Anti-symmetric rank-two Tensor Matter Field on Superspace for $N_T = 2$

Wesley Spalenza$^{a,b,1}$, Wander G. Ney$^{a,b,c}$ and J.A. Helayel-Neto$^{a,b,1}$

$^a$Centro Brasileiro de Pesquisas Físicas - (CBPF) - Rio de Janeiro - RJ
$^b$Grupo de Física Teórica José Leite Lopes - (GFT) - Petrópolis - RJ
$^c$Centro Federal de Educação Tecnológica - (CEFET) - Campos dos Goytacazes - RJ

E-mails
wesley@cbpf.br, wander@cbpf.br, helayel@cbpf.br.

Abstract

In this work, we discuss the interaction between anti-symmetric rank-two tensor matter and topological Yang-Mills fields. The matter field considered here is the rank-2 Avdeev-Chizhov tensor matter field in a suitably extended $N_T = 2$ SUSY. We start off from the $N_T = 2$, $D = 4$ superspace formulation and we go over to Riemannian manifolds. The matter field is coupled to the topological Yang-Mills field. We show that both actions are obtained as $Q$–exact forms, which allows us to write the energy-momentum tensor as $Q$–exact observables.

1 Introduction

Topological field theories such as Chern-Simons and BF-type gauge theories probe space-time in its global structure, and this aspect has a significative relevance in quantum field theories. On the other hand, there is great deal of interest in anti-symmetric rank-2 tensor fields that can be put into two categories: gauge fields or matter fields. In recent years, Avdeev Chizhov [1, 2, 3] proposed a model where the antisymmetric tensor behaves as a matter field.

In a recent work [4], Geyer-Mülsch presented a formulation until then unknown in the literature, which is a construction of the Avdeev-Chizhov action described in the topological formalism [5]. This was built for $N_T = 1$ and generalized for $N_T = 2$. Known the properties of the anti-symmetric rank-two tensor matter field theory, also called Avdeev-Chizhov field [6], the supersymmetric properties and characteristics are presented also in ref. [7]: following this formalism, we shall write this action in the superfield formalism, as presented by Horne [8] in topological theories as a Donaldson-Witten topological theories [9, 5].

Our goal in this work is to discuss the interaction between matter and topological Yang-Mills fields as presented by Geyer-Mülsch [4] for $N_T = 1$ and $N_T = 2$. The matter field considered here is the rank-2 tensor matter field as a complex self-duality condition [6]. Thus, we write this field now as an anti-symmetric rank-two tensor matter superfield in $N_T = 2$ SUSY in the superspace formalism, founded also in [7]. The matter field is coupled to the topological Yang-Mills connection by means of the Blau-Thompson action. We write the Yang-Mills superconnection

---

1Supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico CNPq – Brazil.
as a 2–superform in a superspace with four bosonic dimensions spacetime described by Grassmann-odd coordinates and two fermionic dimensions described by Grassmann-even coordinates, and them construct the action in a superfield formalism following the definitions by Horne \[8\]. Then, we go over to Riemannian manifolds duly described in terms of the vierbein and the spin connection, where we take the gravitation as a background. We introduce and discuss the Wess-Zumino gauge condition induced by the shift supersymmetry better detailed in \[10\]. Then, we arrive at a topological invariant action as the sum of the Avdeev-Chizhov’s action coupled to the topological super-Yang-Mills action; both actions are obtained as \(Q\)–exact forms, and the energy-momentum tensor is shown to be \(Q\)–exact.

2 The \(N_T = 2\) Super-connection, Super-curvature and Shift Algebra

Let us now consider the Donaldson-Witten theory, whose space of solutions is the space of self-dual instantons, \(F = *F\). To follow our superfield formulation, we shall proceed with the definition of the action of Horne \[8\] and Blau-Thompson \[13, 14\]. The \(N_T = 2\) superfield conventions are the ones of \[10\]. The superfields superconnection and its associated superghosts are given as below:

\[
\hat{A} = \hat{A}^a T_a, \quad \hat{C} = \hat{C}^a T_a, \quad (2.1)
\]

whose the generators belonging the Lie algebra:

\[
[T_a, T_b] = i f_{abc} T_c. \quad (2.2)
\]

Expanding the superforms \(2.1\) in component superfields, we have

\[
\hat{A} = A(x, \theta^I) + E_I(x, \theta^I) d \theta^J, \quad \hat{C} = C(x, \theta^I), \quad (2.3)
\]

with \(I = 1, 2\); in component fields, it comes out as below:

\[
A(x, \theta) = a(x) + \theta^I \psi_I(x) + \frac{1}{2} \theta^2 \alpha(x), \quad (2.4)
\]

\[
E_I(x, \theta) = \chi_I(x) + \theta^J \phi_{IJ}(x) + \frac{1}{2} \theta^2 \eta_I(x), \quad (2.5)
\]

\[
C(x, \theta) = c(x) + \theta^I \epsilon_I(x) + \frac{1}{2} \theta^2 \epsilon_F(x). \quad (2.6)
\]

