Absence of Differential Correlations Between the Wave Equations for Upper-Lower One-Index Twistor Fields Borne by the Infeld-van der Waerden Spinor Formalisms for General Relativity

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Abstract

It is pointed out that the wave equations for any upper-lower one-index twistor fields which take place in the frameworks of the Infeld-van der Waerden $\gamma$-formalisms must be formally the same. The only reason for the occurrence of this result seems to be directly related to the fact that the spinor translation of the traditional conformal Killing equation yields twistor equations of the same form. It thus appears that the conventional torsionless devices for keeping track in the $\gamma$-formalism of valences of spinor differential configurations turn out not to be useful for sorting out the typical patterns of the equations at issue.
1 Introduction

Certain calculational techniques were utilized in an earlier paper [1] for working out the twistor equation for contravariant one-index fields in curved spacetimes. The main aim associated to the completion of the relevant procedures was to derive one of the simplest sets of wave equations for conformally invariant spinor fields that should presumably take place in the frameworks of the Infeld-van der Waerden $\gamma\varepsilon$-formalisms [2-4]. A striking feature of these wave equations is that they involve no couplings between the twistor fields and wave functions for gravitons [5-7]. In actuality, the only coupling configurations brought about by the techniques allowed for thereabout take up appropriate outer products carrying the fields themselves along with some electromagnetic wave functions for the $\gamma$-formalism [4, 5]. Loosely speaking, the non-occurrence of $\varepsilon$-formalism couplings stems even in the case of charged fields from the applicability of a peculiar property of partially contracted second-order covariant derivatives of spin-tensor densities which carry only one type of indices as well as suitable geometric attributes [8-10]. Indeed, the electromagnetic curvature contributions that normally enter such derivative expansions really cancel out whenever the non-vanishing entries of the valences of the differentiated densities are adequately related to the respective weights and antiweights [4].

The present paper just brings forward the result that the above-mentioned wave equations possess the same form as the ones for the corresponding lower-index fields. It shall become clear that the legitimacy of this result rests upon the fact that the spinor translation of the classical conformal Killing equation leads to twistor equations which must be formally the same. Consequently, the conventional covariant devices for keeping track of valences of spinor differential configurations in the $\gamma$-formalism [4, 6], turn out not to be useful as regards the attainment of the full specification of the formal patterns for the field and wave equations being considered. We mention, in passing, that such devices had originally been built up in connection with the derivation of a system of sourceless gravitational and electromagnetic wave equations [5], with the pertinent construction having crucially been based upon the implementation of the traditional eigenvalue equations for the $\gamma$-formalism metric spinors [2, 3]. It may be said that the motivations for elaborating upon the situation entertained herein rely on our interest in completing the work of Ref. [1], thereby making up appropriately the set of $\gamma\varepsilon$-wave equations which emerge from the curved-space version of twistor equations for one-index fields.

The paper has been outlined as follows. In Section 2, we exhibit the twistor field equations which are of immediate relevance to us at this stage. We look at the twistor wave equations in Section 3, but the key remarks concerning the lack of differential correlations between them shall be made in Section 4. It will be convenient to employ the world-spin index notation adhered to in Ref. [11]. Without any risk of confusion, we will utilize a torsion-free operator $\nabla_a$ upon taking account of covariant derivatives in each formalism. Likewise, the D'Alembertian
operator for either $\nabla_a$ will be written as $\square$. A horizontal bar will be used once in Section 4 to denote the ordinary operation of complex conjugation. Einstein’s equations should thus be taken as

$$2\Xi_{ab} = \kappa (T_{ab} - \frac{1}{4} T g_{ab}), \quad T \doteq \tilde{T}^{ab} g_{ab},$$

where $T_{ab}$ amounts to the world version of the energy-momentum tensor of some sources, $g_{ab}$ denotes a covariant spacetime metric tensor and $\kappa$ stands for the Einstein gravitational constant. By definition, the quantity $(-2)\Xi_{ab}$ is identified with the trace-free part of the Ricci tensor $R_{ab}$ for the Christoff connexion of $g_{ab}$. The cosmological constant $\lambda$ will be allowed for implicitly through the well-known trace relation

$$R = 4\lambda + \kappa T, \quad R \doteq R^{ab} g_{ab}. $$

Our choice of sign convention for $R_{ab}$ coincides with the one made in Ref. [11], namely,

$$R_{ab} \doteq R_{ahbh},$$

with $R_{abe}^d$ being the corresponding Riemann tensor. We will henceforth assume that the local world-metric signature is $(+ - - -)$. The calculational techniques referred to before shall be taken for granted at the outset.

