ON A RELATION BETWEEN HARMONIC MEASURE AND HYPERBOLIC DISTANCE ON PLANAR DOMAINS

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Abstract. Let $\psi$ be a conformal map of $\mathbb{D}$ onto an unbounded domain and, for $\alpha > 0$, let $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$. If $\omega_0(0, F_\alpha)$ denotes the harmonic measure at 0 of $F_\alpha$ and $d_0(0, F_\alpha)$ denotes the hyperbolic distance between 0 and $F_\alpha$ in $\mathbb{D}$, then an application of the Beurling-Nevanlinna projection theorem implies that $\omega_0(0, F_\alpha) \geq \frac{1}{2} e^{-d_0(0, F_\alpha)}$. Thus a natural question, first stated by P. Poggi-Corradini, is the following: Does there exist a positive constant $K$ such that for every $\alpha > 0$, $\omega_0(0, F_\alpha) \leq K e^{-d_0(0, F_\alpha)}$? In general, we prove that the answer is negative by means of two different examples. However, under additional assumptions involving the number of components of $F_\alpha$ and the hyperbolic geometry of the domain $\psi(\mathbb{D})$, we prove that the answer is positive.

1 Introduction

We will give an answer to a question of P. Poggi-Corradini ([18, p. 36]) about an inequality relating harmonic measure and hyperbolic distance. For a domain $D$, a point $z \in D$ and a Borel subset $E$ of $\overline{D}$, let $\omega_D(z, E)$ denote the harmonic measure at $z$ of $E$ with respect to the component of $D \setminus E$ containing $z$. The function $\omega_D(\cdot, E)$ is exactly the solution of the generalized Dirichlet problem with boundary data $\varphi = 1_E$ (see [1, ch. 3], [8, ch. 1] and [20, ch. 4]). The hyperbolic distance between two points $z, w$ in the unit disk $\mathbb{D}$ (see [1, ch. 1], [2, p. 11-28]) is defined by

$$d_\mathbb{D}(z, w) = \log \left( 1 + \frac{z-w}{1-z\bar{w}} \right).$$

It is conformally invariant and thus it can be defined on any simply connected domain $D \neq \mathbb{C}$ as follows: If $f$ is a Riemann map of $\mathbb{D}$ onto $D$ and $z, w \in D$, then $d_D(z, w) = d_\mathbb{D}(f^{-1}(z), f^{-1}(w))$. Also, for a set $E \subset D$, we define $d_D(z, E) := \inf \{ d_D(z, w) : w \in E \}$.

The Hardy space with exponent $p$, $p > 0$, and norm $\| \cdot \|_p$ (see [5, p. 1-2], [8, p. 435-441]) is defined to be

$$H^p(\mathbb{D}) = \left\{ f \in H(\mathbb{D}) : \| f \|^p_p = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta < +\infty \right\},$$

where $H(\mathbb{D})$ denotes the family of all holomorphic functions on $\mathbb{D}$. The fact that a function $f$ belongs to $H^p(\mathbb{D})$ imposes a restriction on the growth of $f$ and this restriction is stronger as $p$ increases. If $\psi$ is a conformal map on $\mathbb{D}$, then $\psi \in H^p(\mathbb{D})$ for all $p < 1/2$ ([5, p. 50]). Harmonic measure and hyperbolic distance are both conformally invariant and many Euclidean estimates are known for them. Thus, expressing the $H^p$-norms of a conformal map $\psi$ on $\mathbb{D}$ in terms of harmonic measure and hyperbolic distance, we are able to obtain information about the growth of

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the function by looking at the geometry of its image region $\psi(\mathbb{D})$. Indeed, if $\psi$ is a conformal map on $\mathbb{D}$ and $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$ for $\alpha > 0$, then (see [18, p. 33])

$$\psi \in H^p(\mathbb{D}) \iff \int_1^{+\infty} \alpha^{p-1} \omega_\mathbb{D}(0, F_\alpha) d\alpha < +\infty.$$  

(1.1)

Now observe that if $E \subset \mathbb{D} \setminus \{0\}$, then $\omega_\mathbb{D}(0, E)$ and $d_\mathbb{D}(0, E)$ can be related by means of a special case of Beurling-Nevanlinna projection theorem (see [1, p. 43-44], [4, p. 43], [8, p. 105] and [20, p. 120]) which is stated as follows: Let $E \subset \mathbb{D} \setminus \{0\}$ be a closed, connected set intersecting the unit circle. Let $E^* = \{ -|z| : z \in E \} = (-1, -r_0]$, where $r_0 = \min \{|z| : z \in E\}$. Then,

$$\omega_\mathbb{D}(0, E) \geq \omega_\mathbb{D}(0, E^*) = \frac{2}{\pi} \arcsin \left( \frac{1-r_0}{1+r_0} \right).$$

If $\psi$ is a conformal map of $\mathbb{D}$ onto an unbounded domain and $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$ for $\alpha > 0$, then

$$d_\mathbb{D}(0, F_\alpha) = \inf \{ d_\mathbb{D}(0, z) : z \in F_\alpha \} = \log \frac{1+r_0}{1-r_0},$$

where $r_0 = \min \{|z| : z \in F_\alpha\}$. Thus, the Beurling-Nevanlinna projection theorem implies that

(1.2)

$$\omega_\mathbb{D}(0, F_\alpha) \geq \frac{2}{\pi} \arcsin \left( \frac{1-r_0}{1+r_0} \right) = \frac{2}{\pi} \arcsin e^{-d_\mathbb{D}(0, F_\alpha)} \geq \frac{2}{\pi} e^{-d_\mathbb{D}(0, F_\alpha)}.$$

Poggi-Corradini observed that, in general, the opposite inequality fails. But for a sector domain ([18, p. 34-35]),

$$\omega_\mathbb{D}(0, F_\alpha) \leq Ke^{-d_\mathbb{D}(0, F_\alpha)}.$$

So, taking all these results into consideration, he set the following questions ([18, p. 36]):

**Question 1.1.** Let $\psi$ be a conformal map of $\mathbb{D}$ onto an unbounded domain and, for $\alpha > 0$, let $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$. Does there exist a positive constant $K$ such that for every $\alpha > 0$,

$$\omega_\mathbb{D}(0, F_\alpha) \leq Ke^{-d_\mathbb{D}(0, F_\alpha)}?$$

**Question 1.2.** More generally, is it true that

$$\psi \in H^p(\mathbb{D}) \iff \int_1^{+\infty} \alpha^{p-1} e^{-d_\mathbb{D}(0, F_\alpha)} d\alpha < +\infty?$$

In Section [4] we give a negative answer to the first question by mapping, through a conformal map $\psi$, $\mathbb{D}$ onto the simply connected domain $D$ of Fig. 1. Its special feature is that as $\alpha \rightarrow +\infty$, the number of components of $\psi(F_\alpha)$ tends to infinity. This in conjunction with the fact that $d_D(0, \psi(F_\alpha))$ is related to one component of $\psi(F_\alpha)$ whereas $\omega_D(0, \psi(F_\alpha))$ is related to the whole $\psi(F_\alpha)$, made us believe that the choice of $D$ would give a negative answer to the Question 1.1 and so it did.
Consequently, a natural query would be whether the answer to the Question 1.1 is positive in case the number of components of $\psi(F_\alpha)$ is bounded from above by a positive constant for every $\alpha > 0$. However, in Section 5 we prove by mapping $D$ onto the simply connected domain $D'$ of Fig. 2 that the answer is again negative. This is due to the fact that the hyperbolic distance between $\psi(F_\alpha)$ and the geodesic, $\psi(\Gamma_\alpha)$, joining the endpoints of $\psi(F_\alpha)$ in $D'$ tends to infinity, as $\alpha \to +\infty$. These results lead us to set sufficient conditions on the domain $\psi(D)$ in order to give a positive answer to the Question 1.1.

