Instability of 1-loop superstring cosmology

Shinsuke Kawai

Graduate School of Human and Environmental Studies, Kyoto University, Kyoto 606-8501, Japan

Masa-aki Sakagami† and Jiro Soda‡

Department of Fundamental Sciences, FIHS, Kyoto University, Kyoto 606-8501, Japan

A stability analysis is made in the context of the previously discovered non-singular cosmological solution from 1-loop corrected superstring effective action. We found that this solution has an instability in graviton mode, which is shown to have a close relation to the avoidance of initial singularity via energy condition. We also estimate the condition for the breakdown of the background solution due to the overdominance of the graviton.

Tracing back the history of our presently expanding universe, we are naturally led to the era of high temperature and large energy density. When the energy density approaches Planck scale, our general relativistic description of the universe is no longer considered as valid since quantum gravitational processes are thought to be significant at such a high energy scale. The leading candidate for the satisfactory theory in the very early universe is the superstring theory [1], and roughly speaking, two approaches are being made in the attempt to construct the consistent history of the early universe.

Pre-big-bang universe model [2] was proposed to realize an inflation "before" the initial singularity. Due to the scale factor duality [3], the equations of motion in this model have two distinct and disconnected branches of the solution, one corresponding to an ordinary Friedmann solution, and the other corresponding to the so-called super-inflation solution with increasing Hubble parameter. The super-inflationary stage, which is usually put before the Friedmann stage and characterized by the negative-power dependence of the scale factor on time, is regarded as a sort of inflation that lessens the difficulties appearing in the standard big bang model. Although this model is advantageous in that the inflation is naturally explained from one of the most promising theories at present, many difficulties, besides the serious graceful-exit problem [3], have been pointed out [4].

Another approach in string cosmology is the search for non-singular cosmological models [8]. Initial singularity problem is inherent in the standard big bang cosmology [9], and unless we believe the universe was really generated by one sudden blow at infinite temperature, it is natural to think that the classical theory was violated in such an extreme condition and that singular situation never happened. Also, getting rid of the infinitely high temperature suggests possibilities to understand the whole history of our universe without knowing the theory of everything.

Recently, non-singular cosmological model based on 1-loop corrected superstring effective action [11] was proposed [12]. One of the most interesting features of this model is that its solutions include a super-inflationary stage before the ordinary Friedmann-like universe, which is, just like pre-big-bang case, expected to give a natural explanation for the inflation. It should be emphasized that the transition in this model is smooth in contrast with the pre-big-bang scenario. In order to concentrate on the behavior of the metric-modulus sub-system which plays the essential role in the realization of super-inflationary singularity-free solution, a simpler version of this model was proposed by Rizos and Tamvakis [14], in which they generalized the form of the coupling and analytically examined the conditions for the existence of non-singular solutions. In this letter perturbative analysis is made for this metric-modulus system and we show that its non-singular solution is geometrically unstable.

We start with the 1-loop effective action of the heterotic string with orbifold compactification, which is given by

\[ S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} R - \frac{1}{4} (D\Phi)^2 - \frac{3}{4} (D\sigma)^2 + \frac{1}{16} \left[ \lambda_1 \epsilon^\Phi - \lambda_2 \xi(\sigma) \right] R_{GB}^2 \right\}, \] (1)

where \( R, \Phi \) and \( \sigma \) are the Ricci scalar curvature, the dilaton, and the modulus field, respectively [10]. The Gauss-Bonnet

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*E-mail:kawai@phys.h.kyoto-u.ac.jp
†E-mail:sakagami@phys.h.kyoto-u.ac.jp
‡E-mail:jiro@phys.h.kyoto-u.ac.jp

1We will use in this paper the following convention. The signature of the metric: \((- , + , + , + \)). Riemann tensor: \( R^\rho_{\alpha\nu\beta} = \Gamma^\rho_{\alpha\beta,\nu} + \ldots \), Ricci tensor: \( R_{\alpha\beta} = R^\rho_{\alpha\mu\beta} \). Greek indices denote 4 dimensional space-time while Latin indices run from 1 to 3.
Variation of this action gives equations of motion

$$G_{\mu\nu} = \varphi^\mu\varphi_\nu - \frac{1}{2} \varphi^\alpha \varphi_\alpha \delta^\mu_\nu + \frac{\lambda}{2} \left[ H^\mu_{\alpha\beta} \xi^{\alpha\beta} + G^\mu_{\alpha\nu} \xi^{\alpha\nu} - G^\mu_\nu \xi^\alpha_\alpha \right],$$

