HOPF BIFURCATION AND STEADY-STATE BIFURCATION FOR A LESLIE-GOWER PREY-PREDATOR MODEL WITH STRONG ALLEE EFFECT IN PREY

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Abstract. It is well known that the Leslie-Gower prey-predator model (without Allee effect) has a unique globally asymptotically stable positive equilibrium point, thus there is no Hopf bifurcation branching from positive equilibrium point. In this paper we study the Leslie-Gower prey-predator model with strong Allee effect in prey, and perform a detailed Hopf bifurcation analysis to both the ODE and PDE models, and derive conditions for determining the steady-state bifurcation of PDE model. Moreover, by the center manifold theory and the normal form method, the direction and stability of Hopf bifurcation solutions are established. Finally, some numerical simulations are presented. Apparently, Allee effect changes the topology structure of the original Leslie-Gower model.

1. Introduction. The dynamical relationship between prey and predator has been a research hotspot in mathematical biology. So far, a lot of worthwhile prey-predator models have been proposed from theoretical and practical perspectives. In particular, Leslie [11, 12] introduced a prey-predator model where the carrying capacity of predator’s environment is proportional to the number of prey

\[
\begin{align*}
\frac{dH}{dt} &= (r_1 - a_1 P - b_1 H)H, \quad t > 0, \\
\frac{dP}{dt} &= \left( r_2 - \frac{a_2 P}{H} \right) P, \quad t > 0,
\end{align*}
\]

which is known as the second Leslie-Gower prey-predator model [23]. The parameters \( r_1, r_2, a_1, a_2, b_1 \) are positive constants; \( r_1 \) and \( r_2 \) are respectively the growth rate of the prey \( H \) and the predator \( P \); \( b_1 \) measures the strength of competition among individuals of species \( H \); \( a_1 \) is the maximal per capita consumption rate, i.e.,

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the maximum number of prey that can be eaten by a predator in each time unit; \(a_2\) is a measure of food quality that the prey provides for conversion into predator population. The system (1) has a unique coexisting equilibrium \((r_1r_2 + a_2b_1, r_1r_2 + a_2b_1)\), which is globally asymptotically stable [10]. The term \(a_2P/H\) is called the Leslie-Gower term. It measures the loss in the predator population due to rarity (per capita \(P/H\)) of its favorite food. The system (1) assumes that the predator subsistence depends on prey population exclusively. However, in literature [2], the predator \(P\) will switch over to other preys when its favorite food \(H\) is seriously scarce, and its growth will be limited when its favorite food is not available in abundance. This situation can be taken care of by adding a positive constant \(c\) to the denominator. Hence, the second equation of (1) becomes \(dP/dt = \left[r_2 - \frac{a_2P}{(H + c)}\right]P\), which is also known as modified Leslie-Gower model. For this scenario, its unique interior equilibrium still is globally asymptotically stable under certain conditions. The PDE version of the Leslie-Gower model also has been studied extensively, see [7, 17, 24, 29].

In 1931, Warder Clyde Allee proposed that intraspecific cooperation might lead to inverse density dependence. This idea was extended in his famous book [1] on animal ecology in 1949. Allee observed that many animal and plant species suffer a decrease of the per capita birthrate as their populations reach small sizes or low densities, which is now generally known as the Allee effect. This phenomenon has become crucial for population dynamics since in fact it has a surprising number of ramifications towards different branches of ecology [4, 25]. Recently, there have been some works concerning the Allee effect in the classical dynamic population models [5, 8, 13, 18, 19, 26]. Many interesting dynamical properties caused by the Allee effect are found which differ from the original system. Actually, Allee effect induces the appearance of a new equilibrium point which changes the stability of other equilibrium points. It shows that Allee effect changes the dynamic properties of original system significantly. In mathematical terms, Allee effect is expressed by modifying the natural growth function (usually the logistic growth function). The most common mathematical form describing this phenomenon for a single species is given by the equation [3]

\[
\frac{dx}{dt} = r \left(1 - \frac{x}{k}\right)(x - m)x,
\]

where \(x\) is the population size, \(r\) is the intrinsic growth rate and \(k\) is the environmental carrying capacity. The parameter \(m\) is a threshold population level. The equation (2) represents the strong Allee effect and weak Allee effect when \(m > 0\) and \(m \leq 0\), respectively. In this paper, we are interested in Leslie-Gower prey-predator model incorporating the strong Allee effect in prey, which is generally described by the Kolmogorov type differential system:

\[
\begin{cases}
  u_t = u(1-u)(u/b-1) - \beta uv, & t > 0, \\
  v_t = \mu v(1 - v/u), & t > 0, \\
  u(0) = u_0 > 0, & v(0) = v_0 \geq 0,
\end{cases}
\]

where \(u\) and \(v\) represent the densities of prey and predator, respectively. The positive constants \(\beta\) and \(\mu\) have the same meaning with \(a_1\) and \(r_2\) in system (1), respectively. The parameter \(b \in (0, 1)\) represents Allee threshold value.

On the other hand, the spatial component of ecological interactions has been recognized as an important factor in studying how ecological communities are shaped
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Considering spatially inhomogeneous distribution, a reaction-diffusion model corresponding to (3) can be written as follows

\[
\begin{cases}
  u_t = d_1 \Delta u + u(1 - u)(u/b - 1) - \beta uv, & x \in \Omega, \quad t > 0, \\
  v_t = d_2 \Delta v + \mu v(1 - v/u), & x \in \Omega, \quad t > 0, \\
  \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial \Omega, \quad t > 0, \\
  u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0, & x \in \Omega,
\end{cases}
\]

where \( \Omega \) is a bounded domain in \( \mathbb{R}^n(n \geq 1) \) with smooth boundary \( \partial \Omega \); the parameters \( d_1 \) and \( d_2 \) are respectively the diffusion coefficients of \( u \) and \( v \); \( \nu \) is the unit outward normal vector on the boundary \( \partial \Omega \). The homogeneous Neumann boundary condition means that the system (4) is self-contained and has zero population flux across the boundary \( \partial \Omega \). Ni and Wang give some estimates and the dynamical properties of the solution to (3) in [18] and investigate the stabilities of nonnegative constant steady states and the existence and non-existence of non-constant positive steady states of (4) in [19].

For a more comprehensive study of the dynamic properties of systems (3) and (4), motivated by the papers [6, 14, 33, 27], we will carry out the Hopf bifurcation analysis of (3) and (4) and provide the steady-state bifurcation analysis of (4) in this paper. The introduction of Allee effect will lead the systems (3) and (4) to have two equilibrium points at the interior of the first quadrant under certain conditions. Bifurcation analysis for the system with Allee effect have been very rare all the time due to the complexity of this situation. We have known that the Leslie-Gower prey-predator model without Allee effect (i.e. the system (1)) has a unique globally asymptotically stable positive equilibrium point for all parameters [10], thus there is no Hopf bifurcation branching from positive equilibrium point. However, in this paper, we find that the system (3) has Hopf bifurcation from one positive equilibrium point and the system (4) has Hopf bifurcation from two positive equilibrium points. Apparently, Allee effect changes the topology structure of the original Leslie-Gower system.

Now, we state some existing results (see [18, 19]) on systems (3) and (4) to prepare for the later research. For (3) and (4), there are two semi-trivial steady states \((b, 0), (1, 0)\) and some positive constant steady candidates:

(i) if \(1 + b - \beta b > 2\sqrt{b}\), there are two positive equilibria: \((\lambda^{(1)}, \lambda^{(1)}), (\lambda^{(2)}, \lambda^{(2)})\), where

\[
\lambda^{(1)} = \frac{1}{2} \left(1 + b - \beta b + \sqrt{(1 + b - \beta b)^2 - 4b}\right),
\]

\[
\lambda^{(2)} = \frac{1}{2} \left(1 + b - \beta b - \sqrt{(1 + b - \beta b)^2 - 4b}\right),
\]

(ii) if \(1 + b - \beta b = 2\sqrt{b}\), there is a unique positive equilibrium \((\lambda^{(3)}, \lambda^{(3)})\) with \(\lambda^{(3)} = \sqrt{b}\).

Define

\[
A(\lambda^{(k)}) = (1 + 1/b)\lambda^{(k)} - 2(\lambda^{(k)})^2/b, \quad k = 1, 2, 3,
\]

then the following relations hold naturally,

\[
A(\lambda^{(1)}) < \beta \lambda^{(1)}, \quad A(\lambda^{(2)}) > \beta \lambda^{(2)}, \quad A(\lambda^{(3)}) = \beta \lambda^{(3)}.
\]
The local stability of these constant steady states is clear. Actually, for system (3) and (4), steady states (1, 0), (b, 0) and (λ(2), λ(2)) are unstable and the positive equilibrium (λ(0), λ(0)) is unstable when μ < A(λ(0)). For system (3), the singularity (λ(1), λ(1)) is locally asymptotically stable when μ > A(λ(1)). For system (4), the singularity (λ(1), λ(1)) is locally asymptotically stable when μ > \max\{A(λ(1)), d_1^{-1}d_2A(λ(1))\} and it is unstable when μ < A(λ(1)).

Next, we declare some basic but useful facts based on the above results. Owing to the local stability of (λ(1), λ(1)) and the purpose for investigation of Hopf bifurcation, we will suppose that μ ≤ A(λ(1)) and μ < \max\{A(λ(1)), d_1^{-1}d_2A(λ(1))\} corresponding to (3) and (4), respectively. And obviously, above suppositions suggest that 0 < λ(1) < (b + 1)/2. In addition, on account of the existence condition and expressions of (λ(1), λ(1)) and (λ(2), λ(2)), we can get \sqrt{b} < λ(1) < (b + 1)/2 and b < λ(2) < \sqrt{b}. It is noteworthy that the images of A(λ(k)) (k = 1, 2, 3) are located in the same parabola A(λ) and they occupy the disjoint parts of this parabola respectively. In this paper, we shall fix μ and b and use λ(k) as the main bifurcation parameter (equivalently β as a bifurcation parameter) for (λ(k), λ(k)) in sections of Hopf bifurcation analysis while fix β and b and use μ as the bifurcation parameter in section of steady state bifurcation analysis. Throughout this article, we define

\[ b_0 = A(\sqrt{b}) = -2 + \sqrt{b} + \frac{1}{\sqrt{b}}, \quad b^0 = A\left(\frac{b + 1}{4}\right) = \frac{(b + 1)^2}{8b}. \]

Clearly, \( b_0 \leq b^0 \) according to image properties of A(λ).

The rest of the paper is arranged as follows. In Section 2, the existence and stability of Hopf bifurcating periodic solutions to (3) are established. In Section 3, we focus on the existence, stability and direction of Hopf bifurcation solutions to (4). In Section 4, the local structure of the steady-state bifurcation of (4) is presented. Finally, numerical simulations are shown in Section 5.

2. Hopf bifurcation of ODE problem (3). In this section, we shall study the existence, stability and direction for Hopf bifurcation of (3).

The Jacobi matrix of (3) at (λ(k), λ(k)) is

\[ L_0(λ(k)) = \begin{pmatrix} A(λ(k)) & -βλ(k) \\ μ & -μ \end{pmatrix}, \]

where \( A(λ(k)) \) is defined in (5) and \( k = 1, 2, 3 \). The characteristic equation of matrix \( L_0(λ(k)) \) is

\[ σ^2 - T(λ(k))σ + D(λ(k)) = 0, \]

where

\[ T(λ(k)) = A(λ(k)) - μ, \quad D(λ(k)) = μ(βλ(k) - A(λ(k))), \quad k = 1, 2, 3. \]

Analyzing the characteristic roots of \( L_0(λ(k)) \) with (6) and (8), we know that the singularity (λ(2), λ(2)) is a saddle point and \( L_0(λ(3)) \) has one zero eigenvalue at least. Thus, the system (3) can undergo Hopf bifurcation only surrounding (λ(1), λ(1)). Next, we analyze the Hopf bifurcation occurring from (λ(1), λ(1)) by treating λ(1) as bifurcation parameter (equivalently β as a bifurcation parameter).

Denote the eigenvalues of \( L_0(λ(1)) \) by \( σ(λ(1)) = α(λ(1)) ± iω(λ(1)) \) with α(λ(1)) = \( T(λ(1))/2 \) and ω(λ(1)) = \( \sqrt{D(λ(1)) - α^2(λ(1))} \). Let λ_{0,-}(λ(1)) < (b + 1)/4 and λ_{0,+}(λ(1)) > (b + 1)/4 be the possible real roots of \( A(λ(1)) = μ \) in the interval (\sqrt{b}, (b + 1)/2).
Note that $\lambda_{0,-}^{(1)}$ and $\lambda_{0,+}^{(1)}$ may not exist at the same time. If $\lambda^{(1)} = \lambda_{0, \pm}^{(1)}$, then $L_0(\lambda_{0, \pm}^{(1)})$ has a pair of imaginary eigenvalues $\pm i \omega(\lambda_{0, \pm}^{(1)})$.

