Improved Approximation Algorithms for Capacitated Vehicle Routing with Fixed Capacity

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Abstract

The Capacitated Vehicle Routing Problem (CVRP) is one of the most extensively studied problems in combinatorial optimization. According to the property of the demand of customers, we distinguish three variants of CVRP: unit-demand, splittable and unsplittable. We consider $k$-CVRP in general metrics and general graphs, where $k$ is the capacity of the vehicle and all the three versions are APX-hard for each fixed $k \geq 3$.

The ITP algorithm based on a given Hamiltonian cycle is a classic method to solve $k$-CVRP. It gives an $(\alpha + 1 - \Theta(\alpha/k))$-approximate solution for splittable and unit-demand $k$-CVRP and an $(\alpha + 2 - \Theta(\alpha/k))$-approximate solution for unsplittable $k$-CVRP, where $\alpha \approx 3/2$ is the approximation ratio for metric TSP. Very recently, there are two significant progresses: Blauth et al. improved the ratio to $\alpha + 1 - \varepsilon$ for splittable and unit-demand $k$-CVRP and to $\alpha + 2 - 2\varepsilon$ for unsplittable $k$-CVRP, where $\varepsilon > 1/3000$ for $\alpha = 3/2$ and any $k \geq 3$; Friggstad et al. further improved the ratio to $\alpha + 1 + \ln 2 - \varepsilon'$ for the unsplittable case, where $\varepsilon'$ is around $1/3000$ for $\alpha = 3/2$ and any $k \geq 3$.

In this paper, we give a $(5/2 - \Theta(\sqrt{1/k}))$-approximation algorithm for splittable and unit-demand $k$-CVRP and a $(5/2 + \ln 2 - \Theta(\sqrt{1/k}))$-approximation algorithm for unsplittable $k$-CVRP (assume the approximation ratio for metric TSP is $\alpha = 3/2$). Thus, our approximation ratio is better than previous results for sufficient large $k$, say $k \leq 1.7 \times 10^7$.

For small $k$, we design independent algorithms by using more techniques to get further improvements. For splittable and unit-demand cases, we improve the ratio from 1.934 to 1.500 for $k = 3$, and from 1.750 to 1.667 for $k = 4$. For the unsplittable case, we improve the ratio from 2.693 to 1.500 for $k = 3$, from 2.443 to 1.750 for $k = 4$, and from 2.893 to 2.157 for $k = 5$.

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1 Introduction

In the Capacitated Vehicle Routing Problem (CVRP), we are given an undirected complete graph $G = (V \cup \{v_0\}, E)$ with an edge weight $w$ satisfying the symmetric and triangle inequality properties. The $n$ nodes in $V = \{v_1, \ldots, v_n\}$ represent $n$ customers and each customer $v_i$ has a demand $d_i \in \mathbb{Z}_{\geq 1}$. A vehicle with a capacity of $k \in \mathbb{Z}_{\geq 1}$ is initially located at the depot $v_0$. A tour is a walk that begins and ends at the depot and the sum of deliveries to all customers in it is at most $k$. The distance of a tour is the sum of the edge weight in the tour. In CVRP, we wish to find a set of tours to satisfy the demand of every customer with the minimum total distance of all the tours. We use $k$-CVRP to denote the problem where the capacity $k$ is a constant. In the unsplittable version of the problem, the demand of each customer can only be delivered by a single tour. In the splittable version, the demand of each customer can be delivered by more than one tour. If the demand of each customer is unit-demand, it is called the unit-demand version. We can reduce the splittable version to the unit-demand version by considering one customer $v_i$ with demand $d_i$ as $d_i$ identical customers with unit-demand and zero interdistance. Despite this reduction using pseudopolynomial time, we can do it implicitly only for some analysis [16]. So we may simply consider the unit-demand version for the splittable version.

Since CVRP was raised by Dantzig and Ramser [19] in 1959, it has become a very famous problem with numerous applications in combinatorial optimization. It has been widely studied in both theory and application. Readers can refer to a survey [47] for its applications and fast solvers in practice. In theory, it is a rather rich problem in approximation algorithms [16, 15, 42, 43, 18, 24].

When $k = 1$ or $k = 2$, $k$-CVRP can be solved in polynomial time [7]. For each fixed $k \geq 3$, the problem is APX-Hard even for the unit-demand case [6]. A classic algorithm based on a given Hamiltonian cycle, called Iterated Tour Partitioning (ITP), was proposed about 40 years ago [30]. ITP is not only fast in practice but also can achieve a good approximation ratio in theory. Given an $\alpha$-approximation algorithm for metric TSP. For splittable and unit-demand $k$-CVRP, ITP can achieve an approximation ratio of $\alpha + 1 - \alpha/k$ [30]. For unsplittable $k$-CVRP, a modification of ITP, called UITP, can achieve a ratio of $\alpha + 2 - 2\alpha/k$ for even $k$ [2]. For metric TSP, there is a well-known $3/2$-approximation algorithm [17, 46]. Currently, there was a breakthrough on metric TSP by Karlin, Klein and Gharan [35]: a randomized algorithm achieves the ratio $3/2 - \varepsilon$, where the improvement $\varepsilon$ is about $10^{-36}$. Although this is a nice breakthrough, in our algorithms for $k$-CVRP, we will still assume the ratio $\alpha = 3/2$ for metric TSP for ease of comparison since the improvement is too small to make a difference. ITP itself is simple and the $3/2$-approximation metric TSP algorithm is also simple. Due to the simplicity, the ITP algorithm is versatile and has been adapted to other related vehicle routing problems [41]. However, for general metrics, there have been few improvements over the ITP algorithm in approximating $k$-CVRP.

One interesting improvement was done by Bompadre et al. [16] about 20 years ago. For any $\alpha \geq 1$, the ratio was improved by a term of $\frac{1}{3k^3}$ for all the three versions. For $\alpha = 3/2$, the improvement could be $\frac{1}{3k^3}$ for splittable and unit-demand $k$-CVRP, and $\frac{1}{k}$ for unsplittable $k$-CVRP. Before this paper, the results in [16] were still the best approximation ratios for many small constants $k$. Very recently, one significant progress was done by Blauth et al. [15]. They improved the approximation ratio to $\alpha + 1 - \varepsilon$ for splittable and unit-demand $k$-CVRP, and to $\alpha + 2 - 2\varepsilon$ for unsplittable $k$-CVRP, where $\varepsilon$ is a value related to $\alpha$ and $\varepsilon > \frac{1}{30000}$ when $\alpha = 3/2$. This improves the constant part of the approximation ratio ($\varepsilon$ is only related to $\alpha$ and not related to $k$), but the improvement is small and the compensation of other parts is large. As a result, they cannot improve the ratio in [16] for small $k$. Also very recently, Friggstad et al. [25] proposed two further improvements for unsplittable $k$-CVRP: the first is an $(\alpha + 1.75)$-approximation algorithm based on a combinatorial method, and the second is an $(\alpha + \ln 2 + \frac{1}{1-\delta})$-approximation algorithm based
on an LP rounding method with a running time \( n^{O(1/\varepsilon)} \). Both of the ratios can be further improved by a small constant \( \varepsilon' \) by combining the method in [15].

In this paper, we will pay more attention to \( k \)-CVRP with a bounded \( k \). As mentioned in [16, 14], several problems that arise in practice have small \( k \). We also mention a question raised by a logistics company: they need to use a truck that can carry at most six cars to transport the newly produced cars. In this example, the capacity \( k \) is 6. In the benchmark set, the capacities of instances are also usually integers in hundreds or thousands of levels [48].

For small \( k \), there are some improved results. For unit-demand 3-CVRP, the ITP algorithm has an approximation ratio of 2. Based on the 25/33-approximation algorithm for MAX TSP on general graphs with symmetric weights [33], Bazgan et al. [8] proposed an 1.990-approximation algorithm, which is the first improvement to our knowledge. By using the best-known 4/5-approximation algorithm for MAX TSP [22], the ratio of their algorithm can be further improved to 1.934. For unit-demand 4-CVRP, the ITP algorithm has an approximation ratio of 2.125. Anily and Bramel [4] showed an approximation algorithm for capacitated TSP with pickups and delivers. Their algorithm can be applied to unit-demand \( k \)-CVRP and achieve a better ratio only for the case of \( k = 4 \). The ratio is 1.750, which is also the best-known result.

Although there are little improvements on the general metrics, a huge number of contributions have been made for special cases. We list some of them.

Consider unit-demand \( k \)-CVRP. In \( \mathbb{R}^2 \) a PTAS is known for \( k = O(\log \log n) \) [30], \( k = O\left(\frac{\log n}{\log \log n}\right) \) or \( k = \Omega(n) \) [7], and \( k \leq 2^{\log f(\varepsilon)} n \) [1]. In \( \mathbb{R}^l \) where \( l \) is fixed a PTAS is known for \( k = O(\log^{1/2} n) \) [36]. Das and Mathieu proposed a QPTAS in \( \mathbb{R}^2 \) for arbitrary \( k \) with a running time \( n^{\log O(1/\varepsilon)} \) [20]. It has been improved to \( n^{O(\log^{3/2} n/\varepsilon^3)} \) by Jayaprakash and Salavatipour [34], where they also showed a QPTAS for graphs of bounded treewidth, bounded doubling metrics, or bounded highway dimension with arbitrary \( k \).

Consider unit-demand \( k \)-CVRP with \( k = O(1) \). Becker et al. showed a QPTAS for planar and bounded genus graphs in [10], a PTAS in bounded highway dimension and an \( O(n^{tw(Q)}) \) time exact algorithm for graphs with treewidth \( tw \) in [11]. A PTAS in planar graphs was given in [12]. Cohen-Addad et al. showed an EPTAS for graphs of bounded-treewidth, an EPTAS for bounded highway dimension, an EPTAS for bounded genus metrics, and the first QPTAS for minor-free metrics in [18]. Filtser and Le proposed a better EPTAS for planar graphs which runs in almost-linear time, and an EPTAS for minor-free metrics in [24].

For the case of trees, splittable CVRP was shown to be NP-hard [38], and unsplittable CVRP was shown to be NP-hard to approximate better than \( 3/2 \) [28]. For splittable CVRP on trees, Hamaguchi and Katoh [31] designed the first approximation algorithm which has a ratio of \( 3/2 \). The ratio was improved to \( (\sqrt{14} - 4)/4 \) by Asano et al. [5] and further to \( 4/3 \) by Becker [9]. Then, Becker and Paul [13] proposed an \((1, 1 + \varepsilon)\)-bicriteria polynomial time scheme where the capacity of each tour is allowed to be violated by an \( \varepsilon \) fraction. Finally, a PTAS was given by Mathieu and Zhou [42]. For unsplittable CVRP on trees, Labbé et al. [38] proposed the first 2-approximation algorithm. Recently, it has been improved to \((3/2 + \varepsilon)\) which is tight [43].

Very recently, Grandoni et al. gave a \((2 + \varepsilon)\)-approximation algorithm for unsplittable CVRP in \( \mathbb{R}^2 \) [29]. Dufay et al. showed an \((1.692 + \varepsilon)\)-approximation algorithm for distance-constrained CVRP on trees [23]. Mömke and Zhou showed an 1.94-approximation algorithm for graphic CVRP [44].

### 1.1 Our Contributions

We present several approximation algorithms for \( k \)-CVRP and improve the previous ratios for any \( k \leq 1.7 \times 10^7 \). This value is already large. The capacity of all practical and artificial instances as
far as we know is not larger than this value. We summarize our contributions to splittable (also unit-demand) \(k\)-CVRP and unsplittable \(k\)-CVRP separately below.

For splittable and unit-demand \(k\)-CVRP, we have the following contributions.

1. The classic ITP algorithm is based on a given Hamiltonian cycle of the graph. We extend ITP to an algorithm based on any cycle packing, called EX-ITP. One advantage is that an optimal cycle packing is polynomially computable. Based on EX-ITP, we first quickly improve the ratio from 1.934 to 1.500 for 3-CVRP and from 1.750 to 1.667 for 4-CVRP (Section 5). Then we consider general \(k\). We show that for metric TSP ratio \(\alpha \geq 1\), by making a trade-off between ITP and EX-ITP we can achieve an approximation ratio of \(\alpha + 1 - \alpha/k - \Theta(1/k)\), improving the previous ratio of \(\alpha + 1 - \alpha/k - \max\{\Omega(1/k^2), \varepsilon\}\) \([15, 16]\) for small \(k\) (Section 6). Our precise ratio is presented in Theorem 30, and some numerical values for concrete \(k\) under \(\alpha = 3/2\) are shown in Table 1 (the lines for Sec. 6). Note that if the metric TSP ratio \(\alpha\) is improved, we will get better results.

2. Note that the best approximation ratio for metric TSP is still about 3/2. Under \(\alpha = 3/2\), the previous best approximation ratio of \(k\)-CVRP is about 5/2\(-1.5\)-\(\alpha/k\)\(-\max\{\Omega(1/k^2), 1.005/3000\}\) \([15, 16]\). By using some deep analysis on a simple known algorithm, we get an approximation ratio of 5/2\(-\Theta(\sqrt{1/k})\), which is smaller than previous results for \(k \leq 1.7 \times 10^7\) (Section 7). Our precise ratio is presented in Theorem 40, and some numerical values for concrete \(k\) are shown in Table 1 (the lines for Sec. 7).

| \(k\) | 3    | 4    | 5    | 6    | 7    | 8    | 9    | 10   |
|------|------|------|------|------|------|------|------|------|
| Previous | 1.934 | 1.750 | 2.188 | 2.242 | 2.280 | 2.308 | 2.330 | 2.348 |
| Results | [8]  | [4]  | [16] | [16] | [16] | [16] | [16] | [16] |
| Our Results in Sec. 5 | 1.500 | 1.667 | -    | -    | -    | -    | -    | -    |
| Our Results in Sec. 6 (for \(\alpha = 3/2\)) | 1.556 | 1.709 | 1.800 | 1.917 | 2.000 | 2.063 | 2.112 | 2.150 |
| Our Results in Sec. 7 | 1.667 | 1.750 | 1.800 | 1.875 | 1.929 | 1.969 | 2.000 | 2.025 |

| \(k\) | 29    | ...  | 5833 | 5834 | ...  | 1.7 \times 10^7 | 1.8 \times 10^7 |
|------|-------|------|------|------|------|-----------------|-----------------|
| Previous | 2.44795 | ...  | 2.49941 | 2.49941 | ...  | 2.49967         | 2.49967         |
| Results | [15]  | ...  | [15] | [15] | ...  | [15]            | [15]            |
| Our Results in Sec. 6 (for \(\alpha = 3/2\)) | 2.37932 | ...  | 2.49940 | 2.49941 | ...  | 2.50000         | 2.50000         |
| Our Results in Sec. 7 | 2.22414 | ...  | 2.48140 | 2.48141 | ...  | 2.49966         | 2.49967         |

**Table 1:** Splittable and unit-demand \(k\)-CVRP: previous and our approximation ratios for different values of \(k\), where the best ratios are marked in bold.

For unsplittable \(k\)-CVRP, we get similar results. However, we may use different techniques.

1. We refine and extend the UITP algorithm for unsplittable \(k\)-CVRP (Section 8), and analyze several properties for some special cycles in a cycle packing (Section 9). Based on these, we are able to show that the approximation ratio of unsplittable 3-CVRP and 4-CVRP can be improved to 1.500 and 1.750, respectively (Section 10).
2. By combining the extended UITP with the LP-based technique used in [25], we get the LP-UITP algorithm. By running LP-UITP on two initial Hamiltonian cycles and doing a trade-off between them, we obtain an approximation ratio $\alpha + 1 + \ln 2 - 2\alpha/k - \Theta(1/k)$, improving the previous ratio of $\alpha + 1 + \ln 2 - 2\alpha/k - \varepsilon'$ [25] for small $k$ (Section 11). Under $\alpha = 3/2$, the precise ratio is presented in Theorem 50, and some numerical values for concrete $k$ are shown in Table 2 (the lines for Sec. 11). Note that for better metric TSP ratio $\alpha$, we will get better results.

3. For unsplittable $k$-CVRP, we further get a $(5/2 + \ln 2 - \Theta(\sqrt{1/k}))$-approximation algorithm, improving the previous ratio of $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k$ in [25] for $k \leq 1.7 \times 10^7$ (Section 12). Our precise ratio is presented in Theorem 51, and some numerical values for concrete $k$ are shown in Table 2 (the lines for Sec. 12). For unsplittable 5-CVRP, we also further improve the ratio to 2.157 by refining the analysis (Section 13).

| $k$ | Previous Results | Our Results in Sec. 10 & 13 | Our Results in Sec. 11 (for $\alpha = 3/2$) | Our Results in Sec. 12 |
|-----|------------------|-----------------------------|---------------------------------------------|-----------------------|
|     | 2.693            | 1.500                       | 1.906                                       | 1.906                 |
| 3   | 2.433            | 1.750                       | 1.955                                       | 1.955                 |
| 4   | 2.893            | 2.157                       | 2.178                                       | 2.178                 |
| 5   | 2.693            | -                           | 2.163                                       | 2.163                 |
| 6   | 2.979            | -                           | 2.351                                       | 2.343                 |
| 7   | 2.818            | -                           | 2.383                                       | 2.337                 |
| 8   | 3.027            | -                           | 2.537                                       | 2.471                 |
| 9   | 2.893            | -                           | 2.538                                       | 2.448                 |
| 10  |                  |                             |                                             |                       |

| $k$ | Previous Results | Our Results in Sec. 11 (for $\alpha = 3/2$) | Our Results in Sec. 12 |
|-----|------------------|---------------------------------------------|-----------------------|
|     | 3.19256          | 3.19256                                    | 3.17973               |
| 3   | 3.19269          | 3.19256                                    | 3.17981               |
| 4   | 3.19256          | 3.19256                                    | 3.17973               |
| 5   |                 | 3.19282                                    | 3.19282               |
| 6   |                 |                                            |                       |
| 7   |                 |                                            |                       |
| 8   |                 |                                            |                       |
| 9   |                 |                                            |                       |
| 10  |                 |                                            |                       |

Table 2: Unsplittable $k$-CVRP: previous and our approximation ratios for different values of $k$, where the best ratios are marked in bold.

In the above two tables, the previous results we used are $\alpha + 1 - \alpha/k - \max\{\Omega(1/k^3), \varepsilon\}$ and $\alpha + 1 + \ln 2 - 2\alpha/k - \varepsilon'$ for splittable and unsplittable $k$-CVRP, respectively. The ratio $\alpha + 1 - \alpha/k - \Omega(1/k^3)$ can be directly computed using the result in [16].$^1$ The values $\varepsilon$ and $\varepsilon'$ depending on $\alpha$ and $k$ can be calculated using the results in [15, 25]. We adopt $\alpha = 3/2$. For this case, the values $\varepsilon$ and $\varepsilon'$ are smaller than 1.005/3000 and $-\ln(1 - 1.005/3000)$, respectively. Hence, the approximation ratio of splittable and unit-demand $k$-CVRP in [15] is at least $5/2 - 1.005/3000 - 1.5/k$ for $k \geq 3$. $^2$ The approximation ratio of unsplittable $k$-CVRP in [25] is at least $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k$ for even $k \geq 3$. $^3$ Note that for unsplittable $k$-CVRP with odd $k \geq 3$, we need to double the capacity and the demand, and get a slightly worse ratio $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 1.5/k$.

$^1$See the calculation in Appendix A.
$^2$See the calculation in Appendix B.
$^3$See the calculation in Appendix C.
2 Definitions, Assumptions and Notations

A walk in a graph is a succession of edges in the graph, where the same edge can appear more than one time. We will use a sequence of vertices to denote a walk: \((v_1, v_2, v_3, \ldots, v_l)\) means a walk with edges \((v_1, v_2), (v_2, v_3), \text{ and so on.} A path in a graph is a walk such that no vertex appears twice in the sequence, and a cycle is a walk such that only the first and the last vertices are the same. A cycle containing \(l\) edges is called an \(l\)-cycle and the length of it is \(l\). Two subgraphs (or two sets of edges) are vertex-disjoint if they do not have a common vertex. Given an edge-weighted graph, where the number \(n\) of vertices is a multiple of \(k\), a minimum \(k\)-cycle packing is a set of exactly \(n/k\) vertex-disjoint \(k\)-cycles with the minimum total weight of edges in the \(k\)-cycles in the set. A minimum mod-\(k\)-cycle packing is a set of vertex-disjoint cycles such that the length of each cycle is divisible by \(k\), each vertex of the graph appears in exactly one cycle, and the total weight of edges in the cycles in the set is minimized. A minimum cycle packing is a set of vertex-disjoint cycles such that the length of each cycle is at least three, each vertex of the graph appears in exactly one cycle, and the total weight of edges in the cycles in the set is minimized.

2.1 Problem Definitions

We use \(G = (V \cup \{v_0\}, E)\) to denote a complete graph, where the vertex \(v_0\) represents the depot and vertices in \(V\) represent customers. There is a non-negative weight function \(w : E \rightarrow \mathbb{R}_{\geq 0}\) on the edges in \(E\), which denotes the distance between two endpoints of the edge. The weight function \(w\) is a semi-metric function, i.e., it satisfies the symmetric and triangle inequality properties. For any weight function \(w : X \rightarrow \mathbb{R}_{\geq 0}\), we will extend it to subsets of \(X\), i.e., we define \(w(Y) = \sum_{x \in Y} w(x)\) for \(Y \subseteq X\). An itinerary is a walk starting and ending at vertex \(v_0\). An itinerary can be split into several minimal cycles containing \(v_0\), and each such cycle is called a tour. The Capacitated Vehicle Routing Problem (CVRP) can be described as follows.

Definition 1. An instance \((G = (V \cup \{v_0\}, E), w, d, k)\) of CVRP consists of:

- a complete graph \(G\), where \(V = \{v_1, \ldots, v_n\}\) represents the \(n\) customers and \(v_0\) represents the depot,
- a semi-metric weight function on edges \(w: (V \cup \{v_0\}) \times (V \cup \{v_0\}) \rightarrow \mathbb{R}_{\geq 0}\), which represent the distances,
- the demand of each customer \(d = (d_1, \ldots, d_n)\), where \(d_i \in \mathbb{Z}_{\geq 1}\) is the demand required by customer \(v_i \in V\),
- the capacity \(k \in \mathbb{Z}_{\geq 1}\) of the vehicle that initially stays at \(v_0\).

A feasible solution is an itinerary such that

- each tour delivers at most \(k\) of the demand to customers on the tour,
- the union of tours meets the demand of every customer.

