ON THE 2-RANK AND 4-RANK OF THE CLASS GROUP OF SOME REAL PURE QUARTIC NUMBER FIELDS

MBAREK HAYNOU AND MOHAMMED TAOUS

Abstract. Let $K = \mathbb{Q}(\sqrt[4]{pd^2})$ be a real pure quartic number field and $k = \mathbb{Q}(\sqrt{p})$ its real quadratic subfield, where $p \equiv 5 \pmod{8}$ is a prime integer and $d$ an odd square-free integer coprime to $p$. In this work, we calculate $r_2(K)$, the 2-rank of the class group of $K$, in terms of the number of prime divisors of $d$ that decompose or remain inert in $\mathbb{Q}(\sqrt{p})$, then we will deduce forms of $d$ satisfying $r_2(K) = 2$. In the last case, the 4-rank of the class group of $K$ is given too.

1. Introduction and Notations

Let $K$ be a number field and $Cl_2(K)$ its 2-class group, that is the 2-Sylow subgroup of its class group $Cl(K)$. We define the 2-rank and the 4-rank of $Cl(K)$ respectively as follows:

$$r_2(K) = \dim_{\mathbb{F}_2}(Cl_2(K)/Cl_2(K)^2)$$

and

$$r_4(K) = \dim_{\mathbb{F}_2}(Cl_2(K)^2/Cl_2(K)^4)$$

where $\mathbb{F}_2$ is the finite field with 2 elements.

Using genus theory, many mathematicians calculated $r_2(K)$ and $r_4(K)$ whenever $K$ is a number field having a subfield with odd class number. For instances, we mention the following works:

- For biquadratic number fields, we indicate the works [1, 2, 3, 4, 14, 15].
- For cyclic quartic fields, we indicate the works [5, 6].
- For pure quartic number fields, we indicate the works of Parry [11, 12].

In the two last papers, Parry determined the exact power of 2 dividing the class number of some pure quartic number fields.

In this paper, we consider the real pure quartic number fields $K = \mathbb{Q}(\sqrt[4]{pd^2}) = k(\sqrt{d\sqrt{p}})$, where $k = \mathbb{Q}(\sqrt{p})$ with $p \equiv 5 \pmod{8}$ is a prime integer and $d$ is an odd square-free integer satisfying some conditions.

Our first goal is to determine the 2-rank of the class group of $K$, using the ambiguous class number formula in $K/k$ ([7]):

$$\#Am(K/k) = h(k) \cdot 2^{t-1} \frac{[E_k : E_k \cap N_{K/k}(K^\times)]}{[E_k : E_k \cap N_{K/k}(K^\times)]},$$

where

- $t$ is the number of finite and infinite primes ramified in $K/k$,
- $E_k$ is the unit group of $k$,
- $E_k \cap N_{K/k}(K^\times)$ is the subgroup of units that are norms of elements of $K$,
- $h(k)$ is the class number of $k$.

2010 Mathematics Subject Classification. 11R11, 11R16, 11R29, 11R37.

Key words and phrases. class groups, pure quartic number field, ambiguous class number formula.
As $h(k)$ is odd, then the above formula implies that

$$r_2(K) = t - e - 1,$$  \hspace{1cm} (1)

with $e$ is an integer defined by $2^e = [E_k : E_k \cap N_{K/k}(K^\times)]$. Our second goal is to calculate the 4-rank of the class group of fields $K$ satisfying $r_2(K) = 2$. For this, we will use a formula provided by Y. Qin ([14]). This formula states that the 4-rank of the class group of a quadratic extension $k(\sqrt{\delta})$, where the base field $k$ is of odd class number, is given by:

$$r_4(k(\sqrt{\delta})) = t - 1 - \text{rank}(R_{k(\sqrt{\delta})/k}),$$ \hspace{1cm} (2)

where $t$ is the number of ramified primes in $k(\sqrt{\delta})/k$ and $R_{k(\sqrt{\delta})/k}$ is the following matrix:

$$R_{k(\sqrt{\delta})/k} = \begin{pmatrix}
    \left(\frac{a_1; \delta}{\mathcal{P}_1}\right) & \ldots & \left(\frac{a_n; \delta}{\mathcal{P}_1}\right) & \ldots & \left(\frac{a_{n+r}; \delta}{\mathcal{P}_1}\right) \\
    \left(\frac{a_1; \delta}{\mathcal{P}_2}\right) & \ldots & \left(\frac{a_n; \delta}{\mathcal{P}_2}\right) & \ldots & \left(\frac{a_{n+r}; \delta}{\mathcal{P}_2}\right) \\
    \left(\frac{a_1; \delta}{\mathcal{P}_t}\right) & \ldots & \left(\frac{a_n; \delta}{\mathcal{P}_t}\right) & \ldots & \left(\frac{a_{n+r}; \delta}{\mathcal{P}_t}\right)
\end{pmatrix}.$$ \hspace{1cm} (3)

It is a matrix of type $t \times (n+r)$ with coefficients in $\mathbb{F}_2$, called the generalized Rédei-matrix, where $(\mathcal{P}_i)_{1 \leq i \leq t}$ are the primes (finite and infinite) of $k$ which ramify in $k(\sqrt{\delta})$, $(a_j)_{1 \leq j \leq n+r}$ is a family of elements of $k$ defined by Y. Qin in [14, §2, p. 27] and $\left(\frac{-; \delta}{\mathcal{P}_i}\right)$ is the Hilbert symbol on $k$. Note that this matrix is given by replacing the 1’s by 0’s and the -1’s by 1’s. For more details concerning the generalized Rédei-matrix, we refer the reader to [14].

During this paper, we adopt the following notations:

- $k = \mathbb{Q}(\sqrt{p})$ with $p \equiv 5 \pmod{8}$ is a prime integer.
- For $z \in k$, $z'$ denotes the conjugate of $z$ over $\mathbb{Q}$.
- $K = \mathbb{Q}(\sqrt{pd}) = k(\sqrt{d\sqrt{p}})$, where $d$ is an odd square-free integer such that $(p, d) = 1$.
- $\mathcal{O}_k$: the ring of integers of $k$.
- $r_2(K)$: the 2-rank of the class group $Cl(K)$.
- $r_4(K)$: the 4-rank of the class group $Cl(K)$.
- $\delta = d\sqrt{p}$.
- $\mathcal{P}_i$: a prime ideal of $k$ which ramifies in $K$.
- $N_{K/k}(\cdot)$: the relative norm of $K$ to $k$.
- $R_{K/k}$: the generalized Rédei-matrix of $K$.
- $\left(\frac{\cdot}{m}\right)$: the Legendre symbol.
- $\left(\frac{\cdot}{\mathcal{P}_i}\right)$: the quadratic symbol over $k$.
- $\left(\frac{\mathbb{Q}}{\mathcal{P}_i}\right)$: the Hilbert symbol over $k$.
- $\varepsilon_k$: the fundamental unit of $k$.

Our main theorems are as follows. Their proofs will be given in Sections 2 and 3.

**Theorem A.** Suppose that $d = q_1 \cdots q_s q_1' \cdots q_t'$ is an odd square-free integer such that $q_i$ and $q_j'$ are distinct primes satisfying $\left(\frac{p}{q_i}\right) = -1$ and $\left(\frac{p}{q_j'}\right) = 1$ for each $i, j$. Then the 2-rank of $Cl(K)$ is $r_2(K) = 2t + s$. 

