A STABILITY RESULT FOR THE $\infty$-LAPLACE EQUATION

NIKOLAI UBOSTAD

Abstract. We investigate a degenerate elliptic PDE related to the $\infty$-Laplace equation $\Delta_\infty u = 0$. A stability result is derived. The $\Gamma$-convergence of the corresponding functionals is investigated.

1. Introduction

The $\infty$-Laplace equation

$$\Delta_\infty u := \sum_{i,j=1}^n u_{x_i} u_{x_j} u_{x_i x_j} = 0$$

was introduced by Aronsson in [Aro67]. The solutions are called absolutely minimizing Lipschitz extensions of the given boundary values. The equation appears as the Euler-Lagrange equation for the functional $\|Du\|_\infty$, and is the limit of the $p$-Laplace equation

$$\Delta_p u := |Du|^{p-2} (|Du|^2 \Delta u + (p-2) \Delta_\infty u) = 0,$$

as $p \to \infty$. The equation also arises in connection with Tug-of-War games, studied by Peres et al in [PSSW09]. We also mention the applications within image processing discovered by Caselles et al in [CMS98], and glaciology in [Glo03].

The evolutionary counterpart

$$u_t = \Delta_\infty u$$

and related equations have recently received attention, see for example [CW03], or [JK06].

The study of (1.1) is difficult because the equation is both fully nonlinear and very degenerate elliptic. Since the equation cannot be written on divergence form, the concept of viscosity solutions, introduced by Crandall and Lions in [CL83], is required. This approach was taken in [BDM89].

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Several approximation methods have been used, most famously Jensen’s auxiliary equations

\[
\min\{\Delta_\infty v, |Dv| - \sigma\} = 0, \\
\max\{\Delta_\infty u, \sigma - |Du|\} = 0,
\]

to prove the comparison principle in \textbf{[Jen93]}.

An interesting device is the “patching solutions,” introduced by Crandall, Gunnarson and Wang in \textbf{[CGW07]}. There is also the “easy” proof of uniqueness by Armstrong and Smart, see \textbf{[AS10]}.

We introduce a new approach. Attempting to eliminate the domain where the gradient is zero, we minimize variational integrals of the form

\[
\int_\Omega \{|Du|^2 - \sigma\}_{+}^p \, dx, \quad \sigma > 0
\]

A “dead core” where the gradient is less than \(\sigma\) appears. We then use the patching solutions in the dead core, and let \(\sigma \to 0\) and \(p \to \infty\).

Since this approach focuses on the convergence of minimizers of \(p\)-energies, it is natural to consider the \(\Gamma\)-convergence of the corresponding functionals.

The first main result is that, no matter which limit you take first, minimizers of \textbf{(1.3)} converge to viscosity solutions of the \(\infty\)-Laplace equation \textbf{(1.1)}.

**Theorem 1.1.** Let \(u_{p,\sigma}\) denote minimizers of \textbf{(1.3)}, and \(u_\infty\) be a solution to \textbf{(1.1)}. Then the following diagram of convergence commutes

\[
\begin{array}{ccc}
\text{u}_{p,\sigma} & \xrightarrow{p \to \infty} & \text{u}_{\infty,\sigma} \\
\downarrow{\sigma \to 0} & & \downarrow{\sigma \to 0} \\
\text{u}_p & \xrightarrow{p \to \infty} & \text{u}_\infty
\end{array}
\]

where \(u_p\) is the solution to the \(p\)-Laplace equation \textbf{(1.2)}.

In particular, the case \(p = \infty\) leads to the interesting equation

\[
\{|Du|^2 - \sigma\}_{+} \Delta_\infty = 0.
\]

The second is that the functionals corresponding to the minimizers in Theorem 1.1 \(\Gamma\)-converge with respect to uniform convergence in \(C(\overline{\Omega})\) for \(p > n\):
Theorem 1.2. The following diagram of $\Gamma$-convergences with respect to the uniform topology on $C(\Omega)$ commutes

\[
\begin{array}{c}
E_\sigma^p \xrightarrow{p \to \infty} E_\sigma^\infty \\
\downarrow \sigma \to 0 \quad \downarrow \sigma \to 0 \\
E_p \xrightarrow{p \to \infty} E_\infty,
\end{array}
\]

where $E_\sigma^p$, $E_\sigma^\infty$, $E_p$ and $E_\infty$ are the functionals with minimizers as in Theorem 1.1.

The article is structured as follows: In Section 2 we introduce the variational integral (with $\sigma = 1$ for simplicity), and prove the existence of a unique minimizer. A comparison result is established for viscosity solutions of the corresponding Euler-Lagrange equation. In Section 4 we introduce the patching solutions and prove Theorem 1.1. Section 5 is dedicated to the $\Gamma$-convergence of the corresponding functionals, and we prove Theorem 1.2.

1.1. Notation. Let $\{u\}_+$ denote $\max\{u, 0\}$, and $\langle a, b \rangle$ is the Euclidian inner product of the vectors $a, b \in \mathbb{R}^n$. $\Omega$ will denote bounded domains in $\mathbb{R}^n$. $Du = (u_{x_1}, u_{x_2}, \cdots, u_{x_n})$ is the gradient of $u$, and $W^{1,p}(\Omega)$ is the Sobolev space of functions $u$ with the norm

\[
\|u\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |u|^p \, dx \right)^{\frac{1}{p}} + \left( \int_{\Omega} |Du|^p \, dx \right)^{\frac{1}{p}}.
\]

$W^{1,\infty}(\Omega)$ is the space of Lipschitz continuous functions, and

\[
\|u\|_{W^{1,\infty}(\Omega)} = \sup_{\Omega} |u| + \sup_{\Omega} |Du| < \infty.
\]

$W^{1,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the norm (1.4). $\text{USC}(\Omega)$ denotes the upper semicontinuous functions from $\Omega$ to $\mathbb{R} \cup \{+\infty\}$, and $\text{LSC}(\Omega)$ denotes the lower semicontinuous ones. The diameter of a set $\Omega \subset \mathbb{R}^n$ is defined by

\[
\text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y|,
\]

and we let $d_U(x, y)$ denote the Euclidian distance between $x, y \in U$. For integrable functions $u$, we let $u_\Omega$ denote the average of $u$ over $\Omega$.

