Abstract. Using his deep and beautiful idea of cutting with a Hyperplane, Lefschetz explained how the homology groups of a projective smooth variety could be constructed from basic pieces, that he called primitive homology. This idea can be applied every time we have a vector space with a bilinear form (i.e. homology with cup product) and a ±-symmetric nilpotent operator (i.e. cutting with a hyperplane). We will illustrate this in the context of Singularity Theory: A germ of an isolated singular point of a hypersurface defined by \( f = 0 \). We begin with the algebraic setting in the Jacobian (or Milnor) Algebra of the singularity, with Grothendieck pairing as bilinear form and multiplication by \( f \) as a symmetric nilpotent operator. We continue in the topological setting of vanishing cohomology with bilinear form induced from cup product and as nilpotent map the logarithm of the unipotent map of the monodromy as an anti-symmetric operator. We then show how these 2 very different settings are tied up using the Brieskorn lattice as a D-module, on using results of Brieskorn (1970), A. Varchenko (1980s), M. Saito (1989), and C. Hertling (1999, 2004, 2005), inducing a Polarized Mixed Hodge structure at the singularity (Steenbrink, 1976) and bringing the spectrum of the singularity as a deeper invariant than the eigenvalues of the Algebraic Monodromy. In particular, we show how an \( f \)-Jordan chain is obtained from several \( N \)-Jordan chains by gluing them in the Brieskorn lattice.

1. Introduction

Given a germ of a holomorphic function \( f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) with an isolated singularity one can associate to it the following objects:

1) The **algebraic object**, consists of the Jacobian Algebra of \( f \)

\[
A_f := \frac{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}{J_f}
\]

where \( z_0, \ldots, z_n \) are coordinates on \( \mathbb{C}^{n+1} \), \( f_j = \partial f / \partial z_j \), \( \mathcal{O}_{\mathbb{C}^{n+1}, 0} \) is the ring of germs of holomorphic functions at \( 0 \in \mathbb{C}^{n+1} \) and \( J_f := (f_0, f_1, \ldots, f_n) \subset \mathcal{O}_{\mathbb{C}^{n+1}, 0} \) is the Jacobian ideal generated by the partial derivatives of \( f \). The dimension as a \( \mathbb{C} \)-vector space of \( A_f \) is the Milnor number \( \mu := \text{dim}_C A_f < \infty \) ([17]). \( A_f \) carries naturally a non-degenerate bilinear form, the Grothendieck pairing ([7] p. 649)

\[
\text{res}_f : A_f \times A_f \rightarrow \mathbb{C},
\]
The nilpotent operator $M_f : A_f \rightarrow A_f$ given by multiplication with $f$ is $\text{res}_f$-symmetric. The index of nilpotency $m$ of $M_f$ is at most $n$ (i.e. $M_f^{m+1} = 0$ but $M_f^m \neq 0 : A_f \rightarrow A_f$ [1]). We can obtain from this data a flag of ideals in $A_f$:

$$A_f = W_{-m}(A_f, M_f) \supset W_{-m+1}(A_f, M_f) \supset \cdots \supset W_m(A_f, M_f) \supset W_{m+1}(A_f, M_f) = 0$$

with extreme terms at both ends

$$W_{-m+1}(A_f, M_f) := \text{Ann}_{A_f}(f^m) \supset W_m(A_f, M_f) := (f^m)$$

and proceeding to construct the flag from induction with

$$M_f^{m-1} : \frac{\text{Ann}_{A_f}(f^m)}{(f^m)} \rightarrow \frac{\text{Ann}_{A_f}(f^m)}{(f^m)}$$

which now has index of nilpotency $m - 1$.

One then constructs another vector space of dimension $\mu$ which is obtained by “chopping up” $A_f$ according to the filtration $W_s(A_f, M_f)$, and is called the graded module associated to the filtration:

$$\text{Gr}_W^t(A_f, M_f) := \bigoplus_{j=-m}^m \text{Gr}_W^j(A_f, M_f) := \bigoplus_{j=-m}^m \frac{W_j(A_f, M_f)}{W_{j+1}(A_f, M_f)}$$

Grothendieck residue will induce non-degenerate bilinear forms on $\text{Gr}_W^t(A_f, M_f)$ of type $\text{res}_f(M_f^j, \ast)$ with nice orthogonality properties, and we will obtain that each of the graded pieces is made up of “primitive” ones. It is in this graded module $\text{Gr}_W^t(A_f, M_f)$ that we can see the Lefschetz decomposition most clearly in this algebraic setting. Of course, it takes some reflection to understand this Lefschetz decomposition in the graded object with its bilinear form and its relation to the Jacobian Algebra $A_f$ and Grothendieck Residue pairing.

2) The **topological object** consists of the $C^\infty$ locally trivial fibre bundle $\{X_t := f^{-1}(t)\}_{t \in D^*}$ over a punctured disc $D^* \subset \mathbb{C}$ with fibre the “Milnor Fibre”, which is a smooth $2n$ compact orientable differentiable manifold with boundary which is homotopically a bouquet of $\mu \ n$-dimensional spheres (Milnor[17]). The Geometric Monodromy map is the isotopy class of the automorphism of the Milnor Fibre obtained from lifting a loop around $0 \in D^*$ to a diffeomorphism of the fibre of the locally trivial fibre bundle being the identity on the boundary of the fibre. The $n^{th}$ dimensional cohomology groups with integer, rational or complex coefficients $\{H^n(X_t, K)\}_{t \in D^*}$, called the “vanishing cohomology” groups, form a $K$-vector bundle over $D^*$ with a flat connection (the “Gauss-Manin connection” ([2]) whose Algebraic Monodromy map $M_K : H^n(X_t, K) \rightarrow H^n(X_t, K)$ is a fundamental linear invariant of the singularity. The Monodromy Theorem asserts that $M_K$ has roots of unity as eigenvalues, and that the size of the Jordan blocks of $M_K$ is bounded by $n + 1$. For a sufficiently large $s$, $M_K^s$ will be a unipotent integer matrix and the $\mu \times \mu$ matrix

$$N := \frac{1}{s} \text{Log}(M_K^s) = \frac{1}{s} \sum_{j=1}^{n+1} \frac{1}{j} (M_K^s - I \text{Id}_\mu)^j$$

with rational coefficients satisfies that $e^N$ is the unipotent part of the factorization of $M = M_s M_u$ as a semisimple times a unipotent matrix. $N$ is a a nilpotent endomorphism of
\( H^n(X_t, \mathbb{C}) \) that gives rise to a decomposition of the cohomology groups \( H^n(X_t, \mathbb{C}) \) ‘a la Lefschetz’.

The bilinear form \( Q \) in vanishing cohomology comes from cup product on \( V_t \). It is a \( \pm \)-symmetric bilinear form, and \( N \) is antisymmetric with respect to it. All this allows us to give a decomposition of vanishing cohomology as made up of many pieces of basic primitive forms, seen as vector spaces with bilinear forms \( Q(N^j \bullet, J \bullet) \), where \( J \) is an involution whose eigenspaces are described using the mixed Hodge structure of the singularity. Arithmetic appears here, since the bundle and the cup product are actually defined with integer coefficients and the bilinear form has positive definite properties: the “Riemann-Hodge bilinear relations” ([7]). All this gets codified into the “Polarized Mixed Hodge Structure of Vanishing Cohomology”.

3) The differential object consists of the following: Denote as \( H^n(X_\infty, \mathbb{C}) \) the canonical cohomology fiber which is identified with the holomorphic sections associated to the vector bundle

\[
H^n = \bigcup_{t \in D^*} H^n(X_t, \mathbb{C}) := \{H^n(X_t, \mathbb{C})\}_{t \in D^*}.
\]

Let \( \{H_{e^{-2\pi i \beta}}\}_{\beta \in (-1,0) \cap \mathbb{Q}} \) be the set of generalized eigenspaces with respect to the semisimple part of the monodromy map \( M_s \). Hence the elements on each \( H_{e^{-2\pi i \beta}} \) become flat sections with the meromorphic Gauss-Manin connection. Consider the space

\[
H := \bigoplus_{\beta \in (-1,0) \cap \mathbb{Q}} H_{e^{-2\pi i \beta}} t^{\beta - \frac{1}{2\pi i} N}.
\]

The \( V \)-filtration in \( H \) is defined by

\[
V^\gamma := \bigoplus_{\gamma \leq \beta + k} H_{e^{-2\pi i \beta}} t^{\beta + k - \frac{1}{2\pi i} N},
\]

the weight \( W \)-filtration is formed by the topological \( W_N \)-filtration (that is, the weight filtration associated to the nilpotent restriction map \( N : H_{e^{-2\pi i \beta}} \to H_{e^{-2\pi i \beta}} \)) in each summand, the \( VW \) bi-filtration is a lexicographic combination of the two, with a double index, first the \( V \)-filtration followed by the \( W \)-filtration. Introduce the \( \mathbb{C} \)-linear map

\[
\partial_t^{-1} : H \to H,
\]

which is defined on its summands by the relation (essentially, this map is the inverse of the local Gauss-Manin connection):

\[
\partial_t^{-1}(A_{\beta_j + k} t^{\beta_j + k - \frac{1}{2\pi i} N}) := ((\beta_j + k + 1)I - \frac{1}{2\pi i} N)^{-1} A_{\beta_j + k} t^{\beta_j + k + 1 - \frac{1}{2\pi i} N}
\]

for \( k = 0, \ldots, n - 1 \) and 0 for \( k = n \). The nilpotent map is \( \partial_t^{-1} N \) and the bilinear form \( <, >_H \) is the orthogonal direct sum

\[
H = \bigoplus_{k=0,\ldots,(n-1)/2} [H^k \oplus H^{n-k}] \left( \bigoplus H^{n/2} \text{ for } n \text{ even} \right)
\]

where in each summand the bilinear form has the form

\[
\langle vt^k, wt^{n-k} \rangle_H := \frac{(1)^{1+r}}{(2\pi i)^{n+1+r}} Q(\partial_t^k vt^k, (-1)^{n-k} \partial_t^{n-k} wt^{n-k}), \quad v \in H, w \in H, \quad r = -1 \text{ or } 0,
\]

where \( \partial_t^k vt^k \) and \( \partial_t^{n-k} wt^{n-k} \) are flat sections and \( Q \) is the cup product as above. The structure in \( H \) then is topological (coming from \( H, M, N \) and \( Q \)) and analytic due to \( \partial_t \), which can
be interpreted as the Gauss-Manin connection on the sections of the bundle of \( n^{th} \) primitive cohomology.

Consider now the (Poincaré-Gelfand-Leray) residue ([7]). Namely, given a germ \( \omega \in \Omega^{n+1} \) of an \((n+1)\)-differential form at \( 0 \in \mathbb{C}^{n+1} \), its residue \( s(\omega) \) is a section of the cohomology bundle \( H \) over the punctured disk. Expand it in a Laurent type expansion ([16]) and consider the first \((n+1)\)-order terms in the expansion \( s_n : \Omega^{n+1} \rightarrow \mathbb{H} \), with image \( \mathbb{H}' \). Introduce the subspace \( \mathbb{H}' := s_n(df \wedge \Omega^n) \) consisting of those expansions that can be realized by sections of the cohomology bundle \( H \) of the form \( [\eta]_{X_i} \), for \( \eta \in \Omega^n \), where we are taking the de-Rham class in cohomology \( H^n(X_i, \mathbb{C}) \) of the closed \( n\)-form \( \eta|_{X_i} \). The connection between the algebraic and the analytic then comes from the theorem of Varchenko ([28]) that asserts that the annihilator of the restriction of the bilinear form \( <,> \) isomorphism of the non-degenerate bilinear spaces \(\mathbb{H}' \rightarrow \mathbb{H}''\), where we have put in the domain the bilinear form \( res_f \) and in the image the induced non-degenerate bilinear form from \( <,>_{\mathbb{H}} \).

One induces the \(V\)-filtration on \( \mathbb{H}'/\mathbb{H}' \) and then on \( \Omega_f \) via the isomorphism \( s_n \). Namely \( s_n(\omega) \in V^r(\mathbb{H}'/\mathbb{H}') \) if there exists an \( \eta \in \Omega^n \) with \( s_n(\omega + df \wedge \eta) \in V^r(\mathbb{H}) \). The principal term of the form \( \omega \in \Omega^{n+1} \) is the \( gr(V) \)-smallest non-zero coefficient of \( s_n(\omega) \). The form \( \omega \in \Omega^{n+1} \) is original if the principal term of \( s_n(\omega) \) cannot be \( gr(VW) \)-diminished by adding an element of the form \( s_n(df \wedge \eta) \), for \( \eta \in \Omega^n \).

Multiplication by \( f \) in \( \Omega^{n+1} \) satisfies \( s_n(f \omega) = ts_n(\omega) \), and hence \( s_n \) preserves the \( V(\Omega_f) \)-filtration inducing a degree 1 map in the associated graded module:

\[
gr(f) : Gr^*_V \Omega_f \longrightarrow Gr^{*+1}_V \Omega_f.
\]

Another theorem of Varchenko ([30]) asserts that if we consider only the principal terms by considering the graded map \( gr(s_n) \) when cutting with the \( V \)-filtration, then multiplication by \( f \) corresponds to applying \( -\frac{1}{2\pi i}N \). This means that multiplication by \( f \) in the Jacobian module contains more information than applying \( -\frac{1}{2\pi i}N \) to its principal term, since it is acting on \( \Omega_f \) and not only on its principal term \( Gr^*_V \Omega_f \).

This structure was clarified by M. Saito [20, 19] and C. Hertling [11, 12] by choosing a convenient basis of \( \Omega_f \) by original forms \( \omega_1, \ldots, \omega_\mu \in \Omega^{n+1} \), which is adapted to multiplication by \( f \). This basis puts the bilinear form \( res_f \) in a canonical form and the principal part of \( s_n(\omega_j) \) is \( A_j t^{\alpha_j - \frac{1}{2\pi i}N} \in H_{e^{-2\pi i \alpha_j}} t^{\alpha_j - \frac{1}{2\pi i}N} \) which is adapted to multiplication by \( N \) and puts the bilinear form \( S \) into canonical form ([11, Prop. 5.1 and 5.4]). The rational numbers

\[-1 < \alpha_1 \leq \ldots \leq \alpha_\mu \leq n\]

are the spectral values. There is an increasing function \( \nu_N : \{1, \ldots, \mu\} \longrightarrow \{1, \ldots, \mu + 1\} \) and the \( N \)-adaptedness condition is

\[
A_{\nu_N(j)} t^{\alpha_N(j) - \frac{1}{2\pi i}N} = \partial_{\ell}^{-1} N(A_j) t^{\alpha_j + 1 - \frac{1}{2\pi i}N} \quad \text{with} \quad A_{\mu + 1} := 0.
\]

The orbits of \( \nu_N \) split the spectral values into \( N \)-chains of spectral values:

\[
\{\alpha_1, \alpha_N(1) = \alpha_1 + 1, \alpha_N(2) = \alpha_2 + 1, \ldots, \alpha_N(\mu) = \alpha_\mu + 1\};
\]

\[
\{\alpha_2, \alpha_N(2) = \alpha_2 + 1, \alpha_N(3) = \alpha_3 + 1, \ldots, \alpha_N(\mu) = \alpha_\mu + \ell_2\}, \ldots
\]
The adaptedness condition for \( f \) means that, beginning with the original 1-form \( \omega_1 \), its principal term is \( A_1 t^{\alpha_1 - \frac{1}{2} N} \). Now \( f \omega_1 \) is not original, but its original representative is \( f \omega_1 - (\alpha_1 + 1) df \wedge \eta_1 \), where \( df \wedge \eta_1 = \omega_1 \) with principal term \( [(\alpha_1 + 1) I - \frac{1}{2^{\alpha_1}} N]^{-1} N A_1 \), etc. The binding of the spectral \( N \)-chains is produced from an argument as the following: taking the last basis element corresponding to the first \( N \)-chain, apply \( f \) to it and subtract \( (\alpha_i N) + 1) df \wedge \eta_1 \) with \( \omega_\ell N(1) = df \wedge \eta_1 \):

\[
\text{f}\omega_\ell N(1) - (\alpha_i N + 1) df \wedge \eta_1.
\]

Since this element is in the end of an \( N \)-chain, the principal term that one would expect vanishes (i.e. \( N^{\ell+1} A_1 = 0 \)), and so its principal term corresponds to a spectrum point bigger than \( \alpha_i N + 1 \). We continue the \( N \)-chain from this point as before. One has binded in this way the \( 2 \) \( N \)-chains together and one continues to obtain in this manner to build the \( f \)-chains.

