Research Article

Bivariate Kumaraswamy Distribution Based on Conditional Hazard Functions: Properties and Application

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A new class of bivariate distributions is deduced by specifying its conditional hazard functions (hfs) which are Kumaraswamy distribution. The interest of this model is positively, negatively, or zero correlated. Properties and local measures of dependence of the bivariate Kumaraswamy conditional hazard (BKCH) distribution are studied. The estimation of type parameters is considered by used the maximum likelihood and pseudolikelihood of the new class. A simulation study was performed to inspect the bias and mean squared error of the maximum likelihood estimators. Finally, an application is obtained to clarify our results with the maximum likelihood and pseudolikelihood. Also, the results are used to compare BKCH distribution with bivariate exponential conditionals (BEC) and bivariate Lindley conditionals hazard (BLCH) distributions.

1. Introduction

The problem of constructing new families of continuous bivariate distributions is one of the important and continue active research problem in statistics. This is due from one side to the limitations of the existing distributions and its lack in modeling some stochastic phenomena and from the other side to meet the needs of modern and developed applied field. Balakrishnon et al. [1] obtained a new technology to build the bivariate probability density functions (pdfs) with specified conditional hfs, and the development of the models determines many of the several properties, including survival function (sf), hazard bivariate function, the Clayton–Oakes measure, and the measure conditional pdfs and hfs of the marginal and conditional pdfs. Moreover, Navarro and Sarabia [2] studied the reliability properties for the series and parallel systems with component lifetimes having these dependence models.

Therefore, the authors introduce a new form of the bivariate distributions with Kumaraswamy conditional hfs, and we studied some of the characteristics and discussed the estimation procedures of the parameters of this distribution. Moreover, we estimate the parameters of this distribution by two methods, e.g., maximum likelihood estimation (MLE) and maximum pseudolikelihood estimation (MPLE) procedures. Finally, this distribution is compared with other distributions considering a real dataset collected from a secondary source (p. 374 of Johnson and Wichern [3]).

A continuous random variable is Kumaraswamy distribution if its pdf (Kumaraswamy [4]) is

\[ f_X(x) = abx^{a-1}(1-x^a)^{b-1}, \quad 0 < x < 1, \quad a > 0, \quad b > 0. \]  

The corresponding sf is

\[ F(x) = (1 - x^a)^b, \quad 0 < x < 1, \quad a > 0, \quad b > 0. \]
and the hf is
\[ r(x) = \frac{abx^{a-1}}{1-x^b}, 0 < x < 1, a > 0, b > 0. \] (3)

We note that
\[ F(x) = \exp \left( -\int_0^x r(u)du \right), \] (4)
\[ f_X(x) = r(x) \exp \left( -\int_0^x r(u)du \right). \] (5)

This distribution is applicable to some scientific, practical, and hydrological applications (Dey et al. [5]; Ishaq et al. [6]).

2. BKCH Model

Suppose that the conditional hfs (Kumaraswamy) are distributed as
\[ r_1(x|y) = \frac{a_1b_1(y)x^{a_1-1}}{1-x^{b_1}}, 0 < x, y < 1, a_1 \text{ positive constant}, \]
\[ r_2(y|x) = \frac{a_2b_2(x)y^{a_2-1}}{1-y^{b_2}}, 0 < x, y < 1, a_2 \text{ positive constant}, \] (6)

where \( b_1(y) \) and \( b_2(x) \) are function of \( y \) and \( x \), respectively.

Then, the bivariate distribution from (6) will be called a bivariate BKCH distribution.

Theorem 1. The general bivariate distribution with conditional hfs as \( r_1(x|y) \) and \( r_2(x|y) \) is

\[ f_{X,Y}(x, y) = [N(\Theta)]^{-1} x^{(a_1-1)} y^{(a_2-1)} \exp[\theta_1 \log(1 - y^{a_2})] \]
\[ + (\theta_2 + \theta_3 \log(1 - y^{a_2}))(\log(1 - x^{a_1}))], 0 < x, y < 1, -\infty < \theta_i < \infty, \]
\[ i = 1, 2, 3, \] (7)

where \( N(\Theta) = \{\theta_1, \theta_2, \theta_3\} \) is the normalizing constant such that \( f_{X,Y}(x, y) \) integrates to 1.

