(-k)-critical trees and k-minimal trees

Walid Marweni

Faculty of Sciences - Sfax University, Tunisia.

In a graph $G = (V, E)$, a module is a vertex subset $M$ of $V$ such that every vertex outside $M$ is adjacent to all or none of $M$. For example, $\emptyset$, $\{x\} (x \in V)$ and $V$ are modules of $G$, called trivial modules. A graph, all the modules of which are trivial, is prime; otherwise, it is decomposable. A vertex $x$ of a prime graph $G$ is critical if $G - x$ is decomposable. Moreover, a prime graph with $k$ non-critical vertices is called $(-k)$-critical graph. A prime graph $G$ is $k$-minimal if there is some $k$-vertex set of $X$ of vertices such that there is no proper induced subgraph of $G$ containing $X$ is prime. From this perspective, I. Boudabbous proposes to find the $(-k)$-critical graphs and $k$-minimal graphs for some integer $k$ even in a particular case of graphs. This research paper attempts to answer I. Boudabbous’s question. First, it describes the $(-k)$-critical tree. As a corollary, we determine the number of nonisomorphic $(-k)$-critical tree with $n$ vertices where $k \in \{1, 2, \lfloor \frac{n}{2} \rfloor \}$. Second, it provide a complete characterization of the $k$-minimal tree. As a corollary, we determine the number of nonisomorphic $k$-minimal tree with $n$ vertices where $k \leq 3$.

**Keywords:** Graphs, tree, module, prime, critical vertex, minimal.

1 Introduction and presentation of the results

Our work falls within the framework of graph theory with a special focus on decomposition problems. A graph $G = (V(G), E(G))$ (or $G = (V, E)$) consists of a finite set $V$ of vertices together with a set $E$ of pairs of distinct vertices, called edges. On the one hand, let $G = (V, E)$ be a graph. The neighborhood of $x$ in $G$, denoted by $N_G(x)$ or simply $N(x)$, is the set $N_G(x) = \{y \in V \setminus \{x\} : \{x, y\} \in E\}$. The degree of $x$ in $G$, denoted by $d_G(x)$ (or $d(x)$), is the cardinal of $N_G(x)$. A vertex $x$ with degree one is called a leaf, its adjacent vertex is called a support vertex and it is denoted by $x^+$. If $x$ is a support vertex in $G$ admitting a unique leaf neighbor, this leaf is denoted by $x^-$. The set of leaves and support vertices in a graph $G$ is denoted by $L(G)$ and $S(G)$ respectively. An internal vertex is a vertex with a degree greater than or equal to 2. The distance between two vertices $u$ and $v$ in $G$ is the length (number of edge) of the shortest path connecting them and is denoted by $dist_G(u, v)$ or simply $dist(u, v)$. The notation $x \sim Y$ for each $Y \subseteq V \setminus \{x\}$ means $x$ is adjacent to all or none vertex of $Y$. The negation is denoted by $x \not\sim Y$.

On the other hand, given a graph $G = (V, E)$, with each subset $X$ of $V$, the graph $G[X] = (X, E \cap \binom{X}{2})$ is an induced subgraph of $G$. For $X \subseteq V$ (resp. $x \in V$), the induced subgraph $G[V \setminus X]$ (resp. $G[V \setminus \{x\}]$) is denoted by $G - X$ (resp. $G - x$). The notions of isomorphism and embedding are defined in the following way. Two graphs $G = (V, E)$ and $G' = (V', E')$ are isomorphic, which is denoted by $G \simeq G'$, if there is an isomorphism from $G$ onto $G'$, i.e., a bijection from $V$ onto $V'$ such that for

*walid.marweni@gmail.com

ISSN subm. to DMTCS © by the authors by the author(s) Distributed under a Creative Commons Attribution 4.0 International License
all \(x, y \in V\), \(\{x, y\} \in E\) if and only if \(\{f(x), f(y)\} \in E'\). We say that a graph \(G'\) embeds into a graph \(G\) if \(G'\) is isomorphic to an induced subgraph of \(G\). Otherwise, we say that \(G\) omits \(G'\).

Given a graph \(G\) and a one-to-one function \(f\) defined on a set containing \(V(G)\), we denote by \(f(G)\) the graph \((f(V(G)); f(E(G))) = (f(V(G)); \{\{f(x), f(y)\} : \{x, y\} \in E(G)\})\). A nonempty subset \(C\) of \(V\) is a connected component of \(G\) if for \(x \in C\) and \(y \in V \setminus C\), \(\{x, y\} \notin E\) and if for \(x \neq y \in C\), there is a sequence \(x = x_0, \ldots, x_n = y\) of \(C\) elements such that for each \(0 \leq i \leq n - 1\), \(\{x_i, x_{i+1}\} \in E\). A vertex \(x\) of \(G\) is isolated if \(\{x\}\) constitutes a connected component of \(G\). The set of the connected components of \(G\) is a partition of \(V\), denoted \(C(G)\). The graph \(G\) is connected if it has at most one connected component of \(G\). Otherwise, it is called non-connected. For example, a tree is a connected graph in which any two vertices are connected by exactly one path.

In addition, let’s consider a graph \(G = (V, E)\), a subset \(M\) of \(V\) is a module of \(G\) if every vertex outside \(M\) is adjacent to all or none of \(M\). This concept was introduced in Spinrad (1992) and independently under the name interval in Cournier and Ille (1998); Fraisse (1984); Schmerl and Trotter (1993) and an autonomous set in Ehrenfeucht and Rozenberg (1990). The empty set, the singleton sets, and the full set of \(V\) are trivial modules. A graph is indecomposable (or primitive) if all its modules are trivial; otherwise, it is decomposable. Therefore, indecomposable graphs with at least four vertices are prime graphs. This concept was developed in several papers e.g. Ehrenfeucht and Rozenberg (1990); Fraisse (1984); Kelly (1986); Schmerl and Trotter (1993); Sumner (1973)), and is now elaborated in a book by Ehrenfeucht, Harju and Rozenberg (Ehrenfeucht et al., 1997). Properties of the prime substructures of a given prime structure were determined by Schmerl and Trotter (1993) as follows: Each prime graph of order \(n\) is decomposable for each \(n \geq 3\), and if for \(n \geq 4\), \(\{P_n\}\) is prime for \(n \geq 4\). A path with extremities \(x\) and \(y\) is referred to as \((x, y)\)-path. For example, it is easy to verify that each prime graph is connected.

