Learning Latent Networks in Vector Auto Regressive Models

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Abstract

We study the problem of learning the dependency graph between random processes in a vector auto regressive (VAR) model from samples when a subset of the variables are latent. We show that the dependencies among the observed processes can be identified successfully under some conditions on the VAR model. Moreover, we can recover the length of all directed paths between any two observed processes which pass through latent part. By utilizing this information, we reconstruct the latent subgraph with minimum number of nodes uniquely if its topology is a directed tree. Furthermore, we propose an algorithm that finds all possible minimal latent networks if there exists at most one directed path of each length between any two observed nodes through the latent part. Experimental results on various synthetic and real-world datasets validate our theoretical results.

1. Introduction

Identifying causal influences among time series is a problem of interest in many different fields. In macroeconomics, for instance, researchers seek to understand what factors contribute to economic fluctuations and how these factors interact with each other (Lütkepohl & Krätzig, 2004). In neuroscience, extensive body of research focuses on learning the interactions between different regions of brain by analyzing neural spike trains (Roebroeck et al., 2005; Besserve et al., 2010; Kim et al., 2011).

Granger causality (Geiger et al., 2015), transfer entropy (Schreiber, 2000), and directed information (Massey, 1990; Marko, 1973) are some of the most commonly used measures in the literature to capture causation in time series. Measuring the reduction of uncertainty in one variable after observing another variable is the key concept behind such measures. (Eichler, 2012) provides an overview of various definitions of causation for time series.

In this work we study the causal identification problem in VAR models when only a subset of times series is observed. More precisely, we assume that the available measurements are a set of random processes \( \vec{X}(t) \in \mathbb{R}^n \) which, together with another set of latent random processes \( \vec{Z}(t) \in \mathbb{R}^m \), where \( m \leq n \) form a first order VAR model as follows:

\[
\begin{bmatrix}
\vec{X}(t+1) \\
\vec{Z}(t+1)
\end{bmatrix} =
\begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{bmatrix}
\begin{bmatrix}
\vec{X}(t) \\
\vec{Z}(t)
\end{bmatrix} +
\begin{bmatrix}
\vec{\omega}_X(t+1) \\
\vec{\omega}_Z(t+1)
\end{bmatrix}.
\]

(1)

Note that in VAR models due to the functional dependencies among the processes there exists a natural notion of causation. More precisely, the support of the coefficient matrix encodes the causal structure in a VAR model (Granger, 1969; Spirtes et al., 2000; Pearl, 2009).

Contributions: The contributions of this paper are as follows: we propose a learning approach that recovers the observed sub-network (support of \( A_{11} \)) from linear regression on the observed variables \( \vec{X} \) as long as the latent sub-network (support of \( A_{22} \)) is a directed acyclic graph (DAG). We also derive a set of sufficient conditions under which we can uniquely recover the causal influences from latent to observed processes, (support of \( A_{12} \)) and also the causal influences among the latent variables, ( support of \( A_{22} \)). Ad-
ditionally, we propose a sufficient condition under which the complete causal structure can be recovered uniquely.

More specifically, we show that under an assumption on the observed to latent noise power ratio, if none of the sub-matrices \( A_{12} \) and \( A_{21} \) are zero, it is possible to determine the length of all directed latent paths. We refer to this information as linear measurements. This information reveals important properties of the causal structure among the latent and observed processes, i.e., support of \([0, A_{12}; A_{21}, A_{22}]\). We call this causal sub-network of a VAR model unobserved network. We show that in the case that the unobserved network is a directed tree and each latent node has at least two parents and two children, a straightforward application of (Patrinos & Hakimi, 1972) can recover the unobserved network uniquely. Furthermore, we propose Algorithm 1 that recovers the support of \( A_{22} \) and \( A_{12} \) given the linear measurements when only the latent sub-network is a directed tree plus some extra structural assumptions (see Assumption 1 in Section 4.2). Lastly, we study the causal structures of VAR models in more general case in which there exists at most one directed latent path of length \( k \geq 2 \) between any two observed processes (see Assumption 2 in Section 4.3). For such VAR models, we propose Algorithm 2 that can recover all possible unobserved networks with minimum number of latent processes.

Related works: The problem of latent causal structure for time series has been studied in the literature. (Jalali & Sanghavi, 2012) showed that direct causal relations between observed variables can be identified in a VAR model assuming that connections between observed variables are sparse and each latent variable interacts with many observed variables. However, their approach focuses on learning only the observed sub-network. (Boyen et al., 2012) studied a network of processes (not necessary a VAR model) whose underlying structure is a polytree and introduced an algorithm that can learn the entire causal structure (observed and unobserved networks) using a so-called discrepancy measure.

2. Problem Definition

In this part, we review some basic definitions and our notation. Throughout this paper, we use an arrow over the letters to denote vectors. We assume that the time series are stationary and denote the autocorrelation of \( \tilde{X} \) by \( \gamma_X(k) := E[\tilde{X}(t)\tilde{X}(t-k)^T] \). We denote the support of a matrix \( A \) by \( \text{Supp}(A) \) and use \( \text{Supp}(A) \subseteq \text{Supp}(B) \) to indicate \( [A]_{ij} = 0 \) whenever \( [B]_{ij} = 0 \). We also denote the Fourier transform of \( g \) by \( \mathcal{F}(g) \) and it is given by \( \sum_{h=-\infty}^{\infty} g(h)e^{-ht} \).

A directed graph \( G = (V, \tilde{E}) \) is characterized by a set of vertices (or nodes) \( V \) and a set of ordered pairs of vertices, called edges \( \tilde{E} \subseteq V \times V \). We denote the set of parents of a node \( v \) by \( \mathcal{P}_v := \{ u \in V : (u, v) \in \tilde{E} \} \) and the set of its children by \( \mathcal{C}_v := \{ u \in V : (v, u) \in \tilde{E} \} \). The skeleton of a directed graph \( G \) is the undirected graph obtained by removing all the directions in \( G \).

2.1. System Model

Consider the VAR model in (1). Let \( \tilde{\omega}_Z(t) \in \mathbb{R}^m \) be i.i.d random vectors with mean zero. For simplicity, we denote the matrix \([A_{11}, A_{12}; A_{21}, A_{22}]\) by \( A \).

