Dilute magnetic moments in an exactly solvable interacting host

Chandan Setty

Department of Physics & Astronomy, Rice Center for Quantum Materials, Rice University, Houston, Texas 77005, USA

Despite concerted efforts, the problem of dilute local moments embedded in a correlated conduction electron host such as Nd$_{1-x}$Ce$_x$CuO$_2$ persists due to lack of analytically controllable models. Here, we address the question: how do local moments couple to correlated but integrable hosts? We describe the conduction electrons by the model of Hatsugai-Kohmoto (HK) which has undergone a recent resurgence arising from its exact solvability, existence of Luttinger surfaces, and connections to Sachdev-Ye-Kitaev (SYK) thermodynamics. We derive an exact low energy “Kondo-HK” Hamiltonian and show the existence of additional spin-exchange coupling that is relevant in the renormalization group (RG) sense. This term is ferromagnetic and does not vanish at low energies yielding an algebraic enhancement of the Kondo temperature. “Poor man’s” scaling of couplings exhibits an exotic step-like RG flow between UV-IR fixed points attributed to severely restricted scattering phase space. This phenomenon is analogous to the flow of central charge in Zamolodchikov’s diagonal resonance scattering in integrable quantum field theories.

Introduction: When dilute local moments are placed in a metallic host, weakly interacting conduction electrons in the metal can screen the magnetic moment provided interactions between local moments are weak. The Kondo effect [1] as it is termed, occurs due to the formation of a spin-singlet between the local moment and conduction electron spins below a characteristic crossover energy scale called the Kondo temperature given by $T_K \propto e^{-1/(2JN_0)}$ ($J$ is the Kondo exchange coupling between the two spins and $N_0$ is the density of states). From a renormalization group (RG) standpoint, the coupling $J \equiv J(w)$ depends on the energy scale $w$ at which it is probed and corresponding exchange term in the effective Hamiltonian is marginal. Consequently, a perturbative “poor man’s” scaling of the coupling yields a $\beta(J)$ function of the form

$$\frac{dJ}{d\ln w} \equiv \beta(J) = -2J^2 N_0. \quad (1)$$

A key conclusion from the solution to Eq. (1) at low energies is that $J(w)$ renormalizes to infinity (zero) if $J(w_0) > 0$ ($J(w_0) < 0$), where $w_0$ is the original conduction electron bandwidth. Hence the formation of an antiferromagnetic spin-singlet quenches the local moment while no ferromagnetic coupling survives at low energy.

The aforementioned paradigm fails in the heavy fermion cuprate Nd$_{1-x}$Ce$_x$CuO$_2$ [2, 3] as electron correlations in the host render a breakdown of the quasiparticle concept. Hence the question of how local moments couple to strongly correlated hosts is highly non-trivial as one must either incorporate the conduction electron Hubbard or $t$-$J$ terms into the Anderson or Kondo models [3–29], rendering the problem notoriously difficult from the viewpoint of analytical tractability, or settle for reduced dimensionality of the host [27–33].

In this paper, we pursue a simplification by raising the issue of how local moments couple to metallic hosts that are correlated but integrable, and what is the fate of these couplings at low energy? A resolution of this question would be a significant step toward uncovering the thermodynamics of Kondo lattices in Nd$_{1-x}$Ce$_x$CuO$_2$ and its analogues. As the Hubbard model, minimal model applicable to the Cuprate normal state, is unsolved in two or more dimensions, we instead describe the conduction electrons by the Hatsugai-Kohmoto (HK) model [34] (the acronym HK is not to be confused with the Hubbard-Kondo model routinely used in literature). The HK is a minimal model for a doped Mott insulator that is integrable – there exist extensively many conserved quantities, one for each crystal momentum – leading to exact solvability. The model has undergone a revival due to existence of Luttinger surfaces (LSs), contours of Green function zeros in momentum space. It was recently demonstrated that pairing instabilities on a LS and their corresponding fluctuation thermodynamics resemble that of the $O(1)$ Sachdev-Ye-Kitaev free energy [35, 36]. Following these results, Ref. [37] derived a Cooper instability by adding an infinitesimal attractive interaction to the HK model whereas Ref. [38] reproduced the Cuprate Fermi arcs and pseudo gap. The HK interaction term has also been used recently to drive topological transitions [39] and assess stability conditions for LSs to perturbations [40]. In real space, the HK Hamiltonian comprises a quadratic nearest neighbor hopping term and a quartic interaction term that is non-zero only for electrons with a net zero center-of-mass. In momentum space, such a Hamiltonian acquires a particularly simple form

$$H_{HK} = \sum_{k\sigma} \xi_k c_{k\sigma}^\dagger c_{k\sigma} + \hat{U} \sum_k n_{k\uparrow} n_{k\downarrow}, \quad (2)$$