2
Theorem B. If \( d = q \) is an odd prime such that \( \left( \frac{\mathfrak{p}}{q} \right) = 1 \). Then

1. If \( q \equiv 1 \pmod{8} \), then \( r_4(K) = \begin{cases} 0, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = -1, \\ 1, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = - \left( \frac{q}{p} \right)_4 = 1, \\ 2, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = \left( \frac{q}{p} \right)_4 = 1. \end{cases} \)

2. If \( q \equiv 3 \pmod{8} \), then \( r_4(K) = 0 \)

3. If \( q \equiv 5 \pmod{8} \), then \( r_4(K) = \begin{cases} 0, & \text{if } \left( \frac{\mathfrak{p}}{p} \right)_4 = -1, \\ 1, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = 1. \end{cases} \)

4. If \( q \equiv 7 \pmod{8} \), then \( r_4(K) = \begin{cases} 0, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = 1, \\ 1, & \text{if } \left( \frac{\mathfrak{p}}{q} \right)_4 = -1. \end{cases} \)

Theorem C. If \( d = q_1q_2 \) where \( q_1, q_2 \) are odd prime integers such that \( \left( \frac{p}{q_1} \right) = \left( \frac{p}{q_2} \right) = -1 \), then

\[
\begin{align*}
\text{if } (q_1, q_2) \equiv (3, 3) \pmod{4}, \\
1, \\
0, \quad \text{otherwise.}
\end{align*}
\]

We will analyze the behaviour of this non-Galois extension, \( K \), of degree 4 in order to compare, in an other paper in preparation, the results obtained here in this paper with those that can one have when \( K \) is replaced by a biquadratic field. The determination of the 2-rank and 4-rank of the class group of \( K \), can be also among the most important properties used to study the following problems:

1. Characterize the structure of the 2-group \( G = Gal(K^{(2)}/K) \) of \( K \) where \( K^{(2)} \) is its second Hilbert 2-class field.
2. Determine the Hilbert 2-class field tower of pure quartic field \( K \).
3. Study the capitulation of the 2-ideal classes of the pure quartic field \( K \) in the intermediate sub-extensions of \( K^{(1)}_2/K \) where \( K^{(1)}_2 \) is the first Hilbert 2-class field of \( K \).

2. The 2-rank of \( Cl(K) \)

In this section, we assume that \( d = q_1 \cdots q_s q'_1 \cdots q'_t \) where \( q_i \) and \( q'_j \) are distinct primes satisfying \( \left( \frac{\mathfrak{p}}{q_i} \right) = -1, \left( \frac{\mathfrak{p}}{q'_j} \right) = 1 \) for each \( i, j \).

The relative discriminant of \( K/k \) is \( \Delta_{K/k} = (4d\sqrt{p}) \) (see [9]), then the finite primes ramified in \( K/k \) are \( \sqrt{p}, q_1, \ldots, q_s, \pi_1, \ldots, \pi_t, \bar{\pi}_1, \ldots, \bar{\pi}_t \) and \( 2_I \), where \( pO_k = (\sqrt{p})^2, 2_I = 2O_k, \pi_i, q_i = q_iO_k \) and \( \bar{q}_i = q_iO_k \) for each \( i \). Regarding infinite primes ramifying in \( K/k \), \( k \) has two infinite primes \( p_\infty \) and \( \bar{p}_\infty \) which correspond respectively to the two following \( \mathbb{Q} \)-embeddings:

\[
i_{p_\infty} : \sqrt{p} \mapsto -\sqrt{p} \text{ and } i_{\bar{p}_\infty} : \sqrt{p} \mapsto \sqrt{p}.
\]

As \( i_{p_\infty} \) can be extend to the two \( \mathbb{Q} \)-embeddings:

\[
j_{p_\infty} : \sqrt{p} \mapsto i\sqrt{p} \text{ and } j_{p_\infty} : \sqrt{p} \mapsto -i\sqrt{p},
\]

which are complex embeddings, and \( i_{\bar{p}_\infty} \) can be extend to the two \( \mathbb{Q} \)-embeddings:

\[
j_{\bar{p}_\infty} : \sqrt{p} \mapsto \sqrt{\bar{p}} \text{ and } j_{\bar{p}_\infty} : \sqrt{p} \mapsto -\sqrt{\bar{p}},
\]

which are real embeddings, then \( p_\infty \) is the unique infinite prime of \( k \) that ramifies in \( K \).
Lemma 2.1. Keeping previous hypotheses and notations, then
\[ \left( \frac{-1; \delta}{p_\infty} \right) = \left( \frac{\varepsilon_p; \delta}{\sqrt{p}} \right) = \left( \frac{-\varepsilon_p; \delta}{\sqrt{p}} \right) = -1. \]

Proof.

(1) Let \( i_{p_\infty} : \sqrt{p} \mapsto -\sqrt{p} \) be the complex \( \mathbb{Q} \)-isomorphism of \( k \) corresponding to \( p_\infty \), then by definition of Hilbert symbol given in [8, Ch II § 7 Definitions 7.1, p. 195] and [8, Ch II § 7 Definitions 7.3.1, p. 201], we have,
\[ \left( \frac{-1, \delta}{p_\infty} \right) = i_{p_\infty}^{-1}(i_{p_\infty}(-1), i_{p_\infty}(\delta))_{p_\infty} = i_{p_\infty}^{-1}((-1, -\delta)_{p_\infty}) = i_{p_\infty}^{-1}(-1) = -1. \]

(2) Since \( v(\sqrt{p})(\delta) = 1 \) and via the property [11, Lemma V, p. 105],
\[ \left( \frac{\varepsilon_p, \delta}{\sqrt{p}} \right) = \left[ \frac{\varepsilon_p}{\sqrt{p}} \right]^{v(\delta)} = \left[ \frac{\varepsilon_p}{\sqrt{p}} \right] = -1. \]

(3) We have \( \left( \frac{-1, \delta}{\sqrt{p}} \right) = \left[ \frac{-1}{\sqrt{p}} \right] = \left( \frac{-1}{p} \right) = 1 \). Then by applying multiplicative property of Hilbert symbol we get
\[ \left( \frac{-\varepsilon_p, \delta}{\sqrt{p}} \right) = \left( \frac{\varepsilon_p, \delta}{\sqrt{p}} \right) \left( \frac{-1, \delta}{\sqrt{p}} \right) = -1. \]

\[ \square \]

Proof of Theorem A. Firstly, we know from Hasse norm theorem [8, Ch II § 6, Theorem 6.2, p. 179] that an element \( \alpha \) in \( k^\times \) is norm in \( K \) if and only if \( \left( \frac{\alpha, \delta}{p_1} \right) = 1 \) for all primes of \( k \) ramified in \( K \). So, by Lemma (2.1), the units \( -1, \varepsilon_p, -\varepsilon_p \) of \( k \) are not norms of elements of \( K \). Since \( E_k = \langle -1, \varepsilon_p \rangle \) and \( E_k \cap N_{K/k}(K^\times) = \langle \varepsilon_p^2 \rangle \), therefore \( [E_k : E_k \cap N_{K/k}(K^\times)] = 4 \), so \( e = 2 \). Secondly, the finite and infinite primes ramified in \( K/k \) are \( p_\infty, (\sqrt{q}), 2, \tilde{q}_1, \cdots, \tilde{q}_s, \pi_1, \cdots, \pi_t, \pi_1', \cdots, \pi_t' \). Then their number is \( 2t + s + 3 \). Finally, the 2-rank of the class group of \( K \) is computed using formula (1).

\[ \square \]

Corollary 2.2. Keeping previous hypotheses and notations, then \( r_2(K) = 2 \) if and only if one of the following conditions holds:

(a) \( d = q \) is a prime number such that \( \left( \frac{q}{q} \right) = 1 \),

(b) \( d = q_1q_2 \), where \( q_1, q_2 \) are odd primes such that \( \left( \frac{q_1}{q_2} \right) = \left( \frac{q_2}{q_1} \right) = -1 \).

Proof. By Theorem A, \( r_2(K) = 2 \) if and only if \( (t = 1 \text{ and } s = 0) \) or \( (t = 0 \text{ and } s = 2) \). Hence,

(a) if \( t = 1 \text{ and } s = 0 \), then \( d = q \) is a prime number and \( \left( \frac{q}{q} \right) = 1 \),

(b) if \( t = 0 \text{ and } s = 2 \), then \( d = q_1q_2 \) where \( q_1, q_2 \) are odd primes such that \( \left( \frac{q_1}{q_2} \right) = \left( \frac{q_2}{q_1} \right) = -1 \).

\[ \square \]
3. The 4-rank of \( Cl(K) \)

In this section, we compute the 4-rank of \( Cl(K) \) whenever \( r_2(K) = 2 \). At the end of each case, we give some numerical examples which are verified using the Pari/GP calculator (version 2-11-3), [13].