Furthermore, a direct calculation gives

$$a'(\lambda^{(1)}) = \frac{1}{2} A' (\lambda^{(1)}) \begin{cases} > 0 & \text{if } 0 < \lambda^{(1)} < \frac{b+1}{4}, \\ < 0 & \text{if } \frac{b+1}{4} < \lambda^{(1)} < \frac{b+1}{2}. \end{cases} \tag{9}$$

Hence the transversality condition is always satisfied as long as $b \neq (b+1)/4$. By Poincaré-Andronov-Hopf bifurcation Theorem, if $0 < \mu < b^0$ and $\lambda_{0,-}^{(1)} (\lambda_{0,+}^{(1)})$ exists, then the system (3) undergoes a Hopf bifurcation at $(\lambda^{(1)}, \lambda^{(1)})$ when $\lambda^{(1)} = \lambda_{0,-}^{(1)} (\lambda_{0,+}^{(1)})$. Due to the fact that the number of roots to $A(\lambda^{(1)}) = \mu$ is determined by the range of $b$ and $\mu$, the number of Hopf bifurcation points is also related to $b$ and $\mu$. Actually, if $0 < b < 7 - 4\sqrt{3}$ satisfying $\sqrt{6} < (b+1)/4$, then $A(\lambda^{(1)}) = \mu$ may have one or two real roots; if $7 - 4\sqrt{3} < b < 1$ satisfying $\sqrt{6} > (b+1)/4$, then $A(\lambda^{(1)}) = \mu$ may have only one real root (see Table 1).

Table 1: Hopf bifurcation values of ODE problem (3)

| $0 < b < b_1$ | $b_1 < b < 1$ |
|---------------|---------------|
| $0 < \mu < b_0$ | One Hopf bifurcation value $\lambda_{0,+}^{(1)}$ |
| $b_0 < \mu < b^0$ | Two Hopf bifurcation values $\lambda_{0,-}^{(1)}, \lambda_{0,+}^{(1)}$ |
| $\mu > b^0$ | Null |

| $b_1 = 7 - 4\sqrt{3}$ |
|------------------------|
| One Hopf bifurcation value $\lambda_{0,+}^{(1)}$ |

Summarizing the above analysis, we obtain the main result of this section:

**Theorem 2.1.** Assume that the parameters $\beta$, $\mu > 0$ and $0 < b < 1$.

1. If $0 < \mu < b_0$, then the system (3) undergoes a Hopf bifurcation from $(\lambda^{(1)}, \lambda^{(1)})$ when $\lambda^{(1)} = \lambda_{0, \pm}^{(1)}$.

2. If $0 < b < 7 - 4\sqrt{3}$ and $b_0 < \mu < b^0$, then the system (3) undergoes a Hopf bifurcation from $(\lambda^{(1)}, \lambda^{(1)})$ when $\lambda^{(1)} = \lambda_{0,-}^{(1)}$ or $\lambda^{(1)} = \lambda_{0,+}^{(1)}$.

3. If $0 < b < 7 - 4\sqrt{3}$ and $\mu = b_0$, then the system (3) undergoes a Hopf bifurcation from $(\lambda^{(1)}, \lambda^{(1)})$ when $\lambda^{(1)} = \lambda_{0,+}^{(1)}$; if $7 - 4\sqrt{3} \leq b < 1$ and $\mu = b_0$, there is no bifurcation point for (3).
The detailed natures of Hopf bifurcation need further analysis of the normal form of (3). In the following, we adopt the same notations and computations as in [31, 32] to study the direction and stability of Hopf bifurcation.

Theorem 2.2. Suppose that the system (3) undergoes a Hopf bifurcation from \((\lambda^{(1)}), \lambda^{(1)})\) when \(\lambda^{(1)} = \lambda_0\). Then the following statements hold true.

1. When \(0 < \lambda_0 < (b + 1)/4\). If \(a(\lambda_0) < 0\) \((> 0)\), then the bifurcating periodic solutions are stable \((unstable)\) and the direction of Hopf bifurcation is supercritical \((subcritical)\), where \(a(\lambda_0)\) will be given in the proof.

2. When \((b + 1)/4 < \lambda_0 < b + 1\). If \(a(\lambda_0) < 0\) \((> 0)\), then the bifurcating periodic solutions are stable \((unstable)\) and the direction of Hopf bifurcation is subcritical \((supercritical)\).

Proof. Let \(\pm i\omega_0\) be the imaginary eigenvalues of \(L_0(\lambda_0)\). We translate (3) into the following system by \(\hat{u} = u - \lambda^{(1)}\) and \(\hat{v} = v - \lambda^{(1)}\), and still let \(u\) and \(v\) denote \(\hat{u}\) and \(\hat{v}\), respectively. Then we have

\[
\begin{align*}
\begin{cases}
u_t = f(u,v,\lambda^{(1)}), & t > 0, \\
v_t = g(u,v,\lambda^{(1)}), & t > 0,
\end{cases}
\end{align*}
\]

or equivalently

\[
\begin{pmatrix} u_t \\ v_t \end{pmatrix} = L_0(\lambda^{(1)}) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f(u,v,\lambda^{(1)}) - A(\lambda^{(1)})u + \beta\lambda^{(1)}v \\ g(u,v,\lambda^{(1)}) - \mu u + \mu v \end{pmatrix}, \quad t > 0, \quad (10)
\]

where

\[
f(u,v,\lambda^{(1)}) = (u + \lambda^{(1)})(1 - u - \lambda^{(1)})\left(\frac{u + \lambda^{(1)}}{b} - 1\right) - \beta(u + \lambda^{(1)})(v + \lambda^{(1)}),
\]

\[
g(u,v,\lambda^{(1)}) = \mu(v + \lambda^{(1)})\left(1 - \frac{v + \lambda^{(1)}}{u + \lambda^{(1)}}\right).
\]

Define a matrix

\[
B = \begin{pmatrix} 1 & 0 \\ N & M \end{pmatrix}
\]

with \(M = \frac{\omega(\lambda^{(1)})}{\beta\lambda^{(1)}}\), \(N = \frac{A(\lambda^{(1)}) - \alpha(\lambda^{(1)})}{\beta\lambda^{(1)}}\).

Clearly,

\[
B^{-1} = \begin{pmatrix} 1 & 0 \\ N & 1 \\ M & 1 \end{pmatrix}.
\]

By the transformation

\[
\begin{pmatrix} u \\ v \end{pmatrix} = B \begin{pmatrix} x \\ y \end{pmatrix},
\]

the system (10) becomes

\[
\begin{pmatrix} x_t \\ y_t \end{pmatrix} = \begin{pmatrix} \alpha(\lambda^{(1)}) & -\omega(\lambda^{(1)}) \\ \omega(\lambda^{(1)}) & \alpha(\lambda^{(1)}) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} F^1(x,y,\lambda^{(1)}) \\ F^2(x,y,\lambda^{(1)}) \end{pmatrix},
\]

where

\[
\begin{align*}
F^1(x,y,\lambda^{(1)}) &= f(u,v,\lambda^{(1)}) - A(\lambda^{(1)})u + \beta\lambda^{(1)}v, \\
F^2(x,y,\lambda^{(1)}) &= g(u,v,\lambda^{(1)}) - \mu u + \mu v.
\end{align*}
\]
where

\[ F^1(x, y, \lambda^{(1)}) = -\frac{1}{b}x^3 + \left(\frac{1}{b} - \frac{3}{b}\lambda^{(1)} + 1 - \beta N\right)x^2 - \beta Mxy + \cdots, \]

\[ F^2(x, y, \lambda^{(1)}) = \frac{N}{bM}x^3 + \left[\frac{\lambda^{(1)}\beta N^2 - \mu + 2N\mu - \mu N^2}{\lambda^{(1)}M} - \frac{N}{M}\left(\frac{1}{b} - \frac{3}{b}\lambda^{(1)} + 1\right)\right]x^2 \]

\[ + \left(\beta N - \frac{2\mu}{\lambda^{(1)}} + \frac{2\mu}{\lambda^{(1)}}\right)xy - \frac{M\mu}{\lambda^{(1)}}y^2 - \frac{2\mu}{(\lambda^{(1)})^2}x^2y + \cdots. \]

In order to obtain the stability of Hopf bifurcation periodic solutions, we need to calculate the sign of \(a(\lambda_0)\) given by

\[ a(\lambda_0) = \frac{1}{16}\left(F^1_{xxx} + F^1_{xyy} + F^2_{xy} + F^2_{yy}\right) \]

\[ + \frac{1}{\omega_0}\left[F^1_{xy}(F^1_{xx} + F^1_{yy}) - F^2_{xy}(F^2_{xx} + F^2_{yy}) - F^1_{xx}F^2_{xy} + F^1_{yy}F^2_{xy}\right], \]

where all partial derivatives are evaluated at the bifurcation point, i.e., \((x, y, \lambda^{(1)}) = (0, 0, \lambda_0)\). It is easy to calculate that

\[ F^1_{xxx}(0, 0, \lambda_0) = \frac{6}{b}, \quad F^2_{xy}(0, 0, \lambda_0) = -\frac{4\mu}{\lambda_0}, \quad F^1_{xyy}(0, 0, \lambda_0) = F^2_{yy}(0, 0, \lambda_0) = 0, \]

\[ F^1_{xx}(0, 0, \lambda_0) = -\frac{2}{b}\lambda_0, \quad F^1_{x}(0, 0, \lambda_0) = -\frac{\omega_0}{\lambda_0}, \quad F^1_{y}(0, 0, \lambda_0) = 0, \]

\[ F^2_{xx}(0, 0, \lambda_0) = 2\left[\frac{3\mu^2}{\omega_0\lambda_0} - \frac{\beta\mu}{\omega_0} - \frac{\mu^3}{\omega_0\beta\lambda_0^2} - \frac{\mu}{\omega_0}\left(\frac{1}{b} - \frac{3}{b}\lambda_0 + 1\right)\right], \]

\[ F^2_{xy}(0, 0, \lambda_0) = \frac{3\mu}{\lambda_0} - \frac{2\mu^2}{\beta\lambda_0^2}, \quad F^2_{yy}(0, 0, \lambda_0) = -\frac{2\mu\omega_0}{\beta\lambda_0^2}. \]

By tedious but simple calculations, we can obtain

\[ a(\lambda_0) = -\frac{1}{4b} - \frac{\mu}{4\lambda_0} - \frac{b}{8\lambda_0^2}\left(\frac{3\lambda_0}{b} + 1\right) \left(\frac{2\mu^2}{\lambda_0} - \frac{2\mu^2}{\beta\lambda_0^2} + \frac{2\lambda_0}{\mu}\right) \]

\[ + \frac{1}{8\omega_0}\left(\frac{2\mu^2}{\beta\lambda_0^2} - \frac{3\mu^2}{\lambda_0} + \frac{2\lambda_0}{b}\right) \left(\frac{3\mu^2}{\lambda_0} - \frac{\mu^3}{\beta\lambda_0^2} - \beta\mu\right). \]

From (9), the above calculation of \(a(\lambda_0)\) and the Poincaré-Andronov-Hopf Theorem, the desired result can be deduced. □

3. Hopf bifurcation of PDE problem (4). In this section, we shall investigate the existence, stability, and direction of Hopf bifurcation solutions to (4). Here we only consider the one-dimensional case. Without loss of generality, \(\Omega\) is restricted on the one-dimensional space domain \((0, l\pi)\) with \(l \in \mathbb{R}^+\).

Define the real-valued Sobolev space

\[ X = \{(u, v) \in H^2(0, l\pi) \times H^2(0, l\pi) : (u_x, v_x)|_{x=0, l\pi} = (0, 0)\}, \]

and corresponding complexification

\[ X_C = X \oplus iX = \{x_1 + ix_2 : x_1, x_2 \in X\}. \]

Consider the linearization of (4) near \((\lambda^{(k)}, \lambda^{(k)})\):

\[ L(\lambda^{(k)}) = \begin{pmatrix} A(\lambda^{(k)}) + d_1\Delta & -\beta\lambda^{(k)} \\ -\beta\lambda^{(k)} & -\mu + d_2\Delta \end{pmatrix} \]
where $A(\lambda^{(k)})$ is given by (5) and $k = 1, 2, 3$, $n = 0, 1, 2, \ldots$.

