A Perturbative Approach for the Asymptotic Evaluation of the Neumann Value
Corresponding to the Dirichlet Datum of a Single Periodic Exponential for the NLS

Guenbo Hwang
Department of Mathematics, Daegu University,
Gyeongsan Gyeongbuk, 712-714, Korea
ghwang@daegu.ac.kr

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Boundary value problems for the nonlinear Schrödinger equation formulated on the half-line can be analyzed by the Fokas method. For the Dirichlet problem, the most difficult step of this method is the characterization of the unknown Neumann boundary value. For the case that the Dirichlet datum consists of a single periodic exponential, namely, \( a \exp(i\omega t) \), \( a, \omega \) real, it has been shown in [2–4] that if one assumes that the Neumann boundary value is given for large \( t \) by \( c \exp(i\omega t) \), then \( c \) can be computed explicitly in terms of \( a \) and \( \omega \). Here, using the perturbative approach introduced in [16], it is shown that for typical initial conditions, it is indeed the case that at least up to third order in a perturbative expansion the Neumann boundary value is given by \( c \exp(i\omega t) \) and the value of \( c \) is at least up to this order the value found in [2–4].

**Keywords:** Initial-boundary value problem; Generalized Dirichlet to Neumann map; Nonlinear Schrödinger equation.

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1. Introduction

A unified method for analyzing boundary value problems, extending ideas of the so-called inverse scattering transform method, was introduced in [5], see also [6] and the review [7] (as well as the review [12] for the implementation of the unified method to linear PDEs).

For *integrable nonlinear* evolution PDEs, the most difficult step in the implementation of this so-called Fokas method, is the characterization of the Dirichlet to Neumann map. For example, for the nonlinear Schrödinger equation (NLS) on the half-line with given initial \( q(x,0) = q_0(x) \) and Dirichlet \( q(0,t) = g_0(t) \) data, this involves characterizing the unknown Neumann boundary value \( q(0,t) = g_1(t) \) in terms of \( g_0(x) \) and \( g_0(t) \). In this respect we note: (i) for certain particular boundary conditions called *linearizable*, the above characterization can be achieved via explicit formulas and hence for these cases the Fokas method is as effective as the inverse scattering transform method. (ii) If \( g_0(t) \) vanishes as \( t \to \infty \), it is possible to bypass the characterization of the Dirichlet to Neumann map and to obtain the asymptotic form of the solution as \( t \to \infty \) [8–10, 15]. (iii) If \( g_0(t) \) is a periodic function, then in order to obtain the large \( t \) asymptotics of the solution \( q(x,t) \), it is first necessary to determine the asymptotic form of \( g_1(t) \) as \( t \to \infty \).

Pioneering results regarding (iii) have been obtained in a series of papers by Boutet de Monvel and co-authors [2–4]. The final result of these authors is the following: Consider the NLS on the
half-line
\[ iq_t + q_{xx} - 2\lambda |q|^2 q = 0, \quad x > 0, \quad t > 0, \]
\[ (1.1) \]
with \( \lambda = \pm 1 \) and a single periodic exponential as the Dirichlet datum,
\[ q(0,t) = ae^{i\omega t}, \quad a, \omega \text{ real.} \]
\[ (1.2) \]
Assume that the Neumann boundary value is asymptotically periodic, namely,
\[ q_x(0,t) = ce^{i\omega t} + o(1), \quad t \to \infty. \]
\[ (1.3) \]
Then, \( c \) can be obtained explicitly in terms of \( a \) and \( \omega \):
\[ \omega > 0 : \quad c = -a\sqrt{\omega + \lambda a^2}, \]
\[ \omega < 0 : \quad c = ia\sqrt{-\omega - 2\lambda a^2}. \]
\[ (1.4a, 1.4b) \]
An effective characterization of the Dirichlet to Neumann map for the NLS equation on the half-line was recently presented in [11] and [16] (see also [13, 14] for the application of the sine-Gordon and the modified Korteweg-de Vries equations formulated on the half-line) using two different formulations, both of which are based on the analysis of the global relation: the formulation in [11] is based on the eigenfunctions involved in the definition of the spectral functions \( \{A(k), B(k)\} \), whereas the formulation in [16] is based on an extension of the Gelfand-Levitan-Marchenko approach first introduced in [1]. In particular, in [16] a perturbative approach was introduced for the explicit construction of the Neumann boundary value as \( t \to \infty \). Using this approach, we show here that for the Dirichlet datum (1.2) and the initial datum
\[ q(x,0) = ae^{-\eta x}, \quad \eta > 0, \quad 0 < x < \infty, \]
\[ (1.5) \]
the Neumann boundary value is indeed given up to third order in \( a \) by (1.3), where \( c \) satisfies, at least up to this order, equations (1.4).

2. The Main Result

Theorem 2.1. Let
\[ q(x,t) = \varepsilon q_1(x,t) + \varepsilon^2 q_2(x,t) + \cdots, \quad \varepsilon \to 0, \]
be the perturbation solution \( q(x,t) \) of the NLS on the half-line with the initial data
\[ q(x,0) = \varepsilon e^{-\eta x}, \quad \eta > 0 \]
\[ (2.1) \]
and the Dirichlet boundary data
\[ q(0,t) = \varepsilon g_{01}(t) + O(\varepsilon^4), \quad \varepsilon \to 0, \]
\[ (2.2) \]
where
\[ g_{01}(t) = e^{i\omega t}, \quad \omega \in \mathbb{R}. \]
\[ (2.3) \]
Then,
\[ q_x(0,t) = \varepsilon g_{11}(t) + \varepsilon^3 g_{13}(t) + O(\varepsilon^4), \quad \varepsilon \to 0, \]
\[ (2.4) \]
where \( g_{11} \) and \( g_{13} \) are given by the following formulas:
\( (i) \ \omega > 0 \)

\[
\begin{align*}
g_{11}(t) &= -\sqrt{\omega} e^{i\omega t} + o(1), \quad t \to \infty, \\
g_{13}(t) &= -\frac{\lambda}{2\sqrt{\omega}} e^{i\omega t} + o(1), \quad t \to \infty.
\end{align*}
\]  

\( (ii) \ \omega < 0 \)

\[
\begin{align*}
g_{11}(t) &= i\sqrt{-\omega} e^{i\omega t} + o(1), \quad t \to \infty, \\
g_{13}(t) &= -i\frac{\lambda}{\sqrt{-\omega}} e^{i\omega t} + o(1), \quad t \to \infty.
\end{align*}
\]

**Proof.** It is shown in [11] that

\[
q_\lambda(0,t) = \frac{2}{i\pi} \int_{\partial D_3} \left[ i\Phi_1(t,k) + k \Phi_1(t,k) \right] dk + \frac{2q_0(t)}{\pi} \int_{\partial D_3} \left( \Phi_2(t,k) - \Phi_2(t,-k) \right) dk - \frac{4}{i\pi} \int_{\partial D_3} \frac{k e^{-i4\kappa^2 t} b_1(-k) d(-k) \Phi_2(t,-k)dk}{a(-k)},
\]

where \( \partial D_3 \) is the oriented boundary of the third quadrant of the complex \( k \)-plane and \( \{\Phi_1, \Phi_2\} \) solve the following system of equations:

\[
\begin{align*}
\Phi_1(t,k) &= \int_0^1 e^{i4\kappa^2(t-t')} \left[ -i\lambda |q(0,t')|^2 \Phi_1(t',k) + (2kq(0,t') + i\Phi_1(t',k)) \Phi_2(t',k) \right] dt', \\
\Phi_2(t,k) &= 1 + \lambda \int_0^1 \left( 2kq(0,t') - i\Phi_1(t',k) \right) \Phi_1(t',k) + i|q(0,t')|^2 \Phi_2(t',k) dt'.
\end{align*}
\]

Substituting in the above equation (2.2) and (2.4), as well as (2.7b) and (2.7c), we find the following equations:

\[
\begin{align*}
g_{11}(t) &= \frac{2}{i\pi} \int_{\partial D_3} \left\{ k \Phi_{11}(t,k) - \Phi_{11}(t,-k) + ig_{01}(t) \right\} dk - \frac{4}{i\pi} \int_{\partial D_3} \frac{k e^{-i4\kappa^2 t} b_1(-k) d(-k) \Phi_2(t,-k)dk}{a(-k)}, \\
g_{13}(t) &= \frac{2}{i\pi} \int_{\partial D_3} \left\{ k \Phi_{13}(t,k) - \Phi_{13}(t,-k) \right\} dk + \frac{2}{\pi} g_{01}(t) \int_{\partial D_3} \left\{ \Phi_{22}(t,k) - \Phi_{22}(t,-k) \right\} dk \\
&\quad - \frac{4}{i\pi} \int_{\partial D_3} \frac{k e^{-i4\kappa^2 t} b_1(-k) \Phi_2(t,-k) - a_2(-k)} d(-k) \Phi_2(t,-k)dk, \\
\Phi_{11}(t,k) &= e^{-i4\kappa^2 t} \int_0^t e^{i4\kappa^2 \tau} \left( 2kg_{01}(\tau) + ig_{11}(\tau) \right) d\tau, \\
\lambda \Phi_{22}(t,k) &= \int_0^t \left[ 2k\tilde{g}_{01}(\tau) - i\tilde{g}_{11}(\tau) \right] \Phi_{11}(\tau,k) d\tau + i \int_0^t |g_{01}(\tau)|^2 d\tau, \\
\lambda \Phi_{13}(t,k) &= e^{-i4\kappa^2 t} \int_0^t e^{i4\kappa^2 \tau} \left[ -i|g_{01}|^2 \Phi_{11}(\tau,k) + (2kg_{01} + ig_{11}) \lambda \Phi_{22}(\tau,k) \right] d\tau.
\end{align*}
\]

where the functions \( b_1(k) \) and \( a_2(k) \) are given by

\[
\begin{align*}
b_1(k) &= -\int_0^\infty e^{2ikx} e^{-\eta x} dx, \\
a_2(k) &= \lambda \int_0^\infty e^{-\eta x} \int_x^\infty e^{-2ik(x-x')} e^{-\eta x'} dx' dx.
\end{align*}
\]
3. A Perturbative Approach for $\omega > 0$

