TRACE FUNCTIONS OF THE PARAFERMION VERTEX OPERATOR ALGEBRAS

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Abstract. The trace functions for the Parafermion vertex operator algebra associated to any finite dimensional simple Lie algebra $\mathfrak{g}$ and any positive integer $k$ are studied and an explicit modular transformation formula of the trace functions is obtained.

1. Introduction

This paper is a continuation of our study of the Parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated to any finite dimensional simple Lie algebra $\mathfrak{g}$ and positive integer $k$. In particular, we determine the modular transformation formula of the trace functions for the Parafermion vertex operator algebras.

The Parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is the commutant of the Heisenberg vertex operator algebra $M_\mathfrak{h}(k)$ in the affine vertex operator algebra $L\hat{\mathfrak{g}}(k, 0)$, and can also be regarded as the commutant of the lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ in $L\hat{\mathfrak{g}}(k, 0)$ where $Q_L$ is the lattice spanned by the long roots of $\mathfrak{g}$. While the $C_2$-cofiniteness of the Parafermion vertex operator algebras was proved in [20] and [2], the rationality was established in [17] with the help of a result in [5] on the abelian orbifolds for rational and $C_2$-cofinite vertex operator algebras. The irreducible modules of $K(\mathfrak{g}, k)$ were classified in [2] for $\mathfrak{g} = sl_2$ and in [17], [1] for general $\mathfrak{g}$, and the fusion rules were computed in [22] for $sl_2$ and in [1] for general $\mathfrak{g}$ with the help of quantum dimensions. See recent work in [6], [7], [19], [22], [3] and [28] for other topics on Parafermion vertex operator algebras.

The trace functions $Z_M(v, \tau)$ are the main objects in this paper where $M$ is an irreducible $K(\mathfrak{g}, k)$-module and $v \in K(\mathfrak{g}, k)$. Since $K(\mathfrak{g}, k)$ is rational and $C_2$-cofinite, the space spanned by $Z_M(v, \tau)$ for the irreducible modules $M$ affords a representation of the modular group $SL_2(\mathbb{Z})$ [30]. Our goal is to determine this representation explicitly. The irreducible $K(\mathfrak{g}, k)$-modules are labeled by $M^{\Lambda, \lambda}$. Here $\Lambda$ is a dominant weight of $\mathfrak{g}$ such that $\langle \Lambda, \theta \rangle \leq k$, $\langle \cdot, \cdot \rangle$ is the normalized invariant bilinear form on $\mathfrak{g}$ so that the squared length of a long root is 2, $\theta$ is the maximal root, $\lambda \in \Lambda + Q$ modulo $kQ_L$ and $Q$ is the root lattice. In the case $v = 1$ the functions $\chi_{M^{\Lambda, \lambda}}(\tau)$ are a special kind of branching functions studied previously in [29], [30]. Moreover, $\chi_{M^{\Lambda, \lambda}}(\tau)/\eta(\tau)^{\ell}$ is the string function.
in [29], [30] where $l$ is the rank of $\mathfrak{g}$. In fact, explicit modular transformation formulas for branching functions were obtained in [29]. Our main result in this paper is that the same transformation formula for the branching functions is valid for the trace function with 1 replaced by any $v \in K(\mathfrak{g}, k)$.

The main idea is to use the modular transformation formulas for the affine vertex operator algebra $\hat{L}_\mathfrak{g}(k, 0)$ and lattice vertex operator algebras $V_{\sqrt{k}\mathfrak{Q}_L}$. There are two modular transformation formulas for the affine vertex operator algebra $\hat{L}_\mathfrak{g}(k, 0)$ given in [29] explicitly and [35] abstractly. Using a transformation formula for the abstract theta functions studied in [31], [35], one can easily show that these two transformation formulas are the same. Unfortunately, this can only give the modular transformation formula for the character $\chi_{M, \lambda}(\tau)$, not for the one point function $Z_{M, \lambda}(v, \tau)$. Again, results in [31] on abstract theta functions including some vector $w$ help us to solve the problem. To explain what kind of vector $w$ is, we recall from [16] that if $V = \oplus_{n \geq 0} V_n$ is a rational, $C_2$-cofinite, simple vertex operator algebra of CFT type, then $V_1$ is a reductive Lie algebra whose rank (the dimension of a Cartan subalgebra) is less than or equal to the central charge. Fix a Cartan subalgebra $\mathfrak{h}$ of $V_1$, vector $w$ satisfies conditions $h_n w = 0$ for all $h \in \mathfrak{h}$ and $n \geq 0$. In the case $V = L_\mathfrak{g}(k, 0)$, the condition is exactly equivalent to $w \in K(\mathfrak{g}, k)$. The rest of the proof is similar to that given in [29] for the branching functions.

The paper is organized as follows: We review basics on vertex operator algebras and their modules in Section 2 including rationality, $C_2$-cofiniteness and modular transformation results on the trace functions associated to a rational and $C_2$-cofinite vertex operator algebra $V$. Following [35] and [31], we discuss Section 3 the modular transformation formula for generalized theta functions $\chi_i(w, v, q) = \text{tr}_{M^i} e^{2\pi i \sigma(v)} q^{L(0) - c/24}$ where $M^i$ is an irreducible $V$-module and $v \in \mathfrak{h}$ and $w$ is defined as before. In Section 4 we discuss the affine vertex operator algebras $L_\mathfrak{g}(k, 0)$ and the modular transformation formula of the specialized characters of the level $k$ integrable highest weight modules for the affine Kac-Moody algebra $\hat{\mathfrak{g}}$. We recall the various known results on the Parafermion vertex operator algebra $K(\mathfrak{g}, k)$ such as rationality and classification of irreducible modules. Section 6 deals with the modular transformation formula for the 1 point function $Z_{M, \lambda}(v, \tau)$. The $T$-matrix is easy and the most effort is on the $S$-matrix. The idea and method for finding the $S$-matrix is similar to that in [29] and [30].

2. Basics

In this section we review the basic on vertex operator algebra. Let $V = (V, Y, 1, \omega)$ be a vertex operator algebra (cf. [4] and [25]).

A vertex operator algebra $V$ is of CFT type if $V$ is simple, with respect to $L(0)$ one has $V = \oplus_{n \geq 0} V_n$ and $V_0 = \mathbb{C} 1$ [12].
A vertex operator algebra $V$ is called \( C_2 \)-cofinite if $\dim V/C_2(V) < \infty$ where $C_2(V)$ is the subspace of $V$ spanned by $u_{-2}v$ for $u, v \in V$ [36].

