Cosmological String Backgrounds
from Gauged WZW Models

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Abstract
We discuss the four-dimensional target-space interpretation of bosonic strings based on gauged WZW models, in particular of those based on the non-compact coset space $SL(2, \mathbb{R}) \times SO(1, 1)^2/ SO(1, 1)$. We show that these theories lead, apart from the recently broadly discussed black-hole type of backgrounds, to cosmological string backgrounds, such as an expanding Universe. Which of the two cases is realized depends on the sign of the level of the corresponding Kac-Moody algebra. We discuss various aspects of these new cosmological string backgrounds.

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The investigation of strings in curved space-time backgrounds might provide us with some important insight into questions about quantum gravity. Recently, a very intriguing relation between black-hole type of backgrounds [1] in two dimensions and gauged Wess-Zumino-Witten (WZW) models [2] based on the non-compact coset space $SL(2, \mathbb{R})/SO(1, 1)$ was discovered [3]. Subsequently, this discussion was put forward to higher dimensions [4]. All these examples show a singularity in forward times (black-holes, black branes, etc.). In this note we shall argue that the same non-compact cosets, which give rise to black-holes, also lead to cosmological string backgrounds, namely to an expanding Universe. (Cosmological string backgrounds were discussed before in [5], [6], [7]). In this case, the black-hole singularity becomes a singular surface hidden behind the light-cone, which can never be met in future times when one travels inside the light-cone. Even our cosmological string solution is derived from an exact conformal field theory in the semi-classical approximation, we think that it also serves to be an interesting solution in its own right to Einstein’s equations coupled to a non-constant dilaton matter field.

The gauged WZW model based on the coset $G/H$ is described by the action

$$S = \frac{k}{4\pi} \int d^2 z \text{tr}(g^{-1} \partial g^{-1} \partial g) - \frac{k}{12\pi} \int_B \text{tr}(g^{-1} dg \wedge g^{-1} dg \wedge g^{-1} dg)$$

$$+ \frac{k}{2\pi} \int d^2 \text{tr}(A \overline{g} g^{-1} - \overline{A} g^{-1} \partial g - g^{-1} A g A),$$

where the boundary of $B$ is the 2D worldsheet, $g$ is a group element of the group $G$, and $A$ are the gauge fields of $H$ transforming as $A \rightarrow h_L^{-1} (A + \partial) h_L$; $k$ is the level of the Kac–Moody algebra for the group $G$. To be specific, we want to discuss a four-dimensional target-space based on the non-compact coset $SL(2, \mathbb{R}) \times SO(1, 1)^2/SO(1, 1)$. The central charge of this coset CFT (non-compact coset CFT’s were discussed in [8]) is given by

$$c = 4 + \frac{6}{k - 2},$$

where the $k$, being a real number, is the level of the non-compact $SL(2, \mathbb{R})$ Kac–Moody algebra. For the case that the string theory is entirely given by this coset CFT, the condition $c = 26$ implies $k = 25/11$. However, we want to study the semiclassical
limit $k \to \pm \infty$. Therefore we prefer to couple the $\text{SL}(2, \mathbb{R}) \times \text{SO}(1,1)^2/\text{SO}(1,1)$ coset CFT to an internal CFT with central charge

$$c_{\text{int}} = 22 - \delta c, \quad \delta c = \frac{6}{k - 2},$$

(3)

Now we parametrize the group element of $\text{SL}(2, \mathbb{R})$ as $g = \begin{pmatrix} u & a \\ -b & v \end{pmatrix}$ with $uv + ab = 1$. Finally we assume for simplicity that $H = \text{SO}(1,1)$ is entirely inside $\text{SL}(2, \mathbb{R})$. Performing a vector-like gauging, $a = \pm b$, the action (1) can be identified with a $\sigma$-model action of the form

$$S = \int \! d^2 z G_{MN}^\sigma \partial X^M \partial X^N.$$  

(4)

In the semiclassical approximation $k \to \pm \infty$, the corresponding four-dimensional $\sigma$-model metric is given as

$$ds_\sigma^2 = -k \frac{dudv}{1 - uv} + dx_2^2 + dx_3^2.$$  

(5)