2. Variational Integral

For $p > 2$, define the variational integral

\[
E_p^1(u) = \int_{\Omega} \{|Du|^2 - 1\}^\frac{p}{2} \, dx.
\]
**Theorem 2.1.** Given \( f \in W^{1,p}(\Omega) \), there exists at least one minimizer \( u \) of the variational integral (2.1), so that \( u - f \in W^{1,p}_0(\Omega) \).

**Proof.** We shall employ the so-called direct method of Lebesgue, see [Dac07].

Define
\[
m = \inf \left[ \int_{\Omega} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} \, dx \right],
\]
where the infimum is taken over all functions \( u - f \in W^{1,p}(\Omega) \). We want to show that the infimum is, in fact, a minimum. Since
\[
m \leq \int_{\Omega} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} \, dx \leq \int_{\Omega} \left\{ |Df|^2 - 1 \right\}^\frac{p}{2} \, dx
\]
\[
\leq \int_{\Omega} |Df|^p \, dx = ||Df||_p^p < \infty,
\]
the definition of infimum gives the existence of a minimizing sequence \((u_j - f)_j \subset W^{1,p}(\Omega)\) so that, as \( j \to \infty \),
\[
\int_{\Omega} \left\{ |Du_j|^2 - 1 \right\}^\frac{p}{2} \, dx \to m.
\]
We can assume that
\[
\int_{\Omega} \left\{ |Du_j|^2 - 1 \right\}^\frac{p}{2} \, dx \leq m + 1
\]
for all \( j \), and hence the sequence \((||Du_j||_p)_j\) is bounded. Minkowski’s inequality gives the estimate
\[
||u_j||_p \leq ||f||_p + ||f - u_j||_p.
\]
Using first Sobolev’s inequality, then Minkowski’s inequality, we arrive at
\[
||u_j||_p \leq ||f||_p + C_\Omega ||Df - Du_j||_p \leq ||f||_p + C_\Omega ((m + 1)^{1/p} + ||Df||_p),
\]
where the constant \( C_\Omega \) is only dependent upon the domain. The weak compactness of Sobolev spaces guarantees the existence of functions \( u \) and \( w \) such that, for some subsequence we have that
\[
\lim_{i \to \infty} u_{j_i} = u
\]
and
\[
\lim_{i \to \infty} Du_{j_i} = w
\]
weakly in \( L^p(\Omega) \). By the definition of Sobolev derivatives, \( w = Du \). \( \square \)
Having established the existence of at least one minimizer, we turn to the problem of uniqueness. Since
\[ \int_{|Du|<1} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} dx = 0, \]
we have no information where the gradient is less than one, and therefore no uniqueness in this "dead core." We prove uniqueness outside this subset.

Set \( A_p = \{ x \in \Omega : |Du|^2 - 1 < 0 \} \) and \( B_p = \{ x \in \Omega : |Dv|^2 - 1 < 0 \} \).

**Theorem 2.2.** Let \( u, v \) be two minimizers of (2.1), and assume \( A_p = B_p \equiv A \). Then \( u = v \) in \( \Omega \setminus A \).

**Proof.** Let
\[ m = \int_{\Omega} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} dx, \]
and assume that \( u, v \) both minimize (2.1). Then the function \((u+v)/2\) is admissible, and we get
\[ m = \int_{\Omega} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} dx \leq \int_{\Omega} \left\{ \frac{|Du + Dv|^2}{2} - 1 \right\}^\frac{p}{2} dx \]
Jensen’s inequality
\[ \left| \frac{a+b}{2} \right|^2 \leq \frac{1}{2} |a|^2 + \frac{1}{2} |b|^2 \]
is strict, if \( a \neq b \). Using
\[ \left| \frac{|Du + Dv|^2}{2} - 1 \right| \leq \frac{|Du|^2 - 1}{2} + \frac{|Dv|^2 - 1}{2}, \]
we conclude that the integral in (2.2) can be taken over \( \Omega \setminus A \). If \( Du \neq Dv \) in a subset of \( \Omega \setminus A \) of positive measure, we get
\[ m < \int_{\Omega \setminus A} \left( \frac{|Du|^2 - 1 + |Dv|^2 - 1}{2} \right)^\frac{p}{2} dx \]
\[ \leq \int_{\Omega \setminus A} \frac{1}{2} \left( |Du|^2 - 1 \right)^\frac{p}{2} + \frac{1}{2} \left( |Dv|^2 - 1 \right)^\frac{p}{2} dx \]
\[ = \int_{\Omega} \frac{1}{2} \left\{ |Du|^2 - 1 \right\}^\frac{p}{2} + \frac{1}{2} \left\{ |Dv|^2 - 1 \right\}^\frac{p}{2} dx = \frac{1}{2} m + \frac{1}{2} m = m. \]
To avoid the contradiction \( m < m \), we must have \( Du = Dv \) almost everywhere in \( \Omega \setminus A \), and hence in \( \Omega \). Then \( u = v + c \), for a constant
contradicting the fact that $u - v \in W^{1,p}_0(\Omega)$. Hence we must have $u = v$. \hfill \Box

**Theorem 2.3.** Let $u \in W^{1,p}(\Omega)$. Then the conditions

(i) $u$ is minimizing,

$$
\int_{\Omega} \{ |Du|^2 - 1 \}^{\frac{p}{p-1}} dx \leq \int_{\Omega} \{ |Dv|^2 - 1 \}^{\frac{p}{p-1}} dx,
$$

when $v - u \in W^{1,p}_0$.

(ii) the first variation vanishes,

$$
(2.3) \quad \int_{\Omega} \{ |Du|^2 - 1 \}^{\frac{p}{p-1}} \langle Du, D\phi \rangle dx = 0,
$$

for $\phi \in C^\infty_0(\Omega)$,

are equivalent.