Proof. The conditional pdfs from (6) using (5) are
\[ f_{X|Y}(x|y) = \frac{a_1b_1(y)x^{a_1-1}}{1-x^{b_1}} \exp \left( -\int_0^x \frac{a_1b_1(y)u^{a_1-1}}{1-u^{b_1}} du \right), \] (8)
\[ f_{Y|X}(y|x) = \frac{a_2b_2(x)y^{a_2-1}}{1-y^{b_2}} \exp \left( -\int_0^y \frac{a_2b_2(x)u^{a_2-1}}{1-u^{b_2}} du \right). \] (9)

Then, the identity \( f_Y(y) f_{X|Y}(x|y) = f_X(x) f_{Y|X}(y|x) \) yields the following relation:
\[ f_Y(y) \frac{a_1b_1(y)(1-y^{b_2})}{y^{a_2-1}} (1-x^{a_1}) b_1(y)^{-1} \]
\[ = f_X(x) \frac{a_2b_2(x)(1-x^{b_1})}{x^{a_1-1}} (1-y^{a_2}) b_2(x)^{-1}, \] (10)
where \( f_Y(y) \) and \( f_X(x) \) denote the marginal pdfs. Denoting
\[ g(y) = \log \left( f_Y(y) \frac{b_1(y)(1-y^{a_2})}{y^{a_2-1}} \right), \] (11)
\[ h(x) = \log \left( f_X(x) \frac{b_2(x)(1-x^{a_1})}{x^{a_1-1}} \right), \] (12)
then taking logarithms in (9) and using (10) and (11) WITH then taking logarithms in (10), using (11) and (12)

\[ g(y) + (b_1(y) - 1) \log(1 - x^{a_1}) - h(x) \]
\[ - (b_2(x) - 1) \log(1 - y^{a_2}) = 0, \] (13)

which is a functional equation of the form \( \sum_{k=1}^n f_k(x)g_k(y) = 0 \), whose most general solution is given by Aczel [7] as
\[ b_1(y) - 1 = \theta_1 + \theta_2 \log(1 - y^{a_2}), \]
\[ b_2(x) - 1 = \theta_1 + \theta_3 \log(1 - x^{a_1}). \] (14)

From (11) and (12), we have the corresponding marginal pdfs given by
\[ f_X(x) = [N(\Theta)]^{-1} x^{a_1-1} (1-x^{a_1})^{b_1(y)}, 0 < x, y < 1, -\infty < \theta_i < \infty, \]
\[ i = 1, 2, 3, \] (15)

and
\[ f_Y(y) = [N(\Theta)]^{-1} y^{a_2-1} (1-y^{a_2})^{b_2(x)}, 0 < x, y < 1, -\infty < \theta_i < \infty, \]
\[ i = 1, 2, 3. \] (16)

Equation (7) describes the complete class of BKCH distribution that has the three parameters \( \theta_1, \theta_2, \) and \( \theta_3 \).

Figure 1 shows the BKCH given by (7) for the special cases for any \( \theta_1, \theta_2, \) and \( \theta_3 \).
The conditional hfs for the BKCH distribution are

\[ r_1(x|y) = \frac{a_1 x^{\theta_i - 1} (\theta_2 + \theta_3 \log (1 - y^{\theta_i}) + 1)}{1 - x^{\theta_i}}, \quad 0 < x, y < 1, -\infty < \theta_i < \infty, \quad i = 1, 2, 3, \]  

\[ r_2(y|x) = \frac{a_2 y^{\theta_i - 1} (\theta_1 + \theta_3 \log (1 - x^{\theta_i}) + 1)}{1 - y^{\theta_i}}, \quad 0 < x, y < 1, -\infty < \theta_i < \infty, \quad i = 1, 2, 3. \]  

The compatibility of (17) and (18), Balakrishnan et al. [1] secures the existence of BKCH distribution. Figure 2 shows hfs \( r_1(x|y) \) of the BKCH given by (7) for the special cases for \( \theta_2 \) and \( \theta_3 \). Similarly, \( r_2(y|x) \).

