Negative moments
of characteristic polynomials of random matrices:
Ingham-Siegel integral
as an alternative to Hubbard-Stratonovich transformation

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Abstract

We reconsider the problem of calculating arbitrary negative integer moments of the (regularized) characteristic polynomial for $N \times N$ random matrices taken from the Gaussian Unitary Ensemble (GUE). A very compact and convenient integral representation is found via the use of a matrix integral close to that considered by Ingham and Siegel. We find the asymptotic expression for the discussed moments in the limit of large $N$. The latter is of interest because of a conjectured relation to properties of the Riemann $\zeta-$ function zeroes. Our method reveals a striking similarity between the structure of the negative and positive integer moments which is usually obscured by the use of the Hubbard-Stratonovich transformation. This sheds a new light on "bosonic" versus "fermionic" replica trick and has some implications for the supersymmetry method. We briefly discuss the case of the chiral GUE model from that perspective.

1 Introduction

Recently there was an outburst of research activity related to investigating the moments and correlation functions of characteristic polynomials $Z_N(\mu) = \det(\mu I_N - \hat{H})$ for random $N \times N$ matrices $H$ of various types.

There are several, not completely independent, sources of motivation behind studying characteristic polynomials. First is the intriguing conjecture relating limiting distribution of the non-trivial zeroes $s_k = \frac{1}{2} + it_k$ of the Riemann zeta function $\zeta(s)$, on the scale of their mean spacing, to that of (unimodular) eigenvalues of large random unitary matrices. This implies that locally-determined statistical properties of $\zeta(s)$, high up the critical line $\text{Re } s = 1/2$, might be modelled by the corresponding properties of $Z(\mu)$, averaged over the so-called Circular Unitary Ensembles (CUE), i.e., with respect to the normalized Haar measure of the group $U(N)$ of $N \times N$ unitary matrices. Such a line of thought and underlying evidences in favour of the conjecture are explained in detail in the papers by Keating and collaborators, see [1, 2, 3]. In particular, in [3] the authors managed to evaluate arbitrary moments of $|Z_N(\mu = e^{i \theta})|^n$ explicitly:

$$M_{N,2n} = \left< |Z_N(\mu = e^{i \theta})|^{2n} \right> = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j + 2n)}{(\Gamma(j + n))^2}, \quad \lim_{N \to \infty} \frac{M_{N,2n}}{N^{n^2}} = \prod_{j=0}^{n-1} \frac{j!}{(j + n)!} \tag{1}$$

where $\Gamma(z)$ is the Euler gamma-function. The moments as above were derived for $\text{Re } n > -1/2$ but can be analytically continued to the whole complex $n-$plane. The limiting value is presented
for the integer positive $n$. This should be compared with the conjecture\[1, 18]\n
\[
\frac{1}{T} \int_0^T dt \left| \zeta(1/2 + it) \right|^{2n} \sim a_n \prod_{j=0}^{n-1} \frac{j!}{(j+n)!} \left( \log \frac{T}{2\pi} \right)^{n^2}
\]

(2)

for the values of the positive integer moments of the Riemann $\zeta$–function as $T \to \infty$. Here $a_n$ is the number specific for $\zeta$-function\[1], but the rest shows universal features common to both the random matrix calculations and $\zeta$– function. The parameter $\frac{1}{2\pi} \log \left( \frac{T}{2\pi} \right)$ in the above equation plays the role of the inverse spacing between the $\zeta$– function zeroes at a height $T$ and should be identified with $N/2\pi$ of the unitary random matrix calculations\[1].

It is important to have in mind a high degree of universality of the obtained results, as discussed in the work by Brezin and Hikami\[4, 5]. By universality one usually means insensitivity of the spectral characteristics to details of distributions of matrix entries. In particular, the limiting value Eq.(1) of the positive integer moments of the characteristic polynomials for unitary matrices is shared, after an appropriate normalisation, by a broad class of Hermitian random matrices, whose most prominent representative is the Gaussian Unitary Ensemble (GUE).

Other quantities like the distribution of the logarithm of the characteristic polynomial, its derivative, etc. enjoyed thorough investigations as well\[2, 3, 4, 6, 7, 8, 9]. The results were also extended to the ensemble of unitary symmetric matrices (COE- Circular Orthogonal Ensemble), which are related to statistics of zeroes of the so-called $L$– functions\[19]. An updated summary of open questions on relations between the properties of Riemann zeta function and random matrices can be found at web-page of the American Institute of Mathematics\[20\].

What concerns negative moments of the characteristic polynomials, an additional interest in calculating them arose because of a conjectured behaviour of the negative moments of the (regularized) Riemann zeta function:

\[
\frac{1}{T} \int_0^T dt \left| \zeta \left( \frac{1}{2} + \frac{\delta}{\log T} + it \right) \right|^{-2n} \sim \left( \frac{\log \frac{T}{2\pi}}{\delta} \right)^{n^2}, \quad T \to \infty
\]

(3)

put forward in \[3\] for $1 \leq \delta \ll \log T$.

The formula Eq.(3) shows divergency at negative integers $n$ and thus provides one with no explicit answer. Such a divergency is a natural consequence of necessity to regularize characteristic polynomials by adding a small imaginary part to the spectral parameter $\mu$ to avoid singularities due to eigenvalues. When such an imaginary part is comparable with the separation between the neighbouring eigenvalues one again might expect universality of the corresponding expressions.

The Section 8 of the work by Brezin and Hikami \[3\] discusses a possible way of calculating the negative moments of the characteristic polynomials themselves. However, in contrast to the moments of the absolute values those are not divergent and, when taken alone, are insufficient for the sake of comparison with Eq.(3).

Original goal of the present paper was to reconsider the problem of calculating both the negative integer moments of the characteristic polynomials and those of their absolute value. We succeeded in the analysis of correlation functions of the (regularized) characteristic polynomials in the limit of large $N$ and obtained:

\[
\lim_{N \to \infty} \frac{\langle [Z_N(\mu_1)Z_N(\mu_2)]^{-n} \rangle}{\langle [Z_N(\mu_1)]^{-n} \rangle \langle [Z_N(\mu_2)]^{-n} \rangle} = \left[ \frac{2\pi \rho(\mu_1)}{-i(\mu_1 - \mu_2)} \right]^{n^2}
\]

(4)
where the regularizations $\text{Im}\mu_1 > 0, \text{Im}\mu_2 > 0$ as well as the spectral difference $\omega = \text{Re}(\mu_1 - \mu_2)$ were considered to be of the order of mean eigenvalue spacing $\Delta\mu = [N\rho(\mu)]^{-1}$, with $\rho(\mu)$ being the mean eigenvalue density at $\mu = \frac{1}{2}\text{Re}(\mu_1 + \mu_2)$.

Such an expression complements that given in Eq.(1) and is expected to be universal and applicable to the Riemann zeta-function. Indeed, taking into account the nonuniformity of the spectral density for GUE the correspondence between the parameters should be as follows: $\Delta\mu^{-1} = N\rho(\mu) \sim \frac{1}{\pi} \log (T/2\pi)$. We see that the random matrix result Eq.(1) and the conjectured Riemann $\zeta$-function behaviour Eq.(3) agree in the overall parametric dependence.

Another motivation for such a calculation comes from some questions that arose in applications of the random matrix theory to chaotic and disordered quantum systems which we shortly discuss below.

As is well known, eigenvalues of large random matrices\cite{21, 22} played a prominent role in the development of the field of quantum chaos, see e.g.\cite{23}, and in revealing its connections to mesoscopic systems\cite{24} as well as to some aspects of Quantum Chromodynamics\cite{25}. Results on moments and correlation functions of the characteristic polynomials of large random matrices related to various aspects of quantum chaotic systems can be found in \cite{13, 7, 8, 14, 9}, see also \cite{15, 17} for related studies. Characteristic polynomials for chiral random matrix ensembles are used as a model partition function for the phenomenon of chiral symmetry breaking and as such enjoyed thorough considerations, see \cite{25} and references therein.

Intimately connected with the field of quantum chaos is the domain of mesoscopic disordered systems. The paradigmatic example is a single non-relativistic quantum particle moving at zero temperature in a static random potential. The system Hamiltonian is, in essence, equivalent to a matrix with random entries. Moreover, in some limiting case such a matrix belongs to the "domain of universality" of the classical random matrix theory. This fact is of paramount importance and follows from the seminal Efetov’s work, see the book \cite{26}, where the notion of the supermatrix (graded) non-linear $\sigma$−model was introduced for the first time. The latter tool alternative to other techniques in the theory of random matrices expresses expectation values of the (products of) resolvents of random operators in terms of integrals over graded matrices containing both commuting and anticommuting entries. The method proved to be capable of dealing with quantities less accessible by other methods and turned out to be indispensable in establishing links between the theory of random matrices and quantum chaotic/mesoscopic systems, see \cite{26, 27} and references therein.

An alternative technique which enjoyed many applications in theoretical physics of disordered systems of interacting particles is the (in)famous "replica trick". Suppose one likes to calculate the ensemble average $\langle \log Z \rangle$ of a logarithm of some quantity $Z$. The replica trick exploits the relation: $\log Z = \lim_{n \to 0} \frac{1}{n}(Z^n - 1)$ and attempts to extract the averaged logarithm from the behaviour of the moments $\langle Z^n \rangle$, with $n$ being either positive or negative integer. It is clear that in general the limiting procedure suffers from non-uniqueness of the analytical continuation and "mathematicians will throw up their hands in horror or despair, while physicists are much intrigued"\cite{29}. Random matrices provide an important testing ground for the replica calculations, with the absolute value $|Z_N(\mu)|$ of the characteristic polynomial playing the role of $Z$. The advantage here is that one has a better control on results obtained by the ill-defined recipe comparing them against those known from independent calculations.

In particular, the early paper by Verbaarschot and Zirnbauer \cite{28} devoted to the relation between the replica and supermatrix methods revealed inherent problems in the former absent in the latter. They found that the natural analytic continuation $n \to 0$ gave two different answers for the "fermionic" (positive moments) and "bosonic" (negative moments") versions of the replica trick,
neither of them coinciding with the known result. In contrast, the latter is correctly reproduced within the supermatrix approach.

Very recently the verdict of inadequacy of the fermionic replicas was challenged by Kamenev and Mezard [11] and further elaborated by Yurkevich and Lerner [12]. In particular, Kamenev and Mezard discovered a convenient integral representation for the integer positive moments providing one with a better control on analytical structure of the expressions. This allowed them to put forward an ansatz which yielded in the limit $n \to 0$ the correct exact (nonperturbative) result for the GUE matrices, and the correct asymptotic results for other symmetry classes. A critical analysis by Zirnbauer [29] demonstrated in a coherent manner that the proposed ansatz was in no way a well-behaved analytical continuation. Even so, such a critique did not devaluate the recipe itself but rather restricted its domain of applicability to perturbative calculations and called for further investigations. And indeed, the amended fermionic replica trick immediately found applications in the theory of disordered electronic systems with interactions [31, 32] when it was among very few tools actually available. Let us also mention a recent development in the framework of the Calogero-Sutherland model inspired by closely related ideas [6].

The discussed new insights in the nature of the fermionic replica left, however, unclear if one could come forward with a meaningful amendment for their bosonic counterpart within the context of nonlinear $\sigma$-model ideas (see, however, [30] for the replica limit in the context of orthogonal polynomials).

An additional motivation for the present paper was to try to bridge the gap between the cases of positive and negative $n$. Our attempt succeeded in discovering an integral representation for the negative integer moments which is strikingly close to that obtained by Kamenev and Mezard [11] for the positive ones.

Technically, analyticity properties inherent in the negative moments of the absolute value of characteristic polynomials is known to result in the non-compact ("hyperbolic") nature of the integration manifold for the bosonic nonlinear $\sigma$-model discovered by Schäfer and Wegner [35]. In standard considerations such a manifold enters via the so-called Hubbard-Stratonovich transformation (see the Appendix D for more details). It came as quite a surprise to the present author that the Hubbard-Stratonovich transformation turned out to be not only unnecessary, but played, in fact, a misleading role hiding the simple structure of the negative moments. To reveal that structure one should introduce an alternative route via use of the matrix integral close to one considered by Ingham [36] and Siegel [37] many years ago.

As to the replica limit, the fact of close similarity between our integral representation and those in [11, 12] makes it apparent that very the same KMYL recipe "works" for the bosonic version in the same way as for its fermionic counterpart. This should not be considered as contradicting the Zirnbauer's argumentation since both versions of the replica trick are somewhat deficient, in the strict mathematical sense. The result obtained just indicates that accepting one of them we have little reasons for discarding the other.

