Complete Affine Kähler Manifolds

Fang Jia and An-Min Li

Abstract. In this paper we prove that for a complete, connected and oriented Affine Kähler manifold \((M, G)\) of dimension \(n\), if it is affine Kähler Ricci flat or if the affine Kähler scalar curvature \(S \equiv 0\), \((n \leq 5)\), then the affine Kähler metric is flat.

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Introduction

An affine manifold is a manifold which can be covered by coordinate charts so that the coordinate transformations are given by invertible affine transformations. Let \(M\) be such affine manifold. We shall always assume that our coordinate systems are chosen as above and we call it affine coordinates. Let \(M\) be an affine manifold. An affine Kähler metric on \(M\) is a Riemannian metric on \(M\) such that locally, for affine coordinates \((x_1, x_2, \cdots, x_n)\), there is a potential \(f\) such that

\[
G_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}.
\]

The pair \((M, G)\) is called an affine Kähler manifold.

It is easy to see that the tangent bundle of an affine manifold is naturally a complex manifold. For each coordinate chart \((x_1, x_2, \cdots, x_n)\), if we write a tangent vector of \(M\) as \(\sum y_i \frac{\partial}{\partial x_i}\), then

\[
z_i = x_i + \sqrt{-1} y_i, \ i = 1, 2, \cdots, n
\]

are local holomorphic coordinates of \(TM\). The affine Kähler metric naturally extends to be a Kähler metric of the complex manifold. The Ricci curvature and the scalar curvature of this Kähler metric are given respectively by

\[
K_{ij} = -\sum \frac{\partial^2}{\partial x_i \partial x_j} \left(\log \det (f_{kl})\right),
\]

\[
S = -\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} f_{ij} \frac{\partial^2 \log \det (f_{kl})}{\partial x_i \partial x_j},
\]

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where $f_{kl} = \frac{\partial^2 f}{\partial x_k \partial x_l}$. Following Cheng and Yau ([2]) we call $K_{ij}$ and $S$ the affine Kähler Ricci curvature and the affine Kähler scalar curvature of $(M, G)$ respectively. We say that the affine Kähler metric $G$ is Einstein if its Ricci tensor is a scalar multiple of the affine Kähler metric, that is

$$-rac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = af_{ij},$$

where $a$ is a constant. In particular, if $a = 0$ then we call $(M, G)$ affine Kähler Ricci flat.

Our main results can be stated as follows:

**Theorem 1.** Let $(M, G)$ be an $n$-dimensional complete, connected and oriented $C^\infty$ affine Kähler Ricci flat manifold. Then the affine Kähler metric is flat.

**Theorem 2.** Let $(M, G)$ be a complete, connected and oriented $C^\infty$ affine Kähler manifold of dimension $n$. If the affine Kähler scalar curvature $S \equiv 0$, then, for $n \leq 5$, the affine Kähler metric is flat.

As consequences we have

**Theorem 3.** Let $f(x_1, ..., x_n)$ be a smooth and strictly convex function defined in $\Omega \subset \mathbb{R}^n$. If the affine Kähler Ricci curvature is identically 0, and if the Calabi metric is complete, then $f$ must be a quadratic polynomial.

**Theorem 4.** Let $f(x_1, ..., x_n)$ be a smooth and strictly convex function defined in $\Omega \subset \mathbb{R}^n$. If the affine Kähler scalar curvature $S \equiv 0$, and if the Calabi metric is complete, then, for $n \leq 5$, $f$ must be a quadratic polynomial.

**Remark.** This is an unpublished paper that was finished in 2005. Since then, the formula of $\Delta \Phi$ (cf. Proposition 1) has been used frequently in various circumstances. Since it is used in our recent papers again (see [9, 10]), we decide to put this original version (with slight revision) on Arxiv.

