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The $k$-path coloring problem in graphs of bounded treewidth: an application in integrated circuit manufacturing

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Abstract

We investigate the $k$-path coloring problem, a variant of vertex coloring arising in the context of integrated circuit manufacturing. We devise a direct (i.e. not relying on Courcelle’s celebrated theorem) polytime dynamic programming algorithm for graphs of bounded treewidth and we provide computational evidences, on true instances from the company Mentor Graphics, that the corresponding algorithm is suitable for practice, which is quite unusual for “bounded treewidth”-based dynamic programming approaches.

Keywords: $k$-path coloring, dynamic programming, bounded tree-width, Integrated circuit manufacturing

1. Introduction

Integrated circuits are made of several layers. In a nutshell, the bottom layers contains the transistors, while the other layers (called metal layers) are used to connect the different components to comply with the designed functionalities of the device. We usually distinguish two types of components in metal layers: vias and ‘wires’. The former components are used for vertical connections and the latter for horizontal ones, see Fig. 1 for an illustration.

![Figure 1: a 3D view of an integrated circuit (source: https://commons.wikimedia.org/wiki/File:Silicon_chip_3d.png)](https://commons.wikimedia.org/wiki/File:Silicon_chip_3d.png)

Integrated circuit manufacturing involves the production of each layer of the circuit iteratively. The core technique used in production is lithography. In brief, the idea is to etch each layer by exposing a photosensitive material to a light source through a mask: this creates a kind of mould, which is then filled with an appropriate conductor material (the mould is later removed through some chemical process). Optical distortion might however result in the fusion of components if attention is not paid to keep the...

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minimum distance between any two components above a certain threshold, called the lithography distance. When some components are below this distance, the production of the mould is typically decomposed into several rounds of lithography: sub-masks are defined for (feasible) subsets of components, and the mould is produced in sequence, a process called multiple patterning. The process requires the proper alignment of the sub-masks which is a major challenge as the number of rounds increases. Hence multiple patterning induces additional costs and time in the production process and the industry tries to keep the number of patterning steps small. The problem of finding the minimum number of patterning steps readily translates into a vertex coloring problem. Indeed, one can build a graph $G$ whose node set is the set of components and two components are adjacent if they are at a distance less or equal to the lithography distance. This graph is sometimes called the conflict graph. The minimum number of rounds needed to produce the mould is the chromatic number of $G$.

DSA-aware Multiple Patterning (DSA-MP) is a technique that combines Directed Self-Assembly (DSA) technology with lithography in order to go beyond the resolution limit imposed by lithography alone in the fabrication of integrated circuits. Due to the nature of the process, it is particularly useful for the manufacturing of vias. DSA-MP allows to introduce within a same sub-mask some vias that are closer to the lithography distance, under certain conditions. The main idea is to intentionally fuse some vias within larger objects, called guiding patterns, and then to correct the corresponding design with DSA in a second step. More precisely, the mould, possibly obtained after applying several rounds of lithography, is filled with a block copolymer in a random state that self-organize in a structured way after a chemical reaction is triggered: if the guiding patterns are designed appropriately, and thus if the vias that are fused satisfy some specific properties, one of the two polymers assembles into a set of cylinders that can then be turned into vias. We refer the reader to [3] for more details about integrated circuit manufacturing and DSA-aware multiple patterning.

One of the core problem in DSA-MP is again the problem of minimizing the number of lithography steps as the production time and costs are still dominated by lithography. We showed in [3] that several variants are of interest in the industry. One such variant consists in fusing “small chains of vias” and it reduces to the following variant of vertex coloring: given a graph $G = (V, E)$, a subset $F$ of edges of $E$ and an integer $k \geq 0$, color the vertices of $G$ with a minimum number of colors so that each color induces in $G$ a disjoint union of paths of length less or equal to $k$ and such that each arc in the path belongs to $F$. We call the problem the $k$-path coloring problem. Concretely, the node set of $G$ is the set of vias in a layer, the edges correspond to the pairs of vias that are closer than the lithography distance, $F$ are the pairs of vias whose distance is within a certain range specific to the DSA technology used (but smaller or equal to the lithography distance), and $k$ is a small number. The Electronic Design Automation (EDA) industry is especially interested in the 1-path and the 2-path coloring problems as they correspond to the current technological capabilities. Regular (vertex) coloring is equivalent to 0-path coloring, which proves that (i) $k$-path coloring is NP-hard in general and (ii) that we can only improve upon standard multiple patterning by choosing $k \geq 1$. The problem was introduced under the same name in [4] in the special case where $F = E$ (note that some authors use the same terminology for another variant of graph coloring, see for instance [13, 17, 20]). In particular it is known that $k$-path $L$-colorability is already hard for $L = 2$ and $k = 1$ [15] and for $L = 3$ and $k = 2$ [16].