In Section 2 we introduce some preliminary notions and results such as the domain decomposition method studied by N. Papamichael and N.S. Stylianopoulos (see [15], [16], [17]). In Section 3 we present some lemmas required for the proof of the theorem which is stated and proved in Section 4 and gives a negative answer to the Question 1.1 through the study of the domain $\psi(D)$ in order to give a positive answer to the Question 1.1.

First note that if $\psi$ is a conformal map of $D$ onto an unbounded simply connected domain $D$ and $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$ for $\alpha > 0$, then $\psi(F_\alpha)$ is a countable union of open arcs in $D$ which are the intersection of $D$ with the circle $\{ z \in \mathbb{C} : |z| = \alpha \}$ and have two distinct endpoints on $\partial D$. Thus, the preimage of every such arc is also an arc in $D$ with two distinct endpoints on $\partial D$ (see Proposition 2.14 [19, p. 29]). So, in Section 5 we prove the following theorem:

**Theorem 1.1.** Let $\psi$ be a conformal map of $\mathbb{D}$ onto an unbounded simply connected domain $D$ and let $F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \}$ for $\alpha > 0$. If $N(\alpha)$ denotes the number of components of $F_\alpha$ and $F_\alpha^j$ denotes each of these components for $j = 1, 2, \ldots, N(\alpha)$, then we set $z_\alpha^j, z_\alpha'^j$ be the endpoints of $F_\alpha^j$ on $\partial \mathbb{D}$ and $\Gamma_\alpha^j$ be the geodesic joining $z_\alpha^j$ to $z_\alpha'^j$ in $\mathbb{D}$ for $j = 1, 2, \ldots, N(\alpha)$. Suppose that the following conditions are satisfied:

1. There exists a positive constant $c_1$ such that $N(\alpha) \leq c_1$ for every $\alpha > 0$.
2. There exists a positive constant $c_2$ such that $\psi(F_\alpha^j) \subset \{ z \in D : d_D(z, \psi(\Gamma_\alpha^j)) < c_2 \}$ for every $\alpha > 0$ and every $j \in \{1, 2, \ldots, N(\alpha)\}$.
Then there exists a positive constant $K$ such that for every $\alpha > 0$,
$$
\omega_D (0, F_\alpha) \leq K e^{-d_\beta (0, F_\alpha)}.
$$

**Remark 1.1.** The direction “$\Rightarrow$” in Question 1.2 is a simple consequence of (1.1) and (1.2). So, the question actually concerns the other direction. The domain $D$ does not give an answer because by Theorem 4.1 in [9, p. 239], which gives the Hardy number of an unbounded starlike region with respect to $z = 0$, we derive that the corresponding Riemann map belongs in $H^p (D)$ for every $p > 0$. So, (1.1) implies that
$$
\int_1^{+\infty} \alpha^{p-1} \omega_D (0, F_\alpha) d\alpha < +\infty
$$
for every $p > 0$.

**Remark 1.2.** The domain $D'$ does not give an answer to the Question 1.2 because we have found that $\omega_D (0, F_\alpha)$ decreases very rapidly so that
$$
\int_1^{+\infty} \alpha^{p-1} \omega_D (0, F_\alpha) d\alpha < +\infty
$$
for every $p > 0$. This follows from calculations which we don’t present here.

### 2 Preliminary results and notations

#### 2.1 Minda’s reflection principle

Concerning the hyperbolic metric we use the following theorem known as Minda’s Reflection Principle [13, p. 241]. First, note that, if $\Gamma$ is a straight line (or circle), then $R$ is one of the half-planes (or the disk) determined by $\Gamma$ and $\Omega^*$ is the reflection of a hyperbolic region $\Omega$ in $\Gamma$.

**Theorem 2.1.** Let $\Omega$ be a hyperbolic region in $\mathbb{C}$ and $\Gamma$ be a straight line or circle with $\Omega \cap \Gamma \neq \emptyset$. If $\Omega \setminus R \subset \Omega^*$, then
$$
\lambda_{\Omega^*} (z) \leq \lambda_\Omega (z)
$$
for all $z \in \Omega \setminus \overline{R}$. Equality holds if and only if $\Omega$ is symmetric about $\Gamma$.

#### 2.2 Quasi-hyperbolic distance

The hyperbolic distance between $z_1, z_2 \in D$ can be estimated by the quasi-hyperbolic distance, $\delta_D (z_1, z_2)$, which is defined by
$$
\delta_D (z_1, z_2) = \inf_{\gamma : z_1 \rightarrow z_2} \int_{\gamma} \frac{|dz|}{d (z, \partial D)},
$$
where the infimum ranges over all the paths connecting $z_1$ to $z_2$ in $D$ and $d (z, \partial D)$ denotes the Euclidean distance of $z$ from $\partial D$. Then it is proved that $(1/2) \delta_D \leq d_D \leq 2 \delta_D$ (see [2, p. 33-36], [18, p. 8]).

#### 2.3 Extremal length

Another conformally invariant quantity which plays a central role in the proof of Section 4 is the extremal length. We present the definition and the properties we need as they are stated in [1, ch. 4], [4, p. 361-385], [6, ch. 7], [8, ch. 4], [12, p. 88-100] and [14, ch. 2].

**Definition 2.1.** Let $\{C\}$ be a family of curves and $\rho (z) \geq 0$ be a measurable function defined in $\mathbb{C}$. We say $\rho (z)$ is admissible for $\{C\}$ and denote by $\rho \in \text{adm} \{C\}$, if for every rectifiable $C \in \{C\}$, the integral $\int_C \rho (z) |dz|$ exists and $1 \leq \int_C \rho (z) |dz| \leq +\infty$. The extremal length of $\{C\}$, $\lambda \{C\}$, is defined by
$$
\frac{1}{\lambda \{C\}} = \inf_{\rho \in \text{adm} \{C\}} \int \int \rho^2 (z) dxdy.
$$
Note that if all curves of \( \{ C \} \) lie in a domain \( D \), we may take \( \rho ( z ) = 0 \) outside \( D \). The conformal invariance is an immediate consequence of the definition (see [6] p. 90). As a typical example (see [4] p. 366, [8] p. 131), we mention the case in which \( R \) is a rectangle with sides of length \( a \) and \( b \) and \( \{ C \} \) is the family of curves in \( R \) joining the opposite sides of length \( a \). Then \( \lambda \{ C \} = \frac{a}{b} \). Next we state two basic properties of extremal length that we will need (see [1] p. 54-55, [4] p. 363, [6] p. 91, [8] p. 134-135, [14] p. 79).

**Theorem 2.2.** If \( \{ C' \} \subset \{ C \} \) or every \( C' \in \{ C' \} \) contains a \( C \in \{ C \} \), then \( \lambda \{ C \} \leq \lambda \{ C' \} \).

**Theorem 2.3** (The serial rule). Let \( \{ B_n \} \) be mutually disjoint Borel sets and each \( C_n \in \{ C_n \} \) be in \( B_n \). If \( \{ C \} \) is a family of curves such that each \( C \) contains at least one \( C_n \) for every \( n \), then

\[
\lambda \{ C \} \geq \sum_n \lambda \{ C_n \}.
\]

Sometimes it is more convenient to use the more special notion of extremal distance. Let \( D \) be a plane domain and \( E_1, E_2 \) be two disjoint closed sets on \( \partial D \). If \( \{ C \} \) is the family of curves in \( D \) joining \( E_1 \) to \( E_2 \), then the extremal length \( \lambda_D \{ C \} \) is called the extremal distance between \( E_1 \) and \( E_2 \) with respect to \( D \) and is denoted by \( \lambda_D ( E_1, E_2 ) \).