### 3. BKCH Properties

#### 3.1. The Conditional Distributions. The conditional pdfs of the BKCH distribution are

\[ f_{X|Y}(x|y) = a_1 x^{\theta_i - 1} (\theta_2 + \theta_3 \log (1 - y^{\theta_i}) + 1) \]
\[ \cdot (1 - x^{\theta_i}) \left( \theta_2 + \theta_3 \log (1 - y^{\theta_i}) \right), \]

\[ f_{Y|X}(y|x) = a_2 y^{\theta_i - 1} (\theta_1 + \theta_3 \log (1 - x^{\theta_i}) + 1) \]
\[ \cdot (1 - y^{\theta_i}) \left( \theta_1 + \theta_3 \log (1 - x^{\theta_i}) \right), \]

i.e.,

\[ X|Y = y \sim \text{kumaraswamy}(\theta_2 + \theta_3 \log (1 - y^{\theta_i})), \]
\[ Y|X = x \sim \text{kumaraswamy}(\theta_1 + \theta_3 \log (1 - x^{\theta_i})). \]

When \( \theta_1 = \theta_2 \), the conditional pdfs are identical.
3.2. Regression Function. Now, using (19) and (20), the conditional moments are

\[ E(X^k | Y = y) = \frac{\Gamma(a_1 + k/a_1) \Gamma[2 + \theta_2 + \log(1 - y^{a_1}) \theta_3] (1 + \theta_2 + \log(1 - y^{a_1}) \theta_3)}{\Gamma[3 + k/a_1 + \theta_2 + \log(1 - y^{a_1}) \theta_3]}, \]

\[ E(Y^k | X = x) = \frac{\Gamma(a_2 + k/a_2) \Gamma[2 + \theta_1 + \log(1 - x^{a_2}) \theta_3] (1 + \theta_1 + \log(1 - x^{a_2}) \theta_3)}{\Gamma[3 + k/a_2 + \theta_1 + \log(1 - x^{a_2}) \theta_3]} \] (23)

For \( k = 1 \), we have

\[ E(X | Y = y) = \frac{\Gamma(a_1 + 1/a_1) \Gamma[2 + \theta_2 + \log(1 - y^{a_1}) \theta_3] (1 + \theta_2 + \log(1 - y^{a_1}) \theta_3)}{\Gamma[3 + 1/a_1 + \theta_2 + \log(1 - y^{a_1}) \theta_3]}, \] (24)


We notice that the regression function for $X|Y$ is nonlinear increasing for $\theta_1, \theta_2 < 0$ and decreasing for $\theta_1, \theta_2 > 0$ (Figure 3). Additionally, the regression function for $X|Y$ is nonlinear increasing for $\theta_2, \theta_3 < 0$ and decreasing for $\theta_2, \theta_3 > 0$ (Figure 4).

3.3. Stochastic Ordering. Let $f_1(x, y)$ and $f_2(x, y)$ be two-bivariate BKCH distribution, and

$\Delta f_{12}(x, y) = f_1(x, y) - f_2(x, y)$

such that $\Delta f_{12}(x, y) \geq -\Delta f_{12}(y, x)$ and $\Delta f_{12}(x, y) \geq 0$, i.e., $f_1(x, y) \geq f_2(x, y)$ and $\Delta f_{12}(x, y)$ increasing in $x$, for all $x \geq y$. Also, $\Delta f_{12}(x, y)$ decreases in $y$, for all $y \leq x$. Based on the results obtained by Shanthikumar [9], we obtain likelihood ratio order ($X \geq_\text{LR} Y$), hazard rate order ($X \geq_\text{HR} Y$), and stochastic order ($X \geq_\text{ST} Y$) $\iff E[f_1(x, y)] \geq E[f_2(x, y)]$, for all $f_1(x, y)$ and $f_2(x, y)$.

