Kinetic equations for sterile neutrinos from thermal fluctuations

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Abstract

We obtain non-linear kinetic equations for sterile neutrino occupancies and lepton minus baryon numbers by matching real time correlation functions of thermal fluctuations computed in an effective description to those computed in thermal quantum field theory. After expanding in the sterile-neutrino Yukawa couplings, the coefficients in the equations are written as real time correlation functions of Standard Model operators. Our kinetic equations are valid for an arbitrary number of sterile neutrinos of any mass spectrum. They can be used to describe, e.g., low-scale leptogenesis via neutrino oscillations, or sterile neutrino dark matter production in the Higgs phase.

1 Introduction

Sterile neutrinos may play a key role in the evolution of the Universe, e.g. by producing the matter-antimatter asymmetry \cite{1} or by constituting all or part of the dark matter \cite{2}. They appear in the arguably most straightforward extensions of the Standard Model which can account for non-zero masses of the active neutrinos. Having no Standard Model gauge interactions and typically small Yukawa couplings, the sterile neutrinos equilibrate only slowly, if at all, so that they can provide the non-equilibrium conditions required for baryogenesis.

Depending on the masses and couplings of the sterile neutrinos, leptogenesis can be realized in different stages of the evolution of the Universe. Thermal (or high-scale) leptogenesis requires very heavy sterile neutrinos, with masses larger than $10^6$ GeV \cite{3}. If there are two nearly mass-degenerate sterile neutrinos, this bound can be lowered to $10^3$ GeV \cite{4}. While sterile neutrinos with such large masses are well motivated by the see-saw mechanism, they are not detectable in any foreseeable experiment. Leptogenesis through oscillations \cite{5,6} (or low-scale leptogenesis) can work for even smaller masses, below $\sim 5$ GeV, and these sterile neutrinos could in principle be experimentally detected \cite{7}.

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Leptogenesis through oscillations has been described by Boltzmann equations, and in the relativistic case with several flavors with generalizations thereof \cite{8}. The momentum spectrum of sterile neutrinos is non-thermal and it can be important to keep the full momentum dependence \cite{9}, but for parameter-space scans usually momentum space averages are considered (for recent work see, e.g., \cite{10,11}). There have been various approaches which start from first principles to avoid some ad-hoc assumptions inherent in the Boltzmann equation, and to systematically include medium effects \cite{12–15}.

The approach of \cite{16} makes use of the slowness of the sterile neutrino’s interaction right from the start. Since most degrees of freedom equilibrate much faster, one needs to account for only a few non-equilibrium quantities. They can be described by kinetic equations with coefficients which only depend on the temperature and on the chemical potentials of conserved charges. The coefficients can be expressed in terms of finite temperature correlation functions \cite{16}. These relations are valid to all orders in Standard Model couplings, and can thus be used to compute higher order corrections allowing to estimate the accuracy of the approximations \cite{17}. In \cite{16} the washout rate was obtained by matching thermal fluctuations in the effective description and in quantum field theory. It was applied to the production of a single sterile-neutrino species in \cite{18}. Non-linear terms in the kinetic equations may also play an important role \cite{9}. In \cite{19} the rates for leptogenesis including non-linear terms were obtained using a quite different approach. There the sterile-neutrino masses were neglected. In \cite{20} non-linear equations were obtained also for the massive case.

In this paper we generalize the approach of \cite{16} to several species of sterile neutrinos which allows for oscillations, and by including non-linear terms. It is organized as follows. In section 2 we obtain master formulae for the coefficients in the equations of motion in terms of correlators in thermal field theory. In section 3 we define the Lagrangian and the variables used to describe the sterile neutrino densities, and we obtain kinetic equations in terms of correlators of Standard Model operators. In section 4 we discuss the conserved charges in the various temperature regimes and their connection with the chemical potentials in the evolution equations, elaborating on the role played by the lepton number carried by right-handed electrons. We give the expressions for the dissipative contribution from multiple soft scattering and compute some of the dispersive contributions, both in the symmetric phase, in section 5, and we summarize our results in section 6. We describe the perturbative solution of the equations of motion for fluctuations in appendix A, illustrate the coarse-graining method in momentum space needed to apply our method to occupancies in appendix B, summarize some useful formulae concerning Green’s functions in appendix C, and demonstrate the cancellation of some rate coefficients in appendix D.
Notation  We write four-vectors in lower-case italics, \( k \), and the corresponding three-vectors in boldface, \( \mathbf{k} \). Integrals over three-momentum are denoted by \( \int k \equiv (2\pi)^{-3} \int d^3 k \). Four-vectors are denoted as \( k = (k^0, \mathbf{k}) \). In imaginary time the Matsubara frequencies are \( k^0 = i2n\pi T \) for bosonic or \( k^0 = i(2n + 1)\pi T \) for fermionic operators with integer \( n \), \( T \) is the temperature. We denote fermionic Matsubara sums by a tilde, \( \tilde{\sum}_k \). We use the metric with signature \((+, - , - , - )\).

2  Kinetic equations and real time correlations

We consider an out-of-equilibrium system in which some quantities evolve much more slowly than most other degrees of freedom. Their deviations from equilibrium, which we denote by \( y_a \), will depend on time. The non-equilibrium state is determined by the values of the \( y_a \), and by the temperature of the system, as well as the values of the conserved charges. Therefore, the time derivative of \( y_a \) depends only on these quantities. We assume that we can expand

\[
\dot{y}_a = - \gamma_{ab} y_b - \frac{1}{2} \gamma_{abc} y_b y_c - \frac{1}{3!} \gamma_{abcd} y_b y_c y_d - \cdots
\]

(2.1)

The \( y_a \) and the rate coefficients \( \gamma \) are assumed to be real. The \( \gamma \)'s only depend on the temperature and the values of the (strictly) conserved charges. The effective kinetic equations (2.1) are valid for frequencies \( \omega \ll \omega_{\text{fast}} \) where \( \omega_{\text{fast}} \) is the characteristic frequency of the ‘fast,’ or ‘spectator’ processes, which keep the other degrees of freedom in thermal equilibrium.

For sufficiently small \( y_a \) the equations of motion can be linearized. This approximation has been used to obtain equations for thermal leptogenesis in [21]. Non-linear terms in (2.1) can be important even when all \( y_a \) are small, but hierarchical such that, e.g., \( y_1 \) is of similar size as \( y_2 y_3 \). Non-linear terms in the kinetic equations for leptogenesis through oscillations were taken into account in [9]. In resonant dark matter production [22] they naturally occur in the resummed active neutrino propagator.

2.1 Correlators in the effective theory

The thermal fluctuations of the slow variables \( y_a \) satisfy the same type of equations as (2.1), but with an additional Gaussian noise term on the right-hand side, representing the effect of rapidly fluctuating quantities\(^3\).

From these equations one can compute real-time correlation functions of the fluctuations such as

\[
C_{ab}(t) = \langle y_a(t) y_b(0) \rangle
\]

(2.2)

\(^3\)See, e.g., §118 of reference [23] on correlations of fluctuations in time.
by solving these equations and then averaging over the noise and over initial conditions.\footnote{Non-linear terms in the equation of motion could potentially lead to non-vanishing expectation values of $y_\alpha$. Therefore in general one also has to include $y$-independent terms on the right-hand side to ensure that the expectation values vanish. Eventually we want to describe deviations from thermal equilibrium with (2.1). Then the $y_\alpha$ are much larger than their thermal fluctuations, and the $y$-independent term will be small compared to the non-linear terms in (2.1) and can be neglected.}

We solve the equations of motion for the fluctuations by one-sided Fourier transformation

$$y_\alpha^+(\omega) \equiv \int_0^\infty dt \, e^{i\omega t} y_\alpha(t).$$

(2.3)

At linear order we obtain

$$y_\alpha^{+(0)}(\omega) = \left[(-i\omega + \gamma)^{-1}\right]_{ab} y_b(0) + \cdots,$$

(2.4)

where the ellipsis represents a term linear in the noise. Inserting this into the one-sided Fourier transform of (2.2) one obtains \cite{16}

$$C_{ab}^+(\omega) = \left[(-i\omega + \gamma)^{-1}\right]_{ac} \Xi_{cb}.$$  

(2.5)

Here the noise term has dropped out. When averaging over the initial conditions at $t = 0$, one encounters the real and symmetric susceptibility matrix with elements

$$\Xi_{ab} \equiv \langle y_\alpha y_\beta \rangle,$$

(2.6)

i.e., the equal time correlators $C_{ab}(0)$. As in (2.2), the average in (2.6) is canonical, that is, at fixed values of the conserved charges.

The rate matrix $\gamma_{ab}$ can be extracted from (2.5) by considering frequencies $\omega$ which are parametrically much larger than the elements of the matrix $\gamma$. Then one can expand (2.5) in $\gamma/\omega$. For real $\omega$ the leading term in this expansion is purely imaginary. Thus by taking the real part of (2.5) one can extract the next term which is linear in $\gamma$ \cite{16},

$$\text{Re} \, C_{ab}^+(\omega) = \frac{1}{\omega^2} \gamma_{ac} \Xi_{cb} + O(\omega^{-3}) \quad \text{(for real } \omega).$$

(2.7)

Here it is important that we take the one-sided Fourier transform instead of the Fourier transform because the latter only depends on the symmetric part of $\gamma$.