The company Mentor graphics has developed in-house heuristics to solve the problem quickly. A typical design can be made of several hundred thousands of vias per layer and Mentor Graphics’ heuristics can solve (approximately) these instances in less than a second. However the company has interest in fast exact algorithms as well. Initially, their main interest for exact approaches lay in the possibility to assess the quality of their heuristics. However, because of our first encouraging results with integer programming [3], the company understood that exact approaches might actually find their way to production. While testing different integer programming models in [3], we observed that typical industrial instances exhibit some structure. We indeed noticed that most instances are extremely sparse and typically ‘tree-like’. The main reason is that the design of the circuit (and the placement of vias in particular) is made with lithography constraints in mind so that distances between vias is kept as large as possible. In this project, we decided to test whether exact algorithms exploiting the ‘tree-like’ property could be suitable (computationally wise) for production. A well-known measure of “tree-likeness” is the notion of treewidth introduced by Robertson and Seymour [21].

**Definition 1.** (Tree decomposition [21]). Let $G = (V, E)$ be a graph. A tree decomposition of $G$ is a pair $(X, T)$ where $T$ is a tree and $X = \{X_1, \ldots, X_n\}$ is the set of nodes of $T$, called bags such that
Additional definitions and notations

Given a graph $G = (V, E)$ and a subset $E'$ of edges of $E$, we call a $E'$-neighbor of a vertex $v$, a vertex $u$ such that $(u, v) \in E'$. Also a $E'$-path denotes a path with edges in $E'$. For a graph $G$, we sometimes denote by $V(G)$ the set of vertices of $G$, by $E(G)$ the set of edges of $G$, and, for $U \subseteq V(G)$, by $E[U]$ the set of edges of $G$ induced by the vertices in $U$ (that is with both extremities in $U$).

We call a pair $(G, f)$, where $G = (V, E)$ is a graph and $f$ is a positive weight function on its edge set, a weighted graph. We can represent a weighted graph by its (weighted) adjacency matrix, that is a matrix $A$ of size $|V| \times |V|$ such that $A(u, v) = f((u, v)) > 0$ for all $u, v \in V : (u, v) \in E$ and $A(u, v) = 0$ for all $u, v \in V : (u, v) \notin E$. Note that the support of $A$, that is, the $0/1$ matrix of same dimension as $A$ whose ones indicate pairs $(u, v)$ for which $A(u, v) > 0$, is the (standard) adjacency matrix of $G$. A graph $G$ can be considered as a weighted graph with weight function $f = 1$.

Let $P$ be a path of $G$ with node set $v_0, ..., v_p$, for some integer $p \geq 0$, and edge set $\{(v_i, v_{i+1})\}$ for $i = 0, ..., p - 1$ and let $U \subseteq V$. We call $v_0, v_p$ the extremities of $P$. We call a node of $P$ internal if it is not an extremity. Given a path $P$ such that $V(P) \cap U \neq \emptyset$ and a function $f : E(P) \rightarrow \mathbb{R^+} \setminus \{0\}$, we define the trace on $U$ of the weighted path $(P, f)$ as the weighted graph obtained from $P$ by ‘shrinking’ the internal nodes of $P$ not in $U$. More formally, if $i_1 < ... < i_l$ are the indices of the vertices of $U$ on $P$, for some integer $l \geq 1$, the trace of $(P, f)$ on $U$ is the weighted graph $(P', f')$, where $P'$ is the path with vertex set $\{v_0, v_{i_1}, ..., v_l, v_p\}$ and edge set $\{(v_0, v_{i_1}), (v_{i_1}, v_{i_2}), ..., (v_{i_{l-1}}, v_p)\}$ for $j = 1, ..., l - 1$, and for any edge $e$ of $P'$, $f'(e)$ is the length with respect to $f$ of the (sub)path of $P$ in-between the two end
points of \( e \). We define the trace on \( U \) of a union of disjoint paths as the union of the traces of each path intersecting \( U \).

