QUALITATIVE PROPERTIES FOR SOLUTIONS TO CONFORMALLY INVARIANT FOURTH ORDER CRITICAL SYSTEMS

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Abstract. We study properties of nonnegative solutions to a class of conformally invariant coupled system of fourth order equations involving critical exponents. For solutions defined in the punctured space, there exist essentially two cases to analyze. If the origin is a removable singularity, we prove that regular solutions are rotationally symmetric and weakly positive. More precisely, they are the product of a fourth order spherical solution by a unit vector with nonnegative coordinates. If the origin is a non-removable singularity, we show that the solutions are asymptotic radially symmetric and strongly positive. Furthermore, using a Pohozaev-type invariant, we prove the non-existence of semi-singular solutions, i.e., all components equally blow-up in the neighborhood of origin. Namely, they are classified as multiples of the Emden–Fowler solution. Our results are natural generalizations of the famous classification due to L. A. Caffarelli, B. Gidas, and J. Spruck on the classical Yamabe equation.

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1. Description of the results

In this paper, we are concerned with some qualitative properties for nonnegative $p$-map solutions $U : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^p$ of the following vector-valued equation

$$\Delta^2 U = \frac{1}{2^{**}} c(n) D_U |U|^{2**} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \quad (1)$$

where $n \geq 5$, $2^{**} = 2n/(n-4)$ is the critical Sobolev exponent, $\Delta^2$ denotes the bi-Laplacian acting in $p$-maps, $|U|$ is the euclidean norm of the $p$-map $U$ and $D_U$ is the derivation operator with respect to $U$ and

$$c(n) = \frac{n(n-4)(n^2 - 4)}{16}$$

is a normalizing constant. More precisely, writing $U = (u_1, \ldots, u_p)$, where each $u_i$ is called a component solution, we have $\Delta^2 U = (\Delta^2 u_i)_i, |U|^{2**} = \sum_{i=1}^{p} |u_i|^{2**}$ and $\frac{1}{2^{**}} D_U |U|^{2**} = |U|^{2**-2} u_i$ for $i \in I := \{1, \ldots, p\}$. Hence, another way in which (1) can be reformulated is like the conformally invariant elliptic system in the potential form

$$\Delta^2 u_i = c(n) |U|^{2**-2} u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \quad (2)$$

System (2) is strongly coupled by the critical Gross–Pitaevskii-type nonlinearity

$$f_i(U) = c(n) |U|^{2**-2} u_i \quad \text{for} \quad i \in I, \quad (3)$$

where the map $F = (f_1, \ldots, f_p)$ is called the potential map and in our case has critical growth.

Here we consider solutions defined in the punctured space, thus some of its components may develop a non-removable singularity at the origin. In this fashion, a solution $U$ of (2) is said to be regular, if the origin is a removable singularity of $u_i$ for all $i \in I$, that is, $\liminf_{|x| \to 0} |U| < \infty$ (in this case, they can be extended smoothly to the whole domain); fully-singular, if the origin is a non-removable singularity for all $i \in I$. Otherwise, we call $U$ semi-singular. Note that for both type of singular solutions, we have $\lim_{|x| \to 0} |U| = \infty$ (see 25).

Let us notice that if $p = 1$, system (2) becomes the following fourth order equation

$$\Delta^2 u = c(n) u^{2**-1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \quad (4)$$

In this sense, (3) is the more natural strongly coupling term in order to (2) to be the generalization of (4) for the case $p > 1$. Our objective is to present classification results for regular and singular solutions to our conformally invariant system.

Our first main result on (2) is motivated by the fundamental classification theorem due to C. S. Lin [49, Theorem 1.4] for solutions to (4) with a removable singularity at the origin

**Theorem A.** Let $u$ be a nonnegative regular solution of (4). Then, there exist $x_0 \in \mathbb{R}^n$ and $\mu > 0$ such that $u$ is radially symmetric with respect to $x_0$ and

$$u_{\mu, x_0}(x) = \left( \frac{2\mu}{1 + \mu^2 |x - x_0|^2} \right)^{n+4\mu-4}. \quad (5)$$

Let us call $u_{\mu, x_0}$ a fourth order spherical solution.

This $(n+1)$-parameter family of solutions can also be regarded as maximizers for the Sobolev embedding theorem $\mathcal{D}^{2,2}(\mathbb{R}^n) \hookrightarrow L^{2**}(\mathbb{R}^n)$, that is,

$$\|u_{\mu, x_0}\|_{L^{2**}(\mathbb{R}^n)} = S(2, 2, n) \|u_{\mu, x_0}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)}, \quad (6)$$

where $S(2, 2, n)$ is the best Sobolev constant (see (52) below). The existence of extremal functions for (6) was obtained by P.-L. Lions [50, Section V.3]. In addition, these optimizers were found in more general way by E. Lieb [48, Theorem 3.1] using an equivalent dual formulation. Subsequently,
D. E. Edmunds et al. [23, Theorem 2.1] completed the classification. We also refer to X. Xu [67, Theorem 1.1] that reproved this result via a moving spheres method.

On the second main result, we will provide a classification theorem for singular solutions to (2). On this subject we should mention that when the origin is a non-removable singularity C. S. Lin [49, Theorem 1.5] obtained radial symmetry for solutions to (4) using the asymptotic moving planes technique. Recently Z. Guo et al. [35, Theorem 2.5] proved the existence of periodic solutions applying a mountain pass theorem and conjectured that all solutions must be periodic. Later R. L. Frank and T. König [27, Theorem 1.3] answered this conjecture, obtaining more accurate results about the classification for global singular solutions to (4)

**Theorem B.** Let $u$ be a singular solution of (4). Then, $u$ is radially symmetric with respect to the origin and there exist $a \in (0, a_0]$ and $T_a \in (0, T_0]$ such that

$$u_{a,T}(x) = |x|^\frac{n-4}{2} v_a(\ln |x| + T_a).$$

(7)

Here $a_0 = [n(n - 4)/(n^2 - 4)]^{n-4/4}$ and $v_a$ is the unique periodic bounded solution of the fourth order Cauchy problem,

$$\begin{cases}
v_a^{(4)} - K_2v_a^{(2)} + K_0v_a = c(n)v_a^{\frac{n+4}{4}} \\
v_a(0) = a, \ v_a^{(1)}(0) = 0, \ v_a^{(2)}(0) = b, \ v_a^{(3)}(0) = 0,
\end{cases}$$

(8)

where $K_2, K_0$ are constants depending only on the dimension, $T_a$ is the fundamental period and of $v_a$ and $T_0 = T_{a_0}$. We call both $u_{a,T}$ and $v_{a,T}$ Emden–Fowler (or a Delaunay-type) solutions and $a \in (0, a_0)$ its Fowler parameter, which can be chosen satisfying $a = \min_{t>0} v_a(t)$.

Let us remark that differently to Theorem A where solutions can be classified using a $(n + 1)$-parameter family, in Theorem B we have a two-parameter family of solutions. However, we should mention that composing three conformal transformations is possible to construct a $n$-parameter family of deformations for (7), which are called the deformed Emden–Fowler solutions (see [43] for more details). In this sense, the necksize $a$ of a singular solution of (4) plays the same role as the parameter $\mu$ for the regular solutions to (4).

In the light of Theorems A and B, we present our main results

**Theorem 1** (Liouville–type). Let $U$ be a regular solution of (2). Then, $U$ is radially symmetric with respect to some $x_0 \in \mathbb{R}^n$. Moreover, there exists $\Lambda \in \mathbb{S}_{+}^{p-1} = \{x \in \mathbb{S}^{p-1} : x_i \geq 0\}$ and a fourth order spherical solution given by (5) such that

$$U = \Lambda u_{x_0,\mu}.$$  

As an application, we will show that regular solutions classified above are the extremal maps for a higher order Sobolev-type inequality. Moreover, the best constant associated to this embedding coincides with the one when $p = 1$. On this subject, we refer to [4, 42].

**Theorem 2** (Classification). Let $U$ be a singular solution of (2). Then, $U$ is radially symmetric with respect to the origin and decreasing. Moreover, there exists $\Lambda^* \in \mathbb{S}_{+,*}^{p-1} = \{x \in \mathbb{S}^{p-1} : x_i > 0\}$ and an Emden–Fowler solution given by (7) such that

$$U = \Lambda^* u_{a,T}.$$  

Since the singular solutions to (2) are the natural candidates for asymptotic models of the same system in the punctured ball, the last theorem is the first step in describing the local asymptotic behavior for positive singular solutions to

$$\Delta^2 u_i = c(n)|U|^{2^{*} - 2} u_i \quad \text{in} \quad B_1^n \setminus \{0\}.$$  

(9)
This asymptotic analysis would be a version of the celebrated Caffarelli–Gidas–Spruck result for the context of fourth order systems. When \( p = 1 \), we should mention that the subcritical cases of (4) was addressed in [63,68], however the question about the asymptotic local behavior for singular solutions to (4) near the isolated singularity remains unsolved. We will pursue these problems in a different paper.

**Remark 3.** The existence of regular (singular) solutions to (2) follows directly from Theorem A (Theorem B). In fact, for any \( \Lambda \in S^{n-1}_+ \) \((\Lambda^* \in S^{n-1}_{++})\) we observe that \( U = \Lambda u_{x_0,\mu} \) \((U = \Lambda^* u_{a,T})\) is a regular (singular) solution of (2). Roughly speaking, our results classify this solutions as the only possible expressions for nontrivial solutions to (2). We should also mention that in [55] is proved the existence of nontrivial solutions to a more general class of potentials involving polyharmonic operators on Riemannian manifolds.

The main sources of difficulties in seeking for some qualitative properties for (2) are the lack of maximum principle and the failure of truncation methods imposed by the bi-Laplacian operator in the left-hand side of (2), the strongly-coupled setting caused by the Gross–Pitaevskii nonlinearity in the right-hand side of (2), and the non-removable singularity at the origin of punctured space.

One of the first results on classification of solution to second order equations dates back to the seminal work of L. A. Caffarelli et al. [8] (see also [15,43,45]). This challenging analysis of singular PDEs has been motivated by the classical papers [24,26,44] regarding the Lane–Emden–Fowler equation

\[
- \Delta u = u^s \quad \text{in} \quad \mathbb{R}^n \setminus \{0\},
\]

for \( n \geq 3 \) and \( s > 1 \), which models the distribution of mass density in spherical polytropic star in hydrostatic equilibrium (see [11] for more details). In addition, when \( s = 2^* - 1 \), where \( 2^* := \frac{2n}{n-2} \) is the critical Sobolev exponent, (10) corresponds to the conformal scalar curvature equation, a famous problem in differential geometry, which can be set as

\[
- \Delta u = \frac{n(n-2)}{4} u^{2^* - 1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}.
\]

It is well known that (11) is a particular case of the Yamabe problem for non-compact complete Riemannian manifolds \((M,g)\) with simple structure at infinity, that is, there exists \( \bar{M} \) containing \( M \) such that \( M = \bar{M} \setminus Z \), where \( Z \) is a closed subset called the ends of \( M \). Thus this problem can reduced to obtaining positive solution to the Singular Yamabe equation

\[
\begin{cases}
\Delta_g u + R_g u = \frac{n(n-2)}{4} u^{2^* - 1} \quad &\text{in} \quad \bar{M} \setminus Z, \\
\lim_{d(x,Z) \to 0} u(x) = \infty,
\end{cases}
\]

where \( \Delta_g \) is the Laplace-Beltrami operator and \( R_g \) is the scalar curvature. In this way, (11) is a version of (12) when the background metric is conformally flat and \( Z \) is a unique point.

The study of singular solutions to equations of the above type is related to the characterization of the size of the limit set of the image domain in \( S^n \) of the developing map of a locally conformally flat \( n \)-manifold, which has been highlighted by the works of R. Schoen and S.-T. Yau [59–61]. More specifically, positive solutions to (12) give raise to complete conformal metrics \( \bar{g} = u^{4/(n-2)} g \) with constant scalar curvature. Then, by the geometrical point of view the understanding of the local behavior for singular solution near the isolated singularity is equivalent to studying the asymptotic behavior of the conformal metric at the ends of \( \bar{M} \).
In [8], using ODE methods it has been proved that if \( u \) is a regular solution of (11), then there exist \( x_0 \in \mathbb{R}^n \) and \( \mu > 0 \) such that
\[
  u(x) = \left( \frac{2\mu}{1 + \mu^2 |x - x_0|^2} \right)^{\frac{n-2}{2}}.
\] (13)
This classification result can be seen as a complement to the works of T. Aubin [3] and G. Talenti [64] (see also [29,30,56]). Moreover, they as well dealt with the singular case, proving that if \( u \) is a singular solution of (11), there exist \( a \in (0,a_0] \) and \( T_a \in (0,T_0] \) such that
\[
  u(x) = |x|^{\frac{n-2}{2}} v_a(-\ln |x| + T_a),
\] (14)
where \( v_a \) is the unique periodic bounded solution of the following second order problem
\[
  \begin{cases}
    u''(x) - \frac{(n-2)^2}{4} u + \frac{n(n-2)}{4} u^{\frac{n+2}{2}} = 0 \\
    v(0) = a, \quad v(1)(0) = 0,
  \end{cases}
\] (15)
where \( T_a > 0 \) is the minimal period of \( v_a \). In this situation, the asymptotic properties of global solutions to (15) can be inferred using standard ODE methods, such as conservation of energy, phase-plane analysis and Floquet theory.

The following critical second order Gross–Pitaevskii system is the generalization of (11)
\[
  -\Delta u_i = \frac{n(n-2)}{4} |u|^{2^* - 2} u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\} \quad \text{for} \quad i \in I,
\] (16)
As in Remark 3, we observe that the existence of regular (singular) solutions to (16) is a direct consequence of the results due to P.-L. Lions [50] (R. Fowler [26]). Indeed, for every \( \Lambda \in S_+^{p-1} \) (\( \Lambda^* \in S_+^{p-1} \)) unit vector with nonnegative (positive) coordinates and \( u \) a regular (singular) solution of (11), we have that \( U = \Lambda u \) is a regular (singular) solution of (16). Moreover, O. Druet et al. [21, Proposition 1.1] on System (16) proved the Liouville-type theorem stated below. We refer to [16, Theorem 1] for related results on integral systems with critical exponents.

**Theorem C.** Let \( U \) be a regular solution of (16). Then, \( U \) is radially symmetric with respect to some point and \( U = \Lambda u \) for some \( \Lambda \in S_+^{p-1} \), where \( u \) is given by (13).

At this point, a natural question that raises is whether Theorem C holds for the singular case. Recently, R. Caju et al. [9, Theorem 1.2] gave an affirmative answer for this

**Theorem D.** Let \( U \) be a singular solution of (16). Then, \( U \) is radially symmetric with respect to the origin, monotonically decreasing and \( U = \Lambda^* u \) for some \( \Lambda^* \in S_+^{p-1} \), where \( u \) is given by (14).

Strongly coupled fourth order systems appears in several important branches of mathematical physic. For instance, in hydrodynamics, for modeling the behavior of deep-water and Rogue waves in the ocean (for more details, see [22,51]). As well as, in the Hartree–Fock theory for Bose–Einstein double condensates (see for instance [1,25]). Moreover, in conformal geometry (4) can be seen as the limit equation to the \( Q \)-curvature problem for a non-compact complete Riemannian manifold \((M,g)\) with simple structure at infinity. Hence, in the same way of the singular Yamabe problem, solutions to (4) gives raises to complete conformal metrics on \( M \) with a constant \( Q \)-curvature. For more details on the \( Q \)-curvature problem and its applications in general relativity see for instance [12,38].

Motivated by its applications in PDE and differential geometry, classification for singular solutions to PDEs has been a topic of intense study in recent years. We should mention that there also exist a vast literature for related problems arising in conformal geometry. For instance, in prescribing different types of curvature, such as the \( Q^k \)-curvature (see [14,66]), the fractional
curvature (see [7, 19, 69]) and the $\sigma_k$-curvature (see [13, 37, 47]). We emphasize that for each type above, the transformation law relating the curvature of two conformal metrics involves respectively: higher order operators (poly-Laplacian), nonlocal operators (fractional Laplacian) and fully nonlinear operator ($k$-Hessian).

Here is a brief description of our plan for the remaining part of the paper. In Section 2, we will summarize some basic definitions. In Section 3, we will prove that solutions to (2) are regular and weakly positive. We will show that Theorem 1 holds for weak solutions to (2). Hence, we will apply a moving spheres method to provide that classical solutions are weak solutions. In addition, we will also prove that solutions obtained in Theorem 1 are extremal functions for a Sobolev embedding theorem. In Section 4, we will obtain that singular solutions are as well classical. Thus we will employ a asymptotic moving planes method to show they are rotationally invariant with respect to the origin. Therefore, in the singular case (2) is equivalent to a fourth order ODE system in the real line. In this direction, we will use its Hamiltonian energy to define a suitable Pohozaev-type invariant. Finally, we will proceed by a delicate ODE analysis to find a removable-singularity classification for solutions to (2) based in the sign of the Pohozaev invariant. Then, as a direct consequence, we will give the proof of Theorem 2.

2. Basic definitions

In this section, let us introduce some basic definitions that will frequently appear in the remaining part of the text. Here and subsequently, we always deal with nonnegative solutions $U$ of (2), that is, $u_i \geq 0$ for all $i \in I$. First, let us split the index set $I$ in two parts $I_0 = \{i : u_i \equiv 0\}$ and $I_+ = \{i : u_i > 0\}$. Then, following standard notation for elliptic systems, we divide solutions to (2) in two types:

**Definition 4.** Let $U$ be a nonnegative solution of (2). We call $U$ strongly positive if $I_+ = I$. On the other hand, when $I_0 \neq \emptyset$ and $I \setminus I_0 = I_+$, we say that $U$ is weakly positive.

**Remark 5.** For the proof of Theorems 1 and 2, it is crucial to show that any solution of (2) is weakly positive. In fact, we need to guarantee that nontrivial solutions to (2) do not develop zeros in the domain. Namely, our strategy is to prove that the so-called quotient function $q_{ij} = u_i/u_j$ is constant. First, in order to the quotient to be well defined, it is necessary to the denominator to be strictly positive. It has been already observed by E. Hebey [41, Remark 1.1] that contrary to the case $p = 1$, nonnegative solutions to some elliptic coupled systems are not necessarily weakly positive (and thus not strongly positive as well).

**Definition 6.** For $m \in \mathbb{N}$, let us consider the poly-Laplacian operator applied in a smooth function $u$ defined by

$$
\begin{align*}
\Delta^m/2u &= \Delta(\Delta^m/2u), & \text{if } m \text{ is even} \\
\nabla^m/2u &= \nabla(\Delta^m/2u), & \text{if } m \text{ is odd},
\end{align*}
$$

This differential operator can also be referred as the $m/2$-Laplacian.