# 1 Fundamental formulas

Let $(M, G)$ be an affine Kähler manifold. Choose a local affine coordinate system $(x_1, ..., x_n)$. Let $f(x)$ be a local potential function of $G$. Then $f$ is locally strictly convex function and

$$G = \sum_{i,j} f_{ij} dx_i dx_j.$$
We recall some fundamental facts on the Riemannian manifold \((M, G)\) (cf. [9]). The Levi-Civita connection is given by
\[
\Gamma^k_{ij} = \frac{1}{2} \sum f^{kl} f_{ijl}.
\]
The Fubini-Pick tensor is
\[
A_{ijk} = -\frac{1}{2} f_{ijk}.
\]
Then the curvature tensor and the Ricci tensor are
\[
\begin{align*}
R_{ijkl} &= \sum f^{mh}(A_{jkm}A_{hil} - A_{ikm}A_{hjl}) \\
R_{ik} &= \sum f^{mh}f_{jl}(A_{jkm}A_{hil} - A_{ikm}A_{hjl}).
\end{align*}
\]

Let \(\rho = \left[\det(f_{ij})\right]^{-\frac{1}{n+2}}\). Set
\[
\Phi = \frac{||\nabla \rho||^2_G}{\rho^2}.
\]
\[
4n(n-1)J = \sum f^{il} f^{jm} f^{kn} f_{ijkl}. m
\]
It is easy to check that \(\Phi\) and \(J\) are independent of the choice of the affine coordinate systems. Hence they are invariants globally defined on \(M\). If \(\Phi \equiv 0\) then \(\rho = \text{constant}\). It is well known that (see [6])
\[
\Delta J \geq 2(n+1)J^2.
\]

Here and later the Laplacian and the covariant differentiation with respect to the metric \(G\) will be denoted by “\(\Delta\)” and “,” respectively.

## 2 Estimate for \(\Delta \Phi\)

In this section we calculate \(\Delta \Phi\) for affine Kähler manifold with \(S = 0\) and affine Kähler Ricci flat manifold.

**Proposition 1.** Let \((M, G)\) be an \(n\)-dimensional, connected and oriented \(C^\infty\) affine Kähler Ricci flat manifold. Then the following estimate holds

\[
\Delta \Phi \geq \frac{n}{n-1} \sum \frac{||\nabla \Phi||^2_G}{\Phi} + \frac{n^2 - 3n - 10}{2(n-1)} < \nabla \Phi, \nabla \log \rho >_G + \frac{(n+2)^2}{n-1} \Phi^2.
\]

**Proof.** Let \(p \in M\) be any fixed point. Choose an affine coordinate neighborhood \(\{U, \varphi\}\) with \(p \in U\). We have:
\[-\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = (n + 2) \left( \frac{\rho_{ij}}{\rho} - \frac{\rho_{i} \rho_{j}}{\rho^2} \right),\]

where \(\rho_i = \frac{\partial \rho}{\partial x_i}\) and \(\rho_{ij} = \frac{\partial^2 \rho}{\partial x_i \partial x_j}\). Noting that \(-\frac{\partial^2}{\partial x_i \partial x_j} (\log \det (f_{kl})) = 0\), we obtain

\[
\frac{1}{\rho} \sum f^{ij} \rho_{ij} + \frac{n}{\rho^2} \sum f^{ij} \rho_i \rho_j - \frac{n + 1}{\rho^2} \sum f^{ij} \rho_i \rho_j = 0,
\]

where the matrix \((f^{ij})\) denotes the inverse matrix of the matrix \((f_{ij})\). Then we have

\[
\Delta \rho = \frac{n + 4}{2} \left( \frac{\nabla \rho}{\rho} \right)^2 G_{\rho}.
\]

We choose a local orthonormal frame field of the metric \(G\) on \(U\). Then

\[
\Phi = \sum \frac{(\rho_{ij})^2}{\rho^2}, \quad \Phi, i = 2 \sum \frac{\rho_{j} \rho_{ji}}{\rho^2} - 2 \rho, i \sum \frac{(\rho_{ij})^2}{\rho^3},
\]

\[
\Delta \Phi = 2 \sum \frac{(\rho_{ij})^2}{\rho^2} + 2 \sum \frac{\rho_{j} \rho_{ji}}{\rho^2} - 8 \sum \frac{\rho_{j} \rho_{i} \rho_{ji}}{\rho^3} - (n - 2) \left( \frac{\sum (\rho_{ij})^2}{\rho^4} \right).
\]

where we used (5). In the case \(\Phi(p) = 0\), it is easy to get , at \(p\),

\[
\Delta \Phi \geq 2 \sum \frac{(\rho_{ij})^2}{\rho^2}.
\]

