Abstract. In this paper, we establish the sharp $k$-broad estimate for a class of phase functions satisfying the homogeneous convex conditions. As an application, we obtain improved local smoothing estimates for the half-wave operator in dimensions $n \geq 3$. As a byproduct, we also generalize the restriction estimates of Ou–Wang [22] to a broader class of phase functions.

1. Introduction

Let $u$ be the solution to the Cauchy problem
\[
\begin{cases}
(\partial_t - \Delta)u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^n, \\
u(0,x) = f, & \partial_t u(0,x) = 0
\end{cases}
\] (1.1)

where $f$ is a Schwartz function. $u$ can be expressed in terms of the half-wave operator $e^{it\sqrt{-\Delta}}$ as
\[
u(x,t) = \frac{1}{2} \left( e^{it\sqrt{-\Delta}} f + e^{-it\sqrt{-\Delta}} f \right).
\]

This paper is concerned with the $L^p$-regularity estimate of the solution $u$. For fixed time $t$, the classical sharp estimate of Peral [24] and Miyachi [20] reads:
\[
\|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)} \leq C_{t,p} \|f\|_{L^{s_p}(\mathbb{R}^n)}, \quad s_p := \left( n - 1 \right)^{1/2} \left( \frac{1}{p} - \frac{1}{p'} \right) 1 < p < \infty.
\] (1.2)

This estimate trivially leads to the following space-time estimate
\[
\left( \int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)}^p \, dt \right)^{1/p} \lesssim \|f\|_{L^p_{s_p}(\mathbb{R}^n)}.
\] (1.3)

One natural question then arises: can one do better than (1.3)? More precisely, does there exist some $\varepsilon > 0$ such that (1.3) holds with $s_p - \varepsilon$ in place of $s_p$? The following local smoothing conjecture was formulated by Sogge [25].

Conjecture 1.1 (Local smoothing conjecture). For $n \geq 2$, the inequality
\[
\left( \int_1^2 \|e^{it\sqrt{-\Delta}} f\|_{L^p(\mathbb{R}^n)}^p \, dt \right)^{1/p} \lesssim \|f\|_{L^p_{s_p-\varepsilon}(\mathbb{R}^n)}
\] (1.4)

holds for all
\[
\sigma < \begin{cases} \frac{1}{p'} & \text{if } \frac{2n}{n-1} < p < \infty; \\
s_p & \text{if } 2 < p \leq \frac{2n}{n-1}.
\end{cases}
\] (1.5)

In the same paper, Sogge [25] obtained the first partial results on the above conjecture for all $p > 2$ when $n = 2$, which were greatly simplified and further improved in his joint work with Mockenhaupt and Seeger [21], where the square function approach was introduced. In 2000, Wolff [29] proved Conjecture 1.1 for the case $n = 2$ and $p > 74$ by introducing what is now known as the decoupling inequality for the cone. Following Wolff, decoupling inequalities have been studied by many authors [8, 9, 15]. In 2015, Bourgain–Demeter [5] established the full range of sharp $\ell^2$-decoupling inequalities in all dimensions, of which the influence permeates into

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Theorem 1.2. Let $n \geq 3$ and
\[
p > \begin{cases} 
\frac{2n+5}{2n+6} & \text{for } n \text{ odd,} \\
\frac{2n+6}{2n+7} & \text{for } n \text{ even.}
\end{cases} 
\]
Then
\[
\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n \times [1,2])} < C\|f\|_{L^p_{\sigma,p^{-}}} , \text{ for all } \sigma < \frac{2}{p} - \frac{1}{2}.
\] (1.7)

For $n \geq 3$ and $2 < p < \frac{2(n+1)}{n-1}$, the sharp $\ell^p$ decoupling inequality of Bourgain–Demeter [5] and the $L^2$ energy estimate implies
\[
\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n \times [1,2])} < C\|f\|_{L^p_{\sigma,p^{-}}} , \quad \sigma < \frac{n-1}{2} \left(\frac{2}{p} - \frac{1}{2}\right).
\] (1.8)

A direct calculation shows that if $p < \frac{2(n+3)}{n+1}$,
\[
\frac{n-1}{2} \left(\frac{2}{p} - \frac{1}{2}\right) < \frac{2}{p} - \frac{1}{2}.
\]

One can see that Corollary 1.3 below improves the previous best known local smoothing estimate (1.8) in range of $2 < p < \frac{2(n+1)}{n-1}$ for $n \geq 3$. Indeed, by interpolating using (1.7), (1.8) and the trivial $L^2$ bound, we have the following.

Corollary 1.3. Let $n \geq 3$, Then
\[
\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^n \times [1,2])} < C\|f\|_{L^p_{\sigma,p^{-}}} 
\] (1.9)
for $\sigma < \sigma_p$, where if $n \geq 3$ is odd,
\[
\sigma_p = \begin{cases} 
\frac{3n - 3}{4} \left(\frac{1}{2} - \frac{1}{p}\right), & 2 < p \leq \frac{3n + 5}{3n + 1}, \\
n - 1 \left(\frac{3n + 1}{6n + 10} - \frac{1}{p}\right) + \frac{3n - 3}{6n + 10}, & \frac{3n + 5}{3n + 1} < p \leq \frac{n + 1}{n - 1},
\end{cases} 
\] (1.10)
if $n \geq 3$ is even,
\[
\sigma_p = \begin{cases} 
\frac{3n - 2}{4} \left(\frac{1}{2} - \frac{1}{p}\right), & 2 < p \leq \frac{3n + 6}{3n + 2}, \\
n - 2 \left(\frac{3n + 2}{6n + 12} - \frac{1}{p}\right) + \frac{3n - 2}{6n + 12}, & \frac{3n + 6}{3n + 2} < p \leq \frac{n + 1}{n - 1}.
\end{cases} 
\] (1.11)

See Table 1 for a detailed comparison for the improvement at the conjectured critical exponent $p_c = \frac{2n}{n-1}$ for $n = 3, 4, 5, 6$. See Figure 1 for a $\sigma-p^{-1}$ plot for the odd $n$ case.

Local smoothing of the half wave operator has been studied extensively. As discussed above, instead of handling the half wave operator $e^{it\sqrt{-\Delta}}f$ directly, people usually opt to establish decoupling inequalities or square function estimates, and then apply them to the local smoothing problem. At this point, let us briefly review both approaches.

| $n$ | $p_c$ | $\sigma_{p_c}$ (conjectured) | $\sigma_{p_c}$ ([5]) | $\sigma_{p_c}$ (Corollary 1.3) |
|-----|-------|-----------------------------|----------------------|-----------------------------|
| 3   | 3     | 1/3                         | 1/6                  | 2/9                         |
| 4   | 8/3   | 3/8                         | 3/16                 | 9/32                        |
| 5   | 5/2   | 2/5                         | 1/5                  | 3/10                        |
| 6   | 12/5  | 5/12                        | 5/24                 | 1/3                         |

Table 1. Comparing Corollary 1.3 to previous records at $p_c$. number theory, PDEs and geometric measure theory. As a direct consequence, Bourgain and Demeter obtained the sharp local smoothing estimate for all $n \geq 2$ and $p \geq \frac{2(n+1)}{n-1}$. Recently, Guth-Wang-Zhang [12] resolved the local smoothing conjecture for $n = 2$ by establishing the full range sharp square function inequality. The purpose of this paper is to further improve the local smoothing result for dimensions $n \geq 3$ and $2 < p < \frac{2(n+1)}{n-1}$. In particular, we obtain
Define the domain $\Gamma \subset \mathbb{R}^n$ as
$$
\Gamma := \{ (\xi', \xi_n) \in \mathbb{R}^n \setminus \{0\} : 1 \leq \xi_n \leq 2, \, |\xi'| \leq \xi_n \},
$$
and make slab-decomposition with respect to $\Gamma$ in the following way. Assuming $R > 1$, we select $a$ collection of $R^{-1/2}$-maximally separated points $\{ (\xi'_\nu, 1) \}$ in the unit ball $B^{n-1}(0, 1) \times \{1\}$ of the affine hyperplane $\xi_n = 1$. For each $\nu$, we define the $\nu$-slab as
$$
\nu := \{ (\xi', \xi_n) \in \Gamma : |\xi'_\nu - \xi_n| \leq R^{-1/2} \},
$$
Let $\chi_\nu$ be the characteristic function of the $\nu$-plate, and set $f^\nu = \mathcal{F}^{-1}(\hat{f}_\chi_\nu)$.

Under notation above, the $\ell^p$-decoupling inequality due to Bourgain–Demeter is

$$
\left\| \sum_\nu e^{it\sqrt{-\Delta}} f^\nu \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_\varepsilon R^{(n-1)\frac{1}{2} - \frac{1}{p} + \varepsilon} \left( \sum_\nu \| e^{it\sqrt{-\Delta}} f^\nu \|_{L^p(\mathbb{R}^{n+1})}^p \right)^{\frac{1}{p}},
$$

(1.12)

for $p \geq \frac{2(n+1)}{n+1}$. As a direct consequence of (1.12), Conjecture 1.1 has been resolved for the range $p \geq \frac{2(n+1)}{n+1}$. It seems that $\ell^p$-decoupling inequality is well suited for handling local smoothing estimate with larger exponents, whereas, inefficient for tackling the case when $p$ is close to the endpoint $p = \frac{2n}{n-1}$. In contrast, the following conjectured (reverse) square function inequality has been proven to be powerful near the endpoint.

$$
\left\| \sum_\nu e^{it\sqrt{-\Delta}} f^\nu \right\|_{L^p(\mathbb{R}^{n+1})} \leq C_\varepsilon R^{\varepsilon} \left( \sum_\nu | e^{it\sqrt{-\Delta}} f^\nu |^2 \right)^{\frac{1}{2}} \|_{L^p(\mathbb{R}^{n+1})} ; \quad 2 \leq p \leq \frac{2n}{n-1}.
$$

(1.13)

Recently, in the remarkable work of Guth–Wang–Zhang [12], the authors established inequality (1.13) in dimension two. Following the argument in [21], (1.13) leads to the full range of sharp local smoothing estimate for $n = 2$. To be more precise, to handle the local smoothing
problem using the square function inequality, we also need to use the Nikodym maximal function inequality associated to the cone. The sharp result for such Nikodym maximal function inequality was established in [21] for \( n = 2 \), but its higher dimensional counterpart is still wide open, which limits us, to some extent, to advance the research of the local smoothing in higher dimensions. For more discussion about the local smoothing estimate of the half-wave operator, see [16, 17, 14, 27].

In this paper, motivated by the seminal work of Guth [10], we circumvent these problems through handling the operator \( e^{it\sqrt{-\Delta}} \) directly by employing the so-called \( k \)-broad “norm” estimate, which can be seen as a weaker version of the multilinear restriction estimate due to Bennett–Carbery–Tao [4]. It is worth noting that, there is a difference in the results of Theorem 1.2 between odd and even spatial dimensions. This is a common theme in the study of problems related to the restriction conjecture. In particular, for such problems in the variable coefficient setting, certain Kakeya compression phenomena exist. Such phenomena are usually different between odd and even dimensions. One may refer to [2, 11, 19, 28] for more details. Therefore, even though we only work in the Euclidean case in this article, we believe that the methods used in this paper may stimulate the research of the local smoothing estimate for the class of Fourier integral operator satisfying cinematic curvature conditions in higher dimensions.

Let us describe the outline of our proof and the key difficulties that arise. After some routine reductions, we shall perform a multi-scale broad-narrow argument which is inspired by the arguments in [10, 22]. However, there are two main difficulties that we need to overcome. Firstly, unlike the restriction problem for a circular cone, Lorentz rescaling arguments are much more complicated in the local smoothing setting. As a result, we have to prove the \( k \)-broad estimate for general positively curved cones, which in turn requires us to establish several geometric lemmas without the nice symmetry of a circular cone. This is done in Section 4. Secondly, since we have to deal with general cones, we would need a narrow decoupling theorem for them. Unfortunately, however, unlike the parabolic case, the missing narrow decoupling theorem for general cones does not follow directly from the circular cone case via a Pramanik–Seeger approximation argument [23]. To overcome this, instead of proving estimates for a fixed class of cones, we opt for a new induction on scales argument with respect to a whole family of classes of cones \( \Phi(R) \), which are indexed by the physical scale \( R \). To be more precise, the class of cones that we care about will approximate the circular cone better and better as the scale \( R \) grows. The definition of the class \( \Phi(R) \) will be given in Section 2, and the induction argument will be given in Section 7. The authors believe that this approach is novel and may serve an important role in the study of other related problems.

The rest of this paper is organized as follows: In Section 2, we review some preliminaries and basic reductions. In Section 3, we present the wave packet decomposition used in our proof. In Section 4, we establish a geometric lemma for a general class of cones, which plays a crucial role in the proof of \( k \)-broad “norm” estimate. In Section 5, we prove the \( k \)-broad “norm” estimate via polynomial partitioning, in the spirit of [10, 22]. In Section 6, a parabolic rescaling lemma suited for our setting will be established, which is a critical ingredient in our induction on scale argument. This parabolic rescaling lemma is similar to the ones established in [1, 6] as we are dealing with a whole class of phase functions. Finally, we give the proof of Theorem 1.2 and state the restriction estimates for general cones in Section 7.

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want to thank Prof. David Beltran for some friendly communications, and for pointing out that broad and narrow bounds for the circular cone only is not enough for proving local smoothing type estimates. The authors want to thank an anonymous referee for his or her thorough and invaluable feedback.

**Notation.** For non-negative quantities $X$ and $Y$, we will write $X \lesssim Y$ to denote the inequality $X \leq CY$ for some constant $C > 0$. If $X \lesssim Y \lesssim X$, we will write $X \sim Y$. Dependence of the implicit constants on the spatial dimensions or integral exponents such as $p$ will be suppressed; dependence on additional parameters will be indicated using subscripts or parenthesis. For example, $X \lesssim_M Y$ indicates $X \leq CY$ for some $C = C_M$. For a function $A(R)$, we write $A(R) = \operatorname{RapDec}(R)$ if for any $N \in \mathbb{N}$, there is a constant $C_N$ such that

$$|A(R)| \leq C_N R^{-N} \quad \text{for all } R \geq 1.$$ 

Throughout the paper, $\chi_E$ denotes the characteristic function of the set $E$. We usually denote by $B^n_R(a)$, or simply $B_R(a)$, a ball in $\mathbb{R}^n$ with center $a$ and radius $R$. We will also denote by $B^n_R$, or simply $B_R$, a ball of radius $R$ and arbitrary center in $\mathbb{R}^n$. Let $r > 0$, for the sake of convenience, we denote $C_r^{n+1}$ to be the cylinder $B^n_r \times [-r, r]$. Denote by $A(R) := B^n_{2R}(0) \setminus B^n_{R/2}(0)$. We denote $w_{B^n_R(x_0)}$ to be a nonnegative weight function adapted to the ball $B^n_R(x_0)$ such that

$$w_{B^n_R(x_0)}(x) \lesssim (1 + R^{-1}|x - x_0|)^{-M},$$

for some large constant $M \in \mathbb{N}$.

For any subspace $V \subset \mathbb{R}^n$ and $\eta \in \mathbb{R}^n$, we adopt the notation $\operatorname{Ang}(\eta, V)$ to denote the smallest angle between $\eta$ and any given vector $v \in V \setminus \{0\}$. Let $W \subset \mathbb{R}^n$ be another subspace, define $\operatorname{Ang}(V, W)$ as

$$\operatorname{Ang}(V, W) := \min_{v \in V \setminus \{0\}, w \in W \setminus \{0\}} \operatorname{Ang}(v, w).$$

## 2. Preliminaries

### 2.1. Basic reductions and phase function classes.

In this paper, as is standard in rescaling arguments, we shall prove estimates associated to a large scale $R \gg 1$, and an arbitrarily small parameter $\varepsilon > 0$. We say a phase function $\phi$ is (positively) homogeneous of degree one if it satisfies

$$\textbf{H}_1: \phi \in C^\infty(\mathbb{R}^n \setminus \{0\}), \quad \phi(\lambda \xi) = \lambda \phi(\xi), \quad \forall \lambda > 0.$$ 

Moreover, we say $\phi$ satisfies the homogeneous convex conditions if it satisfies the following condition as well.

$$\textbf{H}_2: \text{The Hessian of } \phi \text{ i.e. } \left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)_{i,j,n} \text{ has (n-1)-positive eigenvalues.}$$

A prototypical example for the homogeneous convex function is given by $\phi(\xi) = |\xi|$. We are concerned with the half-wave operator $e^{it\sqrt{-\Delta}}$, and it can be reduced to considering an oscillatory integral involving the phase function $|\xi|$. For technical reasons, we need to employ an induction on scales argument which requires the phase function to stay invariant under certain transformations, while the function $|\xi|$ alone does not ensure this. Thus, we shall work with the following class of phase functions.

