WELL-POSEDNESS OF RENORMALIZED SOLUTIONS FOR A STOCHASTIC $p$-LAPLACE EQUATION WITH $L^1$-INITIAL DATA

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Abstract. We consider a $p$-Laplace evolution problem with stochastic forcing on a bounded domain $D \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions for $1 < p < \infty$. The additive noise term is given by a stochastic integral in the sense of Itô. The technical difficulties arise from the merely integrable random initial data $u_0$ under consideration. Due to the poor regularity of the initial data, estimates in $W^{1,p}_0(D)$ are available with respect to truncations of the solution only and therefore well-posedness results have to be formulated in the sense of generalized solutions. We extend the notion of renormalized solution for this type of SPDEs, show well-posedness in this setting and study the Markov properties of solutions.

1. Introduction.

1.1. Motivation of the study. We are interested in the study of well-posedness for a $p$-Laplace evolution problem with stochastic forcing on a bounded domain $D \subset \mathbb{R}^d$ with homogeneous Dirichlet boundary conditions for $1 < p < \infty$. For $p = 2$, we are in the case of the classical Laplace operator, for arbitrary $1 < p < \infty$, $u \mapsto -\text{div}(|\nabla u|^{p-2}\nabla u)$ is a monotone operator on the Sobolev space $W^{1,p}_0(D)$ that is singular for $p < 2$ and degenerate for $p > 2$. Evolution equations of $p$-Laplace type may appear as continuity equations in the study of gases flowing in pipes of uniform cross sectional areas and in models of filtration of an incompressible fluid through a porous medium (see [3, 14]): In the case of a turbulent regime, a nonlinear version of the Darcy law of $p$-power law type for $1 < p < 2$ is more appropriate (see [14]). Turbulence is often associated with the presence of randomness (see [10] and the references therein). Adding random influences to the model, we also take uncertainties and multiscale interactions into account. Randomness may be introduced as random external force by adding an Itô integral on the right-hand side of the equation and by considering random initial values. Consequently, the equation becomes a stochastic partial differential equation (SPDE) and the solution is then a stochastic process.

For square-integrable initial data $u_0$, the stochastic $p$-Laplace evolution problem can be solved with classical methods for nonlinear, monotone SPDEs (see, e.g.
[26], [24]). Existence and regularity results for stochastic p-Laplacian-type systems have been proposed in [9]. In our contribution, we focus on more general, merely integrable random initial data. The well-posedness of quasilinear, degenerate hyperbolic-parabolic SPDEs with \( L^1 \) random initial data has already been addressed [18] in the framework of kinetic solutions, but, to the best of our knowledge, these results do not apply in our situation. On the other hand, there has been an extensive study on nonlinear evolution PDEs with initial data and right-hand side in \( L^1 \) (see, e.g., [7, 6, 7]). From these results it is well known that the deterministic p-Laplace evolution problem with \( L^1 \)-data is not well-posed in the variational setting for \( 1 < p < d \), where \( d \) is the space dimension. For this reason, the problem is formulated in the framework of renormalized solutions. The notion of renormalization summarizes different strategies to get rid of infinities (see [13]) that may appear in physical models. The notion of renormalized solutions has been introduced to partial differential equations by Di Perna and Lions in the study of Boltzmann equation (see [15]) and then extended to many elliptic and parabolic problems (see, e.g., [16], [4, 7, 5] and the references therein). The main idea is to make a nonlinear change of unknown \( v = S(u) \) in the equation, where \( S \) is chosen in order to remove infinite quantities of the solution \( u \). For SPDEs, this concept has been applied for stochastic transport equations in [1, 11] and for the Boltzmann equation with stochastic kinetic transport in [27]. For many physically relevant singular SPDEs, a slightly different notion of renormalization has recently been developed (see [19, 20] and the references therein). For these cases, solutions may be obtained as limits of classical solutions to regularized problems with addition of diverging correction terms. These counterterms arise from a renormalization group which is defined in terms of an associated regularity structure.

In this contribution, it is our aim to extend the notion of renormalized solutions in the sense of [7] for the stochastic p-Laplace evolution problem with random initial data in \( L^1 \) and to show well-posedness in this framework.

1.2. Statement of the problem and results. Let \((\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t \in [0,T]}, (\beta_t)_{t \in [0,T]})\) be a stochastic basis with a complete, countably generated probability space \((\Omega, \mathcal{F}, P)\), a filtration \((\mathcal{F}_t)_{t \in [0,T]} \subset \mathcal{F}\) satisfying the usual assumptions and a real valued, \( \mathcal{F}_0 \)-Brownian motion \((\beta_t)_{t \in [0,T]}\). Let \( D \subset \mathbb{R}^d \) be a bounded Lipschitz domain, \( T > 0 \), \( Q_T = (0, T) \times D \) and \( p > 1 \). Furthermore, let \( u_0 : \Omega \rightarrow L^1(D) \) be \( \mathcal{F}_0 \)-measurable and \( \Phi \in L^2(\Omega \times Q_T) \) be progressively measurable.

We are interested in well-posedness to the following stochastic p-Laplace evolution problem

\[
\begin{align*}
\frac{du}{dt} - \text{div}(|\nabla u|^{p-2} \nabla u) &= \Phi \, d\beta & \text{in } \Omega \times Q_T, \\
u &= 0 & \text{on } \Omega \times (0, T) \times \partial D, \\
u(0, \cdot) &= u_0 & \in L^1(\Omega \times D).
\end{align*}
\]

Due to the poor regularity of the initial data \( u_0 \), a-priori estimates on \( \nabla u \) are not available and therefore the well-posedness result has to be formulated in the sense of a generalized solution, more precisely in the framework of renormalized solutions.

In Section 2, we show the existence of a unique strong solution to (1) in the case where the initial value \( u_0 \) is an element of \( L^2(\Omega \times D) \). After that, we establish a contraction principle that shows that a sequence of strong solutions is a Cauchy sequence in \( L^1(\Omega; C([0,T]; L^1(D))) \) whenever the sequence of initial values is a Cauchy sequence in \( L^1(\Omega \times D) \). In Section 4 we prove a version of the Itô formula which
makes it possible to define renormalized solutions to equation (1). Section 5 contains the definition of renormalized solutions to (1), in Section 6 we show the existence of such a solution and Section 7 contains the uniqueness result, which is a consequence of the $L^1$-contraction principle for renormalized solutions stated in Theorem 7.1. Another consequence of the contraction principle is the continuous dependence of the renormalized solution on the initial value in $L^1$ which is necessary for the proof of the Markov property. In the Hilbert space setting, these arguments are classical nowadays, see, e.g., [24]. In Section 8 we give a brief overview of the arguments in the $L^1$ setting and study regularity properties of the associated semigroup.

For the sake of clarity of the presentation, we restrict ourselves to the case of a real-valued Brownian motion. However, it is straightforward to extend our results for additive stochastic perturbations by a stochastic integral in the sense of Itô with respect to a cylindrical Wiener process in $L^2(D)$. In this case, the vector space of Hilbert-Schmidt Operators from $L^2(D)$ to $\mathbb{R}$ can be identified with $L^2(D)$. For multiplicative stochastic perturbations, a more general version of Proposition 9.1 is necessary and the study of this case will be subject to a forthcoming work.

2. Strong solutions.

Theorem 2.1. Let the conditions in the introduction be satisfied. Furthermore, let $u_0 \in L^2(\Omega \times D)$ be $\mathcal{F}_0$-measurable. Then there exists a unique strong solution to (1), i.e., an $\mathcal{F}_t$-adapted stochastic process $u : \Omega \times [0,T] \to L^2(D)$ such that $u \in L^p(\Omega; L^p(0,T; W_0^{1,p}(D))) \cap L^2(\Omega; C([0,T]; L^2(D)))$, $u(0,\cdot) = u_0$ in $L^2(\Omega \times D)$ and

$$u(t) - u_0 - \int_0^t \text{div} ([\nabla u|^{p-2}\nabla u]) \, ds = \int_0^t \Phi \, d\beta$$

in $W^{-1,p'}(D) + L^2(D)$ for all $t \in [0,T]$ and a.s. in $\Omega$.

Remark 2.2. Since we know from all terms except the term $\int_0^t \text{div} ([\nabla u|^{p-2}\nabla u]) \, ds$ that these terms are elements of $L^2(D)$ for all $t \in [0,T]$ and a.s. in $\Omega$ it follows that $\int_0^t \text{div} ([\nabla u|^{p-2}\nabla u]) \, ds \in L^2(D)$ for all $t \in [0,T]$ and a.s. in $\Omega$. Therefore this equation is an equation in $L^2(D)$.

Proof. The existence result is a consequence of [23], Chapter II, Theorem 2.1 and Corollary 2.1. Following the notations therein, we set $E = \mathbb{R}$, $H = L^2(D)$, $V = W_0^{1,p}(D) \cap L^2(D)$ for $1 < p < 2$ and $V = W_0^{1,p}(D)$ for $p \geq 2$. In particular, $V \hookrightarrow H \hookrightarrow V'$ with continuous embedding in accordance with [23], p.1251 conditions $a.)$-$d.)$. Moreover, we have $A : V \to V^*$, $A(u) = -\text{div} ([\nabla u|^{p-2}\nabla u])$, $B = \Phi$, $f(t,\omega) = 2 + ||B(t,\omega)||_2^2$ for a.e. $(t,\omega) \in (0,T) \times \Omega$, $z = 0$ and $L_Q(E; H) = L_2(\mathbb{R}, L^2(D)) \cong L^2(D)$. We remark that $A$ does not depend on $(t,\omega) \in [0,T] \times \Omega$ and that $B$ does not depend on $u \in V$. Therefore it follows immediately that conditions (A1)-(A5) of [23] are satisfied for all $1 < p < \infty$. The uniqueness is a consequence of [23], Chapter II, Theorem 3.1, which applies under the same assumptions.

3. Contraction principle.

Theorem 3.1. Let $u_0, v_0 \in L^2(\Omega \times D)$ and $u$ and $v$ strong solutions to the problem (1) with initial value $u_0$ and $v_0$, respectively. Then

$$\sup_{t \in [0,T]} \int_D |u(t) - v(t)| \, dx \leq \int_D |u_0 - v_0| \, dx$$
a.s. in $\Omega$.

Proof. We subtract the equations for $u$ and $v$ and we get

$$u(t) - v(t) - (u_0 - v_0) - \int_0^t \text{div}(|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \, ds = 0$$

for all $t \in [0,T]$ and a.s. in $\Omega$. We apply the Itô formula pointwise a.s. with respect to $x \in D$ with an approximation of the absolute value $N_3$ in (2) which is defined as follows (see, e.g., Proposition 5 in [28]): For $\delta > 0$, let $\rho(r) = c \frac{r}{\delta} - r^2 + \frac{1}{\delta^2} 1_{\{|r| \leq 1\}}$ such that $\int_\mathbb{R} \rho(s) \, ds = 1$ and $\rho_\delta(r) = \frac{1}{\delta} \rho(\frac{r}{\delta})$ be the classical symmetric mollifier sequence approximating the Dirac mass with support $[-\delta,\delta]$. Then, $\eta_\delta(r) = 2 \int_0^r \rho_\delta(s) \, ds$ is a regular nondecreasing Lipschitz approximation of the sign function with Lipschitz constant $\frac{2}{\delta}$. Now, $N_3(r) := \int_0^r \eta_\delta(s) \, ds$. Discarding the nonnegative term coming from the $p$-Laplace and passing to the limit with $\delta \to 0^+$ yields

$$\int_D |u(t) - v(t)| \, dx - \int_D |u_0 - v_0| \, dx \leq 0$$

for all $t \in [0,T]$ and a.s. in $\Omega$. \qed

4. Itô formula and renormalization. In order to find an appropriate notion of renormalized solutions to (1), we prove an Itô formula in the $L^1$-framework. We remark that the combined Itô chain and product rule from [10], Appendix A4 does not apply to our situation for two reasons. Firstly, we take the bounded domain $D \subset \mathbb{R}^d$ into account in our regularizing procedure by adding a cutoff function (see Appendix, Subsection 9.1). Secondly, the spacial regularities are different in our case.

For two Banach spaces $X$, $Y$ let $L(X;Y)$ denote the Banach space of bounded, linear operators from $X$ to $Y$ and $L(X)$ denote the space of bounded linear operators from $X$ to $X$ respectively.

For the sake of completeness, we recall the following regularization procedure which has been introduced similarly by Fellah and Pardoux in [17]:

**Lemma 4.1.** Let $D \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary, $1 \leq p < \infty$. There exists a sequence of operators

$$\Pi_n : W^{-1,p}(D) + L^1(D) \to W^{-1,p}(D) \cap L^\infty(D), \quad n \in \mathbb{N}$$

such that

i.) $\Pi_n(v) \in W^{1,p}(D) \cap C^\infty(D)$ for all $v \in W^{-1,p}(D) + L^1(D)$ and all $n \in \mathbb{N}$

ii.) For any $n \in \mathbb{N}$ and any Banach space $F \in \{W^{1,p}_0(D), L^2(D), L^1(D), W^{-1,p}(D), W^{-1,p}(D) + L^1(D)\}$

$$\Pi_n : F \to F \text{ is a bounded linear operator such that } \lim_{n \to \infty} \Pi_n|_F = I_F$$

pointwise in $F$, where $I_F$ is the identity on $F$.

Proof. See Appendix, Subsection 9.1. \qed

In the following, we refer to [2], p.33 for a definition of the progressively measurable functions on $\Omega \times (0,T)$. For $1 \leq p < \infty$, with a slight abuse of notation we will call a function $f \in L^p(\Omega \times Q_T)$ progressively measurable if it belongs to the set of $dP \otimes dt$-equivalence classes of progressively measurable functions with finite $L^p$-norm.
Proposition 4.2. Let $G \in L^p(\Omega \times Q_T)^d$, $g \in L^2(\Omega \times Q_T)$, $f \in L^1(\Omega \times Q_T)$ be progressively measurable, $u_0 \in L^1(\Omega \times D)$ be $\mathcal{F}_0$-measurable.

Let $u \in L^1(\Omega; C([0,T]; L^p(D))) \cap L^p(\Omega; L^p(0,T; W_0^{1,p}(D)))$ satisfy the equality

$$u(t) - u_0 + \int_0^t (-\text{div} \, G + f) \, ds = \int_0^t g \, d\beta$$

(3) in $L^1(D)$ for all $t \in [0,T]$ and a.s. in $\Omega$.

Then, for all $\psi \in C^\infty([0,T] \times \overline{D})$ and all $S \in W^{2,\infty}(\mathbb{R})$ with $S''$ piecewise continuous such that $S'(0) = 0$ or $\psi(t,x) = 0$ for all $(t,x) \in [0,T] \times \partial D$, we have

$$(S(u(t)), \psi(t))_2 - (S(u_0), \psi(0))_2 + \int_0^t (-\text{div} \, G + f, S'(u) \psi) \, ds$$

$$= \int_0^t (S'(u)g, \psi)_2 \, d\beta + \int_0^t (S(u), \psi_t)_2 \, ds + \frac{1}{2} \int_0^t \int_D S''(u)g^2 \psi \, dx \, ds$$

(4) for all $t \in [0,T]$ and a.s. in $\Omega$, where

$$(-\text{div} \, G + f, S'(u)\psi) = (-\text{div} \, G + f, S'(u)\psi)_{W^{-1,p'}(D), W_0^{1,p}(D) \cap L^\infty(D)}$$

$$= \int_D (G \cdot \nabla S'(u)\psi + fS'(u)\psi) \, dx$$

a.s. in $\Omega \times (0,T)$. In particular, for $\psi \in C^\infty(\overline{D})$ not depending on $t \in [0,T]$ we get

$$\int_D (S(u(t)) - S(u_0))\psi \, dx + \int_0^t \int_D G \cdot \nabla S'(u)\psi \, dx \, ds + \int_0^t \int_D fS'(u)\psi \, dx \, ds$$

$$= \int_0^t \int_D S'(u)g \, dx \, d\beta + \frac{1}{2} \int_0^t \int_D S''(u)g^2 \, dx \, ds$$

(5) for all $t \in [0,T]$ and a.s. in $\Omega$.

Remark 4.3. Since the right-hand side of (3) is in $L^2(D)$ for all $t \in [0,T]$, even if the members on the left-hand are not in $L^2(D)$, (3) holds also in $L^2(D)$.