**Definition 7.** We call $U$ a weak solution of (2) if it belongs to $\mathcal{D}^{2,2}(\Omega, \mathbb{R}^p)$ and satisfies

$$
\int_{\mathbb{R}^n} \Delta u_i \Delta \phi_i dx = \int_{\mathbb{R}^n} c(n)|U|^{2^*-2} u_i \phi_i dx, \quad \forall \Phi \in C_c^{\infty}(\Omega, \mathbb{R}^p),
$$

where $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n \setminus \{0\}$ and $\Phi = (\phi_i)_{i \in I}$
3. Liouville-type theorem for regular solutions

Here we will present the proof of Theorem 1. Using the regularity lifting theorem based from [17],
our strategy is to obtain regularity results for solutions to (2) with a removable singularity at the
origin. Then, employing an induction argument from [67] we will show that regular solutions
to (2) are weakly positive. In addition, adapting a variational technique from O. Druet and E.
Hebey [20], we will prove that the Liouville-type result holds for weak solutions to (2). Therefore,
we will perform a moving spheres technique from O. Druet et al. [21] and Y. Li and L. Zhang [46]
to obtain that solutions to (2) are rotationally invariant with respect to some point. This argument
gives as a by-product an estimate on the Sobolev norm any solution, yielding that classical solutions
to (2) are also weak. Finally, as an application of our main result, we will show that regular
solutions to (2) are indeed extremal maps for the Sobolev embedding of the space $D^{2,2}(\mathbb{R}^n,\mathbb{R}^p)$
to $L^{2^*}(\mathbb{R}^n,\mathbb{R}^p)$. Here, as usual for $\Omega$ a smooth domain in $\mathbb{R}^n$ let us denote by $D^{2,2}(\Omega)$ the
Beppo–Levi space, completion of the space $C^\infty_c(\Omega)$ under the Dirichlet norm, $\|u\|^2 = \int_{\Omega} |\Delta u|^2 \, dx$.

Since the origin is a removable singularity of $U$, System (2) can be modeled in the entire space,
in the sense that solutions can be smoothly extended to be defined in $\mathbb{R}^n$. In this situation, (2) is reduced to
\begin{equation}
\Delta^2 u_i = c(n) |U|^{2^* - 2} u_i \quad \text{in} \quad \mathbb{R}^n.
\end{equation}

Subsequently, the idea is to provide some properties for solutions to (17) by writing this system
as a nonlinear fourth order Schrödinger equation with potential $V : \mathbb{R}^n \to \mathbb{R}$ defined by
$V(x) = c(n) |U(x)|^{2^* - 2}$.

3.1. Regularity. Initially, we prove that weak solutions to (2) are as well classical solutions. We
should mention that De Giorgi–Nash–Moser bootstrap techniques in combination with the Brézis-
Kato method can be used to prove regularity results for second order elliptic PDEs involving critical
growth. Unfortunately, due to the failure of truncation methods for higher order derivatives, this
tool does not work in our critical fourth order setting. More precisely, the nonlinearity in the
right-hand side of (2) has critical growth, so $|U|^{2^* - 2} u_i \in L^{2n/(n+4)}(\mathbb{R}^n)$. Although a priori we
cannot conclude, using the Sobolev embedding theorem, that $|U|^{2^* - 2} u_i$ belongs to $L^q(\mathbb{R}^n)$ for
some $q > 2n/(n+4)$, we are able to overcome this lack of regularity applying the lifting method
due to W. X. Chen and C. Li [17]

Proposition A. Let $Z$ be a Hausdorff topological space, $\|: X, : Y : Z \to [0, \infty]$ extended norms
in $Z$ and $X, Y$ be subspaces defined by $X = \{z \in Z : \|z\|_X < \infty\}$ and $Y = \{z \in Z : \|z\|_Y < \infty\}$.
Suppose that $T$ is a contraction map from $X$ into itself and from $Y$ into itself. If for $f \in X$, there
exists $g \in X \cap Y$ such that $f = T f + g$, then $f \in X \cap Y$.

The next step is to apply Proposition A and show that it is possible to improve the Lebesgue class in which solutions to (17) are defined. Here our strategy is to show that they indeed belong to the Lebesgue space $L^s(\mathbb{R}^n,\mathbb{R}^p)$ for any $s > 2^*$.

Proposition 8. Let $U \in D^{2,2}(\mathbb{R}^n,\mathbb{R}^p)$ be a weak solution of (17). Then, $U \in L^s(\mathbb{R}^n,\mathbb{R}^p)$ for all
$s > 2^*$.

Proof. First, let us consider the $Z = C^\infty(\mathbb{R}^n), X = L^{2n/(n-4)}(\mathbb{R}^n)$ and $Y = L^s(\mathbb{R}^n)$ for
$s > 2n/(n-4)$. Let $\Gamma(x,y) = C_n|x-y|^{4-n}$ be the fundamental solution of $\Delta^2$ in $\mathbb{R}^n$. Hence, it is
well-defined the following inverse operator
\begin{equation}
(Tf)(x) = \int_{\mathbb{R}^n} \Gamma(x,y)f(y) \, dy.
\end{equation}
Hence, using the Hardy–Littlewood–Sobolev inequality (see [48]), it follows that for any $q \in (1, n/4)$,
\[
\|Tf\|_{L^{\frac{nq}{n-q}}(\mathbb{R}^n)} = \|\Gamma * f\|_{L^{\frac{nq}{n-q}}(\mathbb{R}^n)} \leq C\|f\|_{L^q(\mathbb{R}^n)}.
\]

For $M > 0$, let us define $\tilde{V}_M(x) = V(x) - V_M(x)$, where
\[
V_M(x) = \begin{cases} V(x), & \text{if } |V(x)| \geq M, \\ 0, & \text{otherwise}. \end{cases}
\]

Applying the integral operator $T_M f := \Gamma * \tilde{V}_M f$ in both sides of (17), we obtain $u_i = T_M u_i + F_M$, where
\[
(T_M u_i)(x) = \int_{\mathbb{R}^n} \Gamma(x, y)V_M(y)u_i(y)dy \quad \text{and} \quad F_M(x) = \int_{\mathbb{R}^n} \Gamma(x, y)\tilde{V}_M(y)u_i(y)dy.
\]

**Claim 1:** For $n/(n - 4) < s < \infty$ and $M > 0$ sufficiently large, $T_M : L^s(\mathbb{R}^n) \to L^s(\mathbb{R}^n)$ is a contraction.

In fact, for any $s \in (n/(n - 4), \infty)$, there exists $r \in (1, n/4)$ such that $s = nr/(n - 4r)$. Then, by the Hölder inequality, for any $f \in L^s(\mathbb{R}^n)$ we get
\[
\|T_M f\|_{L^s(\mathbb{R}^n)} \leq \|\Gamma * V_M f\|_{L^s(\mathbb{R}^n)} \leq C\|V_M\|_{L^{n/4}(\mathbb{R}^n)}\|f\|_{L^s(\mathbb{R}^n)}.
\]

Since $V_M \in L^{n/4}(\mathbb{R}^n)$ it is possible to choose a large $M > 0$ satisfying $\|V_M\|_{L^{n/4}(\mathbb{R}^n)} < 1/2C$. Therefore, we arrive at $\|T_M f\|_{L^s(\mathbb{R}^n)} \leq 1/2\|f\|_{L^s(\mathbb{R}^n)}$, which yields that $T_M$ is a contraction.

**Claim 2:** For any $n/(n - 4) < s < \infty$, $F_M \in L^s(\mathbb{R}^n)$.

Indeed, for any $n/(n - 4) < s < \infty$, choose $1 < r < n/4$, satisfying $s = nr/(n - 4r)$. Since $\tilde{V}_M$ is bounded we obtain
\[
\|F_M\|_{L^s(\mathbb{R}^n)} = \|\Gamma * \tilde{V}_M u_i\| \leq C_1\|\tilde{V}_M u_i\|_{L^r(\mathbb{R}^n)} \leq C_2\|u_i\|_{L^r(\mathbb{R}^n)}.
\]

However, by Sobolev embedding theorem, we have that $u_i \in L^q(\mathbb{R}^n)$ for $r = 2n/(n - 4)$, which implies $s = 2n/(n - 8)$. Thus we find that $u_i \in L^s(\mathbb{R}^n)$ for
\[
\begin{cases} 1 < s < \infty, & \text{if } 5 \leq n \leq 8 \\ 1 < s \leq \frac{2n}{n-8}, & \text{if } n \geq 9. \end{cases}
\]

Now we can repeat the argument for $r = 2n/(n - 8)$ to obtain that $u_i \in L^s(\mathbb{R}^n)$ for
\[
\begin{cases} 1 < s < \infty, & \text{if } 5 \leq n \leq 12 \\ 1 < s \leq \frac{2n}{n-12}, & \text{if } n \geq 13. \end{cases}
\]

Therefore proceeding inductively as in the last argument, the proof of the claim follows.

Combining Claims 1 and 2, we can apply Proposition A to show that $u_i \in L^s(\mathbb{R}^n)$ for all $s > 2^*$ and $i \in I$. In particular, the proof of the proposition is concluded.

**Corollary 9** (Regularity). Let $U \in \mathcal{D}^{2,2}(\mathbb{R}^n, \mathbb{R}^p)$ be a weak regular solution of (2). Then, $U \in C^{4,\beta}(\mathbb{R}^n, \mathbb{R}^p)$ is a classical solution of (2).

**Proof.** By Morrey embedding theorem, we have that $u_i \in C^{0,\beta}(\mathbb{R}^n)$ for some $\beta > 0$. Finally using Schauder estimates, one concludes $u_i \in C^{4,\beta}(\mathbb{R}^n)$, which provides $U \in C^{4,\beta}(\mathbb{R}^n, \mathbb{R}^p)$.

**Remark 10.** In [65, Proposition 3.1] using a different approach, K. Uhlenbeck and J. Viaclovski proved regularity for solutions to a class of general geometric fourth order PDEs. They could show that for some $\bar{q} > 2^*$ it holds $U \in L^{\bar{q}}(\mathbb{R}^n, \mathbb{R}^p)$. 

\[\Box\]
3.2. Superharmonicity. Our aim is to obtain a strong maximum principle for nonnegative solutions to (2). In this direction, we are inspired in [67, Theorem 2.1] to prove that any component solution of (2) is superharmonic. The main difference in our approach is the appearance of (3) in the right-hand side of (17). This coupled nonlinearity may imply the failure of method for some components. However, we are able to overcome this issue thanks to an inequality involving the norm of the $p$-map solution. Before proving the superharmonicity result, we need to establish two technical lemmas, which proofs are merely calculus argument and can be found in [67, Lemma 2.2 and Lemma 2.3], respectively.

**Lemma A.** Suppose that $l_0 = 2$ and $\{l_j\}_{j \in \mathbb{N}}$ given by the formula $l_{j+1} = sl_j + 4$ for some $s > 1$. Then, for all $j \in \mathbb{N}$,

(i) Recursion formula: $l_{j+1} = \frac{2s^{j+2} + 2s^{j+1} - 4}{s-1}$;

(ii) Upper estimate: $(n + sl_j)(2 + sl_j)(n + 2 + sl_j)(4 + sl_j) \leq (n + 2 + 2s)^4(s+1)$.

**Lemma B.** Suppose that $b_0 = 0$ and define $\{b_j\}_{j \in \mathbb{N}}$ by $b_{j+1} = sb_j + 4(j + 1)$. Then, for all $j \in \mathbb{N}$,

$$b_{j+1} = 4\left[\frac{s^{j+2} - (j + 2)s + j + 1}{s^2}\right].$$

The superharmonicity result can be stated as follows

**Proposition 11.** Let $U$ be a regular solution of (2). Then, $-\Delta u_i \geq 0$ in $\mathbb{R}^n$ for all $i \in I$.

**Proof.** Let us fix $i \in I$ and $x_0 \in \mathbb{R}^n$ satisfying $-\Delta u_i(x_0) < 0$, which exists supposing by contradiction that the proposition does not hold. Since the Laplacian is invariant by translations, we may suppose without loss of generality that $x_0 = 0$. We transform (2) into

$$\begin{cases}
-\Delta u_i = w_i \\
-\Delta w_i = c(n)|\nabla u_i|^{2* - 2}u_i
\end{cases} \text{ in } \mathbb{R}^n. \quad (18)$$

Let $B_r$ be the ball of radius $r$ and $\omega_{n-1}$ be the $(n - 1)$-dimensional surface of the unit sphere, we set $w_i = -\Delta u_i$ and

$$\bar{u}_i = \frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} u_i \mathrm{d}\sigma_r \quad \text{and} \quad \bar{w}_i = \frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} w_i \mathrm{d}\sigma_r,$$

the spherical averages of $u_i$ and $w_i$, respectively. Now integrating system (18) over $\partial B_r$,

$$\Delta \bar{u}_i + \bar{w}_i = 0. \quad (19)$$

Furthermore, performing equally in the right-hand side of (17), we find

$$\frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} c(n)|\nabla u_i(x)|^{2* - 2}u_i(x) \mathrm{d}\sigma_r \leq c(n)|\nabla u_i|^{\frac{n+4}{n-4}},$$

which implies

$$\Delta \bar{w}_i + c(n)|\nabla u_i|^{\frac{n+4}{n-4}} \leq 0. \quad (20)$$

By the definition of spherical average, we have that $\bar{w}_i(0) = w_i(0) < 0$ and

$$\Delta \bar{w}_i \leq 0. \quad (21)$$

Then, multiplying equation (21) by $r^{n-1}$ and integrating, we arrive at

$$r^{n-1}\partial_r[\bar{w}_i] \leq 0.$$

It clearly implies that $\bar{w}_i$ is monotonically decreasing with

$$\bar{w}_i(r) \geq w_i(0), \quad \text{for} \quad r > 0. \quad (22)$$
Substituting (22) into (19) and integrating, we get
\[ \bar{u}_i(r) \geq -\frac{w_i(0)}{2n} r^2. \] (23)

Using (20)–(23) and multiplying both side of inequality by \( r^{n-1} \) and integrating, we obtain
\[ \bar{w}_i(r) \leq -\frac{c_0^2 r^{2s+2}}{(n + 2 + 2s)(2s + 4)}, \] (24)

where \( c_0 = -w_i(0)/2n > 0 \) and \( s = (n+4)/(n-4) \). Then, combining (24) with (19), and repeating the same procedure, it follows
\[ \bar{u}_i(r) \geq \frac{c_0^j r^l_j}{(n + 2 + 2s)^b_j}. \] (25)

Based on (25) and thanks to Lemma B, we may assume that for some \( j \in \mathbb{Z} \) and \( l_j, b_j \in \mathbb{R} \), it holds
\[ \bar{u}_i(r) \geq \frac{c_0^j r^l_j}{(n + 2 + 2s)^b_j}. \] (26)

Again, we can use estimate (26) combined with (19) and (20) as before to find
\[ \bar{w}_i(r) \leq -\frac{c_0^{s+1} r^{l_j+2}}{(n + 2 + 2s)^{b_j}(n + s l_j)(s l_j + 2)} \]

and
\[ \bar{u}_i(r) \geq \frac{c_0^{s+1} r^{l_j+1}}{(n + 2 + 2s)^{b_j+1}}. \] (27)

Setting \( l_{j+1} = s l_j + 4 \), by (ii) of Lemma A, we have that (27) remains true for \( j + 1 \) with \( b_{j+1} = s b_j + 4(j + 1) \). In other words, we have
\[ \bar{u}_i(r) \geq \frac{c_0^{s+1} r^{l_{j+1}}}{(n + 2 + 2s)^{b_{j+1}}}. \]

Assuming that \( c_0 \geq 1 \), we can choose \( r_0 = (n+2s+2)^{4/(s-1)} \) and by Lemmas A and B the following estimates holds
\[ \bar{u}_i(r_0) \geq c_0^{s+1} \left[ (n + 2s + 2)^{4/(s-1)} \right]^{s^{j+2}+2s^{j+1}+(j+2)s-j-5}. \] (28)

Taking the limit as \( j \to \infty \) in (28), we find a contradiction since right-hand side blows up. Therefore, \( \Delta u_i \leq 0 \) for all \( i \in I \). When \( c_0 < 1 \), choosing \( r_0 = c_0^{-1}(n + 2s + 2)^{4/(s-1)} \) the same argument can be applied.

Using the superharmonicity result we can prove that solutions to (2) are weakly positive.

**Corollary 12.** Let \( U \) be a regular solution of (2). Then, for any \( i \in I \), we have that either \( u_i \equiv 0 \) or \( u_i > 0 \). In other terms, \( I = I_0 \cup I_+ \) is a disjoint union.

### 3.3. Lower bound estimates

The fact that component solutions to (2) are superharmonic is useful to provide essential properties required to start the moving spheres method. More precisely, we obtain a lower bound estimate for any component solution. The idea is to use Proposition 11 and the Three-Spheres theorem for bi-Laplacian, which can be stated as follows and is proved in the appendix.
Lemma 13. Let \( \Omega \subset \mathbb{R}^n \) be a region containing two concentric spheres of radii \( r_1 \) and \( r_2 \) and the region between them and \( u : \Omega \to \mathbb{R} \) be a superharmonic function in \( \Omega \). Then, for every \( r > 0 \) such that \( 0 < r_1 < r < r_2 \) we have

\[
\min_{\partial B_r} u \geq \left( \frac{\min_{\partial B_{r_1}} u}{r_2^{n-4} - r_1^{n-4}} + \frac{\min_{\partial B_{r_2}} u}{r_2^{n-4} - r_1^{n-4}} \right). 
\]

Moreover, equality only occurs if for some \( a, b \in \mathbb{R} \) we have \( u(|x|) = a + b|x|^{n-4} \).

Corollary 14. Let \( U \) be a regular solution of (2). Then, given \( 0 < r_0 < r \), for any \( x \in B_r \setminus B_{r_0} \) we have

\[
u_i(x) \geq \left( \frac{r_0}{|x|} \right)^{n-4} \min_{\partial B_{r_0}} u.
\]

Proof. Fix \( 0 < r_0 < r \), by applying Lemma 13, we get

\[
(r_0^{n-4} - r^{n-4}) u_i(x) \geq (|x|^{n-4} - r^{n-4}) \min_{\partial B_{r_0}} u_i.
\]

Thus, letting \( r \to \infty \) gives the conclusion. \( \Box \)

3.4. Kelvin transform. Now we define some type of transform suitable to explore the symmetries of (2), which is called the fourth order Kelvin transform of a \( p \)-map. The Kelvin transform is a device to extend the concept of harmonic (superharmonic or subharmonic) functions by allowing the definition of a function which is harmonic (superharmonic or subharmonic) at infinity. This map is a key ingredient for developing any type of sliding method, namely moving spheres or moving planes techniques (see [5]). In order to define the Kelvin transform, we need to establish the concept of inversion through a sphere \( \partial B_{\mu}(x_0) \), which is a map \( I_{x_0,\mu} : \mathbb{R}^n \to \mathbb{R}^n \setminus \{x_0\} \) given by \( I_{x_0,\mu}(x) = x_0 + K_{x_0,\mu}(x)^2(x - x_0) \), where \( K_{x_0,\mu}(x) = \mu/|x - x_0| \). In particular, when \( x_0 = 0 \) and \( \mu = 1 \), we denote it simply by \( I_{0,1}(x) = x^* \) and \( K_{0,1}(x) = x|x|^{-2} \). For easy reference let us summarize some well-known facts about the inversion map.

Proposition 15. The map \( I_{x_0,\mu} \) has the properties:

(i) It maps \( B_{\mu}(x_0) \) into its complement \( \mathbb{R}^n \setminus \overline{B_{\mu}(x_0)} \), such as \( x_0 \) into \( \infty \);

(ii) It let the boundary \( \partial B_{\mu}(x_0) \) invariant, that is, \( I_{x_0,\mu}(\partial B_{\mu}(x_0)) = \partial B_{\mu}(x_0) \);

(iii) It is conformal invariant, in the sense that \( \langle I_{x_0,\mu}(x), I_{x_0,\mu}(y) \rangle = \langle x, y \rangle \) for all \( x, y \in \mathbb{R}^n \setminus \{x_0\} \).

The next step is a generalization of the Kelvin transform for \( m \)th order operators applied to \( p \)-maps. In what follows, we will often work with the fourth order Kelvin transform.

Definition 16. For any \( U \), let us consider the \( m \)th order Kelvin transform through the sphere with center at \( x_0 \in \mathbb{R}^n \) and radius \( \mu > 0 \) defined by

\[
U_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{n-m} U(I_{x_0,\mu}(x)).
\]

In particular, when \( p = 1 \) let us fix the notation \( u_{x_0,\mu} \).

