WEIGHTED SCHUR FUNCTIONS

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Abstract. We introduce the weighted (factorial) Schur functions and establish the correspondence between those functions and the (equivariant) Schubert classes for the weighted Grassmannians. This generalizes the well-known correspondence between (factorial) Schur functions and (equivariant) Schubert classes in the (equivariant) cohomology of Grassmannians.

1. Introduction

Let \( P(d) \) be the set of partitions with at most \( d \) rows. For every \( \lambda \in P(d) \), the factorial Schur function \( s_\lambda(x|a) \) ([2]) is defined as a polynomial in the variables \( (x_1, \ldots, x_d) \) and \( (a_i)_{i \in \mathbb{N}} \) (see (2.1) for the defining formula). They form a \( \mathbb{Z}[a] \)-module basis of \( \mathbb{Z}[a][x]^{\mathfrak{S}_d} \) which is the ring of symmetric polynomials in \( x \)-variables with the coefficients in the polynomial ring of \( a \)-variables. On the other hand, let \( \text{Gr}(d, n) \) be the Grassmannian of complex \( d \)-dimensional subspaces in the complex \( n \)-plane \( \mathbb{C}^n \). The standard \( n \)-torus \( \mathbb{T}^n \) action on \( \mathbb{C}^n \) induces the \( \mathbb{T}^n \)-action on \( \text{Gr}(d, n) \) and we have the equivariant cohomology \( H^*_{\mathbb{T}^n}(\text{Gr}(d, n); \mathbb{Z}) \) which is an algebra over the polynomial ring \( H^*(BT^*; \mathbb{Z}) = \mathbb{Z}[y_1, \ldots, y_n] \). The Schubert varieties in \( \text{Gr}(d, n) \) are \( \mathbb{T}^n \)-invariant subvarieties indexed by the partitions \( \lambda \in P(d, n) \) where \( P(d, n) \) is the set of the partitions contained in the \( d \times (n - d) \) rectangle. The corresponding equivariant Schubert classes \( \tilde{S}_\lambda \) form a \( \mathbb{Z}[y_1, \ldots, y_n] \)-module basis of the equivariant cohomology. The factorial Schur functions represent the equivariant cohomology rings of Grassmannians in a sense that there is a surjective \( \mathbb{Z}[a] \)-algebra homomorphism

\[
\mathbb{Z}[a][x]^{\mathfrak{S}_d} \to H^*_{\mathbb{T}^n}(\text{Gr}(d, n); \mathbb{Z})
\]

that sends \( s_\lambda(x|a) \) to \( \tilde{S}_\lambda \) if \( \lambda \in P(d, n) \), or 0 otherwise ([8]). Here the \( \mathbb{Z}[a] \)-action on the RHS is given by the projection \( \mathbb{Z}[a] \to \mathbb{Z}[y_1, \ldots, y_n] \) sending \( a_i \mapsto -y_{n+1-i} \) if \( 1 \leq i \leq n \), or 0 otherwise. This picture specializes to the Schur functions and the ordinary cohomology of Grassmannians by setting \( a \)-variables and \( y \)-variables to zero, i.e. there is a surjective \( \mathbb{Z} \)-algebra homomorphism from the ring of symmetric polynomials in \( x \)-variables to the ordinary cohomology

\[
\mathbb{Z}[x]^{\mathfrak{S}_d} \to H^*(\text{Gr}(d, n); \mathbb{Z}).
\]

One of the advantages of these correspondences is that we can study the structure constants by multiplying actual polynomials. Similar examples include the (double/quantum) Schubert polynomials ([4] [9] [10]) for (equivariant/quantum) cohomology of full flag varieties and (factorial) Schur \( Q \)-polynomials ([5] [6] [7]) for (equivariant) cohomology of Lagrangian Grassmannians, e.t.c.

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In this paper, we will introduce the \textit{weighted (factorial) Schur functions} that are obtained as a variant of the factorial Schur functions \( s_{\lambda}(x|a) \). In the sense mentioned above, these functions will present the equivariant cohomology of the weighted Grassmannians introduced by Corti-Reid \[3\]. Below we summarize the results in this paper in detail.

We introduce the new sets of variables \( v = (v_1, \ldots, v_d) \) and \( w = (w_i)_{i \in \mathbb{N}} \). Let \( v_{ch} := v_1 + \cdots + v_d \) and \( x_{ch} = x_1 + \cdots + x_d \). For each partition \( \lambda \in \mathcal{P}(d) \), we define the \textit{weighted factorial Schur function} \( s^{\omega}_\lambda(v; x|a) \) by
\[
s^{\omega}_\lambda(v; x|a) := s_\lambda(x^\omega|a^\omega)
\]
where \( x^\omega = (x_1^\omega, \ldots, x_d^\omega) \) and \( a^\omega = (a_i^\omega)_{i \in \mathbb{N}} \) are the \textit{shifted} sequences of variables defined by
\[
x_i^\omega := x_i - \frac{v_i}{v_{ch}} x_{ch}, \quad i = 1, \ldots, d, \quad \text{and} \quad a_l^\omega := a_l - \frac{w_l}{v_{ch}} x_{ch}, \quad l \in \mathbb{N}.
\]

These functions lie in the ring \( \mathbb{Z}[w_{loc}[a]|v_{loc}[x]] \) of polynomials in \( w, a, v, x \), localized at
\[
v_{ch} \quad \text{and} \quad w_\lambda = \sum_{i=1}^d w_{\lambda,i} \quad \text{for all} \quad \lambda \in \mathcal{P}(d)
\]
where \( \bar{\lambda}_i := b_i^a + (d - i + 1) \) for the number \( b_i^a \) of boxes at the \( i \)-th rows of \( \lambda \). Moreover, \( s^{\omega}_\lambda(v; x|a) \in \mathbb{Z}[w_{loc}[a]|v_{loc}[x]]^{\mathcal{S}_d} \) for all \( \lambda \in \mathcal{P}(d) \). We also define the \textit{weighted Schur functions} by \( s^{\lambda}_\lambda(v; x|a) := s^{\lambda}_\lambda(v; x|0) \). Let \( \mathcal{WSch} \) and \( \mathcal{WSch} \) be the \( \mathbb{Z}[w_{loc}[a]] \)- and \( \mathbb{Z}[w_{loc}] \)-submodules generated by \( s^{\lambda}_\lambda(v; x|a) \) and \( s^{\lambda}_\lambda(v; x) \) respectively, i.e.
\[
\mathcal{WSch} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[w_{loc}[a]] \cdot s^{\lambda}_\lambda(v|a) \subset \mathbb{Z}[w_{loc}[a]|v_{loc}[x]]^{\mathcal{S}_d}
\]
\[
\mathcal{WSch} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[w_{loc}[a]] \cdot s^{\lambda}_\lambda(v; x) \subset \mathbb{Z}[w_{loc}[a]|v_{loc}[x]]^{\mathcal{S}_d}.
\]

Apparently \( \mathcal{WSch} = \mathbb{Z} \otimes_{\mathbb{Z}[a]} \mathcal{WSch} \), but also we can show that \( \mathcal{WSch} = \mathbb{Z}[a] \otimes_{\mathbb{Z}} \mathcal{WSch} \) (Lemma 2.3).

The first part of the main results of this paper is summarized as follows.

\textbf{Theorem A.} \( \mathcal{WSch} \) is a \( \mathbb{Z}[w_{loc}[a]] \)-subalgebra of \( \mathbb{Z}[w_{loc}[a]|v_{loc}[x]]^{\mathcal{S}_d} \) and \( \{ s^{\lambda}_\lambda(v; x|a) \}_{\lambda \in \mathcal{P}(d)} \) is a \( \mathbb{Z}[w_{loc}[a]] \)-module basis of \( \mathcal{WSch} \). Similarly, \( \mathcal{WSch} \) is a \( \mathbb{Z}[w_{loc}] \)-subalgebra of \( \mathbb{Z}[w_{loc}|v_{loc}[x]]^{\mathcal{S}_d} \) and \( \{ s^{\lambda}_\lambda(v; x) \}_{\lambda \in \mathcal{P}(d)} \) is a \( \mathbb{Z}[w_{loc}] \)-module basis of \( \mathcal{WSch} \).