### 2.4 Domain decomposition method

In case of quadrilaterals, the opposite inequality in the serial rule has been studied by N. Papanichael and N.S. Stylianopoulos by means of a domain decomposition method for approximating the conformal modules of long quadrilaterals (see [15], [16], [17]). Before stating the theorem we need, we present the required notation.

Let \( \Omega \) be a Jordan domain in \( \mathbb{C} \) and consider a system consisting of \( \Omega \) and four distinct points \( z_1, z_2, z_3, z_4 \) in counterclockwise order on its boundary \( \partial \Omega \). Such a system is said to be a quadrilateral \( Q \) and is denoted by

\[
Q := \{ \Omega; z_1, z_2, z_3, z_4 \}.
\]

The conformal module \( m ( Q ) \) of \( Q \) is the unique number for which \( Q \) is conformally equivalent to the rectangular quadrilateral

\[
Q' := \{ R_m ( Q ); 0, 1, 1 + m ( Q ) i, m ( Q ) i \},
\]

where \( R_m ( Q ) = \{ x + yi : 0 \leq x < 1, 0 < y < m ( Q ) \} \) (see Fig. 3). Note that \( m ( Q ) \) is conformally invariant and it is equal to the extremal distance between the boundary arcs \( ( z_1, z_2 ) \) and \( ( z_3, z_4 ) \) of \( \Omega \). So, \( \Omega \) and \( Q := \{ \Omega; z_1, z_2, z_3, z_4 \} \) will denote respectively the original domain and the corresponding quadrilateral. Moreover, \( \Omega_1, \Omega_2, \ldots \), and \( Q_1, Q_2, \ldots \), will denote the principle subdomains and corresponding component quadrilaterals of the decomposition under consideration. Now consider the situation of Fig. 3 where the decomposition of \( Q := \{ \Omega; z_1, z_2, z_3, z_4 \} \) is defined by two non-intersecting arcs \( \gamma_1, \gamma_2 \) that join respectively two distinct points \( a \) and \( b \) on the boundary arc \( ( z_2, z_3 ) \) to two points \( d \) and \( c \) on the boundary arc \( ( z_4, z_1 ) \). These two arcs subdivide \( \Omega \) into three non-intersecting subdomains denoted by \( \Omega_1, \Omega_2 \) and \( \Omega_3 \). In addition, the arc \( \gamma_1 \) subdivides \( \Omega \) into \( \Omega_1 \) and another subdomain denoted by \( \Omega_{2,3} \), i.e. we take

\[
\Omega_{2,3} = \Omega_2 \cup \Omega_3.
\]

Similarly, we say that \( \gamma_2 \) subdivides \( \Omega \) into \( \Omega_{1,2} \) and \( \Omega_3 \), i.e. we take

\[
\Omega_{1,2} = \Omega_1 \cup \Omega_2.
\]

Finally, we use the notations \( Q_1, Q_2, Q_3, Q_{1,2} \) and \( Q_{2,3} \) to denote, respectively, the quadrilaterals corresponding to the subdomains \( \Omega_1, \Omega_2, \Omega_3, \Omega_{1,2} \) and \( \Omega_{2,3} \), i.e.

\[
Q_1 := \{ \Omega_1 ; z_1, z_2, a, d \}, \quad Q_2 := \{ \Omega_2 ; d, a, b, c \}, \quad Q_3 := \{ \Omega_3 ; c, b, z_3, z_4 \}
\]
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and

\[ Q_{1,2} := \{ \Omega_{1,2}; z_1, z_2, b, c \}, \quad Q_{2,3} := \{ \Omega_{2,3}; d, a, z_3, z_4 \}. \]

Figure 3. The subdivision of \( \Omega \) into \( \Omega_1, \Omega_2, \Omega_3 \) and the conformal map \( F : Q \to Q' \).

The following theorem was proved by Papamichael and Stylianopoulos in [16, p. 221-222]; see also [7, p. 454-455].

**Theorem 2.4.** Consider the decomposition and the notations illustrated in Fig. 3. With the terminology defined above, we have

\[ |m(Q) - (m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2))| \leq 8.82e^{-\pi m(Q_2)}, \]

provided that \( m(Q_2) \geq 3 \).

**Remark 2.1.** Papamichael and Stylianopoulos proved Theorem 2.4 in case \( \Omega \) is a Jordan domain. However, it follows from the proof that the theorem is still valid if \( \Omega \) is a simply connected domain and its boundary sets \((z_1, z_2)\) and \((z_3, z_4)\) are arcs of prime ends.

2.5 Beurling’s estimates for harmonic measure

A basic property of extremal distance is its connection to harmonic measure as the following theorems, due mainly to Beurling, state (see [3, p. 280], [4, p. 369-372], [8, p. 143-146] and [12, p. 100]).

**Theorem 2.5.** Let \( D \) be a simply connected domain in \( \mathbb{C} \) and \( E \) consist of a finite number of arcs lying on \( \partial D \). Fix \( z_0 \in D \) and choose a curve \( \gamma_0 \) that contains \( z_0 \), lies in \( D \) and joins two points of \( \partial D \) so that \( \gamma_0 \) bounds with \( \partial D \) a domain \( D_0 \) and \( z_0 \) can be joined to \( E \) inside \( D \setminus D_0 \) (see Fig. 4). If \( \lambda_{D\setminus D_0}(\gamma_0, E) > 2 \), then

\[ \omega_D(z_0, E) \leq 3\pi e^{-\pi \lambda_{D\setminus D_0}(\gamma_0, E)}. \]
Theorem 2.6. Let $D$ be a simply connected domain in $\mathbb{C}$ and $E$ be an arc (of prime ends) on $\partial D$. Fix $z_0 \in D$ and map $D$ onto $\mathbb{D}$ by the conformal map $f$ so that $f(z_0) = 0$ and $f(E) = \{ e^{it} : \theta \in [-t,t] \}$ for some $t \in [0,\pi]$. If $\gamma_E := f^{-1}([-1,0])$, then there exists an absolute positive constant $C$ such that
\[
\omega_D(z_0,E) \geq Ce^{-\pi\lambda_D(\gamma_E,E)}.
\]

3 Auxiliary lemmas

Let $\Omega$ be a simply connected domain of the form illustrated in Fig. 5. Note that the positive numbers $\alpha_0, \alpha_1, \alpha_2, \ldots$, are the real parts of the tips of the horizontal boundary segments of $\Omega$. We consider the straight line crosscuts $l_1, l$ of Fig. 5 so that $l$ lies on the vertical line passing through the midpoint of $[\alpha_0, \alpha_1]$ and $l_1$ lies on the vertical line passing through the midpoint of $[\alpha_0, \frac{\alpha_0 + \alpha_1}{2}]$.

We decompose $\Omega$ by means of the straight line crosscuts $l_1, l, l_1'$ into four subdomains $\Omega_1, \Omega_2, \Omega_3, \Omega_4$ so that $\Omega_3$ is the reflection of $\Omega_2$ in $l$.

