Binary words containing infinitely many overlaps

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Abstract

We characterize the squares occurring in infinite overlap-free binary words and construct various $\alpha$ power-free binary words containing infinitely many overlaps.

1 Introduction

If $\alpha$ is a rational number, a word $w$ is an $\alpha$ power if there exists words $x$ and $x'$, with $x'$ a prefix of $x$, such that $w = x^n x'$ and $\alpha = n + |x'|/|x|$. We refer to $|x|$ as a period of $w$. An $\alpha^+$ power is a word that is a $\beta$ power for some $\beta > \alpha$. A word is $\alpha$ power-free (resp. $\alpha^+$ power-free) if none of its subwords is an $\alpha$ power (resp. $\alpha^+$ power). A 2 power is called a square; a $2^+$ power is called an overlap.

Thue [17] constructed an infinite overlap-free binary word; however, Dekking [7] showed that any such infinite word must contain arbitrarily large squares. Shelton and Soni [16] characterized the overlap-free squares, but it is not hard to show that there are some overlap-free squares, such as 00110011, that cannot occur in an infinite overlap-free binary word. In this paper, we characterize those overlap-free squares that do occur in infinite overlap-free binary words.
Shur [15] considered the bi-infinite overlap-free and 7/3 power-free binary words and showed that these classes of words were identical. There have been several subsequent papers [1, 9, 10, 13] that have shown various similarities between the classes of overlap-free binary words and 7/3 power-free binary words. Here we contrast the two classes of words by showing that there exist one-sided infinite 7/3 power-free binary words containing infinitely many overlaps. More generally, we show that for any real number $\alpha > 2$ there exists a real number $\beta$ arbitrarily close to $\alpha$ such that there exists an infinite $\beta^+$ power-free binary word containing infinitely many $\beta$ powers.

All binary words considered in the sequel will be over the alphabet $\{0, 1\}$. We therefore use the notation $\overline{w}$ to denote the binary complement of $w$; that is, the word obtained from $w$ by replacing 0 with 1 and 1 with 0.

2 Properties of the Thue-Morse morphism

In this section we present some useful properties of the Thue-Morse morphism; i.e., the morphism $\mu$ defined by $\mu(0) = 01$ and $\mu(1) = 10$. It is well-known [11, 17] that the Thue-Morse word

$$t = \mu^\omega(0) = 0110100110010110 \cdots$$

is overlap-free.

The following property of $\mu$ is easy to verify.

**Lemma 1.** Let $x$ and $y$ be binary words. Then $x$ is a prefix (resp. suffix) of $y$ if and only if $\mu(x)$ is a prefix (resp. suffix) of $\mu(y)$.

Shur [15] proved the following useful theorem.

**Theorem 2 (Shur).** Let $w$ be a binary word and let $\alpha > 2$ be a real number. Then $w$ is $\alpha$ power-free if and only if $\mu(w)$ is $\alpha$ power-free.

The following sharper version of one direction of this theorem (implicit in [9]) is also useful.

**Theorem 3.** Suppose $\mu(w)$ contains a subword $u$ of period $p$, with $|u|/p > 2$. Then $w$ contains a subword $v$ of length $\lceil |u|/2 \rceil$ and period $p/2$.

Karhumäki and Shallit [9] gave the following generalization of the factorization theorem of Restivo and Salemi [14]. The extension to infinite words is clear.

**Theorem 4 (Karhumäki and Shallit).** Let $x \in \{0, 1\}^*$ be $\alpha$ power-free, $2 < \alpha \leq 7/3$. Then there exist $u, v \in \{\epsilon, 0, 1, 00, 11\}$ and an $\alpha$ power-free $y \in \{0, 1\}^*$ such that $x = u\mu(y)v$. 

3 Overlap-free squares

Let

\[ A = \{00, 11, 010010, 101101\} \]

and let

\[ \mathcal{A} = \bigcup_{k \geq 0} \mu^k(A). \]

Pansiot \[12\] and Brlek \[6\] gave the following characterization of the squares in \( t \).