For the proofs, we rely on the Littlewood-Richardson type formula for the factorial Schur functions obtained by Molev-Sagan \[11\] Theorem 3.1 but particularly we derive and use the generalizations of the classical formulas, namely Weighted Vanishing Lemma (Lemma 2.6) and Weighted Pieri Rules (Lemma 2.7) for the weighted factorial Schur functions.

The second part is to relate the weighted (factorial) Schur functions to the (equivariant) cohomology of weighted Grassmannians. Fix an infinite sequence \( (w_i)_{i \in \mathbb{N}} \) of non-negative integers and a positive integer \( u \). For each \( n > d \), let \( \mathcal{WGr}(d, n) \) be the weighted Grassmannian associated to the weight \( (w_n, \ldots, w_1) \) and \( u \), with the standard torus \( T^u \)-action. The bar in the notation reminds us to use the backward order of the weights. The equivariant cohomology \( H^*_{T^u}(\mathcal{WGr}(d, n); \mathbb{Q}) \) is an algebra over \( H^*(BT^u; \mathbb{Q}) = \mathbb{Q}[y_1, \ldots, y_n] \) and recall that the equivariant weighted Schubert classes \( \mathcal{WSch} \) for all \( \lambda \in \mathcal{P}(d, n) \) introduced in \[11\] form a \( \mathbb{Q}[y_1, \ldots, y_n] \)-module basis. Similarly the weighted Schubert classes \( \mathcal{WSch} \) form a \( \mathbb{Q} \)-basis of \( H^*_{T^u}(\mathcal{WGr}(d, n); \mathbb{Q}) \).
Theorem B. We have the algebra homomorphisms
\[ \tilde{\Phi}_n : w\text{Sch} \to H^*_T(\mathfrak{Gr}(d, n); \mathbb{Q}) \quad \text{and} \quad \Phi_n : w\text{Sch} \to H^* (\mathfrak{Gr}(d, n); \mathbb{Q}) \]
that send (factorial) weighted Schur functions to (equivariant) weighted Schubert classes.

These homomorphisms are as algebras over \( \mathbb{Z}[w]_{\text{loc}}[a] \) and \( \mathbb{Z}[w]_{\text{loc}} \) respectively and their actions on the cohomology rings are given by sending

\[ w_l \mapsto w_l + u/d \quad \text{for all } l \in \mathbb{N} \quad \text{and} \quad a_l \mapsto \begin{cases} -y_{n+1-l} & \text{if } 1 \leq l \leq n \\ 0 & \text{otherwise} \end{cases} \]

We will also present \( \widetilde{w\text{Sch}} \) and \( w\text{Sch} \) as the cohomology of certain inductive limits constructed from weighted Grassmannians. This gives a topological explanation for the existence of the rings \( w\text{Sch} \) and \( w\text{Sch} \). For each \( n \), there is a natural inclusion \( w_{l_n} : \mathfrak{Gr}(d, n) \to \mathfrak{WGr}(d, n+1) \) which is equivariant with respect to \( \rho_n : T^n \to T^{n+1}, (t_1, \ldots, t_n) \to (1, t_1, \ldots, t_n) \) (see Section 3 for the detail). We let

\[ \mathfrak{WGr}(d, \infty) := \lim_{\longrightarrow} \mathfrak{WGr}(d, n) \quad \text{and} \quad \mathfrak{WGr}(d, \infty)_T := \lim_{\longrightarrow} (ET^n \times_T \mathfrak{Gr}(d, n)). \]

By using a general theory of the inductive/projective limits in homological algebras, there exist the weighted Schubert classes defined in the cohomology of these limit spaces. Then we conclude that there are algebra isomorphisms sending (factorial) weighted Schur functions to (equivariant) weighted Schubert classes

\[ \mathbb{Q} \otimes_{\mathbb{Z}[w]_{\text{loc}}} w\text{Sch} \xrightarrow{\sim} H^*(\mathfrak{WGr}(d, \infty); \mathbb{Q}) \quad \text{and} \quad \mathbb{Q}[a] \otimes_{\mathbb{Z}[w]_{\text{loc}}[a]} \widetilde{w\text{Sch}} \xrightarrow{\sim} H^*(\mathfrak{WGr}(d, \infty)_T; \mathbb{Q}), \]

where \( \mathbb{Q}[a] \) is the ring of polynomials in \( a \)-variables that are possibly infinite linear combinations of finite degree monomials and the tensor product is given by evaluating \( w_l \) by \( w_l + u/d \) for all \( l \in \mathbb{N} \).

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2. Weighted (Factorial) Schur Functions

2.1. Preliminaries. Fix a positive integer \( d \). Let \( \mathcal{P}(d) \) be the set of partitions with at most \( d \) rows. For each \( \lambda \in \mathcal{P}(d) \), the number of boxes at the \( i \)-th row is denoted by \( b^i_\lambda \) where \( i = 1, \ldots, d \). Let \( x = (x_1, \ldots, x_d) \) and \( a = (a_l)_{l \in \mathbb{N}} \) be sequences of variables. Let \( \mathbb{Z}[a] \) be the polynomial ring in \( a_l \)'s, by which we mean the ring of finite linear combinations of monomials in \( a_l \)'s with finite degrees. Let \( \mathbb{Z}[x]^S_d \) be the symmetric polynomial ring where \( S_d \) denotes the permutation group on \( d \) letters. Recall that the factorial Schur function \( s_\lambda(x|a) \) is defined as follows (c.f. \( [11] \)). For each \( k > 0 \), let

\[ (y|a)^k := (y - a_1) \cdots (y - a_k). \]
Define, for each partition $\lambda \in \mathcal{P}(d)$,
\[
(2.1) \quad s_\lambda(x|a) := \frac{\det[(x_i|a)^b_i^d - i \leq i, j \leq d]}{\prod_{i < j}(x_i - x_j)}.
\]

Although $s_\lambda(x|a)$ is a rational function a priori, it is actually a positive degree polynomial function that involves finitely many variables. In fact, we have the following combinatorial formula
\[
s_\lambda(x|a) = \sum_T \prod_{\alpha \in \lambda} (x_{T(\alpha)} - a_{T(\alpha) + c(\alpha)})
\]
where $T$ runs over all semi-standard tableaux of the shape $\lambda$ with entries in $\{1, \cdots, d\}$, $T(\alpha)$ is the entry of the box $\alpha$ in $\lambda$, and $c(\alpha) := j - i$ if $\alpha$ is in the $i$-th row and the $j$-th column. The ordinary Schur functions $s_\lambda(x)$ can be obtained by specializing $s_\lambda(x|a)$ at $a_l = 0$ for all $l \in \mathbb{N}$. The factorial Schur functions form a $\mathbb{Z}[a]$-module basis of $\mathbb{Z}[a] \otimes_{\mathbb{Z}} \mathbb{Z}[x]^G$ and the Littlewood-Richardson type formula for the structure constants $c_{\lambda \mu}^\nu(a) \in \mathbb{Z}[a]$ is obtained by Molev-Sagan [11]. They actually computed more general structure constants $c_{\lambda \mu}^\nu(a, b) \in \mathbb{Z}[a, b]$ defined by
\[
(2.2) \quad s_\lambda(x|b) \cdot s_\mu(x|a) = \sum_{\nu \in \mathcal{P}(d)} c_{\lambda \mu}^\nu(a, b)s_\nu(x|a),
\]
where $b = (b_i)_{i \in \mathbb{N}}$ is another infinite sequence of variables.