Lemma 3.1. With the notation above, let $Q = \{ \Omega; z_1, 0, z_2, z_3 \}$. According to the terminology introduced in Section 2, we have for the decomposition defined by $l$,
\[
0 \leq m(Q) - (m(Q_{1,2}) + m(Q_{3,4})) \leq 26.46e^{-\pi m(Q_2)}.
\]
provided that \( m(Q_2) \geq 3 \).

**Proof.** Since \([z_1,0]\) and \([z_2, z_3]\) are arcs of prime ends, there exists a conformal map \( F \) of \( Q \) onto \( F(Q) = \{R_{m(Q)}: 0, 1, 1 + m(Q)i, m(Q)i\} \), where \( R_{m(Q)} = \{x + yi: 0 < x < 1, 0 < y < m(Q)\} \).

By symmetry we have that \( m(Q_{2,3}) = 2m(Q_2) \geq 6 \). So, applying Theorem 2.4 we get

\[
|m(Q) - (m(Q_{1,2,3}) + m(Q_{2,3,4}) - m(Q_{2,3}))| \leq 8.82e^{-\pi m(Q_{2,3})}
\]

or equivalently

\[
(3.1) \quad |m(Q) - (m(Q_{1,2,3}) + m(Q_{2,3,4}) - 2m(Q_2))| \leq 8.82e^{-2\pi m(Q_2)}.
\]

Now consider the quadrilateral \( Q_{2,3,4} \). Since \( m(Q_3) = m(Q_2) \geq 3 \), by applying Theorem 2.4 we deduce that

\[
|m(Q_{2,3,4}) - (m(Q_{2,3}) + m(Q_{3,4}) - m(Q_3))| \leq 8.82e^{-\pi m(Q_3)}
\]

or equivalently

\[
(3.2) \quad |m(Q_{2,3,4}) - (m(Q_2) + m(Q_{3,4}))| \leq 8.82e^{-\pi m(Q_2)}.
\]

Similarly, consider the quadrilateral \( Q_{1,2,3} \). Since \( m(Q_2) \geq 3 \), by applying Theorem 2.4 we deduce that

\[
|m(Q_{1,2,3}) - (m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2))| \leq 8.82e^{-\pi m(Q_2)}
\]

or equivalently

\[
(3.3) \quad |m(Q_{1,2,3}) - (m(Q_2) + m(Q_{1,2}))| \leq 8.82e^{-\pi m(Q_2)}.
\]

By relations (3.1), (3.2), (3.3) and the serial rule

\[
m(Q) \geq m(Q_{1,2}) + m(Q_{3,4}),
\]

we finally get

\[
0 \leq m(Q) - (m(Q_{1,2}) + m(Q_{3,4})) \leq 26.46e^{-\pi m(Q_2)}.
\]

\( \square \)

In the following lemma we use the notation \( D(0, \alpha) \) to denote the disk with center at 0 and radius \( \alpha \).

**Lemma 3.2.** Let \( \Omega \) be a simply connected domain of the form illustrated in Fig. 6 and \( E \) be an arc of prime ends on \( \partial \Omega \cap \partial D(0, \alpha) \). If \( f \) is the conformal map of \( \Omega \) onto \( \mathbb{D} \) such that \( f(0) = 0 \) and \( f(E) = \{e^{i\theta} : \theta \in [-t, t]\} \) for some \( t \in [0, \pi] \), then

\[
\gamma \subset \overline{D}(0, \alpha_0),
\]

where \( \gamma := f^{-1}([-1, 0]) \).
Proof. Set \( C = \{ \alpha_0 e^{i\theta} : \theta \in \left[ \frac{\pi}{2}, 2\pi \right] \} \) and \( z_0 = f^{-1}(-1) \). Since \( D(0, \alpha_0) \subset \Omega \), by Corollary 4.3.9 [20, p. 102] and conformal invariance of harmonic measure, we have
\[
\omega_D(0, f(C)) = \omega_\Omega(0, C) \geq \omega_{D(0,\alpha_0)}(0, C) = \frac{3}{4}.
\]
This, in conjunction with the fact that \( f(C) \cap f(E) = \emptyset \) and \( f(C) \) is a connected arc of \( \partial \mathbb{D} \), implies that
\[
\left\{ e^{i\theta} : \theta \in \left[ \frac{\pi}{2}, \frac{3\pi}{2} \right] \right\} \subset f(C).
\]
So, \( z_0 = f^{-1}(-1) \in C \). Now suppose that \( \gamma \not\subset D(0, \alpha_0) \). Then \( \gamma \) contains a curve \( \gamma_0 \) lying in \( \Omega \setminus D(0, \alpha_0) \) with endpoints \( z_1, z_2 \in \partial D(0, \alpha_0) \) (see Fig. 7, 8).
Since $\gamma$ is the hyperbolic geodesic joining $0$ to $z_0$ in $\Omega$, $\gamma_0$ is the hyperbolic geodesic joining $z_1$ to $z_2$ in $\Omega$. Notice that $\Omega$ is a hyperbolic region in $\mathbb{C}$ such that $\Omega \cap \partial D(0, \alpha_0) \neq \emptyset$ and $\Omega \setminus D(0, \alpha_0) \subset \Omega^*$, where $\Omega^*$ is the reflection of $\Omega$ in the circle $\partial D(0, \alpha_0)$. So, applying Theorem 2.1 we get

$$\lambda_{\Omega^*}(z) < \lambda_{\Omega}(z), \quad z \in \gamma_0$$

and thus

$$\int_{\gamma_0^*} \lambda_{\Omega}(z^*) |dz^*| < \int_{\gamma_0} \lambda_{\Omega}(z) |dz|,$$

where $\gamma_0^*$ is the reflection of $\gamma_0$ in $\partial D(0, \alpha_0)$. But this leads to contradiction because $\gamma_0$ is the hyperbolic geodesic joining $z_1$ to $z_2$ in $\Omega$. So, $\gamma \subset D(0, \alpha_0)$. Note that the same result could come from Jørgensen’s theorem that closed disks in $\Omega$ are strictly convex in the hyperbolic geometry of $\Omega$ (see [11]).

**Lemma 3.3.** Let $\Omega, \gamma, E$ be as in Lemma 3.2 and $z_1, z_2, z_3$ be the points illustrated in Fig. 9. Take $r_1, r_2$ so that $\alpha_0 < r_1 < r_2 < \alpha_1$ and $\log \frac{r_1}{r_2} \geq \frac{3\pi}{2}$. Decomposing $Q = \{\Omega \setminus \gamma; z_1, z_1, z_2, z_3\}$ as in Fig. 9 with the terminology introduced in Section 2, we have

$$|m(Q) - (m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2))| \leq 8.82e^{-\pi m(Q_2)}.$$

In the notation $Q = \{\Omega \setminus \gamma; z_1, z_1, z_2, z_3\}$, by the pair of points $z_1, z_1$, we mean the two different prime ends supported at the point $z_1$.

