EXPANDERS ARE ORDER DIAMETER NON-HYPERBOLIC

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Abstract. We show that expander graphs must have Gromov-hyperbolicity at least proportional to their diameter, with a constant of proportionality depending only on the expansion constant and maximal degree. In other words, expanders contain geodesic triangles which are $\Omega(\text{diam } \Gamma)$-thick.

1. Introduction

An expander graph is a finite graph of bounded degree in which all sets of vertices have large boundary (see e.g., [HLW]). A hyperbolic graph is a graph which behaves like a tree on large scales (see e.g., [Bo]).

It is shown in [Be] that sufficiently large graphs cannot be both expanders and hyperbolic. More precisely, expander graphs $\Gamma$ satisfy

$$\delta_{\Gamma} = \Omega \left( \frac{\log |\Gamma|}{\log \log |\Gamma|} \right),$$

where $\delta_{\Gamma}$ is the Gromov-hyperbolicity constant of the graph $\Gamma$, i.e., the minimal $\delta$ such that all geodesic triangles in $\Gamma$ are $\delta$-thin.

We modify the argument slightly to improve this result to

$$\delta_{\Gamma} = \Omega (\log |\Gamma|).$$

For a more precise statement, see Theorem 3.2.

One new consequence of this result is that a family of expanders cannot have a tree as an asymptotic cone.

A similar result is known for vertex-transitive graphs, that is, if $\Gamma$ is a finite vertex-transitive graph, then

$$\delta_{\Gamma} = \Omega (\text{diam } \Gamma).$$

See, e.g., [BS]. This fact can be used to give an elementary argument that hyperbolic groups only have finitely many finite subgroups, up to conjugacy [Br].

There is also a similar result for random graphs. A random $d$-regular graph with $n$ vertices is asymptotically almost surely an expander, and expanders share many properties with random graphs. In [B+], it is shown that random graphs have large almost-geodesic cycles. (It is open whether the same holds for true geodesic cycles.) Another consequence of their argument is that a random $d$-regular graph with $n$ vertices, $\Gamma$, asymptotically almost surely satisfies

$$\delta_{\Gamma} = \frac{1}{2} \text{diam}(\Gamma) - O(\log \log |\Gamma|) = \frac{1}{2} \text{diam}(\Gamma)(1 - o(1)).$$

The constant $1/2$ is the best possible, since for any graph $\Gamma$, the hyperbolicity constant $\delta_{\Gamma}$ is bounded above by $\text{diam}(\Gamma)/2$.

While an expander cannot be hyperbolic, it is possible that it has a high proportion of thin geodesic triangles. In [LT], Li and Tucci construct a family of expanders...
in which a positive proportion of geodesics pass through a particular vertex. It follows that a positive proportion of geodesic triangles are tripods, i.e., 0-thin. Indeed, their argument constructs expander families in which this proportion is arbitrarily close to 1, at the cost of worse expansion. This leaves the question: is there a fixed \( \delta \) and an expander family \( (\Gamma_i)_i \), in which geodesic triangles are \( \delta \)-thin asymptotically almost surely, i.e., in which the probability that a randomly chosen geodesic triangle in \( \Gamma_i \) is \( \delta \)-thin tends to 1 as \( |\Gamma_i| \to \infty \)?

2. Conventions

A graph is a finite undirected graph, which may have loops and repeated edges. We abuse notation and use \( \Gamma \) to denote both a graph and its set of vertices. A graph \( \Gamma \) is an \( h \)-expander if for every set \( S \subseteq \Gamma \) with \( |S| \leq |\Gamma|/2 \), there are at least \( h|S| \) vertices in \( \Gamma \setminus S \) adjacent to \( S \). A graph has valency bounded by \( d \) if every vertex in the graph has degree at most \( d \). We let \( B_{\Gamma,p}(r) \) denote the ball in \( \Gamma \) with center \( p \) and radius \( r \).

A \( \delta \)-hyperbolic space is a geodesic metric space in which all geodesic triangles are \( \delta \)-thin. That is, each side of a triangle is contained in a \( \delta \)-neighborhood of the other two sides. If \( X \) is a geodesic metric space, and \( p, q \in X \), we let \( \overline{pq} \) denote a geodesic segment between \( p \) and \( q \). This is a slight abuse of notation because such a path need not be unique, but this does not cause any issues in our arguments.