**Definition 2.1.** For each $\lambda \in \mathcal{P}(d)$, let $\tilde{\lambda} = (\tilde{\lambda}_1 > \cdots > \tilde{\lambda}_d)$ be the strictly decreasing sequence of integers defined by
\[
(2.3) \quad \tilde{\lambda}_i := b_i^\lambda + (d - i + 1) \quad \text{for all } i = 1, \cdots, d.
\]

For each $\mu \in \mathcal{P}(d)$, we introduce a $\mathbb{Z}[a]$-algebra homomorphism
\[
(2.4) \quad \psi_\mu : \mathbb{Z}[a] \otimes_{\mathbb{Z}} \mathbb{Z}[x]^G \to \mathbb{Z}[a] \quad \text{by} \quad x_i \mapsto a_{\tilde{\mu}_i} \quad \text{for all } i = 1, \cdots, d.
\]

**Lemma 2.2** (Vanishing Lemma, [12], c.f. [11]). Let $a_\lambda := \sum_{i=1}^d a_{\tilde{\lambda}_i}$ for each $\lambda \in \mathcal{P}(d)$. For each $\lambda, \mu \in \mathcal{P}(d)$, we have
\[
(2.5) \quad \psi_\mu(s_\lambda(x|a)) = s_\lambda(a_{\tilde{\mu}_d}, \cdots, a_{\tilde{\mu}_1}a) = \begin{cases} 0 & \text{if } \mu \not\supseteq \lambda \\ \prod_{\rho \in [\lambda]} (a_\lambda - a_\rho) & \text{if } \mu = \lambda, \end{cases}
\]
where $[\lambda]$ is the set of partitions $\rho$ such that $\rho \subseteq \lambda$ and $\{|\rho_1, \cdots, \rho_d| \cap \{\tilde{\lambda}_1, \cdots, \tilde{\lambda}_d\}| = d - 1$.

2.2. **Definition of Weighted (Factorial) Schur Functions.** In order to define the weighted Schur functions, we introduce new sets of variables $w := (w_l)_{l \in \mathbb{N}}$ and $v = (v_1, \cdots, v_d)$. Let $\mathbb{Z}[w]$ and $\mathbb{Z}[v]$ be the corresponding polynomial rings. Let $\mathbb{Z}[w]_{\mathrm{loc}}$ be the localization of the ring $\mathbb{Z}[w]$ at the multiplicative subset
\[
\{w_\lambda^{(1)} \cdots w_\lambda^{(k)} | \lambda^{(1)}, \cdots, \lambda^{(k)} \in \mathcal{P}(d), k \in \mathbb{Z}_{\geq 0}\}
\]
where $w_\lambda := \sum_{l=1}^d w_{\tilde{\lambda}_l}$ for each $\lambda \in \mathcal{P}(d)$. Similarly let $\mathbb{Z}[v]_{\mathrm{loc}}$ be the localization of $\mathbb{Z}[v]$ at the multiplicative subset $\{v_{ch}^l | l \in \mathbb{Z}_{\geq 0}\}$ where $v_{ch} := v_1 + \cdots + v_d$. We denote $\mathbb{Z}[w]_{\mathrm{loc}}[a] := \mathbb{Z}[w]_{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}[a]$. Let $(\mathbb{Z}[v]_{\mathrm{loc}}[x]^G)^{\mathcal{S}_d}$ be the invariant ring of $\mathbb{Z}[v]_{\mathrm{loc}}[x] := \mathbb{Z}[v]_{\mathrm{loc}} \otimes_{\mathbb{Z}} \mathbb{Z}[x]$ under the simultaneous permutations on the variable sets $x$ and $v$. We denote $\mathbb{Z}[w]_{\mathrm{loc}}[a]((\mathbb{Z}[v]_{\mathrm{loc}}[x]^G)^{\mathcal{S}_d}$ for $\mathbb{Z}[w]_{\mathrm{loc}}[a] \otimes_{\mathbb{Z}} (\mathbb{Z}[v]_{\mathrm{loc}}[x]^G)^{\mathcal{S}_d}$. We adapt the same notational convention for $\mathbb{Z}[w]_{\mathrm{loc}}((\mathbb{Z}[v]_{\mathrm{loc}}[x]^G)^{\mathcal{S}_d}$ and so on.

We introduce the **shifted sequences** $x^w = (x^w_1, \cdots, x^w_d)$ and $a^w = (a^w_i)_{i \in \mathbb{N}}$ by
\[
x^w_i := x_i - \frac{v_i}{v_{ch}} x_{ch} \quad \text{for all } i = 1, \cdots, d \quad \text{and} \quad a^w_i := a_i - \frac{w_i}{w_{ch}} x_{ch} \quad \text{for all } l \in \mathbb{N}
\]
where $x_{ch} := x_1 + \cdots + x_d$. We denote $0_l^{\omega} := a_l^{\omega}|_{a_l=0}, l \in \mathbb{N}$, and the corresponding sequence by $0^\omega$.

**Definition 2.3.** For each $\lambda \in \mathcal{P}(d)$, the weighted factorial Schur function $s_\lambda^w(v; x|a) \in \mathbb{Z}[w_{\text{loc}}[a]|v_{\text{loc}}[x]]$ is defined by

$$s_\lambda^w(v; x|a) := s_\lambda(x^\omega|a^\omega)$$

Similarly for each $\lambda \in \mathcal{P}(d)$, the weighted Schur function $s_\lambda^w(v; x) \in \mathbb{Z}[w_{\text{loc}}[v_{\text{loc}}[x]]]$ is defined by

$$s_\lambda^w(v; x) := s_\lambda(x^\omega|0^\omega).$$

Since $s_\lambda(x|a)$ is invariant under the permutations on $x$-variables, it is not hard to see from this definition that $s_\lambda^w(v; x|a)$ is invariant under the simultaneous permutations on $x$ and $v$ variables, i.e.

$$s_\lambda^w(v; x|a) \in \mathbb{Z}[w_{\text{loc}}[a]|(v_{\text{loc}}[x])^{\tilde{\alpha}_a}] \quad \text{and} \quad s_\lambda^w(v; x) \in \mathbb{Z}[w_{\text{loc}}[v_{\text{loc}}[x]])^{\tilde{\alpha}_a}].$$

**Example 2.4.** We list a few examples of $s_\lambda^w(v; x|a)$ and $s_\lambda^w(v; x)$. Let $w_{ch} := \sum_{i=1}^d w_i$ and $a_{ch} := \sum_{i=1}^d a_i$. Let $\square \in \mathcal{P}(d)$ be the partition given by $b_1^\square = 1$ and $b_2^\square = \cdots = b_d^\square = 0$.

(2.6) \hspace{1cm} s_{\square}^w(v; x|a) = \frac{w_{ch}}{v_{ch}} x_{ch} - a_{ch}

(2.7) \hspace{1cm} s_{\square}^w(v; x) = \frac{w_{ch}}{v_{ch}} x_{ch}

Let $\lambda \in \mathcal{P}(d)$ be a partition with one box only at the first and second row, i.e. $b_1^\lambda = b_2^\lambda = 1$ and $b_3^\lambda = \cdots = b_d^\lambda = 0$. Then

(2.8) \hspace{1cm} s_\lambda^w(v; x|a) = \sum_{1 \leq i < j \leq d} (x_i - a_i - \frac{v_i - w_i}{v_{ch}} x_{ch})(x_j - a_{j-1} - \frac{v_j - w_{j-1}}{v_{ch}} x_{ch}).

**Definition 2.5.** We extend the homomorphism $\psi_\mu$ to a $\mathbb{Z}[w_{\text{loc}}[a]]$-algebra homomorphism $\psi_\mu^\omega$

$$\psi_\mu^\omega : \mathbb{Z}[w_{\text{loc}}[a]|(v_{\text{loc}}[x])^{\tilde{\alpha}_a}] \to \mathbb{Z}[w_{\text{loc}}[a]]; \quad x_i \mapsto a_{\mu_i} \quad \text{and} \quad v_i \mapsto w_{\mu_i} \quad \text{for all} \ i = 1, \ldots, d.$$ Observe that this is well-defined since $v_{ch}$ maps to $w_{ch} \neq 0$. We denote $\psi_\mu^\omega$ by $\psi_\mu$ when there is no confusion.

