RIESZ POTENTIALS, BESSSEL POTENTIALS AND FRACTIONAL DERIVATIVES ON TRIEBEL-LIZORKIN SPACES FOR THE GAUSSIAN MEASURE.

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Abstract. In [7] the boundedness properties of Riesz Potentials, Bessel potentials and Fractional Derivatives were studied in detail on Gaussian Besov-Lipschitz spaces $B^{\alpha p,q}_{\gamma d}$. In this paper we will continue our study proving the boundedness of those operators on Gaussian Triebel-Lizorkin spaces $F^{\alpha p,q}_{\gamma d}$. Also these results can be extended to the case of Laguerre or Jacobi expansions and even further to the general framework of diffusions semigroups.

1. Introduction and Preliminaries

On $\mathbb{R}^d$ let us consider the Gaussian measure

\begin{equation}
\gamma_d(dx) = \frac{e^{-\|x\|^2}}{\pi^{d/2}} dx, \ x \in \mathbb{R}^d
\end{equation}

and the Ornstein-Uhlenbeck differential operator

\begin{equation}
L = \frac{1}{2} \Delta_x - \langle x, \nabla_x \rangle.
\end{equation}

Let $\nu = (\nu_1, \ldots, \nu_d)$ be a multi-index such that $\nu_i \geq 0, i = 1, \ldots, d$, let $\nu! = \Pi_{i=1}^{d} \nu_i!$, $|\nu| = \sum_{i=1}^{d} \nu_i$, $\partial = \frac{\partial}{\partial x}$, for each $1 \leq i \leq d$ and $\partial^\nu = \partial_{\nu_1}^{\nu_1} \cdots \partial_{\nu_d}^{\nu_d}$, consider the normalized Hermite polynomials of order $\nu$ in $d$ variables,

\begin{equation}
h_\nu(x) = \frac{1}{(2^{|\nu|} \nu!)^{1/2}} \Pi_{i=1}^{d} (-1)^{\nu_i} e^{x_i^2} \partial_i^{\nu_i} (e^{-x_i^2}),
\end{equation}

it is well known, that the Hermite polynomials are eigenfunctions of the operator $L$,

\begin{equation}
Lh_\nu(x) = -|\nu| h_\nu(x).
\end{equation}

Given a function $f \in L^1(\gamma_d)$ its $\nu$-Fourier-Hermite coefficient is defined by

\[ \hat{f}(\nu) = \langle f, h_\nu \rangle_{\gamma_d} = \int_{\mathbb{R}^d} f(x) h_\nu(x) \gamma_d(dx). \]

Let $C_n$ be the closed subspace of $L^2(\gamma_d)$ generated by the linear combinations of $\{h_\nu : |\nu| = n\}$. By the orthogonality of the Hermite polynomials with

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respect to $\gamma_d$ it is easy to see that $\{C_n\}$ is an orthogonal decomposition of $L^2(\gamma_d)$,

\[ L^2(\gamma_d) = \bigoplus_{n=0}^{\infty} C_n, \]

this decomposition is called the Wiener chaos.

Let $J_n$ be the orthogonal projection of $L^2(\gamma_d)$ onto $C_n$, then if $f \in L^2(\gamma_d)$

\[ J_n f = \sum_{|\nu|=n} \hat{f}(\nu) h_\nu. \]

Let us define the Ornstein-Uhlenbeck semigroup $\{T_t\}_{t \geq 0}$ as

\[ T_t f(x) = \frac{1}{(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{e^{-2t}(|x|^2 + |y|^2) - e^{-t}(x,y)}{1 - e^{-2t}}} f(y) \gamma_d(dy) \]

\[ = \frac{1}{\pi^{d/2}(1 - e^{-2t})^{d/2}} \int_{\mathbb{R}^d} e^{-\frac{|y - e^{-t}x|^2}{1 - e^{-2t}}} f(y) dy \tag{1.5} \]

The family $\{T_t\}_{t \geq 0}$ is a strongly continuous Markov semigroup on $L^p(\gamma_d)$, $1 \leq p \leq \infty$, with infinitesimal generator $L$. Also, by a change of variable we can write,

\[ T_t f(x) = \int_{\mathbb{R}^d} f(\sqrt{1 - e^{-2t}} u + e^{-t} x) \gamma_d(du). \tag{1.6} \]

Now, by Bochner subordination formula, see Stein \cite{12}, we define the Poisson-Hermite semigroup $\{P_t\}_{t \geq 0}$ as

\[ P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du \tag{1.7} \]

From (1.5) we obtain, after the change of variable $r = e^{-t^2/4u}$,

\[ P_t f(x) = \frac{1}{2\pi^{(d+1)/2}} \int_{\mathbb{R}^d} \int_0^1 \frac{t^2}{(-\log r)^{3/2}} \frac{\exp \left( \frac{1}{r} \frac{|x-y|^2}{1-r^2} \right)}{r} \frac{dr}{r} f(y) dy \]

\[ = \int_{\mathbb{R}^d} p(t, x, y) f(y) dy, \tag{1.8} \]

with

\[ p(t, x, y) = \frac{1}{2\pi^{(d+1)/2}} \int_0^1 \frac{t^2}{(-\log r)^{3/2}} \frac{\exp \left( \frac{1}{r} \frac{|x-y|^2}{1-r^2} \right)}{r} dr. \tag{1.9} \]

Also by the change of variable $s = t^2/4u$ we have,

\[ P_t f(x) = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-u}}{\sqrt{u}} T_{t^2/4u} f(x) du = \int_0^\infty T_s f(x) \mu_t^{(1/2)}(ds), \tag{1.10} \]

where the measure

\[ \mu_t^{(1/2)}(ds) = \frac{t}{2\sqrt{\pi}} \frac{e^{-t^2/4s}}{s^{3/2}} ds, \tag{1.11} \]
is called the one-side stable measure on \((0, \infty)\) of order \(1/2\).

The family \(\{P_t\}_{t \geq 0}\) is a strongly continuous semigroup on \(L^p(\gamma_d)\), \(1 \leq p \leq \infty\), with infinitesimal generator \(-(-L)^{1/2}\), furthermore \(\{P_t\}\) is an analytic semigroup in \(t\). In what follows, often we are going to use the notation

\[ u(x,t) = P_t f(x) \text{ and } u^{(k)}(x,t) = \frac{\partial^k}{\partial t^k} P_t f(x). \]

Observe that by (1.4) we have that

\[ T_t h_{\nu}(x) = e^{-t|\nu|} h_{\nu}(x), \]

and

\[ P_t h_{\nu}(x) = e^{-t\sqrt{|\nu|}} h_{\nu}(x), \]

i.e. the Hermite polynomials are eigenfunctions of \(T_t\) and \(P_t\) for any \(t \geq 0\).

The operators that we are going to consider in this paper are the following:

- For \(\beta > 0\), the Fractional Integral or Riesz potential of order \(\beta\), \(I_\beta\), with respect to the Gaussian measure is defined formally as

\[ I_\beta = (-L)^{-\beta/2} \Pi_0, \]

where, \(\Pi_0 f = f - \int_{\mathbb{R}^d} f(y)\gamma_d(dy),\) for \(f \in L^2(\gamma_d).\) That means that for the Hermite polynomials \(\{h_{\nu}\},\) for \(|\nu| > 0,\)

\[ I_\beta h_{\nu}(x) = \frac{1}{|\nu|^\beta/2} h_{\nu}(x), \]

and for \(\nu = 0 = (0, ..., 0), I_\beta(h_0) = 0.\) Then by linearity, \(I_\beta^j\) can be defined to any polynomial.

