Solving for Best Linear Approximates

Avraham Bourla

January 19, 2023

Abstract

Our goal is to finally settle the persistent problem in Diophantine Approximation of finding best linear approximates. Classical results from the theory of continued fractions provide the solution for the special homogeneous case in the form of a sequence of normal approximates. We develop numeration systems and real expansions allowing this notion of normality to percolate into the general inhomogeneous setting.

1 Introduction

The general linear approximate problem asks for iterations of an irrational slope that are integrally close to some fixed real intercept (we use the borrowed terms ‘slope’ and ‘intercept’ as in [7]). First, we give an elementary introduction for sake of self containment and to familiarize the reader with subsequent definitions and notations. We survey the theory of numeration and integer systems, real expansions, the continued fraction expansion and the sequence of linear approximation coefficients. After subsequent introduction and treatment of the solved homogeneous case, we will set up the various variants of the inhomogeneous linear approximate problem and quote the theorems that construe their solution.

1.1 Numeration Systems and Real Expansions

A numeration system is a one–to–one correspondence between the set of counting numbers $\mathbb{Z}_{\geq 0}$ and the set of all finite sequences of non–negative integer digits that are subject to the basis constraints of the system. A sequence adhering to these constraints is called an admissible digit string. For instance, in the fix–radix base–$N$ systems the basis constraint is the upper bound of $N – 1$ applied uniformly to all digits. Mapping a given counting number using this correspondence gives us its representation, whereas applying the inverse of this correspondence to some admissible digit string gives us its recovery. In our familiar base–10 decimal system, the representation of the counting number $S$ is the output of the following iteration scheme:
Solving for best linear approximates

A. Bourla

**Algorithm 1:** the decimal numeration system

**input:** the counting number \( S \in \mathbb{Z}_{\geq 0} \)

**output:** the decimal digit string \( \langle d_k \rangle_n \)

1. set \( S_0 := S; n := 0; k := 1; \)
2. while \( S_{k-1} \geq 1 \) do
   3. set \( d_k := S_{k-1} \mod 10; \)
   4. set \( S_k := [S_{k-1} - d_k] / 10; \)
   5. set \( k := k + 1; \)
3. end
4. set \( n := k - 1; \)

Unless the input is \( S = 0 \), yielding the vacuous digit string representation \( \langle d_k \rangle_{k=1}^0 = \emptyset \), the iteration of the assignment in line–4 yields the sum

\[
S = S_0 = d_1 + 10S_1 = d_1 + 10d_2 + 100S_2 = \ldots = d_1 \cdot 1 + d_2 \cdot 10 + d_3 \cdot 100 + \ldots + d_n \cdot 10^{n-1}.
\]

Thus, the recovery \( S = \sum_{k=1}^n d_k \cdot 10^{k-1} \) is the dot product of the output digit string \( \langle d_k \rangle_n \) with the truncation of the decimal weight vector \( \langle 10^k \rangle_{\infty}^0 \) to the \( n - 1 \) position.

Next, we want to extend decimal representation to all integers while keeping the same dot product recovery. We do so by negating the ratio 10 in the decimal weight vector as \( \langle [-10]^k \rangle_{\infty}^0 \) and applying the following iteration:

**Algorithm 2:** the decimal integer system

**input:** the integer \( Z \in \mathbb{Z} \)

**output:** the decimal digit string \( \langle d_k \rangle_n \)

1. set \( Z_0 := Z; n := 0; k := 1; \)
2. while \( Z_{k-1} \neq 0 \) do
   3. set \( d_k := [-1]^{k-1} \cdot Z_{k-1} \mod 10; \)
   4. set \( Z_k := [Z_{k-1} + [-1]^k \cdot d_k] / 10; \)
   5. set \( k := k + 1; \)
3. end
4. set \( n := k - 1; \)

Then the recovery in the resulting base–[-10] **integer system** is the valuation of the alternating sum \( \sum_{k=1}^n d_k \cdot [-10]^{k-1} \). For instance, we recover the digit string \( \langle 9, 3, 7, 1 \rangle \) as

\[
\langle 9, 3, 7, 1 \rangle \cdot \langle 1, -10, 100, -1000 \rangle = 9 - 30 + 700 - 1000 = -321,
\]

whereas the base–[-10] representation of 321 is the digit string \( \langle 1, 8, 4 \rangle \).

We may turn any numeration system into a **real expansion**, allowing for the representation of arbitrary real numbers as possibly infinite digit strings. Given \( r \in \mathbb{R} \), we define its
floor \( [r] \) to be the largest integer not greater than \( r \) and apply the following iteration to its modulus \( \alpha = (r) := r \mod 1 = r - [r] \in [0, 1) \). In the decimal system this expansion is output of the following iteration scheme:

**Algorithm 3: the decimal expansion**

**input**: the real number \( \alpha \in [0, 1) 

**output**: the extended decimal digit string \( \langle d_k \rangle_1 \)

1. set \( \alpha_0 := \alpha; \ell := \infty; k := 1; \)
2. while \( \alpha_{k-1} \neq 0 \) do
   3. set \( d_k := \lfloor 10 \cdot \alpha_{k-1} \rfloor; \)
   4. \( \alpha_k := 10 \cdot \alpha_{k-1} - d_k; \)
   5. set \( k := k + 1; \)
3. end
4. set \( \ell := k - 1; \)

Here \( \ell \) is the possibly infinite extended index \( \ell \in \mathbb{Z}^* := \mathbb{Z} \cup \{\infty\} \). The decimal expansion \( (r) = \sum_{k=1}^{\ell} d_k \cdot 10^{-k} \) is the dot product of the digit string \( \langle d_k \rangle_1 \) with the truncation of the inverted decimal weight vector \( \langle 10^{-k} \rangle_0^\infty \) to the \( \ell - 1 \) position (where we take \( \infty - 1 := \infty \)). As for the alternating real expansion, we define the ceiling \( \lceil r \rceil := -\lfloor -r \rfloor \) of \( r \) be the smallest integer greater or equal to \( r \) and compute:

**Algorithm 4: the alternating decimal expansion**

**input**: the real number \( r \in \mathbb{R} 

**output**: the extended decimal digit string \( \langle d_k \rangle_0 \)

1. set \( r_0 := r; \ell := \infty; k := 1; \)
2. while \( r_{k-1} \neq 0 \) do
   3. set \( d_k := \lceil r_{k-1} \rceil; \)
   4. \( r_k := 10[d_k - r_{k-1}]; \)
   5. set \( k := k + 1; \)
3. end
4. set \( \ell := k - 1; \)

The resulting alternating base-[10] real expansion \( r = \sum_{k=1}^{\ell} d_k[-10]^{-k} \) is the dot product of \( \langle d_k \rangle_0 \) with the alternating inverted decimal weight vector \( \langle [-10]^{-k} \rangle_0^\infty \) truncated to the \( \ell - 1 \) position (here \( d_0 \) is not a decimal digit and is allowed to take any integer value). For instance, the expansion of \( 45.123 = 46 - .9 + .02 - .007 \) is the string \( \langle 46, 9, 3, 7 \rangle \), whereas the expansion of \( -45.123 \) is the string \( \langle -45, 2, 8, 3 \rangle \).

The decimal expansion as well as all other fix radix base-\( N \) expansions, exhibit certain non-remedial built-in pathologies. The first is an infinite expansion of certain rational numbers e.g. \( 1/3 = .333 \). The second is that all rational numbers have non–unique representations e.g. \( 1 = .999 \). For our purposes however, the biggest problem is that the
truncation of the expansion in general yields low quality approximations. For instance, the Archimdean $\pi$–approximation $355/113$ has a denominator which is more than 884 times smaller than the decimal expansion of $\pi$ to the fifth digit $3.14159 = 314159/100000$. Comparing the accuracy in approximating $\pi$, we see that the former is also more than 9.94 accurate, thus being of much higher quality. Is there a real expansion that is unique on the entire real line, finitely represent all rational numbers and always produce high quality approximations?

### 1.2 The continued fractions expansion

Given the initial seed $\alpha \in [0,1)$, we expand it as a continued fraction using the iteration scheme:

**Algorithm 5: The continued fraction expansion**

input : the initial seed $\alpha \in [0,1)$  
output: the admissible digit string $\langle a_k \rangle_1^\ell \subseteq \mathbb{Z}_{\geq 1}$

1. set $a_0 := \alpha; \ell := \infty; k := 1$
2. while $\alpha_{k-1} > 0$ do  
3. \hspace{1cm} set $a_k := \lfloor 1/\alpha_{k-1} \rfloor$
4. \hspace{1cm} set $\alpha_k := (1/\alpha_{k-1}) = 1/\alpha_{k-1} - a_k \in [0,1)$
5. \hspace{1cm} set $k := k + 1$
6. end
7. set $\ell := k - 1$
8. if $\ell \geq 2$ and $a_\ell = 1$ then
9. \hspace{1cm} set $\ell := \ell - 1$
10. \hspace{1cm} set $a_\ell := a_\ell + 1$

Unless $\alpha = 0$ and the expansion is vacuous, we use the inequalities

$$\left\lfloor \frac{1}{\alpha_{k-1}} \right\rfloor \alpha_{k-1} \leq 1 < \left\lfloor \frac{1}{\alpha_{k-1}} \right\rfloor + 1 \alpha_{k-1},$$

with the assignment of the digit $a_k$ in line–3 to derive the inequalities

$$a_k \alpha_{k-1} \leq 1 < [a_k + 1] \alpha_{k-1}, \quad 1 < \ell + 1,$$

(where we take $\infty + 1 := \infty$ whenever the extended index $\ell$ is infinite). After we rewrite the assignment in line–4 of the iterate $\alpha_k$ as $\alpha_{k-1} = [a_k + \alpha_k]^{-1}$, we obtain the continued fraction

$$\alpha = \frac{1}{a_1 + \alpha_1} = \frac{1}{a_1 + \frac{1}{a_2 + \alpha_2}} = \ldots = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \ldots}}}. $$
The rational approximation obtained by halting the iteration after \( n < \ell + 1 \) steps is the fraction
\[
\frac{p_n}{q_n} := \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\ddots + \frac{1}{a_n}}}},
\]
called the \( n^{th} \) convergent of \( \alpha \). Given \( r \in \mathbb{R} \), we set \( a_0 = \lfloor r \rfloor \) and apply this algorithm to the modulus \( \alpha := (r) \) to obtain its continued fraction expansion \( \langle a_k \rangle_0^\ell \). This expansion is finite precisely when \( r \) is rational and is unique after we apply the condition in line–8, eliminating finite expansions terminating with the digit ‘1’ e.g. \( 1/(1 + 1/1) = 1/2 \). In the next section, we will see that the convergents construe the best possible rational approximations for \( \alpha \) hence the continued fraction expansion neatly fixes the three pathologies of the fix radix expansions that were raised in the previous section. In our \( \pi \) example, the output of the first three iterations of algorithm applied to \( \pi - 3 \) is the digit string \( \langle 7, 15, 1 \rangle \), yielding the convergents \( 1/7 \),
\[
\frac{1}{7 + \frac{1}{15}} = \frac{15}{106} \quad \text{and} \quad \frac{1}{7 + \frac{1}{15 + \frac{1}{1}}} = \frac{1}{7 + \frac{1}{16}} = \frac{16}{113}.
\]
Adding 3 to this last convergent results in the Archimedean Approximation 355/113 of the previous section (in fact, Archimedes was aware of the previous two convergents 1/7 and 333/106 as well, see [2]). Note that because there are no magnitude constraints on its digits, this basis–free expansion is not an extension of any numeration system.

From now on we take the initial seed \( \alpha \in (0, 1) \) to be irrational, yielding a continued fraction expansion which is always infinite and unique. We start by quoting a few results pertaining the infinite sequence of convergents \( \langle p_k/q_k \rangle \). The first is that this sequence converges to \( \alpha \) with a rate which is governed by the inequality
\[
\left| \alpha - \frac{p_k}{q_k} \right| < \frac{1}{q_k^2}.
\]
The second is that the terms alternate around \( \alpha \) with \( p_k/q_k < \alpha \) precisely when \( k \) is even, that is, satisfying the inequality
\[
\frac{p_{2k}}{q_{2k}} < \alpha < \frac{p_{2k+1}}{q_{2k+1}}, \quad k \geq 0.
\]
We also quote the well known recursive equations
\[
p_{-1} := 1, \quad p_0 = 0, \quad p_k = a_k p_{k-1} + p_{k-2}, \quad k \geq 1
\]
and
\[
q_{-1} = 0, \quad q_0 = 1, \quad q_k = a_k q_{k-1} + q_{k-2}, \quad k \geq 1.
\]
Proofs appear in standard introductory texts on the subject such as [4].

Given \( n \in \mathbb{Z}_{\geq 0} \) we continue to iterate the relationship (5) as

\[
q_n = a_n q_{n-1} + q_{n-2} = a_n q_{n-1} + a_{n-2} q_{n-3} + q_{n-4} = \ldots
\]

and, since \( q_0 = 1 \), we obtain the identity

\[
q_n = \begin{cases} 
1 + \sum_{k=1}^{n/2} a_{2k} q_{2k-1}, & \text{if } n \text{ is even} \\
\sum_{k=0}^{[n-1]/2} a_{2k+1} q_{2k}, & \text{if } n \text{ is odd.}
\end{cases}
\]

We abbreviate this last identity as

\[
q_n = \begin{cases} 
1 + \sum_{k=1}^{n} \rho_{[n+k]} a_k q_k, & \text{if } n \text{ is even} \\
\sum_{k=0}^{[n-1]/2} a_{2k+1} q_{2k}, & \text{if } n \text{ is odd.}
\end{cases}
\]

where the parity \( \rho_n \) of \( n \) is defined to be zero or one precisely when \( n \) is even or odd, that is,

\[
\rho_n := \lceil n/2 \rceil - \lfloor n/2 \rfloor.
\]

We define the alternating sequences

\[
p_k := [-1]^k p_k, \quad q_k := [-1]^k q_k, \quad k \geq -1
\]

and, after using the recursive formulas (4) and (5) derive their corresponding recursive formulas

\[
p_k^* = p_{k-2} - a_k p_{k-1}^*, \quad q_k^* = q_{k-2}^* - a_k q_{k-1}^*, \quad k \geq 1,
\]

after a quick induction argument.

The recursive formula (5) yield the inequality \( q_k \geq F_k \) for all \( k \geq -1 \), where \( \langle F_k \rangle_{-1} = \langle 0, 1, 1, 2, 3, 5, 8, \ldots \rangle \) is the Fibonacci Sequence. From the same recursion, we see that in order to realize the equality \( q_k = F_k \), we must have \( a_k = 1 \) for all \( k \geq 1 \). We label the number for which \( a_k := 1 \) for all \( k \geq 1 \) as \( \phi \) and expand

\[
\phi := \frac{1}{1 + \frac{1}{1 + \frac{1}{1 + \ddots}}} = \frac{1}{1+\phi}.
\]

Solving the resulting quadratic equation \( \phi^2 - \phi - 1 = 0 \), we assert that \( \phi \) is the golden section \( 5[5.5 - 1] \). We apply the assignment in line–4 of algorithm \( 5[5.5 - 1] \) and verify that \( \phi_1 = 1/\phi_0 - 1 = \phi \), so that, by induction, we will always have

\[
\phi_k = \phi, \quad k \geq 1.
\]
Another special number is the silver section $\alpha := \sqrt{2} - 1$, for which line–3 and line–4 of algorithm 5 yield

$$a_1 = \left\lfloor \frac{1}{\sqrt{2} - 1} \right\rfloor = \lfloor \sqrt{2} + 1 \rfloor = 2, \quad \alpha_1 := 1/\alpha - 2 = \sqrt{2} - 1 = \alpha.$$  

Continuing by induction, we see that its continued fraction digit string is the singleton set \{2\}.

### 1.3 The sequence of linear approximation coefficients

Given an irrational number $r$ and a rational number written as the unique quotient of the relatively prime integers $p$ and $q \geq 1$, a fundamental object of interest from Diophantine Approximation is the linear approximation coefficient $\theta = \theta[r, p/q] := qr - p$. Being the product of the approximation error $r - p/q$ with the denominator $q$, we take the linear approximation coefficient as the most basic measure of the quality of $p/q$ as an $r$–approximation. Letting $\langle p_k/q_k \rangle_1^\infty$ be the continued fraction convergents for $\alpha := (r)$, we define the sequence of linear approximation coefficients:

$$\theta_k := \theta \left[ \alpha, \frac{p_k}{q_k} \right] = q_k \alpha - p_k, \quad k \geq -1. \quad \text{(11)}$$

Note that since adding integers to fractions does not change their denominators, $r$ and $\alpha$ will share the same sequence of linear approximation coefficients.