**Lemma 2.6 (Vanishing Lemma).** For each $\lambda, \mu \in \mathcal{P}(d)$, we have

$$\psi_\mu(s_\lambda^w(v; x|a)) = \begin{cases} 0 & \text{if } \mu \nleq \lambda \\ \prod_{\rho \in [\lambda]}\left(1 - \frac{w_{\rho}}{w_{\lambda}} a_{\lambda} - a_{\rho}\right) & \text{if } \mu = \lambda. \end{cases}$$

**Proof.** Let $a^\mu = ((a^\mu)_l)_{l \in \mathbb{N}}$ be the $\mu$-shifted sequence of $a$ defined by

$$(a^\mu)_l := a_l - \frac{w_l}{w_{\mu}} a_{\mu}.$$ Since $\psi_\mu(x_i^\mu) = (a^\mu)_{\mu_i}$ for $i = 1, \ldots, d$ and $\psi_\mu((a^\mu)^l) = (a^\mu)^l$ for $l \in \mathbb{N}$, we find from (2.8) that

$$\psi_\mu(s_\lambda(x^\omega|a^\omega)) = s_\lambda((a^\mu)_{\mu_1}, \ldots, (a^\mu)_{\mu_d}|a^\mu) = \begin{cases} 0 & \text{if } \mu \nleq \lambda \\ \prod_{\rho \in [\lambda]}((a^\lambda)_\rho - (a^\lambda)_{\rho}) & \text{if } \mu = \lambda. \end{cases}$$

We finish the proof by computing

$$(a^\lambda)_\lambda - (a^\lambda)_{\rho} = (a_\lambda - \frac{w_\lambda}{w_{\lambda}} a_{\lambda}) - (a_{\rho} - \frac{w_{\rho}}{w_{\lambda}} a_{\lambda}) = \frac{w_{\rho}}{w_{\lambda}} a_{\lambda} - a_{\rho}.$$ \qed
2.3. Algebras of weighted (factorial) Schur functions. Let \( \widetilde{\text{wSch}} \) be the \( \mathbb{Z}[\text{w}_{\text{loc}}[a]] \)-submodule of \( \mathbb{Z}[\text{w}_{\text{loc}}[a]([\nu]_{\text{loc}}[x])]^{\mathcal{E}_s} \) generated by \( s^w_\lambda(v;x|a) \)'s:
\[
\text{wSch} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[\text{w}_{\text{loc}}[a]] \cdot s^w_\lambda(v;x|a).
\]
Similarly let \( \text{wSch} \) be the \( \mathbb{Z}[\text{w}_{\text{loc}}[a]] \)-submodule of \( \mathbb{Z}[\text{w}_{\text{loc}}[a]([\nu]_{\text{loc}}[x])]^{\mathcal{E}_s} \) generated by \( s^w_\lambda(v;x) \)'s:
\[
\text{wSch} := \sum_{\lambda \in \mathcal{P}(d)} \mathbb{Z}[\text{w}_{\text{loc}}[a]] \cdot s^w_\lambda(v;x).
\]
Our goal in this section is to show that these submodules \( \widetilde{\text{wSch}} \) and \( \text{wSch} \) are actually subalgebras and also to prove that the weighted factorial Schur functions form a \( \mathbb{Z}[\text{w}_{\text{loc}}[a]] \)-module basis of \( \widetilde{\text{wSch}} \). The linear independency of weighted Schur functions will be postponed until Section \[4\].

To begin with, observe that we have
\[
s^w_\lambda(v;x|a') \cdot s^w_\mu(v;x|a) = \sum_{\nu \in \mathcal{P}(d)} c^w_{\lambda\mu}(a^{\nu w}, a'^{\nu w}) s^w_\nu(v;x|a)
\]
by substituting \( a \mapsto a^{\nu w}, a' \mapsto a'^{\nu w} \) and \( x \mapsto x^{\nu} \) in (2.9). This is not an expansion formula of a product of two weighted factorial Schur functions over \( a \)- and \( a' \)-variables: one should notice that each \( c^w_{\lambda\mu}(a^{\nu w}, a'^{\nu w}) \) contains \( x \)-variables. Here, the product is taken in the ring \( \mathbb{Z}[\text{w}_{\text{loc}}[a]] \otimes \mathbb{Z}[\text{w}_{\text{loc}}[a']] \otimes (\mathbb{Z}[\text{w}_{\text{loc}}[x]])^{\mathcal{E}_s} \).

**Lemma 2.7** (Weighted Pieri Rule).
\[
(2.10) \quad s^w_\square(v;x|a') \cdot s^w_\lambda(v;x|a) = \left( \frac{w'_\lambda}{w_\lambda} a_\lambda - a'_\lambda \right) s^w_\lambda(v;x|a) + \sum_{\lambda' \to \lambda} \frac{w'_\lambda}{w_\lambda} s^w_{\lambda'}(v;x|a)
\]
\[
(2.11) \quad s^w_\lambda(v;x) \cdot s^w_\nu(v;x) = \sum_{\lambda' \to \lambda} \frac{w'_\lambda}{w_\lambda} s^w_{\lambda'}(v;x)
\]

**Proof.** From Theorem 3.1. in [11], we find that
\[
s_\square(x|a') \cdot s_\lambda(x|a) = (a_\lambda - a'_\lambda) s_\lambda(x|a) + \sum_{\lambda' \to \lambda} s_{\lambda'}(x|a).
\]
By substituting \( a \mapsto a^{\nu w}, a' \mapsto a'^{\nu w} \) and \( x \mapsto x^{\nu} \) as in (2.9), we obtain
\[
s_\square(x^{\nu}|a'^{\nu w}) \cdot s_\lambda(x^{\nu}|a^{\nu w}) = (a^{\nu w}_\lambda - a'^{\nu w}_\lambda) s_\lambda(x^{\nu}|a^{\nu w}) + \sum_{\lambda' \to \lambda} s_{\lambda'}(x^{\nu}|a^{\nu w})
\]
By the definition of \( a^{\nu w} \) and (2.7), we have
\[
a^{\nu w}_\lambda - a'^{\nu w}_\lambda = (a_\lambda - a'_\lambda) - \frac{w_\lambda}{w'_{\nu}} x \cdot \left( a_\lambda - \frac{w_\lambda}{w'_{\nu}} a'_\lambda \right) - \frac{w_\lambda}{w'_{\nu}} s^w_\lambda(v;x|a')
\]
After substituting this to the previous equation, it is straightforward to obtain the desired formula. \[\square\]

**Lemma 2.8.** As submodules of \( \mathbb{Z}[\text{w}_{\text{loc}}[a]][[\nu]_{\text{loc}}[x]]^{\mathcal{E}_s} \), we have
\[
\widetilde{\text{wSch}} = \mathbb{Z}[a] \otimes \text{wSch}.
\]

**Proof.** First we prove that \( \widetilde{\text{wSch}} \subset \mathbb{Z}[a] \otimes \text{wSch} \), i.e. for each \( \lambda \), \( s^w_\lambda(v;x|a) \in \mathbb{Z}[a] \otimes \text{wSch} \). Setting \( a = 0 \), \( a_i = 0 \) and \( w'_i = w_i \) for all \( i \in \mathbb{N} \), and substituting \( a'_i \mapsto a_i \) for all \( i \in \mathbb{N} \) in (2.9), we obtain
\[
s^w_\lambda(v;x|a) = \sum_{\nu \in \mathcal{P}(d)} c^w_{\lambda\nu}(0^{\nu w}, a^{\nu w}) s^w_{\nu}(v;x).
\]
Since $a_{i}^{\nu w} = a_{i} - w_{t} \frac{x_{t}}{v_{ch}}$ and $0^{\nu w} = -w_{t} \frac{x_{t}}{v_{ch}}$, $c_{\lambda\mu}^{\nu}(a_{i}^{\nu w}, 0^{\nu w})$ is a polynomial in $x_{ch}/v_{ch}$ with coefficients in $\mathbb{Z}[w]_{loc}[a]$. Since we have (2.7), the weighted Pieri rule (2.11) implies that $s_{\lambda}^{w}(v; x|a)$ is a linear combination of $s_{\mu}^{w}(v; x)$ over $\mathbb{Z}[w]_{loc}[a]$. Thus we conclude that $s_{\lambda}^{w}(v; x|a) \in \mathbb{Z}[a] \otimes w\text{Sch}$.