**Proof.** First, applying the conformal map $f(z) = \log z$ (principal branch of the logarithm) on the quadrilateral $Q_2 = \{\Omega_2; r_1 i, r_1, r_2, r_2 i\}$ we take the rectangular quadrilateral

$$f(Q_2) = \{f(\Omega_2); \log r_1 + \frac{\pi}{2}i, \log r_1, \log r_2, \log r_2 + \frac{\pi}{2}i\},$$

**Figure 9.** The decomposition of $Q$ into $Q_1, Q_2, Q_3$. 
where \( f(\Omega) = \{x + yi : \log r_1 < x < \log r_2, 0 < y < \pi/2\} \). Because of the conformal invariance of modules and our assumption about \( r_1, r_2 \),

\[
m(Q) = m(f(Q)) = \frac{\log(r_2/r_1)}{\pi/2} \geq 3.
\]

Since the boundary sets \((z_1, 0, z_1)\) and \(E\) are arcs of prime ends, there exists a conformal map \( F \) of \( Q \) onto

\[
Q' = \{R_m(Q); 0, 1, 1 + m(Q)i, m(Q)i\}
\]

where \( R_m(Q) = \{x + yi : 0 < x < 1, 0 < y < m(Q)\} \). Since \( m(Q) \geq 3 \), Theorem 2.4 implies that

\[
|m(Q) - (m(Q_{1,2}) + m(Q_{2,3}) - m(Q_2))| \leq 8.82e^{-\pi m(Q_2)}.
\]

\[\square\]

4 THE FIRST EXAMPLE

**Theorem 4.1.** There exists an unbounded simply connected domain \( D \) with the following properties:

Let \( \psi \) be a conformal map of \( \mathbb{D} \) onto \( D \) with \( \psi(0) = 0 \). If \( F_\alpha = \{z \in \mathbb{D} : |\psi(z)| = \alpha\} \) for \( \alpha > 0 \), then

1. the number of components of \( \psi(F_\alpha) \) tends to infinity as \( \alpha \to +\infty \) and
2. \( \forall K > 0 \exists \alpha \) such that

\[
\omega_D(0,F_\alpha) \geq Ke^{-d_D(0,F_\alpha)}.
\]

**Proof.** Step 1: If \( \alpha_0 = 1, \alpha_1 = e^{4\pi}, \alpha_2 = e^{8\pi}, \ldots, \alpha_n = e^{4n\pi}, \ldots \), let \( D \) be the simply connected domain of Fig. 10, namely

\[
D = \mathbb{C} \setminus \bigcup_{k=0}^{3} \left[\alpha_0 e^{\frac{i\pi}{2}k}, +\infty\right) \bigcup_{l=1}^{+\infty} \bigcup_{k=0}^{2^{l+1}-1} \left[\alpha_l e^{i\frac{\pi}{2}(\frac{1}{2} + k)}, +\infty\right),
\]

with the notation \([re^{i\theta}, +\infty) = \{se^{i\theta}: s \geq r\}\).

![Figure 10. The simply connected domain D.](image)
The Riemann Mapping Theorem implies that there exists a conformal map \( \psi \) from \( D \) onto \( D \) such that \( \psi(0) = 0 \). Let \( N(\alpha) \) be the number of components of \( \psi(F_\alpha) = D \cap \partial D(0, \alpha) \), then we have

\[
N(\alpha) = \begin{cases} 
1, & \text{if } \alpha \in (0, \alpha_0) \\
2^2, & \text{if } \alpha \in [\alpha_0, \alpha_1) \\
2^3, & \text{if } \alpha \in [\alpha_1, \alpha_2) \\
\vdots \\
2^{n+2}, & \text{if } \alpha \in [\alpha_n, \alpha_{n+1}) \\
\vdots 
\end{cases}
\]

**Step 2:** We fix a real number \( \alpha > \alpha_1 \) and \( \alpha \neq \alpha_n \) for every \( n \in \mathbb{N} \). Then there exists a fixed number \( n \in \mathbb{N} \) such that \( \alpha \in (\alpha_n, \alpha_{n+1}) \) and thus \( N(\alpha) = 2^{n+2} \). Since hyperbolic distance is conformally invariant we have

\[
e^{-d_D(0,F_\alpha)} = e^{-d_D(0,\psi(F_\alpha))} = e^{-d_D(0,\psi(F^*_\alpha))}.
\]

where \( \psi(F^*_\alpha) \) is a component of \( \psi(F_\alpha) \) containing a point \( z_0 \) for which

\[
d_D(0,\psi(F_\alpha)) = \inf \{d_D(0,z) : z \in \psi(F_\alpha) \} = d_D(0,z_0).
\]

Due to the symmetry of \( D \), we may assume without loss of generality that \( \psi(F^*_\alpha) \) lies on the first quartile \( P = \{ z \in \mathbb{C} : \text{Im} \, z > 0, \text{Re} \, z > 0 \} \). By relation (1.2) of Section 1, we infer that

\[
e^{-d_D(0,\psi(F^*_\alpha))} \leq \frac{\pi}{2} \omega_D(0, F^*_\alpha)
\]

which in conjunction with the conformal invariance gives

\[
e^{-d_D(0,\psi(F^*_\alpha))} \leq \frac{\pi}{2} \omega_D(0, \psi(F^*_\alpha)).
\]

Moreover, by Theorem 2.5 we deduce that

\[
\omega_D(0, \psi(F^*_\alpha)) \leq 3\pi e^{-\pi \lambda_{D'}(\gamma, \psi(F^*_\alpha))},
\]

where \( \gamma \) is the arc of the circle passing through the points 0, 1, i such that \( \gamma \) connects i to 1 and \( \gamma \cap P = \emptyset \) and \( D' \) is the subdomain of \( D \) bounded by \( \gamma, [1, \alpha), [i, \alpha i] \), \( P \cap \partial D(0, \alpha) \) and \( \partial D \cap P \cap D(0, \alpha) \) (see Fig. 11). If \( \gamma_0 = \partial \mathbb{D} \cap P \) and \( D'' \) is the subdomain of \( D' \) bounded by \( \gamma_0, [1, \alpha), [i, \alpha i] \), \( P \cap \partial D(0, \alpha) \) and \( \partial D \cap P \cap \partial D(0, \alpha) \) (see Fig. 12), then by Theorem 2.2 we have

\[
\lambda_{D'}(\gamma, \psi(F^*_\alpha)) \geq \lambda_{D''}(\gamma_0, \psi(F^*_\alpha)).
\]

Combining the relations (4.1), (4.2), (4.3) and (4.4), we obtain

\[
e^{-d_D(0,F_\alpha)} \leq \frac{3\pi^2}{2} e^{-\pi \lambda_{D''}(\gamma_0, \psi(F^*_\alpha))}.
\]
Next we consider the crosscuts $\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_{n-1}$ of $D''$, where for $j = 1, 2, 3, \ldots, n - 1, \gamma_j$ is an arc of the circle with center at 0 and radius equal to the midpoint of $[\alpha_j, \alpha_{j+1}]$ as illustrated in Fig. 13.
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Setting
\[ m(Q_1) := \lambda_{D^\nu}(\gamma_0, \gamma_1), \ m(Q_2) := \lambda_{D^\nu}(\gamma_1, \gamma_2), \ldots, \ m(Q_n) := \lambda_{D^\nu}(\gamma_{n-1}, \psi(F_\alpha)) \]
by the serial rule we deduce that
\[ \lambda_{D^\nu}(\gamma_0, \psi(F_\alpha)) \geq m(Q_1) + m(Q_2) + \ldots + m(Q_n) \]
and thus by (4.5) and (4.6),
\[ e^{-d_D(0, F\alpha)} \leq \frac{3\pi^2}{2} e^{-\pi(m(Q_1)+m(Q_2)+\ldots+m(Q_n))}. \]

**Step 3:** Since harmonic measure is conformally invariant and \( N(\alpha) = 2^{n+2} \), we have
\[ \omega_D(0, F\alpha) = \omega_D(0, \psi(F_\alpha)) = \sum_{j=1}^{N(\alpha)} \omega_D(0, \psi(F_\alpha)^j) \geq N(\alpha) \omega_D(0, \psi(F_\alpha)^m) = 2^{n+2} \omega_D(0, \psi(F_\alpha)^m), \]
where \( \psi(F_\alpha)^j, \ j=1,2,3,\ldots,N(\alpha), \) are the components of \( \psi(F_\alpha) \) and
\[ \omega_D(0, \psi(F_\alpha)^m) = \min \left\{ \omega_D(0, \psi(F_\alpha)^j) : j \in \{1,2,3,\ldots,N(\alpha)\} \right\}. \]

