Morse resolutions of powers of square-free monomial ideals of projective dimension one

Susan M. Cooper1 · Sabine El Khoury2 · Sara Faridi3 · Sarah Mayes-Tang4 · Susan Morey5 · Liana M. Šega6 · Sandra Spiroff7

Received: 31 March 2021 / Accepted: 14 October 2021 / Published online: 31 October 2021
© The Author(s), under exclusive licence to Springer Science+Business Media, LLC, part of Springer Nature 2021

Abstract
Let \( I \) be a square-free monomial ideal of projective dimension one. Starting with the Taylor complex on the generators of \( I^r \), we use discrete Morse theory to describe a CW complex that supports a minimal free resolution of \( I^r \). To do so, we concretely describe the acyclic matching on the faces of the Taylor complex.

Keywords Monomial ideal · Projective dimension one · Powers of ideal · Minimal resolution · Discrete Morse Theory

Mathematics Subject Classification Primary 13A15, 13D02, 05E40 · Secondary 13C15

1 Introduction

The powers of an ideal \( I \) in a ring \( R \) have a fascinating but poorly understood structure. Describing these powers, and algebraic invariants associated to them, is a vibrant area of mathematics, with problems and tools arising from algebra, geometry, and combinatorics. Even when the ideal itself is well understood, the powers typically have unexpected behavior, making their classification difficult.

In general there is more known about the asymptotic behavior of the collection \( \{ I^r \} \) as \( r \to \infty \) than about the individual ideals \( I^r \), see M. Brodmann [7], V. Kodiyalam [21], S. D. Cutkosky, J. Herzog, and N. V. Trung [11] and Kodiyalam [22]. On the other hand, relatively little is known about the structure of the resolutions of the individual ideals \( I^r \). Although many invariants of \( I \) are known to stabilize for large powers of \( I \), these invariants can exhibit poor behavior for powers below the stable point.

Sara Faridi
faridi@dal.ca
Extended author information available on the last page of the article
In the case where the ideal $I$ is a monomial ideal (and hence so are all its powers $I^r$), the problem of finding free resolutions can be translated into finding combinatorially described topological objects whose chain maps can be adapted to provide a free resolution of the ideal. D. Taylor’s thesis [28] initiated this approach by encoding the faces of a simplex with the least common multiples of the monomial generators of the ideal. Over the last few decades, Taylor’s construction has been generalized ([4,6,26]) to other topological objects using the same idea: label the vertices of a topological object $\Gamma$ with monomials $m_1, \ldots, m_q$ and look for conditions under which the cellular chain complex of $\Gamma$ can be homogenized to describe a free resolution of $I = (m_1, \ldots, m_q)$. In this case we say $\Gamma$ supports a free resolution of $I$, and $I$ has a cellular resolution, and we use combinatorial and homological information about $\Gamma$ to gain information about invariants of $I$.

This point of view allows one to use homotopy theory to describe free resolutions of ideals as the cellular resolutions of homotopy equivalent topological objects. Discrete Morse theory—an adaptation of Morse theory of manifolds to the discrete setting—is one such homotopy theory. It was proven by R. Forman [16] that a discrete Morse function $f$ on the set of cells of a CW complex $X$ produces a CW complex $X_f$ which is homotopy equivalent to $X$. M. Chari [9] reinterpreted Forman’s Morse function as acyclic matchings. E. Batzies and V. Welker [3,4] showed that if the matchings are homogeneous, then this construction leads to a multigraded resolution. As demonstrated in [4], Morse theory has great potential in the study of resolutions, providing a clever tool to shrink large free resolutions into smaller ones, see also [2], [13], and [19, Chapter 6]. It is important to stress that not all monomial ideals have minimal cellular resolutions; see M. Velasco [30].

Inspired by recent work of J. Àlvarez Montaner, O. Fernández-Ramos, P. Gimenez [1] where acyclic matchings are used to prune extra faces from the Taylor complex in order to achieve a smaller resolution for a given monomial ideal, our approach in this paper is to define homogeneous acyclic matchings on the poset lattice of the Taylor complex of $I^r$ for a square-free monomial ideal $I$ of projective dimension one.

The structure of ideals of projective dimension one has an illustrious history. In 1890 D. Hilbert [18], in a result extended by L. Burch [5], described the structure of these ideals in terms of the minors of a presentation matrix, resulting in the celebrated Hilbert-Burch theorem. (See [12, Theorem 20.15] for the statement and historical context.)

Our motivation for this paper is the classification of resolutions of monomial ideals of projective dimension one in B. Hersey’s Master’s thesis ([15,17]) as cellular resolutions supported on graphs. In this paper, we exploit the structure of the graph supporting a minimal free resolution of $I$ to find a CW complex that supports a free resolution of $I^r$. In addition, we give concrete descriptions of the cells and gradient paths of the resulting Morse complex to show that the resolution is minimal.

Our main results can be summarized in the following statement.

**Theorem 1.1** (Theorems 5.3 and 7.2) *If $I$ is an ideal of projective dimension one in a polynomial ring over a field, minimally generated by $q$ square-free monomials, and $r$ is a positive integer, then there exists a homogeneous acyclic matching on the Taylor complex of $I^r$ (that is explicitly defined using the structure of a graph supporting the...*
resolution of $I$) such that the resulting CW complex supports a minimal free resolution of $I^r$.

As an immediate corollary, we recover an explicit formula for the projective dimensions, or equivalently the depths, of the powers of $I$ (Corollaries 7.3 and 7.4), and we are able to describe the regularity of large powers of $I$ in Corollary 7.6.

While the focus of this paper is on Morse resolutions, it is worth noting that in the case of monomial ideals of projective dimension one, these minimal resolutions can also be obtained from a Koszul complex, see [10]. Our work here, however, aligns with a different type of question (see also Engström and Noren [13]): Starting from a (minimal) free resolution of $I$ supported on a cell (simplicial) complex $X$, how close to a minimal free resolution of $I^r$ can we get by using homogeneous acyclic matchings to reach a smaller CW complex $X^r$ supporting a free resolution of $I^r$?

The paper is organized as follows. Section 2 provides the basic definitions of simplicial and CW complexes, cellular resolutions, and the particular case of ideals of projective dimension one. In Sect. 3 we develop the Morse matching on the faces of the Taylor complex, which will later be used in Sect. 5 to construct the Morse complex supporting a free resolution of $I^r$. In Sect. 4 we develop the necessary monomial labelings for the Morse complex, and in Sect. 6 we describe the cells of the Morse complex supporting the resolution, all of which will allow us to prove, in Sect. 7, that the cellular resolution of the Morse complex homogenizes to a minimal free resolution of $I^r$.

2 Setup

2.1 Simplicial complexes and quasi-trees

We begin by recalling some standard definitions and notations. A simplicial complex $\Delta$ on a vertex set $V$ is a set of subsets of $V$ such that $F \in \Delta$ and $G \subseteq F$ implies that $G \in \Delta$. An element $F$ of $\Delta$ is called a face, and a maximal face of $\Delta$ (under inclusion) is called a facet. A simplicial complex $\Delta$ can be uniquely determined by its facets $F_1, \ldots, F_q$, so we write $\Delta = \langle F_1, \ldots, F_q \rangle$. A simplex is a simplicial complex with only one facet. The dimension of a face $F \in \Delta$ is $\dim(F) = |F| - 1$, and the dimension of $\Delta$ is $\dim(\Delta) = \max\{\dim(F) \mid F \in \Delta\}$. The 0-dimensional faces are called vertices and the face $\emptyset$ has dimension $-1$.

A facet $F$ of $\Delta$ is a leaf [14] if $F$ is the only facet of $\Delta$ or if there is another facet $G$ of $\Delta$, called a joint such that $(F \cap H) \subseteq G$ for every facet $H \neq F$. A free vertex of $\Delta$ is a vertex belonging to exactly one facet of $\Delta$. If $F$ is a leaf of a simplicial complex, then $F$ necessarily has a free vertex [14]. A quasi-forest [32] is a simplicial complex $\Delta$ whose faces can be ordered as $F_1, \ldots, F_q$ such that for each $i \in \{1, \ldots, q\}$, the facet $F_i$ is a leaf of $\langle F_1, \ldots, F_i \rangle$. A connected quasi-forest is called a quasi-tree.

If $\Delta = \langle F_1, \ldots, F_q \rangle$ is a simplicial complex on the vertex set $V$, then the complement of $\Delta$ is the simplicial complex

$$\Delta^c = \langle F_1^c, \ldots, F_q^c \rangle$$
where \( F_i^c = V \setminus F_i \).

These definitions give a variety of ways to construct square-free monomial ideals by means of simplicial complexes and vice versa. While traditionally a simplicial complex was associated to its Stanley-Reisner ring [27, Chapter II], and more recently many authors have studied edge ideals, defined in [31] (or more generally facet ideals defined in [14]), the focus of this paper will be to exploit the connections between the square-free monomial generators of an ideal \( I \) and the structure of \( \Delta^c \).

2.2 Resolutions supported on simplicial complexes

Assume that \( S = k[x_1, \ldots, x_n] \) is a polynomial ring over a field \( k \) and \( I = (m_1, \ldots, m_q) \) is an ideal generated by monomials. Let \( \text{LCM}(I) \) denote the lcm-lattice of \( I \), that is, the poset consisting of the lcm’s of the generating set \( m_1, \ldots, m_q \) ordered by divisibility.

A **graded free resolution of** \( I \) **is an exact sequence of free** \( S \)-**modules of the form:**

\[
\mathbb{F}: 0 \to G_d \to \cdots \to G_i \xrightarrow{\partial_i} G_{i-1} \to \cdots \to G_1 \xrightarrow{\partial_1} G_0
\]

where \( I \cong G_0 / \text{im}(\partial_1) \), and each map \( \partial_i \) is graded, in the sense that it preserves the degrees of homogeneous elements.

If \( \partial_i(G_i) \subseteq (x_1, \ldots, x_n)G_{i-1} \) for every \( i > 0 \), then the free resolution \( \mathbb{F} \) is **minimal**. The grading on each of the free modules \( G_i \) can be further refined by writing \( G_i \) as a direct sum of 1-dimensional free \( S \)-modules of the form \( S(m) \beta_i, m \) indexed by monomials \( m \). In particular, when \( \mathbb{F} \) is a minimal resolution,

\[
G_i \cong \bigoplus_{m \in \text{LCM}(I)} S(m)^{\beta_i, m}
\]

where the \( \beta_i, m \) are invariants of \( I \) called the **multigraded Betti numbers** of \( I \). The length of a minimal free resolution of \( I \) (\( d \) in the case of \( \mathbb{F} \) above) is another invariant of \( I \) called the **projective dimension** and denoted by \( \text{pd}_S(I) \).

One concrete way to calculate a multigraded free resolution is to use chain complexes of topological objects. This approach was initiated by Taylor [28], and further developed by Bayer and Sturmfels [6] and many other researchers.

Given a monomial ideal \( I = (m_1, \ldots, m_q) \), the **Taylor complex** \( \text{Taylor}(I) \) of \( I \) is a simplex on \( q \) vertices, each of which is labeled by a monomial generator of \( I \), and where each face is labeled by the lcm of the monomial labels of its vertices.

Below we provide a simple example of a Taylor complex, chosen as the smallest interesting one. It will be used as a running example throughout the paper for the purpose of demonstrating the subtleties in the later constructions via the diagrams and complexes associated to this ideal and its powers. For the ideal \( I \) below, even \( \text{Taylor}(I^2) \), for instance, is already quite complicated. (See Example 5.4.)
Example 2.1 If $I = (xy, yz, zu)$, then the Taylor complex of $I$ is below.

When $I$ is generated by $q$ monomials, one can observe that the simplicial chain complex of Taylor$(I)$ (which is transformed into a free resolution of $I$) has length equal to the dimension of Taylor$(I)$, which is $(q - 1)$. Since every free resolution contains a minimal free resolution, we can see that pd$_S(I) \leq q - 1$.

2.3 Ideals of projective dimension 1

If $\Gamma$ supports a minimal free resolution of $I$ and pd$(I) = 1$, then dim$(\Gamma) = 1$ and so $\Gamma$ is an acyclic graph and hence a tree. Faridi and Hersey proved that these two conditions are in fact equivalent.

Before stating the theorem, we recall a standard definition. Let $S = k[x_1, \ldots, x_n]$ and $\Delta$ be a simplicial complex whose vertices are labeled with the variables $x_1, \ldots, x_n$. Then the square-free monomial ideal

$I = (x_{i_1} \cdots x_{i_r} \mid \{x_{i_1}, \ldots, x_{i_r}\} \text{ is a facet of } \Delta)$

is called the facet ideal of $\Delta$, denoted by $\mathcal{F}(\Delta)$, and $\Delta$ is called the facet complex of $I$, denoted by $\mathcal{F}(I)$. Similarly, $\mathcal{F}(I)^c$ denotes the complex with facets $F_i^c$ where $F_i$ are the facets of $\mathcal{F}(I)$.

Theorem 2.2 ([15, Theorem 27]) Let $I$ be a square-free monomial ideal in a polynomial ring $S$. Then the following statements are equivalent:

1. pd$_S(I) \leq 1$;
2. $\mathcal{F}(I)^c$ is a quasi-forest;
3. The Alexander dual of the Stanley-Reisner complex of $I$ is a quasi-forest;
4. $S/I$ has a minimal free resolution supported on a graph, which is a tree.

This theorem will be used to associate a graph in part (4) to a monomial ideal of projective dimension one in part (1) and its associated complex in part (2). Part (3) is included to show connections to the broader field, including the theory of Stanley-Reisner. The graph in part (4) of Theorem 2.2 is constructed based on an ordering...
of facets of the quasi-forest $\mathcal{F}(I)^c$ in part (2), or equivalently an ordering of the minimal monomial generating set of $I$. In fact, as seen in Construction 2.3 the minimal generators of $I$ are obtained from the product of the variables that are in the complement of the facets. Since the ordering of facets in a quasi-tree requires that every facet introduces a new variable, it follows that the number of generators of $I$ is less or equal than $n$, see also Eq. (4.0.1). Furthermore, Construction 2.3 is a procedure whose output is the ordered monomial generating set $m_1, \ldots, m_q$ for $I$, which we will use throughout the paper to build a cellular resolution for $I'$. Construction 2.3 also gives the definition of the joint function $\tau$ which will be used heavily in the rest of the paper.