We recognize that the signature of the two-dimensional part of this metric depends just on the sign of the level $k$ of the $\text{SL}(2, \mathbb{R})$ Kac–Moody algebra. In fact, for $k \to +\infty$ ($\delta c > 0$) space-time possesses a singularity in futures times $\tau = u + v$. In this case, $u$ and $v$ are Kruskal-like coordinates of a black-hole metric; more precisely the metric (5) has the causal structure of a four-dimensional black-brane (see figure 1). The singularity at $uv = 1$ originates from the fixed points of the modded vector gauge symmetry $H$ at this curve. However for $k \to -\infty$ ($\delta c < 0$) the causal structure is completely different. Specifically the causal structure for negative $k$ is obtained from the black-hole case by a $90^0$ rotation of the two-dimensional $u, v$-plane (see figure 2). Now one has a forward light-cone with the singularity behind it. (Of course, the two cases $k \to \pm \infty$ are not analytic continuations of each other.) As we will discuss in the following, this new class of string backgrounds with negative $k$ describes an expanding Universe with singularity outside the visible horizon.

The four-dimensional effective string action for the gravitational and dilaton $\Phi$ background fields has the following form:

$$S_{\text{eff}} = \int \! d^4 x \sqrt{G^\sigma} e^\Phi (R^\sigma - (D\Phi)^2 + \Lambda), \quad \Lambda = \frac{2\delta c}{3}.$$  

(6)
Here $R^\sigma$ is the curvature scalar derived from the metric $G^\sigma_{MN}$ in the $\sigma$-model frame. It is not difficult to show that the $\sigma$-model metric eq.(5) together with the dilaton field

$$\Phi = \log(1 - uv)$$

satisfy the equations of motion [9]

$$R^\sigma_{MN} + D_M D_N \Phi = 0,$$

$$R^\sigma + (D_M \Phi)^2 + 2 D_M D^M \Phi = \Lambda,$$  

(8)

for both signs of $k$. (In fact, one has to replace $k$ by $k - 2 = 6/\delta c$ in eq.(5), which is allowed in the large $k$ limit.)

To get contact with standard gravity theory, which possesses a canonical Einstein term, one has to perform a Weyl rescaling of the $\sigma$-model metric eq.(5) by the exponential of the dilaton field.* This Weyl rescaling however does not change the causal structure of the theory. Specifically, the metric in the Einstein frame is given as

$$ds^2 = e^\Phi d\sigma^2 = du dv + (1 - uv)(dx_2^2 + dx_3^2).$$

(9)

Here we have focused on the cosmological case with $k \to -\infty$, where we have absorbed the level $k$ in the metric. The effective action now has the form

$$S_{\text{eff}} = \int d^4x \sqrt{G} \left( \frac{1}{2} R - \frac{1}{4} D_M \Phi D^M \Phi - V(\Phi) \right), \quad V(\Phi) = 2e^{-\Phi}.$$  

(10)

The metric eq.(9) in the Kruskal-type of coordinates $u$ and $v$ leads to a space-time singularity at $uv = 1$. This can be seen by computing the corresponding Ricci tensor:

$$R_{uv} = \frac{uv - 2}{2(1 - uv)^2}.$$

(11)

Second, there is the horizon at $uv = 0$. Thus we see that, introducing the proper time $\tau = u - v$, there is no singularity for future times $\tau$ inside the light-cone $uv < 0$ (region I in figure 2). The singularity is hidden behind the horizon. However signals in regions II, III may hit the singularity, and the singularity may also send signals through regions II, III into region I.

* There is no Weyl rescaling of the metric in two space-time dimensions.
Now we want to show that inside the singularity free region I we have in fact an expanding Universe with metric similar to the Robertson-Walker metric. For this purpose it is useful to introduce coordinates which cover exactly the region I, namely†

\[ u = e^{x_1} t, \quad v = -e^{-x_1} t. \]  

(12)

(The coordinates \( t \) and \( x_1 \) are of course not geodesically complete; the singularity occurs for imaginary times \( t, t^2 < -1 \).) It follows that curves of constant times \( t \) correspond in figure 2 to hyperbolae \( uv = -t^2 \), whereas the curves of constant \( x_1 \) are given by the straight lines \( u/v = -e^{2x_1} \) (see figure 2). In these coordinates the metric now looks like

\[ ds^2 = -dt^2 + t^2 dx_1^2 + (1 + t^2)(dx_2^2 + dx_3^2), \]  

(13)

and the dilaton has the form

\[ \Phi(t) = \log(1 + t^2). \]  

(14)

