Weak cosmic censorship in Born-Infeld electrodynamics and bound on charge-to-mass ratio

Tong-Tong Hu, Yan Song, Shuo Sun, Hong-Bo Li, and Yong-Qiang Wang

Research Center of Gravitation & Institute of Theoretical Physics &
Key Laboratory for Magnetism and Magnetic of the Ministry of Education,
Lanzhou University, Lanzhou 730000, China

Abstract

We construct a class of counterexamples to cosmic censorship in four dimensional Einstein-Born-Infeld theory with asymptotically anti-de Sitter boundary conditions, and investigate the effect of the Born-Infeld parameter $b$ in comparison with the counterpart of Einstein-Maxwell theory. When a charged massive scalar field is included into the action, we find that this class of counterexamples to cosmic censorship would be removed if the charge of scalar fields is above the minimum value of charge $q_{\text{min}}$. In particular, the minimum value of charge required to preserve cosmic censorship increases with the increasing of Born-Infeld parameter. Meanwhile, we also show the lower bounds on charge-to-mass ratio with the different values of Born-Infeld parameter.
1 Introduction

The study of black hole singularities has been an interesting subject since the original work on the weak cosmic censorship conjecture (WCCC) [1], which states naked singularities arising in the solutions of Einstein’s equations [2] must be hidden within event horizons of black hole, and therefore cannot be observed from future null infinity. Although there existed many works of testing the validity of WCCC, lots of counterexamples to cosmic censorship have been found. For example, the collapse of a massive matter cloud with regular initial data results in the formation of a naked singularity [3]. It is an interesting question to ask whether there exists a region of arbitrarily large curvature that is observed to distant observers. In higher dimensions, the event horizon of a class of black holes is not necessarily the case of topologically spherical. Due to the Gregory-Laflamme instability [4], black holes will be unstable under gravitational perturbations and produce a naked singularity, which implies the violation of cosmic censorship [5, 6, 7]. Analysing the non-linear evolution of black holes [8, 9], one found that a naked singularity should form in finite time. In four dimensions, by investigating the model of Einstein gravity coupling to a Maxwell field with a negative cosmological constant [10, 11], one could obtain a class of counterexamples to cosmic censorship with asymptotically anti-de Sitter (AdS) boundary. Instead of adding a Maxwell field, the authors in Ref. [12] constructed smooth stationary solutions with differential rotation to the boundary metric, which provides a possible vacuum counterexamples to weak cosmic censorship in AdS spacetime.

Recently, by introducing a charged scalar field to the Einstein-Maxwell solutions [13], the authors found that when the charge of the scalar field was sufficient large, the static solution of Einstein-Maxwell action would become unstable and a new stable solution could appear with nontrivial charged scalar field, which is analogous to the instability of a charged black hole to develop scalar condensation in the study of holographic superconductor [14, 15, 16]. Meanwhile, there exists a minimum value of charge required to remove the counterexamples and preserve cosmic censorship. It is surprising that the minimum value of charge appears to agree precisely with that proposed in the weak gravity conjecture [17], which states that any consistent quantum theory of gravity must have a stable state whose charge-to-mass ratio is equal to or larger than that of an extremal black hole. In the case of Einstein-Maxwell model, the bound of $q/m$ is equal to
1. To further test the weak gravity - cosmic censorship connection, the author in [18] also studied the static solutions in Einstein theory with a dilaton field and the multi-charged scalar field case.

Considering that Maxwell theory is only the theory of linear electrodynamics, we would like to know whether or not there exists a class of counterexamples to cosmic censorship and the charge-to-mass ratio bound in four dimensional Einstein-nonlinear electrodynamics theory. The Born-Infeld electrodynamics [19, 20, 21] is a nonlinear generalization of the Maxwell’s theory, which can remove the divergence of self-energy of a point-like charge in Maxwell electrodynamics. Moreover, the Lagrangian of Born-Infeld can arise from the low-energy effective theory describing electromagnetism [22, 23]. Many works on the black holes solutions in Einstein-Born-Infeld theory have been studied in [24, 25, 26, 27, 28, 29].