The goal is to find an itinerary \(I\), minimizing the total distances of the succession of edges in the walk, i.e. \(w(I) := \sum_{e \in I} w(e)\).

According to the property of the demand, we can define three different versions of the problem. If the demand of each customer should be delivered in one tour, we call it unsplittable CVRP. If the demand of a customer can be split into several tours, we call it splittable CVRP. If the demand of each customer is a unit, we call it unit-demand CVRP. In our problems, we will also make some assumptions.
2.2 Some Assumptions

**Assumption 2.** Assume that the capacity \( k \) is a constant satisfying \( k \geq 3 \), and in an optimal itinerary, each tour is a simple cycle.

If there exists a tour that is not a simple cycle, we can get a simple cycle by shortcutting.

**Assumption 3.** If the vehicle just goes to place \( A \) and passes place \( B \) without delivering anything to place \( B \), we do not think we have passed place \( B \). So in each tour, the vehicle only visits customers where it delivers.

A tour is called *trivial* if it only visits one customer and *non-trivial* otherwise. We show some properties of optimal itineraries to splittable CVRP.

**Lemma 4 ([21]).** For splittable CVRP, there exists an optimal itinerary where no two tours visit two common customers.

To our knowledge, this property was first used in [21]. We explore some properties.

**Lemma 5.** For splittable CVRP, there exists an optimal itinerary that contains at most \( n - 1 \) non-trivial tours.

**Proof.** Given an optimal itinerary \( I \), we construct an auxiliary graph \( G_I \) on \( V \), where two vertices in \( V \) are adjacent in \( G_I \) if and only if there is at least one tour in \( I \) that visits both of them. By Lemma 4, we assume that no two tours in \( I \) visit two common customers. We further assume that \( I \) is an itinerary to make \( G_I \) have the minimum number of edges. Next, we assume to the contrary that there are more than \( n - 1 \) non-trivial tours in \( I \) and show a contradiction.

For two vertices visited in the same tour \( C_i \), there is an edge between them and we color the edge with \( i \). By the assumption that no two tours in \( I \) visit two common customers, we know that each edge is colored with exactly one color and the set of edges with the same color form a clique. Since \( I \) has more than \( n - 1 \) non-trivial tours, each non-trivial tour will create a clique of size \( \geq 2 \) in \( G_I \) with the same color for the edges in the clique, we know that the graph \( G_I \) contains at least one cycle, denoted by Cycle = \((v_1, v_2, \ldots, v_l, v_1)\), such that each edge has a different color.

Based on the cycle Cycle = \((v_1, v_2, \ldots, v_l, v_1)\), we can assume that tour \( C_i \) visits \( v_i \) and \( v_{i+1} \) for each \( i \in \{1, \ldots, l\} \), and tour \( C_i \) delivers \( x_{i,i} \) and \( x_{i,i+1} \) to customer \( v_i \) and \( v_{i+1} \), respectively, where we let \( v_{i+1} = v_1 \) and \( x_{i,l+1} = x_{l,1} \). Without loss of generality, we assume that \( x_{1,1} = \min_i \{x_{i,i}\} \). we can get another itinerary \( I' \) by making some exchanges on the deliveries: tour \( C_i \) delivers \( (x_{i,i} - x_{1,1}) \) to customer \( v_i \) and delivers \( x_{i,i+1} + x_{l,1} \) to customer \( v_{i+1} \), \( i \in \{1, \ldots, l\} \). In \( I' \), the demand of each customer is still satisfied, and the total delivery in each tour does not change. So, \( I' \) is still an optimal solution. However, tour \( C_i \) in \( I' \) does not need to visit \( v_1 \), and then \( G_{I'} \) will have a less number of edges than \( G_I \), a contradiction.

In an itinerary, if the vehicle always delivers an integer number of demand to each customer in each tour, then we say that the itinerary satisfies the integer property. It is not hard to prove that there is an optimal itinerary satisfying the integer property by using the exchanging argument similar to that in the proof in Lemma 5. So we also assume that

**Assumption 6.** In our problems, we assume in each tour the vehicle delivers an integer amount of the demand to each customer.

**Lemma 7.** For splittable \( k \)-CVRP, if the demand \( d_i \) of a customer \( v_i \) is at least \((n - 1)(k - 1) + 1\), then there is an optimal itinerary that contains a trivial tour visiting \( v_i \).
Proof. By Lemma 5, we assume that there are at most \(n - 1\) non-trivial tours in the itinerary, each of which can deliver at most \(k - 1\) demand to the customer \(v_i\). If the demand \(d_i > (n - 1)(k - 1)\), we know that there is at least one trivial tour visiting \(v_i\).

By Lemma 7, we can iteratively design a trivial tour visiting \(v_i\) until the demand of it is at most \((n - 1)(k - 1)\). So we have the following assumption.

**Assumption 8.** For splittable k-CVRP, we assume that the demand \(d_i\) of each customer \(v_i \in V\) is at most \((n - 1)(k - 1)\).

**Lemma 9.** If \(k = n^{O(1)}\), then there is a polynomial-time reduction from splittable CVRP to unit-demand CVRP.

For a customer with \(d\) of demand in splittable CVRP, we consider it as \(d\) customers with unit-demand as the same size. Thus, we get an instance of unit-demand CVRP. By Assumption 8, we can assume that the demand of each customer is at most \((n - 1)(k - 1)\). However, the reduction is polynomial only when \(k = n^{O(1)}\). We note that this property was also observed in [34]. In this paper, we will always consider \(k\) as a constant. Thus, splittable CVRP can be polynomially reduced to unit-demand CVRP. We will design approximation algorithms for unit-demand CVRP, which also work for splittable CVRP.

**Assumption 10.** For unsplittable k-CVRP, we assume that the demand \(d_i\) of each customer \(v_i \in V\) is less than \(k\).

In an optimal itinerary, if the vehicle always delivers \(k\) of demand in each tour, then we say that the itinerary satisfies the **saturated property**.

**Assumption 11.** For k-CVRP, we assume that there exists an optimal saturated itinerary where each tour delivers exactly \(k\) of the demand.

First, it is easy to see that after adding some customers with unit-demand at the same position of \(v_0\), the new instances are equivalent to the old one. Moreover, if there exists an optimal saturated itinerary, then after adding \(n'\) with \(n'\mod k = 0\) unit-demand customers at the same position of \(v_0\), there also exists an optimal saturated itinerary in the new instance since we can use \(n'/k\) tours to meet the demands of the new \(n'\) unit-demand customers.

Consider unit-demand CVRP. Suppose there exists an optimal itinerary where there are \(m_i\) tours delivering \(i\) of the demand for each \(i \in \{1, 2, \ldots, k\}\). If we add \(\sum_{i=1}^{k}(k-i)m_i\) unit-demand customers at the same position of \(v_0\), it is easy to see that there exists an optimal saturated itinerary in the new instance. Moreover, we have \(\sum_{i=1}^{k}(k-i)m_i + n = \sum_{i=1}^{k} m_i k\) which is divisible by \(k\). But, we do not know the precise values of \(m_i\). We will further add \(n'\) unit-demand customers at the same position of \(v_0\) such that the total number of added customers is \(\sum_{i=1}^{k}(k-i)m_i + n' = k^2n + k - n \mod k\). It is easy to see that \(n' \mod k = 0\). Since \(m_i \leq \lceil n/i \rceil \leq n\), we know that \(\sum_{i=1}^{k}(k-i)m_i \leq \sum_{i=1}^{k-1} kn \leq k^2n\) and then \(n' \geq k - n \mod k \geq 0\). Therefore, if we add \(k^2n + k - n\mod k\) unit-demand customers at the same position of \(v_0\), there will exist an optimal saturated itinerary in the new instance.

For unsplittable CVRP, if we add \(\sum_{i=1}^{n} k^2d_i + k - (\sum_{i=1}^{n} d_i)\mod k\) unit-demand customers at the same position of \(v_0\), by a similar argument, we know that there will exist an optimal saturated itinerary in the new instance.

We need this assumption to simplify some arguments and make some presentations neat.
2.3 Some Important Notations

The following notations are illustrated with the unit-demand case. Most of them will be used to establish some lower bounds for our problems. To get some lower bounds, we will compare an optimal solution $I^*$ with $\Delta, H^*, H_{CS}, M^*, \text{MST, } C^*, C^*_{\text{mod } k}, \text{ and } C^*$, which are defined below.

$I^*$: an optimal solution to our problem;

$\Delta$: the sum of the customer's demand times the weight of the edge from the depot $v_0$ to the customer, i.e. $\sum_{v_i \in V} d_i w(v_0, v_i)$;

$H^*$: a minimum Hamiltonian cycle on $V \cup \{v_0\}$;

$H_{CS}$: the Hamiltonian cycle on $V \cup \{v_0\}$ obtained by the Christofides-Serdyukov algorithm [17, 46];

$M^*$: a minimum perfect matching of $G[V]$;

MST: the total weight of the edges in a minimum spanning tree of $G$;

$C^*$: a minimum cycle packing of $G[V]$;

$C^*_k$: a minimum $k$-cycle packing of $G[V]$;

$C^*_{\text{mod } k}$: a minimum mod-$k$-cycle packing of $G[V]$.

We also mention the follows. A minimum perfect matching can be found in $O(n^3)$ time [26, 39], and a minimum spanning tree can be computed in $O(n^2)$ time [45]. It is NP-hard to compute a minimum $k$-cycle packing for any $k \geq 3$ [37]. There is an $O(n^2 \log n)$-time 2-approximation algorithm for the minimum mod-$k$-cycle packing problem [27]. On the other hand, a minimum cycle packing of a graph can be computed in $O(n^3)$ time [32]. These results will be used in our algorithms.

For an optimal itinerary $I^*$, the edges incident to $v_0$ in $I^*$ are called home-edges. The set of home-edges of $I^*$ is denoted by $h(I^*)$. We will also use $\chi$ to denote the proportion of weights of home-edges in $I^*$, i.e.

$$\chi = \frac{w(h(I^*))}{w(I^*)}.$$ 

Note that we simply assume $w(I^*) \neq 0$ to exclude this trivial case.

3 Lower Bounds on CVRP

In this section, we study lower bounds. First of all, we mention that all the lower bounds in this section hold for all the three versions of CVRP. Unit-demand CVRP can be considered as a relaxed version of unsplittable CVRP and lower bounds on unit-demand CVRP will also hold on unsplittable CVRP. An instance of splittable CVRP can also be transferred into an equivalent unit-demand CVRP instance by replacing each customer with demand $d_i$ as $d_i$ customers with the unit-demand. The optimal solution will not change. So lower bounds for unit-demand CVRP will also hold for splittable CVRP. Next, we assume that the problem considered is unit-demand CVRP and prove some lower bounds.

We will use the concept of $\chi$ to obtain some refined lower bounds, which were not considered in previous lower bounds. The first lower bound related to the minimum Hamiltonian cycle $H^*$ on $V \cup \{v_0\}$ was used in most previous papers.
Lemma 12 ([30, 2]). $w(I^*) \geq w(H^*)$.

Proof. It is easy to see that each vertex in $I^*$ has an even degree. Since $I^*$ is a connected graph, we can obtain an Euler tour and then get a Hamiltonian cycle on $V \cup \{v_0\}$ by shortcutting, which has at least the weight of the minimum Hamiltonian cycle. Thus, the lemma will hold. \hfill \square

Next, we consider the relation to $\Delta$. For unit-demand CVRP, $\Delta$ is exactly the sum weight of all edges incident on $v_0$. It is easy to get $\frac{1}{2}w(I^*) \geq \Delta$ by the triangle inequality. However, we still need to use the middle and tighter relations in the following lemma.

Lemma 13. $\frac{k}{2}w(I^*) \geq (\chi + \frac{k-2}{2})w(I^*) \geq \Delta$.

Proof. Since $0 \leq \chi \leq 1$, we have that $\frac{k}{2} = (1 + \frac{k-2}{2}) \geq (\chi + \frac{k-2}{2})$. Thus, the first inequality holds.

Now we show the second. By Assumption 11, $I^*$ consists of a set of $(k+1)$-cycles. We consider an arbitrary $(k+1)$-cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ in $I^*$. Since $k \geq 3$, then for each $i \in \{2, 3, \ldots, k-1\}$, it holds that $w(C) \geq 2w(v_0, v_i)$ by shortcutting. Thus, we have that $\sum_{i \in \{2, \ldots, k-1\}} w(v_0, v_i) \leq w(h(C)) + \sum_{i \in \{2, \ldots, k-1\}} w(v_0, v_i) \leq w(h(C)) + \frac{k-2}{2}w(C)$. By summing up the above inequality for all cycles in $I^*$, we get that $\sum_{v_i \in V} w(v_0, v_i) \leq \sum_{C \in I^*} w(h(C)) + \frac{k-2}{2}\sum_{C \in I^*} w(C)$, i.e. $\Delta = \sum_{v_i \in V} w(v_0, v_i) \leq \chi w(I^*) + \frac{k-2}{2}w(I^*) = (\chi + \frac{k-2}{2})w(I^*)$. \hfill \square

Lemma 14. $(1 - \frac{1}{2}\chi)w(I^*) \geq \text{MST}$.

Proof. First, it holds that $w(h(I^*)) = \chi w(I^*)$ according to the definition. For each tour in $I^*$, there are exactly two home-edges. We can get a spanning tree of the graph from $I^*$ by deleting the longer home-edge in each tour in $I^*$. The weight of this spanning tree is at least $w(I^*) - \frac{1}{2}\chi w(I^*) = (1 - \frac{1}{2}\chi)w(I^*)$, which is at least the weight of the minimum spanning tree. So, the lemma holds. \hfill \square

Recall that $H_{CS}$ is the Hamiltonian cycle on $V \cup \{v_0\}$ obtained by the Christofides-Serdyukov algorithm. The following bound will be used in our algorithm.

Lemma 15 ([17, 46]). $\text{MST} + \frac{1}{2}w(H^*) \geq w(H_{CS})$.

By Lemmas 12, 14, and 15, it is easy to get the following lemma.

Lemma 16. $\frac{3}{2}w(I^*) \geq w(h_{CS})$.

Lemma 17. $\min\{2(1 - \chi), 1\}w(I^*) \geq w(C^*_k) \geq w(C^*_{\text{mod } k}) \geq w(C^*)$.

Proof. We first show that there exists a $k$-cycle packing of $G[V]$ whose weight is bounded by $\min\{2(1 - \chi), 1\}w(I^*)$. Recall that $I^*$ consists of a set of $(k+1)$-cycles by Assumption 11. We consider an arbitrary $(k+1)$-cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ in $I^*$. Let $C' = (v_1, \ldots, v_k, v_0)$ be the $k$-cycle obtained by shortcutting $v_0$ from $C$. By the triangle inequality, we have $w(v_1, v_k) \leq \min\{w(v_0, v_1) + w(v_0, v_k), w(C) - w(v_0, v_1) - w(v_0, v_k)\}$. So, we have $w(C') = w(C) - w(v_0, v_1) - w(v_0, v_k) + w(v_1, v_k) \leq \min\{w(C), 2w(C') - 2w(v_0, v_1) - 2w(v_0, v_k)\}$. Note that the edges $(v_0, v_1)$ and $(v_0, v_k)$ are home-edges of the cycle $C$. By summing up the above inequality for all tours in $I^*$, we can get the claim. The corresponding $k$-cycle packing $C_k$ of $G[V]$ is obtained from $I^*$ by shortcutting $v_0$ from each tour in $I^*$. Thus, we have that $w(C^*_k) \leq w(C_k) \leq \min\{2(1 - \chi), 1\}w(I^*)$.

Since any $k$-cycle packing is a mod-$k$-cycle packing and any mod-$k$-cycle packing is also a cycle packing, we get that $w(C^*) \leq w(C^*_{\text{mod } k}) \leq w(C^*_k)$.
4 The ITP algorithms

ITP (Iterated Tour Partitioning) is a frequently used technique for (unit-demand) CVRP. The main idea of ITP is to construct feasible solutions for CVRP based on given Hamiltonian cycles: first split the Hamiltonian cycle into several connected pieces of length at most \( k \) and then construct a tour for each piece. The algorithm will consider several different ways to split the Hamiltonian cycle and choose the best one.

There are different versions of the ITP algorithm for CVRP according to the Hamiltonian cycle containing the depot \( v_0 \) or not. We briefly introduce them below and then introduce an extension. Now we assume the problem is unit-demand \( k \)-CVRP.

4.1 The AG-ITP Algorithm

We first review the ITP algorithm introduced by Altinkemer and Gavish [3], where the Hamiltonian cycle needs to go through the depot \( v_0 \). Assume that there is a Hamiltonian cycle \( H = (v_0, v_1, \ldots, v_n, v_0) \) on \( V \cup \{v_0\} \) as a part of the input. The AG-ITP algorithm will select the best solution from the following \( k \) solutions. For each \( 1 \leq i \leq k \), the \( i \)-th solution contains the following \( \left( \left[ \frac{n-1}{k} \right] + 1 \right) \) tours: \((v_0, v_1, \ldots, v_i, v_0), (v_0, v_i+1, \ldots, v_{i+k}, v_0), (v_0, v_{i+k+1}, \ldots, v_{i+2k}, v_0), \ldots, (v_0, v_{i+\left(\left[\frac{n-1}{k}\right]-1\right)k+1}, \ldots, v_n, v_0)\). Except for the first and last tour, each tour contains \( k \) customers. It is easy to see that the algorithm can be carried out in \( O(nk) \) time.

**Lemma 18** ([3]). Given a Hamiltonian cycle \( H \) on \( V \cup \{v_0\} \) as a part of the input, for unit-demand \( k \)-CVRP with any \( k \geq 3 \), the AG-ITP algorithm can use \( O(nk) \) time to output a solution with a weight of at most \( (2/k)\Delta + (1 - 1/k)w(H) \).

The running time \( O(nk) \) is linear when \( k \) is a constant. Using an \( \alpha \)-approximate Hamiltonian cycle on \( V \cup \{v_0\} \), by Lemmas 12, 13 and 18, we can see that the AG-ITP algorithm returns a solution with a weight of at most \( (2/k)\Delta + (1 - 1/k)\alpha w(H^*) \leq w(I^*) + (1 - 1/k)\alpha w(I^*) = (\alpha + 1 - \alpha/k)w(I^*) \).

**Corollary 19.** The approximation ratio of the AG-ITP algorithm is \( \alpha + 1 - \alpha/k \), where \( \alpha \) is the approximation ratio of metric TSP.

4.2 The HR-ITP Algorithm

Now we review the ITP algorithm introduced by Haimovich and Rinnooy Kan [30], where the used Hamiltonian cycle does not go through the depot \( v_0 \). Assume that there is a Hamiltonian cycle \( H = (v_1, v_2, \ldots, v_n, v_1) \) on \( V \) (not passing through the depot \( v_0 \)). The ITP algorithm will select the best solution from the following \( n \) solutions. For each \( 1 \leq i \leq n \), the \( i \)-th solution contains the following \( \left[ n/k \right] \) tours: \((v_0, v_i, \ldots, v_{i+k-1}, v_0), (v_0, v_{i+k}, \ldots, v_{i+2k-1}, v_0), (v_0, v_{i+2k}, \ldots, v_{i+3k-1}, v_0), \ldots, (v_0, v_{i+(\left[ n/k\right]-1)k}, \ldots, v_{i+n-1}, v_0)\) where we let \( v_{n+i'} = v_t \) (\( i' > 0 \)). Except for the last tour, each tour contains exactly \( k \) customers. The last tour contains \( k \) customers only when \( n \mod k = 0 \). The running time is \( O(n^2) \) time.

**Lemma 20** ([30]). Given a Hamiltonian cycle \( H \) on \( V \) as a part of the input, for unit-demand \( k \)-CVRP with any \( k \geq 3 \), the HR-ITP algorithm can use \( O(n^2) \) time to output a solution with a weight of at most \( (2/[n/k]/\Delta + (1 - \left[ n/k \right]/n)w(H) \).

Note that if \( n \) is divisible by \( k \), the HR-ITP algorithm can generate a solution with the same bounded weight of \( (2/k)\Delta + (1 - 1/k)w(H) \) in Lemma 18. Otherwise, the weight maybe worse than \( (2/k)\Delta + (1 - 1/k)w(H) \).
4.3 An Extension of ITP

The above two ITP algorithms are based on given Hamiltonian cycles. In fact, the requirement of Hamiltonian cycles is not necessary. We can replace the Hamiltonian cycle with a cycle packing (a 2-factor of the graph) in the algorithms to construct feasible solutions to CVRP in a similar way.

Given a cycle packing $C$ of the graph $G[V]$ or $G[V \cup \{v_0\}]$ (either containing the depot $v_0$ or not), for each cycle $C \in C$, we call the HR-ITP algorithm on it if $C$ does not contain the depot $v_0$, and call the AG-ITP algorithm on it if $C$ contains the depot $v_0$. After adding all together, we get a feasible solution to CVRP. The quality of the solution is related to the cycle packing. We call the algorithm the EX-ITP algorithm. Although the algorithm is still simple, it will play an important role in our algorithms.

Lemma 21. Given a cycle packing $C$ of $G[V]$ or $G[V \cup \{v_0\}]$ as a part of the input, for unit-demand $k$-CVRP with any $k \geq 3$, the EX-ITP algorithm uses $O(n^2)$ time to output a feasible solution with a weight of at most $2g\Delta + (1-g)w(C)$, where $g = \max_{C \in C} \frac{|C|/k}{|C|}$.

Proof. Define $\Delta_C = \sum_{v \in C} w(v, v_i)$. For the cycle $C \in C$ such that $v_0 \notin C$, the AG-ITP algorithm can generate an itinerary on $C$ with a weight of at most $(2/k)\Delta_C + (1-1/k)w(C)$. For each cycle $C \in C$ such that $v_0 \notin C$, the HR-ITP algorithm can generate an itinerary on $C$ with a weight of at most $(2\lceil|C|/k\rceil/|C|)\Delta_C + (1 - \lceil|C|/k\rceil/|C|)w(C)$. Note that for each cycle $C \in C$, we can get that $(2/k)\Delta_C + (1-1/k)w(C) \leq (2\lceil|C|/k\rceil/|C|)\Delta_C + (1 - \lceil|C|/k\rceil/|C|)w(C) \leq 2g\Delta_C + (1-g)w(C)$, where the inequalities follows from $w(C) \leq 2\Delta_C$ by the triangle inequality. Note that $\sum_{C \in C} \Delta_C = \Delta$. Hence, the total weight is bounded by $\sum_{C \in C} (2g\Delta_C + (1-g)w(C)) = 2g\Delta + (1-g)w(C)$. \qed

In our algorithm, we frequently use a special case that the input cycle packing is a mod-$k$-cycle packing $C_{\text{mod } k}$ of $G[V]$. We have a good approximation ratio for this special case and a good algorithm to find a mod-$k$-cycle packing.