Now we go beyond the linear order. We include the non-linear terms in the equation motion, and expand

$$y = y^{(0)} + y^{(1)} + y^{(2)} + \cdots$$

(2.8)
where $y^{(n)}$ is of order $(y(0))^{n+1}$ and vanishes at the initial time $t = 0$. We will encounter the generalization of (2.6),

$$
\Xi_{a_1 a_2 \cdots a_n} \equiv \langle y_{a_1} y_{a_2} \cdots y_{a_n} \rangle_c,
$$

where the subscript ‘C’ indicates that we only include the connected part, for which we assume

$$(\Xi_{a_1 \cdots a_m})^{1/m} \ll (\Xi_{a_1 \cdots a_n})^{1/n} \quad (m > n \geq 2).$$

(2.10)

This can be seen as a consequence of our assumption that we can expand the right-hand side of equation (2.1), since the time evolution of the fluctuations also determines their equal time correlations (see, e.g., [24]). For the occupancies of sterile neutrinos we have checked the assumption (2.10) in appendix B. The coefficient $\gamma_{abc}$ of the quadratic term in equation (2.1) can then be extracted from the correlation function

$$
\mathcal{C}_{a(bc)}(t) \equiv \langle y_a(t)y_by_c(0) \rangle
$$

as follows (for details see appendix A). We have

$$
\mathcal{C}_{a(bc)}(t) = \langle y_a^{(0)}(t)y_by_c(0) \rangle + \langle y_a^{(1)}(t)y_by_c(0) \rangle.
$$

(2.11)

(2.12)

The first term on the right-hand side is very similar to (2.5), one only has to replace (2.6) with the expectation value of three factors of $y(0)$. Again we take the one-sided Fourier transform. Our assumption (2.10) allows us to neglect the contribution from $\Xi_{abcd}$, which gives

$$
\langle y_a^{(1)}(\omega)y_by_c(0) \rangle_c = \frac{1}{\omega^2} \gamma_{ade} \Xi_{db} \Xi_{ec} + O(\omega^{-3}).
$$

(2.13)

Thus we obtain

$$
\text{Re } \mathcal{C}_{a(bc)}^+(\omega) = \frac{1}{\omega^2} [\gamma_{ade} \Xi_{db} \Xi_{ec} + \gamma_{ai} \Xi_{abc}] + O(\omega^{-3}),
$$

(2.14)

which allows us to extract $\gamma_{abc}$. Similarly we obtain the coefficient multiplying the cubic term in (2.1) by solving the equation of motion for $y_a$ perturbatively up to linear order in $\gamma_{abc}$ and $\gamma_{abcd}$ and successively computing the connected correlation function

$$
\mathcal{C}_{a(bcd)}(t) \equiv \langle y_a(t)y_by_cy_d(0) \rangle_c.
$$

(2.15)

Following the same line of arguments, we obtain

$$
\text{Re } \mathcal{C}_{a(bcd)}^+(\omega) = \frac{1}{\omega^2} \left[ \gamma_{aijk} \Xi_{ib} \Xi_{jc} \Xi_{kd} + \frac{1}{2} \gamma_{aij} \Xi_{ijbcd} + \gamma_{ai} \Xi_{abcd} \right] + O(\omega^{-3}).
$$

(2.16)

Note that (2.11) is connected, because the expectation value of a single $y$ vanishes.
2.2 Correlators in the microscopic theory

The one-sided Fourier transforms of the correlation functions (2.2), (2.11), and (2.15) as well as the susceptibilities (2.9) can also be computed in the microscopic quantum theory. In the range of validity $\omega \ll \omega_{\text{fast}}$ of the effective equations of motion (2.1) they have to match their counterparts in the effective theory. This way the coefficients in (2.1) can be computed from (2.7), (2.14) and (2.16) with the quantum correlators on the left-hand side, evaluated in the regime $\gamma \ll \omega \ll \omega_{\text{fast}}$. In this regime $C_{\text{ab}}^+$ has to match the one-sided Fourier transform of the microscopic correlation function

$$C_{\text{ab}}(t) \equiv \frac{1}{2} \langle \{ y_a(t), y_b(0) \} \rangle. \tag{2.17}$$

Since $\omega_{\text{fast}} \lesssim T$, we are dealing with frequencies $\omega$ much smaller than the temperature. In this regime the one-sided Fourier transform of (2.17) is approximately given by

$$C_{\text{ab}}^+(\omega) = -\frac{i T}{\omega} \left[ \Delta_{\text{ab}}(\omega) - \Delta_{\text{ab}}(0) \right], \tag{2.18}$$

where

$$\Delta_{\text{ab}}(\omega) \equiv \int \frac{d\omega'}{2\pi} \rho_{\text{ab}}(\omega') \frac{\omega}{\omega' - \omega}. \tag{2.19}$$

(2.19) is an analytic function off the real axis, and

$$\rho_{\text{ab}}(\omega) \equiv \int dt e^{i\omega t} \langle [y_a(t), y_b(0)] \rangle \tag{2.20}$$

is the spectral function of the bosonic operators $y_a$ and $y_b$. For real $\omega$, $\Delta_{\text{ab}}(\omega + i0^+)$ equals the retarded two-point function $\Delta_{\text{ret}}(\omega)$ (see (C.6)). Matching $C^+$ and $C^+$, and using (2.7) as well as the fact that $\Delta_{\text{ab}}(0)$ is real one obtains the master formula

$$\gamma_{\text{ab}} = T \omega \text{Im} \Delta_{\text{ret}}(\omega)(\Xi^{-1})_{\text{ab}} \quad (\gamma \ll \omega \ll \omega_{\text{fast}}). \tag{2.21}$$

For real spectral functions it agrees with the Kubo-type relation in [16]. Following the same steps with (2.14) and (2.16) we obtain the master formulas

$$\gamma_{\text{abc}} = \left[ T \omega \text{Im} \Delta_{\text{ret}}(\omega)(\Xi^{-1})_{\text{fde}} \right] (\Xi^{-1})_{\text{db}}(\Xi^{-1})_{\text{ec}}, \tag{2.22}$$

$$\gamma_{\text{abcd}} = \left[ T \omega \text{Im} \Delta_{\text{ret}}(\omega)(\Xi^{-1})_{\text{efg}} \right] - \frac{1}{2} \gamma_{\text{aj}} (\Xi^{-1})_{\text{eb}}(\Xi^{-1})_{\text{fc}}(\Xi^{-1})_{\text{gd}}, \tag{2.23}$$

where in both cases $\gamma \ll \omega \ll \omega_{\text{fast}}$. As in (2.15), we include only the connected piece of the correlator $\Delta_{\text{ret}}^\text{\tiny{2}(efg)}$. In general the operators inside the retarded correlators will not necessarily commute at equal times.
In some cases it is more convenient to compute the correlators of time derivatives of one or both of the operators, and then use

\[
\Delta_{AB}^{\text{ret}}(\omega) = \frac{1}{\omega} \left[ i \Delta_{AB}^{\text{ret}}(\omega) + \langle [A(0), B(0)] \rangle \right],
\]

(2.24)

\[
\Delta_{AB}^{\text{ret}}(\omega) = \frac{1}{\omega^2} \left[ \Delta_{AB}^{\text{ret}}(\omega) + i \langle [A(0), \dot{B}(0)] \rangle + \omega \langle [A(0), B(0)] \rangle \right].
\]

(2.25)

3 Kinetic equations for sterile neutrinos

We now consider the Standard Model extended by \( n_s \) flavors of sterile (or right-handed) neutrinos \( N_i \). The full Lagrangian of the system is given by

\[
\mathcal{L} = \mathcal{L}_{\text{SM}} + \frac{1}{2} \bar{N}(i\sigma^2 \phi - M)N - (\bar{N} h J + \text{H.c.})
\]

(3.1)

with

\[
J \equiv \tilde{\varphi}^\dagger \ell,
\]

(3.2)

where \( \tilde{\varphi} \equiv i\sigma^2 \varphi^* \) with the Pauli matrix \( \sigma^2 \). We describe the sterile neutrinos by the Majorana spinors \( N_i \), in a flavor basis with diagonal mass matrix \( M \). In general, the matrix of Yukawa couplings \( h \) is then non-diagonal with elements \( h_{\alpha \alpha} \) for \( \alpha = e, \mu, \tau \). For \( n_s > 1 \) the Yukawa couplings can violate CP, with an amount which may be much larger than the one in the quark sector of the Standard Model. The latter is way too small for generating the observed baryon asymmetry of the Universe\(^6\)

\[
\frac{n_B}{s} = 8.71(4) \cdot 10^{-11},
\]

(3.3)

and we neglect it in the following.

We consider temperatures at which the muon Yukawa interaction is in equilibrium, which is the case when \( T \ll 10^9 \text{ GeV} \) \(^{27}\). Then there are two types of slow variables we are interested in. The first type are the charges

\[
X_\alpha \equiv L_\alpha - B/3
\]

(3.4)

where \( L_\alpha \) is the lepton number in flavor \( \alpha \) and \( B \) is the baryon number. Unlike \( L_\alpha \) and \( B \), the conservation of \( (3.4) \) is not violated by the chiral anomaly, so that sphaleron processes do not change \( (3.3) \). In the presence of conserved charges the \( X_\alpha \) can have a non-vanishing equilibrium value \( X_\alpha^{\text{eq}} \), so that the \( y_\alpha \) in \( (2.1) \) correspond to \( \delta X_\alpha \equiv X_\alpha - X_\alpha^{\text{eq}} \). We discuss

\(^6\)The quantity \( \Omega_B h^2 \) has been measured by Planck \(^{25}\) to a high precision. It is related to the quantity in \( (3.3) \), where \( n_B \) and \( s \) denote baryon number density and entropy density, respectively, via \( n_B/s = 3.887 \cdot 10^{-9} \Omega_B h^2 \), see, e.g., chapter 5.2 of reference \(^{25}\).
the equilibrium expectation values in section 4. At $T \sim 85$ TeV, when the rate of electron Yukawa interactions is comparable to the Hubble rate \[28\], the lepton number carried by right-handed electrons $L_{eR}$ is a slow variable, and at $T \sim 130$ GeV, when electroweak sphalerons freeze out \[29\], baryon number is slow as well. The conservation of $L_{eR}$ and $B$ is not violated by the sterile-neutrino Yukawa interaction. However, the $X_\alpha$, $L_{eR}$ and $B$ are individually correlated with $U(1)$-hypercharge, so that their evolution equations do not decouple. When $T \gtrsim 10^9$ GeV the muon Yukawa coupling causes slow interactions and additional, flavor-non-diagonal charges have to be taken into account \[12\].

