TENSOR WEIGHT STRUCTURES AND T-STRUCTURES ON DERIVED CATEGORIES OF NOETHERIAN SCHEMES

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Abstract. We give a condition which characterises those weight structures on a derived category which come from a Thomason filtration on the underlying scheme. Weight structures satisfying our condition will be called $\otimes^c$-weight structures. More precisely, for a Noetherian separated scheme $X$, we give a bijection between the set of compactly generated $\otimes^c$-weight structures on $D(Qcoh X)$ and the set of Thomason filtrations of $X$. We achieve this classification in two steps. First, we show that the bijection [SP16, Theorem 4.10] restricts to give a bijection between the set of compactly generated $\otimes^c$-weight structures and the set of compactly generated tensor t-structures. We then use our earlier classification of compactly generated tensor t-structures to obtain the desired result. We also study some immediate consequences of these classifications in the particular case of the projective line. We show that in contrast to the case of tensor t-structures, there are no non-trivial tensor weight structures on $D^b(Coh P^1_k)$.

1. Introduction

Weight structures on triangulated categories were introduced by Bondarko [Bon10] as an important natural counterpart of t-structures with applications to Voevodsky’s category of motives. Pauksztello independently came up with the same notion while trying to obtain a dual version of a result due to Hoshino, Kato and Miyachi; he termed it co-t-structures, see [Pau08]. It has been observed by Bondarko that the two notions, t-structures and weight structures, are connected by interesting relations. In this vein, Štovíček and Pospíšil have proved for a certain class of triangulated categories, the collection of compactly generated t-structures and compactly generated weight structures are in bijection [SP16, Theorem 4.10] with each other, where the bijection goes via a duality at the compact level. In particular, this bijection holds in the derived category of a Noetherian ring $R$ and since in this case, we have the classification of compactly generated t-structures in terms of Thomason filtrations of $Spec R$ [ATJLS10, Theorem 3.11], they obtain a classification of compactly generated weight structures of $D(R)$.

Our aim in this short article is twofold: first to generalize the theorem of Štovíček and Pospíšil [SP16, Theorem 4.15] to the case of separated Noetherian schemes, and second to understand the two types of notions, in the simplest non-affine situation - the derived category of the projective line over a field $k$. Our interest in this special case arose partly from the work of Krause and Stevenson [KS19], where the authors study the localizing subcategories of $D(Qcoh P^1_k)$, and partly from our desire to better understand the general results.

In our earlier work [DS22], we have shown that a t-structure on $D(Qcoh X)$ supported on a Thomason filtration of a Noetherian scheme $X$ satisfies a tensor condition. We call them tensor t-structures. In this article, we introduce the analogous notion of tensor weight structures, also a slightly weaker notion which we call $\otimes^c$-weight structures. We then show that the bijection [SP16, Theorem 4.10] restricts to a bijection between tensor t-structures and $\otimes^c$-weight structures; this can be seen as a consequence of our Lemma 3.5 and Theorem 3.10. Next, we specialize to the case of the derived categories of separated Noetherian schemes and classify compactly generated $\otimes^c$-weight structures in this case, see Theorem 4.4.

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In the last section, we apply all the general theory and the classification results to the derived category of the projective line over a field \(k\). By our Theorem \ref{thm:classification} classifying compactly generated tensor t-structures of \(D(\text{Coh} \mathbb{P}^1_k)\) is equivalent to classifying \(\otimes\)-preaisles of \(D^b(\text{Coh} \mathbb{P}^1_k)\), so we restrict our attention to \(\otimes\)-preaisles of \(D^b(\text{Coh} \mathbb{P}^1_k)\). We give a complete description of the \(\otimes\)-preaisles in Proposition \ref{prop:tensor_weight} and in Proposition \ref{prop:aisles} we determine which of these are aisles or in other words give rise to t-structures on \(D^b(\text{Coh} \mathbb{P}^1_k)\). The result of Proposition \ref{prop:tensor_weight} is not new, it can possibly be deduced from \[B\] from \cite{BBK04}, the authors describe the bounded t-structures on \(D^b(\text{Coh} \mathbb{P}^1_k)\) by using the classification of t-stabilities on \(D^b(\text{Coh} \mathbb{P}^1_k)\). Finally, we consider the same question for tensor weight structures, and to our surprise, we discovered that there are no non-trivial tensor weight structures on \(D^b(\text{Coh} \mathbb{P}^1_k)\).

2. Preliminaries

Let \(\mathcal{T}\) be a triangulated category and \(\mathcal{T}^c\) denote the full subcategory of compact objects. We recall the definition of t-structures which was introduced in \cite{BBK82}.

**Definition 2.1.** A t-structure on \(\mathcal{T}\) is a pair of full subcategories \((\mathcal{U}, \mathcal{V})\) satisfying the following properties:

- t1. \(\Sigma \mathcal{U} \subseteq \mathcal{U}\) and \(\Sigma^{-1} \mathcal{V} \subseteq \mathcal{V}\).
- t2. \(\mathcal{U} \perp \Sigma^{-1} \mathcal{V}\).
- t3. For any \(T \in \mathcal{T}\) there is a distinguished triangle
  \[U \to T \to V \to \Sigma U\]
  where \(U \in \mathcal{U}\) and \(V \in \Sigma^{-1} \mathcal{V}\). We call such a triangle truncation decomposition of \(T\).

Next, we quote the definition of weight structures from \cite{BS18}.

**Definition 2.2.** A weight structure on \(\mathcal{T}\) is a pair of full subcategories \((\mathcal{X}, \mathcal{Y})\) satisfying the following properties:

- w0. \(\mathcal{X}\) and \(\mathcal{Y}\) are closed under direct summands.
- w1. \(\Sigma^{-1} \mathcal{X} \subseteq \mathcal{X}\) and \(\Sigma \mathcal{Y} \subseteq \mathcal{Y}\).
- w2. \(\mathcal{X} \perp \Sigma \mathcal{Y}\).
- w3. For any object \(T \in \mathcal{T}\) there is a distinguished triangle
  \[X \to T \to Y \to \Sigma X\]
  where \(X \in \mathcal{X}\) and \(Y \in \Sigma \mathcal{Y}\). The above triangle is called a weight decomposition of \(T\).