**Proof.** (i) $\implies$ (ii).

Assume that $u$ is minimizing, and define, for any smooth $\phi$,

$$
I(\epsilon) = \int_{\Omega} \{ |(u + \epsilon \phi)|^2 - 1 \}^{\frac{p}{p-1}} dx.
$$

We then calculate

$$
I'(\epsilon) = \frac{p}{2} \int_{\Omega} \{ |(u + \epsilon \phi)|^2 - 1 \}^{\frac{p}{p-1}-1} \langle (Du + \epsilon D\phi), D\phi \rangle dx
$$

Since $u$ is minimizing, $I$ attains a minimum for $\epsilon = 0$. Hence $I'(0) = 0$, and (2.3) follows.

(ii) $\implies$ (i):

Assume that

$$
\int_{\Omega} \{ |Dv|^2 - 1 \}^{\frac{p}{p-1}} \langle Du, D\phi \rangle dx = 0
$$

for every $\phi \in C^\infty_0$. The function $x \mapsto \{ |x|^2 - 1 \}^{p/2}$ is convex. Thus

$$
\{ |x|^2 - 1 \}^{p/2} \geq \{ |y|^2 - 1 \}^{p/2} + p\{ |y|^2 - 1 \}^{p/2-1} y, x - y
$$

for all vectors $x, y$. This implies that

$$
(2.4) \quad \{ |Dv|^2 - 1 \}^{p/2} \geq \{ |Du|^2 - 1 \}^{p/2} + p\{ |Du|^2 - 1 \}^{p/2-1} Du, D(v-u).
$$

We integrate this inequality to obtain

$$
\int_{\Omega} \{ |Dv|^2 - 1 \}^{p/2} dx
$$

$$
\geq \int_{\Omega} \{ |Du|^2 - 1 \}^{p/2} dx + p \int_{\Omega} \{ |Du|^2 - 1 \}^{p/2-1} Du, D(v-u) \ dx.
$$
If (ii) holds, we can take $\phi = v - u$ to see that the last integral vanishes. Thus $u$ is minimizing. □

Theorem 2.4. If $u \in C^2(\Omega)$ and $p > 4$, the minimizers of the variational integral (2.1) have Euler-Lagrange equation

$$L_p u := \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} (|Du|^2 - 1) + \Delta u + (p - 2) \Delta_\infty u = 0.$$ 

Here $\Delta_\infty u$ is the $\infty$-Laplace operator of $u$:

$$\Delta_\infty u = \sum_{i,j=1}^n u_{x_i}u_{x_j}u_{x_i x_j}.$$ 

Proof. A straightforward application of the divergence theorem on (2.3) gives (2.5). □

In the proof of Theorem 2.3 we used a function $\phi \in W^{1,p}_0(\Omega)$ as a test function in the weak formulation. This requires a standard argument.

Lemma 2.5. If (2.3) holds for every function in $C_0^\infty(\Omega)$, so it does for every $\phi \in W^{1,p}_0(\Omega)$.

Proof. Since $W^{1,p}_0(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the Sobolev norm (1.4), we have that given a $w \in W^{1,p}_0(\Omega)$, for every $\epsilon > 0$ there exists a $\phi \in C_0^\infty(\Omega)$ so that $||w - \phi||_{W^{1,p}_0(\Omega)} < \epsilon$. Now the triangle inequality gives

$$\left| \int_\Omega \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} \langle Du, D\phi \rangle \, dx \right| \leq \int_\Omega \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} \langle Du, D(w - \phi) \rangle \, dx,$$

where the first integral is zero, by the definition of weak solutions. Again using the triangle inequality, then Hölder’s inequality:

$$\int_\Omega \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} \langle Du, D(w - \phi) \rangle \, dx \leq \int_\Omega \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} |Du| \cdot |D(w - \phi)| \, dx \leq \left\| \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} |Du| \right\|_{p/(p-1)} \| D(w - \phi) \|_p \leq \left\| |Du|^{p-2} |Du| \right\|_{p/(p-1)} \| D(w - \phi) \|_p \leq \| Du \|_{p-1} \| D(w - \phi) \|_{W^{1,p}_0(\Omega)} < K\epsilon,$$

where we used that

$$\left\| \{ |Du|^2 - 1 \}_{+}^{\frac{p-4}{2}} \right\|_{p/(p-1)} \leq \left\| |Du|^{p-2} \right\|_{p/(p-1)}.$$
Remark. In what follows, we may safely assume that weak solutions of (2.5) are continuous. Indeed, since we later on shall let \( p \to \infty \), we assume that \( p > \max(4, n) \), and hence Morrey’s inequality implies that for every \( u \in W^{1,p}(\Omega) \) there exists a continuous \( u^* \) equal to \( u \) almost everywhere. We can therefore always choose this continuous version of \( u \). See for example [Jos02].

We assume that \( p > 4 \) to ensure that (2.5) makes sense.

**Theorem 2.6 (Comparison).** Assume that \( u, v \in C(\overline{\Omega}) \) are solutions of (2.3), so that \( A_p = B_p \) and that \( u \leq v \) on \( \partial \Omega \). Then \( u \leq v \) in \( \Omega \setminus A_p \).

**Proof.** We argue by contradiction. Assume that \( u \leq v \) on \( \partial \Omega \), but that there exists a subdomain \( \Sigma \subset \Omega \setminus A_p \) with positive measure such that \( u > v \) on \( \Sigma \). Then, \( u = v \) on \( \partial \Sigma \).

Consider the variational integral
\[
\int_{\Sigma} \{|Du|^2 - 1\}^{\frac{p}{2}} \, dx,
\]
where the admissible functions are all \( u, u - v \in W^{1,p}_0 \). From Theorem 2.2, we know that this integral has a unique minimizer \( u \) contradicting the existence of two minimizers there.