Now we need to understand how (2) behaves under the action of the Kelvin transform.

Proposition 17. The Kelvin transform let superharmonic \( p \)-maps invariants.

Proof. This fact is a consequence of the following formula

\[
\Delta u_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{-(n+2)} \Delta u(I_{x_0,\mu}(x)) \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\}, \quad (29)
\]

which proof is included in the appendix. \( \Box \)
Proposition 18. System (2) is conformally invariant in the sense that it is invariant under the action of Kelvin transform, i.e., if $\mathcal{U}$ is a regular solution of (2), then $\mathcal{U}_{x_0,\mu}$ is a solution of

$$\Delta^2 (u_i)_{x_0,\mu} = c(n)|\mathcal{U}_{x_0,\mu}|^{2^{**}-2} (u_i)_{x_0,\mu} \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\} \quad \text{for} \quad i \in I,$$

where $\mathcal{U}_{x_0,\mu} = ((u_1)_{x_0,\mu}, \ldots, (u_p)_{x_0,\mu})$.

Proof. Let us remember the formula

$$\Delta^2 u_{x_0,\mu}(x) = K_{x_0,\mu}(x)^{-(n+4)} \Delta^2 (I_{x_0,\mu}(x)) \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\}.$$

which proof is included in the appendix.

On the other hand, by an easy computation we observe

$$|\mathcal{U}_{x_0,\mu}(x)|^{2^{**}-2} = K_{x_0,\mu}(x)^{-(n+4)} |\mathcal{U}(x)|^{2^{**}-2} u_i(x).$$

Therefore, our conclusion holds as a combination of (31) and (30).

Remark 19. Proposition 18 is not a surprising conclusion, because the Gross–Pitaevskii-type nonlinearity preserves the conformal invariance which the scalar case enjoys. Namely, in the case $p = 1$ with critical growth, (4) is invariant under the action of the conformal euclidean group.

3.5. Variational technique. As an independent part of our work, we use a variational approach to prove a weak version of Theorem 1, which is based in the analysis of quotient functions $q_{ij} = u_i/u_j$, and has been developed by O. Druet and E. Hebey [20]. Before starting our method, we must be cautious about the quotient to be well-defined, since we may have solutions having zeros in the domain or even being identically null. By Proposition 12 we know that the latter situation does not occur. Moreover, we can avoid the former situation by considering component solutions $u_i$ to be strictly positive, that is, $i \in I_+$. Let us notice that Theorem 1 is equivalent to prove that the quotient functions are identically constant or also that component solutions are proportional to each other. For related results on proportionality of components see [54] and the references therein.

Let us state the main result of this part, which is a fourth order version of [20, Proposition 3.1]

Theorem 1'. Let $\mathcal{U}$ be a weak nonnegative regular solution of (2). Then, there exists $\Lambda \in S^{n-1}_{++}$ such that $\mathcal{U} = \Lambda u_{x_0,\mu}$, where $u_{x_0,\mu}$ is a fourth order spherical solution given by (5).

Proof. Let $\mathcal{U}$ be a weak nonnegative regular solution of (2) and $i, j \in I_+$. Considering the aforementioned quotient function $q_{ij}$, we divide the argument in two claims.

The first one provides a universal estimates for any quotient function

Claim 1: $\min_{\partial B_R(0)} q_{ij} \leq q_{ij} \leq \max_{\partial B_R(0)} q_{ij}.$

The proof of the claim is contained in the following two steps

**Step 1**: $\Delta q_{ij} \geq 0$ in $B_R$.

It is straightforward to obtain that

$$\Delta^2 q_{ij} = \frac{u_j \Delta u_i - u_i \Delta u_j}{u_j^2} - 4 \frac{\nabla^{3/2} q_{ij} \nabla u_j - 6 \Delta q_{ij} \Delta u_j - 4 \nabla q_{ij} \nabla^{3/2} u_j}{u_j},$$

which by (2) has first term in the right-hand side being zero. Thus we are left with

$$- \Delta^2 q_{ij} - 6 \frac{\Delta u_j \Delta q_{ij}}{u_j} = 4 \frac{\nabla^{3/2} q_{ij} \nabla u_j + \nabla q_{ij} \nabla^{3/2} u_j}{u_j}.$$

(32)

On the other hand, note that

$$4 \frac{\nabla^{3/2} q_{ij} \nabla u_j + \nabla q_{ij} \nabla^{3/2} u_j}{u_j} \geq 0.$$
Hence, setting \( c(x) = -6u_j^2 \Delta u_j \), (32) can be reformulated as \(-\Delta(\Delta q_{ij}) + c(x) \Delta q_{ij} \geq 0\), which implies that \( \Delta q_{ij} \) is a supersolution of \( L = -\Delta + c \). In addition, by Proposition 11, \( L \) is a nonnegative operator and the weak maximum principle from [31, Theorem 3.1] can be applied to show that \( \min_{\partial B_R} \Delta q_{ij} \leq \Delta q_{ij} \). Therefore, by the definition of \( \Delta q_{ij} \), we get \( \min_{\partial B_R} \Delta q_{ij} = 0 \), which concludes the proof of the first step.

**Step 2:** \( \Delta q_{ij} = 0 \) in \( B_R \).

It is straightforward to observe that \( q_{ij} \) satisfies the following uniformly elliptic second order equation \(-\Delta q_{ij} + \tilde{c}(x)q_{ij} = 0 \) in \( B_R \), where \( \tilde{c}(x) = q_j^{-1} \Delta q_{ij} \). Then, using Step 1 we have that \( \tilde{c}(x) \geq 0 \), which again using the weak maximum principle we conclude the proof.

**Claim 2:** \( \min_{\partial B_R(0)} q_{ij} \rightarrow \lambda_{ij} \) and \( \max_{\partial B_R(0)} q_{ij} \rightarrow \lambda_{ij} \) as \( R \rightarrow \infty \), where

\[
\lambda_{ij} = \frac{\int_{\mathbb{R}^n} |U|^{2^{**}-2} u_i \, dx}{\int_{\mathbb{R}^n} |U|^{2^{**}-2} u_j \, dx}.
\]  

(33)

In fact, we divide the proof in three steps. The first one concerns with the behavior at infinity of component solutions to (17).

**Step 1:** \( |x|^{(n-4)/2} u_i(x) = o_R(1) \) as \( R \rightarrow \infty \).

For \( R > 0 \), let us consider \( W_R = R^{(n-4)/2} U(rx) \) the rescaling of \( U \), which in terms of the component solutions is given by \( (w_R)_i = R^{(n-4)/2} u_i(x) \). Since \( u_i \in L^{2^{**}}(\mathbb{R}^n) \) we get

\[
\Delta^2 (w_R)_i = c(n)|W|^{2^{**}-2}(w_R)_i \quad \text{and} \quad \int_{B_2(0) \setminus B_{1/2}(0)} |W_R|^{2^{**}} \, dx = o_R(1) \quad \text{as} \quad R \rightarrow \infty.
\]

Thus, \( (w_R)_i \rightarrow 0 \) in \( C^\infty_{\text{loc}}(B_{3/2}(0) \setminus B_{3/4}(0)) \) as \( R \rightarrow \infty \).

In the next step, we obtain a precise upper bound for component solutions to (17), which can be applied to obtain an interpolation estimate, namely \( u_i \in L^p(\mathbb{R}^n) \) for \( 2 < p < 2^{**} \).

**Step 2:** For all \( 0 < \varepsilon < 1/2 \), there exists \( C_\varepsilon > 0 \) such that \( u_i(x) \leq C_\varepsilon |x|^{(4-n)(1-\varepsilon)} \).

First, by Step 1 for a given \( 0 < \varepsilon < 1/2 \), there exists \( R_\varepsilon > 0 \) sufficiently large satisfying

\[
\sup_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} |x|^2 |U(x)|^{2^{**}-2} < \frac{(n-4)^2}{2} \varepsilon (1-\varepsilon).
\]  

(34)

For \( R \geq R_\varepsilon \), let us consider \( \sigma(R) = \max_{i \in I_\varepsilon} \max_{\partial B_R(x_0)} u_i \) and the auxiliary function

\[
G_\varepsilon(x) = \sigma(R_\varepsilon) \left( \frac{|x|}{R_\varepsilon} \right)^{(4-n)(1-\varepsilon)} + \sigma(R) \left( \frac{|x|}{R} \right)^{(4-n)\varepsilon}.
\]

Note that by construction, we clearly have \( u_i \leq G_\varepsilon \) on \( \partial B_{R_\varepsilon}(0) \cup \partial B_{R_\varepsilon}(0) \). Let us suppose that there exists \( x_0 \in B_R(0) \setminus \bar{B}_{R_\varepsilon}(0) \) maximum point of \( u_i/G_\varepsilon \). We would have that \( \Delta(u_iG_\varepsilon^{-1}(x_0)) \leq 0 \) and then

\[
\frac{\Delta u_i(x)}{u_i(x)} \geq \frac{\Delta G_\varepsilon^{-1}(x)}{G_\varepsilon^{-1}(x)}.
\]  

(35)

Furthermore, a direct computation implies

\[
\Delta G_\varepsilon^{-1}(x) = G_\varepsilon^{-1}(x) \frac{(n-4)^2}{2} \varepsilon (1-\varepsilon)|x|^{-2}.
\]  

(36)
Therefore, by Proposition 11, we obtain that \( \Delta^2 u_i(x) - \Delta u_i(x) \geq 0 \), which in combination with (35) and (36) yields
\[
|x|^2 |\mathcal{U}(x)|^{2**} = \frac{\Delta^2 u_i(x)}{u_i(x)} \geq \frac{\Delta u_i(x)}{u_i(x)} \geq \frac{\Delta G_{\varepsilon}^{-1}(x)}{G_{\varepsilon}^{-1}(x)} = \frac{(n-4)^2}{2} \varepsilon (1 - \varepsilon).
\]
This is a contradiction with (34) since our choice of \( R_\varepsilon > 0 \). Then, applying a strong maximum principle, for all \( R > R_\varepsilon \), we have
\[
u_i(x) \leq \sigma(R_\varepsilon) \left( \frac{|x|}{R_\varepsilon} \right)^{(4-n)(1-\varepsilon)} + \sigma(R) \left( \frac{|x|}{R} \right)^{(4-n)\varepsilon}
\]
in \( B_R(0) \setminus \bar{B}_{R_\varepsilon}(0) \). Hence we get that (37) in combination with Step 1 and taking the limit as \( R \to \infty \), we get
\[
u_i(x) \leq \sigma(R_\varepsilon) \left( \frac{|x|}{R_\varepsilon} \right)^{(4-n)(1-\varepsilon)} \text{ in } \mathbb{R}^n.
\]
**Step 3:** \( |x|^{4-n}\nu_i(x) = \int_{\mathbb{R}^n} |\mathcal{U}(x)|^{2**-2}\nu_i(x)dx + o_R(1) \) as \( R \to \infty \).
First, since \( \nu_i \in L^{2**} (\mathbb{R}^n) \) we have \( |\mathcal{U}|^{2**-2}\nu_i \in L^{2n/(n+4)} (\mathbb{R}^n) \) for all \( i \in I \), which gives \( |\mathcal{U}|^{2**-2}\nu_i \in W^{-2,2} (\mathbb{R}^n) \). Hence we get that (2) can be reduced to the following integral system,
\[
u_i(x) = C_n \int_{\mathbb{R}^n} |x-y|^{4-n} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy,
\]
From which follows that
\[
|x|^{n-4} \nu_i(x) = C_n \int_{\mathbb{R}^n} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy = C_n(I_1 + I_2),
\]
where
\[
I_1 = \int_{B_R(0)} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy, 
I_2 = \int_{\mathbb{R}^n \setminus B_{R}(0)} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy.
\]
In order to control \( I_1 \), we observe that since
\[
\int_{B_R(0)} \left[ \left( \frac{|x|}{|x-y|} \right)^{n-4} - 1 \right] |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy = o_R(1),
\]
the following asymptotically identity holds
\[
I_1 = \int_{B_R(0)} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy + o_R(1) \text{ as } R \to \infty. \quad (38)
\]
It remains to estimate \( I_2 \). Accordingly, using Step 2, we can write
\[
I_2 = \int_{\mathbb{R}^n \setminus B_R(0)} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy
\]
\[
\leq \int_{B_{|x|/2}(x)} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy + \int_{\mathbb{R}^n \setminus B_{|x|/2}(x)} \left( \frac{|x|}{|x-y|} \right)^{n-4} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy
\]
\[
\leq C_{\varepsilon}^{2**-1} \int_{B_{|x|/2}(x)} \left( \frac{|x|}{|x-y|} \right)^{n-4} \left( \frac{|x|}{2} \right)^{-(n+4)(1-\varepsilon)} dy + 2^n \int_{\mathbb{R}^n \setminus B_R(0)} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy
\]
\[
\leq C_{\varepsilon}^{2**-2} (n+4)(1-\varepsilon)^{-2} \omega_{n-1} n^{-n+4}(1-\varepsilon) + 2^n \int_{\mathbb{R}^n \setminus B_R(x)} |\mathcal{U}(y)|^{2**-2}\nu_i(y)dy,
\]
Choosing $\varepsilon = 4/(n + 4)$ in (39), we obtain that $n - (n + 4)(1 - \varepsilon) \leq 0$ and thus

$$I_2 = o_R(1) \quad \text{as} \quad R \to \infty,$$

which in combination with (39) and (38), concludes the proof of Step 3.

Now using Step 3, we obtain that for all $i, j \in I_+$, it holds

$$\frac{u_i(x)}{u_j(x)} = \frac{|x|^{n-4} u_i(x)}{|x|^{n-4} u_j(x)} = \frac{\int_{\mathbb{R}^n} |U(x)|^{2^{*}-2} u_i(x) dx + o_R(1)}{\int_{\mathbb{R}^n} |U(x)|^{2^{*}-2} u_j(x) dx + o_R(1)},$$

which by taking the limit as $R \to \infty$ yields (33).

Finally, combining Claims 1 and 2, we find that $u_i = \lambda_{ij} u_j$. In particular, for all $i \in I_+$ we have the proportionality $u_i = \lambda_i u_1$ where $\lambda_i = \lambda_{1,i}$, which provides $\Delta^2 u_1 = |\Lambda'|^{2^{*} - 2} u_1^{2^{*} - 1}$ in $\mathbb{R}^n$, where $\Lambda' = (\Lambda_i)_{i \in I_+}$. By Theorem A, for some $x_0 \in \mathbb{R}^n$ and $\mu > 0$, we obtain that $u_1$ has the following form $u_1(x) = |\Lambda'|^{-1} \left( \frac{2\mu}{1 + \mu |x - x_0|^2} \right)^{\frac{n+4}{n-4}}$, which implies that our classification holds for $\Lambda = (\Lambda_1 |\Lambda'|^{-1}, \ldots, \Lambda_p |\Lambda'|^{-1})$; thus the proof of Theorem 1' is completed. \qed

3.6. Moving spheres method. In the next part of our paper, we apply the moving sphere method to obtain that any solution $U$ of (17) is radially symmetric. More specifically, we are able to provide the classification for $|U|$. Subsequently, we use this expression to directly compute the Sobolev norm of $U$ and then conclude that any classical solution of (17) is in fact a weak solution. The method of moving spheres is a variant of the moving planes method, which can be used as an alternative manner of showing radial symmetry or Liouville-type results for solutions to some PDE. For a partial list of references on this variant of the moving planes method, we refer to Y. Li [46] and the references therein. Our inspiration is the moving spheres arguments due to O. Druet et al. [21, Proposition 3.1].

In order to apply the moving spheres technique, we need two classification results which can be found in [46, Lemma 11.1 and Lemma 11.2]

**Proposition B (Weak Liouville-type).** Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$ and $\nu > 0$. Suppose that for every $x \in \mathbb{R}^n$ there exists $\mu(x) > 0$ such that

$$\left( \frac{\mu(x)}{|y - x|} \right)^\nu \left( x + \frac{\mu(x)(y - x)}{|y - x|^2} \right) = f(y) \quad \text{for} \quad y \in \mathbb{R}^n \setminus \{x\}. \quad (40)$$

Then, for some $\mu \geq 0$, $\mu' > 0$ and $x_0 \in \mathbb{R}^n$, we have that $f(x) = \pm \left( \frac{\mu'}{\mu + |x - x_0|^2} \right)^{\nu/2}$.

**Proposition C (Strong Liouville-type).** Let $f \in C^1(\mathbb{R}^n)$, $n \geq 1$ and $\nu > 0$. Suppose that for every $x \in \mathbb{R}^n$ there exists $\mu(x) > 0$ such that

$$\left( \frac{\mu(x)}{|y - x|} \right)^\nu \left( x + \frac{\mu(x)(y - x)}{|y - x|^2} \right) \leq f(y) \quad \text{for} \quad y \in \mathbb{R}^n \setminus \bar{B}_\mu(x). \quad (41)$$

Then, $f$ is constant.

**Remark 20.** In terms of Kelvin transform, conditions (40) and (41) can be rewritten respectively as $f_{x,\mu(x)} = f$ in $\mathbb{R}^n \setminus \{x\}$ and $f_{x,\mu(x)} \leq f$ in $\mathbb{R}^n \setminus B_{\mu(x)}(x)$.

Let us divide the moving spheres process into three lemmas: I, II and III.

First, we show that it is possible to start the process of moving spheres. For this, it is crucial Corollaries 12 and 14.

**Lemma I.** Let $U$ be a regular solution of (2). Then, for any $x_0 \in \mathbb{R}^n$, there exists $\mu_0(x_0) > 0$ such that for $\mu \in (0, \mu_0(x_0))$, $(u_i)_{x_0, \mu} \leq u_i$ in $\mathbb{R}^n \setminus B_{\mu}(x_0)$ for all $i \in I$. 

Proof. By translation invariance, we may take \( x_0 = 0 \). Let us denote \((u_i)_{\mu_i} = (u_i)_{\mu} \) for \( i \in I \).

Claim 1: For any \( i \in I_+ \), there exists \( r_0 > 0 \) such that for \( r \in (0, r_0] \), we have
\[
\partial_r \left( r^{\frac{n-4}{2}} u_i(r \theta) \right) > 0 \quad \text{for} \quad \theta \in \mathbb{S}^{n-1}.
\]
In fact, since \( u_i \) is a continuously differentiable function, for each \( i \in I_+ \) there exists \( r_i > 0 \) satisfying \( \inf_{0 < y \leq r_i} u_i > 0 \) and \( \sup_{0 < y \leq r_i} \nabla u_i < \infty \). Then, choosing
\[
r_i = \min \left\{ \tilde{r}_i, \frac{(n-4)}{2} \sup_{0 < y \leq \tilde{r}_i} \nabla u_i \right\},
\]
for \( 0 < r < r_i \), we get
\[
\partial_r \left( r^{\frac{n-4}{2}} u_i(r \theta) \right) \geq r^{\frac{n-4}{2}} \left( \frac{n-4}{2} u_i(r \theta) - r |\partial_r (r \theta)| \right).
\]
By the choice of \( r_i > 0 \), we obtain that \( \partial_r \left( r^{\frac{n-4}{2}} u_i(r \theta) \right) \geq 0 \), which by taking \( r_0 = \min_{i \in I_+} r_i \) concludes the proof of the claim.

Claim 2: For \( \mu \in (0, r_0] \) and \( x \in B_{r_0} \setminus B_\mu \), \( (u_i)_\mu \leq u_i \) in \( B_{r_0} \setminus B_\mu \).
Indeed, using Claim 1 we observe that \( \rho(r) = r^{(n-4)/2} u_i(r \theta) \) is radially increasing in \( (0, r_0] \) for any \( \theta \in \mathbb{S}^{n-1} \). Hence, taking \( r = 1 \) and \( r' = (\mu/|x|)^2 \), we have \( \rho(r') \leq \rho(1) \), which completes the proof.