4. Local Measures of Dependence

Holland and Wang [10, 11] introduced a local dependence function given by the following formula:

$$
y_i(x, y) = \frac{\partial^2}{\partial x \partial y} \ln f_{X,Y}(x, y),
$$

where $f_{X,Y}(x, y)$ is the joint pdf of $x$ and $y$.

$$
f_{X,Y}(x, y) = [N(\Theta)]^{-1} x^{a_i-1} y^{a_i-1} \exp[\theta_1 \log(1 - x^{a_i}) + \theta_2 \log(1 - y^{a_i})], x > 0, y > 0, -\infty < \theta_1, \theta_2 < \infty, \quad i = 1, 2, \quad a_1, a_2 > 0.
$$

Figure 5 shows the joint pdf given by (29) for the special cases for $\theta_i$ and $\theta_2$.

The conditional hfs for the joint pdf given by (29) are

$$
r_i(y|x) = \frac{a_i x^{a_i-1} (-\theta_i + 1)}{1 - x^{a_i}}, \quad r_2(y|x) = \frac{a_2 y^{a_2-1} (-\theta_2 + 1)}{1 - y^{a_2}}.
$$

and the conditional pdfs are

$$
f_{X|Y}(x|y) = a_1(x^{a_1-1})(1 - \theta_2) \exp(-(\theta_2 + 1) \log(1 - x^{a_1})),
$$

$$
f_{Y|X}(y|x) = a_2(y^{a_2-1})(1 - \theta_1) \exp(-(\theta_1 + 1) \log(1 - y^{a_2})).
$$

Now, the marginal pdfs of $X$ and $Y$ using (29) are

$$
f_X(x) = \frac{[N(\Theta)]^{-1} x^{a_1-1}}{a_2(1 - \theta_2)} \exp(-\theta_2 \log(1 - x^{a_1})),
$$

$$
f_Y(y) = \frac{[N(\Theta)]^{-1} y^{a_2-1}}{a_2(1 - \theta_2)} \exp(-\theta_1 \log(1 - y^{a_2})).
$$

And the moment of joint pdf given by (29) is

$$
E(XY) = \frac{[N(\Theta)]^{-1} \Gamma(a_1)/\Gamma(b_1) \Gamma(1 - \theta_1) \Gamma(1 - \theta_2)}{a^2 b^2 \Gamma(2 + 1/b - \theta_1) \Gamma(2 + 1/a - \theta_2)}.
$$
Figure 3: $E(X|Y = y)$ of the BKCH distribution for selected values. (a) $\theta_1 = 0.5, \theta_3 = .2; a_1 = a_2 = 1$. (b) $\theta_1 = -0.5, \theta_3 = -.2; a_1 = a_2 = 1$.

Figure 4: $E(Y|X = x)$ of the BKCH distribution for selected values of $\theta_2$ and $\theta_3$. (a) $\theta_2 = 0.5, \theta_3 = .2; a_1 = a_2 = 1$. (b) $\theta_2 = -0.5, \theta_3 = -.2; a_1 = a_2 = 1$.

Figure 5: Continued.
5. Estimation of BKCH Parameters

Suppose that \((x_i, y_i), (i = 1, 2, \ldots, n)\) are observed from BKCH distribution.