**Proposition 3.1.** For $\omega > 0$,

$$g_{11}(t) = -\sqrt{\omega}e^{\text{i}\omega t} + \int_{\partial \tilde{D}_3} e^{-4\text{i}k_3^2 t}A(k_3^2)dk_3 + \frac{4}{i\pi} \int_{\partial \tilde{D}_3} \frac{ke^{-4\text{i}k^2 t}}{2ik + \eta}dk,$$

(3.1)

where the contour $\partial \tilde{D}_3$ is depicted in figure 1 and

$$A(k^2) = \frac{8k^2}{\pi(4k^2 + \omega)},$$

(3.2)

Furthermore,

$$\int_{\partial \tilde{D}_3} e^{-4\text{i}k^2 t}A(k^2)dk_3 = O(t^{-3/2}), \quad t \to \infty,$$

(3.3a)

and

$$\int_{\partial \tilde{D}_3} \frac{ke^{-4\text{i}k^2 t}}{2ik + \eta}dk = O(t^{-3/2}), \quad t \to \infty.$$  

(3.3b)

**Proof.** Using (2.13), we find

$$b_1(-k) = -\int_0^\infty e^{-(2ik + \eta)x}dx = -\frac{1}{2ik + \eta},$$

(3.4)

and then substituting (3.4) into (2.8), we obtain the last terms in (3.1).

Note that equation (2.10) yields

$$k[\Phi_{11}(t,k) - \Phi_{11}(t,-k)] = -\frac{4ik^2}{4k^2 + \omega}(e^{\text{i}\omega t} - e^{-4\text{i}k^2 t}).$$

(3.5)

Inserting the rhs of (3.5) into the rhs of (2.8), we find

$$\frac{2}{i\pi} \int_{\partial \tilde{D}_3} \{k[\Phi_{11}(t,k) - \Phi_{11}(t,-k)] + ig_0(t)\}dk = \frac{2}{\pi} \int_{\partial \tilde{D}_3} \frac{4k^2 e^{-4\text{i}k^2 t} + \omega e^{\text{i}\omega t}}{4k^2 + \omega}dk.$$  

(3.6)
Indeed, let us denote the integral in (3.1) by the rhs of (3.7), we find the first two terms in (3.1).

For the second integral in (3.8), let $k = \rho e^{3i\pi/4}$, we find

$$I = \left( \int_{C_1} + \int_{C_-} \right) \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk.$$  \hfill (3.8)

For the first integral in (3.8), letting $k = \rho e^{i\theta}$ with $3\pi/4 \leq \theta \leq \pi$. Thus, for large $R$, we obtain

$$\left| \int_{C_R} \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk \right| \leq \int_{3\pi/4}^{\pi} \frac{R^3 e^{4ik^2t} \sin 2\theta}{4R^2 - |\omega|} d\theta = \frac{\pi R^3}{16R^2 - 4|\omega|} [I_0(4R^2t) - L_0(4R^2t)],$$  \hfill (3.10)

where $I_0(z)$ is the modified Bessel function and $L_0(z)$ is the modified Struve function. Using the asymptotics

$$I_0(z) - L_0(z) \sim -\frac{2}{\pi z}, \quad |z| \to \infty,$$

we find that the rhs of (3.10) vanishes as $R \to \infty$ and hence the second integral in (3.8) is identically zero.

Similarly, we can show that the integral along the part of $\partial \bar{D}_3$ involving the negative imaginary axis yields the same expression as in (3.9). Thus,

$$\int_{\partial \bar{D}_3} \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk = -2ie^{-i\pi/4} \int_0^\infty \frac{\rho^2 e^{-4\rho^2t}}{4\rho^2 - \omega} d\rho.$$  \hfill (3.11)

A stationary point calculation implies that the leading order contribution vanishes. The leading order contribution from integration by parts also vanishes, thus we obtain (3.3a). Equation (3.3b) can be derived in a similar way.

Note that the integrand in the rhs of (3.6) has removable singularities at $k = -i\sqrt{\omega}/2$. In what follows it will be necessary to split the integral in (3.6) in two separate integrals. Thus, before splitting we deform the contour $\partial D_3$ to the contour $\partial \bar{D}_3$. Hence, (3.6) becomes

$$g_1(t) = \frac{2}{\pi} \int_{\partial \bar{D}_3} \frac{e^{i\omega t}}{4k^2 + \omega} dk + \frac{2}{\pi} \int_{\partial D_3} \frac{4k^2 e^{-4ik^2t}}{4k^2 + \omega} dk.$$  \hfill (3.7)

The above splitting is consistent with the fact that $e^{-4ik^2t}$ decays in the second and fourth quadrants of the complex $k$-plane as $t \to \infty$. Using the residue theorem to compute the first integral in the rhs of (3.7), we find the first two terms in (3.1).

It is straightforward to estimate the large $r$ behavior of the integrals appearing in the rhs of (3.1). Indeed, let us denote the integral in (3.1) by $\frac{8}{3} I$. Using the Cauchy theorem, we deform the part of $\partial \bar{D}_3$ involving the negative real axis along the ray $\arg k = 3\pi/4$, which we denote by $C_1$ (cf. figure 3), and along $C_{\infty}$, where $C_{\infty}$ is the limit of $C_R = \{ |k| = R, 3\pi/4 \leq \arg k \leq \pi \}$ as $R \to \infty$:

$$I = \left( \int_{C_1} + \int_{C_-} \right) \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk.$$  \hfill (3.8)

For the first integral in (3.8), letting $k = \rho e^{3i\pi/4}$, we find

$$ie^{3i\pi/4} \int_0^\infty \frac{\rho^2 e^{-4\rho^2t}}{4\rho^2 - \omega} d\rho = -ie^{-i\pi/4} \int_0^\infty \frac{\rho^2 e^{-4\rho^2t}}{4\rho^2 - \omega} d\rho.$$  \hfill (3.9)

For the second integral in (3.8), let $k = Re^{i\theta}$ with $3\pi/4 \leq \theta \leq \pi$. Thus, for large $R$, we obtain

$$\left| \int_{C_R} \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk \right| \leq \int_{3\pi/4}^{\pi} \frac{e^{4ik^2t} \sin 2\theta}{4R^2 - |\omega|} d\theta = \frac{\pi R^3}{16R^2 - 4|\omega|} [I_0(4R^2t) - L_0(4R^2t)],$$  \hfill (3.10)

where $I_0(z)$ is the modified Bessel function and $L_0(z)$ is the modified Struve function. Using the asymptotics

$$I_0(z) - L_0(z) \sim -\frac{2}{\pi z}, \quad |z| \to \infty,$$

we find that the rhs of (3.10) vanishes as $R \to \infty$ and hence the second integral in (3.8) is identically zero.

Similarly, we can show that the integral along the part of $\partial \bar{D}_3$ involving the negative imaginary axis yields the same expression as in (3.9). Thus,

$$\int_{\partial \bar{D}_3} \frac{k^2 e^{-4ik^2t}}{4k^2 + \omega} dk = -2ie^{-i\pi/4} \int_0^\infty \frac{\rho^2 e^{-4\rho^2t}}{4\rho^2 - \omega} d\rho.$$  \hfill (3.11)

A stationary point calculation implies that the leading order contribution vanishes. The leading order contribution from integration by parts also vanishes, thus we obtain (3.3a). Equation (3.3b) can be derived in a similar way.
We write $g_{11}(t)$ as

$$g_{11}(t) = g_{11}^{(1)}(t) + g_{11}^{(2)}(t),$$  

(3.11)

where

$$g_{11}^{(1)}(t) = -\sqrt{4\pi}e^{i\omega t} + \int_{\partial D_3} e^{-4ik^2t}A(k^3)dk_3$$  

(3.12a)

and

$$g_{11}^{(2)}(t) = \frac{4}{i\pi} \int_{\partial D_3} \frac{ke^{-4ik^2t}}{2ik + \eta}dk.$$  

(3.12b)

Also, we denote $\Phi_{11}(t,k)$ by

$$\Phi_{11}(t,k) = \Phi_{11}^{(1)}(t,k) + \Phi_{11}^{(2)}(t,k),$$  

(3.13)

where

$$\Phi_{11}^{(1)}(t,k) = e^{-4ik^2t} \int_0^t e^{4ik^2\tau}(2kg_{01}(\tau) + ig_{11}^{(1)}(\tau))d\tau$$

(3.14a)

and

$$\Phi_{11}^{(2)}(t,k) = ie^{-4ik^2t} \int_0^t e^{4ik^2\tau}g_{11}^{(2)}(\tau)d\tau.$$  

(3.14b)

**Lemma 3.1.** For $\omega > 0$,

$$\Phi_{11}^{(1)}(t,k) = \frac{i(e^{-4ik^2t} - e^{i\omega t})}{2k + i\sqrt{\omega}} + \frac{ie^{-4ik^2t}}{2\sigma k - i\sqrt{\omega}} + \frac{1}{4} \int_{\partial D_3} \frac{e^{-4ik^2t}}{k^2 - k_3^2}A(k^3)dk_3,$$  

(3.15)

where the contour $\partial D_3^\sigma$ is depicted in figure 1 and $\sigma = 1$ if $k \in \partial D_3$, whereas $\sigma = -1$ if $-k \in \partial D_3$.