Our goal is to recover \( \text{Supp}(A) \) from observed data, i.e., \( \{\tilde{X}(t)\} \). Rewrite 1 as follows

\[
\tilde{X}(t+1) = \sum_{k=0}^{t} A_k^{1} \tilde{X}(t-k) + A_{12} A_{22}^T \tilde{Z}(0) + \sum_{k=0}^{t-1} \hat{A}_k \tilde{\omega}_Z(t-k) + \tilde{\omega}_X(t+1),
\]

where \( A_0 := A_{11}, A_k^{1} := A_{12} A_{22}^{k-1} A_{21} \) for \( k \geq 1 \), and \( \hat{A}_k := A_{12} A_{22}^{k-1} \). In the remainder of this paper, we will assume that the \( A_{22} \) is acyclic, i.e., \( \exists 0 < l \leq m \), such that \( A_{22}^l = 0 \). Thus, for \( t \geq l \), the above equation becomes

\[
\tilde{X}(t+1) = \sum_{k=0}^{l} A_k^{1} \tilde{X}(t-k) + \sum_{k=0}^{t-1} \hat{A}_k \tilde{\omega}_Z(t-k) + \tilde{\omega}_X(t+1).
\]

Note that the limits of summations in (3) are changed.

We are interested in recovering the set \( \{\text{Supp}(A_k^{1})\}_{k=0}^{l} \) because it captures important information about the structure of the VAR model. Specifically, \( \text{Supp}(A_0^{1}) = \text{Supp}(A_{11}) \); so it represents the direct causal influences between the
observed variables and \(\text{Supp}(A^*_k)\) for \(k \geq 1\) determines whether at least one directed path of length \(k + 1\) exists between any two observed nodes which goes through the latent sub-network\(^3\). We will make use of this information in our recovery algorithm. We call the set of matrices \(\{\text{Supp}(A^*_k)\}_{k \geq 0}\), linear measurements. In Section 4, we present a set of sufficient conditions under which given the linear measurements, we can recover the entire or most parts of the unobserved network uniquely.

Note that in general, the linear measurements cannot uniquely specify the unobserved network. For example, Figure 1 illustrates two different unobserved networks that both share the same set of linear measurements,

\[
A^*_1 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}, \quad A^*_2 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0
\end{bmatrix},
\]

and \(A^*_k = 0\) for \(k > 2\).

### 3. Identifiability of the Linear Measurements

As we need the linear measurements for our structure learning, in this section, we study the conditions required for recovering the linear measurements from the observed processes \(\{X(t)\}\). To do so, we start off by rewriting Equation (3) as follows

\[
\tilde{X}(t+1) = A\tilde{X}_{t-1:t} + \sum_{k=0}^{l-1} \tilde{A}_k \tilde{\omega}_Z(t-k) + \tilde{\omega}_X(t+1),
\]

where \(A := [A^*_0, \ldots, A^*_l]_{n \times n(t+1)}\), and \(\tilde{X}_{t-1:t} := [\tilde{X}(t); \ldots; \tilde{X}(t-l)]_{n(t-1+1) \times 1}\).

By projecting \(\tilde{A}_k \tilde{\omega}_Z(t-k)\) onto the vector space spanned by the observed processes, i.e., \(\{\tilde{X}(t), \ldots, \tilde{X}(t-l)\}\), we obtain

\[
\tilde{A}_k \tilde{\omega}_Z(t-k) = \sum_{r=0}^{l} C^*_r \tilde{X}(t-r) + \tilde{N}_Z(t-k), \quad 0 \leq k \leq l-1,
\]

where \(\{\tilde{N}_Z(t-k)\}\) denote the residual terms and \(\{C^*_r\}\) are the corresponding coefficient matrices. Substituting (5) into (4) implies

\[
\tilde{X}(t+1) = B\tilde{X}_{t-1:t} + \bar{\theta}(t+1),
\]

where \(B := [B^*_0, \ldots, B^*_l]\), and

\[
B^*_l := A^*_l + \sum_{s=0}^{l-1} C^*_r, \quad \bar{\theta}(t+1) := \tilde{\omega}_X(t+1) + \sum_{k=0}^{l-1} \tilde{N}_Z(t-k).
\]

Note that by this representation, \(\bar{\theta}(t+1)\) is orthogonal to \(\tilde{\omega}_X(t+1)\), i.e., \(\mathbb{E}[\bar{\theta}(t+1)^T \tilde{\omega}_X(t-k)] = 0\), for \(0 \leq k \leq l\). Hence, Equation (6) shows that the minimum mean square error (MMSE) estimator can learn the coefficient matrix \(B\) given the observed processes. More precisely, we have

\[
\mathcal{B} = [\gamma_X(1), \ldots, \gamma_X(l+1)] \times \Gamma_X(l)^{-1},
\]

where \(\Gamma_X(l) := \mathbb{E}[\tilde{X}_{t-1:t}^T \tilde{X}_{t-1:t}]\).

#### Proposition 1

For the stationary VAR model in (1) in which the latent sub-network is a DAG, i.e., \(A^*_{zz} = 0\), we have

\[
\max_{0 \leq k \leq l} ||B^*_k - A^*_k||_1 \leq \sqrt{nlM/L} ||A_{12}||_2.
\]

where \(L := \inf_{\Omega \in [0,2\pi]} \lambda_{\text{min}}(\mathcal{F}(\gamma_X))\) and \(M := \sup_{\Omega \in [0,2\pi]} \lambda_{\text{max}}(\mathcal{F}(\gamma_\omega))\).

#### Proof

See Appendix A.

\[
\square
\]

This result implies that we can asymptotically recover the support of \(A^*_{kk}\), as long as the absolute values of nonzero entries of \(A^*_k\) are bounded away from zero by \(2\sqrt{nlM/\pi L} ||A_{12}||_2\). Note that the direct causal influences among the observed nodes (support of \(A_{11}\)) can be recovered from \(A^*_0\). We will make use of \(\{\text{Supp}(A^*_k)\}_{k \geq 0}\) to recover the unobserved network in the next section.

#### Proposition 2

Let \(\Sigma_X\) and \(\Sigma_Z\) be the autocovariance matrices of \(\tilde{\omega}_X(t)\) and \(\tilde{\omega}_Z(t)\), respectively. Then, the ratio \(M/L\) strictly increases by decreasing \(\sigma_X^2/\sigma_Z^2\) where \(\Sigma_X = \sigma_X^2 I_{n \times n}\) and \(\Sigma_Z = \sigma_Z^2 I_{m \times m}\).

