A local in time existence and uniqueness result of an inverse problem for the Kelvin-Voigt fluids

Pardeep Kumar, Kush Kinra and Manil T Mohan

Department of Mathematics, Indian Institute of Technology Roorkee-IIT Roorkee, Haridwar Highway, Roorkee, Uttarakhand 247667, India
E-mail: pkumar3@ma.iitr.ac.in, kkinra@ma.iitr.ac.in, manilmohan@ma.iitr.ac.in and maniltmohan@gmail.com

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Abstract
In this paper, we consider an inverse problem for three dimensional viscoelastic fluid flow equations, which arises from the motion of Kelvin–Voigt fluids in bounded domains. This inverse problem aims to reconstruct the velocity and kernel of the memory term simultaneously, from the measurement described as an integral overdetermination condition. By using the contraction mapping principle in an appropriate space, a local in time existence and uniqueness result for the inverse problem of Kelvin–Voigt fluids are obtained. Furthermore, using similar arguments, a global in time existence and uniqueness result for an inverse problem for Oseen type equations are also achieved.

Keywords: Kelvin–Voigt fluids equation, inverse problem, memory kernel, integral overdetermination condition, contraction mapping principle

1. Introduction
First we shall discuss the fundamental facts that we come across while dealing with inverse problems. In the literature for the same, it can be noticed that the global in time results are hard to comprehend for inverse problems. Not only claiming the uniqueness and stability results, but also establishing the existence of a solution is also a challenging problem. The general strategy to obtain global in time existence and uniqueness results is to find a priori estimates in appropriate function spaces for the unknowns of the problem with proper assumptions on the data.

1.1. The physical model and direct problem
We, in this paper, shall be studying an inverse problem for Kelvin–Voigt fluid flow equations with memory term. The original direct problem model is the so-called viscoelastic fluid flow
equations arising from the Kelvin–Voigt model for the non-Newtonian fluid flows, which can be described by the following system of integro-differential equations (cf. [18, 21, 22])

\[
\partial_t u - \mu_1 \partial_t \Delta u - \mu_0 \Delta u + (u \cdot \nabla)u - \int_0^t k(t-s) \Delta u(s) ds + \nabla p = f, \quad \text{in } \Omega \times (0, T), \quad (1.1)
\]

and the incompressibility condition

\[
\nabla \cdot u = 0, \quad \text{in } \Omega \times (0, T), \quad (1.2)
\]

where

\[
\mu_1 = \frac{2\kappa_2}{\lambda} \quad \text{and} \quad \mu_0 = \frac{2}{\lambda} \left( \kappa_1 - \kappa_2 \right),
\]

and all quantities are positive. The quantities \( \lambda \) and \( \kappa_j, j \in \{1, 2\} \), denote the relaxation time and retardation time, respectively (cf. [18] for more details). The system (1.1) and (1.2) is supplemented with the following initial and boundary conditions:

\[
u = u_0, \quad \text{in } \Omega \times \{0\}, \quad (1.3)
\]

\[
u = 0, \quad \text{on } \partial \Omega \times [0, T). \quad (1.4)
\]

Here, \( \Omega \) is a bounded domain in \( \mathbb{R}^3 \) with smooth boundary \( \partial \Omega \), \( T > 0 \) is a constant, \( u(x, t) \in \mathbb{R}^3 \) denotes the velocity vector, \( p(x, t) \in \mathbb{R} \) represents the pressure of the fluid and \( f \) stands for the external forcing. The integral term of (1.1) with kernel \( k \) is the memory or hereditary term, which describes the viscoelastic property of the non-Newtonian fluids. One can consider these equations as 3D Navier–Stokes–Voight equations or Oskolkov’s equations (cf. [16]) with a memory term \(-\int_0^t k(t-s) \Delta u(s) ds\). In the literature, some authors have referred the system (1.1)–(1.4) as Kelvin–Voigt fluid flow equations with ‘fading memory’ to differentiate it with Navier–Stokes–Voight equations ([18, 19, 21], etc).

When kernel \( k \) is given (for example \( k(t) = \gamma e^{-\delta t} \), where \( \gamma = \frac{2}{\lambda} \left( \nu - \frac{\kappa_1}{\kappa_2} \right) + \frac{\kappa_2}{\lambda} > 0 \) and \( \delta = \frac{1}{\lambda} > 0 \), \( \nu \) is the coefficient of kinematic viscosity), the global solvability results for the system (1.1)–(1.4) in bounded domains are available in the literature, see [18, 21–23, 26], etc and the references therein. Recently, the author in [18] established the existence and uniqueness of weak as well as strong solutions for the kernel \( k(t) = \gamma e^{-\delta t} \), using a local monotonicity property of the linear and nonlinear operators and a localized version of the Minty Browder technique in bounded and unbounded domains like Poincaré domains. By using an \( m \)-accretive quantization of the linear and nonlinear operators, the author established the existence and uniqueness of strong solution also. The asymptotic behavior of Kelvin–Voigt fluid flow equations is studied in [19] by establishing the existence of a finite Hausdorff and fractal dimensional global attractor.

1.2. The inverse problem

In practical phenomena, the kernel of memory term in (1.1) represents the physical property of non-Newtonian fluids, which is difficult to determine in advance. A significant indirect method to find the memory capability of the material is by putting the material into some stimulation and then measuring the corresponding feedback of the material. We expect this measurement to give the relevant information about the kernel \( k \) so that we can infer kernel from it. Performing this requires us to deal with an inverse problem of determining \( u \) and \( k \) simultaneously, where we have some additional information on the velocity \( u \). This additional information on \( u \) is represented in an integral form and it admits a physical interpretation as a result of measuring
a physical parameter by a perfect sensor. The essence of this problem is that any sensor, due to its finite size, always performs some averaging of a measured parameter over the domain of action. The integral overdetermination condition in our work is given by

$$\int_{O} (I - \mu_1 \Delta) \varphi(x) \cdot u(x, t) dx = r(t), \quad t \in [0, T],$$  (1.5)

where $r(t)$ is the measurement data representing the average velocity on the domain $O$ and $\varphi$ is a given function representing the type of device used to measure the velocity. Further, we can consider $\varphi$ as internal tiny sensor that measures the average velocity in a very small subset of the domain $O$, that is, $O' := \text{supp}(\varphi) \subset O$ may be very small. The physical phenomena behind this mathematical issue shows the averaging effect of measurement data, where both the convolution $\int_{t_0}^{t} k(t-s) \Delta u(s, s) ds$ and the integral $\int_{O} (I - \mu_1 \Delta) \varphi(x) \cdot u(x, t) dx$ stands for some averaging process (cf [24, 25], etc for more details). Throughout this paper, external force $f$ is assumed to be zero. The precise formulation of our inverse problem is as follows:

- Determine the velocity $u$ and kernel of memory $k$ satisfying the system (1.1)–(1.5) with initial pressure $p(x, 0) = p_0$.

Due to the memory term, the inverse problem of reconstructing the kernel of the memory term from (1.1)–(1.5) is a system of partial integro-differential equations. This inverse problem has two types of nonlinearities, first, the convective term $(u \cdot \nabla) u$ and second, the convolution $\int_{t_0}^{t} k(t-s) \Delta u(s, s) ds$, since $k$ is unknown.

For some scalar parabolic equations with a memory term, the inverse problem of reconstructing the kernel of memory term has been well studied by Colombo et al using analytic semigroup theory (see [2–11], etc). In particular, the results pertaining to both local and global in time existence and uniqueness of solution for an evolution equation with the nonlinearity having desirable growth condition has been discussed by Colombo and Guidetti (see [4]). The authors have presented a new method, where the convolution term is rewritten as a sum of two linear terms in the unknown. By this way, they were able to successfully handle the nonlinearities of the convolution term and the source term in the proof of global uniqueness.

For the nonlinear Oskolkov’s system, the inverse problem with unknown source function is studied in [13]. Carleman estimates and its application of proving the Lipschitz stability of an inverse problem for the Kelvin–Voigt model is described in [14]. Taking the similar path as in [4], the hyperbolic type of inverse problems have been investigated by several authors to obtain global in time existence and uniqueness results (see [1, 17, 25], etc). In [1], the authors studied an inverse problem for the strongly damped wave equation with memory to obtain the similar results. A global in time existence and uniqueness result using a fixed point theorem for a one-dimensional integro-differential hyperbolic system, which arises from a simplified model of thermoelasticity, is established in [25]. The author in [17] studied the identification of two locally in time two (smooth) convolution kernels in a fully hyperbolic phase-field system coupling two hyperbolic integro-differential equations. Using Schauder’s fixed point theorem, the solvability results of an inverse problem to the 2D and 3D Navier–Stokes equations with integral overdetermination conditions as well as final overdetermination data is obtained in [24]. Recently, the authors in [15] proved the local solvability of an inverse problem to the Navier–Stokes equation with memory term (Oldroyd models) using fixed point arguments.

Similar to the inverse parabolic problem studied in [4], the difficulties of our inverse problem also arise due to the memory term and the convective (nonlinear) term in the equation (1.1).
Existence and uniqueness of the inverse problem plays a significant role, especially when establishing the reconstruction scheme. The basic ideas to prove the results of this work have been adapted from [4, 15]. Using the contraction mapping principle, we obtain the local in time existence and uniqueness of solutions for our inverse problem for the Kelvin–Voigt fluids. The main hindrance for establishing the global existence and uniqueness is the presence of convective term \((u \cdot \nabla) u\) in the Kelvin–Voigt fluid equation (1.1). In order to show the effectiveness of the method described in [4], we consider a similar inverse problem for Oseen type equations corresponding to Kelvin–Voigt fluids with memory term. First, we obtain local in time existence and uniqueness result for this inverse problem. Then by linearizing the convolution term, that is, by splitting the convolution term into an unknown velocity \(v\), and kernel of memory term \(k\) and then calculating \textit{a priori} estimates for \(v\) and \(k\), we obtain the existence and uniqueness of global in time results for our inverse problem for the Oseen type equations with memory.

The rest of the paper is structured as follows: In the next section, we reformulate the inverse problem in a concrete setting by providing the compatibility and regularity conditions on the data and state some useful technical results. An equivalent form for the inverse problem is formulated in section 3. Using this equivalent form and contraction mapping principle, the main result on the local in time existence and uniqueness of solutions is established in section 4 (theorem 4.1). Oseen type equations corresponding to Kelvin–Voigt fluids with memory term is considered in section 5. As in the case of nonlinear problem, firstly, we reformulate the inverse problem and then provide an equivalent form of this inverse problem. Using this equivalent problem, we prove the results on local and global in time existence and uniqueness of solutions (theorems 5.2, 5.4, 5.5).

2. The concrete version of the inverse problem

In this section, we reformulate the inverse problem mentioned in the introduction into a concrete setting. Then we introduce the compatibility and regularity conditions on the data so that we can obtain the well-posedness of the inverse problem. We assume an additional condition on velocity field \(u(t)\) in the form of an integral representation given in (1.5). This additional condition also helps us to determine the kernel of the memory term. We discuss the necessary preliminary materials needed to obtain the local in time existence and uniqueness result of our inverse problem in three dimensional bounded domains.

2.1. Function spaces

For any integers \(m, p\), let us denote by \(W^{m,p}(\Omega) := W^{m,p}(\Omega; \mathbb{R}^3)\) and \(W^{m,p}(0, T) = W^{m,p}(0, T; \mathbb{R})\) for the usual Sobolev spaces defined for spatial variable and time variable, respectively. For any Banach space \(X\), the space \(L^p(0, T; X)\) consists all Lebesgue measurable functions \(u : [0, T] \to X\) with

\[
\|u\|_{L^p(0,T;X)} := \left( \int_0^T \|u(t)\|_X^p \, dt \right)^{1/p} < \infty, \quad \text{for } 1 \leq p < \infty,
\]

and

\[
\|u\|_{L^\infty(0,T;X)} := \text{ess sup}_{0 \leq t \leq T} \|u(t)\|_X < \infty.
\]
The Sobolev space $W^{m,k}(0, T; X)$ consists of all functions $u \in L^k(0, T; X)$ such that $\partial_t^\beta u$ exists in the weak sense and belongs to $L^k(0, T; X)$, for all $0 \leq \beta \leq m$. Let us represent $H^m(O) := \mathbb{H}^{m-1}(O)$, $H^m(0, T) := W^{m,2}(0, T)$, and $H^m(0, T; X) := W^{m,2}(0, T; X)$. We define $L^2(O) := \{ p \in L^2(O; \mathbb{R}) : \int_O p(x)dx = 0 \}$ and $a \wedge b := \min\{a, b\}$, where $a, b \in \mathbb{R}$.

2.2. The inverse problem

For $T > 0$, the inverse problem is to determine $\tau \in (0, T]$,

$$u \in H^1(0, \tau; H^1_0(O) \cap H^2(O))$$

and $k \in L^2(0, \tau)$, such that $(u, k)$ satisfies the system:

$$
\begin{align*}
\partial_t u - \mu_1 \partial_t \Delta u &= \mu_0 \Delta u + (u \cdot \nabla)u - \int_0^t k(t - s)\Delta u(s)ds + \nabla p = 0, \quad \text{in} \ O \times (0, \tau), \\
\nabla \cdot u &= 0, \quad \text{in} \ O \times (0, \tau), \\
u &= 0, \quad \text{on} \ \partial O \times (0, \tau), \\
u &= \mu_0, \quad \text{in} \ O \times \{0\}, \\
\int_O (I - \mu_1 \Delta)\varphi(x) \cdot u(x, \tau)dx &= r(\tau), \quad \text{in} \ (0, \tau).
\end{align*}
$$

We solve the above inverse problem under the following assumptions on the data:

(A1) $u_0 \in H^1_0(O) \cap H^2(O)$, $\nabla \cdot u_0 = 0$, in $O$;

(A2) $\varphi \in H^1_0(O) \cap H^2(O)$, $\nabla \cdot \varphi = 0$, in $O$;

(A3) $\alpha^{-1} := \int_O \varphi \cdot \Delta u_0 dx \neq 0$;

(A4) $v_0 := (I - \mu_1 \Delta)^{-1}(\mu_0 \Delta u_0 - (u_0 \cdot \nabla)u_0 - \nabla p_0) \in H^1_0(O)$, $\nabla \cdot v_0 = 0$, in $O$, $p_0 \in H^1(O)$;

(A5) $r \in H^1(0, T)$, with

$$
\begin{align*}
\int_O (I - \mu_1 \Delta)\varphi(x) \cdot u_0(x)dx &= r(0), \\
\int_O (\mu_0 \Delta u_0(x) - (u_0(x) \cdot \nabla)u_0(x)) \cdot \varphi(x)dx &= r'(0).
\end{align*}
$$

2.3. Important inequalities and results

The following inequalities and results are used frequently in this paper.