Note that if \((\mathcal{U}, \mathcal{V})\) is a t-structure on \(\mathcal{T}\) then \((\mathcal{V}, \mathcal{U})\) is a t-structure on \(\mathcal{T}^{\text{op}}\). Similarly, \(\mathcal{X}\) is a weight structure on \(\mathcal{T}\) then \((\mathcal{Y}, \mathcal{X})\) is a weight structure on \(\mathcal{T}^{\text{op}}\).

For any subcategory \(\mathcal{U}\) of \(\mathcal{T}\), we denote \(\mathcal{U}^{\perp}\) to be the full subcategory consisting of objects \(B \in \mathcal{T}\) such that \(\text{Hom}(A, B) = 0\) for all \(A \in \mathcal{U}\). Analogously we define \(\mathcal{U}^{\perp}\) to be the full subcategory of objects \(B \in \mathcal{T}\) such that \(\text{Hom}(B, A) = 0\) for all \(A \in \mathcal{U}\).

**Definition 2.3.** We say a t-structure \((\mathcal{U}, \mathcal{V})\) is compactly generated if there is a set of compact objects \(S\) such that \(\mathcal{U} = \perp(S^{\perp})\). A weight structure \((\mathcal{X}, \mathcal{Y})\) is compactly generated if there is a set of compact objects \(S\) such that \(\mathcal{X} = \perp(S^{\perp})\).

**Definition 2.4.** A subcategory \(\mathcal{U}\) of \(\mathcal{T}\) is a preaisle if it is closed under positive shifts and extensions. Dually, we say \(\mathcal{U}\) is a copreaisle of \(\mathcal{T}\), if \(\mathcal{U}\) is a preaisle of \(\mathcal{T}^{\text{op}}\).

A preaisle is called thick if it is closed under direct summands. We say a preaisle is cocomplete if it is closed under coproducts in \(\mathcal{T}\), and complete if it is closed under products. Similarly, we define thick, cocomplete and complete copreaisles.
For a t-structure \((\mathcal{U}, \mathcal{V})\) the subcategory \(\mathcal{U}\) is a cocomplete preaisle of \(\mathcal{T}\), and for a weight structure \((\mathcal{X}, \mathcal{Y})\) the subcategory \(\mathcal{X}\) is a cocomplete copreaisle of \(\mathcal{T}\).

We need the notion of stable derivators to formulate the next theorem but our requirement of the theory of derivators is the bare minimum. We will not go into the precise lengthy definition here, instead, we refer the reader to [SP16, §2.1] and references therein.

**Theorem 2.5** ([SP16, Theorem 4.5]). Let \(\mathcal{T} = D(e)\), where \(D\) is a stable derivator such that for each small category \(I\), \(D(I)\) has all small coproducts. Then,

i. There is a bijection between the set of compactly generated t-structures of \(\mathcal{T}\) and the set of thick preaisles of \(\mathcal{T}^c\) given by

\[
(\mathcal{U}, \mathcal{V}) \mapsto \mathcal{U} \cap \mathcal{T}^c
\]

\[
P \mapsto (\perp (P^\perp), \Sigma P^\perp).
\]

ii. There is a bijection between the set of compactly generated weight structures of \(\mathcal{T}\) and the set of thick copreaisles of \(\mathcal{T}^c\) given by

\[
(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \cap \mathcal{T}^c
\]

\[
P \mapsto (\perp (P^\perp), \Sigma^{-1} P^\perp).
\]

### 3. Tensor weight and t-structures

We recall the definition of tensor triangulated category from [HPS97, Definition A.2.1].

**Definition 3.1.** A tensor triangulated category \((\mathcal{T}, \otimes, 1)\) is a triangulated category with a compatible closed symmetric monoidal structure. This means there is a functor \(- \otimes - : \mathcal{T} \times \mathcal{T} \to \mathcal{T}\) which is triangulated in both the variables and satisfies certain compatibility conditions. Moreover, for each \(B \in \mathcal{T}\) the functor \(- \otimes B\) has a right adjoint which we denote by \(\mathcal{H} \text{om}(B, -)\). The functor \(\mathcal{H} \text{om}(-, -)\) is triangulated in both the variables, and for any \(A, B, C\) in \(\mathcal{T}\) we have natural isomorphisms \(\text{Hom}(A \otimes B, C) \to \text{Hom}(A, \mathcal{H} \text{om}(B, C))\).

**Definition 3.2.** Let \(\mathcal{T}\) be a tensor triangulated category given with a preaisle \(\mathcal{T}^{\leq 0}\) satisfying

\[
\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 0}\text{ and } 1 \in \mathcal{T}^{\leq 0}.
\]

A preaisle \(\mathcal{U}\) of \(\mathcal{T}\) is a \(\otimes\)-preaisle (with respect to \(\mathcal{T}^{\leq 0}\)) if \(\mathcal{T}^{\leq 0} \otimes \mathcal{U} \subset \mathcal{U}\). We say a copreaisle \(\mathcal{X}\) of \(\mathcal{T}\) is a \(\otimes\)-copreaisle (with respect to \(\mathcal{T}^{\leq 0}\)) if \(\mathcal{H} \text{om}(\mathcal{T}^{\leq 0}, \mathcal{X}) \subset \mathcal{X}\). A t-structure \((\mathcal{U}, \mathcal{V})\) is a called a tensor t-structure if \(\mathcal{U}\) is a \(\otimes\)-preaisle, and a weight structure \((\mathcal{X}, \mathcal{Y})\) is a tensor weight structure if \(\mathcal{X}\) is a \(\otimes\)-copreaisle.

Let \(\mathcal{S} \subset \mathcal{T}\) be a class of objects. We denote the smallest cocomplete preaisle containing \(\mathcal{S}\) by \(\langle \mathcal{S} \rangle^{\leq 0}\) and call it the cocomplete preaisle generated by \(\mathcal{S}\). If \(\mathcal{T}\) does not have coproducts we denote \(\langle \mathcal{S} \rangle^{\leq 0}\) to be the smallest preaisle containing \(\mathcal{S}\).