A rooted tree decomposition is a tree decomposition where a bag is chosen as a root and the edges of the tree are oriented in the direction of the root bag. In a rooted tree decomposition \((X, T)\), we can naturally define children and parent bags and then, for a bag \( X_i \), we denote by \( V_i \) the union of the bags in the subtree of \( T \) rooted at \( X_i \), and by \( G_i \) the subgraph of \( G \) induced by the nodes in \( V_i \). It is convenient to present dynamic programming algorithms on nice tree decompositions. We give the definition below. Note that Kloks [18] proved that a tree decomposition can be converted into a nice tree decomposition (with at most four times the number of vertices in \( G \)) in linear time, while preserving the same treewidth.

**Definition 2. (Nice tree decomposition).** Let \( G = (V, E) \) be a graph. A nice tree decomposition is a rooted tree decomposition of \( G \) where every bag \( X_i \) of the tree has at most two children and is one of the four following types:

- **Leaf bag:** \( X_i \) has no children and contains only one vertex \( v \in V \), i.e. \(|X_i| = 1\).
- **Introduce bag:** \( X_i \) has exactly one child bag noted \( X_j \) such that \( X_i = X_j \cup \{v\} \) for some \( v \in V \) (and \( v \not\in V_j \)).
- **Forget bag:** \( X_i \) has exactly one child bag noted \( X_j \) such that \( X_i = X_j \setminus \{v\} \) for some \( v \in X_j \).
- **Join bag:** \( X_i \) has exactly two children noted \( X_{j1} \) and \( X_{j2} \) such that \( X_i = X_{j1} \cup X_{j2} \).

2. Dynamic programming

We now develop a dynamic programming algorithm to solve the \( k \)-path \( L \)-coloring problem on graphs of bounded treewidth. We assume that we are given a nice tree decomposition (by the result of Kloks [18] mentioned above). We start with providing the general idea of the algorithm.

Let \( G = (V, E) \) be a graph, let \( F \subseteq E \), and let us consider a nice tree decomposition \((X, T)\) of \( G \) of width \( w \). The general idea behind a dynamic programming approach to a decision problem on graph of bounded treewidth is to evaluate in each bag \( X_i \) whether there is a solution to the problem restricted to \( G_i \), and to store enough information about the corresponding solutions, to propagate and extend the solutions from the children bag(s) to the parent node iteratively. Hence, in our setting, for each bag \( X_i \), we would like to know if there exists a \( k \)-path \( L \)-coloring of \( G_i \). A \( k \)-path \( L \)-coloring of \( G_i \) is obviously a \( k \)-path \( L \)-coloring of its children bag(s). Hence we can try to identify the \( k \)-path \( L \)-coloring of \( G_i \) by checking which \( k \)-path \( L \)-coloring of the children bag(s) can be extended to a \( k \)-path \( L \)-coloring of \( G_i \). Now, because we want to design an efficient algorithm for testing \( k \)-path \( L \)-colorability of \( G \), we cannot keep a full list of all \( k \)-path \( L \)-colorings of \( G_j \) for all bags \( X_j \) as the list could grow exponentially as we move up the tree. Because each color of a \( k \)-path \( L \)-coloring induces a disjoint union of \( F \)-paths, and because of the structure of a tree decomposition, it is enough, as we will discuss in the proof of Theorem 1, to keep the colors of the vertices in \( X_i \) and the trace on \( X_i \) of the \( F \)-paths in each color. We call such a solution a partial solution (this is the standard terminology in the field) as it can be extended to build a \( k \)-path \( L \)-coloring of \( G_i \). We can encode, for each bag \( X_i \), the (disjoint) union of the traces on \( X_i \) of the \( F \)-paths of each color as a weighted graph with at most \( 3w \) vertices (we call this weighted graph the trace of the solution): indeed each path in the trace contains at least one node from \( X_i \) by definition and thus there are at most \( w \) such (disjoint) paths and, in the worst case, each path contains exactly two additional neighbors representing the extremities outside \( X_i \) (and each path may contain only one node from \( X_i \)). Hence the number of partial solutions is bounded by \( O(L^w \cdot k^{(3w)^2}) \) for each bag, which is constant for \( k \) and \( w \) bounded \((O(k^{(3w)^2})\) corresponds to the number of (weighted) adjacency matrices of a weighted graph with at most \( 3w \) vertices and weights in \( \{0, \ldots, k\} \): we could obviously use better data structures to improve upon this value). The dynamic programming algorithm and the (inductive) proof of its validity is given in Theorem 1.

