Modeling the Electrocardiogram as Oscillatory Almost-Cyclostationary Process

ANTONIO NAPOLITANO1, (Fellow, IEEE)

1University of Napoli “Parthenope”, Department of Engineering, Napoli, Italy (email: antonio.napolitano@uniparthenope.it antnapol@gmail.com URL: https://sites.google.com/site/antnapol)

Corresponding author: Antonio Napolitano (e-mail: antonio.napolitano@uniparthenope.it).

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ABSTRACT A new model for the electrocardiogram (ECG) signal is proposed. Specifically, the ECG signal is modeled as an amplitude-modulated and time-warped version of a cyclostationary process which is the sum of a periodic signal and a zero-mean cyclostationary term. For the proposed model, the second-order characterization is derived in both time and frequency domains. The autocorrelation function is shown to be the superposition of amplitude- and angle-modulated sine waves and the Loève bifrequency spectrum a spread version of that of the underlying cyclostationary process. The signal model belongs the recently introduced class of the oscillatory almost-cyclostationary processes. A procedure for estimating the second-order statistical functions in both time and frequency domains is outlined. The effectiveness of the proposed model and of the estimation procedure is corroborated by measurements on real ECG signals which are in full agreement with the theoretical analytical expressions. The proposed model is shown to be effective with observation intervals much larger than those adopted up to now with the classical cyclostationary model and is suitable to be exploited for arrhythmia modeling and characterization, and for diagnosis and biometric purposes.

INDEX TERMS Electrocardiogram; Cyclostationarity; Time warping.

I. INTRODUCTION

THE analysis of the electrocardiogram (ECG) signal is at the basis of the diagnosis of cardiovascular diseases. A first step in the analysis consists in providing an accurate signal model for the ECG.

Several models have been proposed in the literature. A pole-zero model is presented in [30]. Hidden Markov models are adopted to describe accurately segmented ECG signals in [25]. In [27], a dynamical model based on three coupled ordinary differential equations is introduced which is capable of generating realistic ECG signals. In [39], a model based on wavelet neural networks is proposed. Sequential Bayesian methods are considered in [6]. The ECG signal is modeled as generated by three nonlinear oscillators in [40]. A simple mathematical model based on the sum of two Gaussian functions is proposed in [1]. Surveys of ECG signal analysis techniques are made in [22] and [44]. The usefulness of the discriminative characteristics of the ECG signal in the biometric field is addressed in [42].

Cyclostationary processes are an appropriate model for signals arising from the interaction of periodic and random phenomena. In such a case, signals are not periodic functions but have statistical functions such as distribution, mean, autocorrelation, moments, and cumulants that are periodic functions of time [9, Part II], [10], [35, Chaps. 1,2]. More generally, if more incommensurate periodicities are present in the generating mechanisms, the resulting statistical functions are poly-periodic or almost-periodic.

The structure of the electrocardiogram (ECG) signal is consequence of a periodic generating mechanism, the heart pulsation, and other less predictable phenomena like the propagation of the electrical wave throughout the heart and the body and possibly artifacts. This combination of periodic and random phenomena suggests that the cyclostationary model could be exploited for ECG signal modeling [18], [19], [29], [43], and [46]. The cyclostationary model has been exploited in [17] for the fetal PQRST extraction, in [15] for arrhythmia classification, in [16] for denoising the...
The ECG signal, in [23] and [24] for heart and respiration rate monitoring, in [12] for removing ballistocardiogram artifacts, in [13] for heart and lung sound separation, and in [26] for heart sound selection. In all the works cited above, the length of the data collection time does not exceed 10 seconds. In fact, only within such a time interval, the heart rate can be considered approximately constant, a necessary assumption for the validity of the cyclostationary model. The chirp-cyclostationary model considered in [28] allows one to consider a data-record length of 30 seconds.

A generalization of the cyclostationary processes that accounts for an irregular pace of a not exactly periodic generating mechanism are the amplitude-modulated (AM) time-warped (TW) almost-cyclostationary (ACS) processes [11], [34], [36], [35, Sec. 14.3]. Preliminary results have shown that such a model allows one to consider for the ECG signal data-record lengths larger than those for which the cyclostationary model is valid [34]. Other approaches to model signals with time-varying cyclostationary characteristics are considered in [2], [38].

In this paper, the ECG signals are modeled as AM-TW ACS processes, which are a special case of the recently introduced class of the oscillatory almost-cyclostationary (OACS) processes [32, Sec. 6], [35, Chap. 14]. For an OACS process, the autocorrelation function is the superposition of amplitude- and angle-modulated sine waves whose frequencies are the cycle frequencies of an underlying ACS process. In particular, the ACS process can be cyclostationary. For the ECG signal, the underlying cyclostationary process is modeled as the sum of a periodic signal with period equal to the reciprocal of the average heart rate and a zero-mean cyclostationary process exhibiting the same cyclostationarity period. The second-order characterization of the proposed model is derived in the time domain in terms of time-varying autocorrelation function and in the frequency domain in terms of a smoothed version of the Loeve bifrequency spectrum which is the correlation between two spectral components of the process. For the underlying cyclostationary process, the analytical expressions of cyclic spectra and cyclic autocorrelation functions are derived. Due to the presence of the periodic signal in the ECG signal model, the cyclic autocorrelation functions contain finite-strength additive sine waves and the cyclic spectra contain Dirac impulses. The cyclic covariance and the second-order cyclic polyspectrum of the underlying cyclostationary process are the cyclic autocorrelation and the cyclic spectrum, respectively, of the zero-mean cyclostationary component.

The effectiveness of the proposed model is corroborated by measurements made on several ECG data available in the PhysioNet database https://www.physionet.org [14]. The amplitude-modulation and time-warping functions are estimated by amplitude and angle demodulation of one of the modulated sine waves present in the second-order lag product. Then, their effects are compensated in order to reconstruct the underlying cyclostationary signal and its cyclic spectral analysis is carried out. The proposed model for the underlying cyclostationary signal is confirmed by the presence of sinusoidal terms in the estimated cyclic autocorrelations and of impulses in the estimated cyclic spectra.

The analysis made on the ECG signals of several patients confirms the general validity of the proposed model. In addition, the analysis shows that the shapes of cyclic autocorrelations, spectra, autocovariances, and 2nd-order polyspectra are different for each patient. Therefore, biometric information and information on possible diseases can be deducted from the Fourier coefficients of the periodic component and from the cyclic statistics of the zero-mean cyclostationary component. Transitory effects and arrhythmia are linked to the behavior of the time-warping function.

The paper is organized as follows. In Section II, the proposed signal model for the ECG signal is presented (Sec. II-A) and its 2nd-order characterization in both time (Sec. II-B) and frequency (Sec. II-C) domains is derived. The statistical function estimation is addressed in Section III. Numerical results are presented in Section IV and conclusions are drawn in Section V. Proofs are reported in Appendix A.

### II. MODEL FOR THE ELECTROCARDIOGRAM SIGNAL

#### A. SIGNAL MODEL

In this section, the proposed mathematical model for the ECG signal is presented.

Let $T_0$ be the average cardiac cycle, that is, the reciprocal of the average heart rate $\alpha_0$. In practice, the average is made within the observation interval.

The proposed model for the electrocardiogram is the real-valueed signal

$$y(t) = a(t) x(\psi(t)) .$$

In (2.1),

1) $x(t)$ is a real-valueed cyclostationary process, with cyclostationarity period $T_0$, that can be decomposed into the sum of a periodic component and a zero-mean term. That is,

$$x(t) = x_p(t) + x_r(t)$$

with

$$x_p(t) \triangleq \sum_{h=\infty}^{\hat{h}} x_{h/T_0} e^{j2\pi(h/T_0)t}$$

real-valued (deterministic) periodic signal with period $T_0$ and Fourier coefficients $x_{h/T_0}$ (such that $x_{-h/T_0} = x_{h/T_0}$), and $x_r(t)$ real-valued zero-mean cyclostationary process with cyclostationarity period $T_0$.

2) $\psi(t)$ is a deterministic time-warping function

$$\psi(t) \triangleq t + \epsilon(t)$$

with $\epsilon(t)$ slowly varying, that is, such that

$$\sup_{t} |\dot{\epsilon}(t)| \ll 1$$

where $\dot{\epsilon}(t)$ is the first-order derivative of $\epsilon(t)$. Condition (2.5) means that time variations of $\epsilon(t)$ are small if compared with 1 which is the derivative of $t$. 