**Definition 2.1 (Phase function class $\Phi$).** We say a function $\phi$ lies in the class $\Phi$, if $\phi$ obeys the homogeneous convex conditions $\textbf{H}_1, \textbf{H}_2$, with eigenvalues of $\left( \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right)$ in the interval $[1/2, 2]$, and

$$|\partial^\alpha \phi(\xi)| \leq C_{\text{par}}, \quad \forall |\alpha| \leq N_{\text{par}}, \quad \xi \in N_{\varepsilon_0}(v_n),$$

where $C_{\text{par}} > 0$, $0 < \varepsilon_0 \ll 1$ and $N_{\text{par}} \in \mathbb{N}$ are universal constants, and $N_{\varepsilon_0}(v_n)$ denotes the $\varepsilon_0$-neighborhood of $v_n$.

Moreover, to facilitate the proof of the narrow decoupling theorem in Section 7, we will also consider a more special class of phase functions satisfying a more precised condition which we will define now. Let $K_0 > 0$, $\delta = \delta(\varepsilon) > 0$, both of which will be chosen later in the argument.
\( H_3: \) Let \( K = K_0 R^4 \), and 
\[
\phi_R(\xi) = \frac{\xi_1^2 + \cdots + \xi_n^2}{2\xi_n} + K^{-4}E_R(\xi), \quad \forall \xi \in \mathbb{N}_0(e_n),
\]
where \( E_R(\xi) \) is a homogeneous function of degree 1 and satisfies 
\[
|\partial^\alpha E_R(\xi)| \leq c_{\text{par}}, \quad |\alpha| \leq N_{\text{par}},
\]
for some fixed constant \( 0 < c_{\text{par}} < 1 \).

**Definition 2.2** (Phase function class \( \Phi(R) \)). We say a function \( \phi_R \) is in the class \( \Phi(R) \), if \( \phi_R \) obeys condition \( H_3 \).

Here and throughout the paper, we shall always assume \( \phi \in \Phi \) and \( \phi_R \in \Phi(R) \). It should be noted that one needs to be extra careful when working with the class \( \Phi(R) \), since it depends on the scale \( R \). To be more precise, we need to make sure that after each rescaling step, our new phase function lands in the appropriate class associated to the new scale. In addition, since it is easy to check that \( \Phi(R) \subseteq \Phi \), any statements that we prove for phase functions in the bigger class \( \Phi \) will certainly hold for any \( \phi_R \in \Phi(R) \).

Next, let us collect some useful standard results from previous works.

**2.2. Transference between local and global estimates.** This reduction was used in [7] for a slightly different case. We give the details here for completeness.

Let \( \psi \) be a non-negative smooth function on \( \mathbb{R}^n \) such that 
\[
\text{supp} \hat{\psi} \subset B_1^n(0), \quad \sum_{\ell \in \mathbb{Z}^n} \psi(x - \ell) \equiv 1, \quad \forall x \in \mathbb{R}^n. \tag{2.2}
\]
Define \( \psi_t(x) := \psi(R^{-1}x - \ell) \) and \( f_\ell = \psi_\ell f \).

**Lemma 2.3.** Assume \( \text{supp} \hat{\psi} \subset A(1) \), then for any \( \varepsilon > 0 \), there holds 
\[
|e^{it\sqrt{-\Delta}}f(x)| \lesssim \varepsilon \left| e^{it\sqrt{-\Delta}}(\Psi_{B_1^{n+1}(x_0)})(x) \right| + \text{RapDec}(R) \sum_{|\ell| > R^\varepsilon} \| f|\psi_\ell(\cdot - x_0) \|_{L^p(\mathbb{R}^n)}^{\frac{1}{2}}, \tag{2.3}
\]
for \((x, t) \in B_R^n(x_0) \times [-R, R], 1 < p < \infty \), where 
\[
\Psi_{B_1^{n+1}(x_0)}(x) := \sum_{|\ell| \leq R^\varepsilon} \psi(R^{-1}(x - x_0) - \ell).
\]

**Proof.** Without loss of generality, we may assume that \( x_0 = 0 \). We rewrite \( e^{it\sqrt{-\Delta}}f \) via (2.2) as 
\[
e^{it\sqrt{-\Delta}}f(x) = \sum_{\ell \in \mathbb{Z}^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i((x-y)\cdot \xi + t|\xi|)} \eta(\xi)f_\ell(y) \, d\xi \, dy, \tag{2.4}
\]
where \( \eta(\xi) \in C_c^\infty(B_1^n(0)) \) satisfying that \( \eta(\xi) = 1 \) for \( \xi \in B_1^n(0) \). The associated kernel \( K_\ell(\cdot) \) of the operator \( e^{it\sqrt{-\Delta}} \eta(D) \) is given by 
\[
K_\ell(x) := \int_{\mathbb{R}^n} e^{i(x - \xi)\cdot \xi} \eta(\xi) \, d\xi.
\]
Noting that \( |\ell| \leq R \), by an integration by parts argument, we see that 
\[
|K_\ell(x)| \leq C \chi_{|x| \leq CR} + C_M \frac{\chi_{|x| \geq CR}}{(1 + |x|)^{M}}, \quad \text{for all } M \in \mathbb{N}. \tag{2.5}
\]
Now we decompose \( e^{it\sqrt{-\Delta}}f(x) \) into two parts 
\[
e^{it\sqrt{-\Delta}}f(x) = \sum_{|\ell| \leq R^\varepsilon} e^{it\sqrt{-\Delta}}f_\ell(x) + \sum_{|\ell| > R^\varepsilon} e^{it\sqrt{-\Delta}}f_\ell(x)
\]
\[
= e^{it\sqrt{-\Delta}}(\Psi_{B_1^{n+1}(0)})(x) + \sum_{|\ell| > R^\varepsilon} e^{it\sqrt{-\Delta}}f_\ell(x). \tag{2.6}
\]
To complete the proof it suffices to bound the second term. By Hölder’s inequality, we have
\[
\left| \sum_{|\ell| > R^e} e^{it\sqrt{-\Delta}} f_{\ell}(x) \right| \leq \sum_{|\ell| > R^e} \int_{\mathbb{R}^n} K_\ell(x-y)f_{\ell}(y) \, dy
\]
\[
\leq \sum_{|\ell| > R^e} \int_{\mathbb{R}^n} \left| K_\ell(x-y) \right| \frac{1}{|\ell|} |\psi_\ell(y)| \frac{1}{|\ell|} |f(y)| |K_\ell(x-y)| \frac{1}{|\ell|} \, dy
\]
\[
\leq \sum_{|\ell| > R^e} \left( \int_{\mathbb{R}^n} \left| K_\ell(x-y) \right| \frac{1}{|\ell|} |\psi_\ell(y)| \frac{1}{|\ell|} \, dy \right)^{\frac{p}{p}} \left( \int_{\mathbb{R}^n} |\psi_\ell(y)| \frac{1}{|\ell|} |f(y)| |K_\ell(x-y)| \frac{1}{|\ell|} \, dy \right)^{\frac{1}{p'}}.
\]
For \((x, t) \in B_R^0(0) \times [-R, R],\) using the rapidly decay of \(\psi\) and (2.5), we have
\[
|K_\ell(x-y)\psi_\ell(y)| \lesssim_M \frac{R^{-\epsilon}}{(1 + |R^{-1}y - \ell|)^M}, \quad |\ell| > R^e, \quad \forall \, x \in B_R^0(0), \quad y \in \mathbb{R}^n,
\]
and
\[
|K_\ell(x-y)| \lesssim_M \frac{1}{1 + |y|^2}, \quad \forall \, x \in B_R^0(0), \quad y \in \mathbb{R}^n.
\]
Hence,
\[
\left| \sum_{|\ell| > R^e} e^{it\sqrt{-\Delta}} f_{\ell}(x) \right| \lesssim_M R^{-\epsilon M + \frac{n}{p}} \sum_{|\ell| > R^e} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(0))}.
\]
\]
\]

As a immediate consequence of Lemma 2.3, we obtain the relation between local and global estimates in the spatial variables.

**Corollary 2.4.** Let \(I \subseteq [-R, R]\) be an interval. Suppose supp \(\hat{f} \subset \Lambda(1)\) and
\[
\|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(B_R^n \times I)} \leq CR^n \|f\|_{L^p},
\]
then, given any \(\epsilon > 0\), there exists a constant \(C_\epsilon\) such that
\[
\|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(\mathbb{R}^n \times I)} \leq C_\epsilon R^{n+\epsilon} \|f\|_{L^p}.
\]

**Proof.** Let \(\{B_R^0(x_k)\}_{k \in \mathbb{Z}^n}\) be a covering of \(\mathbb{R}^n\) using balls of radius \(R\) with bounded overlaps. We have
\[
\|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(\mathbb{R}^n \times I)} \leq \sum_{k \in \mathbb{Z}^n} \|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(B_R^0(x_k) \times I)}.
\]
Using Lemma 2.3, we get
\[
\|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(B_R^0(x_k) \times I)} \lesssim_C \left( \sum_{\ell \in \mathbb{Z}^n} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(x_k))} \right)^{\frac{1}{p}} + \text{RapDec}(R) \sum_{|\ell| > R^e} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(x_k))}.
\]
Summing over \(k\), we obtain
\[
\sum_{k} \|e^{it\sqrt{-\Delta}} f\|_{L^p_{x,t}(B_R^0(x_k) \times I)} \lesssim_C \left( \sum_{k} \left( \sum_{|\ell| > R^e} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(x_k))} \right)^{\frac{1}{p}} + \text{RapDec}(R) \sum_{|\ell| > R^e} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(x_k))} \right)^{\frac{1}{p}}.
\]
It follows from (2.7) and the bounded overlaps of \(\{B_R^0(x_k)\}\) that the first term can be estimated by
\[
\sum_{k} \left( \sum_{|\ell| > R^e} \|f|\psi_\ell|^\frac{1}{p}\|_{L^p(w_B_R^0(x_k))} \right)^{\frac{1}{p}} \lesssim R^{sp+\epsilon p} \|f\|_{L^p}^{\frac{p}{p'}}.
\]
To finish the proof, we use Minkowski’s inequality to see that the second term satisfies

\[ \text{RapDec}(R) \sum_k \left( \sum_{|\ell|>R^\epsilon} \|f|v_\ell(\cdot - x_k)\|_{L^p(w_{B^R(x_k)})} \right)^p \lesssim \text{RapDec}(R) \|f\|_{L^p_{\text{w}B}}. \]

\[ \square \]

2.3. Reducing to the class Φ(R). To prove Theorem 1.2, it suffices to show

\[ \|e^{it\sqrt{-\Delta}}f\|_{L^p(B^R \cup [-R,R])} \leq C_\epsilon R^n(1/2 - \frac{1}{p}) \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset A(1). \]  

(2.11)

Indeed, by Corollary 2.4, we have

\[ \|e^{it\sqrt{-\Delta}}f\|_{L^p(R^n \times [1,2])} \leq C_\epsilon R^n(1/4 - 1/p) \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset A(1). \]  

(2.12)

After rescaling, we get

\[ \|e^{it\sqrt{-\Delta}}f\|_{L^p(R^n \times [1,2])} \leq C_\epsilon R^n(1/2 - 1/p) \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset A(R/2). \]  

(2.13)

Now we perform the standard Littlewood–Paley decomposition on \( f \). Let \( \varphi \) be a radial bump function supported on the ball \( |\xi| \leq 2 \) and equal to 1 on the ball \( |\xi| \leq 1 \). For \( N \in 2\mathbb{Z} \), we define the Littlewood–Paley projection operators by

\[ P_{\leq N} f(\xi) := \varphi(\xi/N) \hat{f}(\xi), \]

\[ P_{> N} f(\xi) := (1 - \varphi(\xi/N)) \hat{f}(\xi), \]

\[ P_N f(\xi) := (\varphi(\xi/N) - \varphi(2\xi/N)) \hat{f}(\xi). \]

Then we write,

\[ e^{it\sqrt{-\Delta}}f = e^{it\sqrt{-\Delta}}P_{\leq 1}f + \sum_{N>1} e^{it\sqrt{-\Delta}}P_N f. \]

By the fixed-time estimate (1.2) we have

\[ \|e^{it\sqrt{-\Delta}}P_{\leq 1}f\|_{L^p(R^n \times [1,2])} \lesssim \|P_{\leq 1}f\|_{L^p_{\text{w}B}(R^n)} \lesssim \|f\|_{L^p(R^n)}. \]  

(2.14)

By the triangle inequality and (2.13), Theorem 1.2 is proved.

Recall that \( K = K_0 R^\delta \ll R^\epsilon \), where \( K_0, \delta \) will be chosen to satisfy the requirement of the forthcoming argument. To prove

\[ \|e^{it\sqrt{-\Delta}}f\|_{L^p(B^R \cup [-R,R])} \leq C_\epsilon R^n(1/2 - 1/p) \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset A(1), \]  

(2.15)

it suffices to show

\[ \|e^{it\varphi_N(D)}f\|_{L^p(B^R \cup [-R,R])} \leq C_\epsilon R^n(1/2 - 1/p) \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset N_{\epsilon_0}(e_N), \]  

(2.16)

where \( \phi_R \) is in the class \( \Phi(R) \).

Indeed, since \( \text{supp } \hat{f} \subset A(1) \), we decompose \( A(1) \) into a collection of finitely-overlapping sectors \( \tau \) of radius \( K^{-3} \) in the angular direction. Write \( f = \sum_{\tau} f^\tau \) where \( f^\tau \) is Fourier supported in \( \tau \). Then

\[ \|e^{it\sqrt{-\Delta}}f\|_{L^p(B^R \cup [-R,R])} \leq \sum_{\tau} \|e^{it\sqrt{-\Delta}}f^\tau\|_{L^p(B^R \cup [-R,R])}. \]

Given \( \tau \), let \( A_\tau \) be an orthogonal matrix such that \( A_\tau e_n = \eta_\tau \). By changing of variables:

\[ \xi \mapsto A_\tau \xi, \]

we rotate the sector \( \tau \) to a sector centered at \( e_n \). Correspondingly, we make another change of variables with respect to \( x \), i.e.

\[ x \mapsto A_\tau x. \]

Then after sending

\[ x \mapsto x - t e_n, \]

correspondingly, the phase function becomes

\[ x \cdot \xi + t(|\xi| - \xi_n). \]
Finally we perform another change of variables with respect to $x, t, \xi$ as follows

$$\xi' \to K^{-3}\xi', x' \to K^3 x', t \to K^6 t.$$  

We claim that after the above change of variables, the resulting phase function is now in the class $\Phi(R)$. Indeed, $\phi_R(\xi)$ is given by

$$\phi_R(\xi) = K^6\left(\sqrt{K^{-6}|\xi'|^2 + \frac{\xi_n^2}{\xi_n^2}} - \xi_n\right).$$

By the homogeneity of the phase function, we have

$$\phi_R(\xi) = K^6\xi_n\left(1 + \left|\frac{K^{-3}\xi'}{\xi_n}\right|^2\right)^{\frac{1}{2}} - 1 = \frac{\xi_n^2 + \cdots + \xi_n^{2-1}}{2\xi_n} + K^{-4}E_R(\xi),$$

where

$$E_R(\xi) = -\frac{K^{-2}}{2}\frac{|\xi'|^4}{\xi_n} \int_0^1 (1-s)^\alpha \left(1 + sK^{-6}|\xi'|^2\right)^{-\frac{3}{2}} ds.$$

For fixed $N_{\text{par}} \in \mathbb{N}$, by choosing $K_0$ sufficiently large, we have

$$|\partial^\alpha E_R(\xi)| < c_{\text{par}}, \quad 0 \leq |\alpha| \leq N_{\text{par}}.$$

Thus, it suffices to estimate

$$e^{it\phi_R(D)}g := \int_{\mathbb{R}^n} e^{i(x' \xi' + x_n \xi_n + t\phi_R(\xi))} \hat{g}(\xi', \xi_n) \hat{g}(\xi', \xi_n) d\xi,$$

where

$$\hat{g}(\xi) = \hat{f}(A_t(K^{-3}\xi', \xi_n)).$$

A direct calculation shows that $\text{supp } \hat{g} \subset N_{\epsilon_0}(\epsilon_n)$. We have finished verifying the claim.