**Proof.** Substituting the expression for $g_{11}^{(1)}$ in (3.14a) and integrating with respect to $d\tau$, we obtain

$$\Phi_{11}^{(1)}(t,k) = \frac{i(e^{-4ik^2t} - e^{i\omega t})}{2k + i\sqrt{\omega}} + \frac{1}{4} \int_{\partial D_3} \frac{e^{-4ik^2t} - e^{-4ik^2t}}{k^2 - k_3^2}A(k^3)dk_3.$$  

(3.16)

The integrand in the rhs of (3.16) has removable singularities at $k_3 = k$ and $k_3 = -k$. In what follows it will be necessary to split the integral in (3.16) in two separate integrals. Thus, before splitting we deform the contour $\partial D_3$ to the contour $\partial D_3^{\sigma k}$ shown in figure 1. One of these two integrals can be computed exactly via the residue theorem:

$$-\frac{2e^{-4ik^2t}}{\pi} \int_{\partial D_3} \frac{k_3^2}{(4k_3^3 + \omega)(k^3 - k_3^3)}dk_3$$

$$= -4ie^{-4ik^2t} \left[ -\frac{\sigma k}{2(4k^2 + \omega)} + \frac{k_3}{8(k^2 - k_3^2)} \right]_{k_3 = i\sqrt{\omega}/2} = \frac{ie^{-4ik^2t}}{2\sigma k - i\sqrt{\omega}}.$$  

Substituting this expression in (3.16), where $\partial D_3$ is replaced with $\partial D_3^{\sigma k}$, we find (3.15).

For $k = 0$, a separate analysis shows that the above integral equals $-1/\sqrt{\omega}$, which agrees with the limit of the above expression as $k \to 0$. 

\[\square\]
Lemma 3.2. For $\omega > 0$,
\[
\Phi^{(2)}_{11}(t, k) = \frac{e^{-4ik^2t}}{2i\sigma k + \eta} + \frac{1}{i\pi} \int_{\partial \hat{D}_2} \frac{k_3 e^{-4ik^2t}}{(2ik_3 + \eta)(k^2 - k_3^2)} \, dk_3. \tag{3.17}
\]
Moreover,
\[
\Phi_{11}(t, -\bar{k}) = \Phi^{(1)}_{11}(t, -\bar{k}) = \frac{e^{4ik^2t}}{2i\sigma k - \eta} + \frac{1}{i\pi} \int_{\partial \hat{D}_2} \frac{k_2 e^{4ik^2t}}{(2ik_2 - \eta)(k^2 - k_2^2)} \, dk_2, \tag{3.18}
\]
where the contour $\partial \hat{D}_2$ is depicted in figure 2 and
\[
\Phi^{(1)}_{11}(t, -\bar{k}) = \frac{i(e^{4ik^2t} - e^{-i\omega t})}{2k + i\sqrt{\omega}} - \frac{ie^{4ik^2t}}{2\sigma k + i\sqrt{\omega}} + \frac{1}{4} \int_{\partial \hat{D}_2} e^{4ik^2t} \, dk_2. \tag{3.19}
\]
Proof. Substituting (3.12b) into (3.14b) and integrating the resulting expression with respect to $d\tau$, we find that $\Phi^{(2)}_{11}(t, k)$ is given by
\[
\Phi^{(2)}_{11}(t, k) = \frac{e^{-4ik^2t}}{i\pi} \int_{\partial D_2} \frac{k_3}{2ik_3 + \eta} \left( \frac{e^{4ik^2t}}{k^2 - k_3^2} - 1 \right) \, dk_3. \tag{3.20}
\]
The integrand in (3.20) has removable singularities at $k_3 = \pm k$. Thus, before splitting the integral we deform the contour $\partial D_2$ into $\partial \hat{D}_2$. Note that it is not necessary to avoid $k = -i\sqrt{\omega}/2$. Using the residue theorem, one of these integrals can be computed as
\[
-\frac{e^{-4ik^2t}}{i\pi} \int_{\partial \hat{D}_2} \frac{k_3}{(2ik_3 + \eta)(k^2 - k_3^2)} \, dk_3 = \frac{e^{-4ik^2t}}{2i\sigma k + \eta}
\]
and then we find $\Phi^{(2)}_{11}(t, k)$ given in (3.17).

Regarding $\Phi_{11}(t, -\bar{k})$, we note that
\[
\Phi_{11}(t, -\bar{k}) = -e^{4ik^2t} \int_{0}^{t} e^{-4ik^2\tau} [2ke^{-i\omega \tau} + i\bar{g}_{11}(\tau)] \, d\tau \tag{3.21}
\]
and $\bar{g}_{11}$ is given by
\[
\bar{g}_{11}(t) = \frac{\bar{g}^{(1)}_{11}(t)}{i\pi} + \frac{4}{i\pi} \int_{\partial D_2} \frac{k_2 e^{4ik^2t}}{2ik_2 - \eta} \, dk_2 \tag{3.22}
\]
with
\[
\bar{g}^{(1)}_{11}(t) = -\sqrt{\omega} e^{-i\omega t} + \int_{\partial \hat{D}_2} e^{4ik^2t} A(k_2^2) \, dk_2, \tag{3.23}
\]
where $\partial D_2$ is the oriented boundary of the second quadrant of the complex $k$-plane and the contour $\partial \hat{D}_2$ is depicted in figure 2. Substituting (3.22) and (3.23) into (3.21) and then integrating the resulting expression with respect to $d\tau$, we find
\[
\Phi_{11}(t, -\bar{k}) = \frac{i(e^{4ik^2t} - e^{-i\omega t})}{2k + i\sqrt{\omega}} + \frac{1}{i\pi} \int_{\partial D_2} \frac{A(k_2^2)(e^{4ik^2t} - e^{-4ik^2t})}{k^2 - k_2^2} \, dk_2 + \frac{1}{i\pi} \int_{\partial \hat{D}_2} \frac{k_2(e^{4ik^2t} - e^{-4ik^2t})}{(2ik_2 - \eta)(k^2 - k_2^2)} \, dk_2.
\]
The integrands in the above expression have removable singularities at $k_2 = \pm k$. Thus, before splitting the integrals we deform the contour $\partial D_2$ into $\partial \hat{D}_2$, however it is not necessary to avoid $k = i\sqrt{\omega}/2$, and then we find (3.18) by using the residue theorem. \qed
Let us denote $\lambda \Phi_{22}(t, k)$ by

$$\lambda \Phi_{22}(t, k) = \lambda \Phi_{22}^{(1)}(t, k) + \lambda \Phi_{22}^{(2)}(t, k),$$

(3.24)

where

$$\lambda \Phi_{22}^{(1)}(t, k) = \int_0^t \left[ 2k g_0(\tau) - i g_{11}(\tau) \right] \Phi_{11}^{(1)}(\tau, k) d\tau + i \int_0^t g_0(\tau)^2 d\tau,$$

(3.25a)

and

$$\lambda \Phi_{22}^{(2)}(t, k) = -i \int_0^t g_{11}(\tau) \Phi_{11}^{(1)}(\tau, k) d\tau + \int_0^t \left[ 2k g_0(\tau) - i g_{11}(\tau) \right] \Phi_{11}^{(2)}(\tau, k) d\tau.$$

(3.25b)

**Lemma 3.3.** For $\omega > 0$,

$$\lambda \Phi_{22}^{(1)}(t, k) = \frac{i\alpha_1(k)}{2k - i\sqrt{\omega}} e^{-i(4k^2 + \omega)} + c_1(k) + \int_{\partial\hat{D}^k_2} c_2(k, k_3) e^{-i(4k_3^2 + \omega)} dk_3$$

$$+ \int_{\partial\hat{D}^k_2} c_3(k, k_2) e^{4i(k_2^2 - k^2)} dk_2 + \int_{\partial\hat{D}^k_3} c_4(k, k_2) e^{i(4k_2^2 + \omega)} dk_2$$

$$+ \int_{\partial\hat{D}^k_3} \int_{\partial\hat{D}^k_2} c_5(k, k_2, k_3) e^{4ik_3^2 - k_2^2} dk_3,$$

(3.26)

where the contours $\partial\hat{D}_2$ and $\partial\hat{D}_2^{\tilde{k}}$ are depicted in figure 2, $\tilde{\sigma} = 1$ if $k \in \partial D_2$, $\tilde{\sigma} = -1$ if $-k \in \partial D_2$, and similarly for $\partial\hat{D}_3^{\tilde{k}_3}$, with $\tilde{\sigma} = 1$ if $k_3 \in \partial D_3$, $\tilde{\sigma} = -1$ if $-k_3 \in \partial D_3$. Furthermore, the functions $\alpha_1(k)$ and $\{c_j\}_{j=1}^5$ are defined as follows:

$$\alpha_1(k) = \frac{i}{2k + i\sqrt{\omega}} + \frac{i}{2\sigma k - i\sqrt{\omega}},$$

$$c_1(k) = c_1^{(1)}(k) + c_1^{(2)}(k) + c_1^{(3)}(k),$$

(3.27)

$$c_2(k, k_3) = \frac{1}{4i(k^2 - k_3^2)} \frac{A(k_3)}{2k - i\sqrt{\omega}},$$

$$c_3(k, k_2) = \frac{\alpha_1(k)}{4} \frac{A(k^2_3)}{(k_2^2 - k^2_3)},$$

(3.28)

$$c_4(k, k_2) = \frac{iA(k^2_3)}{2k + i\sqrt{\omega}} (4k_2^2 + \omega),$$

$$c_5(k, k_2, k_3) = -\frac{1}{16} \frac{A(k^2_3)A(k^2_2)}{(k^2_2 - k_3^2)(k^2_2 - k_3^2)},$$

(3.29)
where

\[ c_1^{(1)}(k) = \frac{2k(\sigma + 1)}{(4k^2 + \omega)(2\sigma k - i\sqrt{\omega})} - \frac{1}{2i\sqrt{\omega}} \frac{2k + i\sqrt{\omega}}{(2\sigma k - i\sqrt{\omega})^2} - \frac{1}{2i\sqrt{\omega}(2k + i\sqrt{\omega})}, \]  

