Strong Spatial Mixing for Binary Markov Random Fields

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Abstract

Gibbs distribution of binary Markov random fields on a sparse on average graph is con-
sidered in this paper. The strong spatial mixing is proved under the condition that the
‘external field’ is uniformly large or small. Such condition on ‘external field’ is meaningful
in physics.

Keywords: Strong Spatial Mixing; Self-Avoiding Trees; Binary Markov Random Fields; Ising Models

1. Introduction

Strong spatial mixing property of Gibbs measures is very important in statistical
physics. It roughly says that if there is a modification (or perturbation) on the
boundary conditions, its influence to the Gibbs measure of a single vertex decays
exponentially fast as the distance to the support of the perturbation (the set of
vertices, whose spins are changed) becomes large. In the classic literatures, it is also
required that the support of the perturbation has to be a single vertex[3]. Weitz
considers the support of perturbation to be a set of vertices of arbitrary size. This
generalized definition is equivalent to the one in [3] when the graph grows sub-
exponentially (e.g. integer lattices). In fact, the definition by Weitz has much wider
application. For example, it provides a natural algorithm to calculate the partition
function of Gibbs measures if the strong spatial mixing holds[13]. In this paper, the
definition of strong spatial mixing is in the sense of Weitz.

Recently the strong spatial mixing is also studied through recursive formula. This
approach is introduced by Weitz[13] and Bandyopadhyay, Gamarnik[1] for counting
the number of independent sets and colorings. The key point of this method is

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to build the strong spatial mixing on certain rooted trees. In \cite{13}, the equivalence between the marginal probability of a vertex in a general graph $G$ and that of the root of a tree for hard core model is proved using the \textit{self-avoiding tree} technique. This shows the correlations on any graph decay at least as fast as its corresponding self-avoiding tree. The strong spatial mixing for hard-core model on bounded degree trees is also proved. Later Gamarnik et.al.\cite{5} and Bayati et.al.\cite{2} bypass the construction of a self-avoiding tree. Instead, they create a \textit{computation tree} and establish the strong spatial mixing on the corresponding computation tree for list coloring and matching problems. Considering the Weitz’s motivation of construction of the self-avoiding tree, Jung and Shah\cite{6} and Nair and Tetali \cite{11} generalize Weitz’s work to certain Markov random fields models, and Lu et.al.\cite{8} on TP decoding problem. Very recently Mossel and Sly \cite{10} show that ferromagnetic Ising model exhibits strong correlation decay on ‘sparse on average’ graph under the tight assumption.

We consider the Binary Markov random fields, which are also known as two state spin systems, on a sparse on average graph where the total degrees along each self-avoiding path (a path with distinct vertices) with length $O(\log n)$ is $O(\log n)$ \cite{10}. We prove, for any ‘inverse temperature’ on this graph, Gibbs distribution exhibits strong spatial mixing when the ‘external field’ is uniformly larger than $B(d, \alpha_{\text{max}}, \gamma)$ or smaller than $-B(d, -\alpha_{\text{min}}, \gamma)$. Here, $d$ is ‘maximum average degree’ and $\alpha_{\text{min}}, \alpha_{\text{max}}, \gamma$ are parameters of the system. To the best of our knowledge, this condition on ‘external field’ is first considered for strong spatial mixing. Our proof is based on a well known recursive formula \cite{6} on a tree and the self-avoiding tree technique. We also employed Lipchitz method, which was used in \cite{11,2,5}. The novelty of our proof is that we propose a ‘path’ characterization of Lipchitz method, which enables us to give the ‘external field’ condition in terms of ‘maximum average degree’ for the strong spatial mixing.

The remainder of the paper has the following structure. In Section 2, we present some preliminary definitions and notations. We go on to propose the main result in Section 3. Section 4 is devoted to prove the main theorem. Conclusion and further work are given in Section 5.

\section{Preliminaries}

Let $G = (V, E)$ be a finite graph with vertices $V = \{1, 2, \cdots, n\}$ and edge set $E$, and let $d(u, v)$ denote the distance between $u$ and $v$, for any $u, v \in V$. A path $v_1, v_2, \cdots$ is called a self-avoiding path if $v_i \neq v_j$ for any $i \neq j$. The distance between
a vertex $v \in V$ and a subset $\Lambda \subset V$ is defined as

$$d(v, \Lambda) = \min\{d(v, u) : u \in \Lambda\}.$$  

A set of vertices with distance $l$ to the vertex $v$ is denoted by

$$S(G, v, l) = \{u : d(v, u) = l\}.$$  

Let $\delta_v$ denote the degree of the vertex $v \in G$. The maximal path density of the graph $G$ is given by

