APPROXIMATE TANGENTS, HARMONIC MEASURE AND DOMAINS WITH RECTIFIABLE BOUNDARIES

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In memory of G. I. Chatzopoulos

ABSTRACT. We show that if $E \subset \mathbb{R}^d$, $d \geq 2$ is a closed and weakly lower Ahlfors-David $m$–regular set, then the set of points where there exists an approximate tangent $m$–plane, $m \leq d$, can be written as the union of countably many Lipschitz graphs. This implies that any $m$–rectifiable and weak lower Ahlfors-David $m$–regular set $E$, for which $H^m|_E$ is locally finite, can be written as the union of countably many Lipschitz graphs up to set of $H^m$–measure zero. Moreover, let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$ be a connected domain with weak lower Ahlfors-David $n$–regular and $n$–rectifiable boundary so that $H^n|_E$ be locally finite. If the reduced boundary of $\Omega$ coincides with its topological boundary up to a set of $H^n$–measure zero, then $\partial \Omega$ can be covered $H^n$–almost everywhere by a countable union of Lipschitz domains which are contained in $\Omega$. This implies that in such a domain the Hausdorff measure $H^n$ is absolutely continuous with respect to harmonic measure $\omega_\Omega$.

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1. INTRODUCTION

Determining (mutual) absolute continuity of the harmonic measure associated to the Laplace operator and the $d$-Hausdorff measure in domains with “rough” boundaries has been a hot topic of research in mathematical analysis for almost four decades now. The interest in such questions can be
justified partially by the connection between (a quantitative version of) the
absolute continuity of the harmonic measure and the well-posedness of the
Dirichlet problem with data in some $L^p$ space (even for elliptic operators
of divergence form with merely bounded real coefficients).

Already in 1916, F. and M. Riesz [31] showed that for simply connected
planar domains, bounded by a Jordan curve, whose boundary has finite
length, harmonic measure and arc-length are mutually absolutely continuous.
Their theorem was improved by Lavrentiev [28] demonstrating that in a simply connected domain in the complex plane, bounded by a chord-
arc curve, the harmonic measure is in the $A_\infty$ class of Muckehoupt weights.

Bishop and Jones [12] proved a local version of F. and M. Riesz theorem
by showing that if $\Omega$ is a simply connected planar domain and $\Gamma$ is a curve
of finite length, then $\omega \ll H^1$ on $\partial \Omega \cap \Gamma$, where $\omega$ stands for the harmonic
measure. They also give an example of a domain $\Omega$ whose boundary is con-
tained in a curve of finite length, but $H^1(\partial \Omega) = 0 < \omega(\partial \Omega)$, thus showing
that some sort of connectedness in the boundary is required.

In higher dimensions, the situation is a lot more delicate. The obvious
generalization to higher dimensions is false due to examples of Wu and
Ziemer: they construct topological two-spheres in $\mathbb{R}^3$ with boundaries of fi-
nite Hausdorff measure $H^2$ where either harmonic measure is not absolutely continuous with respect to $H^2$ [33] or $H^2$ is not absolutely continuous with
respect to harmonic measure [34], respectively. In the affirmative direction,
Dahlberg shows in [16] that in a Lipschitz domain, the harmonic measure and the $d$-Hausdorff measure restricted to the boundary are $A_\infty$-equivalent.
The same result was proved by David and Jerison in [18] under the assump-
tions that $\Omega \subset \mathbb{R}^{d+1}$ is an NTA domain and $\partial \Omega$ is Ahlfors-David regular.

Recently, Azzam, Hofmann, Martell, Nyström and Toro [9] showed that
any uniform domain with uniformly rectifiable boundary is an NTA domain and thus, $\omega \in A_\infty$ by [18] (a direct proof of the $A_\infty$-equivalence between $\omega$ and $H^d|_{\partial \Omega}$ in this case was given earlier by Hofmann and Martell [22];
the converse implication is proved in [25] and a stronger version of it in [23]). One can also find similar results for domains with uniformly rectifi-
able boundaries (without the uniformity assumption) in [13]. Hofmann, Martell and Toro [24] recently obtained a characterization of uniform do-
 mains with uniformly rectifiable boundaries via the $A_\infty$ equivalence of the elliptic harmonic measure and the $d$-Hausdorff measure (for second order elliptic operators of divergence form with real, locally Lipschitz coefficients that satisfy a natural Carleson condition).

