Continuous Submodular Function Maximization

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Abstract

Continuous submodular functions are a category of generally non-convex/non-concave functions with a wide spectrum of applications. The celebrated property of this class of functions – continuous submodularity – enables both exact minimization and approximate maximization in polynomial time. Continuous submodularity is obtained by generalizing the notion of submodularity from discrete domains to continuous domains. It intuitively captures a repulsive effect amongst different dimensions of the defined multivariate function.

In this paper, we systematically study continuous submodularity and a class of non-convex optimization problems: continuous submodular function maximization. We start by a thorough characterization of the class of continuous submodular functions, and show that continuous submodularity is equivalent to a weak version of the diminishing returns (DR) property. Thus we also derive a subclass of continuous submodular functions, termed continuous DR-submodular functions, which enjoys the full DR property. Then we present operations that preserve continuous (DR-)submodularity, thus yielding general rules for composing new submodular functions. We establish intriguing properties for the problem of constrained DR-submodular maximization, such as the local-global relation, which captures the relationship of locally (approximate) stationary points and global optima. We identify several applications of continuous submodular optimization, ranging from influence maximization with general marketing strategies, MAP inference for DPPs to mean field inference for probabilistic log-submodular models. For these applications, continuous submodularity formalizes valuable domain knowledge relevant for optimizing this class of objectives. We present inapproximability results and provable algorithms for two problem settings: constrained monotone DR-submodular maximization and constrained non-monotone DR-submodular maximization. Finally, we extensively evaluate the effectiveness of the proposed algorithms on different problem instances, such as influence maximization with marketing strategies and revenue maximization with continuous assignments.

Keywords: Continuous submodularity, Continuous DR-submodularity, Submodular function maximization, Provable non-convex optimization, Revenue maximization

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1. Introduction

Submodularity is a structural property usually associated with set functions, with important implications for optimization (Nemhauser et al., 1978). The general setup requires a ground set \( V \) containing \( n \) items, which could be, for instance, the set of all features in a given supervised learning problem (Das and Kempe, 2011), or the set of all users in the influence maximization problem (Kempe et al., 2003). Usually, we have an objective function that maps a subset of \( V \) to a real value: \( F(X) : 2^V \rightarrow \mathbb{R}_+ \). This function often quantifies utility, coverage, relevance, diversity etc. Equivalently, one can express any subset \( X \) as a binary vector \( x \in \{0,1\}^n \). Hereby, for component \( i \) of \( x \), \( x_i = 1 \) means that item \( i \) is inside \( X \), otherwise item \( i \) is outside of \( X \). This binary representation associates the powerset of \( V \) with all vertices of an \( n \)-dimensional hypercube. Because of this, we also call submodularity of set functions “submodularity over binary domains” or “binary submodularity”.

Over binary domains, there are two well-known definitions of submodularity: the lattice definition and the diminishing returns (DR) definition.

Definition 1 (Lattice definition) A set function \( F : 2^V \mapsto \mathbb{R}_+ \) is submodular iff \( \forall X, Y \subseteq V \), it holds:
\[
F(X) + F(Y) \geq F(X \cup Y) + F(X \cap Y).
\]

Definition 2 (DR definition) A set function \( F(X) : 2^V \mapsto \mathbb{R}_+ \) is submodular iff \( \forall A \subseteq B \subseteq V \) and \( \forall v \in V \setminus B \), it holds:
\[
F(A \cup \{v\}) - F(A) \geq F(B \cup \{v\}) - F(B).
\]

Optimizing submodular set functions has found numerous applications in machine learning, including variable selection (Krause and Guestrin, 2005), dictionary learning (Krause and Cevher, 2010; Das and Kempe, 2011), sparsity inducing regularizers (Bach, 2010), summarization (Gomes and Krause, 2010; Lin and Bilmes, 2011a; Mirzasoleiman et al., 2013) and variational inference (Djolonga and Krause, 2014b). Submodular set functions can be efficiently minimized (Iwata et al., 2001), and there are strong guarantees for approximate maximization (Nemhauser et al., 1978; Krause and Golovin, 2012).

Even though submodularity is most widely considered in the discrete setting, the notion can be generalized to arbitrary lattices (Fujishige, 2005). Of particular interest are lattices over real vectors, which can be used to define submodularity over continuous domains (Topkis, 1978; Bach, 2015; Bian et al., 2017b). But one may wonder: why do we need continuous submodularity?

In summary, there are two motivations for studying continuous submodularity: i) It is an important modeling ingredient for many real-world applications; ii) It captures a subclass of well-behaved non-convex optimization problems, which admits guaranteed optimization with algorithms running in polynomial time. In the following, we will informally illustrate these two aspects.
Natural Prior Knowledge for Modeling. In order to illustrate the first motivation, let us consider a stylized scenario. Suppose you got stuck in the desert one day, and became extremely thirsty. After two days of exploration you found a bottle of water. What is even better is that you also found a bottle of soda.

We will use a two-dimensional function $f([x_1; x_2])$ to quantize the “happiness” gained by having $x_1$ quantity of water and $x_2$ quantity of soda. Let $\delta = [50\text{ml water}; 50\text{ml soda}]$. Now it is natural to see that the following inequality shall hold: $f([1\text{ml}; 1\text{ml}] + \delta) - f([1\text{ml}; 1\text{ml}]) \geq f([100\text{ml}; 100\text{ml}] + \delta) - f([100\text{ml}; 100\text{ml}])$. The LHS of the inequality measures the marginal gain of happiness by having $\delta$ more [water, soda] based on a small context ([1ml; 1ml]), while the RHS means the marginal gain based on a large context ([100ml; 100ml]), this is a typical example of the well-known diminishing returns (DR) phenomenon, which will formally defined in Section 3.1. The DR property models the context sensitive expectation that adding one more unit of resource contributes more in the small context than in a large context.

This example illustrates that diminishing returns effects naturally occur in continuous domains, not only discrete ones. While related to concavity, we will see that continuous submodularity yields complementary means of modeling diminishing returns effects over continuous domains. Real-world examples comprise user preferences in recommender systems, customer satisfaction, influence in social advertisements etc.

Non-Convex Structure enabling Provable Optimization. Non-convex optimization is a core challenge in machine learning, and arises in numerous learning tasks from training deep neural networks (Bottou et al., 2018) to latent variable models (Anandkumar et al., 2014). A fundamental problem in non-convex optimization is to reach a stationary point assuming smoothness of the objective for unconstrained optimization (Sra, 2012; Li and Lin, 2015; Reddi et al., 2016a; Allen-Zhu and Hazan, 2016) or constrained optimization problems (Ghadimi et al., 2016; Lacoste-Julien, 2016). However, without further assumptions, a stationary point may in general be of arbitrary poor objective value. It thus remains a challenging problem to understand which classes of non-convex objectives can be tractably optimized.

In pursuit of solving this challenging problem, we show that continuous submodularity provides a natural structure for provable non-convex optimization. It arises in various important non-convex objectives. Let us look at a simple example by considering a classical quadratic program (QP): $f(x) = \frac{1}{2}x^T H x + h^T x + c$. When $H$ is symmetric, we know that the Hessian matrix is $\nabla^2 f = H$. Let us consider a specific two dimensional example, where $H = [-1, -2; -2, -1]$. One can verify that its eigenvalues are $[1; -3]$. So it is an indefinite quadratic program, which is neither convex, nor concave. However, it will soon be clear that $f$ is a DR-submodular function (see definitions in Section 3). In this paper, we propose polynomial-time solvers for optimizing such objectives with strong approximation guarantees. Further examples of submodular objectives include the Lovász (Lovász, 1983) and multilinear extensions (Calinescu et al., 2007) of submodular set functions, or to the softmax extensions (Gillenwater et al., 2012) for DPP (determinantal point process) MAP inference.

Organization of the Paper. We will present a brief background of submodular optimization, the classical Frank-Wolfe algorithm and existing structures for non-convex optimiza-
tion in Section 2. In Section 3 we give a thorough characterization of the class of continuous submodular and DR-submodular1 functions. Section 4 presents general composition rules that preserve continuous (DR-)submodularity, along with exemplary applications of these rules, such as for designing deep submodular functions. Section 5 discusses intriguing properties for the problem of constrained DR-submodular maximization in both monotone and non-monotone settings, such as the local-global relation. In Section 6 we illustrate representative applications of continuous submodular optimization. In the next two sections we discuss hardness results and algorithmic techniques for constrained DR-submodular maximization in different settings: Section 7 illustrates how to maximize monotone continuous DR-submodular functions, and Section 8 provides techniques for maximizing non-monotone DR-submodular functions with a down-closed convex constraint. We present experimental results on three representative problems in Section 9. Lastly, Section 10 discusses and concludes the paper.

2. Background and Related Work

We give a brief introduction of the background of submodular optimization in this section.

**Notation.** Throughout this work we assume \( V = \{v_1, v_2, ..., v_n\} \) being the ground set of \( n \) elements, and \( e_i \in \mathbb{R}^n \) is the characteristic vector for element \( v_i \) (also the standard \( i \)th basis vector). We use boldface letters \( \mathbf{x} \in \mathbb{R}^V \) and \( \mathbf{x} \in \mathbb{R}^n \) interchangeably to indicate an \( n \)-dimensional vector, where \( x_i \) is the \( i \)th entry of \( \mathbf{x} \). We use a boldface capital letter \( \mathbf{A} \in \mathbb{R}^{m \times n} \) to denote an \( m \) by \( n \) matrix and use \( A_{ij} \) to denote its \( ij \)th entry. By default, \( f(\cdot) \) is used to denote a continuous function, and \( F(\cdot) \) to represent a set function. For a differentiable function \( f(\cdot), \nabla f(\cdot) \) denotes its gradient, and for a twice differentiable function \( f(\cdot), \nabla^2 f(\cdot) \) denotes its Hessian. \([n] := \{1, ..., n\}\) for an integer \( n \geq 1\). \( \|\cdot\| \) means the Euclidean norm by default. Given two vectors \( \mathbf{x}, \mathbf{y}, \mathbf{x} \leq \mathbf{y} \) means \( x_i \leq y_i, \forall i \). \( \mathbf{x} \vee \mathbf{y} \) and \( \mathbf{x} \wedge \mathbf{y} \) denote coordinate-wise maximum and coordinate-wise minimum, respectively. \( \mathbf{x}|_i(k) \) is the operation of setting the \( i \)th element of \( \mathbf{x} \) to \( k \), while keeping all other elements unchanged, i.e., \( \mathbf{x}|_i(k) = \mathbf{x} - x_i e_i + k e_i \).

2.1 Submodularity over Discrete Domains

As a discrete analogue of convexity, submodularity provides computationally effective structure so that many discrete problems with this property can be efficiently solved or approximated. Of particular interest is a \( (1 - 1/e) \)-approximation for maximizing a monotone submodular set function subject to a cardinality, a matroid, or a knapsack constraint (Nemhauser et al., 1978; Vondrák, 2008; Sviridenko, 2004). For maximizing non-monotone submodular functions, a 0.325-approximation under cardinality and matroid constraints (Gharan and Vondrák, 2011), and a 0.2-approximation under a knapsack constraint have been shown (Lee et al., 2009). Another result pertains to unconstrained maximization of non-monotone submodular set functions, for which Buchbinder et al. (2012) propose the deterministic double greedy algorithm with a 1/3 approximation guarantee, and the randomized double greedy algorithm that achieves the tight 1/2 approximation guarantee.

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1. A DR-submodular function is a submodular function with the additional diminishing returns (DR) property, which will be formally defined in Section 3.
Although most commonly associated with set functions, in many practical scenarios, it is natural to consider generalizations of submodular set functions, including bisubmodular functions, $k$-submodular functions, tree-submodular functions, adaptive submodular functions, as well as submodular functions defined over integer lattices.

Golovin and Krause (2011) introduce the notion of adaptive submodularity to generalize submodular set functions to adaptive policies. Kolmogorov (2011) studies tree-submodular functions and presents a polynomial-time algorithm for minimizing them. For distributive lattices, it is well-known that the combinatorial polynomial-time algorithms for minimizing a submodular set function can be adopted to minimize a submodular function over a bounded integer lattice (Fujishige, 2005).

Approximation algorithms for maximizing bisubmodular functions and $k$-submodular functions have been proposed by Singh et al. (2012); Ward and Zivny (2014). Recently, maximizing a submodular function over integer lattices has attracted considerable attention. In particular, Soma et al. (2014) develop a $(1 - 1/e)$-approximation algorithm for maximizing a monotone $\text{DR}$-submodular integer function under a knapsack constraint. For non-monotone submodular functions over the bounded integer lattice, Gottschalk and Peis (2015) provide a $1/3$-approximation algorithm. Recently, Soma and Yoshida (2018) present a continuous non-smooth extension for maximizing monotone integer submodular functions.

2.2 Submodularity over Continuous Domains

Even though submodularity is most widely considered in the discrete realm, the notion can be generalized to arbitrary lattices (Fujishige, 2005). Wolsey (1982) considers maximizing a special class of continuous submodular functions subject to one knapsack constraint, in the context of solving location problems. That class of functions are additionally required to be monotone, piecewise linear and concave. Calinescu et al. (2007) and Vondrák (2008) discuss a subclass of continuous submodular functions, which is termed smooth submodular functions\footnote{A function $f : [0, 1]^n \rightarrow \mathbb{R}$ is smooth submodular if it has second partial derivatives everywhere and all entries of its Hessian matrix are non-positive.}, to describe the multilinear extension of a submodular set function. They propose the continuous greedy algorithm, which has a $(1 - 1/e)$ approximation guarantee for maximizing a smooth submodular function under a down-closed polytope constraint. Bach (2015) considers the problem of minimizing continuous submodular functions, and proves that efficient techniques from convex optimization may be used for minimization (Fujishige, 2005).

Ene and Nguyen (2016) provide an approach for reducing integer $\text{DR}$-submodular function maximization problems to submodular set function maximization problem. This approach suggests a way to approximately optimize continuous submodular functions over simple continuous constraints: Discretize the continuous function and constraint to be an integer instance, and then optimize it using the reduction. However, for monotone DR-submodular function maximization, this method can not handle the general continuous constraints discussed in this work, i.e., arbitrary down-closed convex sets. Moreover, for general submodular function maximization, this method cannot be applied, since the reduction needs the additional diminishing returns property. Therefore we focus on explicitly continuous methods in this work.
Algorithm 1: Classical Frank-Wolfe algorithm for constrained convex optimization
(Frank and Wolfe, 1956)

\begin{algorithm}
\begin{algorithmic}
\State \textbf{Input:} \( \min_{x \in \mathbb{R}^n, x \in \mathcal{D}} f(x) \); \( x^0 \in \mathcal{D} \)
\For {\( t = 0 \ldots T \)}
\State Compute \( s^{t} := \arg \min_{s \in \mathcal{D}} \langle s, \nabla f(x^t) \rangle \); \hfill // LMO
\State Choose step size \( \gamma \in (0, 1] \);
\State Update \( x^{t+1} := (1 - \gamma)x^t + \gamma s^t \);
\EndFor
\State \textbf{Output:} \( x^T \);
\end{algorithmic}
\end{algorithm}

Recently, Niazadeh et al. (2018)\(^3\) present optimal algorithms for non-monotone submodular maximization with a box constraint. Continuous submodular maximization is also well studied in the stochastic setting (Karimi et al., 2017; Hassani et al., 2017; Mokhtari et al., 2018b), online setting (Chen et al., 2018), bandit setting (Dürr et al., 2019) and decentralized setting (Mokhtari et al., 2018a).

2.3 Classical Frank-Wolfe Algorithm

Since the workhorse algorithms for continuous DR-submodular maximization are Frank-Wolfe style algorithms, we give a brief introduction of classical Frank-Wolfe algorithms in this section. The Frank-Wolfe algorithm (Frank and Wolfe, 1956) (also known as Conditional Gradient algorithm or the Projection-Free algorithm) is one of the classical algorithms for constrained convex optimization. It has received renewed interest in recent years due to its projection free nature and its ability to exploit structured constraints (Jaggi, 2013b).

The Frank-Wolfe algorithm solves the following constrained optimization problem:

\[
\min_{x \in \mathbb{R}^n, x \in \mathcal{D}} f(x),
\]

where \( f \) is differentiable with \( L \)-Lipschitz gradients and the constraint \( \mathcal{D} \) is convex and compact.

A sketch of the Frank-Wolfe algorithm is presented in Algorithm 1. It needs an initializer \( x^0 \in \mathcal{D} \). Then it runs for \( T \) iterations. In each iteration it does the following: in Step 2 it solves a linear minimization problem whose objective is defined by the current gradient \( \nabla f(x^t) \). This step is often called the linear minimization/maximization oracle (LMO). In Step 3 a step size \( \gamma \) is chosen. Then it updates the solution \( x \) to be a convex combination of the current solution and the LMO output \( s \).

There are several popular rules to choose the step size in Step 3. For a short summary: i) \( \gamma_t := \frac{2}{t+2} \), which is often called the “oblivious” rule since it does not depend on any information of the optimization problem; ii) \( \gamma_t = \min \{ 1, \frac{g_t}{\| s^t - x^t \|} \} \), where \( g_t := -\langle \nabla f(x^t), s^t - x^t \rangle \) is the so-called Frank-Wolfe gap, which is an upper bound of the suboptimality if \( f \) is convex; iii) Line search rule: \( \gamma_t := \arg \min_{\gamma \in [0, 1]} f(x^t + \gamma (s^t - x^t)) \).

Frank-Wolfe Algorithm for Non-Convex Optimization. Recently, Frank-Wolfe algorithms have been extended for smooth non-convex optimization problems with constraints. Lacoste-Julien (2016) analyzes the Frank-Wolfe method for general constrained problems.

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3. Appeared later than when the paper Bian et al. (2019) was released.
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non-convex optimization problems, where he uses the Frank-Wolfe gap as the non-stationarity measure. Reddi et al. (2016b) study Frank-Wolfe methods for non-convex stochastic and finite-sum optimization problems. They also used the Frank-Wolfe gap as the non-stationarity measure.

2.4 Structures for Non-Convex Optimization

Optimizing non-convex continuous functions has received considerable interest in the last decades. There are two widespread structures for non-convex optimization: quasi-convexity and geodesic convexity, both of them are based on relaxations of the classical convexity definition.

Quasi-Convexity. A function \( f : \mathcal{D} \to \mathbb{R} \) defined on a convex subset \( \mathcal{D} \) of a real vector space is quasi-convex if for all \( x, y \in \mathcal{D} \) and \( \lambda \in [0, 1] \) it holds,

\[
f(\lambda x + (1 - \lambda)y) \leq \max\{f(x), f(y)\}.
\]

(4)

Quasi-convex optimization problems appear in different areas, such as industrial organization (Wolfstetter, 1999) and computer vision (Ke and Kanade, 2007). Quasi-convex optimization problems can be solved by a series of convex feasibility problems (Boyd and Vandenberghe, 2004). Hazan et al. (2015a) study stochastic quasi-convex optimization, where they proved that a stochastic version of the normalized gradient descent can converge to a global minimum for quasi-convex functions that are locally Lipschitz.

Geodesic Convexity. Geodesic convex functions are a class of generally non-convex functions in Euclidean space. However, they still enjoy the nice property that local optimality implies global optimality. Sra and Hosseini (2016) provide an introduction to geodesic convex optimization with machine learning applications. Recently, Vishnoi (2018) study various aspects of geodesic convex optimization.