Proof. Let us assume $S \in C^2(\mathbb{R})$ such that $S'$, $S''$ is bounded, the general result then follows by an approximation argument (see Corollary 9.2 in the Appendix).

We choose the regularizing sequence $(\Pi_n)$ according to Lemma 4.1 and set $u_n := \Pi_n(u)$, $u_0^n := \Pi_n(u_0)$, $G_n := \Pi_n(-\text{div} \, G)$, $f_n := \Pi_n(f)$ and $g_n := \Pi_n(g)$. We apply the operator $\Pi_n$ to both sides of (3). Since $\Pi_n \in L(W^{-1,p'}(D) + L^1(D); W_0^{1,p}(D) \cap L^\infty(D))$, we may conclude

$$u_n(t) - u_0^n + \int_0^t G_n + f_n \, ds = \int_0^t g_n \, d\beta$$

in $D$, for all $t \in [0,T]$ and a.s. in $\Omega$. For $x \in D$ fixed, we apply the classic Itô formula for $h(t,r) := S(r)\psi(t,x)$ with respect to the time variable $t$. Integration over $D$ afterwards and Fubini Theorem yield

$$I_1 + I_2 + I_3 = I_4 + I_5 + \frac{1}{2}I_6,$$

where

$$I_1 = \int_D S(u_n(t))\psi(t) - S(u_0^n)\psi(0) \, dx,$$

$$I_2 = \int_0^t \langle G_n, S'(u_n)\psi \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} \, ds,$$
For any $t \in [0, T]$ and a.s. in $\Omega$. Now, we want to pass to the limit with $n \to \infty$ in $I_1 - I_6$. Since $u^n_0 \to u_0$ and $u_n(t) \to u(t)$ in $L^1(D)$ a.s. in $\Omega$ for any $t \in [0, T]$,

$$\lim_{n \to \infty} I_1 = \int_D S(u(t))\psi(t) - S(u_0)\psi(0) \, dx. \quad (6)$$

For any $s \in (0, t)$ and a.s. in $\Omega$, $G_n(\omega, s) \to -\operatorname{div} G(\omega, s)$ in $W^{-1,p'}(D)$ for $n \to \infty$.

Moreover,

$$\|G_n(\omega, s)\|_{W^{-1,p'}(D)} \leq \|\Pi_n\|_{L(W^{-1,p'}(D))} - \operatorname{div} G(\omega, s)\|_{W^{-1,p'}(D)} \leq C_U - \operatorname{div} G(\omega, s)\|_{W^{-1,p'}(D)},$$

where $C_U \geq 0$ is a generic constant not depending on $n \in \mathbb{N}$ from the Uniform Boundedness Principle. Since $G(\omega, \cdot) \in L^p(Q_T)^d$ for a.e. $\omega \in \Omega$, it follows that

$$\lim_{n \to \infty} G_n = -\operatorname{div} G$$

in $L^p(0, t; W^{-1,p'}(D))$ for every $t \in (0, T)$, a.s. in $\Omega$. For every $s \in (0, t)$ and a.e. $\omega \in \Omega$, from the chain rule for Sobolev functions we get

$$\nabla[S'(u_n(\omega,s))\psi(s)] = S''(u_n(\omega,s))\nabla u_n(\omega,s)\psi(s) + S'(u_n(\omega,s))\nabla\psi(s). \quad (7)$$

For any $s \in [0, t]$, almost every $\omega \in \Omega$, $u_n(\omega, s) \to u(\omega, s)$ in $W^{1,p}_0(D)$ for $n \to \infty$, passing to a (not relabeled) subsequence if necessary (that may depend on $(\omega, s)$), the right-hand side of (7) converges to $S''(u(\omega,s))\nabla u(\omega,s)\psi(s) + S'(u(\omega,s))\nabla\psi(s)$ for $n \to \infty$ a.e. in $D$ and there exists $\zeta \in L^p(D)$, that may depend on $(\omega, s)$, such that

$$|u_n(\omega,s)| + |\nabla u_n(\omega,s)| \leq \zeta(\omega,s)$$

for all $n \in \mathbb{N}$, a.s. in $D$. Consequently, $S'(u_n(\omega,s))\psi(s) \to S'(u(\omega,s))\psi(s)$ for $n \to \infty$ in $W^{1,p}_0(D)$ and this convergence holds for the whole sequence. From the boundedness of $S'$, $S''$ and $\nabla\psi$ it follows that there exist constants $\overline{C}, \overline{C} \geq 0$ not depending on the parameters $n, \omega, s$ such that

$$\|S'(u_n(\omega,s))\psi(s)\|_{W^{1,p}_0(D)} = \|\Pi_n\|_{L(W^{1,p}_0(D))}\|u(\omega,s)\|_{W^{1,p}_0(D)} + \overline{C}$$

and $\|\Pi_n\|_{L(W^{1,p}_0(D))} \leq C_U$ for all $n \in \mathbb{N}$ thanks to the Uniform Boundedness Principle. For these reasons, from Lebesgue’s dominated convergence theorem it follows that

$$\lim_{n \to \infty} S'(u_n)\psi = S'(u)\psi \quad (8)$$
in $L^p(0, t; W^{1,p}_0(D))$ a.s. in $\Omega$ and therefore
\[
\lim_{n \to \infty} I_2 = \int_0^t (- \text{div } G, S'(u)\psi)_{W^{-1,p'}(D), W^{1,p}_0(D)} \, ds
\] (9)
a.s. in $\Omega$. For any $s \in (0, t)$ and a.e. $\omega \in \Omega$, $f_n(\omega, s) \to f(\omega, s)$ in $L^1(D)$. Moreover,
\[
\|f_n(\omega, s)\|_{L^1(D)} \leq C_U \|f(\omega, s)\|_{L^1(D)}
\]
for all $n \in \mathbb{N}$, for all $s \in (0, t)$ and a.e. in $\Omega$. Therefore, from Lebesgue’s dominated convergence theorem it follows that $f_n \to f$ in $L^1((0, t) \times D)$ a.s. in $\Omega$ for $n \to \infty$. On the other hand, since $S'(u_n)\psi$ is bounded with respect to $n \in \mathbb{N}$ in $L^\infty(Q_T)$ and from the convergence (8) in $L^p(0, t; W^{1,p}_0(D))$ it follows that $S'(u_n)\psi \rightharpoonup S'(u)\psi$ in $L^\infty(Q_T)$ a.s. in $\Omega$, therefore
\[
\lim_{n \to \infty} I_3 = \int_0^t \int_D f S'(u)\psi \, dx \, ds
\] (10)
a.s. in $\Omega$. Using Itô isometry we get that
\[
\mathbb{E}\left[ \int_0^t \int_D S'(u_n)\psi g_n - S'(u)\psi g \, dx \, ds \right]^2
\leq |D| \mathbb{E}\left[ \int_0^t \int_D |S'(u_n)\psi g_n - S'(u)\psi g|^2 \, dx \, ds \right]
\leq 2|D| \|\psi\|^2_{L^\infty} \left[ \int_0^t \int_D |S'(u_n)(g_n - g)|^2 \, dx \, ds + \int_0^t \int_D |(S'(u_n) - S'(u))g|^2 \, dx \, ds \right]
\leq 2|D| \|\psi\|^2_{L^\infty} \left[ \|S'\|^2_{L^2(D)} \int_0^t \|g_n - g\|^2_{L^2(D)} \, ds + \int_0^t \int_D |(S'(u_n) - S'(u))g|^2 \, dx \, ds \right].
\] (11)
Since $g_n(\omega, s) \to g(\omega, s)$ for $n \to \infty$ a.s. in $\Omega \times (0, T)$ and
\[
\|g_n(\omega, s) - g(\omega, s)\|^2 \leq 2\|g(\omega, s)\|^2_{L^2(D)}(C_U + 1),
\]
from Lebesgue’s dominated convergence theorem it follows that
\[
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^t \|g_n - g\|^2_{L^2(D)} \, ds \right] = 0.
\] (12)
Since $u_n(\omega, s) \to u(\omega, s)$ for $n \to \infty$ in $L^1(D)$ and
\[
\|u_n(\omega, s)\|_{L^1(D)} \leq C_U \|u(\omega, s)\|_{L^1(D)}
\]
for a.e. $(\omega, s) \in \Omega \times (0, T)$ and all $n \in \mathbb{N}$, from Lebesgue’s dominated convergence theorem it follows that $u_n \to u$ in $L^1(\Omega \times Q_T)$ and, passing to a not relabeled subsequence if necessary, also a.s. in $\Omega \times Q_T$. Consequently, a.s. in $\Omega \times Q_T$, we get
\[
\lim_{n \to \infty} \|S'(u_n(\omega, s, x)) - S'(u(\omega, s, x))\|^2 |g(\omega, s, x)|^2 = 0.
\]
In addition,
\[
|S'(u_n(\omega, s, x)) - S'(u(\omega, s, x))|^2 |g(\omega, s, x)|^2 \leq 2\|S'\|^2_{L^\infty} |g(\omega, s, x)|^2
\]
a.s. in $\Omega \times Q_T$ and from Lebesgue’s dominated convergence theorem it follows that
\[
\lim_{n \to \infty} \mathbb{E}\left[ \int_0^t \int_D |S'(u_n) - S'(u)|^2 |g|^2 \, dx \, ds \right] = 0
\] (13)
for any $t \in [0, T]$. Combining (11), (12) and (13), it follows that

$$
\lim_{n \to \infty} \int_0^t \int_D S'(u_n) \psi \, dx \, d\beta = \int_0^t \int_D S'(u) \psi \, dx \, d\beta
$$

in $L^2(\Omega)$ for any $t \in [0, T]$, and, passing a not relabeled subsequence if necessary, also a.s. in $\Omega$. Hence, up to a not relabeled subsequence,

$$
\lim_{n \to \infty} I_4 = \int_0^t \int_D S(u) \psi \, dx \, d\beta
$$

a.s. in $\Omega$. From the boundedness of $S$ and the convergence of $u_n(\omega, s)$ to $u(\omega, s)$ in $L^1(D)$ for all $s \in (0, t)$, a.s. in $\Omega$, it follows that $S(u_n) \to S(u)$ for $n \to \infty$ in $L^1(Q_T)$, a.s. in $\Omega$ and therefore

$$
\lim_{n \to \infty} I_5 = \int_0^t \int_D S(u) \psi_t \, dx \, ds
$$

a.s. in $\Omega$. According to the convergence properties of $(g_n)$, $g_n^2 \to g^2$ in $L^1((0, t) \times D)$ for $n \to \infty$ a.s. in $\Omega$. On the other hand, from the boundedness and the continuity of $S''$ we get $S''(u_n) \to S''(u)$ in $L^q((0, t) \times D)$ for all $1 \leq q < \infty$ and weak-\* in $L^\infty((0, t) \times D)$ a.s. in $\Omega$, thus it follows that

$$
\lim_{n \to \infty} I_6 = \int_0^t \int_D S''(u) \psi g^2 \, dx \, ds
$$

a.s. in $\Omega$. Summarizing our results in (6), (9), (10), (14), (15) and (16), we get

$$
\int_D S(u(t)) \psi(t) - S(u_0) \psi(0) \, dx
+ \int_0^t \langle (-\text{div} \ G + f), S'(u) \psi \rangle_{W^{-1,p}(D) + L^1(D), W_0^{1,p}(D) \cap L^\infty(D)} \, ds
= \int_0^t \int_D S'(u) \psi g \, dx \, d\beta + \int_0^t \int_D S(u) \psi_t \, dx \, ds + \frac{1}{2} \int_0^t \int_D S''(u) \psi g^2 \, dx \, ds
$$

for all $t \in [0, T]$ and a.s. in $\Omega$. \hfill \Box

5. **Renormalized solutions.** Let us assume that there exists a strong solution $u$ to (1) in the sense of Theorem 2.1. We observe that for initial data $u_0$ merely in $L^1$, the Itô formula for the square of the norm (see, e.g., [26]) can not be applied and consequently the natural a priori estimate for $\nabla u$ in $L^p(\Omega \times Q_T)^d$ is not available. Choosing $g = \Phi$, $f \equiv 0$, $\psi \equiv 1$ and

$$
S(u) = \int_0^u T_k(r) \, dr
$$

in (5), where $T_k : \mathbb{R} \to \mathbb{R}$ is the truncation function at level $k > 0$ defined by

$$
T_k(r) = \begin{cases} 
    r & , \ |r| \leq k, \\
    k \text{sign}(r) & , \ |r| > k,
\end{cases}
$$

we find that there exists a constant $C(k) \geq 0$ depending on the truncation level $k > 0$, such that

$$
\mathbf{E} \int_0^T \int_D |\nabla T_k(u)|^p \, dx \, ds \leq C(k).
$$
As in the deterministic case, the notion of renormalized solutions takes this information into account:

**Definition 5.1.** Let $u_0 \in L^1(\Omega \times D)$ be $\mathcal{F}_0$-measurable. An $\mathcal{F}_t$-adapted stochastic process $u : \Omega \times [0, T] \to L^1(D)$ such that $u \in L^1(\Omega; C([0, T]; L^1(D)))$ is a renormalized solution to (1) with initial value $u_0$, if and only if

(i) $T_k(u) \in L^p(\Omega; L^p(0, T; W^{1,p}_0(D)))$ for all $k > 0$.

(ii) For all $\psi \in C^\infty([0, T] \times \bar{D})$ and all $S \in C^2(\mathbb{R})$ such that $S'$ has compact support with $S'(0) = 0$ or $\psi(t, x) = 0$ for all $(t, x) \in [0, T] \times \partial D$ the equality

$$
\int_D S(u(t))\psi(t) - S(u_0)\psi(0) \, dx + \int_0^t \int_D S''(u)|\nabla u|^p \psi \, dx \, ds
$$

$$
+ \int_0^t \int_D S'(u)|\nabla u|^{p-2} \nabla u \cdot \nabla \psi \, dx \, ds
$$

$$
= \int_0^t \int_D S'(u)\psi \Phi \, dx \, ds + \int_0^t \int_D |S(u)|\psi \, dx \, ds + \frac{1}{2} \int_0^t \int_D S''(u)|\nabla u|^p \Phi^2 \, dx \, ds
$$

(17)

holds true for all $t \in [0, T]$ and a.s. in $\Omega$.

(iii) The following energy dissipation condition holds true:

$$
\lim_{k \to \infty} E \int_{\{k|u| < k+1\}} |\nabla u|^p \, dt = 0.
$$

Several remarks about Definition 5.1 are in order: Let $u$ be a renormalized solution in the sense of Definition 5.1. Since supp $(S') \subset [-M, M]$, it follows that $S$ is constant outside $[-M, M]$ and for all $k \geq M$, $S(u(t)) = S(T_k(u(t)))$ a.s. in $\Omega \times D$ for all $t \in [0, T]$. In particular, we have

$$
S(u) \in L^1(\Omega; C([0, T]; L^1(D))) \cap L^p(\Omega; L^p(0, T; W^{1,p}(D))) \cap L^\infty(\Omega \times Q_T).
$$

From the chain rule for Sobolev functions it follows that

$$
S'(u)|\nabla u|^{p-2} \nabla u = S'(u)\chi_{\{u| < M\}}(|\nabla u|^{p-2} \nabla u)
$$

$$
= S'(T_M(u))(|\nabla T_M(u)|^{p-2} \nabla T_M(u))
$$

(18)

a.s. in $\Omega \times Q_T$ and therefore from (i) it follows that all the terms in (17) are well-defined. In general, for the renormalized solution $u$, $\nabla u$ may not be in $L^p(\Omega \times Q_T)^d$ and therefore (iii) is an additional condition which can not be derived from (ii).

However, for $u \in L^1(\Omega \times Q_T)$ satisfying (i), we can define a generalized gradient (still denoted by $\nabla u$) by setting

$$
\nabla u(\omega, t, x) := \nabla T_k(u(\omega, t, x))
$$

a.s. in $\{|u| < k\}$ for $k > 0$ (see [4] for details). For $u \in L^1(\Omega; C([0, T]; L^1(D)))$ such that $T_k(u) \in L^p(\Omega; L^p(0, T; W^{1,p}_0(D)))$ for all $k > 0$, (ii) is equivalent to

$$
S(u(t)) - S(u(0)) - \int_0^t \text{div} (S'(u)|\nabla u|^{p-2} \nabla u) \, ds + \int_0^t S''(u)|\nabla u|^p \, ds - \frac{1}{2} \Phi^2 \, ds
$$

(19)

or equivalently, in differential form,

$$
dS(u) - \text{div} (S'(u)|\nabla u|^{p-2} \nabla u) \, dt + S''(u)|\nabla u|^p - \frac{1}{2} \Phi^2 \, dt = \Phi S'(u) \, d\beta
$$

(20)
in $W^{-1,p'}(D) + L^1(D)$ for all $t \in [0,T]$, a.s. in $\Omega$ and for any $S \in C^2(\mathbb{R})$ with $\text{supp}(S')$ compact, and, since the right-hand side of (20) is in $L^2(D)$, also in $L^2(D)$.