Applying the recursive formulas (4) and (5) in a quick induction argument yield the recursion

$$\theta_{-1} := -1, \quad \theta_0 := \alpha, \quad \theta_k := \theta_{k-2} + a_k \theta_{k-1}, \quad k \geq 1. \quad \text{(12)}$$

Assuming that $\theta_k = -\theta_{k-1} \alpha_k$ for $k \geq 1$, the assignment of $\alpha_{k+1}$ in line–4 of algorithm 5 and recursive formula (12) yield the identity

$$\theta_{k+1} = \theta_{k-1} + a_{k+1} \theta_k = \theta_{k-1} [1 - a_{k+1} \alpha_k] = \theta_{k-1} \alpha_k \left[ \frac{1}{\alpha_k} - a_{k+1} \right] = -\theta_k \alpha_{k+1}. $$

After verifying the base case $\theta_0 = \alpha = -\theta_{-1} \alpha$, we have just proved by induction that

$$\theta_k = -\theta_{k-1} \alpha_k = [-1]^k \alpha_0 \alpha_1 \ldots \alpha_k, \quad k \geq 0. \quad \text{(13)} $$

Since $0 < \alpha_k < 1$ by its definition in line–4 of algorithm 5, we see that the sequence of linear approximation coefficients is absolutely decreasing to zero while alternating as

$$|\theta_k| = [-1]^k \theta_k, \quad k \geq -1. \quad \text{(14)}$$

In tandem with definition (8) of $p_k^*$ and $q_k^*$, we derive the identity

$$|\theta_k| = q_k^* \alpha - p_k^* \quad k \geq -1, \quad \text{(15)}$$
and in tandem with the recursive formula (9), we obtain the new recursive formula
\[ |\theta_{-1}| = 1, \quad |\theta_0| = \alpha, \quad |\theta_k| = |\theta_{k-2}| - a_k|\theta_{k-1}|, \quad k \geq 1. \quad (16) \]

We use identity (13) to derive the inequality
\[ 2|\theta_1| = 2\alpha_0\alpha_1 = \frac{2\alpha_1}{\alpha_1 + \alpha_1} < \frac{1 + \alpha_1}{\alpha_1 + \alpha_1} < 1, \]
so that \( |\theta_1| < .5 \) and
\[ \|\theta_k\| = |\theta_k|, \quad k \geq 1, \quad (17) \]
(since \( \theta_0 = \alpha \in (0, 1) \) could be greater than .5, this identity need not extend to the initial \( k = 0 \) term).

The base-±\( \alpha \) expansions we are about to define in chapter 3 will compare to the series \( \sum_{k=1}^{\infty} a_k|\theta_{k-1}| \) and \( \sum_{k=1}^{\infty} a_k\theta_{k-1} \), whose convergence we will now discuss. Using the inequality (1) and the identity (13), we write
\[ \frac{a_k|\theta_{k-1}|}{|\theta_{k-2}|} = a_k\alpha_{k-1} < 1 < [a_k + 1]\alpha_{k-1} = \frac{[a_k + 1]|\theta_{k-1}|}{|\theta_{k-2}|}, \quad k \geq 1 \]
and obtain the inequalities
\[ a_k|\theta_{k-1}| < |\theta_{k-2}| < [a_k + 1]|\theta_{k-1}|, \quad k \geq 1. \quad (18) \]

By a simple comparison test, the absolute series \( \sum_{k=1}^{\infty} a_k|\theta_{k-1}| \) will then converge (and as a direct consequence so will the alternating series \( \sum_{k=1}^{\infty} a_k\theta_{k-1} \)), once we assert:

**Proposition 1.1.** The series \( \sum_{k=0}^{\infty} |\theta_k| \) converges for all \( \alpha \).

**Proof.** Multiplying both sides of the inequality (2) by \( q_k \) yields the inequality
\[ |\theta_k| = |q_k\alpha - p_k| < q_k^{-1}, \quad k \geq 0. \]
This inequality and the recursive formula (5) yield the inequalities
\[ \frac{|\theta_{k+1}|}{|\theta_k|} < \frac{q_k}{q_{k+1}} = \frac{q_k}{a_{k+1}q_k + q_{k-1}} < \frac{1}{a_{k+1}}, \quad k \geq 0. \]

Thus, as long as \( \limsup_{k \to \infty} \langle a_k \rangle \geq 2 \), we conclude convergence from the simple comparison and ratio tests. When \( \limsup_{k \to \infty} \langle a_k \rangle = 1 \), then \( \alpha \) must be a noble number, whose continued fraction expansion terminates with a tail of 1’s. By the limit comparison test, we need only establish the convergence for when \( \langle a_k \rangle = \{1\} \) corresponding to the golden section \( \alpha = \phi \). We apply the identities (10), (13) and use the formula for the convergence of infinite geometric series to compute
\[ \sum_{k=0}^{\infty} |\theta_k| = \sum_{k=0}^{\infty} \phi_0\phi_1...\phi_k = \sum_{k=0}^{\infty} \phi^{k+1} = \frac{1}{1 - \phi} = 1 + \phi, \]
thus obtaining convergence for this case as well.
Solving for best linear approximates

A. Bourla

Now that we have absolute and conditional convergence, we may rearranging these sums without changing their value, an effect that will provide us with some important identities. The recursive formula (16) allows us to telescope the partial sum

$$\sum_{k=1}^{n} a_k|\theta_{k-1}| = \sum_{k=1}^{n-2} |\theta_k| - \sum_{k=1}^{n} |\theta_k| = 1 + \alpha - |\theta_{n-1}| - |\theta_n|. \quad (19)$$

Since $|\theta_n| \to 0$ as $n \to \infty$, we derive the expansion

$$1 + \alpha = \sum_{k=1}^{\infty} a_k|\theta_{k-1}| \quad (20)$$

and the tail

$$\sum_{k=n+1}^{\infty} a_k|\theta_{k-1}| = \sum_{k=1}^{\infty} a_k|\theta_{k-1}| - \sum_{k=1}^{n} a_k|\theta_{k-1}| = |\theta_{n-1}| + |\theta_n|. \quad (21)$$

Similarly, the recursive formula (12) allows us to telescope the partial sum

$$\sum_{k=1}^{n} a_k \theta_{k-1} = \sum_{k=1}^{n} [\theta_k - \theta_{k-2}] = -\theta_1 - \theta_0 + \theta_{n-1} + \theta_n = 1 - \alpha + \theta_{n-1} + \theta_n$$

and then take its limit as $n \to \infty$ to derive the sum

$$1 - \alpha = \sum_{k=1}^{\infty} a_k \theta_{k-1}. \quad (22)$$

Subtracting the sum (22) from the sum (20) and dividing by two yields the self expansion

$$\alpha = \sum_{k=1}^{\infty} a_{2k}|\theta_{2k-1}| = -\sum_{k=1}^{\infty} a_{2k} \theta_{2k-1}. \quad (23)$$

Adding the sum (22) to the sum (20) and then dividing by two yields the expansion of unity

$$1 = \sum_{k=0}^{\infty} a_{2k+1} \theta_{2k} = \sum_{k=0}^{\infty} a_{2k+1}|\theta_{2k}|. \quad (24)$$

More applications of the recursive formulas (12) and (16) allow us to expand

$$|\theta_{n-1}| = a_{n+1} |\theta_n| + |\theta_{n+1}| = a_{n+1} |\theta_n| + a_{n+3} |\theta_{n+2}| + |\theta_{n+3}| = \ldots$$

and

$$-\theta_{n-1} = a_{n+1} \theta_n - \theta_{n+1} = a_{n+1} \theta_n + a_{n+3} \theta_{n+2} - \theta_{n+3} = \ldots,$$

obtaining the tail expansions

$$|\theta_{n-1}| = \sum_{k=\lfloor n/2 \rfloor}^{\infty} a_{2k+1 - \rho_n} |\theta_{2k-\rho_n}| \quad (25)$$
and
\[ -\theta_{n-1} = \sum_{k=\lceil n/2 \rceil}^{\infty} a_{2k+1-\rho_n} \theta_{2k-\rho_n}, \]
where \( \rho_n \) is the parity of \( n \) as in definition (7).

### 1.4 The solved homogeneous case

Given an irrational initial seed \( \alpha \), we seek positive integer values \( m \) for which the multiple \( m\alpha \) is close to some integer value, that is, for which the fractional part \( (m\alpha) \) is close to either zero or one. In this context, \( m \) is called a positive two–sided \( \alpha \)–approximate and the value \( (m\alpha) \) is called its iterate. After defining the integral distance function \( \|r\| = \min\{r, 1-r\} \) to be the distance from the real number \( r \) to its nearest integer, we rephrase our problem as seeking positive two-sided \( \alpha \)–approximates \( m \) whose two–sided error \( \|m\alpha\| \) is small. Since \( (m\alpha) = (m\alpha) \), there is no loss of generality by requiring \( \alpha \) to lie in the unit interval. The set of all iterates \( \{(m\alpha)\}_{m=1}^{\infty} \) is the same as the orbit of \( \alpha \) under the irrational rotation map \( x \mapsto x + \alpha \, (\text{mod} \, 1) \). It is well known \([5]\) that for all \( \alpha \), this set is dense in the interval, hence the two–sided error \( \|m\alpha\| \) can attain an arbitrarily small value as long as \( m \) is allowed to increase without a bound.

The homogeneous approximation problem asks for those high quality \( \alpha \)–approximates \( m \) that combine accuracy, measured by the two–sided error \( \|m\alpha\| \), with simplicity, measured by the magnitude of \( m \) itself. This leads us to grade the quality of a positive \( \alpha \)–approximate \( m \) as the value \( m \cdot \|m\alpha\| \); the smaller this value, the better \( m \) is as an \( \alpha \)–approximate. A positive two–sided \( \alpha \)–approximate \( m \) is called normal when it satisfies the inequality \( m \cdot \|m\alpha\| < 1 \). Since for all \( r \in \mathbb{R} \) and \( t \in \mathbb{Z} \) we have \( \|r+t\| = \|r\| \leq |r| \), we apply the inequality (2) and obtain the normality condition
\[ q_k \|q_k\alpha\| = q_k \|q_k\alpha - p_k\| \leq q_k \|q_k\alpha - p_k\| = q_k \left| \frac{\alpha - p_k}{q_k} \right| < 1. \]
Therefore, the denominator \( q_k \) of every convergent \( p_k/q_k \) for the continued fraction expansion of \( \alpha \) is a normal positive two–sided \( \alpha \)–approximate. Furthermore, theorem 16 in \([4]\) asserts that there are no additional normal positive two–sided approximates besides these denominators. Forming a complete set of normal \( \alpha \)–approximates, we take the special sequence \( \langle q_k \rangle_{0}^{\infty} \) as the solution to the positive two–sided linear homogeneous approximate problem. It is worth noting that the resulting sequence of quadratic approximation coefficients \( \langle \Theta_k := q|\theta_k| = q^2|\alpha - p/q| \) exhibits many interesting and surprising relations as well as a beautifully symmetric internal structure, see \([1, 3]\).

Other variants to the homogeneous approximate problem arise from considering other types of linear approximates. First, we would like to consider negative \( \alpha \)–approximates as well, hence we need to adjust the normality condition as
\[ |m| \cdot \|m\alpha\| < 1. \]
Since \( \|m\alpha\| = \|-m\alpha\| \), the vector \( \langle -q_k \rangle \) is the solution for the negative linear approximate variant. Considering both positive and negative approximates and passing from unsigned to signed errors, we say that a non-zero multiple \( m \) is an **over** \( \alpha \)-approximate when \( (m\alpha) < .5 \), otherwise we say that \( m \) is **under**. By the inequalities (3), we see that \( q_k \) is over precisely when \( k \) is even. Thus the subsequences \( \langle q_{2k} \rangle \) and \( \langle q_{2k+1} \rangle \) form complete sets of normal positive over and under \( \alpha \)-approximates. Similarly, since for all \( r \in \mathbb{R} \) we have \( (r) = 1 - (r) \), the negative approximate \( -q_k \) is over precisely when \( k \) is odd. Thus the subsequences \( \langle -q_{2k} \rangle \) and \( \langle -q_{2k+1} \rangle \) form complete sets of normal negative under and over \( \alpha \)-approximates. Combining these observations asserts that the alternating sequences \( \langle q_k^* \rangle \) and \( \langle -q_k^* \rangle \) as in definition (5) solves these variants as they form complete sets of non-zero over and under \( \alpha \)-approximates.

### 1.5 The inhomogeneous linear approximate problem

Given the irrational slope \( \alpha \) and intercept \( \beta \in (0,1) \), we seek non-negative integer values \( m \) for which the two-sided error \( \|\beta - m\alpha\| \) is small. Since \( \|\beta - m\alpha\| = \|(\beta) - m(\alpha)\| \), we may restrict without loss of generality both \( \alpha \) and \( \beta \) to lie in the unit interval. We begin by naively asking if we can extend the results for the homogeneous case \( \beta = 0 \) from the previous section into the general inhomogeneous setting \( \beta > 0 \). The negative answer, as stated by Theorem 26 in [4], asserts that when \( \beta > 0 \), there are no \( (\alpha, \beta) \)-approximates \( m \) that satisfy the non-homogeneous adjustment \( |m| \cdot \|m\alpha - \beta\| < 1 \) of the normality condition (27). Our goal then becomes finding the minimal relaxation of the normality condition necessary to produce ‘almost’ normal approximates.

To further illustrate that solving the inhomogeneous problem will require a novel approach, we provide a specific example. As homogeneous \( \pi \)-approximates, both the over approximate 2 and the under approximate \( -2 \) generate the same two-sided error \( \|2\pi\| \approx .28 \). Since this quantity is approximately \( 1 - (e) \), the number \( -2 \) makes for an excellent negative \( (\pi, e) \)-approximate (abusing the notation for sake of clarity, we are actually taking the modulii \( \alpha := \pi - 3 \) and \( \beta := e - 2 \)). We manually verify that the two-sided error \( \|e + 2\pi\| \) is smaller than the error \( \|e - n\pi\| \) for all \( 0 \leq n \leq 22251 \). We need to go as high as 22252 to generate a positive \( (\pi, e) \)-approximate, which is of higher accuracy than \( -2 \)!

The characterization of inhomogeneous approximates slightly differs from the special homogeneous case. For starts, zero is now a valid \( (\alpha, \beta) \)-approximate and thus we will include it in the set of positive, negative and non-zero approximates, relabeling these as **forwards**, **backwards** and **total**. As for the error-type, the approximate \( m \) is over \( \beta \) when \( (m\alpha) \in [\beta, \beta + .5) \) (mod 1) and is otherwise under. For instance, if \( .1 < \alpha < .11 \) is irrational and \( \beta := .9 \), then the total approximates \( m \in \{0, 1, 2, 3, 4\} \) are all over \( \beta \), whereas \( m = -1 \) and \( m = 5 \) are under.

Given the possibly infinite limit \( \ell \in \mathbb{Z}_{\geq 0}^* \) we say that a sequence \( \langle A_n \rangle_{1}^{\ell} \) of \( (\alpha, \beta) \)-approximates is a **general solution** when:

(i) \( |A_n| \leq q_n, \quad 1 \leq n < \ell + 1 \)
(ii) \(0 < \|\beta - A_n\alpha\| < |\theta_{n-1}|, \quad 1 \leq n < \ell\)

(iii) \(\|\beta - A_\ell\alpha\| = 0\) \hspace{1cm} (28)

When the limit \(\ell\) is infinite, we read the last condition as \(\lim_{n\to\infty} \|\beta - A_n\alpha\| = 0\). Thus, while bounding the magnitude of the general \(n\)th term by the magnitude of its homogeneous counterpart \(q_n\), we relax the error tolerance by a single index offset and bound it using the error of the previous homogeneous term \(q_{n-1}\). In corollaries 3.4 and 3.5 we provide general sequences solutions to the total under and total over variants. The (two–sided positive) general linear approximate problem itself is solved in corollary 3.8 and its negative counterpart is solved in corollary 3.9.

2 Irrational Base Numeration Systems

The first numeration system we are are about to present is close to being a century old, making its first appearance in a paper by Ostrowski [6]. It uses the special solution sequence \(\langle q_k \rangle_0^\infty\) as its weight vector. Following our treatment of the decimal numeration system in the introduction, we will then introduce for the first time in a rigorous manner the corresponding alternating integer system whose weight vector is the alternating sequence \(\langle q_k^* \rangle_0^\infty\) as in definition (8).

2.1 The Irrational Base Absolute Numeration System

Given the irrational base \(\alpha \in (0, 1)\), we use algorithm 5 and the recursive formula (5) to derive the continued fraction digit string \(\langle a_k \rangle_1^\infty\) and the special \(\langle q_k \rangle_0^\infty\). In the next theorem, we prove that we can represent every positive integer \(S\) base–\(\alpha\) as a index \(n \geq 0\) and a digit string \(\langle c_k \rangle_1^n\) with \(c_n \geq 1\), subject to the Markov Conditions:

\[
\begin{align*}
(i) & \quad 0 \leq c_1 \leq a_1 - 1 \\
(ii) & \quad 0 \leq c_k \leq a_k, \quad 2 \leq k \leq n \\
(iii) & \quad c_k = a_k \implies c_{k-1} = 0, \quad 2 \leq k \leq n \hspace{1cm} \text{(29)}
\end{align*}
\]

A digit string that satisfies these conditions is called left–admissible. We then realize the seed \(S\) as the dot product of this digit string with the special sequence \(\langle q_k \rangle\) truncated to the \(n - 1\) position.