The characteristic equation of $L_n(\lambda^{(k)})$ is

$$
\sigma^2 - T_n(\lambda^{(k)})\sigma + D_n(\lambda^{(k)}) = 0, \quad n = 0, 1, 2, \ldots,
$$

where

$$
\begin{align*}
T_n(\lambda^{(k)}) &= A(\lambda^{(k)}) - \mu - (d_1 + d_2)\frac{n^2}{l^2}, \\
D_n(\lambda^{(k)}) &= d_1 d_2 \frac{n^4}{l^4} + (d_1 \mu - d_2 A(\lambda^{(k)}))\frac{n^2}{l^2} + \mu(\beta \lambda^{(k)} - A(\lambda^{(k)})).
\end{align*}
$$

Moreover, the eigenvalues of $L_n(\lambda^{(k)})$ are $\alpha(\lambda^{(k)}) \pm i\omega(\lambda^{(k)})$, where

$$
\alpha(\lambda^{(k)}) = \frac{A(\lambda^{(k)}) - \mu}{2} - \frac{(d_1 + d_2)n^2}{2l^2}, \quad \omega(\lambda^{(k)}) = \sqrt{D_n(\lambda^{(k)}) - \alpha^2(\lambda^{(k)})}.
$$

Furthermore, a routine computation gives rise to

$$
\alpha'(\lambda^{(k)}) = \frac{1}{2} A'(\lambda^{(k)}) \begin{cases}
> 0, & \text{if } 0 < \lambda^{(k)} < \frac{b + 1}{4}, \\
< 0, & \text{if } \frac{b + 1}{4} < \lambda^{(k)} < \frac{b + 1}{2}.
\end{cases}
$$

To determine Hopf bifurcation value $\lambda_0$, we recall the following sufficient condition from [33]:

(H1) There exists an integer $n_0 \in \mathbb{N}$ such that

$$
T_{n_0}(\lambda_0) = 0, \quad D_{n_0}(\lambda_0) > 0 \quad \text{and} \quad T_n(\lambda_0) \neq 0, \quad D_n(\lambda_0) \neq 0 \quad \text{for} \quad n \neq n_0;
$$

and the unique pair of complex eigenvalues near the imaginary axis $\alpha(\lambda) \pm i\omega(\lambda)$ satisfies transversality condition

$$
\alpha'(\lambda_0) \neq 0.
$$

It can be judged from the third equality of (6) and the above condition (H1) that it is hard to occur Hopf bifurcation from $(\lambda^{(3)}, \lambda^{(3)})$ for (4). Next, we will establish existence conditions of Hopf bifurcation branching from positive equilibria $(\lambda^{(1)}, \lambda^{(1)})$ and $(\lambda^{(2)}, \lambda^{(2)})$.

Let $\lambda^{(k)}_n$ be the possible real root of $A(\lambda^{(k)}) - \mu - (d_1 + d_2)\frac{n^2}{l^2} = 0$, $k = 1, 2$, $n = 0, 1, 2, \ldots$. If $0 < \lambda^{(k)}_n < (b+1)/4$ then $\lambda^{(k)}_n = \lambda^{(k)}_{n-}$ and if $(b+1)/4 < \lambda^{(k)}_n < (b+1)/2$ then $\lambda^{(k)}_n = \lambda^{(k)}_{n+}$. Apparently, $\lambda^{(1)}_{n\pm}$ and $\lambda^{(2)}_{n\pm}$ are located in interval $(\sqrt{b} \frac{b+1}{2})$ and $(b, \sqrt{b})$, respectively. As $\lambda^{(1)}_{n\pm}$ in previous section, $\lambda^{(1)}_{n\pm}$ $(\lambda^{(2)}_{n\pm})$ and $\lambda^{(1)}_{n\pm}$ $(\lambda^{(2)}_{n\pm})$ may not exist together, either. Recalling the definition of $b_0$ and $b^0$ in (7), we define the
Theorem 3.1. For the system (4), let the parameters $d_1$, $d_2$, $l$, $\beta$, $\mu > 0$ and
$0 < b < 1$ be fixed.
1. When $0 < b < 7 - 4\sqrt{3}$ and $d_1^{-1}d_2b^0 \leq \mu < b^0$, the system undergoes a Hopf
bifurcation at $\lambda^{(1)} = \lambda^{(1)}_{n,\pm}$ for $n \in \Sigma_1$; the system undergoes a Hopf bifurcation at
$\lambda^{(1)} = \lambda^{(1)}_{n,\pm}$ for $n \in \Sigma_3$. Moreover, there are $s_1 + 2s_3$ Hopf bifurcation points.
2. When $7 - 4\sqrt{3} < b < 1$ and $d_1^{-1}d_2b^0 \leq \mu < b_0$, the system undergoes a Hopf
bifurcation at $\lambda^{(1)} = \lambda^{(1)}_{n,\pm}$ for $n \in \Sigma_2$. Moreover, there are $s_2$ Hopf bifurcation points.

Proof. 1. When $d_1^{-1}d_2b^0 \leq \mu < b^0$, it is easy to check $\Sigma_1 \cup \Sigma_3 \neq \emptyset$ due to $0 \in
\Sigma_1 \cup \Sigma_3$. Without loss of generality, we may assume that $\Sigma_1 \neq \emptyset$ and $\Sigma_3 \neq \emptyset$. For
some $\hat{n} \in \Sigma_1$, the equation $A(\lambda^{(1)}) - \mu - (d_1 + d_2)n^2/l^2 = 0$ must have one root
$\lambda^{(1)}_{\hat{n},\pm}$ in $(\sqrt{b}, (b+1)/2)$, which implies that

$$T_{\hat{n}}(\lambda^{(1)}_{\hat{n},\pm}) = 0, \quad T_n(\lambda^{(1)}_{\hat{n},\pm}) \neq 0 \quad \text{for } n \neq \hat{n}. $$

Similarly, for some $\tilde{n} \in \Sigma_3$, the equation $A(\lambda^{(1)}) - \mu - (d_1 + d_2)n^2/l^2 = 0$ must have
two roots $\lambda^{(1)}_{\tilde{n},-}$ and $\lambda^{(1)}_{\tilde{n},+}$ in $(\sqrt{b}, (b+1)/2)$. Moreover, $\lambda^{(1)}_{\tilde{n},\pm}$ satisfy

$$T_{\tilde{n}}(\lambda^{(1)}_{\tilde{n},\pm}) = 0, \quad T_n(\lambda^{(1)}_{\tilde{n},\pm}) \neq 0 \quad \text{for } n \neq \tilde{n}. $$

On the other hand, by virtue of the assumption $\mu \geq d_1^{-1}d_2b^0$ and the first
inequality of (6), we can check that $D_n(\lambda^{(1)}_{\hat{n},\pm}) > 0$ and $D_n(\lambda^{(1)}_{\tilde{n},\pm}) > 0$ for any
nonnegative integer $n$. Moreover, since $\mu \neq b^0$ and (12), we have $\alpha'(\lambda^{(1)}_{\hat{n},\pm}) \neq 0$ and
$\alpha'(\lambda^{(1)}_{\tilde{n},\pm}) \neq 0$. Hence, the conclusion is followed from (H1) and the definitions of $s_1$
and $s_3$.

2. The condition $d_1^{-1}d_2b^0 \leq \mu < b_0$ implies that the set $\Sigma_2$ is not empty. The
remainder of the argument is analogous to that in case 1, so we omit it. □

Theorem 3.1 gives conditions under which the system (4) occurs Hopf bifurcation from ($\lambda^{(1)}$, $\lambda^{(1)}$). Actually, the similar conclusions for ($\lambda^{(2)}$, $\lambda^{(2)}$) also can be obtained by the same method, but it is very complicated since too many cases of $b$. In addition, note that Theorem 3.1 only gives the number of Hopf bifurcation
points but does not give the order of them. Therefore, we will not repeat above process for ($\lambda^{(2)}$, $\lambda^{(2)}$). In the following, we will use another method to obtain not only more detailed existence conditions of Hopf bifurcation from ($\lambda^{(1)}$, $\lambda^{(1)}$) and ($\lambda^{(2)}$, $\lambda^{(2)}$) but also the exact order of bifurcation points.
Define
\[ l^b_r = r \sqrt{\frac{d_1 + d_2}{b^0 - \mu}}, \quad l^b_m = m \sqrt{\frac{d_1 + d_2}{b^0 - \mu}}, \quad l_k^{-b} = k \sqrt{\frac{d_1 + d_2}{1 - b - \mu}}, \quad r, m, k \in \mathbb{N}. \quad (13) \]

**Theorem 3.2.** Assume that \( d_1, d_2, \mu, l, \beta > 0 \) and \( 0 < b < 1 \). The constants \( l^b_r \), \( l^b_m \) and \( l_k^{-b} \) are defined in \((13)\).

1. When \( 0 < b < 7 - 4\sqrt{3} \), the following states hold true.

   (i) If \( d_1^{-1}d_2b^0 \leq \mu \leq b_0 \), for any \( l \in (l^b_r, l^b_{r+1}] \cap [l^b_m, l^b_{m+1}) \), then there exist
   \[ 2r - m + 1 \] 
   Hopf bifurcation points which are denoted by \( \lambda^{(1)}_{n_+} \) or \( \lambda^{(1)}_{n_-} \) with \( n = 0, 1, \cdots, r \). Moreover, these Hopf bifurcation points satisfy
   \[ \sqrt{b} < \lambda^{(1)}_{n-} < \cdots < \lambda^{(1)}_{r-} < \frac{b + 1}{4} < \lambda^{(1)}_{r+} < \cdots < \lambda^{(1)}_{m+1} < \lambda^{(1)}_{m+1, +} < \cdots < \lambda^{(1)}_{0+} < \frac{b + 1}{2}. \]

   (ii) If \( \max\{d_1^{-1}d_2b^0, b_0\} < \mu < b^0 \), for any \( l \in (l^b_r, l^b_{r+1}] \), then there exist \( 2r + 2 \)
   Hopf bifurcation points which are denoted by \( \lambda^{(1)}_{n+} \) or \( \lambda^{(1)}_{n-} \) with \( n = 0, 1, \cdots, r \).
   Moreover, these Hopf bifurcation points satisfy
   \[ \sqrt{b} < \lambda^{(1)}_{n-} < \cdots < \lambda^{(1)}_{r-} < \frac{b + 1}{4} < \lambda^{(1)}_{r+} < \cdots < \lambda^{(1)}_{m+} < \lambda^{(1)}_{m+1} < \cdots < \lambda^{(1)}_{0+} < \frac{b + 1}{2}. \]

2. When \( 7 - 4\sqrt{3} \leq b < 1 \) and \( d_1^{-1}d_2b^0 \leq \mu \leq b_0 \), for any \( l \in (l^b_m, l^b_{m+1}] \), there exist \( m + 1 \) Hopf bifurcation points which are denoted by \( \lambda^{(1)}_{n+} \) with \( n = 0, 1, \cdots, m \).
   Moreover, these Hopf bifurcation points satisfy
   \[ \frac{b + 1}{4} \leq \sqrt{b} < \lambda^{(1)}_{m+} < \cdots < \lambda^{(1)}_{0+} < \frac{b + 1}{2}. \]

**Proof.** We just give the proof of (i) and that of other cases are similar to (i). When \( d_1^{-1}d_2b^0 \leq \mu \leq b_0 \), for any \( l \in (l^b_r, l^b_{r+1}] \cap [l^b_m, l^b_{m+1}) \), the equation
   \[ A(\lambda^{(1)}) - \mu - (d_1 + d_2) \frac{n^2}{l^2} = 0, \quad n = 0, 1, 2, \cdots, r, \]
   has one root when \( 0 \leq n \leq m \) and two roots when \( m + 1 \leq n \leq r \) in \((\sqrt{b}, (b + 1)/2)\) (see table 2). Therefore, there are \( 2r - m + 1 \) roots satisfying
   \[ \sqrt{b} < \lambda^{(1)}_{m+1, -} \cdots \lambda^{(1)}_{r-} < \frac{b + 1}{4} < \lambda^{(1)}_{r+} \cdots < \lambda^{(1)}_{m+1} < \lambda^{(1)}_{m+1, +} \cdots < \lambda^{(1)}_{0+} < \frac{b + 1}{2}. \]

Clearly,
   \[ T_n(\lambda^{(1)}_{n-}) = 0, \quad \forall m + 1 \leq n \leq r; \quad T_n(\lambda^{(1)}_{n+}) = 0, \quad \forall 0 \leq n \leq r; \]
   and
   \[ T_j(\lambda^{(1)}_{n-}) \neq 0, \quad \forall j \neq n, \quad m + 1 \leq n \leq r \quad \text{and} \quad T_j(\lambda^{(1)}_{n+}) \neq 0, \quad \forall j \neq n, \quad 0 \leq n \leq r. \]

In view of \( \mu \geq d_1^{-1}d_2b^0 \) and the first inequality of \((6)\), for all \( j \in \mathbb{N} \), we get
   \[ D_j(\lambda^{(1)}_{n-}) > 0, \quad \forall m + 1 \leq n \leq r; \quad D_j(\lambda^{(1)}_{n+}) > 0, \quad \forall 0 \leq n \leq r. \]

Furthermore, thanks to \( 0 < b < 7 - 4\sqrt{3} \) satisfying \( \sqrt{b} < (b + 1)/4 \), it follows that \( b_0 < b^0 \) and thus \( \mu \neq b^0 \). Accordingly, based on \((12)\), we know that the transversality condition holds.