In [11], Badger shows that if one merely assumes $H^d|_{\partial \Omega}$ is locally finite and $\Omega \subset \mathbb{R}^{d+1}$ is NTA, then we still have $H^d|_{\partial \Omega} \ll \omega$. He also shows that
\( \omega \ll \mathcal{H}^d|_{\partial \Omega} \ll \omega \) on the set
\[ \{ x \in \partial \Omega : \liminf_{r \to 0} \mathcal{H}^d( B(x, r) \cap \partial \Omega ) / r^d < \infty \}. \]

The question whether NTA-ness of the domain is enough to obtain \( \omega \ll \mathcal{H}^d|_{\partial \Omega} \) was already answered in the negative by Wolff in [32], with the impressive construction of the so-called Wolff snowflakes. Although, there was a question in [11] whether this could be true under the additional assumption that \( \mathcal{H}^d|_{\partial \Omega} \) is locally finite. Recently, Azzam, Tolsa and the author [7] demonstrated that there exists an NTA domain with very flat boundary for which \( \mathcal{H}^d|_{\partial \Omega} \) is locally finite and yet, one can find a set \( E \subset \partial \Omega \) such that \( \omega(E) > 0 = \mathcal{H}^d(E) \).

However, it was left open whether one can show that \( \mathcal{H}^d|_{\partial \Omega} \ll \omega \) relaxing the geometric conditions of the domain. In fact, this was done in [30] and [2] simultaneously. It was proved that \( \mathcal{H}^d \ll \omega \) on \( \partial \Omega \), under the assumption that the domain is uniform and its boundary is Ahlfors-David \( d \)-regular and \( d \)-rectifiable (all the definitions can be found in section 2). In fact, [30] was slightly more general since instead of upper Ahlfors-David regularity, it was assumed that \( \mathcal{H}|_{\partial \Omega} \) is locally finite. It was a real challenge though to weaken the assumptions even further, which we do in the current paper. Let us state our main results.

**Theorem 1.1.** Let \( E \subset \mathbb{R}^d \), \( d \geq 2 \) be a closed and weakly lower Ahlfors-David \( m \)-regular set. For a fixed \( s \in (0, 1/3) \), let \( \mathcal{K} \subset \partial \Omega \) be the set of all points \( x \in E \) for which there exists an \( s \)-approximate tangent \( m \)-plane \( V_x \) for \( \partial \Omega \) at \( x \). Then there exists a countable collection of Lipschitz graphs \( \{ \Gamma_j \}_{j \geq 1} \) so that \( \mathcal{K} = \cup_{j \geq 1} \Gamma_j \). In particular, \( \mathcal{K} \) is \( m \)-rectifiable.

Notice that we do not assume that \( \mathcal{H}^m|_{E} \) is locally finite.

**Corollary 1.2.** Let \( E \subset \mathbb{R}^d \), \( d \geq 2 \) be a closed rectifiable and weakly lower Ahlfors-David \( m \)-regular set. If \( \mathcal{H}^n|_{\partial \Omega} \) is locally finite, then there exists a countable collection of Lipschitz graphs \( \{ \Gamma_j \}_{j \geq 1} \) so that \( E = \cup_{j \geq 1} \Gamma_j \cup F \), where \( \mathcal{H}^m(F) = 0 \).