(3.30a)

\[ c_1^{(2)}(k) = \frac{1}{2\sigma k + i\sqrt{\omega}} \left( \frac{1}{2k + i\sqrt{\omega}} + \frac{1}{2\sigma k - i\sqrt{\omega}} \right), \]  

(3.30b)

\[ c_1^{(3)}(k) = \frac{4k^2}{i\pi(4k^2 + \omega)^2} \left( \log 4 - \log \omega \right) + \frac{2k\log k}{i\pi(4k^2 + \omega)(2k + i\sqrt{\omega})} \]  

\[ + \frac{2k\log(-k)}{i\pi(4k^2 + \omega)(2k - i\sqrt{\omega})} - \frac{1}{i\pi(4k^2 + \omega)} + \frac{4k^2 - \omega}{2(4k^2 + \omega)^2}. \]  

(3.30c)

Proof. We note that

\[ -i\bar{g}_{11}^{(1)}(\tau) + 2k\bar{g}_{01}(\tau) = (2k + i\sqrt{\omega})e^{-i\omega t} - i \int_{\partial D_2} e^{4ik^2\tau} A(k_3^2) dk_2. \]

Let us denote \( \Phi_{11}^{(1)} \) by

\[ \Phi_{11}^{(1)} = \alpha_1(k) e^{-4ik^2\tau} + \alpha_2(k) e^{i\omega t} + \frac{1}{4} \int_{\partial D_3^a} e^{-4ik_3^2\tau} A(k_3^2) dk_3, \]  

(3.31)

with \( \alpha_1(k) \) defined by the first equation in (3.27) and \( \alpha_2(k) \) defined by

\[ \alpha_2(k) = -\frac{i}{2k + i\sqrt{\omega}}. \]  

(3.32)

Thus,

\[ \int_{0}^{t} \left( -i\bar{g}_{11}^{(1)}(\tau) + 2k\bar{g}_{01}(\tau) \right) \Phi_{11}^{(1)}(\tau, k) d\tau \]

\[ = (2k + i\sqrt{\omega}) \int_{0}^{t} e^{-i\omega \tau} \left[ \alpha_1(k) e^{-4ik^2\tau} + \alpha_2(k) e^{i\omega t} \right] + \frac{1}{4} \int_{\partial D_3^a} e^{-4ik_3^2\tau} A(k_3^2) dk_3 \]  

\[ - i \int_{0}^{t} \left\{ \int_{\partial D_2} e^{4ik^2\tau} A(k_3^2) \left[ \alpha_1(k) e^{-4ik^2\tau} + \alpha_2(k) e^{i\omega t} \right] + \frac{1}{4} \int_{\partial D_3^a} e^{-4ik_3^2\tau} A(k_3^2) dk_3 \right\} dk_2. \]

(3.33)

Integrating with respect to \( d\tau \), we find that the rhs of (3.33) equals the following expression:

\[ (2k + i\sqrt{\omega}) \left\{ \alpha_1(k) \frac{e^{-it(4k^2 + \omega)} - 1}{-i(4k^2 + \omega)} + \alpha_2(k) t + \frac{1}{4} \int_{\partial D_3^a} \frac{e^{-it(4k_3^2 + \omega)} - 1}{-i(4k_3^2 + \omega)} A(k_3^2) dk_3 \right\} \]

\[ - i\alpha_1(k) \int_{\partial D_2} \frac{e^{it(4k^2 - k_3^2)} - 1}{4i(k_3^2 - k^2)} A(k_3^2) dk_2 - i\alpha_2(k) \int_{\partial D_2} \frac{e^{it(4k_3^2 + \omega)} - 1}{i(4k_3^2 + \omega)} A(k_3^2) dk_2 \]

\[ - \frac{i}{4} \int_{\partial D_2} \left( \int_{\partial D_3^a} \frac{e^{it(4k^2 - k_3^2)} - 1}{4i(k_3^2 - k^2)} A(k_3^2) dk_3 \right) dk_2. \]

Note that

\[ (2k + i\sqrt{\omega}) \alpha_2(k) t = -it, \]
Regarding the double integral, we deform \( \beta \) axis respectively (see figure 3). In order to compute \( C_1 \) and \( C_2 \) where the single integrals involving \( \partial \) have removable singularities at \( \pm k \); thus, before splitting these integrals we deform \( \partial D_2 \) to \( \partial \tilde{D}_2^k \).

Using the residue theorem, we find

\[
\int_{\partial \tilde{D}_2^k} \frac{A(k_3^2)dk_3}{(k^2 - k_3^2)(4k_3^2 + \omega)} = -\frac{2}{\sqrt{\omega}(2\sigma k - i\sqrt{\omega})^2} := \beta_1(k),
\]
\[
\int_{\partial \tilde{D}_2^k} \frac{A(k^2_3)dk_2}{k^2 - k_3^2} = \frac{4i}{2\sigma k + i\sqrt{\omega}} := \beta_2(k), \quad \int_{\partial D_3} \frac{A(k_3^2)dk_2}{4k_3^2 + \omega} = -\frac{1}{2\sqrt{\omega}}.
\]

Regarding the double integral, we deform \( \partial D_2 \) to \( \partial \tilde{D}_3^{\sigma k} \), where \( \sigma = 1 \) if \( k_3 \in \partial D_3 \) and \( \sigma = -1 \) if \( -k_3 \in \partial D_3 \). Then, using a residue calculation similar to the one used in lemma 3.1, we find the identity

\[
\int_{\partial \tilde{D}_3^{\sigma k}} \frac{A(k_3^2)}{k_3^2 - k_3^2} dk_3 = \begin{cases} 
\frac{-4i}{2k_3 + i\sqrt{\omega}} & \text{if } \sigma = 1, \\
\frac{4i}{2k_3 - i\sqrt{\omega}} & \text{if } \sigma = -1.
\end{cases}
\]

Thus, denoting the double integral with \( \beta_3(k) \), we find

\[
\int_{\partial \tilde{D}_3^{\sigma k}} \left( \int_{\partial \tilde{D}_3^{\sigma k}} \frac{A(k_3^2)A(k_3^2)}{(k_3^2 - k_3^2)(k_3^2 - k_3^2)} dk_3 \right) dk_2 = \beta_3(k) = \frac{32}{i\pi} (\beta_3^{(1)} - \beta_3^{(2)}),
\]

where

\[
\beta_3^{(1)}(k) = \int_{C_1} \frac{k_3^2}{(4k_3^2 + \omega)} \frac{1}{(2k_3 + i\sqrt{\omega})} \frac{1}{(k_3^2 - k_3^2)} dk_3,
\]
\[
\beta_3^{(2)}(k) = \int_{C_2} \frac{k_3^2}{(4k_3^2 + \omega)} \frac{1}{(2k_3 - i\sqrt{\omega})} \frac{1}{(k_3^2 - k_3^2)} dk_3,
\]

and \( C_1 \) and \( C_2 \) denote the parts of \( \partial \tilde{D}_3^{\sigma k} \) involving the negative real axis and the negative imaginary axis respectively (see figure 3). In order to compute \( \beta_3^{(1)} \) and \( \beta_3^{(2)} \), it is slightly more convenient.
to deform $C_1$ and $C_2$ to $\hat{C}_1$ and $\hat{C}_2$ respectively (see figure 3). Thus, using $k_3 = re^{3\pi/4} = -r\alpha$, $\alpha = e^{-i\pi/4}$ and $k_3 = r\alpha$ for $\beta_3^{(1)}$ and $\beta_3^{(2)}$, respectively, we find

$$\beta_3^{(1)} = e^{-i\pi/4} \int_0^\infty \frac{ir^2}{(4ir^2 - \omega)} \frac{1}{(2re^{-i\pi/4} - i\sqrt{\omega})} \frac{1}{(k^2 + ir^2)} \, dr,$$

$$\beta_3^{(2)} = -\beta_3^{(1)}.$$

Thus,

$$\beta_3(k) = \frac{64}{i\pi} \beta_3^{(1)}(k).$$

Computing $\beta_3^{(1)}(k)$ explicitly, we find

$$\beta_3(k) = \frac{64k^2}{i\pi(4k^2 + \omega)^2} \left( \log 4 - \log \omega \right) + \frac{32k \log k}{i\pi(4k^2 + \omega)(2k + i\sqrt{\omega})} + \frac{32k \log(-k)}{i\pi(4k^2 + \omega)(2k - i\sqrt{\omega})} - \frac{16}{i\pi(4k^2 + \omega)} + \frac{8(4k^2 - \omega)}{(4k^2 + \omega)^2}.$$

Using the above formulas, we find (3.26), where $c_1(k)$ is defined by

$$c_1(k) = \frac{2k + i\sqrt{\omega}}{i(4k^2 + \omega)} \alpha_1(k) + \frac{2k + i\sqrt{\omega}}{4i} \beta_1(k) - \frac{\alpha_1(k)}{4} \beta_2(k) - \frac{\alpha_2(k)}{2\sqrt{\omega}} + \frac{1}{16} \beta_3(k). \quad (3.34)$$

Simplifying the rhs of (3.34), we find that $c_1(k)$ is given by the second equation in (3.27) with (3.30).