Definition 3 (Geodesically convex functions) Let \((\mathcal{M}, g)\) be a Riemannian manifold and \(K \subseteq \mathcal{M}\) be a totally convex set with respect to \(g\). A function \(f : K \to \mathbb{R}\) is a geodesically convex function with respect to \(g\) if \(\forall p, q \in K\), and for all geodesic \(\gamma_{pq} : [0, 1] \to K\) that joins \(p\) to \(q\), it holds,

\[
\forall t \in [0, 1], f(\gamma_{pq}(t)) \leq (1 - t)f(p) + tf(q).
\]

(5)

Various applications with non-convex objectives in Euclidean space can be resolved with geodesic convex optimization methods, such as Gaussian mixture models (Hosseini and Sra, 2015), metric learning (Zadeh et al., 2016) and matrix square root (Sra, 2015). By deriving explicit expressions for the smooth manifold structure, such as inner products, gradients, vector transport and Hessian, various optimization methods have been developed. Jejuris et al. (2012) present conjugate gradient, BFGS and trust-region methods. Qi et al. (2010) propose the Riemannian BFGS (RBFGS) algorithm for general retraction and vector transport. Ring and Wirth (2012) prove its local superlinear rate of convergence. Sra and Hosseini (2015) present a limited memory version of RBFGS.
Other Non-convex Structures. Tensor methods have been used in various non-convex problems, e.g., learning latent variable models (Anandkumar et al., 2014) and training neural networks (Janzamin et al., 2015). A fundamental problem in non-convex optimization is to reach a stationary point assuming the smoothness of the objective (Sra, 2012; Li and Lin, 2015; Reddi et al., 2016a; Allen-Zhu and Hazan, 2016). With extra assumptions, certain global convergence results can be obtained. For example, for functions with Lipschitz continuous Hessians, the regularized Newton scheme of Nesterov and Polyak (2006) achieves global convergence results for functions with an additional star-convexity property or with an additional gradient-dominance property (Polyak, 1963). Hazan et al. (2015b) introduce the family of $\sigma$-nice functions and propose a graduated optimization-based algorithm, that provably converges to a global optimum for this family of non-convex functions. However, it is typically difficult to verify whether these assumptions hold in real-world problems.

2.5 Our Contributions

To the best of our knowledge, this work is the first to systematically study continuous submodularity and its maximization algorithms. Our main contributions are:

Thorough characterizations of submodularity. By lifting the notion of submodularity to continuous domains, we identify a subclass of tractable non-convex optimization problems: continuous submodular optimization. We provide a thorough characterization of continuous submodularity, which results in 0th order, 1st order and 2nd order definitions.

Continuous submodularity preserving operations. We study general principles for maintaining continuous (DR-)submodularity. These enable: i) Convenient ways of recognizing new continuous submodular objectives; ii) Generic rules for designing new continuous or discrete submodular objectives, such as deep submodular functions.

Properties of constrained DR-submodular maximization. We discover intriguing properties of the general constrained DR-submodular maximization problem, such as the local-global relation (in Proposition 22), which relates (approximately) stationary points and the global optimum, thus allowing to incorporate progress in the area of non-convex optimization research.

Provable algorithms for DR-submodular maximization. We establish hardness results and propose provable algorithms for constrained DR-submodular maximization in two settings: i) Maximizing monotone functions with down-closed convex constraints; ii) Maximizing non-monotone functions with down-closed convex constraints.

Applications with (DR)-submodular objectives. We formulate representative applications with (DR)-submodular objectives from various areas, such as machine learning, data mining and combinatorial optimization.

Extensive experimental evaluations. We present representative applications with the studied continuous submodular objectives, and extensively evaluate the proposed algorithms on these applications.

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4. This journal paper is partially based on the previous conference papers Bian et al. (2017b), Bian et al. (2017a) also the thesis Bian (2019).
Continuous submodular function λg weak DR ∂ ∇ f or { matrix are non-positive can be applied to X the interval by default, but it is worth noting that the properties introduced in this section also be an interval, which is referred to as a continuous domain. In this section, we consider Specifically, X ∧ where \( x \in X \), it holds for all (\( x, y \)) ∈ X × X, is a compact subset of R, \( (\text{Topkis, 1978; Bach, 2015}) \). A function \( f : \mathcal{X} \to \mathbb{R} \) is submodular iff all off-diagonal entries of its Hessian are non-positive. Lastly, indefinite quadratic functions of the form \( f(x) = \frac{1}{2}x^T H x + h^T x + c \) with all off-diagonal entries of H non-positive are examples of submodular but non-convex/non-concave functions. Interestingly, characterizations of continuous submodular functions are in correspondence to those of convex functions, which are summarized in Table 1.

3. Characterizations of Continuous Submodular Functions

Continuous submodular functions are defined on subsets of \( \mathbb{R}^n \): \( \mathcal{X} = \prod_{i=1}^n \mathcal{X}_i \), where each \( \mathcal{X}_i \) is a compact subset of \( \mathbb{R} \) (Topkis, 1978; Bach, 2015). A function \( f : \mathcal{X} \to \mathbb{R} \) is submodular iff all \( \langle \nabla f(x), y - x \rangle \geq 0 \) (symmetric positive semidefinite).

\[
\forall x \in \mathcal{X}, \quad \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, \quad \forall i \neq j.
\]

The class of continuous submodular functions contains a subset of both convex and concave functions, and shares some useful properties with them (illustrated in Figure 1). Examples include submodular and concave functions of the form \( \phi_{ij}(x_i - x_j) \) for \( \phi_{ij} \) convex; submodular and concave functions of the form \( x \mapsto g(\sum_{i=1}^n \lambda_i x_i) \) for \( g \) concave and \( \lambda_i \) non-negative. Lastly, indefinite quadratic functions of the form \( f(x) = \frac{1}{2}x^T H x + h^T x + c \) with all off-diagonal entries of H non-positive are examples of submodular but non-convex/non-concave functions. Interestingly, characterizations of continuous submodular functions are in correspondence to those of convex functions, which are summarized in Table 1.

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Table 1: Comparison of definitions of continuous submodular and convex functions

| Definitions | Continuous submodular function \( f(\cdot) \) | Convex function \( g(\cdot), \forall \lambda \in [0, 1] \) |
|-------------|---------------------------------|---------------------------------|
| 0\(^\text{th} \) order | \( f(x) + f(y) \geq f(x \lor y) + f(x \land y) \) | \( \lambda g(x) + (1-\lambda) g(y) \geq g(\lambda x + (1-\lambda)y) \) |
| 1\(^\text{st} \) order | weak DR property (Definition 6), or \( \nabla f(\cdot) \) is a weak antitone mapping (Lemma 8) | \( g(y) \geq g(x) + \langle \nabla g(x), y - x \rangle \) |
| 2\(^\text{nd} \) order | \( \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, \forall i \neq j \) | \( \nabla^2 g(x) \succeq 0 \) (symmetric positive semidefinite) |

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5. Notice that an equivalent definition of (6) is that \( \forall x \in \mathcal{X}, \forall i \neq j \) and \( a_i, a_j \geq 0 \) s.t. \( x_i + a_i \in \mathcal{X}_i, x_j + a_j \in \mathcal{X}_j \), it holds \( f(x + a_ie_i) + f(x + a_je_j) \geq f(x) + f(x + a_ie_i + a_je_j) \). With \( a_i \) and \( a_j \) approaching zero, one gets (7).
3.1 The DR Property and DR-Submodular Functions

The Diminishing Returns (DR) property was introduced when studying set and integer functions. We generalize the DR property to general functions defined over $\mathcal{X}$. It will soon be clear that the DR property defines a subclass of submodular functions. All of the proofs can be found in Appendix A.

**Definition 4 (DR/IR property, DR-submodular/IR-supermodular functions)** A function $f(\cdot)$ defined over $\mathcal{X}$ satisfies the diminishing returns (DR) property if $\forall a \leq b \in \mathcal{X}$, $\forall i \in [n]$, $\forall k \in \mathbb{R}_+$ such that $(ke_i + a)$ and $(ke_i + b)$ are still in $\mathcal{X}$, it holds,

$$f(ke_i + a) - f(a) \geq f(ke_i + b) - f(b).$$

This function $f(\cdot)$ is called a DR-submodular function. If $-f(\cdot)$ is DR-submodular, we call $f(\cdot)$ an IR-supermodular function, where IR stands for “Increasing Returns”.

One immediate observation is that for a differentiable DR-submodular function $f(\cdot)$, we have that $\forall a \leq b \in \mathcal{X}$, $\nabla f(a) \geq \nabla f(b)$, i.e., the gradient $\nabla f(\cdot)$ is an antitone mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$. This observation can be formalized below:

**Lemma 5 (Antitone mapping)** If $f(\cdot)$ is continuously differentiable, then $f(\cdot)$ is DR-submodular iff $\nabla f(\cdot)$ is an antitone mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e., $\forall a \leq b \in \mathcal{X}$, $\nabla f(a) \geq \nabla f(b)$.

Recently, the DR property is explored by Eghbali and Fazel (2016) to achieve the worst-case competitive ratio for an online concave maximization problem. The DR property is also closely related to a sufficient condition on a concave function $g(\cdot)$ (Bilmes and Bai, 2017, Section 5.2), to ensure submodularity of the corresponding set function generated by giving $g(\cdot)$ boolean input vectors.

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6. Note that the DR property implies submodularity and thus the name “DR-submodular” contains redundant information about submodularity of a function, but we keep this terminology to be consistent with previous literature on integer submodular functions.
3.2 The Weak DR Property and Its Equivalence to Submodularity

It is well known that for set functions, the DR property is equivalent to submodularity, while for integer functions, submodularity does not in general imply the DR property (Soma et al., 2014; Soma and Yoshida, 2015a,b). However, it was unclear whether there exists a diminishing-return-style characterization that is equivalent to submodularity of integer functions. In this work we give a positive answer to this question by proposing the weak diminishing returns (weak DR) property for general functions defined over $\mathcal{X}$, and prove that weak DR gives a sufficient and necessary condition for a general function to be submodular.

**Definition 6 (Weak DR property)** A function $f(\cdot)$ defined over $\mathcal{X}$ has the weak diminishing returns property (weak DR) if $\forall a \leq b \in \mathcal{X}$, $\forall i \in \mathcal{V}$ such that $a_i = b_i$, $\forall k \in \mathbb{R}_+$ such that $(ke_i + a)$ and $(ke_i + b)$ are still in $\mathcal{X}$, it holds,

$$f(ke_i + a) - f(a) \geq f(ke_i + b) - f(b).$$

(9)

The following proposition shows that for all set functions, as well as integer and continuous functions, submodularity is equivalent to the weak DR property. All the proofs can be found in Appendix A.

**Proposition 7 (submodularity) ⇔ (weak DR)** A function $f(\cdot)$ defined over $\mathcal{X}$ is submodular iff it satisfies the weak DR property.

Given Proposition 7, one can treat weak DR as the first order definition of submodularity: Notice that for a continuously differentiable function $f(\cdot)$ with the weak DR property, we have that $\forall a \leq b \in \mathcal{X}$, $\forall i \in \mathcal{V}$ s.t. $a_i = b_i$, it holds $\nabla_i f(a) \geq \nabla_i f(b)$, i.e., $\nabla f(\cdot)$ is a weak antitone mapping. Formally,

**Lemma 8 (Weak antitone mapping)** If $f(\cdot)$ is continuously differentiable, then $f(\cdot)$ is submodular iff $\nabla f(\cdot)$ is a weak antitone mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$, i.e., $\forall a \leq b \in \mathcal{X}$, $\forall i \in \mathcal{V}$ s.t. $a_i = b_i$, $\nabla_i f(a) \geq \nabla_i f(b)$.

Now we show that the DR property is stronger than the weak DR property, and the class of DR-submodular functions is a proper subset of that of submodular functions, as indicated by Figure 1.

**Proposition 9 (submodular/weak DR) + (coordinate-wise concave) ⇔ (DR)** A function $f(\cdot)$ defined over $\mathcal{X}$ satisfies the DR property iff $f(\cdot)$ is submodular and coordinate-wise concave, where the coordinate-wise concave property is defined as: $\forall x \in \mathcal{X}$, $\forall i \in \mathcal{V}$, $\forall k, l \in \mathbb{R}_+$ s.t. $(ke_i + x), (le_i + x), ((k + l)e_i + x)$ are still in $\mathcal{X}$, it holds,

$$f(ke_i + x) - f(x) \geq f((k + l)e_i + x) - f(le_i + x),$$

(10)

or equivalently (if twice differentiable) $\frac{\partial^2 f(x)}{\partial x_i^2} \leq 0$, $\forall i \in \mathcal{V}$.

Proposition 9 shows that a twice differentiable function $f(\cdot)$ is DR-submodular iff $\forall x \in \mathcal{X}$, $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0$, $\forall i, j \in \mathcal{V}$, which does not necessarily imply the concavity of $f(\cdot)$. Given Proposition 9, we also have the characterizations of continuous DR-submodular functions, which are summarized in Table 2.
Table 2: Summarization of definitions of continuous DR-submodular functions

| Definitions                | Continuous DR-submodular function $f(\cdot)$, $\forall x, y \in X$ |
|----------------------------|---------------------------------------------------------------------|
| 0th order                  | $f(x) + f(y) \geq f(x \lor y) + f(x \land y)$, and $f(\cdot)$ is coordinate-wise concave (see (10)) |
| 1st order DR property      | (Definition 4), or $\nabla f(\cdot)$ is an antitone mapping (Lemma 5) |
| 2nd order                  | $\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0$, $\forall i, j$ (all entries of the Hessian matrix being non-positive) |

### 3.3 A Simple Visualization

Figure 2 shows the contour of a 2-D continuous submodular function $[x_1; x_2] \mapsto 0.7(x_1 - x_2)^2 + e^{-4(2x_1 - \frac{5}{3})^2} + 0.6e^{-4(2x_1 - \frac{5}{3})^2} + e^{-4(2x_2 - \frac{5}{3})^2} + e^{-4(2x_2 - \frac{5}{3})^2}$ and a 2-D DR-submodular function

$$x \mapsto \log \det (\text{diag}(x)(L - I) + I), x \in [0, 1]^2,$$

where $L = [2.25, 3; 3, 4.25]$. We can see that both of them are neither convex, nor concave. Notice that along each coordinate, continuous submodular functions may behave arbitrarily. In contrast, DR-submodular functions are always concave along any single coordinate.

![Figure 2: Left: A 2-D continuous submodular function: $[x_1; x_2] \mapsto 0.7(x_1 - x_2)^2 + e^{-4(2x_1 - \frac{5}{3})^2} + 0.6e^{-4(2x_1 - \frac{5}{3})^2} + e^{-4(2x_2 - \frac{5}{3})^2} + e^{-4(2x_2 - \frac{5}{3})^2}$. Right: A 2-D softmax extension, which is continuous DR-submodular. $x \mapsto \log \det (\text{diag}(x)(L - I) + I), x \in [0, 1]^2$, where $L = [2.25, 3; 3, 4.25]$.](image)

### 4. Operations that Preserve Continuous (DR-)Submodularity

Continuous submodularity is preserved under various operations, e.g., the sum of two continuous submodular functions is submodular, non-negative combinations of continuous submodular functions are still submodular, and a continuous submodular function multiplied by a positive scalar is still submodular. In this section, we will study some general submod-
ularity preserving operations from the perspective of function composition. Then we will look at some exemplary applications resulting from these rules.

**Observation 10 (Bach (2015))** Let $f$ be a DR-submodular function over $X = \prod_{i=1}^{n} X_i$. Let $\tilde{f}$ be the function defined by restricting $f$ on a product of subsets of $X_i$. Then $\tilde{f}$ is DR-submodular.

Observation 10 will be useful when we try to obtain discrete submodular functions by discretizing continuous submodular functions.

### 4.1 Function Composition

Suppose there are two functions $h : \mathbb{R}^m \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$. Consider the composed function $g(x) := f(h(x)) = (f \circ h)(x)$. We are interested in what properties are needed from $f$ and $h$ such that the composed function $g$ is DR-submodular.

Here $h$ is a multivariate vector-valued function. Equivalently, we can express $h$ as $n$ multivariate functions $h^k : \mathbb{R}^m \to \mathbb{R}$, $k = 1, ..., n$. We use $\nabla h$ to denote the $n \times m$ Jacobian matrix of $h$. Let $y = h(x)$, so $y_k = h^k(x)$.

For a vector-valued function, we define its (DR)-submodularity as,

**Definition 11 ((DR-)submodularity for vector-valued functions)** Let $h : \mathbb{R}^m \to \mathbb{R}^n$ be a multivariate vector-valued function, and $h^k : \mathbb{R}^m \to \mathbb{R}$ be the $k$th entry of the output, $k = 1, ..., n$. Then we say $h$ is (DR-)submodular iff $h^k$ is (DR-)submodular, $\forall k \in [n]$.

Assume for simplicity that both $f$ and $h$ are twice differentiable. Applying the chain rule twice, one can verify that

$$\nabla^2 g(x) = \nabla h(x)^\top \nabla^2 f(y) \nabla h(x) + \sum_{k=1}^{n} \frac{\partial f(y)}{\partial y_k} \nabla^2 h^k(x),$$

where the product above is the standard matrix multiplication. After some manipulation, one can see that the $(i,j)\text{th}$ entry of $\nabla^2 g(x)$ is,

$$\frac{\partial^2 g(x)}{\partial x_i \partial x_j} = \sum_{s,t=1}^{n} \frac{\partial^2 f(y)}{\partial y_s \partial y_t} \frac{\partial h^s(x)}{\partial x_i} \frac{\partial h^t(x)}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial f(y)}{\partial y_k} \frac{\partial^2 h^k(x)}{\partial x_i \partial x_j}. \quad (13)$$

Maintaining DR-submodularity (or IR-supermodularity) means maintaining the sign of $\frac{\partial^2 g(x)}{\partial x_i \partial x_j}$. From Equation (13), one can see that if we want $\frac{\partial^2 g(x)}{\partial x_i \partial x_j}$ to be non-positive, $h$ must in general be monotone. $h$ could be either nondecreasing or nonincreasing, in both cases we have $\frac{\partial h^s(x)}{\partial x_i} \frac{\partial h^t(x)}{\partial x_j} \geq 0$.

**Theorem 12 (DR-submodularity preserving conditions on function composition)** Suppose $h : \mathbb{R}^m \to \mathbb{R}^n$ is monotone (nondecreasing or nonincreasing), $f : \mathbb{R}^n \to \mathbb{R}$. The following statements about the composed function $g(x) := f(h(x)) = (f \circ h)(x)$ hold:

1. If $f$ is DR-submodular, nondecreasing, and $h$ is DR-submodular, then $g$ is DR-submodular;
2. If \( f \) is DR-submodular, nonincreasing, and \( h \) is IR-supermodular, then \( g \) is DR-submodular;

3. If \( f \) is IR-supermodular, nondecreasing, and \( h \) is IR-supermodular, then \( g \) is IR-supermodular;

4. If \( f \) is IR-supermodular, nonincreasing, and \( h \) is DR-submodular, then \( g \) is IR-supermodular.

If \( f \) and \( h \) are both twice differentiable, Theorem 12 can be directly proved by examining the \((i,j)\)th entry of \( \nabla^2 g(x) \) in Equation (13). Furthermore, the above conclusions can also be rigorously proved when the functions are non-differentiable. Below we give an exemplar proof of statement 1 in Theorem 12. The other proofs are omitted due to high similarity.