**Construction 2.3 (An order on the generators of $I$)** Let $I$ be a square-free monomial ideal with $\text{pd}_S(I) \leq 1$.

1. Order the facets of $\Delta = \mathcal{F}(I)^c$ as $F_1, F_2, \ldots, F_q$ such that $F_i$ is a leaf of $\Delta_i = \langle F_1, \ldots, F_i \rangle$.
2. Start with the one vertex tree $T_1 = (V_1, E_1)$ where $V_1 = \{v_1\}$ and $E_1 = \emptyset$.
3. If $i = 1$, set $\tau(1) = 1$. For $i = 2, \ldots, q$ do the following:
   - Pick $u < i$ such that $F_u$ is a joint of $F_i$ in $\Delta_i$. Set $\tau(i) = u$;
   - Set $V_i = V_{i-1} \cup \{v_i\}$;
   - Set $E_i = E_{i-1} \cup \{(v_i, v_u)\}$.
4. The result is a tree $T = (V_q, E_q)$ with $q$ vertices. We label the vertex $v_i$ of $T$ with the monomial
   \[ m_i = \prod_{x_t \in F_i^c} x_t \]
   The monomials $m_1, \ldots, m_q$ form a minimal generating set of $I$, ordered as in Step (1) above.

**Example 2.4** If $I = (xy, yz, zu)$ is the ideal in Example 2.1, it is easy to check using a software such as Macaulay 2 [23] that $\text{pd}_S(I) = 1$. Set $\Delta = \mathcal{F}(I)^c$, which is a simplicial complex with facets $\{z, u\}, \{x, u\}, \{x, y\}$. In fact, $\Delta$ is a quasi-tree. Consider the facet order on $\Delta$: $F_1 = \{z, u\}, F_2 = \{x, u\}$ and $F_3 = \{x, y\}$ pictured below.

\[ \Delta : \]

\[ z \quad u \quad x \quad y \]

\[ F_1 \quad F_2 \quad F_3 \]

We also have $\tau(3) = 2$ and $\tau(2) = 1$. Following Construction 2.3 we obtain a tree $G$ with vertices indexed by the monomials generating $I$ which supports a minimal free resolution of $I$.

\[ G : \]

\[ \text{Springer} \]
2.4 Cell complexes and cellular resolutions

As a generalization of the Taylor resolution, one could homogenize the cellular chain complexes of CW complexes to obtain free resolutions of monomial ideals, which has the advantage that often there are fewer faces in each dimension, meaning that the cellular resolution will potentially be closer to a minimal one. This idea was first developed by Bayer and Sturmfels [6] and has been expanded by many authors since. Morse resolutions are an example of cellular resolutions.

To formally define a finite regular CW complex (also known as a finite regular cell complex), we will use [24,25]. Let $B^n$, with $n \geq 1$, denote the $n$-dimensional closed ball

$$B^n = \left\{ (a_1, \ldots, a_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} a_i^2 \leq 1 \right\}.$$

The $n$-dimensional open ball is the interior $\text{int}(B^n)$ of $B^n$. The $(n - 1)$-dimensional sphere $S^{n-1}$ is the boundary of $B^n$.

A topological space is called a(n) (open) cell of dimension $n$, or $n$-cell, if it is homeomorphic to $\text{int}(B^n)$ when $n \geq 1$ and to a point when $n = 0$. A cell decomposition of a space $X$ is a family $\Gamma = \{ c_i : i \in I \}$ of pairwise disjoint subspaces of $X$ such that each $c_i$ is a cell and $X = \bigcup_{i \in I} c_i$. We say that $\Gamma$ is finite when the index set $I$ is finite.

Given a cell decomposition as above and $n \geq 0$, let $\Gamma^n$ denote the set of all $n$-cells in $\Gamma$. The $n$-skeleton of $X$ is the subspace

$$X^n = \bigcup_{0 \leq i \leq n} \bigcup_{c \in \Gamma^i} c.$$

The elements of the 0-skeleton $X^0$ are called vertices.

**Definition 2.5** A finite CW complex is a Hausdorff space $X$, together with a finite cell decomposition $\Gamma$ such that for each $n \geq 0$ and $c \in \Gamma^n$ there is a continuous map $\Phi_c : B^n \to X$ that restricts to a homeomorphism $\Phi_c|_{\text{int}(B^n)} : \text{int}(B^n) \to c$ and takes $S^{n-1}$ into $X^{n-1}$. A finite CW complex is also referred to as a finite cell complex. When the cell decomposition $\Gamma$ is understood (or implied), we will use only the letter $X$ to refer to the CW complex.

As described in [25, Lemma 2.2.6], CW complexes can also be defined by constructing the skeleton sets recursively, in terms of a procedure of adjunction of cells of increasing dimensions, starting with a discrete set of points as the 0-skeleton.

3 Acyclic matchings

In this section we set the foundations for the construction of a Morse resolution of $I^r$ from an acyclic matching on the face poset of the Taylor complex. We define the
Morse complex itself in Theorem 5.3, give a detailed description of it in Theorem 6.1 and prove it supports a minimal resolution of $I^r$ in Theorem 7.2.

We begin with some background in discrete Morse theory.

**Terminology 3.1** Let $V$ be a finite set. We denote by $2^V$ the set of subsets of $V$. Let $Y$ denote a subset of $2^V$. We call the elements $\sigma$ of $Y$ cells. If $\sigma$ has exactly $n$ elements, then it is called an $(n-1)$-cell. A cell with only one element is a 0-cell, and the empty set is a $(-1)$-cell. We note that the Taylor complex is a simplex consisting of the set of all subsets of a finite set $V$ of vertices. When convenient, we view the faces of the Taylor complex as cells of the $2^V$ set.

We define the directed graph $G_Y$ whose vertex set $\{\sigma \mid \sigma \in Y\}$ is the set of cells of $Y$, and with directed edges $E_Y$ consisting of $\sigma \rightarrow \sigma'$ with $\sigma' \subseteq \sigma$ and $|\sigma| = |\sigma'| + 1$. A matching of $G_Y$ is a set $A \subseteq E_Y$ of edges of $G_Y$ with the property that each cell of $Y$ occurs in at most one edge of $A$. A cycle is a series of $n \geq 3$ directed edges $\sigma_0 \rightarrow \sigma_1 \rightarrow \cdots \rightarrow \sigma_n$ with $\sigma_0 = \sigma_n$ and $\sigma_i \neq \sigma_j$ for $0 \leq i < j < n$.

For a matching $A$, let $G_Y^A$ be the graph whose edge set is

$$E_Y^A = (E_Y \setminus A) \cup \{\sigma' \rightarrow \sigma \mid \sigma \rightarrow \sigma' \in A\}.$$

Note the reversal of the direction of the edges in $A$. In this edge set, we think of the oriented edges in $E_Y \setminus A$ as pointing down and the oriented edges $\sigma' \rightarrow \sigma$ with $\sigma \rightarrow \sigma'$ in $A$ as pointing up. In an abuse of language/notation, we will simply say that the edges in $A$ “point up”. Note that an edge that goes down corresponds to a decrease in the cardinality of the cells it connects, while an edge that points up corresponds to an increased cardinality.

By definition, a cycle in $G_Y^A$ cannot have two consecutive edges in $A$ since consecutive edges share a cell of $Y$. Since each edge of $G_Y^A$ connects cells whose cardinalities differ by precisely one, a cycle must contain the same number of edges that point down as edges that point up. Combined, these two observations show that edges of a cycle in $G_Y^A$ alternate between edges in $A$ and edges not in $A$.

The diagram in Fig. 1 describes a cycle in $G_Y^A$ for a matching $A$, where $r > 1$ and for $i = 1, \ldots, r$ with indices mod $r$ we have $\sigma_i \rightarrow \sigma_i'$ are edges in $E_Y \setminus A$ and $\sigma_i \rightarrow \sigma_i'$ are edges in $A$, which are reversed in $E_Y^A$.

We say $A$ is acyclic if $G_Y^A$ is acyclic (i.e., does not contain directed cycles). Given an acyclic matching $A$, the cells of $Y$ that do not appear in the edges of the matching are
called the $A$-critical cells; these are cells which are unmatched. The following lemma shows how acyclic matchings can be contracted or preserved in alternate ambient spaces.

**Lemma 3.2** Let $Y \subseteq 2^V$.

1. Let $Y' \subseteq 2^V$ such that $Y \subseteq Y'$. If $A \subseteq E_Y$, then $A$ is an acyclic matching of $G_{Y'}$ if and only if $A$ is an acyclic matching of $G_Y$.
2. If $A$ is an acyclic matching of $G_Y$ and $A' \subseteq A$, then $A'$ is an acyclic matching of $G_Y$ as well.

**Proof** (1) Assume $A$ is an acyclic matching on $G_Y$. We show that $A$ is an acyclic matching on $G_{Y'}$. We see that $A$ is a matching on $G_{Y'}$ because the cells in $Y' \setminus Y$ do not occur in any of the edges of $A$, since $A \subseteq E_Y$. We need to show: If $G_A$ is acyclic, then $G_{A'}$ is acyclic. Thus, assume $G_A$ is acyclic. If there were a cycle in $G_{A'}$ then it would involve at least one cell $\sigma \in Y' \setminus Y$. Since $A \subseteq E_Y$, there are no edges of $A$ that have $\sigma$ as a vertex. Therefore, the edges of the cycle that contain $\sigma$ are both in $E_{Y'} \setminus A$; hence, they both point down. However, one cannot have in a cycle two consecutive edges that point down, as was previously observed; see Fig. 1.

If $A$ is an acyclic matching on $G_{Y'}$, it is clear that it is also an acyclic matching on $G_Y$.

(2) See for example [3, Lemma 3.2.3].

The next lemma will be used later as a source of acyclic matchings.

**Lemma 3.3** Let $v \in V$. Then

$$A_Y^v = \{\sigma \rightarrow \sigma' \in E_Y \mid v \in \sigma \text{ and } \sigma' = \sigma \setminus \{v\}\}.$$ is an acyclic matching on $G_Y$.

**Proof** Set $A = A_Y^v$. To show that $A$ is a matching, assume $\sigma \in Y$ belongs to two edges in $A$. If the two edges are $\sigma \rightarrow \sigma'$ and $\sigma \rightarrow \sigma''$, then $\sigma' = \sigma'' = \sigma \setminus \{v\}$. If the two edges are $\sigma' \rightarrow \sigma$ and $\sigma'' \rightarrow \sigma$, then $\sigma' = \sigma'' = \sigma \cup \{v\}$. If the edges are $\sigma \rightarrow \sigma'$ and $\sigma'' \rightarrow \sigma$, then we see that $v \in \sigma$ because of the first edge and $v \notin \sigma$ because of the second edge, a contradiction.

To show that the matching is acyclic, we need to show that $G^A_Y$ is acyclic. Assume there is a cycle in $G^A_Y$. Label it as in Fig. 1. With the notation as in Terminology 3.1, for all $i$ we have $\sigma'_i \subseteq \sigma_i$, $|\sigma_i| = |\sigma_{i+1}|$, and $|\sigma'_i| = |\sigma'_{i+1}|$, where $|\sigma_i| = |\sigma'_i| + 1$. Since $\sigma_i \rightarrow \sigma'_i \in A$, we see that $v \in \sigma_i$ and $v \notin \sigma'_i$.

The cardinality of $\sigma'_i$ is one less the cardinality of $\sigma_0$; hence, $\sigma_0 = \sigma'_0 \cup \{v'\}$ for some $v' \in V$. Since $v \in \sigma_0$, but $v \notin \sigma'_1$, we must have $v = v'$. However, $\sigma'_1 \cup \{v\} = \sigma_1$, and hence, $\sigma_1 = \sigma_0$, a contradiction.

The following fact, which is a reformulation of Lemma 4.2 in [20], is essential to Theorem 5.3:
Lemma 3.4 (Cluster Lemma [20, Lemma 4.2]) Let $Y$ be a family of finite subsets of a set $V$. Assume that there exists a partition $Y = \bigcup_{q \in Q} Y_q$ indexed by a poset $Q$ with the following property:

If $\sigma \in Y_q$ and $\sigma' \in Y_{q'}$ satisfy $\sigma' \subseteq \sigma$, then $q' \leq q$.

Let $A_q$ be an acyclic matching of $G_{Y_q}$ for each $q$. Then the union $A = \bigcup_{q \in Q} A_q$ is an acyclic matching of $G_Y$.

Definition 3.5 With notation as in Terminology 3.1, a grading on $Y$ is an order-preserving function $\text{gr}: Y \to P$, where $P$ is a poset, and $Y$ is ordered with respect to inclusion. If $Y$ is equipped with a grading, we say that an acyclic matching $A$ on $G_Y$ is homogeneous provided that $\text{gr}(\sigma) = \text{gr}(\sigma')$ for all $\sigma \rightarrow \sigma' \in A$.

If $I$ is a monomial ideal and $X = \text{Taylor}(I)$ is the Taylor complex of $I$, and $P = \text{LCM}(I)$ is the lcm-lattice of $I$, then we can define a grading on $X$ via the function

$$\text{lcm}: X \to \text{LCM}(I)$$

where $\text{lcm}(\sigma) = \text{lcm}(m_i \mid i \in \sigma)$. Note that this grading is exactly the monomial labeling of the faces of the Taylor complex, as in Example 2.1.

Example 3.6 Let $I = (xy, yz, zu)$ be the ideal in Example 2.1 and let $X$ be the Taylor complex of $I$. The graph $G_X$ is shown below. The only homogeneous acyclic matching in $G_X$ under the lcm grading is the singleton edge $\{b\}$, since $\text{lcm}(xy, yz, zu) = \text{lcm}(xy, zu)$.

Our motivation in what follows is the next statement, which is a special case of [4, Proposition 1.2, Theorem 1.3].