We consider a finite volume $V$ and take $V \to \infty$ in the end. Without the Yukawa interaction, $N$ would be a free field and the equation of motion would give

$$N_i(x) = \sum_{k, \lambda} \frac{1}{\sqrt{2E_{ki}}} \left[ e^{ikx} u_{k\lambda} a_{k\lambda}(t) + e^{-ikx} v_{k\lambda} a_{k\lambda}^\dagger(t) \right]$$

(3.5)

with $a_{k\lambda}(t) = \exp(-iE_{ki}t)a_{k\lambda}(0)$ and $E_{ki} = (k^2 + M_i^2)^{1/2}$. The spinors $u$ and $v$ are chosen such that $a^\dagger_{k\pm}(t)$ creates a sterile neutrino with helicity $\pm 1/2$. The sterile neutrinos cannot be expected to be in kinetic equilibrium since kinetic and chemical equilibration are due to the same processes. Therefore the other type of slow variables consists of the phase space densities, or occupancies, of the sterile neutrinos. For each $k$ and $\lambda$ the occupation number operators form a matrix, called matrix of densities, or density matrix, with elements

$$f_{ij}(k\lambda) \equiv a_{k\lambda}^\dagger a_{kj\lambda}.$$

(3.6)

In the presence of the Yukawa interaction in (3.1) we define the occupation number operators through equations (3.5) and (3.6). Their equilibrium values read

$$f^{eq}_{ij}(k\lambda) = \delta_{ij} f_F(E_{ki}),$$

(3.7)

with the Fermi-Dirac distribution $f_F(E) \equiv 1/(e^{E/T} + 1)$.

The variables appearing in the effective kinetic equations (2.1) are real. Therefore we consider the Hermitian operators

$$f^a_{k\lambda} \equiv T^a_{ij} a_{k\lambda}^\dagger a_{kj\lambda}.$$

(3.8)

The $T^a$ are the Hermitian $U(n_s)$ generators satisfying the normalization and completeness relations

$$\text{tr}(T^a T^b) = \frac{\delta^{ab}}{2}, \quad \sum_a T^a_{ij} T^a_{kl} = \delta_{il}\delta_{jk}.$$  

(3.9)

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7 In the literature there are several conventions for the order of the indices. We use the one of \[19\].
8 This definition slightly differs from the one in \[18\]. The definition in \[18\] and our present definition are equivalent to the first and the second definition in \[19\], respectively.
We write $\delta f \equiv f - f^{\text{eq}}$ for both $(f_{k\lambda})_{ij}$ and $f^a_{k\lambda}$. The $\delta f^a_{k\lambda}$ appear as slow variables $y_a$ in the equations of motion (2.1).

We will expand the kinetic equations (2.1) to order $h^2$, and to second order in the deviations $\delta X$ of the charges (3.4), including the terms of order $(\delta X)^2 \delta f$. Terms with more than one factor of $\delta f$ do not enter the kinetic equations at order $h^2$, as we show in appendix D.

The fluctuations of the occupancies are comparable to the deviation of $f$ from equilibrium, and the higher susceptibilities (2.9) of $f$ do not satisfy (2.10). Strictly speaking, the theory in the preceding sections is therefore not applicable to the occupancy. However, one can coarse-grain the operators $f$ over a certain momentum region, and the resulting operators satisfy the requirements of the framework developed in the above sections. The dependence on the momentum averaging volume drops out in the end. This way we can effectively use the original operators $f$ instead of their smeared versions in our equations. We elaborate on the details of this procedure in appendix B.

3.1 Correlation functions

With a certain degeneracy of the vacuum masses in (3.1), the sterile neutrinos undergo oscillations, which appear already at order $h^0$. They are described by the off-diagonal matrix elements in (3.6). One can obtain the equation of motion simply by taking the time derivative of the operators (3.6) and taking the expectation value. However, it is also instructive to use the Kubo relation (2.21). The equilibrium contribution cancels the disconnected contractions, such that we can replace $\delta f \rightarrow f$ in (3.15) and consider only the connected two-point function

$$\Delta f^a_{k\lambda} f^b_{p\lambda'}(t) \equiv \langle T f^a_{k\lambda}(t) f^b_{p\lambda'}(0) \rangle_c. \quad (3.10)$$

The time $t$ in (3.10) is imaginary, $t = -i\tau$ with real $\tau$, and $T$ denotes time ordering with respect to $\tau$. We encounter the 2-point functions of the operators appearing in (3.5), for which we find (for both positive and negative $\tau$)

$$\langle T a^\dagger_{ik\lambda}(-i\tau) a_{jp\lambda'}(0) \rangle = \delta_{ij} \delta_{kp} \delta_{\lambda\lambda'} T \sum_{p_0} \frac{e^{p_0\tau}}{p_0 - E_{ik}}. \quad (3.11)$$

The retarded correlators appearing in (2.21) are obtained by Fourier transforming (3.10) with imaginary bosonic Matsubara frequency $\omega = i\omega_n \equiv i2\pi n T$, and then continuing $\omega$ to the real axis, $\Delta_{\text{ret}}(\omega) = \Delta(\omega + i0^+)$ with real $\omega$. One encounters factors like $1/(\omega + E_{ki} - E_{kj})$. To give a contribution to (2.21) this has to be approximately $1/\omega$ when $\omega \ll \omega_{\text{fast}}$. This requires

$$|\delta M_{ij}^2|/E_{ki} \sim |\delta M_{ij}^2|/T \ll \omega \quad (3.12)$$
with

$$\delta M_{ij}^2 \equiv M_i^2 - M_j^2, \quad (3.13)$$

which means that the frequency for the oscillations between sterile flavors $i$ and $j$ has to be small compared to $\omega_{\text{fast}}$. After a simple computation we obtain at order $h^0$

$$\omega \Delta_{f_{k\lambda} f_{p\lambda'}}^{\text{ret}} (\omega) = - \delta_{kp} \delta_{\lambda \lambda'} \sum_{(ij)} \frac{f_k^i (E_{ki})}{2 E_{ki}} \frac{\delta M_{ij}^2}{E_{ji}} T_{ji}^a T_{ji}^b + O(\omega, h^2, \delta M^4), \quad (3.14)$$

The notation $(ij)$ indicates that we only sum over indices with $|\delta M_{ij}^2|/T \ll \omega_{\text{fast}}$. After expanding in $\delta M^2$ and $h$, the retarded correlator entering (2.21) no longer knows about the scale $\gamma$, and we can put $\omega \to 0$.

At order $h^2$ only the first terms in square brackets in (2.22) and (2.23) survive in the kinetic equations (3.38) and (3.39), if we expand to quadratic order in chemical potentials. Therefore we only discuss these terms in the following. Terms with $\Xi_{XXX}$ are canceled once the slow charges are expressed through their chemical potentials, see (3.35), and the terms containing $\Xi_{fff}$ or $\Xi_{ffff}$ lead to cancellation of the coefficients $\gamma_{fff}$ and $\gamma_{ffff}$, which we demonstrate in appendix D. Since they are defined as the connected pieces, the contributions from the susceptibilities (2.9) with mixed indices $f$ and $X$ vanish at leading order in $h$, since the correlators they multiply in the master formulae are already of order $h^2$.

We keep the dependence on the absolute Majorana mass scale in all expressions. This is important in order to be able to obtain kinetic equations describing light sterile neutrino dark matter production during the QCD epoch, because here the $O(h^2)$ terms actually go like $h^2 M^2$ [30]. This dependence emerges from the retarded active neutrino self-energy, which appears in the kinetic equations in the Higgs phase. At order $h^2$ we neglect terms of order $\delta M^2$, because both $h^2$ and $\delta M^2$ are small quantities.

To determine the coefficients $\gamma_{fff}$, $\gamma_{ffX}$, and $\gamma_{ffXX}$ we employ (2.21) through (2.23) directly, without making use of (2.25). The latter relation turns out to be very inconvenient here if there is more than one sterile flavor due to an UV divergent contribution from the commutators, which only cancels against a divergence in the first term in square brackets of (2.25). Thus we consider

$$\Delta_{f_{k\lambda} (p_{\lambda 1}, X_{a_1} \cdots X_{a_n})} (t) \equiv \langle T f_{k\lambda}^a (t) (f_{p\lambda}^b X_{a_1} \cdots X_{a_n}) (0) \rangle_c. \quad (3.15)$$

Again, we were able to replace $\delta f \to f$ and here also $\delta X \to X$ on the right-hand side of (3.15), since we need only the connected correlator. We adopt this procedure in the remainder of this section, understanding that all expectation values are connected. As
in (3.10), \( t \) is imaginary. \( f_{\beta \lambda}^b \) and the charge operators \( X_{\alpha} \) commute at equal times. The \( \mathcal{O}(h^2) \) contribution to (3.15) becomes

\[
\Delta f^a_{\lambda, k}(f_{\beta \lambda}^b X_{\alpha_1} \cdots X_{\alpha_n})(t) = \int d^4x_1 d^4x_2 \times \text{tr} \left\{ h^1 \left\langle \mathcal{T} N(x_2) \bar{N}(x_1) f^a_{\lambda, k}(t) f_{\beta \lambda}^b(0) \right\rangle \right. \\
\left. \times h \left\langle \mathcal{T} J(x_1) \bar{J}(x_2)(X_{\alpha_1} \cdots X_{\alpha_n})(0) \right\rangle \right\} ,
\]

where the trace refers to both spinor and active flavor indices. Since we consider the leading order in \( h \), we can neglect the sterile-neutrino Yukawa interaction in the expectation values on the right-hand side of (3.16). Our definition of \( a \) and \( a^\dagger \) allows us to substitute them for \( N \) and \( \bar{N} \) in the path integral, and then work with (3.11). The second expectation value in (3.16) is now a correlation function containing only Standard Model fields. When \( h \) is neglected, the charges \( X_\alpha \) are conserved, and one can introduce chemical potentials \( \mu_{X_\alpha} \) such that

\[
\left\langle \mathcal{T} J(x_1) \bar{J}(x_2)(X_{\alpha_1} \cdots X_{\alpha_n})(0) \right\rangle = \left[ T \frac{\partial}{\partial \mu_{X_{\alpha_1}}} \cdots T \frac{\partial}{\partial \mu_{X_{\alpha_n}}} \Delta_{J\bar{J}}(x_1 - x_2, \mu) \right]_{\mu_X = 0}
\]

with

\[
\Delta_{J_\alpha \bar{J}_\beta}(x, \mu) \equiv Z^{-1} \text{tr} \left\{ T J_\alpha(x) \bar{J}_\beta(0) \exp \left[ \frac{1}{T} \left( \sum_\gamma \mu_{X_\gamma} X_\gamma - H_{\text{SM}} \right) \right] \right\},
\]

where \( Z \equiv \text{tr} \exp \left[ \left( \sum_\gamma \mu_{X_\gamma} X_\gamma - H_{\text{SM}} \right) / T \right] \) is the partition function at finite chemical potentials of the slowly varying charges, and \( H_{\text{SM}} \) is the Hamiltonian containing all Standard Model interactions which are in equilibrium at the temperature of interest. The traces in (3.18) and in \( Z \) run over states with definite values of the conserved charges, which is why only the slow charges appear in the exponential. Note that one can introduce chemical potentials for the \( X_\gamma \) only after expanding in \( h \). Therefore one cannot write \( \Delta_{J\bar{J}}(\omega, \mu) \). For vanishing \( h \), (3.18) is diagonal in the active flavor indices, so that we can write

\[
\Delta_{J_\alpha \bar{J}_\beta} \equiv \delta_{\alpha\beta} \Delta_{\alpha}. \quad (3.19)
\]