Combining the above estimates and (2.16), we have

$$\|e^{it\sqrt{-\Delta}}f\|_{L^p(B_R^R \times [-\bar{R}, R])} = K^{O(1)}\|e^{it\phi_R(D)}g\|_{L^p(B_R^R \times [-\bar{R}, R])} \leq R^{n(\frac{1}{2} - \frac{1}{p}) + \epsilon}\|f\|_{L^p(\mathbb{R}^n)}.$$

Let $Q_p(R)$ denote the smallest constant such that the following inequality holds for all phase functions $\phi_R$ in the class $\Phi(R)$,

$$\|e^{it\phi_R(D)}f\|_{L^p(B_R^R \times [-\bar{R}, R])} \leq Q_p(R)R^n(\frac{1}{2} - \frac{1}{p})\|f\|_{L^p}, \quad \text{supp } \hat{f} \subset N_{\epsilon_0}(\epsilon_n).$$

(2.17)

To prove (2.11), it suffices to show

$$Q_p(R) \lesssim R^\epsilon.$$

**Remark 2.5.** We want to emphasize that reducing to the class $\Phi(R)$ is only needed for proving the narrow decoupling estimate in Section 7. The statements in Section 3 through 5, including our $k$-broad “norm” estimates, hold true for all general phase functions $\phi$ in the class $\Phi$. Moreover, in the above reductions, we start with the standard circular cone given by the phase function $|\xi|$, while it can be easily seen that a similar argument works for any phase function satisfying conditions $H_1$ and $H_2$. Indeed, one can see from the following formula

$$\phi(\xi', \xi_n) - \nabla \phi(\epsilon_n) \cdot (\xi', \xi_n) = \frac{\langle \phi_{\xi'}(\epsilon_n), \xi_n \rangle}{2\xi_n} + \sum_{|\alpha| = 3} \frac{3}{\alpha!} \int_0^1 (1-s)^2 (\partial^\alpha \phi)(s\xi_n, 1) ds \frac{\xi_n^\alpha}{\xi_n}.$$

Thus our local smoothing bounds in Theorem 1.2 also hold true for such phase functions.
3. Wave packet decomposition

3.1. Construction of wave packet. In this section, we present the wave packet decomposition and collect some useful properties that we shall need from [22]. In this section and the next, we shall consider a phase function $\phi$ in the bigger class $\Phi$. Same arguments would work for any $\phi_R$ in the smaller class $\Phi(R)$.

Fix a large constant $R \gg 1$. We cover the annulus $A(1)$ using a collection of $1 \times R^{-1/2} \cdots \times R^{-1/2}$ sectors $\nu$ with finite overlaps. Let $\{\psi_\nu\}$ be a smooth partition of unity subordinate to this cover, and write $f = \sum_\nu f_\nu$, where $f_\nu := \psi_\nu f$.

Next, we further decompose $f_\nu$ on the physical side. Cover $\mathbb{R}^n$ by a collection of finitely overlapping balls $B_w := B^\nu_{R^{1+\delta}/2}(w)$ of radius $R\delta/2$ centered at $w \in R\delta/2 \mathbb{Z}^n$, where $\delta > 0$ is a fixed small constant. Let $\eta_\nu$ be a smooth partition of unity subordinate to this cover, write $f = \sum_\nu, w f_{\nu, w}$, where $f_{\nu, w} := (\eta_\nu \psi_\nu f)^w$. For given $\nu, w$, we further decompose the ball $B_w$ into $R^{(1+\delta)/2}$ plates $\{P^\nu_{\nu, w}\}$ of dimension 1 in the direction parallel to $\partial_\nu \phi(\xi_\nu)$ and $R^{1+\delta}$ in all the other directions, where $\xi_\nu \in S^{n-1}$ denotes the direction of the center-line of the sector $\nu$. Let $\eta^w_\nu$ be a smooth partition of unity subordinate to this cover. We write $f = \sum_\nu, w, \ell \eta^w_\nu \psi_\nu (\eta^w_\nu \psi_\nu f)^w$.

Finally, Let $\tilde{\psi}_\nu$ be a smooth function which is essentially support on $\nu$, and $\tilde{\psi}_\nu = 1$ on the $cR^{-1/2}$-neighborhood of the support of $\psi_\nu$, where $c > 0$ is a small constant. Define

$$f^\ell_{\nu, w} := \tilde{\psi}_\nu (\eta^w_\nu \psi_\nu f)^w,$$

then it is straightforward to check that

$$\|f^\ell_{\nu, w} - (\eta^w_\nu \psi_\nu f)^w\|_{L^\infty} \leq \text{RapDec}(R)\|f\|_{L^2}.$$

We may decompose $f$ as follows

$$f = \sum_{\nu, w, \ell} f^\ell_{\nu, w} + \text{RapDec}(R)\|f\|_{L^2}.$$

The functions $\{f^\ell_{\nu, w}\}$ are orthogonal in the sense that: for a set $\mathcal{T}$ of triplets $(\nu, w, \ell)$, we have

$$\left\|\sum_{(\nu, w, \ell) \in \mathcal{T}} f^\ell_{\nu, w}\right\|_{L^2}^2 \sim \sum_{(\nu, w, \ell) \in \mathcal{T}} \left\|f^\ell_{\nu, w}\right\|_{L^2}^2.$$

Now we define the associated tube $T^\ell_{\nu, w}$ by

$$T^\ell_{\nu, w} := \{(x, t) \in \mathbb{R}^{n+1}, |t| \leq R : |\Pi_\nu (x - w_t + t \partial_\nu \phi(\xi_\nu))| \leq CR^\delta, |\Pi_{\nu, \perp} (x - w_t + t \partial_\nu \phi(\xi_\nu))| \leq CR^{1+\delta}\},$$

(3.1)

where $\Pi_\nu, \Pi_{\nu, \perp}$ denote the orthogonal projection operator defined by $\Pi_\nu(\xi) := (\xi \cdot \xi_\nu) \xi_\nu, \Pi_{\nu, \perp}(\xi) := \xi - \Pi_\nu(\xi)$ and $w_t \in \mathbb{R}^n$ is the center of the plate $P^\nu_{\nu, w}$. Define

$$L(\nu) := \frac{1}{\sqrt{1 + |\nabla \phi(\xi_\nu)|^2}} (-\nabla \phi(\xi_\nu), 1).$$

From (3.1), one can see that $T^\ell_{\nu, w}$ intersects the hyperplane $t = 0$ at $P^\ell_{\nu, w}$ and satisfies

$$T^\ell_{\nu, w} \subset N_{CR^\delta}(P^\ell_{\nu, w} + tL(\nu)), |t| \leq CR.$$

We define the extension operator associated to the general cone $(\xi, \phi(\xi))$ by

$$Ef(x, t) := \int_{A(1)} e^{i(x - \xi + t\phi(\xi))} f(\xi) d\xi.$$
The following lemma shows that each wave packet $Ef_{\nu,w}^\ell$ is essentially localized to the tube $T_{\nu,w}^\ell$ in physical space.

**Lemma 3.1.** If $(x,t) \in B_R^{n+1}(0) \setminus T_{\nu,w}^\ell$, then
\[
|Ef_{\nu,w}^\ell(x,t)| \leq \text{RapDec}(R)\|f\|_{L^2}.
\]

The proof of Lemma 3.1 is standard, and for instance, can be obtained by slightly modifying the proof of Lemma 2.1 in [22].

### 3.2. Comparing wave packet at different scales

Suppose $B_R^{n+1}(y) \subset B_R^{n+1}(0)$ for some radius $R^{1/2} < \rho < R$. We need to decompose $f$ into wave packets over the ball $B_\rho^{n+1}(y)$ at this smaller spatial scale $\rho$.

We apply a transformation $z = y + \tilde{z}$ to recenter $B_\rho^{n+1}(y)$, here $z := (x,t)$, $\tilde{z} := (\tilde{x}, \tilde{t})$. Define
\[
\phi_y(\xi) := y^* \cdot \xi + y_{n+1}\phi(\xi), \quad y = (y', y_{n+1})
\]
then,
\[
\tilde{f}(z) = E\tilde{f}(\tilde{z}),
\]
where $\tilde{f}(y) = e^{i\phi_y(y)}f(y)$. We now perform wave packet decomposition with respect to $\tilde{f}$ at scale $\rho$. Following the construction in the last section, we write
\[
\tilde{f} = \sum_{\nu,w,\ell} \tilde{f}_{\nu,w,\ell}^\ell,
\]
where $\nu \subset A(1)$ is a sector of width about $\rho^{-1/2}$ in the angular direction and length about 1 in the radial direction, $\tilde{w} \in \rho^{1/2+\delta}Z^n$. The Fourier support of $\tilde{f}_{\nu,w,\ell}$ is essentially contained in a thin plate $P_{\nu,w,\ell}^\ell$ of side length $\rho^{1/2+\delta}$ and thickness $\rho^\delta$ in the ball of radius $\rho^{1/2+\delta}$ centered at $\tilde{w}$. $Ef_{\nu,w}^\ell$ is essentially supported in the tube $T_{\nu,w}^\ell$ with
\[
P_{\nu,w,\ell}^\ell + R\frac{C}{\rho}L(\nu) \subset \tilde{T}_{\nu,w,\ell}^\ell \subset P_{\nu,w,\ell}^\ell + CR\ell^\delta, \quad C > 0 \text{ sufficiently large}.
\]

A natural question then appears: how this new wave packet decomposition at a smaller scale $\rho$, $\tilde{f} = \sum_{\nu,w,\ell} \tilde{f}_{\nu,w,\ell}^\ell$, relates to the original wave packet decomposition $f = \sum_{\nu,w,\ell} f_{\nu,w}^\ell$ at scale $R$? To be more precise, for a given $(\nu, w, \ell)$, which $(\tilde{\nu}, \tilde{w}, \tilde{\ell})$ contributes significantly to the wave packet $f_{\nu,w}^\ell$? To answer this question, we first define
\[
\tilde{T}_{\nu,w,\ell}^\ell := \{ (\tilde{\nu}, \tilde{w}, \tilde{\ell}) : \text{Ang}(\nu, \tilde{\nu}) \lesssim \rho^{-1/2}, \text{Dist}(P_{\nu,w,\ell}^\ell, P_{\nu,w,\ell} + P_{\nu} - \partial_y \phi_y(\xi_0)) \lesssim R^\delta \},
\]
where $P_{\nu}$ is given by
\[
P_{\nu} := \{ x \in \mathbb{R}^n : |\Pi_{\nu}(x)| \leq C\rho^\delta, |\Pi_{\nu^\perp}(x)| \leq C\rho^{1/2+\delta} \}.
\]
The following lemma shows the relationship between wave packet decomposition at different scales.

**Lemma 3.2.** $(f_{\nu,w}^\ell)^\ell$ is essentially made of small wave packets from $\tilde{T}_{\nu,w,\ell}^\ell$. In other words,
\[
(f_{\nu,w}^\ell)^\ell = \sum_{(\tilde{\nu}, \tilde{w}, \tilde{\ell}) \in \tilde{T}_{\nu,w,\ell}^\ell} \left( (f_{\nu,w}^\ell)^\ell \right)_{\tilde{\nu}, \tilde{w}, \tilde{\ell}} + \text{RapDec}(R)\|f\|_{L^2}.
\]

Next, we explore a geometric features of a tube $T_{\nu,w,\ell}^\ell$ with $(\tilde{\nu}, \tilde{w}, \tilde{\ell}) \in \tilde{T}_{\nu,w,\ell}^\ell$.

**Lemma 3.3.** For any $(\tilde{\nu}, \tilde{w}, \tilde{\ell}) \in \tilde{T}_{\nu,w,\ell}^\ell$, there holds
\[
\text{Ang}(\nu, \tilde{\nu}) \lesssim \rho^{-1/2},
\]
and
\[
\text{Dist}([T_{\nu,w} \cap B_\rho^{n+1}(y)] + 2P_{\nu}, T_{\nu,w,\ell}^\ell) \lesssim R^\delta.
\]
The proof of Lemma 3.2 and Lemma 3.3 are similar to that of Lemma 5.3 and Lemma 5.4 in [22]. We omit the proof here.

Next, we will group large and small wave packets into different sub-collections. Let \( \tilde{\nu}_0 \) be a sector in \( A(1) \) of dimensions \( \rho^{-1/2} \) in the angular direction and \( \sim 1 \) in the radial direction, and \( w_0 \in \mathbb{R}^{2+2}_+ \cap B(0, \rho) \). We define the set \( \mathcal{T}_{\tilde{\nu}_0, w_0} \) and \( \tilde{T}_{\tilde{\nu}_0, w_0} \) respectively as follows:

\[
\tilde{T}_{\tilde{\nu}_0, w_0} := \{(\tilde{\nu}, \tilde{\mu}, \ell) : \text{Ang}(\tilde{\nu}, \tilde{\mu}) \lesssim \rho^{-1/2}, \quad F_{\tilde{\nu}, \tilde{\mu}} \subset B(w_0, R^{1/2+2d})\},
\]

and

\[
\mathcal{T}_{\tilde{\nu}_0, w_0} := \{(\nu, w, \ell) : \text{Ang}(\nu, \tilde{\nu}_0) \lesssim \rho^{-1/2}, \quad T_{\nu, w} \cap B_\rho^{n+1}(y) \subset B(w_0, R^{1/2+2d}) + \rho L(\tilde{\nu}_0) + \{y\}\}.
\]

For any given \( \tilde{\nu}_0, w_0 \) and function \( g \), we define \( \tilde{g}_{\tilde{\nu}_0, w_0} \) and \( g_{\tilde{\nu}_0, w_0} \) respectively as follows:

\[
\tilde{g}_{\tilde{\nu}_0, w_0} := \sum_{(\tilde{\nu}, \tilde{\mu}, \ell) \in \mathcal{T}_{\tilde{\nu}_0, w_0}} \tilde{g}_{\tilde{\nu}, \tilde{\mu}, \ell}, \quad g_{\tilde{\nu}_0, w_0} := \sum_{(\nu, w, \ell) \in \mathcal{T}_{\tilde{\nu}_0, w_0}} g_{\nu, w, \ell}.
\]

Correspondingly, we have the wave packets decomposition for \( g \) and \( \tilde{g} \) in the sense that

\[
g = \sum_{(\tilde{\nu}_0, w_0)} \tilde{g}_{\tilde{\nu}_0, w_0} + \text{RapDec}(R)\|f\|_{L^2}, \quad \tilde{g} := \sum_{(\tilde{\nu}_0, w_0)} \tilde{g}_{\tilde{\nu}_0, w_0} + \text{RapDec}(R)\|f\|_{L^2}.
\]

Furthermore, we have the following \( L^2 \)-orthogonality property:

\[
\|g\|^2_{L^2} \sim \sum_{(\tilde{\nu}_0, w_0)} \|\tilde{g}_{\tilde{\nu}_0, w_0}\|^2_{L^2}, \quad \|\tilde{g}\|^2_{L^2} \sim \sum_{(\tilde{\nu}_0, w_0)} \|\tilde{g}_{\tilde{\nu}_0, w_0}\|^2_{L^2}.
\]

With the definition above, for any \( (\tilde{\nu}_0, w_0) \), these two collections \( \mathcal{T}_{\tilde{\nu}_0, w_0}, \tilde{T}_{\tilde{\nu}_0, w_0} \) are related in the sense that

\[
\tilde{T}_{\tilde{\nu}_0, w_0} = \bigcup_{(\nu, w, \ell) \in \mathcal{T}_{\tilde{\nu}_0, w_0}} \tilde{T}_{\nu, w, \ell}.
\]

Finally, we have

**Lemma 3.4.** If \( g \) is concentrated on large wave packets in \( \mathcal{T}_{\tilde{\nu}_0, w_0} \), then \( \tilde{g} \) is concentrated on small wave packets in \( \tilde{T}_{\tilde{\nu}_0, w_0} \). On the other hand, if \( \tilde{g} \) is concentrated on small wave packets in \( \tilde{T}_{\tilde{\nu}_0, w_0} \), then \( g \) is concentrated on large wave packets on \( \mathcal{T}_{\tilde{\nu}_0, w_0} \).

### 4. A Geometric lemma

In this section, we establish a geometric lemma associated with the cone given by \( (\xi, \phi(\xi)) \), for any given phase functions \( \phi \) in the class \( \Phi \). First, let us give a slightly different version of Lemma 5.8 in [22] for the model case \( \phi(\xi) = |\xi| \), which will shed light on the case of general cones.