**Lemma 3.3.** Let \(\mathcal{T}^{\leq 0}\) be generated by a set of objects \(\mathcal{K}\), that is, \(\mathcal{T}^{\leq 0} = \langle \mathcal{K} \rangle^{\leq 0}\). Then

i. a cocomplete preaisle \(\mathcal{U}\) of \(\mathcal{T}\) is a \(\otimes\)-preaisle if and only if \(\mathcal{K} \otimes \mathcal{U} \subset \mathcal{U}\).

ii. a complete copreaisle \(\mathcal{X}\) of \(\mathcal{T}\) is a \(\otimes\)-copreaisle if and only if \(\mathcal{H} \text{om}(\mathcal{K}, \mathcal{X}) \subset \mathcal{X}\).

**Proof.** Part (i). Suppose \(\mathcal{K} \otimes \mathcal{U} \subset \mathcal{U}\). We define \(\mathcal{B} = \{X \in \mathcal{T}^{\leq 0} \mid X \otimes \mathcal{U} \subset \mathcal{U}\}\). Since \(\mathcal{U}\) is a cocomplete preaisle we can observe that \(\mathcal{B}\) is also a cocomplete preaisle. Now, by our assumption \(\mathcal{K} \subset \mathcal{B}\) so we get \(\mathcal{T}^{\leq 0} \subset \mathcal{B}\) which proves \(\mathcal{U}\) is \(\otimes\)-preaisle. The converse is immediate. Part (ii). Let \(\mathcal{B} = \{X \in \mathcal{T}^{\leq 0} \mid \mathcal{H} \text{om}(\mathcal{X}, \mathcal{X}) \subset \mathcal{X}\}\). Since \(\mathcal{X}\) is a copreaisle we can see that \(\mathcal{B}\) is a preaisle. Completeness of \(\mathcal{X}\) implies \(\mathcal{B}\) is cocomplete. Now, by following a similar argument as in (i) we get \(\mathcal{X}\) is \(\otimes\)-copreaisle.

□
An immediate consequence of the above lemma is if $T^{\leq 0} = \langle 1 \rangle^{\leq 0}$ then every cocomplete preaisle of $T$ is a $\otimes$-preaisle and every complete copreaisle of $T$ is a $\otimes$-copreaisle. In particular, for a commutative ring $R$, all cocomplete preaisle and complete copreaisle of $D(R)$ satisfy the tensor condition.

**Definition 3.4.** We say an object $X \in T$ is rigid or strongly dualizable if for each $Y \in T$ the natural map $\mu : \mathcal{H}om(X, 1) \otimes Y \to \mathcal{H}om(X, Y)$ is an isomorphism. A tensor triangulated category $(T, \otimes, 1)$ is rigidly compactly generated if the following conditions hold:

i. $T$ is compactly generated;

ii. $1$ is compact;

iii. every compact object is rigid.

Let $T$ be a rigidly compactly generated tensor triangulated category. For such a triangulated category the tensor product on $T$ restricts to $T^c$ therefore $(T^c, \otimes, 1)$ is also a tensor triangulated category. Suppose $T$ is given with a preaisle $T^{\leq 0}$ satisfying the condition of Definition 3.2 then so does the preaisle $T^c \cap T^{\leq 0}$ of $T^c$. So we can define $\otimes$-preaisles and $\otimes$-copreaisles of $T^c$ with respect to $T^c \cap T^{\leq 0}$.

**Lemma 3.5.** Let $T$ be a rigidly compactly generated tensor triangulated category given with a preaisle $T^{\leq 0}$ satisfying the condition of Definition 3.2.

Then, there is a one-to-one correspondence between the set of $\otimes$-preaisles and the set of $\otimes$-copreaisles of $T^c$.

**Proof.** Let $U$ be a full subcategory of $T^c$. We denote by $U^*$ the full subcategory

$$U^* = \{ X \in T^c \mid X \cong \mathcal{H}om(Y, 1) \text{ for some } Y \in U \}.$$  

The assignment $U \mapsto U^*$ induces an equivalence between the preaisles and copreaisles of $T^c$; see [SP16, Lemma 4.9]. We only need to show that the above assignment preserves the tensor condition.

Let $U$ be a preaisle of $T^c$, $X \in U^*$ and $T \in T^c \cap T^{\leq 0}$. We have

$$\mathcal{H}om(T, X) \cong \mathcal{H}om(T, \mathcal{H}om(Y, 1)) \cong \mathcal{H}om(T \otimes Y, 1).$$

If we assume $U$ is a $\otimes$-preaisle then $T \otimes Y \in U$. Hence $\mathcal{H}om(T, X) \in U^*$, this proves $U^*$ is a $\otimes$-copreaisle.

Now, suppose $U$ is a copreaisle. Let $X \in U^*$ and $T \in T^c \cap T^{\leq 0}$. Then,

$$T \otimes X \cong T \otimes \mathcal{H}om(Y, 1)$$

$$\cong \mathcal{H}om(Y, T)$$

$$\cong \mathcal{H}om(Y \otimes \mathcal{H}om(T, 1), 1)$$

$$\cong \mathcal{H}om(\mathcal{H}om(T, Y), 1).$$

If we assume $U$ is a $\otimes$-copreaisle then $\mathcal{H}om(T, Y) \in U$. We get, $T \otimes X \in U^*$, which proves $U^*$ is a $\otimes$-preaisle.

□

**Definition 3.6.** A preaisle $U$ is compactly generated if $U = \langle S \rangle^{\leq 0}$ for a set of compact objects $S$.

**Definition 3.7.** We say a triangulated category $T$ has the property $(\ast)$ if:

i. $T$ is rigidly compactly generated;

ii. $T$ has a preaisle $T^{\leq 0}$ satisfying $T^{\leq 0} \otimes T^{\leq 0} \subseteq T^{\leq 0}$ and $1 \in T^{\leq 0}$;

iii. $T^{\geq 0}$ is compactly generated, that is, $T^{\geq 0} = \langle T^c \cap T^{\geq 0} \rangle^{\leq 0}$.