3.1. Case: \( d = q \) is a prime number and \( \left( \frac{q}{p} \right) = 1 \). Let \( p \) and \( q \) be two different odd prime numbers such that \( p \equiv 5 \pmod{8} \), \( \left( \frac{q}{p} \right) = 1 \) and \( \pi_1, \tilde{\pi}_1 \) are prime ideals of \( \mathcal{O}_k \) lying above \( q \), the ideals \( \pi_1^h, \tilde{\pi}_1^h \) are principal ideals of \( \mathcal{O}_k \), put \( \pi_1^h = (x + y \sqrt{p})/2 \) and \( \tilde{\pi}_1^h = (x - y \sqrt{p})/2 \), where \( h = h(k) \) is the class number of \( k = \mathbb{Q}(\sqrt{p}) \). Without loss of generality, we can suppose that \( (x - y \sqrt{p}) > 0 \) and \( (x + y \sqrt{p}) > 0 \), because

- if \( (x - y \sqrt{p}) < 0 \) and \( (x + y \sqrt{p}) < 0 \), we replace \( (x + y \sqrt{p}) \) by \( -(x + y \sqrt{p}) \) and \( (x - y \sqrt{p}) \) by \( -(x - y \sqrt{p}) \). Which are also positive generators for the ideals \( \pi_1^h \) and \( \tilde{\pi}_1^h \) respectively.
- if \( (x - y \sqrt{p}) > 0 \) and \( (x + y \sqrt{p}) < 0 \), we have \( N_{k/\mathbb{Q}}(\varepsilon_p) = \varepsilon_p \varepsilon_p' = -1 \) and \( \varepsilon_p' < 0 \), then we can replace \( (x + y \sqrt{p}) \) by \( \varepsilon_p(x + y \sqrt{p}) \) and \( (x - y \sqrt{p}) \) by \( \varepsilon_p'(x - y \sqrt{p}) \), which are also positive generators for the ideals \( \pi_1^h, \tilde{\pi}_1^h \) respectively.

Furthermore, applying absolute norm map to the ideal \( \pi_1^h = (x + y \sqrt{p})/2 \), one gets \( \pm 4q^h = x^2 - py^2 \). But if we consider \( x + y \sqrt{p} > 0 \) and \( x - y \sqrt{p} > 0 \) this equation becomes \( 4q^h = x^2 - py^2 \). So we can reduce our study to the case when the ideals \( \pi_1^h \) and \( \tilde{\pi}_1^h \) have positive generators.

To compute the 4-rank of the class group of \( K \) by using the generalized Rédei-matrix, we need the following lemmas.

Lemma 3.2. Keeping previous hypotheses and notations, then

\[
(a) \quad \left( \frac{\varepsilon_p \delta}{p_\infty} \right) = \left( \frac{\sqrt{p} \delta}{p_\infty} \right) = -1.
\]

\[
(b) \quad \left( \frac{2 \delta}{p_\infty} \right) = \frac{(2(x + \sqrt{p}), \delta)}{p_\infty} = \frac{(2(x - \sqrt{p}), \delta)}{p_\infty} = 1.
\]

Proof. By the definition of Hilbert symbol given in [8, Ch II, § 7 Definitions 7.1, p. 195] and [8, Ch II § 7 Definitions 7.3.1, p. 201], we have

\[
(a) \quad \left( \frac{\varepsilon_p \delta}{p_\infty} \right) = \left( \frac{i^-_p((i_{p_\infty}(\varepsilon_p), i_{p_\infty}(\delta))_{p_\infty})}{p_\infty} \right) = \left( \frac{i^-_p((\varepsilon_p', -\delta)_{p_\infty})}{p_\infty} \right) = i^-_p(1) = -1,
\]

since the fundamental unit of \( k \) satisfy \( \varepsilon_p > 0 \) and \( N_{k/\mathbb{Q}}(\varepsilon_p) = \varepsilon_p \varepsilon_p' = -1 \), then \( \varepsilon_p' < 0 \).

\[
\left( \frac{\sqrt{p} \delta}{p_\infty} \right) = i^-_p((i_{p_\infty}((\sqrt{p}), i_{p_\infty}(\delta))_{p_\infty})) = i^-_p((-\sqrt{p}, -\delta)_{p_\infty}) = i^-_p(1) = -1.
\]

\[
(b) \quad \left( \frac{2 \delta}{p_\infty} \right) = \left( \frac{(i_{p_\infty}(2), i_{p_\infty}(\delta))_{p_\infty}}{p_\infty} \right) = \left( \frac{(2, -\delta)_{p_\infty}}{p_\infty} \right) = i^-_p(1) = 1.
\]

\[
\left( \frac{2(x + \sqrt{p}), \delta}{p_\infty} \right) = i^-_p((i_{p_\infty}(2(x + \sqrt{p}), i_{p_\infty}(\delta))_{p_\infty})) = i^-_p((2(x - \sqrt{p}), -\delta)_{p_\infty}) = i^-_p(1) = 1.
\]

\[
\left( \frac{2(x - \sqrt{p}), \delta}{p_\infty} \right) = i^-_p((i_{p_\infty}(2(x - \sqrt{p}), i_{p_\infty}(\delta))_{p_\infty})) = i^-_p((2(x + \sqrt{p}), -\delta)_{p_\infty}) = i^-_p(1) = 1.
\]

\[\square\]
Lemma 3.3. Let $p$ and $q$ be distinct odd primes with $p \equiv 1 \pmod{4}$. Then if $(x, y)$ is an integer solution of the equation $u^2 - pv^2 = 4q^h$, then

$$\left(\frac{x}{p}\right) = \left(\frac{2}{p}\right) \left(\frac{q}{p}\right)_4.$$ 

Moreover, if $p \equiv 5 \pmod{8}$, then

$$\left(\frac{x}{q}\right) = \begin{cases} 
\left(\frac{q}{q}\right)_4, & \text{if } q \equiv 1 \pmod{4}, \\
-\left(\frac{2}{p}\right)_4, & \text{if } q \equiv 3 \pmod{4}.
\end{cases}$$

Proof. Let $(x, y)$ be a solution of the equation $4q^h = u^2 - pv^2$, then $x^2$ and $y^2$ have same parity, so $x$ and $y$ have same parity too. Hence $4q^h \equiv x^2 \pmod{p}$. As $p \equiv 1 \pmod{4}$, so

$$\left(\frac{x}{p}\right) \equiv \left(x^2\right)^{\frac{p-1}{2}} \pmod{p}$$

$$\equiv \left(\frac{x^2}{p}\right)_4$$

$$\equiv \left(\frac{q}{p}\right)_4$$

$$\equiv \left(\frac{a}{p}\right)_4 \left(\frac{q}{p}\right)_4$$

$$\equiv \left(\frac{2}{p}\right)_4 \left(\frac{q}{p}\right)_4.$$ 

For the second equality, put $x = 2^e x'$ and $y = 2^e y'$ such that $2 \nmid x'$ and $2 \nmid y'$.

(a) If $q \equiv 1 \pmod{4}$, then

$$\left(\frac{x}{q}\right) = \left(x^2\right)^{\frac{q-1}{2}} = \left(x^2\right)^{\frac{q-1}{2}} \left(\frac{q}{q}\right)$$

$$= \left(\frac{x}{q}\right) \left(\frac{q}{q}\right)_4$$

$$= \left(\frac{2}{q}\right)_4 \left(\frac{q}{q}\right)_4$$

$$= \left(\frac{2}{q}\right)_4 \left(\frac{q}{q}\right)_4.$$ 

- If $q \equiv 1 \pmod{8}$ or $y$ is odd, then $\left(\frac{2}{q}\right)_4 = \left(\frac{q}{q}\right)_4$.
- If $q \equiv 5 \pmod{8}$ and $x, y$ are even, then $x = 2a$ and $y = 2b$, so $q^h = a^2 - p b^2$. Hence by calculation modulo 8 and taking into account the fact that $h$ is odd, we get the following table:

| $a$   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|-------|---|---|---|---|---|---|---|---|
| $a^2$ | 0 | 1 | 4 | 1 | 0 | 1 | 4 | 1 |
| $5 - a^2$ | 5 | 4 | 1 | 4 | 5 | 4 | 1 | 4 |