The associated supercurvature is defined as

\[
\hat{F} = \hat{d}\hat{A} + \hat{A}^2 = (dA + A^2 + (\partial_I A + D_A E_I)) d \theta^J + \frac{1}{2} (\partial_I E_J + \partial_J E_I + [E_I, E_J]) d \theta^I d \theta^J, \quad (2.7)
\]

which can also be expressed as: \(\hat{F} = F + \Psi_I d \theta^I + \Phi_{IJ} d \theta^I d \theta^J\), whose components read as follows:

\[
F = f - \theta^I D_a \psi_I + \frac{1}{2} \theta^2 (D_a \alpha + \frac{1}{2} \varepsilon^{IJ} [\psi_I, \psi_J]), \quad (2.8)
\]

\[
\Psi_I = \psi_I + D_a \chi_I + \theta^J (\varepsilon_{IJ} \alpha - \theta^D \phi_{IJ} + \theta^D [\psi_J, \chi_I]) + \theta^2 (\frac{1}{2} D_a \eta_I - \frac{1}{2} \varepsilon^{JL} [\psi_K, \phi_{KL}] + \frac{1}{2} [\alpha, \chi_I]), \quad (2.9)
\]

\[
\Phi_{IJ} = \frac{1}{2} \{\phi_{IJ} + \phi_{JI} + [\chi_I, \chi_J] + \theta^K (\varepsilon_{KL} \eta_J + \varepsilon_{JK} \eta_I + [\chi_I, \phi_{JK}] + [\phi_{IK}, \chi_J]) + \frac{1}{2} \theta^2 ([\chi_I, \eta_J] + [\eta_I, \chi_J] - \varepsilon^{KL} [\phi_{KL}, \phi_{IJ}])\}, \quad (2.10)
\]

2
where \( f = da + a^2 \) and the covariant derivatives in \( a \) being given by \( D_a(\cdot) = d(\cdot) + [a, (\cdot)] \); the symbol \( (\cdot) \) represents any field which the derivative act upon. This formalism with \( N_T = 2 \), it can be found as an example in the work [11].

The SUSY number, \( s \), is defined by attributing \(-1\) to \( \theta \). Thus, the supersymmetry generators, \( Q \), have \( s = 1 \). The BRST transformation of the superconnection (2.12) is \( sA = -dC - [A, C] = -\hat{D}A \) and component superfields, is given by

\[
\begin{align*}
 sA &= -dC - [A, C] = -D_A C, \\
 sE_I &= -\partial_I C - [E_I, C] = -D_I C, \\
 sC &= -C^2,
\end{align*}
\]

which in components take the form:

\[
\begin{align*}
 sa &= -da - [a, c] = -D_a c, \\
 s\psi & = [c, \psi] - D_ac_I, \\
 s\alpha & = [c, \alpha] - D_a c_F + \varepsilon IJ [c_I, \psi_J], \\
 s\chi I & = [c, \chi I] - c_I, \\
 s\phi IJ & = [c, \phi IJ] - \varepsilon IJ c_F + [\chi I, c_J], \\
 s\eta I & = [c, \eta I] - [c_F, \chi I] + \varepsilon JK [c_J, \phi IK], \\
 sc & = -c^2, \\
 sc_I & = [c, c_I], \\
 sc_F & = [c, c_F] + \frac{1}{2} \varepsilon IJ [c_I, c_J].
\end{align*}
\]

and the super-covariant derivative is decomposed as: \( \hat{D}A = DA + d\theta^I D_I \).

The supersymmetry transformations or shift symmetry transformations are defined as:

\[
Q_I A = \partial_I A, \quad Q_I E_J = \partial_I E_J, \quad Q_I C = \partial_I C;
\]

in components, they read as follows:

\[
\begin{align*}
 Q_I a &= \psi_I, \quad Q_I \psi_J = -\varepsilon IJ \alpha, \quad Q_I \alpha = 0, \\
 Q_I \chi I &= \phi IJ, \quad Q_I \phi Jk = -\varepsilon IJK \eta J, \quad Q_I \eta J = 0, \\
 Q_I c_I &= c_I, \quad Q_I c_I = -\varepsilon IJ c_F, \quad Q_I c_F = 0.
\end{align*}
\]

Next, we believe it is interesting to introduce and discuss a sort of Wess-Zumino gauge choice associated to the shift symmetry above, which is the topological BRST transformation. The Wess-Zumino \(^2\) gauge seen in [12] is here defined by the condition

\[
\chi I = 0 \quad \text{and} \quad \phi(IJ) = 0,
\]

due to the linear shift in the transformations (2.12) for scalar fields \( \chi I \) and \( \phi(IJ) \) respectively, with parameters given by the ghost fields, \( c_I \) and \( c_F \). There exists now, only the symmetric field \( \phi(IJ) \), that we write from now on simply as \( \phi_I \). This condition is not SUSY-invariant under \( Q_I \), and it can be defined in terms of the infinitesimal fermionic parameter \( \varepsilon I \) as

\[
\tilde{Q} = \varepsilon I \tilde{Q}_I.
\]