The corollary above follows from Theorems [1.1] and [2.8]

**Theorem 1.3.** Let \( \Omega \subset \mathbb{R}^d \), \( d \geq 2 \) be an open set with weak lower Ahlfors-David \( (d-1) \)-regular boundary \( \partial \Omega \) so that \( \mathcal{H}^n|_{\partial \Omega} \) is locally finite and \( \mathcal{H}^{d-1}(\partial \Omega \setminus \partial^* \Omega) = 0 \). If \( \{ \Gamma_j \}_{j \geq 1} \) is the collection of the Lipschitz graphs constructed in Theorem [1.1] then for each \( j \geq 1 \), there is a bounded Lipschitz domain \( \Omega_{\Gamma_j} \subset \Omega \), so that \( \Gamma_j \subset \partial \Omega_{\Gamma_j} \).

**Theorem 1.4.** Let \( \Omega \subset \mathbb{R}^{n+1} \), \( n \geq 1 \) be an open connected set with \( n \)-rectifiable and weak lower Ahlfors-David \( n \)-regular boundary \( \partial \Omega \). If \( \mathcal{H}^n(\partial \Omega \setminus \partial^* \Omega) = 0 \) and \( \mathcal{H}^n|_{\partial \Omega} \) is locally finite, then \( \mathcal{H}^n|_{\partial \Omega} \ll \omega^p \).
The following corollary follows from Theorem 1.4 and the main theorem of [5].

**Corollary 1.5.** Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 1$ be an open connected set with $n$-rectifiable and weak lower Ahlfors-David $n$-regular boundary $\partial \Omega$. If $\mathcal{H}^n(\partial \Omega \setminus \partial^* \Omega) = 0$ and $\mathcal{H}^n|_{\partial \Omega}$ is locally finite, then $\mathcal{H}^n|_{\partial \Omega} \ll \omega^p$ if and only if $\partial \Omega$ is $n$-rectifiable.

While this manuscript was in a preliminary form, a preprint by Akman, Bortz, Hofmann and Martell [3] appeared on arxiv, where the authors obtained independently very similar results with the ones we proved in the current paper under slightly weaker assumptions. We would like to emphasize now that our original proofs and results were stated for locally lower Ahlfors-David regular sets but after reading the statements of the theorems in [3], we realized that the same proofs work under the weak lower Ahlfors-David regularity assumption. Also, we had stated Theorem 1.1 only for boundaries of domains, but the same was true for any set $E \subset \mathbb{R}^{n+1}$. Finally, we had falsely proved Theorem 1.3 under the following wrong claim: if the geometric outer normal unit vector exists at a point of the topological boundary $\partial \Omega$ then the measure-theoretic outer normal unit vector exists as well. The discussion in [3] helped us realize that we needed to assume in addition that the reduced boundary of the domain $\Omega$ coincides with its topological boundary apart from a set of $\mathcal{H}^n$-measure zero. Our original proof already contained the use of the properties of the reduced boundary and it was false without this additional assumption. We happily acknowledge the impact of [3] on the improvement of this manuscript.

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2. **Background material**

- If $A, B \subset \mathbb{R}^d$, we let
  
  \[ \text{dist}(A, B) = \inf \{|x - y| : x \in A, y \in B\}, \quad \text{dist}(x, A) = \text{dist}(\{x\}, A), \]

- $B(x, r)$ stands for the open ball of radius $r$ which is centered at $x$. We also denote by $\lambda B(x, r) = B(x, \lambda r)$.

- We will write $p \lesssim q$ if there is $C > 0$ so that $p \leq C q$ and $p \lesssim_M q$ if the constant $C$ depends on the parameter $M$. We write $p \sim q$ to mean $p \lesssim q \lesssim p$ and define $p \sim_M q$ similarly.
• $G(d, m)$ is the Grassmannian manifold of all $m$-dimensional linear subspaces of $\mathbb{R}^d$.
• We denote by $\pi_V : \mathbb{R}^d \to V$ the orthogonal projection on $V \in G(d, m)$.
• $V^\perp \in G(d, d - m)$ is the orthogonal complement of $V \in G(d, m)$.
• $f : E \subset \mathbb{R}^d \to \mathbb{R}^d$ is $L$-Lipschitz if for all $x, y \in E$, $|f(x) - f(y)| \leq L|x - y|$.
• $f : E \subset \mathbb{R}^d \to \mathbb{R}^d$ is called $L$-bi-Lipschitz if for all $x, y \in E$, $L^{-1}|x - y| \leq |f(x) - f(y)| \leq L|x - y|$.

