Abstract

In an earlier paper (hep-th/0101127), we developed heat kernel techniques in $\mathcal{N} = 2$ harmonic superspace for the calculation of the low-energy effective action of $\mathcal{N} = 4$ SYM theory, which can be considered as the most symmetric $\mathcal{N} = 2$ SYM theory. Here, the results are extended to generic $\mathcal{N} = 2$ SYM theories. This involves a prescription for computing the variation of the hypermultiplet effective action. Integrability of this variation allows the hypermultiplet effective action to be deduced. This prescription permits a very simple superfield derivation of the perturbative holomorphic prepotential. Explicit calculations of the prepotential and the leading non-holomorphic correction are presented.
1. In conventional quantum field theory, powerful techniques have been developed for the computation of low-energy effective actions. These often involve a combination of (i) the **background field method** to allow the maintenance of manifest gauge invariance throughout the calculation; and (ii) **heat kernel techniques**, which effectively allow in a single step the summation of an infinite set of Feynman diagrams with increasing number of insertions of the background field.

Many of the remarkable properties of supersymmetric Yang-Mills theories, and indeed of superstring theories, are related to their supersymmetry. In the computation of low-energy effective actions for these theories, the challenge is therefore to use the background field method and heat kernel techniques in a manner which also preserves manifest supersymmetry. This is most efficiently achieved by formulating the theory in an appropriate superspace. In the case of \( \mathcal{N} = 1 \) supersymmetric Yang-Mills theories, their gauge structure can be incorporated into the superspace formulation in a remarkably simple and elegant geometric manner (see [1] for a review). The background field method and heat kernel techniques for these theories are well-developed (see [2, 3] for reviews).

For \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories, the harmonic superspace approach [4, 5] provides a universal setting for the description of their dynamics. The background field method in harmonic superspace was elaborated in refs. [6, 7]. Following this, two important applications of the method were given: (i) the first rigorous proof of the \( \mathcal{N} = 2 \) non-renormalization theorem [8]; (ii) the harmonic-superspace computation of the leading non-holomorphic quantum corrections in \( \mathcal{N} = 4 \) super Yang-Mills theory [7, 9]. However, until recently, heat kernel techniques in \( \mathcal{N} = 2 \) superspace remained almost totally undeveloped, with the result that very little has been achieved in the way of explicit computations of low-energy effective actions for \( \mathcal{N} = 2 \) supersymmetric Yang-Mills theories. A significant development in this direction was achieved in [10]. For the one-loop effective action of the \( \mathcal{N} = 4 \) super Yang-Mills theory (which can be viewed the most symmetric \( \mathcal{N} = 2 \) model), we obtained an \( \mathcal{N} = 2 \) superfield representation which is free of coinciding harmonic singularities and which permits a straightforward evaluation of low-energy quantum corrections in the framework of an \( \mathcal{N} = 2 \) superfield heat kernel technique.

In the present letter, we extend the one-loop results of [10] to the case of generic \( \mathcal{N} = 2 \) super Yang-Mills theories. Essentially, such an extension is equivalent to developing heat kernel techniques for computing the effective action of a hypermultiplet coupled to a background \( \mathcal{N} = 2 \) vector multiplet. Before proceeding, we justify this statement and sketch the most important aspects of the harmonic superspace formulation in this context.

The \( \mathcal{N} = 2 \) harmonic superspace \( \mathbb{R}^{4|8} \times S^2 \) extends conventional superspace, with
coordinates \( z^M = (x^m, \theta^\alpha_i, \bar{\theta}^\dot{\alpha}_i) \), by the two-sphere \( S^2 = SU(2)/U(1) \) parametrized by harmonics, i.e., group elements

\[
(u_i^-, u_i^+) \in SU(2), \quad u_i^+ = \varepsilon_{ij} u_{-j}, \quad \bar{u}_{i+} = u_{i-}, \quad u_{i+} u_{i-} = 1.
\]