Now we assume that \(\Phi(p) \neq 0\). Choose a local orthonormal frame field of the metric \(g\) on \(U\) such that

\[
\rho_{1}(p) = \left( \frac{\nabla \rho}{\rho} \right)_{G} (p) > 0, \quad \rho_{i}(p) = 0 \quad for \ all \ i > 1.
\]

Then

\[
\Delta \Phi = 2 \sum \frac{(\rho_{ij})^2}{\rho^2} + 2 \sum \frac{\rho_{j} \rho_{ji}}{\rho^2} - 8 \frac{(\rho_{1})^2}{\rho^3} (\rho_{1}) - (n - 2) \frac{(\rho_{1})^4}{\rho^4}.
\]

Applying an elementary inequality

\[
a_1^2 + a_2^2 + \cdots + a_{n-1}^2 \geq \frac{(a_1 + a_2 + \cdots + a_{n-1})^2}{n - 1}
\]

and (5), we obtain

\[
2 \sum \frac{(\rho_{ij})^2}{\rho^2} \geq 2 \frac{(\rho_{11})^2}{\rho^2} + 4 \sum_{i>1} (\rho_{i1})^2 + 2 \sum_{i>1} (\rho_{ii})^2
\]
\[ \geq 2\frac{(\rho_{11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{1i})^2}{\rho^2} + \frac{2(\Delta \rho - \rho_{11})^2}{n - 1} \rho^2 \]

\[ \geq \frac{2n}{n - 1} \frac{(\rho_{11})^2}{\rho^2} + 4\frac{\sum_{i>1}(\rho_{1i})^2}{\rho^2} - 2\frac{n + 4(\rho_1)^2\rho_{11}}{n - 1} \rho^3 + \frac{(n + 4)^2(\rho_1)^4}{2(n - 1)} \rho^4. \]  

(8)

An application of the Ricci identity shows that

\[ \frac{2}{\rho^2} \sum \rho_{ij} \rho_{jii} = 2(n + 4)\frac{(\rho_1)^2\rho_{11}}{\rho^3} - (n + 4)\frac{(\rho_1)^4}{\rho^4} + 2R_{11}\frac{(\rho_1)^2}{\rho^2}. \]  

(9)

Substituting (8) and (9) into (7) we obtain

\[ \Delta \Phi \geq \frac{2n}{n - 1} \sum \frac{(\rho_{11})^2}{\rho^2} + \left(2n - 2\frac{n + 4}{n - 1}\right)\frac{(\rho_1)^2\rho_{11}}{\rho^3} + 2R_{11}\frac{(\rho_1)^2}{\rho^2} + \left(\frac{(n + 4)^2}{2(n - 1)} - 2(n + 1)\right)\frac{(\rho_1)^4}{\rho^4} + 4\sum_{i>1}\frac{(\rho_{1i})^2}{\rho^2}. \]  

(10)

Note that

\[ \sum \frac{(\Phi_{,i}^2)}{\Phi} = 4\sum \frac{(\rho_{1i})^2}{\rho^2} - 8\frac{(\rho_1)^2\rho_{11}}{\rho^3} + 4\frac{(\rho_1)^4}{\rho^4}, \]  

(11)

Then (10) and (11) together give us

\[ \Delta \Phi \geq \frac{n}{2(n - 1)} \sum \frac{(\Phi_{,i}^2)}{\Phi} + \left(2n - 8\frac{n - 1}{n - 1} + 2n\right)\frac{(\rho_1)^2\rho_{11}}{\rho^3} + 2R_{11}\frac{(\rho_1)^2}{\rho^2} + \left(\frac{(n + 4)^2}{2(n - 1)} - 2(n + 1) - \frac{2n}{n - 1}\right)\frac{(\rho_1)^4}{\rho^4}. \]  

(12)