**Theorem 5 (Pansiot; Brlek).** The set of squares in \( t \) is exactly the set \( A \).

We can use this result to prove the following.

**Proposition 6.** For any position \( i \), there is at most one square in \( t \) beginning at position \( i \).

**Proof.** Suppose to the contrary that there exist distinct squares \( x \) and \( y \) that begin at position \( i \). Without loss of generality, suppose that \( x \) and \( y \) begin with 0. Then by Theorem 5 \( x = \mu^p(u) \) and \( y = \mu^q(v) \), for some \( p, q \) and \( u, v \in \{00, 010010\} \). Suppose \( p \leq q \) and let \( w = \mu^{q-p}(v) \). By Lemma 1 either \( u \) is a proper prefix of \( w \) or \( w \) is a proper prefix of \( u \), neither of which is possible for any choice of \( u, v \in \{00, 010010\} \).

The set \( \mathcal{A} \) does not contain all possible overlap-free squares. Shelton and Soni \[16\] characterized the overlap-free squares (the result is also attributed to Thue in \[4\]).

**Theorem 7 (Shelton and Soni).** The overlap-free binary squares are the conjugates of the words in \( A \).

Some overlap-free squares cannot occur in any infinite overlap-free binary word, as the following lemma shows.

**Lemma 8.** Let \( x = \mu^k(z) \) for some \( k \geq 0 \) and \( z \in \{010111, 100100\} \). Then \( xa \) contains an overlap for all \( a \in \{0, 1\} \).

**Proof.** It is easy to see that \( x = uvuvuv \) for some \( u, v \in \{0, 1\}^* \), where \( u \) and \( v \) begin with different letters. Thus one of \( uvuvva \) or \( vva \) is an overlap. \( \square \)

We can characterize the squares that can occur in an infinite overlap-free binary word. Let

\[ B = \{001001, 110110\} \]

and let

\[ \mathcal{B} = \bigcup_{k \geq 0} \mu^k(B). \]

**Theorem 9.** The set of squares that can occur in an infinite overlap-free binary word is \( \mathcal{A} \cup \mathcal{B} \). Furthermore, if \( w \) is an infinite overlap-free binary word containing a subword \( x \in \mathcal{B} \), then \( w \) begins with \( x \) and there are no other occurrences of \( x \) in \( w \).
Proof. Let \( w \) be an infinite overlap-free binary word beginning with a square \( yy \notin A \cup B \). Suppose further that \( yy \) is a smallest such square that can be extended to an infinite overlap-free word. If \( |y| \leq 3 \), then \( yy \notin A \cup B \) is one of 011011 or 100100, neither of which can be extended to an infinite overlap-free word by Lemma 8.

We assume then that \( |y| > 3 \). Since, by Theorem 7, \( yy \) is a conjugate of a word in \( A \), we have two cases.

Case 1: \( yy = \mu(zz) \) for some \( z \in \{0,1\}^* \). By Theorem 4, \( w = \mu(zzw') \) for some infinite \( w' \), where \( zzw' \) is overlap-free. Thus \( zz \) is a smaller square not in \( A \cup B \) that can be extended to an infinite overlap-free word, contrary to our assumption.

Case 2: \( yy = a\mu(zz')\pi \) for some \( a \in \{0,1\} \) and \( z, z' \in \{0,1\}^* \). By Theorem 4, \( yy \) is followed by \( a \) in \( w \), and so \( yya \) is an overlap, contrary to our assumption.

Since both cases lead to a contradiction, our assumption that \( yy \notin A \cup B \) must be false.

To see that each word in \( A \cup B \) does occur in some infinite overlap-free binary word, note that Allouche, Currie, and Shallit [2] have shown that the word \( s = 001001 \) is overlap-free.