$$m = m(G, v, l) = \max_{\Gamma} \sum_{u \in \Gamma} \delta_u,$$

where the maximum is taken over all self-avoiding paths $\Gamma$ starting at $v$ with length at most $l$. The maximum average path degree $\delta(G, v, l)$ is defined by

$$\delta(G, v, l) = (m(G, v, l) - \delta_v)/l, \quad l \geq 1.$$  

The maximum average degree of $G$ is defined as

$$\Delta(G, l) = \max_{v \in V} \delta(G, v, l).$$

For any order of all the vertices in $G$ given, an associated partial order of $E$ based on the order of $V$ defined as $(i, j) > (k, l)$ if and only if $(i, j)$ and $(k, l)$ share a common vertex and $i + j > k + l$. In binary Markov random fields (BMRF) on $G$, each vertex $i \in V$ is associated with a random variable $X_i$ with range $\Omega = \{\pm 1\}$ (briefly $\pm$).

**Definition 1.** The Gibbs measure of BMRF on $G$ is defined by the joint distribution of the random variable $X = \{X_1, X_2, \cdots, X_n\}$

$$P_G(X = \sigma) = \frac{1}{Z(G)} \exp\left( \sum_{(i,j) \in E} \beta_{ij}(\sigma_i, \sigma_j) + \sum_{i \in V} h_i(\sigma_i) \right),$$

where $h_i : \Omega \to R$ and $\beta_{ij} : \Omega^2 \to R$. Here $Z(G)$ is called the partition function of the system.

Note that the Gibbs measure would satisfy $\sum_{\sigma \in \Omega^n} P_G(X = \sigma) = 1$. We use notation $\beta_{ij}(a, b) = \beta_{ji}(b, a)$. For any $\Lambda \subseteq V$, $\sigma_{\Lambda}$ denotes the set $\{\sigma_i, i \in \Lambda\}$. With a little abuse of notation, $\sigma_{\Lambda}$ also denotes the condition or configuration that $i$ is fixed $\sigma_i$, for any $i \in \Lambda$. Let $Z(G, \Phi)$ denote the partition function under the condition $\Phi$, e.g. $Z(G, X_1 = +)$ represent the partition function under the condition the vertex
A self-avoiding walk (SAW) is a sequence of moves (on a graph) which does not visit the same point more than once. The following gives an important tool in proving our results. It is introduced in [13].

**Definition 2.** (Self-Avoiding Tree) The self-avoiding tree \( T_{\text{Saw}}(v) \) (for simplicity denoted by \( T_{\text{Saw}}(v) \)) corresponding to the vertex \( v \) of \( G \) is the tree with root \( v \) and generated through the self-avoiding walks originating at \( v \). A vertex closing a cycle is included as a leaf of the tree and is assigned to be +, if the edge ending the cycle is larger than the edge starting the cycle, and − otherwise.

**Remark:** Given any configuration \( \sigma_\Lambda \) of \( G, \Lambda \subset V \), the self-avoiding tree is constructed the same as the above procedure except that, the vertex which is a copy of the vertex \( i \) in \( \Lambda \) is fixed to the same spin \( \sigma_i \) as \( i \) and the subtree below it is not constructed due to the Markov property, see Figure 1 for example, where vertex 5 is fixed + in \( G \).

To generalize the strong spatial mixing property on trees to general graph, we need to utilize the remarkable property of the self-avoiding tree, one of two main results of [13], and explicitly stated in [6]. For any configuration \( \sigma_\Lambda \) of \( G, \Lambda \subset V \), we also use \( \sigma_\Lambda \) to denote the configuration of \( T_{\text{Saw}(v)} \) obtained by imposing the condition corresponding to \( \sigma_\Lambda \).
Proposition 1. For BMRF on $G = (V, E)$, for any configuration $\sigma_\Lambda$ on $G$, $\Lambda \subset V$ and any vertex $v \in V$, then

$$P_G(X_v = +|\sigma_\Lambda) = P_{T_{saw}(v)}(X_v = +|\sigma_\Lambda).$$

In order to study results to the sparse on average graph, their following properties are useful. The proof is based on induction and can be found in [10].