Hence, the conclusion (i) is evident from what we have proved. \( \square \)
Remark 1. If $0 < l \leq l_*$, then the system (4) only have homogeneous Hopf bifurcation points $\lambda_{0,-}^{(1)}$ or $\lambda_{0,+}^{(1)}$, where

$$l_* = \frac{2\sqrt{2b(d_1 + d_2)}}{b + 1}$$

is called minimal spatial size for the system (4) to have a periodic spatial pattern. It implies that the system needs enough space to occur non-homogeneous Hopf bifurcation and more periodic patterns are showed as $l$ grows.

Table 2: Hopf bifurcation values for $(\lambda^{(1)}, \lambda^{(1)})$ in PDE problem (4)

| $\lambda^{(1)}$ | $d_1^{-1}d_2b^0 < \mu < \lambda_0$ | $\max(d_1^{-1}d_2b^0, \lambda_0) < \mu < b^0$ | $\mu > b^0$ |
|-----------------|----------------------------------|----------------------------------|-------------|
| $0 < b < b_1$   | ![Graph 1](image1.png)            | ![Graph 2](image2.png)            | Null        |
| $b_1 < b < 1$   | ![Graph 3](image3.png)            | ![Graph 4](image4.png)            | Null        |
| $b_1 = 7 - 4\sqrt{3}$, $b_2 = \mu + (d_1 + d_2)^2/12$ | ![Graph 5](image5.png)            | ![Graph 6](image6.png)            | Null        |

From the proof of Theorem 3.2, we find that the assumption $\mu \geq d_1^{-1}d_2b^0$ and the first inequality of (6) ensure $D_j(\lambda^{(1)}_n) > 0$. However, the second inequality of (6) does not ensure $D_j(\lambda^{(2)}_n) > 0$ for $(\lambda^{(2)}, \lambda^{(2)})$. Thus, for $(\lambda^{(2)}, \lambda^{(2)})$, we need to verify whether $D_j(\lambda^{(2)}_n) \neq 0$ for $j \in \mathbb{N}$, and in particular, $D_j(\lambda^{(2)}_n) > 0$.

In fact, there is no spatially homogeneous Hopf bifurcation branching from $(\lambda^{(2)}, \lambda^{(2)})$ because of $D_0(\lambda^{(2)}_0) < 0$. Here we drive a condition on the parameters so that $D_j(\lambda^{(2)}_n) > 0$ for any $j \geq 1$. Obviously, if $D_1(\lambda^{(2)}_n) > 0$ then $D_j(\lambda^{(2)}_n) > 0$ for any $j > 1$. Combining with $A(\lambda^{(2)}_n) - \mu - (d_1 + d_2)n^2/l^2 = 0$ and the expression and range of $\lambda^{(2)}$, it is easy to have

$$D_1(\lambda^{(2)}_n) = d_1d_2 \frac{1}{l^4} + [d_1\mu - d_2A(\lambda^{(2)}_n)] \frac{1}{l^2} + \mu[\beta(\lambda^{(2)}_n)^2 - A(\lambda^{(2)}_n)]$$

$$= d_1d_2 \frac{1}{l^4} + \left[d_1\mu - d_2 \left(\mu + (d_1 + d_2)\frac{n^2}{l^2}\right) \right] \frac{1}{l^2} + \mu \left[\frac{1}{b}(\lambda^{(2)}_n)^2 - 1\right]$$

$$\geq d_1d_2 \frac{1}{l^4} + \left[d_1\mu - d_2 \left(\mu + (d_1 + d_2)\frac{n^*2}{l^2}\right) \right] \frac{1}{l^2} + \mu(b - 1),$$
Table 3: Hopf bifurcation values for \((\lambda^{(2)}, \lambda^{(2)})\) in PDE problem (4)

| \(0 < b < b_1\) | \(0 < \mu < 1 - b\) | \(1 - b < \mu < b_0\) | \(b_0 < \mu < b^0\) |
|---|---|---|---|
| \(m - k \) Hopf bifurcation values | \(m \) Hopf bifurcation values | \(\) Null |

\(b_1 = 7 - 4\sqrt{3}, \ b_2 = 3 - 2\sqrt{3}, \ h_1 = \mu + (d_1 + d_2)^2 / l^2\)

Table 4: Hopf bifurcation values for \((\lambda^{(2)}, \lambda^{(2)})\) in PDE problem (4)

| \(b_2 < b < \frac{1}{4}\) | \(0 < \mu < b_0\) | \(b_0 < \mu < 1 - b\) | \(1 - b < \mu < b^0\) |
|---|---|---|---|
| \(2r - m - k \) Hopf bifurcation values | \(2r - m \) Hopf bifurcation values | \(2r \) Hopf bifurcation values |

\(b_2 = 3 - 2\sqrt{3}, \ h_1 = \mu + (d_1 + d_2)^2 / l^2\)

\(\frac{1}{4} < b < 1\)

| \(k - m \) Hopf bifurcation values | \(k \) Hopf bifurcation values | \(\) Null |

\(h_2 = \mu + (d_1 + d_2)^2 / l^2\)

where

\[ n^* = \left\lfloor \sqrt{\frac{b^0 - \mu}{d_1 + d_2}} \right\rfloor, \]

and \([\cdot]\) is integral function, i.e., \([x]\) is the largest integer less or equal to \(x\). Therefore, for any \(j \geq 1\), \(D_j(\lambda_0^{(2)}) > 0\) holds if

\[ d_1 d_2 \frac{1}{l^2} + \left[ d_1 \mu - d_2 \left( \mu + (d_1 + d_2) \frac{n^*}{l^2} \right) \right] \frac{1}{l^2} + \mu(b - 1) > 0. \] (14)
Similar to Theorem 3.2, based on the above analysis, we can get the following Theorem 3.3 about Hopf bifurcation from \((\lambda^{(2)}, \lambda^{(2)})\) with help of Table 3 and Table 4.

**Theorem 3.3.** Suppose that \(d_1, d_2, \beta, l, \mu > 0\) and \(0 < b < 1\) satisfy inequality (14) and \(t_k^0, t_k, l_{k+1}^1\) are defined in (13), then the following statements are true.

1. When \(0 < b \leq 7 - 4\sqrt{3}\).
   (i) If \(\mu < 1 - b\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(m - k\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) with \(n = k + 1, \ldots, m\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{l+1,-}^{(2)} < \cdots < \lambda_{m,-}^{(2)} < \sqrt{b}.\]
   (ii) If \(1 - b \leq \mu < b_0\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(m\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) with \(n = 1, \ldots, m\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{1,-}^{(2)} < \cdots < \lambda_{m,-}^{(2)} < \sqrt{b}.\]

2. When \(7 - 4\sqrt{3} < b \leq 3 - 2\sqrt{3}\).
   (i) If \(\mu < 1 - b\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(2r - m - k\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) or \(\lambda_{n,+}^{(2)}\) with \(n = k + 1, \ldots, r\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{k+1,-}^{(2)} < \cdots < \lambda_{m+1,-}^{(2)} < \cdots < \lambda_{r,-}^{(2)} < \frac{b + 1}{4} < \lambda_{r,+}^{(2)} < \cdots < \lambda_{m+1,+}^{(2)} < \sqrt{b}.\]
   (ii) If \(1 - b \leq \mu < b_0\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(2r - m\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) or \(\lambda_{n,+}^{(2)}\) with \(n = 1, \ldots, r\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{1,-}^{(2)} < \cdots < \lambda_{m+1,-}^{(2)} < \cdots < \lambda_{r,-}^{(2)} < \frac{b + 1}{4} < \lambda_{r,+}^{(2)} < \cdots < \lambda_{m+1,+}^{(2)} < \sqrt{b}.\]
   (iii) If \(b_0 \leq \mu < b_0\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(2r\) Hopf bifurcation points which are denoted by \(\lambda_{n,\pm}^{(2)}\) with \(n = 1, \ldots, r\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{1,-}^{(2)} < \cdots < \lambda_{r,-}^{(2)} < \frac{b + 1}{4} < \lambda_{r,+}^{(2)} < \cdots < \lambda_{1,+}^{(2)} < \sqrt{b}.\]

3. When \(3 - 2\sqrt{2} < b < 1/3\).
   (i) If \(\mu < b_0\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(2r - m - k\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) or \(\lambda_{n,+}^{(2)}\) with \(n = m + 1, \ldots, r\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{k+1,-}^{(2)} < \cdots < \lambda_{r,-}^{(2)} < \frac{b + 1}{4} < \lambda_{r,+}^{(2)} < \cdots < \lambda_{m+1,+}^{(2)} < \sqrt{b}.\]
   (ii) If \(b_0 \leq \mu < 1 - b\), for any \(l \in (t_{k+1}^0, t_{k+1}^0] \cap [l_{k+1}^1, l_{k+1}^1]\), then there exist \(2r - k\) Hopf bifurcation points which are denoted by \(\lambda_{n,-}^{(2)}\) or \(\lambda_{n,+}^{(2)}\) with \(n = 1, \ldots, r\). Moreover, these Hopf bifurcation points satisfy
   \[b < \lambda_{k+1,-}^{(2)} < \cdots < \lambda_{r,-}^{(2)} < \frac{b + 1}{4} < \lambda_{r,+}^{(2)} < \cdots < \lambda_{k+1,+}^{(2)} < \cdots < \lambda_{1,+}^{(2)} < \sqrt{b}.\]
and respectively by two following weaker conditions

For cases 1 and 4 in Theorem 3.3, the condition (14) can be replaced

Remark 2. For cases 1 and 4 in Theorem 3.3, the condition (14) can be replaced respectively by two following weaker conditions

\[
\frac{b+1}{4} < \lambda_{r_+}^{(2)} < \lambda_{r_+}^{(2)} + \frac{b}{4} < \lambda_{r_+}^{(2)} + \frac{b+1}{4} < \sqrt{b}.
\]

4. When $1/3 \leq b < 1$.

(i) If $\mu < b_0$, for any $l \in (l_{k-1}^{(1)}, l_{k+1}^{(1)}) \cap (l_{m-1}^{(1)}, l_{m+1}^{(1)})$, then there exist $k - m$ Hopf bifurcation points which are denoted by $\lambda_{n+1}^{(2)}(2)$ with $n = 1, \ldots, k$. Moreover, these Hopf bifurcation points satisfy

\[
\frac{b+1}{4} < \lambda_{k_+}^{(2)} < \lambda_{n+1}^{(2)} < \sqrt{b},
\]

(ii) If $b_0 \leq \mu < 1 - b$, for any $l \in (l_{k-1}^{(1)}, l_{k+1}^{(1)})$, then there exist $k$ Hopf bifurcation points which are denoted by $\lambda_{n+1}^{(2)}(2)$ with $n = 1, \ldots, k$. Moreover, these Hopf bifurcation points satisfy

\[
\frac{b+1}{4} < \lambda_{k_+}^{(2)} < \lambda_{n+1}^{(2)} < \sqrt{b}.
\]

Now, we adopt the method and the same notations in [9, 33] to calculate the bifurcation direction and stability of bifurcating periodic solutions from $(\lambda^{(1)}, \lambda^{(1)})$.

To cast our discussion into the framework in [33], we translate (4) into the following system by the translation $\hat{u} = u - \lambda^{(1)}$, $\hat{v} = v - \lambda^{(1)}$, and still denote $\hat{u}$ and $\hat{v}$ by $u$ and $v$, respectively. Then we obtain

\[
\begin{align*}
& u_t = d_1 \Delta u + f(u, v, \lambda^{(1)}), \\
& v_t = d_2 \Delta v + g(u, v, \lambda^{(1)}), \\
& u_x(0, t) = u_x(l, t) = 0, u_x(l, t) = v_x(l, t) = 0, \\
& u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq 0,
\end{align*}
\]

where $f(u, v, \lambda^{(1)})$ and $g(u, v, \lambda^{(1)})$ are defined in (10).

**Theorem 3.4.** For the system (4), the following statements hold true.

1. If $\text{Re}(c_1(\lambda^{(1)})) < 0$ (resp. $> 0$), then the Hopf bifurcation at $\lambda^{(1)} = \lambda_{0,+}^{(1)}$ is subcritical (resp. supercritical) and the spatially homogeneous bifurcating periodic solutions are stable (resp. unstable), where $\text{Re}(c_1(\cdot))$ will be given in the proof.
2. If \( \text{Re}(c_1(\lambda^{(1)}_{0,-})) < 0 \) (resp. \( > 0 \)), then the Hopf bifurcation at \( \lambda^{(1)} = \lambda^{(1)}_{0,-} \) is supercritical (resp. subcritical) and the spatially homogeneous bifurcating periodic solutions are stable (resp. unstable).