Finally, to see that any occurrence of \( x \in B \) in \( w \) must occur at the beginning of \( w \), we note that by an argument similar to that used in Lemma 8, \( ax \) contains an overlap for all \( a \in \{0,1\} \), and so \( x \) occurs at the beginning of \( w \).\[\square\]

4 Words containing infinitely many overlaps

In this section we construct various infinite \( \alpha \) power-free binary words containing infinitely many overlaps. We begin by considering the infinite 7/3 power-free binary words.

Proposition 10. For all \( p \geq 1 \), an infinite 7/3 power-free word contains only finitely many occurrences of overlaps with period \( p \).

Proof. Let \( x \) be an infinite 7/3 power-free word containing infinitely many overlaps with period \( p \). Let \( k \geq 0 \) be the smallest integer satisfying \( p \leq 3 \cdot 2^k \). Suppose \( x \) contains an overlap \( w \) with period \( p \) starting in a position \( \geq 2^{k+1} \). Then by Theorem 4 we can write

\[
x = u_1\mu(u_2) \cdots \mu^{k-1}(u_k)\mu^k(y),
\]

where each \( u_i \in \{\epsilon, 0, 1, 00, 11\} \). The overlap \( w \) occurs as a subword of \( \mu^k(y) \). By Lemma 8, \( y \) contains an overlap with period \( p/2^k \leq 3 \). But any overlap with period \( \leq 3 \) contains a 7/3 power. Thus, \( x \) contains a 7/3 power, a contradiction. \[\square\]

The following theorem provides a striking contrast to Shur’s result [15] that the bi-infinite 7/3 power-free words are overlap-free.

Theorem 11. There exists a 7/3 power-free binary word containing infinitely many overlaps.
Proof. We define the following sequence of words: \( A_0 = 00 \) and \( A_{n+1} = 0\mu^2(A_n) \), \( n \geq 0 \). The first few terms in this sequence are

\[
\begin{align*}
A_0 &= 00 \\
A_1 &= 001100110 \\
A_2 &= 0011001101001100101100110100110010110
\end{align*}
\]

We first show that in the limit as \( n \to \infty \), this sequence converges to an infinite word \( a \). It suffices to show that for all \( n \), \( A_n \) is a prefix of \( A_{n+1} \). We proceed by induction on \( n \). Certainly, \( A_0 = 00 \) is a prefix of \( A_1 = 0\mu^2(00) = 001100110 \). Now \( A_n = 0\mu^2(A_{n-1}) \), \( A_{n+1} = 0\mu^2(A_n) \), and by induction, \( A_{n-1} \) is a prefix of \( A_n \). Applying Lemma 1, we see that \( A_n \) is a prefix of \( A_{n+1} \), as required.

Note that for all \( n \), \( A_{n+1} \) contains \( \mu^{2n}(A_1) \) as a subword. Since \( A_1 \) is an overlap with period 4, \( \mu^{2n}(A_1) \) contains \( 2^n \) overlaps with period \( 2^{n+2} \). Thus, \( a \) contains infinitely many overlaps.

We must show that \( a \) does not contain a \( 7/3 \) power. It suffices to show that \( A_n \) does not contain a \( 7/3 \) power for all \( n \geq 0 \). Again, we proceed by induction on \( n \). Clearly, \( A_0 = 00 \) does not contain a \( 7/3 \) power. Consider \( A_{n+1} = 0\mu^2(A_n) \). By induction, \( A_n \) is \( 7/3 \) power-free, and by Theorem 2 so is \( \mu^2(A_n) \). Thus, if \( A_{n+1} \) contains a \( 7/3 \) power, such a \( 7/3 \) power must occur as a prefix of \( A_{n+1} \). Note that \( A_{n+1} \) begins with 00110011. The word 00110011 cannot occur anywhere else in \( A_{n+1} \), as that would imply that \( A_{n+1} \) contained a cube 000 or 111, or the \( 5/2 \) power 1001100110. If \( A_{n+1} \) were to begin with a \( 7/3 \) power with period \( \geq 8 \), it would contain two occurrences of 00110011, contradicting our earlier observation. We conclude that the period of any such \( 7/3 \) power is less than 8. Checking that no such \( 7/3 \) power exists is now a finite check and is left to the reader.

