Pseudo-Hermitian coherent states under the generalized quantum condition with position-dependent mass

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Abstract

In the context of the factorization method, we investigate the pseudo-Hermitian coherent states and their Hermitian counterpart coherent states under the generalized quantum condition in the framework of a position-dependent mass. By considering a specific modification in the superpotential, suitable annihilation and creation operators are constructed in order to reproduce the Hermitian counterpart Hamiltonian in the factorized form. We show that by means of these ladder operators, we can construct a wide range of exactly solvable potentials as well as their accompanying coherent states. Alternatively, we explore the relationship between the pseudo-Hermitian Hamiltonian and its Hermitian counterparts, obtained from a similarity transformation, to construct the associated pseudo-Hermitian coherent states. These latter preserve the structure of Perelomov’s states and minimize the generalized position–momentum uncertainty principle.

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1. Introduction

Motivated by the fact that not only the quantum systems with a constant mass but also the quantum systems endowed with position-dependent mass appear as the dynamical algebra, it is of physical and mathematical interest to construct the various coherent states associated with such quantum systems. Recently, the study of the Schrödinger equation with position-dependent effective mass has attracted a lot of attention [1–8]. This is because such systems have found wide applications in various fields such as the study of electronic properties of the...
semiconductors [9], $^3$He clusters [10], quantum wells, wires and dots [11], quantum liquids [12], graded alloys and semiconductor heterostructures [13], etc.

The coherent states have been one of the fastest developing areas in mathematical physics in the last four decades. It was Glauber [14] who showed that the coherent states can be used to describe the electromagnetic correlation functions in the context of quantum optics. As a result, the constructions of such states are defined in three different ways but are all equivalent for the harmonic oscillator: (i) they are eigenstates of the annihilation operator [15], (ii) they minimize the position–momentum uncertainty relation [16] and (iii) they are displaced versions of the ground wavefunction [17]. Several approaches and techniques have been used in order to construct the coherent states, namely the Nieto–Simmons method [18], the irreducible representation of a Lie group [19], the algebraic method [20], supersymmetric quantum mechanics [21–23] and the mixed supersymmetric-algebraic method [24]. Recently, coherent states endowed with position-dependent mass have been constructed using the intertwining operator [25, 26].

On the other hand, there has been a great deal of interest in the properties of pseudo-Hermiticity [27] and $\mathcal{PT}$-symmetric [28] Hamiltonians since it was shown that some of them may have a real and positive spectrum and are considered as prototypes for solvable models of Lie algebraic type [29]. By definition, a Hamiltonian $H$ is said to be pseudo-Hermitian if it satisfies $H^\dagger = \zeta H \zeta^{-1}$, where $\zeta$ is a positive-definite Hermitian operator. However, it was shown that such a Hamiltonian is equivalent to a Hermitian counterpart Hamiltonian $h$ according to $h = \rho H \rho^{-1}$, with $\rho = \sqrt{\zeta}$ [30].

In this context, Jones [31] and Bagchi et al [32] demonstrated that a non-Hermitian Hamiltonian, proposed earlier by Swanson [33],

$$H^{(\alpha, \beta)} = \omega (\eta^\dagger \eta + \frac{1}{2}) + \alpha \eta^2 + \beta \eta^4,$$  

(1.1)

where $\omega$, $\alpha$ and $\beta$ are three real parameters, admits an equivalent Hermitian Hamiltonian $\tilde{h}^{(\alpha, \beta)}$ where $\eta$ and $\eta^\dagger$ obey the standard commutation relation; i.e. $[\eta, \eta^\dagger] = 1$.

The aim of this paper is the construction of coherent states for pseudo-Hermitian systems with position-dependent mass which is expected to enrich this branch of physics further. We plan to show that for those non-Hermitian Hamiltonians that can be expressed as (1.1), we can construct formally a set of position-dependent mass coherent states for Hermitian counterpart Hamiltonians (HCS) $\tilde{H}^{(\kappa)}$ under a generalized quantum condition $[\tilde{\eta}_\kappa, \tilde{\eta}_\kappa^\dagger] = (\kappa + 1) F[x] I_\delta$, where $F[x]$ is some generating functional acting on the identity operator belonging to the Hilbert space $H$. Afterwards, we explore a similarity transformation, which maps the original Hamiltonian $\tilde{H}^{(\kappa)}$ onto $\tilde{h}^{(\kappa)}$, to deduce the associated pseudo-Hermitian coherent states (PHCS).

Indeed the ladder operators $\eta$ and $\eta^\dagger$ in (1.1) based upon the deformed momentum operator $\Pi = U(x) p U(x)$ lead to a corresponding Hermitian counterpart Hamiltonian with a coordinate dependence in mass; however, it does not reproduce the position-dependent mass in the factorized form. In other words, they do not act as ladder operators on the eigenfunctions. Fortunately, we will see that this inadequacy of the usual approach may be circumvented by invoking a modification in the superpotential provided with the constraint $\beta = 0$ (or equivalently $\alpha = 0$ due to the symmetric nature of the new Hermitian Hamiltonian $\tilde{h}^{(\kappa)}$, where $\kappa$ is either $\alpha$ or $\beta$.) Then the new ladder operators $\tilde{\eta}_\kappa$ and $\tilde{\eta}_\kappa^\dagger$ are nothing but annihilation and creation operators endowed within the generally deformed oscillator algebra. The corresponding coherent states are then constructed with the modified superpotential under the generalized quantum condition. It is found that the analytical expressions of such HCS (i.e. PHCS) preserve the structure of Perelomov’s approach and minimize the generalized
position–momentum uncertainty principle. This method of solution is applied to a wide range of exactly solvable potentials and the corresponding HCS are constructed.