Clearly, the metric (13) describes an expanding Universe in region I with two different scale factors \( R_1(t) = t, R_{2,3}(t) = \sqrt{1 + t^2} \), where \( t \) is the cosmological time coordinate. For small \( t \), the Universe expands unisotropically. However, for large times, one approaches an isotropic, linear expansion of the Friedmann-Robertson-Walker type with \( R_i(t) = t \) \( (i = 1, 2, 3) \). Thus we see that for large \( t \) this background, derived from the coset CFT \( SL(2, \mathbb{R}) \times SO(1, 1)^2/ SO(1, 1) \), asymptotically approaches the linearly expanding Universe considered in ref. [6], which is based on a free scalar CFT plus a dilaton of the form \( \Phi = 2 \log t \). However our solution has no initial singularity at \( t = 0 \) in contrast to the standard isotropic Robertson-Walker Universe. (The Ricci tensor in the coordinates (12) takes the form \( R_{tt} = \frac{2}{(1+t^2)^2}, R_{ij} = -\frac{2}{1+t^2} \delta_{ij} \).)

Let us briefly derive the energy momentum tensor of the dilaton matter field. Specifically consider the classical Einstein equations

\[ R_{MN} - \frac{1}{2} G_{MN} R = -T_{MN}. \]  

(15)

† Alternatively, one can change the coordinates in the following way: \( u = e^{x_1} \sqrt{1-t}, v = -e^{-x_1} \sqrt{1-t} \) in region I and \( u = \pm e^{x_1} \sqrt{1-t}, v = \pm e^{-x_1} \sqrt{1-t} \) in regions II, III. Then the metric takes the form \( ds^2 = -\frac{1}{4t(t-1)} dt^2 + (t-1)dx_1^2 + t(dx_2^2 + dx_3^2). \) In region I this metric again leads to an expanding Universe, isotropic for large \( t \).
The corresponding energy-momentum tensor from the dilaton matter field has the form
\[ T_{MN} = \frac{1}{2} D_M \Phi D_N \Phi - G_{MN} \left( \frac{1}{4} G^{PQ} D_P \Phi D_Q \Phi + V(\Phi) \right). \] (16)

Then we obtain, with \( \Phi = \log(1 + t^2) \) and \( V(\Phi) = \frac{2}{1 + t^2} \), that
\[ T_{tt} = \frac{(\partial_t \Phi)^2}{4} + V = \rho = \frac{3t^2 + 2}{(1 + t^2)^2}, \]
\[ T_{ij} = \left( \frac{(\partial_i \Phi)^2}{4} - V \right) G_{ij} = pG_{ij} = -\frac{t^2 + 2}{(1 + t^2)^2} G_{ij}. \] (17)

Here \( \rho \) is the energy density of the dilaton matter system and \( p \) is its pressure. Now it is easy to see that the quantity
\[ \rho + 3p = -\frac{4}{(1 + t^2)^2} \] (18)
is negative for all \( t \). It is interesting to observe that the form of \( \rho + 3p \), being always negative, violates an assumption by Hawking and Ellis \cite{10} on the form of the matter energy-momentum tensor, which, being satisfied, would always lead to a singular space-time. Thus, the absence of an initial singularity in the cosmological region I can be understood from this point of view.

It is also interesting to discuss the target space duality properties of this cosmological string background. In the gauged WZW model, the duality operation corresponds to the exchange of gauging the axial \( SO(1, 1) \) subgroup of \( SL(2, \mathbb{R}) \) instead of gauging the vector-like subgroup \( SO(1, 1) \) \cite{11}. Seen from the target space point of view it is the exchange of the regions I and IV in figures 1 and 2, whereas the regions II and III are mapped onto themselves \cite{12}, \cite{13}. In fact, one can show explicitly \cite{14} how the metric (5) is transformed under the duality transformation to a new metric, which also satisfies the field equations (8) after a suitable redefinition of the dilaton field. For this purpose it is convenient to introduce another set of coordinates like
\[ u = e^{x_1 \sinh x_0}, \quad v = -e^{-x_1 \sinh x_0}. \] (19)

Then the cosmological string background of region I in the sigma-model frame has the form*
\[ ds^2_\sigma = -dx_0^2 + \tanh^2 x_0 dx_1^2 + dx_2^2 + dx_3^2. \] (20)

