Normalized ground states for Kirchhoff equations in \( \mathbb{R}^3 \) with a critical nonlinearity

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Abstract

This paper is concerned with the existence of ground states for a class of Kirchhoff type equation with combined power nonlinearities

\[- \left( a + b \int_{\mathbb{R}^3} |\nabla u(x)|^2 \right) \Delta u = \lambda u + |u|^{p-2}u + u^5 \quad \text{for some } \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^3,\]

with prescribed \( L^2 \)-norm mass

\[\int_{\mathbb{R}^3} u^2 = c^2\]

in Sobolev critical case and proves that the equation has a couple of solutions \((u_c, \lambda_c) \in S(c) \times \mathbb{R}\) for any \(c > 0\), \(a, b > 0\) and \(\frac{14}{3} \leq p < 6\), where \(S(c) = \{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} u^2 = c^2 \}\).

**Keywords:** Kirchhoff type equation; Critical nonlinearity; Normalized ground states

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1 Introduction

In this paper, we study the existence of normalized ground states for the following Kirchhoff equation:

\[- \left( a + b \int_{\mathbb{R}^3} |\nabla u(x)|^2 \right) \Delta u = \lambda u + |u|^{p-2}u + u^5 \quad \text{for some } \lambda \in \mathbb{R}, \quad x \in \mathbb{R}^3,\]

with prescribed mass

\[\int_{\mathbb{R}^3} u^2 = c^2,\]

where \(a, b > 0\) are constants and \(\frac{14}{3} \leq p < 6\). The weak solutions for the problem correspond to the critical points for the energy functional

\[E(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{b}{4} \left( \int_{\mathbb{R}^3} |\nabla u|^2 \right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6\]

on the constraint manifold

\[S(c) = \{ u \in H^1(\mathbb{R}^3) : \Psi(u) = \frac{1}{2}c^2 \},\]

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where $\Psi(u) = \frac{1}{2} \int_{\mathbb{R}^3} u^2$. Equation (1.1) is viewed as being nonlocal because of the appearance of the term $b \int_{\mathbb{R}^3} |\nabla u(x)|^2 \Delta u$, which indicates that equation (1.1) is no longer a pointwise identity. The nonlocal term also results in lack of weak sequential continuity of the energy function associated to (1.1), even we remove the critical term $u^5$. If $\mathbb{R}^3$ is replaced by a bounded domain $\Omega \subset \mathbb{R}^3$, then equation (1.1) describes the stationary state of the Kirchhoff type equation of the following type:

$$
\begin{cases}
  u_{tt} - (a + b \int_{\Omega} |\nabla u(x)|^2) \Delta u = f(x, u), & x \in \Omega, \\
  u = 0, & x \in \partial \Omega,
\end{cases}
$$

(1.2)

which is presented by Kirchhoff in [8]. It is an extension of D’Alembert’s wave equation by considering the effects of the length of strings during vibrations.

Problem (1.2) has received much attention after Lions [10] proposed an abstract framework to deal with the problem. We refer the readers to [1,2,3,4,5,6,11] and the work [2] seems to be the first one studying the critical Kirchhoff problem.

More recently, normalized solutions for elliptic equations have attracted considerable attentions, e.g. see [3,4,7,11,12,14-19] and the references therein. The work [7] is the first paper to deal with the existence of normalized solutions for a second order Schrödinger equation with a Sobolev sub-critical and $L^2$-supercritical nonlinearity and the papers [14-17] deal with the existence of normalized solutions for the problem in bounded domains. When $b = 0$, problem (1.1) becomes

$$
-\Delta u = \lambda u + |u|^{p-2} u + u^5, \quad x \in \mathbb{R}^3,
$$

with prescribed mass

$$
\int_{\mathbb{R}^3} u^2 = c^2,
$$

which was recently investigated by Soave in [18], where in case of $p \in (2,6)$ the author studied the existence and properties of the ground states for the problem.

For Kirchhoff type problems with a prescribed mass, it is shown that $p = \frac{14}{3}$ is the $L^2$-critical exponent for the minimization problem (1.2). The papers [9,22,23] consider the existence and properties of the $L^2$-subcritical constrained minimizers. In the case of $p \in (\frac{14}{3},6)$, the corresponding functional is unbounded from below on $S(c)$, [13] proved that there are infinitely many critical points by using a minimax procedure. However, few literature is concerned with normalized solutions for critical Kirchhoff problem. Inspired by [18], in this paper we attempt to study the critical Kirchhoff problem (1.1).

Our main result is the following:

**Theorem 1.1.** Let $a,b > 0$ and $\frac{14}{3} \leq p < 6$. Then problem (1.1) has a couple of solutions $(u_c, \lambda_c) \in S(c) \times \mathbb{R}$ for any $c > 0$. Moreover,

$$
E(u_c) = \inf_{u \in V(c)} E(u),
$$

(1.3)

where $V(c)$ is the Pohozaev manifold defined in lemma 2.1.
2 Preliminaries

To prove our theorem, we need some notations and useful preliminary results. Throughout this paper, we denote $B_r(z)$ the open ball of radius $r$ with center at $z$, and $\| \cdot \|_p$ the usual norm of space $L^p(\mathbb{R}^3)$. Let $H = H^1(\mathbb{R}^3)$ with the usual norm of space $H^1(\mathbb{R}^3)$. Generic positive constant is denoted by $C$, $C_1$, or $C_2$..., which may change from line to line. Let $H = H^1(\mathbb{R}^3)$ with the usual norm of space $H^1(\mathbb{R}^3)$. We denote the best constant of $D_{1,2}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ by $S = \inf_{u \in D_{1,2}^1(\mathbb{R}^3) \setminus \{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2}$.

In [20], we know that $S$ is achieved by $U_\varepsilon(x) = C_{\varepsilon}^{\frac{1}{2}} e^{\frac{3}{2} \varepsilon^2 s^2 u(x)}$.