We organized our paper as follows. In section 2, we apply the concept of pseudo-Hermiticity to Hamiltonian (1.1) deducing the corresponding Hermitian Hamiltonian $H^{(\alpha, \beta)}$. A new Hermitian counterpart Hamiltonian $\hat{H}^{(\alpha, \beta)}$ is reproduced in the factorized form through the modified creation and annihilation operators. In section 3, the HCS are constructed under a generalized quantum condition in the context of Perelomov’s states and exploring the relationship between the pseudo-Hermitian Hamiltonian and its Hermitian counterpart, obtained from the similarity transformation, a set of PHCS is constructed straightforwardly. Applying the procedure of section 3, various exactly solvable potentials as well as their HCS are constructed in section 4. Finally, we present our conclusions in the last section.

2. Modified Hermitian counterpart within generalized quantum condition

In units wherein $\hbar = m_0 = 1$, the general first-order differential forms for $\eta$ and $\eta^\dagger$ are given by [25]

$$\eta = \frac{1}{\sqrt{2}} \left( U(x) \frac{d}{dx} U(x) + W(x) \right), \quad \eta^\dagger = \frac{1}{\sqrt{2}} \left( -U(x) \frac{d}{dx} U(x) + W(x) \right),$$  \hspace{1cm} (2.1)

where $U(x)$ and $W(x)$ are the real functions. Here the generalized quantum condition yields

$$[\eta, \eta^\dagger] = U^2(x) W'(x) \mathbf{1}_\beta,$$  \hspace{1cm} (2.2)

where the prime denotes the derivative with respect to $x$. Substituting (2.1) into Hamiltonian (1.1), the corresponding eigenvalue equation reads

$$\left( -\frac{\Omega^{(-)}}{2} U^4(x) \frac{d^2}{dx^2} + K_{\alpha, \beta}(x) \frac{d}{dx} + R_{\alpha, \beta}(x) \right) \psi(x) = E \psi(x),$$  \hspace{1cm} (2.3)

with

$$K_{\alpha, \beta}(x) = (\alpha - \beta) U^2(x) W(x) - 2 \Omega^{(-)} U^3(x) U'(x)$$  \hspace{1cm} (2.4)

$$R_{\alpha, \beta}(x) = \frac{1}{2} \left[ \omega [1 - U^2(x) W'(x)] + \Omega^{(+)} W^2(x) + (\alpha - \beta)[U^2(x) W(x)]' - \Omega^{(-)} U^2(x) [2 U^2(x) + U(x) U''(x)] \right]$$  \hspace{1cm} (2.5)

and $\Omega^{(\pm)} = \omega \pm \alpha \pm \beta$.

In order to bring (2.3) to a Schrödinger equation in the Hermitian form, it is straightforward to remove the first-derivative term. For this purpose, define a similarity transformation

$$\psi(x) = \rho_{\alpha, \beta}^{-1}(x) \chi(x)$$  \hspace{1cm} [32],

where $\rho_{\alpha, \beta}(x)$ is defined as

$$\rho_{\alpha, \beta}(x) = A'(x) \exp \left[ - \int^x \frac{K_{\alpha, \beta}(y)}{U^2(y)} dy \right] = A'(x) \left( U(x) \right)^{2\Omega^{(-)}} \exp \left[-(\alpha - \beta) \int^x \frac{W(y)}{U^2(y)} \frac{dy}{U'(y)} \right],$$  \hspace{1cm} (2.6)

with $A(x)$ an unknown function and $s$ a parameter to be determined. Without loss of generality, we set $\omega = \alpha + \beta + 1$, i.e. $\Omega^{(-)} \equiv 1$. To determine $A(x)$ and $s$, we substitute (2.4) and (2.6) into (2.3) to obtain $A(x) = U(x)$ and $s = -2$; then a Hermitian Hamiltonian $H^{(\alpha, \beta)} = \rho_{\alpha, \beta}(x) H^{(\alpha, \beta)} \rho_{\alpha, \beta}^{-1}(x)$ is equivalent to $H^{(\alpha, \beta)}$. 

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Therefore, \( \rho_{\alpha, \beta}(x) \), \( h^{(\alpha, \beta)} \) and the associated effective potential \( V_{\text{eff}}^{(\alpha, \beta)}(x) \) read explicitly

\[
\rho_{\alpha, \beta}(x) = \exp \left[ -(\alpha - \beta) \int^x W(y) \frac{dy}{U^2(y)} \right],
\]
\[
h^{(\alpha, \beta)} = -\frac{1}{2} \frac{d}{dx} U^2(x) \frac{d}{dx} + V_{\text{eff}}^{(\alpha, \beta)}(x),
\]
\[
V_{\text{eff}}^{(\alpha, \beta)}(x) = \frac{\alpha^2 - 4\alpha\beta}{2} W^2(x) - \frac{\alpha}{2} U^2(x) W'(x) + \frac{\alpha}{2} + V_{\gamma}(x),
\]

where \( V_{\gamma}(x) = -U^2(x) U'(x) - U^3(x) U''(x)/2 \) is a dependent dimensionless mass potential.

We need to note that \( h^{(\alpha, \beta)} \) is symmetric with respect to the parameters \( \alpha \) and \( \beta \), and the condition \( \rho_{\alpha, \beta}(x) = \rho_{\beta, \alpha}^{-1}(x) \) holds [32], so that \( H^{(\alpha, \beta)^T} = \zeta_{\alpha, \beta} H^{(\alpha, \beta)} \zeta_{\alpha, \beta}^{-1} \) is pseudo-Hermitian, with

\[
\zeta_{\alpha, \beta} \equiv \rho_{\beta, \alpha}^{-1}(x) \rho_{\alpha, \beta}(x) = \exp \left[ -2(\alpha - \beta) \int^x W(y) \frac{dy}{U^2(y)} \right] > 0,
\]

for \( U(x) > 0 \). The second equation in (2.7) clearly reveals that, for coordinate dependent mass, \( U^4(x) \) could be identified with the inverse of a certain mass function, i.e. \( U(x) = m^{-1/4}(x) \), with \( M(x) = m_0 m(x) \) the dimensionless mass. A consequence of this is that \( h^{(\alpha, \beta)} \) is not factorized under the ladder operators \( \eta_x \) and \( \eta^+_x \) because they do not give a closed algebra.

