Connection between GRW "spontaneous collapse" and Mensky’s "restricted path integral" models

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May 26, 2013

Abstract

In this paper we will show how Mensky’s model of "restricted path integrals" can be derived from GRW "spontaneous collapse" model.

1. Introduction

According to Mensky’s "restricted path integral" model, a probability that a certain classical trajectory "takes place" is defined by the path integral that is "restricted" to some "corridor" that "surrounds" said trajectory. The width of such corridor roughly corresponds to the transition scale between quantum and classical regimes. Thus, on quantum scales, the corridor is too wide for us to know that it exists, which is why we "falsely" assume that path integral is over entire space. On the other hand, on classical scale, the corridor is so narrow that we mistake it for a single "classical" trajectory which satisfies the principle of least action (deviation from this trajectory by "classically significant" amounts would render restricted path integral close to zero).

Unfortunately, a lot of physicists view "restricted path integral" as an empirical device. Thus, they don’t specify the exact "size" of said "quantum corridor" and, instead, claim that it is situational. For example, if we are describing an electron in the lab, then the size of the corridor is the "size of the laboratory". But then the question is: where did said laboratory come from? The existence of walls of laboratory is due to the fact that its particles have been "measured". And if we appeal to this very laboratory as the entity that "measured" the molecules on its walls, our theory of its origin becomes circular. Thus, the logical way out is to invent some other measuring entity that is explicitly separate from the particles we are trying to measure (this means that it can not consist of particles, and, therefore, can not be laboratory or any kind of device).

One example of such proposal is GRW model of spontaneous collapse. According to this model, an isolated electron is still being subject to "mini-measurements" (known as "hits").
Such hits, however, only modify the wave function slightly, and such change is not noticeable. In order to actually obtain $\delta$-function one needs a very large number of such "hits". It turns out that in the case of entanglement it is easier to obtain $\delta$-function with fewer hits since each electron is being affected by the "hits" that act on all the other electrons in an entangled state. But, be it as it may, such hits do not originate in surrounding matter, which allows us to claim that matter is being "produced" by means of such "hits" without facing circular argument.

In this paper we will explicitly show how GRW model leads to "restricted path integral" on the time scale of large number of such "hits". However, the size of "quantum corridor" is no longer "the size of laboratory" but instead it is strictly a function of parameters used in GRW model.

**Review of GRW model**

According to GRW model, wave function evolves according to Schrodinger’s equation, and this evolution is interrupted by discrete changes of wave function known as "hits" at times $\{\cdots, t_{-1}, t_0, t_1, \cdots\}$. In particular,

$$i\frac{\partial \psi}{\partial t} = V\psi - \frac{1}{2m}\nabla^2 \psi, \ t \notin \{\cdots, t_{-1}, t_0, t_1, \cdots\}$$  \hspace{1cm} (1)

$$\psi(t_n^+, \vec{x}) = N(\psi, \vec{x}_n)e^{-\frac{\alpha}{2}||\vec{x} - \vec{x}_n||^2}\psi(t_n^-, \vec{x})$$  \hspace{1cm} (2)

One good thing about Gaussians is that their product is also a Gaussian. Even if said Gaussians are centered around different points (possibly far away from each other), still their product would have well defined center:

$$\prod_{k=m+1}^{n} e^{-\frac{\alpha}{2}||\vec{x} - \vec{x}_k||^2} = \exp \left( -\frac{\alpha}{2} \sum_{k=m+1}^{n} ||\vec{k} - \vec{x}_k||^2 \right) = e^{-\frac{\alpha}{2}(n-m)||\vec{x} - \langle \vec{x} \rangle_{mn}||^2}$$  \hspace{1cm} (3)

where

$$\langle \vec{x} \rangle_{mn} = \frac{1}{n-m} \sum_{k=m+1}^{n} \vec{x}_k$$  \hspace{1cm} (4)