A weak $V$-module $M = (M, Y_M)$ is a vector space equipped with a linear map

$$Y_M : V \to (\text{End } M)[[z^{-1}, z]]$$

$$v \mapsto Y_M(v, z) = \sum_{n \in \mathbb{Z}} v_n z^{-n-1} \quad (v_n \in \text{End } M) \text{ for } v \in V$$

satisfying the following conditions for $u, v \in V$, $w \in M$:

$$v_n w = 0 \quad \text{for } n \in \mathbb{Z} \text{ sufficiently large;}$$

$$Y_M(1, z) = 1;$$

$$z_0^{-1} \delta \left( \frac{z_1 - z_2}{z_0} \right) Y_M(u, z_1) Y_M(v, z_2) - z_0^{-1} \delta \left( \frac{z_2 - z_1}{-z_0} \right) Y_M(v, z_2) Y_M(u, z_1)$$

$$= z_2^{-1} \delta \left( \frac{z_1 - z_0}{z_2} \right) Y_M(Y(u, z_0)v, z_2).$$

An (ordinary) $V$-module is a weak $V$-module $M$ which is $\mathbb{C}$-graded

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_\lambda$$

such that $\dim M_\lambda$ is finite and $M_{\lambda+n} = 0$ for fixed $\lambda$ and $n \in \mathbb{Z}$ small enough, where $M_\lambda$ is the eigenspace for $L(0)$ with eigenvalue $\lambda$:

$$L(0)w = \lambda w, \quad w \in M_\lambda.$$ 

Let $M$ be an ordinary $V$-module. We denote $v_{n-1}$ by $o(v)$ for $v \in V_n$ and extend to $V$ linearly. Then $o(v)M_\lambda \subset M_\lambda$ for all $\lambda \in \mathbb{C}$.

An admissible $V$-module is a weak $V$-module $M$ which carries a $\mathbb{Z}_+$-grading

$$M = \bigoplus_{n \in \mathbb{Z}_+} M(n)$$

($\mathbb{Z}_+$ is the set all nonnegative integers) such that if $r, m \in \mathbb{Z}$, $n \in \mathbb{Z}_+$ and $a \in V_r$ then

$$a_m M(n) \subseteq M(r + n - m - 1).$$

Note that any ordinary module is an admissible module.

A vertex operator algebra $V$ is called rational if any admissible module is a direct sum of irreducible admissible modules [9].

The following result is proved in [10] and [11].

**Theorem 2.1.** Assume that $V$ is rational.
(1) There are only finitely many inequivalent irreducible admissible modules $V = M^0, ..., M^p$ and each irreducible admissible module is an ordinary module. Each $M^i$ has weight space decomposition

$$M^i = \bigoplus_{n \geq 0} M^i_{\lambda_i + n}$$

where $\lambda_i \in \mathbb{C}$ is a complex number such that $M^i_{\lambda_i} \neq 0$ and $M^i_{\lambda_i + n}$ is the eigenspace of $L(0)$ with eigenvalue $\lambda_i + n$. The $\lambda_i$ is called the weight of $M^i$.

(2) If $V$ is both rational and $C_2$-cofinite, then $\lambda_i$ and central charge $c$ are rational numbers.

In the rest of this paper we assume that $V$ is a strong rational vertex operator algebra. That is $V$ satisfies the following:

(V1) $V = \bigoplus_{n \geq 0} V_n$ is a simple vertex operator algebra of CFT type,

(V2) $V$ is $C_2$-cofinite and rational,

(V3) The conformal weights $\lambda_i$ are nonnegative and $\lambda_i = 0$ if and only if $i = 0$.

Using the assumption (V3) we know that $V$ and its contragredient module $V'$ [24] are isomorphic $V$-modules. From [16] we know $V_1$ is a reductive Lie algebra with $[u, v] = u_0 v$ for $u, v \in V_1$. Moreover, the rank $V_1$ is less than or equal to the central charge $c$ of $V$. For any $u \in V_1$, $u_0$ is a derivation of $V$ in the sense that $[u_0, Y(v, z)] = Y(u_0 v, z)$ for any $v \in V$ and $u_0 \omega = 0$. Furthermore, $e^{u_0}$ is an automorphism of $V$.

Recall from [24] that a bilinear from $(\cdot, \cdot)$ on $V$ is called invariant if

$$(Y(u, z)v, w) = (u, Y(e^{z L(1)}(-z^{-2})L(0))v, z^{-1})w)$$

for $u, v, w \in V$. From our assumptions there is unique nondegenerate invariant bilinear form on $V$ [33]. We shall fix a bilinear form $(\cdot, \cdot)$ on $V$ so that $(u, v) = u_1 v$ for $u, v \in V_1$. It is clear from the definition that $(gu, gv) = (u, v)$ for any automorphism $g$ and $u, v \in V_1$.

The modular transformations of trace functions of irreducible modules of vertex operator algebras [36] are of main importance in this paper. Another vertex operator algebra structure $(V, Y[\cdot, z], 1, \omega - c/24)$ is defined on $V$ in [36] with grading

$$V = \bigoplus_{n \geq 0} V_{[n]}.$$ 

For $v \in V_{[n]}$ we write $\text{wt}[v] = n$. For $i = 0, ..., p$ we set

$$Z_i(v, q) = \text{tr}_{M^i} o(v) q^{L(0) - c/24}$$

which is a formal power series in variable $q$. The $Z_i(1, q)$ which is also denoted by $\text{ch}_q M^i$ is called the $q$-character of $M^i$.

The series $Z_i(v, q)$ converges to a holomorphic function $Z_i(v, \tau)$ on $\mathbb{H} = \{ \tau \in \mathbb{C} \mid \text{Im} \tau > 0 \}$ where $q = e^{2\pi i \tau}$ with $\tau \in \mathbb{H}$ [36].
Recall the modular group $SL_2(\mathbb{Z})$ and let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

**Theorem 2.2.** Let $V$ be a strong rational vertex operator algebra.

1. There is a group homomorphism $\rho : SL_2(\mathbb{Z}) \to GL_{p+1}(\mathbb{C})$ with $\rho(\gamma) = (\gamma_{ij})$ such that for any $0 \leq i \leq p$ and $v \in V$,

$$Z_i(v, \tau) = (c\tau + d)^{wt[v]} \sum_{j=0}^{p} \gamma_{ij} Z_j(v, \tau).$$

2. Each $Z_i(v, \tau)$ is a modular form of weight $wt[v]$ over a congruence subgroup.

Part (1) of the Theorem was obtained in [36] and Part (2) was established in [13]. The matrices $S = \rho_V \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right)$ and $T = \rho_V \left( \begin{array}{cc} 1 & 1 \\ 0 & 1 \end{array} \right)$ are respectively called the genus one $S$- and $T$-matrices of $V$.

### 3. Modular Invariance of the Generalized Theta Functions

We review the modular transformation formula of the theta functions defined on vertex operator algebra given in [35] and [31] for studying the modular invariance of trace functions for the parafermion vertex operator algebras.