From \( \frac{\partial^2}{\partial x_i \partial x_j} \log \det(f_{kl}) = 0 \) we easily obtain

\[ \rho_{,ij} = \rho_{ij} + A_{ij1}\rho_{,1} = \frac{\rho_{,i}\rho_{,j}}{\rho} + A_{ij1}\rho_{,1}. \]

Thus we get

\[ \Phi_{,i} = \frac{2\rho_{11}\rho_{1i}}{\rho^2} - 2\frac{\rho_{,i}(\rho_1)^2}{\rho^3} = 2A_{111}\frac{(\rho_1)^2}{\rho^2}, \]  

(13)

\[ \sum \frac{(\Phi_{,i}^2)}{\Phi} = 4\sum (A_{111})^2 \frac{(\rho_1)^2}{\rho^2}, \quad \sum \Phi_{,i} \frac{\rho_{,i}}{\rho} = 2A_{111} \frac{(\rho_1)^3}{\rho^3}. \]  

(14)

By the same argument of (8) we have

\[ \sum (f_{ml1})^2 \geq (f_{111})^2 + 2\sum_{i>1} (f_{i11})^2 + \sum_{i>1} (f_{ii1})^2 \]
Proposition 2.

Let \((M, G)\) be an \(n\)-dimensional, connected and oriented \(C^\infty\) affine Kähler manifold with \(S \equiv 0\). We have

\[
\nabla \Phi \geq \frac{n}{2(n-1)} \|\nabla \Phi\|^2 + \frac{n^2 - 4}{n-1} < \nabla \Phi, \nabla \log \rho >_G + \frac{(n + 2)^2}{2} \left( \frac{1}{n-1} - \frac{n-1}{4n} \right) \Phi^2. \tag{18}
\]

Proof. From the proof of Proposition 1 we see that the equality (12) remains hold. On the other hand

\[
2R_{11}(p) \left( \frac{\rho_1}{\rho} \right)^2 \geq \frac{1}{2} \sum (f_{k1})^2 \left( \frac{\rho_1}{\rho^2} \right)^2 + \frac{n + 2}{2} f_{111} \left( \frac{\rho_1}{\rho^3} \right)^3
\]

\[
\geq \frac{1}{2} \left[ f_{111} + \frac{1}{n-1} \left( f_{111} + (n+2) \right) \frac{\rho_1}{\rho} \right]^2 \frac{\rho_1^2}{\rho^2} + \frac{n + 2}{2} f_{111} \frac{\rho_1^3}{\rho^3}
\]

\[
\geq \frac{(n + 2)^2}{2(n-1)} \Phi^2 - \frac{(n + 2)^2(n + 1)^2}{8n(n-1)} \Phi^2 \geq - \frac{(n + 2)^2(n - 1)(\rho_1)^4}{8n} \frac{1}{\rho^4}.
\]

This combined with (12) yields

\[
\nabla \Phi \geq \frac{n}{2(n-1)} \sum \left( \frac{\Phi, i}{\Phi} \right)^2 + \left( \frac{2n - 8}{n-1} + 2n \right) \left( \frac{\rho_1}{\rho^3} \right)^3
\]

\[
+ 2R_{11}(p) \left( \frac{\rho_1}{\rho} \right)^2 + \left( \frac{(n + 4)^2}{2(n-1)} - 2(n+1) - \frac{2n}{n-1} \right) \left( \frac{\rho_1}{\rho} \right)^4
\]

\[
\geq \frac{n}{2(n-1)} \sum \left( \frac{\Phi, i}{\Phi} \right)^2 + \frac{n^2 - 4}{n-1} \sum \Phi, j \frac{\rho, j}{\rho} + \frac{(n + 2)^2}{2} \left( \frac{1}{n-1} - \frac{n-1}{4n} \right) \left( \frac{\rho_1}{\rho} \right)^4.
\]
3 Proof of Theorems

It is well known that an affine complete, parabolic affine hypersphere must be a quadratic polynomial. Using (4) and the same argument we can get

Lemma 1. Let \((M, G)\) be a complete, connected and oriented \(C^\infty\) affine Kähler manifold of dimension \(n\). If \(\Phi \equiv 0\), then any local potential function \(f\) of \(G\) must be a quadratic polynomial.