It should be pointed out that it is difficult to obtain suitable ladder operators as defined in conventional quantum mechanics which permit the factorization of \( h^{(\alpha, \beta)} \) in (2.7). The idea is to develop the scheme of a factorization method by reducing the number of parameters \( \alpha \) and \( \beta \) to one real parameter \( \kappa \) by making use of the modified superpotential, so that the new annihilation operator, \( \tilde{\eta}_x \), annihilates the ground state.

To this end, we can recast \( h^{(\alpha, \beta)} \) in the new form

\[
h^{(\alpha, \beta)} \rightarrow \tilde{h}^{(\kappa, \beta)} = \tilde{\eta}_x \tilde{\eta}_x + \delta_x,
\]

where \( \kappa \) is either \( \alpha \) or \( \beta \), while \( \tilde{\eta}_x \) and \( \tilde{\eta}_x^+ \) are expressed in terms of the modified superpotential \( \tilde{W}_x(x) = p W(x) + q U'(x) \) as

\[
\tilde{\eta}_x = \frac{1}{\sqrt{2}} \left( U(x) \frac{d}{dx} U(x) + \tilde{W}_x(x) \right),
\]
\[
\tilde{\eta}_x^+ = \frac{1}{\sqrt{2}} \left( -U(x) \frac{d}{dx} U(x) + \tilde{W}_x(x) \right),
\]

with \( W(x) = W(x) + q_1 U'(x) \) and \( p, q \) and \( q_1 \) are some real constants depending on \( \kappa \).

Substituting (2.10) into (2.9) and comparing with (2.7), we obtain two equivalent solutions with respect to the parameters \( \alpha \) and \( \beta \), as well as a constraint controlling the parameters \( p, q \) and \( q_1 \),

\[
p = \begin{cases} 
\alpha + 1, & \text{for } \beta = 0 \\
\beta + 1, & \text{for } \alpha = 0 \end{cases} \quad \text{and} \quad pq_1 + q = 0.
\]

As a consequence, it is found that the modified superpotential is \( \tilde{W}_x(x) = (\kappa + 1) W(x) \), no matter what the explicit form of \( q \) and \( q_1 \) is. Then inserting (2.10) into (2.9), the general form of the potential \( \tilde{V}_x(x) \) is given by

\[
\tilde{V}_x(x) = \frac{(\kappa + 1)^2}{2} W^2(x) - \frac{\kappa + 1}{2} U^2(x) W'(x) + \delta_x,
\]
where $\delta_\kappa = (\kappa + 1)/2$. Then it is easy to verify that $\tilde{\eta}_\kappa$, $\tilde{\eta}_\kappa^\dagger$ and the Hamiltonian $\tilde{H}^{(\kappa)}$ satisfy the mutually commutation relations

$$
\left[ \tilde{\eta}_\kappa, \tilde{\eta}_\kappa^\dagger \right] = (\kappa + 1) U^2(x) W'(x) 1_{\delta_\kappa},
\left[ \tilde{H}^{(\kappa)}, \tilde{\eta}_\kappa^\dagger \right] = (\kappa + 1) \tilde{\eta}_\kappa U^2(x) W'(x),
\left[ \tilde{H}^{(\kappa)}, \tilde{\eta}_\kappa \right] = - (\kappa + 1) U^2(x) W'(x) \tilde{\eta}_\kappa.
$$

This algebra is nothing but the generally deformed oscillator algebra with the structure function $(\kappa + 1) U^2(x) W'(x)$. As a consequence, the simplest form of a (shifted) harmonic oscillator can be taken for $\left[ \tilde{\eta}_\kappa, \tilde{\eta}_\kappa^\dagger \right] = \kappa + 1$ which leads to express $W(x) = \mu(x) + \text{const}$, where hereafter we adopt the definition $\mu(x) = \int^x dy/U^2(y)$.

Now by (2.9) acting on the state $|v\rangle \in \mathcal{F}_\delta$, one obtains the Hamiltonian in the factorized form

$$
\tilde{\eta}_\kappa^\dagger \tilde{\eta}_\kappa |v\rangle = \Delta E^{(\kappa)}_{\tilde{\eta}_\kappa} |v\rangle,
$$

where $\Delta E^{(\kappa)}_{\tilde{\eta}_\kappa} = E_v - \delta_\kappa$.

It is worth noting that the energy eigenvalues expressed in (2.14) are chosen such that $\delta_\kappa$ corresponds to the ground-state energy $\delta_\kappa = \epsilon_{0,\kappa}$. Then, if $\epsilon_{0,\kappa} \neq 0$ (i.e. $\kappa \neq -1$, which is as it should be since the potential in (2.12) vanishes), then, due to (2.13), $\tilde{\eta}_\kappa$ and $\tilde{\eta}_\kappa^\dagger$ are the ladder operators of the position-dependent mass system.

3. Construction of Hermitian and pseudo-Hermitian coherent states

Our key aim now is to construct a set of position-dependent mass Hermitian coherent states (HCS) $|\xi_\kappa\rangle$ for the Hamiltonian $\tilde{H}^{(\kappa)}$. By exploring that there exists a similarity transformation which maps the pseudo-Hermitian Hamiltonian adjointly to a Hermitian Hamiltonian, we deduce the PHCS $|\Xi_\kappa\rangle$ for the Hamiltonian $\tilde{H}^{(\kappa)}$ with regard to the properties of the corresponding metric $\rho^{(-1)}(\kappa)$.

Naturally, we expect to find many shapes for $|\Xi_\kappa\rangle$ due to the feature that the similarity transformation is not unique when the only requirement for $\tilde{H}^{(\kappa)}$ is its Hermiticity [29]. However, the determination of PHCS may be achieved by assuming some conditions which apply to functions associated with coherent states.