In the present paper, we would like to construct a class of counterexamples to cosmic censorship in four dimensional Einstein-Born-Infeld theory with asymptotically anti-de Sitter boundary. Furthermore, comparing with the counterpart of Einstein-Maxwell theory, we investigate the counterexample of Einstein-Born-Infeld theory for several values of Born-Infeld parameter $b$. Besides, introducing a charged massive scalar field into Einstein-Born-Infeld theory, we find that this class of counterexamples to cosmic censorship would be removed, and we study the value of the lower bound on the scalar field charge required to preserve cosmic censorship in the case of Born-Infeld action.

The paper is organized as follows. In Sec. 2 we introduce the model of Einstein-Born-Infeld coupling to a complex, charged scalar field and the numerical DeTurck method. In Sec. 3 we explore the ansatz of metric and matter field, and analyze the boundary conditions. Numerical results of a class of counterexamples to cosmic censorship and static solutions with charged scalar condensation are shown in Sec. 4. The conclusion and discussion are given in the last section.

2 Set up

Let us begin with the action of the Born-Infeld field and a charged complex scalar field in the four-dimensional Einstein gravity spacetime with a negative cosmological constant, which is written as

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[ R + \frac{6}{L^2} + \mathcal{L}_{BI} - 4 (\mathcal{D}_a \Phi)(\mathcal{D}^a \Phi) - 4 m^2 \Phi \Phi^\dagger \right], \quad (2.1)$$

where $\mathcal{L}_{BI} = \frac{1}{b}(1 - \sqrt{1 + \frac{bF^2}{2}})$ with the field strength of the U(1) gauge field $F = F_{ab} F^{ab}$, and $L$ is the radius of asymptotic AdS spacetime. The constants $m$ and $q$ represent the mass and the charge of the complex scalar field, respectively. The constant $b$ is the Born-Infeld parameter and the Born-Infeld field will reduce to the Maxwell case when $b \to 0$. Where $\mathcal{D}_a = \nabla_a - iqA_a$ is the gauge covariant derivative with respect to $A_a$. Note that the values of $m^2$ must satisfy the Breitenlohner-Freedman (BF) bound $m^2 \geq -9/4$ [30] for the (3+1)-dimensional spacetime.

The motion equations can derived from Eqs. (2.1)

$$R_{ab} + \frac{3}{L^2} g_{ab} = T_{ab}, \quad (2.2a)$$

$$\nabla_a (-\mathcal{F} F^a_b) = i q \left[ (\mathcal{D}_b \Phi) \Phi^\dagger - (\mathcal{D}_b \Phi)^\dagger \Phi \right], \quad (2.2b)$$

$$\mathcal{D}_a \mathcal{D}^a \Phi = m^2 \Phi, \quad (2.2c)$$
with the energy-momentum tensor of matter field

\[ T_{ab} = 2 \left( -\mathcal{F} F_a \epsilon F_b + \frac{g_{ab}}{4} \mathcal{L}_{BI} \right) + 2(\mathcal{D}_a \Phi)(\mathcal{D}_b \Phi)^\dagger + 2(\mathcal{D}_a \Phi)^\dagger(\mathcal{D}_b \Phi) + 2 m^2 g_{ab} \Phi \Phi^\dagger, \]

where \( \mathcal{F} = \frac{\partial \mathcal{L}_{BI}}{\partial F} \), and it equals to -1 in Maxwell condition.

If the complex, massive scalar field \( \psi \) vanishes, the solution of Einstein equations (2.2), which can describe the asymptotically spherically black hole with charge, is the well-known Born-Infeld AdS black hole. In terms of spherical coordinates, the metric of Born-Infeld AdS black hole is the following form

\[ ds^2 = -f(r) dt^2 + f^{-1}(r) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2), \]

with

\[ f(r) = \frac{1}{-2M} + \frac{r^2}{L^2} + \frac{4Q^2 F_1(1, 1/2, 1/4, 5/4, \frac{-b Q^2}{r^4})}{3r^2} + \frac{2r^2}{3b} \left( 1 - \sqrt{\frac{bQ^2}{r^4} + 1} \right), \]

where \( M \) and \( Q \) are the ADM mass and the electric charge of BI AdS black hole, respectively, and \( _2F_1 \) is a hypergeometric function [32]. The gauge potential is

\[ A = -\frac{Q_2 F_1(1, 1/2, 1/4, 5/4, \frac{-b Q^2}{r^4})}{r} dt. \]