The Gaussian on the right hand side of Eq 3 is a lot narrower than the Gaussians on the left hand side that we were taking the product of. This is what accounts for the measurement effect. What "interrupts" this process is quantum evolution *between* said "hits": if Schrodinger’s evolution makes the wave function "spread out" by the same amount as (or more than) the "hits" make it "narrow", then the effects of "hits" would never accumulate and the "narrow peak" would never occur. On the other hand, if somehow the Schrodinger’s evolution can "further enhance" the effects of hits then, in fact, localization will occur. The latter takes place during entanglement: each given electron is being "localized" in some ways by the "hits" affecting every *other* electron it is being entangled *with*, which makes the effects of hits a lot stronger than it would have been if an electron was by itself. Thus, GRW model agrees with other models that entanglement leads to collapse of wave function.
Now, if the collapse is to occur, we would like it to obey Born’s rule. If we ”approximate” the position of resulting ”point” up to the size of the peak, the statement ”Born’s rule will hold once pick is formed” happens to be correct, albeit rather silly. Therefore, our only task is to show that the two ”induction steps” have to work for the sequence of ”hits” taking place at times \(\{\ldots, t_{-1}, t_0, t_1, \ldots\}\):

a) If Born’s rule holds at time \(t^+_k\), it should also hold at time \(t^-_k\)

b) If Born’s rule holds at \(t^-_k\) it should also hold at \(t^+_{k-1}\)

As far as part ”a” is concerned, it strictly depends on us selecting the ”right” probability distribution of the choice of \(\vec{x}_k\). If we ”know” that Born rule holds at \(t^+_k\), and the probability distribution of selection of \(\vec{x}_k\) is \(\sigma_k\), then the probability distribution of ”eventual outcome” of localization based on information available at \(t^-_k\) is

\[
\rho(\psi, t^-_k) = \left| \psi(\vec{x}, t^-_k) \right|^2 \int d^3x_k \sigma_k(\vec{x}_k) N(\vec{x}_k, \psi) e^{-\frac{\alpha}{2} |\vec{x}_k - \vec{\xi}|^2} \psi(\vec{x}, t^-_k) \right|^2 \tag{5}
\]

where \(N(\vec{x}_k, \psi)\) is a normalization factor in Eq [2]. In other words, we are taking into account the impact of all possible locations of a ”hit”, \(\vec{x}_k\), on the probability that afore-given point \(\vec{x}\) wins at the end. Since we are integrating over \(\vec{x}_k\) rather than \(\vec{x}\), we can pull \(\left| \psi(\vec{x}) \right|^2\) out of integral and obtain

\[
\rho(\psi, t^-_k) = \left| \psi(\vec{x}, t^-_k) \right|^2 \int d^3x_k \sigma(\vec{x}_k) N^2(\vec{x}_k, \psi) e^{-\alpha |\vec{x}_k - \vec{x}|^2} \tag{6}
\]

This matches Born’s rule if \(\sigma\) is set to

\[
\sigma(\vec{x}_k) = \left( \frac{\alpha}{\pi} \right)^{3N/2} N^{-2}(\vec{x}_k, \psi) \tag{7}
\]

This can be understood from the point of view of ”gambler’s ruin” suggested by Pearl (see [1] and [2]). Suppose two players are throwing a coin and if it lands on one side then player 1 passes 50 cents to player 2, and if it lands on the other side then player 2 passes 50 cents to player 1; whoever runs out of money loses. It can be easily proven that the probability of win is proportional to the initial amount of money. The same will be true if the coin is biased, provided that the amount of money passed is biased in the opposite direction. For example, if a coin has \(2/3\) chance to land in favor of player 1, then player 2 should pass to player 1 ”only” 33 cents whenever this happens, while player 1 should pass 66 cents to player 2 when ”less likely” outcome occurs. This adjustment is necessary in order for the probability of win to continue to be proportional to initial amount of money the players have. Now, \(\sigma(\vec{x}, \psi)\) is analogous to biased coin, while \(N^2(\vec{x}, \psi)\) is analogous to the amount of money being passed. That is why they are inversely proportional to each other.