## 2 Twistor equations

The differential patterns borne by the original formulation of twistor equations in a curved spacetime [12-14] may be thought of as arising in either formalism from

$$\nabla^{(AA’KBB’)} = \frac{1}{4} (\nabla_{CC’KCC’}) M^{AB} M^{A’B’},$$

and

$$\nabla^{(AA’KB’)} = \frac{1}{4} (\nabla_{CC’KCC’}) M^{AB} M^{A’B’},$$

where the $K$-objects amount to nothing else but the Hermitian spinor versions of a null conformal Killing vector, and the kernel letter $M$ accordingly stands for either $\gamma$ or $\varepsilon$.

It should be emphatically observed that the genuineness of (1) and (2) as a system of equivalent field equations lies behind a general covariant-constancy property of the Hermitian connecting objects for both formalisms [2, 3]. Thus, these equations can be obtained from one another on the basis of the metric-compatibility requirements

$$\nabla_a (M^{AB} M^{A’B’}) = 0 \iff \nabla_a (M_{AB} M^{A’B’}) = 0. $$

\[1\] The symmetry operation involved in Eqs. (1) and (2) must be applied to the index pairs.
Hence, by putting into effect the elementary outer-product prescription
\[ K^{AA'} = \xi^A \xi^{A'}, \tag{4} \]
along with its lower-index version, after accounting for some manipulations, we get the statements
\[ \nabla^{A'}(A \xi^B) = 0, \quad \nabla^{A'}(A \xi_B) = 0, \tag{5} \]
which, when combined together with their complex conjugates, bring out the typical form of twistor equations. We stress that solutions to twistor equations are generally subject to strong consistency conditions (see, for instance, Ref. [1]).

Either \( \xi \)-field of (5) bears conformal invariance [13, 14], regardless of whether the underlying spacetime background bears conformal flatness. In the \( \gamma \)-formalism, the entries of the pair \((\xi^A, \xi_A)\), and their complex conjugates, come into play as spin vectors under the action of the Weyl gauge group of general relativity [15], whereas their \( \varepsilon \)-formalism counterparts appear as spin-vector densities of weights \((+1/2, -1/2)\) and antiweights \((+1/2, -1/2)\), respectively.

### 3 Wave equations

In the \( \gamma \)-formalism, \( \xi^A \) shows up [1] as a solution to the wave equation
\[ (\Box - \frac{R}{12})\xi^A = \frac{2i}{3} \phi^A_B \xi^B, \tag{6} \]
with \( \phi^A_B \) denoting a wave function for Infeld-van der Waerden photons [16-18]. In order to derive in a manifestly transparent manner the \( \gamma \)-formalism wave equation for the lower-index field \( \xi_A \), we initially recast the second of the statements (5) into
\[ 2\nabla^{A'} A \xi_B = \gamma_{AB}^L \gamma_{LM} \nabla^{A'} L \xi_M, \tag{7} \]
and then operate on (7) with \( \nabla^A \). It follows that, calling upon the splitting [5]
\[ \nabla^A \nabla_A = \frac{1}{2} \gamma_{AC} \Box - \Delta_{AC}, \tag{8} \]
together with the definition
\[ \Delta_{AC} \doteq -\nabla_A \gamma_{(A \nabla_C)A'} \tag{9} \]
and the property [4]
\[ \nabla_a (\gamma_{AB} \gamma_{LM}) = 0, \tag{10} \]
we arrive at
\[ \Box \xi_A - \frac{2}{3} \Delta_{A}^B \xi_B = 0. \tag{11} \]
The explicit calculation of the \( \Delta \)-derivative of (11) gives
\[ \Delta_{A}^B \xi_B = \frac{R}{8} \xi_A + i \phi^B_A \xi_B, \tag{12} \]
whence, fitting pieces together suitably, yields

\[ (\Box - \frac{R}{12})\xi_A = \frac{2i}{3} \phi_A B \xi_B. \]  