When there exists a non-trivial configuration of the complex scalar field, it is obvious that we should solve the equations of motion (2.2) numerically instead of seeking the analytical solutions. We will use DeTurk method to solve these equations, which provides a good tool for solving Einstein equations in these papers. By adding a gauge fixing term to Einstein equation (2.2a), we could obtain a set of elliptic equations, which are known as Einstein-DeTurk equation

\[ R_{ab} + \frac{3}{L^2} g_{ab} - \nabla (a \xi_b) = T_{ab}, \]

where \( \xi^a = g^{bc}(\Gamma^a_{bc}[g] - \Gamma^a_{bc}[\tilde{g}]) \) is the Levi-Civita connection associated with a reference metric \( \tilde{g} \), which should be choose to be as same boundary and horizon structure as \( g \).

### 3 Ansatz and boundary conditions

In order to construct static, axisymmetric solutions with a timelike Killing vector and an axisymmetric Killing vector, we also adopt the same axisymmetric metric as that in Refs. [31, 10, 13, 18] with the following ansatz

\[
\begin{align*}
    ds^2 &= \frac{L^2}{(1 - x^2)^2} \left[ \frac{(1 - y^2)^2}{y^2 (2 - y^2)} U_1 dt^2 + \frac{4 U_4}{2 - x^2} \left( dx + \frac{U_3}{1 - y^2} dy \right)^2 
    + \frac{4 U_2}{y^2 (1 - y^2)^2 (2 - y^2)^2} + x^2 (2 - x^2) U_5 d\phi^2 \right],
\end{align*}
\]
where the functions \( U_i \) \( (i = 1, 2, 3, 4, 5) \) depend on the variables \( x \) and \( y \). Both of variables take values in \([0, 1]\). When \( U_1 = U_2 = U_4 = U_5 = 1 \) and \( U_3 = 0 \), the metric (3.1) can reduce to

\[
ds^2 = \frac{L^2}{(1-x^2)^2} \left[ -\frac{(1-y^2)^2}{y^2(2-y^2)} \frac{dt^2}{2-x^2} + \frac{4}{2-x^2} \frac{dy^2}{y^2(1-y^2)^2(2-y^2)^2} + x^2 \left( 2-x^2 \right) d\phi^2 \right].
\]  

(3.2)

When we use a new coordinate system,

\[
z = \frac{y\sqrt{2-y^2}}{1-y^2}(1-x^2),
\]  

(3.3a)

\[
r = \frac{y\sqrt{2-y^2}}{1-y^2}x\sqrt{2-x^2},
\]  

(3.3b)

the line element (3.2) in new coordinates becomes

\[
ds^2 = \frac{L^2}{z^2} \left[ -dt^2 + dr^2 + r^2 d\phi^2 + dz^2 \right],
\]  

(3.4)

which is just the pure Anti-de Sitter spacetime in Poincaré coordinates. The Poincaré horizon is now located at \( y = 1 \), and the axis of rotation is located at \( x = 0 \). The conformal boundary is located at \( x = 1 \), and \( y = 0 \) denotes the intersection of the conformal boundary with the axis of symmetry. We can choose the reference metric \( \tilde{g} \) given by the line element of pure Anti-de Sitter spacetime (3.2).

Considering the above metric (3.1), an ansatz of matter fields should be described as below

\[
A = LU_6 dt, \quad \Phi = (1-x^2)^\Delta y^\Delta (2-y^2)^{\frac{\Delta}{2}} U_7
\]  

(3.5)

with

\[
\Delta \equiv 3/2 + \sqrt{9/4 + m^2},
\]  

(3.6)

where the function \( U_6 \) and \( U_7 \) are the function of \( x \) and \( y \). According to AdS/CFT duality, we take \( \langle O_2 \rangle = (1-y^2)^2U_7 \) to describe the scalar condensation. To simplify, we only choose \( m^2 = -2 \) in our paper.