To prove that $w\text{Sch} \supset \mathbb{Z}[a] \otimes w\text{Sch}$, it suffices to show that $s_{\lambda}^{w}(v; x) \in w\text{Sch}$ for each $\lambda$. We use the similar argument as above. After setting $\mu = \emptyset$ and $a_{i} = 0$ for all $l \in \mathbb{N}$ in (2.9), the equation (2.10) and the weighted Pieri rule (2.11) imply that $s_{\lambda}^{w}(v; x) \in w\text{Sch}$. □

**Proposition 2.9.** $w\text{Sch}$ is a $\mathbb{Z}[w]_{loc}$-subalgebra of $\mathbb{Z}[w]_{loc}([v]_{loc}[x])^{\mathbb{S}_{4}}$. In particular, $w\text{Sch}$ is a $\mathbb{Z}[w]_{loc}[a]$-subalgebra of $\mathbb{Z}[w]_{loc}[a]([v]_{loc}[x])^{\mathbb{S}_{4}}$.

**Proof.** By evaluating $a_{i} = a_{i}' = 0$ and $w_{t}' = w_{t}$ for all $t \in \mathbb{N}$ in (2.10), we have

$$s_{\lambda}^{w}(v; x) \cdot s_{\mu}^{w}(v; x) = \sum_{\nu \in \mathcal{P}(d)} c_{\lambda\mu}^{\nu}(0^{\nu w}, 0^{w}) s_{\mu}^{w}(v; x).$$

Since $c_{\lambda\mu}^{\nu}(a, a)$ is a polynomial in $\{a_{k} - a_{l}; k, l \in \mathbb{N}\}$, $c_{\lambda\mu}^{\nu}(0^{w}, 0^{w})$ is a polynomial in $-w_{k} x_{ch} - \left(-w_{u} x_{ch} - \left(-w_{k} x_{ch} - \left(-w_{t} x_{ch}\right)\right)\right) = -(w_{k} - w_{t}) x_{ch} = -\left(w_{k} - w_{t}\right) s_{\mu}^{w}(v; x)$. Therefore by the weighted Pieri rule (2.11), the product $s_{\lambda}^{w}(v; x) \cdot s_{\mu}^{w}(v; x)$ is a linear combination of $\{s_{\mu}^{w}(v; x)\}_{\nu \in \mathcal{P}(d)}$ over $\mathbb{Z}[w]_{loc}$. Now the latter claim follows from Lemma 2.8 □

**Proposition 2.10.** $\{s_{\lambda}^{w}(v; x|a)\}_{\lambda \in \mathcal{P}(d)}$ is a $\mathbb{Z}[w]_{loc}[a]$-basis of $w\text{Sch}$.

**Proof.** By the definition of $w\text{Sch}$, it is sufficient to show the linear independency. Suppose

$$\sum_{\lambda \in \mathcal{P}(d)} f_{\lambda}(w, a) \cdot s_{\lambda}^{w}(v; x|a) = 0$$

for some $f_{\lambda}$’s in $\mathbb{Z}[w]_{loc}[a]$. Let $\mu$ be a minimal (with respect to the inclusion) partition among those $\lambda$ such that $f_{\lambda}$ is not identically zero, i.e. there is no $\lambda$ such $\mu \subset \lambda$ and $f_{\lambda} \neq 0$ except $\mu$ itself. Thus by the Vanishing Lemma 2.6 we have

$$0 = \psi_{\mu} \left( \sum_{\lambda \in \mathcal{P}(d)} f_{\lambda} \cdot s_{\lambda}^{w}(v; x|a) \right) = f_{\mu} \psi_{\mu}(s_{\mu}^{w}(v; x|a)) = f_{\mu} \prod_{\rho \in [\mu]} \left( \frac{w_{\rho}}{w_{\mu}} a_{\mu} - a_{\rho} \right).$$

Since $\mathbb{Z}[w]_{loc}[a]$ has no zero divisor, we have $f_{\lambda} = 0$. This is a contradiction. □

3. **(Equivariant) Cohomology of Weighted Grassmannians**

In the rest of the paper, all cohomologies are over $\mathbb{Q}$-coefficients unless otherwise specified.

### 3.1. Review of Weighted Grassmannians and Weighted Schubert Classes.

Let us fix an infinite sequence $\{w_{i}\}_{i \in \mathbb{N}}$ of non-negative integers and a positive integer $u$. In this section, we recollect from [1] a few facts about the cohomology of the weighted Grassmannians $\mathbb{W}Gr_{d}(d, n)$ with the weight $(\tilde{w}_{1}, \ldots, \tilde{w}_{n}) = (w_{n}, \ldots, w_{1})$.

For a natural number $n > d$, let $\mathcal{P}(d, n)$ be the set of partitions that are contained in the $d \times (n - d)$ rectangle. Upon a choice of $n$, we identify $\mathcal{P}(d, n)$ with the set $\left\{\rho_{\frac{n}{d}}\right\}$ of subsets of $\{1, \ldots, n\}$ with cardinality $d$ by

$$\lambda \mapsto \{\lambda_{1} < \cdots < \lambda_{d}\} \quad \text{where} \quad \lambda_{i} := n + 1 - \tilde{\lambda}_{i}, \quad i = 1, \ldots, d$$

where $\tilde{\lambda}_{i}$ is defined at (2.3).
Let \( \{e_1, \ldots, e_n\} \) be the standard basis of \( \mathbb{C}^n \) for each \( n > d \). We define \( \text{aPl}(d,n) \) to be the image of
\[
\mathbb{C}^n \times \cdots \times \mathbb{C}^n \to \Lambda^d \mathbb{C}^n, \quad (\alpha_1, \ldots, \alpha_d) \mapsto \alpha_1 \wedge \cdots \wedge \alpha_d
\]
and let \( \text{aPl}(d,n)^\times := \text{aPl}(d,n) \setminus \{0\} \). The standard action of the \( n \)-dimensional complex torus \( T^n \) on \( \mathbb{C}^n \) induces an action of \( T^n \) on \( \text{aPl}(d,n)^\times \). The twisted diagonal subgroup \( \mathfrak{m}_C \) of \( T^n \) is defined by
\[
\mathfrak{m}_C := \{(t^{d\bar{w}_1}u, \ldots, t^{d\bar{w}_n}u) \mid t \in \mathbb{C}^\times\}.
\]
The weighted Grassmanian \( \mathfrak{Gr}(d,n) \) for the weights \( (\bar{w}_1, \ldots, \bar{w}_n) \) is defined by
\[
\mathfrak{Gr}(d,n) := \text{aPl}(d,n)^\times / \mathfrak{m}_C.
\]
The real subtorus \( T^n \) in \( T^n \) acts on \( \mathfrak{Gr}(d,n) \) through the quotient map \( T^n \to \mathfrak{wR}^{n-1} := T^n / T^n \cap \mathfrak{m}_C \). There is the bijection between \( P(d,n) \) and the fixed point set \( F_n \) for the \( T^n \)-action on \( \mathfrak{Gr}(d,n) \) sending \( \lambda \in \{\underline{d}\} = P(d,n) \) to the equivalent class of \( e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} \) which we denote by \([\lambda] \).