The Pohozaev identity plays an important role in our discussion. We give it in the following lemma; for more details, see [7].
Lemma 2.1. Let \((u, \lambda) \in S(c) \times \mathbb{R}\) be a weak solution of equation (1.1). Then \(u\) belongs to the set
\[
V(c) \triangleq \{ u \in H : P(u) = 0 \}
\]
where
\[
P(u) = a \| \nabla u \|_2^2 + b \| \nabla u \|_4^4 - \beta_p \| u \|_p^p - \| u \|_6^6.
\]
For any \(u \in S(c)\) and \(s \in \mathbb{R}\), we define \(\Phi_u(s) \triangleq I(u, s)\). Then
\[
(\Phi_u)'(s) = a e^{s \frac{1}{2}} \| \nabla u \|_2^2 + b e^{4s \frac{1}{2}} \| \nabla u \|_4^4 - \beta_p e^{p \frac{1}{2}} \| u \|_p^p - e^{6s \frac{1}{2}} \| u \|_6^6
\]
\[
= a \| \nabla H(u, s) \|_2^2 + b \| \nabla H(u, s) \|_4^4 - \beta_p \| H(u, s) \|_p^p - \| H(u, s) \|_6^6.
\]
Therefore, we have
Lemma 2.2. For any \(u \in S(c), s \in \mathbb{R}\) is a critical point for \(\Phi_u(s)\) if and only if \(H(s, u) \in V(c)\).

Remark 2.3. The map \((u, s) \in \mathbb{H} \mapsto H(u, s) \in H\) is continuous; (2.4) see \[4, Lemma3.5\].

Lemma 2.4. \[11, Lemma 3.3\] For \(t, s > 0\), the following system
\[
\begin{align*}
x(t, s) &= t - a S(t + s)^{\frac{1}{2}} = 0 \\
y(t, s) &= s - b S^2 (t + s)^{\frac{3}{2}} = 0
\end{align*}
\]
has a unique solution \((t_0, s_0)\). Moreover, if \(x(t, s) \geq 0\) and \(y(t, s) \geq 0\), then \(t \geq t_0, t \geq t_0\).

3 Characterization of mountain pass level

As in \[7,12\], we firstly prove that \(I(u, s)\) has the mountain pass geometry on \(S(c) \times \mathbb{R}\) in the following lemmas.

Lemma 3.1. Assume that \(a, b > 0\) and \(\frac{14}{3} \leq p < 6\). Let \(u \in S(c)\) be arbitrary fixed. Then
\[
(1) \int_{\mathbb{R}^3} |\nabla H(u, s)|^2 \to 0, \text{ and } I(u, s) \to 0^+, \text{ as } s \to -\infty;
\]
\[
(2) \int_{\mathbb{R}^3} |\nabla H(u, s)|^2 \to +\infty, \text{ and } I(u, s) \to -\infty, \text{ as } s \to +\infty.
\]

Proof. The proof is trivial from the facts
\[
\| \nabla H(s, u) \|_2^2 = e^{2s} \| \nabla u \|_2^2
\]
and
\[
I(u, s) = E(H(u, s)) = \frac{a}{2} e^{2s} \| \nabla u \|_2^2 + \frac{b}{4} e^{4s} \| \nabla u \|_4^4 - \frac{e^{p \frac{1}{2}}}{p} \| u \|_p^p - \frac{e^{6s \frac{1}{2}}}{6} \| u \|_6^6.
\]

\[\square\]
Lemma 3.2. Let $a, b, c > 0$ and $\frac{14}{3} \leq p < 6$. Then there exists $K_c > 0$ such that

$$P(u), E(u) > 0 \text{ for all } u \in A_c, \quad \text{and } \quad 0 < \sup_{u \in A_c} E(u) < \inf_{u \in B_c} E(u)$$

with

$$A_c = \{ u \in S_c : \int_{\mathbb{R}^3} |\nabla u|^2 \leq K_c \}, \quad B_c = \{ u \in S_c : \int_{\mathbb{R}^3} |\nabla u|^2 = 2K_c \}.$$ 

Proof. Let $K > 0$ be arbitrary fixed and suppose that $u, v \in S(c)$ are such that $\|\nabla u\|^2_2 \leq K$ and $\|\nabla v\|^2_2 = 2K$. Then, for $K > 0$ small enough, using (2.3) and $p\beta \geq 4$, there exist two constants $C_1$ and $C_2$ such that

$$P(u) \geq a\|\nabla u\|^2_2 + b\|\nabla u\|^4_2 - C_1\|\nabla u\|^p\beta_2 - C_2\|\nabla u\|^{6}_2,$$

$$E(u) \geq \frac{a}{2}\|\nabla u\|^2_2 + \frac{b}{4}\|\nabla u\|^4_2 - C_1\|\nabla u\|^p\beta_2 - C_2\|\nabla u\|^6_2$$

and

$$E(v) - E(u) \geq E(v) - \frac{a}{2}\|\nabla u\|^2_2 - \frac{b}{4}\|\nabla u\|^4_2$$

$$\geq \frac{aK}{2} + \frac{bK^2}{2} - C_1\|\nabla v\|^p\beta_2 - C_2\|\nabla v\|^6_2$$

$$\geq \frac{aK}{2} + \frac{bK^2}{2} - C_1 K^\frac{p\beta}{2} - C_2 K^3.$$

Therefore, by the above inequalities, it follows that there exists $K_c$ small enough such that

$$P(u), E(u) > 0 \text{ for all } x \in A_c, \quad \text{and } \quad 0 < \sup_{u \in A_c} E(u) < \inf_{u \in B_c} E(u)$$

with

$$A_c = \{ u \in S_c : \int_{\mathbb{R}^3} |\nabla u|^2 \leq K_c \}, \quad B_c = \{ u \in S_c : \int_{\mathbb{R}^3} |\nabla u|^2 = 2K_c \}. \quad \square$$

Next, we give a characterization of mountain pass level for $I(u, s)$ and $E(u)$. $E^d$ denotes the set $\{ u \in S_c : E(u) \leq d \}$.