For a triangulated category $T$ having the property $(\ast)$, we define a weaker notion than $\otimes$-(co)preaisle.
Definition 3.8. Let $\mathcal{T}$ have the property $(\ast)$. 
A preaisle $\mathcal{U}$ of $\mathcal{T}$ is a $\otimes^c$-preaisle if for any $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ and $U \in \mathcal{U}$ we have $T \otimes U \in \mathcal{U}$. Similarly a copreaisle $\mathcal{X}$ of $\mathcal{T}$ is a $\otimes^c$-copreaisle if for any $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ and $X \in \mathcal{X}$ we have $\mathcal{H}om(T,X) \in \mathcal{X}$.

A $t$-structure $(U,V)$ on $\mathcal{T}$ is a $\otimes^c$-$t$-structure if $U$ is a $\otimes^c$-preaisle and a weight structure $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{T}$ is a $\otimes^c$-weight structure if $\mathcal{X}$ is a $\otimes^c$-copreaisle.

Remark 3.9. This weaker notion gives something new only for preaisles(resp. copreaisles) which are not cocomplete(resp. complete) since by Lemma 3.3 it can easily be observed that for $\mathcal{T}$ having the property $(\ast)$: (i) every cocomplete $\otimes^c$-preaisle of $\mathcal{T}$ is a $\otimes$-preaisle of $\mathcal{T}$, and (ii) every complete $\otimes^c$-copreaisle of $\mathcal{T}$ is a $\otimes$-copreaisle of $\mathcal{T}$.

With this weaker notion, we now prove the tensor analogue of Theorem 2.5.

Theorem 3.10. Let $\mathcal{T}$ have the property $(\ast)$ (see Definition 3.7) and $\mathcal{T} = \mathcal{D}(e)$, where $\mathcal{D}$ is a stable derivator such that for each small category $I$, $\mathcal{D}(I)$ has all small coproducts. Then,

i. There is a bijective correspondence between the set for compactly generated $t$-structures of $\mathcal{T}$ and the set of thick $\otimes^c$-preaisles of $\mathcal{T}^c$ given by

$$ (\mathcal{U}, \mathcal{V}) \mapsto \mathcal{U} \cap \mathcal{T}^c $$

$$ \mathcal{P} \mapsto (\perp(\mathcal{P}^\perp), \Sigma \mathcal{P}^\perp). $$

ii. There is a bijective correspondence between the set for compactly generated $\otimes^c$-weight structures of $\mathcal{T}$ and the set of thick $\otimes^c$-copreaisles of $\mathcal{T}^c$ given by

$$ (\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \cap \mathcal{T}^c $$

$$ \mathcal{P} \mapsto (\perp(\mathcal{P}^\perp), \Sigma^{-1} \mathcal{P}^\perp). $$

Before proving the theorem, we will make some comments about the lack of symmetry in the above statement:

Remark 3.11. It is easy to observe that given a subcategory $\mathcal{P}$ the subcategory $\perp(\mathcal{P}^\perp)$ is always cocomplete, that is, closed under coproducts. Therefore, by Remark 3.9 saying $\perp(\mathcal{P}^\perp)$ is a $\otimes^c$-preaisle of $\mathcal{T}$ is equivalent to saying it is a $\otimes$-preaisle.

Remark 3.12. As $\mathcal{T}$ has coproducts and is compactly generated, by Brown representability it has products. However, it is not clear to us whether $\perp(\mathcal{P}^\perp)$ is closed under products, hence unlike the preaisle case, we can not claim it is $\otimes$-copreaisle.

Proof of Theorem 3.10. Part(i). It has already been shown in [SP10] Theorem 4.5(i) that the above assignments are bijections between the set of compactly generated $t$-structures of $\mathcal{T}$ and the set of thick preaisles of $\mathcal{T}^c$. We only need to show that the assignments preserve the tensor conditions.

From the definition of $\otimes^c$-preaisle, it is easy to observe that if $\mathcal{U}$ is a $\otimes^c$-preaisle of $\mathcal{T}$ then $\mathcal{U} \cap \mathcal{T}^c$ is a $\otimes$-preaisle of $\mathcal{T}^c$. Suppose $\mathcal{P}$ is a $\otimes$-preaisle of $\mathcal{T}^c$. Since $\perp(\mathcal{P}^\perp)$ is a cocomplete preaisle of $\mathcal{T}$ by Remark 3.9 it is enough to show $\perp(\mathcal{P}^\perp)$ is a $\otimes^c$-preaisle of $\mathcal{T}$. Let $\mathcal{B} = \{X \in \perp(\mathcal{P}^\perp) \mid (\mathcal{T}^c \cap \mathcal{T}^{\leq 0}) \otimes X \subset \perp(\mathcal{P}^\perp)\}$. We note that $\mathcal{B}$ is a cocomplete preaisle containing $\mathcal{P}$. Since $\perp(\mathcal{P}^\perp)$ is the smallest cocomplete preaisle containing $\mathcal{P}$ by [DS22] Lemma 1.9 we get $\mathcal{B} = \perp(\mathcal{P}^\perp)$.