If \( D^* \) is the subdomain of \( D \) bounded by \( [\frac{1}{2}, \alpha], \ [\frac{1}{2}, \alpha_i], \ \partial D \left(0, \frac{1}{2}\right) \setminus P, \ \partial D \left(0, \alpha\right) \cap P \) and \( \partial D \cap D(0, \alpha) \cap P \) as illustrated in Fig. 14, then applying Corollary 4.3.9 [20, p. 102] we obtain
\[ \omega_D(0, \psi(F_\alpha)^m) \geq \omega_{D^*}(0, \psi(F_\alpha)^m) \]
and hence
\[ \omega_D(0, F_\alpha) \geq 2^{n+2} \omega_{D^*}(0, \psi(F_\alpha)^m). \]

**Figure 14.** The simply connected domain \( D^* \) in case \( \alpha \in (\alpha_4, \alpha_5) \).
**Step 4:** If $h_m$ is the conformal map of $D^*$ onto $\mathbb{D}$ such that $h_m(0) = 0$ and $h_m(\psi(F_\alpha)^m) = \{e^{i\theta} : \theta \in [-t, t]\}$ for some $t \in [0, \pi]$ and $\gamma_m := h_m^{-1}([-1, 0])$, then by Theorem 2.6 there exists a positive constant $C_0$ such that

$$\omega_{D^*}(0, \psi(F_\alpha)^m) \geq C_0 e^{-\pi \lambda_{D^*}(\gamma_m, \psi(F_\alpha)^m)}.$$  

Furthermore, by Lemma 3.2 we infer that $\gamma_m \subset \overline{D}(0, \frac{1}{2})$. So, taking the crosscuts $\gamma_0 = \overline{D} \cap \partial D$ and $\gamma'_0 = \overline{D} \cap \partial D(0, e^{\frac{3\pi}{2}})$ of $D^*$ (see Fig. 15) and applying Lemma 3.3 we obtain

$$\lambda_{D^*}(\gamma_m, \psi(F_\alpha)^m) \leq 8.82 e^{-3\pi} - 3 + \lambda_{D^*}(\gamma_0, \gamma'_0) + \lambda_{D^*}(\gamma_0, \psi(F_\alpha)^m).$$

where $\lambda_{D^*}(\gamma_0, \gamma'_0) = 3$ and $\lambda_{D^*}(\gamma_m, \gamma'_0)$ is bounded from above by a positive constant $C_1$ for every $\alpha > 0$ and every $m$ (see [4, p. 370-371] for a similar estimate). By relations (4.8), (4.9) and (4.10), we get

$$\omega_{\mathbb{D}}(0, F_\alpha) \geq 2^{n+2} C_0 e^{(3-8.82e^{-3\pi})\pi} e^{-\pi \lambda_{D^*}(\gamma_0, \psi(F_\alpha)^m)}.$$  

![Figure 15](image1.png)  

**Figure 15.** The crosscuts $\gamma_0, \gamma'_0$ of $D^*$ and $\gamma_m$ in case $\alpha \in (\alpha_4, \alpha_5)$.

![Figure 16](image2.png)  

**Figure 16.** The crosscuts $\gamma_0, \gamma'_0$ of $D^*$ and $\gamma_m$ in magnification.

**Step 5:** Now we concentrate on $e^{-\pi \lambda_{D''}(\gamma_0, \psi(F_\alpha)^m)}$ or equivalently on $e^{-\pi \lambda_{D''}(\gamma_0, \psi(F_\alpha)^m)}$. First we take the crosscuts $\gamma_1^m, \gamma_2^m, \gamma_3^m, \ldots, \gamma_n^m$ of $D''$, where, for $j = 1, 2, 3, \ldots, n - 1$, $\gamma_j^m$ is an arc of the circle with center at 0 and radius equal to the midpoint of $[\alpha_j, \alpha_j+1]$ as illustrated in Fig. 17 and set

$$m(Q_1^m) := \lambda_{D''}(\gamma_0, \gamma_1^m), \ m(Q_2^m) := \lambda_{D''}(\gamma_1^m, \gamma_2^m), \ldots, \ m(Q_n^m) := \lambda_{D''}(\gamma_{n-1}^m, \psi(F_\alpha)^m).$$
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Figure 17. The simply connected domain $D''$ and the crosscuts $\gamma_1^m, \gamma_2^m, \gamma_3^m, \ldots, \gamma_{n-1}^m$ in case $\alpha \in (\alpha_4, \alpha_5)$.

Applying the conformal map $f(z) = \log z$ on $D''$ we take $f(D''), f(\gamma_0), f(\psi(F_\alpha)^m)$ and $f(\gamma_1^m), \ldots, f(\gamma_{n-1}^m), f(Q_1^m), \ldots, f(Q_n^m)$ illustrated in Fig. 18. The conformal invariance of extremal length implies that for every $j = 1, 2, 3, \ldots, n$,

$$\lambda_{D''}(\gamma_0, \psi(F_\alpha)^m) = \lambda_{f(D'')} (f(\gamma_0), f(\psi(F_\alpha)^m)), \quad m(Q_j^m) = m(f(Q_j^m)).$$

Figure 18. The image of $D''$ under the map $f(z) = \log z$ and its decomposition in case $\alpha \in (\alpha_4, \alpha_5)$. 
This leads us to consider the crosscuts $l_1, l_1', l_2, l_2', \ldots, l_{n-1}, l_{n-1}'$ of $f(D')$ so that, for $j = 1, 2, \ldots, n - 1$, each of $l_j$ is a segment which lies on a vertical line passing through the midpoint of $[f(\alpha_j), \frac{f(\alpha_j) + f(\alpha_{j+1})}{2}] = [4j\pi, 2(2j + 1)\pi]$ and $l_j'$ is the reflection of $l_j$ in $f(\gamma^m_j)$ (see Fig. 18). Now let $R_j$ be the rectangle formed by $\partial f(D')$ and $l_j, f(\gamma^m_j)$ for $j = 1, 2, 3, \ldots, n - 1$ as illustrated in Fig. 18. Since

$$m(R_1) = \frac{\pi}{\pi/4} = 2^2 > 3$$

$$m(R_2) = \frac{\pi}{\pi/8} = 2^3$$

$$\vdots$$

$$m(R_{n-1}) = \frac{\pi}{\pi/2^n} = 2^n,$$

we can apply Lemma 3.1 successively and obtain

$$0 \leq \lambda_{D'}(\gamma_0, \psi(F_{\alpha})^m) - m(Q_{n-2}^m) + m(Q_1^m) \leq 26.46 e^{-\pi m(R_1)}$$

$$0 \leq m(Q_1^m) - m(Q_2^m) + m(Q_2^m) \leq 26.46 e^{-\pi m(R_2)}$$

$$\vdots$$

$$0 \leq m(Q_{n-2}^m) + m(Q_{n-1}^m) \leq 26.46 e^{-\pi m(R_{n-1})},$$

where $m\left(\left(Q_j^m\right)^c\right)$ denotes the extremal length between $\gamma_j^m$ and $\psi(F_{\alpha})^m$ in $D'$ for every $j = 1, 2, \ldots, n - 1$ and thus $m\left(\left(Q_{n-1}^m\right)^c\right) = m(Q_n^m)$. Adding the inequalities above, we deduce that