#### 4.1 A geometric lemma for the circular cone \( (\xi, |\xi|) \)

For convenience, we will use \( \mathcal{C} \) to denote the truncated cone, i.e., \( \mathcal{C} := \{(\xi, |\xi|) : 1/2 \leq |\xi| \leq 1\} \). Let \( V \subset \mathbb{R}^{n+1} \) be an \( m \) dimensional affine subspace given by

\[
V := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} a_{i,j} x_j = b_i, \quad i = 1, \ldots, n + 1 - m\}.
\]

For a given point \( (\xi, |\xi|) \in \mathcal{C} \), the unit normal vector of \( \mathcal{C} \) at \( (\xi, |\xi|) \) is \( n_\xi = \frac{1}{\sqrt{2}}(-\xi, 1) \). Therefore, all the points on the cone \( \mathcal{C} \) of which the normal vectors are parallel to \( V \) lie in an affine subspace \( \mathcal{V} \) defined by

\[
\mathcal{V} := \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} a_{i,j} x_j - a_{i,n+1} x_{n+1} = 0, \quad i = 1, \ldots, n + 1 - m\}.
\]
Assume that the unit normal $n_\xi$ is parallel to $V$, i.e.
\[
\sum_{j=1}^{n} a_{i,j} \frac{\xi_j}{|\xi|} - a_{i,n+1} = 0, \quad i = 1, \ldots, n+1-m,
\]
which implies that
\[
\text{rank} \left( \begin{array}{ccc} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n+1-m,1} & \cdots & a_{n+1-m,n} \end{array} \right) = n+1-m. \quad (4.1)
\]
We denote by $V^- \subset \mathbb{R}^n$ a subspace defined as
\[
V^- := \{ (x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{j=1}^{n} a_{i,j} x_j = 0, \quad i = 1, \ldots, n+1-m \}.
\]
From (4.1), we see that $V^-$ is an $(m-1)$-dimensional subspace.

For any $m$-dimensional linear subspace $\bar{V}$, if $\bar{V} \cap C \neq \emptyset$, then $\bar{V}$ intersects the light cone either tangentially or transversally, which is demonstrated in Figure 2. Based on the above observation, we start with two geometric lemmas concerning the circular cone, which corresponds to Lemma 5.8 in [22].

**Lemma 4.1.** Assume $\eta \in S^{n-1}$. If $(\eta, 1) \in \bar{V}$ and $\text{Ang}(\eta, V^-) > \frac{\pi}{2} - K^{-2}$, then $\bar{V} \cap C \subset \{ t(\xi, 1) : t \in \mathbb{R}, \xi \in S^{n-1}, \text{Ang}(\xi, \eta) \lesssim K^{-2} \}$.

**Proof.** Let $\alpha_i = (a_{i,1}, \cdots, a_{i,n})$. Since $\text{Ang}(\eta, V^-) > \frac{\pi}{2} - K^{-2}$, there exists a vector $\bar{\eta} \in S^{n-1}$ such that
\[
\bar{\eta} = (\bar{\eta}_1, \cdots, \bar{\eta}_n) = \sum_{i=1}^{n+1-m} \lambda_i \alpha_i, \quad \lambda_i \in \mathbb{R}, \quad \text{Ang}(\eta, \bar{\eta}) \leq K^{-2}. \quad (4.2)
\]
Note that $(\eta, 1) \in \bar{V}$, we have
\[
\sum_{j=1}^{n} \left( \sum_{i=1}^{n+1-m} \lambda_i a_{i,j} \right) \eta_j - \left( \sum_{i=1}^{n+1-m} \lambda_i a_{i,n+1} \right) = 0.
\]
By (4.2), we obtain
\[
\sum_{j=1}^{n} \bar{\eta}_j \cdot \eta_j - \left( \sum_{i=1}^{n+1-m} \lambda_i a_{i,n+1} \right) = 0,
\]
Note that $\text{Ang}(\eta, \bar{\eta}) \leq K^{-2}$, thus $\bar{\eta} \cdot \eta > 1 - K^{-4}$. It follows that
\[
\bar{\eta}_{n+1} := \sum_{i=1}^{n+1-m} \lambda_i a_{i,n+1} > 1 - K^{-4}.
\]
For all $(\xi, |\xi|) \in \mathcal{C} \cap \bar{V}$, since $(\bar{\eta}, -\bar{\eta}_{n+1}) \in (\bar{V})^\perp$, we have
\[
\xi \cdot \bar{\eta} - |\xi|\bar{\eta}_{n+1} = 0.
\]
Note that $|\eta| = 1$ and $\bar{\eta}_{n+1} > 1 - K^{-4}$, we obtain $\text{Ang}(\xi, \bar{\eta}) \lesssim K^{-2}$.

Decompose $\mathbb{R}^{n+1} = \bar{V} \oplus W$, that is, $W$ is the orthogonal complement subspace of $\bar{V}$ in $\mathbb{R}^{n+1}$.

**Lemma 4.2.** If for each $(\eta, |\eta|) \in \mathcal{C} \cap \bar{V}$, $\text{Ang}(\eta, V^-) \leq \frac{\pi}{2} - K^{-2}$, then $W$ and $V$ are transversal in the sense that $\text{Ang}(V, W) \gtrsim K^{-4}$.

**Proof.** Let $\{\beta_1, \ldots, \beta_{m-1}\}$ be an orthogonal basis for $V^-$, we may choose another unit vector $\beta_m$ such that $\{\beta_1, \ldots, \beta_{m-1}, \beta_m\}$ forms a unit orthogonal basis for $V$. Assume $\eta \in S^{n-1}$, since $\beta_m := \frac{1}{\sqrt{2}}(\eta, 1) \in \bar{V}$, it is easy to verify that $\{\beta_1, \ldots, \beta_{m-1}, \beta_m\}$ forms a basis for the subspace $\bar{V}$. To prove $\text{Ang}(V, W) \gtrsim K^{-4}$, it suffices to show for any unit vector $v$ parallel to $V$, $|\text{Proj}_V v| \gtrsim K^{-4}$. To achieve this, we subdivide the vectors parallel $V$ into two categories.

**Case I:** $|\text{Proj}_V v| > \frac{1}{2}$. Since that $V^- \subset \bar{V}$, we have $|\text{Proj}_V v| > \frac{1}{2}$

**Case II:** $|\text{Proj}_V v| \leq \frac{1}{2}$. In this case, we have $|\text{Proj}_{\beta_m} v| > \frac{1}{2}$. To prove $|\text{Proj}_V v| \gtrsim K^{-4}$, it suffices to show $|\text{Proj}_{\beta_m} \beta_m| \gtrsim K^{-4}$, since $|\text{Proj}_V v| \gtrsim |\text{Proj}_{\beta_m} \beta_m| \gtrsim K^{-4}$.

By our construction $\beta_m \perp V^-$, there exists $(\lambda_1, \ldots, \lambda_{n+1-m}) \in \mathbb{R}^{n+1-m}$ such that
\[
(\sum_{i=1}^{n+1-m} \lambda_i a_{i,1}, \ldots, \sum_{i=1}^{n+1-m} \lambda_i a_{i,n}, c_{n+1}) = \beta_m.
\]

For convenience, denote $c_j = \sum_{i=1}^{n+1-m} \lambda_i a_{i,j}$ with $\sum_{j=1}^{n+1} c_j^2 = 1$.

Take $\tilde{c}_{n+1} \in \mathbb{R}$ such that $(c_1, \ldots, \tilde{c}_{n+1}) \in V^\perp$. Since $\left(-\frac{\xi}{|\xi|}, 1\right) \parallel V$, it follows that
\[
c_1 \frac{\xi_1}{|\xi|} + \cdots + c_n \frac{\xi_n}{|\xi|} - \tilde{c}_{n+1} = 0.
\]

(4.3)

By Cauchy-Schwarz inequality, we have
\[
c_{n+1}^2 \leq \sum_{j=1}^{n} c_j^2.
\]

(4.4)

Since $(c_1, \ldots, c_{n+1}) \in V$, $(c_1, \ldots, \tilde{c}_{n+1}) \in V^\perp$, we have
\[
\sum_{j=1}^{n} c_j^2 + c_{n+1} \tilde{c}_{n+1} = 0.
\]

Together with (4.4), which implies that
\[
\sum_{j=1}^{n} c_j^2 \leq c_{n+1}^2.
\]

(4.5)
Therefore, we have
\[ c_{n+1} \geq \frac{\sqrt{2}}{2}, \quad \left(\sum_{j=1}^{n} c_j^2\right)^{1/2} \leq \frac{\sqrt{2}}{2}. \]
The magnitude of the projection of \( \beta_m \) onto \( \hat{\beta}_m \) equals
\[ \frac{1}{\sqrt{2}} |c_1 \eta_1 + \cdots + c_n \eta_n + c_{n+1}|. \]
Recall that \( \text{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2} \), it follows that
\[ |c_1 \eta_1 + \cdots + c_n \eta_n| < \frac{\sqrt{2}}{2} (1 - CK^{-4}). \]
Since \( c_{n+1} \geq \frac{\sqrt{2}}{2} \), we obtain
\[ \frac{1}{2} |c_1 \eta_1 + \cdots + c_n \eta_n + c_{n+1}| > CK^{-4}. \]
\[ \square \]
We summarize the above discussion as follows:

**Lemma 4.3.** Decompose \( \mathbb{R}^{n+1} = \hat{V} \bigoplus W \) such that \( \hat{V} \perp W \). We have the following dichotomy.

- If for each \((\eta, |\eta|) \in C \cap \hat{V}, \text{Ang}(\eta, V^-) \leq \frac{\pi}{2} - K^{-2} \), then \( W \) and \( V \) are transversal in the sense that \( \text{Ang}(V, W) \gtrsim K^{-4} \);
- If there exists \((\eta, |\eta|) \in C \cap \hat{V} \) such that \( \text{Ang}(\eta, V^-) > \frac{\pi}{2} - K^{-2} \), then the projection of \( C \cap \hat{V} \) onto \( \mathbb{R}^n \) is contained in a slab of dimensions \( \sim 1 \times K^{-2} \times \cdots \times K^{-2} \).

### 4.2. Generalization of Lemma 4.3 to a general class of cones.

Now, we generalize the above lemma to the class of cones which are in the class \( \Phi \) as defined in Section 2.

We define a set \( L \) by
\[ L := \{ \xi \in A(1) : \sum_{j=1}^{n} a_{i,j} \partial_{\xi_j} \phi(\xi) - a_{i,n+1} = 0; \ i = 1, \ldots, n + 1 - m \}. \]
If we choose \( \phi(\xi) = |\xi| \), then \( \{ (\xi, |\xi|) : \xi \in L \} \) lies in the subspace \( \hat{V} \). Therefore, if \( L \) is not empty, the dimension of \( L \) depends on whether \( \hat{V} \) intersects the light cone \( (\xi, |\xi|) \) tangentially or transversally. To be more precise, if \( L \) intersects the light cone tangentially, then \( \text{dim} L = 1 \), otherwise \( \text{dim} L = m - 1 \). In this section, we shall prove that this fact can be generalized to general cones satisfying the homogeneous convex conditions.

**Lemma 4.4.** If \( L \) is not empty, then \( \text{dim} L = 1 \) or \( \text{dim} L = m - 1 \).

**Proof.** Denote \( a_i := (a_{i,1}, \ldots, a_{i,n}) \) and \( F = (F_1, \ldots, F_{n+1-m}) \) where
\[ F_1 := \sum_{j=1}^{n} a_{i,j} \partial_{\xi_j} \phi(\xi) - a_{i,n+1}. \]
For convenience, we will use \( \text{Hess}(\phi) \) to denote the Hessian matrix of \( \phi \), i.e.
\[ \text{Hess}(\phi) = \left( \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} \right)_{n \times n}. \]
By the homogeneous convex conditions, we see that the 0-eigenspace of \( \text{Hess}(\phi) \) at \( \eta \) is spanned by the vector \( \eta \), i.e.
\[ \text{Hess}(\phi)|_{\xi=\eta} \eta = 0 \eta = 0. \] (4.6)
If \( \eta \in L \) and \( \eta \not\in \text{span} \{ a_1, \ldots, a_{n+1-m} \} \), then
\[ \text{rank} \left( \text{Hess}(\phi)|_{\xi=\eta} \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n+1-m,1} & \cdots & a_{n+1-m,n} \end{pmatrix} \right) = n + 1 - m. \]
By the implicit function theorem, we have

$$\dim(L) = m - 1.$$ 

If $\eta \in L$ and $\eta \in \text{span}\{\alpha_1, \cdots, \alpha_{n+1-m}\}$, by the homogeneity of $\phi$, we may assume that $\eta \in S^{n-1}$. If there is another vector $\bar{\eta} \in S^{n-1} \cap L$ not parallel to $\eta$, a simple calculation using the definition of $L$ gives that

$$\left(\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})\right) \cdot \eta = 0. \tag{4.7}$$

Using Taylor’s expansion formula, we obtain

$$\left(\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})\right) \cdot \eta = \langle \eta - \bar{\eta}, \text{Hess}(\phi)\rangle_{\xi=\eta} + \frac{1}{2} \partial_{\xi \xi}^2(\partial_\xi \phi, \eta)\bigg|_{\xi=\eta} \langle \eta - \eta, \eta \rangle^2 + O(\bar{\eta} - \eta)^3. \tag{4.8}$$

By (4.6), we have

$$\langle \eta - \eta, \text{Hess}(\phi)\rangle_{\xi=\eta} = 0.$$

By the homogeneity of $\phi$, it follows

$$\partial_{\xi \xi}^2(\partial_\xi \phi, \eta)\bigg|_{\xi=\eta} = -\left(\partial_{\xi \xi}^2 \phi(\eta)\right)_{\xi=\eta},$$

since $\bar{\eta}$ is not parallel to $\eta$, using the homogeneity convex conditions of $\phi$, it follows that

$$\left|\frac{1}{2} \partial_{\xi \xi}^2(\partial_\xi \phi, \eta)\right|_{\xi=\eta} \langle \eta - \eta, \eta \rangle^2 \sim |\eta - \eta|^2,$$

which contradicts (4.7). Therefore, if $\eta \in L$ and $\eta \in \text{span}\{\alpha_1, \cdots, \alpha_{n+1-m}\}$, then

$$L \subset \{t\eta : t \in \mathbb{R}\}.$$

Let $V, V^-$ be as defined in Section 4.1. Next, we will generalize Lemma 4.1 and 4.2 to general cones.

**Case I: Tangential case.**

**Lemma 4.5.** Let $\eta \in S^{n-1}$. If $\eta \in L$ and $\text{Ang}(\eta, V^-) > \frac{n}{2} - K^{-2}$, then $L$ is contained in the set

$$\{\xi \in \mathbb{R}^n : \text{Ang}(\xi, \eta) \lesssim K^{-2}\}.$$

**Proof.** Let $\eta \in L \cap S^{n-1}$ with $\text{Ang}(\eta, V^-) > \frac{n}{2} - K^{-2}$. Consider another unit vector $\bar{\eta} \in L \cap S^{n-1}$, we need to show that

$$\text{Ang}(\eta, \bar{\eta}) \lesssim K^{-2}.$$

By the definition of $L$, we have

$$\sum_{j=1}^n a_{i,j}(\partial_{\xi_j} \phi(\eta) - \partial_{\xi_j} \phi(\bar{\eta})) = 0, \quad i = 1, \cdots, n + 1 - m.$$ 

Since $\text{Ang}(\eta, V^-) > \frac{n}{2} - K^{-2}$ and $(\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})) \in V^-$, it follows that

$$\left|(\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})) \cdot \eta\right| \lesssim |\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})| K^{-2}. \tag{4.9}$$

As in (4.8),

$$\left|\frac{\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})}{\eta} \right| = \left(\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta}), \text{Hess}(\phi)\right)_{\xi=\eta} + \frac{1}{2} \partial_{\xi \xi}^2(\partial_\xi \phi, \eta)\bigg|_{\xi=\eta} \langle \eta - \eta, \eta \rangle^2 + O(\bar{\eta} - \eta)^3. \tag{4.10}$$

Again by using the homogeneous convex conditions we have

$$|\eta - \bar{\eta}| \lesssim |\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})|^2 K^{-1},$$

while

$$|\partial_\xi \phi(\eta) - \partial_\xi \phi(\bar{\eta})| \sim |\eta - \bar{\eta}|,$$

therefore,

$$|\eta - \bar{\eta}| \lesssim K^{-2}.$$
Case II: Transversal case. In general, \( \{ (\xi, \phi(\xi)) : \xi \in L \} \) may not lie in an affine subspace, actually \( L \) can be a curved submanifold. To generalize Lemma 4.2, we should construct the associated affine subspace \( \tilde{V} \) in our setting. The main idea is that we will approximate \( L \) by the tangent space of a given point, which lies in a slab of sufficiently small scale in the angular direction. Now, let us establish a geometric lemma associated to a fixed point.