We now recall some elements from geometric measure theory following closely [29].

For $A \subset \mathbb{R}^d$ and $s \in (0, d]$ we set $H^s_\delta(A) = \inf \left\{ \sum r_i^s : A \subset \bigcup B(x_i, r_i), x_i \in \mathbb{R}^d \right\}$.

Define the $s$-dimensional Hausdorff measure as $H^s(A) = \lim_{\delta \to 0} H^s_\delta(A)$ and the $s$-dimensional Hausdorff content as $H^s_\infty(A)$.

**Definition 2.1.** Let $0 \leq s < \infty$, $E \subset \mathbb{R}^d$ and $x \in \mathbb{R}^d$. The upper and lower $s$-densities of $E$ at $x$ are defined by

$$
\Theta^*(E, x) = \limsup_{r \to 0} \frac{H^s(E \cap B(x, r))}{r^s}
$$

$$
\Theta_*(E, x) = \liminf_{r \to 0} \frac{H^s(E \cap B(x, r))}{r^s}.
$$

If they agree, their common value is called the $s$-dimensional density of $E$ at $x$ and denoted by $\Theta(E, x)$.

**Definition 2.2.** We say that a set $E \subset \mathbb{R}^d$ is Ahlfors-David $s$-regular (s-ADR) if there is $C \geq 1$ so that

$$
r^s/C \leq H^s(B(x, r)) \leq Cr^s \text{ for all } x \in E, 0 < r < \text{diam } E. \tag{2.1}
$$

If a set $E \subset \mathbb{R}^d$ satisfies only the lower (resp. upper) bound we shall call it lower (resp. upper) Ahlfors-David $s$-regular.

**Definition 2.3.** We say that a set $E \subset \mathbb{R}^d$ satisfies the weak lower Ahlfors-David $s$-regular condition (WLADR) if for $H^s$-a.e. every $x \in E$, there exists $\rho_x > 0$ such that

$$
\inf_{(y, r) \in B(x, r) \times (0, \rho_x)} \frac{H^s(B(y, r))}{r^s} > 0. \tag{2.2}
$$
Definition 2.4. Let $\xi \in E \subset \mathbb{R}^d$ and $V \in G(d, m)$. If $\xi \in \mathbb{R}^d$, $s \in (0,1)$ and $0 < r < \infty$, we say that the set
\[
X(\xi, V, s) = \{ x \in \mathbb{R}^d : \text{dist}(x - \xi, V) < s|x - \xi| \}
\]
is a cone around $\xi + V$ with vertex $\xi$ and aperture $s$. We also set
\[
X(\xi, V, s, r) = X(\xi, V, s) \cap B(\xi, r).
\]
for the truncated cone at height $r$.

Remark 2.5. Notice that the complement $\complement X(\xi, V, s)$ is actually the closure of the cone $X(\xi, V^\perp, \sqrt{1 - s^2})$.

Definition 2.6. Let $E \subset \mathbb{R}^d$, $\xi \in \mathbb{R}^d$ and $V \in G(d, m)$. For fixed $s \in (0,1)$, we say that $V$ is an $s$-approximate tangent $m$-plane for $E$ at $\xi$ if
\[
\Theta^{s,m}(E, \xi) > 0, \quad \lim_{r \to 0} \frac{\mathcal{H}^m(E \cap B(\xi, r) \setminus X(\xi, V, s))}{r^m} = 0.
\]
(2.5)
If this holds for all $s \in (0,1)$ then we just say that $V$ is an approximate tangent $m$-plane for $E$ at $\xi$. We write $\text{ap-Tan}^m(E, \xi)$ for the set of all approximate tangent $m$-planes for $E$ at $\xi$.

