On the sector counting lemma

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Abstract
In this short note, we prove a sector counting lemma for a class of Fermi surface on the plane which are $C^2$-differentiable and strictly convex. This result generalizes the one proved in Feldman et al. (Rev Math Phys 15(9):1121–1169, 2003) for the class of $C^{2+r}$-differentiable, $r \geq 3$, strictly convex and strongly asymmetric Fermi surfaces, and the one proved in Benfatto et al. (Ann Henri Poincaré 7:809–898, 2006) and Feldman et al. (Helv Phys Acta 65:679–721, 1992), for the class of $C^2$-differentiable, strictly convex and central symmetric Fermi surfaces. This new sector counting lemma can be used to construct interacting many-fermion models for the doped graphene, in which the Fermi surface is extended and quasi-symmetric.

Keywords  Fermi surface · Sector counting · Renormalization group

Mathematics Subject Classification  82B10 · 82B28

1 Introduction and main results

1.1 The Fermi surface problem

The Landau theory of the Fermi liquid [10–12] is one of the most important achievements in quantum many-body theory. It essentially states that, in a $d$-dimensional crystal, the excitations of an infinitely large collection of strongly interacting particles can be described as an equally large collection of weakly interacting quasi-particles, which carry the same quantum numbers as the original particles, and are characterized by a definite band structure $\varepsilon_0(\mathbf{k}) := e(\mathbf{k}) - \nu$ on $\mathbb{R}^d$, in which $e(\mathbf{k})$ is the dispersion relation and $\nu \in \mathbb{R}$ is the chemical potential. One important feature of a Fermi liquid is the existence of the Fermi surface (F.S.), which is defined as the zero set of the band

\[ \varepsilon_0(\mathbf{k}) = 0 \]

Therefore, the study of the Fermi surface is crucial in the understanding of Fermi liquids.
structure: \( \mathcal{F}_0 = \{ \mathbf{k} \in \mathbb{R}^d \mid \varepsilon_0(\mathbf{k}) = 0 \} \). It is a compact hyper-surface in \( \mathbb{R}^d \), across which the quasi-particle density function \( n(\mathbf{k}) \) is not continuous but has a jump.

A major difficulty in the rigorous study of an interacting many-fermion system is that interaction produces a deformation of the Fermi surface. Consider a \( d \geq 2 \)-dimensional interacting many-fermions system at temperature \( T > 0 \), defined by the grand-canonical Hamiltonian:

\[
H = \sum_\sigma \int \frac{d^d \mathbf{k}}{(2\pi)^d} \varepsilon_0(\mathbf{k}) a_{\mathbf{k},\sigma}^+ a_{\mathbf{k},\sigma} + \sum_{\sigma, \tau} \frac{1}{2} \int \prod_{i=1}^4 \frac{d^d \mathbf{k}_i}{(2\pi)^d} (2\pi)^d \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3 - \mathbf{k}_4) \\
\times \hat{u}(\mathbf{k}_1 - \mathbf{k}_3)a_{\mathbf{k}_1,\sigma}^+ a_{\mathbf{k}_2,\tau}^+ a_{\mathbf{k}_4,\tau} a_{\mathbf{k}_3,\sigma},
\]

in which \( a^\pm \) are the fermionic creation and annihilation operators defined on \( \mathbb{R}^d \times \{ \uparrow, \downarrow \} \), with \( \sigma, \tau \in \{ \uparrow, \downarrow \} \) the spin indices. \( \hat{u} \) is the Fourier transform of the two-body interaction potential. Let \( k_0 = 2\pi T (n_0 + \frac{1}{2}) \), \( n_0 \in \mathbb{Z} \), be the Fourier dual of the (imaginary) time variable \( x_0 \in [0, \frac{1}{T}] \). When \( \hat{u} = 0 \), the free (non-interacting) propagator is \( \hat{S}_0(\mathbf{k}_0, \mathbf{k}) = (i k_0 - \varepsilon_0(\mathbf{k}))^{-1} \). Obviously, the set of singularities of \( \hat{S}_0(\mathbf{k}_0, \mathbf{k}) \) at \( k_0 = 0 \) is exactly \( \mathcal{F}_0 \). When \( \hat{u} \neq 0 \), the interacting propagator is given by \( \hat{S}(\mathbf{k}_0, \mathbf{k}) = [i k_0 - \varepsilon_0(\mathbf{k}) - \Sigma((\mathbf{k}_0, \mathbf{k}), \varepsilon_0)]^{-1} \), in which \( \Sigma((\mathbf{k}_0, \mathbf{k}), \varepsilon_0) \) is called the self-energy function, which is a highly non-trivial function of the band structure \( \varepsilon_0 \) and the interaction \( \hat{u} \). For \( ||\hat{u}|| \) small under a suitable norm \( ||\cdot|| \), \( \hat{S}(\mathbf{k}_0, \mathbf{k}) \) and \( \Sigma(\mathbf{k}_0, \mathbf{k}) \) can be calculated by perturbation expansions. The *interacting fermi surface* \( \mathcal{F} \) is defined as the set of singularities of \( \hat{S}(\mathbf{k}_0, \mathbf{k}) \) at \( k_0 = 0 \):

\[
\mathcal{F}_{\varepsilon_0} = \{ \mathbf{k} \in \mathbb{R}^d \mid \varepsilon_0(\mathbf{k}) + \Sigma((0, \mathbf{k}), \varepsilon_0) = 0 \},
\]

which is in general different from \( \mathcal{F}_0 \). This shift in the Fermi surface, also called the moving-Fermi surfaces problem, is a major difficulty in Quantum many-fermion problem and may cause divergence of many coefficients in the naive perturbation expansions.