Theorem 3.7 (Resolutions from acyclic matchings) If $I$ is a monomial ideal, $X$ is the Taylor complex of $I$ graded by the lcm function, and $A$ is a homogeneous acyclic matching of $G_X$, then there is a CW complex $X_A$ which supports a multigraded free resolution of $I$. The $i$-cells of $X_A$ are in one-to-one correspondence with the $A$-critical $i$-cells of $X$.

Remark 3.8 In [4], the authors define the notion of acyclic matchings in the context of CW complexes. Since we will only apply the results of [4] in the case of a simplex, we opted to give the less general definitions presented here, as it is simpler to work with them.
4 Powers of monomial ideals of projective dimension one

We now turn our attention to the monomial labelings of the faces of the Taylor complex of $I^\tau$ when $I$ is a square-free monomial ideal of projective dimension 1 with a minimal resolution supported on a tree $G$. In other words, we investigate the monomials in the lcm-lattice of $I^\tau$. The definitions and results established here provide the technical machinery used to define matchings and show they are homogeneous under the lcm labelings in subsequent sections.

In Construction 2.3 we detailed the construction of the graph $G$ from the (ordered) generators $m_1, \ldots, m_q$ of $I$. In the last step of Construction 2.3, note that since $F_i$ is a leaf of $\Delta_i$, there is a free vertex $x \in F_i$ such that $x \notin F_j$ for all $j < i$. Equivalently,

$$x \not\in F_i^c \text{ and } x \in F_j^c \text{ for all } j < i.$$ 

Therefore, for every $i \in \{1, \ldots, q\}$ there exists a variable $x \in \{x_1, \ldots, x_n\}$ such that

$$x \nmid m_i \quad \text{and} \quad x \mid m_j \quad \text{for all} \quad j < i. \quad (4.0.1)$$

We also notice that

$$\text{lcm}(m_i, m_j) = \prod_{x_t \in F_i^c \cup F_j^c} x_t.$$ 

In particular, we have $m_u \mid \text{lcm}(m_i, m_j)$ if and only if $F_u^c \subseteq F_i^c \cup F_j^c$, or equivalently, $F_i \cap F_j \subseteq F_u$. In view of Construction 2.3, since $F_i \cap F_j \subseteq F_{\tau(i)}$ whenever $j < i$, we have

$$\text{lcm}(m_i, m_j, m_{\tau(i)}) = \text{lcm}(m_i, m_j) \quad \text{for all } j < i,$$

and in particular,

$$m_{\tau(i)} \mid \text{lcm}(m_i, m_j) \quad \text{for all } j < i. \quad (4.0.2)$$

Let $\{m_1, \ldots, m_q\}$ be the square-free monomial generating set for $I$. If $a = (a_1, \ldots, a_q) \in (\mathbb{N} \cup \{0\})^q$, we set

$$|a| = a_1 + \cdots + a_q \quad \text{and} \quad m^a = m_1^{a_1} \cdots m_q^{a_q}.$$ 

For each $r \geq 0$, we further set

$$\mathcal{N}_r = \{a \in (\mathbb{N} \cup \{0\})^q \mid |a| = r\} \quad \text{and} \quad \mathcal{M}_r = \{m^a \mid a \in \mathcal{N}_r\}.$$ 

Since $I$ is generated by $m_1, \ldots, m_q$, the ideal $I^\tau$ is generated by monomials in $\mathcal{M}_r$. We show that if pd$_S(I) = 1$, then $\mathcal{M}_r$ is a minimal generating set for $I^\tau$. 
Proposition 4.1 (Uniqueness of generators) Assume $I$ is a square-free monomial ideal with $\text{pd}_S(I) = 1$ and let $\{m_1, \ldots, m_q\}$ be the square-free monomial generating set for $I$. Suppose $a, b \in \mathcal{N}_r$ for some $r > 0$. Then

$$m^a = m^b \iff a = b.$$ 

In particular, $\mathcal{M}_r$ forms a minimal generating set for $I^r$.

Proof One direction of the statement is clear. For the other direction, suppose without loss of generality the generators $m_1, \ldots, m_q$ are ordered as in In view of Construction 2.3,

$$a = (a_1, \ldots, a_q) \neq b = (b_1, \ldots, b_q) \quad \text{and} \quad m^a = m^b.$$ 

If we set

$$a \cap b = (\min(a_1, b_1), \ldots, \min(a_q, b_q)),$$

then by cancelling $m^{a \cap b}$ from both sides of the equation $m^a = m^b$, we can assume without loss of generality that for all $1 \leq j \leq q$, if $a_j \neq 0$ then $b_j = 0$ and vice versa.

Let $k = \max\{j \mid a_j \neq 0 \text{ or } b_j \neq 0\}$, and suppose without loss of generality that $a_k \neq 0$. In particular, it follows that

$$b_j = 0 \text{ for all } j \geq k.$$ 

By (4.0.1), there exists $x \in \{x_1, \ldots, x_n\}$ such that

$$x \nmid m_k \text{ and } x \mid m_j \text{ for all } j < k.$$ 

So $x \mid m_j$ for every $j$ such that $b_j \neq 0$, and therefore, $x^{\lceil b\rceil} \mid m^b$. Since $|a| = |b| = r$, this implies that $x^{|a|} \mid m^a$. But since $a_k \neq 0$ and each $m_i$ is square-free then $x \nmid m_k$, which is impossible. \hfill $\square$

The definitions below will be used in the sequel. It may be helpful to the reader to consider Example 4.3, which illustrates some of the concepts.

Definition 4.2 Let $I$ be a square-free monomial ideal of projective dimension 1 and let $\{m_1, \ldots, m_q\}$ be the square-free monomial generators of $I$ ordered as in Construction 2.3. Suppose $a, b \in \mathcal{N}_r$, where

$$a = (a_1, \ldots, a_q) \quad \text{and} \quad b = (b_1, \ldots, b_q).$$

- The support of $m^a$, or of $a$, is the set

$$\text{Supp}(m^a) = \text{Supp}(a) = \{j \mid a_j \neq 0\} \subseteq \{1, 2, \ldots, q\}.$$ 

- Define $\prec$ by $m^b \prec m^a$ or $b \prec a$ if $b_j < a_j$ for the largest $j$ such that $b_j \neq a_j$. 

Springer
Define \( \partial(m^a, m^b) \) or \( \partial(a, b) \) to be the largest index where \( a \) and \( b \) differ when \( a \neq b \). In other words,

\[
\partial(m^a, m^b) = \partial(a, b) = \max\{ j \mid a_j \neq b_j \}.
\]

If \( a = b \), set \( \partial(m^a, m^b) = \partial(a, b) = -\infty \).

Since \( |a| = |b| \), we note that \( \partial(a, b) > 1 \) when \( a \neq b \).

Recall that a function \( \tau : \{1, 2, \ldots, q\} \rightarrow \{1, 2, \ldots, q\} \) satisfying \( \tau(1) = 1 \) and \( \tau(i) < i \) for \( i > 1 \) is defined in Construction 2.3. If \( j \in \text{Supp}(a) \), and \( e_1, \ldots, e_q \) denote the standard basis vectors for \( \mathbb{R}^q \), we set

\[
\pi_j(a) = a + e_{\tau(j)} - e_j \quad \text{and} \quad \pi_j(m^a) = m^{\pi_j(a)} = m^a \cdot m_{\tau(j)} / m_j.
\]

and we set

\[
\Pi(m^a) = \{ \pi_j(m^a) \mid j \in \text{Supp}(m^a) \} \cup \{ m^a \}. \tag{4.2.1}
\]

Note that if \( j \neq j' \), then \( \pi_j(a) \neq \pi_{j'}(a) \) and therefore \( \pi_j(m^a) \) and \( \pi_{j'}(m^a) \) are distinct.

**Example 4.3** If \( I = (xy, yz, zu) \) is the ideal in Example 2.1, then we label our generators as

\[
m_1 = xy \quad m_2 = yz \quad m_3 = zu.
\]

The generators of \( I^2 \) are uniquely written as \( m^a \) where \( a \in N_2 \). The monomial \( m = xyzu \in I^2 \), for example, corresponds uniquely to the vector \( (1, 0, 1) \).

Following the notation in Example 2.4, since the joint of \( F_3 \) in \( \Delta \) is \( F_2 \), we have \( \tau(3) = 2 \), and thus,

\[
\Pi(xyzu) = \Pi(m^{(1,0,1)}) = \{ m^{(1,0,1)}, m^{(1,1,0)} \} = \{ m_1m_3, m_1m_2 \} = \{ xyzu, xy^2z \}.
\]

We close this section with two technical results which are necessary for our main result later in the paper.

**Lemma 4.4** If \( a \in N_r \), and \( 1 < j < k \in \text{Supp}(a) \), then

\[
\pi_k(a) < \pi_j(a) < a.
\]

Moreover \( \partial(\pi_j(a), \pi_k(a)) = k \) and \( \partial(a, \pi_j(a)) = j \).

**Proof** Suppose \( a = (a_1, \ldots, a_q) \), \( \pi_j(a) = (b_1, \ldots, b_q) \), and \( \pi_k(a) = (c_1, \ldots, c_q) \).

We have

\[
b_i = a_i \text{ if } i > j; \quad b_i = a_i = c_i \text{ if } i > k; \quad \text{and } c_k = a_k - 1 < a_k = b_k \text{ and } b_j = a_j - 1 < a_j
\]

which settles our claim. \( \square \)
Proposition 4.5 Let $m^a, m^b \in \mathcal{M}_r$ such that $b < a$. If $k = \varrho(m^a, m^b)$ then
\[ \text{lcm}(m^a, m^b, \pi_k(m^a)) = \text{lcm}(m^a, m^b). \]

Proof The inequality $b = (b_1, \ldots, b_q) < (a_1, \ldots, a_q) = a$ implies that
\[ a_k > b_k, \quad a_{k+1} = b_{k+1}, \ldots, a_q = b_q. \]

We must show
\[ \pi_k(m^a) | \text{lcm}(m^a, m^b). \]

In particular, we need to show that for every variable $x$ and any integer $s$ such that $x^s | \pi_k(m^a)$, either $x^s | m^a$ or $x^s | m^b$.

If $x$ does not divide $m_{\tau(k)}$ and $x^s | \pi_k(m^a)$ we see from the equality
\[ \pi_k(m^a)m_k = m^a m_{\tau(k)} \]
that $x^s | m^a$.

Assume now $x | m_{\tau(k)}$ and $s$ is maximal such that $x^s | \pi_k(m^a)$. If $x | m_k$, then $x^{s+1} | \pi_k(m^a)m_k$; hence, $x^{s+1} | m^a m_{\tau(k)}$. Since $m_{\tau(k)}$ is square-free, we conclude that $x^s | m^a$. Assume now that $x$ does not divide $m_k$. By (4.0.2), $m_{\tau(k)} | \text{lcm}(m_k, m_j)$ for all $j < k$. We conclude that $x | m_j$ for all $j < k$. We claim that $x^s | m^b$ in this case.

Let $t$ denote the largest integer such that $x^t | m_{\tau(k)}^{a_k} \cdots m_q^{a_q}$. As noted in Definition 4.2, necessarily $k > 1$, and hence, $\tau(k) < k$, which implies that $m_{\tau(k)} \in \{m_1, \ldots, m_{k-1}\}$. Since $x$ does not divide $m_k$ but it divides $m_1, \ldots, m_{k-1}$ and all these monomials are square-free, we have
\[ s = t + \sum_{i=1}^{k-1} a_i + 1. \]

Further, since $a_i = b_i$ for all $i > k$, we see that the largest integer $s'$ such that $x^{s'} | m^b$ is
\[ s' = t + \sum_{i=1}^{k-1} b_i. \]

To show $x^s | m^b$ we need to show $s \leq s'$. Assuming $s' < s$, we have
\[ \sum_{i=1}^k b_i = \sum_{i=1}^{k-1} b_i + b_k \leq \sum_{i=1}^{k-1} a_i + b_k < \sum_{i=1}^{k-1} a_i + a_k = \sum_{i=1}^k a_i \]

The strict inequality above contradicts the fact that $|a| = |b| = r$. \qed
5 $I^r$ has a resolution supported on a CW complex

This section contains our main result. Recall that our setting is a polynomial ring $S = k[x_1, \ldots, x_n]$ and an ideal $I$ with $\text{pd}_S(I) = 1$ generated by square-free monomials \{m_1, \ldots, m_q\} in $S$, ordered as in Construction 2.3. Given a positive integer $r$, we denoted the generating set of $I^r$ by $\mathcal{M}_r$, and showed that each element of $\mathcal{M}_r$ can be uniquely written as $m_a = m_{a_1}^{a_1} \cdots m_{a_q}^{a_q}$ where $a = (a_1, \ldots, a_q) \in (\mathbb{N} \cup \{0\})^q$ is such that $|a| = a_1 + \cdots + a_q = r$ (Proposition 4.1), in other words $a \in \mathcal{N}_r$. Let $X$ be the Taylor complex of $I^r$, with the lcm as the grading function. Recall that $G_X$ denotes the directed graph on the set of faces of $X$, with edge set $E_X$; see Terminology 3.1.

Before stating our main theorem, we extend two functions, which appeared in Definition 4.2, to all faces of $X$.