**Remark 5.2.** If $u$ is a renormalized solution to (1), thanks to (20), the Itô formula from Proposition 4.2 still holds true for $S(u)$ for any $S \in C^2(\mathbb{R})$ with $\text{supp}(S')$ compact such that $S(u) \in W^{-1,p}(D)$ a.s. in $\Omega \times (0,T)$. Indeed, in this case (3) is satisfied for the progressively measurable functions

$$
\tilde{u} = S(u) \in L^1(\Omega; C([0,T]; L^1(D))) \cap L^p(\Omega; L^p(0,T; W^{-1,p}(D))),
$$

$$
G = S'(u)|\nabla u|^{p-2}\nabla u \in L^p(\Omega \times Q_T)^d,
$$

$$
f = S''(u)|\nabla u|^p - \frac{1}{2}\Phi^2 \in L^1(\Omega \times Q_T),
$$

$$
g = \Phi S'(u) \in L^2(\Omega \times Q_T).
$$

**Remark 5.3.** Let $u$ be a renormalized solution to (1) with $\nabla u \in L^p(\Omega \times Q_T)^d$. For fixed $l > 0$, let $h_l : \mathbb{R} \to \mathbb{R}$ be defined by

$$
h_l(r) = \begin{cases} 
0 & |r| \geq l + 1 \\
nl & \quad |r| < l + 1 \\
1 & |r| \leq l.
\end{cases}
$$

Taking $S(u) = \int_0^u h_l(r) \, dr$ as a test function in (66), we may pass to the limit with $l \to \infty$ and we find that $u$ is a strong solution to (1).

6. **Existence of renormalized solutions.** In this Section, we fix $u_0 \in L^1(\Omega \times D) \mathcal{F}_0$-measurable. Let $(u_0^n)_n \subset L^2(\Omega \times D)$ be an $\mathcal{F}_0$-measurable sequence such that $|u_0^n| \leq |u_0|$ for all $n \in \mathbb{N}$ and $\lim_{n \to \infty} u_0^n = u_0 \in L^1(\Omega \times D)$ and in $L^1(D)$ for a.e. $\omega \in \Omega$. A possible choice is $u_0^n = T_n(u_0)$, $n \in \mathbb{N}$.

**Theorem 6.1.** There exists a renormalized solution to (1) with initial datum $u_0$.

Theorem 6.1 will be proved successively in the following Lemmas.

**Lemma 6.2.** For $n \in \mathbb{N}$, let $u_n$ be a strong solution to (1) with respect to the initial value $u_0^n$. Then there exists an $\mathcal{F}_t$-adapted stochastic process $u : \Omega \times [0,T] \to L^1(D)$ such that $u \in L^1(\Omega; C([0,T]; L^1(D)))$ and

$$
\lim_{n \to \infty} u_n = u \text{ in } L^1(\Omega; C([0,T]; L^1(D))) \text{ and in } C([0,T]; L^1(D)) \text{ a.s. in } \Omega. \quad (21)
$$

**Proof.** By assumption, $(u_0^n)_n$ is a Cauchy sequence in $L^1(\Omega \times D)$ and in $L^1(D)$ a.s. in $\Omega$. Theorem 3.1 yields

$$
\sup_{t \in [0,T]} \int_D |u_n(t) - u_m(t)| \, dx \leq \int_D |u_0^n - u_0^m| \, dx
$$

for all $n, m \in \mathbb{N}$ a.s. in $\Omega$ and in $L^1(\Omega)$. Thus, $(u_n)_n$ is a Cauchy sequence both in $C([0,T]; L^1(D))$ and in $L^1(\Omega; C([0,T]; L^1(D)))$. Consequently, there exists $u \in L^1(\Omega; C([0,T]; L^1(D)))$ satisfying (21). In particular, we have $u_n(t) \to u(t)$ in $L^1(D)$ a.s. in $\Omega$ and for all $t \in [0,T]$. As a limit function of a sequence of $\mathcal{F}_t$-measurable functions we may conclude that $u(t)$ is $\mathcal{F}_t$-measurable. \[\square\]

In the following, we will show that the process $u$ from Lemma 6.2 is the renormalized solution to (1) with initial datum $u_0$ in the sense of Definition 5.1.
**Lemma 6.3.** For \( n \in \mathbb{N} \), let \( u_n \) be a strong solution to (1) with respect to the initial value \( u_0^n \). Let \( u \) be defined as in Lemma 6.2. Then, \( u \) satisfies (i) and (ii) from Definition 5.1.

**Proof.** Now, let \( u_n \) be a strong solution to (1) with initial value \( u_0^n \), i.e.,

\[
u_n(t) - u_0^n - \int_0^t \text{div} (|\nabla u_n|^{p-2} \nabla u_n) \, ds = \int_0^t \Phi \, dB
\]

for all \( t \in [0, T] \) and a.s. in \( \Omega \). We apply the Itô formula introduced in Proposition 4.2 to equality (22). Therefore we know that for all \( \psi \in C^\infty([0, T] \times \bar{\Omega}) \) and all \( S \in W^{2,\infty}(\mathbb{R}) \) such that \( S' \) is piecewise continuous and \( S'(0) = 0 \) or \( \psi(t, x) = 0 \) for all \( (t, x) \in [0, T] \times \partial \Omega \),

\[
\int_D S(u_n(t))\psi(t) - S(u_0^n)\psi(0) \, dx + \int_0^t \int_D S''(u_n)|\nabla u_n|^p \psi \, dx \, ds
\]

\[
+ \int_0^t \int_D S'(u_n)|\nabla u_n|^{p-2}\nabla u_n \cdot \nabla \psi \, dx \, ds
\]

\[
= \int_0^t \int_D S'(u_n)\psi \Phi \, dx \, ds + \int_0^t \int_D S(u_n)\psi_t \, dx \, ds + \frac{1}{2} \int_0^t \int_D S''(u_n)\psi^2 \, dx \, ds
\]

holds true for all \( t \in [0, T] \) and a.s. in \( \Omega \). In the following, passing to a suitable, not relabeled subsequence if necessary, and taking the limit for \( n \to \infty \), we will show that (23) is also satisfied by \( u \) and \( u_0 \) respectively and therefore \((ii)\) from Definition 5.1 holds. First, we plug \( S(r) = \int_0^r T_k(\tau) \, d\tau \) and \( \psi = 1 \) into (23) and taking expectation we get

\[
E \int_D \int_0^{u_n(t)} T_k(r) \, dr \, dx + E \int_0^t \int_D |\nabla T_k(u_n)|^p \, dx \, ds
\]

\[
= \frac{1}{2} E \int_0^t \int_D T_k(u_n)\Phi^2 \, dx \, ds + E \int_D T_k(0) \, dr \, dx
\]

for all \( k > 0 \), all \( t \in [0, T] \) and a.s. in \( \Omega \). The first term on the left hand side of (24) is nonnegative. The right-hand side of (24) can be majorized by

\[
C(k) := \frac{1}{2} \|\Phi\|^2_{L^2(\Omega \times Q_T)} + k\|u_0\|_{L^1(\Omega \times \bar{\Omega})}
\]

(25)

and from (24) and (25) it follows that

\[
\|\nabla T_k(u_n)\|^p_{L^p(\Omega \times Q_T)} \leq C(k).
\]

(26)

Since \( u_n \to u \) in \( L^1(\Omega; C([0, T]; L^1(D))) \), for \( n \to \infty \), passing to a not relabeled subsequence if necessary, \( u_n \to u \) a.s. in \( \Omega \times Q_T \) for \( n \to \infty \). Since \( |T_k(u_n)| \leq k \) a.s. in \( \Omega \times Q_T \) for all \( n \in \mathbb{N} \) and any \( k > 0 \), from Lebesgue’s dominated convergence theorem it follows that \( T_k(u_n) \rightarrow T_k(u) \) for all \( k > 0 \) in \( L^p(\Omega \times Q_T) \) for \( n \to \infty \). Recalling (26), we may conclude that, for \( k > 0 \) fixed, \( (T_k(u_n))_n \) is bounded in \( L^p(\Omega; L^p(0, T; W_0^{1,p}(D))) \). From the convergence of \( (T_k(u_n))_n \) and from this boundedness we may conclude, passing to a not relabeled subsequence if necessary, that \( T_k(u_n) \rightarrow T_k(u) \) in \( L^p(\Omega; L^p(0, T; W_0^{1,p}(D))) \) for \( n \to \infty \), and any fixed \( k > 0 \) which claims \((i)\) of Definition 5.1.

Furthermore, from (26) it follows that \( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \) is bounded in \( L^p(\Omega \times Q_T)^d \). Consequently, for any fixed \( k > 0 \) there exists a not relabeled subsequence of \( n \) such that \( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \to \sigma_k \) in \( L^p(\Omega \times Q_T)^d \).
for all $k > 0$. This will be done in the following technical lemmas that are inspired by Theorem 2 and Lemma 2 in [6].

\[
T_k(u_n) \to T_k(u) \text{ in } L^p(\Omega; L^p(0,T; W^{1,p}_0(D)))
\]

Remark 6.4. In Lemma 6.5 and in the following we use the notation $\lim_{n,m \to \infty} F_{n,m}$ if $n$ and $m$ tend successively to $\infty$ and

\[
\lim_{n \to \infty} \lim_{m \to \infty} F_{n,m} = \lim_{n \to \infty} \lim_{m \to \infty} F_{n,m}.
\]

Lemma 6.5. For $n \in \mathbb{N}$, let $u_n$ be a strong solution to (1) with respect to the initial value $u_0^n$. Let $H$ and $Z$ be two real valued functions belonging to $W^{2,\infty}(\mathbb{R})$ such that $H''$ and $Z''$ are piecewise continuous, $H'$ and $Z'$ have compact supports and $Z(0) = Z'(0) = 0$ is satisfied. Then

\[
\lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D H''(u_n)Z(u_n - u_m)|\nabla u_n|^p \, dx \, dt = 0.
\]

Proof. Using the Itô product rule (see Proposition 9.1) yields

\[
\int_D Z(u_n(t) - u_m(t))H(u_n(t)) \, dx = \int_D Z(u_n^0 - u_m^0)H(u_0^0) \, dx
\]

\[
- \int_0^t \int_D |\nabla u_n|^p - |\nabla u_m|^p \nabla (Z(u_n - u_m)H'(u_n)) \, dx \, ds
\]

\[
+ \frac{1}{2} \int_0^t \int_D H''(u_n)Z(u_n - u_m)\Phi^2 \, dx \, ds
\]

\[
+ \int_0^t \int_D H'(u_n)Z(u_n - u_m)\Phi \, dx \, d\beta
\]

\[
- \int_0^t \int_D (|\nabla u_n|^p - |\nabla u_m|^p)(\nabla (Z'(u_n - u_m)H(u_n))) \, dx \, ds
\]

for all $t \in [0,T]$ and a.s. in $\Omega$. Using $t = T$ and passing to the limit yields

\[
\lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D H''(u_n)Z(u_n - u_m)|\nabla u_n|^p \, dx \, dt
\]

\[
= - \lim_{n \to \infty} L_{n,m}^{n,m} - \lim_{m \to \infty} M_{n,m}^{n,m} - \lim_{n,m \to \infty} N_{n,m}^{n,m},
\]

where

\[
L_{n,m}^{n,m} = \mathbb{E} \int_0^T \int_D Z''(u_n - u_m)H(u_n)(|\nabla u_n|^p - |\nabla u_m|^p) \cdot \nabla (u_n - u_m) \, dx \, dt,
\]

\[
M_{n,m}^{n,m} = \mathbb{E} \int_0^T \int_D Z'(u_n - u_m)H'(u_n)(|\nabla u_n|^p - |\nabla u_m|^p) \cdot \nabla u_n \, dx \, dt,
\]

\[
N_{n,m}^{n,m} = \mathbb{E} \int_0^T \int_D Z'(u_n - u_m)H'(u_n)|\nabla u_n|^p \cdot \nabla (u_n - u_m) \, dx \, dt.
\]

The rest of the proof is the same as the proof of [6], Theorem 2. \qed
Lemma 6.6. For \( n \in \mathbb{N} \), let \( u_n \) be a strong solution to (1) with respect to the initial value \( u_0^n \). Let \( u \) be defined as in Lemma 6.2. Then,

\[
\lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u_m)|^{p-2} \nabla T_k(u_m) \right) \cdot (\nabla T_k(u_n) - \nabla T_k(u_m)) \, dx \, ds = 0.
\]

Especially, we have

\[
\nabla T_k(u_n) \to \nabla T_k(u) \text{ in } L^p(\Omega \times Q_T)^d
\]

and

\[
T_k(u_n) \to T_k(u) \text{ in } L^p(\Omega; L^p(0, T; W^{1,p}_0(D)))
\]

for \( n \to \infty \) and for all \( k > 0 \).

Proof. Since \( u_n \) and \( u_m \) are strong solutions to (1), we consider the difference of the corresponding equations. Using \( T_k(u_n - u_m) \) as a test function it yields

\[
\int_D \tilde{T}_k(u_n(T) - u_m(T)) \, dx
+ \int_0^T \int_D \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m \right) \cdot \nabla T_k(u_n - u_m) \, dx \, dt
= \int_D \tilde{T}_k(u_0^n - u_0^m) \, dx
\]

a.s. in \( \Omega \) and for all \( k > 0 \), where \( \tilde{T}_k(s) := \int_0^s T_k(r) \, dr \) for all \( s \in \mathbb{R} \). Since \( \tilde{T}_k \) is nonnegative we may conclude that

\[
\lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u_m)|^{p-2} \nabla T_k(u_m) \right) \cdot (\nabla T_k(u_n) - \nabla T_k(u_m)) \, dx \, dt = 0
\]

(29)

for all \( k > 0 \). We set

\[
\int_0^T \int_D \left( |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) - |\nabla T_k(u_m)|^{p-2} \nabla T_k(u_m) \right) \cdot (\nabla T_k(u_n) - \nabla T_k(u_m)) \, dx \, dt = I_{k, n, m} + J_{k, n, m} + J_{k, m, n}
\]

a.s. in \( \Omega \), where

\[
I_{k, n, m} = \int_{\{|u_n| \leq k \} \cap \{|u_m| \leq k \}} (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) \cdot \nabla (u_n - u_m) \, dx \, dt,
\]

\[
J_{k, n, m} = \int_{\{|u_n| \leq k \} \cap \{|u_m| > k \}} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n \, dx \, dt
\]

a.s. in \( \Omega \). \( J_{k, m, n} \) is the same as \( J_{k, n, m} \) where the roles of \( n \) and \( m \) are reversed. Therefore these two terms can be treated simultaneously.

Since \( \{|u_n| \leq k \} \cap \{|u_m| \leq k \} \subset \{|u_n - u_m| \leq 2k \} \), we get

\[
0 \leq \lim_{n,m \to \infty} \mathbb{E} I_{k, n, m}
\]

\[
\leq \lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D \left( |\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m \right) \cdot \nabla T_2k(u_n - u_m) \, dx \, dt = 0
\]

for all \( k > 0 \) by (29). Now we set

\[
0 \leq J_{k, n, m} = J_{1, k, k'}^{n, m} + J_{2, k, k'}^{n, m}
\]
where

\[
J_{1,k,k'}^{n,m} = \int_{\{ |u_n| \leq k \} \cap \{ |u_m| > k \} \cap \{ |u_n - u_m| \leq k' \}} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n \, dx \, dt,
\]

\[
J_{2,k,k'}^{n,m} = \int_{\{ |u_n| \leq k \} \cap \{ |u_m| > k \} \cap \{ |u_n - u_m| > k' \}} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n \, dx \, dt
\]

for all \( k' > k > 0 \), a.s. in \( \Omega \). Firstly, we focus on \( J_{1,k,k'}^{n,m} \). It is \( \{ |u_n| \leq k \} \cap \{ |u_m| > k \} \cap \{ |u_n - u_m| \leq k' \} \subset \{ |u_n| \leq k \} \cap \{ k < |u_m| \leq k + k' \} \). Therefore we can estimate

\[
0 \leq J_{1,k,k'}^{n,m} = \lim_{n,m \to \infty} J_{1,1,k,k'}^{n,m}
\]

\[
\leq \lim_{n,m \to \infty} \mathbb{E} \int_0^T \int_D (|\nabla u_n|^{p-2} \nabla u_n - |\nabla u_m|^{p-2} \nabla u_m) \cdot \nabla T_{k+k'}(u_n - u_m) \, dx \, dt = 0.
\]

It is

\[
\mathbb{E} J_{1,2,k,k'}^{n,m} = \mathbb{E} \int_0^T \int_D |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \cdot \nabla T_{k+k'}(u_m) \chi_{\{ |u_m| \leq k+k' \}} \, dx \, dt.
\]

Let us define \( \theta_k^k(r) := T_{k+k'}(r) - T_k(r) \).