Clearly, our way of dealing with negative moments suggests certain revision of the supermatrix method whose underlying technical idea is a simultaneous uniform treatment of both types of the moments (positive and negative). In fact, we show that after the disorder average is performed treating "fermionic" and "bosonic" sectors differently can be of some advantage.

The structure of the paper is as follows. In the section II we expose our method on the simplest example of negative integer moments of the characteristic polynomials and analyse the obtained expressions in the limit $N \to \infty$. Then in the section III we proceed through the calculation for the negative moments of the absolute value of the polynomial (in fact, a correlation function). In the section IV we comment on the replica trick and illustrate our statements by addressing briefly
the case considered in [33] - the chiral GUE model - from that perspective. Finally, in the section V we present the simplest nontrivial example of extention of our method to the general type of the correlation (generating) function containing simultaneously both positive and negative moments of the characteristic polynomials of GUE/chiral GUE matrices. The open questions are summarized in the Conclusion. Technical details are presented in the appendices.

2 Negative Moments of the Characteristic Polynomial

Let \( \hat{H} \) be \( N \times N \) random Hermitian matrix with characterized by the standard (GUE) joint probability density:

\[
P(\hat{H}) = C_N \exp \left( -\frac{N}{2} \text{Tr} \hat{H}^2 \right), \quad C_N = (2\pi)^{-\frac{N(N+1)}{2}} N^{-N^2/2}
\]

with respect to the measure \( d\hat{H} = \prod_{i=1}^{N} dH_{ii} \prod_{i<j} dH_{ij} dH_{ij}^* \). Here we use \( \ast \) to denote complex conjugation and denote: \( dzdz^* \equiv 2d\text{Re}z d\text{Im}z \).

Regularizing the characteristic polynomial \( Z_N(\mu) = \det (\mu 1_N - \hat{H}) \) by considering the spectral parameter \( \mu \) such that \( \text{Im} \mu > 0 \) one represents negative integer powers of the determinant as the Gaussian integral:

\[
\left[ Z_N(\mu)^{-n} \right] = \frac{1}{(4\pi i)^n N} \int \prod_{k=1}^{n} d^2S_k \exp \left\{ \frac{i}{2} \mu \sum_{k=1}^{n} S_k^\dagger S_k - \frac{i}{2} \sum_{k=1}^{n} S_k^\dagger \hat{H} S_k \right\}
\]

where for \( k = 1, 2, ..., n \) we introduced complex \( N \)-dimensional vectors \( S_k = (s_k, 1, ..., s_k) \) so that \( d^2S_k = \prod_{i=1}^{N} ds_{k,i} ds_{k,i}^\ast \) and \( T, \dagger \) stand for the transposition and Hermitian conjugation, respectively.

Denoting by \( \langle ... \rangle \) the expectation value with respect to the distribution Eq.(5) we are interested in calculating the negative integer moments of the two types:

\[
K^{(1)}_{N,n}(\mu_1) = \langle [Z_N(\mu_1)]^{-n} \rangle \quad \text{(7)}
\]

as well as

\[
K^{(2)}_{N,n}(\mu_1, \mu_2) = \langle [Z_N(\mu_1)Z_N(\mu_2)]^{-n} \rangle \quad \text{(8)}
\]

assuming the regularization \( \text{Im}(\mu_1) = \text{Im}(\mu_2) > 0 \). In particular, when \( \text{Re}\mu_1 = \text{Re}\mu_2 \), the latter quantity amounts to the negative moment of the absolute value of the characteristic polynomial.

Let us start our consideration with the simplest of the two. Performing the ensemble averaging in the standard way one finds for the moments of the first type:

\[
K^{(1)}_{N,n}(\mu_1) = \frac{1}{(4\pi i)^n N} \int \prod_{k=1}^{n} d^2S_k \exp \left\{ \frac{i}{2} \mu_1 \sum_{k=1}^{n} S_k^\dagger S_k - \frac{1}{8N} \sum_{k,l=1}^{n} (S_k^\dagger S_l) (S_l^\dagger S_k) \right\}
\]

Further introducing a \( n \times n \) Hermitian matrix \( \hat{Q} \) with the matrix elements \( \hat{Q}_{kl} = S_k^\dagger S_l \) the integrand is conveniently rewritten as:

\[
\exp \left\{ \frac{i}{2} \mu_1 \text{Tr} \hat{Q} - \frac{1}{8N} \text{Tr} \hat{Q}^2 \right\}
\]
The standard trick suggested to deal with the apparent problem of the non-Gaussian integral above is to employ the famous Hubbard-Stratonovich transformation amounting to:

$$\exp \left\{ -\frac{1}{8N} \text{Tr} \hat{Q}^2 \right\} = \int d\hat{Q} \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}^2 - \frac{i}{2} \text{Tr} \hat{Q} \hat{Q} \right\}$$

(10)

thus trading the integration over $n \times n$ Hermitian matrices $\hat{Q}$ for a possibility to perform the Gaussian integration over the vectors $S_k$. Then the resulting matrix integral is amenable to the saddle-point treatment in the limit $N \to \infty$.

However one may notice a possibility of an alternative route. Its starting point is similar to the method employed in [38, 39] where it was suggested to rewrite the integral Eq.(9) introducing the matrix $\delta$-distribution as the product of $\delta$-distributions of all relevant matrix elements. Then, obviously,

$$K^{(1)}_{N,n} \propto \int d\hat{Q} e^{-\frac{N}{8} \text{Tr} \hat{Q}^2} I_n(\hat{Q})$$

(11)

where

$$I_n(\hat{Q}) = \prod_{k=1}^n d^2 S_k e^{\frac{i}{2} \sum_{k=1}^n S_k^\dagger S_k} \prod_{k \leq l} \delta \left( \hat{Q}_{k,l} - S_k^\dagger S_l \right)$$

(12)

and the $\delta$-distribution for complex variables is understood as the product of the $\delta$-distributions for their real and imaginary parts. From now on we do not take care explicitly of multiplicative constants in front of the integrals. We will show how to restore the constants on a later stage using the normalisation condition.

To evaluate the last expression we employ the Fourier integral representation for each of the delta-functions involved and combine the Fourier variables into a single $n \times n$ Hermitian matrix $\hat{F}$. This allows us to proceed as follows:

$$I_n(\hat{Q}) \propto \int \prod_{k=1}^n d^2 S_k e^{\frac{i}{2} \sum_{k=1}^n S_k^\dagger S_k} \int d\hat{F} \exp \left\{ \frac{i}{2} \text{Tr} \left( \hat{F} \hat{Q} \right) - \frac{i}{2} \sum_{k \leq l} F_{kl} \left( S_k^\dagger S_l \right) \right\}$$

(13)

$$\propto \int d\hat{F} e^{\frac{i}{2} \text{Tr} (\hat{F} \hat{Q})} \left[ \det \left( \hat{F} - \mu_1 1_n \right) \right]^{-N}$$

Up to this point our consideration was, in fact, parallel to that employed in [38, 39]. We however suggest to go one step further by noticing that the last matrix integral is quite close to the distinguished one considered originally by Ingham [36] and Siegel [37]:

$$J_{p,n}^{IS}(\hat{Q}) = \int_{\hat{F} > 0} d\hat{F} e^{-\frac{i}{2} \text{Tr} (\hat{F} \hat{Q})} \left[ \det \left( \hat{F} - \mu_1 1_n \right) \right]^p = \left( 2\pi \right)^{\frac{n(n-1)}{2}} p! (p+1)!...(p+n-1)! \det Q^{-(p+n)}$$

(14)

where both $\hat{F}$ and $\text{Re}\hat{Q}$ are positive definite Hermitian of the size $n$ and the formula is valid for $p \geq 0$. The Ingham-Siegel integral can be viewed as a direct generalisation of the Euler gamma-function integral: $\Gamma(p+1)q^{-p+1} = \int_{f>0} df f^p e^{-f/q}$ to the Hermitian matrix argument and paved a way to the theory of special functions of matrix arguments which is nowadays an active field of research in mathematics and statistics, see e.g [40].

1In fact, Ingham and Siegel considered the set of real symmetric matrices $\hat{F}$ rather than their Hermitian counterparts and found the result: $(\pi)^{\frac{n(n-1)}{2}} \prod_{k=1}^n \Gamma \left( p + \frac{k-1}{2} \right) \det M^{-(p+\frac{n-1}{2})}$. However, their method is equally applicable to both cases.
It is an easy matter to adopt our method to calculating our integral which is a matrix-
argument generalisation of the formula: \( f_\infty f \frac{e^{izq}}{(1 - z)^n} = \frac{2\pi i}{\Gamma(N)(iq)^{N-1}e^{i\mu}} \) for \( q > 0 \) and zero otherwise, provided \( \operatorname{Im} \mu > 0 \). Performing the calculation (Appendix A) we find for \( N \geq n \):

\[
I_{n,N}(\hat{Q} > 0) = \int d\hat{Q} e^{\frac{i}{2} \text{Tr}(\hat{Q})} \left[ \det \left( \hat{F}_\mu \right) \right]^{-N} = C_{N,n} \det \hat{Q}^{-N-n} e^{\frac{i}{2} \mu \text{Tr}\hat{Q}}
\]

(15)

with \( C_{N,n} = i^n 2^{\frac{n(n+1)}{2}} \prod_{\nu = n+1}^{n+N} (\nu - 1) \) and \( I_{n,N}(\hat{Q}) = 0 \) whenever at least one of the eigenvalues of \( Q \) is negative (we recall our choice \( \operatorname{Im} \mu_1 > 0 \)).

As a result we arrive (after rescaling the integration variable: \( \hat{Q} \rightarrow 2N\hat{Q} \)) to the following integral representation for the negative integer moments of the characteristic polynomial in terms of the integral over the matrices \( \hat{Q} \):

\[
k_{n,N,n}^{(1)} = C_{N,n}^{(1)} \int_{Q > 0} d\hat{Q} e^{-N\left[-i\mu_1 \text{Tr} \hat{Q} + \frac{i}{2} \text{Tr} \hat{Q}^2\right]} \det \hat{Q}^{-N-n}
\]

(16)

provided \( N \geq n \).

The overall constant \( C_{N,n}^{(1)} \) can be restored by noticing that for \( \operatorname{Re} \mu_1 \rightarrow \infty \) the moments tend asymptotically to \( \mu_1^{-n} \). On the other hand, it is easy to understand that such a limit is equivalent to discarding the quadratic in \( \hat{Q} \) term in the exponent of Eq.(16). The resulting integral is precisely the Ingham-Siegel one, Eq.(14), and comparison yields the required constant:

\[
C_{N,n}^{(1)} = (-iN)^{Nn}(2\pi)^{\frac{n(n-1)}{2}} \prod_{j=N-n+1}^{N-1} j!
\]

The last step of the procedure we choose eigenvalues \( q_1, \ldots, q_n \) and the corresponding eigenvectors of (positive definite) Hermitian matrix \( \hat{Q} \) as new integration variables. This corresponds to the change of the volume element: \( d\hat{Q} = G_n \Delta^2(\hat{q}) \prod_{i=1}^{n} dq_i d\mu(U_n) \) where the factor \( \Delta^2(\hat{q}) = \prod_{i<j}(q_i - q_j)^2 \) is the squared Vandermonde determinant, \( G_n = (2\pi)^{\frac{n(n-1)}{2}} \prod_{i=1}^{\frac{n}{2}} j! \) and \( d\mu(U_n) \) stands for the normalized invariant measure on the unitary group \( U(n) \). The integrand is obviously \( U(n) \) invariant and we obtain:

\[
k_{n,N,n}(\mu_1) = \left( \left[ \det (\mu_1 1_N - \hat{H}) \right]^{-n} \right) = C_{N,n}^{(1)} \int_{q_i > 0} \prod_{i=1}^{n} (dq_i q_i^{-n}) \Delta^2(\hat{q}) \exp -N \sum_{i=1}^{n} A(q_i)
\]

(17)

where

\[
C_{N,n}^{-1} = (-iN)^{Nn} \prod_{j=N-n+1}^{N-1} j! \prod_{j=1}^{n} j! \quad \text{and} \quad A(q) = \frac{1}{2} q^2 - i\mu_1 q - \ln q
\]

(18)

The last integral representation is our main result for the negative moments of the first type: \( k_{n,N,n}(\mu_1) \), valid for arbitrary \( N > n \). One can further play with the formulae for finite \( N \) and \( n \), expressing, for example, the negative moments as \( n \times n \) determinants:

\[
\left[ \det (\mu_1 1_N - \hat{H}) \right]^{-n} \propto \det \left[ \Phi_{j,k} \right]_{j,k=1}^{n}
\]

(19)

\[
\Phi_{j,k} = \int_{0}^{\infty} dq q^{N-n} \pi_j^{(1)}(q) \pi_k^{(2)}(q) e^{N[i\mu_1 q - \frac{i}{2} q^2]}
\]

\[\text{We suggest to call such an integral "the Ingham-Siegel integral of second type".}\]
where $\pi_j^{(1)}(q), \pi_j^{(2)}(q)$ are any monic polynomials of degree $j$ in the variable $q$, compare with the case of positive moments in [10].