**Proof of statement 1 in Theorem 12 when the functions are non-differentiable**

Proof [Proof of Theorem 12 when the functions are non-differentiable]

To prove the DR-submodularity of \( g \), it suffices to show that:

\[ \forall x \leq y, \forall i \in [m], \forall k \geq 0, g(x + ke_i) - g(x) \geq g(y + ke_i) - g(y). \quad (14) \]

Due to DR-submodularity of \( h \),

\[ h(x + ke_i) - h(x) \geq h(y + ke_i) - h(y) \quad (15) \]

I) Let us consider the case when \( h \) is nondecreasing. It holds,

\[ h(x) \leq h(y) \quad (16) \]

Then,

\[ g(x + ke_i) - g(x) = f[h(x + ke_i)] - f[h(x)] \quad (17) \]
\[ \geq f[h(x) + h(y + ke_i) - h(y)] - f[h(x)] \quad (15) \text{ and } f \text{ is non-decreasing} \quad (19) \]
\[ \geq f[h(y + ke_i)] - f[h(y)] \quad (16) \text{ and } f \text{ is DR-submodular} \quad (20) \]
\[ = g(y + ke_i) - g(y). \quad (21) \]

Thus we prove Equation (14), i.e., the DR-submodularity of \( g \).

II) Let us consider the case when \( h \) is nonincreasing. It holds,

\[ h(x + ke_i) \geq h(y + ke_i) \quad (22) \]

Thus,

\[ g(y) - g(y + ke_i) = f[h(y)] - f[h(y + ke_i)] \quad (23) \]
\[ \geq f[h(y + ke_i) + h(x) - h(x + ke_i)] - f[h(y + ke_i)] \quad (15) \text{ & } f \text{ is nondecreasing} \quad (25) \]
\[ \geq f[h(x)] - f[h(x + ke_i)] \quad (22) \text{ and } f \text{ is DR-submodular} \quad (26) \]
Figure 3: Layers $l - 1$ and $l$ of the DSF.

$$= g(x) - g(x + ke_i).$$  \(27\)

By examining the $(i,j)^{th}$ entry of $\nabla^2 g(x)$ in Equation (13), we can also prove the following conclusion:

**Lemma 13** Suppose $h$ is monotone (nondecreasing or nonincreasing). In addition, assume $h$ is separable, that is, $m = n$ and $h^k(x) = h^k(x_k), k = 1, ..., n$. Then $f(h(x))$ maintains submodularity (supermodularity) of $f$.

The detailed proof can be found in Appendix B.1. It is worth noting that under the same setting as in Lemma 13, $f(h(x))$ might not maintain DR-submodularity (IR-supermodularity) of $f$.

### 4.2 Examples of Function Composition

**Design deep submodular (set or integer) functions (DSFs).** Using conclusions in this section, we obtain a general way of composing discrete DSFs: i) We make a continuous deep submodular function $f : \mathcal{X} \rightarrow \mathbb{R}$ utilizing the composition rules; ii) By restricting $f$ to the binary lattice $\{0, 1\}^n$, we obtain a deep submodular set function. Similarly, by restricting $f$ on the integer lattice $\{0, 1, 2, ..., k\}^n$, we get a deep submodular integer function. This step is ensured by the restriction rule (observation 10).

For a specific example, we can easily prove that the DSFs composed by nesting SCMMs with concave functions (Bilmes and Bai, 2017) are binary submodular.

Firstly, we can prove that the continuous function composed by nesting SCMMs with concave functions (Bilmes and Bai, 2017) are continuous submodular. Let the original input vector be $x \in \mathbb{R}^n$, which serves as the input vector of the 0th layer. As shown by Figure 3, let the output of the $i^{th}$ neuron in the $l^{th}$ layer be $o_i^l$. So

$$o_i^l = \sigma(W_{1,1}^l o_1^{l-1} + W_{1,2}^l o_2^{l-1} + ... + W_{d,l}^l o_d^{l-1}).$$  \(28\)
where we assume there are $d^l$ neurons in layer $l$, and use $W^l \in \mathbb{R}^{d^l \times d^{l-1}}$ to denote the weight matrix between layer $l-1$ and layer $l$. $\sigma$ is the activation function which is concave and nondecreasing in the positive orthant.

Now let us proceed by induction. When $l = 0$, $o^0_i(x) = x_i$. Let us assume for the $(l-1)^{th}$ layer that $o^{l-1}_i(x)$ is DR-submodular and nondecreasing wrt. $x$. According to Equation (28), $o^l_i(x) = \sigma(W^l_{1,1}o^{l-1}_1(x)+W^l_{1,2}o^{l-1}_2(x)+...+W^l_{1,d^{l-1}}o^{l-1}_{d^{l-1}}(x))$. Since $W$ is non-negative, the function $h(x) = W^l_{1,1}o^{l-1}_1(x)+W^l_{1,2}o^{l-1}_2(x)+...+W^l_{1,d^{l-1}}o^{l-1}_{d^{l-1}}(x)$ is DR-submodular and nondecreasing. $\sigma$ is also DR-submodular and nondecreasing. According to Theorem 12, $o^l_i(x)$ is also DR-submodular wrt $x$. Thus we finish the induction.

Given that continuous DSFs are continuous submodular, by means of the restriction operation, we obtain binary or integer deep submodular functions.

However, the above principles offer more general ways of designing DSFs, other than nesting SCMMs with concave activation functions (as proposed by Bilmes and Bai (2017)). As long as the resultant continuous map is DR-submodular, the discrete function obtained by restriction will be DR-submodular. With the principles proved in this section, one can immediately recognize that the following applications enjoy continuous submodular objectives (more details will be discussed in the corresponding sections).

**Influence Maximization with Marketing Strategies.** One can easily see that the objective in Equation (46) is the composition of a nondecreasing multilinear extension and a monotone activation function. So it is DR-submodular according to Theorem 12.

**Revenue Maximization with Continuous Assignments.** One can also verify that the revenue maximization problem in Equation (52) is the composition of a non-monotone DR-submodular multilinear extension and a separable monotone function, so it is still DR-submodular according to Lemma 13.

## 5. Properties of Constrained DR-Submodular Maximization

In this section, we first formulate the constrained DR-submodular maximization problem, and then establish several properties of it. In particular, we show properties related to concavity of the objective along certain directions, and establish the relation between locally stationary points and the global optimum (thus called “local-global relation”). These properties will be used to derive guarantees for the algorithms in the following sections. All omitted proofs are in Appendix B.

### 5.1 The Constrained (DR-)Submodular Maximization Problem

The general setup of constrained continuous submodular function maximization is,

$$\max_{x \in P \subseteq \mathcal{X}} f(x), \quad (P)$$

where $f : \mathcal{X} \to \mathbb{R}$ is continuous submodular or DR-submodular, $\mathcal{X} = [\underline{u}, \bar{u}]$ (Bian et al., 2017b). One can assume $f$ is non-negative over $\mathcal{X}$, since otherwise one just needs to find a lower bound for the minimum function value of $f$ over $\mathcal{X}$ (because box-constrained submodular minimization can be solved to arbitrary precision in polynomial time (Bach, 2015)).
Let the lower bound be \( f_{\min} \), then working on a new function \( f'(x) := f(x) - f_{\min} \) will not change the solution structure of the original problem (P).

The constraint set \( \mathcal{P} \subseteq \mathcal{X} \) is assumed to be a down-closed convex set, since without this property one cannot reach any constant factor approximation guarantee of the problem (P) (Vondrák, 2013). Formally, down-closedness of a convex set is defined as follows:

**Definition 14 (Down-closedness)** A down-closed convex set is a convex set \( \mathcal{P} \) associated with a lower bound \( u \in \mathcal{P} \), such that:

1. \( \forall y \in \mathcal{P}, u \leq y \);
2. \( \forall y \in \mathcal{P}, x \in \mathbb{R}^n, u \leq x \leq y \) implies that \( x \in \mathcal{P} \).

Without loss of generality, we assume \( \mathcal{P} \) lies in the positive orthant and has the lower bound \( 0 \). Otherwise we can always define a new set \( \mathcal{P}' = \{ x \mid x = y - u, y \in \mathcal{P} \} \) in the positive orthant, and a corresponding continuous submodular function \( f'(x) := f(x + u) \), and all properties of the function are still preserved.

The diameter of \( \mathcal{P} \) is \( D := \max_{x, y \in \mathcal{P}} \| x - y \| \), and it holds that \( D \leq \| \bar{u} \| \). We use \( x^* \) to denote the global maximum of (P). In some applications we know that \( f \) satisfies the monotonicity property:

**Definition 15 (Monotonicity)** A function \( f(\cdot) \) is monotone nondecreasing if,

\[
\forall a \leq b, f(a) \leq f(b). \tag{29}
\]

In the sequel, by “monotonicity”, we mean monotone nondecreasing by default.

We also assume that \( f \) has Lipschitz gradients,

**Definition 16 (Lipschitz gradients)** A differentiable function \( f(\cdot) \) has \( L \)-Lipschitz gradients if for all \( x, y \in \mathcal{X} \) it holds that,

\[
\| \nabla f(x) - \nabla f(y) \| \leq L \| x - y \|. \tag{30}
\]

According to Nesterov (2013, Lemma 1.2.3), if \( f(\cdot) \) has \( L \)-Lipschitz gradients, then

\[
| f(x + v) - f(x) - \langle \nabla f(x), v \rangle | \leq \frac{L}{2} \| v \|^2. \tag{31}
\]

For Frank-Wolfe style algorithms, the notion of curvature usually gives a tighter bound than just using the Lipschitz gradients.

**Definition 17 (Curvature of a continuously differentiable function)** The curvature of a differentiable function \( f(\cdot) \) w.r.t. a constraint set \( \mathcal{P} \) is,

\[
C_f(\mathcal{P}) := \sup_{x, v \in \mathcal{P}, \gamma \in (0,1]} \frac{2}{\gamma} \left[ f(y) - f(x) - (y - x)^\top \nabla f(x) \right]. \tag{32}
\]

If a differentiable function \( f(\cdot) \) has \( L \)-Lipschitz gradients, one can easily show that \( C_f(\mathcal{P}) \leq LD^2 \), given Nesterov (2013, Lemma 1.2.3).
5.2 Properties Along Non-negative/Non-positive Directions

Though in general a DR-submodular function $f$ is neither convex, nor concave, it is concave along some directions:

**Proposition 18 (Bian et al. (2017b))** A continuous DR-submodular function $f(\cdot)$ is concave along any non-negative direction $v \geq 0$, and any non-positive direction $v \leq 0$.

Notice that DR-submodularity is a stronger condition than concavity along directions $v \in \pm \mathbb{R}^n$: for instance, a concave function is concave along any direction, but it may not be a DR-submodular function.

**Strong DR-submodularity.** DR-submodular objectives may be strongly concave along directions $v \in \pm \mathbb{R}^n$, e.g., for DR-submodular quadratic functions. We will show that such additional structure may be exploited to obtain stronger guarantees for the local-global relation.

**Definition 19 (Strong DR-submodularity)** A function $f$ is $\mu$-strongly DR-submodular ($\mu \geq 0$) if for all $x \in X$ and $v \in \pm \mathbb{R}^n$, it holds that,

$$f(x + v) \leq f(x) + \langle \nabla f(x), v \rangle - \frac{\mu}{2} \|v\|^2.$$  

(33)

5.3 Local-Global Relation: Relation Between Approximately Stationary Points and Global Optimum

We know that for unconstrained optimization problems, $\|\nabla f(x)\|$ is often used as the non-stationarity measure of the point $x$. What should be a proper non-stationarity measure of a general constrained optimization problem? We advocate the non-stationarity measure proposed by Lacoste-Julien (2016) and Reddi et al. (2016b), which can be calculated for free within Frank-Wolfe-style algorithms (e.g., Algorithm 2).

**Non-stationarity measure.** For any constraint set $Q \subseteq X$, the non-stationarity of a point $x \in Q$ is,

$$g_Q(x) := \max_{v \in Q} \langle v - x, \nabla f(x) \rangle$$  

(non-stationarity).

(34)

It always holds that $g_Q(x) \geq 0$, and $x$ is defined to be a stationary point in $Q$ if $g_Q(x) = 0$, so (34) is a natural generalization of the non-stationarity measure for unconstrained optimization problems.

We start with the following proposition involving the non-stationarity measure.

**Proposition 20 (Bian et al. (2017a))** If $f$ is $\mu$-strongly DR-submodular, then for any two points $x, y$ in $X$, it holds:

$$(y - x)^\top \nabla f(x) \geq f(x \lor y) + f(x \land y) - 2f(x) + \frac{\mu}{2} \|x - y\|^2.$$  

(35)

Proposition 20 implies that if $x$ is stationary in $P$ (i.e., $g_P(x) = 0$), then $2f(x) \geq f(x \lor y) + f(x \land y) + \frac{\mu}{2} \|x - y\|^2$, which gives an implicit relation between $x$ and $y$.

As the following statements show, $g_Q(x)$ plays an important role in characterizing the local-global relation in both monotone and non-monotone setting.
5.3.1 Local-Global Relation in the Monotone Setting

**Corollary 21 (Local-Global Relation: Monotone Setting)** Let \( x \) be a point in \( \mathcal{P} \) with non-stationarity \( g_\mathcal{P}(x) \). If \( f \) is monotone nondecreasing and \( \mu \)-strongly DR-submodular, then it holds that,

\[
 f(x) \geq \frac{1}{2} [f(x^*) - g_\mathcal{P}(x)] + \frac{\mu}{4} \|x - x^*\|^2.
\]  

Corollary 21 indicates that any stationary point is a \( 1/2 \) approximation, which is also found by Hassani et al. (2017) (with \( \mu = 0 \)). Furthermore, if \( f \) is \( \mu \)-strongly DR-submodular, the quality of \( x \) will be improved considerably: if \( x \) is close to \( x^* \), it should be close to being optimal since \( f \) is smooth; if \( x \) is far away from \( x^* \), the term \( \frac{\mu}{4} \|x - x^*\|^2 \) will boost the approximation bound significantly. We provide here a very succinct proof based on Proposition 20.

**Proof** [Proof of Corollary 21] Let \( y = x^* \) in Proposition 20, one can easily obtain

\[
 f(x) \geq \frac{1}{2} [f(x^* \lor x) + f(x^* \land x) - g_\mathcal{P}(x)] + \frac{\mu}{4} \|x - x^*\|^2.
\]  

Because of monotonicity and \( x^* \lor x \geq x^* \), we know that \( f(x^* \lor x) \geq f(x^*) \). From non-negativity, \( f(x^* \land x) \geq 0 \). Then we reach the conclusion. \( \blacksquare \)

5.3.2 Local-Global Relation in the Non-Monotone Setting

**Proposition 22 (Local-Global Relation: Non-Monotone Setting)** Let \( x \) be a point in \( \mathcal{P} \) with non-stationarity \( g_\mathcal{P}(x) \), and \( \mathcal{Q} := \mathcal{P} \cap \{y | y \leq \bar{u} - x\} \). Let \( z \) be a point in \( \mathcal{Q} \) with non-stationarity \( g_\mathcal{Q}(z) \). It holds that,

\[
 \max\{f(x), f(z)\} \geq \frac{1}{4} \left[ f(x^*) - g_\mathcal{P}(x) - g_\mathcal{Q}(z) \right] + \frac{\mu}{8} \left( \|x - x^*\|^2 + \|z - z^*\|^2 \right),
\]  

where \( z^* := x \lor x^* - x \).

Figure 4 provides a two-dimensional visualization of Proposition 22. Notice that the smaller constraint \( \mathcal{Q} \) is generated after the first stationary point \( x \) is calculated.

**Proof sketch of Proposition 22:** The proof uses Proposition 20, the non-stationarity in (34) and a key observation in the following Claim. The detailed proof is deferred to Appendix B.4.

**Claim 23** Under the setting of Proposition 22, it holds that,

\[
 f(x \lor x^*) + f(x \land x^*) + f(z \lor z^*) + f(z \land z^*) \geq f(x^*).
\]  

Note that Chekuri et al. (2014); Gillenwater et al. (2012) propose a similar relation for the special cases of the multilinear/softmax extensions by mainly proving the same conclusion as in Claim 23. Their relation does not incorporate the properties of non-stationarity or
strong DR-submodularity. They both use the proof idea of constructing a specialized auxiliary set function tailored to specific DR-submodular functions (the considered extensions). We present a different proof method by directly utilizing the DR property on carefully constructed auxiliary points (e.g., \((x+z) \vee x^\star\) in the proof of Claim 23), which is arguably more succinct and straightforward than that of Chekuri et al. (2014); Gillenwater et al. (2012).

6. Exemplary Applications of Continuous Submodular Optimization

Continuous submodularity naturally finds applications in various domains, ranging from influence and revenue maximization, to DPP MAP inference and mean field inference of probabilistic graphical models. We discuss several concrete problem instances in this section.

6.1 Submodular Quadratic Programming (SQP)

Non-convex/non-concave QP problems of the form \(f(x) = \frac{1}{2}x^\top Hx + h^\top x + c\) under convex constraints naturally arise in many applications, including scheduling (Skutella, 2001), inventory theory, and free boundary problems. A special class of QP is the submodular QP (the minimization of which was studied in Kim and Kojima (2003)), in which all off-diagonal entries of \(H\) are required to be non-positive. Price optimization with continuous prices is a DR-submodular quadratic program (Ito and Fujimaki, 2016).

Another representative class of DR-submodular quadratic objectives arise when computing the stability number \(s(G)\) of a graph \(G = (V, E)\), \(s(G)^{-1} = \min_{x \in \Delta} x^\top (A + I)x\), where \(A\) is the adjacency matrix of the graph \(G\), \(\Delta\) is the standard simplex (Motzkin and
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This instance is a convex-constrained monotone DR-submodular maximization problem.

### 6.2 Continuous Extensions of Submodular Set Functions

The Lovász extension (Lovász, 1983) used for submodular set function minimization is both submodular and convex (see Appendix A of Bach (2015)).

The multilinear extension (Calinescu et al., 2007) is extensively used for submodular set function maximization. It is the expected value of \( F(S) \) under the fully factorized surrogate distribution \( q(S|x) := \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j) \):

\[
f_{\text{mt}}(x) := \mathbb{E}_{q(S|x)}[F(S)] = \sum_{S \subseteq V} F(S) \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j).
\]

\( f_{\text{mt}}(x) \) is DR-submodular and coordinate-wise linear (Bach, 2015). The partial derivative of \( f_{\text{mt}}(x) \) can be expressed as,

\[
\nabla_i f_{\text{mt}}(x) = \mathbb{E}_{q(S|x_i=1)}[F(S)] - \mathbb{E}_{q(S|x_i=0)}[F(S)]
\]

\[
= f_{\text{mt}}(x|_i(1)) - f_{\text{mt}}(x|_i(0))
\]

\[
\sum_{S \subseteq V, S \ni i} F(S) \prod_{j \in S \setminus \{i\}} x_j \prod_{j' \notin S} (1 - x_{j'})
\]

\[
- \sum_{S \subseteq V \setminus \{i\}} F(S) \prod_{j \in S \setminus \{i\}} x_j \prod_{j' \notin S, j' \neq i} (1 - x_{j'}).
\]

At first glance, evaluating the multilinear extension in Equation (40) costs an exponential number of operations. However, when used in practice, one often uses sampling techniques to estimate its value and gradient. Furthermore, it is worth noting that for several classes of practical submodular set functions, their multilinear extensions \( f_{\text{mt}}(\cdot) \) admit closed form expressions. We present details in the following.

