The cell-dispensability obstruction
for spaces and manifolds

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Abstract

We compare two properties for a CW-space $X$ of finite type: (1) being homotopy equivalent to a CW-complex without $j$-cells for $k \leq j \leq \ell$ ($(k,\ell)$-cellfree) and (2) $H^j(X; R) = 0$ for any $\mathbb{Z}\pi_1(X)$-module $R$ when $k \leq j \leq \ell$ (cohomology $(k,\ell)$-silent). Using the technique of Wall’s finiteness obstruction, we show that a connected CW-space $X$ of finite type which is cohomology $(k,\ell)$-silent determines a cell-dispensability obstruction $w_k(X) \in K_0(\mathbb{Z}\pi_1(X))$ which vanishes if and only if $X$ is $(k,\ell)$-cellfree ($k \geq 3$). Any class in $K_0(\mathbb{Z}\pi)$ may occur as the cell-dispensability obstruction $w_k(X)$ for a CW-space $X$ with $\pi_1(X)$ identified with $\pi$. Using projective surgery, a similar theory is obtained for manifolds, replacing “cells” by “handles” (antisimple manifolds).

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1 Introduction

In the first part of this paper, we are interested in the following possible properties for a connected CW-space $X$.

Definitions A. (1) $X$ is $(k,\ell)$-cellfree if $X$ is homotopy equivalent to a CW-complex having finite skeleta and without $j$-cells for $k \leq j \leq \ell$.
(2) $X$ is cohomology silent in degree $j$ if $H^j(X; R) = 0$ for any $\mathbb{Z}\pi_1(X)$-module $R$.

(3) $X$ is cohomology $(k, \ell)$-silent if $X$ is cohomology silent in degree $j$ for $k \leq j \leq \ell$.

Note that a $(k, \ell)$-cellfree CW-space $X$ is of finite type (i.e. homotopy equivalent to a complex with finite skeleta). But the definition is stronger than being of finite type and being homotopy equivalent to a complex without $j$-cells for $k \leq j \leq \ell$. Indeed, the two conditions must be realized simultaneously (see Remark 2.29).

Obviously, a CW-space which is $(k, \ell)$-cellfree is cohomology $(k, \ell)$-silent. The converse is known to be true for simply connected CW-spaces of finite type, using a minimal cell decomposition [7, Proposition 4C.1] together with the universal coefficient theorem for cohomology. In Section 2, we show that a connected CW-space $X$ of finite type which is cohomology $(k, \ell)$-silent determines a cell-dispensability obstruction $w_k(X) \in \tilde{K}_0(\mathbb{Z}\pi_1(X))$ which vanishes if and only if $X$ is $(k, \ell)$-cellfree ($k \geq 3$). Any class in $\tilde{K}_0(\mathbb{Z}\pi)$ may occur as the cell-dispensability obstruction $w_k(X)$ for a CW-space $X$ with $\pi_1(X)$ identified with $\pi$ (by somehow fixing the $(k-1)$-skeleton).

The definition of $w_k(X)$ is akin to that of the Wall finiteness obstruction, but the two obstruction are different, since $w_k(X)$ might not vanish for a finite complex $X$. Among the similarities is a product formula. As a consequence, if $X$ is cohomology $(k, \ell)$-silent ($k \geq 3$), then $X \times S^1$ is $(k+1, \ell)$-cellfree.

In Section 3 we apply the above material to smooth manifolds, using the following

**Definitions B.** Let $M$ be a smooth compact manifold of dimension $r \geq 2k$.

(1) $M$ is $k$-antisimple if $M$ admits a handle decomposition without handles of index $j$ for $k \leq j \leq r-k$.

(2) $M$ is cohomology $k$-antisimple if it is cohomology $(k, r-k)$-silent. This definition is also used for a Poincaré complex of formal dimension $r$.

If $M$ is cohomology $k$-antisimple, its cell-dispensability obstruction $w_k(M) \in \tilde{K}_0(\mathbb{Z}\pi_1(M))$ is defined. When $M$ is closed of dimension $r \geq 6$, we prove that $w_k(M) = 0$ if and only if $M$ is $k$-antisimple. The obstruction $w_k(M)$ is $r$-self-dual, i.e $w_k(M) = (-1)^{r+1}w_k(M)^*$ (the involution $*$ on $\tilde{K}_0(\mathbb{Z}\pi_1 M)$ involves the orientation character $\omega: \pi_1(M) \to \{\pm 1\}$ of $M$). It thus defines a class $[w_k(M)]$ in the Tate cohomology group $H^{r+1}(\mathbb{Z}_2; \tilde{K}_0(\mathbb{Z}\pi_1 M, \omega))$. We prove that $[w_k(M)] = 0$ if and only if $M$ is cobordant, in an appropriate sense, to a $k$-antisimple manifold.

A whole section (§3.3) is devoted to realize an element $P \in \tilde{K}_0(\mathbb{Z}\pi)$ as the cell-dispensability obstruction for a Poincaré space or a closed manifold with fundamental group identified with $\pi$ (via fixing the normal $(k-1)$-type). Using the technique of ends of spaces [22, 10], we construct for any $P \in \tilde{K}_0(\mathbb{Z}\pi)$ a finitely dominated Poincaré space $\mathbb{P}$ of formal dimension $r \geq 6$, which is
cohomology $k$-antisimple with $w_k(\mathcal{P}) = \mathcal{P}$. Its Wall finiteness obstruction $\text{Wall}(\mathcal{P})$ is equal to $\mathcal{P} + (-1)^r \mathcal{P}^*$. In particular, when $\mathcal{P}$ is $r$-self-dual, $\mathcal{P}$ is homotopy equivalent to a finite Poincaré complex. Using then the projective surgery theory of \cite{22}, we prove that $\mathcal{P}$ may be replaced by a closed $r$-dimensional manifold when $[w_k(M)]$ belongs to the kernel of the homomorphism $\delta_R: H^{r+1}(\mathbb{Z}_2; \hat{K}_0(\mathbb{Z}\pi, \omega)) \to L^h_r(\pi, \omega)$ occurring in the Ranicki exact sequence (see (3.37)). Examples are discussed in §3.5. Finally, Section 4 contains some results about antisimple manifolds, shedding some new light on previous works of the author \cite{9,11,12}.

The terminology antisimple was introduced in \cite{10} for manifolds related to high dimensional knots. Antisimple manifolds have been studied in \cite{9}, with applications in \cite{11,12}. In \cite{5,33}, cohomology antisimple manifolds are called antisimple manifold.

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2 Cohomology silent complexes

2.1 Preliminaries

2.1. Cell complexes. A CW-complex is also called a cell complex or just a complex. We denote by $X^k$ the $k$-skeleton of a complex $X$. The universal covering of $X$ is denoted by $\tilde{X}$. Maps between topological spaces are always supposed to be continuous.

When a cell complex $X$ is connected, we usually assume that it has only one 0-cell: $X^0 = \{x_0\}$ and we denote by $\pi(X)$ the fundamental group $\pi_1(X, x_0)$, often abbreviated by $\pi$ when the context is clear. The group $\pi$ acts on the left on $\tilde{X}$ by deck transformations.

A CW-space is a topological space $X$ which is homotopy equivalent to a CW-complex. Such a CW-space is

- of finite $k$-type if $X$ is homotopy equivalent to a CW-complex with finite $k$-skeleton.

- of finite type if $X$ is of finite $k$-type for all $k \geq 0$.

- finitely dominated if there exists a finite CW-complex $A$ and maps $f: A \to X$ and $s: X \to A$ such that $f \circ s$ is homotopic to $\text{id}_A$ ($f$ is called a (homotopy) domination).

2.2. r-connectivity and weak r-connectivity. A map $f: Y \to X$ between connected CW-spaces is called $r$-connected if $\pi_j(f) = 0$ for $j \leq r$. When $r \geq 2$, this is equivalent to $f$ inducing an isomorphism on fundamental groups and $H_j(\tilde{f}) = 0$ for $j \leq r$, where $\tilde{f}: \tilde{Y} \to \tilde{X}$ is the induced map on the universal covers. A pair $(X, Y)$ of connected CW-spaces is called $r$-connected if the inclusion map $Y \hookrightarrow X$ is $r$-connected.
A map $f:Y \to X$ is called weakly $r$-connected if $f_*:\pi_j(Y) \to \pi_j(X)$ is an isomorphism for $j \leq r - 1$. When this is the case, one can attach trivial $r$-cells to $Y$, getting a cell complex $Y'$, so that $f$ extends to $f':Y' \to X$ which is $r$-connected. Again, a pair $(X,Y)$ of connected CW-spaces is called weakly $r$-connected if the inclusion map $Y \hookrightarrow X$ is weakly $r$-connected. Note that “weakly $r$-connected” implies “$(r-1)$-connected”.

Well known by specialists, the following lemma will be useful.

**Lemma 2.3.** Let $f:Y \to X$ be a map between connected CW-complexes. Suppose that $f$ is $r$-connected. Then, there exists a CW-complex $Y'$ obtained by attaching to $Y$ cells of dimension $> r$, such that $f$ extends to a homotopy equivalence $f':Y' \to X$. If $X$ and $Y$ have finite skeleta, then $Y'$ may be chosen to have finite skeleta.

**Proof.** It suffices to prove that there exists a CW-complex $Y'$ obtained by attaching to $Y$ cells of dimension $r + 1$, finite in number if $X$ and $Y$ have finite skeleta, such that $f$ extends to an $(r + 1)$-connected $f':Y' \to X$. The required couple $(Y',f')$ will be obtained by iteration of this process.

One can replace $f$ by an injection of a subcomplex. It is classical that the couple $(\hat{Y},f)$ may be obtained by attaching $(r + 1)$-cells using a set of generators of $\pi_{r+1}(X,Y)$ (see [28, p. 59, comments preceding Lemma 1.2 and its proof]). By the homotopy sequence of the triple $(X,Y,Y')$, the $\mathbb{Z}\pi_1(X)$-module $\pi_{r+1}(X,Y)$ is a quotient of $\pi_{r+1}(X,Y')$. The latter is a finitely generated $\mathbb{Z}\pi_1(X)$-module when $X$ and $Y$ are complexes with finite skeleta (see [28, Theorem A]).

2.4. Reference maps. Let $Y$ be a connected CW-complex of dimension $r - 1$. An $Y$-reference map for a CW-space $X$ is a map $g: X \to Y$ which is $r$-connected. The space $X$ together with the map $g$ (or the pair $(X,g)$) is called an $Y$-referred space.

**Lemma 2.5.** Let $g:X \to Y$ be an $Y$-reference map. Then, there exists a CW-complex $X'$ containing $Y$ as a subcomplex and a homotopy equivalence $h:X' \to X$ such that the $Y$-reference map $g \circ h$ is a retraction of $X'$ onto $Y$. The pair $(X',Y)$ is weakly $r$-connected.

In short, a reference map may be realized up to homotopy equivalence by a reference retraction. Note that the pair $(X',Y)$ is not $r$-connected in general (e.g. $X' = X = Y \vee S^r$).

**Proof.** The reference map $g$ may be, up to homotopy equivalence, replaced by a Serre fibration $\hat{g}: \hat{X} \to Y$, whose fiber is then $(r-1)$-connected. As $Y$ is of dimension $r - 1$, there is no obstruction to get a section of $\hat{g}$, which provides a homotopy section $\gamma:Y \to X$ for $g$. This map $\gamma$ may be, up to homotopy equivalence, replaced by a CW-inclusion $J:Y \hookrightarrow X'$ and the map $g$ provides a reference map $g'_1: X' \to Y$, such that $g'_1 \circ J$ is homotopic to $\text{id}_Y$. By the homotopy extension property, there is a homotopy from $g'_1$ to $g':X' \to Y$ such that $g' \circ \gamma = \text{id}_Y$, namely $g$ is a retraction. Since $g$ is $r$-connected, the pair $(X',Y)$ is weakly $r$-connected.

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2.6. Cohomology with local coefficients. Let $X$ be a connected cell complex with fundamental group $\pi$. We consider the cellular chain complex $(C_*(\tilde{X}), \partial)$, with $C_k(\tilde{X}) = H_k(\tilde{X}^k, \tilde{X}^{k-1})$. The latter is a is a free (left) $\mathbb{Z}\pi$-module with basis in bijection with the $k$-cells of $X$ (by choosing, for each $k$-cell $e$, an orientation and a lifting $\tilde{e}$ in $\tilde{X}$ of $e$). The cycles $Z_k(\tilde{X})$ and the boundaries $B_{k-1}(\tilde{X})$ are, as usual, defined as the $\mathbb{Z}\pi$-modules

$$ Z_k(\tilde{X}) = \ker(\partial: C_k(\tilde{X}) \to C_{k-1}(\tilde{X})) $$
$$ B_{k-1}(\tilde{X}) = \text{Image}(\partial: C_k(\tilde{X}) \to C_{k-1}(\tilde{X})). $$

Modules of $\mathbb{Z}\pi$ are left modules, unless mention of the contrary. If $R$ is a such a $\mathbb{Z}\pi$-module, the cohomology $H^*(X; R)$ is the homology of the cochain complex $C^*(\tilde{X}; R) = \text{hom}(C_*(\tilde{X}), R)$, where the coboundary $\delta$ is defined by

$$ \delta(\alpha) = (-1)^k \alpha \circ \partial \quad (\alpha \in C^k(\tilde{X})). $$

Homology with coefficient in a $\mathbb{Z}\pi$-module $R$ will also be used, but only in the framework of manifolds or Poincaré spaces (see §5.1).

2.2 The cell-dispensability obstruction

Recall from the introduction (Definition A) that a connected CW-space $X$ is cohomology silent in degree $k$ if $H^k(X; R) = 0$ for any $\mathbb{Z}\pi_1(X)$-module $R$.

Lemma 2.7. Let $X$ be a connected CW-complex and let $k \geq 1$ be an integer. The following conditions are equivalent.

(a) $X$ is cohomology silent in degree $k$.

(b) The following two conditions hold true:

(b1) the inclusion $B_{k-1}(\tilde{X}) \hookrightarrow C_{k-1}(\tilde{X})$ admits a retraction of $\mathbb{Z}\pi(X)$-modules (in consequence $B_{k-1}(\tilde{X})$ is projective), and

(b2) $H_k(\tilde{X}) = 0$.

Proof. We set $C_j = C_j(\tilde{X})$, $Z_j = Z_j(\tilde{X})$, $B_j = B_j(\tilde{X})$ and $\pi = \pi_1(X)$. The proof proceeds in three steps.

Step 1: (a) $\Rightarrow$ (b1). The boundary homomorphism $\partial: C_k \to B_{k-1}$ may be seen as an element of $Z^k(X; B_{k-1}(\tilde{X}))$. If (a) holds true, there is $\alpha \in \text{hom}(C_{k-1}, B_{k-1})$ such that $\partial = \delta(\alpha)$, i.e. $\partial(u) = \alpha \circ \partial(\tilde{u})$ for all $u \in C_k$. Thus, $\alpha$ is a retraction of $C_{k-1}$ onto $B_{k-1}$.

Step 2: (a) $\Rightarrow$ (b2). By Step 1 already proven, $B_{k-1}$ is projective and thus the exact sequence $0 \to Z_k \to C_k \to B_{k-1} \to 0$ splits, implying that $Z_k$ is a direct summand of $C_k$. Therefore, the surjective homomorphism $Z_k \to H_k(\tilde{X})$ extends to a homomorphism $\alpha: C_k \to H_k(\tilde{X})$, giving an element of $C^k(X; H_k(\tilde{X}))$ satisfying $\alpha \circ \partial = 0$, i.e. $\alpha \in Z^k(X; H_k(\tilde{X}))$. By (a), $\alpha = \tilde{\alpha} \circ \partial$ for some homomorphism $\tilde{\alpha}: C_{k-1} \to H_k(\tilde{X})$. The homomorphism $\tilde{\alpha}$ is surjective since so is $\alpha$. But $\partial(Z_k) = 0$, which implies that $H_k(\tilde{X}) = 0$. 

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Step 3: (b) implies (a). Let $b ∈ H^k(X; R)$, represented by a homomorphism $β: C_k(\tilde{X}) → R$ which is a cocycle, i.e. $β|_{B_k} = 0$. But $B_k = Z_k$ since $H_k(\tilde{X}) = 0$ by (b2). Since $k ≥ 1$, one has $β = β_o∂$ for some morphism of $Zπ$-module $β: B_{k-1} → R$. By (b1), $β$ extends to an element $\hat{β} ∈ C^{k-1}(X; R)$ and thus $β = δ(\hat{β})$, which proves that $b = 0$. Therefore, $H^k(X, R) = 0$ for any $Zπ_1(X)$-module $R$, which proves (a).

We now prepare the definition of the obstruction $w_k(X)$ associated to a cohomology silent cell complex in degree $k$.

**Lemma 2.8.** Let $X$ be a connected CW-complex which is cohomology silent in some degree $k ≥ 3$. Let $φ: K → X$ be a $(k-1)$-connected map, where $K$ is a $(k-1)$-dimensional CW-complex. Then, the $Zπ$-module $π_k(φ)$ is projective.

**Proof.** Up to homotopy equivalence, we may suppose that $φ$ is the inclusion of the $(k-1)$-skeleton $K = X^{k-1} → X$ (see Lemma [2.3]). Since $k ≥ 3$, the map $φ$ induces an isomorphism on the fundamental groups and is covered by a map $\hat{φ}: \tilde{K} → \tilde{X}$ on the universal coverings. By the relative Hurewicz isomorphism theorem, one has

$$π_k(φ) ≈ H_k(\hat{φ}) ≈ H_k(\tilde{X}, \tilde{X}^{k-1}).$$

We now claim that there is an isomorphism of $Zπ$-modules between $B_{k-1}(\tilde{X})$ and $H_k(\tilde{X}, \tilde{X}^{k-1})$. Lemma 2.8 will then follow from Lemma 2.7.

To establish the claim, recall that the boundary homomorphism $∂: C_k(\tilde{X}) → C_{k-1}(\tilde{X})$ is the composition

$$∂: H_k(\tilde{X}^k, \tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^{k-2}).$$

The right arrow is injective since $H_{k-1}(\tilde{X}^{k-2}) = 0$. Hence, $B_{k-1}(\tilde{X})$ may be seen as a submodule of $H_{k-1}(\tilde{X}^{k-1})$:

$$B_{k-1}(\tilde{X}) ≈ \text{Im} \left( H_k(\tilde{X}^k, \tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^{k-1}) \right) = \ker \left( H_{k-1}(\tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^k) \right).$$

As $H_{k-1}(\tilde{X}^k) ≈ H_{k-1}(\tilde{X})$, one has

$$B_{k-1}(\tilde{X}) ≈ \ker \left( H_{k-1}(\tilde{X}^{k-1}) → H_{k-1}(\tilde{X}) \right) = \text{Im} \left( H_k(\tilde{X}, \tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^{k-1}) \right).$$

But

$$\text{Im} \left( H_k(\tilde{X}, \tilde{X}^{k-1}) → H_{k-1}(\tilde{X}^{k-1}) \right) ≈ H_k(\tilde{X}, \tilde{X}^{k-1})$$

since $H_k(\tilde{X}) = 0$ by Lemma 2.7. □

Let $X$ be a connected CW-complex which is cohomology silent in some degree $k ≥ 3$. Let $φ: K^{k-1} → X$ be a $(k-1)$-connected map. By Lemma 2.8, $π_k(φ)$ is a projective $Zπ$-module. Suppose that $K$ is a finite complex and that $X$ is of finite type. By the proof of Lemma 2.8, the $Zπ$-module $π_k(φ) ≈ H_k(\hat{φ})$ is finitely generated and therefore defines a class

$$w_k(X) = (-1)^k [π_k(φ)] ∈ \hat{K}_0(Zπ),$$

(2.1)
where $K_0(\mathbb{Z}\pi)$ denotes the group of finitely generated projective $\mathbb{Z}\pi$-modules modulo the stably free ones. That the map $\varphi$ does not appear in the notation $w_k(X)$ makes sense in view of the following lemma.