The study of the hereditary aspect of the primality in the graphs revolves around the following general question. Given a prime graph \(G\), is there always a proper prime subgraph in \(G\)? Addressing this problematic led to the publication of numerous papers. A first result in this direction dates back to D. P. Sumner (1973) who asserted that for every prime graph \(G\), there exists \(X \subseteq V(G)\) such that \(|X| = 4\) and \(G[X]\) is prime. In 1990, A. Ehrenfeucht and G. Rozenberg (Ehrenfeucht and Rozenberg, 1990) reported also that the prime graphs have the following ascendant hereditary property. Let \(X\) be a subset of a prime graph \(G\) such that \(G[X]\) is prime. If \(|V(G) \setminus X| \geq 2\), then there exist \(x \neq y \in V(G) \setminus X\) such that \(G[X \cup \{x, y\}]\) is prime. The latter result was improved in 1993 by J. H. Schmerl and W. T. Trotter (Schmerl and Trotter, 1993) as follows: Each prime graph of order \(n\) \((n \geq 7)\), embeds a prime graph of order \(n - 2\). It is then natural to raise the next question. Given a prime graph \(G\) of order \(n\), is there always a prime subgraph of \(G\) of order \(n - 1\)? The answer to this question is negative and the prime graph \(G\) such that \(G - x\) is decomposable for each \(x \in V(G)\), referred to as critical graph, is the counterexample. In 1993, J.H. Schmerl and W.T. Trotter (Schmerl and Trotter, 1993) characterized the critical graphs.

Consider now a prime graph \(G = (V, E)\). A vertex \(x\) of \(G\) is said to be critical if \(G - x\) is decomposable. Otherwise, \(x\) is a non-critical vertex. The set of the non-critical vertices of \(G\) is denoted by \(\sigma(G)\). Moreover, if \(G\) admits \(k\) non-critical vertices, it is then called a \((-k)\)-critical graph. Recently, Y. Bouabed and Ille (Boudabbous and Ille, 2009) asked about the description of the \((-1)\)-critical graphs. Their ques-
tion was answered by H. Belkhechine, I. Boudabbous and M. Baka Elayech [Belkhechine et al. (2010)] in the case of graph. More recently, I. Boudabbous and W. Marweni described the triangle-free prime graphs having at most two non critical vertices [Boudabbous and Marweni (2020)].

Another intrinsic tool in this work is the notion of minimal graphs defined as follows. A prime graph $G$ is minimal for a vertex subset $X$, or $X$-minimal, if no proper induced subgraph of $G$ containing $X$ is prime. A graph $G$ is $k$-minimal if it is minimal for some $k$-element set of $k$ elements. A. Cournier and P. Ille [Courrier and Ille (1998)] in 1998 characterized the 1-minimal and 2-minimal graphs. Recently, M. Alzohairi and Y. Boudabbous characterized the 3-minimal triangle-free graphs [Alzohairi and Boudabbous (2014)]. In 2015, M. Alzohairi characterized the triangle-free graphs which are minimal for some nonstable 4-vertex subset [Alzohairi (2015)].

Motivated by these two fundamental notions, I. Boudabbous proposes to find the $(-k)$-critical graphs and $k$-minimal graphs for some integer $k$ even in a particular case of graphs. This work resolves what is requested by I. Boudabbous. For this reason, we shall describe the prime tree having exactly $k$ non-critical vertices. Recall that $[x]$ denotes the greatest integer $\leq x$. Therefore, we obtain:

**Theorem 1.1** Let $T = (V, E)$ be a tree with at least 5 vertices and $\{x_1, \ldots, x_k\}$ be a vertex subset of $G$ where $k$ is an integer. $T$ is $(-k)$-critical and $\sigma(T) = \{x_1, \ldots, x_k\}$ (see Figure 1(a)), if and only if $T$ satisfies the four assertions.

1. For each $x \neq y \in L(T)$, $dist(x, y) \geq 3$.
2. $\{x_1, \ldots, x_k\} \subseteq L(T)$ and $1 \leq k \leq \lceil \frac{n}{2} \rceil$.
3. For each $x \in L(T) \setminus \{x_1, \ldots, x_k\}$, there is a unique $i \in \{1, \ldots, k\}$ such that $dist(x, x_i) = 3$ and $d(x^+) = 2$.
4. If $d(x^+) = 2$ where $i \in \{1, \ldots, k\}$, then for all $x \in L(T) \setminus \{x_i\}$, $dist(x_i, x) \geq 4$.

Moreover, we shall describe the $k$-minimal trees. As a matter of fact, we obtain:

**Theorem 1.2** Let $T = (V, E)$ be a tree with at least 5 vertices and let $\{x_1, \ldots, x_k\}$ be a vertex subset of $G$ where $k$ is a strictly positive integer. $T$ is minimal for $\{x_1, \ldots, x_k\}$ (see Figure 1(b)) if and only if $T$ satisfies the three assertions.

1. For each $x \neq y \in L(T)$, $dist(x, y) \geq 3$.
2. For each $x \in L(T)$, $\{x, x^+\} \cap \{x_1, \ldots, x_k\} \neq \emptyset$.
3. If $x_i \in S(T)$ and $x_i^+ \notin \{x_1, \ldots, x_k\}$ where $i \in \{1, \ldots, k\}$, then $d(x_i) = 2$ and there is $j \neq i \in \{1, \ldots, k\}$ such that $x_j \in L(T)$ and $d(x_i, x_j) = 2$.