By Claim 2 and Proposition 11, the hypothesis in Corollary 14 are satisfied. Consequently, for any \( r > r_0 \) and \( i \in I \),
\[
u_i(x) \geq \left( \frac{r_0}{|x|} \right)^{n-4} \min_{\partial B_{r_0}} u_i \quad \text{in} \quad B_r \setminus B_{r_0}.
\]
Setting \( \mu_0 = r_0 \min_{i \in I_+} \left( \frac{\min_{B_{r_0}} u_i}{\max_{B_{r_0}} u_i} \right)^{4-n} \), for any \( \mu \in (0, \mu_0) \), \( x \in \mathbb{R}^n \setminus B_{r_0} \) and \( i \in I \),
\[
(u_i)_\mu(x) \leq \left( \frac{\mu_0}{|x|} \right)^{n-4} \max_{B_{r_0}} u_i \leq \left( \frac{r_0}{|x|} \right)^{n-4} \min_{\partial B_{r_0}} u_i,
\]
which in combination with Claim 1 completes the proof.

After this lemma let us introduce a well-defined radius \( \mu(x_0) > 0 \). For this radius, we obtain that \( \mathcal{U} \) and its Kelvin transform \( \mathcal{U}_{\mu(x_0)} \) have the same Euclidean norm.

Definition 21. For any \( x_0 \in \mathbb{R}^n \), let us define
\[
\bar{\mu}(x_0) = \sup \{ \mu > 0 : (u_i)_{x_0, \mu} \leq u_i \quad \text{in} \quad \mathbb{R}^n \setminus B_\mu(x_0) \}.
\]

The second lemma states that if (42) is finite the moving spheres process must stop and the euclidean norm of solution to (2) are invariant under Kelvin transform.

Lemma II. Let \( \mathcal{U} \) be a regular solution of (2). If \( \bar{\mu}(x_0) < \infty \), then \( \mathcal{U}_{x_0, \bar{\mu}(x_0)} \equiv \mathcal{U} \) in \( \mathbb{R}^n \setminus \{ x_0 \} \).

Proof. Without loss of generality, we may take \( x_0 = 0 \). We denote \( \bar{\mu}(0) = \bar{\mu} \). By the definition of \( \bar{\mu} \), when \( \bar{\mu} < \infty \), we get that for any \( \mu \in (0, \bar{\mu}] \) and \( i \in I \), it holds
\[
(u_i)_\mu \leq u_i \quad \text{in} \quad \mathbb{R}^n \setminus B_\mu(0).
\]
Then, there exist \( i_0 \in I \) and \( (\mu_j)_{j \in \mathbb{N}} \) in \( [\bar{\mu}, \infty) \) satisfying \( \mu_j \to \bar{\mu} \), and such that (43) does not hold for \( i = i_0 \) and \( \mu = \mu_j \). For \( \mu > 0 \), let us define \( \omega_\mu = (u_{i_0})_\mu - (u_{i_0})_{\mu} \).

Claim 1: \( \omega_\mu \) is superharmonic.
Indeed, as a combination of (17) and Lemma 18, we obtain
\[
\begin{cases}
\Delta^2 \omega_\mu(x) = c_\mu(x) \omega_\mu & \text{in } \mathbb{R}^n \setminus B_\mu(0) \\
\Delta \omega_\mu(x) = \omega_\mu(x) = 0 & \text{on } \partial B_\mu(0),
\end{cases}
\]
where
\[
c_\mu = \frac{c(n)|\mathcal{U}|^{2^* - 2} u_{i_0} - c(n)|\mathcal{U}_\mu|^{2^* - 2} (u_{i_0})_\mu}{u_{i_0} - (u_{i_0})_\mu} > 0 \quad \text{in } \mathbb{R}^n \setminus B_\mu(0).
\]
Therefore, by Claim 1 we can use the strong maximum principle in [31, Theorem 3.5] to conclude
\[
\min_{\mathbb{R}^n \setminus B_\mu(0)} \omega_\mu = \min_{\partial B_\mu(0)} \omega_\mu.
\]

**Claim 2:** \( \omega_\mu \equiv 0. \)

Supposing that \( \omega_\mu \) is not equivalently zero in \( \mathbb{R}^n \setminus B_\mu(0) \), by Hopf Lemma [31, Lemma 3.4], we have that \( \partial_\nu \omega_\mu > 0 \) in \( \partial B_\mu(0) \). Moreover, by the continuity of \( \nabla u_{i_0} \), one can find \( r_0 > \bar{\mu} \) such that for any \( \bar{\mu} \in (\mu, r_0) \), we get
\[
\omega_\mu > 0 \quad \text{in } B_{r_0}(0) \setminus B_\mu(0).
\]
Again applying Proposition 13, we obtain
\[
\omega_\mu \geq \left( \frac{r_0}{|x|} \right)^{n-4} \omega_\mu.
\]
On the other hand, by the uniform continuity of the \( u_{i_0} \) on \( B_{r_0}(0) \), there exists \( \varepsilon > 0 \) such that for any \( \mu \in [\bar{\mu}, \mu + \varepsilon] \) and \( x \in \mathbb{R}^n \setminus B_{r_0}(0) \), it follows
\[
|\omega_\mu(x) - \omega_\mu(x)| = |(u_{i_0})_\mu(x) - (u_{i_0})_\mu(x)| \leq \frac{1}{2} \left( \frac{r_0}{|x|} \right)^{n-4} \min_{\partial B_{r_0}(0)} \omega_\mu.
\]
Therefore, a combination of (44)–(45) yields \( \omega_\mu \geq 0 \) in \( \mathbb{R}^n \setminus B_\mu(0) \) for any \( \mu \in [\bar{\mu}, \bar{\mu} + \varepsilon] \). This is a contradiction with the definition of \( \bar{\mu} \), thus \( \omega_\mu \equiv 0 \) in \( \mathbb{R}^n \setminus B_\mu(0) \). Moreover, let us define
\[
\omega_\mu(x) = - \left( \frac{\mu}{|x|} \right)^{n-4} \omega_\mu \left( \left( \frac{\mu}{|x|} \right)^2 x \right).
\]
Hence it follows that \( \omega_\mu \equiv 0 \) in \( \mathbb{R}^n \setminus \{0\} \). Since \( u_{i_0} \) cannot be identically zero without contradicting the definition of \( \bar{\mu} \), by Proposition 12 \( u_{i_0} \) is nowhere vanishing. Consequently, we obtain that \( |\mathcal{U}| = |\mathcal{U}| \) in \( \mathbb{R}^n \setminus \{0\} \).

In the last lemma, we show that the moving spheres process only stops if \( \mathcal{U} \) is the trivial solution.

**Lemma III.** Let \( \mathcal{U} \) be a regular solution of (2). If \( \bar{\mu}(x_0) < \infty \) for some \( x_0 \in \mathbb{R}^n \), then \( \mathcal{U} \equiv 0. \)

**Proof.** By definition of \( \bar{\mu}(x_0) \), if \( \bar{\mu}(x_0) = \infty \), we get that for any \( \mu > 0 \) and \( i \in I \), \( (u_i)_{x_0, \mu} \leq u_i \) in \( \mathbb{R}^n \setminus B_{\mu}(x_0) \). Moreover, assuming that \( x_0 = 0 \), by (43), we find
\[
\mu^{n-4} \leq \liminf_{|x| \to \infty} \left[ |x|^{n-4} u_i(x) \right],
\]
which by passing to limit as \( \mu \to \infty \) provides that for \( i \in I \), either \( u_i(0) = 0 \) and \( |x|^{n-4} u_i(x) \to 0 \) as \( |x| \to \infty \). Using that \( u_i(0) = 0 \) for all \( i \in I \), by Propositions 11 and 12, we conclude that \( u_i \equiv 0 \). Therefore, we may assume \( |x|^{n-4} u_i(x) \to \infty \) as \( |x| \to \infty \) for all \( i \in I_+ \).

**Claim 1:** \( \bar{\mu} = \infty \) for all \( y \in \mathbb{R}^n \).

Indeed, when \( \bar{\mu}(y) < \infty \) for some \( y \in \mathbb{R}^n \), using Lemma II, we obtain
\[
|x|^{n-4} |\mathcal{U}(x)| = |x|^{n-4} |\mathcal{U}_{y, \bar{\mu}(y)}(x)| \to \bar{\mu}(y)^{n-4} |\mathcal{U}(y)| \quad \text{as } |x| \to \infty,
\]
which is a contradiction.

Combining Claim 1 and Proposition C, we have that \( \mathcal{U} \) is constant. Since \( \mathcal{U} \) satisfies (2), it follows that \( \mathcal{U} \equiv 0 \).

3.7. Proof of Theorem 1. Now using Lemma III and Proposition B, we have enough conditions to classify \( |\mathcal{U}| \). From this classification we can compute the \( D^{2,2}(\mathbb{R}^n, \mathbb{R}^p) \)-norm of any classical solution of (17). This enables us to conclude that classical solutions are weak solutions and consequently Theorem 1 can be applied to complete the proof.

**Proof.** By Lemma III, we may assume \( \bar{\mu}(y) < \infty \) for any \( y \in \mathbb{R}^n \). Moreover, using Proposition B, there exist \( x_0 \in \mathbb{R}^n \) and \( \mu, \mu' > 0 \) such that

\[
|\mathcal{U}(x)| = \left( \frac{\mu'}{\mu + |x - x_0|^2} \right)^{\frac{n-4}{2}}. \tag{46}
\]

Let us consider a smooth cut-off function satisfying \( \eta \equiv 1 \) in \([0, 1]\), \( 0 \leq \eta \leq 1 \) in \([1, 2]\) and \( \eta \equiv 0 \) in \([2, \infty)\). For \( R > 0 \), setting \( \eta_R(x) = \eta(R^{-1}x) \) and multiplying the equation (2) by \( \eta_R u_i \), we obtain \( \Delta^2 u_i \eta_R u_i = |\mathcal{U}|^{2^* - 2} \eta_R u_i^2 \), which gives

\[
\sum_{i=1}^p \Delta^2 u_i \eta_R u_i = \sum_{i=1}^p |\mathcal{U}|^{2^* - 2} \eta_R u_i^2 = |\mathcal{U}|^{2^*} \eta_R.
\]

Thus,

\[
\int_{\mathbb{R}^n} \sum_{i=1}^p \Delta^2 u_i \eta_R u_i \, dx = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^*} \eta_R \, dx.
\]

Using integration by parts in the left-hand side,

\[
\int_{\mathbb{R}^n} \sum_{i=1}^p \Delta^2 u_i \eta_R u_i \, dx = \sum_{i=1}^p \int_{\mathbb{R}^n} u_i \Delta^2 (\eta_R u_i) \, dx. \tag{47}
\]

Applying the formula for the bi-Laplacian of the product in the right-hand side of (47),

\[
\sum_{i=1}^p \int_{\mathbb{R}^n} u_i \Delta^2 (\eta_R u_i) \, dx = \sum_{i=1}^p \int_{\mathbb{R}^n} \left[ u_i \Delta^2 (\eta_R) u_i + 4 u_i \nabla^{3/2} \eta_R \nabla u_i \right] \, dx
\]
\[+ \sum_{i=1}^p \int_{\mathbb{R}^n} \left[ 6 u_i \Delta \eta_R \Delta u_i + 4 u_i \nabla \eta_R \nabla^{3/2} u_i + u_i \eta_R \Delta^2 u_i \right] \, dx,
\]

which combined with (47) provides

\[
\sum_{i=1}^p \int_{\mathbb{R}^n} \left[ u_i \Delta^2 (\eta_R) u_i + 4 u_i \nabla^{3/2} \eta_R \nabla u_i + 6 u_i \Delta \eta_R \Delta u_i + 4 u_i \eta_R \nabla^{3/2} u_i \right] \, dx = 0. \tag{48}
\]

Again, we use integration by parts in (48) to find

\[
\sum_{i=1}^p \left[ \int_{\mathbb{R}^n} u_i^2 \Delta \eta_R \, dx - 4 \left( \int_{\mathbb{R}^n} \Delta \eta_R |\nabla u_i|^2 \, dx + \int_{\mathbb{R}^n} u_i \Delta \eta_R \Delta u_i \, dx \right) \right.
\]
\[+ 6 \int_{\mathbb{R}^n} u_i \Delta \eta_R \Delta u_i \, dx - 4 \left( \int_{\mathbb{R}^n} u_i \eta_R \Delta^2 u_i \, dx + \int_{\mathbb{R}^n} \eta_R \nabla u_i \nabla^{3/2} u_i \, dx \right) \bigg] = 0,
\]

which completes the proof.
which yields
\begin{equation}
4 \sum_{i=1}^{p} \int_{\mathbb{R}^n} \Delta^2 u_i \eta_R u_i dx = \int_{\mathbb{R}^n} (u_i)^2 (\Delta^2 \eta_R) dx - 4 \int_{\mathbb{R}^n} \Delta \eta_R |\nabla u_i|^2 dx \tag{49}
+ 2 \int_{\mathbb{R}^n} u_i \Delta \eta \Delta u_i dx + 4 \int_{\mathbb{R}^n} \Delta u_i \nabla \eta \nabla u_i dx + 4 \int_{\mathbb{R}^n} \eta_R |\Delta u_i|^2 dx.
\end{equation}

As a result of (49) and (3.7), we obtain
\begin{equation}
\int_{\mathbb{R}^n} |\mathcal{U}|^{2^*} \eta_R dx = \frac{1}{4} \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R dx - \int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 \Delta \eta_R dx \tag{50}
+ \frac{1}{2} \int_{\mathbb{R}^n} \mathcal{U} \Delta \mathcal{U} \Delta \eta_R dx + \int_{\mathbb{R}^n} \Delta \nabla \mathcal{U} \nabla \eta_R dx + \int_{\mathbb{R}^n} |\Delta \mathcal{U}|^2 \eta_R dx.
\end{equation}

Moreover, we have
\begin{equation*}
\int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R dx = \mathcal{O}(R^{4-n}) \quad \text{as} \quad R \to \infty.
\end{equation*}

Indeed, we observe
\begin{equation*}
\left| \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R dx \right| \leq \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R dx
\leq \|\Delta^2 \eta_R\|_{C^0(\mathbb{R}^n)} \int_{B_{2R}(0) \setminus B_R(0)} |\mathcal{U}|^2 dx
\leq \frac{\|\Delta^2 \eta\|_{C^0(\mathbb{R}^n)}}{R^4} \int_{R}^{2R} |\mathcal{U}(r)|^2 r^{n-1} dr
\leq \frac{\|\Delta^2 \eta\|_{C^0(\mathbb{R}^n)} \|\mathcal{U}\|_{L^\infty(\mathbb{R}^n)}^2}{R^4} \int_{R}^{2R} r^{n-1} dr
= C(n)R^{n-4}.
\end{equation*}

Analogously to the others terms, we get the following estimates
\begin{equation*}
\int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 \Delta \eta_R dx = \mathcal{O}(R^{2-n}), \quad \int_{\mathbb{R}^n} \mathcal{U} \Delta \mathcal{U} \Delta \eta_R dx = \mathcal{O}(R^{1-n}) \quad \text{and} \quad \int_{\mathbb{R}^n} \Delta \nabla \mathcal{U} \nabla \eta_R dx = \mathcal{O}(R^{1-n}).
\end{equation*}

Therefore, taking \(R \to \infty\) in (50), we find that \(\eta_R \to 1\) in the \(C^0(\mathbb{R}^n)\)-topology and
\begin{equation*}
\int_{\mathbb{R}^n} |\Delta \mathcal{U}|^2 dx = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^*} dx < \infty.
\end{equation*}

Since \(|\mathcal{U}|\) has the classification (46), a direct computation yields
\begin{equation*}
\int_{\mathbb{R}^n} |\mathcal{U}|^{2^*} dx = S(2, 2, n)^{-n},
\end{equation*}

where \(S(2, 2, n)\) is the best constant of Sobolev defined in (52). Hence, \(\mathcal{U}\) is a weak solution of (2) and the proof follows as a direct application of Theorem 1.

\begin{remark}
System (17) is equivalent to the following integral system
\begin{equation}
\mathcal{U}(x) = C_n \int_{\mathbb{R}^n} |x - y|^{4-n} f_i(\mathcal{U}(y)) dy \quad \text{in} \quad \mathbb{R}^n. \tag{51}
\end{equation}
\end{remark}

In the sense that every solution of (17) is a solution (51) plus a constant and the reciprocal also holds. W. X. Chen and C. Li in [16, Theorem 3] used the moving planes method in its integral form to classify solutions to a class of system like (51) involving more general nonlinearities. Let us mention that this approach can be extended to study higher order systems.
3.8. Maximizers for a vectorial Sobolev inequality. In order to motivate this part of the work, we observe that $U = \Lambda u_{x_0,\mu}$ also satisfy the non-coupled system
\[ \Delta^2 u_i = \lambda_i u_i^{2^*+1} \quad \text{for} \quad i \in I, \]
which is equivalent (up to a constant $c(n)$) to find the extremal function for each inequality $\|u_i\|_{D^{2,2}(\mathbb{R}^n)} \leq C(n,i)\|u_i\|_{L^2(\mathbb{R}^n)}$ for $i \in I$. In order to attack this problem, we use a variational framework based in the Sobolev space $D^{2,2}(\mathbb{R}^n, \mathbb{R}^p)$. We show that solutions obtained in Theorem 1 are indeed the extremal $p$-maps for a type of Sobolev embedding (see for instance [2,4,6,40,42]).

As usual let us denote by $D^{k,q}(\mathbb{R}^n, \mathbb{R}^p)$ the Beppo–Levi space defined as the completion of $C_c^\infty(\mathbb{R}^n, \mathbb{R}^p)$ with respect to the norm $\|U\|_{D^{k,m}(\mathbb{R}^n, \mathbb{R}^p)} = \sum_{i=1}^p \|u_i\|_{D^{k,m}(\mathbb{R}^n)}$. Notice that if $q = 2$, then $D^{k,2}(\mathbb{R}^n, \mathbb{R}^p)$ is a Hilbert space with scalar product given by $\langle U, V \rangle = \sum_{i=1}^p \langle u_i, v_i \rangle_{D^{k,2}(\mathbb{R}^n)}$. Moreover, for the higher order critical Sobolev exponent $q_k^* = (n + q)/(n - kq)$, we have the continuous embedding, $D^{k,q}(\mathbb{R}^n, \mathbb{R}^p) \hookrightarrow L^{q_k^*}(\mathbb{R}^n, \mathbb{R}^p)$ with
\[ \|U\|_{L^{q_k^*}(\mathbb{R}^n, \mathbb{R}^p)} \leq S(k, q, n, p)\|U\|_{D^{k,q}(\mathbb{R}^n, \mathbb{R}^p)}. \]
In this fashion, a natural problem to consider is obtaining extremal functions and best constants for the inequality above.