**Lemma 2.9.** Let $X$ be a connected CW-complex which is of finite type and cohomology silent in degree $k \geq 3$. Let $\varphi_i: K^{k-1}_i \to X$ ($i = 1, 2$) be two $(k-1)$-connected maps, where $K_i$ are finite complexes. Then $[\pi_k(\varphi_1)] = [\pi_k(\varphi_2)]$ in $K_0(\mathbb{Z}\pi)$.

**Remarks 2.10.** (1) When $X$ has the homotopy type of a $k$-dimensional complex, then the Wall finiteness obstruction $\text{Wall}(X)$ of $X$ is defined \[28\] and is equal to $w_k(X)$. See also Section 2.4.

(2) Taking for $\varphi: K \to X$ the inclusion of $K = X^{k-1}$ into $X$, one has $w_k(X) = (-1)^k[B_{k-1}(X)]$ (see the proof of Lemma 2.8). □

**Proof of Lemma 2.9.** We essentially follow the proof of \[28\] Lemma 3.2, as detailed in \[27\] Lemma 2.9, p. 155. Up to homotopy equivalence, the map $\varphi_i$ may be replaced by a Serre fibration $\theta_i$, with fiber $F_i$. As $\varphi_i$ is $(k-1)$-connected, one has $\pi_j(F_i) = 0$ for $j < k - 1$, and thus the first obstruction to construct a section of $\theta_i$ belongs to $H^k(X; \pi_{k-1}(F_i))$. The latter vanishes since $X$ is cohomology silent in degree $k$. Using obstruction theory \[34\] Chapter VI], we can thus construct map $s^i: X^k \to K_i$ such that $\varphi_i \circ s^i$ is homotopic to the inclusion $X^k \to X$. This induces a morphism of $\mathbb{Z}\pi$-module $s^i_*: H_{k-1}(X^k) \to H_{k-1}(K_i)$ which splits the exact sequence

\[
0 \longrightarrow H_k(\tilde{\varphi}_i) \longrightarrow H_{k-1}(\tilde{K}_i) \stackrel{\psi_i}{\longrightarrow} H_{k-1}(\tilde{X}) \longrightarrow 0
\]

(2.2)

(the morphism $H_k(\tilde{\varphi}_i) \to H_{k-1}(\tilde{K}_i)$ is indeed injective since $H_k(\tilde{X}) = 0$ by Lemma 2.7). By cellularity, one may assume that $\varphi_i(K_i) \subset X^{k-1}$. We can thus define a map $\psi: K_1 \to K_2$ as $\psi = s^2 \circ \varphi_1$.

Since $\varphi_i$ is $(k-1)$-connected and since $K_i$ is of dimension $k-1$, one has $H_j(\tilde{\psi}) = 0$ when $j \neq k - 1, k$ and the horizontal line in the following diagram is an exact sequence.

\[
\begin{array}{ccc}
H_k(\tilde{\psi}) & \longrightarrow & H_{k-1}(\tilde{K}_1) \\
& \searrow \psi_* & \nearrow (\tilde{\phi}_1)_* \\
& & s^2_* \\
H_{k-1}(\tilde{X}) & \longrightarrow & H_{k-1}(\tilde{K}_2) \longrightarrow H_{k-1}(\tilde{\psi})
\end{array}
\]

Therefore, using (2.2), one gets isomorphisms of $\mathbb{Z}\pi$-modules

\[
H_k(\tilde{\varphi}_1) \approx H_k(\tilde{\psi}) \quad \text{and} \quad H_k(\tilde{\varphi}_2) \approx H_{k-1}(\tilde{\psi})
\]

(2.3)

With the abbreviations $C_j = C_j(\tilde{\psi})$, $Z_j = Z_j(\tilde{\psi})$ and $B_j = b_j(\tilde{\psi})$, the cellular chain complex $C_*(\tilde{\psi})$ gives the following exact sequences.
Since Proposition 2.13 of the introduction are used. thanks to the following proposition, in which the concepts of Definitions A of finite type. Suppose that \( X \) is a CW-complex with finite skeleta. If \( X \) has no \( k \)-cells, then \( B_{k-1}(X) = 0 \) and thus \( w_k(X) = 0 \) by Part (2) of Remark 2.12. Therefore, (b) implies (a).

Proof. Using Remark 2.12 we assume that \( X \) is a CW-complex with finite skeleta. If \( X \) has no \( k \)-cells, then \( B_{k-1}(X) = 0 \) and thus \( w_k(X) = 0 \) by Part (2) of Remark 2.11. Therefore, (b) implies (a)

\[ Z_1 \leftrightarrow C_1 \rightarrow C_0 \Rightarrow B_1 = Z_1 \text{ is stably free} \]

\[ Z_2 \leftrightarrow C_2 \rightarrow B_1 \Rightarrow B_2 = Z_2 \text{ is stably free, etc.} \]

This shows that \( B_j = Z_j \) is stably free for \( j < k - 1 \) and that \( Z_{k-1} \) is stably free. Then one has

\[ B_{k-1} \leftrightarrow Z_{k-1} \rightarrow H_{k-1}(\tilde{\psi}) \quad Z_k \leftrightarrow C_k \rightarrow B_{k-1} \quad Z_k \xrightarrow{\sim} H_k(\tilde{\psi}) \]  

(2.4)

Since \( X \) is cohomology silent in degree \( k \), the \( \mathbb{Z}_\pi \)-modules \( H_{k-1}(\tilde{\psi}) \) and \( H_k(\tilde{\psi}) \) are projective by Lemma 2.7 and (2.3). By (2.4), \( B_{k-1} \) and \( Z_k \) are also projective and, by (2.3) again,

\[ [H_k(\tilde{\psi})] = [H_{k-1}(\tilde{\psi})] = -[B_{k-1}] = [Z_k] = [H_k(\tilde{\psi})] = [H_k(\tilde{\psi})]. \]

A slight modification of the proof of Lemma 2.9 shows that \( w_k(X) \) is an invariant of the \( k \)-type of \( X \). More precisely, one has the following

Lemma 2.11. Let \( X_i \) \((i=1,2)\) be two connected CW-complexes of finite type and \( k \geq 3 \) be an integer. Suppose that, for \( i = 1, 2 \), \( X_i \) is cohomology silent in degree \( k \). Let \( h: X_1 \to X_2 \) be a \( k \)-connected map. Then, \( w_k(X_2) = h_*(w_k(X_1)) \), where \( h_*: \tilde{K}_0(\mathbb{Z}_\pi_1(X_1)) \xrightarrow{\sim} \tilde{K}_0(\mathbb{Z}_\pi_1(X_2)) \) is the isomorphism induced by \( h \).

Proof. We assume that \( h \) is cellular. We follow the proof of Lemma 2.9 with the map \( \phi_i: K_i \to X_i \) being the inclusion of \( K_i = X_i^{k-1} \) into \( X_i \). For \( i = 1, 2 \), we get a map \( s_i: X_i^k \to K_i \) such that \( \phi_i \circ s_i \) is homotopic to the inclusion \( X_i^k \to X_i \). We then just need to replace the map \( \psi \) of the proof of Lemma 2.9 by \( \psi = s_2 \circ h \circ \phi_1 \).

Remark 2.12. Lemma 2.11 enables us to define \( w_k(X) \in \tilde{K}_0(\pi_1(X)) \) for a CW-space \( X \) of finite \( k \)-type which is cohomology silent in degree \( k \). Just choose a homotopy equivalence \( f: X' \to X \) where \( X' \) is a CW-complex with finite \( k \)-skeleton and define \( w_k(X) = f_*(w_k(X')) \). Lemma 2.11 guarantees that this is independent of the choice of \( (X', f) \).

The element \( w_k(X) \) will be called the cell-dispensability obstruction of \( X \), thanks to the following proposition, in which the concepts of Definitions A of the introduction are used.

Proposition 2.13. Let \( 3 \leq k \leq \ell \) be integers. Let \( X \) be a connected CW-space of finite type. Suppose that \( X \) is cohomology \((k, \ell)\)-silent. Then, the following conditions are equivalent.

(a) \( w_k(X) = 0 \).

(b) \( X \) is \((k, \ell)\)-cellfree.

Proof. Using Remark 2.12 we assume that \( X \) is a CW-complex with finite skeleta. If \( X \) has no \( k \)-cells, then \( B_{k-1}(X) = 0 \) and thus \( w_k(X) = 0 \) by Part (2) of Remark 2.11. Therefore, (b) implies (a).
Conversely, suppose that (a) holds. As seen in Step 1 of the proof of Lemma 2.7, the module $B_{k-1}(\tilde{X})$ is a retract of $C_{k-1}(\tilde{X})$. We may thus write $C_{k-1}(X) \cong B_{k-1}(\tilde{X}) \oplus T$. If $w_k(X) = 0$, then $B_{k-1}(X)$ is a finitely generated stably free $\mathbb{Z}\pi$-module, and so is $T$. By wedging if necessary $X$ with a finite collection of $D^{k-1}s$ (canceling pairs of $(k-2)$ and $(k-1)$-cells), one may assume that $T$ is $\mathbb{Z}\pi$-free, with basis $\mathcal{B} = \{t_1, \ldots, t_r\}$.

As $k \geq 3$, the complexes $\tilde{X}^{k-1}$ and $X^{k-2}$ are connected and thus the Hurewicz homomorphism

$$\pi_{k-1}(X^{k-1}, X^{k-2}) \xrightarrow{h} H_{k-1}(\tilde{X}^{k-1}, \tilde{X}^{k-2}) = C_{k-1}(\tilde{X})$$

is surjective (see [34, Theorem 7.2, p. 178]). One can thus lift $T$ as a free submodule of $\pi_{k-1}(X^{k-1}, X^{k-2})$ and represent $t_j$ by a map of pair $\tau_j: (D^{k-1}, S^{k-2}) \to (X^{k-1}, X^{k-2})$. Let $K$ be the finite CW-complex of dimension $k - 1$ obtained by adding $(k - 1)$-cells $e_1, \ldots, e_r$ to $X^{k-2}$, the cell $e_j$ being attached by the map $\tau_j: S^{k-2} \to X^{k-2}$. Thus, $C_{k-1}(K) = T$. The inclusion $X^{k-2} \to X$ then extends to a map $f: K \to X$, using the map $\tau_j$ on the cell $e_j$. One checks that $f$ induces an isomorphism on the fundamental groups and on the homology groups $H_i(K) \xrightarrow{\cong} H_i(X)$ for $i \leq k - 1$. But, by Lemma 2.7, $H_j(\tilde{X}) = 0$ for $k \leq j \leq \ell$. Therefore, $f$ is $\ell$-connected and, by Lemma 2.3, there exists a homotopy equivalence $f': K' \to X$ where $K'$ has finite skeleta and no $j$-cells for $k \leq j \leq \ell$.

\begin{proof}[Remark 2.14] In Proposition 2.13 one may wonder that Condition (b) is just $w_k(X) = 0$ and not $w_j(X) = 0$ for $k \leq j \leq \ell$. But, actually, $w_k(X) = w_j(X)$ for $k \leq j \leq \ell$. Indeed, as $H_j(\tilde{X}) = 0$ for $k \leq j \leq \ell$, one has $B_j = Z_j$ and thus an exact sequences $0 \to B_j \to C_j \to B_j \to 0$ for $k \leq j \leq \ell$.
\end{proof}

\begin{proof}[Remark 2.15] Multiple gaps. Let $3 \leq k_1 \leq \ell_1 < k_2 \leq \ell_2 < \cdots < k_q \leq \ell_q$ be integers. One may consider a CW-space of finite type which is cohomology $(k_j, \ell_j)$-silent for all $j$ with $1 \leq j \leq q$. Then the cell-dispensability obstructions $w_{k_j}(X) \in K_0(\mathbb{Z}\pi_1(X))$ are defined for $1 \leq j \leq q$ and Proposition 2.13 is valid for all $j$ simultaneously. For instance, if $w_{k_j}(X) = 0$ for all $j$, then $X$ is homotopy equivalent to a CW-complex having finite skeleta and without $r$-cells for $k_j \leq r \leq \ell_j$, for all $j$.
\end{proof}

We finish this section with the following lemma, which will be used in Section 3.3.

\begin{lemma} Let $(X, T)$ be a $(k - 1)$-connected pair of CW-spaces of finite $k$-type $(k \geq 2)$. Suppose that $X$ and $T$ are cohomology silent in degree $k$. Then, $H_k(\tilde{T}, \tilde{X})$ is a finitely generated projective $\mathbb{Z}\pi$-module $(\pi = \pi_1(X) \cong \pi_1(T))$ and

$$w_k(T) = w_k(X) + [H_k(\tilde{T}, \tilde{X})].$$

\end{lemma}

\begin{proof} We suppose that $(X, T)$ is a pair of CW-complexes with finite $k$-skeleta. Let $K = X^{k-1}$. Then $(X, K)$ and $(T, K)$ are $(k - 1)$-connected and, by 2.1, one has

$$w_k(X) = [H_k(\tilde{X}, \tilde{K})] \quad \text{and} \quad w_k(T) = [H_k(\tilde{T}, \tilde{K})].$$

\end{proof}
Let us consider the homology exact sequence of the triple \((\tilde{T}, \tilde{X}, \tilde{K})\). One has \(H_{k-1}(\tilde{K}, \tilde{X}) = 0\) since \((X, K)\) is \((k - 1)\)-connected. The connecting homomorphism \(H_{k+1}(\tilde{T}, \tilde{X}) \to H_k(\tilde{X}, \tilde{K})\) factors through \(H_k(\tilde{X})\) which vanishes by Lemma \(\ref{lemma:vanishing}\). We thus get a short exact sequence

\[0 \to H_k(\tilde{X}, \tilde{K}) \to H_k(\tilde{T}, \tilde{K}) \xrightarrow{\beta} H_k(\tilde{T}, \tilde{X}) \to 0.\]

We claim that this sequence splits, which will prove the lemma. To construct a section \(\sigma\) of \(\beta\), let us consider the commutative diagram

\[
\begin{array}{ccc}
H_k(\tilde{T}, \tilde{K}) & \longrightarrow & H_{k-1}(\tilde{K}) \\
\beta \downarrow & & \downarrow \phi \\
H_k(\tilde{T}, \tilde{X}) & \longrightarrow & H_{k-1}(\tilde{X})
\end{array}
\]

in which the horizontal lines are exact. The left hand horizontal arrows are injective since \(H_k(\tilde{T}) = 0\) by Lemma \(\ref{lemma:injectivity}\). A seen in the proof of Lemma \(\ref{lemma:vanishing}\) the inclusion \(\varphi: \tilde{K} \to X\), considered as a Serre fibration, admits a section over \(X^k\), whence the section \(\sigma_0\) of \(\tilde{\varphi}\). A chasing argument in Diagram \(\ref{diagram:section}\) then produces the required section \(\sigma\) of \(\beta\).

\[\square\]

### 2.3 Realization of cell-dispensability obstructions

We now give two propositions concerning the realization of elements of \(\tilde{K}_0(\mathbb{Z}\pi)\) as the obstruction \(w_k(X)\) for some cohomology \(k\)-silent complex \(X\) with an identification \(\pi_1(X) \approx \pi\). This identification will be achieved via a reference map to a fixed finite cell complex \(Y\) (see \(\ref{subsection:identification}\)).

**Proposition 2.17.** Let \(Y\) be a connected finite CW-complex of dimension \(k - 1 \geq 2\) and let \(\ell > k\) be an integer. Let \(P \in \tilde{K}_0(\mathbb{Z}\pi_1(Y))\). Then there exists a \(Y\)-referred finite CW-complex \((X_k^\ell, g^\ell)\) of dimension \(\ell + 1\) such that \(X_k^\ell\) is cohomology \((k, \ell)\)-silent and satisfies \(g^\ell(w_k(X)) = P\).

**Proof.** Let us represent the class \((-1)^k P\) by the image \(P\) of a projector \(\text{pr}_P: F \to F\), where \(F\) is a finitely generated free \(\mathbb{Z}\pi_1(Y)\)-module. Then \(F\) admits a supplementary projector \(\text{pr}_Q: F \to F\) with image \(Q = \ker \text{pr}_P\), and \(F = P \oplus Q\). Let \(B = \{b_1, \ldots, b_r\}\) be a basis of \(F\). We introduce some precise notations which will also be used in the proof of Proposition \(\ref{proposition:realization}\). Let

\[
S = S_{k-1}^1 \vee \cdots \vee S_{k-1}^r, \quad D = D_k^1 \amalg \cdots \amalg D_k^r
\]

where \(S_{k-1}^j\) and \(D_k^j\) are copies of the sphere \(S^{k-1}\) and of the disk \(D^k\).

We shall add cells to \(Y\) to get skeleta \(X^j\) of a cell complex \(X\), together with a \(Y\)-reference retraction \(g^\ell: X^j \to Y\), such that \((X^\ell, g^\ell) = (X_k^\ell, g^\ell)\) when \(\ell > k\) (recall from Lemma \(\ref{lemma:reference}\) that an \(Y\)-reference map is anyway, up to homotopy equivalence, equal to a retraction).

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The \((k-1)\)-skeleton of \(X\) is defined to be

\[
X^{k-1} = Y \lor S,
\]

with the \(Y\)-reference retraction \(g^{k-1}: X^{k-1} \to Y\) sending \(S\) to the base point.

Since \(k \geq 3\), \(\pi = \pi_1(Y) \approx \pi_1(X^{k-1})\) and \(\pi_{k-1}(X^{k-1}, Y) \approx H_{k-1}(\tilde{X}^{k-1}, \tilde{Y})\)

is identified with \(F\) (the inclusion \(S^{k-1}_j \to S \to X^{k-1}\) representing the element \(b_j \in F\)). Since \(g^{k-1}\) is a retraction, the homotopy exact sequence of the pair \((X^{k-1}, Y)\) splits

\[
\begin{array}{rccccccccc}
0 & \rightarrow & \pi_{k-1}(Y) & \xrightarrow{g^*_{k-1}} & \pi_{k-1}(X^{k-1}) & \xrightarrow{\partial} & \pi_{k-1}(X^{k-1}, Y) & \rightarrow & 0.
\end{array}
\]

Let

\[
\sigma : \pi_{k-1}(X^{k-1}, Y) \rightarrow \pi_{k-1}(X^{k-1})
\]

be the section sending \(\pi_{k-1}(X^{k-1}, Y)\) isomorphically onto \(\ker g^*_{k-1}\). Form the cell complex

\[
X^k = X^{k-1} \cup gD
\]

where the restriction to \(\partial(D^k_j)\) of the attaching map \(\theta : \partial(D) \to X^{k-1}\) represents \(\sigma \circ \text{pr}_p(b_j) \in \pi_{k-1}(X^{k-1})\). As \(g^*_{k-1} \circ \sigma = 0\), the retraction \(g^{k-1}\) extends to a retraction \(g^k : X^k \to Y\). Again, \(\pi_k(X^k, X^{k-1}) \approx H_k(\tilde{X}^k, \tilde{X}^{k-1})\) is identified with \(F\). One has a diagram whose horizontal lines are exact:

\[
\begin{array}{ccccccccc}
\pi_k(Y) & \xrightarrow{g^*_{k-1}} & \pi_k(X^{k-1}) & \xrightarrow{\pi_k} & \pi_k(X^k, X^{k-1}) & \xrightarrow{\pi_k} & \pi_{k-1}(X^{k-1})
\end{array}
\]

The right square being commutative, there exist homomorphisms \(\tau : Q \rightarrow \pi_k(X^k)\)

making the left square commutative. Since \(g^*_{k-1}\) is onto, there is such a \(\tau\) such that \(g^*_{k} \circ \tau = 0\). Form the cell complex

\[
X^{k+1} = X^k \cup e^{k+1}_1 \cup \cdots \cup e^{k+1}_r,
\]

where the \((k+1)\)-cell \(e^{k+1}_j\) is attached by the element \(\tau \circ \text{pr}_p(b_j) \in \pi_k(X^k)\). Since \(g^*_{k} \circ \tau = 0\), the retraction \(g^k\) extends to a retraction \(g^{k+1} : X^{k+1} \to Y\). One checks that \(H_k(X^{k+1}) = 0\) (one had \(H_k(\tilde{X}^k) \approx Q\)) and that

\[
C_{k-1}(\tilde{X}^{k+1}) \approx C_{k-1}(Y) \oplus Q \oplus B_{k-1}(\tilde{X}^k) \approx C_{k-1}(Y) \oplus Q \oplus P.
\]

By Lemma \[2.7\], the cell complex \(X^{k+1}\) is cohomology silent in degree \(k\) and \(g^{k+1}(w_k(X)) = P\) by Remark \[2.10\]. Thus, for \(\ell = k + 1\), we can just define \(X^{\ell}_k = X^{k+1}\).
Now, since $B_k(\tilde{X}^{k+1}) \approx Q$ is projective, one has isomorphisms of $\mathbb{Z}\pi$-modules

$$
\begin{array}{ccc}
C_{k+1}(\tilde{X}^{k+1}) & \approx & B_k(\tilde{X}^{k+1}) \oplus Z_{k+1}(\tilde{X}^{k+1}) \\
F & \approx & Q \oplus P
\end{array}
$$

Exactly like above, one may attach $r$ cells of dimension $k + 2$ to $X^{k+1}$ to obtain $X^{k+2}$ and the retraction $g^{k+2}$ onto $Y$, so that the boundary homomorphism

$$
\partial : F \approx C_{k+2}(\tilde{X}^{k+2}) \to C_{k+1}(\tilde{X}^{k+1}) \approx Q \oplus P
$$

coincides with $pr_P$. Therefore, $H^{k+1}(\tilde{X}^{k+2}) = 0$. Together with (2.9), this implies by Lemma 2.7 that $X^{k+2}$ is cohomology silent in degree $k + 1$.