#### 6.2.1 Gibbs Random Fields

Let us use \( v \in \{0, 1\}^V \) to equivalently denote the \( n \) binary random variables in a Gibbs random field. \( F(v) \) corresponds to the negative energy function in Gibbs random fields. If the energy function is parameterized with a finite order of interactions, i.e., \( F(v) = \sum_{s \in V} \theta_s v_s + \sum_{(s,t) \in V \times V} \theta_{s,t} v_s v_t + \ldots + \sum_{(s_1,s_2,...,s_d)} \theta_{s_1,s_2,...,s_d} v_{s_1} \cdots v_{s_d}, d < \infty \), then one can verify that its multilinear extension has the following closed form,

\[
f_{\text{mt}}(x) = \sum_{s \in V} \theta_s x_s + \sum_{(s,t) \in V \times V} \theta_{s,t} x_s x_t + \ldots
\]

\[
+ \sum_{(s_1,s_2,...,s_d)} \theta_{s_1,s_2,...,s_d} x_{s_1} \cdots x_{s_d}.
\]

The gradient of this expression can also be easily derived. Given this observation, one can quickly derive the multilinear extensions of a large category of energy functions of Gibbs random fields, e.g., graph cut, hypergraph cut, Ising models, etc. Specifically,
Undirected MaxCut. For undirected MaxCut, its objective is \( F(v) = \frac{1}{2} \sum_{(i,j) \in E} w_{ij}(v_i + v_j - 2v_i v_j), \ v \in \{0, 1\}^V \). One can verify that its multilinear extension is \( f_{\text{mt}}(x) = \frac{1}{2} \sum_{(i,j) \in E} w_{ij}(x_i + x_j - 2x_i x_j), \ x \in [0, 1]^V \).

Directed MaxCut. For directed MaxCut, its objective is \( F(v) = \sum_{(i,j) \in E} w_{ij} v_i(1 - v_j), \ v \in \{0, 1\}^V \). Its multilinear extension is \( f_{\text{mt}}(x) = \sum_{(i,j) \in E} w_{ij} x_i(1 - x_j), \ x \in [0, 1]^V \).

Ising models. For Ising models (Ising 1925) with non-positive pairwise interactions (antiferromagnetic interactions), \( F(v) = \sum_{s \in V} \theta_s v_s + \sum_{(s,t) \in E} \theta_{st} v_s v_t, \ v \in \{0, 1\}^V \), this objective can be easily verified to be submodular. Its multilinear extension is:

\[
f_{\text{mt}}(x) = \sum_{s \in V} \theta_s x_s + \sum_{(s,t) \in E} \theta_{st} x_s x_t, \ x \in [0, 1]^V.
\] (43)

6.2.2 Facility Location and FLID (Facility Location Diversity)

FLID is a diversity model (Tschiatschek et al. 2016) that has been designed as a computationally efficient alternative to DPPs (Kulesza et al. 2012). It is based on the facility location objective. Let \( \mathbf{W} \in \mathbb{R}^{|V| \times D} \) be the weights, each row corresponds to the latent representation of an item, with \( D \) as the dimensionality. Then

\[
F(S) := \sum_{i \in S} u_i + \sum_{d=1}^D (\max_{i \in S} W_{i,d} - \sum_{i \in S} W_{i,d})
\]  
\[
= \sum_{i \in S} u'_i + \sum_{d=1}^D \max_{i \in S} W_{i,d};
\]  (44)

which models both coverage and diversity, and \( u'_i = u_i - \sum_{d=1}^D W_{i,d} \). If \( u'_i = 0 \), one recovers the facility location objective. The computational complexity of evaluating its partition function is \( \mathcal{O}(|V|^{D+1}) \) (Tschiatschek et al., 2016), which is exponential in terms of \( D \).

We now show the technique such that \( f_{\text{mt}}(x) \) and \( \nabla_i f_{\text{mt}}(x) \) can be evaluated in \( \mathcal{O}(Dn^2) \) time. Firstly, for one \( d \in [D], \) let us sort \( W_{i,d} \) such that \( W_{i_d(1),d} \leq W_{i_d(2),d} \leq \cdots \leq W_{i_d(n),d} \). After this sorting, there are \( D \) permutations to record: \( i_d(l), l = 1, \ldots, n, \forall d \in [D] \). Now, one can verify that

\[
f_{\text{mt}}(x) = \sum_{i \in [n]} u'_i x_i + \sum_d \sum_{S \subseteq V} \max_{i \in S} W_{i,d} \prod_{m \in S} x_m \prod_{m' \in S} (1 - x_{m'})
\]  
\[
= \sum_{i \in [n]} u'_i x_i + \sum_d \sum_{l=1}^n W_{i_d(l),d} x_{i_d(l)} \prod_{m=l+1}^n (1 - x_{i_d(m)}).
\]

Sorting costs \( \mathcal{O}(Dn \log n) \), and from the above expression, one can see that the cost of evaluating \( f_{\text{mt}}(x) \) is \( \mathcal{O}(Dn^2) \). By the relation that \( \nabla_i f_{\text{mt}}(x) = f_{\text{mt}}(x_i|1) - f_{\text{mt}}(x_i|0) \), the cost is also \( \mathcal{O}(Dn^2) \).

6.2.3 Set Cover Functions

Suppose there are \( |C| = \{c_1, \ldots, c_{|C|}\} \) concepts, and \( n \) items in \( V \). Give a set \( S \subseteq V \), \( \Gamma(S) \) denotes the set of concepts covered by \( S \). Given a modular function \( m : 2^C \mapsto \mathbb{R}_+ \),
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the set cover function is defined as \( F(S) = m(\Gamma(S)) \). This function models coverage in maximization, and also the notion of complexity in minimization problems (Lin and Bilmes, 2011b). Let us define an inverse map \( \Gamma^{-1} \), such that for each concept \( c \), \( \Gamma^{-1}(c) \) denotes the set of items \( v \) such that \( \Gamma^{-1}(c) \ni v \). So the multilinear extension is,

\[
\text{f}_{\text{mt}}(x) = \sum_{i \in V} m(\Gamma(S)) \prod_{m \in S} x_m \prod_{m' \notin S} (1 - x_{m'})
\]

\[
= \sum_{c \in C} m_c \left[ 1 - \prod_{i \in \Gamma^{-1}(c)} (1 - x_i) \right].
\]

The last equality is achieved by considering the situations where a concept \( c \) is covered. One can observe that both \( \text{f}_{\text{mt}}(x) \) and \( \nabla_i \text{f}_{\text{mt}}(x) \) can be evaluated in \( O(n|C|) \) time.

6.2.4 General Case: Approximation by Sampling

In the most general case, one may only have access to the function values of \( F(S) \). In this scenario, one can use a polynomial number of sample steps to estimate \( \text{f}_{\text{mt}}(x) \) and its gradients.

Specifically: 1) Sample \( k \) times \( S \sim q(S|x) \) and evaluate function values for them, resulting in \( F(S_1), ..., F(S_k) \). 2) Return the average \( \frac{1}{k} \sum_{i=1}^{k} F(S_i) \). According to the Hoeffding bound (Hoeffding, 1963), one can easily derive that \( \frac{1}{k} \sum_{i=1}^{k} F(S_i) \) is arbitrarily close to \( \text{f}_{\text{mt}}(x) \) with increasingly more samples: With probability at least \( 1 - \exp(-k\epsilon^2/2) \), it holds that \( \left| \frac{1}{k} \sum_{i=1}^{k} F(S_i) - \text{f}_{\text{mt}}(x) \right| \leq \epsilon \max_{S} |F(S)| \), for all \( \epsilon > 0 \).

6.3 Influence Maximization with Marketing Strategies

Kempe et al. (2003) propose a general marketing strategy for influence maximization. They assume that there exists a number \( m \) of different marketing actions \( M_i \), each of which may affect some subset of nodes by increasing their probabilities of being activated. A natural requirement would be that the more we spend on any one action, the stronger should be its effect. Formally, one chooses \( x_i \) investments to marketing action \( M_i \), so a marketing strategy is an \( m \)-dimensional vector \( x \in \mathbb{R}^m \). Then the probability that node \( i \) will become activated is described by the activation function: \( a^i(x) : \mathbb{R}^m \to [0, 1] \). This function should satisfy the DR property by assuming that any marketing strategy is more effective when the targeted individual is less “marketing-saturated” at that point.

Now we search for the expected size of the final active set, which is the expected influence. We know that given a marketing strategy \( x \), a node \( i \) becomes active with probability \( a^i(x) \), so the expected influence is:

\[
f(x) = \sum_{S \subseteq V} F(S) \prod_{i \in S} a^i(x) \prod_{j \notin S} (1 - a^j(x)).
\]

\( F(S) \) is the influence with the seeding set as \( S \). It is submodular for many influence models, such as the Linear Threshold model and Independent Cascade model of Kempe et al. (2003). One can easily see that Equation (46) is DR-submodular by viewing it as a composition of the multilinear extension of \( F(S) \) and the activation function \( a(x) \).
6.3.1 Realizations of the Activation Function $a(x)$

For the activation function $a^i(x)$, we consider two realizations:

1. Independent marketing action.

Here we provide one action for each customer, and different actions are independent. So we have $m = |V|$ actions, and for customer $i$, there exists an activation function $a^i(x_i)$, which is a one dimensional nondecreasing DR-submodular function. A specific instance is that $a^i(x_i) = 1 - (1 - p_i)x_i$, $p_i \in [0,1]$ is the probability of customer $i$ becoming activated with one unit of investment.

2. Bipartite marketing actions.

Suppose there are $m$ marketing actions and $|V|$ customers. The influence relationship among actions and customers are modeled as a bipartite graph $(M,V,W)$, where $M$ and $V$ are collections of marketing actions and customers, respectively, and $W$ is the collection of weights. The edge weight, $p_{st} \in W$, represents the influence probability of action $s$ to customers $t$ by providing one unit of investment to action $s$. So with a marketing strategy as $x$, the probability of a customer $t$ being activated is $a^i(x) = 1 - \prod_{(s,t) \in W} (1 - p_{st})^{x_s}$. This is a nondecreasing DR-submodular function.

One may notice that the independent marketing action is a special case of bipartite marketing action.

6.4 Optimal Budget Allocation with Continuous Assignments

Optimal budget allocation is a special case of the influence maximization problem. It can be modeled as a bipartite graph $(S,T;W)$, where $S$ and $T$ are collections of advertising channels and customers, respectively. The edge weight, $p_{st} \in W$, represents the influence probability of channel $s$ to customer $t$. The goal is to distribute the budget (e.g., time for a TV advertisement, or space of an inline ad) among the source nodes, and to maximize the expected influence on the potential customers (Soma et al., 2014; Hatano et al., 2015).

The total influence of customer $t$ from all channels can be modeled by a proper monotone DR-submodular function $I_t(x)$, e.g., $I_t(x) = 1 - \prod_{(s,t) \in W} (1 - p_{st})^{x_s}$ where $x \in \mathbb{R}_+^S$ is the budget assignment among the advertising channels. For a set of $k$ advertisers, let $x^i \in \mathbb{R}_+^S$ be the budget assignment for advertiser $i$, and $x := [x^1, \cdots, x^k]$ denote the assignments for all the advertisers. The overall objective is,

$$g(x) = \sum_{i=1}^k \alpha_i f(x^i)$$

with

$$f(x^i) := \sum_{t \in T} I_t(x^i), \quad 0 \leq x^i \leq \bar{u}^i, \forall i = 1, \ldots, k,$$

which is monotone DR-submodular.

A concrete application arises when advertisers bid for search marketing, i.e., where vendors bid for the right to appear alongside the results of different search keywords. Here, $x^i_s$ is the volume of advertisement space allocated to the advertiser $i$ to show his ad alongside query keyword $s$. The search engine company needs to distribute the budget (advertising space) to all vendors to maximize their influence on the customers, while respecting various
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constraints. For example, each vendor has a specified budget limit for advertising, and the ad space associated with each search keyword cannot be too large. All such constraints can be formulated as a down-closed polytope $P$, hence the Submodular FW algorithm (Algorithm 4 in Section 7) can be used to find an approximate solution for the problem $\max_{x \in P} g(x)$.

Note that one can flexibly add regularizers in designing $I_t(x^i)$ as long as it remains monotone DR-submodular. For example, adding separable regularizers of the form $\sum_i \phi(x^i)$ does not change off-diagonal entries of the Hessian, and hence maintains submodularity. Alternatively, bounding the second-order derivative of $\phi(x^i)$ ensures DR-submodularity.

6.5 Softmax Extension for DPPs

Determinantal point processes (DPPs) are probabilistic models of repulsion, which have been used to model diversity in machine learning (Kulesza et al., 2012). The constrained MAP (maximum a posteriori) inference problem of a DPP is an NP-hard combinatorial problem in general. Currently, the methods with the best approximation guarantees are based on either maximizing the multilinear extension (Calinescu et al., 2007) or the softmax extension (Gillenwater et al., 2012), both of which are continuous DR-submodular functions.

The multilinear extension is given as an expectation over the original set function values, thus evaluating the objective of this extension requires expensive sampling in general. In contrast, the softmax extension has a closed form expression, which is more appealing from a computational perspective. Let $L$ be the positive semidefinite kernel matrix of a DPP, its softmax extension is:

$$f(x) = \log \det [\text{diag}(x)(L - I) + I], x \in [0, 1]^n,$$

where $I$ is the identity matrix, $\text{diag}(x)$ is the diagonal matrix with diagonal elements set as $x$. Its DR-submodularity can be established by directly applying Lemma 3 of Gillenwater et al. (2012), which immediately implies that all entries of $\nabla^2 f$ are non-positive, so $f(x)$ is continuous DR-submodular.

The problem of MAP inference in DPPs corresponds to the problem $\max_{x \in P} f(x)$, where $P$ is a down-closed convex constraint, e.g., a matroid polytope or a matching polytope.

6.6 Mean Field Inference for Probabilistic Log-Submodular Models

Probabilistic log-submodular models (Djolonga and Krause, 2014a) are a class of probabilistic models over subsets of a ground set $V = [n]$, where the log-densities are submodular set functions $F(S)$. The partition function $Z = \sum_{S \subseteq V} \exp(F(S))$ is typically hard to evaluate. One can use mean field inference to approximate $p(S)$ by some factorized distribution $q(S|x) = \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j), x \in [0, 1]^n$, by minimizing the distance measured w.r.t. the Kullback-Leibler divergence between $q$ and $p$, i.e.,

$$\sum_{S \subseteq V} q(S|x) \log \frac{q(S|x)}{p(S)}.$$  

It is,

$$\text{KL}(x) = - \sum_{S \subseteq V} F(S) \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j) + \sum_{i=1}^n [x_i \log x_i + (1 - x_i) \log(1 - x_i)] + \log Z.$$  

25
KL(\(x\)) is IR-supermodular w.r.t. \(x\). To see this: The first term is the negative of a multilinear extension, so it is IR-supermodular. The second term is separable, and coordinate-wise convex, so it will not affect the off-diagonal entries of \(\nabla^2 KL(x)\), it will only contribute to the diagonal entries. Now, one can see that all entries of \(\nabla^2 KL(x)\) are non-negative, so KL(\(x\)) is IR-supermodular w.r.t. \(x\). Minimizing the Kullback-Leibler divergence KL(\(x\)) amounts to maximizing a DR-submodular function.

### 6.7 Revenue Maximization with Continuous Assignments

Given a social connection graph with nodes denoting \(n\) users and edges encoding their connection strength, the viral marketing suggests to choose a small subset of buyers to give them some product for free, to trigger a cascade of further adoptions through “word-of-mouth” effects, in order to maximize the total revenue (Hartline et al., 2008). For some products (e.g., software), the seller usually gives away the product in the form of a trial, to be used for free for a limited time period. In this task, except for deciding whether to choose a user or not, the sellers also need to decide how much the free assignment should be, in which the assignments should be modeled as continuous variables. We call this problem **revenue maximization with continuous assignments**.

We use a directed graph \(G = (V, E; W)\) to represent the social connection graph. \(V\) contains all the \(n\) users, \(E\) is the edge set, and \(W\) is the adjacency matrix. We treat the undirected social connection graph as a special case of the directed graph, by taking one undirected edge as two directed edge with the same weight.

#### 6.7.1 A Variant of the Influence-and-Exploit (IE) Strategy

One model with “discrete” product assignments is considered by Soma and Yoshida (2017) and Dürr et al. (2019), motivated by the observation that giving a user more free products increases the likelihood that the user will advocate this product. It can be treated as a simplified variant of the Influence-and-Exploit (IE) strategy of Hartline et al. (2008). Specifically:

- **Influence** stage: Each user \(i\) that is given \(x_i\) units of products for free becomes an advocate of the product with probability \(1 - q^{x_i}\) (independently from other users), where \(q ∈ (0, 1)\) is a parameter. This is consistent with the intuition that with more free assignment, the user is more likely to advocate the product.

- **Exploit** stage: suppose that a set \(S\) of users advocate the product while the complement set \(V \setminus S\) of users do not. Now the revenue comes from the users in \(V \setminus S\), since they will be influenced by the advocates with probability proportional to the edge weights. We use a simplified concave graph model (Hartline et al., 2008) for the value function, i.e., \(v_j(S) = \sum_{i ∈ S} W_{ij}, j ∈ V \setminus S\). Assume for simplicity that the users of \(V \setminus S\) are visited independently with each other. Then the revenue is:

\[
R(S) = \sum_{j ∈ V \setminus S} v_j(S) = \sum_{j ∈ V \setminus S} \sum_{i ∈ S} W_{ij}. \tag{51}
\]

Notice that \(S\) is a random set drawn according to the distribution specified by the continuous assignment \(x\).
With this Influence-and-Exploit (IE) strategy, the expected revenue is a function $f : \mathbb{R}_+^V \to \mathbb{R}_+$, as shown below:

$$f(x) = \mathbb{E} [R(S)] = \mathbb{E} \left[ \sum_{i \in S} \sum_{j \in V \setminus S} W_{ij} \right]$$

$$= \sum_{i \in V} \sum_{j \in V \setminus \{i\}} W_{ij} (1 - q^{x_i}) q^{x_j}. \quad (52)$$

According to Lemma 13, one can see that the above objective is submodular, since it is composed by the multilinear extension of $R(S)$ (which is continuous submodular) and the separable function $h : \mathbb{R}^V \to \mathbb{R}^V$, where $h^i(x_i) = 1 - q^{x_i}$.

### 6.7.2 An Alternative Model

In addition to the Influence-and-Exploit (IE) model, we also consider an alternative model. Assume there are $q$ products and $n$ buyers/users, let $x^i \in \mathbb{R}_+^n$ be the assignments of product $i$ to the $n$ users, let $x := [x^1, \ldots, x^q]$ denote the assignments for the $q$ products. The revenue can be modeled as $g(x) = \sum_{i=1}^q f(x^i)$ with

$$f(x^i) := \alpha_i \sum_{s:x^i_s=0} R_s(x^i) + \beta_i \sum_{t:x^i_t \neq 0} \phi(x^i_t) + \gamma_i \sum_{t:x^i_t \neq 0} \bar{R}_t(x^i), \quad 0 \leq x^i \leq \bar{u}^i, \quad (53)$$

where $x^i_t$ is the assignment of product $i$ to user $t$ for free, e.g., the amount of free trial time or the amount of the product itself. $R_s(x^i)$ models revenue gain from user $s$ who did not receive the free assignment. It can be some non-negative, non-decreasing submodular function. $\phi(x^i_t)$ models revenue gain from user $t$ who received the free assignment, since the more one user tries the product, the more likely he/she will buy it after the trial period. $\bar{R}_t(x^i)$ models the revenue loss from user $t$ (in the free trial time period the seller cannot get profits), which can be some non-positive, non-increasing submodular function. For products with continuous assignments, usually the cost of the product does not increase with its amount, e.g., the product as a software, so we only have the box constraint on each assignment. The objective in Equation (53) is generally non-concave/non-convex, and non-monotone submodular (see Appendix D for more details).

