Macorealism from entropic Leggett-Garg inequalities

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We formulate entropic Leggett-Garg inequalities, which place constraints on the statistical outcomes of temporal correlations of observables. The information theoretic inequalities are satisfied if macrorealism holds. We show that the quantum statistics underlying correlations between time-separated spin component of a quantum rotor mimics that of spin correlations in two spatially separated spin-s particles sharing a state of zero total spin. This brings forth the violation of the entropic Leggett-Garg inequality by a rotating quantum spin-s system in similar manner as does the entropic Bell inequality (Phys. Rev. Lett. 61, 662 (1988)) by a pair of spin-s particles forming a composite spin singlet state.

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Conflicting foundational features like non-locality [1], contextuality [2] mark how quantum universe differs from classical one. Non-locality rules out that spatially separated systems have their own objective properties prior to measurements and they do not get influenced by any local operations by the other parties. Violation of Clauser-Horne-Shimony-Holt (CHSH) - Bell correlation inequality [3] by entangled states reveals that local realism is untenable in the quantum scenario. On the other hand, quantum contextuality states that the measurement outcome of an observable depends on the set of compatible observables that are measured alongside it. In this sense, non-locality turns out to be a reflection of contextuality in spatially separated systems.

Yet another foundational concept of classical world that is at variance with the quantum description is macrorealism [4]. The notion of macrorealism rests on the classical world view that (i) physical properties of a macroscopic object exist independent of the act of observation and (ii) measurements are non-invasive i.e., the measurement of an observable at any instant of time does not influence its subsequent evolution. Quantum predictions differ at a foundational level from these two contentions. In 1985, Leggett and Garg (LG) [4] designed an inequality (which places bounds on certain linear combinations of temporal correlations of a dynamical observable) to test whether a single macroscopic object exhibits macrorealism or not. The Leggett-Garg correlation inequality is satisfied by all macrorealistic theories and is violated if quantum law governs. Debates on the emergence of macroscopic classical realm from the corresponding quantum domain continue and it is a topic of current experimental and theoretical research [3,5].

The CHSH-Bell (LG) inequalities were originally formulated for dichotomic observables and they constrain certain linear combinations of correlation functions of spatially (temporally) separated states. Information theoretic considerations indicate that classical Shannon entropies associated with correlation outcomes obey certain constraints, violations of which would imply non-existence of a legitimate joint probability for all the measured quantities – which need not be dichotomic. Braunstein and Caves (BC) [6] developed information theoretic Bell inequality applicable to any pair of spatially separated systems. They showed that the inequality is violated by two spatially separated spin-s particles sharing a state of zero total angular momentum. More recently, Kurzyński et. al. [11] constructed an entropic inequality to investigate failure of non-contextuality in a single quantum three level system and they identified optimal measurements revealing violation of the inequality. It is highly relevant to address the question “Does the macrorealistic tenet encrypted in the form of classical entropic inequality get defeated in the quantum realm?” In this paper we formulate entropic LG inequalities to investigate the notion of macrorealism of a single system. We show that the entropic inequality is violated by a spin-s quantum rotor (prepared in a completely random state) in a manner similar to the information theoretic BC inequality for a counter propagating entangled pair of spin-s particles in a spin-singlet state.

We start with some basic elements of probabilities and the associated information content in order to develop the entropic LG inequality. Consider a macrorealistic system in which $Q(t_i)$ is a dynamical observable at time $t_i$. Let the outcomes of measurements of the observable $Q(t_i)$ be denoted by $q_i$ and the corresponding probabilities $P(q_i)$. In a macrorealistic theory, the outcomes $q_i$ of the observables $Q(t_i)$ at all instants of time exist irrespective of their measurement; this feature is mathematically validated in terms of a joint probability distribution $P(q_1, q_2, \ldots)$ characterizing the statistics of the outcomes; the joint probabilities yield the marginals $P(q_i)$ of individual observations at time $t_i$. Further, mea-
measurement invasiveness implies that the act of observation of $Q(t_i)$ at an earlier time $t_i$ has no influence on its subsequent value at a later time $t_j > t_i$. This demands that the joint probabilities are expressed as a convex combination of product of probabilities $P(q_i|\lambda)$, averaged over a hidden variable probability distribution $\rho(\lambda)$ [9, 12]:

$$P(q_1, q_2, \ldots, q_n) = \sum_\lambda \rho(\lambda) P(q_1|\lambda)P(q_2|\lambda)\ldots P(q_n|\lambda),$$

$$0 \leq \rho(\lambda) \leq 1, \sum_\lambda \rho_\lambda = 1; \quad 0 \leq P(q_i|\lambda) \leq 1, \sum_{q_i} P(q_i|\lambda) = 1.$$ (1)