The chemical potential of the operator \( J_\alpha \) (in the sense of (C.3)) is \( \mu_{J_\alpha} = -\mu_{X_\alpha} \). Therefore the function \( e^{\mu_{X_\alpha}} \Delta_{\alpha}( -i\tau, x, \mu ) \) is anti-periodic in \( \tau \), see (C.3), and the Fourier decomposition of (3.18) reads:

\[
\Delta_\alpha( -i\tau, x, \mu ) = T \sum_{p^0} e^{ - (p^0 + \mu_{X_\alpha}) \tau } \Delta_\alpha( p^0, x, \mu ).
\]

See, e.g., chapter 8.1 of reference [31].
The Matsubara correlator on the right-hand side of (3.20) can be expressed through its spectral function\(^{10}\) via
\[
\Delta_\alpha(-it_\tau, \mu) = \int \frac{dw'}{2\pi} e^{-w't_\tau} \rho_\alpha(w', \mu) \left\{ \Theta(\tau) \left[ 1 - f_F(w' - \mu_{X_\alpha}) \right] \right. \\
\left. - \Theta(-\tau) f_F(w' - \mu_{X_\alpha}) \right\}. \tag{3.21}
\]

The spectral function satisfies
\[
\rho_\alpha(k, \mu) = \frac{1}{i} \left[ \Delta_\alpha^{\text{ret}}(k, \mu) - \Delta_\alpha^{\text{adv}}(k, \mu) \right], \tag{3.22}
\]
and according to (C.7), the retarded and advanced correlators are given by
\[
\Delta_\alpha^{\text{ret,adv}}(k, \mu) = \Delta_\alpha(k + u[-\mu_{X_\alpha} \pm i0^+], \mu) \tag{3.23}
\]
with real \(k^0\), where \(u = (1, 0)\) is the four-velocity of the plasma.

After summing over the Matsubara frequencies we analytically continue \(\omega\) towards the real axis which gives the \(\mathcal{O}(\hbar^2)\) contribution to the retarded correlator
\[
\omega \Delta_{j_k\lambda}^{(f_{X_\alpha X_{\alpha_1}} \cdots X_{\alpha_n})}(\omega) = -\delta_{k\rho} \delta_{\lambda\gamma} \sum_{\beta(ij)} \frac{f_F(E_{ki})}{2E_{ki}} \left[ T \frac{\partial}{\partial \mu_{X_{\alpha_1}}} \cdots T \frac{\partial}{\partial \mu_{X_{\alpha_n}}} \right] \tag{3.24}
\]
\[
\times \left\{ h_{ij}^t T_{ij}^a h_{ij} b X_{\alpha \lambda} \Delta_{ij}^{\text{ret}}(k, \mu) s_{kl\lambda} - h_{ij}^t T_{ij}^a h_{ij} b X_{\alpha \lambda} \Delta_{ij}^{\text{adv}}(k, \mu) s_{kl\lambda} \right. \\
\left. + h_{ij}^t T_{ij}^a h_{ij} b X_{\alpha \lambda} \Delta_{ij}^{\text{ret}}(-k, \mu) s_{kl\lambda} - h_{ij}^t T_{ij}^a h_{ij} b X_{\alpha \lambda} \Delta_{ij}^{\text{adv}}(-k, \mu) s_{kl\lambda} \right\}_{\mu_X=0}
\]
with \(k_j \equiv (E_{kj}, k)\). Here and in the following we take \(\omega \to 0\) on the right-hand side, as we did in equation (3.14). Since
\[
\left[ \bar{\nu}_{k\lambda\gamma} \Delta_{j_k\lambda}^{\text{ret}}(q, \mu) s_{k\lambda\gamma} \right]^* = \bar{\nu}_{k\lambda\gamma} \Delta_{j_k\lambda}^{\text{adv}}(q, \mu) s_{k\lambda\gamma}, \tag{3.25}
\]
the right-hand side of (3.24) is purely imaginary.

The computation of the correlator entering \(\gamma_{fX}\) is analogous to the one relevant for \(\gamma_{fXX}\), with a few less extra steps. In the computation of the latter we make use of (2.24), where the commutator vanishes, and obtain the contribution
\[
\omega \text{Im} \Delta_{j_k(X_{\alpha X_{\beta}})}^{\text{ret}}(\omega) = -\text{Re} \left[ \Delta_{j_k(X_{\alpha X_{\beta}})}^{\text{ret}}(\omega) + \Delta_{j_k(X_{\alpha X_{\beta}})}^{\text{ret}}(\omega) + \Delta_{j_k([X_{\alpha}, X_{\beta}])}^{\text{ret}}(\omega) \right] \tag{3.26}
\]
to the master formula (2.22). Since
\[
\left[ \Delta_{AB}^{\text{ret}}(\omega) \right]^* = \Delta_{AB}^{\text{ret}}(-\omega) \tag{3.27}
\]

\(^{10}\)The spectral function for fermionic operators is defined as in (2.24), but with an anticommutator instead of the commutator.
for bosonic operators $A$ and $B$, the rightmost term in (3.26) vanishes for $\omega \to 0$, and we are left with the two terms in which the charge $X$ to the right of $\dot{X}$ below (3.16) after having expanded the correlators to quadratic order in $h^1$, relating averages like $\langle T J \dot{J} X \rangle$ to derivatives with respect to chemical potentials of $\langle T J \rangle_{\mu}$. The time derivative of the charge is obtained from the Heisenberg equation of motion and reads

$$
\dot{X}_\alpha(t) = i \sum_j \int d^3x \left[ \dddot{N}_j(t, x) h_{j\alpha} J_\alpha(t, x) - \text{H.c.} \right]. \tag{3.28}
$$

Using (3.25) and omitting terms of order $\delta M^2$, we obtain (for $n = 0, 1$)

$$
\text{Re} \Delta_{\Delta_{\mu}}(\chi_{\alpha}X_{\beta})(\omega) = - \sum_{i,j} \frac{1}{4E_{ki}} \frac{f_F(E_{ki})}{f_F(E_{ki})} T_{ij} \left[ \left( T \frac{\partial}{\partial \mu X_{\beta}} \right)^n \right. \left. \times \left\{ \left( 1 + e^{-\mu x_i/T} \right) f_F(E_{ki} - \mu X_{\beta}) h^\dagger_{\alpha i} h_{j\alpha} \bar{\mu}_{k\lambda} \rho_{\alpha}(k_i, \mu) u_{k\lambda} 

- \left( 1 + e^{\mu x_i/T} \right) f_F(E_{ki} + \mu X_{\beta}) h^\dagger_{\alpha i} h_{j\alpha} \bar{\mu}_{k\lambda} \rho_{\alpha}(-k_i, \mu) v_{k\lambda} \right\} \right]_{\mu = 0}. \tag{3.29}
$$

Analogously, we use (2.24) to relate $\Delta_{\chi_{(X\cdots)}}^\text{ret}$ to $\Delta_{\chi_{(X\cdots)}}^\text{ret}$, where the commutator vanishes once again. Using (3.25) and (3.22) we obtain the correlators

$$
\omega \Delta_{\chi_{(X\cdots)}}^\text{ret}(\mu; \nu; \cdots; \nu) = i \sum_{\gamma(i,j)} \frac{f_F(E_{ki})}{2E_{ki}^2} T_{ij} \left[ \frac{\partial}{\partial \mu X_{\beta_1}} \cdots \frac{\partial}{\partial \mu X_{\beta_n}} \right. \left. \times \left\{ h^\dagger_{\alpha i} h_{j\alpha} \bar{\mu}_{k\lambda} \rho_{\alpha}(k_i, \mu) u_{k\lambda} - h^\dagger_{\alpha i} h_{j\alpha} \bar{\mu}_{k\lambda} \rho_{\alpha}(-k_i, \mu) v_{k\lambda} \right\} \right]_{\mu = 0}, \tag{3.30}
$$

where once again we have dropped terms of order $\delta M^2$.