Proposition 2. Let $j$, $l$ be positive integers. Then one has

$$m(G, v, jl) \leq j \max_{u \in G}\{m(G, u, l) - \delta_u\} + \delta_v$$

and

$$|S(T_{saw(v)}, v, l + 1)| \leq \delta_v(\delta(G, v, l) - 1)^l.$$

Definition 3. (Strong Spatial Mixing) The Gibbs distribution of BMRF exhibits strong spatial mixing if and only if there exist positive numbers $a$, $b$, $c$ independent of $n$, for any vertex $v \in V$, subset $\Lambda \subset V$, any two configurations $\sigma_\Lambda$ and $\eta_\Lambda$ on $\Lambda$, denote perturbation set $\Theta = \{v \in \Lambda : \sigma_v \neq \eta_v\}$ and $t = d(v, \Theta)$, when $t = ka \log n + 1, k = 1, 2, \cdots$

$$|P_G(X_v = +|\sigma_\Lambda) - P_G(X_v = +|\eta_\Lambda)| \leq f(t),$$

where decay function $f(t) = b \exp(-ct)$.

3. Main Results

In the binary Markov random fields, it is well known that if $\beta_{ij}(\sigma_i, \sigma_j) = J_{ij}\sigma_i\sigma_j$ and $h_i = B_i\sigma_i$ for all the edge $(i, j) \in E$ and vertex $i \in V$, and $J_{ij}$ is uniformly positive (or negative) for all $(i, j) \in E$, the BMRF is called ferromagnetic (or antiferromagnetic) Ising model. For simplicity, we use the following notations. Let $J_{ij} = \frac{\beta_{ij}(+, +) + \beta_{ij}(-, -) - \beta_{ij}(-, +) - \beta_{ij}(+, -)}{4}$, and $B_i = \frac{h_i(+) - h_i(-)}{2}$ for all edges and vertices. We call $J_{ij}$ and $B_i$ 'inverse temperature' and 'external field' of BMRF. Let $J = \max_{(i, j) \in E}|J_{ij}|$, $B_{\min} = \min_{i \in V}B_i$. 

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and \( B_{\max} = \max_{i \in V} B_i \). Denote

\[
\alpha_{\max} = \max_{(i,j) \in E} \{\beta_{ij}(-, -) - \beta_{ij}(+,-), \beta_{ij}(-, +) - \beta_{ij}(+ ,+}\}
\]

and

\[
\alpha_{\min} = \min_{(i,j) \in E} \{\beta_{ij}(-, -) - \beta_{ij}(+,-), \beta_{ij}(-, +) - \beta_{ij}(+ ,+}\}.
\]

Let

\[
\gamma_{ij} = \max_{(i,j) \in E} \left\{ \frac{|b_{ij}c_{ij} - a_{ij}d_{ij}|}{a_{ij}c_{ij}}, \frac{|b_{ij}c_{ij} - a_{ij}d_{ij}|}{b_{ij}d_{ij}} \right\},
\]

and \( \gamma = \max_{(i,j) \in E} \{\gamma_{ij}\} \), where

\[
\begin{align*}
a_{ij} &= \exp(\beta_{ij}(+, +)), \\
b_{ij} &= \exp(\beta_{ij}(+, -)), \\
c_{ij} &= \exp(\beta_{ij}(-, +)), \\
d_{ij} &= \exp(\beta_{ij}(-, -)).
\end{align*}
\]

**Theorem 1.** Let \( G = (V, E) \) be a graph with \( n \) vertices. There exist two positive numbers \( a > 0 \) and \( d > 0 \) such that \( \Delta(G, a \log n) \leq d \), and \( (d-1) \tanh J \geq 1 \). Assume

\[
B_{\min} > B(d, \alpha_{\max}, \gamma) \quad \text{or} \quad B_{\max} < -B(d, -\alpha_{\min}, \gamma)
\]

where

\[
B(d, \alpha, \gamma) = \frac{(d-1)\alpha}{2} + \log\left(\frac{\sqrt{\gamma(d-1)} + \sqrt{\gamma(d-1) - 4}}{2}\right).
\]

Then the Gibbs distribution of BMRF exhibits exponential strong spatial mixing.

**Remark:** In Theorem 1, by the definition of \( \gamma_{ij} \), one has

\[
\gamma_{ij}^2 \geq \frac{(|b_{ij}c_{ij} - a_{ij}d_{ij}|)}{a_{ij}c_{ij}} (|c_{ij}d_{ij} - a_{ij}b_{ij}|) = (e^{2J_{ij}} - e^{-2J_{ij}})^2,
\]

hence,

\[
\gamma_{ij} \geq |e^{2J_{ij}} - e^{-2J_{ij}}| = (e^{J_{ij}} + e^{-J_{ij}})^2 \tanh J_{ij} \geq 4 \tanh J_{ij}.
\]

