MULTIPLE CHERN-SIMONS FIELDS ON A TORUS

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Intertwined multiple Chern-Simons gauge fields induce matrix statistics among particles. We analyse this theory on a torus, focusing on the vacuum structure and the Hilbert space. The theory can be mimicked, although not completely, by an effective theory with one Chern-Simons gauge field. The correspondence between the Wilson line integrals, vacuum degeneracy and wave functions for these two theories are discussed. Further, it is obtained in both of these cases that the two total momenta and Hamiltonian commute only in the physical Hilbert space.

1. Introduction

It is known that Chern-Simons gauge theory coupled to non-relativistic matter fields is equivalent to anyon quantum mechanics on a plane and any Riemann surface. The widespread interest lately in anyon quantum mechanics is due primarily to its many applications, for instance, to the fractional quantum Hall effect and possibly high-$T_c$ superconductivity, but also due to the rich and interesting mathematical properties it possesses [1].

There is growing interest in a theory in which not just one kind, but multiple kinds of Chern-Simons gauge interactions are introduced among particles [2–5]. It has been known that multiple Chern-Simons interactions induce matrix statistics which generalizes ordinary fractional statistics in the space of particle species.

Interest has been renewed recently [6–10] by a possible application of the theory to double-layered Hall systems for which new experiments are now available [11]. It also has been noticed that there is a connection with the Halperin wave functions [12].

In another direction of developments, it has been noted that in Chern-Simons gauge theory on multiply-connected spaces, non-integrable phases of Wilson line integrals along
non-contractible loops become physical degrees of freedom [13–18]. The dynamics of these phases lead to rich physical consequences, generating an interesting degenerate vacuum structure. The theory [14] naturally leads to a representation of the Braid group on multiply connected spaces [19]. Also, on $T^2$ for example, the analysis can be mathematically rigorous, with none of the infrared divergence or ambiguity in boundary conditions at space infinity which so plague the analysis of the theory on a plane.

In this paper we shall examine multiple Chern-Simons theory on a torus. We expect that the study of multiple Chern-Simons fields should shed light on the physics of multiple-layered Hall systems, and have relevance to other condensed-matter situations. Furthermore we shall see that the theory has a rather beautiful mathematical structure which by itself deserves special attention.

In Section 2 we consider the coupling of multiple ($M$) Chern-Simons fields to matter fields, finding the statistics phases generated therefrom. We give several examples for the case $M = 2$, known to be relevant to condensed-matter systems. In Sections 3 and 4 we turn to $T^2$ and investigate the topologically-induced vacuum structure, deriving expressions for a general wave function and basis of vacuum states. We then examine the result of the action of the Wilson line operators on this basis, and consider several examples. In Section 5 we derive results for the wave functions and action of Wilson line integrals in the effective theory of $M$ Chern-Simons fields. In Section 6 we compare and make the connection between the original and effective theories, finding their results to be almost (but not quite) equivalent. In Section 7 we show that the Hamiltonian and total momenta commute among themselves only in the physical Hilbert space. Section 8 consists of some concluding remarks and comments.

2. Matrix statistics

We consider theories with $M$ distinct Chern-Simons gauge fields, $a^I_\mu (I = 1, \cdots, M)$ and nonrelativistic matter fields. The Lagrangian is given by $\mathcal{L} = \mathcal{L}_{\text{CS}} + \mathcal{L}_{\text{matter}}$. The Chern-Simons term $\mathcal{L}_{\text{CS}}$ is given by

$$\mathcal{L}_{\text{CS}} = \frac{1}{4\pi} \sum_{IJ} K_{IJ} a^I \varepsilon \partial a^J \quad (I, J = 1, \cdots, M) ,$$

(2.1)

where $K_{IJ}$ is a $M$-by-$M$ real symmetric matrix.

The meaning of the $K$-matrix becomes definite only when the matter coupling is specified. We shall consider two typical couplings. Firstly we suppose that there are $M$ species of matter fields $\psi_I(x) (I = 1, \cdots, M)$, and that $\psi_I$ couples only to $a^{I}_\mu$.

$$\mathcal{L}_1 = \sum_{I=1}^{M} \left\{ i\bar{\psi}_I D_0^I \psi_I - \frac{1}{2m_I} (D^I_k \bar{\psi}_I)(D^I_k \psi_I) \right\} ,$$

(2.2)

$$D^I_\mu = \partial_\mu + i a^I_\mu .$$
Secondly we suppose that there is only one matter field \( \psi(x) \), and that \( \psi \) couples to all the \( a^I_\mu \)'s:

\[
\mathcal{L}_2 = i\psi_\dagger D_0\psi - \frac{1}{2m} (D_k\psi)^\dagger(D_k\psi),
\]

\[
D_\mu = \partial_\mu + i\sum_{I=1}^M a^I_\mu.
\]

We shall see shortly that \( \psi \) in (2.3) can be viewed as a bound state of \( \psi_1, \cdots, \psi_M \) in (2.2). As is obvious, only the combination \( a_\mu = \sum_{I=1}^M a^I_\mu \) is relevant in the coupling (2.3).

To clarify the meaning of the matter couplings above, we first diagonalize the \( K \)-matrix appearing in the Lagrangian. Thus

\[
K \mathbf{\bar{v}}^I = \lambda^I \mathbf{\bar{v}}^I, \quad \mathbf{\bar{v}}^I \cdot \mathbf{\bar{v}}^J = \delta_{IJ} \quad (I, J = 1, \cdots, M),
\]

\[
K = O^I \Lambda O,
\]

\[
O^I = \left( \mathbf{\bar{v}}^1, \cdots, \mathbf{\bar{v}}^M \right), \quad \Lambda = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_M \end{pmatrix}.
\]

Note that

\[
O_{IJ} = (\mathbf{\bar{v}}^I)_J = v^I_J, \quad v^I_L v^J_L = v^L_I v^L_J = \delta_{IJ}.
\]

Correspondingly, we introduce a new basis for the Chern-Simons fields:

\[
a^I_\mu = O_{IJ} a^J_\mu, \quad a^J_\mu = O_{JI} a^I_\mu.
\]

Then the Chern-Simons Lagrangian (2.1) becomes

\[
\mathcal{L}_{CS} = \sum_{I=1}^M \frac{\lambda^I}{4\pi} \alpha^I \cdot \partial \alpha^I.
\]

Without loss of generality we suppose that \( \det K \neq 0 \) and \( \lambda_I \neq 0 \).

In the first coupling \( \mathcal{L}_1 \) we denote by \( \theta_s^{(I,J)} \) the statistics phase acquired by the wave function when particles of the \( I \)-th and \( J \)-th kinds are interchanged. (For \( I \neq J \) the phase acquired by \( 2\pi \)-rotation is defined to be \( 2\theta_s^{(I,J)} \).) Noting that \( a^I = v^I_\alpha L \), one finds

\[
\theta_s^{(I,J)} = \sum_L \frac{\pi}{\lambda_L} v^I_\alpha L \cdot v^J_\alpha L = \pi K_{IJ}^{-1}.
\]

In other words, multiple Chern-Simons gauge theory induces matrix statistics among particles [2]. (In some of the recent literature \( \theta_s^{(I,J)} \) is called a mutual or relative statistics phase [8-10]).