This problem can be solved mainly in two approaches, one is to fix the interacting Fermi surface \([6]\) such that it coincides with the non-interacting one \( \mathcal{F}_0 \), by introducing a suitable counter-term \( \sum_\sigma \int \frac{d^d \mathbf{k}}{(2\pi)^d} \delta \varepsilon_0(\mathbf{k}) a_{\mathbf{k},\sigma}^+ a_{\mathbf{k},\sigma} \) to the interaction potential. But this approach raises another difficulty, which is called the *inversion problem* (cf., e.g., \([7]\)):
given the band structure \( \varepsilon(\mathbf{k}) := \varepsilon_0(\mathbf{k}) + \delta \varepsilon_0 \), whether the counter-term \( \delta \varepsilon_0 \) can be uniquely decided, and how to determine \( \varepsilon_0 \) from \( \varepsilon \). This problem has not been solved non-perturbatively. The other approach \([1]\) is to use the renormalized interacting propagator \( \hat{S} \), whose singular set defines the interacting Fermi surface, in the perturbation expansions. Either approach needs the renormalization group (RG) analysis.

In order to perform the RG analysis, one needs to decompose the support of the propagators in \( \mathbb{R}^d \) into a set of rectangles, called the *sectors*. The decomposition is performed in two successive steps. First of all, one decomposes the region in \( \mathbb{R}^d \) close to the Fermi surface into shells surrounding the Fermi surface, with size of the shell
depending on the temperature. But this is not enough to obtain the desired decaying behavior of the propagator, due to the mismatch of the volume form in momentum space and the position space [5]. One has to further decompose each shell into a set of sectors. This complicates the RG analysis, as in addition to the scaling indices labeling the shells, one needs to sum over properly the sector indices while taking into account the conservation of momentum. This is called the sector counting problem, which lies at the heart of Fermi liquid theory. A key step for proving the sector counting lemma is to estimate the flexibility of the constraint imposed by the conservation of momentum; namely, for any given $q \in \mathbb{R}^2$ and any Fermi surface $\mathcal{F}$, the cardinality of the set $\{k_1, k_2 | k_1, k_2 \in \mathcal{F}, k_1 + k_2 = q\}$. This is equivalent to consider the inverse image of the mapping $\Phi_1: \mathcal{F} \times \mathcal{F} \to \mathbb{R}^2$, $\Phi_1: (k_1, k_2) \mapsto q$, in which $\Phi_1$ is differentiable but not necessarily injective. The systematic way of estimating this number is cumulated into the parallelogram lemmas.

The sector counting lemmas for the convex and central symmetric Fermi surfaces have been proved by [1,2,5]. In particular, the authors of [1] have solved the inversion problem for the doped Hubbard model on the square lattice, following the second approach. But the sector counting lemma of [1] cannot be applied to more general Fermi surfaces. The sector counting lemma for the strictly convex and strongly asymmetric Fermi surfaces has been proved in [3]. Based on this lemma and by introducing the counter-term, the authors constructed an interacting many-fermions model which exhibits Fermi liquid behaviors at zero temperature. But they cannot provide a non-perturbative solution of the inversion problem. The sector counting lemma for Fermi surfaces that are not strictly convex but have flat edges has been proved in [13–15].

1.2 The main results

In this paper, we consider the 2-d many-fermions models with two classes of Fermi surfaces: the quasi-asymmetric Fermi surface (cf. Definition 2.3) and the quasi-symmetric Fermi surfaces (Definition 2.4), both are required to be strongly convex and $C^2$ differentiable. The first main result is Lemma 3.1, in which we prove a parallelogram lemma for these Fermi surfaces. Since a quasi-symmetric Fermi surface is neither strongly asymmetric (see [3]) nor central symmetric (see [1,2,5]), our result is an important generalization of these results. What is more, the parallelogram lemma is valid for any $C^2$-differentiable strongly convex Fermi surfaces, and for the class of concave Fermi surfaces such that at any point on the Fermi surface there exists only finite number of antipodal points (cf. Definition 2.2). Based on this result, we prove the sector counting lemma for a single scale (Theorem 4.1) and for multi-scales (Theorem 4.2).

Inspired by [3], we prove the main results by identifying the zero measure set $\text{Gra}(a) =: \{(k, a(k)) | k \in \mathcal{F}\}$, the pairs of antipodal points, on which the mapping $\Phi$ fails to be injective. Since $\Phi$ is injective on the domain $\mathcal{F} \times \mathcal{F} \setminus \text{Gra}(a)$, the implicit function theorem can be applied. Then, we prove that the zero measure set does not change the result of sector counting.