This operator leaves the conditions (2.14) invariant, and it is built up by the combinations of \( Q \) with the BRST transformations in the Wess-Zumino gauge, such that

\[
\tilde{Q} = (s + Q)|_{c_I = \varepsilon I \phi_I, \ c_F = \frac{1}{2} \varepsilon I \eta J}.
\]

\(^2\)This name is given since we are dealing with a linear gauge and scalar ghost field.
The results in terms of component fields are displayed below:

\[ \tilde{Q}_a = -D_a c + \epsilon^I \psi_I, \]
\[ \tilde{Q}_I = -[c, \psi_I] - \epsilon^J D_a \phi_{IJ} + \epsilon_I \alpha, \]
\[ \tilde{Q}_\alpha = -[c, \alpha] + \epsilon^J \epsilon^K [\phi_{IK}, \psi_J] - \frac{1}{2} \epsilon^I D_a \eta_I, \]
\[ \tilde{Q}_{\phi_{IJ}} = -[c, \phi_{IJ}] + \frac{1}{2} (\epsilon_I \eta_J + \epsilon_J \eta_I), \]
\[ \tilde{Q}_\eta = -[c, \eta_I] + \epsilon^J \epsilon^K [\phi_{JM}, \phi_{IK}], \]
\[ \tilde{Q}_c = -c^2 + \epsilon^I \epsilon^J \phi_{IJ}. \]

in agreement with the transformation found in the works of \cite{15, 14}; the nilpotence reads as

\[ (\tilde{Q})^2 \propto \delta_{\phi_{IJ}}, \]  

that is an infinitesimal transformation of $\phi_{IJ}$. With the result of the previous section, we are ready to write down the Blau-Thompson action, which is the invariant Yang-Mills action for the topological theory.

### 3 The Blau-Thompson action

The associated action for $N_T = 2$, $D = 4$ is the Witten action \cite{8, 15, 16}, described in $N_T = 2$ by the Blau-Thompson action \cite{13, 14}, with gauge completely fixed in terms of the superfield. For the construction of this action, we wish a Lagrange multiplier that couples to the topological super-Yang-Mills so as to manifest its self-duality: $F = \ast F$. We then define a 2-form-superfield Lagrange multiplier, with the property of anti-self-duality and super-gauge covariant: $sK = -[C, K]$, such that

\[ K(x, \theta) = k(x) + \theta^I k_I(x) + \frac{1}{2} \theta^2 \kappa(x). \]

We still wish a quadratic term in the last component field of $K$. Still, we need a 0-form-superfield to complete the gauge-fixing for $\Psi_I$, which is defined as:

\[ H_I(x, \theta) = h_I(x) + \theta^J h_{IJ}(x) + \frac{1}{2} \theta^2 \rho_I(x). \]  

To fix the super-Yang-Mills gauge, we define an anti-ghost superfield for $C$, being a 0-form-superfield of fermionic nature

\[ \overline{c}(x, \theta) = c(x) + \theta^I c_I(x) + \frac{1}{2} \theta^2 c_\Phi(x), \]  

we define a 0-form-superfield Lagrange multiplier

\[ B(x, \theta) = b(x) + \theta^I b_I(x) + \frac{1}{2} \theta^2 \beta(x). \]  

Their BRST transformations are $s\overline{c} = B$, $sB = 0$, and in components they reads

\[ s\overline{c} = b, \quad s\overline{c}_I = b_I, \quad s\overline{c}_\Phi = \beta, \]
\[ sb = 0, \quad sb_I = 0, \quad s\beta = 0. \]  

Therefore the complete Blau-Thompson action in superspace takes the form

\[ S_{BT} = \int d^2 \theta \sqrt{g} Tr \{ K \ast F + \zeta K \ast D_0^2 K + \epsilon^I \epsilon^J H_I D_A \ast \Psi_J + s(\overline{c} d \ast A) \}. \]  


In this 4-dimensional Riemannian manifold, we find the following properties:

$$S_{BT} = \int \sqrt{g} Tr\left[ \frac{1}{2} \kappa \ast f + \kappa \kappa + \zeta \varepsilon^{IJ}(k * [\eta_I, k_J] + [k_J, \eta_I] * k) - \zeta \phi^{IJ} \phi_{IJ} k \ast k \right]$$

$$- \frac{1}{2} \varepsilon^{IJKL}k_I D_a \psi_J + \frac{1}{2} k \ast D_a \alpha + \frac{1}{4} k \ast \varepsilon^{IJKL}[\psi_I, \psi_J]$$

$$+ \varepsilon^{IJKL}[\frac{1}{2} \rho_I D_a * \psi_J + \frac{1}{2} h_{J1} D_a * \alpha - \frac{1}{2} \varepsilon^{KL} h_{K1} D_a * D_a \phi_{JL}]$$