#### Proof

See Appendix B.

\[
\square
\]

When only a finite number of samples from the observed processes are available, say \(\{\tilde{X}(t)\}_{t=1}^T\), we can estimate the coefficient matrix \(B\), using an empirical estimator for \(\Gamma_X(l), \{\gamma_X(h)\}\), and then applying (7). Denote the result
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of this estimation by $\mathcal{B}_T$. It can be shown that (Lütkepohl & Krätzig, 2004)
\[
\sqrt{T}\text{vec}(\mathcal{B}_T - \mathcal{B}) \xrightarrow{d} N(0, \Gamma_X^{-1}(l) \otimes \Sigma).
\]
where $\xrightarrow{d}$ denotes convergence in distribution. Matrix $\Sigma$ is
given by $(\Sigma_X + \sum_{k=0}^{n-1}(A_{k}\Sigma Z_{k}^T A_{k}^T))$. The vec(.) operator transforms a matrix to a vector by stacking its columns and
$\otimes$ is the Kronecker product.

4. Learning the Unobserved Network

Recall that we refer to $\text{Supp}([0, A_{12}; A_{21}, A_{22}])$ as the unobserved network and $\text{Supp}(A_{22})$ as the latent sub-
network. We present three algorithms that take linear measurements $\{\text{Supp}(A_{k}^2)\}_{k \geq 0}$ as their input. First algorithm
recovers the entire unobserved network uniquely as long as it is a directed tree and each latent node has at least two
parents and two children. The output of the second algorithm is $\text{Supp}([0, A_{12}; A_{21}, A_{22}])$, where $\text{Supp}(A_{22}) \subseteq \text{Supp}(A_{22})$. This output is guaranteed whenever the latent
sub-network is a directed tree and some extra conditions are satisfied on how the latent and observed variables are con-
ected (see Assumption 1 in Section 4.2). Third algorithm finds the set of all possible networks that are consistent with
the measurements and have the minimum number of latent
nodes. This algorithm is able to do so when there exists at most one directed “latent path” of any arbitrarily length between two observed nodes (see Assumption 2 in Section
4.3). A directed path is called latent if all the intermediate
variables on that path are latent.

4.1. Unobserved Network is a Directed Tree

(Patrinos & Hakimi, 1972) introduced a necessary and
sufficient condition and also an algorithm to recover a
weighted directed tree uniquely\(^4\) from a valid distance ma-
trix $D$ defined on the observed nodes. The condition is
as follows: every latent node must have at least two par-
ents and two children. A matrix $D$, in (Patrinos & Hakimi,
1972), is a valid distance matrix over a weighted directed
tree, when $[D]_{ij}$ equals the sum of all the weights of those
edges that belong to the directed path from $i$ to $j$, and
$[D]_{ij} = 0$, if there is no directed path from $i$ to $j$.

The algorithm in (Patrinos & Hakimi, 1972) has two
phases. In the first phase, it creates a directed graph among
the observed nodes with the adjacency matrix $\text{Supp}(D)$.
In the second phase, it recursively finds and removes the
circuits\(^3\) by introducing latent nodes for each circuit. For
more details see (Patrinos & Hakimi, 1972).

In order to adopt (Patrinos & Hakimi, 1972)’s algorithm
for learning the unobserved network, we introduce a valid
distance matrix using our linear measurements as follows,
\[
[D]_{ij} = \begin{cases} k + 1 & \text{if } [\text{Supp}(A_{k}^2)]_{ij} \neq 0, \\ 0 & \text{Otherwise}. \end{cases}
\]
Recall that $[\text{Supp}(A_{k}^2)]_{ij}$ indicates whether there exists a
directed latent path from $i$ to $j$ of length $k + 1$ in the un-
observed network. From theorem 8 in (Patrinos & Hakimi,
1972), it is easy to show that the unobserved network can be
recovered uniquely from above distance matrix if its topol-
yogy is a directed tree.

4.2. Latent Sub-network is a Directed Tree

We need the following definition to present our results.

**Definition 1.** We denote the subset of observed nodes that
are parents of a latent node $h$ by $\mathcal{P}_h^\mathcal{O}$ and denote the subset of observed nodes that $h$ is their parent, by $\mathcal{C}_h^\mathcal{O}$. We further
denote the set of all leaves in the latent sub-network by $\mathcal{L}$.

We consider learning an unobserved network $G$ that satis-
ifies the following assumptions.

**Assumption 1.** Assume that the latent sub-network of $G$ is
directed tree. Furthermore, for any latent node $h \in G$; (i) $\mathcal{P}_h^\mathcal{O} \subseteq \cup_{h \neq j} \mathcal{P}_j^\mathcal{O}$ and if $h$ is a leaf of the latent sub-network,
then (ii) $\mathcal{C}_h^\mathcal{O} \subseteq \cup_{i \in \mathcal{L}, i \neq h} \mathcal{C}_i^\mathcal{O}$.

This assumption states that the latent sub-network of $G$
must be a directed tree such that each latent node in $G$
has at least one unique parent in the set of observed nodes.
That is, a parent who is not shared with any other latent
node. Furthermore, each latent leaf has at least one unique
child among the observed nodes. For instance, when $\text{Supp}(A_{22})$ represents a directed tree and both $\text{Supp}(A_{12})$
and $\text{Supp}(A_{21})$ contain identity matrices, Assumption 1
holds.

Figure 3a illustrates a simple network that satisfies Ass-
sumption 1 in which the unique parents of latent nodes
$a, b, c,$ and $d$ are $\{1\}$, $\{3\}$, $\{2\}$, and $\{4\}$, respectively. The
unique children of latent leaves $c$ and $d$ are $\{5\}$ and $\{2, 4\}$, respectively.

**Theorem 1.** Among all unobserved networks that are consis-
tent with the linear measurements induced from (1),
graph $G$ that satisfies Assumption 1 has the minimum num-
ber of latent nodes.

**Proof.** See Appendix C.

Note that if Assumption 1 is violated, one can find many
unobserved networks that are consistent with the linear
measurements but are not minimum (in terms of the num-
ber of latent nodes). For example, the network in Figure

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\(^4\)The skeleton of the recovered tree is the same as the original one but not necessary the weights.