A graph can be thought of as a geodesic space in which each edge is a segment of length 1. A finite graph \( \Gamma \) is always \( \delta \)-geodesic for some \( \delta \), e.g., \( \delta \) equal to the diameter of the graph. We write \( \delta_\Gamma \) for the minimal such \( \delta \).

3. Proof of Main Result

In [Be], the argument proceeds by showing that, in an expander, removing a ball does not significantly affect expansion, so the diameter of the graph does not increase much. On the other hand, removing a ball in a hyperbolic graph increases distances by an amount exponential in the radius of the ball.

More precisely, let \( D \) be the diameter of an expander \( \Gamma \), i.e., the maximal distance between two points \( p, q \in \Gamma \). Note that \( D = \Theta(\log |\Gamma|) \). Removing a ball of radius \( \Theta(D) \) between \( p \) and \( q \) shrinks boundaries of balls centered at \( p \) and \( q \) by an additive amount \( \exp(\Theta(D)) \), which increases \( d(p,q) \) by at most a constant factor. However, in a \( \delta \)-hyperbolic graph, removing such a ball must increase \( d(p,q) \) by \( \exp(\Omega(D/\delta)) \). Hence,

\[
D + \exp(\Omega(D/\delta)) = O(D),
\]

and therefore

\[
\delta = \Omega(D/\log D).
\]

The \( \log D \) factor arises because we compare a multiplicative increase of \( d(p,q) \) in an expander to an additive increase of \( d(p,q) \) in a hyperbolic graph. We eliminate this factor by using a \( \Theta(D) \)-neighborhood of a \( \Theta(D) \)-length path between \( p \) and \( q \), instead of a ball of radius \( \Theta(D) \). With an appropriate choice of constants, removing such a cylinder from an expander still changes \( d(p,q) \) by at most a constant factor. However, in a \( \delta \)-hyperbolic graph, removing such a cylinder increases \( d(p,q) \) by a multiplicative factor of \( \exp(\Omega(D/\delta)) \). So we obtain

\[
D \exp(\Omega(D/\delta)) = O(D),
\]
and therefore
\[ \delta = \Omega(D). \]

The key lemma, then, is the increase of distances in a hyperbolic graph caused
by removing a cylinder. We prove this lemma using some elementary facts about
hyperbolic spaces, whose statements and proofs we defer to section 4.

**Lemma 3.1.** Let \( X \) be a \( \delta \)-hyperbolic space, and let \( p, q \in X \). Define \( D = d(p, q) \),
and choose \( p', q' \) on the geodesic segment \( \overline{pq} \) so that the points \( p, p', q', q \) are evenly spaced, that is,
\[ d(p, p') = d(p', q') = d(q', q) = D/3. \]
Let \( \gamma \) be a path in \( X \) which does not intersect the \( R \)-neighborhood of the geodesic
segment \( \overline{p'q'} \). Then
\[ \ell(\gamma) \geq DB^{2R/\delta} \]
where \( B \) is a universal constant, and \( R/\delta, D/\delta \) are taken sufficiently large.

**Proof of Lemma 3.1.** Consider the set of all pairs \((x, r)\) with \( x \) on the path \( \gamma \) and \( r \) on the geodesic \( \overline{pq} \), for which
\[ d(x, r) = \min_{s \in \overline{pq}} d(x, s), \]
i.e., \( r \) is the nearest point of \( \overline{pq} \) to \( x \). By Lemma 4.2, there is a constant \( C \) such
that every interval of length \( C\delta \) in \( \overline{pq} \) contains some such \( r \). If we let
\[ N = \left\lfloor \frac{D/3}{C\delta + K_0\delta} \right\rfloor, \]
we can find \( N \) intervals of length \( C\delta \) in \( \overline{p'q'} \), spaced more than \( K_0\delta \) apart, and pick
such a point \( r \) in each interval. So, there is a collection of \( N \) such pairs \((r_i, x_i)\) for
which the \( r_i \) are more than \( K_0\delta \) apart from each other. We may rearrange these
pairs so that the \( x_i \) occur in order of appearance on \( \gamma \).