2 Proof of Theorem 1.1:

We recall the characterization of the prime tree set forward by to M. Alzohairi and Y. Boudabbous.

**Lemma 2.1** [Alzohairi and Boudabbous (2014)]

1. If $M$ is a nontrivial module in a decomposable tree $T$, then $M$ is a stable set of $T$. Moreover, the elements of $M$ are leaves of $T$. 
2. A tree with at least four vertices is prime if and only if any two distinct leaves do not have the same neighbor.

As an immediate consequence of Lemma 2.1, we have the following result.

**Corollary 2.2** Let $T = (V, E)$ be a tree. $T$ is prime if and only if $d(x, y) \geq 3$, for each $x \neq y \in \mathcal{L}(T)$.

The following observation follows immediately from Lemma 2.1.

**Observation 2.3** Let $T = (V, E)$ be a prime tree with $n$ vertices. Then, $|\mathcal{S}(T)| = |\mathcal{L}(T)| \leq \left\lfloor \frac{n}{2} \right\rfloor$.

Now, we establish the next lemma that will be needed in the sequel.

**Lemma 2.4** Let $T = (V, E)$ be a prime tree and $x \in \mathcal{L}(T)$. If $T - x$ is decomposable, then there is $y \in \mathcal{L}(T) \setminus \{x\}$ such that $\{y, x^+\}$ is the unique module of $T - x$.

**Proof:** Consider a prime tree $T = (V, E)$ and $x \in \mathcal{L}(T)$. Assume that $T - x$ is a decomposable tree. Resting upon Lemma 2.1, there exist two distinct leaves of $T - x$, said $y$ and $z$, which have the same neighbor. Hence, $\{y, z\}$ is a module of $T - x$. Since $T$ is a prime tree, $x \not\sim \{y, z\}$. Thus, $x^+ \in \{y, z\}$.

Without loss of generality, we may assume that $x^+ = z$ and we have $y \in \mathcal{L}(T)$. Since $T$ is prime and $y \in \mathcal{L}(T)$, then $\text{dist}(y, u) \geq 3$ for each $u \neq y \in \mathcal{L}(T)$. Therefore, $\{y, x^+\}$ is the unique module of $T - x$.

**Proof of Theorem 1.1** Consider a tree $T = (V, E)$ with $n$ vertices where $n \geq 5$ and $\{x_1, ..., x_k\}$ is a subset of $V$ where $k$ is a strictly positive integer. Assume that $T$ is $(-k)$-critical and $\sigma(T) = \{x_1, ..., x_k\}$. Since $T$ is prime, by Corollary 2.2, we have for each $x \neq y \in \mathcal{L}(T)$, $\text{dist}(x, y) \geq 3$. Hence, $T$ satisfies the condition (1) of Theorem 1.1.

Moreover, let $x \in V \setminus \mathcal{L}(T)$, $x$ is an internal vertex of $T$ and $T - x$ is a non-connected graph. Then, $T - x$ is decomposable and $x \notin \sigma(T)$. Thus, $\{x_1, ..., x_k\} \subseteq \mathcal{L}(T)$. As $T$ is prime, based on Observation 2.3, we have $1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor$. Hence, $T$ satisfies condition (2) of Theorem 1.1.

Now, consider $x \in \mathcal{L}(T) \setminus \{x_1, ..., x_k\}$. Then, $T - x$ is a decomposable tree. By Lemma 2.4, there is $y \in \mathcal{L}(T) \setminus \{x\}$ such that $\{y, x^+\}$ is the only module of $T - x$. Clearly, $\text{dist}(x, y) = 3$ and $d(x^+) = 2$.

Now, prove that $y \in \sigma(T)$. To the contrary, suppose that $y \notin \sigma(T)$, implying that $T - y$ is a decomposable
tree. Using again Lemma 2.4 there is \( z \in \mathcal{L}(T) \setminus \{ y \} \) such that \( \{ z, y^+ \} \) is the unique module of \( T - y \). Thus, \( d(y^+) = 2 \) and \( \text{dist}(y, z) = 3 \). This implies that \( z = x \) and we obtain that \( T \) is with 4 vertices; which contradicts the fact that \( T \) is a tree having at least 5 vertices. Hence, \( y \in \sigma(T) \). Therefore, \( T \) satisfies the condition (3) of Theorem 1.1.

Besides, assume that there is \( i \in \{ 1, ..., k \} \) such that \( d(x_i^+) = 2 \). Then, \( T - x_i \) is a prime tree and \( x_i^+ \in \mathcal{L}(T - x_i) \). Referring to Corollary 2.2 for all \( y \in \mathcal{L}(T - x_i) \), \( \text{dist}(x_i^+, y) \geq 3 \). Since \( \mathcal{L}(T - x_i) \setminus \{ x_i^+ \} = \mathcal{L}(T) \setminus \{ x_i \} \), then for each \( y \neq x_i \in \mathcal{L}(T) \), \( \text{dist}(x_i, y) \geq 4 \). Hence, \( T \) satisfies condition (4) of Theorem 1.1.

Conversely, assume that \( T \) satisfies the conditions (1)-(4) of Theorem 1.1. Proving that, \( T \) is \((-k)\)-critical and \( \sigma(T) = \{ x_1, ..., x_k \} \). Since for each \( x \neq y \in \mathcal{L}(T) \), \( \text{dist}(x, y) \geq 3 \) and by Corollary 2.2 \( T \) is prime. Clearly, if \( x \in V \setminus \mathcal{L}(T) \), \( T - x \) is a non-connected graph. Thus, \( T - x \) is decomposable and hence \( x \) is a critical vertex.