(13)

It should be evident that the equality (11) remains formally valid in the \( \varepsilon \)-formalism as well. Therefore, since the \( \varepsilon \)-formalism field \( \xi_A \) is a covariant one-index spin-vector density of weight \(-1/2\), the \( \varepsilon \)-counterpart of the derivative (12) has to be expressed as the purely gravitational contribution\(^2\)

\[ \Delta_A B \xi_B = \frac{R}{8} \xi_A. \]  

(14)

Hence, the \( \varepsilon \)-formalism statement corresponding to (13) must be spelt out as

\[ (\Box - \frac{R}{12})\xi_A = 0. \]  

(15)

4 Concluding remarks and outlook

The formulae shown in Section 3 supply the entire set of wave equations for one-index conformal Killing spinors that should be tied in with the context of the \( \gamma \varepsilon \)-frameworks. It is worth pointing out that the common overall sign on the right-hand sides of (6) and (13), is due to the \( \gamma \)-formalism metric relationship between the differential configuration (12) and

\[ \Delta_A B \xi^B = -\left(\frac{R}{8} \xi^A + i \phi_A B \xi^B\right), \]

with the aforesaid relationship actually coming about when we invoke the well-known derivatives [4]

\[ \Delta_{AB} \gamma_{CD} = 2 i \phi_{AB} \gamma_{CD}, \quad \Delta_{AB} \gamma^{CD} = -2 i \phi_{AB} \gamma^{CD}. \]

What happens with regard to it is, in effect, that the pieces of those contracted \( \Delta \xi \)-derivatives somehow compensate for each other while producing the formal commonness feature of the apposite couplings through

\[ \Delta_A B \xi_B + \Delta_{AB} B = 2 i \phi_A B \xi_B. \]

At first sight, one might think that a set of differential correlations between the \( \gamma \)-formalism wave equations for \( \xi^A \) and \( \xi_A \) could at once arise out of utilizing the devices [4, 5]

\[ \Box \xi^A = \gamma^{AB} \Box \xi_B + (\Box \gamma^{AB}) \xi_B + 2(\nabla^h \gamma^{AB}) \nabla^h \xi_B, \]

and

\[ \Box \xi_A = \gamma_{BA} \Box \xi_B + (\Box \gamma_{BA}) \xi_B + 2(\nabla^h \gamma_{BA}) \nabla^h \xi_B, \]

\(^2\)For a similar reason, the \( \varepsilon \)-formalism version of (6) reads \( (\Box - \frac{R}{12})\xi^A = 0 \). It will become manifest later in Section 4 that the relation (14) is compatible with this assertion.
in conjunction with the eigenvalue equations \[2-4\]
\[
\nabla_a \gamma_{AB} = i \beta^a \gamma_{AB}, \quad \nabla_a \gamma^{AB} = (-i \beta^a) \gamma^{AB},
\]
and
\[
\square \gamma^{AB} = -\Theta \gamma^{AB}, \quad \square \gamma_{AB} = -\Theta \gamma_{AB},
\]
where
\[
\Theta = \beta^h \beta_h + i \nabla_h \beta^h,
\]
and \(\beta^a\) is a gauge-invariant real world vector. If any such raising-lowering device were implemented in a straightforward way, then a considerable amount of "strange" information would thereafter be brought into the picture whilst some of the contributions involved in the intermediate steps of the calculations that give rise to the characteristic statements
\[
\nabla^{(A'(A'K')B')} = 0, \quad \nabla_{(A'(A'K')B')} = 0,
\]
would eventually be ruled out. We can conclude that any attempt at making use of a metric prescription to recover either of (6) and (13) from the other, would visibly carry a serious inadequacy in that the twistor equations (5) could not be brought forth simultaneously. It is obvious that the property we have deduced ultimately reflects the absence of index contractions from twistor equations.

It would be worthwhile to derive the \(\gamma_{\varepsilon}\)-wave equations for twistor fields of arbitrary valences. This issue will probably be considered further elsewhere.

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