In the second coupling \( \mathcal{L}_2 \) one has \( \sum_I a^I = \sum_I v^I_\alpha L \alpha^L \). Hence the phase \( \theta_s \) picked up by interchanging two particles is

\[
\theta_s = \sum_L \frac{\pi}{\lambda_L} \left( \sum_I v^I_\alpha L \right)^2 = \pi \sum_{I,J} K_{IJ}^{-1} \equiv \frac{\pi}{\kappa_{eff}}.
\]
κ_{eff} turns out to be the Chern-Simons coefficient in the effective theory for a, and ψ. In the application to multi-layer Hall systems, it is related to the total filling factor ν by ν = κ_{eff}^{-1}.

Now consider a bound state B composed of M particles, ψ₁, ⋯, ψ_M in the first coupling L₁. If two B’s are interchanged, then the statistics phase acquired is, from the preceding relations,

\[ \sum_{I,J} \theta_{s}^{(IJ)} = \theta_s \quad . \]  

(2.9)

So as advertised above, we see that the second coupling may be viewed as the effective theory for boundstates B in the first coupling.

We next consider several illustrative and relevant examples, for the case M = 2:

**Example 1**

\[ K = \begin{pmatrix} p & q \\ q & p \end{pmatrix}, \]

\[ \Lambda = \begin{pmatrix} p + q & p - q \\ p - q & p + q \end{pmatrix}, \quad O = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \]

\[ \theta_{s}^{(11)} = \theta_{s}^{(22)} = \frac{\pi p}{p^2 - q^2}, \quad \theta_{s}^{(12)} = -\frac{\pi q}{p^2 - q^2} \]

\[ \theta_s = \frac{2\pi}{p + q} . \]

(2.10)

For specific values of p and q we find the following.

**Example 2**

\[(p,q) = (3, 2)\]

\[ \theta_{s}^{(11)} = \theta_{s}^{(22)} = \frac{3\pi}{5}, \quad \theta_{s}^{(12)} = -\frac{2\pi}{5} \]

\[ \theta_s = \frac{2\pi}{5} . \]

(2.11)

This case has been discussed as an alternative way of describing the first daughter state in the fractional quantum Hall effect [5, 2].

**Example 3**

\[(p,q) = (3, 1)\]

\[ \theta_{s}^{(11)} = \theta_{s}^{(22)} = \frac{3\pi}{8}, \quad \theta_{s}^{(12)} = -\frac{\pi}{8} \]

\[ \theta_s = \frac{\pi}{2} . \]

(2.12)

It has been argued that this corresponds to the structure found in double-layered Hall systems at a half filling ν = 1/2 [6–10].
Example 4

\[(p, q) = (0, 2)\]

\[\theta_s^{(11)} = \theta_s^{(22)} = 0, \quad \theta_s^{(12)} = \frac{\pi}{2}\]

\[\theta_s = \pi \quad .\]  

Wilczek has proposed this case for the anyon superconductivity with P and T invariance [9]. Indeed, if one supposes that under P (or T), \(\alpha_1^\mu\) and \(\alpha_2^\mu\) are interchanged in addition to the ordinary transformation, then the theory becomes manifestly invariant. The model should exhibit no P or T violation effect.

3. The vacuum structure on \(T^2\)

On a torus, \(T^2\), for each C-S field there are two nonintegrable phases of Wilson line integrals along non-contractible loops, \(C_j\). Take a rectangular torus with coordinates \(0 \leq x_j \leq L_j, j = 1, 2\). The constant parts of \(a_I^j(x)\) are new degrees of freedom undetermined by the matter content:

\[
\exp \left\{ i \int_{C_j} dx_k a_k^I \right\} \Rightarrow W_j^I = e^{i\theta_j^I} \quad .
\]

The Wilson line phases \(\theta_j^I\)'s introduce an interesting structure into the theory, which we are going to clarify in the following.

Inserting \(a_I^j = \theta_I^j / L_j + \cdots\) into the Lagrangian \(L_{CS}\), one finds that

\[L_{CS} \Rightarrow \frac{1}{4\pi} K_{IJ}(\theta_I^j \dot{\theta}_J^j - \theta_I^j \dot{\theta}_I^j) \quad .\]

Hence one sees that \(\theta_1^I\)'s and \(\theta_2^I\)'s define conjugate pairs:

\[[\theta_1^I, \theta_2^J] = 2\pi i K_{IJ}^{-1} \quad .\]  

(3.2)

In the \(\theta_1\)-representation

\[
\theta_2^I = -2\pi i K_{IJ}^{-1} \frac{\partial}{\partial \theta_1^J} \quad , \quad K_{IJ} \theta_2^J = -2\pi i \frac{\partial}{\partial \theta_1^I} \quad .
\]  

(3.3)

The system is invariant under large gauge transformations which shift the Wilson line phases by multiples of \(2\pi\):

\[U^I_j : \ \theta_I^j \rightarrow \theta_I^j + 2\pi \quad .\]  

(3.4)

More explicitly

\[a_\mu^I(x) \rightarrow a_\mu^I(x) + \partial_\mu \beta_{IJ}^I(x) \quad ,\]

\[\beta_{IJ}^I(x) = \delta_{IJ} \frac{2\pi x_j}{L_j} \quad .\]  

(3.5)
Accordingly the matter field changes, in the case of the first coupling, for instance, as
\[ \psi^J(x) \rightarrow e^{-i\beta J_I(x)} \psi^J(x) \, . \] (3.6)

To be precise, boundary conditions for the fields have to be specified to define the theory on a torus. In general the fields are not single-valued. After translation along a noncontractible loop the fields return to their original values up to a gauge transformation. This problem has been analysed in detail in the case of one Chern-Simons field in [13, 15], and can be straightforwardly extended to the cases under consideration. We only note here that transformations (3.5) and (3.6) leave the boundary conditions unaltered.

Unitary operators inducing the transformation (3.4) or (3.5) are given by
\[ U_{Ij} = e^{+i\epsilon K_{IJ}} \theta^I J_k \, . \] (3.7)

The transformation of matter fields, (3.6), is induced by
\[ U_{I}^{\text{matter}, J} = \exp \left\{ 2\pi i \int dx \frac{x_J}{L_J} \psi^J_I(x) \right\} \, . \] (3.8)

Since these \( U_{I}^{\text{matter}, J} \)'s commute among themselves, they will not play an important role in the following discussions.

The two sets of operators, \( \{ U_{Ij} \} \) and \( \{ W_{Ij} \} \), are complimentary. They satisfy relations
\[ U_{Ij} U_{I'j'} = e^{-2\pi i K_{JJ'}} U_{I'j'} U_{Ij} \, , \]
\[ W_{Ij} W_{I'j'} = e^{-2\pi i K_{J'} J} W_{I'j'} W_{Ij} \, , \]
\[ U_{Ij} W_{I'j'} = W_{I'j'} U_{Ij} \, . \] (3.9)

Note that they do not commute with each other in general. The algebra is invariant under the interchange of \( U_{Ij} \) and \( W_{Ij} \) supplemented by the replacement of \( K_{IJ} \) by \( K_{IJ}^{-1} \). This suggests that there is a duality between the theories with the Chern-Simons coefficient matrix \( K \) and with \( K^{-1} \).