**Lemma 2.1 (Gagliardo–Nirenberg interpolation inequality, theorem 1, [20]).** Let $O \subset \mathbb{R}^n$ and $u \in \mathbb{W}^{m,p}(O)$, $p \geq 1$ and fix $1 \leq p, q \leq \infty$ and a natural number $m$. Suppose also that a real number $\theta$ and a natural number $j$ are such that

$$\theta = \left( \frac{j}{n} + \frac{1}{q} - \frac{1}{r} \right) \left( \frac{m}{n} - \frac{1}{p} + \frac{1}{q} \right)^{-1}$$

and $\frac{1}{m} \leq \theta \leq 1$. Then for any $u \in \mathbb{W}^{m,p}(O)$, we have

$$\|\nabla^j u\|_{L^q} \leq C \|u\|_{\mathbb{W}^{m,p}}^{\theta} \|u\|_{L^p}^{1-\theta}.$$

(2.4)
where the constant $C$ depends upon the domain $O$, $m,n$.

Let us take $j = 0, m = 2, r = n = 3$ and $p = q = 2$ in (2.3) to get $\theta = \frac{1}{4}$ and
\[
\|u\|_{L^3} \leq C\|u\|_{L^3}^{1/4}\|u\|_{L^2}^{3/4}.
\]
Now if we consider $j = 1, m = 2, r = 2, n = 3$ and $p = q = 2$ in (2.3), then we find $\theta = \frac{1}{2}$ and
\[
\|\nabla u\|_{L^2} \leq C\|u\|_{L^1}^{1/2}\|u\|_{L^2}^{1/2}.
\]

**Lemma 2.2 (Theorem 4.4, [4])**. Let $X$ be a Banach space, $p \in (1, \infty)$, $\tau \in \mathbb{R}^+$, $k \in L^p(0, \tau)$, and $f \in L^p(0, \tau; X)$. Then $k * f \in L^p(0, \tau; X)$ and
\[
\|k * f\|_{L^p(0, \tau; X)} \leq \tau^{1-1/p}\|k\|_{L^p(0, \tau)}\|f\|_{L^p(0, \tau; X)},
\]
where $(k * f)(t) := \int_0^t k(t - s)f(s)ds$.

**Proof.** Using Young’s inequality for convolution and Hölder’s inequality, we have
\[
\|k * f\|_{L^p(0, \tau; X)} \leq \|k\|_{L^p(0, \tau)}\|f\|_{L^p(0, \tau; X)} \leq \tau^{1-1/p}\|k\|_{L^p(0, \tau)}\|f\|_{L^p(0, \tau; X)},
\]
which completes the proof. \(\square\)

**Lemma 2.3 (Theorem 4.5, [4])**. Let $X$ be a Banach space, $p \in (1, \infty)$, $\tau \in \mathbb{R}^+$, $z \in W^{1,p}(0, \tau; X)$ with $z(0) = 0$. Then
\[
\|z\|_{L^\infty(0, \tau; X)} \leq \tau^{1-1/p}\|\partial_t z\|_{L^p(0, \tau; X)},
\]
\[
\|z\|_{L^p(0, \tau; X)} \leq \tau\|\partial_t z\|_{L^p(0, \tau; X)}.
\]

The proof can be easily concluded from Hölder’s inequality and Young’s inequality for convolution, using the formula $z = 1 * \partial_t z$.

3. The equivalent problem

In this section, we transform the original inverse problem (2.2a)–(2.2e) into an equivalent system, using which we provide the proof of main result in the next section. Assuming the compatibility and regularity conditions (A1)–(A5), such a transformation is possible, and the following theorem gives an equivalent formulation.

**Theorem 3.1.** Let the assumptions (A1)–(A5) hold. Let $(u, k)$ be a solution of the system (2.2a)–(2.2e) defined up to $T$ such that
\[
u \in H^1(0, T; H^2_0(O) \cap H^1_0(O)) \quad \text{and} \quad k \in L^2(0, T).
\]

Then $\nu := \partial_t u$ and $k$ verify the conditions
\[
\nu \in H^1(0, T; H^2_0(O) \cap H^1_0(O)) \quad \text{and} \quad k \in L^2(0, T),
\]
and solve the system
\[ \partial_t v - \mu_1 \partial_t \Delta v - \mu_0 \Delta v - k \Delta u_0 - \int_0^t k(t-s) \Delta v(s) ds + (v \cdot \nabla) \left( u_0 + \int_0^t v(s) ds \right) \]
\[ + \left( \left( u_0 + \int_0^t v(s) ds \right) \cdot \nabla \right) v + \nabla \partial_t p = 0, \quad \text{in } \mathcal{O} \times (0, T), \quad (3.3a) \]
\[ \nabla \cdot v = 0, \quad \text{in } \mathcal{O} \times (0, T), \quad (3.3b) \]
\[ v = 0, \quad \text{on } \partial \mathcal{O} \times [0, T), \quad (3.3c) \]
\[ v = v_0, \quad \text{in } \mathcal{O} \times \{0\}, \quad (3.3d) \]

with

\[ k(t) = \alpha \left\{ \frac{r''(t)}{s} - \mu_0 \int_{\mathcal{O}} v \cdot \Delta \varphi \, dx - \int_{\mathcal{O}} \int_0^t k(t-s) v \cdot \Delta \varphi \, dx \, ds \right\} \]
\[ - \int_{\mathcal{O}} \left( (v \cdot \nabla) \varphi \right) \cdot \left( u_0 + \int_0^t v(s) ds \right) \, dx - \int_{\mathcal{O}} \left[ \left( \left( u_0 + \int_0^t v(s) ds \right) \cdot \nabla \right) \varphi \right] \cdot v \, dx \right\}. \quad (3.3e) \]

On the other hand, under the setting \( u(t) := u_0 + \int_0^t v(s) ds \), if \((v, k)\) satisfies (3.2) and is a solution to the system (3.3a)–(3.3e), then under the above setting, \((u, k)\) satisfies (3.1) and is a solution to the system (2.2a)–(2.2e).

**Proof.** The proof constitutes of mainly two steps.

**Step 1.** Suppose that the system (2.2a)–(2.2e) has a solution \((u, k)\) satisfying

\[ u \in H^2(0, T; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O})) \quad \text{and} \quad k \in L^2(0, T). \]

Then, it is easy to see that \( v := \partial_t u \) and \( k \) satisfy the equations (3.2), (3.3b) and (3.3c). From the equation (2.2a), we have

\[ \partial_t u = (1 - \mu_1 \Delta)^{-1} \left( \mu_0 \Delta u - (u \cdot \nabla) u + \int_0^t k(t-s) \Delta u(s) ds - \nabla p \right), \]
\[ v(x, 0) = (1 - \mu_1 \Delta)^{-1} \left( \mu_0 \Delta u_0 - (u_0 \cdot \nabla) u_0 - \nabla p_0 \right) = v_0. \]

Therefore, we obtain (3.3d) using assumption (A4). Taking \( \partial_t \) to the equation (2.2a), we get

\[ \partial_t v - \mu_1 \partial_t \Delta v - \mu_0 \Delta v - k \Delta u_0 - \int_0^t k(t-s) \Delta v(s) ds + (v \cdot \nabla) \left( u_0 + \int_0^t v(s) ds \right) \]
\[ + \left( \left( u_0 + \int_0^t v(s) ds \right) \cdot \nabla \right) v + \nabla \partial_t p = 0, \quad \text{in } \mathcal{O} \times (0, T), \]

and equation (3.3a) follows. Taking the inner product in (3.3a) with \( \varphi \) and using assumptions (A2) and (A3), we obtain
which is (2.2e) and it completes the proof.

**Step 2.** Suppose that the system (3.3a)–(3.3e) has a solution \((v, k)\) satisfying

\[ v \in H^1(0, T; \mathbb{H}^1_0(\Omega) \cap \mathbb{H}^2(\Omega)) \quad \text{and} \quad k \in L^2(0, T). \]

It can be easily shown that (3.1) and (2.2b)–(2.2d) hold. Since \(v = \partial_t u\), the equation (3.3a) can be rewritten as

\[ \partial_t^2 u - \mu_1 \partial_t \Delta u - \mu_0 \Delta u + \left( u \cdot \nabla \right) u - \int_0^t k(t-s) \Delta u(s) \, ds + \nabla p = 0. \]

Integrating the above equation, we obtain

\[ \partial_t u - \mu_1 \partial_t \Delta u - \mu_0 \Delta u + \left( u \cdot \nabla \right) u - \int_0^t k(t-s) \Delta u(s) \, ds + \nabla p = c_1, \]

where \(c_1\) is the constant of integration. Taking \(t = 0\), we find

\[ (I - \mu_1 \Delta) v_0 - \mu_0 \Delta u_0 + \left( u_0 \cdot \nabla \right) u_0 + \nabla p_0 = c_1. \]

Making use of assumption (d), we get \(c_1 = 0\) and (2.2a) follows. The equation (3.3e) for \(k\) can be rewritten as

\[ r''(t) = \partial_t \left( \mu_0 \int_{\Omega} \varphi \cdot \Delta u \, dx - \int_{\Omega} \left( \left( u \cdot \nabla \right) u \right) \cdot \varphi \, dx + \int_0^t \int_{\Omega} k(t-s) \varphi \cdot \Delta u \, dx \, ds \right). \]

Integrating this equation with respect to \(t\), we obtain

\[ r'(t) = \mu_0 \int_{\Omega} \varphi \cdot \Delta u \, dx - \int_{\Omega} \left( \left( u \cdot \nabla \right) u \right) \cdot \varphi \, dx + \int_0^t \int_{\Omega} k(t-s) \varphi \cdot \Delta u \, dx \, ds + c_2. \]

Taking \(t = 0\) and using assumption (A5), we get \(c_2 = 0\), so that

\[ r'(t) = \mu_0 \int_{\Omega} \varphi \cdot \Delta u \, dx - \int_{\Omega} \left( \left( u \cdot \nabla \right) u \right) \cdot \varphi \, dx + \int_0^t \int_{\Omega} k(t-s) \varphi \cdot \Delta u \, dx \, ds. \]

Using (2.2a) in the above equation along with assumption (A2), we deduce

\[ r'(t) = \int_{\Omega} (I - \mu_1 \Delta) \partial_t u \cdot \varphi \, dx + \int_{\Omega} \nabla p \cdot \varphi \, dx \]

\[ = \int_{\Omega} (I - \mu_1 \Delta) \partial_t u \cdot \varphi \, dx + \partial_t \left[ \int_{\Omega} (I - \mu_1 \Delta) \varphi \cdot u \, dx \right]. \]

Integrating the above equation, taking \(t = 0\) and using assumption (A5), we arrive at

\[ r(t) = \int_{\Omega} (I - \mu_1 \Delta) \varphi \cdot u \, dx, \]

which is (2.2e) and it completes the proof. \(\square\)
4. Local in time existence

From theorem 3.1, we know that solving the inverse problem (2.2a)–(2.2e) with solution \((u, k)\) is equivalent to solving the system (3.3a)–(3.3e) with solution \((v, k)\). Therefore, instead of proving results of the existence and uniqueness of solutions to the system (2.2a)–(2.2e), we prove similar results for the system (3.3a)–(3.3e). The proof is based on contraction mapping principle. It is important to note that having existence of unique \((v, k)\) to the system (3.3a)–(3.3e) is equivalent to having existence of unique \((u, k)\) to the inverse problem (2.2a)–(2.2e).