3.1. Position-dependent mass HCS

In the last section, a position-dependent mass is shown to be factorized by two adjoint operators $\tilde{\eta}_\kappa$ and $\tilde{\eta}_\kappa^\dagger$; then the related coherent states $|\xi_\kappa\rangle$ for the Hamiltonian $\tilde{H}^{(\kappa)}$ are constructed as eigenstates of the annihilation operator $\tilde{\eta}_\kappa |\xi_\kappa\rangle = \xi_\kappa |\xi_\kappa\rangle$ and thus $\tilde{\eta}_\kappa$ annihilates the ground state $\tilde{\eta}_\kappa |0\rangle_\kappa = 0$.

The ground state $|0\rangle_\kappa$ can be calculated by integrating the first equation of (2.10)

$$
|0\rangle_\kappa = U^{-1}(x) \exp \left[ - (\kappa + 1) \int^{\mu(x)} W(y) \, d\mu(y) \right],
$$

where we have used the definition of $\mu(x)$ quoted above. Here the presence of $\kappa$ in our previous results suggests that the complex parameter $\xi_\kappa$ depends on some real parametric function $\gamma(\kappa)$ such as $\xi_\kappa = \gamma(\kappa) \xi$. The choice of the function $\gamma(\kappa)$ in the sense of this paper corresponds to fixing the form of HCS and PHCS as reviewed below.

As usual, the coherent states can be frequently introduced by using the displacement and unitary operator

$$
|\xi_\kappa\rangle = \mathcal{D}(\xi_\kappa) |0\rangle_\kappa.
$$
To this purpose, let us assume that $\mathcal{D}(\xi_\kappa)$ acts on the ladder operators $\hat{n}_k$ and $\hat{n}_k^\dagger$ according to the generalized scheme
\begin{equation}
\mathcal{D}(\xi_\kappa)^\dagger \hat{n}_k \mathcal{D}(\xi_\kappa) = \hat{n}_k + \xi_\kappa \left[ \hat{n}_k, \hat{n}_k^\dagger \right],
\end{equation}
and inserting (3.3) into (3.2), we obtain the usual (shifted) harmonic oscillator coherent states.

In order to construct the coherent states $|\xi\rangle_\kappa$ of (3.2), we look for $\mathcal{D}(\xi_\kappa)$ as [25]
\begin{equation}
\mathcal{D}(\xi_\kappa) = \exp [i S(\xi_\kappa)],
\end{equation}
where $S(\xi_\kappa) = S_\kappa(\xi) = \gamma(\kappa) S(\xi)$ is a real and linear function on $\xi_\kappa$ and verifies the relation $S_\kappa(\xi) = -i \xi_\kappa \hat{Q}_\kappa$, where $\hat{Q}_\kappa$ is an operator to be determined. Moreover, as $\mathcal{D}(\xi_\kappa)$ is a unitary one leads to identify that both $S_\kappa(\xi)$ and $\hat{Q}_\kappa$ are Hermitian if $\xi_\kappa = -\xi_\kappa^*$, i.e. $\Re(\xi_\kappa) = 0$. Indeed by imposing the requirements
\begin{equation}
[S_\kappa(\xi), \hat{n}_k] = i \xi_\kappa \left[ \hat{n}_k, \hat{n}_k^\dagger \right],
\end{equation}
and substituting $S_\kappa(\xi) = -i \xi_\kappa \hat{Q}_\kappa$ into (3.5), a brief examination yields two solutions for $\hat{Q}_\kappa$: either $\hat{n}_k^\dagger$ or $\hat{n}_k + \hat{n}_k^\dagger$. It is obvious that the second solution is that in which we are interested, because the first solution will be omitted in order to avoid the ill-defined Hermiticity condition imposed on $\hat{Q}_\kappa$, while the second one combined with the condition $\xi_\kappa = -\xi_\kappa^*$ yields the well-known deformed displacement operator $\exp \left[ \xi_\kappa \hat{n}_k^\dagger - \xi_\kappa^* \hat{n}_k \right]$. At this stage, it is important to prove that (3.5) verify (3.3), respectively. To this end, let us consider the operator $\hat{P}_\kappa = [\hat{n}_k, \mathcal{D}(\xi_\kappa)]$. Expanding $\exp [i S_\kappa(\xi)]$ in the development of the Taylor series
\begin{equation}
\hat{P}_\kappa = -\sum_{k=0}^\infty \frac{i^k}{k!} \left[ S_\kappa^k(\xi), \hat{n}_k \right],
\end{equation}
and using (3.5) a straightforward calculation leads to a recursion relation which is satisfied by the commutators
\begin{equation}
\left[ S_\kappa^k(\xi), \hat{n}_k \right] = i k \xi_\kappa S_\kappa^{k-1}(\xi) \left[ \hat{n}_k, \hat{n}_k^\dagger \right],
\end{equation}
and inserting (3.7) into (3.6), we have
\begin{equation}
\hat{P}_\kappa = \xi_\kappa \exp^{i S_\kappa(\xi)} \left[ \hat{n}_k, \hat{n}_k^\dagger \right].
\end{equation}
On the other hand, starting from $\hat{P}_\kappa = [\hat{n}_k, \mathcal{D}(\xi_\kappa)]$ and multiplying both sides of the expression on the left by $\mathcal{D}(\xi_\kappa)^\dagger$ and comparing the result with (3.3), we obtain (3.8). Then the calculations performed agree with our assertion.