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denotes the cyclic autocorrelation at cycle frequency $R_{\langle \cdot \rangle t}$. The cyclic autocorrelation function is the autocorrelation function of the cyclostationary zero-mean term $x_{r}(t)$. In (2.7),

$$R_{x_{r}}^{k/T_{0}}(\tau) \triangleq \left\langle E\{x_{r}(t + \tau) x_{r}(t)\} e^{-j2\pi(k/T_{0})t}\right\rangle_{t}$$

denotes the cyclic autocorrelation at cycle frequency $k/T_{0}$ of $x_{r}(t)$. It is the Fourier coefficient at frequency $k/T_{0}$ of the periodic function $t \mapsto E\{x_{r}(t + \tau) x_{r}(t)\}$ [10, Sec. 3.2.1]. In (2.8), $\langle \cdot \rangle_{t}$ denotes infinite time average.

From (2.2) and accounting for (2.3) and (2.7), it follows that the cyclic autocorrelation function of $x_{r}(t)$ is given by

$$R_{x_{r}}^{\alpha}(\tau) \triangleq \left\langle E\{x(t + \tau) x(t)\} e^{-j2\pi\alpha t}\right\rangle_{t}$$

denoted as the additive periodic term in the cyclic autocorrelation (2.9b).

The signal $x(t)$ is cyclostationary with cycle frequencies $\alpha = k/T_{0}, k \in \mathbb{Z}$. The presence of the additive periodic term in the expression of $x(t)$ reflects into the presence of sinusoidal terms in the cyclic autocorrelation (2.9b).

The cyclic autocovariance of $x_{r}(t)$ is given by

$$C_{x_{r}}^{\alpha}(\tau) \triangleq \left\langle E\{[x(t + \tau) - E\{x(t + \tau)\}] [x(t) - E\{x(t)\}\} e^{-j2\pi\alpha t}\right\rangle_{t} = R_{x_{r}}^{\alpha}(\tau)$$

and $Q_{x_{r}}^{\alpha}(t, \tau)$ is shown that $|x_{r}(t)|$ is negligible for $|h| > h_{\text{max}}$. In Appendix A, it is shown that

$$Q_{x_{r}}^{\alpha}(t, \tau) \simeq R_{x_{r}}^{k/T_{0}}(\tau) \quad |\tau| < \tau_{\text{max}}$$

where $B_{r}$ is the bandwidth of $x_{r}(t)$. Let $h_{\text{max}} > 0$ be the order of the maximum significant harmonic of the periodic signal $x_{p}(t)$. That is, $h_{\text{max}}$ is such that $|x_{p}(t)|$ is negligible for $|h| > h_{\text{max}}$. In Appendix A, it is shown that

$$Q_{x_{r}}^{\alpha}(t, \tau) \simeq \sum_{|h|<h_{\text{max}}} x_{h}/T_{0} x_{r}(h-k)/T_{0} e^{j2\pi(h/T_{0})\tau} \quad |\tau| < \tau_{\text{max}}$$

with $\tau_{\text{max}}$ such that

$$2\pi \frac{h_{\text{max}}}{T_{0}} \sup_{t} |\epsilon(t)| \tau_{\text{max}} \ll 1.$$
Such a condition, accounting for (2.5), reduces to
\[
2\pi \frac{h_{\text{max}}}{T_0} \tau_{\text{max}} = O(1) .
\]  
(2.18)

Under the assumption that \(a(t)\) is slowly varying with respect to \(x_p(t)\) and \(x_r(t)\), we have \(a(t + \tau) \simeq a(t)\) for \(|\tau| < \min(\tau_{\text{max}}, 1/B_\tau)\). Thus, in (2.12) it results
\[
a(t + \tau) a(t) \simeq a^2(t) \quad |\tau| < \min(\tau_{\text{max}}, 1/B_\tau) .
\]  
(2.19)

Therefore, under conditions (2.5) and (2.17), and accounting for (2.15), (2.16), and (2.19),
\[
Q^a_r(t, \tau) + Q^p_r(t, \tau) \simeq R^a_r(\tau)
\]  
(2.20)

with \(R^a_r(\tau)\) cyclic autocorrelation of \(x(t)\) given in (2.9b), and for the oscillatory cyclic autocorrelation \(\rho^a_y(t, \tau)\) in (2.11) the following approximate expression hold
\[
\rho^a_y(t, \tau) = m^a_y(t) R^a_x(\tau) \quad \alpha = \frac{k}{T_0}, \ k \in \mathbb{Z}
\]  
(2.21)

where
\[
m^a_y(t) \triangleq a^2(t) e^{j2\pi \alpha t} .
\]  
(2.22)

By replacing (2.21) and (2.22) into (2.11), one obtains the following approximate expression for the autocorrelation of the ECG signal \(y(t)\)
\[
E \{y(t + \tau) y(t)\} = \sum_{k=-\infty}^{+\infty} a^2(t) R^a_y(t) e^{j2\pi(k/T_0)(t+\epsilon(t))}
\]  
(2.23)

This expression can be interpreted as the autocorrelation of a cyclostationary process with "time-varying cyclic autocorrelations" and "time-varying instantaneous cycle frequencies". Thus, the instantaneous cycle frequency for \(k = 1\) in (2.23) provides the time-varying heart-rate of the proposed ECG signal model
\[
\text{heart rate} = \frac{1}{2\pi} \frac{d}{dt} \left[ 2\pi \frac{1}{T_0} (t + \epsilon(t)) \right] = \alpha_0 (1 + \epsilon(t)) .
\]  
(2.24)

C. SECOND-ORDER FREQUENCY-DOMAIN CHARACTERIZATION

The cyclic spectrum of the underlying cyclostationary process \(x(t)\) is the Fourier transform of the cyclic autocorrelation (2.9b):
\[
S^\alpha_x(f) \triangleq \int_{\mathbb{R}} R^a_x(\tau) e^{-j2\pi f \tau} d\tau
\]  
(2.25a)

\[
= \left\{ \begin{array}{ll}
S_{x^a}^{k/T_0}(f) + \sum_{h=-\infty}^{+\infty} x_{h/T_0} x^*_{(h-k)/T_0} \delta(f - h/T_0) \\
\quad \alpha = \frac{k}{T_0}, \ k \in \mathbb{Z} \\
\quad 0 \ 	ext{otherwise}
\end{array} \right.
\]  
(2.25b)

where \(S_{x^a}^{k/T_0}(f)\) is the cyclic spectrum of the zero-mean term \(x^a(t)\), that is, the Fourier transform of \(R^{k/T_0}_{x^a}(\tau)\), and the presence of Dirac deltas is consequence of the additive periodic term in (2.2).

For a finite-memory or practically finite-memory process \(x(t)\), the cyclic autocorrelation function is summable. Hence, the second-order cyclic polyspectrum of \(x(t)\), which is the Fourier transform, defined in ordinary sense, of the cyclic autocovariance (2.10a) of \(x(t)\)
\[
S_{x^a}^{\alpha}(f) \triangleq \int_{\mathbb{R}} C^\alpha_x(\tau) e^{-j2\pi f \tau} d\tau
\]  
(2.26a)

\[
\triangleq S_{x^a}(f)
\]  
(2.26b)

is coincident with the cyclic spectrum of \(x(t)\) and does not contain Dirac impulses [35, Sec. 1.4].

The Loève bifrequency spectrum of \(x(t)\) and its rotated version are given by [35, App. A.3]
\[
S_x(f_1, f_2) \triangleq E \{X(f_1) X^*(f_2)\}
\]  
(2.27a)

\[
= \sum_{k=-\infty}^{+\infty} S_{x}^{k/T_0}(f_1) \delta(f_1 - f_2 - k/T_0)
\]  
(2.27b)

and
\[
\overline{S}_x(f, \delta f) \triangleq S_x(f_1, f_2)|_{f_1 = f, f_2 = f - \delta f}
\]  
(2.28a)

\[
= \sum_{k=-\infty}^{+\infty} S_{x}^{k/T_0}(f) \delta(\delta f - k/T_0)
\]  
(2.28b)

respectively. In (2.27a), \(X(f)\) formally denotes the Fourier transform of \(x(t)\) defined in a generalized sense [31, Sec. 4.2.1], [35, Sec. A.3].

The rotated Loève bifrequency spectrum of the process \(y(t)\) can be expressed as the double Fourier transform
\[
\overline{S}_y(f, \delta f) \triangleq \int_{\mathbb{R}^2} E \{y(t + \tau) y(t)\} e^{-j2\pi f \tau} e^{-j2\pi \delta t} dt d\tau
\]  
(2.29a)

\[
= S_y(f_1, f_2)|_{f_1 = f, f_2 = f - \delta f}
\]  
(2.29b)

where
\[
S_y(f_1, f_2) \triangleq E \{Y(f_1) Y^*(f_2)\} .
\]  
(3.30)

is the Loève bifrequency spectrum of \(y(t)\). In (3.30), \(Y(f)\) formally denotes the Fourier transform of \(y(t)\).