where $\xi_k$ is the band dispersion relative to the Fermi energy $\epsilon_F$, $c_{k\sigma}$ destroys an electron at momentum $k$, spin $\sigma$ and follows fermion anti-commutation relations, $n_{k\sigma}$ is the number operator, and $\hat{U}$ is the HK four fermion interaction which, unlike the Hubbard model, is local in momentum instead of position. A notable feature of Eq. (2) is that kinetic and interaction terms commute.
and the particle number is conserved for each \( k \) rendering exact solvability of the model. The ground state can be written in the particle number representation as \( | \psi_G, \{ \sigma_j \} \rangle = \prod_{k \in \Omega_1} c_{k \sigma}^\dagger \prod_{k \in \Omega_2} c^\dagger_{k \sigma} | 0 \rangle \) where \( \Omega_i \) is the region in momentum space with occupation number \( i \). The model exhibits a Mott gap (correlated metallic behavior) for \( \bar{U} > W \) \((\bar{U} < W)\) where \( W \) is the non-interacting bandwidth. In addition, the ground state has a large degeneracy resulting from all possible spin configurations \( \{ \sigma_j \} \) in the singly occupied regions of the Brillouin zone. The exact Green function is
\[
G_\sigma(k, i\omega_n) = \frac{1 - \langle n_{k\sigma} \rangle}{i\omega_n - \xi_k} + \frac{\langle n_{k\sigma} \rangle}{i\omega_n - (\xi_k + U)},
\]
where \( \langle n_{k\sigma} \rangle \) is the average occupation of state \( k \) with opposite spin \( \sigma \). An examination of Eq. 3 reveals a crucial simplification of the HK model – \( G_\sigma(k, i\omega_n) \) is additive in doubly occupied and unoccupied sectors of the theory with no entanglement between them. This is a consequence of the ground state being an eigenstate of the number operator which leads to exact cancellation of the mixing terms. A physical implication of this cancellation is the separation of Mott gap formation for \( \bar{U} > W \) and dynamical spectral weight transfer, two important ingredients of Mottness. While the two are intrinsically intertwined in the Hubbard model, the HK model harbors only a Mott gap with no dynamical spectral weight transfer thus making it considerably more tractable. Nevertheless, the lack of a quasiparticle picture due to ‘doublon-holon’ excitations, the presence of a Mott spectral gap, divergence of the self-energy leading to LSs, and violation of the Luttinger sum rule make the model irreconcilable with a Fermi liquid for any \( \bar{U} \neq 0 \). Our results below demonstrate that an inclusion of HK interaction term into the Anderson impurity model leads to fundamental differences with the Kondo effect in a Fermi liquid. Applying the Schrieffer-Wolf transformation to the Anderson-HK model we derive a low energy “Kondo-HK” model which – in addition to the marginal Kondo spin exchange \( J \) – acquires a relevant spin exchange term \( (D) \) as well as a six-fermion term \( (E) \) formed by coupling the electron density to spins. Unlike the conventional spin exchange coupling which is antiferromagnetic \( (J > 0) \), \( D \) is always ferromagnetic \( (D < 0) \) but does not vanish at low energy. With only the \( D \) term present, the Kondo temperature is enhanced by interactions yielding a power law dependence on the coupling \( (T_K \sim D) \) as opposed to an exponential. When \( J \) and \( E \) terms are turned on, only the coupling between \( J \) and \( D \) survives among the mixed terms and the Kondo-log in \( J \) is cut-off at low energies. Furthermore, the existence of extensively many conservation laws for the number operator forces exotic ‘step-like’ RG flows analogous to diagonal \( S \)-matrix resonance scattering previously studied in well known integrable field theories in high energy physics. Our calculations present novel insights into an age-old but persistent question of how local moments couple to correlated hosts where a quasiparticle picture is absent, and into the nature of thermodynamics and transport in dirty correlated systems. 