**Theorem 2.1.** There exists a one to one correspondence between \(\mathbb{Z}_{\geq 0}\) and the set of all finite left–admissible digit strings. The resulting base–\(\alpha\) numeration system admits to every counting number \(S \in \mathbb{Z}_{\geq 0}\) an index \(n\) satisfying \(q_{n-1} \leq S \leq q_n - 1\) and a left–admissible digit string \(\langle c_k \rangle_1^n\) such that \(S = \sum_{k=1}^{n} c_k q_{k-1}\).
Proof. Given the input seed \( S \in \mathbb{Z}_{\geq 0} \), apply the following iteration scheme:

**Algorithm 6:** the base–\( \alpha \) numeration system

| input : the counting number \( S \in \mathbb{Z}_{\geq 0} \) | output: the left–admissible digit string \( \langle c_k \rangle_n \) |
|-----------------------------------------------------------|----------------------------------------------------------|
| 1 set \( S_0 := S \), \( m := 0 \);                      |                                                          |
| 2 while \( S_m \geq 1 \) do                               |                                                          |
| 3 let \( n_m \) be such that \( q_{n_m-1} \leq S_m \leq q_{n_m} - 1 \); |                                                          |
| 4 set \( c_{n_m} := \lfloor S_m / q_{n_m-1} \rfloor \);  |                                                          |
| 5 set \( S_{m+1} := S_m - c_{n_m} q_{n_m-1} \);         |                                                          |
| 6 set \( m := m + 1 \);                                  |                                                          |
| 7 end                                                     |                                                          |
| 8 set \( M := m \), \( n := n_0 \), \( c_k := 0 \) for all \( k \notin \{ n_m \}_{m=0}^M \); |                                                          |

If \( S = 0 \), then \( 0 = q_{-1} \leq S \leq q_0 - 1 = 0 \) and the algorithm yields the vacuous representation \( n = 0 \) and \( \langle c_k \rangle = \emptyset \). Otherwise, from the assignment of the index \( n_0 \) in line–3 during the first \( m = 0 \) step of this iteration we have \( q_{n_0-1} \leq S \leq q_{n_0} - 1 \). After the assignment \( n := n_0 \) in line–8, we deduce that the index \( n \) is chosen such that

\[
q_{n-1} \leq S \leq q_n - 1 \tag{30}
\]

as in the hypothesis (note that if \( S = a_1 = q_1 = 1 \), we will set \( n := 2 \)). We rewrite the assignment of line–4 as the inequalities

\[
c_{n_m} q_{n_m-1} \leq S_m < [c_{n_m} + 1] q_{n_m-1} \tag{31}
\]

and then apply the assignment of line–5 to deduce the inequalities

\[
S_{m+1} = S_m - c_{n_m} q_{n_m-1} < [c_{n_m} + 1] q_{n_m-1} - c_{n_m} q_{n_m-1} = q_{n_m-1}.
\]

Hence, during the next \( m + 1 \) step, we will choose in line–3 the index \( n_{m+1} \) such that

\[
q_{n_{m+1}-1} \leq S_{m+1} < q_{n_{m-1}} \leq S_m,
\]

so that this iteration scheme must eventually terminate with the subindex \( M \geq 0 \) such that \( S_M = 0 \) and the chain of index inequalities

\[
0 \leq n_M < \ldots < n_1 < n_0 = n.
\]

After we assign \( c_k := 0 \) in line–8 for all \( 1 \leq k \leq n \) such that \( k \notin \{ n_m \}_{m=0}^{M-1} \), we apply the assignment of line–5 again and obtain the sum

\[
S = S_0 = c_{n_0} q_{n_0-1} + S_1 = c_{n_0} q_{n_0-1} + c_{n_1} q_{n_1-1} + S_2 = \ldots
\]

\[
= \sum_{m=0}^{M-1} c_{n_m} q_{n_m-1} + S_M = \sum_{m=0}^{M-1} c_{n_m} q_{n_m-1} = \sum_{k=1}^{n} c_k q_{k-1},
\]

13
as stated in the hypothesis.

We now show that the output digit string \( (c_k)^n \) is left–admissible. We verify condition–(i) by looking at the terminal subindex \( M \). If \( n_M \geq 2 \) then \( 1 \notin \{ n_m \}_{m=0}^{M-1} \) so in line–8 we will assign \( c_1 := 0 \) and if \( n_M = 1 \), the inequality (31) and the assignment in line–3 assert that

\[
c_1 = c_1q_0 \leq S_1 \leq q_1 - 1 = a_1 - 1.
\]

To verify condition–(ii), we assume by contradiction that for some \( k \) we have \( c_k \geq a_k + 1 \). Since \( c_k = 0 \) for all \( k \notin \{ n_m \}_{m=0}^{M-1} \), there exists a subindex \( m \) such that \( c_{n_m} \geq a_{n_m} + 1 \). Then the recursive formula (5), the inequality (31) and the fact that the sequence \( \langle q_k \rangle_0^\infty \) is strictly increasing will lead us to the contradiction

\[
S_m < q_{n_m} = a_{n_m}q_{n_m-1} + q_{n_m-2} < [a_{n_m} + 1]q_{n_m-1} \leq c_{n_m}q_{n_m-1} \leq S_m.
\]

To assert condition–(iii), suppose by contradiction that for some \( k \geq 2 \), we have \( c_k = a_k \) yet \( c_{k-1} \geq 1 \). Since the line–4 assignment of the two successive positive coefficients \( c_{k-1} \) and \( c_k \) was carried through two successive iterations, there is some index \( m \) for which \( k - 1 = n_m \) and \( k = n_m + 1 = n_{m+1} \). The recursive formula (5), the inequality (31) and the assignment of line–5 will now leads us to the contradiction

\[
S_m < q_{n_m} = q_{k-1} < q_k = q_k - S_{m+1} + S_{m+1} = q_k - c_{n_m+1}q_{n_m} + S_{m+1} = q_k - c_kq_{k-1} + S_{m+1} = q_k - a_kq_{k-1} + S_{m+1} = q_{k-2} + S_{m+1} \leq c_{k-1}q_{k-2} + S_{m+1} = c_{n_m}q_{n_m-1} + S_{m+1} = S_m.
\]

Conclude that \( \langle c_k \rangle_1^n \) is left–admissible.

We have just seen that algorithm 6 is a well defined map between the input set of counting numbers and the output set of left–admissible digit strings. To complete the proof, we need to show that this map is, in fact, a one to one correspondence. This map is clearly onto for if we desire the left–admissible digit string \( \langle c_k \rangle_1^n \) as output, we just plug in \( S := \sum_{k=1}^n c_kq_{k-1} \) as input. To prove that it is also one to one, we suppose that the two left–admissible digit strings \( \langle c_k \rangle_1^n \) and \( \langle c_k \rangle_1^n \) represents the same number, that is, that

\[
\sum_{k=1}^n c_kq_{k-1} = \sum_{k=1}^n \hat{c}_kq_{k-1}.
\]

The inequality (30) asserts that \( n = \hat{n} \). If we now suppose by contradiction that the sequences \( \langle c_k \rangle_1^n \) and \( \langle \hat{c}_k \rangle_1^n \) are not identical, then we let \( 1 \leq j \leq n \) be the largest index for which \( c_j \neq \hat{c}_j \) so that the tails \( \sum_{k=j+1}^n c_kq_{k-1} \) and \( \sum_{k=j+1}^n \hat{c}_kq_{k-1} \) are equal. We assume, without loss of generality, that \( c_j + 1 \leq \hat{c}_j \), apply the inequality (30) to assert that \( \sum_{k=1}^{j-1} c_kq_{k-1} < q_{j-1} \), and reach the contradiction

\[
\sum_{k=1}^n c_kq_{k-1} = \sum_{k=1}^{j-1} c_kq_{k-1} + c_jq_{j-1} + \sum_{k=j+1}^n c_kq_{k-1} < q_{j-1} + c_jq_{j-1} + \sum_{k=j+1}^n c_kq_{k-1} \leq \sum_{k=1}^n \hat{c}_kq_{k-1} = \sum_{k=1}^n c_kq_{k-1}.
\]

Therefore the algorithm 6 is a one to one correspondence between \( \mathbb{Z}_{\geq 0} \) and the set of left–admissible digit strings.
For example, when \( \alpha = \phi \) is the golden section, we have seen that the special sequence \( \langle q_k \rangle \) is the Fibonacci Sequence. Applying this theorem using the golden section as base is the Zeckendorf Theorem, stating that every positive integer can be uniquely written as the sum of nonconsecutive terms in the Fibonacci Sequence \( \langle 1, 2, 3, 5, 8, \ldots \rangle \). When \( \alpha = \sqrt{2} - 1 \) is the silver section for which \( a_k = 2 \) for all \( k \geq 1 \), then formula (5) will then assert that \( \langle q_k \rangle_0^3 = \langle 1, 2, 5, 12 \rangle \). The following table display how the digits behave when we count to twenty four using this base:

| \( N \) | \( q_3 = 12 \) | \( q_2 = 5 \) | \( q_1 = 2 \) | \( q_0 = 1 \) |
|--------|---------------|---------------|---------------|---------------|
| 1      | 1             | 0             | 0             | 0             |
| 2      | 0             | 0             | 1             | 0             |
| 3      | 0             | 0             | 1             | 1             |
| 4      | 0             | 0             | 2             | 0             |
| 5      | 0             | 1             | 0             | 0             |
| 6      | 0             | 1             | 0             | 1             |
| 7      | 0             | 1             | 1             | 0             |
| 8      | 0             | 1             | 1             | 1             |
| 9      | 0             | 1             | 2             | 0             |
| 10     | 0             | 2             | 0             | 0             |
| 11     | 0             | 2             | 0             | 1             |
| 12     | 1             | 0             | 0             | 0             |

\[
\begin{array}{cccc}
N & q_3 = 12 & q_2 = 5 & q_1 = 2 & q_0 = 1 \\
\hline
13 & 1 & 0 & 0 & 1 \\
14 & 1 & 0 & 1 & 0 \\
15 & 1 & 0 & 1 & 1 \\
16 & 1 & 0 & 2 & 0 \\
17 & 1 & 1 & 0 & 0 \\
18 & 1 & 1 & 0 & 1 \\
19 & 1 & 1 & 0 & 0 \\
20 & 1 & 1 & 1 & 0 \\
21 & 1 & 1 & 2 & 0 \\
22 & 1 & 2 & 0 & 0 \\
23 & 1 & 2 & 0 & 1 \\
24 & 2 & 0 & 0 & 0 \\
\end{array}
\]

### 2.2. The Irrational Base Alternating Integer System

Given the irrational base \( \alpha \in (0, 1) \), we use algorithm 5 and the recursive formula (5) to derive the continued fraction digit string \( \langle a_k \rangle_1^\infty \) and the special sequence \( \langle q_k \rangle_1^\infty \). In the next theorem, we prove that we can represent every integer \( T \) base \(-[-\alpha]\) as an index \( n \geq 0 \) and a digit string \( \langle b_k \rangle_1^n \) with \( b_n \geq 1 \), subject to the variant of the Markov Conditions:

\[
(i) \ 0 \leq b_k \leq a_k, \quad 1 \leq k \leq n \\
(ii) \ b_k = a_k \implies b_{k+1} = 0, \quad 1 \leq k \leq n - 2
\]  

(32)

A digit string that satisfies these conditions is called **right-admissible**. We then realize the seed \( T \) as the dot product of this digit string with the truncation of the alternating special sequence \( \langle q_k \rangle \) of definition (5) to the \( n - 1 \) position. Define the sequence of sets

\[
I_n^* := [1 - q_{n-\rho_n}, q_{n-\rho_n}] \cap \mathbb{Z}, \quad n \geq 0,
\]

(33)

where \( \rho_n \) is the parity of \( n \) as in definition (7). Since \( q_{k+1} > q_k \), this sequence is nested as \( I_k^* \subset I_{k+1}^* \) and since for all \( \alpha \) we have \( q_{-1} = 0 \) and \( q_0 = 1 \), we will always have \( I_{-1}^* := \emptyset \) and \( I_0^* = \{0\} \).

**Theorem 2.2.** There exists a one to one correspondence between \( \mathbb{Z} \) and the set of all finite right-admissible digit strings. The resulting base \(-[-\alpha]\) integer system admits to every integer \( T \) the index \( n \) satisfying \( T \in I_n^* \setminus I_{n-1}^* \) and a right-admissible digit string \( \langle b_k \rangle_1^n \) such that \( b_n \geq 1 \) and \( T = \sum_{k=1}^{n} b_k q_{k-1}^* \).
Proof. Given $T \in \mathbb{Z}$, define the indicator function

$$\chi_{<0}[T] := \begin{cases} 1, & T \leq -1 \\ 0, & \text{otherwise.} \end{cases}$$

The base–$[-\alpha]$ representation $T$ is the output of the following iteration scheme:

**Algorithm 7:** the base–$[-\alpha]$ integer system

**input:** the integer $T \in \mathbb{Z}$

**output:** the right–admissible digit string $\langle b_k \rangle_1^n$

1. set $T_0 := T$, $m := 0$;
2. while $T_m \notin \{0, 1\}$ do
   3. let $n'_m \geq 0$ be such that $q_{n'_m-1} < |T_m| + \chi_{<0}[T_m] \leq q_{n'_m}$;
   4. if $[-1]n'_m T_m > 0$ then
      5. set $n_m := n'_m$;
      6. set $b'_{n_m} := \lceil |T_m|/q_{n_m-1} \rceil$;
      7. if $|T_m - b'_{n_m} q_{n_m-1}| + \chi_{<0}[T_m - b'_{n_m} q_{n_m-1}] \leq q_{n_m-2}$ then
         8. set $b_{n_m} := b'_{n_m}$;
      9. else
         10. set $b_{n_m} := b'_{n_m} + 1$;
   11. end
   12. else
      13. set $n_m := n'_m + 1$;
      14. set $b_{n_m} := 1$;
   15. end
   16. set $T_{m+1} := T_m - b_{n_m} q_{n_m-1}$;
   17. set $m := m + 1$;
3. set $M := m$, $n_M := 0$, $n := n_0$;
4. set $b_k := 0$ for all $1 \leq k \leq n$ where $k \notin \{n_m\}_{m=0}^M$;
5. reset $b_1 := b_1 + T_M$;

• **step-(i):** We will show that during step $m$, the index $n_m$ is chosen such that

$$T_m \in I_{n_m}^* \setminus I_{n_m-1}^*, \quad 0 \leq m \leq M. \quad (34)$$

If $T_m = 0$, then in line–19 we will set $M := m$ and $n_M := n_m := 0$ so that $T_m = 0 \in \{0\} = I_{n_m}^* \setminus I_{n_m-1}^*$. We will consider the cases $T_m \geq 1$ and $T_m \leq -1$ separately:

If $T_m \geq 1$, then $|T_m| + \chi_{<0}[T_m] = T_m$, so that the choice of the preindex $n'_m \geq 0$ in line–3 will satisfy the inequality

$q_{n'_m-1} < T_m \leq q_{n'_m}$.

If the preindex $n'_m$ is odd, then the condition of line–4 holds and in line–5 we assign the odd index $n_m := n'_m$, so that

$$0 \leq q_{[n_m-1]-[n_m-2]} = q_{n_m-2} = q_{n'_m-2} < q_{n'_m-1} < T_m$$
and
\[ T_m \leq q'n_m = q_n = q_{n_{m-\rho n_{m-1}}} \].

If the preindex \( n'_m \) is even, then the condition of line 4 fails and in line 13 we assign the odd index \( n_m := n'_m + 1 \), so that
\[ 0 \leq q_{n_{m-1}-\rho n_{m-2}} = q_{n_m-2} = q_{n'_m-1} < T_m \]
and
\[ T_m \leq q'n_m = q_{n_m-1} < q_n = q_{n_{m-\rho n_{m-1}}} \].

In either case we have
\[ q_{n_{m-1}-\rho n_{m-2}} < T_m \leq q_{n_{m-\rho n_{m-1}}} \],
so that, by the definition (33) of \( I^*_n \), we conclude the desired containment \( T_m \in I^*_n \setminus I^*_{n-1} \).

If, on the other hand \( T_m \leq -1 \), then \( \chi_{<0}[T_m] + |T_m| = 1 - T_m \), so that the choice of the preindex \( n'_m \geq 0 \) in line 3 will satisfy the inequality
\[ 1 - q'n_m \leq T_m < 1 - q{n'_m-1} \].

If the preindex \( n'_m \) is odd, then the condition of line 4 fails and in line 13 we assign the even index \( n_m := n'_m + 1 \), so that
\[ 1 - q_{n_{m-\rho n_{m}}} = 1 - q_n = 1 - q'n'_m + 1 < 1 - q'n_m \leq T_m \]
and
\[ T_m < 1 - q'n_{m-1} = 1 - q_{n_m-2} = 1 - q_{n_{m-1}-\rho n_{m-1}} \leq 0. \]

If the preindex \( n'_m \) is even, then the condition of line 4 holds and in line 5 we assign the even index \( n_m = n'_m \), so that
\[ 1 - q_{n_{m-\rho n_{m}}} = 1 - q_n = 1 - q'n_m \leq T_m \]
and
\[ T_m < 1 - q'n_{m-1} = 1 - q_{n_m-1} < 1 - q_{n_m-2} = 1 - q_{n_{m-1}-\rho n_{m-1}} \leq 0. \]

In either case we have
\[ 1 - q_{n_{m-\rho n_{m}}} \leq T_m < 1 - q_{n_{m-1}-\rho n_{m-1}} \],
so that, by the definition (33) of \( I^*_n \), we conclude the desired containment \( T_m \in I^*_n \setminus I^*_{n-1} \).

**step-(ii):** We will assert the inequality
\[ T_m q_{n_{m-1}} > 0. \]  \hspace{1cm} (35)

For if, in line 3, we choose the preindex \( n'_m \) such that \([-1]^{n'_m-1} T_m > 0\), then the condition of line 4 holds and, in line 5, we will set the index \( n_m = n'_m \). We then apply the definition (8) of \( q^*_m \) to obtain the inequality
\[ T_m q_{n_{m-1}} = [-1]^{n_{m-1}} T_m q_{n_{m-1}} = [-1]^{n'_m-1} T_m q_{n_{m-1}} > 0. \]
If, on the other hand, the preindex \( n'_m \) is chosen such that \([-1]^{n'_m-1}T_m < 0\), then the condition of line–4 fails and, in line–13, we will set the index \( n_m := n'_m + 1 \). We will then have the inequality

\[
T_m q_{n_m-1}^* = [-1]^{n_m-1}T_m q_{n_m-1} = -[-1]^{n'_m-1}T_m q_{n_m-1} > 0.
\]

In either case, we deduce the desired inequality (35).

- **step-(iii):** We will prove that during step \( m + 1 \), we choose an index \( n_{m+1} \) which is smaller than the current index \( n_m \). This means that this iteration will terminate with a finite subindex \( M \) such that \( T_M = 0 \) and with the chain of index inequalities

\[
0 = n_M < n_{M-1} < \ldots < n_1 < n_0 = n.
\]

After we assign \( b_k := 0 \) in line–20 for all \( 1 \leq k \leq n \) such that \( k \notin \{n_m\}_{m=0}^M \), we apply the assignment of line–16 again and obtain the sum

\[
T = T_0 = b_{n_0}q_{n_0-1}^* + T_1 = b_{n_0}q_{n_0-1}^* + b_{n_1}q_{n_1-1}^* + T_2 = \ldots = \sum_{k=1}^n b_kq_k^* - 1,
\]

as stated in the hypothesis. We will prove that \( n_{m+1} < n_m \) by considering both cases of the condition in line–4 separately.