The following theorem is the main result of this section:

**Theorem 4.1 (Local in time existence).** Let the assumptions (A1)–(A5) hold. Then there exists \(\tau \in (0, T]\), such that the inverse problem (2.2) has a unique solution

\[
(u, k) \in H^2(0, \tau; H^1_0(O) \cap H^2(\Omega)) \times L^2(0, \tau).
\]

**Proof.** Let us define the space

\[
\mathcal{V}(\tau, L) := \left\{ (\tilde{v}, \tilde{k}) \in H^1(0, \tau; H^1_0(O) \cap H^2(\Omega)) \times L^2(0, \tau) : \nabla \cdot \tilde{v} = 0, \quad \text{in } O \times (0, \tau),
\right.

\[
\tilde{v} = 0, \quad \text{on } \partial O \times (0, \tau), \quad \tilde{v} = v_0, \quad \text{in } O \times \{0\} \quad \text{and}
\]

\[
\|\tilde{v}\|_{H^1(0, \tau; H^1_0(O))} + \|\tilde{k}\|_{L^2(0, \tau)} \leq L,
\]

where \(L\) is a positive constant, which will be determined later. We also define the mapping \(\Psi : \mathcal{V}(\tau, L) \to \mathcal{V}(\tau, L)\) such that \((\tilde{v}, \tilde{k}) \mapsto (v, k)\) through

\[
k(t) := \alpha \left\{ r'' - \mu_0 \int_0^t \tilde{v} \cdot \Delta \varphi \, dx - \int_0^t \int_O \tilde{k}(t - s) \tilde{v} \cdot \Delta \varphi \, dx \, ds
\]

\[
- \int_O ((\tilde{v} \cdot \nabla) \varphi) \cdot \left( u_0 + \int_0^t \tilde{v}(s) \, ds \right) \, dx - \int_O \left[ \left( u_0 + \int_0^t \tilde{v}(s) \, ds \right) \cdot \nabla \varphi \right] \cdot \tilde{v} \, dx \right\},
\]

(4.1)

and the initial boundary value problem

\[
\partial_t v - \mu_1 \partial_t \Delta v - \mu_4 \Delta v - k \Delta u_0 - \int_0^t \tilde{k}(t - s) \Delta \tilde{v}(s) \, ds + (\tilde{v} \cdot \nabla) \left( u_0 + \int_0^t \tilde{v}(s) \, ds \right)
\]

\[
+ \left( u_0 + \int_0^t \tilde{v}(s) \, ds \right) \cdot \nabla \nabla \cdot \tilde{v} + \nabla \partial_t p = 0, \quad \text{in } O \times (0, \tau),
\]

\[
\nabla \cdot v = 0, \quad \text{in } O \times (0, \tau),
\]

\[
v = 0, \quad \text{on } \partial O \times [0, \tau),
\]

\[
v = v_0, \quad \text{in } O \times \{0\}.
\]

(4.2)

In order to complete the proof of theorem 4.1, we need to show that the mapping \(\Psi : \mathcal{V}(\tau, L) \to \mathcal{V}(\tau, L)\) is a contraction map. For simplicity, hereafter \(C\) represents a generic constant depending only on \(O\).
Step 1. Firstly, we show that the map $\Psi$ is well defined for an appropriate choice of $L$ and $\tau$. From (4.1), we have

$$
\|k\|_{L^2(0,\tau)} \leq \alpha \|\varphi''\|_{L^2(0,\tau)} + \alpha \mu_0 \|\tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))} \|\nabla \varphi\|_{L^2(\Omega)}
$$

$$
+ \alpha \left( \int_0^\tau (t-s) \left( \int_0^t \tilde{v} \cdot \Delta \varphi \, dx \right) \, ds \right)
$$

$$
+ 2\alpha \left( \int_0^\tau \|u_0\|_{L^2(\partial\Omega)} + \int_0^\tau \|\tilde{v}(s)\|_{L^\infty(\partial\Omega)} \right)^2 \|\nabla \varphi\|_{L^2(\partial\Omega)}^2 \right)^{1/2}.
$$

(4.3)

Applying lemma 2.2, we find

$$
\left\| \int_0^\tau (t-s) \left( \int_0^t \tilde{v} \cdot \Delta \varphi \, dx \right) \, ds \right\|_{L^2(0,\tau)} \leq \left\| \tilde{k} \right\|_{L^1(0,\tau)} \left\| \int_0^t \tilde{v} \cdot \Delta \varphi \, dx \right\|_{L^2(0,\tau)}
$$

$$
\leq \tau^{1/2} \left\| \tilde{k} \right\|_{L^2(0,\tau)} \left( \int_0^\tau \|\tilde{v}(t)\|_{L^2(\partial\Omega)} \|\Delta \varphi\|_{L^2(\partial\Omega)} \, dt \right)^{1/2}
$$

$$
= \tau^{1/2} \left\| \tilde{k} \right\|_{L^2(0,\tau)} \|\Delta \varphi\|_{L^2(\partial\Omega)} \|\tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))}.
$$

Substituting the above estimate in (4.3), we obtain

$$
\|k\|_{L^2(0,\tau)} \leq \alpha \|\varphi''\|_{L^2(0,\tau)} + \alpha \left\{ \mu_0 \|\Delta \varphi\|_{L^2(\partial\Omega)} + \tau^{1/2} \left\| \tilde{k} \right\|_{L^2(0,\tau)} \|\Delta \varphi\|_{L^2(\partial\Omega)}
$$

$$
+ 2 \left( \|u_0\|_{L^\infty(\partial\Omega)} + \int_0^\tau \|\tilde{v}(s)\|_{L^\infty(\partial\Omega)} \, ds \right) \left\| \nabla \varphi\|_{L^2(\partial\Omega)} \right\| \tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))}.
$$

(4.4)

Using the Sobolev embedding theorem, we deduce that

$$
\|u_0\|_{L^\infty(\partial\Omega)} + \int_0^\tau \|\tilde{v}(s)\|_{L^\infty(\partial\Omega)} \, ds \leq C \left( \|u_0\|_{H^2(\partial\Omega)} + \int_0^\tau \|\tilde{v}(s)\|_{H^2(\partial\Omega)} \, ds \right)
$$

$$
\leq C \left( \|u_0\|_{H^2(\partial\Omega)} + \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau; H^2(\partial\Omega))} \right).
$$

(4.5)

Applying lemma 2.3, we get

$$
\|\tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))} \leq \|\tilde{v} - v_0\|_{L^2(0,\tau; L^2(\partial\Omega))} + \|v_0\|_{L^2(0,\tau; L^2(\partial\Omega))}
$$

$$
\leq \tau \|\partial_t (\tilde{v} - v_0)\|_{L^2(0,\tau; L^2(\partial\Omega))} + \tau^{1/2} \|v_0\|_{L^2(\partial\Omega)}
$$

$$
\leq \tau \|\partial_t \tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))} + \tau^{1/2} \|v_0\|_{L^2(\partial\Omega)}.
$$

(4.6)

Using (4.5) and (4.6) in (4.4), and then by the definition of the space $\mathcal{V}(\tau, L)$, we arrive at

$$
\|k\|_{L^2(0,\tau)} \leq \alpha \|\varphi''\|_{L^2(0,\tau)} + \alpha \left\{ \mu_0 \|\Delta \varphi\|_{L^2(\partial\Omega)} + \tau^{1/2} \left\| \tilde{k} \right\|_{L^2(0,\tau)} \|\Delta \varphi\|_{L^2(\partial\Omega)}
$$

$$
+ C \left( \|u_0\|_{H^2(\partial\Omega)} + \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau; H^2(\partial\Omega))} \right) \|\nabla \varphi\|_{L^2(\partial\Omega)}
$$

$$
\times \left( \tau \|\partial_t \tilde{v}\|_{L^2(0,\tau; L^2(\partial\Omega))} + \tau^{1/2} \|v_0\|_{L^2(\partial\Omega)} \right).
$$
\[ \leq \alpha \|\nabla u\|_{L^2(0,\tau)} + \alpha \left\{ \mu \| \Delta v\|_{L^2(\Omega)} + \tau^{1/2} L \| \Delta v\|_{L^2(\Omega)} \right\} \]

\[ + C \left( \| u_0 \|_{H^2(\Omega)} + \tau^{1/2} L \right) \left\| \nabla v\|_{L^2(\Omega)} \right\| \tau^{1/2} \left( \tau^{1/2} L + \| v_0 \|_{L^2(\Omega)} \right). \]  

(4.7)

Using the energy estimates for the linear problem (4.2), we estimate

\[ \left\| v\right\|_{H^1(0,\tau; \Delta u; \Omega)} \leq C \left( \| v_0 \|_{H^2(\Omega)} + \| G\|_{L^2(0,\tau; \Delta u; \Omega)} \right), \]

where

\[ G := k \Delta u_0 + \int_0^\tau k(t-s) \Delta \bar{v}(s) ds - \left( \bar{v} \cdot \nabla \right) \left( u_0 + \int_0^\tau \bar{v}(s) ds \right) \]

\[ = G_1 + G_2 + G_3 + G_4 \]

(4.9)

Next, we estimate each \( G_i \) \( (i = 1, 2, 3, 4) \) separately as follows:

\[ \left\| G_1 \right\|_{L^2(0,\tau; L^2(\Omega))} = \| k \|_{L^1(0,\tau)} \| \Delta u_0 \|_{L^2(\Omega)} \leq C \| k \|_{L^1(0,\tau)} \| u_0 \|_{H^2(\Omega)}. \]

(4.10)

Applying lemma 2.2, we obtain

\[ \left\| G_2 \right\|_{L^2(0,\tau; L^2(\Omega))} \leq \| k \|_{L^1(0,\tau)} \| \Delta \bar{v}\|_{L^2(0,\tau; L^2(\Omega))} \]

\[ \leq C \tau^{1/2} \| k \|_{L^1(0,\tau)} \| \bar{v}\|_{L^2(0,\tau; L^2(\Omega))}. \]

(4.11)

In order to estimate \( G_3 \), we split it into two parts

\[ G_3 = (\bar{v} \cdot \nabla) u_0 + (\bar{v} \cdot \nabla) \left( \int_0^\tau \bar{v}(s) ds \right) =: G_{31} + G_{32}. \]

(4.12)

Applying H"older’s inequality, Sobolev embedding theorem and Gagliardo–Nirenberg’s inequality, we have

\[ \left\| G_{31} \right\|_{L^2(0,\tau; L^2(\Omega))} = \left( \int_0^\tau \left( \| \bar{v} \cdot \nabla u_0 \|_{L^2(\Omega)} \right)^2 ds \right)^{1/2} \]

\[ \leq \| \nabla u_0 \|_{L^2(\Omega)} \left( \int_0^\tau \| \bar{v}(s) \|_{L^2(\Omega)}^2 ds \right)^{1/2} \]

\[ \leq C \| u_0 \|_{H^2(\Omega)} \| \bar{v}\|_{L^2(0,\tau; L^2(\Omega))}^{1/4} \| \bar{v}\|_{L^2(0,\tau; L^2(\Omega))}^{3/4}. \]

(4.13)

Similarly for \( G_{32} \), we find
we obtain the following estimate
\[ \|G_2\|_{L^2(0,\tau;L^2(\Omega))} \leq \|\tilde{v}\|_{L^2(\Omega)} \left( \int_0^\tau \left\| \nabla \tilde{v}(s) \right\|_{L^2(\Omega)} \, ds \right) \]
\[ \leq \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \left( \int_0^\tau \left\| \nabla \tilde{v}(s) \right\|_{L^2(\Omega)} \, ds \right) \]
\[ \leq C \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \int_0^\tau \|\tilde{v}(s)\|_{H^1(\Omega)} \, ds \]
\[ \leq C \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \]  \hspace{1cm} (4.14)

Finally, we estimate \( G_4 \) as
\[ \|G_4\|_{L^2(0,\tau;L^2(\Omega))} \leq \left\| u_0 + \int_0^\tau \tilde{v}(s) \, ds \right\|_{L^2(0,\tau;L^2(\Omega))} \]
\[ \leq \left\| u_0 + \int_0^\tau \tilde{v}(s) \, ds \right\|_{L^2(0,\tau;L^2(\Omega))} \]
\[ \leq C \left( \|u_0\|_{H^2(\Omega)} + \int_0^\tau \|\tilde{v}(s)\|_{H^2(\Omega)} \, ds \right) \|\nabla \tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \]
\[ \leq C \left( \|u_0\|_{H^2(\Omega)} + \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \right) \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \]  \hspace{1cm} (4.15)

Combining (4.9)–(4.15) and making use of (4.6), and the definition of the space \( \mathbb{V}(\tau, L) \), we obtain the following estimate
\[ \|G\|_{L^2(0,\tau;L^2(\Omega))} \]
\[ \leq C \left\{ \left( \|u_0\|_{H^2(\Omega)} + \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \right) \|\tilde{k}\|_{L^2(\Omega)} \right. \]
\[ + \|u_0\|_{H^2(\Omega)} \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \right. \]
\[ + \left. \left( \|u_0\|_{H^2(\Omega)} + \tau^{1/2} \|\tilde{v}\|_{L^2(0,\tau;\mathbb{H}_0^2(\Omega))} \right) \|\tilde{v}\|_{L^2(0,\tau;L^2(\Omega))} \right\} \]
\[ \leq C \left\{ \left( \|u_0\|_{H^2(\Omega)} + \tau^{1/2} \right) \|\tilde{k}\|_{L^2(\Omega)} \right. \]
\[ + \left. \left( \tau^{3/8} L^{1/4} \|u_0\|_{H^2(\Omega)} + \tau^{3/8} L^{5/4} \right) \left( \|u_0\|_{L^2(\Omega)} + \tau^{1/2} \right) \right\} \]  \hspace{1cm} (4.16)

If we take \( \tau > 0 \) small enough such that
\[ \tau(1 + L + L^2) \leq 1, \quad (4.17) \]

estimates (4.7)–(4.16), transform to
\[
\|v\|_{H^1_0(\Omega)\cap H^2(O)} + \|k\|_{L^2(\Omega)} \\
\leq C \left( \|v_0\|_{H^1_0(\Omega)\cap H^2(O)} + \|G\|_{L^2(\Omega\cap \Omega)} + \|k\|_{L^2(\Omega)} \right) \\
\leq C \left( \|v_0\|_{H^1_0(\Omega)\cap H^2(O)} + (\|u_0\|_{H^2(\Omega)} + 1) \|k\|_{L^2(\Omega)} + (\|u_0\|_{H^2(\Omega)} + 1) \left( \|v_0\|_{L^2(\Omega)} + 1 \right)^{3/4} \right. \\
+ \left( \|u_0\|_{H^2(\Omega)} + 1 \right) \left( \|v_0\|_{L^2(\Omega)} + 1 \right)^{1/2} \right) \\
\leq C \left( \|v_0\|_{H^1_0(\Omega)\cap H^2(O)} + \alpha (\|u_0\|_{H^2(\Omega)} + 1) \right. \\
\times \left\{ \|v''\|_{L^2(\Omega)} + [\mu_0 \|\Delta \varphi\|_{L^2(\Omega)} + \|\Delta \varphi\|_{L^2(\Omega)} \right) \\
\left. + \left( \|u_0\|_{H^2(\Omega)} + 1 \right) \|\nabla \varphi\|_{L^2(\Omega)} \right\} (\|v_0\|_{L^2(\Omega)} + 1) \right) \\
+ \left( \|u_0\|_{H^2(\Omega)} + 1 \right) \left( \|v_0\|_{L^2(\Omega)} + 1 \right)^{3/4} + \left( \|u_0\|_{H^2(\Omega)} + 1 \right) \left( \|v_0\|_{L^2(\Omega)} + 1 \right)^{1/2} \right] =: L_0.
\]

We notice that \( L_0 = L_0(\tau) \) is non-decreasing. Therefore, we can define a mapping \( \Psi : \mathcal{V}(\tau, L) \rightarrow \mathcal{V}(\tau, L) \) by choosing \( L \geq L_0 \) and \( \tau \) small enough as in (4.17).