Then \( \nabla \theta_k^k(u_m) = \nabla T_{k+k'}(u_m) \chi_{\{ |u_m| < k+k' \}} \) and \( \nabla \theta_k^k(u_m) \rightarrow \nabla \theta_k^k(u) \) in \( L^p(\Omega \times Q_T)^d \). Now we can estimate

\[
0 \leq \lim_{n,m \to \infty} \mathbb{E} J_{1,2,k,k'}^{n,m} = \mathbb{E} \int_0^T \int_D \sigma_k \nabla \theta_k^k(u) \, dx \, dt.
\]

We show that \( \sigma_k = \chi_{\{ |u| < k \}} \sigma_{k+1} \) a.e. on \( \{ |u| \neq k \} \). If we do so it follows that \( \sigma_k = 0 \) a.e. on \( \{ |u| > k \} \). Since \( \nabla \theta_k^k(u) = 0 \) on \( \{ |u| \leq k \} \) it follows \( \lim_{n,m \to \infty} \mathbb{E} J_{1,2,k,k'}^{n,m} = 0 \).

Let \( \psi \in L^p(\Omega \times Q_T)^d \). Then

\[
\lim_{n \to \infty} \mathbb{E} \int_{Q_T} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \cdot \psi \cdot \chi_{\{ |u| \neq k \}} \, dx \, dt = \mathbb{E} \int_{Q_T} \sigma_k \psi \cdot \chi_{\{ |u| \neq k \}} \, dx \, dt.
\]

On the other hand we know that \( u_n \to u \) a.e. in \( \Omega \times Q_T \). Hence, we have \( \chi_{\{ u_n < k \}} \to \chi_{\{ u < k \}} \) a.e. in \( \{ |u| \neq k \} \). Therefore the theorem of Lebesgue yields

\[
\chi_{\{ |u_n| < k \}} \cdot \chi_{\{ |u| \neq k \}} \cdot \psi \to \chi_{\{ |u| < k \}} \cdot \chi_{\{ |u| \neq k \}} \cdot \psi \text{ in } L^p(\Omega \times Q_T)^d.
\]
We may conclude that
\[
\lim_{n \to \infty} \mathbb{E} \int_{Q_T} |\nabla T_k(u_n)|^{p-2} \nabla T_k(u_n) \cdot \psi \cdot \chi_{\{|u| \neq k\}} \, dx \, dt = \lim_{n \to \infty} \mathbb{E} \int_{Q_T} |\nabla T_{k+1}(u_n)|^{p-2} \nabla T_{k+1}(u_n) \cdot \psi \cdot \chi_{\{|u| \neq k\}} \cdot \chi_{\{|u| < k\}} \, dx \, dt
\]
\[
= \mathbb{E} \int_{Q_T} \sigma_{k+1} \psi \chi_{\{|u| \neq k\}} \cdot \chi_{\{|u| < k\}} \, dx \, dt.
\]
It follows that \(\sigma_k = \chi_{\{|u| < k\}} \sigma_{k+1}\) a.e. on \(\{|u| \neq k\}\) and therefore
\[
\lim_{n,m \to \infty} \mathbb{E} J_{n,m}^{1,2,k,k'} = 0.
\]
Now let us consider \(J_{n,m}^{1,3,k,k'}\). Since \(|\nabla \theta_{k_{k}}(u_n)|^{p-2} \nabla \theta_{k_{k}}(u_n) \to \tilde{\sigma}_{k'}^{k'}\) in \(L^p(\Omega \times Q_T)\) for a subsequence we have
\[
\lim_{n,m \to \infty} \mathbb{E} J_{n,m}^{1,3,k,k'} = \mathbb{E} \int_0^T \int_D \tilde{\sigma}_{k'}^{k'} \cdot \nabla T_k(u) \, dx \, dt.
\]
Since \(\nabla \theta_{k'}^{k'}(v) = \chi_{\{|k'_{k} < |u| < k + k'\}} \nabla \theta_{k_{k}-1}(v)\) for all \(v \in L^p(\Omega; L^p(0,T; W^{1,p}_0(D)))\), we can show by similar arguments as before that \(\tilde{\sigma}_{k'}^{k'} = \chi_{\{|k'_{k} < |u| < k + k'\}} \tilde{\sigma}_{k_{k}-1}^{k_{k}+2}\) a.e. on \(\{|u| \neq k\} \cup \{|u| \neq k + k'\}\). Therefore it follows that
\[
\lim_{n,m \to \infty} \mathbb{E} J_{n,m}^{1,3,k,k'} = 0.
\]
It is left to show that
\[
\lim_{n,m \to \infty} \mathbb{E} J_{n,m}^{1,2,k,k'} = 0.
\]
Using Lemma 6.5, (27) with \(H = H_{\delta}^{k}\) for \(\delta, k > 0\) such that
\[
(H_{\delta}^{k})''(r) = \begin{cases} 1, & |r| < k, \\ -k\delta, & k \leq |r| < k + \frac{1}{2}, \\ 0, & |r| > k + \frac{1}{2}. \end{cases}
\]
yields
\[
\limsup_{n,m \to \infty} \mathbb{E} \int_{\{|u| \leq k\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt 
\]
\[
\leq k\delta \limsup_{n,m \to \infty} \mathbb{E} \int_{\{|k \leq |u| \leq k + \frac{1}{2}\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt 
\]
\[
\leq \delta \cdot k \|Z\|_{\infty} \limsup_{n \to \infty} \mathbb{E} \int_{\{|k \leq |u| \leq k + \frac{1}{2}\}} |\nabla u_n|^p \, dx \, dt.
\]
Now applying Proposition 4.2 with \(S = \int_0^T \theta_{k}^{\frac{1}{r}} =: \tilde{\theta}_{k}^{\frac{1}{r}}, \psi = 1, g = \Phi\) and \(f = 0\) and taking expectation yields
\[
\mathbb{E} \int_D \tilde{\theta}_{k}^{\frac{1}{r}}(u_n(T)) \, dx + \mathbb{E} \int_0^T \int_D \chi_{\{|k \leq |u| \leq k + \frac{1}{2}\}} |\nabla u_n|^p \, dx \, dt
\]
\[
= \mathbb{E} \int_D \tilde{\theta}_{k}^{\frac{1}{r}}(u_0^n) \, dx + \frac{1}{2} \mathbb{E} \int_0^T \int_D \chi_{\{|k \leq |u| \leq k + \frac{1}{2}\}} \Phi^2 \, dx \, dt.
\]
The first term on the left hand side is nonnegative and the integrand of the second term on the right hand side can be estimated as follows
\[ \chi_{\{k \leq |u_n| \leq k + \frac{1}{n}\}} \Phi^2 \leq \Phi^2 \in L^1(\Omega \times Q_T). \]
Multiplying by \( \delta \) and passing to the limit with \( n \to \infty \) yields
\[ \delta \cdot \limsup_{n \to \infty} \mathbb{E} \int_0^T \int_D \chi_{\{k \leq |u_n| \leq k + \frac{1}{n}\}} |\nabla u_n|^p \, dx \, dt \]
\[ \leq \mathbb{E} \int_D \delta \Theta_k^\frac{1}{p} (u_0) \, dx + \frac{1}{2} \delta \|\Phi\|_{L^2(\Omega \times Q_T)}^2. \]
We can estimate that \( \delta \Theta_k^\frac{1}{p} (u_0) \to 0 \) a.e. in \( \Omega \times D \) as \( \delta \to 0 \) and \( |\delta \Theta_k^\frac{1}{p} (u_0)| \leq u_0 + C \) for a constant \( C > 0 \). Therefore Lebesgue’s Theorem yields
\[ \lim_{\delta \to 0} \limsup_{n \to \infty} \mathbb{E} \int_{\{k \leq |u_n| \leq k + \frac{1}{n}\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt = 0. \]
Therefore we may conclude
\[ \lim_{n,m \to \infty} \mathbb{E} \int_{\{|u_n| \leq k\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt \]
\[ = \limsup_{n \to \infty} \limsup_{m \to \infty} \mathbb{E} \int_{\{|u_n| \leq k\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt = 0. \]
 Choosing \( Z \) such that \( Z(r) = 1 \) for \( |r| \geq k' \) and \( Z \geq 0 \) on \( \mathbb{R} \) such that \( Z(0) = Z'(0) = 0 \), it follows
\[ 0 \leq \lim_{n,m \to \infty} \mathbb{E} J_{k,k'}^{n,m} \]
\[ = \lim_{n,m \to \infty} \mathbb{E} \int_{\{|u_n| \leq k\} \cap \{|u_m| > k\} \cap \{|u_n - u_m| > k'\}} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n \, dx \, dt \]
\[ \leq \lim_{n,m \to \infty} \mathbb{E} \int_{\{|u_n| \leq k\}} Z(u_n - u_m) |\nabla u_n|^p \, dx \, dt = 0, \]
which finally shows the validity of equality (28). Since equality (28) holds true, it follows that
\[ \lim_{n \to \infty} \mathbb{E} \int_0^T |\nabla T_k(u_n)|_p^{p-2} \nabla T_k(u_n) \cdot \nabla T_k(u_n) \, dx \, dt = \mathbb{E} \int_0^T \int_D \sigma_k \cdot \nabla T_k(u) \, dx \, dt. \]
Minty’s trick yields \( \sigma_k = |\nabla T_k(u)|^{p-2} \nabla T_k(u) \). We may conclude by using equality (30) that
\[ \lim_{n \to \infty} \|\nabla T_k(u_n)\|_{L^p(\Omega \times Q_T)}^p = \|\nabla T_k(u)\|_{L^p(\Omega \times Q_T)}^p. \]
Since \( L^p(\Omega \times Q_T) \) is uniformly convex and \( \nabla T_k(u_n) \to \nabla T_k(u) \) in \( L^p(\Omega \times Q_T) \) it yields
\[ \nabla T_k(u_n) \to \nabla T_k(u) \text{ in } L^p(\Omega \times Q_T). \]
which ends the proof of Lemma 6.6. \( \square \)

For the proof of Theorem 6.1 is left to show that the energy dissipation condition (iii) from Definition 5.1 holds true.
Lemma 6.7. For \( n \in \mathbb{N} \), let \( u_n \) be a strong solution to (1) with respect to the initial value \( u_0^n \). Let \( u \) be defined as in Lemma 6.2. Then,

\[
\limsup_{k \to \infty} \limsup_{n \to \infty} E \int_{\{k < |u_n| < k+1\}} |\nabla u_n|^p \, dx \, dt = 0.
\]

(31)

Proof. For fixed \( l > 0 \), let \( h_l : \mathbb{R} \to \mathbb{R} \) be defined as in Remark (5.2). We plug \( S(r) = \int_0^r h_l(r)(T_{k+1}(r) - T_k(r)) \, dr \) and \( \Psi \equiv 1 \) in (23) and take expectation to obtain

\[
I_1 + I_2 + I_3 = I_4 + I_5,
\]

(32)

where

\[
I_1 = \mathbb{E} \int_D \int_{u_0^n}^{u_n(t)} h_l(r)(T_{k+1}(r) - T_k(r)) \, dr \, dx,
\]

\[
I_2 = \mathbb{E} \int_{\{t < |u_n| < l+1\}} -\text{sign}(u_n)(T_{k+1}(u_n) - T_k(u_n))|\nabla u_n|^p \, dx \, ds,
\]

\[
I_3 = \mathbb{E} \int_{\{k < |u_n| < k+1\}} h_l(u_n)|\nabla u_n|^p \, dx \, ds,
\]

\[
I_4 = \frac{1}{2} \mathbb{E} \int_{\{t < |u_n| < l+1\}} -\text{sign}(u_n)(T_{k+1}(u_n) - T_k(u_n))\Phi^2 \, dx \, ds,
\]

\[
I_5 = \frac{1}{2} \mathbb{E} \int_{\{k < |u_n| < k+1\}} h_l(u_n)\Phi^2 \, dx \, ds
\]

for all \( t \in [0, T] \). We can pass to the limit with \( l \to \infty \) in (32) by Lebesgue's Dominated Convergence theorem. We obtain

\[
J_1 + J_2 = J_3,
\]

(33)

where

\[
J_1 = \mathbb{E} \int_D \int_{u_0^n}^{u_n(t)} T_{k+1}(r) - T_k(r) \, dr \, dx,
\]

\[
J_2 = \mathbb{E} \int_{\{k < |u_n| < k+1\}} |\nabla u_n|^p \, dx \, ds,
\]

\[
J_3 = \frac{1}{2} \mathbb{E} \int_{\{k < |u_n| < k+1\}} \Phi^2 \, dx \, ds.
\]

Since \( u_n \to u \) in \( L^1(\Omega; C([0, T]; L^1(D))) \) and \( u_0^n \to u_0 \) in \( L^1(\Omega \times D) \), for \( n \to \infty \), it follows that

\[
\lim_{k \to \infty} \lim_{n \to \infty} \int_D \int_{u_0^n}^{u(t)} T_{k+1}(r) - T_k(r) \, dr \, dx = 0.
\]

(34)

Now, the term \( J_3 \) desires our attention. For any \( \sigma > 0 \) we have

\[
J_3 = \frac{1}{2} \mathbb{E} \int_0^t \int_{\{k < |u_n| < k+1\}} \left( \chi_{\{\Phi^2 > \sigma\}} + \chi_{\{\Phi^2 \leq \sigma\}} \right) \Phi^2 \, dx \, ds
\]

\[
\leq \frac{1}{2} \mathbb{E} \int_0^t \int_{\{k < |u_n| < k+1\}} \sigma \, dx \, ds + \frac{1}{2} \mathbb{E} \int_0^t \int_{\{\Phi^2 > \sigma\}} \Phi^2 \, dx \, ds
\]

\[
\leq \frac{\sigma}{2k} ||u_n||_{L^1(\Omega \times Q_T)} + \mathbb{E} \int_0^t \int_{\{\Phi^2 > \sigma\}} \Phi^2 \, dx \, ds.
\]

(35)
Thanks to the convergence of \((u_n)\), there exists a constant \(C \geq 0\) not depending on the parameters \(k\), \(n\) and \(\sigma\) such that
\[
\|u_n\|_{L^1(\Omega \times Q_T)} \leq C.
\]
Thus,
\[
\limsup_{k \to \infty} \limsup_{n \to \infty} J_3 \leq \mathbb{E} \int_0^t \int_{\{\Phi^2 > \sigma\}} \Phi^2 \, dx \, ds
\]
and therefore, passing to the limit with \(\sigma \to \infty\), from (36) and the nonnegativity of \(J_3\) it follows that
\[
\limsup_{k \to \infty} \limsup_{n \to \infty} J_3 = 0.
\]
Combining (33), (34) and (37), and using the nonnegativity of \(J_2\), we arrive at (31).

