Buoyancy induced Couette-Poiseuille flow in a vertical microchannel

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Abstract. The fully developed buoyancy-induced (natural convective) Couette-Poiseuille flow in a vertical microchannel is investigated with the velocity slip and temperature jump boundary conditions. Closed form analytical solutions for the velocity and temperature fields are obtained. The effects of the fluid-wall interaction parameter, wall-ambient temperature difference ratio, Knudsen number, mixed convection parameter, and the dimensionless pressure gradient on the velocity, temperature, volume flow rate, heat flux between the plates and the Nusselt number have been discussed in detail through graphs. The outcomes of the investigation indicate that the volume flow rate increases with increasing values of mixed convection parameter, wall-ambient temperature difference ratio, and Knudsen number.

1. Introduction
In recent years, demands from all around the world calling for more functions and increased portability devices has driven the technology advancement towards devices miniaturization. This process has led to the generation of micro-electro-mechanical systems (MEMS) such as micro-reactors, micro-engines, micro-opto-electromechanical systems, micro-fluidic devices and biomedical applications that involve DNA sequencing and bio-MEMS. All microfluidic devices consist of various microchannel geometries whose characteristic dimension range from 1 micrometre to 1 millimetre. Generally, a parameter called Knudsen number is used to measure the effect of small length scale of microgeometries on the flow condition. Knudsen number represents the ratio of mean free path to the characteristic dimension of the system which was defined in four flow regimes by Beskok and Karniadakis [1] as $\text{Kn} < 0.001$: continuum regime, $0.001 < \text{Kn} < 0.1$: slip regime, $0.1 < \text{Kn} < 10$: transition regime, and $\text{Kn} > 10$: free molecular regime. As the Kn increases, rarefaction effects become much more significant and it is inappropriate to address the microscale rarefied gas flow problems by applying the macroscale solution due to the great behavioural difference. The no-slip velocity and no-temperature-jump boundary conditions are no longer valid for microscale devices. In the present study, gas no longer reaches the temperature of the two plates and the velocity of the surface due to the moving hot plate and a stationary cold plate. Hence, velocity slip and temperature jump boundary conditions are considered. Chen and Weng [2] investigated the natural convection flow in a vertical microchannel with asymmetric wall temperatures. They discovered that the volume flow rate was higher and the heat transfer rate was lower in microscale than in macroscale. Dongari et al. [3] used the second-order slip boundary condition to investigate the gas flow in long microchannels. It was proven that employing the second order accurate boundary condition in the study of gas flow through microchannels provide a better insight of the behavior of pressure and volume flux compared...
to the first order. Avci and Aydin [4] examined the mixed convective heat transfer in a vertical microchannel. In this study, they discovered that an increase of mixed convection parameter increased the velocity near the hot wall while it decreased the velocity near the cold wall. Avci and Aydin [5] presented the exact analysis of the mixed convection flow in a microchannel with asymmetric wall heat fluxes. Their results indicated that the velocity slip increased and the maximum velocity decreased with increased Knudsen number. Aydin and Avci [6] also studied the laminar forced convection in a microchannel for two types of thermal boundary conditions namely constant heat flux (CHF) and constant wall temperature (CWT). They found that the Nusselt number was decreased with the increase of Knudsen number in both cases. The similarity solutions were derived by Wang et al. [7] for flow and heat transfer in microchannels. They employed the similarity transformation to transform the governing equations and Homotopy analysis method (HAM) to solve the resulting ordinary differential equations. Kushwaha and Sahu [8] analyzed the gaseous flow in a micropipe by using the second order velocity slip and temperature jump boundary conditions to solve the governing equations. They reported that Nusselt number has lower value in the case of second-order slip model as compared to the first order slip model. Buonuomo and Manca [9] conducted the study of transient natural convection in microchannel by using a finite-difference method. They identified that mass flow rate increased with increased Knudsen number while the Nusselt number decreased with increased Knudsen number. Moslehi and Saghaﬁan [10] numerically examined the developing laminar mixed convection flow in a microchannel under the inﬂuence of a uniform magnetic ﬁeld by employing the ﬁnite volume method. They found that an increase in the Hartmann number decreased the velocity for both Knudsen number equals to 0 and 0.1, and for all values of mixed convection parameter.