One now proceeds to analyse the bilinear forms \( S \) and \( \text{res}_f(\cdot, \cdot) \) in Saito’s basis. There is a function \( \kappa : \{1, \ldots, \mu\} \rightarrow \{1, \ldots, \mu\} \) which is \( \kappa(j) = \mu + 1 - j \) except for some values where \( \kappa(j) = j \), with \( \alpha_j = (n - 1)/2 \). We have ([11, Prop. 5.1 and Prop. 4.4])

\[
S(L_j A_j, L_\ell A_\ell) = (-1)^{j+1+r} \delta_{\kappa(j), \ell}(2\pi i)^{n+1+r}, \quad r = -1 \text{ or } 0, \quad \text{res}_f([\omega_j], [\omega_\ell]) = \delta_{\kappa(j), \ell}.
\]

We proceed to state the main results of this paper. Indeed, for a germ of a holomorphic function \( f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) with an isolated critical point at \( 0 \in \mathbb{C}^{n+1} \) we will show:

1) Theorem 3 which states that the bilinear forms \( \text{res}_f(f^j, \cdot) \) in the Jacobian module obtained by Grothendieck duality and multiplication by \( f^j \) in one factor can be expressed as a sum of the principal topological bilinear form \( Q(N^j, J) \) and weaker bilinear forms \( Q_j \); being \( J \) an involution whose eigenspaces are expressed in terms of the Mixed Hodge structure of the Jacobian module.

2) Corollary 2 from which we obtain that the nature of the bilinear forms \( Q_j \) arises from the bindings of the \( N \)-chains into the \( f \)-chains.

The signatures of these bilinear forms enter as normalizing constants of a formula for computing indices of vector fields tangent to a hypersurface, when working over the real numbers ([6]).

We will use the simplified framework expounded in Hertling ([11], [10]) of work by Varchenko ([28], [29]), K. and M. Saito ([18], [20]), Scherk and Steenbrink ([26]). Recent related material are van Straten ([27]) and M. Saito ([19]).

2. Weight filtration on the Jacobian Algebra \( A_f \)

2.1. Choosing a basis for \( A_f \) adapted to the Jordan block structure of \( M_f \). Let \( f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0) \) be a germ of a holomorphic function with an isolated singularity at \( 0, z_0, \ldots, z_n \) coordinates on \( \mathbb{C}^{n+1} \), \( f_j = \partial f / \partial z_j \) and \( \mathcal{O}_{\mathbb{C}^{n+1}, 0} \) the ring of germs of holomorphic functions at \( 0 \in \mathbb{C}^{n+1} \). Denote the Jacobian ideal \( J_f := (f_0, f_1, \ldots, f_n) \subset \mathcal{O}_{\mathbb{C}^{n+1}, 0} \) of \( f \) and the Jacobian Algebra

\[
A_f := \frac{\mathcal{O}_{\mathbb{C}^{n+1}, 0}}{J_f}
\]

with Milnor number \( \mu := \dim_{\mathbb{C}} A_f < \infty \). Let

\[ M_f : A_f \rightarrow A_f, \quad [g] \rightarrow [fg] \]
be the operator defined by multiplication with \( f \), where \([g] \) denotes the class of \( g \in \mathcal{O}_{\mathbb{C}^{n+1},0} \) in \( A_f \). Multiplication by \( f \) is a symmetric operator with respect to the Algebra structure of \( A_f \):

\[
[M_f(a)][b] = [a][M_f(b)] \iff (fa)b = a(fb).
\]

\( M_f \) is nilpotent with index of nilpotency \( m_0 \leq n \) (i.e. \( M_f^{m_0} \neq 0 \), \( M_f^{m_0+1} = 0 \), [1]).

We will choose an ordered basis (as a \( \mathbb{C} \)-vector space) of \( A_f \) adapted to \( M_f \): Choose \( g_1 \in A_f - \text{Ann}_{A_f}(f^{m_0}) \) and the basis begins with

\[
g_1, g_2 := fg_1, \ldots, g_{m_0+1} := f^{m_0}g_1.
\]

Imagine a horizontal chain of length \( 2m_0 \) with \( m_0 + 1 \) spheres located at distance 2 in the chain beginning with the extremes. Put on each of these spheres the element of the above partial basis beginning on the left with \( g_1, fg_1, f^2g_1, \ldots \) The mapping \( M_f \) is a movement of the chain a step to the right of length 2 (each sphere moves to the next sphere to the right on the chain), and the extreme right sphere disappears under \( M_f \) (\( f^{m_0+1}g_1 = 0 \)). We will call this a chain of size \( m_0 + 1 \), the number of basis vectors (or spheres) in the chain.

![Fig. 1. Representation of the largest \( M_f \)-Jordan chain in \( A_f \).](image-url)

Continue by choosing \( g_{m_0+2} \in \text{Ann}_{A_f}(f^{m_1}) - \text{Ann}_{A_f}(f^{m_1-1}) \), with largest possible \( m_1 \leq m_0 + 1 \) so that it is independent of the already chosen basis elements \( g_1, \ldots, g_{m_0+1} \), so as to have the new partial basis

\[
g_1, \ldots, g_{m_0+1}, g_{m_0+2}, \ldots, g_{m_0+m_1+2} := f^{m_1}g_{m_0+2},
\]

etc. Continue in this way we obtain a basis \( \{g_k\} \), that we fix. Denote by \( \ell_j \) the number of chains of size \( j \) obtained. The numbers \( (\ell_{m_0+1}, \ldots, \ell_1) \) determine the Jordan-block type of \( M_f : A_f \to A_f \), and for the Milnor number we have

\[
\mu = (m_0 + 1)\ell_{m_0+1} + \cdots + 2\ell_2 + \ell_1,
\]

since there are \( \ell_1 \) chains of size 1, \( \ell_2 \) of size 2, etc.

Organize all the chains in such a way that the center of each chain is located over the point 0 and the negative numbers are to the left, as well as by size: smaller chains go on top. In this way we obtain a pyramid like structure, which by definition is symmetric with respect to the involution on the vertical line over 0.
We define the \( m \) platform of the pyramid as formed by those basis elements which are part of a Jordan chain of size \( m \) (so \( m \)-spheres, \( m - 1 \) arrows, length \( 2(m - 1) \)) and the height of the \( m \)-platform is \( \ell_m \), the number of Jordan chains of size \( m \). In this image we are developing the \( \ell_m \) Jordan blocks of size \( m \) form the \( m \)-platform. Imagine this platform as made up of \( m - 1 \) blocks, each one of horizontal length 2 and vertical height \( \ell_m \) where the spheres constituting the platform are vertically evenly distributed and horizontally are separated at distance 2. The only different platform is the 1-platform, which looks more like an antena, since its horizontal length is 0, and it has vertical length \( \ell_1 \).

The weight \( w(g_j) \) of a basis element \( g_j \) who is a member of a chain of size \( m \) and occupies the \( k^{th} \) place in the chain, \( k = 1, \ldots, m \), is defined as \(-m + 2k - 1\). It is the value of the projection of its sphere to the segment \([-m_0 + 1, m_0 - 1]\) at the base of the pyramid in the above construction. Its weight is 0 if it is in the middle of a chain of size \( m \) and it is \(-m + 1\) (or
$m - 1$) if it is on the extreme right (or left) of a chain of size $m$. The weight $w(g)$ of a general element $g = \sum_{i=1}^{\mu} a_i g_i \in A_f$, is

$$w(g) := \min\{w(g_i) / a_i \neq 0\}.$$

Since each block of the pyramid has length 2, and the platform on top has 1 chain less, then this length 2 is divided into 2 segments of length 1 on each side of the platform. So the blocks are not just ‘piled up’, but they have a weaving structure:

![Fig. 4. Weaving Structure of the Pyramid of $(A_f, M_f)$.

The $j^{th}$-column of the pyramid consists of those basis elements with weight $j$. The height of the columns are, beginning on the left (or on the right):

$$\ell_{m_0}, \ell_{m_0-1}, \ell_{m_0} + \ell_{m_0-2}, \ell_{m_0-1} + \ell_{m_0-3}, \ldots, \ell_{m_0} + \ell_{m_0-2} + \cdots + \ell_{m_0-2\kappa}, \ldots, \ell_{m_0} + \ell_{m_0-2}, \ell_{m_0-1}, \ell_{m_0}$$

since the columns at distance 2 to the right and left of a given column consists of adding or deleting an additional piece of the column, corresponding to the spheres on its uppermost platform, as can be clearly seen by drawing the pyramid.

![Fig. 5. Platform and Column Structure of the Pyramid of $(A_f, M_f)$.

We reorder the above basis $g_1, \ldots, g_\mu$ of $A_f$ to obtain a basis $\omega_1, \ldots, \omega_\mu$: Begin the ordering with the basis elements on the columns to the left, and on each column we begin the ordering with the elements in the blocks of the lower part of the column and moving up till the top block of that column, before proceeding to the next column.
We remember the collection of basis elements that appear in each of the columns

\[
\{\omega_1, \ldots, \omega_\mu\} := \{\{\omega_1, \ldots, \omega_{\ell m_0}\}, \{\omega_{\ell m_0+1}, \ldots, \omega_{\ell m_0+\ell m_1}\},
\{f\omega_1, \ldots, f\omega_{\ell m_0}, \omega_{2\ell m_0+\ell m_1+1}, \ldots, \omega_{2\ell m_0+\ell m_1+\ell m_2}\}, \ldots\};
\]

forming \(2m_0+1\) blocks, that we number by the integer points in \([-m_0, m_0]\). The above grouping of the basis will give decompositions of endomorphisms \(M_f\) of \(A_f\) in the form of block matrices.

Define the strictly increasing function

\[
\nu_f : \{1, \ldots, \mu\} \rightarrow \{1, \ldots, \mu, \mu + 1\}
\]

by the condition \(M_f(h_\ell) = \omega_{\nu(\ell)}\), with \(\omega_{\mu+1} := 0\). This function codifies, in the above ordering of the basis, the effect of multiplication by \(f\): The matrix expression of \(M_f\) in this basis is \((\delta_{j,\nu_f(j)})\).

**2.2. The Weight Filtration of the Jacobian Algebra.** This structure on the Jacobian Algebra appears due to the existence of a canonical filtration \(W(f)\) of \(A_f\) induced by \(M_f\), called the weight filtration ([8], [21]):

\[
A_f \supset W_{-m_0}(M_f) \supset W_{-m_0+1}(M_f) \supset \cdots \supset W_0(M_f) \supset \cdots \supset W_{m_0-1}(M_f) \supset W_{m_0}(M_f) \supset 0.
\]

**Definition 1.** The element \(W_j(M_f)\) of the weight filtration \(W(M_f)\) of \(A_f\) is defined as the vector space spanned by those basis elements whose weight is greater than or equal to \(j\), i.e. those which are located to the right of \(j\), in the above pyramid like structure of \(A_f\).

Note that the ordering of the basis is constructed in such a way that certain increasing segments of them \(g_{k_1}, \ldots, g_{k_2}\) generate transversals to \(W_j(M_f)\) in \(W_{j-1}(M_f)\).

**Lemma 1.** The weight filtration is independent of the chosen basis.
Proof. We give an intrinsic definition of the weight filtration: It proceeds by induction on the maximal length \( m_0 \) of the Jordan chains of \( M_f \), defining the extreme elements of the filtration by

\[
W_{m_0}(M_f) := \text{Ann}(f^{m_0}) \quad \text{and} \quad W_{-m_0}(M_f) := (f^{m_0}).
\]

For the induction step one considers the map induced by \( M_f \) on \( \text{Ann}(f^{m_0})/(f^{m_0}) \). If \( M_f \) has type \((\ell_1, \ldots, \ell_r)\), then the induced map in \( \text{Ann}(f^{m_0})/(f^{m_0})\) will have type \((\ell_1, \ldots, \ell_{r-2} + \ell_r, \ell_{r-1})\), since we have removed only the 2-extreme blocks of the \( r \)-platform, so the remaining blocks will be of size \( r - 2 \), and are incorporated to the \((r-2)\)-platform of height \( \ell_{r-2} \), to give the new platform of height \( \ell_{r-2} + \ell_r \). Now we only have Jordan chains of length strictly less than \( m_0 \), so induction hypothesis apply. We pull back to \( \text{Ann}(f^{m_0}) \) the obtained flag in \( \text{Ann}(f^{m_0})/(f^{m_0}) \), and we complete with the already chosen \( W_{\pm m_0}(f) \). This completes the proof. \( \square \)

2.3. The \( W(f) \)-Graded Jacobian \( \text{Gr}_{W(f)}^*(A_f, M_f) \).

Definition 2. The \( W(f) \)-graded Jacobian is a \( \mathbb{C} \)-vector space of dimension \( \mu \) defined by

\[
\text{Gr}_{W(f)}^*(A_f, M_f) := \bigoplus_{j=-m_0}^{m_0} \text{Gr}_{W(f)}^j(A_f, M_f) \quad \quad \text{Gr}_{W(f)}^j(A_f, M_f) := \frac{W_j(M_f)}{W_{j+1}(M_f)}.
\]

We have natural projections

\[
gr_j : W_j(f) \longrightarrow \frac{W_j(f)}{W_{j+1}(f)} \subset \text{Gr}_{W(f)}^*(A_f, M_f)
\]

that may be thought as taking the ‘principal part’ (with respect to the weight filtration \( W(f) \)). \( \text{Gr}_{W(f)}^*(A_f, M_f) \) inherits a graded basis \( \{gr(\omega_1), \ldots, gr(\omega_{\mu})\} \). The map \( M_f \) sends \( W_j(M_f) \) to \( W_{j+2}(M_f) \), so there is an induced degree 2 map

\[
gr(M_f) : \text{Gr}_{W(f)}^*(A_f, M_f) \longrightarrow \text{Gr}_{W(f)}^*(A_f, M_f) \quad \quad gr(M_f)(gr(\omega_\ell)) = gr(f\omega_\ell) = gr(\omega_{\nu(\ell)}).
\]

It has in the graded basis the same matrix form as \( M_f \): \( (\delta_{\nu(j)}, \nu(j)) \).

For \( j = -m_0, \ldots, 0 \) define the primitive spaces

\[
\text{Prim}^j(A_f, M_f) \subset \text{Gr}_{W(f)}^j(A_f, M_f)
\]

as the vector spaces generated by \( \{gr(\omega_j) \mid j \notin \text{Im}(\nu_f)\} \). They correspond to the basis elements on the top block of the columns on the left side of the pyramid, or the basis elements most to the left on each of the platforms. They are the elements of the basis which are not divisible by \( f \).
We obtain from this chain description for \( j = -m_0, \ldots, m_0 \) the direct sum

\[
Gr_j^W(M_f)(A_f, M_f) = \bigoplus_{-m_0 \leq j-2k \leq 0} M_k^j Prim^{j-2k}(A_f, M_f),
\]

which describes the columns \( Gr_j^W(M_f)(A_f, M_f) \) formed by piling up primitive blocks. The difference between one column and the column 2 steps to the left or right is the addition or deletion of one of the corresponding primitive block on top.