We have
\[
\chi\{k < |u_n| < k + 1\} \chi\{|u| \neq k\} \chi\{|u| \neq k + 1\} \to \chi\{k < |u| < k + 1\} \chi\{|u| \neq k\} \chi\{|u| \neq k + 1\}
\]
for \(n \to \infty\) in \(L^r(\Omega \times Q_T)\) for any \(1 \leq r < \infty\) and a.e. in \(\Omega \times Q_T\). From Lemma 6.6 we recall that for any \(k > 0\),
\[
\nabla T_k(u_n) \to \nabla T_k(u) \text{ in } L^p(\Omega \times Q_T)^d
\]
for \(n \to \infty\), thus, passing to a not relabeled subsequence if necessary, also a.s. in \(\Omega \times Q_T\). Since \(\nabla T_k(u) = 0\) a.s. on \(|u| = m\) for any \(m \geq 0\), from Fatou’s Lemma it follows that
\[
\liminf_{n \to \infty} \mathbb{E} \int_{\{k < |u_n| < k + 1\}} |\nabla u_n|^p \, dx \, dt
\]
\[
\geq \liminf_{n \to \infty} \mathbb{E} \int_{\{k < |u_n| < k + 1\}} |\nabla u_n|^p \chi\{|u| \neq k\} \chi\{|u| \neq k + 1\} \, dx \, dt
\]
\[
\geq \mathbb{E} \int_{\{k < |u| < k + 1\}} |\nabla u|^p \chi\{|u| \neq k\} \chi\{|u| \neq k + 1\} \, dx \, dt
\]
\[
= \mathbb{E} \int_{\{k < |u| < k + 1\}} |\nabla u|^p \, dx \, dt
\]
and the energy dissipation condition (iii) follows combining (31) with (38).

7. Uniqueness of renormalized solutions. In the following, we formulate a contraction principle that yields immediately both uniqueness and continuous dependence on the initial values for renormalized solutions.

**Theorem 7.1.** Let \(u, v\) be renormalized solutions to (1) with initial data \(u_0 \in L^1(\Omega \times D)\) and \(v_0 \in L^1(\Omega \times D)\), respectively. Then we get
\[
\int_D |u(t) - v(t)| \, dx \leq \int_D |u_0 - v_0| \, dx
\]
a.s. in \(\Omega\), for all \(t \in [0, T]\).
Proof. This proof is inspired by the uniqueness proof in [8]. We know that $u$ satisfies the SPDE

$$dS(u) - \text{div}(S'(u)|\nabla u|^p \nabla u) dt + S''(u)|\nabla u|^p dt = \Phi S'(u) d\beta + \frac{1}{2} S''(u) \Phi^2 dt$$

(40)

for all $S \in C^2(\mathbb{R})$ such that supp $S'$ compact. Moreover, $v$ satisfies an analogous SPDE. Subtracting both equalities yields

$$S(u(t)) - S(v(t)) =
S(u_0) - S(v_0) + \int_0^t \text{div}[S'(u)|\nabla u|^p \nabla u - S'(v)|\nabla v|^p \nabla v] ds$$

$$- \int_0^t (S''(u)|\nabla u|^p - S''(v)|\nabla v|^p) ds + \int_0^t \Phi(S'(u) - S'(v)) d\beta$$

(41)

$$+ \frac{1}{2} \int_0^t \Phi^2 (S''(u) - S''(v)) ds$$
in $W^{-1,p'}(D) + L^1(D)$ for all $t \in [0,T]$, a.s. in $\Omega$.

Now we set $S(r) := T^\sigma_s(r)$ for $r \in \mathbb{R}$ and $s, \sigma > 0$ and define $T^\sigma_s$ as follows: Firstly, we define for all $r \in \mathbb{R}$

$$(T^\sigma_s)'(r) = \begin{cases}
1, & \text{if } |r| \leq s, \\
\frac{1}{s}(s + \sigma - |r|), & \text{if } s < |r| < s + \sigma, \\
0, & \text{if } |r| \geq s + \sigma.
\end{cases}$$

Then we set $T^\sigma_s(r) := \int_0^r (T^\sigma_s)'(r) d\tau$. Furthermore we have the weak derivative

$$(T^\sigma_s)''(r) = \begin{cases}
-\frac{1}{s} \text{sign}(r), & \text{if } s < |r| < s + \sigma, \\
0, & \text{otherwise}.
\end{cases}$$

Applying Proposition 4.2 to equality (41) with $S(r) = \frac{1}{k} \tilde{T}_k(r) = \frac{1}{k} \int_0^r \tilde{T}_k(\tau) d\tau$ and $\psi \equiv 1$ yields (see also Remark 5.2)

$$\int_D \left( \frac{1}{k} \tilde{T}_k(T^\sigma_s(u(t)) - T^\sigma_s(v(t))) - \frac{1}{k} \tilde{T}_k(T^\sigma_s(u_0) - T^\sigma_s(v_0)) \right) dx$$

$$- \int_0^t \left( \text{div}((T^\sigma_s)'(u)|\nabla u|^p \nabla u - (T^\sigma_s)'(v)|\nabla v|^p \nabla v), \frac{1}{k} \tilde{T}_k(T^\sigma_s(u) - T^\sigma_s(v)) \right) dx$$

$$= \int_D \int_0^t \left( - ((T^\sigma_s)''(u)|\nabla u|^p - (T^\sigma_s)''(v)|\nabla v|^p) + \frac{1}{2} \Phi^2 ((T^\sigma_s)''(u) - (T^\sigma_s)''(v)) \right) dx$$

$$\cdot \frac{1}{k} \tilde{T}_k(T^\sigma_s(u) - T^\sigma_s(v)) dr dx$$

$$+ \int_D \int_0^t \Phi((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} \tilde{T}_k(T^\sigma_s(u) - T^\sigma_s(v)) d\beta dx$$

$$+ \frac{1}{2} \int_D \int_0^t \Phi^2((T^\sigma_s)'(u) - (T^\sigma_s)'(v))^2 \frac{1}{k} \chi_{(|T^\sigma_s(u) - T^\sigma_s(v)| < k)} dr dx$$

(42)

a.s. in $\Omega$ for any $t \in [0,T]$. We write equality (42) as

$$I_1 + I_2 = I_3 + I_4 + I_5.$$
For \( \omega \in \Omega \) and \( t \in [0, T] \) fixed, we pass to the limit with \( \sigma \to 0 \) firstly, then we pass to the limit \( k \to 0 \) and finally we let \( s \to \infty \). Before we do so, we have to give some remarks on \( T^\sigma_s \).

By definition of \( (T^\sigma_s)' \) we see immediately that \( (T^\sigma_s)'(r) \to \chi_{\{|r| \leq s\}} \) pointwise for all \( r \in \mathbb{R} \) as \( \sigma \to 0 \). Since \(|(T^\sigma_s)'| \leq 1 \) on \( \mathbb{R} \) we have,

\[
(T^\sigma_s)'(u) \to \chi_{\{|u| \leq s\}}
\]

in \( L^1(Q_T) \) a.s. in \( \Omega \) and a.e. in \( \Omega \times Q_T \) as \( \sigma \to 0 \). An analoguous result holds true for \( v \) instead of \( u \).

For \( 0 < \sigma < 1 \) and fixed \( s > 0 \), we have \( \text{supp}(T^\sigma_s)' \subset [-s - 1, s + 1] \). Therefore \( T^\sigma_s \) is bounded in \( L^\infty(\mathbb{R}) \) for fixed \( s \) and we may conclude \( T^\sigma_s(u) \to T_s(u) \) a.e. in \( \Omega \times Q_T \) and in \( L^1(Q_T) \) a.s. in \( \Omega \) as \( \sigma \to 0 \). Furthermore, we have \( \nabla T^\sigma_s(u) = \nabla T^\sigma_s(T_{s+1}(u)) = \nabla T_{s+1}(u)(T^\sigma_s)'(u) \). Since \(|(T^\sigma_s)'| \leq 1 \) on \( \mathbb{R} \), we have

\[
\nabla T_{s+1}(u)(T^\sigma_s)'(u) \to \nabla T_{s+1}(u)\chi_{\{|u| \leq s\}} \text{ in } L^p(Q_T)^d
\]

for \( \sigma \to 0 \) a.s. in \( \Omega \). Since \( \nabla T_{s+1}(u)\chi_{\{|u| \leq s\}} = \nabla T_s(u) \) we have

\[
\nabla T^\sigma_s(u) \to \nabla T_s(u) \text{ in } L^p(Q_T)^d
\]

(43)

for \( \sigma \to 0 \) a.s. in \( \Omega \).

Let us consider \( I_1 \). By Lebesgue’s Theorem it follows

\[
\lim_{\sigma \to 0} I_1 = \int_D \left( \frac{1}{k} \tilde{T}_k(T_s(u(t)) - T_s(v(t))) - \frac{1}{k} \tilde{T}_k(T_s(u_0) - T_s(v_0)) \right) dx
\]

a.s. in \( \Omega \), for all \( t \in [0, T] \). Since \( \frac{1}{k} \tilde{T}_k \to |\cdot| \) pointwise in \( \mathbb{R} \) as \( k \to 0 \), we may conclude

\[
\lim_{k \to 0} \lim_{\sigma \to 0} I_1 = \int_D |T_s(u(t)) - T_s(v(t))| - |T_s(u_0) - T_s(v_0)| dx
\]

a.s. in \( \Omega \), for all \( t \in [0, T] \). Finally, since \( u(t), v(t), u_0, v_0 \in L^1(D) \) for all \( t \in [0, T] \), a.s. in \( \Omega \) again Lebesgue’s Theorem yields

\[
\lim_{s \to \infty} \lim_{k \to 0} \lim_{\sigma \to 0} I_1 = \int_D |u(t) - v(t)| - |u_0 - v_0| dx
\]

(44)

a.s. in \( \Omega \), for all \( t \in [0, T] \).

Next we want to show that

\[
\liminf_{\sigma \to 0} I_2 \geq 0
\]

a.s. in \( \Omega \).

Since \( \text{supp}(T^\sigma_s)' \subset [-s - 1, s + 1] \) for \( 0 < \sigma < 1 \) we can estimate

\[
I_2 = \int_D \int_0^t \left| (T^\sigma_s)'(u) \right| |\nabla u|^{p-2} \nabla u - (T^\sigma_s)'(v) |\nabla v|^{p-2} \nabla v | \right| \frac{1}{k} \nabla T_k(T^\sigma_s(u) - T^\sigma_s(v)) \right| \right) dx dt
\]

\[
= \int_D \int_0^t \left( \left| (T^\sigma_s)'(u) \right| |\nabla T_{s+1}(u)|^{p-2} \nabla T_{s+1}(u) - (T^\sigma_s)'(v) |\nabla T_{s+1}(v)|^{p-2} \nabla T_{s+1}(v) \right) \cdot \frac{1}{k} \nabla T_k(T^\sigma_s(u) - T^\sigma_s(v)) \right| \right) dx dt.
\]

Hence

\[
\lim_{\sigma \to 0} I_2 = \frac{1}{k} \int_{\{|T_s(u) - T_s(v)| < k\}} \left( |\nabla T_s(u)|^{p-2} \nabla T_s(u) - |\nabla T_s(v)|^{p-2} \nabla T_s(v) \right) \cdot \nabla (T_s(u) - T_s(v)) \right| \right) dx dt \geq 0
\]

(45)
a.s. in \( \Omega \), for all \( t \in [0, T] \).

We write

\[
I_3 = I_3^1 + I_3^2,
\]

where

\[
I_3^1 = - \int_0^t \int_0^1 \left( (T_s^\sigma)^{''}(u) |\nabla u|^p - (T_s^\sigma)^{''}(v) |\nabla v|^p \right) \frac{1}{k} T_k((T_s^\sigma)(u) - T_s^\sigma(v)) \, dr \, dx,
\]

\[
I_3^2 = \int_0^t \int_0^1 \frac{1}{2} \Phi^\sigma((T_s^\sigma)^{''}(u) - (T_s^\sigma)^{''}(v)) \frac{1}{k} T_k(T_s^\sigma(u) - T_s^\sigma(v)) \, dr \, dx.
\]

For \( 0 < \sigma < 1 \) we have

\[
|I_3^1| \leq \frac{1}{\sigma} \int_{\{s < |u| < s + \sigma\}} |\nabla u|^p \, dr \, dx + \frac{1}{\sigma} \int_{\{s < |v| < s + \sigma\}} |\nabla v|^p \, dr \, dx
\]

and we want to show that there exists a sequence \( (s_j)_{j \in \mathbb{N}} \subset \mathbb{N} \) such that \( s_j \to \infty \) as \( j \to \infty \) and

\[
limit_{j \to \infty} \limsup_{\sigma \to 0} \frac{1}{\sigma} \int_{\{s < |u| < s + \sigma\}} |\nabla u|^p \, dr \, dx + \frac{1}{\sigma} \int_{\{s < |v| < s + \sigma\}} |\nabla v|^p \, dr \, dx = 0.
\]

According to Lemma 6 in [8] it is sufficient to show that for any \( s \in \mathbb{N} \) there exists a nonnegative function \( F \in L^1(\Omega \times (0, t) \times D) \) such that for a.e. \( \omega \in \Omega \)

\[
\frac{1}{\sigma} \int_{\{s < |u| < s + \sigma\}} |\nabla u|^p \, dr \, dx + \frac{1}{\sigma} \int_{\{s < |v| < s + \sigma\}} |\nabla v|^p \, dr \, dx \\
\leq \frac{1}{\sigma} \left( \int_{\{s \leq |u| \leq s + \sigma\}} F \, dr \, dx + \int_{\{s \leq |v| \leq s + \sigma\}} F \, dr \, dx \right) + \epsilon(\sigma, s, \omega),
\]

with \( \limsup_{\sigma \to 0} \epsilon(\sigma, s, \omega) \to 0 \) for almost every \( \omega \in \Omega \) as \( s \to \infty \) to conclude that

\[
\lim_{j \to \infty} \limsup_{k \to 0} \limsup_{\sigma \to 0} |I_3^1| = 0,
\]

where \( s \) is exchanged by \( s_j \) in \( I_3^1 \).

For symmetry reasons, we only have to show the existence of a nonnegative function \( F \in L^1(\Omega \times (0, t) \times D) \) such that for fixed \( \omega \in \Omega \)

\[
\frac{1}{\sigma} \int_{\{s < |u| < s + \sigma\}} |\nabla u|^p \, dr \, dx \leq \frac{1}{\sigma} \int_{\{s \leq |u| \leq s + \sigma\}} F \, dr \, dx + \epsilon(\sigma, s, \omega).
\]

To this end, we plug \( S(u(t)) = \frac{1}{T} \int_0^T h_l(\tau)(T_{s+\sigma}(\tau) - T_s(\tau)) \, d\tau \) and \( \Psi \equiv 1 \) in the renormalized formulation for \( u \) to obtain

\[
L_1 + L_2 + L_3 = L_4 + L_5 + L_6
\]

a.s. in \( \Omega \), where

\[
L_1 := \frac{1}{\sigma} \int_{D} \int_{u_0}^{u(t)} h_l(\tau)(T_{s+\sigma}(\tau) - T_s(\tau)) \, d\tau \, dx,
\]

\[
L_2 := \frac{1}{\sigma} \int_{D} \int_{0}^{t} - |\text{sign}(u)\chi_{\{t < |u| < t+1\}}(T_{s+\sigma}(u) - T_s(u))| \nabla u|^p \, dr \, dx,
\]

\[
L_3 := \frac{1}{\sigma} \int_{D} \int_{0}^{t} h_l(u)\chi_{\{s < |u| < s + \sigma\}} |\nabla u|^p \, dr \, dx,
\]
\[ L_4 := \frac{1}{\sigma} \int_0^t \int_D h_i(u)(T_{s+\sigma}(u) - T_s(u)) \Phi \, dx \, d\beta, \]
\[ L_5 := \frac{1}{2\sigma} \int_0^t \int_D \frac{-\text{sign}(u)\chi_{\{|u|<t+1\}}(T_{s+\sigma}(u) - T_s(u))\Phi^2}{2} \, dx \, dr, \]
\[ L_6 := \frac{1}{2\sigma} \int_0^t \int_D h_i(u)\chi_{\{|u|<s+\sigma\}}\Phi^2 \, dr \, dx. \]

It is straightforward to pass to the limit with \( l \to \infty \) for a.e. \( \omega \in \Omega \) in \( L_1, L_3, L_4, L_5 \) and \( L_6 \). We have
\[ |L_2| \leq \frac{\sigma}{\sigma} \int_{\{l<|u|<l+1\}} |\nabla u|^p \, dx \]
a.s. in \( \Omega \). In order to pass to the limit with \( l \to \infty \) in \( L_2 \), we recall that from the energy dissipation condition \((iii)\) it follows that, passing to a not relabeled subsequence if necessary,
\[ \lim_{l \to \infty} \int_{\{l<|u|<l+1\}} |\nabla u|^p \, dx = 0 \]
a.s. in \( \Omega \) and therefore \( \lim_{l \to \infty} L_2 = 0 \) a.s. in \( \Omega \). After this passage to the limit the remaining terms are
\[ J_2 = -J_1 + J_3 + J_4 \]
a.s. in \( \Omega \), where
\[ J_1 := \frac{1}{\sigma} \int_D \int_{u_0}^{u(t)} (T_{s+\sigma}(\tau) - T_s(\tau)) \, d\tau \, dx, \]
\[ J_2 := \frac{1}{\sigma} \int_{\{s<|u|<s+\sigma\}} |\nabla u|^p \, dx \]
\[ J_3 := \frac{1}{\sigma} \int_0^t \int_D (T_{s+\sigma}(u) - T_s(u)) \Phi \, dx \, d\beta, \]
\[ J_4 := \frac{1}{\sigma} \int_{\{s<|u|<s+\sigma\}} \frac{1}{2} \Phi^2 \, dx \.