**Lemma 24** If $R_s(x^i)$ is non-decreasing submodular and $\bar{R}_t(x^i)$ is non-increasing submodular, then $f(x^i)$ in Equation (53) is submodular.

### 6.8 Applications Generalized from the Discrete Setting

Many discrete submodular problems can be naturally generalized to the continuous setting with continuous submodular objectives. The maximum coverage problem and the problem of text summarization with submodular objectives are among the examples (Lin and Bilmes, 2010). We put details in the sequel.
6.8.1 Text Summarization

Submodularity-based objective functions for text summarization perform well in practice (Lin and Bilmes, 2010). Let $C$ be the set of all concepts, and $V$ be the set of all sentences. As a typical example, the concept-based summarization aims to find a subset $S$ of the sentences to maximize the total credit of concepts covered by $S$. Soma et al. (2014) considered extending the submodular text summarization model to one that incorporates “confidence” of a sentence, which has a discrete value, and modeling the objective to be an integer submodular function. It is perhaps even more natural to consider continuous confidence values $x_i \in [0, 1]$. Let us use $p_i(x_i)$ to denote the set of covered concepts when selecting sentence $i$ with confidence level $x_i$, it can be a monotone covering function $p_i : \mathbb{R}_+ \rightarrow 2^C$, $\forall i \in V$. Then the objective function of the extended model is $f(x) = \sum_{j \in \bigcup_{i \in V} p_i(x_i)} c_j$, where $c_j \in \mathbb{R}_+$ is the credit of concept $j$. It can be verified that this objective is a monotone continuous submodular function.

6.8.2 Maximum Coverage

In the maximum coverage problem, there are $n$ subsets $C_1, ..., C_n$ from the ground set $V$. One subset $C_i$ can be chosen with “confidence” level $x_i \in [0, 1]$, the set of covered elements when choosing subset $C_i$ with confidence $x_i$ can be modeled with the following monotone normalized covering function: $p_i : \mathbb{R}_+ \rightarrow 2^V$, $i = 1, ..., n$. The target is to choose subsets from $C_1, ..., C_n$ with confidence level to maximize the number of covered elements $|\bigcup_{i=1}^n p_i(x_i)|$, at the same time respecting the budget constraint $\sum_i c_i x_i \leq b$ (where $c_i$ is the cost of choosing subset $C_i$). This problem generalizes the classical maximum coverage problem. It is easy to see that the objective function is monotone submodular, and the constraint is a down-closed polytope.

6.8.3 Sensor Energy Management

For cost-sensitive outbreak detection in sensor networks (Leskovec et al., 2007), one needs to place sensors in a subset of locations selected from all the possible locations $V$, to quickly detect a set of contamination events $E$, while respecting the cost constraints of the sensors. For each location $v \in V$ and each event $e \in E$, a value $t(v, e)$ is provided as the time it takes for the placed sensor in $v$ to detect event $e$. Soma and Yoshida (2015a) considered the sensors with discrete energy levels. It is natural to model the energy levels of sensors to be a continuous variable $x \in \mathbb{R}_+ V$. For a sensor with energy level $x_v$, the success probability it detects the event is $1 - (1 - p)^{x_v}$, which models that by spending one unit of energy one has an extra chance of detecting the event with probability $p$. In this model, beyond deciding whether to place a sensor or not, one also needs to decide the optimal energy levels. Let $t_{\infty} = \max_{e \in E, v \in V} t(v, e)$, let $v_e$ be the first sensor that detects event $e$ ($v_e$ is a random variable). One can define the objective as the expected detection time that could be saved,

$$f(x) := \mathbb{E}_{e \in E} \mathbb{E}_{v_e} [t_{\infty} - t(v_e, e)],$$

(54)

which is a monotone DR-submodular function. Maximizing $f(x)$ w.r.t. the cost constraints pursues the goal of finding the optimal energy levels of the sensors, to maximize the expected detection time that could be saved.
6.8.4 Multi-Resolution Summarization

Suppose we have a collection of items, e.g., images \( V = \{v_1, ..., v_n\} \). We follow the strategy to extract a representative summary, where representativeness is defined w.r.t. a submodular set function \( F : 2^V \rightarrow \mathbb{R} \). However, instead of returning a single set, our goal is to obtain summaries at multiple levels of detail or resolution. One way to achieve this goal is to assign each item \( v_i \) a nonnegative score \( x_i \). Given a user-tunable threshold \( \tau \), the resulting summary \( S_\tau = \{v_i | x_i \geq \tau\} \) is the set of items with scores exceeding \( \tau \). Thus, instead of solving the discrete problem of selecting a fixed set \( S \), we pursue the goal to optimize over the scores, e.g., to use the following continuous submodular function,

\[
f(x) = \sum_{i \in V} \sum_{j \in V} \phi(x_j) s_{i,j} - \sum_{i \in V} \sum_{j \in V} x_i x_j s_{i,j},
\]

where \( s_{i,j} \geq 0 \) is the similarity between items \( i, j \), and \( \phi(\cdot) \) is a non-decreasing concave function.

6.8.5 Facility Location with Scales

The classical discrete facility location problem can be generalized to the continuous case where the scale of a facility is determined by a continuous value in interval \([0, \bar{u}]\). For a set of facilities \( V \), let \( x \in \mathbb{R}^V_+ \) be the scale of all facilities. The goal is to decide how large each facility should be in order to optimally serve a set \( T \) of customers. For a facility \( s \) of scale \( x_s \), let \( p_{st}(x_s) \) be the value of service it can provide to customer \( t \in T \), where \( p_{st}(0) = 0 \). Assuming each customer chooses the facility with highest value, the total service provided to all customers is \( f(x) = \sum_{t \in T} \max_{s \in V} p_{st}(x_s) \). It can be shown that \( f \) is monotone submodular.

7. Algorithms for Monotone DR-Submodular Maximization

In this section, we present two classes of algorithms for maximizing a monotone continuous DR-submodular function subject to a down-closed convex constraint. The detailed proofs can be found in Appendix C. Even despite the monotonicity assumption, solving the problem to optimality is still a very challenging task. In fact, we prove the following hardness result:

**Proposition 25 (Hardness and Inapproximability)** The problem of maximizing a monotone nondecreasing continuous DR-submodular function subject to a general down-closed polytope constraint is NP-hard. For any \( \epsilon > 0 \), it cannot be approximated in polynomial time within a ratio of \( (1 - 1/e + \epsilon) \) (up to low-order terms), unless \( RP = NP \).

Proposition 25 can be proved by the reduction from the problem of maximizing a monotone submodular set function subject to cardinality constraints. The proof relies on the techniques of multilinear extension (Calinescu et al., 2007; Calinescu et al., 2011) and p-page rounding (Ageev and Sviridenko, 2004), and also the hardness results of Feige (1998); Calinescu et al. (2007).

**Remark 26** Due to the NP-hardness of converging to the global optimum for Problem (P), in the following by “convergence” we mean converging to a solution point which has a constant factor approximation guarantee with respect to the global optimum.
7.1 Algorithms based on the Local-Global Relation: Non-convex FW and PGA

The first class of algorithms directly utilize the local-global relation of Corollary 21. We know that any stationary point is a 1/2 approximate solution. Thus any solver that obtains a stationary point yields a solution with a 1/2 approximation guarantee. We give two concrete examples below.

7.1.1 The Non-convex FW Algorithm

For sake of completeness, we summarize the Non-convex FW algorithm in Algorithm 2.

**Algorithm 2: Non-convex FW \((f, \mathcal{P}, K, \epsilon, x^0)\) (Lacoste-Julien, 2016) for maximizing a smooth objective**

**Input:** \(\max_{x \in \mathcal{P}} f(x)\), \(f\): a smooth function, \(\mathcal{P}\): convex set, \(K\): number of iterations, \(\epsilon\): stopping tolerance

1. for \(k = 0, \ldots, K\) do
2. find \(v^k\) s.t. \(\langle v^k, \nabla f(x^k) \rangle \geq \max_{v \in \mathcal{P}} \langle v, \nabla f(x^k) \rangle\); // LMO
3. \(d^k \leftarrow v^k - x^k, g_k := \langle d_k, \nabla f(x^k) \rangle\); // \(g_k\): non-stationarity measure
4. if \(g_k \leq \epsilon\) then return \(x^k\);
5. Option I: \(\gamma_k \in \arg \min_{\gamma \in [0,1]} f(x^k + \gamma d^k)\),
6. Option II: \(\gamma_k \leftarrow \min\{\frac{g_k}{C_f(\mathcal{P})}, 1\}\) for \(C \geq C_f(\mathcal{P})\);
7. \(x^{k+1} \leftarrow x^k + \gamma_k d^k\);

**Output:** \(x^{k'}\) and \(g^{k'} = \min_{0 \leq k \leq K} g_k\); // modified output solution compared to that of Lacoste-Julien (2016)

Algorithm 2 is modified from Lacoste-Julien (2016). The only difference lies in the output: we return the solution \(x^{k'}\) with the minimum non-stationarity, which is needed to invoke the local-global relation. In contrast, Lacoste-Julien (2016) outputs the solution from the last iteration. Since \(C_f(\mathcal{P})\) is generally hard to evaluate, we use the classical oblivious step size rule \((\frac{2}{k + 2})\) and the Lipschitz step size rule \((\gamma_k = \min\{1; \frac{g_k}{L\|d^k\|}\})\), where \(g_k\) is the so-called Frank-Wolfe gap) in the experiments (Section 9).

Hassani et al. (2017) show that the Projected Gradient Ascent algorithm (PGA) with constant step size \((1/L)\) can converge to a stationary point, so it has a 1/2 approximation guarantee. We can also show that the Non-convex FW of Lacoste-Julien (2016) has a 1/2 approximation guarantee according to the local-global relation:

**Corollary 27** The non-convex Frank-Wolfe algorithm (abbreviated as Non-convex FW) of Lacoste-Julien (2016) has a 1/2 approximation guarantee, and \(1/\sqrt{k}\) rate of convergence for solving Problem \((\mathcal{P})\) when the objective is monotone nondecreasing.

7.1.2 The PGA Algorithm

Algorithm 3 is reproduced from Hassani et al. (2017) for completeness. It takes a smooth DR-submodular function \(f\), and a convex constraint \(\mathcal{P}\). Then it runs for \(K\) iterations. In each iteration, we firstly choose a step size \(\gamma_k\), then we update the current solution using the current gradient to get a point \(y^{k+1}\). Lastly, we projects \(y^{k+1}\) onto the convex set \(\mathcal{P}\), which
Algorithm 3: PGA for maximizing a monotone DR-submodular objective (Hassani et al. 2017)

Input: \( \max_{x \in \mathcal{P}} f(x) \), \( f \): a smooth DR-Submodular function, \( \mathcal{P} \): convex set, \( K \): number of iterations, \( x^0 \in \mathcal{P} \)

1 for \( k = 0, \ldots, K - 1 \) do
2     Set step size \( \gamma_k \); // i): “Lipschitz” rule \( \frac{1}{L} \); ii): adaptive rule: \( C/\sqrt{k} \)
3     \( y^{k+1} \leftarrow x^k + \gamma_k \nabla f(x^k) \);
4     \( x^{k+1} \leftarrow \arg \min_{x \in \mathcal{P}} \|x - y^{k+1}\| \); // Projection
Output: \( x^{k'} \) with \( k' = \arg \max_{0 \leq k \leq K} f(x^k) \); // modified output compared to that of Hassani et al. (2017)

amounts to solving a constrained quadratic program. After \( K \) iterations, we output the solution with the maximal function value, which is slightly different from that of Hassani et al. (2017).

The resulting algorithm has a 1/2 approximation guarantee and sublinear rate of convergence:

Theorem 28 (Hassani et al. (2017)) For Algorithm 3, if one chooses \( \gamma_k = 1/L \), then after \( K \) iterations,

\[
f(x^K) \geq f(x^*) - \frac{D^2 L}{2K}.
\]

(56)

It is worth noting that, in general the smoothness parameter \( L \) is difficult to estimate, so the “Lipschitz” step size rule \( \gamma_k = 1/L \) poses a challenge for implementation. In experiments, Hassani et al. (2017) also suggest the adaptive step size rule \( \gamma_k = C/\sqrt{k} \), where \( C \) is a constant.

7.2 Submodular FW: Follow Concave Directions

For DR-submodular maximization, one key property is that while being non-convex/non-concave in general, they are concave along any non-negative directions (c.f., Proposition 18). Thus, if we design an algorithm such that it follows a non-negative direction in each update step, we ensure that it achieves progress in a concave direction. As a consequence, its function value is guaranteed to grow by a certain increment. Based on this intuition, we present the Submodular FW algorithm, which is a generalization of the continuous greedy algorithm of Vondrák (2008), and the classical Frank-Wolfe algorithm (Frank and Wolfe, 1956; Jaggi, 2013a).

Algorithm 4 summarizes the details. Since it is a variant of the convex Frank-Wolfe algorithm for DR-submodular maximization, we call it Submodular FW. In iteration \( k \), it uses the linearization of \( f(\cdot) \) as a surrogate, and moves in the direction of the maximizer of this surrogate function, i.e., \( v^k = \arg \max_{v \in \mathcal{P}} \langle v, \nabla f(x^k) \rangle \). Intuitively, it searches for the direction in which one can maximize the improvement in the function value and still remain feasible. Finding such a direction requires maximizing a linear objective at each iteration. Meanwhile, it eliminates the need for projecting back to the feasible set in each
Algorithm 4: Submodular FW for monotone DR-submodular maximization (Bian et al. 2017b)

| Input: max_{x \in P} f(x), P is a down-closed convex set in the positive orthant with lower bound 0; prespecified step size γ \in (0, 1]; Error tolerances α and δ. # of iterations K. |
| x^0 \leftarrow 0, t \leftarrow 0, k \leftarrow 0; // k : iteration index, t: cumulative step size |
| 1 \textbf{while } t < 1 \textbf{ do} |
| 2 \quad \text{find step size } γ_k \in (0, 1], \text{ e.g., } γ_k \leftarrow γ; \text{ set } γ \leftarrow \min\{γ_k, 1 - t\}; |
| 3 \quad \text{find } v^k \text{ s.t. } \langle v^k, \nabla f(x^k) \rangle \geq α \max_{v \in P} \langle v, \nabla f(x^k) \rangle - \frac{1}{2} δγ_k LD^2 ; // α \in (0, 1] \text{ is the multiplicative error level, } δ \in [0, \bar{δ}] \text{ is the additive error level} |
| 4 \quad x^{k+1} \leftarrow x^k + γ_k v^k, t \leftarrow t + γ_k, k \leftarrow k + 1; |
| Output: x^K; |

iteration, which is an essential step for methods such as projected gradient ascent (PGA). The Submodular FW algorithm updates the solution in each iteration by using step size γ_k, which can simply be set to a prespecified constant γ.

Note that Submodular FW can tolerate both multiplicative error α and additive error δ when solving the LMO subproblem (Step 4 of Algorithm 4). Setting α = 1 and δ = 0 would recover the error-free case.

Remark 29 The main difference of Submodular FW in Algorithm 4 and the classical Frank-Wolfe algorithm in Algorithm 1 lies in the update direction being used: For Algorithm 4, the update direction (in Step 5) is v^k, while for classical Frank-Wolfe it is v^k - x^k, i.e., x^{k+1} \leftarrow x^k + γ_k(v^k - x^k).

To prove the approximation guarantee, we first derive the following lemma.

**Lemma 30** The output solution x^K lies in P. Assuming x^* to be the optimal solution, one has,

\[
\langle v^k, \nabla f(x^k) \rangle \geq α[f(x^*) - f(x^k)] - \frac{1}{2} δγ_k LD^2, \forall k = 0, ..., K - 1. \tag{57}
\]

**Theorem 31 (Approximation guarantee)** For error levels α ∈ (0, 1], δ ∈ [0, \bar{δ}], with K iterations, Algorithm 4 outputs x^K ∈ P such that,

\[
f(x^K) \geq (1 - e^{-α}) f(x^*) - \frac{LD^2(1 + \delta)}{2} \sum_{k=0}^{K-1} γ_k^2 + e^{-α} f(0). \tag{58}
\]

Theorem 31 gives the approximation guarantee for any step size γ_k. By observing that \( \sum_{k=0}^{K-1} γ_k = 1 \) and \( \sum_{k=0}^{K-1} γ_k^2 \geq K^{-1} \) (see the proof in Appendix C.5), with constant step size, we obtain the following “tightest” approximation bound,

**Corollary 32** For a fixed number of iterations K, and constant step size γ_k = γ = K^{-1}, Algorithm 4 provides the following approximation guarantee:

\[
f(x^K) \geq (1 - e^{-α}) f(x^*) - \frac{LD^2(1 + \delta)}{2K} + e^{-α} f(0). \tag{59}
\]
Corollary 32 implies that with a constant step size \( \gamma \), 1) when \( \gamma \to 0 \) \( (K \to \infty) \), Algorithm 4 will output the solution with the worst-case guarantee \( (1 - 1/e)f(x^*) \) in the error-free case if \( f(0) = 0 \); and 2) The Submodular FW has a sub-linear convergence rate for monotone DR-submodular maximization over any down-closed convex constraint.

**Remarks on computational cost.** It can be seen that when using a constant step size, Algorithm 4 needs \( O(\frac{1}{\epsilon}) \) iterations to get \( \epsilon \)-close to the best-possible function value \( (1 - e^{-1})f(x^*) \) in the error-free case. When \( P \) is a polytope in the positive orthant, one iteration of Algorithm 4 costs approximately the same as solving a positive LP, for which a nearly-linear time solver exists (Allen-Zhu and Orecchia, 2015).

8. Algorithms for Non-Monotone DR-Submodular Maximization

In this section we present algorithms for the problem of non-monotone DR-submodular maximization, all omitted proofs can be found in Appendix E. Non-monotone DR-submodular maximization is strictly harder than the monotone setting. For the simple situation with only one hyperrectangle constraint \( (P = [0,1]^n) \), we have the following hardness result:

**Proposition 33 (Hardness and Inapproximability)** The problem of maximizing a generally non-monotone continuous DR-submodular function subject to a hyperrectangle constraint is NP-hard. Furthermore, there is no \( (\frac{1}{2} + \epsilon) \)-approximation for any \( \epsilon > 0 \), unless \( RP = NP \).

The above results can be proved through the reduction from the problem of maximizing an unconstrained non-monotone submodular set function. The proof depends on the techniques of Calinescu et al. (2007); Buchbinder et al. (2012) and the hardness results of Feige et al. (2011); Dobzinski and Vondrák (2012).

We propose two algorithms: The first is based on the local-global relation, and the second is a Frank-Wolfe variant adapted for the non-monotone setting. All the omitted proofs are deferred to Appendix E.

8.1 Two-Phase Algorithm: Applying the Local-Global Relation

**Algorithm 5:** The Two-Phase Algorithm (Bian et al., 2017a)

| Input: max\(_{x \in P} f(x)\), stopping tolerances \( \epsilon_1, \epsilon_2 \), #iterations \( K_1, K_2 \) |
|---|
| 1 \( x \leftarrow \text{Non-convex Frank-Wolfe}(f, P, K_1, \epsilon_1, x^0) \); \( x^0 \in P \) |
| 2 \( Q \leftarrow P \cap \{y \in \mathbb{R}^n_+ | y \leq \bar{u} - x\} \); |
| 3 \( z \leftarrow \text{Non-convex Frank-Wolfe}(f, Q, K_2, \epsilon_2, z^0) \); \( z^0 \in Q \) |
| Output: arg max\{\( f(x) \), \( f(z) \) \} |

By directly applying the local-global relation in Section 5.3, we present the Two-Phase algorithm in Algorithm 5. It generalizes the “two-phase” method of Chekuri et al. (2014); Gillenwater et al. (2012). It invokes a non-convex solver (we use the Non-convex FW by Lacoste-Julien (2016); pseudocode is included in Algorithm 2 of Section 7.1.1) to find approximately stationary points in \( P \) and \( Q \), respectively, then returns the solution with the larger function value.
Though we use Non-convex FW as a subroutine here, it is noteworthy that any algorithm that is guaranteed to find an approximately stationary point can be plugged into Algorithm 5 as a subroutine. We give an improved approximation bound by considering more properties of DR-submodular functions. Building on the results of Lacoste-Julien (2016), we obtain the following.

