A SHORT SIMPLICIAL h-VECTOR AND THE UPPER BOUND THEOREM

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Abstract. We verify the Upper Bound Conjecture (UBC) for a class of odd-dimensional simplicial complexes that in particular includes all Eulerian simplicial complexes with isolated singularities. The proof relies on a new invariant of simplicial complexes — a short simplicial h-vector.

1. Introduction

The goal of this note is to prove an extension of the Upper Bound Theorem for (simplicial) polytopes. The main tool in the proof is a certain new invariant of simplicial complexes, which is a simplicial analog of a short cubical h-vector introduced by Adin.

We start by recalling several definitions. A (finite) simplicial complex $\Delta$ is pure if each maximal face of $\Delta$ has the same dimension. A pure simplicial complex $\Delta$ is Eulerian if for every face $F$ of $\Delta$ (including the empty face) the Euler characteristic of its link is equal to the Euler characteristic of the sphere of the same dimension, that is,

$$\chi(\text{lk}F) = 1 + (-1)^{\dim(\text{lk}F)}.$$ 

In particular, by Poincaré duality, every odd-dimensional homology manifold is Eulerian. (Recall that a simplicial complex $\Delta$ is a homology manifold if its geometric realization $X$ possesses the following property: for every $p \in X$ and every $i < \dim X$, $H_i(X, X - p) = 0$, while $H_{\dim X}(X, X - p) \cong \mathbb{Z}$. Here $H_i(X, X - p)$ denotes the $i$-th relative singular homology with coefficients $\mathbb{Z}$.)

The Upper Bound Conjecture (abbreviated UBC) proposed by Motzkin in 1957 (see [1]) asserts that among all $d$-dimensional (simplicial) polytopes with $n$ vertices, the number of $i$-dimensional faces (for every $i = 1, \ldots, d - 1$) is maximized by the cyclic polytope $C_d(n)$. Over the last 40 years this conjecture has been treated extensively by many mathematicians: in 1970, McMullen proved the UBC for polytopes; McMullen’s result was preceded in 1964 by a surprising work of Klee, where he verified that the UBC holds for all Eulerian complexes with a sufficiently large number of vertices, and conjectured that it holds for all Eulerian complexes [4]; in 1975 Stanley proved the UBC for arbitrary triangulations of spheres [3],[11], and in 1998 Novik verified the UBC for triangulations of odd-dimensional manifolds and several classes of even-dimensional manifolds [7].

In this note we will prove the UBC for a class of odd-dimensional simplicial complexes that in particular includes all odd-dimensional Eulerian complexes whose geometric realization has isolated singularities. More precisely, we obtain the following theorem in which $f_i(\Delta)$ denotes the number of $i$-dimensional faces of a complex $\Delta$, the values $\beta_i(\Delta) = \dim(\bar{H}_i(\Delta))$ denote the reduced Betti numbers...
of $\Delta$ over a field of characteristic 0, and $C_d(n)$ is a $d$-dimensional cyclic polytope on $n$ vertices.

**Theorem 1.** Let $\Delta$ be a pure $(2k+1)$-dimensional simplicial complex on $n$ vertices, such that for every vertex $v$ of $\Delta$, the link of $v$ is either a homology manifold whose Euler characteristic is 2, or an oriented homology manifold satisfying the following condition

$$\beta_k(\text{lk } v) \leq 2\beta_{k-1}(\text{lk } v) + 2 \sum_{i=0}^{k-3} \beta_i(\text{lk } v).$$

Then $f_i(\Delta) \leq f_i(C_{2k+2}(n))$ for $i = 1, 2, \ldots, 2k + 1$.

The main ingredient in the proofs is a new invariant of simplicial complexes, $\tilde{h}(\Delta) = (\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{\dim(\Delta)})$, which is a simplicial analog of the short cubical $h$-vector introduced by Adin (see [1]). We give its definition and list some of its properties in the next section. Section 3 is devoted to a proof of Theorem 1. Section 4 contains several remarks and additional results on the UBC and the $\tilde{h}$-vector.