In practice, however, we are mostly interested in the limit of large matrix sizes where one expects the results to show universality as was discussed in much detail in the Introduction. To extract the leading asymptotics as $N \to \infty$ when keeping moment order $n$ fixed one should employ the saddle-point method and find the saddle points of $A(q_i)$.

Before doing this we observe that the structure of the derived expressions show striking similarity to those obtained for the positive moments of the characteristic polynomials found in [11], see also [4, 10]:

\[
\langle \left[ \det (\mu_1 \mathbf{1}_N - \hat{H}) \right]^n \rangle = \tilde{C}_{N,n}^{(1)} e^{N\mu_1^2} \int_{-\infty}^{\infty} \prod_i dq_i \Delta^2 \{ \hat{q} \} \exp -N \sum_{i=1}^{n} A(q_i) \tag{20}
\]

where

\[
\tilde{C}_{N,n}^{(1)} = (-i)^N N^{n^2/2} \left( \frac{1}{(2\pi)^n} \right) \frac{1}{\prod_{j=1}^{n} j!}
\]

and the expression for $A(q)$ is the same as in Eq. (18).

The only essential difference between the two representations (apart from that in the multiplicative constants and a slight change of the power of the determinant: $N - n$ rather than just $N$, which is anyway irrelevant for large $N$) is the range of integration. For the positive moments one integrates over the whole real axis $-\infty < q_i < \infty$ whereas it is over the positive semiaxis $0 < q_i < \infty$ for the negative moments.

Thus, we need to consider the saddle points of $A_{\pm}(q)$. It is convenient for further reference to define $\mu_1 = \mu + \frac{N\omega}{2} + i\delta$, with $\mu, \omega, \delta$ -real, and consider $N\omega, N\delta$ to be fixed when $N \to \infty$. Then one can replace $\mu_1$ with $\mu$ in the saddle-point calculations. The saddle points are obviously given by equations:

\[
q_i - i\mu - \frac{1}{q_i} = 0 \tag{21}
\]

where $i = 1, 2, \ldots, n$. Each of these equations has two solutions:

\[
q^\pm = \frac{i\mu \pm \sqrt{4 - \mu^2}}{2} \tag{22}
\]

We would like to choose the spectral parameter $\mu$ to satisfy $|\mu| < 2$ in accordance with the idea of considering the bulk of the spectrum for GUE matrices of large size. Then only for $q^+$ the real parts are positive and the corresponding saddle points contribute to the integral over the positive semiaxis: $q > 0$. Consequently, among $2^n$ possible sets of saddle points $(q_1^\pm, \ldots, q_n^\pm)$ only the choice

\[
q^+ = \text{diag}(q_1^+, \ldots, q_n^+) \tag{23}
\]

should be considered as relevant. This feature constitutes a considerable difference from the case of positive moments where all $2^n$ saddle-points yield, in principle, non-trivial contributions, albeit of different order of magnitude in powers of the small parameter $N^{-1}$. For example, for $n = 2K$ the leading order contribution in the later case comes from the choice of half of saddle-points to be $q^+$, the rest being $q^-$, with the combinatorial factor $\binom{2K}{K}$ counting the number of such sets [4].
Presence of the Vandermonde determinants makes the integrand vanish at the saddle-point sets of the exponent and thus care should be taken when calculating the saddle point contribution to the integral. This part of the procedure uses explicitly the so-called Selberg integral:

\[ Z_n(t) = \int_{-\infty}^{\infty} \prod_{k=1}^{n} d\xi_k \prod_{k_1 < k_2} (\xi_{k_1} - \xi_{k_2})^2 e^{-\frac{t}{2} \sum_{k=1}^{n} \xi_k^2} = (2\pi)^{n/2} t^{-n^2/2} \prod_{j=1}^{n} j! \]

for \( t > 0 \), see the paper by Kamenev and Mezard [1] for more details. General points of their analysis are applicable for our case without any modification.

Expanding around the relevant saddle-points: \( q_k = q^+ + \xi_k \) and performing the required calculations we find in a straightforward way the asymptotic expressions for the negative moments:

\[
K_{N,n}^{(1)}(\mu_1) = (-i)^N N^{N - \frac{n^2}{2}} \frac{(2\pi)^{n/2}}{\prod_{j=N-n}^{N-1} j!} \left[ \frac{i\mu + \sqrt{4 - \mu^2}}{2} \right]^{N_n + \frac{n^2}{2}} (4 - \mu^2)^{-\frac{n^2}{2}} \exp \left\{ \frac{i\omega N_n}{4} (i\mu + \sqrt{4 - \mu^2}) - \frac{N n}{2} \left( 1 + \frac{\mu^2 - i\mu \sqrt{4 - \mu^2}}{2} \right) \right\}
\]

The formula for \( K_{N,n}^{(1)}(\mu_2) \) where \( \mu_2 = \frac{\mu}{2} - i\delta \) can be obtained from the above expression by taking its complex conjugate and changing \( \omega \rightarrow -\omega \). Taking the product of the two expressions we finally find:

\[
K_{N,n}^{(1)}(\mu_1)K_{N,n}^{(1)}(\mu_2) = N^{2N - n^2} \frac{(2\pi)^n}{\prod_{j=N-n}^{N-1} j!^2} [2\pi \rho(\mu)]^{-n^2} \exp \left\{ \frac{N n}{2} \left[ i\mu \rho(\mu) - \left( 1 + \frac{\mu^2}{2} \right) \right] \right\}
\]

where we used the known expression for the (semicircular) mean density of GUE eigenvalues: \( \rho(\mu) = \frac{1}{\pi} \sqrt{4 - \mu^2} \).

This completes the calculation of the denominator in the formula Eq.(1). To find the corresponding numerator we proceed to derivation of the analogous expressions for the moments of the second type.

### 3 Correlation functions for the negative moments of the characteristic polynomials.

To this end, we consider the product of the expression Eq.(1) with its complex conjugate at a different value of the spectral parameter and average it over the GUE probability density. From now on we use the index \( \sigma = 1, 2 \) to label the N-component vectors \( S_\sigma \) stemming from the first/second set of the integrals. To write the resulting expression in a compact form it is again convenient to introduce \( 2n \times 2n \) Hermitian matrix \( \hat{Q} \) with the matrix elements \( \hat{Q}_{kl}^{\sigma_1,\sigma_2} = S_{\sigma_1,k}^\dagger S_{\sigma_2,l} \), with \( k \) and \( l \) taking the values \( 1, \ldots, n \). In terms of such a matrix we have:

\[
K_{N,n}^{(2)}(\mu_1, \mu_2) \propto \int \prod_{k=1}^{n} d^2S_{1,k} d^2S_{2,k} \exp \left\{ \frac{i}{2} \mu_1 \sum_{k=1}^{n} S_{1,k}^\dagger S_{1,k} - \frac{i}{2} \mu_2 \sum_{k=1}^{n} S_{2,k}^\dagger S_{2,k} - \frac{1}{8N} \text{Tr} \left( \hat{Q} \hat{L} \hat{Q} \hat{L} \right) \right\}
\]
where \( \hat{L} = \text{diag}(1_n, -1_n) \).

Again, the standard way is to use a variant of the Hubbard-Stratonovich transformation Eq. (10) allowing to convert the term quadratic in \( \hat{Q} \) (quartic in \( S \)) to that linear in \( \hat{Q} \) (quadratic in \( S \)) and integrate out the vectors \( S \). However, presence of the matrix \( \hat{L} \) and the requirement of convergency of the Gaussian integrals necessitates introducing this time a rather non-trivial domain (the so-called "hyperbolic manifold", \[34\]) for the integration over \( \hat{Q} \), to make such a "decoupling" well-defined. This problem comprehensively discussed e.g. in \[23, 41\] makes the whole procedure technically involved. For a good pedagogical introduction see \[23\], the outline of the procedure is presented in the Appendix D of the present paper.

For the method suggested in the present paper such problem does not arise at all. The \( 2n \times 2n \) matrix \( \hat{Q} \) is a Hermitian positive definite and the whole procedure at this stage does not require any modification. Employing the Ingham-Siegel integral of second type yields in this case:

\[
K_{N,n}^{(2)}(\mu_1, \mu_2) = C_{N,n}^{(2)} N \int_{\hat{Q} > 0} d\hat{Q} e^{-N[ -i \text{Tr} \hat{M} \hat{Q} + \frac{1}{4} \text{Tr}(\hat{L} \hat{Q} \hat{L})]} \det \hat{Q}^{N-2n} \hat{M} = \text{diag}(\mu_1 1_n, -\mu_2 1_n)
\]

provided \( N \geq 2n \), with the overall constant

\[
C_{N,n}^{(2)} = (N)^{2Nn} (2\pi)^{-n(2n-1)} \prod_{j=N-2n+1} \frac{1}{j!}
\]

Clearly, such a uniform applicability can be considered as a technical advantage. Nevertheless hyperbolic structure, in fact, lurks in the expression above and manifests itself at the next stage. Namely, equation Eq. (28) differs from its analogue Eq. (10) in one important aspect: it is now presented in the Appendix D of the present paper. We therefore arrive to the following expression:

\[
K_{N,n}^{(2)} \propto \int_{\hat{P}_1 > 0} \int_{\hat{P}_1 < 0} d\hat{P}_1 d\hat{P}_2 \text{I} \left( \hat{M}, \hat{P}_1, \hat{P}_2 \right) \prod_{k_1,k_2} (q_{1,k_1} - q_{2,k_2})^2 \det \hat{P}_1^{N-2n} \det \left( -\hat{P}_2 \right)^{N-2n} e^{\frac{1}{4} \text{Tr}(\hat{P}_1^2 + \hat{P}_2^2)}
\]
where

\[
I(\hat{M}, \hat{P}_1, \hat{P}_2) = \int d\mu(T) \exp \left\{ iN \text{Tr} \left( \begin{array}{cc}
\mu_1 & 1_n \\
\mu_2 & 1_n
\end{array} \right) \hat{T}_0 \left( \begin{array}{c}
\hat{P}_1 \\
\hat{P}_2
\end{array} \right) \hat{T}_0^{-1} \right\}
\]

\[
\propto [-i(\mu_1 - \mu_2)]^{-n^2} \prod_{k_1, k_2} (q_{1, k_1} - q_{2, k_2}) \exp(iN \text{Tr}(\mu_1 \hat{q}_1 + \mu_2 \hat{q}_2))
\]  

The calculation of the above integral is presented in the Appendix C. We see that its value depends only on the eigenvalue matrices \( \hat{q}_1 \) and \( \hat{q}_2 \). As a final step we change \( \hat{P}_2 \to -\hat{P}_2 \) and again introduce those eigenvalues (and corresponding eigenvectors) of the Hermitian matrices \( \hat{P}_1 > 0 \) and \( \hat{P}_2 > 0 \) as the integration variables. This results in the following expression for the correlation function of negative moments of the characteristic polynomial:

\[
K_{N,n}^{(2)}(\mu_1, \mu_2) = \left\langle \left[ \text{det}(\mu_1 \mathbf{1}_N - \hat{H}) \text{det}(\mu_2 \mathbf{1}_N - \hat{H}) \right]^{-n} \right\rangle
\]

\[
= C_{N,n}^{(2)} \left( \prod_{k_1, k_2} (q_{1, k_1} + q_{2, k_2}) \right) e^{-N \sum_{i=1}^n A_1(q_{1, i}) - N \sum_{i=1}^n A_2(q_{2, i})}
\]

where

\[
C_{N,n}^{(2)} = \frac{N^{2Nn-n^2}}{\prod_{j=1}^{N-n} j!(j-n)!} \left[ \prod_{j=1}^n j! \right]^{2}
\]

and

\[
A_1(q) = \frac{1}{2} q^2 - i \mu_1 q - \ln q \quad , \quad A_2(q) = \frac{1}{2} q^2 + i \mu_2 q - \ln q
\]

The constant \( C_{N,n}^{(2)} \) given above is most easily checked by considering the limit \( \mu_1 \gg \mu_2 \gg 1 \) in both sides of Eq. \( (32) \) and using the identity:

\[
\int_0^\infty \prod_{i=1}^n dq_i q_i^p \Delta^2(\hat{q}) e^{-\beta \sum_{i=1}^n q_i} = \beta^{-n(n+p)} \prod_{j=1}^n j! \prod_{j=p}^{p+n-1} j!
\]

valid for \( p \geq 0 \) and \( Re\beta > 0 \). Such a formula is an immediate consequence of Eq. \( (34) \) when going to eigenvalues of the matrix \( \hat{F} \) as integration variables and considering: \( Q = \beta \mathbf{1}_n \).