**Theorem 34** The output of Algorithm 5 satisfies,

\[
\begin{align*}
\max\{f(x), f(z)\} & \geq \frac{\mu}{8} \left( \|x - x^*\|^2 + \|z - z^*\|^2 \right) \\
& \quad + \frac{1}{4} \left[ f(x^*) - \min \left\{ \frac{\max\{2h_1, C_f(P)\}}{\sqrt{K_1 + 1}}, \epsilon_1 \right\} - \min \left\{ \frac{\max\{2h_2, C_f(Q)\}}{\sqrt{K_2 + 1}}, \epsilon_2 \right\} \right],
\end{align*}
\]

where \(h_1 := \max_{x \in P} f(x) - f(x^0), \ h_2 := \max_{x \in Q} f(z) - f(z^0)\) are the initial suboptimalities, \(C_f(P) := \sup_{x, v \in P, \gamma \in (0, 1], y = x + \gamma(v - x)} \frac{\gamma^2}{\gamma} \left( f(y) - f(x) - (y - x)^\top \nabla f(x) \right)\) is the curvature of \(f\) w.r.t. \(P\), and \(z^* = x^\top x^* - x\).

Theorem 34 indicates that Algorithm 5 has a 1/4 approximation guarantee and 1/\(\sqrt{K}\) rate of convergence. However, it has good empirical performance as demonstrated by the practical experiments. Informally, this can be partially explained by the term \(\frac{\mu}{8} \left( \|x - x^*\|^2 + \|z - z^*\|^2 \right)\) in (60): if \(x\) strongly deviates from \(x^*\), then this term will augment the bound; if \(x\) is close to \(x^*\), by the smoothness of \(f\), it should be close to optimal.

### 8.2 Shrunken FW: Follow Concavity and Shrink Constraint

**Algorithm 6**: The Shrunken FW Algorithm for Non-monotone DR-submodular Maximization (Bian et al., 2017a)

**Input**: \(\max_{x \in P} f(x)\); \#iterations \(K\); step size \(\gamma = 1/K\).

1. \(x^0 \leftarrow 0, t^0 \leftarrow 0, k \leftarrow 0; \quad // \ k: \text{iteration index}, t^k: \text{cumulative step size}
2. while \(t^k < 1\) do
3. \(v^k \leftarrow \arg \max_{v \in P, v \leq u - x^k} \langle v, \nabla f(x^k) \rangle; \quad // \text{shrunken LMO}
4. \text{use uniform step size } \gamma_k = \gamma; \text{ set } \gamma_k \leftarrow \min\{\gamma_k, 1 - t^k\};
5. \(x^{k+1} \leftarrow x^k + \gamma_k v^k, t^{k+1} \leftarrow t^k + \gamma_k, k \leftarrow k + 1;
\)

**Output**: \(x^K\); \quad // suppose there are \(K\) iterations in total

Algorithm 6 summarizes the Shrunken FW variant, which is inspired by the unified continuous greedy algorithm in Feldman et al. (2011) for maximizing the multilinear extension of a submodular set function.

It initializes the solution \(x^0\) to be 0, and maintains \(t^k\) as the cumulative step size. At iteration \(k\), it maximizes the linearization of \(f\) over a “shrunken” constraint set \(\{v \mid v \in P, v \leq u - x^k\}\), which is different from the classical LMO of Frank-Wolfe-style algorithms (hence we refer to it as the “shrunken LMO”). Then it employs an update step in the direction \(v^k\) chosen by the LMO with a uniform step size \(\gamma_k = \gamma\). The cumulative step size \(t^k\) is used to ensure that the overall step sizes sum to one, thus the output solution \(x^K\) is a convex combination of the LMO outputs, hence also lies in \(P\).
The shrunken LMO (Step 3) is the key difference compared to the Submodular FW variant in Bian et al. (2017b) (detailed in Algorithm 4). Therefore, we call Algorithm 6 **Shrunken FW**. The extra constraint \( v \leq \bar{u} - x^k \) is added to prevent too rapid growth of the solution, since in the non-monotone setting such fast increase may hurt the overall performance.

The next theorem states the guarantees of **Shrunken FW** in Algorithm 6.

**Theorem 35** Consider Algorithm 6 with uniform step size \( \gamma \). For \( k = 1, \ldots, K \) it holds that,

\[
f(x^k) \geq t^k e^{-t^k} f(x^*) - \frac{LD^2}{2} k\gamma^2 - O(\gamma^2) f(x^*). \tag{61}
\]

By observing that \( t^K = 1 \) and applying Theorem 35, we get the following Corollary:

**Corollary 36** The output of Algorithm 6 satisfies

\[
f(x^K) \geq \frac{1}{e} f(x^*) - \frac{LD^2}{2K} - O\left(\frac{1}{K^2}\right) f(x^*). \tag{62}
\]

Corollary 36 shows that Algorithm 6 enjoys a sublinear convergence rate towards some point \( x^K \) inside \( \mathcal{P} \), with a \( 1/e \) approximation guarantee.

**Proof sketch of Theorem 35:** The proof is by induction. To prepare the building blocks, we first of all show that the growth of \( x^k \) is indeed bounded,

**Lemma 37 (Bounding the growth of \( x^k \))** Assume \( x^0 = 0 \). For \( k = 0, \ldots, K - 1 \), it holds,

\[
x^k_i \leq \bar{u}_i [1 - (1 - \gamma)^t^k / \gamma], \forall i \in [n]. \tag{63}
\]

Then the following Lemma provides a lower bound, which depends on the global optimum,

**Lemma 38 (Generalized from Lemma 7 of Chekuri et al. (2015))** Given \( \theta \in (0, \bar{u}] \), let \( \lambda' = \min_{i \in [n]} \frac{\bar{u}_i}{\bar{u}} \). Then for all \( x \in [0, \theta] \), it holds,

\[
f(x \lor x^*) \geq (1 - \frac{1}{\lambda'}) f(x^*). \tag{64}
\]

Then the key ingredient for induction is the relation between \( f(x^{k+1}) \) and \( f(x^k) \) indicated by:

**Claim 39** For \( k = 0, \ldots, K - 1 \) it holds,

\[
f(x^{k+1}) \geq (1 - \gamma) f(x^k) + \gamma (1 - \gamma)^{t^k / \gamma} f(x^*) - \frac{LD^2}{2} \gamma^2. \tag{65}
\]

It is derived by a combination of the quadratic lower bound in Equation (31), Lemma 37 and Lemma 38.
8.3 Remarks on the Two Algorithms.

Notice that though the Two-Phase algorithm has an inferior guarantee compared to Shrunken FW, it is still of interest: i) It preserves flexibility in using a wide range of existing solvers for finding an (approximately) stationary point. ii) The guarantees that we present rely on a worst-case analysis. The empirical performance of the Two-Phase algorithm is often comparable or better than that of Shrunken FW. This suggests to explore more properties in concrete problems that may favor the Two-Phase algorithm.

9. Experimental Evaluation

9.1 Influence Maximization with Marketing Strategies

Follow the application in Section 6.3, we consider the following simplified influence model for experiments. The resulted problem is an instance of the monotone DR-submodular maximization problem.

9.1.1 Experimental Setup

**Simplified Influence Model for Experiments.** For general influence models, it is hard to evaluate Equation (46). To simplify the experiments, we consider $F(S)$ to be a facility location objective, for which the expected influence has a closed-form expression, as shown by Bian et al. (2019, Section 4.2). Here each customer may represent an “opinion leader” in social networks, and there is a bipartite graph describing the influence strength of each opinion leader to the population.

**Dataset.** We used the UC Irvine forum dataset as the real-world bipartite graph. It is a bipartite network containing user posts to forums. The users are students at the University of California, Irvine. An edge represents a forum message on a specific forum. It has in total 899 users, 522 forums and 33,720 edges (posts on the forum).

For a specific (user, forum) pair, we determine the edge weight as the number of posts from that user on the forum. This weighting indicates that the more one user has posted on a forum, the more he has influenced that particular forum. With this processing, we have 7,089 unique edges between users and forums.

We experimented with the independent marketing actions in Section 6.3.1 for simplicity. For a customer $i$, we set the parameter $p_i \in [0, 1]$ based on the following heuristic: Firstly, we calculate the “degree” of customer $i$ as the number of forums he has posted on: $d_i = \|W_i\|_0$. Then we set $p_i = \sigma(-d_i)$, $\sigma(\cdot)$ is the logistic sigmoid function. Remember that $p_i$ is the probability of customer $i$ becoming activated with one unit of investment, so this heuristic means that the more influence power a user has, the more difficult it is to activate him, because he might charge more than other users with less influence power. Since it is too time consuming to experiment on the whole bipartite graph, we experimented on different subgraphs of the original bipartite graph.
Figure 5: Expected influence w.r.t. iterations of different algorithms on real-world subgraphs with (a) 50 users; (b) 100 users; (c) 150 users; (d) 200 users. **Submodular FW** has a stable performance. It does not need to tune the step sizes or any hyperparameters. **PGA** algorithms are sensitive to quality of tuned step sizes. **Non-convex FW** with the Lipschitz step size rule also needs a careful tuning of the Lipschitz parameter.
9.1.2 Experimental Results

Figure 5 documents the trajectories of expected influence of different algorithms. We can see that Submodular FW has a very stable performance: It can always reach a fairly good solution, no matter what kind of setting you have. And it does not need to tune the step sizes or any hyperparameters. One drawback is that it converges relatively slowly in the beginning.

For PGA algorithms, we tested with two step size rules: the Lipschitz rule \(1/L\) which has the \(1/2\) approximation guarantee; the diminishing step size rule \((C/\sqrt{k} + 1)\), which does not have a formal theoretical guarantee. One general observation is that both step size rules need a careful tuning of hyperparameters, and the performance crucially depends on the quality of hyperparameters. For example, for PGA, if the step size is too small, it may converge too slowly; if the step sizes are too large, it tends to fluctuate.

For Non-convex FW algorithms, we also tested two step size rules: the “oblivious” rule \((2/(k + 2))\) and the Lipschitz rule. Apparently the Lipschitz step size rule needs a careful tuning of the Lipschitz parameter \(L\), while the oblivious rule does not. With a careful tuning of \(L\), both Non-convex FW variants converge very fast and converge to the highest function value.

9.2 Maximizing Softmax Extensions

Maximizing Softmax extensions of DPP MAP inference is an important instance of non-monotone DR-submodular maximization problem. One can obtain the derivative of the softmax extension in Equation (49) as:

\[
\nabla_i f(x) = \text{tr}([\text{diag}(x)(L - I) + I]^{-1}[(L - I)_i]), \forall i \in [n],
\]

where \((L - I)_i\) denotes the matrix obtained by zeroing all entries except for the \(i\)th row of \((L - I)\). Let \(C := (\text{diag}(x)(L - I) + I)^{-1}, D := (L - I)_i\), one can see that \(\nabla_i f(x) = D_i^T C_i\),\(^8\) which gives an efficient way to calculate the gradient \(\nabla f(x)\).

Results on Synthetic Data. We generate the softmax objectives (see (49)) in the following way: first generate the \(n\) eigenvalues \(d \in \mathbb{R}^n_+\), each evenly distributed in \([0, 10]\), and set \(D = \text{diag}(d)\). After generating a random unitary matrix \(U\), we set \(L = UDU^T\). One can verify that \(L\) is positive semidefinite and has eigenvalues as the entries of \(d\). Then we generate one cardinality constraint in the form of \(Ax \leq b\), where \(A = 1^{1 \times n}\) and \(b = 0.5n\).

Function value trajectories returned by different solvers on problem instances with different dimensionalities are shown in Figure 6. One can observe that Two-Phase FW has the fastest convergence. Shrunken FW converges slower, however it always eventually returns a high function value. The performance of PGA highly depends on the hyperparameters of the step sizes.

9.3 Revenue Maximization with Continuous Assignments

We experiment with the model from Section 6.7.1 on several real-world graphs. Note that the objective of the simplified revenue maximization model is in general continuous submod-

\(^7\) http://konect.uni-koblenz.de/networks/opsahl-ucforum

\(^8\) where \(D_i\) means the \(i\)th row of \(D\) and \(C_i\) indicates the \(i\)th column of \(C\).
Figure 6: Trajectories of different solvers on Softmax instances with one cardinality constraint. Left: $n = 50$; Middle: $n = 130$; Right: $n = 210$. Two-Phase FW has the fastest convergence. Shrunken FW converges slower, yet it always eventually returns a high function value. The performance of PGA highly depends on the hyperparameters of the step sizes.

9.3.1 Experimental Setting

The real-world graphs are from the Konect network collection (Kunegis, 2013)\(^9\) and the SNAP\(^{10}\) dataset. The graph datasets and corresponding experimental parameters are recorded in Table 3. We tested with the constraint that is the interaction of one box constraint ($0 \leq x_i \leq u$) and one cardinality constraint $1^\top x \leq b$.

Table 3: Graph datasets and the corresponding experimental parameters. $n$ is the number of nodes, $q$ is the parameter of the model in Section 6.7.1, $u$ is the upper bound of the box constraint, and $b$ is the budget.

| Dataset name             | $n$  | #edges      | $q$  | $u$  | budget $b$ |
|--------------------------|------|-------------|------|------|------------|
| “Reality Mining”         | 96   | 1,086,404 (multiedge) | 0.75 | 10   | 0.2nu      |
| “Residence hall”         | 217  | 2,672       | 0.75 | 10   | 0.4nu      |
| “Infectious”             | 410  | 17,298      | 0.7  | 20   | 0.2nu      |
| “U. Rovira i Virgili”    | 1,133| 5,451       | 0.8  | 20   | 0.2nu      |
| “ego Facebook”           | 4,039| 88,234      | 0.9  | 40   | 0.1nu      |

\(^9\) http://konect.uni-koblenz.de/networks

\(^{10}\) http://snap.stanford.edu/
For a specific example, the “Reality Mining” (Eagle and Pentland, 2006) dataset\(^{11}\) contains the contact data of 96 persons through tracking 100 mobile phones. The dataset was collected in 2004 over the course of nine months and represents approximately 500,000 hours of data on users’ location, communication and device usage behavior. Here one contact could mean a phone call, Bluetooth sensor proximity or physical location proximity. We use the number of contacts as the weight of an edge, by assuming that the more contacts happen between two persons, the stronger the connection strength should be.

9.3.2 Experimental Results

Results on a Small Graph for Visualization. First, we tested on a small graph, in order to clearly visualize the results. We select a subgraph from the “Reality Mining” dataset by taking the first five users/nodes, the nodes and number of contacts amongst nodes are shown in Figure 7a. For illustration, we label the five users as “A, B, C, D, E”. One can see that there are different level of contacts between different users, for example, there are 22,194 contacts between A and B, while there are only 82 contacts between E and C.

Figure 7b traces the trajectories of different algorithms when maximizing the revenue objective. They were all run for 20 iterations. One can see that Shrunken FW and Two-Phase FW reach higher revenue than PGA algorithms. Notice that Shrunken FW and Two-Phase FW with oblivious step sizes do not need to tune any hyperparameters, while the others need to adapt the Lipschitz parameter \(L\) and the constant \(C\) to determine the step sizes.

![The “Reality Mining” subgraph.](image)

![Trajectories of algorithms with 20 iterations.](image)

Figure 7: Results on the “Reality Mining” subgraph with one cardinality constraint, where \(u = 10, b = 0.2 \times n \times u\).

One may ask the question: How does the assignment look like for different algorithms? In order to show this behavior, we visualize the assignments in Figure 8. One can see that Shrunken FW assigns user A the most free products (6.1), followed by user C (3.3), then user E (0.6). All other users get 0 assignment. This is consistent with the intuition: one can

\[^{11}\text{http://konect.uni-koblenz.de/networks/mit and http://realitycommons.media.mit.edu/realitymining.html}\]
observe that user A most strongly influences others users (with total contacts as 22,194 + 410 + 143), while user D exerts zero influence on others. Two-Phase FW provides similar result, while PGA is conservative in assigning free products to users.

Figure 8: Assignments to the users returned by different algorithms. PGA is more conservative in terms of assigning free products to users than the other two algorithms: Shrunken FW and Two-Phase FW.

Results on Big Graphs. Then we looked at the behavior of the algorithms on the original big graph, which is plotted in Figure 9, for real-world graphs with at most $n = 4,039$ nodes.

One can observe that usually Two-Phase FW algorithm achieves the highest objective value, and also converges with the fastest rate. Shrunken FW converges slower than Two-Phase FW, but it always reaches competitive function value. PGA algorithms need to tune parameters for the step size, and converges to lower objective values.

10. Conclusion

In this work, we have systematically studied continuous submodularity and the problem of continuous (DR)-submodular maximization. With rigorous characterizations and study of composition rules, we established important properties of this class of functions. Based on geometric properties of continuous DR-submodular maximization, we proposed provable algorithms for both the monotone and non-monotone settings. We also identified representative applications and demonstrated the effectiveness of the proposed algorithms on both synthetic and real-world experiments.
Figure 9: Trajectory of different algorithms on real-world graphs. Usually Two-Phase FW achieves the highest objective value, and also converges with the fastest rate. Shrunken FW converges slower than Two-Phase FW, but it always reaches competitive function value. PGA algorithms need to tune parameters for the step size, and converges to lower objective values.
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Appendix

Appendix A. Proofs of Characterizations of Continuous Submodular Functions

Since $X_i$ is a compact subset of $\mathbb{R}$, we denote its lower bound and upper bound to be $u_i$ and $\bar{u}_i$, respectively.

A.1 Proofs of Lemma 5 and Lemma 8

**Proof [Proof of Lemma 5]**

**Sufficiency:** For any dimension $i$, 
\[
\nabla_i f(a) = \lim_{k \to 0} \frac{f(ke_i + a) - f(a)}{k} \geq \lim_{k \to 0} \frac{f(ke_i + b) - f(b)}{k} = \nabla_i f(a). \tag{67}
\]

**Necessity:**

Firstly, we show that for any $c \geq 0$, the function $g(x) := f(c + x) - f(x)$ is monotonically non-increasing.

\[
\nabla g(x) = \nabla f(c + x) - \nabla f(x) \leq 0. \tag{68}
\]

Taking $c = ke_i$, since $g(a) \leq g(b)$, we reach the DR-submodularity definition.

**Proof [Proof of Lemma 8]** Similar as the proof of Lemma 5, we have the following:

**Sufficiency:** For any dimension $i$ s.t. $a_i = b_i$,

\[
\nabla_i f(a) = \lim_{k \to 0} \frac{f(ke_i + a) - f(a)}{k} \geq \lim_{k \to 0} \frac{f(ke_i + b) - f(b)}{k} = \nabla_i f(a). \tag{69}
\]

**Necessity:**

We show that for any $k \geq 0$, the function $g(x) := f(ke_i + x) - f(x)$ is monotonically non-increasing.

\[
\nabla g(x) = \nabla f(ke_i + x) - \nabla f(x) \leq 0. \tag{70}
\]

Since $g(a) \leq g(b)$, we reach the weak DR definition.