Let \( \eta \in L \cap S^{n-1} \), with \( \text{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2} \). Define \( \tilde{V} \) to be the \((n+1-m)\)-dimensional linear subspace spanned by the vectors \( \gamma_1, \cdots, \gamma_{n+1-m} \) given by\[ \gamma_i := \text{Hess}(\phi)|_{\xi=\eta} \alpha_i, \quad \alpha_i = (a_i, \cdots, a_i, n), \quad i = 1, \cdots, n+1-m, \]
The assumption \( \text{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2} \) ensures that \( \gamma_i, i = 1, \cdots, n+1-m \) are linearly independent. Let \( \tilde{V}^- \) be the orthogonal complement of \( \tilde{V} \) in \( \mathbb{R}^n \), i.e.
\[ \mathbb{R}^n = \tilde{V} \oplus \tilde{V}^- . \]
Let \( \bar{V} \) be the linear subspace spanned by \( \tilde{V}^- \) and \( e_{n+1} \). Define \( W \) to be the orthogonal complement space of \( \tilde{V} \) in \( \mathbb{R}^{n+1} \), i.e.
\[ \mathbb{R}^{n+1} = \tilde{V} \oplus W . \]
We remark that, unlike the circular cone case, all linear spaces \( \bar{V}, \tilde{V}, \tilde{V}^-, W \) defined above depend on the choice of \( \eta \). In fact, we define \( \tilde{V}^- \) in such a way that it represents a certain linearization of \( L \) at the point \( \eta \).

Lemma 4.6. Let \( \eta \in S^{n-1} \cap L \). If \( \text{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2} \), then \( W \) and \( V \) are transversal in the sense that \( \text{Ang}(V, W) \geq K^{-4} \).

Proof. Let \( \{ \beta_1, \cdots, \beta_{m-1} \} \) be an orthonormal basis for \( V^- \), we may choose another unit vector \( \beta_m \) such that \( \{ \beta_1, \cdots, \beta_{m-1}, \beta_m \} \) forms an orthonormal basis for \( V \). To prove
\[ \text{Ang}(V, W) \geq K^{-4} , \]
it suffices to show for any unit vector \( v \) parallel to \( V \),
\[ |\text{Proj}_V v| \geq K^{-4} . \tag{4.11} \]
To prove (4.11), we subdivide the set vectors of \( v \) which are parallel \( V \) into two categories.

Case Ia: \( |\text{Proj}_{\beta_m} v| > K^{-4} \).

Assume that \( v = \sum_{i=1}^{m-1} a_i \beta_i + a_m \beta_m \), then we have \( |a_m| > K^{-4} \). Therefore
\[ |\text{Proj}_V v| \geq |\text{Proj}_{e_{n+1}} v| = |a_m| |\text{Proj}_{e_{n+1}} \beta_m| \geq K^{-4} , \]
where we used the fact that\[ |\text{Proj}_{e_{n+1}} \beta_m| \geq 1 . \]

Case Ib: \( |\text{Proj}_{\beta_m} v| \leq K^{-4} \).

In this case
\[ |\text{Proj}_{V^+ v}| \geq 1/2 . \]
Then (4.11) will follow from the following claim:

Claim: for each unit vector \( v_1 \in V^- \), \( v_2 \in V^+ \),
\[ \text{Ang}(v_1, v_2) \leq \frac{\pi}{2} - CK^{-4} . \tag{4.12} \]

Since \( \alpha_i, i = 1, \cdots, n+1-m \) form a basis of a subspace which is orthogonal to \( V^- \) in \( \mathbb{R}^n \), and \( \text{Hess}(\phi)|_{\xi=\eta} \alpha_i, i = 1, \cdots, n+1-m \) form a basis of \( \tilde{V} \) which is orthogonal to \( \tilde{V}^- \). Thus it suffices to show
\[ \langle \alpha_i, \text{Hess}(\phi)|_{\xi=\eta} \alpha_i \rangle \geq K^{-4} . \tag{4.13} \]
Note that
\[ \text{Ang}(\eta, V^-) < \frac{\pi}{2} - K^{-2} , \]
which implies that
\[ \text{Ang}(\eta, \alpha_i) > K^{-2}, \quad i = 1, \cdots, n+1-m . \tag{4.14} \]
Let \( \alpha_i = a_1 \eta + a_2 \tilde{\eta} \), where \( \tilde{\eta} \) lies in the subspace \( \eta^\perp \), which is the orthogonal complement of \( \{ \eta, t \in \mathbb{R} \} \) in \( \mathbb{R}^n \). (4.14) implies \( a_2 > cK^{-2} \). Since \( \eta \in \ker \text{Hess}(\phi) \big|_{\xi = \eta} \), we have

\[
\left\langle \alpha_i, \text{Hess}(\phi) \big|_{\xi = \eta} \alpha_i \right\rangle = (a_2)^2 \left\langle \tilde{\eta}, \text{Hess}(\phi) \big|_{\xi = \eta} \tilde{\eta} \right\rangle \gtrsim K^{-4}. \tag{4.15}
\]

In the last inequality, we have used the fact that \( \text{Hess}(\phi) \big|_{\xi = \eta} \) is nondegenerate when restricted to the subspace \( \eta^\perp \).

We summarize the above discussion below.

**Lemma 4.7.** Let \( L, \tilde{V}, V^- \) and \( W \) are defined as above and \( \eta \in L \). We have the following dichotomy:

a) If \( \text{Ang}(\eta, V^-) \leq \frac{\pi}{2} - K^{-2} \), then \( W \) and \( V \) are transversal in the sense that \( \text{Ang}(V, W) \gtrsim K^{-4} \);

b) If \( \text{Ang}(\eta, V^-) > \frac{\pi}{2} - K^{-2} \), then \( L \) is contained in a slab of dimensions \( \sim 1 \times K^{-2} \times \cdots \times K^{-2} \).

5. \( k \)-broad “norm” estimate

In this section, we prove \( k \)-broad “norm” estimates associated to a general phase function \( \phi \) in the class \( \Phi \). As discussed earlier, the same estimates should hold for any \( \phi_R \in \Phi(R) \). Recall that we have \( 1 \ll K \ll R^\ell \). We partition \( A(1) \), in the angular direction, into a collection of \( (k-1) \)-dimensional linear subspaces \( \tau \) and \( \nu \) of dimensions \( 1 \times R^{-1} \times \cdots \times R^{-1} \) and \( 1 \times K^{-1} \times \cdots \times K^{-1} \) respectively. In this part, we write \( f_\nu := f_{\chi_\nu} \) and choose \( V \subset \mathbb{R}^{n+1} \) to be a \((k-1)\)-dimensional linear subspace. The set \( G(\nu) \) consisting of the unit normal vectors of the cone associated with the slab \( \nu \) is defined by

\[
G(\nu) := \left\{ \frac{1}{\sqrt{1 + |\nabla \phi|^2}} (- \nabla \phi(\xi), 1) : \xi \in \nu \right\}.
\]

Similarly, we define

\[
G(\tau) := \bigcup_{\nu \subset \tau} G(\nu).
\]

We denote by \( \text{Ang}(G(\nu), V) \) the smallest angle between non-zero vectors \( v \in V \) and \( v' \in G(\nu) \).

For each \( B_{K^2}^{n+1} \subset C_R^{n+1} \), we define \( \mu_{EF}(B_{K^2}^{n+1}) \) by

\[
\mu_{EF}(B_{K^2}^{n+1}) := \min_{V_1, \ldots, V_n} \max_{\text{Ang}(G(\nu), V_2) \gg K^{-2}} \left( \int_{B_{K^2}^{n+1}} |E_{f_\nu}|^p \, dx \, dt \right).
\]

Let \( \{ B_{K^2}^{n+1} \} \) be a collection of finitely overlapping balls which forms a cover of \( C_R^{n+1} \). Then we define the \( k \)-broad “norm” as

\[
\| Ef \|^p_{BL^p_{k,A}(C_R^{n+1})} := \sum_{B_{K^2}^{n+1} \subset C_R^{n+1}} \mu_{EF}(B_{K^2}^{n+1}).
\]

Next, we will record some useful properties of the broad “norm” of which the proof can be found in [10].

**Lemma 5.1** (Triangle inequality). Suppose that \( 1 \leq p < \infty \), \( f = g + h \) and \( A = A_1 + A_2 \), where \( A, A_1, A_2 \) are nonnegative integers. Then

\[
\| Ef \|_{BL^p_{k,A}(U)} \lesssim \| Eg \|_{BL^p_{k,A_1}(U)} + \| Eh \|_{BL^p_{k,A_2}(U)}. \tag{5.1}
\]

**Lemma 5.2** (Hölder’s inequality). Suppose that \( 1 \leq p, p_1, p_2 < \infty \), \( 0 \leq \alpha_1, \alpha_2 \leq 1 \) satisfy \( \alpha_1 + \alpha_2 = 1 \) and

\[
\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}.
\]

Suppose that \( A = A_1 + A_2 \), then

\[
\| Ef \|_{BL^p_{k,A}(U)} \leq \| Ef \|_{BL^p_{k,A_1}(U)}^{\alpha_1} \| Ef \|_{BL^p_{k,A_2}(U)}^{\alpha_2}. \tag{5.2}
\]
In the argument we will need to choose $A$ sufficiently large to ensure that the above inequalities may be applied quite a few (but finitely many) times. At the end of the argument, the reader shall see that the relation between the parameters $K, A, R$ can be described by the following inequalities:

$$1 \ll A \lesssim K^\epsilon \lesssim R^2.$$ 

In this part, we aim to prove that the following broad-"norm" estimate.

**Theorem 5.3.** For any $2 \leq k \leq n + 1$ and $\epsilon > 0$, there exists a large constant $A$ such that

$$\|E f\|_{BL^\epsilon_{k, A}(C^n_{R^2})} \lesssim_{\epsilon, \phi} R^2 \|f\|_{L^2(A(1))}, \quad \text{supp } f \subset A(1),$$

(5.3)

for $p \geq \frac{2n+k+1}{n+k-1}$, where the implicit constant depends on the derivatives of $\phi$ up to finite orders and the eigenvalues of the hessian matrix of $\phi$.

As a direct consequence of Theorem 5.3 and Lemma 2.3, we have the following $L^p$ estimate.

**Corollary 5.4.** For any $2 \leq k \leq n + 1$ and $\epsilon > 0$, there is a large constant $A$ such that

$$\|e^{it\phi(D)} f\|_{BL^\epsilon_{k, A}(C^n_{R^2})} \lesssim_{\epsilon} R^{n(\frac{1}{2} - \frac{1}{k} + \epsilon)} \|f\|_{L^p(R^n)}, \quad \text{supp } \hat{f} \subset A(1),$$

(5.4)

for $p \geq \frac{2n+k+1}{n+k-1}$.

Proof. We decompose $f$ in spatial space, and obtain by Lemma 2.3

$$|e^{it\phi(D)} f(x)| \leq |e^{it\phi(D)}(\Psi_{B^m_{R^2}} f)| + \text{RapDec}(R)\|f\|_{L^p}.$$ 

Then (5.4) follows from Theorem 5.3 and Hölder’s inequality. \hfill \Box

To prove Theorem 5.3, we need to employ the polynomial partitioning argument. Let us first collect some useful results from [10] and [22].

**Definition 5.5** (Transverse complete intersection). Fix an integer $m \in [1, n+1]$ and let $P_1, \cdots, P_{n+1-m}$ be polynomials on $\mathbb{R}^{n+1}$ whose common zero set is denoted by $Z(P_1, \cdots, P_{n+1-m})$. We use $D_Z$ to denote the degree of $Z(P_1, \cdots, P_{n+1-m})$ which is the highest degree among those polynomials $P_i$’s. The variety $Z(P_1, \cdots, P_{n+1-m})$ is called a transverse complete intersection if

$$\nabla P_1(x) \wedge \cdots \wedge \nabla P_{n+1-m}(x) \neq 0, \forall x \in Z(P_1, \cdots, P_{n+1-m}).$$

The following theorem is essentially proved in Section 8.1 of [10] while not explicitly stated there.

**Theorem 5.6** ([10]). Let $r \gg 1, d \in \mathbb{N}$ and $F \in L^1(\mathbb{R}^n)$ be non-negative and supported on $B^m \cap N_{1/2+\delta} Z$ for some $0 < \delta \ll 1$, where $Z$ is an $m$-dimensional transverse complete intersection of degree $D_Z = d$. Then, at least one of the following cases holds:

1. **Cellular case:** There exists a polynomial $P : \mathbb{R}^n \to \mathbb{R}$ of degree $D = D(d)$ such that there exists $\sim D^m$ cells $O_i \subset Z \setminus N_{1/2+\delta} Z(P)$ with $O_i \subset B^m$ and

$$\int_{B^m} F \sim D^{-m} \int_{\mathbb{R}^n} F, \quad \text{for all } O.$$ 

Furthermore, each tube of length $r$ and radius $r^{1/2+\delta}$ intersects at most $O(D)$ cells.

2. **Algebraic case:** There exists an $(m-1)$-dimensional transverse complete intersection $Y$ of degree at most $O(D)$ such that

$$\int_{B^m \cap N_{1/2+\delta} Z} F \lesssim \int_{B^m \cap N_{1/2+\delta} Y} F.$$ 

**Proposition 5.7.** Let $T$ be a cylinder of radius $r$ with central line $\ell$ and suppose that $Z = Z(P_1, \cdots, P_{n+1-m}) \subset \mathbb{R}^{n+1}$ is a transverse complete intersection, where the polynomials $P_j$ have degree at most $D$. For any $\alpha$, define

$$Z_{>\alpha} := \{z \in Z : \text{Angle}(T_z Z, \ell) > \alpha\}.$$ 

Then $Z_{>\alpha} \cap T$ is contained in a union of $\lesssim D^n$ balls of radius $\lesssim r \alpha^{-1}$. 


Definition 5.8. Let $Z$ be an $m$-dimensional variety in $\mathbb{R}^{n+1}$. A tube $T_{\nu,w}^\ell$ is said to be $\gamma$-tangent to $Z$ in $B_R^{n+1}$ if
\[ T_{\nu,w}^\ell \subset N_\gamma R(Z) \cap B_R^{n+1}, \]
and for all $z \in Z \cap N_\gamma R(T_{\nu,w}^\ell)$ there holds
\[ \text{Ang}(T_z Z, L(\nu)) \leq \gamma. \]

Definition 5.9. Let $Z$ be a transverse complete intersection of degree $D \sim O(1)$ and dimension $m$ inside $B_R^{n+1}$. Define
\[ \mathcal{T}_Z := \{ (\nu, w, \ell) : T_{\nu,w}^\ell \text{ is } R^{-1/2+\delta_m}\text{-tangent to } Z \text{ in } B_R^{n+1} \}, \]
where $\delta_m \geq 0$ is a fixed small parameter for each dimension $m$.

Theorem 5.3 can be deduced from the following proposition.

Proposition 5.10. For $\varepsilon > 0$, there are small parameters
\[ 0 < \delta \ll \delta_n \ll \delta_{n-1} \ll \cdots \ll \delta_1 \ll \delta_0 \ll \varepsilon, \]
and a large constant $A$ such that the following holds. Let $1 \leq m \leq n+1$ and $Z = Z(P_1, \cdots, P_{n+1-m})$ be a transverse complete intersection with degree $D_Z$. Suppose that $f$ is concentrated on a union of wave packets coming from $\mathcal{T}_Z$. Then, for any $1 \leq A \leq A$ and radius $R \geq 1$, we have
\[ \| Ef \|_{BL^p_{k,A}(C_R^{n+1})} \leq C(K, \varepsilon, m, D_Z)R^{\alpha x}R^{(\log A - \log A)}R^{-e+1/2}\| f \|_{L^2} \quad (5.5) \]
for all
\[ 2 \leq p \leq \bar{p}(k, m), \]
where
\[ e := \frac{1}{2}\left( \frac{1}{2} - \frac{1}{p} \right)(n + 1 + k) \]
and
\[ \bar{p}(k, m) := \begin{cases} 2 - \frac{m+k}{2m}, & k < m; \\ 2 - \frac{m}{m-1}, & k = m. \end{cases} \]

When $m = n + 1$, by taking $Z = \mathbb{R}^{n+1}$ and choosing $A = A$ and $p = \bar{p}(k, n + 1)$, we have $-e + \frac{1}{2} = 0$. Therefore, Theorem 5.3 follows from Proposition 5.10.