Model: We begin by including the HK interaction term in Eq. 2 into the Anderson correlated impurity model. The total Hamiltonian for the Anderson-HK model is given by
\[
H = H_0 + H_v
\]
\[
H_0 = \sum_{k \sigma} \xi_k c^\dagger_{k \sigma} c_{k \sigma} + \bar{U} \sum_k n_{k\uparrow} n_{k\downarrow}
+ \sum_\sigma \epsilon_d d^\dagger_\sigma d_\sigma + U n_{d\uparrow} n_{d\downarrow},
\]
\[
H_v = \sum_{k \sigma} V_k \left( \epsilon_{k\sigma} d_\sigma + d^\dagger_\sigma c_{k\sigma} \right),
\]
where the local moment with spin \( \sigma \) at the origin is created by the operator \( d_\sigma \) which follows fermion anticommutation relations, and \( n_{k\sigma} \) is the corresponding number operator. We have defined the bare conduction electron dispersion \( \xi_k = \epsilon_k - \epsilon_F \) and local moment energy \( \epsilon_d \), measured with respect to \( \epsilon_F \). \( U \) is the onsite local moment repulsion and \( V_k \equiv V \) is the hybridization between the conduction electrons and local moments which is chosen to be a constant. Throughout the manuscript, we choose \( \bar{U} < W \) so that the conduction electrons are in the correlated metal phase. For a Kondo-type interaction between the local moment and conduction electrons, we require a well-defined magnetic moment to exist. To his end, we choose to solve the problem in limit where \( \bar{U}, V \ll U \) so that the condition for local moment formation is not drastically altered from free electron case. This is ensured because the condition \( \bar{U} < W \) only weakly affects the density of states at the local moment energy \( (\epsilon_d < \epsilon_F) \), and local moment broadening still satisfies \( \Delta \ll U \). Despite the above hierarchy of scales and small \( \bar{U} \), the conduction electrons cannot be described by a Fermi liquid as long as \( \bar{U} \neq 0 \). We now derive the Kondo-HK model from the Anderson-HK model Eq. 4 by performing the Schrieffer-Wolf transformation on \( H \) to eliminate double occupancy. The effective low-energy Kondo-HK Hamiltonian is given as \( H_K = e^S H e^{-S} \) where the operator \( S \) is obtained by imposing the condition that linear in \( V \) terms vanish in \( H_K \). This condition is given as \( H_v + [S, H_0] = 0 \), so that the Kondo-HK Hamiltonian becomes \( H_K = H_0 + \frac{1}{2} [S, H_0] + O(V^3) \) to quadratic order in \( V \). To find the operator \( S \), we choose the ansatz
\[
S = V \sum_{k \sigma} \left( A'_k - B'_k n_{d\sigma} + D_k n_{k\sigma} + E'_k n_{k\sigma} n_{d\sigma} \right)
\left( c^\dagger_{k\sigma} d_\sigma - d_\sigma c_{k\sigma} \right),
\]
and substitute it into the condition \( H_v + [S, H_0] = 0 \). Solving for the primed coefficients we have \( A'_k = (\xi_k - \epsilon_d)^{-1}, B'_k = -U(\epsilon_d - \xi_k)^{-1}(U + \epsilon_d - \xi_k)^{-1} \),
Here $\sigma$ is the vector of $c_{k\sigma}$ electron Pauli matrices, $S_i(k)$ and $S_d$ are the spin operators for the conduction electrons and local moment respectively, $S^\perp(k)$, $S^\parallel(k)$ are the corresponding components of the ladder operator, $D_k \equiv V^2 D_k'$, $E_k \equiv V^2 E_k'$ and $J(k,k') \equiv -\frac{V^2 k^2}{(\xi - \epsilon_d - \epsilon_k + \epsilon_{k'}) + k - k'}$. The first two terms in Eq. 6 form the HK Hamiltonian, Eq. 2. The $J(k,k')$ term is the familiar spin-exchange Kondo term in a Fermi liquid. The terms proportional to $D_k$ and $E_k$ are entirely due to the HK interaction term and are absent in the Fermi liquid Kondo effect. Like the $J(k,k')$ term, $D_k$ is a four-fermion coupling but with two key distinctions. First, the presence of Kronecker delta function strongly constrains the scattering phase space and is attributed to integrability of the HK interaction. Second, for a quasiparticle with energy $\xi_k$ above $\epsilon_F$ and the existence of a well defined local moment, we require the local moment $U$ to be large enough so that we have $\epsilon_d + U > \xi_k$. This, in addition to the fact that $\xi_k - \epsilon_d > 0$, implies that $J(k,k') > 0$ (antiferromagnetic) while $D_k < 0$ (ferromagnetic). Finally, the term proportional to $E_k$ is a novel six-fermion term that is formed by the coupling of electron spins and density. The sign of $E_k$ can be negative for values of $U, \bar{U}$ such that $U + \epsilon_d > \bar{U} + \xi_k$ and $U + 2\epsilon_d < \bar{U} + 2\xi_k$. This condition is satisfied for intermediate values of $U, \bar{U} > 0$. While methods employed in Refs. [57-60] can, in principle, be generalized to seek exact solutions of $H_K$ in Eq. 6 we leave this for future work and focus rest of the paper on performing a perturbative $T$-matrix expansion using the fully polarized HK ground state and derive scaling equations and RG flow.

Scaling dimensions: Before we set out to calculate the scattering matrix elements, it is instructive to evaluate scaling dimensions of the various terms appearing in Eq. 6. This requires determining the scaling dimension of $c_{k\sigma}$ and $d_\sigma$ operators which can be done in two ways: setting the scaling dimensions of either the quadratic $\xi_k$ or the interacting $\bar{U}$ term to be zero. Written as an action, $H_K$ has a Matsubara time variable $\tau$ that scales as $\tau \rightarrow s^{-1}\tau$ whereas the momenta scale as $k_\parallel \rightarrow sk_\parallel$, $k_\perp \rightarrow k_\perp$ for $s \rightarrow 0$. Here $k_\parallel, k_\perp$ are components of momentum parallel and perpendicular to normal vector of the zero energy surface (Fermi or Luttinger surface). Fixing then the non-interacting term to be dimensionless yields $c_k \rightarrow \frac{1}{\sqrt{\xi}}c_k$ and $d_\sigma \rightarrow d_\sigma$. From this scheme we conclude that while the $J(k,k')$ term is marginal (scales as $s^0$), $D_k$ and $E_k$ terms are relevant ($s^{-1}$) as $s$ is taken to zero. If on the other hand we fix the HK interaction term $\bar{U}$ to be dimensionless, the $c_{k\sigma}$ operator also scales as $s^0$. In this scheme, the $J(k,k')$ and $E_k$ terms are irrelevant ($s^1$) while the $D_k$ term is marginal ($s^0$) as $s$ is taken to zero. From both these schemes, it is evident that the following poor man’s scaling analysis will be controlled by the four-fermion $D_k$ term regardless of whether the perturbation is considered about the gaussian or the strongly coupled fixed point.