(5)

$$\varphi^\alpha_{\alpha} = \frac{\lambda}{16} \xi \varphi R_{GB}^2.$$  

(6)

Here, $$\varphi$$ denotes derivative with respect to $$\varphi$$, $$G_{\mu\nu}$$ is the Einstein tensor and

$$H^\mu_{\alpha\beta} := R^\mu_{\alpha\beta} - R_{\alpha\beta} \delta^\mu_\nu + R_{\alpha\mu} \delta^\mu_\beta.$$  

(7)

Assuming the homogeneous and isotropic flat metric

$$ds^2 = -dt^2 + a^2 \delta_{ij} dx^i dx^j,$$

(8)

the equations of motion are rewritten using $$\varphi$$ and the Hubble parameter $$H := \dot{a}/a$$ as

$$\ddot{\varphi} = 6H^2(1 - \frac{\lambda}{2} H \dot{\xi})$$

(9)

$$(2\dot{H} + 5H^2)(1 - \frac{\lambda}{2} H \dot{\xi}) + H^2(1 - \frac{\lambda}{2} \ddot{\xi}) = 0$$

(10)

$$\ddot{\varphi} + 3H \dot{\varphi} + \frac{3\lambda}{2} \left(H + H^2\right) H^2 \xi \varphi = 0.$$  

(11)

Dot (\dot{\cdot}) is the derivative with respect to the physical time $$t$$. Thus, along with $$\frac{d}{dt} a = aH$$ and $$\frac{d}{dt} \varphi = \dot{\varphi}$$, we have four first order differential equations for four variables $$a$$, $$H$$, $$\varphi$$ and $$\dot{\varphi}$$, and there is one constraint (11).

As long as we consider the spatially flat universe, the system has two degrees of freedom for initial conditions since the scale factor is unimportant except its relative change. Furthermore, because we are only interested in the region $$\varphi \sim 0$$ where the energy condition is violated and the initial singularity is avoided [14], the initial conditions actually have only one degree of freedom. Since it is shown in [14] that $$\dot{\varphi}$$ and $$H$$ do not change their signs during their evolution, we choose $$\dot{\varphi}$$ and $$H$$ to be positive in order to describe the expanding universe. The action (11) is symmetric under the change of $$\varphi$$'s sign so that $$\dot{\varphi}$$ can be set to be positive without a loss of generality. We also assume $$\lambda$$ to be positive, since otherwise the system has no singularity-free solutions [12,14].

Homogeneous, isotropic and spatially flat solutions leading to Friedmann-like universe (decelerating expansion) in the future are shown in fig.1. Since $$\dot{\varphi}$$ is always positive throughout the evolution of the solutions, larger value of $$\varphi$$ means later in time, i.e. time flows from left to right. We here put $$\lambda = 1$$, which fixes the time scale. The equations of motion are numerically integrated from the future to the past. In fig.1, we fix the initial (future) value of $$\varphi$$ and take several different initial values of $$H$$. The initial values of $$\dot{\varphi}$$ are determined by $$H$$ and $$\varphi$$ through the constraint (11). We see that there are 2 classes of solutions: singular and non-singular. The singular solutions (a and b in fig. 1) lead to an initial singularity as is the case for the usual big-bang universe. The non-singular solutions (c, d and e) are free from the initial singularity, and approach a flat space in the infinite past. It is shown in [14] that all solutions that inhabit in the $$\varphi < 0 \ H > 0$$ quarter plane regularly continue to $$\varphi > 0 \ H > 0$$ quarter plane.

Asymptotic solution of the system can be studied by assuming a power-law behavior for $$H$$ and $$\dot{\varphi}$$ as
\[ H \sim \omega_1 |t|^\beta, \]
\[ \varphi \sim \varphi_0 + \omega_2 \ln |t|, \]
\[ \dot{\varphi} \sim \frac{\omega_2}{t}, \]

in the asymptotic region \( t \to \pm \infty \). Using equations (9)(11) and the asymptotic form of the function \( \xi \)
\[ \frac{\partial \xi}{\partial \varphi} \sim \text{sign}(\varphi) \frac{\pi}{3} e^{|\varphi|}, \]
we have following asymptotic solutions for \( t \to \infty \) and \( t \to -\infty \) regions respectively:
\[ A_{\infty} : \beta = -1, \omega_1 = \frac{1}{3}, \omega_2 = \sqrt{\frac{2}{3}}, \]
\[ A_{-\infty} : \beta = -2, \omega_1 > 0, \omega_2 = -5. \]