**Step 2.** In this step, we prove that \( \Psi \) is a contraction mapping. For \( j \in \{1, 2\} \), let \((\tilde{v}_j, \tilde{k}_j) \in \mathcal{V}(\tau, L)\) and define \( k_j \) and \((v_j, p_j)\) by (4.1) and (4.2) with \((\tilde{v}, \tilde{k}) = (\tilde{v}_j, \tilde{k}_j)\), respectively. Then, \((v_1, p_1, k_1)\) and \((v_2, p_2, k_2)\) satisfies

\[
\begin{aligned}
&\partial_t(v_1 - v_2) - \mu_1 \partial \Delta (v_1 - v_2) - \mu_0 \Delta (v_1 - v_2) - (k_1 - k_2) \Delta u_0 \\
&- \int_0^t (k_1 - k_2)(t - s) \Delta \tilde{v}_1(s) ds - \int_0^t k_2(t - s) \Delta (\tilde{v}_1 - \tilde{v}_2)(s) ds \\
&+ ((\tilde{v}_1 - \tilde{v}_2) \cdot \nabla) \left[ u_0 + \int_0^t \tilde{v}_1(s) ds \right] + (\tilde{v}_2 \cdot \nabla) \left( \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) ds \right) \\
&+ \left( \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) ds \cdot \nabla \right) \tilde{v}_1 + \left[ \left( u_0 + \int_0^t \tilde{v}_2(s) ds \right) \cdot \nabla \right] (\tilde{v}_1 - \tilde{v}_2) \\
&+ \nabla \partial_t (p_1 - p_2) = 0, \quad \text{in } \Omega \times (0, \tau), \\
&\nabla \cdot (v_1 - v_2) = 0, \quad \text{in } \Omega \times (0, \tau), \\
&v_1 - v_2 = 0, \quad \text{on } \partial \Omega \times [0, \tau), \\
&v_1 - v_2 = 0, \quad \text{in } \Omega \times \{0\},
\end{aligned}
\]

and

\[ (4.18) \]
\[
(k_1 - k_2)(t) = -\alpha \left\{ \mu_0 \int_0^t (\tilde{v}_1 - \tilde{v}_2) \cdot \Delta \varphi \, dx \, ds + \int_0^t (\tilde{k}_1 - \tilde{k}_2)(t-s) \tilde{v}_1 \cdot \Delta \varphi \, dx \, ds \right. \\
+ \int_0^t \int_\Omega \tilde{k}_2(t-s) (\tilde{v}_1 - \tilde{v}_2) \cdot \Delta \varphi \, dx \, ds + \left. \int_\Omega [(\tilde{v}_1 - \tilde{v}_2) \cdot \nabla \varphi] \cdot (u_0 + \int_0^t \tilde{v}_1 \, ds) \, dx \right. \\
+ \int_\Omega [(\tilde{v}_2 \cdot \nabla) \varphi] \cdot \left( \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) \, ds \right) \, dx + \int_\Omega \left[ \left( \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) \, ds \cdot \nabla \right) \varphi \right] \cdot \tilde{v}_1 \, dx \\
+ \int_\Omega \left[ \left( u_0 + \int_0^t \tilde{v}_2(s) \, ds \right) \cdot \nabla \varphi \right] \cdot (\tilde{v}_1 - \tilde{v}_2) \, dx \right) .
\tag{4.19}
\]

Let us now estimate \((k_1 - k_2)(\cdot)\) from the equation (4.19). For this, our aim is to bound each individual term of \((k_1 - k_2)(\cdot)\), that is, we estimate \(B_i, i = 1, \ldots, 7\). Applying Hölder’s inequality, we have

\[
\|B_1\|_{L^2(0, T)} \leq \mu_0 \left( \int_0^T \| (\tilde{v}_1 - \tilde{v}_2)(s) \|_{L^2(\Omega)}^2 \| \Delta \varphi \|_{L^2(\Omega)}^2 \, ds \right)^{1/2} \\
= \mu_0 \| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(0, T; L^2(\Omega))} \| \Delta \varphi \|_{L^2(\Omega)}.
\]

Using lemma 2.2, we obtain

\[
\|B_2\|_{L^2(0, T)} \leq \| \tilde{k}_1 - \tilde{k}_2 \|_{L^1(0, T)} \left\| \int_\Omega \tilde{v}_1 \cdot \Delta \varphi \, dx \right\|_{L^2(0, T)} \\
\leq \tau^{1/2} \| \tilde{k}_1 - \tilde{k}_2 \|_{L^2(0, T)} \| \tilde{v}_1 \|_{L^2(0, T; L^2(\Omega))} \| \Delta \varphi \|_{L^2(\Omega)},
\]

and

\[
\|B_3\|_{L^2(0, T)} \leq \tau^{1/2} \| \tilde{k}_1 - \tilde{k}_2 \|_{L^2(0, T)} \| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(0, T; L^2(\Omega))} \| \Delta \varphi \|_{L^2(\Omega)}.
\]

Using Hölder’s inequality and Sobolev embedding theorem, we get

\[
\|B_4\|_{L^2(0, T)} \leq \left\| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(\Omega)} \| \nabla \varphi \|_{L^2(\Omega)} \right\| u_0 + \int_0^T \tilde{v}_1(s) \, ds \right\|_{L^\infty(\Omega)} \| \tilde{v}_1 \|_{L^2(0, T)} \\
\leq \| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(0, T; L^2(\Omega))} \| \nabla \varphi \|_{L^2(\Omega)} \left\| u_0 + \int_0^T \tilde{v}_1(s) \, ds \right\|_{L^\infty(0, T; L^\infty(\Omega))} \\
\leq C \left( \| u_0 \|_{H^2(\Omega)} + \int_0^T \| \tilde{v}_1(s) \|_{H^2(\Omega)} \, ds \right) \\
\| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(0, T; L^2(\Omega))} \| \nabla \varphi \|_{L^2(\Omega)} \\
\leq C \left( \| u_0 \|_{H^2(\Omega)} + \tau^{1/2} \| \tilde{v}_1 \|_{L^2(0, T; H^2(\Omega))} \right) \\
\| \tilde{v}_1 - \tilde{v}_2 \|_{L^2(0, T; L^2(\Omega))} \| \nabla \varphi \|_{L^2(\Omega)}
\]

\[14\]
and
\[ \|B_5\|_{L^2(0,\tau)} \leq \left\| \int_0^\tau (\tilde{v}_1 - \tilde{v}_2)(s) ds \right\|_{L^2(0,\tau)} \]
\[ \leq \left\| \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \int_0^\tau (\tilde{v}_1 - \tilde{v}_2)(s) ds \right\|_{L^2(0,\tau, L^2(0,\tau))} \]
\[ \leq \left( \int_0^\tau \| (\tilde{v}_1 - \tilde{v}_2)(s) \|_{L^2(0,\tau)} ds \right) \left\| \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \]
\[ \leq C \tau^{1/2} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))}. \]

Applying similar arguments as in the estimate of \( B_5 \), we deduce
\[ \|B_6\|_{L^2(0,\tau)} \leq \left\| \int_0^\tau (\tilde{v}_1 - \tilde{v}_2)(s) ds \right\|_{L^2(0,\tau)} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 \right\|_{L^2(0,\tau)} \]
\[ \leq C \tau^{1/2} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 \right\|_{L^2(0,\tau)}. \]

Similar arguments as in the estimate for \( B_4 \) yield
\[ \|B_7\|_{L^2(0,\tau)} \leq C \left( \|u_0\|_{H^1(0,\tau)} + \tau^{1/2} \|\tilde{v}_2\|_{L^2(0,\tau, L^2(0,\tau))} \right) \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))}. \]

Combining the estimates for \( B_4 \), one can conclude that
\[ \|k_1 - k_2\|_{L^2(0,\tau)} \leq \alpha \left\{ \|u_0\| \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} + \tau^{1/2} \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \right\} \]
\[ \times \left\{ \|u_0\| \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} + \tau^{1/2} \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \right\}
\[ + C \left( \|u_0\|_{H^1(0,\tau)} + \tau^{1/2} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \right) \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \]
\[ + \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \].

Using lemma 2.3 and the definition of the space \( \mathcal{V}(\tau, L) \), we obtain the following estimate
\[ \|k_1 - k_2\|_{L^2(0,\tau)} \leq \alpha \left\{ \|u_0\| \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} + \tau^{1/2} L \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{k}_1 - \tilde{k}_2 \right\|_{L^2(0,\tau)} \right\}
\[ \left\{ \|u_0\| \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} + \tau^{1/2} L \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \right\}
\[ + C \left( \|u_0\|_{H^1(0,\tau)} + \tau^{1/2} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \right) \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \]
\[ + C \tau^{1/2} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} \right\}
\[ \leq \alpha \tau^{1/2} \left\{ \|u_0\| \left\| \Delta \varphi \right\|_{L^2(0,\tau)} + C \left( \tau^{1/2} \left\| \nabla \varphi \right\|_{L^2(0,\tau)} + \tau L + L \right) \left\| \nabla \varphi \right\|_{L^2(0,\tau)} \right\}
\[ \times \left\| \tilde{v}_1 - \tilde{v}_2 \right\|_{L^2(0,\tau, L^2(0,\tau))} + \alpha \tau^{1/2} L \left\| \Delta \varphi \right\|_{L^2(0,\tau)} \left\| \tilde{k}_1 - \tilde{k}_2 \right\|_{L^2(0,\tau)} \right\}. \]
Again, using the estimates for the linear problem (4.18), we estimate

$$
\|v_1 - v_2\|_{H^1(0,T;L^2(\Omega))} \leq C \|E\|_{L^2(0,T;L^2(\Omega))},
$$

(4.21)

where

$$
E := (k_1 - k_2) \Delta u_0 + \int_0^t (k_1 - k_2)(t-s) \Delta \tilde{v}_1(s) ds \int_0^t k_2(t-s) \Delta (\tilde{v}_1 - \tilde{v}_2)(s) ds
$$

\begin{align*}
&= \left[ ((\tilde{v}_1 - \tilde{v}_2) \cdot \nabla) u_0 + \int_0^t (\tilde{v}_1(s) ds) - (\tilde{v}_2 \cdot \nabla) \left( \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) ds \right) \\
&= \int_0^t (\tilde{v}_1 - \tilde{v}_2)(s) ds \cdot \nabla \tilde{v}_1 - \left[ u_0 + \int_0^t \tilde{v}_2(s) ds \right] \cdot \nabla (\tilde{v}_1 - \tilde{v}_2) \right].
\end{align*}

For $E_j (j = 1, 2, \ldots, 7)$, we obtain the following estimates:

$$
\|E_j\|_{L^2(0,T;L^2(\Omega))} \leq C \|k_1 - k_2\|_{L^1(0,T)} \|u_0\|_{L^2(\Omega)}.
$$

Using lemma 2.2, we get

$$
\|E_2\|_{L^2(0,T;L^2(\Omega))} \leq C \tau^{1/2} \|k_1 - k_2\|_{L^1(0,T)} \|\tilde{v}_1\|_{L^2(0,T;H^2(\Omega))},
$$

and

$$
\|E_3\|_{L^2(0,T;L^2(\Omega))} \leq C \tau^{1/2} \|k_2\|_{L^1(0,T)} \|\tilde{v}_1 - \tilde{v}_2\|_{L^2(0,T;H^2(\Omega))}.
$$