For the scalar case, the celebrated papers [3,64] contain the sharp Sobolev constant for $k = 1$ as follows
\[ S(1, q, n) = \begin{cases} \pi^{-\frac{1}{2}} n^{-\frac{1}{q}} \left(\frac{q-1}{n-q}\right)^{\frac{1}{2}} \left(\frac{\Gamma(1+\frac{n}{q})\Gamma(n)}{\Gamma(\frac{n}{q})\Gamma(n+1-\frac{n}{q})}\right)^{\frac{1}{2}}, & \text{if } 1 < q < n \\ \pi^{-\frac{1}{2}} n^{-1} \left(1+\frac{n}{2}\right)^{-\frac{1}{2}}, & \text{if } q = 1, \end{cases} \]
with extremal given by the spherical functions, i.e., for some $\mu > 0$ and $x_0 \in \mathbb{R}^n$,
\[ u(x) = \left(\frac{2\mu}{1+\mu^2|x-x_0|^{q-1}}\right)^{\frac{n-q}{q}}. \]
In particular, when $q = 2$, we get
\[ S(1, 2, n) = \left(\frac{4}{n(n-2)\omega_n^{2/n}}\right)^{1/2} \quad \text{and} \quad u_{x_0,\mu}(x) = \left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-2}{2}}. \]
On the fourth order case, $k = 2$ and $q = 2$, C. S. Lin [49] found the best constant and characterized the set of maximizers,
\[ S(2, 2, n) = \left(\frac{16}{(n-4)(n^2-4)\omega_n^{4/n}}\right)^{1/2} \quad \text{and} \quad u_{x_0,\mu}(x) = \left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-4}{2}}. \]

In the vectorial case, we quote the second order Sobolev inequality
\[ \|U\|_{L^{2^*}(\mathbb{R}^n, \mathbb{R}^p)} \leq S(1, 2, n, p)\|U\|_{D^{1,2}(\mathbb{R}^n, \mathbb{R}^p)}, \quad (53) \]
where the extremal maps are the multiples of the second order spherical functions and $S(1, 2, n, p) = S(1, 2, n)$ for all $p > 1$. Let us also consider the fourth order case of (53) as
\[ D^{2,2}(\mathbb{R}^n, \mathbb{R}^p) \hookrightarrow L^{2^*}(\mathbb{R}^n, \mathbb{R}^p). \quad (54) \]
On main result here states that the solutions to (17) are the extremal functions for
\[ \|U\|_{D^{2,2}(\mathbb{R}^n, \mathbb{R}^p)} \leq S(2, 2, n, p)\|U\|_{L^{2^*}(\mathbb{R}^n, \mathbb{R}^p)}. \quad (55) \]
Remarkably, the best constant for (54) is the same as in the scalar case, that is, $S(2, 2, n, p) = S(2, 2, n)$ for all $p > 1$. In other words, the number of equations of the system has no effects.
in the best Sobolev constant for product spaces. In what follows, let us fix the notation\( S(2, 2, n, p) = S(n, p) \).

**Proposition 23.** Let \( U \) be the spherical solution of (17). Then, \( U \) are the extremal functions for the Sobolev inequality (55), that is,

\[
\|U\|_{D^{2, 2}(\mathbb{R}^n)} = S(n, p)\|U\|_{L^{2**}(\mathbb{R}^n)}.
\]

Moreover,\( S(n, p) = S(n) \) for all \( p > 1 \).

**Proof.** Initially, we observe

\[
S(n, p)^{-2} = \inf_{\mathcal{H}^p(\mathbb{R}^n)} \sum_{i=1}^{p} \int_{\mathbb{R}^n} |\Delta u_i|^2 \, dx,
\]

where \( \mathcal{H}^p(\mathbb{R}^n) = \{ U \in D^{2, 2}(\mathbb{R}^n) : \|U\|_{L^{2**}(\mathbb{R}^n)} = 1 \} \).

When \( p = 1 \) our result is a consequence of Theorem A with best constant \( S(n) \) presented in (52).

**Claim 1:** \( S(n, p) = S(n) \) for all \( p > 1 \).

In fact, taking \( u \in D^{2, 2}(\mathbb{R}^n) \) such that \( \|u\|_{L^{2**}(\mathbb{R}^n)} = 1 \), we have that \( U = ue_1 \) belongs to \( \mathcal{H}^p(\mathbb{R}^n) \), where \( e_1 = (1, 0, \ldots, 0) \). Substituting \( U \) in (56), we get that \( S(n, p) \leq S(n) \). Conversely, we obtain

\[
\left( \sum_{i=1}^{p} \int_{\mathbb{R}^n} |u_i|^{2**} \, dx \right)^{2/2**} \leq \left( \sum_{i=1}^{p} \left( S(n, p)^{-1} \int_{\mathbb{R}^n} |\Delta u_i|^2 \, dx \right)^{2**/2} \right)^{2/2**}
\]

\[
\leq S(n)^{-1} \sum_{i=1}^{p} \int_{\mathbb{R}^n} |\Delta u_i|^2 \, dx.
\]

Therefore, by (57) we find that \( S(n, p)^{-1} \leq S(n)^{-1} \), which gives the proof of the claim.

Finally, using the direct computation

\[
\frac{\|U\|_{D^{2, 2}(\mathbb{R}^n)}}{\|U\|_{L^{2**}(\mathbb{R}^n)}} = \frac{\|u_{x_0, \mu}\|_{D^{2, 2}(\mathbb{R}^n)}}{\|u_{x_0, \mu}\|_{L^{2**}(\mathbb{R}^n)}} = S(n),
\]

we conclude the proof of the proposition. \( \square \)

### 4. Classification result for singular solutions

The objective of this section is to present the proof of Theorem 2. We will show that singular solutions to (2) are radially symmetry with respect to the origin. Initially, using an argument from F. Catrina and Z.-Q. Wang [10], we will transform (2) into an PDE system in the cylinder. Then, we obtain radial symmetry via an asymptotic moving planes technique due to [8] (see also [9, 18, 49]); this property turns (2) into an fourth order ODE system. Eventually, we are able to define a Pohozaev-type invariant by integrating the Hamiltonian energy of the associated Emden-Fowler system (for related references, we refer to [27, 28, 35, 43, 52, 68]).

Moreover, we will prove that the sign of the Pohozaev invariant provides a removable-singularity classification for solutions to (2), which in combination with a delicate ODE analysis as in [27] completes our argument.

Since in this section we are dealing with singular solutions to (2), it might happen that some component solutions have a non-removable singularity at the origin.

**Definition 24.** Let us define the blow-up set by \( I_{\infty} = \{ i \in I : \liminf_{r \to 0} u_i(r) = \infty \} \).

It is easy to observe that \( U \) being a singular solution of (2) is equivalent to \( I_{\infty} \neq \emptyset \). Hence, based in the blow-up set, we divide singular solutions to (2) in two types:

**Definition 25.** Let \( U \) be a singular solution of (2). We say that \( U \) is fully-singular if \( I_{\infty} = I \). Otherwise, if \( I_{\infty} \neq I \) we call \( U \) a semi-singular.
4.1. Cylindrical transformation. Let us introduce the so-called cylindrical transformation (see [10]). Using this device, we will convert singular solutions to (2) in the punctured space into a regular solutions in a cylinder. In fact, considering spherical coordinates denoted by (r, σ), we can rewrite (2) as the nonautonomous nonlinear system,

$$\Delta_{\text{sph}} u_i = c(n)|U|^{2^* - 2 - 2} u_i \quad \text{in} \quad \tilde{C} \quad \text{for} \quad i \in I,$$

Here $\tilde{C} = (0, \infty) \times S^{n-1}$ is the half-cylinder and $\Delta_{\text{sph}}$ is the bi-Laplacian in spherical coordinates given by

$$\Delta_{\text{sph}} = \partial_r^4 + \frac{2(n-1)}{r} \partial_r^3 + \frac{(n-1)(n-3)}{r^2} \partial_r^2 - \frac{(n-1)(n-3)}{r^3} \partial_r,$$

$$\Delta_{\text{sph}} = \frac{1}{r^4} \Delta_{\sigma}^2 + \frac{2}{r^2} \partial_r^2 \Delta_{\sigma} + \frac{2(n-3)}{r^3} \partial_r \Delta_{\sigma} + \frac{2(n-4)}{r^4} \Delta_{\sigma},$$

where $\Delta_{\sigma}$ denotes the Laplace–Beltrami operator in $S^{n-1}$.

Moreover, the vectorial Emden–Fowler change of variables (or logarithm coordinates) given by $V(t, \theta) = r^n U(r, \sigma)$, where $r = |x|$, $t = \ln r$, $\theta = x/|x|$ and $\gamma = (n - 4)/2$ is the Fowler rescaling exponent, sends the problem to the entire cylinder $\tilde{C} = \mathbb{R} \times S^{n-1}$. In the geometric setting, this change of variables corresponds to the conformal diffeomorphism between the cylinder $C$ and the punctured space $\phi : (C, g_{cyl}) \to (\mathbb{R}^n \setminus \{0\}, \delta)$ defined by $\phi(t, \sigma) = e^{-t} \sigma$. Here $g_{cyl} = dt^2 + d\sigma^2$ stands for the cylindrical metric and $d\delta = e^{2t}(dt^2 + d\sigma^2)$ for its volume element obtained via the pullback $\phi^*\delta$, where $\delta$ is the standard flat metric. Using this coordinate system and performing a lengthy computation (see [35, 68]), we arrive at the following fourth order nonlinear PDE in the cylinder,

$$\Delta_{\text{cyl}} v_i = c(n)|V|^{2^* - 2} v_i \quad \text{in} \quad C \quad \text{for} \quad i \in I.$$  

Here $V = (v_i)_{i \in I}$ and $\Delta_{\text{cyl}}$ is the bi-Laplacian in cylindrical coordinates given by

$$\Delta_{\text{cyl}} = \partial_t^4 - K_2 \partial_t^2 + K_0 + \Delta_{\theta}^2 - J_1 \Delta_{\theta} + 2 \partial_t^2 \Delta_{\theta},$$

where $K_0, K_2, J_1$ are constants depending only in the dimension defined by

$$K_0 = \frac{n^2(n-4)^2}{16}, \quad K_2 = \frac{n^2 - 4n + 8}{2} \quad \text{and} \quad J_1 = \frac{n(n-4)}{4}. $$

Along this lines let us consider the cylindrical transformation of a p-map as follows

$$\mathfrak{F} : C^\infty_c(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to C^\infty_c(C, \mathbb{R}^p) \quad \text{given by} \quad \mathfrak{F}(U) = r^n U(r, \sigma).$$

Remark 26. The transformation $\mathfrak{F}$ is a continuous bijection with respect to the Sobolev norms $\| \cdot \|_{D^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p)}$ and $\| \cdot \|_{H^2(C, \mathbb{R}^p)}$, respectively. Furthermore, this transformation sends singular solutions to (2) into solutions to (59) and by density, we get $\mathfrak{F} : D^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to H^2(C, \mathbb{R}^p)$.

4.2. Regularity. As in Proposition 9, a important question is whether weak solutions to (2) are as well classical solutions. Since the method of regularity lifting used in Proposition 9 does not directly apply in the punctured space, we use $\mathfrak{F}$ to convert (2) into a PDE in a cylinder. Then, we are able to perform a regularity method for complete non-compact manifolds in order to prove our main proposition. Here we use some result from [32].

Proposition 27. Let $U \in D^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p)$ be a weak singular solution of (2). Then, $U \in C^{4,\beta}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p)$ is a classical solution.

Proof. Note that $U \in C^{4,\beta}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p)$ is a classical solution of (2) if and only if $V \in C^{4,\beta}(C, \mathbb{R}^p)$ is a classical solution of (59). Moreover, since the cylinder is a non-compact complete Riemannian manifold with nonnegative Ricci curvature, using the continuous higher order Sobolev embedding
\[ H^2(\mathcal{C}) \hookrightarrow L^q(\mathcal{C}) \] for \( 2 \leq q \leq 2^* \) and adapting the proof of Proposition 9, we can conclude the proof of our result.

4.3. Asymptotic moving planes technique. Here using a variant of moving planes technique, we prove that singular solutions to (2) are radially symmetric with respect to the origin. The first work proving radial symmetry for solutions to PDEs via this method is due to J. Serrin [62] (see also [29,30]); his approach has been based in the reflection method developed earlier by A. D. Aleksandrov to study embedded surfaces of constant mean curvature. In our case solutions are singular at the origin, thus in order to show that they are rotationally invariant, we need to perform an adaptation of Aleksandrov’s method, which is called asymptotic moving planes method and has first appeared in [8]. Furthermore, by its simplicity this technique can be extended to fourth order problems as in [18] and also for second order singular systems (see for instance [9,49]). Our result seems to be the first one for fourth order systems.

In order to prove our main result, we require three background lemmas from [8, Section 3]

Lemma C. Let \( \vartheta \) be a harmonic function and consider \( (\vartheta)_{z,1} = |x|^{2-n} \vartheta \left( z + x \right) \) the second order Kelvin transform of \( \vartheta \), which for simplicity it is denoted by \( (\vartheta)_{z,1} = \tilde{\vartheta} \). Then, \( \tilde{\vartheta} \) is harmonic in a neighborhood at infinity and satisfies the asymptotic expansion,

\[
\begin{align*}
\tilde{\vartheta}(x) &= a_0 |x|^{2-n} + a_j x_j |x|^{-n} + O(|x|^{-n}) \\
\partial_{x_j} \tilde{\vartheta}(x) &= (2-n)a_0 x_j |x|^{-n} + O(|x|^{-n}) \\
\partial_{x_j x_j} \tilde{\vartheta}(x) &= O(|x|^{-n}).
\end{align*}
\]

Lemma D. Let \( \vartheta \) be a positive function defined in a neighborhood at infinity satisfying the asymptotic expansion (61). Then, there exist \( \lambda < 0 \) and \( R > 0 \) such that \( \vartheta(x) > \vartheta(x,\lambda) \), for \( \lambda \leq \lambda_0, |x| \geq R \) and \( x \in \Sigma_\lambda \).

Lemma E. Let \( \vartheta \) satisfy the assumptions of Lemma D with \( \vartheta(x) = \vartheta(x,\lambda) \), for some \( x \in \Sigma_\lambda \). Then, \( \vartheta(x) \) is a positive function defined in a neighborhood at infinity satisfying the asymptotic expansion (61). Then, there exist \( \lambda < 0 \) and \( R > 0 \) satisfying
\[ \begin{align*}
(i) \quad &\vartheta(x_n) > 0, \text{ if } |x_n - \lambda_0| < \varepsilon \text{ and } |x| > R; \\
(ii) \quad &\vartheta(x) > \vartheta(x,\lambda), \text{ if } x_n \geq \lambda_0 + \varepsilon/2 > \lambda \text{ and } |x| > R, \text{ for all } x \in \Sigma_\lambda, \lambda \leq \lambda_0, \text{ with } |\lambda_0 - \tilde{\lambda}| < C_0 \varepsilon,
\end{align*}\]
where \( C_0 > 0 \) is small and depends on \( \lambda \) and \( \varepsilon \).

We also require a maximum principle for singular domains contained in [15, Lemma 2.1]

Proposition D. Let \( \Omega \) be a smooth bounded domain in \( \mathbb{R}^n \) and \( Z \) be a compact set in \( \mathbb{R}^n \) with \( \text{cap}(Z) = 0 \). Assume that \( \vartheta(x), h(x) \) are nonnegative continuous functions in \( \Omega \setminus Z \) satisfying
\[ -\Delta \vartheta(x) + h(x) \leq 0 \quad \text{in} \quad \Omega \setminus Z \]
in the distributional sense. Then,
\[ \vartheta(x) \geq \int_E G(x,y)h(y)dy + \int_{\partial E} \partial_n G(x,y)\vartheta(y)ds_y \quad \text{in} \quad \Omega \setminus Z, \]
where \( G(x,y) \) is the Green function of \( -\Delta \) in \( \Omega \) with Dirichlet boundary condition. In particular,
\[ \vartheta(x) \geq \inf_{\partial(\Omega\setminus Z)} \vartheta. \]

Proposition 28. Let \( \mathcal{U} \) be a singular solution of equation (2). Then, \( \mathcal{U} \) is radially symmetric with respect to the origin and monotonically decreasing.
Proof. Since \( \mathcal{U} \) is a singular solution, we may suppose without loss of generality that 0 is a non-removable singularity of \( u_1 \). Fixing \( z \neq 0 \) a regular point of \( \mathcal{U} \), that is, \( \lim_{|x| \to z} |\mathcal{U}| < \infty \), we perform the fourth order Kelvin transform with center at the \( z \) and radius one, we have

\[
(u_i)_{z,1}(x) = |x|^{4-n}u_i \left( z + \frac{x}{|x|^2} \right) \quad \text{for} \quad i \in I.
\]

Denoting \( \tilde{u}_i = (u_i)_{z,1} \), we observe that \( \tilde{u}_1 \) is singular at zero and \( z_0 = -z/|z|^2 \), whereas the others components are singular only at zero. Furthermore, using the conformal invariance of (2), we get

\[
\Delta^2 \tilde{u}_i = |\tilde{u}|^{2* - 2}\tilde{u}_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0, z_0\}.
\]

Let us set \( \vartheta_i(x) = -\Delta \tilde{u}_i(x) \), thus \( \vartheta_i(x) = \mathcal{O}(|x|^{2-n}) \) as \( |x| \to \infty \). Using Lemma C, we get that \( \vartheta_i \) has the following harmonic asymptotic expansion at infinity,

\[
\begin{aligned}
&\vartheta_i(x) = a_{i0}|x|^{2-n} + a_{ij}x_j|x|^{-n} + \mathcal{O}(|x|^{-n}) \\
&\partial_{x_j}\vartheta_i(x) = (2-n)a_{i0}x_j|x|^{-n} + \mathcal{O}(|x|^{-n}) \\
&\partial_{x_kx_j}\vartheta_i(x) = \mathcal{O}(|x|^{-n}),
\end{aligned}
\]

where \( a_{i0} = \vartheta_i(0) \) and \( a_{ij} = \partial_{x_j}\vartheta_i(0) \).

Considering the axis defined by 0 and \( z \) as the reflection direction, we can suppose that this axis is orthogonal to the positive \( x_n \) direction, that is, \( e_n = (0, 0, \ldots, 1) \). For \( \lambda > 0 \), we construct the sets

\[
\Sigma_\lambda := \{ x \in \mathbb{R}^n : x_n > \lambda \} \quad \text{and} \quad T_\lambda := \partial \Sigma_\lambda
\]

and the reflection through the plane \( T_\lambda \) given by

\[
x = (x_1, \ldots, x_{n-1}, x_n) \mapsto x_\lambda = (x_1, \ldots, x_{n-1}, 2\lambda - x_n).
\]

Let us introduce the notation \((w_i)_\lambda(x) = \tilde{u}_i(x) - \vartheta_i)_\lambda(x)\), where \((\tilde{u}_i)_\lambda(x) = \tilde{u}_i(x_\lambda)\). Then, showing radial symmetry with respect to the origin for singular solutions to (2) is equivalent to prove that

\[
(w_i)_\lambda \equiv 0 \quad \text{for} \quad \lambda = 0.
\] (62)

Subsequently we divide the proof of (62) into three claims.