To this end, as $b^2 \equiv 0, 1$ or 4 (mod 8), so $3b^2 \equiv 0, 3$ or 4 (mod 8). On the other hand, since $q^h - a^2 \equiv 5 - a^2 \equiv 3b^2$ (mod 8) one deduces that $3b^2 \equiv 4$ (mod 8), thus multiplying by the inverse of 3 (mod 8) we get $b^2 \equiv 4$ (mod 8). Hence $e = 2$, this in turn yields that

$$\left(\frac{x}{q}\right) = \left(\frac{p}{q}\right)_4.$$
(b) If \( q \equiv 3 \pmod{4} \), then by quadratic reciprocity we get
\[
\left( \frac{a}{q} \right) = \left( \frac{2 \sqrt{q}}{q} \right) = \left( \frac{2}{q} \right)^r \left( -\frac{q}{a} \right) = \left( \frac{2}{q} \right)^r \left( \frac{q}{a} \right) = \left( \frac{2}{q} \right)^r \left( \frac{q}{p} \right)^2 (-1)^{r+1}.
\]
- If \( q \equiv 3 \pmod{8} \) or \( x \) is odd, then \( \left( \frac{a}{q} \right) = -\left( \frac{2}{p} \right)^2 \).
- If \( q \equiv 7 \pmod{8} \) and \( x \) and \( y \) are even \( x = 2a \) and \( y = 2b \), so \( q^h = a^2 - pb^2 \), hence by calculation modulo 8 we get the following table:

| \( b \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|--------|---|---|---|---|---|---|---|---|
| \( b^2 \) | 0 | 1 | 4 | 1 | 0 | 1 | 4 | 1 |
| \( 7 + 5b^2 \) | 7 | 4 | 3 | 4 | 7 | 4 | 3 | 4 |

with the following congruences: \( b^2 \equiv 0, 1 \) or \( 4 \pmod{8} \) and \( a^2 \equiv q^h + pb^2 \equiv 7 + 5b^2 \), we find that \( a^2 \equiv 4 \pmod{8} \), then \( r = 2 \). Which also means that
\[
\left( \frac{x}{q} \right) = -\left( \frac{q}{p} \right)^2.
\]

\[\square\]

**Lemma 3.4.** Let \( p \) and \( q \) be two different odd prime numbers such that \( p \equiv 5 \pmod{8} \) and \( \left( \frac{2}{q} \right) = 1 \), then
(a) \( \left( \frac{2 \delta}{\sqrt{p}} \right) = -1 \).
(b) \( \left( \frac{\sqrt{p} \delta}{\sqrt{p}} \right) = -\left( \frac{\epsilon \delta}{\sqrt{p}} \right) = 1 \).
(c) \( \left( \frac{2(x+\sqrt{p} \delta)}{\sqrt{p}} \right) = \left( \frac{2(x-\sqrt{p} \delta)}{\sqrt{p}} \right) = \left( \frac{2}{p} \right) \left( \frac{2}{p} \right) = \left( \frac{2}{p} \right)^2 \). A similar argument shows that
\[
\left( \frac{2}{\sqrt{p}} \right) = \left( \frac{2}{p} \right)^4.
\]

\[\square\]

**Lemma 3.5.** Keep hypotheses and notations of Lemma 3.4, then
(a) \( \left( \frac{-1 \delta}{\pi_1} \right) = \left( \frac{-1 \delta}{\pi_1} \right) = \begin{cases} 1, & \text{if } q \equiv 1 \pmod{4}, \\ -1, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \)
(b) \( \left( \frac{\epsilon \delta}{\pi_1} \right) = \left( \frac{\epsilon \delta}{\pi_1} \right) = \left( \frac{p}{q} \right)^4 \left( \frac{q}{p} \right)^4 \).
- If \( q \equiv 1 \pmod{4} \), then \( \left( \frac{\epsilon \delta}{\pi_1} \right) = \left( \frac{\epsilon \delta}{\pi_1} \right) = \left( \frac{p}{q} \right)^4 \left( \frac{q}{p} \right)^4 \).
- If \( q \equiv 3 \pmod{4} \), then \( \left( \frac{\epsilon \delta}{\pi_1} \right) = -\left( \frac{\epsilon \delta}{\pi_1} \right) = -\left( \frac{y}{q} \right) \).
(c) \( \left( \frac{2 \delta}{\pi_1} \right) = \left( \frac{2 \delta}{\pi_1} \right) = \left( \frac{2}{q} \right) \).
(d) • If \( q \equiv 1 \pmod{4} \), then \( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = \left( \frac{\sqrt{q} \cdot \delta}{\pi_1} \right) = \left( \frac{\nu}{q} \right)_4 \).

• If \( q \equiv 3 \pmod{4} \), then \( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = -\left( \frac{\sqrt{q} \cdot \delta}{\pi_1} \right) = \left( \frac{\nu}{q} \right)_4 \).

(e) • If \( q \equiv 1 \pmod{4} \), then \( \left( \frac{2(x+y \sqrt{p}) \cdot \delta}{\pi_1} \right) = \left( \frac{2(y-x \sqrt{p}) \cdot \delta}{\pi_1} \right) = 1 \).

• If \( q \equiv 3 \pmod{4} \), then \( \left( \frac{2(x+y \sqrt{p}) \cdot \delta}{\pi_1} \right) = -\left( \frac{2(y-x \sqrt{p}) \cdot \delta}{\pi_1} \right) = \left( \frac{\nu}{q} \right)_4 \).

(f) \( \left( \frac{2(x+y \sqrt{p}) \cdot \delta}{\pi_1} \right) = \left( \frac{2(y-x \sqrt{p}) \cdot \delta}{\pi_1} \right) = \begin{cases} \left( \frac{\nu}{q} \right)_4, & \text{if } q \equiv 1 \pmod{4}, \\ -\left( \frac{\nu}{q} \right)_4, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \)

**Proof.** The main tools to prove these equalities are the properties cited in [8, Ch II § 7 Proposition 7.4.3, p. 205] and by applying the bilinear property of Hilbert symbol, we get

(a) \( \left( \frac{-1, \delta}{\pi_1} \right) = \left( \frac{-1, \delta}{\pi_1} \right) = \left( \frac{-1, \delta}{\pi_1} \right) = \begin{cases} 1, & \text{if } q \equiv 1 \pmod{4}, \\ -1, & \text{if } q \equiv 3 \pmod{4}. \end{cases} \)

(b) • Suppose \( q \equiv 1 \pmod{4} \). Since \( q \equiv p \equiv 1 \pmod{4} \) and \( \left( \frac{\nu}{q} \right)_4 = 1 \), then by Scholz’s reciprocity law ( [10, Ch 5 § 5.2 Proposition 5.8, p. 160]), we have

\[
\left( \frac{\epsilon_\nu \cdot \delta}{\pi_1} \right) = \left( \frac{\epsilon_\nu}{\pi_1} \right) = \left( \frac{\nu}{q} \right)_4 \left( \frac{\nu}{p} \right)_4, \text{ moreover } \left( \frac{\epsilon_\nu}{\pi_1} \right) = \left( \frac{-1}{q} \right)_4 = 1, \text{ then } \left( \frac{\epsilon_\nu \cdot \delta}{\pi_1} \right) = \left( \frac{\nu}{q} \right)_4 \left( \frac{\nu}{p} \right)_4.
\]

• If \( q \equiv 3 \pmod{4} \) we find that \( \left( \frac{\epsilon_\nu \cdot \delta}{\pi_1} \right) = \left( \frac{\epsilon_\nu}{\pi_1} \right) \text{ and } \left( \frac{\epsilon_\nu \cdot \delta}{\pi_1} \right) = \left( \frac{\epsilon_\nu}{\pi_1} \right) \).

Consequently, \( \left[ \frac{\epsilon_\nu}{\pi_1} \right] \left[ \frac{\epsilon_\nu}{\pi_1} \right] = \left[ \frac{\epsilon_\nu}{\pi_1} \right] \left[ \frac{\epsilon_\nu}{\pi_1} \right] = \left[ \frac{\epsilon_\nu}{\pi_1} \right] \left[ \frac{-\epsilon_\nu}{\pi_1} \right] \left[ \frac{-\epsilon_\nu}{\pi_1} \right] = \left( \frac{-1}{q} \right)_4 = -1. \)

As \( \left[ \frac{\epsilon_\nu \sqrt{p}}{\pi_1} \right] = -\left( \frac{\nu}{p} \right)_4 \) (via [16]) and \( \left[ \frac{\epsilon_\nu \sqrt{p}}{\pi_1} \right] = \left[ \frac{\epsilon_\nu}{\pi_1} \right] \left[ \frac{-\epsilon_\nu}{\pi_1} \right] = \left( \frac{-1}{q} \right)_4 \left( \frac{\nu}{q} \right)_4 \left( \frac{\nu}{p} \right)_4 \left( \frac{\nu}{p} \right)_4 \).