**Theorem 1.** There exists an algorithm that, given a graph \( G = (V, E) \), a set of edges \( F \subseteq E \), and a nice tree decomposition \((X, T)\) of \( G \) of width \( w \) solves the \( k \)-path \( L \)-coloring problem in time \( O(L^w \cdot k^{2(3w)^2} \cdot (3w)^2 \cdot |T|) \).
Proof. Let $G = (V, E)$ be a graph, let $F \subseteq E$, and let us consider a nice tree decomposition $(X, T)$ of $G$ of width $w$. For conciseness, we simultaneously give the dynamic programming algorithm and the (inductive) proof that the procedure is correct. We need to distinguish according to the different types of bag.

- **Leaf bag**: In a leaf bag $X_i$, we have $|X_i| = \{v\}$ for some $v \in V$ and we can enumerate all $k$-path $L$-coloring of $G_i$ by enumerating all possible colorings of $v$. The partial solutions coincide with the solutions for $G_i$ so the trace is trivial in this case: it consists in the graph with vertex set $\{v\}$ and edge set $\{\}$, and any weight function as the set of edges is empty.

- **Introduce bag**: In an introduce bag $X_i$ we add a vertex $v$ from a child bag $X_j$, i.e., $X_i = X_j \cup \{v\}$ for some $v \in V$ (and $v \notin V_j$). In order to check whether a $k$-path $L$-coloring $S_i$ of $G_i$ can be extended to a $k$-path $L$-coloring of $G_i$, we only need to check which coloring $c(v)$ of $v$ is compatible with $S_j$. Adding $v$ to a color set adds edges in the subgraph of $G$ induced by the nodes of color $c(v)$. We want the resulting graph to be a union of disjoint $F$-paths of $G_i$ of length at most $k$. Let $G_i^{c(v)}$ (resp. $G_j^{c(v)}$) be the subgraph of $G_i$ (resp. $G_j$) induced by the node of color $c(v)$. Because of the structure of a tree decomposition, $v$ is only adjacent to vertices of $X_i$ in $G_i$. It follows that in order to check whether $S_j$ can be extended, it is enough to check whether the set of edges $E \subseteq E$ incident to both $v$ and a node of color $c(v)$ in $X_i$ are all in $F$ and that adding $v$ and $E$ (with weight one) to the trace $(H, f)$ of $G_j^{c(v)}$ on $X_j$ yields a weighted graph $(H', f')$ whose support $H'$ is a union of disjoint paths of weight at most $k$ with respect to $f'$ (note that adding $v$ might merge two previously disjoint paths). This can be checked in linear time by checking that: (i) $v$ has no more than two neighbors of color $c(v)$ in $X_i$ (and that are linked by an edge of $F$), (ii) those neighbors are the extremities of different paths in the trace graph $(H, f)$, and (iii) the weight of the $F$-path containing $v$ is no more than $k$ (for this we need to know the weight of the path(s) starting at the neighbors of $v$ in $(H, f)$, which can be easily computed by traversing the edges of the corresponding path(s) in $(H, f)$). In case of a positive result, substituting the trace $(H', f')$ by $(H, f)$ actually yields the trace on $X_i$ of the extension of $S_j$ to $G_i$.

- **Forget bag**: In a forget bag $X_i$, we delete a vertex $v$ from a child bag $X_j$, i.e., $X_i = X_j \setminus \{v\}$ for some $v \in V$. In such a node of the tree decomposition, any partial solution $P_j$ for $X_j$ yields a partial solution $P_i$ for $X_i$. Indeed, any $k$-path $L$-coloring for $G_j$ that would be consistent with $P_j$ would again be a $k$-path $L$-coloring of $G_i$ as $G_i$ and $G_j$ coincide. The traces of the solution on $X_i$ and $X_j$ differ though, but we can easily recover the trace on $X_i$ from the trace on $X_j$. Indeed, we can update the coloring and the trace as follows: the coloring is kept identical but it is restricted to the nodes of $X_i$ and the new trace is simply the trace on $X_i$ of the trace in $P_j$ (that is we simply ‘shrink’ $v$).

