Research on Stiffness Measurement of Spring Tubes Based on Three-Dimensional Conformal Contacts Model

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Abstract. A contact model and corresponding numerical method are proposed in this paper to study the influences of contact deformations on stiffness measuring precision of spring tubes. Firstly, measuring principle and the force analysis are presented. Then, the contact model is set up by simplifying the contact force and considering the real contact condition. Lastly, a numerical method is developed to solve the contact equation, and quantizing relationships between measurement errors and the contact deformation are derived subsequently. It’s proved by simulation and tests that the contact model is more accurate than Herzian contact theory and the contact deformation can make stiffness values lower than its true ones.

1. Introduction
Spring tubes are one kind of thin-wall spring component. Their stiffness has direct relation with the performance of a control system, such as control accuracy and sensitivity, etc. It is unable to set up an accurate mathematical model for their stiffness. Therefore, measuring is the only way to guarantee the stiffness. Now, there are mainly three methods to measure the spring tubes’ stiffness at the field of production: manual measurement; semi-automatic measurement, which is proposed by Tao [1]; and Wang [2] proposed the automatic measurement method on the basis of Tao. The later two methods have taken the place of manual measurement little by little for their high precision and efficient in measuring.

The stiffness-measuring instrument designed by Wang is taken to study in this paper. Firstly, we introduce the principle of measurement. Secondly, a contact model is set up by taking account of surface roughness and one numerical method is proposed to solve the system of contact equations. Thirdly, the error compensation model is built in order to obtain the relation between the contact deformation and the stiffness. Lastly, through the simulation and tests, it’s proved that the contact model and error compensation model are effectual. And the contact model and corresponding numerical algorithm also can be applied to solve the conformal contact problem between holes and shifts with similar dimension.

2. Principle of stiffness measurement and analysis of contact force
The principle of spring tube stiffness measurement is shown in Figure 1. The flange of the spring tube 3 is fastened on the mounting clamp (not shown), and the head of the spring tube is installed in the clamping bar 2, whose stiffness is much greater than the spring tube. The position of the application point is guaranteed by the spacer pin 4. By using a torque spanner to screw up the carrier rod 6, the head of spring tube is clamped into the clamping bar under the action of the footstock 7. Exerting the
load on iron ball 1 automatically and choosing \( n \) points for measuring, we can achieve the Torque-Angle stiffness computational formula of the spring tubes

\[
K = \frac{\sum_{i=1}^{n} F_i S_i - \frac{1}{n} \left( \sum_{i=1}^{n} S_i \right) \left( \sum_{i=1}^{n} F_i \right)}{\left( \sum_{i=1}^{n} S_i^2 - \frac{1}{n} \left( \sum_{i=1}^{n} S_i \right)^2 \right) \times L^2} \text{ (Nm/rad)}
\]  

(1)

Where \( F_i \) is the measuring force applied on the \( i \)-th point, which is detected by the force sensor. \( S_i \) is deformation of spring tube’s thin-wall at the \( i \)-th point’s, which is picked up by displacement sensor. \( i = 1,2,\cdots,n \). And \( L \) is the arm of the force.

During the measurement, the head of spring tube is keeping on contacting with the hole of clamping bar and the contact forces are mainly caused by both the measuring force \( F \) detected by displacement sensor and pretightening force \( F^* \), in which \( F^* \) is caused by the screw down torque \( T \). So we can acquire the distributed forces caused by \( F \) and \( F^* \) as shown in Figure 2.

3. Contact model and its numerical solution

As is shown in Figure 3, by the action of the load \( P \) at the center of the head of the spring, the head of spring comes into contact with the hole of the clamping bar. The total displacement can be represented by a rigid-body motion plus a superposed deformation which decays rapidly with distance from the region of contact. We assume that the rigid-body motion of spring tubes’ heads relative to clamping bars’ holes consists of a translation through distance \( \delta \) in the radial direction. The quantity \( \delta \) is called the rigid-body approach. Because the difference of the radius \( \Delta R \) is far less than \( R_1 \) and \( R_2 \) the contacts between them belong to closely conformal problem [3-5].

Let us consider two points \( M_1 \) and \( M_2 \) on the surfaces of two contact bodies. After deformation occurs, points \( M_1 \) and \( M_2 \) move to the same point \( M \). If \( \bar{u}_{r1} \) and \( \bar{u}_{r2} \) present the elastic placement of \( M_1 \) and \( M_2 \), then the conformal contact equation can be written as by considering the friction surfaces

\[
\bar{u}_{r2} - \bar{u}_{r1} = \delta - \Delta R(1 - \cos \phi) - h(r, \theta)
\]  

(2)

where \( h(r,\theta) = z_1(r,\theta) + z_2(r,\theta) \) is the clearance of contact surfaces caused by asperities of roughened surfaces. While \( z_1(r,\theta) \) and \( z_2(r,\theta) \) are the asperity heights functions of two contact surfaces.

Contact stress \( \sigma \) and deformations \( \delta \) are solved with classical Boussinesq formula [5]. Now, equation (2) will be

\[
\frac{1}{\pi E} \int_{A_c} \frac{p(\xi,\eta)}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} d\xi d\eta = \delta \cos \phi - \Delta R(1 - \cos \phi) - h(x, y)
\]  

(3)

where \( A_c \) is the projection of \( \sigma \) on the \( x-y \) plane, which is called the contact region.