Intrinsically we have

\[
Prim^j(A_f, M_f) := Ker[M_f^{-j+1} : Gr_j^W(M_f)(A_f, M_f) \rightarrow Gr_{j}^{W(M_f)}(A_f, M_f)] \simeq \frac{Ann_{A_f}(f) \cap (f^j)}{Ann_{A_f}(f) \cap (f^{j+1})},
\]
since on a platform, the left block corresponds bijectively with the right block, by applying $M_f$ an adequate number of times.

Define the even and odd pieces of $Gr^*_W(A_f, M_f)$ by

$$Gr^*_{W(M_f)}(A_f, M_f) = \bigoplus_k Gr^{2k}_{W(M_f)}(A_f, M_f), \quad Gr^*_{W(M_f)}(A_f, M_f) = \bigoplus_k Gr^{2k+1}_{W(M_f)}(A_f, M_f)$$

and the induced maps

$$gr(M_f) : Gr^*_{W(M_f)}(A_f, M_f) \to Gr^*_{W(M_f)}(A_f, M_f), \quad gr(M_f) : Gr^*_{W(M_f)}(A_f, M_f) \to Gr^*_{W(M_f)}(A_f, M_f)$$

We may summarize our conclusion in a way which is independent of the chosen basis as:

**Proposition 1.** There is a filtration of $A_f$, called the $f$-weight filtration, canonically induced from the nilpotent map $M_f$ in the Jacobian Algebra $A_f$:

$$A_f = W_{-m_0}(A_f, M_f) \supset W_{-m_0+1}(A_f, M_f) \supset \cdots \supset W_{m_0}(A_f, M_f) \supset W_{m_0+1}(A_f, M_f) = 0$$

with the properties:

1) $M_f(W_j(A_f, M_f)) \subset W_{j+2}(A_f, M_f)$

2) Denote the associated graded objects by

$$Gr_j(A_f, M_f) := \frac{W_j(A_f, M_f)}{W_{j+1}(A_f, M_f)}, \quad Gr^*_j(A_f, M_f) := \bigoplus_{j=-m_0}^{m_0} Gr_j(A_f, M_f)$$

$$Gr_{even}(A_f, M_f) := \bigoplus Gr_{2j}(A_f, M_f), \quad Gr_{odd}(A_f, M_f) := \bigoplus Gr_{2j+1}(A_f, M_f),$$

and the +2-graded maps induced from $M_f$:

$$M_f : Gr_{even}(A_f, M_f) \to Gr_{even}(A_f, M_f), \quad M_f : Gr_{odd}(A_f, M_f) \to Gr_{odd}(A_f, M_f)$$

with primitive pieces for $j = 0, \ldots, m_0$:

$$Prim_{-j}(A_f, M_f) := \text{Ker}[M_f^{j+1}] : Gr_{-j}(A_f, M_f) \to Gr_{j+2}(A_f, M_f)$$

of dimension $p_j$ and giving rise to a Lefschetz-type decomposition for $j = -m_0, \ldots, m_0$:

$$(6) \quad Gr_j(A_f, M_f) = \bigoplus M_f^k Prim_{j-2k}(A_f, M_f)$$

and isomorphisms

$$M_f^j : Gr_{n-j}(A_f, M_f) \to Gr_{n+j}(A_f, M_f).$$

3) The interval $I := \{1, \ldots, \mu\}$ may be divided into subintervals $I = \{I_{-m_0}, \ldots, I_{m_0}\}$ and each one further $I_j = \{I_{j,k}\}$ with $p_{j-2k}$ elements, for $j = m_0, \ldots, 0$, and we may choose a basis $g_1, \ldots, g_\mu$ of $A_f$ with an increasing map $\alpha : I \to \{j, \mu + 1\}$, $g_{\mu+1} := 0$ such that:

a) $M_f(g_k) = g_{\alpha(k)}$,

b) $\{g_\ell\}$ for $\ell \in \{I_j, \ldots, I_{m_0}\}$ form a basis of $W_j(A_f, M_f)$,

c) $gr(g_\ell)$ with $\ell \in I_{j,k}$ are a basis of $M_f^k Prim_{j-2k}(A_f, M_f)$. 


2.4. Grothendieck’s Bilinear Form in the Jacobian Algebra. Define the linear transformation

\[ L : \mathcal{O}_{\mathbb{C}^{n+1},0} \rightarrow \mathbb{C}, \quad L(h) := \left( \frac{1}{2\pi i} \right)^{n+1} \int_{\Gamma} \frac{h dz}{f_0 \cdots f_n} \]

where \( dz = dz_0 \wedge \ldots \wedge dz_n \) and \( \Gamma \) is the \((n+1)\)-real cycle

\[ \Gamma := \{ z \in \mathbb{C}^{n+1} : |f_j(z)| = \varepsilon, 0 \leq j \leq n \}, \quad d(\arg f_0) \wedge \cdots \wedge d(\arg f_n) > 0, \quad \varepsilon \ll 1. \]

On using Stokes’ formula, one has \( L(h) = 0 \) for \( h \in J_f \). So \( L \) defines a \( \mathbb{C} \)-linear map \( L : A_f \rightarrow \mathbb{C} \).

Definition 3. Grothendieck’s bilinear pairing is defined by

\[ (7) \quad res_f : A_f \times A_f \rightarrow \mathbb{C}, \quad res_f([h_1], [h_2]) = L([h_1 h_2]) = \left( \frac{1}{2\pi i} \right)^{n+1} \int_{\Gamma} \frac{h_1 h_2 dz}{f_0 \cdots f_n}. \]

It is a nondegenerate pairing, by Grothendieck Local Duality ([7], p. 659). The class \( Hess(f) \) of the Hessian determinant of \( f \) generates the socle in \( A_f \): the minimal non-zero ideal in \( A_f \) ([5]). The fundamental property of \( L \) that is used to obtain a non-degenerate bilinear form from the algebra structure of \( A_f \) is that \( L(Hess(f)) \neq 0 \).

3. The Bilinear Form in Cohomology for Hypersurfaces in Projective Space with an Isolated Singularity

3.1. Pencils of Hypersurfaces in Projective Space. Let \( f : \mathbb{C}^{n+1} \rightarrow \mathbb{C} \) be a polynomial such that \( f^{-1}(0) \) extends to the hyperplane at infinity, \( V_0 := \overline{f^{-1}(0)} \) as a smooth \( n \) dimensional variety except at 0, where it has an isolated singularity. Then \( V_t := \overline{f^{-1}(t)} \) is a projective manifold of dimension \( n \) for \( t \in \Delta - \{0\} \), a sufficiently small punctured disk. The interesting part of the cohomology algebra \( H^*(V_t, \mathbb{Q}) \) of \( V_t \) is in \( H^n(V_t, \mathbb{Q}) \). We have a non-degenerate bilinear form given by cup product:

\[ Q : H^n(V_t, \mathbb{Q}) \times H^n(V_t, \mathbb{Q}) \rightarrow H^{2n}(V_t, \mathbb{Q}) \cong \mathbb{Q}, \]

which is unimodular, symmetric for \( n \) even and antisymmetric for \( n \) odd. Extending the coefficients to \( \mathbb{Q} \oplus i\mathbb{Q} \) we obtain 2 types of extensions of the bilinear form \( Q \), the \( \mathbb{C} \)-linear and the Hermitian:

\[ (u_0 + iu_1)Q(v_0 + iv_1), \quad (u_0 + iu_1)Q(v_0 - iv_1) \]

which receive block expressions, respectively:

\[ (8) \quad Q_{\mathbb{C}} := \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} + i \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}, \quad Q_{\mathbb{C}} := \begin{pmatrix} Q & 0 \\ 0 & -Q \end{pmatrix} + i \begin{pmatrix} 0 & Q \\ Q & 0 \end{pmatrix}. \]

The first 3 matrices are symmetric if \( Q \) is symmetric and the last one is symmetric if \( Q \) is antisymmetric. If \( n \) is even, then \( Q_{\mathbb{C}} \) is Hermitian symmetric, and if \( n \) is odd

\[ iQ_{\mathbb{C}} = \begin{pmatrix} 0 & Q \\ -Q & 0 \end{pmatrix} + i \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \]

is Hermitian symmetric. If we extend the coefficients to \( \mathbb{C} \) the matrix expressions of the extended bilinear forms is the same (see [9]).
If we use the de Rham complex of $C^\infty$-differential forms on $V_t$ to represent $H^n(V_t, \mathbb{C})$, the bilinear form is obtained by cup product of two closed $n$-differential forms, and integrating over $V_t$ the resulting $2n$-form. The symmetry or anti-symmetry is then a consequence of the alternating nature of the exterior algebra (see [24]).

3.2. The Semi-simple Decomposition of $H^n(V_t, \mathbb{C})$. The monodromy map $M : H^n(V_t, \mathbb{Q}) \to H^n(V_t, \mathbb{Q})$ is the effect on cohomology of going around $t = 0$ for the map $f$. It is a $Q$-automorphism: $Q(Mu, Mv) = Q(u, v)$ due to the functoriality of the cup product. Consider the map $M_s : H^n(V_t, \mathbb{C}) \to H^n(V_t, \mathbb{C})$ defined by multiplication by $\lambda$ restricted to the generalized $\lambda$-eigenspace $H^n(V_t, \mathbb{C})_\lambda$ of $M$, and $M_u := M_s^{-1}M$ the unipotent part of $M$. The monodromy automorphism $M$ is the product of its semisimple and unipotent part $M = M_uM_s$.

Since $M$ is real we have

$$H^n(V_t, \mathbb{C})_\lambda = H^n(V_t, \mathbb{C})_{\overline{\lambda}}$$

so that if

$$H^n(V_t, \mathbb{R})_{\lambda, \overline{\lambda}} = [H^n(V_t, \mathbb{C})_\lambda \bigoplus H^n(V_t, \mathbb{C})_{\overline{\lambda}}] \bigcap H^n(V_t, \mathbb{R})$$

we have a $Q$-orthogonal direct sum, $M$-invariant decomposition

$$H^n(V_t, \mathbb{R}) = H^n(V_t, \mathbb{R})_1 \bigoplus H^n(V_t, \mathbb{R})_{-1} \bigoplus \bigoplus_{Im\lambda > 0} H^n(V_t, \mathbb{R})_{\lambda, \overline{\lambda}}$$

(9)

$$H^n(V_t, \mathbb{Q}) = H^n(V_t, \mathbb{Q})_1 \bigoplus H^n(V_t, \mathbb{Q})_{1, Q},$$

$$H^n(V_t, \mathbb{Q})_{1, Q} := H^n(V_t, \mathbb{Q})_{-1} \bigoplus \bigoplus_{Im\lambda > 0} H^n(V_t, \mathbb{Q})_{\lambda, \overline{\lambda}}$$

3.3. The Unipotent Decomposition of $H^n(V_t, \mathbb{Q})$. Define

$$N_V := log(M_u) := \sum_{j \geq 1} \frac{(-1)^{j+1}(M_u - Id)^j}{j}.$$ 

deleting the subscript $V$, we have which is $Q$-antisymmetric: $Q(N\bullet, \bullet) = -Q(\bullet, N\bullet)$ and nilpotent, say $N^{r_0 + 1} = 0$. Since $M_sM_u = M_uM_s$, the description that follows may be applied to each summand in (9) and then take the direct summand, without being explicit about this.

There is a canonical filtration $W(N)$ of $H^n(V_t, \mathbb{Q})$ induced by $N$, the weight filtration ([21])

$$W(N)_{-r_0} \subset W(N)_{-r_0 + 1} \subset \cdots \subset W(N)_{0} \subset \cdots \subset W(N)_{r_0 - 1} \subset W(N)_{r_0} \subset H^n(V_t, \mathbb{Q}).$$

(10)

We may visualize the weight filtration by choosing a Jordan basis of $H^n(V, \mathbb{Q})$ with respect to $N$: To each $\ell$-Jordan block associate a horizontal chain of length $2\ell$, marking the even integer points on the chain and putting on each point an element of the basis beginning on the left $A, NA, \ldots, N^{\ell - 1}A$. The mapping $N$ is a movement of the points on the chain 2 steps to the right, and the extreme right points disappear under $N$. Organize all the chains in such a way that the center of each chain is located over the point 0. The element $W_j(N)$ of the weight filtration are then spanned by those elements of the basis which are located to the right of $j$.

Order the above basis of $H^n(V_t, \mathbb{Q})$ beginning with a basis for $W_{-r_0}$, then completing it to a basis of $W_{-r_0 + 1}$, etc.: $$\{\{A_1, \ldots, A_{\ell_1}\}, \ldots, \{\ldots, A_{\ell}\}\}.$$ We remember the collection of
basis elements that appear in each step, so as to obtain representations of endomorphisms of
\( H^n(V_t, \mathbb{Q}) \) in the form of block matrices. There is a decreasing function
\[ \nu_N : \{1, \ldots, \ell \} \to \{0, 1, \ldots, \ell \} \]
with the property that \( N(A_k) = A_{\nu_N(k)} \), with \( A_0 := 0 \). The matrix of \( N \) in this basis is \( (\delta_{j,\nu_N(j)}) \).

Introduce the \( W(N) \)-graded Cohomology Algebra

\[
Gr_{W(N)}(H^n(V_t, \mathbb{Q}), N) := \bigoplus_{j=-r_0}^{r_0} Gr^j_{W(N)}(H^n(V_t, \mathbb{Q}), N),
\]
with graded basis \( \{gr(A_1), \ldots, gr(A_\ell)\} \). The induced map
\[
gr(N) : Gr^n(V_t, \mathbb{Q}), N) \to Gr^n(V_t, \mathbb{Q}), N),
\]
has in the graded basis the same matrix form as \( N : (\delta_{j,\nu_N(j)}) \).

For \( j = 0, \ldots, m_0 \) the primitive spaces
\[
Prim_j(H^n(V_t, \mathbb{Q}), N) \subset Gr^j_{W(N)}(H^n(V_t, \mathbb{Q}), N)
\]
are generated by \( \{gr(A_j) / j \notin \text{Im}(\nu_N)\} \) and they are the beginning of the chains description of \( Gr_{W(N)}(H^n(V_t, \mathbb{Q}), N) \). We obtain from the chain image for \( j = -m_0 \ldots, m_0 \) the direct sum
\[
Gr^j_{W(N)}(H^n(V_t, \mathbb{Q}), N) = \bigoplus_{0 \leq j + 2k \leq m_0} N^k Prim_{j+2k}(H^n(V_t, \mathbb{Q}), N),
\]
which describes the columns \( Gr^j_{W(N)}(H^n(V_t, \mathbb{Q}), N) \) formed by piling up primitive blocks. The difference between one column and the column 2 steps to the left or right is the addition or deletion of one of the corresponding primitive block on top. Intrinsically we have
\[
Prim_j(V) := Ker[N^{j+1} : Gr^j_{W(N)}(H^n(V_t, \mathbb{Q}), N)) \to Gr^{-j-2}_{W(N)}(H^n(V_t, \mathbb{Q}), N)] \cong Ann_{H^n(V_t, \mathbb{Q})}(N) \cap (N^{j+1}).
\]

3.4. Vanishing Cohomology and its Non-degenerate Pairing. Let \( f : \mathbb{C}^{n+1} \to \mathbb{C} \) be a germ of a holomorphic function at 0 with an isolated singularity. Following Brieskorn [2] and Scherk [25] (cf. [26]) one can make an analytic change of coordinates in such a way that \( f \) is a polynomial map of sufficiently large degree and such that \( V_t := \overline{f^{-1}(l)} \) its closure in \( \mathbb{C}P^{n+1} \) is a smooth hypersurface at infinity, and such that denoting \( X_t := V_t \cap B \) with \( B \) a sufficiently small ball in \( \mathbb{C}^{n+1} \) the restriction map gives rise to the exact sequence
\[
0 \to Ker(M - Id) \to H^n(V_t, \mathbb{Q}) \to H^n(X_t, \mathbb{Q}) \to 0
\]
Since \( Ker(M - Id) = Ker(N) \) ([25]) we have an isomorphism
\[ \phi : H^n(X_t, \mathbb{Q}) \to H^n(V_t, \mathbb{Q}) / Ker(N) \]
and we define the Hertling-Steenbrink polarization (see [9], [10] and [11]) bilinear form
\[
Q_{X_t} : H^n(X_t, \mathbb{Q}) \times H^n(X_t, \mathbb{Q}) \to \mathbb{Q}
\]
as in (8).
4. The Differential Description of Grothendieck duality

The Brieskorn lattices $H''_0$ and $H'_0$ are defined by the exact sequence of $\mathbb{C}$-vector spaces:

$$0 \rightarrow H'_0 := \frac{df \wedge \Omega^n_{\mathbb{C}^{n+1},0}}{df \wedge d\Omega^{n+1}_{\mathbb{C}^{n+1},0}} \rightarrow H''_0 := \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge d\Omega^{n+1}_{\mathbb{C}^{n+1},0}} \rightarrow \Omega_f := \frac{\Omega^{n+1}_{\mathbb{C}^{n+1},0}}{df \wedge d\Omega^{n+1}_{\mathbb{C}^{n+1},0}} \rightarrow 0.$$

The cohomology bundle over a punctured disk $R^nf_*\mathbb{C}_{\mathbb{C}^{n+1},0}$ will be denoted by $H^n$. It is a $\mu$-dimensional flat bundle with the Gauss-Manin connection $\partial_t$ whose fiber over $t$ is the vanishing cohomology group $H^n(X_t, \mathbb{C})$.