Since minimizers are unique by Theorem 2.2 and Theorem 2.3 establishes that minimizers are precisely weak solutions to (2.5), we get the following:

**Theorem 2.7.** Equation (2.5) for \( p > n \) has a unique, continuous solution \( u \in W^{1,p}(\Omega) \).

3. **Two Limit Procedures**

We move on to viscosity solutions of (2.5). The following definition is found in [CIL92].

**Definition.** We say that \( u \in USC(\Omega) \) is a viscosity supersolution of (2.5) if, whenever \( u - \phi \) has a maximum at \( x_0 \) we have that
\[
L_p \phi(x_0) \leq 0
\]
Also, \( u \in LSC(\Omega) \) is a viscosity subsolution if, whenever \( u - \phi \) has a minimum at \( x_0 \) we have that
\[
L_p \phi(x_0) \geq 0
\]
We agree that \( |D\phi(x_0)| \leq 1 \) in the dead core.

A viscosity solution is both a viscosity sub- and supersolution.
Note that since a viscosity solution is both upper and lower semi-continuous, it is continuous by definition.

**Theorem 3.1.** Weak solutions are viscosity solutions.

**Proof.** Let \( x_0 \in \Omega \). If \( x_0 \in A_p \) there is nothing to prove, since we only know that \( u \) satisfies \( |Du| < 1 \) here. For \( x_0 \in \Omega \setminus A_p \), we make the following antithesis:

Assume that \((u - \phi)(x_0) > (u - \phi)(x)\) for all \( x \in \Omega \setminus \{x_0\} \),

while \( L_p \phi(x_0) < 0 \).

We note that since we assumed that \( L_p \phi(x_0) < 0 \), we have by definition that \( |D\phi(x_0)| > 1 \).

Since \( \phi \in C^2(\Omega) \), we must have that \( L_p \phi < 0 \) in a ball \( B_r(x_0) \subset \Omega \setminus A_p \). Let \( m_r = \max_{\partial B_r(x_0)} \{u - \phi\} \). Clearly \( m_r > 0 \) by our assumption. Define

\[
\bar{\phi} = \phi + m_r/2.
\]

Then \( L_p \bar{\phi} < 0 \), and \( \bar{\phi} > u \) on \( \partial B_r(x_0) \). Let \( U = \{x \mid \bar{\phi}(x) > u(x)\} \subset B_r(x_0) \). The Comparison Principle Theorem 2.6 gives that \( U \) is empty, a contradiction. \( \square \)

We derive the limit equation of (2.5) as \( p \to \infty \).

**Theorem 3.2.** As \( p \to \infty \), the viscosity solution \( u_p \) of (2.5) converges to a viscosity solution \( u_\infty \) of

\[
\{ |Du_\infty|^2 - 1 \} _+ \Delta_\infty u_\infty = 0.
\]

**Proof.** Let \( u_p \) solve

\[
\begin{cases}
\{ |Du_p|^2 - 1 \} _+ (\{ |Du_p|^2 - 1 \} _+ \Delta u_p + (p-2) \Delta_\infty u_p) = 0 & \text{in } \Omega \\
u_p = f & \text{on } \partial \Omega.
\end{cases}
\]

We have then that \( u_p - f \in W_0^{1,p}(\Omega) \), and since we can assume that \( p > n \) and \( u_p \in W^{1,p}(\Omega) \cap C(\Omega) \). Fix \( q < p \). Hölder’s inequality gives

\[
\left( \frac{1}{|\Omega|} \int_{\Omega} \{ |Du_p|^2 - 1 \} _+^{q/2} \ dx \right)^{2/q} \leq \left( \frac{1}{|\Omega|} \int_{\Omega} \{ |Du_p|^2 - 1 \} _+^{p/2} \ dx \right)^{2/p} \leq \frac{1}{|\Omega|} \int_{\Omega} \{ |Df|^2 - 1 \} _+^{p/2} \ dx \right)^{2/p} \leq K,
\]

since \( u_p \) is minimizing, and hence \( ||Du_p||_q \) is bounded independently of \( p \). Using Minkowski’s inequality and Sobolev’s inequality, we get

\[
||u_p||_q \leq ||u_p - f||_q + ||f||_q \\
\leq C||Du_p - Df||_q + M \leq C(K + M) + M.
\]
This implies that for every \( p \), the weak compactness of the \( L^q \)-spaces guarantees that the existence of a subsequence, also named \( u_p \) that converges weakly to a function \( u_\infty \) in every \( L^q, q < p \), while the Rellich-Kondrachov Compactness criterion gives pointwise convergence a.e. in \( \Omega \).

Theorem 7.17 in [GT01] gives the following inequality, valid for all cubes \( Q \subset \Omega \) and \( p > q \):

\[
|u_p(x) - u_p(y)| \leq \frac{2qn}{q - n} |x - y|^{1 - n/q} ||Du_p||_q.
\]

This implies the bound

\[
|u_p(x) - u_p(y)| \leq 2nK \text{diam}(\Omega)^{1 - n/q}.
\]

Since we also have the inequality

\[
||u_p||_q \leq c(n, q)|\Omega|^{1/n - 1/q} ||Du_p||_q \leq Kc(n, q)|\Omega|^{1/n - 1/q},
\]

this shows that \( u_p \) is equicontinuous in \( \Omega \), and Arzelà-Ascoli’s Theorem gives the existence of a subsequence, again labeled \( u_p \), that converges locally uniformly to a continuous function \( u_\infty \).

Assume that \( u_p \) is a viscosity supersolution to (2.5). We want to show that, as \( p \to \infty \), \( u_p \) converge to a viscosity supersolution of (3.1).

Assume therefore that \( u_\infty - \phi \) has a maximum at \( x_0 \in \Omega \). We want to show that this implies

\[
(3.2) \quad \{|D\phi(x_0)|^2 - 1\} + \Delta_\infty \phi(x_0) \leq 0.
\]

Let \( u_p - \phi \) have a maximum at \( x_p \in \Omega \). By definition of viscosity subsolution, we then have

\[
\{|D\phi(x_p)|^2 - 1\} \leq (p - 2) \Delta_\infty \phi(x_p) + \Delta \phi(x_p) + (|D\phi(x_p)|^2 - 1).
\]

If \( |D\phi(x_p)| - 1 \leq 0 \), for all \( p \) greater than some \( N \), we have that (3.2) is trivially true.