The main conceptual advantage of harmonic superspace is that the \( \mathcal{N} = 2 \) vector multiplet and hypermultiplets can be described by \emph{unconstrained} superfields over the analytic subspace of \( \mathbb{R}^{4|8} \times S^2 \) parametrized by the variables \( \zeta^M \equiv (x^m_A, \theta^{+\alpha}_i, \bar{\theta}^{i\dot{\alpha}}, u_{i+}^j, u_{i-}^j) \), where the so-called analytic basis is defined by

\[
x^m_A = x^m - 2i \theta^i \sigma^m \bar{\theta} \theta^j u_{i+}^j, \quad \theta^{+\alpha}_i = u_{i+}^{\alpha} \theta^i, \quad \bar{\theta}^{i\dot{\alpha}} = u_{i-}^{\dot{\alpha}} \bar{\theta}_{i}.
\]

and represents a generalization of the chiral superspace basis in \( \mathcal{N} = 1 \) supersymmetry. The \( \mathcal{N} = 2 \) vector multiplet is described by a real analytic superfield \( V^{++} = V^{++}_I(\zeta) T_I \) taking its values in the Lie algebra of the gauge group. A hypermultiplet transforming in a representation \( R \) of the gauge group is described by an analytic superfield \( q^+ (\zeta) \) and its conjugate \( \bar{q}^+ (\zeta) \). The classical action for a generic \( \mathcal{N} = 2 \) super Yang-Mills theory is

\[
S_{\mathcal{N}=2YM} = \frac{1}{2g^2} \int d^4x d^4\theta \text{tr} W^2 - \int d\zeta^{(-4)} \bar{q}^+ D^{++} q^+,
\]

where \( W \) is the \( \mathcal{N} = 2 \) covariantly chiral superfield strength \( [\mathbf{1}] \), \( d\zeta^{(-4)} \) denotes the analytic subspace integration measure, and \( D^{++} = D^{++} + iV^{++} \) is the analyticity-preserving covariant derivative. The first term in (3), which is the action of the \( \mathcal{N} = 2 \) pure super Yang-Mills theory \([11]\), can be expressed as a gauge-invariant functional of \( V^{++} \) \([12]\). The second term in (3) is the action of a massless hypermultiplet. The massive case simply corresponds to allowing a constant expectation value for \( V^{++} \) along an Abelian subalgebra of the gauge algebra (see, e.g., \([13, 14, 15]\)).

The hypermultiplet effective action \( \Gamma^{(R)}_H \) is defined by

\[
\exp \left( i \Gamma^{(R)}_H [V^{++}] \right) = \int [dq^+] [dq^+] \exp \left( - i \int d\zeta^{(-4)} \bar{q}^+ D^{++} q^+ \right).
\]

It can be shown \([7]\) that the one-loop effective action of the theory (3) is

\[
\Gamma_{\mathcal{N}=2YM} = \Gamma_{\mathcal{N}=4YM} + \Gamma^{(R)}_H - \Gamma^{(ad)}_H,
\]

with \( \Gamma_{\mathcal{N}=4YM} \) the one-loop effective action of \( \mathcal{N} = 4 \) super Yang-Mills theory. The low-energy structure of \( \Gamma_{\mathcal{N}=4YM} \) has been studied in \([\mathbf{11}]\). Therefore, it remains to analyse the hypermultiplet effective action. It is worth pointing out that here we are only interested in
the functional dependence of the effective action on the $\mathcal{N} = 2$ vector multiplet, and ignore all quantum corrections with hypermultiplet external legs. The latter were discussed, to some extent, in [13, 14].

The outline of the paper is as follows. In section 2, an expression for the variation of the hypermultiplet effective action is derived, and this is used in section 3 to calculate the perturbative holomorphic prepotential. The technique for computing non-holomorphic higher-derivative quantum corrections is described in section 4. This is followed by a short conclusion.

2. As determined by (4), the formal definition of the hypermultiplet effective action reads

$$\Gamma_H = i \text{Tr} \ln \mathcal{D}^{++} = - i \text{Tr} \ln G^{(1,1)} ,$$

with $G^{(1,1)}(\zeta_1, \zeta_2)$ the hypermultiplet Green function:

$$\mathcal{D}^{++} G^{(1,1)}(\zeta_1, \zeta_2) = \delta^{(3,1)}(\zeta_1, \zeta_2) ,$$

$$G^{(1,1)}(\zeta_1, \zeta_2) = - \frac{1}{\Omega_1} (\mathcal{D}_1^{++})^4 (\mathcal{D}_2^{++})^4 \delta^{12}(z_1 - z_2) \frac{1}{(u_1^+ u_2^+)^2} ,$$

where $\square$ is the analytic d’Alembertian

$$\square = \mathcal{D}^m \mathcal{D}_m + \frac{i}{2} (\mathcal{D}^{+\alpha} \mathcal{W}) \mathcal{D}_m^{-\alpha} + \frac{i}{2} (\bar{\mathcal{D}}_m^{+\bar{\alpha}} \bar{\mathcal{W}}) \bar{\mathcal{D}}^{-\bar{\alpha}}_m - \frac{i}{4} (\mathcal{D}^{+\alpha} \mathcal{D}_m^{+\bar{\alpha}} \mathcal{W}) \mathcal{D}^{-\beta} + \frac{i}{8} [\mathcal{D}^{+\alpha_1} \mathcal{D}_m^{-\alpha_2}] \mathcal{W} + \frac{1}{2} \{ \bar{\mathcal{W}}, \mathcal{W} \} .$$