Definition 2.7. If $E \subset \mathbb{R}^d$ is a Borel set, we say that $E$ is $m$-rectifiable if $\mathcal{H}^m(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i) = 0$ where $\Gamma_i = f_i(E_i), E_i \subseteq \mathbb{R}^m$, and $f_i : E_i \to \mathbb{R}^d$ is Lipschitz.

The criterion for rectifiability which will be most useful for us is the following.

Theorem 2.8. [29, Theorem 15.19] Let $E \subset \mathbb{R}^d$ be a $\mathcal{H}^m$-measurable set so that $\mathcal{H}^m|_E$ is locally finite. Then the following are equivalent:

1. $E$ is $m$-rectifiable.
2. For $\mathcal{H}^m$ almost every point $\xi \in E$, there is a unique approximate tangent $m$-plane for $E$ at $\xi$.
3. For $\mathcal{H}^m$ almost every point $\xi \in E$, there is some approximate tangent $m$-plane for $E$ at $\xi$.

Definition 2.9. A function $f \in L^1_{\text{loc}}(U)$ has locally bounded variation in an open set $U \subset \mathbb{R}^{n+1}$ and we write $f \in BV_{\text{loc}}(U)$, if for each open set $V \subset U$,
\[
\sup \left\{ \int_V f \text{ div} \phi \, d\mathcal{L}^{n+1} : \phi \in C_c^\infty(V; \mathbb{R}^{n+1}), |\phi| \leq 1 \right\} < \infty,
\]
where $\mathcal{L}^{n+1}$ stands for the $(n + 1)$-dimensional Lebesgue measure. An $\mathcal{L}^{n+1}$-measurable set $E \subset \mathbb{R}^{n+1}$ has \textit{locally finite perimeter} in $U$ if $\chi_E \in BV_{loc}(U)$.

**Definition 2.10.** For each $x \in \partial^* E$ we define the \textit{hyperplane}

$$H(x) = \{ y \in \mathbb{R}^{n+1} : \nu_E(x) \cdot (y - x) = 0 \}$$

and the \textit{half-spaces}

$$H^+(x) = \{ y \in \mathbb{R}^{n+1} : \nu_E(x) \cdot (y - x) \geq 0 \},$$

$$H^-(x) = \{ y \in \mathbb{R}^{n+1} : \nu_E(x) \cdot (y - x) \leq 0 \}.$$

A unit vector $\nu_E(x)$ is called the \textit{measure theoretic unit outer normal} to $E$ at $x$ if

$$\lim_{r \to 0} \frac{\mathcal{L}^{n+1}(B(x, r) \cap E \cap H^+(x))}{r^{n+1}} = 0$$

and

$$\lim_{r \to 0} \frac{\mathcal{L}^{n+1}((B(x, r) \setminus E) \cap H^-(x))}{r^{n+1}} = 0.$$

**Definition 2.11.** Let $x \in \mathbb{R}^{n+1}$. We say that $x \in \partial_x E$, the \textit{measure theoretic boundary} of $E$, if

$$\limsup_{r \to 0} \frac{\mathcal{L}^{n+1}(B(x, r) \cap E)}{r^{n+1}} > 0$$

and

$$\limsup_{r \to 0} \frac{\mathcal{L}^{n+1}((B(x, r) \setminus E)}{r^{n+1}} > 0.$$

**Remark 2.12.** Note that $\partial^* E \subset \partial_x E$ and $\mathcal{H}^n(\partial_x E \setminus \partial^* E) = 0$ (see [21, p. 208]). Moreover, if $E$ has locally finite perimeter, then $\|\partial E\| = \mathcal{H}^n|_\partial E$ (see [21, p. 205]).

A useful criterion that allows us to determine whether a set has locally finite perimeter, whose proof can be found in [21, p. 222], is the following:

**Theorem 2.13.** If $E \subset \mathbb{R}^{n+1}$ is $\mathcal{L}^{n+1}$–measurable, then it has locally finite perimeter if and only if $\mathcal{H}^n(K \cap \partial_x E) < \infty$, for each compact set $K \subset \mathbb{R}^{n+1}$.