Part(ii). In view of [SP10] Theorem 4.5(ii), again we only need to show that the assignments preserve the appropriate tensor conditions. If $\mathcal{X}$ is a $\otimes^c$-copreaisle of $\mathcal{T}$ then it is easy to observe that $\mathcal{X} \cap \mathcal{T}^c$ is a $\otimes$-copreaisle of $\mathcal{T}^c$. Suppose $\mathcal{P}$ is a $\otimes$-copreaisle of $\mathcal{T}^c$ we need to show that $\perp(\mathcal{P}^\perp)$ is a $\otimes^c$-copreaisle of $\mathcal{T}$. By [SP16] Theorem 3.7 an object $A \in \mathcal{T}$ belongs to $\perp(\mathcal{P}^\perp)$ if and only if $A$ is a summand of a homotopy colimit of a sequence

$$ 0 = Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} \cdots $$
Let $X$ be a Noetherian separated scheme. $D(Qcoh \ X)$ denotes the derived category of complexes of quasi coherent $O_X$-modules. The derived category $(D(Qcoh \ X), \otimes_{O_X}^L, O_X)$ is a tensor triangulated category with the derived tensor product $\otimes_{O_X}^L$ and the structure sheaf $O_X$ as the unit. The full subcategory of complexes whose cohomologies vanish in positive degree $D^{\leq 0}(Qcoh \ X)$ is a preaisle of $D(Qcoh \ X)$ satisfying the conditions of Definition 4.2. We define the $\otimes$-preaisles and $\otimes$-copreaisles of $D(Qcoh \ X)$ with respect to $D^{\leq 0}(Qcoh \ X)$. Similarly, the $\otimes$-preaisles and $\otimes$-copreaisles of $\text{Perf}(X)$ are defined with respect to $\text{Perf}^{\leq 0}(X)$. Note that $D(Qcoh \ X)$ has the property $(\ast)$ (see Definition 3.7), so we can define $\otimes^c$-(co)preaisles of $D(Qcoh \ X)$.

**Definition 4.1.** A subset $Z$ is a specialization closed subset of $X$ if for each $x \in Z$ the closure of the singleton set $\{x\}$ is contained in $Z$, that is, $\overline{\{x\}} \subset Z$. Note that a specialization closed subset is a union of closed subsets of $X$.

A subset $Y$ is a Thomason subset of $X$ if $Y = \bigcup_\alpha Y_\alpha$ is a union of closed subsets $Y_\alpha$ such that $X \setminus Y_\alpha$ is quasi compact. Note that if $X$ is Noetherian then the two notions coincide.

**Definition 4.2.** A Thomason filtration of $X$ is a map $\phi: Z \to 2^X$ such that $\phi(i)$ is a Thomason subset of $X$ and $\phi(i) \supset \phi(i+1)$ for all $i \in \mathbb{Z}$.

In our earlier work we have mentioned without proof (see [DS22 Remark 4.13]) about the following result, here we explicitly state it for future reference. This is a generalization of Thomason’s classification [Tho97 Theorem 3.15] of $\otimes$-ideals to $\otimes$-preaisles of $\text{Perf}(X)$, for separated Noetherian scheme $X$.

**Proposition 4.3.** Let $X$ be a separated Noetherian scheme. The assignment sending a Thomason filtration $\phi$ to $S_\phi = \{E \in \text{Perf}(X) \mid \text{Supp}(H^i(E)) \subset \phi(i)\}$ provides a one-to-one correspondence between the following sets:

i. Thomason filtrations of $X$;
ii. Thick $\otimes$-preaisles of $\text{Perf}(X)$.

**Proof.** In [DS22 Theorem 4.11] we have shown that sending $\phi$ to

$$U_\phi = \{E \in D(Qcoh \ X) \mid \text{Supp}(H^i(E)) \subset \phi(i)\}$$

provides a bijection between the set of Thomason filtrations and the set of compactly generated tensor $t$-structure of $D(Qcoh \ X)$. From part (i) of Theorem 3.10 we conclude that the above assignment provides a bijection between Thomason filtrations of $X$ and thick $\otimes$-preaisles of $\text{Perf}(X)$.

**Theorem 4.4.** Let $X$ be a separated Noetherian scheme. There is a one-to-one correspondence between the following sets:

i. Thomason Filtrations of $X$;
ii. Compactly generated $\otimes^c$-weight structures of $D(Qcoh \ X)$.

The assignment is given by

$$\phi \mapsto (A_\phi, B_\phi)$$

where

$$B_\phi = \{B \in D(Qcoh \ X) \mid \text{Hom}(O_X, S \otimes_{O_X}^L B) = 0 \text{ for all } S \in S_\phi, \}$$

where each $f_i$ occurs in a triangle $Y_i \to Y_{i+1} \to S_i \to \Sigma Y_i$ with $S_i \in \text{Add } P$.

First, we observe that for any compact object $T$ the functor $\mathcal{H}om(T,-)$ preserves small coproducts therefore $\mathcal{H}om(T,-)$ takes homotopy sequences to homotopy sequences. Since $P$ is $\otimes$-copreaisle of $T^c$ for any $T \in T^c \cap T^{\leq 0}$ we have $\mathcal{H}om(T, S_i) \in \text{Add } P$. Thus applying [SP16 Theorem 3.7] again we get $\mathcal{H}om(T, A) \in \perp (P^\perp)$.

\[\square\]
Since $B$ point $P_Coh\langle S\rangle$ in $Coh_P D$ preaisle containing $U$ recall that for any subcategory the direct sum of its cohomology sheaves, see [GKR04, Proposition 6.1]. Also, every coherent sheave is a preaisle of $D_O$. Proof. Suppose Lemma 5.1. A thick preaisle $A$ of $D^b(Coh P_k^1)$ is a $\otimes$-preaisle if and only if $O(-1) \otimes A \subset A$.

Proof. Suppose $A$ is a $\otimes$-preaisle then $O(-1) \otimes A \subset A$ is true by definition. Conversely, suppose $A$ is a preaisle of $D^b(Coh P_k^1)$. Take $B := \{ B \in D^{b,\leq 0}(Coh P_k^1) \mid B \otimes A \subset A \}$. From our assumption, we have $O(-1) \in B$. It is now easy to see that for every $n \geq 0$ we have $O(-n) \in B$.

As $Coh P_k^1$ has homological dimension one, every complex of $D^b(Coh P_k^1)$ is quasi isomorphic to the direct sum of its cohomology sheaves, see [GKR04, Proposition 6.1]. Also, every coherent sheave over $P_k^1$ is the direct sum of line bundles and torsion sheaves. Since $B$ is a preaisle, to show $B = D^{b,\leq 0}(Coh P_k^1)$ it is enough to show that $B$ contains all line bundles and torsion sheaves.