$$+ \frac{1}{2} h_{I} D_a * D_a \eta_J - \varepsilon^{KL} h_{I} D_a * \left[ \psi_K, \phi_{JL} \right] - \frac{1}{2} [h_I, \psi_J] * \alpha$$

$$- \frac{1}{2} \varepsilon^{KL} \left[ \psi_K, \psi_I \right] D_a * \phi_{JL} + \frac{1}{2} \varepsilon^{KL} \left[ \psi_K, h_{I1} \right] * \psi_J + \left[ \alpha, h_I \right] * \psi_J$$

$$+ \frac{1}{2} [bd, B] + \frac{1}{2} \varepsilon^{IJKL}[\psi_J, c_J] - \frac{1}{2} \varepsilon^{IJKL}d * \left( \psi_J, c_J \right) + \frac{1}{2} \varepsilon^{IJKL}d * \left( a, c_J \right)$$

$$+ \frac{1}{2} \varepsilon^{IJKL} \left( d * \left( \psi_J, c_J \right) - \frac{1}{2} \varepsilon^{IJKL}d * \left( a, c_J \right) \right). \quad (3.6)$$

where $g$ is the background metric of the Riemannian manifold.

In the next section, we shall discuss the Avdeev-Chizhov action in a general Riemannian manifold with the same background metric.

## 4 Tensorial Matter in a General Riemannian Manifold

To couple the theory above to the Avdeev-Chizhov model, we start describing the Avdeev-Chizhov action through the complex self-dual field $\varphi$, initially written in the 4-dimensional Minkowskian manifold, whose indices are: $m, n, \ldots$. We write this action, according to the work of [6], as

$$S_{\text{matter}} = \int d^4x \{ (D^m \varphi_{mn}) \dagger (D^p \varphi^{pn}) + q (\varphi^\dagger \varphi) \}. \quad (4.1)$$

Here $q$ is a coupling constant for the self-interaction, and the covariant derivative $D^m \varphi_{mn} = \partial^m \varphi_{mn} - [a^m, \varphi_{mn}]$; $a^m$ is the Lie-algebra-valued gauge potential and we assume $\varphi_{mn}$ to belong to a given representation of the gauge group $G$. This action is invariant under the following transformations:

$$\delta_G(\omega) a_m = D_m \omega, \quad \delta_G(\omega) \varphi_{mn} = \varphi_{mn} \omega, \quad \delta_G(\omega) \varphi^\dagger_{mn} = -\omega \varphi^\dagger_{mn}, \quad (4.2)$$

with $\varphi$ given by

$$\varphi_{mn} = T_{mn} + i T_{mn}, \quad (4.3)$$

which exhibit the properties $\varphi_{mn} = i \varphi_{mn}$, $\varphi^\dagger_{mn} = -\varphi_{mn}$, where the duality is defined by $\varphi_{mn} = \frac{1}{2} \varepsilon_{mnpq} \varphi^\dagger_{pq}$.

To treat this theory, in a general Riemannian manifold as a topological theory, Geyer-Mülsch [4] rewrite the field in a four-dimensional Riemannian manifold, endowed of the vierbein $e_{\mu}^m$ and a spin-connection $\omega_{\mu}^{mn}$, i.e., the tensorial matter read as $\varphi_{\mu\nu} = e_{\mu}^m e_{\nu}^n \varphi_{mn}$, where the action (4.1) is given by

$$S_{\text{matter}} = \int d^4x \sqrt{g} \{ (\nabla_{\mu} \varphi^\dagger_{\mu\nu}) (\nabla_{\rho} \varphi^\rho_{\nu}) + q (\varphi^\dagger_{\mu\nu} \varphi^\rho_{\mu\nu} \varphi^\dagger (\mu \lambda) \varphi_{\rho\lambda}) \}. \quad (4.4)$$

In this 4-dimensional Riemannian manifold, we find the following properties:

$$\sqrt{g} e_{\mu\nu\rho\lambda} \varepsilon^{mnpq} = \varepsilon^\mu_{[m} e_{n}^p e_{\rho}^q e_{\lambda]}, \quad (4.5)$$
\[ e^\mu_m e^n_{\nu} g^{\mu\nu} = \eta^{mn}, \quad e^\mu_m e^n_{\nu} \eta_{mn} = g_{\mu\nu}. \]  

(4.6)

The covariant derivative in the Riemannian manifold is now written in terms of the spin-connection:

\[ \nabla_{\mu} = D_{\mu} + \omega_{\mu}, \]  

(4.7)

where \( \omega_{\mu} = \frac{1}{2} \omega_{\mu}{}^{mn} \sigma_{mn} \), being \( \sigma_{mn} \) the generator of the holonomy Euclidean group \( SO(4) \), also we have: \( D_{\mu} = (D_{\alpha})_{\mu} \), where, \( \alpha \), is the Yang-Mills connection.