Joint Shannon information entropy associated with the measurement statistics of the observable at two different times $t_k, t_{k+1}$ is defined as, $H(Q_k, Q_{k+1}) = -\sum_{q_k, q_{k+1}} P(q_k, q_{k+1}) \log_2 P(q_k, q_{k+1})$. The conditional information carried by the observable $Q_{k+1}$ at time $t_{k+1}$, given that it had assumed the values $Q_k = q_k$ at an earlier time is given by, $H(Q_{k+1}|Q_k = q_k) = -\sum_{q_{k+1}} P(q_{k+1}|q_k) \log_2 P(q_{k+1}|q_k)$. Here, $P(q_{k+1}|q_k) = P(q_k, q_{k+1})/P(q_k)$ denotes the conditional probability of finding the outcome $Q_{k+1} = q_{k+1}$, given that the result $Q_k = q_k$ has been realized at an earlier instant. The mean conditional information entropy is given by

$$H(Q_{k+1}|Q_k) = \sum_{q_k} P(q_k) H(Q_{k+1}|Q_k = q_k)$$

$$= -\sum_{q_k, q_{k+1}} P(q_k, q_{k+1}) \log_2 P(q_{k+1}|q_k)$$

$$= H(Q_k, Q_{k+1}) - H(Q_k).$$ (2)

The classical Shannon information entropies obey the inequality [10]:

$$H(Q_{k+1}|Q_k) \leq H(Q_{k+1}) \leq H(Q_k, Q_{k+1}),$$

$$H(Q_k, Q_{k+1}) \leq H(Q_k, Q_{k+1}, Q_{k+2}) = H(Q_{k+1}|Q_k) + H(Q_k),$$

$$H(Q_{k+1}|Q_k) \leq H(Q_k, Q_{k+1}, Q_{k+2}) = H(Q_{k+1}|Q_k) + H(Q_k)$$

$$H(Q_{k+1}|Q_k) \leq H(Q_k, Q_{k+1}, Q_{k+2}) = H(Q_{k+1}|Q_k) + H(Q_k),$$

$$H(Q_{k+1}|Q_k) \leq H(Q_k, Q_{k+1}, Q_{k+2}) = H(Q_{k+1}|Q_k) + H(Q_k).$$ (3)

The macrorealsim information underlying the statistical outcomes of the observable at $n$ different times must be consistent with the information associated with pairwise non-invasive measurements as given in [5].

Note that for even values of $n$, there is a one-to-one correspondence between the entropic inequality [5] of a single system and the information theoretic BC inequality [9] for two spatially separated parties (Alice and Bob). More specifically, let us consider $n = 4$ in [5] and associate temporal observable $Q_i$ with Alice’s (Bob’s) observables $A’, A’’$ as $Q_1 \leftrightarrow B$, $Q_2 \leftrightarrow A’, Q_3 \leftrightarrow B’, Q_4 \leftrightarrow A$ to obtain the BC inequality for a set of four correlations: $H(A|B) \leq H(A|B’) + H(B’|A’) + H(A’|B)$, which is satisfied by any local realistic model of spatially

$$H(Q_n|Q_1) \leq H(Q_n|Q_{n-1}) + H(Q_{n-1}|Q_{n-2}) + \ldots + H(Q_2|Q_1).$$ (5)
separated pairs.

We proceed to show that LG entropic inequality is violated by a quantum spin-s system. Consider a quantum rotor prepared initially in a maximally mixed state

\[ \rho = \frac{1}{2s+1} \sum_{m=-s}^{s} |s,m\rangle\langle s,m| \]  

(6)

where \( |s,m\rangle \) are the simultaneous eigenstates of the squared spin operator \( S^2 = S_x^2 + S_y^2 + S_z^2 \) and the z-component of spin \( S_z \) (with respective eigenvalues \( s(s+1) \) and \( mh \)); \( I \) denotes the \((2s+1) \times (2s+1)\) identity matrix. We consider the Hamiltonian

\[ H = \omega S_y, \]  