**the condition in line–4 holds:** we have \([-1]^{n'_m-1}T_m > 0\) and, in line–5, we will set the index \( n_m = n'_m \). The assignment of line–6 will now yield the inequality

\[
b'_m q_{n_m-1} \leq |T_m| < [b'_{n_m} + 1]q_{n_m-1}. \tag{36}
\]

We will consider the two cases \( T_m > 0 \) and \( T_m < 0 \) separately. If \( T_m > 0 \), then the identity (35) tells us that \( n'_m - 1 = n_m - 1 \) is even. If the condition of line–7 holds, that is, if \( b_{n_m} = b'_{n_m} \), then these inequalities and the assignment of line–16 now yield the inequalities

\[
0 = b'_m q_{n_m-1} - b_{n_m}q_{n_m-1} \leq |T_m| - b_{n_m}q_{n_m-1} = T_m - b_{n_m}q_{n_m-1} = T_{m+1}
\]

and

\[
T_{m+1} = T_m - b_{n_m}q_{n_m-1} = |T_m| - b'_m q_{n_m-1} < [b'_m + 1]q_{n_m-1} - b'_m q_{n_m-1} = q_{n_m-1},
\]

so that

\[
|T_{m+1}| + \chi < 0|T_{m+1}| = T_{m+1} \leq q_{n_m-1}.
\]

We will then choose the index \( n'_{m+1} \) in line–3 during the next step \( m + 1 \) such that \( n'_{m+1} \leq n_m - 1 \). If \( n'_{m+1} \leq n_m - 2 \), then we will always have

\[
n_{m+1} \leq n'_{m+1} + 1 \leq n_m - 1.
\]

If \( n'_{m+1} = n_m - 1 \), then the condition of line–3 will imply that

\[
q_{n'_m - 1} \leq |T_{m+1}| + \chi < 0|T_{m+1}| = T_{m+1}.
\]
Equality must be realized for if \( \eta_{m+1} < T_{m+1} \) then the assignment of line–16 and the condition of line–7, will yield the contradiction

\[ g_{m+2} = g_{m+1} < T_{m+1} = |T_{m+1}| = |T_m - b'_{n_m} q_{m-1}| + \chi<0[T_m - b'_{n_m} q_{m-1}] \leq q_{m-2}. \]

Then \( T_{m+1} = q_{m+1} = q_{m+2} \) and \( n_{m+1} \leq n_{m+1} + 1 = n_m - 1. \)

If the condition of line–7 fails, that is, if

\[ |T_m - b'_{n_m} q_{m-1}| + \chi<0[T_m - b'_{n_m} q_{m-1}] \geq q_{m-2} + 1 \]

and \( b_{n_m} = b'_{n_m} + 1, \) then the assignment in line–16 and the inequality \([36]\) yield the inequalities

\[ q_{m+2} - q_{m+1} - 1 \leq |T_m - b'_{n_m} q_{m-1}| + \chi<0[T_m - b'_{n_m} q_{m-1}] - q_{m-1} \]

\[ = T_m - b'_{n_m} q_{m-1} - q_{m-1} = T_m - b_{n_m} q_{m-1} = T_m - b_{n_m} q_{m-1} = T_{m+1} \]

and

\[ T_{m+1} = T_m - b_{n_m} q_{m-1} = T_m - b_{n_m} q_{m-1} = T_m - |b_{n_m} + 1| q_{m-1} < 0. \]

so that

\[ |T_m + 1| + \chi<0[T_m + 1] = -T_{m+1} + 1 \leq q_{m-1} - q_{m-2}. \]

If \( q_{m-2} = q_{-1} = 0, \) then \( q_{m-1} = q_0 = 1 \) and this inequality implies that \( T_{m+1} = 0, \) so that the iteration will terminate the next step with \( M = m + 1. \) We may assume then that \( q_{m-2} \geq 1, \) hence \( |T_{m+1}| + \chi<0[T_{m+1}] < q_{m-1}. \) We will then choose the index \( n'_{m+1} \) in line–3 during the next step \( m+1 \) such that \( n'_m \leq n_m - 1. \) If \( n'_m \leq n_m - 2, \) then we will always have

\[ n_{m+1} \leq n'_m + 1 \leq n_m - 1. \]

If \( n'_m = n_m - 1 \) then since \( n_m - 1 \) is even and since \( T_{m+1} < 0, \) the condition of line–4 will hold and we will assign \( n_{m+1} = n'_m = n_m - 1 \) in line–5.

Similarly, if \( T_m < 0, \) then the identity \([35]\) tells us that \( n'_m - 1 = n_m - 1 \) is odd. If the condition of line–7 holds, that is, if \( b_{n_m} = b'_{n_m}, \) then these inequalities and the assignment of line–16 now yield the inequalities

\[ -q_{m-1} = -|b'_{n_m} + 1| q_{m-1} + b'_{n_m} q_{m-1} - |T_m| + b_{n_m} q_{m-1} = T_m - b_{n_m} q_{m-1} = T_{m+1} \]

and

\[ T_{m+1} = -|T_m| + b_{n_m} q_{m-1} \leq -b'_{n_m} q_{m-1} + b_{n_m} q_{m-1} = 0. \]

This means that unless \( T_{m+1} = 0 \) and the iteration terminates, we have \( T_{m+1} < 0 \) and

\[ |T_{m+1}| + \chi<0[T_{m+1}] = -T_{m+1} + 1 < q_{m-1} + 1 \leq q_{m-1}. \]

Then in line–3 during the next step \( m + 1, \) we will choose the index \( n'_{m+1} \leq n_m - 1. \) If \( n'_m \leq n_m - 2, \) then we will always have

\[ n_{m+1} \leq n'_m + 1 \leq n_m - 1. \]
Solving for best linear approximates

A. Bourla

If $n_{m+1}' = n_m - 1$, then the condition of line–3 will imply that

$$q_{m+1}^* - 1 \leq |T_{m+1}| + \chi < 0 [T_{m+1}] = -T_{m+1} + 1.$$  

Equality must be realized for if $q_{m+1}' - 1 < -T_{m+1} + 1$ then the assignment of line–16 and the condition of line–7, will yield the contradiction

$$q_{m-2} = q_{m+1}' - 1 < -T_{m+1} + 1 = |T_m - b_m n_{m+1}' q_{m+1}'| + \chi < 0 [T_m - b_m n_{m+1}' q_{m+1}'] \leq q_{m-2}.$$  

If the condition of line–7 fails, that is, if

$$q_{m-2} < |T_m - b_m n_{m+1}' q_{m+1}'| + \chi < 0 [T_m - b_m n_{m+1}' q_{m+1}'] = |T_m + b_m n_{m+1}' q_{m+1}'| + \chi < 0 [T_m + b_m n_{m+1}' q_{m+1}']$$

and $b_n = b_{n+1}'$, then the assignment in line–16 and the inequality (36) yield the inequalities

$$0 < -|T_m| + [b_m n_{m+1}' q_{m+1}'] = T_m + b_m n_{m+1}' q_{m+1}' = T_m - b_m n_{m+1}' q_{m+1}' = T_{m+1}$$

and

$$T_{m+1} = T_m - b_m n_{m+1}' q_{m+1}' = -|T_m| + b_m n_{m+1}' q_{m+1}' \leq -b_m n_{m+1}' q_{m+1}' + b_m n_{m+1}' q_{m+1}' = q_{m-1},$$

so that

$$|T_{m+1}| + \chi < 0 [T_{m+1}] = T_{m+1} < q_{m-1}.$$  

We will then choose the index $n_{m+1}'$ in line–3 during the next step $m + 1$ such that $n_{m+1}' \leq n_m - 1$. If $n_{m+1}' \leq n_m - 2$, then we will always have

$$n_{m+1} \leq n_{m+1}' + 1 \leq n_m - 1.$$  

If $n_{m+1}' = n_m - 1$ then since $n_m - 1$ is odd and since $T_{m+1} > 0$, the condition of line–4 will hold and we will assign $n_{m+1} = n_{m+1}' = n_m - 1$ in line–5. We thus conclude the desired inequality for when the condition in line–4 holds

the condition in line–4 fails: we have $[-1] n_{m+1}' - 1 T_m < 0$ and, in line–5, we will set the index $n_m = n_{m+1}' + 1$ and line–14, we set the coefficient $b_{n+1} = 1$. We will consider the two cases $T_m > 0$ and $T_m < 0$ separately. If $T_m > 0$, then the identity (35) tells us that $n_{m} = n_{m} - 1$ is even. By the assignment of line–3, we have

$$q_{m-2} + 1 = q_{m-1} + 1 \leq |T_m| + \chi < 0 [T_m] = T_m \leq q_{m}' = q_{m-1}.$$  

so that by the assignment of line–16, we obtain the inequalities

$$q_{m-2} - q_{m-1} + 1 \leq T_m - q_{m-1} = T_m - b_{n_{m+1}} q_{m+1}' = T_{m+1}$$

and

$$T_{m+1} = T_m - b_{n_{m+1}} q_{m+1}' = T_m - q_{m-1} \leq 0.$$  

Thus we deduce the inequality

$$|T_{m+1}| + \chi < 0 [T_{m+1}] = -T_{m+1} + 1 \leq q_{n-1} - q_{n-2} \leq q_{n-1},$$

20
so that, during step $m+1$, we will choose the index $n_{m+1}$ in line-3 such that $n_{m+1}' \leq n_m - 2$. If $n_{m+1}' \leq n_m - 2$, then we will always have

$$n_{m+1} \leq n_{m+1}' + 1 \leq n_m - 1.$$  

If $n_{m+1}' = n_m - 1$, then since $n_m - 1$ is even, we have

$$[-1]^{n_{m+1}'} - 1 = [-1]^{n_m} < 0.$$  

Since we have also seen that $T_{m+1} \leq 0$, we deduce that $[-1]^{n_{m+1}'} T_{m+1} > 0$, so that the condition of line-4 will hold and we will assign in line-5 the index $n_{m+1} = n_{m+1}' = n_m - 1$.

Similarly, if $T_m < 0$, then the identity (33) tells us that $n_m' = n_m - 1$ is odd. By the assignment of line-3, we have

$$q_{n_m - 2} = q_{n_m - 1} < |T_m| + \chi_{<0}[T_m] = 1 - T_m \leq q_{n_m}' = q_{n_m - 1},$$

so that by the assignment of line-16, we obtain the inequalities

$$1 \leq T_m + q_{n_m - 1} = T_m - b_{n_m} q_{n_m - 1} = T_{m+1}$$

and

$$T_{m+1} = T_m + q_{n_m - 1} \leq 1 - q_{n_m - 2} + q_{n_m - 1}.$$  

Thus we deduce the inequality

$$|T_{m+1}| + \chi_{<0}[T_{m+1}] = T_{m+1} \leq q_{n_m - 1} - q_{n_m - 2} \leq q_{n_m - 1}.$$  

so that, during step $m+1$, we will choose the index $n_{m+1}$ in line-3 such that $n_{m+1}' \leq n_m - 1$. If $n_{m+1}' \leq n_m - 2$, then we will always have

$$n_{m+1} \leq n_{m+1}' + 1 \leq n_m - 1.$$  

If $n_{m+1}' = n_m - 1$, then since $n_m - 1$ is odd, we have

$$[-1]^{n_{m+1}'} - 1 = [-1]^{n_m} > 0.$$  

Since we have also seen that $T_{m+1} \geq 1$, we deduce that $[-1]^{n_{m+1}'} T_{m+1} > 0$, so that the condition of line-4 will hold and we will assign in line-5 the index $n_{m+1} = n_{m+1}' = n_m - 1$.

**step-(iv)**: We will prove that the digit string $\langle b_k \rangle_1^n$ satisfies condition-(i). Since $b_k \geq 1$ for all $k \in \{n_m\}_{M=0}^{M-1}$ and is otherwise zero, we have $b_n = b_{n_0} \geq 1$ and need only prove that $0 \leq b_{n_m} \leq a_{n_m}$ for all subindices $0 \leq m \leq M - 1$. This is clear whenever $n_m = n_m' + 1$ for by the assignment of line-13, we have $b_{n_m} = 1$. When $n_m = n_m'$, we cannot have $b_{n_m}' \geq a_{n_m} + 1$, for then we would use the recursive formula (5) and the inequalities of line-3 and (36) to obtain the contradiction

$$|T_m| \leq q_{n_m} - \chi_{<0}[T_m] \leq q_{n_m} = a_{n_m} q_{n_m - 1} + q_{n_m - 2} \leq |b_{n_m}' - 1| q_{n_m - 1} + q_{n_m - 2}$$

21
We will proceed according to the parity of $k$ the index

\[ b_{nm} = a_{nm} \text{ and } T_m > 0, \text{ then the inequality } (35) \text{ asserts that } q_{nm-1}^* = q_{nm-1}, \text{ so that, by the inequality, } (36) \text{ we have} \]

\[ T_m - b_{nm}^* q_{nm-1} = |T_m| - b_{nm}^* q_{nm-1} \geq 0. \]

The recursive formula (5) and the inequality of line–3 will now yield the inequality

\[ |T_m - b_{nm}^* q_{nm-1}| + \chi < 0 [T_m - b_{nm}^* q_{nm-1}] = |T_m - b_{nm}^* q_{nm-1}| \]

\[ = T_m - b_{nm}^* q_{nm-1} = T_m - a_{nm} q_{nm-1} \leq q_{nm} - a_{nm} q_{nm-1} = q_{nm-2}. \]

Similarly, if $b_{nm}^* = a_{nm}$ and $T_m < 0$, then the inequality (35) asserts that $q_{nm-1}^* = -q_{nm-1}$ so that, by the inequality (36), we have

\[ T_m - b_{nm}^* q_{nm-1} = -|T_m| + b_{nm}^* q_{nm-1} \leq 0. \]

The recursive formula (5) and the inequality of line–3 will now yield the inequality

\[ |T_m - b_{nm}^* q_{nm-1}| + \chi < 0 [T_m - b_{nm}^* q_{nm-1}] \leq -[T_m - b_{nm}^* q_{nm-1}] + 1 \]

\[ = |T_m| + b_{nm}^* q_{nm-1} + 1 \leq q_{nm} - \chi < 0 [T_m] + b_{nm}^* q_{nm-1} + 1 \]

\[ = q_{nm} - 1 - a_{nm} q_{nm-1} + 1 = q_{nm} - a_{nm} q_{nm-1} = q_{nm-2}. \]

In either case the condition in line–7 is satisfied, hence in line–8 we will assign $b_{nm} = b_{nm}^* = a_{nm}$. Conclude that $0 \leq b_k \leq a_k$ for all $k$, which is condition–(i).

**step-(v):** To assert condition–(ii), we assume by contradiction that for some running index $1 \leq k \leq M$, we have the two consecutive digits $b_k = a_k$ and $b_{k+1} \geq 1$. Since in line–20 we have assigned $b_k = 0$ for all $k \notin \{n_m\}_{m=0}^M$ and since the sequence of indices $\langle n_m \rangle_{m=0}^M$ is decreasing, we deduce that in two consecutive steps $m - 1$ and $m$ of this iteration we have assigned the indices $n_m$ and $n_m$ such that $n_m - 1 = n_m + 1 = k + 1$ and $n_m = k$. Since $n_{m+1} \leq n_m - 1 = k - 1$, the definition (33) of $I_n^*$ yields the containment

\[ T_{m+1} \in I_{n_{m+1}} \subset I_{n_m} = I_{k-1}^*. \] (37)

We will proceed according to the parity of $k$:

the index $k = n_m$ is odd; by definition (8) of $q_k^*$ and the inequality (35), we see that $T_m > 0$ and $T_{m-1} < 0$. We will consider the value of preindex $n_{m-1}'$ which, by the assignments of line–5 and line–13 is either $n_{m-1}$ or $n_{m-1} - 1$.