The process may be repeated one degree higher, using the diagram like (2.9) with $k$ replaced by $k + 1$ and $P$ and $Q$ being exchanged. Carrying on this procedure, one gets the $Y$-referred pair $(X^\ell_k, g^\ell)$ for $\ell > k$ enjoying the required properties.

Remark 2.18. The pair $(X^\ell_k, Y)$ as constructed in the above proof is weakly $(k - 1)$-connected but not $(k - 1)$-connected. Indeed, the retraction $g^\ell$ splits the homology exact sequence of $(\tilde{X}^\ell_k, \tilde{Y})$

$$
0 \to H_j(\tilde{Y}) \to H_j(\tilde{X}^\ell_k) \to H_j(\tilde{X}^\ell_k, \tilde{Y}) \to 0
$$

and $H_j(\tilde{X}^\ell_k) = 0$ for $k \leq j < \ell$ by Lemma 2.7. Therefore, if $H_{k-1}(\tilde{X}^\ell_k, \tilde{Y}) = 0$, the connectivity of the pair $(X^\ell_k,Y)$ would be $\ell - 1 \geq k$ and thus $w_k(X^\ell_k) = 0$.

Passing to the limit for $\ell \to \infty$, Proposition 2.17 and its proof give rise to the following result which will be used in Section 3.3.

**Proposition 2.19.** Let $Y$ be a connected finite CW-complex of dimension $k - 1 \geq 2$. Let $P \in K_0(\mathbb{Z}\pi_1(Y))$. Then there exists an $Y$-referred space $(X_k, g)$ such that

(a) $X_k$ is of finite type.

(b) $X_k$ is cohomology silent in degree $j$ for all integers $j \geq k$ and satisfies $g_*(w_k(X_k)) = P$.

(c) $X_k$ is homotopy equivalent to a countable CW-complex $L$ of dimension $k$.

(d) For $\ell > k$, the inclusion $X^\ell_k \hookrightarrow X_k$ is a domination. In particular, $X_k$ and $L$ are finitely dominated.

**Proof.** Define $X_k$ as the union of $X^\ell_k$ for $\ell \geq k$, endowed with the weak topology (note that $X^\ell_k \subset X^{\ell+1}_k$). The rejections $g^\ell : X^\ell_k \to Y$ induce the required $Y$-reference retraction $g : X_k \to Y$ and the pair $(X_k, g)$ clearly satisfies (a) and (b).
To prove (d), we replace up to homotopy equivalence the inclusion \( \psi^k: X^k \hookrightarrow X \) by a Serre fibration \( \tilde{\psi}^k: \tilde{X}^k \to X \). Its fiber \( F \) being \((\ell - 1)\)-connected, the obstructions to get a section of \( \tilde{\psi}^\ell \) belong to \( H^i(X; \pi_{-1}(F)) \) for \( i > \ell \). If \( \ell > k \), the latter vanishes by (b). The map \( \tilde{\psi}^k \) thus admits a section and is therefore a domination.

It remains to prove (c). We use the notations of the proof of Proposition 2.17 where \( K \) will now define. One has \( \psi \) the latter vanishes by (b). The map \( \psi \) thus admits a section and is therefore a domination.

The CW-complex \( L \) is then obtained by attaching \( k \)-cells to \( L^{k-1} \):

\[
L = L^{k-1} \cup_{\theta_0,1} D_{0,1} \cup_{\theta_1,2} D_{1,2} \cup \cdots, \tag{2.13}
\]

where \( D_{r,s} \) is a copy of \( D \), with the attaching map \( \theta_{r,s}: \partial(D_{r,s}) \to L^{k-1} \) which we will now define. One has \( \pi_{k-1}(L^{k-1}, Y) \approx F_0 \oplus F_1 \oplus \cdots \), where \( F_j \) is a copy of \( F \). Since the pair \( (L^{k-1}, Y) \) is weakly \((k-2)\)-connected, the homomorphism \( \pi_{k-1}(L^{k-1}) \to \pi_{k-1}(L^{k-1}, Y) \) is onto and one can choose a section \( \sigma_L: \pi_{k-1}(L^{k-1}, Y) \to \pi_{k-1}(L^{k-1}) \) whose restriction to \( F_j \) is denoted by \( \sigma_j \). For \( \sigma_0 \), we make our choice so that the following diagram

\[
\begin{array}{ccc}
F & \approx & \pi_{k-1}(L^{k-1}, Y) \\
\downarrow & & \downarrow \\
F_0 & \approx & \pi_{k-1}(L^{k-1}, Y)
\end{array}
\]

\[
\sigma_0\pi_{k-1}(L^{k-1}) 
\]

is commutative, where the vertical arrows are induced by the inclusion \( K^{k-1} \to L^{k-1} \) and \( \sigma \) is the section of \( \pi_{k-1}(L^{k-1}, Y) \approx F_0 \oplus F_1 \oplus \cdots \). The attaching map \( \theta_{r,s}: \partial(D_{r,s}) \to L^{k-1} \) is now chosen so that its restriction to \( \partial(D_{r,s}^k) \) represents the class \( \sigma_s \circ \text{pr}_F(b_j) + \sigma_s \circ \text{pr}_Q(b_j) \in \pi_{k-1}(L^{k-1}) \). One can thus write

\[
C_k(\tilde{L}) \approx F \oplus F \oplus \cdots \approx (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \tag{2.14}
\]

so that the boundary homomorphism \( \partial: C_k(\tilde{L}) \to C_{k-1}(\tilde{L}) \) sends the \( j \)-th term \((P + Q)\) of \( C_k(\tilde{L}) \) isomorphically, through \( \text{pr}_P \oplus \text{pr}_Q \), to the corresponding
The inclusion chain reflects the very definition of \( \varphi \). This boundary homomorphism being injective, one has \( H_k(\tilde{L}) = 0 \).

The above definitions of \( K \) and \( L \) guaranty the existence of a map \( f: L \to K = X_k \) extending the obvious identification \( Y \lor S_0 \cong Y \lor S \) and, for \( j \geq 1 \), sending \( S_j \) and \( D_{j,j+1} \) to the base point of \( K \). The pairs \((X_k,Y)\) and \((L,Y)\) being weakly \((k-1)\)-connected, so is the map \( f \). The homomorphism \( C_{k-1}f: C_{k-1}(\tilde{L}) \to C_{k-1}(\tilde{K}) \) sends the first summands \( C_{k-1}(\tilde{Y}) \oplus Q \)

of \( (2.11) \) isomorphically those of \( (2.10) \). But these summands provide isomorphisms \( H_{k-1}(\tilde{Y}) \oplus Q \cong H_{k-1}(\tilde{L}) \) and \( H_{k-1}(\tilde{Y}) \oplus Q \cong H_{k-1}(\tilde{K}) \). Therefore, \( H_{k-1}f: H_{k-1}(\tilde{L}) \to H_{k-1}(\tilde{K}) \) is an isomorphism. Since \( H_j(\tilde{L}) = 0 = H_j(\tilde{K}) \) for \( j \geq k \), the map \( f \) is a homotopy equivalence.

\[ \square \]

Here are a few remarks about Proposition 2.19 and its proof.

**Remark 2.20.** When \( P \neq 0 \), the complex \( X_k \) of Proposition 2.19 is of finite type and is homotopy equivalent to a complex without \( j \)-cells for \( k \leq j \leq \ell \) (but not simultaneously). So \( X_k \) is not \((k,\ell)\)-cellfree. \( \square \)

**Remark 2.21.** The inclusion \( \beta: K^{k-1} \hookrightarrow L \) is \((k-1)\)-connected and \( K^{k-1} \) is a finite complex. Since \( H_k(\tilde{L}) = 0 \), one has

\[ 0 \to H_k(\tilde{\beta}) \to H_{k-1}(\tilde{K}^{k-1}) \to H_{k-1}(\tilde{L}) \to 0 \]

Therefore, \( \pi_k(\beta) \approx H_k(\tilde{\beta}) \approx P \) and, in \( \tilde{K}_0(\mathbb{Z}\pi) \), one gets

\[ w_k(L) = (1)^k[\pi_k(\beta)] = (1)^k[P] = P \]

as expected. \( \square \)

**Remark 2.22.** The CW-complex \( L \) (or \( X_k \)) fulfills the conditions for the Wall finiteness obstruction \( \text{Wall}(L) \in \tilde{K}_0(\mathbb{Z}\pi_1(L)) \) to be defined \([28, \text{Theorem F}]\). Using Remark 2.21, the obstruction \( \text{Wall}(L) \) is equal to \( P \), thus equal to \( w_k(L) \), as claimed in Remark 2.10. \( \square \)

**Remark 2.23.** The CW-complex \( L \) may be obtained as the direct limit of a nested system

\[ Y \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \ldots \hookrightarrow L_\infty = L \overset{\varphi}{\to} Y \quad (2.15) \]

where

\[ L_1 = Y \lor S \quad \text{and} \quad L_{j+1} = L_j \lor S \cup D \quad \text{for} \quad j \geq 1 \]

and such that the composition of maps in \( 2.15 \) is the identity of \( Y \). Indeed, the inclusion chain reflects the very definition of \( L \). Let \( f: L \to X_k \) be the homotopy equivalence constructed to prove Part (c) of Proposition 2.19. Then, the composition of \( f \) with the \( Y \)-reference map \( g: X_k \to Y \) gives an \( Y \)-reference retraction from \( L \) onto \( Y \), also called \( g: L \to Y \). Its restriction to \( L_j \) provides an \( Y \)-reference retraction \( g_j: L_j \to Y \). This material is bound to be used in Section 3.3. \( \square \)
2.4 Relationship with Wall’s finiteness obstruction

A certain relationship between the cell-dispensability obstruction \(w_k(X)\) and the Wall’s finiteness obstruction was already noticed in Remark 2.10. The following variant of Proposition 2.19 may shed more light about this matter.

**Proposition 2.24.** Let \(X\) be a finite CW-complex of dimension \(\ell\) which is cohomology \((k,\ell)\)-silent. Then, there is a finitely dominated CW-complex \(X'\) containing \(X\) as a subcomplex, such that \((X',X)\) is \(\ell\)-connected and \(X'\) is homotopy equivalent to a countable CW-complex of dimension \(k\). Two such complexes \(X'_1\) and \(X'_2\) are homotopy equivalent relative \(X\). Moreover, the image of \(w_k(X)\) into \(\tilde{K}_0(X')\) coincides with \(\text{Wall}(X')\).

**Proof.** Starting with \(X\) we apply the process of the proof of Propositions 2.17 and 2.19, ending with the space \(X'_1 = X_k\) of Proposition 2.19, which enjoys the required properties. If \(X'_1\) and \(X'_2\) are two such complexes, as \(X'_1\) is cohomology silent in degree \(\geq k\), there is no obstruction to extend \(\text{id}_X\) to a map \(f: X'_1 \to X'_2\), which will be a homotopy equivalence, since \(H_j(X'_1) = 0 = H_j(X'_2)\) for \(j \geq k\) (by Lemma 2.7). Finally, the complex \(X'\) fulfills the conditions for the Wall finiteness obstruction \(\text{Wall}(X') \in \tilde{K}_0(\mathbb{Z}\pi_1(X'))\) to be defined [28, Theorem F] and the homological algebra definitions of \(w_k(X)\) and of \(\text{Wall}(X')\) make them to be equal modulo the identification \(\pi_1(X) \approx \pi_1(X')\).

Another consequence of the formally identical homological algebra definitions of \(w_k(X)\) and of \(\text{Wall}(X')\) is the following product formula. For the Wall obstruction, this is a particular case of the product formula proven in Siebenmann’s thesis [23, Theorem 7.2] (see also [27, Theorem 3.14, p. 169]).

**Proposition 2.25** (**Product formula**). Let \(X\) be a connected CW-space of finite type which is cohomology \((k,\ell)\)-silent for \(k \geq 3\). Let \(A\) be a connected finite CW-complex of dimension \(a \leq \ell - k\). Then \(X \times A\) is cohomology \((k + a,\ell)\)-silent and the equality

\[w_{k+a}(X \times A) = \chi(A) i_*(w_k(X))\]

holds in \(\tilde{K}_0(\mathbb{Z}\pi_1(X \times A)))\), where \(\chi(A)\) is the Euler characteristic of \(A\) and \(i_*: \tilde{K}_0(\mathbb{Z}\pi_1(X)) \to \tilde{K}_0(\mathbb{Z}\pi_1(X \times A))\) is induced by the inclusion \(i: \pi_1(X) \to \pi_1(X \times A) \approx \pi_1(X) \times \pi_1(A)\) sending \(u \in \pi_1(X)\) to \((u,1)\).

**Proof.** The homological algebra’s arguments used in [27, pp. 168–170] to prove the same formula for the Wall finiteness obstruction work as well to establish the proposition.

**Corollary 2.26.** Let \(X\) be a connected CW-space of finite type which is cohomology \((k,\ell)\)-silent for \(k \geq 3\). Then \(X \times S^1\) is \((k + 1,\ell)\)-cellfree.

For more applications of Proposition 2.25 see Proposition 3.32.
3 Antisimplicity

3.1 Preliminaries

3.1. Homology with local coefficients. While cohomology with local coefficients was defined for an arbitrary pair (see [2.6]), homology with local coefficients is defined for a pair over \( \mathbb{R} P^\infty \). A space over \( \mathbb{R} P^\infty \) is a space \( X \) together with a (homotopy class of) map \( w: X \to \mathbb{R} P^\infty \). A pair \( (X, Z) \) is said to be over \( \mathbb{R} P^\infty \) if both \( X \) and \( Z \) are over \( \mathbb{R} P^\infty \) with \( w_Z \) being the restriction of \( w_X \) to \( Z \).

For example, the space \( BO \) is over \( \mathbb{R} P^\infty \) by the classifying map \( \omega_{BO}: BO \to B\pi_1(BO) \simeq B\{\pm 1\} \simeq \mathbb{R} P^\infty \).

Thus, a stable vector bundle \( \eta \) over \( X \), classified by \( \Phi_\eta: X \to BO \), makes \( X \) a space over \( \mathbb{R} P^\infty \) by the map \( \omega_\eta = \omega_{BO} \circ \Phi_\eta \), called the orientation character of \( \eta \).

In the case of a (smooth) manifold \( M \), the orientation character of its stable normal (or tangent) bundle gives the orientation character \( \omega_M: M \to \mathbb{R} P^\infty \) of \( M \).

When \( X \) is a CW-space, the functor \( \pi_1 \) provides a bijection \([X, \mathbb{R} P^\infty] \to \text{hom}(\pi_1(X), \{\pm 1\}) \simeq \mathbb{Z} \pi \) (see [13, p. 15]). For example, let \((M, \partial M)\) be a compact manifold pair, with \( M \) connected. When \( R = \mathbb{Z} \) with trivial \( \pi_1(M) \)-action, then \( H_*(M, \partial M; \mathbb{Z}) \) is the ordinary homology of \((M, \partial M)\) with integral coefficients when \( M \) is orientable, and that of the orientation cover \((M^{or}, \partial M^{or})\) otherwise. In any case, \( H_n(M, \partial M; \mathbb{Z}) \approx \mathbb{Z} \). A compact connected manifold of dimension \( n \) is always supposed to be equipped with a generator \( [M] \) of \( H_n(M, \partial M; \mathbb{Z}) \approx \mathbb{Z} \), called the fundamental class of \( M \).

3.2. Poincaré spaces. Let \( P \) be a finitely dominated CW-space over \( \mathbb{R} P^\infty \), together with a class \([P] \in H_n(P; \mathbb{Z})\). We say that \( P \) is a Poincaré space of formal dimension \( n \) if the the cap product with \([P]\)

\[
\cup [P]: H^r(P, Z; R) \xrightarrow{\cup} H_{n-r}(P; R) \tag{3.16}
\]
is an isomorphism for any \( \mathbb{Z}\pi_1(P) \)-module \( R \). More generally, let \( (P,Q) \) be a pair over \( \mathbb{R}P^\infty \) of finitely dominated CW-spaces, together with a class \([P] \in H_n(P,Q)\). We say that \((P,Q)\) is a Poincaré pair of formal dimension \( n \) if the following two conditions hold true

(a) the cap product with \([P]\)

\[
- \smile [P] : H^r(P,Q; R) \xrightarrow{\approx} H_{n-r}(P; R)
\]  

(3.17)

is an isomorphism for any \( \mathbb{Z}\pi_1(P) \)-module \( R \), and

(b) the space \( Q \) together with the class \([Q] = \partial_*([P])\) is a Poincaré space of formal dimension \( n-1 \).

We may also say that \( P \) is a Poincaré space of formal dimension \( n \), with boundary \( \partial P = Q \). This boundary is then a Poincaré space without boundary.

Finally, by a Poincaré cobordism \((P,Q_1,Q_2)\), we mean a pair \((P,Q)\) of finitely dominated CW-spaces with \( Q = Q_1 \sqcup Q_2 \), together with a class \([P] \in H_n(P,Q)\) such that (a) above is satisfied and such that \( Q_1 \) is a Poincaré space with fundamental class \([Q_1]\) such that \( \partial_*([P]) = [Q_1] - [Q_2] \).

**Remarks 3.3.** (1) A compact \( n \)-manifold pair \((M,\partial M)\) is a Poincaré space of formal dimension \( n \). This follows from the standard proof of the Poincaré-Lefschetz duality using a \( C^n \)-triangulation of \( M \) and its dual cell decomposition. (2) By [13] Theorem B], Condition (a) is equivalent to

(a') the cap product with \([P]\)

\[
- \smile [P] : H^r(P; R) \xrightarrow{\approx} H_{n-r}(P; Q; R)
\]  

(3.18)

is an isomorphism for any \( \mathbb{Z}\pi_1(P) \)-module \( R \).

Hence, if (a) and/or (a') is true, one has a sign-commutative diagram for the (co)homology in a \( \mathbb{Z}\pi_1(P) \)-module \( R \):

\[
\begin{array}{cccccc}
H^j(P,Q) & \xrightarrow{-[P]} & H^j(P) & \xrightarrow{-[P]} & H^j(Q) & \xrightarrow{-[P]} & H^{j+1}(P,Q) & \xrightarrow{-[P]} & H^{j+1}(P) \\
\approx & & \approx & & \approx & & \approx & & \approx & \\
H_{n-j}(P) & \xrightarrow{-[P]} & H_{n-j}(P,Q) & \xrightarrow{\approx} & H_{n-1-j}(Q) & \xrightarrow{\approx} & H_{n-1-j}(P) & \xrightarrow{\approx} & H_{n-1-j}(P,Q) \\
\end{array}
\]

By the five lemma, \( - \smile [Q]: H^j(Q; R) \to H_{n-1-j}(Q; R) \) is an isomorphism for any \( \mathbb{Z}\pi_1(P) \)-module \( R \), but this may be not the case if \( R \) is a \( \mathbb{Z}\pi_1(Q) \)-module (see [13] Theorem A]). Thus, Conditions (a) and (b) are independent in general. Note however, that (a) implies (b) when \( \pi_1(Q) \approx \pi_1(P) \).