For $p = 2$, Proposition 5.10 follows directly from the trivial $L^2$ estimate:
\[ \| Ef \|_{L^2(C_R^{n+1})} \leq CR^{1/2}\| f \|_{L^2}. \]
Thus, by interpolation and Hölder’s inequality of the broad norm, Proposition 5.10 is reduced to the endpoint case $p = \bar{p}(k, m)$. We prove Proposition 5.10 by an induction argument. In particular, we will induct on the dimension $m$, the radius $R$, and the parameter $A$. We start by checking the base case of the induction. If $R$ is small, the desired estimate can be deduced by choosing $C(K,\varepsilon,m,D_Z)$ sufficiently large. If $A = 1$, we may choose $A$ sufficiently large such that $R^{(\log A - \log 1)} = R^{10n}$, then the desired estimate will follow from the following trivial estimate
\[ \| Ef \|_{BL^p_{k,1}(C_R^{n+1})} \leq |C_R^{n+1}|\| f \|_{L^2}. \]
Finally, we will check the base case for the dimension $m$. This can be deduced from the following lemma from [10, 22].

Lemma 5.11 ([10, 22]). If $Ef$ is $R^{-1/2+\delta}$-tangent to a variety $Z$ of degree $O(1)$ and dimension $m \leq k-1$, then
\[ \| Ef \|_{BL^p_{k,A}(B_R^{n+1})} \leq \text{RapDec}(R)\| f \|_{L^2}. \quad (5.6) \]
If \( m = k - 1 \), then by Lemma 5.11, we have
\[
\|Ef\|_{BL^p_{\nu,w}(C^{n+1}_R)} \leq \text{RapDec}(R)\|f\|_{L^2}.
\]

Next, we assume that Proposition 5.10 holds if we decrease the dimension \( m \), the radius \( R \), or the value of \( A \). We proceed the inductive steps.

By invoking Theorem 5.6 with
\[
F = \frac{1}{|B|} \sum_{B_{\nu,w} \subset C_{R}^{n+1}} \mu_{Ef}(B_{\nu,w}^{n+1})\chi_{B_{\nu,w}^{n+1}},
\]
we see that there are two cases, that is, either the mass of \( \mu_{Ef} \) can be concentrated in a small neighborhood of a lower-dimensional variety or we can reduce the estimate of \( \mu_{Ef} \) to smaller cells. We say we are in the \textit{algebraic case}, if there is a transverse complete intersection \( Y \subset Z \) of dimension \( m - 1 \), defined by using polynomials of degree \( \leq D(\epsilon, D_Z) \) with
\[
\mu_{Ef}(N_{R^{1/2}+\delta_{m-1}}(Y) \cap C_{R}^{n+1}) \gtrsim \mu_{Ef}(C_{R}^{n+1}).
\] (5.7)

Otherwise, we say we are in the \textit{cellular case} with
\[
\mu_{Ef}(C_{R}^{n+1}) \lesssim \sum_{i=1}^{D^m} \mu_{Ef}(O_i).
\] (5.8)

5.1. The \textbf{cellular case}. Assume that we are in cellular case, that is \( \sum_i \mu_{Ef}(O_i) \sim \mu_{Ef}(C_{R}^{n+1}) \). For a given \( i \), define \( f_i = \sum_{(\nu,w,\ell) \in T_i} f_{\nu,w,\ell} \), where
\[
T_i := \{(\nu, w, \ell) : T_{\nu, w}^\ell \cap O_i \neq \emptyset \}.
\]

By a pigeonholing argument, we may choose a cell \( O_i \) such that
\[
\|f_i\|_{L^2} \lesssim \frac{D^m}{D^{m-1}} \|f\|_{L^2}.
\] (5.9)

By covering \( O_i \) by a family of finitely-overlapping balls of radius \( R/2 \), we can prove (5.5) by inducting on \( R \) as follows
\[
\mu_{Ef}(C_{R}^{n+1}) \lesssim D^m \mu_{Ef}(O_i) \lesssim D^m \sum_{B_{\nu,w}^{n+1} \subset C_{R}^{n+1}} \mu_{Ef}(O_i \cap B_{\nu,w}^{n+1})
\]
\[
\lesssim R^\epsilon D^m \|f_i\|_{L^2}^p \lesssim R^\epsilon D^{m-(\frac{m-1}{2})} \|f\|_{L^2}^p.
\] (5.10)

The induction closes for \( p > \frac{2m}{m-1} \) if we choose \( D(\epsilon, D_Z) \) sufficiently large to control the implicit constant.

5.2. The \textbf{algebraic case}. By definition, there exists a transverse complete intersection \( Y \) of dimension \( m - 1 \) such that
\[
\mu_{Ef}(N_{R^{1/2}+\delta_{m-1}}(Y)) \gtrsim \mu_{Ef}(C_{R}^{n+1}).
\]

In this case, we subdivide \( C_{R}^{n+1} \) into balls \( \{B_j\}_j \) of radius \( \rho \), with \( R^{1/2} \ll \rho \ll R \) and \( \rho^{1/2+\delta_{m-1}} = R^{1/2+\delta_m} \). Define \( f_j = \sum_{(\nu,w,\ell) \in T_j} f_{\nu,w,\ell} \), where
\[
T_j := \{(\nu, w, \ell) : T_{\nu, w}^\ell \cap N_{R^{1/2}+\delta_{m-1}}(Y) \cap B_j \neq \emptyset \}.
\]

We further decompose \( T_j \) into tubes that are transverse to \( Y \) and tubes that are transverse to \( Y \). We say \( T_{\nu, w}^\ell \) is tangent to \( Y \) in \( B_j \) if
\[
T_{\nu, w}^\ell \cap 2B_j \subset N_{R^{1/2}+\delta_{m-1}}(Y) \cap 2B_j = N_{\rho^{1/2+\delta_{m-1}}} \cap 2B_j.
\]

and for any \( x \in T_{\nu, w}^\ell \) and \( Y \cap 2B_j \) with \( |x - y| \lesssim R^{1/2+\delta_m} = \rho^{1/2+\delta_{m-1}}, \)
\[
\text{Ang}(G(\nu), T_j Y) \lesssim \rho^{-1/2+\delta_{m-1}}.
\]
Define the collection of tangential wave packets \( T_{j,\text{tang}} \) as
\[
T_{j,\text{tang}} = \{ (\nu, w, \ell) \in T_j : T_{\nu,w}^\ell \text{ is tangent to } Y \text{ in } B_j \},
\]
and the transverse wave packets \( T_{j,\text{trans}} \) by
\[
T_{j,\text{trans}} := T_j \setminus T_{j,\text{tang}}.
\]
Correspondingly, we define
\[
f_{j,\text{tang}} := \sum_{(\nu, w, \ell) \in T_{j,\text{tang}}} f_{\nu,w,\ell}^j, f_{j,\text{trans}} := \sum_{(\nu, w, \ell) \in T_{j,\text{trans}}} f_{\nu,w,\ell}^j.
\]
Note that \( Ef \) is essentially equal to \( Ef_j \) on the ball \( B_j \) in the sense that \( Ef = Ef_j + \text{RapDec}(R)\|f\|_{L^2} \). By the triangle inequality, we have
\[
\sum_j \|Ef\|_{B_k^{p,A}(B_j)}^p \leq \sum_j \|Ef_j\|_{B_k^{p,A}(B_j)}^p + \sum_j \|Ef_{j,\text{trans}}\|_{B_k^{p,A}(B_j)}^p + \text{RapDec}(R)\|f\|_{L^2}^p.
\]
Therefore, it remains to prove Proposition 5.10 for both the tangential case and the transversal case.

5.3. The tangential case. Assuming the tangential part dominates, we will prove Proposition 5.10 by induction on the dimension \( m \) and \( A \). Since we are now working with a ball of radius \( \rho \ll R \), in order to match our assumption, we need to redo the wave packet decomposition at scale \( \rho \). For the sake of simplicity, define \( g = f_{j,\text{tang}} \) and
\[
\tilde{g} = \sum_{\nu, w, \ell} \tilde{g}_{\nu,w,\ell}^j + \text{RapDec}(R)\|f\|_{L^2}.
\]
In order to perform the induction on dimension argument, we have to verify that \((\tilde{\nu}, \tilde{w}, \tilde{\ell})\) corresponds to tubes which are tangent to \( Y \) in \( B_j \). To be more precise, we need to show that
\[
T_{\tilde{\nu},\tilde{w}}^\ell \subset N_{\rho^{1/2+\delta_{m-1}}} (Y) \cap B_j,
\]
and for any \( x \in T_{\tilde{\nu},\tilde{w}}^\ell, y \in Y \cap B_j \) with \( |x - y| \leq \rho^{1/2+\delta_{m-1}} \),
\[
\text{Ang}(G(\tilde{\nu}), T_y Y) \leq \rho^{-1/2+\delta_{m-1}},
\]
which can be deduced from Lemma 3.3. By induction on \( m \) and \( A \), we have
\[
\|E\tilde{g}\|_{B_k^{p,A}(B_j)}^p \leq C(K, \varepsilon, m, D(\varepsilon, D_Z))\rho^{(m-1)\varepsilon} \rho^{A(\log A - \log A/2)} \rho^{-\varepsilon+1/2}\|f_{j,\text{tang}}\|_{L^2}.
\]
for
\[
2 \leq p \leq \bar{p}(k, m - 1).
\]
Since there are \( R^O(\delta_{m-1}) \) many \( B_j \)'s, by summing over the balls and noting that
\[
\rho^{1/2+\delta_{m-1}} = R^{1/2+\delta_{m}},
\]
finally we have
\[
\|E\tilde{g}\|_{B_k^{p,A}(C_{R}^{m+1})}^p \leq C(K, \varepsilon, m, D(\varepsilon, D_Z)) R^{O(\delta_{m-1})} R^{(m-1)\varepsilon} R^{A(\log A - \log A)} \rho^{-\varepsilon+1/2}\|f_{j,\text{tang}}\|_{L^2}.
\]
Using the fact that \( \delta_{m-1} \ll \varepsilon \), we close the induction.
5.3.1. The transversal case. Unlike the circular cone case studied in [22], \( \{ (\xi, \phi(\xi)) : \xi \in L \} \) may not lie in an affine subspace. To overcome this difficulty, we work with small sector \( \tau \) of dimension \( \rho^{-1/2+\delta_m} \times \cdots \times \rho^{-1/2+\delta_m} \times 1 \) with \( R^{1/2} \ll \rho \ll R \). At this scale, an affine subspace \( V \) can be constructed using the tangent space of a point on the surface \( \{ (\xi, \phi(\xi)) : \xi \in L \} \), so that \( \{ (\xi, \phi(\xi)) : \xi \in L \cap \tau \} \) lies in a \( R^{-1/2+\delta_m} \)-neighborhood of \( V \).

Given \( \eta \in L \), we use \( T_\eta L \) to denote the tangent space of \( L \) at \( \eta \). By the definition of \( L \), it is easy to check that \( T_\eta L \) is orthogonal to the vectors

\[
\left( \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} \right)_{\xi=\eta} a_i, \; i = 1, \cdots, n+1-m.
\]

Let

\[
\hat{V} := T_\eta L + \epsilon_{n+1}.
\]

This definition of \( \hat{V} \) is consistent with that in subsection 4.2.

**Lemma 5.12.** Let \( \tau \) be a cap of dimension \( 1 \times \rho^{-1/2+\delta_m} \times \cdots \times \rho^{-1/2+\delta_m} \) with \( \eta \in \tau \), then

\[
\{ (\xi, \phi(\xi)) : \xi \in L \cap \tau \} \subset N_{C R^{-1/2+\delta_m}} \hat{V}.
\]

where \( C > 0 \) is a large constant.

**Proof.** To prove (5.14), it suffices to show

\[
\{ \xi : \xi \in L \cap \tau \} \subset N_{C R^{-1/2+\delta_m}} T_\eta L.
\]

By the homogeneity of \( \phi \), this can be further reduced to showing that if \( \xi, \eta \in L \cap S^{n-1} \) with \( |\xi - \eta| \lesssim \rho^{-1/2+\delta_m} \), then

\[
\{ \xi : \xi \in L \cap S^{n-1} \cap \tau \} \subset N_{C R^{-1/2+\delta_m}} T_\eta L.
\]

Since \( |\xi - \eta| \lesssim \rho^{-1/2+\delta_m} \) we may construct a curve \( \{ \xi(t) \} \subset L \cap S^{n-1} \) connecting \( \xi \) and \( \eta \) with \( \xi(0) = \eta, \xi(\rho^{-1/2+\delta_m}) = \xi \) and

\[
|\xi'(t)| \leq C, \; 0 \leq t \leq \rho^{-1/2+\delta_m}, \; t \leq 3.
\]

It remains to show

\[
|\text{Proj}_{T_\eta (L \cap S^{n-1})}(\xi(t) - \xi(0))| \lesssim R^{-1/2+\delta_m}.
\]

Note that

\[
\left( \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} \right)_{\xi=\eta} a_i, \; i = 1, \cdots, n+1-m,
\]

are orthogonal to \( T_\eta (L \cap S^{n-1}) \), and by our construction \( \xi'(0) \in T_\eta (L \cap S^{n-1}) \). Therefore we have

\[
\left| (\xi(\rho) - \xi(0)) \cdot \left( \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} \right)_{\xi=\eta} a_i \right| = \rho \xi'(0) \cdot \left( \frac{\partial^2 \phi}{\partial \xi_i \partial \xi_j} \right)_{\xi=\eta} a_i + O(\rho^{-1+2\delta_m}) \lesssim R^{-1/2+\delta_m}.
\]

Here we have used the assumption \( R^{1/2} \ll \rho \ll R \). The proof is complete.

Fix a ball \( B \) of radius \( R^{1/2+\delta_m} \). Let \( V \) be the tangent space to \( Z \) at some point in \( B \cap Z \). By Definition 5.8, it does not matter which point we choose. Define two sets \( T_{B,Z} \) and \( T_{B,Z,\tau} \) respectively as follows:

\[
T_{B,Z} := \{ (\nu, w, \ell) : T_{\nu, w}^\ell \text{ is } R^{-1/2+\delta_m} \text{ tangent to } Z, T_{\nu, w}^\ell \cap B \neq \emptyset \},
\]

\[
T_{B,Z,\tau} := \{ (\nu, w, \ell) : T_{\nu, w}^\ell \text{ is } R^{-1/2+\delta_m} \text{ tangent to } Z, T_{\nu, w}^\ell \cap B \neq \emptyset, \nu \cap 2\tau \neq \emptyset \}.
\]

Let \( h_B \) related to \( T_{B,Z} \) be defined by

\[
h_B := \sum_{(\nu, w, \ell) \in T_{B,Z}} h_{\nu, w}^\ell.
\]
Similarly, define $h_{B,\tau}$ related to $T_{B,\pi,\tau}$ as

$$h_{B,\tau} := \sum_{(v, w, \ell) \in T_{B,\pi,\tau}} h_{v, w}^\ell.$$ 

Fix $B_j = B_{\rho}^{R+1}(y)$ and cover $B_j$ by balls $B$ of radius $R^{1/2+\delta_m}$. Since $V$ is determined by $B$, on account of Lemma 4.7, we may sort the balls $B$ into two classes $X_a$ and $X_b$ according to whether case a) or case b) in Lemma 4.7 holds. Now partition $N_{R^{1/2+\delta_m}}(Z) \cap B_j \subset X_a \cup X_b$, where $X_a$, $X_b$ are the union of balls $B$ in case a) or in case b) respectively. First assume that $B \in X_b$. Since the support of $h_B$ is contained in $O(1)$ slabs, we have

$$\|E_h B\|^3_{\mathcal{B}_{L^\infty}^p(B)} = \text{Rapdec}(R)\|h_B\|^2_{L^2}. \quad (5.17)$$

Otherwise, we have

**Lemma 5.13.** Let $h_{B,\tau} = \sum_{(v, w, \ell) \in T_{B,\pi,\tau}} h_{v, w}^\ell$ and $B \in X_a$. Then for any $\rho \leq R$,

$$\int_{B \cap N_{\rho^{1/2+\delta_m}}(Z)} |E h_{B,\tau}|^2 \lesssim R^{O(\delta_m)} \left( \frac{R^{1/2}}{\rho^{1/2+2\delta_m}} \right)^{(n+1-m)} \int_{2B} |E h_{B,\tau}|^2 + \text{Rapdec}(R)\|h_B\|^2_{L^2}. \quad (5.18)$$

**Proof.** Since $V$ is the tangent space of $Z$ at some point in $B \cap Z$, $T_{B,\pi,\tau} \subset T_{B,\pi} = \{ (v, w, \ell) : T_{\nu,\omega}^\ell \cap B \neq \emptyset \text{ and } \text{Ang}(\nu, V) \lesssim R^{-1/2+\delta_m} \}$. By Lemma 5.12, $(E h_{B,\tau})^\ell$ is supported in $N_{R^{-1/2+\delta_m}}(Z)$. Consider an $(n+1-m)$-dimensional plane $\Pi$ parallel to $W$ passing through $B$. If we restrict $E h_{B,\tau}$ to the plane $\Pi$, then its Fourier transform is supported in a ball of radius $\lesssim R^{-1/2+\delta_m}$. Therefore, by Lemma 6.4 in [10] we have

$$\int_{B(\bar{x}, \rho^{1/2+2\delta_m}) \cap \Pi} |E h_{B,\tau}|^2 \lesssim \left( \frac{R^{1/2-2\delta_m}}{\rho^{1/2+2\delta_m}} \right)^{(n+1-m)} \int_{\Pi} w_{B(x, \rho^{1/2+2\delta_m})}|E h_{B,\tau}|^2,$$

for any point $\bar{x} \in \mathbb{R}^{n+1}$.