Again we see that the structure of the derived expressions is strikingly similar to those obtained for the correlation functions of the positive moments of the characteristic polynomials \( \mathbf{1}_n \):

\[
\left\langle \left[ \text{det}(\mu_1 \mathbf{1}_N - \hat{H}) \text{det}(\mu_2 \mathbf{1}_N - \hat{H}) \right]^{n} \right\rangle
\]

\[
\propto C_{N,n}^{(2)} e^{\frac{\partial}{\partial \mu_1^2 + \partial \mu_2^2}} \int_0^\infty \prod_{i=1}^n dq_{1,i} \Delta^2(\hat{q}_1) \int_0^\infty \prod_{i=1}^n dq_{2,i} \Delta^2(\hat{q}_1)
\]

\[
\times \prod_{k_1, k_2} (q_{1, k_1} + q_{2, k_2}) e^{-N \sum_{i=1}^n A_1(q_{1, i}) - N \sum_{i=1}^n A_2(q_{2, i})}
\]
where

\[
\tilde{C}_{N,n}^{(2)} = \left[ \tilde{C}_{N,n}^{(1)} \right]^2 \frac{1}{[-i(\mu_1 - \mu_2^*)]^n} ,
\]

the constant \( \tilde{C}_{N,n}^{(1)} \) is defined earlier in Eq. (24) and expressions for \( A_1(q) \), \( A_2(q) \) are the same as for the negative moments, Eq. (23).

Now, however, the difference between the domains of integration has more important consequences. Namely, the negative moments of the absolute value of the characteristic polynomial are truly divergent for \((\mu_1 - \mu_2^*) \to 0\), as represented by the factor \((\mu_1 - \mu_2^*)^{-n}\) in the corresponding formula. For their positive counterparts such a singularity is fake and is compensated when performing the integration along the whole real axis.

Again, we would like to perform the asymptotic analysis for \( N \to \infty \). As discussed in the Introduction the most interesting ”local” universal regime is to occur when one keeps the difference \( \text{Re}(\mu_1 - \mu_2) = \omega \) and the regularisation \( \delta \) so small as to ensure \( N\max(\omega,\delta) < \infty \) in such a limit, whereas \( \mu = \text{Re}(\mu_1 + \mu_2) \) is kept in the range \(|\mu| < 2\). To short our notations we include the regularization \( \delta \) into \( \omega \), so that \( \mu_{1,2} = \mu \pm \omega/2 \). Then we can write:

\[
N \sum_{i=1}^n A_1(q_{1,i}) + N \sum_{i=1}^n A_2(q_{2,i}) = \frac{i}{2} N \omega \sum_{n=1}^n (q_{1,i} + q_{2,i}) + N \left[ \sum_{i=1}^n A_+(q_{1,i}) + \sum_{i=1}^n A_-(q_{2,i}) \right] ,
\]

where the functions \( A_\pm(q) \) are obtained from \( A_{1,2}(q) \) by setting \( \mu_1 = \mu_2 = \mu \).

The stationary points of \( A_\pm(q) \) which are obviously given by the equations:

\[
q_{1,i} - i\mu - \frac{1}{q_{1,i}} = 0 \quad \text{and} \quad q_{2,i} + i\mu - \frac{1}{q_{2,i}} = 0
\]

where \( i = 1,2,\ldots,n \). Each of these two equations has two solutions:

\[
q_{1,i}^\pm = \frac{i\mu \pm \sqrt{4 - \mu^2}}{2} \quad \text{and} \quad q_{2,i}^\pm = -\frac{i\mu \pm \sqrt{4 - \mu^2}}{2} ,
\]

but only for \( q_{1,2}^\pm \) the real parts are positive and the corresponding saddle points contribute to the integral over the positive semiaxis: \( q_{1,i} > 0 \) or \( q_{2,i} > 0 \). Consequently, among \( 2^{2n} \) possible sets of stationary point \((q_{1,1}^\pm,\ldots,q_{1,n}^\pm,q_{2,1}^\pm,\ldots,q_{2,n}^\pm)\) only the choice

\[
\hat{q}^+ = \text{diag}(q_{1,1}^+,\ldots,q_{1,n}^+,q_{2,1}^+,\ldots,q_{2,n}^+)
\]

should be considered as relevant.

Taking care of the Vandermonde determinants via the Selberg integral Eq. (24) and calculating in this way the fluctuations around the chosen saddle points we find the asymptotic expression for the negative moments of the second type:

\[
\left\langle \left[ \det (\mu_1 1_N - \hat{H}) \det (\mu_2 1_N - \hat{H}) \right]^{-n} \right\rangle
\]

\[
= (2\pi)^n \left( \frac{1}{-i(\mu_1 - \mu_2^*)} \right)^n N^{2n(N-n)} \frac{1}{\prod_{j=N-n}^{N-1} j!} e^{-nN(1+\mu^2/2)+iNn\pi\omega\rho(\mu)}
\]

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which enters the numerator of Eq. (4). Dividing this expression by that presented in Eq. (26) and taking into account:

\[
\prod_{N-n}^{N-1} \frac{j!}{(j-n)!} \sim N^n \quad \text{as } N \to \infty
\]

we arrive at the announced formula Eq. (4).

4 Replica limit. Chiral GUE models

Let us now briefly consider implications of the derived negative moments representations for performing the replica limit \( n \to 0 \). We recall the main steps of the scheme for the positive moments ("fermionic replica") as suggested by Kamenev and Mezard [11], see also Yurkevich and Lerner [12].

Given the expression Eq. (35), one takes into account two types of stationary points: the "maximally symmetric" one \( q_{1,i} = q_{2,i} = q^+ \) as well as all possible sets where exactly one of \( q_{1,i} \) and exactly one of \( q_{2,i} \) are taken to be equal to \( q^- \), the rest \( 2n - 2 \) being equal to \( q^+ \) as before. It was demonstrated that taking the factors arising from multiplicity of the saddle-points and gaussian fluctuations around them (Selberg integrals) into account only those two possibilities produce leading order contributions nonvanishing in the replica limit.

Let us stress clearly the bizarre nature of this prescription, as compared with the well-defined stationary point procedure for the integer values \( n = 1, 2, ..., \). First of all, for \( n = 0 \) one takes only one of two "mostly symmetric" saddle points discarding its partner \( q_{1,i} = q_{2,i} = q^- \). At the same time, for the positive integer moments the latter produced exactly the same contribution at vanishing imaginary part \( \text{Im} \mu \to 0 \). The step is dictated by "causality arguments" [11], i.e. by necessity to break analyticity inherent in the positive moments, see [29].

Second, the saddle-point sets containing admixture of two \( q^- \) contribute now to the same leading order as the "fully symmetric" one, whereas for any positive integer \( n \) the two contributions were different by the factor \( 1/N \). All this is to remind the reader that presently the replica trick is more a kind of art rather than science (or, rather a kind of alchemist’s wisdom than regular chemistry). For the present author it is however in no way an intimidating characteristics but rather a challenge to imagination.

Let us now turn our attention to the negative integer moments as described by Eq. (31) and compare them to Eq. (35). A little inspection shows that all the factors that make those two expressions different are immaterial in the replica limit. For example, \( \lim_{n \to 0} \prod_{N-n}^{N-1} j! = \lim_{n \to 0} \prod_{i=1}^{N-1} j_i ! = 1 \) and the same is valid for \( |\det \tilde{q}|^n \) and other factors. Thus, in the limit \( n \to 0 \) the two expressions are indistinguishable on the level of saddle point sets and expansions around them. The only essential deviation which seems to persist is the difference in the domain of integration, which is half the real axis for all the negative integer moments. The latter feature is really dictated by analyticity (or causality), which, being a meaningful notion for all negative (but not for the positive!) moments, dictates only one saddle point to be operative - that with all \( q^+ \).

At the same time, there is no obvious reason why other stationary points should be excluded from a consideration in the replica limit. All the experience of working with the replicated expressions suggests that saddle-points irrelevant for integer \( n \) could be most relevant for \( n = 0 \), and vice versa. As a distinguished example one can invoke the famous Sherrington-Kirkpatrick model of spin glasses where the saddle-points dominating in the replica limit are, in fact, local maxima rather than minima of the corresponding functionals. Moreover, formally dominant contributions
in that case seem to come from the boundaries of the integration domains, but are discarded as "unphysical" in favour of the mentioned maxima, see e.g. discussion in p.869 of the reference [42].

We therefore suggest that a sensible recipe to perform the replica limit for negative moments of the characteristic polynomials is as follows: (i) Find an integral representation for the moments with help of the Ingham-Siegel-like integrals (ii) Evaluate the resulting integral as a sum over the stationary points, starting with the most symmetric set as dictated by analytical structure, and adding to it those discovered by Kamenev and Mezard irrespective of the constraints on the integration domain.

To illustrate that such suggestion makes sense beyond the GUE model let us briefly consider one more example. This is the so-called chiral GUE introduced to provide a background for calculating the universal part of the microscopic level density for the QCD Dirac operator, see [33] and references therein. The quantity to be calculated are negative moments of the spectral determinant:

\[
I_{N,n}^{(b)}(m) = A_N \int dJdJ^\dagger e^{-N \text{Tr} J^\dagger J} \left[ \det \left( \begin{array}{c|c} m1_N & iJ^\dagger \\ \hline iJ & m1_N \end{array} \right) \right]^{-n} \quad (41)
\]

where \( m > 0 \) is a parameter proportional to the quark mass and, in the simplest case of zero topological charge, \( J \) is a complex random \( N \times N \) gaussian matrix, with \( J^\dagger \) being its conjugate and \( A_N \) being the normalisation constant.

We relegate the details of consideration of this interesting and important model, as well as its close relative - that of non-Hermitian random matrices - to a separate publication [43] and present here only a brief account.

Application of our method based on the use of the Ingham-Siegel type integral Eqs. (14,15) as an alternative to the conventional Hubbard-Stratonovich transformation results in the following simple formula:

\[
I_{N,n}^{(b)}(m) = C_{n,N}^{ch} \int_{\hat{Q}>0} d\hat{Q} e^{-mN \text{Tr} \hat{Q}} \left[ \det \left( 1_n + m\hat{Q}^{-1} \right) \right]^{-N} \left[ \det \hat{Q} \right]^{-n} \quad (42)
\]

which is exact for arbitrary \( N \geq n \). Here \( \hat{Q} \) is a positive definite \( n \times n \) Hermitian and the constant is given by:

\[
C_{n,N}^{ch} = N^N n^n \frac{1}{(2\pi)^{n(n-1)/2} \prod_{j=N-n}^{N-1} j!}.
\]

In the thermodynamic chiral limit one considers \( m \to 0 \), \( N \to \infty \) but keeping the product \( mN = \frac{x^2}{2} \) fixed. This results in reducing the above expression to:

\[
I_{N,n}^{(b)}(m) = C_{n,N}^{ch} \int_{\hat{Q}>0} d\hat{Q} e^{-\frac{x^2}{2} \text{Tr} (\hat{Q} + \hat{Q}^{-1})} \left[ \det \hat{Q} \right]^{-n} \quad (43)
\]

This should be compared with the corresponding formula for the positive moments [33]:

\[
I_{N,n}^{(f)}(m) = A_N \int dJdJ^\dagger e^{-N \text{Tr} J^\dagger J} \det \left( \begin{array}{c|c} m1_N & iJ^\dagger \\ \hline iJ & m1_N \end{array} \right) \left[ \det \hat{Q} \right]^{-n} \quad (44)
\]

\[
\propto \int_{U(n)} d\mu(\hat{U}) \exp \left\{ \frac{x^2}{2} \text{Tr} \left( \hat{U} + \hat{U}^{-1} \right) \right\}
\]

where the integration goes over the unitary group \( U(n) \).
The correspondence between the two integrals is very similar to the GUE case discussed by us earlier in this paper. Considering $x \to \infty$ as the parameter justifying the saddle-point approximation one finds the saddle-point sets of both integrands coincide: they are given by matrices with the eigenvalues $\pm 1$. Again, in view of the constraint $Q > 0$ any negative integer moment is dominated by the most symmetric set with all eigenvalues being equal unity, whereas positive moments are given by the sum of contributions of many such sets. The Kamenev-Mezard-Yurkevich-Lerner limiting procedure which uses the most symmetric $+1$ configuration as the reference point was shown to produce sensible results for the integral Eq.(44) \cite{3}. Taking into account that the difference between the two integrands is immaterial in the replica limit, we again arrive at the conclusion that we can not help but adopt the same scheme for proceeding from the negative moments.