**Definition 2.14.** Let $E$ be a set of locally finite perimeter in $\mathbb{R}^{n+1}$ and $x \in \mathbb{R}^{n+1}$. We say that $x \in \partial^* E$, the \textit{reduced boundary} of $E$, if

1. $\|\partial E\|(B(x, r)) > 0$, for all $r > 0$,
2. $\lim_{r \to 0} \frac{1}{\|\partial E\|(B(x, r))} \int_{B(x, r)} \nu_E(y) d\|\partial E\| = \nu_E(x)$, and
3. $|\nu_E(x)| = 1$. 

3. Proof of Theorems 1.1, 1.3 and 1.4

By hypothesis, for a fixed $s \in (0, 1)$ and $\xi \in \mathcal{K}$ so that (2.2) holds, there exists an $m$-plane $V_\xi$ passing through the origin so that

$$\lim_{r \to 0} \frac{\mathcal{H}^m(E \cap B(\xi, r) \cap cX(\xi, V_\xi, s))}{r^m} = 0. \quad (3.1)$$

Lemma 3.1. There exists $r_\xi > 0$ so that $cX(\xi, V_\xi, 2s) \cap B(\xi, r_\xi) \cap E = \{\xi\}$.

Proof. Let $\rho_\xi$ be the radius from the definition of weak Ahlfors-David regularity for the point $\xi$. Let us assume that we can find a sequence of radii $r_i < \rho_\xi$ so that for each $i \geq 1$, there exists $x_i \in cX(\xi, V_\xi, 2s) \cap B(\xi, r_i) \cap E$. We may choose $r_i$ so that $c_0 r_i \leq |x_i - \xi| < r_i$, for a constant $c_0 \sim 1$ to be fixed momentarily. Moreover, we can find a constant $\delta \sim s$ 1 so that

$$B(x_i, \delta r_i) \subset cX(\xi, V_\xi, s) \cap B(\xi, 2r_i). \quad (3.2)$$

Indeed, since $\pi_{V_\xi^\perp}$ is a linear, 1-Lipschitz map, for any $y \in B(x_i, \delta r_i)$, it holds

$$|\pi_{V_\xi^\perp}(y - \xi)| = |\pi_{V_\perp}(x_i - \xi) + \pi_{V_\xi^\perp}(y - x_i)| \geq |s|x_i - \xi| - |y - x_i|$$

$$\geq 2c_0 s r_i - \delta r_i \geq \frac{2c_0 s - \delta}{1 + \delta} |y - \xi| = s|y - \xi|,$$

if we choose $\delta = \frac{(2s^2 - 1)s}{1 + s}$ and $c_0$ so that $\delta \sim 1$. The fact that $B(x_i, \delta r_i) \subset B(\xi, 2r_i)$ is trivial. By (3.2) and (3.1), we have that

$$\inf_{(y, r) \in B(\xi, r) \times (0, \rho)} \frac{\mathcal{H}^m(B(y, r))}{r^m} \leq \lim_{r_i \to 0} \frac{\mathcal{H}^m(E \cap B(x_i, \delta r_i))}{r_i^m}$$

$$\leq \lim_{r_i \to 0} \frac{\mathcal{H}^m(E \cap cX(\xi, V_\xi, s) \cap B(\xi, 2r_i))}{r_i^m} = 0,$$

which by the weak lower Ahlfors-David $m$-regularity of $\partial \Omega$ is a contradiction. This concludes our lemma. \hfill \square

For $V, W \in G(d, d - m)$, we define $d(V, W) = \|\pi_V - \pi_W\|$, where $\|\cdot\|$ is the usual operator norm for linear maps. With this metric $G(d, d - m)$ is a compact metric space and thus, for any fixed number $s \in (0, 1/3)$, there is a finite subset of $G(d, d - m)$, say $\mathcal{P}_m(s) = \{V_j\}^{N(s)}_{j=1}$, such that the following holds: for any $V \in G(d, d - m)$, there exists $V_{j_0} \in \mathcal{P}_m$ so that $d(V, W) < s$.