If, during step $m - 1$, we have assigned the odd preindex $n_{m-1}' = n_{[m-1]} - 1 = k$, then, since $T_{m-1} < 0$, we rewrite the first inequality of line–3 as

\[ q_{k-1} = q_{n_{m-1}'} - 1 < |T_{m-1}| + \chi < 0 |T_{m-1}| = 1 - T_{m-1} \]
and deduce that
\[ T_{m-1} \leq -q_{k-1}. \]  
(38)

Since \( n_{m-1} \neq n'_{m-1} \), then during step \( m - 1 \) the condition of line–4 failed, resulting in the assignment in line–14 of the digit \( b_{k+1} = 1 \), so that we obtain the inequality
\[ T_m = T_{m-1} - b_{k+1}q_k^* = T_{m-1} + q_k \leq q_k - q_{k-1}. \]

On the other hand, from the containment \( (37) \) we have \( 1 - q_{k-1} \leq T_{m+1} \), so that, during step \( m \), the setting in line–16 of the iterate \( T_{m+1} \) will yield the identity
\[ 1 + [a_k - 1]q_{k-1} \leq T_{m+1} = T_{m+1} + q_k q_{k-1} = T_m. \]

We combine the last two inequalities as the single working formula
\[ 1 + [a_k - 1]q_{k-1} \leq T_m \leq q_k - q_{k-1}. \]  
(39)

We will now consider the value of the preindex \( n'_{m} \), which is assigned during step \( m \) in line-3. If we had assigned \( n'_m := n_m - 1 = k - 1 \), then, since \( T_m > 0 \), we have \( T_m = |T_m| + \chi_{<0}[T_m] \) and thus we rewrite the inequalities of line–3 as
\[ q_{k-2} = q_n_{m-1} < T_m \leq q_{n'_m} = q_{k-1}. \]

But since \( k \) is odd and \( T_m > 0 \), we have \([-1]^{n_{m-1}}T_m = [-1]^{k-2}T_m < 0 \) so that the condition in line–4 must have failed. This results in the assignment in line–14 of the digit \( b_k = a_k = 1 \), so that the inequality \( (39) \) now yield the contradiction
\[ T_m \leq q_k - q_{k-1} = [a_k - 1]q_{k-1} + q_{k-2} = [b_k - 1]q_{k-1} + q_{k-2} = q_{k-2} = q_{n'_m - 1} < T_m. \]

Thus we must have assigned in line–5 the index \( n'_m = n_m = k \), so that the condition of line–4 must have held. This means that in line–6, we have assigned the predict \( b'_k \) such that \( b'_kq_{k-1} \leq |T_m| \), which, since \( k \) is odd and \( T_m > 0 \) means that
\[ b_k q_{k-1}^* = b'_k q_{k-1} \leq |T_m| = T_m. \]  
(40)

If \( b'_k = b_k = a_k \), then the recursive formula \( (5) \), the assignment in line–16 of the iterate \( T_m > 0 \), the fact that \( b_{k+1} = 1 \) and the inequality \( (38) \) will result in the inequality
\[ q_k - q_{k-2} = a_kq_{k-1} = b'_k q_{k-1} \leq T_m = T_{m-1} - b_{k+1}q_k^* = T_{m-1} + q_k \leq -q_{k-1} + q_k. \]

This yields the inequality \( q_{k-1} \leq q_{k-2} \), which is only possible when \( k = 2 \) (and \( q_1 = a_1 = 1 \)), contradicting the fact that \( k \) is odd. Then we must have \( b_k = b'_k + 1 \), which means that the condition of line–7 must have failed. By the inequality \( (40) \), the failure of this condition correspond to the inequality
\[ T_m - b'_k q_{k-1}^* = |T_m - b'_k q_{k-1}^*| + \chi_{<0}[T_m - b'_k q_{k-1}^*] > q_{k-2}. \]

Using this inequality, the inequality \( (39) \) and the fact that \( a_k = b_k = b'_k + 1 \), we have reached the contradiction
\[ q_{k-2} < T_m - b'_k q_{k-1}^* = T_m - b_k q_{k-1} + q_{k-1} \leq q_k - b_k q_{k-1} = q_k - a_k q_{k-1} = q_{k-2}. \]
This asserts condition–(ii) whenever the preindex \( n'_m - 1 = n_m - 1 = k \) is odd.

If, during step \( m - 1 \), we have assigned the even preindex \( n'_m - 1 = n_m - 1 = k + 1 \), then, the condition of line–4 must have held, so that, in line–6, we would have assigned the predigit \( b'_{n_m - 1} = b'_{k+1} \) satisfying the inequality \( b'_{k+1}q_k \leq |T_{m-1}| \). Since \( T_{m-1} < 0 \) and \( k \) is odd, we deduce the inequality

\[
-b'_{k+1}q_k = b'_{k+1}q_k \leq |T_{m-1}| = -T_{m-1}
\]

(41)

The digit \( b_{k+1} \) could not be equal to \( b'_{k+1} \), for then this inequality, the assignment of the iterate \( T_m \) line–16 and the facts that \( k \) is odd and \( T_m > 0 \) will yield the contradiction

\[
-b_{k+1}q_k < T_m - b_{k+1}q_k = T_m + b_{k+1}q_k = T_{m-1} = -|T_{m-1}| \leq -b'_{k+1}q_k = -b_{k+1}q_k.
\]

Thus, we must have that \( b_{k+1} = b'_{k+1} + 1 \), so that the condition in line–7 have failed. By the inequality (41), we have

\[
1 + q_{k-1} \leq \chi < |T_{m-1} - b'_{k+1}q_k^*| + |T_{m-1} - b'_{k+1}q_k^*| = 1 - T_{m-1} + b_{k+1}q_k^* = 1 - T_{m-1} - b'_{k+1}q_k,
\]

so that, the assignment of the iterate \( T_m \) in line–16 will now imply that

\[
T_m = -q_{k-1} + T_m - 1 + [1 + q_{k-1}] \leq -q_{k-1} + [T_m - T_{m-1}] - b'_{k+1}q_k
\]

\[
= -q_{k-1} - b_{k+1}q_k^* - b'_{k+1}q_k = -q_{k-1} + b_{k+1}q_k - [b_{k+1} + 1]q_k = q_k - q_{k-1}.
\]

(42)

Since \( n_m = n'_m = k \) is odd and since \( T_m > 0 \), we see that

\[
[-1]n'_m - 1 T_m = [-1]^{k-1} T_m > 0,
\]

so that, during step \( m \), the condition of line–4 must have held. Then in line–5 we would assign the predigit \( b_k \) such that

\[
b_k q_{k-1} \leq |T_m| = T_m
\]

(43)

We will now consider the two possible outcomes of the condition in line–7.

If, during step \( m \), the condition of line–7 held, we would have assigned in line–8 the digit \( b_k := b'_{k} \) satisfying the inequality

\[
b_k q_{k-1} = b'_{k}q_{k-1} \leq |T_m| = T_m.
\]

Using the recursive formula (5) and the inequality (42), we would obtain the inequality

\[
q_{k} = q_{k-2} + a_{k}q_{k-1} = q_{k-2} + b_{k}q_{k-1} \leq q_{k-2} + T_m \leq q_{k-2} - q_{k-1} + q_{k},
\]

hence would deduce that \( q_{k-1} \leq q_{k-2} \). This is only possible when \( k = 2 \) (and \( q_1 = a_1 = 1 \)), which contradicts the fact that \( k \) is odd. On the other hand, if, during step \( m \), the condition of line–7 failed, that is, if

\[
|T_m - b'_{k}q_{k-1}| + \chi < 0[T_m - b'_{k}q_{k-1}] = |T_m - b'_{k}q_{k-1}^*| + \chi < 0[T_m - b'_{k}q_{k-1}^*] > q_{k-2},
\]

then

\[
|T_m - b'_{k}q_{k-1}| + \chi < 0[T_m - b'_{k}q_{k-1}] = |T_m - b'_{k}q_{k-1}^*| + \chi < 0[T_m - b'_{k}q_{k-1}^*] > q_{k-2},
\]

(44)
then we would assign in line–10 the digit \( b_k := b'_k + 1 \). The fact that \( b_k = a_k \), that \( T_m > 0 \) and the inequality (43) will now imply that
\[
q_{k-2} < |T_m - b'_k q_{k-1}| + \chi < 0[T_m - b'_k q_{k-1}] = T_m - b'_k q_{k-1} = T_m - [a_k - 1]q_{k-1},
\]
so that, after applying the recursive formula (5) and inequality (42), we would obtain the contradiction
\[
q_{k-2} < T_m - [a_k - 1]q_{k-1} < -q_{k-1} + q_k - [a_k - 1]q_{k-1} = q_{k-2}.
\]
This contradiction asserts condition–(ii) whenever we have assigned during step \( m - 1 \) the preindex \( n'_{m-1} = n_{m-1} = k + 1 \). In tandem with the previous paragraph we conclude the assertion of condition–(ii) whenever \( k \) is odd.

If, during step \( m - 1 \), we have assigned the even preindex \( n'_{m-1} = n_{[m-1]} - 1 = k \), then, since \( T_{m-1} > 1 \), we rewrite the last inequality of line–3 as
\[
q_{k-1} = q_{n'_{m-1} - 1} < |T_{m-1}| + \chi < 0[T_{m-1}] = T_{m-1},
\]
and deduce that
\[
q_{k-1} + 1 \leq T_{m-1}. \tag{44}
\]

Since \( n_{m-1} \neq n'_{m-1} \), then during step \( m - 1 \) the condition of line–4 failed, resulting in the assignment in line–14 of the digit \( b_{k+1} = 1 \), so that we obtain the inequality
\[
1 + q_{k-1} - q_k \leq T_{m-1} - q_k - T_{m-1} - b_{k+1} q^*_k = T_m
\]
On the other hand, from the containment (37) we have \( T_{m+1} \leq q_{k-1} \), so that, during step \( m \), the setting in line–16 of the iterate \( T_{m+1} \) will yield the identity
\[
T_m = T_{m+1} + b_k q_{k-1}^* = T_{m+1} - a_k q_{k-1} \leq [1 - a_k]q_{k-1}.
\]
We combine the last two inequalities as the single working formula
\[
1 + q_{k-1} - q_k \leq T_m \leq [1 - a_k]q_{k-1}. \tag{45}
\]
We will now consider the value of the the preindex \( n'_m \), which is assigned during step \( m \) in line-3. If we had assigned \( n'_m := n_m - 1 = k - 1 \), then, since \( T_m < 0 \), we have \( 1 - T_m = |T_m| + \chi < 0[T_m] \) and thus we rewrite the first inequality of line–3 as
\[
q_{k-2} = q_{n'_m - 1} < 1 - T_m
\]
But since \( k \) is even and \( T_m < 0 \), we have \([1]_{n'_m - 1} T_m = [-1]^{k-1} T_m > 0 \) so that the condition in line–4 must have failed. This results in the assignment in line–14 of the digit \( b_k = a_k = 1 \), so that the inequality (39) now yields the contradiction
\[
T_m < 1 - q_{k-2} = [1 - a_k]q_{k-1} + 1 - q_{k-2} = 1 + q_{k-1} - q_k \leq T_m.
\]
Thus we must have assigned in line–5 the index \( n'_m = n_m = k \), so that the condition of line–4 must have held. This means that in line–6, we have assigned the predigit \( b'_k \) such that \( b'_k q_{k-1} \leq |T_m| \), which, since \( k \) is even and \( T_m < 0 \) means that
\[
T_m = -|T_m| \leq -b'_k q_{k-1} = b_k q_{k-1}^*.
\]

We cannot have \( b'_k = b_k = a_k \), for then the recursive formula \((5)\), the assignment in line–16 of the iterate \( T_m < 0 \), the fact that \( b_{k+1} = 1 \) and the inequality \((38)\) will result in the inequality
\[
q_k - q_{k-2} = a_k q_{k-1} = b'_k q_{k-1} \leq |T_m| = -T_m = -T_{m-1} + b_{k+1} q_k^*
\]
\[
= -T_{m-1} + q_k < 1 - T_{m-1} + q_k \leq -q_{k-1} + q_k,
\]
which yields the contradiction \( q_{k-1} < q_{k-2} \). Then we must have \( b_k = b'_k + 1 \), which means that the condition of line–7 must have failed, which means that
\[
1 - [T_m - b'_k q_{k-1}^*] = \chi < 0 |T_m - b'_k q_{k-1}^*| + |T_m - b_k q_{k-1}^*| > q_{k-2}.
\]

Using this inequality, the inequality \((45)\) and the fact that \( a_k = b_k = b'_k + 1 \), we have reached the contradiction
\[
q_{k-2} < 1 - T_m + b'_k q_{k-1}^* = 1 - T_m - [b_k - 1] q_{k-1} \leq +q_k - b_k q_{k-1} = q_k - a_k q_{k-1} = q_{k-2}.
\]

This asserts condition–(ii) whenever the preindex \( n'_{m-1} = n_{m-1} - 1 = k \) is even.

If, during step \( m - 1 \), we have assigned the odd preindex \( n'_{m-1} = n_{m-1} = k + 1 \), then, the condition of line–4 must have held, so that, in line–6, we would have assigned the predigit \( b'_{n_{m-1}} = b'_{k+1} \) satisfying the inequality \( b'_{k+1} q_k \leq |T_{m-1}| \). Since \( T_{m-1} > 0 \) and \( k \) is even we deduce the inequality
\[
b'_{k+1} q_k = b'_{k+1} q_k \leq |T_{m-1}| = T_{m-1}
\]

The digit \( b_{k+1} \) could not be equal to \( b'_{k+1} \), for then this inequality, the assignment of the iterate \( T_m \) line–16 and the facts that \( T_m < 0 \) and \( k \) is even will yield the contradiction
\[
b_{k+1} q_k = b'_{k+1} q_k \leq T_{m-1} = T_m + b_{k+1} q_k^* T_m + b_{k+1} q_k = b_{k+1} q_k.
\]

Thus, we must have that \( b_{k+1} = b'_{k+1} + 1 \), so that the condition in line–7 have failed. By the inequality \((47)\), we have
\[
q_{k-1} < \chi < 0 |T_{m-1} - b'_{k+1} q_k^*| + |T_{m-1} - b'_{k+1} q_k^*| = T_{m-1} - b'_{k+1} q_k = T_{m-1} - b_{k+1} q_k,
\]
so that, the assignment of the iterate \( T_m \) in line–16 will now imply that
\[
-T_m = -T_m + q_k - q_{k-1} < [-T_m + T_{m-1}] - b'_{k+1} q_k - q_{k-1}
\]
\[
b_{k+1} q_k^* - b'_{k+1} q_k - q_{k-1} = b_{k+1} q_k - [b_{k+1} - 1] q_k - q_{k-1} = q_k - q_{k-1}.
\]
Since \( n_m = n'_m = k \) is even and since \( T_m < 0 \), we see that
\[
[-1]^{n'_m-1}T_m = [-1]^{k-1}T_m > 0,
\]
so that, during step \( m \), the condition of line–4 must have held. Then in line–5 we would assign the predigit \( b'_m \) such that
\[
b'_m q_{k-1} \leq |T_m| = -T_m
\] (49)

We will now consider the two possible outcomes of the condition in line–7.

If, during step \( m \), the condition of line–7 held, we would have assigned in line–8 the digit \( b_k := b'_k \) satisfying the inequality
\[
b_k q_{k-1} = b'_k q_{k-1} \leq |T_m| = -T_m.
\]
Using the recursive formula (5) and the inequality (48), we would obtain the inequality
\[
q_k = q_{k-2} + a_k q_{k-1} = q_{k-2} + b_k q_{k-1} \leq q_{k-2} - T_m < q_{k-2} + q_k - q_{k-1},
\]
which would deduce that \( q_{k-1} < q_{k-2} \), which is impossible. On the other hand, if, during step \( m \), the condition of line–7 failed, that is, if
\[
q_{k-2} < |T_m - b'_k q_{k-1}^*| + \chi < 0[T_m - b'_k q_{k-1}^*]|T_m + b'_k q_{k-1} + \chi < 0[T_m + b'_k q_{k-1}],
\]
we would assign in line–10 the digit \( b_k := b'_k + 1 \). The fact that \( b_k = a_k \), that \( T_m < 0 \) and the inequality (49) will now imply that
\[
1 + q_{k-2} \leq |T_m + b'_k q_{k-1}| + \chi < 0[T_m + b'_k q_{k-1}] = 1 - [T_m + b'_k q_{k-1}] = 1 - T_m - [a_k - 1]q_{k-1},
\]
so that, after applying the recursive formula (5) and the inequality (48), we would obtain the contradiction
\[
q_{k-2} \leq -T_m - [a_k - 1]q_{k-1} < q_k - q_{k-1} - [a_k - 1]q_{k-1} = q_{k-2}.
\]
This contradiction asserts condition–(ii) whenever we have assigned during step \( m - 1 \) the preindex \( n'_{m-1} = n_{m-1} = k + 1 \). In tandem with the previous paragraph we conclude the assertion of condition–(ii) whenever \( k \) is even.

- **Step-(vi):** We will show that the map which is algorithm 7 is a well defined integer system by proving it is one to one and onto the set of all finite right–admissible digit strings. To prove injectivity, we need to show that this representation is unique. We do so by unzipping the representation according to the parity of the running index into its positive and odd negative parts and invoking the uniqueness of the positive base–\( \alpha \) system of the previous section. Define the two positive integers
\[
T^+ := \sum_{k=0}^{[n/2]} b_{2k+1} q_{2k}^* = \sum_{k=0}^{[n/2]} b_{2k+1} q_{2k}, \quad T^- := -\sum_{k=1}^{[n/2]} b_{2k} q_{2k-1}^* = \sum_{k=1}^{[n/2]} b_{2k} q_{2k-1},
\]
so that $T = T^+ - T^-$. If $\left(\hat{b}_k\right)_{k=1}^{\hat{n}}$ is another right-admissible digit string with $T = \sum_{k=1}^{\hat{n}} \hat{b}_k q_{k-1}$, then the containment (34) asserts that $n = \hat{n}$. For all $1 \leq k \leq \lceil n/2 \rceil$, define the digits

$$c_k := \begin{cases} 0, & k \text{ is odd} \\ b_k, & k \text{ is even} \end{cases} \quad \hat{c}_k := \begin{cases} 0, & k \text{ is odd} \\ \hat{b}_k, & k \text{ is even}. \end{cases}$$

Since every other digit in the resulting digit strings $\langle c_k \rangle_{k=1}^{\lceil n/2 \rceil}$ and $\langle \hat{c}_k \rangle_{k=1}^{\lceil n/2 \rceil}$ is zero, we verify, according to definition (29), that both are left-admissible. Thus

$$\sum_{k=1}^{\lceil n/2 \rceil} c_k q_{k-1} = \sum_{k=1}^{\lceil n/2 \rceil} b_{2k} q_{2k-1} = T^- = T^+ - T = \sum_{k=0}^{\lceil n/2 \rceil} b_{2k+1} q_{2k} - \sum_{k=1}^{\lceil n/2 \rceil} \hat{b}_k q_{k-1}$$

$$= \sum_{k=0}^{\lceil n/2 \rceil} \left[b_{2k+1} - \hat{b}_{2k+1}\right] q_{2k} + \sum_{k=1}^{\lceil n/2 \rceil} \hat{b}_{2k} q_{2k-1} + \sum_{k=0}^{\lceil n/2 \rceil} \left[b_{2k+1} - \hat{b}_{2k+1}\right] q_{2k} + \sum_{k=1}^{\lceil n/2 \rceil} \hat{c}_k q_{k-1}.$$

Then the one to one correspondence of the map defined in algorithm 6 as stated in theorem 2.1 now guarantees that

$$0 = c_k = b_k - \hat{b}_k, \quad k \text{ is odd}$$

and

$$b_k = c_k = \hat{c}_k = \hat{b}_k, \quad k \text{ is even}.$$ Conclude that $b_k = \hat{b}_k$ for all $1 \leq k \leq n$ so that this representation must be unique. After proving that this map is one to one, proving it is also onto is straightforward. For if we desire the right-admissible digit string $\langle b_k \rangle_0^n$ as output, we just plug in $T := \sum_{k=1}^{n} b_k q_{k-1}$ as input. Therefore there is a one to one correspondence between $\mathbb{Z}$ and the set of right-admissible digit strings as claimed. This completes the proof.