If, on the other hand, \( |D\phi(x_p)| - 1 > 0 \) for \( p > N \), we divide through by \( (|D\phi(x_p)|^2 - 1) \frac{p - 4}{p - 2} (p - 2) > 0 \) to get

\[
\frac{|D\phi(x_p)|^2 - 1}{p - 2} \Delta \phi(x_p) + \Delta_\infty \phi(x_p) \leq 0
\]

Letting \( p \to \infty \), (possibly along a subsequence) we have that \( x_p \to x_0 \) uniformly in \( \Omega \), and that \( D\phi(x_p) \to D\phi(x_0) \) and \( D^2 \phi(x_p) \to D^2 \phi(x_0) \). This implies that

\[
\Delta_\infty \phi(x_0) \leq 0,
\]

whenever \( u_\infty - \phi \) has a maximum at \( x_0 \). This implies (3.2), and that \( u_\infty \) is a viscosity supersolution of (3.1).

The proof for subsolutions is similar.
It is clear that all the calculations up to this point remain valid if $1$ is replaced by $\sigma$. The next step is to show that as $\sigma \to 0$, we retrieve the $p$-Laplace equation.

**Theorem 3.3.** As $\sigma \to 0$, the viscosity solutions $u_\sigma$ of

$$S_\sigma^p u_\sigma := \{D u_\sigma |^2 - \sigma\}^\frac{p-2}{p} \{D u_\sigma |^2 \}^\frac{p-2}{p} (\{D u_\sigma |^2 - \sigma\} + \Delta u_\sigma + (p-2)\Delta_\infty u_\sigma) = 0$$

converge to the viscosity solution $u$ of the $p$-Laplace equation

$$\Delta_p u = 0$$

in $\Omega$.

**Proof.** The first step is to show the existence of a uniformly convergent subsequence $(u_\sigma_j)$. Since $u_\sigma$ minimizes the variational integral

$$\int_\Omega \{D u_\sigma |^2 - \sigma\}^\frac{p}{p+\varepsilon} dx,$$

we have that

$$\int_\Omega \{D u_\sigma |^2 - \sigma\}^\frac{p}{p+\varepsilon} dx \leq \int_\Omega \{D f |^2 - \sigma\}^\frac{p}{p+\varepsilon} dx \leq ||D f||_p \leq K,$$

and hence $||D u_\sigma||_p$ is bounded. Now Minkowski’s and Sobolev’s inequalities give

$$||u_\sigma||_p \leq ||u_\sigma - f||_p + ||f||_p \leq C_\Omega ||Du_\sigma - Df||_p + C < \infty.$$  

Hence the sequence $(u_\sigma)_\sigma$ is bounded in $W^{1,p}(\Omega)$, and the weak compactness gives the existence of a subsequence $(u_\sigma_j)_j$ that converges weakly to a function $u$. Furthermore, the Rellich-Kondrachov Compactness criterion gives that it converges pointwise almost everywhere.

Further, we again have from [GT01]

$$|u_\sigma(x) - u_\sigma(y)| \leq \frac{2pn}{p-n} ||Du_\sigma||_p |x - y|^{1-\frac{n}{p}} \leq 2nK \text{diam}(\Omega)^{1-\frac{n}{p}}$$

and so $(u_\sigma_j)_j$ is equicontinuous in $\Omega$. The Arzelà-Ascoli theorem gives the existence of a subsequence, which with a slight abuse of notation we shall also name $(u_\sigma_j)_j$, that converges locally uniformly to $u$.

Assume that $u - \phi$ has a maximum at $x_0 \in \Omega$. We want to show that this implies

$$\Delta_p \phi(x_0) \leq 0.$$  

for every $\phi \in C^2(\Omega)$. 

Let \( u_{\sigma_j} - \phi \) have a maximum at \( x_j \in \Omega \). By definition of viscosity subsolution, this implies that
\[
S_p^m \phi(x_j) \leq 0,
\]
and since \( u_{\sigma_j} \) converges uniformly to \( u \), we must have that \( x_j \to x_0 \).
This implies that \( D\phi(x_j) \to D\phi(x_0) \) and \( D^2\phi(x_j) \to D^2\phi(x_0) \). Furthermore, since
\[
\{|D\phi(x_j)|^2 - \sigma_j\}_+ \to |D\phi(x_0)|^2,
\]
we get, as \( \sigma_j \to 0 \),
\[
\Delta_p \phi(x_0) \leq 0
\]
whenever \( u - \phi \) has a maximum at \( x_0 \).

Now it could happen that upon choosing another subsequence of \( (u_{\sigma})_\sigma \) we would end up with a different limit function \( u^* \), but uniqueness of viscosity solutions of \( (3.4) \) prohibits this. \( \square \)

4. Patching

Consider the equation
\[
\{|Du_{\sigma}|^2 - \sigma|\}_+ \Delta_{\infty} u_{\sigma} = 0
\]
in \( \Omega \), and let \( A_\sigma = \{x \in \Omega \mid |Du(x)| < \sigma\} \). If \( A_\sigma \neq \emptyset \), the only information about the solution \( u_{\sigma} \) we have in \( A_\sigma \) is that \( |Du_{\sigma}|^2 < \sigma \). Since there are infinitely many such functions, uniqueness certainly fails here. It is natural to choose one of these solutions, namely the “biggest,” that is the one where \( |Du_{\sigma}|^2 = \sigma \). This leads to the “patching” solutions introduced by Crandall, Gunmarsson and Wang in [CGW07].

**Definition.** Define the patched solution \( h_\sigma \) as follows: Let \( h_\sigma \) be the unique viscosity solution of
\[
\Delta_{\infty} h_\sigma = 0 \text{ in } \Omega \setminus A_\sigma,
\]
\[
h_\sigma(x) = \sup_{y \in \partial U} (h(y) - \sigma d_U(x,y))
\]
for each connected component \( U \) of \( A_\sigma \).