We would like to determine vacuum wave functions satisfying (3.9). From now on we shall suppose that all \( K_{IJ} \)'s are integers so that all the \( U_{Ij} \)'s commute among themselves. We may thus simultaneously diagonalize these operators and take
\[ U_{Ij}^{\dagger} |\Psi\rangle = e^{i\gamma J} |\Psi\rangle \, . \] (3.10)

For convenience we introduce vector notation: \( \tilde{\theta}_j = (\theta_1^j, \cdots, \theta_M^j) \), \( \tilde{\gamma}_j = (\gamma_1^j, \cdots, \gamma_M^j) \), etc.

We write the wave functions
\[ u(\tilde{\theta}_1) \equiv \langle \tilde{\theta}_1 | \Psi \rangle \, , \]
\[ v(\tilde{\theta}_2) \equiv \langle \tilde{\theta}_2 | \Psi \rangle = \int d\tilde{\theta}_1 \langle \tilde{\theta}_2 | \tilde{\theta}_1 \rangle \ u(\tilde{\theta}_1) \, , \] (3.11)
and proceed to find a general \( u(\vec{\theta}_1) \).

First consider \( U^\dagger_1 |\Psi\rangle = e^{i\gamma_1^I} |\Psi\rangle \). This implies that

\[
e^{2\pi(\partial/\partial \theta^I_1)} u(\vec{\theta}_1) = u(\cdots, \theta^I_1 + 2\pi, \cdots) = e^{i\gamma_1^I} u(\vec{\theta}_1) .
\]

Thus one can express \( u(\vec{\theta}_1) \) as an \( M \)-dimensional series,

\[
u(\vec{\theta}_1) = e^{i\vec{\gamma}_1^I/2\pi} \sum_{\vec{n}} d(\vec{n}) e^{i\vec{\eta} \cdot \vec{\theta}_1} , \quad \vec{n} \in \mathbb{Z}^M , \tag{3.12}
\]

where \( \vec{n} \) is an \( M \)-dimensional integer vector. Secondly, \( U^\dagger_2 |\Psi\rangle = e^{i\gamma_2^J} |\Psi\rangle \) gives

\[
e^{-i(K\vec{\theta}_1)^J} u(\vec{\theta}_1) = e^{i\gamma_2^J} u(\vec{\theta}_1) . \tag{3.13}
\]

We introduce here two sets of vectors by

\[
K = \begin{pmatrix} \vec{k}_1, & \cdots, & \vec{k}_M \end{pmatrix} = \begin{pmatrix} \vec{k}_1^t \\ \vdots \\ \vec{k}_M^t \end{pmatrix} , \tag{3.14}
\]

\[
K^{-1} = \begin{pmatrix} \vec{l}_1, & \cdots, & \vec{l}_M \end{pmatrix} = \begin{pmatrix} \vec{l}_1^t \\ \vdots \\ \vec{l}_M^t \end{pmatrix} .
\]

They satisfy

\[
\vec{k}_I \cdot \vec{l}_J = \delta_{IJ} . \tag{3.15}
\]

Then we have

\[
e^{-i(K\vec{\theta}_1)^J} \sum_{\vec{n}} d(\vec{n}) e^{i\vec{\eta} \cdot \vec{\theta}_1} = \sum_{\vec{n}} d(\vec{n}) e^{i(\vec{n} - \vec{k}_I) \vec{\eta}_1}
\]

\[
= \sum_{\vec{n}} d(\vec{n} + \vec{k}_I) e^{i\vec{\eta} \cdot \vec{\theta}_1},
\]

so that (3.13) leads to

\[
d(\vec{n} + \vec{k}_I) = e^{i\gamma_2^J} d(\vec{n}) \quad (I = 1, \cdots, M) . \tag{3.16}
\]

To find the general wave functions (3.12) satisfying (3.16), we first note that

\[
\vec{\gamma}_2^J \cdot K^{-1} \vec{k}_I = \vec{\gamma}_2^J (\vec{l}_J \cdot \vec{k}_I) = \gamma_2^I .
\]

Therefore

\[
u(\vec{\theta}_1) = e^{i\vec{\gamma}_1^I/2\pi} \sum_{\vec{n}} e^{i\vec{\gamma}_2^J \vec{n}} \tilde{d}(\vec{n}) e^{i\vec{\eta} \cdot \vec{\theta}_1} , \tag{3.17}
\]

\[
\tilde{d}(\vec{n} + \vec{k}_I) = \tilde{d}(\vec{n}) .
\]
Consider the $M$-dimensional lattice, $\vec{n} \in \mathbb{Z}^M$. The relation $\tilde{d}(\vec{n} + \vec{K}_I) = \tilde{d}(\vec{n})$ in (3.17) implies that the lattice contains $det K = r$ inequivalent points, each corresponding to a different ground state of the system. We introduce $\{\vec{f}_1, \ldots, \vec{f}_r\}$. We may thus parameterize $\vec{n}$ as

$$\vec{n} = \vec{f}_a + \sum_{I=1}^{M} p_I \vec{K}_I \quad (a = 1, \ldots, r, \ p_I \in \mathbb{Z}) \ . \quad (3.18)$$

The wave function is expressed as

$$u(\vec{\theta}_1) = e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \sum_{a=1}^{r} d_a \sum_{p} e^{i(\vec{f}_a + p_1 \vec{K}_1 + \ldots + p_M \vec{K}_M) \cdot (\vec{\theta}_1 + K^{-1} \vec{\gamma}_2)} \ , \quad (3.19)$$

Since $\sum_I p_I \vec{K}_I \cdot (\vec{\theta}_1 + K^{-1} \vec{\gamma}_2) = \vec{p} \cdot (K\vec{\theta}_1 + \vec{\gamma}_2)$, one then finds

$$u(\vec{\theta}_1) = e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \sum_{a=1}^{r} d_a \sum_{p} e^{i\vec{f}_a \cdot (\vec{\theta}_1 + K^{-1} \vec{\gamma}_2)} e^{i\vec{p} \cdot (K\vec{\theta}_1 + \vec{\gamma}_2)} \quad \quad (3.20)$$

$$= (2\pi)^M e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \sum_{a=1}^{r} d_a \ e^{i\vec{K}_a \cdot (\vec{\theta}_1 + K^{-1} \vec{\gamma}_2)} \delta_{2\pi}[K\vec{\theta}_1 + \vec{\gamma}_2] \ .$$

Let us define an equivalence relation $\sim$ among vectors $\vec{r}$ in $R^M$:

$$\vec{h} \sim \vec{g} \iff \vec{h}^I = \vec{g}^I \ (mod\ 2\pi) \ \ I = 1, \ldots, M \ . \quad (3.21)$$