The calculation of the double Fourier transform (2.29a) of the autocorrelation function (2.11) with (2.12)–(2.14) substituted into is a formidable problem. However, a smoothed version of the rotated Loève bifrequency spectrum can be easily obtained starting from (2.11) and substituting for (2.21). It should be noted that the approximate expression (2.21) holds for \(|\tau| < \min(\tau_{\text{max}}, 1/B_\tau)\).

Let \(w(\tau)\) be a time window such that \(w(\tau) = 0\) for \(|\tau| > \min(\tau_{\text{max}}, 1/B_\tau)\) and let \(W(f)\) be the Fourier transform of \(w(\tau)\).

By doubly Fourier transforming with respect to \(t\) and \(\tau\) the windowed autocorrelation
\[
E \{y(t + \tau) y(t)\} w(\tau)
\]  
(2.30)
one obtains the following smoothed version of the rotated Loève bifrequency spectrum (Appendix A)
\[
\overline{S}_y(f, \delta f)_{\text{smooth}} = \sum_{k=-\infty}^{+\infty} \left[ S_x^{k/T_0}(f) \otimes W(f) \right] M_y^{k/T_0}(\delta f - k/T_0)
\]  
(2.31)

where
\[
M_y^\alpha(f) \triangleq \int_{\mathbb{R}} m_y^\alpha(t) e^{-j2\pi ft} \, dt
\]  
(2.32a)
\[
= \int_{\mathbb{R}} a^2(t) e^{j2\pi \alpha \epsilon(t)} e^{-j2\pi ft} \, dt.
\]  
(2.32b)

A smoothed version of the Loève bifrequency spectrum of \( y(t) \) is given by
\[
\overline{S}_y(f_1, f_2)_{\text{smooth}} = \overline{S}_y(f, \delta f)_{\text{smooth}} \bigg|_{f=f_1, \delta f=f_1-f_2}
\]  
(2.33a)
\[
= \sum_{k=-\infty}^{+\infty} \left[ S_x^{k/T_0}(f_1) \otimes W(f_1) \right] M_y^{k/T_0}(f_1 - f_2 - k/T_0).
\]  
(2.33b)

By comparing the expression of \( \overline{S}_x(f_1, f_2) \) in (2.28b) and \( \overline{S}_y(f_1, f_2)_{\text{smooth}} \) in (2.31), it follows that the smoothed rotated Loève bifrequency spectrum (2.31) of the ECG signal \( y(t) \) is given by the sum of the cyclic spectra of the underlying cyclostationary signal \( x(t) \) smoothed in \( f \) and spread in the \( \delta f \) dimension around the cycle frequencies \( k/T_0 \). The spreading is due to the amplitude- and angle-modulation of the sine waves in the autocorrelation function. For this reason, Dirac deltas in (2.28b) are replaced by the functions \( M_y^\alpha(\cdot) \) in (2.31) that have nonzero bandwidth and depend on \( \alpha \). The smoothing along the \( f \) dimension is due to the convolution of the cyclic spectra with the spectral window \( W(f) \) as a consequence of the windowing of the autocorrelation.

### III. STATISTICAL FUNCTION ESTIMATION

In this section, the problem of estimating the second-order statistical functions of the ECG signal modeled as AM-TW ACS process is addressed.

The autocorrelation function and the Loève bifrequency spectrum are not directly estimated. Rather, the functions that appear in their analytical description are estimated separately. Then, they are used to build estimates of the autocorrelation function and the Loève bifrequency spectrum.

Specifically, the functions involved in the time- and frequency-domain characterizations in Sections II-B and II-C are derived as follows:

(a) The amplitude- and angle-modulation functions \( a(t) \) and \( \epsilon(t) \) (or \( \psi(t) = t + \epsilon(t) \)) are estimated (Sec. III-A);  
(b) Using the estimates of \( a(t) \) and \( \epsilon(t) \), amplitude modulation is compensated and de-warping is performed in order to reconstruct an estimate \( \hat{x}(t) \) of the underlying cyclostationary signal \( x(t) \) (Sec. III-B);  
(c) Cyclic spectral analysis is carried out on the signal \( \hat{x}(t) \) obtaining estimates of the cyclic autocorrelation \( R_x^\alpha(\tau) \) in (2.9a), cyclic autocovariance \( C_x^\alpha(\tau) \) in (2.10a), cyclic spectrum \( S_x^\alpha(f) \) in (2.25a), and 2nd-order cyclic polyspectrum \( P_x^\alpha(f) \) in (2.26a) of the underlying cyclostationary process \( x(t) \) (Sec. III-C).

Starting from the estimates obtained in steps (a)–(c), the estimate of the autocorrelation function (2.11) of \( y(t) \) is obtained as follows:

(i) The estimates of \( a(t) \) and \( \epsilon(t) \) provide the estimates of amplitude and phase (divided by \( 2\pi \alpha \)) respectively, of the function \( m_y^\alpha(\tau) \) in (2.22);  
(ii) The estimate of \( m_y^\alpha(\tau) \), jointly with the estimate of \( R_x^\alpha(\tau) \), provides an estimate of the oscillatory cyclic autocorrelation \( \rho_y^\alpha(t, \tau) \) in (2.21);  
(iii) The estimates of the functions \( \rho_y^\alpha(t, \tau) \), considered for a sufficient number of cycle frequencies \( \alpha = k/T_0 \), provide an estimate of the autocorrelation function (2.11) of \( y(t) \).

Starting from the estimates obtained in steps (a)–(c), the estimate of the smoothed Loève bifrequency spectrum (2.33b) of \( y(t) \) is obtained as follows:

(iv) The estimate of the Fourier transform \( M_y^\alpha(f) \) in (2.32a) is obtained by Fourier transforming the estimate of \( m_y^\alpha(\tau) \) (see (i));  
(v) The estimate of the cyclic spectrum \( S_x^\alpha(f) \) is obtained at step (c);  
(vi) The estimates of \( M_y^\alpha(f) \) and \( S_x^\alpha(f) \) considered for a sufficient number of cycle frequencies \( \alpha = k/T_0 \), provide an estimate of thesmoothed Loève bifrequency spectrum (2.33b) of \( y(t) \).

The above outlined procedures allow the estimation of the autocorrelation function and the Loève bifrequency spectrum of \( y(t) \). However, it is worthwhile to underline that for the objectives of the numerical analysis conducted in Section IV, only the estimates obtained in steps (a)–(c) are exploited.

### A. AMPLITUDE- AND ANGLE-DEMODULATION

From (2.11)–(2.14) with \( \tau = 0 \) we have
\[
E \{ y^2(t) \} = a^2(t) \left[ \sum_{k=-\infty}^{+\infty} P_{z_x}^{k/T_0}(0) + \sum_{k=-\infty}^{+\infty} \left( \sum_{h=-\infty}^{+\infty} x_{h/T_0} x_{(h-k)/T_0}^* \right) e^{j2\pi(k/T_0)(t+\epsilon(t))} \right]
\]  
(3.1)

This function is the superposition of amplitude- and angle-modulated sine waves centered around the frequencies \( k/T_0 \), \( k \) integer.

Let us assume that the bandwidths of these modulated terms are such that their power spectra do not significantly
overlap the power spectrum of the modulated sine wave with \( k = 1 \). Such an assumption has been verified in all the observed ECG signals (see Section IV).

Let \( \tilde{\alpha}_0 \) be an estimate of the average heart rate \( \alpha_0 = 1/T_0 \) obtained by counting the number \( n(T) \) of pulses in the observation interval of width \( T \)

\[
\tilde{\alpha}_0 \triangleq \frac{n(T)}{T} \tag{3.2}
\]

and consider the signal

\[
\begin{align*}
\tilde{z}(\tilde{\alpha}_0, W)(t) & \triangleq \left[ y^2(t) e^{-2\pi \tilde{\alpha}_0 t} \right] \ast h_W(t) \tag{3.3a} \\
\tilde{a}^2(t) & \simeq a^2(t) \left[ R_{\tilde{x}}^{1/T_0}(0) + \sum_{h=-\infty}^{+\infty} x_{h/T_0} x^*_{h-1/T_0} e^{j2\pi \tilde{\alpha}_0 h} \right] ^{1/2} \tag{3.3b}
\end{align*}
\]

In (3.3a), \( h_W(t) \) is the impulse-response function of a low-pass filter whose bandwidth \( W \) is such that the term in (3.3b) passes unaltered and all the other terms in \( y^2(t) e^{-2\pi \tilde{\alpha}_0 t} \) are filtered out.

Under the mild assumption that the constant term in the square brackets in (3.3b) is non zero, the time-varying amplitude \( a(t) \) can be estimated as

\[
\hat{a}(t) = \left| \tilde{z}(\tilde{\alpha}_0, W)(t) \right|^{1/2} \tag{3.4}
\]

but for a nonzero constant multiplicative factor which is the magnitude of the (unknown) term in the square brackets in (3.3b).