Claim 1: There exists \( \lambda_0 < 0 \) such that \((w_i)_\lambda > 0 \) in \( \Sigma_\lambda \) for all \( \lambda < \lambda_0 \) and \( i \in I \). In fact, note that \((w_i)_\lambda \) satisfies

\[
\begin{aligned}
&\Delta^2 (w_i)_\lambda = (b_i)_\lambda (w_i)_\lambda \quad \text{in} \quad \Sigma_\lambda \\
&\Delta (w_i)_\lambda = (w_i)_\lambda = 0 \quad \text{on} \quad T_\lambda,
\end{aligned}
\] (63)

where

\[
(b_i)_\lambda = \frac{c(n)|\tilde{u}_\lambda|^{2* - 2}(\vartheta_i)_\lambda - c(n)|\tilde{u}|^{2* - 2}\vartheta_i}{\tilde{u}_i - (\vartheta_i)_\lambda} > 0 \quad \text{in} \quad \Sigma_\lambda.
\]

Then, as a consequence of Lemma D, there exist \( \tilde{\lambda} < 0 \) and \( R > |z_0| + 10 \) such that

\[
\Delta (w_i)_\lambda(x) = (\vartheta_i)_\lambda(x) - \vartheta_i(x) < 0 \quad \text{for} \quad x \in \Sigma_\lambda, \quad \lambda \leq \tilde{\lambda} \quad \text{and} \quad |x| > R.
\] (64)

In addition, by Proposition D we can find \( C > 0 \) satisfying

\[
\vartheta_i(x) \geq C \quad \text{for} \quad x \in B_R \setminus \{0, z_0\}.
\] (65)

Since \( v_i \to 0 \) as \( |x| \to \infty \), combining (64) and (65), there exists \( \tilde{\lambda}_0 < \tilde{\lambda} \) such that

\[
\Delta (w_i)_\lambda(x) = (\vartheta_i)_\lambda(x) - \vartheta_i(x) < 0 \quad \text{for} \quad x \in \Sigma_\lambda \quad \text{and} \quad \lambda \leq \tilde{\lambda}_0.
\] (66)

Using that \( \lim_{|x| \to \infty} (w_i)_\lambda(x) = 0 \), we can apply the strong maximum principle to conclude that \((w_i)_\lambda(x) > 0 \) for all \( \lambda \leq \lambda_0 \) and \( i \in I \), which implies the proof of the claim.
Now thanks to Claim 1 we can define the critical parameter given by
\[ \lambda^* = \sup\{ \lambda > 0 : (66) \text{ holds for } \lambda \geq \lambda^* \}. \]

**Claim 2:** \((w_i)_{\lambda^*} \equiv 0 \) for all \( i \in I \).

Fix \( i \in I \) and suppose by contradiction that \((w_i)_{\lambda^*}(x_0) \neq 0 \) for some \( x_0 \in \Sigma_{\lambda^*} \). By continuity, we have \( \Delta(w_i)_{\lambda^*} \leq 0 \) in \( \Sigma_{\lambda^*} \). Since \( \lim_{|x| \to \infty}(w_i)_{\lambda^*}(x) = 0 \), a strong maximum principles yields \((w_i)_{\lambda^*} > 0 \) in \( \Sigma_{\lambda^*} \). Moreover, by (2) we get \( \Delta^2(w_i)_{\lambda^*} = c(n)\tilde{U}^{2^* - 2 - 2}\tilde{u}_i - c(n)\mathcal{U}|^{2^* - 2}(\tilde{u}_i)_{\lambda^*}(x) > 0 \). Hence \( \Delta(w_i)_{\lambda^*} \) is a subharmonic function. By employing again a strong maximum principle, we obtain that \( \Delta(w_i)_{\lambda} < 0 \). In addition, by the definition of \( \lambda^* \), there exists a sequence \( \{\lambda_j\}_{j \in \mathbb{N}} \) such that, \( \lambda_j \not\to \lambda^* \) and \( \sup_{\Sigma_{\lambda_j}} \Delta(w_i)_{\lambda_j}(x) > 0 \). Noting that \( \lim_{|x| \to \infty} \Delta(w_i)_{\lambda_j}(x) = 0 \), we can find \( x_j \in \Sigma_{\lambda_j} \) satisfying
\[ \Delta(w_i)_{\lambda_j}(x_j) = \sup_{\Sigma_{\lambda_j}} \Delta(w_i)_{\lambda_j}(x). \]  

By Lemma E, we observe that \( \{x_j\}_{j \in \mathbb{N}} \) is bounded. Thus, up to subsequence, we may assume that \( x_j \to x_0 \). If \( x_0 \in \Sigma_{\lambda^*} \), passing to the limit in (67), we obtain \( \Delta(w_i)_{\lambda^*}(x_0) = 0 \), which is a contradiction with \( \Delta(w_i)_{\lambda^*}(x) \leq 0 \). If \( x_0 \in T_{\lambda^*} \), we have that \( \nabla(\Delta(w_i)_{\lambda^*}(x_0)) = 0 \). This contradicts the Hopf boundary Lemma, because \( \Delta(w_i)_{\lambda^*} \) is negative and subharmonic in \( \Sigma_{\lambda^*} \), which completes the proof of the claim.

**Claim 3:** \( \lambda^* = 0 \).

Let us assume that the claim is not true, that is, \( \lambda^* < 0 \). Then, for \( \lambda = \lambda^* \), it holds \( \Delta w_{\lambda^*}(x) < 0 \). Since \( \lim_{|x| \to 0} u_1(x) = \infty \), we observe that \( u_1 \) cannot be invariant under the reflection \( x_{\lambda^*} \). Thus, using a strong maximum principle in (63), we conclude
\[ \tilde{u}_i(x) < u_1(x) \quad \text{for} \quad x \in \Sigma_{\lambda^*} \quad \text{and} \quad x_{\lambda^*} \notin \{0, z_0\}. \]  

Notice that as a consequence of \( \lambda^* < 0 \), we have that \( \{0, z_0\} \notin T_{\lambda^*} \). Whence applying the Hopf boundary Lemma, we get
\[ \partial_{x_j}(\tilde{u}_i(x_{\lambda^*}) - \tilde{u}_i(x)) = -2\partial_{x_j}\tilde{u}_i(x) > 0. \]  

Now choose \( \lambda_j \not\to \lambda^* \) and \( x_j \in \Sigma_{\lambda_j} \) such that \( \tilde{u}_1(x_{\lambda_j}) < \tilde{u}_1(x_j) \). Then, by Lemma D, we obtain that \( \{x_j\}_{j \in \mathbb{N}} \) is bounded. Whence, \( x_j \to \bar{x} \in \Sigma_{\lambda^*} \) with \( \tilde{u}_1(\bar{x}_{\lambda^*}) < \tilde{u}_1(\bar{x}) \). By (68) we know that \( \bar{x} \in \partial \Sigma_{\lambda^*} \) and then \( \partial_{x_j}\tilde{u}_1(\bar{x}) > 0 \), a contradiction with (69), which proves (62).

### 4.4. Superharmonicity

Here using the radial symmetry we prove that any component of a singular solution of (2) is superharmonic.

**Proposition 29.** Let \( \mathcal{U} \) be a singular solution of (2). Then, \( -\Delta u_i \geq 0 \) in \( \mathbb{R}^n \setminus \{0\} \) for all \( i \in I \).

**Proof.** Let us recall that \( u_i(r) = r^{-\gamma}v_i(\ln r) \) thus \( u_i(r) \geq C_1 r^{-\gamma} \), which together with (2) implies
\[ 0 < \omega_{n-1} \partial_r \Delta u_i(r) = \int_{B_r} c(n)|\mathcal{U}|^{2^* - 2}u_i dx, \]  

for \( r > 0 \) sufficiently small. Then, we get
\[ \lim_{r \to 0^+} r^{n-1} \partial_r \Delta u_i(r) = 0. \]  

Moreover, \( u_i \) satisfies
\[ \partial_r \left[r^{n-1} \partial_r \Delta u_i(r)\right] = r^{n-1} c(n)|\mathcal{U}|^{2^* - 2}u_i, \]  

which combined with (70) gives \( \partial_r \Delta u_i(r) > 0 \). Therefore, \( \Delta u_i(r) \) is strictly increasing and by the relation between \( u_i \) and \( v_i \) we find that \( \lim_{r \to \infty} \Delta u_i(r) = 0 \), which completes the proof. \( \square \)

As a direct consequence of Proposition 29, we show that singular solutions to (2) are weakly positive. Again, this property is fundamental to the study of the quotient function \( q_{i,j} = u_i/u_j \).
Proposition 30. Let \( \mathcal{U} \) be a singular solution of (2). Then, \( \mathcal{U} \) is weakly positive.

Proof. It follows direct by Proposition 29 and the strong maximum principle. \( \Box \)

Later on, we will prove that singular solutions are more than weakly positive, indeed they are strongly positive (see Corollary 48). In this case, either \( I_0 = \emptyset \) or \( I_+ = \emptyset \).

4.5. Fourth order Emden–Fowler system. Since we already know that solutions are rotationally invariant, the cylindrical transformation converts (2) into a fourth order ODE system with constant coefficients. More specifically, using Proposition 28, we eliminate the angular components in expression (58), thus we arrive at

\[
v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|v|^{2n-2} v_i \quad \text{in} \quad \mathbb{R} \quad \text{for} \quad i \in I
\]

with initial conditions given by

\[
v_i(0) = a_i, \quad v_i^{(1)}(0) = 0, \quad v_i^{(2)}(0) = b_i, \quad v_i^{(3)}(0) = 0.
\]

Remark 31. It is essential to ask about the existence of solution to the IVP (71)–(72). One can see that a necessary condition for this to hold is that the initial values must be the same for all components, that is, \( a_i = a \) and \( b_i = b \) for all \( i \in I \). Indeed, by (71) for all \( i, j \in I \) we have

\[
\frac{v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i}{v_i} = \frac{v_j^{(4)} - K_2 v_j^{(2)} + K_0 v_j}{v_j}.
\]

Then, for any \( C < 0 \) both component solution \( v_i \) and \( v_j \) satisfy the linear eigenvalue problem

\[
v^{(4)} - K_2 v^{(2)} + K_0 v = C v.
\]

Moreover, since \( K_2^2 - 4K_0 > 0 \), we have that the characteristic equation associated to (73) has four different roots \( \lambda_1 < \lambda_2 < 0 < \lambda_3 < \lambda_4 \). Thus any solution of (73) has the form

\[
\bar{v}(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + C_3 e^{\lambda_3 t} + C_4 e^{\lambda_4 t}
\]

for some \( C_1, C_2, C_3, C_4 \) constant depending on the initial values, which implies that for any \( v \) solution of (73), \( V = (1, \ldots, 1)\bar{v} \) is a solution of (71). Then, \( a_i = \bar{v}(0) \) and \( b_i = \bar{v}^{(2)}(0) \) for \( i \in I \).

4.6. Pohozaev invariant. The Pohozaev invariant is a homological constant related to the existence and classification to solutions to a vast class of PDEs. Its first appearance dates back the classical paper of S. Pohozaev [57]. Later on, N. Korevaar et al. [43] used this idea together with rescaling analysis to prove removable-singularity theorems for solutions to the singular Yamabe equation for flat background metric. For the case of a generic non-flat metric, we quote the approach of F. C. Marques [52]. In the context of second order systems we refer to the reader to [9, 18, 28].

The existence of the Pohozaev-type invariant is closely related to the conservation of the Hamiltonian energy of the ODE system associated to (2). For our fourth order setting, we define a energy which is conserved in time for all \( p \)-map solutions \( \mathcal{V} \) of system (71) (see also [27, 35, 68]).

Definition 32. For any \( \mathcal{V} \) solution of (71), let us consider its Hamiltonian Energy given by

\[
\mathcal{H}(t, \mathcal{V}) = -\langle \mathcal{V}^{(3)}(t), \mathcal{V}^{(1)}(t) \rangle + \frac{1}{2} |\mathcal{V}^{(2)}(t)|^2 - \frac{K_2}{2} |\mathcal{V}^{(1)}(t)|^2 + \frac{K_0}{2} |\mathcal{V}(t)|^2 - c(n)|\mathcal{V}(t)|^{2n},
\]

\[
\text{for} \quad t > 0.
\]

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\]

\[
\text{for} \quad t > 0.
\]
or explicitly in components,
\[
\mathcal{H}(t, \mathcal{V}) = -\left[\sum_{i=1}^{p} v_i^{(3)}(t)v_i^{(1)}(t)\right] + \frac{1}{2} \left[\sum_{i=1}^{p} v_i^{(2)}(t)^2\right] - \frac{K_2}{2} \left[\sum_{i=1}^{p} v_i^{(1)}(t)^2\right] \\
+ \frac{K_0}{2} \left[\sum_{i=1}^{p} v_i(t)^2\right] - \hat{c}(n) \left[\sum_{i=1}^{p} v_i(t)^2\right]^{2n/2},
\]
where \(\hat{c}(n) = (2n)^{-1}c(n)\).

Let us remark that
\[
\partial_t \mathcal{H}(t, \mathcal{V}) = 0. \tag{75}
\]
In other words, the Hamiltonian energy is invariant on the variable \(t\). In addition, we can integrate (74) over \(\theta\) to define another conserved quantity.

**Definition 33.** For any \(\mathcal{V}\) solution of (71), let us define the cylindrical Pohozaev integral by
\[
\Psi(t, \mathcal{V}) = \int_{S_1^{n-1}} \mathcal{H}(t, \mathcal{V}) \, d\theta.
\]
Here \(S_1^{n-1} = \{t\} \times S_1^{n-1}\) is the cylindrical ball with volume element given by \(d\theta = e^{2t}d\sigma\), where \(d\sigma_r\) is the volume element of the ball of radius \(r\) in \(\mathbb{R}^n\).

By definition, \(\Psi\) also does not depend on \(t\). Then, let us consider the cylindrical Pohozaev invariant \(\Psi(\mathcal{V}) := \Psi(t, \mathcal{V})\). Hence, applying the inverse of cylindrical transformation, we can recover the classical spherical Pohozaev integral defined by \(\mathcal{P}(r, \mathcal{U}) := \Psi \circ \mathcal{F}^{-1}(\mathcal{V})\).

**Remark 34.** We are not computing explicitly the formula for the spherical Pohozaev, because it is too lengthy and is not required for the rest of our studies. In fact, the cylindrical Pohozaev-invariant is enough to proceed our methods. Indeed, fixing \(\mathcal{H}(t, \mathcal{V}) = H\) and \(\mathcal{P}(\mathcal{U}) = P\), we get that \(\omega_{n-1} \mathcal{H} = P\). This show that the Hamiltonian energy \(H\) and spherical Pohozaev invariant \(P\) have the same sign. For an expression of the Pohozaev invariant in the spherical, we refer to [33, Proposition 3.3].

**Remark 35.** There exists a natural relation between the derivatives of \(\mathcal{P}\) and \(\mathcal{H}\) respectively,
\[
\partial_r \mathcal{P}(r, \mathcal{U}) = r \partial_t \mathcal{H}(t, \mathcal{V}).
\]
Thus for any solution \(\mathcal{U}\), the value \(\mathcal{P}(r, \mathcal{U})\) is also radially invariant.

**Definition 36.** For any \(\mathcal{U}\) solution of (2), let us define the Spherical Pohozaev Invariant given by \(\mathcal{P}(r, \mathcal{U}) := \mathcal{P}(\mathcal{U})\).

**Remark 37.** For easy reference let us summarize:
(i) A relation between the cylindrical and spherical Pohozaev invariants \(\mathcal{P}(\mathcal{U}) = \omega_{n-1} \Psi(\mathcal{V}), \omega_{n-1}\) is the \((n-1)\)-dimensional Hausdorff measure of the unit sphere;
(ii) The Pohozaev invariant of the vectorial solutions are equal to the Pohozaev invariant in the scalar case, which can be defined in a similar manner using the Hamiltonian energy of (4). More precisely, let us define \(P(u) = \Psi(r^7u)\), where
\[
\Psi(v) = \int_{S_1^{n-1}} \left[-v^{(3)}v^{(1)} + \frac{1}{2}|v^{(2)}|^2 - \frac{K_2}{2}|v^{(1)}|^2 + \frac{K_0}{2}|v|^2 - \hat{c}(n)|v|^{2n/3}\right] \, d\theta.
\]
Hence, if the regular solution is \(\mathcal{U} = \Lambda u_{x_0, \mu}\) for some \(\Lambda \in S_{n-1}\) and \(u_{x_0, \mu}\) a spherical solution from Theorem A, we obtain that \(\mathcal{P}(\mathcal{U}) = P(u_{x_0, \mu}) = 0\). Analogously, if the singular solution has the form \(\mathcal{U}_0 = \Lambda u_{a, T}\) for some \(\Lambda \in S_{n-1}\) and \(u_{a, T}\) a Emden–Fowler solution from Theorem B, we get that \(\mathcal{P}(\mathcal{U}_0) = P(u_{a, T}) < 0\).
4.7. ODE system analysis. Here we perform an asymptotic analysis process due to Z. Chen and C. S. Lin [27, Section 3]. Our analysis is based in the sign of the Pohozaev invariant, which combination with some results from [18, 35] determines whether a solution of (2) has a removable or a non-removable singularity at the origin. Before studying how this invariant classifies solutions to (2) we require a set of background results concerning the asymptotic behavior for solutions to (71) and its derivatives.

Definition 38. For \( V \) solution of (71), let us define the asymptotic set given by

\[
A(V) := \bigcup_{i=1}^{p} A(v_i) \subset \mathbb{R} = [0, \infty], \quad \text{where} \quad A(v_i) := \left\{ l \in [0, \infty] : \lim_{t \to \pm \infty} v_i(t) = l \right\}.
\]

In other words, \( A(V) \) is the set of all possible limits at infinity of the component solutions \( v_i \).

The first of our lemmas states that the asymptotic set of \( V \) is quite simple, in the sense that it does not depend on \( i \in I \) and coincides with the one in the scalar case.

Lemma 39. Let \( V \) be a solution of (71). Suppose that for all \( i \in I \) there exists \( l_i \in [0, \infty] \) such that \( \lim_{t \to \pm \infty} v_i(t) = l_i \). Therefore, \( l_i \in \{0, l^*\} \), where \( l^* = p^{-1} K_0 \frac{n+4}{n-4} \). In other terms, \( A(V) = \{0, l^*\} \).

Moreover, if \( \Psi(V) \geq 0 \), then \( l^* = 0 \).

Proof. Here it is only necessary to consider the case \( t \to \infty \) since when \( t \to -\infty \), taking \( \tau = -t \) and observing that \( \hat{V}(\tau) := V(t) \) also satisfies (71), the result follows equally.

Suppose by contradiction that the lemma does not hold. Thus, for some fixed \( i \in I \) one of the following two possibilities shall happen: either the asymptotic limit of \( v_i \) is a finite constant \( l_i > 0 \), which does not belong to the asymptotic set \( A \), or the limit blows-up, that is, \( l_i = +\infty \). Subsequently we analyze these two cases:

Case 1: \( l_i \in [0, \infty) \setminus \{0, l^*\} \).