By Lemma 4.7, $\text{Ang}(V, W) \gtrsim K^{-4}$, we have

$$\Pi \cap N_{\rho^{1/2+\delta_m}}(Z) \cap B \subset \Pi \cap B(x_0, \rho^{1/2+25\delta_m}),$$

for some point in $B$.

Therefore, modulo a rapidly decaying error, we obtain

$$\int_{\Pi \cap N_{\rho^{1/2+\delta_m}}(Z) \cap B} |E h_B|^2 \leq \int_{\Pi \cap B(x_0, \rho^{1/2+2\delta_m})} |E h_B|^2 \lesssim R^{O(\delta_m)} \left( \frac{R^{1/2}}{\rho^{1/2+2\delta_m}} \right)^{(n+1-m)} \int_{\Pi} w_{B}|E h_B|^2.$$

Integrating over all $\Pi$ that is parallel to $W$ and passing through $B$, one obtains the desired results. 

Define $g_{\text{ess}}$ and $g_{\text{tail}}$ to be the essential part and tail part of $f_{j,\text{trans}}$ respectively by

$$g_{\text{ess}} := \sum_{(\bar{\nu}_0, \omega_0) \in \mathcal{T}_{\text{ess}}} g_{\bar{\nu}_0, \omega_0},$$

$$g_{\text{tail}} := \sum_{(\bar{\nu}_0, \omega_0) \in \mathcal{T}_{\text{tail}}} g_{\bar{\nu}_0, \omega_0},$$

where

$$\mathcal{T}_{\text{ess}} := \{ (\bar{\nu}_0, \omega_0) : \exists (\nu, \omega, \ell) \in \mathcal{T}_{\nu,\omega} \text{ with } T_{\nu,\omega} \cap X_a \neq \emptyset \},$$

$$\mathcal{T}_{\text{tail}} := \{ (\bar{\nu}_0, \omega_0) : \forall (\nu, \omega, \ell) \in \mathcal{T}_{\nu,\omega}, T_{\nu,\omega} \cap X_a = \emptyset \}.$$
By the triangle inequality and (5.17), we have
\[ \|Eg\|_{BL^p_{k,A}(B_j)} \leq \|Eg_{\text{ess}}\|_{BL^p_{k,A/2}(B_j)} + \|Eg_{\text{tail}}\|_{BL^p_{k,A/2}(B_j)} \]
\[ \leq \|Eg_{\text{ess}}\|_{BL^p_{k,A/2}(B_j)} + \text{RapDec}(R)\|f\|_{L^2}. \]

Next, through an appropriate reduction, it suffices to consider a direction \( b \) such that \( |b| \leq R^{1/2+\delta_m} \) and \( b \) is transversal to \( T_jZ \) for all points in \( z \in Z \cap B_j \). Indeed, we will show that \( L^2\) norm of \( g_{\text{ess}} \) is equidistributed along different choices of \( b \) in \( N_{R^{1/2+\delta_m}}(Z) \cap B_j \). To this end, we need a useful reversed Hörmander’s \( L^2 \) bound which can be found in [10].

**Lemma 5.14** (Lemma 3.4 in [10]). Suppose that \( h \) is a function concentrated on a set of wave-packets \( T \) and for every \( T^d_{\nu,w} \in T, T^d_{\nu,w} \cap B_r(z) \neq \emptyset \) for some radius \( r \geq R^{1/2+\delta_m} \). Then
\[ \|Eh\|_{L^2(B^{R^{1/2+\delta_m}}(z))} \sim r\|h\|_{L^2}. \]

As a direct consequence of Lemma 5.14, it follows that for any \( B \subset X_a \) such that \( B \cap T^d_{\nu,w} \neq \emptyset \), where \( (\nu, w, \ell) \in T_{\text{ess}} \) for some \( (\tilde{\nu}_0, w_0) \in T_{\text{ess}} \), we have
\[ \|g_{\tilde{\nu}_0,w_0}\|_{L^2} \sim R^{-1/2-\delta_m}\|Eg_{\tilde{\nu}_0,w_0}\|_{L^2(40B)}. \]

Let \( b \in B_{R^{1/2+\delta_m}} \). Decompose
\[ \tilde{g} = \sum_{\tilde{\nu}, \tilde{w}, \tilde{\ell}} \tilde{g}_{\tilde{\nu}, \tilde{w}, \tilde{\ell}} + \text{RapDec}(\rho)\|f\|_{L^2}. \]

A key observation is that, for any \( (\tilde{\nu}, \tilde{w}, \tilde{\ell}) \), if \( \tilde{T}^d_{\tilde{\nu}, \tilde{w}} \) intersects \( N_{R^{1/2+\delta_m}}(Z+b) \cap B_j \), then according to Lemma 3.3, \( \tilde{T}^d_{\tilde{\nu}, \tilde{w}} \) is \( p^{-1/2+\delta_m} \)-tangent to \( Z+b \) in \( B_j \). Define
\[ \tilde{T}_{Z+b} := \{(\tilde{\nu}, \tilde{w}, \tilde{\ell}) : \tilde{T}^d_{\tilde{\nu}, \tilde{w}} \text{ is tangent to } Z+b \text{ in } B_j \}, \tilde{g}_b := \sum_{(\tilde{\nu}, \tilde{w}, \tilde{\ell}) \in \tilde{T}_{Z+b}} \tilde{g}_{\tilde{\nu}, \tilde{w}, \tilde{\ell}}. \]

Define
\[ \tilde{g}_{\text{ess}, b} = \sum_{(\tilde{\nu}_0, w_0) \in T_{\text{ess}}} \sum_{(\tilde{\nu}, \tilde{w}, \tilde{\ell}) \in \tilde{T}_{Z+b} \cap T_{\tilde{\nu}_0, w_0}} \tilde{g}_{\tilde{\nu}, \tilde{w}, \tilde{\ell}}. \]

Therefore, \( \tilde{g}_{\text{ess}, b} \) is tangent to \( Z+b \) in \( B_j \).

With the above notations, to finish the proof of the transverse case, we need the following important transverse equidistribution estimate, the proof of which can be obtained by carrying over the proof of Lemma 5.13 in [22].

**Lemma 5.15.** Let \( g_{\text{ess}} \) and \( \tilde{g}_{\text{ess}, b} \) be defined as above, then
\[ \|\tilde{g}_{\text{ess}, b}\|_{L^2} \leq R^{O(\delta_m)} \left( \frac{R^{1/2}}{p^{1/2}} \right)^{(n+1-m)} \|g_{\text{ess}}\|_{L^2}. \]

Following the approach in [10], we may choose a finite set of vectors \( \mathcal{B} = \{ b \} \) where \( b \in B_{R^{1/2+\delta_m}} \) such that for each \( B_j \), we have
\[ \|Eg_{\text{ess}}\|_{BL^p_{k,A/2}(B_j)} \leq (\log R) \sum_{b \in \mathcal{B}} \|Eg_{\text{ess}, b}\|_{BL^p_{k,A/2}(B_j)}, \]
where
\[ f_{j, \text{trans}, b} = e^{-i\varphi_b(t)}g_{\text{ess}, b}, \]
and for different choices of \( b \in \mathcal{B} \), the corresponding sets \( B_j \cap N_{R^{1/2+\delta_m}}(Z+b) \) have finite overlaps.

Thus, one has
\[ \|Eg_{\text{ess}}\|_{BL^p_{k,A}(B_R)} \leq (\log R) \sum_{j} \sum_{b \in \mathcal{B}} \|Eg_{\text{ess}, b}\|_{BL^p_{k,A/2}(B_j)}, \]
and
\[ \sum_{b \in \mathcal{B}} \|g_{\tilde{\nu}_0, w_0, b}\|_{L^2} \lesssim \|g_{\tilde{\nu}_0, w_0}\|_{L^2}. \]
Finally, by the equidistribution estimate (5.18), one has
\[
\max_{|\xi| \leq B} \| F^{ess}_{j,trans,b} \|_{L^2} \leq R^{O(\delta_m)} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-(n+1-m)} \| g^{ess} \|_{L^2}.
\]
Now, we may employ an induction on scales argument to complete the proof. By our assumption on \( B_j \), we have
\[
\| F^{ess}_{j,trans,b} \|_{BL^p_{k,A/2}(B_j)} \leq C(K, \varepsilon, m, D_Z) p^{m \varepsilon} \rho^{(A-\log(A/2)) \rho^{-e+1/2}} \| F^{ess}_{j,trans,b} \|_{L^2}
\]
\[
\leq C(K, \varepsilon, m, D_Z) R^{p \varepsilon} p^{m \varepsilon} R^{(A-\log(A)) \rho^{-e+1/2}} \| F^{ess}_{j,trans,b} \|_{L^2}.
\]
Combing the above estimates together, we have
\[
\| F \|_{BL^p_{k,A}(B_R)} \lesssim \log R \sum_{j} \sum_{b \leq B} \| F^{ess}_{j,trans,b} \|_{BL^p_{k,A/2}(B_j)}^{p}\]
\[
\lesssim R^{O(\delta_m)} (C(K, \varepsilon, m, D_Z) p^{m \varepsilon} R^{(A-\log(A)) \rho^{-e+1/2}} \rho^{-e+1/2}) \sum_{j,b} \| F^{ess}_{j,trans,b} \|_{L^2}^{p}\]
\[
\lesssim R^{O(\delta_m)} (C(K, \varepsilon, m, D_Z) p^{m \varepsilon} R^{(A-\log(A))} \rho^{-e+1/2}) \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-(n+1-m)(p/2-1)} \| F \|_{L^2}^{p}.
\]
If \( p = p(m, k) = 2 \frac{m+k}{m+k-1} \), then
\[
\rho^{-(e+1/2)p} \left( \frac{R^{1/2}}{\rho^{1/2}} \right)^{-(n+1-m)(p/2-1)} = R^{-(e+1/2)p},
\]
hence,
\[
\| F \|_{BL^p_{k,A}(B_R)} \leq C(\varepsilon, D_Z) R^{O(\delta_m)} (R/\rho)^{-mp\varepsilon} \left( C(K, \varepsilon, m, D_Z) R^{m \varepsilon} R^{(A-\log(A))} R^{-e+1/2} \right)^p \| F \|_{L^2}^{p}.
\]
Note \( R/\rho = R^{O(\delta_m-1)} \), by choosing \( \delta_m \ll \varepsilon \delta_{m-1} \) such that
\[
C(\varepsilon, D_Z) R^{O(\delta_m)} (R/\rho)^{-mp\varepsilon} \leq 1,
\]
then the induction closes and the proof is complete.

6. Parabolic rescaling

To prove Theorem 1.2, we will employ an induction on scales argument. To fulfill the argument, a crucial ingredient is a parabolic rescaling lemma which connects estimates at different scales and facilitates the induction argument.

For the cone restriction setting in [22], one can use the standard Lorentz transformation to tilt the light cone \((\xi, |\xi|)\) into the form \((\xi, \xi_{\xi}^2 + \cdots + \xi_{\xi}^2 - \xi_{\xi}^2 / 2 \xi_{\xi})\) which is well-suited for performing the parabolic rescaling argument, since each vertical slice of this cone is parabolic. However, in our setup of the local smoothing problem for the operator \( e^{it\sqrt{-\Delta}} \), the Lorentz transformation is not readily available unless the right-hand side is \( L^2 \)-based. We can get around this difficulty using the reductions shown in Section 2. It then suffices to consider phase functions in the class \( \Phi(R) \), which depends on the scale \( R \).

Lemma 6.1. Suppose \( \nu \) is a slab of dimension \( 1 \times K^{-1} \times \cdots \times K^{-1} \) with central line lying in the direction \( \nu_\xi \), then
\[
\| e^{it\phi_{\nu}(D)} f^{\nu} \|_{L^p(B_R \times [-R,R])} \leq Q_p(R/K^2) K^{-2n(\frac{1}{4} - \frac{1}{4}) + \frac{3}{2} - \varepsilon} R^{n(\frac{1}{4} - \frac{1}{4}) + \varepsilon} \| f^{\nu} \|_{L^p+\text{RapDec}(R)} \| f \|_{L^p}.
\]

Proof. Without loss of generality, we may assume any \( \xi \in \nu \) satisfies
\[
\left| \frac{\xi}{\xi_n} - \xi' \right| \leq \varepsilon_0 K^{-1}, \quad \xi' \in \mathbb{R}^{n-1}, \quad |\xi'| \leq \varepsilon_0.
\]
First, we make a change of variables with respect to \( \xi' \) to locate \( \nu \) in a neighborhood of \( c_n \)
\[
\xi' \rightarrow \xi_n \xi'_\nu + \xi',
\]
correspondingly, the phase function becomes
\[
\frac{|\xi'|^2}{2\xi_n} + \xi' \cdot \xi' + \frac{|\xi'|^2}{2\xi_n} + K^{-4}E_R(\xi_n, \xi').
\]
Using the homogeneity of $E_R$ and Taylor’s formula, we have
\[
E_R(\xi_n, \xi') = \xi_n E_R(\xi', \xi_n) = \xi_n (E_R(\xi', 1) + \partial_{\xi'} E_R(\xi', 1) \frac{\xi'}{\xi_n} + E_R(\xi)).
\]
Here $\tilde{E}_R(\xi)$ denotes the remainder coming from Taylor’s formula. Taking spatial variables into account, the associate phase function reads
\[
(\chi' + tK^{-4}\partial_{\xi'} E_R(\xi', 1) ) \cdot \xi' + (x_n + t\frac{|\xi'|^2}{2} + tK^{-4}E_R(\xi', 1))\xi_n + t\left(\frac{|\xi'|^2}{2\xi_n} + K^{-4}\tilde{E}_R(\xi)\right).
\]
Now we perform the change of variables in $(x, t)$ by:
\[
\begin{align*}
x' &= x', \\
x_n + t\frac{|\xi'|^2}{2} + tK^{-4}E_R(\xi', 1) &= x_n, \\
t &\to t.
\end{align*}
\]
Then under the new coordinates, $B_0^0 \times [-R, R]$ is transformed into a subset of $B_{C R/K}^{n-1} \times (-CR, CR) \times (-CR/K^2, CR/K^2)$. Now the new phase function is given by
\[
\bar{\phi}_R(\xi) = \frac{\xi'^2 + \cdots + \xi'^{n-1}}{2\xi_n} + K^{-4}\tilde{E}_R(\xi), \quad \text{where} \quad \tilde{R} := R/K^2, \quad \bar{K} := K_0\tilde{R},
\]
where
\[
\tilde{E}_R(\xi) = \tilde{K}^{-\frac{sl}{1-2s}} \xi_n^2 K^2 \left(E_R(\xi_n + K^{-1} \frac{\xi'}{\xi_n}, 1) - E_R(\xi', 1) - K^{-1} \partial_{\xi'} E_R(\xi', 1) \frac{\xi'}{\xi_n}\right).
\]
If we invoke Taylor’s formula with the remainder of integral form, we have
\[
\bar{E}_R(\xi) = \tilde{K}^{-\frac{sl}{1-2s}} \left(\frac{1}{2} \frac{d^n \tilde{E}_R(\xi_n, \xi')}{\xi_n} + K^{-1} \sum_{|\alpha|=3} \frac{3}{\alpha!} \frac{(\xi')^\alpha}{\xi_n} \int_0^1 (1-t)^2 (\partial^\alpha E_R)(tK^{-1} \frac{\xi'}{\xi_n} + \xi', 1) dt\right).
\]
It is then easy to see that derivatives of $\bar{E}_R$ do not blow up in $K$. If we set
\[
\hat{g}(\xi) = \hat{g}'(\xi_n, \xi'),
\]
it is then easy to verify that sup $\hat{g} \subset N_{\epsilon_n}(\epsilon_n)$. By choosing $K_0$ sufficiently large, it is straightforward to check that $\bar{E}_R(\xi)$ satisfies condition $H_3$, and therefore $\phi_R \in \Phi(\tilde{R})$.