The correlators containing only charges $X$ are obtained similarly to the steps that yield (3.29). Equation (2.25) gives

$$
\Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega) = \frac{1}{\omega^2} \left[ \Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega) + i \langle [X_{\alpha}, (X_{\beta} X_{\gamma})] \rangle + \omega \langle [X_{\alpha}, (X_{\beta} X_{\gamma})] \rangle \right]. \tag{3.31}
$$

The first commutator on the right-hand side drops out when taking the imaginary part in (2.22). The second one vanishes because the charges (3.4) commute. We rewrite the first term on the right-hand side as

$$
\Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega) = \Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega) + \Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega) + \Delta_{\chi_{(X\cdots)}}^\text{ret}(\omega), \tag{3.32}
$$

11In contrast to the computation of (3.21), here only one additional interaction is needed, because $\dot{X}$ is already of order $h$, see (3.28).
where the third term on the right-hand side of (3.32) does not contribute to (2.22) due to the relation (3.27). Since $\dot{X} = O(h)$, at order $h^2$ the first two terms can be obtained from

$$\Delta^\text{ret}_{X\alpha(X_{\beta}X_{\gamma})}(\omega) = \left[ T \frac{\partial}{\partial \mu_{X\alpha}} \Delta^\text{ret}_{X\alpha X_{\beta}}(\omega, \mu) \right]_{\mu_x = 0}. \quad (3.33)$$

Then we take the thermodynamic limit replacing $\sum_k \rightarrow V \int k$ and we find (the trace runs over spinor indices)

$$\frac{1}{\omega} \text{Im} \Delta^\text{ret}_{X\alpha X_{\beta}}(\omega, \mu) = -\delta_{\alpha\beta} V \int \sum_i |h_{ia}|^2 \frac{1}{4E_{ki}} f'_F(E_{ki}) f_F(E_{ki}) \mu_{X\alpha} \rho_{\alpha}(k_i, \mu) \times \text{tr} \left\{ \hat{k}_{i} \left[ (1 + e^{-\mu_{X\alpha}/T}) f_F(E_{ki} - \mu_{X\alpha}) \rho_{\alpha}(k_i, \mu) + (1 + e^{\mu_{X\alpha}/T}) f_F(E_{ki} + \mu_{X\alpha}) \rho_{\alpha}(-k_i, \mu) \right] \right\}. \quad (3.34)$$

### 3.2 Kinetic equations

In [16] the washout rate was written in terms of charges. Here we express all rates in terms of the chemical potentials $\mu_{X\alpha}$ by making use of

$$\delta X_{\alpha} = \Xi_{X_{\alpha}X_{\beta}} \frac{\mu_{X\beta}}{T} + \frac{1}{2} \Xi_{X_{\alpha}X_{\beta}X_{\gamma}} \frac{\mu_{X\beta} \mu_{X\gamma}}{T} + O(\mu^3). \quad (3.35)$$

When $L_{\alpha R}$ is slowly evolving, (3.35) contains an additional term $\Xi_{X_{\alpha}L_{\alpha R}} \mu_{L_{\alpha R}}/T$, which does, however, not show up in the kinetic equations for $f_{\lambda\lambda}$ and $X_{\alpha}$, since $\sum_{\alpha} (\Xi^{-1})_{X_{\beta}X_{\alpha}} \Xi_{X_{\alpha}L_{\alpha R}} = \delta_{X_{\beta}L_{\alpha R}} = 0$. The same argument holds for baryon number $B$ in the temperature regime where its evolution is slow. The $\Xi_{XXX}$ vanish, unless there is some conserved charge which is nonzero.

In order to close the set of equations we need to specify the susceptibilities in (3.35). Furthermore, to evaluate the correlation functions (3.18) one has to switch from an ensemble in which the conserved charges have fixed values to a grand-canonical one in which all charges $Q_{a}$ fluctuate, not just the slowly varying ones. Then we have a relation similar to (3.35), but for all charges,

$$Q_{a} = \sum_{b} \chi_{ab} \frac{\mu_{Qb}}{T} + O(\mu^3). \quad (3.36)$$

$\chi$ is the susceptibility matrix in the full grand-canonical ensemble, in which the charges are odd functions of the chemical potentials. Therefore, unlike in (3.35), no terms of order $\mu^2$ and no equilibrium values appear in (3.36).

Combining relations (3.35) and (3.36) leads to $(\Xi^{-1})_{ab} = (\chi^{-1})_{ab}$, where $a$ and $b$ label only slow charges, and this way the equilibrium expectation values of the charges can be obtained. The matrix $\chi$ depends on the temperature regime, see section 4.
From now on we understand the $\Delta_\alpha(k,\mu)$ to be defined in a grand-canonical description, where the $\mu$ label chemical potentials associated with all charges.

We expand our kinetic equations to quadratic order in slowly varying chemical potentials, then the higher order terms in (3.35) do not contribute. The term with $\Xi_{XXX}$ on the right-hand side of (3.35) cancels the second term in square brackets in (2.14). The corresponding susceptibility of the occupancies $\Xi_{fff}$, which appears only in (2.22), leads to cancellation of the coefficient $\gamma_{fff}$, see appendix [D] and $\Xi_{ffff}$ in (2.23) does the same with the coefficient $\gamma_{ffff}$. At order $h^0$ all other $\Xi_{abc}$ vanish identically, and the only other nonzero $\Xi_{abcd}$ are those with four charges $\delta X$, which, however, enter (2.23) only for a coefficient multiplying three factors of $\delta X$ in (2.1), which is beyond our expansion to order $\mu^2$.

The susceptibilities of the sterile-neutrino occupancy read (without sum over $k$ or $\lambda$)

$$\Xi_{f姥姥k姥姥f} = T_{ij}T_{ji} f_F(E_k)[1 - f_F(E_k)]$$

Plugging (3.14), (3.24), and (3.29) into the respective master formulae (2.21)–(2.23) we obtain the kinetic equations

$$\dot{f}_{k\lambda}^{mn} = \frac{i}{2E_{km}} \left\{ \delta M_{mn}^2 (f_{k\lambda})_{mn} \right. \right.$$ 

$$+ \sum_l \left[ \bar{\nu}_{kl\lambda} \left( h_{ma} \Delta_{a}^{\text{ret}}(k_l,\mu) h_{al}^\dagger [(f_{k\lambda})_{ml} - \delta_{ml} f_F(E_{kl} - \mu_X)] \right. \right.$$ 

$$- h_{la} \Delta_{a}^{\text{adv}}(k_l,\mu) h_{al}^\dagger [(f_{k\lambda})_{ln} - \delta_{ln} f_F(E_{kl} - \mu_X)]] \right. u_{kl\lambda} \right.$$ 

$$+ \bar{\nu}_{kl\lambda} \left( h_{ma} \Delta_{a}^{\text{ret}}(-k_l,\mu) h_{al}^\dagger [(f_{k\lambda})_{ln} - \delta_{ln} f_F(E_{kl} + \mu_X)] \right.$$ 

$$- h_{la} \Delta_{a}^{\text{adv}}(-k_l,\mu) h_{al}^\dagger [(f_{k\lambda})_{ml} - \delta_{ml} f_F(E_{kl} + \mu_X)]] \right. \right.$$ 

$$+ O(\mu^3, h^2 \delta M^2, \delta M^4, h^4),$$

for those elements of the occupancy matrix for which $|\delta M_{mn}^2|/T \ll \omega_{\text{fast}}$ (including, of course, the diagonal elements). For the other elements the right-hand side vanishes in our approximation. The sum is over indices $l$ for which $|\delta M_{ml}^2|/T \ll \omega_{\text{fast}}$. $f_{k+}$ does not appear on the right-hand side of the kinetic equation for $f_{k-}$, and vice versa.

In equation (3.35) we express $\bar{k}$ through the completeness relation of the $u$ or $v$ spinors. Together with equation (3.30), and taking the limit $V \to \infty$, we obtain the kinetic
equation for the charge density $n_{X\alpha} \equiv X_{\alpha}/V$

$$\dot{n}_{X\alpha} = \sum_{(ij)\lambda} \int_k \frac{1}{2E_{ki}} \left\{ \tau_{k\lambda} h_{ia}\rho_{\alpha}(k_{ii},\mu)h_{aj}^t u_{k\lambda i} \left[ (f_{k\lambda})_{ij} - \delta_{ij} f_F(E_{ki} - \mu_{X\alpha}) \right] \right. $$

$$- \left. \tau_{k\lambda} h_{ja}\rho_{\alpha}(-k_{ii},\mu)h_{ai}^t v_{k\lambda i} \left[ (f_{k\lambda})_{ij} - \delta_{ij} f_F(E_{ki} + \mu_{X\alpha}) \right] \right\}$$

$$+ \mathcal{O}(\mu^3, h^2\delta M^2, h^4).$$

(3.39)

When additional processes are slow, one needs additional kinetic equations. Around $T \sim 130$ GeV, this is the case for the $B+L$ violating electroweak sphaleron processes [29]. Then it is convenient to include a kinetic equation for $B$ [32][33], since $B$ is not violated by the sterile-neutrino Yukawa interaction so that only the sphaleron rate enters this equation.

When $T \sim 85$ TeV, the lepton number carried by right-handed electrons evolves slowly, and one has to include the kinetic equation for $L_{eR}$ [28].

Using a different approach from ours, an equation similar to (3.38) was derived previously in [19] assuming $M_i \ll T$ for all Majorana masses, so that the condition (3.12) is satisfied. In [19] the chemical potential for $L_{\alpha}$ appears in $f_F$ instead of the one for $X_{\alpha}$, which is nevertheless consistent, see section 4. Unlike the equation in [19], our (3.38) contains not only scattering contributions, but also dispersive ones, see section 5.1. In [19] the latter are incorporated at a later stage. Aside from that, the first term in the curly bracket in (3.38) (which contains the $u$-spinors) is equivalent to the corresponding one in [19]. For vanishing Majorana masses (3.39) coincides with the corresponding equation in [19]. Furthermore, for non-vanishing Majorana masses, our contributions containing the $u$-spinors also appear there. The $v$-spinor contribution is equivalent, after the replacement described in footnote 13.

In reference [30] kinetic equations for the spin-averaged occupancies of sterile neutrinos without near mass-degeneracy and for lepton numbers in the Higgs phase have been obtained. There the spin-asymmetry of the sterile neutrinos has been neglected. In reference [20] the same authors have obtained equations for a hierarchical system with one light and two heavy sterile neutrinos in the Higgs phase. There the kinetic equations are given in terms of the retarded correlator of $J$ as a function of slowly varying chemical potentials, like in our (3.38). The terms multiplying these correlators are given to linear

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12 See equation (2.29) of reference [19].
13 The helicity diagonal contribution containing the $v$-spinors in [19] is not consistent with our equation (3.38). However, it becomes consistent after applying (2.22) of [19].
14 See equation (2.32) of reference [19].
15 See equations (2.21) and (2.24) of reference [30].
order in slowly varying quantities. Using the relation
\[ \bar{v}_{ki\lambda} \Delta^\text{ret}_{\alpha}(q, -\mu) v_{ki\lambda} = - \bar{u}_{ki,-\lambda} \Delta^\text{adv}_{\alpha}(q, -\mu) u_{ki,-\lambda}, \] (3.40)
(no sum over repeated indices) which is valid when the Standard-Model CP violation is neglected, together with (3.25), we can reproduce the kinetic equations for the light flavor and the heavy ones as well as the one for the lepton asymmetries.