Proposition 3.3. Under assumptions that $a, b > 0$ and $\frac{14}{3} \leq p < 6$, let

$$\tilde{\gamma}_c = \inf_{\tilde{h} \in \Gamma_c} \max_{t \in [0,1]} I(\tilde{h}(t))$$

where

$$\tilde{\Gamma}_c = \{ \tilde{h} \in C([0,1], S(c) \times \mathbb{R}) : \tilde{h}(0) \in (A_c, 0), \tilde{h}(1) \in (E^0, 0) \},$$

and

$$\tilde{\Gamma}_c = \Gamma_c \cup \{ (\tilde{h}(0), \tilde{h}(1)) : \tilde{h} \in \tilde{\Gamma}_c \}.$$
and
\[ \gamma_c = \inf_{h \in \Gamma_c} \max_{t \in [0,1]} E(h(t)) \]
where
\[ \Gamma_c = \{ h \in C([0,1], S(c)) : h(0) \in A_c, h(1) \in E^0 \}. \]

Then we have
\[ \tilde{\gamma}_c = \gamma_c. \]

Proof. Since \( \Gamma_c \times \{0\} \subseteq \tilde{\Gamma}_c \), we have \( \tilde{\gamma}_c \leq \gamma_c \). So, it remains to prove that \( \tilde{\gamma}_c \geq \gamma_c \). For any \( \tilde{h}(t) = (\tilde{h}_1(t), \tilde{h}_2(t)) \in \tilde{\Gamma}_c \), we set \( h(t) = H(\tilde{h}_1(t), \tilde{h}_2(t)) \). Then \( h(t) \in \Gamma_c \) and
\[ \max_{t \in [0,1]} I(h(t)) = \max_{t \in [0,1]} E(H(\tilde{h}_1(t), \tilde{h}_2(t))) = \max_{t \in [0,1]} E(h(t)), \]
which shows that \( \tilde{\gamma}_c \geq \gamma_c \). \( \square \)

In the following proposition, we give the existence of \((PS)_{\tilde{\gamma}_c}\) sequence for \( I(u, s) \). Its proof is by a standard argument using the Ekeland’s Variational principle and constructing pseudo-gradient flow.

**Proposition 3.4.** Let \( \{g_n\} \subset \tilde{\Gamma}_c \) be such that
\[ \max_{t \in [0,1]} I(g_n(t)) \leq \tilde{\gamma}_c + \frac{1}{n}. \]
Then there exists a sequence \( \{(u_n, s_n)\} \subset S(c) \times \mathbb{R} \) such that

(1) \( I(u_n, s_n) \in [\gamma_c - \frac{1}{n}, \gamma_c + \frac{1}{n}] \);

(2) \( \min_{t \in [0,1]} \| (u_n, s_n) - g_n(t) \|_{H} \leq \frac{1}{\sqrt{n}} \);

(3) \( \| I'|_{S(c) \times \mathbb{R}}(u_n, s_n) \| \leq \frac{2}{\sqrt{n}} \) i.e.
\[ |\langle I'(u_n, s_n), z \rangle_{H^{-1} \times H} | \leq \frac{2}{\sqrt{n}} \| z \|_H \]
for all \( z \in \tilde{T}_{(u_n, s_n)} \triangleq \{ (z_1, z_2) \in H : \langle u_n, z_1 \rangle_{L^2} = 0 \} \).

**Proposition 3.5.** Under the assumptions \( a, b > 0 \) and \( \frac{4}{3} \leq p < 6 \), there exists a sequence \( \{v_n\} \subset S(c) \) such that

(1) \( E(v_n) \to \gamma_c \), as \( n \to \infty \);

(2) \( P(v_n) \to 0 \), as \( n \to \infty \);
\( (3) \) \( E'|_{S(c)}(v_n) \to 0, \) as \( n \to \infty \) i.e.

\[
\langle E'(v_n), h \rangle_{H^{-1} \times H} \to 0
\]

uniformly for all \( h \) satisfying

\[
\|h\|_H \leq 1 \quad \text{where} \quad h \in T_{v_n} \triangleq \{h \in H : \langle v_n, h \rangle_{L^2} = 0\}.
\]

Proof. By Proposition \[3.3\] \( \tilde{\gamma}_c = \gamma_c \). Pick \( \{g_n = ((g_n)_1, 0)\} \subset \tilde{\Gamma}_c \) such that

\[
\max_{t \in [0,1]} I(g_n(t)) \leq \tilde{\gamma}_c + \frac{1}{n}.
\]

It follows from Proposition \[3.3\] that there exists a sequence \( \{(u_n, s_n)\} \subset S(c) \times \mathbb{R} \) such that, as \( n \to \infty \), one has

\[
\begin{align*}
I(u_n, s_n) &\to \gamma_c, \quad \text{(3.1)} \\
 s_n &\to 0, \quad \text{(3.2)} \\
 \partial_s I(u_n, s_n) &\to 0. \quad \text{(3.3)}
\end{align*}
\]

Let \( v_n = H(u_n, s_n) \). Then \( E(v_n) = I(u_n, s_n) \) and, by (3.1), (1) holds. For the proof of (2), we notice that

\[
\partial_s I(u_n, s_n) = ae^{2s_n} \|\nabla u_n\|_2^2 + be^{4s_n} \|\nabla u_n\|_2^4 - e^{p\beta s_n} \|u_n\|_p^p - e^{6s_n} \|u_n\|_6^6
\]

\[
= a \|\nabla v_n\|_2^2 + b \|\nabla v_n\|_2^4 - \beta \|v_n\|_p^p - \|v_n\|_6^6
\]

\[
= P(v_n),
\]

which implies (2) by (3.3).
For the proof of (3), let \( h_n \in T_{v_n} \). We have

\[
\langle E'(v_n), h_n \rangle_{H^{-1} \times H} = a \int_{\mathbb{R}^3} \nabla v_n(x) \nabla h_n(x) + b \int_{\mathbb{R}^3} |\nabla v_n(x)|^2 \int_{\mathbb{R}^3} \nabla v_n(x) \nabla h_n(x)
\]

\[
- \int_{\mathbb{R}^3} |v_n(x)|^{p-2} v_n(x) h_n(x) - \int_{\mathbb{R}^3} (v_n(x))^5 h_n(x)
\]