In fact, we can prove the following stronger statement.

**Theorem 12.** There exist uncountably many \( 7/3 \) power-free binary words containing infinitely many overlaps.

**Proof.** For a finite binary sequence \( b \), we define an operator \( g_b \) on binary words recursively by

\[
\begin{align*}
g_b(w) &= w \\
g_{0b}(w) &= \mu^2(g_b(w)) \\
g_{1b}(w) &= 0\mu^2(g_b(w)).
\end{align*}
\]

Note that \( g_b(0) \) always starts with a 0, so that for any finite binary words \( p \) and \( b \), \( g_p(0) \) is always a prefix of \( g_{pb}(0) \). Since \( g_0(0) \) is not a prefix of \( g_1(0) \), \( g_{p0}(0) \) is not a prefix of \( g_{pb}(1) \) for any \( p \), so that distinct \( b \) give distinct words. Given an infinite binary sequence \( b = b_1b_2b_3\ldots \) where the \( b_i \in \{0,1\} \), define an infinite binary sequence \( w_b \) to be the limit of

\[
g_{e}(00), g_{b_1}(00), g_{b_1b_2}(00), g_{b_1b_2b_3}(00), \ldots
\]
By an earlier argument, each \( w_b \) is \( 7/3 \) power-free. Since \( g_1(00) = 001100110 \) is an overlap, \( g_{b_1}(00) = g_b(001100110) \) ends with an overlap for any finite word \( b \). Thus, each 1 in \( b \) introduces an overlap in \( w_b \). Since uncountably many binary sequences contain infinitely many 1’s, uncountably many of the \( w_b \) are \( 7/3 \) power-free words containing infinitely many overlaps. \( \square \)

Next, we show that the sequence \( a \) constructed in the proof of Theorem 11 is an automatic sequence (in the sense of [3]).

**Proposition 13.** The sequence \( a \) is 4-automatic.

**Proof.** We show that \( a = g(h^\omega(0)) \), where \( h \) and \( g \) are the morphisms defined by

\[
\begin{align*}
h(0) &= 0134 & g(0) &= 0 \\
h(1) &= 2134 & g(1) &= 0 \\
h(2) &= 3234 & g(2) &= 0 \\
h(3) &= 2321 & g(3) &= 1 \\
h(4) &= 3421 & g(4) &= 1 
\end{align*}
\]

We make some observations concerning 2-letter subwords: The sequence \( h^\omega(0) \) clearly does not contain any of the words 11, 14, 22, 24, 31, 33, 41 or 44. In fact, neither 12 nor 43 appears as a subword either: Words 12 and 43 do not appear internally in \( h(i) \), \( 0 \leq i \leq 4 \); therefore, if 43 appears in \( h^n(0) \), it must ‘cross the boundary’ in one of \( h(12), h(14), h(22) \) or \( h(24) \). Since 14, 22 and 24 do not appear in \( h^\omega(0) \), word 43 can only appear in \( h^n(0) \) as a descendant of a subword 12 in \( h^{n-1}(0) \). However, the situation is symmetrical; word 12 can only appear in \( h^n(0) \) as a descendant of a subword 43 in \( h^{n-1}(0) \). By induction, neither 43 nor 12 ever appears.

The point of the previous paragraph is that

- \( h(0) \) always occurs in the context \( h(0)2 \)
- \( h(1) \) always occurs in the context \( h(1)2 \)
- \( h(2) \) always occurs in the context \( h(2)2 \)
- \( h(3) \) always occurs in the context \( h(3)3 \)
- \( h(4) \) always occurs in the context \( h(4)3 \)