Even though there are plenty of studies that focused on the microchannel flows, studies that focus on shear-driven microchannel flows remains limited. Kabov et al. [11] experimentally investigated the flow of a locally heated shear-driven liquid ﬁlm moving under the friction of gas in a channel. Kabov et al. [12] also extended the experimental study to the evaporation and ﬂow dynamics of thin, shear-driven liquid ﬁlms in microgap channels. Through the two experiments, it was found that shear-driven ﬂuid serves as a more suitable option for cooling applications compared to gravity-driven liquid ﬁlms as the critical heat ﬂux of the shear-driven ﬂuid is signiﬁcantly higher than gravity-driven ﬂuid. Recently, Narahari and Rajashekhar [13] investigated analytically the natural convective Couette ﬂow in a vertical microchannel. Their results showed that the ﬂuid velocity as well as the volume ﬂow rate increased with increased Knudsen number. However, the shear-driven mixed convection ﬂow in a microchannel has not been addressed in the literature even though such kind of ﬂow is encountered in micromotors, comb mechanisms, and microbearings. The objective of the present paper is to investigate the fully developed natural convective Couette-Poiseuille ﬂow of a rarefied gas in a vertical parallel plate microchannel with a moving hot plate and a stationary cold plate. The effects of the ﬂuid-wall interaction parameter (In), wall-ambient temperature difference ratio (ξ), Knudsen number (Kn), mixed convection parameter (Gr/Re), and the dimensionless pressure gradient (P1) on the velocity (U), temperature (θ), volume ﬂow rate (M), heat ﬂux between the plates (Q) and the Nusselt number (Nu) are determined and discussed graphically.

2. Mathematical analysis

Consider the laminar rarefied gaseous ﬂow in a vertical microchannel with cooler and hotter plates temperatures maintained at T1 and T2, respectively. The x-axis is taken along the plates in the upward direction and the y-axis is normal to the plates. The cooler plate located at y = 0 is ﬁxed while the hotter plate at y = b is moving with uniform velocity u0 in the upward direction in its own plane. Both ends of the channel are open to ambient of temperature T0 and density ρ0 as shown in figure 1. The ﬂuid properties are assumed to be constant, internal heat generation and viscous dissipation are neglected. Then under the Boussinesq approximation, the hydrodynamically fully developed steady state ﬂow can be represented by the following equations (Avci and Aydin [4]):
\[ 0 = -\frac{dp}{dx} + \rho g \beta (T - T_0) + \mu \frac{\partial^2 u}{\partial y^2} \]  
\[ u \frac{\partial T}{\partial x} = \frac{k}{\rho c_p} \frac{\partial^2 T}{\partial y^2} \]  

The corresponding velocity slip and temperature jump boundary conditions (Chen and Weng [2]) are

\[ u = \frac{2 - F_v}{F_v} \frac{\lambda}{\partial y} \text{ at } y = 0 \]  
\[ u = u_p - \frac{2 - F_v}{F_v} \frac{\lambda}{\partial y} \text{ at } y = b \]  
\[ T = T_1 + \frac{2 - F_i}{F_i} \frac{2 \gamma_s}{\gamma_s + 1 \text{ Pr} \ \partial y} \frac{\lambda}{\partial T} \text{ at } y = 0 \]  
\[ T = T_2 - \frac{2 - F_i}{F_i} \frac{2 \gamma_s}{\gamma_s + 1 \text{ Pr} \ \partial y} \frac{\lambda}{\partial T} \text{ at } y = b \]  