Lemma 3.4. For $\omega > 0$,

$$\lambda \Phi_{22}^\beta(t, k) = \frac{ie^{-i(4k^2 + \omega)}}{(2i\sigma k + \eta)(2k - i\sqrt{\omega})} + d_1(k) + \int_{\partial D^4} d_2(k, k_3)e^{-i(4k_3^2 + \omega)} \, dk_3$$

$$+ \int_{\partial D^4} d_3(k, k_2)e^{i(4k_2^2 + \omega)} \, dk_2 + \int_{\partial D^4} d_4(k, k_2)e^{-4it(k^2 - k_2^2)} \, dk_2$$

$$+ \int_{\partial D^4} \left( \int_{\partial D^4} d_5(k, k_2, k_3)e^{-4it(k_3^2 - k_2^2)} \right) \, dk_2, \quad (3.35)$$

where the functions $\{d_j\}_1^5$ are defined by

$$d_1(k) = d_1^{(1)}(k) + d_1^{(2)}(k) + d_1^{(3)}(k),$$

$$d_2(k, k_3) = \frac{k_3}{\pi(2ik_3 + \eta)(2k - i\sqrt{\omega})(k^2 - k_3^2)},$$

$$d_3(k, k_2) = \frac{k_2}{\pi(2ik_2 - \eta)} \left( \frac{\alpha_1(k)}{2ik_2 - \eta} + \frac{1}{2i\sigma k_2 + \eta} \right) + \frac{4k_2}{A(k_2^2)},$$

$$d_4(k, k_2) = \frac{k_2}{\pi(2ik_2 - \eta)} \left( \frac{\alpha_1(k)}{2ik_2 - \eta} + \frac{1}{2i\sigma k_2 + \eta} \right) + \frac{4(2i\sigma k + \eta)(k^2 - k_2^2)}{4(2i\sigma k + \eta)(k^2 - k_2^2)},$$

and
\[ d_5(k, k_2, k_3) = \frac{1}{\pi(k^2 - k_3^2)(k^2 - k_3^2)} \left[ k_3 \frac{k_2}{2ik_2 + \eta} \left( \frac{k_2}{\pi(2ik_3 - \eta)} + \frac{A(k_3^2)}{4i(2ik_2 - \eta)} \right) - k_2A(k_3^2) \right], \quad (3.36) \]

with

\[ d_1^{(1)}(k) = -\frac{i}{\eta + \sqrt{\omega}} \left( \frac{1}{2k + i\sqrt{\omega}} + \frac{1}{2k - i\sqrt{\omega}} \right), \quad (3.37a) \]
\[ d_1^{(2)}(k) = -\frac{i}{2i\sigma k - \eta} \left( \frac{1}{2k + i\sqrt{\omega}} + \frac{1}{2\sigma k - i\sqrt{\omega}} \right) + \frac{i(\eta - \sqrt{\omega})}{(2i\sigma k + \eta)(2\sigma k - \eta)(2i\sigma k + i\sqrt{\omega})}, \quad (3.37b) \]
\[ d_1^{(3)}(k) = \frac{2k^2(\pi + 2i)}{\pi(4k^2 + \eta^2)} - \frac{\eta^2(\pi - 2i)}{2\pi(4k^2 + \eta^2)^2} - \frac{2ik\log(-k)}{\pi(2k + i\eta)(2k - i\eta)^2} - \frac{2ik\log k}{\pi(2k - i\eta)(2k + i\eta)^2} - \frac{8ik^2\log(2\eta)}{\pi(4k^2 + \eta^2)(\eta^2 - \omega)} - \frac{2i\omega k\log \omega}{4\pi k^2 - 2\pi\eta \sqrt{\omega} - 16ik^2\log 2} - \frac{2k(\eta + \sqrt{\omega})(\log(-k) - \log k)}{\pi(4k^2 + \eta^2)} + \frac{8ik^2(\log(-k) + \log k)}{\pi(4k^2 + \eta^2)} - \frac{4i\eta^2\log \eta}{\pi(4k^2 + \eta^2)(\eta^2 - \omega)}. \quad (3.37c) \]

**Proof.** Recalling (3.12b), (3.14a) and (3.14b), as well as (3.1), we find that \( \lambda \Phi^{(2)}_{22}(t, k) \) is given by

\[ \lambda \Phi^{(2)}_{22}(t, k) = \int_0^t 2ke^{-i\omega} \left[ \frac{e^{-4ik^2\tau}}{2i\sigma k + \eta} + \frac{1}{i\pi} \int_{\partial D^2} \frac{k_3e^{-4ik^2\tau}}{(2ik_3 + \eta)(k^2 - k_3^2)} dk_3 \right] d\tau \]

\[ -4 \frac{1}{i\pi} \int_0^t \int_{\partial D^2} \frac{k_2e^{4ik^2\tau}}{2ik_2 - \eta} \left[ \Phi^{(1)}_{11}(\tau, k) + \frac{1}{i\pi} \int_{\partial D^2} \frac{k_3e^{-4ik^2\tau}}{(2ik_3 + \eta)(k^2 - k_3^2)} dk_3 \right] dk_2 d\tau \]

\[ -i \int_0^t \Phi^{(1)}_{11}(\tau) \left[ \frac{e^{-4ik^2\tau}}{2i\sigma k + \eta} + \frac{1}{i\pi} \int_{\partial D^2} \frac{k_3e^{-4ik^2\tau}}{(2ik_3 + \eta)(k^2 - k_3^2)} dk_3 \right] d\tau. \quad (3.38) \]

Integrating the first line in (3.38) with respect to \( d\tau \), we find

\[ \frac{2ik}{2i\sigma k + \eta} \left( \frac{e^{-i(4k^2 + \omega)} - 1}{4k^2 + \omega} \right) + \frac{2k}{\pi} \int_{\partial D^2} \frac{k_3(e^{-i(4k^2 + \omega)} - 1)}{(2ik_3 + \eta)(4k_3^2 + \omega)(k^2 - k_3^2)} dk_3. \quad (3.39) \]

Evaluating one of the integrals with respect to \( dk_3 \), (3.39) yields

\[ \frac{2i\omega e^{-i(4k^2 + \omega)} - 2ik}{2i\sigma k + \eta} - \frac{2ik}{(\eta + \sqrt{\omega})(4k^2 + \omega)} + \frac{2k}{\pi} \int_{\partial D^2} \frac{k_3e^{-i(4k^2 + \omega)}}{(2ik_3 + \eta)(4k_3^2 + \omega)(k^2 - k_3^2)} dk_3, \quad (3.40) \]

where we have used the identity

\[ \int_{\partial D^2} \frac{k_3}{(2ik_3 + \eta)(4k_3^2 + \omega)(k^2 - k_3^2)} dk_3 = -\frac{i\pi}{4k^2 + \omega} \left( \frac{1}{2i\sigma k + \eta} - \frac{1}{\eta + \sqrt{\omega}} \right) := \beta_4(k). \]
Using (3.31) for \( \Phi_{11}^{(1)} \) and then integrating the resulting expression with respect to \( d\tau \), the second line in (3.38) becomes

\[
-\frac{4}{\pi} \int_{\partial D_2} \frac{k_2}{2ik_3 - \eta} \left\{ \alpha_1(k) e^{-i t (k^2 - k_3^2)} - 1 \right\} \alpha_2(k) e^{i t (k_3^2 + \omega)} - 1 \frac{1}{i (4k_3^2 + \omega)} + \frac{1}{4} \int_{\partial D_3^i} \frac{k_3}{k^2 - k_3^2} \left( e^{i t (k_3^2 - k_3^2)} - 1 \right) \right\} dk_3 - \frac{e^{-i t (k_3^2 - k_3^2)} - 1}{4i(2i\sigma k + \eta)(k^2 - k_3^2)} \frac{k_3}{k_3^2 - \eta} \right) \right\} dk_3 = \beta_5(k),
\]

(3.43a)

\[
\int_{\partial D_3^i} \frac{k_3}{k^2 - k_3^2} \left( e^{i t (k_3^2 - k_3^2)} - 1 \right) \right\} dk_3 - \frac{e^{-i t (k_3^2 - k_3^2)} - 1}{4i(2i\sigma k + \eta)(k^2 - k_3^2)} \frac{k_3}{k_3^2 - \eta} \right) \right\} dk_3 = \beta_6(k),
\]

(3.43b)

\[
\int_{\partial D_3^i} \frac{k_3}{k^2 - k_3^2} \left( e^{i t (k_3^2 - k_3^2)} - 1 \right) \right\} dk_3 - \frac{e^{-i t (k_3^2 - k_3^2)} - 1}{4i(2i\sigma k + \eta)(k^2 - k_3^2)} \frac{k_3}{k_3^2 - \eta} \right) \right\} dk_3 = \beta_7(k),
\]

(3.43c)

Thus equation (3.41) involves the following explicit terms

\[
-\frac{\alpha_1(k)}{2i\sigma k - \eta} + \frac{\alpha_2(k)}{\eta + \sqrt{\omega}} - \frac{1}{(2i\sigma k + \eta)(2i\sigma k - \eta)} + \frac{\beta_5(k)}{4i\pi} - \frac{\beta_6(k)}{\pi^2} = \beta_7(k).
\]

(3.44)

Similarly, for (3.42), we find

\[
-\frac{\sqrt{\omega}}{2i\sigma k + \eta} \left( e^{-i t (4k^2 + \omega)} - 1 \right) + \frac{\sqrt{\omega}}{i\pi} \beta_5(k) - \frac{\beta_2(k)}{4(2i\sigma k + \eta)} + \frac{\beta_7(k)}{4i\pi}.
\]

(3.45)

The double integrals in (3.43) can be evaluated in a similar way as the term \( \beta_5(k) \) in lemma 3.3; tedious but straightforward calculations yield

\[
\frac{1}{4i\pi} (\beta_5(k) + \beta_7(k)) - \frac{\beta_6(k)}{\pi^2} = d_1^{(3)}(k),
\]

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where \( d_1^{(3)} \) is given in (3.37c).