2. $\tilde{h}$-vector

In this section we introduce the notion of $\tilde{h}$-vector for pure simplicial complexes and list some of its properties. Let us begin by recalling definitions of $f$-vectors and $h$-vectors. For a $(d-1)$-dimensional simplicial complex $\Delta$, its $f$-vector, denoted $f(\Delta)$, is a vector $(f_{d-1}, f_0, f_1, \ldots, f_{d-1})$ where $f_i$ counts the number of $i$-dimensional faces. In particular, $f_{d-1} = 1$, $f_0$ is the number of vertices of $\Delta$, and $f_1$ is the number of edges. The $h$-vector of $\Delta$, denoted $h(\Delta)$, is a vector $(h_0, h_1, \ldots, h_d)$ where

$$h_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta), \quad i = 0, 1, \ldots, d. \quad (1)$$

Equivalently,

$$f_{j-1}(\Delta) = \sum_{i=0}^{j} \binom{d-i}{d-j} h_i(\Delta), \quad j = 0, 1, \ldots, d. \quad (2)$$

Ron Adin [1, eq. (1), (11)] defined for any cubical complex $C$ its short cubical $h$-vector, denoted $h^{(sc)}(C) = (h_0^{(sc)}, h_1^{(sc)}, \ldots, h_{\dim(C)}^{(sc)})$. It was later observed by G. Hetyei that if $C$ is pure, then $h^{(sc)}(C) = \sum_{v \in V} h(\text{lk } v)$, where $V$ is the set of vertices of $C$. (Note that the links of the vertices in a cubical complex are simplicial complexes, and hence the $h$-vector $h(\text{lk } v)$ is well-defined.)

Similarly to the short cubical $h$-vector, we define a short simplicial $h$-vector, denoted $\tilde{h}$, as follows.

**Definition 1.** Let $\Delta$ be a pure $(d-1)$-dimensional simplicial complex on the vertex set $V$. Define

$$\tilde{h}(\Delta) = (\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{d-1}) := \sum_{v \in V} h(\text{lk } v),$$

so in particular $\tilde{h}_i(\Delta) := \sum_{v \in V} h_i(\text{lk } v)$.

The next lemma gives several properties of $\tilde{h}$.
Lemma 1. (i) Let $\Delta$ be a pure $(d-1)$-dimensional simplicial complex. Then
\[ \tilde{h}_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} (j+1) \binom{d-1-j}{d-1-i} f_j(\Delta) \quad (0 \leq i \leq d-1) \] and
\[ f_j(\Delta) = (j+1)^{-1} \sum_{i=0}^{j} \binom{d-1-i}{d-1-j} \tilde{h}_i(\Delta) \quad (0 \leq j \leq d-1). \]

In particular, the $f$-numbers of a simplicial complex are non-negative linear combinations of its $\tilde{h}$-numbers.

(ii) If $\Delta$ is a pure $(2k+1)$-dimensional simplicial complex such that the link of every vertex is a homology manifold, then the $f$-numbers of $\Delta$ are non-negative linear combinations of $\tilde{h}_0, \tilde{h}_1, \ldots, \tilde{h}_{k+1}$. In other words,
\[ f_j(\Delta) = \sum_{i=0}^{k+1} b_j^i \tilde{h}_i(\Delta) \quad 0 \leq j \leq 2k+1, \]
where the coefficients $b_j^i$ are independent of $\Delta$ and are non-negative.

Proof: Since every $j$-dimensional simplex has $j+1$ vertices, it follows that
\[ \sum_{v \in V} f_{j-1}(\text{lk} v) = (j+1)f_j(\Delta), \]
where $V$ is the set of vertices of $\Delta$. This equation together with relations (1) and (2) (applied to the links of vertices) implies part (i).

Part (ii) is a consequence of equation (3) and [7, Lemma 6.1], which asserts that the $f$-numbers of a $2k$-dimensional homology manifold are non-negative linear combinations of its $h$-numbers $h_0, h_1, \ldots, h_{k+1}$. \qed

3. The proof of the Upper Bound Theorem

In this section we prove Theorem [1]. This will require the following facts and definitions.

Definition 2. A simplicial complex $\Delta$ is $l$-neighborly if each set of $l$ of its vertices forms a face in $\Delta$.