\[ \begin{align*} 
X &= \frac{x}{\text{Re} \ b}, \\
Y &= \frac{y}{b}, \\
U &= \frac{u}{u_p}, \\
\theta &= \frac{T - T_0}{T_2 - T_0}, \\
Gr &= \frac{g \beta (T_2 - T_0) b^3}{\nu^2}, \\
\text{Pr} &= \frac{\mu c_p}{k}, \\
P &= \frac{p - \rho u_p^2}{\rho}, \\
\text{Re} &= \frac{u_p b}{\nu}, \\
\beta_v &= \frac{2 - F_v}{F_v}, \\
\beta_i &= \frac{2 - F_i}{F_i} \frac{\gamma_s}{\gamma_s + 1 \text{ Pr} b}, \\
\text{Kn} &= \frac{\lambda}{b}, \\
\xi &= \frac{T_1 - T_0}{T_2 - T_0}. 
\end{align*} \]  

The dimensionless form of equations (1) to (6) are, respectively, as follows

\[ \frac{dP}{dX} = Gr \frac{\theta + \frac{d^2 U}{dY^2}}{\text{Re}} \]  

**Figure 1.** Schematic diagram of the physical model.
where $p = p' - p^*$ is the pressure difference, $p'$ is the static pressure, $p^*$ is the hydrostatic pressure, $x$ and $y$ are rectangular coordinates, $\rho$ is the density, $g$ is the gravitational acceleration, $\beta$ is the thermal expansion coefficient, $T$ is the fluid temperature, $T_0$ is the inlet fluid temperature, $\mu$ is the dynamic viscosity, $u$ is the velocity component in the $x$-direction, $k$ is the thermal conductivity, $c_p$ is the specific heat at constant pressure, $F_t$ is the thermal accommodation coefficient, $F_v$ is the tangential momentum accommodation coefficient, $\lambda$ is the molecular mean free path, $b$ is the channel width, $u_p$ is the plate velocity, $T_1$ and $T_2$ are the cold and hot walls temperatures, respectively, $\gamma_s = c_p / c_v$ is the ratio of specific heats, $c_v$ is the specific heat at constant volume, $Pr$ is the Prandtl number, $X$ and $Y$ are dimensionless rectangular coordinates, $U$ is the dimensionless velocity component in $X$-direction, $P$ is the dimensionless pressure, $Gr$ is the Grashof number, $Re$ is the Reynolds number, $\nu$ is the kinematic viscosity, $\theta$ is the dimensionless temperature, $\beta_t$ and $\beta_v$ are the dimensionless variables, $\xi$ is the wall-ambient temperature different ratio, $Kn = \lambda / b$ is the Knudsen number.

For fully developed thermal flow, equation (9) becomes

$$0 = \frac{\partial^2 \theta}{\partial Y^2}$$

Integrating equation (14) with respect to the boundary conditions (12) and (13), we have,

$$\theta(Y) = A_1 Y + A_2$$

where

$$A_1 = \frac{1 - \xi}{1 + 2 \beta_v Kn In}$$

$$A_2 = \frac{\xi + (\xi + 1) \beta_v Kn In}{(1 + 2 \beta_v Kn In)}$$

and $In = \beta_t / \beta_v$ is the fluid-wall interaction parameter.

Solving the momentum equation (8) subject to the boundary conditions (10) and (12), gives

$$U = \frac{PY^2}{2} - \frac{Gr}{Re} \left( \frac{A_1 Y^3}{6} + \frac{A_2 Y^2}{2} \right) + B_1 Y + B_2$$
where

\[
P_1 = \frac{dP}{dX}
\]

\[
B_1 = \frac{1}{(1 + 2 \beta, Kn)} - \frac{P_1}{2(1 + 2 \beta, Kn)} \frac{Gr}{6Re} \left[ A_1 \left( \frac{1 + 3 \beta, Kn}{1 + 2 \beta, Kn} \right) + 3A_2 \right]
\]

\[
B_2 = \frac{\beta, Kn}{(1 + 2 \beta, Kn)} - \frac{P_1 \beta, Kn}{2(1 + 2 \beta, Kn)} + \frac{\beta, Kn Gr}{6Re} \left[ A_1 \left( \frac{1 + 3 \beta, Kn}{1 + 2 \beta, Kn} \right) + 3A_2 \right]
\]