The future solution \( A_{\infty} \) is obtained by balancing the terms which do not contain Gauss-Bonnet contribution, and therefore describe the situation where the 1-loop effect is negligible. On the contrary, \( A_{-\infty} \) corresponds to the case in which the Gauss-Bonnet contribution is dominant, and requires \( \lambda > 0 \), which is acceptable for our non-singular solution. These asymptotic solutions are uniquely determined under the conditions mentioned above, that is, the solutions are singularity-free and \( H > 0, \dot{\varphi} > 0 \). To sum it up, the non-singular universe we are considering here begins with an asymptotically flat space at the past infinity, and experiences super-inflation until \( \varphi \) crosses zero, and then enters into a stage of ordinary Friedmann-like expansion where Gauss-Bonnet effect is negligible.

For the analysis of perturbation we now look into a typical non-singular solution more closely. To characterize the equations of state of the system we introduce a new variable \( \Gamma \) defined by
\[ p_{\text{eff}} = (\Gamma - 1)\rho_{\text{eff}}. \]

Effective energy density \( \rho_{\text{eff}} \) and effective pressure \( p_{\text{eff}} \) used here are defined by the components of Einstein tensor as \( \rho_{\text{eff}} = -G^{00} \) and \( p_{\text{eff}} = \frac{1}{3}G^{ii} \), respectively. Using the background variables we can express \( \Gamma \) in a more explicit form:
\[ \Gamma = -\frac{2H}{3H}. \]

Evolutions of \( H, \varphi \) and \( \Gamma \) in a typical non-singular solution are shown in the fig.2. The friction \( 3H + \dot{\alpha}/\alpha \) is for later discussion. In the far future \( \Gamma \) approaches 2, which indicates that the equation of state for this system in later time is nearly that of stiff matter, i.e. free scalar field. On the other hand, \( \Gamma \) takes large negative values in the far past. This peculiar behavior of \( \Gamma \) has a close relation to the non-singular nature of the solution. By means of this \( \Gamma \), weak and strong energy conditions \( \rho_{\text{eff}} + p_{\text{eff}} \geq 0 \) and \( \rho_{\text{eff}} + 3p_{\text{eff}} \geq 0 \) can be written as \( \Gamma \geq 0, \Gamma \geq 2/3 \), respectively. Thus, as can be seen from the fig.2, violation of those energy conditions, which is necessary to avoid the initial singularity, occurs near and before the peak of the Hubble parameter \( H \) at \( \varphi \sim 0 \).

We now consider the perturbation of our model (4) to analyze the stability of the non-singular solutions. We include only the tensor perturbation here, and write the full metric as
\[ ds^2 = -dt^2 + a^2(\delta_{ij} + h_{ij})dx^i dy^j. \]

For \( h_{ij} \) we assume the transverse-traceless conditions \( h^i_i = 0, h_{ij}^j = 0 \). Equations of motion for \( h_{ij} \) are obtained by perturbing the Einstein-like equation (2), and non-trivial one comes from the spatial part. Expanding \( h_{ij} \) with transverse-traceless basis tensors as
\[ h_{ij} = h_+(k,t)e_+_{ij}(k) + h_-(k,t)e_-_{ij}(k), \]
we obtain the same equation for each polarization mode

\[^3\text{We have also analyzed scalar and vector part of the perturbation using the gauge-invariant method \cite{15}, but no instability was found.}\]
\[ \ddot{h} + (3H + \frac{\dot{\alpha}}{\alpha})\dot{h} + \frac{k^2}{a^2 \alpha} (1 - \frac{\lambda}{2} \dot{\xi}) h = 0, \]  
(22)

where \( \alpha \) is a function of the background variables defined by

\[ \alpha := 1 - \frac{\lambda}{2} \dot{H} \dot{\xi} = \frac{\dot{\varphi}^2}{6H^2}. \]  
(23)

Equation (22) describes the evolution of the geometrical fluctuation in our non-singular model. Of course, this is nothing but the equation for the gravitational wave in FRW background if \( \lambda = 0 \). The background equation (14) and the definition of \( \Gamma \) lead to \( 1 - \frac{4}{3} \dot{\xi} = (3\Gamma - 5)\alpha \) so that the perturbation equation (22) can be expressed as

\[ \ddot{h} + (3H + \frac{\dot{\alpha}}{\alpha})\dot{h} + \frac{k^2}{a^2 \alpha} (3\Gamma - 5) h = 0, \]  
(24)

which is suitable to discuss the stability of the system in relation to the energy conditions. In this expression the third term determines the stability of the system, and the second term is regarded as a friction. One can see that the system is unstable for tensor perturbation when

\[ \Gamma < \frac{5}{3}, \]  
(25)

and that strong instability appears in small scale. We can combine this result with the discussion about the energy conditions to reach the statement, that is, Cosmological solution of this model is geometrically unstable as long as it is non-singular. In order to avoid the initial singularity, the strong energy condition \( \Gamma \geq 2/3 \) must be violated. Then, since \( \Gamma < 2/3 < 5/3 \), condition (25) is satisfied and the system becomes unstable for tensor perturbation. Therefore, the non-singular nature of the background space-time inevitably requires its geometrical instability, which is characterized by the condition (25).