## 5 Supersymmetrization of the Avdeev-Chizhov Action

From now on, we can write the action (4.4) in terms of superfields, mentioning the conventions of the works [10] [8]. The superfield that accommodates the rank-two anti-symmetric tensorial matter field, is similar to the one defined in [7], being now expressed as a linear fermionic. This is defined as a rank-two anti-symmetric tensor in the 4-dimensional Riemannian manifold, and with the topological fermionic index \( I \) referring to the topological SUSY index:

\[ \Sigma^I_{\mu \nu}(x, \theta) = \lambda^I_{\mu \nu}(x) + \theta^I \varphi_{\mu \nu}(x) + \frac{1}{2} \theta^2 \zeta^I_{\mu \nu}(x), \]  

(5.1)

where \( \varphi_{\mu \nu}(x) \) is the Avdeev-Chizhov field. The super-manifold is composed by Riemannian manifold and the \( N_T = 2 \) topological manifold.

The superfield is defined under the SUSY transformations

\[ Q_I \Sigma_{\mu \nu; J} = \partial_I \Sigma_{\mu \nu; J}, \]  

(5.2)

and in components:

\[ Q_I \lambda_{\mu \nu; J} = \varepsilon_{IJ} \varphi_{\mu \nu}, \quad Q_I \varphi_{\mu \nu} = -\zeta_{IJ} \varphi_{\mu \nu}, \quad Q_I \zeta_{\mu \nu; J} = 0 \]  

(5.3)

Based on the work of ref. [6], we rewrite the BRST transformations, referring the non-Abelian Avdeev-Chizhov model, in terms of the transformations:

\[ s \varphi^i_{mn} = ic^a(T^a)^{ij} \varphi^i_{mn}, \quad s \varphi^i_{mn} = -ic^a \lambda_{mn}^{ij} (T^a)^{ij}, \]

\[ s(\nabla_m \varphi_{mn})^i = ic^a(T^a)^{ij} (\nabla_m \varphi_{mn})^j, \quad s(\nabla_m \varphi_{mn})^i = -ic^a (\nabla_m \varphi_{mn})^j (T^a)^{ij}, \]

where \( \mathfrak{g} \) is the Lie algebra. We wish to write the BRST—transformation for a supergauge transformation, generalizing the transformations for the Avdeev-Chizhov fields, according to

\[ s(\Sigma^I_{\mu \nu}) = ic(\Sigma^I_{\mu \nu}), \quad s(\Sigma^I_{\mu \nu}) = ic(\Sigma^I_{\mu \nu}); \]  

(5.4)

in components, we get:

\[ s \lambda^I_{\mu \nu} = ic \lambda^I_{\mu \nu}, \quad s \lambda^I_{\mu \nu} = -ic \lambda^I_{\mu \nu}, \]

\[ s \varphi_{\mu \nu} = ic \varphi_{\mu \nu} + ic \lambda_{\mu \nu I}, \quad s \varphi_{\mu \nu} = -ic \varphi_{\mu \nu} - ic \lambda_{\mu \nu I}, \]

\[ s \zeta^I_{\mu \nu} = ic \varphi_{\mu \nu} - ic \varphi_{\mu \nu} + ic \lambda_{\mu \nu I}, \quad s \zeta^I_{\mu \nu} = -ic \varphi_{\mu \nu} + ic \varphi_{\mu \nu} - ic \lambda_{\mu \nu I}. \]  

(5.5)

The super-derivative of the \( 4 \) is covariant under the BRST—transformation, where now, the covariant super-derivative is

\[ D_{\mu}(\cdot) = (D_A)_{\mu}(\cdot) + \omega_{\mu}(\cdot) = \nabla_{\mu}(\cdot) + \theta^I [\psi_{I \mu}(\cdot) + \frac{1}{2} \theta^2 [\alpha_{I \mu}(\cdot)], \]  

6
according to (4.3), then gives

\[ s(D_\mu \Sigma^I_{\mu\nu}) = C (D_\mu \Sigma^I_{\mu\nu}), \]
\[ s(D_I \Sigma^I_{\mu\nu}) = C (D_I \Sigma^I_{\mu\nu}), \]

where we chose here, \( s\omega = 0 \).

By now performing BRST-transformation on the components that survive in the \( N_T = 2 \) Wess-Zumino gauge (2.15), we find:

\[ Q\lambda_{\mu I} = e^I \epsilon_{JJ} \phi_{\mu J} + ic\lambda_{\mu I}, \]
\[ Q\lambda^I_{\mu J} = -ic\epsilon_{\mu J} - ic\lambda^I_{\mu J}, \]
\[ Q\phi_{\mu J} = ic\lambda_{\mu I} + i\epsilon J \lambda^I_{\mu J}, \]
\[ Q\phi^I_{\mu J} = -ic\epsilon_{\mu J} - ic\phi_{\mu J} \lambda^I_{\mu J}, \]
\[ Q\psi_{\mu J} = ic\epsilon_{\mu J} - ic\phi_{\mu J} \lambda^I_{\mu J}, \]
\[ Q\eta_{\mu J} = -ic\epsilon_{\mu J} - ic\phi_{\mu J} \lambda^I_{\mu J}, \]

in agreement to (2.16).