The following Lemma is contained in [CGW07].

**Lemma 4.1.** Let \( h_\sigma \) defined as in (4.1). Then
1) \( \Delta_{\infty} h_\sigma \geq 0 \) in the viscosity sense.
3) \( h_\sigma = h \text{ in } \Omega \setminus A_\sigma \) and \( h_\sigma \leq h \text{ in } \Omega \).
4) \( h_\sigma \) is the (unique) viscosity solution of the Eikonal equation
\[
\begin{cases}
|Dh_\sigma|^2 = \sigma^2 \text{ in } U, \\
h_\sigma = h|_{\partial U},
\end{cases}
\]
for each connected component \( U \) of \( A_\sigma \).
If we let $z_\sigma$ be the viscosity solution of Jensen’s lower equation
\begin{equation}
\min\{\Delta_\infty z_\sigma, |Dz_\sigma|^2 - \sigma^2\} = 0,
\end{equation}
with $z_\sigma|_{\partial\Omega} = h_\sigma|_{\partial\Omega}$, we get the following

**Lemma 4.2.** Let $h_\sigma$ be defined by the viscosity solution of (4.1). Then $h_\sigma = z_\sigma$.

**Proof.** Let $\phi \in C^2(\Omega)$ be so that $h_\sigma - \phi$ has a minimum at $x_0 \in \Omega$.

Case 1: $x_0 \in \Omega \setminus A_\sigma$. Then by definition $h_\sigma(x_0) = h(x_0)$. Also, since $h_\sigma \leq h$ in $\Omega$, we have that $h - \phi$ has a minimum at $x_0$. Since $h$ satisfies $\Delta_\infty h = 0$, we must have
\[\min\{\Delta_\infty \phi(x_0), |D\phi(x_0)|^2 - \sigma^2\} \leq \Delta_\infty \phi(x_0) \leq 0.\]

Case 2: $x_0 \in A_\sigma$. Then we must have that $|Dh_\sigma(x_0)| - \sigma \leq 0$ and again we see that
\[\min\{\Delta_\infty \phi(x_0), |D\phi(x_0)|^2 - \sigma^2\} \leq 0,
\]
proving that $h_\sigma$ is a viscosity subsolution to (4.2).

On the other hand, assume that $h_\sigma - \phi$ has a maximum at $x_0$. To prove that $h_\sigma$ is a viscosity subsolution to (4.2), we need to prove that $|D\phi(x_0)|^2 - \sigma^2 > 0$ and $\Delta_\infty \phi(x) > 0$.

The second inequality is true everywhere by the Lemma 4.1, whilst for the other inequality we have
\[\max_{B_\epsilon(x)} \phi - \phi(x_0) \geq \max_{B_\epsilon(x)} h_\sigma - h_\sigma(x_0) \geq rL(h_\sigma, x) \geq r\sigma.\]

Dividing by $r$ and letting $r \to 0$ gives that $|D\phi(x_0)|^2 - \sigma^2 > 0$ in $\Omega \setminus A_\sigma$. Hence $h_\sigma$ is a viscosity subsolution to (4.2). The fact that $h_\sigma = z_\sigma$ comes from the uniqueness of solutions of (4.2).

We turn to the last part of the proof of Theorem 1.1. The proof that $u_\sigma$ converges to the $\infty$-harmonic function in $\Omega$ runs similar to Theorem 2.1 in [CGW07].

**Theorem 4.3.** Let $u_\sigma$ be defined as above. the $u_\sigma \to u$ as $\sigma \to 0$.

**Proof.** If $|Du(x)| > \sigma$ then by construction $u_\sigma(x) = u(x)$ so there is nothing to prove. On the other hand, if $|Du(x)| = 0$, we let $N = \{x \in \Omega : |Du(x)| = 0\}$. Let $y \in \partial N$ be the closest point on $\partial N$ to $x$. Then, since $u$ by definition is constant here, $u(x) = u(y)$ and by construction of $u_\sigma(x)$ we have that $|u_\sigma(x) - u_\sigma(y)| \leq \sigma|x - y|$, we have
\[|u_\sigma(x) - u(x)| \leq |u_\sigma(x) - u_\sigma(y)| + |u_\sigma(y) - u(y)| + |u(y) - u(x)| \leq \sigma|x - y| + |u_\sigma(y) - u(y)| + 0.
\]

Letting $\sigma \to 0$ we see that $|u_\sigma(x) - u(x)| \to 0$, since $u_\sigma(y) \to u(y)$. \(\Box\)
We know from [BDM89] and [Jen93] that as $p \to \infty$ we have that viscosity solutions $u_p$ of the $p$-Laplace equation converge uniformly to the $\infty$-harmonic function. This, combined with Theorem 3.2, Theorem 3.3 and Theorem 4.3, we get the following diagram of convergence:

$$
\begin{array}{c}
\xymatrix{ u_{p,\sigma} \ar[r]^{p \to \infty} & u_\sigma \\
 u_p \ar[r]^{p \to \infty} & u_\infty \ar[d]_{\sigma \to 0} \\
 & \sigma \to 0
}\end{array}
$$

proving Theorem 1.1.

5. $\Gamma$-CONVERGENCE

In this Section we prove Theorem 1.2. This establishes that (1.3) is the "correct" approximation to the functional $||Du||_\infty$. The following definition is found in [Bra02].