\(^3\)In a directed graph, a circuit is a cycle after removing all the directions.
Consider an unobserved network $G$ with one less latent node.

**Theorem 2.** Consider an unobserved network $G$ with adjacency matrix $\text{Supp}(0, A_{12}; A_{21}, A_{22})$. If $G$ satisfies Assumption 1, then its corresponding linear measurements uniquely identify $G$ up to $\text{Supp}(0, A_{12}; \tilde{A}_{21}, \tilde{A}_{22})$, where $\text{Supp}(A_{22}) \subseteq \text{Supp}(\tilde{A}_{22})$.

**Proof.** See Appendix D.

Figure 3a gives an example of a network satisfying Assumption 1 and an alternate network, Figure 3b, with the same linear measurements which departs from the Figure 3a in $A_{21}$ component.

Next, we propose the directed tree recovery (DTR) algorithm that takes the linear measurements of an unobserved network $G$ satisfying Assumption 1 and recovers $G$ up to the limitation in Theorem 2. This algorithm consists of three main loops. Recall that Assumption 1 implies that each latent node has at least one unique observed parent. The first loop finds all the unique observed parents for each latent node (lines: 3-9). The second loop reconstructs $\text{Supp}(A_{22})$ and $\text{Supp}(A_{12})$ (lines: 10-15). And finally, the third loop constructs $\text{Supp}(\tilde{A}_{21})$ such that $\text{Supp}(A_{22}) \subseteq \text{Supp}(\tilde{A}_{22})$ (lines: 16-20).

The following lemma shows that the first loop of Algorithm 1 can find all the unique observed parents from each latent node. To present the lemma, we need the following definitions.

**Definition 2.** For a given observed node $i$, we define

\[
    l_i := \max\{k : [A_{k-1}^*]_{si} \neq 0, \text{ for some } s\}, \quad (8)
\]

\[
    R_i := \{j : [A_{i-1}^*]_{ji} \neq 0\}, \quad (9)
\]

\[
    M_i := \{(j, r) : [A_{r-1}^*]_{ji} \neq 0\}. \quad (10)
\]

In the above equations, $l_i$ denotes the length of longest directed latent path that connects node $i$ to any other observed node. $R_i$ is the set of all observed nodes that can be reached by $i$ with a directed latent path of length $l_i$ and set $M_i$ consists of all pairs $(j, r)$ such that there exists a directed latent path from $i$ to $j$ with length $r$.

**Lemma 1.** Under Assumption 1, an observed node $i$ is the
Algorithm 2 The Node-Merging (NM) Algorithm

1: **Initialization:** Construct graph $G_0$.
2: $G_0 := G_0, G_s := \emptyset, \forall s > 0$
3: $k := 0$
4: while $G_k \neq \emptyset$ do
5: for $G \in G_k$ do
6: for $i', j' \in G$ do
7: if Check($G, i', j'$) then
8: $G_{k+1} := G_{k+1} \cup \text{Merge}(G, i', j')$.
9: end if
10: end for
11: end for
12: $k := k + 1$
13: end while
14: **Output:** $G_{out} := G_{k-1}$

Definition 3. (Merging) We define merging two nodes $i'$ and $j'$ in graph $G$ as follows: remove node $j'$ and the edges between $i'$ and $j'$, then give all the parents and children of $j'$ to $i'$. We denote the resulting graph after merging $i'$ and $j'$ by $\text{Merge}(G, i', j')$. We say that two nodes $i'$ and $j'$ are mergeable if $\text{Merge}(G, i', j')$ is consistent with the linear measurements of $G$.

Definition 4. (Contentedness) Consider an undirected graph $G$ over the observed nodes which is constructed as follows: there is an edge between two nodes $i$ and $j$ in $G$, if there exists $k \geq 1$ s.t. $\text{Supp}([A^*_{k}]_{ij}) = 1$ or $\text{Supp}([A^*_{k}]_{ji}) = 1$; We say that two observed nodes $i$ and $j$ are “connected” if there exist a path between them in $G$.

It can be seen that if pairs $i, j$ and $j, k$ are connected then node $i$ and $k$ are also connected. Thus, we can define a connected class. That is, a subset of observed nodes in which any two nodes are connected.

The Node-Merging algorithm has two phases: initialization and merger.

Initialization: We first find the set of all connected classes, say $S_1, S_2, ..., S_C$. For each class $S_c$, we create a directed graph $G_{0,c}$ that is consistent with the linear measurements. To do so, for any two observed nodes $i, j \in S_c$, if $[A^*_{i,j}]_{ji} \neq 0$, we construct a directed path with length $r + 1$ from node $i$ to node $j$ by adding $r$ new latent nodes to $G_{0,c}$.

Merger: In this phase, for any $G_{0,c}$ from the initialization phase, we merge its latent nodes iteratively until no further latent pairs can be merged. Since order of mergers leads to different networks with minimum number of latent nodes, the output of this phase will be the set of all such networks.

Algorithm 2 summarizes the steps of the NM algorithm. In this algorithm, subroutine Check($G, i', j'$) checks whether two nodes $i'$ and $j'$ are mergeable.

Theorem 4. Under Assumption 2, the NM algorithm re-

unique parent of a latent node if and only if for any other observed node $j$ s.t. $l_j = l_j$, we have

$$(R_j \not\subseteq R_i) \lor (R_j = R_i \land M_i \subseteq M_j).$$

Proof. See Appendix E.

The second loop recovers $\text{Supp}(A_{22})$ based on the following observation. If a latent node $h_k$ is the parent of latent node $h_s$, then $h_k$ can reach all the observed nodes in $R_s$, i.e., $R_s \subseteq R_k$ and $l_k = l_s + 1$ (line: 11). Furthermore, $\text{Supp}(A_{12})$ can be recovered using the fact that an observed node $j$ is a children of a latent node $h_s$, if a unique parent of $h_s$, e.g., $s$ can reach $j$ by a directed latent path of length 2 (line: 14). Finally, the third loop reconstructs $\text{Supp}(A_{21})$ by adding an observed node $i$ to the parent set of latent node $h_j$, if $i$ can reach all the observed nodes that a unique parent of $h_j$, e.g., $j$ reaches (lines: 17-18).