A.2 Alternative Formulation of the weak DR Property

First of all, we will prove that **weak DR** has the following alternative formulation, which will be used to prove Proposition 7.

**Lemma 40 (Alternative formulation of weak DR)** The weak DR property (Equation (9), denoted as Formulation I) has the following equivalent formulation (Equation (71), denoted as Formulation II): \forall a \leq b \in X, \forall i \in \{i' | a_{i'} = b_{i'} = u_{i'}\}, \forall k' \geq l' \geq 0 s.t. (k'e_i + a), (l'e_i + a), (k'e_i + b), and (l'e_i + b) are still in $X$, the following inequality is satisfied,

\[
f(k'e_i + a) - f(l'e_i + a) \geq f(k'e_i + b) - f(l'e_i + b). \tag{71}
\]
Proof

Let $D_1 = \{i|a_i = b_i = u_i\}$, $D_2 = \{i|u_i < a_i = b_i < \bar{u}_i\}$, and $D_3 = \{i|a_i = b_i = \bar{u}_i\}$.

1) Formulation II $\Rightarrow$ Formulation I

When $i \in D_1$, set $l' = 0$ in Formulation II one can get $f(k'e_i + a) - f(a) \geq f(k'e_i + b) - f(b)$.

When $i \in D_2$, for all $k \geq 0$, let $l' = a_i - u_i = b_i - u_i > 0$, $k' = k + l' = k + (a_i - u_i)$, and let $\hat{a} = (a_i|l_i(u_i))$, $\hat{b} = (b_i|l_i(u_i))$. It is easy to see that $\hat{a} \leq b_i$ and $\bar{a}_i = \bar{b}_i = u_i$. Then from Formulation II,

$$f(k'e_i + \hat{a}) - f(l'e_i + \hat{a}) = f(k'e_i + a) - f(l'e_i + a) \geq f(k'e_i + \hat{b}) - f(l'e_i + \hat{b}) = f(k'e_i + b) - f(l'e_i + b).$$

When $i \in D_3$, Equation (9) holds trivially.

The above three situations proves the Formulation I.

2) Formulation II $\Leftarrow$ Formulation I

Let $\forall a \leq b \in \mathcal{X}$, $\forall i \in D_1$, $\forall k' \geq l'$, $\forall l' \geq 0$, let $\hat{a} = l'e_i + a$, $\hat{b} = l'e_i + b$, let $k = k' - l'$, it can be verified that $\hat{a} \leq \hat{b}$ and $\bar{a}_i = \bar{b}_i$, from Formulation I,

$$f(k'e_i + \hat{a}) - f(l'e_i + \hat{a}) = f(k'e_i + a) - f(l'e_i + a) \geq f(k'e_i + \hat{b}) - f(l'e_i + \hat{b}) = f(k'e_i + b) - f(l'e_i + b).$$

which proves Formulation II.

A.3 Proof of Proposition 7

Proof

1) submodularity $\Rightarrow$ weak DR:

Let us prove the Formulation II (Equation (71)) of weak DR, which is,

$\forall a \leq b \in \mathcal{X}$, $\forall i \in \{i'|a_{i'} = b_{i'} = u_{i'}\}$, $\forall k' \geq l'$, $\forall l' \geq 0$, the following inequality holds,

$$f(k'e_i + a) - f(l'e_i + a) \geq f(k'e_i + b) - f(l'e_i + b).$$

And $f$ is a submodular function iff $\forall x, y \in \mathcal{X}$, $f(x) + f(y) \geq f(x \vee y) + f(x \wedge y)$, so $f(y) - f(x \wedge y) \geq f(x \vee y) - f(x)$.

Now $\forall a \leq b \in \mathcal{X}$, one can set $x = l'e_i + b$ and $y = k'e_i + a$. It can be easily verified that $x \wedge y = l'e_i + a$ and $x \vee y = k'e_i + b$. Substituting all the above equalities into $f(y) - f(x \wedge y) \geq f(x \vee y) - f(x)$ one can get $f(k'e_i + a) - f(l'e_i + a) \geq f(k'e_i + b) - f(l'e_i + b)$.

2) submodularity $\Leftarrow$ weak DR:

Let us use Formulation I (Equation (9)) of weak DR to prove the submodularity property.

$\forall x, y \in \mathcal{X}$, let $D := \{e_1, \ldots, e_d\}$ be the set of elements for which $y_e > x_e$, let $k_{e_i} := y_{e_i} - x_{e_i}$. Now set $a^0 := x \wedge y$, $b^0 := x$ and $a^i = (a^{i-1}|_{e_i}(y_{e_i})) = k_{e_i}e_i + a^{i-1}, b^i = (b^{i-1}|_{e_i}(y_{e_i})) = k_{e_i}e_i + b^{i-1}$, for $i = 1, \ldots, d$.

One can verify that $a^i \leq b^i, a^k_{e_i} = b^k_{e_i}$ for all $i' \in D, i = 0, \ldots, d$, and that $a^d = y, b^d = x \vee y$. 

52
Applying Equation (9) of the weak DR property for \( i = 1, \ldots, d \) one can get
\[
\begin{align*}
f(k_{e_1} e_1 + a^0) - f(a^0) & \geq f(k_{e_1} e_1 + b^0) - f(b^0) \\
f(k_{e_2} e_2 + a^1) - f(a^1) & \geq f(k_{e_2} e_2 + b^1) - f(b^1) \\
\cdots \\
f(k_{e_d} e_d + a^{d-1}) - f(a^{d-1}) & \geq f(k_{e_d} e_d + b^{d-1}) - f(b^{d-1}).
\end{align*}
\] (75)
(76)
(77)

Taking a sum over all the above \( d \) inequalities, one can get
\[
\begin{align*}
f(k e_d e_d + a^{d-1}) - f(a^0) & \geq f(k e_d e_d + b^{d-1}) - f(b^0) \\
\iff f(y) - f(x \wedge y) & \geq f(x \vee y) - f(x) \\
\iff f(x) + f(y) & \geq f(x \vee y) + f(x \wedge y),
\end{align*}
\] (78)
(79)
(80)
which proves the submodularity property.

A.4 Proof of Proposition 9

Proof

1) submodular + coordinate-wise concave \( \Rightarrow \) DR:
From coordinate-wise concavity we have \( f(a + k e_i) - f(a) \geq f(a + (b_i - a_i + k)e_i) - f(a + (b_i - a_i)e_i) \). Therefore, to prove DR it suffices to show that
\[
f(a + (b_i - a_i + k)e_i) - f(a + (b_i - a_i)e_i) \geq f(b + k e_i) - f(b). \tag{81}
\]

Let \( x := b, y := (a + (b_i - a_i + k)e_i), \) so \( x \wedge y = (a + (b_i - a_i)e_i), x \vee y = (b + k e_i). \) From submodularity, one can see that inequality (81) holds.

2) DR \( \Rightarrow \) submodular + coordinate-wise concave:
From DR property, the weak DR (Equation (9)) property is implied, which equivalently proves the submodularity property.
To prove coordinate-wise concavity, one just need to set \( b := a + l e_i, \) then we have \( f(a + k e_i) - f(a) \geq f(a + (k + l)e_i) - f(a + l e_i). \)

Appendix B. Proofs for Properties of Continuous DR-Submodular Maximization

B.1 Proof of Lemma 13

Proof [Proof of Lemma 13]
Suppuse for simplicity that \( f \) and \( h \) are both twice differentiable. Note that when \( f \) and \( h \) are not differentiable, one can similarly prove the conclusion using zero\(^\text{th}\) order definition of continuous submodularity.

53
Without loss of generality, let us prove that \( f(h(x)) \) maintains submodularity of \( f \). One just need to show that the term \( \frac{\partial^2 g_i(x)}{\partial x_i \partial x_j} \) in Equation (13) is non-positive when \( i \neq j \).

Firstly, let us consider the term \( \sum_{k=1}^n \frac{\partial f(y) \frac{\partial^2 h_k(x)}{\partial x_i \partial x_j}}{\partial y_k} \). Since \( h \) is separable as stated above, \( \frac{\partial^2 h_k(x)}{\partial x_i \partial x_j} \) is always zero, so \( \sum_{k=1}^n \frac{\partial f(y) \frac{\partial^2 h_k(x)}{\partial x_i \partial x_j}}{\partial y_k} \) is always zero.

Then it remains to show that the term \( \sum_{s,t=1}^n \frac{\partial^2 f(y) \frac{\partial h_s(x)}{\partial x_i} \frac{\partial h_t(x)}{\partial x_j}}{\partial y_s \partial y_t} \) is non-positive. There are two situations: 1) \( s = t \). Since \( i \neq j \), there must be one term out of \( \frac{\partial h_s(x)}{\partial x_i} \) and \( \frac{\partial h_t(x)}{\partial x_j} \) that are zero (because \( h \) is separable). 2) \( s \neq t \). Since \( f \) is submodular, it holds that \( \frac{\partial^2 f(y)}{\partial y_s \partial y_t} \leq 0 \). Because \( h \) is monotone, it also holds that \( \frac{\partial h_s(x)}{\partial x_i} \frac{\partial h_t(x)}{\partial x_j} \geq 0 \). So the term \( \sum_{s,t=1}^n \frac{\partial^2 f(y) \frac{\partial h_s(x)}{\partial x_i} \frac{\partial h_t(x)}{\partial x_j}}{\partial y_s \partial y_t} \) is non-positive in the above two situations.

Now we reach the conclusion that \( f(h(x)) \) maintains submodularity of \( f \). \[ \blacksquare \]

B.2 Proof of Proposition 18

**Proof** [Proof of Proposition 18] Consider a univariate function

\[
g(\xi) := f(x + \xi v^*), \xi \geq 0, v^* \geq 0. \tag{82}
\]

We know that

\[
\frac{dg(\xi)}{d\xi} = (v^*, \nabla f(x + \xi v^*)). \tag{83}
\]

It can be verified that:

\[
g(\xi) \text{ is concave } \iff \frac{d^2 g(\xi)}{d\xi^2} = (v^*)^T \nabla^2 f(x + \xi v^*) v^* = \sum_{i \neq j} v_i^* v_j^* \nabla_{ij}^2 f + \sum_i (v_i^*)^2 \nabla_{ii}^2 f \leq 0. \tag{84}
\]

The non-positiveness of \( \nabla_{ij}^2 f \) is ensured by submodularity of \( f(\cdot) \), and the non-positiveness of \( \nabla_{ii}^2 f \) results from the coordinate-wise concavity of \( f(\cdot) \).

The proof of concavity along any non-positive direction is similar, which is omitted here. \[ \blacksquare \]

B.3 Proof of Proposition 20

**Proof** [Proof of Proposition 20] Since \( f \) is DR-submodular, so it is concave along any direction \( v \in \pm \mathbb{R}_+^n \). We know that \( x \vee y = x \geq 0 \) and \( x \wedge y = x \leq 0 \), so from the strong DR-submodularity in (33),

\[
f(x \vee y) - f(x) \leq \langle \nabla f(x), x \vee y - x \rangle - \frac{\mu}{2} \| x \vee y - x \|^2, \tag{85}
\]

\[
f(x \wedge y) - f(x) \leq \langle \nabla f(x), x \wedge y - x \rangle - \frac{\mu}{2} \| x \wedge y - x \|^2. \tag{86}
\]
Summing the above two inequalities and notice that $x \vee y + x \wedge y = x + y$, we arrive,

$$(y - x)^\top \nabla f(x) \geq f(x \vee y) + f(x \wedge y) - 2f(x) + \frac{\mu}{2} ||x \vee y - x||^2 + ||x \wedge y - x||^2$$

(87)

$$= f(x \vee y) + f(x \wedge y) - 2f(x) + \frac{\mu}{2} ||y - x||^2,$$

(88)

the last equality holds since $||x \vee y - x||^2 + ||x \wedge y - x||^2 = ||y - x||^2$.  

B.4 Proof of Proposition 22

Proof [Proof of Proposition 22] Consider the point $z^* := x \vee x^* - x = (x^* - x) \vee 0$. One can see that: 1) $0 \leq z^* \leq x^*$; 2) $z^* \in \mathcal{P}$ (down-closedness); 3) $z^* \in Q$ (because of $z^* \leq u - x$).

From Proposition 20,

$$(x^* - x, \nabla f(x)) + 2f(x) \geq f(x \vee x^*) + f(x \wedge x^*) + \frac{\mu}{2} ||x - x^*||^2;$$

(89)

$$(z^* - z, \nabla f(z)) + 2f(z) \geq f(z \vee z^*) + f(z \wedge z^*) + \frac{\mu}{2} ||z - z^*||^2.$$  

(90)

Let us first of all prove the following key Claim.

Claim 23 Under the setting of Proposition 22, it holds that,

$$f(x \vee x^*) + f(x \wedge x^*) + f(z \vee z^*) + f(z \wedge z^*) \geq f(x^*).$$

(39)

Proof [Proof of Claim 23] Firstly, we are going to prove that

$$f(x \vee x^*) + f(z \vee z^*) \geq f(z^*),$$

(91)

which is equivalent to $f(x \vee x^*) - f(z^*) \geq f((x + z) \vee x^*) - f(z \vee z^*)$. It can be shown that $x \vee x^* - z^* = (x + z) \vee x^* - z \vee z^*$. Combining this with the fact that $z^* \leq z \vee z^*$, and using the DR property (see Definition 4) implies (91). Then we establish,

$$x \vee x^* - z^* = (x + z) \vee x^* - z \vee z^*.$$  

(92)

We will show that both the RHS and LHS of the above equation are equal to $x$: for the LHS of (92) we can write $x \vee x^* - z^* = x \vee x^* - (x \vee x^* - x) = x$. For the RHS of (92) let us consider any coordinate $i \in [n],$

$$(x_i + z_i) \vee x^*_i - z_i \vee z^*_i = (x_i + z_i) \vee x^*_i - ((x_i + z_i) - x_i) \vee ((x_i \vee x^*_i) - x_i) = x_i,$$

(93)

where the last equality holds easily for the two situations: $(x_i + z_i) \geq x^*_i$ and $(x_i + z_i) < x^*_i$.

Next, we are going to prove that,

$$f(z^*) + f(x \wedge x^*) \geq f(x^*) + f(0).$$

(94)
It is equivalent to \( f(z^*) - f(0) \geq f(x^*) - f(x \wedge x^*) \), which can be done similarly by the DR property: Notice that
\[
x^* - x \wedge x^* = x \vee x^* - x = z^* - 0 \text{ and } 0 \leq x \wedge x^*.
\] (95)

Thus (94) holds from the DR property. Combining (91) and (94) one can get,
\[
f(x \vee x^*) + f(z \vee z^*) + f(x \wedge x^*) + f(z \wedge z^*) \geq f(x^*) + f(0) + f((x + z) \vee x^*) + f(z \wedge z^*)
\] (96)
\[\geq f(x^*). \text{ (non-negativity of } f)\]

Combining (89) and (90) and Claim 23 it reads,
\[
\langle x^* - x, \nabla f(x) \rangle + \langle z^* - z, \nabla f(z) \rangle + 2(f(x) + f(z)) \geq f(x^*) + \frac{\mu}{2} (\|x - x^*\|^2 + \|z - z^*\|^2).
\] (97)
\[\geq f(x^*) + \frac{\mu}{2} (\|x - x^*\|^2 + \|z - z^*\|^2).
\] (98)

From the definition of non-stationarity in (34) one can get,
\[
g_P(x) := \max_{v \in P} \langle v - x, \nabla f(x) \rangle \geq \langle x^* - x, \nabla f(x) \rangle, \quad (99)
g_Q(z) := \max_{v \in Q} \langle v - z, \nabla f(z) \rangle \geq \langle z^* - z, \nabla f(z) \rangle. \quad (100)
\]

Putting together Equations (97), (99) and (100) we can get,
\[
2(f(x) + f(z)) \geq f(x^*) - g_P(x) - g_Q(z) + \frac{\mu}{2} (\|x - x^*\|^2 + \|z - z^*\|^2).
\] (101)

So it arrives
\[
\max \{f(x), f(z)\} \geq \frac{1}{4} [f(x^*) - g_P(x) - g_Q(z)] + \frac{\mu}{8} (\|x - x^*\|^2 + \|z - z^*\|^2).
\] (102)

\[\geq \frac{1}{4} [f(x^*) - g_P(x) - g_Q(z)] + \frac{\mu}{8} (\|x - x^*\|^2 + \|z - z^*\|^2).
\] (103)

Appendix C. Additional Details for Monotone DR-Submodular Maximization

C.1 Proof of Proposition 25

Proof [Proof of Proposition 25]

On a high level, the proof idea follows from the reduction from the problem of maximizing a monotone submodular set function subject to cardinality constraints.

56
Let us denote $\Pi_1$ as the problem of maximizing a monotone submodular set function subject to cardinality constraints, and $\Pi_2$ as the problem of maximizing a monotone continuous DR-submodular function under general down-closed polytope constraints. Following Calinescu et al. (2011), there exist an algorithm $\mathcal{A}$ for $\Pi_1$ that consists of a polynomial time computation in addition to polynomial number of subroutine calls to an algorithm for $\Pi_2$. For details on $\mathcal{A}$ see the following.

First of all, the multilinear extension (Calinescu et al., 2007) of a monotone submodular set function is a monotone continuous submodular function, and it is coordinate-wise linear, thus falls into a special case of monotone continuous DR-submodular functions. Evaluating the multilinear extension and its gradients can be done using sampling methods, thus resulted in a randomized algorithm.

So the algorithm $\mathcal{A}$ shall be: 1) Maximize the multilinear extension of the submodular set function over the matroid polytope associated with the cardinality constraint, which can be achieved by solving an instance of $\Pi_2$. We call the solution obtained the fractional solution; 2) Round the fractional solution to a feasible integral solution using polynomial time rounding technique in Ageev and Sviridenko (2004); Calinescu et al. (2007) (called the pipage rounding). Thus we prove the reduction from $\Pi_1$ to $\Pi_2$.

Our reduction algorithm $\mathcal{A}$ implies the NP-hardness and inapproximability of problem $\Pi_2$.

For the NP-hardness, because $\Pi_1$ is well-known to be NP-hard (Calinescu et al., 2007; Feige, 1998), so $\Pi_2$ is NP-hard as well.

For the inapproximability: Assume there exists a polynomial algorithm $\mathcal{B}$ that can solve $\Pi_2$ better than $1 - 1/e$, then we can use $\mathcal{B}$ as the subroutine algorithm in the reduction, which implies that one can solve $\Pi_1$ better than $1 - 1/e$. Now we slightly adapt the proof of inapproximability on max-k-cover of Feige (1998), since max-k-cover is a special case of $\Pi_1$. According to the proof of Theorem 5.3 in Feige (1998) and our reduction $\mathcal{A}$, we have a reduction from approximating 3SAT–5 to problem $\Pi_2$. Using the rest proof of Theorem 5.3 in Feige (1998), we reach the result that one cannot solve $\Pi_2$ better than $1 - 1/e$, unless RP = NP.

C.2 Proof of Corollary 27

Proof [Proof of Corollary 27] Firstly, according to Theorem 1 of Lacoste-Julien (2016), non-convex FW is known to converge to a stationary point with a rate of $1/\sqrt{k}$.

Then according to Corollary 21, any stationary point is a $1/2$ approximate solution.

C.3 Proof of Lemma 30

Proof It is easy to see that $x^K$ is a convex combination of points in $\mathcal{P}$, so $x^K \in \mathcal{P}$.

Consider the point $v^* := (x^* \lor x) - x = (x^* - x) \lor 0 \geq 0$. Because $v^* \leq x^*$ and $\mathcal{P}$ is down-closed, we get $v^* \in \mathcal{P}$.