As far as ”part b” is concerned we can, again, use ”gambler’s ruin” argument while pointing out that Schrodinger’s evolution preserves normalization naturally without any need of normalization constant. This time, the ”random coin” comes from the fact that the specific structure of the molecules of the screen – which ”in principle” can be computed – is simply ”too complicated” for us to compute. Thus, despite the fact that Schrodinger's
evolution is deterministic, we can treat it as "random coin" in a "classical" sense. Now, the wave function outside the screen is not "complicated". That is why if the duration of the measurement is long enough for electron to fly around (that is, the number of hits necessary for measurement to occur is so large that we have to "wait" for a long time for them to occur) then we can not use a simple Born's rule and, instead, we have to take into account the Schrodinger's evolution of wave function outside the screen throughout this time, perhaps with some modifications due to co-occurring "measurement". But, if the measurement occurs very fast, then we can be sure that the only part of the wave function that is the most relevant is the one of the molecules of the screen and that is where "random coin" would come from. Furthermore, our "knowledge" of wave function outside of screen "forces us" to "admit" the non-trivial \( N(\psi, \vec{x}) \) (which can be easily computed) and thus we "have to" introduce \( \sigma(\psi, \vec{x}) \) to "cancel" it. On the other hand, our "lack of knowledge" of internal structure of the screen allows us to "get away" without having to introduce another "\( \sigma \)" for the latter case (which we wouldn't have been able to do even if we wanted to).

Finally, one might ask why should we introduce "extra" random coin of "part a" if we "already have" the random coin of part b, regardless? The answer is that none of the "random coins" would result in one single point winning. Now, the random coin in part a has a property that the "winning points" will end up being close to each other (within narrow Gaussian) while random coin in part b doesn't have that property. So if we only had the latter, we might end up having "winners" spread throughout the space instead of being localized.

Emergence of Mensky from GRW

As we have discussed earlier, in GRW model there is a bias in the selection of \( \vec{x}_k \), and that bias is based on the specifics of wave function. Furthermore, in a special case where wave function forms a single peak, the subsequent \( \vec{x}_k \) is almost certain to be found within that peak. This allows us to re-interpret \( \vec{x}_k \) as the actual location of a point-like particle (or an ensamble of particles in case of configuration space). One unconventional thing this would imply is that "point particle" is localized even if the wave function is not. In the latter case, however, \( \vec{x}_k \) and \( \vec{x}_{k+1} \) might be "far away" from each other; this means that the location of the "particle" is inconsequential since it can jump arbitrary far. On the other hand, when wave function itself is localized, then the "particle" will almost certainly localized within the region of localization of wave function, which would make \( \vec{x}_k \) and \( \vec{x}_{k+1} \) close to each other and, therefore, of interest. Another unconventional thing that needs to be noticed is that said "particle" only exists at times \( \{ \cdots, t_{-1}, t_0, t_1, \cdots \} \). In other words, the trajectory of the particle is discrete, while the evolution of the wave function is continuous.

With this interpretation in mind, we can ask ourselves what is a probability of the entire trajectory that a particle takes (as opposed to the probability of any given location). Suppose we know the wave function at \( t^+_M \), and we would like to ask ourselves what is the probability that we will "hit" the "afore-given" values of \( \{ \vec{x}_{M+1}, \cdots, \vec{x}_N \} \). Suppose we have already
By substituting Eq 9 into Eq 10 we obtain
\[ f_k(\vec{x}_k) = \left( \frac{\alpha}{\pi} \right)^{3/2} \int d^3 x' e^{-\alpha|\vec{x}' - \vec{x}_k|^2} |\psi(\vec{x}', t^-)|^2 \] (8)