Combining (3.40), (3.44) and (3.45), \( \lambda \Phi_2^{(2)}(t,k) \) is given in (3.35) with (3.36). \( \square \)

**Proposition 3.2.** For \( \omega > 0 \),

\[
\lambda g_{13}(t) = -\frac{1}{2\sqrt{\omega}} e^{i\omega t} + o(1), \quad t \to \infty.
\]  
(3.46)

**Proof.** First, note that (2.9) can be written in the form

\[
\lambda g_{13}(t) = \lambda g_{13}^{(1)}(t) + \lambda g_{13}^{(2)}(t) + \lambda g_{13}^{(3)}(t),
\]  
(3.47)

where

\[
\lambda g_{13}^{(1)}(t) = \frac{2}{i\pi} \int_{\partial D_1} k [\lambda \Phi_1(t,k) - \lambda \Phi_1(t,-k)] dk,
\]  
(3.48)

\[
\lambda g_{13}^{(2)}(t) = \frac{2}{i\pi} e^{i\omega t} \int_{\partial D_1} [\lambda \Phi_2(t,k) - \lambda \Phi_2(t,-k)] dk
\]  
(3.49)

and

\[
\lambda g_{13}^{(3)}(t) = -\frac{4}{i\pi} \int_{\partial D_1} ke^{-4ik^2t} b_1(-k) [\lambda \Phi_2(t,-k) - \lambda a_2(-k)] dk.
\]  
(3.50)

The expression for \( \Phi_2(t,k) \) needed in (3.49) is given by (3.24) with (3.26) and (3.35) with \( \sigma = 1 \); similarly \( \Phi_2(t,-k) \) can be obtained by evaluating the rhs of (3.24) with (3.26) and (3.35) for \( \sigma = -1 \) and then replacing \( k \) by \(-k\). In what follows, we will determine the terms involving \( e^{i\omega t} \) in (3.47).

In order to compute \( \lambda g_{13}^{(3)}(t) \) involving \( e^{i\omega t} \), we will first show that

\[
\frac{4}{i\pi} \int_{\partial D_1} ke^{-4ik^2t} b_1(-k) \lambda a_2(-k) dk = O(t^{-3/2}), \quad t \to \infty.
\]  
(3.51)

Recalling (2.14), we find

\[
\lambda a_2(-k) = \int_0^\infty e^{(2ik-\eta)x} \int_x^\infty e^{-(2ik+\eta)x'} dx' dx = \frac{1}{2\eta(2ik+\eta)}.
\]

Thus, the lhs of (3.51) with (3.4) yields

\[
-\frac{2}{i\eta \pi} \int_{\partial D_1} \frac{ke^{-4ik^2t}}{(2ik+\eta)^2} dk.
\]  
(3.52)

Using a similar analysis as in proposition 3.1, we can show that (3.52) is of \( O(t^{-3/2}) \) as \( t \to \infty \). We next consider the following integral

\[
-\frac{4}{i\pi} \int_{\partial D_1} ke^{-4ik^2t} b_1(-k) [\lambda \Phi_2(t,-k)] dk,
\]  
(3.53)

where from (2.11), \( \lambda \Phi_2(t,-k) \) is given by

\[
\lambda \Phi_2(t,-k) = \int_0^t [-2ke^{i\omega \tau} + ig_{11}(\tau)] \Phi_{11}(\tau,-k) d\tau - t \int_0^t |g_{01}|^2 d\tau.
\]  
(3.54)
From (3.18), we write \( \Phi_{11}(t, -\vec{k}) \) as
\[
\Phi_{11}(t, -\vec{k}) = \tilde{\alpha}_1(k) e^{ik\text{t}} + \alpha_2(k) e^{i\omega t} + \frac{1}{4} \int_{\partial \hat{D}_1} A(k^2) e^{4ik\text{t}} dk_2
\]
\[
+ \frac{1}{i\pi} \int_{\partial \hat{D}_1} \frac{k_2 e^{4ik\text{t}}}{2i(k_2 - \eta)(k^2 - k_2^2)} dk_2,
\]
(3.55)
where \( \alpha_2(k) \) is given in (3.32) and
\[
\tilde{\alpha}_1(k) = \frac{i}{2k + i\sqrt{\omega}} - \frac{i}{2\sigma k + i\sqrt{\omega}} - \frac{1}{2\sigma k - \eta}.
\]
(3.56)
Substituting (3.55) into (3.54) and using a similar calculation as in lemma 3.3, the term involving \( e^{i\omega t} \) in (3.54) arises from
\[
\frac{i\tilde{\alpha}_1(k)}{2k - i\sqrt{\omega}}
\]
and (3.53) implies that the relevant contribution is
\[
\frac{4}{\pi} \int_{\partial \hat{D}_3} \frac{k\tilde{\alpha}_1(k)}{(2k - i\sqrt{\omega})(2ik + \eta)} dk = \frac{4}{\pi} \left\{ \int_{C_1} \frac{k}{(2ik - \eta)(2ik + \eta)(2i\omega - \eta)} \right\}
\]
\[
- \int_{C_2} \frac{k}{(2ik + \eta)(2k - i\sqrt{\omega})} \left( \frac{4ik}{4k^2 + \omega} + \frac{1}{2ik + \eta} \right) dk \right\}.
\]
Deforming each contour into \( \hat{C}_1 \) and \( \hat{C}_2 \) and letting \( k = -r\alpha \) and \( k = r\alpha \), respectively, we can evaluate the rhs of the above equation and we obtain
\[
\lambda g^{(3)}_{13}(t) = \left( -\frac{1}{\eta + \sqrt{\omega}} - \frac{2}{i\pi(\eta - \sqrt{\omega})} + \frac{(\eta^2 + \omega) \log(\eta^2 / \omega)}{i\pi(\eta - \sqrt{\omega})(\eta^2 - \omega)} \right) e^{i\omega t} + o(1), \quad t \to \infty.
\]
(3.57)
We next compute the term in \( \lambda g^{(2)}_{13}(t) \) involving \( e^{i\omega t} \). Recalling (3.13), the term involving \( e^{i\omega t} \) arises from
\[
\frac{2}{\pi} \int_{\partial \hat{D}_3} [c_1(k) - c_1(-k)] dk + \frac{2}{\pi} \int_{\partial \hat{D}_3} [d_1(k) - d_1(-k)] dk.
\]
(3.58)
Note that \( c_1(k) = c_1^{(1)}(k) + c_1^{(2)}(k) + c_1^{(3)}(k) \) with \( c_1^{(2)}(-k) = 0 \) and \( c_1^{(3)}(k) = c_1^{(3)}(-k) \). Hence, for the first term in (3.58) we need to evaluate the integral
\[
\frac{2}{\pi} \int_{\partial \hat{D}_3} [c_1^{(1)}(k) - c_1^{(1)}(-k)] dk + \frac{2}{\pi} \int_{\partial \hat{D}_3} c_1^{(2)}(k) dk.
\]
(3.59)
For the first integral in (3.59), we find
\[
c_1^{(1)}(k) - c_1^{(1)}(-k) = \frac{4ik}{\sqrt{\omega(4k^2 + \omega)}},
\]
which has simple poles at \( k = -i\sqrt{\omega}/2 \) and \( k = \infty \). Hence, the first integral in (3.59) is given by
\[
\frac{2}{\pi} \int_{\partial \hat{D}_3} [c_1^{(1)}(k) - c_1^{(1)}(-k)] dk = -\frac{1}{\sqrt{\omega}}.
\]
In order to compute the second integral in (3.59), we introduce as before \( C_1 \) and \( C_2 \) as the parts of \( \partial \hat{D}_3 \) involving the negative real axis and the negative imaginary axis respectively (cf. figure 3).
Thus, the second integral in (3.59) can be written in the form

\[ \frac{2}{\pi} \int_{\partial B_3} c^{(2)}_1(k) dk = \frac{8}{\pi} \left( c^{(1)}_1 - c^{(2)}_1 \right), \]

where

\[ c^{(1)}_1 = \int_{C_1} \frac{k}{(2k+i\sqrt{\omega})^2(2k-i\sqrt{\omega})} dk, \quad c^{(2)}_1 = \int_{C_2} \frac{k}{(2k+i\sqrt{\omega})(2k-i\sqrt{\omega})} dk. \]

In order to evaluate the above integrals, we deform \( C_1 \) and \( C_2 \) to \( \tilde{C}_1 \) and \( \tilde{C}_2 \), respectively, and then use \( k_3 = -r\alpha, \alpha = e^{-i\pi/4} \) and \( k_3 = r\alpha \) for \( c^{(1)}_1 \) and \( c^{(2)}_1 \), respectively (cf. figure 3); this yields \( \tilde{c}^{(1)}_1 = c^{(2)}_1 \) and hence,

\[ \frac{2}{\pi} \int_{\partial B_3} c^{(2)}_1(k) dk = 0. \]

Thus, we find

\[ \frac{2}{\pi} \int_{\partial B_3} \left[ c_1(k) - c_1(-k) \right] dk = -\frac{1}{\sqrt{\omega}}. \tag{3.61} \]

For the second term in (3.58), we note that \( d^{(3)}_1(k) = d^{(3)}_1(-k) \), and hence we need to compute

\[ \frac{2}{\pi} \int_{\partial B_3} \left[ d^{(1)}_1(k) - d^{(1)}_1(-k) \right] dk + \frac{2}{\pi} \int_{\partial B_3} \left[ d^{(2)}_1(k) - d^{(2)}_1(-k) \right] dk. \tag{3.62} \]

The integrand of the first integral in (3.62) can be simplified to

\[ d^{(1)}_1(k) - d^{(1)}_1(-k) = -\frac{8ik}{(\eta + \sqrt{\omega})(4k^2 + \omega)}, \]

and then we find

\[ \frac{2}{\pi} \int_{\partial B_3} \left[ d^{(1)}_1(k) - d^{(1)}_1(-k) \right] dk = \frac{2}{\eta + \sqrt{\omega}}. \tag{3.63} \]

where we have used the fact that the integrand has poles at \( k = -i\sqrt{\omega}/2 \) and at \( k = \infty \). The second integral in (3.62) can be written as

\[ \frac{2}{\pi} \int_{\partial B_3} \left[ d^{(2)}_1(k) - d^{(2)}_1(-k) \right] dk = \frac{8}{\pi} \left\{ \int_{C_1} \frac{k}{(2ik-\eta)(4k^2+\omega)} dk - \int_{C_2} \frac{k}{(2ik+\eta)(4k^2+\omega)} dk \right\}. \]