Proposition 3. Suppose network $G$ satisfies Assumption 1. Then given its corresponding linear measurements, Algorithm 1 recovers $G$ up to the limitation in Theorem 2.

Proof. See Appendix F.

4.3. Learning More General Unobserved Networks with Minimum Number of Latent Nodes

In general, there may not be a unique minimal unobserved network consistent with the linear measurements (see Fig. 1). Hence, we try to find an efficient approach for recovering all possible minimal unobserved networks under some conditions. In fact, without any extra conditions, finding a minimal unobserved network is NP-hard.

Theorem 3. Finding an unobserved network that is both consistent with a given linear measurements and has minimum number of latent nodes is NP-hard.

Proof. See Appendix G.

In the reminder of this section, after some definitions, we propose the Node-Merging (NM) algorithm. This algorithm returns all possible unobserved networks with minimum number of latent nodes that are consistent with the linear measurements if we consider the following assumption.

Assumption 2. Assume that there exists at most one directed latent path of each length between any two observed nodes.

For example, the graph in Figure 3b satisfies this assumption but not the one in Figure 3a. This is because there are two directed latent paths of length 2 from node 5 to node 4.

Theorem 4. Under Assumption 2, the NM algorithm re-
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(a) The average normalized error versus number of observed nodes.

(b) The average of maximum estimation error versus OLNR.

Figure 4. Average error in computing linear measurements.

turns the set of all networks that are consistent with the linear measurements and have minimum number of latent nodes.

Proof. See Appendix H.

5. Experimental Results

5.1. Synthetic Data:

We considered a directed random graph denoted by DRG($p, q$), such that there exists a directed link from an observed node to a latent node and vice versa independently, with probability $p$. Furthermore, there is a directed link from a latent node to any other latent node with probability $q$. If there is a link between two nodes, we set the weight of that link uniformly from $\{-a, a\}$.

In order to evaluate how well we can estimate the linear measurements, we generated 1000 instances of DRG(0.4, 0.4) with $n + m = 100$, $E\{\|w_X(t)\|^2\} = E\{\|w_Z(t)\|^2\} = 0.1$, and $a = 0.1$. The length of time series was set to $T = 1000$. We considered two cases for estimating $A_{11}$ using linear regression in (6) with lag length $l = 1$ and $l = 3$. Let $\hat{A}_{11}$ be the output of linear regression. We computed $Supp(\hat{A}_{11})$ by setting entry $(i, j)$ to one if $|\hat{a}_{ij}| > a/2$. In Figure 4a, the expected estimation error, i.e. $\|Supp(\hat{A}_{11}) - Supp(A_{11})\|_F^2/n^2$, is computed where $\|\cdot\|_F$ is the Frobenius norm. As it can be seen, the estimation error decreases as we increase the lag length.

We also studied the effect of observed to latent noise power ratio (OLNR), $E\{\|w_X(t)\|^2\}/E\{\|w_Z(t)\|^2\}$, in estimating the linear measurements. We generated 1000 instances of DRG(0.1, 0.1) with $n = 10$, $m = 5$, and $a = 0.5$. Figure 4b illustrates $\max_{k} \|Supp(\hat{A}_{k}) - Supp(A_{k})\|_F^2$, as a function of OLNR. As it can be seen, the average of maximum estimation error decreases as OLNR increases which is expected from Proposition 2.

We investigated what percentage of instances of random graphs satisfy Assumption 1. We generated 1000 instances of DRG($p, 1/n$) with $n = 100$, and $p \in [0.04, 0.2]$. In Figure 5, the probability of satisfying Assumption 1, $P_{sat.}$, is depicted versus $p$ for different number of latent variables in the VAR model. As it can be seen, for large value of $m$, the probability $P_{sat.}$ decreases. This is because it becomes less likely to see a unique observed parent for each latent node. For a fixed number of latent nodes, the same event will occur if we increase $p$. Furthermore, for small $p$, there might exist some latent nodes that have no observed parent or no observed children.

We also evaluated the performance of the NM algorithm in random graphs. We generated 1000 instances of DRG($1/2n, 1/2n$) with $n = 10, 20, ..., 100$, $m = n/2$, and computed the linear measurements. If for a class of connected nodes, the number of latent nodes generated in the initial phase exceeds 40, we assumed that the corresponding instance cannot be recovered efficiently in time and did not proceed to the merging phase. In Figure 6a, we depicted the percentage of instances in which the algorithm can recover all possible minimal unobserved networks. As it is shown, large portion of instances (at least 96.9%) can be recovered even for the case $n = 100$. In Figure 6b, the average run time of the algorithm is depicted\(^6\). This plot shows that we can recover all possible minimal unobserved networks for a large portion of instances efficiently even in relatively large networks. This observation is not surprising since we know that the size of each connected class nodes is of order $\log(n)$ in sparse random graphs (Erdos & R´enyi, 1960).

\(^6\)This experiment was performed on a on a Mac Pro with $2 \times 2.4$ GHz 6-Core Intel Xeon processor and 32 GB of RAM.
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Figure 6. Recovering the minimal unobserved network: Results are averaged over 1000 instances of DRG(p,q) where \( n = 10, 20, \cdots, 100 \), \( m = n/2 \), and \( p = q = 1/(2n) \).

Figure 7. The histogram of \( ||\text{Supp}(\hat{A}_{11}) - \text{Supp}(A_{11})||^2 \) for high power and low power conditions.

5.2. US Macroeconomic Data:

We considered the following set of time series from the quarterly US macroeconomic data for the period from 31-Mar-1947 to 31-Mar-2009 collected from the St. Louis Federal Reserve Economic Database (FRED) (http://research.stlouisfed.org/fred2/): gross domestic product (GDP), gross domestic product price deflator (GDPDEF), paid compensation of employees (COE), non-farm business sector index of hours worked (HOANBS), three-month treasury bill yield (TB3MS), personal consumption expenditures (PCEC), and gross private domestic investment (GPDI).

We selected any four times series as observed processes and computed \( \text{Supp}(\hat{A}_{11}) \) with lag length \( l = 3 \). We divided the \( \binom{7}{4} = 35 \) possible selections into two classes: 1) High power: \( \text{tr}(\epsilon^2(\omega_X(t)\omega_X(t)^T)) > \tau \) for a fixed threshold \( \tau \). 2) Low power: \( \text{tr}(\epsilon^2(\omega_X(t)\omega_X(t)^T)) < \tau \). In this experiment, we set \( \tau = 0.02 \). In Figure 7, we plotted the histograms of \( ||\text{Supp}(\hat{A}_{11}) - \text{Supp}(A_{11})||^2 \) for these two classes. As it can be seen, in the high power regime, most of the possible selections have small estimation error.