We set \( F := \frac{1}{2}\Phi^2 \).

It is left to show that \( \lim_{s \to \infty} \limsup_{\sigma \to 0} \epsilon(\sigma, s, \omega) = 0 \), a.s. in \( \Omega \), where
\[ \epsilon(\sigma, s, \omega) = -J_1 + J_3. \]

Since \( \frac{1}{\sigma}|\frac{1}{\sigma}(T_{s+\sigma}(\tau) - T_s(\tau))| \leq 1 \) for \( 0 < \sigma \leq 1 \), it follows that
\[ \left| \int_{u_0}^{u(t)} \frac{1}{\sigma}(T_{s+\sigma}(\tau) - T_s(\tau)) \, d\tau \right| \leq |u(t) - u_0| \in L^1(\Omega \times D). \]

Moreover,
\[ \frac{1}{\sigma}(T_{s+\sigma}(\tau) - T_s(\tau)) \to \text{sign}(\tau)\chi_{\{\tau \geq s\}} \]
a.e. in \( \{|\tau| \neq s\} \) and from Lebesgue’s dominated convergence theorem it follows that
\[
\lim_{\sigma \to 0} \int_D \int_{u_0} \frac{1}{\sigma} (T_{s+\sigma}(\tau) - T_s(\tau)) \, d\tau \, dx = \lim_{\sigma \to 0} \int_D \int_{u_0} \frac{1}{\sigma} (T_{s+\sigma}(\tau) - T_s(\tau)) \chi_{\{|\tau| \neq s\}} \, d\tau \, dx
\]
\[
= \int_D \int_{u_0} \text{sign}(\tau) \chi_{\{|\tau| \geq s\}} \chi_{\{|\tau| \neq s\}} \, d\tau \, dx \to 0
\]
as \( s \to \infty \) a.s. in \( \Omega \).

Using the Itô isometry and Hölder inequality we obtain
\[
\lim_{\sigma \to 0} \mathbb{E} \left| \int_0^t \int_D \left( \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi - \Phi \chi_{\{|u| > s\}} \right) \, dx \, d\beta \right|^2
\]
\[
= \lim_{\sigma \to 0} \mathbb{E} \int_0^t \int_D \left( \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \chi_{\{|u| > s\}} - \Phi \chi_{\{|u| > s\}} \right) \, dx \, d\beta
\]
\[
\leq |D| \lim_{\sigma \to 0} \mathbb{E} \int_0^t \int_D \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \chi_{\{|u| > s\}} - \Phi \chi_{\{|u| > s\}} \, dx \, d\beta.
\]
Since \( \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \chi_{\{|u| > s\}} - \Phi \chi_{\{|u| > s\}} \) converges to 0 a.e. in \( \Omega \times Q_T \) for \( \sigma \to 0 \) and
\[
\left| \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \chi_{\{|u| > s\}} - \Phi \chi_{\{|u| > s\}} \right|^2 \leq 4\Phi^2 \in L^1(\Omega \times Q_T),
\]
from Lebesgue’s dominated convergence theorem it follows that
\[
\lim_{\sigma \to 0} \int_0^t \int_D \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \, dx \, d\beta = \int_0^t \int_D \Phi \chi_{\{|u| > s\}} \, dx \, d\beta
\]
in \( L^2(\Omega) \). Moreover, from Itô isometry and Hölder inequality we get
\[
\lim_{s \to \infty} \left| \int_0^t \int_D \Phi \chi_{\{|u| > s\}} \, dx \, d\beta \right|^2 \leq \lim_{s \to \infty} |D| \int_0^t \int_D \Phi^2 \chi_{\{|u| > s\}} \, dx \, d\beta.
\]
Since \( \Phi^2 \chi_{\{|u| > s\}} \to 0 \) a.e. in \( \Omega \times Q_T \) for \( s \to \infty \) and since \( \Phi^2 \chi_{\{|u| > s\}} \leq \Phi^2 \in L^1(\Omega \times Q_T) \) we get from Lebesgue’s dominated convergence theorem that
\[
\lim_{s \to \infty} \int_0^t \int_D \Phi \chi_{\{|u| > s\}} \, dx \, d\beta = 0
\]
in \( L^2(\Omega) \). Therefore it follows that
\[
\lim_{s \to \infty} \lim_{\sigma \to 0} \int_0^t \int_D \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \, dx \, d\beta = 0
\]
in \( L^2(\Omega) \). Passing to suitable subsequences in \( \sigma \) and \( s \) we may conclude
\[
\lim_{s \to \infty} \lim_{\sigma \to 0} \int_0^t \int_D \frac{1}{\sigma} (T_{s+\sigma}(u) - T_s(u)) \Phi \, dx \, d\beta = 0
\]
a.s. in \( \Omega \). It follows that
\[
\lim_{s \to \infty} \lim_{\sigma \to 0} \epsilon(\sigma, s, \omega) = 0.
\]
Therefore we get
\[
\lim_{j \to \infty} \limsup_{k \to 0} \limsup_{\sigma \to 0} |I_3^3| = 0. \tag{47}
\]

Now let us consider the integrand of \( I_3^3 \) pointwise in \( Q_t \) for a fixed \( w \in \Omega \). We have \((T^\sigma_s)'(u) \to 0 \) a.e. in \( Q_T \) as \( \sigma \to 0 \). Hence the whole integrand of \( I_3^3 \) tends to \( 0 \) a.e. in \( Q_t \) as \( \sigma \to 0 \). W.l.o.g. assume that \( u \geq v \) at some point in \( Q_t \). Then we have \( \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \geq 0 \). Furthermore, we have \((T^\sigma_s)'(u) - (T^\sigma_s)'(v) > 0 \) if and only if \( s < v < s + \sigma < u \) or \( v < -s - \sigma < u < -s \). In both cases we can estimate easily that \( T^\sigma_s(u) - T^\sigma_s(v) \leq \frac{s}{2} \). Since \(|(T^\sigma_s)'(u) - (T^\sigma_s)'(v)| \leq \frac{s}{2} \) the results above yield
\[
\frac{1}{2} \Phi^2((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \leq \frac{1}{2k} \Phi^2
\]
a.e. in \( Q_t \) and \( \frac{1}{2k} \Phi^2 \in L^1(Q_t) \). Therefore Fatou’s Lemma yields
\[
\limsup_{\sigma \to 0} I_3^3 \leq 0 \tag{48}
\]
a.s. in \( \Omega \).

Now we consider the term \( I_4 \). We claim that
\[
\lim_{s \to \infty} \lim_{k \to 0} \lim_{\sigma \to 0} \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx = 0 \tag{49}
\]
in \( L^2(\Omega) \) for every \( t \in [0, T] \). Indeed, by Itô isometry and Lebesgue’s Theorem it follows with similar arguments as in the proof of (46) that
\[
\lim_{\sigma \to 0} \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx
\]
\[
= \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx, \tag{50}
\]
\[
\lim_{k \to 0} \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx
\]
\[
= \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \text{sign}_0(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx \tag{51}
\]
and
\[
\lim_{s \to \infty} \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \text{sign}_0(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx = 0 \tag{52}
\]
in \( L^2(\Omega) \), for every \( t \in [0, T] \). Now, combining (50), (51) and (52) yields (49).

Passing to suitable subsequences in \( \sigma, k \) and \( s \) from (49) it follows that
\[
\lim_{s \to \infty} \lim_{k \to 0} \lim_{\sigma \to 0} \int_D \int_0^t \Phi \cdot ((T^\sigma_s)'(u) - (T^\sigma_s)'(v)) \frac{1}{k} T_k(T^\sigma_s(u) - T^\sigma_s(v)) \, d\beta\, dx = 0 \tag{53}
\]
a.s. in \( \Omega \), for all \( t \in [0, T] \).
We have
\[
\limsup_{\sigma \to 0} I_5 \leq \limsup_{\sigma \to 0} I_5^{\varphi_{k,s}}
\]
\[
= \frac{1}{2k} \int_D \int_0^t \Phi^2((T_s)'(u) - (T_s)'(v))^2 \chi_{\{|T_s(u) - T_s(v)| \leq k\}} \ dr \ dx
\]
\[
= I_{5,1}^{k,s} + I_{5,2}^{k,s},
\]
where
\[
I_{5,1}^{k,s} = \frac{1}{2k} \int_D \int_0^t \Phi^2((T_s)'(u) - (T_s)'(v))^2 \chi_{\{|T_s(u) - T_s(v)| \leq k\}} \chi_{\{|u| > s\}} \chi_{\{|v| < s\}} \ dr \ dx,
\]
\[
I_{5,2}^{k,s} = \frac{1}{2k} \int_D \int_0^t \Phi^2((T_s)'(u) - (T_s)'(v))^2 \chi_{\{|T_s(u) - T_s(v)| \leq k\}} \chi_{\{|u| < s\}} \chi_{\{|v| > s\}} \ dr \ dx.
\]
In the following we show that there exists a subsequence \((s_j)_{j \in \mathbb{N}} \subset \mathbb{N}\) with \(\lim_{j \to \infty} s_j = +\infty\) that may vary with \(\omega \in \Omega\) such that

\[
\lim_{j \to \infty} \limsup_{k \to 0} I_{5,1}^{k,s_j} = \limsup_{j \to \infty} \limsup_{k \to 0} I_{5,2}^{k,s_j} = 0
\]
a.s. in \(\Omega\) and therefore

\[
\lim_{j \to \infty} \limsup_{k \to 0} \limsup_{\sigma \to 0} I_5^{\varphi_{k,s_j}} = 0
\] (54)
a.s. in \(\Omega\). We show the argumentation for \(I_{5,1}^{k,s}\). For symmetry reasons, the analogous result follows for \(I_{5,2}^{k,s}\) by interchanging the roles of \(u\) and \(v\). For \(\omega \in \Omega\) fixed, we have

\[
\limsup_{k \to 0} \left| I_{5,1}^{k,s} \right| = \limsup_{k \to 0} \frac{1}{2k} \int_D \int_0^t \Phi^2 \chi_{\{|T_s(u) - T_s(v)| \leq k\}} \chi_{\{||u| > s\}} \chi_{\{||v| < s\}} \ dr \ dx
\]
\[
\leq \limsup_{k \to 0} \frac{1}{2k} \int_{\{s-k \leq |u| \leq s\}} \Phi^2 \ dr \ dx.
\]
Since \(\Phi^2(\omega)\) is a nonnegative, integrable function for a.e. \(\omega \in \Omega\), the assertion follows from Lemma 6 in [8]. From (44) - (54) it follows that

\[
\int_D |u(t) - v(t)| \ dr \ dx \leq \int_D |u_0 - v_0| \ dr \ dx
\]
a.s. in \(\Omega\), for all \(t \in [0, T]\).

8. Markov property. Note that it is possible to replace the starting time 0 by a starting time \(r \in [0, T]\). In this case, we consider the filtration starting at time \(r\), i.e., \((\mathcal{F}_t)_{t \in [r, T]}\). Then, \(\tilde{\beta}_t := \beta_t - \beta_r, t \in [r, T]\), is a Brownian motion with respect to \((\mathcal{F}_t)_{t \in [r, T]}\) such that \(\sigma(\tilde{\beta}_t, t \geq r)\) is independent of \(\mathcal{F}_r\) (see, e.g., Remark 3.2 in [2]). Moreover, the augmentation \(\mathcal{F}_r\) of \(\sigma(\tilde{\beta}_t, t \geq r)\) is right-continuous and independent of \(\mathcal{F}_r\). Furthermore, we have \(d\tilde{\beta}_t = d\beta_t\) and all results and arguments still hold true in the case of a starting time \(r \in [0, T]\) and \(\mathcal{F}_r\)-measurable initial conditions \(u_r \in L^1(\Omega \times D)\). In this section, we denote by \(u(t, r, u_r)\) the unique renormalized solution of (1) starting in \(u_r\) at time \(r\) for \(t, r \in [0, T]\) with \(r \leq t\) and \(u_r \in L^1(\Omega \times D)\) \(\mathcal{F}_r\)-measurable.
Proposition 8.1. For all \( r, s, t \in [0, T] \) with \( r \leq s \leq t \) and all \( u_r \in L^1(\Omega \times D) \) \( \mathcal{F}_r \)-measurable we have
\[
 u(t, s, u(s, r, u_r)) = u(t, r, u_r)
\]
as in \( \Omega \).

Proof. The proof is similar to the proof in [2] or [24]. Let \( r \in [0, T], \psi \in C^\infty([r, T] \times \bar{D}) \) and \( S \in C^2(\mathbb{R}) \) such that \( S' \) has compact support with \( S'(0) = 0 \) or \( \psi(t, x) = 0 \) for all \( (t, x) \in [r, T] \times \partial D \). Now we fix \( s, t \in [0, T] \) with \( r \leq s \leq t \) and \( u_r \in L^1(\Omega \times D) \) \( \mathcal{F}_r \)-measurable. Since \( u(\cdot, r, u_r) \) is the unique renormalized solution to (1) starting in \( u_r \) at time \( r \) we have
\[
\int_D S(u(t, r, u_r)) \psi(t) \, dx = \int_D S(u_r) \psi(r) \, dx - \int_r^t \int_D S''(u(\tau, r, u_r)) |\nabla u(\tau, r, u_r)|^p \psi \, dx \, d\tau
- \int_r^t \int_D S'(u(\tau, r, u_r)) |\nabla u(\tau, r, u_r)|^{p-2} \nabla u(\tau, r, u_r) \cdot \nabla \psi \, dx \, d\tau
+ \int_r^t \int_D S'(u(\tau, r, u_r)) \psi \Phi \, dx \, d\beta + \int_r^t \int_D S(u(\tau, r, u_r)) \psi_t \, dx \, d\tau
+ \frac{1}{2} \int_r^t \int_D S''(u(\tau, r, u_r)) \psi^2 \, dx \, d\tau
= \int_D S(u(s, r, u_r)) \psi(s) \, dx - \int_s^t \int_D S''(u(\tau, r, u_r)) |\nabla u(\tau, r, u_r)|^p \psi \, dx \, d\tau
- \int_s^t \int_D S'(u(\tau, r, u_r)) |\nabla u(\tau, r, u_r)|^{p-2} \nabla u(\tau, r, u_r) \cdot \nabla \psi \, dx \, d\tau
+ \int_s^t \int_D S'(u(\tau, r, u_r)) \psi \Phi \, dx \, d\beta + \int_s^t \int_D S(u(\tau, r, u_r)) \psi_t \, dx \, d\tau
+ \frac{1}{2} \int_s^t \int_D S''(u(\tau, r, u_r)) \psi^2 \, dx \, d\tau
\]
as in \( \Omega \). Therefore \( u(t, r, u_r) \) is a renormalized solution to (1) starting in \( u(s, r, u_r) \) at time \( s \). Uniqueness yields the result.

\[\text{Theorem 8.2.} \quad \text{Let} \ u_r \in L^1(\Omega \times D), \ r \in [0, T] \ \text{be} \ \mathcal{F}_r \text{-measurable. The unique renormalized solution} \ u(t) = u(t, r, u_r), \ t \in [r, T], \ \text{of (1) starting in} \ u_r \ \text{at time} \ r \ \text{satisfies the Markov property in the following sense:}
\]
For every bounded and \( \mathcal{B}(L^1(D)) \)-measurable function \( G : L^1(D) \to \mathbb{R} \) and all \( s, t \in [r, T] \) with \( s \leq t \) we have
\[
\mathbb{E}[G(u(t)) | \mathcal{F}_s](\omega) = \mathbb{E}[G(u(t, s, u(s, r, u_r)(\omega)))]
\]
for a.e. \( \omega \in \Omega \).