Hence, substituting the expressions of $S_\kappa(\xi)$ and $\hat{Q}_\kappa$ into (3.4) and this latter into (3.2) including (3.1), the HCS for Hamiltonian (2.9) can be specified by the general formula
\begin{equation}
|\xi\rangle_\kappa = \mathcal{D}(\xi_\kappa) |0\rangle_\kappa = m^{1/4}(x) e^{i \gamma(x) \xi W(x)} \exp \left[ -\left( \kappa + 1 \right) \int W(y) \mathrm{d}\mu(y) \right],
\end{equation}
where $\hat{Q}_\kappa \equiv \hat{n}_k + \hat{n}_k^\dagger = \sqrt{2} \left( \kappa + 1 \right) W(x)$ is deduced from (2.10) and for coordinate dependent mass, we have $U^{-1}(x) = m^{1/4}(x)$.\]


Now using the condition $\xi_\kappa = -\xi_\kappa^*$, a simple inspection of (3.9) shows that $\kappa \langle \xi | \xi \rangle_\kappa = \kappa \langle 0 | 0 \rangle_\kappa = 1$ and, following [24, 25], it is easy to verify that these states minimize the generalized position–momentum uncertainty relation which yields

$$\langle \Delta \tilde{W}_\kappa \rangle^2 \langle \Delta \Pi_\kappa \rangle^2 = \frac{1}{4} \kappa \langle \xi | \left[ \tilde{n}_\kappa, \tilde{n}^*_\kappa \right] | \xi \rangle^2 = \frac{(\kappa + 1)^2}{4} \kappa \langle \xi | U^2(x) W'(x) | \xi \rangle^2,$$

(3.10)

where $\kappa$ is the deformed momentum operator defined as $\Pi_\kappa \equiv -i(\tilde{n}_\kappa - \tilde{n}^*_\kappa)/\sqrt{2} = U(x)p U(x)$ with $p = -i\partial/\partial x$.

### 3.2. Position-dependent mass PHCS

As mentioned in [32], the associated non-Hermitian Hamiltonian $\tilde{H}^{(\alpha, \beta)}$ has the same real spectrum as the ground state $|0\rangle = \rho^{-1}_{\alpha, \beta}(x)|0\rangle$.

As should be expected, we are thus in a position to construct PHCS, which to this order is given by a similarity transformation

$$|\Xi\rangle = \rho^{-1}_{\alpha, \beta}(x)|\xi\rangle.$$

(3.11)

However, the requirement that the only knowledge for $\tilde{H}^{(\kappa)}$ is its Hermiticity leads to suggest that a similarity transformation is not unique [29]; in other words, the determination of PHCS can be achieved using some restrictions imposed on the metric $\rho_{\alpha, \beta}(x)$. To this end, it is worth mentioning the existence of an underlying indicial symmetry that explains how a sufficient condition for the positivity of $\xi_{\alpha, \beta}$ may be provided by interchanging $\alpha$ and $\beta$. This is clearly taking into account that $\rho_{\alpha, \beta}(x) = \rho_{\beta, \alpha}(x)$, so that the following symmetry in terms of $\kappa$ is kept:

$$\rho_{-\kappa}(x) = \rho_{\kappa}^{-1}(x).$$

(3.12)

The most straightforward assumption to consider for $\rho_{\kappa}(x)$ is an exponential form such as the one chosen in (3.9)

$$\rho_{\kappa}(x) = \exp \left[ -f(\kappa) \int^{\mu(x)} W(y) \, d\mu(y) \right],$$

(3.13)

where $f(\kappa)$ is some unknown function to be determined. Besides restriction (3.13), condition (3.12) leads to identify $f(\kappa)$ as an odd function, i.e. $f(-\kappa) = -f(\kappa)$.

Then inserting (3.9) into (3.11), PHCS $|\Xi\rangle_\kappa$ are re-expressed as

$$|\Xi\rangle_\kappa = \rho^{-1}_{\kappa}(x)|\xi\rangle_\kappa = m^{1/4}(x) e^{i\int^{\mu(x)} W(y) \, d\mu(y)} \exp \left[ -z(\kappa) \int^{\mu(x)} W(y) \, d\mu(y) \right],$$

(3.14)

where $z(\kappa) = \kappa + 1 - f(\kappa)$. PHCS (3.14) are ideally suited with regard to the determination of $f(\kappa)$. To start with, we impose on $|\Xi\rangle_\kappa$ by demanding it to be of the form of $|\xi\rangle_\kappa$. This means that we may assume the equality

$$z(\kappa) \equiv (\kappa + 1) \gamma(\kappa),$$

(3.15)

which is enough to determine both $\gamma(\kappa)$ and $f(\kappa)$. Note that when we impose the constraint $f(-\kappa) = -f(\kappa)$ on equality (3.15), we find that this latter is solved solely by demanding on $\gamma(\kappa)$ to be an odd function too\(^1\), i.e.

$$\gamma(\kappa) = \frac{1}{\kappa} \quad \Rightarrow \quad f(\kappa) = \kappa - \frac{1}{\kappa},$$

(3.16)

\(^1\) Indeed if $\gamma(\kappa)$ is an even function, then $f(\kappa) = 0$, which we will neglect.
Thus, by making use of the constraining equation (3.16), we can re-express the HCS |ξ⟩κ in (3.9) and PHCS |Ξ⟩κ in (3.14) purely as a function of $\bar{W}_κ(x)$,

$$\vert \xi \rangle_κ \equiv m^{1/4}(x) e^{\sqrt{2} \xi} e^x \exp \left[ -\kappa \int_{\mu(x)}^{\eta(x)} \bar{W}_κ(y) d\mu(y) \right],$$  

(3.17)

and

$$\vert \Xi \rangle_κ \equiv m^{1/4}(x) e^{\sqrt{2} \xi} e^x \exp \left[ -\int_{\mu(x)}^{\eta(x)} \bar{W}_κ(y) d\mu(y) \right],$$  

(3.18)

where

$$\bar{W}_κ(x) \equiv \frac{1}{\kappa} \tilde{W}_κ(x) = \frac{\kappa + 1}{\kappa} W(x).$$  

(3.19)

Identities (3.17) and (3.18) are our main results.