Scattering rate: We now rewrite Eq. 6 as $H_K = H_{HK} + H_e$ where $H_e$ consists of all the exchange terms arising from the coupling of local moments and the correlated host. For simplicity, we will also henceforth treat the couplings to be independent of the density. The sign of $\xi_k$ above $\epsilon_F$ is set close to $\epsilon_F$. We will also ignore the effect of the coupling $E$ in the current study. This can happen if we choose $\bar{U}$ to be small and made to satisfy the condition $\bar{U} \sim U - 2\epsilon_F + 2\epsilon_d$ with $D \neq 0$. In addition, an evaluation of the six-fermion diagrams (see Fig. 3) shows that $E$ is effectively decoupled from $J$ and $D$ channels at low temperatures. To evaluate the scattering rate, we write the Lippmann-Schwinger expansion of the $T$-matrix given by

$$T = H_e + H_{\bar{U}} \frac{1}{E_x - H_{HK} + i\eta} H_e + \ldots,$$

(7)

where $E_x$ is the excitation energy. To further evaluate the matrix elements $T_{k\sigma,k'\sigma'} = \langle k|T|k'\sigma'\rangle$ between in state $|k\sigma\rangle$ and out state $|k'\sigma'\rangle$, we define a ‘holon’ excitation $|k'\sigma'\rangle = c_{k'\sigma'}^\dagger (1 - n_{k'\sigma'})|\Omega\rangle$ at an energy $\xi_0$ close to $\epsilon_F$, where $|\Omega\rangle$ is the ground state of the HK model. As previously mentioned, the HK has a huge ground state degeneracy due to all possible spin configurations in the single occupied regions of the Brillouin zone [24]. For the purposes of calculations, however, we choose the ‘pure’ state that is maximally polarized with spin $\sigma$ as the ground state given by $|\Omega\rangle = \prod_{k \in \Omega} c_{k\uparrow}^\dagger c_{k\downarrow}^\dagger \prod_{k \in \Omega} n_{k\sigma}$ [61]. In addition, to calculate the matrix elements we note that ground state is specified by the density operator and can hence be described by the Fock basis $|\{n_j\}\rangle$. The matrix elements can therefore be factorized [57, 61, 62] in terms of an occupation average in the ground state

$$\langle n_{k\sigma}\rangle = \frac{e^{-\beta \xi_k} + e^{-\beta(2\xi_k + \bar{U})}}{1 + 2e^{-\beta \xi_k} + e^{-\beta(2\xi_k + \bar{U})}},$$

(8)

which at zero temperature equals $\frac{1}{2}$ for $\bar{U} < \xi_k < 0$.
and 0 (1) for $\xi_k > 0$ ($\xi_k < -\bar{U}$). To start, we keep only $D \neq 0$ and set all other couplings to zero. Calculating the leading order $T_{kk'}^{(1)}$ and next-to-leading order $T_{kk'}^{(2)}$ contributions to the $T$-matrix at low temperatures yields the diagonal ‘elastic’ scattering matrix

$$T_{kk'}^{(1)} + T_{kk'}^{(2)} = D (S_d \cdot \sigma) \left( 1 - \frac{2D}{E_x - E_0} \right) \delta_{kk'} \quad (9)$$

where $E_0$ is the renormalized ground state energy obtained by successive actions of $H_{HK}$ on $|\Omega\rangle$. To obtain the scaling equation we integrate only within an energy shell $[-w, -w + \delta w] \cup [w - \delta w, w]$. The second order correction to the coupling $D$ is given by

$$\delta D = -\frac{D^2 \delta(w - \xi_0)}{E_x - E_0} \delta w. \quad (10)$$

The Dirac delta function signifies the resonance condition at the energy of the excitation. This occurs entirely due to the severely restricted phase space for scattering in the $D$ term in Eq. (9). As a consequence, the in and out states have the same $k$ (diagonal $T$-matrix) thereby constraining the internal loop momenta and energy as well. Integrating $\delta D$ from an IR ($w'$) to UV ($w$) scale gives the running coupling

$$D(w') = \frac{D(w)}{1 - D(w) \Theta(\xi_0 - w')(E_x - E_0)^{-1}}, \quad (11)$$

where $\Theta(x)$ is the Heaviside function. Thus, $D(w')$ behaves as a step-like function with $\xi_0$ marking the point of discontinuity and does not vanish at the lowest energies despite being ferromagnetic. Instead the system simply slides between two fixed points at $w = \xi_0$ (see red curve in Fig. 1). The Kondo temperature when only $D \neq 0$ ($T_{KD}$) can therefore be determined by a diverging $D(w')$ which, for $w' < \xi_0$, gives $E_x - E_0 \approx T_{KD} \sim D$. The modified $\beta$ function for only $D \neq 0$ is therefore given as

$$\frac{dD}{dw} |_{\ln w} = \begin{cases} 0, & w \neq \xi_0 \\ -\frac{N_0 D^2}{T_{KD}}, & w = \xi_0. \end{cases}$$

Turning on $J$ leads to a term that resembles Kondo effect in a Fermi liquid as well as mixing between the two couplings. The set of coupled differential equations for $J(w)$ and $D(w)$ are given by

$$J' = -\frac{N_0 J^2}{w} - \frac{-2JD \delta(w - \xi_0)}{T_{KD}} \quad (12)$$

$$D' = -\frac{D^2 \delta(w - \xi_0)}{T_{KD}^2} - \frac{-2DJ \delta(w - \xi_0)}{T_{KD}} \quad (13)$$

where $J'$ and $D'$ are derivatives of the couplings with respect to $w$. The solutions to the above set of differential equations is shown in Fig. 1 with initial conditions $J(w) > 0$ and $D(w) < 0$ in the UV limit. In the absence of any mixing between the two (dark blue and red curves in Fig. 1), $J(w)$ exhibits the usual Kondo log divergence and $D(w)$ shows a step-like flow as indicated above. With an increase in mixing between the couplings (lighter curves in Fig. 1), $J(w)$ also inherits a step-like RG flow at $w = \xi_0$ and the $w = 0$ divergence is cut-off to a finite value. Therefore, unlike the Fermi liquid Kondo case where perturbation theory breaks down at lowest energies due a divergence in $J(w)$, a well-defined expansion is still valid in the Kondo-HK model.