Next, we will discuss the boundary conditions. At conformal boundary, located at \( x=1 \), the metric must reduce to pure AdS spacetime, so we must take

\[
U_1(1, y) = U_2(1, y) = U_4(1, y) = U_5(1, y) = 1, \quad U_3(1, y) = 0,
\]  

(3.7)

and the boundary of matter fields

\[
\partial_x U_7(1, y) = 0, \quad U_6(1, y) = a(1-y^2)^n, \quad n = 2, 4, 6, \ldots,
\]  

(3.8)

where \( a \) is the amplitude. At \( x = 0 \), we could impose the Dirichlet boundary conditions on \( U_3(1, y) = 0 \) and

\[
\partial_x U_1(0, y) = \partial_x U_2(0, y) = \partial_x U_4(0, y) = \partial_x U_5(0, y) = \partial_x U_6(0, y) = \partial_x U_7(0, y) = 0,
\]  

(3.9)

Neumann boundary conditions on the other functions. Moreover, the asymptotic behavior of the equations of motion near \( x = 0 \) gives the condition \( U_4(0, y) = U_5(0, y) \).
At \( y=0 \), i.e. the intersection of the conformal boundary with the axis of symmetry, we set the boundary conditions

\[
U_1(x, 0) = U_2(x, 0) = U_4(x, 0) = U_5(x, 0) = 1, \quad U_3(x, 0) = \partial_y U_7(x, 0) = 0, \quad U_6(x, 0) = a, \quad (3.10)
\]

and at Poincaré horizon \( y=1 \), Dirichlet boundary conditions are imposed on

\[
U_1(x, 1) = U_2(x, 1) = U_4(x, 1) = U_5(x, 1) = 1, \quad U_3(x, 1) = U_6(x, 1) = U_7(x, 1) = 0. \quad (3.11)
\]

## 4 Numerical results

In this section, we will numerically solve the coupled system of nonlinear partial differential equations (2.2) with the ansatzs (3.1) and (3.5). We employ finite element methods in the integration region \( 0 \leq x \leq 1 \) and \( 0 \leq y \leq 1 \) defined on non-uniform grids, allowing the grids to be more finer grid points near the boundary of \( y = 0 \) and \( y = 1 \). Our iterative process is the Newton-Raphson method. The relative error for the numerical solutions in this work is estimated to be below \( 10^{-5} \). In order to keep good agreement with the aforementioned error, the grid size has to be increased and typically a \( 120 \times 200 \) to \( 120 \times 300 \) grid was used.

We would study the following two cases with scalar field \( \Phi = 0 \) and \( \Phi \neq 0 \), respectively. For convenience, in the following results, we will set \( L = 1 \).

### 4.1 With charged scalar field \( \Phi = 0 \)

In this subsection, we will analyze the solutions with charged scalar field \( \Phi = 0 \) and present the evidence for counterexample to cosmic censorship. When one obtains a solution of Einstein equations, it is important to investigate whether or not the spacetime of solution is regular. In general, one of the most useful ways is to check for the finiteness of the Kretschmann scalar, which is also called Riemann tensor squared and written as

\[
K = R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}, \quad (4.1)
\]

where \( R_{\alpha\beta\gamma\delta} \) is the Riemann curvature tensor. Therefore, Kretschmann scalar is a sum of squares of tensor components and a quadratic invariant.

Numerical results are presented in Fig 1. In the left panel we present the distribution of Kretschmann scalar \( K \) as a function of \( x \) and \( y \) coordinate with \( b = 0.25, n = 8 \) and \( a = 6 \). and it is obvious that the spacetime is not flat, and the value of the Kretschmann scalar in the purple area is about to \( 24/L^4 \), which means one recover the result of \( AdS_4 \) spacetime. In addition, the maximum of the Kretschmann scalar appears at the boundary \( x = 0 \). In the right panel, we exhibit the maximum of the Kretschmann scalar \( K \) versus the amplitude \( a \) with the fixed Born-Infeld parameters \( b = 0, 0.25, 0.5 \), represented by the black, blue and green lines, respectively, and the vertical black, blue and cyan dashed gridlines indicate \( a_{\text{max}} = 8.06, 6.56 \) and 5.94, respectively, and the horizontal red line is the value of \( K = 24 \) for \( AdS_4 \) spacetime. From the figure, we could find that when the Born-Infeld parameters \( b \neq 0 \), there still exists the growth of the maximum of Kretschmann scalar with the increasing of \( a \), which indicates the formation of a curvature singularity similar to the case of Maxwell action in [13].
Figure 1: Left: The distribution of Kretschmann scalar as a function of \(x\) and \(y\) coordinate with \(b = 0.25\), \(n = 8\) and \(a = 6\). Right: The maximum of the Kretschmann scalar as a function of \(a\) with the fixed Born-Infeld parameters \(b = 0\), 0.25, 0.5, represented by the black, blue and cyan lines, respectively. The vertical black, blue and cyan dashed gridlines indicate \(a_{\text{max}} = 8.06\), \(a_{\text{max}} = 6.56\) and \(a_{\text{max}} = 5.94\), respectively, and the horizontal red line is the value of \(K = 24\) for AdS\(_4\) spacetime.