Recall that $V_1$ is a reductive Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$ of $V_1$. Then the abelian Lie algebra $\mathfrak{h}$ acts on $M^i$ semisimply for all $i$. Following [35], we define the generalized theta functions as

$$T_i(v, u, q) = tr_{M^i} e^{2\pi i (u_0 + (u, u)/2)} q^{L(0) + u_0 + (u, u)/2 - c/24}$$

for $u, v \in \mathfrak{h}$. The bilinear form on $V_1$ used in [35] is the negative of the bilinear form used in this paper. So our $T_i(v, u, q)$ defined here is exactly the $Z_{M^i}(v, u, q)$ in [35]. Based on Theorem 2.2, a modular transformation law was obtained in [35] with the convergence of $Z_i(v, u, q)$ proved in [14].

**Theorem 3.1.** Let $u, v \in \mathfrak{h}$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then $T_i(v, u, q)$ converges to a holomorphic function in the upper half plane with $q = e^{2\pi i \tau}$ and

$$T_i(v, u, \frac{a\tau + b}{c\tau + d}) = \sum_{j=0}^{p} \gamma_{ij} T_j(dv + bu, cv + au, \tau)$$

where $\gamma_{ij}$ is the same as in Theorem 2.2.

Set $\chi_i(v, \tau) = T_i(v, 0, \tau)$. Using Theorem 3.1 one can easily show the following result [14].
Proposition 3.2. Assume that \( V \) is a strong rational vertex operator algebra. Then for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \), \( v \in \mathfrak{h} \)

\[
\chi_s\left( \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = e^{\pi i (c(v,v)/(c\tau + d))} \sum_{j=0}^{P} \gamma_{ij} \chi_j(v, \tau).
\]

As usual we set \( o(w) = w_{\omega w^{-1}} \) for homogeneous \( w \in V \) and extend to whole \( V \). Now take \( w \in V \) such that \( h_n w = 0 \) for all \( h \in \mathfrak{h} \) and \( n \geq 0 \). Also define

\[
\chi_i(w, v, q) = \text{tr}_{M_i o(w)} e^{2\pi i o(v)} q^{L(0) - c/24}.
\]

The following result \cite{31} generalizes Proposition 3.2.

Theorem 3.3. Let \( v \in \mathfrak{h} \). We assume that \( \chi_i(w, v, q) \) converges to a holomorphic function \( \chi_i(w, v, \tau) \) in \( \mathbb{H} \) with \( q = e^{2\pi i \tau} \) for any \( v \in \mathfrak{h} \). Then for \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \),

\[
\chi_i(w, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) = (c\tau + d)^{\text{wt}[v]} e^{\pi i c(v,v)/(c\tau + d)} \sum_{j=0}^{P} \gamma_{ij} \chi_j(w, \tau).
\]

4. Affine vertex operator algebras

In this section we discuss the affine vertex operator algebra \( L_{\hat{g}}(k, 0) \) associated to the level \( k \) integrable highest weight module for affine Kac-Moody algebra \( \hat{g} \) and its irreducible modules.

Let \( \mathfrak{g} \) be a finite dimensional simple Lie algebra with a Cartan subalgebra \( \mathfrak{h} \). We denote the corresponding root system by \( \Delta \) and the root lattice by \( Q \). Fix an invariant symmetric nondegenerate bilinear form \( \langle , \rangle \) on \( \mathfrak{g} \) such that \( \langle \alpha, \alpha \rangle = 2 \) if \( \alpha \) is a long root, where we have identified \( \mathfrak{h} \) with \( \mathfrak{h}^* \) via \( \langle , \rangle \). We denote the image of \( \alpha \in \mathfrak{h}^* \) in \( \mathfrak{h} \) by \( t_\alpha \). That is, \( \alpha(h) = \langle t_\alpha, h \rangle \) for any \( h \in \mathfrak{h} \). Fix simple roots \( \{\alpha_1, \ldots, \alpha_l\} \) and let \( \Delta_+ \) be the set of corresponding positive roots. Denote the highest root by \( \theta \), and the Weyl group by \( W \). Also let \( \rho \) be the half sum of positive roots.

Recall that the weight lattice \( P \) of \( \mathfrak{g} \) consists of \( \lambda \in \mathfrak{h}^* \) such that \( \frac{2\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z} \) for all \( \alpha \in \Delta \). It is well-known that \( P = \bigoplus_{i=1}^{l} \mathbb{Z} \Lambda_i \) where \( \Lambda_i \) are the fundamental weights defined by the equation \( \frac{2\langle \Lambda_i, \alpha \rangle}{\langle \alpha, \alpha \rangle} = \delta_{i,j} \) for \( 1 \leq i, j \leq l \). Let \( P_+ \) be the subset of \( P \) consisting of the dominant weight \( \Lambda \in P \) in the sense that \( \frac{2\langle \Lambda, \alpha \rangle}{\langle \alpha, \alpha \rangle} \) is nonnegative for all \( j \). For any nonnegative integer \( k \) we also let \( P_+^k \) be the subset of \( P_+ \) consisting of \( \Lambda \) satisfying \( \langle \Lambda, \theta \rangle \leq k \).

Let \( Q = \sum_{i=1}^{l} \mathbb{Z} \alpha_i \) be the root lattice and \( Q_L \) be the sublattice of \( Q \) spanned by the long roots. Recall that the dual lattice \( Q_L^\circ \) consists \( \lambda \in \mathfrak{h}^* \) such that \( \langle \lambda, \alpha \rangle \in \mathbb{Z} \) for all \( \alpha \in Q_L \). Then \( P \) is the dual lattice of \( Q_L^\circ \).
Let $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ be the affine Lie algebra. Fix a nonnegative integer $k$. For any $\Lambda \in P_+^k$ let $L(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight $\Lambda$ and $L_\hat{\mathfrak{g}}(k, \Lambda)$ be the unique irreducible $\hat{\mathfrak{g}}$-module such that $L_\hat{\mathfrak{g}}(k, \Lambda)$ is generated by $L(\Lambda)$ and $\mathfrak{g} \otimes t^n L(\Lambda) = 0$ for $t > 0$ and $K$ acts as constant $k$. The following result is well known (cf. [26], [32]):

**Theorem 4.1.** The $L_\hat{\mathfrak{g}}(k, 0)$ is a simple, rational and $C_2$-cofinite vertex operator algebra with central charge $\frac{k \dim \mathfrak{g}}{k + h^{\vee}}$ and whose irreducible modules are $L_\hat{\mathfrak{g}}(k, \Lambda)$ for $\Lambda \in P_+^k$ and the weight $n_{\Lambda}$ of $L_\hat{\mathfrak{g}}(k, \Lambda)$ is $\langle \Lambda + 2\rho, \Lambda \rangle$. 