Since \( a^2(t) > 0 \), from (3.3b) it follows that

\[
\arg \left[ z(\tilde{\alpha}_0, W)(t) \right] = q + 2\pi (\alpha_0 - \tilde{\alpha}_0) + 2\pi \tilde{\alpha}_0 \epsilon(t) \mod 2\pi \tag{3.5}
\]

where \( q \) denotes the phase of the (unknown) term in the square brackets in (3.3b). The affine term \( mt + q \), with \( m = 2\pi (\alpha_0 - \tilde{\alpha}_0) \), accounts for the error in the cycle frequency estimate \( \tilde{\alpha}_0 \).

The function \( \epsilon(t) \) can be estimated but for a constant additive term as

\[
\tilde{\epsilon}(t) = \frac{1}{2\pi \alpha_0} \arg_{\text{unw}} \left[ z(\tilde{\alpha}_0, W)(t) \right] - (\hat{m} t + \hat{q}) \tag{3.6}
\]

where \( \arg_{\text{unw}}[\cdot] \) denotes the unwrapped phase. In (3.6), \( \hat{m} \) and \( \hat{q} \) are estimates of \( m \) and \( q \) obtained by least-squares linear regression on the available data \( \arg \left[ z(\tilde{\alpha}_0, W)(nT_s) \right] \), \( n = 0, 1, \ldots, N-1 \), where \( T_s \) is the sampling period and \( T = NT_s \) is the data-record length.

By letting \( \hat{\alpha}_0 \triangleq 2\pi (\alpha_0 - \tilde{\alpha}_0) \), the following refined cycle frequency estimate is obtained

\[
\hat{\alpha}_0 = \tilde{\alpha}_0 + \frac{\hat{m}}{2\pi}. \tag{3.7}
\]

By replacing \( \tilde{\alpha}_0 \) with \( \hat{\alpha}_0 \) into (3.3a)–(3.6), refined estimates of \( \hat{a}(t) \) and \( \tilde{\epsilon}(t) \) are obtained [35, Sec. 14.3.3].

The estimate \( \hat{\alpha}_0 \) in (3.7) of the average heart rate and the estimate \( \tilde{\epsilon}(t) \) in (3.6) allow one to estimate the time-varying heart rate (2.24b) as

\[
\text{estimated heart rate} = \hat{\alpha}_0 \left( 1 + \tilde{\epsilon}(t) \right) \tag{3.8}
\]

B. AMPLITUDE-MODULATION COMPENSATION AND DE-WARPING

Let us consider the signal model (2.1). If time warping never reverses time, then the warping-function \( \psi(t) \) is invertible [11]. Denoting \( \psi^{-1}(t) \) as its inverse, \( x(t) \) can be obtained by compensating the amplitude modulation in \( y(t) \) and then de-warping the result. Specifically, from (2.1) it follows that

\[
x(t) = \frac{y(\psi^{-1}(t))}{a(\psi^{-1}(t))} \tag{3.9}
\]

for all values of \( t \) such that the denominator is non zero. An estimate \( \hat{x}(t) \) of the underlying ACS signal \( x(t) \) is obtained by replacing \( \psi^{-1}(t) \) with their estimates.

Let us consider \( \psi(t) = t + \epsilon(t) \) with \( \epsilon(t) \) slowly varying (see (2.5)). Let \( \tilde{a}(t) \) and \( \tilde{\epsilon}(t) \) be the estimates of \( a(t) \) and \( \epsilon(t) \) obtained by the procedure described in Section III-A. It can be shown that, for the purpose of de-warping, an estimate of \( \psi^{-1}(t) \) is given by \( t - \tilde{\epsilon}(t) \) [35, Sec. 14.8]. Thus, accounting for (3.9), an estimate of the signal \( x(t) \) is obtained by \( y(t) \) as

\[
\hat{x}(t) = \frac{y(t - \tilde{\epsilon}(t))}{\tilde{a}(t - \tilde{\epsilon}(t))} \tag{3.10}
\]

provided that \( a(t)/\tilde{a}(t) \simeq 1 \), that is, the estimate \( \hat{a}(t) \) is sufficiently accurate, and the estimation error on \( \epsilon(t) \) is sufficiently small in the sense that

\[
\sup_t |\epsilon(t) - \tilde{\epsilon}(t)| \ll 1/B \tag{3.11}
\]

with \( B \) bandwidth of \( x(t) \).

The samples of the time-warped versions of \( y(\cdot) \) and \( \hat{a}(\cdot) \) in (3.10) are obtained from those of \( y(\cdot) \), \( \tilde{\epsilon}(\cdot) \), and \( \hat{a}(\cdot) \) by interpolation [31, Sec. 7.7.4]. For example, the samples with sampling period \( T_s \) of the numerator in (3.10) are computed as

\[
y(t - \tilde{\epsilon}(t))|_{t=nT_s} = \sum_{k=0}^{N-1} y(kT_s) \text{sinc}(n - \tilde{\epsilon}(nT_s)/T_s - k) \tag{3.12}
\]

where \( N \) is the number of available samples of \( y(\cdot) \).

C. STATISTICAL FUNCTION ESTIMATION OF THE UNDERLYING CYCLOSTATIONARY PROCESS

Estimates of statistical functions of the underlying ACS signal \( x(t) \) are obtained through measurements of those of the de-warped signal \( \hat{x}(t) \) obtained by (3.10). Specifically (see [8], [9, Chap. 13]):

1) The cyclic autocorrelation of \( x(t) \) is estimated by the cyclic correlogram of \( \hat{x}(t) \)

\[
R_x^{(T)}(\alpha, \tau) \triangleq \frac{1}{T} \int_{-T/2}^{T/2} \hat{x}(t+\tau) \hat{x}(t) e^{-j2\pi \alpha t} dt \tag{3.13}
\]
2) The cyclic spectrum of \( x(t) \) is estimated by the thye \textit{frequency-smoothed cyclic periodogram} of \( \hat{x}(t) \)

\[
S_{\hat{x}}^{(T,\Delta f)}(\alpha, f) \triangleq \frac{1}{\Delta f} \int_{f-\Delta f/2}^{f+\Delta f/2} \frac{1}{T} \hat{X}_T(\lambda)\hat{X}_T^{*}(-\lambda-\alpha)d\lambda
\]

(3.14)

where \( \Delta f \) is the width of the frequency-smoothing window and is coincident with the spectral frequency resolution, and

\[
\hat{X}_T(f) \triangleq \int_{-T/2}^{T/2} \hat{x}(s) e^{-j2\pi fs} ds
\]

(3.15)

is the finite-time Fourier transform of \( \hat{x}(t) \) on the data-record of length \( T \). Reliable estimates are obtained for \( T\Delta f \gg 1 \) [8], [9, Chap. 13], [35, Chap. 5].

The additive periodic component \( x_p(t) \) in the signal \( x(t) \) (see (2.2) and (2.3)) gives rise to first-order cyclostationarity and, at second-order, reflects into the presence of finite-strength additive sine-wave components in the cyclic autocorrelation (see (2.9b)) and Dirac deltas in the cyclic spectrum (see (2.25b)). Consequently, the frequency-smoothed cyclic periodogram of \( x(t) \) presents spikes whose width is equal to the frequency-smoothing window-width \( \Delta f \).

Estimates of the cyclic autocovariance (2.10a) and the second-order cyclic polyspectrum (2.26a) are obtained by removing the effects of the additive periodic component. This is realized by removing the spikes in the frequency-smoothed cyclic periodogram of \( x(t) \) by a sliding median-filtering window [33, Sec. 3], [35, Sec. 5.2.5]. This method allows one to remove the effects of an almost-periodic additive component with unknown frequencies and was originally proposed in [37] in the context of cyclic polyspectrum estimation.