Lemma 3.2. Assume that $\varepsilon > 0$. For any $\xi \in \mathcal{K}$, there exists $j = j(\xi, \varepsilon) \in \mathbb{N}$, such that $V_j \in \mathcal{P}_m$ and $cX(\xi, V_j^\perp, 2s + \varepsilon) \subset cX(\xi, V_\xi, 2s)$. 

Proof. For fixed \( \varepsilon > 0 \) and \( \xi \in \mathcal{K} \), there exists \( V_j \in \mathcal{P}_m(\varepsilon) \), so that 
\[ d(V_\xi, V_j) < \varepsilon. \]
If \( y \in cX(\xi, V_j, 2s + \varepsilon) \), we have that
\[
\pi_{V_\xi}(y - \xi) \geq \pi_{V_j}(y - \xi) - |(\pi_{V_j} - \pi_{V_\xi})(y - \xi)| \\
\geq (2s + \varepsilon - \varepsilon)|y - \xi| \\
= 2s|y - \xi|.
\]
This readily shows that \( y \in cX(\xi, V_\xi, 2s) \) and finishes our proof. \( \square \)

We set
\[
S_{j,k} = \{ \xi \in \mathcal{K} : j = j(\xi, s) \text{ and } cX(\xi, V_j, 3s) \cap B(\xi, k^{-1}) \cap E = \{ \xi \} \}.
\]
Let us fix \( j \in \{1, 2, \ldots, N(s)\} \) and \( k \in \mathbb{N} \). Since \( S_{j,k} \) is separable it has a countably dense subset \( \{ x_\ell \}_{\ell=1}^\infty \). Therefore, for each \( \xi \in S_{j,k} \), there exists \( \ell \) so that \( |x_\ell - \xi| < (4k)^{-1} \). Notice that there might be more than one \( \ell \) for each \( \xi \). Although, to any fixed \( \xi \in S_{j,k} \), we assign once and for all a unique \( \ell(\xi) \) with the requirement that \( |x_{\ell(\xi)} - \xi| < (4k)^{-1} \). If we set
\[
S_{j,k,\ell} = \{ \xi \in S_{j,k} : \ell(\xi) = \ell \},
\]
then we get that
\[
\mathcal{K} = \bigcup_{j} \bigcup_{k} \bigcup_{\ell} S_{j,k,\ell}. \tag{3.3}
\]

Fix now \( j, k \) and \( \ell \) and denote \( S = S_{j,k,\ell} \). Without loss of generality we may assume that \( V_j = \mathbb{R}^{d-m} \) and \( V_j^\perp = \mathbb{R}^m \) since projections are invariant under rotations.

Lemma 3.3. \( S \) is contained in the graph of a (possibly rotated) \((3s)^{-1}\)-Lipschitz function \( \varphi : \mathbb{R}^m \to \mathbb{R}^{d-m} \).

Proof. Let \( \xi \in S \). Note that if \( |\pi_{\mathbb{R}^m}(\xi) - \pi_{\mathbb{R}^m}(\xi')| < \sqrt{1 - (3s)^2} |\xi - \xi'| \) and \( |\xi - \xi'| < k^{-1} \), then \( \xi' \in X(\xi, \mathbb{R}^{d-m}, \sqrt{1 - (3s)^2} \cap B(\xi, k^{-1})) \), or else, \( \xi' \in \pi_{\mathbb{R}^m}(\xi, \mathbb{R}^m, 3s) \cap B(\xi, k^{-1}) \). By hypothesis, this means that \( \xi' \notin \partial \Omega \) and thus, \( |\pi_{\mathbb{R}^m}(x) - \pi_{\mathbb{R}^m}(\xi)| \geq 3s|x - \xi| \), for any \( x, \xi \in S \). This implies that \( \pi_{\mathbb{R}^m}|_S \) is a \( 3s \)-bi-Lipschitz map with \((3s)^{-1}\)-Lipschitz inverse
\[
\tilde{f} = (\pi_{\mathbb{R}^m}|_S)^{-1} : \pi_{\mathbb{R}^m}(S) \to \mathbb{R}^d.
\]
Note that \( S = \tilde{f}(\pi_{\mathbb{R}^m}(S)) \). By Kirszbraun’s theorem, we may extend \( \tilde{f} \) to a globally defined \((3s)^{-1}\)-Lipschitz function \( f : \mathbb{R}^m \to \mathbb{R}^d \) with \( f|_{\pi_{\mathbb{R}^m}(S)} = \tilde{f} \). If we set \( \phi = \pi_{\mathbb{R}^{d-m}} \circ f \), then it is clear that \( \phi \) is \((3s)^{-1}\)-Lipschitz and every \( x \in S \) belongs to the graph \( \Gamma_{\phi} := \{(y, \phi(y)) : y \in \mathbb{R}^m \} \). \( \square \)