The above definition is purely formal, since the operator $\mathcal{D}^{++}$ in (8) maps the space of analytic superfields $q^+ \text{ with } U(1) \text{ charge } +1 \text{ onto a space of analytic superfields possessing } U(1) \text{ charge } +3$, and therefore its determinant $\text{Det} \mathcal{D}^{++}$ is ill-defined\(^1\). However, the expression for an arbitrary variation of the effective action

$$\delta \Gamma_H = - \text{Tr} \left\{ \delta \mathcal{V}^{++} G^{(1,1)} \right\}$$

is well-defined. On formal grounds, this variation is integrable, since

$$\delta_2 \delta_1 \Gamma_H = i \text{Tr} \left\{ \delta \mathcal{V}_1^{++} G^{(1,1)} \delta \mathcal{V}_2^{++} G^{(1,1)} \right\} = \delta_1 \delta_2 \Gamma_H .$$

Our goal in the present paper is first to develop a covariant heat kernel technique for computing $\delta \Gamma_H$, and then to integrate this variation to yield $\Gamma_H$.

\(^1\)This is similar to the well-known situation for chiral fermions.
Using Schwinger’s proper-time representation,
\[
\frac{1}{\Box} = i \int_0^\infty ds \, e^{-i s \Box} ,
\]  
we introduce a regularized variation of the effective action
\[
\delta \Gamma_{H, \varepsilon} = \mu^{2 \varepsilon} \text{tr} \int_0^\infty d(is) \, (is)^\varepsilon \int d\zeta \delta^{(-4)} \delta V^{++} 
\times e^{-i s \Box_1} (D^+_1)^4 (D^+_2)^4 \delta^{12}(z_1 - z_2) \bigg|_{1=2} ,
\]
with \(\varepsilon\) the ultraviolet regularization parameter, set to zero at the end of calculations, and \(\mu\) the normalization point. The expression for \(\delta \Gamma_{H, \varepsilon}\) can be brought to a more useful form by applying the identity
\[
(D^+_1)^4 (D^+_2)^4 \delta^{12}(z_1 - z_2)
\]
\[
= (D^+_1)^4 \left\{ (D^-_1)^4 (u_1^+ u_2^+) - \frac{i}{2} \Delta^- (u_1^- u_2^+) - \Box_1 (u_1^- u_2^+)^2 \right\} \delta^{12}(z_1 - z_2) ,
\]
where
\[
\Delta^- = D^\alpha \bar{D}^\alpha + \frac{1}{2} \mathcal{W}(D^-)^2 + \frac{1}{2} \mathcal{W} \mathcal{W} - (D^- \mathcal{W}) D^- + (\mathcal{W} D^- \mathcal{W}) + \frac{1}{2} (D^- D^- \mathcal{W}) .
\]

The two-point function in the first line of (13) contains a harmonic distribution which is singular at coincident points, \(u_1 = u_2\). On the right hand side of (13), it is only the third term which contains a potential coinciding harmonic singularity. However, this singular term does not contribute to \(\delta \Gamma_{H, \varepsilon}\), and therefore \(\delta \Gamma_{H, \varepsilon}\) is free of harmonic singularities. To see this, note that the expression
\[
\Upsilon_\varepsilon = \mu^{2 \varepsilon} \text{tr} \int_0^\infty d(is) \, (is)^\varepsilon e^{-i s \Box_1} (D^+_1)^4 \delta^{12}(z_1 - z_2) \bigg|_{z_1 = z_2} ,
\]
contains no divergences in \(\varepsilon\). This fact and the identity \((D^+)^4 \Box = \Box (D^+)^4\) then imply
\[
\lim_{\varepsilon \to 0} \Upsilon_\varepsilon = \text{tr} \int_0^\infty d(is) \, e^{-i s \Box_1} \delta^{12}(z_1 - z_2) \bigg|_{z_1 = z_2} = 0 ,
\]
\footnote{See ref. [5] for a detailed discussion of harmonic delta-functions and harmonic distributions of the general form \((u_1^+ u_2^+)^{-n}\), where \(n > 0\).}
since at least eight spinor derivatives are required to produce a non-vanishing result when acting on the Grassmann delta-function $\delta^8(\theta_1 - \theta_2)$ before setting $\theta_1 = \theta_2$. As a result, we arrive at the following expression for $\delta \Gamma_{H,\varepsilon}$:

$$
\delta \Gamma_{H,\varepsilon} \equiv \text{tr} \int d\zeta \left( \frac{-4}{\sqrt{2}} \delta V^{++} \right) e^{-i s \bar{\Delta}_{1}^{-} (\bar{u}_1 u_2)} \frac{1}{2} \Delta_{1}^{-} (u_1^{-} u_2^{-}) \left| \delta_{12} (z_1 - z_2) \right|_{1=2} .
$$

(17)

This is our working formula for the computation of $\delta \Gamma_{H,\varepsilon}$.