Note that $L_\hat{\mathfrak{g}}(k, \Lambda)$ has a decomposition with respect to the action of $h$:

$$L_\hat{\mathfrak{g}}(k, \Lambda) = \bigoplus_{\lambda \in \Lambda^+} L_\hat{\mathfrak{g}}(k, \Lambda)(\lambda)$$

where $h(0)$ acts on $L_\hat{\mathfrak{g}}(k, \Lambda)(\lambda)$ as constant $\lambda(h) = \langle \lambda, h \rangle$ for $h \in \mathfrak{h}$. Following [29], define the character

$$\chi_\Lambda(h, \tau) = \text{tr}_{L_\hat{\mathfrak{g}}(k, \Lambda)} e^{2\pi i h(0)} q^{L(0) - c/24}$$

for $h \in \mathfrak{h}$. Note that the character defined in [29] has an extra factor $e^{2\pi i ku}$ with $u$ being a complex number. This extra factor makes the modular transformation formula more beautiful. But from the point of view of vertex operator algebra, this extra factor is not necessary. From [29] we have

**Theorem 4.2.** Let $\mathfrak{g}$ and $k$ be as before.

1. \{\chi_\Lambda(h, \tau) | \Lambda \in P_+^k \} are linearly independent functions on $\mathfrak{h} \times \mathbb{H}$.
2. For $\Lambda, \Lambda' \in P_+^k$, set

$$S_{\Lambda, \Lambda'} = i^{l(\Lambda)} P/(k + h^{\vee}) Q L^{-1/2} \sum_{w \in W} (-1)^{l(w)} e^{-\frac{2\pi}{k + h^{\vee}} (w(\Lambda + \rho), \Lambda') - \frac{c}{24}}$$

where $l(w)$ is the length of $w$. Then

$$\chi_\Lambda\left(\frac{h}{\tau}, \frac{-1}{\tau}\right) = e^{\pi i h(0)/\tau} \sum_{\Lambda' \in P_+^k} S_{\Lambda, \Lambda'} \chi_{\Lambda'}(h, \tau).$$

Recall Theorem [2.2] and Proposition [3.2].

**Lemma 4.3.** The $S$-matrices in Theorem [2.2] and Theorem [4.2] are the same.

**Proof.** Let $V = L_\hat{\mathfrak{g}}(k, 0)$. Then $V$ satisfies the assumptions in Theorem [2.2]. Recall that we have defined a bilinear form on $V_1$ such that $(u, v) = u_1 v$. Let $u = a(-1)1$ and $v = b(-1)1$ for $a, b \in \mathfrak{g}$. Then $(u, v) = a(1)b(-1)1 = k\langle a, b \rangle$. Identify Lie algebra $V_1$ with $\mathfrak{g}$ we see
that \((a, b) = k\langle a, b \rangle\). We denote the \(S\)-matrix in Theorem 2.2 by \(\langle S_{Lg(k,\Lambda)}, Lg(k,\Lambda') \rangle\). Then from Proposition 3.2 we know

\[
\chi_{A}(\frac{\hbar}{\tau}, \frac{-1}{\tau}) = e^{2\pi i (h, h)/\tau} \sum_{\lambda' \in Lg(k,\Lambda)} \chi_{A'}(h, \tau).
\]

The result now is an immediate consequence of Theorem 4.2 (1). \qed

5. Parafermion vertex operator algebras

In this section we recall the definition of a parafermion vertex operator algebra \(K(\mathfrak{g}, k)\) associated to any finite dimensional simple Lie algebra \(\mathfrak{g}\) and a positive integer \(k\). We also discuss some known results on \(K(\mathfrak{g}, k)\) from [17].

Let \(\lambda_i \in P\) such that \(\lambda_i = \frac{(\theta, \theta)}{2} \Lambda_i\) for \(i = 1, \ldots, l\). Then \(\langle \alpha_i, \lambda_j \rangle = \delta_{i,j}\) for all \(i, j\) and \(Q^0 = \bigoplus_{i=1}^{l} \mathbb{Z}\lambda_i\). The following result is immediate from the relation between \(\Lambda_i\) and \(\lambda_i\).

**Lemma 5.1.** \(P/Q^0\) is a group of order

\[
|P/Q^0| = \begin{cases}
1 & A_l, D_l, E_6, E_7, E_8 \\
2 & B_l \\
2^{l-1} & C_l \\
2^2 & F_4 \\
3 & G_2
\end{cases}
\]

**Lemma 5.2.** For any simple Lie algebra \(\mathfrak{g}\) and any positive integer \(k\), \(\frac{1}{k}P/Q^0\) and \(Q/kQ_L\) are dual groups, so that for any \(\beta \in Q\), \(g_{\beta}(\alpha) = e^{2\pi i (\beta, \alpha)}\) defines an irreducible character for \(\frac{1}{k}P/Q^0\). In particular, \(\frac{1}{k}P/Q^0\) and \(Q/kQ_L\) are isomorphic groups.

**Proof.** Clearly, \(g_{\beta}\) defines an irreducible character of \(\frac{1}{k}P/Q^0\) as \(Q_L = P\). Also, \(g_{\beta_1} = g_{\beta_2}\) for if and only if \(\beta_1 - \beta_2 \in kQ_L\). So \(Q/kQ_L\) is a subgroup of the dual group of \(\frac{1}{k}P/Q^0\). To finish the proof, it is enough to show that \(|P/Q^0| = |Q/Q_L|\). This is obvious if \(\mathfrak{g}\) is a Lie algebra of type \(A, D, E\). If \(\mathfrak{g}\) is a Lie algebra of other type, we verify the result case by case using the root systems given in [27]. We have already known \(|P/Q^0|\) from Lemma 5.1. So we only need to compute \(|Q/Q_L|\).

1. Type \(B_l\). Let \(E = \mathbb{R}^l\) with the standard orthonormal basis \(\{\epsilon_1, \ldots, \epsilon_l\}\). Then

\[
\Delta = \{\pm \epsilon_i, \pm (\epsilon_i \pm \epsilon_j); i \neq j\}.
\]

Then \(Q = \sum_i \mathbb{Z}\epsilon_i\) and \(Q_L = \sum_{i \neq j} \mathbb{Z}(\epsilon_i + \epsilon_j) + \mathbb{Z}(\epsilon_i - \epsilon_j)\). It is evident that \(2Q \subset Q_L\) and \(|Q/Q_L| = 2\) with coset representatives 0 and \(\epsilon_1\).

2. Type \(C_l\). In this case, \(\Delta = \{\pm \sqrt{2}\epsilon_i, \pm \frac{1}{\sqrt{2}}(\epsilon_i \pm \epsilon_j); i \neq j\}\).
Then $Q = \frac{1}{\sqrt{2}} \sum_{i \neq j} (Z(\epsilon_i + \epsilon_j) + Z(\epsilon_i - \epsilon_j))$ and $Q_L = \sqrt{2} \sum_{i=1}^{l} Z \epsilon_i$. Thus $|Q/Q_L| = 2^{l-1}$ with coset representatives $a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_{l-1} \alpha_{l-1}$ for $a_i = 0, 1$ and $a_i = \frac{1}{\sqrt{2}}(\epsilon_i - \epsilon_{i+1})$.