**Fig. 1.** Scaling RG flow of Kondo couplings $J(w)$ and $D(w)$. The lighter shades correspond to higher couplings between $J(w)$ and $D(w)$. The shaded region corresponds to resonance energy of quasiparticle denoted by $\xi_0$ in the text.

**Discussions:** As alluded to above, integrability of the HK model is at the heart of the step-like RG flow pattern shown in Fig. 1. In the canonical Kondo effect, the internal loop quantum numbers label virtual states and are hence arbitrary. In the Kondo-HK model, however, the Kronecker delta functions appearing in Eq. (6) strongly constrain these quantum numbers to values set by the in and out states with no role for virtual states; therefore, when only $D \neq 0$, the scattering matrix is diagonal in momentum. This is a direct consequence of extensively many conserved density operators, one for each $k$ and a characteristic feature of integrability [63]. The step-like discontinuity in the flow occurs when the scaling energy equals the energy of the quasiparticle that undergoes scattering while the couplings determine the ground state and excitation gap at that energy. Thus properties of the system drastically differ if they are probed at an energy slightly above (UV) or below (IR) this scale, i.e., if the scattered particle is effectively massless or massive with respect to the probe energy.

An analogy can be drawn between the physics described above and diagonal resonance scattering in relativistic integrable quantum field theories put forward by Zamolodchikov and others [43, 55]. An example is the homogenous Sine-Gordon model [43, 55] (for a gentler introduction to these topics, the reader is encouraged to follow Refs. [66, 67]) which is an integrable deformation of a conformal field theory (CFT). The model describes
scattering of solitonic solutions of the Sine-Gordon model labelled by certain quantum numbers (mass, resonance parameters etc). Like earlier, integrability of the model implies dynamics are constrained resulting in a diagonal “elastic” scattering matrix. The ground state energy is determined by the effective central charge \( c(w) \) – a quantity that flows with the RG scale and is evaluated from solutions of the Thermodynamic Bethe Ansatz (TBA). The effective central charge hence plays the analogous role of HK-Kondo couplings. Crucially, there is a step-like jump in \( c(w) \) when the RG scale equals the particle mass determined by the \( in \) and \( out \) solitonic states. More generally, multiple plateaux are possible (“staircase” models) when several particles can scatter to give rise to well separated mass scales; in this scenario, the system slides from one CFT to the next depending on the probe energy and eventually settles into an IR fixed point. It will be interesting to reproduce such staircase patterns in the Kondo-HK model when multi-particle scattering is taken into account and will be the subject of future work.

* email for correspondence: csetty@rice.edu

[1] Piers Coleman, *Introduction to many-body physics* (Cambridge University Press, 2015).
[2] T Brugger, T Schreiner, G Roth, P Adelmann, and G Czjzek, “Heavy-fermion-like excitations in metallic nd 2-y ce y cuo 4,” Physical review letters 71, 2481 (1993).
[3] P Fulde, V Zevin, and G Zwicknagl, “Model for heavy-fermion behavior of nd 1.8 ce 0.2 cuo 4,” Zeitschrift für Physik B Condensed Matter 92, 133–135 (1993).
[4] Tom Schork and Peter Fulde, “Interaction of a magnetic impurity with strongly correlated conduction electrons,” Physical Review B 50, 1345 (1994).
[5] J. Igarashi, T. Tonegawa, M. Kaburagi, and P. Fulde, “Magnetic impurities coupled to quantum antiferromagnets in one dimension,” Phys. Rev. B 51, 5814–5823 (1995).