It implies, for instance, that $\delta_{2\pi}[\vec{h}] = \delta_{2\pi}[\vec{g}]$ if $\vec{h} \sim \vec{g}$. A set of vectors $\mathcal{H}(K) = \{\vec{h}_a\}$, which are independent in the coset space $\mathcal{R}^M/\sim$ in the sense that $\vec{h}_a \not\sim \vec{h}_b$ iff $a \neq b$, is defined by

$$\mathcal{H}(K) = \{ \vec{h}_a \in \mathcal{R}^M, \ (a = 1, \ldots, r) ; \ K \vec{h}_a \sim 0 \} \ . \quad (3.22)$$

With this definition one has

$$\delta_{2\pi}[K\vec{\phi}] = \sum_{b=1}^{r} c_b \ \delta_{2\pi}[\phi - \vec{h}_b] \ . \quad (3.23)$$

Hence our wave function is

$$u(\vec{\theta}_1) = (2\pi)^M e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \sum_{a=1}^{r} d_a \ \sum_{b=1}^{r} c_b \ e^{i\vec{K}_a \cdot \vec{h}_b} \delta_{2\pi}[\vec{\theta}_1 + K^{-1} \vec{\gamma}_2 - \vec{h}_b] \quad \quad (3.24)$$

$$= e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \sum_{b=1}^{r} c_b \ \delta_{2\pi}[\vec{\theta}_1 + K^{-1} \vec{\gamma}_2 - \vec{h}_b] \ .$$

As an independent basis of vacua one can thus choose

$$\langle \vec{\theta}_1 | 0_a \rangle \equiv u_a(\vec{\theta}_1) = e^{i\vec{\gamma}_1 \vec{\theta}_1/2\pi} \delta_{2\pi}[\vec{\theta}_1 + K^{-1} \vec{\gamma}_2 - \vec{h}_a] \ ,$$

$$(a = 1, \ldots, r = det K) \ . \quad (3.25)$$
The fact that there are $r$ independent vectors $\vec{h}_a \in \mathcal{H}(K)$ has been proven a posteriori.

The action of Wilson lines on the vacuum is evaluated as follows.

$$
\langle \vec{\theta}_1 | W_1 | 0_a \rangle = e^{i\theta_1} u_a(\vec{\theta}_1) = e^{-i(K_1^{-1}\vec{\gamma}_2) + i2\pi a} u_a(\vec{\theta}_1)
$$

$$
= e^{-i\vec{\gamma}_2 + i2\pi a} u_a(\vec{\theta}_1),
$$

$$
\langle \vec{\theta}_1 | W_2 | 0_a \rangle = e^{2\pi(K_1^{-1}\partial / \partial \vec{\theta}_1) + i2\pi a} u_a(\vec{\theta}_1)
$$

$$
= e^{-i\vec{l}_I \vec{\gamma}_2 + i2\pi a} u_a(\vec{\theta}_1),
$$

(3.26)

To see that $W_1^I$ induces a mapping $I(a)$ among the vacua, we note that

$$
K(\vec{h}_a - 2\pi \vec{l}_I) \sim 0
$$

so that $\vec{h}_a - 2\pi \vec{l}_I \in \mathcal{H}$. Hence we have

$$
I(a) : \vec{h}_a \to \vec{h}_{I(a)} \sim \vec{h}_a - 2\pi \vec{l}_I.
$$

(3.27)

Then clearly

$$
\langle \vec{\theta}_1 | W_1^I | 0_a \rangle = e^{i\vec{\gamma}_I} u_I(\vec{\theta}_1).
$$

(3.28)

In summary,

$$
W_1^I | 0_a \rangle = e^{-i\vec{l}_I \vec{\gamma}_2 - i2\pi a} | 0_a \rangle,
$$

$$
W_2^I | 0_a \rangle = e^{i\vec{l}_I \vec{\gamma}_1} | 0_I \rangle.
$$

(3.29)

4. Examples

We apply the results in the previous section to two examples.

Case 1. $K = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}$

This is the second example in Section 2. This $K$ gives

$$
K^{-1} = \frac{1}{5} \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}, \quad \text{det } K = 5, \quad \kappa_{\text{eff}} = \frac{5}{2},
$$

(4.1)

where $\kappa_{\text{eff}}$ has been defined in (2.8). The basis $\{\vec{f}_a\}$ is given by

$$
\vec{f}_a = a \vec{l} \quad (a = 0, \cdots, 4) \quad \text{where} \quad \vec{l} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
$$

(4.2)

The corresponding $\mathcal{H} = \{\vec{h}_a\}$ is

$$
\vec{h}_a = \frac{2\pi a}{5} \vec{l} \quad (a = 0, \cdots, 4).
$$

(4.3)

Thus the vacua are given by

$$
u_a(\vec{\theta}_1) = e^{i\vec{\gamma}_I \vec{\theta}_1 / 2\pi} \delta_{2\pi[\vec{\theta}_1 + K^{-1}\vec{\gamma}_2 - \frac{2\pi a}{5} \vec{l}]}, \quad (a = 0, \cdots, 4).
$$

(4.4)
They satisfy
\[ |a + 5\rangle = |a\rangle \] (4.5)

Vectors \( \vec{l}_I \)'s are given by
\[ \vec{l}_1 = \frac{1}{5} \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \quad \vec{l}_2 = \frac{1}{5} \begin{pmatrix} -2 \\ 3 \end{pmatrix}. \]

Since
\[ \vec{h}_a - 2\pi \vec{l}_I \sim \vec{h}_{a+2}, \] (4.6)

one finds the mapping \( I(a) \) of (3.27) to be
\[ |I(a)\rangle = |a + 2\rangle. \] (4.7)

Hence we have
\[ W^I_1 |a\rangle = e^{-i\vec{l}_I \vec{\gamma}_2 + 2\pi ia/5} |a\rangle, \]
\[ W^I_2 |a\rangle = e^{+i\vec{l}_I \vec{\gamma}_1} |a + 2\rangle. \] (4.8)

Case 2. \[ K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \]

This is the third example in Section 2. This \( K \) gives
\[ K^{-1} = \frac{1}{8} \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad \text{det } K = 8, \quad \kappa_{\text{eff}} = 2. \] (4.9)

A choice of a basis for \( \{ \vec{f}_a \} \) is given by
\[ \vec{f}_a = a \vec{1} - b \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (a = 0 \sim 3, \ b = 0, 1). \] (4.10)

The corresponding \( \mathcal{H} = \{ \vec{h}_{ab} \} \) is
\[ \vec{h}_{ab} = \frac{\pi a}{2} \vec{1} + \frac{\pi b}{4} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (a = 0 \sim 3, \ b = 0, 1). \] (4.11)

The vacua are thus given by
\[ u_{ab}(\vec{h}_1) = e^{i\vec{\gamma}_1 \vec{h}_1/2\pi} \delta_{2\pi}[ \vec{h}_1 + K^{-1} \vec{\gamma}_2 - \vec{h}_{ab}]. \] (4.12)

We note that
\[ |a + 4, b\rangle = |a, b\rangle, \]
\[ |a, b \pm 2\rangle = |a \pm 1, b\rangle. \] (4.13)