In this paper the following representation of \(I_\beta^j\) is crucial to get the results. If \(f\) is a polynomial,

\[ I_\beta f(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} (P_t f(x) - P_\infty f(x)) \, dt, \]

where \(P_\infty f(x) = \int_{\mathbb{R}^d} f(y)\gamma_d(dy).\) The proof is very simple but we include it because we will refer to this calculation several times. By linearity it is enough to do it for the Hermite polynomials. Let \(|\nu| > 0,\) otherwise the Gaussian measure is defined trivially, then by orthogonality and the change of variable \(u = t\sqrt{|\nu|},\) we have

\[ \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} t^{\beta-1}(P_t h_{\nu}(x) - P_\infty h_{\nu}(x)) \, dt \]

\[ = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} t^{\beta-1} e^{-t\sqrt{|\nu|}} h_{\nu}(x) \, dt \]

\[ = \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} (\frac{u}{\sqrt{|\nu|}})^{\beta-1} e^{-u} \frac{du}{\sqrt{|\nu|}} h_{\nu}(x) = \frac{1}{|\nu|^\beta/2} h_{\nu}(x). \]
Observation 1.1. P. A. Meyer’s multiplier theorem shows that $I_{\beta}$ has a continuous extension to $L^p(\gamma d)$, $1 < p < \infty$, and then (1.16) can be extended for $f \in L^p(\gamma d)$, by the density of the polynomials, see [1].

- The Bessel Potential of order $\beta > 0$, $J_{\beta}$, associated to the Gaussian measure is defined formally as

$$J_{\beta} = (I + \sqrt{-L})^{-\beta},$$

meaning that for the Hermite polynomials we have,

$$J_{\beta} h_\nu(x) = \frac{1}{(1 + \sqrt{\nu})^\beta} h_\nu(x).$$

Again by linearity $J_{\beta}$ can be extended to any polynomial. By similar calculation as above (1.16), the Bessel potentials can be represented

$$J_{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_0^{+\infty} t^\beta e^{-t} P_t f(x) \frac{dt}{t};$$

Observation 1.2. P. A. Meyer’s theorem shows that $J_{\beta}$ is a continuous operator on $L^p(\gamma d)$, $1 < p < \infty$, and again (1.18) can be extended for $f \in L^p(\gamma d)$, by the density of the polynomials there.

- The Riesz fractional derivative of order $\beta > 0$ with respect to the Gaussian measure $D^\beta$, is defined formally as

$$D^\beta = (-L)^{\beta/2},$$

meaning that for the Hermite polynomials, we have

$$D^\beta h_\nu(x) = |\nu|^{\beta/2} h_\nu(x),$$

thus by linearity can be extended to any polynomial, see [8] and [9].

Now, if $f$ is a polynomial, by the linearity of the operators $I_{\beta}$ and $D^\beta$, (1.15) and (1.20), we get

$$\Pi_0 f = I_{\beta}(D^\beta f) = D^\beta(I_{\beta} f).$$

In the case of $0 < \beta < 1$ we have the following integral representation, for $f$ a polynomial,

$$D^\beta f(x) = \frac{1}{c_\beta} \int_0^{+\infty} t^{-\beta-1}(P_t - I) f(x) dt,$$

where $c_\beta = \int_0^{+\infty} u^{-\beta-1}(e^{-u} - 1) du$.

Now, if $\beta \geq 1$, let $k$ be the smallest integer greater than $\beta$ i.e. $k - 1 \leq \beta < k$, then the fractional derivative $D^\beta$ can be represented as

$$D^\beta f = \frac{1}{c_\beta^k} \int_0^{+\infty} t^{-\beta-1}(P_t - I)^k f dt,$$

where $c_\beta^k = \int_0^{+\infty} u^{-\beta-1}(e^{-u} - 1)^k du$ and $f$ a polynomial, see [11].
We also define a Bessel fractional derivative $D^\beta$, defined formally as

$$D^\beta = (I + \sqrt{-L})^\beta,$$

which means that for the Hermite polynomials, we have

$$D^\beta h_\nu(x) = (1 + \sqrt{\nu})^\beta h_\nu(x),$$

In the case of $0 < \beta < 1$ we have the following integral representation,

$$D^\beta f = \frac{1}{c_\beta} \int_0^\infty t^{-\beta-1}(e^{-t}P_t - I) f dt,$$

where, as before, $c_\beta = \int_0^\infty u^{-\beta-1}(e^{-u} - 1) du$ and $f$ a polynomial.

Moreover, if $1 \leq \beta < k$, we have the following representation of $D^\beta f$

$$D^\beta f = \frac{1}{c^k_\beta} \int_0^\infty t^{-\beta-1}(e^{-t}P_t - I)^k f dt,$$

where $c^k_\beta = \int_0^\infty u^{-\beta-1}(e^{-u} - 1)^k du$ and $f$ a polynomial, see [11].

The Gaussian Triebel-Lizorkin $F^\alpha_{p,q}(\gamma_d)$ spaces were introduced in [10], see also [9], as follows

**Definition 1.1.** Let $\alpha \geq 0$, $k$ be the smallest integer greater than $\alpha$, and $1 \leq p, q < \infty$. The Gaussian Triebel-Lizorkin space $F^\alpha_{p,q}(\gamma_d)$ is the set of functions $f \in L^p(\gamma_d)$ for which

$$\left\| \left( \int_0^\infty (t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| t) \right)^{q/4} dt \right\|_{p,\gamma_d} < \infty.$$  

The norm of $f \in F^\alpha_{p,q}(\gamma_d)$ is defined as

$$\|f\|_{F^\alpha_{p,q}} := \|f\|_{p,\gamma_d} + \left\| \left( \int_0^\infty (t^{k-\alpha} \left| \frac{\partial^k P_t f}{\partial t^k} \right| t) \right)^{q/4} dt \right\|_{p,\gamma_d}.$$  

The definition of $F^\alpha_{p,q}(\gamma_d)$ does not depend on which $k > \alpha$ is chosen and the resulting norms are equivalent, for the proof of this result and other properties of these spaces see [10].

For Gaussian Triebel-Lizorkin spaces we have the following inclusion result,

**Proposition 1.1.** The inclusion $F^\alpha_{p,q_1}(\gamma_d) \subset F^\alpha_{p,q_2}(\gamma_d)$ holds for $\alpha_1 > \alpha_2 > 0$ and $q_1 \geq q_2$.

For the proof of this see [10], Proposition 2.2.
In what follows we will need Hardy’s inequalities, so for completeness we will write them here, see [12] page 272,

$$\int_{0}^{+\infty} \left( \int_{0}^{x} f(y)dy \right)^{p} x^{-r-1} dx \leq \frac{p}{r} \int_{0}^{+\infty} (yf(y))^{p} y^{-r-1} dy,$$

and

$$\int_{0}^{+\infty} \left( \int_{x}^{\infty} f(y)dy \right)^{p} x^{r-1} dx \leq \frac{p}{r} \int_{0}^{+\infty} (yf(y))^{p} y^{r-1} dy,$$

where $f \geq 0, p \geq 1$ and $r > 0$.

Finally, in [6] Gaussian Lipchitz spaces $Lip_{\alpha}(\gamma_{d})$ were considered and the boundedness of Riesz Potentials, Bessel potentials and Fractional Derivatives on them and in [7] the boundedness of those operators were studied on Gaussian Besov-Lipschitz spaces $B_{p,q}^{\alpha}(\gamma_{d})$. In the next section we are going to study the boundedness properties of those operators for Gaussian Triebel-Lizorkin spaces $F_{p,q}^{\alpha}(\gamma_{d})$. As usual in what follows $C$ represents a constant that is not necessarily the same in each occurrence.

2. Main results and Proofs

**Theorem 2.1.** Let $\alpha \geq 0, \beta > 0$, $1 < p < \infty, 1 \leq q < \infty$ then $I_{\beta}$ is bounded from $F_{p,q}^{\alpha}(\gamma_{d})$ into $F_{p,q}^{\alpha+\beta}(\gamma_{d})$.