**3.4. Spivak fibrations.** Let \( BG \) be the classifying space for stable spherical fibration. Let \((X,Y)\) be a pair over \( BG \), i.e. a pair \((X,Y)\) together with a map \( g: X \to BG \). Consider the graded abelian group

\[
H_*(X,Y) = \pi_n^s(T(\xi), T(\xi(Q))) , \quad T() = \text{Thom spectrum ,}
\]

17
where $\xi$ is the pull-back by $g$ of the universal stable spherical fibration over $BG$. Then $H_\ast$ is a homology theory for spaces over $BG$.

The natural maps $BG \to \mathbb{R}P^\infty$ makes a space over $BG$ a space over $\mathbb{R}P^\infty$. Therefore, the homology $H_\ast(X,Y;R)$ with coefficient in a $\mathbb{Z}\pi_1(X)$-module $R$, in the sense of 3.1, is defined. When $R = \mathbb{Z}$ with trivial $\pi_1(X)$-action, one has degree 0 graded homorphism

$$t : H_\ast(X,Y) \to H_\ast(X,Y;\mathbb{Z}) \quad (3.19)$$

obtained by the composition of the Hurewicz map with the Thom isomorphism.

Let $(P,Q)$ be a Poincaré pair of formal dimension $n$. A Spivak fibration for $(P,Q)$ is a stable spherical fibration $\xi$ over $P$, classified by a lifting $\omega_P$ of $\omega_{\partial P}$, such that there exists a class $\xi_j \in H_n(X,Y)$ satisfying $t(\xi_j) = [P]$. We call $(P)$ a Spivak class. In our context of (finitely dominated) Poincaré spaces, Spivak fibrations do exist and, if $\xi_j (j = 1, 2)$ are two such fibrations with Spivak classes $(P_j)$, there is a fibre homotopy equivalence from $\xi_1$ to $\xi_2$ sending $(P_1)$ to $(P_2)$ [30, Corollary 3.6].

**Remark 3.5.** Although our definitions are inspired by those of [13], there are some differences. What we call a Poincaré space is called a PD-space there. A Poincaré space in [13] is a PD-space together with a Spivak class. The present paper is self-contained with respect to [13].

3.6. $\eta$-normal maps. Let $Y$ be a connected CW-space and let $\eta$ be a stable vector bundle over $Y$, with characteristic map $\Phi_\eta : Y \to BO$.

**Definitions 3.7.** (A) A map $q : Z \to Y$, from a manifold $Z$ to $Y$ is called $\eta$-normal if $\Phi_\eta \circ q$ classifies the stable normal bundle $\nu_Z$ of $Z$.

(B) If $(P,Q)$ is a Poincaré pair, a map $q : P \to Y$ is called $\eta$-normal if the composite

$$P \xrightarrow{q} Y \xrightarrow{\Phi_\eta} BO \to BG$$

classifies a Spivak fibration for $(P,Q)$.

The following result is well known.

**Lemma 3.8.** (1) A map from a compact manifold $M$ to $Y$ which is $\eta$-normal in the sense of (A) is $\eta$-normal in the sense of (B) for the pair $(M, \partial M)$.

(2) Let $f : (P', Q') \to (P, Q)$ be a map of degree 1 (i.e. $f_*([P']) = [P]$) between Poincaré pairs and let $q : P \to Y$ be a map. If the map $q \circ f$ is $\eta$-normal, so is the map $q$.

(3) If $q : P \to Y$ is $\eta$-normal for the Poincaré pair $(P,Q)$, then its restriction to the Poincaré space $Q$ is $\eta$-normal.

**Proof.** The Thom-Pontryagin class $(M)_{TP}$ is a Spivak class for $(M, \partial M)$, which proves (1). For (2), we note that the class $f_*([P])$ is a Spivak class for $(P,Q)$. For (3), we use that the restriction over $Q$ of a Spivak fibration for $(P,Q)$ is a Spivak fibration for $Q$, taking $(Q) = \partial_*([P])$. \[\square\]
In most of our applications, \( Y \) will be a finite complex of dimension \( k - 1 \) and \( q \) will be an \( Y \)-reference map. If \( q \) is \( \eta \)-normal, we call \( q \) an \( (Y, \eta) \)-reference map.

Let \( (P, Q) \) be a Poincaré pair of formal dimension \( n \) and let \( q: P \to Y \) be an \( \eta \)-normal map. The Spivak fibration \( \xi \) over \( (P, Q) \) is thus endowed with the vector bundle reduction \( q^* \eta \). This corresponds to a surgery problem (normal map of degree one) \( \alpha: (M, \partial M) \to (P, Q) \), where \( M \) is a compact \( n \)-manifold and \( q \circ \alpha \) is \( \eta \)-normal. Such a data determines a surgery obstruction in the Wall group \( L^n_\eta(\pi_1(P), \pi_1(Q), \omega_P) \), vanishing if and only if \( \alpha \) is normally cobordant to a homotopy equivalence. When \( Q = \emptyset \), we denote by \( \sigma(P, q) \in L^n_\eta(\pi_1(Y), \omega_\eta) \) the image by \( q^* \) of this surgery obstruction.

3.9. \( \eta \)-normal Poincaré decompositions. Let \( (X_{\pm}, \partial X_{\pm}) \) be two CW-pairs with \( \partial X_{-} = Z_{-} \amalg Z_{0} \) and \( \partial X_{+} = Z_{0} \amalg Z_{+} \). Form the CW-pair \((X, \partial X)\) where

\[
X = X_{-} \cup_{Z_{0}} X_{+}, \quad \partial X = Z_{-} \amalg Z_{+}.
\] (3.20)

For the sake of simplicity, we assume that the homomorphisms \( \pi_1(\partial X_{\pm}) \to \pi_1(X_{\pm}) \) and \( \pi_1(Z_{0}) \to \pi_1(X_{\pm}) \) induced by the inclusions are all isomorphisms.

Let \( Y \) be a connected CW-space and let \( \eta \) be a stable vector bundle over \( Y \). Let \( q: X \to Y \) be a map, restricting to \( q_{\pm}: X_{\pm} \to Y \). These maps make \( X \), \( \partial X \), \( X_{\pm} \), etc, spaces over \( \text{BO} \), \( \text{BG} \) and \( \mathbb{R}P^\infty \). Consider their homology \( H_{n}(()) \) with coefficients in \( \mathbb{Z} \) with trivial \( \mathbb{Z}\pi_1(Y) \)-action. Define \( \beta_{H}: H_{n}(X, \partial X) \to H_{n}(X_{-}, \partial X_{-}) \oplus H_{n}(X_{+}, \partial X_{+}) \) by the following diagram

\[
H_{n}(X, \partial X) \xrightarrow{\beta_{H}} H_{n}(X; \partial X \amalg Z_{0}) \xrightarrow{\approx \text{excis.}} H_{n}(X_{-}, \partial X_{-}) \oplus H_{n}(X_{+}, \partial X_{+})
\] (3.21)

**Definition 3.10.** We say that \((X, \partial X)\) is an \( \eta \)-normal Poincaré decomposition if \((X, \partial X)\) and \((X_{\pm}, \partial X_{\pm})\) are Poincaré pairs of formal dimension \( n \) satisfying \( \beta_{H}([X]) = ([X_{-}], [X_{+}]) \) and such that \( q \) and \( q_{\pm} \) are \( \eta \)-normal.

We now give results to deduce from weaker hypotheses that a decomposition like in \((X, \partial X)\) is an \( \eta \)-normal Poincaré decomposition. By excision, one has the isomorphism

\[
H_{*}(\partial X) \xrightarrow{\approx} H_{*}(\partial X \amalg Z_{0}, Z_{0}).
\] (3.22)

Diagram \((3.21)\) and isomorphism \((3.22)\) may be considered as well for the homology theory \( H_{*} \) over \( BG \). Therefore, the exact sequences of the triple \((X, \partial X \amalg Z_{0}, Z_{0})\) and the homomorphism \( t \) of \((3.19)\) give rise to a morphism of exact sequences

\[
H_{n}(X, \partial X) \xrightarrow{\beta_{H}} H_{n}(X_{-}, \partial X_{-}) \oplus H_{n}(X_{+}, \partial X_{+}) \xrightarrow{\partial_{H}} H_{n-1}(Z_{0})
\]

\[
H_{n}(X, \partial X) \xrightarrow{\beta_{H}} H_{n}(X_{-}, \partial X_{-}) \oplus H_{n}(X_{+}, \partial X_{+}) \xrightarrow{\partial_{H}} H_{n-1}(Z_{0})
\] (3.23)
Proposition 3.11. Let \((X) \in \mathcal{H}_n(X, \partial X)\) such that \(\beta_{\mathcal{H}}((X)) = ((X_), (X_+)).\)

(i) Suppose that \((X_+, \partial X_+)\) are Poincaré pairs of formal dimension \(n\) with \([X_+] = t((X_+))\) (in other words, \(q_+\) is \(\eta\)-normal and \((X_+)\) are Spivak classes).

Then \((X, \partial X)\) is a Poincaré pair of formal dimension \(n\) with \([X] = t((X))\).

(ii) Suppose that \((X, \partial X)\) and \((X_-, \partial X_-)\) are Poincaré pairs of formal dimension \(n\) with \([X] = t((X))\) and \([X_-] = t((X_-))\). Then \((X_+, \partial X_+)\) is a Poincaré pair of formal dimension \(n\) with \([X_+] = t((X_+))\).

Consequently, in both cases, (3.20) is \(\eta\)-normal Poincaré decomposition.

Proof. The requirements about finite dominations are guaranteed by [25, Complement 6.6]. The Poincaré duality statements are proven in [30, Theorem 2.1 and its addendum]. The assertions about the Spivak classes are then deduced from the commutativity of the left square in Diagram (3.23), using for (i) that \(\beta_{\mathcal{H}}\) is injective.

Corollary 3.12. Let \((X_+) \in \mathcal{H}_n(X, \partial X)\) such that \(\partial_{\mathcal{H}}((X_-), (X_+)) = 0\). Suppose that \((X_+, \partial X_+)\) are Poincaré pairs of formal dimension \(n\) with \([X_+] = t((X_+))\). Then \((X, \partial X)\) is a Poincaré pair of formal dimension \(n\) and \(q\) is \(\eta\)-normal. Consequently, (3.20) is \(\eta\)-normal Poincaré decomposition.

Proof. Since the top line in (3.23) is exact, there is a (possibly non-unique) class \((X) \in \mathcal{H}_n(X, \partial X)\) such that \(\beta_{\mathcal{H}}((X)) = ((X_), (X_+))\). The corollary then follows from Part (i) of Proposition 3.11 (note that the fundamental class \([X]\) is unique since \(\beta_{\mathcal{H}}\) is injective).

Part (ii) of Proposition 3.11 may be shortened in the following

Corollary 3.13. Suppose that \((X, \partial X)\) and \((X_-, \partial X_-)\) are Poincaré pairs of formal dimension \(n\) and that \(q\) and \(q_-\) are \(\eta\)-normal. Then \((X_+, \partial X_+)\) is a Poincaré pairs of formal dimension \(n\) and \(q_+\) is \(\eta\)-normal. In other words, (3.20) is an \(\eta\)-normal Poincaré decomposition.

3.14. Stable \(\eta\)-thickenings. Let \(Y\) be a connected finite CW-complex of dimension \(k - 1 \geq 2\) and let \(\eta\) be a stable vector bundle over \(Y\). An \((Y, \eta)\)-referred manifold \((N, g)\) is called an \(\eta\)-thickening of \(Y\) of dimension \(n\) if

- \(N\) is a compact connected manifold of dimension \(n\) and \(g\) is a simple homotopy equivalence;
- the pair \((N, \partial N)\) is \((n - k)\)-connected (with \(n - k \geq 2\)).

(Recall that \(\dim Y = k - 1\).) Two \(\eta\)-thickenings \((N_1, g_1)\) and \((N_2, g_2)\) of \(Y\) are regarded as equivalent if there is a degree-one diffeomorphism \(h: N_1 \overset{\approx}{\to} N_2\) such that \(g_2 \circ h\) is homotopic to \(g_1\). In the stable range \(n > 2(k - 1)\), one has the following

Proposition 3.15 (Wall). Let \(Y\) be a connected finite cell complex of dimension \(k - 1\) and let \(\eta\) be a stable vector bundle over \(Y\). Then, if \(n > 2(k - 1) \geq 6\), there is an unique equivalence class \(\Xi_{n}(Y, \eta)\) of \(\eta\)-thickenings of \(Y\) in dimension \(n\). \(\Box\)
A representative \((N, g)\) of \(\Xi_n(Y, \eta)\) is called a (or sometimes the) stable \(\eta\)-thickening of \(Y\) of dimension \(n\).

**Proof.** This follows from Wall’s classification of stable thickenings by their stable tangent bundle \([29]\) Proposition 5.1, whence by their stable normal bundle. More precisely for the uniqueness, Wall’s theorem provides a diffeomorphism between any two stable thickenings, which may be not of degree one. But a stable thickening of \(N\) is of the form \(N_− \times I\) (see Remark 3.16 below) and thus \(N\) admits a self-diffeomorphism of degree \(-1\).

**Remark 3.16.** When \(n > 2k - 1 \geq 6\), the boundary \(\partial N\) of a stable \(\eta\)-thickening \((N, g)\) is a \(k\)-antisimple manifold (See Definition B in the introduction). Indeed, let \(\gamma_0: Y \to N\) be a homotopy inverse of \(g\). As \((N, \partial N)\) is \((n-k)\)-connected and \(n-k > k - 1\), the map \(\gamma_0\) is homotopic to a map \(\gamma: Y \to \partial N\). Up to homotopy equivalence, one may assume that \(Y\) is a finite simplicial complex and that \(\gamma\) is an embedding into a \(C^\infty\)-triangulation of \(\partial N\). Then, \(\gamma(Y)\) has a smooth regular neighborhood \(N_−\) \([15]\) and \((N_−, g_1|N_−)\) is a stable \(\eta\)-thickening of \(Y\) of dimension \(n-1\). By the s-cobordism theorem, the manifold \(N\) is diffeomorphic to \(N_− \times [0, 1]\) and one thus gets a manifold triad \((N, N_− \cup \partial N_− \times [0, 1], N_− \times \{1\})\). But the pair \((N_−, \partial N_−)\) is \((n-k-1)\)-connected and thus admits a handle decomposition with handles of index \(\leq k - 1\). Therefore, \(\partial N\) is \(k\)-antisimple.

The uniqueness of the Spivak class admits a different form for stable thickenings.

**Lemma 3.17.** Let \((N_j, g_j)\) \((j = 1, 2)\) be two stable \(\eta\)-thickenings of \(Y\) of dimension \(n > 2(k-1) \geq 6\), with Spivak class \((N_j) \in H_a(N_j, \partial N_j)\). Then there exists a simple homotopy equivalence \(f: (N_1, \partial N_1) \to (N_2, \partial N_2)\) satisfying \(g_1 \simeq f \circ g_2\) and \(H_a f((N_1)) = (N_2)\).

**Proof.** The data \((N_j, g_j, (N_j))\) determine surgery problems \(\alpha_j: (N_j, \partial N_j) \to (N_j, \partial N_j)\) such that \(H_a \alpha_j((N_j)_{TP}) = (N_j)\), where \((N_j)_{TP} \in H_a(N_j, \partial N_j)\) is the Thom-Pontryagin class of \(N_j\). By Wall’s \((\pi - \pi)\)-theorem, one may suppose that \(f_j\) are simple homotopy equivalences. Hence, \((N_j, g_j \circ f_j)\) are stable \(\eta\)-thickenings of \(Y\). By Proposition 3.14 there is a diffeomorphism \(h: N_1 \to N_2\) such that \(g_2 \circ h \circ \alpha_1 = \alpha_2\). Being a diffeomorphism of degree one, \(h\) satisfies \(H_a h((N_1)_{TP}) = (N_2)_{TP}\). Therefore the map \(f = \alpha_2 \circ h \circ \alpha_1\) satisfies the conclusion of Lemma 3.17.

**3.18.** Ends of manifolds. A general reference for ends of spaces is \([10]\). We restrict to the case of a \(\sigma\)-compact finitely dominated open manifolds \(U\) with one end \(\epsilon\). Such an end is called tame \([10],\) Definition 10, p. xiii if it admits a sequence \(U \supset Z_1 \supset Z_2 \supset \cdots\) of finitely dominated closed neighborhoods with

\[
\bigcap_{j=1}^{\infty} Z_j = \emptyset , \quad \pi_1(Z_1) \approx \pi_1(Z_2) \approx \cdots \approx \pi_1(\epsilon).
\]
Recall that, according to our definition in 3.2, a Poincaré pair is finitely dominated.

**Proposition 3.19.** Let \( U \) be a connected \( \sigma \)-compact finitely dominated \( n \)-dimensional manifold with compact boundary \( \partial U \) and with one end \( \epsilon \), which is tame and satisfies \( \pi_1(\epsilon) \cong \pi_1(U) \). Set \( P = Wall(U) \in \tilde{K}_0(\pi_1(U), \omega_U) \). Then there exists a Poincaré cobordism \( (U, \partial U, P) \), with \( \pi_1(P) \approx \pi_1(U) \), together with a homotopy equivalence \( \alpha: U \to \mathbb{U} \). Moreover, with the identification \( \pi_1(P) \approx \pi_1(U) \approx \pi_1(\mathbb{U}) \) obtained using \( \alpha \), the formulae

\[
Wall(U) = P \quad \text{and} \quad Wall(P) = P + (-1)^nP^*
\]

hold true in \( \tilde{K}_0(\pi_1(U), \omega_U) \).

**Proof.** We use the space of ends \( e(U) \) of \( U \), which is the space of proper maps \( c: [0, \infty) \to U \) [16, Definition 1.2]. It is equipped with the origin map \( \vartheta: e(U) \to U \), given by \( \vartheta(c) = c(0) \). Define \( \mathbb{U} \) to be the mapping cylinder of \( \vartheta \):

\[
\mathbb{U} = e(U) \times [0, 1] \sqcup U / \{(c, 1) \sim c(0)\},
\]

which retracts by deformation onto \( U \), so the inclusion \( \alpha: U \to \mathbb{U} \) is a homotopy equivalence. Set \( P = e(U) \).

The conclusions of the proposition come from [16, Proposition 10.5]. We just need to check that the hypotheses of the latter, i.e. forward and reverse tameness of \( \epsilon \), are implied by our tameness hypothesis. This is guaranteed by [16, Proposition 8.9 and 10.13]. \( \square \)

### 3.2 Cohomology anti-simple manifolds

Let \( M \) be a compact connected (smooth or PL)-manifold of dimension \( n \) and let \( k \) be an integer with \( n \geq 2k \). We shall compare the two Definitions B of the introduction, i.e. \( M \) is \( k \)-antisimple if it admits a handle decomposition without handles of index \( j \) for \( k \leq j \leq n - k \) and \( M \) is cohomology \( k \)-antisimple if it is cohomology \((k, n-k)\)-silent.

Obviously, a \( k \)-antisimple-manifold is cohomology \( k \)-antisimple. Only the empty manifold is cohomology 0-antisimple. A closed manifold is 1-antisimple if and only if it is a homotopy sphere.

The above definition may also be used for Poincaré spaces (with or without boundary) of formal dimension \( n \). It will be mainly used for closed manifold. In this case, “cohomology antisimple” could be called “homology antisimple” thanks to the following

**Proposition 3.20.** Let \( M \) be a closed connected manifold of dimension \( n \) (or a Poincaré space of formal dimension \( n \) without boundary). The following condition are equivalent.