Definition 5.1 For $\sigma \in X$, $\sigma \neq \emptyset$, let $\text{max}\prec \sigma$ denote the largest monomial label of the vertices of $\sigma$ with respect to the order defined in Definition 4.2. Then if $\text{max}\prec \sigma = m_a$, define

$$d(\sigma) = \begin{cases} -\infty & \text{if } \sigma \subseteq \Pi(m_a) \\ \max\{d(m_a, m_b) \mid m_b \in \sigma \setminus \Pi(m_a)\} & \text{otherwise} \end{cases} \quad \sigma \not\subseteq \Pi(m_a).$$

When $d(\sigma) \neq -\infty$, we set

$$\pi(\sigma) = \pi_{d(\sigma)}(m_a).$$

Note that $\pi(\sigma)$ may not be in $\sigma$. Also, if $a \neq b$ and $m_a, m_b \in I^r$, then

- $1 < d(m_a, m_b) \leq q$;
- (see also Lemma 4.4) for every $m_a \in \mathcal{M}_r$ and $k \in \text{Supp}(a)$,

$$d(m_a, \pi_k(m_a)) = k.$$

Example 5.2 In our running example $I = (xy, yz, zu)$, label the ordered generators as $m_1 = xy, m_2 = yz$ and $m_3 = zu$. In view of Proposition 4.1, we use the monomial labels of the vertices of the Taylor complex to represent these vertices. If

$$\sigma = \{xyz, x^2y^2, y^2z^2, xy^2z\} = \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}, m^{(1,1,0)}\},$$

which is a face of the Taylor complex of $I^2$, then $\text{max}\prec \sigma = m^{(1,0,1)} = xyzu$, and clearly

$$\sigma \not\subseteq \Pi(m^{(1,0,1)}) = \{m^{(1,0,1)}, m^{(1,1,0)}\}$$

as calculated in Example 4.3. So we have

$$\sigma \setminus \Pi(m^{(1,0,1)}) = \{m^{(2,0,0)}, m^{(0,2,0)}\}$$. 
and
\[ d(m^{(1,0,1)}, m^{(2,0,0)}) = d(m^{(1,0,1)}, m^{(0,2,0)}) = 3, \]
which results in \( d(\sigma) = 3 \). Finally \( \pi(\sigma) = \pi_3(m^{(1,0,1)}) = m^{(1,1,0)} \), see Example 2.4.

The main theorem of this section defines a matching \( A \) and its critical cells using the sets \( \Pi(m^a) \) for \( m^a \in I' \). To easily find the non-empty critical cells of this matching, for each \( m^a \in I' \), identify all subsets of \( \Pi(m^a) \) that contain \( m^a \). These will be the critical cells. The remaining cells are matched in pairs that have the same largest monomial label. These pairs are described using the invariants in Definition 5.1, which we note are dependent on the tree structure in Construction 2.3. When confusion is not likely, for \( \sigma \in X \) we will use the vertices of \( \sigma \) and the labels of those vertices interchangeably.

**Theorem 5.3** (Main Theorem) Let \( I = (m_1, \ldots, m_q) \) be a square-free monomial ideal, with \( q \) generators, of projective dimension one in \( S = k[x_1, \ldots, x_n] \). Let \( r \in \mathbb{N} \) and let \( X \) be the Taylor complex of \( I' \). Let \( A \) be the following subset of \( E_X \)
\[ A = \{ \sigma \rightarrow (\sigma \setminus \{\pi(\sigma)\}) \mid \sigma \neq \emptyset, d(\sigma) \neq -\infty, \pi(\sigma) \in \sigma \}. \] (5.3.1)

Then:
1. **The set** \( A \) **is a homogeneous acyclic matching of** \( G_X \);
2. **The set of critical cells for the matching** \( A \) **consists of**
\[ \{\emptyset\} \cup \bigcup_{m^a \in \mathcal{M}_r} \{\sigma \cup \{m^a\} \mid \sigma \subseteq \Pi(m^a)\}; \]
3. **There is a CW complex** \( X_A \) **that supports a free resolution of** \( I' \), **and whose** \( i \)-**cells are in one-to-one correspondence with the** \( A \)-**critical** \( i \)-**cells of** \( X \).

The proof will be given in a series of steps. For an illustration of the notation and concepts used in the proof, see Example 5.4, at the end of the section, which details many of the constructions herein.

**Proof** The proof consists of creating and refining a partition on the Taylor complex \( X \) associated to \( I' \), and then describing carefully crafted matchings on each of the sets in the refined partition, which are then combined to produce the desired matching.

**Step 1.** We construct a partition on \( X \).
For each \( m^a \in \mathcal{M}_r \), set
\[ X_{m^a} = \{ \sigma \in X \mid \text{max}_{<} \sigma = m^a \}. \]
By Lemma 4.4, we have \( \Pi(m^a) \in X_{m^a} \). The sets \( X_{m^a} \), together with \( \{\emptyset\} \), form a partition of \( X \). Notice that for \( m^a, m^b \in \mathcal{M}_r, \sigma \in X_{m^a}, \) and \( \sigma' \in X_{m^b} \) we have:
\[ \sigma' \subseteq \sigma \Rightarrow m^b \subseteq m^a. \] (5.3.2)
To see this, suppose \( \sigma' \subseteq \sigma \) for \( \sigma \in X_{m^a} \) and \( \sigma' \in X_{m^b} \). We have \( m^b \in \sigma' \subseteq \sigma \); hence, \( m^b \in \sigma \). Since \( \sigma \in X_{m^a} \) and \( m^b \in \sigma \), we then have \( m^b \leq m^a \).

**Step 2.** Let \( m^a \in M_r \). We construct a partition of \( X_{m^a} \). For \( k \in \{2, \ldots, q\} \) set

\[
Y_{m^a,k} = \{ \sigma \in X_{m^a} \mid \mathcal{D}(\sigma) = k \}, \quad \text{and set}
Y_{m^a,-\infty} = \{ \sigma \in X_{m^a} \mid \mathcal{D}(\sigma) = -\infty \} = \{ \sigma \in X_{m^a} \mid \sigma \subseteq \Pi(m^a) \}.
\]

Note that \( Y_{m^a,k} = \emptyset \) is possible. Also if \( k \neq k' \), then \( Y_{m^a,k} \) and \( Y_{m^a,k'} \) are disjoint. We have thus a partition of \( X_{m^a} \) that consists of the sets \( Y_{m^a,k} \) that are not empty. In particular, we have

\[
\bigcup_{k=2}^q Y_{m^a,k} = X_{m^a} \setminus Y_{m^a,-\infty}.
\]

Assume \( \sigma \in Y_{m^a,k} \) and \( \sigma' \in Y_{m^a,k'} \) such that \( \sigma' \subseteq \sigma \). By definition, \( \mathcal{D}(\sigma) = k \) and \( \mathcal{D}(\sigma') = k' \). In particular, for some \( m^b \in \sigma' \setminus \Pi(m^a) \), \( \mathcal{D}(m^a, m^b) = k' \). Since \( \sigma' \subseteq \sigma \), we also have \( m^b \in \sigma \setminus \Pi(m^a) \), and hence, \( k' \leq \mathcal{D}(\sigma) = k \). Thus, for \( k, k' \in \{2, \ldots, q\} \), \( \sigma \in Y_{m^a,k} \), and \( \sigma' \in Y_{m^a,k'} \), we have

\[
\sigma' \subseteq \sigma \Rightarrow k' \leq k.
\]

**Step 3.** Let \( k \in \{2, \ldots, n\} \) and \( m^a \in M_r \). We construct a homogeneous acyclic matching (Definition 3.5) on \( G_{Y_{m^a,k}} \) when \( Y_{m^a,k} \neq \emptyset \).

If \( Y_{m^a,k} \neq \emptyset \), we define a subset of \( E_{Y_{m^a,k}} \) as follows:

\[
A_{m^a,k} = \left\{ \sigma \rightarrow \sigma' \mid \sigma \in Y_{m^a,k}, \pi_k(m^a) \in \sigma, \sigma' = \sigma \setminus \{\pi_k(m^a)\} \right\}.
\]

**Claim** \( A_{m^a,k} \) is a homogeneous acyclic matching on \( G_{Y_{m^a,k}} \) that has no critical cells.

**Proof of Claim.** Notice that if \( \sigma \in Y_{m^a,k} \), then \( \sigma' = \sigma \setminus \{\pi_k(m^a)\} \in Y_{m^a,k} \) as well. It follows directly from Lemma 3.3 that \( A_{m^a,k} \) is an acyclic matching. To see that it is homogeneous, we need to see that

\[
lcm(\sigma) = lcm(\sigma') \quad \text{for all} \quad \sigma \rightarrow \sigma' \in A_{m^a,k}.
\]

Since \( \sigma \in Y_{m^a,k} \), we know by definition there is \( m^b \in \sigma \setminus \Pi(m^a) \) such that \( \mathcal{D}(m^a, m^b) = k \). By Proposition 4.5

\[
lcm(m^b, m^a, \pi_k(m^a)) = lcm(m^b, m^a),
\]

hence, the matching is homogeneous.

Finally, let \( \tilde{\sigma} \in Y_{m^a,k} \). If \( \pi_k(m^a) \in \tilde{\sigma} \), then we can take \( \sigma = \tilde{\sigma} \) in (5.3.5). Otherwise, we take \( \sigma = \tilde{\sigma} \cup \{\pi_k(m^a)\} \in Y_{m^a,k} \). In either case, \( \tilde{\sigma} \) is a vertex of an edge \( \sigma \rightarrow \sigma' \) in \( A_{m^a,k} \); hence, the matching \( A_{m^a,k} \) has no critical cells.
Step 4. Let \( m^a \in M_r \). We construct an acyclic matching on \( G_{X_{m^a}} \) with set of critical cells equal to \( Y_{m^a, -\infty} \).

In view of (5.3.4), Lemma 3.4 gives that the set

\[
A_m = \bigcup_{k=2}^q A_{m^a, k}
\]

is a homogeneous acyclic matching on \( G_{\bigcup_{k=2}^q Y_{m^a, k}} \). Since each matching \( A_{m^a, k} \) on \( G_{Y_{m^a, k}} \) has no critical cells, this matching has no critical cells either.

Since \( \bigcup_{k=2}^q Y_{m^a, k} \subseteq X_{m^a} \), Lemma 3.2(1) gives that \( A_{m^a} \) is also a homogeneous acyclic matching on \( G_{X_{m^a}} \). The set of critical cells is \( X_{m^a} \setminus \bigcup_{k=2}^q Y_{m^a, k} \), which is equal to \( Y_{m^a, -\infty} \), by (5.3.3).

Conclusion. In view of (5.3.2), Lemma 3.4 gives that \( \bigcup_{m^a \in M_r} A_{m^a} \) is a homogeneous acyclic matching on \( G_{\bigcup_{m^a \in M_r} X_{m^a}} \). Note that \( \bigcup_{m^a \in M_r} A_{m^a} \) is precisely the set \( A \) described in part (1) of the statement of the theorem.

The set of critical cells is

\[
Y_{-\infty} = \bigcup_{m^a \in M_r} Y_{m^a, -\infty} = \{ \sigma \in X \mid \sigma \neq \emptyset, \sigma \subseteq \Pi(\max_{\prec} \sigma) \}.
\]

This is precisely the set whose cells are described in part (2) of the statement of the theorem. Since

\[
\bigcup_{m^a \in M_r} X_{m^a} = X \setminus \{ \emptyset \} \subseteq X,
\]

Lemma 3.2(1) implies that \( \bigcup_{m^a \in M_r} A_{m^a} \) is an acyclic matching on \( G_X \); the set of critical cells is \( Y_{-\infty} \cup \{ \emptyset \} \).

By [4, Theorem 1.3] there is a CW complex \( X_A \) (the Morse complex of \( X \)) whose \( i \)-cells are in one-to-one correspondence with the \( A \)-critical \( i \)-cells of \( X \), and \( X_A \) supports a free resolution of \( I^r \). \( \square \)

The following example illustrates many of the notations and concepts used above.

Example 5.4 In the running example \( I = (xy, yz, zu) \), label the generators as before, namely

\[
m_1 = xy, m_2 = yz, m_3 = zu.
\]

By Proposition 4.1, the generators of \( I^2 \) are uniquely written as \( m^a \) where \( a \in N_2 \).

The Taylor complex \( X \) for \( I^2 \) is a 5-dimensional simplex whose vertices are labeled with the 6 generators of \( I^2 \). The monomial \( m = xyzu \in I^2 \) corresponds to the vertex \((1, 0, 1) \). The set \( X_m \) from Step 1 of Theorem 5.3 consists of all faces of the Taylor complex containing \((1, 0, 1) \) and any subset of vertices satisfying the constraints

\[
(a, b, c) < (1, 0, 1) \quad \text{and} \quad a + b + c = 2,
\]

\( \square \) Springer
which are precisely the vertices \((2, 0, 0), (0, 2, 0), (1, 1, 0)\). Thus, using the same monomial labeling convention as in Example 5.2, we get

\[
X_{xyzu} = X_m^{(1,0,1)}
\]

\[
= \{m^{(1,0,1)} \cup U \mid U \subseteq \{m^{(2,0,0)}, m^{(0,2,0)}, m^{(1,1,0)}\}\}
\]

\[
= \{xyzu, x^2y^2, xy^2z, y^2z^2\}, \{xyzu, x^2y^2, xy^2z, x^2y^2, y^2z^2\},
\]

\[
\{xyzu, x^2y^2, y^2z^2\}, \{xyzu, x^2y^2, y^2z^2\}, \{xyzu, y^2z^2\}, \{xyzu\}\}.
\]

Next, we calculate the sets \(Y_{m,k}\) for various \(k\), as per Step 2 of Theorem 5.3. From Example 4.3 we know

\[
\Pi(xyzu) = \{m^{(1,0,1)}, m^{(1,1,0)}\} = \{xyzu, xy^2z\}.
\]

For each \(\sigma \in X_{xyzu}\), \(\max_{\prec} \sigma = m^{(1,0,1)}\). If

\[
\sigma = \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}, m^{(1,1,0)}\},
\]

then clearly \(\sigma \nsubseteq \Pi(\max_{\prec} \sigma)\). For this \(\sigma\), \(\partial(\sigma) = 3\). In fact, \(\partial(\sigma) = 3\) for any \(\sigma \in X_{xyzu}\) as long as

\[
\sigma \nsubseteq \Pi(\max_{\prec} \sigma) = \{m^{(1,0,1)}, m^{(1,1,0)}\}.
\]

Therefore,

\[
Y_{xyzu,3} = Y_{m^{(1,0,1)},3}
\]

\[
= \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}, m^{(1,1,0)}\},
\]

\[
\{m^{(1,0,1)}, m^{(2,0,0)}, m^{(1,1,0)}\}, \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}\},
\]

\[
\{m^{(1,0,1)}, m^{(0,2,0)}, m^{(1,1,0)}\}, \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(1,1,0)}\}, \{m^{(1,0,1)}, m^{(0,2,0)}\}\}.
\]

In other words,

\[
Y_{xyzu,3} = \{xyzu, x^2y^2, xy^2z, y^2z^2\}, \{xyzu, x^2y^2, xy^2z, x^2y^2, y^2z^2\},
\]

\[
\{xyzu, x^2y^2, y^2z^2\}, \{xyzu, x^2y^2, y^2z^2\}, \{xyzu, y^2z^2\}, \{xyzu\}\}.
\]

Viewed another way, \(Y_{xyzu,3} = X_{xyzu} - \{\{xyzu, xy^2z\}, \{xyzu\}\}\). Note that

\[
Y_{xyzu,2} = \emptyset
\]

and

\[
Y_{xyzu,-\infty} = \{m^{(1,0,1)}, m^{(1,1,0)}, m^{(1,0,1)}\} = \{\{xyzu, xy^2z\}, \{xyzu\}\}.
\]
Hence, recalling that $\pi_3(m^{(1,0,1)}) = m^{(1,1,0)}$ (see Example 2.4), the elements in $A_{m,3}$ consist of the following directed edges:

$\{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}, m^{(1,1,0)}\} \rightarrow \{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}\}$

$\{m^{(1,0,1)}, m^{(2,0,0)}, m^{(1,1,0)}\} \rightarrow \{m^{(1,0,1)}, m^{(2,0,0)}\}$

$\{m^{(1,0,1)}, m^{(0,2,0)}, m^{(1,1,0)}\} \rightarrow \{m^{(1,0,1)}, m^{(0,2,0)}\}$.