The effective current $J^{++}_\varepsilon$ in (17) should be analytic,

$$
\mathcal{D}^+_\alpha J^{++}_\varepsilon = \bar{\mathcal{D}}^+_\alpha J^{++}_\varepsilon = 0.
$$

(18)

The requirement of gauge invariance of the effective action is equivalent to the conservation equation

$$
\mathcal{D}^{++} J^{++}_\varepsilon = 0.
$$

(19)

Relation (17) simplifies considerably when the background gauge superfield satisfies the classical equation of motion

$$
\mathcal{D}^{(i)j} \mathcal{W} = \bar{\mathcal{D}}^{(i)\bar{j}} \bar{\mathcal{W}} = 0.
$$

(20)

Then $\delta \Gamma_{H}$ turns out to be free of ultraviolet divergences, and eq. (17) takes the form

$$
\delta \Gamma_{H} = -\frac{1}{2} \text{tr} \int ds \int d\zeta \left( \frac{-4}{\sqrt{2}} \delta V^{++} e^{-i s \bar{\Delta}_{1}^{-} (\bar{u}_1 u_2)} \right) \left| \delta_{12} (z_1 - z_2) \right|_{z_1 = z_2} .
$$

(21)

This simplified variation is useful for the computation of non-holomorphic corrections to the effective action, which will be done in section 4.

3. To this point, our results are applicable for an arbitrary non-Abelian gauge group. In the remainder of the paper, we specialize to the case of a $U(1)$ gauge group in order to illustrate the techniques for computing the effective action. The Abelian gauge field and chiral field strength will be denoted by $V^{++}$ and $W$ respectively.

In the Abelian case, quantum corrections to the effective action can be computed in the framework of a derivative expansion. On general grounds, the part of $\Gamma_{H}$ which does not contain space-time derivatives of $W$ and $\bar{W}$ should have the following structure (17):

$$
\Gamma_{H} = \int d^4 x d^4 \theta \mathcal{F}(W) + \int d^4 x d^4 \bar{\theta} \bar{\mathcal{F}}(\bar{W}) + \int d^4 x d^4 \theta d^4 \bar{\theta} \left\{ c \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda} + \ln \frac{W}{\Lambda} \Sigma(\bar{\Psi}) + \ln \frac{\bar{W}}{\Lambda} \Sigma(\Psi) + \Omega(\Psi, \bar{\Psi}) \right\} ,
$$

(22)
where
\[
\tilde{\Psi} = \frac{1}{W^2} D^4 \ln \frac{W}{\Lambda}, \quad \Psi = \frac{1}{\bar{W}^2} \bar{D}^4 \ln \frac{\bar{W}}{\Lambda},
\]
with $\Lambda$ a formal scale which in fact drops out from all structures listed. Here $F$ is a holomorphic function (often called the perturbative prepotential) which is the starting point of the Seiberg-Witten theory \[18\]; it encodes the quantum corrections with at most two derivatives in components (super $F^2$ terms). The first term in the second line of (22) comprises the leading higher-derivative quantum corrections (super $F^4$ terms) and is known to be one-loop exact \[19\]. Finally, the holomorphic ($\Sigma$) and real analytic ($\Omega$) functions determine higher-derivative quantum corrections (super $F^6$ and higher order terms). We will demonstrate how these quantum corrections can be computed in the framework of equation (17).

It is worth making a few comments on the explicit structure of the low-energy action (22). The classical action (3) is invariant under the $N = 2$ superconformal group (see \[20\] for a discussion of superconformal transformations in harmonic superspace). The superconformal symmetry is known to be anomalous at the quantum level, and the anomaly is completely determined by the holomorphic prepotential. The non-holomorphic part of the effective action should be superconformally invariant. This dictates the structure in the second line of (22); see \[17\] for more details.