Furthermore, if \( x \in \mathcal{L}(T) \setminus \{ x_1, ..., x_k \} \), by assertion (3), there is a unique \( i \in \{ 1, ..., k \} \) such that \( \text{dist}(x, x_i) = 3 \) and \( d(x_i^+) = 2 \). Then, \( \{ x_i^+ \} \) is a module of \( T - x \). Hence for each \( x \in \mathcal{L}(T) \setminus \{ x_1, ..., x_k \} \), \( T - x \) is decomposable and then \( x \) is a critical vertex.

Now, given \( i \in \{ 1, ..., k \} \); if \( d(x_i^+) \geq 3 \), then \( \mathcal{L}(T) \setminus \{ x_i \} = \mathcal{L}(T - x_i) \). According to first hypothesis of Theorem 1.1 for each \( x \neq y \in \mathcal{L}(T) \), \( \text{dist}(x, y) \geq 3 \). By Corollary 2.2 \( T - x_i \) is a prime tree.

Assume now that \( d(x_i^+) = 2 \). Suppose that \( T - x_i \) is a decomposable tree. Then, by Lemma 2.4 there is a unique \( y \neq x_i \in \mathcal{L}(T) \) such that \( \{ y, x_i^+ \} \) is the unique module of \( T - x_i \). As a matter of fact, \( \text{dist}(y, x_i) = 3 \); which contradicts the hypothesis 4 of Theorem 1.1. Hence, \( T - x_i \) is prime. Consequently, \( T \) is \((-k)\)-critical and \( \sigma(T) = \{ x_1, ..., x_k \} \).

Our second objective in this section lies in determining the number of nonisomorphic \((-k)\)-critical trees with \( n \geq 5 \) vertices where \( k \in \{ 1, 2, \lfloor \frac{n}{2} \rfloor \} \). According to the characterization of critical graphs Schmerl and Trotter [1993], \( P_4 \) is the a unique critical trees. To specify the the number of nonisomorphic \((-k)\)-critical trees where \( k \in \{ 1, 2, \lfloor \frac{n}{2} \rfloor \} \), we introduce for all \( n \in \mathbb{N} \), the one-to-one function:

\[
T_n : \mathbb{N} \to \mathbb{N} \\
p \mapsto p + n
\]

Now, we introduce also the following trees.

- For integers \( m \geq 2 \), let \( A_{2m+1} \) be the tree defined on \( \{ 0, ..., 2m \} \) and \( E(A_{2m+1}) = \{ \{ 0, i \}, \{ i, i+m \} : 1 \leq i \leq m \} \) (see Figure 2).

- For integers \( k \geq 4, t \geq 1 \), let \( P_{k,t} \) be the tree defined on \( \{ 1, ..., 2t + k \} \) and \( E(P_{k,t}) = E(T_{2t}(P_k)) \cup \{ \{ 2i - 1, 2i \} : 1 \leq i < t \} \cup \{ \{ 2t + 2, 2i \} : 1 \leq i < t \} \) (see Figure 3).

- For integers \( m \geq 4, n_1 \geq 1, n_2 \geq 1 \), for each \( p \in \{ 1, 2 \} \), \( s_p = n_1 + ... + n_p \). Let \( P_{m,n_1,n_2} \) be the tree defined on \( \{ 1, ..., 2s_2 + m \} \) and \( E(P_{m,n_1,n_2}) = E(T_{2s_2}(P_m)) \cup \{ \{ 2i - 1, 2i \} : 1 \leq i \leq s_2 \} \cup \{ \{ 2s_2 + 2, 2i \}, \{ 2s_2 + m - 1, 2j \} : 1 \leq i < n_1 \text{ and } n_1 < j \leq s_2 \} \) (see Figure 4).

**Proposition 2.5**

1. Up to isomorphisms, the \((-1)\)-critical trees with \( n \) vertices are the tree \( P_4, n \geq 6 \) where \( n \) is an even integer.

2. Up to isomorphisms, the \((-\lfloor \frac{n}{2} \rfloor)\)-critical trees with \( n \) vertices are the tree \( A_n \) where \( n \) is an odd integer \( \geq 5 \).
Proof:

1. Clearly, departing from Theorem 1.1, $P_{4,\frac{n+4}{2}}$ is a \((-1)\)-critical tree where $n \geq 6$ and $\sigma(P_{4,\frac{n+4}{2}}) = \{n - 3\}$. Now, we consider a \((-1)\)-critical tree $T$ with $n \geq 5$ vertices such that $\sigma(T) = \{x_1\}$. By Theorem 1.1, $x_1 \in L(T)$. If $x \neq x_1 \in L(T)$, then by assertion (3) of Theorem 1.1, $dist(x, x_1) = 3$ and $d(x^+) = 2$. On the contrary, suppose that $|L(T)| = 2$. Since $T$ is prime, then $T$ is isomorphic to $P_4$; which contradicts the fact that $T$ has at least 5 vertices. Hence, $|L(T)| \geq 3$. By assertion (3), for each $y \in L(T)$, $dist(y, x_1) = 3$ and $d(y^+) = 2$. Thus, $T$ is isomorphic to $P_{4,\frac{n+4}{2}}$ where $n \geq 6$ is an even integer.