By an argument similar to the estimate of $\|G_3\|_{L^2(0,T;L^2(\Omega))}$, we obtain

$$
E_4 = ((\tilde{v}_1 - \tilde{v}_2) \cdot \nabla) u_0 + ((\tilde{v}_1 - \tilde{v}_2) \cdot \nabla) \int_0^t \tilde{v}_1(s) ds,
$$

where we bound $E_{41}$ and $E_{42}$ as

$$
\|E_{41}\|_{L^2(0,T;L^2(\Omega))} \leq C \|u_0\|_{H^2(\Omega)} \|\tilde{v}_1 - \tilde{v}_2\|^{1/4}_{L^2(0,T;H^2(\Omega))} \|\tilde{v}_1 - \tilde{v}_2\|^{3/4}_{L^2(0,T;L^2(\Omega))},
$$

$$
\|E_{42}\|_{L^2(0,T;L^2(\Omega))} \leq \left\| \|\tilde{v}_1 - \tilde{v}_2\|_{L^1(\Omega)} \left\| \int_0^t \nabla \tilde{v}_1(s) ds \right\|_{L^1(\Omega)} \right\|_{L^2(0,T)}
$$

\begin{align*}
&\leq \|\tilde{v}_1 - \tilde{v}_2\|_{L^2(0,T;L^1(\Omega))} \left\| \int_0^t \nabla \tilde{v}_1(s) ds \right\|_{L^2(0,T;L^2(\Omega))}
\leq C \|\tilde{v}_1 - \tilde{v}_2\|^{1/4}_{L^2(0,T;H^1(\Omega))} \|\tilde{v}_1 - \tilde{v}_2\|^{3/4}_{L^2(0,T;L^2(\Omega))} \int_0^t \|\tilde{v}_1(s)\|_{H^1(\Omega)} ds
\leq C \tau^{1/2} \|\tilde{v}_1\|_{L^2(0,T;H^2(\Omega))} \|\tilde{v}_1 - \tilde{v}_2\|^{1/4}_{L^2(0,T;H^1(\Omega))} \|\tilde{v}_1 - \tilde{v}_2\|^{3/4}_{L^2(0,T;L^2(\Omega))}.
\end{align*}
Also, we have
\[
\|E_5\|_{L^2(0,\tau;L^2(O))} \leq C \left( \|\tilde{u}_1 - \tilde{u}_2\|_{H^2(O)} \right) \|\tilde{v}_1 - \tilde{v}_2\|_{H^1(O)} \|\tilde{v}_1\|_{H^1(O)}
\]
\[
\leq C \left( \|\tilde{u}_1 - \tilde{u}_2\|_{H^2(O)} \right) \|\tilde{v}_1 - \tilde{v}_2\|_{H^1(O)} \|\tilde{v}_1\|_{H^1(O)}
\]
where we have used Hölder’s inequality and Sobolev embedding theorem. Similarly, we find
\[
\|E_6\|_{L^2(0,\tau;L^2(O))} \leq C \left( \|\tilde{u}_1 - \tilde{u}_2\|_{H^2(O)} \right) \|\tilde{v}_1 - \tilde{v}_2\|_{H^1(O)} \|\tilde{v}_1\|_{H^1(O)}
\]
Using Hölder’s inequality, Sobolev embedding theorem and Gagliardo–Nirenberg’s inequality, we get
\[
\|E_7\|_{L^2(0,\tau;L^2(O))} \leq C \left( \|\tilde{u}_0\|_{H^2(O)} + \|\tilde{v}_1\|_{H^2(O)} \right) \|\tilde{v}_1 - \tilde{v}_2\|_{L^2(0,\tau;L^2(O))}
\]
\[
\leq C \left( \|\tilde{u}_0\|_{H^2(O)} + \|\tilde{v}_1\|_{H^2(O)} \right) \|\tilde{v}_1 - \tilde{v}_2\|_{L^2(0,\tau;L^2(O))}
\]
Combining the estimates for $E_7$, one can conclude that
\[
\|E\|_{L^2(0,\tau;L^2(O))} \leq C \left( \|u_0\|_{H^2(O)} + \|v_1\|_{H^2(O)} \right) \|v_1 - v_2\|_{L^2(0,\tau;L^2(O))}
\]
Thus, combining (4.20)–(4.22), we finally arrive at

\[
\|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(0,T;H^2(O))} + |k_1 - k_2|_{L^2(0,T)} \\
\leq C\left(\|E\|_{L^1(0,T;L^2(O))} + |k_1 - k_2|_{L^2(0,T)} + \tau^{1/2}L + 1\right)\|k_1 - k_2\|_{L^2(0,T)} \\
+ \tau^{1/2}\left[3L + \left(\|u_0\|_{H^2(O)} + \tau^{1/2}L\right)(\tau^{1/4} + 1)\right] \times \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(0,T;H^2(O))} \\
\leq C\tau^{1/2}\left(\|u_0\|_{H^2(O)} + \tau^{1/2}L + 1\right) + \alpha \left(\mu_0^{1/2} + \tau L\right)\|\Delta \varphi\|_{L^2(O)} \\
+ \left(\tau^{1/2}|u_0|_{H^2(O)} + \tau L + L\right)\|\nabla \varphi\|_{L^2(O)} \\
+ \left[3L + \left(\|u_0\|_{H^2(O)} + \tau^{1/2}L\right)(\tau^{1/4} + 1)\right] \times \|\mathbf{v}_1 - \mathbf{v}_2\|_{H^1(0,T;H^2(O))} \\
+ C\alpha\tau^{1/2}\left(\|u_0\|_{H^2(O)} + \tau^{1/2}L + 1\right) \times \Delta \varphi\|_{L^2(O)}\|\mathbf{k}_1 - \mathbf{k}_2\|_{L^2(0,T)}.
\]

If we take \(T > 0\) small enough, then we have \(\Psi: \mathcal{V}(\tau, L) \to \mathcal{V}(\tau, L)\) is a contraction mapping. Thus, from the contraction mapping theorem, one can conclude that for a small time \(T\), there exists a unique solution \((\mathbf{v}, k) \in H^1(0,T;H^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0,T)\) to the equivalent system (3.3a)–(3.3e) in \([0,T]\), and the proof is completed.

**Remark 4.2.** Due to the presence of convective (nonlinear) term \((\mathbf{u} \cdot \nabla)\mathbf{u}\) in the Kelvin–Voigt fluid equation (1.1), we are not able to prove the global existence and uniqueness of solutions. The authors in [4] obtained global solvability results for bounded nonlinearities.
only (cf lemma 8.3, [4]). Thus, one cannot apply the same techniques used in [4] to obtain the global solvability results. This problem will be addressed in a future work.

5. Oseen type equations

In order to make use of the techniques available in [4] for the global solvability results, we consider an inverse problem for the Oseen type equations corresponding to Kelvin–Voigt fluids with memory in this section. Let us first consider the following Oseen type equations:

\[
\partial_t u - \mu_1 \partial_t \Delta u - \mu_0 \Delta u + (u_\infty \cdot \nabla) u - \int_0^t k(t-s) \Delta u(s) \, ds + \nabla p = 0, \quad \text{in } O \times (0, T), \\
\nabla \cdot u = 0, \quad \text{in } O \times (0, T), \\
u = 0, \quad \text{on } \partial O \times [0, T), \\
u = u_0, \quad \text{in } O \times \{0\},
\]

where \(u_\infty\) is a divergence free vector field with \(u_\infty \in H^1_0(O) \cap H^2(O)\).

5.1. The inverse problem

For \(T > 0\), the inverse problem is to determine \(\tau \in (0, T]\),

\[
u \in H^2(0, \tau; H^1_0(O) \cap H^2(O)) \quad \text{and} \quad k \in L^2(0, \tau),
\]

such that \((\nu, k)\) satisfies the system:

\[
\partial_t \nu - \mu_1 \partial_t \Delta \nu - \mu_0 \Delta \nu + (u_\infty \cdot \nabla) \nu - \int_0^t k(t-s) \Delta \nu(s) \, ds + \nabla p = 0, \quad \text{in } O \times (0, \tau), \\
\nabla \cdot \nu = 0, \quad \text{in } O \times (0, \tau), \\
\nu = 0, \quad \text{on } \partial O \times [0, \tau), \\
\nu = u_0, \quad \text{in } O \times \{0\},
\]

\[
\int_O (I - \mu_1 \Delta) \varphi(x) \cdot \nu(x, t) \, dx = r(t), \quad \text{in } (0, \tau).
\]

We solve the inverse problem under the following assumptions on the data:

(H1) \(u_0 \in H^1_0(O) \cap H^2(O), \quad \nabla \cdot u_0 = 0, \quad \text{in } O\).

(H2) \(\varphi \in H^1_0(O) \cap H^2(O), \quad \nabla \cdot \varphi = 0, \quad \text{in } O\).

(H3) \(\alpha^{-1} := \int_O \varphi \cdot \Delta u_0 \, dx \neq 0\).

(H4) \(v_0 := (I - \mu_1 \Delta)^{-1} (\mu_0 \Delta u_0 - (u_\infty \cdot \nabla) u_0 - \nabla p_0) \in H^1_0(O), \quad \nabla \cdot v_0 = 0, \quad \text{in } O, p_0 \in H^1(O)\).

(H5) \(r \in H^2(0, T), \quad \text{with}
\]

\[
\int_O (I - \mu_1 \Delta) \varphi(x) \cdot u_0(x) \, dx = r(0),
\]

\[
\int_O (\mu_0 \Delta u_0(x) - (u_\infty(x) \cdot \nabla) u_0(x)) \cdot \varphi(x) \, dx = r'(0).
\]
The following theorem provides an equivalent form of the system (5.6a)–(5.6e).

**Theorem 5.1.** Let the assumptions (H1)–(H5) hold. Let \((u, k)\) be a solution of the system (5.6a)–(5.6e) up to \(T\) such that
\[
\begin{align*}
u & \in H^2(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{and} \quad k \in L^2(0, T). \tag{5.7}
\end{align*}
\]
Then \(v := \partial_t u\) and \(k\) verify the conditions
\[
\begin{align*}
u & \in H^1(0, T; H_0^1(\Omega) \cap H^2(\Omega)) \quad \text{and} \quad k \in L^2(0, T), \tag{5.8}
\end{align*}
\]
and solve the system
\[
\begin{align*}
\partial_t v - \mu_1 \partial_t \Delta v - \mu_0 \Delta u_0 - \int_0^t k(t-s) \Delta v(s) ds + (u_\infty \cdot \nabla) v \\
+ \nabla \partial_t p & = 0, \quad \text{in } \Omega \times (0, T), \tag{5.9a}
\end{align*}
\]
\[
\begin{align*}
\nabla \cdot v & = 0, \quad \text{in } \Omega \times (0, T), \tag{5.9b}
\end{align*}
\]
\[
\begin{align*}
v & = 0, \quad \text{on } \partial \Omega \times (0, T), \tag{5.9c}
\end{align*}
\]
\[
\begin{align*}
v & = v_0, \quad \text{in } \Omega \times \{0\}. \tag{5.9d}
\end{align*}
\]

with
\[
k(t) = \alpha \left\{ r'(t) - \mu_0 \int_\Omega v \cdot \Delta \varphi \, dx - \int_\Omega ((u_\infty \cdot \nabla) \varphi) \cdot v \, dx - \int_0^t \int_\Omega k(t-s) v \cdot \Delta \varphi \, dx \, ds \right\}.
\]

On the other hand, under the setting \(u(t) := u_0 + \int_0^t v(s) ds\), if \((v, k)\) satisfies (5.8) and is a solution to the system (5.9a)–(5.9e), then considering to the above setting, \((u, k)\) satisfies (5.7) and is a solution to the system (5.6a)–(5.6e).

**Proof.** The proof constitutes similar arguments as given in the proof of theorem 3.1. \(\square\)

The following theorem provided similar results for the system (5.6a)–(5.6e), as we have proved in theorem 4.1 for the system (2.2a)–(2.2e).

**Theorem 5.2 (Local in time existence).** Let the assumptions (H1)–(H5) hold. Then there exists \(\tau \in (0, T]\), such that the inverse problem (5.1) has a unique solution
\[
(u, k) \in H^2(0, \tau; H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(0, \tau).
\]

**Proof.** Let us define the space
\[
V(\tau, M) := \left\{ (\tilde{v}, \tilde{k}) \in H^1(0, \tau; H_0^1(\Omega) \cap H^2(\Omega)) \times L^2(0, \tau): \right.
\]
\[
\begin{align*}
\nabla \cdot \tilde{v} & = 0, \quad \text{in } \Omega \times (0, \tau), \tilde{v} = 0, \quad \text{on } \partial \Omega \times [0, \tau), \\
\tilde{v} & = v_0, \quad \text{in } \Omega \times \{0\} \quad \text{and} \\
\|\tilde{v}\|_{H^1(0, \tau; H_0^1(\Omega) \cap H^2(\Omega))} + \|\tilde{k}\|_{L^2(0, \tau)} & \leq M \right\},
\]

where $M$ is a positive constant, which will be determined later. We also define the mapping $\Gamma: \mathcal{V}(\tau, M) \to \mathcal{V}(\tau, M)$ such that $(\hat{v}, \hat{k}) \mapsto (v, k)$ through

$$k(t) := \alpha \left\{ r'' - \mu_0 \int_\mathcal{O} \hat{v} \cdot \Delta \varphi \, dx - \int_0^t \int_\mathcal{O} (k(t-s) \hat{v} \cdot \Delta \varphi) \, dx \, ds - \int_\mathcal{O} ((u_\infty \cdot \nabla) \varphi) \cdot \hat{v} \, dx \right\}.$$  

(5.10)

and the initial boundary value problem

$$\begin{aligned}
\partial_t v - \mu_1 \partial_t \Delta v - \mu_0 \Delta v - k \Delta u_0 + (u_\infty \cdot \nabla) \hat{v} - \int_0^t (k(t-s) \Delta \hat{v} \, ds) + \nabla \partial_t p &= 0, \quad \text{in } \mathcal{O} \times (0, \tau), \\
\nabla \cdot v &= 0, \quad \text{in } \mathcal{O} \times (0, \tau), \\
v &= 0, \quad \text{on } \partial \mathcal{O} \times [0, \tau), \\
v &= v_0, \quad \text{in } \mathcal{O} \times \{0\}.
\end{aligned}$$  

(5.11)

In order to complete the proof of theorem 5.2, we need to show that the mapping $\Gamma: \mathcal{V}(\tau, M) \to \mathcal{V}(\tau, M)$ is a contraction map. The proof constitutes similar arguments as given in the proof of theorem 4.1.

$\square$

Next, we prove the global uniqueness result for our inverse problem (5.1). Let $0 < \tau < \tau' \leq (2\tau) \wedge T$ and let $(\hat{v}, \hat{k})$ be the solution of the system (5.9a)–(5.9e) obtained in $[0, \tau]$ from theorem 5.2 for small $\tau > 0$. We will prove that $(\hat{v}, \hat{k})$ is extensible to a solution $(v, k) \in H^1(0, \tau'; \mathbb{H}^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O})) \times L^2(0, \tau')$ in $[0, \tau']$. We start with the following lemma which is very useful in the proof of theorem 5.4.