Since the \( r_i \) are sufficiently far apart, Lemma 4.3 guarantees that each geo-
desic \( \overline{x_ix_{i+1}} \) passes within \( K_1\delta \) of \( \overline{p'q'} \). However, the path \( \gamma \) does not pass within \( R \)
of \( \overline{p'q'} \). Hence, some point on \( \overline{x_ix_{i+1}} \) is at distance at least \( R - K_1\delta \) from \( \gamma \). So, by
Lemma 4.3, the segment of \( \gamma \) between \( x_i \) and \( x_{i+1} \) has length at least \( \delta 2^{R/\delta - K_1} \).

There are \( N - 1 \) such segments, so the total length of \( \gamma \) is at least
\[ (N - 1) \delta 2^{R/\delta - K_1} \geq D \left( \frac{1}{3(C + K_0)} - \frac{2}{D/\delta} \right) 2^{R/\delta}. \]
Taking, e.g., \( B = (4(C + K_0))^{-1} \) gives us
\[ \ell(\gamma) \geq DB^{2R/\delta}, \]
for sufficiently large \( D/\delta \). \qed

We can now prove our result by arguing that in an expander, removing such a
cylinder does not increase distances too much.

**Theorem 3.2.** For any positive real number \( h \) and any positive integer \( d \), there is
a constant \( C_{d,h} \) such that for any \( h \)-expander \( \Gamma \) which has valency bounded by \( d \), we have
\[ \delta \Gamma > C_{d,h} \log |\Gamma|. \]
Proof. Choose $\alpha$ small enough that
\[(1 + h)^{1/3 - \alpha} < d^\alpha\]

Pick $p, q \in \Gamma$ such that $d(p, q)$ is divisible by 3 and as large as possible, and let $D = d(p, q)$. Let $p', q'$ be points on the geodesic segment from $p$ to $q$ such that $p, p', q, q'$ are equally spaced, i.e.
\[d(p, p') = d(p', q') = d(q', q) = D/3.\]

Let $C$ be the $(\alpha D)$-neighborhood of the geodesic segment $\overline{p'q'}$, and let $\tilde{\Gamma} = \Gamma \setminus C$ be the graph $\Gamma$ with this cylinder removed.

We consider the growth of the balls $B_{\tilde{\Gamma}, p}(r)$ in $\tilde{\Gamma}$ centered at $p$. Roughly speaking, by the time these balls reach the cylinder $C$, they are large enough that their growth is not significantly slowed down by the cylinder.

As long as $|B_{\tilde{\Gamma}, p}(r - 1)| < |\tilde{\Gamma}|/2$, the ball $B_{\tilde{\Gamma}, p}(r - 1)$ has a large boundary in $\tilde{\Gamma}$. When $r < D/3 - \alpha D$, the ball has not yet reached the cylinder $C$, so this boundary lies entirely in $\tilde{\Gamma}$, and we have
\[|B_{\tilde{\Gamma}, p}(r)| \geq |B_{\tilde{\Gamma}, p}(r - 1)|(1 + h)\]
and in particular, by the time the ball reaches the cylinder, we have
\[|B_{\tilde{\Gamma}, p}(D/3 - \alpha D - 1)| \geq (1 + h)^{D/3 - \alpha D - 1}.\]

For larger $r$, the growth of the ball in $\tilde{\Gamma}$ is reduced by at most the size of the cylinder, so we have
\[|B_{\tilde{\Gamma}, p}(r)| \geq |B_{\tilde{\Gamma}, p}(r - 1)|(1 + h) - |C|.\]

Hence, as long as we still have $|B_{\tilde{\Gamma}, p}(r - 1)| < |\tilde{\Gamma}|/2$, and the cylinder is small enough compared to the ball, say,
\[(1) \quad |C| < |B_{\tilde{\Gamma}, p}(r - 1)|h/2,
the balls continue to grow:
\[(2) \quad |B_{\tilde{\Gamma}, p}(r)| \geq |B_{\tilde{\Gamma}, p}(r - 1)|(1 + h/2),\]