**Definition.** We say that the functional $E_n$ $\Gamma$-converges to $E$ if

1) (The $\Gamma$-lim inf)

Whenever $u_n \to u$ in $X$, we have

$$
\liminf_{n \to \infty} E_n(u_n) \geq E(u)
$$

2) (The $\Gamma$-lim sup)

For every $u \in X$, there exists a sequence $u_n$ (called the recovery sequence) so that $u_n \to u$ and

$$
\limsup_{n \to \infty} E_n(u_n) \leq E(u)
$$

Define

$$
E_p^\sigma(u) = \left( \int_\Omega \{|Du|^2 - \sigma\}^{p/2} dx \right)^{1/p} = \|\{|Du|^2 - \sigma\}^{1/2}\|_p
$$

$$
E_p(u) = \|Du\|_p,
$$

$$
E_\infty^\sigma(u) = \|\{|Du|^2 - \sigma\}^{1/2}\|_\infty,
$$

and

$$
E_\infty(u) = \|Du\|_\infty.
$$

**Theorem 5.1.** $\|\{|Du|^2 - \sigma\}^{1/2}\|_p$ $\Gamma$-converges to $\|\{|Du| - \sigma\}^{1/2}\|_\infty$ with respect to uniform convergence in $C(\Omega)$. 
Proof. Assume that \((u_p)_p \subset W^{1,p}(\Omega)\) is such that \(\|Du_{p_j}\|_{p_j} \leq C\) for some subsequence \(p_j \rightarrow \infty\) as \(j \rightarrow \infty\). Our goal is to extract a subsequence of \(u_p\) that converges uniformly to a function \(u \in C(\Omega)\).

Fix \(q > 2\). Then Hölder’s inequality gives the estimate
\[
\|\{ |Du_{p_j}|^2 - \sigma \}_+^{1/2}\|_q \leq |\Omega|^{1/2} \sup_{j} \|\{ |Du_{p_j}|^2 - \sigma \}_+^{1/2}\|_{p_j}
\]
for all \(p_j \geq q\). Further, the Poincaré inequality gives
\[
\|u_{p_j} - u_{\Omega, p_j}\|_{W^{1,q}(\Omega)} \leq C \|Du_{p_j}\|_{W^{1,q}},
\]
so that the sequence \(\|u_{p_j} - u_{\Omega, p_j}\|_{W^{1,q}(\Omega)}\) is bounded. Hence the weak compactness of the \(W^{1,q}\)-spaces implies the existence a subsequence that converges weakly in \(W^{1,q}(\Omega)\) to some \(u_q\). A diagonal procedure now gives a new subsequence, labeled \(u_k\) for convenience, so that
\[
(u_k - u_{k,\Omega}) \rightarrow u \text{ as } k \rightarrow \infty
\]
weakly in \(W^{1,q}(\Omega)\) for all \(2 < q < \infty\). This implies that the limit function \(u\) is in \(W^{1,q}(\Omega)\) for all \(q\). The lower semi-continuity of the \(q\)-norm gives
\[
\|\{ |Du|^2 - \sigma \}_+^{1/2}\|_q \leq \liminf_{k \rightarrow \infty} \|\{ |Du_k|^2 - \sigma \}_+^{1/2}\|_q,
\]
which together with the estimate (5.7) gives
\[
\|\{ |Du|^2 - \sigma \}_+^{1/2}\|_q \leq |\Omega|^{1/2} \liminf_{k \rightarrow \infty} \left( |\Omega|^{-1/2} \|\{ |Du_k|^2 - \sigma \}_+^{1/2}\|_k \right)
\]
\[
= |\Omega|^{1/2} \liminf_{k \rightarrow \infty} \|\{ |Du_k|^2 - \sigma \}_+^{1/2}\|_k.
\]
As \(q \rightarrow \infty\), we get
\[
(5.8) \quad \|\{ |Du|^2 - \sigma \}_+^{1/2}\|_{\infty} \leq \liminf_{k \rightarrow \infty} \|\{ |Du_k|^2 - \sigma \}_+^{1/2}\|_k.
\]
We see that we have \(\{ |Du|^2 - \sigma \}_+^{1/2} \in L^{\infty}(\Omega)\), and so \(u \in W^{1,\infty}(\Omega)\) and \(u \in C(\Omega)\).

Fix a \(q > n\), and let \(V \subset \Omega\) be a sub-domain with regular boundary. Morrey’s inequality and then Poincaré inequality gives
\[
\|u_k - u_{k,\Omega}\|_{C^{0,1-\frac{n}{q}}(V)} \leq C(q, n)\|u_k - u_{k,\Omega}\|_{W^{1,q}(V)}
\]
\[
\leq C(q, n)\|u_k - u_{k,\Omega}\|_{W^{1,q}(\Omega)}
\]
\[
\leq C(q, n)\|Du_k\|_q \leq K < \infty,
\]
for all \(k \geq q\). Thus there exists a subsequence of \(u_k\) that converges in \(L^{\infty}(V)\) to \(u\). Exhausting \(\Omega\) with an increasing sequence of regular sets, a diagonal argument gives
\[
(5.9) \quad (u_k - u_{k,\Omega}) \rightarrow u \text{ in } L^{\infty}(\Omega).
\]
We shall prove the $\Gamma$-lim inf property, that is that for every sequence $u_p$ that converges uniformly to $u$ in $C(\Omega)$, we have that

$$E^\sigma_\infty(u) \leq \liminf_{p \to \infty} E^\sigma_p(u_p).$$

This follows directly from the estimate (5.8), together with the uniform convergence of $u_p$ in (5.9).

Further we prove the $\Gamma$-lim sup property, that is for every $u \in C(\Omega)$ there exists a sequence $u_p$ (called the recovery sequence of $u$) converging uniformly to $u$ so that

$$E^\sigma_\infty(u) \geq \limsup_{p \to \infty} E^\sigma_p(u_p).$$

Since $\|f\|_{L^\infty(\Omega)} = \lim_{p \to \infty} \|f\|_{L^p(\Omega)}$ holds for all measurable functions $f$, the $\Gamma$-lim sup property follows immediately with $f = \{|Du| - \sigma\}^\frac{1}{2}$ and $u_p = u$ for all $p$. □

The fundamental theorem properties of $\Gamma$-convergence gives, see [Bra02]:

1. If $\lim_{p \to \infty}(E^\sigma_p(u_p) - \min E^\sigma_p) = 0$ then $u_p \to u$ in $C(\Omega)$, and $E^\sigma_\infty(u) = \min E^\sigma_\infty$.