By Lemma 21 and the previous analysis, we can directly get that

Corollary 22. Given a mod-$k$-cycle packing $C_{\text{mod } k}$ of $G[V]$ as a part of the input, for unit-demand $k$-CVRP with any $k \geq 3$, the EX-ITP algorithm can use $O(nk)$ time to output a feasible solution with a weight of at most $(2/k)\Delta + (1-1/k)w(C_{\text{mod } k})$.

5 Applications of the EX-ITP algorithm

In this section, we demonstrate that the EX-ITP algorithm can be used to design improved approximation algorithms for splittable and unit-demand $k$-CVRP. We show as examples, the approximation ratio can be significantly improved from 1.934 [8] to $3/2 = 1.500$ for 3-CVRP, and from 1.750 [4] to $5/3 < 1.667$ for 4-CVRP.

5.1 Unit-demand and Splittable 3-CVRP

The idea of our algorithm is to call the EX-ITP algorithm on a minimum cycle packing $C^*$ on $V$.

Lemma 23. For unit-demand 3-CVRP, there is a polynomial-time algorithm that can generate a solution with a weight of at most $\Delta + \frac{1}{2}w(C^*)$.

Proof. Our algorithm first computes a minimum cycle packing $C^*$ of $G[V]$ in polynomial time, and then call the EX-ITP algorithm on $C^*$. 

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Next, we analyze the quality of the solution. Since $|C| \geq 3$ for each cycle $C \in \mathcal{C}^*$, it is easy to get that
\[ g = \max_{C \in \mathcal{C}^*} \frac{|C|/3}{|C|} \leq \max_{|C| \geq 3} \frac{|C|/3}{|C|} = \frac{4/3}{4} = \frac{1}{2}. \]
By Lemma 21, the weight of the solution is at most $\Delta + \frac{1}{2}w(C^*)$. \qed

By Lemmas 13 and 17, we have
\[ \Delta + \frac{1}{2}w(C^*) \leq \left( \chi + \frac{1}{2} \right) w(I^*) + \frac{1}{2} \min\{2(1 - \chi), 1\} w(I^*) \leq \frac{3}{2} w(I^*). \]
Note that we have $\chi = \frac{1}{2}$ in the worst case. Therefore, we can get that

**Theorem 24.** For splittable and unit-demand 3-CVRP, there is a $\frac{3}{2}$-approximation algorithm.

### 5.2 Unit-demand and Splittable 4-CVRP
The idea of the algorithm is similar. However, we will first construct a good mod-2-cycle packing $\mathcal{C}_{\text{mod 2}}$ on $V$, instead of using a minimum cycle packing on $V$, and then call the EX-ITP algorithm.

We compute the mod-2-cycle packing $\mathcal{C}_{\text{mod 2}}$ in this way: first find a minimum perfect matching $\mathcal{M}^*$ in graph $G[V]$, then find a minimum perfect matching $\mathcal{M}^{**}$ in graph $G[V] \setminus \mathcal{M}^*$, and then let $\mathcal{C}_{\text{mod 2}} = \mathcal{M}^* \cup \mathcal{M}^{**}$. It is easy to see that $\mathcal{C}_{\text{mod 2}}$ is a mod-2-cycle packing without 2-cycles since $\mathcal{M}^*$ and $\mathcal{M}^{**}$ are two disjoint perfect matchings of the graph. Recall that we use $\mathcal{C}_4^*$ to denote a minimum 4-cycle packing of $G[V]$. We have the following lemma.

**Lemma 25.** $w(\mathcal{C}_{\text{mod 2}}) \leq w(\mathcal{C}_4^*)$.

**Proof.** We first prove the following property.

**Claim 1.** Given a minimum 4-cycle packing $\mathcal{C}_4^*$ and a minimum perfect matching $\mathcal{M}^*$, there is a way to color edges in $\mathcal{C}_4^*$ with red and blue such that
1. the blues (resp., red) edges form a perfect matching $\mathcal{M}_b$ (resp., $\mathcal{M}_r$);
2. $\mathcal{C}_4^* = \mathcal{M}_b \cup \mathcal{M}_r$;
3. $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-2-cycle packing without 2-cycles.

For each 4-cycle $C = (v_1, v_2, v_3, v_4, v_1)$ in $\mathcal{C}_4^*$, we color the four edges in it by considering three cases.

**Case 1.** $|C \cap \mathcal{M}^*| = 0$: We color the edges $(v_1, v_2)$ and $(v_3, v_4)$ with blue and the edges $(v_1, v_4)$ and $(v_2, v_3)$ with red. Now blue edges (resp., red edges) are vertex-disjoint.

**Case 2.** $|C \cap \mathcal{M}^*| = 1$: Without loss of generality, we assume $C \cap \mathcal{M}^* = \{(v_1, v_2)\}$. Then, we color the edges $(v_1, v_2)$ and $(v_3, v_4)$ with red and the edges $(v_1, v_4)$ and $(v_2, v_3)$ with blue.

**Case 3.** $|C \cap \mathcal{M}^*| = 2$: Without loss of generality, we assume $C \cap \mathcal{M}^* = \{(v_1, v_2), (v_3, v_4)\}$. Then, we color the edges $(v_1, v_2)$ and $(v_3, v_4)$ with red and the edges $(v_1, v_4)$ and $(v_2, v_3)$ with blue.

It is easy to see that the set of blues edges $\mathcal{M}_b$ and the set of red edges $\mathcal{M}_r$ are two perfect matchings and $\mathcal{M}_b \cup \mathcal{M}_r = \mathcal{C}_4^*$. Moreover, we know that $\mathcal{M}_b \cap \mathcal{M}^* = \emptyset$, and hence $\mathcal{M}_b \cup \mathcal{M}^*$ is a mod-2-cycle packing without 2-cycles. Thus, Claim 1 holds.

By Claim 1, we get that
\[ w(\mathcal{M}_b \cup \mathcal{M}^*) \leq w(\mathcal{M}_b \cup \mathcal{M}_r) = w(\mathcal{C}_4^*). \]

Since the mod-2-cycle packing $\mathcal{C}_{\text{mod 2}} = \mathcal{M}^* \cup \mathcal{M}^{**}$ is the minimum mod-2-cycle packing containing the minimum perfect matching $\mathcal{M}^*$, we have that
\[ w(\mathcal{C}_{\text{mod 2}}) = w(\mathcal{M}^* \cup \mathcal{M}^{**}) \leq w(\mathcal{M}_b \cup \mathcal{M}^*). \]
Thus, $w(C_{\text{mod } 2}) \leq w(C_4^*)$ and the lemma holds. \hfill \square

**Lemma 26.** For unit-demand 4-CVRP, there is a polynomial-time algorithm that can generate a solution with a weight of at most $\frac{2}{3}\Delta + \frac{2}{3}w(C_4^*)$.

**Proof.** We call the EX-ITP algorithm on $C_{\text{mod } 2}$. Since $|C|$ is even and $|C| \geq 4$ for each cycle $C \in C_{\text{mod } 2}$, we can get that

$$g = \max_{C \in C_{\text{mod } 2}} \frac{|C|/4}{|C|} \leq \max_{|C| \in \{4,6,\ldots\}} \frac{|C|/4}{|C|} = \frac{6/4}{6} = \frac{1}{3}. $$

By Lemmas 21 and 25, we know that the weight of the solution is at most $\frac{2}{3}\Delta + \frac{2}{3}w(C_{\text{mod } 2}) \leq \frac{2}{3}\Delta + \frac{2}{3}w(C_4^*)$. \hfill \square

By Lemmas 13 and 17, we have

$$\frac{2}{3}\Delta + \frac{2}{3}w(C_4^*) \leq \frac{2}{3}(\chi + 1)w(I^*) + \frac{2}{3}\min\{(2 - 2\chi), 1\}w(I^*) \leq \frac{5}{3}w(I^*).$$

Note that we have $\chi = \frac{1}{2}$ in the worst case. Therefore, we can get that

**Theorem 27.** For splittable and unit-demand 4-CVRP, there is a $\frac{5}{3}$-approximation algorithm.

### 6 An Improvement for Splittable $k$-CVRP

In this section, we show how to use the EX-ITP algorithm to improve splittable $k$-CVRP. Given an $\alpha$-approximation algorithm for metric TSP, where $1 \leq \alpha \leq 3/2$. We will prove a ratio of $\alpha + 1 - \alpha/k - \Theta(1/k)$ for splittable $k$-CVRP, which improves the ratio $\alpha + 1 - \alpha/k - \Omega(1/k^3)$ in [16]. To get the claimed ratio, we need to make a trade-off among three algorithms. We will use $\rho_1(\chi)$, $\rho_2(\chi)$, and $\rho_3(\chi)$ to denote the ratio of the three algorithms, respectively. They are functions on the parameter $\chi$. Our final ratio is $\max_{0 \leq \chi \leq 1} \min\{\rho_1(\chi), \rho_2(\chi), \rho_3(\chi)\}$.

Our first algorithm is to call the EX-ITP algorithm on a mod-$k$-cycle packing $C_{\text{mod } k}$ on $V$. We use the following three steps to compute $C_{\text{mod } k}$.

**Step 1.** Compute a mod-$k$-tree packing $T_{\text{mod } k}$ of $G[V]$ using the prime-dual algorithm in [27].

**Step 2.** Find a minimum matching $M$ on odd-degree vertices of $T_{\text{mod } k}$.

**Step 3.** Construct a mod-$k$-cycle packing $C_{\text{mod } k}$ of $G[V]$ from $T_{\text{mod } k} \cup M$ by shortcutting.

The second algorithm is to call the AG-ITP algorithm on any $\alpha$-approximate Hamiltonian cycle on $V \cup \{v_0\}$. The third algorithm is to call the AG-ITP algorithm on the Hamiltonian cycle $H_{CS}$ on $V \cup \{v_0\}$.

We first analyze some properties of the first algorithm. The mod-$k$-tree packing computed in the first step has the following property.

**Lemma 28 ([27]).** There is a polynomial-time algorithm that can generate a mod-$k$-tree packing $T_{\text{mod } k}$ of $G[V]$ such that $w(T_{\text{mod } k}) \leq w(C_4^*)$.

In Steps 2 and 3, we generate a mod-$k$-cycle packing based on the mod-$k$-tree packing $T_{\text{mod } k}$ and a minimum perfect matching $M$ on odd-degree vertices of $T_{\text{mod } k}$. It also has a good property in the following lemma.

**Lemma 29.** There is a polynomial-time algorithm that generates a mod-$k$-cycle packing $C_{\text{mod } k}$ of $G[V]$ such that $w(C_{\text{mod } k}) \leq w(C_4^*) + \frac{1}{2}w(H^*)$. 
Recall that we will make a trade-off between three algorithms. The first algorithm uses the cycle $H$ and the degree of each vertex is even. Therefore, by shortcutting each component, we can get a set of components, where in each component the number of vertices is divisible by $k$ and the degree of each vertex is even. Therefore, by shortcutting each component, we can get a mod-$k$-cycle packing $C_{mod\,k}$ such that $w(C_{mod\,k}) \leq w(T_{mod\,k}) + w(M) \leq w(C_{mod\,k}^\ast) + \frac{1}{2}w(H^\ast)$. □

Now, we are ready to analyze the three algorithms.

**Theorem 30.** Suppose there is an $\alpha$-approximate Hamiltonian cycle $H$ on $V \cup \{v_0\}$, then there is an approximation algorithm for splittable and unit-demand $k$-CVRP such that

- For $1 \leq \alpha \leq 7/6$ and $k \geq 3$: the ratio is $\alpha + 1 - \alpha/k - (\alpha - 1)/k$;
- For $7/6 \leq \alpha \leq 3/2$ and $3 \leq k \leq 5$: the ratio is $(13k - 11)/(6k)$;
- For $7/6 \leq \alpha \leq 3/2$ and $k \geq 6$: the ratio is $\alpha + 1 - \alpha/k - 4(\alpha - 1)/k$.

**Proof.** Recall that we will make a trade-off between three algorithms. The first algorithm uses the mod-$k$-cycle packing $C_{mod\,k}$ of $G[V]$ in Lemma 29, the second uses an $\alpha$-approximate Hamiltonian cycle $H$ on $V \cup \{v_0\}$, and the third uses the Hamiltonian cycle $H_{CS}$ on $V \cup \{v_0\}$.

Using the mod-$k$-cycle packing $C_{mod\,k}$, by Corollary 22 and Lemmas 12, 13, 17, and 29, the EX-ITP algorithm can generate a solution with a weight of at most

$$\frac{2}{k}\Delta + \left(1 - \frac{1}{k}\right)w(C_{mod\,k}) \leq \frac{2}{k}\Delta + \frac{k - 1}{k} \left(w(C_{mod\,k}^\ast) + \frac{1}{2}w(H^\ast)\right) \leq \frac{2}{k}\left(\chi + \frac{k - 2}{2}\right)w(I^\ast) + \frac{k - 1}{k} \left(2(1 - \chi) + \frac{1}{2}\right)w(I^\ast) = \left(\frac{2k - 4}{k}\chi + \frac{7k - 9}{2k}\right)w(I^\ast).$$

The ratio, denoted by $\rho_1(\chi)$, satisfies $\rho_1(\chi) = -\frac{2k - 4}{k}\chi + \frac{7k - 9}{2k}$.

Using the Hamiltonian cycle $H$, by Lemmas 12, 13, and 18, the AG-ITP algorithm can generate a solution with a weight of at most

$$\frac{2}{k}\Delta + \left(1 - \frac{1}{k}\right)w(H) \leq \frac{2}{k}\Delta + \frac{k - 1}{k}\alpha w(H^\ast) \leq \frac{2}{k}\left(\chi + \frac{k - 2}{2}\right)w(I^\ast) + \frac{k - 1}{k}\alpha w(I^\ast) = \left(\frac{2k - 4}{k}\chi + \frac{(k - 1)\alpha + k - 2}{k}\right)w(I^\ast).$$

The ratio, denoted by $\rho_2(\chi)$, satisfies $\rho_2(\chi) = \frac{2}{7}\chi + \frac{(k - 1)\alpha + k - 2}{k}$.

Using the Hamiltonian cycle $H_{CS}$, by Lemmas 13, 17, and 18, the AG-ITP algorithm can generate a solution with a weight of at most

$$\frac{2}{k}\Delta + \left(1 - \frac{1}{k}\right)w(H_{CS}) \leq \frac{2}{k}\left(\chi + \frac{k - 2}{2}\right)w(I^\ast) + \frac{k - 1}{k} \cdot \frac{3 - \chi}{2} w(I^\ast) = \left(-\frac{k - 5}{2k}\chi + \frac{5k - 7}{2k}\right)w(I^\ast).$$

The ratio, denoted by $\rho_3(\chi)$, satisfies $\rho_3(\chi) = -\frac{k - 5}{4k}\chi + \frac{5k - 7}{4k}$.
By making a trade-off, we can get the approximation ratio \(\max_{0 \leq \chi \leq 1} \min \{\rho_1(\chi), \rho_2(\chi), \rho_3(\chi)\}\). The results can be briefly summarized as follows.

**Case 1.** When \(1 \leq \alpha \leq 7/6\) and \(k \geq 3\), the best ratio is
\[
\max_{0 \leq \chi \leq 1} \min \{\rho_1(\chi), \rho_2(\chi)\} = \alpha + 1 - \alpha/k - (\alpha - 1)/k.
\]

**Case 2.** When \(7/6 \leq \alpha \leq 3/2\) and \(3 \leq k \leq 5\), the best ratio is
\[
\max_{0 \leq \chi \leq 1} \min \{\rho_1(\chi), \rho_3(\chi)\} = (13k - 11)/(6k).
\]

**Case 3.** When \(7/6 \leq \alpha \leq 3/2\) and \(k \geq 6\), the best ratio is
\[
\max_{0 \leq \chi \leq 1} \min \{\rho_2(\chi), \rho_3(\chi)\} = \alpha + 1 - \alpha/k - 4(\alpha - 1)/k.
\]

We can improve the ratio \(\alpha + 1 - \alpha/k\) by a term of at least \(0.5/k\) (when \(\alpha = 1\), the improvement is \(0.5/k\)), which is larger than the improvement \(\frac{1}{3}\) in [16]. For larger \(\alpha\), we will get a larger improvement. For \(\alpha = 3/2\), the ratio is \(5/2 - 3.5/k\) and the improvement is \(2/k\), which is also larger than the improvement \(\frac{1}{3k}\) in [16]. For some detailed comparisons, readers can refer to Table 1.

### 7 A Further Improvement for Splittable \(k\)-CVRP

The current approximation ratio for metric TSP is about \(\alpha = 3/2\). The approximation ratio of splittable and unit-demand \(k\)-CVRP is \(5/2 - 3.5/k = 5/2 - \Theta(1/k)\) in the previous section. In this section, we will further improve the ratio to \(5/2 - \Theta(\sqrt{1/k})\).

Our algorithm is indeed a known algorithm that is to call the classic AG-ITP algorithm on the Hamiltonian cycle \(H_{CS}\) on \(V \cup \{v_0\}\). We will give a deep analysis for this algorithm.

Using the Hamiltonian cycle \(H_{CS}\), by Lemmas 15 and 18, the AG-ITP algorithm can generate an itinerary \(I\) with
\[
w(I) \leq (2/k)\Delta + (1-1/k)H_{CS} \leq (2/k)\Delta + (1-1/k)\text{MST} + (1/2)(1-1/k)w(H^*). \tag{1}
\]

We first analyze the structure in more details. By Assumption 11, \(I^*\) consists of a set of \((k+1)\)-cycles. We consider an arbitrary \((k+1)\)-cycle \(C = (v_0, v_1, \ldots, v_k, v_0)\) in \(I^*\). First, we use \(\Delta_C\) to denote the sum weight of the edges between vertices \(v_0\) and \(v_i\) for \(1 \leq i \leq k\), i.e. \(\Delta_C = \sum_{i=1}^{k} w(v_0, v_i)\). Note that \(\Delta = \sum_{C \in I^*} \Delta_C\). Then, we use \(T_C\) to denote the spanning tree of \(G[C]\) obtained by deleting the most weighted edge in \(C\), i.e. \(w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \mod (k+1)})\). After we delete the most weighted edge for each cycle \(C \in I^*\), the remaining graph is a spanning tree of \(G\). So, we have \(\text{MST} \leq \sum_{C \in I^*} w(T_C)\).

When \(k\) is odd (resp., even), the number of edges in the cycle \(C\) is even (resp., odd). Due to different structure properties, these two cases have to be handled separately. We first consider the case that \(k\) is odd. For the sake of presentation, we also let \(m = \lceil k/2 \rceil\).

**Case 1. \(k\) is odd:** Note that \(m = \lceil k/2 \rceil = \frac{k+1}{2}\). We define parameters
\[
a_i = \frac{w(v_{i-1}, v_i) + w(v_{k+1-i}, v_{(k+2-i) \mod (k+1)})}{w(C)},
\]
where \(1 \leq i \leq m\). It is easy to see that \(\sum_{i=1}^{m} a_i = 1\) and \(a_i \geq 0\) for each \(1 \leq i \leq m\). Note that we assume \(w(C) \neq 0\). Otherwise, we will show that our conclusion holds trivially. See Figure 1 for an illustration.
Figure 1: An illustration of the cycle \( C = (v_0, v_1, \ldots, v_k, v_0) \) with the case of odd \( k \), where \( a_i \) \((1 \leq i \leq m)\) measures the weight proportion of the blue edges compared to the cycle \( C \)

**Lemma 31.** When \( k \) is odd, for any cycle \( C \in I^* \), we have \( w(T_C) \leq (1 - \max_{1 \leq i \leq m} \frac{1}{2}a_i)w(C) \).

**Proof.** By the definition of \( T_C \), we have that \( w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \mod (k+1)}) \). Therefore, we have that

\[
w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \mod (k+1)}) \leq w(C) - \max_{1 \leq i \leq m} \frac{1}{2} \left( w(v_{i-1}, v_i) + w(v_{k+1-i}, v_{(k+2-i) \mod (k+1)}) \right) = \left( 1 - \max_{1 \leq i \leq m} \frac{1}{2}a_i \right) w(C),
\]

where the inequality can be obtained directly.

**Lemma 32.** When \( k \) is odd, for any cycle \( C \in I^* \), we have \( \Delta_C \leq (\frac{k+2}{2} - \sum_{i=1}^{m} i a_i)w(C) \).

**Proof.** First, by the definition of \( \Delta_C \), it is easy to see that

\[
\Delta_C = \sum_{i=1}^{k} w(v_0, v_i) = \sum_{i=1}^{m-1} (w(v_0, v_i) + w(v_0, v_{k+1-i})) + w(v_0, v_m).
\]

Then, by the triangle inequality, we can get that

\[
w(v_0, v_i) \leq \sum_{j=1}^{i} w(v_{j-1}, v_j) \quad \text{and} \quad w(v_0, v_{k+1-i}) \leq \sum_{j=1}^{i} w(v_{k+1-j}, v_{(k+2-j) \mod (k+1)}).
\]

See Figure 2 for an illustration of these two inequalities. Especially, we have \( w(v_0, v_m) \leq \frac{1}{2}w(C) \).

Figure 2: An illustration of the two inequalities: there are two cycles (denoted by directed edges): \((v_0, v_1, \ldots, v_i, v_0)\) and \((v_0, v_{k+1-i}, \ldots, v_k, v_0)\); for each cycle, the weight of the only red edge is at most the sum weight of the blue edges by the triangle inequality.

Recall that \( w(v_{j-1}, v_j) + w(v_{k+1-j}, v_{(k+2-j) \mod (k+1)}) = a_j w(C) \). Hence, we can get that

\[
\Delta_C \leq \sum_{i=1}^{m-1} \sum_{j=1}^{i} a_j w(C) + \frac{1}{2}w(C) = \sum_{i=1}^{m} (m-i)a_i w(C) + \frac{1}{2}w(C) = \left( \frac{k+2}{2} - \sum_{i=1}^{m} i a_i \right) w(C),
\]

where the last equality follows from \( \sum_{i=1}^{m} ma_i w(C) = mw(C) = \frac{k+1}{2}w(C) \).

\[\square\]
Lemma 33. When $k$ is odd, for any cycle $C \in \mathcal{I}^*$, we have
\[
\frac{2}{k} \Delta + \left(1 - \frac{1}{k}\right) w(T_C) \leq \max_{a_1, a_2, \ldots, a_m \geq 0} \sum_{i=1}^{m} \left\{ \frac{2k+1}{k} - \frac{m}{k} \right\} w(C),
\]
where $c_i^0 = \frac{k+3}{2k}$ and $c_i^0 = \frac{2i}{k}$ for $2 \leq i \leq m$.