Giniyat Khaliullin and Peter Fulde, “Magnetic impurity in a system of correlated electrons,” Physical Review B 52, 9514 (1995).
[7] J. Igarashi, K. Murayama, and P. Fulde, “Magnetic impurity coupled to a strongly correlated electron system in two dimensions,” *Phys. Rev. B* 52, 15966–15974 (1995).
[8] Tom Schork, “Magnetic impurity coupled to interacting conduction electrons,” Physical Review B 53, 5626 (1996).
[9] Samuel Moukouri, Liang Chen, and Laurent G Caron, “Local moments coupled to a strongly correlated electron chain,” Physical Review B 53, R488 (1996).
[10] K Itai and P Fazekas, “Interaction effect in the kondo energy of the periodic anderson-hubbard model,” Physical Review B 54, R752 (1996).
[11] Naokazu Shibata, Tomotoshi Nishino, Kazuo Ueda, and Chikara Ishii, “Spin and charge gaps in the one-dimensional kondo-lattice model with coulomb interaction between conduction electrons,” Physical Review B 53, R8828 (1996).
[12] Piers Coleman and AM Tsvelik, “Local moments in an interacting environment,” *Physical Review B* 57, 12757 (1998).
[13] Benny Davidovich and V Zevin, “Magnetic impurity in a metal with correlated conduction electrons: An infinite-dimensions approach,” *Physical Review B* 57, 7773 (1998).
[14] Walter Hofstetter, Ralf Bulla, and Dieter Vollhardt, “Anderson impurity in a correlated conduction band,” Physical review letters 84, 4417 (2000).
[15] Ryu Takayama and Osamu Sakai, “Single impurity anderson model with coulomb repulsion between conduction electrons on the nearest-neighbour ligand orbital,” Journal of the Physical Society of Japan 67, 1844–1847 (1998).
[16] M Neef, S Tornow, V Zevin, and G Zwicknagl, “Kondo effect in a metal with correlated conduction electrons: Diagrammatic approach,” Physical Review B 68, 035114 (2003).
[17] P Lederer and Marcelo Javier Rozenberg, “Impurity scattering in a strongly correlated host,” *EPL (Europhysics Letters)* 81, 67002 (2008).
[18] Tom Schork and Stefan Blawid, “Periodic anderson model with correlated conduction electrons,” Physical Review B 56, 6559 (1997).
[19] Tom Schork, Stefan Blawid, and Jun-ichi Igarashi, “Kondo lattice model with correlated conduction electrons,” Physical Review B 59, 9888 (1999).
[20] Ryota Sato, Takuma Ohashi, Akihisa Koga, and Norio Kawakami, “Periodic anderson model with degenerate orbitals: linearized dynamical mean field theory approach,” Journal of the Physical Society of Japan 73, 1864–1869 (2004).
[21] S Nishimoto and P Fulde, “Single magnetic impurity in a correlated electron system: Density-matrix renormalization group study,” Physical Review B 76, 035112 (2007).
[22] Robert Peters and Thomas Pruschke, “Magnetic phases in the correlated kondo-lattice model,” Physical Review B 76, 245101 (2007).
[23] Tsuneiya Yoshida, Takuma Ohashi, and Norio Kawakami, “Effects of conduction electron correlation on heavy-fermion systems,” Journal of the Physical Society of Japan 80, 064710 (2011).
[24] I Hagymási, K Itai, and J Sólyom, “Periodic anderson model with correlated conduction electrons: variational and exact diagonalization study,” Physical Review B 85, 235116 (2012).
[25] JPL Faye, MN Kiselev, P Ram, B Kumar, and D Sénéchal, “Phase diagram of the hubbard-kondo lattice model from the variational cluster approximation,” Physical Review B 97, 235151 (2018).
[26] Bikash Padhi, Apoorv Tiwari, Chandan Setty, and Philip W Phillips, “Log-rise of the resistivity in the holo- graphic kondo model,” Physical Review D 97, 066012 (2018).
[27] Dung-Hai Lee and John Toner, “Kondo effect in a luttinger liquid,” Physical review letters 69, 3378 (1992).
[28] Akira Furusaki and Naoto Nagaosa, “Kondo effect in a tomonaga-luttinger liquid,” Physical review letters 72, 892 (1994).
[29] YM Li, “Effect of conduction-electron interactions on anderson impurities,” Physical Review B 52, R6979 (1995).
[30] Philip Phillips and Nancy Sandler, “Enhanced local moment formation in a chiral luttinger liquid,” Physical Review B 53, R468 (1996).
[31] Per Fröjd and Henrik Johansson, “Magnetic impurity in a lattice liquid: A conformal field theory approach,” Phys. Rev. B 53, 3211–3236 (1996).

[32] Avraham Schiller and Kevin Ingersent, “Renormalization-group study of a magnetic impurity in a lattice liquid,” EPL (Europhysics Letters) 39, 645 (1997).

[33] David Withoff and Eduardo Fradkin, “Phase transitions in gapless fermi systems with magnetic impurities,” Physical review letters 64, 1835 (1990).

[34] Yasuhiro Hatsugai and Mahito Kohmoto, “Exactly solvable model of correlated lattice electrons in any dimensions,” Journal of the Physical Society of Japan 61, 2056–2069 (1992).

[35] Chandan Setty, “Pairing instability on a lattice surface: A non-fermi liquid to superconductor transition and its sachdev-ye-kitaev dual,” Physical Review B 101, 184506 (2020).

[36] Chandan Setty, “Superconductivity from lattice surfaces: Emergent sachdev-ye-kitaev physics with infinite-body interactions,” Physical Review B 103, 014501 (2021).

[37] Philip W Phillips, Luke Yeo, and Edwin W Huang, “Exact theory for superconductivity in a doped mott insulator,” Nature Physics 16, 1175–1180 (2020).

[38] Kun Yang, “Exactly solvable model of fermi arcs and pseudogap,” Physical Review B 103, 024526 (2021).

[39] Huai-Shuang Zhu, Zhidan Li, Qiang Han, and ZD Wang, “Topological s-wave superconductors driven by electron correlation,” Physical Review B 103, 024514 (2021).

[40] Edwin Huang, Gabriele La Nave, and Philip W Phillips, “Doped mott insulators break z2 symmetry of a fermi liquid: Stability of strongly coupled fixed points,” arXiv preprint arXiv:2103.03256 (2021).

[41] Philip Phillips, “Colloquium: Identifying the propagating charge modes in doped mott insulators,” Reviews of Modern Physics 82, 1719 (2010).

[42] John R Schrieffer and Peter A Wolff, “Relation between the anderson and kondo hamiltonians,” Physical Review 149, 491 (1966).

[43] Al B Zamolodchikov, Resonance factorized scattering and roaming trajectories, Tech. Rep. (1991).

[44] Al B Zamolodchikov, “Thermodynamic bethe ansatz for rsos scattering theories,” Nuclear Physics B 358, 497–523 (1991).