Taking the universal covering $e : D_\infty \rightarrow D^*$, $\xi \mapsto \exp 2\pi i \xi$ of $D^*$ and the inclusion map $i : D^* \rightarrow D$ we have the canonical Milnor fibre given by the pullback

$$X_\infty := X^* \times_{\Delta^*} \Delta_\infty \xymatrix{ \ar[r] & X^* \ar[l]_f \ar[d]^f } \ar[d]_e \xymatrix{ \ar[r] & D_\infty \ar[l]_e \ar[r]^i & D^*},$$

where $f : X^* \rightarrow \Delta^*$ is a $\mathcal{C}_\infty$ fiber bundle whose fibres $X_t := f^{-1}(t) \cap X$ (see [10, 11]).

So we have homotopy equivalences given by the inclusions

$$X_{u(\tau)} \simeq (X_\infty)_\tau \xymatrix{ \ar[r] & X_\infty, } \quad x \mapsto \varsigma_\tau(x) = (x, \tau),$$

and so isomorphisms

$$(14) \quad H^n(X_\infty, \mathbb{C}) \xymatrix{ \ar[r]^-{\varsigma_\tau} & H^n(X_t, \mathbb{C}), } \quad H_n(X_t, \mathbb{C}) \xymatrix{ \ar[r]^-{(\varsigma_\tau)_*} & H_n(X_\infty, \mathbb{C}), }$$

where $t = e(\tau)$ (see [13]).

The bundle $H^n$ has a natural $\partial_t$-invariant non-degenerate bilinear form obtained by gluing the bilinear form explained in section 3.4 for each $X_t$. Such bilinear form induces, up to conjugation by the isomorphisms in (14), an equivalent bilinear form defined on $H^n(X_\infty, \mathbb{C})$ which without loss of generality we also denote as $Q$. On the other hand, we have monodromy maps $M_t$ on $H^n(X_t, \mathbb{C})$ which induce the monodromy map $M$ on $H^n(X_\infty, \mathbb{C})$:

$$H^n(X_\infty, \mathbb{C}) \xymatrix{ \ar[r]^-M & H^n(X_\infty, \mathbb{C}) } \quad \xymatrix{ \ar[d]^{\varsigma_\tau} & \ar[d]^{\varsigma_\tau} } \quad H^n(X_t, \mathbb{C}) \xymatrix{ \ar[r]^-{M_t} & H^n(X_t, \mathbb{C}). }$$

With respect to the decomposition $M = M_s M_u = M_u M_s$ into semisimple and unipotent parts. there is a eigenspace decomposition

$$H := H^n(X_\infty, \mathbb{C}) = \bigoplus_{\lambda} H_\lambda,$$

with respect to $M_s$, i.e., $H_\lambda := \ker(M_s - \lambda I)$. Set $H_{\neq 1} := \bigoplus_{\lambda \neq 1} H_\lambda$ and let $N := -\frac{1}{2\pi i} \log M_u \in \text{End}_\mathbb{C}(H)$ be the logarithm of the unipotent part of the monodromy which is nilpotent (by monodromy theorem) with nilpotence index $r_0 \leq n + 1$ (see [10] and [13].) Hence, we have a similar description as shown in subsection 3.3. In particular, we have an application $\nu_N$ as in
that encodes the Jordan blocks of $N$ in terms of a basis that describes the corresponding weight filtration which we also denotes as $W_*(N)$.

A class of holomorphic (univalued) sections of $H^n$ may be represented by Laurent-type series expansions

$$V^{> -\infty} := \left\{ \sum_{j=1}^{r} \frac{t^{[(\beta_j+k)I-\frac{1}{2\pi i} N]}}{k} A_{j,k}(t) \right\},$$

where $A_{j,k}(t)$ is a Gauss-Manin (multivalued) flat section which takes values in $H_{\leq 2\pi i \beta_j} \cdot \beta_j \in (-1, 0]$. The convergence of the corresponding series holds in each sector $a < \arg t < b$ with $|t|$ small. The $V$-filtration is defined by

$$V^{\beta(t)} := \left\{ \sum_{j=1}^{r} \frac{t^{[(\beta_j+k)I-\frac{1}{2\pi i} N]}}{k} A_{j,k}(t), \quad \beta_j + k \geq (>) \beta \right\}$$

and we denote its graded pieces by

$$C_{\beta_j+k} := \text{Gr}_V^{\beta_j+k}(V^{> -\infty}) = \frac{V^{\beta_j+k}}{V^{> \beta_j+k}} \xrightarrow{\sim} H^n_{e^{2\pi i \beta_j}}, \text{ where } \psi_{\beta_j+k}(A_{j,k}) := \frac{t^{[(\beta_j+k)I-\frac{1}{2\pi i} N]}}{k} A_{j,k}(t).$$

The $V$-weight $v(A)$ of $A \in V^{-\infty}$ is the smallest $\beta_j + k$ such that $A \in V^{\beta_j+k}$, and so, it is the smallest rational number where $A_{\beta_j,k} \neq 0$ in the expansion of $A$.

The expression of the Gauss-Manin connection is $\partial_t : V^{-\infty} \rightarrow V^{-\infty}$ that has the direct sum expression

$$\partial_t = \psi_{\beta_j+k-1} \left( ((\beta_j + k)I - \frac{1}{2\pi i} N) \psi_{\beta_j+k}^{-1} : C_{\beta_j+k} \rightarrow C_{\beta_j+k-1} \right)$$
on using Leibniz rule in the computation:

$$\partial_t e^{[(\beta_j+k)I-\frac{1}{2\pi i} N] \log(t)} A_{j,k}(t) := \partial_t e^{[(\beta_j+k)I-\frac{1}{2\pi i} N] \log(t)} A_{j,k}(t) = e^{[(\beta_j+k)I-\frac{1}{2\pi i} N]} \left( \frac{(\beta_j+k)I - \frac{1}{2\pi i} N}{t} \right) A_{j,k}(t),$$

and the maps $\partial_t^k$ correspond to the linear maps $L_k : C_{\beta_j+k} \rightarrow C_{\beta_j}$ up to conjugation by $\psi$:

$$L_k = [(\beta_j + 1)I - \frac{1}{2\pi i} N] \cdots [((\beta_j+k)I - \frac{1}{2\pi i} N) \psi_{\beta_j+k}^{-1},$$

that is,

$$L_k = [(\beta_j + 1)I - \frac{1}{2\pi i} N] \cdots [((\beta_j+k)I - \frac{1}{2\pi i} N].$$

Introduce the spaces

$$\mathbb{H}_0 := \frac{V^{> -\infty}}{V^0}, \quad \mathbb{H}_k := \frac{V^{k-1}}{V^k}, \quad k = 1, \ldots, n, \quad \mathbb{H} = \bigoplus_{k=0}^{n} \mathbb{H}_k = \frac{V^{> -\infty}}{V^n}.$$
by applying the maps in (15). Each \( \mathbb{H}_k, k = 1, \ldots, n, \) is isomorphic to vanishing cohomology \( H^n(X_\infty, \mathbb{C}) \):

\[
H^n(X_\infty, \mathbb{C}) \xrightarrow{\psi^{-1}} \mathbb{H}_0 \oplus \mathbb{C}_0 \xrightarrow{\partial_k^{-1}} \mathbb{H}_k,
\]

where \( \psi := \bigoplus_{-1 < \beta_i \leq 0} \psi_{\beta_j}, \) and \( \mathbb{H}_0 \) is isomorphic to \( H^n_{\neq 1} := \bigoplus_{-1 < a < 0} H^n_{e^{-2\pi i a}}. \)

Introduce in \( \mathbb{H} \) the non-degenerate bilinear form \( \langle \cdot, \cdot \rangle_\mathbb{H} \) as the orthogonal decomposition

\[
\mathbb{H} = [(\mathbb{H}_0 \oplus \mathbb{C}_0) \oplus \mathbb{H}_n] \bigoplus \bigoplus_{\ell=1, \ldots, n/2} \mathbb{H}_\ell \oplus \mathbb{H}_{n-\ell} \bigoplus \mathbb{H}_{n/2}
\]

and in each factor it is defined as

\[
\langle v_1 t^\ell, v_2 t^{n-\ell} \rangle_\mathbb{H} = \frac{(-1)^{1+|\beta_j|}}{(2\pi i)^{n+1+|\beta_j|}} Q(L_k V_1, (-1)^{n-\ell} L_{n-k} V_2), \quad v_1 t^\ell \in Gr^{|\beta_j|+\ell}_V, \quad v_2 t^{n-\ell} \in Gr^{-|\beta_j|-1+n-\ell}_V
\]

where

\[
L_k V_1 \in H^{n-2\pi i \beta_j}, \quad L_{n-\ell} V_2 \in H^{n-2\pi i \beta_j+1}_V
\]

are such that

\[
\psi_{-1}^{-1} \partial_k^{-1} v_1 t^\ell = L_k \psi_{-1}^{-1} (v_1 t^\ell) = L_k V_1,
\]

\[
\psi_{-1}^{-1} \partial_k^{-1} v_1 t^{n-\ell} = L_{n-\ell} \psi_{-1}^{-1} (v_2 t^{n-\ell}) = L_{n-\ell} V_2,
\]

and \( |\beta_j| = -1, 0, \) and 0 otherwise. \( Q \) is the cup product in flat sections.

The Gelfand-Leray residue defines a map

\[
\Omega^{n+1}_{\mathbb{C}^{n+1}, 0} \rightarrow H^n \quad \text{,} \quad s(\omega)(t) := \text{res}_{X_t} \left[ \frac{\omega}{f-t} \right] \in H^n(X_t, \mathbb{C})
\]

and it induces the period map

\[
s : H''_0 \rightarrow V^{-1}
\]

whose restriction to \( H'_0 \) has the expression

\[
s(df \wedge \eta) = [\eta|_{X_t}] \in H^n(X_t, \mathbb{C}), \quad \eta \in \Omega^n_{\mathbb{C}^{n+1}, 0}.
\]

The period map \( s \) is injective and it satisfies \( s(H''_0) \supset s(H'_0) \supset V^n. \) We identify the Brieskorn lattice \( H''_0 \) with its image in \( V^{>1} \). The map \( \partial^{-1}_t : V^{>1} \rightarrow V^{>0} \) is bijective and it defines an injective map

\[
\partial^{-1}_t : H''_0 \rightarrow H'_0 \quad \text{,} \quad s(\omega) := s(df \wedge \eta) \rightarrow s(df \wedge \eta) \quad \text{,} \quad \partial^{-1}_t H''_0 = H'_0,
\]

hence providing an isomorphism

\[
(16) \quad s : \Omega_f \rightarrow \frac{H''_0}{\partial^{-1}_t H''_0}.
\]

Introduce the subspaces

\[
\mathbb{H}_0' := \frac{H'_0}{H'_0 \cap V^n} \subset \mathbb{H}_0'' := \frac{H''_0}{H''_0 \cap V^n} \subset \mathbb{H}, \quad \Omega_f \cong \frac{\mathbb{H}_0''}{\mathbb{H}_0'},
\]

whose elements consist of the coefficients in the expansion between \( \langle -1, n \rangle \) which are realized by differential forms in \( df \wedge \Omega^n \) or in \( \Omega^{n+1} \), respectively. The last isomorphism is induced by \( s \) in (16).
The next theorem is essentially due to Varchenko [28] and its variant that we present here is following Hertling [11, Proposition 4.4]:

**Theorem 1 (Grothendieck duality).** The radical of the restriction of the bilinear form $< \ , \ >_{\mathbb{H}}$ to $\mathbb{H}_0'$ is $\mathbb{H}_0'$, and the induced non-degenerate bilinear form in $\mathbb{H}_0'$ via the identification with $\Omega_f$ is Grothendieck residue $\text{res}_f$.

We may induce the descending filtration $V(V^{-1})$ to a filtration $V(\mathbb{H}_0')$ and then to a filtration $V(\Omega_f)$ in the Jacobian module. Explicitly: $[\omega] \in V^\beta(\Omega_f)$ if $\beta$ is the smallest rational number such that there is $\eta \in \Omega^n_{\mathbb{C}^{n+1},0}$ such that $v_0(s(\omega + df \wedge \eta)) = \beta$.

The spectrum $\{ \alpha_1 \leq \ldots \leq \alpha_\mu \}$ is formed by those rational numbers in $(-1, n) \cap \mathbb{Q}$ where $Gr_{\omega}^\alpha(\Omega_f) \neq 0$. The spectrum can be interpreted as choosing logarithms of the eigenvalues of the monodromy. The choosing of the logarithms is unveiled by the differential description.

5. **Normal form of the bilinear forms** $\text{res}_f( f^j \ , \ )$ **in Saito–Hertling basis**

In this section we will give a normal form for the higher bilinear forms $\text{res}_{f,0}( f^j \ , \ )$, $1 \leq j \leq n + 1$ in terms of the Saito-Hertling basis.

First, following Hertling [11] (cf. [20]), one introduces the Hodge filtration in vanishing cohomology:

$$H^n(X_{\infty},\mathbb{C}) \supset F^0 \supset F^1 \supset \ldots \supset F^n \supset 0.$$ 

It is compatible with the semi-simple decomposition in vanishing cohomology, and it is defined for $p = 0, \ldots, n$ by

$$F^p(H^n_{e^{-2\pi i \alpha_j}}) = \psi^{-1} \left( \frac{(V^{\alpha_j} \cap \partial_t^{n-p} H^0_n) + V^{> \alpha_j}}{V^{> \alpha_j}} \right).$$

**Theorem 2** (C. Hertling [10, 11], cf. [24]). Let $f : (\mathbb{C}^{n+1},0) \to (\mathbb{C},0)$ be a holomorphic germ with an isolated singularity. Then, the vanishing cohomology $H^n(X_{\infty},\mathbb{Q})$ has a polarized mixed Hodge Structure (PMHS): $(F^*, W_*, S)$. This means that there is a PMHS of weight $n$ on $H^n(X_{\infty},\mathbb{Q})_{\neq 1}$ and a PMHS of weight $n + 1$ on $H^n(X_{\infty},\mathbb{Q})_{1}$.

Now, following Hertling [11] (cf. M. Saito[20]), it is positive to choose a convenient basis of $\Omega_f$ such that their representatives whose principal terms into its asymptotical expansion are inside $H^n_0 \subset V^{> \alpha_1} \subset V^{> -1}$ form a Jordan basis for the nilpotent part of the monodromy.