The linear inclusion \( \wedge^d \mathbb{C}^n \to \wedge^d \mathbb{C}^{n+1} \) sending \( e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} \mapsto e_{\lambda_1} \wedge \cdots \wedge e_{\lambda_d} + e_{\lambda_{d+1}} \) for each \( \lambda \in \{\underline{d}\} \) induces a map \( \iota_\lambda : \text{aPl}(d,n)^\times \to \text{aPl}(d,n+1)^\times \). This is equivariant under the homomorphism \( \rho_n : T^n \to T_{n+1}^n \) which sends \((t_1, \ldots, t_n) \) to \((1, t_1, \ldots, t_n)\), and hence induces the \( \rho_n \)-equivariant map
\[
w_{n} : \mathfrak{Gr}(d,n) \to \mathfrak{Gr}(d,n+1).
\]
This restricts to the inclusion \( F_n \to F_{n+1} \) which corresponds to the natural inclusion \( P(d,n) \subset P(d,n+1) \) and the inclusion \( \{\underline{d}\} \subset \{\underline{d+1}\} \) given by \( \lambda \mapsto \lambda + 1 \). Let \( y_1, \ldots, y_n \) be the standard basis of the weight lattice \( \text{Lie}(T^n) \) for every \( n \). We identify \( H^*(B T^n) = \mathbb{Q}[y_1, \ldots, y_n] \) and \( \rho_n \) induces the projection
\[
\rho_n : \mathbb{Q}[y_1, \ldots, y_{n+1}] \to \mathbb{Q}[y_1, \ldots, y_n]; \quad y_i \mapsto 0 \quad \text{and} \quad y_i \to y_{i-1} \quad \text{for all} \quad 2 \leq i \leq n + 1.
\]

Recall that for each \( \lambda \in P(d,n) \), the equivariant weighted Schubert class \( \mathfrak{wS}_{\lambda}^n \) is defined as a class in \( H_{\mathfrak{wR}^{n-1}}(\mathfrak{Gr}(d,n)) \) and that they form a \( H^*(B \mathfrak{wR}^{n-1}) \)-basis. We use the same symbol \( \mathfrak{wS}_{\lambda}^n \) for the image of \( \mathfrak{wS}_{\lambda}^n \) under the map \( H_{\mathfrak{wR}^{n-1}}(\mathfrak{Gr}(d,n)) \to H_{\mathfrak{T}_0}(\mathfrak{Gr}(d,n)) \) induced by the quotient map \( T^n \to \mathfrak{wR}^{n-1} \). The following are easy to check and will be used in the rest of the paper.

(i) The equivariant weighted Schubert classes \( \mathfrak{wS}_{\lambda}, \lambda \in P(d,n) \) form a \( \mathbb{Q}[y_1, \ldots, y_n] \)-module basis of \( H^*_\mathfrak{T}_0(\mathfrak{Gr}(d,n)) \).

(ii) The restriction map
\[
H^*_\mathfrak{T}_0(\mathfrak{Gr}(d,n)) \to H^*_\mathfrak{T}_0(F_n) = \bigoplus_{[e_\mu] \in F_n} H^*_\mathfrak{T}_0([e_\mu]) = \bigoplus_{\mu \in P(d,n)} \mathbb{Q}[y_1, \ldots, y_n]
\]
is a \( \mathbb{Q}[y_1, \ldots, y_n] \)-algebra homomorphism, and it is in fact injective. The image \( \mathfrak{wS}_{\lambda}^n \mid_\mu \) of \( \mathfrak{wS}_{\lambda}^n \) at the fixed point \([e_\mu] \in F_n \) is computed in [1, Proposition 6.1] ;
\[
\mathfrak{wS}_{\lambda}^n \mid_\mu = s_{\lambda}(-(y_\mu)^n)_{\mu,1}, \ldots, -(y_\mu)^n_{\mu,n} - (y_\mu_{\mu})_1, \quad (y_\mu)^n_i = y_i - \frac{\bar{w}_i y_\mu}{w_\mu},
\]
where \( y_\mu := \sum_{i=1}^d y_{\mu,i} \) and \( \bar{w}_\mu := \sum_{i=1}^d \bar{w}_{\mu,i} \).
The pullback map $\tilde{w}_n^*: H^*_T(\text{Gr}(d, n+1)) \to H^*_T(\text{Gr}(d, n))$ is a $\mathbb{Q}[y_1, \cdots, y_{n+1}]$-algebra homomorphism with respect to $\rho^*_n$ and for each $\mu \in \mathcal{P}(d, n)$, we have

$$
\tilde{w}_n^*(\mathbb{W}\mathbb{S}_\lambda) = \begin{cases} 
\mathbb{W}\mathbb{S}_\lambda & \text{if } \lambda \in \mathcal{P}(d, n), \\
0 & \text{if } \lambda \notin \mathcal{P}(d, n).
\end{cases}
$$

In particular, $\tilde{w}_n^*$ is surjective.

For each $\lambda \in \mathcal{P}(d, n)$, the weighted Schubert class $\mathbb{W}\mathbb{S}_\lambda$ is the image of $\mathbb{W}\mathbb{S}_\lambda$ under the natural map $H^*_T(\text{Gr}(d, n)) \to H^*(\text{Gr}(d, n))$ and they form a $\mathbb{Q}$-basis of $H^*(\text{Gr}(d, n))$. The pullback map $w_n^*: H^*(\text{Gr}(d, n+1)) \to H^*(\text{Gr}(d, n))$ satisfies

$$
\tilde{w}_n^*(\mathbb{W}\mathbb{S}_\lambda) = \begin{cases} 
\mathbb{W}\mathbb{S}_\lambda & \text{if } \lambda \in \mathcal{P}(d, n), \\
0 & \text{if } \lambda \notin \mathcal{P}(d, n).
\end{cases}
$$

In particular, $w_n^*$ is surjective.

3.2. Cohomology of $\mathbb{W}\text{Gr}(d, \infty)$. By using the inclusions $\{w_n, n \in \mathbb{N}\}$, we define

$$
\mathbb{W}\text{Gr}(d, \infty) := \lim_{\leftarrow} \mathbb{W}\text{Gr}(d, n) = \bigcup_{n \in \mathbb{N}} \mathbb{W}\text{Gr}(d, n).
$$

Since $w_n^*$ is surjective for each $n$, we have $\lim_{\leftarrow}^1 H^k(\mathbb{W}\text{Gr}(d, n)) = 0$ for each $k$. Therefore there is the $\mathbb{Q}$-linear isomorphism

$$
H^k(\mathbb{W}\text{Gr}(d, \infty)) \to \lim_{\leftarrow} H^k(\mathbb{W}\text{Gr}(d, n)) \quad \text{for each } k \geq 0.
$$

The cup products on $H^*(\mathbb{W}\text{Gr}(d, n))$ for all $n \in \mathbb{N}$ canonically define a structure of $\mathbb{Q}$-algebra on the direct sum of $\lim_{\leftarrow} H^k(\mathbb{W}\text{Gr}(d, n))$ over all $k \geq 0$ and we have

**Proposition 3.1.** The inclusions induces a canonical $\mathbb{Q}$-algebra isomorphism

$$
H^*(\mathbb{W}\text{Gr}(d, \infty)) \to \bigoplus_{k \geq 0} \lim_{\leftarrow} H^k(\mathbb{W}\text{Gr}(d, n)).
$$

The property (iv), together with this proposition, defines the classes $\mathbb{W}\mathbb{S}_\lambda^\infty$ in $H^*(\mathbb{W}\text{Gr}(d, \infty))$ such that the pullback $H^k(\mathbb{W}\text{Gr}(d, \infty)) \to H^k(\mathbb{W}\text{Gr}(d, n))$ sends $\mathbb{W}\mathbb{S}_\lambda^\infty$ to $\mathbb{W}\mathbb{S}_\lambda^n$ if $\lambda \in \mathcal{P}(d, n)$ and 0 otherwise.

**Proposition 3.2.** $\{\mathbb{W}\mathbb{S}_\lambda^\infty\}_{\lambda \in \mathcal{P}(d)}$ forms a $\mathbb{Q}$-basis of $H^*(\mathbb{W}\text{Gr}(d, \infty))$.