1. \( M \) is cohomology \( k \)-antisimple.
2. \( H_j(M; R) = 0 \) for any \( \mathbb{Z}\pi_1(M) \)-module \( R \) when \( k \leq j \leq n - k \).
Proof. Since, by Poincaré duality, $H^j(M; R) \approx H_{n-j}(M; R)$.}

By Section 2.2, a cohomology $k$-antisimple compact manifold $M$ determines a cell-dispensability obstruction $w_k(M) \in \tilde{K}_0(\pi_1(M))$. We call it the antisimple obstruction for $M$, thanks to the following

**Proposition 3.21.** Let $M$ be a closed connected manifold of dimension $n \geq 2k \geq 6$ which is cohomology $k$-antisimple. The following conditions are equivalent.

(a) $M$ is $k$-antisimple.

(b) $w_k(M) = 0$.

Proof. If $M$ is $k$-antisimple, then $M$ has the homotopy type of a finite CW-complex $X$ having no cells in dimension $j$ for $k \leq j \leq n - k$. By Lemma 2.11 one has $0 = w_k(X) = w_k(M)$, which proves (a) ⇒ (b).

Conversely, suppose that $w_k(M) = 0$. As $M$ is cohomology $k$-antisimple and $k \geq 3$, there is by Proposition 2.13 a homotopy equivalence $f: Y \to M$, where $Y$ is a finite CW-complex without $j$-cells for $k \leq j \leq n - k$. Let $K$ be a finite simplicial complex homotopy equivalent to $Y^{k-1}$. As $n \geq 2k$ one may suppose that $f$ is an PL-embedding of a $C^\infty$-triangulation of $M$. The complex $f(K)$ admits a smooth regular neighborhood $T$, which admits a handle decomposition with handles of index $\leq k - 1$. Let $V = M - \operatorname{int} T$, which we see as a cobordism between $N = \operatorname{Bd} T$ and the empty set. As $n \geq 2k \geq 6$, one has $\pi_1(N) \approx \pi_1(V)$ and $H_j(V, \bar{N}) \approx H_j(M, \bar{T}) = 0$ for $j \leq n - k$. The procedure of eliminating handles in a cobordism (see [17]) then produces a handle decomposition of $(V, N)$ with only handles of index $> n - k$, thus proving (a).}

We now turn our attention to realizing elements of $\tilde{K}_0(\mathbb{Z}\pi)$ as antisimple obstructions $w_k(M)$ for a cohomology $k$-antisimple manifold $M$ with $\pi_1(M) = \pi$. We first establish necessary conditions to be fulfilled by $w_k(M)$. If $P$ is a finitely generated projective $\mathbb{Z}\pi$-module, so is its dual $P^* = \operatorname{hom}_{\mathbb{Z}\pi}(P, \mathbb{Z}\pi)$, endowed with the left $\mathbb{Z}\pi$-action $(a\beta)(u) = \beta(a)\bar{u}$ (recall that the involution $a \mapsto \bar{a}$ on $\mathbb{Z}\pi$ involves the orientation character $\omega$ of $M$; see [33]). This induces an involution $P \mapsto P^*$ on $\tilde{K}_0(\mathbb{Z}\pi)$. An element $P \in \tilde{K}_0(\mathbb{Z}\pi)$ is called $n$-self-dual, i.e. $P = (-1)^{n+1}P^*$.

**Proposition 3.22.** Let $M$ be a closed connected manifold of dimension $n \geq 6$ which is cohomology $k$-antisimple for $k \geq 3$. Then $w_k(M)$ is $n$-self-dual.

Proof. Consider a handle decomposition $\mathcal{H}$ for $M$

$$\mathcal{H} : D^n = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_n = M,$$

where $\mathcal{H}_j$ is the union of handles of index $\leq j$ in $\mathcal{H}$. This handle decomposition makes $M$ homotopy equivalent to a cell complex and thus gives rise to a chain complex of free $\mathbb{Z}\pi$-modules $C_j(\mathcal{H}) = H_j(\mathcal{H}_j, \mathcal{H}_{j-1})$ whose homology is that of $M$. As seen in Lemma 2.8 and its proof, one has

$$w_k(M) = (-1)^k[B_{k-1}(\mathcal{H})] \in \tilde{K}_0(\mathbb{Z}\pi).$$

(3.24)
We now use the “dual” handle decomposition $H^*$ of $H$ (see e.g. [20, p. 394]), producing a chain complex $C_j(H^*)$, whose homology is also that of $\tilde{M}$. By Remark 2.14, one has

$$w_k(M) = w_{n-k}(M) = (-1)^{n-k}[B_{n-k-1}(H^*)] \in \tilde{K}_0(\mathbb{Z}\pi). \quad (3.25)$$

The correspondence $\Theta$ associating to a $j$-handle $e$ its dual $(n-j)$-handle $\Theta(e)$ produces a chain isomorphism

$$\begin{align*}
C_{n-k+1}(H^*) & \xrightarrow{\partial} C_{n-k}(H^*) \xrightarrow{\partial} C_{n-k-1}(H^*) \\
\approx & \xrightarrow{\Theta} \approx \xrightarrow{\Theta} \approx \\
C_{k-1}(H)^* & \xrightarrow{\partial^*} C_k(H)^* \xrightarrow{\partial^*} C_{k+1}(H)^*
\end{align*}$$

Therefore,

$$B_{n-k-1}(H^*) \approx \text{Image} (\partial^*: C_k(H)^* \rightarrow C_{k+1}(H)^*). \quad (3.26)$$

Since $B_{k-1}(H)$ is projective, the exact sequence

$$0 \rightarrow Z_k(H) \rightarrow C_k(H) \xrightarrow{\partial} B_{k-1}(H) \rightarrow 0 \quad (3.27)$$

splits. Passing to the dual modules, we get the split exact sequence

$$0 \rightarrow B_{k-1}(H)^* \xrightarrow{\partial^*} C_k(H)^* \rightarrow Z_k(H)^* \rightarrow 0. \quad (3.28)$$

Therefore,

$$(-1)^{n-k}w_k(M) = [B_{n-k-1}(H^*)] \quad \text{by (3.25)}$$

$$= [Z_k(H)^*] \quad \text{by (3.26) and (3.28)}$$

$$= [Z_k(H)]^*$$

$$= -[B_{k-1}(H)]^* \quad \text{by (3.27)}$$

$$= (-1)^{k+1}w_k(M)^* \quad \text{by (3.24)},$$

proving that $w_k(M) = (-1)^{n+1}w_k(M)^*$. □

### 3.3 Realization of antisimple obstructions

Let $\pi$ be a finitely presented group and let $\omega: \pi \rightarrow \{\pm 1\}$ be a homomorphism. In this section, we shall realize classes in $\tilde{K}_0(\mathbb{Z}\pi, \omega)$ as antisimple obstruction of a cohomology $k$-antisimple closed manifold $M$. Making sense of this question requires some identification of $(\pi_1(M), \omega_M)$ with $(\pi, \omega)$. More strongly, we will fix the stable normal $(k-1)$-type of $M$, using the notion of an $(Y, \eta)$-referred manifold (see 3.6). We will thus consider the following
Problem 3.23. Let $Y$ be a connected finite cell complex of dimension $k - 1$ and let $\eta$ be a stable vector bundle over $Y$. Let $P \in K_0(\mathbb{Z}\pi_1(Y), \omega_\eta)$. Does there exist an $(Y, \eta)$-referred closed manifold $(M, g)$ such that $M$ is cohomology $k$-antisimple and satisfies $g_*(w_k(M)) = P$?

Problem 3.23 will partly answered in Theorem 3.27 below (see also Proposition 3.25). Before going to this, we show that a $(Y, \eta)$-referred compact manifold enjoys a decomposition involving a stable $\eta$-thickening of $Y$ in the sense of 3.14.

Proposition 3.24. Let $Y$ be a connected finite cell complex of dimension $k - 1$ and let $\eta$ be a stable vector bundle over $Y$. Let $(M, g)$ be an $(Y, \eta)$-referred compact manifold of dimension $r \geq 2k \geq 6$. Then, there are codimension-$0$ compact submanifolds $N$ and $T$ of $M$ giving a decomposition

$$M = N \cup T , \ N \cap T = \partial N , \ \partial T = \partial M \cap \partial N ,$$

(3.29)

such that $(N, g|N)$ is a stable $\eta$-thickening of $Y$ and $g|T$ is an $(Y, \eta)$-referred map. Moreover, for any $\mathbb{Z}\pi_1(Y)$-module $R$, the restriction homomorphism $\rho^j : H^j(M, T; R) \rightarrow H^j(M; R)$ satisfies the following properties:

(i) $\rho^j$ is an isomorphism for $k \leq j \leq r - k - 1$ and $\rho^{r-k}$ is injective;

(ii) if $M$ is a closed manifold, then $\rho^j$ is an isomorphism for $k \leq j \leq r - k$. Consequently, $M$ is cohomology $k$-antisimple if and only if $T$ is cohomology $k$-antisimple, in which case $g_*(w_k(M)) = (g|T)_*(w_k(T))$ in $K_0(\mathbb{Z}\pi_1(Y), \omega_\eta)$.

Proof. The $(Y, \eta)$-reference map $g : M \rightarrow Y$ admits a homotopy section $\gamma : Y \rightarrow M$ (see Lemma 2.5 and its proof). Up to homotopy equivalence, one may assume that $Y$ is a finite simplicial complex and that $\gamma$ is an embedding into a $C^\infty$-triangulation of $M$ with $\gamma(Y) \cap \partial M = \emptyset$. Then, $\gamma(Y)$ has a smooth regular neighborhood $N$ in $M \setminus \partial M$ [15]. Hence, the embedding $\gamma$ factors through an embedding $\tilde{\gamma} : Y \rightarrow N$ which is a simple homotopy equivalence, and $g|N : N \rightarrow Y$ is a homotopy inverse of $\tilde{\gamma}$. It follows that $g|N$ is a homotopy equivalence, which is simple by the composition rule for the Whitehead torsion [20] Lemma 7.8. Since $g$ is $\eta$-normal, so is $g|N$. Since $r \geq 2k \geq 6$, one has $\pi_1(\partial N) \approx \pi_1(N)$ and, by Poincaré duality, $H_j(\tilde{N}, \partial \tilde{N}) \approx H^{r-j}(N; \mathbb{Z}\pi_1(N)) \approx H^{r-j}(Y; \mathbb{Z}\pi_1(Y)) = 0$ for $j \leq n - k$. Therefore, the pair $(N, \partial N)$ is $(r - k)$-connected and $(N, g|N)$ is a stable $\eta$-thickening of $Y$. We thus get the decomposition (3.29) by setting $T = M \setminus \text{int}N$.

By van Kampen theorem, $\pi_1g|T : \pi_1(T) \rightarrow \pi_1(Y)$ is an isomorphism. Since $H_j(M, \partial M) \approx H_j(\tilde{N}, \partial \tilde{N})$ the pair $(M, T)$ is $(r - k)$-connected. Therefore, $g|T$ is an $Y$ reference map, which is $\eta$-normal since $g$ is so.

By excision and Poincaré duality, one has isomorphisms

$$H^j(M, T; R) \xrightarrow{\approx} H^j(N, \partial N; R) \xrightarrow{\approx} H^{r-j}(N; R) = 0 \text{ for } j \leq r - k ,$$

(3.30)

which proves (i).
When \( \partial M = \emptyset \), we claim that the restriction homomorphism \( \beta: H^*(M, T; R) \to H^*(M; R) \) is injective. Indeed, (3.30) sits in a commutative diagram

\[
\begin{array}{ccc}
H^j(M, T; R) & \xrightarrow{\text{excis}} & H^j(N, \partial N; R) \\
\downarrow \beta & & \downarrow \sigma_* \\
H^j(M; R) & \xrightarrow{\text{PD}} & H_{r-j}(M; R)
\end{array}
\]

where \( \sigma_* \) is induced by the inclusion \( \sigma: N \to M \). As the reference map \( g: M \to Y \) produces a homotopy retraction of \( \sigma \), the homomorphism \( \sigma_* \) is injective. Therefore, the cohomology sequence of the pair \((M, T)\) splits into short exact sequences

\[
0 \to H^j(M, T; R) \to H^j(M; R) \to H^j(T; R) \to 0.
\]

This, together with (3.30), proves the first assertion of (ii). Finally, one has \( H_*(\tilde{M}, \tilde{T}) \approx H_*(\tilde{N}, \partial N) \), so the pair \((M, T)\) is \( (r-k) \)-connected. As \( r-k \geq k \), one has \( g_* (w_k(M)) = (g_{r-j})_* (w_k(T)) \) by Lemma 2.11.

Our first result concerning Problem 3.23 is an affirmative answer in the category of Poincaré spaces. Recall that, according to our definition in 3.2, a Poincaré space or pair is finitely dominated.

**Proposition 3.25.** Let \( Y \) be a connected finite cell complex of dimension \( k-1 \geq 2 \) and let \( \eta \) be a stable vector bundle over \( Y \). Let \( \mathcal{P} \in \tilde{K}_0(\mathbb{Z}[\pi_1 Y], \omega_\eta) \) and let \( r \geq 2k \) be an integer. Then, there exists an \((Y, \eta)\)-referred Poincaré space \( (\mathcal{P}, q) \) of formal dimension \( r \), such that \( \mathcal{P} \) is cohomology \( k \)-antisimple satisfying \( w_k(\mathcal{P}) = \mathcal{P} \) and \( \text{Wall}(\mathcal{P}) = \mathcal{P} + (-1)^r \mathcal{P}^* \) (see Convention 3.26 below). Moreover, \( \mathcal{P} \) admits up to homotopy an \( \eta \)-normal Poincaré decomposition (see (77))

\[
\mathcal{P} \approx N \cup T, \quad N \cap T = \partial N = \partial T,
\]

where \( (N, q|_N) \) is a stable \( \eta \)-thickening of dimension \( r \) of \( Y \). The Poincaré space \( T \) is cohomology \( k \)-antisimple and satisfies \( w_k(T) = w_k(\mathcal{P}) \) and \( \text{Wall}(T) = \text{Wall}(\mathcal{P}) \).

**Convention 3.26.** Let \( Y \) be a connected finite cell complex. Let \( (A, \alpha) \) be an \((Y, \eta)\)-referred space. Then, \( \tilde{K}_0 \)-equalities are understood to hold in \( \tilde{K}_0(\pi_1 Y, \omega_\eta) \), using the map \( \alpha \) and its restrictions to subspaces of \( A \). For instance, the equation \( w_k(T) = w_k(\mathcal{P}) \) in Proposition 3.25 should be understood as \( (q_T)_* (w_k(T)) = q_* (w_k(\mathcal{P})) \).

**Proof.** We set \((\pi, \omega) = (\pi_1 Y, \omega_\eta)\) and use Convention 3.26 throughout the proof without notice. Consider the infinite \( k \)-dimensional CW-complex \( L \) of Proposition 2.19. By Remark 2.23, \( L \) is the direct limit \( L_\infty \) of a direct system

\[
Y \hookrightarrow L_1 \hookrightarrow L_2 \hookrightarrow \cdots \hookrightarrow L_\infty = L \xrightarrow{g} Y
\]

of finite cell complexes of dimension \( k \). The \( Y \)-referred map \( g: L \to Y \) is the direct limit of \( Y \)-referred maps \( g_j: L_j \to Y \). The composed map in (3.32) is \( \text{id}_Y \).
We now see that Direct system (3.32) may be realized up to homotopy equivalence by a system of \((r + 1)\)-dimensional manifolds (compact with boundary when \(j < \infty\)), getting a commutative diagram

\[
\begin{array}{ccccccccc}
Y & \xrightarrow{\phi_0} & L_1 & \xrightarrow{\phi_1} & L_2 & \xrightarrow{\phi_2} & \cdots & \xrightarrow{\phi_{k-1}} & L_\infty = L & \xrightarrow{g} & Y \\
U_0 & \xrightarrow{\alpha_0} & U_1 & \xrightarrow{\alpha_1} & U_2 & \xrightarrow{\alpha_2} & \cdots & \xrightarrow{\alpha_{k-1}} & U_\infty = U
\end{array}
\tag{3.33}
\]

where the vertical arrows are homotopy equivalences (simple for \(j < \infty\)). We start with \((U_0, \phi_0)\) being a stable \((Y, g_0^* \eta)\)-thickening of dimension \(r + 1\). We can thus choose a stable isomorphism from the stable normal bundle on \(U_0\) and \(\phi_0^* (g_0^* \eta)\), or, equivalently, a stable trivialization \(\mathcal{F}_0\) of \(TU_0 \oplus \phi_0^* (g^* \eta)\) (i.e. \((\phi_0, \mathcal{F}_0)\) is a normal map in the language of [31]). Recall that the inclusions in (3.32) are obtained by attaching cells of dimension \(k - 1\) and \(k\). By surgery below the middle dimension on \(U_0\), [31] Chapter 1, the data \((U_0, \phi_0, \mathcal{F}_0)\) thus determines \((U_1, \phi_1, \mathcal{F}_1)\), then \((U_2, \phi_2, \mathcal{F}_2)\), etc. Note that \((U_j, \phi_j)\) is the stable \((L_j, g_j^* \eta)\)-thickening of dimension \(r + 1\) and \(\mathcal{F}_j\) (obtained from \(\mathcal{F}_0\) without further choice) determines the embedding \(\alpha_j: U_j \to U_{j+1}\).

At the limit when \(j \to \infty\), one gets an open manifold \(U\) together with an \((Y, \eta)\)-reference map \(\gamma \circ \phi: U \to Y\). The manifold \(U\) has one end \(\epsilon\), determined by the cofinal system of neighborhoods \(E_j = U \setminus \text{int} U_j\). The inclusions \(E_{j+1} \hookrightarrow E_j \hookrightarrow U\) induce an isomorphism on the fundamental groups. As \(U \simeq L\) is finitely dominated by Proposition 2.19 and \(\partial E_j\) is a closed manifold, the open manifold \(E_j\) is finitely dominated (see [25] Complement 6.6). Therefore, the end \(\epsilon\) is tame (see 3.13).

As mentioned in Remark 2.22, \(w_k(L) = w_k(U)\) is Wall’s finiteness obstruction \(\text{Wall}(U)\). One has \(\text{Wall}(U) = \text{Wall}(Z)\) by the sum theorem [25] Theorem 6.5] and thus, by [25] Proposition 6.11, \(\text{Wall}(U)\) coincides with \(\sigma(\epsilon)\), the Siebenmann obstruction to complete \(U\) into a compact manifold with boundary.

By Proposition 3.19 there exists a Poincaré pair \((U, P)\), with \(\pi_1(P) \approx \pi_1(U)\), together with a homotopy equivalence \(\alpha: U \to U\). Recall from Proposition 2.19 that \(U \simeq L\) is is cohomology silent in degrees \(\geq k\). By Poincaré duality, one has

\[
H_j(U, P) \approx H_{j+1}(U, P; \mathbb{Z}_2) \approx H^{r+1-j}(U; \mathbb{Z}_2) = 0 \quad \text{for} \quad j \leq n + 1 - k.
\]

Hence, the pair \((U, P)\) is \((r + 1 - k)\)-connected. Since \(U \simeq U\), the map \(g \circ \phi: U \to Y\) extends to an \(Y\)-reference map \(q: U \to Y\). By the connectivity of \((U, P)\), the restriction of \(q\) to \(P\) (also called \(q\)) is an \(Y\)-reference map. Also, this connectivity implies that \(P\) is cohomology \(k\)-antisimple and that \(w_k(P) = w_k(L) = P\). As \(\alpha: U \to U\) is a homotopy equivalence, one has \(\text{Wall}(U) = \text{Wall}(U) = \text{Wall}(L) = P\). The formula

\[
\text{Wall}(P) = P + (-1)^r P^*
\tag{3.34}
\]

is asserted in Proposition 3.19.