| \(a_{\text{max}}\) | \(n\) | 4     | 6     | 8     | 10    |
|-----------------|------|-------|-------|-------|-------|
| 0.00            | 4.95 | 6.53  | 8.06  | 9.13  |
| 0.25            | 4.22 | 5.51  | 6.56  | 7.45  |
| 0.5             | 3.82 | 4.99  | 5.94  | 6.76  |
| 1               | 3.35 | 4.36  | 5.20  | 5.91  |

Table 1: Maximum amplitude \(a_{\text{max}}\) for several values of \(n\) and \(b\).

When the amplitude reaches to a maximum value, there will appear a singularity, which means producing arbitrarily large curvature in spacetime. This maximum is depended on both \(n\) and \(b\). We show our result in the following table.

Form Table 1, we could see that the maximum amplitude \(a_{\text{max}}\) increases with the increasing of \(n\), meanwhile, it decreases with the increasing of the Born-Infeld parameter \(b\). When Maxwell field is extended to Born-Infeld model, the static solutions could still become singular at a finite value of \(a\).

4.2 With charged scalar field \(\Phi \neq 0\)

In the last subsection, we obtain a family of Einstein-Maxwell solutions with charged scalar field \(\Phi = 0\), we would like to know whether or not there exist the charge static solutions with no-
vanished scalar field, which is analogous to the case of holographic superconductor studied in 
[33, 34, 16]. When fixed the parameter $b$ and $a$, there is a critical value of $q$, below which 
the scalar field vanishes and the solution is the simple planar. Above this value, the charge static 
solution becomes unstable to develope scalar hair. Before numerically solving the full dynamic 
equations of motion including scalar field in Eqs. (2.2), we would follow the same method as in 
[35, 13], and find the critical value of $q$. First, we solve the time-independent scalar field equations 
at a fixed background.

$$(\nabla_a \nabla^a - m^2)\Phi = q^2 A_a A^a \Phi,$$ \hspace{1cm} (4.2)

which one could recognize as an eigenvalue problem with the eigenfunction $\Phi$ and eigenvalue $q^2$. 
We will use $q_{\text{min}}$ to denote the smallest eigenvalue, which is the critical charge and needed as a 
function of the amplitude $a$ for a zero-mode.

![Figure 2](image)

Figure 2: The critical charge $q_{\text{min}}$ as a function of the amplitude $a$. \textit{Left:} For the fixed $n = 8$, the 
curves from top to bottom correspond to $b = 1, 0.5, 0.25, 0$, respectively. \textit{Right:} For the fixed 
$b = 0.25$, the curves from top to bottom correspond to $n = 10, 8, 6, 4$, respectively.

The results are shown in Fig 2. In left panel, with the fixed $n = 8$, we plot critical charge 
$q_{\text{min}}$ as a function of the amplitude $a$ for the corresponding values of $b = 1, 0.5, 0.25$ and 0, 
represented by cyan, blue, red and black lines, respectively. The vertical dashed gridlines indicate 
the maximum amplitude $a_{\text{max}} = 8.06$ (black), 6.56 (red), 5.94 (blue) and 5.20 (cyan), respectively, 
which corresponds to the values of $n = 8$ shown in Table 1. For each curve, the critical charge 
$q_{\text{min}}$ decreases with the increasing amplitude $a$. We see that for the fixed amplitude $a$, the critical 
charge $q_{\text{min}}$ increases with increasing Born-Infeld parameter $b$. Furthermore, when the amplitude 
a decreases, the critical charge $q_{\text{min}}$ tends to the same value for several values of $b$.