3.3 Small Majorana masses

In low-scale leptogenesis the sterile-neutrino masses are small compared to the temperature, so that typically \( M_i \ll |k| \). Then they can be neglected in the terms containing their (also small) Yukawa couplings, so that the helicity eigenspinors \( u \) and \( v \) are purely right- and left-handed. Since the operator \( J \) is purely left-handed, the terms with \( v^+ \) or \( u^- \) drop out. Then we also have \( u_{ki+} \bar{u}_{ki+} = v_{ki-} \bar{v}_{ki-} = P_R \delta \) with the chiral projector \( P_R \equiv (1 + \gamma^5)/2 \). Furthermore, condition (3.12) is trivially satisfied, so that the kinetic equations simplify to

\[ \dot{f}_{k+} = \frac{i}{2|k|} \left\{ M^2 f_{k+} - \sum_{\alpha l} \text{tr} \left[ \bar{h} \left( h_{\alpha l} \Delta^\text{ret}_{\alpha}(k, \mu) h^\dagger_{\alpha l} \left[ (f_{k+})_{ml} - \delta_{ml} f_F(|k| - \mu X_\alpha) \right] 
- h_{\alpha l} \Delta^\text{adv}_{\alpha}(k, \mu) h^\dagger_{\alpha m} \left[ (f_{k+})_{ln} - \delta_{ln} f_F(|k| - \mu X_\alpha) \right] \right) \right\}, \] (3.41)

\[ \dot{f}_{k-} = \frac{i}{2|k|} \left\{ M^2 f_{k-} - \sum_{\alpha l} \text{tr} \left[ \bar{h} \left( h_{\alpha l} \Delta^\text{ret}_{\alpha}(k, \mu) h^\dagger_{\alpha l} \left[ (f_{k-})_{ln} - \delta_{ln} f_F(|k| + \mu X_\alpha) \right] 
- h_{\alpha l} \Delta^\text{adv}_{\alpha}(k, \mu) h^\dagger_{\alpha m} \left[ (f_{k-})_{ml} - \delta_{ml} f_F(|k| + \mu X_\alpha) \right] \right) \right\}, \] (3.42)

and (again \( n_{X_\alpha} \equiv X_\alpha/V \))

\[ \dot{n}_{X_\alpha} = \sum_{(ij)} \int \frac{1}{2|k|} \text{tr} \left[ \bar{h} \left\{ h_{ija} \rho_\alpha(k, \mu) h^\dagger_{ij} \left[ (f_{k+})_{ij} - \delta_{ij} f_F(|k| - \mu X_\alpha) \right] 
- h_{jia} \rho_\alpha(-k, \mu) h^\dagger_{ij} \left[ (f_{k-})_{ij} - \delta_{ij} f_F(|k| + \mu X_\alpha) \right] \right\} \right]. \] (3.43)

\(^{16}\)Since we have made the transition to the grand-canonical description, \( \mu \) in equation (3.40) now denotes the chemical potentials of all charges.

\(^{17}\)See equations (2.5) and (2.6) of reference [20].

\(^{18}\)See equation (2.4) of reference [20].
In (3.41)–(3.43) we have $k^0 = |\mathbf{k}|$, the traces refer to spinor indices, and we have neglected terms of order $\mu^3$, as well as terms of order $\hbar^2 M^2$, $M^4$ and $h^4$. Similar equations have been obtained in reference [34]. Keeping in mind that they use the index convention of reference [8], we can reproduce their kinetic equation\(^\text{[19]}\) for the sterile neutrino occupancies by setting $\text{tr} [\mathbf{k} \Delta^{\text{ret}}_\alpha (k, \mu)] = \text{tr} [\mathbf{k} \Delta^{\text{adv}}_\alpha (k, \mu)]^* \rightarrow -T^2/4 + i |\mathbf{k}| (\gamma^{(0)} + \mu_\alpha \gamma^{(2)})$ in our (3.41) and (3.42), and neglecting terms quadratic in chemical potentials. Recalling (3.22), we can also reproduce their kinetic equation\(^\text{[20]}\) corresponding to our (3.43) in the same way. Setting $\mu = 0$ in the spectral function $\rho_\alpha$ in (3.43) and expanding the Fermi distribution to linear order in $\mu X_\alpha$, one obtains the washout term which was found in [16].

4 Susceptibilities and right-handed electron number

The computation of the susceptibilities in (3.36) in the symmetric phase is described in [16], where the leading order (in Standard Model couplings) contributions to the pressure have been obtained. There also the $\mathcal{O}(g)$ corrections and some of the $\mathcal{O}(g^2)$ contributions can be found, and the remaining $\mathcal{O}(g^2)$ contributions have been obtained in [35]. In the symmetric phase, the only conserved charge that is correlated with the $X_\alpha$ at all temperatures is the $U(1)$-hypercharge $Y$. The zero-momentum mode of the hypercharge gauge field plays the role of the corresponding chemical potential $\mu_Y$ in the path integral formalism [32] where it ensures hypercharge neutrality of the plasma.

When electroweak sphalerons are in equilibrium, our statistical operator which determines $\Delta_\alpha$ contains $\mu_{X_\alpha} X_\alpha$, but no separate chemical potential for baryon number because the latter is not conserved. In reference [19] $\mu_\alpha L_\alpha + \mu_B B$ appears. However, using the equilibrium conditions one can match the coefficients which gives $\mu_\alpha = \mu_{X_\alpha}$, and $\mu_B = -\sum_\alpha \mu_{X_\alpha}/3$. Therefore the chemical potentials appearing in the distribution functions in our kinetic equations are consistent with those in [19].

In the temperature range $8.5 \cdot 10^4 \text{ GeV} \ll T \ll 10^9 \text{ GeV}$ the lepton number $L_{eR}$ carried by right-handed electrons is not yet efficiently violated by the electron Yukawa coupling [28] so that it constitutes an additional conserved charge, and we have to introduce a corresponding chemical potential in (3.36). The relation between all charges and their chemical potentials has been obtained in [28]\(^\text{[21]}\). Hypercharge neutrality implies

$$\mu_Y = \frac{1}{11} \mu_{L_{eR}} + \frac{8}{33} \sum_\alpha \mu_{X_\alpha},$$

\(^\text{19}\)See equation (2.14) of reference [34].
\(^\text{20}\)See equation (2.18) of reference [34].
\(^\text{21}\)See equation (A.2) of reference [28].
and the relation between $L_{eR}$ and its chemical potential reads

$$\mu_{L_{eR}} = -\frac{5}{6}\mu_{X_e} + \frac{4}{15}(\mu_{X_\mu} + \mu_{X_\tau}) + \frac{33}{5T^2V}L_{eR}. \quad (4.2)$$

Around $T \sim 8.5 \cdot 10^4$ GeV the interactions mediated by the electron Yukawa coupling happen at a rate comparable to the Hubble rate, so that $L_{eR}$ is slowly varying according to the evolution equation in reference [28]. The relation between charges and chemical potentials remains the same as above, and also (4.1) and (4.2) are valid. In particular, even though the sterile-neutrino interactions do not violate the conservation of $L_{eR}$, and the electron Yukawa coupling does not violate the one of $X_\alpha$, the evolution equations of $L_{eR}$ and the $X_\alpha$ are coupled through the matrix of susceptibilities.

At much lower temperatures the right-handed electron lepton number is in equilibrium, and no chemical potential $\mu_{L_{eR}}$ is included. Between $160 \text{ GeV} \ll T \ll 8.5 \cdot 10^4$ GeV, all Standard Model interactions are in equilibrium, and the relation between charges and chemical potentials is found in [16]. Imposing hypercharge neutrality leads to the relation $\mu_Y = \mu_Y(\mu_{X_\alpha})$ given in [19].

Around $T \sim 130$ GeV electroweak sphalerons freeze out [29], so that baryon number is a slow variable, and we have to introduce a corresponding chemical potential $\mu_B$. The relation between charges and chemical potentials can be found in [36], where also the developing Higgs expectation value is taken into account. Hypercharge neutrality yields the relation $\mu_Y(\mu_B, \mu_{X_\alpha})$ found in [19]. In this temperature regime our statistical operator contains $\mu_{X_\alpha}X_\alpha + \mu_BB$, while the one in reference [19] is the same as in the high-temperature regime discussed above. Matching the chemical potentials again yields $\mu_\alpha = \mu_{X_\alpha}$, and this time $\mu_B^{\text{here}} = \mu_B^{\text{there}} - \sum_\alpha \mu_{X_\alpha}/3$, so that again the chemical potentials in the distribution functions appearing in the kinetic equations coincide.

Deep in the Higgs phase ($T \ll 130$ GeV) the susceptibilities have a non-trivial dependence on the temperature. They have been studied in [37].

5 Standard Model correlators in the symmetric phase

Deep in the symmetric phase one has to distinguish two temperature regimes. When $M_i \gg gT$, at leading order in the Standard Model couplings the dissipative (imaginary) part of $\Delta^{\text{ret}}_\alpha$ is determined by hard $2 \leftrightarrow 2$ scattering processes. For $M_i \lesssim gT$, nearly collinear $1 \leftrightarrow 2$ decays and inverse decays involving a Higgs boson, a SM lepton and a

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22See equation (63) of reference [16].
23See equation (4.2) of reference [19].
24See equations (3.8) and (3.9) of reference [36].
25See equations (4.5) and (4.6) of reference [19].
26See appendix A of reference [37].
sterile neutrino, plus the same process with additional soft scatterings, which are sometimes referred to as $1n \leftrightarrow 2n$ processes, also contribute at leading order [38].