Then \( \left[ \frac{\epsilon_\nu}{\pi_1} \right] = -\left( \frac{-1}{q} \right)_4 \left( \frac{\nu}{q} \right)_4 \left( \frac{\nu}{p} \right)_4 \left( \frac{\nu}{p} \right)_4 \). This completes the proof.

(c) \( \left( \frac{2 \cdot \delta}{\pi_1} \right) = \left[ \frac{2}{\pi_1} \right] = \left( \frac{2}{q} \right)_4 \) and \( \left( \frac{2 \cdot \delta}{\pi_1} \right) = \left[ \frac{2}{\pi_1} \right] = \left( \frac{2}{q} \right)_4 \).

(d) • If \( q \equiv 1 \pmod{4} \), then

\( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = \left[ \frac{\sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{2x - 2(x+y \sqrt{p})}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{2x}{\pi_1} \right] = \left( \frac{\nu}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{\nu}{q} \right)_4 \).

And

\( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = \left[ \frac{\sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{2x - 2(x - y \sqrt{p})}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{-2x}{\pi_1} \right] = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{x}{q} \right)_4.
\)

• In the same way, we show that if \( q \equiv 3 \pmod{4} \), then

\( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = \left[ \frac{\sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{2x \sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{-2x}{\pi_1} \right] = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{x}{q} \right)_4.
\)

\( \left( \frac{\sqrt{p} \cdot \delta}{\pi_1} \right) = \left[ \frac{\sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{2x \sqrt{p}}{\pi_1} \right] = \left[ \frac{2y}{\pi_1} \right] \left[ \frac{-2x}{\pi_1} \right] = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{-1}{q} \right)_4 \left( \frac{x}{q} \right)_4 = \left( \frac{x}{q} \right)_4.
\)
• We also have that if \( q \equiv 1 \pmod{4} \), then
\[
\frac{2(x+y\sqrt{q})}{\pi_1} = \left( \frac{2(x+y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}(x-y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}(x-y\sqrt{q})}{\pi_1} \right) \\
= \left( \frac{-8q}{\pi_1}, \delta \right) \left( -\frac{x}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}}{\pi_1} \right) \\
= \left( -\delta, \delta \right) \left( -\frac{2\sqrt{q}(x-y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{2\sqrt{q}(x-y\sqrt{q})}{\pi_1} \right) \\
= \left( \frac{\sqrt{q}}{\pi_1} \right) \left( \frac{x}{\pi_1} \right) \left( \frac{\sqrt{q}}{\pi_1} \right) \left( \frac{x}{\pi_1} \right) \left( \frac{\sqrt{q}}{\pi_1} \right) \\
= \left( \frac{\sqrt{q}}{\pi_1} \right)^3 \left( \frac{x}{\pi_1} \right)^2 = 1.
\]

• If \( q \equiv 3 \pmod{4} \), then
\[
\frac{2(x+y\sqrt{q})}{\pi_1} = \left( \frac{2(x+y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}(x-y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}(x-y\sqrt{q})}{\pi_1} \right) \\
= \left( \frac{-4q}{\pi_1}, \delta \right) \left( -\frac{x}{\pi_1}, \delta \right) \left( -\frac{\sqrt{q}}{\pi_1} \right) \\
= \left( -\delta, \delta \right) \left( -\frac{2\sqrt{q}(x-y\sqrt{q})}{\pi_1}, \delta \right) \left( -\frac{2\sqrt{q}(x-y\sqrt{q})}{\pi_1} \right) \\
= \left( \frac{\sqrt{q}}{\pi_1} \right) \left( \frac{4x}{\pi_1} \right) \left( \frac{\sqrt{q}}{\pi_1} \right) \left( \frac{4x}{\pi_1} \right) \left( \frac{\sqrt{q}}{\pi_1} \right) \\
= \left( \frac{\sqrt{q}}{\pi_1} \right)^3 \left( \frac{x}{\pi_1} \right)^2 = 1.
\]

(f) With similar calculations we prove that

• If \( q \equiv 1 \pmod{4} \), then
\[
\frac{2(x+y\sqrt{q})}{\pi_1} = \left[ \frac{2(x+y\sqrt{q})}{\pi_1} \right] = \left[ \frac{4x}{\pi_1} \right] = \left( \frac{x}{q} \right) = \left( \frac{p}{q} \right)_4,
\]

and
\[
\frac{2(x-y\sqrt{q})}{\pi_1} = \left[ \frac{2(x-y\sqrt{q})}{\pi_1} \right] = \left[ \frac{4x}{\pi_1} \right] = \left( \frac{x}{q} \right) = \left( \frac{p}{q} \right)_4.
\]
• If \( q \equiv 3 \) (mod 4), then
\[
\left( \frac{2(x+y\sqrt{p})}{\pi_1} \right) = \left( \frac{2(x+y\sqrt{p})}{\bar{\pi}_1} \right) = \left( \frac{4x}{\pi_1} = \left( \frac{x}{q} \right) = \left( \frac{2}{p} \right) \left( \frac{p}{q} \right)_4 = -\left( \frac{q}{p} \right)_4, \\
\text{and}
\left( \frac{2(x-y\sqrt{p})}{\pi_1} \right) = \left( \frac{2(x-y\sqrt{p})}{\bar{\pi}_1} \right) = \left( \frac{4x}{\pi_1} = \left( \frac{x}{q} \right) = \left( \frac{2}{p} \right) \left( \frac{p}{q} \right)_4 = -\left( \frac{q}{p} \right)_4.
\]

\[\square\]

**Lemma 3.6.** Keeping previous hypotheses and notations, then

(a) \( \left( \frac{\varepsilon_p\delta}{2} \right) = \begin{cases} 1, & \text{if } q \equiv 1 \text{ (mod 4)}, \\ -1, & \text{if } q \equiv 3 \text{ (mod 4)}. \end{cases} \)

(b) \( \left( \frac{2\delta}{2I} \right) = \left( \frac{-1, \delta}{2I} \right) = -1, \text{ and } \left( \frac{\sqrt{\pi}\delta}{2I} \right) = \begin{cases} -1, & \text{if } q \equiv 1 \text{ (mod 4)}, \\ 1, & \text{if } q \equiv 3 \text{ (mod 4)}. \end{cases} \)

(c) • If \( q \equiv 1 \) (mod 4), then \( \left( \frac{2(x+y\sqrt{p})\delta}{2I} \right) = \left( \frac{2(x-y\sqrt{p})\delta}{2I} \right) = \left( \frac{\varepsilon}{q} \right)_4 \left( \frac{2}{p} \right)_4. \)

• If \( q \equiv 3 \) (mod 4), then \( \left( \frac{2(x+y\sqrt{p})\delta}{2I} \right) = -\left( \frac{2(x-y\sqrt{p})\delta}{2I} \right) = -\left( \frac{\varepsilon}{q} \right)_4. \)

**Proof.** We prove these results by using the product formula for Hilbert symbol and previous lemmas. \[\square\]

**Remark 3.7.** The above results can be summed up in the following tables. For \( q \equiv 1 \) (mod 4), we have

| \( 2(x+y\sqrt{p}) \) | \( \pi_1 \) | \( \bar{\pi}_1 \) | \( 2I \) | \( \sqrt{p} \) | \( p_{\infty} \) |
|---------------------|----------|----------|-------|----------|-------|
| \( 2(x+y\sqrt{p}) \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) |
| \( 2(x-y\sqrt{p}) \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) |
| \( \sqrt{p} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) |
| \( \varepsilon_p \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) | \( \frac{\varepsilon}{q} \) |

For \( q \equiv 3 \) (mod 4), we have \( p^{(q+1)/8} = p^{(q+1)/2} = p^{(q-1)/2} = p^{(q-1)/2 + 1} = \left( \frac{\varepsilon}{q} \right)_4 p \equiv p \) (mod \( q \)),

then \( p^{(q+1)/8} \) is a solution to \( x^4 \equiv p \) (mod \( q \)), hence \( \left( \frac{\varepsilon}{q} \right)_4 = 1. \)
Proof of Theorem B.