**Proof.**

Let $k > \alpha + \beta + 1$ an integer fixed and $f \in F_{p,q}^{\alpha}(\gamma_{d})$. Using the integral representation of Riesz Potentials (1.16), the semigroup property of $\{P_{t}\}$ and the fact that $P_{\infty}f(x)$ is a constant, we get

$$P_{t}(I_{\beta}f)(x) = \frac{1}{\Gamma(\beta)} \int_{0}^{+\infty} s^{\beta-1} P_{t}(P_{s}f(x) - P_{\infty}f(x)) ds$$

$$= \frac{1}{\Gamma(\beta)} \int_{0}^{+\infty} s^{\beta-1} (P_{t+s}f(x) - P_{\infty}f(x)) ds.$$  

(2.1)

Then using again that $P_{\infty}f(x)$ is a constant and the chain rule,

$$\frac{\partial^{k}}{\partial t^{k}} P_{t}(I_{\beta}f)(x) = \frac{1}{\Gamma(\beta)} \int_{0}^{+\infty} s^{\beta-1} \frac{\partial^{k}}{\partial t^{k}} (P_{t+s}f(x) - P_{\infty}f(x)) ds$$

$$= \frac{1}{\Gamma(\beta)} \int_{0}^{+\infty} s^{\beta-1} u^{(k)}(x, t+s) ds.$$  

(2.2)
i) Case $\beta \geq 1$: Using (2.2), the change of variable $r = t + s$, $dr = ds$ and Hardy’s inequality (1.30), we have
\[
\left( \int_0^{+\infty} \left( t^{k/(\alpha + \beta)} \frac{\partial^k P_k(I_\beta f)(x)}{\partial t^k} \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta))} \left( \int_0^{+\infty} s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \\
= \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta))} \left( \int_t^{+\infty} (r - t)^{\beta - 1} |u(k)(x, r)| dr \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta))} \left( \int_t^{+\infty} r^{\beta - 1} |u(k)(x, r)| dr \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{\Gamma(\beta)} \frac{1}{k - (\alpha + \beta)} \left( \int_0^{+\infty} \left( r^{k/\alpha} |u(k)(x, r)| \right)^q \frac{dr}{r} \right)^{1/q},
\]
and therefore
\[
\| \left( \int_0^{+\infty} \left( t^{k/(\alpha + \beta)} \frac{\partial^k P_k(I_\beta f)}{\partial t^k} \right)^q \frac{dt}{t} \right)^{1/q} \|_{p, \gamma} \\
\leq C_{k, \alpha, \beta} \left( \int_0^{+\infty} \left( t^{k/(\alpha + \beta)} \left( \int_0^{+\infty} s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \right)^{\frac{1}{q}} \|_{p, \gamma} < \infty,
\]
since $f \in F_{p, q}^{\alpha}$. By Observation 1.1 and the previous estimate,
\[
\| I_\beta f \|_{F_{p, q}^{\alpha + \beta}} \leq C \| f \|_{F_{p, q}^{\alpha}}.
\]

ii) Case $0 < \beta < 1$: Again using (2.2),
\[
\left( \int_0^{+\infty} \left( t^{k/(\alpha + \beta)} \frac{\partial^k P_k(I_\beta f)(x)}{\partial t^k} \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta))} \left( \int_0^{+\infty} s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \\
\leq \frac{C}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta)) - 1} \left( \int_0^t s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \\
+ \frac{C}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k/(\alpha + \beta)) - 1} \left( \int_0^{+\infty} s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \\
\quad = (I) + (II).
\]
Writing $s^{\beta - 1} = \frac{s^{\beta - 1}}{s^q} + \frac{s^{\beta - 1}}{s^{q'}}$, $\frac{1}{q} + \frac{1}{q'} = 1$, and using Hölder’s inequality in the internal integral,
\[
(I) \leq \frac{C_{\beta}}{\beta q - 1} \left( \int_0^{+\infty} t^{q(k/\alpha) - 1} \left( \int_0^t s^{\beta - 1} |u(k)(x, t + s)| ds \right)^q \frac{dt}{t} \right)^{1/q}.
\]
By Fubini-Tonelli’s theorem and using that $q(k - \alpha) - \beta - 1 > 0$, as $k > \alpha + \beta + 1$, we get

$$
(I) \leq \frac{C_{\beta}}{\beta^{q-1}} \left( \int_{0}^{+\infty} s^{\beta-1} \int_{s}^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^{q} dt \, ds \right)^{1/q}
$$

$$
\leq \frac{C_{\beta}}{\beta^{q-1}} \left( \int_{0}^{+\infty} s^{\beta-1} \int_{s}^{+\infty} (t+s)^{q(k-\alpha)-\beta-1}|u^{(k)}(x,t+s)|^{q} dt \, ds \right)^{1/q}.
$$

Then, by the change of variable $r = t + s$ and Hardy’s inequality \((1.30)\),

$$
(II) \leq \frac{C_{\beta}}{\beta^{q-1}} \left( \int_{0}^{+\infty} s^{\beta-1} \int_{2s}^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^{q} dr \, ds \right)^{1/q}
$$

$$
\leq \frac{C_{\beta}}{\beta^{q-1}} \left( \int_{0}^{+\infty} s^{\beta-1} \int_{s}^{+\infty} t^{q(k-\alpha)-\beta-1}|u^{(k)}(x,r)|^{q} dr \, ds \right)^{1/q}
$$

$$
\leq \frac{C_{\beta}}{\beta^{q-1}} \frac{1}{\beta^{1/q}} \left( \int_{0}^{+\infty} (s^{k-\alpha}|u^{(k)}(x,r)|)^{q} dr \right)^{1/q}.
$$

Therefore,

$$
\| \left( \int_{0}^{+\infty} (r^{k-\alpha+\beta}) \left| \frac{\partial^{k} P_{\beta} f(x)}{\partial x^{k}} \right|^{q} dr \right)^{1/q} \| \leq C_{k,\alpha,\beta} \| \left( \int_{0}^{+\infty} (s^{k-\alpha}) \left| \frac{\partial^{k} P_{\beta} f(x)}{s^{k}} \right|^{q} ds \right)^{1/q} \| < \infty.
$$

as $f \in F_{p,q}^{\alpha}$. By Observation \(1.1\) and the previous estimate, we get

$$
\| I_{\beta} f \|_{F_{p,q}^{\alpha+\beta}} \leq C \| f \|_{F_{p,q}^{\alpha}}.
$$

Next we want to study the boundedness properties of the Bessel potentials on Triebel-Lizorkin spaces. In \([10]\), Theorem 2.4, the following result was proved, for completeness the proof will be given here too.

**Theorem 2.2.** Let $\alpha \geq 0, \beta > 0$ then for $1 < p < \infty, 1 \leq q < \infty J_{\beta}$ is bounded from $F_{p,q}^{\alpha}(\gamma_{d})$ into $F_{p,q}^{\alpha+\beta}(\gamma_{d})$.\]
Proof.

Let $k > \alpha + \beta + 1$ a fixed integer and $f \in F_{p,q}^\alpha{\gamma_d}$. Observe that

$$|\frac{\partial^k P_t}{\partial t^k}(J_\beta f)(x)| \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta e^{-s} |u^{(k)}(x, t + s)| \frac{ds}{s} \leq \frac{1}{\Gamma(\beta)} \int_0^{+\infty} s^\beta |u^{(k)}(x, t + s)| \frac{ds}{s},$$

and therefore

$$\left( \int_0^{+\infty} \left( t^{k-(\alpha+\beta)} |\frac{\partial^k}{\partial t^k} P_t(J_\beta f)(x)| \right)^q \frac{dt}{t} \right)^{1/q} \leq \frac{1}{\Gamma(\beta)} \left( \int_0^{+\infty} t^{q(k-(\alpha+\beta))} \left( \int_0^{+\infty} s^\beta |u^{(k)}(x, t + s)| \frac{ds}{s} \right)^q \frac{dt}{t} \right)^{1/q}.$$  

But this is the same term studied in the proof of Theorem 2.1, so the argument (divided in two cases $\beta \geq 1$ and $0 < \beta < 1$) is totally analogous in this case.

We will study next the boundedness of the Riesz fractional derivative $D^\beta$ on Triebel-Lizorkin spaces.

**Theorem 2.3.** Let $0 < \beta < \alpha < 1$, Let $1 \leq p, q < \infty$ then $D^\beta$ is bounded from $F_{p,q}^\alpha{\gamma_d}$ into $F_{p,q}^{\alpha-\beta}{\gamma_d}$.