2. By Theorem 1.1, $A_{2m+1}$ is \((-\left\lfloor \frac{2m+1}{2} \right\rfloor)\)-critical and $\sigma(A_{2m+1}) = L(A_{2m+1})$ where $m \geq 2$. Now, we consider a \((-\left\lfloor \frac{n}{2} \right\rfloor)\)-critical tree $T$ with $n \geq 5$ vertices. Using Theorem 1.1, $\sigma(T) \subseteq L(T)$ implies that $|\sigma(T)| = \left\lfloor \frac{n}{2} \right\rfloor \leq |L(T)|$. By Observation 2.3, $|L(T)| = |S(T)| \leq \left\lfloor \frac{n}{2} \right\rfloor$ and therefore
Proposition 2.6. k-critical trees and σ

Proof. By Theorem 1.1, n ≥ 1. Now, we shall prove that n is odd. To the contrary, suppose that n is even. Then, |L(T)| = |S(T)| = n/2. Hence, V(T) = L(T) ∪ S(T). Since T[S(T)] is a tree, there exists a vertex y ∈ S(T) with d_T[S(T)](y) = 1. Hence, d_T(y) = 2. We may assume that N(y) = {y, x} where x ∈ S(T). Thus, {y, x} is a module of T − y; which contradicts the fact that y is not a critical vertex. Accordingly, n is odd, |L(T)| = |S(T)| = n/2, and V(T) = L(T) ∪ S(T) ∪ {z}. We can assume that L(T) = {x_1, ..., x_{n−1}} and S(T) = {x_1^+, ..., x_{n−1}^+}. Since T[S(T) ∪ {z}] is a tree, there exists a vertex x_i^+ ∈ S(T) where 1 ≤ i ≤ n − 1/2 with d_T[S(T)](x_i^+) = 1 and hence d_T(x_i^+) = 2. By Theorem 1.1, dist(x_i, x_j) ≥ 4 for each j ≠ i ∈ {1, ..., n−1/2}. Hence, T is isomorphic to A_n where n ≥ 5.

As a consequence of Theorem 1.1, we get the following result.

Proposition 2.6. Up to isomorphisms, the (−2)-critical trees with n ≥ 5 vertices are the trees P_n, P_{k,t} where k ≥ 4, t ≥ 1 and n = k + 2t, and P_{m,n_1,n_2} where m ≥ 4, n_1, n_2 ≥ 1 and n = m + 2(n_1 + n_2).

Proof. By Theorem 1.1, P_n, P_{k,t} where k ≥ 4, t ≥ 1, and P_{m,n_1,n_2} where m ≥ 4, n_1, n_2 ≥ 1 are (−2)-critical trees and σ(P_n) = {1, n}, σ(P_{k,t}) = {2t + 1, 2t + k}, and σ(P_{m,n_1,n_2}) = {2s_2 + 1, 2s_2 + m}. Now, assume that T is a (−2)-critical tree with n ≥ 5 vertices such that σ(T) = {x_1, x_2}. By Theorem 1.1, x_1, x_2 ∈ L(T). As T is a prime tree, then the (x_1, x_2)-path is isomorphic to P_k where k ≥ 4. If |L(T)| = 2, then T is isomorphic to P_n and n = k. Assume that |L(T)| ≥ 3, then by Theorem 1.1 for each x ∈ L(T) \ {x_1, x_2}; there is a unique i ∈ {1, 2} such that dist(x, x_i) = 3 and d(x_i^+) = 2. Hence, T is isomorphic to P_{k,t} where k ≥ 4, t ≥ 1 and n = k + 2t or T is isomorphic to P_{k,n_1,n_2} where k ≥ 4, n_1, n_2 ≥ 1 and n = k + 2(n_1 + n_2). □
Theorem 2.7 The number of nonisomorphic $(-2)$-critical trees with $n$ vertices equals:

- $\frac{n^2}{4} - 1$ if $n \equiv 0 \pmod{4}$.
- $\frac{n^2}{4}$ if $n \equiv 1 \pmod{4}$.
- $\frac{n}{4} \left(\left\lceil \frac{n}{4} \right\rceil + 1 \right) - 1$ if $n \equiv 2 \pmod{4}$.
- $\frac{n}{4} \left(\left\lceil \frac{n}{4} \right\rceil + 1 \right)$ otherwise.

Proof: At the beginning, it is not difficult to verify that there are not two isomorphic different trees in the union $\{P_m : m \geq 5\} \cup \{P_{k,t} : k \geq 5, t \geq 1\} \cup \{P_{m,n_1,n_2} : m \geq 4, n_1 \geq 1 \text{ and } n_2 \geq 1\}$.

By Proposition 2.6, $P_5$ is the unique $(-2)$-critical tree with five vertices and $P_6$ is the unique $(-2)$-critical tree with six vertices. As a matter of fact, the result holds.

Now, assume that $n \geq 7$. By Proposition 2, the nonisomorphic $(-2)$-critical trees with $n$ vertices are $P_n$, the family of $P_{k,t}$ where $t \geq 1, k \geq 5$, and $n = 2t + k$, or the family of $P_{m,n_1,n_2}$ where $1 \leq n_1 \leq n_2$, $m \geq 4$, and $n = 2(n_1 + n_2) + m$. Therefore, it is sufficient to determine the number of the family of $P_{k,t}$ and the number of the family of $P_{m,n_1,n_2}$.

Let $S_m = \{(n_1,n_2) \in \mathbb{N} \times \mathbb{N} : 1 \leq n_1 \leq n_2, \ n_1 + n_2 = \frac{n-m}{2}\}$, where $4 \leq m \leq n - 4$ and let $C_t = \{k \in \mathbb{N} : 5 \leq k \text{ and } k = n - 2t\}$, where $1 \leq t \leq \frac{n-5}{2}$. Since $n - m = 2(n_1 + n_2)$, it is obvious that $n$ and $m$ are of the same parity. Hence, we distinguish two cases.

Case 1: If $n = 2p$ where $4 \leq p$ and $m = 2q$ where $2 \leq q \leq p - 2$.