Proof. We apply Lemma 4.1 in [2] (The freezing Lemma). To this end we set for fixed \( r, t, s \in [0, T] \) with \( r \leq s \leq t \), \( u_r \in L^1(\Omega \times D) \) \( \mathcal{F}_r \)-measurable and a fixed bounded and \( \mathcal{B}(L^1(D)) \)-measurable function \( G : L^1(D) \to \mathbb{R} : D = \mathcal{F}_s, G = \mathcal{F}_t, E = L^1(D), \mathcal{E} = \mathcal{B}(L^1(D)), X = u(s) = u(s, r, u_r) \) and \( \psi : L^1(D) \times \Omega \to \mathbb{R}, \psi(x, \omega) = G(u(t, s, u(s, r, u_r)(\omega))) \).

It is only left to prove that \( \psi \) is \( \mathcal{B}(L^1(D)) \otimes \mathcal{F}_t \)-measurable. Since \( G \) is \( \mathcal{B}(L^1(D)) \)-measurable it is left to show that \( \phi : L^1(D) \times \Omega \to L^1(D), \phi(x, \omega) = u(t, s, x)(\omega) \) is
Corollary 8.3. For all \( \Omega \ni \omega \mapsto \phi(x, \omega) \) is \( \tilde{\mathcal{F}}_t \)-measurable for all \( x \in L^1(D) \),

(i) \( \Omega \ni \omega \mapsto \phi(x, \omega) \) is \( \tilde{\mathcal{F}}_t \)-measurable for all \( x \in L^1(D) \),

(ii) \( L^1(D) \ni x \mapsto \phi(x, \omega) \) is continuous for almost every \( \omega \in \Omega \).

Since it is possible to choose the filtration \( \tilde{\mathcal{F}}_t \) instead of the filtration \( (\mathcal{F}_t)_{t \in [s,T]} \), Theorem 6.1 yields that for fixed \( x \in L^1(D) \) the function \( u(t, s, x) \) is \( \mathcal{F}_t \)-measurable.

Moreover, Theorem 7.1 yields that the mapping in (ii) is a contraction for almost every \( \omega \in \Omega \), especially it is continuous.

Now, Lemma 4.1. in [2] is applicable and yields the assertion. \( \square \)

For \( s, t \in [0,T] \), \( s \leq t \) and \( x \in L^1(D) \) we set \( P_{s,t} : B_b(L^1(D)) \to B_b(L^1(D)) \),

\[
P_{s,t}(\varphi)(x) = \mathbb{E}[^t_s \varphi(u(t, s, x))],
\]

where \( B_b(L^1(D)) \) denotes the space of all bounded Borel functions from \( L^1(D) \) to \( \mathbb{R} \). Moreover, we set \( P_t := P_{0,t} \).

As a consequence of Theorem 8.2 we obtain the Chapman-Kolmogorov property:

**Corollary 8.3.** For \( r, s, t \in [0,T] \), \( r \leq s \leq t \), \( x \in L^1(D) \) and \( \varphi \in B_b(L^1(D)) \) we have

\[
P_{r,t}(\varphi)(x) = P_{r,s}(P_{s,t}(\varphi))(x). \tag{55}
\]

**Proof.** Let \( r, s, t \in [0,T] \), \( r \leq s \leq t \), \( x \in L^1(D) \) and \( \varphi \in B_b(L^1(D)) \). From Theorem 8.2 it follows that

\[
P_{r,t}(\varphi)(x) = \mathbb{E}[^t_s \mathbb{E}[^r_s \varphi(u(t, r, x))]|\mathcal{F}_s] = \mathbb{E}[\mathbb{E}[^t_s \varphi(u(t, s, u(s, r, x)))]|\mathcal{F}_s]
\]

\[
= \mathbb{E}[P_{r,s}(\varphi)(u(s, r, x))] = P_{r,s}(P_{s,t}(\varphi))(x).
\]

\( \square \)

**Corollary 8.4.** For all \( s, t \in [0,T] \), \( s \leq t \), we get

\[
P_{s,t} = P_{0,t-s}.
\]

In particular, \( (P_t)_{t \in [0,T]} \) is a semigroup.

**Proof.** Similar as in Proposition 8.1 we can show that \( u(\tau + s, s, x) = u^\beta(\tau, 0, x) \) for \( s \in [0,T] \), \( \tau \in [0, T-s] \), where \( \dot{\beta}(\tau) = \beta(\tau + s) - \beta(s) \) and \( u^\beta(\tau, 0, x) \) is the unique renormalized solution to (1) with respect to the Brownian motion \( \dot{\beta} \) and initial value \( x \in L^1(D) \). Since renormalized solutions to (1) are pathwise unique, they are jointly unique in law (see, e.g., [25], Theorem 2) and therefore we have

\[
P_{s,s+\tau} = P_{0,\tau}.
\]

Setting \( t = \tau + s \) yields the assertion. \( \square \)

Now we show that \( P_{s,t} \) is Feller and \( (P_t)_{t \in [0,T]} \) is a Feller semigroup (see, e.g., [12], p. 247):

**Proposition 8.5.** For all \( s, t \in [0,T] \), \( s \leq t \) we have \( P_{s,t}(C_b(L^1(D))) \subset C_b(L^1(D)) \).

**Proof.** Let \( s, t \in [0,T] \), \( s \leq t \) and \( \varphi \in C_b(L^1(D)) \). Let \( (x_n) \subset L^1(D) \) such that \( x_n \to x \) in \( L^1(D) \). Theorem 7.1 and the continuity of \( \varphi \) yields

\[
\varphi(u(t, s, x_n)) \to \varphi(u(t, s, x))
\]
a.e. in $\Omega$. Since $\varphi$ is bounded this convergence is also a convergence in $L^1(\Omega)$ by Lebesgue’s Theorem. Therefore we have

$$P_{s,t}(\varphi(x)) = \mathbb{E}(\varphi(u(t,s,x_n))) \to \mathbb{E}(\varphi(u(t,s,x))) = P_{s,t}(\varphi).$$

Since $|P_{s,t}(\varphi)(x)| \leq \|\varphi\|_\infty < \infty$ for all $x \in L^1(D)$ we may conclude $P_{s,t}(\varphi) \in C_b(L^1(D))$.

**Proposition 8.6.** The family $P_{s,t}$, $s,t \in [0,T]$, $s \leq t$, has the $e$-property in the sense of [22], i.e.: For all $\varphi \in \text{Lip}_b(L^1(D))$, $x \in L^1(D)$ and $\epsilon > 0$ there exists $\delta > 0$ such that for all $z \in B(x,\delta)$ and all $0 \leq s \leq t \leq T$:

$$|P_{s,t}(\varphi)(x) - P_{s,t}(\varphi)(z)| < \epsilon,$$

where $\text{Lip}_b(L^1(D))$ denotes the space of all bounded Lipschitz continuous functions from $L^1(D)$ to $\mathbb{R}$.

**Proof.** Let $\varphi \in \text{Lip}_b(L^1(D))$, $x \in L^1(D)$ and $\epsilon > 0$ and let $L > 0$ be a Lipschitz constant of $\varphi$. We set $\delta := \frac{\epsilon}{L}$. Then, for all $z \in B(x,\delta)$ and all $0 \leq s \leq t \leq T$ Theorem 7.1 yields

$$|P_{s,t}(\varphi)(x) - P_{s,t}(\varphi)(z)| = |\mathbb{E}(\varphi(u(t,s,x)) - \varphi(u(t,s,z))|$$

$$\leq L \cdot \mathbb{E}[\|u(t,s,x) - u(t,s,z)\|_1]$$

$$\leq L \cdot \|x - z\|_1 < L \cdot \delta = \epsilon. \quad \Box$$

As in [24] we define for $x \in L^1(D)$

$$\mathbb{P}_x := P \circ (u(\cdot,0,x))^{-1},$$

i.e., $\mathbb{P}_x$ is the distribution of the unique renormalized solution to (1) with initial condition $x \in L^1(D)$, defined as a probability measure on $C([0,T];L^1(D))$. We equip $C([0,T];L^1(D))$ with the $\sigma$-Algebra

$$\mathcal{G} := \sigma(\pi_s, \ s \in [0,T])$$

and filtration

$$\mathcal{G}_t := \sigma(\pi_s, \ s \in [0,t]), \ t \in [0,T],$$

where $\pi_t : C([0,T];L^1(D)) \to L^1(D), \pi_t(w) := w(t)$. Finally we can prove the following property of $\mathbb{P}_x$:

**Proposition 8.7.** The measure-valued process $(\mathbb{P}_x)_{x \in L^1(D)}$ is a time-homogenous Markov process on $C([0,T];L^1(D))$ with respect to the filtration $(\mathcal{G}_t)_{t \in [0,T]}$, i.e., for all $s,t \in [0,T]$ such that $s + t \leq T$ and all $\varphi \in B_b(L^1(D))$ we have

$$\mathbb{E}_x(\varphi(\pi_{s+t})|\mathcal{G}_s) = \mathbb{E}_{\pi_s}(\varphi(\pi_t)) \quad (56)$$

$\mathbb{P}_x$-a.s., where $\mathbb{E}_x$ and $\mathbb{E}_x(\cdot|\mathcal{G}_s)$ denote the expectation and the conditional expectation with respect to $\mathbb{P}_x$, respectively.

**Proof.** We start the proof by showing the right-hand side of (56) to be $\mathcal{G}_s$-measurable. This follows by applying the monotone class argument to the set

$$\mathcal{H} := \{ \varphi : L^1(D) \to \mathbb{R}, \ \mathbb{E}_{\pi_s}(\varphi(\pi_t)) : C([0,T];L^1(D)) \to \mathbb{R} \text{ is } \mathcal{G}_s \text{ - measurable.} \}$$
For the rest of the proof we follow the ideas in [12]. Let, for arbitrary \( n \in \mathbb{N} \), 

\( G : L^1(D)^n \rightarrow \mathbb{R} \) be a bounded \( \otimes_{i=1}^n \mathcal{B}(L^1(D)) \)-measurable function and \( 0 \leq t_1 < ... < t_n \leq s \). Then from Theorem 8.2 and Corollary 8.4 it follows

\[
\mathbb{E}_x [G(\pi_{t_1}, ..., \pi_{t_n}) \varphi(\pi_{t+s})] = \mathbb{E}_x \left[ G(\pi_{t_1}, ..., \pi_{t_n}) \mathbb{E}_{\pi_x} [\varphi(\pi_t)] \right].
\]

This yields the assertion. \( \square \)

9. Appendix.

9.1. Proof of Lemma 4.1. For \( n \in \mathbb{N} \), we define the following disjoint subdivision of \( D \):

\[
D_n := \{ x \in D \mid \text{dist}(x, \partial D) \geq \frac{2}{n} \},
\]

\[
B_n := \{ x \in D \mid \text{dist}(x, \partial D) \leq \frac{1}{n} \},
\]

\[
H_n := \{ x \in D \mid \frac{1}{n} < \text{dist}(x, \partial D) < \frac{2}{n} \}.
\]

In particular, \((D_n)_{n \in \mathbb{N}}\) is an increasing sequence of domains in \( D \) such that \( D_n \subset \subset D_{n+1} \subset D \) for all \( n \in \mathbb{N} \) with

\[
\bigcup_{n \in \mathbb{N}} D_n = D.
\]

We choose a sequence of cutoff functions \((\varphi_n)_{n \in \mathbb{N}} : D \rightarrow \mathbb{R}\) such that \( \varphi_n \in C_0^\infty(D) \), \( 0 \leq \varphi_n \leq 1 \) in \( D \), \( \varphi_n \equiv 1 \) on \( D_n \), \( \varphi_n \equiv 0 \) on \( B_n \) and \( |\nabla \varphi_n| \leq 2n \) for all \( n \in \mathbb{N} \). Let \((\rho_n)_{n \in \mathbb{N}} \subset C_0^\infty(\mathbb{R}^d)\) be a sequence of symmetric mollifiers with support in \([-\frac{1}{n}, \frac{1}{n}]\).

For \( n \in \mathbb{N} \) we define the linear operator

\[
\Pi_n : W^{-1,\varphi'}(D) + L^1(D) \rightarrow W_0^{1,p}(D) \cap L^\infty(D),
\]

\[
v \mapsto (\varphi_n v) * \rho_n.
\]

We recall that \( v \in W^{-1,\varphi'}(D) + L^1(D) \) iff there exist \( G \in L^{\varphi'}(D)^d \), \( f \in L^1(D) \) such that \( v = -\text{div } G + f \) in \( D\) and, according to the multiplication and convolution of distributions (see, e.g., [29], Def. 1.5., p. 15 and Def. 1.6., p. 20)

\[
\Pi_n(v)(x) = ((\varphi_n f) * \rho_n)(x) + (-\text{div } G, \varphi_n(\cdot)\rho_n(x-\cdot))_{W^{-1,\varphi'}(D),W_0^{1,p}(D)}
\]

\[
= \int_D \rho_n(x-y) \varphi_n(y) f(y) \, dy + \int_D G(y) \nabla y [\rho_n(x-y) \varphi_n(y)] \, dy \tag{57}
\]

for all \( x \in \mathbb{R}^d \). From the definition of \( \Pi_n \) it follows immediately that \( \Pi_n \) is linear and from (57) we get that \( \Pi_n(v) \) is a smooth function with \( \Pi_n(v) = 0 \) on \( D^C \) for all \( n \in \mathbb{N} \). A straightforward calculation shows that, for arbitrary \( v = -\text{div } G + f \in W^{-1,\varphi'}(D) + L^1(D) \), there exists a constant \( C \geq 0 \) not depending on \( f \) and \( G \) that may depend on \( n \in \mathbb{N} \), such that

\[
\| \Pi_n v \|_{W_0^{1,p}(D) \cap L^\infty(D)} =
\max(\| \Pi_n(v) \|_{L^\infty(D)}, \| \Pi_n(v) \|_{W_0^{1,p}(D)}) \leq C(\| f \|_{L^1(D)} + \| -\text{div } G \|_{W^{-1,\varphi'}(D)}) \tag{58}
\]

and, passing to the infimum over all \( f \in L^1(D) \), \( G \in L^{\varphi'}(D)^d \) such that \( v = f - \text{div } G \) in (58), we get that \( \Pi_n \) is a bounded linear operator from \( W^{-1,\varphi'}(D) + L^1(D) \) into \( W_0^{1,p}(D) \cap L^\infty(D) \) for any \( n \in \mathbb{N} \). For \( F \in \{ W_0^{1,p}(D), L^2(D), L^1(D) \} \) and every \( v \in F \subset W^{-1,\varphi'}(D) + L^1(D) \), from the classical properties of the convolution and
Young inequality it follows that $\Pi_n \in L(F)$ for any $n \in \mathbb{N}$ and $\Pi_n(v) \to v$ in $F$ for $n \to \infty$. For arbitrary $v \in W^{-1,p'}(D) + L^1(D)$,

$$\lim_{n \to \infty} \|\Pi_n(v) - v\|_{W^{-1,p'}(D) + L^1(D)} = 0$$

if

$$\lim_{n \to \infty} \left(\|\Pi_n(f) - f\|_{L^1(D)} + \|\Pi_n(-\div G) - (-\div G)\|_{W^{-1,p'}(D)}\right) = 0 \quad (59)$$

for all $f \in L^1(D)$, $G \in L^{p'}(D)^d$ such that $v = f - \div G$. Thus, to conclude the proof, the convergence of $\Pi_n(-\div G) - -\div G$ for $n \to \infty$ in $W^{-1,p'}(D)$ for arbitrary $G \in L^{p'}(D)^d$ deserves our attention. For $g \in W_0^{1,p}(D)$, we have

$$\left|\langle \Pi_n(-\div G), g \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)}\right|$$

$$= \left|\int_D g(x) (-\div G, \varphi_n(\cdot)\rho_n(x - \cdot))_{W^{-1,p'}(D), W_0^{1,p}(D)} dx\right|$$

$$= \left|\int_D \int_D G(y) \cdot (\nabla_y \varphi_n(y))\rho_n(x - y) + (\nabla_y \rho_n(x - y))\varphi_n(y) dy g(x) dx\right|$$

$$\leq I_1^n + I_2^n,$$

where

$$I_1^n = \left|\int_D \int_D G(y) \cdot (\nabla_y \varphi_n(y))\rho_n(x - y) dy g(x) dx\right|,$$

$$I_2^n = \left|\int_D \int_D G(y) \cdot (\nabla_y \rho_n(x - y))\varphi_n(y) dy g(x) dx\right|.$$