By assumption, we have

\[
\lim_{t \to \infty} \left( c(n) |V| \frac{n}{n-4} v_i(t) - K_0 v_i(t) \right) = \kappa(n), \quad \text{where} \quad \kappa(n) := c(n) p \frac{n+4}{n-4} - K_0 l_i \neq 0, \tag{76}
\]

which implies

\[
\frac{c(n) |V|}{n-4} v_i(t) - K_0 v_i(t) = v_i^{(4)}(t) - K_2 v_i^{(2)}(t). \tag{77}
\]

A combination of (76) and (77) implies that for any \( \varepsilon > 0 \) there exists \( T_i > 0 \) sufficiently large satisfying

\[
\kappa(n) - \varepsilon < v_i^{(4)}(t) - K_2 v_i^{(2)}(t) < \kappa(n) + \varepsilon. \tag{78}
\]

Now, integrating (78), we obtain

\[
\int_{T_i}^{t_i} (\kappa(n) - \varepsilon) \, d\tau < \int_{T_i}^{t_i} \left[ v_i^{(4)}(\tau) - K_2 v_i^{(2)}(\tau) \right] \, d\tau < \int_{T_i}^{t_i} (\kappa(n) + \varepsilon) \, d\tau, \tag{79}
\]

which provides

\[
(\kappa(n) - \varepsilon)(t - T_i) + C_1(T_i) < v_i^{(3)}(t) - K_2 v_i^{(1)}(t) < (\kappa(n) + \varepsilon)(t - T_i) + C_1(T_i), \tag{80}
\]

where \( C_1(T_i) > 0 \) is a constant. Defining \( \delta := \sup_{t \geq T_i} |v_i(t) - v_i(T_i)| < \infty \), we obtain

\[
\left| \int_{T_i}^{t_i} K_2 v_i^{(1)}(\tau) \, d\tau \right| \leq |K_2| \delta.
\]

Hence, integrating (80) provides

\[
\frac{(\kappa(n) - \varepsilon)}{2} (t - T_i)^2 + L(t) < v_i^{(2)}(t) < \frac{(\kappa(n) + \varepsilon)}{2} (t - T_i)^2 + R(t), \tag{81}
\]
where $L(t), R(t) \in \mathcal{O}(t^2)$, namely

$$L(t) = C_1(T_t)(T_t - t) - |K_2|\delta + C_2(T_t) \quad \text{and} \quad R(t) = C_1(T_t)(T_t - t) + |K_2|\delta + C_2(T_t).$$

Repeating the same integration procedure in (81), we find

$$\frac{(\kappa(n) - \varepsilon)}{24}(t - T_t)^4 + \mathcal{O}(t^4) < v_i(t) < \frac{(\kappa(n) + \varepsilon)}{2}(t - T_t)^4 + \mathcal{O}(t^4). \quad (82)$$

Therefore, since $\kappa(n) \neq 0$ we can choose $\varepsilon > 0$ sufficiently small such that $\kappa(n) - \varepsilon$ and $\kappa(n) + \varepsilon$ have the same sign. Finally, passing to the limit in inequality (82), we obtain that $v_i$ blows-up and $l_i = \infty$, which is contradiction.

**Case 2:** $l_i = \infty$.

This case is more delicate and requires an argument from [53]. More precisely, let $\phi_0 \in C^\infty([0, \infty))$ be a nonnegative function satisfying $\phi_0 > 0$ in $[0, 2)$,

$$\phi_0(z) = \begin{cases} 1, & \text{for } 0 \leq z \leq 1, \\ 0, & \text{for } z \geq 2, \end{cases}$$

and for $j \in \{1, 2, 3, 4\}$, let us fix the positive constants

$$M_j := \int_0^2 \frac{|\phi^{(j)}_0(z)|}{|\phi_0(z)|} \, dz. \quad (83)$$

Using the contradiction hypothesis, we may assume that there exists $T_i > 0$ such that for $t > T_i$, it follows

$$v_i^{(4)}(t) - K_2v_i^{(2)}(t) = c(n)|V(t)|^{\frac{n}{n+4}}v_i(t) - K_0v_i(t) \geq c(n)v_i(t)^{\frac{n+4}{n-4}} - K_0v_i(t) \geq \frac{1}{2} c(n)v_i(t)^{\frac{n+4}{n-4}} \quad (84)$$

and

$$v_i^{(3)}(t) - K_2v_i^{(1)}(t) = \frac{1}{2} \int_{T_i}^t c(n)v_i(\tau)^{\frac{n+4}{n-4}} \, d\tau + C_1(T_i). \quad (85)$$

From (85) we can find $T^*_i > T_i$ satisfying $v_i^{(3)}(T^*_i) - K_2v_i^{(1)}(T^*_i) := v > 0$. In addition, since (71) is autonomous we may suppose without loss of generality that $T^*_i = 0$. Hence, multiplying inequality (84) by $\phi(t) = \phi_0(t/t)$ and integrating,

$$\int_{T_i}^{T'} v_i^{(4)}(\tau)\phi(\tau) \, d\tau - K_2\int_{T_i}^{T'} v_i^{(2)}(\tau)\phi(\tau) \, d\tau \geq \frac{1}{2} \int_{T_i}^{T'} v_i(\tau) \, d\tau,$$

where $T' = 2T$. Moreover, integration by parts combined with $\phi^{(j)}(T') = 0$ for $j = 0, 1, 2, 3$ implies

$$\int_{T_i}^{T'} v_i(\tau)\phi^{(4)}(\tau)v_i(\tau) \, d\tau - K_2\int_{T_i}^{T'} v_i(\tau)\phi^{(2)}(\tau)v_i(\tau) \, d\tau \geq \frac{1}{2} \int_{T_i}^{T'} c(n)v_i(\tau)^{\frac{n+4}{n+4}} \, d\tau + v. \quad (86)$$

On the other hand, applying the Young inequality in the right-hand side of (86), it follows

$$v_i(\tau)|\phi^{(j)}(\tau)| = \varepsilon v_i^{\frac{n+4}{n+4}}(\tau)|\phi(\tau)| + C_\varepsilon \frac{\phi^{(j)}(\tau)|\phi^{(j)}(\tau)|}{\phi(\tau)^{\frac{n+4}{n+4}}} \quad (87)$$

Hence, combining (87) and (86), we have that for $\varepsilon > 0$ sufficiently small, it holds

$$\tilde{C}_1 \int_{T_i}^{T'} \left[ \frac{|\phi^{(4)}(\tau)|^{\frac{n+4}{n-4}}}{\phi(\tau)^{\frac{n-4}{4}}} + \frac{|\phi^{(2)}(\tau)|^{\frac{n+4}{n-4}}}{\phi(\tau)^{\frac{n-4}{4}}} \right] \, d\tau \geq \frac{1}{4} \int_{T_i}^{T'} c(n)v_i(\tau)^{\frac{n+4}{n-4}} \, d\tau + v$$

for some $\tilde{C}_1 > 0$. Now by (83) we obtain

$$\tilde{C}_2 \left( M_4T^{-\frac{n+2}{2}} - M_2T^{-\frac{n}{4}} \right) \geq \frac{1}{4} \int_0^T c(n)v_i(\tau)^{\frac{n+4}{n-4}} \, d\tau \quad (88)$$
for some $\tilde{C}_2 > 0$. Therefore, passing to the limit in (88) the left-hand side converges whereas the right-hand side blows-up, which is a contradiction.

For proving of the second part, let us notice that

$$\lim_{t \to \infty} \Psi(t, V) = |S_{t-1}^{n-1}| \left( \frac{K_0}{2} |l^*|^2 - \tilde{c}(n) |l^*| \frac{2n}{n+1} \right) \geq 0,$$

which implies $l^* = 0$ and $\Psi(V) = 0$. □

The next lemma shows that if a component solution of (2) blows-up, then it shall be in finite time. In this fashion, we provide an accurate higher order asymptotic behavior for singular solutions $V$ of (71). Namely, $\bigcup_{j=1}^{\infty} \mathcal{A}(\mathcal{V}^{(j)}) = \{0\}$.

**Lemma 40.** Let $V$ be a solution of (71) such that $\lim_{t \to \pm \infty} v_i(t) \in \mathcal{A}$ for all $i \in I$. Then, for any $j \in \mathbb{N}$, we have that $\lim_{t \to \pm \infty} v_i^{(j)}(t) = 0$.

**Proof.** As before we only consider the case $t \to \infty$. Since $\mathcal{A} = \{0, l^*\}$ we must divide our approach in two cases:

**Case 1:** $\lim_{t \to \pm \infty} v_i(t) = 0$.

For each ordinary derivative case $j = 1, 2, 3, 4$, we construct one step. The case $j \geq 5$ follows directly from the previous ones and its proof is omitted. Let us begin with $j = 2$,

**Step 1:** $\mathcal{A}(v_i^{(2)}) = 0$.

By assumption $v_i(t) < l^*$ for $t \gg 1$ large, then $v_i^{(4)} - K_2 v_i^{(2)} = (c(n)|V|^n v_i - K_0 v_i) < 0$.

Defining $B_i(t) = v_i^{(2)}(t) + K_0 v_i(t)$, we have that $B_i^{(2)}(t) < 0$ for all $t \in \mathbb{R}$ and thus $B_i$ is concave near infinity, which implies $\mathcal{A}(B_i) \neq \emptyset$. Hence there exists $b^* \in [0, \infty]$ such that $b^* := \lim_{t \to \infty} B_i(t)$ and $b^* := \lim_{t \to \infty} v_i^{(2)}(t)$.

Supposing that $b^*_i \neq 0$, there exist three possibilities: First, if we assume $b^*_i = \infty$, then we get that $\lim_{t \to \infty} v_i^{(1)}(t) = \infty$, which is contradiction with $\lim_{t \to \infty} v_i(t) = 0$.

Second, if $\infty > b^*_i > 0$, it implies that for $t > 0$ sufficiently large, we find that $v_i^{(2)}(t) > b^*_i t/2$; thus $v_i^{(1)}(t) > b^*_i t/4$, which is also a contradiction with the hypothesis. Third, $b^*_i < 0$; thus using the same argument as before, we obtain that $v_i^{(1)}(t) \leq b^*_i t/4$, leading to the same contradiction. Therefore $b^*_i = 0$, which concludes the proof.

**Step 2:** $\mathcal{A}(v_i^{(1)}) = 0$.

Indeed, for $t$ sufficiently large, there exists $\tau \in [t, t+1]$ such that $v_i(t+1) - v_i(t) = v_i^{(1)}(t) + \frac{1}{2} v_i^{(2)}(\tau)$, which by taking the limit and observing that when $t \to \infty$, we have that $\tau \to \infty$, $v_i(t+1) \to 0$ and $v_i(t) \to 0$ provides $\lim_{t \to \infty} v_i^{(2)}(\xi) \to 0$. Consequently, it follows that $v_i^{(1)}(t) \to 0$.

**Step 3:** $\mathcal{A}(v_i^{(3)}) = 0$.

Since $H_i$ is concave for sufficiently large $t$ and $B_i(t) \to \infty$ as $t \to \infty$, we find $\lim_{t \to \infty} B_i^{(1)}(t) = 0$. Consequently, $v_i^{(3)}(t) \to \infty$ as $t \to \infty$.

**Step 4:** $\mathcal{A}(v_i^{(4)}) = 0$.

By equation (71) and by Step 1, we observe that $v_i^{(4)}(t) \to \infty$ as $t \to \infty$.

As a combination of Step 1–4, we finish the proof of Case 1.

The second case has an additional difficulty. Namely, since $v_i(t) \to l^*$ as $t \to \infty$ for sufficiently large $t$, there exist two possibilities: either $v_i$ is eventually decreasing or $v_i$ is eventually increasing. In both situations proofs are similar, thus we only present the proof of the first one.

**Case 2:** $\lim_{t \to \infty} v_i(t) = l^*$.

Here we proceed as before.

**Step 1:** $\mathcal{A}(v_i^{(2)}) = 0$. 


Since we are considering $v_i$ is eventually decreasing, there exists a large $T_i > 1$ such that $v_i(t) > l^*$ for $t > T_i$ and we get that $v_i^{(1)}(t) - K_2 v_i^{(2)}(t) = \left(c(n)|V|\right)^{\frac{4n}{2n+1}} v_i - K_0 v_i \geq 0$. In this case, we observe that $B_i$ is convex for sufficiently large $t > 0$. Hence, $A(B_i) \neq \emptyset$ and there exists $b_3^* = \lim_{t \to \infty} B_i(t)$. Observing that $v_i(t) \to l^*$ as $t \to \infty$, we get $\lim_{t \to \infty} v_i^{(2)}(t) = b_3^*$, where $b_3^* = b_3^* - K_2 l^*$. Now repeating the same procedure as before, we obtain $b_3^* = 0$ and thus $\lim_{t \to \infty} B_i(t) = K_2 l^*$. Therefore, $A(v_i^{(2)}) = 0$.

The remaining steps of the proof of Claim 2 follow similarly to Claim 1.

In order to continue our analysis, it is essential to show that any solution of (71) is bounded. Indeed, this is the content of the following lemma.

**Lemma 41.** Let $\mathcal{V}$ be a solution of (71). Then, for all $i \in I$, we have $v_i(t) < l^*$. In particular, $|\mathcal{V}|$ is bounded.

**Proof.** For $i \in I$, let us define the set $Z_i = \left\{ t \geq 0 : v_i^{(1)}(t) = 0 \right\}$. We divide the proof of the lemma in two cases:

- **Case 1:** $Z_i$ is bounded.

  In this case, we have that $v_i$ is monotone for large $t > 0$ and $A(v_i) \neq \emptyset$. Therefore, using Lemma 39 we obtain that $v_i$ bounded by $l^*$ for $t > 0$ sufficiently large.

- **Case 2:** $Z_i$ is unbounded.

  Fixing $H > 0$, we define $F(\tau) = \hat{c}(n)|\tau|^{\frac{2n}{2n+1}} - \frac{1}{2}|\tau|^2$, which provides that $\lim_{\tau \to \infty} F(\tau) = \infty$. Therefore, there exists $R_i > |v_i(0)|$ satisfying $F(\tau) > H$ for $\tau > R_i$.

  **Claim 1:** $|v_i| < R_i$ on $[0, \infty)$.

  Supposing by contradiction that $M_{R_i} = \{ t \geq 0 : |v_i(t)| \geq R_i \}$ is non-empty, we can define $t_i^* = \inf_{M_{R_i}} v_i$, which is strictly positive by the choice of $R_i$. Thus we obtain that $v_i(t_i^*) = R_i$ and also $v_i^{(1)}(t_i^*) \geq 0$. In addition, since $Z_i$ is unbounded, we have that $Z_i \cap [t_i^*, \infty) \neq \emptyset$. Therefore, let us consider $T_i^* = \inf_{Z_i \cap [t_i^*, \infty)} v_i$. Hence, a combination of $v_i^{(1)}(T_i^*) = 0$ and Proposition 27 implies that $v_i^{(1)}(t) \geq 0$ for all $t \in [t_i^*, T_i^*]$. Eventually, we get $v_i(T_i^*) > R_i$ and $H(T_i^*, \mathcal{V}) = \frac{1}{2} |\mathcal{V}|^{(2)}(T_i^*)^2 + F(|\mathcal{V}(T_i^*)|) > H$, which is a contradiction with (75). In order to complete the proof lemma, it is straightforward to check $R_i = l^*$ for all $i \in I$.

**Lemma 42.** Let $\mathcal{V}$ be a solution of (71). Then, for all $i \in I$ and $t \in \mathbb{R}$ we have $v_i^{(1)}(t) < \gamma v_i(t)$, where we recall that $\gamma = (n - 4)/2$ is the Fowler exponent.

**Proof.** Let us define

$$\tilde{\gamma} = \sqrt{\frac{K_2}{2} - \sqrt{\frac{K_2^2}{4} - K_0}}.$$  

Then, by a direct computation, we get $\tilde{\gamma} = \gamma$. Setting

$$\lambda_1 = \frac{K_2}{2} - \sqrt{\frac{K_2^2}{4} - K_0} \quad \text{and} \quad \lambda_2 = \frac{K_2}{2} + \sqrt{\frac{K_2^2}{4} - K_0},$$

we get that $\lambda_1 + \lambda_2 = K_2$ and $\lambda_1 \lambda_2 = K_0$. Defining the auxiliary function $\phi_i(t) = v_i^{(2)}(t) - \lambda_2 v_i(t)$, we observe $\phi_i^{(2)}(t) - \lambda_2 \phi_i = |\mathcal{V}|^{\frac{4n}{2n+1}} v_i$ and $-\phi_i^{(2)} + \lambda_2 \phi_i \leq 0$. Hence, since $\mathcal{V}$ is nonnegative solution of (8) by a strong maximum principle, we get $\phi_i < 0$ and so $w_i = v_i^{(1)}/v_i$ satisfies

$$w_i^{(1)} = -w_i + \frac{\phi_i}{v_i} \quad \text{and} \quad \frac{v_i^{(2)}}{v_i} = \lambda_1 + \frac{\phi_i}{v_i}.$$  

(89)
Moreover, by Lemma 39 there exists $t_0 \in \mathbb{R}$ such that $v_i^{(1)}(t_0) = 0$, consequently $w_i(t_0) = 0$.
Setting $M := \{ t > t_0 : w_i(t) \geq \sqrt{\lambda_i} \}$, the proof of the lemma is reduced to the next claim.

**Claim 1:** $M = \emptyset$.

Indeed, supposing the claim is not true, we set $t_1 = \inf M$. Note that $t_1 > t_0$ and $w_i^{(1)}(t_1) \geq 0$ and $w_i(t_1) = \sqrt{\lambda_i}$. On the other hand, by (89) we have $w_i^{(1)}(t_1) = \frac{2_i(t_1)}{v_i(t_1)} < 0$, which is contradiction and the claim is proved. \(\square\)

As an application of Lemma 42, we complete the proof of Proposition 28, which states that $U$ is radially decreasing.

**Corollary 43.** Let $U$ be a singular solution of (2). Then, $\partial_r u_i(r) < 0$ for all $r > 0$ and $i \in I_+$.

**Proof.** By a direct computation, we have $\partial_r u_i(r) = -r^{-\gamma-1} \left[v_i^{(1)}(t) - \gamma v_i(t)\right]$. Then, it is easy to see that the proof follows from Lemma 42. \(\square\)

In order to continue our analysis, we need to define some auxiliary functions. For $i \in I$, let us set $\varphi_i : \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\varphi_i(t) = v_i^{(3)}(t)v_i^{(1)}(t) - \frac{1}{2} v_i^{(2)}(t)^2 + \frac{K_2}{2} |v_i^{(1)}(t)|^2 - \frac{K_0}{2} |v_i(t)|^2 + c(n)|v_i(t)|^{\frac{2n}{n-4}}.$$  

**Remark 44.** By Lemma 40, we observe that

$$\varphi_i^{(1)}(t) = \left(c(n)|\nabla(t)|^{\frac{n}{n-4}} - |v_i(t)|^{\frac{n}{n-4}}\right) v_i(t)v_i^{(1)}(t).$$

Since $|\nabla| \geq |v_i|$, we have that $\text{sgn}(\varphi_i^{(1)}) = \text{sgn}(v_i^{(1)})$. In other terms, the monotonicity of $f_i$ is the same of component function $v_i$. Moreover, it holds $\sum_{i=1}^p \varphi_i(t) = -H$.

4.8. **Removable singularity classification.** After establishing those previous lemmas concerning the asymptotic behavior of global solutions to the ODE system (71), we are able to prove the main results of the section. Namely, the removable-singularity classification and the non-existence of semi-singular solutions to (2). These results are the basis for proving Theorem 2. More precisely, we show that the Pohozaev invariant of any solution is always nonpositive and is zero if and only if the origin is a non-removable, otherwise for singular solutions to (2) this invariant is negative.

**Proposition 45.** Let $U$ be a solution of (2). Then, $\mathcal{P}(U) \leq 0$ and $\mathcal{P}(U) = 0$ if and only if $U \in C^4(\mathbb{R}^n, \mathbb{R}^p)$.

**Proof.** Let us divide the proof in two claims as follows. The first one concerns about the sign of the Pohozaev invariant. Namely, we show it is always nonpositive.

**Claim 1:** If $\mathcal{P}(U) \geq 0$, then $\mathcal{P}(U) = 0$.

Indeed, let us define the sum function $v_\Sigma : \mathbb{R} \rightarrow \mathbb{R}$ given by $v_\Sigma(t) = \sum_{i=1}^p v_i(t)$. Hence by Lemma 39, for any $v_i$ there exists a sufficient large $t_i \geq 0$ such that $v_i^{(1)}(t_i) = 0$. Furthermore, by Lemma 40 for any $i \in I$, we can find a sufficiently large $t_i \geq t_i$ such that $v_i^{(1)}(t_i) < 0$ for all $t > t_i$. Then, choosing $t > \max_{i \in I} \{t_i\}$, we have $v_\Sigma^{(1)}(t) < 0$, which implies $\lim_{t \rightarrow \infty} v_i(t) = 0$. Consequently, by Lemma 42, we conclude that $\mathcal{P}(U) = 0$.