Therefore, \( \gamma(d-1) - 4 \geq 0 \) under the condition \( (d-1) \tanh J \geq 1 \). The case of \( (d-1) \tanh J < 1 \) is discussed separately in [14] with totally different method. The decay function corresponding to the above two conditions are respectively

\[
f(t) = \frac{\delta_i \gamma}{4} \left( \frac{(d-1) \gamma \exp({2B_{\min} - (d-1) \alpha_{\max}})}{1 + \exp(2B_{\min} - (d-1) \alpha_{\max})^2} \right)^{t-1},
\]

and

\[
f(t) = \frac{\delta_i \gamma}{4} \left( \frac{(d-1) \gamma \exp({2B_{\max} - (d-1) \alpha_{\min}})}{1 + \exp(2B_{\max} - (d-1) \alpha_{\min})^2} \right)^{t-1}.
\]
4. Proofs

Theorem 1 is proved with the recursive formula[6]. The technique used is Lipchitz method, which is well known. A ‘path’ version of it is presented first. We use the following notations for simplicity. Let $T = (V, E)$ be a tree rooted at 0 with vertices $V = \{0, 1, 2, \cdots, n\}$, edge set $E$ and BMRF on it. For each edge $(i, j) \in E$, recall the notation in Theorem 1,

$a_{i,j} = e^{\beta_{ij}(+)}$, $b_{i,j} = e^{\beta_{ij}(-)}$, $c_{i,j} = e^{\beta_{ij}(-,+)}$, and $d_{i,j} = e^{\beta_{ij}(-,-)}$.

Let $M_{ij} = c_{ij} - d_{ij}$ and $N_{ij} = a_{i,j} - b_{i,j}$. Define

$f_{ij}(x) = \frac{M_{ij}x + d_{ij}}{N_{ij}x + b_{ij}}$ and $h_{ij}(x) = \frac{a_{ij}d_{ij} - b_{ij}c_{ij}}{(M_{ij}x + d_{ij})(N_{ij}x + b_{ij})}.$

For any $i \in V$, let $T_i$ denote the subtree rooted at $i$ and there is a associated BMRF on $T_i$ restricted by BMRF on $T$. Recall $B_i = \frac{h_i(+)-h_i(-)}{2}$ is the external field. Denote $\lambda_i = e^{-2B_i}$, and let $\Gamma_{ij}$ be the unique self-avoiding path from $i$ to $j$ on $T$.

Lemma 1. For any $(i, j) \in E$, $\max_{x \in [0,1]} |h_{ij}(x)| \leq \gamma_{ij}$.

Proof. Since $M_{ij}x + d_{ij} \geq 0$ and $N_{ij}x + b_{ij} \geq 0$, $\forall x \in [0,1]$, thus all we need is to show

$$\min_{x \in [0,1]} w(x) = \min(a_{ij}c_{ij}, b_{ij}d_{ij}),$$

where $w(x) = (M_{ij}x + d_{ij})(N_{ij}x + b_{ij})$. The case $M_{ij}N_{ij} = 0$ is trivial. Hence without loss of generality, suppose $M_{ij}N_{ij} \neq 0$. Noting $x_l = \frac{-d_{ij}N_{ij}+b_{ij}M_{ij}}{2M_{ij}N_{ij}}$ is an extremum of $w(x)$ on $R$. There are three cases needed to be discussed.

Case 1. $M_{ij}N_{ij} < 0$, then $w(x)$ reaches its minimum at boundary. Then $\min_{x \in [0,1]} w(x) \leq \min(w(0), w(1)) = \min(a_{ij}c_{ij}, b_{ij}d_{ij})$.

Case 2. $M_{ij} > 0, N_{ij} > 0$, then $x_l \leq 0$, $w(x)$ is increasing on $[0, 1]$, then $\min_{x \in [0,1]} w(x) = w(0) = b_{ij}d_{ij}$.

Case 3. $M_{ij} < 0, N_{ij} < 0$, then $x_l \geq 1$, $w(x)$ is decreasing on $[0, 1]$, hence $\min_{x \in [0,1]} w(x) = w(1) = a_{ij}c_{ij}$. \qed

With Lemma 1, we present a ‘path’ version of Lipchitz approach.

Lemma 2. Let $\Lambda \subset V$ , $\zeta_\Lambda$ and $\eta_\Lambda$ be any two configurations on $\Lambda$. Let $\Theta =$
\{i : \zeta_i \neq \eta_i, i \in \Lambda\}, \ t = d(0, \Theta) \text{ and } S(T, 0, t) = \{i : d(0, i) = t, i \in T\}. \text{ Then}

\begin{align*}
|P_T(X_0) = +|\zeta_\Lambda| - P_T(X_0) = +|\eta_\Lambda| & \leq \gamma^n \sum_{k \in S(T, 0, t) \ in \Gamma_{\delta k} \ i \neq k} g_i(z_i)(1 - g_i(z_i)) \\
\end{align*}

where \(z_i\) are constant vectors with elements in \([0, 1]\), and \(g_i(x_i) = (1 + \lambda_i \prod_{i, j \in T} f_{ij}(x_{ij}))^{-1}\), \(x_i = (x_{i1}, x_{i2}, \cdots, x_{i\delta_{i-1}})\).