**Proof.** For each $k \geq 0$, the pull back gives us an isomorphism

$$
H^k(\mathbb{W}\text{Gr}(d, \infty)) \cong H^k(\mathbb{W}\text{Gr}(d, n)) = \bigoplus_{\lambda \in \mathcal{P}(d, n), l(\lambda) = k} \mathbb{Q} \cdot \mathbb{W}\mathbb{S}_\lambda^n
$$

for a sufficiently large $n > k$ where $l(\lambda) = \sum_{i=1}^d b_i^\lambda$ is the number of boxes in $\lambda$. By the definition of the elements $\{\mathbb{W}\mathbb{S}_\lambda^\infty\}$, they correspond to $\{\mathbb{W}\mathbb{S}_\lambda^n\}$ which is a $\mathbb{Q}$-basis of the image, the claim follows. \qed
3.3. Cohomology of equivariant analogue of \( \mathfrak{wGr}(d, \infty) \). Let \( ET^n \to BT^n \) is a universal principal \( T^n \)-bundle in which \( ET^n \) is contractible. We choose a \( \rho_n \)-equivariant continuous map \( ET^n \to ET^{n+1} \) for each \( n \). Hence we have the induced maps \( \rho_n^*: BT^n \to BT^{n+1} \) whose cohomology pullbacks are exactly the surjection \( \rho_n^* \). Let \( BT_\infty := \lim_{\to} BT^n \) be the corresponding inductive limit. As in the last section, there is no \( \lim^1 \) and hence we have a \( \mathbb{Q} \)-algebra isomorphism

\[
H^*(BT_\infty) \cong \bigoplus_{k \geq 0} \lim_{\to} \mathbb{Q}[y_1, \ldots, y_n]^{(k)}
\]

where \( \mathbb{Q}[y_1, \ldots, y_n]^{(k)} \) is the component of the cohomological degree \( 2k \). Let \( (\overline{y}_l)_{l \in \mathbb{N}} \) be an infinite sequence of variables and let \( \mathbb{Q}[\overline{y}] \) be the ring of polynomials in \( \overline{y}_l \)'s which are possibly infinite linear combinations of finite degree monomials. Then the RHS of (3.2) can be identified with \( \mathbb{Q}[\overline{y}] \) through the homomorphisms

\[
\theta_n: \mathbb{Q}[\overline{y}] \to \mathbb{Q}[y_1, \ldots, y_n]; \quad \overline{y}_l \mapsto \begin{cases} y_{n+1-l} & \text{if } 1 \leq l \leq n \\ 0 & \text{if } l > n. \end{cases}
\]

By using the \( \rho_n \)-equivariant map \( \tilde{w}_n \), define

\[
\mathfrak{wGr}(d, \infty)_T := \lim_{\to} (ET^n \times_{T^n} \mathfrak{wGr}(d, n)),
\]

then we have the commutative diagrams for all \( n \):

\[
\begin{array}{ccc}
\mathfrak{wGr}(d, \infty)_T & \xrightarrow{\rho_n} & ET^n \times_{T^n} \mathfrak{wGr}(d, n) \\
\downarrow & & \downarrow \\
BT_\infty & \xrightarrow{\tilde{w}_n} & BT^n
\end{array}
\]

Note that the cohomology \( H^*(\mathfrak{wGr}(d, \infty)_T) \) and the pullback \( \tilde{w}_n^* \) do not depend on the choices of \( ET^n \)'s and the maps \( ET^n \to ET^{n+1} \) up to isomorphisms.

As in the last section, we have no \( \lim^1 \) term for the propjective system \( \{ H^k_{T^n}(\mathfrak{wGr}(d, n)), \tilde{w}_n^* \} \) for each \( k \), therefore the top map in the above diagram induces the \( \mathbb{Q} \)-algebra isomorphism

\[
H^*(\mathfrak{wGr}(d, \infty)_T) \xrightarrow{\bigoplus_{k \geq 0}} \lim_{\to} H^k_{T^n}(\mathfrak{wGr}(d, n)).
\]

Since the right vertical maps of (3.3) commute with \( \rho_n^* \) and \( \tilde{w}_n^*: ET^n \times_{T^n} \mathfrak{wGr}(d, n) \to ET^{n+1} \times_{T^{n+1}} \mathfrak{wGr}(d, n+1) \), the ring \( \mathbb{Q}[\overline{y}] \) acts on the RHS of (3.4). Thus, by the commutativity of (3.3), the map (3.4) is actually a \( \mathbb{Q}[\overline{y}] \)-algebra isomorphism. With the property (iii), the isomorphism (3.4) defines the classes \( \mathfrak{w}S^\infty_\lambda \) in \( H^*(\mathfrak{wGr}(d, \infty)_T) \) such that the pullback \( H^*(\mathfrak{wGr}(d, \infty)_T) \to H^*_T(\mathfrak{wGr}(d, n)) \) sends \( \mathfrak{w}S^\infty_\lambda \) to \( \mathfrak{w}S^\infty_\lambda \) if \( \lambda \in \mathcal{P}(d, n) \) and 0 otherwise. Finally it is not difficult to see from the RHS of (3.4) that \( \mathfrak{w}S^\infty_\lambda, \lambda \in \mathcal{P}(d) \) form a \( \mathbb{Q}[\overline{y}] \)-module basis of \( H^*(\mathfrak{wGr}(d, \infty)_T) \). We conclude this section by summarizing above as follows.

**Proposition 3.3.** \( H^*(\mathfrak{wGr}(d, \infty)_T) \) is a \( \mathbb{Q}[\overline{y}] \)-algebra and there is a \( \mathbb{Q}[\overline{y}] \)-module basis \( \{ \mathfrak{w}S^\infty_\lambda, \lambda \in \mathcal{P}(d) \} \) such that \( \mathfrak{w}S^\infty_\lambda \) maps to \( \mathfrak{w}S^\infty_\lambda \) under the pullback \( H^*(\mathfrak{wGr}(d, \infty)_T) \to H^*_T(\mathfrak{wGr}(d, n)) \) for each \( n \).
4. CORRESPONDENCES OF FUNCTIONS AND COHOMOLOGY

Recall from Proposition 2.9 w̃Sch is a \( \mathbb{Z}[w]_{\text{loc}}[a] \)-subalgebra of \( \mathbb{Z}[w]_{\text{loc}}[a](v)_{\text{loc}}[x] \)\(^{\Theta_{\mu}} \). Let \( \{ w_l \} \in \mathbb{N} \subset \mathbb{Z}_{\geq 0} \) and \( u \in \mathbb{Z}_{\geq 0} \). For each \( n \in \mathbb{N} \), define a ring homomorphism

\[
\varphi^w_n : \mathbb{Z}[w]_{\text{loc}}[a] \to \mathbb{Q}[y_1, \cdots, y_n]; \quad w_l \mapsto w_0 + u/d \quad \text{and} \quad a_l \mapsto \begin{cases} -y_{n+l} & \text{if } 1 \leq l \leq n \\ 0 & \text{if } l > n \end{cases}
\]

where \( l \) runs all the natural numbers \( \mathbb{N} \). It is well-defined since

\[
w_{\mu} = \sum_{i=1}^{d} w_{\bar{\mu}_i} \mapsto \bar{w}_{\mu} = \sum_{i=1}^{d} \bar{w}_{\mu_i} + u \neq 0.
\]

These \( \varphi^w_n \)'s induce the algebra homomorphism

\[
\varphi^w : \mathbb{Z}[w]_{\text{loc}}[a] \to \mathbb{Q}[y]; \quad a_l \mapsto -\bar{y}_l \quad \text{and} \quad w_l \mapsto w_l + u/d \quad \text{for all } l \in \mathbb{N}.
\]

**Theorem 4.1.** There is a \( \mathbb{Z}[w]_{\text{loc}}[a] \)-algebra homomorphism

\[
\widetilde{\Phi}_n : w\text{Sch} \to H^*_T(\overline{\text{Gr}}(d, n)); \quad s^w_{\lambda}(v, x|a) \mapsto \begin{cases} \overline{w\text{Sch}}_{\lambda} & \text{if } \lambda \in \mathcal{P}(d, n), \\ 0 & \text{otherwise}, \end{cases}
\]

where the action of \( \mathbb{Z}[w]_{\text{loc}}[a] \) on the RHS is given by \( \varphi^w_n \). In particular, this defines a \( \mathbb{Z}[w]_{\text{loc}}[a] \)-algebra homomorphism

\[
\widetilde{\Phi}_\infty : w\text{Sch} \to H^*(\overline{\text{Gr}}(d, \infty)_T); \quad s^w_{\lambda}(v, x|a) \mapsto \overline{w\text{Sch}}_{\lambda}^\infty
\]

where the action \( \mathbb{Z}[w]_{\text{loc}}[a] \) on the RHS is given by \( \varphi^w_\infty \).