The multiple soft scatterings need to be resummed, which is referred to as Landau-Pomeranchuk-Migdal (LPM) [39–41] resummation. The result gives an imaginary contribution to $\Delta^\text{ret}_\alpha(k_j, \mu)$ which can be computed along the lines of [38,42]. It can be expressed in terms of the 2-component vector function $f_j(b)$ and the scalar function $\psi_j(b)$, which both depend on the 2-component impact parameter vector $b$. They can be obtained by solving the differential equations in [38] with $M_N$ replaced by $M_j$. It is straightforward to generalize the analysis of [38] to non-zero chemical potentials which gives

$$\Delta^\text{ret, LPM}_\alpha(k_j, \mu) = \frac{i}{2} \int \frac{dp_\parallel}{2\pi} \frac{1}{|k| - p_\parallel} \left[ f_B \left( p_\parallel - |k| + \frac{\mu Y}{2} \right) + f_F \left( p_\parallel - \mu X_\alpha + \frac{\mu Y}{2} \right) \right]$$

$$\times P_L \lim_{b \to 0} \left\{ \left( \gamma^0 - \hat{k} \cdot \gamma \right) \text{Re} \psi_j(b) + \frac{1}{8p_\parallel^2} \left( \gamma^0 + \hat{k} \cdot \gamma \right) \text{Im} \nabla_b \cdot f_j(b) \right\}.$$ (5.1)

Here $f_B(E) \equiv 1/(e^{E/T} - 1)$ is the Bose-Einstein distribution, $P_L \equiv (1 - \gamma_5)/2$ is a chiral projector, and $\hat{k} \equiv k/|k|$. The second term in the curly bracket is of order $g^2$ times the first. Nevertheless, it has to be kept because when sandwiched between the $u$- and $v$-spinors, the first term gets multiplied by $M_j^2$ which here is assumed to be order $g^2 T^2$ or smaller. The LPM contribution was computed in [19], where the result does not contain a chiral projector. Otherwise it is consistent with equation (5.1) (the second term in the curly bracket differs from the corresponding one in [19] only by higher orders in $g$). Chiral projectors were correctly included in [19] when the result was sandwiched between the $u$ and $v$ spinors which makes it consistent with ours (cf. footnote 13 on page 16). The $2 \to 2$ scattering contributions to the rate coefficients have been computed in [19].

5.1 Dispersive contributions

The imaginary parts of the 2-point functions in (3.41), (3.42) have been computed in [19] at nonzero chemical potentials. Here we compute the real part in the symmetric phase which modifies the dispersion relations of the sterile neutrinos. We include the chemical potentials to linear order and we work at leading order in Standard Model couplings, assuming $M_i \ll |k|$. The leading order is contained in the 1-loop contribution, which reads

$$\Delta_\alpha(k^0, k, \mu) = T \sum_{p^0} \int \frac{2 P_L(p - \mu \ell_i \hat{\gamma})}{(p - \mu \ell_i u)^2 \left[(k - p - \mu \varphi u)^2 - m^2_\varphi\right]}$$ (5.2)

\(^{27}\)See equation (3.4) of reference [19].
in imaginary-time. The factor 2 is the dimension of the representation of the weak SU(2). The chemical potentials in (5.2) are the ones carried by the field operators, cf. (C.4). They have the opposite sign compared to the chemical potentials carried by the particles which they annihilate. Note that in (5.2)

\[ \mu_\varphi = -\frac{\mu_Y}{2}, \]

appears, rather than \( \mu_\bar{\varphi} \). The relation between the hypercharge chemical potential \( \mu_Y \) and the chemical potentials of the slowly varying charges depends on the temperature, see section 4.

The leading contribution from (5.2) is due to soft Higgs momenta, which are cut off by the thermal Higgs mass \[ m_\varphi^2 = \frac{1}{16} (3g^2 + g'^2 + 4ht^2 + 8\lambda) (T^2 - T_0^2) \]

in the Higgs propagator, which gives rise to an infrared enhancement. Here \( T_0 = 160 \) GeV, and \( g \) and \( g' \) are the weak SU(2) and hypercharge U(1) gauge couplings, respectively. Our normalization is such that covariant derivatives are \( D_\mu = \partial_\mu + iy_\alpha g'B_\mu + \cdots \) with the hypercharge gauge field \( B \), and \( y_\varphi = 1/2 \) for the Higgs field \( \varphi \). Furthermore, \( h_t \) is the top quark Yukawa coupling, and the quartic term in the Higgs potential is given by \( \lambda(\varphi^\dagger \varphi)^2 \).

After summing over the imaginary fermionic frequency \( p^0 \) we analytically continue to real \( k^0 \) according to (3.23), and obtain

\[ \text{Re} \left[ \phi \Delta^\text{ret}_\alpha(k, \mu) \right] = -\frac{T^2}{4} + \frac{m_\varphi T}{4\pi|k|}\mu_\varphi + \mathcal{O}(g^2\mu, \mu^2) \]

with \( k^0 = |k| \). The \( g \) in the higher order terms in (5.5) stands for a generic Standard Model coupling. The first term on the right-hand side of (5.5) gives rise to the thermal mass. The \( \mu \)-dependent contribution is not simply a correction of the thermal mass, but it depends on momentum. In particular, it is enhanced at small \( k \). The leading correction from chemical potentials is independent of \( \alpha \). In reference [37] an expression corresponding to (5.5) in the broken phase has been obtained.

### 6 Summary

In this paper we have obtained non-linear kinetic equations which describe the time evolution of sterile-neutrino phase space densities and charge densities carried by Standard

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\[ ^{28}\text{Equation (5.3) is consistent with [19] and [28], where the Higgs chemical potential is defined as a particle chemical potential.} \]

\[ ^{29}\text{See equation (5.7) of reference [37].} \]
Model particles by generalizing the approach of reference [16] to include non-linear terms. To determine the coefficients in these equations we have matched not only real-time two-point functions in the effective kinetic equations for thermal fluctuations to those in thermal field theory, like in [16], but also higher point functions. The sterile neutrinos have been integrated out using a path integral over their Fourier coefficients, which correspond to their creation and annihilation operators. We have included only the leading order in their Yukawa coupling and in their Majorana mass squared differences. This way we have obtained relations between the rate coefficients and real-time correlation functions of Standard Model fields, evaluated at finite temperature and chemical potentials for charges which are conserved or slowly violated (in the case of $L_{eR}$ or $B$ for certain temperatures) by the Standard Model interactions. The rate coefficients are infrared safe in the sense that they are well behaved when a parameter characterizing a slow interaction vanishes.

The kinetic equations and the relations for the rate coefficients found in this paper are mostly consistent with the ones obtained in reference [19] the authors of which use a different starting point by making an ansatz with a non-equilibrium density matrix which contains the chemical potentials from the very start, even though we differ at intermediate steps.

We have computed the leading order correction of departures from equilibrium of the charges to the dispersion relation of the sterile neutrinos in the symmetric phase. There we have considered only the leading order contribution from Standard Model couplings. Our equations can be applied to low-scale leptogenesis and to sterile-neutrino dark matter production.

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A Perturbative solution of the equations of motion for fluctuations

The fluctuations of $y_a$ satisfy (2.1) with an additional Gaussian noise $\zeta$, and a $y$-independent term that does not play a role once we consider (2.1) for departures from equilibrium. We solve the equation of motion by one-sided Fourier transformation. Neglecting non-linear terms and expanding as in (2.8), one obtains (2.4). Now inserting (2.8) up to $y^{(1)}$
into (2.1), including the term with $\gamma_{abc}$ and dropping the one with $\gamma_{abcd}$, we obtain

$$y_a^{+(1)}(\omega) = -\frac{1}{2} \left[ ( -i\omega + \gamma )^{-1} \right]_{ab} \gamma_{bcd} \int \frac{d\omega'}{2\pi} y_{c}^{+(0)}(\omega') y_{d}^{+(0)}(\omega - \omega'), \quad (A.1)$$

where we have used that

$$y_a(t) = \int \frac{d\omega'}{2\pi} e^{-i\omega't} y_a^{+(\omega')}.$$

(A.2)

Considering now the correlator $C^+_{a(bc)}$ and inserting (2.4) yields averages like $\langle \zeta y(0) \rangle$ which vanish. We then consider frequencies much larger than the rates $\gamma_{ab}$, approximating $[( -i\omega + \gamma )^{-1}]_{ab} \approx i\delta_{ab}(\omega + i0^+)^{-1}$. The disconnected contribution vanishes, and we obtain

$$C^+_{a(bc)}(\omega) \equiv -\frac{i}{2\omega} \gamma_{ajk} \left[ \Xi_{jb} \Xi_{kc} + \Xi_{jc} \Xi_{kb} \right] \int \frac{d\omega'}{2\pi} \frac{i}{(\omega' + i0^+)} \frac{i}{(\omega - \omega' - i0^+)} \quad (A.3)$$

along with a contribution from $\gamma_{ab}$. Closing the contour in the upper half-plane gives only a contribution from the second pole and using the symmetry $\gamma_{ajk} = \gamma_{akj}$, we arrive at (2.13).

The perturbation caused by $\gamma_{abcd}$ is obtained by inserting (2.8) into (2.1), this time keeping also $y^{(2)}$. Having obtained already $y^{+(0)}$ and $y^{+(1)}$ we can now solve for $y^{+(2)}$, which reads

$$y_a^{+(2)}(\omega) = -\frac{1}{3!} \left[ ( -i\omega + \gamma )^{-1} \right]_{ab} \gamma_{bede} \times \int \frac{d\omega'}{2\pi} \int \frac{d\omega''}{2\pi} y_{c}^{+(0)}(\omega') y_{d}^{+(0)}(\omega'') y_{e}^{+(0)}(\omega - \omega' - \omega''). \quad (A.4)$$

Considering the same limit $\omega \gg \gamma$, we find contributions of orders 1, $\gamma_{ab}$, $\gamma_{abc}$ and $\gamma_{abcd}$ in the correlator (2.15). The $O(1)$ contribution is time-independent and does not contribute to the real part. The contributions from $\gamma_{ab}$ and $\gamma_{abc}$ are obtained in the same manner as before. The contribution from $\gamma_{abcd}$ reads

$$C^+_{a(bc)}(\omega) \equiv -\frac{i}{3!\omega} \gamma_{ajkl} \left[ \Xi_{jb} \Xi_{kc} \Xi_{ld} + \Xi_{jb} \Xi_{kd} \Xi_{lc} + \Xi_{jc} \Xi_{kb} \Xi_{ld} 
+ \Xi_{jc} \Xi_{kd} \Xi_{lb} + \Xi_{jd} \Xi_{kb} \Xi_{lc} + \Xi_{jd} \Xi_{kc} \Xi_{lb} \right] \times \int \frac{d\omega'}{2\pi} \frac{d\omega''}{2\pi} \frac{i}{(\omega' + i0^+)} \frac{i}{(\omega'' + i0^+)} \left( \omega - \omega' - \omega'' + i0^+ \right). \quad (A.5)$$

Because $C_{a(bc)}$ contains only the connected pieces, no contractions like $\Xi_{jk}\Xi_{lb}\Xi_{cd}$ appear in the square brackets in (A.5), and because of (2.10) we can neglect the connected part of the six-point function $\langle y_y y_yc y_jy_jy_ky_l \rangle$. Making use of the symmetry of $\gamma_{ajkl}$ under permutations of the last three indices, (A.5) can be solved for the rate coefficient. Carrying out the integrals and collecting the contributions from $\gamma_{ab}$ and $\gamma_{abc}$ eventually leads to (2.16).
B Smeared occupancies

The fluctuations of the occupancies are not small, $\Xi_{ff} \sim \delta f$. In order for the approach in sections 2.1 and 2.2 to be applicable, we consider occupancies averaged over a certain momentum space region $\Omega_k$ around $k$,

$$F_k \equiv \frac{(2\pi)^3}{V|\Omega_k|} \sum_{p \in \Omega_k} f_p.$$  \hfill (B.1)

The volume of this region $|\Omega_k|$ is taken to be independent of the spatial volume $V$, with $(2\pi)^3/V \ll |\Omega_k| \ll T^3$. The susceptibilities (2.9) of $F_k$ are of order $(V|\Omega_k|)^{-n+1}$. Now we have $\sqrt{\Xi_{F_kF_k}} \ll \delta F_k$, and the assumption (2.10) is satisfied.