In fact, using the duality properties that the Deligne splitting inherits from the PMHS on $H^n(X_{\infty},\mathbb{Q})$ we begin by choosing a $\mathbb{C}$-basis $A_1, \ldots, A_\mu$ of $H^n(X_{\infty},\mathbb{C})$ such that:

(a) it corresponds with $s_1, \ldots, s_\mu$, $s_i \in C_{\alpha_i}$, $1 \leq i \leq \mu$, by the relation

$$\partial_t^{k_i} s_i = \psi(A_i), \quad \text{i.e.} \quad L_{k_i} s_i = \psi^{-1} \partial_t^{k_i} s_i = A_i$$

where $k_i$ is such that $\alpha_i - k_i \in (-1, 0]$, i.e., $s_i \in \mathbb{H}_{k_i}$.
(b) $s_1, \ldots, s_\mu$ project onto a basis of $\bigoplus_{-1 < \alpha < n} \text{Gr}_V^\alpha(H_0''/\partial_t^{-1}H_0'') \simeq \bigoplus_{-1 < \alpha < n} \text{Gr}_V^\alpha(\Omega_f)$ satisfying that

\begin{equation}
 s_{\nu(i)} = (\text{Id} - (\alpha_i + 1)\partial_t^{-1})s_i = \partial_t^{-1}Ns_i = N\partial_t^{-1}s_i
\end{equation}

where $\nu$ is such that

\begin{equation}
 A_{\nu(j)} = \frac{-1}{2\pi\sqrt{-1}}NA_j, \quad A_{\mu + 1} = 0.
\end{equation}

Actually, we may suppose that $\nu$ coincides with $\nu_N$.

(c) there exists an involution $\kappa : \{1, \ldots, \mu\} \rightarrow \{1, \ldots, \mu\}$ with $\kappa(i) = \mu + 1 - i$ if $\alpha_i \neq \frac{1}{2}(n - 1)$ and $\kappa(i) = \mu + 1 - i$ or $\mu(i) = i$ otherwise and satisfying the orthogonality relations $\langle s_i, s_j \rangle_H = \delta_{\kappa(i), j}$.

Essentially this comes from the fact that

\begin{equation}
 \delta_{\kappa(i), l} = \begin{cases}
 (-1)^{\nu_i}\left(\frac{1}{2\pi\sqrt{-1}}\right)^n Q(A_i, A_l) & \text{if } \lambda_{\alpha_i} = \bar{\lambda}_{\alpha_l} \neq 1 \\
 (-1)^{\nu_l+1}\left(\frac{1}{2\pi\sqrt{-1}}\right)^{n+1} Q(A_i, A_l) & \text{if } \lambda_{\alpha_i} = \lambda_{\alpha_l} = 1,
\end{cases}
\end{equation}

where $\alpha_i, \alpha_l \in sp(f)$ and $\delta_{\kappa(i), l}$ is the Kronecker delta. Hence the paring $\langle , \rangle_H$ acquires the normal form $[\langle , \rangle_H] = S_1$, where $S_1$ is the anti-diagonal $\mu \times \mu$ matrix

\[ S_1 = \begin{pmatrix}
 1 & \cdots & 1 \\
 0 & \ddots & \vdots \\
 \vdots & \ddots & 1 \\
 0 & \cdots & 1
\end{pmatrix}. \]

d) Following [11, Lemma 5.2], from $s_1, \ldots, s_\mu$ a $\mathbb{C}\{\partial_t^{-1}\}$-basis $h_1, \ldots, h_\mu$ for Brieskorn lattice $H_0''$ can be constructed

\begin{equation}
 h_i \in H_0'' \cap (C_{\alpha_i} \oplus \sum_{j,p: \alpha_i + p < \alpha_j \in sp(f)} \mathbb{C} \cdot \partial_t^p s_j), \quad i = 1, \ldots, \mu;
\end{equation}

which induces the corresponding basis for $\mathbb{H}_0''$ whose principal part of each $h_i$ is $s_i$.

Finally, one chooses forms $\eta_1, \ldots, \eta_\mu \in \Omega^{n+1}_{\mathbb{C}^{n+1}, 0}$ for which $h_1, \ldots, h_\mu$ are the corresponding images under $s$. 

5.1. **The normal form of map multiplication by \( f \) in the Saito–Hertling basis.** Following [11, Proposition 5.4] we have a description for the map multiplication by \( t \):

\[
t : \mathbb{H}_0'' \to \mathbb{H}_0''
\]

that corresponds to the map multiplication by \( f \) via the relation \( s[f \bullet] = ts[\bullet] \).

From this result we may obtain an induced normal form for the map multiplication by \( f \),

\[
M_f : \Omega_f \longrightarrow \Omega_f, [\omega] \mapsto M_f[\omega] := [f\omega],
\]

In fact, (choosing coordinates \((z_0, \ldots, z_n)\) on \((\mathbb{C}^{n+1}, 0))\) it will be determined by

\[
[g dz_0 \wedge dz_1 \wedge \cdots \wedge d z_n] \mapsto [(fg) dz_0 \wedge dz_1 \wedge \cdots \wedge d z_n].
\]

Hence, we may use that \( A_f \) is isomorphic to \( \Omega_f \), via the map \( 1 \mapsto [dz_0 \wedge dz_1 \wedge \cdots \wedge d z_n] \), to obtain the following commutative diagram of \( O_{\mathbb{C}^{n+1}, 0} \)-modules:

\[
\begin{array}{ccc}
A_f & \xrightarrow{M_f} & A_f \\
\cong & & \cong \\
\Omega_f & \xrightarrow{M_f} & \Omega_f
\end{array}
\]

and therefore we have up to this conjugation (and up to choosing holomorphic coordinates) the map given by (1). And therefore, we have that

\[
(22) \quad f[\eta_i] = s^{-1}(th_i) = s^{-1}(h_{\nu_N(i)}) + \sum_{j=1}^{\mu} c_{ij} s^{-1}(h_j) \in s^{-1}(\mathbb{H}_0''/\mathbb{H}_0') = \Omega_f,
\]

where \( c_{ij} \) are constants determined by the spectral values in such a way that

\[
c_{ij} := \begin{cases} 
0 & \text{if } \alpha_i + 1 \geq \alpha_j \\
(\alpha_j - 1 - \alpha_i) c_{ij}^{(1)} & \text{if } \alpha_i + 1 < \alpha_j.
\end{cases}
\]

From (19) we have that the \( \mu_i \)'s depend on the length of the Jordan chain of \( A_i \) with respect to \( \nu_N \), that is, \( \mu_i \leq \mu - \nu_N(i) < \mu \) for each \( i = 1, \ldots, \mu \). Hence, this determines a \( \mu \times \mu \) matrix \( N_1 \) using the terms of order greater than \( \alpha_i + 1, 1 \leq i \leq \mu \) according the expansions in (22); this is done by using the commutative diagram

\[
\begin{array}{ccc}
\mathbb{C}^{\mu} & \longrightarrow & \mathbb{C}^{\mu} \\
\cong & & \cong \\
\mathbb{H}_0''/\mathbb{H}_0' & \longrightarrow & \mathbb{H}_0''/\mathbb{H}_0'
\end{array}
\]

given by \([h]\)-coordinates, in such a way that \( N_1 \) is equals to the transpose matrix of \((c_{ij})\).

Let us use the notation \( K_f := (, )_{\mathbb{H}} \).
Definition 4. Let $N_{\text{top}}$ be the $\mu \times \mu$ matrix given by
\[
N_{\text{top}} = [t]_{[\mathcal{H}_0]} - N_1,
\]
where $[t]_{[\mathcal{H}_0]}$ is the induced $\mathbb{C}$-basis for $\mathbb{H}_0$ coming from basis $\{h_j\}$. Define the endomorphism $N_1 : H^n(X_\infty, \mathbb{C})$ such that its $\Delta$-matrix expression is given by $N_1$.

Lemma 2. Let $1 \leq i, j \leq \mu$. For any integers $p, q$ such that $0 \leq p \leq r_i$ and $q \leq r_j$,
\[
K_f(\partial_t^p s_i, \partial_t^q s_j) = (-1)^{-q} K_f(s_i, s_j) \cdot \partial_t^{p+q}
\]
where $r_i, r_j$ are the levels of $s_i, s_j$, respectively.

Proof. From definition of $K_f$,
\[
(23) \quad K_f(s_i, s_j) = (-1)^{r_j} K_f(\partial_t^p s_i, \partial_t^q s_j) \cdot \partial_t^{-r_i-r_j},
\]
where $r_i, r_j$ are the levels of $s_i, s_j$, respectively. In the same way,
\[
K_f(\partial_t^p s_i, \partial_t^q s_j) = (-1)^{-q} K_f(\partial_t^{p-r} \partial_t^p s_i, \partial_t^{q-r} \partial_t^q s_j) \cdot \partial_t^{-r_i-r_j+p+q}
\]
\[
(24) \quad = (-1)^{r_j-q} K_f(\partial_t^r s_i, \partial_t^r s_j) \cdot \partial_t^{-r_i-r_j+p+q}
\]
Finally, the claim follows from (23), (24), and (20). \hfill \Box

Let $[\text{res}_{f,0}(\bullet, \bullet)]_{[\mathcal{H}]}$ be the $[\mathcal{H}]$-matrix of the Grothendieck pairing which is induced by (21).

Proposition 2. \hfill (1) For any $i, j \in \{1, \ldots, \mu\}$,
\[
K_f(h_i, h_j) = \delta_{\kappa(i), j} \cdot \partial_t^{-n-1} \in \mathbb{C} \cdot \partial_t^{-n-1} = K_f(s_i, s_j).
\]
In particular,
\[
(26) \quad [\text{res}_{f,0}(\bullet, \bullet)]_{[\mathcal{H}]} = S_1.
\]
(2) Suppose that $h_{\nu(i)} \neq 0$. Then, for any $j \in \{1, \ldots, \mu\}$,
\[
K_f(h_{\nu(i)}, h_j) = K_f(s_{\nu(i)}, s_j) = K_f(\partial_t^{-1} \tilde{N}_\alpha, s_i, s_j).
\]
(3) The $(i, j)$ entry of matrix $[N]_{\Delta}$ of $S_1$ is equals to
\[
(28) \quad \partial_t^{n+1} K_f(h_{\nu(i)}, h_j).
\]
Or, equivalently,
\[
(29) \quad N_{\text{top}} = [N]_{\Delta},
\]
where $[N]_{\Delta}$ be the $\mu \times \mu$ constant $\Delta$-matrix associated to the operator $N$ satisfying (19) and (20).
Proof. Note that (27) follows from (25) and (18). Also, (26) follows from (25) and (ii) in Theorem 1. On the other hand, (3) follows from (27) and by noting that

\[ K_f(s_{\nu(i)}, s_l) = \begin{cases} (-1)^{r_i} \left( \frac{1}{2\pi \sqrt{-1}} \right)^n Q(NA_i, A_l) \cdot \partial_t^{-n-1} & \text{if } (\alpha_i - r_i) + (\alpha_l - r_l) = -1 \\ (-1)^{r_i+1} \cdot \left( \frac{1}{2\pi \sqrt{-1}} \right)^{n+1} Q(NA_i, A_l) \cdot \partial_t^{-n-1} & \text{if } (\alpha_i - r_i) = (\alpha_l - r_l) = 0 \end{cases} \]

Finally, we will prove (25). Let \( 1 \leq i, l \leq \mu \). Assuming

\[ h_i = s_i + \sum_{\begin{subarray}{c} j \geq 1 \\ p \geq 1 \\ \alpha_i + p < \alpha_j \end{subarray}}^{\mu} c_{ij}^{(p)} \cdot \partial^p_s s_j \quad \text{and} \quad h_l := s_l + \sum_{\begin{subarray}{c} l_1 \geq 1 \\ q \geq 1 \\ \alpha_l + q < \alpha_i \end{subarray}}^{\mu} c_{ll_1}^{(q)} \cdot \partial^q_s s_{l_1}, \]

it follows that

\[
K_f(h_i, h_j) = K_f \left( s_i + \sum_{\begin{subarray}{c} j \geq 1 \\ p \geq 1 \\ \alpha_i + p < \alpha_j \end{subarray}}^{\mu} c_{ij}^{(p)} \cdot \partial^p_s s_j , \; s_l + \sum_{\begin{subarray}{c} l_1 \geq 1 \\ q \geq 1 \\ \alpha_l + q < \alpha_i \end{subarray}}^{\mu} c_{ll_1}^{(q)} \cdot \partial^q_s s_{l_1} \right) \]

\[
= K_f(s_i, s_l) + \sum_{\begin{subarray}{c} j \geq 1 \\ p \geq 1 \\ \alpha_i + p < \alpha_j \end{subarray}}^{\mu} \sum_{\begin{subarray}{c} l_1 \geq 1 \\ q \geq 1 \\ \alpha_l + q < \alpha_i \end{subarray}}^{\mu} c_{ij}^{(p)} \cdot c_{ll_1}^{(q)} \cdot K_f \left( \partial^p_s s_j , \; \partial^q_s s_{l_1} \right). \]

So, by equation (20) and Lemma 2

\[
K_f(h_i, h_j) = \delta_{\kappa(i), l} \cdot \partial_t^{-n-1} + \sum_{\begin{subarray}{c} j \geq 1 \\ p \geq 1 \\ \alpha_i + p < \alpha_j \end{subarray}}^{\mu} \sum_{\begin{subarray}{c} l_1 \geq 1 \\ q \geq 1 \\ \alpha_l + q < \alpha_i \end{subarray}}^{\mu} c_{ij}^{(p)} \cdot c_{ll_1}^{(q)} \cdot (-1)^{q} \delta_{\kappa(j), l_1} \cdot \partial_t^{-n-1+p+q}. \]

Since \( 1 \leq p, q \leq n \), it satisfies that \(-(n - 1) \leq n + 1 - p - q \leq n - 1 \). So, by (i) in Theorem 1, it follows that the second summand of (30) is equal to zero. This complete the proof. \( \square \)

Proposition 2 implies the following corollary.