This paper is organized as follows. In Sect. 2, we recall basic definitions, notations of the Fermi surfaces. In Sect. 3, we prove the parallelogram lemma (Lemma 3.1) for the $C^2$ strictly convex F.S. which are quasi-asymmetric or quasi-symmetric. This lemma
is valid up to a measure zero set. In Sect. 4, we prove that the measure zero set does not change the sector counting and present the sector counting lemmas for a single scale and for multi-scales of sectorizations. We expect this result can be used to solve the inversion problem for the model considered by [4] without introducing counter-term, as well as constructing models with more general FS, like the honeycomb Hubbard model for the study of graphene. Some results proved in this paper have analogues in [3] and their proofs are almost identical. So we simply omit these proofs and ask the interested readers to consult [3] for details.

2 Preliminary

2.1 Fermi surfaces and the sectors

Consider a many-fermions model on a \( d = 2 \) lattice.

**Definition 2.1** A Fermi surface is defined as the zero set of the dispersion relation. It is a closed curve in \( \mathbb{R}^2 \) which may contain several connected components, each of which is called a Fermi curve (F.C.). A Fermi curve (also denoted by \( \mathcal{F} \)) is called \( C^r \) differentiable, \( r \geq 1 \), if the dispersion relation \( \epsilon(\mathbf{k}) \) is a \( C^r \)-differentiable function in a neighborhood of \( \mathbf{k} \), for all \( \mathbf{k} \in \mathcal{F} \); it is called strictly convex if its curvature is bounded away from zero.

Let us choose an orientation for the fermi curve \( \mathcal{F} \): For any \( \mathbf{k} \in \mathcal{F} \), let \( \mathbf{t}_k \) be the unit tangent vector to \( \mathcal{F} \) at \( \mathbf{k} \) and \( \mathbf{n}_k \) the inward pointing unit normal vector to \( \mathcal{F} \) at \( \mathbf{k} \). There is a differentiable function \( \phi_k : I_{0,a} \to \mathbb{R} \), where \( I_{0,a} \) is an interval in \( \mathbb{R} \) centered at 0 with size \( a \), such that \( s \mapsto \mathbf{k} + s\mathbf{t}_k + \phi_k(s)\mathbf{n}_k \) is an oriented parametrization of \( \mathcal{F} \) near \( \mathbf{k} \). By construction, we have \( \phi_k(0) = \phi'_k(0) = 0 \) and \( \phi''_k(0) \) is the curvature of \( \mathcal{F} \) at \( \mathbf{k} \).

**Definition 2.2** Let \( \mathbf{k} \in \mathcal{F} \) be any point on the Fermi curve, an antipodal point \( a(\mathbf{k}) \) of \( \mathbf{k} \), \( a(\mathbf{k}) \neq \mathbf{k} \), is a point in \( \mathcal{F} \) such that the tangent vector \( \mathbf{t}_{a(\mathbf{k})} \) to \( \mathcal{F} \) at \( a(\mathbf{k}) \) is parallel or anti-parallel to \( \mathbf{t}_k \).

Remark that, since \( \mathcal{F} \) is strictly convex, each point \( \mathbf{k} \in \mathcal{F} \) has a unique antipodal point.

**Definition 2.3** Let \( \mathcal{F} \) be a \( C^r \)-differentiable Fermi curve. It is called strongly asymmetric (also called \( C^r \)-strongly asymmetric) if there is \( n_0 \in \mathbb{N}, n_0 \leq r \), such that for each \( \mathbf{k} \in \mathcal{F} \), there exists \( p \leq n_0 \) such that

\[
\phi_k^{(p)}(0) \neq \phi_{a(\mathbf{k})}^{(p)}(0). \tag{3}
\]

A Fermi curve \( \mathcal{F} \) is called \( C^r \)-quasi asymmetric if it is \( C^r \)-strongly asymmetric at almost all points on the Fermi surface, up to a zero measure set.

**Definition 2.4** A Fermi curve \( \mathcal{F} \) is called quasi-symmetric if it is strictly convex and the isometric group is the dihedral group \( D_{2n+1}, n \geq 1, n \in \mathbb{N} \), i.e., the symmetry
Fig. 1 A quasi-symmetric Fermi curve $\mathcal{F}$ with $D_3$ symmetry and the parallelogram on $\mathcal{F}$

group of regular polygons with $2n + 1$ sides. It is called $C^r$-quasi-symmetric if it is $C^r$-differentiable.

See Fig. 1 for a quasi-symmetric Fermi curve with symmetry group $D_3$, which can be considered as the F.C. for the Graphene system with doping [14].

**Definition 2.5** (Shells and sectors) By introducing suitable cutoff functions (e.g., the Gevrey class of functions, cf. [5,14]), the support of the free propagator in the momentum space can be decomposed into shells labeled by indices $j$, $j \geq 2$. Let $S^j = \{k \in \mathbb{R}^2 | M^{-j} \leq |\Omega(k) - \mu| \leq M^{-j+1}\}$ be the $j$-th shell in the momentum space. Let $I$ be an interval on the Fermi curve $\mathcal{F}$ and $\pi_{\mathcal{F}}(k)$ is the orthogonal projection of $k$ on the Fermi curve. Then

$$s = \{k \in S^j | \pi_{\mathcal{F}}(k) \in I\}$$

is called a sector of length $|I|$ at scale $j$. Two different sectors $s$ and $s'$ are called neighbors if $s \cap s' \neq \emptyset$.