4. Exactly solvable potentials and their HCS

Here we illustrate the procedure by which a wide range of exactly solvable potentials endowed with position-dependent mass can be recovered as well as their corresponding coherent states. The forms must satisfy the commutation relation (2.13) where the explicit expression $U^2(x)W(x)$ is equal to a certain generating functional $F[x]$ to be determined. A deeper insight is necessary if we are interested to find the solution of (2.13). For convenience, we introduced the auxiliary function $\varphi(x)$ governed by some differential equations and related to $W(x)$. The strategy consists in choosing these differential equations in such a way that $U^2(x)\varphi'(x)$ contains some terms that allow us to get rid of derivatives and, as a consequence, the solutions in $\varphi(x)$; i.e. $W(x)$, can be explicitly carried out by a simple Euler-type integration.

Our solutions fall into three classes and differ slightly from those proposed in [2], with which we can obtain the Hermitian potentials, the ground-state energy eigenvalue and the accompanying HCS. They are identified with differential equations,

- class 1:

$$W(x) = k_0 \varphi(x) + k_1,$$

$$U^2(x)\varphi'(x) = a \varphi^2(x) + b \varphi(x) + c,$$

(4.1)

- class 2:

$$W(x) = k_0 \varphi(x) + \frac{k_1}{\varphi(x)},$$

$$U^2(x)\varphi'(x) = a \varphi^2(x) + b,$$

(4.2)

- class 3:

$$W(x) = \frac{k_0 \varphi(x) + k_1}{\sqrt{a^2 \varphi^2(x) + b^2}},$$

$$U^2(x)\varphi'(x) = (c \varphi(x) + d)\sqrt{a^2 \varphi^2(x) + b^2},$$

(4.3)

where $k = (k_0, k_1)$ and $a = (a, b, c, d)$ are two sets of parameters which determine the quantum system completely.
Applications. As one can see, for lack of space, we have not exhibited the detailed results of our calculations which can be easily determined from (2.12) and (4.1)–(4.3). The corresponding HCS are deduced for the Hermitian counterpart Hamiltonian using (3.17), while (3.18) is used to convert them to PHCS:

(i) shifted harmonic oscillator (class 1) \((a = b = 0, c = 1)\):
\[
\psi(x) = \mu(x), \quad W(x) = k_0 \mu(x) + k_1,
\]
\[
\tilde{V}_\xi(x) = \frac{\alpha_x^2}{2} \left( \mu(x) - \frac{\lambda_x}{\omega_x} \right)^2, \quad \epsilon_{0,\xi} = \frac{\alpha_x}{2},
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp\left\{ \frac{\sqrt{2}}{2} (\omega_x \mu(x) - \lambda_x) \xi_x \right\} \exp\left\{ - \frac{\alpha_x}{2} \mu^2(x) + \lambda_x \mu(x) \right\}, \quad (4.4)
\]
where \(\omega_x = (\kappa + 1)k_0\) and \(\lambda_x = -(\kappa + 1)k_1\). The HCS of (4.4) are eigenstates of \(\tilde{n}_x\) which minimize the uncertainty relation (2.13) for \([\tilde{n}_x, \tilde{n}_x^\dagger]\) = \(\omega_x = \text{const}\).

(ii) Morse potential (class 1) \((a = 0, c = -b, k_1 = 0)\):
\[
\psi(x) = 1 - \frac{1}{c} e^{-\gamma \mu(x)}, \quad W(x) = k_0 - \frac{k_0}{c} e^{-\gamma \mu(x)},
\]
\[
\tilde{V}_\xi(x) = \frac{\alpha_x^2 c^2}{2} e^{-2\gamma \mu(x)} + j_x \lambda_x e^{-\gamma \mu(x)}, \quad \epsilon_{0,\xi} = \frac{c^2}{8} (2j_x + 1)^2,
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp\left\{ \sqrt{2} \left[ \lambda_x c^2 + \left( j_x + \frac{1}{2} \right) e^{-\gamma \mu(x)} \right] \xi_x \right\} e^{-\lambda_x c^2 \mu(x)} \exp\{-\lambda_x e^{-\gamma \mu(x)}\},
\]
\[
(4.5)
\]
where \(\lambda_x = (\kappa + 1)k_0/c^2\) and \(j_x = -(\kappa c + 1/2)\). The HCS of (4.5) minimize the uncertainty relation \([\tilde{n}_x, \tilde{n}_x^\dagger]\) = \(\lambda_x c^2 e^{-\gamma \mu(x)}\).

(iii) Radial Coulomb potential (class 1) \((b^2 = 4ac, k_1 = 0)\):
\[
\psi(x) = -\frac{1}{a \mu(x)} - \frac{b}{2a}, \quad W(x) = -\frac{k_0}{a \mu(x)} - \frac{bk_0}{2a},
\]
\[
\tilde{V}_\xi(x) = -\frac{Ze^2}{\mu(x)} + \frac{l_x (l_x + 1)}{2 \mu^2(x)} \xi_x \left| \frac{\lambda_x}{\omega_x} \right|^2, \quad \epsilon_{0,\xi} = -\frac{1}{2} \left( \frac{Ze^2}{l_x + 1} \right)^2,
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp\left\{ \sqrt{2} \left[ -\frac{l_x + 1}{\mu(x)} + \frac{Ze^2}{l_x + 1} \right] \xi_x \right\} \left| \mu(x) \right|^{l_x+1} \exp\left\{ -\frac{Ze^2}{l_x + 1} \mu(x) \right\},
\]
\[
(4.6)
\]
where \(l_x = (\kappa + 1)k_0/a - 1\) and \(Ze^2 = -b(l_x + 1)^2/2\). The HCS of (4.6) are eigenstates of \(\tilde{n}_x\) which minimize the uncertainty relation for \([\tilde{n}_x, \tilde{n}_x^\dagger]\) = \((l_x + 1)/\mu^2(x)\).