By monotonicity, $f(x + v^*) = f(x^* \lor x) \geq f(x^*)$.  

57
Consider the function $g(\xi) := f(x + \xi v^*)$, $\xi \geq 0$. $\frac{dg(\xi)}{d\xi} = \langle v^*, \nabla f(x + \xi v^*) \rangle$. From Proposition 18, $g(\xi)$ is concave, hence

$$
g(1) - g(0) = f(x + v^*) - f(x) \leq \frac{dg(\xi)}{d\xi} \bigg|_{\xi = 0} \times 1 = \langle v^*, \nabla f(x) \rangle. \quad (104)$$

Then one can get

$$
\langle v, \nabla f(x) \rangle \overset{(a)}{\geq} \alpha \langle v^*, \nabla f(x) \rangle - \frac{1}{2}\delta\gamma LD^2 \geq \alpha(f(x + v^*) - f(x)) - \frac{1}{2}\delta\gamma LD^2, \quad (106)
$$

where (a) is resulted from the LMO step of Algorithm 4.

C.4 Proof of Theorem 31

Proof [Proof of Theorem 31] From the Lipschitz assumption of $f$ (Equation (30)):

$$
f(x^{k+1}) - f(x^k) = f(x^k + \gamma_k v^k) - f(x^k) \geq \gamma_k \langle v^k, \nabla f(x^k) \rangle - \frac{L}{2}\gamma_k^2 \|v^k\|^2 \quad \text{(Lipschitz smoothness)}
$$

$$
\geq \gamma_k \alpha[f(x^*) - f(x^k)] - \frac{1}{2}\gamma_k^2 \delta LD^2 - \frac{L}{2}\gamma_k^2 D^2. \quad \text{(Lemma 30)}
$$

After rearrangement,

$$
f(x^{k+1}) - f(x^*) \geq (1 - \alpha\gamma_k)[f(x^k) - f(x^*)] - \frac{LD^2\gamma_k^2(1 + \delta)}{2}. \quad (108)
$$

Therefore,

$$
f(x^K) - f(x^*) \geq \prod_{k=0}^{K-1} (1 - \alpha\gamma_k)[f(0) - f(x^*)] - \frac{LD^2(1 + \delta)}{2} \sum_{k=0}^{K-1} \gamma_k^2. \quad (109)
$$

One can observe that $\sum_{k=0}^{K-1} \gamma_k = 1$, and since $1 - y \leq e^{-y}$ when $y \geq 0$,

$$
f(x^*) - f(x^K) \leq [f(x^*) - f(0)]e^{-\alpha \sum_{k=0}^{K-1} \gamma_k} + \frac{LD^2(1 + \delta)}{2} \sum_{k=0}^{K-1} \gamma_k^2 \quad (110)
$$

$$
= [f(x^*) - f(0)]e^{-\alpha} + \frac{LD^2(1 + \delta)}{2} \sum_{k=0}^{K-1} \gamma_k^2. \quad (111)
$$

After rearrangement, we get,

$$
f(x^K) \geq (1 - 1/e^\alpha)f(x^*) - \frac{LD^2(1 + \delta)}{2} \sum_{k=0}^{K-1} \gamma_k^2 + e^{-\alpha} f(0). \quad (112)
$$

\[\square\]
C.5 Proof of Corollary 32

Proof [Proof of Corollary 32] Fixing $K$, to reach the tightest bound in Equation (58) amounts to solving the following problem:

$$
\min_{\gamma} \sum_{k=0}^{K-1} \gamma_k^2 \quad \text{s.t.} \quad \sum_{k=0}^{K-1} \gamma_k = 1, \gamma_k \geq 0.
$$

(113)

Using Lagrangian method, let $\lambda$ be the Lagrangian multiplier, then

$$
L(\gamma_0, \cdots, \gamma_{K-1}, \lambda) = \sum_{k=0}^{K-1} \gamma_k^2 + \lambda \left[ \sum_{k=0}^{K-1} \gamma_k - 1 \right].
$$

(114)

It can be easily verified that when $\gamma_0 = \cdots = \gamma_{K-1} = K^{-1}$, $\sum_{k=0}^{K-1} \gamma_k^2$ reaches the minimum (which is $K^{-1}$). Therefore we obtain the tightest worst-case bound in Corollary 32. ■

Appendix D. Details of Revenue Maximization with Continuous Assignments

D.1 More Details About the Model

As discussed in the main text, $R_s(x^i)$ should be some non-negative, non-decreasing, submodular function; therefore, we set $R_s(x^i) := \sqrt{\sum_{t:x^i_t \neq 0} x^i_t w_{st}}$, where $w_{st}$ is the weight of edge connecting users $s$ and $t$. The first part in R.H.S. of Equation (53) models the revenue from users who have not received free assignments, while the second and third parts model the revenue from users who have gotten the free assignments. We use $w_{tt}$ to denote the “self-activation rate” of user $t$: Given certain amount of free trial to user $t$, how probable is it that he/she will buy after the trial. The intuition of modeling the second part in R.H.S. of Equation (53) is: Given the users more free assignments, they are more likely to buy the product after using it. Therefore, we model the expected revenue in this part by $\phi(x^i_t) = w_{tt} x^i_t$; The intuition of modeling the third part in R.H.S. of Equation (53) is: Given the users more free assignments, the revenue could decrease, since the users use the product for free for a longer period. As a simple example, the decrease in the revenue can be modeled as $\gamma \sum_{t:x^i_t \neq 0} -x^i_t$.

D.2 Proof of Lemma 24

Proof

First of all, we prove that $g(x) := \sum_{s:x_s=0} R_s(x)$ is a non-negative submodular function.

It is easy to see that $g(x)$ is non-negative. To prove that $g(x)$ is submodular, one just need,

$$
g(a) + g(b) \geq g(a \lor b) + g(a \land b), \quad \forall a, b \in [0, \bar{u}].
$$

(115)
Let $A := \text{supp}(a)$, $B := \text{supp}(b)$, where $\text{supp}(x) := \{i| x_i \neq 0\}$ is the support of the vector $x$. First of all, because $R_s(x)$ is non-decreasing, and $b \geq a \land b$, $a \geq a \land b$,

$$\sum_{s \in A \setminus B} R_s(b) + \sum_{s \in B \setminus A} R_s(a) \geq \sum_{s \in A \setminus B} R_s(a \land b) + \sum_{s \in B \setminus A} R_s(a \land b). \quad (116)$$

By submodularity of $R_s(x)$, and summing over $s \in \mathcal{V}(A \cup B)$,

$$\sum_{s \in \mathcal{V}(A \cup B \setminus B)} R_s(a) + \sum_{s \in \mathcal{V}(A \cup B \setminus A)} R_s(b) \geq \sum_{s \in \mathcal{V}(A \cup B \setminus B)} R_s(a \lor b) + \sum_{s \in \mathcal{V}(A \cup B \setminus A)} R_s(a \land b). \quad (117)$$

Summing Equations 116 and 117 one can get

$$\sum_{s \in \mathcal{V}(A \setminus B)} R_s(a) + \sum_{s \in \mathcal{V}(B \setminus A)} R_s(b) \geq \sum_{s \in \mathcal{V}(A \cup B \setminus B)} R_s(a \lor b) + \sum_{s \in \mathcal{V}(A \cup B \setminus A)} R_s(a \land b)$$

which is equivalent to Equation (115).

Then we prove that $h(x) := \sum_{t: x_t \neq 0} \bar{R}_t(x)$ is submodular. Because $\bar{R}_t(x)$ is non-increasing, and $a \leq a \lor b$, $b \leq a \lor b$,

$$\sum_{t \in A \setminus B} \bar{R}_t(a) + \sum_{t \in B \setminus A} \bar{R}_t(b) \geq \sum_{t \in A \setminus B} \bar{R}_t(a \lor b) + \sum_{t \in B \setminus A} \bar{R}_t(a \lor b). \quad (118)$$

By submodularity of $\bar{R}_t(x)$, and summing over $t \in A \cap B$,

$$\sum_{t \in A \cap B} \bar{R}_t(a) + \sum_{t \in A \cap B} \bar{R}_t(b) \geq \sum_{t \in A \cap B} \bar{R}_t(a \lor b) + \sum_{t \in A \cap B} \bar{R}_t(a \land b). \quad (119)$$

Summing Equations 118, 119 we get,

$$\sum_{t \in A} \bar{R}_t(a) + \sum_{t \in B} \bar{R}_t(b) \geq \sum_{t \in A \cup B} \bar{R}_t(a \lor b) + \sum_{t \in A \cup B} \bar{R}_t(a \land b) \quad (120)$$

which is equivalent to $h(a) + h(b) \geq h(a \lor b) + h(a \land b)$, $\forall a, b \in [0, \bar{u}]$, thus proving the submodularity of $h(x)$.

Finally, because $f(x)$ is the sum of two submodular functions and one modular function, so it is submodular.

---

**Appendix E. Proofs for Non-Monotone DR-Submodular Maximization**

**E.1 Proof for Hardness and Inapproximability**

**Proof** [Proof of Proposition 33] The main proof follows from the reduction from the problem of maximizing an unconstrained non-monotone submodular set function.

Let us denote $\Pi_1$ as the problem of maximizing an unconstrained non-monotone submodular set function, and $\Pi_2$ as the problem of maximizing a box constrained non-monotone continuous DR-submodular function. Following the Appendix A of Buchbinder et al. (2012),
there exist an algorithm $A$ for $\Pi_1$ that consists of a polynomial time computation in addition to polynomial number of subroutine calls to an algorithm for $\Pi_2$. For details see the following.

Given a submodular set function $F : 2^V \to \mathbb{R}_+$, its multilinear extension (Calinescu et al., 2007) is a function $f : [0, 1]^V \to \mathbb{R}_+$, whose value at a point $x \in [0, 1]^V$ is the expected value of $F$ over a random subset $R(x) \subseteq V$, where $R(x)$ contains each element $e \in V$ independently with probability $x_e$. Formally, $f(x) := \mathbb{E}[R(x)] = \sum_{S \subseteq V} F(S) \prod_{e \in S} x_e \prod_{e' \not\in S} (1 - x_{e'})$. It can be easily seen that $f(x)$ is a non-monotone DR-submodular function.

Then the algorithm $A$ can be: 1) Maximize the multilinear extension $f(x)$ over the box constraint $[0, 1]^V$, which can be achieved by solving an instance of $\Pi_2$. Obtain the fractional solution $\hat{x} \in [0, 1]^n$; 2) Return the random set $R(\hat{x})$. According to the definition of multilinear extension, the expected value of $F(R(\hat{x}))$ is $f(\hat{x})$. Thus proving the reduction from $\Pi_1$ to $\Pi_2$.

Given the reduction, the hardness result follows from the hardness of unconstrained non-monotone submodular set function maximization.

The inapproximability result comes from that of the unconstrained non-monotone submodular set function maximization in Feige et al. (2011) and Dobzinski and Vondrák (2012).

E.2 Proof of Theorem 34

Proof [Proof of Theorem 34]

Let $g_P(x), g_Q(z)$ to the non-stationarity of $x$ and $z$, respectively. Since we are using the Non-convex FW (Algorithm 2) as subroutine, according to Lacoste-Julien (2016, Theorem 1), one can get,

$$g_P(x) \leq \min \left\{ \frac{\max\{2h_1, C_f(P)\}}{\sqrt{K_1 + 1}}, \epsilon_1 \right\}, \quad (121)$$

$$g_Q(z) \leq \min \left\{ \frac{\max\{2h_2, C_f(Q)\}}{\sqrt{K_2 + 1}}, \epsilon_2 \right\}. \quad (122)$$

Plugging the above into Proposition 22 we reach the conclusion in (60).

E.3 Detailed Proofs for Theorem 35

E.3.1 Proof of Lemma 37

Lemma 37 (Bounding the growth of $x^k$) Assume $x^0 = 0$. For $k = 0, ..., K - 1$, it holds,

$$x^k_i \leq \bar{u}_i[1 - (1 - \gamma)^{t^k_i} / \gamma], \forall i \in [n]. \quad (63)$$

Proof [Proof of Lemma 37] We prove by induction. First of all, it holds when $k = 0$, since $x^0_i = 0$, and $t^0 = 0$ as well. Assume it holds for $k$. Then for $k + 1$, we have

$$x^{k+1}_i = x^k_i + \gamma v^k_i \quad (123)$$
≤ x_i^k + \gamma (\bar{u}_i - x_i^k) \quad \text{(constraint of shrunken LMO)} \quad (124)
= (1 - \gamma) x_i^k + \gamma \bar{u}_i
≤ (1 - \gamma) \bar{u}_i [1 - (1 - \gamma)^{t^k / \gamma}] + \gamma \bar{u}_i \quad \text{(induction)} \quad (125)
= \bar{u}_i [1 - (1 - \gamma)^{t^{k+1} / \gamma}].

E.3.2 Proof of Lemma 38

Lemma 38 (Generalized from Lemma 7 of Chekuri et al. (2015)) Given \( \theta \in (0, \bar{u}] \), let \( \lambda' = \min_{i \in [n]} \frac{\bar{u}_i}{\theta_i} \). Then for all \( x \in [0, \theta] \), it holds,
\[
f(x \lor x^*) \geq (1 - \frac{1}{\lambda'}) f(x^*).
\] (64)

Proof [Proof of Lemma 38]
Consider \( r(\lambda) = x^* + \lambda (x \lor x^* - x^*) \), it is easy to see that \( r(\lambda) \geq 0, \forall \lambda \geq 0 \).

Notice that \( \lambda' \geq 1 \). Let \( y = r(\lambda') = x^* + \lambda' (x \lor x^* - x^*) \), it is easy to see that \( y \geq 0 \), it also holds that \( y \leq \bar{u} \): Consider one coordinate \( i \), 1) if \( x_i \geq x_i^* \), then \( y_i = x_i^* + \lambda' (x_i - x_i^*) \leq \lambda' x_i \leq \lambda' \theta_i \leq \bar{u}_i \); 2) if \( x_i < x_i^* \), then \( y_i = x_i^* \leq \bar{u}_i \). So \( f(y) \geq 0 \).

Note that
\[
x \lor x^* = (1 - \frac{1}{\lambda'}) x^* + \frac{1}{\lambda'} y = (1 - \frac{1}{\lambda'}) r(0) + \frac{1}{\lambda'} r(\lambda'),
\] (126)
since \( f \) is concave along \( r(\lambda) \), so it holds that,
\[
f(x \lor x^*) \geq (1 - \frac{1}{\lambda'}) f(x^*) + \frac{1}{\lambda'} f(y) \geq (1 - \frac{1}{\lambda'}) f(x^*). \] (127)

E.3.3 Proof of Theorem 35

Proof [Proof of Theorem 35]
First of all, let us prove the Claim:

Claim 39 For \( k = 0, ..., K - 1 \) it holds,
\[
f(x^{k+1}) \geq (1 - \gamma) f(x^k) + \gamma (1 - \gamma)^{t^k / \gamma} f(x^*) - \frac{LD^2}{2} \gamma^2. \] (65)

Proof [Proof of Claim 39] Consider a point \( z^k := x^k \lor x^* - x^k \), one can observe that: 1) \( z^k \leq \bar{u} - x^k \); 2) since \( x^k \geq 0, x^* \geq 0 \), so \( z^k \leq x^* \), which implies that \( z^k \in P \) (from down-closedness of \( P \)). So \( z^k \) is a candidate solution for the shrunken LMO (Step 3 in Algorithm 6). We have,
\[
f(x^{k+1}) - f(x^k) \geq \gamma \langle \nabla f(x^k), v^k \rangle - \frac{L}{2} \gamma^2 \| v^k \|^2 \quad \text{(Quadratic lower bound of (31))} \] (128)
where the last equality comes from setting $\theta := \frac{u}{\bar{u}} = (1 - (1 - \gamma)^{tk}/\gamma)^{-1}$.

After rearrangement, we reach the claim.

Then, let us prove Theorem 35 by induction.

First of all, it holds when $k = 0$ (notice that $t^0 = 0$). Assume that it holds for $k$.

Then for $k + 1$, considering the fact $e^{-t} - O(\gamma) \leq (1 - \gamma)^{tk}/\gamma$ when $0 < \gamma \leq t \leq 1$ and Claim 39 we get,

\[
\begin{align*}
f(x^{k+1}) & \geq (1 - \gamma)f(x^k) + (1 - \gamma)^{tk}/\gamma f(x^*) - \frac{LD^2}{2}\gamma^2 \\
& \geq (1 - \gamma)f(x^k) + [e^{-tk} - O(\gamma)]f(x^*) - \frac{LD^2}{2}\gamma^2 \\
& \geq [(1 - \gamma)e^{-tk} + \gamma e^{-tk}]f(x^*) - \frac{LD^2}{2}\gamma^2[(1 - \gamma)k + 1] - [(1 - \gamma)O(\gamma^2) + \gamma O(\gamma)]f(x^*) \\
& \geq (1 - \gamma)^{tk+1}e^{-tk+1}f(x^*) - \frac{LD^2}{2}\gamma^2(k + 1) - O(\gamma^2)f(x^*). \tag{138}
\end{align*}
\]

Let us consider the term $[(1 - \gamma)tk e^{-tk} + \gamma e^{-tk}]f(x^*)$. We know that the function $g(t) = te^{-t}$ is concave in $[0, 2]$, so $g(tk + \gamma) - g(tk) \leq \gamma g'(tk)$, which amounts to,

\[
[(1 - \gamma)tk e^{-tk} + \gamma e^{-tk}]f(x^*) \geq (tk + \gamma)e^{-(tk+\gamma)}f(x^*) \geq t^{k+1}e^{-tk+1}f(x^*). \tag{139}
\]

Plugging Equation (140) into Equation (138) we get,

\[
f(x^{k+1}) \geq t^{k+1}e^{-tk+1}f(x^*) - \frac{LD^2}{2}\gamma^2(k + 1) - O(\gamma^2)f(x^*). \tag{141}
\]

Thus proving the induction, and proving the theorem as well.
Appendix F. Miscellaneous Results

F.1 Verifying DR-Submodularity of the Objectives

**Softmax extension.** For softmax extension, the objective is,

$$f(x) = \log \det (\text{diag}(x)(L - I) + I), \ x \in [0, 1]^n.$$ 

Its DR-submodularity can be established by directly applying Lemma 3 in (Gillenwater et al., 2012): Gillenwater et al. (2012, Lemma 3) immediately implies that all entries of $\nabla^2 f$ are non-positive, so $f(x)$ is DR-submodular.

**Multilinear extension.** The DR-submodularity of multilinear extension can be directly recognized by considering the conclusion in Appendix A.2 of Bach (2015) and the fact that multilinear extension is coordinate-wise linear.

**KL($x$).** The Kullback-Leibler divergence between $q_x$ and $p$, i.e., $\sum_{S \subseteq V} q_x(S) \log \frac{q_x(S)}{p(S)}$ is,

$$\text{KL}(x) = -\sum_{S \subseteq V} \prod_{i \in S} x_i \prod_{j \notin S} (1 - x_j) F(S) + \sum_{i=1}^{n} [x_i \log x_i + (1 - x_i) \log(1 - x_i)] + \log Z.$$ 

The first term is the negative of a multilinear extension, so it is DR-supermodular. The second term is separable, and coordinate-wise convex, so it will not affect the off-diagonal entries of $\nabla^2 \text{KL}(x)$, it will only contribute to the diagonal entries. Now, one can see that all entries of $\nabla^2 \text{KL}(x)$ are non-negative, so $\text{KL}(x)$ is DR-supermodular w.r.t. $x$. 