**Proof.**

Let $f \in F_{p,q}^\alpha{\gamma_d}$, using Hardy’s inequality $\text{[1,29]}$ with $p = 1$, and the Fundamental Theorem of Calculus,

$$|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |P_s f(x) - f(x)| ds$$

$$\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} \int_0^s |\frac{\partial}{\partial r} P_r f(x)| dr ds$$

$$\leq \frac{1}{c_\beta^2} \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r}. \tag{2.3}$$

Thus,

$$\|D^\beta f\|_{p,\gamma} \leq C_\beta \left( \int_0^{+\infty} r^{1-\beta} |\frac{\partial}{\partial r} P_r f(x)| \frac{dr}{r} \right) \|f\|_{p,\gamma}$$

$$\leq C_\beta \|f\|_{F_{p,q}^\alpha} < \infty, \tag{2.4}$$
because $F^\alpha_{p,q}(\gamma_d) \subset F^\beta_{p,1}(\gamma_d)$ ($\alpha > \beta$ and $q \geq 1$). Now, by analogous argument using Hardy’s inequality \[1.29\) with $p = 1$,

\[
\frac{\partial}{\partial t} P_t(D^\alpha f)(x) \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta - 1} \frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x) ds \\
\leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta - 1} \int_0^s |u^{(2)}(x, t + r)| dr ds \\
\leq \frac{1}{c_\beta s} \int_0^{+\infty} r^{-\beta} |u^{(2)}(x, t + r)| dr 
\]

This implies that

\[
\int_0^\infty \left( t^{1-(a-b)} |\frac{\partial}{\partial t} P_t(D^\alpha f)(x)| \right)^q dt \\
\leq \frac{1}{c_\beta s} \int_0^{+\infty} \left( t^{1-(a-b)} \int_0^{+\infty} r^{-\beta} |u^{(2)}(x, t + r)| dr \right)^q dt \\
= C_\beta \int_0^{+\infty} \left( t^{1-(a-b)} \int_0^t r^{-\beta} |u^{(2)}(x, t + r)| dr \right)^q \frac{dt}{t} \\
+ C_\beta \int_0^{+\infty} \left( t^{1-(a-b)} \int_t^{+\infty} r^{-\beta} |u^{(2)}(x, t + r)| dr \right)^q \frac{dt}{t} \\
= (I) + (II).
\]

Writing $s^{\beta - 1} = s^{\frac{\beta - 1}{\beta}} + \frac{1}{\alpha} + \frac{1}{\alpha} = 1$, and using Hölder’s inequality in the internal integral, we have

\[
(I) \leq C_\beta (1 - \beta)^{1-q} \int_0^\infty t^{(2-\alpha)q - 2+\beta} \int_0^t r^{-\beta} |u^{(2)}(x, t + r)|^q dr dt, \\
\]

then by Fubini-Tonelli’s theorem, we get

\[
(I) \leq C_\beta (1 - \beta)^{1-q} \int_0^{+\infty} r^{-\beta} \int_r^{+\infty} t^{(2-\alpha)q - 2} |u^{(2)}(x, t + r)|^q dt dr.
\]

It is easy to prove that $(2 - \alpha)q + \beta - 2 > -1$. We need to study two cases: Case #1: If $(2 - \alpha)q + \beta - 2 < 0$: as $r < t$ and by the change of variable $w = t + r$, we have

\[
(I) \leq C_\beta (1 - \beta)^{1-q} \int_0^{+\infty} r^{(2-\alpha)q - 2} \int_r^{+\infty} |u^{(2)}(x, t + r)|^q dt dr \\
\leq C_\beta (1 - \beta)^{1-q} \int_0^{+\infty} r^{[(2-\alpha)q - 1]-1} \int_{2r}^{+\infty} |u^{(2)}(x, w)|^q dw dr \\
\leq C_\beta (1 - \beta)^{1-q} \int_0^{+\infty} r^{[(2-\alpha)q - 1]-1} \int_r^{+\infty} |u^{(2)}(x, w)|^q dw dr.
\]

Then by Hardy’s inequality \[1.30\) as $(2 - \alpha)q - 1 > 0$

\[
(I) \leq C_\beta (1 - \beta)^{1-q} \frac{1}{(2 - \alpha)q - 1} \int_0^{+\infty} (w^{2-\alpha} |u^{(2)}(x, w)|)^q \frac{dw}{w}.
\]
Case #2: If \((2 - \alpha)q + \beta - 2 \geq 0\): By the change of variable \(w = t + r\), we get

\[
(I) \leq C_\beta (1 - \beta)^{-q} \int_0^\infty r^{-\beta} \int_r^\infty (t + r)^{(2-\alpha)q + \beta - 2} |u^{(2)} (x, t + r)|^q dt dr
= C_\beta (1 - \beta)^{-q} \int_0^\infty r^{-\beta} \int_0^r w^{(2-\alpha)q + \beta - 2} |u^{(2)} (x, w)|^q dw dr
\leq C_\beta (1 - \beta)^{-q} \int_0^\infty r^{-\beta} \int_0^{2r} w^{(2-\alpha)q + \beta - 2} |u^{(2)} (x, w)|^q dw dr,
\]

and by Hardy’s inequality \(\text{[1.30]}\),

\[
(I) \leq \frac{C_\beta}{(1 - \beta)^q} \int_0^\infty (w^{2-\alpha} |u^{(2)} (x, w)|)^{\frac{q}{\alpha}} \frac{dw}{w}.
\]

Therefore, in both cases we have

\[
(I) \leq C_\beta \int_0^\infty (w^{2-\alpha} |u^{(2)} (x, w)|)^{\frac{q}{\alpha}} \frac{dw}{w}.
\]

To estimate \((II)\) observe \(r^{-\beta} < t^{-\beta}\), for \(r > t\) and \(\beta > 0\), then we use the same argument used before to estimate \((I)\) case #1, doing the change of variable \(w = t + r\), and using we get Hardy’s inequality \(\text{[1.30]}\) to get

\[
(II) \leq \frac{C_\beta}{1 - \alpha} \int_0^\infty (w^{2-\alpha} |u^{(2)} (x, w)|)^{\frac{q}{\alpha}} \frac{dw}{w}.
\]

Finally,

\[
\| (\int_0^\infty (t^{1-(\alpha-\beta)} \frac{\partial}{\partial t} P_t (D^\beta f)) \frac{dt}{t})^{1/q} \|_{p,\gamma}
\leq C \| (\int_0^\infty (t^{1-(\alpha-\beta)} \int_0^{r-\beta} |u^{(2)} (\cdot, t + r)| dt) \frac{dt}{t})^{1/q} \|_{p,\gamma}
\leq C \| (\int_0^\infty (w^{2-\alpha} |u^{(2)} (\cdot, w)|)^{\frac{q}{\alpha}} \frac{dw}{w})^{1/q} \|_{p,\gamma} < \infty,
\]

as \(f \in F_{p,q}^{\alpha}(\gamma_d)\). By the previous estimate and \(\text{[2.4]}\)

\[
\| D^\beta f \|_{F_{p,q}^{\alpha-\beta}} \leq C \| f \|_{F_{p,q}^{\alpha}}.
\]

\[
\square
\]

In following theorem we will study the boundedness of the Bessel fractional derivative on Triebel-Lizorkin spaces, for the case \(0 < \beta < \alpha < 1\),

**Theorem 2.4.** Let \(0 < \beta < \alpha < 1\), \(1 \leq p, q < \infty\) then \(D^\beta\) is bounded from \(F_{p,q}^{\alpha}(\gamma_d)\) into \(F_{p,q}^{\alpha-\beta}(\gamma_d)\).
Proof.
Let \( f \in L^p(\gamma_d) \), using the Fundamental Theorem of Calculus we can write,
\[
|D^\beta f(x)| \leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} |e^{-s} P_s f(x) - f(x)| ds
\]
\[
\leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} e^{-s} |P_s f(x) - f(x)| ds + \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1||f(x)| ds
\]
\[
\leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} \int_0^s \frac{\partial}{\partial r} P_r f(x) dr ds + \frac{1}{c_{\beta}} |f(x)| \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr ds.
\]