**Proof.** For any vertex \(i\) in \(T\), recall \(T_i\) denote the subtree rooted at \(i\) with BMRF induced on \(T_i\) by \(T\). Let \(p_i^{\zeta_{\Lambda}} \equiv P_T(X_i = +|\zeta_{\Lambda_i})\) and \(R_i^{\zeta_{\Lambda}} \equiv P_T(X_i = +|\zeta_{\Lambda_i}) / P_T(X_i = -|\zeta_{\Lambda_i})\), where \(\zeta_{\Lambda_i}\) is configuration by restriction of \(\zeta_{\Lambda}\) on \(T_i\). Let \(\Omega_{T_i}\) denote the configuration space in \(T_i\) under the condition \(\zeta_{\Lambda_i, i = 1, 2, \cdots, n}\). \(\Omega_0\) denotes the configuration space of \(T_0\) under the condition \(\zeta_{\Lambda} \cup \{\sigma_0\}\). Let \(0_1, 0_2, \cdots, 0_q\) be the neighbors connected to \(0\), \(q = \delta_0\) (the degree of the root). Now we present the recursive formula,

\begin{align*}
R_0^{\zeta_{\Lambda}} &= \frac{Z(T_0, X_0 = +, \zeta_{\Lambda})}{Z(T_0, X_0 = -, \zeta_{\Lambda})} \\
&= e^{h_0(+)} \sum_{\sigma \in \Omega_0} e^{\sum_{i=1}^q (\beta_{00}(+|\sigma_0)+ \sum_{(k, l) \in T_{0_i}} \beta_{kl}(\sigma_k, \sigma_l) + \sum_{k \in T_0} h_k(\sigma_k))} \\
&= e^{h_0(-)} \sum_{\sigma \in \Omega_0} e^{\sum_{i=1}^q (\beta_{00}(-|\sigma_0)+ \sum_{(k, l) \in T_{0_i}} \beta_{kl}(\sigma_k, \sigma_l) + \sum_{k \in T_0} h_k(\sigma_k))} \\
&= e^{2B_0} \prod_{i=1}^q \sum_{\sigma \in \Omega_{T_{0_i}}} e^{\beta_{00}(+|\sigma_0)+ \sum_{(k, l) \in T_{0_i}} \beta_{kl}(\sigma_k, \sigma_l) + \sum_{k \in T_0} h_k(\sigma_k)} \\
&= e^{2B_0} \prod_{i=1}^q \frac{a_{00} Z(T_{0_i}, X_i = +, \zeta_{\Lambda_i}) + b_{00} Z(T_{0_i}, X_i = -, \zeta_{\Lambda_i})}{c_{00} Z(T_{0_i}, X_i = +, \zeta_{\Lambda_i}) + d_{00} Z(T_{0_i}, X_i = -, \zeta_{\Lambda_i})} \\
&= e^{2B_0} \prod_{i=1}^q \frac{a_{00} R_{0i}^{\zeta_{\Lambda}} + b_{00} i}{c_{00} R_{0i}^{\zeta_{\Lambda}} + d_{00} i}. \\
\end{align*}
Then we have the following equality
\[
p_0^{\zeta_\Lambda} = P_T(X_0 = + | \zeta_\Lambda) = \frac{1}{1 + \frac{P_T(X_0 = - | \zeta_\Lambda)}{P_T(X_0 = + | \zeta_\Lambda)}} = \frac{1}{1 + 1/R_0^{\zeta_\Lambda}}
\]
\[
= \frac{1 + \lambda_0 \prod_{(0,0_j) \in T} \frac{c_{00_j} R_0^{\zeta_\Lambda} + d_{00_j}}{a_{00_j} R_0^{\zeta_\Lambda} + b_{00_j}}}{1 + \lambda_0 \prod_{(0,0_j) \in T} \frac{M_{00_j} p_{0_j}^{\zeta_\Lambda} + d_{00_j}}{N_{00_j} p_{0_j}^{\zeta_\Lambda} + b_{00_j}}}
\]
\[
= g_0(x_0),
\]
where \(x_0 = (p_{0_1}^{\zeta_\Lambda}, p_{0_2}^{\zeta_\Lambda}, \ldots, p_{0_{b_0}}^{\zeta_\Lambda})\). First, note that for any \(x = (x_1, x_2, \ldots, x_q)\) and \(y = (y_1, y_2, \ldots, y_q)\), first order Taylor expansion at \(y\) gives that there exists a \(\theta \in [0, 1]\) such that
\[
g_0(x) - g_0(y) = \nabla g_0(y + \theta(x - y))(x - y)^T,
\]