(7)

resulting in the unitary evolution \( U(t) = e^{-i\omega t S_y/\hbar} \) of the system (which corresponds to a rotation about the y-axis by an angle \( \omega t \)). We choose z-component of spin \( Q(t) = S_z(t) = U^\dagger(t)S_yU(t) \) as the dynamical observable for our investigation of macrorealism. Let us suppose that the observable \( Q_k = S_z(t_k) \) takes the value \( m_k \) at time \( t_k \). Correspondingly, at a later instant of time \( t_{k+l} \) if the spin component \( S_z(t_{k+l}) \) assumes the value \( m_{k+l} \), the quantum mechanical joint probability is given by

\[ P(m_k, m_{k+l}) = p_{m_k}(t_k) q(m_{k+l}, t_{k+l}|m_k,t_k). \]  

(8)

Here, \( p_{m_k}(t_k) = \text{Tr}[\rho \Pi_{m_k}(t_k)] \) is the probability of obtaining the outcome \( m_k \) at time \( t_k \); \( q(m_{k+l}, t_{k+l}|m_k,t_k) = \text{Tr}[\Pi_{m_k}(t_k)\rho \Pi_{m_{k+l}}(t_{k+l})\Pi_{m_{k+l}}(t_{k+l})/p_{m_k}(t_k)] \) denotes the conditional probability of obtaining the outcome \( m_{k+l} \) for the spin component \( S_z \) at time \( t_{k+l} \), given that it had taken the value \( m_k \) at an earlier time \( t_k \); \( \Pi_m(t) = U^\dagger(t)|s,m\rangle\langle s,m|U(t) \) is the projection operator measuring the outcome \( m \) for the spin component at time \( t \). For the maximally mixed initial state \( \rho \), we obtain the quantum mechanical joint probabilities as,

\[ P(m_k, m_{k+l}) = \frac{1}{2s+1} |\langle s, m_{k+l}|e^{-i\omega(t_{k+l}-t_k)}S_y|s, m_k\rangle|^2 \]  

(9)

where \( d_{m_{k+l}m_k}^{s}(\theta_{kl}) = \langle s, m'|e^{-i\theta_{kl} S_y/\hbar}|s, m \rangle \) are the matrix elements of the \( 2s+1 \) dimensional irreducible representation of rotation \( [13] \) about y-axis by an angle \( \theta_{kl} = \omega(t_{k+l}-t_k) \). The marginal probability of the outcome \( m_k \) for the observable \( Q_k \) is readily obtained by making use of the unitarity property of \( d \) matrices: \( P(m_k) = \sum_{m_{k+l}} P(m_k, m_{k+l}) = \frac{1}{2s+1} \sum_{m_{k+l}} |d_{m_{k+l}m_k}^{s}(\theta_{kl})|^2 \)  

Clearly, the temporal correlation probability \( [9] \) of quantum rotor is similar to the quantum mechanical pair probability \( [10] \)

\[ P(m_a, m_b) = \left[ \hat{a} \langle s, m_a | \otimes \hat{b} \langle s, m_b | \right] |\Psi_{AB}\rangle = \frac{1}{2s+1} |d_{m_a, m_b}^{s}(\theta_{ab})|^2 \]  

(10)

that Alice’s measurement of spin component \( \vec{S} \cdot \hat{a} \) yields the value \( m_a \) and Bob’s measurement of \( \vec{S} \cdot \hat{b} \) results in the outcome \( m_b \) in a spin singlet state \( |\Psi_{AB}\rangle = \frac{1}{\sqrt{2s+1}} \sum_{m=-s}^{s} (-1)^{s-m} |s, m \rangle \otimes |s, -m \rangle \) of a spatially separated pair of spin-s particles. (Here \( \theta_{ab} \) is the angle between the unit vectors \( \hat{a} \) and \( \hat{b} \).) In other words, quantum statistics of temporal correlations in a single spin-s rotor mimics that of spatial correlations in an entangled counter propagating pair of spin-s particles.

Let us consider measurements at equidistant time intervals \( \Delta t = t_{k+1} - t_k \), \( k = 1, 2, \ldots n \) and denote \( \theta = (n-1)\omega\Delta t \). The quantum mechanical information entropy depends only on the time separation, specified by the angle \( \theta \) and is given by,

\[ H(Q_k|Q_{k+1}) \equiv H[\theta/(n - 1)] = -\frac{1}{2s+1} \sum_{m_k, m_{k+1}} |d_{m_{k+1}m_k}^{s} [\theta/(n - 1)]|^2 \log_2 |d_{m_{k+1}m_k}^{s} [\theta/(n - 1)]|^2. \]  