$$0 \leq \lambda_{D'}(\gamma_0, \psi(F_{\alpha})^m) \leq 26.46 \sum_{j=1}^{n-1} e^{-\pi m(R_j)} + m(Q_1^m) + m(Q_2^m) + \ldots + m(Q_n^m),$$

where $m(R_j) = 2^{j+1}$. So,

$$e^{-\lambda_{D'}(\gamma_0, \psi(F_{\alpha})^m)} \geq e^{-26.46 \pi \sum_{j=1}^{n-1} e^{-2^{j+1}} \pi} e^{-\pi (m(Q_1^m) + m(Q_2^m) + \ldots + m(Q_n^m))}.$$
Setting
\[ C_3 := \frac{3\pi^2}{2} \frac{K}{C_0 C_2} e^{C_1 \pi} e^{-(3 - 8.82e^{-3\pi})\pi} \]
and using (4.7), (4.13) and (4.14), we infer that
\[ 2^{n+2} e^{-\pi(m(Q_1) + m(Q_2) + \ldots + m(Q_n))} \leq C_3 e^{-\pi(m(Q_1) + m(Q_2) + \ldots + m(Q_n))} \]
or equivalently
\[ 2^{n+2} \leq C_3 \]
for every \( n \in \mathbb{N} \). Finally, taking limits in (4.15) as \( n \to +\infty \), that is \( \alpha \to +\infty \), we obtain the contradiction
\[ \lim_{n \to +\infty} 2^{n+2} \leq C_3 < +\infty. \]
So, \( \forall K > 0 \exists \alpha \) such that
\[ \omega_D (0, F_\alpha) \geq K e^{-d_D(0, F_\alpha)}. \]
\[ \square \]

5 The second example

The main feature of \( D \) which plays a central role in the proof of Theorem 4.1 is that as \( \alpha \to +\infty \), the number of components of \( \psi(F_\alpha) \) tends to infinity. Next we prove that even if the number of components of \( \psi(F_\alpha) \) is bounded from above by a positive constant for every \( \alpha > 0 \), the answer to the Question 1.1 is still negative. To verify this, we need the following lemma whose proof is straightforward and thus is omitted.

Lemma 5.1. Let \( \Gamma \) be the geodesic between two points \( z_1, z_2 \in \partial D \) in \( \mathbb{D} \). Then
\[ e^{-d_\mathbb{D}(0, \Gamma)} \leq \omega_D (0, \Gamma) \leq \frac{4}{\pi} e^{-d_\mathbb{D}(0, \Gamma)}. \]

Theorem 5.1. There exists an unbounded simply connected domain \( D \) with the following properties:
Let \( \psi \) be a conformal map of \( \mathbb{D} \) onto \( D \). If \( F_\alpha = \{ z \in \mathbb{D} : |\psi(z)| = \alpha \} \) for \( \alpha > 0 \), then
(1) \( \psi(F_\alpha) \) is a connected set for every \( \alpha > 0 \) and
(2) \( \forall K > 0 \exists \alpha \) such that
\[ \omega_D (0, F_\alpha) \geq K e^{-d_\mathbb{D}(0, F_\alpha)}. \]

Proof. Let \( D \) be the simply connected domain of Fig. 19, namely
\[ D = \{ z \in \mathbb{C} \setminus \mathbb{D} : |\text{Arg} \, z| < 1 \} \setminus \bigcup_{n=1}^{+\infty} \left\{ z \in \partial D (0, e^n) : \frac{1}{40^n} \leq |\text{Arg} \, z| < 1 \right\}. \]
The Riemann Mapping Theorem implies that there exists a conformal map \( \psi \) from \( \mathbb{D} \) onto \( D \) such that \( \psi(0) = e^{\frac{i}{4}} \). Let \( \alpha_n = e^{\pi - \frac{1}{10n}} \) for every \( n \in \mathbb{N} \) and take the arcs \( \partial D(0, \alpha_n) \cap \overline{D} \) as illustrated in Fig. 19. Now fix a number \( n > 1 \). If \( \Gamma_{\alpha_n} \) is the geodesic joining \( \psi^{-1}(\alpha_ne^{-i}) \) to \( \psi^{-1}(\alpha_ne^i) \) in \( \mathbb{D} \) and \( S_{\alpha_n} \) denotes the arc of \( \partial \mathbb{D} \) between \( \psi^{-1}(\alpha_ne^{-i}) \) and \( \psi^{-1}(\alpha_ne^i) \) (see Fig. 20), then by Lemma 5.1 and [4, p. 370] we get

\[
\omega_{\mathbb{D}}(0, F_{\alpha_n}) \geq \omega_{\mathbb{D}}(0, S_{\alpha_n}) = \frac{1}{2} \omega_{\mathbb{D}}(0, \Gamma_{\alpha_n}) \geq \frac{1}{2} e^{-d_{\mathbb{D}}(0, \Gamma_{\alpha_n})} = \frac{1}{2} e^{-d_{\mathbb{D}}(e^{\frac{i}{4}}, \psi(\Gamma_{\alpha_n}))}.
\]
Since $\psi$ preserves the geodesics and $D$ is symmetric with respect to the real axis, we deduce that 
\[ d_D\left( e^{\frac{1}{4}}, \psi\left( \Gamma_{\alpha_n} \right) \right) = d_D\left( e^{\frac{1}{4}}, b_n \right), \]
where $b_n \in (e^{n-1}, e^n)$. So, 
\[ \omega_D\left( 0, F_{\alpha_n} \right) \geq \frac{1}{2} e^{-d_D\left( e^{\frac{1}{4}}, b_n \right)}. \]

Notice that if $g_D\left( e^{\frac{1}{4}}, z \right)$ denotes the Green’s function for $D$ (see [8, p. 41-43], [20, p. 106-115]), then 
\[ d_D\left( e^{\frac{1}{4}}, z \right) = \log \frac{1 + e^{-g_D\left( e^{\frac{1}{4}}, z \right)}}{1 - e^{-g_D\left( e^{\frac{1}{4}}, z \right)}} \]
(see [2, p. 12-13] and [20, p. 106]). Consider the conformal map $h(z) = \log z$ of $D$ onto $D' := h(D)$ (see Fig. 21). For every $\alpha'_n \in \psi\left( F_{\alpha_n} \right) \setminus \{\alpha_n\}$, we infer, by a symmetrization result, that 
\[ g_D\left( e^{\frac{1}{4}}, \alpha_n \right) = g_{D'}\left( e^{\frac{1}{4}}, \log \alpha_n \right) \geq g_{D'}\left( e^{\frac{1}{4}}, \log \alpha'_n \right) = g_D\left( e^{\frac{1}{4}}, \alpha'_n \right), \]
(see Lemma 9.4 [10, p. 659]). Since 
\[ f(x) = \log \frac{1 + e^{-x}}{1 - e^{-x}} \]
is a decreasing function on $(0, +\infty)$, we have that $d_D\left( e^{\frac{1}{4}}, \psi(F_{\alpha_n}) \right) = d_D\left( e^{\frac{1}{4}}, \alpha_n \right)$. Thus, 
\[ e^{-d_D\left( 0, F_{\alpha_n} \right)} = e^{-d_D\left( e^{\frac{1}{4}}, \psi(F_{\alpha_n}) \right)} = e^{-d_D\left( e^{\frac{1}{4}}, \alpha_n \right)}. \]