The non-dimensional volume flow rate \( M \) is given by

\[
M = \frac{M'}{Re u_p b^2} = \frac{1}{0} U dY = \frac{P_1}{6} - \frac{Gr}{Re} \left( \frac{A_1}{24} + \frac{A_2}{6} \right) + \frac{B_1}{2} + B_2
\]

where \( M' \) is the volume flow rate. The total heat absorbed by the fluid while traversing the channel (or) the vertical heat flux is given by

\[
Q = \frac{Q'}{\rho c_Y u_p (T_2 - T_0)b} = \frac{1}{0} U \theta dY = \frac{P_1}{24} \left( 3A_1 + 4A_2 \right) + \frac{A_1 B_1}{3} + A_2 B_3 + \frac{1}{2} (A_1 B_2 + A_2 B_1)
\]

\[
- \frac{Gr}{6Re} \left( \frac{A_1^2}{5} + A_2^2 + A_1 A_2 \right)
\]

where \( Q' \) is the vertical heat flux and \( Q \) is the dimensionless vertical heat flux. The non-dimensional mean/bulk temperature \( \theta_m \) is given by

\[
\theta_m = \frac{T_m - T_0}{T_2 - T_0} = \frac{1}{0} U \theta dY = \frac{Q}{M}
\]

The convective heat transfer coefficient \( h \) is given by

\[
h = \frac{k}{\rho c_Y u_p (T_2 - T_0)b} = \frac{k}{b} \frac{\partial \theta}{\partial Y} \bigg|_{Y=1} = \frac{k}{b} \frac{\partial \theta}{\partial Y} \bigg|_{Y=1}
\]

which can be obtained from the following Nusselt number (Nu),

\[
Nu = \frac{hh}{k} = \frac{1}{(1 - \theta_m)} \frac{\partial \theta}{\partial Y} \bigg|_{Y=1} = \frac{MA_1}{M-Q}
\]

3. Results and Discussion

Using the analytical approach, closed-form solutions are obtained for the velocity and temperature fields. In the present investigation, the range of Knudsen number \( Kn \) is \( 0 \leq Kn \leq 0.1 \), where \( Kn = 0 \) represent the macroscale case while \( Kn > 0 \) represent the microscale case. The effects of the fluid-
wall interaction parameter \((In)\), wall-ambient temperature difference ratio \((\xi)\), Knudsen number \((Kn)\), mixed convection parameter \((Gr/Re)\), and the dimensionless pressure gradient \((P_1)\) on the velocity \((U)\), volume flow rate \((M)\), heat flux between the plates \((Q)\) and the Nusselt number \((Nu)\) are projected through graphs. The effect of Knudsen number \((Kn)\) on the velocity distribution along the \(y\)-axis is shown in figure 2. The velocity is distributed the most at the center of two walls. \(Kn\) appeared to have a higher impact on the velocity distribution and increases the flow velocity as well as slip velocity at the plates with increasing \(Kn\). Thus the rarefaction effect is to increase the slip velocity at both plates in the natural convective Couette-Poiseuille flow. Figure 3 illustrates the effects of dimensionless pressure gradient on the velocity distribution along the \(y\)-axis. A similar trend can be observed where velocity is the highest in the middle of the channel. The velocity increases with increasing negative value of \(P_1\) (pressure assisted flow) and an opposite trend can be observed with increasing positive value of \(P_1\) (pressure opposed flow). From figure 4 it is clear that the velocity distribution under the effect of wall-ambient temperature difference ratio is increasing and the maximum velocity is shifting towards the middle of the channel with the increase of \(\xi\). When \(\xi = 0\), the peak of the velocity is distributed nearer to the moving hot wall. Also, the slip velocity is increasing with increasing value of \(\xi\) and it is slightly low at the moving hot plate as compared to the fixed cold plate. Figure 5 shows the effect of mixed convection parameter \((Gr/Re)\) on the velocity distribution. It can be seen that the velocity is slightly higher at the moving hot wall. However, as the \(Gr/Re\) rises from 0 to 500 the velocity rises sharply and the highest velocity is distributed at the center of the microchannel. Further, the velocity slip is increasing considerably with increasing value of \(Gr/Re\).