This time we have
\[ \vec{l}_1 = \frac{1}{8} \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \quad \vec{l}_2 = \frac{1}{8} \begin{pmatrix} -1 \\ 3 \end{pmatrix}. \] (4.14)
so that

\[ \vec{h}_{ab} - 2\pi\vec{l}_1 \sim \vec{h}_{a-1,b-1} , \]
\[ \vec{h}_{ab} - 2\pi\vec{l}_2 \sim \vec{h}_{a,b+1} . \]

Hence the action of Wilson line operators is given by

\[ W_1^I \mid a,b \rangle = e^{-i\vec{l}_1 \gamma_2 - i\vec{h}_{ab}} \mid a,b \rangle , \]
\[ W_2^{(1)} \mid a,b \rangle = e^{+i\vec{l}_2 \gamma_1} \mid a-1,b-1 \rangle , \]
\[ W_2^{(2)} \mid a,b \rangle = e^{+i\vec{l}_2 \gamma_1} \mid a,b+1 \rangle . \]

5. Effective theory

In Section 2 we have seen that in the case of the second matter coupling \( L_2 \), (2.3), the induced statistics phase \( \theta_s \) is given by (2.8). Since the matter field \( \psi \) couples only to \( a_\mu = \sum_I a^I_\mu \), one can integrate all Chern-Simons gauge fields but the degree \( a_\mu \), getting an effective theory for \( a_\mu \) exactly with a Chern-Simons coefficient \( \kappa_{\text{eff}} \) given by (2.8).

On a torus, or more generally on a Riemann surface with a genus \( \geq 1 \), however, there might arise a difference between the original multiple Chern-Simons theory and the effective theory. This is because the vacuum structures are different in general. In the next section we shall examine the correspondence between the two theories in the cases discussed in the previous section.

Let us recall that the effective Chern-Simons coefficient is given by

\[ \frac{1}{\kappa_{\text{eff}}} = \sum_{I,J} K^{-1}_{IJ} . \]

Hence, even if all \( K_{IJ} \)’s are integers, \( \kappa_{\text{eff}} \) is, in general, a rational number,

\[ \kappa_{\text{eff}} = \frac{p}{q} , \]

where \( p \) and \( q \) are two mutually prime integers. In this section we clarify the vacuum structure of this theory on a torus. Previously it has been studied by Lee [17] and by Polychronakos [18]. Our derivation of vacuum wavefunctions proceeds along the same line as in Section 3 (We remark that our basis is slightly different from that given in [17] (cf. [15]).

We denote operators in the effective theory with bars on the top. The two non-integrable phases in the theory, \( \bar{\theta}_1, \bar{\theta}_2 \), their corresponding Wilson line operators, \( \vec{W}_j = e^{i\bar{\theta}_j} \), and the generators of large gauge transformation, \( \vec{U}_j = e^{ie^{i\phi}p\bar{\theta}_j/q} \) satisfy the following commutator relations:

\[ [\bar{\theta}_1, \bar{\theta}_2] = \frac{2\pi i q}{p} , \]
\[ \vec{U}_1 \vec{U}_2 = e^{-2\pi ip/q} \vec{U}_2 \vec{U}_1 , \]
\[ \vec{W}_1 \vec{W}_2 = e^{-2\pi iq/p} \vec{W}_2 \vec{W}_1 . \]
Note that $\tilde{U}_j$ do not commute for $q \neq 1$. But since $\tilde{U}_1^q$ commute with $\tilde{U}_2^q$, we can diagonalize $\tilde{U}_j^q$: \[
\tilde{U}_j^q | \Psi \rangle = e^{i\gamma_j} | \Psi \rangle . \quad (5.4)
\]
To find $u(\tilde{\theta}_1) = \langle \tilde{\theta}_1 | \Psi \rangle$, we start with \[
\langle \tilde{\theta}_1 | \tilde{U}_1^q | \Psi \rangle = e^{2\pi q (\partial / \partial \tilde{\theta}_1)} u(\tilde{\theta}_1) = u(\tilde{\theta}_1 + 2\pi q) = e^{i\gamma_1} u(\tilde{\theta}_1) . \quad (5.5)
\]
There are $q$ independent solutions to Eq. (5.5): \[
u_s(\tilde{\theta}_1) = \sum_m c_{sm} e^{i(m+ps/q+\gamma_1/2\pi q)\tilde{\theta}_1} , \quad s = 0, 1, \ldots, q - 1 . \quad (5.6)
\]
Secondly we have \[
\langle \tilde{\theta}_1 | \tilde{U}_2^q | \Psi \rangle = e^{-ip\tilde{\theta}_1} u(\tilde{\theta}_1) = e^{i\gamma_2} u(\tilde{\theta}_1) . \quad (5.7)
\]
Substitution of (5.6) into (5.7) gives the condition on $c_{s,m}$: $c_{s,m+p} = e^{i\gamma_2} c_{s,m}$, which is solved by \[
\begin{align*}
c_{s,m} &= e^{i\gamma_2/p} A_s d_m , \quad d_{m+p} = d_m . \quad (5.8)
\end{align*}
\] The $s$-dependent constant $A_s$ will be determined later. 

Now we can rewrite $u_s(\tilde{\theta}_1)$ by making use of (5.7) and writing $m = lp + r$. Noting that $d_m = dl_{p+r} = dr$, we have \[
\begin{align*}
u_s(\tilde{\theta}_1) &= A_s \sum_{r=0}^{p-1} d_r \sum_{l=-\infty}^{\infty} \exp \left\{ i \left( l + \frac{r}{p} \right) \tilde{\gamma}_2 + i \left( lp + r + \frac{ps}{q} + \frac{\tilde{\gamma}_1}{2\pi q} \right) \tilde{\theta}_1 \right\} \\
&= 2\pi A_s \sum_{r=0}^{p-1} d_r e^{i(r\tilde{\gamma}_1 + \tilde{\gamma}_2/p + (ps/q + \tilde{\gamma}_1/2\pi q)\tilde{\theta}_1)} \sum_{a=0}^{p-1} \delta_{2\pi p} [p\tilde{\theta}_1 + \tilde{\gamma}_2 - 2\pi a q] . \quad (5.9)
\end{align*}
\] In the second line we have expressed $\delta_{2\pi}[p\tilde{\theta}_1 + \tilde{\gamma}_2]$ in terms of $\delta_{2\pi}[\cdot | \cdot]$, taking advantage of the mutually prime nature of $p$ and $q$. Further \[
\begin{align*}
u_s(\tilde{\theta}_1) &= 2\pi A_s e^{i(ps/q + \tilde{\gamma}_1/2\pi q)\tilde{\theta}_1} \sum_{r=0}^{p-1} \sum_{a=0}^{p-1} d_r e^{i2\pi pa/p} \delta_{2\pi p} [p\tilde{\theta}_1 + \tilde{\gamma}_2 - 2\pi a q] \\
&= 2\pi A_s e^{i(ps/q + \tilde{\gamma}_1/2\pi q)\tilde{\theta}_1} \sum_{a=0}^{p-1} f_a \delta_{2\pi} \left[ \tilde{\theta}_1 + \frac{1}{p} (\tilde{\gamma}_2 - 2\pi a q) \right] . \quad (5.10)
\end{align*}
\] We choose $A_s$ such that $u_{s+q}(\tilde{\theta}_1) = u_s(\tilde{\theta}_1)$, which leads to the condition on $A_s$: $A_{s+q} = e^{i\gamma_2} A_s$. Again, this is solved by $A_s = e^{i\gamma_2/q}$. Hence as a linearly independent basis one can take \[
u_{a,s}(\tilde{\theta}_1) = e^{i\gamma_1/2\pi q} \exp \left\{ i(s/(q+\tilde{\theta}_1/q) \delta_{2\pi} \left[ \tilde{\theta}_1 + \frac{1}{p} (\tilde{\gamma}_2 - 2\pi a q) \right] \right\} , \quad (a = 0, \ldots, p - 1 , \quad s = 0, \ldots, q - 1 ) . \quad (5.11)
\]
Let us denote the corresponding vacuum by $|a, s\rangle$. They satisfy
\begin{align}
|a + p, s\rangle &= |a, s\rangle, \\
|a, s + q\rangle &= |a, s\rangle.
\end{align}
(5.12)