We now prove that the reference map \(q: U \to Y\) is \(\eta\)-normal (and so is \(q: P \to Y\) by Part (3) of Lemma 3.8). We use the ideas of Pedersen-Ranicki [22] proof of Lemma 3.1] (with different notations).
By Siebenmann’s famous theorem [25, Theorem 7.5], the open manifold $U \times S^1$ is the interior of a compact manifold $A$ with boundary $V = \partial A$. The Poincaré duality argument used above to prove that the pair $(U, P)$ is $(r+1-k)$-connected proves that $(A, V)$ is $(r+1-k)$-connected. The projections of $U \times S^1$ onto $S^1$ together with the map $g_{0}: U \to Y$ gives rise to a map $g_{A}: A \to Y$ which is $\eta$-normal and a map $p_{A}: A \to S^1$. Let $(\hat{A}, \hat{V}) \to (A, V)$ be the infinite cyclic cover associated to the map $p_{A}$. It is proven in [22, proof of Lemma 3.1] that there are homotopy equivalences 

1. $h_{1}: (A, V) \xrightarrow{\simeq} (U, P) \times S^1$
2. $h_{2}: \hat{V} \xrightarrow{\simeq} \mathbb{P}$.

The composite map

$$A \xrightarrow{h_{1}} U \times S^1 \xrightarrow{h'_{1}} A \xrightarrow{g_{A}} Y,$$

where $h'_{1}$ is a homotopy inverse of $h_{1}$, is homotopic to $g_{A}$ and is thus $\eta$-normal. By Lemma 3.8 we deduce that $g_{A} \circ h'_{1}$ is $\eta$-normal, and so is its restriction to $U \times pt$. But this restriction is homotopic to the $Y$-reference map $q: U \to Y$, which is thus $\eta$-normal.

It remains to construct the decomposition of (3.31). We check that the map $\tilde{g}_{A} = (g_{A}, p_{A}): A \to Y \times S^1$ is an $(Y \times S^1, \tilde{\eta})$-reference map for $A$, where $\tilde{\eta}$ is induced from $\eta$ by the projection to $Y$. As the pair $(A, V)$ is $(r+1-k)$-connected, the restriction of $\tilde{g}_{A}$ to $V$ is an $(Y \times S^1, \tilde{\eta})$-reference map for $V$. By Proposition 3.21 there is a manifold decomposition $V = N_{0} \cup T_{0}$, where $(N_{0}, \tilde{g}_{A})$ is a stable $\tilde{\eta}$-thickening of $Y \times S^1$. As $\tilde{g}_{A}|U \times S^1 \simeq q \times \text{id}_{S^1}$, Proposition 3.15 implies that $N_{0}$ is diffeomorphic to $N \times S^1$, where $(N, q)$ is a stable $\eta$-thickening of $Y$.

Using the homotopy equivalence $h_{2}: \hat{V} \to \mathbb{P}$, one thus gets a decomposition $\mathbb{P} \simeq (N \times \mathbb{R}) \cup T_{0}$. The manifold $N$ is diffeomorphic to $N \cup_{\partial N} C$ where $C = \partial N \times [0, 1]$. Form the space

$$T = (C \times \{0\}) \cup \hat{T}_{0} \subset (N \times \mathbb{R}) \cup \hat{T}_{0},$$

containing a subspace $\partial T = \partial (N \times \{0\})$. One has the decomposition

$$\mathbb{P} \simeq (N \times \{0\}) \cup T \simeq N \cup T.$$

As the inclusions $\partial N \hookrightarrow N \hookrightarrow \mathbb{P}$ induce isomorphisms $\pi_{1}(\partial N) \approx \pi_{1}(N) \approx \pi_{1}(\mathbb{P})$, it follows from van Kampen’s theorem and the property of amalgamated products (see [19, Theorem 2.6, p. 187]) that $\pi_{1}(\partial N) \approx \pi_{1}(T) \approx \pi_{1}(\mathbb{P})$. By excision, $H_{n}(N, \partial N) \approx H_{n}(\mathbb{P}, \hat{T})$. Therefore, the pair $(\mathbb{P}, T)$ is $(r-k)$-connected and thus the map of $q|_{T}: T \to Y$ is an $Y$-reference map.

As the map $q: N \to Y$ is $\eta$-normal, that (3.31) is an $\eta$-normal Poincaré decomposition follows from Corollary 3.13.

Finally, the equation $\text{Wall}(T) = \text{Wall}(\mathbb{P})$ comes from the sum theorem [25, Theorem 6.5]. That $T$ is cohomology $k$-antisimple and the equation $w_{k}(T) = w_{k}(\mathbb{P})$ are proven by the corresponding arguments given in the proof of Part (ii) of Proposition 3.21.
We now turn our attention to Problem 3.23 as it is stated, that is in the category of closed manifolds. Let $Y$ be a connected finite cell complex of dimension $k-1$ and let $\eta$ be a stable vector bundle over $Y$. Set $(\pi, \omega) = (\pi_1(Y), \omega_\eta)$. Let $(M, g)$ be an $(Y, \eta)$-referred closed manifold of dimension $r$ such that $M$ is cohomology $k$-antisimple. By Proposition 3.22 its antisimple obstruction is $r$-self-dual, i.e. satisfies $g_*(w_k(M)) = (-1)^{r+1}g_*(w_k(M))^*$. It thus defines a class

$$[g_*(w_k(M))] \in H^{r+1}(\mathbb{Z}_2; \overline{K}_0(\mathbb{Z}_\pi, \omega))$$

(3.36)

in the Tate cohomology group

$$H^{r+1}(\mathbb{Z}_2; \overline{K}_0(\mathbb{Z}_\pi, \omega)) = \{ P \in \overline{K}_0(\mathbb{Z}_\pi, \omega) | P = (-1)^{r+1}P^* \}/\{ P + (-1)^{r+1}P^* \}.$$ 

This Tate cohomology group is an abelian group of exponent two, which occurs in the Ranicki exact sequence

$$L^r_{\pi+1}(\pi, \omega) \rightarrow L^p_{\pi+1}(\pi, \omega) \rightarrow H^{r+1}(\mathbb{Z}_2; \overline{K}_0(\mathbb{Z}_\pi, \omega)) \xrightarrow{\delta_R} L^h_r(\pi, \omega) \rightarrow L^p_\pi(\pi, \omega)$$

(3.37)

where $L^p_j$ (respectively: $L^h_j$) are the surgery obstruction groups for surgery data with target a finite (respectively: finitely dominated) Poincaré complex. This sequence was first obtained by Ranicki using his algebraic setting of surgery [23 Theorem 4.3]. A geometrical version of Sequence (3.37) will be used, which was provided by Pedersen-Ranicki [22, p. 243]. Our partial solution of Problem 3.23 is the following

**Theorem 3.27.** Let $Y$ be a connected finite cell complex of dimension $k-1 \geq 2$ and let $\eta$ be a stable vector bundle over $Y$. Set $(\pi, \omega) = (\pi_1(Y), \omega_\eta)$ and let $r \geq 2k$ be an integer. Let $P \in \overline{K}_0(\mathbb{Z}_\pi, \omega)$ be an $r$-self-dual element. Suppose that $[P] \in H^{r+1}(\mathbb{Z}_2; \overline{K}_0(\mathbb{Z}_\pi, \omega))$ belongs to the kernel of the homomorphism $\delta_R$ of (3.37). Then, there exists an $(Y, \eta)$-referred closed manifold $(M, q)$ of dimension $r$ such that $M$ is cohomology $k$-antisimple and satisfies $q_*(w_k(M)) = P$.

**Proof.** We consider the Poincaré pair $(\mathbb{U}, P)$ of the proof of Proposition 3.25 equipped with its $(Y, \eta)$-reference map $q: \mathbb{U} \rightarrow Y$ and its restriction $q: P \rightarrow Y$. One has Wall$(\mathbb{U}) = P$ and Wall$(P) = P + (-1)^rP^*$ (using Convention 3.26). Since $P = (-1)^{r+1}P^*$ by hypothesis, one has Wall$(P) = 0$ and, by [23 Theorem F], $P$ is homotopy equivalent to a finite complex. Thus, by changing the pair $(\mathbb{U}, P)$ by a homotopy equivalence, one may suppose that $P$ is a finite Poincaré space.

Being $\eta$-normal, the map $q$ determines a surgery problem with target $\mathbb{U}$ (see 3.6). In this language and using the point of view of [22, pp. 242-44], the data $(\eta, q)$ represents a class in the geometric L-group $L^{1, h}_{\pi+1}(Y)$. The definition of the latter is akin to that of $L^1_{\pi+1}(Y) = L^{1, h}_{\pi+1}(Y)$ also working for $L^{1, h}_{\pi+1}(Y)$ given in [31 Chapter 9]. With these geometric L-groups, Pedersen and Ranicki obtain, for $r \geq 5$, an isomorphism $\sigma_*$ of exact sequences

$$
\begin{align*}
L^{1, h}_{\pi+1}(Y) \rightarrow & L^{1, p}_{\pi+1}(Y) \rightarrow L^{1, p, h}_{\pi+1}(Y) \xrightarrow{\delta_{PR}} L^h_\pi(Y) \rightarrow L^p_\pi(Y) \\
L^h_\pi(\pi) \rightarrow & L^p_\pi(\pi) \rightarrow H^{r+1}(\mathbb{Z}_2; \overline{K}_0(\mathbb{Z}_\pi)) \xrightarrow{\delta_R} L^h_\pi(\pi) \rightarrow L^p_\pi(\pi)
\end{align*}
$$

(3.38)

where the bottom line is Ranicki’s exact sequence [33], writing $L^*_\pi(\pi)$ for $L^*_\pi(\pi, \omega)$ and $\overline{K}_0(\mathbb{Z}_\pi)$ for $(\overline{K}_0(\mathbb{Z}_\pi), \omega)$. The arrows in [33] satisfy the following properties


- \( \sigma^{p,h}(U,q) = [\text{Wall}(U)] \);
- \( \delta_{PR} \) sends the class of \((U,q)\) to that of \((P,q)\).
- \( \sigma^h \) is the surgery obstruction measured in \( L^h_0(\pi) \).

Hence, in the language of 3.6 one has

\[
\sigma(P,q) = \sigma^h \circ \delta_{PR}(U,q) = \delta_R \circ \sigma^{p,h}(U,q) = \delta_R(P).
\]

By hypothesis, \([\text{Wall}(U)] = [P] \in \ker \delta_R \), hence, \( \sigma(P,q) = 0 \). This means that there exists a \( q^* \eta \)-normal homotopy equivalence \( \theta : M \to P \) where \( M \) is a closed \( r \)-manifold. The latter thus admits the \((Y,\eta)\)-reference map \( q \circ \theta \). Being homotopy equivalent to \( P \), the closed manifold \( M \) is cohomology \( k \)-antisimple and \( w_k(M) = w_k(P) = P \) by Lemma 2.11

### 3.4 Cohomology antisimple cobordisms

Let \( n \geq 2k \geq 6 \) be integers. A (compact) cobordism \((W,M,M')\) is called cohomology \( k \)-antisimple if the following three conditions hold.

- \( M \) and \( M' \) are closed connected manifold of dimension \( n \) which are cohomology \( k \)-antisimple;
- The pairs \((W,M)\) and \((W,M')\) are weakly \((k-1)\)-connected (see 2.2);
- \( W \) is cohomology \((k,n-k)\)-silent.

For example, an \( h \)-cobordism between closed connected cohomology \( k \)-antisimple manifolds is cohomology \( k \)-antisimple.

Recall from (3.30) that a closed connected manifold \( M \) of dimension \( n \) which is cohomology \( k \)-antisimple defines a class \([w_k(M)] \in H^{n+1}(\mathbb{Z}^2;K_0(\mathbb{Z} \pi_1(M),\omega_M)) \).

**Proposition 3.28.** Let \((W,M,M')\) be a cohomology \( k \)-antisimple cobordism between closed connected cohomology \( k \)-antisimple manifolds of dimension \( n \geq 2k \geq 6 \). Then \([w_k(M)] = [w_k(M')] \) in \( H^{n+1}(\mathbb{Z}^2;K_0(\mathbb{Z} \pi_1(W),\omega_W)) \).

**Proof.** The proof involves two steps.

**Step 1: Reduction to the case where \((W,M)\) and \((W,M')\) are \((k-1)\)-connected.** The cobordism \((W,M,M')\) admits a handle decomposition of the form

\[
M \times I = \mathcal{H}_{k-1} \subset \mathcal{H}_k \subset \cdots \subset \mathcal{H}_{n+2-k} = W,
\]

where \( \mathcal{H}_j \) is the union of handles of index \( \leq j \). Indeed, since the pair \((W,M)\) is weakly \((k-1)\)-connected, it is \((k-2)\)-connected and the procedure of eliminating handles in a cobordism (see 17), applied to a given handle decomposition \( \mathcal{H} \), permits us to to get rid of the handles of index \( < k-1 \) and \( > n+2-k \). The same procedure applied to the dual handle decomposition \( \mathcal{H}' \) makes it of the form

\[
M' \times I = \mathcal{H}'_{k-1} \subset \mathcal{H}'_k \subset \cdots \subset \mathcal{H}'_{n+2-k} = W.
\]

The manifold \( V = \mathcal{H}_{k-1} \) and \( V' = \mathcal{H}'_{k-1} \) give cobordisms \((V,M,M_1)\) and \((V',M',M'_1)\) having handle decomposition involving only handles of index \( k-1 \). One has \( W = V \cup W_1 \cup V' \), defining a cobordism \((W_1,M_1,M'_1)\). The inclusions of the above manifolds into \( W \) all induce isomorphisms on fundamental groups with orientation characters, so all these will be identified with \((\pi,\omega) = (\pi_1(W),\omega_W) \). One has the following facts.
(1) \( V \) is cohomology \((k,n-k)\)-silent. Indeed, for any \( \mathbb{Z}_p \)-module \( R \), the exact sequence \( H^j(V,M;R) \to H^j(V;R) \to H^j(M;R) \) implies the claim, since \( M \) is cohomology \( k \)-antisimple and \( H^j(V,M;R) = 0 \) if \( j \neq k \).

(2) \( w_k(V) = w_k(M) \). Let \( N \) be the union of handles of index \( \leq k - 1 \) in a handle decomposition of \( M \) and let \( T = M \cap \text{int} N \). Since the pair \((M,N)\) is \((k-1)\)-connected, the \((k-1)\)-handles of \( V \) might be attached on \( N \), producing a cobordism \( N_+ \) from \( N \), and \( V = N_+ \cup_{\partial N \times I} (T \times I) \). The pair \((V,N_+)\) is \((k-1)\)-connected and

\[
H_k(\bar{V},\bar{N}_+) \approx H_k(\bar{T},\partial \bar{N}) \approx H_k(M,N).
\]

As \( w_k(V) = [H_k(\bar{V},\bar{N}_+)] \) and \( w_k(M) = [H_k(\bar{M},\bar{N})] \), the claim follows.

(3) \( M_1 \) is cohomology \( k \)-antisimple and \( w_k(M_1) = w_k(M) \). This follows from (1) and (2), since \((V,M_1)\) is \((n+2-k)\)-connected and \( n+2-k > n-k > k \).

(4) \( W_1 \) is cohomology \((k,n-k)\)-silent. As \((W,W_1)\) is \((n+2-k)\)-connected, this follows from the corresponding property of \( W \).

Points (1)–(3) also hold true for \((V',M')\). Therefore, it is equivalent to prove Proposition 3.28 for \((W,M,M')\) or for \((W_1,M_1,M'_1)\), so we may assume that \((W,M)\) and \((W,M')\) are \((k-1)\)-connected.

**Step 2: Proof of Proposition 3.28 when \((W,M)\) and \((W,M')\) are \((k-1)\)-connected.** Applying Lemma 2.16 to both ends of \( W \) gives the equalities

\[
w_k(M) + [H_k(W,M)] = w_k(W) = w_k(M') + [H_k(W,M')]. \tag{3.39}
\]

We will compute \([H_k(W,M')]\). As in Step 1, one checks that the cobordism \((W,M,M')\) admits a handle decomposition of the form

\[
M \times I = H_k \subset H_{k+1} \subset \cdots \subset H_{n+1-k} = W,
\]

where \( H_j \) is the union of handles of index \( \leq j \). Since \( n \geq 2k \), one has \( n+2-k > k \).

The chain complex \( C_*(W,M) \) is thus the equivalent to

\[
0 \to C_{n+1-k}(\bar{H}) \to \cdots \to C_k(\bar{H}) \to 0.
\]

Write \( C_j = C_j(\bar{H}) \), \( Z_j = Z_j(\bar{H}) \) and \( B_j = B_j(\bar{H}) \). For \( k < j < n+1-k \), one has \( H_j(W) = 0 = H_j(\bar{M}) \) by Lemma 2.7. Therefore, for \( k < j < n+1-k \), \( H_j(W,M) = 0 \) and thus \( Z_j = B_j \). One then has exact sequences

\[
0 \to B_k \to C_k \to H_k(\bar{W},\bar{M}) \to 0
\]

\[
0 \to B_{k+1} \to C_{k+1} \to B_k \to 0
\]

\[
\cdots \text{ etc} \cdots
\]

\[
0 \to H_{n+1-k}(\bar{W},\bar{M}) \to C_{n+1-k} \to B_{n-k} \to 0
\]

As \( H_k(\bar{W},\bar{M}) \) is finitely generated and projective, so are all the above modules and one has

\[
[H_k(\bar{W},\bar{M})] = -[B_k] = [B_{k+1}] = -[B_{k+2}] = \cdots = (-1)^{r+1}[B_{k+r}].
\]

As \([H_{n+1-k}(\bar{W},\bar{M})] = -B_{n-k}\), one has

\[
[H_{n+1-k}(\bar{W},\bar{M})] = (-1)^n[H_k(\bar{W},\bar{M})]. \tag{3.40}
\]
Now, by Poincaré duality,

\[ H_{n+1-k}(\tilde{W}, \tilde{M}) = H_{n+1-k}(W, M; \mathbb{Z}_\pi) \approx H^k(W, M'; \mathbb{Z}_\pi). \]

To compute the latter, one uses the handle decomposition \( H \) of \((W, M')\) which is the dual handle decomposition to \( \mathcal{H} \). Then, \( \mathcal{H}' \) has no handle of index \( k \). Therefore

\[ H^k(W, M'; \mathbb{Z}_\pi) \approx \ker \left( C_k(\mathcal{H}')^* \xrightarrow{\delta} C_{k+1}(\mathcal{H}')^* \right) \]

\[ = \{ \alpha \in C_k(\mathcal{H}')^* | \alpha \circ \partial = 0 \} \]

\[ \approx H_k(W, M')^*. \]

Thus, \( H_k(\tilde{W}, \tilde{M}') \approx H^k(W, M'; \mathbb{Z}_\pi)^* \) and, by (3.30) and (3.40), one has

\[ w_k(M') = w_k(M) + [H_k(\tilde{W}, \tilde{M})] - [H_k(\tilde{W}, \tilde{M}')] \]

\[ = w_k(M) + [H_k(\tilde{W}, \tilde{M})] - [H^k(W, M'; \mathbb{Z}_\pi)]^* \]

\[ = w_k(M) + [H_k(W, M)] - [H_{n+1-k}(W, M)]^* \]

\[ = w_k(M) + [H_k(W, M)] + (-1)^{n+1}[H_k(W, M)]^*, \]

which proves the proposition. \( \square \)

The rest of this section is devoted to the proof of the following result.

**Theorem 3.29.** Let \( M \) be a closed connected cohomology \( k \)-antisimple manifold of dimension \( n \geq 2k \geq 6 \). The following conditions are equivalent.

(a) \( M \) is cohomology \( k \)-antisimply cobordant to an \( k \)-antisimply closed manifold.

(b) \( [w_k(M)] = 0 \) in \( H^{n+1}(\mathbb{Z}_2; K_0(\mathbb{Z}_\pi_1(M), \omega_M)) \).

We need some preliminary results before starting the proof. For two integer \( s \leq t \), we denote by \( CS(t, s) \) the class of CW-spaces which are cohomology \((s, t)\)-silent.

**Lemma 3.30.** Let \( X = X_- \cup X_+ \), \( X_- \cap X_+ = X_0 \) be a CW-decomposition. Suppose that \( X_\pm \) and \( X_0 \) are in \( CS(s, t) \) and that \( H^{s-1}(X_+; R) \rightarrow H^{s-1}(X_0; R) \) is onto for any \( \mathbb{Z}_\pi_1(X) \)-module \( R \). Then \( X \in CS(s, t) \).