We first turn to the computation of holomorphic quantum corrections to $\Gamma_H$ of the general form $S_{\text{eff}} = \text{Re} \int d^4xd^4\theta F(W)$. Under an arbitrary variation $V^{++} \to V^{++} + \delta V^{++}$ of the analytic gauge field,

\[
\delta S_{\text{eff}} = \frac{1}{4} \int d\zeta^{-4} \delta V^{++} \left\{ (D^{+})^2 F'(W) + (\bar{D}^{+})^2 \bar{F}'(\bar{W}) \right\},
\]

see, e.g., \[14\] for a derivation. Eq. (23) indicates that to compute $F(W)$, one has to retain all terms in $\delta \Gamma_{H,\varepsilon}$ which involve exactly two spinor derivatives of the field strengths.

Both terms in (17) contribute nontrivially to $\delta S_{\text{eff}}$. The first term contains exactly eight spinor derivatives to annihilate the Grassmann delta-function, via the identity

\[
(D^{+})^4 (D^{-})^4 \delta^8(\theta - \theta') \bigg|_{\theta = \theta'} = 1.
\]

Due to harmonic identities $(u_1^+ u_2^+)_{1=2} = 0$ and $(u_1^- u_2^+)_{1=2} = -1$, this term produces a non-vanishing contribution only if we pick up a factor of $(D^+ D^+ W) D^{- -}$ in the decomposition of $\exp(-i s \bar{\Box})$ and use it to act on $(u_1^+ u_2^+)$ (the harmonic derivative $D^{- -}$ is defined by $D^{- -} u^+ = u^-$, $D^{- -} u^- = 0$). Since we are working to second order in spinor derivatives, it is sufficient for our purposes to approximate $\exp(-i s \bar{\Box}) \approx$
As a result, the first term in (17) contributes as follows (with the analytic subspace integral omitted):

\[
\frac{\mu^2}{(8\pi)^2} \delta V^{++} (D^+ D^+ W) \int_0^\infty d\tau \tau^{\varepsilon-1} e^{-\tau W \bar{W}}
\]

\[
= \frac{1}{(8\pi)^2} \delta V^{++} \left( \frac{1}{\varepsilon} (D^+ D^+ W) - \ln \frac{W}{\mu} (D^+ D^+ W) - \ln \frac{\bar{W}}{\mu} (\bar{D}^+ \bar{D}^+ \bar{W}) \right) + O(\varepsilon),
\]

where we have applied Schwinger’s rotation \( s \rightarrow -i \tau \) and used the Bianchi identity

\[
D^+ D^+ W = \bar{D}^+ \bar{D}^+ \bar{W}.
\]

It is worth pointing out that the \( 1/\varepsilon \) term in (25) completely determines the divergent part of \( \Gamma_H \); there are no other contributions.

In order for the second term in (17) to contribute to \( \delta S_{\text{eff}} \), we have to accumulate a product of two spinor derivatives from the expansion of \( \exp(-i s \slashed{\nabla}) \) to have enough spinor derivatives to annihilate the Grassmann delta-function. The result reads

\[
- \frac{1}{(8\pi)^2} \delta V^{++} (\bar{W} D^+ D^+ W + W \bar{D}^+ \bar{D}^+ \bar{W}) \int_0^\infty d\tau e^{-\tau W \bar{W}} + O(\varepsilon)
\]

\[
= - \frac{1}{(8\pi)^2} \delta V^{++} \left( \frac{D^+ W D^+ W}{W} + \frac{\bar{D}^+ \bar{W} \bar{D}^+ \bar{W}}{W} \right) + O(\varepsilon).
\]

From eqs. (23), (25) and (27) we read off the divergent part of the effective action

\[
\Gamma_{H, \text{div}} = \frac{1}{32\pi^2 \varepsilon} \int d^4 x d^4 \theta W^2
\]

and the perturbative holomorphic prepotential

\[
\mathcal{F}(W) = - \frac{1}{32\pi^2} W^2 \ln \frac{W}{\mu}.
\]
4. To compute the higher-derivative quantum corrections, which appear in the second line of (22), it is sufficient to evaluate $\delta \Gamma_H$ for an on-shell background vector multiplet, defined by eq. (20). In this case, $\delta \Gamma_H$ takes the simplified form (21), and the problem reduces to computing the kernel (30) (strictly speaking the coincidence limit of a kernel)

$$K^{--}(z; s) = \lim_{z \to z'} e^{-i s \tilde{\Box}} \Delta^{--}(D^+)^4 \delta^{12}(z - z').$$

(30)

Since we ignore the corrections to $\Gamma_H$ containing space-time derivatives of $W$ and $\bar{W}$, we can further restrict the background vector multiplet by the conditions

$$W|_{\theta = 0} = \text{const}, \quad D^\alpha_i W|_{\theta = 0} = \text{const}, \quad D^\alpha_i \bar{D}_\beta i W|_{\theta = 0} \equiv 8 F_{\alpha \beta} = \text{const}.$$  

(31)

Then, the kernel (30) can be computed exactly by adapting techniques developed in [23]. Below we outline the main steps in the calculation of $K^{--}(z; s)$.