(11)
The $n$-term entropic inequality (5) for observations at equidistant time steps assumes the form,

$$(n - 1) H[\theta/(n - 1)] - H(\theta) = -\frac{1}{2s+1} \sum_{m_k,m_{k+1}} [(n - 1) |d^n_{m_k,m_{k+1}}[\theta/(n - 1)]|^2 \log_2 |d^n_{m_k,m_{k+1}}[\theta/(n - 1)]|^2$$

$$-|d^n_{m_k,m_{k+1}}(\theta)|^2 \log_2 |d^n_{m_k,m_{k+1}}(\theta)|^2 \geq 0$$

(12)

We introduce information deficit, measured in units of $\log_2(2s + 1)$ bits, as

$$D_n(\theta) = \frac{(n - 1) H[\theta/(n - 1)] - H(\theta)}{\log_2(2s + 1)}$$

so that the violation of the LG entropic inequality (12) is implied by negative values of $D_n(\theta)$. The units $\log_2(2s + 1)$ for the quantity $D_n(\theta)$ imply that the base of the logarithm for evaluating the entropies of a spin $s$ system is chosen appropriately to be $(2s + 1)$. For a spin-1/2 rotor, it is in bits.

In Fig. 1, we have plotted information deficit $D_n(\theta)$ for $n = 3$ (Fig. 1a) and $n = 6$ (Fig. 1b) as a function of $\theta = (n - 1) \omega \Delta t$ for spin values $s = 1/2, 1, 3/2$ and 2. The results illustrate that the information deficit assumes negative values, though the range of violation (i.e., the value of the angle $\theta$ for which the violation occurs) and also the strength (maximum negative value of $D_n(\theta)$) of the entropic violation reduces (14) with the increase of spin $s$. This implies the emergence of macrorealism for the dynamical evolution of a quantum rotor in the limit of large spin $s$. It may be noted that Kolfer and Brukner [7] had shown, violation of the correlation LG inequality – corresponding to the measurement outcomes of a dichotomic parity observable in the example of a quantum rotor – persists even for large values of spin if the eigenvalues of spin can be experimentally resolved by sharp quantum measurements. However, under the restriction of coarse-grained measurements classical realm emerges in the large spin limit.

Macrorealism requires that a consistently larger information content $H[\theta/(n - 1)]$ has to be carried by the system, when number of observations $n$ is increased and small steps of time interval are employed; however, quantum situation does not comply with this constraint. To see this explicitly, consider the limit of $n \to \infty$ and infinitesimal time steps $\omega \Delta t \to 0$. Quantum statistics leads to vanishingly small information i.e., $H(\frac{\theta}{n-1}) \to 0$ – a signature of quantum Zeno effect. In this limit, the information deficit (see (13)) $D_n(\theta) \to -\frac{H(\theta)}{\log_2(2s + 1)}$ is negative – thus violating the entropic LG inequality. The entropic test clearly brings forth the severity of macrorealistic demands towards knowing the observable in a non-invasive manner under such miniscule time scale observations.

In conclusion, we have formulated entropic LG inequality, which places bounds on the amount of information associated with non-invasive measurement of a macroscopic observable. This entropic formulation can be applied to any observables – not necessarily dichotomic ones – and it puts to test macrorealism i.e., a combined demand of the pre-existence of definite values of the measurement outcomes of a given dynamical observable at different instants of time – together with the assumption that act of observation at an earlier instant does not influence the subsequent evolution. Macrorealism requires that the statistical outcomes of measuring an observable at consecutive time intervals is characterized by a valid joint
probability distribution of the form 1. Non-existence of such a relevant joint probability distribution in the quantum scenario can show up as a violation of correlation as well as the entropic LG inequality. To demonstrate violation of the entropic inequality, we have considered the dynamical evolution of a quantum spin system prepared initially in a maximally mixed state. We have shown that the entropic violation in a quantum rotor system is similar to that of a spatially separated pair of spin-s particles sharing a state of total spin zero 10. Further, the information content of a rotor of large spin is shown to be consistent with the requirement of macrorealism.

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[14] The maximum negative value of information deficit (strength of violation of the entropic inequality) – evaluated in bits – increases with spin-s. However, in units of log2(2s + 1) it reveals a reverse trend – thus suggesting the emergence of macrorealism in the classical limit. Similar feature was identified in the violation of entropic BC inequalities in the large spin limit. See A. R. Usha Devi, J. Phys. A: Math. Gen. 33, 227 (1999).