It is well-known that all $d$-dimensional cyclic polytopes are $\lfloor d/2 \rfloor$-neighborly, and that all $\lfloor d/2 \rfloor$-neighborly $d$-dimensional polytopes with $r$ vertices have the same $h$-vector:
\[ h_i = h_{d-i} = \binom{r-d+i-1}{i} \quad \text{for } 0 \leq i \leq \lfloor d/2 \rfloor. \]

In the proof of Theorem [1] we will also use the following version of the Upper Bound Theorem for even-dimensional homology manifolds.

Lemma 2. Let $K$ be a $2k$-dimensional homology manifold on $r$ vertices. Furthermore, let us assume that either $\chi(K) = 2$, or $K$ is an oriented homology manifold such that
\[ \beta_k(K) \leq 2\beta_{k-1}(K) + 2 \sum_{i=0}^{k-3} \beta_i(K). \]

Then
\[ h_i(K) \leq h_i(C_{2k+1}(r)) \quad \text{for } 0 \leq i \leq k+1. \]
Proof: In the case of $\chi(K) = 2$, the lemma follows from [3, Theorem 6.6] and the Dehn-Sommerville relations for Eulerian complexes [3]. In the second case, the result is a part of the proof of [3, Theorem 6.7]. □

We are now ready to verify Theorem 1. The argument is very similar to the proof of a special case of the cubical upper bound conjecture (see [2, Theorem 4.3]). The only difference is that we use the $\tilde{h}$-vector instead of the short cubical $h$-vector.

Proof of Theorem 1: Let $\Delta$ be a simplicial complex satisfying the conditions of the theorem. By Lemma 1(ii), it suffices to check that $\tilde{h}_i(\Delta) \leq \tilde{h}_i(C_{2k+2}(n))$ for $0 \leq i \leq k + 1$. To this end, note that for every vertex $v$ of $\Delta$, $\text{lk}_v$ is a simplicial complex on at most $n - 1$ vertices that is either a homology manifold with Euler characteristic 2, or an oriented homology manifold satisfying condition (4). Thus, by Lemma 2,

$$h_i(\text{lk}_v) \leq h_i(C_{2k+2}(n)) \quad \text{for } 0 \leq i \leq k + 1.$$  

Since $C_{2k+2}(n)$ is a $(k + 1)$-neighborly polytope, it follows that the link of every vertex of $C_{2k+2}(n)$ is a $k$-neighborly $(2k+1)$-dimensional polytope on $n - 1$ vertices. Hence,

$$\tilde{h}_i(\Delta) = \sum_v h_i(\text{lk}_v) \leq \sum_v h_i(C_{2k+1}(n - 1)) = \tilde{h}_i(C_{2k+2}(n)) \quad \text{for } 0 \leq i \leq k + 1,$$

implying the theorem. □

Corollary 1. Let $\Delta$ be a $(2k + 1)$-dimensional oriented pseudomanifold on $n$ vertices such that the link of every vertex is either a $2k$-dimensional homology manifold with vanishing middle homology, or it is a $2k$-dimensional homology manifold whose Euler characteristic $\chi$ satisfies $(-1)^k(\chi - 2) \leq 0$. Then

$$f_i(\Delta) \leq f_i(C_{2k+2}(n)) \quad \text{for } 1 \leq i \leq 2k + 1.$$  

Proof: Any such complex $\Delta$ satisfies the assumptions of Theorem 1. □

4. Additional remarks and results

1. Theorem 1 proves a special case of Gil Kalai’s conjecture [7, Section 7] that the UBC holds for all simplicial complexes having the property that every link (of a face) of dimension $2k$ ($k = 1, 2, \ldots$) satisfies condition (4).