As we recall, \( \psi \) evolved according to Equations 1 and 2. If we now rewrite Eq 1 in terms of path integral, we obtain
\[ \psi(\vec{x}, t_{j+1}^-) = A_{j+1} \int_{\vec{x}'(t_{j+1}) = \vec{x}} [Dx'] \psi(\vec{x}', t_{j+1}^0) \exp \left( -\frac{\alpha}{2} |\vec{x} - \vec{x}_{j+1}|^2 + i \int_{t_j}^{t_{j+1}} dt L(\vec{x}', t') \right) \] (9)

By substituting Eq 9 into Eq 10 we obtain
\[ \psi(\vec{x}, t_{j+1}^+) = A_{j+1} B_{j+1} \int_{\vec{x}'(t_{j+1}) = \vec{x}} [Dx'] \psi(\vec{x}', t_{j+1}^0) \exp \left( -\frac{\alpha}{2} |\vec{x} - \vec{x}_{j+1}|^2 + i \int_{t_j}^{t_{j+1}} dt L(\vec{x}', t') \right) \] (11)

By substituting \( \psi(\vec{x}, t_{j+1}^+) \) into \( \psi(\vec{x}, t_{j+2}^+) \) we can now compute \( \psi(\vec{x}, t_{j+2}^+) \) as follows:
\[ \psi(\vec{x}, t_{j+2}^+) = A_{j+2} B_{j+2} \int_{\vec{x}'(t_{j+2}) = \vec{x}} [Dx'] \psi(\vec{x}', t_{j+1}^0) \exp \left( -\frac{\alpha}{2} |\vec{x} - \vec{x}_{j+2}|^2 + i \int_{t_j}^{t_{j+2}} dt L(\vec{x}', t') \right) = A_{j+1} A_{j+2} B_{j+1} B_{j+2} \int_{\vec{x}'(t_{j-2}) = \vec{x}} [Dx'] \left[ \exp \left( -\frac{\alpha}{2} |\vec{x} - \vec{x}_{j+2}|^2 + i \int_{t_j}^{t_{j+2}} dt L(\vec{x}', t') \right) \right. \]
\[ \times \left. \int_{\vec{x}'(t_{j+1}) = \vec{x}'(t_{j+1})} [Dx'''] \psi(\vec{x}''', t_{j+1}^0) \exp \left( -\frac{\alpha}{2} |\vec{x}'(t_{j+1}) - \vec{x}_{j+1}|^2 + i \int_{t_j}^{t_{j+1}} dt L(\vec{x}'', t') \right) \right] \] (12)

In the above expression, \( \vec{x}' \) runs through \( t_{j+1} \leq t < t_{j+2} \) and \( \vec{x}''' \) runs through \( t_j \leq t < t_{j+1} \). We can therefore combine them into a single \( \vec{x}' \), running through \( t_j \leq t < t_{j+2} \). In this case the condition of the second integral, \( \vec{x}'''(t_{j+1}) = \vec{x}' \), can be dropped, since it is a simple consequence of continuity of \( \vec{x}' \). The two integrals combine into one to produce
\[ \psi(\vec{x}, t_{j+2}^+) = A_{j+1} A_{j+2} B_{j+1} B_{j+2} \times \] (13)

By induction, one can see that
\[ \psi(\vec{x}, t_k^-) = \left( \prod_{i=M+1}^{k} A_i \right) \left( \prod_{j=M+1}^{k-1} B_j \right) \times \] (14)
\[ \times \int_{\vec{x}'(t_k) = \vec{x}} [Dx'] \psi(\vec{x}', t_{M+1}^0) \exp \left( -\frac{\alpha}{2} \sum_{j=M+1}^{k-1} |\vec{x} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' L(\vec{x}', t') \right) \]
Since $A_j$'s and $B_j$'s were selected in such a way that $\psi$ was normalized properly at any given step, we know by induction that their product will produce correct normalization at the end. This allows us to "know" the product in question without explicitly computing each of its ingredients. Therefore, we obtain

$$
\psi_{M,k-1}(\vec{x}, t) =
$$

$$
= \frac{\int_{\mathcal{X}(t_k) = \vec{x}} [Dx' \psi(\vec{x}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{x} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{x}', t') \right) - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{y} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{y}, t') \right) \right)^2}{\int d^3y \int_{\mathcal{Y}(t_k) = \vec{y}} [Dy' \psi(\vec{y}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{y} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{y}', t') \right) \right)^2}
$$