There are 6 critical cells in $M_2$ which correspond to Fig. 2 listed in Example 6.8. Note that $\{m^{(1,0,1)}, m^{(0,2,0)}\}$ is not a critical cell; i.e., it is not an edge in the diagram. However, $\{m^{(1,0,1)}, m^{(2,0,0)}, m^{(0,2,0)}\}$ is a critical cell. Since the complex that supports the resolution does not contain $\{m^{(1,0,1)}, m^{(0,2,0)}\}$ then it is not a simplicial complex, and hence, the resolution is not simplicial.

6 The Morse complex $X_A$

Theorem 5.3 states that there is a CW complex $X_A$ supporting a free resolution of $I^r$, where $X$ is the Taylor complex of $I$ and $A$ is an acyclic matching on the poset graph $G_X$ of $X$. Moreover, the $i$-cells of $X_A$ are in one-to-one correspondence with the $A$-critical $i$-cells of $X$. Following the notation in [4], if $\sigma$ is an $A$-critical cell of $X$, we denote by $\sigma_A$ the unique corresponding cell of $X_A$. We use the notation $\sigma_A' \leq \sigma_A$ to say that the cell $\sigma_A'$ is contained in the closure of the cell $\sigma_A$.

Our goal in this section is to determine, given $A$-critical cells $\sigma$ and $\sigma'$ of $X$, under what conditions on $\sigma$ and $\sigma'$ do we get

$$\sigma_A' \leq \sigma_A?$$

It is not always immediately clear from the definition of a closure which cells are contained in the closure of other cells. Batzies and Welker [4] characterized the cell ordering $\sigma_A' \leq \sigma_A$ in the Morse complex $X_A$ in terms of certain paths in the directed graph $G_X^A$, called "gradient paths" (see (6.2.2)).

In this section, we will focus on the structure of the gradient paths in our setting, and show, in Theorem 6.1, exactly what the cell order $\sigma_A' \leq \sigma_A$ in the Morse complex $X_A$ means in terms of the $A$-critical cells $\sigma$ and $\sigma'$ of $X$. Given the technical nature
of the discussions and the fact that they apply only to this particular proof, we have chosen to state Theorem 6.1 early on. What follows after are all the components that go into its proof.

In the statement of Theorem 6.1, we denote the $A$-critical $i$-cells of $X$ (as described in Theorem 5.3(2)) by

$$\sigma(m^a, D) = \{m^a\} \cup \{\pi_j(m^a) \mid j \in D\} \subseteq \Pi(m^a)$$

where $a \in \mathbb{N}_r$, $D \subseteq \text{Supp}(a) \setminus \{1\}$ and $|D| = i$.

We are now ready to state the main result of this section. Note that part (1) of Theorem 6.1 follows immediately from Theorem 5.3, but we have included it in order to have a complete statement for the characterization of the cells of $X_A$.

**Theorem 6.1** (The cells of $X_A$) Let $X_A$ be the Morse complex of the matching $A$ on the Taylor complex $X$ as described in (5.3.1), and let $i > 0$. Then:

1. (Theorem 5.3) For every $i$-cell $c$ of $X_A$ there is a unique $A$-critical cell $\sigma = \sigma(m^a, D)$ of $X$ where $m^a \in \mathcal{M}_r$, $D \subseteq \text{Supp}(m^a) \setminus \{1\}$ and $|D| = i$, such that $c = \sigma_A$.

2. If $c'$ is an $(i - 1)$-cell of $X_A$, then $c' \leq c$ if and only if $c' = \sigma'_A$ where $\sigma' = \sigma(m^a, D \setminus \{k\})$ or $\sigma' = \sigma(\pi_k(m^a), D \setminus \{k\})$ for some $k \in D$.

Theorem 6.1 gives a concrete description of the ordering of cells in the Morse complex (see (6.2.2)) in terms of the $A$-critical cells of the Taylor complex. To start our way towards the proof, we further develop the notation used for critical cells.

**Notation 6.2** Let $a \in \mathbb{N}_r$ and let $D = \{i_1, \ldots, i_s\}$ be a subset of $\text{Supp}(a)$. Set

$$\pi_D(a) = a + \sum_{i \in D} e_{\tau(i)} - \sum_{i \in D} e_i \quad \text{and}$$

$$\pi_D(m^a) = m^a_{\pi_D(a)} = \frac{m^a m_{\tau(i_1)} m_{\tau(i_2)} \cdots m_{\tau(i_s)}}{m_{i_1} m_{i_2} \cdots m_{i_s}}.$$ 

When $D = \emptyset$ we set $\pi_{\emptyset}(a) = a$ and $\pi_{\emptyset}(m^a) = m^a$. Further, we set

$$\sigma(m^a, D) = \{m^a\} \cup \{\pi_i(m^a) \mid i \in D\} \subseteq \Pi(m^a) \quad \text{and}$$

$$\overline{\sigma}(m^a, D) = \{m^a_{\tau(L)} \mid L \subseteq D\}.$$ 

Since $\pi_{\emptyset}(m^a) = m^a$, we have $m^a \in \overline{\sigma}(m^a, D)$ for all $D$. Also, note that $\sigma(m^a, \emptyset) = \overline{\sigma}(m^a, \emptyset) = \{m^a\}.$
Observe that \( \sigma(\mathbf{m}^a, D) \subseteq \sigma(\mathbf{m}^a, D) \), and if \( D = D' \cup D'' \) with \( D' \cap D'' = \emptyset \), then
\[
\pi_D(\mathbf{m}^a) = \pi_{D'}(\pi_{D''}(\mathbf{m}^a)) = \pi_{D''}(\pi_{D'}(\mathbf{m}^a)).
\] (6.2.1)

Note that the \( A \)-critical \( i \)-cells of \( X \) appearing in Theorem 5.3(2) are precisely the cells \( \sigma(\mathbf{m}^a, D) \) with \( a \in \mathcal{N}_r, D \subseteq \text{Supp}(a) \smallsetminus \{1\} \) and \( |D| = i \).

We now describe the cell ordering in the Morse complex in terms of gradient paths, following the authors in [4]. A gradient path in the graph \( G^A_X \) (defined in Terminology 3.1) is a directed path
\[
P: \sigma_0 \to \cdots \to \sigma_n
\]
where \( \sigma_0 \) is the initial point and \( \sigma_n \) is the end point; see, e.g., [4, p. 165]. For cells \( \sigma, \sigma' \) of \( X \), the set of all gradient paths in \( G^A_X \) with initial point \( \sigma \) and end point \( \sigma' \) is denoted by \( \text{GradPath}_A(\sigma, \sigma') \).

By [4, Proposition 7.3] if \( \sigma'', \sigma \) are \( A \)-critical cells of \( X \) of dimensions \( i - 1 \) and \( i \), respectively, with \( \sigma''_A \) and \( \sigma_A \) the corresponding cells in \( X_A \) of dimensions \( i - 1 \) and \( i \), respectively, then
\[
\sigma''_A \leq \sigma_A \iff \begin{cases} 
\sigma'' \subseteq \sigma \\
\text{GradPath}_A(\sigma', \sigma'') \neq \emptyset \text{ for some } \sigma' \subseteq \sigma \text{ with } \dim(\sigma') = i - 1.
\end{cases}
\] (6.2.2)

**Discussion 6.3** (Gradient paths) A gradient path between two \((i - 1)\)-cells \( \sigma' \) and \( \sigma'' \), where \( \sigma'' \) is \( A \)-critical, can be visualized in terms of edges pointing up or pointing down:

\[
\sigma' = \sigma_0 \to \sigma_1 \uparrow \sigma_2 \ldots \sigma_{u-2} \downarrow \sigma_{u-1} \uparrow \sigma_u = \sigma''.
\]

To see this, recall (Terminology 3.1) that an edge points down if it corresponds to an inclusion between cells and an edge points up if it is the reverse of an edge in \( A \). Therefore, there are no edges that point up and end with \( \sigma'' \) since \( \sigma'' \) is an \( A \)-critical cell, so \( \sigma'' \) can only be reached via a down arrow. Also, recall that one cannot have two consecutive edges pointing up, because every cell appears only once in the matching \( A \). So every up arrow must be followed by a down arrow and once a (down) arrow hits a critical cell, then the gradient path must stop, since there is no up arrow from a critical cell.

Our discussion applied to the gradient path drawn above forces the following:

- \( \sigma' = \sigma_0, \ldots, \sigma_{u-1} \) are not critical cells since they are included in edges pointing up.
- \( \sigma_u = \sigma'' \) is the first critical cell along this path, and the path stops here.
• The dimensions of the cells along the path are \( i - 1, i, i - 1, i, \ldots, i - 1 \), since the path is a series of up (dimension goes up by one) and down (dimension goes down by one) arrows.

• An upward arrow \( \sigma_j \uparrow \sigma_{j+1} \) indicates

\[
\sigma_{j+1} = \sigma_j \cup \{\pi_k(m^b)\},
\]

where \( m^b = \max(\sigma_j) \) and \( k = d(\sigma_j) \) by Theorem 5.3. In particular, \( \max(\sigma_j) = \max(\sigma_{j+1}) \) in this case.

• A downward arrow \( \sigma_j \downarrow \sigma_{j+1} \) indicates \( \sigma_{j+1} \subseteq \sigma_j \) and \( \max(\sigma_{j+1}) \leq \max(\sigma_j) \).

• The previous two items show that \( \max(\sigma_j) \) does not increase as one proceeds through the gradient path.

We now start proving part (2) of Theorem 6.1. We first describe the cells of \( X_A \) and the order relation on the cells, in terms of the faces of \( X \) and possible gradient paths between them. Lemma 6.4 establishes that there is always a gradient path of the type needed for Theorem 6.1, and Lemma 6.7 says that those are the only gradient paths.

**Lemma 6.4** Let \( m^a \in M_r \) and \( \sigma = \sigma(m^a, D) \) for some \( D \subseteq \text{Supp}(m^a) \setminus \{1\} \) with \( |D| \geq 2 \), and let \( k \in D \). Then there is a gradient path in \( G_X^A \) from \( \sigma \setminus \{1\} \) to \( \sigma(\pi_k(m^a), D \setminus \{k\}) \).

**Proof** Let \( \sigma' = \sigma \setminus \{m^a\} \), \( \sigma'' = \sigma(\pi_k(m^a), D \setminus \{k\}) \) and \( D = \{d_1, \ldots, d_s\} \) where \( d_1 < \cdots < d_s \).

Then by Lemma 4.4 we have

\[
\pi_{d_s}(m^a) < \pi_{d_{s-1}}(m^a) < \cdots < \pi_{d_2}(m^a) < \pi_{d_1}(m^a).
\]

We start by constructing a gradient path in the most basic case \( k = d_1 = \min(D) \) carefully. Following the rules in Discussion 6.3 we build the following gradient path in \( G_X^A \), where the arrow from the noncritical cell \( \sigma' \) must be an up arrow, and the arrow to the critical cell \( \sigma'' \) must be a down arrow. We use \( u \) to denote the number of steps in the gradient path.

\[
\sigma' = \sigma_0 \uparrow \sigma_1 \downarrow \sigma_2 \ldots \downarrow \sigma_{u-2} \uparrow \sigma_{u-1} \downarrow \sigma_u = \sigma''.
\]

Using Lemma 4.4 we observe that \( \max(\sigma_0) = \pi_{d_1}(m^a) \) and \( d(\sigma_0) = d_s \), so by (6.3.1), we have only one choice for the upward arrows. Thus,

\[
\sigma_1: \sigma_1 = \sigma_0 \cup \{\pi_{d_1}(m^a)\} = \{\pi_{d_1}(m^a), \ldots, \pi_{d_s}(m^a), \pi_{\{d_s,d_1\}}(m^a)\}.
\]

\[
\sigma_2: \text{The next downward arrow is a simple elimination, and here we choose the option}
\]

\[
\sigma_2 = \sigma_1 \setminus \{\pi_{d_1}(m^a)\} = \{\pi_{d_1}(m^a), \ldots, \pi_{d_{s-1}}(m^a), \pi_{\{d_s,d_1\}}(m^a)\}.
\]
\( \sigma_3: \) Once again, since \( \max(\sigma_2) = \pi_{d_1}(m^a) \) and \( d(0) = d_{s-1}, \)

\[
\sigma_3 = \sigma_2 \cup \{ \pi_{d_{i-1}}(\pi_{d_i}(m^a)) \} \\
= \{ \pi_{d_1}(m^a), \ldots, \pi_{d_{s-1}}(m^a), \pi_{[d_s,d_1]}(m^a), \pi_{[d_{s-1},d_1]}(m^a) \}.
\]

We continue in this manner, with every downward arrow eliminating the largest remaining \( d_i, \) and finally we arrive at

\( \sigma_{u-1}: \) \( \sigma_{u-1} = \{ \pi_{d_1}(m^a), \pi_{d_2}(m^a), \pi_{[d_s,d_1]}(m^a), \ldots, \pi_{[d_{s-1},d_1]}(m^a) \} \)

\( \sigma_u: \) \( \sigma_u = \sigma_{u-1} \setminus \pi_{d_2}(m^a) = \sigma(\pi_{d_1}(m^a), D \setminus \{d_1\}) = \sigma'' \).