2. If $E^\sigma_\infty(u) = \min E^\sigma_\infty$, then there exists a sequence $u_p$ with $u_p \to u$ as $p \to \infty$ so that $\lim_{p \to \infty}(E^\sigma_p(u_p) - \min E^\sigma_p) = 0$.

This implies that any sequence $u_p$ of viscosity solutions of (2.5) accumulate at a minimiser of $E^\sigma_\infty$. Using this, we can prove the following analogue to the classical Absolutely Minimizing Lipschitz Extension property of $\infty$-harmonic functions described in [BDM89].

**Theorem 5.2.** Let $u$ be the limit of minimizers. Then for every $V \subset \Omega \setminus \{|Du|^2 < \sigma\}$ we have that

$$\|\{|Du| - \sigma\}^\frac{1}{2}\|_{L^\infty(V)} \leq \|\{|Dw| - \sigma\}^\frac{1}{2}\|_{L^\infty(V)}$$

for every $w \in W^{1,\infty}(\Omega) \cap C(\Omega)$, $u = w$ on $\partial V$.

**Proof.** The proof mimics [BDM89].

Let $w \in W^{1,\infty}(V) \cap C(\overline{V})$ be given, and consider $\{w > u\} \subset V$. Fix $\epsilon > 0$ so that $\{w > u + \epsilon\}$ is an open, non-empty subset of $\{w > u\}$. In view of uniform convergence of $u_p$, fix $p$ big enough so that

$$\{w > u + \epsilon\} \subset \{w > u_p + \epsilon/2\} \subset \{w > u\}$$
We get
\[
\int_{\{w > u + \epsilon\}} \{\|Du\|^2 - \sigma\}^{p/2} dx \leq \int_{\{w > u + \epsilon/2\}} \{\|Du\|^2 - \sigma\}^{p/2} dx
\]
\[
\leq \int_{\{w > u + \epsilon/2\}} \{|D(w - \epsilon/2)|^2 - \sigma\}^{p/2} dx
\]
\[
\leq \|\{|Du|^2 - \sigma\} + \|L^\infty(\{w > u + \epsilon\})\|\{\{w > u + \epsilon\}\}
\]
Raising both sides of the inequality to $1/p$ and $\lim \inf_{p \to \infty}$, we get
\[
\|\{|Du|^2 - \sigma\} + \|L^\infty(\{w > u + \epsilon\})\|\{\{w > u + \epsilon\}\}
\]
Since $\epsilon$ was arbitrary, we get
\[
\|\{|Du|^2 - \sigma\} + \|L^\infty(\{w > u\})\|\{\{w > u\}\}
\]
and the argument above can be repeated with the set $\{w < u\}$. Since $Du = Dw$ in $\{u = w\}$, we have
\[
\|\{|Du|^2 - \sigma\} + \|L^\infty(\{w > u\})\|\{\{w > u\}\}
\]

The proof that $E_p \Gamma$-converges to $E_\infty$ is very similar. All the arguments in the proof of Theorem 5.1 is true for $\sigma = 0$, and so we get

**Theorem 5.3.** As $p \to \infty$,
\[
E_p \Gamma \to E_\infty,
\]
with respect to uniform convergence.

We prove that as $\sigma \to 0$, we retrieve the well-known $p$-energy functionals related to the $p$-Laplace equation.

**Theorem 5.4.** Let $p > n$. Then
\[
E_p^\sigma \Gamma \to E_p \quad \text{as} \quad \sigma \to 0,
\]
in the uniform convergence topology on $C(\overline{\Omega})$. 
Proof. We have from before that \( \|Du_\sigma\|_p \) and \( \|u_\sigma\|_p \) are bounded. Since \( p > n \), Morrey’s inequality implies that \( u \in C(V) \) for a regular \( V \subset \Omega \). Well-known bounds give that the sequence \( u_\sigma \) is equicontinuous on \( V \), and Arzelá-Ascoli compactness criterion implies that \( u_\sigma \to u \) as \( \sigma \to 0 \) uniformly on \( V \). Exhausting \( \Omega \) with regular sets, a diagonal procedure gives \( u_\sigma \to u \) as \( \sigma \to 0 \) uniformly on \( \Omega \).

To prove the \( \Gamma \)-lim inf property (5.1) we must show that for every \( u_\sigma \) that converges uniformly to \( u \) we have

\[
(5.13) \quad \|Du\|_p \leq \liminf_{\sigma \to 0} \|\{|Du_\sigma|^2 - \sigma\}^{1/2}\|_p.
\]

Clearly \( \{|Du_\sigma|^2 - \sigma\}^{p/2}_+ \to |Du|^p \) as \( \sigma \to 0 \), and so Fatou’s lemma gives (5.13).

For the \( \Gamma \)-lim sup, we define our recovery sequence by \( u_\sigma = u \) for all \( \sigma > 0 \). Clearly \( u_\sigma \to u \), and

\[
(5.14) \quad \{|Du|^2 - \sigma\}^{1/2}_+ \leq |Du|,
\]

so raising both sides to the power \( p/2 \), integrating over \( \Omega \) and taking \( \limsup \), we get

\[
\limsup_{\sigma \to 0} \int_{\Omega} \{|Du|^2 - \sigma\}^{p/2}_+ \, dx \leq \int_{\Omega} |Du|^p \, dx,
\]

showing that property (5.2) holds, and so \( E_\sigma^p \) \( \Gamma \)-converges to \( E_p \) with respect to uniform convergence. \( \square \)

For the last convergence in Theorem 1.2, we note that since (5.13) holds for all \( p > n \), it also holds in the limit \( p \to \infty \). This combined with (5.14), shows that the following Theorem holds.

**Theorem 5.5.** As \( \sigma \to 0 \),

\[
E_\sigma^\infty \Gamma \to E_\infty,
\]

with respect to uniform convergence.

We have that Theorem 5.1, Theorem 5.3, Theorem 5.5 and Theorem 5.10 together prove Theorem 1.2.

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