To see the necessary inequality (1), observe that the cylinder cannot be too large,
\[|C| \leq (D/3 + 1)d^\alpha D,\]
and as we noted earlier, for $r \geq D/3 - \alpha D$, the ball has size at least
\[|B_{\tilde{\Gamma}, p}(r)| \geq (1 + h)^{D/3 - \alpha D - 1}.\]

Hence,
\[|C|/|B_{\tilde{\Gamma}, p}(r - 1)| \leq (D/3 + 1)d^\alpha D(1 + h)^{-D/3 + \alpha D} = (D/3 + 1) \left(d^\alpha (1 + h)^{-1/3 + \alpha} \right)^D\]

By our choice of $\alpha$, this goes to 0 as $D \to \infty$. So when $D$ is sufficiently large, we have
\[|C|/|B_{\tilde{\Gamma}, p}(r - 1)| < h/2,\]
as desired.
From equation (2), we have that
\[ |B_{\Gamma,p}(r)| \geq \min \left( (1 + h/2)^r, |\Gamma|/2 \right), \]
and hence if \( r \geq \log |\Gamma|/\log(1 + h/2) \), we have
\[ |B_{\Gamma,p}(r)| \geq |\Gamma|/2. \]
The same holds for \( |B_{\Gamma,q}(r)| \), so \( B_{\Gamma,p}(r) \) and \( B_{\Gamma,q}(r) \) intersect. Hence,
\[ d_{\Gamma}(p, q) \leq 2 \log |\Gamma|/\log(1 + h/2). \]
However, by Lemma 3.1, if \( \Gamma \) is \( \delta \)-hyperbolic we have two possibilities. Either \( D/\delta \) is not sufficiently large, in which case and \( \delta \) is bounded below by a constant times \( D \),
\[ d_{\Gamma}(p, q) \leq DB2^{\alpha D/\delta}. \]
And therefore,
\[ \frac{D}{\delta} \leq \frac{1}{\alpha \log 2} \log \left( \frac{1}{B \log(1 + h/2)} \frac{\log|\Gamma|}{D} \right) \]
However, \( D \) is bounded below by \( \log|\Gamma|/\log d \), so the right hand side is bounded above by a constant. Hence, there are constants \( K, K' > 0 \) such that when \( |\Gamma| \) is sufficiently large we have
\[ \delta \geq KD \geq K' \log|\Gamma|. \]

4. Elementary Lemmas about Hyperbolic Spaces

We now state and prove the necessary basic lemmas about \( \delta \)-hyperbolic spaces. These results appear in the literature, but we include proofs for completeness.

**Lemma 4.1.** Let \( X \) be a \( \delta \)-hyperbolic space, and let \( x, y, z \in X \) such that
\[ d(z, x) = d(z, y) = \min_{w \in [xy]} d(z, w), \]
that is, \( x \) and \( y \) are both points of minimal distance to \( z \) on a geodesic segment \( xy \) between them. Then \( d(x, y) < K\delta \), where \( K \) is some universal constant.

**Proof.** Since every point on \( xy \) is within \( \delta \) of either \( x \) or \( y \), by continuity, there must be a point \( w \) on \( xy \) within \( \delta \) of both \( x \) and \( y \).
\[ d(z, x) \geq d(z, w) + d(w, x) - 2\delta \geq d(z, x) + d(w, x) - 2\delta. \]
Hence \( d(w, x) \leq 2\delta \). Similarly, \( d(w, y) \leq 2\delta \), so \( d(x, y) \leq 4\delta. \)

**Lemma 4.2.** There is a universal constant \( C \) such that the following is true.
Let \( X \) be a \( \delta \)-hyperbolic space, and let \( p, q \) be points in \( X \). Let \( \beta \) be some path from \( p \) to \( q \). For any interval of length \( C\delta \) in the geodesic segment \( pq \), there is a pair \( (x, r) \) such that \( r \) is in that interval, \( x \) is on the path \( \beta \), and \( r \) is the closest point on \( pq \) to \( x \).
Proof. We write $\alpha : [0, L] \to X$ for the parametrization of $pq$ by arclength. We reparametrize $\beta$ if necessary so we have $\beta : [0, 1] \to X$, with

$$\alpha(0) = \beta(0) = p \quad \text{and} \quad \alpha(L) = \beta(1) = q.$$ 

Consider the set of pairs

$$S = \left\{ (t, s) \in [0, 1] \times [0, L] \mid d(\beta(t), \alpha(s)) = \min_{s' \in [0, L]} d(\beta(t), \alpha(s')) \right\}$$