By substituting this into Eq ?????, we find that

$$
\left( \frac{\pi}{\alpha} \right)^{3/2} f_k(\vec{x}_k) =
$$

$$
= \frac{\int d^3x e^{-\alpha|\vec{x} - \vec{x}_k|^2} \left| \int_{\mathcal{X}(t_k) = \vec{x}} [Dx' \psi(\vec{x}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{x} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{x}', t') \right) \right|^2}{\int d^3y \left| \int_{\mathcal{Y}(t_k) = \vec{y}} [Dy' \psi(\vec{y}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{y} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{y}', t') \right) \right|^2}
$$

By absorbing $e^{-\alpha|\vec{x} - \vec{x}_k|^2}$ into the summation of $|\vec{x} - \vec{x}_j|^2$, we obtain

$$
\left( \frac{\pi}{\alpha} \right)^{3/2} f_k(\vec{x}_k) =
$$

$$
= \frac{\int d^3x \left| \int_{\mathcal{X}(t_k) = \vec{x}} [Dx' \psi(\vec{x}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k} |\vec{x} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{x}', t') \right) \right|^2}{\int d^3y \left| \int_{\mathcal{Y}(t_k) = \vec{y}} [Dy' \psi(\vec{y}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{y} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{y}', t') \right) \right|^2}
$$

Now, from Eq ????? we know that

$$
\psi(\vec{y}, t_k^+) = A_k \int_{\mathcal{X}(t_k^+) = \vec{x}} [Dx' \psi(\vec{x}', t_k^+) \exp \left( i \int_{t_{k-1}}^{t_k} dt' \mathcal{L}(\vec{y}', t') \right) \right)
$$

This implies that if we are going to replace "integral from $t_M$ to $t_k^+$" with "integral from $t_M$ to $t_{k-1}$" in the denominator of Eq ????? we will change the ratio by the factor of $A_k$:

$$
A_k \left( \frac{\pi}{\alpha} \right)^{3/2} f_k(\vec{x}_k) =
$$

$$
= \frac{\int d^3x \left| \int_{\mathcal{X}(t_k) = \vec{x}} [Dx' \psi(\vec{x}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k} |\vec{x} - \vec{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\vec{x}', t') \right) \right|^2}{\int d^3y \left| \int_{\mathcal{Y}(t_k) = \vec{y}} [Dy' \psi(\vec{y}', t_M') \exp \left( - \frac{a}{2} \sum_{j=M+1}^{k-1} |\vec{y} - \vec{x}_j|^2 + i \int_{t_M}^{t_{k-1}} dt' \mathcal{L}(\vec{y}', t') \right) \right|^2}
$$
We can rewrite it as

\[ A_k \left( \frac{\pi}{\alpha} \right)^{3/2} f_k(\bar{x}_k) = \frac{I_{M,k}}{I_{M,k-1}} \quad (20) \]

where

\[ I_{M,k} = \int d^3 \bar{x} \left| \int_{\bar{x}'(t_k) = \bar{x}} [D \bar{x}'] \psi(\bar{x}', t_M^+) \exp \left( -\frac{\alpha}{2} \sum_{j=M+1}^{k} |\bar{x} - \bar{x}_j|^2 + i \int_{t_M}^{t_k} dt' \mathcal{L}(\bar{x}', t') \right) \right|^2 \quad (21) \]

Now that we have computed the probability of any individual hit, we can go back to our original goal of computing the probability of sequence of hits, \((\bar{x}_{M+1}, t_{M+1}), \cdots, (\bar{x}_N, t_N)\).