In the next claim, we use some arguments from [35, Lemma 2.4] to show that solutions with zero Pohozaev invariant have a removable singularity at the origin.

**Claim 2:** If $\mathcal{P}(U) = 0$, then $U \in C^4(\mathbb{R}^n, \mathbb{R}^p)$.

In fact, note that $v_\Sigma$ satisfies

$$v_\Sigma^{(4)} - K_2 v_\Sigma^{(2)} + K_0 v_\Sigma = c(n)|\nabla|^{\frac{n}{n-4}} v_\Sigma.$$

(90)
Setting $\tilde{f}(V) = c(n)|V|^\frac{n}{n-4}v_\Sigma$, since $v_i(t) \to 0$ as $t \to \infty$, it follows that $\lim_{t \to \infty} \tilde{f}(V(t)) = 0$. Then, we define $\tau = -t$ and $\tilde{v}_\Sigma(\tau) = v_\Sigma(t)$, which implies that $\tilde{v}_\Sigma$ also satisfies (90). Moreover, $\lim_{\tau \to -\infty} v_\Sigma(t) = \lim_{\tau \to -\infty} \tilde{v}_\Sigma(\tau) = 0$ and also
\[
\lim_{\tau \to -\infty} \tilde{f}(\tilde{V}(\tau)) = 0.
\] Consequently, by ODE theory (see, for instance [34, 39]), we can find sufficiently large $T > 1$ satisfying
\[
\tilde{v}_\Sigma(\tau) = A_1 e^{\lambda_1 \tau} + A_2 e^{\lambda_2 \tau} + A_3 e^{\lambda_3 \tau} + A_4 e^{\lambda_4 \tau}
+ B_1 \int_T^\tau e^{\lambda_1 (\tau - t)} \tilde{f}(\tilde{V}(t)) \, dt + B_2 \int_T^\tau e^{\lambda_2 (\tau - t)} \tilde{f}(\tilde{V}(t)) \, dt
- B_3 \int_\tau^\infty e^{\lambda_3 (\tau - t)} \tilde{f}(\tilde{V}(t)) \, dt - B_4 \int_\tau^\infty e^{\lambda_4 (\tau - t)} \tilde{f}(\tilde{V}(t)) \, dt,
\]
where $A_1, A_2, A_3, A_4$ are constants depending on $T$, $B_1, B_2, B_3, B_4$ are constants not depending on $T$ and
\[
\lambda_1 = -\frac{n}{2}, \quad \lambda_2 = -\frac{n-4}{2}, \quad \lambda_3 = \frac{n}{2} \quad \text{and} \quad \lambda_4 = -\frac{n-4}{2}
\]
are the solutions to the characteristic equation $\lambda^4 - K_2 \lambda^2 + K_0 \lambda = 0$. In addition, by (91) we obtain that $A_3 = A_4 = 0$. Hence we use the same ideas in [36, Theorem 3.1] to arrive at
\[
\tilde{v}_\Sigma(\tau) = O(e^{\frac{-n+2}{2} \tau}) \quad \text{as} \quad \tau \to \infty \quad \text{or} \quad v_\Sigma(t) = O(e^{\frac{-n+2}{2} t}) \quad \text{as} \quad t \to -\infty.
\]
Eventually, undoing the cylindrical transformation, we have that $u_\Sigma(r) = O(1)$ as $r \to 0$, which finishes the proof of the claim.

Therefore, using the last claim we get $u_\Sigma$ is uniformly bounded in a neighborhood of the origin. It implies $u_i \in C^0(\mathbb{R}^n)$ for all $i \in I$. Finally, standard elliptic regularity theory provides $u_i \in C^4(\mathbb{R}^n)$ for all $i \in I$, which concludes the proof of the proposition.

**Proposition 46.** Let $U$ be a singular solution of (2). Suppose $\mathcal{P}(U) < 0$, then $U$ is fully-singular.

**Proof.** Suppose by contradiction $U$ is semi-singular, that is, there exists some $i_0 \in I \setminus I_\infty$, which gives
\[
\lim_{r \to 0 \atop i \neq i_0} u_i(r) = \infty \quad \text{and} \quad \liminf_{r \to 0 \atop i \neq i_0} u_i(r) = C_{i_0} < \infty.
\] **Claim 1:** $\lim_{r \to 0 \atop i \neq i_0} v_{i_0}(t) = \infty$.

Indeed, by Lemma 42 we get $\gamma^{-1}|v_{i_0}^{(1)}(t)| \leq v_{i_0}(t) \leq C_i e^{-\gamma t}$ for all $i \in I \setminus \{i_0\}$, which provides $u_{i_0}(t) \to 0$ as $t \to \infty$. Hence since $P < 0$, we get $H < 0$, which in combination with Remark 44 yields $\sum_{i \neq 1} \varphi_i(t) = -H$. Let us divide the rest of the proof in two steps:

**Step 1:** For each $i \in I \setminus \{i_0\}$, there exists $C_i > 0$ such that $u_i(r) \geq C_i r^{-\gamma}$ for all $r \in (0,1]$, which is equivalent to the existence of some $C_i > 0$ satisfying $\inf_{t \geq 0} v_{i_0}(t) \geq C_i$. In fact, assume by contradiction that it does not hold true. Then, there exists a sequence $t_n \to \infty$ satisfying $v_i(t_n) \to 0$. Moreover, using Lemma 42 for all $i \in I$ one obtains $0 \leq \gamma^{-1}|v_i^{(1)}(t_n)| \leq v_i(t_n) \to 0$. Therefore, one concludes $\varphi_i(t_n) \to 0$, which is a contradiction and the proof of Step 1 is finished.

**Step 2:** There exists $C > 0$ such that $u_{i_0}(r) \geq Cr^{-n}$.

Indeed, writing the Laplacian in spherical coordinates, we get
\[
\partial_r \left[ r^{n-1} \partial_r u_{i_0}(r) \right] = r^{n-1} c(n) |U|^\frac{n}{n-4} u_{i_0}.
\]
Now use the estimates in Step 1 to obtain,

\[ \partial_r \left[ r^{n-1} \partial_r u_{i_0}(r) \right] \geq \hat{C}_1 r^{n-5}, \]

which implies

\[ \partial_r \left[ r^{n-1} \partial_r u_{i_0}(r) \right] \geq \hat{C}_2 r^{n-4}. \]

Then, proceeding as in (93), we get

\[ r^{n-1} \partial_r u_{i_0}(r) \geq \hat{C}_3 r^{-2}. \]

Therefore, by isolating and integrating, we conclude the proof of Step 2.

Eventually, passing to the limit as \( r \to 0 \) in Claim 1, we obtain that \( u_{i_0} \) blows-up at the origin, which is a contradiction with (92). Hence semi-singular cannot exists. \( \square \)

**Remark 47.** We highlight that Proposition 46 is such a surprising result since for the type of singular system considered in [18], it is only possible to obtain the same conclusion with some restriction on the dimension. In fact, this better behavior is due for the symmetries our nonlinearity enjoys.

**Corollary 48.** Let \( \mathcal{U} \) be a singular solution of (2). Then, \( \mathcal{U} \) is strongly positive.

**Proof.** We already know by Proposition 30 that \( \mathcal{U} \) is weakly positive. Suppose by contradiction that \( \mathcal{U} \) is not strongly positive, then there exists some \( i_0 \in I_0 \), that is, \( u_{i_0} \equiv 0 \) and so regular at the origin. Thus, by Proposition 46 all the others components must also be regular at the origin. Therefore, \( I_\infty = \emptyset \), which is contradiction since \( \mathcal{U} \) is a singular solution of (2). \( \square \)

4.9. **Proof of Theorem 2.** We have conditions to connect the information obtained so far to prove our classification result. Our idea is to use the analysis of the Pohozaev invariant and ODE methods together with Theorem 2, Propositions 45 and 46. We can summarize as follows

**Theorem 2’** Let \( \mathcal{U} \) be a solution of (2). There exist two possibilities:

(i) If \( \mathcal{P}(\mathcal{U}) = 0 \), then \( \mathcal{U} = \Lambda u_{x_0,\mu} \) (Spherical solution);

(ii) If \( \mathcal{P}(\mathcal{U}) < 0 \), then \( \mathcal{U} = \Lambda^* u_{a,T} \) (Emden–Fowler solution).

**Proof.** (i) It follows directly by Proposition 45 and Theorem 1.

(ii) Initially, note that by Corollary 48, we find \( I_+ = I_\infty = I \). Subsequently we show that for all \( i,j \in I \) the quotient functions \( q_{ij} = v_i/v_j \) are constants. Indeed, let us define the auxiliary function \( \xi_{ij}(t) = \left( v_i^{(1)} v_j - v_i v_j^{(1)} \right)(t) \). Notice that \( v_i \) and \( v_j \) satisfy

\[
\begin{align*}
\left\{ v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|\mathcal{V}|^{2^* - 2} v_i \\
\left( v_j^{(4)} - K_2 v_j^{(2)} + K_0 v_j = c(n)|\mathcal{V}|^{2^* - 2} v_j,
\right.
\end{align*}
\]

which provides

\[ (v_i^{(4)} v_j - v_i^{(4)} v_j) + K_2 \left( v_i^{(2)} v_j - v_i v_j^{(2)} \right) = 0. \]

Furthermore, by a direct computation, we have

\[ \begin{cases}
\left( v_i^{(2)} v_j - v_i v_j^{(2)} = \xi_{ij}^{(1)} \\
\left( v_i^{(4)} v_j - v_i v_j^{(4)} v_i = \xi_{ij}^{(3)} + R_{ij},
\right)
\end{cases} \]

where \( R_{ij} = \left( v_i^{(3)} v_j^{(1)} - v_j^{(3)} v_i^{(1)} \right) \).

Continuing our method, we must eliminate \( R_{ij} \), for this we apply the following claim:

**Claim 1:** If \( \mathcal{P}(\mathcal{U}) < 0 \), then \( R_{ij} \equiv 0 \) for all \( i,j \in I \).

In fact, let us suppose by contradiction that for some \( t_0 \in \mathbb{R} \), we have \( R_{ij}(t_0) \neq 0 \). Thus it is easy to observe that \( \mathcal{H}(t_0, \mathcal{V}) \neq \mathcal{H}(t, \mathcal{V}) \), which is a contradiction with (75).
Substituting (95) in (94), by Claim 1 and (72), we arrive at
\[
\begin{aligned}
&\left\{\begin{array}{l}
\xi_{ij}^{(3)} + K_2\xi_{ij}^{(1)} = 0 \\
\xi_{ij}^{(2)}(0) = \xi_{ij}^{(1)}(0) = \xi_{ij}(0) = 0.
\end{array}\right.
\end{aligned}
\] (96)

Moreover, using the Picard–Lindelöf theorem, we conclude that \(\xi_{ij} \equiv 0\) is the unique solution of (96), which provides \(\left(\frac{v}{v_j}\right)^{(1)} = 0\). Consequently, we have that \(v_i = \Lambda_{i,j}v_j\), for some \(\Lambda_{i,j} > 0\).

Finally, we use the same argument in the proof of Theorem 1' to find \(\Lambda \in S^{p-1}_{+,\ast}\) such that \(V(t) = \Lambda v_{a,T}(t)\), where \(v_{a,T}\) is the global Emden–Fowler solution of (8).

Now combining Theorem 2 and Lemma 41 imply a sharp global estimate for the blow-up rate of singular solutions to (2).

**Corollary 49.** Let \(U\) be a singular solution of (2). Then, there exist \(C_1, C_2 > 0\) such that
\[
C_1|x|^{\frac{4-n}{2}} \leq |U(x)| \leq C_2|x|^{\frac{4-n}{2}}
\] for \(x \in \mathbb{R}^n \setminus \{0\} \).

**Appendix A. Some basic proofs**

In this appendix, we present the proofs of some results used in the part of our text.

**Proof of Lemma 13.** Our argument are similar in spirit to the one in [58]. In order to simplify the notations, let us denote \(m(r) = \min_{\partial B_r} u\). Furthermore, suppose that for some \(a, b \in \mathbb{R}\), we have that \(\phi(r) = a + br^{4-n}\). Note that choosing \(a, b \in \mathbb{R}\) such that \(\phi(r_1) = m(r_1)\) and \(\phi(r_2) = m(r_2)\), it yields
\[
\phi(r) = \frac{m(r_1) \left(r_2^{-4} - r_1^{-4}\right) + m(r_2) \left(r_2^{-4} - r_1^{-4}\right)}{r_2^{-4} - r_1^{-4}}.
\]
Defining \(v(x) = u(x) - \phi(|x|)\), we conclude that \(\Delta u \leq 0\). Moreover, since \(-\Delta \phi \leq 0\), it holds that \(\Delta v \leq 0\) in \(B_1 \setminus B_2\), and \(v \geq 0\) on \(\partial (B_1 \setminus B_2)\). Thus, the strong minimum principle yields \(v \geq 0\), equivalently \(u(x) \geq \phi(|x|)\) for all \(r_1 \leq |x| \leq r_2\). Therefore, \(m(r) \geq \phi(r)\), which proves the inequality. \(\square\)

**Proof of (29).** Let us define a new coordinate system given by \(x = I_{x_0,p}(x)\) where \(I_{x_0,p}\) is the inversion through \(\partial B_p(x_0)\) defined previously. Thus \(\xi = (\xi_1, \ldots, \xi_n)\) is a system of orthogonal curvilinear coordinates with respect to \(x_0 \in \mathbb{R}^n\) and we have that the metric tensor is given by \(g_{ik} = |\xi|^{-4} \delta_{jk}\). This is a consequence of the following identities,
\[
\partial_{x_k} \xi_j = |x|^{-2} \left(\delta_{jk} - \frac{2x_jx_k}{|x|^2}\right)
\] and \(\sum_{l=1}^n \partial_{x_j} \xi_l \partial_{x_k} \xi_l = |x|^{-4} \delta_{jk}\).

In addition, we can compute the Lamé coefficients associated to metric
\[
h_{ij} = \sqrt{g_{ij}} = |\xi|^{-2}.
\] (97)
Using the Lamé coefficients, let us quote the formula for the Laplacian of a function
\[
\Delta u = \frac{1}{\prod_{k=1}^n h_k} \sum_{j=1}^n \partial_{x_j} \xi_j \left(\sum_{k=1}^n \frac{h_k}{h_j^2} \partial_{x_k} u\right),
\]
which in combination with (97) provides
\[
\Delta u(x) = |\xi|^{2n} \sum_{j=1}^n \partial_{x_j} \left(|\xi|^{4-2n} \partial_{x_j} u\right).
\] (98)
Moreover, we have
\[ |\xi|^{2n} \sum_{j=1}^{n} \partial_{\xi_j} \left( |\xi|^{4-2n} \partial_u \xi_j \right) = |\xi|^{n+2} \sum_{j=1}^{n} \partial_{\xi_j}^{(2)} \left( |\xi|^{2-n} u \right), \] (99)
which yields
\[ \sum_{j=1}^{n} \partial_{\xi_j}^{(2)} \left( |\xi|^{2-n} u \right) = u \sum_{j=1}^{n} \partial_{\xi_j}^{(2)} \left( |\xi|^{2-n} \right) + 2 \sum_{j=1}^{n} \partial_{\xi_j} \left( |\xi|^{2-n} u \right) \partial_{\xi_j} u + |\xi|^{2-n} \sum_{j=1}^{n} \partial_{\xi_j}^{(2)} u, \]
Finally, combining (98) and (99) the proof is concluded. \( \square \)

**Proof of (30).** Since the bi-Laplacian is invariant under translations and dilations, we may suppose without loss of generality that \( x_0 = 0 \) and \( \mu = 1 \), thus \( K_{0,1}(x) = |x|^{-1} \). Setting \( I_{0,1}(x) = x^* \) and \( u_{x_0,\mu} = \tilde{u} \), we can write \( \tilde{u}(x) = |x|^2 |x|^{2-n} u(x^*) \), which provides
\[ \Delta \left( |x|^2 |x|^{2-n} u(x^*) \right) = \Delta(|x|^2) |x|^{2-n} u(x^*) \] (100)
\[ \quad + 2 \nabla \left( |x|^2 \right) \nabla \left( |x|^{2-n} u(x^*) \right) + |x|^2 \Delta \left( |x|^{2-n} u(x^*) \right). \]
In addition, we get \( \Delta(|x|^2) = 2n \), which combined with (100) gives
\[ \Delta \tilde{u}(x) = 2n |x|^{2-n} u(x^*) + 2 \nabla \left( |x|^2 \right) \nabla \left( |x|^{2-n} u(x^*) \right) + \Delta \left( |x|^{-n} u(x^*) \right). \]
Again applying the Laplacian, we find
\[ \Delta^2 \tilde{u}(x) = \Delta \left( 2n |x|^{-n} u(x^*) \right) + \Delta \left( 2 \nabla \left( |x|^2 \right) \nabla \left( |x|^{n-2} u(x^*) \right) \right) + \Delta \Delta \left( |x|^{-n} u(x^*) \right) \] (101)
\[ := L_1 + L_2 + L_3. \]
Subsequently let us estimate each term in (101):
(i) For the \( L_1 \), using (29), we have
\[ \Delta \left( 2n |x|^{2-n} u(x^*) \right) = 2n |x|^{-n+2} \Delta u(x^*). \] (102)
(ii) For \( L_3 \), we get
\[ \Delta \Delta \left( |x|^{-n} u(x^*) \right) \] (103)
\[ = \Delta \left( |x|^{-2} \right) \Delta \left( |x|^{2-n} u(x^*) \right) + 2 \nabla \left( |x|^{-2} \right) \nabla \left( \Delta \left( |x|^{2-n} u(x^*) \right) \right) + |x|^{-2} \Delta \left( \Delta \left( |x|^{2-n} u(x^*) \right) \right). \]
Hence, (103) and (29) implies
\[ \Delta \left( |x|^{-n} u(x^*) \right) = 8 |x|^{-2(n+2)} \Delta \left( |x|^{2-n} u(x^*) \right) - 2n |x|^{-n+2} \Delta \left( |x|^{2-n} u(x^*) \right) \] (104)
\[ + 4 |x|^{-4} \nabla \left( \Delta \left( |x|^{2-n} u(x^*) \right) \right) + |x|^{-2} \Delta \left( \Delta \left( |x|^{2-n} u(x^*) \right) \right). \]
(iii) For \( L_2 \), we have
\[ \Delta \left( 2 \nabla \left( |x|^2 \right) \nabla \left( |x|^{2-n} u(x^*) \right) \right) = 8 \nabla x \nabla \left( |x|^2 \right) \nabla \nabla \left( |x|^{2-n} u(x^*) \right) + 4 |x|^{-n-2} \Delta u(x^*). \] (105)
On the other hand, by (29) we arrive at
\[ \nabla x \nabla \left( |x|^2 \right) \nabla \left( \nabla \left( |x|^{2-n} u(x^*) \right) \right) = \Delta \left( |x|^{2-n} u(x^*) \right) = |x|^{-(n+2)} \Delta u(x^*) \] (106)
and
\[ \nabla \left( |x|^{-(n+2)} \Delta u(x^*) \right) = 4 |x|^{-(n+4)} \Delta u(x^*) + |x|^4 \nabla \left( |x|^{2-n} \Delta u(x^*) \right). \] (107)
Thus, combining (106), (107) and (105), we obtain
\[ \Delta \left( 2 \nabla \left( |x|^2 \right) \nabla \left( |x|^{2-n} u(x^*) \right) \right) = -4 |x|^{-(n+4)} \Delta u(x^*) + 4 |x|^4 \nabla \left( |x|^{2-n} \Delta u(x^*) \right). \] (108)
Eventually, by substituting (102), (104) and (108) in (101), we conclude the proof.

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