The actions of $\widetilde{U}_j$ and $\widetilde{W}_j$ on these vacua are found to be:
\begin{align}
\widetilde{U}_1 |a, s\rangle &= e^{i(\gamma_1 + 2\pi p)/q} |a, s\rangle, \\
\widetilde{U}_2 |a, s\rangle &= e^{i\gamma_2/q} |a, s - 1\rangle, \\
\widetilde{W}_1 |a, s\rangle &= e^{-i(\gamma_2 - 2\pi a)/p} |a, s\rangle, \\
\widetilde{W}_2 |a, s\rangle &= e^{i\gamma_1/p} |a - 1, s\rangle.
\end{align}
(5.13)

Notice that $\widetilde{U}_j$ acts on the space of the $s$ index, whereas $\widetilde{W}_j$ on that of the $a$ index.

There can be two possible interpretations of (5.13). One can view that $\widetilde{U}_2$ generates gauge copies so that there are only $p$ physically distinct vacuum states. This viewpoint has been adopted by Polychronakos [18]. An alternative interpretation is possible. In applications to condensed-matter systems, vacua here correspond to ground states. Wen and Niu [20] have argued that additional interactions such as a tunneling effect in condensed-matter systems generally lift the degeneracy in the $s$ index, which forces us to regard all $p \cdot q$ states physically distinct. At the level of an idealized Chern-Simons gauge theory under consideration, both interpretations are acceptable.

6. The correspondence between the two theories

There is a correspondence between multiple Chern-Simons theory and single Chern-Simons theory with the corresponding effective coupling. We work out this correspondence in two typical cases discussed in Section 4.

First
\begin{align}
\text{case 1 : } K &= \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}, \\
\kappa_{\text{eff}} &= \frac{p}{q} = \frac{5}{2}.
\end{align}
(6.1)

Recall that
\begin{align}
det K &= 5, \\
pq &= 10.
\end{align}
(6.2)

Hence in the $K$-theory there are 5 distinct vacua, whereas in the effective theory there are 10 vacua (including gauge copies in Polychronakos’ interpretation).

Applying the results in the previous section, one finds that in the effective theory
\begin{align}
\widetilde{U}_1 |a, s\rangle &= e^{i\gamma_1/2} |a, s\rangle, \\
\widetilde{U}_2 |a, s\rangle &= e^{i\gamma_2/2} |a, s - 1\rangle, \\
\widetilde{W}_1 |a, s\rangle &= e^{-i(\gamma_2 - 4\pi a)/5} |a, s\rangle, \\
\widetilde{W}_2 |a, s\rangle &= e^{i\gamma_1/5} |a - 1, s\rangle.
\end{align}
(6.3)
On the other hand in the $K$-theory we have seen in Section 3 that

\[
U_j^{(1)} U_j^{(2)} | a \rangle = e^{i(\gamma_j^1 + \gamma_j^2)} | a \rangle,
\]
\[
W_j^{(1)} W_j^{(2)} | a \rangle = e^{4\pi i a/5 - i(\ell_1 + \ell_2)\gamma_j^2} | a \rangle = e^{4\pi i a/5 - i(\gamma_j^1 + \gamma_j^2)^2/5} | a \rangle ,
\]
\[
W_j^{(2)} | a \rangle = e^{+i(\gamma_j^1 + \gamma_j^2)^2/5} | a - 1 \rangle .
\]

(6.4)

Comparing (6.3) and (6.4), one finds an exact correspondence:

| $K$ | $\kappa_{\text{eff}}$ |
|-----|------------------|
| $\theta_j^{(1)} + \theta_j^{(2)}$ | $\bar{\theta}_j$ |
| $U_j^{(1)} U_j^{(2)}$ | $(\bar{U}_j)^2$ |
| $W_j^{(1)} W_j^{(2)}$ | $\bar{W}_j$ |
| $\gamma_j^{(1)} + \gamma_j^{(2)}$ | $\bar{\gamma}_j$ |
| $| a \rangle$ | $| a, s \rangle$ |

(6.5)

Note that both $| a, s = 0 \rangle$ and $| a, s = 1 \rangle$ in the effective theory correspond to $| a \rangle$ in the multiple Chern-Simons theory.

The second case is

**case 2**: $K = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$, $\kappa_{\text{eff}} = \frac{p}{q} = 2$.

(6.6)

We recall that

\[
det K = 8 , \quad p q = 2 .
\]

(6.7)

Hence in the $K$-theory there are 8 distinct vacua, whereas in the effective theory there are only two vacua. Nevertheless there exists correspondence.

In the effective theory we have found that

\[
\bar{U}_j | c \rangle = e^{i\bar{\gamma}_j} | c \rangle ,
\]
\[
\bar{W}_1 | c \rangle = e^{i\pi c - i\bar{\gamma}_2/2} | c \rangle ,
\]
\[
\bar{W}_2 | c \rangle = e^{i\bar{\gamma}_1/2} | c - 1 \rangle ,
\]

(6.8)

where $| c + 2 \rangle = | c \rangle$. In the $K$-matrix theory of Section 3

\[
U_j^{(1)} U_j^{(2)} | a, b \rangle = e^{i(\gamma_j^1 + \gamma_j^2)} | a, b \rangle ,
\]
\[
W_j^{(1)} W_j^{(2)} | a, b \rangle = e^{i(\pi a - \pi b)/2 - i(\gamma_j^1 + \gamma_j^2)^2/5} | a, b \rangle ,
\]
\[
W_j^{(2)} | a, b \rangle = e^{+i(\gamma_j^1 + \gamma_j^2)^2/4} | a - 1, b \rangle ,
\]

(6.9)
with the identities (4.13).