Replacing the delta-function by its Fourier representation

$$(D^+)^4 \delta^{12}(z - z') = \int d\eta^{(+4)} e^{i k_a (x - x')^a} e^{i \epsilon^-(\theta - \theta')} e^{i \bar{\epsilon}^-(\bar{\theta} - \bar{\theta}')^-}$$

with $\int d\eta^{(+4)} = 16 \int \frac{d^4 k}{(2\pi)^4} \int d^2 \epsilon^- \int d^2 \bar{\epsilon}^-$, the kernel can be expressed

$$K^{--}(z; s) = \int d\eta^{(+4)} e^{-i s \tilde{\Box}} \left( X^{\dot{\alpha} \alpha} X^{- \dot{\alpha}} + \frac{1}{2} W(X^-)^2 + \frac{1}{2} \bar{W}^{\dot{\alpha}} X^{- \dot{\alpha}} + (D^{- \alpha} W) X^{- \dot{\alpha}} + (\bar{D}_{\dot{\alpha}} \bar{W}) X^{\dot{\alpha}} + W \bar{W} \right).$$

(32)

where

$$X_a = D_a + i k_a, \quad X^{- \dot{\alpha}} = D^{- \dot{\alpha}} + i \epsilon^{- \dot{\alpha}}, \quad X^{- \dot{\alpha}} = D^{- \dot{\alpha}} + i \bar{\epsilon}^{- \dot{\alpha}};$$

$$\tilde{\Box} = X^a X_a + \frac{1}{2} (D^+ a W) X^{- \dot{\alpha}} + \frac{1}{2} (\bar{D}_{\dot{\alpha}} \bar{W}) X^{\dot{\alpha}} + W \bar{W}.$$

The “shifted” operators $X$’s satisfy the same algebra as the corresponding operators $D$’s.

Commuting $W$, $\bar{W}$ and their spinor derivatives in (32) to the left through $\exp(-i s \tilde{\Box})$, and noting that $D^{- \alpha} W$ and $\bar{D}_{\dot{\alpha}} \bar{W}$ commute with $\tilde{\Box}$ on-shell, we obtain

$$K^{--}(z; s) = K_X^{\alpha \dot{\alpha}} X^{- \dot{\alpha}} X^{- \dot{\alpha}}(z; s) + (D^{- \alpha} W) K_X^{\alpha}(z; s) + (\bar{D}_{\dot{\alpha}} \bar{W}) K_X^{- \dot{\alpha}}(z; s)$$

$$+ \frac{1}{2} \left\{ W + \frac{1}{2} (D^+ a W) \left( \frac{e^{-s N} - 1}{N} \right)^\alpha \right\} \beta (D^{- \beta} W) K_{\chi^{- \beta}}(z; s)$$

$$+ \frac{1}{2} \left\{ W - \frac{1}{2} (D^+ \dot{\alpha} W) \left( \frac{e^{-s N} - 1}{N} \right)^{\dot{\alpha}} \right\} \beta (D^{- \beta} \dot{\alpha} W) K_{\chi^{- \beta}}(z; s),$$

(33)

When restricting the background vector multiplet to satisfy eq. (20), we completely lose the corrections to $\delta \Gamma_H$ containing factors of $D^\pm D^\pm W = D^\pm D^\pm \bar{W}$. The role of such corrections, however, is just to complete the surviving contributions in such a way that the effective current $J^{++}$ is analytic.
where we have introduced generalized Gaussian moments

\[ K_\hat{O}(z; s) = \int d\eta^{(+4)} e^{-is\hat{O}} \hat{O} \]

and defined

\[ N^\alpha_\beta = \frac{1}{2} D^{-\alpha} D^\beta W, \quad \bar{N}^\alpha_\beta = -\frac{1}{2} \bar{D}^{-\alpha} \bar{D}^\beta \bar{W}. \]