We also considered the following six time series of US macroeconomic data during 1-Jun-2009 to 31-Dec-2016 from the same database: GDP, GPDI, PCEC, TBSMS, effective federal funds rate (FEDFUND), and ten-year treasury bond yield (GS10). We obtained the causal structure among these six time series using a linear regression with lag length \( l = 1 \) and considered the result as our ground truth (see Figure 8). Then, we removed GPDI from the dataset and considered the remaining five time series as observe processes. We performed a linear regression with lag length \( l = 2 \) to obtain the linear measurements and detected non-zero entries of linear measurements by considering a threshold of 2.2. Algorithm 1 recovered the ground truth in Figure 8 correctly.

5.3. Dairy Prices and West German Macroeconomic Data:

A collection of three US dairy prices has been observed monthly from January 1986 to December 2016 (http://future.aae.wisc.edu/tab/prices.html): milk price, butter price, and cheese price. We performed a linear regression with lag length \( l = 1 \) on the whole time series and considered the resulting graph as our ground truth (see Figure 9a). We used 0.25 as the threshold to detect the non-zero entries of the coefficient matrix. Next, we omitted the butter prices from the dataset and considered the milk price and cheese prices as observed processes. We performed the linear regression with lag length \( l = 2 \) and detected the nonzero entries with a threshold of 0.15. The linear measurements were: \( \text{Supp}(A^*) = \text{Supp}(A_{11}) = [1, 1; 1, 0] \) and \( \text{Supp}(A^*_2) = [0, 0; 1, 0] \). Algorithm 1 recovered correctly the true causal graph using this linear measurements.

We also considered the quarterly West German consumption expenditures \( X_1 \), fixed investment \( X_2 \), and disposable income \( X_3 \) during 1960-1982 (http://www.jmulti.de/data_imtsa.html). Similar to the previous experiment with dairy prices, we found entire causal structure among \( \{X_1, X_2, X_3\} \) using a threshold of 0.2. Figure 9b depicts the resulting graph. Next, we considered \( X_3 \) to be latent and used \( \{X_1, X_2\} \) to estimate the linear
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Figure 9. The true causal structure.

measurements $\text{Supp}(A^*_0) = \text{Supp}(A_{11}) = [0, 0; 1, 1]$ and $\text{Supp}(A^*_1) = [1, 0; 1, 0]$, where the threshold for detecting nonzero entries was set to 0.1. Using this linear measurements, Algorithm 1 recovered correctly the true network in Figure 9b.

6. Conclusion

We considered the problem of causal inference from observational time series data. Our approach consisted of two parts: First, we studied sufficient conditions under which the causal properties of the underlying system is identifiable. Second, we proposed two algorithms that recover the causal structures that satisfy the sufficient conditions from the first part.

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A. Proof of Proposition 1

The set of equation in (5), can be written in a matrix form as follows
\[
\begin{bmatrix}
\tilde{A} \\
\vdots \\
\tilde{A}_2(t-1) \\
\end{bmatrix} = C \begin{bmatrix}
\tilde{X}(t) \\
\vdots \\
\tilde{X}(t) \\
\end{bmatrix} + \begin{bmatrix}
\tilde{N}_Z(t) \\
\vdots \\
\tilde{N}_Z(t) \\
\end{bmatrix},
\]
where \( \tilde{A} = diag(\tilde{A}_0, ..., \tilde{A}_{l-1}) \), and C a block matrix with \( C_s \) as its \((s, r)\)th block for \( s = 0, ..., l-1 \) and \( k = 0, ..., l \). Since \( \tilde{N}_Z \) and \( \tilde{X} \) are orthogonal, we imply
\[
\|\tilde{A}\Gamma_{\omega_2}(l-1)\tilde{A}^T\|_2 \geq \|\Gamma_X(l)C^T\|_2.
\]
Using (12) and the relationship between \( \ell_2 \) and \( \ell_1 \) norms of a matrix, we obtain
\[
\lambda_{\max}(\Gamma_{\omega_2}(l-1)) \|\tilde{A}\|_2^2 \geq \lambda_{\min}(\Gamma_X(l)) \|\Gamma\|_1^2/(nl)
\]
where \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) denote the minimum and maximum eigenvalues of a given matrix, respectively. Since \( \Gamma_X(l) \) and \( \Gamma_{\omega_2}(l-1) \) are block-Toeplitz matrices, their eigenvalues can be bounded as follows (Gutiérrez-Gutiérrez et al., 2012):
\[
L := \inf_{\Omega \in [0, 2\pi]} \lambda_{\min}(\mathcal{F}(\gamma_X)) \leq \lambda_{\min}(\Gamma_X(l)),
\]
\[
M := \sup_{\Omega \in [0, 2\pi]} \lambda_{\max}(\mathcal{F}(\gamma_{\omega_2})) \geq \lambda_{\max}(\Gamma_{\omega_2}(l-1)),
\]
where \( j \) denotes \( \sqrt{-1} \). Using (13)-(15) and the fact that \( \tilde{A} \) is diagonal and \( \|A_{22}\|_2 < 1 \), we obtain
\[
\sqrt{nl/M} \|A_{12}\|_2 \geq \sqrt{nl/M} \max_{\nu \leq j-1} \|A_{22}\|_2 \|\Gamma\|_1.
\]

From (6), we have \( B_1^* - A_k = \sum_{i=1}^{l-1} C_{i1} \), where the right side hand side can be obtained by summing up the appropriate columns of matrix \( C \). This implies that \( \max_{0 \leq k < l} \|B_k^* - A_k\|_1 \leq \|\Gamma\|_1 \). Combining this inequality and the bound in (16) concludes the result.