Numerical solutions of perturbation for different wave numbers in the flat background are shown in fig.3. The ordinate is in logarithmic scale, and we can easily recognize that the instability arises in the super-inflationary phase. In these calculations the initial value of \( h \) is set to unity so that \( \log \dot{\varphi} \) starts from zero. Our statement can be confirmed by comparing fig.2 and fig.3. One interesting point we can notice is that the effect of the friction term \( 3H + \dot{\alpha}/\alpha \) is not necessarily negligible. The friction term is negative before the peak of the Hubble parameter and this negative friction enhances the instability. After the peak, the friction turns to be positive and it weakens the growth of the perturbation. This is why the instability seems to cease before \( \Gamma \) gets greater than \( 5/3 \).

One might wonder whether this instability continues from the infinite past or not. To answer this question we can use the asymptotic background solution \( A_\infty \), which tells us the asymptotic form of \( \Gamma \) in the infinite past. Substituting (17) into (14), we have the asymptotic form of \( \Gamma \) as \( \Gamma \sim 4t/3\omega_1 \). Since \( \omega_1 \) is positive, \( \Gamma \) decreases indefinitely in the past and the energy conditions can never be recovered. From the condition (25), therefore, the geometrical instability never disappears even if we go to the infinite past. To obtain the asymptotic solution of \( h \) in the past asymptotic region, we estimate each term of the equation (24). As \( \dot{\alpha}/\alpha \) behaves like \( 1/|t| \) and \( H \sim \omega_1/|t|^2 \) in the far past, the friction term in (24) can be neglected. In the third term, the scale factor \( a \) behaves like \( a = \exp \int H dt \sim \exp(\omega_0 - \frac{\omega_1}{2}) \sim \exp \omega_0 =: a_0 \) (\( \omega_0 \) and \( a_0 \) are constants). Therefore, (24) becomes

\[ \ddot{h} + \frac{4k^2}{\omega_1 a_0^2 \tau^2} h = 0. \]  
(26)

and the asymptotic solution of \( h \) can be expressed using Airy functions as \( h(k, t) \sim C_1 Ai(\tau) + C_2 Bi(\tau) \), where \( \tau = \sqrt{\frac{4k^2}{\omega_1 a_0^2 \tau^2}} |t| \).

The fact that the perturbative instability continues from the infinite past does not necessarily mean the failure of the background solution. To see the criterion for the breakdown of the background solution, we assume a state of the perturbation at some finite past time \( t = t_i \) as \( h(k, t_i) = h_i(k) \) and \( h'(k, t_i) = h'_i(k) \) (primed denotes differentiation with respect to \( \tau \)). Then, \( C_1 \) and \( C_2 \) are determined and we can write

\[ h(k, t) = \pi (Bi'(\tau_i)h_i(k) - Bi(\tau_i)h'_i(k)) Ai(\tau) - \pi (Ai'(\tau_i)h_i(k) - Ai(\tau_i)h'_i(k)) Bi(\tau), \]  
(27)

where \( \tau_i = \sqrt{\frac{4k^2}{\omega_1 a_0^2 |t_i|}} \). In order to keep the perturbation smaller than the background metric through the evolution of the solution, \( h(k, t) \) has to be smaller than unity when \( t \sim 0 \), which is sufficient since the growth of the perturbation terminates around the Hubble peak. Then, discarding the decaying mode and picking up the dominant term \( h_i(k) \)
and $h_i(k)$ are assumed to be comparable in magnitude), the background solution breaks down if and only if $h_i(k,0) = \pi B_i'(\tau_0)h_i(k)Ai(0) \gg O(1)$. We can use the asymptotic form of the Airy function to put this breaking down condition expressed as a condition for the value of the perturbation at time $t_i$

$$h_i(k) \gg \pi^{-\frac{1}{2}}Ai(0)^{-1}\left(\frac{4k^2}{\omega_1^2 a_0^2}\right)^{\frac{1}{4}} |t_i|^{-\frac{1}{4}} \exp \left(-\frac{4|k|}{3a_0\omega_1} |t_i|^{\frac{3}{4}}\right),$$

(28)

where $Ai(0)^{-1} \approx 2.817 \sim O(1)$. Therefore, premising the existence of some natural cut-off for $k$, we can say that the background system breaks down due to the overdominant graviton if the amplitude $h_i(k)$ of at least one mode $k$ satisfies the condition (28) at time $t_i$.