5.1. MLE of BKCH Parameters and Asymptotic Confidence Intervals. From the BKCH distribution, the log-likelihood function for the \((x_i, y_i) (i = 1, 2, \ldots, n)\) sample is

\[
I(\theta) = -n \log(N(\theta)) + (a_1 - 1) \sum_{i=1}^{n} \log x_i + (a_2 - 1) \sum_{i=1}^{n} \log y_i \\
+ \theta_1 \sum_{i=1}^{n} \log(1 - y_i^{a_1}) + \theta_2 \sum_{i=1}^{n} \log(1 - x_i^{a_2}) \\
+ \theta_3 \sum_{i=1}^{n} \log(1 - x_i^{a_1}) \log(1 - y_i^{a_2}).
\]

The likelihood equations are

\[
\frac{\partial N(\theta)}{\partial \theta_1} = \frac{1}{n} \sum_{i=1}^{n} \log(1 - y_i^{a_1}), \\
\frac{\partial N(\theta)}{\partial \theta_2} = \frac{1}{n} \sum_{i=1}^{n} \log(1 - x_i^{a_2}), \\
\frac{\partial N(\theta)}{\partial \theta_3} = \frac{1}{n} \sum_{i=1}^{n} \log(1 - x_i^{a_1}) \log(1 - y_i^{a_2}).
\]

The MLE \(\hat{\theta} = \{\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3\}\) can be obtained numerically using systems (35). For the asymptotic confidence interval (CI), the normal approximation of the MLE can be used to construct asymptotic CIs for the parameters \(\theta\) when the sample size is large enough. A two-sided \((1 - \alpha)\)100% CIs for \(\theta\) are \((\hat{\theta} \pm Z_{\alpha/2} \sqrt{\text{Var}(\hat{\theta})})\), where \(\text{Var}(\hat{\theta})\) are the asymptotic variances of \(\hat{\theta}\).

5.2. MPLE of BKCH Parameters. In [13, 14], the study of alternative estimation technique is not based on normalizing constant. The MPLE for joint probability mass function (pmf) or joint pdf is based on the maximization of the product conditional pmfs or conditional pdfs, respectively. It is clear from (19) and (20). The log pseudolikelihood (log PL(\(\theta\))) function is

\[
\text{logPL}(\theta) = n \log(a_1 a_2) + \sum_{i=1}^{n} \log(x_i^{(a_1 - 1)} y_i^{(a_2 - 1)}) \\
+ \sum_{i=1}^{n} \log(\theta_2 + \theta_3 \log(1 - y_i^{a_1})) + 1 \\
+ \sum_{i=1}^{n} \log(\theta_2 + \theta_3 \log(1 - y_i^{a_2})) \log(1 - x_i^{a_1}) + 1 \\
+ \sum_{i=1}^{n} \log(\theta_1 + \theta_3 \log(1 - x_i^{a_1})) + 1 \\
+ \sum_{i=1}^{n} \log(\theta_1 + \theta_3 \log(1 - x_i^{a_2})) \log(1 - y_i^{a_2}).
\]
The MLE of BKCH parameters can be obtained by solving the equations:

\[
\frac{\partial \log \text{PL}(\Theta)}{\partial \theta_1} = \sum_{i=1}^{n} \frac{1}{\theta_1 + \theta_2 \log(1 - x_i^{\theta_2}) + 1} + \sum_{i=1}^{n} \log(1 - y_i^{\theta_1}),
\]

\[
\frac{\partial \log \text{PL}(\Theta)}{\partial \theta_2} = \sum_{i=1}^{n} \frac{1}{\theta_2 + \theta_3 \log(1 - y_i^{\theta_3}) + 1} + \sum_{i=1}^{n} \log(1 - x_i^{\theta_2}),
\]

\[
\frac{\partial \log \text{PL}(\Theta)}{\partial \theta_3} = \sum_{i=1}^{n} \frac{\log(1 - y_i^{\theta_3})}{\theta_2 + \theta_3 \log(1 - x_i^{\theta_2}) + 1} + \sum_{i=1}^{n} \frac{\log(1 - x_i^{\theta_2})}{\theta_1 + \theta_3 \log(1 - x_i^{\theta_2}) + 1} + 2 \sum_{i=1}^{n} \frac{\log(1 - y_i^{\theta_3}) \log(1 - x_i^{\theta_2})}{\theta_1 + \theta_3 \log(1 - x_i^{\theta_2}) + 1}.
\]

(37)

The MLEs decrease to zero as \( n \to \infty \). This shows the consistency of the estimators.