**Proof.** For ease of notations, we denote \( \lambda_0 \) as the spatially homogeneous Hopf bifurcation value. Obviously, \( \lambda_0 = \lambda^{(1)}_{0,-} \) and \( \lambda_0 = \lambda^{(1)}_{0,+} \) when \( 0 < \lambda_0 < (b + 1)/4 \) and \( (b + 1)/4 < \lambda_0 < (b + 1)/2 \), respectively. From Theorem 2.1 in [33], we need compute \( \text{Re}(c_1(\lambda_0)) \) to analyze the direction and stability of Hopf bifurcation solutions. Adopting the notations and calculations in [33], we set \( \omega_0 = \sqrt{\mu(\beta\lambda_0 - \mu)} \),

\[
q := \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} = \begin{pmatrix} \frac{1}{\beta\lambda_0} - \frac{\omega_0}{\mu}i \\ \frac{\omega_0}{\mu}i \end{pmatrix} \quad \text{and} \quad q^* := \begin{pmatrix} a_0^* \\ b_0^* \end{pmatrix} = \begin{pmatrix} \frac{1}{2\pi} + \frac{\mu}{\beta\lambda_0}i \\ -\frac{\beta\lambda_0}{2\pi\omega_0}i \end{pmatrix}
\]

satisfying

\[
L(\lambda_0)q = i\omega_0q, \quad L^*(\lambda_0)q^* = -i\omega_0q^*, \quad \langle q^*, q \rangle = 1 \quad \text{and} \quad \langle q^*, \bar{q} \rangle = 0,
\]

where \( \langle \cdot \rangle \) is the complex-valued \( L^2 \) inner product on Hilbert space \( X_C \) and defined as follows:

\[
\langle U_1, U_2 \rangle = \int_0^{l\pi} (\bar{u}_1u_2 + \bar{v}_1v_2)dx,
\]

where \( U_i = (u_i, v_i)^T \in X_C \) \( (i = 1, 2) \).

Recalling the definition of \( f(u, v, \lambda) \) and \( g(u, v, \lambda) \) in (10) and doing some directly calculations, then we can obtain that

\[
c_0 = f_{uuu}a_0^2 + 2f_{uuv}a_0b_0 + f_{uvb_0}^2 = 2 \left( \frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \frac{\mu}{\lambda_0} + \frac{\omega_0}{\lambda_0}i \right)
\]

\[
d_0 = g_{uuu}a_0^2 + 2g_{uuv}a_0b_0 + g_{uvb_0}^2 = -2\frac{\mu}{\lambda_0}(1 - b_0)^2
\]

\[
e_0 = f_{uuu}|a_0|^2 + f_{uuv}(a_0\bar{b}_0 + \bar{a}_0b_0) + f_{uvb_0}|b_0|^2 = 2 \left( \frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \frac{\mu}{\lambda_0} \right)
\]

\[
f_0 = g_{uuu}|a_0|^2 + g_{uuv}(a_0\bar{b}_0 + \bar{a}_0b_0) + g_{uvb_0}|b_0|^2 = \frac{2\mu}{\lambda_0} \left( 1 - \frac{\mu}{\lambda_0} \right) + \frac{\omega_0^2}{\beta^2\lambda_0^2} = -\frac{2\omega_0^2}{\beta^2\lambda_0^2}
\]

\[
g_0 = f_{uuuu}|a_0|^2 a_0 + f_{uuvu}(2|a_0|^2 b_0 + a_0^2\bar{b}_0) + f_{uvuu}(2|b_0|^2 a_0 + b_0^2\bar{a}_0) + f_{uuuvb_0}|b_0|^2 b_0 = \frac{6}{b}
\]

\[
h_0 = g_{uuuu}|a_0|^2 a_0 + g_{uuvu}(2|a_0|^2 b_0 + a_0^2\bar{b}_0) + g_{uvuu}(2|b_0|^2 a_0 + b_0^2\bar{a}_0) + g_{uuuvb_0}|b_0|^2 b_0 = \frac{6\mu}{\lambda_0^2} \left( 1 - \frac{\mu}{\beta\lambda_0} \right)^2 + \frac{2\mu\omega_0^2}{\beta^2\lambda_0^2} + \frac{4\omega_0^3}{\beta^2\lambda_0^3}i
\]

Denote

\[
Q_{qq} = (c_0, d_0)^T, \quad Q_{q\bar{q}} = (c_0, f_0)^T, \quad C_{qq\bar{q}} = (g_0, h_0)^T.
\]
Then we have
\[
\langle q^*, Q_{qq} \rangle = \left( -\frac{3}{b} \lambda_0 + \frac{1}{b} + 1 + \frac{2\omega_0^2}{\beta \lambda_0^2} \right) \frac{\mu}{\omega_0} \left( -\frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \mu \lambda_0 - \frac{2\omega_0^2}{\beta \lambda_0^2} \right) i,
\]
\[
\langle q^*, Q_{q\bar{q}} \rangle = \left( -\frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \frac{\mu}{\lambda_0} \right) - \frac{\mu}{\omega_0} \left( -\frac{3}{b} \lambda_0 + \frac{1}{b} + 1 + \beta - \frac{2\mu}{\lambda_0} \right) i,
\]
\[
\langle \bar{q}, Q_{qq} \rangle = \frac{\mu}{\omega_0} \left( \frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \frac{2\mu}{\lambda_0} + \beta + \frac{\omega_0^2}{\lambda_0 \mu \beta \lambda_0^2} - \frac{2\omega_0^2}{\beta \lambda_0^2} \right) i,
\]
\[
\langle q^*, C_{qq\bar{q}} \rangle = \left( -\frac{3}{b} - \frac{2\omega_0^2}{\beta \lambda_0^2} \right) + \left[ \frac{3\mu}{8\omega_0} + \frac{3\beta \mu}{\lambda_0 \omega_0} \left( 1 - \frac{\mu}{\beta \lambda_0} \right)^2 + \frac{\mu \omega_0}{\beta \lambda_0^2} \right] i,
\]
which imply that
\[
H_{20} = Q_{qq} - \langle q^*, Q_{qq} \rangle q - \langle q^*, Q_{q\bar{q}} \rangle \bar{q} = 0,
\]
\[
H_{11} = Q_{q\bar{q}} - \langle q^*, Q_{q\bar{q}} \rangle q - \langle q^*, Q_{q\bar{q}} \rangle \bar{q} = 0.
\]
Hence
\[
\omega_{20} = \omega_{11} = 0 \quad \text{and} \quad \langle q^*, Q_{w_{11}q} \rangle = \langle q^*, Q_{w_{00}q} \rangle = 0.
\]
Then we have
\[
\Re(c_1(\lambda_0)) = \Re \left\{ \frac{1}{2\omega_0} \langle q^*, Q_{qq} \rangle \cdot \langle q^*, Q_{q\bar{q}} \rangle + \frac{1}{2} \langle q^*, C_{qq\bar{q}} \rangle \right\}
\]
\[
= \frac{\mu}{\omega_0^2} \left( \frac{3}{b} \lambda_0 + \frac{1}{b} + 1 \right) \left( -\frac{3}{b} \lambda_0 + \frac{1}{b} + 1 - \frac{2\mu}{\lambda_0} + \beta \right) + \frac{\mu^2}{2\lambda_0 \omega_0} - \frac{\beta}{2b}.
\]
Based on $T_n(\lambda_0) < 0$ and $D_n(\lambda_0) > 0$ for any $n \geq 1$, it follows that all other eigenvalues of $L(\lambda_0)$ have negative real parts. On the other hand, since $0 < \lambda_{0,-} < (b+1)/4$ and $(b+1)/4 < \lambda_{0,0} < (b+1)/2$, it is easy to check that $\alpha'(\lambda_{0,-}) > 0$ and $\alpha'(\lambda_{0,0}) < 0$. The desired result can be deduced by Theorem 2.1 in [33].

For the spatially non-homogeneous periodic solutions in Theorem 12, we have

**Theorem 3.5.** For system (4), the spatially non-homogeneous periodic solutions bifurcating from $(\lambda^{(1)}, \lambda^{(1)})$ are all unstable. Moreover,

1. the Hopf bifurcation at $\lambda^{(1)} = \lambda^{(1)}_{n,+}$ is subcritical (supercritical) if $\Re(c_1(\lambda^{(1)}_{n,+})) < 0$ ($> 0$);

2. the Hopf bifurcation at $\lambda^{(1)} = \lambda^{(1)}_{n,-}$ is supercritical (subcritical) if $\Re(c_1(\lambda^{(1)}_{n,-})) < 0$ ($> 0$), where $n \neq 0$ and $\Re(c_1(\lambda^{(1)}_{n,\pm}))$ is defined in Appendix.

**Proof.** The bifurcation direction is obtained by Theorem 2.1 in [33] and (12). The calculation of $\Re(c_1(\lambda^{(1)}_{n,\pm}))$ is lengthy, and we will give it in Appendix. When $\lambda^{(1)} = \lambda^{(1)}_{n,\pm}$ ($n \neq 0$), the steady state $(\lambda^{(1)}, \lambda^{(1)})$ is always unstable due to the fact that $L(\lambda^{(1)}_{n,\pm})$ has at least two characteristic roots with positive real part. Actually, for any $n \neq 0$, $L_0(\lambda^{(1)}_{n,\pm})$ has two characteristic roots with positive real part because of $T_0(\lambda^{(1)}_{n,\pm}) > 0$ and $D_0(\lambda^{(1)}_{n,\pm}) > 0$. It follows that the spatially non-homogeneous periodic solutions from $(\lambda^{(1)}, \lambda^{(1)})$ are clearly unstable. \qed
All of spatially non-homogeneous periodic solutions surrounding \((\lambda^{(2)}, \lambda^{(2)})\) are also unstable because \((\lambda^{(2)}, \lambda^{(2)})\) is always unstable. The calculation of bifurcation direction is similar to \((\lambda^{(1)}, \lambda^{(1)})\), thus we omit it.

4. Steady state bifurcation of problem (4). In this section, we consider the steady state bifurcations of the system (4) with \(\Omega = (0, l\pi)\) and regard \(\mu\) as bifurcation parameter. The non-negative steady state solutions of (4) satisfy the following semilinear elliptic system:

\[
\begin{aligned}
-d_1u_{xx} &= u(1-u)\left(\frac{u}{b} - 1\right) - \beta uv, \quad x \in (0, l\pi), \\
-d_2v_{xx} &= \mu v \left(1 - \frac{v}{u}\right), \quad x \in (0, l\pi), \\
u_x(x) &= v_x(x) = 0, \quad x = 0, l\pi.
\end{aligned}
\]  

(16)

Clearly, the system (16) still has spatially homogeneous solutions \((b,0), (1,0), (\lambda^{(k)}, \lambda^{(k)}), k = 1, 2, 3\). For every positive equilibrium \((\lambda^{(k)}, \lambda^{(k)})\), we identify steady state bifurcation value \(\mu = \mu_0\) which satisfies the steady state bifurcation condition (H2) based on [33]:

(H2) There exists \(n_0 \in \mathbb{N}\) such that

\[
D_{n_0}(\mu_0, \lambda^{(k)}) = 0, T_{n_0}(\mu_0, \lambda^{(k)}) \neq 0; \quad T_n(\mu_0, \lambda^{(k)}) \neq 0, D_n(\mu_0, \lambda^{(k)}) \neq 0 \quad \text{for} \ n \neq n_0;
\]

and

\[
\frac{d}{d\mu} D_{n_0}(\mu_0, \lambda^{(k)}) \neq 0,
\]

where

\[
D_n(\mu, \lambda^{(k)}) = d_1d_2 \frac{n^4}{l^4} + (d_1\mu - d_2A(\lambda^{(k)})) \frac{n^2}{l^2} + \mu(\beta\lambda^{(k)} - A(\lambda^{(k)})),
\]

\[
T_n(\mu, \lambda^{(k)}) = A(\lambda^{(k)}) - \mu - (d_1 + d_2) \frac{n^2}{l^2}.
\]

We first analyze the possibility of the bifurcation from \((\lambda^{(3)}, \lambda^{(3)})\). By the third equality of (6), \(D_0(\mu, \lambda^{(3)}) = 0\) and \(dD_0(\mu, \lambda^{(3)}) = 0\), accordingly, the condition (H2) does not hold. It is hard to expect the steady state bifurcation from \((\lambda^{(3)}, \lambda^{(3)})\). Next, we will investigate the existence of steady state bifurcations from \((\lambda^{(1)}, \lambda^{(1)})\) and \((\lambda^{(2)}, \lambda^{(2)})\). For the sake of convenience to express, we define

\[
T(\mu, p, \lambda^{(k)}) = A(\lambda^{(k)}) - \mu - (d_1 + d_2) p,
\]

\[
D(\mu, p, \lambda^{(k)}) = d_1d_2 p^2 + (d_1\mu - d_2A(\lambda^{(k)})) p + \mu(\beta\lambda^{(k)} - A(\lambda^{(k)})).
\]

where \(p = n^2/l^2\) and \(k = 1, 2\).