* This metric is exactly of the form of the cosmological backgrounds discussed in \cite{15}.
The duality transformation now acts as

$$R_1(x_0) = \tanh x_0 \rightarrow \frac{1}{R_1(x_0)} = \coth x_0. \quad (21)$$

Therefore the dual metric just describes region IV with a singularity at $x_0 = 0$. The field equations (8) are invariant under this duality transformation, provided that the dilaton transforms as

$$\Phi(x_0) = 2 \log \cosh x_0 \rightarrow \Phi(x_0) + 2 \log R_1(X_0) = 2 \log \sinh x_0. \quad (22)$$

Finally, in the coordinates eq.(12), the duality transformation is expressed as

$$t^2 \rightarrow -1 - t^2. \quad (23)$$

Thus we see again that the cosmological region I is mapped to the cosmological region IV, which requires an analytic continuation to imaginary $t$ values.

Let us discuss briefly some alternative models. For example consider the gauged WZW model based on the coset $SL(2, \mathbb{R}) \times SU(1,1)/U(1)$.† The central charge of this CFT is given by

$$c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 + 2} - 2. \quad (24)$$

The corresponding background in the Einstein frame has the following form:

$$ds^2 = -k_1 (1 + z \overline{z}) du dv + k_2 (1 - uv) dz d\overline{z}. \quad (25)$$

Here $z$ is a complex parameter of an $SU(2)$ group element, $z = x_2 + i x_3$. For the black-hole case with positive $k_1$ (the $SU(2)$ Kac–Moody level $k_2$ should always be a positive integer) one can set $k_1 = k_2 + 4$. Then one obtains that $c = 4$ regardless of the value of $k_1$. Thus $\delta c = 0$ for all $k_1$ and one could expect that the four-dimensional background of the semiclassical approximation is valid for all $k_1$. In addition, since $c_{\text{int}} = 22$ for all $k_1$, the internal CFT could be simply given by (compactified) free bosons.

† This model is relevant for curved backgrounds with target-space supersymmetries as discussed in ref. [17].
For the cosmological case with negative $k_1$ we cannot set $k_1 = k_2 + 4$. As an alternative one could couple $SL(2, \mathbb{R})/SO(1,1)$ with negative $k_1$ to an Euclidean two-dimensional ‘black-hole’ based on the coset $SL(2, \mathbb{R})/U(1)$ with positive Kac–Moody level $k_2$. (A twisted version of the product of an Euclidean and Minkowskian black-hole coset CFT was already considered in ref. [16].) The central charge of this model is

$$c = \frac{3k_1}{k_1 - 2} + \frac{3k_2}{k_2 - 2} - 2.$$  \hspace{1cm} (26)$$

Again one obtains $\delta c = 0$ for $k_1 = -k_2 + 4$. It would be interesting to study further alternatives to this model.

In summary we have shown that gauged WZW-models have a completely different target space interpretation when one changes the sign of the level of the underlying non-compact Kac–Moody algebra. For both signs, black-holes as well as cosmological backgrounds, the metric of the target space leads to space-time singularities. In the cosmological case the singularity is hidden behind the light-cone. However the singularity could send signals into the light-cone and therefore influence the expansion of the Universe. This is very similar to the initial singularity (Big Bang) in the standard Friedmann-Robertson–Walker Universe. It is important to stress that, seen from the CFT point of view, the singularities in the black-hole as well as in the cosmological frameworks have exactly the same origin, namely in the existence of fixed points of the modded (vector) ‘gauge’ symmetry $H$. Therefore one could expect that the same type of quantum gravity effects are relevant near the black-hole as well as in the early Universe at times shortly after the initial singularity. These quantum gravity effects should in principle be determined from the underlying coset CFT. (For considerations in this directions for the black-hole case, see ref. [13].) Finally we want to mention that the gauged WZW model based on the non-compact coset $SL(2, \mathbb{R})/SO(1,1)$ possesses, for both signs of $k^*$, still many problems which are presently not completely understood. In particular the elimination of the negative-norm states is unclear. This question should be addressed together with the spectrum of the additional part of the CFT. Moreover, the construction of a modular invariant partition function is still problematic (for a discussion on this issue see [17]).

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* For negative $k$, this coset CFT shows many parallels to the CFT based on the compact space $SU(2)/U(1)$. 

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Figure Captions

**Figure 1:** The causal structure of the two-dimensional slice of the black-hole metric equation (5) with positive Kac–Moody level $k$.

**Figure 2:** The causal structure of the two-dimensional slice of the cosmological metric equation (5) with negative Kac–Moody level $k$. 