\[
= ae^{\frac{5\alpha_n}{2}} \int_{\mathbb{R}^3} \nabla u_n(e^{\alpha_n} x) \nabla h_n(x) - e^{\frac{15\alpha_n}{2}} \int_{\mathbb{R}^3} (u_n(e^{\alpha_n} x))^5 h_n(x)
\]

\[
+ be^{\frac{15\alpha_n}{2}} \int_{\mathbb{R}^3} |\nabla u_n(e^{\alpha_n} x)|^2 \int_{\mathbb{R}^3} \nabla u_n(e^{\alpha_n} x) \nabla h_n(x)
\]

\[
- e^{\frac{3(p-1)\alpha_n}{2}} \int_{\mathbb{R}^3} |u_n(e^{\alpha_n} x)|^{p-2} u_n(e^{\alpha_n} x) h_n(x)
\]

\[
= ae^{2\alpha_n} \int_{\mathbb{R}^3} \nabla u_n(x) e^{-\frac{3\alpha_n}{2}} \nabla h_n(e^{-\alpha_n} x)
\]

\[
+ be^{4\alpha_n} \int_{\mathbb{R}^3} |\nabla u_n(x)|^2 \int_{\mathbb{R}^3} \nabla u_n(x) e^{-\frac{5\alpha_n}{2}} \nabla h_n(e^{-\alpha_n} x)
\]

\[
- e^{p\alpha_n} \int_{\mathbb{R}^3} |u_n(x)|^{p-2} u_n(x) e^{-\frac{3\alpha_n}{2}} h_n(e^{-\alpha_n} x)
\]

\[
- e^{6\alpha_n} \int_{\mathbb{R}^3} (u_n(x))^5 e^{-\frac{3\alpha_n}{2}} h_n(e^{-\alpha_n} x).
\]

Setting \( \hat{h}_n(x) = e^{-\frac{3\alpha_n}{2}} h_n(e^{-\alpha_n} x) \), then

\[
\langle I'(u_n, s_n), (\hat{h}_n, 0) \rangle_{H^{-1} \times H} = \langle E'(v_n), h_n \rangle_{H^{-1} \times H}.
\]

It is easy to see that

\[
\langle u_n(x), \hat{h}_n(x) \rangle_{L^2} = \int_{\mathbb{R}^3} u_n(x) e^{\frac{3\alpha_n}{2}} h_n(e^{-\alpha_n} x)
\]

\[
= \int_{\mathbb{R}^3} u_n(e^{\alpha_n} x) e^{\frac{3\alpha_n}{2}} h_n(x)
\]

\[
= \int_{\mathbb{R}^3} v_n(x) h_n(x) = 0.
\]

So, we have that \( (\hat{h}_n(x), 0) \in \tilde{T}_{(u_n, s_n)} \). On the other hand,

\[
\|\hat{h}_n(x), 0\|_{H^{-1}}^2 = \|\hat{h}_n(x)\|_H^2
\]

\[
= \|h_n(x)\|_H^2 + e^{-2\alpha_n} \|\nabla h_n(x)\|_2^2
\]

\[
\leq C\|\hat{h}_n(x)\|_H^2,
\]

where the last inequality holds by (3.3). Thus, (3) is proved. \( \square \)

In the following lemma, we give an upper bound estimate for the mountain pass level \( \gamma_c \).
Lemma 3.6. Under assumptions $a, b > 0$ and $\frac{14}{3} \leq p < 6$, then $\gamma_c < \gamma_c^* \triangleq \frac{abS^4}{4} + \frac{b^3S^6}{24} + \frac{(4a + b^2S^2)^2}{24}$, where $S$ is defined in (2.1).

Proof. Let $\varphi(x) \in C_0^\infty(B_2(0))$ be a radial cut-off function such that $0 \leq \varphi(x) \leq 1$ and $\varphi(x) \equiv 1$ on $B_1(0)$. Then we take $u_\varepsilon = \varphi(x)U_\varepsilon$ ($U_\varepsilon$ defined in (2.2)) and

$$v_\varepsilon = \frac{u_\varepsilon}{\|u_\varepsilon\|_2} \in S(c) \cap H^1_\varepsilon.$$

We take $\varepsilon = 1$ and define

$$K_1 \triangleq \|\nabla U_1\|_2^2, \quad K_2 \triangleq \|U_1\|_6^6, \quad K_3 \triangleq \|U_1\|_p^p.$$

According to [18 Appendix A], we have

$$\begin{aligned}
K_1/K_2 &= S, \quad \|\nabla u_\varepsilon\|_2^2 = K_1 + O(\varepsilon), \quad \|u_\varepsilon\|_6^2 = K_2 + O(\varepsilon^2), \\
\|u_\varepsilon\|_p^p &= \varepsilon^{3-\frac{p}{2}} (K_3 + O(\varepsilon^{p-3})), \quad \|u_\varepsilon\|_2^2 = O(\varepsilon^2) + \omega \left( \int_0^2 \varphi(r)dr \right) \varepsilon,
\end{aligned} \tag{3.4}$$

where $\omega$ is the area of the unit sphere in $\mathbb{R}^3$. Define

$$\Psi_{v_\varepsilon}(s) \triangleq \frac{a}{2} \varepsilon^{2s}\|\nabla v_\varepsilon\|_2^2 + \frac{b}{4} \varepsilon^{4s}\|\nabla v_\varepsilon\|_2^4 - \frac{e^{6s}}{6}\|v_\varepsilon\|_6^6.$$

Then

$$(\Psi)'_{v_\varepsilon}(s) = ae^{2s}\|\nabla v_\varepsilon\|_2^2 + be^{4s}\|\nabla v_\varepsilon\|_2^4 - e^{6s}\|v_\varepsilon\|_6^6.$$