So we have shown the existence of the gradient path when \( k = d_1. \)

If \( k = d_e \in D \) and \( 1 < e < s, \) we construct a gradient path from \( \sigma' \) to \( \sigma'' \) below. For a cleaner picture we use the product of indices in \( L \) to denote the monomial \( \pi_L(m^a) \) for \( L \subseteq D, \) and we keep track of all these indices in the table below. We start with \( \sigma_0 = \{ \pi_{d_1}(m^a), \ldots, \pi_{d_1}(m^a) \}. \)

| Row | Add | Delete | Add | Delete | Add | Delete | Remaining \( \pi_L(m^a) \) |
|-----|-----|--------|-----|--------|-----|--------|--------------------------|
| 1   | \( d_1d_e \) | \( d_1d_{e-1} \) | \( d_1d_{e+1} \) | \( d_1d_e \) | \( d_1d_e \) | \( d_2, \ldots, d_e, \) |
|     | \( d_s \) | \( d_{s-1} \) | \( d_{s+1} \) | \( d_s \) | \( d_s \) | \( d_1d_e, d_2d_e, \) |
|     | \( d_1d_s \) | \( d_1d_{s-1} \) | \( d_1d_{s+1} \) | \( d_1d_s \) | \( d_1d_s \) | \( d_2d_e, d_3d_e, \) |
|     | \( d_{e-2}d_s \) | \( d_{e-2}d_{s-1} \) | \( d_{e-2}d_{s+1} \) | \( d_{e-2}d_s \) | \( d_{e-2}d_s \) | \( d_2d_e, d_3d_e, \) |
| \( e-1 \) | \( d_{e-1}d_s \) | \( d_{e-1}d_{s-1} \) | \( d_{e-1}d_{s+1} \) | \( d_{e-1}d_s \) | \( d_{e-1}d_s \) | \( d_{e-1}d_e, d_{e-1}d_e, \) |
|     | \( d_{e-1}d_s \) | \( d_{e-1}d_{s-1} \) | \( d_{e-1}d_{s+1} \) | \( d_{e-1}d_s \) | \( d_{e-1}d_s \) | \( d_{e-1}d_e, d_{e-1}d_e, \) |
|     | \( d_{e-1}d_s \) | \( d_{e-1}d_{s-1} \) | \( d_{e-1}d_{s+1} \) | \( d_{e-1}d_s \) | \( d_{e-1}d_s \) | \( d_{e-1}d_e, d_{e-1}d_e, \) |
|     | \( d_{e-1}d_s \) | \( d_{e-1}d_{s-1} \) | \( d_{e-1}d_{s+1} \) | \( d_{e-1}d_s \) | \( d_{e-1}d_s \) | \( d_{e-1}d_e, d_{e-1}d_e, \) |
|     | \( d_{e-1}d_s \) | \( d_{e-1}d_{s-1} \) | \( d_{e-1}d_{s+1} \) | \( d_{e-1}d_s \) | \( d_{e-1}d_s \) | \( d_{e-1}d_e, d_{e-1}d_e, \) |

**Row 1:** Recall that each up arrow adds a monomial and each down arrow deletes one, so the first series of arrows in the gradient path will start from \( \sigma_0 \) and do the following sequence of additions and deletions:

\[
\text{add } \pi_{[d_1,d_1]}(m^a), \ \text{delete } \pi_{d_1}(m^a), \ \text{add } \pi_{[d_1,d_2]}(m^a), \ \text{delete } \pi_{d_2}(m^a), \ldots, \ \text{add } \pi_{[d_1,d_{e+1}]}(m^a), \ \text{delete } \pi_{d_{e+1}}(m^a), \ \text{add } \pi_{[d_1,d_e]}(m^a), \ \text{delete } \pi_{d_1}(m^a).
\]
The final down arrow in this row eliminates \( \pi_{d_1}(m^a) \), making the monomial \( \pi_{d_2}(m^a) \) the largest one in the remaining cell, whose elements are the monomials \( \pi_L(m^a) \) where \( L \) ranges over index sets listed at the end of Row 1.

**Middle rows:** In Row \( i \), \( 1 < i < e \), by Lemma 4.4 \( \pi_{d_i}(m^a) \) is the largest remaining monomial at that spot in the path, and we go through the same moves:

\[
\begin{align*}
\text{add } & \pi_{d_i}(m^a), \text{ delete } \pi_{d_i}(m^a), \text{ add } \pi_{d_i}(m^a), \text{ delete } \pi_{d_i}(m^a), \\
& \ldots, \text{ add } \pi_{d_i}(m^a), \text{ delete } \pi_{d_i}(m^a), \text{ add } \pi_{d_i}(m^a), \text{ delete } \pi_{d_i}(m^a),
\end{align*}
\]

eliminating \( \pi_{d_i}(m^a) \) at the very end to make \( \pi_{d_{i+1}}(m^a) \) the largest monomial.

**Row e:** The final row of the table starts from a point in the gradient path where \( \pi_{d_e}(m^a) \) is the maximal label, and we go through the same moves as Row \( i \), with \( i = e \), but we skip the last two steps: we do not add \( \pi_{d_e}(m^a) \) (since we already have it) and we do not delete \( \pi_{d_e}(m^a) \). Now the very last set of indices in the table are those of \( \pi_L(m^a) \) appearing in \( \sigma'' \), and hence, we have built a gradient path between \( \sigma' \) and \( \sigma'' \).

Finally, if \( k = d_s \) or, in other words, \( e = s \) we build a more compact version of the path above. The path below with initial point \( \sigma' \) and end point \( \sigma'' \) shows only the indices \( L \) appearing in each \( \sigma_i \) along the way.

\[
\begin{align*}
\{d_1, d_1d_s\} & \quad \ldots \quad \{d_1d_s\} \\
\{d_1d_s, d_s-1d_s\} & \quad \ldots \quad \{d_1d_s, d_s-1d_s\}
\end{align*}
\]

We next show that Lemma 6.4 is reversible, that is, we prove that all gradient paths connect cells of the given form. In order to do so, we first show in Lemma 6.5 that all critical cells contained in \( \overline{\sigma}(m^a, D) \) of dimension one less either contain \( m^a \) or have the form given in Lemma 6.4, and then we establish in Lemma 6.6 a few basic facts about the cells and their orders.

**Lemma 6.5** Let \( m^a \in \mathcal{M}_e \) and \( \emptyset \neq D \subseteq \text{Supp}(m^a) \setminus \{1\} \). If \( \sigma' \) is a \((|D| - 1)\)-dimensional A-critical cell contained in \( \overline{\sigma}(m^a, D) \) and \( m^a \notin \sigma' \), then \( \sigma' = \sigma(\pi_k(m^a), D \setminus \{k\}) \) for some \( k \in D \).

**Proof** Suppose that \( |D| = 1 \). Then \( \overline{\sigma}(m^a, D) = \sigma(m^a, D) = \{m^a, \pi_k(m^a)\} \) where \( D = \{k\} \). If \( \sigma' \) is a 0-critical cell in this set and \( m^a \notin \sigma' \), then \( \sigma' = \{\pi_k(m^a)\} \); hence, \( \sigma' = \sigma(\pi_k(m^a), \emptyset) \).
Now assume that $|D| \geq 2$, and let $\sigma' = \sigma(m^b, D')$ for some $m^b \in M_r$ and $D' \subseteq \text{Supp}(m^b) \setminus \{1\}$ with $|D'| = |D| - 1$ and with $m^a \not\in \sigma'$. Then

$$m^b \in \sigma(m^b, D') = \sigma' \subseteq \overline{\sigma}(m^a, D).$$

Since $m^b \in \overline{\sigma}(m^a, D)$ and $m^a \not\in m^b$, for some nonempty $L \subseteq D$ we have

$$m^b = \pi_L(m^a).$$

**Claim** $D' \subseteq D \setminus L$.

**Proof of Claim.** Let $d \in D'$. Then

$$\pi_d(\pi_L(m^a)) = \pi_d(m^b) \in \sigma(m^b, D') \subseteq \overline{\sigma}(m^a, D).$$

Consequently, for some $L' \subseteq D$

$$\pi_d(\pi_L(m^a)) = \pi_{L'}(m^a). \quad (6.5.1)$$

By Notation 6.2 and Definition 4.2, this implies that

$$\sum_{i \in L} e_i - \sum_{i \in L} e_{\tau(i)} + e_d - e_{\tau(d)} = \sum_{i \in L'} e_i - \sum_{i \in L'} e_{\tau(i)}.$$

All indices in $L \cap L'$ appear on both sides equally, so we can cancel those out and assume that $L \cap L' = \emptyset$. Now let $j = \max\{L \cup L' \cup \{d\}\}$. If $j \neq d$, then considering the fact that $\tau(i) < i$ for each $i \neq 1$, $j \in L \cap L'$ (since $e_j$ must appear on both sides of the equation). This is impossible as we assumed $L \cap L' = \emptyset$. Therefore, $j = d$. It follows that $d \in L' \subseteq D$, and since $L \cap L' = \emptyset$, it also follows that $d \notin L$. Therefore, $D' \subseteq D \setminus L$, which proves our claim.

Since $\sigma'$ is a $(|D| - 1)$-cell, $\sigma'$ has exactly $|D|$ elements; on the other hand, $\sigma' = \sigma(m^b, D')$ has $|D'| + 1$ elements. Therefore, $|D'| = |D| - 1$. Under the assumption that $D' \subseteq D \setminus L$, it follows that $L$ must have just one element, which gives

$$m^b = \pi_k(m^a) \quad \text{and} \quad D' = D \setminus \{k\}$$

for some $k \in D$, which ends our argument.

**Lemma 6.6** Let $a, b, c \in N_r$, such that $m^b, m^c \in \overline{\sigma}(m^a, D)$, where $D \subseteq \text{Supp}(a) \setminus \{1\}$. Then there exist $L, L' \subseteq D$ such that

$$m^b = \pi_{L'}(m^a), \quad m^c = \pi_L(m^a).$$

Set $k = \max\left((L \cup L') \setminus (L \cap L')\right)$. Then

$$m^c < m^b \iff k \in L$$

and in this case $\overline{\sigma}(m^b, m^c) = k$. In particular, $\pi_k(m^b) \in \overline{\sigma}(m^a, D)$. 

$\square$ Springer
Proof The existence of $L, L'$ follows directly from the definition of $\overline{\sigma}(m^a, D)$ in Notation 6.2. By assumption we know $b = \pi_L(a)$ and $c = \pi_L(a)$. Let $b = (b_1, \ldots, b_q)$ and $c = (c_1, \ldots, c_q)$. By Definition 4.2, $c < b$ if and only if $b_j - c_j > 0$ where $j$ is the largest index with $b_j \neq c_j$. Moreover

$$b - c = \pi_{L'}(a) - \pi_L(a)$$

$$= \left( a + \sum_{i \in L'} e_{\tau(i)} - \sum_{i \in L} e_i \right) - \left( a + \sum_{i \in L} e_{\tau(i)} - \sum_{i \in L} e_i \right)$$

$$= \sum_{i \in L' \setminus L \cap L'} e_{\tau(i)} + \sum_{i \in L \setminus L \cap L'} e_i - \left( \sum_{i \in L' \setminus L \cap L'} e_i + \sum_{i \in L \setminus L \cap L'} e_{\tau(i)} \right).$$

Since $\tau(i) < i$ when $i \neq 1$, if $k \in L$, then it follows immediately that $c < b$ and $d(b, c) = k$. If $k \in L'$ a similar equation for $c - b$ shows that $b < c$ and $d(c, b) = k$. $\square$

We are now ready to prove a converse statement of Lemma 6.4, which is the final ingredient needed to prove Theorem 6.1.

Lemma 6.7 Let $m^a \in M_r, D \subseteq \text{Supp}(m^a) \setminus \{1\}$ and $\sigma = \sigma(m^a, D)$ an $A$-critical cell of $X$ of dimension $|D|$, and $\sigma' = \sigma \setminus \{m^a\}$ a noncritical cell of $\sigma$. Suppose $\sigma''$ is an $A$-critical cell of $X$ of dimension $|D| - 1$ which is connected to $\sigma'$ via the gradient path

$$\mathcal{P}: \sigma' = \sigma_0 \to \sigma_1 \to \cdots \to \sigma_{h-1} \to \sigma_h = \sigma''$$

in $G_A^X$. Then for some $k \in D$

$$\sigma'' = \sigma(\pi_k(m^a), D \setminus \{k\}).$$

Proof We will show that for each $i$, if $\sigma_i \subseteq \overline{\sigma}(m^a, D)$, then $\sigma_{i+1} \in \overline{\sigma}(m^a, D)$. The arrow

$$\sigma_i \to \sigma_{i+1}$$

in $\mathcal{P}$ could be one of two forms. If it is a downward arrow, then by Discussion 6.3 we have an inclusion $\sigma_{i+1} \subseteq \sigma_i \subseteq \overline{\sigma}(m^a, D)$.

If the arrow is pointing up, from Discussion 6.3 we know

$$\sigma_{i+1} = \sigma_i \cup \{\pi_k(m^b)\},$$

where $m^b = \max(\sigma_i)$ and $k = d(\sigma_{i+1}) \neq -\infty$. We have thus $k = d(m^b, m^c)$ for some $m^c \in \sigma_{i+1} \setminus \Pi(m^b)$ with $m^c < m^b$. In particular, we have $m^c \in \sigma_i$. Since we assumed
σ_i \subseteq \overline{σ}(m^a, D), we know m^b, m^c \in \overline{σ}(m^a, D). By Lemma 6.6, π_k(m^b) \in \overline{σ}(m^a, D) and so

\[ \sigma_{i+1} \subseteq \overline{σ}(m^a, D). \]

Since σ_0 = σ' \subseteq \overline{σ}(m^a, D), the above argument, applied inductively over i implies that

\[ \sigma'' \subseteq \overline{σ}(m^a, D). \]

Note that m^a \notin σ'', since max(σ_i) is non-increasing along P (see Discussion 6.3), m^a \notin σ', and m^a is larger than all elements of σ', so it cannot be added along the gradient path. By Lemma 6.5 for some k ∈ D,

\[ \sigma'' = \sigma(π_k(m^a), D \setminus \{k\}). \]

□

Now that the gradient paths and the forms of the associated critical cells have been established, the proof of Theorem 6.1 follows.