Specifically, the 2nd-order cyclic polyspectrum of \( x(t) \), that is, the cyclic spectrum of the zero-mean component \( x_z(t) \) (see (2.26b)), is estimated by the \textit{median-filtered frequency-smoothed cyclic periodogram} of \( \hat{x}(t) \) [37], [33, Sec. 3]. [35, Sec. 5.2.5]

\[
P_{\hat{x}}^{(T,\Delta f,Df)}(\alpha, f) \triangleq \text{med}_{\nu \in J(f,Df)} \left[ S_{\hat{x}}^{(T,\Delta f)}(\alpha, \nu) \right]
\]

(3.16)

In (3.16), \( \text{med}_{\nu \in J(f,Df)} \text{[}Z(f)\text{]} \) denotes the median value of the complex-valued function \( Z(f) = \text{Re}(Z(f)) + j\text{Im}(Z(f)) \) in the interval \( J(f,Df) \triangleq [f-Df/2, f+Df/2] \) defined as

\[
\text{med}_{\nu \in J(f,Df)} \text{[}Z(\nu)\text{]} \triangleq \text{med}_{\nu \in J(f,Df)} \left[ \text{Re}(Z(\nu)) \right]
\]

\[
+ j \text{med}_{\nu \in J(f,Df)} \left[ \text{Im}(Z(\nu)) \right].
\]

(3.17)

The \textit{inverse Fourier transform} of the \textit{median-filtered frequency-smoothed cyclic periodogram} of \( \hat{x}(t) \) is estimated as estimator of the cyclic autocovariance of \( x(t) \), that is, the cyclic autocorrelation function of the zero-mean component \( x_z(t) \) (see (2.10b)). Reliable estimates are obtained for \( T\Delta f \gg 1 \) and \( Df > \Delta f \) (the value \( Df = 6 \Delta f \) is adopted in the numerical experiments of Section IV)

Alternative techniques for removing the effects of an additive periodic component are proposed and analyzed in [20], [21], [45].

The spectral density of the rotated Loève bifrequency spectrum is obtained by considering cyclic spectrum estimates over a grid of values of \( \alpha \) (see (3.21)).

The knowledge of the estimates \( \hat{a}(t) \), \( \hat{c}(t) \), and of the cyclic correlogram of \( \hat{x}(t) \), provides an estimate of the autocorrelation function of the amplitude-modulated time-warped ACS signal \( y(t) \).

D. EFFECTS OF NOISE AND OTHER DISTURBANCES ON THE MODEL AND THE ESTIMATION PROCEDURE

In this section, the effects of noise and other disturbances on the proposed model and the estimation procedure are briefly discussed.

1) Additive Noise

Assume that the disturbance can be modeled as additive stationary noise independent of the ECG signal. Thus, as a consequence of the well-known signal selectivity property of cyclostationarity-based signal processing algorithms [35, Sec. 9.2], the noise, also after the dewarping procedure of Section III-B, influences the estimated cyclic statistical functions only at \( \alpha = 0 \). In particular, it does not influence these functions at cycle frequency equal to the heart rate. The smallest signal-to-noise ratio (SNR) that can be tolerated in practice depends on the considered data-record length [35, Sec. 9.2].

The same result holds under mild assumptions if the additive noise is nonstationary. In fact, the only case in which the noise has cyclic features that overlap those of the underlying cyclostationary signal \( x(t) \) is when the noise second-order lag product contains finite-strength additive sine-wave components with the same cycle frequencies of \( x(t) \) that have experienced the same time-warping of \( x(t) \).

2) Disturbance on the Amplitude- and Angle-Modulation Functions

Baseline wander and electrode movements are expected to influence the amplitude modulation function \( a(t) \) and the angle modulation function \( \psi(t) \) in model (2.1), mainly if the data-record length is large.

The recovery of the underlying cyclostationary process \( x(t) \) can be carried out, independently of the causes of the amplitude and angle modulations, by the procedures outlined in Sections III-A and III-B, provided that the made assumptions are valid. In particular, the angle modulation introduced by \( \psi(t) \) must be not too deep in order to have that condition (2.5) holds. In such a case, the statistical functions of \( x(t) \) can be still estimated as explained in Section III-C.

The estimated functions \( a(t) \) and \( \psi(t) \), however, are not representative only of physiological factors but also of arti-
facts. Thus, they cannot be exploited for arrhythmia or other
disease analysis and diagnosis.

IV. NUMERICAL RESULTS

In this section, experiments are conducted aimed at
corroborating the effectiveness of the mathematical
model of ECG signal proposed in Section II. Several
ECG data available in the PhysioNet database
https://www.physionet.org [14] have been ana-
yzed.

The analysis is carried out on observation intervals of
several minutes. In such a case, the data-record length is such that
the ACS model is not appropriate and the time-warping
must be accounted for in the model. Subsampling analysis of
cyclic statistic estimates [3], [4], [5] is made in both cases of
non compensated and compensated time warping.

A. EXPERIMENT 1

The ECG signal is extracted from the data file m001.dat
of the CEBS database (combined measurement of ECG,
breathing and seismocardiogram) available at the URL
https://physionet.org/physiobank/database
/cebsdb/ [7]. Signals in m001.dat are recorded from
Subject 001, a 30 years Caucasian male, non-smoker, with
sedentary lifestyle, presumed healthy, after recent coffee
intake, while listening music.

Data was acquired by a Biopac MP36 data acquisition
system (Santa Barbara, CA, USA). The considered signal is
channel 1 (lead I) which is devoted to measure the conventional
ECG with a bandwidth between 0.05 Hz and 150 Hz.
For the ECG measurement, monitoring electrodes were used
with foam tape and sticky gel (3M Red Dot 2560). More
details on the database description are available at the above
mentioned URL and in [7].

The temporal mean value \( \langle y(t) \rangle _t \) is removed, that is, the
signal \( y(t) - \langle y(t) \rangle _t \) is analyzed. Data in m001.dat are
obtained with a sampling frequency of 5kHz. These data
are decimated by a factor 25. Thus, the considered digital-
ized signal \( y(t) \) is converted to a sampling frequency
\( f_s = 1/T_s = 200 \) Hz (sampling period \( T_s = 5\) ms). The data
is converted into Matlab/Octave format by the wfdb toolbox
[41].

In the experiment, \( N = 160 \cdot 10^3 \) samples are taken. This
 corresponds to a data-record length \( T = NT_s = 800 \) s. The
first 3000 samples are drawn in Fig. 1. The considered data-
record length is significantly larger than that adopted in [18],
[29], [43], and [46] (less than 10 s), where a cyclostationary
model is assumed for the ECG signal \( y(t) \).

A coarse estimate of the average heart rate \( \alpha_0 \triangleq \frac{1}{T_0} \) is
obtained by counting the spikes within the observation
interval (see (3.2)). In this case, one obtains \( \alpha_0 = 0.0055 f_s \). Thus, the estimated average cardiac cycle is
\( T_0 = 1/\alpha_0 \simeq 182 T_s \simeq 0.91 \) s. This corresponds to \( 60/T_0 \simeq 66 \) beats per
minute.

The discrete-time counterparts of the cyclic correlogram
\( R_y^{(\tau)}(\alpha, \tau) \) as a function of \( \alpha/f_s \) and \( \tau/T_s \) and of the
frequency-smoothed cyclic correlogram \( S_y^{(T, \Delta f)}(\alpha, f) \) as a function of \( \alpha/f_s \) and \( f/f_s \) are computed. The width of
the frequency-smoothing window is \( \Delta f = f_s/256 \). The magnitude of the estimates are reported in Fig. 2. According
to the theoretical results for the autocorrelation (2.11) and
rotated Loève bifrequency spectrum (2.31) derived for the
proposed model, the ECG signal \( y(t) \) (which is not ACS)
exhibits cyclic features which are spread around the cycle
frequencies \( k/T_0 \) of the underlying cyclostationary process
\( x(t) \). The estimate of the time-averaged autocorrelation
and power spectral density (PSD) of \( y(t) \) are the slices for \( \alpha = 0 \)
in Fig. 2 left and right, respectively. The time-averaged
autocorrelation of \( y(t) \) is not periodic in \( \tau \) since \( y(t) \) does not
contain an additive periodic component but, rather, contains
modulated sine waves.

In Fig. 3, the energy of the cyclic correlogram as a function of
\( \alpha/f_s \) (left)

\[
\lambda_y^{(T)}(\alpha) \triangleq \int_T |R_y^{(T)}(\alpha, \tau)|^2 \, d\tau
\]  

and the energy of the frequency-smoothed cyclic peri-
odogram as a function of \( \alpha/f_s \) (right)

\[
\lambda_y^{(T, \Delta f)}(\alpha) \triangleq \int_B |S_y^{(T, \Delta f)}(\alpha, f)|^2 \, df
\]

are reported, where \( T \) is a set in which the cyclic correlogram
is significantly non zero and \( B = (-f_s/2, f_s/2) \). If the signal
\( y(t) \) were cyclostationary, one should obtain functions
\( \lambda_y^{(T)}(\alpha) \) and \( \lambda_y^{(T, \Delta f)}(\alpha) \) highly concentrated around
a discrete set of cycle frequencies with the widths of the
peaks of the order of \( 1/T \) which is the cycle frequency
resolution of cyclic statistic estimators based on a data-record
length \( T \) [35, Secs. 5.2.1 and 5.2.3]. In contrast, the width
of the peaks in Fig. 3 is significantly larger than \( 1/T \) and is
compatible with irregular cyclicity, that is, with the presence
of additive amplitude- and/or angle modulated sine waves
in the autocorrelation function (see (2.11)). Since \( a(t) \simeq 1 \) (see
Fig. 6 right), for \( \alpha = 0 \) we have \( M_0^y(f) \simeq \delta(f) \) and there is
no spread of the PSD around the line \( \alpha = 0 \).