The latter is

\[ f_{M+1}(\bar{x}_{M+1}), \cdots, f_N(\bar{x}_N) = \left( \prod_{k=M+1}^{N} A_k \right) \left( \frac{\alpha}{\pi} \right)^{3/2(N-M)} I_{M,M+1} \cdots I_{M,N} = \left( \prod_{k=M+1}^{N} A_k \right) \left( \frac{\alpha}{\pi} \right)^{3/2(N-M)} I_{M,M} \quad (22) \]

Now, it is easy to see that in computing \(I_M\) the expression under exponent is zero. This means that the normalization of \(\psi\) at \(t_M\) implies that

\[ I_{M,M} = 1 \quad (23) \]

Therefore, we can rewrite the above as

\[ f_{M+1}(\bar{x}_{M+1}), \cdots, f_N(\bar{x}_N) = \left( \prod_{k=M+1}^{N} A_k \right) \left( \frac{\alpha}{\pi} \right)^{3/2(N-M)} I_{M,M+1} \cdots I_{M,N} = \left( \prod_{k=M+1}^{N} A_k \right) \left( \frac{\alpha}{\pi} \right)^{3/2(N-M)} \times \]

\[ \times \int d^3 \bar{x} \left| \int_{\bar{x}'(t_k) = \bar{x}} [D \bar{x}'] \psi(\bar{x}', t_M^+) \exp \left( -\frac{\alpha}{2} \sum_{j=M+1}^{N} |\bar{x} - \bar{x}_j|^2 + i \int_{t_M}^{t_N} dt' \mathcal{L}(\bar{x}', t') \right) \right|^2 \quad (24) \]

Thus, we have a regular path integral with a ”weight factor”

\[ w(x) = \exp \left( -\frac{\alpha}{2} \sum_{j=M+1}^{N} |\bar{k} - \bar{x}_j|^2 \right) \quad (25) \]

Now, if we make \(N - M\) larger and larger but, at the same time, change time scale so that on our ”revised” scale the time interval \(t_N - t_M\) is the same, then we obtain

\[ w(x) \longrightarrow_{N-M \to \infty, t_N-t_M = \text{const}} \exp \left( -\frac{\alpha}{2} \int |\bar{x}(t) - \bar{x}_d(t)|^2 dt \right) \quad (26) \]

provided that either the selection of ”hits” happens to look continuous on a large scale (which is highly unlikely) or we have some notion of integral of discontinuous curve. Be it as it may, at least intuitively it ”looks” like Mensky’s path integral where instead of corridor with ”sharp” boundaries we have ”smoothed out” ”Gaussian” corridor. Still, the ”width” of ”smoothed out” corridor can be claimed to be something of the order of \(1/\sqrt{\alpha}.\)
Conclusion

We have shown that Mensky’s quantum corridors can be derived as an emergent outcome of GRW model on a time scale in which several different ”hits” occur. This makes intuitive sense. If we take ”very large” time scale, it would look like hits are ”spaced very closely” to each other, thus approximating smooth ”corridor”. We had to rethink, however, the meaning of trajectory. In particular, the trajectory now corresponds to the sequence of locations of the ”hits” (which was earlier viewed as merely sources of modification of wave function). This makes sense: since we are on a time scale on which ”hits” are ”very close” to each other, their locations do, in fact, ”merge” into the curve. The curve, however, is arbitrary discontinuous since the location of the subsequent hit is independent of the location of previous one. Be it as it may, the fact that the ”size” of the corridor is approximately the scale of quantum-classical transition, it is ”small” as far as our ”classical” observations are concerned. Thus, classically, we are not able to ”see” all this ”jumping around”; instead, the entire ”corridor” looks like a single curve, and this sequence of hits ”trace out” this curve.

References

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