Now, using Hardy’s inequality (1.29), with \( p = 1 \), in both integrals, we have
\[
|D^\beta f(x)| \leq \frac{1}{c_{\beta}} \int_0^{+\infty} r^{1-\beta} |u^{(1)}(x, r)| dr + \frac{1}{c_{\beta}} \Gamma(1 - \beta)|f(x)|.
\]

Thus,
\[
|D^\beta f(x)| \leq \frac{1}{c_{\beta}} \int_0^{+\infty} r^{1-\beta} |u^{(1)}(x, r)| dr + \frac{1}{c_{\beta}} \Gamma(1 - \beta)|f(x)|.
\]

Therefore, if \( f \in F_{p,q}^\alpha(\gamma_d) \), we get
\[
\|D^\beta f\|_{p,\gamma} \leq \frac{1}{c_{\beta}} \| \int_0^{+\infty} r^{1-\beta} |u^{(1)}(\cdot, r)| dr \|_{p,\gamma} + \frac{1}{c_{\beta}} \Gamma(1 - \beta)\| f \|_{p,\gamma}
\]
(2.7)
\[
\leq C_{\beta}\| f \|_{F_{p,1}^\beta} \leq C_{\beta}^p\| f \|_{F_{p,q}^\alpha},
\]
since \( F_{p,q}^\alpha(\gamma_d) \subset F_{p,1}^\beta(\gamma_d) \), as \( \alpha > \beta \), and \( q \geq 1 \).

On the other hand, using a similar argument as above (the Fundamental Theorem of Calculus and Hardy’s inequality (1.29) with \( p = 1 \)), we have,
\[
\left| \frac{\partial}{\partial t} P_t(D^\beta f)(x) \right| \leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} |e^{-s} \frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)| ds
\]
\[
\leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} e^{-s} |\frac{\partial}{\partial t} P_{t+s} f(x) - \frac{\partial}{\partial t} P_t f(x)| ds
\]
\[
+ \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} |e^{-s} - 1||\frac{\partial}{\partial t} P_t f(x)| ds
\]
\[
\leq \frac{1}{c_{\beta}} \int_0^{+\infty} s^{-\beta-1} \int_0^s |u^{(2)}(x, t + r)| dr ds
\]
\[
+ \frac{1}{c_{\beta}} |u^{(1)}(x, t)| \int_0^{+\infty} s^{-\beta-1} \int_0^s e^{-r} dr ds,
\]
\[
\leq \frac{1}{c_{\beta}} \int_0^{+\infty} r^{-\beta} |u^{(2)}(x, t + r)| dr + \frac{1}{c_{\beta}} \Gamma(1 - \beta)|u^{(1)}(x, t)|
\]
Therefore,
\[ |\frac{\partial}{\partial t} P_t(D^\beta f(x))| \leq \frac{1}{\beta c_\beta} \int_0^\infty r^{-\beta}|u^{(2)}(x,t + r)|dr + \frac{\Gamma(1 - \beta)}{\beta c_\beta}|u^{(1)}(x,t)|, \]
and then, we have
\[ \| (\int_0^\infty (t^{1-(a-\beta)}|\frac{\partial}{\partial t} P_t(D^\beta f)|)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
\[ \leq \frac{C}{\beta c_\beta} \| (\int_0^\infty (t^{1-(a-\beta)} \int_0^\infty r^{-\beta}|u^{(2)}(\cdot,t + r)|dr)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
\[ + \frac{C}{\beta c_\beta} \Gamma(1 - \beta) \| (\int_0^\infty (t^{1-(a-\beta)}|u^{(1)}(\cdot,t)|)^q \frac{dt}{t})^{1/q} \|_{p,\gamma}. \]
Since the exponential factor has been remove, the first term can be estimate as in the proof of Theorem 2.3, estimates (2.5) and (2.6),
\[ \| (\int_0^\infty (t^{1-(a-\beta)} \int_0^\infty r^{-\beta}|u^{(2)}(\cdot,t + r)|dr)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
\[ \leq C \| (\int_0^\infty (t^{2-a} |\frac{\partial^2}{\partial t^2} P_t f|)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
which is finite as \( f \in \mathcal{F}_{p,q}^{\alpha}(\gamma_d) \), and for the second term, we have
\[ \| (\int_0^\infty (t^{1-(a-\beta)}|u^{(1)}(\cdot,t)|)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
\[ \leq C \| f \|_{\mathcal{F}_{p,q}^{\alpha-\beta}} \leq C \| f \|_{\mathcal{F}_{p,q}^{\alpha}}, \]
as \( \mathcal{F}_{p,q}^{\alpha}(\gamma_d) \subset \mathcal{F}_{p,q}^{\alpha-\beta}(\gamma_d) \), thus,
\[ \| (\int_0^\infty (t^{1-(a-\beta)}|\frac{\partial}{\partial t} P_t(D^\beta f)|)^q \frac{dt}{t})^{1/q} \|_{p,\gamma} \]
\[ \leq C \| f \|_{\mathcal{F}_{p,q}^{\alpha}}. \]
Therefore, \( D^\beta f \in \mathcal{F}_{p,q}^{\alpha-\beta}(\gamma_d) \) and moreover, by previous estimate and (2.7)
\[ \| D^\beta f \|_{\mathcal{F}_{p,q}^{\alpha-\beta}} \leq C \| f \|_{\mathcal{F}_{p,q}^{\alpha}}. \]

To study the general case for fractional derivatives (removing the condition that the indexes must be less than 1), we need to consider forward differences. For a given function \( f \), the \( k \)-th order forward difference of \( f \) starting at \( t \) with increment \( s \) is defined as,
\[ \Delta^k_s(f,t) = \sum_{j=0}^k \binom{k}{j} (-1)^j f(t + (k - j)s). \]
We will need the following technical results

**Lemma 2.1.** For any positive integer \( k \)

i) \( \Delta^k_s(f,t) = \Delta^{k-1}_s(f_s(\cdot),t) = \Delta_s(\Delta^{k-1}_s(f,\cdot),t) \)

ii) \( \Delta^k_s(f,t) = \int_t^{t+s/v_1} \int_{v_1}^{v_1+s/v_1} ... \int_{v_{k-1}}^{v_{k-1}+s/v_{k-1}} \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k)dv_kdv_{k-1}...dv_2dv_1 \)
iii) For any positive integer \( k \),

\[
\frac{\partial}{\partial s}(\Delta_{s}^{k}(f, t)) = k \Delta_{s}^{k-1}(f', t + s),
\]

and for any integer \( j > 0 \),

\[
\frac{\partial^{j}}{\partial t^{j}}(\Delta_{s}^{k}(f, t)) = \Delta_{s}^{k}(f^{(j)}, t).
\]

The proof of this known lemma can also be found in an appendix in [6].

**Lemma 2.2.** (Hardy’s type inequality) Let \( f \geq 0, r > 0, p \geq 1 \) and \( k \in \mathbb{N} \) then

\[
\left( \int_{0}^{+\infty} \left( \int_{0}^{x} \left( \int_{0}^{x} f(r_{1}, ..., r_{k})dr_{1}...dr_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p} \leq \int_{0}^{1} \left( \int_{0}^{1} \left( \int_{0}^{+\infty} \left( f(x^{k}v_{1}, ..., xv_{k}) x^{k}dv_{1}...dv_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p} dv_{1}...dv_{k}
\]

**Proof.** Taking \( r_{1} = xv_{1}, ..., r_{k} = xv_{k} \), we get

\[
\left( \int_{0}^{+\infty} \left( \int_{0}^{x} \left( \int_{0}^{x} f(r_{1}, ..., r_{k})dr_{1}...dr_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p} = \left( \int_{0}^{+\infty} \left( \int_{0}^{1} \left( \int_{0}^{1} f(xv_{1}, ..., xv_{k}) x^{k}dv_{1}...dv_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p}
\]

Now, consider the spaces \( L^{p}((0, +\infty), x^{-r-1}) \) and \( L^{p}((0,1)^{k}) \); then by Minkowski’s inequality,

\[
\left( \int_{0}^{+\infty} \left( \int_{(0,1)^{k}} f(xv_{1}, ..., xv_{k}) x^{k}dv_{1}...dv_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \leq \int_{(0,1)^{k}} \left( \int_{0}^{+\infty} \left( f(xv_{1}, ..., xv_{k}) x^{k} dv_{1}...dv_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} dv_{1}...dv_{k}
\]