Consider $S = \bigcup_{q=2}^{p-2} S_{2q}$ and $C = \bigcup_{t=1}^{p-3} C_t$. First, it is clear that the number of the family of $P_{k,t}$ is the cardinality of the set $C$. Moreover, it is clear that $|C_t| = 1$ where $1 \leq t \leq p - 3$. Hence, $|C| = \sum_{t=1}^{p-3} |C_t| = p - 3$. Second, obviously the number of the family of $P_{m,n_1,n_2}$ is the cardinality of the set $S$. Furthermore, we have $|S| = \sum_{q=2}^{p-2} |S_{2q}|$. It is noticeable that for each $2 \leq q \leq p - 2$, $|S_{2q}| = P_2(\frac{n - 2q}{2})$, where $P_i(j)$ is the number of partitions of $j$ to $i$ parts. Recall that for an integer $k \geq 3$, $P_2(k) = \left\lfloor \frac{k}{2} \right\rfloor$ where $\lfloor x \rfloor$ is the
(-k)-critical trees and k-minimal trees

greatest integer \leq x^{\text{Anderson} (2006)}. We get then

\[ |S| = \sum_{q=2}^{p-2} P_2 \left( \frac{n - 2q}{2} \right) \]

\[ = \sum_{q=2}^{p-2} \left\lfloor \frac{n - 2q}{4} \right\rfloor \]

\[ = \sum_{q=2}^{p-2} \left\lfloor \frac{n}{2} - \frac{q}{2} \right\rfloor \]

\[ = \sum_{i=0}^{p-2} \left\lfloor \frac{i}{2} \right\rfloor . \]

\[ = \begin{cases} (k-1)^2 & \text{if } p = 2k, \\ (k-1)k & \text{if } p = 2k + 1. \end{cases} \]

**Case 2:** Let \( n = 2p + 1 \) where \( 4 \leq p \) and \( m = 2q + 1 \) where \( 2 \leq q \leq p - 2 \). Let \( S = \bigcup_{q=2}^{p-2} S_{2q+1} \)

and \( C = \bigcup_{t=1}^{p-2} C_t \). Clearly, the number of the family of \( P_{k,t} \) is the cardinality of the set \( C \). Hence,

\[ |C| = \sum_{t=1}^{p-2} |C_t| = p - 2. \]

In addition, the number of the family of \( P_{m,n_1,n_2} \) is the cardinality of the set \( S \).

Therefore, we have \[ |S| = \sum_{q=2}^{p-2} |S_{2q+1}|. \]

Since for each \( 2 \leq q \leq p - 2 \), \( |S_{2q+1}| = P_2 \left( \frac{(n-1)-2q}{2} \right) \). Proceeding in the same manner as case 1, if \( p = 2k \) where \( k \geq 2 \), then \[ |S| = (k-1)^2. \] Otherwise, \( |S| = (k-1)k. \)

Consequently, the number of nonisomorphic (-2)-critical trees with \( n \) vertices equals:

\[ \begin{cases} \left\lfloor \frac{n}{4} \right\rfloor^2 - 1 & \text{if } n \equiv 0 \pmod{4}, \\ \left\lfloor \frac{n}{4} \right\rfloor^2 & \text{if } n \equiv 1 \pmod{4}, \\ \left\lfloor \frac{n}{4} \right\rfloor \left( \left\lfloor \frac{n}{4} \right\rfloor + 1 \right) - 1 & \text{if } n \equiv 2 \pmod{4}, \\ \left\lfloor \frac{n}{4} \right\rfloor \left( \left\lfloor \frac{n}{4} \right\rfloor + 1 \right) & \text{if } n \equiv 3 \pmod{4}. \end{cases} \]
3 Proof of Theorem 1.2:

We set two major objectives throughout this section. First, to characterize the $k$-minimal trees. Second, to determine the number of nonisomorphic $k$-minimal trees with $n$ vertices where $k \in \{1, 2, 3\}$.

**Proof of Theorem 1.2.** Let $T = (V, E)$ be a tree with $n \geq 5$ vertices and let $\{x_1, ..., x_k\}$ be a vertex subset of $G$ where $k$ is a strictly positive integer. Assume that $T$ is minimal for $\{x_1, ..., x_k\}$. Since $T$ is prime, it satisfies the first condition of Theorem 1.2. Suppose, on the contrary, that there is $y \in \mathcal{L}(T)$ such that $\{y, y^+\} \cap \{x_1, ..., x_k\} = \emptyset$. Since $T - y$ is a decomposable tree, by Lemma 2.4, there is $x \in \mathcal{L}(T)$ such that $\{x, y^+\}$ is a module of $T - y$. Thus, $d(y^+) = 2$ and $d(x, y) = 3$. If $x \notin \{x_1, ..., x_k\}$, then $T - x$ is a decomposable tree. By using again Lemma 2.4, $d(x) = 2$ and so $T$ is isomorphic to $P_4$; which contradicts the fact that $n \geq 5$.

Moreover, assume that $x \in \{x_1, ..., x_k\}$ and $d(x^+) = 3$, then $\mathcal{L}(T) \setminus \{y\} = \mathcal{L}(T \setminus \{y, y^+\})$. By Lemma 2.1, $T - \{y, y^+\}$ is a prime tree containing $\{x_1, ..., x_k\}$; which contradicts the fact that $T$ is minimal for $\{x_1, ..., x_k\}$. Hence, for each $x \in \mathcal{L}(T)$, $\{x, x^+\} \cap \{x_1, ..., x_k\} = \emptyset$ and $T$ satisfies the second condition of Theorem 1.2.

Now, assume that there is $x_i \in S(T)$ and $x_i^- \notin \{x_1, ..., x_k\}$ where $1 \leq i \leq k$. On the contrary, suppose that $d(x_i^+) \geq 3$, then $\mathcal{L}(T) \setminus \{x_i^-\} = \mathcal{L}(T - x_i^-)$. By Lemma 2.1, $T - x_i^-$ is a prime tree containing $\{x_1, ..., x_k\}$; which is impossible. Hence, $d(x_i) = 2$. On the contrary, suppose that for each $j \neq i \in \{1, ..., k\}$ such that $x_j \in \mathcal{L}(T)$, $d(x_i, x_j) \geq 3$. Since $T - x_i^-$ is a decomposable tree, then by Lemma 2.4 there is $y \in \mathcal{L}(T)$ such that $d(x_i, y) = 2$ and hence $y \notin \{x_1, ..., x_k\}$. By Condition 2, $y^+ \in \{x_1, ..., x_k\}$ and so $d(y^+) = 2$. Thus, $T$ is isomorphic to $P_4$; which is impossible. Therefore, $T$ satisfies the third condition.