**Lemma 5.3.** Let the assumptions (H1)–(H5) hold. Let $0 < \tau < \tau' \leq (2\tau) \wedge T$ and let $(u, k)$ be a solution of the system (5.6a)–(5.6e) in $[0, \tau']$, with the regularity

$$u \in H^2(0, \tau'; \mathbb{H}^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O})), \quad k \in L^2(0, \tau').$$

We set

(F1) $v := \partial_t u,$

(F2) $\hat{v} := v|_{[0, \tau]},$

(F3) $v_r(t) := v(\tau + t), \; t \in [0, \tau' - \tau],$

(F4) $k := k|_{[0, \tau]},$

(F5) $k_r(t) := k(\tau + t), \; t \in [0, \tau' - \tau],$

(F6) $h(t) := - \int_0^t k(\tau + t - s) \Delta \hat{v}(s) \, ds + \nabla \partial_t p(\tau + t), \; t \in [0, \tau' - \tau],$

(F7) $u_r := u(\tau),$

(F8) $r_r(t) := r(\tau + t), \; t \in [0, \tau' - \tau].$

Then we have the following obvious conditions:

(I) $v_r \in H^1(0, \tau' - \tau; \mathbb{H}^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O})),$

(II) $k_r \in L^2(0, \tau' - \tau),$

(III) $u_r \in \mathbb{H}^1(\mathcal{O}) \cap \mathbb{H}^2(\mathcal{O}).$
(IV) \( r_\tau \in H^2(0, \tau' - \tau), \)
(V) \( h \in L^2(0, \tau' - \tau; L^2(\Omega)), \)
(VI) \((v_\tau, k_\tau)\) solves the system:

\[
\begin{aligned}
\partial_t v_\tau - \mu_1 \partial_t \Delta v_\tau - \mu_0 \Delta v_\tau - k_\tau \Delta u_0 - \int_0^t k_\tau (t-s) \Delta \hat{v}(s) ds \\
- \int_0^t \tilde{k}(t-s) \Delta v_\tau(s) ds + (u_\infty \cdot \nabla) v_\tau + h(t) = 0, \quad \text{in } \Omega \times [0, \tau' - \tau], \\
\nabla \cdot v_\tau = 0, \quad \text{in } \Omega \times [0, \tau' - \tau], \\
v_\tau = 0, \quad \text{on } \partial \Omega \times [0, \tau' - \tau], \\
v_\tau = u_\tau, \quad \text{in } \Omega \times \{0\}, 
\end{aligned}
\]

(5.12)

and

\[
\begin{aligned}
k_\tau(t) = \alpha \left[ r_\tau''(t) - \left\{ \mu_0 \int_\Omega v_\tau \cdot \Delta \varphi dx + \int_0^t \int_\Omega \tilde{k}(t-s) \nabla \cdot \nabla \varphi dx ds \\
+ \int_0^t \int_\Omega \tilde{k}(t-s) \nabla \cdot \nabla \varphi dx ds + \int_\Omega ((u_\infty \cdot \nabla) \varphi) \cdot v_\tau dx \right\} \\
+ \int_0^t \int_\Omega \tilde{k}(t-s) \nabla \cdot \nabla \varphi dx ds \right], \quad t \in [0, \tau' - \tau].
\end{aligned}
\]

(5.13)

(VII) On the other hand, if \((\hat{v}, \hat{k}) \in H^1(0, \tau; H^1_0(\Omega) \cap H^2(\Omega)) \times L^2(0, \tau),\) solves the system (5.9a)–(5.9e) and \((v_\tau, k_\tau) \in H^1(0, \tau' - \tau; H^1_0(\Omega) \cap H^2(\Omega)) \times L^2(0, \tau' - \tau)\) solves the system (5.12) and (5.13). Set for \(t \in [0, \tau'],\)

\[
(v(t), k(t)) = \begin{cases} 
(\hat{v}(t), \hat{k}(t)), & \text{if } t \in [0, \tau], \\
(v(t-\tau), k_\tau(t-\tau)), & \text{if } t \in [\tau, \tau']. 
\end{cases}
\]

(5.14)

Then, \((v, k) \in H^1(0, \tau'; H^1_0(\Omega) \cap H^2(\Omega)) \times L^2(0, \tau')\) and solves the system (5.9a)–(5.9e) in \([0, \tau'].\)

**Proof.** The assertions (I)–(V) and (VII) can be easily verified. Let us now prove (f). First of all, from the system (5.9a)–(5.9e), we have

\[
\begin{aligned}
\partial_t v(\tau + t) - \mu_1 \partial_t \Delta v(\tau + t) - \mu_0 \Delta v(\tau + t) - k(\tau + t) \Delta u_0 - \int_0^{\tau+\tau} k(\tau + t-s) \Delta v(s) ds \\
+ (u_\infty \cdot \nabla) v(\tau + t) + \nabla \partial_t p(\tau + t) = 0, \quad \text{in } \Omega \times [0, \tau' - \tau], \\
\nabla \cdot v(\tau + t) = 0, \quad \text{in } \Omega \times [0, \tau' - \tau], \\
v(\tau + t) = 0, \quad \text{on } \partial \Omega \times [0, \tau' - \tau], \\
v(\tau) = u_\tau, \quad \text{in } \Omega \times \{0\},
\end{aligned}
\]

(5.15)
Then, if $\tau' < \tau$, for every $\tau$.

Theorem 5.4 (Global in time uniqueness).

We have proved in theorem 5.2 that for every $\tau$.

Next, as $\tau' - \tau \leq \tau$,

Substituting (5.17) in (5.15) and (5.16), we immediately get (5.12) and (5.13).

Theorem 5.4 (Global in time uniqueness). Let the assumptions (H1)–(H5) hold. Then, if $\tau \in (0, T]$, and the inverse problem (5.1) has two solutions

$$(u_j, k_j) \in H^2(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \times L^2(0, \tau), \quad j \in \{1, 2\},$$

then $(u_1, k_1) = (u_2, k_2)$.

Proof. Let $(u_j, k_j) \in H^2(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \times L^2(0, \tau), \quad j \in \{1, 2\}$, be solutions of the system (5.6a)–(5.6e), for some $\tau \in (0, T]$. We claim that $u_1 = u_2$ and $k_1 = k_2$. We set $v_j := \partial_t u_j, \quad j \in \{1, 2\}$. Then, from theorem 5.1, we have $(v_j, k_j) \in H^1(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \times L^2(0, \tau)$ and solves the system (5.9a)–(5.9e). It suffices to show that $v_1 = v_2$ and $k_1 = k_2$. Set $j \in \{1, 2\}$, $u_j(t) := u_0(t) + \int_0^t v_j(s)ds$, (as in theorem 5.1). We need to prove

$$\|v_1 - v_2\|_{H^1(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}))} + \|k_1 - k_2\|_{L^2(0, \tau)} = 0. \quad (5.18)$$

We have proved in theorem 5.2 that for every $M \in \mathbb{R}^+$, there exists $\tau(M) \in (0, T]$ such that for all $\tau \in (0, \tau(M)]$, system (5.9a)–(5.9e) has a unique solution

$$(v, k) \in H^1(0, \tau(M); H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \times L^2(0, \tau(M)),$$

such that

$$\|\hat{v}\|_{H^1(0, \tau(M); H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}))} + \|\hat{k}\|_{L^2(0, \tau(M))} \leq M.$$

This implies that, if $(v_j, k_j) \in H^1(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O})) \times L^2(0, \tau), \quad j \in \{1, 2\}$, are solutions of the system (5.9a)–(5.9e), then there exists $\tau_1 \in (0, \tau)$, such that $(v_1, k_1)$ and $(v_2, k_2)$ coincide on $[0, \tau_1]$. In fact, we can set

$$M_1 := \max \left\{\|\hat{v}\|_{H^1(0, \tau; H^1_0(\mathcal{O}) \cap H^2(\mathcal{O}))} + \|\hat{k}\|_{L^2(0, \tau)} : j \in \{1, 2\} \right\}$$
and obtain that \((v_1, k_1)\) and \((v_2, k_2)\) coincide in \([0, \tau(M_1) \land \tau]\). Let us define
\[
\tau_1 := \inf \left\{ t \in (0, \tau) : \|v_1 - v_2\|_{H^1(0, \tau; \mathbb{C})^J} + \|k_1 - k_2\|_{L^2(0, \tau)} > 0 \right\}. 
\] (5.19)

If (5.18) is not true, then it is obvious that \(\tau_1\) is well defined and we have \(\tau_1 \in (0, \tau)\). Set, for \(t \in [0, \tau - \tau_1], \ j \in \{1, 2\},\)
\[
v_j'(t) = v_j(\tau_1 + t), \quad k_j'(t) = k_j(\tau_1 + t).
\]

Keeping in mind that \(v_1 = v_2\) and \(k_1 = k_2\) a.e. if \(t \in [0, \tau_1]\), From the conditions (I)–(VI) in lemma 5.3, we obtain
\[
\begin{align*}
\partial_t(v_1^* - v_2^*) - \mu_1 \partial_s \Delta(v_1^* - v_2^*) - \mu_0 \Delta(v_1^* - v_2^*) - (k_1^* - k_2^*)\Delta u_0 \\
- \int_0^t (k_1^* - k_2^*)(t-s)\Delta v_1(s)ds - \int_0^t k_1(t-s)\Delta(v_1^* - v_2^*)(s)ds \\
+ (u_\infty \cdot \nabla)(v_1^* - v_2^*) + \nabla \partial_t (p_1 - p_2) = 0, \quad \text{in } \mathcal{O} \times [0, \tau_1 \land (\tau - \tau_1)], \\
\n\\n
\n\begin{align*}
\n\n\n(\nu_1^* - \nu_2^*) = 0, \quad \text{in } \mathcal{O} \times [0, \tau_1 \land (\tau - \tau_1)], \\
(\nu_1^* - \nu_2^*) = 0, \quad \text{in } \mathcal{O} \times \{0\},
\end{align*}
\]

and
\[
\begin{align*}
(k_1^* - k_2^*)(t) &= -\alpha \left\{ \mu_0 \int_{\mathcal{O}} (v_1^* - v_2^*) \cdot \Delta \varphi \, dx + \int_0^t \int_{\mathcal{O}} (k_1^* - k_2^*)(t-s)v_1 \cdot \Delta \varphi \, dx \, ds \\
+ \int_0^t \int_{\mathcal{O}} k_1(t-s)(v_1^* - v_2^*) \cdot \Delta \varphi \, dx \, ds + \int_{\mathcal{O}} (u_\infty \cdot \nabla) \varphi \cdot (v_1^* - v_2^*) \, dx. 
\end{align*}
\] (5.20)

Assume that \(\delta \in (0, \tau_1 \land (\tau - \tau_1)]\). Applying the same arguments as in theorem 4.1, we estimate \((k_1^* - k_2^*)\) from equation (5.21) as
\[
\|k_1^* - k_2^*\|_{L^2(0, \tau)} \leq \alpha \left\{ \mu_0 \|v_1^* - v_2^*\|_{L^2(0, \tau; \mathbb{C})^J} \|\Delta \varphi\|_{L^2(\mathcal{O})} \\
+ \delta^{1/2} \|k_1^* - k_2^*\|_{L^2(0, \tau)} \|\Delta \varphi\|_{L^2(\mathcal{O})} \|v_1\|_{L^2(0, \tau; \mathbb{C})^J} \\
+ \delta^{1/2} \|k_1\|_{L^2(0, \tau)} \|v_1^* - v_2^*\|_{L^2(0, \tau; \mathbb{C})^J} \|\Delta \varphi\|_{L^2(\mathcal{O})} \\
+ C \|u_\infty\|_{L^2(\mathcal{O})} \|\nabla \varphi\|_{L^2(\mathcal{O})} \|v_1^* - v_2^*\|_{L^2(0, \tau; \mathbb{C})^J} \right\} \\
\leq C(\delta) \|v_1^* - v_2^*\|_{L^2(0, \tau; \mathbb{C})^J} + C\delta^{1/2} \|k_1^* - k_2^*\|_{L^2(0, \tau)}.
\]

Choose \(\delta > 0\) such that \(C\delta^{1/2} \leq \frac{1}{4}\), we obtain
\[
\|k_1^* - k_2^*\|_{L^2(0, \tau)} \leq C(\delta) \|v_1^* - v_2^*\|_{L^2(0, \tau; \mathbb{C})^J}. 
\] (5.22)

Taking the inner product with \((v_1^* - v_2^*)(\cdot)\) to the first equation in (5.20) and then using integration by parts, we derive that
\[
\frac{1}{2} \frac{d}{dt} \left( \| (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \mu_1 \| \nabla (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 \right) + \mu_0 \| \nabla (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 \\
= -((u_\infty \cdot \nabla)(v_1^* - v_2^*)(t), (v_1^* - v_2^*)(t)) - ((k_1^* - k_2^*) \nabla v_1(t), \nabla (v_1^* - v_2^*)(t)) \\
- (k_1^* - k_2^*) \Delta u_0, \nabla (v_1^* - v_2^*)(t) - (k_1 \nabla (v_1^* - v_2^*)(t), \nabla (v_1^* - v_2^*)(t)) \\
\leq \left\| u_\infty \right\|_{L^\infty(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \\
+ \left\| (k_1^* - k_2^*) \nabla v_1(t) \right\|_{L^2(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \\
+ |(k_1^* - k_2^*)| \left\| \nabla u_0 \right\|_{L^2(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \\
+ |k_1 | \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \\
\leq C \left\| u_\infty \right\|_{L^2(\Omega)} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} + \frac{\mu_0}{2} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2 \\
+ \frac{3}{2\mu_0} \left\| (k_1^* - k_2^*) \nabla v_1(t) \right\|_{L^2(\Omega)}^2 + \frac{3}{2\mu_0} \left\| (k_1^* - k_2^*) \nabla u_0 \right\|_{L^2(\Omega)}^2 \\
+ \frac{3}{2\mu_0} \left\| k_1 \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2,
\]
for a.e. \( t \in [0, \delta] \). Integrating the above inequality from 0 to \( t \), we find

\[
\left\| (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2 + \mu_1 \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2 + \mu_0 \int_0^t \left\| \nabla (v_1^* - v_2^*)(s) \right\|_{L^2(\Omega)}^2 \, ds \leq \mu_1 \left\| \nabla (v_1^* - v_2^*)(0) \right\|_{L^2(\Omega)}^2 + C \left\| u_\infty \right\|_{L^2(\Omega)} \times \int_0^t \left\| \nabla (v_1^* - v_2^*)(s) \right\|_{L^2(\Omega)}^2 \, ds \\
+ \frac{3}{\mu_0} \left\| (k_1^* - k_2^*) \right\|_{L^2(\Omega)}^2 \int_0^t \left\| \nabla v_1(s) \right\|_{L^2(\Omega)}^2 \, ds + \frac{3}{\mu_0} \left\| (k_1^* - k_2^*) \right\|_{L^2(\Omega)}^2 \left\| \nabla u_0 \right\|_{L^2(\Omega)}^2 \\
+ \frac{3}{\mu_0} \left\| k_1 \right\|_{L^2(\Omega)}^2 \int_0^t \left\| \nabla (v_1^* - v_2^*)(s) \right\|_{L^2(\Omega)}^2 \, ds \\
\leq C \left\| \nabla (v_1^* - v_2^*)(0) \right\|_{L^2(\Omega)}^2 + C(\delta) \left\| v_1^* - v_2^* \right\|_{L^2(\Omega)}^2 \int_0^t \left\| \nabla v_1(s) \right\|_{L^2(\Omega)}^2 \, ds \\
+ C \left( \left\| u_\infty \right\|_{L^2(\Omega)} + \left\| (k_1^* - k_2^*) \right\|_{L^2(\Omega)} \right) \int_0^t \left\| \nabla (v_1^* - v_2^*)(s) \right\|_{L^2(\Omega)}^2 \, ds \\
+ C(\delta) \left\| v_1^* - v_2^* \right\|_{L^2(\Omega)}^2 \left\| \nabla u_0 \right\|_{L^2(\Omega)}^2,
\]
for all \( t \in [0, \delta] \), where we have used equation (5.22). An application of Gronwall’s inequality gives

\[
\sup_{t \in [0, \delta]} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)} \leq C \left( \left\| \nabla (v_1^* - v_2^*)(0) \right\|_{L^2(\Omega)} , \delta \right).
\]