(iv) Pöschl–Teller II potential (class 1) \((a = -c, b = k_1 = 0)\):
\[
\psi(x) = \tanh a \mu(x), \quad W(x) = k_0 \tanh a \mu(x),
\]
\[
\tilde{V}_\xi(x) = -\frac{\alpha_x^2}{2} \left( j_x + 1 \right) \sech^2 a \mu(x), \quad \epsilon_{0,\xi} = -\frac{\alpha_x^2}{2} j_x^2,
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp\left\{ \sqrt{2} j_x a \xi_x \tanh a \mu(x) \right\} \left[ \cosh a \mu(x) \right]^{-j_x},
\]
\[
(4.7)
\]
where \(j_x = (\kappa + 1)k_0/a\). The HCS of (4.7) minimize the uncertainty relation \([\tilde{n}_x, \tilde{n}_x^\dagger]\) = \(j_x a^2 \sech^2 a \mu(x)\).
(v) Eckart potential (class 1) \((a = c, b = 0)\):
\[
\psi(x) = \coth a\mu(x), \quad W(x) = -k_0 \coth a\mu(x) + k_1,
\]
\[
\tilde{V}_x(x) = \frac{a^2}{2} \lambda_x (\lambda_x - 1) \coth^2 a\mu(x) - v_x a^2 \coth a\mu(x), \quad \epsilon_{0,\kappa} = -\frac{a^2}{2} \left( \lambda^2_x + \frac{v^2_x}{\lambda^2_x} \right),
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp \left\{ \sqrt{2} a \left[ -\lambda_x \coth a\mu(x) + \frac{v_x}{\lambda_x} \right] \xi \right\} \exp \left\{ -\frac{av_x}{\lambda_x} \mu(x) \right\} \times \exp \left\{ -\frac{av_x}{\lambda_x} \mu(x) \right\},
\]
where \(\lambda_x = (\kappa + 1)k_0/a\) and \(a^2v_x = (\kappa + 1)^2k_0k_1\). The HCS of (4.8) are eigenstates of \(\tilde{H}_x\) which minimize the uncertainty relation for \(\tilde{H}_x, \tilde{H}_x^*\) = \(\lambda_x a^2 \coth^2 a\mu(x)\).

(vi) Rosen–Morse I potential (class 1):
\[
\psi(x) = \cot a\mu(x), \quad W(x) = k_0 \cot a\mu(x) - k_1,
\]
\[
\tilde{V}_x(x) = \frac{a^2}{2} \lambda_x (\lambda_x + 1) \cot^2 a\mu(x) - v_x a^2 \cot a\mu(x), \quad \epsilon_{0,\kappa} = -\frac{a^2}{2} \left( \lambda^2_x - \frac{v^2_x}{\lambda^2_x} \right),
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp \left\{ \sqrt{2} a \left[ \lambda_x \cot a\mu(x) - \frac{v_x}{\lambda_x} \right] \xi \right\} \exp \left\{ \frac{av_x}{\lambda_x} \mu(x) \right\} \left[ \sin a\mu(x) \right]^{-\lambda_x},
\]
where \(\lambda_x = (\kappa + 1)k_0/a\) and \(a^2v_x = (\kappa + 1)^2k_0k_1\). The HCS of (4.9) are eigenstates of \(\tilde{H}_x\) which minimize the uncertainty relation for \(\tilde{H}_x, \tilde{H}_x^*\) = \(-\lambda_x a^2 \cot^2 a\mu(x)\).

(vii) Manning–Rosen potential (class 1) \((a = -1, b > 0, c = 0)\):
\[
\psi(x) = \frac{be^{-b\mu(x)}}{1 - e^{-b\mu(x)}}, \quad W(x) = -\frac{k_0b e^{-b\mu(x)}}{1 - e^{-b\mu(x)}} + k_1,
\]
\[
\tilde{V}_x(x) = \frac{b^2}{2} \frac{J_0(J_0 - 1) e^{-2b\mu(x)}}{(1 - e^{-b\mu(x)})^2} - \frac{b^2}{2} \lambda_x e^{-b\mu(x)}, \quad \epsilon_{0,\kappa} = -\frac{b^2}{2} \left( \frac{\lambda^2_x - 1}{\lambda^2_x} \right),
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp \left\{ \sqrt{2} \left[ \frac{\lambda_x}{2\lambda_x} - \frac{1}{2} - \frac{bJ_0}{1 - e^{-b\mu(x)}} \right] \xi \right\} \left( 1 - e^{-b\mu(x)} \right)\mu(x),
\]
where \(J_0 = (\kappa + 1)k_0\) and \(\lambda_x = [2(\kappa + 1)k_0/b + 1]J_0\). The HCS of (4.10) minimize the uncertainty relation \(\tilde{H}_x, \tilde{H}_x^*\) = \(b^2 \frac{\lambda_x}{2}\cot^2 a\mu(x)\).

(viii) Hulthén potential (class 1): it is well known that the Manning–Rosen potential reduces to the Hulthén potential by setting \(J_0 = 1\), i.e. \(k_0 = 1/(\kappa + 1)\). Thus,
\[
\tilde{V}_x(x) = -\frac{Z\varepsilon^2 b e^{-b\mu(x)}}{1 - e^{-b\mu(x)}}, \quad \epsilon_{0,\kappa} = -\frac{b^2}{2} \left( \frac{Z\varepsilon^2}{b} - 1/2 \right)^2,
\]
\[
|\xi\rangle_x = m^{1/4}(x) \exp \left\{ \sqrt{2} \left[ \frac{Ze^2}{b} - \frac{1}{2} - \frac{b e^{-b\mu(x)}}{1 - e^{-b\mu(x)}} \right] \xi \right\} \left( 1 - e^{-b\mu(x)} \right)\mu(x),
\]
where \(Ze^2 = b\lambda_x/2\). The HCS of (4.11) minimize the uncertainty relation \(\tilde{H}_x, \tilde{H}_x^*\) = \(b^2 \frac{\lambda_x}{2}\cot^2 a\mu(x)\). We note also that it is possible to recover the Yukawa potential and the accompanying HCS by setting \(b \to 0\).
(ix) Radial harmonic oscillator potential (class 2) \((a = -1, \ b = 0)\):