4.1. Steady state bifurcation from \((\lambda^{(1)}, \lambda^{(1)})\). For system (4), to investigate the existence of nonconstant steady state solutions branching from \((\lambda^{(1)}, \lambda^{(1)})\), the condition of unstable \((\lambda^{(1)}, \lambda^{(1)})\) is necessary. Recalling related content in Section 1, the positive constant coexistence steady state \((\lambda^{(1)}, \lambda^{(1)})\) is locally asymptotically stable when \(\mu > \max\{A(\lambda^{(1)}), d_1^{-1}d_2A(\lambda^{(1)})\}\). From the expression of \(D_n(\mu, \lambda^{(1)})\), we know that, for any \(n \in \mathbb{N}\), if \(\mu \geq d_1^{-1}d_2A(\lambda^{(1)})\) then \(D_n(\mu, \lambda^{(1)}) > 0\), which conflicts with the condition (H2). To guarantee \(T_n(\mu, \lambda^{(1)}) \neq 0\), especially
$T_n(\mu, \lambda^{(1)}) < 0$, we set $\mu > A(\lambda^{(1)})$. Based on the above analysis, we always assume $A(\lambda^{(1)}) < \mu < d_1^{-1}d_2A(\lambda^{(1)})$ in this subsection.

To find $n_0$ satisfying (17), we solve $p$ from $D(\mu, p, \lambda^{(1)}) = 0$ and get

$$p = p_\pm(\mu) = \frac{d_2A(\lambda^{(1)}) - d_1\mu \pm \sqrt{g_1(\mu)}}{2d_1d_2},$$

where

$$g_1(\mu) = [d_1\mu - d_2A(\lambda^{(1)})]^2 - 4d_1d_2[\beta\lambda^{(1)} - A(\lambda^{(1)})].$$

We hope $g_1(\mu) \geq 0$ in order to obtain real $p_\pm(\mu)$. Through direct analysis, it is obvious that there exist two positive constants $\mu_*$ and $\mu^*$ satisfying $\mu_* < d_1^{-1}d_2A(\lambda^{(1)}) < \mu^*$, $g_1(\mu_*) = g_1(\mu^*) = 0$ and $g_1(\mu) > 0$ for any $\mu \in (0, \mu_*)$ and $(\mu^*, \infty)$, see Fig. 1. We only discuss the case of $\mu \in (0, \mu_*)$ since we have assumed

$$\mu < d_1^{-1}d_2A(\lambda^{(1)}).$$

Thus, any potential bifurcation point $\mu_0$ must exist in the interval $(0, \mu_*]$. Thanks to the above assumption of $\mu > A(\lambda^{(1)})$, we must make sure $A(\lambda^{(1)}) \in (0, \mu_*)$. Next, we will give a condition to guarantee this. By direct computing, we get

$$g_1(A(\lambda^{(1)})) = [d_1A(\lambda^{(1)}) - d_2A(\lambda^{(1)})]^2 - 4d_1d_2A(\lambda^{(1)})[\beta\lambda^{(1)} - A(\lambda^{(1)})]$$

$$= (d_1 + d_2)^2A^2(\lambda^{(1)}) - 4d_1d_2[\beta A(\lambda^{(1)}))\lambda^{(1)}]$$

$$= 4d_1d_2A^2(\lambda^{(1)}) \left( \frac{d_1}{4d_2} + \frac{1}{2} + \frac{d_2}{4d_1} - \frac{\beta \lambda^{(1)}}{A(\lambda^{(1)})}\right).$$

In fact, if we let parameters $d_2$, $b$, $\beta$ be fixed, then there exists a small enough $d_1$ such that

$$A(\lambda^{(1)}) < \frac{d_2}{d_1}A(\lambda^{(1)}) \quad \text{and} \quad \frac{1}{2} + \frac{d_2}{4d_1} > \frac{\beta \lambda^{(1)}}{A(\lambda^{(1)})}. \quad (20)$$

For any $\mu$, a direct computation gives rise to $\frac{dD(\mu, p, \lambda^{(1)})}{d\mu} = d_1P + \beta\lambda^{(1)} - A(\lambda^{(1)}) > 0$ due to the first inequality of (6). Then the determination of steady state bifurcation points $\mu_0$ reduces to describing the following set

$$\Lambda := \{\mu_0 \in A(\lambda^{(1)}, \mu_*] : \text{for some } n_0 \in \mathbb{N}, D_n(\mu_0, \lambda^{(1)}) = 0, D_n(\mu_0, \lambda^{(1)}) \neq 0 \text{ if } n \neq n_0\}.$$  

To determine $\Lambda$, we analyze the basic properties of $p_\pm(\mu)$ defined in $(0, \mu_*]$ in the following lemma.
Lemma 4.1. Suppose that \( \mu \in (0, \mu_*] \), then \( p_+(\mu) \) is decreasing in \((0, \mu_*] \) and \( p_-(\mu) \) is increasing in \((0, \mu_*] \). Moreover

\[
p_-(0) = 0, \quad p_+(0) = \frac{A(\lambda^{(1)})}{d_1}, \quad p_-(\mu_*) = p_+(\mu_*) = \frac{d_2A(\lambda^{(1)}) - d_1\mu_*}{2d_1d_2}. \tag{21}
\]

Proof. By directly computing, we have

\[
\frac{d}{d\mu} p_+(\mu) = \frac{1}{2d_1d_2} \left( -d_1 + \frac{d_1(d_1\mu - d_2A(\lambda^{(1)})) - 2d_1d_2(\beta \lambda^{(1)} - A(\lambda^{(1)}))}{\sqrt{g_1(\mu)}} \right).
\]

Based on the preceding assumption \( \mu < d_1^{-1}d_2A(\lambda^{(1)}) \) and the first inequality of (6), it follows that \( \frac{d}{d\mu} p_+(\mu) < 0 \), accordingly, \( p_+(\mu) \) is decreasing in \((0, \mu_*] \), see Fig. 2. Similarly, a direct calculation gives

\[
\frac{d}{d\mu} p_-(\mu) = \frac{1}{2d_1d_2} \left( -d_1 + \frac{d_1[(d_2A(\lambda^{(1)}) - d_1\mu) + 2d_2(\beta \lambda^{(1)} - A(\lambda^{(1)}))]}{\sqrt{g_1(\mu)}} \right).
\]

Direct comparison shows that for any \( \mu \in (0, \mu_*] \) the following inequality

\[
d_2A(\lambda^{(1)}) - d_1\mu + 2d_2(\beta \lambda^{(1)} - A(\lambda^{(1)})) > \sqrt{g_1(\mu)}
\]

is always valid, which implies that \( \frac{d}{d\mu} p_-(\mu) > 0 \) and \( p_-(\mu) \) is increasing in \((0, \mu_*] \). A simple calculation gives (21). The proof is completed. \( \square \)

Lemma 4.1 shows that the graph \((\mu, p_{\pm}(\mu))\) has three critical points, see Fig. 2. We define

\[
p_+ := \sup_{A(\lambda^{(1)})<\mu\leq \mu_*} p(\mu) = p_+(A(\lambda^{(1)})), \quad p_- := \inf_{A(\lambda^{(1)})<\mu\leq \mu_*} p(\mu) = p_-(A(\lambda^{(1)})). \tag{22}
\]

If \( p_- < n^2/l^2 < p_+ \), then there exists \( \mu_n \in (A(\lambda^{(1)}), \mu_*] \) such that \( p_+(\mu_n) = n^2/l^2 \) or \( p_-(\mu_n) = n^2/l^2 \) and accordingly \( D_n(\mu_n) = 0 \). Define \( \tilde{l}_{n,\pm} = n/\sqrt{p_{\pm}} \), then for any \( l \in (\tilde{l}_{n,\pm}, \tilde{l}_{n,-}) \), there exists \( \mu_n \) such that \( D_n(\mu_n) = 0 \).

By direct application of Theorem 3.1 in [33] and the above demonstration, we are now ready to state the main result in this subsection:

**Theorem 4.2.** Suppose that the constants \( d_1, d_2, \beta > 0 \) and \( 0 < b < 1 \) satisfy (20), \( p_{\pm}(\mu) \) are defined as (19). If for some \( n \in \mathbb{N} \), \( l \in (\tilde{l}_{n,+}, \tilde{l}_{n,-}) \), there exists exactly one point \( \mu_n \in (A(\lambda^{(1)}), \mu_*] \) such that \( p_+(\mu_n) = n^2/l^2 \) or \( p_-(\mu_n) = n^2/l^2 \). Then there is a smooth curve \( \Gamma_n \) of positive solution of (16) bifurcating from \( (\mu, u, v) = (\mu_n, \lambda^{(1)}, \lambda^{(1)}) \) and \( \Gamma_n \) is contained in global branch \( C_n \) of the positive solution of (16). Moreover, the smooth curve \( \Gamma_n \) can be expressed as \( \Gamma_n = \{ (\mu(s), u(s), v(s)) : s \in (-\varepsilon, \varepsilon) \} \), where

\[
\left\{ \begin{array}{l}
u(s) = \lambda^{(1)} + s a_n \cos(nx/l) + s \psi_1(s), \quad s \in (-\varepsilon, \varepsilon), \\
u(s) = \lambda^{(1)} + s b_n \cos(nx/l) + s \psi_2(s), \quad s \in (-\varepsilon, \varepsilon),
\end{array} \right.
\]

and \( \psi_1, \psi_2 : (-\varepsilon, \varepsilon) \to \mathbb{R} \) are \( C^1 \) functions such that \( \psi_1(0) = \psi_2(0) = 0 \). Here \( Z = Z_1 \times Z_1 \), with \( Z_1 = \{ u : \int_0^\pi u(x) \cos(nx/l)dx = 0 \} \), \( a_n \) and \( b_n \) satisfy \( L_n(\mu_n)(a_n, b_n)^2 = (0, 0)^2 \).
4.2. Steady state bifurcation from \((\lambda^{(2)}, \lambda^{(2)})\). Similar to the analysis in subsection 4.1, we firstly solve \(p\) from \(D(\mu, p, \lambda^{(2)}) = 0\), then we have

\[
p = p_{\pm}(\mu) = \frac{d_2 A(\lambda^{(2)}) - d_1 \mu \pm \sqrt{g_2(\mu)}}{2d_1 d_2},
\]

where

\[
g_2(\mu) = [d_1 \mu - d_2 A(\lambda^{(2)})]^2 - 4d_1 d_2 \mu [\beta \lambda^{(2)} - A(\lambda^{(2)})].
\]

Thanks to \(\beta \lambda^{(2)} < A(\lambda^{(2)})\), \(p_{-}(\mu)\) is always negative. Accordingly, we only consider \(p = p_{+}(\mu)\).

A direct calculation gives

\[
\frac{d}{d\mu} p_{+}(\mu) = \frac{1}{2d_1 d_2} \left( \frac{-d_1 (d_1 \mu - d_2 A(\lambda^{(2)})) - 2d_1 d_2 (\beta \lambda^{(2)} - A(\lambda^{(2)})) \sqrt{g_2(\mu)}}{\sqrt{g_2(\mu)}} \right)
\]

Comparing \(\sqrt{g_2(\mu)}\) and \(d_1 \mu - d_2 A(\lambda^{(2)}) - 2d_2 (\beta \lambda^{(2)} - A(\lambda^{(2)}))\), we have \(\frac{d}{d\mu} p_{+}(\mu) < 0\). Thus, for any \(\mu \in (0, +\infty)\), \(p_{+}(\mu)\) is decreasing in \(\mu\) (see Fig. 3) and satisfies

\[
p_{+}(0) = \frac{A(\lambda^{(2)})}{d_1}, \quad \lim_{\mu \to +\infty} p_{+}(\mu) = \frac{A(\lambda^{(2)}) - \beta \lambda^{(2)}}{d_1} := \tilde{p}.
\]

In order to guarantee \(T_n(\mu, \lambda^{(2)}) \neq 0\), especially \(T_n(\mu, \lambda^{(2)}) < 0\), we always assume \(\mu > A(\lambda^{(2)})\) in this subsection. Define

\[
p^{*} := p(A(\lambda^{(2)}))
\]

then \(\tilde{p} < p(\mu) < p^{*}\) for any \(\mu \in (A(\lambda^{(2)}), +\infty)\).