Therefore, it suffices to estimate
\[
e^{i\phi_R(D)} g := \int_{\mathbb{R}^n} e^{i(x \xi + \phi_R(\xi))} \hat{g}(\xi, \xi_n) \hat{g}(\xi', \xi_n) d\xi.
\]
We decompose $B_{C R/K}^{n-1} \times (-CR, CR) \times (-CR/K^2, CR/K^2)$ into a family of finitely overlapping balls of scale $\tilde{R}$, i.e.
\[
B_{C R/K}^{-1} \times (-CR, CR) \times (-CR/K^2, CR/K^2) \subset \bigcup_y Q_{y, \tilde{R},}
\]
where $y \in \mathbb{R}^{n+1}$ is the center of $Q_{y, \tilde{R}}$. 
Finally, by a localization argument as in Lemma 2.3, we have
\[
\|e^{it\phi_R(D)} f\|_{L^p(B(R) \cap [-R,R], R)} \lesssim K^{(n+1)-(n-1)p} \sum_y \|e^{it\phi_R(D)} g\|_{L^p(Q_y, \mathbb{R})} + \text{RapDec}(R) \|f\|_{L^p}.
\]
\[
\lesssim K^{(n+1)-(n-1)p} \sum_y \|\Psi_B^{n, R} (\tilde{\phi}_R^{y/(k+1), n}) g\|_{L^p(Q_y, \mathbb{R})} + \text{RapDec}(R) \|f\|_{L^p}.
\]
\[
\lesssim K^{(n+1)-(n-1)p} \sum_y \|\Psi_B^{n, R} (\tilde{\phi}_R^{y/(k+1), n}) g\|_{L^p(R^n)} Q_p^p (R/K^2) \left( R/K^2 \right)^{np(\frac{1}{2} - \frac{1}{p})} + \text{RapDec}(R) \|f\|_{L^p}.
\]
\[
\lesssim K^{(n+1)-(n-1)p} \|g\|_{L^p} Q_p^p (R/K^2) \left( R/K^2 \right)^{np(\frac{1}{2} - \frac{1}{p}) + \epsilon} + \text{RapDec}(R) \|f\|_{L^p}.
\]
\[
\lesssim K^{(n+1)-(n-1)p} \|f\|_{L^p} R^p (\frac{1}{2} - \frac{1}{p}) + \epsilon + \text{RapDec}(R) \|f\|_{L^p}.
\]

\[
\square
\]

7. PROOF OF THE MAIN THEOREM

Roughly speaking, the strategy of proving Theorem 1.2 is to decompose \(e^{it\phi_R(D)} f\) into two terms: a “narrow” term and a “broad” term. The narrow term comes from the caps of which the normal vectors make a small angle with some \((k-1)\)-plane. The broad part comes from the remaining caps. The broad term can be bounded via the broad norm estimate established in Section 5. To bound the narrow term, we need a narrow decoupling theorem.

Let \(\delta > 0, \nu\) be a slab of width \(K^{-1}\) defined as usual. We use \(\nu_\delta\) to denote the \(\delta\) neighborhood of the corresponding slab on the cone defined by

\[
\nu_\delta := \{ (\eta, \eta_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : \text{dist}(\eta, \eta_{n+1}), (\xi, \phi_R(\xi)) \leq \delta, \text{ for some } \xi \in \nu \},
\]

where \(\phi_R\) is a phase function in the class \(\Phi(R)\).

**Theorem 7.1** (Narrow decoupling theorem). Let \(k \geq 3\) and \(F = \sum_{\nu} F_{\nu}\) be a sum over \(K^{-1}\) slabs with \(\nu_{\delta} \subset \nu_{K^{-2}}\). Assume that there is a \((k-1)\)-dimensional vector space \(V\), such that each cap \(\nu_{K^{-2}}\) contains a point with normal lying in a \(K^{-2}\) neighborhood of \(V\). Then for any \(\epsilon > 0\),

\[
\|F\|_{L^p(B_{K^{-2}}^{n+1})} \leq C_{\epsilon} K^{-\epsilon} \left( \sum_{\nu} \|F_{\nu}\|_{L^p(\mathbb{R}^{n+1})} \right)^{1/2}, \quad 2 \leq p \leq \frac{2(k-1)}{k-3}.
\]

(Theorem 7.1) can be deduced from the Theorem 2.3 in [13] for the case when the phase function is the circular cone. By Lorentz transformation, it is also valid for the phase function of the form \((\xi_1^2 + \cdots + \xi_{n-1}^2) / 2 \xi_n\). Since \(\phi_R\) is in the class \(\Phi(R)\), which is \(K^{-4}\)-close to the above standard form, Theorem 7.1 is a immediate corollary of Theorem 2.3 in [13].

Theorem 1.2 can be deduced from the following proposition.

**Proposition 7.2.** Let \(k \geq 2\). For all \(K, \epsilon > 0\), and \(\bar{p}(k, n) \leq p \leq 2 \frac{k-1}{k-2}\), where

\[
\bar{p}(k, n) = \begin{cases} 
\frac{(n+1)}{2} & k = 2, \\
\frac{2n - k + 5}{2n - k + 3} & k \geq 3.
\end{cases}
\]

If

\[
\|e^{it\phi_R(D)} f\|_{L^p(\mathbb{R}^{n+1})} \lesssim K^{\epsilon} R^{n(\frac{1}{2} - \frac{1}{p}) + \epsilon} \|f\|_{L^p},
\]

then we have

\[
\|e^{it\phi_R(D)} f\|_{L^p(C_{\mathbb{R}^{n+1}})} \lesssim R^{n(\frac{1}{2} - \frac{1}{p}) + \epsilon} \|f\|_{L^p}.
\]

**Proof of Theorem 1.2.** Recall that it suffices to prove the following estimate

\[
\|e^{it\phi_R(D)} f\|_{L^p(C_{\mathbb{R}^{n+1}})} \lesssim R^{n(\frac{1}{2} - \frac{1}{p}) + \epsilon} \|f\|_{L^p}, \quad \text{supp} \tilde{f} \subset A(1) \cap N_{\epsilon_n}(e_n).
\]

(7.3)
By Corollary 5.4 and Proposition 7.2, we obtain (7.3) if
\[ p > \min_{2 \leq k \leq n+1} \max \left\{ \frac{n + k + 1}{n + k - 1}, \bar{p}(k, n) \right\}. \]
In particular, if we choose
\[ k = \begin{cases} \frac{n + 5}{2} & \text{if } n \text{ is odd,} \\ \frac{n + 4}{2} & \text{if } n \text{ is even,} \end{cases} \]
then the range of \( p \) matches the requirement of Theorem 1.2. \( \square \)

**Proof of Proposition 7.2.** We invoke the broad-narrow argument to induct on the scale \( R \). The base case is trivial to check. Assume that for any \( \varepsilon > 0 \), \( 0 < R' < R/2 \), we have
\[ Q_p(R') \leq C_\varepsilon R^\varepsilon. \]
we need to prove that
\[ Q_p(R) \leq C_\varepsilon R^\varepsilon. \]
For a given ball \( B_{K+1}^{n+1} \subset C_{n+1}^{K+1} \), let \( V_1 \cdots V_A \) be \((k-1)\)-dimensional linear subspaces which achieve the minimum in the definition of the \( k \)-broad “norm”, we obtain
\[
\int_{B_{K+1}^{n+1}} |e^{it\phi_N(D)} f(x)|^p \, dx \, dt \lesssim K^{O(1)} \max_{\tau \in V_i} \int_{B_{K+1}^{n+1}} |e^{it\phi_N(D)} f^\tau(x)|^p \, dx \, dt
\]
\[ + \sum_{\ell=1}^A \int_{B_{K+1}^{n+1}} \left| \sum_{\tau \in V_i} e^{it\phi_N(D)} f^\tau(x) \right|^p \, dx \, dt. \]
Summing over balls \( \{B_{K+1}^{n+1}\} \) yields
\[
\int_{C_{n+1}^{K+1}} |e^{it\phi_N(D)} f(x)|^p \, dx \, dt \lesssim K^{O(1)} \sum_{B_{K+1}^{n+1} \subset C_{n+1}^{K+1}} \min_{V_1 \cdots V_A} \max_{\tau \in V_i} \int_{B_{K+1}^{n+1}} |e^{it\phi_N(D)} f^\tau(x)|^p \, dx \, dt
\]
\[ + \sum_{B_{K+1}^{n+1} \subset C_{n+1}^{K+1}} \sum_{\ell=1}^A \int_{B_{K+1}^{n+1}} \left| \sum_{\tau \in V_i} e^{it\phi_N(D)} f^\tau(x) \right|^p \, dx \, dt. \]
Invoking Corollary 5.4, we have
\[
\sum_{B_{K+1}^{n+1} \subset C_{n+1}^{K+1}} \min_{V_1 \cdots V_A} \max_{\tau \in V_i} \int_{B_{K+1}^{n+1}} |e^{it\phi_N(D)} f^\tau(x)|^p \, dx \, dt \lesssim C(\varepsilon, A, K) R^{\bar{p}(\frac{1}{2} - \frac{1}{p} + \varepsilon p/2)\|f\|_p^p}. \]
Next, we use Theorem 7.1 and Lemma 6.1 to estimate the contribution of the second term in the right-hand side of (7.4). It follows from Theorem 7.1 that for any \( \delta > 0 \)
\[
\sum_{\ell=1}^A \int_{B_{K+1}^{n+1}} \left| \sum_{\tau \in V_i} e^{it\phi_N(D)} f^\tau(x) \right|^p \, dx \, dt
\]
\[ \lesssim C(\delta, A) K^3 \max\{1, K^{(k-3)(\frac{1}{2} - \frac{1}{p})}\} \sum_{\tau} \int_{B_{K+1}^{n+1}} w_{B_{K+1}^{n+1}} |e^{it\phi_N(D)} f^\tau(x)|^p \, dx \, dt,
\]
where we have used the fact that
\[ \#\{\tau : \tau \in V_i\} \lesssim \max\{1, K^{k-3}\}. \]
Summing over $B_{K^2}^{n+1}$ in both sides of the above inequality, we obtain
\[
\sum_{B_{K^2}^{n+1} \subset C_{n+1}} \sum_{\tau=1}^{A} \int_{B_{K^2}^{n+1}} \sum_{\tau \in \mathcal{V}_n} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt
\]
\[
\leq C(\delta, A) K^\delta \max \{1, K^{(k-3)(\frac{1}{2} - \frac{1}{p})}\} \sum_{\tau \in \mathcal{V}_n} \int_{B_{K^2}^{n+1}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt. \tag{7.7}
\]
Using the rapidly decaying property of the weight function, we have
\[
\int_{B_{K^2}^{n+1}} w_{C_{n+1}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt \leq \int_{C_{n+1}^{n+4}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt + \text{RapDec}(R) f \bigg|^{p}.
\]
For $\varepsilon > 0$, we see that by Lemma 6.1
\[
\int_{C_{n+1}^{n+4}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt \leq \sum_{C_{n+1}^{n+4} \subset C_{n+1}^{n+4}} \int_{C_{n+1}^{n+4}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt
\]
\[
\leq \varepsilon_1 K^{-2n(\frac{1}{2} - \frac{1}{p}) + 2 - \varepsilon} Q_p^p \left( \frac{R}{K^2} \right) R^{n p(\frac{1}{2} - \frac{1}{p}) + \varepsilon_1 + (n+1)\delta} \bigg| f \bigg|^{p} + \text{RapDec}(R) f \bigg|^{p}.
\]
Summing over $\tau$ and noting that
\[
\sum_{\tau} \bigg| f \bigg|^{p} \leq C \bigg| f \bigg|^{p}, \quad \text{for } 2 \leq p \leq \infty,
\]
we have
\[
\sum_{\tau} \int_{R^n} w_{C_{n+1}^{n+4}} e^{i t \phi_n(D)} f^\tau(x) \bigg|^{p} \, dx dt
\]
\[
\leq \varepsilon_1 K^{-2n(\frac{1}{2} - \frac{1}{p}) + 2 - \varepsilon} Q_p^p \left( \frac{R}{K^2} \right) R^{n p(\frac{1}{2} - \frac{1}{p}) + \varepsilon_1 + (n+1)\delta} \bigg| f \bigg|^{p} + \text{RapDec}(R) f \bigg|^{p}. \tag{7.8}
\]
Collecting the estimates (7.5)-(7.8) and inserting them into (7.4), we obtain
\[
\int_{C_{n+1}^{n+1}} |e^{i t \phi_n(D)} f(x)|^p \, dx dt \leq C(\varepsilon, A, K) R^{n p(\frac{1}{2} - \frac{1}{p}) + \varepsilon/2} \| f \|_{L^p}^p
\]
\[
+ C(\delta, \varepsilon_1, A) K^{\delta} R^{n p(\frac{1}{2} - \frac{1}{p}) + \varepsilon_1 + (n+1)\delta - \varepsilon(p, k, n) - \varepsilon_1} Q_p^p \left( \frac{R}{K^2} \right) \| f \|_{L^p}^p,
\]
where
\[
e(p, k, n) := \max \left\{ 2n(\frac{1}{2} - \frac{1}{p})p - 2, 2n(\frac{1}{2} - \frac{1}{p})p - 2 - (k - 3)(\frac{1}{2} - \frac{1}{p})p - 2 \right\}.
\]
Note that $e(p, k, n) \geq 0$, if
\[
p \geq \begin{cases} \frac{2(n + 1)}{2n - k + 5} & k = 2, \\ \frac{2n - k + 5}{2n - k + 3} & k \geq 3, \end{cases}
\]
therefore by the definition of $Q_p^p(R)$, we have
\[
Q_p^p(R) \leq C(\varepsilon, A, K) R^{\frac{p}{2} + n + 1} + C(\delta, \varepsilon_1, A) K^{\delta} R^{n + 1 + \delta + \varepsilon_1} K^{-\varepsilon_1} Q_p^p \left( \frac{R}{K^2} \right).
\]
Invoking the induction hypothesis, we have
\[
Q_p^p(R) \leq C(\varepsilon, A, K) R^{\frac{p}{2} + n + 1} + C \varepsilon R^2 C(\delta, \varepsilon_1, A) R^{n + 1 + \delta + \varepsilon_1} K^{\delta - \varepsilon_1 - 2\varepsilon}.
\]
The first term is harmless. If we can choose suitable $\delta, \tilde{\delta}, \varepsilon_1$ such that
\[
C(\delta, \varepsilon_1, A) R^{n + 1 + \delta + \varepsilon_1} K^{\delta - \varepsilon_1 - 2\varepsilon} < \frac{1}{2}
\]
then the induction closes. Fixing $p > \overset{\circ p(k, n)}{p}$, recall that $K = K_0 R^{\delta}$, the right hand side of (7.9) is bounded above by
\[
C(\delta, \varepsilon_1, A) R^{n + 1 + \delta + \varepsilon_1 + \delta (\delta - \varepsilon_1 - 2\varepsilon)} \tag{7.10}
\]
By choosing \( \delta = \varepsilon_1 = \frac{\varepsilon^2}{10n} \), \( \tilde{\delta} = \varepsilon \), we see that (7.10) is bounded above by
\[
C(\varepsilon, A) R^{-\varepsilon^2},
\]
which is less than \( \frac{1}{2} \) if \( R \gg 1 \), the induction closes. \( \square \)

Lastly, recall that our extension operator \( E \) is defined with respect to a cone of the form \((\xi, \phi(\xi))\) with \( \phi \in \Phi \). After reducing to the smaller class \( \Phi(R) \), one may run the argument of Ou–Wang [22] to see that our \( k \)-broad and narrow decoupling bounds imply the following generalization of Theorem 1 in [22] to such a general cone.

**Theorem 7.3.** For any \( n \geq 2 \) and
\[
p > \begin{cases} 
4 & \text{if } n = 2, \\
\frac{3n + 4}{2} & \text{if } n > 2 \text{ is even}, \\
\frac{3n + 3}{3n - 1} & \text{if } n > 2 \text{ is odd},
\end{cases}
\]
then
\[
\|Ef\|_{L^p(\mathbb{R}^{n+1})} \leq C_p \|f\|_{L^p(A(1))}.
\]

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