(3) Type $F_4$. Let $\mathbb{E} = \mathbb{R}^4$. Then

$$\Delta = \{ \pm \epsilon_i, \pm (\epsilon_i \pm \epsilon_j), \pm \frac{1}{\sqrt{2}} (\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4) | i \neq j \}.$$  

Then $|Q/Q_L| = 2^2$ with coset representatives $a \epsilon_3 + b \frac{1}{\sqrt{2}} (\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$ for $a, b = 0, 1$.

(4) Type $G_2$. Let $\mathbb{E}$ be the subspace of $\mathbb{R}^3$ orthogonal to $\epsilon_1 + \epsilon_2 + \epsilon_3$. Then

$$\Delta = \pm \frac{1}{\sqrt{3}} \{ \epsilon_i - \epsilon_j, 2 \epsilon_i - \epsilon_2 - \epsilon_3, 2 \epsilon_2 - \epsilon_1 - \epsilon_3, 2 \epsilon_3 - \epsilon_1 - \epsilon_2 | i \neq j \}.$$  

Then $|Q/Q_L| = 3$ with coset representatives $a \frac{1}{\sqrt{3}} (\epsilon_1 - \epsilon_2)$ for $a = 0, 1, 2$. The proof is complete.

Let $M_{\mathfrak{h}}(k)$ be the vertex operator subalgebra of $L_{\mathfrak{g}}(k,0)$ generated by $h(-1)1$ for $h \in \mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, denote by $M_{\mathfrak{h}}(k, \lambda)$ the irreducible highest weight module for $\widehat{\mathfrak{h}}$ with a highest weight vector $e^\lambda$ such that $h(0)e^\lambda = \lambda(h)e^\lambda$ for $h \in \mathfrak{h}$. The parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is the commutant [26] of $M_{\mathfrak{h}}(k)$ in $L_{\mathfrak{g}}(k,0)$. We have the following decomposition

$$L_{\mathfrak{g}}(k, \lambda) = \bigoplus_{\lambda \in Q + \Lambda} M_{\mathfrak{h}}(k, \lambda) \otimes M^{\Lambda, \lambda} = \bigoplus_{\alpha \in Q} M_{\mathfrak{h}}(k, \lambda + \alpha) \otimes M^{\Lambda, \lambda + \alpha}$$  

(5.1)  

as $M_{\mathfrak{h}}(k) \otimes K(\mathfrak{g}, k)$-module. Moreover, $M^{0,0} = K(\mathfrak{g}, k)$ and $M^{\Lambda, \lambda}$ is an irreducible $K(\mathfrak{g}, k)$-module [17]. Recall equation (5.1). It is easy to see that

$$L_{\mathfrak{g}}(k, \lambda)(\lambda) = M_{\mathfrak{h}}(k, \lambda) \otimes M^{\Lambda, \lambda}.$$  

(5.2)

It is proved in [21] that the lattice vertex operator algebra $V_{\sqrt{Q_L}}$ is a vertex operator subalgebra of $L_{\mathfrak{g}}(k,0)$ and the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is also a commutant of $V_{\sqrt{Q_L}}$ in $L_{\mathfrak{g}}(k,0)$. This gives us another decomposition

$$L_{\mathfrak{g}}(k, \lambda) = \bigoplus_{i \in Q/kQ_L} V_{\sqrt{Q_L} + \frac{1}{\sqrt{3}}(\lambda + \beta_i)} \otimes M^{\Lambda, \lambda + \beta_i}$$  

(5.3)  

as modules for $V_{\sqrt{Q_L}} \otimes K(\mathfrak{g}, k)$, where $M^{\Lambda, \lambda}$ is as before, $Q = \cup_{i \in Q/kQ_L} (kQ_L + \beta_i)$, and $\beta_i \in Q$ is a representative of $i$. Moreover, for any $i \in Q/kQ_L$, we have

$$V_{\sqrt{Q_L} + \frac{1}{\sqrt{3}}(\lambda + \beta_i)} \otimes M^{\Lambda, \lambda + \beta_i} = \bigoplus_{\alpha \in kQ_L} M_{\mathfrak{h}}(k, \lambda + \alpha + \beta_i) \otimes M^{\Lambda, \lambda + \alpha + \beta_i}$$

where we have used the isomorphism between $K(\mathfrak{g}, k)$-modules $M^{\Lambda, \lambda + \alpha + \beta_i}$ and $M^{\Lambda, \lambda + \beta_i}$ for any $\alpha \in kQ_L$ [17]. So for any $h \in \mathfrak{h}$, $h(0) \in \widehat{\mathfrak{g}}$ acts as $\sqrt{k}h(0)$ or $\langle h, k\alpha + \Lambda + \beta_i \rangle$ on $e^{\sqrt{k}h(0) + \frac{1}{\sqrt{3}}(\lambda + \beta_i)}$.

Let $\theta = \sum_{i=1}^{l} a_i \alpha_i$. According to [33], [34], if $a_i = 1$ then $L_{\mathfrak{g}}(k, k\lambda_i)$ is a simple current and $L_{\mathfrak{g}}(k, k\Lambda_i) \otimes L_{\mathfrak{g}}(k, \Lambda) = L_{\mathfrak{g}}(k, \Lambda^{(i)})$ for any $\Lambda \in P^k_+$ where $\Lambda^{(i)} \in P^k_+$ is uniquely
determined by Λ and i. Then \( L_0(k, \Lambda) \) and \( L_0(k, \Lambda^{(i)}) \) are isomorphic \( K(g, k) \)-modules \[17\].

Here are some main results on \( K(g, k) \) from \[20\], \[2\], \[3\], \[17\], \[1\].

**Theorem 5.3.** Let \( g \) be a simple Lie algebra and \( k \) a positive integer.

1. The \( K(g, k) \) is a rational, simple and \( C_2 \)-cofinite vertex operator algebra of CFT type.
2. For any \( \Lambda \in P_+^k \), \( \lambda \in \Lambda + Q \) and \( \alpha \in Q_L \), \( M^{\Lambda, \lambda} = M^{\Lambda, \lambda + k\alpha} \).
3. For each \( i \in I \), \( \Lambda \in P_+^k \) there exists a unique \( \Lambda^{(i)} \in P_+^k \) such that for any \( \lambda \in \Lambda + Q \), \( M^{\Lambda, \lambda} = M^{\Lambda^{(i)}, \lambda + k\Lambda} \).
4. Any irreducible \( K(g, k) \)-module is isomorphic to \( M^{\Lambda, \lambda} \) for some \( \Lambda \in P_+^k \) and \( \lambda \in \Lambda + Q \).
5. The \( K(g, k) \) has exactly \( \frac{|P_+^k| |Q/kQ_L|}{|P/Q|} \) inequivalent irreducible modules.