By assuming \( q \) is a prime number satisfying \( \left( \frac{q}{p} \right) = 1 \), the finite primes of \( k \) ramifying in \( K \) are: \( \pi_1, \tilde{\pi}_1, 2_I \) and \( (\sqrt{p}) \). So there exist prime ideals \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \) of \( K \) such that

\[
\mathcal{H}_1^2 = \pi_1O_K, \mathcal{H}_2^2 = \tilde{\pi}_1O_K, \mathcal{H}_3^2 = 2_I O_K, \mathcal{H}_4^2 = \sqrt{p}O_K.
\]

Therefore, we have

- \( \mathcal{H}_1^2 \sigma(\mathcal{H}_1^2) = \mathcal{H}_1^2 \mathcal{H}_1^2 O_K = \pi_1^2 O_K = ((x + y \sqrt{p})/2)O_K \),
- \( \mathcal{H}_2^2 \sigma(\mathcal{H}_2^2) = \mathcal{H}_2^2 \mathcal{H}_2^2 O_K = \tilde{\pi}_1^2 O_K = ((x - y \sqrt{p})/2)O_K \),
- \( \mathcal{H}_3^2 \sigma(\mathcal{H}_3^2) = \mathcal{H}_3^2 \mathcal{H}_3^2 O_K = 2_I O_K = 2O_K \),
- \( \mathcal{H}_4^2 \sigma(\mathcal{H}_4^2) = \mathcal{H}_4^2 \mathcal{H}_4^2 O_K = \sqrt{p}O_K = \sqrt{p}O_K \).

where \( \sigma \) is the generator of the Galois group of \( K/k \), and \( h \) the class number of \( k \).

Moreover, we know from Lemma 2.1 that \(-1, \varepsilon_p, -\varepsilon_p \notin E_k \cap N_{K/k}(K)\). Then the \( 5 \times (4 + 2) \) generalized Rédei’s matrix of Hilbert symbols (3), can be written as

\[
R_{K/k} = \begin{pmatrix}
\frac{2(x + y \sqrt{p})}{\pi_1} & \frac{2(x - y \sqrt{p})}{\tilde{\pi}_1} & \frac{n}{\mathcal{H}_1} & \frac{n}{\mathcal{H}_2} & \frac{n}{\mathcal{H}_3} & \frac{n}{\mathcal{H}_4} \\
\frac{2(x + y \sqrt{p})}{2_I} & \frac{2(x - y \sqrt{p})}{2_I} & \frac{n}{\mathcal{H}_1} & \frac{n}{\mathcal{H}_2} & \frac{n}{\mathcal{H}_3} & \frac{n}{\mathcal{H}_4} \\
\frac{2(x + y \sqrt{p})}{p_{\infty}} & \frac{2(x - y \sqrt{p})}{p_{\infty}} & \frac{n}{\mathcal{H}_1} & \frac{n}{\mathcal{H}_2} & \frac{n}{\mathcal{H}_3} & \frac{n}{\mathcal{H}_4} \\
\end{pmatrix}
\]

We consider the above matrix with coefficients in \( \mathbb{F}_2 \) by replacing 1 by 0 and \(-1 \) by 1. Consequently, we can fill the Rédei matrix according to the values of the prime number \( q \).

Suppose that \( q \equiv 1 \pmod{8} \), then \( \left( \frac{2}{q} \right) = 1 \).

\[
R_{K/k} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}, \quad R_{K/k} = \begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
\]
If \((p/q)^4 = - (q/p)^4 = 1\)

\[
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 0 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Consequently, \(\text{rank}(R_{K/k}) = 4\), if \((p/q)^4 = - (q/p)^4 = -1\),

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Thus from the 4-rank formula of \(C\ell(K)\) (see, (2)), we obtain

\[
r_4(K) = \begin{cases} 
0, & \text{if } (p/q)^4 = -1, \\
1, & \text{if } (p/q)^4 = - (q/p)^4 = 1, \\
2, & \text{if } (p/q)^4 = (q/p)^4 = 1.
\end{cases}
\]

Suppose that \(q \equiv 5 \pmod{8}\), then \((2/q) = -1\).

If \((p/q)^4 = - (q/p)^4 = 1\)

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Consequently, \(\text{rank}(R_{K/k}) = \begin{cases} 
3, & \text{if } (p/q)^4 = 1, \\
4, & \text{if } (p/q)^4 = -1.
\end{cases}\)

Then \(r_4(K) = \begin{cases} 
1, & \text{if } (p/q)^4 = 1, \\
0, & \text{if } (p/q)^4 = -1.
\end{cases}\)

Suppose that \(q \equiv 3 \pmod{8}\), then \((2/q) = -1\).

If \((p/q) = 1\)

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1
\end{pmatrix}
\]

Consequently, \(\text{rank}(R_{K/k}) = \begin{cases} 
3, & \text{if } (p/q)^4 = 1, \\
4, & \text{if } (p/q)^4 = -1.
\end{cases}\)

Then \(r_4(K) = \begin{cases} 
1, & \text{if } (p/q)^4 = 1, \\
0, & \text{if } (p/q)^4 = -1.
\end{cases}\)
If \( \left( \frac{q}{p} \right)_4 = 1 \)
\[
R_{K/k} = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix},
\]

\[
R_{K/k} = \begin{pmatrix}
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
\]

If \( \left( \frac{q}{p} \right)_4 = -1 \)
\[
R_{K/k} = \begin{pmatrix}
1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix},
\]

\[
R_{K/k} = \begin{pmatrix}
1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.
\]

So rank\( (R_{K/k}) = 4 \) and thus \( r_4(K) = 0 \).

Suppose that \( q \equiv 7 \pmod{8} \), then \( \left( \frac{2}{q} \right) = 1 \).

If \( \left( \frac{2}{q} \right) = 1 \)
\[
R_{K/k} = \begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix},
\]

\[
R_{K/k} = \begin{pmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
\]

If \( \left( \frac{2}{q} \right) = -1 \)
\[
R_{K/k} = \begin{pmatrix}
1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix},
\]

\[
R_{K/k} = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.
\]

Consequently, rank\( (R_{K/k}) = \begin{cases} 
4, & \text{if } \left( \frac{2}{p} \right)_4 = 1, \\
3, & \text{if } \left( \frac{2}{p} \right)_4 = -1.
\end{cases} \)

Then \( r_4(K) = \begin{cases} 
0, & \text{if } \left( \frac{2}{p} \right)_4 = 1, \\
1, & \text{if } \left( \frac{2}{p} \right)_4 = -1.
\end{cases} \)

□
Example 3.8. Keep previous hypotheses and notations. Here we give examples when $q \equiv 1 \pmod{8}$.

(i) For $p = 173$ and $q = 41$, we have $173 \equiv 5 \pmod{8}$ and $(\frac{173}{41}) = 1$. From Corollary ((2.2)-(a)), we obtain $r_3(K) = 2$. Also $41 \equiv 1 \pmod{8}$ and $(\frac{173}{41})_4 = -1$, so the condition of Theorem ((B)-(1)) are satisfied. Hence, $r_4(K) = 0$, i.e., $Cl_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(ii) For $p = 101$ and $q = 17$, we have $101 \equiv 5 \pmod{8}$ and $(\frac{101}{17}) = 1$. From Corollary((2.2)-(a)), we obtain $r_2(K) = 2$. As $17 \equiv 1 \pmod{8}$ and $(\frac{101}{17})_4 = -1$, so the condition of Theorem ((B)-(1)) are satisfied. Hence, $r_4(K) = 1$. In fact, by Pari/GP we get $Cl_2(K) \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(iii) For $p = 157$ and $q = 17$, we have $157 \equiv 5 \pmod{8}$ and $(\frac{157}{17}) = 1$. From Corollary((2.2)-(a)), we obtain $r_2(K) = 2$. Also $17 \equiv 1 \pmod{8}$ and $(\frac{157}{17})_4 = 1$, so the condition of Theorem ((B)-(1)) are satisfied. Hence, $r_4(K) = 2$. Precisely, $Cl_2(K) \cong \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$.