We build up rank-two anti-symmetric tensorial matter field in a super space formulation, leaving the superfield with the same properties as shown in (7); this is invariant under gauge transformations (5.6) and SUSY transformations. The kinetic term is proposed as

\[ S_{\text{kin}} = \int d^4x d^2\theta \sqrt{g} \epsilon^{IJ} \{ (D_\mu \Sigma^I_{\mu\nu})^I (D_\rho \Sigma^0_{\rho\nu}) \}. \]

In components, we get:

\[ S_{\text{kin}} = \int d^4x \sqrt{g} \left[ \frac{1}{2} (\nabla_\mu \phi^\mu) (\nabla_\rho \phi^\rho) + \frac{1}{2} (\nabla_\mu \lambda^\mu) (\nabla_\rho \lambda^\rho) \right] 
+ \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \phi^\mu) (\nabla_\rho \phi^\rho) 
+ \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \lambda^\mu) (\nabla_\rho \lambda^\rho) 
+ \epsilon^{IJ} \left[ \alpha_{\mu J} \mu \lambda_{I} + [\psi_{\mu J}, \phi^I_{\nu J}] \right] \right) \]

The interaction term has the peculiarity of presenting two derivatives of the Grassmann coordinates; it should also be invariant under the gauge transformations (5.6) and supersymmetry. We write it as

\[ S_{\text{int}} = \int d^4x d^2\theta \sqrt{g} \left\{ \epsilon^{IJ} \epsilon^{LM} (\Sigma_{\mu\nu})^I (D^K (\Sigma_{\rho\nu}^\rho) (\Sigma_L^\lambda)^I (D_K (\Sigma_{\rho\lambda} M)) \right\} \]

where \( D_K(\cdot) = \partial K(\cdot) + [E_K, \cdot] \); in components,

\[ S_{\text{int}} = \frac{1}{2} \int d^4x \sqrt{g} \left\{ (\phi_{\mu\nu}^I \phi_{\rho\lambda}) (\phi_{\mu\nu}^I \phi_{\rho\lambda}) - \epsilon^{IJ} (\lambda_{\mu I} \epsilon_{\nu J} + \epsilon_{\mu J} \lambda_{\nu I}) \right\} \]

\[ + \epsilon^{IJ} \left[ \alpha_{\mu J} \mu \lambda_{I} + [\psi_{\mu J}, \phi^I_{\nu J}] \right] \]

\[ + \epsilon^{IJ} \left[ \alpha_{\mu J} \mu \lambda_{I} + [\psi_{\mu J}, \phi^I_{\nu J}] \right] \]

The total action is being determined for: \( S_{\text{kin}} + qS_{\text{int}} \), such that

\[ S_{\text{kin}} = -\int d^4x d^2\theta \sqrt{g} \left\{ \epsilon^{IJ} (D_\mu \Sigma^I_{\mu\nu})^I (D_\rho \Sigma^0_{\rho\nu}) + q \epsilon^{IJ} \epsilon^{LM} (\Sigma_{\mu\nu})^I (D^K (\Sigma_{\rho\nu}^\rho) (\Sigma_L^\lambda)^I (D_K (\Sigma_{\rho\lambda} M)) \right\} . \]
where $q$ is a quartic coupling constant. In components, we have the Avdeev-Chizhov action plus its partness:

\[
S_{AC} = \int d^4x \sqrt{g} \left( \frac{1}{2} \left( \nabla_\mu \varphi^{\mu\nu} \right)^\dagger (\nabla_\nu \varphi^{\mu\nu}) + \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \lambda_I^{\mu\nu})^\dagger (\nabla_\nu \lambda^{\mu\nu}_J) + \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \xi^{\mu}_I)^\dagger (\nabla_\nu \xi^{\mu}_J) \right) + \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \rho^{\mu}_I)^\dagger (\nabla_\nu \rho^{\mu}_J) + \frac{1}{2} \epsilon^{IJ} (\nabla_\mu \phi^{\mu}_I)^\dagger (\nabla_\nu \phi^{\mu}_J)
\]

\[+ [\psi_{\mu I}, \varphi^{\mu\nu}] (\nabla_\rho \rho^{\mu}_J) + \epsilon^{IJ} (\nabla_\mu \lambda^{\mu\nu}_I) (\nabla_\nu \lambda^{\mu\nu}_J) + [\lambda_{\mu I}, \lambda^{\mu\nu}_J] + [\rho_{\mu I}, \rho^{\mu}_J] + [\phi^{\mu\nu}_{\mu I}, \phi^{\mu\nu}_J]
\]