Use of the identity \(0 = \int d\eta^{(+4)} \partial/\partial \epsilon_\alpha \left[ \exp(-is\hat{O}) X_\beta X_\alpha \right]\) yields

\[ K_{X_\alpha}(z; s) = \frac{1}{4} (D_\beta^+ W) \left( \frac{e^{-sN} - 1}{N} \right)^\alpha_\beta K_{(X^-)^2}(z; s), \quad (34) \]

and then (33) collapses to

\[ K^{-}(z; s) = K_{X^{\dagger}\alpha X_\alpha \hat{X}_\alpha}(z; s) + \frac{1}{2} W K_{(X^-)^2}(z; s) + \frac{1}{2} \bar{W} K_{(\bar{X}^-)^2}(z; s). \tag{35} \]

As in the derivation of (34), more general identities of the form

\[ 0 = \int d\eta^{(+4)} \frac{\partial}{\partial k_{\alpha\dagger}} \left[ e^{-is\hat{O}} \hat{O} \right], \quad 0 = \int d\eta^{(+4)} \frac{\partial}{\partial \epsilon_\alpha} \left[ e^{-is\hat{O}} \hat{O} \right] \tag{36} \]

allow one to express all the moments in (34) via a single one, \(K_{(X^-)^2}(z; s)\), as follows:

\[ K_{(X^-)^2} = \frac{1}{16} (\bar{D}^+ \bar{W} \bar{D}^+ \bar{W}) \text{tr} \left( \frac{1 - \cosh s\bar{N}}{N^2} \right) K_{(X^-)^2}(\bar{X}^-)^2, \tag{37} \]

\[ K_{X^{\dagger}\alpha X_\alpha \hat{X}_\alpha} = \frac{1}{16} (D_\alpha^+ W) \left( \frac{e^{-sN} - 1}{N} \right)_{\alpha\beta} (D_\beta^+ \bar{W}) \left( \frac{e^{-s\bar{N}} - 1}{\bar{N}} \right)_{\beta\bar{\alpha}} S^{\alpha\beta \bar{\beta}} K_{(X^-)^2}(\bar{X}^-)^2, \]

where (with \(i F_{\alpha\dagger \dagger} = [D_{\alpha\dagger\dagger}, D_{\beta\bar{\beta}}]\))

\[ S_{\alpha\dagger \dagger} = -\frac{1}{2} \sum_{n=1}^{\infty} \sum_{p=1}^{n+1} \frac{s^{n+1}}{(n+1)!} \left[ \left( \frac{e^{sF} - 1}{F} \right)^{-1} (-F)^{n-p} \right]_{\alpha\dagger \dagger} \beta\bar{\beta} \]

\[ \times \left\{ (D_\rho^+ W) (e^{-sN} N^{p-1})^{\rho\beta} (D_\beta^+ \bar{W}) - (D_\rho^+ \bar{W}) (e^{-s\bar{N}} \bar{N}^{p-1})_{\rho\bar{\beta}} (D_\bar{\beta}^+ W) \right\}. \tag{38} \]

The remaining step is the computation of \(K_{(X^-)^2}(\bar{X}^-)^2(z; s)\). This proceeds via the differential equation

\[ i \frac{d}{ds} K_{(X^-)^2}(\bar{X}^-)^2 = \frac{1}{2} K_{X^{\dagger}\alpha X_\alpha (X^-)^2(\bar{X}^-)^2} + \int d\eta^{(+4)} e^{-is\hat{O}} \bar{W} W (X^-)^2 (\bar{X}^-)^2. \quad (39) \]

The trick is to express the right hand side in terms of \(K_{(X^-)^2}(\bar{X}^-)^2\), thus establishing a linear differential equation for \(K_{(X^-)^2}(\bar{X}^-)^2\). This is easily done for the second term in (39).
by commuting $W\bar{W}$ to the left. For the first term, it is necessary to proceed via the first identity (36) with $\hat{O} = X_{\beta\dot{\beta}}(X^-)^2(\bar{X}^-)^2$ and $\hat{O} = (X^-)^2(\bar{X}^-)^2$, with the result

$$K_{X^{\dot{\alpha}\alpha} X_{\alpha\dot{\alpha}}(X^-)^2(\bar{X}^-)^2} = \left\{ -i \left( \frac{F}{e\sigma^F - 1} \right) \tilde{\alpha}\tilde{\alpha} + S_{\alpha\dot{\alpha}} S^{\alpha\dot{\alpha}} \right\} K_{(X^-)^2(\bar{X}^-)^2} ; \quad (40)$$

see [20] for more details.