A further confirmation of the presence of irregular cyclic-
ity in the autocorrelation function is given by the subsam-
pling analysis [3], [4], [5] of the cyclic correlogram. The
block size is \( b = 100T_0 \) (with \( T_0 \) estimated cardiac cycle) and
the block-overlap factor is 1/4. In Fig. 4, (left) real part and
(right) imaginary part of the cyclic correlogram \( R_y^{(T)}(\alpha, \tau) \) as a function of \( \tau/T_0 \) at \( \alpha = \alpha_0 \) (estimated average heart
rate) are reported (thick line). The shaded area is the region
within 95% confidence interval. The high variability of the
estimate is in accordance with the fact that the (regular)
cyclostationary model is not appropriate for the ECG signal
\( y(t) \) if the data-record length exceeds few seconds.

In order to estimate \( e(t), a(t) \) and then to recover an estimate
of \( x(t) \) by de-warping, the procedure described in Sec-

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The condition that the amplitude- and angle-modulated sine waves with $k \neq 1$ in (3.1) have spectra that practically do not overlap the power spectrum of the modulated sine wave with $k = 1$ (so that (3.3b) holds) is verified.

The estimates $\hat{\alpha}(t)$ and $\hat{\alpha}(t)$ obtained by (3.6) and (3.4) are reported in Fig. 6 left and right, respectively. From this figure, we see that $\sup_t |\epsilon(t)| \simeq 500 T_s$ which is greater than $T_0$. That is, the time-warping is relevant and cannot be neglected.

The refined cycle frequency estimate obtained at the end of the angle-demodulation procedure (see (3.7)) is $\hat{\alpha}_0 = 1/T = 0.00558018 f_s$.

The estimate $\hat{x}(t)$ of the underlying cyclostationary signal $x(t)$ is obtained by amplitude-modulation compensation and de-warping from $y(t)$ by (3.10) (Section III-B).

In order to get estimates of the cyclic autocorrelation and the cyclic spectrum of $x(t)$, the discrete-time counterparts of the cyclic correlogram $R_x^\tau(\alpha, \tau)$ as a function of $\alpha/f_s$ and $\tau/T_s$ and of the frequency-smoothed cyclic periodogram $S_x^{(T,\Delta f)}(\alpha, f)$ as a function of $\alpha/f_s$ and $f/T_s$ are computed. The width of the frequency-smoothing window is $\Delta f = f_s/256$. The magnitude of the estimates are reported in Fig. 7. Unlike the ECG signal $y(t)$ (Fig. 2), $\hat{x}(t)$ has cyclic features well concentrated around the cycle frequencies $k/T_0$ by confirming the conjectured model (2.1) with underlying cyclostationary signal $x(t)$.

In Fig. 8, the energy of the cyclic correlogram of $\hat{x}$ as a function of $\alpha/f_s$ (left) and the energy of the frequency-smoothed cyclic periodogram of $\hat{x}$ as a function of $\alpha/f_s$ (right) defined according to (4.1) and (4.2), respectively, with $y$ replaced by $\hat{x}$, are reported. As further confirmation of the proposed cyclostationary model for $x(t)$, unlike the ECG signal $y(t)$ (Fig. 3), $\hat{x}(t)$ has cyclic features well concentrated around the cycle frequencies $k/T_0$.

The slices at $\alpha = \hat{\alpha}_0$ of the cyclic correlogram and of the frequency-smoothed cyclic periodogram are reported in Fig. 9. The presence of an additive periodic component in $R_x^\tau(\alpha, \tau)$ (Fig. 9 left) is in agreement with (2.9b). The spikes in $S_x^{(T,\Delta f)}(\alpha, f)$ (Fig. 9 right), whose shape is the frequency-smoothing window with width $\Delta f$, correspond to the Dirac pulses in (2.25b).

The estimates of the cyclic autocorrelation and the cyclic spectrum of the zero-mean cyclostationary component $x_s(t)$ are obtained by the median-filtering procedure described in Sec. III-C and are reported in Fig. 10. They are obtained as the estimate $C_x^{(T,\Delta f, B f)}(\alpha, \tau)$ of the autocovariance and the estimate $F_x^{(T,\Delta f, D f)}(\alpha, f)$ of the 2nd-order cyclic polyspectrum of the underlying cyclostationary signal $x(t)$.

As a further confirmation of the effectiveness of the proposed model for the ECG signal $y(t)$ with an underlying cyclostationary signal $x(t)$, a subsampling analysis of the cyclic correlogram of $\hat{x}(t)$ is carried out. The block size is $b = 100 T_0$ and the block-overlap factor is $1/4$. In Fig. 11, (left) real part and (right) imaginary part of the cyclic correlogram $R_x^\tau(\alpha, \tau)$ as a function of $\tau/T_s$ at $\alpha = \hat{\alpha}_0$ (estimated average heart rate) are reported. The shaded area is the region within 95% confidence interval. The very low variability of the estimate confirms the (regular) cyclostationary model for $x(t)$.

### B. EXPERIMENT 2

In order to corroborate the effectiveness of the proposed model for the ECG signal, the analysis made in Section IV-A is repeated for all the subjects whose ECG measurements are available in the CEBS database. For the sake of brevity, only results of three subjects are reported here. Results for the other subjects are similar.

The analysis parameters are the same as those in the experiment of Section IV-A. The only difference is in the value of the estimated average heart rate $\hat{\alpha}_0$ since it depends on the subject.

In Figs. 12–14, results are reported for the subjects 002, 004, and 010. For all the subjects the results are in accordance with the proposed theoretical model described in Section II.

In the left-top sub-figures of Figs. 12–14, the thick line represents the PSD of $y^2(t) e^{-j2\pi\hat{\alpha}_0 t}$ as a functions of $f/f_s$.
and the dotted line delimits the portion of spectrum within the band of the low-pass filter \( h_W(t) \). For all the subjects, the spectral region corresponding to the modulated first harmonic in the autocorrelation is well separated from the spectral regions of the neighboring harmonics and can be easily extracted by the filtering procedure (3.3a), (3.3b). In the left-bottom sub-figures, the estimated \( \epsilon(t) \) as a function of \( t/T_s \) is reported. Spectral analysis of the estimated \( \epsilon(t) \) is suitable to be exploited for arrhythmia diagnosis and characterization.

In the middle-top sub-figures of Figs. 12–14, the energy \( \lambda^{(T)}_\nu(\alpha) \) (see (4.1)) of the cyclic correlogram of the ECG signal \( y(t) \) as a function of \( \alpha/f_s \) is reported. According to (2.31), the cyclic features are spread around the cycle frequencies of the underlying cyclostationary signal \( x(t) \). In the middle-bottom sub-figures, the energy \( \lambda^{(T)}_x(\alpha) \) of the cyclic correlogram of the estimated underlying cyclostationary signal \( \hat{x}(t) \) as a function of \( \alpha/f_s \) is reported. For all the subjects, the cyclic features of \( \hat{x}(t) \) are well concentrated around the integer multiples of the estimated average heart rate \( \hat{\alpha}_0 \) confirming the validity of the proposed model for the ECG signal.

In the right-top sub-figures of Figs. 12–14, the magnitude of the estimate (3.13) of the cyclic autocorrelation of \( \hat{x}(t) \) at cycle frequency \( \alpha = \hat{\alpha}_0 \) as a function of \( \tau/T_s \) is reported. In all plots, that is, for all the subjects, an approximately periodic pattern can be recognized, with superimposed a function which is nonzero around \( \tau = 0 \). This behavior is in agreement with the analytical model (2.9a)–(2.10b). In fact, the cyclic autocorrelation \( R_x^\alpha(\tau) \) is the sum (see (2.9b)) of a periodic function and the autocorrelation \( R_{x_0}(\tau) \) of the residual term which is coincident with the autocovariance \( C_{x_0}(\tau) \) (see (2.10b)). The function \( C_{x_0}(\tau) \) is summable and, hence, approaches zero for large values of \( |\tau| \).