**Step 1:** It is easy to see that $\Psi_{v_\varepsilon}(s)$ has a unique critical point $s_0$, which is a strict maximum point such that

$$e^{2s_0} = \frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|_6^6}}{2\|v_\varepsilon\|_6^6} \tag{3.5}$$

and the maximum level of $\Psi_{v_\varepsilon}(s)$ is

$$\begin{aligned}
\Psi_{v_\varepsilon}(s_0) &= \frac{a}{2}\|\nabla v_\varepsilon\|_2^2 \left( \frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|_6^6}}{2\|v_\varepsilon\|_6^6} \right) \\
&\quad + \frac{b}{4}\|\nabla v_\varepsilon\|_2^4 \left( \frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|_6^6}}{2\|v_\varepsilon\|_6^6} \right)^2 \\
&\quad - \frac{1}{6}\|v_\varepsilon\|^6_6 \left( \frac{b\|\nabla v_\varepsilon\|_2^4 + \sqrt{b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|_6^6}}{2\|v_\varepsilon\|_6^6} \right)^3 \\
&= \frac{ab\|\nabla v_\varepsilon\|_2^6}{4\|v_\varepsilon\|_6^6} + \frac{b^3\|\nabla v_\varepsilon\|_2^{12}}{24\|v_\varepsilon\|_6^{12}} + \frac{(b^2\|\nabla v_\varepsilon\|_2^8 + 4a\|\nabla v_\varepsilon\|_2^4\|v_\varepsilon\|_6^6)^{\frac{3}{2}}}{24\|v_\varepsilon\|_6^{12}}. \tag{3.6}
\end{aligned}$$
By (3.4), we conclude that
\[
\frac{ab \| \nabla v_\varepsilon \|_6^6}{4 \| v_\varepsilon \|_6^6} = \frac{ab \| \nabla u_\varepsilon \|_6^6}{4 \| u_\varepsilon \|_6^6} = \frac{ab}{4} \left( \frac{K_1 + O(\varepsilon)}{K_2 + O(\varepsilon^2)} \right)^3 = \frac{ab}{4} S^3 + O(\varepsilon);
\]
\[
\frac{b^3 \| \nabla v_\varepsilon \|_6^{12}}{24 \| v_\varepsilon \|_6^6} = \frac{b^3 \| \nabla u_\varepsilon \|_6^{12}}{24 \| u_\varepsilon \|_6^6} = \frac{b^3}{24} \left( \frac{K_1 + O(\varepsilon)}{K_2 + O(\varepsilon^2)} \right)^6 = \frac{b^3}{24} S^6 + O(\varepsilon).
\]

For the last term in (3.6)
\[
\left( \frac{b^2 \| \nabla v_\varepsilon \|_2^2}{24 \| v_\varepsilon \|_6^6} + 4a \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6 \right)^\frac{3}{2} = \frac{1}{24} \left( (b^2 K_1 / K_2 + O(\varepsilon))^4 + 4a (K_1 / K_2 + O(\varepsilon)) \right)^\frac{3}{2}
\]
\[
= \frac{1}{24} \left( 4a S + b^2 S^4 + O(\varepsilon) \right)^\frac{3}{2}
\]
\[
= \frac{1}{24} (4a S + b^2 S^4)^\frac{\beta}{2} + O(\varepsilon).
\]

By the above estimates, one has
\[
\Psi_{v_\varepsilon}(s_0) = \frac{ab S^3}{4} + \frac{b^3 S^6}{24} + \frac{4a S + b^2 S^4}{24}^\frac{\beta}{2} + O(\varepsilon).
\]

(3.7)

Step 2: We give an upper bound estimate for \( \Phi_{v_\varepsilon}(s) = I(v_\varepsilon, s) \). Note that
\[
(\Phi_{v_\varepsilon})'(s) = ae^{2s} \| \nabla v_\varepsilon \|_2^2 + be^{4s} \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6 \beta - \beta_p e^{p\beta \varepsilon s} \| v_\varepsilon \|_p^p - e^{6s} \| v_\varepsilon \|_6^6.
\]

Obviously, \( \Phi_{v_\varepsilon}(s) \) has a unique critical point \( s_1 \) and
\[
e^{2s_1} \leq e^{2s_0} = \frac{b \| \nabla v_\varepsilon \|_2^2 + \sqrt{b^2 \| \nabla v_\varepsilon \|_2^2 + 4a \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6}}{2 \| v_\varepsilon \|_6^6},
\]

Since
\[
(\Phi_{v_\varepsilon})'(s_1) = e^{2s_1} \| \nabla v_\varepsilon \|_2^2 + e^{4s_1} \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6 \beta - \beta_p e^{p\beta \varepsilon s_1} \| v_\varepsilon \|_p^p - e^{6s_1} \| v_\varepsilon \|_6^6 = 0,
\]
in view of the definition of \( \beta_p \) with \( p \beta \geq 4 \) and (3.4), we have
\[
e^{2s_1} = \frac{b \| \nabla v_\varepsilon \|_2^2}{\| v_\varepsilon \|_6^6} + e^{-2s_1} a \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6 - \beta_p e^{2s_1} \| v_\varepsilon \|_p^p \| v_\varepsilon \|_6^6
\]
\[
\geq \frac{b \| \nabla u_\varepsilon \|_2^2}{\| v_\varepsilon \|_6^6} + e^{-2s_1} a \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6 - \beta_p e^{2s_1} \| v_\varepsilon \|_p^p \| v_\varepsilon \|_6^6
\]
\[
= \frac{b}{c^2} \| \nabla u_\varepsilon \|_2^2 \| u_\varepsilon \|_6^6 \| v_\varepsilon \|_6^6 + \frac{2 \| v_\varepsilon \|_6^6}{\| v_\varepsilon \|_p^p} \left( \frac{b \| \nabla v_\varepsilon \|_2^2 + \sqrt{b^2 \| \nabla v_\varepsilon \|_2^2 + 4a \| \nabla v_\varepsilon \|_2^2 \| v_\varepsilon \|_6^6}}{2 \| v_\varepsilon \|_6^6} \right)^{\frac{(p \beta - 4)}{2}}
\]
\[
= C_1 \varepsilon + O(\varepsilon^2) + C_2 \varepsilon + O(\varepsilon^2) - C_3 (\varepsilon + O(\varepsilon^2)) \frac{\beta}{2} - \frac{4p}{\beta}.
\]