**Proof.** Consider the \( \mathbb{Z}[w]_{\text{loc}}[a] \)-algebra homomorphism

\[
\begin{align*}
\overline{w\text{Sch}} & \xrightarrow{(\varphi^w_n)_{n \in \mathcal{P}(d)}} \prod_{\mu \in \mathcal{P}(d)} \mathbb{Z}[w]_{\text{loc}}[a] \xrightarrow{\prod_{\mu \in \mathcal{P}(d, n)} Q[y_1, \cdots, y_n]}
\end{align*}
\]

where the second map is given by \( \varphi^w_n \) if \( \mu \in \mathcal{P}(d, n) \) and a trivial map if otherwise. If \( \lambda \notin \mathcal{P}(d, n) \), then for all \( \mu \in \mathcal{P}(d, n) \), we have \( \mu \not\subseteq \lambda \) and therefore the image of \( s^w_{\lambda}(v, x|a) \) under \( \prod \) is 0. If \( \lambda \in \mathcal{P}(d, n) \), then for all \( \mu \in \mathcal{P}(d, n) \), under the map \( \prod \) we have

\[
s^w_{\lambda}(v, x|a) \mapsto s_{\lambda}(a^\mu_1, \cdots, (a^\mu_\lambda) | a^\mu), \quad (a^\mu)_l = a_l - \frac{w_l}{w_{\bar{\mu}}},
\]

Here for the second map, we have used the fact that \( s_{\lambda}(x|a) \) does not involve \( a_l \) for all \( i > n \) if \( \lambda \in \mathcal{P}(d, n) \) by definition. The property (ii) in Section 3.1 implies that the image of \( s^w_{\lambda}(v, x|a) \) is exactly the restriction of \( \overline{w\text{Sch}}_{\lambda} \) to the fixed points if \( \lambda \in \mathcal{P}(d, n) \) and 0 otherwise. Moreover it follows from the injectivity in (ii) that the map \( \prod \) factors through the desired map \( \widetilde{\Phi}_n : w\text{Sch} \to H^*_T(\overline{\text{Gr}}(d, n)) \). By introducing \( \deg x_i = \deg a_l = 2 \) (and \( \deg v_i = \deg w_l = 0 \)) for all \( i = 1, \cdots, d \) and \( l \in \mathbb{N} \), the map \( \widetilde{\Phi}_n \) is a homomorphism as graded \( \mathbb{Q} \)-algebras. The obvious commutativity \( \overline{w\text{Sch}}_{\lambda} \circ \widetilde{\Phi}_{n+1} = \widetilde{\Phi}_n \) allow us to take the projective limit of the maps \( \widetilde{\Phi}_n \) on each degree, and taking their direct sum, we obtain the desired map \( \widetilde{\Phi}_\infty \).

By tensoring \( \mathbb{Z} \) over \( \mathbb{Z}[a] \) with respect to the homomorphism \( \mathbb{Z}[a] \to \mathbb{Z}, (a_i) \mapsto 0 \), we obtain the \( \mathbb{Z}[w]_{\text{loc}}[a] \)-algebra homomorphism \( \Phi_n := \mathbb{Z} \otimes_{\mathbb{Z}[a]} \widetilde{\Phi}_n \)

\[
\Phi_n : \mathbb{Z} \otimes \mathbb{Z}[a] \overline{w\text{Sch}} \to \mathbb{Z} \otimes \mathbb{Z}[a] H^*_T(\overline{\text{Gr}}(d, n)).
\]
The LHS is exactly wSch by Lemma 2.8. The action of \( \mathbb{Z}[a] \) on \( H^*_\mathfrak{z}(\mathbb{W}Gr(d, n)) \) is through \( \mathbb{Z}[a] \to \mathbb{Q}[y_1, \cdots, y_n] \) that sends \( a_i \) to \( y_{n+1-i} \) if \( 1 \leq i \leq n \) and 0 otherwise. Therefore the RHS is \( H^*(\mathbb{W}Gr(d, n)) \). Thus the following is an immediate consequence.

**Theorem 4.2.** For each \( n \in \mathbb{N} \), the map \( \tilde{\Phi}_n \) induces the \( \mathbb{Z}[w]_{\text{loc}} \)-algebra homomorphism

\[
\Phi_n : \text{wSch} \to H^*(\mathbb{W}Gr(d, n)); \quad s^w_\lambda(v; x) \mapsto \begin{cases} 
\mathbb{W}S_\lambda^\infty & \text{if } \lambda \in \mathcal{P}(d, n), \\
0 & \text{otherwise}.
\end{cases}
\]

In particular, this defines a \( \mathbb{Z}[w]_{\text{loc}} \)-algebra homomorphism

\[
\Phi_\infty : \text{wSch} \to H^*(\mathbb{W}Gr(d, \infty)); \quad s^w_\lambda(v; x) \mapsto \mathbb{W}S_\lambda^\infty.
\]

**Proposition 4.3.** The weighted Schur functions \( s^w_\lambda(v; x) \), \( \lambda \in \mathcal{P}(d) \), form a \( \mathbb{Z}[w]_{\text{loc}} \)-module basis of wSch.

**Proof.** It is enough to check the linear independency. Suppose that

\[
\sum_{\lambda \in \mathcal{P}(d)} f_\lambda(w)s^w_\lambda(v; x) = 0
\]

for some \( f_\lambda(w) \in \mathbb{Z}[w]_{\text{loc}} \). There exists \( n \) such that, for all \( \lambda \) appearing in the sum, \( f_\lambda(w) \) involves only \( w_1, \cdots, w_n \). Then \( \tilde{\Phi}_n \) send the equality to

\[
\sum_{\lambda \in \mathcal{P}(d)} f_\lambda(w_1 + u/d, \cdots, w_n + u/d)\mathbb{W}S_\lambda^w = 0.
\]

This holds for arbitrary \( w_1, \cdots, w_n \in \mathbb{Z}_{>0} \) and \( u \in \mathbb{Z}_{>0} \) and since \( \{\mathbb{W}S_\lambda^w\}_{\lambda \in \mathcal{P}(d, n)} \) is a linearly independent set, we conclude that \( f_\lambda(w) \) is identically 0 for all \( \lambda \) appearing in the sum. \( \square \)

**Remark 4.4.** The homomorphisms \( \tilde{\Phi}_\infty \) in Theorem 4.1 and \( \Phi_\infty \) in Theorem 4.2 can be made into isomorphisms by evaluating the w-variables. Namely, let

\[
\text{wSch} := \mathbb{Q} \otimes_{\mathbb{Z}[w]_{\text{loc}}} \text{wSch} \subset (\mathbb{Q}[v]_{\text{loc}}[x])^{\mathcal{S}_d}
\]

\[
\text{wSch} := \mathbb{Q}[a] \otimes_{\mathbb{Z}[w]_{\text{loc}}[a]} \text{wSch} \subset \mathbb{Q}[a][[v]_{\text{loc}}[x])^{\mathcal{S}_d}.
\]

where the tensor products are given by \( \mathbb{Z}[w]_{\text{loc}} \to \mathbb{Q} \) (\( w_i \mapsto w_i + u/d \)). Then clearly \( \Phi_\infty \) and \( \tilde{\Phi}_\infty \) induce the isomorphisms

\[
\Phi^w_\infty : \text{wSch} \xrightarrow{\cong} H^*(\mathbb{W}Gr(d, \infty)) \quad \text{and} \quad \tilde{\Phi}^w_\infty : \text{wSch} \xrightarrow{\cong} H^*(\mathbb{W}Gr(d, \infty)\tau).
\]

Here \( \Phi^w_\infty \) is an algebra isomorphism with respect to \( \mathbb{Q}[a] \cong \mathbb{Q}[\overline{\mathcal{F}}] \) defined by \( a_l \mapsto -\overline{a_l} \) for all \( l \in \mathbb{N} \). Then these isomorphisms send the evaluated weighted (factorial) Schur functions

\[
s'^w_\lambda(v; x) := s^w_\lambda(v; x)|_{w_1=w_1+u/d, l \in \mathbb{N}} \quad \text{and} \quad s'^w_\lambda(v; x|a) := s^w_\lambda(v; x|a)|_{w_1=w_1+u/d, l \in \mathbb{N}}
\]

to \( \mathbb{W}S_\lambda^\infty \) and \( \mathbb{W}S_\lambda^\infty \) respectively.

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