For example, when $\alpha$ is the golden section, we have $\langle q_k \rangle_0^\infty := \langle 1, -1, 2, -3, 5, ... \rangle$ and are thus able to extend the Zeckendorf’s Theorem to include all integers. When $\alpha = \sqrt{2} - 1$ is the silver section, we have $\langle q_k \rangle_0^\infty := \langle 1, -2, 5, -12, 29, ... \rangle$ and $I_1^* = \{0, 1, 2\}$, $I_2^* = \{-4, ..., 2\}$, $I_3^* = \{-4, ..., 12\}$ and $I_4^* = \{-29, ..., 12\}$. The following tables displays how the digits behave when counting from -24 to 24 using this base:
Solving for best linear approximates

The $\alpha$–based real expansions we are about to introduce for the first time in a rigorous manner will utilize the conditional and absolute sequence of linear approximation coefficients as weight vectors. Interestingly enough, when we extend the base–$\pm \alpha$ systems to expand real numbers, the alternating base–$[\pm \alpha]$ system leads to an absolute expansion whereas the absolute base–$\alpha$ leads to an alternating expansion.

### 3.1 The Irrational Base Absolute Expansion

The absolute base–$\alpha$ expansion of the intercept $\beta \in [0, \alpha)$ is the limit $\ell \in \mathbb{Z}_{\geq 0}^*$ and the sum $\sum_{k=1}^{\ell} b_k \theta_{k-1}$, where the digit string $\langle b_k \rangle_{1}$ satisfies the following augmentation of the right–admissible conditions (32):

(i) $0 \leq b_k \leq a_k$ and $b_{\ell} \geq 1$
(ii) \( b_k = a_k \implies b_{k+1} = 0 \) for all \( 1 \leq k < \ell - 1 \)

(iii) If \( \ell = \infty \), then \( b_k \leq a_k - 1 \) for infinitely many odd and even indices \( k \)

In case the limit \( \ell \) is infinite, we read the condition \( b_\ell \geq 1 \) as the admissibility condition \( \limsup b_k \geq 1 \) of not ending with a tail of zeros. The next theorem asserts that there is a one to one correspondence between intercepts and right–admissible digit strings.

**Theorem 3.1.** Given any limit \( \ell \in \mathbb{Z}_{\geq 0}^* \) and any \( \alpha \)-admissible digit strings \( \langle b_k \rangle_\ell \), we have \( \sum_{k=1}^\ell b_k|\theta_{k-1}| \in [0, 1) \). Furthermore, for every intercept \( \beta \in [0, 1) \), there exists a unique limit \( \ell \in \mathbb{Z}_{\geq 0}^* \) and an right \( \alpha \)-admissible digit string \( \langle b_k \rangle_1 \) such that \( \beta = \sum_{k=1}^\ell b_k|\theta_{k-1}| \).

**Proof.** Given a limit \( \ell \in \mathbb{Z}_{\geq 0}^* \) and a right–admissible digit string \( \langle b_k \rangle_1 \), we first show that \( \sum_{k=1}^\ell b_k|\theta_{k-1}| \in [0, 1) \). If \( \ell \) is finite, then unless \( \ell = \beta = 0 \) is the vacuous expansion, we first pad this sequence an infinite tail of zeros by setting \( b_k := 0 \) for all \( k > \ell \). If \( b_1 \leq a_1 - 1 \), then we use identity (20) as well as right admissibility conditions–(i) and (iii) to obtain the inequality

\[
0 < \sum_{k=1}^\ell b_k|\theta_{k-1}| = \sum_{k=1}^\infty b_k|\theta_{k-1}| \leq \sum_{k=2}^\infty b_k|\theta_{k-1}| - [a_1 - 1]|\theta_0| + \sum_{k=1}^\infty b_k|\theta_{k-1}|
\]

\[
< [a_1 - 1]|\theta_0| + \sum_{k=2}^\infty a_k|\theta_{k-1}| = -\alpha - \sum_{k=1}^\infty a_k|\theta_{k-1}| = -\alpha + [1 + \alpha] = 1.
\]

If \( b_1 = a_1 \), then by condition–(ii), we must have \( b_2 = 0 \). Let \( n \geq 1 \) be the first index for which \( b_{2n+1} \leq a_{2n+1} - 1 \), so that \( b_{2k-1} = a_{2k-1} \) and \( b_{2k} = 0 \) for all \( 1 \leq k \leq n \) (the existence of such \( n \) is guaranteed by condition–(iii)). Then by identities (16), (19) and (21), we have

\[
0 < \sum_{k=1}^\infty b_k|\theta_{k-1}| = \sum_{k=1}^{2n} a_{2k-1}|\theta_{2k-2}| + b_{2n+1}|\theta_{2n}| + \sum_{k=2n+2}^\infty b_k|\theta_{k-1}|
\]

\[
\leq \sum_{k=0}^n a_{2k-1}|\theta_{2k-2}| + [a_{2n+1} - 1]|\theta_{2n}| + \sum_{k=2n+2}^\infty b_k|\theta_{k-1}|
\]

\[
< \sum_{k=0}^n a_{2k-1}|\theta_{2k-2}| + a_{2n+1}|\theta_{2n}| - |\theta_{2n}| + \sum_{k=2n+2}^\infty a_k|\theta_{k-1}|
\]

\[
= [1 - |\theta_{2n-1}|] + [\theta_{2n-1} - |\theta_{2n+1}|] - |\theta_{2n}| + [|\theta_{2n}| + |\theta_{2n+1}|] = 1.
\]

This asserts that \( \sum_{k=1}^\ell b_k|\theta_{k-1}| \in [0, 1) \).
The base–\(\alpha\) expansion of the intercept \(\beta \in [0, 1)\) is the output of the iteration scheme:

\begin{algorithm}
\begin{algorithmic}
\STATE \textbf{input} : the intercept \(\beta \in [0, 1)\)
\STATE \textbf{output}: the right–admissible digit string \(\langle b_k \rangle^\ell_1\)
\STATE set \(\beta_0 := \beta\), \(\ell := \infty\), \(k = 1\);
\WHILE {\(\beta_{k-1} > 0\)}
\STATE set \(b_k := \lfloor \beta_{k-1}/|\theta_{k-1}| \rfloor\);
\STATE set \(\beta_k := \beta_{k-1} - b_k|\theta_{k-1}|\);
\STATE set \(k := k + 1\);
\ENDWHILE
\STATE set \(\ell := k - 1\);
\end{algorithmic}
\end{algorithm}

This iteration may terminate with a finite value for \(\ell\) or continue indefinitely in which case \(\ell = \infty\). The assignment of \(b_k\) in line–4 yields the inequality

\[ b_k|\theta_{k-1}| \leq \beta_{k-1} < [b_k + 1]|\theta_{k-1}|, \quad k < \ell + 1, \quad (50) \]

From the assignment of line–4, we see that

\[ \frac{\beta_{k-1}}{|\theta_{k-1}|} = b_k + \frac{\beta_k}{|\theta_{k-1}|}, \quad k < \ell + 1, \]

that is, \(b_k\) and \(\beta_k\) are the quotient and remainder of the division of \(\beta_{k-1}\) by \(|\theta_{k-1}|\), hence

\[ \beta_k < |\theta_{k-1}|. \quad (51) \]

We cannot have \(b_k \geq a_k + 1\), for then this inequality and the inequalities \((18)\) and \((50)\) would imply the contradiction

\[ |\theta_{k-2}| < [a_k + 1]|\theta_{k-1}| \leq b_k|\theta_{k-1}| < \beta_{k-1} < |\theta_{k-2}| \quad k \geq 1. \]

Thus \(0 \leq b_k \leq a_k\) for all \(k < \ell + 1\). Unless \(\ell = \beta = 0\) is the vacuous expansion, then since the sequence \(\langle |\theta_k| \rangle^\infty_0\) is strictly decreasing to zero, we must have \(b_k \geq 1\) at least once. Thus condition–(i) is satisfied. A simple comparison of the sum \(\sum_{k=1}^{\ell} b_k|\theta_{k-1}|\) to the convergent series \(\sum_{k=1}^{\infty} a_k|\theta_{k-1}|\) of proposition \[1.1\] establishes its convergence. The assignment of line–4 provide us with the identity

\[ \beta_n = \sum_{k=n+1}^{\ell} b_k|\theta_{k-1}|, \quad 0 \leq n < \ell. \quad (52) \]

and confirms that \(\beta = \beta_0 = \sum_{k=1}^{\ell} b_k|\theta_{k-1}|\). We have already proved that the digit string \(\langle b_k \rangle^\ell_1\) satisfies condition–(i), to show it is right–admissible, we verify it satisfies conditions–(ii) and (iii) as well.
To establish condition–(ii), suppose by contradiction that \( b_k = a_k \) yet \( b_{k+1} \geq 1 \). Then we use the recursive formula (12) and the assignment of line–4 and the inequality (51) to obtain the contradiction

\[
\beta_k = -b_{k+1}|\theta_k| + \beta_{k-1} \leq -|\theta_k| + \beta_{k-1} < -|\theta_k| + |\theta_{k-2}| < -|\theta_k| + |\theta_{k-2}| + \beta_{k+1} = a_k|\theta_{k-1}| + \beta_{k+1} = b_k|\theta_{k-1}| + \beta_{k+1} = \beta_k.
\]

This contradiction asserts condition–(ii).

To assert condition–(iii), first suppose by contradiction that \( \ell = \infty \) and \( b_k \leq a_k - 1 \) for only finitely many odd indices \( k \). Then there must exist some index \( n \geq 1 \) for which \( b_{2k-1} = b_{2k-1} \) for all \( k \geq n \). Condition–(ii) now implies that \( b_{2k} = 0 \) for all \( k \geq n + 1 \). By the recursive formula (12), the identity (25), the inequality (51) and the expansion (52) we arrive at the contradiction

\[
\beta_{2n+1} < |\theta_{2n}| = \sum_{k=n+1}^{\infty} a_{2k-1}|\theta_{2k-1}| = \sum_{k=n+1}^{\infty} b_{2k-1}|\theta_{2k-1}| = \sum_{k=n+1}^{\infty} b_{2k}|\theta_{2k-1}| = \sum_{k=2n+1}^{\infty} b_k|\theta_{k-1}| = \beta_{2n+1}.
\]

These contradictions establishes condition–(iii) and completes the proof.

**Corollary 3.2.** Every real number \( r \) is uniquely expanded base \( \alpha \) as the sum \( r = \sum_{k=0}^{\ell} b_k|\theta_{k-1}| \), where \( \ell \in \mathbb{Z}_{\geq 0}^\infty \), \( b_0 \in \mathbb{Z} \) and \( \langle b_k \rangle_1^\ell \) is right-admissible.

**Proof.** If \( r \) is an integer, we set \( \ell := 0 \), \( b_0 := r \) so that, by the definition of \( \theta_{-1} := -1 \) in the recursive formula (12), we obtain the expansion \( r = b_0|\theta_{-1}| \). Otherwise, we set \( b_0 := \lfloor r \rfloor \), apply the theorem to

\[
\beta_0 := r - b_0|\theta_{-1}| = r - \lfloor r \rfloor \in (0, 1)
\]

and obtain the desired expansion. If \( \langle b_k \rangle_0^\ell \) is another \( \alpha \)-expansion for \( r \), then \( \sum_{k=1}^{\ell} b_k|\theta_{k-1}| \in (0, 1) \), hence we must have \( b_0 = 0 = \lfloor r \rfloor \). The uniqueness of this expansion now guarantees that \( \ell = \ell' \) and \( b_k = b_k' \) for all \( 1 \leq k < \ell \).

**Theorem 3.3.** Fix the base \( \alpha \in (0, 1)\setminus\mathbb{Q} \) and let the sequences \( \langle p_k/q_k \rangle_1^\infty \) and \( \langle \theta_k \rangle_1^\infty \) be as in the recursive formulas (4), (5) and the definition (11). Apply algorithm \( 8 \) to the intercept \( \beta \in [0, 1) \) and obtain the index \( \ell \) and the right-admissible digit string \( \langle b_k \rangle_1^\ell \) such that \( \sum_{k=1}^{\ell} b_k|\theta_{k-1}| \). Define the terms

\[
T_n^\ell := \sum_{k=1}^{n} b_kq_k^{n-1}, \quad 1 \leq n < \ell + 1.
\]
Then for all \( n \), we have the containment \( T_n^\alpha \in I_n^* \) as in definition \( 33 \) and the inequality

\[
\beta - (T_n^\alpha) = \min_{T \in I_n} \{ \beta - (T \alpha) \}.
\]

That is, \( T_n^\alpha \) is the most accurate total under \( (\alpha, \beta) \)-approximate in the range \( I_n^* \) of approximates.

**Proof.** We will first the recursive formula \( 5 \) to verify that \( T_n^\alpha \in I_n^* \). When \( n = 0 \), we have \( T_0^\alpha = 0 \in \{0\} = I_0^* \). Otherwise, when \( n \geq 1 \) is odd we have the identity

\[
1 - q_{n-1} = q_0 - q_{n-1} = \sum_{k=1}^{[n-1]/2} a_{2k}[-q_{2k-1}] = \sum_{k=1}^{[n-1]/2} a_{2k}q_{2k-1}^* - \sum_{k=1}^{n/2} a_{2k}q_{2k-1}^* \leq \sum_{k=1}^{[n-1]/2} a_{2k}q_{2k-1}^* + \sum_{k=1}^{n/2} a_{2k-1}q_{2k-2}^* = \sum_{k=1}^{n} a_{k}q_{k-1}^* = T_n^\alpha \leq \sum_{k=1}^{[n+1]/2} a_{2k-1}q_{2k-2} = \sum_{k=2}^{n/2} a_{2k-1}q_{2k-2} = q_n.
\]

On the other hand, when \( n \geq 1 \) is even we have the identity

\[
1 - q_n = q_0 - q_n = \sum_{k=1}^{n/2} a_{2k}[-q_{2k-1}] = \sum_{k=1}^{n/2} a_{2k}q_{2k-1}^* - \sum_{k=1}^{n/2} a_{2k}q_{2k-1}^* + \sum_{k=2}^{n/2} a_{2k-1}q_{2k-2} = \sum_{k=1}^{n} a_{k}q_{k-1}^* = T_n^\alpha \leq \sum_{k=2}^{n/2} a_{2k-1}q_{2k-2} = \sum_{k=2}^{n/2} a_{2k-1}q_{2k-2} = q_{n-1}.
\]

In either case we assert that \( T_n^\alpha \in I_n^* \).

Next, we will show that the sequence \( (T_n^\alpha) \) converges to \( \beta \). From the identity \( 52 \), we know that for all \( n \geq 0 \) we have \( 0 \leq \beta_n \leq \beta \). In tandem with identity \( 15 \) we write

\[
T_n^\alpha + \beta_n = \sum_{k=1}^{n} b_kq_{k-1}^* + \sum_{k=n+1}^{\ell} b_k[q_{k-1}^* - p_{k-1}^*] = \sum_{k=1}^{\ell} b_k[\alpha - p_{k-1}^*] + \sum_{k=1}^{n} b_kp_{k-1}^* = \sum_{k=1}^{n} b_k[\theta_{k-1}] + \sum_{k=1}^{n} b_kp_{k-1}^* = \beta + \sum_{k=1}^{n} b_kp_{k-1}^*,
\]

hence

\[
T_n^\alpha = \beta + \sum_{k=1}^{n} b_kp_{k-1}^*.
\]

Since the last term is an integer and since \( 0 \leq \beta_n \leq \beta < 1 \), we deduce that

\[
(T_n^\alpha) = (\beta - \beta_n) = \beta - \beta_n.
\]

Since the sequence \( \{\beta_n\} \) is decreasing to zero, we conclude that the sequence \( \{\langle T_n^\alpha \rangle_{n=1}^\ell \} \) is increasing to \( \beta \).
Now suppose by contradiction that there exists a positive integer \( n \) and an integer \( T^\ast \in I^*_n \) satisfying
\[
0 < (T^\ast_n \alpha) < (T^\ast \alpha) < \beta. \tag{55}
\]
By theorem 2.2, there is a sequence of right \( \alpha \)-admissible digits \( \langle b'_k \rangle_n^1 \) such that \( T^\ast = \sum_{k=1}^{n} b'_k q_{k-1}^* \). We thus have
\[
(T^\ast \alpha) = \left( \sum_{k=1}^{n} b'_k q_{k-1}^* \alpha \right) = \left( \sum_{k=1}^{n} b'_k [q_{k-1}^* \alpha - p_{k-1}^*] \right) = \sum_{k=1}^{n} b'_k \theta_{k-1}.
\]
Because \( T^\ast \neq T^* \), the uniqueness of the base–\( \alpha \) numeration system guarantees that there are indices \( 1 \leq k \leq n \) for which \( b_k \neq b'_k \). We let \( m \) be the smallest index for which \( b_m \neq b'_m \).