Next, we check the transversality condition (18),

\[
\frac{d}{d\mu} D(\mu, p, \lambda^{(2)}) = d_1 p + \beta \lambda^{(2)} - A(\lambda^{(2)}) > 0.
\]

From above analysis, we can give the existence result of steady-state bifurcation surrounding \((\lambda^{(2)}, \lambda^{(2)})\):
Theorem 4.3. Let the constants $d_1, d_2, \beta > 0$ and $0 < b < 1$ be fixed and $p_+(\mu)$ is defined as (23). Define $\tilde{l}_n = n/\sqrt{p}$, and $\tilde{l}_n = n/\sqrt{p}^*$. If for some $n \in \mathbb{N}$, $l \in (\tilde{l}_n, \tilde{l}_n^*)$, and $\tilde{p} + (\mu_n) = n^2/l^2$. Then there exists exactly one point $\mu_n \in (A(\lambda^{(2)}), +\infty)$ such that $p_+(\mu_n) = n^2/l^2$. Then there is a smooth curve $\Gamma_n$ of positive solution of (16) bifurcation from $(\mu, u, v) = (\mu_n, \lambda^{(2)}, \lambda^{(2)})$ and $\Gamma_n$ is contained in a global branch $C_n$ of the positive solution of (16). Moreover, the smooth curve $\Gamma_n$ can be expressed as $\Gamma_n = \{ (\mu(s), u(s), v(s)) : s \in (-\varepsilon, \varepsilon) \}$, where

\[
\begin{align*}
u(s) &= \lambda^{(2)} + sc_n \cos(nx/l) + s\psi_1(s), \quad s \in (-\varepsilon, \varepsilon), \\
u(s) &= \lambda^{(2)} + sd_n \cos(nx/l) + s\psi_2(s), \quad s \in (-\varepsilon, \varepsilon),
\end{align*}
\]

where $\psi_1, \psi_2 : (-\varepsilon, \varepsilon) \to Z$ are $C^1$ functions such that $\psi_1(0) = \psi_2(0) = 0$. Here $Z = Z_1 \times Z_1$, with $Z_1 = \{ u : \int_0^{\pi} u(x) \cos(nx/l)dx = 0 \}$. $c_n$ and $d_n$ satisfy

$L_n(\mu_n)(c_n, d_n)^T = (0, 0)^T$.

5. Simulation and discussion. In this section, we will make some simulations for systems (3) and (4) to discuss parameters’ practical significance. For the system (3), the simulations’ results suggest that many properties of (3) are neglected in [18]. In addition, a local Hopf bifurcation surrounding $(\lambda^{(1)}, \lambda^{(1)})$ can be seen in Fig. 4. For the system (4), Hopf bifurcations from $(\lambda^{(1)}, \lambda^{(1)})$ are showed easily by numerical simulation while ones from $(\lambda^{(2)}, \lambda^{(2)})$ are hard to see because they are unstable. However, we will give parameters’ values such that the system (4) occurs Hopf bifurcation from $(\lambda^{(2)}, \lambda^{(2)})$.

In the system (3), we take $b = 0.5$, $\beta = 0.1714$ and $\mu = 0.1$ satisfying $1 + b - \beta b > 2\sqrt{b}$. Then the system (3) undergoes an unstable Hopf bifurcation branching from $(\lambda^{(1)}, \lambda^{(1)})$. The bifurcating limit cycle is so small that we have to enlarge Fig. 4 to see it. At the moment, $(\lambda^{(2)}, \lambda^{(2)})$ is also unstable. From Fig. 4, most of the solutions with the initial $(u_0, v_0) \in (0, 1) \times (0, 1)$ will go to the origin. The same phenomenon also happens under the condition $1 + b - \beta b = 2\sqrt{b}$, see Fig. 5.
For the system (3), we take $b = 0.5$, $\beta = 0.13$ and $\mu = 0.1$ satisfying $1 + b - \beta b > 2\sqrt{b}$. As we all know that $b$ is the Allee threshold value, Fig. 6 shows that if $u_0 < b$, then the prey $u$ dies out and so does the predator $v$ due to food shortage. Theorem 3.2 of [18] concluded that the solutions $(u, v)$ would converge to $(0, 0)$ provided $u_0 < \min\{v_0, \lambda^{(2)}\}$. However, Fig. 6 shows that there are some $u_0$ which does not satisfy the condition $u_0 < \min\{v_0, \lambda^{(2)}\}$ such that the solution with initial value $(u_0, v_0)$ also converges to $(0, 0)$. Fig. 6 also reveals that the region enclosed by the parabola $v = (1 - u)(u/b - 1)/\beta$ and $v = 0$ is a part of stable region of $(\lambda^{(1)}, \lambda^{(1)})$. Moreover, we can infer that the stable region of $(\lambda^{(1)}, \lambda^{(1)})$ is on the right side of the tangent line of parabola $v = (1 - u)(u/b - 1)/\beta$ to $(\lambda^{(2)}, \lambda^{(2)})$.

![Fig. 5: Most of solutions to (3) converge to (0, 0) when $\beta = 0.17157288$.](image)

For the system (4), we give some examples of spatially homogeneous and spatially non-homogeneous Hopf bifurcation branching from $(\lambda^{(1)}, \lambda^{(1)})$. Taking $d_1 = 0.2$, $d_2 = 0.1$, $b = 0.01$, $\mu = 10$, $l = 1$, $u(x, 0) = 0.1352 + 0.1\sin(x)$ and $v(x, 0) = 0.1361 + 0.1\cos(x)$, then the system (4) has eight Hopf bifurcation points:

$\lambda_{0, -}^{(1)} = 0.1352129589, \lambda_1^{(1)} = 0.1417920509, \lambda_2^{(1)} = 0.1644304252, \lambda_3^{(1)} = 0.2364921894, \lambda_{3, +}^{(1)} = 0.2685078106, \lambda_{2, +}^{(1)} = 0.3405695748, \lambda_{1, +}^{(1)} = 0.3632079491, \lambda_{0, +}^{(1)} = 0.3697870411.$

Fig. 7 shows that the system (4) occurs spatially homogeneous Hopf bifurcation at $\lambda^{(1)} = \lambda_{0, -}^{(1)}$ (equivalently $\beta = 61.3170$). Meanwhile, there is another spatially homogeneous Hopf bifurcation when $\lambda^{(1)} = \lambda_{0, +}^{(1)}$ (equivalently $\beta = 80.0830$).

Fig. 8 shows that the system (4) undergoes a spatially non-homogeneous Hopf bifurcation at $\lambda^{(1)} = \lambda_{2, -}^{(1)}$ (equivalently $\beta = 78.4754$). Actually, the system (4) also
Fig. 6: The system (3) has two positive equilibrium points $(\lambda^{(1)}, \lambda^{(1)})$ and $(\lambda^{(2)}, \lambda^{(2)})$. The former is stable and the later unstable.

Fig. 7: Spatially homogeneous Hopf bifurcation of (4) when $\beta = 61.3170$ and $n = 0$.

occurs spatially non-homogeneous Hopf bifurcation at $\lambda^{(1)} = \lambda_{2,+}^{(1)}$ or $\lambda^{(1)} = \lambda_{n,\pm}^{(1)}$, $n = 1, 3, 4$. As all situations are similar, we omit their figures.

The periodic solutions bifurcating from $(\lambda^{(2)}, \lambda^{(2)})$ are very hard to see because they are unstable. But we give four groups of parameters’ values satisfying (14) such that the system (4) occurs Hopf bifurcation from $(\lambda^{(2)}, \lambda^{(2)})$, see Table 5.
Fig. 8: Spatially non-homogeneous Hopf bifurcation of (4) when $\beta = 78.4754$ and $n = 2$.

Table 5: Parameters’ values of Hopf bifurcation for $(\lambda^{(2)}, \lambda^{(2)})$

|   | $b$   | $\mu$ | $\beta$ | $d_1$ | $d_2$ | $l$ |
|---|-------|-------|--------|------|------|---|
| 1 | 0.03  | 0.1   | 22.44329 | 1    | 0.1  | 1  |
| 2 | 0.05  | 0.1   | 11.46339 | 1    | 0.1  | 1  |
| 3 | 0.06  | 0.1   | 9.485507 | 1    | 0.1  | 1  |
| 4 | 0.06  | 0.1   | 6.305220 | 1    | 0.1  | 1  |

In Table 6, we give four groups of parameters’ values for (4) such that the system (4) undergoes steady-state bifurcation. Moreover, under the first and second group of parameters’ values, the steady-state bifurcation bifurcates from $(\lambda^{(1)}(1), \lambda^{(1)}(1))$; under the third and forth group of parameters’ values, the steady-state bifurcation bifurcates from $(\lambda^{(2)}(1), \lambda^{(2)}(1))$.

Table 6: Parameters’ values for steady-state bifurcation

|   | $b$   | $\mu$ | $\beta$ | $d_1$ | $d_2$ | $l$ |
|---|-------|-------|--------|------|------|---|
| 1 | 0.25  | 0.292 | 0.972  | 0.5  | 3    | 0.531 |
| 2 | 0.062 | 2.431 | 8.667  | 0.5  | 2    | 1.283 |
| 3 | 0.25  | 1     | 0.667  | 1    | 1    | 1    |
| 4 | 0.062 | 1     | 10     | 1    | 1    | 2    |

Appendix. Bifurcation direction of spatially non-homogeneous periodic solutions from $(\lambda^{(1)}, \lambda^{(1)})$. In this appendix, we compute $\text{Re}(c_1(\lambda^{(1)}_{n, \pm}))$ $(n \neq 0)$. We adopt the same notations in [33]. When $\lambda^{(1)} = \lambda^{(1)}_{n, \pm}$ $(n \neq 0)$, we set

\[
q := \cos \frac{nx}{l} (a_n, b_n)^T = \cos \frac{nx}{l} \left( 1, \frac{\mu l^2 + d_2 n^2}{\beta l^2 \lambda^{(1)}_{n, \pm} - \omega_0 i, \frac{\beta \lambda^{(1)}_{n, \pm}}{l \omega_0 i} \right)^T,
\]

\[
q^* := \cos \frac{nx}{l} (a_n^*, b_n^*)^T = \cos \frac{nx}{l} \left( 1, \frac{1}{l^2 \pi}, \frac{\mu l^2 + d_2 n^2}{l^2 \pi \omega_0 i, \frac{\beta \lambda^{(1)}_{n, \pm}}{l \pi \omega_0 i} \right)^T,
\]

where

\[
\omega_0 = \sqrt{\beta \mu \lambda^{(1)}_{n, \pm} - \mu^2 - 2d_2 \mu n^2 l^2 - d_2 n^4 l^4}.
\]
When \( n \neq 0 \), noticing that
\[
\int_0^{2\pi} \cos^3 \frac{n\pi}{I} \, dx = 0,
\]
and by calculation, we have \( \langle q^*, Q_{qq} \rangle = \langle q^*, Q_{qq} \rangle = 0 \). Thus, in order to calculate \( \text{Re}(c_1(\lambda^{(1)}_{n,\pm})) \), it remains to calculate \( \langle q^*, Q_{w1} q \rangle, \langle q^*, Q_{w2} q \rangle \) and \( \langle q^*, C_{qqq} \rangle \).

It is straightforward to compute that
\[
c_n = f_{uu} + 2f_{uv}(\mu + d_2 n^2 / I^2) / (\beta \lambda^{(1)}_{n,\pm}) - 2f_{uv} \omega_0 i / (\beta \lambda^{(1)}_{n,\pm}),
\]
\[
d_n = g_{uu} + 2g_{uv} \text{Re}(b_n) + \text{Re}^2(b_n) - \text{Im}^2(b_n) + 2[g_{uv} \text{Im}(b_n) + \text{Re}(b_n) \text{Im}(b_n)]i,
\]
\[
e_n = f_{uu} + 2f_{uv} \text{Re}(b_n), \quad g_n = f_{uuu},
\]
\[
f_n = g_{uu} + 2g_{uv} \text{Re}(b_n) + g_{vv}[\text{Re}^2(b_n) + \text{Im}^2(b_n)],
\]
\[
h_n = g_{uuu} + 3g_{uuu} \text{Re}(b_n) + g_{vvv}[3\text{Re}^2(b_n) + \text{Im}^2(b_n)] + [g_{uuu} \text{Im}(b_n) + 2g_{uvv} \text{Re}(b_n) \text{Im}(b_n)]i,
\]
and
\[
\begin{cases}
  f_{uu} = -\frac{6(u + \lambda^{(1)})}{b} + \frac{2}{b} + 2, & f_{uv} = -\beta, & f_{uuu} = -\frac{6}{b}, \\
  f_{vv} = f_{uuv} = f_{uvv} = f_{vvv} = 0,
  \\
  g_{uu} = -\frac{2\mu(v + \lambda^{(1)})^2}{(u + \lambda^{(1)})^3}, & g_{uv} = \frac{2\mu(v + \lambda^{(1)})}{(u + \lambda^{(1)})^2},
  \\
  g_{vv} = -\frac{2\mu}{u + \lambda^{(1)}}, & g_{uuv} = \frac{6\mu(v + \lambda^{(1)})^2}{(u + \lambda^{(1)})^4}, & g_{vvv} = 0,
  \\
  g_{uuv} = -\frac{4\mu(v + \lambda^{(1)})}{(u + \lambda^{(1)})^3}, & g_{uvv} = \frac{2\mu}{(u + \lambda^{(1)})^2}.
\end{cases}
\]

Here and in the following we always assume that all the partial derivatives of \( f \) and \( g \) are evaluated at \((0, 0, \lambda^{(1)}_{n,\pm})\).