2. In his proof of the UBC for spheres [9, 11], R. Stanley showed that if $K$ is a $(d - 1)$-dimensional homology sphere on $n$ vertices then

$$h_i(K) \leq h_i(C_d(n)) \quad \text{for } 0 \leq i \leq d - 1.$$  

Since the $f$-numbers of any simplicial complex $\Delta$ are non-negative combinations of its $h$-numbers (by Lemma 1(i)), arguing exactly as in the proof of Theorem 1, but using (3) instead of Lemma 2, we obtain a new proof of the UBC for odd-dimensional homology manifolds. This proof is shorter and more elementary than the one presented in [7, Theorem 1.4]. (It does not use any facts about Buchsbaum complexes!)
3. It would be interesting to clarify whether for a \((2k+1)\)-dimensional complex \(\Delta\) satisfying the assumptions of Theorem 3, the inequality \(h_i(\Delta) \leq h_i(C_{2k+2}(n))\) \((0 \leq i \leq k+1)\) necessarily holds. We have the expression

\[
h_r(\Delta) = \sum_{j=0}^{r} (-1)^{r-j} \binom{2k+2-j}{2k+2-r} f_j(\Delta)
\]

\[
= (-1)^r \binom{2k+2}{r} + \sum_{i=0}^{r-1} \tilde{h}_i(\Delta) \binom{2k+1-i}{2k+2-r} \sum_{j=i+1}^{r} \frac{1}{j} (-1)^{r-j} \binom{r-i-1}{r-j}
\]

\[
= (-1)^r \binom{2k+2}{r} + \sum_{i=0}^{r-1} \tilde{h}_i(\Delta) \binom{2k+1-i}{2k+2-r} \int_0^1 x^i(x-1)^{r-i-1} \, dx.
\]

Hence the coefficients of \(\tilde{h}\)-numbers in the expression for \(h_r\) alternate in sign so that short simplicial \(h\)-vectors are not sufficient to resolve this question.

4. Lower bounds. Let \(\Delta\) be a simplicial complex, let \(\text{Skel}_i(\Delta)\) denote its \(i\)-dimensional skeleton, and let \(\chi_i(\Delta) := \chi(\text{Skel}_i(\Delta)) = \sum_{j=0}^{i} (-1)^j f_j(\Delta)\) denote the Euler characteristic of \(\text{Skel}_i(\Delta)\). It was shown in 8 that if \(\Delta\) is a \((2k-1)\)-dimensional manifold, then \((-1)^i \chi_i(\Delta) \geq 0\) for \(0 \leq i \leq 2k-1\). The proof relied on several facts about Buchsbaum complexes. Using \(h\)-numbers we provide a short proof of the following related result.

Proposition 1. Let \(\Delta\) be a \((d-1)\)-dimensional Buchsbaum simplicial complex (i.e. a pure simplicial complex such that for every vertex \(v \in \Delta\) the link of \(v\) is Cohen-Macaulay). Then \((-1)^i \chi_i(\Delta) \geq 0\) for \(0 \leq i \leq \lfloor (d-1)/2 \rfloor\).

Proof: Since for every vertex \(v \in \Delta\), \(\text{lk} v\) is Cohen-Macaulay, it follows that \(h_i(\text{lk} v) \geq 0\) for \(i = 0, 1, \ldots, d-1\), and hence, \(\tilde{h}_i(\Delta) \geq 0\) for \(i = 0, 1, \ldots, d-1\). Expressing the \(f\)-numbers of \(\Delta\) in terms of its \(h\)-numbers (Lemma 3), we obtain

\[
(-1)^i \chi_i(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} f_j = \sum_{j=0}^{i} \left( \sum_{j=0}^{i} (-1)^{i-j} \frac{1}{j+1} \binom{d-1-l}{i-j} \right) \tilde{h}_i,
\]

It is straightforward to show that if \(0 \leq i \leq \lfloor (d-1)/2 \rfloor\) and \(0 \leq l \leq i\), then

\[
\frac{1}{i+1} \binom{d-1-l}{i+1} \geq \frac{1}{i} \binom{d-1-l}{i} \geq \frac{1}{i} \binom{d-1-l}{d-1-i} \geq \frac{1}{l+1} \binom{d-1-l}{d-1-l}.
\]

Hence for any \(0 \leq i \leq \lfloor (d-1)/2 \rfloor\), all coefficients of \(\tilde{h}\)-numbers in equation (6) are non-negative, implying the proposition.

5. Semi-Eulerian complexes. One may also use short simplicial \(h\)-vectors and the Dehn-Sommerville relations to give a new proof of the fact that all odd-dimensional semi-Eulerian simplicial (or regular cell) complexes are Eulerian. This result was proven more generally for posets in 10, Exercise 3.69(c)] by a very different approach.

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