**Corollary 1.** The matrix expression of \( f \) in the basis \([\eta_1], \ldots, [\eta_\mu] \in \Omega_f \) is

\[ [M_f]_2 = [N]_\Delta + N_1 = N_{top} + N_1, \]

where \( N \) is in canonical Jordan Form with respect to basis \( \Delta = \{A_j\} \) and \( N_1 = (c_{ij})^{tr} \) accordingly to (17) and (22) .
Explicitly, we have the following normal form for the map multiplication by \( f \):

\[
[M_f] \equiv 
\begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
n_{\nu(1)} & 0 & 0 & 0 \\
0 & n_{\nu(2)} & 0 & 0 \\
0 & 0 & \cdots & n_{\nu(\mu-1)} & 0 \\
0 & 0 & \cdots & 0 & n_{\nu(\mu)} \\
c_{11} & c_{21} & \cdots & c_{\mu-11} & c_{\mu1} \\
c_{12} & c_{22} & \cdots & c_{\mu-12} & c_{\mu2} \\
c_{1\mu_1} & c_{2\mu_2} & \cdots & c_{\mu\mu-1} & c_{\mu\mu} \\
\end{pmatrix}
\]

\[= [N]_{\{A_j\}} + N_1,\]

\[
con
n_{\nu(i)} = \begin{cases} 
1 & \text{if } \nu(i) \neq \mu + 1 \\
0 & \text{if } \nu(i) = \mu + 1.
\end{cases}
\]

5.2. Main results.

**Theorem 3** (First main result). There exist an isomorphism of \( \mathbb{C} \)-vector spaces \( \varphi : \Omega_f \to H^n(X, \mathbb{C}) \) and an automorphism \( J : H^n(X, \mathbb{C}) \to H^n(X, \mathbb{C}) \) such that:

(a) The bilinear forms \( \text{res}_f \) on \( \Omega_f \) and \( Q(\bullet, J\bullet) \) on \( H^n(X, \mathbb{C}) \) are equivalent, that is,

\[\text{res}_f(\bullet, \bullet) = Q(\varphi \bullet, J\varphi \bullet).\]

(b) For \( j = 1, \ldots, n \), the \( \mathbb{C} \)-bilinear spaces
\[
(\Omega_f, \text{res}(f^j \bullet, \bullet)) \text{ and } \left( H^n(X, \mathbb{C}), Q((N + N_1)^j \bullet, J\bullet) \right)
\]

are equivalent, that is,

\[\text{res}(f^j \bullet, \bullet) = Q((N + N_1)^j \varphi \bullet, J\varphi \bullet).\]

(c) If \( f \) has finite monodromy, then for \( j = 1, \ldots, n \), the \( \mathbb{C} \)-bilinear spaces
\[
(\Omega_f, \text{res}(f^j \bullet, \bullet)) \text{ and } \left( H^n(X, \mathbb{C}), Q((N_1)^j \bullet, J\bullet) \right)
\]

are equivalent, that is,

\[\text{res}(f^j \bullet, \bullet) = Q(N_1^j \varphi \bullet, J\varphi \bullet).\]

**Proof.** From Proposition 2 one has the orthogonality relations

\[
\langle s_j, s_t \rangle_{\mathbb{H}} = \delta_{\kappa(j), t} = \langle [h_j], [h_t] \rangle_{\mathbb{H}} = \text{res}_f([\eta_j], [\eta_t]),
\]

\[\langle s_{\nu N(j)}, s_t \rangle_{\mathbb{H}} = \delta_{\kappa(\nu N(j)), t} = \langle [h_{\nu N(j)}], [h_t] \rangle_{\mathbb{H}} = \text{res}_f([\eta_{\nu N(j)}], [\eta_t]).
\]

By definition of \( \langle \cdot, \cdot \rangle_{\mathbb{H}} \) on pairs

\[
(s_j, s_t) \in Gr_V^1 \times Gr_V^{-1-\alpha_j+n} = C_{(\alpha_j-k_j)+k_j} \times C_{-1-(\alpha_j-k_j)+(n-k_j)} = C_{\beta_j+k_j} \times C_{-1-\beta_j+(n-k_j)},
\]

\[
\]
where $\beta_j = \alpha_j - k_j \in (-1, 0] \cap \mathbb{Q}$, we have

$$\delta_{\mu(j), \ell} = \langle s_j, s_\ell \rangle_H = \frac{(-1)^{1+|\beta_j|}}{(2\pi i)^{n+1+|\beta_j|}} Q(L_{k_j} s_j, (-1)^{n-k_j} L_{n-k_j} s_\ell)$$

$$= \frac{(-1)^{1+|\beta_j|}}{(2\pi i)^{n+1+|\beta_j|}} Q(A_j, (-1)^{n-k_j} A_\ell).$$

Therefore, we can be defined on each generalized eigenspace $H^n (X_\infty, \mathbb{C})_{e^{-2\pi i \alpha_j}}$ an automorphism $J_j$ by the condition

$$J_j(A_j) := (-1)^{n-k_j} \frac{(-1)^{1+|\beta_j|}}{(2\pi i)^{n+1+|\beta_j|}} A_j \quad \quad |\beta_j| = -1, 0.$$

Define an automorphism in $H^n$ as the direct sum $J := \bigoplus_{j=1}^\mu J_j$. The automorphism $J$ does not depend on the basis $A_1, \ldots, A_\mu$ but only on the Hodge filtration $F$. Indeed, for each $\alpha_j \notin \mathbb{Z}$, by definition $A_j \in F^{p(j)}(H^n_{e^{-2\pi i \alpha_j}})$ if and only if $s_j \in (\nu^* \circ \partial^{n-p(j)} H_{\alpha_j}^n) = \partial^n \circ \partial^{p(j)} H_{\alpha_j}^n$, and the fact that the $s_1, \ldots, s_\mu$ projects onto a basis of $\text{Gr}^{\alpha_j}_H (H^n_{e^{-2\pi i \alpha_j}}) \supset \text{Gr}^{\alpha_j}_H (H^n_{e^{-2\pi i \alpha_j}})$ implies that $\text{Gr}^{\alpha_j}_H (J) = (\nu^* \circ \partial^{n-p(j)} H_{\alpha_j}^n)$. Then, automorphism $J$ is such that $Q(\bullet, J \bullet)$ is symmetric on $H^n (X_\infty, \mathbb{C})$.

Theorem 1, (22) and (31) prove the theorem by choosing bases $\{[\eta_j] : j = 1, \ldots, \mu\}$ for $\Omega_f$ and $\{A_j : j = 1, \ldots, \mu\}$ for $H^n (X_\infty, \mathbb{C})$, and the isomorphism $\varphi : \Omega_f \cong H^n (X_\infty, \mathbb{C})$ is given by

$$\Omega_f \xrightarrow{\varphi} H^n (X_\infty, \mathbb{C})$$

for each $1 \leq j \leq \mu$. 

This theorem endows the higher bilinear forms $\text{res}_{f, 0}(f^j \bullet, \bullet)$ with a normal form given by the right expressions. This will be illustrated in section 5.3 and the examples below.

**Corollary 2** (Second main result). The Jordan Chains of multiplication by $f$ in $\Omega_f$ are obtained by binding of Jordan chains of $N$.

**Proof.** In order to simplify the notation, from now on we set $\nu = \nu_N$ as in (19).

Take a Jordan chain adapted to the wight filtration $W_\bullet(f) \subset \Omega_f \simeq A_f$ associated to the nilpotent operator multiplication by $f$ (see subsection 2.1):

$$\frac{\Omega_f^{n+1, 0}}{\Omega_f^{n+1, 0}} \supset C(v_j) : [v] \longrightarrow [fv] \longrightarrow [f^2 v] \longrightarrow [f^3 v] \longrightarrow \cdots \longrightarrow [f^{\ell-1} v] \longrightarrow [f^\ell v] \longrightarrow 0.$$

We can see this Jordan chain via the isomorphism $s$ as
If we assume that \( g \) is a Jordan basis of \((\Omega f, f)\) we may follow its evolution with respect to the Saito-Hertling basis

\[ h := h_1, \ldots, h_\mu \]

Since,

\[ h_\ell = s_\ell + \sum_{j,p} c^{(p)}_{\ell j} \partial_t^p s_j, \]

(33)

\[ th_\ell = h_{v(\ell)} + \sum_{\alpha_\ell+1 < \alpha_j} c^{(1)}_{\ell j} (\alpha_j - \alpha_\ell - 1) h_j, \quad \ell = 1, \ldots, \mu. \]

In fact, to simplify the idea we will suppose there is a Jordan chain for \( M_f \) with length 3:

\[ C(v) : s[v]_0 \rightarrow s[fv]_0 \leftarrow s[f^2v]_0 \rightarrow 0. \]

Since \( s[v]_0 \in \frac{H'_0}{0} \) there are unique constants \( a_1, \ldots, a_\mu \in \mathbb{C} \) such that \( s[v]_0 \) is an \( \mathbb{C} \)-linear combination

\[ s[v]_0 = a_1 h_1 + \cdots + a_\mu h_\mu. \]

Without loss of generality, we may keep with the summands that effectively contribute to the \( h \)-linear expansion, that is, we do not contemplate the constants \( a_j = 0 \), and therefore, we assume that \( a_1, \ldots, a_r \neq 0 \), \( 1 \leq r \leq \mu \) and

(34)

\[ s[v]_0 = a_1 h_1 + \cdots + a_r h_r. \]

This linear combination will denote the first Step 0 which encodes the beginnings of topological chains according to (33), that is, we consider

(35)

\[ s_1, \ldots, s_r, \]

which result from not to apply \( \{ f \} \), that is, we apply the identity map \( \{ f \}^0 := I \) to \( s[v]_0 \).

Analyzing carefully Step 0, one has that the begins of Jordan \( f \)-chains in (35) are already descended from someone, and therefore they may be of the following types:

i) \( s_j \) with \( A_j \in \text{Im} \ N \setminus \text{Ker} \ N \) is a bind of a chain:

\[ C(N) : \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow A_j \rightarrow \cdots \rightarrow \bullet \rightarrow 0 \]

ii) \( s_j \) with \( A_j \in \text{Im} \ N \cap \text{Ker} \ N:

\[ C(N) : \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow A_j \rightarrow 0. \]

iii) \( s_j \) with \( A_j \in \text{Ker} \ N \setminus \text{Im} \ N:

\[ C(N) : A_j \rightarrow 0. \]
iv) $s_j$ with $A_j \notin Ker N \cup Im N$ that means that the $A_j$ is a begin of a $N$-chain:

$$C(N) : A_j \rightarrow \cdots \rightarrow \cdots \rightarrow 0.$$ 

On the other hand, Step 1 will codify the fact to kill the elements of $\ker(f) \simeq \ker(t)$ which appear in Step 0. In fact, from (33) we assume that we have only an expansion with basis elements $h_{i_\ell} \notin \ker(t)$, $\ell = 1, \ldots, r_0 \leq r$ with the non zero constants $a_{i_1}, \ldots, a_{i_{r_0}}$. Without loss of generality assume that $r_0 = r$ and $i_1 < \ldots < i_{r_0}$ whose order respects the ordered basis $h$ as in (33), such that

$$s[fv]_0 = a_{i_1} h_{i_1} + \cdots + a_{i_r} h_{i_r} = a_{i_1} h_{\nu(i_1)} + a_{i_2} h_{\nu(i_2)} + \cdots + a_{i_r} h_{\nu(i_r)} + \sum_{i_1, \ldots, i_\mu} d_{i_1, \ldots, i_\ell} \cdot h_{j(i_\ell)}$$

where $d_{i_1, \ldots, i_\ell}$ are constants and each subindex $j(i_1), \ldots, j(i_\ell)$ denotes the dependence on $i_1, \ldots, i_\ell$, according to (33), respectively. Continuing in this manner we prove the corollary. \n
5.3. Canonical form for The Bilinear Form $res_f(M_f \cdot, \cdot)$. We will explain a more details for the bilinear form of order $j = 1$ according to Theorem 3. On $H^n(X_\infty, \mathbb{C})$ one has a canonical (polarized) mixed Hodge structure whose polarization is induced by $Q_{X_1}$, that is, $Q \simeq Q_{X_1}$ and therefore, we have the induced orthogonal decomposition

$$H^n(X_\infty, \mathbb{C}) = H^n(X_\infty, \mathbb{C})_{1, 0} \bigoplus H^n(X_\infty, \mathbb{C})_{0, 1},$$

$$H^n(X_\infty, \mathbb{C})_{1, 0} := H^n(X_\infty, \mathbb{C})_{-1} \bigoplus \bigoplus_{\lambda > 0} H^n(X_\infty, \mathbb{C})_{\lambda, 0} \lambda \lambda$$

which is defined over $\mathbb{Q}$.

From results in Theorem 3 we may interpret the mixed polarized Hodge structure as the one simply described using the basis of Saito-Hertling. Associated to the function $f$ there is its spectrum, $-1 < \alpha_1 \leq \ldots \leq \alpha_\mu < 0$ who are logarithms of the eigenvalues of the monodromy and a basis $A_1, \ldots, A_\mu$ of $H^n(X_\infty, \mathbb{Q})$ such that $A_j$ is an eigenvector of $M_s$ with eigenvalue $e^{-2\pi i \alpha_j}$ forming a Jordan basis for the nilpotent operator $N$ (i.e. $N(A_j) = A_{\nu(j)}$) and puts the bilinear form $Q$ into canonical form: In fact, organizing the basis $\{A_j\}$ so that first we put
those corresponding to $\alpha_j$ an integer, and then the rest, the matrix expression of $Q$ in this basis is, for $n$ even or odd, respectively:

\[
Q = \begin{pmatrix}
(2\pi i)^{n+1} & 0 & 0 \\
0 & (2\pi i)^n & 0 \\
0 & 0 & (2\pi i)^n
\end{pmatrix}, \quad \text{or} \quad Q = \begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & (2\pi i)^n
\end{pmatrix},
\]

being the bilinear form $Q (-1)^{n+1}$-symmetric in the first factor term of (36) and $(-1)^n$-symmetric in the other $Q$-orthogonal factor. And the endomorphism $N$ has the normal form:

\[
[N]_A = \begin{pmatrix}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
n_{\nu(1)} & 0 & \cdots & 0 & 0 \\
0 & n_{\nu(2)} & \cdots & 0 & 0 \\
0 & 0 & \cdots & n_{\nu(\mu-1)} & 0 \\
0 & 0 & \cdots & 0 & n_{\nu(\mu)} \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0
\end{pmatrix}
\]

where

\[
n_{\nu(i)} = \begin{cases}
1 & \text{si } \nu(i) \neq \mu + 1 \\
0 & \text{si } \nu(i) = \mu + 1,
\end{cases}
\]

where the columns with a one generate the Image of $[N]$ en the cero columns generate $\text{ker}[N]$.

The Hodge flags $F^kH^n(X_\infty, \mathbb{C})_1$ and $F^kH^n(X_\infty, \mathbb{C})_{\neq 1}$ consists of those vector spaces whose basis is the set of $A_j$ with $\alpha_j > k$, and the weight filtration comes from the Jordan block structure of $N$ codified in $\nu$.

On the other hand, the Saito-Hertling basis comes also with the selection of $\eta_1, \ldots, \eta_\mu \in \Omega^{n+1}_{n+1,0}$ satisfying that the principal term in the asymptotic expansion of $s(\eta_j)$ is $\partial_{t^{k_j}} t^{\alpha_j + N} A_j$ corresponding to the flat section

\[
\left( ((\alpha_j - k_j + 1)I - \frac{1}{2\pi i}N) \cdots ((\alpha_j - k_j + k_j)I - \frac{1}{2\pi i}N) \right)^{-1} A_j \in H^n(X_\infty, \mathbb{C})_{e^{-2\pi i \alpha_j}}
\]

where $k_j \in \mathbb{Z}_{>0}$ and $\alpha_j - k_j = \beta_j \in (0, -1] \cap \mathbb{Q}$ (see equations (15) and (17)). Its classes $\{[\eta_j]\}$ form a basis of the Jacobian module $\Omega_f$. In the basis $\{[\eta_j]\}$, Grothendieck bilinear form receives the corresponding expressions:
\[
[\text{res}_{f,0}] = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0
\end{pmatrix} \quad \text{or} \quad [\text{res}_{f,0}] = \begin{pmatrix}
0 & 1 \\
1 & 0 \\
0 & 0
\end{pmatrix}
\]

If we denote by \( S \) this matrix, then the relation between \( S \) and \( Q \) can be expressed as \( S = QJ \) where the matrix of automorphism \( J \) becomes of the form:

\[
[J] = \begin{pmatrix}
J_1 & 0 \\
0 & J_{\neq 1}
\end{pmatrix} := \begin{pmatrix}
\frac{1}{(2\pi i)^n+1} & 0 \\
0 & \frac{1}{(2\pi i)^n} & 0
\end{pmatrix}
\]

and \( J_1 \) (resp. \( J_{\neq 1} \)) is \( \frac{1}{(2\pi i)^n} \text{Id} \) (resp. \( \frac{-1}{(2\pi i)^n} \text{Id} \)) on the basis elements corresponding to even elements of the Hodge flag, and \( J_1 \) (resp. \( J_{\neq 1} \)) is \( \frac{-1}{(2\pi i)^n} \text{Id} \) (resp. \( \frac{1}{(2\pi i)^n} \text{Id} \)) on the basis elements corresponding to odd elements.