Thus,

\[
\left( \int_{0}^{+\infty} \left( \int_{0}^{x} \left( \int_{0}^{x} f(r_{1}, ..., r_{k})dr_{1}...dr_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p} \leq \int_{0}^{1} \left( \int_{0}^{1} \left( \int_{0}^{+\infty} \left( f(xv_{1}, ..., xv_{k}) x^{k}dv_{1}...dv_{k} \right)^{p} x^{-r-1}dx \right)^{1/p} \right)^{1/p} dv_{1}...dv_{k}
\]

\( \square \)
Lemma 2.3. For any positive integer $k$,

$$
\Delta^k_s(f, t) = \int_t^{t+s} \int_{v_1}^{v_1+s} \cdots \int_{v_{k-1}}^{v_{k-1}+s} f^{(k)}(v_k) dv_k \, dv_1 \cdots dv_k
$$

$$
= \int_0^s \cdots \int_0^s f^{(k)}(t + v_1 + \cdots + v_k) dv_k \, dv_1 \cdots dv_k
$$

Lemma 2.4. Let $t \geq 0, \beta > 0$ and let $f$ be the smallest integer greater than $\beta$ and let $f$ differentiable up to order $k$, then

$$
\int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(f, t)| ds \leq C_{\beta,k} \int_0^{+\infty} w^{k-\beta-1} |f^{(k)}(t + w)| dw
$$

where $C_{\beta,k} = \int_0^1 \cdots \int_0^1 (v_1 + \cdots + v_k)^{\beta-k} dv_k \cdots dv_k$

**Proof.** From Lemmas 2.2 and 2.3 we have,

$$
\int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(f, t)| ds \leq \int_0^{+\infty} s^{-\beta-1} \int_0^s \cdots \int_0^s |f^{(k)}(t + v_1 + \cdots + v_k)| dv_1 \cdots dv_k ds
$$

$$
\leq \int_0^1 \cdots \int_0^1 (\int_0^{+\infty} (s^k |f^{(k)}(t + s(v_1 + \cdots + v_k)|) s^{-\beta-1} ds) dv_1 \cdots dv_k
$$

$$
= \int_0^1 \cdots \int_0^1 (\int_0^{+\infty} (s^{k-\beta-1} |f^{(k)}(t + s(v_1 + \cdots + v_k)|) ds) dv_1 \cdots dv_k
$$

taking $r = s(v_1 + \cdots + v_k)$ then $dr = (v_1 + \cdots + v_k) ds$,

$$
\int_0^{+\infty} s^{k-\beta-1} |f^{(k)}(t + s(v_1 + \cdots + v_k)| ds = \int_0^{+\infty} r^{k-\beta} |f^{(k)}(t + r)| \frac{dr}{r} (v_1 + \cdots + v_k)^{\beta-k}
$$

$$
= \int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t + r)| dr (v_1 + \cdots + v_k)^{\beta-k}.
$$

Therefore,

$$
\int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(f, t)| ds
$$

$$
\leq \int_0^1 \cdots \int_0^1 (\int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t + r)| dr (v_1 + \cdots + v_k)^{\beta-k}) dv_1 \cdots dv_k
$$

$$
= (\int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t + r)| dr) \int_0^1 \cdots \int_0^1 (v_1 + \cdots + v_k)^{\beta-k} dv_1 \cdots dv_k
$$

$$
= C_{\beta,k} \left( \int_0^{+\infty} r^{k-\beta-1} |f^{(k)}(t + r)| dr \right),
$$

where $C_{\beta,k} = \int_0^1 \cdots \int_0^1 (v_1 + \cdots + v_k)^{\beta-k} dv_1 \cdots dv_k < \infty$. □
Observation 2.1. Using the Binomial Theorem and the semigroup property of \( \{P_s\} \), we have

\[
(P_s - I)^k f(x) = \sum_{j=0}^{k} \binom{k}{j} P_{s-j}(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j P_{s-j} f(x)
\]

\[
= \sum_{j=0}^{k} \binom{k}{j} (-1)^j (P_{s-j}) f(x) = \sum_{j=0}^{k} \binom{k}{j} (-1)^j u(x, (k - j)s)
\]

(2.10)

where as usual, \( u(x, s) = P_s f(x) \) and furthermore,

(2.11)

\[
P_t (P_s - I)^k f(x) = \Delta_s^k (u(x, t), 0).
\]

The following result extends Theorem 2.3 to the general case \( 0 < \beta < \alpha \),

Theorem 2.5. Let \( 0 < \beta < \alpha \), \( 1 \leq p, q < \infty \) then \( D^\beta \) is bounded from \( F^{\alpha}_{p,q}(\gamma d) \) into \( F^{\alpha - \beta}_{p,q}(\gamma d) \).

Proof. Let \( f \in F^{\alpha}_{p,q}(\gamma d) \), using Observation 2.1 and Lemma 2.4,

\[
|D^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |(P_s - I)^k f(x)| ds
\]

\[
= \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta_s^k (u(x, \cdot), 0)| ds
\]

\[
\leq C_{\beta,k} \int_0^{+\infty} r^{k-\beta-1} |u^{(k)}(x, r)| dr,
\]

then

\[
\|D^\beta f\|_{p,\gamma} \leq C_{\beta,k} \| \int_0^{+\infty} r^{k-\beta-1} |u^{(k)}(\cdot, r)| dr \|_{p,\gamma} < \infty,
\]

since \( F^{\alpha}_{p,q}(\gamma d) \subset F^{\beta}_{p,1}(\gamma d) \), \( (\alpha > \beta \text{ and } 1 \leq q < \infty) \).

On the other hand, let \( n \in \mathbb{N}, n > \alpha \); by Observation 2.1 and Lemma 2.4, we get,

\[
\left| \frac{\partial^n}{\partial t^n} P_t (D^\beta f)(x) \right| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta_s^k (u^{(n)}(x, \cdot), t)| ds
\]

\[
\leq \frac{1}{c_\beta} C_{\beta,k} \int_0^{+\infty} r^{k-\beta-1} |u^{(n+k)}(x, t + r)| dr.
\]

The rest of the proof follows the argument used in Theorem 2.3. We will write the details for the sake of completeness.
\[
\left( \int_0^\infty \left( t^{n-(\alpha-\beta)} |\frac{\partial^n}{\partial t^n} P_t(D_{\beta} f)(x)| \right)^\frac{q}{t} \right)^{1/q} \\
\leq C_{\beta,k} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^{+\infty} r^{k-\beta-1} |u^{(n+k)}(x,t+r)| dr \right)^\frac{q}{t} \right)^{1/q} \\
\leq C_{\beta,k} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_0^t r^{k-\beta-1} |u^{(n+k)}(x,t+r)| dr \right)^\frac{q}{t} \right)^{1/q} \\
\quad + C_{\beta,k} \left( \int_0^\infty \left( t^{n-(\alpha-\beta)} \int_t^{+\infty} r^{k-\beta-1} |u^{(n+k)}(x,t+r)| dr \right)^\frac{q}{t} \right)^{1/q} \\
\leq (I) + (II).
\]

Writing \( s^{\beta-1} = s^{\frac{\beta-1}{q}} + \frac{2-\beta}{q}, \frac{1}{q} + \frac{1}{2\beta} = 1 \), and using Hölder’s inequality in the internal integral and Fubini-Tonelli’s theorem,

\[
(I) \leq \frac{(k - \beta)^{1/q}}{k - \beta} \left( \int_0^\infty \left( t^{(n+k-\alpha)q + \beta - k - 1} \right)^q r^{k-\beta-1} dt \right)^{1/q}
\leq C \left( \int_0^\infty r^{k-\beta-1} \left( \int_r^{+\infty} t^{(n+k-\alpha)q + \beta - k - 1} |u^{(n+k)}(x,t+r)| dt \right)^q dr \right)^{1/q}
\]

Now, it is easy to see that \((n + k - \alpha)q + \beta - k - 1 > -1\). Then we need to study two cases:

Case# 1: If \((n + k - \alpha)q + \beta - k - 1 < 0\) then as \(w < t\) and taking the change of variable \(v = r + t\),

\[
(I) \leq (k - \beta)^{1/q - 1} \left( \int_0^\infty r^{((n+k-\alpha)q-1)-1} \int_0^r |u^{(n+k)}(x,v)|^q dv dr \right)^{1/q}
\leq (k - \beta)^{1/q - 1} \left( \int_0^\infty r^{((n+k-\alpha)q-1)-1} \int_r^\infty |u^{(n+k)}(x,v)|^q dv dr \right)^{1/q}.
\]

Therefore, by Hardy’s inequality as \((n + k - \alpha)q - 1 > 0\)

\[
(I) \leq (k - \beta)^{1/q - 1} \left( \frac{1}{(n + k - \alpha)q - 1} \right)^{1/q} \left( \int_0^\infty \left( s^{n+k-\alpha} |u^{(n+k)}(x,s)| \right)^q \frac{ds}{s} \right)^{1/q}
\]
Therefore, as $r^{k-\beta-1} \leq t^{k-\beta-1}$ for $r \geq t$, since $k - \beta - 1 \leq 0$, and taking the change of variable $v = t + r$,

$$ (II) \leq C \left( \int_{0}^{\infty} (s^{n+k-\alpha} |u^{(n+k)}(x,s)| q \frac{ds}{s})^{1/q} \right)^{1/q}. $$

Thus, by Hardy’s inequality, as $k - \beta > 0$,

$$ (I) \leq (k - \beta)^{1/q-1} \left( \int_{0}^{\infty} r^{k-\beta-1} \left( \int_{r}^{+\infty} t^{n+k-\alpha} |u^{(n+k)}(x,t+r)|^q dt \right)^{1/q} dr \right)^{1/q} $$

$$ \leq (k - \beta)^{1/q-1} \left( \int_{0}^{\infty} r^{k-\beta-1} \left( \int_{r}^{+\infty} u^{(n+k-\alpha)}(x,t+r)^q dt \right)^{1/q} dr \right)^{1/q} $$

$$ = (k - \beta)^{1/q-1} \left( \int_{0}^{\infty} r^{k-\beta-1} \left( \int_{2r}^{+\infty} u^{(n+k-\alpha)}(x,s)^q ds \right)^{1/q} dr \right)^{1/q} $$

$$ \leq (k - \beta)^{1/q-1} \left( \int_{0}^{\infty} s^{n+k-\alpha} |u^{(n+k)}(x,s)|^q ds \right)^{1/q} $$

Thus in both cases, we have

$$ (I) \leq C \left( \int_{0}^{\infty} (s^{n+k-\alpha} |u^{(n+k)}(x,s)| q \frac{ds}{s})^{1/q} \right)^{1/q}. $$

As $r^{k-\beta-1} \leq t^{k-\beta-1}$ for $r \geq t$, since $k - \beta - 1 \leq 0$, and taking the change of variable $v = t + r$,

$$ (II) \leq C \left( \int_{0}^{\infty} (t^{n+k-\alpha-1} \int_{t}^{+\infty} u^{(n+k)}(x,t+r)^q dr \frac{dt}{t})^{1/q} \right)^{1/q} $$

$$ = C \left( \int_{0}^{\infty} (t^{n+k-\alpha-1} \int_{2t}^{+\infty} u^{(n+k)}(x,s)^q ds \frac{dt}{t})^{1/q} \right)^{1/q} $$

$$ \leq C \left( \int_{0}^{\infty} t^{(n+k-\alpha-1)q-1} \left( \int_{t}^{+\infty} u^{(n+k)}(x,s)^q ds \right)^{1/q} dt \right)^{1/q}, $$

then, by Hardy’s inequality

$$ (II) \leq \frac{C}{n+k-\alpha-1} \left( \int_{0}^{\infty} (s^{n+k-\alpha} |u^{(n+k)}(x,s)| q \frac{ds}{s})^{1/q} \right)^{1/q}. $$

Therefore,

$$ \left\| \left( \int_{0}^{\infty} (t^{n-(\alpha-\beta)} \frac{\partial^n}{\partial t^n} P_t D_{\beta} f) \right)^q \frac{dt}{t} \right\|_{p,\gamma}^{1/q} $$

$$ \leq C \left( \int_{0}^{\infty} (t^{n-(\alpha-\beta)} \int_{0}^{+\infty} r^{k-\beta-1} |u^{(n+k)}(\cdot,t+r)|^q dr \frac{dt}{t})^{1/q} \right)^{1/q} $$

$$ (2.12) \leq C \left( \int_{0}^{\infty} (s^{n+k-\alpha} |u^{(n+k)}(\cdot,s)| q \frac{ds}{s})^{1/q} \right)^{1/q} \right\|_{p,\gamma}^{1/q} $$

$$ \leq C \|f\|_{F_{p,q}^{\alpha,\beta}}. $$

Therefore, $D_{\beta} f \in F_{p,q}^{\alpha,\beta}(\gamma_d)$ and moreover,

$$ \|D_{\beta} f\|_{F_{p,q}^{\alpha,\beta}} \leq C \|f\|_{F_{p,q}^{\alpha,\beta}}. $$

$\square$
Finally, the following result extends Theorem 2.4 to the general case $0 < \beta < \alpha$.

**Theorem 2.6.** Let $0 < \beta < \alpha$, $1 < p < \infty$ and $1 \leq q < \infty$, then $\mathcal{D}^\beta$ is bounded from $F_{p,q}^\alpha(\gamma_d)$ into $F_{p,q}^{\alpha-\beta}(\gamma_d)$.

**Proof.** Let $f \in F_{p,q}^\alpha(\gamma_d)$ and $v(x, r) = e^{-r}u(x, r)$, using Lemma 2.4 and Leibnitz’s differentiation rule for the product

$$|\mathcal{D}^\beta f(x)| \leq \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |(e^{-s} P_s - I)^k f(x)| ds = \frac{1}{c_\beta} \int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(v(x, \cdot), 0)| ds$$

$$\leq C_{\beta,k} \int_0^{+\infty} r^{-\beta}|v^{(k)}(x, r)| \frac{dr}{r} \leq C_{\beta,k} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} r^{-\beta} |u^{(k-j)}(x, r)| \frac{dr}{r}$$

$$= C_{\beta,k} \sum_{j=0}^{k-1} \binom{k}{j} \int_0^{+\infty} r^{-\beta} |u^{(k-j)}(x, r)| \frac{dr}{r} + C_{\beta,k} \int_0^{+\infty} r^{-\beta} |u(x, r)| \frac{dr}{r},$$

then

$$\|\mathcal{D}^\beta f\|_{p,q} \leq C_{\beta,k} \sum_{j=0}^{k-1} \binom{k}{j} \|\int_0^{+\infty} r^{-\beta} |u^{(k-j)}(\cdot, r)| \frac{dr}{r}\|_{p,q} + C_{\beta,k} \|\int_0^{+\infty} r^{-\beta} |u(\cdot, r)| \frac{dr}{r}\|_{p,q}$$

$$\leq C_{\beta,k} \sum_{j=0}^{k-1} \binom{k}{j} \|\int_0^{+\infty} r^{-\beta} |u^{(k-j)}(\cdot, r)| \frac{dr}{r}\|_{p,q} + C_{\beta,k} \int_0^{+\infty} r^{-\beta} \|u(\cdot, r)\|_{p,q} \frac{dr}{r}$$

$$\leq C_{\beta,k} \sum_{j=0}^{k-1} \binom{k}{j} \|\int_0^{+\infty} r^{-\beta} |u^{(k-j)}(\cdot, r)| \frac{dr}{r}\|_{p,q} + C_{\beta,k} \|f\|_{p,q} \Gamma(k - \beta)$$

$$\leq C \|f\|_{F_{p,q}^\alpha},$$

because $F_{p,q}^\alpha(\gamma_d) \subset F_{p,q}^{\beta-j}(\gamma_d)$, as $\alpha > \beta \geq \beta - j \geq 0$, for $j = 0, \ldots, k - 1$ and $q \geq 1$.