B. Proof of Proposition 2

The spectral density of matrix \( \gamma_X(h) \) can be computed as follows:
\[
\mathcal{F}(\gamma_X) = \sigma_Z^2 F_X(\Omega) F_X(\Omega)^H + \sigma_Z^2 F_Z(\Omega) F_Z(\Omega)^H
\]
where \( F_X(\Omega) = [e^{j\Omega t_{\text{ini}}} - A_{11} - \sum_{k=0}^{l-1} A_k e^{-j\Omega t_{\text{ini}}}]^{-1}, \)
\( F_Z(\Omega) = F_X(\Omega)(A_{12} \sum_{k=0}^{l-1} A_k e^{-j\Omega t_{\text{ini}}}) \), and \( H \) denotes Hermitian of a matrix.

We define the function \( \psi_{\omega, x}(\cdot), \psi_{\omega, y}(\cdot) := \|v\| \mathcal{F}(\gamma_X)\|v\|^{-2} \)
where \( \|v\| \) is a unit vector. Suppose that \( (\Omega^*, \bar{v}^*) \) minimizes the function \( \psi_{\omega, x}(\cdot) \). By the definition of \( L \) and \( M \), the ratio \( M/L \) is equal to 1/\( \psi_{\omega, x}(\Omega^*, \bar{v}^*) \). Now if we decrease \( \frac{\sigma_Z^2}{\sigma_x^2} \) to \( \frac{\sigma_Z^2}{\sigma_x^2} \), then we have: \( \psi_{\omega, x}(\Omega^*, \bar{v}^*) < \psi_{\omega, x}(\Omega^*, \bar{v}^*) \).

Moreover, for the optimal solution \( (\Omega^*, \bar{v}^*) \) of \( \psi_{\omega, x}(\cdot) \), we know that: \( \psi_{\omega, x}(\Omega^*, \bar{v}^*) < \psi_{\omega, x}(\Omega^*, \bar{v}^*) \). Thus, we can conclude that: \( 1/\psi_{\omega, x}(\Omega^*, \bar{v}^*) \geq 1/\psi_{\omega, x}(\Omega^*, \bar{v}^*) \).

C. Proof of Theorem 1

First, we show such \( G \) has minimum number of latent nodes. We do this by means of contradiction. But first observe that since the latent subnetwork of \( G \) is a directed tree, we can assign a non-negative integer \( l_h \) to latent node \( h \) that represents the length of longest directed path from \( h \) to its latent descendants. Clearly, all such descendants are leaves which we denote them by \( L_h \). For instance, if the latent subnetwork of \( G \) is \( a \rightarrow b \rightarrow c \), then \( l_a = 2 \) and \( L_a = \{c\} \).

Suppose that \( G \) contains \( m \) latent nodes \( \{h_1, ..., h_m\} \) and there exists another network \( G_1 \) (not necessary with tree-structure induced latent subgraph), with \( m_1 < m \) number of latent nodes that it is also consistent with the same linear measurements as \( G \). Due to assumption (i), there is at least \( m \) distinct observed nodes that have out-going edges to the latent subnetwork. More precisely, each \( h_i \) has at least a unique observed node as its parent. We denote a unique observed parent of node \( h_i \) by \( o_i \).

Because \( m_1 < m \), there exists at least one observed node in \( \bar{O} := \{o_1, ..., o_m\} \) that has shared its latent children with some other latent nodes in \( G_1 \). Among all such observed nodes, let \( o_{i_1} \) to be the one\(^7\) that its corresponding latent node in \( G, (h_{i_1}) \) has maximum \( l_{h_{i_1}} \). Furthermore, let \( \bar{I}_{i_1} \subset \{1, ..., m\} \setminus \{i_1\} \) to be the index-set of those observed nodes that \( o_{i_1} \) has shared a latent child with them in \( G_1 \).

By the choice of \( o_{i_1} \), we know that \( h_{i_1} \leq h_{i_2} \), for all \( j \in \bar{I}_{i_1} \) and if for some \( 1 \leq k \leq m \), \( l_{h_{i_k}} > l_{h_{i_1}} \), then \( o_{i_1} \) has not shared its latent child in \( G_1 \) with any other observed nodes in \( \bar{O} \). Moreover, there should be at least a latent node \( h_{j_1} \) where \( j_1 \in \bar{I}_{i_1} \) such that \( h_{j_1} = h_{i_1} \). Otherwise, \( G_1 \) will not be consistent with the linear measurements of \( G \). Let \( \bar{I}_{i_1} := \{j : h_{j} = h_{i_1}\} \cap \bar{I}_{i_1} \). Because, \( o_{i_1} \) shares its latent children with \( \cup_{j \in \bar{I}_{i_1}} o_j \) in \( G_1 \) and the fact that both \( G \) and \( G_1 \) consistent with the same linear measurements, then the following holds in graph \( G \),

\[
\mathcal{C}_{\bar{O}}^{\bar{O}}(G) \subseteq \bar{U}_{j \in \bar{I}_{i_1}} \mathcal{C}_{\bar{O}}^{\bar{O}}(G).
\]

\(^7\)If there are several such observed node, let \( o_{i_1} \) to be one of them.
where \( C^O_{h_i}(G) \) indicates the set of observed children of the set \( L_{h_i} \). This indeed contradicts with assumption (ii).

**D. Proof of Theorem 2**

First, we require the following definition. For a network \( G \) with corresponding latent sub-network that is a tree, we define \( U_k(G) := \{ h \in G : h_1 = k \} \). To prove the equivalence, suppose there exists another network \( G_2 \) such that its latent sub-network is a tree and has minimum number of latent nodes. Let \( \{ h_1, ..., h_m \} \) to denote the latent nodes in \( G \). Since \( G \) satisfies Assumption (i), for every latent node \( h_i \) there exists a unique observed node \( o_i \) such that \( o_i \in P_i^O(G) \) and \( o_j \notin P_i^O(G) \) for all \( j \neq i \).