(3.8)
In view of
\[
\Phi_{v_\varepsilon}(s_1) = \Psi_{v_\varepsilon}(s_1) - \frac{e^{p\beta p} \varepsilon}{p} \|v_\varepsilon\|_p^p
\]
\[
\leq \Psi_{v_\varepsilon}(s_0) - \frac{e^{p\beta p} \varepsilon}{p} \|v_\varepsilon\|_p^p
\]
and
\[
\|v_\varepsilon\|_p^p = C_4 \varepsilon^{3-p} + O(1),
\]
letting \(\varepsilon \to 0\), we have
\[
\Phi_{v_\varepsilon}(s_1) \leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24} + O(\varepsilon) - \frac{e^{p\beta p} \varepsilon}{p} \|v_\varepsilon\|_p^p
\]
\[
\leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24} + O(\varepsilon) - (e^{2s_1})^{p\beta p} \frac{\|v_\varepsilon\|_p^p}{p}
\]
\[
\leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24} + O(\varepsilon) - C_5 \varepsilon^{(4-p)/p}
\]
\[
< \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24},
\]
by (3.7) and (3.8).

**Step 3:** Take \(\varepsilon\) small enough such that \(v_\varepsilon\) satisfies the above inequality. By Lemma 3.1 and (2.4), there exist \(s^- \ll -1\) and \(s^+ \gg 1\) such that
\[
h_{v_\varepsilon} : \tau \in [0, 1] \mapsto H(v_\varepsilon, (1 - \tau)s^- + \tau s^+) \in \Gamma.
\]
Therefore,
\[
\gamma_c \leq \max_{\tau \in [0, 1]} E(h_{v_\varepsilon}(\tau)) \leq \Phi_{v_\varepsilon}(s_1) \leq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24}.
\]
By letting \(\gamma_c^* \triangleq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4aS + b^2S^4)^{3/2}}{24}\), the conclusion follows from the above inequality.

Let \(m(c) \triangleq \inf_{u \in V(c)} E(u)\), where \(V(c)\) is the Pohozaev manifold. We have the following relationship between \(\gamma_c\) and \(m(c)\).

**Lemma 3.7.** Under the assumptions \(a, b > 0\) and \(\frac{14}{3} \leq p < 6\), we have that
\[
m(c) = \gamma_c > 0.
\]

**Proof.** **Step 1:** We claim that for every \(u \in S(c)\), there exists a unique \(t_u \in \mathbb{R}\) such that \(H(u, t_u) \in V(c)\), where \(t_u\) is a strict maximum point for \(\Phi_u(s)\) at a positive level.
The existence of $t_u$ follows from Lemma 2.2. The uniqueness is from the following reasoning. Noticing that

$$(\Phi u)'(s) = 2ae^2\|\nabla u\|^2_2 + 4be^4\|\nabla u\|^4_2 - p\beta_p^2 e^{p\beta_p s}\|u\|_p^p - 6e^{6s}\|u\|_6^6$$

$$(\Phi u)''(s) = 2a\|\nabla H(u, s)\|^2_2 + 4b\|\nabla H(u, s)\|^4_2 - p\beta_p^2 \|H(u, s)\|_p^p - 6\|H(u, s)\|_6^6,$$

combining with $(\Phi u)'(t_u) = 0$, we have

$$(\Phi u)''(t_u) = -2a\|\nabla H(u, t_u)\|^2_2 - \beta_p(p\beta_p - 4)\|H(u, t_u)\|_p^p - 2\|H(u, t_u)\|_6^6 < 0.$$
The function $\tilde{P}(\tau)$ is continuous by (2.4), and hence we deduce that there exists $\tilde{\tau} \in (0, 1)$ such that $\tilde{P}(\tilde{\tau}) = 0$, which implies that $H(\tilde{h}_1(\tilde{\tau}), \tilde{h}_2(\tilde{\tau})) \in V(c)$ and

$$\max_{\tau \in [0, 1]} I(\tilde{h}(\tau)) = \max_{\tau \in [0, 1]} E(H(\tilde{h}_1(\tau), \tilde{h}_2(\tau))) \geq \inf_{u \in V(c)} E(u).$$

So, we infer that $\tilde{\gamma}_c = \gamma_c \geq m(c)$.

**Step 4:** At last, we will prove that $\gamma_c > 0$.

If $u \in V(c)$, then $P(u) = 0$ and by GNS inequality (2.3), we deduce that

$$a \|
abla u\|^2_2 + b \|
abla u\|^4_2 \leq C \|
abla u\|^\beta_p + C \|
abla u\|^6_2,$$

which implies that there exists $\delta > 0$ such that $\inf_{V(c)} \|
abla u\|^2_2 \geq \delta$. For any $u \in V(c)$, in view of $p\beta_p \geq 4$, we obtain that

$$E(u) = E(u) - \frac{1}{4} P(u) = \frac{a}{4} \|
abla u\|^2_2 + \left(\frac{\beta_p}{4} - \frac{1}{p}\right) \|u\|^p_p + \frac{1}{12} \|u\|^6_6 \geq \frac{a\delta}{4} > 0.$$

Thus, $\gamma_c > 0$.

**4 Proof of Theorem 1.1**

Choosing a PS sequence $\{v_n\}$ as in Proposition 3.5 and applying Lagrange multipliers rule to (3) of Proposition 3.5, there exists a sequence $\{\lambda_n\} \subset \mathbb{R}$ such that

$$E'(v_n) - \lambda_n \Psi'(v_n) \rightarrow 0 \quad \text{in} \quad H^{-1}. \tag{4.1}$$

**Step 1:** We claim that $v_n$ is bounded in $H$ and up to a subsequence, $v_n \rightharpoonup v$ in $H$. Since $P(v_n) \rightarrow 0$, we have

$$E(v_n) + o_n(1) = E(v_n) - \frac{1}{4} P(v_n) = \frac{a}{4} \|
abla v_n\|^2_2 + \left(\frac{\beta_p}{4} - \frac{1}{p}\right) \|v_n\|^p_p + \frac{1}{12} \|v_n\|^6_6.$$

Then, using the fact $p\beta_p \geq 4$, we deduce that

$$\frac{a}{4} \|
abla v_n\|^2_2 \leq \gamma_c + o_n(1).$$

Thus, $v_n$ is bounded in $H$. Since the embedding $H^1_1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ is compact for $q \in (2, 6)$, we deduce that there exists $v \in H^1_1(\mathbb{R}^3)$ such that, up to a subsequence, $v_n \rightharpoonup v$ in $H$.