- **Join bag**: In a join bag $X_i$, we want to check, for every pair $(P_j_1, P_j_2)$ of partial solutions for $X_{j_1}$ and $X_{j_2}$, whether the union of any possible extensions of $P_j_1, P_j_2$ to solutions of $G_{j_1}$ and $G_{j_2}$ yield a solution for $G_i$. More precisely, let $P_{j_1}$ and $P_{j_2}$ be two partial solutions for $X_{j_1}$ and $X_{j_2}$ respectively and $S_{j_1}, S_{j_2}$ any two extensions. We first need to check that common vertices in $G_{j_1}$ and $G_{j_2}$ are colored in the same way in both $P_{j_1}, P_{j_2}$. Because common vertices are vertices in $X_j$ (by the property of tree decompositions), it is enough to check that the partial solutions agree on the colors of the vertices in $X_j$. Then we need to check that the union of the $F$-paths induced by a same color $c$ in $S_{j_1}$ and $S_{j_2}$ is a (disjoint) union of $F$-paths of length at most $k$ (the union of the $F$-paths is actually the graph induced by color $c$ in $G_i$ as the only edges in common are edges in $E[X_i]$ - which are in both $G_{j_1}$ and $G_{j_2}$). Now it is enough to look at the multi-union of the traces associated with color $c$ in $P_{j_1}$ and $P_{j_2}$ respectively (formally, the multi-union is defined as follows: we take the union of the two corresponding traces (union of the vertices and edges, with same weight), and we allow parallel edges, except for ‘original edges’ in $E[X_i]$ - that is, edges that are in both trace graphs and of weight 1: edges of the traces of weight 2 or more are not edges from $E[X_i]$). Indeed we simply need to check that the multi-union of the traces induces a disjoint union of paths of weight at most $k$. Note that we do not consider parallel edges for ‘original edges’ in $E[X_i]$ in the definition of the multi-union as, in contrast with the other edges in the trace (which cannot appear as ‘subpaths’ in both solutions $S_{j_1}$ and $S_{j_2}$ because they include a vertex in $V_j \setminus X_i$ or
$V_{j_2} \setminus X_i$ that was shrunk (and thus cannot appear in the other solution because of the properties of tree decompositions), they correspond to edges that could (and will) be in the $F$-paths associated with color $c$ in both $S_{j_1}$ and $S_{j_2}$. In case of a positive outcome, the trace of the solution for the join bag $X_i$ is simply the union of the two traces (as defined above).

The discussion above shows that we do not miss any partial solution as we move up the tree (and that each partial solution we build is actually valid). Hence there exist a $k$-path coloring of $G$ if and only if there exists a partial solution in the root node of the tree decomposition when applying the procedure above.

The computational time at each node of the nice tree decomposition is dominated by the join bag case. For each coloring of the vertices of $X_i$ we then need to check whether the union of the support graph of the trace of each possible partial coloring for $X_{j_1}$ and $X_{j_2}$ are compatible: in the worst case we need to check $L^w$ coloring, and $k(3w)^2 \times k(3w)^2$ pairs of traces, and checking that the union of the traces in each color still yields a union of disjoint paths of length at most $k$ can be done in time $O((3w)^2)$, by first building the weighted adjacency matrix of the union of the traces (and checking that it does not contain multi-edges) and by then adapting the depth first search algorithm, since each weighted adjacency matrix has size at most $(3w)^2$. The overall complexity is thus bounded by $O(L^w \cdot k^2(3w)^2 \cdot (3w)^2 \cdot |T|)$.

\[\square\]

3. Numerical Experiments

In [6], Arnborg et al have proved that deciding if the treewidth of a graph $G$ is at most $w$, where $w \geq 0$, is NP-complete. However, there are good heuristics to determine a tree decomposition of a given graph $G$ with a width ‘close to’ the treewidth. Different heuristics are presented and compared in [10]. Moreover, Bodlaender [8] showed that for every fixed value of $w$, there is a linear-time algorithm that finds a tree decomposition of width $w$ (if it exists).

We propose to solve DSA-MP on real instances arising from integrated circuit manufacturing by first using a heuristic to get a ‘small’ tree-decomposition of the graph, and by then using the dynamic programming algorithm described in the previous section to solve the problem (actually, we tailored the algorithms to the case where $k = 1$ and $k = 2$ to make them slightly simpler to implement, see [2] for the details about the corresponding implementations). We used D-FLAT to implement the corresponding algorithm [1]. D-FLAT is a software system that allows for rapid prototyping of dynamic programming algorithms for graphs of ‘small’ treewidth. It has the advantage to implement different state-of-the-art heuristics to find a close-to-optimal nice tree decomposition and offers a generic language (Answer Set Programming) to describe how to extend solutions for each type of nodes of the nice tree decomposition.