6. **Trace functions for parafermion vertex operator algebras**

In this section we determine the \( S \)-matrix for parafermion vertex operator algebra \( K(g, k) \) associated to any finite dimensional simple Lie algebra \( g \) and positive integer \( k \) from \[17\].

From decomposition (5.1) we have

\[
\chi_{L_0(k, \Lambda)}(\tau) = \sum_{\lambda \in Q + \Lambda} \chi_{M_0(k, \lambda)}(\tau) \chi_{M^{\Lambda, \lambda}}(\tau) = \sum_{\lambda \in Q + \Lambda} q^{(\lambda, \Lambda)/2k} \eta(\tau)^{l(\lambda)} \chi_{M^{\Lambda, \lambda}}(\tau)
\]

where \( \eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \). The

\[
c^{\Lambda}_\lambda(\tau) = \eta(\tau)^{-l(\lambda)} \chi_{M^{\Lambda, \lambda}}(\tau)
\]

is called the string function and \( \chi_{M^{\Lambda, \lambda}}(\tau) \) is called the branching function denoted by \( b^{\Lambda}_\lambda(\tau) \) in \[29\]. The modular transformation formulas of branching functions were also given in \[29\].

**Theorem 6.1.** Let \( \Lambda \in P_+^k \) and \( i \in Q/kQ_L \). Then

1. \( \chi_{M^{\Lambda, \lambda + \beta_i}}(\tau + 1) = e^{2\pi i (\Lambda + 2\rho, \Lambda)/2k} \chi_{M^{\Lambda, \lambda + \beta_i}}(\tau) \),
2. \( \chi_{M^{\Lambda, \lambda + \beta_i}}(\frac{-1}{\tau}) = \sum_{\lambda' \in P_+^k} S_{(\Lambda, \lambda + \beta_i), (\Lambda, \lambda + \beta_i)}^{(\Lambda', \lambda', \lambda')} \chi_{M^{\Lambda', \lambda', \lambda'}}(\tau) \), where

\[
S_{(\Lambda, \lambda + \beta_i), (\Lambda, \lambda + \beta_i)}^{(\Lambda', \lambda', \lambda')} = |P/kQ_L|^{-1/2} S_{\Lambda, \lambda} e^{2\pi i (\Lambda + \beta_i, \Lambda + \beta_i)/k}
\]

and \( P \) stands for Parafermion.

The main result in this paper is that the transformation formula for the branching functions remains valid for the trace functions \( Z_{M, \lambda}(w, \tau) \) with \( w \in K(g, k) \):
Theorem 6.2. Let $\Lambda \in P_+^k$ and $i \in Q/kQ_L$. Then for any $w \in K(g, k)$

1. $Z^{\Lambda, \Lambda + \beta_i}(w, \tau + 1) = e^{2\pi i \frac{\langle \Lambda + 2\beta_i, \Lambda \rangle}{k \sin \tau} - \frac{k \sin \tau}{24} + \frac{1}{24}} Z^{\Lambda, \Lambda + \beta_i}(w, \tau),$

2. $Z^{\Lambda, \Lambda + \beta_i}(w, \frac{-1}{\tau}) = \tau^{\text{wt}[w]} \sum_{N' \in P_+^k, j \in Q/kQ_L} S_{N', N' + \beta_i}^{\Lambda, \Lambda + \beta_i} Z^{\Lambda, N' + \beta_i}(w, \tau).$

Proof. (1) is straightforward. The proof (2) is divided in several steps.

(a) $\chi_{Lq(k, \Lambda)}(w, h, q)$ converges to a holomorphic function $\chi_{Lq(k, \Lambda)}(w, h, \tau)$ for any $w \in K(g, k)$ and $h \in \mathfrak{h}$ where we have identified $x \in g$ with $x(-1)1 \in L(g, k)_0$. We denote the Virasoro vectors of $Lq(k, 0)$, $V_{\sqrt{k}Q_L}$ and $K(g, k)$ by $\omega^a$, $\omega^I$ and $\omega^p$ and denote the corresponding components of the vertex operators by $L^a(n), L^I(n)$ and $L^p(n)$ respectively for $n \in \mathbb{Z}$. Also denote the Virasoro central charges of $Lq(k, 0)$ and $K(g, k)$ by $c^a$ and $c^p$ respectively. Then $c^a = l + c^p$.

Recall decomposition (3.3). Then

$$\chi_{Lq(k, \Lambda)}(w, h, q) = \text{tr}_{Lq(k, \Lambda)} o(w) e^{2\pi i h(0) q L^a(0) - c^a/24} = \sum_{i \in Q/kQ_L} \text{tr}_{V_{\sqrt{k}Q_L} + \frac{1}{\sqrt{k}}(i + \beta_i)_{\otimes M^{\Lambda, \Lambda + \beta_i}}} o(w) e^{2\pi i h(0) q L^a(0) - c^a/24} = \sum_{i \in Q/kQ_L} \text{tr}_{V_{\sqrt{k}Q_L} + \frac{1}{\sqrt{k}}(i + \beta_i)_{\otimes M^{\Lambda, \Lambda + \beta_i}}} e^{2\pi i h(0) q L^a(0) - c^a/24} \text{tr}_{M^{\Lambda, \Lambda + \beta_i}} o(w) q L^p(0) - c^p/24$$

$$= \sum_{i \in Q/kQ_L} \chi_{V_{\sqrt{k}Q_L} + \frac{1}{\sqrt{k}}(i + \beta_i)_{\otimes M^{\Lambda, \Lambda + \beta_i}}} (h, q) Z^{\Lambda, \Lambda + \beta_i}(w, q).$$

Note that $\chi_{V_{\sqrt{k}Q_L} + \frac{1}{\sqrt{k}}(i + \beta_i)_{\otimes M^{\Lambda, \Lambda + \beta_i}}} (h, q)$ converges to a holomorphic function $\chi_{V_{\sqrt{k}Q_L} + \frac{1}{\sqrt{k}}(i + \beta_i)_{\otimes M^{\Lambda, \Lambda + \beta_i}}} (h, \tau)$ and $Z^{\Lambda, \Lambda + \beta_i}(w, q)$ converges to a holomorphic function $Z^{\Lambda, \Lambda + \beta_i}(w, \tau)$. Thus $\chi_{Lq(k, \Lambda)}(w, h, q)$ converges.

For short, we set $\chi_{A}(w, h, \tau) = \chi_{Lq(k, \Lambda)}(w, h, \tau)$.