where \((x - y)^T\) denotes the transportation of the vector \((x - y)\). Careful calculations give the following
\[
\frac{\partial g_0(x)}{\partial x_i} = -\frac{\lambda_0 \prod_{j=1}^q f_{00_j}(x_j) \left( \frac{d \log(f_{00_j}(x_i))}{dx_i} \right)}{(1 + \lambda_0 \prod_{j=1}^q f_{00_j}(x_j))^2} = -g_0(x)(1 - g_0(x)) \left( \frac{M_{00_i}}{M_{00_i} x_i + d_{00_i}} - \frac{N_{00_i}}{N_{00_i} x_i + b_{00_i}} \right)
\]
\[
= g_0(x)(1 - g_0(x)) \left( \frac{a_{00_i} d_{00_i} - b_{00_i} c_{00_i}}{M_{00_i} x_i + d_{00_i})(N_{00_i} x_i + b_{00_i})} \right)
\]
\[
= g_0(x)(1 - g_0(x)) h_{00_i}(x_i).
\]
Hence, let \(x_0 = (p_{0_1}^{\zeta_\Lambda}, p_{0_2}^{\zeta_\Lambda}, \ldots, p_{0_{b_0}}^{\zeta_\Lambda})\) and \(y_0 = (p_{0_1}^{n_\Lambda}, p_{0_2}^{n_\Lambda}, \ldots, p_{0_{b_0}}^{n_\Lambda})\), then there exits \(\theta_0 \in [0, 1]\) such that
\[
|p_0^{\zeta_\Lambda} - p_0^{n_\Lambda}| \leq \sum_{j=1}^q |g_0(z_0)(1 - g_0(z_0)) h_{00_j}(x_j)||p_{0_j}^{\zeta_\Lambda} - p_{0_j}^{n_\Lambda}|
\]
\[
\leq \sum_{j=1}^q g_0(z_0)(1 - g_0(z_0)) |\gamma_{00_j}| |p_{0_j}^{\zeta_\Lambda} - p_{0_j}^{n_\Lambda}| \leq \gamma \sum_{j=1}^q g_0(z_0)(1 - g_0(z_0))|p_{0_j}^{\zeta_\Lambda} - p_{0_j}^{n_\Lambda}|,
\]
where \( z_0 = x_0 + \theta_0(x_0 - y_0) \) and the second inequality follows by Lemma 1. Now repeat the procedure on the subtree \( T_{0j} \) for \( |p_{0j}^\Lambda - p_{0j}^\eta|, j = 1, 2, \cdots, q \) and so on. We can see that the summation is over all the self-avoiding paths starting at the root 0. For each path \( \Gamma \), if the end point of \( \Gamma \) is a leave \( j \) with \( d(0, j) \leq t - 1 \) or there is a vertex \( i \) on \( \Gamma \) with \( d(0, i) \leq t - 1 \) being fixed, the contribution of the path to the summation is zero since \( p_i^\Lambda - p_i^\eta = p_j^\Lambda - p_j^\eta = 0 \). Hence the remaining path with length \( t \) is in the set \( \{ \Gamma_{0k} : k \in S(T, 0, t) \} \). This completes the proof of lemma 2. \( \square \)

In order to complete the proof of Theorem 1, we need the following lemma.

**Lemma 3.** Let \( \lambda_i \geq 0, i = 1, 2, \cdots, n \). Then

\[
\prod_{i=1}^{n} (1 + \lambda_i) \geq (1 + \sqrt[n]{\prod_{i=1}^{n} \lambda_i})^n.
\]

**Proof.** Consider

\[
\prod_{i=1}^{n} (1 + \lambda_i) = 1 + \sum_{k=1}^{n} \left( \sum_{i_1 < i_2 < \cdots < i_k} \prod_{j=1}^{k} \lambda_{i_j} \right) \\
\geq 1 + \sum_{k=1}^{n} (C_n^k \left( \prod_{i=1}^{n} \lambda_i \right)^{\frac{k}{n!}}) \\
= 1 + \sum_{k=1}^{n} (C_n^k \left( \prod_{i=1}^{n} \lambda_i \right)^{\frac{k}{n}}) \\
= (1 + \sqrt[n]{\prod_{i=1}^{n} \lambda_i})^n,
\]

where \( C_n^k = \frac{n!}{k!(n-k)!} \). The first inequality uses the arithmetic-geometric average inequality.