**Step 2:** We will prove that, up to a subsequence, $\lambda_n \rightarrow \lambda < 0$.

Since $v_n$ is bounded in $H$, by (1.1), we know that

$$E'(v_n)v_n - \lambda_n \Psi'(v_n)v_n = o_n(1).$$

Therefore,

$$\lambda_n \|v_n\|^2_2 = a \|
abla v_n\|^2_2 + b \|
abla v_n\|^4_2 - \|v_n\|^p_p - \|v_n\|^6_6. \tag{4.2}$$
Using the fact that $\|v_n\|_2^2 = c^2$ and $\{v_n\}$ is bounded in $H$, we deduce that $\{\lambda_n\}$ is bounded. Hence, up to a subsequence, $\lambda_n \to \lambda \in \mathbb{R}$. Putting $P(v_n) \to 0$ into (4.2), we obtain that

$$
\lambda c^2 = \lim_{n \to \infty} a \|\nabla v_n\|_2^2 + b \|\nabla v_n\|_2^4 - \|v_n\|_p^p - \|v_n\|_6^6 = (\beta_p - 1)\|v\|_p^p \leq 0.
$$

(4.3)

Hence, $\lambda = 0$ if and only if $v \equiv 0$. Therefore, we only need to prove that $v \not\equiv 0$. We assume by contradiction that $v \equiv 0$. Up to a subsequence, let $\|\nabla v_n\|_2^2 \to l \in \mathbb{R}$. Since $P(v_n) \to 0$ and $v_n \to 0$ in $L^p(\mathbb{R}^3)$, we have

$$
\|v_n\|_6^6 \to a l + b l^2.
$$

By (2.1), one has

$$
(al + bl^2) \frac{1}{l} \leq \frac{l}{S}.
$$

So, we have

$$
l = 0 \quad \text{or} \quad l \geq \frac{b S^3}{2} + \sqrt{\frac{b^2 S^6}{4} + a S^3}.
$$

If $l > 0$, then we have

$$
m(c) + o_n(1) = E(v_n) = E(v_n) - \frac{1}{6} P(v_n) + o_n(1)
$$

$$
= \frac{a}{3} \|\nabla v_n\|_2^2 + \frac{b}{12} \|\nabla v_n\|_2^4 + o_n(1)
$$

$$
= \frac{a}{3} l + \frac{b}{12} l^2 + o_n(1).
$$

Hence $m(c) = \frac{a}{6} l + \frac{b}{12} l^2$. On the other hand, by $l \geq \frac{b S^3}{2} + \sqrt{\frac{b^2 S^6}{4} + a S^3}$, we infer that

$$
m(c) = \frac{a}{3} l + \frac{b}{12} l^2 \geq \frac{a}{3} \left( \frac{b S^3}{2} + \sqrt{\frac{b^2 S^6}{4} + a S^3} \right) + \frac{b}{12} \left( \frac{b S^3}{2} + \sqrt{\frac{b^2 S^6}{4} + a S^3} \right)^2
$$

$$
= \frac{a b S^3}{4} + \frac{b^3 S^6}{24} + \frac{(4a S + b^2 S^4)^{\frac{1}{2}}}{24} = \gamma^*.
$$

By Lemma 3.6, this contradicts to $m(c) = \gamma_c < \gamma^*$. If $l = 0$, then we have

$$
\|\nabla v_n\|_2^2 \to 0, \quad \|v_n\|_6^6 \to 0,
$$

implying that $E(v_n) \to 0$, which is a contradiction to $\gamma_c > 0$.

**Step 3:** Since $\lambda < 0$, we can define an equivalent norm of $H$

$$
\|u\|^2 = a \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx - \lambda \int_{\mathbb{R}^3} |u(x)|^2 dx.
$$

Up to a subsequence, let $\lim_{n \to \infty} \|\nabla v_n\|_2^2 = A^2 > 0$. Then $v$ satisfies

$$
a \int_{\mathbb{R}^3} \nabla v \nabla \phi + b A^2 \int_{\mathbb{R}^3} \nabla v \nabla \phi - \lambda \int_{\mathbb{R}^3} v \phi - \int_{\mathbb{R}^3} |v|^{p-2} v \phi - \int_{\mathbb{R}^3} v^5 \phi = 0
$$

(4.5)
for any $\phi \in H^1_0(\mathbb{R}^3)$. Let

$$J_{\lambda}(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{\lambda}{2} \int_{\mathbb{R}^3} |u|^2 + \frac{bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p - \frac{1}{6} \int_{\mathbb{R}^3} |u|^6.$$ 

Then we have $J'_\lambda(v) = 0$ and $\{v_n\}$ is a PS sequence of $J_\lambda(u)$. The Pohozaev identity associated with (4.3) is

$$P_\lambda(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{3\lambda}{2} \int_{\mathbb{R}^3} |u|^2 + \frac{bA^2}{2} \int_{\mathbb{R}^3} |\nabla u|^2 - \frac{3}{p} \int_{\mathbb{R}^3} |u|^p - \frac{1}{2} \int_{\mathbb{R}^3} |u|^6.$$ 

Hence, we have

$$J_\lambda(v) = J_\lambda(v) - \frac{1}{3} P_\lambda(v) = \frac{a + bA^2}{3} \int_{\mathbb{R}^3} |\nabla v|^2$$

$$= \frac{a + bA^2}{4} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{1}{4} (J_\lambda(v) - P_\lambda(v)). \quad (4.6)$$