The word \( h^\omega(0) \) can thus be parsed in terms of a new morphism \( f \):

\[
\begin{align*}
f(0) &= 1342 \\
f(1) &= 1342 \\
f(2) &= 2342 \\
f(3) &= 3213 \\
f(4) &= 4213.
\end{align*}
\]
The parsing in terms of \( f \) works as follows: If we write \( h^\omega(0) = 0w \), then \( w = f(0w) \). It is useful to rewrite this relation in terms of the finite words \( h^n(0) \). For non-negative integer \( n \) let \( x_n \) be the unique letter such that \( h^n(0)x_n \) is a prefix of \( h^\omega(0) \). Thus \( x_0 = 1, x_1 = 2, \) etc. We then have

\[
h^n(0)x_n = 0f(h^{n-1}(0)), \quad n \geq 1. \tag{1}
\]

Since for all \( a \in \{0, 1, 2, 3, 4\} \), \( g(f(a)) = \mu^2(g(a)) \), we have \( g(f(u)) = \mu^2(g(u)) \) for all words \( u \). Therefore, applying \( g \) to (1)

\[
g(h^n(0)x_n) = g(0f(h^{n-1}(0))) = g(0)g(f(h^{n-1}(0))) = 0\mu^2(g(h^{n-1}(0))), \quad n \geq 1.
\]

From this relation we show by induction that \( A_n \) is the prefix of \( g(h^{n+1}(0)) \) of length \((4^n+1+3 \cdot 4^n-1)/3\). Certainly, \( A_0 = 00 \) is the prefix of length 2 of \( g(h(0)) = 0011 \). Consider \( A_n = 0\mu^2(A_{n-1}) \). We can assume inductively that \( A_{n-1} \) is the prefix of \( g(h^n(0)) \) of length \((4^n+3 \cdot 4^{n-1}-1)/3\). Writing \( g(h^n(0)) = A_{n-1}z \) for some \( z \), we have

\[
g(h^{n+1}(0)x_{n+1}) = 0\mu^2(g(h^n(0))) = 0\mu^2(A_{n-1}z) = A_n\mu^2(z),
\]

for some \( x_{n+1} \), whence \( A_n \) is a prefix of \( g(h^{n+1}(0)) \). Since \( |A_n| = 4|A_{n-1}| + 1 \), we have \( |A_n| = (4^n+1+3 \cdot 4^n-1)/3 \), as required.

The result of Theorem 11 can be strengthened even further.

**Theorem 14.** For every real number \( \alpha > 2 \) there exists a real number \( \beta \) arbitrarily close to \( \alpha \), such that there is an infinite \( \beta^+ \) power-free binary word containing infinitely many \( \beta \) powers.

**Proof.** Let \( s \geq 3 \) be a positive integer, and let \( r = |\alpha + 1| \). Let \( t \) be the largest positive integer such that \( r - t/2^s > \alpha \), and such that the word obtained by removing a prefix of length \( t \) from \( \mu^s(0) \) begins with 00. Let \( \beta = r - t/2^s \). Since \( \alpha \geq r - 1 \), we have \( t < 2^s \). Also, \( \mu^3(0) = 01101001 \) and \( \mu^3(1) = 10010110 \) are of length 8, and both contain 00 as a subword; it follows that \( |\alpha - \beta| \leq 8/2^s \), so that by choosing large enough \( s \), \( \beta \) can be made arbitrarily close to \( \alpha \).

We construct sequences of words \( A_n, B_n \) and \( C_n \). Define \( C_0 = 00 \). For each \( n \geq 0 \):

1. Let \( A_n = 0\mu^2(C_n) \).
2. Let \( B_n = \mu^s(A_n) \).
3. Remove the first \( t \) letters from \( B_n \) to obtain a new word \( C_{n+1} \) beginning with 00.
Since each $A_n$ begins with the $r$ power $0^r$, each $B_n = \mu^s(A_n)$ begins with an $r$ power of period $2^s$. Removing the first $t$ letters ensures that $C_{n+1}$ commences with an $(r2^s - t)/2^s$ power, viz., a $\beta$ power. The limit of the $C_n$ gives the desired infinite word. Let us check that this limit exists:

Let $w$ be the word consisting of the first $t$ letters of $\mu^s(0)$. Since all the $A_n$ commence with 0 by construction, all the $B_n$ commence with $\mu^s(0)$, and hence with $w$. This means that $B_n = wC_{n+1}$ for each $n$.