With Lemma 2 and 3, it is sufficient to prove Theorem 1.

**Proof of Theorem 1.** Following the notation of Lemma 2, let \( s = |S(T, 0, t)| \), we have
\[ |p_0^{\zeta \Lambda} - p_0^{\eta \Lambda}| \leq \gamma^t \sum_{k \in S(T, 0, t)} \prod_{i \in \Gamma_{0k} \atop i \neq k} g_i(z_i)(1 - g_i(z_i)) \]

\[ \leq s \gamma^t \max_{k \in S(T, 0, t)} \prod_{i \in \Gamma_{0k} \atop i \neq k} g_i(z_i)(1 - g_i(z_i)) \]

\[ \leq s \frac{\gamma^t}{4} \max_{(0, 0) \in T} \prod_{k \in S(T, 0, t) \atop i \in \Gamma_{0k}} g_i(z_i)(1 - g_i(z_i)). \]

For each \( \Gamma_{0, k} \), where \((0, 0) \in T, k \in S(T, 0, t), \)

\[ \prod_{i \in \Gamma_{0j} \atop i \neq k} g_i(z_i)(1 - g_i(z_i)) = \prod_{i \in \Gamma_{0j} \atop i \neq k} \lambda_i \prod_{(i, i) \in T_i} f_{ii}(z_{ii}) \]

\[ \leq (\frac{r_{jk}}{(1 + r_{jk})^2})^{t-1}, \]

where \( r_{jk} = (\prod_{i \in \Gamma_{0j} \atop i \neq k} \lambda_i \prod_{(i, i) \in T_i} f_{ii}(z_{ii}))^{1/(t-1)} \) and the inequality above follows from Lemma 3. A simple calculation gives that \( e^{\alpha_{\text{min}}} \leq f_{ij}(x) \leq e^{\alpha_{\text{max}}} \), for any \((i, j) \in T, \)

Hence,

\[ e^{\alpha_{\text{min}}(\delta(T, 0, t-1) - 1)} \leq \left( \prod_{i \in \Gamma_{0j} \atop (i, i) \in T_i} f_{ii}(z_{ii}) \right)^{1/(t-1)} \leq e^{\alpha_{\text{max}}(\delta(T, 0, t-1) - 1)}. \]

Now we prove the exponential strong spatial mixing under assumption of Theorem 1. Suppose \( \Gamma \) is a self-avoiding path of \( G \). Noting that each self-avoiding path on \( T \) by removing the ending point is also a self-avoiding path on \( G \). From proposition 1, we know 0 is a vertex of \( G \) and let \( p_0^{\zeta \Lambda} = P_G(X_0 = +|\zeta \Lambda) \). By proposition 2, we know \( \delta(T, 0, t-1) \leq \Delta(G, t-1) \leq d \) when \( t = ka \log n + 1, k = 1, 2, \cdots \). If \( B_{\text{min}} > B(d, \alpha_{\text{max}}, \gamma) \), then

\[ \gamma(d - 1) \exp(2B_{\text{min}} - \alpha_{\text{max}}(d - 1)) \]

\[ \frac{(1 + \exp(2B_{\text{min}} - \alpha_{\text{max}}(d - 1)))^2}{(1 + \exp(2B_{\text{min}} - \alpha_{\text{max}}(d - 1)))^2} < 1. \]

By proposition 2, we know \( s \leq \delta_0(d - 1)^{t-1} \). Noting \( \prod_{i \in \Gamma_{0j} \atop i \neq k} \lambda_i^{1/(t-1)} \leq e^{-2B_{\text{min}}} \),
now we can see
\[
|p_0^{\xi} - p_0^{\eta}| = |p_0^{\xi} - p_0^{\eta}| \leq \delta_0 \gamma \frac{\gamma^t (r_{jk})^2}{4 (1 + r_{jk})^2} t^{-1},
\]
where the first equality follows from the proposition 1. The similar case holds for
\[
B_{\max} < -B(d, -\alpha_{\min}, \gamma).
\]
This completes the proof. 

From the proof above, we can see if the graph is bounded degree with maximum
degree is $d$, the condition for ‘external field’ can be relaxed to $B_i > B(d, \alpha_{\max}, \gamma)$ or
$B_i < -B(d, -\alpha_{\min}, \gamma)$ for any $i \in V$, which does not require that ‘external field’ is
uniformly large or uniformly small as in Theorem 1.