Let $w_n = v_n - v$. Then we have $w_n \rightharpoonup 0$ in $H$ and $w_n \to 0$ in $L^q(\mathbb{R}^3)$ for $2 < q < 6$. Using Brezis-Lieb lemma,

$$\begin{cases}
\|w_n\|^2 = \|v_n\|^2 - \|v\|^2 + o_n(1); \\
\|w_n\|^6_6 = \|v_n\|^6 - \|v\|^6_6 + o_n(1); \\
A^2 = \|\nabla v_n\|^2 + o_n(1) = \|\nabla w_n\|_2^2 + \|\nabla v\|_2^2 + o_n(1). 
\end{cases} \quad (4.7)$$

Since $\{v_n\}$ is a PS sequence of $J_\lambda$, it follows (4.7) that

$$o_n(1) = \langle J'_\lambda(v_n), v_n \rangle$$

$$= \langle J'_\lambda(v), v \rangle + (a + bA^2) \int_{\mathbb{R}^3} |\nabla w_n|^2 - \lambda \int_{\mathbb{R}^3} |w_n|^2 - \int_{\mathbb{R}^3} |w_n|^6 + o_n(1)$$

$$= \|w_n\|^2 + b\|\nabla w_n\|^2 + b \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 - \|w_n\|^6_6 + o_n(1),$$

which yields

$$\|w_n\|^2 + b\|\nabla w_n\|^2 + b \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 - \|w_n\|^6_6 = o_n(1). \quad (4.8)$$

Up to a subsequence, we assume that

$$\|w_n\|^2 \to l_1 \geq 0, \quad b\|\nabla w_n\|^2 + b \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 \to l_2 \geq 0, \quad \|w_n\|^6_6 \to l_3 \geq 0.$$ 

Then we have $l_1 + l_2 = l_3$. If $l_1 = 0$, we obtain $v_n \to v$ in $H$, which completes the proof. If $l_1 > 0$, by Sobolev inequality $\|v\| \leq S$, we have

$$l_1 \geq aS(l_1 + l_2)\frac{1}{2} \quad \text{and} \quad l_2 \geq bS^5(l_1 + l_2)^{\frac{5}{2}}. \quad (4.9)$$

By Lemma 2.4 we have

$$l_1 \geq \frac{abS^3 + \sqrt[3]{a^2b^2S^6 + 4aS^4}}{2} \quad \text{and} \quad l_2 \geq \frac{abS^3 + \sqrt[3]{b^3S^6 + b^2S^3\sqrt{b^2S^6 + 4aS^4}}}{2}. \quad (4.9)$$
From (4.7) and (4.8), we deduce that
\[
J_\lambda(v_n) = J_\lambda(v) + \frac{(a + bA^2)}{2} \int_{\mathbb{R}^3} |\nabla w_n|^2 - \frac{\lambda}{2} \int_{\mathbb{R}^3} |w_n|^2 - \frac{1}{6} \int_{\mathbb{R}^3} |w_n|^6 + o_n(1)
\]
\[
= J_\lambda(v) + \frac{1}{2} \|w_n\|^2 + \frac{b}{4} \left( \|\nabla w_n\|^4 + \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 \right)
\]
\[-\|w_n\|^6 + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w_n|^2 + o_n(1)
\]
\[
= J_\lambda(v) + \frac{1}{3} \|w_n\|^2 + \frac{b}{12} \left( \|\nabla w_n\|^4 + \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 \right)
\]
\[+ \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w_n|^2 + o_n(1).
\] (4.10)

On the other hand, since \(E(v_n) \to \gamma_c\), we have
\[
J_\lambda(v_n) = \gamma_c - \frac{\lambda}{2} \|v_n\|^2 + \frac{b}{4} A^2 + o_n(1).
\] (4.11)

In view of (4.6), (4.9), (4.10) and (4.11), we infer that
\[
\gamma_c - \frac{\lambda}{2} \|v_n\|^2 + \frac{b}{4} A^2 = \frac{1}{3} \|w_n\|^2 + \frac{b}{12} \left( \|\nabla w_n\|^4 + \int_{\mathbb{R}^3} |\nabla w_n|^2 \int_{\mathbb{R}^3} |\nabla v|^2 \right)
\]
\[+ \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w_n|^2 + o_n(1)
\]
\[\geq \frac{abS^3}{4} + \frac{b^3S^6}{24} + \frac{(4as + b^2S^4)^3}{24} + J_\lambda(v) + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w_n|^2 + o_n(1)
\]
\[= \gamma_c^* + \frac{a + bA^2}{4} \int_{\mathbb{R}^3} |\nabla v|^2 + \frac{bA^2}{4} \int_{\mathbb{R}^3} |\nabla w_n|^2
\]
\[+ \frac{\lambda}{4} \|v\|^2 + \frac{1}{2p} \|v\|^p + \frac{1}{12} \|v_n\|^6 + o_n(1)
\]
\[\geq \gamma_c^* + \frac{b}{4} A^2 + \gamma_c^* + \frac{\lambda}{4} \|v\|^2 + \frac{1}{2p} \|v\|^p + o_n(1),
\]
which implies that
\[
\gamma_c \geq \gamma_c^* + \frac{\lambda}{4} \|v\|^2 + \frac{1}{2p} \|v\|^p + \frac{\lambda}{2} \|v_n\|^2 + o_n(1)
\]
\[
\geq \gamma_c^* + \frac{3\lambda}{4} \|v_n\|^2 + \frac{1}{2p} \|v\|^p + o_n(1)
\]
\[= \gamma_c^* + \frac{3\lambda}{4} c^2 + \frac{1}{2p} \|v\|^p
\]
\[= \gamma_c^* + \left( \frac{3}{4}(\beta_p - 1) + \frac{1}{2p} \right) \|v\|^p \geq \gamma_c^*,
\]
where the last inequality is from (4.3) and \(p\beta_p \geq 4\). This is a contradiction to \(\gamma_c < \gamma_c^*\). So we obtain \(v_n \to v\) in \(H\). And \(E(v) = \inf_{u \in V(c)} E(u)\) follows from lemma 3.7. \(\Box\)

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