**Proof of Theorem 6.1** Part (1) of the statement follows directly from Theorem 5.3(3) and the description of the A-critical cells as σ(m^a, D). For part (2) suppose c = σ_A and c' = σ'_A are i and (i - 1) cells of X_A, respectively. If c' \leq c, then by (6.2.2) there are two possibilities. The first one is σ' \subseteq σ, in which case, since σ' is A-critical, for some k ∈ D, σ' = σ(m^a, D \setminus \{k\}). Otherwise, there is a gradient path from an (i - 1)-cell σ'' of σ to σ'. Then by Discussion 6.3 σ'' must be a non-critical cell of σ. So σ'' = σ \setminus \{m^a\} and by Lemma 6.7 we must have σ' = σ(π_k(m^a), D \setminus \{k\}) for some k ∈ D.

Conversely, if σ' = σ(m^a, D \setminus \{k\}) for some k ∈ D, then by (6.2.2) c' \leq c. Suppose σ' = σ(π_k(m^a), D \setminus \{k\}). If |D| = 1, we have D = \{k\}, σ' = σ(π_k(m^a), D) and σ = σ(m^a, \{k\}) = \{m^a, π_k(m^a)\}. In this case, σ' \subseteq σ and c' \leq c by (6.2.2). If |D| ≥ 2, then by Lemma 6.4 there is a gradient path from σ \setminus \{m^a\} to σ', which again by (6.2.2) implies that c' \leq c. This concludes the proof. □

**Example 6.8** With I as in Example 5.4, the critical cells for any m^a ∈ M_2 are depicted in Fig. 2. Note that in all cases |Supp(m^a) \setminus \{1\}| ≤ 2, so we can say exactly what the cell order in the Morse complex look like from Theorem 6.1. For example in the case of m^{(0,1,1)}, we note that Supp(m^{(0,1,1)}) = \{2, 3\}, so we let D = \{2, 3\}, then σ = σ(m^{(0,1,1)}, D) is a critical 2-cell of X corresponding to a 2-cell σ_A of the Morse complex X_A. The 1-cells of X_A contained in σ_A will be σ'_A where σ' ranges over the following critical 1-cells of X:

σ(m^{(0,1,1)}, \{2\}), σ(m^{(0,1,1)}, \{3\}), σ(π_2(m^{(0,1,1)}), \{3\}), σ(π_3(m^{(0,1,1)}), \{2\}).
We list the cell order in the Morse complex in the table below, using critical cells.

| $m^a$ | $\Pi(m^a)$ corresponding to an $i$–cell of $X_A$ | subsets of $\Pi(m^a)$ corresponding to $(i - 1)$–sub-cells in $X_A$ |
|-------|-----------------------------------------------|-------------------------------------------------|
| $m^{(0,1,1)}$ | {$m^{(0,1,1)}, m^{(1,0,1)}, m^{(0,2,0)}$} | {$m^{(0,1,1)}, m^{(1,0,1)}, m^{(0,2,0)}$} |
| $m^{(1,1,0)}$ | {$m^{(1,1,0)}, m^{(2,0,0)}$} | {$m^{(1,1,0)}, m^{(2,0,0)}$} |
| $m^{(1,0,1)}$ | {$m^{(1,0,1)}, m^{(1,1,0)}$} | {$m^{(1,0,1)}, m^{(1,1,0)}$} |
| $m^{(0,2,0)}$ | {$m^{(0,2,0)}, m^{(1,1,0)}$} | {$m^{(0,2,0)}, m^{(1,1,0)}$} |
| $m^{(0,0,2)}$ | {$m^{(0,0,2)}, m^{(0,1,1)}$} | {$m^{(0,0,2)}, m^{(0,1,1)}$} |
| $m^{(2,0,0)}$ | {$m^{(2,0,0)}$} | $\emptyset$ |

One can show that the 2-cell of the Morse complex can be represented by the whole square in Fig. 2 of Example 5.4.

**Example 6.9** Consider the ideal $I = (xyz, xyw, yuww, xuww)$ in the polynomial ring $S = k[x, y, z, u, v, w]$, and let $\Delta$ be the simplicial complex $\mathcal{F}(I)^c$, pictured on the left in Fig. 3. Then $\Delta$ is a quasi-tree with facet ordering $F_1, F_2, F_3, F_4$ (this is one of several possible orders). By Theorem 2.2 $\text{pd}(I) = 1$. Using the above facet order we set $\tau(2) = 1$ and $\tau(3) = \tau(4) = 2$. We label the monomial generators of $I$ accordingly as

$$m_1 = xyz, m_2 = xyw, m_3 = yuww, m_4 = xuww.$$
By Theorem 5.3, $I^2$ has a free resolution supported on a CW complex whose cells are in one-to-one correspondence with the $A$-critical cells of the Taylor complex of $I^2$. We record these critical cells in the right picture in Fig. 3. Note that the faces
\[ \{m^{(1,0,1,0)}, m^{(0,2,0,0)}\}, \{m^{(1,0,0,1)}, m^{(0,2,0,0)}\}, \{m^{(0,1,0,1)}, m^{(0,1,1,0)}\} \]
are not $A$-critical cells, hence they are not edges in the diagram.

Theorem 6.1 allows us to exactly determine to which 2-cell of the Morse complex each of the shaded triangles which are missing an edge corresponds. We focus on one such triangle coming from the critical cell for the monomial generator
\[ m^{(0,0,1,1)} = m_3m_4 = xyu^2v^2w^2 \]
of $I^2$. The 2-dimensional critical cell of the Taylor complex of $I^2$
\[ \sigma = \prod(m^{(0,0,1,1)}) = \sigma(m^{(0,0,1,1)}, \{3, 4\}) = \{m^{(0,0,1,1)}, m^{(0,1,0,1)}, m^{(0,1,1,0)}\} \]
corresponds to a 2-cell $\sigma_A$ of the Morse complex which, by Theorem 6.1, contains the 1-cells
\[ (\sigma_1)_A, (\sigma_2)_A, (\sigma_3)_A, (\sigma_4)_A \]
where
\[ \sigma_1 = \{m^{(0,0,1,1)}, m^{(0,1,0,1)}\}, \quad \sigma_2 = \{m^{(0,0,1,1)}, m^{(0,1,1,0)}\}, \quad \sigma_3 = \{m^{(0,1,0,1)}, m^{(0,2,0,0)}\}, \quad \sigma_4 = \{m^{(0,1,1,0)}, m^{(0,2,0,0)}\}. \]

The 2-cell $\sigma_A$ can therefore be visualized as the following shaded square.

\[ \begin{array}{c}
\begin{array}{c}
\text{m}^{(0,0,1,1)} \\
\text{m}^{(0,1,0,1)}
\end{array} \\
\begin{array}{c}
\text{m}^{(0,1,1,0)} \\
\text{m}^{(0,2,0,0)}
\end{array}
\end{array} \]

### 7 Minimality of the Morse resolution

This section is devoted to establishing the minimality of the free resolution supported by the Morse complex $X_A$ described in Theorem 5.3. What we need to establish ([4, Lemma 7.5]) is that if $c' \leq c$ are two cells of the CW complex $X_A$ with $\dim(c') = \dim(c) - 1$, then $\text{lcm}(c) \neq \text{lcm}(c')$.

In our case, the critical cells are of the form $\sigma(m^a, D)$ where $m^a \in \mathcal{M}_r$ and $D \subseteq \text{Supp}(a) \setminus \{1\}$ using Notation 6.2. By applying Theorem 6.1 we proceed to
prove that no two embedded cells of $X_A$ come from $A$-critical cells of $X$ with the same lcm label.

Our first statement describes the monomial label of the face $\sigma(m^a, D)$ of the Taylor complex. To find the label, multiply $m^a$ by the free vertices of the leaf $F_j$ of the quasi-tree $(F_1, \ldots, F_j)$ in Construction 2.3 for all $j \in D$.

**Proposition 7.1** Let $m^a \in M, D \subseteq \text{Supp}(a) \setminus \{1\}$. Then

$$\text{lcm} \sigma(m^a, D) = m^a \prod_{j \in D} \prod_{x \in F_j \setminus F_{\tau(j)}} x.$$ 

**Proof** We use induction on $d = |D|$. If $d = 0$, then $D = \emptyset$ and by Notation 6.2 we have $\sigma(m^a, D) = m^a$, so there is nothing to prove. If $d = 1$, then $D = \{k\}$ and (see Construction 2.3)

$$\text{lcm} \sigma(m^a, \{k\}) = \text{lcm} (m^a, \pi_k(m^a)) = \text{lcm} \left( m^a, \frac{m^a \cdot m_{\tau(k)}}{m_k} \right)$$

$$= m^a \cdot \prod_{x|m_{\tau(k)}, x|m_k} x = m^a \cdot \prod_{x \neq F_{\tau(k)}, x \in F_k} x = m^a \cdot \prod_{x \in F_k \setminus F_{\tau(k)}} x.$$

If $d > 1$, and $k = \text{max}(D)$, let $D' = D \setminus \{k\}$. Then, once again using Construction 2.3 as in the base case, plus the induction hypothesis on $|D'|$, we have

$$\text{lcm} \sigma(m^a, D) = \text{lcm} \left( \text{lcm} \sigma(m^a, D'), \pi_k(m^a) \right)$$

$$= \text{lcm} \left( m^a \prod_{j \in D'} \prod_{x \in F_j \setminus F_{\tau(j)}} x, \frac{m^a \cdot m_{\tau(k)}}{m_k} \right)$$

$$= m^a \prod_{j \in D'} \prod_{x \in F_j \setminus F_{\tau(j)}} x \cdot \prod_{x \in F_k \setminus F_{\tau(k)}} x$$

$$= m^a \prod_{j \in D} \prod_{x \in F_j \setminus F_{\tau(j)}} x.$$

$$\square$$

**Theorem 7.2** (The resolution supported on $X_A$ is minimal) The free resolution that is supported on the CW Morse complex $X_A$, with $A$ being the acyclic matching as in (5.3.5), is minimal.

**Proof** By [4, Lemma 7.5] the resolution is minimal if and only if for $\sigma_A$ an $i$-cell and $\sigma'_A$ an $(i-1)$-cell of $X_A$ such that $\sigma'_A \leq \sigma_A$, then $\text{lcm}(\sigma) \neq \text{lcm}(\sigma')$.

Using Theorem 6.1 we know that, if $\sigma = \sigma(m^a, D)$ with $m^a \in M$ and $D \subseteq \text{Supp}(m^a \setminus \{1\})$, then $\sigma' = \sigma(m^a, D \setminus \{k\})$ or $\sigma' = \sigma(\pi_k(m^a), D \setminus \{k\})$ for some $k \in D$. 

\[ Springer \]
If $\sigma' = \sigma(m^a, D \setminus \{k\})$, then by Proposition 7.1
\[
\text{lcm } \sigma = \text{lcm } \sigma' \cdot \prod_{x \in F_k \setminus F_{\tau(k)}} x.
\]
Since $F_k \setminus F_{\tau(k)}$ consists of the free vertices of the leaf $F_k$ of the quasi-tree $\langle F_1, \ldots, F_k \rangle$, we have $F_k \setminus F_{\tau(k)} \neq \emptyset$. Thus, $\text{lcm}(\sigma) \neq \text{lcm}(\sigma')$.

If $\sigma' = \sigma(\pi_k(m^a), D \setminus \{k\})$, then suppose $\text{lcm}(\sigma) = \text{lcm}(\sigma')$. From Proposition 7.1 it follows that
\[
\pi_k(m^a) \prod_{j \in D \setminus \{k\}} \prod_{x \in F_j \setminus F_{\tau(j)}} x = m^a \prod_{j \in D} \prod_{x \in F_j \setminus F_{\tau(j)}} x.
\]
Replacing $\pi_k(m^a)$ in the equation and simplifying both sides, we have
\[
\frac{m^a \cdot m_{\tau(k)}}{m_k} = m^a \prod_{x \in F_k \setminus F_{\tau(k)}} x,
\]
which with further simplification results in
\[
m_{\tau(k)} = m_k \prod_{x \in F_k \setminus F_{\tau(k)}} x.
\]
This last equality implies that $m_k \mid m_{\tau(k)}$ which (see Construction 2.3) means that $F_{\tau(k)} \subseteq F_k$, a contradiction to $F_{\tau(k)}$ and $F_k$ both being facets of a simplicial complex, and $\tau(k) \leq k$.

Note that an alternative approach to showing minimality is to explicitly show that the Morse complex is isomorphic to the convex cellular complex defined in [10].

**Corollary 7.3** (The projective dimension of $I^r$) If $I$ is generated by $q$ square-free monomials in a polynomial ring $S$, $I$ has projective dimension one, and $r$ is a positive integer, then
\[
\text{pd}_S(I^r) = \begin{cases} q - 1 & \text{if } r \geq q - 1 \\ r & \text{if } r < q - 1 \end{cases} \quad (7.3.1)
\]

**Proof** By Theorems 5.3 and 7.2 we know that the Morse complex $X_A$ from Theorem 5.3 supports a minimal free resolution of $I^r$ and for any $k$, the $k$-cells of $X_A$ are in one-to-one correspondence with the $A$-critical $k$-cells of $X = \text{Taylor}(I)$.