We show that $A_n$ is always a prefix of $A_{n+1}$ by induction. Certainly $A_0$ is a prefix of $A_1$. Assume that $A_{n-1}$ is a prefix of $A_n$. Since $A_n = 0^{r-2}C_n$ and $A_{n+1} = 0^{r-2}C_{n+1}$, $A_n$ is a prefix of $A_{n+1}$ if $C_n$ is a prefix of $C_{n+1}$. We conclude that $A_n$ is a prefix of $A_{n+1}$.

It follows that $C_n$ is a prefix of $C_{n+1}$ for $n \geq 0$, so that the limit of the $C_n$ exists. It will thus suffice to prove the following claim:

**Claim:** The $A_n$, $B_n$, and $C_n$ satisfy the following:

1. The word $C_n$ contains no $\beta^+$ powers.
2. The only $\beta^+$ power in $A_n$ is $0^r$.
3. Any $\beta^+$ powers in $B_n$ appear only in the prefix $\mu^s(0^r)$.

Certainly $C_0$ contains no $\beta^+$ powers, and since $\beta > r - 1$, the only $\beta^+$ power in $A_0$ is $0^r$.

Now suppose that $B_n = \mu^s(0^{r-2})\mu^s(C_n)$ contains a $\beta^+$ power $u$ with period $p$. Since $C_n$ contains no $\beta^+$ powers, Theorem 3 ensures that $\mu^s(C_n)$ contains no $\beta^+$ powers. We can therefore write $B_n = xu\overline{y}$ where $|x| < |\mu^s(0^{r-2})|$. In other words, $u$ overlaps $\mu^s(0^{r-2})$ from the right. By Theorem 3, the preimage of $B_n$ under $\mu$, i.e., $\mu^{-1}(A_n)$, contains a $\beta^+$ power of length at least $|u|/2$ and period $p/2$. In fact, iterating this argument, $A_n$ contains a $\beta^+$ power of period $p/2^s$ of length at least $|u|/2^s$. Since the only $\beta^+$ power in $A_n$ is $0^r$, with period 1, we see that $p/2^s = 1$, whence $p = 2^s$ and $|u| \leq r2^s$.

Recall that $B_n$ has a prefix $\mu^s(0^r)$ which also has period $2^s$, and that this prefix is overlapped by $u$. It follows that all of $xu$ is a $\beta^+$ power with period $p = 2^s$. However, as just argued, this means that $|xu| \leq r2^s = |\mu^s(0^r)|$, so that $u$ is contained in $\mu^s(0^r)$ and part 3 of our claim holds for $B_n$. We now show that parts 1 and 2 hold for $C_{n+1}$ and $A_{n+1}$ respectively, and the truth of our claim will follow by induction.

Part 1 follows immediately from part 3.

Now suppose that $A_{n+1}$ contains a $\beta^+$ power $u$. Recall that $A_{n+1} = 0^{r-2}C_{n+1}$, and $C_{n+1}$ begins with 00, but contains no $\beta^+$ powers. It follows that $u$ is not a subword of $C_{n+1}$. Therefore, 000 must be a prefix of $u$. If $u = 0^q$ for some integer $q$, then $q \leq r$ by the construction of $A_{n+1}$, and

$$r \geq q > \beta > \alpha > r - 1.$$  

This implies that $q = r$, and $u = 0^r$, as claimed. If we cannot write $u = 0^q$, then $|u| \geq 1$. Because $u$ is a $2^+$ power, 000 must appear twice in $u$ with a 1 lying somewhere between the
two appearances. This implies that 000 is a subword of $C_{n+1}$, and hence of $B_n = \mu^s(A_n)$. However, no word of the form $\mu(w)$ contains 000. This is a contradiction. □

We conclude by presenting the following open problem.

Does there exist a characterization (in the sense of [5, 8]) of the infinite 7/3 power-free binary words?

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