Proof. By Lemmas 31 and 32, we know that
\[
\frac{2}{k} \Delta + \left(1 - \frac{1}{k}\right) w(T_C)
\leq \max_{a_1, a_2, \ldots, a_m \geq 0} \sum_{i=1}^{m} \left\{ \frac{2k+1}{k} - \frac{m}{k} \right\} w(C)
\]
\[
= \max_{a_1, a_2, \ldots, a_m \geq 0} \sum_{i=1}^{m} \left\{ \frac{2k+1}{k} - \frac{m}{k} \right\} w(C).
\]

Note that it is sufficient to prove $a_1 \geq a_2 \geq \cdots \geq a_m$ since we can also get $\max_{1 \leq i \leq m} a_i = a_1$.

Assume that there exists $a_p < a_q$ for some $p < q$. If we exchange their values, we note that the value $\max_{1 \leq i \leq m} a_i$ does not change. However, since the coefficients of $a_p$ and $a_q$ satisfy that $0 > -2p/k > -2q/k$, we can get a bigger solution which causes a contradiction. \qed

Next, we need to consider the case that $k$ is even. The analysis is almost the same.

Case 2. $k$ is even: Note that $m = \lfloor \frac{k+1}{2} \rfloor = \frac{k+2}{2}$. We define parameters
\[
b_i = \frac{w(v_i, v_{i-1}) + w(v_{k+1-i}, v_{k+2-i} \mod (k+1))}{w(C)},
\]
where $1 \leq i \leq m - 1$. We also define $b_m = w(v_{k/2}, v_{k/2+1})/w(C)$. Then, we have $\sum_{i=1}^{m} b_i = 1$. See Figure 3 for an illustration.

Figure 3: An illustration of the cycle $C = (v_0, v_1, \ldots, v_k, v_0)$ with the case of even $k$, where $b_i$ $(1 \leq i < m)$ measures the weight proportion of the blue edges compared to the cycle $C$ and $b_m$ measures the weight proportion of the red edge compared to the cycle $C$.

Lemma 34. When $k$ is even, for any cycle $C \in \mathcal{I}^*$, we have $w(T_C) \leq (1 - \max_{1 \leq i \leq m} \frac{1}{2} b_i) w(C)$. 

Proof. Recall that $w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \mod (k+1)})$. Therefore, we have that
\[
w(T_C) = w(C) - \max_{0 \leq i \leq k} w(v_i, v_{(i+1) \mod (k+1)}) \leq w(C) - \max_{1 \leq i \leq m} \left\{ \frac{1}{2} (w(v_{i-1}, v_i) + w(v_{k+1-i}, v_{k+2-i} \mod (k+1))), w(v_{k/2}, v_{k/2+1}) \right\}
\]
\[
= \left(1 - \max_{1 \leq i \leq m} \left\{ \frac{1}{2} b_i, b_m \right\} \right) w(C)
\]
\[
\leq \left(1 - \max_{1 \leq i \leq m} \frac{1}{2} b_i \right) w(C),
\]

where the inequalities can be obtained directly. □

**Lemma 35.** When \( k \) is even, for any cycle \( C \in I^* \), we have \( \Delta_C \leq (\frac{k+2}{2} - \sum_{i=1}^{m} ib_i)w(C) \).

**Proof.** First, by the definition of \( \Delta_C \), we have that

\[
\Delta_C = \sum_{i=1}^{k} w(v_0, v_i) = \sum_{i=1}^{m} (w(v_0, v_i) + w(v_0, v_{k+1-i})).
\]

Then, by the triangle inequality, we can get that

\[
w(v_0, v_i) \leq \sum_{j=1}^{i} w(v_{j-1}, v_j) \quad \text{and} \quad w(v_0, v_{k+1-i}) \leq \sum_{j=1}^{i} w(v_{k+1-j}, v_{(k+2-j) \mod (k+1)}).
\]

See Figure 4 for an illustration of these two inequalities. The analysis is the same.

![Figure 4: An illustration of the two inequalities: there are two cycles (denoted by directed edges): (v_0, v_1, \ldots, v_i, v_0) and (v_0, v_{k+1-i}, \ldots, v_k, v_0); for each cycle, the weight of the only red edge is at most the sum weight of the blue edges by the triangle inequality](image)

Recall that \( w(v_{j-1}, v_j) + w(v_{k+1-j}, v_{(k+2-j) \mod (k+1)}) = b_jw(C) \). Hence, we can get that

\[
\Delta_C \leq \sum_{i=1}^{m-1} \sum_{j=1}^{i} b_jw(C) = \sum_{i=1}^{m} (m-i)b_iw(C) = \left( \frac{k+2}{2} - \sum_{i=1}^{m} ib_i \right) w(C),
\]

where the last equality follows from \( \sum_{i=1}^{m} mb_iw(C) = mw(C) = \frac{k+2}{2}w(C) \). □

Both Lemmas 34 and 35 have the same form as that in Lemmas 31 and 32. Hence, by a similar argument with the proof in Lemma 33, we can get the following lemma.

**Lemma 36.** When \( k \) is even, for any cycle \( C \in I^* \), we have

\[
\frac{2}{k} \Delta_C + \left( 1 - \frac{1}{k} \right) w(T_C) \leq \max_{\substack{b_1 \geq b_2 \geq \ldots \geq b_m \geq 0 \\ b_1 + b_2 + \ldots + b_m = 1}} \left( \frac{2k+1}{k} - \sum_{i=1}^{m} c^*_i b_i \right) w(C),
\]

where \( c^*_i = \frac{k+3}{2k} \) and \( c^*_i = \frac{2i}{k} \) for \( 2 \leq i \leq m \).

By Lemmas 33 and 36, we can see that these two cases can be combined as follows.

**Lemma 37.** When \( k \geq 3 \), for any cycle \( C \in I^* \), we have

\[
\frac{2}{k} \Delta_C + \left( 1 - \frac{1}{k} \right) w(T_C) \leq \max_{\substack{x_1 \geq x_2 \geq \ldots \geq x_m \geq 0 \\ x_1 + x_2 + \ldots + x_m = 1}} \left( \frac{2k+1}{k} - \sum_{i=1}^{m} c_i x_i \right) w(C),
\]

where \( c_1 = \frac{k+3}{2k} \) and \( c_i = \frac{2i}{k} \) for \( 2 \leq i \leq m \).
For the sake of analysis, we generalize some definitions and define variables \( \{x_i \mid i \in \mathbb{Z}_{\geq 1}\} \) where \( c_1 = \frac{k+3}{2k} \) and \( c_i = \frac{2i}{k} \) for \( i \geq 2 \), then we have

\[
\min_{x_1 \geq x_2 \geq \cdots \geq 0} \sum_{i=1}^{\infty} c_i x_i \leq \min_{x_1 \geq x_2 \geq \cdots \geq x_m \geq 0} \sum_{i=1}^{m} c_i x_i,
\]

since the latter is a special case of the former. Then, we take an analysis on the former LP.

**Lemma 38.** For the following LP,

\[
\min_{x_1 \geq x_2 \geq \cdots \geq 0} \sum_{i=1}^{\infty} c_i x_i,
\]

where \( c_1 = \frac{k+3}{2k} \) and \( c_i = \frac{2i}{k} \) for \( i \geq 2 \). There exists an optimal solution, denoted by SOL\((x_1^*, x_2^*, \ldots)\), satisfying that \( x_1^* = x_2^* = \cdots = x_l^* = \frac{1}{l} \) and \( x_i^* = 0 \) for all \( i > l \), where \( l = \lceil \frac{\sqrt{2k+1}-1}{2} \rceil \). Moreover, it holds that SOL\((x_1^*, x_2^*, \ldots)\) = \( \frac{2l^2 + 2l + k - 1}{2k} \).

**Proof.** Since \( \lim_{i \to +\infty} c_i = +\infty \), we can get that the number of nonzero variables is limited for any optimal solution. Hence, we will consider the optimal solution SOL\((x_1^*, x_2^*, \ldots)\) with the number of nonzero variables minimized. Let \( x_l^* \) be the last nonzero variable. We define \( c_l = \frac{1}{l} \sum_{i=1}^{l} c_i \) for \( i > 0 \). We will show that \( x_1^* = x_2^* = \cdots = x_l^* = \frac{1}{l} \) and \( l = \arg\min_{i \in \mathbb{Z}_{\geq 1}} \{c_{i+1} \geq \frac{\sqrt{l}}{l}\} \).

Since the number of nonzero variables is minimized, we have \( c_2 \geq c_l = c_1 \) if and only if \( l = 1 \). Hence, we know that \( l = \arg\min_{i \in \mathbb{Z}_{\geq 1}} \{c_{i+1} \geq \frac{\sqrt{i}}{i}\} \) holds when \( c_2 \geq c_l = c_1 \). In the following, we consider \( c_2 < c_l \) and we have \( l > 1 \).

**Claim 1.** For all \( 1 < i \leq l \), we have \( c_i < c_{i-1} \).

Suppose there exists \( n_0 \in \mathbb{Z}_{> 1} \) such that \( 1 < n_0 \leq l \) and \( c_{n_0} \geq c_{n_0-1} \), then we let \( x_i^{**} = x_i^* + \frac{1}{n_0-1} \sum_{i'=n_0}^{l} x_i^{*} \) for \( 1 \leq i < n_0 \) and \( x_i^{**} = 0 \) for \( n_0 \leq i \leq l \). Then, we have that

\[
\sum_{i=1}^{n_0-1} c_i x_i^{**} = \sum_{i=1}^{n_0-1} c_i \left( x_i^* + \frac{1}{n_0-1} \sum_{i'=n_0}^{l} x_i^{*}\right)
= \sum_{i=1}^{n_0-1} c_i x_i^* + \frac{l}{n_0-1} \sum_{i=1}^{n_0-1} c_i x_i^*
= \sum_{i=1}^{n_0-1} c_i x_i^* + \sum_{i'=n_0}^{l} \frac{c_{n_0-1}}{n_0-1} x_i^{*}
\leq \sum_{i=1}^{n_0-1} c_i x_i^* + \sum_{i'=n_0}^{l} c_{i'} x_i^{*}
\leq \sum_{i=1}^{n_0-1} c_i x_i^* + \sum_{i'=n_0}^{l} c_{i'} x_i^{*}
= \sum_{i=1}^{l} c_i x_i^{*},
\]

where the first inequality follows from \( c_{n_0-1} \leq c_{n_0} \) and the second is due to \( c_{n_0} \leq c_{i'} \) for \( i' \geq n_0 > 1 \) since \( c_i = \frac{2i}{k} \) when \( i > 1 \). Note that the number of nonzero variables in the new solution is \( n_0-1 < l \), which is a contradiction.
**Claim 2.** For all $1 < i \leq l$, we have $x_i^* = x_{i-1}^*$.

Suppose there exists (minimized) $n_o \in \mathbb{Z}_{>1}$ such that $1 < n_o \leq l$ and $x_1^* = \cdots = x_{n_o-1}^* > x_{n_o}^*$, we simply let $x_1^* = \cdots = x_{n_o}^* = \frac{1}{n_o} \sum_{i=1}^{n_0} x_i^*$. Note that the new solution is still a feasible solution since $x_{n_o}^* > x_{n_0}^* \geq x_{n_0+1}^*$. We can get

$$\sum_{i=1}^{n_0} c_i x_i^* = \sum_{i=1}^{n_0-1} c_i x_i^* + c_{n_0} x_{n_0}^*$$

$$= (n_0 - 1) c_{n_0-1} x_{n_0}^* + c_{n_0} x_{n_0}^*$$

$$= (n_0 - 1) c_{n_0-1} x_{n_0}^* + c_{n_0} (x_{n_0}^* - x_{n_0}^*) + c_{n_0} x_{n_0}^*$$

$$< (n_0 - 1) c_{n_0-1} x_{n_0}^* + c_{n_0-1} (x_{n_0}^* - x_{n_0}^*) + c_{n_0} x_{n_0}^*$$

$$= c_{n_0-1} (n_0 x_{n_0}^* - x_{n_0}^*) + c_{n_0} x_{n_0}^*$$

$$= c_{n_0-1} \sum_{i=1}^{n_0-1} x_i^* + c_{n_0} x_{n_0}^*$$

$$= \sum_{i=1}^{n_0} c_i x_i^*,$$

where the inequality follows from $c_{n_0} < \frac{1}{c_{n_0-1}}$ by Claim 1. The new solution is better than the optimal solution, a contradiction.

**Claim 3.** $c_{l+1} \geq \overline{c}$.

Otherwise, if $c_{l+1} < \overline{c}$, we let $x_1^* = \cdots = x_{l+1}^* = \frac{1}{l+1} \sum_{i=1}^{l} x_i^* = x^*$. Note that $x_1^* = \cdots = x_l^*$ by Claim 2, and then we have

$$\sum_{i=1}^{l+1} c_i x_i^* = \sum_{i=1}^{l+1} c_i x_i^* = t c_{l+1} x_i^* + c_{l+1} x_i^* < \overline{c} (l+1) x_i^* = \overline{c} \sum_{i=1}^{l} x_i^* = \sum_{i=1}^{l} c_i x_i^*.$$

The new solution is better than the optimal solution, a contradiction.

By Claims 1 and 3, we have

$$l = \arg \min_{l' \in \mathbb{Z}_{\geq 1}} \{ c_{l'+1} \geq \overline{c} \} = \arg \min_{l' \in \mathbb{Z}_{\geq 1}} \{ l'^2 + l' - \frac{k-1}{2} \geq 0 \} = \lfloor \frac{\sqrt{2k-1}}{2} \rfloor.$$

So, we have SOL($x_1^*, x_2^*, \ldots \) = $\sum_{i=1}^{l} c_i x_i^* = \frac{1}{\overline{c}} \sum_{i=1}^{l} c_i = \overline{c} = \frac{2 l^2 + 2 l + k - 1}{2 kl}$, where $l = \lfloor \frac{\sqrt{2k-1}}{2} \rfloor$. □

By Lemmas 37 and 38, we have

**Lemma 39.** When $k \geq 3$, for any cycle $C \in I^*$, we have that

$$\frac{2}{k} \Delta_C + \left( 1 - \frac{1}{k} \right) w(T_C) \leq \left( 2 - \frac{2 l^2 + k - 1}{2 kl} \right) w(C),$$

where $l = \lfloor \frac{\sqrt{2k-1}}{2} \rfloor$.

Recall that we assume $w(C) \neq 0$. If $w(C) = 0$, we have $\Delta_C = w(T_C) = 0$, and then Lemma 39 holds trivially. Next, we are ready to analyze the AG-ITP algorithm.

**Theorem 40.** For splittable and unit-demand $k$-CVRP, the AG-ITP algorithm admits an approximation ratio of $\frac{\sqrt{2k} - \sqrt{2k-1}}{2k} < \frac{\sqrt{2k}}{2} - \sqrt{2/k}$, where $l = \lfloor \frac{\sqrt{2k-1}}{2} \rfloor$. □
Proof. Using the Hamiltonian cycle $H_{CS}$, by (1), the AG-ITP algorithm can output a solution with a weight of at most $(2/k)\Delta + (1 - 1/k)\text{MST} + (1/2)(1 - 1/k)w(H^*)$. By Lemmas 12 and 39, we have

$$
\frac{2}{k} \Delta + \left(1 - \frac{1}{k}\right) \text{MST} + \frac{1}{2} \left(1 - \frac{1}{k}\right) w(H^*)
$$

$$
\leq \sum_{C \in I^*} \left(\frac{2}{k} \Delta_C + \frac{k - 1}{k} w(T_C)\right) + \frac{k - 1}{2k} w(I^*)
$$

$$
\leq \sum_{C \in I^*} \left(2 - \frac{2l^2 + k - 1}{2kl}\right) w(C) + \frac{k - 1}{2k} w(I^*)
$$

$$
= \left(2 - \frac{2l^2 + k - 1}{2kl}\right) \sum_{C \in I^*} w(C) + \frac{k - 1}{2k} w(I^*)
$$

$$
= \left(2 - \frac{2l^2 + k - 1}{2kl} + \frac{k - 1}{2k}\right) w(I^*)
$$

$$
= \left(\frac{5}{2} - \frac{2l^2 + k + l - 1}{2kl}\right) w(I^*).
$$

To show the ratio $\frac{5}{2} - \frac{2l^2 + k + l - 1}{2kl} < \frac{5}{2} - \sqrt{2/k}$ holds for any $k \geq 3$, it is sufficient to prove

$$
\frac{2l^2 + k + l - 1}{2kl} > \sqrt{2/k} \iff 2l^2 - (2\sqrt{2k} - 1)l + k - 1 > 0.
$$

The latter holds since the discriminant of the quadratic equation satisfies $(2\sqrt{2k} - 1)^2 - 8(k - 1) = 9 - 4\sqrt{2k} < 0$ for any $k \geq 3$. \hfill \square

In the previous section, when $\alpha = 3/2$ and $k > 5$, the approximation ratio is $5/2 - 3.5/k$. It is easy to see that the new result $5/2 - \Theta(\sqrt{1/k})$ is strictly better for any $k > 5$. The result is also better than $5/2 - 1.005/3000 - 1.5/k$ for any $k \leq 1.7 \times 10^7$. Please see Table 1.

When $3 \leq k \leq 5$, the approximation ratio equals $2 - 1/k$, which is tight since it has been shown that the AG-ITP algorithm has an approximation ratio of at least $2 - 1/k$ on general metrics even using an optimal Hamiltonian cycle on $V \cup \{v_0\}$ [40]. It is worth noting that Theorem 40 still holds even if we drop Assumption 11 since in the worst case almost all tours in $I^*$ have a length of $k+1$.

8 ITP for Unsplittable CVRP

From this section, we consider unsplittable CVRP. The ITP-based algorithms play an important role in solving splittable $k$-CVRP. A natural idea is to extend ITP-based algorithms for unsplittable $k$-CVRP. In order to generate a feasible solution without splitting the demand of one customer into two tours, we need to modify the initial solution obtained by ITP. One of the most famous algorithms for unsplittable CVRP is the ITP-based algorithm introduced by Altinkemer and Gavish [2]. The main idea of the algorithm is as follows. Given a Hamiltonian cycle $H$ on $V \cup \{v_0\}$ as a part of the input, the algorithm first uses the AG-ITP algorithm to generate a solution for unit-demand $k/2$-CVRP with a weight of at most $(4/k)\Delta + (1 - 2/k)w(H)$. Then, in the initial solution, if the demand of one customer is delivered by two tours, then set all the demand to one tour to get a feasible solution for unsplittable $k$-CVRP without increasing the weight of the solution. By using Lemmas 12 and 13, we have

$$(4/k)\Delta + (1 - 2/k)\alpha w(H^*) \leq 2w(I^*) + (1 - 2/k)\alpha w(I^*) = (\alpha + 2 - 2\alpha/k)w(I^*).$$
The approximation ratio is $\alpha + 2 - 2\alpha/k$. Note that in this algorithm $k$ should be an even number. For odd $k$, there are two possible ways to solve it. The first one is to double the capacity and all the demands and call the above algorithm for $2k$. For this case, we may get a ratio of $(\alpha + 2 - \alpha/k)$. The second idea is to solve unit-demand $[k/2]$-CVRP in the first step. For the second method, we may get an even worse ratio.

Next, we give a refined algorithm for unsplittable CVRP based on ITP. The refined AG-UITP algorithm has two advances: it has the uniform form for both odd $k$ and even $k$, and the approximation ratio will be improved.

A customer is called big (resp., small) if the demand of it is greater than (resp., less than or equal to) $[k/2]$. Let $V_{\text{big}}$ (resp., $V_{\text{small}}$) denote the set of big customers (resp., small customers). Define $\Delta_{\text{big}} = \sum_{v_i \in V_{\text{big}}} d_i w(v_0, v_i)$, and $\Delta_{\text{small}} = \sum_{v_i \in V_{\text{small}}} d_i w(v_0, v_i)$.

The refined AG-UITP algorithm works as follows. First, for each big customer, we assign a single trivial tour. Second, we compute a Hamiltonian cycle $H_{\text{small}}$ on small customers and $v_0$ by shortcutting all the big customers in a Hamiltonian cycle $H$ of $G$. Third, we call the AG-ITP algorithm for unit-demand $([k/2] + 1)$-CVRP on $H_{\text{small}}$. Last, if the demand of one customer is delivered by two tours in the solution of the third step, then adjust all the demand to one tour: if one customer $v_1$ is split into two tours, say $T_1 = (v_0, \ldots, v_1, v_0)$, and $T_2 = (v_0, v_1, \ldots, v_0)$, we simply let $T_2$ deliver all the demand for $v_1$ without exceeding the capacity and delete $v_1$ from $T_1$: if one customer $v_1$ is split into three tours, say $T_1 = (v_0, \ldots, v_1, v_0), T_2 = (v_0, v_1, v_0)$, and $T_3 = (v_0, v_1, \ldots, v_0)$, we simply take $T_2$ as a trivial tour delivering all the demand of $v_1$, and remove $v_1$ from $T_1$ and $T_2$.