Therefore, the value of $\text{pd}_S(I^r)$ is equal to the largest size of a critical cell. Recall that the critical cells have the form $\sigma(m^a, D)$ with $m^a \in M_r$ and $D \subseteq \text{Supp}(a) \setminus \{1\}$, and thus, the largest cell has $D = \text{Supp}(a) \setminus \{1\}$. We have then
\[
\text{pd}_S(I^r) = \max\{|\text{Supp}(a) \setminus \{1\}| \mid a \in (\mathbb{N} \cup \{0\})^q, a_1 + a_2 + \cdots + a_q = r\}.
\]
The maximum in the expression above is obtained by choosing \( a_i \) such that \( a_i = 1 \) for as many values of \( i > 1 \) as possible. When \( q \leq r + 1 \), the maximum is obtained, for example, when \( a = (0, 1, \ldots, 1, r - q + 2) \); the maximum is \( q - 1 \) in this case. When \( q > r + 1 \), the maximum is achieved, for example, when \( a = (0, 1, \ldots, 1, 0, \ldots, 0) \), where precisely \( r \) entries are non-zero; the maximum is thus \( r \) in this case. \( \square \)

It is worth noting that the precise formula for the projective dimension of all powers also provides a precise formula for the depths and in particular, the formula above pinpoints exactly where this sequence stabilizes. This stabilizing point is referred to as the bf index of depth stability in the literature and is denoted \( d_{\text{stab}}(I) \). Finding bounds for \( d_{\text{stab}}(I) \) is an active area.

**Corollary 7.4** If \( I \) is generated by \( q \) square-free monomials and has projective dimension 1, then \( d_{\text{stab}}(I) = n - q + 1 \).

**Proof** This follows immediately from Corollary 7.3 and the Auslander-Buchsbaum formula (see [12, Theorem 19.9]). \( \square \)

Other invariants that can be read from an explicit minimal graded free resolution of a graded module are maximal shifts and Castelnuovo–Mumford regularity. In Sect. 2.2 we defined the multi-graded Betti numbers of a monomial ideal \( I \), denoted \( \beta_{i,m}(I) \) for each integer \( i \) and each \( m \in \text{LCM}(I) \). For each integer \( i \), the invariant

\[
\tau^S_i(I) = \sup \{ j \in \mathbb{Z} \mid \beta_{i,m}(I) \neq 0 \text{ for some } m \in \text{LCM}(I) \text{ with } \deg(m) = j \}
\]

(7.4.1)

is precisely the **maximal shift** (with respect to total degree) of a free module that occurs in the \( i \)th component of a minimal graded free resolution of \( I \). The **Castelnuovo–Mumford regularity** of an ideal \( I \) is defined by

\[
\text{reg}_S(I) = \sup_{i \geq 0} \{ \tau^S_i(I) - i \}.
\]

(7.4.2)

We establish a formula for maximal shifts and regularity in the case of interest for our paper.

**Corollary 7.5** If \( I \) has projective dimension 1 and is minimally generated by square-free monomials \( m_1, \ldots, m_q \) in a polynomial ring \( S \) over a field, then for all \( i \leq r \) we have:

\[
\tau^S_i(I^r) = \max \left\{ \sum_{j \in D} \deg(\text{lcm}(m_j, m_{\tau(j)})) \mid D \subseteq [q] \setminus \{1\}, |D| = i \right\}
\]

\[
+ (r - i) \max \{ \deg(m_j) \mid j \in [q] \}.
\]

In particular, \( r \mapsto \tau^S_i(I^r) \) is a linear function when \( r \geq i \).
Proof The description of the minimal free resolution of $I^r$ as the free resolution supported on the CW complex $X_A$ (Theorems 5.3 and 7.2) shows

$$\beta_{i,m}(I^r) \neq 0 \iff m = \lcm (m^a, D) \text{ for some } m^a \in \mathcal{M}_r \text{ and } D \subseteq \text{Supp}(a) \setminus \{1\} \text{ with } |D| = i$$

and hence, using Proposition 7.1, we have

$$t_i^S(I^r) = \max \{ \deg (\lcm (m^a, D)) \mid m^a \in \mathcal{M}_r, D \subseteq \text{Supp}(a) \setminus \{1\}, |D| = i \}$$

$$= \max \{ \deg \left( m^a \prod_{j \in D} \prod_{x \in F_j \setminus F_{\tau(j)}} x \right) \mid m^a \in \mathcal{M}_r, D \subseteq \text{Supp}(a) \setminus \{1\}, |D| = i \}.$$

Let $u \in [q]$ be such that

$$\deg (m_u) = \alpha = \max \{ \deg (m_j) \mid j \in [q] \}.$$ 

Now let $a \in \mathcal{M}_r$ and $D = \{k_1, \ldots, k_i\} \subseteq \text{Supp}(a) \setminus \{1\}$. If $e_1, \ldots, e_q$ is the standard basis of $\mathbb{R}^q$, set

$$b = a - \sum_{j \in D} e_j \in \mathcal{M}_{r-i} \text{ so that } m^a = m_{k_1} \ldots m_{k_i} m^b.$$

Using Proposition 7.1, we see that

$$\deg (\lcm (m^a, D)) = \deg (m^a) + \sum_{j \in D} |F_j \setminus F_{\tau(j)}|$$

$$= \deg (m^b) + \sum_{j \in D} (\deg (m_j) + |F_j \setminus F_{\tau(j)}|)$$

$$\leq \alpha (r - i) + \sum_{j \in D} (\deg (m_j) + |F_j \setminus F_{\tau(j)}|).$$

On the other hand, if we let

$$m^c = m_{k_1} \ldots m_{k_i} (m_u)^{r-i} \in \mathcal{M}_r$$

we observe that $D \subseteq \text{Supp}(m^c)$ and, by Proposition 7.1,

$$\alpha (r - i) + \sum_{j \in D} (\deg (m_j) + |F_j \setminus F_{\tau(j)}|) = \deg (\lcm (m^c, D)).$$

\[ \square \] Springer
This implies that
\[ t^S_i(I^r) = \alpha(r - i) + \max\left\{ \sum_{j \in D} (\deg(m_j) + |F_j \setminus F_{\tau(j)}|) : D \subseteq [q] \setminus \{1\}, |D| = i \right\}. \]

The formula in the statement follows by noting that, by Proposition 7.1 for all \( j \in [q] \)
\[ \deg(\text{lcm}(m_j, m_{\tau(j)})) = \deg(m_j) + |F_j \setminus F_{\tau(j)}|. \]

\[ \square \]

It is known that the regularities of the powers of a homogeneous ideal in a polynomial ring are asymptotically linear ([8, Theorem 4.7], [11,22,29]). In other words, there are integers \( a, b \) such that
\[ \text{reg}(I^r) = ar + b \]
for all \( r \gg 0 \). Corollary 7.6 below fine tunes this fact for square-free monomial ideals of projective dimension 1.

**Corollary 7.6** (The regularity of \( I^r \)) If \( I \) has projective dimension 1 and is minimally generated by square-free monomials \( m_1, \ldots, m_q \) in a polynomial ring \( S \) over a field, then
\[ \text{reg}_S(I^r) = \alpha r + (1 - q)\alpha + \text{reg}_S(I^{q-1}) \]
for all \( r \geq q - 1 \) where \( \alpha = \max\{\deg(m_j) \mid j \in [q]\} \).

**Proof** With \( \alpha = \max\{\deg(m_j) \mid j \in [q]\} \), Corollary 7.5 gives that for all \( r > 0 \) and all \( i \leq r \) there exists an integer \( c_i \) such that
\[ t^S_i(I^r) = \alpha r + c_i. \] (7.6.1)

In view of Corollary 7.3, we have
\[ \text{reg}_S(I^r) = \sup_{0 \leq i \leq q-1} \{ t^S_i(I^r) - i \}. \] (7.6.2)

If \( r \geq q - 1 \), then for any \( i \) with \( 0 \leq i \leq q - 1 \) we also have \( i \leq r \), and (7.6.1) gives
\[ t^S_i(I^r) = \alpha r + c_i = (r - q + 1)\alpha + ((q - 1)\alpha + c_i) = (r - q + 1)\alpha + t^S_i(I^{q-1}). \] (7.6.3)

The desired conclusion follows from (7.6.2) and (7.6.3).

\[ \square \]
Acknowledgements The research leading to this paper was initiated during the week-long workshop “Women in Commutative Algebra” (19w5104) which took place at the Banff International Research Station (BIRS). The authors would like to thank the organizers and acknowledge the hospitality of BIRS and the additional support provided by the National Science Foundation (NSF), DMS-1934391. For this work Liana Șega and Sandra Spiroff were supported in part by grants from the Simons Foundation (#354594, #584932, respectively), and Susan Cooper and Sara Faridi were supported by Natural Sciences and Engineering Research Council of Canada (NSERC). The authors are grateful to Volkmar Welker for useful background information. The computations for this project were done using the computer algebra software Macaulay2 [23]. Finally, the authors thank both referees for carefully reading the paper and providing many insightful comments. For the last author this material is based upon work supported by and while serving at the National Science Foundation. Any opinion, findings, and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

Data availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. Álvarez Montaner, J., Fernández-Ramos, O., Gimenez, P.: Pruned cellular free resolutions of monomial ideals. J. Algebra 541, 126–145 (2020)
2. Barile, M., Macchia, A.: Minimal cellular resolutions of the edge ideals of forests. Electron. J. Combin. 27(2), P2.41 (2020)
3. Batzies, E.: Discrete Morse theory for cellular resolutions, Ph.D. Thesis, University of Marburg, (2002)
4. Batzies, E., Welker, V.: Discrete Morse theory for cellular resolutions. J. Reine Angew. Math. 543, 147–168 (2002)
5. Burch, L.: On ideals of finite homological dimension in local rings. Proc. Cambridge Philos. Soc. 64, 941–948 (1968)
6. Bayer, D., Sturmfels, B.: Cellular resolutions of monomial modules. J. Reine Angew. Math. 503, 123–140 (1998)
7. Brodmann, M.: The asymptotic nature of the analytic spread. Math. Proc. Cambridge Philos. Soc. 86(1), 35–39 (1979)
8. Carlini, E., Hà, T., Harbourne, B., Van Tuyl, A.: Ideals of powers and powers of ideals. Intersecting algebra, geometry, and combinatorics, Lecture Notes of the Unione Matematica Italiana, 27. Springer, Cham, (2020)
9. Chari, M.: On discrete Morse functions and combinatorial decompositions. Discrete Math. 217(1–3), 101–113 (2000)
10. Cooper, S., El Khoury, S., Faridi, S., Mayes-Tang, S., Morey, S., Șega, L.M., Spiroff, S.: Powers of graphs and applications to resolutions of powers of monomial ideals (submitted). arXiv:210807703
11. Cutkosky, S.D., Herzog, J., Trung, N.V.: Asymptotic behaviour of the Castelnuovo–Mumford regularity. Compos. Math. 118(3), 243–261 (1999)
12. Eisenbud, D.: Commutative Algebra. With a View toward Algebraic Geometry, Graduate Texts in Mathematics, vol. 150. Springer, New York (1995)
13. Engström, A., Noren, P.: Cellular resolutions of powers of monomial ideals, arXiv preprint arXiv:1212.2146 (2012)
14. Faridi, S.: The facet ideal of a simplicial complex. Manuscr. Math. 109(2), 159–174 (2002)
15. Faridi, S., Hersey, B.: Resolutions of monomial ideals of projective dimension 1. Comm. Algebra 45(12), 5453–5464 (2017)
16. Forman, R.: Morse theory for cell complexes. Adv. Math. 134(1), 90–145 (1998)
17. Hersey, B.: Resolutions of Monomial Ideals via Quasi-Trees, MSc Thesis, Dalhousie University (2015)
18. Hilbert, D.: Über die Theorie der algebraischen Formen. Math. Ann. 36(4), 473–534 (1890)
19. Jöllenbeck, M., Welker, V.: Minimal resolutions via algebraic discrete Morse theory. Mem. Am. Math. Soc. 197, 923 (2009)
20. Jonsson, J.: Simplicial Complexes of Graphs. Lecture Notes in Mathematics, vol. 1928. Springer, Berlin (2008)
21. Kodiyalam, V.: Homological invariants of powers of an ideal. Proc. Amer. Math. Soc. 118(3), 757–764 (1993)
22. Kodiyalam, V.: Asymptotic behaviour of Castelnuovo–Mumford regularity. Proc. Amer. Math. Soc. 128(2), 407–411 (2000)
23. Grayson, D.R., Stillman, M.E.: Macaulay2, a software system for research in algebraic geometry, Available at http://www.math.uiuc.edu/Macaulay2/
24. Massey, I.: Singular Homology Theory, Graduate Texts in Mathematics, vol. 70. Springer, New York (1980)
25. Orlik, P., Welker, V., Algebraic combinatorics. Lectures from the Summer School held in Nordfjordeid, Universitext, p. 2007. Springer, Berlin (2003)
26. Peeva, I.: Graded Syzygies, Algebra and Applications, vol. 14. Springer, London (2011)
27. Stanley, R.P.: Combinatorics and Commutative Algebra, Progress in Mathematics, vol. 41, 2nd edn. Birkhäuser, Boston (1996)
28. Taylor, D.: Ideals generated by monomials in an $R$-sequence, Ph.D. Thesis, University of Chicago (1966)
29. Trung, N.V., Wang, H.J.: On the asymptotic linearity of Castelnuovo–Mumford regularity. J. Pure Appl. Algebra 201(1–3), 42–48 (2005)
30. Velasco, M.: Minimal free resolutions that are not supported by a CW complex. J. Algebra 319(1), 102–114 (2008)
31. Villarreal, R.H.: Cohen-Macaulay graphs. Manuscr. Math. 66(3), 277–293 (1990)
32. Zheng, X.: Resolutions of facet ideals. Comm. Algebra 32(6), 2301–2324 (2004)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Authors and Affiliations

Susan M. Cooper$^1$ · Sabine El Khoury$^2$ · Sara Faridi$^3$ · Sarah Mayes-Tang$^4$ · Susan Morey$^5$ · Liana M. Şega$^6$ · Sandra Spiroff$^7$

$^1$Department of Mathematics, University of Manitoba, 520 Machray Hall, 186 Dysart Road, Winnipeg, MB R3T 2N2, Canada
$^2$Department of Mathematics, American University of Beirut, Bliss Hall 315, P.O. Box 11-0236, Beirut 1107-2020, Lebanon