From Theorem 3, for any \( u \in H^n(X_\infty, \mathbb{C}) \), \( [u] \in \mathbb{C}^n \) denotes the \( A \)-coordinate column vector associated to \( u \). Hence, given \( u \in H^n(X_\infty, \mathbb{C}) \), there exists a unique \( \omega := \varphi^{-1}(u) \in \Omega_f \) such that, in term of coordinates, \( [u] = [\omega] \in \mathbb{C}^n \). Then, we have the following bilinear forms

\[
B^{\text{top}}, B^{\text{alg}} : H^n(X_\infty, \mathbb{C}) \times H^n(X_\infty, \mathbb{C}) \longrightarrow \mathbb{C}
\]

defined respectively by

\[
(u, v) \longmapsto [u]^T \cdot (N^{\text{top}}) SJ \cdot [v], \tag{38}
\]

\[
(u, v) \longmapsto [u]^T \cdot (N_{\text{alg}}) SJ \cdot [v]. \tag{39}
\]

What Theorem 3 says is that the bilinear form (of order one) is described in terms of the bilinear forms \( B^{\text{top}}, B^{\text{alg}} \) in such a way that for any \( \omega, \eta \in \Omega_f \):

\[
\text{res}_{f,0}(f\omega, \eta) = B^{\text{top}}(\varphi(\omega), \varphi(\eta)) + B^{\text{alg}}(\varphi(\omega), \varphi(\eta)).
\]

Equivalently, for any \( u, v \in H^n(X_\infty, \mathbb{C}) \):

\[
\text{res}_{f,0}(f \varphi^{-1}(u), \varphi^{-1}(v)) = B^{\text{top}}(u, v) + B^{\text{alg}}(u, v). \tag{40}
\]

The bilinear form \( B^{\text{top}} \) (resp. \( B^{\text{alg}} \)) will be called the topological (resp. algebraic) part of the bilinear form.

Let \( \text{Gr}_V \Omega_f \longrightarrow H^n(X_\infty, \mathbb{C}) \) be the \( \mathbb{C} \)-isomorphism defined in such a way that

\[
\text{Gr}_V \Omega_f \xrightarrow{\cong} \text{Gr}_V \left( \frac{H^n}{H^n_0} \right) \xrightarrow{\cong} \text{Gr}_V \Psi \cong H^n(X_\infty, \mathbb{C})
\]

\[
gr_V[\eta_j] \xrightarrow{[s_j]} A_j,
\]

for each \( 1 \leq j \leq \mu \). Then we obtain the following result which has a graded flavor, where we recover a Varchenko’s Lemma [29] that the maps \( \text{Gr}_V \{ f \} \) and \( N \) have the same Jordan
canonical normal form and also a consequence by looking at the bilinear form $Gr_V \text{res}_{f,0}(f \bullet, \bullet)$ which is induced on the graded space $Gr_V \Omega_f$.

**Corollary 3.** If we restrict to the graded space, with respect to the $V$-filtration, $Gr_V(\mathbb{H}^n_0/\mathbb{H}^l_0)$ and consider the graded endomorphism $Gr_V\{f\}$ on $Gr_V \Omega_f$, then

$$[Gr_V\{f\}] = [N]$$

and for any $u, v \in H^n(X, \mathbb{C})$:

$$\text{res}_{f,0}(Gr_V\{f\}(\Psi \circ s)^{-1}(u), (\Psi \circ s)^{-1}(v)) = B^{\text{top}}(u, v).$$

where $s$ is the basis of $Gr_V(\mathbb{H}^n_0/\mathbb{H}^l_0)$ induced by $\{s_i\}_{1 \leq i \leq \mu}$.

If the germ $f$ has finite monodromy, then $N = 0$ and hence $B^{\text{top}} = 0$. Hence, we have next corollary which shows that the bilinear form $\text{res}_{f,0}(f \bullet, \bullet)$ has a little more information than the topological one given by the bilinear form $B^{\text{top}}$. In the examples below we will illustrate this aspect.

**Corollary 4.** If $f$ has finite monodromy, then

$$\text{res}_{f,0}(f \omega, \eta) = B_{\text{alg}}(\varphi(\omega), \varphi(\eta)).$$

Equivalently, for any $u, v \in H^n(X, \mathbb{C})$:

$$(41) \quad \text{res}_{f,0}(f \varphi^{-1}(u), \varphi^{-1}(v)) = B_{\text{alg}}(u, v).$$

6. **Examples**

We will give examples illustrating Theorem 3 through Corollary 1 and, equations (38), (39) and (40); some calculations have been done using the computer algebra software SINGULAR [3] by means of libraries gmssing.lib [22] and monodromy.lib [23].

**Example 1.** Let $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be the germ of Isolated Hypersurface Singularity given by the semi-quasi-homogeneous polynomial $f = x^5 + y^6 + x^4 y$; hence (using singular [22] with command monodromy) we have that $M_f \neq 0$, but $M_f^2 \equiv 0$; $f$ has finite monodromy, i.e., $N \equiv 0$. The Milnor number is $\mu = 19$ and the spectrum $\text{sp}(f)$ is

$$-5/8, \ -11/24, \ -5/12, \ -7/24, \ -1/4, \ -5/24, \ -1/8, \ -1/12, \ -1/24,$$

$$0, \ 1/24, \ 1/12, \ 1/8, \ 5/24, \ 1/4, \ 7/24, \ 5/12, \ 11/24, \ 5/8.$$

Notice that $\alpha_1 = -5/8$, $\alpha_19 = 5/8$ and $\alpha_i \neq \alpha_j$, $i \neq j$ since $\alpha_i$ has multiplicity $d_{\alpha_i} = 1$ and $\alpha_i < \alpha_{i+1}$, $\forall i = 1, \ldots, 18$. In this example, $n = 1$, hence the Saito-Hertling basis $h$ for Brieskorn lattice $H^0_0$ is given by

$$h_1 = s_1 + \sum_{j=17}^{19} c^{(1)}_{ij} \partial_i s_j, \quad h_2 = s_2 + c^{(1)}_{2,19} \partial_i s_{19}, \quad h_3 = s_3 + c^{(1)}_{3,19} \partial_i s_{19}, \quad h_j = s_j, \quad 4 \leq j \leq 19.$$

and one obtains a $\mathbb{C}$-basis $[h]$ for $H^0_0/\partial_i^1 H^0_0$ so that $M_f$ in Corollary 1 has the matrix form:

$$[M_f]_{[h]} = N_1 \text{ (since } N \equiv 0),$$
and

\[
N_1 = \begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & 0 & 0 & \cdots & 0 \\
\frac{1}{24} c^{(1)}_{1,17} & 0 & 0 & 0 & \cdots & 0 \\
\frac{2}{24} c^{(1)}_{1,18} & \frac{1}{24} c^{(1)}_{1,17} & 0 & 0 & \cdots & 0 \\
\frac{6}{24} c^{(1)}_{1,19} & \frac{2}{24} c^{(1)}_{1,18} & \frac{1}{24} c^{(1)}_{1,17} & 0 & \cdots & 0
\end{pmatrix}_{19\times19}
\]

which is symmetric with respect to the antidiagonal. The Tjurina number of \( f \) is \( \tau = 17 \), hence the rank of matrix (42) is \( \mu - \tau = 2 \). Therefore at least one of the constants \( c^{(1)}_{1,18} \) or \( c^{(1)}_{1,17} \) is not equal to zero. Since the multiplicity of \( \alpha_{10} = 0 = (n-1)/2 \) is \( d_0 = 1 \), the \([\cdot]_\eta\)-matrix expression for \( \text{res}_{f,0} \) is

\[
[\text{res}_{f,0}]_{\eta} = \begin{pmatrix}
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 1 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
0 & \cdots & 1 & 0 & 0 \\
1 & 0 & \cdots & 0 & 0
\end{pmatrix}_{19\times19}
\]

Finally, the \([\cdot]_\eta\)-matrix for \( \text{res}_{f,0}(f\bullet, \bullet) : \Omega_f \times \Omega_f \to \mathbb{C} \) is

\[
[\text{res}_{f,0}(M_f \bullet, \bullet)]_{\eta} = \begin{pmatrix}
\frac{6}{24} c^{(1)}_{1,19} & \frac{2}{24} c^{(1)}_{1,18} & \frac{1}{24} c^{(1)}_{1,17} & 0 & \cdots & 0 \\
\frac{2}{24} c^{(1)}_{1,18} & 0 & 0 & 0 & \cdots & 0 \\
\frac{1}{24} c^{(1)}_{1,17} & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & \cdots & 0
\end{pmatrix}_{19\times19}
\]

which is symmetric and has rank 2. Since \( n = 1 \), the involution \( \kappa \) in (20) is given by

\[
\kappa : \{1, \ldots, 19\} \to \{1, \ldots, 19\}
\]

\[
k(j) = \begin{cases}
20 - j, & \text{if } \alpha_j \neq (n-1)/2 = 0, \\
10, & \text{if } \alpha_{10} = (n-1)/2 = 0.
\end{cases}
\]

More over the levels of each \( s_j \) in the \( V \)-filtration flag are

\[
r_j = \begin{cases}
1, & \text{for } 1/24, 1/12, 1/8, 5/24, 1/4, 7/24, 5/12, 11/24, 5/8, \\
0, & \text{for } -5/8, -11/24, -5/12, -7/24, -1/4, -5/24, -1/8, -1/12, -1/24, 0
\end{cases}
\]

and the level for each \( A_j \) in the Hodge flag is

\[
p(j) = p(\alpha_j) = \begin{cases}
0, & \text{for } 1/24, 1/12, 1/8, 5/24, 1/4, 7/24, 5/12, 11/24, 5/8 \\
1, & \text{for } -5/8, -11/24, -5/12, -7/24, -1/4, -5/24, -1/8, -1/12, -1/24, 0
\end{cases}
\]
Hence, if we arrange the basis \( \{ A_j \} \) first integers and then the rest,

\[
S = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}_{18 \times 18}
\]

and

\[
J = \frac{1}{2\pi i} \begin{pmatrix}
-\frac{1}{2\pi i} & 1 & 0 & 0 \\
0 & -1 & 1 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 0 & -1
\end{pmatrix}_{18 \times 18} = \begin{pmatrix}
\frac{1}{(2\pi i)^2} & 0 \\
0 & \frac{1}{(2\pi i)} (I - I)
\end{pmatrix}
\]

Hence,

\[
Q = (2\pi i)^{1} \begin{pmatrix}
-2\pi i & 0 & 0 \\
1 & 0 \\
0 & -1 \\
1 & 0 \\
0 & 0 & -1 \\
1 & 0 \\
0 & -1 \\
1 & 0
\end{pmatrix}_{18 \times 18} = \begin{pmatrix}
(2\pi i)^2(-1) & 0 \\
0 & (2\pi i)(I - I)
\end{pmatrix}
\]

It verifies that \( S = QJ \). Since \( N \equiv 0 \), the bilinear form \( Q(N\bullet, J\bullet) \) is trivial and we verify the formula \( \text{res}_{f,0}(M_f\bullet, \bullet) = Q(N_1, J) \).

**Example 2.** Consider the germ of Isolated Hypersurface Singularity given by the semi-quasi-homogeneous polynomial \( f = x^4 + y^5 + xy^4 \), for which we can check (using singular [22] with command monodromy ) that \( N \equiv 0 \). And that the corresponding Milnor and Tjurina number are \( \mu = 12 \) and \( \tau = 11 \); hence the rank of \( M_f \) is \( \text{rank}(M_f) = 1 \) and \( M_f \neq 0 \). Since \( n = 1 \), \( M_f^2 \equiv 0 \); we also may compute the spectrum \( \text{sp}(f) \):

\[
\alpha_1 = -11/20, \ -7/20, \ -3/10, \ -3/20, \ -1/10, \ -1/20, \\
1/20, \ 1/10, \ 3/20, \ 3/10 \ 7/20, \ \alpha_{12} = 11/20.
\]
Notice that each $\alpha_i$ has multiplicity $d_{\alpha_i} = 1$ and $\alpha_i < \alpha_{i+1}$ for $i = 1, \ldots, 11$. The Saito-Hertling basis $h$ for Brieskorne lattice $H''_0$ is given by

$$h_1 = s_1 + c_{1,12}^{(1)} \partial_t s_{12}, \quad h_j = s_j, \quad 2 \leq j \leq 12.$$ 

In the corresponding $C$-basis $[h]$ for $H''_0/\partial_t^{-1} H''_0$, $M_f$ has the matrix form:

$$[M_f]_{[\eta]} = N_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{10} c_{1,12}^{(1)} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{12 \times 12}$$

the $[\eta]$-matrix expression for $\text{res}_{f,0}$ is

$$[\text{res}_{f,0}]_{[\eta]} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}_{12 \times 12}$$

Finally, the $[\eta]$-matrix for $\text{res}_{f,0}(M_f \bullet, \bullet) : \Omega_f \times \Omega_f \to C$ is

$$[\text{res}_{f,0}(M_f \bullet, \bullet)]_{[\eta]} = \begin{pmatrix} \frac{1}{10} c_{1,12}^{(1)} & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}_{12 \times 12}$$

which is symmetric and has rank 1. Since $n = 1$ and $\alpha \neq (n-1)/2 = 0$ for all $\alpha \in \text{sp}(f)$, the involution $\kappa : \{1, \ldots, 12\} \to \{1, \ldots, 12\}$ is given by $k(j) = 13 - j$.

More over the levels of each $s_j$ in the $V$-filtration flag are

$$r_j = \begin{cases} 1, & \text{for } 1/20, 1/10, 3/20, 3/10, 7/20, 11/20 \\ 0, & \text{for } -11/20, -7/20, -3/10, -3/20, -1/10, -1/20 \end{cases}$$

and the level for each $A_j$ in the Hodge flag is

$$p(j) = p(\alpha_j) = \begin{cases} 0, & \text{for } 1/20, 1/10, 3/20, 3/10, 7/20, 11/20 \\ 1, & \text{for } -11/20, -7/20, -3/10, -3/20, -1/10, -1/20 \end{cases}$$

Notice that for any $\alpha \in \text{sp}(f)$, $\alpha \neq 0$. Hence, $H^1(X_\infty, C) = H^1(X_\infty, C) \neq 1$; up to arranging basis $\{A_j\}$,

$$S = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{12 \times 12}$$
and

\[
J = \left( \frac{1}{2\pi i} \right) \begin{pmatrix}
1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & -1 \\
\end{pmatrix} = \left( \frac{1}{2\pi i} \right) \begin{pmatrix} I & 0 \\
0 & -I \\
\end{pmatrix}
\]

Hence,

\[
Q = (2\pi i) \begin{pmatrix}
0 & -1 & 0 \\
1 & 0 & 0 \\
\end{pmatrix}
= (2\pi i) \begin{pmatrix} 0 & -I \\
I & 0 \\
\end{pmatrix}
\]

It verifies that \( S = QJ \) and \( \text{res}_{f,0}(M_f \bullet, \bullet) = Q(N_1 \bullet, J \bullet) \) since the bilinear form \( Q(N \bullet, J \bullet) \) is trivial.