Taking the inner product with \( \partial_t (v_1^* - v_2^*) \) to the first equation in (5.20), we obtain

\[
\| \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \mu_1 \left\| \nabla \partial_t (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2 + \frac{\mu_0}{2} \frac{d}{dt} \left\| \nabla (v_1^* - v_2^*)(t) \right\|_{L^2(\Omega)}^2 \\
= -((u_\infty \cdot \nabla)(v_1^* - v_2^*)(t), \partial_t (v_1^* - v_2^*)(t)) - ((k_1^* - k_2^*) \nabla v_1(t), \nabla \partial_t (v_1^* - v_2^*)(t)) \\
+ ((k_1^* - k_2^*) \Delta u_0, \partial_t (v_1^* - v_2^*)(t)) - (k_1 \nabla (v_1^* - v_2^*)(t), \nabla \partial_t (v_1^* - v_2^*)(t))
\]
for a.e. \( t \in [0, \delta] \). Integrating the above inequality from 0 to \( t \), we get

\[
\mu_0 \| \nabla(v_1^* - v_2^*)(t) \|^2_{L^2(\Gamma)} + \int_0^t \| \partial_t(v_1^* - v_2^*)(s) \|^2_{L^2(\Gamma)} ds + \mu_1 \int_0^t \| \nabla \partial_t(v_1^* - v_2^*)(s) \|^2_{L^2(\Gamma)} ds \\
\leq \mu_0 \| \nabla(v_1^* - v_2^*)(0) \|^2_{L^2(\Gamma)} + C \| u_\infty \|^2_{H^1(\Omega)} \int_0^t \| (v_1^* - v_2^*)(s) \|^2_{L^2(\Gamma)} ds \\
+ \frac{3}{\mu_1} \| k_1^* - k_2^* \|^2_{L^2(\Gamma)} \int_0^t \| \nabla v_1(s) \|^2_{L^2(\Gamma)} ds + \| k_1^* - k_2^* \|^2_{L^2(\Gamma)} \Delta u_0 \|^2_{L^2(\Gamma)} \\
+ \frac{3}{\mu_1} \| k_1 \|^2_{L^2(\Gamma)} \int_0^t \| \nabla(v_1^* - v_2^*)(s) \|^2_{L^2(\Gamma)} ds,
\]

for all \( t \in [0, \delta] \). Thus, it is immediate from equation (5.22) that \( \partial_t(v_1^* - v_2^*) \in L^2(0, \delta; H^1(\Omega)) \).

Taking divergence of the first equation in (5.20), we deduce that

\[
\partial_t(p_1 - p_2) = (-\Delta)^{-1} \left[ \nabla \cdot [(u_\infty \cdot \nabla)(v_1^* - v_2^*)] \right],
\]

in the weak sense. Therefore, we easily have

\[
\sup_{t \in [0, \delta]} \| \nabla \partial_t(p_1 - p_2)(t) \|_{L^2(\Gamma)} \leq C \sup_{t \in [0, \delta]} \| (u_\infty \cdot \nabla)(v_1^* - v_2^*)(t) \|_{L^2(\Gamma)} \\
\leq C \sup_{t \in [0, \delta]} \| (u_\infty \|_{L^\infty(\Gamma)} \| \nabla(v_1^* - v_2^*)(t) \|_{L^2(\Gamma)} \),
\]

which is finite and hence \( \nabla \partial_t(p_1 - p_2) \in L^\infty(0, \delta; L^2(\Omega)) \). Taking the inner product with \(-\Delta(v_1^* - v_2^*)(t)\) to the first equation in (5.20), we obtain

\[
\frac{1}{2} \frac{d}{dt} \left( \| \nabla(v_1^* - v_2^*)(t) \|^2_{L^2(\Gamma)} + \mu_1 \| \Delta(v_1^* - v_2^*)(t) \|^2_{L^2(\Gamma)} \right) \\
+ \mu_0 \| \Delta(v_1^* - v_2^*)(t) \|^2_{L^2(\Gamma)} \\
= ((u_\infty \cdot \nabla)(v_1^* - v_2^*)(t), \Delta(v_1^* - v_2^*)(t)) - (k_1^* - k_2^*) \Delta u_0, \Delta (v_1^* - v_2^*)(t) \\
- (k_1^* - k_2^*) \Delta u_0, \Delta (v_1^* - v_2^*)(t) - (k_1^* - k_2^*) \Delta (v_1^* - v_2^)(t) \\
+ (\nabla \partial_t(p_1 - p_2)(t), \Delta(v_1^* - v_2^*)(t)) \\
\leq \| u_\infty \|_{L^\infty(\Gamma)} \| \nabla(v_1^* - v_2^*)(t) \|_{L^2(\Gamma)} \| \Delta(v_1^* - v_2^*)(t) \|_{L^2(\Gamma)} \\
+ \| (k_1^* - k_2^*) \Delta u_0 \|_{L^2(\Gamma)} \| \Delta(v_1^* - v_2^*)(t) \|_{L^2(\Gamma)} + \| (k_1^* - k_2^*) \|_{L^2(\Gamma)} \| \Delta u_0 \|_{L^2(\Gamma)},
\]
\[
\begin{align*}
&\times \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} + \|k_1 \ast \Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} + \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} \\
&\quad + \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} + \mu_0 \int_0^t \|\Delta(v^*_1 - v^*_2(s))\|_{L^2(\Omega)} ds \\
&\leq C\|u_\infty\|_{L^\infty(\Omega)}^2 \|\nabla(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} + \frac{\mu_0}{2} \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 - k_2\|_{L^2(\Omega)}^2 \|\Delta v_1(t)\|_{L^2(\Omega)} + \frac{5}{2\mu_0} \|k_1 - k_2\|_{L^2(\Omega)}^2 \|\Delta u_0\|_{L^2(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 \ast \Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 + \frac{5}{2\mu_0} \|\Delta v_1(t)\|_{L^2(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 \ast \Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 + \frac{5}{2\mu_0} \|\Delta u_0\|_{L^2(\Omega)}^2,
\end{align*}
\]
for a.e. \(t \in [0, \delta]\). Integrating the above inequality from 0 to \(t\), we find

\[
\begin{align*}
\|\nabla(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 + \mu_1 \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 + \mu_0 \int_0^t \|\Delta(v^*_1 - v^*_2(s))\|_{L^2(\Omega)}^2 ds \\
&\leq \|\nabla(v^*_1 - v^*_2(0))\|_{L^2(\Omega)}^2 + \mu_1 \|\Delta(v^*_1 - v^*_2(0))\|_{L^2(\Omega)}^2 + C\|u_\infty\|_{L^\infty(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 - k_2\|_{L^2(\Omega)}^2 \|\Delta v_1(s)\|_{L^2(\Omega)}^2 ds + \frac{5}{2\mu_0} \|k_1 - k_2\|_{L^2(\Omega)}^2 \|\Delta u_0\|_{L^2(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 \ast \Delta(v^*_1 - v^*_2(s))\|_{L^2(\Omega)}^2 + C\sup_{t \in [0, \delta]} \|\nabla \partial_t(p_1 - p_2)\|_{L^\infty(\Omega)} \|v^*_1 - v^*_2\|_{L^2(\Omega)}^2 \\
&\quad + \frac{5}{2\mu_0} \|k_1 \ast \Delta(v^*_1 - v^*_2(s))\|_{L^2(\Omega)}^2 + C\sup_{t \in [0, \delta]} \|\nabla \partial_t(p_1 - p_2)\|_{L^\infty(\Omega)} \|\Delta u_0\|_{L^2(\Omega)}^2,
\end{align*}
\]
for all \(t \in [0, \delta]\), where we have used equation (5.22). An application of Gronwall’s inequality gives

\[
\sup_{t \in [0, \delta]} \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} \leq C \left(\|v^*_1 - v^*_2(0)\|_{L^2(\Omega)}, \|\Delta(v^*_1 - v^*_2(0))\|_{L^2(\Omega)}, \delta\right).
\]

Finally, taking the inner product with \(-\Delta \partial_t(v^*_1 - v^*_2)\) to the first equation in (5.20), we obtain

\[
\begin{align*}
\|\nabla \partial_t(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 + \mu_1 \|\Delta \partial_t(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 \\
&\quad + \frac{\mu_0}{2} \frac{d}{dt} \|\Delta(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}^2 \\
&= \langle (u_\infty \ast \nabla)(v^*_1 - v^*_2(t)), \Delta \partial_t(v^*_1 - v^*_2(t)) \rangle \\
&\quad - \langle (k^*_1 - k^*_2) \ast \Delta v_1(t), \Delta \partial_t(v^*_1 - v^*_2(t)) \rangle \\
&\quad - \langle (k^*_1 - k^*_2) \ast \Delta u_0, \Delta \partial_t(v^*_1 - v^*_2(t)) \rangle \\
&\quad - \langle (k^*_1 - k^*_2) \ast \Delta u_0, \Delta \partial_t(v^*_1 - v^*_2(t)) \rangle \\
&\quad + \langle \nabla \partial_t(p_1 - p_2(t), \Delta \partial_t(v^*_1 - v^*_2(t)) \rangle \\
&\leq \|u_\infty\|_{L^\infty(\Omega)} \|\Delta \partial_t(v^*_1 - v^*_2(t))\|_{L^2(\Omega)} \|\nabla(v^*_1 - v^*_2(t))\|_{L^2(\Omega)}.
\end{align*}
\]
Using (5.22) in the above estimate, we obtain for a.e. $k$

$$+ \| (k_1 - k_2)^* \Delta v_1(t) \|_{L^2(\Omega)} \| \Delta \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}$$
$$+ \| (k_1 - k_2)^* (t) \|_{L^2(\Omega)} \| \Delta \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}$$
$$+ \| k_1 \Delta (v_1^* - v_2^*)(t) \|_{L^2(\Omega)} \| \Delta \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}$$
$$+ \| \nabla \partial_t (p_1 - p_2)(t) \|_{L^2(\Omega)} \| \Delta \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}$$
$$\leq C \| u_\infty \|_{L^2(\Omega)}^2 \| \nabla (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \frac{\mu_1}{2} \| \nabla \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2$$
$$+ \frac{5}{2 \mu_1} \| (k_1 - k_2)^* \Delta v_1(t) \|_{L^2(\Omega)}^2 \| \Delta u_0 \|_{L^2(\Omega)}^2$$
$$+ \frac{5}{2 \mu_1} \| k_1 \Delta (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \frac{5}{2 \mu_1} \| \nabla \partial_t (p_1 - p_2)(t) \|_{L^2(\Omega)}^2,$$

for a.e. $t \in [0, \delta]$. From the above inequality, one can easily deduce

$$\mu_1 \| \Delta \partial_t (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \mu_0 \| \Delta (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2$$
$$\leq C \| u_\infty \|_{L^2(\Omega)}^2 \| \nabla (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \frac{\mu_1}{2} \| (k_1 - k_2)^* \Delta v_1(t) \|_{L^2(\Omega)}^2$$
$$+ \frac{5}{\mu_1} \| (k_1 - k_2)^* \Delta v_1(t) \|_{L^2(\Omega)}^2 \| \Delta u_0 \|_{L^2(\Omega)}^2$$
$$+ \frac{5}{\mu_1} \| k_1 \Delta (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2 + \frac{5}{\mu_1} \| \nabla \partial_t (p_1 - p_2)(t) \|_{L^2(\Omega)}^2,$$

for a.e. $t \in [0, \delta]$. Integrating the above inequality from $0$ to $t$, we find

$$\mu_1 \int_0^t \| \Delta \partial_t (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 ds + \mu_0 \| \Delta (v_1^* - v_2^*)(t) \|_{L^2(\Omega)}^2$$
$$\leq \mu_0 \| \Delta (v_1^* - v_2^*)(0) \|_{L^2(\Omega)}^2 + C \| u_\infty \|_{L^2(\Omega)}^2 \int_0^t \| \nabla (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 ds$$
$$+ \frac{5}{\mu_1} \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 \int_0^t \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 ds$$
$$+ \frac{5}{\mu_1} \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 \int_0^t \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 ds$$
$$+ \frac{5}{\mu_1} \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 \int_0^t \| \Delta (v_1^* - v_2^*)(s) \|_{L^2(\Omega)}^2 ds + C \| \nabla \partial_t (p_1 - p_2)(t) \|_{L^2(\Omega)}^2.$$
Theorem 5.5 (Global in time existence and uniqueness). Let the assumptions (H1)–(H5) hold. Let \( T > 0 \). Then the inverse problem (5.1) has a unique solution

\[
(u, k) \in H^2(0, T; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0, T).
\]

To prove theorem 5.5, that is, the global solvability of the inverse problem (5.6a)–(5.6e), we prove the result for its equivalent form, that is, for any given time \( T > 0 \), there exists a solution to the system (5.9a)–(5.9e). We start with the following lemmas, which will be useful in the sequel.