The linear differential equation for $K_{(X^-)^2(\bar{X}^-)^2}$ can be solved exactly, and the solution will be given in a separate publication. Here we only reproduce the expansion of $K^{--}$ to sixth order in spinor derivatives of $W$. After making Schwinger’s rotation $s \to -i \tau$ of the proper-time parameter, $K^{--}$ reads

$$K^{--}(z; \tau) = \frac{i}{32\pi^2} e^{-\tau W\bar{W}} (D^+W D^+W) \left\{ \bar{W} - \frac{i\tau}{2} (1 - \frac{\tau}{2} W\bar{W}) (\bar{D}_{\dot{\beta}}^+ \bar{W})(\bar{D}_{\beta}^- \bar{W}) \right. \right.

$$

$$+ \frac{\tau^2}{24} \bar{W} \bar{N}_{\dot{\alpha} \dot{\gamma}} \bar{N}_{\dot{\beta} \dot{\alpha}} + \frac{\tau^2}{6} (1 - \frac{\tau}{2} W\bar{W}) (\bar{D}_{\dot{\alpha}}^+ \bar{W}) \bar{N}_{\dot{\alpha} \dot{\beta}} (\bar{D}_{\beta}^- \bar{W}) \right. \right.

$$

$$- \frac{\tau^3}{16} W(1 - \frac{\tau}{4} W\bar{W}) (\bar{D}_{\dot{\alpha}}^+ \bar{W}) (\bar{D}_{\dot{\alpha}}^- \bar{W}) \right. \right.

$$

$$+ \frac{\tau^3}{96} (D^+W D^+W) (D^-W D^W) \} + \text{conjugate} . \quad (41)$$

Then, performing the integral over $\tau$ gives

$$\delta \Gamma_H = - \frac{1}{(8\pi)^2} \int d\zeta^{(-4)} \delta V^{++} (D^+W D^+W) \left\{ \frac{1}{W} + \frac{1}{12} \frac{\bar{N}_{\dot{\alpha} \dot{\beta}} \bar{N}_{\dot{\beta} \dot{\alpha}}}{W^3W^2} \right. \right.

$$

$$- \frac{1}{6} \frac{(D_{\dot{\alpha}}^+ \bar{W}) \bar{N}_{\dot{\alpha} \dot{\beta}} (D^-\bar{W})}{(W\bar{W})^3} + \frac{1}{16} \frac{(D^+W D^+W) (D^-W D^-W)}{W^3W^4} \} + \text{c.c.} \quad (42)$$

Let us analyse the expression for $\delta \Gamma_H$ derived above. The first term on the right hand side of (12) is quadratic in derivatives of $W$. This term, which coincides with the earlier result (27), is generated by the on-shell variation of the holomorphic superpotential (29). The variation $\delta \Gamma_H$ does not involve any terms of fourth order in derivatives of $W$ and $\bar{W}$, although such terms do appear at intermediate stages of the calculation. The reason all such terms cancel out in the final expression for $\delta \Gamma_H$ is very simple. As was already mentioned, the effective current $J^{++}$ should be analytic. But there exists no analytic superfield which carries harmonic $U(1)$ charge +2 and is fourth order in spinor derivatives of $W$ and $\bar{W}$. Apart from the first term on the right hand side of (12), the rest of the terms are sixth order in derivatives of $W$ and $\bar{W}$. They are generated by the on-shell variation of the leading non-holomorphic correction in (22). Indeed, under
an arbitrary variation $V^{++} \rightarrow V^{++} + \delta V^{++}$ of the analytic gauge field, a real functional
\[ \delta \int d^{12}z \mathcal{H}(W, \bar{W}) = \frac{1}{64} \int d\zeta (-4) \delta V^{++} (\bar{D}^+)^2(\bar{D}^+)^2(\bar{D}^-)^2 \frac{\partial \mathcal{H}(W, \bar{W})}{\partial W} + \text{c.c.} \] (43)

The terms in the on-shell variation \[ ] of sixth order in spinor derivatives are generated by the superfield
\[ \mathcal{H}(W, \bar{W}) = \frac{1}{192\pi^2} \ln \frac{W}{\Lambda} \ln \frac{\bar{W}}{\Lambda} . \] (44)

This quantum correction was previously computed in [17] with the use of $\mathcal{N} = 1$ superfield techniques and $\mathcal{N} = 2$ superconformal considerations.

5. In this paper, we have presented a method for computing the hypermultiplet effective action using $\mathcal{N} = 2$ harmonic superspace heat kernel techniques. Combined with the results of our earlier paper [10], this provides a prescription for the analysis of the effective action in generic $\mathcal{N} = 2$ super Yang-Mills theories. Explicit calculations were presented only for the case of $U(1)$ background; these can be extended to arbitrary non-Abelian backgrounds in a fairly straightforward manner. It would also be of interest to develop a method to calculate the hypermultiplet effective action directly rather than by integrating its variation.

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