**Proof.** Let \( R \) be \( \mathbb{Z}_\pi_1(X) \)-module and consider the Mayer-Vietoris sequence

\[ H^{j-1}(X_0; R) \xrightarrow{\delta^{j-1}} H^j(X; R) \xrightarrow{\delta^j} H^j(X_-; R) \oplus H^j(X_+; R) \]

By our hypotheses, \( X_0 \in CS(s, t) \) and \( \delta^{j-1} = 0 \), which implies the lemma. \( \square \)

**Lemma 3.31.** Let \( Y \) be a connected finite cell complex of dimension \( k - 1 \geq 2 \) and let \( \eta \) be a stable vector bundle over \( Y \). Let \( n \geq 2k \) be an integer. Let \( P \in K_0(\mathbb{Z}_\pi(Y), \omega_\eta) \) being \( n \)-self-dual. Suppose that \( P \) represents \( 0 \) in \( H^{n+1}(\mathbb{Z}_2; K_0(\mathbb{Z}_\pi(Y), \omega_\eta)) \). Then, there exists an \((Y, \eta)\)-referred compact \((n + 1)\)-dimensional manifold \((A, q)\) (see below) such that

(a) \( M = \partial A \) is cohomology \( k \)-antisimple and satisfies \( q_*(w_k(M)) = P \).

(b) \( A \in CS(k, n - k) \).

Here, an \((Y, \eta)\)-referred manifold \((A, q)\) means that \( q: A \rightarrow Y \) and as well as \( q|_{\partial A} \) are both \((Y, \eta)\)-reference maps.
Proof. We set \((\pi, \omega) = (\pi_1(Y), \omega_\eta)\). All spaces are equipped with a \((Y, \eta)\)-reference map, so we use Convention \ref{convention:reference_maps} throughout the proof. Reference maps are often dropped from the notation.

We refer to the proof of Proposition \ref{proposition:reference_map_properties} for \(r = n\) and use its notations. There, a Poincaré pair \((U, \mathcal{P})\) of formal dimension \(n + 1\) is constructed, together with an \((Y, \eta)\)-reference map \(q: U \to Y\), such that \(\text{Wall}(U) = \mathcal{P}\) and \(\text{Wall}(\mathcal{P}) = \mathcal{P} + (-1)^n \mathcal{P}^*\). Moreover, \(\mathcal{P}\) is cohomology \(k\)-antisimple with \(w_k(\mathcal{P}) = \mathcal{P}\). The space \(U\) is a completion of the one-end open manifold \(U\) of \textbf{\ref{proposition:one_end_manifolds}}, satisfying \(\text{Wall}(U) = \mathcal{P}\). We mentioned that \(U = U_0 \cup E_0\) where \((U_0, q_0U_0)\) is a stable \(\eta\)-thickening of \(Y\). Thus, \(U = U_0 \cup E\), where \((E, \partial U_0, \mathcal{P})\) is a Poincaré cobordism with \(\text{Wall}(E) = \mathcal{P}\). As the pair \((U, E)\) is highly connected, the restriction \(q_E: E \to Y\) of \(q\) to \(E\) is an \(Y\)-reference map. By Corollary \ref{corollary:reference_map_properties}, \(q_E\) is \(\eta\)-normal.

The hypothesis that \(\mathcal{P}\) represents 0 in \(H^{n+1}(\mathbb{Z}; K_0(\mathbb{Z}, \pi, \omega))\) means that 
\[ \mathcal{P} = -Q - (-1)^{n+1} Q^* \]
for some \(Q \in K_0(\mathbb{Z}, \pi, \omega)\). We now use Proposition \ref{proposition:reference_map_properties} for \(r = n + 1\), replacing \(\mathcal{P}\) by \(Q\). One gets a \((Y, \eta)\)-referred Poincaré complex (without boundary) \(\((\mathbb{E}_1, q_1)\)\) of formal dimension \(n + 1\), having an \(\eta\)-normal Poincaré decomposition

\[ \mathbb{E}_1 \simeq N_1 \cup T_1, \quad N_1 \cap T_1 = \partial N_1 = \partial T_1, \]

where \((N_1, q_1|_{N_1})\) is a stable \(\eta\)-thickening of \(Y\) in dimension \(n + 1\) and \(T_1\) is an \((Y, \eta)\)-referred Poincaré space with boundary \(\partial(T_1) = \partial N_1\) such that \(\text{Wall}(T_1) = Q + (-1)^{n+1} Q^*\).

As \(q\) and \(q_1\) are \(\eta\)-normal, one can choose Spivak classes \((U) \in H_{n+1}(U, \partial U)\) and \((P_1) \in H_{n+1}(P_1)\). By Proposition \ref{proposition:spivak_classes}(ii), they induce Spivak classes \((U_0) \in H_{n+1}(U_0, \partial U_0)\) and \((N_1) \in H_{n+1}(N_1, \partial N_1)\). By Proposition \ref{proposition:spivak_classes} there exists a simple homotopy equivalence \(f: (U_0, \partial U_0) \to (N_1, \partial N_1)\) commuting with the \(Y\)-reference maps and satisfying \(\pi_* f((U_0)) = (N_1)\). Form the space

\[ \mathbb{A} = E \cup_{\partial(U_0)} C(f) \cup_{\partial N_1 = \partial T_1} T_1, \]

where \(C(f)\) is the mapping cylinder of \(f, \partial U_0\):

\[ C(f) = \partial U_0 \times [0, 1] \sqcup \partial N_1 \setminus \{(x, 1) \sim f(x)\}. \]

Set \(E^+ = E \cup C(f)\) and \(\partial E^+ = \mathcal{P} \sqcup \partial T_1\).

That \(f\) commutes with the reference maps on \(E^+ \simeq E\) and \(T_1\) implies that these references maps extend to a map \(q_E: \mathbb{A} \to Y\). This map \(q_E\) is an \(Y\)-reference map. Indeed, all fundamental groups under consideration are isomorphic and identified with \(\pi\). One has a morphism of of Mayer-Vietoris sequences

\[
\begin{array}{cccccc}
H_j(\partial N_1) & \longrightarrow & H_j(\mathbb{E}^+) & \oplus & H_j(T_1) & \longrightarrow & H_j(\mathbb{A}) \\
\downarrow{(q_N)_*} & & \downarrow{(q_E)_*} & & \downarrow{(q_{T_1})_*} & & \downarrow{(q_{\mathbb{A}})_*} \\
H_j(\bar{Y}) & \longrightarrow & H_j(\bar{Y}) & \oplus & H_j(\bar{Y}) & \longrightarrow & H_j(\bar{Y})
\end{array}
\]

where the bottom line is the Mayer-Vietoris sequence of \(Y = Y \cup Y, Y \cap Y = Y\). The homomorphism \((q_N)_*: H_j(\partial N_1) \to H_j(\bar{Y})\) is an isomorphism for \(j \leq n + 2 - k > k\), whence \(\partial M_{\mathbb{A}} = 0\) on this range. As \(q_E\) and \(q_{T_1}\) are \(Y\)-reference maps, one deduces that \(q_E\) is an \(Y\)-reference map.
We now refer to Diagram (3.23) and Proposition 3.11 with $X_- = E^+$, $X_+ = T_1$ and $Z_0 = \partial T_1$. Since $H_*(f_* (U_0)) = (N_1)$, one has $\partial H((E^+), -(T_1)) = 0$. By Corollary 3.12 $(\alpha, \beta)$ is a Poincaré pair of formal dimension $n+1$ and $q_0$ is an $(Y, \eta)$-reference map. We claim that $\alpha \in CS(k, n-k)$. Indeed, one has the following

- $E^+ \in CS(k, n-k)$. For, $E_0 \in CS(k, n-k)$ by Proposition 3.24 (for $r = n+1$), so $E^+ \simeq E_0 \in CS(k, n-k)$.

- $T_1 \in CS(k, n+1-k)$ by Proposition 3.25 (for $r = n+1$).

- $\partial U_0 \in CS(k, n-k)$ by Remark 3.16

As the inclusion $\partial U_0 \to T_1$ homotopy commutes with the $Y$-reference maps, it induces a homomorphism $H^{k-1}(T_1; R) \to H^{k-1}(\partial U_0; R)$ which is onto for any $\mathbb{Z}_{21}(X)$-module $R$. By Lemma 3.30, we deduce that $\alpha \in CS(k, n-k)$.

By Siebenmann’s sum theorem [25, Theorem 6.5], one has

$$Wall(\alpha) = Wall(E) + Wall(T_1) = P + Q + (-1)^{n+1}Q^* = 0.$$ 

On the other hand, one has $Wall(P) = P + (-1)^{n+1}P^*$ by Proposition 3.26. Since $P = (-1)^{n+1}P^*$ by hypothesis, one has $Wall(P) = 0$. By [25] Theorem F], $(\alpha, \beta)$ has the homotopy type of a finite Poincaré pair. The surgery obstruction $\sigma(\alpha, q_0)$ (see 3.6) is thus defined in $L_{n+1}^k(\pi, \pi, \omega) = 0$ by Wall’s $(\pi - \pi)$-theorem. Thus, up to homotopy equivalence, one may replace $(\alpha, \beta)$ is an $(Y, \eta)$-referred compact manifold pair $(A, M)$ of dimension $n+1$. The manifold pair $(A, M)$ has the required properties for Lemma 3.31

**Proof of Theorem 3.29.** That $(b) \Rightarrow (a)$ is guaranteed by Proposition 3.28. For the converse, let us start with some preliminary constructions. Choose a handle decomposition of $M$ and let $N$ be the union of handles of index $\leq k - 1$. Then, $N$ is a stable $n$-thickening of a connected finite subcomplex $Y$ of $M$ of dimension $k - 1$. By the classification of stable thickening (see the proof of Theorem 3.15), $N$ is determined up to diffeomorphism by the the stable vector bundle $\nu$ over $Y$, i.e. the restriction of the stable normal bundle of $M$ over $Y$. Let $(\pi, \omega) = (\tau_1(Y), \omega_\nu)$. The inclusion $Y \subset M$, which is $(k-1)$-connected, induces an identification $(\pi_1(M), \omega_M) \approx (\pi, \omega)$. Using this identification, let $P = w_k(M) \subset K_0^*(\pi, \omega)$. Set $T = M \setminus int N$.

Let $(A, q)$ be an $(Y, \nu)$-referred compact manifold of dimension $n+1$, as produced by Lemma 3.31 so that $\partial A$ is a closed connected cohomology $k$-antisimple manifold with $w_k(\partial A) = -P \in K_0(\pi, \omega)$ (identification via $q$). Let $M_0 = \partial A$. [Note the asymmetry of the pairs $(M, Y)$ and $(M_0, Y)$: the pair $(M, Y)$ is $(k-1)$-connected but there is no retraction of $M$ onto $Y$, while $M_0$ retracts onto $Y$ but the pair $(M_0, Y)$ is only weakly $(k-1)$-connected (see Lemma 2.9)]. By Proposition 3.24 there is a manifold decomposition

$$M_0 = N_0 \cup T_0 \quad , \quad N_0 \cap T_0 = \partial T_0 \cap \partial N_0 = \partial T_0$$

where $(N_0, q_N)$ is a stable $\nu$-thickening of $Y$ and $T_0$ is cohomology $k$-antisimple with $w_k(T_0) = w_k(\partial A)$. Using a collar neighborhood of $M_0$ in $A$, the inclusion $N_0 \hookrightarrow M_0$ may be extended to an embedding $\alpha: N_0 \times [0, 2] \to A$ with $\alpha^{-1}(\partial A) = N_0 \times \{0\}$. Let $A_0 = A_0(\text{int} N_0 \times [0, 2])$. Note that $A_0$ is homeomorphic to $A$ (diffeomorphic modulo rounding corners).

By Proposition 3.15 there is a degree-one diffeomorphism $h: N \to N_0$. Let $h_0: \partial N \to \partial N_0$ be restriction of $h$ to $\partial N$. Form the space

$$W = T \times [0, 1] \cup _B A_0,$$
where $\beta: \partial T \times [0, 1] \to \partial T_0 \times [0, 1]$ is the diffeomorphism given by $\beta(x, t) = (\alpha \circ h_0(x), t)$. After rounding the corners (see appendix of [3]), $W$ is a smooth cobordism between the manifolds
\[ M = T \cup h_0 T_0 \quad \text{and} \quad M_1 \approx T \cup h_0 N_0 \]
(since $\partial N_0 \times [1, 2] \cup N_0 \times \{2\} \approx N_0$). Theorem 3.29 will follow from the three assertions
(A) $M$ is $k$-antisimple.
(B) $M_1$ is diffeomorphic to $M$.
(C) $W$ is a cohomology $k$-antisimple cobordism.

We now start proving the above assertions. There is an embedding $Y \hookrightarrow \partial N$. We leave to the reader to prove, using Van Kampen’s theorem, that the various fundamental groups $\pi_1(W)$, $\pi_1(M)$, are thus all identified with $\pi_1(Y)$.

Proof of (A). We first prove that $M$ is cohomology $k$-antisimple (i.e. $M \in \text{CS}(k, n - k)$). Let $R$ be a $\mathbb{Z}$-$\pi$-module. Given that $T_0 \in \text{CS}(k, n - k)$ and the exact sequence
\[ H^{j-1}(T_0; R) \xrightarrow{\delta} H^j(M; T_0; R) \to H^j(M; R) \to H^j(T_0; R), \]
it is enough to prove that $\delta$ is surjective when $k \leq j \leq n - k$. By excision, one has
\[ H^j(M; T_0; R) \approx H^j(T, \partial N; R) \approx H^j(M, N; R). \]
As $M \in \text{CS}(k, n - k)$ and $\dim Y = k - 1$, one deduces from the exact sequence
\[ H^{j-1}(N; R) \to H^j(M, N; R) \to H^j(M; R) \]
that $H^j(M, T_0; R) \approx H^{j-1}(N; R) = 0$ for $j > k$. For $j = k$, one uses the following commutative diagram
\[
\begin{array}{ccc}
H^{k-1}(T_0; R) & \xrightarrow{a} & H^{k-1}(\partial T; R) \\
\downarrow{\delta} & & \downarrow{\delta_1} \\
H^k(M, T_0; R) & \xrightarrow{\approx} & H^k(T, \partial T; R) \\
\end{array}
\]
where the horizontal arrows are induced by inclusions, via the identifications $\partial T = \partial N \approx \partial N_0 = \partial T_0$. The homomorphism $\delta_2$ is onto since $H^k(M; R) = 0$. This implies that $\delta_1$ is onto. Now, the homomorphism $a$ may be identified with the homomorphism $a_0$ in the commutative diagram
\[
\begin{array}{ccc}
H^{k-1}(M_0; R) & \xrightarrow{(\sigma_0)^*} & H^{k-1}(N_0; R) \\
\downarrow{b} & & \downarrow{b} \\
H^{k-1}(T_0; R) & \xrightarrow{a_0} & H^{k-1}(\partial T_0; R) \\
\end{array}
\]
where $H^{k-1}(N_0, \partial N_0; R) \approx H_{n-k}(N_0; R) = 0$. The homomorphism $b$ induced by the inclusion is also onto for, by Poincaré duality
\[ H^k(N_0, \partial N_0; R) \xrightarrow{\approx} H_{n-k}(N_0; R) = 0, \]
since \( n - k > k - 1 \). Therefore, \( a \) is onto in Diagram (3.41), implying that \( \delta \) is onto, which proves that \( \mathcal{M} \) is cohomology \( k \)-antisimple.

We now compute the antisimple obstruction \( w_k(\mathcal{M}) \). By excision, one has isomorphisms

\[
H_*(\tilde{M}, \tilde{\partial N}) \xrightarrow{\approx} H_*(\tilde{T}, \tilde{\partial N}) \oplus H_*(\tilde{T}_0, \tilde{\partial N}) \xrightarrow{\approx} H_*(\tilde{M}, \tilde{N}) \oplus H_*(\tilde{M}_0, \tilde{N}_0)
\]

(3.42)

By construction, the pair \((M, N)\) is \((k - 1)\)-connected and so is the pair \((\tilde{M}_0, \tilde{N}_0)\) by Lemma 2.5. Therefore, (3.42) implies that \((\tilde{M}, \tilde{\partial N})\) is \((k - 1)\)-connected. The first isomorphism of (3.42) for \(* = k\) then implies that \(\pi_k(M, N) \cong \pi_k(\tilde{M}_0, \tilde{N}_0)\), which, by Definition (2.1), implies that

\[
w_k(\mathcal{M}) = w_k(M) + w_k(\mathcal{M}_0) = \mathcal{P} - \mathcal{P} = 0.
\]

As \( n \geq 6 \), Assertion (A) follows from Proposition 3.21.

Proof of (B). Let \( e : \partial N \times [0, 1) \to N \) and \( c_0 : \partial N_0 \times [0, 1) \to N_0 \) be embeddings of open collar neighborhoods of \( \partial N \) and \( \partial N_0 \) in \( N \) and \( N_0 \). Let \( T^1 = T \cup_{\partial T = \partial N} \partial N \times (0, 1) \).

As smooth manifolds, \( M \) and \( \mathcal{M}_1 \) may be presented as follows

\[
M = T^1 \sqcup N / \{(x, t) \sim c(x, t) \} \quad \text{for} \quad (x, t) \in \partial N \times (0, 1)
\]

and

\[
\mathcal{M}_1 = T^1 \sqcup N_0 / \{(x, t) \sim c_0(h_\delta(x), t) \} \quad \text{for} \quad (x, t) \in \partial N_0 \times (0, 1).
\]

Thus, a diffeomorphism \( s : M \xrightarrow{\approx} \mathcal{M}_1 \) may be defined as

\[
s(x) = \begin{cases} 
  x & \text{if } x \in T^1 \\
  h(x) & \text{if } x \in \text{int } N.
\end{cases}
\]

Proof of (C). We consider the morphism of Mayer-Vietoris sequences for the inclusion of 4-ads

\[
(\tilde{M}, \tilde{T}, \tilde{T}_0, \tilde{\partial N}) \hookrightarrow (\tilde{W}, \tilde{T} \times [0, 1], \tilde{A}_0, \tilde{\partial N} \times [0, 1]).
\]

For \( j \leq k - 1 \), all the homomorphisms, except possibly \( H_j(\mathcal{M}) \to H_j(\mathcal{W}) \), are isomorphisms. But then the latter is an isomorphism by the five lemma. The same happens for the inclusion

\[
(\tilde{M}_1, \tilde{T}, \tilde{N}, \tilde{\partial N}) \hookrightarrow (\tilde{W}, \tilde{T} \times [0, 1], \tilde{A}_0, \tilde{\partial N} \times [0, 1]).
\]

Therefore, both pairs \((\mathcal{W}, \mathcal{M})\) and \((\mathcal{W}, \mathcal{M}_1)\) are weakly \((k - 1)\)-connected. The proof that \( \mathcal{W} \in \text{CS}(k, n - k) \) is the same as that of \( \mathcal{M} \in \text{CS}(k, n - k) \) (see the proof of (A)): just replace \( \mathcal{M} \) by \( \mathcal{W} \) and \( T_0 \) by \( A_0 \).

\[\square\]

3.5 Examples

Let \( Y \) be a connected finite cell complex of dimension \( k - 1 \geq 2 \) and let \( \eta \) be a stable vector bundle over \( Y \). Let \( (\pi, \omega) = (\pi_1(Y), \omega_0) \). Set \( H^+ = H^{2j} = H^{2j}(\mathbb{Z}_2, \mathbb{K}_0(\pi, \omega)) \) and \( H^- = H^{2j+1} = H^{2j+1}(\mathbb{Z}_2, \mathbb{K}_0(\pi, \omega)) \).

Let \( \mathcal{P} \) be a \( \pm \)-symmetric element of \( \mathbb{K}_0(\mathbb{Z}, \omega) \), i.e. \( \mathcal{P} \) satisfies \( \mathcal{P} = \pm \mathcal{P}^* \). Suppose that \( \mathcal{P} \) represents zero in \( H^\pm \). By Theorem 3.27 there exist \((Y, \eta)\)-referred closed connected manifolds \((M, g)\) of dimension \( r \geq 2k \) such that \( M \) is cohomology \( k \)-antisimple.
with \( g_\ast(w_M(M)) = \mathcal{P} \). By Theorem 3.29 when \( r \geq 6 \), such manifolds are cohomology \( k \)-antisimply cobordant to an \( k \)-antisimple closed manifolds.