\( \text{Figure 21. The simply connected domain } D'. \)
Applying Beurling-Nevanlinna projection theorem \[1\] p. 43, we get
\[
\omega_D \left( \frac{1}{40^n}, \left\{ iy : \frac{1}{40^n} \leq |y| \leq 1 \right\} \right) \geq \omega_D \left( \frac{1}{40^n}, \left[ -1, -\frac{1}{40^n} \right] \right) \geq \omega_D \left( \frac{1}{40}, \left[ -1, -\frac{1}{40} \right] \right),
\]
where
\[
\omega_D \left( \frac{1}{40}, \left[ -1, -\frac{1}{40} \right] \right) = \frac{2}{\pi} \arcsin \left( \frac{1}{1 + \frac{1}{40}} \right) = 0.719987303 > 0.7
\]
Therefore,
\[
\omega_D (\alpha_n, \psi (S_{\alpha_n})) > 0.7 > 0.5 = \omega_D (b_n, \psi (S_{\alpha_n}))
\]
which implies that \( b_n < \alpha_n \) and thus
\[
(5.3)
\]

Now suppose there exists a positive constant \( K \) such that for every \( \alpha > 0 \),
\[
\omega_D (0, F_\alpha) \leq K e^{-d_D (0, F_\alpha)}.
\]
Combining the relations (5.1), (5.2), (5.3) and (5.4), we infer that
\[
\frac{1}{2} e^{-d_D (e^{\frac{1}{4}} \cdot b_n)} \leq \omega_D (0, F_\alpha) \leq K e^{-d_D (0, F_\alpha)} = K e^{-d_D (e^{\frac{1}{4}} \cdot b_n)} e^{-d_D (b_n, \alpha_n)}
\]
or
\[
(5.5)
\]
for every \( n > 1 \). However, using the quasi-hyperbolic distance defined in Section 2 we get
\[
d_D (b_n, \alpha_n) = d_D (\log b_n, \log \alpha_n) \geq \frac{1}{2} d_D (\log b_n, \log \alpha_n) = \frac{1}{2} \int_{\log b_n}^{\log \alpha_n} \frac{dx}{d(x, \partial D')} 
\]
\[
\geq \frac{1}{2} \int_{\log b_n}^{\log \alpha_n} \frac{dx}{\sqrt{\left( \frac{1}{10^n} \right)^2 + (n - x)^2}} = -\frac{1}{2} \arcsinh \left( 40^n (n - x) \right) \bigg|_{\log b_n}^{\log \alpha_n}
\]
\[
= -\frac{1}{2} \arcsinh (1) + \frac{1}{2} \arcsinh (40^n (n - \log b_n))
\]
\[
(5.6)
\]
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where $k$ is a positive constant whose existence comes from the fact that for every $n > 1$,

$$n - \log b_n \geq \log \alpha_n - \log b_n$$

and

$$\omega_{D'} (\log b_n, h \circ \psi (S_{\alpha_n})) = \frac{1}{2} < 0.7 < \omega_{D'} (\log \alpha_n, h \circ \psi (S_{\alpha_n})), $$

as we proved before. Finally, taking limits in (5.6) as $n \to +\infty$, we obtain the contradiction to the relation (5.5),

$$\lim_{n \to +\infty} d_D (b_n, \alpha_n) = +\infty.$$

So, $\forall K > 0 \exists \alpha$ such that

$$\omega_D (0, F_\alpha) \geq Ke^{-d_D (0, F_\alpha)}.$$  

□

6 Proof of Theorem 1.1

Proof. Because of the assumption (1) and the additivity of harmonic measure we may assume that $N (\alpha) = 1$. We map conformally $D$ onto the strip $S = \{ z \in \mathbb{C} : |\text{Im} z | < 1 \}$ so that

$$0 \mapsto z_0 \in i \mathbb{R}^+, \Gamma_\alpha \mapsto \mathbb{R}.$$  

Let $F'_\alpha$ be the image of $F_\alpha$. By assumption (2), there exists a positive constant $c = c (c_2)$ such that $c < |z_0|$ and $F'_\alpha \subset \{ z \in S : |\text{Im} z | < c \}$ for every $\alpha > 0$ (see Fig. 23).

![Diagram](image-url)  

**Figure 23.** The conformal mapping of $D$ onto the strip $S$.

Set $S_1 = \{ z \in S : \text{Im} z = c \}$ and $S_2 = \{ z \in S : \text{Im} z = -c \}$. Then we have

$$\omega_D (0, F_\alpha) = \omega_S (z_0, F'_\alpha) \leq \omega_S (z_0, S_1).  \tag{6.1}$$

Notice that by symmetry, for every $z \in S_1$,

$$\omega_S (z, S_2) = \omega_S (ic, S_2) = \frac{1 - c}{1 + c},$$

where the second equality comes from [20] p. 100]. Therefore, the strong Markov property for harmonic measure (see [3] p. 282]) implies that

$$\omega_S (z_0, S_2) = \int_{S_1} \omega_S (z, S_2) \omega_S (z_0, dz) = \omega_S (z_0, S_1) \omega_S (ic, S_2)$$

or

$$\omega_S (z_0, S_1) = \frac{1 + c}{1 - c} \omega_S (z_0, S_2). \tag{6.2}$$
Combining the relations (6.1) and (6.2), we get

\[ \omega_D (0, F_\alpha) \leq \frac{1 + c}{1 - c} \omega_S (z_0, S_2) \leq \frac{1 + c}{1 - c} \omega_S (z_0, \mathbb{R}). \]

Conformal invariance and Lemma 5.1 imply that

\[ \omega_S (z_0, \mathbb{R}) \leq \frac{4}{\pi} e^{-d_S(z_0, \mathbb{R})} \]

which in conjunction with (6.3) leads to

\[ \omega_D (0, F_\alpha) \leq \frac{4}{\pi} \frac{1 + c}{1 - c} e^{-d_S(z_0, \mathbb{R})} \leq \frac{4}{\pi} \frac{1 + c}{1 - c} e^{-d_S(z_0, S_1)}. \]

But

\[ d_S (z_0, S_2) = d_S (z_0, -ic) = d_S (z_0, ic) + d_S (ic, -ic) \]

\[ = d_S (z_0, S_1) + d_S (-ic, ic), \]

where by \[2\] p. 31,

\[ d_S (-ic, ic) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{\cos t} = \log \frac{1 + \sin (c\pi/2)}{1 - \sin (c\pi/2)}. \]

Combining the relations (6.5) and (6.6), we infer that

\[ -d_S (z_0, S_1) = -d_S (z_0, S_2) + \log \frac{1 + \sin (c\pi/2)}{1 - \sin (c\pi/2)} \leq -d_S (z_0, F_\alpha) + \log \frac{1 + \sin (c\pi/2)}{1 - \sin (c\pi/2)}. \]

This together with (6.4) give

\[ \omega_D (0, F_\alpha) \leq \frac{4}{\pi} \frac{1 + c + 1 + \sin (c\pi/2)}{1 - c - \sin (c\pi/2)} e^{-d_S(z_0, F_\alpha')} = \frac{4}{\pi} \frac{1 + c + 1 + \sin (c\pi/2)}{1 - c - \sin (c\pi/2)} e^{-d_D(0,F_\alpha)}. \]

Thus, setting \( K := \frac{4}{\pi} \frac{1 + c + 1 + \sin (c\pi/2)}{1 - c - \sin (c\pi/2)} \), we finally get that for every \( \alpha > 0 \),

\[ \omega_D (0, F_\alpha) \leq Ke^{-d_D(0,F_\alpha)}. \]
On a relation between harmonic measure and hyperbolic distance on planar domains

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