**Definition 2.6** A sectorization of length $l$ at scale $j$ around a Fermi curve $\mathcal{F}$ is a set $\Sigma_{\mathcal{F}}$ of sectors of length $l$ and at scale $j$ that obeys

- the set of sectors covers the Fermi curve;
- each sector $s$ has precisely two neighbors in $\Sigma_{\mathcal{F}}$;
- if $s, s' \in \Sigma_{\mathcal{F}}$ are neighbors then $\frac{1}{16}l \leq |s \cap s' \cap \mathcal{F}| \leq \frac{1}{8}l$.

From the above definition, we can easily find that there are at most $2\ell(\mathcal{F})/l$ sectors in $\Sigma_{\mathcal{F}}$, where $\ell(\mathcal{F})$ is the length of $\mathcal{F}$.

**Definition 2.7** Let $\mathcal{M}$ be a sub-manifold of $\mathbb{R}^d$ and $\mu$ be the volume measure on $\mathcal{B}(\mathcal{M})$, the Borel $\sigma$-algebra over $\mathcal{M}$. We say that a subset $A \subset \mathcal{M}$ has $\mu$-measure zero if for every smooth chart $(U, \phi)$ for $\mathcal{M}$, the subset $\phi(A \cap U) \subset \mathbb{R}^d$ has $m$-measure zero, where $m$ is the Lebesgue measure on $\mathbb{R}^d$. Let $\Phi : \mathcal{M} \to \mathbb{R}^d$ be a measurable mapping of class $C^1$ and is not necessarily injective. For any measurable $E \subset \mathcal{M}$ and $y \in \mathbb{R}^n$, define the function $\#(E, y) = \{|x \in E | \Phi(x) = y\}$. In case $\Phi(x) = y$ for infinitely many $x \in E$, then we define $\#(E, y) = \infty$.

The Jacobian of the mapping $\Phi$ and the number $\#(E, y)$ is related by the following theorem:
Theorem 2.1 Let $\mathcal{M}$ and $\Phi$ be defined as above. Let $J(x) = \Phi'(x)$ be the Jacobian at $x \in \mathcal{M}$. Then for any measurable $E \subset \mathcal{M}$, $(E, y)$ is a measurable function of $y \in \mathbb{R}^n$, and
\[
\int_E |J(x)|dx = \int_{\mathbb{R}^n} #(E, y)dy.
\]

This theorem is already known. The interested reader could consult, e.g., [9], pages 505–510, for a complete proof and we do not repeat it here.

Feldman, Knörrer and Trubowitz proved in [3] the following parallelogram lemma concerning the strictly asymmetric Fermi curves.

Lemma 2.1 (Lemma XX.7 of [3]) Let $\mathcal{F} \subset \mathbb{R}^2$ be a strongly asymmetric, strictly convex Fermi surface that is $C^{2+r}$ differentiable, $r \geq 3$. Let $\mathcal{B}(\mathcal{F} \times \mathcal{F})$ be the Borel sigma algebra over $\mathcal{F} \times \mathcal{F}$ and $\mu$ be the volume measure on $\mathcal{B}(\mathcal{F} \times \mathcal{F})$. Define the measurable mapping $\Phi : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^2$, $(k_1, k_2) \mapsto k_1 + k_2$, which is not necessarily injective. Then, there exists $X \subset \mathcal{F} \times \mathcal{F}$, for which $\mu(\mathcal{F} \times \mathcal{F} \setminus X) = 0$, and a constant $C$, which depends on the geometry of $\mathcal{F}$, such that for any measurable subset $E \subset X$ and any $q \in \mathbb{R}^2$, we have
\[
\#((E, q)) \leq C.
\]

**Proof** The full measure set $X$ can be chosen as $X = \{(k_1, k_2) | k_1, k_2 \in \mathcal{F}, k_1 \neq k_2\}$. The mapping $\Phi$ is injective on $X$ except at the points $\{(k, a(k))\}$, which form a measure zero set in $\mathcal{F} \times \mathcal{F}$. A detailed proof has been given in [3], so we do not repeat it here.

\[\Box\]

3 The parallelogram lemma

As a generalization of Lemma 2.1, we can prove the following parallelogram lemmas.

Lemma 3.1 Let $\mathcal{F} \subset \mathbb{R}^2$ be a strictly convex Fermi curve which is $C^2$ quasi-asymmetric or $C^2$ quasi-symmetric; let $\mu$ be the volume measure on $\mathcal{B}(\mathcal{F} \times \mathcal{F})$. Define the mapping $\Phi : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^2$, $\Phi : (k_1, k_2) \mapsto k_1 + k_2$ which is not necessarily injective. Then, there exists $X_1 \subset \mathcal{F} \times \mathcal{F}$, with $\mu(\mathcal{F} \times \mathcal{F} \setminus X_1) = 0$, and a constant $C$, which depends on the geometry of $\mathcal{F}$, such that for any measurable subset $E \subset X_1$ and for almost every $q \in \mathbb{R}^2$, we have
\[
\#((E, q)) \leq C.
\]