Proof of Theorem 1. By Lemma 1 it suffices to prove that \(\Phi \equiv 0\). Consider the function

\[
F = (a^2 - r^2)^2 \Phi
\]

defined on \(B_a(p_0)\). Obviously, \(F\) attains its supremum at some interior point \(p^*\) of \(B_a(p_0)\). Then, at \(p^*\),

\[
\frac{\Phi_i}{\Phi} - 2 \frac{(r^2)_i}{a^2 - r^2} = 0. \tag{19}
\]

\[
\frac{\Delta \Phi}{\Phi} - \sum \frac{(\Phi_i)^2}{\Phi^2} - 2 \sum \frac{(r^2)_i^2}{(a^2 - r^2)^2} - 2 \frac{\Delta (r^2)}{a^2 - r^2} \leq 0. \tag{20}
\]

Inserting (19) into (20) we get

\[
\frac{\Delta \Phi}{\Phi} \leq 24 \frac{r^2}{(a^2 - r^2)^2} + \frac{4}{a^2 - r^2} + 4 \frac{r \Delta r}{a^2 - r^2}. \tag{21}
\]

Denote by \(a^* = r(p_0, p^*)\). In the case \(p^* \neq p_0\) we have \(a^* > 0\). Let

\[
B_{a^*}(p_0) = \{p \in M | r(p_0, p) \leq a^*\}.
\]

By (17) we have

\[
\max_{p \in B_{a^*}(p_0)} \Phi(p) = \max_{p \in \partial B_{a^*}(p_0)} \Phi(p).
\]

On the other hand, we have \(a^2 - r^2 = a^2 - a^*^2\) on \(\partial B_{a^*}(p_0)\), it follows that

\[
\max_{p \in B_{a^*}(p_0)} \Phi(p) = \Phi(p^*).
\]

Let \(p \in B_{a^*}(p_0)\) be any point. Then from the definition of \(R_{ik}\), we get

\[
R_{ii}(p) = \frac{1}{4} \sum f^{jl} f^{hm} (f_{hil} f_{mjl} - f_{hii} f_{mjl}) \geq - \frac{(n + 2)^2}{16} \Phi(p) \geq - \frac{(n + 2)^2}{16} \Phi(p^*). \tag{7}
\]
Thus, by Laplacian comparison theorem (see [6] Appendix 2), we obtain

\[ r \Delta r \leq (n - 1) \left( 1 + \frac{n + 2}{4} \sqrt{\Phi(p^*)} \right), \]

(22)

In the case \( p^* = p_0 \), we have \( r(p_0, p^*) = 0 \). Consequently, from (21) and (22), it follows that

\[ \frac{\Delta \Phi}{\Phi} \leq \left( 24 + \frac{(n - 1)^2(n + 2)^2}{4} \right) \frac{r^2}{(a^2 - r^2)^2} + \frac{4n}{a^2 - r^2} + \Phi. \]

(23)

On the other hand, by (17) we have

\[ \frac{\Delta \Phi}{\Phi} \geq -\frac{(n^2 - 3n - 10)^2}{(n - 1)^2} \frac{a^2}{(a^2 - r^2)^2} + \left( \frac{(n + 2)^2}{n - 1} - 1 \right) \Phi. \]

(24)

where we used (19). Inserting (24) into (23) we get

\[ (a^2 - r^2)^2 \Phi \leq C_1(n)a^2, \]

where \( C_1(n) \) is a constant depending only on \( n \). Hence, at any interior point of \( B_{\frac{a^2}{r}}(p_0) \), we have

\[ \Phi \leq \frac{16C_1(n)}{9a^2}. \]

Let \( a \to \infty \), then \( \Phi \equiv 0 \). We complete the proof of Theorem 1. ■

Applying a similar method and using the differential inequality (18) we can prove theorem 2.

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An-Min Li              Fang Jia
Department of Mathematics      Department of Mathematics
Sichuan University    Sichuan University
Chengdu, Sichuan      Chengdu, Sichuan
P.R.CHina             P.R.China
e-mail:math-li@yahoo.com.cn e-mail:jiafangscu@yahoo.com.cn