Conversely, let $T = (V, E)$ be a tree with $n \geq 5$ vertices. Since for each $x \neq y \in \mathcal{L}(T)$, $d(x, y) \geq 3$, $T$ is a prime tree. Let $X$ be a subset of $V$ such that $\{x_1, ..., x_k\} \subseteq X$ and $T[X]$ is prime. Consider $x \in \mathcal{L}(T)$. If $x \notin \{x_1, ..., x_k\}$, then $x \in X$.

Now, assume that $x \notin \{x_1, ..., x_k\}$. On the contrary, suppose that $x \notin X$. By assertion (2) of Theorem 1.2, $x^+ \in \{x_1, ..., x_k\}$. Since $T$ satisfies assertion (3) of Theorem 1.2, then $d(x^+) = 2$ and there is $i \in \{1, ..., k\}$ such that $x_i \in \mathcal{L}(T)$ and $d(x_i, x^+) = 2$. Thus, $\{x^+, x_i\}$ is a module of $T[X]$; which is impossible. Hence, $x \in X$. We conclude that $\mathcal{L}(T) \subseteq X$. Since $T[X]$ is a prime, it is connected. Therefore, $T[X]$ is a tree containing $\mathcal{L}(T)$. Hence, $X = V$. Thus, $T$ is minimal for $\{x_1, ..., x_k\}$. 

The following corollary is an immediate consequence of Theorem 1.2.

**Corollary 3.1** For any distinct vertices $x_1, x_2, ..., x_k$ in a prime tree $H$, there is an induced subtree $T$ of $H$ that contains $\{x_1, x_2, ..., x_k\}$, and satisfies the assertions of Theorem 1.2.

Our second objective is to determine the number of nonisomorphic $k$-minimal trees with $n$ vertices where $k \in \{1, 2, 3\}$. According to the characterization of 1-minimal and 2-minimal graphs, $P_4$ is the unique 1-minimal tree and $P_k$, where $k \geq 4$, is the unique 2-minimal tree [Cournier and Ille (1998)].

To specify the number of nonisomorphic 3-minimal trees with $n$ vertices, we introduce the following tree.

- For positive integers $k, m, n$ with $k \leq m \leq n$, let $S_{k,m,n}$ be the $(k + m + n + 1)$-vertex tree with the union of the paths of lengths $k, m$, and $n$ having common endpoint $r$. Let $a_1, ..., a_k, b_1, ..., b_m, c_1, ..., c_n$ denote the other vertices on these paths, indexed by their distance from $r$ (see Figure 5).

As an immediate consequence of Theorem 1.2, we get the following result which was already obtained by M. Alzohairi and Y. Boudabbous in [Alzohairi and Boudabbous (2014)].
Corollary 3.2 Let $x$, $y$, and $z$ be distinct vertices in a tree $T$. The tree $T$ is minimal for $\{x, y, z\}$ if and only if $T$ and $\{x, y, z\}$ have one of the following forms:

1. $T \simeq P_4$.
2. $T \simeq P_k$ with $k \geq 5$ such that $\{x, y, z\}$ contains the leaves.
3. $T \simeq S_{k,m,n}$ with $m \geq 2$ such that $\{x, y, z\}$ are the leaves.
4. $T \simeq S_{1,2,n}$ such that $\{x, y, z\} = \{a_1, b_1, c_n\}$.
5. $T \simeq S_{1,2,2}$ such that $\{x, y, z\} = \{a_1, b_1, c_1\}$.

Proposition 3.3 The number of nonisomorphic 3-minimal trees with $n$ vertices equals:

- 1 if $n \in \{4, 5\}$.
- 2 if $n = 6$.
- $\left\lfloor \frac{(n-1)^2}{12} \right\rfloor - \left\lfloor \frac{n-4}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor - 1$ if $n \geq 7$, where $[x]$ is the nearest integer from $x$.

Proof: At the beginning, it is not difficult to verify that there are not two isomorphic different graphs in the union $\{P_k : k \geq 4\} \cup \{S_{k,m,n} : m = 2\}$.

By Corollary 3.2, $P_4$ is the unique 3-minimal tree with four vertices and $P_5$ is the unique 3-minimal tree with five vertices. In addition, the only 3-minimal tree with six vertices are $P_6$ and $S_{1,2,2}$. Therefore, the result holds for $n \in \{4, 5, 6\}$.

Now, assume that $n \geq 7$. By Corollary 3.2, the non isomorphic 3-minimal $n$-vertex trees are $P_n$ and the family of $S_{k,m,t}$, where $k \leq m \leq t$, $m \geq 2$, and $k + m + t + 1 = n$. Therefore, it is sufficient to prove that the cardinality of the set $S = \{(k,m,t) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 1 \leq k \leq m \leq t, m \geq 2, k + m + t = n - 1\}$ equals

$$\left\lfloor \frac{(n-1)^2}{12} \right\rfloor - \left\lfloor \frac{n-4}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor - 2.$$
Let $S_2 = \{(k, m, t) \in S : k \geq 2\}$. It is easy to infer that $|S - S_2| = P_2(n - 2) - 1$. Notice that $|S_2| = |\{(p, q, r) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 1 \leq p \leq q \leq r, p + q + r = n - 4\}| = P_3(n - 4)$. Moreover, grounded on Anderson (2006), the number of partitions of $k$ with at most 3 parts is equal to $\binom{(k+3)^2}{12}$. It follows that $P_3(k) = \left\lceil \frac{(n-1)^2}{12} \right\rceil - \left\lfloor \frac{n-4}{2} \right\rfloor - 1$. Therefore,

$$|S_2| = \left\lceil \frac{(n-1)^2}{12} \right\rceil - \left\lfloor \frac{n-4}{2} \right\rfloor - 1$$

and

$$|S - S_2| = \left\lfloor \frac{n-2}{2} \right\rfloor - 1.$$

Thus,

$$|S| = \left\lceil \frac{(n-1)^2}{12} \right\rceil - \left\lfloor \frac{n-4}{2} \right\rfloor + \left\lfloor \frac{n-2}{2} \right\rfloor - 2.$$

\[ \square \]