\[+ \epsilon^{IJ} \left( [\alpha_{\mu I}, \lambda^{\mu\nu}_I] + [\psi_{\mu I}, \varphi^{\mu\nu}_J] \right) (\nabla_\rho \lambda^{\mu\nu}_J)
\]

\[+ q (\varphi_{\mu I}^{\mu\nu} \varphi^{\mu\nu} \lambda^{\mu\nu}_I \rho^{\mu\nu}_J - \epsilon^{IJ} [\lambda^{\mu\nu}_I \xi^{\mu\nu}_J + \xi^{\mu\nu}_I \lambda^{\mu\nu}_J] \varphi^{\mu\nu}_I \rho^{\mu\nu}_J
\]

\[+ \epsilon^{IJ} \left( \lambda^{\mu\nu}_I \zeta^{\mu\nu}_J + \zeta^{\mu\nu}_I \lambda^{\mu\nu}_J \right) + \epsilon^{IJ} \frac{\delta}{\delta g_{\mu\nu}} (S_{BT} + S_{AC}) \right)\]

(5.13)

It is invariant under conformal transformations. Therefore, the total gauge invariant action can be written as: $S_{AC} + S_{BT}$. We could also have replace $S_{BT}$ by the super-$BF$ action described in the work of ref. [11].

The $Q$–exactness of the total action above is also true for $N_T = 2$ SUSY as in [4]; this is so because the fermionic volume element $Q^2 \propto Q_1 Q_2$, which means the exactness in the charge $Q_1$, $Q_2$ of this action. This proof for $N_T = 1$ and general $N_T$, is given in the works [10], where the total action is also $s$–exact. According to Blau-Thompson in their review [17], the energy-momentum tensor $\Theta_{\mu\nu}$ is also $Q$–exact,

\[O = \langle 0 | \Theta_{\mu\nu} | 0 \rangle = \langle 0 | \frac{2}{\sqrt{g}} \frac{\delta}{\delta g^{\mu\nu}} (S_{BT} + S_{AC}) | 0 \rangle = \langle 0 | Q \ U_{\mu\nu} | 0 \rangle
\]

(5.14)

ensuring the topological nature of the theory, where we shall just use the Avdeev-Chizhov kinetic term, because the interaction term carries the coupling constant $q$, which is irrelevant for the attainment of the observables of the theory [4].

Concluding Remarks

The main goal of this paper is the settlement of a topological superspace formulation for the investigation of the coupling between the rank-two Avdeev-Chizhov matter field and Yang-Mills fields. It comes out that the stress tensor is $Q$–exact. This opens us the way for the identification of a whole class of observables that we are trying to classify [19].

It is worthwhile to draw the attention here to the shift symmetry that allows us to detect the ghost mode of the Avdeev-Chizhov field. On the other hand, it is known that there appears a ghost mode in the spectrum of excitations of our tensor matter field [1]. The connection between these two observations remain to be clarified. The fact that the Avdeev-Chizhov field manifest itself as a ghost guide future developments in the quest for a consistent mechanism to systematically decouple the unphysical mode mentioned above.

We are also trying to embed the tensor field in the framework of a gauge theory with Lorentz symmetry breaking [15]. We expect that this breaking may identify the right ghost mode present among the two spin 1 components of the Avdeev-Chizhov field.
Acknowledgments. We thank Álvaro Nogueira, Clisthenis P. Constantinidis, José.L. Boldo, Daniel H.T. Franco for many useful discussions and the Prof. Olivier Piguet for the the great help and encouragement.

Appendices

A Conventions

The topological fermionic index: $I = 1, 2$, is lowered and raised by the anti-symmetric Levi-Civita tensor: $\varepsilon_{IJ}$, with $\varepsilon^{12} = - \varepsilon^{12} = 1$. The $\theta-$coordinates definitions: $\theta^I = \varepsilon^{IJ} \theta_J$, $\theta_I = \varepsilon_{IJ} \theta^J$, the quadratic forms are:

$$\theta^2 = \theta^I \theta^I = \theta_I \theta^I = \frac{1}{2} \varepsilon^{IJ} \theta^2, \quad \theta_I \theta_J = \frac{1}{2} \varepsilon_{IJ} \theta^2,$$

with $\varepsilon_{IK} \varepsilon^{KJ} = \delta^I_J$. The derivatives in the $\theta-$coordinates are defined by

$$\partial_I = \frac{\partial}{\partial \theta^I}, \quad \partial^I = \frac{\partial}{\partial \theta_I} \quad \text{and} \quad \partial_I \theta^J \overset{Def}{=} \delta_I^J.$$

(A.1)

thus we have

$$\partial_I f(x, \theta) = \varepsilon_{IJ} \partial^J f(x, \theta),$$

with $f(x, \theta)$ a any superfunction. Deriving the $\theta-$coordinates gives

$$\partial^I \theta^J = - \varepsilon^{IJ}, \quad \partial_I \theta_J = - \varepsilon_{IJ}$$