For any $m \geq 0$ consider the following triangle coming from the corresponding short exact sequence in $Coh P_k^1$, see for instance [GKR04 Equation 6.3],

\[ O(-2)^{\oplus (m+1)} \rightarrow O(-1)^{\oplus (m+2)} \rightarrow O(m) \rightarrow O(-2)[1]. \]

Since $B$ is closed under extension and positive shifts we have $O(m) \in B$.

Next, for any indecomposable torsion sheave of degree $d$ say $T_x$, which is supported on a closed point $x \in P_k^1$, consider the following triangle coming from the corresponding short exact sequence in $Coh P_k^1$; see [GKR04 Equation 6.5],

\[ O(-2)^{\oplus d} \rightarrow O(-1)^{\oplus d} \rightarrow T_x \rightarrow O(-2)[1]. \]

Again using the fact that $B$ is closed under extension and positive shifts we have $T_x \in B$.

Recall that for a set of objects $S$ of $T$ we denote the smallest cocomplete preaisle containing $S$ by $\langle S \rangle^{\leq 0}$. If $T$ does not have coproducts, for instance $D^b(Coh P_k^1)$, we denote $\langle S \rangle^{\leq 0}$ to be the smallest preaisle containing $S$. Similarly we denote $\langle S \rangle^{\geq 0}$ to be the smallest copreaisle containing $S$. Also recall that for any subcategory $U$ we denote $U^*$ the full subcategory $U^* = \{ X \in T \mid X \cong Hom(Y, 1) \text{ for some } Y \in U \}$. 

5. IN THE CASE OF PROJECTIVE LINE

In this section, we will specialize to the case of projective line $P_k^1$ over a field $k$. By the results of earlier sections, classifying compactly generated tensor t-str uctures of $D(Qcoh P_k^1)$ is equivalent to classifying thick $\otimes$-preaisles of $Perf(P_k^1)$. For any smooth Noetherian scheme $X$ the inclusion functor from $Perf(X)$ to the derived category of bounded complexes of coherent sheaves $D^b(Coh X)$ is an equivalence. Therefore, we restrict our attention to $D^b(Coh P_k^1)$. Note that we define $\otimes$-preaisles of $D^b(Coh P_k^1)$ with respect to the standard preaisle $D^{b,\leq 0}(Coh P_k^1) := \{ E \in D^b(Coh P_k^1) \mid H^i(E) = 0 \forall i > 0 \}$.

Lemma 5.1. A thick preaisle $A$ of $D^b(Coh P_k^1)$ is a $\otimes$-preaisle if and only if $O(-1) \otimes A \subset A$. 

Proof. Suppose $A$ is a $\otimes$-preaisle then $O(-1) \otimes A \subset A$ is true by definition. Conversely, suppose $A$ is a preaisle of $D^b(Coh P_k^1)$. Take $B := \{ B \in D^{b,\leq 0}(Coh P_k^1) \mid B \otimes A \subset A \}$. From our assumption, we have $O(-1) \in B$. It is now easy to see that for every $n \geq 0$ we have $O(-n) \in B$.

As $Coh P_k^1$ has homological dimension one, every complex of $D^b(Coh P_k^1)$ is quasi isomorphic to the direct sum of its cohomology sheaves, see [GKR04, Proposition 6.1]. Also, every coherent sheave over $P_k^1$ is the direct sum of line bundles and torsion sheaves. Since $B$ is a preaisle, to show $B = D^{b,\leq 0}(Coh P_k^1)$ it is enough to show that $B$ contains all line bundles and torsion sheaves.

For any $m \geq 0$ consider the following triangle coming from the corresponding short exact sequence in $Coh P_k^1$, see for instance [GKR04 Equation 6.3],

\[ O(-2)^{\oplus (m+1)} \rightarrow O(-1)^{\oplus (m+2)} \rightarrow O(m) \rightarrow O(-2)[1]. \]

Since $B$ is closed under extension and positive shifts we have $O(m) \in B$.

Next, for any indecomposable torsion sheave of degree $d$ say $T_x$, which is supported on a closed point $x \in P_k^1$, consider the following triangle coming from the corresponding short exact sequence in $Coh P_k^1$; see [GKR04 Equation 6.5],

\[ O(-2)^{\oplus d} \rightarrow O(-1)^{\oplus d} \rightarrow T_x \rightarrow O(-2)[1]. \]

Again using the fact that $B$ is closed under extension and positive shifts we have $T_x \in B$. 

Recall that for a set of objects $S$ of $T$ we denote the smallest cocomplete preaisle containing $S$ by $\langle S \rangle^{\leq 0}$. If $T$ does not have coproducts, for instance $D^b(Coh P_k^1)$, we denote $\langle S \rangle^{\leq 0}$ to be the smallest preaisle containing $S$. Similarly we denote $\langle S \rangle^{\geq 0}$ to be the smallest copreaisle containing $S$. Also recall that for any subcategory $U$ we denote $U^*$ the full subcategory $U^* = \{ X \in T \mid X \cong Hom(Y, 1) \text{ for some } Y \in U \}$. 

5. IN THE CASE OF PROJECTIVE LINE
Example 5.2. For a fixed \( n \in \mathbb{Z} \) we denote
\[
B_n := \langle \mathcal{O}(n) \rangle^{\leq 0}; \quad \text{and} \quad C_n := \langle \mathcal{O}(n), \mathcal{O}(n + 1) \rangle^{\leq 0}.
\]

Using Lemma 5.1 we can check that \( B_n \) and \( C_n \) are not \( \otimes \)-preaisles of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \). Similarly, \( B_n^* \) and \( C_n^* \) provide examples of copreaisles of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \) which are not \( \otimes \)-copreaisles. This can be observed using Lemma \( \text{X}.3 \).

Recall that a Thomason filtration of \( X \) is a map \( \phi : \mathbb{Z} \to 2^X \) such that \( \phi(i) \) is a Thomason subset of \( X \) and \( \phi(i) \supset \phi(i + 1) \) for all \( i \in \mathbb{Z} \). We say \( \phi \) is type-1 if \( \bigcup_i \phi(i) \neq X \); and we say \( \phi \) is type-2 if \( \bigcup_i \phi(i) = X \) but not all \( \phi(i) = X \).