On the other hand, let $n \in \mathbb{N}, n > \alpha$ and $w(x, t) = e^{-t}u^{(n)}(x, t)$, by Lemma 2.4 we get

$$\left| \frac{\partial^n}{\partial t^n} P_t(\mathcal{D}^\beta f(x)) \right| \leq e^t \int_0^{+\infty} s^{-\beta-1} |\Delta^k_s(w(\cdot, \cdot), t)| ds$$

$$\leq e^t C_{\beta,k} \int_0^{+\infty} s^{-\beta-1} |u^{(k)}(x, t+s)| ds.$$
for all $r > 0$. Thus

$$|\frac{\partial^n}{\partial t^n} P_t(D^\beta f)(x)| \leq C_{\beta,k} \sum_{j=0}^{k} \binom{k}{j} \int_0^{+\infty} s^{k-\beta-1} e^{-s} |u(k+n-j)(x, t+s)| ds.$$ 

Therefore,

$$\left( \int_0^{\infty} \left( t^{\alpha-\beta} \frac{\partial^n}{\partial t^n} P_t(D^\beta f)(x) \right)^q \frac{dt}{t} \right)^{1/q} \leq C_{\beta,k} \sum_{j=0}^{k} \binom{k}{j} \left( \int_0^{\infty} t^{\alpha-\beta} \int_0^{+\infty} s^{k-j-(\beta-j)-1} e^{-s} |u(k+n-j)(x, t+s)| ds \right)^q \frac{dt}{t} \right)^{1/q}$$

For $0 \leq j \leq k-1$, we have $\beta - j \geq \beta - (k - 1) \geq 0$ using the same argument as in the proof of Theorem 2.5, see (2.12), using Hardy’s inequality we get

$$\| \left( \int_0^{\infty} \left( t^{\alpha-\beta} \int_0^{+\infty} s^{k-j-(\beta-j)-1} e^{-s} |u(k+n-j)(x, t+s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \|_{p, \gamma} \leq C \| f \|_{F^\alpha_{p,q}}$$

for $0 \leq j \leq k - 1$.

Unfortunately the remaining case $j = k$ requires a special argument, that uses the following known estimate for the Poisson-Hermite semigroup

$$(2.13) \quad \left| \frac{\partial^n}{\partial t^n} P_t f(x) \right| \leq C T^* f(x) t^{-n},$$

where $T^* f$ is the maximal function of the Ornstein-Uhlenbeck semigroup. The proof of this estimate can be found in [10], Lemma 2.1.

$$\left( \int_0^{\infty} \left( t^{\alpha-\beta} \int_0^{+\infty} s^{k-\beta-1} e^{-s} |u(n)(\cdot, t+s)| ds \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left( \int_0^{\infty} \left( t^{\alpha-\beta} \int_0^{t} s^{k-\beta-1} e^{-s} |u(n)(x, t+s)| ds \right)^q \frac{dt}{t} \right)^{1/q} + C \left( \int_t^{\infty} \left( t^{\alpha-\beta} \int_t^{+\infty} s^{k-\beta-1} e^{-s} |u(n)(x, t+s)| ds \right)^q \frac{dt}{t} \right)^{1/q}$$

$$= (I) + (II).$$

We consider first the case $k \leq \alpha$. The term (I) is estimated as term (I) in the proof of Theorem 2.5

$$(I) \leq C \left( \int_0^{\infty} \left( t^{\alpha-k} |u(n)(x, t)| \right)^q \frac{dt}{t} \right)^{1/q}.$$
Since $\beta \geq k - 1$, making the change of variable $v = t + s$ we get

$$(II) \geq C\left( \int_0^\infty \int_t^{t+\infty} |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$= C\left( \int_0^\infty \int_2t^{+\infty} |u^{(n)}(x, r)| dr \right)^q dt \frac{1}{q}$$

$$\leq C\left( \int_0^\infty \int_t^{t+\infty} |u^{(n)}(x, r)| dr \right)^q dt \frac{1}{q}.$$

Therefore, by Hardy’s inequality (1.30), we have

$$(II) \leq \frac{C}{n + k - \alpha - 1} \left( \int_0^\infty (r^{n-(\alpha-k)} |u^{(n)}(x, r)|)^q \frac{dr}{r} \right)^{1/q}.$$

Next consider the case $k > \alpha$. In this case, using inequality (2.13) and Hardy’s inequality (1.29), we have

$$(I) \leq C_n |T^n f(x)| \left( \int_0^\infty \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$\leq C_n |T^n f(x)| \left( \int_0^\infty \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$= C_n |T^n f(x)| \left( \frac{1}{\alpha - \beta} \right)^{1/(k - \alpha)} \Gamma((k - \alpha)q)^{1/q}.$$

On the other hand,

$$(II) \leq \left( \int_0^{+\infty} \int_1 \int_0^{+\infty} e^{-s} |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$+ \left( \int_1^\infty \int_0^{+\infty} e^{-s} |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$= (III) + (IV).$$

By the usual argument using the change of variable $v = t + s$ and Hardy’s inequality (1.30), we get

$$(III) \leq \left( \int_0^{+\infty} \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$\leq \left( \int_0^{+\infty} \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$= \left( \int_0^{+\infty} \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$\leq \left( \int_0^{+\infty} \int_0^\infty |u^{(n)}(x, t + s)| ds \right)^q dt \frac{1}{q}$$

$$\leq \frac{1}{n - 1} \left( \int_0^\infty (r^n |u^{(n)}(x, r)|)^q \frac{dr}{r} \right)^{1/q}.$$
Finally using again inequality \((2.13)\), we get

\[
(IV) \leq \left( \int_{1}^{\infty} t^{(n+k-\alpha-1)q-1} \left( \int_{t}^{+\infty} e^{-\varepsilon t} C_{n} |T^{*} f(x)| t^{-n} ds \right)^{q} dt \right)^{1/q}
\]

\[
= C_{n} |T^{*} f(x)| \left( \int_{1}^{\infty} t^{(k-\alpha-1)q-1} \varepsilon^{-tq} dt \right)^{1/q} \leq C_{n} |T^{*} f(x)| \left( \frac{1}{(\alpha + 1 - k)q} \right)^{1/q}.
\]

Hence, in both cases, we get that

\[
\| \left( \int_{0}^{\infty} (t^{n-(\alpha-\beta)}) \left( \frac{\partial^{n}}{\partial t^{n}} P_{t}(D^{\beta} f) \right) \right) dt \|_{p,\gamma} \leq C \| f \|_{F_{p,q}^{\alpha}}.
\]

Observation Let us observe that if instead of considering the Ornstein-Uhlenbeck operator \((1.2)\) and the Poisson-Hermite semigroup \((1.7)\) we consider the Laguerre differential operator in \(\mathbb{R}^{d}_{+}\),

\[
L^{\alpha} = \sum_{i=1}^{d} \left[ x_{i} \frac{\partial^{2}}{\partial x_{i}^{2}} + (\alpha_{i} + 1 - x_{i}) \frac{\partial}{\partial x_{i}} \right],
\]

and the corresponding Poisson-Laguerre semigroup, or if we consider the Jacobi differential operator in \((-1,1)^{d}\),

\[
L^{\alpha,\beta} = -\sum_{i=1}^{d} \left[ (1 - x_{i}^{2}) \frac{\partial^{2}}{\partial x_{i}^{2}} + (\beta_{i} - \alpha_{i} - (\alpha_{i} + \beta_{i} + 2) x_{i}) \frac{\partial}{\partial x_{i}} \right],
\]

and the corresponding Poisson-Jacobi semigroup (for details we refer to \([16]\)), the arguments are completely analogous. That is to say, we can defined in analogous manner Laguerre-Triebel-Lizorkin spaces, and Jacobi-Triebel-Lizorkin spaces then prove that the corresponding notions of Fractional Integrals and Fractional Derivatives behave similarly. In order to see this it is more convenient to use the representation \((1.7)\) of \(P_{t}\) in terms of the one-sided stable measure \(\mu_{t}^{(1/2)}(ds)\), see \([10]\).

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