A source of such examples is given by the odd torsion \( T_{odd} \) of \( K_0(\mathbb{Z}_\pi) \). Indeed, \( T_{odd} \) splits under the involution \( * \) into a direct sum of eigenspaces \( T_{odd}^\lambda \) for the eigenvalues \( \pm 1 \). This produces \( \pm \)-symmetric element of \( K_0(\mathbb{Z}_\pi, \omega) \) which represent zero in \( H^\pm \), since the latter is a 2-group. The group \( T_{odd} \) is not zero for many finite groups, such as \( p \)-groups for \( p \) an odd prime, or the symmetric groups (see 21 for a survey and references).

More essential examples may arise when \( \pi \) is a finite 2-group. For example, when \( \pi = (\mathbb{Z}_2)^3 \), then \( K_0(\mathbb{Z}_\pi) \approx \mathbb{Z}_2 \) ([2] Theorem 12,9). Thus, \( H^\pm \approx \mathbb{Z}_2 \). On the other hand, the homomorphism

\[
\delta^{(n)} = \delta^{(n)}_R : H^{n+1} \to L^h_\ast(\pi,\omega)
\]

of (3.43) vanishes for \( n \equiv 2 \pmod{4} \) [2] Theorem C] (warning: the conventions for \( H^\pm \) in [2] are the opposite of ours, since the authors use there the involution \( \mathcal{P} \rightarrow -\mathcal{P}^* \)). By Theorem 3.27 and Proposition 3.28 for \( n = 4j + 2 \geq 2k \), there exists an \((Y,\eta)\)-referred closed connected \( n \)-manifolds \((M,g)\) which is cohomology \( k \)-antisimple but not cohomology \( k \)-antisimply cobordant to an \( k \)-antisimple manifold. As [2] Theorem C] applies to a large class of finite 2-groups, one might be able to get other examples.

When \( \pi \) is the generalized quaternion group \( Q_{2s} \) (\( s \geq 3 \)), one sees in [6, § 6 and (1.5)] that \( H^\pm \approx \mathbb{Z}_2 \). On the other hand, the homomorphism \( \delta^{(n)} \) of (3.43) vanishes for \( \not\equiv 1 \pmod{4} \) [2] Theorem D]. As above, one deduces from Theorem 3.27 and Proposition 3.28 that, for \( 2k \leq n \not\equiv 4j + 1 \), there an \((Y,\eta)\)-referred closed connected \( n \)-manifolds \((M,g)\) which is cohomology \( k \)-antisimple but not cohomology \( k \)-antisimply cobordant to an \( k \)-antisimple manifold.

Finally, Proposition 3.31 Theorem 3.29 together with Proposition 2.25 implies the following

**Proposition 3.32.** Let \( M \) be a closed connected \( n \)-dimensional manifold \((n \geq 6)\) which is cohomology \( k \)-antisimple for \( k \geq 3 \). Let \( A \) be a closed connected manifold of dimension \( a \) with \( a \leq n - 2k \). Then the manifold \( M \times A \) is cohomology \((k+a)\)-antisimple and

1. if \( \chi(A) = 0 \), then \( M \times A \) is \((k+a)\)-antisimple
2. if \( \chi(A) \) is even, then \( M \times A \) is cohomology \((k+a)\)-antisimply cobordant to an \((k+a)\)-antisimple closed manifold. \( \Box \)

### 4 Referred antisimple manifolds

Through this section \( Y \) will be a fixed connected finite cell complex of dimension \( k - 1 \geq 2 \) equipped with a stable vector bundle \( \eta \) over it. We set \((\pi,\omega) = (\pi_1(Y),\omega_\eta)\). We denote by \( Wh(\pi,\omega) \) the Whitehead group \( Wh(\pi) \) endowed with the involution \( \alpha \rightarrow \alpha^* \) using the orientation character \( \omega \). When we consider an \((Y,\eta)\)-referred manifold \((X,g)\), its fundamental group and orientation character \( (\pi_1(X),\omega_X) \) is thus is identified via \( g_* \) with \((\pi,\omega)\). This gives an identification \( Wh(\pi_1(X),\omega_X) \approx Wh(\pi,\omega) \). Equalities between elements of \( Wh(\pi_1(X'),\omega_{X'}) \) and \( Wh(\pi_1(X''),\omega_{X''}) \) for \((Y,\eta)\)-manifolds \((X',g')\) and \((X'',g'')\) should be understood to hold in \( Wh(\pi,\omega) \) via the identifications (like in Convention 3.26). The intervalle \([0,1]\) is denoted by \( I \).

As done in the proof of Proposition 3.24 if \((M,g)\) be an \((Y,\eta)\)-referred closed manifold of dimension \( r \geq 2k \geq 6 \), a homotopy section \( \gamma : Y \rightarrow M \) of \( g \) gives rise to a
manifold decomposition $N \cup T \to M$ such that $(N, g|_N)$ is a stable $\eta$-thickening of $Y$. When $M$ is $k$-antisimple, we get a richer decomposition.

**Proposition 4.1.** Let $(M, g)$ be an $(Y, \eta)$-referred $k$-antisimple closed manifold of dimension $r \geq 2k \geq 6$. Then,

1. a homotopy section $\gamma: Y \to M$ of $g$ gives rise to a manifold decomposition
   \[ M = N \cup V \cup N^* , \quad N \cap N^* = \emptyset , \quad V \cap N = \partial N , \quad V \cap N^* = \partial N^* \]
   such that
   - $(N, g|_N)$ and $(N^*, g|_{N^*})$ are stable $\eta$-thickenings of $Y$ of dimension $r$.
   - the compact $r$-manifold $V$ is an $h$-cobordism between $\partial N$ and $\partial N^*$.
2. The Whitehead torsion $\tau(V, \partial N) \in Wh(\pi, \omega)$ is $r$-self-dual, i.e. satisfies $\tau(V, \partial N) = (-1)^{r+1}(\tau(V, \partial N^*))$.
3. The class $[\tau(V, \partial N)] \in H^{r+1}(\mathbb{Z}; Wh(\pi_1(Y)))$ does not depend on the section $\gamma$.

Thanks to (3) above, the class of $[\tau(V, \partial N)] \in H^{r+1}(\mathbb{Z}; Wh(\pi_1(Y)))$ will be denoted by $\tau(M, g)$.

**Proof.** As in the proof of Proposition 3.24 one may suppose that $\gamma: Y \to M$ is an embedding. Let $N$ be a smooth regular neighborhood of $\gamma(Y)$. We thus get the manifold decomposition $M = N \cup T$ of Proposition 3.24. By general position, $\gamma$ is isotopic to an embedding $\gamma^*: Y \to T$. Let $N^*$ be a smooth regular neighborhood of $\gamma^*(Y)$ in $\text{int}(T)$. Let $V = M \setminus \text{int}(N \cup N^*)$.

To prove that the cobordism $(V, \partial N, \partial N^*)$ is an $h$-cobordism, let us consider a handle decomposition $\mathcal{H}$ for $M$

\[ \mathcal{H} : D^n = \mathcal{H}_0 \subset \mathcal{H}_1 \subset \cdots \subset \mathcal{H}_n = M , \]

where $\mathcal{H}_j$ is the union of handles of index $\leq j$ in $\mathcal{H}$. We suppose that $\mathcal{H}$ is $k$-antisimple, i.e. $\mathcal{H}_{k-1} = \mathcal{H}_{n-k+1}$ and that $\mathcal{H}$ has only one $r$-dimensional handle. Consider the dual handle decomposition $\mathcal{H}^*$ of $\mathcal{H}$ (see e.g. [20, p. 394]).

By general position, one may assume that $\gamma(Y) \subset N \subset \text{int}\mathcal{H}_{k-1}$. Let $X = \mathcal{H}_{k-1} \setminus \text{int} N$. The pairs $(M, N)$ and $(M, \mathcal{H}_{k-1})$ are both weakly $(k-1)$-connected. Therefore, the pair $(\mathcal{H}_{k-1}, N)$ is weakly $(k-1)$-connected. Since the homotopy dimensions of $Y$ and of $\mathcal{H}_{k-1}$ are both $\leq k-1$, one deduces that $\gamma: Y \to \mathcal{H}_{k-1}$ is a homotopy equivalence. It follows easily that $(X, \partial N, \partial \mathcal{H}_{k-1})$ is an $h$-cobordism (see e.g. [14, proof of Proposition 4.6]).

Similarly, one may suppose that $N^* \subset \text{int} \mathcal{H}_{k-1}^*$ and get an h-cobordism $(X^*, \partial N^*, \partial \mathcal{H}_{k-1}^*)$. One has that $V \cap V^* = \partial \mathcal{H}_{k-1} = \partial \mathcal{H}_{k-1}^*$, so the cobordism $V = X \cup X^*$ is an h-cobordism.

To prove (2), recall from Remark 3.16 that $N \approx N_- \times I$ and thus $\partial N \approx N_- \cup N_+ \times \{0\}$. Analogously, $N^* \approx N_-^* \times I$ and $\partial N^* \approx N_-^* \cup N_+^* \times \{1\}$. For $t = 0, 1$, consider the embedding $\gamma_t: Y \to N_- \times \{t\}$ homotopic in $N$ to $\gamma$. We do the same for $\gamma_t^*: Y \to N_-^* \times \{t\}$ homotopic in $N^*$ to $\gamma^*_t$.

The two embeddings $\gamma_0$ and $\gamma_0^*$ are isotopic in $M$. By general position, they are isotopic in $V$. Therefore, $V \approx (N_- \times I) \cup Z$, where $(Z, N_- \times 1, N_-^* \times 1)$ is an h-cobordism. Set $\tilde{\gamma}_1: Y \to Z$ to be $\gamma_1$ post-composed with the inclusion $N_- \times \{1\} \to Z$ and $\tilde{\gamma}_1^*: Y \to Z$ to be $\gamma_1^*$ post-composed with the inclusion $N_-^* \times \{1\} \to Z$ One has $\tau(V, \partial N) = \tau(Z, N_- \times \{1\}) = \tau(\tilde{\gamma}_1)$.
and 
\[(−1)^{r+1}\tau(V,∂N) = \tau(V,∂N^*) = \tau(Z, N^* × \{1\}) = \tau(γ_1)\].

Since γ_1 and γ_1^* are homotopic in ∂Z, this proves (2).

To prove (3), let δ: Y → M be another homotopy section of g, giving rise to a manifold decomposition M ≈ N(δ) ∪ V(δ) ∪ N^*(δ) which we have to compare with the decomposition M ≈ N(γ) ∪ V(γ) ∪ N^*(γ). As above, by general position and given the connectivity of (M, N(γ)) and (M, N(δ)), there is a self-homotopy equivalence φ of Y such that γ = δ ∘ φ. Therefore, we shall have N(δ) ⊂ int N(γ) and N^*(δ) ⊂ int N^*(γ), whence h-cobordisms (S, ∂N(δ), ∂N(γ)) and (S^*, ∂N^*(δ), ∂N^*(γ)). One has V(δ) = S ∪ V(γ) ∪ S^*. Therefore

\[\tau(V(δ), ηN(δ)) = \tau(V(γ), ηN(γ)) + \tau(S, ηN(δ)) + η(S^*, ηN^*(γ)) = \tau(V(γ), ηN(γ)) + \tau(S, ηN(δ)) + (−1)^{r+1}\tau(S, ηN(δ))^*,\]

which proves (3).

\[\square\]

**Remark 4.2.** In case τ(M, g) = 0, then, by the s-cobordism theorem, V ∪ N^* is diffeomorphic to N and M is diffeomorphic to the gluing of two copies of N by a self-diffeomorphism of ∂N. We call such an antisimple manifold a twisted double. \[\square\]

Two (Y, η)-referred manifold (M_i, g_i) (i = 0, 1) are called h-cobordant if there is an h-cobordism (W, M_0, M_1) admitting a k-connected η-normal map G: W → Y extending g_0 and g_1.

**Proposition 4.3.** Let (M, g) be an (Y, η)-referred k-antisimple closed manifold of dimension r ≥ 2k ≥ 6. Then,

1. The class τ(M, g) ∈ H^{r+1}(Z_2; Wh(π_1(Y))) is an invariant of the h-cobordism class of (M, g).
2. τ(M, g) = 0 if and only if (M, g) is h-cobordant to a twisted double.

**Proof.** Let (W, G) be an (Y, η)-referred h-cobordism between (M_0, g_0) and (M_1, g_1). Let Γ: Y → W be a homotopy section of G. Composed with the projections on M_i, it gives rise to homotopy sections γ_i: Y → M_i of g_i. One can also construct a homotopy Γ*: Y ∪ I → W between γ_0 and γ_1 and a homotopy Γ**: Y × I → W between γ_0 and γ_1. By general position, this can be done by embeddings and we thus get a decomposition

\[W ≈ (N × I) ∪ Z ∪ (N^* × I)\]

where (Z, V(γ_0), V(γ_1)) is an h-cobordism. One has

\[\tau(V(γ_0), ηN(γ_0)) + \tau(Z, V(γ_0)) = \tau(V(γ_1), ηN(γ_1)) + \tau(Z, V(γ_1)) = \tau(V(γ_1), ηN(γ_1)) + (−1)^r\tau(Z, V(γ_0))^*\]

(4.44)

which proves (1).

If (M, g) is h-cobordant to a twisted double, then τ(M, g) = 0 by (1) already proven. Conversely, suppose that τ(M, g) = 0. This means that there is a manifold decomposition M ≈ N ∪ V ∪ N^* with (V, ∂N, ∂N^*) being an h-cobordism with τ(V, ∂N) = α + (−1)^{r+1}α^* for some α ∈ Wh(π, ω). Let (Z, M, M') be an h-cobordism with τ(Z, M) = −α. Using (4.44), this proves that τ(V', N') = 0 and thus M' is a twisted double. \[\square\]
We now turn our attention to the construction of antisimple manifolds which are not h-cobordant to twisted doubles. Analogously to Ranicki’s exact sequence \[3.37\], the main ingredient is the *Rothenberg exact sequence* \[4.1\]

\[
L^b_{r+1}(\pi, \omega) \rightarrow L^b_{r+1}(\pi, \omega) \xrightarrow{\tau} H^{r+1}(\mathbb{Z}_2; Wh(\pi, \omega)) \xrightarrow{\delta_{\text{Rot}}} L^b_r(\pi, \omega)
\]  

**Proposition 4.4.** Let \(a \in H^{r+1}(\mathbb{Z}_2; Wh(\pi, \omega))\) such that \(\delta_{\text{Rot}}(a) = 0\). Then, for \(r \geq 2k \geq 6\) there exists an \((Y, \eta)\)-referred \(k\)-antisimple closed manifold \((M, g)\) with \(\tau(M, g) = a\).

**Proof.** The map \(T\) in \[(4.45)\] may be geometrically described as follows (see \[24, p. 313\]). Let \((N, g)\) be a stable \(\eta\)-thickening of \(Y\) of dimension \(r\). Set \(\partial(N \times I) \approx N_+ \cup N_-\), where \(N_- = N \times \{0\} \cup \partial N \times [0, 1/2]\) and \(N_+ = N \times \{1\} \cup \partial N \times [1/2, 1]\). An element \(\sigma \in L^b_{r+1}(\pi, \omega)\) may be represented as the surgery obstruction of a degree one normal map \(f: (X, X_-, X_+) \rightarrow (N \times I, N_+, N_-)\), where \(X\) is a compact \((r+1)\)-dimensional manifold with \(\partial X = X_- \cup X_+\), such that \(f_- = f|_{X_-}: X_- \rightarrow N_-\) is a diffeomorphism and \(f_+ = f|_{X_+}: X_+ \rightarrow N_+\) is a homotopy equivalence. One has \(T(\sigma) = \sigma(f_+)\) (via the identification by \(g_a\)).

As mentioned in Remark \[3.16\], \(N = N^{(n-1)}\) is itself an \(s\)-cobordism between an \(\eta\)-thickening \(N^{(n-1)}\) of \(Y\) and itself. It follows that \(X_+\) is an \(h\)-cobordism between two copies \(N_+^{(n-1)}\) of \(N^{(n-1)}\), with \(\tau(N_+, N^{(n-1)}) = -\tau(f)\). One deduces that \(M = \partial X\) is an \(k\)-antisimple closed manifold, \((Y, \eta)\)-referred via \(g\), with \(\tau(M, g) = a\) (signs are irrelevant since \(H^{r+1}(\mathbb{Z}_2; Wh(\pi, \omega))\) is a group of exponent 2).

**Examples 4.5.**

1. \((1)\) \(L^0_{\text{odd}}(\pi) = 0\) if \(\pi\) is a finite group of odd order \([1]\) \((\omega\) is then trivial). Hence, by Proposition \[4.3\] any class \(a \in H^2(\mathbb{Z}_2; Wh(\pi))\) may be realized as \(\tau(M, g)\) for an \((Y, \eta)\)-referred closed \(k\)-antisimple manifold of dimension \(n = 2j \geq 6\). It follows from \([2, \S 4]\) that \(H^2(\mathbb{Z}_2; Wh(\pi)) \neq 0\) if \(\pi\) is finite abelian, of exponent \(\neq 1, 2, 3, 4\) or 6.

2. Let \(Y_1, \ldots, Y_s\) be finite CW-complexes of dimension \(\leq k - 1\), such that \(\pi_1(\tilde{Y}_i) = G_i\) is a finite group of odd order. Let \(Y = Y_1 \vee \cdots \vee Y_s\) and \(G = \pi_1(Y) \approx G_1 \ast \cdots \ast G_s\). One has \(L^b_{\text{odd}}(G) = 0\) by \([1, \text{Theorem 5}]\). Hence, as in \((1)\) above, any class \(a \in H^2(\mathbb{Z}_2; Wh(G))\) may be realized as \(\tau(M, g)\) for an \((Y, \eta)\)-referred closed \(k\)-antisimple manifold of dimension \(n = 2j \geq 6\). Recall that \(H^2(\mathbb{Z}_2; Wh(G)) \approx \bigoplus_{i=1}^s H^2(\mathbb{Z}_2; Wh(G_i))\) by \([20]\).

3. When \(Y\) is acyclic, an \((Y, \eta)\)-referred closed \(k\)-antisimple manifold is a homology sphere. As an example, consider the Poincaré 3-sphere minus a disk, which collapses to an acyclic 2-dimensional finite complex \(Y\) with

\[
\pi_1(Y) = \Delta = \langle a, b \mid a^5 = b^3 = (ab)^2 \rangle,
\]

the binary icosahedral group. Let \(C_5 = \langle t \mid t^5 = 1 \rangle\) be the cyclic group of order 5 with a given generator \(t\). There is a homomorphism \(\nu: C_5 \rightarrow \Delta\) with \(\nu(t) = a^2\). One can prove that the induced homomorphism

\[
\mathbb{Z}_2 \approx H^2(\mathbb{Z}_2; Wh(C_5)) \xrightarrow{\nu^*} H^2(\mathbb{Z}_2; Wh(\Delta))
\]

is injective \([8, \text{Chapter 5}]\). By naturality of the Rothenberg exact sequence and Example \((1)\) above, this produces 3-antisimple homology spheres in dimension \(r = 2j \geq 6\) which are not h-cobordant to a twisted double. Using \((1)\) and \((2)\) above, we can get such homology spheres with fundamental group a free product of finitely many copies of \(\Delta\).

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(4) Suppose that $Y$ is the lens space $L(p, q)$ with $p \equiv 3 \mod 4$, or $(p, q) = (5, 2)$ and take for $\eta$ the trivial bundle. Then, any $(Y, \eta)$-referred $k$-antisimple closed manifold $M$ of dimension $r = 2j + 1 \geq 2k \geq 6$ is a twisted double. Indeed, in the decomposition of Proposition 4.1, the h-cobordism $(V, \partial N, \partial N^*)$ is inertial, i.e. $\partial N^*$ is diffeomorphic to $\partial N$. But, there is no non-trivial inertial h-cobordism starting from $\partial N = Y \times S^2$ under our hypotheses [12, Theorem 6.1 and Lemma 6.5]. Hence $M$ is a twisted double by Remark 4.2.

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