Deforming each contour into \( \tilde{C}_1 \) and \( \tilde{C}_2 \) and letting \( k = -r\alpha \) and \( k = r\alpha \), respectively, we find that the integral in the rhs of the above equation is identically zero. Thus, from (3.63), we find

\[ \frac{2}{\pi} \int_{\partial B_3} \left[ d_1(k) - d_1(-k) \right] dk = \frac{2}{\eta + \sqrt{\omega}}. \tag{3.64} \]

Therefore, combining (3.64) with (3.61), the coefficient of the term \( e^{i\omega t} \) for \( \lambda g^{(2)}_{13}(t) \) is given by

\[ \lambda g^{(2)}_{13}(t) = \left( -\frac{1}{\sqrt{\omega}} + \frac{2}{\eta + \sqrt{\omega}} \right) e^{i\omega t} + o(1), \quad t \to \infty. \tag{3.65} \]
In order to compute the analogous contributions from $\lambda g_{13}^{(1)}(t)$, we note that according to (2.12), $\lambda \Phi_{13}(t,k)$ involves two contributions. Noting that

\[
\int_0^t -i[g_{01}]^2 e^{4ik^2 \tau} \Phi_{11}(\tau,k) d\tau = -i \int_0^t e^{4ik^2 \tau} \left[ \alpha_1(k)e^{-4ik^2 \tau} + \alpha_2(k)e^{i\omega \tau} \right] + \frac{1}{4} \int_{\partial D_k} \frac{e^{-4ik^2 \tau}}{\sqrt{k^2 - k^2_1}} \Phi_{12}(\tau,k) d\tau,
\]

it follows that the $\lambda g_{13}^{(1)}(t)$ involves the following explicit terms:

\[
I_1(t,k) = -i e^{-4ik^2 t} \left\{ \alpha_1(k)t + \frac{t}{2i\sigma k + \eta} + \alpha_2(k) \frac{e^{i(4k^2 + \omega)} - 1}{i(4k^2 + \omega)} \right\}. \tag{3.66}
\]

Furthermore, noting that

\[
\int_0^t [2kg_{01}(\tau) + ig_{11}(\tau)] e^{4ik^2 \tau} \lambda \Phi_{22}(\tau,k) d\tau
= \int_0^t e^{4ik^2 \tau} \left[ 2ke^{i\omega \tau} + i\sqrt{\omega} e^{i\omega \tau} + i \int_{\partial D_k} e^{-4ik^2 \tau} \Phi_{12}(\tau,k) d\tau + \frac{4}{i\pi} \int_{\partial D_k} \frac{ke^{-4ik^2 \tau}}{2ik + \eta} d\tau \lambda \Phi_{22}(\tau,k) d\tau,
\]

it follows that $\lambda g_{13}^{(1)}(t)$ also involves the following explicit terms:

\[
I_2(t,k) = e^{-4ik^2 t} \left\{ i\alpha_1(k)t + \frac{it}{2i\sigma k + \eta} + (c_1(k) + d_1(k)) \frac{e^{i(4k^2 + \omega)} - 1}{i(2k + i\sqrt{\omega})} \right\}. \tag{3.67}
\]

The expressions $I_1$ and $I_2$ yield the following asymptotic contributions:

\[
\lambda g_{13}^{(1)}(t) \sim \frac{2}{i\pi} \int_{\partial D_k} k \left[ I_1(t,k) \big|_{\sigma = 1} - I_1(t,-k) \big|_{\sigma = -1} \right] d\sigma
\]

\[
+ \frac{2}{i\pi} \int_{\partial D_k} k \left[ I_2(t,k) \big|_{\sigma = 1} - I_2(t,-k) \big|_{\sigma = -1} \right] d\sigma. \tag{3.68}
\]

We next compute the contribution in (3.68) of terms involving $e^{i\omega t}$. These contributions arise from the third term in $I_1$ and from the third term in $I_2$. The coefficient of $e^{i\omega t}$ from the third term of $I_1$ leads to

\[
- \frac{\alpha_2(k)}{4k^2 + \omega} = \frac{i}{2k + i\sqrt{\omega}(4k^2 + \omega)}
\]

and the relevant contribution in the first term of (3.68) equals

\[
\frac{2}{i\pi} \int_{\partial D_k} \frac{k}{4k^2 + \omega} \left[ \frac{i}{2k + i\sqrt{\omega}} + \frac{i}{2k - i\sqrt{\omega}} \right] d\sigma = -\frac{1}{2\sqrt{\omega}}, \tag{3.69}
\]

where we have used the fact that the above integrand has a double poles at $k = -i\sqrt{\omega}/2$. 

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The coefficient of $e^{i\omega t}$ from the third term in $I_2$ is given by
\[ c_1(k) + d_1(k) \over i(2k + i\sqrt{\omega}). \]
Thus, the relevant contribution equals
\[ {2 \over i\pi} \oint_{\partial D_1} k \left[ {c_1(k) + d_1(k) \over i(2k + i\sqrt{\omega})} + {c_1(-k) + d_1(-k) \over i(2k - i\sqrt{\omega})} \right] dk. \quad (3.70) \]
Recall that
\[ c_1(k) = c_1^{(1)}(k) + c_1^{(2)}(k) + c_1^{(3)}(k), \quad d_1(k) = d_1^{(1)}(k) + d_1^{(2)}(k) + d_1^{(3)}(k). \]
For $c_1^{(1)}(k)$, we find
\[ {kc_1^{(1)}(k) \over 2k + i\sqrt{\omega}} + {kc_1^{(1)}(-k) \over 2k - i\sqrt{\omega}} = {8k^2 \over (4k^2 + \omega)^2}, \]
which has a pole at $k = -i\sqrt{\omega}/2$, and hence
\[ -{2 \over \pi} \int_{\partial D_1} k \left[ {c_1^{(1)}(k) \over 2k + i\sqrt{\omega}} + {c_1^{(1)}(-k) \over 2k - i\sqrt{\omega}} \right] dk = {1 \over \sqrt{\omega}}. \quad (3.71) \]
For $c_1^{(2)}(k)$, we obtain
\[ -{2 \over \pi} \int_{\partial D_1} {kc_1^{(2)}(k) \over 2k + i\sqrt{\omega}} dk = {8 \over \pi} \int_{C_1} {k^2 \over (4k^2 + \omega)(2k + i\sqrt{\omega})^2} dk + {8 \over \pi} \int_{C_1} {k^2 \over (4k^2 + \omega)^2} dk. \quad (3.72) \]
The integrals in the rhs of (3.72) can be computed in a similar way as before; we find
\[ -{2 \over \pi} \int_{\partial D_1} {kc_1^{(2)}(k) \over 2k + i\sqrt{\omega}} dk = -{\pi - 4i \over 8\pi\sqrt{\omega}}. \quad (3.73) \]
For the term involving $c_1^{(3)}$, using $c_1^{(3)}(k) = c_1^{(3)}(-k)$, we find
\[ -{2 \over \pi} \int_{\partial D_1} k \left[ {1 \over 2k + i\sqrt{\omega}} + {1 \over 2k - i\sqrt{\omega}} \right] c_1^{(3)}(k) dk = {\pi - 4i \over 8\pi\sqrt{\omega}}, \quad (3.74) \]
where we have used the fact that the integrand has a pole at $k = -i\sqrt{\omega}/2$. Combining (3.61), (3.69), (3.71), (3.73) and (3.74), we obtain the coefficient of $e^{i\omega t}$ as
\[ {2 \over i\pi} \oint_{\partial D_1} k \left[ {c_1(k) \over i(2k + i\sqrt{\omega})} + {c_1(-k) \over i(2k - i\sqrt{\omega})} \right] dk = {1 \over \sqrt{\omega}}. \]
We will evaluate the integrals involving the term $d_1(k)$ in (3.70). First, noting that
\[ k \left[ {d_1^{(1)}(k) \over 2k + i\sqrt{\omega}} + {d_1^{(1)}(-k) \over 2k - i\sqrt{\omega}} \right] = -{8\sqrt{\omega} \over (\eta + \sqrt{\omega}) (4k^2 + \omega)^2}, \]
we find
\[ -{2 \over \pi} \int_{\partial D_1} k \left[ {d_1^{(1)}(k) \over 2k + i\sqrt{\omega}} + {d_1^{(1)}(-k) \over 2k - i\sqrt{\omega}} \right] dk = -{1 \over \eta + \sqrt{\omega}}, \quad (3.75) \]
where we have used the fact that the integrand has a double pole at $k = -i\sqrt{\omega}/2$.\[ ]
For $d_1^{(2)}$, we note that

$$-\frac{2}{\pi} \int_{\partial D_3} k \left[ \frac{d_1^{(2)}(k)}{2k + i \sqrt{\omega}} + \frac{d_1^{(2)}(-k)}{2k - i \sqrt{\omega}} \right] dk$$

$$= \frac{8i}{\pi} \left\{ \int_{C_1} \frac{k^2}{(2k + i \sqrt{\omega})(2ik - \eta)(4k^2 + \omega)} dk - \int_{C_2} \frac{k^2}{(2k - i \sqrt{\omega})(2ik + \eta)(4k^2 + \omega)} dk \right\}$$

$$+ \frac{8(\eta - \sqrt{\omega})}{i\pi} \left\{ \int_{C_1} \frac{k^2}{(2ik - \eta)(2ik + \eta)(2k + i \sqrt{\omega})(4k^2 + \omega)} dk \right\} + \frac{8i}{\pi} \left\{ \int_{C_2} \frac{k^2}{(2ik + \eta)^2(2k - i \sqrt{\omega})(4k^2 + \omega)} dk \right\}$$.