Example 3.9. Now an example when $q \equiv 3 \pmod{8}$.

For $p = 53$ and $q = 43$, we have $53 \equiv 5 \pmod{8}$ and $(\frac{53}{43}) = 1$. From Corollary((2.2)-(a)), we obtain $r_2(K) = 2$, and as $43 \equiv 3 \pmod{8}$, with the condition of Theorem ((B)-(2)), hence $r_4(K) = 0$. In other words, $Cl_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 3.10. The following two examples illustrate the 3rd case of Theorem B.

(i) For $p = 269$ and $q = 53$, we have $269 \equiv 5 \pmod{8}$ and $(\frac{269}{53}) = 1$. The Corollary((2.2)-(a)) implies that, $r_3(K) = 2$ and since the condition of Theorem ((B)-(3)) are satisfied, so, $r_4(K) = 0$, i.e., $Cl_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

(ii) For $p = 293$ and $q = 109$, we can easily verify that $293 \equiv 5 \pmod{8}$ and $(\frac{293}{109}) = 1$, $109 \equiv 5 \pmod{8}$ and $(\frac{293}{109})_4 = 1$, then the Theorem ((B)-(3)) implies that $r_4(K) = 1$, exactly, $Cl_2(K) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Example 3.11. We end with an example for the last case of Theorem B.

(i) For $p = 173$ and $q = 23$, we have $173 \equiv 5 \pmod{8}$ and $(\frac{173}{23}) = 1$, $23 \equiv 7 \pmod{8}$ and $(\frac{173}{23})_4 = 1$. According to Corollary((2.2)-(a)) and Theorem ((B)-(4)), we have $r_4(K) = 0$.

(ii) Similarly for $p = 149$ and $q = 7$, we have $149 \equiv 5 \pmod{8}$, $(\frac{149}{7}) = 1$, $7 \equiv 7 \pmod{8}$ and $(\frac{149}{7})_4 = -1$, we find that $r_4(K) = 1$. By Pari/GP, we find $Cl_2(K) \cong \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

3.12. Case: $d = q_1q_2$ with $q_1$, $q_2$ are distinct prime numbers and $(\frac{p}{q_1}) = (\frac{p}{q_2}) = -1$.

Let $p$, $q_1$, $q_2$ be different odd prime numbers such that $p \equiv 5 \pmod{8}$, $(\frac{p}{q_1}) = (\frac{p}{q_2}) = -1$. Then the finite primes of $k$ ramifying in $K$, in this case, are: $\tilde{q}_1, \tilde{q}_2, 2t$ and $(\sqrt{p})$, where $\tilde{q}_1 = q_1O_k$, $\tilde{q}_2 = q_2O_k$, $pO_k = (\sqrt{p})^2$ and $2t = 2O_k$. To compute the 4-rank of the class group of $K$, by using the generalized Rédei-matrix, we need the following lemma:

Lemma 3.13. Keeping previous hypotheses and notations, then

a. For $i = 1, 2$ we have $(\frac{q_i, \delta}{q_i}) = (\frac{-1}{q_i})$ and $(\frac{2, \delta}{q_i}) = (\frac{-1, \delta}{q_i}) = 1$. 
b. \( \frac{q_i \delta}{q_j} = - \left( \frac{-1}{q_i} \right) \text{ and } \frac{q_j \delta}{q_i} = 1 \text{ for } i \neq j \in \{1, 2\} \).

Proof. We have

a. \( \frac{-1, \delta}{q_i} = \left[ \frac{-1}{q_i} \right] = \left( \frac{1}{q_i} \right) = 1 \).

b. As \( N_{k/Q}(\epsilon_p) = -1 \), so by (\cite[Ch 4 § 4.1 Proposition 4.2, p. 112]{10}),
\[
\left( \frac{\epsilon_p \delta}{q_i} \right) = \left( \frac{\epsilon_p}{q_i} \right) = \left( \frac{N_{k/Q}(\epsilon_p)}{q_i} \right) = \left( \frac{-1}{q_i} \right), \text{ and } \left( \frac{\epsilon_p \delta}{q_1} \right) = \left( \frac{-1}{q_1} \right).
\]

By the same argument, we get \( \frac{-1, \delta}{q_2} = \left[ \frac{-1}{q_2} \right] = \left( \frac{1}{q_2} \right) = 1 \).

b. \( \left( \frac{\epsilon_p \delta}{p} \right) = \left[ \frac{\epsilon_p \delta}{p} \right] = 1 \), and
\[
\left( \frac{q_1 \delta}{q_1} \right) = \left( \frac{q_1 \delta}{q_1} \right) = \left( \frac{q_1 - q_2 \sqrt{\mathcal{P}}}{q_1} \right) = \left( \frac{q_1 - q_2 \sqrt{\mathcal{P}}}{q_1} \right) = \left( \frac{q_2 \sqrt{\mathcal{P}}}{q_1} \right) = \left( \frac{1}{q_1} \right) \left( \frac{q_2}{q_1} \right) = \left( \frac{-1}{q_1} \right).
\]

By the same argument, we get \( \frac{q_2 \delta}{q_1} = 1 \) and \( \frac{q_2 \delta}{q_2} = -\left( \frac{-1}{q_2} \right) \).

\[ \square \]

Remark 3.14. By similar calculations as in Lemma (3.2), we get
\[
\left( \frac{q_1 \delta}{\sqrt{\mathcal{P}} / p} \right) = \left( \frac{q_2 \delta}{\sqrt{\mathcal{P}} / p} \right) = -1 \text{ and } \left( \frac{q_1 \delta}{p \infty} \right) = \left( \frac{q_2 \delta}{p \infty} \right) = 1.
\]
The symbols \( \left( \frac{\delta, \delta}{2I} \right) \) can be computed by using product formula except for \( \left( \frac{\sqrt{\mathcal{P}} / p}{p} \right) \) which can be calculated by the formula \( \left( \frac{\sqrt{\mathcal{P}} / p}{p_i} \right) = \left( \frac{-1, \delta}{p_i} \right) \left( \frac{q_1 \delta}{p_i} \right) \left( \frac{q_2 \delta}{p_i} \right) \) consequence of \( \left( \frac{-\delta, \delta}{p_i} \right) = 1 \) for \( p_i = \sqrt{\mathcal{P}}, 2I, \sqrt{\mathcal{P}} \) or \( p \infty \).

If \( d = q_1 q_2 \), \( \left( \frac{p}{q_1} \right) = \left( \frac{p}{q_2} \right) = -1 \), then we can summarise the results in the following table:

| element | prime | \( q_1 \) | \( q_2 \) | \( 2I \) | \( \sqrt{\mathcal{P}} \) | \( p \infty \) |
|---------|-------|---------|---------|--------|----------------|--------------|
| \( q_1 \) | \( -\left( \frac{-1}{q_1} \right) \) | 1 | \( \left( \frac{-1}{q_1} \right) \) | -1 | 1 |
| \( q_2 \) | 1 | \( -\left( \frac{-1}{q_2} \right) \) | \( \left( \frac{-1}{q_2} \right) \) | -1 | 1 |
| 2 | 1 | 1 | -1 | -1 | 1 |
| \( \sqrt{\mathcal{P}} \) | \( -\left( \frac{-1}{q_1} \right) \) | \( -\left( \frac{-1}{q_2} \right) \) | \( -\left( \frac{-1}{q_2} \right) \) | 1 | -1 |
| \( \epsilon_p \) | \( -\left( \frac{-1}{q_1} \right) \) | \( -\left( \frac{-1}{q_2} \right) \) | \( -\left( \frac{-1}{q_2} \right) \) | -1 | -1 |

Proof of Theorem C. If \( d = q_1 q_2 \) with \( q_1, q_2 \) are two different prime numbers such that \( \left( \frac{p}{q_1} \right) = \left( \frac{p}{q_2} \right) = -1 \), by corollary (2.2) we have \( r_2(K) = 2 \). As the finite primes of \( k \) which ramify in \( K \) are: \( \sqrt{\mathcal{P}}, 2I, (\sqrt{\mathcal{P}}) \) where \( \sqrt{\mathcal{P}} = q_1 O_K, \sqrt{\mathcal{P}} = q_2 O_K, 2I = 2 O_K \) and \( (\sqrt{\mathcal{P}})^2 = p O_K \).