Directly computations give rise to
\[
[2i\omega_0 I - L_{2n}(\lambda^{(1)}_{n,\pm})]^{-1} = (\alpha_1 + \alpha_2)^{-1} \begin{pmatrix}
  2i\omega_0 + \mu + \frac{4d_2 n^2}{I^2} & -\beta \lambda^{(1)}_{n,\pm} \\
  \mu & 2i\omega_0 - A(\lambda^{(1)}_{n,\pm}) + \frac{4d_1 n^2}{I^2}
\end{pmatrix},
\]
\[
[2i\omega_0 I - L_0(\lambda^{(1)}_{n,\pm})]^{-1} = (\alpha_3 + \alpha_4)^{-1} \begin{pmatrix}
  2i\omega_0 + \mu & -\beta \lambda^{(1)}_{n,\pm} \\
  \mu & 2i\omega_0 - \mu - \frac{(d_1 + d_2) n^2}{I^2}
\end{pmatrix},
\]
\[
L_{2n}^{-1} = \frac{1}{\alpha_5} \begin{pmatrix}
  -\mu - \frac{4d_2 n^2}{I^2} & \beta \lambda^{(1)}_{n,\pm} \\
  -\mu & \mu + \frac{(d_2 - 3d_1) n^2}{I^2}
\end{pmatrix},
\]
\[
L_0^{-1} = \frac{1}{\alpha_6} \begin{pmatrix}
  -\mu & \beta \lambda^{(1)}_{n,\pm} \\
  -\mu & \mu + \frac{(d_1 + d_2) n^2}{I^2}
\end{pmatrix}.
\]
where
\[
\begin{align*}
\alpha_1 & := \frac{(12d_1d_2 - 3d_2^2)n^4 + 3\mu^2n^2(d_1 - d_2) - 3\omega_0^2l^4}{l^4}, \\
\alpha_2 & := \frac{6\omega_0(d_1 + d_2)n^2}{l^2}, \\
\alpha_3 & := \frac{\mu(d_2 - d_1)l^2n^2 + d_2^2n^4 - 3\omega_0^2l^4}{l^4}, \\
\alpha_4 & := -\frac{2\omega_0(d_1 + d_2)n^2}{l^2}, \\
\alpha_5 & := \frac{(12d_1d_2 - 3d_2^2)n^4 + 3\mu(d_1 - d_2)l^2n^2 + l^4\omega_0^2}{l^4}, \\
\alpha_6 & := \frac{d_2^2n^4 + \mu(d_2 - d_1)l^2n^2 + l^4\omega_0^2}{l^4}.
\end{align*}
\]

When \( n \neq 0 \), we have
\[
\begin{align*}
\tilde{w}_{20} &= \begin{pmatrix} \tau_1 \\ \tau_2 \end{pmatrix} \cos \frac{2n}{l} x + \begin{pmatrix} \tau_3 \\ \tau_4 \end{pmatrix}, \\
\tilde{w}_{11} &= \begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} \cos \frac{2n}{l} x + \begin{pmatrix} \sigma_3 \\ \sigma_4 \end{pmatrix},
\end{align*}
\]

where
\[
\begin{align*}
\tau_1 &= \frac{(\alpha_1 + \alpha_2i)^{-1}}{2} \left[ \left( 2i\omega_0 + \mu + \frac{4d_2n^2}{l^2} \right) e_n - \beta \lambda d_n \right], \\
\tau_2 &= \frac{(\alpha_1 + \alpha_2i)^{-1}}{2} \left[ \mu e_n + \left( 2i\omega_0 - \mu + \frac{(3d_1 - d_2)n^2}{l^2} \right) d_n \right], \\
\tau_3 &= \frac{(\alpha_3 + \alpha_4i)^{-1}}{2} \left[ (2i\omega_0 + \mu) e_n - \beta \lambda d_n \right], \\
\tau_4 &= \frac{(\alpha_3 + \alpha_4i)^{-1}}{2} \left[ \mu e_n + \left( 2i\omega_0 - \mu + \frac{(d_1 + d_2)n^2}{l^2} \right) d_n \right], \\
\sigma_1 &= \frac{1}{2\alpha_5} \left[ \left( \mu + \frac{4d_2n^2}{l} \right) e_n - \beta \lambda f_n \right], \\
\sigma_2 &= \frac{1}{2\alpha_5} \left[ \mu e_n + \left( \frac{3d_1 - d_2)n^2}{l^2} - \mu \right) f_n \right], \\
\sigma_3 &= \frac{1}{2\alpha_6} (\mu e_n - \beta \lambda f_n), \\
\sigma_4 &= \frac{1}{2\alpha_6} \left( \mu e_n - \mu f_n - \frac{(d_1 + d_2)n^2}{l^2} f_n \right).
\end{align*}
\]

Then we have
\[
\begin{align*}
Q_{W_{20,q}} &= \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \cos \frac{2nx}{l} \cos \frac{nx}{l} + \begin{pmatrix} \xi_3 \\ \xi_4 \end{pmatrix} \cos \frac{nx}{l},
\end{align*}
\]

and
\[
\begin{align*}
Q_{W_{11,q}} &= \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \cos \frac{2nx}{l} \cos \frac{nx}{l} + \begin{pmatrix} \eta_3 \\ \eta_4 \end{pmatrix} \cos \frac{nx}{l}.
\end{align*}
\]
where
\[
\xi_1 = f_{uu} \tau_1 + f_{uv} \tau_1 \overline{b_n} + f_{uv} \tau_2, \quad \xi_2 = g_{uu} \tau_1 + g_{uv} \tau_1 \overline{b_n} + g_{uv} \tau_2 \overline{b_n}, \\
\xi_3 = f_{uu} \tau_3 + f_{uv} \tau_3 \overline{b_n} + f_{uv} \tau_4, \quad \xi_4 = g_{uu} \tau_3 + g_{uv} \tau_3 \overline{b_n} + g_{uv} \tau_4 \overline{b_n}, \\
\eta_1 = f_{uu} \sigma_1 + f_{uv} \sigma_1 \overline{b_n} + f_{uv} \sigma_2, \quad \eta_2 = g_{uu} \sigma_1 + g_{uv} \sigma_1 \overline{b_n} + g_{uv} \sigma_2 + g_{uv} \sigma_2 \overline{b_n}, \\
\eta_3 = f_{uu} \sigma_3 + f_{uv} \sigma_3 \overline{b_n} + f_{uv} \sigma_4, \quad \eta_4 = g_{uu} \sigma_3 + g_{uv} \sigma_3 \overline{b_n} + g_{uv} \sigma_4 + g_{uv} \sigma_4 \overline{b_n}.
\]

Notice that for any \( n \neq 0 \),
\[
\int_0^{l \pi} \cos^2 \frac{2 \pi x}{l} \, dx = \frac{1}{2} l \pi, \quad \int_0^{l \pi} \cos \frac{2 \pi x}{l} \cos^2 \frac{2 \pi x}{l} \, dx = \frac{1}{4} l \pi, \quad \int_0^{l \pi} \cos^4 \frac{2 \pi x}{l} \, dx = \frac{3}{8} l \pi,
\]
we have
\[
(q^*, Q_{w_0 \overline{q}}) = \frac{l \pi}{4} \left( a_n^* \xi_1 + b_n^* \xi_2 \right) + \frac{l \pi}{2} \left( -a_n^* \xi_3 + b_n^* \xi_4 \right), \\
(q^*, Q_{w_1 \overline{q}}) = \frac{l \pi}{4} \left( a_n^* \eta_1 + b_n^* \eta_2 \right) + \frac{l \pi}{2} \left( -a_n^* \eta_3 + b_n^* \eta_4 \right), \\
(q^*, C_{qqq}) = \frac{3 \pi}{8} \left( a_n^* g_n + b_n^* h_n \right).
\]

It follows that
\[
\text{Re}(q^*, Q_{w_0 \overline{q}}) = \frac{1}{4} \left[ f_{uu} (\tau_1^R + 2 \tau_3^R) + f_{uv} \left( b_n^R (\tau_1^R + 2 \tau_3^R) + b_n^I (\tau_1^I + 2 \tau_3^I) + \tau_2^R + 2 \tau_4^R \right) \right] \\
+ \frac{l^2 \mu + d_2 n^2}{4 l^2 \omega_0} \left[ f_{uu} (\tau_1^I + 2 \tau_3^I) + f_{uv} \left( b_n^R (\tau_1^I + 2 \tau_3^I) - b_n^I (\tau_1^R + 2 \tau_3^R) + \tau_2^I + 2 \tau_4^I \right) \right] \\
- \frac{\beta \lambda^{(1)}_{n \pm}}{4 \omega_0} \left[ g_{uu} (\tau_1^R + 2 \tau_3^R) + g_{uv} \left( b_n^R (\tau_1^R + 2 \tau_3^R) - b_n^I (\tau_1^R + 2 \tau_3^R) + \tau_2^I + 2 \tau_4^I \right) \right] \\
- \frac{\beta \lambda^{(1)}_{n \pm}}{4 \omega_0} \left[ g_{uv} \left( b_n^R (\tau_2^R + 2 \tau_4^R) - b_n^I (\tau_2^R + 2 \tau_4^R) \right) \right],
\]
\[
\text{Re}(q^*, Q_{w_1 \overline{q}}) = \frac{1}{4} \left[ f_{uu} (\sigma_1^R + 2 \sigma_3^R) + f_{uv} \left( b_n^R (\sigma_1^R + 2 \sigma_3^R) - b_n^I (\sigma_1^I + 2 \sigma_3^I) + \sigma_2^R + 2 \sigma_4^R \right) \right] \\
+ \frac{l^2 \mu + d_2 n^2}{4 l^2 \omega_0} \left[ f_{uu} (\sigma_1^I + 2 \sigma_3^I) + f_{uv} \left( b_n^R (\sigma_1^I + 2 \sigma_3^I) + b_n^I (\sigma_1^R + 2 \sigma_3^R) + \sigma_2^I + 12 \sigma_4^I \right) \right] \\
- \frac{\beta \lambda^{(1)}_{n \pm}}{4 \omega_0} \left[ g_{uu} (\sigma_1^R + 2 \sigma_3^R) + g_{uv} \left( b_n^R (\sigma_1^R + 2 \sigma_3^R) + b_n^I (\sigma_1^R + 2 \sigma_3^R) + \sigma_2^R + 2 \sigma_4^R \right) \right] \\
- \frac{\beta \lambda^{(1)}_{n \pm}}{4 \omega_0} \left[ g_{uv} \left( b_n^R (\sigma_2^R + 2 \sigma_4^R) + b_n^I (\sigma_2^R + 2 \sigma_4^R) \right) \right]
\]
and
\[
\text{Re}(q^*, C_{qqq}) = \frac{3}{8} \left[ f_{uu} + g_{uu} + \frac{2 g_{uvv}}{\beta \lambda^{(1)}_{n \pm}} \left( \mu + \frac{d_2 n^2}{l^2} \right) \right],
\]
where \( \tau_j^R = \text{Re} \tau_j, \sigma_j^R = \text{Re} \sigma_j, \tau_j^I = \text{Im} \tau_j, \sigma_j^I = \text{Im} \sigma_j, j = 1, 2, 3, 4. \)
So far, we have
\[ \text{Re}(c_1(1, \lambda_n)) = \text{Re}(q^*, Qw_{1q}) + \frac{1}{2} \text{Re}(q^*, Qw_{20q}) + \frac{1}{2} \text{Re}(q^*, Cqq). \]

In addition, thanks to \( 0 < \lambda_n < (b + 1)/4 \) and \( (b + 1)/4 < \lambda_n < 1 \), we have \( \alpha'(\lambda_n^+) > 0 \) and \( \alpha'(\lambda_n^-) < 0 \) for any \( n \neq 0 \). The conclusion is followed from Theorem 8 in [33].

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