**Lemma 41.** Given a Hamiltonian cycle $H$ on $V \cup \{v_0\}$ as a part of the input, for unsplittable $k$-CVRP with any $k \geq 3$, the refined AG-UITP algorithm uses $O(nk)$ time to output a feasible solution of weight at most

\[
\frac{2}{[k/2] + 1} \Delta + \left(1 - \frac{1}{[k/2] + 1}\right) w(H).
\]

**Proof.** First, we analyze the quality of the solution before modifying the demands of customers.

For all big customers, we assign a single trivial tour. The total weight is at most $\frac{2}{[k/2] + 1} \Delta_{\text{big}}$. For all small customers, we can get a Hamiltonian cycle $H_{\text{small}}$ by shortcutting all the big customers in $H$. Then, we use the AG-ITP algorithm with a capacity of $[k/2] + 1$. By Lemma 18, the total weight is at most $\frac{2}{[k/2] + 1} \Delta_{\text{small}} + (1 - \frac{1}{[k/2] + 1}) w(H_{\text{small}})$. Note that $w(H_{\text{small}}) \leq 2 \Delta_{\text{small}}$ and $w(H_{\text{small}}) \leq w(H)$ by the triangle inequality. We have $\frac{2}{[k/2] + 1} \Delta_{\text{small}} + (1 - \frac{1}{[k/2] + 1}) w(H_{\text{small}}) \leq \frac{2}{[k/2] + 1} \Delta_{\text{small}} + (1 - \frac{1}{[k/2] + 1}) w(H)$. Therefore, the total weight is bounded by $\frac{2}{[k/2] + 1} (\Delta_{\text{big}} + \Delta_{\text{small}}) + (1 - \frac{1}{[k/2] + 1}) w(H) = \frac{2}{[k/2] + 1} \Delta + (1 - \frac{1}{[k/2] + 1}) w(H)$.

Next, we analyze the modification in the last step of our algorithm. If a customer is split into three tours, the modification clearly erases an infeasible case without increasing the weight. If a customer $v_j$ is split into two consecutive tours $T_1 = (v_0, v_h, \ldots, v_j, v_0)$ and $T_2 = (v_0, v_i, \ldots, v_j, v_0)$ in the clockwise direction of the Hamiltonian cycle, we modify them by letting $T_1^* = (v_0, v_h, \ldots, v_{i-1}, v_0)$ and $T_2^* = (v_0, v_i, \ldots, v_j, v_0)$. Then, we show the new itinerary is a feasible solution with a non-increasing weight for unsplittable $k$-CVRP.

By the triangle inequality, it holds $w(T_1^*) + w(T_2^*) \leq w(T_1) + w(T_2)$. So, the weight is non-increasing. To prove the feasibility, we only need to prove that the total demand in $T_1^*$ and $T_2^*$ is bounded by $k$. For $T_1^*$, the total demand becomes less and hence is feasible. For $T_2^*$, the total demand is bigger than the total demand of $T_2$. Note that the tour $T_2$ has a total demand of at most $[k/2] + 1$. We only need to analyze the newly added demand. Note that any small customer
has a demand of at most $|k/2|$. Thus, the newly added demand is at most $|k/2| - 1$. Therefore, for $T_2$, the total demand is bounded by $(|k/2| + 1) + (|k/2| - 1) = k$. Thus, after modifying all two consecutive conflict tours, we can get a feasible itinerary with a non-increasing total weight. □

Using an $\alpha$-approximate Hamiltonian cycle on $V \cup \{v_0\}$, by Lemmas 12 and 41, the refined AG-UITP algorithm satisfies that $\frac{2}{k/2+1} \Delta + \left(1 - \frac{1}{k/2+1}\right)\alpha w(H^*) \leq \left(\alpha + \frac{k}{k/2+1} - \frac{k}{k/2}w(I^*)\right)$. Hence, the approximation ratio is $\alpha + \frac{k}{k/2+1} - \frac{k}{k/2}$ for any $k \geq 3$, which is even better than the ratio $\alpha + 2 - 2a/k - \Omega(1/k^3)$ in [16].

In the refined AG-UITP algorithm, we call the AG-ITP algorithm with a capacity of $[k/2]+1$ on the Hamiltonian cycle of small customers and then we can modify them by adjusting the demands. The idea can be used on a cycle packing of small customers. We can call the EX-ITP algorithm with a capacity of $[k/2]+1$ on the cycle packing of small customers and then modify the conflicting tours in the same way. The algorithm is called the EX-UITP algorithm. By Lemma 21, we can get

**Lemma 42.** Given a cycle packing $C$ of $G[V]$ or $G[V \cup \{v_0\}]$ as a part of the input, assuming all customers are small customers, for unsplittable-demand $k$-CVRP with any $k \geq 3$, the EX-UITP algorithm uses $O(n^2)$ time to output a feasible solution with a weight of at most $2g\Delta + (1 - g)w(C)$, where $g = \max_{C \in C} \left(\frac{|C|}{\left(|k/2| + 1\right)}\right) / |C|$. Note that $|C| = \sum_{v_i \in C} d_i$ and all customers are small customers. The lemma can be used to analyze the algorithms based on a cycle packing, where we may first assign a trivial tour to each big customer and then call the EX-UITP algorithm on a cycle packing on small customers.

## 9 Some Structural Properties

The EX-UITP algorithm provides the framework of our algorithm. In this section, we prove some properties that may be used for some location structures.

For unsplittable $k$-CVRP, given a cycle $C$, we let $\Delta_C = \sum_{v_i \in C} d_i w(v_0, v_i)$ and $|C| = \sum_{v_i \in C} d_i$. Recall that the demand of a small customer is less than or equal to $|k/2|$ and the demand of a big customer is bigger than $|k/2|$.

**Lemma 43.** For unsplittable $k$-CVRP, if there is a cycle $C$ with $1 \leq |C| \leq k$, we can assign a single tour on $C$ with a weight of at most $2g\Delta_C + (1 - g)w(C)$, where $g = 1/|C|$. Proof. We first generate a tour by using the EX-ITP algorithm with a capacity of $k$. By Lemma 21, we know the weight is at most $2g\Delta_C + (1 - g)w(C)$, where $g = |C|/k / |C| = 1/|C|$ since $|C| \leq k$. However, the tour may be infeasible because a customer may appear more than once in the tour. In this case, we can modify the tour by shortcutting so that each customer appear exactly once. The new tour is feasible since $|C| \leq k$ and the weight is non-increasing by the triangle inequality. □

**Lemma 44.** For unsplittable $k$-CVRP, if there is a cycle $C$ with $|C| > k \geq 4$ and there is no big customer in $C$, we can assign $\left(\frac{|C| - 1}{\left(|k/2| + 1\right)}\right)$ tours on $C$ with a weight of at most $2g\Delta_C + (1 - g)w(C)$, where $g = \left(\frac{|C| - 1}{\left(|k/2| + 1\right)}\right) / |C|$. Proof. Let $m = g|C| = \left(\frac{|C| - 1}{\left(|k/2| + 1\right)}\right)$. Note that $m \geq 2$ since $|C| > k \geq 4$. We first show that we can assign $m$ tours on $C$ with a weight of at most $2g\Delta_C + (1 - g)w(C)$.

The first tour has a capacity of $\left[k/2\right] + 2$, the middle $m - 2$ tours has a capacity of $\left[k/2\right] + 1$, and the last tour has a capacity of $|C| - \left(\left[k/2\right] + 2\right) - (m - 2)\left(\left[k/2\right] + 1\right)$. Note that $0 \leq |C| - \left(\left[k/2\right] + 2\right) - (m - 2)\left(\left[k/2\right] + 1\right) \leq |k/2| + 1$ since $(|C| - 1)/\left(|k/2| + 1\right) \leq m = \left(\frac{|C| - 1}{\left(|k/2| + 1\right)}\right) <
with the demand of modified into feasible tours. We will consider the following two cases.

First, we call the EX-ITP algorithm with a capacity of \( \frac{k}{2} + 2 \) and the demand of \( v \) is at most \( 2g\Delta \). The total capacity is exactly \( |C| \). Hence, we can generate \( m \) such tours on \( C \). Similar to the HR-ITP algorithm, by considering \( |C| \) solutions and selecting the best one, we can get a solution on \( |C| \) with a weight of at most \((2m/|C|)\Delta + (1-m/|C|)w(C) = 2g\Delta + (1-g)w(C)\).

Then, we show these tours can be modified to a feasible solution with a non-increasing weight.

The idea is similar to the analysis in Lemma 41. Note that the tour has a capacity of at most \( \frac{k}{2} \) in Lemma 41. However, in the \( m \) tours, the first tour, denoted by \( T_1 \), has a capacity of \( \frac{k}{2} + 2 \), while the other tours, denoted by \( T_2, \ldots, T_m \) in the clockwise direction of the cycle \( C \), have a capacity of less than or equal to \( |k/2 + 1 \). The main difference is due to the tour \( T_1 \). We will use a similar modification in Lemma 41 while making sure the tour \( T_1 \) is feasible.

We consider the following three cases.

**Case 1.** There is no tour having a conflict with \( T_1 \): we can simply modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_i-1, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \). The modification is the same as that in Lemma 41. In this case, the tour \( T_1 \) does not have an effect on modification. For other tours, using a similar argument in Lemma 41, it is easy to see the modification is feasible.

If there exist conflict tours with \( T_1 \), using the same modification in Lemma 41 may cause the tour \( T_1 \) infeasible. For example, we let \( T_m = (v_0, \ldots, v_i, v_0) \) and \( T_1 = (v_0, v_i, \ldots, v_0) \), where the delivered demands of \( v_i \) in \( T_m \) and \( T_1 \) are \( \lfloor k/2 \rfloor - 1 \) and 1, respectively. Using the same modification in Lemma 41, we have \( T_m = (v_0, \ldots, v_i-1, v_0) \) and \( T_1 = (v_0, v_i, \ldots, v_0) \). The original demand of \( T_1 \) is \( \lceil k/2 \rceil + 2 \) and the newly added demand is \( \lfloor k/2 \rfloor - 1 \). The total demand is \( k + 1 \) and hence infeasible. Actually, in this case, we can modify the tours \( T_m \) and \( T_1 \) by letting \( T_m^* = (v_0, \ldots, v_i, v_0) \) and \( T_1^* = (v_0, v_i+1, \ldots, v_0) \) in a different direction. Hence, we will consider two following cases.

**Case 2.** There is only one tour having a conflict with \( T_1 \): if the tour is \( T_2 \), we modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_i-1, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \); if the tour is \( T_m \), we modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i+1, \ldots, v_j, v_0) \). The directions of these two modifications are opposite. The modified tour \( T_1 \) will always have a less demand and hence will always be feasible. For other tours, using a similar argument in Lemma 41, we know the modification is safe.

**Case 3.** There are two tours having conflicts with \( T_1 \): we can simply modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_i-1, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \). The modification is the same with that in Lemma 41. We only need to prove \( T_1^* \) is feasible. The original demand of \( T_1 \) is at most \( \lceil k/2 \rceil + 2 \). Since \( T_1 \) has a conflict with the tour \( T_m \), the newly added demand of \( T_1^* \) in the modification is at most \( \lfloor k/2 \rfloor - 1 \) since all small customers have a demand of at most \( \lfloor k/2 \rfloor \). Note that \( T_1 \) also has a conflict with the tour \( T_2 \), and hence, it reduces at least 1 demand according to the modification. The demand of \( T_1^* \) is at most \((\lfloor k/2 \rfloor + 2) + (\lfloor k/2 \rfloor - 1) - 1 = k \). Hence, the tour \( T_1 \) will always be feasible. For other tours, using a similar argument in Lemma 41, we know the modification is safe.

**Lemma 45.** For unsplittable \( k \)-CVRP, if there exists a cycle \( C \) with \( |C| > k \geq 4 \) and there is only one big customer and the demand of it is \( \lfloor k/2 \rfloor + 1 \) in \( C \), we can assign \( |C|/\lfloor (k/2) + 1 \rfloor \) tours on \( C \) with a weight of at most \( 2g\Delta + (1-g)w(C) \), where \( g = |C|/\lfloor (k/2) + 1 \rfloor/|C| \).

**Proof.** First, we call the EX-ITP algorithm with a capacity of \( \lfloor k/2 \rfloor + 1 \) on \( C \). By Lemma 21, we know the weight is at most \( 2g\Delta + (1-g)w(C) \).

Next, we modify the conflict tours. Note that the only difference is that there is one big customer with the demand of \( \lfloor k/2 \rfloor + 1 \). We only need to prove the tours involving the big customer can be modified into feasible tours. We will consider the following two cases.
Case 1. The big customer is not split: we can simply modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_j, v_0) \) and \( T_{i+1} = (v_0, v_l, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_{i-1}, v_0) \) and \( T_{i+1} = (v_0, v_l, \ldots, v_j, v_0) \). The modification is the same with that in Lemma 41. In this case, the big customer does not have an effect on modification. Hence, the modification is feasible.

Case 2. The big customer is split in two tours. Suppose the big customer is \( v_1 \) and is split in tours \( T_1 = (v_0, \ldots, v_1, v_0) \) and \( T_2 = (v_0, v_1, \ldots, v_0) \). The delivered demand of \( v_1 \) in \( T_1 \) (resp., \( T_2 \)) is denoted by \( x_1 \) (resp., \( x_2 \)). If \( x_1 \leq x_2 \), we modify every conflict tours \( T_i = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_{i-1}, v_0) \) and \( T_{i+1} = (v_0, v_i, \ldots, v_j, v_0) \) by letting \( T_i^* = (v_0, v_h, \ldots, v_i, v_0) \) and \( T_{i+1} = (v_0, v_{i+1}, \ldots, v_j, v_0) \). The directions of these two modifications are opposite. Due to symmetry, we only consider \( x_1 \leq x_2 \). The big customer is entirely contained in the tour \( T_2^* \) after modification. We only need to prove \( T_2^* \) is feasible. The demand of the original tour \( T_2 \) has a demand of at most \([k/2]+1\). The newly added demand is \( x_1 \leq \lfloor k/2 \rfloor - 1 \) since \( x_1 + x_2 \leq \lfloor k/2 \rfloor + 1 \), \( x_1 \leq x_2 \) and \( k \geq 4 \). Therefore, the demand of \( T_2^* \) is at most \((\lfloor k/2 \rfloor +1) + (\lfloor k/2 \rfloor -1) = k \) and hence feasible.

10 Improvements on Unsplittable 3-CVRP and 4-CVRP

In this section, we demonstrate that the techniques in the above two sections can be used to improve unsplittable 3-CVRP and 4-CVRP.

10.1 Unsplittable 3-CVRP

We improve the ratio from \( 3/2 + \ln(3/2) < 1.906 \) to \( 3/2 = 1.500 \) for unsplittable 3-CVRP. Note that this ratio is already the ratio for splittable 3-CVRP. Any further improvement on unsplittable 3-CVRP will also improve splittable 3-CVRP.

We can assume that no customer’s demand is greater than 2. Thus, the demand of each customer is either 1 or 2. Our algorithm contains three steps. First, we find a cycle packing \( C^* \) on all customers \( V \), where \( |C_i| = \sum_{v_j \in C_i} d_j \geq 3 \) for each \( C_i \in C^* \). The cycle packing \( C^* \) is computed in the following way: we consider the instance as a unit-demand instance \( G' \) by taking each customer \( v_j \) as \( d_j \) customers with the unit-demand as the same place. Then, we use the polynomial time algorithm in [32] to find a minimum cycle packing \( C^* \) in \( G'[V] \). Note that \( C^* \) may not be a cycle packing for the original graph \( G[V] \). For a vertex \( v_i \), there will be \( 2d_i \) edges incident on it in \( C^* \). However, by shortcutting in \( C^* \), we can get a cycle packing \( C^{**} \) on \( V \) such that \( w(C^{**}) \leq w(C^*) \). Second, we will deal with all 2-demand customers: if a cycle \( C_i \in C^{**} \) contains no 1-demand customer or at least two 1-demand customers, assign a trivial tour for each 2-demand customer in \( C_i \) and obtain a small cycle \( C'_i \) from \( C_i \) by shortcutting all 2-demand customers (having been delivered); if a cycle \( C \in C^{**} \) contains exactly one 1-demand customer \( u \), then there is at least one 2-demand customer \( u' \) in \( C \), and we assign a tour \( T \) visiting the two customers \( u \) and \( u' \) only and a trivial tour for each other 2-demand customer in \( C \) except \( u' \). Now we get a cycle packing \( C' = \{C'_i\} \) on all customers that have not been delivered yet. These customers are 1-demand customers. Third, we apply the EX-ITP algorithm on \( C' \).

We analyze the quality of this solution. We partition the tours constructed in the algorithm into three parts.

The first part contains all trivial tours visiting one 2-demand customer. Let \( V_1 \) denote this set of 2-demand customers and \( \Delta_1 = \sum_{v_i \in V_1} d_i w(v_0, v_i) = 2 \sum_{v_i \in V_1} w(v_0, v_i) \). Thus, the total weight of the tours in the first part is \( \Delta_1 \).
Let \( C^{*}_{2} \subseteq C^{*} \) be the set of cycles containing exact one 1-demand customer. For each cycle \( C \in C^{*}_{2} \), the algorithm will construct a tour \( T = (v_0, u, u', v_0) \) visiting one 1-demand customer \( u \) and one 2-demand customer \( u' \) in \( C \) in the second step of the algorithm. The second part consists of all these tours, and we let \( V_2 \) denote the set of customers visited by these tours. Let \( \Delta_2 = \sum_{v_i \in V_2} d_i w(v_0, v_i) \). The weight of \( T_0 = w(v_0, u) + w(v_0, u') + w(u, u') \), where \( w(u, u') \) is at most the half of \( w(C) \). Thus, the total weight of the tours in the second part is at most \( \Delta_2 + \frac{1}{2} \sum_{\Delta \in C^{*}_{2}} w(C) \).

The third part is the tours generated by the EX-ITP algorithm in the last step. Let \( V_3 \) denote the set of 1-demand customers visiting in the tours in the third part and \( \Delta_3 = \sum_{v_i \in V_3} d_i w(v_0, v_i) = \sum_{v_i \in V_3} w(v_0, v_i) \). Note that for each cycle \( C' \in C' \) there are at least two 1-demand customers and hence we have \( |C'| \geq 2 \). By Lemma 21, we know that the weight of tours in the third part is at most \( \Delta_3 + \frac{1}{2} w(C') (g \leq \frac{1}{2} \text{ in Lemma 21}) \), where \( w(C') \leq w(C^{*} \setminus C^{*}_{2}) \).

Recall that \( w(C^{*}_2) \leq w(C^{*}) \). By adding the three parts together, we know the weight of the solution is at most

\[
\Delta_1 + \left( \Delta_2 + \frac{1}{2} w(C^{*}_2) \right) + \left( \Delta_3 + \frac{1}{2} w(C') \right) \leq \Delta + \frac{1}{2} w(C^{*}_2) + \frac{1}{2} w(C^{*} \setminus C^{*}_{2}) = \Delta + \frac{1}{2} w(C^{*}) \leq \Delta + \frac{1}{2} w(C^{*}).
\]

By Lemmas 13 and 17, we have

\[
\Delta + \frac{1}{2} w(C^{*}) \leq \left( \chi + \frac{1}{2} \right) w(I^{*}) + \frac{1}{2} \min\{2(1 - \chi), 1\} w(I^{*}) \leq \frac{3}{2} w(I^{*}).
\]

Note that we have \( \chi = \frac{1}{2} \) in the worst case. Therefore, we can get that

**Lemma 46.** For unsplittable 3-CVRP, there is a \( \frac{3}{2} \)-approximation algorithm.

### 10.2 Unsplittable 4-CVRP

We improve the approximation ratio from \( 5/3 + \ln(4/3) < 1.955 \) to \( 7/4 = 1.750 \) for unsplittable 4-CVRP. We can assume that the demand of each customer is 1, 2 or 3. Our algorithm will first compute a cycle packing \( C_{\text{mod} \ 2} \), where \( \text{mod} \ 2 \) means the total demand of customers in each cycle in the packing is even. We will simply call this cycle packing a mod-2-cycle packing. It is computed in the following way: we consider the instance as a unit-demand instance \( G' \) by taking each customer \( v_i \) as \( d_i \) customers with the unit-demand as the same place. First, we compute a minimum perfect matching \( M^{*} \) in \( G'[V'] \) and then compute a minimum perfect matching \( M^{**} \) in \( G'[V'] \setminus M^{*} \). Let \( C'_{\text{mod} \ 2} = M^{*} \cup M^{**} \). Note that \( C'_{\text{mod} \ 2} \) may not be a cycle packing for the original graph \( G[V] \).

Similar to the analysis in previous section, by shortcutting in \( C'_{\text{mod} \ 2} \) on \( V \). We will use this mod-2-cycle packing \( C_{\text{mod} \ 2} \) for unsplittable 4-CVRP. We have that \( w(C_{\text{mod} \ 2}) \leq w(C'_{\text{mod} \ 2}) \leq w(C'_{\text{mod} \ 2}) \) by Lemma 25 and it holds that \( |C| = \sum_{v_j \in C} d_j \geq 4 \) and \( |C| \mod 2 = 0 \) for each \( C \in C_{\text{mod} \ 2} \). The next steps of our algorithm are based on \( C_{\text{mod} \ 2} \). For different types of cycles in \( C_{\text{mod} \ 2} \), we have different operations. Consider a cycle \( C \) in \( C_{\text{mod} \ 2} \). Let \( x \) be the sum of the demands of all 1-demand and 2-demand customers in \( C \). We classify the cycles in \( C_{\text{mod} \ 2} \) into several types \( C_{<x>} \) according to the value of \( x \). We will use \( V_i \) to denote the set of customers contained in a cycle in \( C_{<i>} \) and let \( \Delta_i = \sum_{v_j \in V_i} d_j w(v_0, v_j) \). Note that we can get \( w(C_{<i>}) \leq 2\Delta_i \) by the triangle inequality. Then, for any \( g_1 \) and \( g_2 \) with \( 0 < g_1 \leq g_2 \leq 1 \), we have

\[
2g_1 \Delta_i + (1 - g_1) w(C_{<i>}) \leq 2g_2 \Delta_i + (1 - g_2) w(C_{<i>}).
\]

We may use the above inequality frequently. Now, we are ready to introduce our algorithm.
**Type 0:** Cycles with \( x = 0 \), the set of which is denoted by \( C_{<0>} \). For each cycle \( C \in C_{<0>} \), there is no 1-demand or 2-demand customer and we assign a trivial tour for each 3-demand customer in \( C \). The weight of trivial tours is \( \frac{2}{3}\Delta_0 \leq \frac{3}{4}\Delta_0 + \frac{5}{8}w(C_{<0>}). \)

**Type 1:** Cycles with \( x \bmod 3 = 0 \) and \( x \geq 3 \), the set of which is denoted by \( C_{<1>} \). For each cycle \( C \in C_{<1>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by shortcutting all 3-demand customers. The set of cycles \( C' \) is denoted by \( C'_{<1>} \). Then we apply the EX-UITP algorithm on \( C'_{<1>} \).