And \( \frac{1}{E^*} = \frac{1-v_2^2}{E_2} - \frac{1-v_1^2}{E_1} \).
The contacting interface region is discretized into a mesh of small elements over the domain of entire contact area [6] and a piecewise interpolation of contact pressure distribution can be used inside each element. The various Lagrange polynomials can be used for piecewise interpolation of contact pressure inside each element. However, when a mesh element is small enough, the contact pressure at each element can be treated as a constant. Due to the relationships between the load $P$ and the total stress on the elements the equations set solving the contact problem can be expressed as

$$
\begin{align*}
&\frac{c}{E'} \sum_{k=1}^{M} \sum_{l=1}^{N} C_{kl} p_{kl} = \delta - \Delta R(1 - \cos \phi) - h_{ij} \\
&P = A \sum_{k=1}^{M} \sum_{l=1}^{N} p_{kl}
\end{align*}
$$

(4)

where roughness heights $h_{ij}$ can be obtained by Patir method [7].

We solve the equations set with overrelaxation iteration method. Then modify the stress solution acquired according as equation (4) and replace the stress values smaller than zero with zero until the iteration precision reaches to appointed accuracy. Lastly, the contact deformation $\delta$ and stress distribution will be output as shown in Figure 4.

4. Relations between contact deformations and measurement errors

Now, we establish the relations between the measurement result and contact deformation. Firstly, we analyze how $\delta'$, the deformation generated by $F'$, affects the measurement accuracy. Deformation $\delta'$ will shorten the length of the arm of force so that it will make the measuring results lower than its true value. So the valid arm $L'$ will be

$$
L' = L - \delta'
$$

(5)

Now, let’s study the contact deformation $\delta(h)$ caused by $F$. $\delta(h)$ can directly enlarge the deformation $S_i$ detected by displacement sensor at and measuring points $(S_i, F_i)$, to calculate the contact deformations $\delta_{max}$. Then the deformation detected by sensor at the measuring points $(F_i, S_i)$ can be modified as $S_i'$, whose form is as follows

$$
S_i' = S_i - \frac{2L}{h} \delta_{max}
$$

(6)

Now, by replacing $L$ and $S_i$ in equation (1) with $L'$ and $S_i'$, the true value of the stiffness $K'$ can be written as

$$
K' = \frac{\sum_{i=1}^{n} F_i (S_i - \frac{2(L - \delta')}{h} \delta_{i, max}) - \frac{1}{n} \sum_{i=1}^{n} (S_i - \frac{2(L - \delta')}{h} \delta_{i, max}) \sum_{i=1}^{n} F_i}{\sum_{i=1}^{n} (S_i - \frac{2(L - \delta')}{h} \delta_{i, max})^2 - \frac{1}{n} \left( \sum_{i=1}^{n} (S_i - \frac{2(L - \delta')}{h} \delta_{i, max}) \right)^2} \times (L - \delta')^2 \text{ (Nm/rad)}
$$

(7)
The measuring errors \( \varepsilon \) caused by total contact deformations can be defined as

\[
\varepsilon = \frac{K' - K}{K} \times 100\% \tag{8}
\]

Replacing \( K \) and \( K' \) with equation (1) and equation (7), we can obtain the relationships between errors \( \varepsilon \) and contact deformations.

Table 1. A comparison of different method obtained stiffness (“Origin” stands for the method proposed in section 2, “Hertian” stands for the Hertian compensation, “Conformal” stands for compensation proposed in this paper and “New” stands for the method without contact effect).

| Method   | Stiffness measured by 5 clamping repeatedly (Nm(rad)) | Average       |
|---------|---------------------------------------------|---------------|
| Origin  | 75.932                                      | 76.247        |
| Hertian | 82.767                                      | 83.237        |
| Conformal | 79.171                                      | 79.439        |
| New     | 80.653                                      | 80.422        |

5. Experimental results and conclusions

From equation (7), we can know that \( \delta_{\text{max}} \) will decrease the true values and \( \delta' \) will enlarge them. In practice, the measuring data prove that the non-repeatability errors increase so high that it can not guarantee the design index for some relatively great stiffness of spring tubes. So we developed a new method that can avoid the influences of contact deformations. Regarding the values measured by new method as true values, we choose one type of spring tubes for studying the relationships between errors and contact deformations, which are calculated by Hertian method and the contact model proposed in this paper respectively. Testing data and compensating results by these different methods are shown in Table 1. And normalized pressure of two method for conformal contact are shown in Figure 5. Figure 6 presents how \( \Delta k \), which is the difference of \( K' \) and \( K \), varies with the distributed pressure. From the test data, we can get the following conclusions:

1. The contact deformations are linearly proportional to the distributed pressure when the pressure is small for both Hertzian and Conformal methods and the deformations will make the measuring values lower than its true values.

2. With increase of the distributed force \( \Delta k \) increases in parabolic way, which means that effects of contact deformations on measuring results for the spring tubes whose stiffness is small are smaller than those with big values. So we have to compensate the measuring results or take new measuring method for those whose stiffness is big.

3. The analytical solution for conformal cylindrical contact proposed in this paper is more accurate because the hypothesis of Herzian contact disagrees with the true contact region.

![Figure 5](image1.png)  ![Figure 6](image2.png)
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