By comparing the right-top subfigures of Figs. 12–14 (that correspond to different subjects), it can be observed that different subjects are characterized by different shapes of the cyclic autocovariance and different periodic patterns. Thus, these features can be exploited for biometric purposes. In the right-bottom subfigures, the magnitude of the estimate (3.16) of the second-order cyclic polyspectrum of \( \hat{x}(t) \) at cycle frequency \( \alpha = \hat{\alpha}_0 \) as a function of \( f/f_s \) is reported. Also this feature is different for different subjects and is suitable to be exploited for biometric purposes.
FIGURE 2. ECG signal $y(t)$. (Left) magnitude of the cyclic correlogram $R^{(T)}_y(\alpha, \tau)$ as a function of $\alpha/f_s$ and $\tau/T_s$. (Right) magnitude of the frequency-smoothed cyclic periodogram $S^{(T,\Delta f)}_y(\alpha, f)$ as a function of $\alpha/f_s$ and $f/f_s$. Cyclic features of the underlying cyclostationary signal $x(t)$ are spread around the cycle frequencies $k/T_0$ of $x(t)$ according to (2.31).

FIGURE 3. ECG signal $y(t)$. (Left) energy of the cyclic correlogram $R^{(T)}_y(\alpha, \tau)$ as a function of $\alpha/f_s$. (Right) energy of the frequency-smoothed cyclic periodogram $S^{(T,\Delta f)}_y(\alpha, f)$ as a function of $\alpha/f_s$. Cyclic features of the underlying cyclostationary signal $x(t)$ are spread around the cycle frequencies $k/T_0$ of $x(t)$ according to (2.31).

FIGURE 4. ECG signal $y(t)$. Thick line: Cyclic correlogram $R^{(T)}_y(\alpha, \tau)$ as a function of $\tau/T_s$ at $\alpha = \hat{\alpha}_0 = \text{estimated average heart rate}$. Shaded area: region within 95% confidence interval. (Left) real part and (right) imaginary part. For each $\tau$, estimates made on blocks of length $b$ are spread around the estimate made on the whole data-record length $T$. 

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FIGURE 5. ECG signal $y(t)$. Power spectral density of $y^2(t) e^{-j2\pi \alpha_0 t}$. The condition that the amplitude- and angle-modulated sine waves in (3.1) have spectra that practically do not overlap (so that (3.3b) holds) is verified. The thin line delimits the portion of spectrum in the band of the low-pass filter $h_W(t)$ aimed at extracting the signal $z^{(\alpha_0; W)}(t)$ (see (3.3a) and (3.3b)).

FIGURE 6. ECG signal $y(t)$. (Left) $\epsilon(t)$ estimated by (3.6); (right) $a(t)$ estimated by (3.4).

FIGURE 7. Estimated underlying cyclostationary signal $\tilde{x}(t)$. (Left) magnitude of the cyclic correlogram $R_x^{(T)}(\alpha, \tau)$ as a function of $\alpha/f_s$ and $\tau/T_s$. (Right) magnitude of the frequency-smoothed cyclic periodogram $S_x^{(T, \Delta f)}(\alpha, f)$ as a function of $\alpha/f_s$ and $f/f_s$. Unlike the ECG signal $y(t)$ (Fig. 2), $\tilde{x}(t)$ has cyclic features concentrated around the cycle frequencies $k/T_0$. 
FIGURE 8. Estimated underlying cyclostationary signal \( \hat{x}(t) \). (Left) energy of the cyclic correlogram \( R^{(T)}(\alpha, \tau) \) as a function of \( \alpha/f_s \). (Right) energy of the frequency-smoothed cyclic periodogram \( S^{(T, \Delta f)}(\alpha, f) \) as a function of \( \alpha/f_s \). Unlike the ECG signal \( y(t) \) (Fig. 3), \( \hat{x}(t) \) has cyclic features concentrated around the cycle frequencies \( k/T_0 \).

FIGURE 9. Estimated underlying cyclostationary signal \( \hat{x}(t) \). (Left) magnitude of the cyclic correlogram \( R^{(T)}(\alpha, \tau) \) as a function of \( \tau/T_s \) at \( \alpha = \hat{\alpha}_0 = \) estimated average heart rate. An additive periodic component is present in accordance with (2.9b). (Right) magnitude of the frequency-smoothed cyclic periodogram \( S^{(T, \Delta f)}(\alpha, f) \) as a function of \( f/f_s \) at \( \alpha = \hat{\alpha}_0 \). Spikes correspond to Dirac deltas in (2.25b).

FIGURE 10. Estimated underlying cyclostationary signal \( \hat{x}(t) \). (Left) magnitude of the estimated cyclic covariance \( C^{(T, \Delta f, Df)}(\alpha, \tau) \) as a function of \( \tau/T_s \) for \( \alpha = \hat{\alpha}_0 = \) estimated average heart rate. (Right) magnitude of the estimated second-order cyclic polyspectrum \( P^{(T, \Delta f, Df)}(\alpha, f) \) as a function of \( f/f_s \) at \( \alpha = \hat{\alpha}_0 \).
FIGURE 11. Estimated underlying cyclostationary signal \( \hat{z}(t) \). Thick line: Cyclic correlogram \( R_T^{(x)}(\alpha, \tau) \) as a function of \( \frac{\tau}{T} \), at \( \alpha = \hat{\alpha}_0 = \) estimated heart rate. Shaded area: region within 95% confidence interval. (Left) real part and (right) imaginary part. For each \( \tau \), estimates made on blocks of length \( b \) are concentrated around the estimate made on the whole data-record length \( T \).

FIGURE 12. Subject m002. Left Top: Power spectral density of \( y^2(t) e^{-j2\pi \hat{\alpha}_0 t} \) (thick line) as a functions of \( \frac{f}{f_s} \). The dotted line delimits the portion of spectrum within the band of the low-pass filter \( h_W(t) \). The condition that the amplitude- and angle-modulated sine waves with \( k \neq 1 \) in (3.1) have spectra that practically do not overlap the power spectrum of the sine wave with \( k = 1 \) (so that (3.3b) holds) is verified. Left Bottom: Estimated \( \epsilon(t) \) as a function of \( \frac{t}{T_s} \). Middle Top: Energy \( \lambda^{(T)}_0(\alpha) \) (see (4.1)) of the cyclic correlogram of the ECG signal \( y(t) \) as a function of \( \alpha/f_s \). Cyclic features of the underlying cyclostationary signal \( x(t) \) are spread around cycle frequencies \( k/T_0 \) of \( x(t) \) according to (2.31). Middle Bottom: Energy \( \lambda^{(T)}_{\hat{\alpha}_0}(\alpha) \) of the cyclic correlogram of the estimated underlying cyclostationary signal \( \hat{z}(t) \) as a function of \( \alpha/f_s \). Cyclic features are concentrated around the cycle frequencies \( k/T_0 \). Right Top: Magnitude of the estimate (3.13) of the cyclic autocorrelation of \( \hat{z}(t) \) at cycle frequency \( \hat{\alpha} = \hat{\alpha}_0 \) as a function of \( \frac{\tau}{T_s} \). The pattern of the cyclic autocovariance is superimposed to the periodic pattern due to the periodic term in the signal model for \( x(t) \). Right Bottom: Magnitude of the estimate (3.16) of the second-order cyclic polyspectrum of \( \hat{z}(t) \) at cycle frequency \( \hat{\alpha} = \hat{\alpha}_0 \) as a function of \( \frac{f}{f_s} \). \( \hat{\alpha}_0 = 0.005365 f_s \).

FIGURE 13. Subject m004. Same caption of Fig. 12. \( \hat{\alpha}_0 = 0.005557 f_s \).
FIGURE 14. Subject m010. Same caption of Fig. 12. $\alpha_0 = 0.004952 f_s$. 
V. CONCLUSION

A new model for the electrocardiogram (ECG) signal is proposed. The ECG signal is modeled as an amplitude-modulated and time-warped cyclostationary process. The underlying cyclostationary process is decomposed into the sum of a deterministic periodic signal and a zero-mean cyclostationary term. The period of the periodic signal and the cyclostationarity period of the zero-mean term are both equal to the reciprocal of the average heart rate. The time-warping function describes the heart rate variability due to different factors like physical activity, emotions and arrhythmia. Both amplitude-modulating and time-warping functions are also consequence of variations in the propagation of the electrical wave throughout the heart.

For the proposed model, the second-order characterization is derived in both time and frequency domains. The autocorrelation function is found to be the superposition of amplitude- and angle-modulated sine waves whose frequencies are the cycle frequencies of the periodic autocorrelation function of the underlying cyclostationary signal. Thus, the proposed ECG signal model belongs to the recently introduced class of the oscillatory almost-cyclostationary processes. Since the derivation of the Loève bifrequency spectrum is a formidable problem, a smoothed version is derived. The presence of the additive periodic term in the signal model reflects into the presence of finite-strength additive sine waves in the cyclic autocorrelations and Dirac impulses in the cyclic spectra of the underlying cyclostationary signal.

Estimators of the amplitude- and angle-modulation functions are derived and a de-warping and amplitude-modulation compensation procedure for estimating the underlying cyclostationary signal is outlined. Thus, for such a signal, cyclic autocorrelation, autocovariance, spectrum, and 2nd-order polyspectrum are estimated.