5 Implications for the supersymmetry method

A curious point which is worth mentioning is that for the case of positive moments the use of the Hubbard-Stratonovich transformation is, in contrast, very natural and effective. Attempting to use our method for that problem encounters with the difficulty of dealing with diverging integrals. The latter hide, in essence, necessity to work with higher derivatives of the $\delta-$ distributions. All this is suggestive of a certain duality between the two methods: working with Grassmann integrations requires the use of the Hubbard-Stratonovich transformation but dealing with commuting variables is much facilitated by avoiding that route. This observation has certain implications for the supersymmetry method treating both types of the variables on equal basis. From this point it is obvious an analytic one in $(\mu_1,f)\mu_2,f)$ plane. It turns out to be technically convenient to change: $\mu_1f \to -i\mu_1f ,\mu_2f \to -i\mu_2f$ when performing the ensemble averaging, and restore the original generating function by a simple analytical continuation.

To calculate the average we first use the standard “supersymmetrisation” procedure and represent each of the two characteristic polynomials in the denominator as the Gaussian integrals, Eq.(44), whereas those two in the numerator are represented as the Gaussian integrals over the Grassmannian $N-$component vectors $\chi_1,\chi_1^\dagger$ and $\chi_2,\chi_2^\dagger$, see e.g. \cite{28}. The ensemble average is then easy to perform and after a straightforward manipulations the generating function can be brought to the following form:

$$C_N(\hat{\mu}_B,\hat{\mu}_F) = \left\langle \frac{Z_N(\mu_1f)Z_N(\mu_2f)}{Z_N(\mu_{1b})Z_N(\mu_{2b})} \right\rangle_{\text{GUE}}$$

where $\hat{\mu}_B = \text{diag}(\mu_{1b},\mu_{2b}^*)$, $\hat{\mu}_F = \text{diag}(\mu_1f,\mu_2f)$ and $\text{Im}(\mu_{1b},\mu_{2b}) > 0$. That generating function is obviously an analytic one in $(\mu_1f,\mu_2f)$ plane. It turns out to be technically convenient to change: $\mu_1f \to -i\mu_1f ,\mu_2f \to -i\mu_2f$ when performing the ensemble averaging, and restore the original generating function by a simple analytical continuation.

To calculate the average we first use the standard ”supersymmetrisation” procedure and represent each of the two characteristic polynomials in the denominator as the Gaussian integrals, Eq.(44), whereas those two in the numerator are represented as the Gaussian integrals over the anticommuting (Grassmannian) $N-$component vectors $\chi_1,\chi_1^\dagger$ and $\chi_2,\chi_2^\dagger$, see e.g. \cite{28}. The ensemble average is then easy to perform and after a straightforward manipulations the generating function can be brought to the following form:

$$C_N(\hat{\mu}_B,\hat{\mu}_f) \propto \int d\chi_1 d\chi_1^\dagger \int d\chi_2 d\chi_2^\dagger \exp \left\{ \frac{1}{2} \left( \mu_1f\chi_1^\dagger \chi_1 + \mu_2f\chi_2^\dagger \chi_2 \right) + \frac{1}{8N} \text{Tr} \left( \hat{Q}_F^2 \right) \right\} \times \int d^2S_1 \int d^2S_2 \exp \left\{ \frac{i}{2} \mu_1S_1^\dagger S_1 - \frac{i}{2} \mu_2S_2^\dagger S_2 - \frac{1}{8N} \text{Tr} \left( \hat{Q}_B\hat{L}\hat{Q}_B\hat{L} \right) \right\}$$
\[
\times \exp \left\{ -\frac{1}{4N} \left( \chi_1^\dagger \chi_2^\dagger \right) \left( \begin{array}{c c}
S_1 \otimes S_1^\dagger & S_2 \otimes S_2^\dagger \\
0 & S_1 \otimes S_1^\dagger - S_2 \otimes S_2^\dagger
\end{array} \right) \right\}
\]

where we introduced the 2 \times 2 matrices
\[
\hat{Q}_F = \left( \begin{array}{c c}
\chi_1^\dagger \chi_1 & \chi_1^\dagger \chi_2 \\
\chi_2 \chi_1 & \chi_2 \chi_2
\end{array} \right), \quad \hat{Q}_B = \left( \begin{array}{c c}
S_1 \otimes S_1 & S_1 \otimes S_2 \\
S_2 \otimes S_1 & S_2 \otimes S_2
\end{array} \right)
\]

and \( \hat{L} = \text{diag}(1, -1) \).

Now, following the main idea outlined in the beginning of the present section we employ the Hubbard-Stratonovich identity:
\[
\exp \left[ \frac{1}{8N} \text{Tr} \hat{Q}_F^2 \right] \propto \int d\hat{Q}_F \exp \left[ -\frac{1}{2} \text{Tr} \hat{Q}_F^2 - \frac{1}{2} \frac{1}{\sqrt{N}} (\chi_1^\dagger, \chi_2^\dagger) \hat{Q}_F^T \left( \begin{array}{c}
\chi_1 \\
\chi_2
\end{array} \right) \right]
\]

where the integration goes over the manifold of 2 \times 2 Hermitian matrices \( \hat{Q}_F = \left( \begin{array}{c c}
q_{11} & q_{12} \\
q_{12} & q_{22}
\end{array} \right) \).

Changing the order of integrations and exploiting the above identity one performs the (Gaussian) Grassmannian integral explicitly in a simple way and brings the expression to the form:
\[
C_N(\hat{\mu}_b, \hat{\mu}_f) \propto \int d\hat{Q}_F e^{-\frac{1}{2} \text{Tr} \hat{Q}_F^2} \int d^2S_1 d^2S_2 \exp \left\{ \frac{i}{2} \mu_1 S_1^\dagger S_1 - \frac{i}{2} \mu_2 S_2^\dagger S_2 - \frac{1}{8N} \text{Tr} (\hat{Q}_B \hat{L} \hat{Q}_B \hat{L}) \right\}
\]

\[
\times \det \left( \begin{array}{c c}
\mu_1 - \frac{1}{\sqrt{N}} q_{11} & 1_N - \hat{B} \\
-\frac{1}{\sqrt{N}} q_{12} 1_N & \mu_2 - \frac{1}{\sqrt{N}} q_{22} \end{array} \right) \left( \begin{array}{c c}
1_N - \hat{B} & -\frac{1}{\sqrt{N}} q_{12} 1_N \\
-\frac{1}{\sqrt{N}} q_{11} 1_N & 1_N - \hat{B}
\end{array} \right)
\]

where we introduced the \( N \times N \) matrix \( \hat{B} = \frac{1}{2N} \left[ S_1 \otimes S_1^\dagger - S_2 \otimes S_2^\dagger \right] \).

Observing that \( \text{Tr} \hat{B}^n \propto \text{Tr}(\hat{Q}_B \hat{L})^n \) for any integer \( n \), and introducing \( \lambda_1, \lambda_2 \) as two (real) eigenvalues of the (Hermitian) matrix \( Q_F^\mu = \hat{\mu}_F - \frac{1}{\sqrt{N}} Q_F \) one can satisfy oneself that the determinant factor in the integrand of Eq.\((48)\) is equal to:
\[
(\det Q_F)^{N-2} \det \left[ \lambda_1 1_2 - \frac{1}{2N} \hat{Q}_B \hat{L} \right] \det \left[ \lambda_2 1_2 - \frac{1}{2N} \hat{Q}_B \hat{L} \right]
\]

Next step is to evaluate the integral over \( S_1, S_2 \). Since the integrand depends on those variables only via the matrix \( \hat{Q}_B \) the calculation can be done exactly the same way as elsewhere in the paper. That involves the \( \delta \)-function representations and the subsequent Ingham-Siegel second type integration, see Eqs.\((13, 15, 16)\). The resulting expression is given by:
\[
C_N(\hat{\mu}_b, \hat{\mu}_f) \propto \int d\hat{Q}_F (\det Q_F)^{N-2} e^{-\frac{1}{2} \text{Tr} \hat{Q}_F^2} \int d\hat{Q}_B \det Q_B^{N-2} \exp \left\{ -\frac{1}{8N} \text{Tr} (\hat{Q}_B \hat{L})^2 + \frac{i}{2} \text{Tr} [\hat{\mu}_B \hat{Q}_B \hat{L}] \right\}
\]

\[
\times \det \left[ \lambda_1 1_2 - \frac{1}{2N} \hat{Q}_B \hat{L} \right] \det \left[ \lambda_2 1_2 - \frac{1}{2N} \hat{Q}_B \hat{L} \right]
\]
Further steps involve: (i) changing: $\tilde{Q}_F \to \sqrt{N}\tilde{Q}_F$, $\mathcal{Q}_B \to 2N\mathcal{Q}_B$ (ii) diagonalizing $\mathcal{Q}_B\mathcal{L} = \mathcal{T}\text{diag}(p_1,p_2)\mathcal{T}^{-1}$, where $p_1 > 0, p_2 < 0$ (iii) integrating over the "hyperbolic" manifold of the pseudounitary matrices $\mathcal{T}$ in the same way as it is done in Eqs. (24), (30) (iv) shifting the integration over $\tilde{Q}_F$ to that over $\tilde{Q}_F^{(\mu)}$ and introducing the (real) eigenvalues $q_1, q_2$ of the latter matrix (and the corresponding unitary matrices of eigenvectors $\hat{\mathcal{Q}}$) as new integration variables. (v) performing the integration over the unitary group $U_F$ by the standard Itzykson-Zuber-Harish-Chandra formula, see Appendix C. Continuing finally analytically as $(\mu_1, \mu_2) \to (i\mu_1, i\mu_2)$ we arrive at

$$
\mathcal{C}_N(\tilde{\mu}_b,\tilde{\mu}_f) \propto e^{\frac{N}{2}(\mu_1^2 + \mu_2^2)} \frac{1}{(\mu_1^{(b)} - \mu_2^{(b)})(\mu_1^{(f)} - \mu_2^{(f)})} \times \int_{-\infty}^{\infty} dq_1 \int_{-\infty}^{\infty} dq_2 \frac{(q_1 - q_2)}{(q_1 q_2)^2} \exp\{-N[\mathcal{L}_1(q_1) + \mathcal{L}_2(q_2)]\} \times \int_{0}^{\infty} dp_1 \int_{0}^{\infty} dp_2 \frac{(p_1 + p_2)}{(p_1 p_2)^2} \exp\{-N[\mathcal{L}_1(b) + \mathcal{L}_2(b)]\}
$$

where

$$
\mathcal{L}_1(q) = \frac{1}{2}q^2 - i\mu_1 q - \ln q \quad \mathcal{L}_2(q) = \frac{1}{2}q^2 - i\mu_2 q - \ln q
$$

$$
\mathcal{L}_1(b) = \frac{1}{2}b^2 - i\mu_1 b - \ln p \quad \mathcal{L}_2(b) = \frac{1}{2}b^2 + i\mu_2 b - \ln p
$$

So far all the formulæ were exact for any integer $N \geq 2$. Let us stress an apparent symmetry of the resulting equation with respect to $q$ and $p$ variables, apart from the anticipated difference in the integration domain. Consider now the limit $N \to \infty$, where the integrals are dominated by the saddle-points:

$$
q_1^{\pm} = \frac{1}{2}i\mu_1 \pm \sqrt{4 - \mu_1^2} \quad q_2^{\pm} = \frac{1}{2}i\mu_2 \pm \sqrt{4 - \mu_2^2}
$$

$$
p_1 = \frac{1}{2}i\mu_1 + \sqrt{4 - \mu_1^2} \quad p_2 = \frac{1}{2}i\mu_2 + \sqrt{4 - \mu_2^2}
$$