**Proof** Since $\mathcal{F}$ is strictly convex, for any $k \in \mathcal{F}$ there exists a unique antipodal point $a(k)$. Obviously $a : \mathcal{F} \to \mathcal{F}$, $a : k \mapsto a(k)$ is an isomorphism of $\mathcal{F}$ and the graph of the mapping $\text{Gra}(a) := \{(k, a(k)) | k \in \mathcal{F}, a(k) \in \mathcal{F}\}$ is a $\mu$-measure zero set in $\mathcal{F} \times \mathcal{F}$. Since $\Phi$ is injective on $\mathcal{F} \times \mathcal{F} \setminus \text{Gra}(a)$, we can identify the full measure set $X_1$ as $X_1 = X \setminus \text{Gra}(a)$, in which $X = \{(k_1, k_2) | k_1, k_2 \in \mathcal{F}, k_1 \neq k_2\}$. By construction, no pair of antipodal points is contained in $X_1$. Then, we only need to prove (7) for pairs of vectors that are not antipodal of each other. This proof can be found in Lemma XX.7 in [3], and we do not repeat it here.

\[\Box\]
Remark 3.1 It is important to notice that the conclusion of this lemma depends only on the convexity of the Fermi curve $\mathcal{F}$ but not on the global symmetry of the F.C., i.e., if it is quasi-asymmetric or quasi symmetric, and this conclusion holds only up to a measure zero set. We will prove in the next section that the measure zero sets do not change the result of the sector counting. This can also be seen by the following simple argument. Let $\Sigma_1$ be any sectorization of $\mathcal{F}$, whose elements are positive measure sets. Then, any measure zero set in $\mathcal{F} \times \mathcal{F}$ must be contained in $\Sigma_1 \times \Sigma_1$. So the existence of measure zero set does not change the result of counting sectors.

Remark 3.2 Remark that Lemma 3.1 can be generalized to the case of concave Fermi curves $\mathcal{F}$ for which the antipodal points of any point $k \in \mathcal{F}$ form a finite set.

4 The sector counting lemma

In this part, we shall consider the sector counting lemma for $2^n$ sectors constrained by the conservation of momentum. As a first step, let us consider the case of 4 sectors.

4.1 Parallelogram lemma for two sectors

Consider 4 vectors $k_1, \ldots, k_4$ that belongs to the four sectors $s_1, \ldots, s_4$, respectively. We are interested in counting the cardinality of the configuration set of sectors $\{(s_1, \ldots, s_4) \mid \sum_i k_i = 0, k_i \in s_i\}$ that are compatible with conservation of momentum. Let $k_3 + k_4 = -q$, this problem is equivalent to counting the number of decompositions of a subset $A \subseteq \mathbb{R}^2$, to which $q$ belongs, into sectors. The result is also called the parallelogram lemma for two sectors.

Definition 4.1 (Vector sum of sectors) Let $\Sigma_1$ be a sectorization of a F.C. $\mathcal{F}$, i.e., a set of sectors whose union form an $\epsilon$-neighborhood $\mathcal{F}(\epsilon)$ of $\epsilon$. Let $s_1, s_2 \in \Sigma_1$ be two sectors and $k_1 \in s_1$, $k_2 \in s_2$ be two vectors varying in the two sectors, respectively. Define a differentiable mapping $\Phi : \Sigma_1 \times \Sigma_1 \rightarrow \mathbb{R}^2$, $\Phi : k_1 + k_2 \mapsto q$. The image of the mapping $\Phi(s_1 \times s_2)$, noted by $\Phi(s_1 \times s_2) := s_1 + s_2$, is called the vector sum of the sectors $s_1$ and $s_2$. In the same way, we can define the vector sum for any $n \geq 2$ sectors.

We are mainly interested in inverse problem of the vector sum of sectors, given a sectorization $\Sigma_1$ of a Fermi curve $\mathcal{F}$ and a measurable subset $A \subseteq \mathbb{R}^2$, the cardinality of the set $\{(s_1, s_2) \mid s_1, s_2 \in \Sigma_2, s_1 + s_2 \subseteq A\}$. A general solution to this problem is called the parallelogram lemma for sectors, which is the simplest but most important example of the sector counting lemma. Instead of using techniques from planar differential geometry, Feldman, Knörrer and Trubowitz proposed in [3] a new method which reduces the counting problems to the problems of estimating volumes of sets in momentum space that are constrained by the conservation of momentum. Before proceeding, let us introduce the following definitions from Riemannian geometry.

Definition 4.2 ([8]) Let $(M, d)$ be a $d$-dimensional Riemannian manifold with metric function $d$. Given $\epsilon > 0$, a subset $\Gamma$ of $M$ is called $\epsilon$-separated if for any two different elements $\gamma, \gamma' \in \Gamma$, $d(\gamma, \gamma') \geq \epsilon$. 
Feldman, Knörrer and Trubowitz proved in [3] that:

**Lemma 4.1** (Lemma XX.4 in [3]) Let \((M, d)\) be a \(d\)-dimensional Riemannian manifold, \(\Phi : M \to \mathbb{R}^d\) be a differentiable mapping. Let \(B_r(x) = \{ y \in M | d(x, y) < r \}\) be an open ball of radius \(r\) around \(x\), let \(\mu\) be the volume measure on \(\mathcal{B}(M)\), and

\[
V_{M, \epsilon} = \inf_{x \in M, 0 < r \leq \epsilon} \frac{\mu(B_r(x))}{r^2}.
\] (8)

Then for all \(\epsilon_0 > 0\), \(A \subset \mathbb{R}^d\) with \(\epsilon\)-neighborhood \(A'(\epsilon)\), \(0 < \epsilon < \epsilon_0\), and all \(\epsilon\)-separated subsets \(\Gamma \subset M\), we have:

\[
\#(\Phi^{-1}(A) \cap \Gamma) \leq \frac{1}{\epsilon^n V_{M, \epsilon_0}} \mu(\Phi^{-1}(A'(\epsilon))).
\] (9)

Taking \(M = \mathcal{F} \times \mathcal{F}\) and \(d = 2\), using Lemma 3.1, we can prove the following lemma, which is a key step for proving the sector counting lemma.