[45] Al B Zamolodchikov, “Tba equations for integrable perturbed su (2) k× su (2) lsu (2) k+l coset models,” Nuclear Physics B 366, 122–132 (1991).

[46] Al B Zamolodchikov, “From tricritical ising to critical ising by thermodynamic bethe ansatz,” Nuclear Physics B 358, 524–546 (1991).

[47] Al B Zamolodchikov, “Two-point correlation function in scaling lee-yang model,” Nuclear Physics B 348, 619–641 (1991).

[48] Marcio Jose Martins, “Renormalization-group trajectories from resonance factorized s matrices,” Physical review letters 69, 2461 (1992).

[49] Patrick Dorey and Francesco Ravanini, “Staircase models from affine toba field theory,” International Journal of Modern Physics A 8, 873–893 (1993).

[50] OA Castro-Alvaredo, A Fring, C Korff, and JL Miramontes, “Thermodynamic bethe ansatz of the homogeneous sine-gordon models,” Nuclear Physics B 575, 535–560 (2000).

[51] Andre Leclair, Jose Maria Roman, and German Sierra, “Russian doll renormalization group and kosterlitz–thouless flows,” Nuclear physics B 675, 584–606 (2003).

[52] OA Castro-Alvaredo, J Dreißig, and A Fring, “Integrable scattering theories with unstable particles,” The European Physical Journal C-Particles and Fields 35, 393–411 (2004).

[53] Patrick Dorey, Guy Siviour, and Gábor Takács, “Form factor realocalisation and interpolating renormalisation group flows from the staircase model,” Journal of High Energy Physics 2015, 54 (2015).

[54] Elias Kiritsis, Francesco Nitti, and Leandro Silva Pimenta, “Exotic rg flows from holography,” Fortschritte der Physik 65, 1600120 (2017).

[55] Chanyong Park and Jung Hun Lee, “Exotic rg flow of entanglement entropy,” Physical Review D 101, 086008 (2020).

[56] H Alloul, J Bobroff, M Gabay, and PJ Hirschfeld, “Defects in correlated metals and superconductors,” Reviews of Modern Physics 81, 45 (2009).

[57] Natan Andrei, “Diagonalization of the kondo hamiltonian,” Physical Review Letters 45, 379 (1980).

[58] Natan Andrei, K Furuya, and JH Lowenstein, “Solution of the kondo problem,” Reviews of modern physics 55, 331 (1983).

[59] PB Vigman, “Exact solution of sd exchange model at t=0,” JETP Lett 31, 364 (1980).

[60] AM Tsvelick and PB Wiegmann, “Exact results in the theory of magnetic alloys,” Advances in Physics 32, 453–713 (1983).

[61] Pawel Danielewicz, “Quantum theory of nonequilibrium processes, i,” Annals of Physics 152, 239–304 (1984).

[62] Philip W Phillips, Chandan Setty, and Shuyi Zhang, “Absence of a charge diffusion pole at finite energies in an exactly solvable interacting flat-band model in d dimensions,” Physical Review B 97, 195102 (2018).

[63] Benjamin Doyon, “Introduction to integrable quantum field theory,” Lecture notes for a course given at Durham University (2008).

[64] Carlos R Fernandez-Pousa and J Luís Miramontes, “Semi-classical spectrum of the homogeneous sine-gordon theories,” Nuclear Physics B 518, 745–769 (1998).

[65] J Luís Miramontes and CR Fernández-Pousa, “Integrable quantum field theories with unstable particles,” Physics Letters B 472, 392–401 (2000).

[66] Patrick Dorey and J Luís Miramontes, “Aspects of the homogeneous sine-gordon models,” arXiv preprint hep-th/0211174 (2002).

[67] Jesús Cuevas-Maraver, Panayotis G Kevrekidis, and Floyd Williams, “The sine-gordon model and its applications,” Nonlinear Systems and Complexity (Switzerland: Springer) (2014).
APPENDIX

FIG. 2. Feynman diagrams contributing to scattering probabilities between interacting conduction electrons (solid lines) and impurity spins (dashed lines). The four (six) fermion vertices are denoted by open circles (solid disks).

In the Appendix, we expand on details of calculations appearing in the main text. Fig. 2 shows a summary of distinct Feynman diagrams contributing to the scattering amplitude at second order in the perturbation. The top row contains only four-fermion vertices (open circles) like those that appear in the canonical Kondo effect for a Fermi liquid. These diagrams contribute to $D(w)^2$, $J(w)^2$ or $D(w)J(w)$ terms in the RG equations. The second and third rows are mixed diagrams that contain both six-fermion (solid disks) and four-fermion vertices. These diagrams contribute to mixing terms of the form $D(w)E(w)$ or $J(w)E(w)$. Finally, the bottom two rows contain purely six-fermion vertices and contribute to $E(w)^2$ terms in the $T$-matrix. To understand the step-like RG flow of $D(w)$ outlined in the main text, we recall the original perturbative argument for magnetic moments in a Fermi liquid. The leading order contribution to the impurity-conduction electron coupling is