Here we took into account the restrictions $\Re p_{1,2} \geq 0$ and also set $\Im \mu_{(1,2)b} = 0$. In fact, in the limit $N \to \infty$ we are interested in the "local regime" : $\mu_{(1,2)b} = \mu \pm \frac{i\rho}{\pi}$, $\mu_{(1,2)f} = \mu \pm \frac{i\rho}{\pi}$ where $|\mu| \leq 2$ and $\omega, \omega_f = O(1/N)$. We find it convenient to define:

$$
\mu_{(1,2)b} = 2\sin \phi_{(1,2)b} \quad \mu_{(1,2)f} = 2\sin \phi_{(1,2)f}
$$

Then $\omega_{b,f} = \mu_{1(b,f)} - \mu_{2(b,f)} \sim (\phi_{(1,f)} - \phi_{(2,f)})2\cos \phi$, where $2\cos \phi = \sqrt{4 - \mu^2} = \pi \rho(\mu)$ is proportional to the mean spectral density. Thus we have:

$$
\phi_{1b} - \phi_{2b} = \frac{\omega_b}{\pi \rho(\mu)} \quad \phi_{1f} - \phi_{2f} = \frac{\omega_f}{\pi \rho(\mu)}
$$

The saddle-point values, Eqs. (63) are given by:

$$
p_1 = e^{i\phi_{1b}}, \quad p_2 = e^{-i\phi_{2b}}, \quad q_1^{\pm} = e^{\pm i\phi_{1f}}, \quad q_2^{\pm} = e^{\pm i\phi_{2f}}
$$

One should take into account a symmetry of the integrand with respect to the transformation: $q_1 \to q_2$ to simplify the expression.
which shows that:

\[ q_1^\pm - q_2^\pm = e^{\pm i\phi_1^f - \phi_2^f} = 0 \]

The latter relation makes it clear that only two saddle point sets \((q_1^+, q_2^+)\) give dominant contribution whereas the other two: \((q_1^-, q_2^-)\) are suppressed due to the factor \((q_1 - q_2)\) in the integrand of Eq. (50).

Expanding around the relevant saddle points and picking up the leading order contributions we obtain after a standard set of manipulations:

\[
C_{N \gg 1}(\hat{\mu}_b, \hat{\mu}_f) \propto \frac{1}{(\mu_{1b} - \mu_{2b})(\mu_{1f} - \mu_{2f})} \left\{ \frac{\delta}{\omega_{b\omega_f}} \left[ e^{iN\pi p(\mu)(\omega_b + \omega_f)}(\omega_b - \omega_f)^2 - e^{iN\pi p(\mu)(\omega_b - \omega_f)}(\omega_b + \omega_f)^2 \right] \right\}^{1/2}
\]

The latter expression coincides with the particular case of the result obtained in [13] by a different method.

The method based on the combination of the Ingham-Siegel and Hubbard-Stratonovich transformations proves also to be a direct way to derive the mean density of eigenvalues for the chiral GUE ensemble discussed in the previous section. Let us outline the corresponding calculation for the (quenched) case of zero topological charge.

The generating function to be calculated is given by (cf. Eq. (41)):

\[
\mathcal{I}_N(m_f, m_b) \propto \int dJ dJ^\dagger e^{-N \text{Tr} j j^\dagger} p(\mu)(\omega_b + \omega_f) \left( \omega_b - \omega_f \right)^2 - e^{iN\pi p(\mu)(\omega_b - \omega_f)}(\omega_b + \omega_f)^2 \right\}
\]

where \(m_f, m_b > 0\).

Representing the determinant in the numerator and that in the denominator as the Gaussian integrals over anticommuting and commuting variables respectively, we easily perform the ensemble averaging, and then employ the Hubbard-Stratonovich transformation for the terms quartic in Grassmannians whereas using the Fourier representation of \(\delta\)-function for their commuting counterparts. This allows to integrate out Grassmann variables exactly and after the Ingham-Siegel integration to arrive to the following expression:

\[
\mathcal{I}_N(m_f, m_b) \propto \int dq dq^* (q^* q)^{N-1} \exp \left\{ -\frac{N}{4} \left\{ q^* q - \frac{m_f^2}{2}(q + q^*) + \frac{1}{4} m_f^2 \right\} \right\} \]

Changing now to the polar coordinates \(R, \theta\) in the complex \(q, q^*\) plane and next to the variable \(t = p_1 p_2\) one can perform the \(\theta\) and \(p_1\)-integrations explicitly finding:

\[
\mathcal{I}_N(x_f, x_b) \propto e^{-\frac{x^2}{2}} \int_0^\infty dr \int_0^\infty dt (rt)^{N-1} \left( r - t \right) e^{-N(r+t)} I_0(2x \sqrt{r}) K_0(2x \sqrt{t}) \]
where $x_{b,f} = N m_{b,f}/4$ and $I_0(z), K_0(z)$ stand for the modified Bessel and Macdonald functions, respectively. In the most interesting chiral limit we set $N \to \infty$ while keeping $x_{b,f}$ fixed. The saddle-point values are obviously given by $r = t = 1$ and expanding around them up to the first non-vanishing term finally yields:

$$I_{\text{chiral}}(x_f, x_b) \propto \left[ x_f I_1(2x_f) K_0(2x_b) + x_b I_0(2x_f) K_1(2x_b) \right]$$

(58)

in full agreement with known results, see [33, 45].

6 Conclusions and Perspectives

In the present paper we suggested a systematic way of evaluating negative integer moments of the (regularized) characteristic polynomials. Using the standard representation of those moments in terms of the Gaussian integrals as the starting point we found a route avoiding the use of the ubiquitous Hubbard-Stratonovich transformation. Instead, we advocated the exploitation of the matrix integral Eq.(15) similar to that considered long ago by Ingham and Siegel, Eq.(14). The advantage of the procedure is that the emerging structures are attractively simple and, in essence, very close to those derived earlier for the positive moments. We evaluated the resulting integrals in the limit $N \to \infty$ by the stationary phase method and extracted the leading asymptotics. The limiting value of the correlation functions for the negative moments is presented in Eq.(4) and expressed in terms suggesting universality. In fact, by using the character expansion method we were able to reproduce the formula (4) for unitary random matrices [48]. We therefore conjecture that it should be equally applicable, mutatis mutandis, to the behaviour of the Riemann zeta function in the close vicinity of the critical line.

Our analysis may raise a few questions which deserve further discussion. First of all, one may wish to know if it is possible to arrive to the same representations via the standard (Hubbard-Stratonovich) method. The answer is of course affirmative as is demonstrated in the Appendix D on the simplest non-trivial example. The general case can be treated along the same lines. We however insist that such a way is hardly natural for the present problem and, in fact, obscures the simple structures arising.

This fact has important implications for the "supersymmetry method" of treating generating functions involving simultaneously characteristic polynomials both in positive and negative powers. We presented a few simple non-trivial examples of such calculations in the text. In fact, we found it natural to treat "bosonic" and "fermionic" auxiliary integrations differently, which was a serious departure from the standard spirit of "supersymmetry".

One should mention that the correlation functions calculated above can be also calculated in the framework of the standard Efetov’s supermatrix approach. However, present method, to our mind, has some technical advantages. For GUE calculations, for example, the two terms appearing in the final result [54] on equal footing have quite a different origin in the Efetov’s method. Namely, one of them appears in the form of the so-called "anomalous", or "boundary term". When trying to calculate the correlation functions of the higher orders the boundary terms turn out to be quite difficult to keep track of, making the $\sigma$–model calculation a daunting job.

In fact, the authors of [13] announced the result for the most general expression of the ratios of integer powers of characteristic polynomials. They arrived to it by a method generalizing that suggested in the paper by Gulrh [19] and employing an analog of the Itzykson-Zuber integral over a supermanifold. Our method provides an alternative way of derivation of the higher correlation function, the corresponding calculation will be presented in a detailed form elsewhere [44].
For the case of chiral GUE ensemble the application of the standard Efetov’s method for
the quenched model requires quite a cumbersome calculation [45], whereas along our route the
calculations are rather short and elementary. Moreover, we found [43] that the same method
proved to be applicable for the most interesting ”unquenched” situation when the Efetov’s method
encountered with unsurmountable difficulties.

Finally, it is interesting to explore if the Ingham-Siegel integrals and their natural generalisations
could provide a serious alternative to the Hubbard-Stratonovich transformation in the whole
class of problems in the domain of random matrices and disordered systems. To this end two
aspects are worth mentioning: (i) the Ingham-Siegel integrals of both types are known for all sym-
metry classes, see Appendix A and (ii) performing the saddle-point calculation directly on the level
of Eq.(28) yields the standard non-linear \( \sigma \)− model representation[28] for the negative moments.
Further work along these lines is under the way[43, 44] but a general affirmative or negative answer
to the questions requires more efforts.

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APPENDICES

A Calculation of the integral Eq.(15)

Our goal is to calculate the integral

\[
I_{n,N}(\hat{Q}_n) = \int d\hat{F}_n e^{i\text{Tr}(\hat{F}_n \hat{Q}_n)} \left[ \det \left( \hat{F}_n - \mu 1_n \right) \right]^{-N}
\]

(59)

where both \( \hat{F}_n \) and \( \hat{Q}_n \) are Hermitian \( n \times n \) matrices. First notice that the integrand is invariant
with respect to the unitary rotations \( F \rightarrow \hat{U} F \hat{U}^{-1} \), hence the result of the integration can depend
only on the eigenvalues of \( \hat{Q} \). Then, following \[36, 37\] one can take \( \hat{Q} \) to be diagonal from the very
beginning and separate the first eigenvalue from the rest:

\[
\hat{Q} = \text{diag}(q_1, q_2, ..., q_n) \equiv \text{diag}(q_1, \hat{Q}_{n-1})
\]

Accordingly decompose the matrix \( \hat{F}_n \) as

\[
\hat{F}_n = \begin{pmatrix} f_{11} & f_{1\dagger} \\ f & \hat{F}_{n-1} \end{pmatrix}, \quad d\hat{F}_n = df_{11} df_{1\dagger} df\hat{F}_{n-1}
\]

(60)

where \( f_{1\dagger} = (f_{21}^*, f_{31}^*, ..., f_{n1}^*) \) is a \( n-1 \) component complex vector.
Next step is to use the well-known property of the determinants:

\[
\det \left( \hat{F}_n - \mu 1_n \right) = \det \left( \hat{F}_{n-1} - \mu 1_{n-1} \right) \left( f_{11} - \mu - f_{1\dagger} \left[ \hat{F}_{n-1} - \mu 1_{n-1} \right]^{-1} f \right)
\]

20
which gives:

$$I_{n,N}(\hat{Q}_n) = \int d\hat{F}_{n-1} e^{\frac{i}{2} \text{Tr}(\hat{F}_{n-1} \hat{Q}_{n-1})} \left[ \det \left( \hat{F}_{n-1} - \mu \mathbf{1}_{n-1} \right) \right]^{-N}$$

(61)

$$\times \int d\hat{F}^\dagger d\hat{f} \int_{-\infty}^{\infty} df \hat{f}^{1,n} \frac{1}{(\hat{f}_{11} - \mu - \hat{f}^\dagger \left[ \hat{F}_{n-1} - \mu \mathbf{1}_{n-1} \right]^{-1} \hat{f})^N}$$

The last integral over $f_{11}$ is evaluated by the residue theorem taking into account $\text{Re} \mu > 0$, the result of the integration being:

$$\frac{2\pi i}{\Gamma(N)} \theta(q_1) \left( \frac{i}{2} \right)^{N-1} \exp \left\{ \frac{i}{2} q_1 \left( \mu + \hat{f}^\dagger \left[ \hat{F}_{n-1} - \mu \mathbf{1}_{n-1} \right]^{-1} \hat{f} \right) \right\}$$

(62)

where $\theta(x) = 1$ for $x > 0$ and zero otherwise and we assumed $N \geq 1$. Now the gaussian integration over $d\hat{f}^\dagger d\hat{f}$ can be easily performed, yielding the factor:

$$\left( \frac{-2\pi}{(i q_1/2)} \right)^{n-1} \det \left( \hat{F}_{n-1} - \mu \mathbf{1}_{n-1} \right)$$

so that we arrive at the recurrence relation:

$$I_{n,N}(\hat{Q}_n) = \frac{(-2\pi)^n}{\Gamma(N)} \left( \frac{i}{2} \right)^{N-n} \theta(q_1) e^{\frac{i}{2}q_1} I_{n-1,N-1}(\hat{Q}_{n-1})$$

(63)

which immediately produces the desired formula:

$$I_{n,N}(\hat{Q}_n) = \left( \frac{2\pi}{\Gamma(N)} \right)^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} \theta(q_j) \det \left[ \frac{i}{2} \hat{Q} \right]^{N-n} e^{\frac{i}{2} \mu \text{Tr} \hat{Q}}$$

(64)

assuming $N \geq n$.