In the right of Fig 2 with the fixed $b = 0.25$, we plot critical charge $q_{\text{min}}$ as a function of the 
amplitude $a$ for the corresponding values of $n = 10, 8, 6$ and 4, represented by cyan, blue, red 
and black lines, respectively. For each curve, the critical charge $q_{\text{min}}$ decreases with the increasing
amplitude $a$, which are similar to the behavior in the left panel. For the fixed amplitude $a$, the critical charge $q_{\text{min}}$ increases with the increasing of $n$.

Next, we could solve the coupled Einstein-Born-Infeld-scalar equations (2.2) to find the solutions for the above critical charge $q < q_{\text{min}}$. We show the profiles of the expectation value for the operator dual to $\Phi$ as a function of the boundary radial coordinate $r$ in Fig. 3. In the left panel, with the fixed $n = 8$ and $q = 2.7$, the curves from top to bottom correspond to $b = 0$, 0.5, 1, represented by black, red and blue lines, respectively. For each curve, the scalar condensation $\langle O_2 \rangle$ decreases with the increasing amplitude $r$. When the amplitude $r$ increases, the scalar condensation tends to zero value for several values of $b$. Moreover, at the origin $r = 0$, there exists the largest value of the scalar condensation. Furthermore, for the fixed amplitude $r$, the scalar condensation $\langle O_2 \rangle$ decreases with the increasing value of Born-Infeld parameter $b$.

In the right of Fig. 3, with the fixed $b = 0.25$ and $q = 2.22$, we plot the scalar condensation $\langle O_2 \rangle$ as a function of $r$ for several values of $a$. The curves from top to bottom correspond to $a = 10$, 9, 8, 7 and 5.8, represented by black, red, blue, cyan and purple lines, respectively. For each curve, the scalar condensation decreases with the increasing $r$, which is similar to the behavior in the left panel. For the fixed amplitude $b$, the scalar condensation $\langle O_2 \rangle$ increases with the increasing value of the parameter $a$.

In order to further understand how increasing the amplitude $a$ affects the maximum of the condensation with the different values of $b$, we show in the left panel of Fig. 4 the condensate $\langle O_2 \rangle$ at $r = 0$ as a function of the amplitude $a$ with the fixed parameter $n = 8$, $q = 2.7$. From top to bottom, the curves correspond to the values of $b = 0$, $b = 0.5$, and $b = 1$, respectively. In addition, the vertical dashed gridlines indicate the critical amplitude $a_c = 4.74$ (black), $a_c = 4.82$ (red) and $a_c = 4.98$ (blue), respectively. As one can see that there exists a critical amplitude $a_c$ above which
the condensate appears, then rises as the system is imposed with much larger amplitudes $a$. This behaviour of $b \neq 0$ is qualitatively similar to that in Maxwell theory with $b = 0$.

To investigate the connection between cosmic censorship and the weak gravity conjecture in Born-Infeld electrodynamics, we now study whether or not there exists a critical $q_{w}$, above which weak cosmic censorship could still be preserved. It is noteworthy that below the critical $q_{w}$ there still exists the scalar field condensate before reaching a singularity. Because now the weak gravity conjecture in Born-Infeld electrodynamics is still not clear, one could not determine whether the minimal value of charge-to-mass ratio is equal to weak gravity conjecture. However, we could still obtain a class of static solutions with charged scalar condensate in the case of Born-Infeld electrodynamic, which would prevent the violation of the weak cosmic censorship if sufficient charged particles were present.

After obtaining the numerical solution with charged scalar condensate, we present in the right panel of Fig. 4 the typical maximal value of Kretschmann scalar $K$ as a function of the charge $q$ with $b = 0.5$, $n = 8$ and $a = 6$, and the vertical black dashed gridlines indicate the $q_{c} = 2.276$ minimal value. From this plot, we can see that the full nonlinear solutions with the scalar condensate are in the range of $q > q_{c}$. It is obvious that there exists the growth of the maximum of Kretschmann scalar with the decreasing of $q$, which indicates the formation of a curvature singularity at some critical value of $q = q_{c}$, and is similar to the case of Maxwell case.