The correspondence is found to be:

| $K$ | $\kappa_{\text{eff}}$ |
|-----|------------------|
| $\theta_j^{(1)} + \theta_j^{(2)}$ | $\tilde{\theta}_j$ |
| $U_j^{(1)}U_j^{(2)}$ | $(\tilde{U}_j)^2$ |
| $W_j^{(1)}W_j^{(2)}$ | $\tilde{W}_j$ |
| $\frac{1}{2}(\gamma_1^{(1)} + \gamma_1^{(2)})$ | $\tilde{\gamma}_1$ |
| $\frac{1}{2}(\gamma_2^{(1)} + \gamma_2^{(2)}) + \pi b$ | $\tilde{\gamma}_2$ |
| $\left\{ |c, b\rangle, |c + 2, b\rangle \right\}$ | $|c\rangle$ |

Note that for each $b$ ($0$ or $1$) the four states $|a, b\rangle$ ($a = 0 \sim 3$) close under the operations above. Also, $|a, b\rangle$ and $|a + 2, b\rangle$ in the $K$-theory correspond to the same state in the effective theory. Indeed, in terms of $\theta_1^{(1)} \pm \theta_1^{(2)} \equiv \theta_1^{\pm}$, the vacuum wave function in the $K$-theory yields:

$$u_{ab}(\tilde{\theta}_1) \rightarrow e^{i(\gamma_1^{(1)} + \gamma_1^{(2)})\theta_1^{\pm}/4\pi} \delta_{2\pi}\left[ \theta_1^{\pm} + \frac{1}{4}(\gamma_2^{(1)} + \gamma_2^{(2)}) - \pi a + \frac{\pi b}{2} \right].$$

(6.11)

Both $u_{a,b}(\tilde{\theta}_1)$ and $u_{a+2,b}(\tilde{\theta}_1)$ give the same $\theta_1^{\pm}$ wave function. The $\theta_1^{-}$ parts are different, but the effective theory is blind to them.

7. The Hilbert space

It has been shown that the Hamiltonian and two total momenta in theories defined on a torus with a single Chern-Simons field commute among themselves only in the physical Hilbert space [15]. In this section we examine the Hilbert space of multiple Chern-Simons theory with two kinds of matter couplings, $\mathcal{L}_1$ and $\mathcal{L}_2$ defined in Section 2.

The argument proceeds as in ref. [15]. We first solve field equations to express Chern-Simons fields in terms of Wilson line phases and matter fields. Then we examine commutators among the Hamiltonian and momenta.

First consider the first matter coupling $\mathcal{L}_1$, (2.2). Chern-Simons field equations are:

$$\frac{1}{4\pi} K_{IJ} \epsilon^{\mu \nu \rho} f_{I\mu}^J = j_I^\mu,$$

(7.1)

where $j_I^\mu$ is a current of the $I$-th matter field $\psi_I$. Solving them in the radiation gauge, one finds:

$$a_I^J(x) = \frac{\theta_I^J(t)}{L_j} - \frac{\Phi_I}{2L_1L_2} e^{jk} x_k + \hat{a}_I^J(x),$$

$$\hat{a}_I^J(x) = \int dy \ e^{ik} \nabla_k^I G(x - y) \left\{ \frac{2\pi K_{IJ}^{-1} \psi_j^I(x) + \Phi_I}{L_1L_2} \right\}.$$  

(7.2)
Here $\Phi_I = -\int dx f^I_{12}$ and $G(x)$ is the periodic Green’s function on a torus satisfying $\Delta G(x) = \delta(x) - (1/L_1 L_2)$.

The field equations (7.1) are solved by (7.2) except for

$$K_{IJ}^{-1} Q_J + \frac{\Phi_I}{2\pi} \approx 0 \quad (I = 1, \cdots, M) \quad (7.3)$$

where $Q_I = \int dx f^I_0$. The conditions (7.3) have to be imposed as constraints on physical Hilbert states. (We have adopted the notation $\approx$ to indicate this.) We note that $Q_I$ is conserved thanks to the gauge invariance, or the field equation for $\psi_I$, whereas $\Phi_I$ is a constant fixed by boundary conditions for $a^I_\mu$.

The Hamiltonian and momenta are given by

$$H = \int dx \sum_I \frac{1}{2m_I} (D^I_k \psi_I)^*(D^I_k \psi_I) ,$$

$$P^I_j = -i \int dx \sum_I \psi^\dagger_I D^I_j \psi_I = -i \int dx \sum_I \psi^\dagger_I D^I_j \psi_I , \quad (7.4)$$

$$\bar{D}^I_j = \nabla_j - i \frac{\theta^I_j}{L_j} + i \frac{\Phi_I}{2L_1 L_2} e^{ikx_k} .$$

In $D^I_j \psi_I$ in the expression above, (7.2) is to be substituted. Note that the Hamiltonian is not completely normal-ordered. The ordering of $\psi_I$ operators must be taken as it appears.

Useful relations are

$$[P^k, \theta^I_j] = -2\pi i e^{kj} \frac{1}{L_k} K^{-1}_{I N} Q_N ,$$

$$[H, \theta^I_j] = -2\pi i e^{kj} \frac{1}{L_k} J^k_{IN} , \quad (7.5)$$

where $J^k_I = \int dx j^k_I(x)$. Further

$$[P^I_j, \psi_I] = i \bar{D}^I_j \psi ,$$

$$\left( \begin{array} {c} [P^I_j, D^I_k \psi_I] \\ [P^I_j, \bar{D}^I_k \psi_I] \end{array} \right) = \left( \begin{array} {c} i \bar{D}^I_j D^I_k \psi_I - e^{jk} \frac{\Phi_I}{L_1 L_2} \psi_I - e^{ik} \frac{2\pi}{L_1 L_2} K^{-1}_{I J} \psi_I Q_J \end{array} \right) . \quad (7.6)$$

With the aid of (7.5) and (7.6) it is straightforward to find

$$[P^I_j, P^k] = i e^{ij} \frac{2\pi}{L_1 L_2} Q_I \left( K^{-1}_{I N} Q_N + \frac{\Phi_I}{2\pi} \right) ,$$

$$[P^I_j, H] = i e^{ij} \frac{2\pi}{L_1 L_2} J^k_I \left( K^{-1}_{I N} Q_N + \frac{\Phi_I}{2\pi} \right) . \quad (7.7)$$

Here we have made use of the fact that gauge invariant quantities are single-valued on a torus. We observe that $H$ and $P^I_j$ commute among themselves only in the physical Hilbert space defined by (7.3).
Similarly commutators between $W_I^j$ and $(H, P^j)$ are evaluated. We note that
\begin{align}
[W_I^j, Q_N] &= 0 \quad , \\
[W_I^j, J_N^k] &= \epsilon^{jk} \frac{2\pi}{m_N L_k} K_{1N}^{-1} Q_N W_I^j \quad . (7.8)
\end{align}

We find that
\begin{align}
[W_I^j, P^k] &= \epsilon^{jk} \frac{\pi}{L_k} K_{1N}^{-1} \{Q_N, W_I^j\} \quad , \\
[W_I^j, H] &= \epsilon^{jk} \frac{\pi}{L_k} K_{1N}^{-1} \{J_N^k, W_I^j\} \quad . (7.9)
\end{align}

Next we consider the second matter coupling $\mathcal{L}_2$, (2.3). Field equations are
\begin{equation}
\frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\rho} f^I_{\mu\nu} = j^\mu \quad . (7.10)
\end{equation}