We analyze the weight of the tours assigned in this step. Let \( V' \) be the set of 1-demand and 2-demand customers in \( V_1 \) and \( V''_1 = V_1 \setminus V'_1 \) be the set of 3-demand customers in \( V_1 \). Let

\[
\Delta'_1 = \sum_{v_j \in V'_1} d_j w(v_0, v_j) \quad \text{and} \quad \Delta''_1 = \sum_{v_j \in V''_1} d_j w(v_0, v_j).
\]

Note that for each cycle \( C' \in C'_{<1>} \) we have \( |C'| = x \geq 3 \). For the tours generated by the EX-UITP algorithm on \( C'_{<1>} \), by Lemma 42, the total weight is at most \( \frac{3}{4}\Delta'_1 + \frac{3}{8}w(C'_{<1>}) \leq \frac{3}{4}\Delta'_1 + \frac{5}{8}w(C_{<1>}) \leq \frac{3}{4}\Delta'_1 + \frac{5}{8}w(C_{<1>}) \) by the triangle inequality. For trivial tours visiting a 3-demand customer, the total weight is \( \frac{3}{4}\Delta'_1 \leq \frac{3}{4}\Delta''_1 \). In total, the weight is at most \( \frac{3}{4}\Delta'_1 + \frac{3}{8}w(C'_{<1>}) = \frac{3}{4}\Delta_1 + \frac{5}{8}w(C_{<1>}). \)

**Type 2:** Cycles with \( x \bmod 3 = 1 \) and \( x \geq 16 \), the set of which is denoted by \( C_{<2>} \). For each cycle \( C \in C_{<2>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by shortcutting all 3-demand customers. The set of cycles \( C' \) is denoted by \( C'_{<2>} \). Then we apply the EX-UITP algorithm on \( C'_{<2>} \).

By the similar analysis for the type 1, we can get that the weight of tours in this step is at most \( \frac{3}{4}\Delta_2 + \frac{5}{8}w(C_{<2>}) \).

**Type 3:** Cycles with \( x \bmod 3 = 2 \) and \( x \geq 8 \), the set of which is denoted by \( C_{<3>} \). For each cycle \( C \in C_{<3>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by shortcutting all 3-demand customers. The set of cycles \( C' \) is denoted by \( C'_{<3>} \). Then we apply the EX-UITP algorithm on \( C'_{<3>} \).

By the similar analysis for the type 1, we can get that the weight of tours in this step is at most \( \frac{3}{4}\Delta_3 + \frac{5}{8}w(C_{<3>}) \).

We still have several cases, where \( x \) is a small constant.

**Type 4:** Cycles with \( x = 1 \), the set of which is denoted by \( C_{<4>} \). Each cycle \( C \in C_{<4>} \) contains exactly one 1-demand customer \( u \) and at least one 3-demand customer \( u' \) since the total demand in the cycle is even. We assign a tour \( T \) visiting two customers \( u \) and \( u' \) only and a trivial tour for each other 3-demand customer in \( C \) except \( u' \). We can obtain a small cycle \( C' \) from \( C \) by shortcutting the 3-demand customers except \( u' \). Let \( V'_4 \) be the set of customers contained in \( T \) and let \( \Delta'_4 = \sum_{v_j \in V'_4} d_j w(v_0, v_j) \). Note that \( |C'| = d_u + d_{u'} = 1 + 3 = 4 \). By Lemma 43, we can assign the tour \( T \) with a weight of at most \( \frac{1}{4}\Delta'_4 \leq \frac{1}{4}\Delta'_4 + \frac{3}{8}w(C') \leq \frac{1}{4}\Delta'_4 + \frac{5}{8}w(C') \leq \frac{1}{4}\Delta'_4 + \frac{5}{8}w(C) \) by the triangle inequality. Let \( V''_4 = C \setminus V'_4 \) and \( \Delta''_4 = \sum_{v_j \in V''_4} d_j w(v_0, v_j) \). For trivial tours visiting the other 3-demand customers in \( V''_4 \), the total weight is \( \frac{3}{4}\Delta''_4 \leq \frac{3}{4}\Delta''_4 \leq \frac{3}{4}\Delta''_4 \). Thus, by summing all cycles in \( C_{<4>} \), the weight of tours in this step is at most \( \frac{3}{4}\Delta_4 + \frac{5}{8}w(C_{<4>}) \).

**Type 5:** Cycles with \( x = 4 \), the set of which is denoted by \( C_{<5>} \). For each cycle \( C \in C_{<5>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by shortcutting all 3-demand customers. Then we assign the last tour \( T \) according to \( C' \). Let \( V'_5 \) be the set of customers contained in \( T \) and let \( \Delta'_5 = \sum_{v_j \in V'_5} d_j w(v_0, v_j) \). By Lemma 43, we can assign the tour \( T \) with a weight of at most \( \frac{1}{4}\Delta'_5 + \frac{3}{8}w(C') \leq \frac{1}{4}\Delta'_5 + \frac{5}{8}w(C') \leq \frac{1}{4}\Delta'_5 + \frac{5}{8}w(C) \) by the triangle inequality. Let \( V''_5 = C \setminus V'_5 \) and \( \Delta''_5 = \sum_{v_j \in V''_5} d_j w(v_0, v_j) \). For trivial tours visiting the 3-demand customers in \( V''_5 \), the total weight is \( \frac{3}{4}\Delta''_5 \leq \frac{3}{4}\Delta''_5 \). Thus, the weight of tours in this step is at most \( \frac{3}{4}\Delta_5 + \frac{5}{8}w(C_{<5>}) \).

**Type 6:** Cycles with \( x = 7 \), the set of which is denoted by \( C_{<6>} \). For each cycle \( C \in C_{<6>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by
shortcutting all 3-demand customers. For \( C' \), we assign two tours \( T_1 \) and \( T_2 \) with the capacity 4 and 3 respectively. Define \( V'_6, \Delta'_6, V''_6 \) and \( \Delta''_6 \) in the similar way as above. By Lemma 44, we can assign the tours \( T_1 \) and \( T_2 \) with a total weight of at most \( w(T_1) + w(T_2) \leq \frac{3}{4}\Delta'_6 + \frac{5}{8} w(C') \leq \frac{3}{4}\Delta'_6 + \frac{5}{8} w(C) \) by the triangle inequality. For trivial tours visiting the 3-demand customers in \( V''_6 \), the total weight is \( \frac{5}{8} \Delta''_6 \leq \frac{3}{4} \Delta'_6 \). Thus, the weight of tours in this step is at most \( \frac{3}{4} \Delta'_6 + \frac{5}{8} w(C_{<6}) \).

**Type 7:** Cycles with \( x = 10 \), the set of which is denoted by \( C_{<7>}. \) For each cycle \( C \in C_{<7>} \), we assign a trivial tour for each 3-demand customer in \( C \) and obtain a small cycle \( C' \) from \( C \) by shortcutting all 3-demand customers. For \( C' \), we assign three tours \( T_1, T_2 \) and \( T_3 \) with the capacity 4, 3 and 3, respectively. Define \( V'_7, \Delta'_7, V''_7 \) and \( \Delta''_7 \) in the similar way as above. By Lemma 44, we can assign the tours \( T_1, T_2 \) and \( T_3 \) with a total weight of at most \( \frac{3}{4} \Delta'_7 + \frac{5}{8} w(C') \leq \frac{3}{4} \Delta'_7 + \frac{5}{8} w(C) \leq \frac{3}{4} \Delta'_7 + \frac{5}{8} w(C) \) by the triangle inequality. For trivial tours visiting the 3-demand customers, the total weight is \( \frac{5}{8} \Delta''_7 \leq \frac{3}{4} \Delta'_7 \). Thus, the weight of tours in this step is at most \( \frac{3}{4} \Delta'_7 + \frac{5}{8} w(C_{<7>}) \).

**Type 8:** Cycles with \( x = 5 \), the set of which is denoted by \( C_{<8>}. \) Each cycle \( C \in C_{<8>} \) contains at least one 3-demand customer \( u' \) because the total demand in the cycle \( C \) is even. We assign a trivial tour for each other 3-demand customer in \( C \) except \( u' \). We get a small cycle \( C' \) by shortcutting the 3-demand customers having been delivered. Define \( V'_8, \Delta'_8, V''_8 \) and \( \Delta''_8 \) in the similar way as above. For \( C' \), the total demand of which is 5+3=8, by Lemma 45, we can generate a solution on \( C' \) with a weight of at most \( \frac{3}{4} \Delta'_8 + \frac{5}{8} w(C') \leq \frac{3}{4} \Delta'_8 + \frac{5}{8} w(C) \) by the triangle inequality. For trivial tours visiting the 3-demand customers, the total weight is \( \frac{5}{8} \Delta''_8 \leq \frac{3}{4} \Delta'_8 \). Thus, the weight of tours in this step is at most \( \frac{3}{4} \Delta'_8 + \frac{5}{8} w(C_{<8>}) \).

**Type 9:** Cycles with \( x = 13 \), the set of which is denoted by \( C_{<9>}. \) Each cycle \( C \in C_{<9>} \) contains at least one 3-demand customer \( u' \) because the total demand in the cycle \( C \) is even. We assign a trivial tour for each other 3-demand customer in \( C \) except \( u' \). We get a small cycle \( C' \) containing only 1-demand and 2-demand customers and \( u' \) by shortcutting the 3-demand customers having been delivered. Define \( V'_9, \Delta'_9, V''_9 \) and \( \Delta''_9 \) in the similar way as above. For \( C' \), the total demand of which is 13+3=16, by Lemma 45, we can generate a solution on \( C' \) with a weight of at most \( \frac{3}{4} \Delta'_9 + \frac{5}{8} w(C') \leq \frac{3}{4} \Delta'_9 + \frac{5}{8} w(C) \) by the triangle inequality. For trivial tours visiting the 3-demand customers, the total weight is \( \frac{5}{8} \Delta''_9 \leq \frac{3}{4} \Delta'_9 \). Thus, the weight of tours in this step is at most \( \frac{3}{4} \Delta'_9 + \frac{5}{8} w(C_{<9>}) \).

**Type 10:** Cycles with \( x = 2 \), the set of which is denoted by \( C_{<10>}. \) Each cycle \( C \in C_{<10>} \) contains one 2-demand customer \( u \) or two 1-demand customers \( u_1 \) and \( u_2 \) and at least two 3-demand customers \( u' \) and \( u'' \) because the total demand in the cycle \( C \) is even. We assign a trivial tour for each other 3-demand customer in \( C \) except \( u' \) and \( u'' \). We get a small cycle \( C' \) containing only \( \{u, u', u''\} \) or \( \{u_1, u_2, u', u''\} \) after shortcutting the 3-demand customers having been delivered. For \( C' \), we assign three tours \( T_1, T_2 \) and \( T_3 \) with the capacity 3, 3 and 2, respectively. Define \( V'_{10}, \Delta'_{10}, V''_{10} \) and \( \Delta''_{10} \) in the similar way as above. Note that \( |C'| = 8 \). By considering all cases, we can easily get that we can assign the tours \( T_1, T_2 \) and \( T_3 \) with a total weight of at most \( \frac{3}{4} \Delta'_{10} + \frac{5}{8} w(C') \leq \frac{3}{4} \Delta'_{10} + \frac{5}{8} w(C) \) by the triangle inequality. For trivial tours visiting the other 3-demand and 4-demand customers in \( V''_{10} \), the total weight is at most \( \frac{3}{4} \Delta''_{10} \leq \frac{3}{4} \Delta'_{10} \). Thus, by summing all cycles in \( C_{<10>} \), the weight of tours in this step is at most \( \frac{3}{4} \Delta'_{10} + \frac{5}{8} w(C_{<10>}) \).

Recall that \( w(C_{\text{mod} 2}) \leq w(C'_{4}). \)

**Lemma 47.** Given the mod-2-cycle packing \( C_{\text{mod} 2} \), for unsplittable 4-CVRP, there is a polynomial-time algorithm to generate an itinerary \( I \) such that \( w(I) \leq \frac{3}{4} \Delta + \frac{5}{8} w(C_{\text{mod} 2}) \leq \frac{3}{4} \Delta + \frac{5}{8} w(C'_{4}). \)

By Lemmas 13 and 17, we have

\[
\frac{3}{4} \Delta + \frac{5}{8} w(C'_{4}) \leq \frac{3}{4} \left( \chi + \frac{1}{2} \right) w(I^*) + \frac{5}{8} \min\{2(1 - \chi), 1\} w(I^*) \leq \frac{7}{4} w(I^*).
\]
Note that we have $\chi = \frac{1}{2}$ in the worst case. Therefore, we can get that

**Theorem 48.** For unsplittable 4-CVRP, there is a $\frac{7}{4}$-approximation algorithm.

## 11 An Improvement for Unsplittable $k$-CVRP

In this section, we consider unsplittable $k$-CVRP. We will give an $(\alpha + 1 + \ln 2 - 2\alpha/k - \Theta(1/k))$-approximation algorithm for unsplittable $k$-CVRP, where $1 \leq \alpha \leq 3/2$ is the ratio for metric TSP. Our algorithm will use several techniques. The first one is the LP-based technique used in [25].

### 11.1 An LP-based Algorithm

When $k$ is a constant, there are at most $n_k = n^{O(1)}$ different tours in total, denoted by $\mathcal{T}$. The idea of the LP-based algorithm is to define a variable $x_T$ for each tour $T \in \mathcal{T}$. Then, we can solve the following LP in $n_k = n^{O(1)}$ time.

\[
\begin{align*}
\text{minimize:} & \quad \sum_{T \in \mathcal{T}} w(T) \cdot x_T \\
\text{subject to:} & \quad \sum_{T \in \mathcal{T}: v \in T} x_T \geq 1 \quad \forall \ v \in V \\
& \quad x_T \geq 0 \quad \forall \ T \in \mathcal{T}
\end{align*}
\]

Note that the idea of the above LP is the same as the LP of set cover. Each vertex $v \in V$ can be seen as an element. Each tour $T \in \mathcal{T}$ can be seen as a set which contains all vertices $v \in T$ and has a weight of $w(T)$. We need to find a minimum weight collection of sets to cover all vertices.

Next, we give a simple analysis of the LP-based algorithm. The details can be seen in [25].

Using a constant $\gamma \geq 0$, we can make a randomized rounding for each tour with a probability of $\min\{1, \gamma \cdot x_T\}$. The chosen tours will form a partial itinerary $I'$ with an expected weight of $\mathbb{E}[w(I')] \leq \gamma \cdot w(I^*)$. The customer in $I'$ is denoted by $V'$. Then, for the undelivered customers in $V'' = V \setminus V'$, we use the UITP algorithm to compute an itinerary $I''$ such that $\mathbb{E}[w(I'')] \leq e^{-\gamma} \cdot (4/k) \Delta + (1 - 2/k) w(H'')$ where $H''$ is a Hamiltonian cycle on $V'' \cup \{v_0\}$. Note that given a Hamiltonian cycle $H$ on $V \cup \{v_0\}$ we can get a Hamiltonian cycle $H''$ on $V'' \cup \{v_0\}$ by shortcutting such that $w(H'') \leq w(H)$. We have $\mathbb{E}[I] = \mathbb{E}[I' \cup I''] \leq \gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k) \Delta + (1 - 2/k) w(H)$.

By combining the LP-based method with the refined AG-UITP algorithm, we can an LP-UITP algorithm.

**Lemma 49.** Given a Hamiltonian cycle $H$ on $V \cup \{v_0\}$, for unsplittable $k$-CVRP with any $k \geq 3$ and any constant $\gamma \geq 0$, the LP-UITP algorithm can use $n^{O(k)} = n^{O(1)}$ time to generate a solution with a weight at most

\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{k/2 + 1} \Delta + \left(1 - \frac{1}{k/2 + 1}\right) w(H).
\]

Note that the LP-UITP algorithm will become the refined AG-UITP algorithm if setting $\gamma = 0$.

### 11.2 Trade-off Between Two Results

Next, we show that by using Lemma 49 and two initial Hamiltonian cycles, we can get the desired ratio by doing trade-off. The main idea is that. We use two initial Hamiltonian cycles $H$ and $H_{CS}$
on \( V \cup \{v_0\} \), where \( H \) is an arbitrary \( \alpha \)-approximate solution to metric TSP and \( H_{CS} \) is the cycle obtained by the Christofides-Serdyukov algorithm, as the input to run the LP-UITP algorithm. We choose the better one. The following theorem shows the ratio of the algorithm.

**Theorem 50.** Given an \( \alpha \)-approximate Hamiltonian cycle \( H \) on \( V \cup \{v_0\} \) with \( 1 \leq \alpha \leq 3/2 \), then there is an approximation algorithm for unsplittable \( k \)-CVRP such that

- When \( 3 \leq k \leq 5 \), the ratio is \( \frac{2[k/2]+1}{[k/2]+1} + \ln \frac{k}{[k/2]+1} \).
- When \( k = 6 \) and \( \frac{7}{6} \leq \alpha \leq \frac{5}{3} \), the ratio is \( \frac{15}{8} + \ln \frac{4}{3} < 2.163 \).
- When \( k = 7 \) and \( \frac{17}{12} \leq \alpha \leq \frac{3}{2} \), the ratio is \( \frac{33}{16} + \ln \frac{4}{3} < 2.351 \).
- When \( k = 6 \) and \( 1 \leq \alpha \leq \frac{7}{6} \) or \( k = 7 \) and \( 1 \leq \alpha \leq \frac{17}{12} \) or \( k \geq 8 \), the ratio is \( \frac{\alpha+1}{\lfloor k/2 \rfloor+1} + \ln \frac{k-4(\alpha-1)}{\lfloor k/2 \rfloor+1} \).

**Proof.** Given two Hamiltonian cycles \( H \) and \( H_{CS} \), we can use the LP-UITP algorithm to generate two solutions. We will select the better one. In the worst case, we assume \( \frac{2}{\lfloor k/2 \rfloor+1} (\chi + \frac{k-2}{2}) \geq 1 \).

We will show that the ratio will be better if \( \frac{2}{\lfloor k/2 \rfloor+1} (\chi + \frac{k-2}{2}) < 1 \).

Using the Hamiltonian cycle \( H \), by Lemmas 12, 13, and 49, the LP-UITP algorithm can generate an itinerary \( I_1 \) such that

\[
w(I_1) \leq \min_{\gamma \geq 0} \left\{ \gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{\lfloor k/2 \rfloor+1} + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) w(H) \right\} \\
\leq \min_{\gamma \geq 0} \left\{ \gamma + e^{-\gamma} \cdot \frac{2}{\lfloor k/2 \rfloor+1} \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \right\} w(I^*) \\
\leq \left( 1 + \ln \frac{2}{\lfloor k/2 \rfloor+1} + \ln \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \right) w(I^*). 
\]

The approximation ratio is \( 1 + \ln \frac{2}{\lfloor k/2 \rfloor+1} + \ln \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \).

Using the Hamiltonian cycle \( H_{CS} \), by Lemmas 12, 16, 13, 14, and 49, the LP-UITP algorithm can generate an itinerary \( I_2 \) such that

\[
w(I_2) \leq \min_{\gamma \geq 0} \left\{ \gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{\lfloor k/2 \rfloor+1} + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) w(H_{CS}) \right\} \\
\leq \min_{\gamma \geq 0} \left\{ \gamma + e^{-\gamma} \cdot \frac{2}{\lfloor k/2 \rfloor+1} \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \right\} w(I^*) \\
\leq \left( 1 + \ln \frac{2}{\lfloor k/2 \rfloor+1} + \ln \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \right) w(I^*). 
\]

The approximation ratio is \( 1 + \ln \frac{2}{\lfloor k/2 \rfloor+1} + \ln \left( \chi + \frac{k-2}{2} \right) + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) \alpha \).

We can make a trade-off between them. The results can be briefly summarized as follows.

**Case 1.** When \( 3 \leq k \leq 5 \), we have \( \chi = 1 \) in the worst case, and the ratio is

\[
1 + \ln \frac{k}{\lfloor k/2 \rfloor+1} + \left( 1 - \frac{1}{\lfloor k/2 \rfloor+1} \right) = \frac{2[k/2]+1}{[k/2]+1} + \ln \frac{k}{[k/2]+1}. 
\]

**Case 2.** When \( k = 6 \) and \( \frac{7}{6} \leq \alpha \leq \frac{5}{3} \), we have \( \chi = \frac{2}{3} \), and the ratio is \( \frac{15}{8} + \ln \frac{4}{3} < 2.163 \).

**Case 3.** When \( k = 7 \) and \( \frac{17}{12} \leq \alpha \leq \frac{3}{2} \), we have \( \chi = \frac{1}{6} \), and the ratio is \( \frac{33}{16} + \ln \frac{4}{3} < 2.351 \).
Case 4. When \( k = 6 \) and \( 1 \leq \alpha \leq \frac{7}{6} \), \( k = 7 \) and \( 1 \leq \alpha \leq \frac{17}{12} \), or \( k \geq 8 \), we have \( \chi = 3 - 2\alpha \), and the ratio is
\[
1 + \ln \frac{k - 4(\alpha - 1)}{\left\lceil k/2 \right\rceil + 1} + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) \alpha = \frac{(\alpha + 1)\left\lceil k/2 \right\rceil + 1}{\left\lceil k/2 \right\rceil + 1} + \ln \frac{k - 4(\alpha - 1)}{\left\lceil k/2 \right\rceil + 1}.
\]

Now we consider the case \( \frac{2}{\left\lceil k/2 \right\rceil + 1} \left( \chi + \frac{k - 2}{2} \right) < 1 \). It is sufficient to show that we can get a better ratio in this case.

On one hand, setting \( \gamma = 0 \), by Lemmas 12 and 13, we get
\[
w(I_1) \leq \frac{2}{\left\lceil k/2 \right\rceil + 1} \Delta + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) w(H)
\leq \left( \frac{2}{\left\lceil k/2 \right\rceil + 1} \left( \chi + \frac{k - 2}{2} \right) + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) \alpha \right) w(I^*)
\leq \left( 1 + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) \alpha \right) w(I^*).
\]
The ratio is better than the ratio in Case 4. Moreover, when \( k = 6 \) or \( k = 7 \), the ratio is at most
\[
1 + (1 - \frac{1}{4}) \cdot \frac{3}{2} < 2.125
\]
which is also better than the ratios in Cases 2 and 3.