Since both \( G \) and \( G_2 \) are consistent with the same linear measurement, it is easy to observe that if \( h_i \in U_k(G) \), then \( o_i \) must have at least a latent child in \( G_2 \), say \( h' \), such that \( l_{h_i} = l_{h'} \). Note that \( l_{h_i} \) is computed in \( G \) and \( l_{h'} \) in \( G_2 \). Moreover, we must have:

\[
C^O_{L_{h_i}}(G) = \bigcup_{h' \in H'(o_i) \cap U_{h_i}(G_2)} C^O_{L_{h'}}(G_2),
\]

where \( H'(o_i) \) denotes the set of latent nodes in \( G_2 \) that have \( o_i \) as their observed parent. In other words, observed nodes that can be reached by a directed path of length \( l_i + 2 \) from \( o_i \) should be the same in both graph \( G \) and \( G_2 \). This results plus the fact that \( G \) satisfies Assumption (ii) imply:

I) For every \( h_i \in U_k(G) \), there exists a unique latent node \( h'_i \in U_k(G_2) \), such that \( o_i \in P_i^O(G_2) \) and \( o_j \notin P_i^O(G_2) \) for all \( j \neq i \), and

\[
C^O_{L_{h_i}}(G) = C^O_{L_{h'_i}}(G_2).
\]

Using I) and knowing that both \( G \) and \( G_2 \) have the same number of latent nodes, we obtain:

II) \( |U_k(G)| = |U_k(G_2)| \), for all \( k \).

Using I) and II), we can define a bijection \( \phi \) between the latent subnetworks of \( G \) and \( G_2 \) as follows \( \phi(h_i) = h' \). Using this bijection and Assumption (ii) of \( G \) conclude that if \( h \in U_k(G) \) is the common parent of \( \{ h_{j_1}, ..., h_{j_s} \} \subseteq U_{k-1}(G) \), then \( \phi(h) \in U_k(G_2) \) should be the common parent of \( \{ \phi(h_{j_1}), ..., \phi(h_{j_s}) \} \subseteq U_{k-1}(G_2) \) and the proof is complete.

**E. Proof of Lemma 1**

Suppose that \( o_i \) is the unique observed node of a latent node \( h_i \). Then, for any \( o_j \) such that \( l_i = l_j \), if \( h_i \) is not a child of \( o_j \), then from assumption ii we have \( R_j \subseteq R_i \). If \( h_i \) is a child of \( o_j \), then we know that \( l_i = l_j \), then \( M_i \subseteq M_j \) and \( R_i = R_j \).

Now, suppose that the observed node \( o_i \) satisfies conditions but it is not unique parent of any latent node. Let \( h_i \) and \( h'_i \) be children of \( o_i \). At least one of them, say node \( h_i \), can reach an observed node by a path of length \( l_i - 1 \). If \( h'_i \) has the same property, then consider the unique observed parent of \( h'_i \), say node \( o_j \). Based on Assumption (ii), we have \( R_j \subseteq R_i \), which is in contradiction with the assumption that node \( o_i \) satisfies conditions of Lemma. Moreover, if \( h'_i \) does not have a path to observed node with a length of \( l_i - 1 \), then for any observed parent of \( h_i \), one of the conditions in the Lemma is not satisfied. Thus, the proof is complete.

**F. Proof of Proposition 3**

Notice that the first loop in Algorithm 1 uses the result of Lemma 1 and finds all the latent nodes and their corresponding unique observed parents. The next loop uses the fact that the latent sub-network is a tree and also it satisfies Assumption 1. Hence, if there exist two latent nodes \( h \) and \( h' \), one with depth \( i \) and the other one with depth \( i+1 \), such that \( R_h \subseteq R_{h'} \), then \( h' \) must be the parent of \( h \) in the latent sub-network.

Moreover, since each latent node has a unique observed parent, using \( A_{12}^1 \), Algorithm 1 can identify all the observed children of a latent node. Finally, the last loop in this algorithm locates the rest of observed nodes as the input of the right latent nodes. The algorithm does it by using the fact that if an observed node \( i \) shares a latent child with another observed node \( j \in U \), then \( M_j \subseteq M_i \). Clearly, if the true unobserved network satisfies Assumption 1, the output of this algorithm will have a latent sub-network that is a tree and consistent with the linear measurement. Thus, by the result of Theorem 1, it will be the same as the true unobserved network up to some permutations in \( Supp(A_{21}) \).

**G. Proof of Theorem 3**

Consider the instance of the problem where \( A_{22} = 0_{m \times m} \). Without loss of generality, we can assume that entries of \( A_{12} \) and \( A_{21} \) are just zero or one. Thus, we need to find \( [A_{12}]_{n \times k} \) and \( [A_{21}]_{k \times n} \) such that \( Supp(A_{12}A_{21}) = Supp(A_k^1) \) and \( k \) is minimum. We will show that the set basis problem (Johnson, 1985) can be reduced to the decision version of finding the minimal unobserved network which we call it the latent recovery problem. But before that, we define the set basis problem:

*The Set Basis Problem* (Johnson, 1985): given a collection \( C \) of subsets of a finite set \( U = \{1, \cdots, n\} \) and an integer \( k \), decide whether or not there is a collection \( B \subseteq 2^U \) of at most \( k \) sets such that for every set \( C \in C \), there exists a collection \( B_C \subseteq B \) where \( \bigcup_{B_C \subseteq B} B = C \).

Any instance of the basis problem can be reduced to an instance of latent recovery problem. To do so, we encode
any set $C$ in collection $C$ to a row of $A_1^n = A_{12}A_{21}$ where $i$-th entry is equal to one if $i \in C$, and otherwise zero. It is easy to verify that the rows of matrix $A_{21}$ correspond to sets in collection $B$ if there exist a solution for the basis problem. Since the basis problem is NP-complete, we can conclude that finding the minimal unobserved network is NP-hard.

H. Proof of Theorem 4

Consider a minimal unobserved network $G_{\text{min}}$. Pick any latent node $i'$ which its in-degree or out-degree is greater than one. Let $V_i^-$ and $V_i^+$ be the sets of nodes that are going to and incoming from node $i'$, respectively. We omit the node $i'$ and create $|V_i^-| \times |V_i^+|$ latent nodes $\{i'_jk' | j' \in V_i^-, k' \in V_i^+\}$. We also add a direct link from node $j' \in V_i^-$ to $i'_jk'$ and from $i'_jk'$ to $k' \in V_i^+$ in order to be consistent with measurements. We continue this process until there is no latent node with in-degree or out-degree greater than one. Since there exists at most one path with length $k$ from any observed node to another observed node, the resulted graph is exactly equal to graph $G_0$. Hence we can construct the minimal graph $G_{\text{min}}$ just by reversing the process of generating latent nodes from $G_{\text{min}}$ to merging latent nodes from $G_0$. But the NM algorithm consider all the sequence of merging operations. Thus, $G_{\text{min}}$ would be in the set $G_{\text{out}}$ and the proof is complete.