Theorem 3.1 readily follows from the above lemmas.
proof of Theorem 1.3. We apply Theorem 1.1 for $m = d - 1$ and $E = \partial^* \Omega \cap K$ that satisfy the requirements of Theorem 1.3 and obtain $\partial^* \Omega \cap K = \bigcup \Gamma_i$, where $\Gamma_i$ are $(3s)^{-1}$-Lipschitz graphs. Let us fix $j, k, \ell$ as before and denote $S = S_{j,k,\ell}$. We let $\nu(x_\ell)$ be the unit vector perpendicular to $\mathbb{R}^{d - 1}$ that emanates from $x_\ell$, so that the endpoint $x$ of the vector $(2k)^{-1} \nu(x_\ell)$ is in $\Omega$. The existence of such a point can be easily derived from [21, Corollary 1, p.203] by simple volume considerations. Recall that for every $\xi \in S$, $|\xi - x_\ell| < (4k)^{-1}$ and thus, \begin{equation} \pi_{\mathbb{R}}(\xi - x_\ell) \leq |\xi - x_\ell| \leq (4k)^{-1}, 
end{equation} for any $\xi \in S$. Set \begin{equation} W = \{ y \in \mathbb{R}^d : (x - y) \cdot \nu(x_\ell) = 0 \}, \end{equation} which is a $(d - 1)$-plane perpendicular to $\nu(x_\ell)$ that contains $x$. Set also \begin{equation} C(\nu(x_\ell), (4k)^{-1}) \end{equation} to be the infinite cylinder with axis $\nu(x_\ell)$ and radius $(4k)^{-1}$. Define now $\Omega_S$ to be the part of $C(\nu(x_\ell), (4k)^{-1})$ which is contained between $W$ and the Lipschitz graph $\Gamma_\Phi$. By (3.4) and the fact that for every $\xi \in S$ there holds $e_X(\xi, V_j, 3s) \cap B(\xi, k^{-1}) \cap \partial \Omega = \{ \xi \}$, it is easy to see that $\Omega_S \subset \Omega$ (using also the definition of the Lipschitz extension from Kirszbraun’s theorem). Moreover, by construction, $\Omega_S$ is a bounded Lipschitz domain and $\partial \Omega \subset \Gamma_\Phi \cap \partial^* \Omega \subset \partial \Omega_S$. This finishes our proof. \hfill \Box

proof of Theorem 1.5. Let $F \subset \partial \Omega$ such that $\mathcal{H}^n|_{\partial \Omega}(F) > 0$. Then, there exists $\Omega_j$ constructed in Theorem 1.3 so that $\mathcal{H}^n|_{\partial \Omega_j}(F) > 0$. Let $p_j \in \Omega_j$ be a Corkscrew point for $\Omega_j$. Then, by Dahberg’s result and maximum principle, we have that $\omega^p_{\Omega_j}(F) > 0$. Since $\Omega$ is connected, we can connect $p_j$ with $p$ by Harnack chains (which consist from a possibly very large but finite numbers of balls) and our result follows from Harnack’s inequality. \hfill \Box

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