If \( b'_m - b_m \leq -1 \) we use the identity (21) to derive the inequality
\[
(T^\ast \alpha) - (T^* \alpha) = \sum_{k=1}^{n} [b'_k - b_k] \langle \theta_{k-1} \rangle < -|\theta_{m-1}| + \sum_{k=m+1}^{n} [b'_k - b_k] |\theta_{k-1}|
\]
\[
\leq -|\theta_{m-1}| + \sum_{k=m+1}^{n} a_k |\theta_{k-1}| < -|\theta_{m-1}| + |\theta_{m-1}| = 0,
\]
contradicting our assumption for \( T^\ast \) in the inequality (55). Conclude that we must have \( b'_m > b_m \), hence
\[
(T^\ast \alpha) - (T^* \alpha) = \sum_{k=1}^{n} [b'_k - b_k] \theta_{k-1} > |\theta_{m-1}|.
\]
This inequality in tandem with The inequalities (51) and (55) as well as the identity (54) will now yield the contradiction
\[
|\theta_{n-1}| \leq |\theta_{m-1}| < (T^\ast \alpha) - (T^* \alpha) < \beta - (T^* \alpha) = \beta_n < |\theta_{n-1}|.
\]
We conclude that \( T^* \) is the best total under \( \alpha \)-approximate to \( \beta \) in \( I^*_n \).

**Corollary 3.4.** Assuming the hypothesis of the previous theorem, the sequence \( \langle T^\ast_n \rangle^\ell_1 \) of definition (53) is the solution to the total under linear approximate problem.

**Proof.** Since \( T^\ast_n \in I^*_n \), condition–(i) of the solution definition (28) is satisfied by the definition (33) of the interval \( I^*_n \). If \( n < \ell \), then the validity of the iteration condition in line–2 of algorithm 8 asserts that \( \beta_n > 0 \). The identity (54) and the inequality (51) now establishes the inequalities
\[
0 < \beta_n = \beta - (T^\ast_n \alpha) = \beta_n < |\theta_{n-1}|,
\]
which is condition–(ii). Finally, if \( \ell \) is finite, then from the termination of this iteration condition and the assignment in line–7 of algorithm 8 we see that \( \beta_\ell = 0 \). The identity (54) now asserts the identity
\[
\beta - (T^{\ast}_\ell \alpha) = \beta - [\beta - \beta_\ell] = 0,
\]
so that this sequence satisfies condition–(iii) as well.
The proof for the over approximate variant is an immediate consequence:

**Corollary 3.5.** Apply algorithm 8 with input base $\alpha$ and seed $1 - \beta$ and obtain the output limit $\hat{\ell}$ and right–admissible digit string $\hat{b}_k$. Define the terms $T_n := -\sum_{k=1}^{n} \hat{b}_k q_{k-1}^*$ for all $0 \leq n < \ell + 1$. Then the sequence $\langle T_n \rangle_0^\ell$ is the solution sequence to the total over $(\alpha, \beta)$–linear variant.

**Proof.** By the previous corollary, the sequence $\langle -T_n \rangle = \sum_{k=1}^{n} \hat{b}_k q_{k-1}^*$ is the solution sequence to the total under $(\alpha, 1-\beta)$–approximate variant. Thus the iterates $(-T_n^\ell \alpha)$ monotonically increase to $1 - \beta$ and the approximate sequence $\langle -T_n \rangle$ satisfies the conditions (28) in the definition of a general solution. Since for all $r \in \mathbb{R}$, we have $\| -r \| = \| r \|$, the terms $\langle T_n \rangle$ must also satisfy these conditions. Since $(-r) = 1 - (r)$, we have

$$(T_n^\ell \alpha) = 1 - (-T_n^\ell \alpha),$$

hence these terms also decrease to $\beta$. Therefore the sequence $\langle T_n \rangle$ is the desired solution sequence.

3.2 The Irrational Base Alternating Expansion

The alternating base–$[-\alpha]$ expansion of the shifted intercept $\gamma \in [-\alpha, 1 - \alpha)$ is the limit $\ell \in \mathbb{Z}_{\geq 0}$ and the sum $\sum_{k=1}^{\ell} c_k \theta_{k-1}$, where the digit string $\langle c_k \rangle_1^\ell$ satisfies the following augmentation of the left–admissible conditions (29):

(i) $0 \leq c_k \leq a_k$ for all $k < \ell + 1$ and $c_\ell \geq 1$

(ii) $c_k = a_k \implies c_{k-1} = 0$, $2 \leq k < \ell + 1$

(iii) $c_1 \leq a_1 - 1$ and if $\ell = \infty$ then $c_k \leq a_k - 1$ for infinitely many odd indices $k$

In case the limit $\ell$ is infinite, we read the condition $c_{\ell} \geq 1$ as the admissibility condition $\limsup \langle c_k \rangle \geq 1$ of not ending with a tail of zeros. The next theorem asserts that there is a one to one correspondence between shifted intercepts and left–admissible digit strings.

**Theorem 3.6.** Given any limit $\ell \in \mathbb{Z}_{\geq 0}^*$ and any left–admissible digit strings $\langle c_k \rangle_1^\ell$, we have $\sum_{k=1}^{\ell} c_k \theta_{k-1} \in [-\alpha, 1 - \alpha)$. Furthermore, for every shifted intercept $\gamma \in [-\alpha, 1 - \alpha)$, there exists a unique limit $\ell \in \mathbb{Z}_{\geq 0}^*$ and a $(-\alpha)$–admissible digit string $\langle c_k \rangle_1^\ell$ such that $\gamma = \sum_{k=1}^{\ell} c_k \theta_{k-1}^\ell$.

**Proof.** Given a limit $\ell \in \mathbb{Z}_{\geq 0}^*$ and a left–admissible digit string $\langle c_k \rangle_1^\ell$, we first show that $\sum_{k=1}^{\ell} c_k \theta_{k-1} \in [-\alpha, 1 - \alpha)$. If $\ell$ is finite, then unless $\ell = \gamma = 0$ is the vacuous expansion, we first pad this sequence an infinite tail of zeros by setting $c_k := 0$ for all $k > \ell$. Using the left–admissibility condition–(i) and the identity (23), we obtain the inequality

$$-\alpha = \sum_{k=1}^{\infty} a_{2k} \theta_{2k-1} = -\sum_{k=1}^{\infty} a_{2k} |\theta_{2k-1}| \leq -\sum_{k=1}^{\infty} c_{2k} |\theta_{2k-1}|$$
Solving for best linear approximates

\[ \leq - \sum_{k=1}^{\infty} c_{2k}|\theta_{2k-1}| + \sum_{k=0}^{\infty} c_{2k+1}|\theta_{2k}| = \sum_{k=1}^{\infty} c_k \theta_{k-1} = \sum_{k=1}^{\ell} c_k \theta_{k-1}, \]

whereas, condition (iii) and the identity (24) yield the inequality

\[ \sum_{k=1}^{\ell} c_k \theta_{k-1} = \sum_{k=1}^{\infty} c_k \theta_{k-1} \leq \sum_{k=1}^{\infty} c_{2k+1} |\theta_{2k}| \leq \sum_{k=0}^{\infty} c_{2k+1} |\theta_{2k}| \leq [a_1 - 1]|\theta_0| + \sum_{k=0}^{\infty} a_{2k+1} |\theta_{2k}| = -\alpha + 1. \]

This asserts that \( \sum_{k=1}^{\ell} c_k \theta_{k-1} \in [-\alpha, 1 - \alpha) \).

The base–[-\alpha] expansion of the shifted intercept \( \gamma \in [-\alpha, 1 - \alpha) \) is the output of the iteration scheme:

**Algorithm 9: the alternating base–[-\alpha] expansion**

**input**: the shifted intercept \( \gamma \in [-\alpha, 1 - \alpha) \)

**output**: the left–admissible digit string \( \langle c_k \rangle_{\ell} \)

1. set \( \gamma_0 := \gamma, \ell := \infty, k := 1; \)
2. while \( \gamma_{k-1} \neq 0 \) do
3.     if \( \gamma_{k-1} \theta_{k-1} > 0 \) then
4.         set \( c_k' := \lfloor \gamma_{k-1}/\theta_{k-1} \rfloor; \)
5.         if \( |\gamma_{k-1} - c_k' \theta_{k-1}| > |\theta_k| \) then
6.             set \( c_k := c_k' + 1; \)
7.         else
8.             set \( c_k := c_k'; \)
9.     else
10.    set \( c_k := 0; \)
11.    set \( \gamma_k := \gamma_{k-1} - c_k \theta_{k-1}; \)
12.    set \( k := k + 1; \)
13. end
14. set \( \ell := k - 1; \)

This iteration may terminate with a finite value for \( \ell \) or continue indefinitely in which case \( \ell = \infty \). We first prove by induction that the iterate \( \gamma_k \) will satisfy the containment

\[ \gamma_k \in [-\theta_{k-\rho_k}, -\theta_{k-1+\rho_k}) \quad 0 \leq k < \ell + 1, \]

which, in tandem with inequality (17), provide us with the inequality

\[ \|\gamma_n\| = |\gamma_n| \leq |\theta_{n-1}|, \quad 1 \leq n < \ell + 1. \]

Using the definitions \( \theta_{-1} := -1 \) and \( \theta_0 := \alpha \) in the recursive formula (12), we verify the base \( k = 0 \) case as the hypothesis condition \( \gamma_0 = \gamma \in [-\alpha, 1) \). After assuming this containment for the index \( k - 1 \geq 0 \), we prove its validity for the index \( k \) by considering its parity.
\[ \rho_k = 1: \text{ formula (14) implies the inequalities } \theta_k < 0 < \theta_{k-1} \text{ and the induction assumption implies the inequality } -\theta_{k-1} \leq \gamma_{k-1}. \text{ If } \gamma_{k-1} \text{ is negative, then the condition in line–5 will fail so that, in line–10, we will assign the coefficient } c_k := 0 \text{ and in line–11 the iterate } \gamma_k := \gamma_{k-1} \in [-\theta_{k-1}, 0). \text{ If } \gamma_{k-1} \text{ is positive, then } 0 \leq |\gamma_{k-1}/\theta_{k-1}| < \gamma_{k-1}/\theta_{k-1}, \text{ hence } 0 < \gamma_{k-1} - |\gamma_{k-1}/\theta_{k-1}| \theta_{k-1}. \text{ This allows us to rewrite the condition in line–5 as}
\]

"if \[ \gamma_{k-1} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| \theta_{k-1} \geq -\theta_k \] then"

If this condition fails, then in line–6 we will assign the coefficient \( c_k := c'_k = |\gamma_{k-1}/\theta_{k-1}| \), so that after the assignment in line–11, we will derive the inequalities

\[ 0 \leq \gamma_k := \gamma_{k-1} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| \theta_{k-1} < -\theta_k. \]

If this condition holds, then we assign in line–6 the coefficient \( c_k := c'_k + 1 \), asserting the containment

\[ \frac{\gamma_k}{\theta_{k-1}} = \frac{\gamma_{k-1} - c_k \theta_{k-1}}{\theta_{k-1}} = \frac{\gamma_{k-1}}{\theta_{k-1}} - |c'_k + 1| = \frac{\gamma_{k-1}}{\theta_{k-1}} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| - 1 = \left( \frac{\gamma_{k-1}}{\theta_{k-1}} \right) \in [-1, 0) \]

and deduce the inequalities \(-\theta_{k-1} \leq \gamma_k < 0\). For all these cases, we see that \( \gamma_k \in [-\theta_{k-1}, -\theta_k) \), which is the desired containment (56) applied to the odd parity \( \rho_k = 1 \).

\[ \rho_k = 0: \text{ formula (14) implies the inequalities } \theta_{k-1} < 0 < \theta_k \text{ and the induction assumption implies the inequality } \gamma_{k-1} < -\theta_{k-1}. \text{ If } \gamma_{k-1} \text{ is positive, then the condition in line–3 will fail so that, in line–10, we will assign the coefficient } c_k := 0. \text{ The assignment of line–11 will then yield the iterate}
\]

\[ \gamma_k := \gamma_{k-1} \in (0, -\theta_{k-1}). \]

If \( \gamma_{k-1} \text{ is negative, then } 0 \leq |\gamma_{k-1}/\theta_{k-1}| < \gamma_{k-1}/\theta_{k-1}, \text{ hence } 0 < \gamma_{k-1} - |\gamma_{k-1}/\theta_{k-1}| \theta_{k-1}. \) This allows us to rewrite the condition in line–5 as

"if \[ \gamma_{k-1} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| \theta_{k-1} \geq \theta_k \] then"

If this condition fails, then in line–6 we will assign the coefficient \( c_k := c'_k = |\gamma_{k-1}/\theta_{k-1}| \), so that after the assignment in line–11, we will derive the inequalities

\[ 0 \leq \gamma_k := \gamma_{k-1} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| \theta_{k-1} < \theta_k. \]

If this condition holds, then we assign in line–6 the coefficient \( c_k := c'_k + 1 \), asserting the containment

\[ \frac{\gamma_k}{\theta_{k-1}} = \frac{\gamma_{k-1} - c_k \theta_{k-1}}{\theta_{k-1}} = \frac{\gamma_{k-1}}{\theta_{k-1}} - |c'_k + 1| = \frac{\gamma_{k-1}}{\theta_{k-1}} - \left| \frac{\gamma_{k-1}}{\theta_{k-1}} \right| - 1 = \left( \frac{\gamma_{k-1}}{\theta_{k-1}} \right) \in [-1, 0) \]

and deduce the inequalities \( 0 < \gamma_k \leq -\theta_{k-1}. \) For all these cases, we see that \( \gamma_k \in [-\theta_k, -\theta_{k-1}) \), which is the desired containment (56) applied to the even parity \( \rho_k = 0 \),
finishing the proof.

To establish condition–(i), we first prove that \( c_k \geq 0 \). This is true if the condition in line–3 holds and \( 0 < \gamma_{k-1} \theta_{k-1} \), for then we must have \( c_k \geq c'_k = [\gamma_{k-1} \theta_{k-1}] \geq 0 \). Otherwise the failure of this condition leads to the assignment \( c_k := 0 \) in line–10. To show that \( c_k \leq a_k \), we first prove that \( c'_k \leq a_k \). For suppose by contradiction that \( c'_k \geq a_k + 1 \), then the recursion formula (16), the fact that \( \langle |\theta_k| \rangle \) is decreasing and the inequality (57) yield the inequality

\[
|\gamma_{k-1}| \leq \theta_{k-2} = a_k |\theta_{k-1}| + |\theta_k| < [a_k + 1] |\theta_{k-1}| \leq c'_k |\theta_{k-1}| = [\gamma_{k-1}/\theta_{k-1}] |\theta_{k-1}|,
\]

leading us to the contradiction \( \gamma_{k-1}/\theta_{k-1} < [\gamma_{k-1}/\theta_{k-1}] \). If \( c'_k \leq a_k - 1 \), then \( c_k \leq c'_k + 1 \leq a_k \), so we need only verify that if \( c'_k = a_k \) then \( c_k = a_k \) as well. We do so separately for each of the parities of \( k \).

\( \rho_k = 1 \): by the identity (14), we see that \( \theta_{k-1} > 0 \), hence the success of the condition in line–3 implies that \( \gamma_{k-1} > 0 \) as well. The assignment of line–4 now implies that

\[
a_k \theta_{k-1} = c'_k \theta_{k-1} = \frac{\gamma_{k-1}}{\theta_{k-1}} \theta_{k-1} < \gamma_{k-1},
\]

so that the recursive formula (12) and the inequalities (56) will yield the inequality

\[
|\gamma_{k-1} - c'_k \theta_{k-1}| = \gamma_{k-1} - a_k \theta_{k-1} < -\theta_{k-2} - a_k \theta_{k-1} = -\theta_k = |\theta_k|.
\]

\( \rho_k = 0 \): by the identity (14), we see that \( \theta_{k-1} < 0 \), hence the success of the condition in line–3 implies that \( \gamma_{k-1} < 0 \) as well. The assignment of line–4 now implies that

\[
a_k [-\theta_{k-1}] = c'_k [-\theta_{k-1}] = \frac{\gamma_{k-1}}{\theta_{k-1}} [-\theta_{k-1}] < -\gamma_{k-1},
\]

so that the recursive formula (12) and the inequalities (56) will yield the inequality

\[
|\gamma_{k-1} - c'_k \theta_{k-1}| = -\gamma_{k-1} + a_k \theta_{k-1} < \theta_{k-2} + a_k \theta_{k-1} = \theta_k = |\theta_k|.
\]

For both parities we see that the condition of line–5 must fail, leading us to assign in line–8 the digit \( c_k := c'_k = a_k \) as desired.