Real ECG signals taken from the CEBS database of the PhysioNet database are analyzed. The assumptions made to develop the de-warping procedure are found to be satisfied by all the analyzed ECG signals. Thus, the de-warping procedure is successfully adopted to restore the underlying cyclostationary signal and its cyclic statistical functions are estimated. The measurement results are in full agreement with the proposed theoretical model. In particular, the estimates of the cyclic autocorrelation functions contain additive sinusoidal terms and, accordingly, the estimates of the cyclic spectra contain spikes. The estimates of the cyclic autocovariance and of the cyclic second-order polyspectrum, that are coincident with the cyclic autocorrelation and cyclic spectrum of the zero-mean residual term, respectively, are also obtained.

The considered data-record lengths are 800 seconds. They are much larger than those of few seconds adopted up to now with the classical (regular) cyclostationary model. The larger data-record length, which is compatible with the proposed ECG signal model, allows the analysis of diseases like the arrhythmia.

The subsampling analysis of the cyclic autocorrelation estimates provides a further confirmation of the effectiveness of the proposed oscillatory almost-cyclostationary model for data-record lengths exceeding few seconds. In fact, the estimate of the cyclic autocorrelation made on the original ECG signal exhibits a wide 95% confidence interval. It is consequence of the large variability of the estimate along the blocks in accordance with the fact that the classical cyclostationary model is compatible only with a constant heart rate within the observation interval. In contrast, if time-warping is compensated and the estimate is made on the reconstructed underlying cyclostationary signal, a narrow 95% confidence interval for the cyclic autocorrelation estimate is found.

Measurements made on the ECG signals of different subjects give rise to underlying cyclostationary signals whose additive periodic components have different Fourier coefficients and whose zero-mean terms have different cyclic statistical functions. Thus, these cyclic features are suitable to be exploited for biometric purposes.

Further numerical results not presented here have confirmed that the cyclic features of the underlying cyclostationary process remain the same for the same subject when measured in different time intervals. In contrast, the amplitude-modulation and time-warping functions, which are consequence of physical activity, emotions, and arrhythmia, depend on the time interval in which they are measured.

Future work will explore the spectral analysis of the time-warping function for the arrhythmia diagnosis and characterization and the exploitation of the cyclostationary features for biometric applications.
Let us consider the 1st-order Taylor series expansion
\[ \epsilon(t+\tau) = \epsilon(t) - \dot{\epsilon}(\bar{t}) \tau \] (A.3)
where \( \bar{t} \in (t, t+\tau) \) if \( \tau > 0 \) and \( \bar{t} \in (t+\tau, t) \) if \( \tau < 0 \). The time-varying term \( \epsilon(t+\tau) - \epsilon(t) \) in the argument of \( R_{x_x}^\alpha(\cdot) \) in (2.13) can be neglected provided that
\[ \sup_t|\dot{\epsilon}(t)| |\tau| < \frac{1}{B_\tau}. \] (A.4)
In (A.4), \( B_\tau \) denotes the bandwidth of \( x_\tau(t) \). Thus, 1/\( B_\tau \) is the approximate width of the time averaged autocorrelation \( R_{x_x}^\alpha(\cdot) \) and, hence, of \( R_{x_x}^\alpha(\cdot), \alpha \neq 0 \).
For \( |\tau| < 1/B_\tau \), condition (A.4) is satisfied provided that (2.5) holds. Thus, if condition (2.5) is satisfied,
\[ R_{x_x}^\alpha(\tau + e(t+\tau) - e(t)) \approx R_{x_x}^\alpha(\tau + \dot{\epsilon}(\bar{t}) \tau) \approx R_{x_x}^\alpha(\tau). \] (A.5)

**DERIVATION OF (2.16)–(2.17)**
Using (A.3) into the generic term of the sum over \( \tau \) in (2.14), one has
\[
\begin{align*}
  x_{h/T_0} x^*_{(h-k)/T_0} e^{j2\pi(h/T_0)(\tau + e(t+\tau) - e(t))} \\
  = x_{h/T_0} x^*_{(h-k)/T_0} e^{j2\pi(h/T_0)(\tau + \dot{\epsilon}(\bar{t}) \tau)} \\
  \approx x_{h/T_0} x^*_{(h-k)/T_0} e^{j2\pi(h/T_0)\tau}
\end{align*}
\] (A.6)
where the approximate equality holds provided that the phase term \( e^{j2\pi(h/T_0)\dot{\epsilon}(\bar{t})} \) can be neglected, that is for \( |\tau| < \tau_{\text{max}} \) with \( \tau_{\text{max}} \) such that
\[ \frac{2\pi |h|}{T_0} \sup_t|\dot{\epsilon}(t)| \tau_{\text{max}} < 1. \] (A.7)
Therefore, since only terms with \( |h| < h_{\text{max}} \) give non negligible contribution in (2.14), under condition (2.17) one obtains (2.16).

**DERIVATION OF (2.31)**
By using the autocorrelation expression (2.11) with (2.21) replaced into, one has
\[ \int_{\mathbb{R}^2} E\{y(t+\tau) y(t)\} w(\tau) e^{-j2\pi f\tau} e^{-j2\pi f\delta t} d\tau dt \]
\[
= \int_{\mathbb{R}^2} \sum_{k=-\infty}^{+\infty} m^k_{y/T_0}(t) R^k_{x_x}(\tau) e^{j2\pi(k/T_0)\tau} \cdot w(\tau) e^{-j2\pi f\tau} e^{-j2\pi f\delta t} d\tau dt \\
= \sum_{k=-\infty}^{+\infty} \int_{\mathbb{R}} R^k_{x_x}(\tau) w(\tau) e^{-j2\pi f\tau} d\tau \\
\cdot \int_{\mathbb{R}} m^k_{y/T_0}(t) e^{-j2\pi(\delta f-k/T_0)\tau} d\tau \] (A.8)
Equation (2.31) follows using (2.32a) and the convolution theorem for Fourier transforms.
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ANTONIO NAPOLITANO (M’95–SM’07–F’16) was born in Napoli, Italy, in 1964. He received the Ph.D. in Electronic Engineering and Computer Science in 1994, from the University of Napoli Federico II. In 1997, he was with the Department of Electrical and Computer Engineering at the University of California, Davis, USA, as a Post-doctorate Research Associate. From 1995 to 2001 he has been Assistant Professor and from 2001 to 2005 Associate Professor at the University of Napoli Federico II. Since 2005 he is Full Professor of Telecommunications at the University of Napoli “Parthenope”. Since 2020 he has the Affiliate Status at the James Watt School of Engineering, University of Glasgow, UK. He was Visiting Professor at Institute de Recherche Mathematique de Rennes, Haute Bretagne, France (2005); Laboratoire d’Analyse des Signaux & des Processus Industriels, Universite Jean-Monnet, Roanne, France (2010, 2013); Ecole Nationale Superieure d’Electronique, d’Electrotechnique, d’Informatique et d’Hydraulique et Telecommunications, Toulouse, France (2014, 2015); Politechnika Krakowska, Krakow, Poland (2021–2022). He held visiting appointments at Universite de Nice–Sophia Antipolis, France (1994); University of North Carolina, Chapel Hill, NC, USA (1997,2001,2002); Technical University of Wroclaw, Poland (1999, 2001); Centro de Investigacion en Matematicas, Guanajuato, Mexico (2000–2002); Econometric Department, Wyszsa Szkola Biznesu, Nowy Sacz, Poland (2000–2007); School of Electrical and Information Engineering, University of South Australia (2010); Beijing Institute of Technology (2019). In 1996 he received the Best Paper of the Year Award from the European Association for Signal Processing (EURASIP) for a paper on higher-order cyclostationarity. In 2007 was recipient of the EURASIP Best Paper Award for a paper on the functional approach in signal analysis. In 2008 he received from Elsevier the Most Cited Paper Award for a review article on cyclostationarity. From 2006 to 2009 and from 2011 to 2015 he has been Associate Editor of the IEEE Transactions on Signal Processing. He is in the Editorial Board of Signal Processing (Elsevier) since 2008 and Digital Signal Processing (Elsevier) since 2015. From 2008 to 2013 and from 2022 he has been Elected Member of the Signal Processing Theory and Method Technical Committee (SPTM-TC) of the IEEE Signal Processing Society. From 2017 he has been Elected Member of the Sensor Array and Multichannel Technical Committee (SAM-TC) of the IEEE Signal Processing Society. He is EURASIP Local Liaison Officer. He is the author of the books “Generalizations of Cyclostationary Signal Processing: Spectral Analysis and Applications,” John Wiley & Sons Ltd. – IEEE Press, 2012 and “Cyclostationary Processes and Time Series: Theory, Applications, and Generalizations,” Elsevier, 2019.