(3.76)

As before, deforming each contours into $\tilde{C}_1$ and $\tilde{C}_2$ and letting $k = -r\alpha$ and $k = r\alpha$, (3.76) yields

$$-\frac{2}{\pi} \int_{\partial D_3} k \left[ \frac{d_1^{(2)}(k)}{2k + i \sqrt{\omega}} + \frac{d_1^{(2)}(-k)}{2k - i \sqrt{\omega}} \right] dk$$

$$= \frac{\eta}{2(\eta + \sqrt{\omega})} \frac{2\eta - \sqrt{\omega}}{\pi(\eta^2 - \omega)} - \frac{\eta(\eta^2 - \eta \sqrt{\omega} + \omega)}{i\pi(\eta^2 - \omega)^2} \log(\eta^2 / \omega).$$

(3.77)

Regarding $d_1^{(3)}$, using $d_1^{(3)}(k) = d_1^{(3)}(-k)$, we find

$$-\frac{2}{\pi} \int_{\partial D_3} k \left[ \frac{1}{2k + i \sqrt{\omega}} + \frac{1}{2k - i \sqrt{\omega}} \right] d_1^{(3)}(k) dk$$

$$= -\frac{\eta}{2(\eta + \sqrt{\omega})^2} - \frac{3i\sqrt{\omega}}{\pi(\eta^2 - \omega)} - \frac{\sqrt{\omega}(2\eta^2 + \omega)}{i\pi(\eta^2 - \omega)^2} \log(\eta^2 / \omega),$$

(3.78)

where we have used the fact that the integrand has poles at $k = -i\sqrt{\omega}/2$ and at $k = -i\eta/2$. Thus, combining (3.75), (3.77) and (3.78), we obtain

$$\frac{2}{i\pi} \int_{\partial D_3} k \left[ \frac{d_1(k)}{i(2k + i \sqrt{\omega})} + \frac{d_1(-k)}{i(2k - i \sqrt{\omega})} \right] dk$$

$$= -\frac{1}{\eta + \sqrt{\omega}} + \frac{2}{i\pi(\eta - \sqrt{\omega})} - \frac{(\eta^2 + \omega) \log(\eta^2 / \omega)}{i\pi(\eta - \sqrt{\omega})(\eta^2 - \omega)}$$

and then we find

$$\lambda g_{13}^{(1)}(t) = \left( -\frac{1}{2\sqrt{\omega}} + \frac{1}{i(\eta + \sqrt{\omega})} + \frac{2}{i\pi(\eta - \sqrt{\omega})} \right) e^{i\omega t} + o(1), \quad t \to \infty.$$
4. A Perturbative Approach for $\omega < 0$

In a similar way as in Section 3, we can obtain the analogous results for the case of $\omega < 0$. For simplicity, here we present the results with a vanishing initial datum, that is, $q(x,0) = 0$. In this case, $a(k) = 1$ and $b(k) = 0$.

**Proposition 4.1.** For $\omega < 0$,

$$g_{11}(t) = i\sqrt{-\omega}e^{i\omega t} + \int_{\partial D_3} e^{-4ik_3^2t}A(k_3^2)dk_3,$$

where the contour $\partial D_3$ is depicted in figure 4 and

$$A(k^2) = \frac{8k^2}{\pi(4k^2 + \omega)}.$$

**Lemma 4.1.** For $\omega < 0$,

$$\Phi_{11}(t,k) = \frac{i(e^{-4ik_3^2t} - e^{i\omega t})}{2k + \sqrt{-\omega}} + \frac{ie^{-4ik_3^2t}}{2\sigma k - \sqrt{-\omega}} + \frac{1}{4} \int_{\partial D_3^\sigma} \frac{e^{-4ik_3^2t}}{k^2 - k_3^2}A(k_3^2)dk_3,$$

where the contour $\partial D_3^\sigma$ is depicted in figure 4 and $\sigma = 1$ if $k \in \partial D_3$, whereas $\sigma = -1$ if $-k \in \partial D_3$.

**Lemma 4.2.** For $\omega < 0$,

$$\lambda \Phi_{22}(t,k) = \frac{i\alpha_1(k)}{2k + \sqrt{-\omega}}e^{-i(4k^2 + \omega)} + \frac{2i\sqrt{-\omega} t}{2k + \sqrt{-\omega}} + c_1(k) + \int_{\partial D_3^\sigma} c_2(k,k_3)e^{-i(4k_3^2 + \omega)}dk_3$$

$$+ \int_{\partial D_3^\sigma} c_3(k,k_2)e^{4ik_2^2 - k_2^2}dk_2 + \int_{\partial D_2} c_4(k,k_2)e^{i(4k_2^2 + \omega)}dk_2$$

$$+ \int_{\partial D_3} \left( \int_{\partial D_2} c_5(k,k_2,k_3)e^{4ik_2^2 - k_2^2}dk_2 \right)dk_3,$$

where $\lambda = 1$ if $\omega < 0$.
where the contours \( \partial \hat{D}_2 \) and \( \partial \hat{D}_2^{\phi k} \) are depicted in figure 5 and the functions \( \alpha_1(k) \) and \( \{c_j\}_1^5 \) are defined as follows:

\[
\begin{align*}
\alpha_1(k) &= \frac{i}{2k + \sqrt{-\omega}} + \frac{i}{2\sigma k - \sqrt{-\omega}}, \\
c_1(k) &= c_1^{(1)}(k) + c_1^{(2)}(k) + c_1^{(3)}(k), \\
c_2(k, k_3) &= -\frac{1}{4i(k^2 - k_3^2)} \frac{A(k_3^2)}{2k + \sqrt{-\omega}}, \\
c_3(k, k_2) &= \frac{\alpha_1(k) A(k_2^2)}{4(k^2 - k_3^2)}, \\
c_4(k, k_2) &= \frac{iA(k_2^2)}{(2k + \sqrt{-\omega})(4k_2^2 + \omega)}, \\
c_5(k, k_2, k_3) &= -\frac{1}{16} \frac{A(k_2^2)A(k_3^2)}{(k^2 - k_3^2)(k^2 - k_3^2)}. 
\end{align*}
\]

with

\[
\begin{align*}
c_1^{(1)}(k) &= \frac{2k(\sigma + 1)}{(4k^2 + \omega)(2\sigma k + \sqrt{-\omega})} - \frac{1}{2\sqrt{-\omega}(2\sigma k - \sqrt{-\omega})^2} + \frac{1}{2\sqrt{-\omega}(2k + \sqrt{-\omega})}, \\
c_1^{(2)}(k) &= \frac{1}{2\sigma k - \sqrt{-\omega}} \left( \frac{1}{2k + \sqrt{-\omega}} + \frac{1}{2\sigma k + \sqrt{-\omega}} \right), \\
c_1^{(3)}(k) &= \frac{4k^2}{i\pi(4k^2 + \omega)^2} \left( \log 4 - \log(-\omega) \right) + \frac{2k \log k}{i\pi(4k^2 + \omega)(2k - \sqrt{-\omega})} \\
&\quad + \frac{2k \log(-k)}{i\pi(4k^2 + \omega)(2k + \sqrt{-\omega})} - \frac{1}{2(2k^2 + \omega)}.
\end{align*}
\]

Proposition 4.2. For \( \omega < 0 \),

\[
\lambda_{g_{13}}(t) = \frac{1}{i\sqrt{-\omega}} e^{i\theta t} + o(1), \quad t \to \infty.
\]

**Proof.** Using (4.1), (4.3) and (4.4), we can derive (4.9). However, \( \lambda \Phi_{22}(t, k) \) contains the term involving \( te^{i\theta t} \). Here, we prove that the contributions from the terms involving \( te^{i\theta t} \) cancel. The coefficient of \( te^{i\theta t} \) in \( \lambda_{g_{13}}^{(2)} \) arises from the second term of (4.4) and the relevant contribution is

\[
\frac{4i\sqrt{-\omega}}{\pi} \int_{D_k} \left( \frac{1}{2k + \sqrt{-\omega}} + \frac{1}{2k - \sqrt{-\omega}} \right) \, dk = -2\sqrt{-\omega}.
\]
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where we have used the fact that the integrand has poles at $k = -\sqrt{-\omega}/2$ and at $k = \infty$. Regarding the coefficient of $t e^{i\omega t}$ in $\lambda g^{(1)}_{13}$, we note that $I_2$ for $\omega < 0$ is given by

$$I_2(t, k) = e^{-4ik^2t} \left\{ \alpha_1(k) \frac{i(2k - \sqrt{-\omega})t}{2k + \sqrt{-\omega}} + c_1(k) \frac{e^{i(4k^2 + \omega)} - 1}{i(2k + \sqrt{-\omega})} \right\} + \frac{2i\sqrt{-\omega}}{(2k + \sqrt{-\omega})^2} \left( \frac{e^{i(4k^2 + \omega)} - 1}{4k^2 + \omega} - it e^{i(4k^2 + \omega)} \right). \quad (4.11)$$

Hence, the coefficient of $t e^{i\omega t}$ in $\lambda g^{(1)}_{13}$ arises from the last term in (4.11) and the relevant contribution is

$$\frac{4\sqrt{-\omega}}{i\pi} \int_{\partial\mathcal{B}_3} k \left( \frac{1}{(2k + \sqrt{-\omega})^2} - \frac{1}{(2k - \sqrt{-\omega})^2} \right) dk = 2\sqrt{-\omega}, \quad (4.12)$$

where we have used the fact that the integrand has poles at $k = -\sqrt{-\omega}/2$ and at $k = \infty$. From (4.10) and (4.12), it follows that the terms involving $t e^{i\omega t}$ cancel.

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