We have shown that the non-singular cosmological model proposed by Antoniadis, Rizos and Tamvakis [12] is unstable for tensor perturbation, and that the energy condition breaking required to obtain the non-singular solutions is a sufficient condition for this instability. We also estimated the condition for the break down of the background solution due to the overdominant graviton [23].

Let us conclude this letter by giving some comments on our result. First, we would like to comment on the nature of the instability found here. One can say that this instability has an opposite nature to Jeans instability, in the sense that it appears in small scales rather than in large scales. Apart from its characteristic spectrum, this instability is distinctive in that it appears only in the tensor part. Since it involves no growth of scalar field perturbation [15], this instability is purely a “geometrical” one. If we assume that the effective action (1) is valid in the early stages of our universe, and if none of the $h_i(k)$ satisfies the condition (28) at time $t_i$ so that the background solution is unaffected by the perturbation, this small scale instability may indicate generation of many small primordial black holes, typically of the Planckian scale. Next, although we have treated only the spatially flat case so far, our claim also holds for the metric with non-zero spatial curvature. However, if the background universe has positive or negative spatial curvature, the asymptotic behavior of the system in the infinite past is not the same as in the flat case. Therefore, if the energy conditions should be recovered in the finite past, the instability might no longer be eternal. Lastly, we have to mention the possibility that the effective action we use here (1) might not be suitable for the model of the early universe. Since the 1-loop correction changes the tree-level solutions drastically, it is sensible to consider that the higher loop effect might change the nature of the cosmological solutions from that of 1-loop action. Our results of rather pathological feature of the instability might be a consequence of our neglecting higher corrections. To take all these loop effects into account, we have to wait for the accomplishment of a non-perturbative string theory.

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In that case, $\Gamma = -\frac{2}{3} \frac{\dot{H}}{H} + \frac{K}{a^2}$ and $k^2$ in (24) is replaced by $k^2 + 2K$, where $K = +1, -1$ for closed and open universes, respectively.
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Figure captions

Fig. 1
Flat(\( \mathcal{K} = 0 \)) background solutions that lead to decelerating expansion in the future. Numerical integration is performed from future to past, and the figure shows the phase diagram of Hubble parameter \( H \) versus modulus field \( \varphi \). Time flows from left to right since \( \dot{\varphi} \) is always positive. There are two classes of solutions: singular solutions (a and b), and non-singular solutions (c,d,e). One can see that the avoidance of the initial singularity takes place when \( \varphi \sim 0 \). For detailed discussions see references [12] and [14].

Fig. 2
Evolutions of (i) \( H \), (ii) \( \varphi \), (iii) \( 3H + \dot{\alpha}/\alpha \) and (iv) \( \Gamma \) in a typical super-inflationary flat solution. The origin of time \( t \) is chosen so that \( \varphi \) crosses zero when \( t = 0 \). \( H \) has a peak at \( t \sim 0 \). \( \varphi \) accelerates from the asymptotically flat space, and decelerates at \( \varphi \sim 0 \). The friction term \( 3H + \dot{\alpha}/\alpha \) in the perturbation equation of motion evolves from a small negative value, increases in its magnitude, and changes its sign at \( t = 0 \), and then decreases in the future. \( \Gamma \) varies from a large negative value in the past to 2 in the future.

Fig. 3
Evolution of the tensor perturbation in a flat background. Note that the tensor perturbation is plotted with logarithmic scale. We set \( a = 1, h(k) = 1 \) at the onset of the integration (\( t = -2 \)), and \( t = 0 \) is the time when \( \varphi \) crosses zero. One can notice that growing modes appear in the initial super-inflationary phase. The smaller the scale of perturbation is, the larger the growth rate becomes. The depression of \( k = 20 \) solution around \( t \sim 1.25 \) indicates the oscillatory behavior in the Friedmann-like phase.
Fig. 1

Hubble vs $\varphi$

Legend:
- a
- b
- c
- d
- e
Fig. 3

$\log_{10}|h(k)|$

t

$k=0.1$
$k=1.0$
$k=5.0$
$k=10.0$
$k=20.0$