Let \( x \in \mathbb{P}^1_k \) be a closed point. We denote the simple torsion sheaf supported on \( x \) by \( k(x) \). Now, we give an explicit description of the \( \otimes \)-preaisles of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \) in terms of simple torsion sheaves and line bundles.

Proposition 5.3. Any proper thick \( \otimes \)-preaisle of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \) is one of the following forms:

i. \( \langle k(x)[-i] \mid x \in \phi(i) \rangle^{\leq 0} \);

where \( \phi \) is a type-1 Thomason filtration of \( \mathbb{P}^1_k \).

ii. \( \langle \mathcal{O}(n)[-i_0], k(x)[-i] \mid \forall n \in \mathbb{Z} \text{ and } x \in \phi(i) \rangle^{\leq 0} \)

where \( \phi \) is a type-2 Thomason filtration of \( \mathbb{P}^1_k \) and \( i_0 \) a fixed integer.

Proof. Suppose \( \mathcal{A} \) is a thick \( \otimes \)-preaisle of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \). By Proposition \( \text{L.3} \) there is a unique Thomason filtration \( \phi \) such that
\[
\mathcal{A} = \{ E \in \text{D}^b(\text{Coh} \mathbb{P}^1_k) \mid \text{Supp} H^i(E) \subset \phi(i) \}.
\]

Since \( \text{Coh} \mathbb{P}^1_k \) has homological dimension one, every complex of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \) is quasi isomorphic to the direct sum of its cohomology sheaves. Therefore, we can write \( \mathcal{A} \) in terms of coherent sheaves alone,
\[
\mathcal{A} = \{ F[-i] \mid F \in \text{Coh} \mathbb{P}^1_k \text{ and } \text{Supp} F \subset \phi(i) \}^{\leq 0}.
\]

Case 1. ( \( \phi \) is type-1 ) Note that \( \phi(i) \subseteq \mathbb{P}^1_k \) for all \( i \). Every coherent sheaf over \( \mathbb{P}^1_k \) is the direct sum of line bundles and torsion sheaves. Since the support of any line bundle is whole \( \mathbb{P}^1_k \). In this case, \( \mathcal{A} \) only contains torsion sheaves. As torsion sheaves can be generated by simple torsion sheaves we have,
\[
\mathcal{A} = \langle k(x)[-i] \mid x \in \phi(i) \rangle^{\leq 0}.
\]

Case 2. ( \( \phi \) is type-2 ) Since \( \bigcup_i \phi(i) = X \) there is an integer \( i_0 \) such that \( \phi(i_0) \) contains the generic point of \( \mathbb{P}^1_k \). We can take \( i_0 \) to be the largest such integer. Observe that \( \phi(i) = \mathbb{P}^1_k \) for all \( i \leq i_0 \) and \( \phi(i_0 + 1) \subseteq \mathbb{P}^1_k \).

Here we can check that
\[
\mathcal{A} = \langle \mathcal{O}(n)[-i_0], k(x)[-i] \mid \forall n \in \mathbb{Z} \text{ and } x \in \phi(i) \rangle^{\leq 0}.
\]

Next, we will show which of these \( \otimes \)-preaisles of \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \) are \( t \)-structures on \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \). First, we prove a few lemmas.

Lemma 5.4. Let \( A \in \text{Coh} \mathbb{P}^1_k \) be a torsion sheaf and \( \mathcal{L} \) be a line bundle. Let \( \delta : A \to \mathcal{L}[1] \) be any map in \( \text{D}^b(\text{Coh} \mathbb{P}^1_k) \). Then, \( \text{cone}(\delta) \notin A^\perp \).

Proof. As we know \( \text{Hom}(A, \mathcal{L}[1]) \cong \text{Ext}^1(A, \mathcal{L}) \), a map \( \delta : A \to \mathcal{L}[1] \) corresponds to an element of the group \( \text{Ext}^1(A, \mathcal{L}) \). By abuse of notation, we denote the corresponding element in \( \text{Ext}^1(A, \mathcal{L}) \) by \( \delta \).

Now we take the short exact sequence corresponding to \( \delta \in \text{Ext}^1(A, \mathcal{L}) \), say
This gives rise to a distinguished triangle

\[ \mathcal{L} \longrightarrow B \longrightarrow A \longrightarrow [1] \]

Hence, \( \text{cone}(\delta) \cong B[1] \). From the short exact sequence, we observe that \( B \) can not be a torsion sheaf, so it must have a torsion free summand. Therefore, \( \text{Hom}(A, B[1]) = \text{Ext}^1(A, B) \neq 0 \).

Recall that a preaisle \( A \) is an aisle if \( (A, A^+[1]) \) a t-structure.

**Lemma 5.5.** Let \( A \) be a \( \otimes \)-preaisle of \( D^b(\text{Coh } \mathbb{P}_k^1) \) and \( \phi \) its corresponding Thomason filtration. If \( A \) is an aisle and \( \phi(i) \neq \emptyset \) for some \( i \), then \( \phi(i - 1) = \mathbb{P}_k^1 \).

**Proof.** Without loss of generality, we may assume \( i = 0 \). If \( \phi(0) = \mathbb{P}_k^1 \) then \( \phi(-1) = \mathbb{P}_k^1 \) and there is nothing to prove. Now suppose \( \phi(0) \subseteq \mathbb{P}_k^1 \) then there is a closed point \( x \in \phi(0) \). We will prove our claim by showing a contradiction.

Let \( \mathcal{L} \) be a line bundle on \( \text{Coh } \mathbb{P}_k^1 \). If \( \phi(-1) \neq \mathbb{P}_k^1 \) then \( \mathcal{L}[1] \notin A \). Since \( \text{Ext}^1(k(x), \mathcal{L}) \neq 0 \) we also have \( \mathcal{L}[1] \notin A^+ \). Now, as \( A \) is given to be an aisle we must have a t-decomposition of \( \mathcal{L}[1] \). But Lemma 5.4 says such a decomposition is not possible.