In fact, the derivation is straightforwardly repeated for the case of real symmetric matrices $\hat{F}_n$ and $\hat{Q}_n$. The recurrence relation in that case is:

$$I_{r.s.,n,N}(\hat{Q}_n) = 2^{\frac{n^2}{2}} \pi^{\frac{n(n+1)}{2}} \prod_{j=1}^{n} \theta(q_j) \det \left[ \frac{i}{2} \hat{Q} \right]^{N-n} e^{\frac{i}{2} \mu \text{Tr} \hat{Q}}$$

(65)

which yields the result:

$$I_{r.s.,n,N}(\hat{Q}_n) = 2^{\frac{n^2}{4}} \pi^{\frac{n(n+1)}{4}} \prod_{j=1}^{n} \theta(q_j) \det \left[ \frac{i}{2} \hat{Q} \right]^{N-n} e^{\frac{i}{2} \mu \text{Tr} \hat{Q}}$$

(66)

for $N \geq \frac{n+1}{2}$.

\section*{B \ Properties of the matrices $\hat{Q}_L$}

In this Appendix we consider the manifold of $2n \times 2n$ matrices $\hat{Q}_L = \hat{Q} \hat{L}$ where $\hat{Q}^\dagger = \hat{Q} > 0$ and $\hat{L} = \text{diag}(1_n, -1_n)$. In fact, this set of matrices is closely related to the object known as a regular matrix pencil.  \[10\]}
We begin with proving that all eigenvalues of such non-Hermitian matrices are real and half of them positive, the rest being negative. In doing this we can safely assume that all eigenvalues are different since matrices with degenerate eigenvalues form a manifold of lower dimension and as such will not contribute when we integrate over the whole manifold of ĤL.

The characteristic polynomial for the eigenvalues  of the matrix ĤL can be written as:

\[ \det (q\mathbf{1}_{2n} - \hat{Q}\hat{L}) = \det \hat{Q}^{1/2} (q\mathbf{1}_{2n} - \hat{Q}^{1/2}\hat{L}\hat{Q}^{1/2}) \hat{Q}^{-1/2} = \det (q\mathbf{1}_{2n} - \hat{Q}^{1/2}\hat{L}\hat{Q}^{1/2}) \]

where we used that  \( \hat{Q}^{1/2} > 0 \) is a nonsingular Hermitian matrix. Then all eigenvalues of  \( \hat{Q}_L \) coincide with those of the Hermitian  \( \hat{Q}^{1/2}\hat{L}\hat{Q}^{1/2} \) and therefore are all real. Moreover, the number of positive and negative eigenvalues of any Hermitian matrix  \( \hat{H} \) stays invariant under transformations  \( \hat{H} \rightarrow T^\dagger\hat{H}T \), where  \( T \) is an arbitrary nonsingular matrix [46]. We arrive at the conclusion that the number of positive and negative eigenvalues of  \( \hat{Q}^{1/2}\hat{L}\hat{Q}^{1/2} \) is the same as that for  \( \hat{L} \), thus proving the statement.

Let  \( q_j \) be an eigenvalue of  \( \hat{Q}_L \) and denote the corresponding (right) eigenvectors as  \( e_j \):

\[ \left( \hat{Q}\hat{L} \right) e_j = q_j e_j, \quad e_j^\dagger \left( \hat{L}\hat{Q} \right) = q_j e_j^\dagger \]

Multiplying the first of these relations with  \( e_k^\dagger \hat{L} \) from the left and the second relation with  \( \hat{L} e_j \) from the right we have:

\[ e_k^\dagger \left( \hat{L}\hat{Q}\hat{L} \right) e_j = q_j e_k^\dagger \hat{L} e_j = q_k e_k^\dagger \hat{L} e_j \]

showing that  \( e_k^\dagger \hat{L} e_j = \delta_{kj} q_j e_j^\dagger \hat{L} e_j \)

Now,  \( \hat{L} \) is a unitary matrix, hence  \( \hat{L}\hat{Q}\hat{L} > 0 \) so that  \( q_j e_j^\dagger \hat{L} e_j > 0 \). Introduce now the "normalized" eigenvectors  \( \hat{e}_j = e_j / \sqrt{\sgn(q_j)(e_j^\dagger \hat{L} e_j)} \), where  \( \sgn(x) \) stands for the sign function. Then it is easy to see that

\[ \hat{e}_j^\dagger \hat{L} e_j = \sgn(q_j) \quad \text{and} \quad \hat{e}_j^\dagger \hat{Q}\hat{L} e_j = |q_j| \delta_{kj} \]

Further introduce the matrix  \( \hat{T} \) whose columns are vectors  \( \hat{e}_j \) for  \( j = 1, 2, ..., 2n \), and consider  \( \hat{T}_L = \hat{T}^\dagger \hat{L} \). It is immediately clear that

\[ \hat{T}_L\hat{T} = \text{diag} (\sgn(q_1), ..., \sgn(q_{2n})) \quad \text{and} \quad \hat{T}_L\hat{Q}\hat{T} = \text{diag} (|q_1|, ..., |q_{2n}|) = \hat{T}_L\hat{T}\text{diag}(q_1, ..., q_{2n}) \]

The vectors  \( \hat{e}_j \) are obviously linearly independent, hence the matrices  \( \hat{T}_L \) and  \( \hat{T} \) are nonsingular. This immediately shows that the matrices  \( \hat{Q}_L \) can be diagonalized by a similarity transformation:

\[ \hat{Q}\hat{L} = \hat{T}\text{diag}(q_1, ..., q_n)\hat{T}^{-1} \]

where  \( \hat{T} \) satisfies:

\[ \hat{T}^\dagger\hat{L}\hat{T} = \text{diag} (\sgn(q_1), ..., \sgn(q_{2n})) \]

The last matrix is essentially  \( \hat{L} \) up to a permutation of its entries on the main diagonal. This completes the proof.
C Evaluation of the integral Eq.(30)

To evaluate the quoted integral one needs to employ an explicit parametrisation of the matrices $\hat{T}_0 \in \frac{U(n,n)}{U(n) \times U(n)}$. We follow the paper\cite{2} where it was suggested that the following parametrisation is especially convenient:

$$\hat{T}_0 = \begin{pmatrix} \sqrt{1 + t^2} & t \\ t & \sqrt{1 + t^2} \end{pmatrix}$$

in terms of complex $n \times n$ matrices $t$, $t^\dagger$. The reason for such a choice is dictated by an especially simple form of the integration measure: $d\mu(t, t^\dagger)$.

Next step is to diagonalise $t^\dagger$ with help of two unitary rotations: $t^\dagger = u_{A,B}^{-1} t u_{A,B}$ where $u_{A,B} \in U(n)$ and $\tilde{\tau} = \text{diag}(\tau_1, \ldots, \tau_n)$, so that $t = u_{A,B}^{-1} \tilde{\tau} u_{A,B}$. It is convenient to write the modulus and the phase of $\tau_k$ explicitly: $\tau_k = \sinh \psi_k e^{i\phi_k}$, $0 < \psi < \infty$, $0 < \phi < 2\pi$, $k = 1, \ldots, n$.

The matrices $\hat{T}_0$ take the form:

$$\hat{T}_0 = \begin{pmatrix} u_A^{-1} & 0 \\ 0 & u_R^{-1} \end{pmatrix} \begin{pmatrix} \cosh \hat{\psi} & e^{i\hat{\phi}} \sinh \hat{\psi} \\ e^{-i\hat{\phi}} \sinh \hat{\psi} & \cosh \hat{\psi} \end{pmatrix} \begin{pmatrix} \hat{u}_A & 0 \\ 0 & \hat{u}_R \end{pmatrix}$$

and $\hat{T}_0^{-1}$ is correspondingly given by:

$$\hat{T}_0^{-1} = \begin{pmatrix} u_A^{-1} & 0 \\ 0 & u_R^{-1} \end{pmatrix} \begin{pmatrix} \cosh \hat{\psi} & -e^{i\hat{\phi}} \sinh \hat{\psi} \\ -e^{-i\hat{\phi}} \sinh \hat{\psi} & \cosh \hat{\psi} \end{pmatrix} \begin{pmatrix} \hat{u}_A & 0 \\ 0 & \hat{u}_R \end{pmatrix}$$

One can straightforwardly calculate the integration measure in the new variables and find:

$$d\mu d\psi d\phi \propto \prod_{k=1}^n \sinh 2\psi_k \prod_{k_1 < k_2} (\sinh^2 \psi_{k_1} - \sinh^2 \psi_{k_2}) \prod_{k=1}^n d\psi_k d\phi_k \times d\mu(\hat{u}_A) d\mu(\hat{u}_R)$$

where $d\mu(\hat{u}_{A,R})$ are normalised invariant measures on $U(n)$ and we introduced: $\lambda_k = \cosh 2\psi_k \in [1, \infty)$ as new variables.

Now we can use the cyclic permutation of the matrices under the trace sign to rewrite the expression in the exponent of Eq.(30) in terms of the introduced variables as follows:

$$\text{Tr} \left[ \left( \begin{array}{cc} \mu_1 1_n & \mu_2 1_n \\ \mu_2^\ast 1_n & \mu_1^\ast 1_n \end{array} \right) \hat{T}_0 \left( \begin{array}{cc} \hat{P}_1 & 0 \\ 0 & \hat{P}_2 \end{array} \right) \hat{T}_0^{-1} \right]$$

$$= \text{Tr} \left[ \left( \begin{array}{cc} \mu_1 1_n & \mu_2 1_n \\ \mu_2^\ast 1_n & \mu_1^\ast 1_n \end{array} \right) \left( \begin{array}{cc} \cosh \hat{\psi} & e^{i\hat{\phi}} \sinh \hat{\psi} \\ e^{-i\hat{\phi}} \sinh \hat{\psi} & \cosh \hat{\psi} \end{array} \right) \left( \begin{array}{cc} \hat{P}_A & 0 \\ 0 & \hat{P}_B \end{array} \right) \left( \begin{array}{cc} \cosh \hat{\psi} & -e^{i\hat{\phi}} \sinh \hat{\psi} \\ -e^{-i\hat{\phi}} \sinh \hat{\psi} & \cosh \hat{\psi} \end{array} \right) \right]$$

$$= \text{Tr} \left[ \hat{P}_A \left( \mu_1 \cosh^2 \hat{\psi} - \mu_2^\ast \sinh^2 \hat{\psi} \right) \right] + \text{Tr} \left[ \hat{P}_B \left( \mu_2^\ast \cosh^2 \hat{\psi} - \mu_1 \sinh^2 \hat{\psi} \right) \right]$$

$$= \frac{1}{2}(\mu_1 + \mu_2) \text{Tr} \left( \hat{P}_A + \hat{P}_B \right) + \frac{1}{2}(\mu_1 - \mu_2) \left( \text{Tr} \hat{P}_A \cosh 2\hat{\psi} + \text{Tr} \hat{P}_B \cosh 2\hat{\psi} \right)$$

where we introduced matrices $\hat{P}_A = \hat{u}_A \hat{P}_1 \hat{u}_A^{-1}$, $\hat{P}_B = \hat{u}_B \hat{P}_2 \hat{u}_B^{-1}$ having the same eigenvalues $\hat{q}_1 = \text{diag}(q_{1,1}, \ldots, q_{1,n})$ and $\hat{q}_2 = \text{diag}(q_{2,1}, \ldots, q_{2,n})$ as the matrices $\hat{P}_{1,2}$. 