### 6. Simulation Study

In this section, the MLE and MPLE approaches are used to estimate the parameters \( \theta_1, \theta_2, \) and \( \theta_3 \) of the BKCH distribution. The population parameters are generated using software Mathematica package. The sampling distributions are obtained for different sample sizes \( n = 50, 100, 200, 300, 400, \) and 500 from \( N = 500 \) repetitions. This study presents an assessment of the properties for both MLE and MPLE techniques in terms of bias and mean square error (MSE). A general form to generate \( x \) from one marginal \( f_X(x) \) and then simulate a corresponding a bivariate vector using the conditional density is \( f_{Y|X}(y|x) \). The MLEs are reported in Table 1 for BKCH (1.5, 3.5, 0.05) and Table 2BKCH (1.5, 2.5, -0.05) with \( a_1 = a_2 = 0.5 \).

### 7. Application of Real Data

Suppose that \( X \) and \( Y \) are Dominant Ulna and Ulna bones. Then, the bivariate data in Table 3 represent the bone mineral density (BMD); after one year of birth, measure in \( \text{gm/cm}^2 \) for 24 kids [3].

The statistic measures for the given data are \( \mu_x = 0.71267, \sigma_x^2 = 0.011103, \mu_y = 0.68904, \sigma_y^2 = 0.01242, \) and \( \rho_{xy} = 0.62813 \). The results in Table 4 represent the MLE and MPLE of BKCH parameters.

The joint pdf of bivariate exponential conditionals \( BPC (\lambda_1, \lambda_2, \lambda_3) \) distribution is [15]

\[
f_{X,Y}(x,y) = \exp(C - \lambda_1 y - \lambda_2 x + \lambda_3 xy), \quad x > 0, y > 0, \lambda_1 > 0, \lambda_2 > 0, \lambda_3 \leq 0,
\]

(38)

where \( C \) is the normalizing constant.

The joint pdf of bivariate Lindley conditionals hazard \( BLCH (\lambda_1, \lambda_2, \lambda_3) \) distribution is [16]

\[
f_{X,Y}(x,y) = N(\lambda_1, \lambda_2, \lambda_3)^{-1}(1 + x)(1 + y)\exp(\lambda_1 y - (\lambda_2 + \lambda_3 y)x), \quad x > 0, y > 0, \lambda_1 < 0, \lambda_2 > 0, \lambda_3 \geq 0,
\]

(39)

where \( N(\lambda_1, \lambda_2, \lambda_3) \) is the normalized constant. \( BLCH (\lambda_1, \lambda_2, \lambda_3) \) is \( TN_2^2 \); also, the conditional pdfs \( X|Y = y \) and \( Y|X = x \) are Lindley distribution with parameter \( \lambda_2 + \lambda_3 y \) and \( -\lambda_1 + \lambda_3 x \), respectively. Table 5 includes log-likelihood and AIC and BIC of BKCH with other models.
We note that AIC and BIC of the BKCH model more than the corresponding of the BLCH and BEC models which means that BKCH distribution is better to fit for the given data. The approximate 95% two-sided CI of the parameters $\theta_1$, $\theta_2$, and $\theta_3$ are given, respectively, as $[-0.7548, 1.7128]$, $[-0.381, 1.2282]$, and $[1.394, 3.5379]$.

### 8. Conclusion

In this study, we introduced a BKCH distribution based on specified conditional hfs of Kumaraswamy distribution and its local measures of dependence. In addition, the methods of MLE and MPLE of BKHC parameters are present. In view of results, Theorem 2, the interest of BKHC is positively, negatively, or zero correlated; this indicates the generality of the distribution. Furthermore, the simulation results showed that MLE operates quite uniformly, and it can be used to estimate the BKHC parameters. In particular, in Table 4, the MPLE is better than MLE because the MPLE technique uses the conditional pdfs which does not contain the normalizing constant. Therefore, Table 5 shows the BKCH distribution is a better fit for the given data compared to the BLCH and BEC distributions.

### Data Availability

All data are available within the article.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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