**Definition 5.6.** We say \( \phi \) is a one-step Thomason filtration of \( \mathbb{P}_k^1 \) if there is an integer \( i_0 \) and a Thomson subset \( Z_{i_0} \) such that

\[
\phi(j) = \begin{cases} 
\mathbb{P}_k^1 & \text{if } j < i_0; \\
Z_{i_0} & \text{if } j = i_0; \\
\emptyset & \text{if } j > i_0.
\end{cases}
\]

**Proposition 5.7.** A \( \otimes \)-preaisle of \( D^b(\text{Coh } \mathbb{P}_k^1) \) is an aisle if and only if the corresponding Thomason filtration is a one-step filtration.

**Proof.** If \( A \) is a \( \otimes \)-preaisle which is also an aisle then by Lemma 5.5 the corresponding filtration is a one-step filtration. Conversely, suppose the filtration is one step, we will show that every complex of \( D^b(\text{Coh } \mathbb{P}_k^1) \) can be decomposed into a triangle where the first term is in \( A \) and the third term is in \( A^+ \). Without loss of generality we may assume the one step occurs at \( i_0 = 0 \), and \( \phi(0) = Z_0 \) a Thomason subset.

If \( Z_0 = \mathbb{P}_k^1 \), then \( A = D^{b, \leq 0}(\text{Coh } \mathbb{P}_k^1) \) and we get standard t-structure. Now, suppose \( Z_0 \neq \mathbb{P}_k^1 \). Since the filtration is one step we only need to show sheaves at degree zero have t-decompositions, all other shifted sheaves have obvious t-decompositions. The functor \( \Gamma_{Z_0}(\cdot) \) gives a t-decomposition of sheaves at degree zero.

Next, we give an explicit description of \( \otimes \)-copreaisles of \( D^b(\text{Coh } \mathbb{P}_k^1) \) in terms of simple torsion sheaves and line bundles.

**Proposition 5.8.** Any proper thick \( \otimes \)-copreaisle of \( D^b(\text{Coh } \mathbb{P}_k^1) \) is one of the following forms:

i. \( \langle k(x)^*[i] \mid x \in \phi(i) \rangle_{\geq 0} \),

where \( \phi \) is a type-1 Thomason filtration of \( \mathbb{P}_k^1 \).

ii. \( \langle \mathcal{O}(n)[i_0], k(x)^*[i] \mid \forall n \in \mathbb{Z} \text{ and } x \in \phi(i) \rangle_{\geq 0} \)

where \( \phi \) is a type-2 Thomason filtration of \( \mathbb{P}_k^1 \) and \( i_0 \) a fixed integer.
Proof. By the proof of Lemma 3.3, we know that every $\otimes$-copreaisle of $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$ is of the form $\mathcal{A}^*$ where $\mathcal{A}$ is $\otimes$-preaisle. Now using the description given in Proposition 5.3, we conclude our result.

The trivial $\otimes$-copreaisles $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$ and $0$ give rise to tensor weight structures on $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$. In contrast to the case of $t$-structures (see Proposition 5.7), the next result shows that, there are no other tensor weight structures on $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$.

**Proposition 5.9.** The trivial weight structures are the only tensor weight structures on $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$.

Proof. Suppose $\mathcal{A}$ is a $\otimes$-copreaisle of $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$ which induces a weight structure. We claim that $\mathcal{A}$ can not be a copreaisle containing only torsion sheaves. Indeed, by Lemma 5.3, any line bundle $\mathcal{L}[1]$ can not have a weight decomposition, so $\mathcal{A}$ must contain line bundles up to shifts. If $\mathcal{L}[i] \in \mathcal{A}$ for some $\mathcal{L}$, then for any line bundle $\mathcal{M}$, $\mathcal{M}[i] = \mathcal{H}om(\mathcal{M} \otimes \mathcal{L}, \mathcal{L}[i]) \in \mathcal{A}$.

Now, there are two cases: (1) either there is an integer $i$ such that $\mathcal{L}[i] \in \mathcal{A}$ and $\mathcal{L}[i+1] \notin \mathcal{A}$, or (2) there is no such $i$ and $\mathcal{A}$ contains all line bundles and their shifts.

Case 1. Suppose there is an integer $i$, then without loss of generality we can assume $i = 0$. By our assumption, for any line bundle $\mathcal{M}$, $\mathcal{M}[1] \notin \mathcal{A}$. Since by tensor condition $\mathcal{A}$ contains all line bundles we can choose $\mathcal{L}$ such that $\text{Ext}^1(\mathcal{L}, \mathcal{M}) \neq 0$, therefore, $\mathcal{M}[1] \notin \mathcal{A}^*$. Then by a similar argument as in Lemma 5.3, $\mathcal{M}[1]$ can not have a weight decomposition (since any distinguished triangles with $\mathcal{M}[1]$ in the middle will result in a third term having a summand isomorphic to $\mathcal{M}[1]$, and $\mathcal{M}[1]$ is not in $\mathcal{A}^*$). Therefore, $\mathcal{A}$ can not induce a weight structure.

Case 2. Suppose $\mathcal{A}$ contains all line bundles and their shifts. In particular, it contains $\mathcal{O}(1)$, $\mathcal{O}(-1)$, and all their shifts. Now, for any indecomposable torsion sheave of degree $d$ say $T_x$ supported on a closed point $x \in \mathbb{P}^1_\mathbb{k}$, consider the following triangle coming from the corresponding short exact sequence in $\text{Coh } \mathbb{P}^1_\mathbb{k}$,

$$\mathcal{O}(-2)^{\oplus d} \rightarrow \mathcal{O}(-1)^{\oplus d} \rightarrow T_x \rightarrow \mathcal{O}(-2)[1].$$

As $\mathcal{A}$ is closed under extensions, $T_x \in \mathcal{A}$. This proves $\mathcal{A}$ contains all torsion sheaves and their shifts. Therefore, $\mathcal{A}$ must be equal to $D^b(\text{Coh } \mathbb{P}^1_\mathbb{k})$.

\[\square\]

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