**Lemma 4.2** Let \(\mathcal{F}\) be a \(C^2\)-differentiable strictly convex Fermi curve. Let \(\omega_1\) and \(\omega_2\) be two positive real numbers such that \(0 < \omega_1 < \frac{1}{2} \omega_2\). For any \(p \in \mathcal{F}\), define the set

\[
\tilde{M} = \{(k_1, k_2) \in \mathcal{F} \times \mathcal{F} | \min[d(k_1, k_2), d(a(k_1), k_2)] \geq \omega_1
\]

and \(\min[d(k_i, p), d(a(k_i), p)] \leq \omega_2\) for \(i = 1, 2\),

\[
\tag{10}
\]

and the mapping \(\Phi : \mathcal{F} \times \mathcal{F} \to \mathbb{R}^2, \Phi(k_1, k_2) = k_1 + k_2\). Then, there exist positive constants \(K\) depending only on the geometry of \(\mathcal{F}\), such that for all measurable subset \(A \subset \mathbb{R}^2\),

\[
\mu(\Phi^{-1}(A) \cap \tilde{M}) \leq \frac{\text{const}}{\omega_1} m(A),
\] (11)

where \(\mu\) is the volume measure on \(\mathcal{B}(\mathcal{F} \times \mathcal{F})\) and \(m\) is the Lebesgue measure on \(\mathbb{R}^2\).

**Proof** First of all, we can calculate explicitly the Jacobian \(J(k_1, k_2), (k_1, k_2) \in \tilde{M}\), for the mapping \(\Phi\). Let \(\theta(k_1, k_2)\) be the angle between the normal vectors to \(\mathcal{F}\) at \(k_1\) and \(k_2\), by simple calculations we find that

\[
J(k_1, k_2) = \sin \theta(k_1, k_2).
\] (12)

From the definition of \(\tilde{M}\), we find that:

\[
| \sin \theta(k_1, k_2) | \geq \text{const} \min[d(k_1, k_2), d(a(k_1), k_2)] \geq \text{const} \omega_1.
\] (13)

Now we consider the following two cases:

(i) \(\Phi^{-1}(A) \cap \tilde{M} = \emptyset\). Then (11) is obviously true.

(ii) if \(M_A := \Phi^{-1}(A) \cap \tilde{M} \neq \emptyset\), then we have \(\int_A #(M_A, y) dy = \int_{M_A'} |J(x)| dx\), by Theorem 2.1. Let \(\text{Gra}(a) = \{(k, a(k)) | k \in \mathcal{F}\} \subset \mathcal{F} \times \mathcal{F}\) be the set of antipodal pairs and \(M'_A = M_A \setminus \text{Gra}(a)\), we have \(\#(M_A, y) = \#(M'_A, y)\), for a.e. \(y \in A\) (cf. [9], pages 505–510).
Let $J_{\text{min}}$ and $J_{\text{max}}$ be the minimal and maximal value of the Jacobian $|J(x)|$; we have
\[
\int_{M_A'} |J| \, dx \geq J_{\text{min}} \mu(M_A') \geq \text{const } \omega_1 \mu(M_A') = \text{const } \omega_1 \mu(M_A),
\] (14)
where for the last equality we used the fact that $M_A = M_A' + \text{Gra}(a)$ and $\text{Gra}(a)$ is a zero measure set. On the other hand, we have
\[
\int_A \#(M_A, y) \, dy = \int_A \#(M_A', y) \, dy \leq \#(M_A', y)_{\text{max}} m(A),
\] (15)
where $\#(M_A', y)_{\text{max}}$ is the maximal number of pre-images of any $y \in A$. Since $F \setminus N_F$ is strongly asymmetric, we have $\#(M_A', y)_{\text{max}} < n$, for a finite $n \in \mathbb{N}$. So we have
\[
\#(M_A', y)_{\text{max}} m(A) \geq \int_A \#(M_A, y) \, dy = \int_{M_A'} |J| \, dx \geq \text{const } \omega_1 \mu(M_A).
\] (16)
and
\[
\mu(\Phi^{-1}(A) \cap \tilde{N}) = \mu(M_A) \leq \frac{\text{const}}{\omega_1} m(A).
\] (17)

Combining the above two lemmas, we can prove the following lemma, which is very similar to Lemma XX.8 of [3], except that the Fermi surface $F$ now has different geometric properties.