The effects of \(Gr/Re\) and \(Kn\) on the volume flow rate are depicted in figure 6. The volume flow rate is constant with \(Kn\) when \(Gr/Re = 0\) and for other values of \(Gr/Re\) the volume flow rate is increasing with increasing \(Gr/Re\) and \(Kn\). However, the rises of volume flow rate with \(Kn\) is more significant at higher values of \(Gr/Re\). Figure 7 shows the effects of \(In\), \(\xi\) and \(Kn\) on the volume flow rate \((M)\). The figure clearly reveals that \(M\) increases monotonically as \(Kn\) and \(\xi\) increases. Also, the volume flow rate is constant at any value of \(In\). Figure 8 depicts the variation of \(Gr/Re\) and \(Kn\) on the vertical heat flux. It is observed that when \(Gr/Re\) is equal to zero, the vertical heat flux remain constant even with the increase of \(Kn\). The vertical heat flux starts to increase linearly with \(Kn\) when \(Gr/Re\) increases and the vertical heat flux pronounced more at higher values of \(Gr/Re\). The impact of \(In\) on \(Q\) is shown in figure 9. The vertical heat flux \((Q)\) slightly decreases with increasing values of fluid-wall interaction parameter \((In)\), however, this decrease is negligible at higher values of \(In\).
Figure 10 illustrate the interaction of wall-ambient temperature difference ratio ($\xi$) on vertical heat flux ($Q$) and it can be seen that an increase in $\xi$ substantially increases the vertical heat flux. Also, the rise of vertical heat flux with Kn pronounced more at higher values of $\xi$. Figure 11 depicts the Nusselt number under the impact of Gr/Re and Kn. The figure clearly reveals a declining curve of Nu with increasing values of both Kn and Gr/Re. Nu is significantly higher when Gr/Re = 0 as compared to Gr/Re = 100, 250 and 500. The decrease of Nu becomes almost negligible at higher values of Gr/Re. Figure 12 depicts the influence of dimensionless pressure gradient on the Nusselt number. It is clear that Nu decreases with increasing negative value of $P_1$ (pressure assisted flow) and Nu increases with increasing positive value of $P_1$ (pressure opposed flow). Thus the pressure assisted flow causes a decrease in the Nusselt number while pressure opposed flow increases the Nusselt number. The effect of $\xi$ on the Nusselt number is presented in figure 13. It is observed that the Nusselt number is maximum at $\xi = 0$ and it decreases with increasing values of $\xi$ as well as Kn. Thus the rarefaction effect is to decrease the Nusselt number in the buoyancy induced Couette-Poiseuille microchannel flow.
Figure 8. Vertical heat flux for different values of $Gr/Re$.

Figure 9. Vertical heat flux for different values of $In$.

Figure 10. Vertical heat flux for different values of $\xi$.

Figure 11. Nusselt number for different values of $Gr/Re$.

Figure 12. Nusselt number for different values of $P_1$.

Figure 13. Nusselt number for different values of $\xi$. 
4. Conclusions
In the present paper, fully developed natural convective Couette-Poiseuille flow in a microchannel is investigated analytically by taking into account the velocity slip and temperature jump conditions. The important findings are listed below:

- Velocity slip increases with increasing Knudsen number, wall-ambient temperature difference ratio, mixed convection parameter and aiding pressure gradient but it decreases with opposing pressure gradient.
- Volume flow rate increases with the escalation of mixed convection parameter, wall-ambient temperature difference ratio and Knudsen number but it is invariant with the fluid-wall interaction parameter.
- Vertical heat flux increases with increasing mixed convection parameter, wall-ambient temperature difference ratio and Knudsen number but it decreases with increasing fluid-wall interaction parameter.
- The Nusselt number decreases with the rise of mixed convection parameter, wall-ambient temperature difference ratio, aiding pressure gradient and Knudsen number but the Nusselt number increases with the opposing pressure gradient.

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