Lemma 5.6. Let assumptions (H1)–(H5) hold. Let \( \tau \in (0, T) \) and

\[
(\tilde{v}, \tilde{k}) \in H^1(0, \tau; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0, \tau)
\]
solves the system (5.9a)–(5.9e) on \([0, \tau]\). Then there exists \( \delta \in (0, T - \tau] \), such that the solution \((\tilde{v}, \tilde{k})\) can be extended to a solution \((v, k)\) on \([0, \tau + \delta]\), such that

\[
(v, k) \in H^1(0, \tau + \delta; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0, \tau + \delta).
\]

Proof. From the conditions (I)–(VI), it suffices to show that the system (5.12) and (5.13), with \( h, u, \tau \) be as defined in lemma 5.3, has a solution

\[
(v_\tau, k_\tau) \in H^1(0, \delta; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0, \delta),
\]
for some \( \delta \in (0, T - \tau] \). We define a space

\[
V(\delta, M) := \left\{ (\tilde{v}, \tilde{k}) \in H^1(0, \delta; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O)) \times L^2(0, \delta): \right. \\
\left. \begin{array}{l}
\nabla \cdot \tilde{v} = 0, \quad \text{in } O \times (0, \delta), \\
\tilde{v} = u, \quad \text{in } O \times \{0\}, \\
\|\tilde{v}\|_{H^1(0, \delta; \mathbb{H}^1_0(O) \cap \mathbb{H}^2(O))} + \|\tilde{k}\|_{L^2(0, \delta)} \leq M
\end{array} \right\},
\]

where \( M \) is a positive constant, which will be determined later. We also define the mapping \( \Gamma : V(\delta, M) \to V(\delta, M) \) such that \((\tilde{v}, \tilde{k}) \mapsto (v, k)\) through

\[
\begin{align}
\partial_t v - \mu_1 \Delta v - \mu_0 \Delta u - k_1 \Delta u_0 - \int_0^t k_1(t - s) \Delta \tilde{v}(s) ds - \int_0^t \tilde{k}(t - s) \Delta \tilde{v}(s) ds \\
+ (u_\infty \cdot \nabla) \tilde{v} + h(t) = 0, \quad \text{in } O \times [0, \delta], \\
\nabla \cdot v = 0, \quad \text{in } O \times [0, \delta], \\
v = 0, \quad \text{in } \partial O \times [0, \delta], \\
v = u, \quad \text{in } O \times \{0\},
\end{align}
\]

(5.23)
and

\[
\begin{aligned}
&k_r(t) = \alpha \left[ r''_i(t) - \left\{ \mu_0 \int_0^t \hat{v} \cdot \Delta \varphi \, dx + \int_0^t \int_0^t \hat{k}(t-s) \hat{v} \cdot \Delta \varphi \, dx \, ds \right. \\
&\left. + \int_0^t \int_0^t \hat{k}(t-s) \hat{v} \cdot \Delta \varphi \, dx \, ds + \int_0^t (\mathbf{u}_\infty \cdot \nabla) \varphi \cdot \hat{v} \, dx \right] + \int_0^t \hat{k}(\tau+t-s) \hat{v} \cdot \Delta \varphi \, dx \, ds \right], \quad t \in [0, \delta].
\end{aligned}
\]

(5.24)

We just need to show that the map \( \Gamma : \mathcal{V}(\delta, M) \to \mathcal{V}(\delta, M) \) is a contraction map. The proof constitutes similar arguments as given in the proof of theorem 4.1.

\[ \square \]

**Lemma 5.7.** Let \( T \in R^+ \), \( \tau \in (0, T) \). Let \( \beta_1, \beta_2 \in R^+ \) and \( v \in H^1(0, \tau; H^1(\Omega) \cap H^2(\Omega)) \) be such that, for all \( t \in [0, \tau] \),

\[
\|v\|_{H^1(0, \tau; H^2(\Omega))} \leq \beta_1 + \beta_2 \|v\|_{L^2(0, \tau; H^1(\Omega))}.
\]

Then,

\[
\|v\|_{H^1(0, \tau; H^2(\Omega))} \leq \frac{C (\beta_1, \beta_2, \|v\|_{L^2(0, \tau; H^1(\Omega)), T})}{\|v\|_{L^2(0, \tau; H^1(\Omega))}}.
\]

where \( C (\beta_1, \beta_2, \|v\|_{L^2(0, \tau; H^1(\Omega)), T}) \in R^+ \).

**Proof.** Applying Gagliardo–Nirenberg’s and Young’s inequalities, we obtain, for all \( \varepsilon \in R^+ \),

\[
\|v\|_{L^2(0, \tau; H^1(\Omega))} \leq C \|v\|_{L^2(0, \tau; H^1(\Omega))} \|v\|_{L^2(0, \tau; H^1(\Omega))} \|v\|_{L^2(0, \tau; H^1(\Omega))}.
\]

(5.25)

\[
\leq C \left( \|v\|_{L^2(0, \tau; H^1(\Omega))}^2 + C(\varepsilon) \|v\|_{L^2(0, \tau; H^1(\Omega))}^2 \right).
\]

(5.26)

Substituting (5.27) in (5.25), we have

\[
\|v\|_{H^1(0, \tau; H^2(\Omega))} \leq \beta_1 + \beta_2 C \|v\|_{L^2(0, \tau; H^1(\Omega))} + \beta_2 C \|v\|_{L^2(0, \tau; H^1(\Omega))}.
\]

(5.28)

for every \( \varepsilon \in R^+ \). Choosing \( \varepsilon \) in such a way that \( \beta_2 C \varepsilon \leq 1/2 \), we get

\[
\|v\|_{H^1(0, \tau; H^2(\Omega))} \leq 2 \beta_1 + C(\beta_2) \|v\|_{L^2(0, \tau; H^1(\Omega))}.
\]

(5.29)

Since \( v \in H^1(0, \tau; H^1(\Omega) \cap H^2(\Omega)), v \) is absolutely continuous (see section 5.9, theorem 2, [12]). Hence, for all \( s \in (0, \tau) \),

\[
\|v(s)\|_{L^2(\Omega)}^2 \leq 2 \|v(0)\|_{L^2(\Omega)}^2 + 2 \left\| \int_0^s \partial_s v(s) \, ds \right\|_{L^2(\Omega)}^2.
\]

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Lemma 5.8. Let the assumptions (H1)–(H5) hold. Let

\( (\hat{v}, \hat{k}) \in H^1(0, \tau; \mathbb{R}^d(O) \cap H^2(O)) \times L^2(0, \tau) \)

be a solution of the system (5.9a)–(5.9e) in \([0, \tau]\) with \(0 < \tau < T\). If

\( (v, k) \in H^1(0, \tau + \delta; \mathbb{R}^d(O) \cap H^2(O)) \times L^2(0, \tau + \delta) \)

is a solution of the system (5.9a)–(5.9e) in \([0, \tau + \delta]\). Then, we can bound \((v, k)\) and there exists \(C > 0\), such that, for all \(\delta \in (0, \tau \wedge (T - \tau))\),

\[
\|v\|_{H^1(0,\tau+\delta;\mathbb{R}^d(O)\cap H^2(O))} + \|k\|_{L^2(0,\tau+\delta)} \leq C.
\]

Proof. Owing to lemma 5.3 and theorem 5.4, if we set

\( v_\tau(t) := v(\tau + t), \quad k_\tau(t) := k(\tau + t), \quad t \in [0, \delta] \),

we get \((v_\tau, k_\tau)\) solves the system (5.12) and (5.13). Let \(k = \hat{k}\). Applying the same arguments to estimate \(v_\tau\) from the system (5.12) as in theorem 5.4, we obtain the following estimate:

\[
\|v_\tau\|_{H^1(0,\tau+\delta;\mathbb{R}^d(O)\cap H^2(O))} \leq C \left( u_0, \hat{k}, \hat{v}, T \right) \left( \|u_0\|_{L^2(0,0)}^2 + \|k_\tau\|_{L^2(0,\tau)}^2 \right),
\]
for all $t \in [0, \delta]$. Similarly, we estimate $k_\tau$ from the equation (5.13) as
\[
\|k_\tau\|_{L^2(0, \delta)}^2 \leq C(\hat{\nu}, \hat{\nu}, T) \left( \|\nu_{\sigma}^2\|_{L^2(0, \delta)}^2 + \|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 \right)
\]
\[
\leq C(\hat{\nu}, \hat{\nu}, T) \left( \|\nu_{\sigma}^2\|_{L^2(0, \delta)}^2 + \|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 \right) + C(\varepsilon)\|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 .
\]
(5.35)

Substituting (5.35) in (5.34), for all $t \in [0, \delta]$, we obtain
\[
\|\nu_t\|_{L^2(0, \delta)}^2 \leq C \left( u_0, \hat{\nu}, \hat{\nu}, T \right) \left( \|u_t^2\|_{H^2(\Omega)} + \|\nu_t^2\|_{L^2(0, \delta)}^2 + \|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 + \varepsilon\|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 + \varepsilon\|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 \right) .
\]
(5.36)

Choosing $\varepsilon$ to be sufficiently small, so that $\varepsilon C \left( u_0, \hat{\nu}, \hat{\nu}, T \right) \leq 1/2$, from (5.36), we obtain
\[
\|\nu_t\|_{L^2(0, \delta)}^2 \leq C \left( u_0, \hat{\nu}, \hat{\nu}, T \right) \left( \|u_t^2\|_{H^2(\Omega)} + \|\nu_t^2\|_{L^2(0, \delta)}^2 + \|\nu_{\tau}^2\|_{L^2(0, \delta)}^2 \right) .
\]
(5.37)

Thus, we conclude from lemma 5.7, (5.37) and (5.35) that
\[
\|\nu\|_{H^1(0, \tau + \delta; \mathbb{R}^2(\Omega))} + \|k\|_{L^2(0, \tau + \delta)}^2 \leq C,
\]
which completes the proof. \qed

Let us now prove theorem 5.5.

**Proof of Theorem 5.5.** Global in time uniqueness follows from theorem 5.4. It remains to prove existence. For this, having theorem 5.1 in hand, it suffices to show the existence of solution to the system (5.9a)–(5.9e), say $(\nu, k) \in H^1(0, T; \mathbb{H}^1(\Omega) \times \mathbb{H}^2(\Omega)) \times L^2(0, T)$. To obtain the same, let us set
\[
\mathcal{Z} := \left\{ \tau \in (0, T] : (\nu, k) \in H^1(0, \tau; \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega)) \times L^2(0, \tau) \text{ solves the system (5.9a)–(5.9e)} \right\} .
\]

Obviously $\mathcal{Z} \neq \emptyset$ from theorem 5.2. Owing to theorem 5.4, for all $\tau \in \mathcal{Z}$, the solution is unique and will be denoted by $(\nu_{\tau}, k_{\tau})$. Let $\tau_1$ and $\tau_2$ be any two elements of $\mathcal{Z}$ such that $\tau_1 < \tau_2$. Then, we have $\nu_{\tau_2}$ and $k_{\tau_2}$ are extensions of $\nu_{\tau_1}$ and $k_{\tau_1}$, respectively. Set $T' := \sup(\mathcal{Z})$. Of course, $0 < T' \leq T$. We need to prove $T' = T$ for the global existence. Let us define
\[
\nu : [0, T') \to \mathbb{H}^1(\Omega) \cap \mathbb{H}^2(\Omega), \quad \nu(t) = \nu_{\tau}(t), \text{ if } t \leq \tau,
\]
and
\[
k : [0, T') \to \mathbb{R}, \quad k(t) = k_{\tau}(t), \text{ a.e. if } t \leq \tau.
\]

Then, $(\nu, k)$ is the unique solution of the system (5.9a)–(5.9e) in $[0, \tau]$, for all $\tau \in (0, T')$. Let $T'' := T'/2$. Then, by lemma 5.8, there exists $C > 0$, such that, for all $\tau \in [T'', T')$,
\[ \|v\|_{L^2((0, \tau'; T); H^1(\Omega) \cap H^2(\Omega))} + \|k\|_{L^2(0, \tau')} \leq C. \quad (5.38) \]

As \( \tau \to T' \) in (5.38), we obtain
\[ \|v\|_{L^2((0, T'; T); H^1(\Omega) \cap H^2(\Omega))} + \|\partial_t v\|_{L^2((0, T'; T); H^1(\Omega) \cap H^2(\Omega))} + \|k\|_{L^2(0, T')} \leq C. \]

Therefore, we conclude that \( (v, k) \in H^1(0, T'; H^1_0(\Omega) \cap H^2(\Omega)) \times L^2(0, T') \) and solves the system (5.9a)–(5.9e) in \([0, T']\). But this implies that \( T' = T \). If not, by lemma 5.6, we could extend \((v, k)\) to a solution with domain \([0, T' + \delta]\), but this is not compatible with the definition of \( T' \), which completes the proof. \( \square \)

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No new data were created or analysed in this study.

ORCID iDs

Manil T Mohan \( \text{https://orcid.org/0000-0003-3197-1136} \)

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