\[
\psi(x) = \frac{1}{\mu(x)}, \quad W(x) = -\frac{k_0}{\mu(x)} + k_1 \mu(x)
\]

\[
\tilde{V}_c(x) = \frac{\omega_0^2}{2} \cdot \mu^2(x) + \frac{l_c (l_c + 1)}{2 \mu^2(x)}, \quad \epsilon_{0,c} = \omega_c \left( \frac{l_c}{2} + \frac{3}{2} \right),
\]

\[
|\xi\rangle_c = m^{1/2}(x) \exp \left\{ \sqrt{\frac{2}{\mu(x)}} \left[ -l_c + \frac{l_c + 1}{\mu(x)} + \omega_c \mu(x) \right] \xi \right\} |\mu(x)|^{1/2} \exp \left\{ -\frac{\omega_c}{2} \mu^2(x) \right\}.
\]

(4.12)

where \(l_c = (\kappa + 1)k_0 - 1\) and \(\omega_c = (\kappa + 1)k_1\). The HCS of (4.12) are eigenstates of \(\tilde{\eta}_c\) which minimize the uncertainty relation for \(\tilde{\eta}_c, \tilde{\eta}_c^*\) = \(\omega_c + k_{3,c} \mu(x)\).

(x) Generalized Pöschl–Teller II potential (class 2) \((a = -b)\):

\[
\psi(x) = \tan a \mu(x), \quad W(x) = k_0 \tan a \mu(x) + k_1 \coth a \mu(x),
\]

\[
\tilde{V}_c(x) = -\frac{a^2}{2} \left[ (m_c + \lambda_c)^2 - \frac{1}{4} \right] \cosh^2 a \mu(x) + \frac{a^2}{2} \left[ (m_c - \lambda_c)^2 - \frac{1}{4} \right] \sech^2 a \mu(x),
\]

\[
\epsilon_{0,c} = -\frac{a^2}{2} (2m_c - 1)^2,
\]

\[
|\xi\rangle_c = m^{1/2}(x) \exp \left\{ \sqrt{\frac{2}{a \mu(x)}} \left[ \Lambda^{(+)}_c \tan a \mu(x) + \Lambda^{(-)}_c \coth a \mu(x) \right] \xi \right\}
\]

\[
\times \left[ \cosh a \mu(x) \right]^{-\Lambda^{(+)}_c} \left[ \sinh a \mu(x) \right]^{-\Lambda^{(-)}_c}.
\]

(4.13)

where \(\Lambda^{(+)}_c = m_c \pm \lambda_c - 1/2\), while \(m_c = (\kappa + 1)(k_0 + k_1)/(2a) \pm 1/2\) and \(\lambda_c = (\kappa + 1)(k_0 - k_1)/(2a)\). The HCS of (4.13) minimize the uncertainty relation \([\tilde{\eta}_c, \tilde{\eta}_c]\) = \(a^2 \Lambda^{(+)}_c \sech^2 a \mu(x) - a^2 \Lambda^{(-)}_c \coth^2 a \mu(x)\).

(xi) Scarf II potential (class 3) \((a = b = d = 1, \ c = 0)\):

\[
\psi(x) = \sinh a \mu(x), \quad W(x) = \frac{k_0}{a} \tan a \mu(x) - \frac{k_1}{a} \sech a \mu(x),
\]

\[
\tilde{V}_c(x) = -\frac{a^2}{2} \left[ \lambda_c^2 - v_c^2 - v_c \right] \sech^2 a \mu(x) - \frac{a^2}{2} \lambda_c (2v_c + 1) \tanh a \mu(x) \sech a \mu(x),
\]

\[
\epsilon_{0,c} = -\frac{a^2}{2} v_c^2.
\]

\[
|\xi\rangle_c = m^{1/2}(x) \exp \left\{ \sqrt{2} a v_c \tanh a \mu(x) - \lambda_c \sech a \mu(x) \xi \right\} [\cosh a \mu(x)]^{-v_c}
\]

\[
\times \exp \left\{ -\lambda_c \arctan \sinh a \mu(x) \right\}.
\]

(4.14)

where \(\lambda_c = (\kappa + 1)k_1/a^2\) and \(v_c = (\kappa + 1)k_0/a^2\). The HCS of (4.14) are eigenstates of \(\tilde{\eta}_c\) which minimize the uncertainty relation for \([\tilde{\eta}_c, \tilde{\eta}_c] = \frac{v_c}{2} \sech^2 a \mu(x) + \frac{a^2}{2} \lambda_c \tanh a \mu(x) \sech a \mu(x)\).

5. Conclusion

The main aim of this paper was to investigate the pseudo-Hermitian coherent states as well as their Hermitian counterpart coherent states, in the context of the factorization method under the generalized quantum condition in the framework of position-dependent mass. We have indicated the difficulty these factorizations pose mainly due to the feature that the ladder operators do not give a closed algebra. However, considering a specific modification in the superpotential instead circumvents these difficulties and, as a result, a new Hermitian counterpart Hamiltonian is deduced and factorized as a product of two adjoint ladder operators.
which are nothing but annihilation and creation operators for a system. As a consequence, the residual algebra is nothing but the generally deformed oscillator algebra.

Considering these ladder operators, we were able to establish a general scheme to construct HCS and PHCS as well as a wide range of exactly solvable potentials, the associated ground-state energy and a simultaneous derivation of their corresponding HCS is then possible under a generalized quantum condition term

\[ U^2(x)W'(x) = F[x]. \]

It is found that this term is rather simple to deduce by making use of (4.1)–(4.3). Alternatively, by imposing a similarity transformation between the pseudo-Hermitian Hamiltonian and its Hermitian counterpart, we have constructed their accompanying PHCS.

We have shown that HCS and PHCS are the same analytical form as Perelomov's states and thus minimize a generalized position–momentum uncertainty principle.

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