The set $S$ is closed, hence compact. By lemma 4.1, each set

$$S_t = \{ s : (t, s) \in S \}$$

has diameter at most $K\delta$. Given any $s \in [0, L]$, the sets

$$\{ t \in [0, 1] : S_t \cap [0, s] \neq \emptyset \} \quad \text{and} \quad \{ t \in [0, 1] : S_t \cap [s, L] \neq \emptyset \}$$

are closed, nonempty, and cover $[0, 1]$. So some $t$ belongs to both, that is, there is a value $t$ such that $S_t$ intersects both $[0, s]$ and $[s, L]$. Since $S_t$ has diameter at most $K\delta$, it contains some element of $[s - K\delta, s + K\delta]$. Thus, some element of $[s - K\delta, s + K\delta]$ belongs to $S_t$.

So, every closed interval of length $2K\delta$ in $[0, L]$ intersects some $S_t$, as desired. □

Lemma 4.3. Let $X$ be a $\delta$-hyperbolic space and let $\gamma$ be a path from $p$ to $q$ in $X$. If some point $r$ on $pq$ is at distance $R$ from $\gamma$, then $\ell(\gamma) \geq \delta 2^{R/\delta}$.

Proof. We prove this by induction on $|R/\delta|$. By assumption, some point on $pq$ is at distance $R$ from both $p$ and $q$, so $\ell(\gamma) \geq d(p, q) \geq 2R$. In particular, if $|R/\delta| = 0$, then $\ell(\gamma) \geq 2R \geq \delta 2^{R/\delta}$.

For the inductive step, let $x$ be the point on $\gamma$ which is halfway between $p$ and $q$. Then $r \in pq$ is within $\delta$ of either $qx$ or $xp$. We may suppose without loss of generality that it is $qx$. Then some point on $qx$ has distance at least $R - \delta$ from $\gamma$. It follows that $\ell(\gamma)/2 \geq \delta 2^{(R-\delta)/\delta}$, and therefore $\ell(\gamma) \geq \delta 2^{R/\delta}$. □

Lemma 4.4. There are universal constants $K_0$ and $K_1$ such that the following is true.

Let $X$ be a $\delta$-hyperbolic space, and let $p, q, x, y$ be points in $X$, where we suppose $d(p, q) > K_0\delta$. If the nearest points on $pq$ to $x$ and $y$ are $p$ and $q$, respectively, then $pq$ passes within $K_1\delta$ of $pq$.

Proof. We take $K_0 = 12$ and $K_1 = 2$.

The geodesic $pq$ stays within a $\delta$-neighborhood of $pq \cup pq$, and the geodesic $pq$ stays within a $\delta$-neighborhood of $pq \cup pq$, hence $pq$ stays within a $2\delta$-neighborhood of $pq \cup pq \cup pq$. If $pq$ does not pass within $K_1\delta = 2\delta$ of $pq$, then there must be a point $z$ on $pq$ which is within $2\delta$ of both $pq$ and $pq$.

In particular, $d(x, z) \leq d(x, p) - d(z, p) + 4\delta$. There is a point $r$ on $pq$ which is within $\delta$ of both $pq$ and $pq$, so $d(z, r) \leq d(z, p) - d(r, p) + 2\delta$. Hence,

$$d(x, r) \leq d(x, z) + d(z, r)$$

$$\leq d(x, p) - d(z, p) + 4\delta + d(z, p) - d(r, p) + 2\delta$$

$$= d(x, p) - d(r, p) + 6\delta$$

$$\leq d(x, r) - d(r, p) + 6\delta.$$

Hence, $d(r, p) \leq 6\delta$. Similarly, $d(r, q) \leq 6\delta$ so $d(p, q) \leq 12\delta = K_0\delta$. So, if $pq$ does not pass within $K_1\delta$ of $pq$ then $d(p, q) \leq K_0\delta$, as desired. □
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