To study the properties of the bound on charge-to-mass ratio, in Fig. 5 we exhibit the phase diagram of the minimal charge $q_{min}$ versus the amplitude $a$ with the Born-Infeld parameter $b = 0.25$, $0.5$, $1$, represented by the red, blue and orange lines, respectively. The black line represents the curve in the Maxwell model which has been discussed in [13]. Comparing with the curve of $q_{min} - a$ curves, we can see that the minimum value of $q_{min}$ increases with the increasing of the
Born-Infeld parameter $b$, Moreover, the asymptotical value $q_{min}$ of the curve appears to approach some bound $q_w$ as $a$ increases. More details are shown in the inset, and the horizontal black, red, blue and orange dashed gridlines indicate the values of $q_w = 2, 2.23, 2.43, 2.708$, respectively. It is noteworthy that for the curve of Maxwell model in the solutions with $\Phi \neq 0$, the critical $q_{min}$ decreases firstly with the decreasing of the amplitude $a$, and then it reaches a minimal point. Further increasing $a$ to $a_{max}$, the value of $q_{min}$ continues to decrease, and a second branch with lower $q_{min}$ is obtained. When the curve with lower $q_{min}$ moves toward $a_{max}$, the numerical error begins to increase and a finer mesh is required to calculate. Though the curve of solutions with $\Phi = 0$ should connect with the curve with $\Phi \neq 0$, it is difficult to handle numerical calculation near the value of $a_{max}$. However, for larger value of $b$, it is relatively easy to obtain the connected curve near the value of $a_{max}$.

5 Conclusions

In this paper, we have presented the static solutions of Einstein gravity coupled with Born-Infeld electrodynamic and a free, massive scalar field in four-dimensional AdS spacetime. We find there exists a critical value of charge $q_{min}$, below which the static solution with zero scalar condensation is stability. However, when the charge of the scalar field was above $q_{min}$, the static solution of Born-Infeld action will become unstable and a new stable solution with nontrivial charged scalar field could appear, which is analogous to the instability of a charged static solution to develop scalar condensation in the case of Maxwell theory \[13\]. Meanwhile, there exists the asymptotical value of charge $q_w$ required to remove the counterexamples and preserve cosmic censorship. Comparing
with the Einstein-Maxwell model, the bound of charge-to-mass ratio \( q/m \) is larger than 1 and increases with the Born-Infeld parameter \( b \).

In the case of Maxwell theory \[13\], it is surprising to find that the minimum value of the charge to mass ratio which is necessary to preserve weak cosmic censorship is precisely equal to the weak gravity bound. For the nonlinear Born-Infeld theory, we also obtain the the minimum value of the charge to mass ratio necessary to preserve weak cosmic censorship, but, because now the weak gravity conjecture in Born-Infeld electrodynamics is still not clear, one could not determine whether the minimal value of charge-to-mass ratio is equal to weak gravity conjecture. However, it is sure that a class of static solution with charged scalar condensate could prevent the violation of the weak cosmic censorship, and the bound of charge-to-mass ratio could also be obtained in the case of Born-Infeld electrodynamics.

It will be interesting to investigate the several further researches. First, since we have studied the static solution with a free scalar condensate in Born-Infeld electrodynamics, we would like to investigate how self-interactions of the scalar field to prevent the violation of the weak cosmic censorship and the bound of charge-to-mass ratio. The second extension of our study is to construct generalized multi-scalar hair configurations, such as two coexisting states of the charge scalar field presented in Born-Infeld electrodynamics. Finally, we are planning to study the model of the Einstein-Born-Infeld-vector model and construct the static solution with charged vector hair necessary to preserve weak cosmic censorship in future work.

**Acknowledgement**

We would like to thank Yu-Xiao Liu, Jie Yang and Li Zhao for helpful discussion. YQW would like to thank J. Santos for discussions and correspondence about the numerical results. Some computations were performed on the shared memory system at institute of computational physics and complex systems in Lanzhou university. This work was supported by the Fundamental Research Funds for the Central Universities (Grants No. lzujbky-2017-182).

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