\(23\)
We see that the integral of interest is expressed now as:

\[ I(\hat{M}, \hat{P}_1, \hat{P}_2) = \int d\mu(\hat{T}) \exp \left\{ i N \text{Tr} \left( \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix} \right) \hat{T}_0 \left( \begin{pmatrix} \hat{P}_1 & 0 \\ 0 & \hat{P}_2 \end{pmatrix} \right) \hat{T}_0^{-1} \right\} \]

\[ \propto \int_1^\infty \prod_k d\lambda_k \Delta^2(\hat{\lambda}) e^{\frac{i}{2} N(\mu_1 - \mu_2^2) \text{Tr}(\hat{q}_1 + \hat{q}_2)} \times \int d\mu(\hat{u}_R) e^{\frac{i}{2} N(\mu_1 - \mu_2^2) \text{Tr} \hat{u}_R \hat{u}_R^{-1} \hat{\lambda}} \int d\mu(\hat{u}_L) e^{\frac{i}{2} N(\mu_1 - \mu_2^2) \text{Tr} \hat{u}_L \hat{u}_L^{-1} \hat{\lambda}} \]

where we used \( \hat{\lambda} = \text{diag}(\lambda_1, ..., \lambda_n) \) and the symbol \( \Delta(\hat{\lambda}) \) for the corresponding Vandermonde determinant.

Two integrals over the (normalized) Haar measure on the unitary group are given by Harish Chandra-Itzykson-Zuber formula [17]:

\[ \int d\mu(\hat{u}) e^{i \beta \text{Tr} \hat{u} \hat{u}^{-1} \hat{\lambda}} = \left( \prod_{j=1}^{n-1} j! \right) \beta^{-\frac{n(n-1)}{2}} \frac{\det [e^{i \lambda_k q_{ij}}]}{\Delta(\hat{\lambda}) \Delta(q)} \]

where in our case \( \beta = \frac{i}{2} N(\mu_1 - \mu_2^2) \) for the first integral, and for the second one \( \beta \to -\beta \). This gives:

\[ I(\hat{M}, \hat{P}_1, \hat{P}_2) \propto (\mu_1 - \mu_2^2)^{-n(n-1)} \frac{1}{\Delta(q_1) \Delta(q_2)} e^{\frac{i}{2} N(\mu_1 + \mu_2^2) \text{Tr}(\hat{q}_1 + \hat{q}_2)} \times \int_1^\infty \prod_k d\lambda_k \det [e^{\frac{i}{2} N(\mu_1 - \mu_2^2) \lambda_k q_{1l}}]^n_k, l=1 \]

The last integral can be easily calculated by expanding each of the two determinants as:

\[ \det [e^{\pm i \lambda_k q_{1l}}]^n_k, l=1 = \sum_{[S]} (-1)^{|S|} \exp \{ \pm i \sum_{k=1}^N \lambda_k q_{rk} \} \]

where \([S]\) stands for a permutation \( r_1, r_2, ..., r_n \) of the index set \( 1, 2, ..., n \). Product of two such expansions can be integrated term by term, and the integrals are convergent due to \( \text{Re} \beta > 0, q_{1l} > 0, q_{2l} < 0 \). This gives:

\[ \int_1^\infty \prod_k d\lambda_k \det [e^{i \lambda_k q_{1l}}]^n_k, l=1 \det [e^{-i \lambda_k q_{2l}}]^n_k, l=1 = [-\beta]^{-n} \sum_{[S_1],[S_2]} (-1)^{|S_1| + |S_2|} e^{\beta \sum_{k=1}^N q_{1k} r_{k} - \beta \sum_{k=1}^N q_{2k} r_{k}} \prod_{k=1}^n (q_{1k} - q_{2k}) \]

where \([S_1],[S_2]\) are two independent permutations \( (r_1, ..., r_n) \) and \( (l_1, ..., l_n) \) of the index set \( 1, 2, ..., n \).

Clearly, one can restrict the summation to be taken over the relative permutations of the two index sets and multiply the result by \( n! \). The exponential above is invariant with respect to any index permutation, so it can be taken out of the summation sign and one recognizes the so-called Cauchy determinant:

\[ \det \left( \frac{1}{q_{1i} - q_{2j}} \right)_{i,j=1}^n = \frac{\Delta(\hat{q}_1) \Delta(\hat{q}_2)}{\prod_{k_1,k_2} (q_{1,k_1} - q_{2,k_2})} \]

in the remaining sum. Collecting all the relevant factors together we arrive at the final formula:

\[ I(\hat{M}, \hat{P}_1, \hat{P}_2) \propto [-i(\mu_1 - \mu_2^2)]^{-n^2} \frac{1}{\prod_{k_1,k_2} (q_{1,k_1} - q_{2,k_2})} e^{i N \text{Tr}(\mu_1 \hat{q}_1 + \mu_2^2 \hat{q}_2)} \]

up to a constant factor, which can be fixed by normalization in the corresponding equations.
D Negative moments by the Hubbard-Stratonovich transformation

Let us satisfy ourself that the standard Hubbard-Stratonovich transformation over the hyperbolic manifold \([23][24]\) produces the same formula Eq. (61). We concentrate on the simplest nontrivial case \( n = 1 \) for the sake of clarity. That case was used for a pedagogical introduction into the Hubbard-Stratonovich method in the author’s lectures in the book \([24]\) and the notations mainly follow those lectures.

Our starting point is Eq. (27) for \( n = 1 \). We introduce the matrices \( \hat{A} = \hat{L}^{1/2} \hat{Q} \hat{L}^{1/2} \) so that \( \text{Tr} \hat{A}^2 = \text{Tr} \hat{Q} \hat{L} \hat{Q} \hat{L} \) and \( \text{Tr} A = S_1^+ S_1 - S_2^+ S_2 \) , \( \text{Tr} AL = S_1^+ S_1 + S_2^+ S_2 \). Remembering \( \mu_{1,2} = \mu \pm (\omega/2 + i\delta) \) we can express all terms appearing in the exponent of Eq. (27) in terms of \( \hat{A} \):

\[
\frac{i}{2} \left( \mu_1 S_1^+ S_1 - \mu_2 S_2^+ S_2 \right) = \frac{i}{2} \mu \text{Tr} \hat{A} + \frac{1}{2}(\delta - i\omega/2)\text{Tr} \hat{A} \hat{L}
\]

The Hubbard-Stratonovich transformation is the identity:

\[
\begin{align*}
\exp \left\{ -\frac{1}{8N} \text{Tr} \hat{A}^2 - \frac{1}{2}(\delta - i\omega/2)\text{Tr} \hat{A} \hat{L} \right\} & \quad \propto e^{N(\delta - i\omega/2)^2} \int d\hat{Q} \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}^2 - iN(\delta - i\omega/2)\text{Tr}(\hat{Q} \hat{L}) - \frac{i}{2} \text{Tr} \hat{Q} \hat{A} \right\} \tag{73}
\end{align*}
\]

Despite looking as an innocent gaussian integral the identity is very nontrivial, since the convergence arguments force one to choose the following ”hyperbolic manifold” of the matrices \( \hat{Q} \) as the integration domain:

\[
\hat{Q} = \hat{T}^{-1} \left( \begin{array}{cc} p_1 & 0 \\ 0 & p_2 \end{array} \right) \hat{T} \quad , \quad \hat{T} = \left( \begin{array}{cc} \cosh \theta & e^{i\phi} \sinh \theta \\ e^{-i\phi} \sinh \theta & \cosh \theta \end{array} \right) \tag{74}
\]

\[
d\hat{Q} \propto (p_1 - p_2)^2 \sinh 2\theta dp_1 dp_2 d\phi, \quad -\infty < \text{Re}(p_{1,2}) < \infty , \quad \text{Im}(-p_1, p_2) > 0 , \quad 0 \leq \theta < \infty , \quad 0 < \phi < 2\pi
\]

A detailed discussion of the convergence problems and of the above identity Eq. (73) can be found, e.g., in the book \([23]\) and in the mentioned lectures \([24]\).

Substituting such an identity back to Eq. (27) and changing the order of integrations over \( \hat{Q} \) and \( S_{1,2} \) we see that it can be processed as follows:

\[
\begin{align*}
e^{N(\delta - i\omega/2)^2} \int d\hat{Q} \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}^2 - iN(\delta - i\omega/2)\text{Tr}(\hat{Q} \hat{L}) \right\} \quad \propto & \quad e^{N(\delta - i\omega/2)^2} \int dSdS^t \exp \left\{ \frac{i}{2} \mu S^+ L S - \frac{i}{2} S^+ \hat{L} \hat{L}^{1/2} \hat{Q} \hat{L}^{1/2} S \right\} \tag{75}
\end{align*}
\]

\[
\times \int \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}^2 - iN(\delta - i\omega/2)\text{Tr}(\hat{Q} \hat{L}) \right\} \propto e^{N(\delta - i\omega/2)^2} \int dp_1 \int dp_2 \frac{(p_1 - p_2)^2}{(\mu - p_1)(\mu - p_2)} \exp \left\{ -\frac{N}{2} \text{Tr} \hat{Q}^2 - iN(\delta - i\omega/2)\text{Tr}(\hat{Q} \hat{L}) \right\}
\]

\[
\times \int dp_1 \int dp_2 \frac{(p_1 - p_2)^2}{(\mu - p_1)(\mu - p_2)} \exp \left\{ -\frac{N}{2} (p_1^2 + p_2^2) \right\} \times \int_0^{2\pi} d\phi \int_0^{\infty} d\theta \sinh 2\theta \exp \left\{ -iN(\delta - i\omega/2)\text{Tr} \left[ \left( \begin{array}{cc} p_1 & 0 \\ 0 & p_2 \end{array} \right) \hat{T} \hat{L} \hat{T}^{-1} \right] \right\}
\]

\]
where we introduced the notation $S = (S_1, S_2)$ and used $\text{Tr} \hat{Q} \hat{A} = S^\dagger \hat{L}^{1/2} \hat{Q}^* \hat{L}^{1/2} S$.

Using the explicit parametrisation for $\hat{T}$ it is easy to verify that:

$$\text{Tr} \left[ \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \hat{T} \hat{L}^{-1} \right] = (p_1 - p_2) \cosh 2\theta$$

which allows one to perform the integration over $\theta$. The expression above is therefore reduced to:

$$\frac{e^{N(\delta-i\omega/2)^2}}{\delta - i\omega/2} \int dp_1 \int dp_2 \frac{p_1 - p_2}{[\mu - p_1][\mu - p_2]^N} \exp \left\{ -\frac{N}{2}(p_1^2 + p_2^2) - iN(\delta - i\omega/2)(p_1 - p_2) \right\}$$ (76)

Remembering $\text{Im} p_1 < 0$, $\text{Im} p_2 > 0$ we can use the identity:

$$[(\mu - p_1)(\mu - p_2)]^{-N} \propto \int_0^\infty dq_1 \int_0^\infty dq_2(q_1q_2)^{N-1} \exp\{i[(\mu - p_1)q_1 - (\mu - p_2)q_2]\}$$

and further rewrite the integral Eq.(76) as:

$$\propto \frac{e^{N(\delta-i\omega/2)^2}}{\delta - i\omega/2} \int_0^\infty dq_1 \int_0^\infty dq_2(q_1q_2)^{N-1} e^{i\mu(q_1 - q_2)} \times \int dp_1 \int dp_2 (p_1 - p_2) \exp \left\{ -\frac{N}{2}(p_1^2 + p_2^2) - iN(\delta - i\omega/2)(p_1 - p_2) - i(p_1q_1 - p_2q_2) \right\}$$

The next step is to use the chain of identities:

$$\int dp_1 \int dp_2 (p_1 - p_2) \exp \left\{ -\frac{N}{2}(p_1^2 + p_2^2) - iN(\delta - i\omega/2)(p_1 - p_2) - i(p_1q_1 - p_2q_2) \right\}$$

$$\propto \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) \int dp_1 \int dp_2 \exp \left\{ -\frac{N}{2}(p_1^2 + p_2^2) - ip_1 [N(\delta - i\omega/2) + q_1] + ip_2 [N(\delta - i\omega/2) + q_2] \right\}$$

$$\propto \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) \exp \left\{ -\frac{1}{2N}(q_1^2 + q_2^2) - (\delta - i\omega/2)(q_1 + q_2) - N(\delta - i\omega/2)^2 \right\}$$

and observe that

$$e^{i\mu(q_1 - q_2)} \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) F(q_1, q_2) \propto \left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right) \left[ e^{i\mu(q_1 - q_2)} F(q_1, q_2) \right]$$

The last formula allows one to integrate by parts over $q_1, q_2$ and in this way to get rid of the derivatives. The boundary terms vanish for $N > 1$, the application of the operator $\left( \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \right)$ to $(q_1q_2)^{N-1}$ produces the term $(q_1q_2)^{N-2}(q_1 + q_2)$ and the resulting expression coincides with that given in eq.(73) for $n = 1$.

For $n > 1$ the equivalence can be shown along essentially the same lines, but calculations become cumbersome and require the use of the procedure similar to that outlined in the Appendix C.

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