrictions \(\mathcal{L}_x\) as above corresponds to multiplication by \(\alpha \neq 1\). Therefore the above proof works word for word.

Now we turn to the proof of Theorem 1.6. We start by the following general remark.

**Remark 2.3.** If \(S\) is an \(s\)-dimensional stratum in a Whitney stratification of \(V\) such that \(x \in S\) and \(W_0\) is transversal to \(V\) at \(x\), then, due to the local product structure, the \(q\)-th reduced local Alexander polynomial \(\Delta_q(t)_x\) is the same as that of the hypersurface singularity \(V \cap T\) obtained by cutting the germ \((V, x)\) by an \((n + 1 - s)\)-dimensional transversal \(T\). It follows that these reduced local Alexander polynomials \(\Delta_q(t)_x\) are all trivial except for \(q \leq n - s\). It is a standard fact that, in the local situation of a hypersurface singularity, the Alexander polynomials can be defined either from the link or as the characteristic polynomials of the corresponding the monodromy operators. Indeed, the local Milnor fiber is homotopy equivalent to the corresponding infinite cyclic covering.

Let \(i : M_W \to Z \setminus W_1\) and \(j : Z \setminus W_1 \to Z\) be the two inclusions. Then one has \(\mathcal{L}_\alpha[n + 1] \in \text{Perv}(M_W)\) and hence \(\mathcal{F} = Ri_*(\mathcal{L}_\alpha[n + 1]) \in \text{Perv}(Z \setminus W_1)\), exactly as above.

Extend now the natural morphism \(Rj_!\mathcal{F} \to Rj_*\mathcal{F}\) to a distinguished triangle

\[Rj_!\mathcal{F} \to Rj_*\mathcal{F} \to G.\]

Applying Theorem 6.4.13 in [3] to this situation, and recalling the above use of Theorem 4.2 in [5], all we have to check is that \(H^m(M_x, \mathcal{L}_x) = 0\) for all points \(x \in W_1\) and \(m < n + 1 - \sigma\). For \(x \in W_1 \setminus W_0\), this claim is clear by the assumptions made.
The case when \( x \in W_1 \cap W_0 \) can be treated exactly as above, using the product structure, and the fact that the monodromy of \((W_1, x)\) is essentially the same as that of \((W_1 \cap W_0, x)\), see our remark above.

This completes the proof of Theorem 1.6.

**Remark 2.4.** Here is an alternative explanation for some of the bounds given in Theorem 4.2 in [10]. Assume that \( \lambda \) is a root of the Alexander polynomial \( \Delta_i(t) \) for some \( i < n + 1 \). Then it follows from Proposition 1.6 the existence of a point \( x \in W_1 \) and of an integer \( \ell \leq i \) such that \( \lambda \) is a root of the local Alexander polynomial \( \Delta_\ell(t)_x \). If \( x \in S \), with \( S \) a stratum of dimension \( s \), then by Remark 2.3 we have \( \ell \leq n - s \). This provides half of the bounds in Theorem 4.2 in [10]. The other half comes from the following remark. Since \( \lambda \) is a root of the Alexander polynomial \( \Delta_i(t) \), it follows that \( H^i(M_W, L_\lambda) \neq 0 \). This implies via an obvious exact sequence that \( \mathbb{H}^{i-n-1}(W_1, G) \neq 0 \). Using the standard spectral sequence to compute this hypercohomology group, we get that some of the groups \( H^p(W_i, \mathcal{H}^{i-n-1-p}G) \) are non zero. This can hold only if \( p \leq 2dim(Supp \mathcal{H}^{i-n-1-p}G) \). Since \( \mathcal{H}^{i-n-1-p}G_x = H^{i-p}(M_x, L_x) \) this yields the inequality \( p = i - \ell \leq 2s \) in Theorem 4.2 in [10].

**Remark 2.5.** Let \( \lambda \in \mathbb{C}^* \) be such that \( \lambda^d = 1 \), where \( d \), the quotient of \( \sum_{j=1,m} d_j \) by \( d_0 \), is assumed to be an integer. Let \( L_\lambda \) denotes the corresponding local system on \( M_W \). The fact that the associated monodromy about the divisor \( W_0 \) is trivial can be restated as follows. Let \( L_\lambda' \) be the rank one local system on \( M_V = Z \setminus V \) associated to \( \lambda \). Let \( j : M_W \to M_V \) be the inclusion. Then

\[
\sum_{j=1,m} d_j \quad \text{by} \quad d_0, \quad \text{is assumed to be an integer. Let} \quad L_\lambda \quad \text{denotes the corresponding local system on} \quad M_W. \quad \text{The fact that the associated monodromy about the divisor} \quad W_0 \quad \text{is trivial can be restated as follows. Let} \quad L_\lambda' \quad \text{be the rank one local system on} \quad M_V = Z \setminus V \quad \text{associated to} \quad \lambda. \quad \text{Let} \quad j : M_W \to M_V \quad \text{be the inclusion. Then}
\]

\[
\mathcal{L}_\lambda = j^{-1} \mathcal{L}_\lambda'.
\]

Let moreover \( \mathcal{L}_\lambda'' \) denote the restriction to \( \mathcal{L}_\lambda' \) to the smooth divisor \( W_0 \setminus (V \cap W_0) \).

Then we have the following Gysin-type long exact sequence

\[
\cdots \to H^q(M_V, \mathcal{L}_\lambda') \to H^q(M_W, \mathcal{L}_\lambda) \to H^{q-1}(W_0 \setminus (V \cap W_0), \mathcal{L}_\lambda'') \to H^{q+1}(M_V, \mathcal{L}_\lambda') \to \cdots
\]

exactly as in [3], p.222.

The cohomology groups \( H^*(M_V, \mathcal{L}_\lambda') \) and \( H^*(W_0 \setminus (V \cap W_0), \mathcal{L}_\lambda'') \) being usually simpler to compute than \( H^*(M_W, L_\lambda) \), this exact sequence can give valuable information on the latter cohomology groups.

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