(A.2)

A superfield is expanded as: $F(x, \theta) = f(x) + \theta^I f_I(x) + \frac{1}{2} \theta^2 F$, obeying the transformation $Q_I F(x, \theta) \overset{Def}{=} \partial_I F(x, \theta)$. In components, we have:

$$Q_I f = f_I; \quad Q_I f_J = - \varepsilon_{IJ} f_F; \quad Q_I f_F = 0.$$

(A.3)

Characteristics table of the superconnection fields:

| Charge | $\psi^I$ | $\alpha^I$ | $\chi^I$ | $\phi^{IJ}$ | $\eta^I$ | $c$ | $c^I$ | $c_F$ |
|--------|---------|---------|---------|---------|---------|----|-------|-------|
| $s$    | $-1$    | $0$     | $1$     | $2$     | $1$     | $2$ | $3$   | $0$   | $1$   |
| $g$    | $1$     | $0$     | $0$     | $0$     | $0$     | $0$ | $1$   | $1$   | $1$   |
| $p$    | $0$     | $1$     | $1$     | $1$     | $0$     | $0$ | $0$   | $0$   | $0$   |
| $P_{grs}$ | $+$    | $-$     | $-$     | $-$     | $+$     | $-$ | $-$   | $+$   | $-$   |

(A.4)

where $s$: susy number, $g$: ghost number, $p$: degree form, $P_{grs}$: Grassmann parity.

B Rules for Topological Grassmannian integration

The definition of integration in this topological SUSY representation is

$$\int d\theta^I \overset{Def}{=} \partial_I.$$

(B.1)
This result is applied to a superfunction $f(x, \theta)$, so that the volume element is

$$
\int d^2 \theta f(x, \theta) \overset{\text{def}}{=} \frac{1}{4} \epsilon^{IJ} \partial_I \partial_J f(x, \theta); \quad (B.2)
$$

therefore, the square of the supersymmetric charge operator (shift operator) is defined by:

$$
Q^2 = Q^I Q_I = \partial^I \partial_I = 4 \int d^2 \theta,
$$

which is a volume element too.

References

[1] L.V. Avdeev and M.V. Chizhov, *A queer reduction of degrees of freedom*, preprint JINR Dubna, [hep-th/9407067](http://arxiv.org/abs/hep-th/9407067)

[2] M.V. Chizhov, Phys.Lett. B 381 (1996) 359, [hep-ph/9511287](http://arxiv.org/abs/hep-ph/9511287)

[3] L. V. Avdeev and M. V. Chizhov, Phys. Lett. B 321 (1994) 212, [hep-th/9312062](http://arxiv.org/abs/hep-th/9312062)

[4] B. Geyer and D. Mülsch, Phys. Lett. B 535 (2002) 349;

[5] E. Witten, Int.J. Mod. Phys. A6 (1991) 2775, Commun. Math. Phys. 117 (1988) 353;

[6] V.E.R. Lemes, R. Renan, S.P. Sorella, Phys. Lett. B 352 (1995) 37;

[7] V.E.R. Lemes, A.L.M.A. Nogueira and J.A. Helayel-Neto, Int. Jorn. Mod. Phy. A 13, No. 18 (1998) 3145, [hep-th/9508045](http://arxiv.org/abs/hep-th/9508045)

[8] J.H. Horne, Nucl. Phys. B 318 (1989) 22;

[9] S. Donaldson, J. Diff. Geom. 30 (1983) 289, Topology 29 (1990) 257;

[10] J. L. Boldo, C.P. Constantinidis, F. Gieres, M. Lefrançois and O. Piguet, [hep-th/0303053](http://arxiv.org/abs/hep-th/0303053)

[11] C.P. Constantinidis, O. Piguet and W. Spalenza, [hep-th/0310184](http://arxiv.org/abs/hep-th/0310184)

[12] Wess-Bagger, *Supersymmetry and Supergravity*, Second Edition, Princeton University Press, New Jersey, 1992;

[13] M. Blau and G. Thompson, Commun. Math. Phys. 152 (1993) 41, [hep-th/9112012](http://arxiv.org/abs/hep-th/9112012)

[14] M. Blau and G. Thompson, Nucl.Phys. B492 (1997) 545, [hep-th/9612143](http://arxiv.org/abs/hep-th/9612143)

[15] B. Geyer and D. Mülsch, Nucl. Phys. B 616 (2001) 476;

[16] C. Vafa and E. Witten, Nucl.Phys. B431 (1994) 3, [hep-th/9408074](http://arxiv.org/abs/hep-th/9408074)

[17] D. Birmingham, M. Blau, M. Rakowski and G. Thompson, Phys. Rep. 209 (1991) 129;

[18] J.A. Helayel-Neto, W.G. Ney, W. Spalenza, work in progress;

[19] J.L. Boldo, J. A. Nogueira, C.P. Constantinidis, O. Piguet and W. Spalenza, work in progress.