$$T^{(1)}_{kk'} = J(S_d, \hat{\sigma})(1 - f_k)(1 - f_{k'}).$$

To second order we have

$$T^{(2)}_{kk'} = J^2 \sum_i \sigma^i_{\sigma_\alpha, \sigma_\alpha} S^i_{\alpha d} S^j_{d} \left( \frac{1 - f_k}{E_x - E_0 - \xi_p} \right) - J^2 \sum_i \sigma^i_{\sigma_\alpha, \sigma_\alpha} S^i_{\alpha d} S^j_{d} \left( \frac{1 - f_k}{E_x - E_0 - \xi_k + \xi_p - \xi_{k'}} \right)$$

where $E_0$ is the ground state energy. We next break up $p$ integral into states below and above the Fermi energy and perform the relevant integrals. Utilizing the relations $[\sigma^i, \sigma^j] = 2i\epsilon_{ijk}\sigma^k$ and $2iJ^2S_{\alpha}^i S_{d}^j \epsilon_{ijk} = -4J^2(S_{d}^i \hat{\sigma})$ where $\epsilon_{ijk}$ are the Levi-Civita constants, we can combine the two terms proportional to $S_{\alpha}^i S_{d}^j$. At low temperatures and $k, k'$ states above the fermi energy we can use $f_k, f_{k'} \sim 0$ and write

$$T^{(1)}_{kk'} + T^{(2)}_{kk'} = J(S_d, \hat{\sigma}) \left( 1 + 2N_0J\ln \left| \frac{w}{E_x - E_0} \right| \right).$$

The perturbative correction to the original coupling $J$ is hence proportional to $J^2$, i.e., $\delta J = -2J^2N_0d\ln w$. We now apply a similar argument for the coupling $D$. For the $T$-matrix contributions from the $D$ terms, we have additional Kronecker functions appearing in the second line of Eq. 6. These constrain the incoming, outgoing and internal loop momenta as $p = k = k'$ making the $p$ integral trivial. Therefore, the first and second order terms are

$$T^{(1)}_{kk'} = D(S_d, \hat{\sigma})(1 - f_k)^2$$

$$T^{(2)}_{kk'} = \delta_{kk'}D^2 \sum_i \sigma^i_{\sigma_\alpha, \sigma_\alpha} S^i_{\alpha d} S^j_{d} \left( \frac{1 - f_k}{E_x - E_0} \right)^3 - \delta_{kk'}D^2 \sum_i \sigma^i_{\sigma_\alpha, \sigma_\alpha} S^i_{\alpha d} S^j_{d} \left( \frac{1 - f_k)^2}{E_x - E_0} \right).$$

$E_{\sigma}$ perform the relevant integrals. Utilizing the relations $f_{\sigma}$ states above the fermi energy we can use $\epsilon_{\sigma}$ to the impurity-conduction electron coupling is
For an excitation with energy $\xi_k \geq 0$, we have $f_k \sim 0$ at low temperatures and only the first term in $T_{kk'}^{(2)}$ above contributes. Using the relation $\sigma^i \sigma^j = \delta_{ij} + i \epsilon_{ijl} \sigma^l$ for the $\sigma$ operator (similarly for the $S_d$ operator) and keeping only the terms proportional to $(S_d, \vec{\sigma})$, we have Eq. 9 of the main text

$$T_{kk'}^{(1)} + T_{kk'}^{(2)} = D(S_d, \vec{\sigma}) \left( 1 - 2 \frac{D}{E_x - E_0} \right) \delta_{kk'}.$$  \hspace{1cm} (19)

To derive the scaling equations, we rewrite Eq. 18 by explicitly inserting a Dirac delta function and converting the momentum sum into an energy integration over a narrow energy window $[-w, -w + \delta w] \cup [w - \delta w, w]$. Since only the first term contributes, we have

$$T_{kk'}^{(2)} = \delta_{kk'} D^2 \int_{w-\delta w}^{w} d\xi_p \sum_{\alpha} \sigma^i_{\alpha \sigma} \sigma^j_{\alpha \sigma} S^i_d S^j_d \frac{(1 - f_k)^2(1 - f_p)}{E_x - E_0} \delta(\xi_p - \xi_k).$$  \hspace{1cm} (20)

The integral above is non-zero only when $w - \delta w < \xi_p = \xi_k \equiv \xi_0 < w$. Using spin identities like before and collecting terms proportional to $(S_d, \vec{\sigma})$ and we are left with

$$T_{kk'}^{(2)} = -\delta_{kk'} D^2 \frac{S_d(\vec{\sigma})}{E_x - E_0} \Theta(w - \xi_0) \Theta(\xi_0 - w + \delta w) \approx -\delta_{kk'} D^2 \frac{S_d(\vec{\sigma})}{E_x - E_0} \delta(\xi_0 - w) \delta w.$$  \hspace{1cm} (21)

As described in the main text, this implies the correction to the original coupling $D$ due to the perturbation can be expressed as a differential equation using a Kronecker function given by

$$\frac{dD}{w} \mid_{\ln w} = \begin{cases} 
0, & w \neq \xi_0 \\
-\frac{N_0 D^2}{E_x - E_0}, & w = \xi_0.
\end{cases}$$

The solution to the differential equation above can be obtained readily by integrating $D(w)$ from a low energy scale $w'$ to a high energy scale $w'$ resulting in Eq. 11 of the main text. A similar argument follows for the mixed terms proportional to $J(w)D(w)$ when $J(w)$ is turned on. The perturbative correction to $D(w)$ ($J(w)$) due to $J(w)$ ($D(w)$) looks like Eq. 18 where $D(w)^2$ is simply replaced by the product $J(w)D(w)$. 