So there exist prime ideals \( \mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4 \) of \( K \) such that
\[
\mathcal{H}_1^2 = \sqrt{\mathcal{P}} O_K, \mathcal{H}_2^2 = \sqrt{\mathcal{P}} O_K, \mathcal{H}_3^2 = 2I O_K, \mathcal{H}_4^2 = \sqrt{\mathcal{P}} O_K.
\]
Moreover, $E_k = \langle -1, \varepsilon_p \rangle$ and $-1, \varepsilon_p, -\varepsilon_p \notin E_k \cap N_{K/k}(K)$ (Lemma 2.1). Then the $5 \times (4 + 2)$ generalized Rédei’s matrix $R_{K/k}$ is

$$R_{K/k} = \begin{pmatrix}
\begin{array}{cccc}
q_1 & q_2 & \sqrt{2} & \sqrt{2} \\
q_1 & q_2 & \sqrt{2} & \sqrt{2} \\
q_1 & q_2 & \sqrt{2} & \sqrt{2} \\
q_1 & q_2 & \sqrt{2} & \sqrt{2} \\
q_1 & q_2 & \sqrt{2} & \sqrt{2} \\
\end{array}
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\begin{pmatrix}
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\varepsilon & \varepsilon & \varepsilon & \varepsilon \\
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}.$$

We consider the above matrix with coefficients in $\mathbb{F}_2$ by replacing 1 by 0 and $-1$ by 1.

| (q_1, q_2) (mod 4) | (1, 1) | (1, 3) |
|---------------------|--------|--------|
| $R_{K/k}$           |        |        |
| 1 0 0 1 0 0         |        | 1 0 0 1 0 0 |
| 0 1 0 1 0 0         |        | 0 0 0 0 0 1 |
| 0 0 1 1 1 0         |        | 0 1 1 0 1 1 |
| 1 1 1 0 0 1         |        | 1 1 1 0 0 1 |
| 0 0 0 1 1 1         |        | 0 0 0 1 1 1 |

| (q_1, q_2) (mod 4) | (3, 1) | (3, 3) |
|---------------------|--------|--------|
| $R_{K/k}$           |        |        |
| 0 0 0 0 0 1         |        | 0 0 0 0 0 1 |
| 0 1 0 1 0 0         |        | 0 0 0 0 0 1 |
| 1 0 1 0 1 1         |        | 1 1 1 1 1 0 |
| 1 1 1 0 0 1         |        | 1 1 1 0 0 1 |
| 0 0 0 1 1 1         |        | 0 0 0 1 1 1 |

So

$$\text{rank}(R_{K/k}) = \begin{cases} 
3, & \text{if } (q_1, q_2) \equiv (3, 3) \pmod{4}, \\
4, & \text{otherwise}.
\end{cases}$$

Then

$$r_4(K) = \begin{cases} 
1, & \text{if } (q_1, q_2) \equiv (3, 3) \pmod{4}, \\
0, & \text{otherwise}.
\end{cases}$$

Example 3.15. We finish this work by illustrating the last Theorem with some examples:

(i) For $p = 5$, $q_1 = 3$ and $q_2 = 7$, we have $p = 5 \equiv 5 \pmod{8}$ and $(\frac{5}{3}) = (\frac{5}{7}) = -1$. From Corollary ((2.2)-(b)), we obtain $r_2(K) = 2$. As $(q_1, q_2) = (3, 7) \equiv (3, 3) \pmod{4}$, then, $r_4(K) = 1$. In fact, $\text{Cl}_2(K) \cong \mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.  

\[\square\]
(ii) Likewise if \( p = 13, q_1 = 19 \) and \( q_2 = 83 \), we obtain \( r_2(K) = 2 \) and \( r_4(K) = 1 \), but this time, \( Cl_2(K) \cong \mathbb{Z}/16\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

For the case \((q_1, q_2) \not\equiv (3, 3) \pmod{4}\), we have the following examples:

(iii) Take \( p = 37, q_1 = 17 \) and \( q_2 = 29 \), we have \( p = 37 \equiv 5 \pmod{8} \) and \( \left(\frac{37}{17}\right) = \left(\frac{37}{29}\right) = -1 \). From Corollary ((2.2)-(b)), we obtain \( r_2(K) = 2 \). Also \((q_1, q_2) = (17, 29) \equiv (1, 1) \pmod{4}\), so the condition of Theorem (C) are satisfied. Hence, \( r_4(K) = 0 \) and \( Cl_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

(iv) For \( p = 37, q_1 = 17 \), we have \( p = 37 \equiv 5 \pmod{8} \), \( \left(\frac{37}{17}\right) = \left(\frac{37}{31}\right) = -1 \) and \((q_1, q_2) = (17, 31) \equiv (1, 3) \pmod{4}\). we also find \( Cl_2(K) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \).

ACKNOWLEDGMENT

We would like to thank the unknown referee for his/her several helpful suggestions and for calling our attention to the missing details.

REFERENCES

[1] A. Azizi et M. Taous, Détermination des corps \( k = \mathbb{Q}(\sqrt{d}, i) \) dont les 2-groupes de classes sont de type \((2, 4) \text{ ou } (2, 2, 2)\), Rend. Istit. Mat. Univ. Trieste. 40, (2008), 93-116.

[2] A. Azizi et A. Mouhib, Sur le rang du 2-groupe de classes de \( \mathbb{Q}(\sqrt{m}, \sqrt{d}) \) où \( m = 2 \) ou un premier \( p \equiv 1 \pmod{4} \), Trans. Amer. Math. Soc. 353, No 7, (2001), 2741-2752.

[3] A. Azizi et A. Mouhib, Le 2-rang du groupe de classes de certains corps biquadratiques et applications, Internat. J. Math. 15, No. 02, (2004), 169-182.

[4] A. Azizi, M. Taous and A. Zekhnini, On the unit index of some real biquadratic number fields, Turk. J. Math. (2018), no. 42, 703-715.

[5] E. Brown and Ch. J. Parry, The 2-class group of certain biquadratic number fields, J. reine angew. Math. 295, (1977), 61-71.

[6] E. Brown and Ch. J. Parry, The 2-class group of certain biquadratic number fields II, Pacific J. Math. 78, No. 1, (1978), 61-71.

[7] C. Chevalley, Sur la théorie du corps de classes dans les corps finis et les corps locaux, J. Fac. Sc. Tokyo, Sect. 1, t. 2, (1933), 365-476.

[8] G. Gras, Class field theory, from theory to practice, Springer Verlag 2003.

[9] J. A. Hymo and C. J. Parry, On relative integral bases for pure quartic fields, Indian J. Pure Appl. Math. 23, 1992, 359-376.

[10] F. Lemmermeyer, Reciprocity Laws. From Euler to Eisenstein, Springer Monographs in Math. (2000).

[11] C. J. Parry, Pure quartic number fields whose class numbers are even, J. Reine Angew. Math. 264(1975), 102-112.

[12] C. J. Parry, A genus theory for quartic fields, J. Reine Angew. Math. 314 (1980), 40-71.

[13] The PARI Group, PARI/GP version 2 − 11 − 3, Univ. Bordeaux, 2020, http://pari.math.u-bordeaux.fr.

[14] Y. Qin, The generalized Rédei-matrix, Math. Z. 261 (2009), 23-37.

[15] M. Taous, Capitalisation des classes d’idéaux de certains corps \( \mathbb{Q}(\sqrt{d}, i) \) de type \((2, 4)\), thèse, Université. Mohammed Premier Faculté des Science, Oujda. 2008.

[16] H. C. Williams, The quadratic character of a certain quadratic surds, Utilitas Math. 5, (1974), 49-55.

MOULAY ISMAIL UNIVERSITY OF MEKNES. FACULTY OF SCIENCES AND TECHNOLOGY, P.O. Box 509-BOUTALAMINE, 52 000 ERRACHIDIA

E-mail address: haynou.mbarek@hotmail.com
E-mail address: taousm@hotmail.com