**Lemma 4.3:** Let $0 < \varepsilon < \omega_1/4$ and let $\Gamma$ be an $\varepsilon$-separated subset of $F$. Let $A$ be a rectangle in $\mathbb{R}^2$ having one pair of sides parallel to $n$ with length $L_1$ and a second pair of sides perpendicular to $n$ of length $L_2$. Then, we have
\[
\#(\Phi^{-1}(A) \cap (F \times F) \cap (\Gamma \times \Gamma)) \leq \frac{\text{const}}{\omega_1 \varepsilon^2}(L_1 + \varepsilon \omega_2)(L_2 + \varepsilon).
\] (18)

**Proof** Using the results of Lemmas 4.1, we have
\[
\#(\Phi^{-1}(A) \cap (F \times F) \cap (\Gamma \times \Gamma)) \leq \frac{\text{const}}{\varepsilon^2} \mu(\Phi^{-1}(A'(\varepsilon))).
\] (19)

Then using Lemma 4.2, we have $\mu(\Phi^{-1}(A'(\varepsilon))) \leq \frac{\text{const}}{\omega_1} m(A'(\varepsilon))$ and the fact that $m(A'(\varepsilon)) \leq (L_1 + \varepsilon \omega_2)(L_2 + \varepsilon)$, the result follows. $\square$

**4.2 The sector counting lemma**

In the previous section, we proved that any measure zero sets in $F \times F$ does not change the result of sector counting. This result can be easily generalized to the case of any $2n$ sectors, as the vector sum of $2n$ sectors can be reduced to the parallelogram lemma for $n$-sectors, which can be further reduced to the one for 2 sectors inductively, by
consider the vector sum of \( n - 1 \) sectors as a single sector. In this section, we consider the sector counting lemma for general \( 2n \) sectors. This part largely follows [3]. Since the difference of the Fermi curves considered in [3] and the ones considered in this paper is also a zero measure set, many results stated in Sections XX and XXI of [3] can be adapted to the current paper. So we mainly present the results without proof. The interested readers are invited to consult [3] for details.

Definition 4.3 Let \( \Sigma_{F} \) be a sectorization of \( F \), in which each sector \( s_{\Lambda,l} \) is a rectangle of length \( l \) and width \( \Lambda \), such that \( 0 \leq \Lambda \leq l, \Lambda \geq l^{2} \). Let \( p \in F \) and \( \Gamma \) be a an \( l \)-separated subset of \( F \). Define

\[
Mom_{2n-1}(\Gamma, p) = \{(k_{1}, \ldots, k_{n}) \in \Gamma^{2n-1} | \exists x_{i} \in s_{\Lambda,l}(k_{i}), i = 1, \ldots, 2n - 1, \] such that \( x_{1} + \cdots x_{2n-1} \in s_{\Lambda,l}(p) \}.
\]

Definition 4.4 The tuple \( (s_{1}, \ldots, s_{n}) \in \Sigma_{F}^{\otimes n} \) is called a configuration of sectors. A configuration of sectors is said to be consistent with the conservation of momentum if the tuple of vectors \( (k_{1}, \ldots, k_{n}) \), with \( k_{i} \in s_{i} \), for \( i = 1, \ldots, n \), satisfies \( \sum_{i=1}^{n} k_{i} = 0 \).

Following exactly the same procedures as in [3] and the same techniques employed in proof of Lemma XX.9, Proposition XX.10, we can prove following proposition:

Proposition 4.1 Let \( F \) be a \( C^{2} \)-differentiable, strictly convex planar Fermi curve. Let \( n \geq 2, \delta \geq l \) and let \( I_{1}, \ldots, I_{2n-1} \) be intervals of length \( \delta \) in \( F \). Assume that

\[
\frac{1}{3} \omega = \max_{1 \leq i \neq j \leq 2n-1} \min(\text{dist}(I_{i}, I_{j}), \text{dist}(I_{i}, a(I_{j}))) > \max(\delta, 4l).
\]

There exists a constant \( K \), which depends on the geometry of the Fermi curve but is independent of the size of sectors, such that for all \( l \)-separated subsets \( \Gamma \) of \( F \), all \( p \in F \),

\[
#Mom_{2n-1}(\Gamma, p) \cap (I_{1} \times \cdots \times I_{2n-1}) \leq Kn^{2}\left(\frac{\delta}{l} + 1\right)^{2n-3}\left(1 + \frac{\Lambda}{l\omega}\right).
\]

Remark the numerical factor 1/3 is inessential and can be replaced by any other fractional number between 0 and 1 but not very close to 0 or 1.

Example 4.1 As an example, consider an anisotropic sectorization \( \Sigma_{F}^{(j)} \) of \( F \) for the single scale \( j \geq 2 \) (cf., e.g., [5]), such that each sector \( s_{\Lambda,l} \) is a rectangle of length \( \gamma^{-j} \) and width \( M^{-j/2} \), in which \( M \geq 10 \) is a fixed constant. Then, we have \( \delta \sim O(1)M^{-j/2} \), for some order 1 constant \( O(1) \). The centers of the sectors form an \( l \)-separated set with \( l = M^{-j/2} \). We have \( \delta/l \sim O(1), \Lambda/l\omega \sim O(1) \), and

\[
#Mom_{2n-1}(\Gamma, p) \cap (I_{1} \times \cdots \times I_{2n-1}) \leq \text{const} n^{2} O(1)^{2n-3},
\]

which is bounded for any \( n \).
The sector counting lemma is simply a reformulation of the above proposition:

**Theorem 4.1** [The sector counting lemma] Let $F$ be a quasi-asymmetric or quasi-symmetric Fermi curve which is $C^2$ differentiable. Let $I_1, \ldots, I_{2n}$ be intervals on the Fermi curve, each of which has length $M^{-j} \leq \delta \leq M^{-j/2}$, $j \in \mathbb{Z}_+$, $j \geq 2$. Let $k_i \in \mathbb{R}^2$ and $k_i'$ be the corresponding projection in $F$. Let $K$ be numerical constants which depends on the band structure. Let $s_1$ be a fixed sector. Let $S_{2, n-1}$, $n \geq 2$, be a set of $(2n-1)$-tuples of sectors $(s_2, \ldots, s_{2n})$ such that there exist $k_i \in \mathbb{R}^2$, $i = 1, \ldots, 2n$ satisfying

$$k_i' \in s_i \cap I_i, \quad |k_i - k_i'| \leq KM^{-j}, \quad i = 1, \ldots, 2n$$

and

$$|k_1 + \cdots + k_{2n}| \leq KM^{-j}.$$  

Then, the cardinality of the set $S_{2, n-1}$, noted by $\#S_{2, n-1}$, is bounded by

$$K^{2n} \left( \frac{\delta}{M^{-j/2}} \right)^{2n-3}. \quad (26)$$

Now we consider the sector counting problem with two scales.

**Definition 4.5** Let $j > i \geq 2$ be two scaling indices. Let $\frac{1}{M^{j-\frac{1}{2}}} \leq l \leq \frac{1}{M^{j-1}}$ and $\frac{1}{M^{i-\frac{1}{2}}} \leq l' \leq \frac{1}{M^{i-1}}$. Let $\Sigma^{(j)}_F$ and $\Sigma^{(i)}_F$ be two sectorizations of length $l$ at scale $j$ and length $l'$ at scale $i$, respectively. Define $\#\text{Cons}(s^{(j)}_1, \ldots, s^{(j)}_m; s^{(i)}_{m+1}, \ldots, s^{(i)}_n)$, in which $s^{(j)}_p$, $p = 1, \ldots, m$, are sectors in $\Sigma^{(j)}_F$ and $s^{(i)}_q$, $q = m + 1, \ldots, n$ are sectors in $\Sigma^{(i)}_F$, as the set of all sectors $(s^{(j)}_{m+1}, \ldots, s^{(j)}_n) \in \Sigma^{(j)}_F \otimes (n-m)$, such that $s^{(j)}_i \cap s^{(i)}_j \neq \emptyset$ for $i = m + 1, \ldots, n$, and the sectors $(s_1, \ldots, s_n)$ are consistent with conservation of momentum.

Following [3], Section XXI, we can prove the sector counting lemma for changing of scales.

**Theorem 4.2** Let $F$ be a $C^2$-differentiable, strictly convex planar Fermi curve. Let $j > i \geq 2$, let $\Sigma^{(j)}_F$ and $\Sigma^{(i)}_F$ be two sectorizations of $F$ defined as above, such that $l < \frac{1}{4} l'$. Let $\omega' \geq 4l'$, and let $s_1 \in \Sigma$ and $s'_2, \ldots, s'_n \in \Sigma'$ such that $\text{dist}(s'_k, s'_l) \geq \omega'$ and $\text{dist}(s'_k, a(s'_j)) \geq \omega'$ for some $2 \leq k \neq l \neq n$. Then, there exists a positive constant $K$, which is independent of the size of the sectors, such that

$$\#\text{Cons}(s_1; s'_2 \cdots s'_n) \leq K \left( \frac{l'}{l} \right)^{n-3} \left( 1 + \frac{1}{M^{j-1}l\omega'} \right). \quad (27)$$

**Proof** The proof of this theorem is technically identical to that of Lemma X XI.4 in [3]. So we do not repeat it here. \hfill \Box

**Example 4.2** Consider two anisotropic sectorizations $\Sigma^{(j)}_F$ and $\Sigma^{(i)}_F$, $j \geq i \geq 2$, of $F$ introduced above. The sectors $s^{(j)} \in \Sigma^{(j)}_F$ are of length $l = M^{-j/2}$ and width
\( \Lambda = M^{-j} \), and the sectors \( s^{(i)} \in \Sigma_{\mathcal{F}}^{(i)} \) are of length \( l' = M^{-i/2} \) and width \( \Lambda' = M^{-i} \). Then, we have \( \omega' \sim O(1)M^{-j/2} \) and

\[
#\text{Cons}(s_1^{(j)}; s_2^{(i)} \cdots s_n^{(i)}) \leq \text{const} \left( \frac{l'}{l} \right)^{n-3} \left( 1 + \frac{1}{M^{j-1}l\omega'} \right) \leq O(1)'M^{(j-i)/2}, \tag{28}
\]

in which \( O(1)' \) is another positive constant of order 1.

5 Conclusions and perspectives

In this paper, we proved the parallelogram lemma and sector counting lemma for any \( C^2 \) differentiable strictly convex Fermi curves, as a generalization of that considered in [1,3,5]. We expect that this important result can be used to solve the inversion problem for more general Fermi surfaces.

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