We ran D-FLAT iteratively to solve the $k$-path $L$-colorability problem starting from $L = 2$ and increasing $L$ until a solution was found. All tests were done on a machine equipped with an Intel(R) Xeon(R) CPU E5-2640 2.60 GHz and a memory of 529GB. As already observed, the typical industrial designs are made of several hundred thousands of vias, but because the placement of the vias is made as to anticipate as much as possible conflicts that may arise from lithography, the ‘conflict graph’ is usually extremely sparse and contains only hundreds or thousands of different connected components. Since the optimization of each connected component can be parallelized, we focus attention on the computational time for each connected component individually.

We report computational experiments on 23 connected components of increasing size arising from true industrial instances in Table 1. We can see that the linear time complexity is confirmed experimentally. All instances could be solved in less than 30 seconds.

The 23 industrial instances used in this study share similar properties with some pseudo-industrial instances generated in [3] (instances where the resolution limit is set to 31nm). We thus compared the dynamic programming approach developed in this paper to the best integer programming model developed in [3] on the same set of instances (and on the same machine) and the results are reported in Table 2.

It appears that the best integer programming formulations from [3] outperform the dynamic programming approach on these instances. However, from an application point of view, the computation times of the dynamic programming approach are very satisfactory (both on the true industrial instances and on the pseudo-industrial ones). Indeed the manufacturing process is defined before the launch of
TW(G) is the width of the tree decomposition returned by D-FLAT heuristics, and DFLAT $k$ time (sec) represents the time used by DFLAT (resp. IP) to solve the $k$-path coloring problem, and $\chi_{\text{path}}^k$ is the $k$-path chromatic number.

| Instance name | $|V|$ | $|E|$ | $\omega(G)$ | $\Delta(G)$ | DFLAT TW(G) | $\chi_{\text{path}}^1$ | CPU time (sec) | $\chi_{\text{path}}^2$ | CPU time (sec) |
|---------------|------|------|-------------|-------------|-------------|----------------|---------------|----------------|---------------|
| industrial_1  | 191  | 242  | 3           | 5           | 3           | 2.74          | 2             | 0.89           | 2             |
| industrial_2  | 139  | 188  | 3           | 5           | 3           | 21.15         | 2             | 64.89          | 2.11          |
| industrial_3  | 96   | 117  | 3           | 4           | 3           | 0.8          | 2             | 1.69           | 0.34          |
| industrial_4  | 120  | 147  | 3           | 3           | 3           | 0.87          | 2             | 0.95           | 0.46          |
| industrial_5  | 170  | 213  | 3           | 5           | 3           | 5.45          | 2             | 9.57           | 1.52          |
| industrial_6  | 185  | 229  | 3           | 5           | 3           | 2.0          | 2             | 38.45          | 1.89          |
| industrial_7  | 203  | 256  | 3           | 3           | 3           | 8.83          | 2             | 92.05          | 0.73          |
| industrial_8  | 122  | 160  | 3           | 5           | 3           | 0.83          | 2             | 2.85           | 1.35          |
| industrial_9  | 152  | 191  | 3           | 3           | 3           | 2.8           | 2             | 31.13          | 1.96          |
| industrial_10 | 159  | 175  | 4           | 3           | 3           | 0.91          | 2             | 2.3            | 0.64          |

Table 2: Pseudo-industrial instances characteristics and results: $\omega(G)$ is the size of the maximum clique in $G$, $\Delta(G)$ is the maximum degree of $G$, DFLAT TW(G) is the width of the tree decomposition returned by D-FLAT heuristics, and $\chi_{\text{path}}^k$ is the $k$-path chromatic number.

Table 1: Industrial instances characteristics and results: $\omega(G)$ is the size of the maximum clique in $G$, $\Delta(G)$ is the maximum degree of $G$, DFLAT TW(G) is the width of the tree decomposition returned by D-FLAT heuristics, and $\chi_{\text{path}}^k$ is the $k$-path chromatic number.

the production and obtaining optimal solutions in a few minutes is perfectly fine. Besides, the dynamic programming approach offers the additional benefit of not requiring the use of expensive licenses (the license fees for advanced programming software such as CPLEX or Gurobi, can reach prohibitive amounts when the algorithm must run in parallel on thousands of instances and this has not been considered a viable alternative by the company). The algorithm is now implemented in the company’s toolbox. They run the corresponding algorithms both to compare and improve their heuristics and also to find the optimal solution in cases where the algorithm ends within a certain time (beyond the limit, they use their heuristics).

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