Recalling that $\nabla_y \rho_n(x - y) = -\nabla_x \rho_n(x - y)$ using Fubini’s theorem and Young’s inequality it follows that

$$I_2^n = \left|\int_D \varphi_n(y)G(y) \cdot \int_D \nabla_x \rho_n(x - y)g(x) dx dy\right|$$

$$= \left|\int_D \varphi_n(y)G(y) \cdot \nabla_y [\rho_n * g](y) dy\right|$$

$$\leq \|G\|_{L^{p'}(D)^d} \|\nabla g\|_{L^p(D)} \quad (60)$$

for all $n \in \mathbb{N}$. Thanks to Fubini’s theorem and to the properties of $\nabla \varphi_n$ and using Hölder and Young’s inequality we get

$$I_1^n = \left|\int_{H_n} \int_D \rho_n(x - y)g(x) dx G(y) \cdot (\nabla_y \varphi_n(y)) dy\right|$$

$$\leq \int_{H_n} |n(\rho_n * g)(y)||G(y)| dy$$

$$\leq \|G\|_{L^{p'}(H_n)^d} \|n(\rho_n * g)||_{L^p(H_n)}$$

$$\leq \|G\|_{L^{p'}(H_n)^d} \left[\int_{H_n} \left(\frac{|g(y)|}{\frac{2}{n}}\right)^p dy\right]^{1/p}. \quad (61)$$

Recalling that for all $y \in H_n$ we have $\text{dist}(y, \partial D) < \frac{2}{n}$ it follows that

$$I_1^n \leq 2^p \|G\|_{L^{p'}(H_n)^d} \left[\int_D \left(\frac{|g(y)|}{\text{dist}(y, \partial D)}\right)^p dy\right]^{1/p}. \quad (62)$$
Now, using Hardy’s inequality we conclude that there exists a constant $C \geq 0$ not depending on $n \in \mathbb{N}$ such that
\[
I^n_1 \leq C \|G\|_{L^{p'}(H_n)} \|\nabla g\|_{L^p(D)}^d.
\]
From (60) and (62) it follows that
\[
\Pi_n(\text{div } G)_{W^{-1,p'}(D)} \leq \|G\|_{L^{p'}(D)}^d + C \left( \int_{H_n} |G(y)|^{p'} dy \right)^{1/p'}
\]
for all $n \in \mathbb{N}$, and therefore $\|\Pi_n(\text{div } G)\|_{W^{-1,p'}(D)}$ is bounded with respect to $n \in \mathbb{N}$ for any $v = -\text{div } G \in W^{-1,p'}(D)$. The proof of Theorem 4.15 in [21] yields that
\[
\Pi_n(\text{div } G) \to -\text{div } G
\]
in $D'(D) = (C_0^\infty(D))^\ast$. Hence by density of $C_0^\infty(D)$ in $W_0^{1,p}(D)$ and boundedness of $\Pi_n(\text{div } G)$ in $W^{-1,p'}(D)$ we get
\[
\lim_{n \to \infty} \Pi_n(\text{div } G) = -\text{div } G
\]
weakly in $W^{-1,p'}(D)$. Finally, we remark that from (63) we also get
\[
\limsup_{n \to \infty} \|\Pi_n(\text{div } G)\|_{W^{-1,p'}(D)} \leq \|\text{div } G\|_{W^{-1,p'}(D)}.
\]
Now, from (65) and the uniform convexity of $W^{-1,p'}(D)$ it follows that (64) holds strongly in $W^{-1,p'}(D)$ and therefore (59) holds true. In particular, we have obtained $\Pi_n \in L(F)$ and $\Pi_n(v) \to v$ for $v \in F$ and $n \to \infty$ in the case $F = W^{-1,p'}(D)$ and $F = W^{-1,p'}(D) + L^1(D)$.

9.2. The Itô product rule. In the well-posedness theory of renormalized solutions in the deterministic setting (see, e.g., [6]), the product rule is a crucial part. In the following lemma, we propose an Itô product rule for strong solutions to (1). In the following, we will call a function $f : \mathbb{R} \to \mathbb{R}$ piecewise continuous, iff it is continuous except for finitely many points.

**Proposition 9.1.** For $1 < p < \infty$, $u_0, v_0 \in L^2(\Omega \times D)$ $\mathcal{F}_0$-measurable let $u$ be a strong solution to (1) with initial datum $u_0$ and $v$ be a strong solution to (1) with initial datum $v_0$ respectively. Then, for any $H \in C^2_b(\mathbb{R})$ and any $Z \in W^{2,\infty}(\mathbb{R})$ with $Z''$ piecewise continuous such that $Z(0) = Z'(0) = 0$

\[
(Z((u - v)(t)), H(u(t)))_2 = (Z(u_0 - v_0), H(u_0))_2 + \int_0^t \langle \Delta_p(u) - \Delta_p(v), H(u)Z'(u - v) \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} ds + \int_0^t \langle \Delta_p(u), H'(u)Z(u - v) \rangle_{W^{-1,p'}(D), W_0^{1,p}(D)} ds + \int_0^t \langle \Phi H'(u), Z(u - v) \rangle_2 d\beta + \frac{1}{2} \int_0^t \int_D \Phi^2 H''(u) Z(u - v) dx ds
\]

for all $t \in [0,T]$, a.s. in $\Omega$.

**Proof.** We fix $t \in [0,T]$. Since $u, v$ are strong solutions to (1), it follows that

\[
u(t) = u_0 + \int_0^t \Delta_p(u) ds + \int_0^t \Phi d\beta,
\]

(67)
\[ v(t) = v_0 + \int_0^t \Delta_p(v) \, ds + \int_0^t \Phi \, d\beta \]

and consequently

\[ (u - v)(t) = u_0 - v_0 + \int_0^t \Delta_p(u) - \Delta_p(v) \, ds \]  \hspace{1cm} (68)

holds in \( L^2(D) \), a.s. in \( \Omega \). For \( n \in \mathbb{N} \) we define \( \Pi_n \) according to Lemma 4.1 and set \( \Phi_n := \Pi_n(\Phi), \ u_0^n := \Pi_n(u_0), \ v_0^n := \Pi_n(v_0), \ u_n := \Pi_n(u), \ v_n := \Pi_n(v), \ U_n := \Pi_n(\Delta_p(u)), \ V_n := \Pi_n(\Delta_p(v)) \). Applying \( \Pi_n \) on both sides of (68) yields

\[ (u_n - v_n)(t) = u_0^n - v_0^n + \int_0^t U_n - V_n \, ds \]  \hspace{1cm} (69)

and applying \( \Pi_n \) on both sides of (67) yields

\[ u_n(t) = u_0^n + \int_0^t U_n \, ds + \int_0^t \Phi_n \, d\beta \]  \hspace{1cm} (70)

in \( W_0^{1,p}(D) \cap L^2(D) \cap C^\infty(D) \) a.s. in \( \Omega \). The pointwise Itô formula in (69) and (70) leads to

\[ Z(u_n - v_n)(t) = Z(u_0^n - v_0^n) + \int_0^t (U_n - V_n)Z'(u_n - v_n) \, ds \]  \hspace{1cm} (71)

and

\[ H(u_n)(t) = H(u_0^n) + \int_0^t U_nH'(u_n) \, ds + \int_0^t \Phi_nH'(u_n) \, d\beta + \frac{1}{2} \int_0^t \Phi_n^2H''(u_n) \, ds \]  \hspace{1cm} (72)

in \( D \), a.s. in \( \Omega \). From (71), (72) and the product rule for Itô processes, which is just an easy application of the classic two-dimensional Itô formula (see, e.g., [2], Proposition 8.1, p. 218), applied pointwise in \( t \) for fixed \( x \in D \) it follows that

\[
\begin{align*}
Z(u_n - v_n)(t)H(u_n)(t) & = Z(u_0^n - v_0^n)H(u_0^n) \\
& + \int_0^t (U_n - V_n)Z'(u_n - v_n)H(u_n) \, ds + \int_0^t U_nH'(u_n)Z(u_n - v_n) \, ds \\
& + \int_0^t \Phi_nH'(u_n)Z(u_n - v_n) \, d\beta + \frac{1}{2} \int_0^t \Phi_n^2H''(u_n)Z(u_n - v_n) \, ds
\end{align*}
\]  \hspace{1cm} (73)

in \( D \), a.s. in \( \Omega \). Integration over \( D \) in (73) yields

\[ I_1 = I_2 + I_3 + I_4 + I_5 + I_6, \]  \hspace{1cm} (74)

where

\[
\begin{align*}
I_1 & = (Z((u_n - v_n)(t)), H((u_n)(t)))_2, \\
I_2 & = (Z(u_0^n - v_0^n), H(u_0^n))_2, \\
I_3 & = \int_0^t \int_D (U_n - V_n)Z'(u_n - v_n)H(u_n) \, dx \, ds, \\
I_4 & = \int_0^t \int_D U_nH'(u_n)Z(u_n - v_n) \, dx \, ds,
\end{align*}
\]
bounded functions, it follows that $v$ and $a.s. \in \Omega$. For any fixed $\in \mathbb{L}_n$ and from the properties of $\Pi n$ dominated convergence theorem it follows that 

$$\lim_{n \to \infty} I_1 = (Z((u - v)(t)), H'(u(t)))_2,$$

$$\lim_{n \to \infty} I_2 = (Z(u_0 - v_0), H'(u_0))_2$$

in $\mathbb{L}^2(\Omega)$ and a.s. in $\Omega$. Note that 

$$I_3 = \int_0^t \langle (U_n - V_n), Z'(u_n - v_n)H(u_n) \rangle_{W^{-1,p'}(D), W^{1,p}(D)} \, ds$$

a.s. in $\Omega$ and from the properties of $\Pi n$ it follows that 

$$\lim_{n \to \infty} U_n(\omega, s) - V_n(\omega, s) = \Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s))$$

in $W^{-1,p'}(D)$ for all $s \in [0,t]$ and a.e. $\omega \in \Omega$. Recalling the convergence result for $(\Pi n)$ from Lemma 4.1, there exists a constant $C_1 \geq 0$ not depending on $s, \omega$ and $n \in \mathbb{N}$ such that 

$$\|U_n(\omega, s) - V_n(\omega, s)\|_{W^{-1,p'}(D)} = \|\Pi n(\Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s)))\|_{W^{-1,p'}(D)} \leq C_1\|\Delta_p(u(\omega, s)) - \Delta_p(v(\omega, s))\|_{W^{-1,p'}(D)}.$$

Since the right-hand side of the above equation is in $\mathbb{L}^p(\Omega \times (0,t))$, from Lebesgue’s dominated convergence theorem it follows that 

$$\lim_{n \to \infty} U_n - V_n = \Delta_p(u) - \Delta_p(v)$$

in $\mathbb{L}^p(\Omega \times (0,t); W^{-1,p'}(D))$ and, with a similar reasoning, also in $\mathbb{L}^p(0,t; W^{-1,p'}(D))$ a.s. in $\Omega$. From the chain rule for Sobolev functions it follows that 

$$\nabla(Z'(u_n - v_n)H(u_n)) = Z''(u_n - v_n)\nabla(u_n - v_n)H(u_n) + Z'(u_n - v_n)H'(u_n)\nabla u_n$$

(77)

a.s. in $(0,t) \times \Omega$.

Moreover, there exists a constant $C_2 = C_2(\|Z'\|_\infty, \|Z''\|_\infty, \|H\|_\infty, \|H'\|_\infty) \geq 0$ such that 

$$\int_0^t \|\nabla(Z'(u_n - v_n)H(u_n))\|^p_p \, ds \leq C_2 \int_0^t (\|\nabla u\|^p_p + \|\nabla v\|^p_p) \, ds$$

(78)

a.s. in $\Omega$. Consequently, for almost every $\omega \in \Omega$ there exists $\chi(\omega) \in \mathbb{L}^p(0,t; W^{1,p}_0(D))$ such that, passing to a not relabeled subsequence that may depend on $\omega \in \Omega$, 

$$Z'(u_n - v_n)H(u_n) \to \chi(\omega)$$

(79)

weakly in $\mathbb{L}^p(0,t; W^{1,p}_0(D))$. Since in addition, 

$$\lim_{n \to \infty} Z'(u_n - v_n)H(u_n) \to Z'(u - v)H(u)$$

This completes the proof. 

in $L^p((0, t) \times D)$ a.s. in $\Omega$, we get
\[ \chi(\omega) = Z'(u - v) H(u) \]  
(80)
in $L^p(0, t; W^{1,p}_0(D))$ a.s. in $\Omega$ and the weak convergence in (79) holds for the whole sequence. Therefore,
\[ Z'(u_n - v_n) H(u_n) \to Z'(u - v) H(u) \]
for $n \to \infty$ weakly in $L^p(0, t; W^{1,p}_0(D))$ for almost every $\omega \in \Omega$. Resuming the above results it follows that
\[ \lim_{n \to \infty} I_3 = \int_0^t \langle \Delta_p(u) - \Delta_p(v), Z'(u - v) H(u) \rangle_{W^{-1,p'}(D), W^{1,p}(D)} \, ds \]  
(81)
a.s. in $\Omega$. With analogous arguments we get
\[ \lim_{n \to \infty} I_4 = \int_0^t \langle \Delta_p(u), H'(u) Z(u - v) \rangle_{W^{-1,p'}(D), W^{1,p}(D)} \, ds \]  
(82)
a.s. in $\Omega$. By Itô isometry,
\[ \mathbb{E} \left| \int_0^t \int_D \Phi_n H'(u_n) Z(u_n - v_n) - \Phi H'(u) Z(u - v) \, dx \, d\beta \right|^2 \]
\[ = \mathbb{E} \int_0^t \int_D |\Phi_n H'(u_n) Z(u_n - v_n) - \Phi H'(u) Z(u - v)|^2 \, dx \, ds. \]
From the convergence
\[ \Phi_n H'(u_n) Z(u_n - v_n) \to \Phi H'(u) Z(u - v) \]
in $L^2(D)$ for $n \to \infty$ a.s. in $\Omega \times (0, t)$ and since, for almost any $(\omega, s)$, there exists a constant $C_3 \geq 0$ not depending on the parameters $n, s, \omega$ such that
\[ \|\Phi_n(\omega, s) H'(u_n(\omega, s)) Z(u_n(\omega, s) - v_n(\omega, s))\|_2 \leq C_3 \|\Phi(\omega, s)\|_2 \]
for all $n \in \mathbb{N}$, a.s. in $\Omega \times (0, t)$, it follows that
\[ \lim_{n \to \infty} \Phi_n H'(u_n) Z(u_n - v_n) = \Phi H'(u) Z(u - v) \]
in $L^2(\Omega \times (0, t) \times D)$ and consequently
\[ \lim_{n \to \infty} I_5 = \int_0^t \int_D \Phi H'(u) Z(u - v) \, dx \, d\beta \]  
(83)
in $L^2(\Omega)$ and, passing to a subsequence if necessary, also a.s. in $\Omega$. According to the properties of $(\Pi_n)$, $\Phi_n^* \to \Phi^*$ in $L^1((0, t) \times D)$ for $n \to \infty$ a.s. in $\Omega$. From the boundedness and the continuity of $H''$ and $Z$ we get
\[ \lim_{n \to \infty} H''(u_n) Z(u_n - v_n) = H''(u) Z(u - v) \]
in $L^q((0, t) \times D)$ for all $1 \leq q < \infty$ and weak-* in $L^\infty((0, t) \times D)$ a.s. in $\Omega$, thus it follows that
\[ \lim_{n \to \infty} I_6 = \frac{1}{2} \int_0^t \int_D \Phi^2 H''(u) Z(u - v) \, dx \, ds \]  
(84)
a.s. in $\Omega$. Passing to a subsequence if necessary, taking the limit in (73) for $n \to \infty$ a.s. in $\Omega$ the assertion follows from (73)-(84).

Corollary 9.2. Proposition 9.1 still holds true for $H \in W^{2,\infty}(\mathbb{R})$ such that $H''$ is piecewise continuous.
Proof. There exists an approximating sequence \((H_\delta)_{\delta>0} \subset C^2_0(\mathbb{R})\) such that \(\|H_\delta\|_\infty \leq \|H\|_\infty, \|H''_\delta\|_\infty \leq \|H''\|_\infty\) for all \(\delta > 0\) and \(H_\delta \to H, H'_\delta \to H'\) uniformly on compact subsets, \(H''_\delta \to H''\) pointwise in \(\mathbb{R}\) for \(\delta \to 0\). With this convergence we are able to pass to the limit with \(\delta \to 0\) in (66). □

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