Since \( 0 \leq c_k \leq a_k \), a simple comparison of the sum

\[
\gamma = \gamma_0 = c_1 \theta_0 + \gamma_1 = c_1 \theta_0 + c_2 \theta_1 + \gamma_2 = \ldots = \sum_{k=1}^{\ell} c_k \theta_{k-1}, \tag{58}
\]

to the convergent series in proposition (14) establishes the convergence of this series to \( \gamma \). The assignment of line–4 provides us with the tail

\[
\gamma_n = \sum_{k=n+1}^{\ell} c_k \theta_{k-1}, \quad 0 \leq n < \ell + 1. \tag{59}
\]
To verify condition–(ii), we suppose that \( c_k \geq 1 \) and prove that we must have \( c_{k+1} \leq a_{k+1} - 1 \). We again proceed according to the parity of \( k \):

\( \rho_k = 1 \): According to the identity (14), we have \( \theta_k < 0 < \theta_{k-1} \). Because \( c_k \) is not zero, the condition in line–3 must have held, implying that \( \gamma_{k-1} > 0 \). Multiplying the inequalities \( 0 \leq \lceil \gamma_{k-1}/\theta_{k-1} \rceil < \gamma_{k-1}/\theta_{k-1} \) by \( \theta_{k-1} > 0 \) and applying the assignment in line–4 of \( c'_k \) will then yield the inequality

\[
0 < \gamma_{k-1} - \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor \theta_{k-1} = \gamma_{k-1} - c'_k \theta_{k-1}.
\]

If the condition in line–5 fails, we will assign in line–8 the digit \( c_k = c'_k \) and then, in line–11, the positive iterate \( \gamma_k := \gamma_{k-1} - c'_k \theta_{k-1} \). In this case the inequality \( \gamma_k \theta_{k-1} < 0 \) means that during the next iteration, the condition in line–3 will fail leading us to assign in line–10 the digit \( c_{k+1} := 0 \). On the other hand, if the condition in line–5 holds, we rewrite it as the inequality

\[
\gamma_{k-1} - c'_k \theta_{k-1} = \lceil \gamma_{k-1} - c'_k \theta_{k-1} \rceil > |\theta_k| = -\theta_k
\]

and, along with the assignment in line–6 of the digit \( c_k := c'_k + 1 \) and the assignment of the iterate \( \gamma_k \) in line–11, we obtain the inequality

\[
\gamma_k = \gamma_{k-1} - c_k \theta_{k-1} = \gamma_{k-1} - c'_k \theta_{k-1} - \theta_{k-1} > -\theta_{k-1} - \theta_k. \tag{60}
\]

After dividing this inequality by \( \theta_k < 0 \), we apply the recursion (12) and the inequalities \( 0 < -\theta_{k+1}/\theta_k < 1 \) to obtain the inequality

\[
\frac{\gamma_k}{\theta_k} < -\frac{\theta_{k-1}}{\theta_k} - 1 = \frac{a_{k+1} \theta_k - \theta_{k+1}}{\theta_k} - 1 = a_{k+1} - \frac{\theta_{k+1}}{\theta_k} - 1 < a_{k+1},
\]

so that \( c'_{k+1} := \lceil \gamma_k/\theta_k \rceil \leq a_{k+1} - 1 \). Therefore, we need only consider the possibility \( c'_{k+1} = a_{k+1} - 1 \) and prove that in this case the condition of line–5 must fail with the resulting assignment in line–8 of the digit \( c_{k+1} = c'_{k+1} = a_{k+1} - 1 \). Multiplying the inequality \( 0 \leq \lceil \gamma_k/\theta_k \rceil < \gamma_k/\theta_k \) by \( \theta_k < 0 \), applying the inequality (60) and the recursion (12) will yield the inequality

\[
|\gamma_k - c'_{k+1} \theta_k| = |\gamma_k - \left\lfloor \frac{\gamma_k}{\theta_k} \right\rfloor \theta_k| = c'_{k+1} \theta_k - \gamma_k < c'_{k+1} \theta_k + \theta_k + \theta_{k-1} = [a_{k+1} - 1] \theta_k + \theta_k + \theta_{k-1} = \theta_{k+1} = |\theta_{k+1}|.
\]

This inequality is precisely the desired failure of the condition in line–5, thus leading us to conclude that for all cases we have \( c_{k+1} \leq a_{k+1} - 1 \).

\( \rho_k = 0 \): According to the identity (14), we have \( \theta_{k-1} < 0 < \theta_k \). Because \( c_k \) is not zero, the condition in line–3 must have held, implying that \( \gamma_{k-1} < 0 \). Multiplying the inequality \( 0 \leq \lceil \gamma_{k-1}/\theta_{k-1} \rceil < \gamma_{k-1}/\theta_{k-1} \) by \( \theta_{k-1} < 0 \) and applying the assignment in line–4 of \( c'_k \) will then yield the inequality

\[
\gamma_{k-1} - c'_k \theta_{k-1} = \gamma_{k-1} - \left\lfloor \frac{\gamma_{k-1}}{\theta_{k-1}} \right\rfloor \theta_{k-1} < 0.
\]
If the condition in line–5 fails, we will assign in line–8 the digit $c_k = c_k'$ and then, in line–11, the negative iterate $\gamma_k := \gamma_{k-1} - c_k \theta_{k-1}$. In this case the inequality $\gamma_k \theta_k < 0$ means that during the next iteration, the condition in line–3 will fail leading us to assign in line–10 the digit $c_{k+1} := 0$. On the other hand, if the condition in line–5 holds, we rewrite it as the inequality
\[ c_k \theta_{k-1} - \gamma_{k-1} = |\gamma_{k-1} - c_k \theta_{k-1}| > |\theta_k| = \theta_k \]
and, along with the assignment in line–6 of the digit $c_k := c_k' + 1$ and the assignment of the iterate $\gamma_k$ in line–11, we obtain the inequality
\[ \gamma_k = \gamma_{k-1} - c_k \theta_{k-1} = \gamma_{k-1} - c_k \theta_{k-1} - \theta_{k-1} < \theta_k - \theta_{k-1}. \]  
(61)

After dividing this inequality by $\theta_k > 0$, we apply the recursion (12) and the inequalities $0 < -\theta_{k+1}/\theta_k < 1$ to obtain the inequality
\[ \frac{\gamma_k}{\theta_k} < -\frac{\theta_k}{\theta_k} - 1 = \frac{a_{k+1} \theta_k - \theta_{k+1}}{\theta_k} - 1 = a_{k+1} - \frac{\theta_{k+1}}{\theta_k} - 1 < a_{k+1}, \]
so that $c_{k+1} := [\gamma_k/\theta_k] \leq a_{k+1} - 1$. Therefore, we need only consider the possibility $c_{k+1} = a_{k+1} - 1$ and prove that in this case the condition of line–5 must fail with the resulting assignment in line–8 of the digit $c_{k+1} = c_{k+1}' = a_{k+1} - 1$. Multiplying the inequality $0 \leq [\gamma_k/\theta_k] < \gamma_k/\theta_k$ by $\theta_k > 0$, applying the inequality (61) and the recursion (12) will yield the inequality
\[ |\gamma_k - c_{k+1}' \theta_k| = |\gamma_k - \frac{\gamma_k}{\theta_k} \theta_k| = |\gamma_k - c_{k+1}' \theta_k| < \theta_k - \theta_{k-1} - c_{k+1}' \theta_k = \theta_k - \theta_{k-1} - |a_{k+1} - 1| \theta_k = -\theta_{k+1} = |\theta_{k+1}|. \]
This inequality is precisely the desired failure of the condition in line–5, thus leading us to conclude that for all cases we have $c_{k+1} \leq a_{k+1} - 1$. This asserts condition–(ii).

To establish condition–(iii), we suppose by contradiction that $\ell = \infty$ and $c_k \leq a_k - 1$ for only finitely many odd indices $k$. Then there must exist some lower bound $n \geq 1$ for which $c_{2k-1} = a_{2k-1}$ for all indices $k \geq n$. By condition–(ii), this implies that $c_{2k} = 0$ for all $k \geq n - 1$. By the tail expansion (26), the inequality (56) and the identity (59), we have reached the contradiction
\[ \gamma_{2n-1} < -\theta_{2n-1} = \sum_{k=n}^{\infty} c_{2k+1} \theta_{2k} = \sum_{k=n}^{\infty} c_{2k+1} \theta_{2k} + \sum_{k=n}^{\infty} c_{2k} \theta_{2k-1} = \sum_{k=2n}^{\infty} c_k \theta_{k-1} = \gamma_{2n-1}. \]
Finally, to prove this expansion is unique, we split $\gamma$ into its positive and negative parts and invoke the uniqueness of the absolute expansion. More precisely, suppose $\langle c_k \rangle_1^\ell$ is a left–admissible sequence such that $\gamma = \sum_{k=1}^{\ell} c_k \theta_{k-1}$. We first pad this sequence with an infinite tail of zeros whenever $\ell$ is finite and then unzip the resulting infinite sequence into its odd and even terms:
\[ b_k^0 := \begin{cases} c_k/2, & \rho_k = 0 \\ 0, & \rho_k = 1 \end{cases}, \quad b_k^1 := \begin{cases} 0, & \rho_k = 0 \\ c_{|k+1|}/2, & \rho_k = 1 \end{cases}. \]
Since the sequence \( \langle c_k \rangle \) is \((-\alpha)\)-admissible, both the sequences \( \langle b^{0}_{k} \rangle_{k=1}^{\infty} \) and \( \langle b^{1}_{k} \rangle_{k=1}^{\infty} \) satisfy condition–(i) and condition–(iii) of right–admissibility. Furthermore, since every other term in both the sequences is zero, they also satisfy condition–(ii). Thus both these sequences are right–admissible, leading to the unique representation of the factors

\[
\gamma^+ := \sum_{k=0}^{\infty} c_{2k+1} \theta_{2k} = \sum_{k=1}^{\infty} b^1_k |\theta_{k-1}|, \quad \gamma^- := -\sum_{k=1}^{\lfloor \ell/2 \rfloor} c_{2k} \theta_{2k-1} = \sum_{k=1}^{\infty} b^0_k |\theta_{k-1}|.
\]

If \( \langle c'_k \rangle_{k=1}^{\ell'} \) is another left–admissible sequence such that \( \gamma = \sum_{k=1}^{\ell'} c'_k |\theta_{k-1}| \), then, we also pad it with a tail of zeros when applicable. Then the uniqueness of the absolute expansion implies that

\[
\gamma^+ = \sum_{k=0}^{\infty} c_{2k+1} |\theta_{2k}| = \sum_{k=1}^{\infty} b^1_k |\theta_{k-1}| = \sum_{k=0}^{\infty} c'_{2k+1} |\theta_{2k}|.
\]

This means that

\[
\sum_{k=0}^{\infty} c_{2k} |\theta_{2k-1}| = \gamma + \gamma^+ = \gamma^- = \sum_{k=1}^{\infty} b^0_k |\theta_{k-1}| = \sum_{k=0}^{\infty} c'_{2k} |\theta_{2k-1}|,
\]

hence \( \langle c_k \rangle_{k=1}^{\infty} = \langle c'_k \rangle_{k=1}^{\infty} \). Furthermore, we must have \( \ell = \ell' \) for otherwise we will obtain two distinct representations for either

\[
\gamma^+ = \sum_{k=0}^{\lfloor \ell/2 \rfloor} c_{2k+1} \theta_{2k} = \sum_{k=0}^{\lfloor \ell'/2 \rfloor} c_{2k+1} \theta_{2k} \quad \text{or} \quad \gamma^- = \sum_{k=1}^{\lfloor \ell/2 \rfloor} c_{2k} \theta_{2k-1} = \sum_{k=1}^{\lfloor \ell'/2 \rfloor} c_{2k} \theta_{2k-1},
\]

contrary to the uniqueness of the absolute expansion, completing the proof. \( \blacksquare \)

**Corollary 3.7.** Every real number \( r \) can is uniquely expanded as the sum \( r = \sum_{k=0}^{\ell} c_k \theta_{k-1} \), where \( \ell \in \mathbb{Z}_{\geq 0}, c_0 \in \mathbb{Z} \) and \( \langle c_k \rangle_{k=1}^{\ell} \) is left–admissible.

**Proof.** If \( r \) is an integer we set \( \ell := 0, c_0 := -r \) and use the definition of \( \theta_{-1} := -1 \) in the recursive formula (12) to obtain the expansion \( r = c_0 \theta_{-1} \). Otherwise, we set \( c_0 := \lfloor r + \alpha \rfloor \) and apply algorithm [3.6] with the input

\[
\gamma_0 := r - \lfloor r + \alpha \rfloor = [r + \alpha - \lfloor r + \alpha \rfloor] - \alpha \in [-\alpha, 1 - \alpha).
\]

We thus obtain the left–admissible digit string \( \langle c_k \rangle_{k=1}^{\ell} \) with \( r = \sum_{k=0}^{\ell} c_k \theta_{k-1} \), which then yields the expansion

\[
r = -\lfloor r + \alpha \rfloor + \lfloor r - \lfloor r + \alpha \rfloor \rfloor = -c_0 + \gamma_0 = c_0 \theta_{-1} + \gamma_0 = \sum_{k=0}^{\ell} c_k \theta_{k-1}.
\]

\( \blacksquare \)
Corollary 3.8. Fix the base $\alpha \in (0, 1) \setminus \mathbb{Q}$ and let the sequences $\langle p_k/q_k \rangle_1^\infty$ and $\langle \theta_k \rangle_1^\infty$ be as in definitions (4), (5) and (11). Given the intercept $\beta \in [0, 1)$, apply algorithm 9 to the shifted intercept

$$\gamma := \beta - \alpha \in [-\alpha, 1 - \alpha]$$

and obtain the output index $\ell$ and the left–admissible digit string $\langle c_k \rangle_1^\ell$ such that $\gamma = \sum_{k=1}^\ell c_k \theta_{k-1}$. For all $0 \leq n < \ell + 1$, define the terms

$$S_n := 1 + \sum_{k=1}^n c_k q_{k-1} \geq 1.$$

Then the sequence $\langle S_n \rangle_0^\ell$ is a general solution sequence as in definition (28) to the forward two–sided $(\alpha, \beta)$ linear approximate problem.

Proof. The sum $\sum_{k=1}^n c_k q_{k-1}$ is of index $n$ whereas the unique representation of the denominator $q_n = \sum_{k=1}^{n+1} \delta_{[k,n+1]} q_{k-1}$ (where $\delta_{[i,j]} = 1$ when $i = j$ and is otherwise zero) is of index $n + 1$. By the uniqueness and monotonicity criteria of the base–$\alpha$ representation, as stated in the hypothesis of theorem 2.1, we thus have $S_n \leq q_n - 1$, which is condition–(i) in the solution definition (28).

By the identities (11), (58) and (59), we derive the equation

$$[S_n - 1] \alpha - \sum_{k=1}^n c_k p_{k-1} = \sum_{k=1}^n c_k q_{k-1} (\alpha - p_{k-1}) = \sum_{k=1}^n c_k \theta_{k-1} = \gamma - \gamma_n. \tag{62}$$

Using the fact that $\beta = \alpha + \gamma$ and that the sum $\sum_{k=1}^n c_k p_{k-1}$ is an integer, we have

$$\|\beta - S_n \alpha\| = \|\gamma - [S_n - 1] \alpha\| = \|\gamma - [S_n - 1] \alpha - \sum_{k=1}^n c_k p_{k-1}\| = \|\gamma_n\|.$$  

If $1 \leq n < \ell$, then the validity of iteration condition in line–2 of algorithm 9 asserts that $0 < \|\gamma_n\|$ and the inequality (57) asserts that $\|\gamma_n\| < |\theta_{n-1}|$. Thus

$$0 < \|\gamma_n\| = \|\beta - S_n \alpha\| < |\theta_{n-1}|, \quad 1 \leq n < \ell,$$

and the sequence $\langle S_n \rangle_0^\ell$ also satisfies condition–(ii). Finally, if $\ell$ is finite, then from the termination of iteration condition of line–2 and the assignment of line–14 in algorithm 9, we have $\gamma_\ell = 0$ so that from equation (62) we have

$$\|\beta - S_\ell \alpha\| = \|\gamma - [S_\ell - 1] \alpha\| = \|\gamma - [S_\ell - 1] \alpha - \sum_{k=1}^\ell c_k p_{k-1}\|$$

$$= \|\gamma_\ell\| = 0.$$

This asserts condition–(iii) in the solution definition (28) and the result follows. \hfill \blacksquare

Corollary 3.9. Apply algorithm 9 with the input $\gamma := 1 - \beta$ and obtain the output limit $\ell$ and left–admissible digit string $\langle \tilde{c}_k \rangle_1^\ell$. For all $1 \leq n < \ell + 1$, set the iterate $B_n := -1 - \sum_{k=1}^n \tilde{c}_k q_{k-1}$. Then the sequence $\langle B_n \rangle_1^\ell$ is the general solution sequence, as in definition (28), to the negative two–sided linear $(\alpha, \beta)$–approximate problem.
Proof. By the previous corollary, the sequence \( \left\langle \hat{S}_n := -B_n \right\rangle_1^\ell \) is the general solution sequence to the positive two-sided linear \((\alpha, 1 - \beta)\)-approximate variant, satisfying both conditions in the solution definition \(28\). Since \(|B_n| = \hat{S}_n \leq q_{n-1} - 1\), the sequence \( \langle B_n \rangle_1^\ell \) satisfies condition–(i) in the solution definition \(28\). Since for all real numbers \(r\) we have \(\| -r \| = \| r \| = \| 1 + r \|\), we apply the definition of the approximate \(B_n\) and use the result of the previous corollary to obtain the inequalities

\[
0 < \| \beta - B_n \alpha \| = \| -\beta + B_n \alpha \| = \| [1 - \beta] - \hat{S}_n \alpha \| < \| q_{n-1} \alpha \|, \quad 1 \leq n < \ell
\]

This asserts condition–(ii) as well. Finally, if \(\ell = \hat{\ell} < \infty\), then by the previous corollary we obtain the equality

\[
\| \beta - B_\ell \alpha \| = \| -\beta + B_\ell \alpha \| = \| [1 - \beta] - \hat{S}_\ell \alpha \| = 0,
\]

which is condition–(iii).

Acknowledgments:

This work could have not been completed without the guidance, encouragement and good company of Robbie Robinson from George Washington University.

References

[1] – Symmetry in the Sequence of Approximation Coefficients, The Proceedings of the AMS (141), 2013.

[2] Dario Castellanos, The Ubiquitous Pi, Mathematics Magazine (61), 1988, 148–163.

[3] K. Dajani, Ergodic Theory of Numbers, The Carus Mathematical Monographs volume (29), Mathematical Association of America, 2002.

[4] A. Ya. Khintchine, Continued fractions, P. Noordhoff Ltd. Groningen, 1963.

[5] S. Ito, H. Nakada, Approximations of real numbers by the sequence \(\{na\}\) and their metrical theory, Acta. Math. Hung. 52:1–2 1988, 91–100.

[6] A. Ostrowski, Bemerkungen zur theorie der diophantischen approximationen, Abh. Math. Semin. Hamburg Univ, 1:77–98, 1922.

[7] A. Robinson, Sturmian Expansions and Entropy, Integers (electronic journal) 11B, 2011.