(b) By Theorems 3.3, 1.2 and Lemma 4.3 we have

$$\chi_{A}(w, \frac{h}{\tau}, -1) = \tau^{\text{wt}[w]} e^{2\pi i k(h,h)/\tau} \sum_{N' \in P_+^k, \Lambda'} S_{\Lambda, \Lambda'} \chi_{A'}(w, h, \tau).$$

(c) Let $L$ be a positive definite even lattice of rank $l$ with bilinear form $(, )$. Recall from [4] and [25] the lattice vertex operator algebra $V_L = M(1) \otimes \mathbb{C}[L]$ and its irreducible modules $V_{L + \lambda_i} = M(1) \otimes \mathbb{C}[L + \lambda_i]$ where $L^0 = \cup_{i \in L^*/L} (L + \lambda_i).$ Then $V_L$ is a vertex operator algebra satisfying V1-V3. In this case

$$\chi_{V_L + \lambda_i}(h, \tau) = \text{tr}_{V_L + \lambda_i} e^{2\pi i h(0) q L^0(0) - l/24} = \frac{\theta_{L + \lambda_i}(h, \tau)}{\eta(\tau)^l}$$

where

$$\theta_{L + \lambda_i}(h, \tau) = \sum_{\lambda \in L + \lambda_i} e^{2\pi i (h, \lambda) q (\lambda, \lambda)/2}.$$
It is well known that \( \{ \theta_{L+\lambda} | L + \lambda \in L^o/L \} \) are linearly independent functions on \( \mathfrak{h} \times \mathbb{H} \) where \( \mathfrak{h} = \mathbb{C} \otimes \mathbb{Z} L \). Thus, \( \{ \chi_{L+\lambda} | L + \lambda \in L^o/L \} \) are linearly independent functions on \( \mathfrak{h} \times \mathbb{H} \). Using the transformation formula

\[
\theta_{L+\lambda} \left( \frac{h}{\tau}, -\frac{1}{\tau} \right) = (-i\tau)^{1/2} |L^o/L|^{-1/2} e^{\pi i (h,h)/\tau} \sum_{\lambda' + L \in L^o/L} e^{-2\pi i (\lambda, \lambda')} \theta_{L+\lambda'} (h, \tau)
\]

and

\[
\eta(-1/\tau) = (-i\tau)^{1/2} \eta(\tau)
\]

we see that

\[
\chi_{L+\lambda} \left( \frac{h}{\tau}, -\frac{1}{\tau} \right) = |L^o/L|^{-1/2} e^{\pi i (h,h)/\tau} \sum_{\lambda' \in L^o/L} e^{-2\pi i (\lambda, \lambda')} \chi_{L+\lambda'} (h, \tau).
\]

Set \( S_{L+\lambda, L+\lambda'} = |L^o/L|^{-1/2} e^{-2\pi i (\lambda, \lambda')} \) for \( L + \lambda, L + \lambda' \in L^o/L \). Also set

\[
S_L = (S_{L+\lambda, L+\lambda'})_{L + \lambda, L + \lambda' \in L^o/L}
\]

which is the \( S \)-matrix for the lattice vertex operator algebra \( V_L \).

(d) Let \( L = \sqrt{\tau} Q_L \). Then \( L^o = \frac{1}{\sqrt{k}} P^k \). As in [29] we consider column vector

\[
\overline{\chi(w, h, \tau)} = (\chi_{\lambda})_{\lambda \in P^k_+}, \quad \overline{\chi_{\sqrt{\tau} Q_L}(h, \tau)} = (\chi_{L+\lambda})_{L + \lambda \in L^o/L}.
\]

Let \( S_A = (S_{\lambda, \lambda'})_{\lambda, \lambda' \in P^k_+} \) which is the \( S \)-matrix for affine vertex operator algebra \( L_{\hat{\mathfrak{g}}}(k, 0) \) (see Lemma 4.3). Also consider the matrix

\[
Z(w, \tau) = (Z_{M, \lambda})_{\lambda \in P^k_+, L + \lambda \in L^o/L}
\]

where \( Z_{M, \lambda} = 0 \) if \( \lambda \) does not lie in \( \Lambda + Q \). From the discussion above we see that

\[
\overline{\chi(w, h, \tau)} = Z(w, \tau) \overline{\chi_{\sqrt{\tau} Q_L}(h, \tau)}.
\]

Performing the transformation of both sides by matrix

\[
\begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}
\]

we see that

\[
t^{\omega[w]} e^{\pi i k (h,h)/\tau} S_A \overline{\chi(w, h, \tau)} = Z(w, \frac{-1}{\tau}) e^{\pi i k (h,h)/\tau} S_L \overline{\chi_{\sqrt{\tau} Q_L}(h, \tau)}.
\]

Or equivalently,

\[
t^{\omega[w]} S_A Z(w, \tau) \overline{\chi_{\sqrt{\tau} Q_L}(h, \tau)} = Z(w, \frac{-1}{\tau}) S_L \overline{\chi_{\sqrt{\tau} Q_L}(h, \tau)}.
\]

The linear independence of functions \( \{ \chi_{L+\lambda} | L + \lambda \in L^o/L \} \) implies that

\[
t^{\omega[w]} S_A Z(w, \tau) = Z(w, \frac{-1}{\tau}) S_L.
\]
It is well known from [29] that $S_L$ is symmetric, unitary and $S_L^{-1} = S_L$. In fact, these properties hold for the $S$-matrix associated to any strong rational vertex operator algebra [13]. Finally we deduce

$$Z(w, -\frac{1}{\tau}) = \tau^{wt[w]} S_\Lambda Z(w, \tau) S_L.$$ 

(c) Comparing the $(\Lambda, \lambda)$-entries of the both sides gives

$$Z_{M, \lambda}(w, -\frac{1}{\tau}) = \tau^{wt[w]} \sum_{\Lambda' \in P^L, \Lambda \in P/kQ_L} S_{\Lambda, \lambda} S_{\lambda, \lambda'} Z_{M, \lambda'}(w, \tau)$$

$$= \tau^{wt[w]} |P/kQ_L|^{-1/2} \sum_{\Lambda' \in P^L, \lambda \in P/kQ_L} S_{\Lambda, \lambda} \tau^{2\pi i (\lambda, \lambda')} Z_{M, \lambda'}(w, \tau).$$