**Corollary 1.** Let $G = (V, E)$ be a bounded graph and with maximum degree $d$ and
BMRF on it, and $\tanh J(d-1) \geq 1$. If $B_i > B(d, \alpha_{\max}, \gamma)$ or $B_i < -B(d, -\alpha_{\min}, \gamma)$
for any $i \in V$. Then the Gibbs distribution exhibits strong spatial mixing.

**Proof.** Following the notations above, by the formula (1) in Lemma 2, we have
\[
|p_0^{\xi} - p_0^{\eta}| \leq \gamma \sum_{j=1}^{q} g_0(z_0) (1 - g_0(z_0)) |p_0^{\xi} - p_0^{\eta}|.
\]
Without loss of generality, suppose the degree of 0 is $d - 1$. Then
\[
|p_0^{\xi} - p_0^{\eta}| \leq \gamma (d - 1) \max_{i \in T} (g_0(z_i) (1 - g_0(z_i))) |p_0^{\xi} - p_0^{\eta}|.
\]
If $B_i > B(d, \alpha_{\max}, \gamma)$ or $B_i < -B(d, -\alpha_{\min}, \gamma)$, we know $\gamma(d-1) g_i(z_i) (1 - g_i(z_i)) < 1$
for any $i \in T$. Hence by induction on the height $t$, we get
\[
|p_0^{\xi} - p_0^{\eta}| \leq (\gamma(d-1) \max_{i \in T} (g_0(z_i) (1 - g_0(z_i))))^t.
\]
Since the degree of 0 is at most $d$. Then
\[
|p_0^{\xi} - p_0^{\eta}| \leq \gamma d g_0(z_0) (1 - g_0(z_0)) (\gamma(d-1) \max_{i \in T} (g_0(z_i) (1 - g_0(z_i))))^{t-1}
\leq \gamma d \frac{\gamma^t (r_{jk})^2}{4 (1 + r_{jk})^2} t^{-1}.
\]
Applying proposition 1 completes the proof. 

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Figure 2: The red line and the blue line denote the “external field” curves for uniqueness of Gibbs measures and for strong spatial mixing on three regular trees $T$ respectively, where $\beta_{ij}(\sigma_i, \sigma_j) = J\sigma_i\sigma_j$ and $h_i(\sigma_i) = B\sigma_i$, for any $i \in T$ and $(i, j) \in T$.

Remark: We emphasize that the tighter bound of $f_{ij}(x)$ is the key to improve the result since better bound of $f_{ij}(x)$ will give better bound for $g_i(x)$. We do not optimize the parameter here. We are not aware that Lipchitz method can make $B(d, \alpha_{\text{max}}, \gamma)$ or $-B(d, -\alpha_{\text{min}}, \gamma)$ optimally approximate the critical point of ‘external field’ for uniqueness of Gibbs measures if they does exit. Note that the critical points of ‘external field’ for ferromagnetic and antiferromagnetic Ising model are different on an infinite $d$ regular tree with degree $d$ for each vertex [4]). We do not expect the critical external field for Ising model on $d$ regular tree for uniqueness of Gibbs measures is the optimal external field for strong spatial mixing. The intuition for this is that the uniqueness of Gibbs measures on the tree is equivalent to weak spatial mixing (see [3][13] for definitions) in some sense[9]. If some configurations are close to the root(note some configurations may be at the hight 2 or 3(see Figure 1) when self-avoiding tree is constructed), the perturbation of the boundary condition changes the Gibbs measures at the root radically. More precisely, strong spatial mixing can be deducted to the weak spatial mixing by removing the support of
unmodified boundary configuration and changing the external field of some vertices (see Lemma 2 in [14]). Hence, the critical external field condition for weak spatial mixing does not hold for strong spatial mixing. Figure 2 illustrates the curve of external field under our condition for strong spatial mixing and the critical external field for uniqueness of Gibbs measures on infinite $d$ regular tree, where $d = 3$.

5. Conclusion and Further Work

The Gibbs distribution on a graph $G = (V, E)$ with ‘maximum average degree’ $d$ is considered in this paper. The (exponential) strong spatial mixing is proved for such systems, when the ‘external field’ $B_1$ is uniformly larger than $B(d, \alpha_{\max}, \gamma)$ or smaller than $-B(d, -\alpha_{\min}, \gamma)$. Here $B(d, \alpha, \gamma)$ is a function with parameter $d$, $\alpha$, $\gamma$. It is not difficult to apply our results to Erdő-Rényi random graph $G(n, d/n)$, where each edge is chosen independently with probability $d/n$ [10].

For future work, some improvements to the condition on ‘external field’ should be possible. We have emphasized the essential key points in the remark of last section. However, we believe that it requires other method other than Lipchitz method. The fixed point method in [7] may be a possible approach.

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