On the other hand, we have \( 0 \leq \chi \leq \frac{\left\lceil k/2 \right\rceil + 3}{2} \leq \frac{k}{2} \) when \( 3 \leq k \leq 5 \). Setting \( \gamma = 0 \), when \( 3 \leq k \leq 5 \), by Lemmas 13 and 16, we get
\[
w(I_2) \leq \frac{2}{\left\lceil k/2 \right\rceil + 1} \Delta + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) w(H_{CS})
\leq \left( \frac{2}{\left\lceil k/2 \right\rceil + 1} \left( \chi + \frac{k - 2}{2} \right) + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) \frac{3 - \chi}{2} \right) w(I^*)
\leq \frac{5}{4} + \frac{k - 9}{4} \left\lceil k/2 \right\rceil \left( 1 + \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) w(I^*),
\]
where the last inequality follows from \( \chi \leq \frac{1}{2} \), \( 3 \leq k \leq 5 \), and \( (\frac{2}{\left\lceil k/2 \right\rceil + 1} - \frac{1}{2}(1 - \frac{1}{\left\lceil k/2 \right\rceil + 1})) \chi \leq (\frac{2}{\left\lceil k/2 \right\rceil + 1} - \frac{3}{2}(1 - \frac{1}{\left\lceil k/2 \right\rceil + 1})) \cdot \frac{1}{2} \). When \( k = 3 \), \( k = 4 \), and \( 5 \), the new ratios are 1.625, 1.834, and 2.167 respectively, which are better than the ratios 1.906, 1.955, and 2.178 respectively in Case 1.

Note that the trade-off does not work when \( \alpha = 1 \), i.e., there is no improvement using the trade-off. However, for unsplittable \( k \)-CVRP with \( k \geq 6 \), by Theorem 50, the ratio is \( \frac{2\left\lceil k/2 \right\rceil + 1 + \ln \frac{k}{\left\lceil k/2 \right\rceil + 1}}{\left\lceil k/2 \right\rceil + 1} \) which can still achieve \( \alpha + 1 + \ln 2 - 2\alpha/k - \Theta(1/k) \). Also note that in this case, we have \( \chi = 1 \) and by Lemma 17, we may use a good mod-\( k \)-cycle packing to get some further improvements.

### 12 A Further Improvement for Unsplittable \( k \)-CVRP

In the above section, the ratio of unsplittable \( k \)-CVRP is \( \frac{5\left\lceil k/2 \right\rceil + 2}{2\left\lceil k/2 \right\rceil + 2} + \ln \frac{k}{\left\lceil k/2 \right\rceil + 1} = 5/2 + \ln 2 - \Theta(1/k) \) for \( k \geq 8 \) under the ratio \( \alpha = 3/2 \) for metric TSP. In this section, we will further improve the ratio to less than \( 5/2 + \ln 2 - \sqrt{2}/k \).

Our algorithm is that. We run LP-UITP on the Hamiltonian cycle \( H_{CS} \) on \( V \cup \{v_0\} \). By Lemma 41, for unsplittable \( k \)-CVRP with any constant \( \gamma \geq 0 \) and \( k \geq 3 \), the LP-UITP algorithm can generate a feasible itinerary \( I \) such that
\[
w(I) \leq \gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{\left\lceil k/2 \right\rceil + 1} \Delta + \left( 1 - \frac{1}{\left\lceil k/2 \right\rceil + 1} \right) w(H_{CS}).
\]
We first show a simple analysis.

By setting $\gamma = \ln 2$, we can get that $w(I) \leq 2 \cdot w(I^*) + \frac{1}{(k/2)+1} \Delta + (1 - \frac{1}{(k/2)+1}) w(H_{CS}) \leq 2 \cdot w(I^*) + \frac{1}{(k/2)+1} \Delta + (1 - \frac{1}{(k/2)+1}) w(H_{CS}) \leq 2 \cdot w(I^*) + \frac{2}{k} \Delta + (1 - \frac{1}{k}) w(H_{CS})$, where the last inequality follows from $w(H_{CS}) \leq 2 \Delta$ by the triangle inequality. Note that the lower bounds of unit-demand $k$-CVRP are also the lower bounds of unsplittable $k$-CVRP. Therefore, by (1) and the previous analysis in Theorem 40, we have $\frac{2}{k} \Delta + (1 - \frac{1}{k}) w(H_{CS}) < (5/2 - \sqrt{2/k}) w(I^*)$. Hence, we have $w(I) \leq 2 w(I^*) + \frac{2}{k} \Delta + (1 - \frac{1}{k}) w(H_{CS}) < (5/2 + \ln 2 - \sqrt{2/k}) w(I^*)$. A simple analysis has already lead to an approximation ratio of less than $5/2 + \ln 2 - \sqrt{2/k}$.

Next, we give a more refined analysis with a better result.

Define $m = \lceil \frac{k+1}{2} \rceil$. Note that $\sum_{C \subseteq T} w(C) = w(I^*)$. By Lemmas 32, 35, 31, and 34, we know

$$\Delta = \sum_{C \subseteq T^*} \Delta_C \leq \left( \frac{k+2}{2} - \sum_{i=1}^{m} i x_i \right) w(I^*) \text{ and } \text{MST} \leq \sum_{C \subseteq T^*} T_C \leq \left( 1 - \max_{1 \leq i \leq m} \frac{1}{2} x_i \right) w(I^*),$$

where $\sum_{i=1}^{m} x_i = 1$ and $x_i \geq 0$ for each $1 \leq i \leq m$. Hence, the approximation ratio satisfies

$$\max_{x_1, x_2, \ldots, x_m \geq 0} \min_{x_1 + x_2 + \cdots + x_m = 1} \left\{ \gamma + e^{-\gamma} \cdot \frac{2}{k/2 + 1} \left( \frac{k+2}{2} - \sum_{i=1}^{m} i x_i \right) + \left( 1 - \frac{1}{k/2 + 1} \right) \left( \frac{3}{2} - \max_{1 \leq i \leq m} \frac{1}{2} x_i \right) \right\}.$$ 

By a similar argument with the proof in Lemma 33, we know that it holds $x_1 \geq x_2 \geq \cdots \geq x_m$ in the worst case. Hence, we have $\max_{1 \leq i \leq m} x_i = x_1$ and the ratio satisfies

$$\max_{x_1, x_2, \ldots, x_m \geq 0} \min_{x_1 + x_2 + \cdots + x_m = 1} \left\{ \gamma + e^{-\gamma} \cdot \frac{2}{k/2 + 1} \left( \frac{k+2}{2} - \sum_{i=1}^{m} i x_i \right) + \left( 1 - \frac{1}{k/2 + 1} \right) \left( \frac{3}{2} - \frac{1}{2} x_1 \right) \right\}.$$ 

In the worst case, one can easily see that we have $\frac{2}{k/2 + 1} (\frac{k+2}{2} - \sum_{i=1}^{m} i x_i) \geq 1$ as in the proof of Lemma 50. Hence, by setting $\gamma = \ln \frac{2}{k/2 + 1} + \ln \left( \frac{k+2}{2} - \sum_{i=1}^{m} i x_i \right)$, the approximation ratio satisfies

$$\max_{x_1, x_2, \ldots, x_m \geq 0} \min_{x_1 + x_2 + \cdots + x_m = 1} \left\{ 1 + \ln \frac{2}{k/2 + 1} + \ln \left( \frac{k+2}{2} - \sum_{i=1}^{m} i x_i \right) + \left( 1 - \frac{1}{k/2 + 1} \right) \frac{3 - x_1}{2} \right\}.$$ 

Taking that $x_1$ is a constant, by a similar argument with the proof in Lemma 38, we can get that $x_2 = x_3 = \cdots = x_l = \frac{1 - x_l}{l - 1}$, where $l = \lceil \frac{k+1}{2} \rceil \leq m$. Note that we cannot get $x_1 = x_2$ directly. Since $\sum_{i=1}^{m} i x_i = x_1 + \sum_{i=2}^{l} \frac{1}{i-1} x_1 = \frac{1}{2} (1 - x_1) + 1$, then the approximation ratio is bounded by

$$\max_{0 < x \leq 1} \min_{\frac{1}{1/x}} \left\{ 1 + \ln \left( \frac{k - (1 - x)}{|k/2 + 1|} \right) + \left( 1 - \frac{1}{|k/2 + 1|} \right) \frac{3 - x}{2} \right\}.$$ 

Next, we compute the approximation ratio. For the sake of analysis, we will only consider $k \geq 7$.

**Theorem 51.** Given any fixed $k \geq 7$, for unsplittable $k$-CVRP, the approximation ratio satisfies

- For any odd $k \geq 7$, the ratio is max\{\rho_o([l_o]), \rho_o([l_o])\}, where $l_o = \frac{\sqrt{(k-1)^2 + 8(k-1)(k+1)} - (k-1)}{4(k+1)}$ and $\rho_o(l_o) = \frac{5}{2} + \ln 2 + \ln \frac{k+1-l_o}{k+1} - \frac{k+6l_o}{2(k+1)}$.

- For any even $k \geq 7$, the ratio is max\{\rho_e([l_e]), \rho_e([l_e])\}, where $l_e = \frac{\sqrt{k^2 + 8k(k+1)(k+2)} - k}{4(k+2)}$ and $\rho_e(l_e) = \frac{5}{2} + \ln 2 + \ln \frac{k+1-l_e}{k+2} - \frac{k+6l_e}{2(k+2)}$. 


Proof. Let

\[ f(x) = 1 + \ln \left( \frac{k - (1 - x) \lfloor \frac{k}{2} \rfloor}{\lfloor k/2 \rfloor + 1} \right) + \left(1 - \frac{1}{\lfloor k/2 \rfloor + 1}\right) \frac{3 - x}{2}, \quad x \in (0, 1], \quad \lfloor \frac{1}{x} \rfloor \leq m. \]

Assume that \( f(x_0) = \max_{0 < x \leq 1} f(x) \), i.e. the approximation ratio. For any \( k \geq 7 \), we will first show that \( 1/x_0 \) must be an integer. Then, let

\[ g(x) = 1 + \ln \left( \frac{k - (1 - \frac{1}{x}) x}{\lfloor k/2 \rfloor + 1} \right) + \left(1 - \frac{1}{\lfloor k/2 \rfloor + 1}\right) \frac{3 - \frac{x}{2}}{2}, \quad x \geq 1. \]

We can determine \( \max_{x \in \mathbb{Z} \geq 1} g(x) \), which is an upper bound of the approximation ratio \( f(x_0) \). Then, we can get the results of this theorem.

**Claim.** \( 1/x_0 \) must be an integer.

Consider \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \), where \( u \in \mathbb{Z}_{\geq 1} \). We have \( \lfloor \frac{1}{x} \rfloor = u + 1 \). We can get

\[ f'(x) = \frac{u + 1}{k - (1 - x)(u + 1)} - \frac{1}{2} \left(1 - \frac{1}{\lfloor k/2 \rfloor + 1}\right), \quad x \in \left[\frac{1}{u+1}, \frac{1}{u}\right). \]

It is easy to see that there are three cases for \( f'(x)\).

**Case 1.** \( f'(x) \geq 0 \) for \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \): We will show that \( x_0 \notin \left[\frac{1}{u+1}, \frac{1}{u}\right) \).

In this case, for \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \), we have \( f(x) \leq \lim_{x \to \left(\frac{1}{u}\right)^+} f(x) = f\left(\frac{1}{u}\right) \). Hence, \( x_0 \notin \left[\frac{1}{u+1}, \frac{1}{u}\right) \).

**Case 2.** \( f'(x) \leq 0 \) for \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \): We will show that \( 1/x_0 \) is an integer if \( x_0 \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \).

In this case, for \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \), we have \( f(x) \leq f\left(\frac{1}{u+1}\right) \). It is easy to see that \( 1/x_0 = u + 1 \) if \( x_0 \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \).

**Case 3.** \( f'(x) \geq 0 \) for \( x \in \left[\frac{1}{u+1}, x_u\right] \) and \( f'(x) \leq 0 \) for \( \left[\frac{1}{u}, \frac{1}{u+1}\right) \), where \( \frac{1}{u+1} < x_u < \frac{1}{u} \): We will show that \( f(x_u) < f\left(\frac{1}{u}\right) \), and hence \( x_0 \notin \left[\frac{1}{u+1}, \frac{1}{u}\right) \).

In this case, we have \( f'(x_u) = 0 \). Then, we have \( x_u = 3 + \frac{2}{\lfloor k/2 \rfloor} - \frac{k}{u+1} \). Note that \( \lfloor \frac{1}{x} \rfloor = u + 1 \leq m = \lfloor k + 1 \rfloor \). Then, \( u \leq \lfloor k + 1 \rfloor - 1 \leq \frac{k}{2} \). Moreover, since \( x_u \geq \frac{1}{u+1} \), we have

\[ \frac{k}{2} \geq u \geq \frac{k + 1}{3 + \frac{2}{\lfloor k/2 \rfloor}} - 1 \geq \frac{k^2 - 3k - 2}{3k + 1}. \]

It is easy to see that \( f(x_u) \leq f'\left(\frac{1}{u+1}\right)(x_u - \frac{1}{u+1}) + f\left(\frac{1}{u+1}\right) \leq f'\left(\frac{1}{u+1}\right)(\frac{1}{u} - \frac{1}{u+1}) + f\left(\frac{1}{u+1}\right) \) since \( f(x) \) is a concave function for \( x \in \left[\frac{1}{u+1}, \frac{1}{u}\right) \). To prove \( f(x_u) < f\left(\frac{1}{u}\right) \), it is sufficient to prove

\[ f'\left(\frac{1}{u+1}\right)\left(\frac{1}{u} - \frac{1}{u+1}\right) + f\left(\frac{1}{u+1}\right) < f\left(\frac{1}{u}\right), \quad \frac{k^2 - 3k - 2}{3k + 1} \leq u \leq \frac{k}{2}. \]

By \( f(x) \) and \( f'(x) \), we know that it is sufficient to prove

\[ \ln \left(1 + \frac{1}{k - u}\right) - \frac{1}{(k - u)u} > 0, \quad \frac{k^2 - 3k - 2}{3k + 1} \leq u \leq \frac{k}{2}. \]

Hence, it is sufficient to prove that \( \ln(1 + \frac{1}{k - u_0}) - \frac{1}{u_0(k - u_0)} > 0 \), where \( u_0 = \frac{k^2 - 3k - 2}{3k + 1} \). Note that \( \ln(1 + x) > x(1 - \frac{x}{2}) \) for \( 0 < x < 1 \). Since \( 0 < \frac{1}{k - u_0} < 1 \), we have \( \ln(1 + \frac{1}{k - u_0}) > \frac{1}{k - u_0}(1 - \frac{1}{2(k - u_0)}) \).

Moreover, since \( k \geq 7 \), we have \( \frac{1}{u_0} = \frac{3k + 1}{k^2 - 3k - 2} \leq \frac{11}{13} \) and \( \frac{1}{2(k - u_0)} = \frac{3k + 1}{4(k+1)^2} \leq \frac{11}{128} \). Hence, we have

\[ \ln \left(1 + \frac{1}{k - u_0}\right) > \frac{1}{k - u_0} \left(1 - \frac{1}{2(k - u_0)}\right) > \frac{1}{k - u_0} \left(\frac{11}{13} + \frac{11}{128} - \frac{1}{2(k - u_0)}\right) \geq \frac{1}{u_0(k - u_0)}. \]
According to the above three cases, we know that the claim holds.

Then, we are ready to calculate \( \max_{x \in \mathbb{Z}_{\geq 1}} g(x) \), which is an upper bound of the approximation ratio. We will consider two cases of \( k \).

**Case 1.** \( k \) is odd: We have

\[
g(x) = \frac{5}{2} + \ln 2 + \ln \frac{k + 1 - x}{k + 1} - \frac{k - 1 + 6x}{2(k + 1)x} \quad \text{and} \quad g'(x) = \frac{-1}{k + 1 - x} + \frac{k - 1}{2(k + 1)x^2}.
\]

When \( x = l_o = \sqrt[4]{\frac{(k-1)^2 + 8(k-1)(k+1)^2 - (k-1)}{4(k+1)}} \), we can get that \( g'(x) = 0 \). Hence, it is easy to see that

\[
\max_{x \in \mathbb{Z}_{\geq 1}} g(x) = \max\{g([l_o]), g([l_o])\} = \max\{\rho_o([l_o]), \rho_o([l_o])\}.
\]

**Case 2.** \( k \) is even: We have

\[
g(x) = \frac{5}{2} + \ln 2 + \ln \frac{k + 1 - x}{k + 2} - \frac{k + 6x}{2(k + 2)x} \quad \text{and} \quad g'(x) = \frac{-1}{k + 1 - x} + \frac{k}{2(k + 2)x^2}.
\]

When \( x = l_e = \sqrt[4]{\frac{k^2 + 8k(k+1)(k+2) - k}{4(k+2)}} \), we can get that \( g'(x) = 0 \). Hence, it is easy to see that

\[
\max_{x \in \mathbb{Z}_{\geq 1}} g(x) = \max\{g([l_e]), g([l_e])\} = \max\{\rho_e([l_e]), \rho_e([l_e])\}.
\]

It is worth noting that for \( k \geq 7 \) we have \( x_1 = x_2 = \cdots = x_l = 1/l \) in the worst case. For \( k < 7 \), e.g., \( k = 6 \), we may have \( 1 > x_1 > x_2 = \cdots = x_l \) in the worst case. Theorem 51 only considers \( k \geq 7 \) for the sake of analysis. In fact, for \( k < 7 \), the ratio is the same as in Theorem 50.

### 13 A Further Improvement for Unsplittable 5-CVRP

In this section, we further improve the approximation ratio from \( 5/3 + \ln(5/3) < 2.178 \) to 2.157 for unsplittable 5-CVRP. Different from our algorithms for unsplittable 3-CVRP and 4-CVRP, we will use LP-UITP for unsplittable 5-CVRP. This is the reason why we do not put our algorithms for unsplittable 3-CVRP, 4-CVRP and 5-CVRP together.

For unsplittable 5-CVRP, we will consider two algorithms and we make a trade-off between them. In the first algorithm, we first construct a mod-5-cycle packing and then generate a feasible solution based on it. The second algorithm is to call the refined LP-UITP algorithm on the Hamiltonian cycle \( H_{CS} \).

We assume that the demand of each customer is 1, 2, 3 or 4. Our first algorithm will first compute a cycle packing \( C_{mod 5} \), where “mod 5” means the total demand of customers in each cycle in the packing is divisible by 5. We will simply call this cycle packing a mod-5-cycle packing. It is computed in the following way: we consider the instance as a unit-demand instance \( G' \) by taking each customer \( v_i \) as \( d_i \) customers with the unit-demand as the same place. Then, we call the polynomial-time algorithm in Lemma 28 to generate a mod-5-k-tree packing \( T_{mod 5} \) in \( G'[V'] \) such that \( w(T_{mod 5}) \leq w(C_{mod 5}^*) \). By doubling the mod-5-k-tree packing \( T_{mod 5} \) and then shortcutting, we can get a mod-5-cycle packing \( C_{mod 5}^* \) in \( G[V] \) such that \( w(C_{mod 5}^*) \leq 2w(C_{mod 5}^*) \). Note that \( C_{mod 5} \) may not be a mod-5-cycle packing for the original graph \( G[V] \). By shortcutting in \( C_{mod 5}^* \), we can get a cycle packing \( C_{mod 5}^* \) on \( V \). Moreover, the total demand of all customers in each cycle in \( C_{mod 5}^* \) is divisible by 5. We will use this mod-5-cycle packing \( C_{mod 5}^* \) for unsplittable 5-CVRP.

We have that \( w(C_{mod 5}) \leq 2w(C_{mod 5}^*) \) and it holds that \( |C| \mod 5 = \sum_{v_j \in C} d_j \mod 5 = 0 \) for each \( C \in C_{mod 5} \). The next steps of our algorithm are based on the packing \( C_{mod 5}^* \). For different types of cycles in \( C_{mod 5} \), we have different operations. Consider a cycle \( C \in C_{mod 5} \). Let \( x \) be the sum of the demands of all 1-demand, 2-demand and 3-demand customers in \( C \). We classify the cycles in \( C_{mod 5} \) into several types \( C_{<i>} \) according to the value of \( x \). We will use \( V_i \) to denote the set of customers contained in a cycle in \( C_{<i>} \) and let \( \Delta_i = \sum_{v_j \in V_i} d_j w(v_0, v_j) \).
Type 0: Cycles with $x = 0$, the set of which is denoted by $C_{<0>}$. For each cycle $C$ in $C_{<0>}$, there is no 1-demand or 2-demand customer and we assign a trivial tour for each 3-demand or 4-demand customer in $C$. The weight of trivial tours is at most $\frac{2}{3} \Delta_0 \leq \frac{2}{3} \Delta_0 + \frac{2}{3} w(C_{<0>})$.

Type 1: Cycles with $x \geq 6$, the set of which is denoted by $C_{<1>}$. For each cycle $C$ in $C_{<1>}$, we assign a trivial tour for each 3-demand or 4-demand customer in $C$ and obtain a small cycle $C'$ from $C$ by shortcutting all 3-demand and 4-demand customers. The set of cycles $C'$ is denoted by $C_{<1>}$. Then we apply the EX-UITP algorithm on $C'_{<1>}$.

We analyze the weight of the tours assigned in this step. Let $V'_1$ be the set of 1-demand and 2-demand customers in $V_1$ and $V'_2 = V_1 \setminus V'_1$ be the set of 3-demand and 4-demand customers in $V_1$. Let $\Delta'_1 = \sum_{v_j \in V'_1} d_j w(v_0, v_j)$ and $\Delta'_2 = \sum_{v_j \in V'_2} d_j w(v_0, v_j)$. Note that for each cycle $C'$ in $C'_{<1>}$ we have $|C'| = x \geq 6$. For the tours generated by the EX-UITP algorithm on $C'_{<1>}$, by Lemma 42, the total weight is at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C'_{<1>}) \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C_{<1>})$ by the triangle inequality. For trivial tours visiting a 3-demand or 4-demand customer, the total weight is at most $\frac{2}{3} \Delta'_1$. In total, the weight is at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} \Delta'_2 + \frac{2}{3} w(C_{<1>}) = \frac{2}{3} \Delta_1 + \frac{2}{3} w(C_{<1>})$.

We still have several cases, where $x$ is a small constant.