**Example 3.** Let \( f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0) \) be the germ of Isolated Hypersurface Singularity given by the not semi-quasi-homogeneous polynomial \( f = x^5 + y^5 + x^2y^2 \). The Milnor and Tjurina numbers are \( \mu = 11 \) and \( \tau = 10 \); hence the rank of \( M_f \) is \( \text{rk}(M_f) = 1 \) and \( M_f \neq 0 \), but \( M_f^2 \equiv 0 \) since \( n = 1 \). It can be checked (using Singular [23] with command monodromy) that \( f \) has a non finite monodromy, more over, \( \nu : H^1(X_\infty, \mathbb{C}) \rightarrow H^1(X_\infty, \mathbb{C}) \) has a Jordan block of size \( 2 \times 2 \) corresponding to eigenvalue \( \lambda = -1 \), and then \( \nu \neq 0 \), but \( \nu^2 \equiv 0 \) by the Monodromy Theorem. The spectrum \( \text{sp}(f) \) is:

\[
(\alpha, d_\alpha) : (-1/2, 1), (-3/10, 2), (-1/10, 2), (0, 1), (1/10, 2), (3/10, 2), (1/2, 1).
\]

Here \( \alpha_1 = -1/2 \) and \( \alpha_{11} = 1/2 \). Each \( \alpha_i \neq -1/2, 1/2 \) has multiplicity \( d_{\alpha_i} = 2 \) and \( d_{\alpha_{11}} = d_{\alpha_1} = 1 \). Notice that \( \alpha_1 + 1 = \alpha_{11} \) and \( \alpha_i + 1 \notin \text{sp}(f) \subset (-1, 1) \). Hence, the Saito-Hertling basis for Brieskorn lattice \( H_0'' \) is such that

\[
\begin{align*}
&h_i = s_i, \quad 1 \leq i \leq \mu, \\
h_{11} = h_{\nu(1)} = s_{\nu(1)} \neq 0, & \quad h_{\nu(i)} = 0, \quad 2 \leq i \leq 11, \\
&th_i = (\alpha_i + 1)\partial_t^{-1}h_i + h_{\nu(i)}, \quad 1 \leq i \leq 11, \\
&[th_i] = \begin{cases} [h_{\nu(1)}] = [h_{11}] \in H_0''/\partial_t^{-1}H_0'' \\
0, & \text{if } i \neq 1. \end{cases}
\end{align*}
\]
Hence, the $[\omega]$-matrix for $M_f$ is given by

$$[M_f]_{[\omega]} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{11 \times 11} = [N]_A$$

Here $N_1 \equiv 0$ and one has the trivial bilinear form $N_1^T S J = 0$. The involution $\kappa_f$ is given by

$$\kappa_f : \{1, \ldots, 11\} \longrightarrow \{1, \ldots, 11\}, \quad \kappa_f(i) = 12 - i.$$ 

and

$$[\text{res}_f,0]_{[\omega]} = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \end{pmatrix}_{11 \times 11}$$

since the multiplicity of $\alpha_6 = 0 = (n - 1)/2$ is $d_0 = 1$.

On the other hand, the levels of each $s_j$ in the V-filtration flag are

$$r_j = \begin{cases} 
1, & \text{for } (1/10, 2), (3/10, 2), (1/2, 1) \\
0, & \text{for } (-1/2, 1), (-3/10, 2), (-1/10, 2), (0, 1) 
\end{cases}$$

and the level for each $A_j$ in the Hodge flag is

$$p(j) = p(\alpha_j) = \begin{cases} 
0, & \text{for } (1/10, 2), (3/10, 2), (1/2, 1) \\
1, & \text{for } (-1/2, 1), (-3/10, 2), (-1/10, 2), (0, 1) 
\end{cases}$$

Hence, if we arrange the basis $\{A_j\}$ first integers and then the rest,

$$S = \begin{pmatrix} 
1 & 0 \\
0 & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}_{10 \times 10} \end{pmatrix}_{11 \times 11}$$

and

$$J = \frac{1}{2\pi i} \begin{pmatrix} 
-(\frac{1}{2\pi i}) & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & \cdots & \cdots \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 
\frac{1}{(2\pi i)^2}(-1) & 0 \\ 0 & \frac{1}{(2\pi i)}(I \ 0) \end{pmatrix}$$
Hence,

\[
Q = (2\pi i)^1 \begin{pmatrix}
-(2\pi i) & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot \\
0 & 1 & 0 & 0 \\
1 & 0 & -1 & 0 \\
\end{pmatrix} = \begin{pmatrix}
((2\pi i)^2(-1)) & 0 & 0 \\
0 & (2\pi i)(0 & -I) \\
\end{pmatrix}
\]

It verifies that \( S = QJ \) and \( \text{res}_{f,0}(M_f \bullet, \bullet) = Q(N \bullet, J \bullet) \) since the bilinear form \( Q(N_1, J) \) is trivial.

**Example 4.** We will consider the M. Saito example as in [19, p.18] which was stated by using the Tom-Sebastiani type theorem ([26]). Set

\[ f = g + g' \text{ with } g = x^{10} + y^3 + x^2y^2, \quad g' = z^6 + w^5 + z^4w^3. \]

Using Singular [22] we may compute that Milnor and Tjurina numbers are \( \mu = 280 \) and \( \tau = 248 \), and the spectrum \( \text{sp}(f) \) is:

\[
(\alpha, d_\alpha) : (-2/15, 1), (-1/30, 1), \\
(1/30, 1), (1/15, 2), (2/15, 1), (1/6, 2), (1/5, 2), (7/30, 2), (4/15, 3), (3/10, 1), (1/3, 2), (11/30, 6), \\
(2/5, 3), (13/30, 3), (7/15, 5), (1/2, 2), (8/15, 7), (17/30, 8), (3/5, 4), (19/30, 5), (2/3, 6), (7/10, 6), \\
(11/15, 9), (23/30, 9), (4/5, 5), (5/6, 6), (13/15, 10), (9/10, 7), (14/15, 10), (29/30, 9), \\
(1, 4), (31/30, 9), (16/15, 10), (11/10, 7), (17/15, 10), (7/6, 6), (6/5, 5), (37/30, 9), (19/15, 9), \\
(13/10, 6), (4/3, 6), (41/30, 5), (7/5, 4), (43/30, 8), (22/15, 7), (3/2, 2), (23/15, 5), (47/30, 3), (8/5, 3), \\
(49/30, 6), (5/3, 2), (17/10, 1), (26/15, 3), (53/30, 2), (9/5, 2), (11/6, 2), (28/15, 1), (29/15, 2), (59/30, 1), \\
(61/30, 1), (32/15, 1)
\]

The rank of \( M_f \) is \( \text{rk } M_f = 32 \); with Singular [22, 23] (with command **jacoblift**) it can be checked that \( M_f \neq 0, M_f^2 \neq 0, \) but \( M_f^3 \equiv 0; \) and (with command **monodromy**) one obtains that \( N \) only has Jordan blocks of size \( 1 \times 1 \) and several Jordan blocks of (maximal) size \( 2 \times 2; \) hence \( N \neq 0, N^2 \neq 0, \) but \( N^3 \equiv 0. \) Our interest in the present example is the following M. Saito result [19]. Let \( H^m_0(f), H^m_0(g), H^m_0(g') \) be the Brieskorn lattices of \( f, g, g', \) respectively. Hence, there are canonical isomorphisms (cf. [26]):

\[
H^m_0(f) \cong H^m_0(g) \otimes_R H^m_0(g'), \quad \frac{\partial r^{-1}H^m_0(f)}{\partial t^{-1}H^m_0(g)} \cong \frac{H^m_0(g)}{\partial t^{-1}H^m_0(g)} \otimes_C \frac{H^m_0(g')}{\partial t^{-1}H^m_0(g')},
\]

\((R := \mathbb{C}\{\{\partial^t\}^{-1}\})\) is the ring of micro-differential operators with constant coefficients) such that the action of \( t \) on the left hand side is identified with \( t \otimes \text{Id} + \text{Id} \otimes t. \) There is a subspace of rank 6, denoted by \( H^m_0(f)' \subset H^m_0(f), \) such that the \( [\hbar] \)-matrix of \( t \) on the corresponding class subset
\( H''_0(f)^\prime \subset H''_0(f)/\partial t^{-1}H''_0(f) \) is given by
\[
[t]_{[\eta]} = \left( \begin{array}{cccccc}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 \\
a & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & a & 1 & 0
\end{array} \right) = [N] + N_1
\]
with \( a \in \mathbb{C} \setminus \{0\} \). As a consequence, the \([\eta]\)-matrix for \( M_f \) is also \([N] + N_1\). Notice that if we restrict to \( \ker(N) \), the bilinear form \( Q(N_1, J) \) is non trivial on \( \varphi^{-1}(\ker(N)) \cap (H''_0(f)^\prime) \).

7. CONCLUSIONS

Making use of the Saito-Hertling basis which is constructed from the Deligne splitting associated to the Steenbrink-Hertling Polarized Mixed Hodge Structure with respect to polarization \( Q \), we may choose a Jordan basis for vanishing canonical cohomology fiber \( H^n(X_\infty, \mathbb{C}) \) adapted to the weight filtration \( W(N) \), in such a way that we can describe a normal form for the bilinear forms and maps:

- Endomorphisms on \( H^n(X_\infty, \mathbb{C}) \): \( N \) and \( N_1 \);
- Map multiplication by \( f \) on \( \Omega^j \): \( M_f \);
- The isomorphism \( \varphi \) from \( \Omega^j \) to \( H^n(X_\infty, \mathbb{C}) \);
- The automorphism on \( H^n(X_\infty, \mathbb{C}) \): \( J \);
- Polarization on \( H^n(X_\infty, \mathbb{C}) \): \( Q(\bullet, \bullet) \);
- The Lefschetz bilinear forms with topological setting on \( H^n(X_\infty, \mathbb{C}) \): \( Q(N^\ell \bullet, \bullet), \ 0 \leq \ell \leq n \);
- The topological weighted bilinear forms on \( H^n(X_\infty, \mathbb{C}) \): \( Q(N^\ell \bullet, J_\bullet), \ 0 \leq \ell \leq n \);
- The algebraic bilinear forms on \( H^n(X_\infty, \mathbb{C}) \): \( Q(N^\ell \bullet, J_\bullet), \ 0 \leq \ell \leq n \);
- The Grothendieck paring: \( \text{res}_{f,0}(\bullet, \bullet) \);
- The higher bilinear forms on the Jacobian module \( \Omega_f \): \( \text{res}_{f,0}(M^j_f \bullet, \bullet), \ 0 \leq j \leq n \).

Our main result given by Theorem 3, uses these normal forms to relate the higher bilinear forms in \( \Omega_f \) to the weighted topological and algebraic higher bilinear forms on \( H^n(X_\infty, \mathbb{C}) \). Essentially, these results are produced using the normal form that the Grothendieck pairing inherits from the Saito-Hertling basis for the Brieskorn lattice. Hence we can note that, the bilinear forms in \( \Omega_f \) have more information than the weighted topological which is detected assuming that the germ has finite monodromy. The interesting part arises from the fact that such weights are determined by the authomorphism \( J \) which is constructed from the Hodge filtration by means of the spectrum of singularity germ \( f \).

Besides, our results also state that the Jordan chains of \( M_f \) can be obtained as a binding of topological \( N \)-spectral chains, without taking the grading that the \( V \)-filtration produces.

The interest in the analysis of the bilinear forms in the Milnor algebra (or the Jacobian module) arises from the attempt of the second author and collaborators [6] for the understanding of it geometrical meaning, since there is a relationship between the signature of these higher bilinear forms to indices of vector fields whenever the germ \( f \) also is real analytic.
Finally, it is worth mentioning that in [4] using a weaker approach and without using the Saito-Hertling basis, the author shows that the bilinear forms \( res_f(f^j\bullet, \bullet) \) have an additive expansion in terms of the bilinear forms \( Q(N^j\bullet, \bullet) \). Such additive expansions depend only on the asymptotic expansions for elements on the Jacobian module, which are induced by the \( V \)-filtration.

References

[1] Briançon, J., Skoda, H, Sur la cloture integrable d’un ideale de germes de fonctions holomorphes en un point de \( \mathbb{C}^n \), C. R. Acad. Sci (Paris) (1974), 278–949.

[2] Brieskorn, E., La monodromie des singularités isolées d’hypersurfaces, Manuscripta Math. 2 (1970), 103–161.

[3] Decker, W., Greuel, G-M., Pfister G., Schönemann, H., Singular 4.0.2 A computer Algebra System for Polynomial Computations, Center for Computer Algebra, University of Kaiserslautern. http://www.singular.uni-kl.de (2015).

[4] Dela-Rosa, M. A., On a lemma of Varchenko and higher bilinear forms induced by Grothendieck duality on the Milnor algebra of an isolated hypersurface singularity. Bull. Braz. Math. Soc. (N.S.) 49(4) (2018), 715–741.

[5] Eisenbud, D., Levine, H. I., Teissier, B., An Algebraic Formula for the Degree of a \( C^\infty \) Map Germ / Sur Une Inégalité à La Minkowski Pour Les Multiplicités. Ann. of Math. 106(1) (1977), 19-44.

[6] Giraldo, L., Gómez-Mont, X., Mardešić, P., Flags in zero dimensional complete intersection algebras and indices of real vector fields, Math. Z. 260 (2008), no. 1, 77-91.

[7] Griffiths, P., Harris, J., Principles of Algebraic Geometry, Wiley (1978).

[8] Griffiths, P., Schmid, W., Recent Developments in Hodge Theory: A Discussion of Techniques and Results, Discrete subgroups of Lie groups and applications to moduli, Oxford Univ. Press, Bombay, (1975), 31–127.

[9] Hertling, C., Formes bilinéaires et hermitiennes pour des singularités: un aperçu. In: Singularités (ed. D. Barlet), Institut Élie Cartan Nancy 18 (2005).

[10] Hertling, C., Frobenius manifolds and moduli spaces for singularities, Cambridge Tracts in Math. 151 (2004).

[11] Hertling, C., Classifying Spaces for Polarized Mixed Hodge Structures and Brieskorne Lattices, Compositio Mathematica. 116 (1999), 1-37.

[12] Hertling, C. and Stahlke, C., Bernstein Polynomial and Tjurina Number, Geometriae Dedicata. 75 (1999), 137-176.

[13] Kulikov, Va. S., Mixed Hodge Structures and Singularities, Cambridge Tracts in Math. 132 (1998).

[14] Looijenga E., Isolated singular points on complex hypersurfaces, Ann. Math., Stud. Princ., Univ. Press, Vol. 77 (1984).

[15] Leray, J., Le calcul différentiel et intégral sur une variété analytique complexe (Problème de Cauchy, III), Bull. Soc. Math. France (1959), 81-180.

[16] Malgrange, B., Integrales Asymptotiques et Monodromie, Ann. Scient. Éc.Norm. Sup. 4e série, t.7 (1974), 405-430.

[17] Milnor, J., Singular Points on Complex Hypersurfaces, Ann. Math., Stud. Princ., Univ. Press, Vol. 61 (1968).

[18] Saito, K., The higher residue pairings \( K^{(k)}_F \) for a family of hypersurface singular points, Proceedings of Symposia in Pure Mathematics Vol. 40 (1983), part 2, 441-463

[19] Saito, M., On the structure of Brieskorn lattices, II. Journal of Singularities, volume 18 (2018), 248-271.

[20] Saito, M., On the Structure of Brieskorn Lattices, Ann. Inst. Fourier Grenoble. 39 (1989), 27-72.

[21] Schmid, W. Variation of Hodge Structure: the singularities of the period mapping. Invent. Math. 22 (1973), 211-319

[22] Schulze, M: gmsing.lib. A SINGULAR 4-0-2 library for computing invariants related to the Gauss-Manin system of an isolated hypersurface singularity. (2015).

[23] Schulze, M: mondromy.lib. A SINGULAR 4-0-2 library for computing the monodromy of an isolated hypersurface singularity. (2015).
[24] Steenbrink, J. *Mixed Hodge structure on the vanishing cohomology*, Nordic Summer School, Symposium in Mathematics, Oslo (1976), 525-563.

[25] Scherk, J., *On the Monodromy Theorem for Isolated Hypersurface Singularities*, Inventiones Math. 58 (1980), 289-301.

[26] Scherk, J., Steenbrink, J., *On the Mixed Hodge Structure on the Cohomology of the Milnor Fibre*, Math. Ann. 271 (1985), 641-665.

[27] van Straten, D., *From Briançon-Skoda to Scherk-Varchenko*. Commutative Algebra and Noncommutative Algebraic Geometry 1 (2015), 347-370.

[28] Varčenko, A. *On the local residue and the intersection form on the vanishing cohomology*, Math USSR Izvestiya, Vol. 26 (1986).

[29] Varčenko, A. *On the monodromy operator in vanishing cohomology and the operator of multiplication by f in the local ring*, Soviet Math. Dokl., Vol. 24 (1981).

[30] Varčenko, A. *The asymptotics of holomorphic forms determine a Mixed Hodge Structure*, Soviet Math. Dokl., Vol. 22 (1980), 772-775.

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