We should also consider smeared occupancies in the microscopic correlators appearing in section 3.1. However, since the volume $|\Omega_k|$ is small compared to characteristic momentum scales over which the correlators in (2.21)–(2.23) vary, they can to a good approximation be replaced by

$$\Delta_{ret}^{F_k} \equiv \frac{(2\pi)^3}{V|\Omega_k|} \Delta_{ret}^{f_k},$$  \hfill (B.2)

so that the dependence on $V|\Omega_k|$ drops out when plugging (B.2) and $\Xi_{FF}$ into the master formula (2.21), and one can effectively use the unsmeared occupancies $f$.

C Green’s functions at finite temperature and chemical potentials

Here we present slight generalizations of some relations in [31] for imaginary time correlators which we use in our calculation. In the presence of one conserved charge $Q$ the 2-point function of operators $A, B$ reads

$$\Delta_{AB}(-i\tau) \equiv Z^{-1} \text{tr} \{ e^{\beta(\mu Q-H)} T [A(-i\tau)B(0)] \}$$  \hfill (C.1)

with $\beta \equiv 1/T$, the partition function $Z \equiv \text{tr} e^{\beta(\mu Q-H)}$, and the time ordering $T$ is with respect to $\tau$. (C.1) is defined for $-\beta \leq \tau \leq \beta$. We assume that $A$ carries a definite charge,

$$[Q, A] = q_A A.$$  \hfill (C.2)

Then

$$\Delta_{AB}(t + i\beta) = \pm e^{-\beta q_A} \Delta_{AB}(t)$$  \hfill (C.3)
where the chemical potential of \( A \) is defined as
\[
\mu_A \equiv q_A \mu. \tag{C.4}
\]

Therefore the function \( e^{-\mu_A \tau} \Delta_{AB}(-i\tau) \) is (anti-) periodic, and can be expanded in a Fourier series with coefficients
\[
\Delta_{AB}^M(i\omega_n) \equiv \int_0^\beta d\tau e^{i\omega_n - \mu_A \tau} \Delta_{AB}(-i\tau). \tag{C.5}
\]

(C.5) can be analytically continued to arbitrary complex frequencies off the real axis, and we denote the resulting function by \( \Delta_{AB}^M \). We need to calculate the retarded correlator
\[
\Delta_{AB}^{\text{ret}}(\omega) = i \int_0^\infty dt e^{i\omega t} \langle [A(t), B(0)] \rangle \tag{C.6}
\]
where \( \omega \) is real. It can be analytically continued to the complex plane. We denote the resulting function by \( \Delta_{AB} \), and then we have \( \Delta_{AB}^{\text{ret}}(\omega) = \Delta_{AB}(\omega + i\delta) \). The two analytic continuations are related by
\[
\Delta_{AB}^M(\omega) = \Delta_{AB}(\omega - \mu_A). \tag{C.7}
\]

**D Cancellation of the rates \( \gamma_{fff} \) and \( \gamma_{ffff} \)**

Here we demonstrate that the coefficients \( \gamma_{fff} \) and \( \gamma_{ffff} \) in the equation of motion for \( f \) vanish at order \( \hbar^2 \). For simplicity we will assume that the \( a \) and \( a^\dagger \) appearing in all occupancies in this appendix correspond to sterile neutrino generations satisfying (3.12). First consider \( \gamma_{fff} \). According to (2.22) it consists of two pieces containing only \( f \) operators (suppressing momentum indices \( k \)),
\[
\gamma_{f_a f_b f_c} = T \omega \text{Im} \left[ \Delta_{f_a f_b f_c}^{\text{ret}}(\omega) - \Delta_{f_a f_b}(\omega)(\Xi^{-1})_{f_a f_b} \Xi_{f_a f_b f_c} \right] (\Xi^{-1})_{f_a f_b} (\Xi^{-1})_{f_a f_c}. \tag{D.1}
\]

Classically, the kinematic variables commute at equal times, which is not the case in the microscopic theory. Therefore one should replace the product \( f^d f^e \) in (D.1) by its symmetrization \( \{ f^d, f^e \} / 2 \). In turn we demonstrate the cancellation of the terms with the ordering as in (D.1), the one with \( d \leftrightarrow e \) is analogous.

We obtain the generalized susceptibility
\[
\Xi_{f_a f_b f_c} = f_F \left[ 1 - f_F \right] \left\{ [1 - f_F] \text{tr} (T^a T^d T^e) - f_F \text{tr} (T^a T^c T^b) \right\}, \tag{D.2}
\]
where \( f_F = f_F(E_k) \). The mass in \( E_k \) is one of the relevant (nearly) degenerate masses, and a change in this mass gives only a correction of order \( \hbar^2 \delta M^2 \), which we neglect. We now consider the correlators as a function of imaginary time \( t = -i\tau \), before Fourier
transformation and analytic continuation to real frequency. Then, using \( (3.9) \) and the susceptibility \( (3.37) \), the second expression contains a term

\[
\Delta f^a_{f^a f^c}(t)(\Xi^{-1})_{f^a f^c f^d} \Xi_{f^d f^e f^f} \geq \frac{1}{4V E_k} T_{ij} \left\{ \left[ 1 - f_F(E_k) \right] (T^d T^e)_{lm} - f_F(E_k) (T^e T^d)_{lm} \right\}
\]

\[
\int_{0}^{1/T} dt_1 dt_2 \left\{ \langle a_{t_1}^i(t_1)a_{m}(0) \rangle \langle T a_{r}(t_2) a_{j}^i(t) \rangle \langle a_{j}(t_1) a_{i}^j(0) \rangle \right. \\
\left. - \langle T a_{r}(t_1) a_{j}(t) \rangle \langle a_{j}(t_2) a_{i}^j(0) \rangle \langle a_{i}^j(t) a_{m}(0) \rangle \right\}
\]

\[
\int d^3x_1 d^3x_2 \, \bar{u}_{q+} T_{q+} \langle J_{\alpha}(t_1, x_1) \bar{J}_{\beta}(t_2, x_2) \rangle h_{\beta r}^i u_{r+}.
\]  

(D.3)

To meaningfully define the object \( \Delta f^a_{f^a f^c}(t) \) in the path integral over the sterile neutrino fields, we separate the quantities at \( t = 0 \) by replacing \( f^d(0) \) by \( \lim_{t' \to 0^+} f^d(t') \), taking the limit in the end when the ambiguities have resolved, which is the case after the expectation value has been reduced using Wick’s theorem. The equivalent expression to the one in \( (D.3) \) now contains 8 terms which fall in either of the following two categories: (i) A product of 4 two-point functions of operators \( a \) and \( a^\dagger \), in which exactly one operator is at time \( t = 0 \), and the other one at a time that is integrated over, or (ii) a product of 3 two-point functions in which the operators are at different times, multiplied by one \( f_F \) or \( [1 - f_F] \).[30]

The 4 expressions of type (i) are \( t \)-independent, since operators at \( t \) are always to the left of those at time 0 so that they are not affected by time ordering, and the time evolutions of \( a_j(t) \) and \( a_j^\dagger(t) \) cancel up to effects of order \( h^2 \delta M^2 \), which we neglect. Time-independent parts do not contribute to our master formula. The remaining four terms of type (ii) are canceled by the terms in \( (D.3) \). The other contributions which we have not written in \( (D.3) \) are canceled in the same way, and we obtain \( \gamma_{fff} = 0 \).

The master formula for the coefficient \( \gamma_{fff} \) also contains only \( f \) operators at order \( h^2 \). The contribution from \( \gamma_{fff} \) vanishes, and the two remaining terms read

\[
\gamma_{f^a f^c f^d f^e} = T \omega \text{Im} \left[ \Delta_{f^a f^c f^d f^e}(\omega) - \Delta_{f^a f^c f^d f^e}^{\text{ret}}(\omega)(\Xi^{-1})_{f^a f^c f^d f^e} \Xi_{f^d f^e f^f f^g} \right]
\]

\[
\times \langle (\Xi^{-1})_{f^a f^c f^d f^e}(\Xi^{-1})_{f^f f^g} \rangle.
\]  

(D.4)

Using \( (3.9) \), we obtain the generalized susceptibilities

\[
\Xi_{f^a f^c f^d f^e} = f_F [1 - f_F] \left\{ [1 - f_F] T^{aT^bT^cT^d} - f_F [1 - f_F] T^{aT^bT^cT^d} \right\}
\]

\[
- f_F [1 - f_F] T^{aT^bT^cT^d} - f_F [1 - f_F] T^{aT^bT^cT^d} \right\}.
\]

(D.5)

[30] Here the time-ordering decides which one of the two expressions \( f_F \) or \( [1 - f_F] \) is generated, and the temporal separation of \( f^d \) and \( f^e \) plays a role.
The cancellation is now analogous to the one above, after replacing \( f(0) \) by \( \lim_{t'' \to 0^+} \lim_{t' \to 0^+} f(t'') f(t') \) with \( t'' > t' \).

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