All Chern-Simons fields couple to the same current $j^\mu$. Inverting (7.10), one finds that the $I$-th Chern-Simons field $a_I^j$ has a coupling strength $\kappa_I$ to $\psi$:
\begin{equation}
\frac{1}{\kappa_I} = \sum_J K_{IJ}^{-1} \quad , \quad \frac{1}{\kappa_{\text{eff}}} = \sum_I \frac{1}{\kappa_I} \quad . (7.11)
\end{equation}

Hence
\begin{align}
a_I^j(x) &= \frac{\theta_I^j(t)}{L_j} - \frac{\Phi_I}{2L_1 L_2} \epsilon^{jk} x_k + \hat{a}_I^j(x) \quad , \\
\hat{a}_I^j(x) &= \int dy \ e^{jk} \nabla_k G(x - y) \left\{ \frac{2\pi}{\kappa_I} \psi^I(y) + \frac{\Phi_I}{L_1 L_2} \right\} \quad . (7.12)
\end{align}

There still are $M$ constraints:
\begin{equation}
\frac{Q}{\kappa_I} + \frac{\Phi_I}{2\pi} \approx 0 \quad \left(I = 1, \cdots, M\right) \quad . (7.13)
\end{equation}

Summing over $I$ in the above, one has
\begin{equation}
\frac{Q}{\kappa_{\text{eff}}} + \frac{\Phi_{\text{tot}}}{2\pi} \approx 0 \quad , \quad \Phi_{\text{tot}} = \sum_I \Phi_I \quad . (7.14)
\end{equation}

The Hamiltonian and momenta are given by
\begin{align}
H &= \int dx \ \frac{1}{2m} (D_k \psi)^\dagger (D_k \psi) \quad , \\
P^j &= -i \int dx \ \psi^\dagger \bar{D}_j \psi \quad , \\
\bar{D}_j &= \nabla_j - i \sum_I \frac{\theta_I^j}{L_j} + i \frac{\Phi_{\text{tot}}}{2L_1 L_2} \epsilon^{jk} x_k \quad . (7.15)
\end{align}
Relations corresponding to (7.5) and (7.6) are
\[
\begin{align*}
\{P^k, \theta_j^I\} &= -2\pi i \epsilon^{kj} \frac{1}{L_k} \kappa_I Q, \\
\{H, \theta_j^I\} &= -2\pi i \epsilon^{kj} \frac{1}{L_k} J^k, \\
\{P^k, \psi\} &= i \bar{D}_k \psi, \\
\{P^j, D_k \psi\} &= \frac{\epsilon^{jk}}{L_1 L_2} \Phi_{\text{tot}} - \frac{\epsilon^{jk}}{L_1 L_2} \kappa_{\text{eff}} \psi Q, \\
\{P^j, \bar{D}_k \psi\} &= \frac{\epsilon^{jk}}{L_1 L_2} \psi - \frac{\epsilon^{jk}}{L_1 L_2} \kappa \psi Q.
\end{align*}
\]

(7.16)

It is easy to see now that
\[
\begin{align*}
\{P^j, P^k\} &= i \epsilon^{jk} \frac{2\pi}{L_1 L_2} Q \left( \frac{Q}{\kappa_{\text{eff}}} + \frac{\Phi_{\text{tot}}}{2\pi} \right), \\
\{P^j, H\} &= i \epsilon^{jk} \frac{2\pi}{L_1 L_2} J^k \left( \frac{Q}{\kappa_{\text{eff}}} + \frac{\Phi_{\text{tot}}}{2\pi} \right).
\end{align*}
\]

(7.17)

Further,
\[
\begin{align*}
\{W_j^I, Q_N\} &= 0, \\
\{W_j^I, J_N^k\} &= \epsilon^{jk} \frac{2\pi}{m_N L_k} \kappa_{\text{eff}} Q_N W_j^I,
\end{align*}
\]

(7.18)

and
\[
\begin{align*}
\{W_j^I, P^k\} &= \epsilon^{jk} \frac{\pi}{L_k} \kappa_I \{Q, W_j^I\}, \\
\{W_j^I, H\} &= \epsilon^{jk} \frac{\pi}{L_k} \kappa_I \{J^k, W_j^I\}.
\end{align*}
\]

(7.19)

Note that (7.17) and (7.19) are formally obtained from (7.7) and (7.9) by identifying $Q_I = Q$ and $J^k_I = J^k$.

Finally we define
\[
\bar{W}_j = \prod_I W_j^I.
\]

(7.20)

This corresponds to the Wilson line operator in the effective theory discussed in Section 5. In the second matter coupling $L_2$, commutators between $\bar{W}_j$ and $(H, P^k)$ are evaluated from (7.19). For $H$, one has
\[
\begin{align*}
\{\bar{W}_j, H\} &= \epsilon^{jk} \frac{\pi}{L_k} \kappa_I \sum_{l} \frac{1}{\kappa_l} W_j^{l-1} (W_j^l J^k + J^k W_j^l) W_j^{l+1} \cdots W_j^M \\
&= \epsilon^{jk} \frac{\pi}{L_k} \kappa_I \left\{ \bar{W}_j J^k - \frac{2\pi}{m_L} \sum_{n=1}^{M-1} \frac{1}{\kappa_n} Q \bar{W}_j \\
&\quad + J^k \bar{W}_j \right\}
\end{align*}
\]

(7.21)
In the second equality we have made use of (7.18). The evaluation of the commutator $[\tilde{W}_j, P^k]$ is simpler as $W^I_j$ and $Q$ commute. To summarize one finds

$$[\tilde{W}_j, P^k] = \epsilon^{jk} \frac{\pi}{L_k \kappa_{\text{eff}}} \{ Q, \tilde{W}_j \},$$

$$[\tilde{W}_j, H] = \epsilon^{jk} \frac{\pi}{L_k \kappa_{\text{eff}}} \{ J^k, \tilde{W}_j \}. \quad (7.22)$$

The algebra defined by (7.17) and (7.22) is exactly the algebra of the effective theory which has been previously obtained in ref. [15]. We also remark that the same commutator relations among $H$, $P^k$, and $W^I_j$ hold in relativistic Dirac theories, as well.

8. Conclusion

In this paper we have unveiled some of the rich mathematical structure of Chern-Simons gauge theory on a torus. In addition to inducing matrix statistics among particles, the theory of multiple Chern-Simons gauge fields enriches the vacuum structure. Depending on the Chern-Simons coefficient matrix $K$, sometimes the theory can be mapped to an effective theory with just one Chern-Simons gauge field, but in general it contains greater degeneracy in the vacua. We have worked out two typical examples, finding rather interesting correspondence between the two theories.

Also we have examined the algebra generated by the Hamiltonian, momenta, and Wilson line operators in the multiple Chern-Simons theory. We have found that the Hamiltonian and momenta commute among themselves only in the physical Hilbert space.

Having clarified the vacuum structure of multiple Chern-Simons theory, one might ask what the Schrödinger equation for many particle systems is and whether or not there exists a singular gauge transformation which eliminates all interactions among particles except for the effect of the matrix statistics. Further, the multiple Chern-Simons theory must lead to representations of a generalized Braid group algebra. These problems are reserved for future investigation.

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