Now we take $\lambda = \Lambda + \beta_i$ for $i \in Q/kQ_L$. Note that $M^\lambda$ is nonzero if and only if $\lambda' + kQ_L = \Lambda' + \beta_j + kQ_L$ for some $j \in Q/kQ_L$. As a result, we see that

$$Z_{M, \lambda+\beta_i}(w, -\frac{1}{\tau}) = \tau^{wt[w]} \sum_{\Lambda' \in P^L, \lambda \in Q/kQ_L} S^P_{(\Lambda, \lambda+\beta_i), (\Lambda', \lambda'+\beta_j)} Z_{M, \lambda'+\beta_j}(w, \tau)$$

and the proof is complete. \hfill \Box

7. Connection with orbifold theory

Set $H = \frac{1}{k} P$. For $\alpha \in H$ we define $g_\alpha = e^{2\pi i \alpha(0)}$ where we have identify $\mathfrak{h}$ with $\mathfrak{h}^*$ via the bilinear form $(\cdot, \cdot)$. Then $g_\alpha$ acts on $L_{\mathfrak{g}}(k, \Lambda)$ for any $\Lambda \in P^L$ such that

$$g_\alpha Y(u, z) g_\alpha^{-1} = Y(g_\alpha u, z)$$

for $u \in L_{\mathfrak{g}}(k, 0)$. In particular, $g_\alpha$ is an automorphism of vertex operator algebra $L_{\mathfrak{g}}(k, 0)$. Moreover, $g_\alpha = 1$ on $L_{\mathfrak{g}}(k, 0)$ if and only if $\alpha \in Q^o$. That is, $G = H/Q^o$ is an automorphism group of $L_{\mathfrak{g}}(k, 0)$. For each $\beta \in Q$ we define an irreducible character $\mu_\beta$ of $G$ such that $\mu_\beta(g_\alpha) = g_\alpha(\beta)$. Following [15] and [8] we use $L_{\mathfrak{g}}(k, 0)^{\mu_\beta}$ to denote the subspace of $L_{\mathfrak{g}}(k, 0)$ which is a sum of irreducible $G$-submodule with character $\mu_\beta$. Recall that $Q = \bigcup_{i \in Q/kQ_L} (kQ_L + \beta_i.$ By Lemma [22] $\{\mu_\beta | i \in Q/kQ_L\}$ gives a complete list of inequivalent irreducible characters of $G$. The following result is immediate from [8].

**Lemma 7.1.** The $L_{\mathfrak{g}}(k, 0)$ is a completely reducible $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$-module

$$L_{\mathfrak{g}}(k, 0) = \bigoplus_{i \in Q/kQ_L} L_{\mathfrak{g}}(k, 0)^{\mu_{\beta_i}},$$

and $L_{\mathfrak{g}}(k, 0)^{\mu_{\beta_i}} = V_{\sqrt{k}Q_L + \beta_i} \otimes M^{0, \beta_i}$ is an irreducible module for $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$. Moreover, if $i \neq j$ then $L_{\mathfrak{g}}(k, 0)^{\mu_{\beta_i}}$ and $L_{\mathfrak{g}}(k, 0)^{\mu_{\beta_j}}$ are inequivalent.
If $\Lambda \in P_+^k$ is not zero, $L_{\widehat{g}}(k,\Lambda)$ is still a module for $H$ but not a module for $G$ unless $\Lambda \in Q$. However, $\alpha \mapsto \tilde{g}_{\alpha} = g_{\alpha}e^{-2\pi i(\alpha,\Lambda)}$ gives a $G$-module structure on $L_{\widehat{g}}(k,\Lambda)$. It is clear that

$$\tilde{g}_{\alpha}Y(u, z)\tilde{g}_{\alpha}^{-1} = Y(g_{\alpha}u, z)$$  \hspace{1cm} (5.1)

on $L_{\widehat{g}}(k,\Lambda)$. A generalization of Lemma 7.1 is the following:

**Lemma 7.2.** The $L_{\widehat{g}}(k,\Lambda)$ is a completely reducible $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$-module

$$L_{\widehat{g}}(k,\Lambda) = \bigoplus_{i \in \frac{Q}{kQ_L}} L_{\widehat{g}}(k,\Lambda)^{\mu_{\beta_i}},$$

and $L_{\widehat{g}}(k,\Lambda)^{\mu_{\beta_i}} = V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda + \beta_i)} \otimes M^{\Lambda + \beta_i}$ is an irreducible module for $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$. Moreover, if $i \neq j$ then $L_{\widehat{g}}(k,\Lambda)^{\mu_{\beta_i}}$ and $L_{\widehat{g}}(k,\Lambda)^{\mu_{\beta_j}}$ are inequivalent.

We can strengthen Lemma 7.2

**Proposition 7.3.** For $\Lambda \in P_+^k$ and $i \in \frac{Q}{kQ_L}$, $L_{\widehat{g}}(k,\Lambda)^{\mu_{\beta_i}}$ are inequivalent irreducible $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$-modules.

**Proof.** From equation (5.1) we know that $L_{\hat{g}}(k,\Lambda) \circ g_{\alpha}$ and $L_{\hat{g}}(k,\Lambda)$ are isomorphic $L_{\hat{g}}(k,0)$-modules where $L_{\hat{g}}(k,\Lambda) \circ g_{\alpha} = L_{\hat{g}}(k,\Lambda)$ as vector spaces and $Y_{L_{\hat{g}}(k,\Lambda)g_{\alpha}}(v, z) = Y_{L_{\hat{g}}(k,\Lambda)}(g_{\alpha}v, z)$ for $v \in L_{\hat{g}}(k,0)$. According to a general result in orbifold theory [23, 18], $L_{\hat{g}}(k,\Lambda)^{\mu_{\beta_i}}$ are inequivalent irreducible $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g},k)$-modules. \hfill $\Box$

We can now express the trace functions $Z_{M^{\Lambda + \beta_i}}(w, \tau)$ in terms of $\chi_{\Lambda}(w, \alpha, \tau)$ and the characters of irreducible modules for lattice vertex operator algebra $V_L$ with $L = \sqrt{k}Q_L$. For $w \in K(\mathfrak{g},k)$ we have

$$Z_{V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda + \beta_i)} \otimes M^{\Lambda + \beta_i}}(w, \tau) = \frac{1}{|Q/kQ_L|} \sum_{\alpha \in G} \operatorname{tr}_{L_{\hat{g}}(k,\Lambda)O}(w)\tilde{g}_{\alpha}q^{L(0)-c/24}e^{-2\pi i(\alpha,\beta_i)}$$

$$= \frac{1}{|Q/kQ_L|} \sum_{\alpha \in G} \operatorname{tr}_{L_{\hat{g}}(k,\Lambda)O}(w)e^{2\pi a(0)}q^{L(0)-c/24}e^{-2\pi i(\alpha,\beta_i + \Lambda)}$$

$$= \frac{1}{|Q/kQ_L|} \sum_{\alpha \in G} e^{-2\pi i(\alpha,\beta_i + \Lambda)}\chi_{\Lambda}(w, \alpha, \tau).$$

This implies the following:

**Proposition 7.4.** For $\Lambda \in P_+^k$, $i \in \frac{Q}{kQ_L}$ and $w \in K(\mathfrak{g},k)$ we have

$$Z_{M^{\Lambda + \beta_i}}(w, \tau) = \frac{1}{|Q/kQ_L|} \eta(\tau)^l \sum_{\alpha \in G} e^{-2\pi i(\alpha,\beta_i + \Lambda)}\chi_{\Lambda}(w, \alpha, \tau).$$
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