Type 2: Cycles with $x = 1$, the set of which is denoted by $C_{<2>}$. Each cycle $C \in C_{<2>}$ contains exactly one 1-demand customer $u$ and at least one 3-demand (or 4-demand) customer $u'$ since the total demand in the cycle is divisible by 5. We assign a tour $T$ visiting two customers $u$ and $u'$ only and a trivial tour for each other 3-demand or 4-demand customer in $C$ except $u'$. We can obtain a small cycle $C'$ from $C$ by shortcutting the 3-demand and 4-demand customers except $u'$. Let $V'_2$ be the set of customers contained in $T$ and let $\Delta'_2 = \sum_{v_j \in V'_2} d_j w(v_0, v_j)$. Note that $4 \leq |C'| = d_u + d_{u'} \leq 5$. By Lemma 43, we can assign the tour $T$ with a weight of at most $\frac{2}{3} \Delta'_1 + (1 - \frac{1}{|C'|}) w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C')$. For trivial tours visiting the other 3-demand and 4-demand customers in $V''$, the total weight is at most $\frac{2}{3} \Delta'_2$. Thus, by summing all cycles in $C_{<2>}$, the weight of tours in this step is at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C_{<2>})$.

Type 3: Cycles with $x = 2$, the set of which is denoted by $C_{<3>}$. Each cycle $C \in C_{<3>}$ contains exactly two 1-demand customers $u_1$ and $u_2$ or exactly one 2-demand customer $u$ and at least one 3-demand (or 4-demand) customer $u'$ since the total demand in the cycle is divisible by 5.

If the demand of $u'$ is 3, we assign a single tour $T$ visiting $\{u_1, u_2, u'\}$ (or $\{u, u'\}$) and a trivial tour for each other 3-demand or 4-demand customer in $C$ except $u'$. We can obtain a small cycle $C'$ from $C$ by shortcutting the 3-demand and 4-demand customers except $u'$. Let $V''$ be the set of customers contained in $T$ and let $\Delta'' = \sum_{v_j \in V''} d_j w(v_0, v_j)$. Note that $|C'| = 5$. By Lemma 43, we can assign the tour $T$ with a weight of at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C')$. For trivial tours visiting the other 3-demand and 4-demand customers in $V''$, the total weight is at most $\frac{2}{3} \Delta''$. Thus, by summing all cycles in $C_{<3>}$, the weight of tours in this step is at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C_{<3>})$.

If the demand of $u'$ is 4, we assign two tours $T_1$ and $T_2$ with the capacity 2 and 4 visiting $\{u_1, u_2, u'\}$ (or $\{u, u'\}$) and a trivial tour for each other 3-demand or 4-demand customer in $C$ except $u'$. We can obtain a small cycle $C'$ from $C$ by shortcutting the 3-demand and 4-demand customers except $u'$. Define $V'_3$, $\Delta'_3$, $V''$ and $\Delta''$ in the similar way as above. Note that $|C'| = 6$. By considering all cases, we can easily get that we can assign the tours $T_1$ and $T_2$ with a weight of at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C') \leq \frac{2}{3} \Delta'_1 + \frac{2}{3} w(C')$. For trivial tours visiting the other 3-demand and 4-demand customers in $V''$, the total weight is at most $\frac{2}{3} \Delta''$. Thus, by summing all cycles in $C_{<3>}$, the weight of tours in this step is at most $\frac{2}{3} \Delta'_1 + \frac{2}{3} w(C_{<3>})$.

Type 4: Cycles with $3 \leq x \leq 5$, the set of which is denoted by $C_{<4>}$. For each cycle $C \in C_{<4>}$, we assign a trivial tour for each 3-demand or 4-demand customer in $C$ and obtain a small cycle
Lemma 43, we can assign the tour or just classic algorithms. They are simple and easy to implement. Our modified algorithms, such as EX-ITP, may also have potentials to be applied in fast experimental and heuristic algorithms.

Although our algorithms are simple, the analysis is technically involved. We need to carefully analyze the structure of the solutions and most of our analysis is based on pure combinatorial analysis. We believe that analyzing better theoretical bounds for simple and classic algorithms is an interesting and important task in algorithms.

There are two algorithms.

Using the mod-5-cycle packing algorithm, by Lemma 52, we can generate an itinerary \( I \) such that
\[
\frac{2}{3} \Delta + \frac{4}{3} w(C_{\text{mod } 5}) \leq \frac{2}{3} \left( \chi + \frac{3}{2} \right) w(I) + \frac{4}{3} (2 - 2 \chi) w(I^*) = \frac{11}{3} - \frac{6 \chi}{3} w(I^*).
\]

We will set \( \gamma = \frac{2}{3} \chi + 1 \). Since \( \Delta \leq (\chi + \frac{3}{2}) w(I^*) \) by Lemma 13, we have that
\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{3} \Delta \leq \gamma \cdot w(I) + e^{-\gamma} \cdot \frac{2}{3} \left( \chi + \frac{3}{2} \right) w(I^*) = \left( 1 + \ln \frac{3 + 2 \chi}{3} \right) w(I^*).
\]

Recall that \( w(H_{CS}) \leq \frac{3 - \chi}{2} \) by Lemma 16. Therefore, we have that
\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot \frac{2}{3} \Delta + \frac{2}{3} w(H_{CS}) \leq \left( \frac{6 - \chi}{3} + \ln \frac{3 + 2 \chi}{3} \right) \frac{11}{3} - \frac{6 \chi}{3} w(I^*).
\]

We will select the better one between these two itineraries. The final ratio is
\[
\max_{0 \leq \chi \leq 1} \min \left\{ \frac{11 - 6 \chi}{3}, \frac{6 - \chi}{3} + \ln \frac{3 + 2 \chi}{3} \right\} = \frac{11 - 6 x_0}{3} < 2.157,
\]
where \( x_0 > 0.755 \) in the worst case, the unique root of the equation \( \frac{5}{3} (x - 1) + \ln \frac{3 + 2 x}{3} = 0 \).

14 Concluding Remarks

In this paper, we consider \( k \)-CVRP in general metrics and improve the approximation ratio for \( k \) being sufficient large, say \( k \leq 1.7 \times 10^7 \). Most of our algorithms are modified from known algorithms or just classic algorithms. They are simple and easy to implement. Our modified algorithms, such as EX-ITP, may also have potentials to be applied in fast experimental and heuristic algorithms. Although our algorithms are simple, the analysis is technically involved. We need to carefully analyze the structure of the solutions and most of our analysis is based on pure combinatorial analysis. We believe that analyzing better theoretical bounds for simple and classic algorithms is an interesting and important task in algorithms.
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A The Approximation Ratio of Bompadre et al.’s Algorithm

Lemma 54 (Bompadre et al. [16], Section 3.2.3). For splittable and unit-demand \( k \)-CVRP with \( k \geq 3 \), given an \( \alpha \)-approximation algorithm for metric TSP with \( \alpha \geq 1 \), there is an approximation algorithm with a ratio of \( 1 - \frac{\alpha}{\beta} + (1 - \beta)\alpha = \alpha + 1 - \alpha/k - \Omega(1/k^3) \), where \( \beta \) is the positive root of the following quadratic equation

\[
0 = (2k^2 - 2k)\beta^2 + \left( 2 - \alpha + \frac{\alpha}{k} - \frac{2}{k+1} + 2k^2 - 2k \right) \beta + \left( 1 - \alpha + \frac{\alpha}{k} - \frac{2}{k+1} \right).
\]

B Calculating the Ratio for Splittable Case

In this section, we will prove the approximation ratio of splittable and unit-demand \( k \)-CVRP in [15] is at least \( 5/2 - 1.005/3000 - 1.5/k \).

B.1 The Main Idea of the Algorithm

The idea in [15] is to distinguish two kinds of instances.

- If \( (1 - \varepsilon)w(I^*) \geq (2/k)\Delta \), the instance is simple.
- Otherwise, the instance is difficult.

Lemma 55 ([15]). For hard instances, there is a function \( f : \mathbb{R}_{>0} \rightarrow \mathbb{R}_{>0} \) with \( \lim_{\varepsilon \rightarrow 0} f(\varepsilon) = 0 \) and a polynomial-time algorithm to get a Hamiltonian cycle \( H \) on \( V \cup \{v_0\} \) such that \( w(H) \leq (1 + f(\varepsilon))w(I^*) \).

Note that the function \( f \) can be seen in Appendix D. The above lemma shows that we can find a good Hamiltonian cycle in the difficult instances if \( \varepsilon \) is small. Hence, using the AG-ITP algorithm, there will always be an improvement by making a trade-off between the two kinds of instances.

Given a Hamiltonian cycle \( H \) on \( V \cup \{v_0\} \), the weight of the AG-ITP algorithm in [3] for splittable and unit-demand \( k \)-CVRP is at most

\[
(2/k)\Delta + (1 - 1/k)w(H) \leq (2/k)\Delta + w(H).
\]

B.2 A Simple Analysis

In this subsection, we use the weaker result \( (2/k)\Delta + w(H) \) of the AG-ITP algorithm. We will show the ratio in [15] is at least \( 5/2 - 1.005/3000 \).

Case 1. For easy instances, by the definition, we have \( (1 - \varepsilon)w(I^*) \geq (2/k)\Delta \). Using an \( \alpha \)-approximate Hamiltonian cycle on \( V \cup \{v_0\} \), by Lemma 12, we can get that

\[
(2/k)\Delta + \alpha w(H^*) \leq (1 - \varepsilon)w(I^*) + \alpha w(I^*) = (\alpha + 1 - \varepsilon)w(I^*).
\]

Case 2. For hard instances, we have \( w(I^*) \geq (2/k)\Delta \) by Lemma 13. Using the Hamiltonian cycle \( H \) on \( V \cup \{v_0\} \) in Lemma 55, we can get that

\[
(2/k)\Delta + w(H) \leq w(I^*) + (1 + f(\varepsilon))w(I^*) = (2 + f(\varepsilon))w(I^*).
\]

Therefore, the approximation ratio of splittable and unit-demand \( k \)-CVRP is

\[
\min_{\varepsilon > 0} \max \{ \alpha + 1 - \varepsilon, 2 + f(\varepsilon) \} = \alpha + 1 - \varepsilon^*.
\]
where $\varepsilon^*$ is the maximum value satisfying $f(\varepsilon) + \varepsilon \leq \alpha - 1$. When $\alpha = 3/2$, we have

$$1.005/3000 - 6 \times 10^{-9} < \varepsilon^* < 1.005/3000 - 5 \times 10^{-9}.$$ 

Therefore, the ratio is at least $5/2 - 1.005/3000$ and it holds $\varepsilon^* < 1.005/3000$.

### B.3 A Tighter Analysis

In the previous subsection, we know the ratio is at least $5/2 - 1.005/3000$. Since we consider $k = O(1)$, using the stronger result $(2/k)\Delta + (1 - 1/k)w(H)$ of the AG-ITP algorithm, the ratio may be better under a tighter analysis. However, we will show the improvement is at most $1.5/k$, i.e., the ratio is at least $5/2 - 1.005/3000 - 1.5/k$.

**Case 1.** For easy instances, by the definition, we have $(1 - \varepsilon)w(I^*) \geq (2/k)\Delta$. Using an $\alpha$-approximate Hamiltonian cycle on $V \cup \{v_0\}$, by Lemma 12, we can get that

$$(2/k)\Delta + (1 - 1/k)\alpha w(H^*) \leq (1 - \varepsilon)w(I^*) + (1 - 1/k)\alpha w(I^*) = (\alpha + 1 - \varepsilon - \alpha/k)w(I^*).$$

**Case 2.** For hard instances, we have $w(I^*) \geq (2/k)\Delta$ by Lemma 13. Using the Hamiltonian cycle $H$ on $V \cup \{v_0\}$ in Lemma 55, we can get that

$$(2/k)\Delta + (1 - 1/k)w(H) \leq w(I^*) + (1 - 1/k)(1 + f(\varepsilon))w(I^*) = (1 + (1 - 1/k)(1 + f(\varepsilon))w(I^*).$$

Therefore, the approximation ratio of splittable and unit-demand $k$-CVRP is

$$\min_{\varepsilon > 0} \max\{\alpha + 1 - \varepsilon - \alpha/k, 1 + (1 - 1/k)(1 + f(\varepsilon))\} = \alpha + 1 - \varepsilon^{**} - \alpha/k,$$

where $\varepsilon^{**}$ is the maximum value satisfying $(1 - 1/k)f(\varepsilon) + \varepsilon \leq (1 - 1/k)(\alpha - 1)$.

Recall that $\varepsilon^*$ is the maximum value satisfying $f(\varepsilon) + \varepsilon \leq \alpha - 1$. Hence, we have $f(\varepsilon^*) + \varepsilon^* = \alpha - 1$. Therefore, we can get $(1 - 1/k)f(\varepsilon^*) + \varepsilon^* = (1 - 1/k)(\alpha - 1 - \varepsilon^*) + \varepsilon^* > (1 - 1/k)(\alpha - 1)$. So, when $\alpha = 3/2$, we have $\varepsilon^{**} < \varepsilon^* < 1.005/3000$. The ratio satisfies that

$$\alpha + 1 - \varepsilon^{**} - \alpha/k > \alpha + 1 - \varepsilon^* - \alpha/k > 5/2 - 1.005/3000 - 1.5/k.$$ 

Therefore, a tighter analysis shows that the ratio in [15] is at least $5/2 - 1.005/3000 - 1.5/k$. To show our result $5/2 - \Theta(\sqrt{1/k})$ is better for any $k \leq 1.7 \times 10^7$, it is sufficient to show that it is better than $5/2 - 1.005/3000 - 1.5/k$ for any $k \leq 1.7 \times 10^7$.

### C Calculating the Ratio for Unsplittable Case

In this section, we will prove the approximation ratio of unsplittable $k$-CVRP in [25] is at least $5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k$.

Note that the constant part $\ln(1 - 1.005/3000)$ is obtained by using the method in [15]. Hence, the idea of analysis is similar. Fixing a constant $\varepsilon > 0$, there are two kinds of instances.

- If $(1 - \varepsilon)w(I^*) \geq (2/k)\Delta$, the instance is simple.
- Otherwise, the instance is difficult.

Given a Hamiltonian cycle on $V \cup \{v_0\}$, the weight of the AG-UITP algorithm in [2] for unsplittable $k$-CVRP with even $k \geq 3$ is at most

$$(4/k)\Delta + (1 - 2/k)w(H) \leq (4/k)\Delta + w(H).$$

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C.1 A Simple Analysis

In this subsection, we use the weaker result \((4/k)\Delta + w(H)\) of the AG-UITP algorithm. Then, the LP-based algorithm in [25] has weight of at most

\[
\min_{\gamma \geq 0}\{\gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k)\Delta + w(H)\}.
\]

We will show the ratio is at least \(5/2 + \ln 2 + \ln(1 - 1.005/3000)\).

**Case 1.** For easy instances, by the definition, we have \((1 - \varepsilon)w(I^*) \geq (2/k)\Delta\). Using an \(\alpha\)-approximate Hamiltonian cycle on \(V \cup \{v_0\}\), by Lemma 12, we can get that

\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k)\Delta + \alpha w(I^*) \leq \gamma \cdot w(I^*) + e^{-\gamma} \cdot 2(1 - \varepsilon)w(I^*) + \alpha w(I^*) = (\gamma + e^{-\gamma} \cdot 2(1 - \varepsilon) + \alpha)w(I^*).
\]

The ratio is

\[
\min_{\gamma \geq 0}\{\gamma + e^{-\gamma} \cdot 2(1 - \varepsilon) + \alpha\} = \alpha + 1 + \ln 2 + \ln(1 - \varepsilon).
\]

**Case 2.** For hard instances, we have \(w(I^*) \geq (2/k)\Delta\) by Lemma 13. Using the Hamiltonian cycle \(H\) on \(V \cup \{v_0\}\) in Lemma 55, we can get that

\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k)\Delta + w(H) \leq \gamma \cdot w(I^*) + e^{-\gamma} \cdot 2w(I^*) + (1 + f(\varepsilon))w(I^*) = (\gamma + e^{-\gamma} \cdot 2 + 1 + f(\varepsilon))w(I^*).
\]

The ratio is

\[
\min_{\gamma \geq 0}\{\gamma + e^{-\gamma} \cdot 2 + 1 + f(\varepsilon)\} = 2 + \ln 2 + f(\varepsilon).
\]

Therefore, the ratio is

\[
\min_{\varepsilon > 0} \max\{\alpha + 1 + \ln 2 + \ln(1 - \varepsilon), 2 + \ln 2 + f(\varepsilon)\} = \alpha + 1 + \ln 2 + \ln(1 - \varepsilon^*),
\]

where \(\varepsilon^*\) is the maximum value satisfying \(f(\varepsilon) - \ln(1 - \varepsilon) \leq \alpha - 1\). When \(\alpha = 3/2\), we also have

\[
1.005/3000 - 6 \times 10^{-9} < \varepsilon^* < 1.005/3000 - 5 \times 10^{-9}.
\]

Therefore, the ratio is at least \(5/2 + \ln 2 + \ln(1 - 1.005/3000)\) and it holds \(\varepsilon^* < 1.005/3000\).

C.2 A Tighter Analysis

In this subsection, using the stronger result \((4/k)\Delta + (1 - 2/k)w(H)\) of the AG-ITP algorithm, we will show the ratio is at least \(5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k\).

**Case 1.** For easy instances, by the definition, we have \((1 - \varepsilon)w(I^*) \geq (2/k)\Delta\). Using an \(\alpha\)-approximate Hamiltonian cycle on \(V \cup \{v_0\}\), by Lemma 12, we can get that

\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k)\Delta + (1 - 2/k)\alpha w(H^*) \leq (\gamma + e^{-\gamma} \cdot 2(1 - \varepsilon) + (1 - 2/k)\alpha)w(I^*).
\]

The ratio is

\[
\min_{\gamma \geq 0}\{\gamma + e^{-\gamma} \cdot 2(1 - \varepsilon) + (1 - 2/k)\alpha\} = \alpha + 1 + \ln 2 + \ln(1 - \varepsilon) - 2\alpha/k.
\]

**Case 2.** For hard instances, we have \(w(I^*) \geq (2/k)\Delta\) by Lemma 13. Using the Hamiltonian cycle \(H\) on \(V \cup \{v_0\}\) in Lemma 55, we can get that

\[
\gamma \cdot w(I^*) + e^{-\gamma} \cdot (4/k)\Delta + (1 - 2/k)w(H) \leq (\gamma + e^{-\gamma} \cdot 2 + (1 - 2/k)(1 + f(\varepsilon)))w(I^*).
\]
The ratio is
\[ \min_{\gamma \geq 0} \{ \gamma + e^{-\gamma} \cdot 2 + (1 - 2/k)(1 + f(\varepsilon)) \} = 1 + \ln 2 + (1 - 2/k)(1 + f(\varepsilon)). \]

Therefore, the ratio is
\[ \min_{\varepsilon > 0} \max\{\alpha + 1 + \ln 2 + \ln(1 - \varepsilon) - 2\alpha/k, 1 + \ln 2 + (1 - 2/k)(1 + f(\varepsilon))\} = \alpha + 1 + \ln 2 + \ln(1 - \varepsilon^{**}) - 2\alpha/k, \]
where \( \varepsilon^{**} \) is the maximum value satisfying \((1 - 2/k)f(\varepsilon) - \ln(1 - \varepsilon) \leq (1 - 2/k)(\alpha - 1)\).

Recall that \( \varepsilon^* \) is the maximum value satisfying \( f(\varepsilon) - \ln(1 - \varepsilon) \leq \alpha - 1 \). Hence, we have \( f(\varepsilon^*) - \ln(1 - \varepsilon^*) = \alpha - 1 \). Therefore, we can get \((1 - 2/k)f(\varepsilon^*) - \ln(1 - \varepsilon^*) = (1 - 2/k)(\alpha - 1 + \ln(1 - \varepsilon^*)) - \ln(1 - \varepsilon^*) > (1 - 2/k)(\alpha - 1)\). So, when \( \alpha = 3/2 \), we have \( \varepsilon^{**} < \varepsilon^* < 1.005/3000 \). Therefore, the ratio is
\[ \alpha + 1 + \ln 2 + \ln(1 - \varepsilon^{**}) - 2\alpha/k > 5/2 + \ln 2 + \ln(1 - \varepsilon^*) - 3/k > 5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k. \]

Therefore, a tighter analysis shows the ratio in [25] is at least \( 5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k \) for unsplittable \( k \)-CVRP with even \( k \geq 3 \). Note that for odd \( k \geq 3 \), using a similar argument, we know the ratio is at least \( 5/2 + \ln 2 + \ln(1 - 1.005/3000) - 1.5/k \). To show our result \( 5/2 + \ln 2 - \Theta(\sqrt{1/k}) \) is better for any \( k \leq 1.7 \times 10^7 \), it is sufficient to show that it is better than \( 5/2 + \ln 2 + \ln(1 - 1.005/3000) - 3/k \) for even \( k \leq 1.7 \times 10^7 \) and better than \( 5/2 + \ln 2 + \ln(1 - 1.005/3000) - 1.5/k \) for odd \( k \leq 1.7 \times 10^7 \).

## D The Function

Letting
\[ \zeta = \frac{3\rho + \tau - 4\tau \cdot \rho}{1 - \rho} + \frac{\varepsilon \cdot (1 - \tau \cdot \rho - 3\rho + \tau - 4\tau \cdot \rho)}{1 - \rho}. \]

There are two functions \( f \) and \( f' \) in [15]. The first function \( f \) proposed in an LP-based algorithm has a better result but runs in a slow polynomial time. Note that we only use the function \( f \). The function \( f \) satisfies that
\[
f(\varepsilon) = \min_{0 < \theta < 1-\tau, \theta \leq 1/6} \left\{ \frac{1 + \zeta}{\theta} + \frac{1 - \tau - \theta}{\theta \cdot (1 - \tau)} + \frac{3\varepsilon}{1 - \theta} + \frac{3\rho}{(1 - \rho) \cdot (1 - \tau)} \right\} - 1.
\]

Note that in their paper \( \theta \) is let to be \( 1 - \tau \) [15]. But it is slightly better to be
\[ \theta = \min \left\{ \frac{1}{\sqrt{\frac{3\varepsilon}{2\tau\zeta} + 1}}, 1 - \tau \right\}. \]

The second result proposed in a combinatorial algorithm has a slightly worse result but runs in a faster \( O(n^3) \) time. The function \( f' \) satisfies that
\[
f'(\varepsilon) = \min_{\gamma > 2} \left\{ \frac{1 + \zeta}{1 - \tau} + \frac{8}{\gamma - 2} + \varepsilon \cdot \left( 4 + \frac{8}{\gamma - 2} + 2\gamma - \frac{\tau}{\tau - 1} \right) + \frac{3\rho}{(1 - \rho) \cdot (1 - \tau)} \right\} - 1.
\]

We can determine \( \gamma \) to be
\[ \gamma = 2 \sqrt{1 + \frac{1}{\varepsilon} + 2}. \]