4 Conclusion

The problems of finding the $(-k)$-critical graphs and the $k$-minimal graphs seem to be challenging where $k$ is an integer ($k \geq 2$). At least, we solve these problems in the particular case of trees. In addition, we determine the number of nonisomorphic $(-k)$-critical trees with $n \geq 5$ vertices where $k \in \{1, 2, \left\lfloor \frac{n}{2} \right\rfloor\}$. Besides, we specify the number of nonisomorphic 3-minimal trees with $n$ ($n \geq 4$) vertices.

References

M. Alzohairi. Triangle-free graphs which are minimal for some nonstable 4-vertex subset. *Arab Journal of Mathematical Sciences*, 21(2):159 – 169, 2015. ISSN 1319-5166. doi: https://doi.org/10.1016/j.ajmsc.2015.02.003. URL [http://www.sciencedirect.com/science/article/pii/S1319516615000092](http://www.sciencedirect.com/science/article/pii/S1319516615000092).

M. Alzohairi and Y. Boudabbous. 3-minimal triangle-free graphs. *Discrete Mathematics*, 331:3 – 8, 2014. ISSN 0012-365X. doi: https://doi.org/10.1016/j.disc.2014.04.020. URL [http://www.sciencedirect.com/science/article/pii/S0012365X14001605](http://www.sciencedirect.com/science/article/pii/S0012365X14001605).

I. Anderson. Andrews, g. e. and eriksson, k. integer partitions (cambridge university press, 2004), x 141 pp., 0 521 60090 1 (paperback), £15.99 (24.99), 0521841186 (hardback), 40(70). *Proceedings of the Edinburgh Mathematical Society*, 49(3):753–754, 2006. doi: 10.1017/S0013091506214937.

H. BELKHECHINE and I. BOUDABBOUS. Indecomposable tournaments and their indecomposable subtournaments on 5 and 7 vertices, 2010.

H. Belkhechine, I. Boudabbous, and J. Dammak. Morphologie des tournois (-1)-critiques. *Comptes Rendus Mathematique*, 345:663–666, 2007.
(-k)-critical trees and k-minimal trees

H. Belkhechine, I. Boudabbous, and M. B. Elayech. Les graphes (-1)-critiques, 2010.

H. Bouchaala, Y. Boudabbous, and M. B. Elayech. Finite tournaments with a nonempty diamonds’ support. Graphs and Combinatorics, 29:1653–1673, 2013.

I. Boudabbous. Tournaments whose indecomposability graph admits a vertex cover of size 2. Ars Comb., 128:63–82, 2016.

I. Boudabbous and W. Marweni. Description of the triangle-free prime graphs having at most two non critical vertices. J. Multiple Valued Log. Soft Comput., 34(1-2):77–103, 2020. URL https://www.oldcitypublishing.com/journals/mvlsc-home/mvlsc-issue-contents/mvlsc-volume-34-number-1-2-2020/mvlsc-34-1-2-p-77-103/.

Y. Boudabbous and P. Ille. Indecomposability graph and critical vertices of an indecomposable graph. Discrete Mathematics, 309(9):2839 – 2846, 2009. ISSN 0012-365X. doi: https://doi.org/10.1016/j.disc.2008.07.015. URL http://www.sciencedirect.com/science/article/pii/S0012365X08004652.

A. Boussaïri and A. Chaïchaâ. Les graphes 2-reconstructibles indécomposables. Comptes Rendus Mathématique, 345:1–4, 2007.

A. Breiner, J. Deogun, and P. Ille. Partially critical indecomposable graphs. Contributions Discret. Math., 3, 2008.

A. Cournier and P. Ille. Minimal indecomposable graphs. Discret. Math., 183:61–80, 1998.

A. Ehrenfeucht and G. Rozenberg. Primitivity is hereditary for 2-structures. Theor. Comput. Sci., 70:343–358, 1990.

A. Ehrenfeucht, T. Harju, and G. Rozenberg. The theory of 2-structures - a framework for decomposition and transformation of graphs. In Handbook of Graph Grammars, 1997.

M. B. Elayech, A. Salhi, and H. S. Kaddour. The (-1)-critically duo-free tournaments. Ars Comb., 119:391–402, 2015.

R. Fraïssé. L’intervalle en theorie des relations; ses generalisations filtre intervallaire et cloture d’une relation. North-holland Mathematics Studies, 99:313–341, 1984.

P. Ille. Indecomposable graphs. Discret. Math., 173:71–78, 1997.

D. Kelly. Invariants of finite comparability graphs. Order, 3:155–158, 1986.

M. Pouzet and I. Zaguia. On minimal prime graphs and posets. Order, 26:357–375, 2009.

M. Y. Sayar. Partially critical tournaments and partially critical supports. Contributions Discret. Math., 6, 2011.

J. H. Schmerl and W. Trotter. Critically indecomposable partially ordered sets, graphs, tournaments and other binary relational structures. Discret. Math., 113:191–205, 1993.
J. Spinrad. P4-trees and substitution decomposition. *Discret. Appl. Math.*, 39:263–291, 1992.

D. Sumner. Graphs indecomposable with respect to the x-join. *Discret. Math.*, 6:281–298, 1973.