Inflation: Homogeneous Limit

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Abstract
I review the motivation for the early-time cosmic acceleration stage in expanding universe and discuss simple inflationary scenarios. Preheating and reheating are considered in great detail. This is a sample chapter from my book "Physical Foundations of Cosmology" published by Cambridge University Press (2005).

Matter is distributed very homogeneously and isotropically on scales larger than a few hundred Mpc. The microwave background gives us a “photograph” of the early universe, which shows that at recombination the universe was extremely homogeneous and isotropic (with accuracy $\sim 10^{-4}$) on all scales up to the present horizon. Given that the universe evolves according to the Hubble law, it is natural to ask which initial conditions could lead to such homogeneity and isotropy.

To obtain an exhaustive answer to this question we have to know the exact physical laws which govern the evolution of the very early universe. However, as long as we are interested only in the general features of the initial conditions it suffices to know a few simple properties of these laws. We will assume that inhomogeneity cannot be dissolved by expansion. This natural surmise is supported by General Relativity (see Part II of this book for details). We will also assume that nonperturbative quantum gravity does not play an essential role at sub-Planckian curvatures. On the other hand, we are nearly certain that nonperturbative quantum gravity effects become very important when the curvature reaches Planckian values and the notion of classical spacetime breaks down. Therefore we address the initial conditions at the Planckian time $t_i = t_{Pl} \sim 10^{-43}$ s.

In this chapter we discuss the initial conditions problem we face in a decelerating universe and show how this problem can be solved if the universe undergoes a stage of the accelerated expansion known as inflation.

1 Problem of initial conditions

There are two independent sets of initial conditions characterizing matter: a) its spatial distribution, described by the energy density $\varepsilon(x)$ and b) the initial field of velocities. Let us determine them given the current state of the universe.
Homogeneity, isotropy (horizon) problem. The present homogeneous, isotropic domain of the universe is at least as large as the present horizon scale, $ct_0 \sim 10^{28}$ cm. Initially the size of this domain was smaller by the ratio of the corresponding scale factors, $a_i/a_0$. Assuming that inhomogeneity cannot be dissolved by expansion, we may safely conclude that the size of the homogeneous, isotropic region from which our universe originated at $t = t_i$ was larger than

$$l_i \sim ct_0 \frac{a_i}{a_0}. \quad (1)$$

It is natural to compare this scale to the size of a causal region $l_c \sim ct_i$:

$$\frac{l_i}{l_c} \sim \frac{t_0}{t_i} \frac{a_i}{a_0}. \quad (2)$$

To obtain a rough estimate of this ratio we note that if the primordial radiation dominates at $t_i \sim t_{Pl}$, then its temperature is $T_{Pl} \sim 10^{32}$ K. Hence

$$(a_i/a_0) \sim (T_0/T_{Pl}) \sim 10^{-32}$$

and we obtain

$$\frac{l_i}{l_c} \sim \frac{10^{17}}{10^{-43}} 10^{-32} \sim 10^{28}. \quad (3)$$

Thus, at the initial Planckian time, the size of our universe exceeded the causality scale by 28 orders of magnitude. This means that in $10^{84}$ causally disconnected regions the energy density was smoothly distributed with fractional variation not exceeding $\delta\varepsilon/\varepsilon \sim 10^{-4}$. Because no signals can propagate faster than light, no causal physical processes can be responsible for such an unnaturally fine-tuned matter distribution.

Assuming that the scale factor grows as some power of time, we can use an estimate $a/t \sim \dot{a}$ and rewrite formula (2) as

$$\frac{l_i}{l_c} \sim \frac{\dot{a}_i}{\dot{a}_0}. \quad (4)$$

Thus, the size of our universe was initially larger than that of a causal patch by the ratio of the corresponding expansion rates. Assuming that gravity was always attractive and hence was decelerating the expansion, we conclude from (4) that the homogeneity scale was always larger than the scale of causality. Therefore, the homogeneity problem is also sometimes called the horizon problem.

Initial velocities (flatness) problem. Let us suppose for a minute that someone has managed to distribute matter in the required way. The next question concerns initial velocities. Only after they are specified is the Cauchy problem completely posed and can the equations of motion be used to predict the future of the universe unambiguously. The initial velocities must obey the Hubble law because otherwise the initial homogeneity is very quickly spoiled. That this has to occur in so many causally disconnected regions further complicates the horizon problem. Assuming that it has, nevertheless, been achieved,
we can ask how accurately the initial Hubble velocities have to be chosen for a
given matter distribution.
Let us consider a large spherically symmetric cloud of matter and compare
its total energy with the kinetic energy due to Hubble expansion, $E_k$. The total
energy is the sum of the positive kinetic energy and the negative potential energy
of the gravitational self-interaction, $E_p$. It is conserved:
$$E_{\text{tot}} = E^k_i + E^p_i = E^k_0 + E^p_0.$$ 

Because the kinetic energy is proportional to the velocity squared,
$$E^k_i = E^k_0 \left( \dot{a}_i / \dot{a}_0 \right)^2$$
and we have
$$\frac{E_{\text{tot}}^i}{E^k_i} = \frac{E^k_i + E^p_i}{E^k_i} = \frac{E^k_0 + E^p_0}{E^k_0} \left( \frac{\dot{a}_0}{\dot{a}_i} \right)^2.$$ (5)

Since $E^k_0 \sim |E^p_0|$ and $\dot{a}_0 / \dot{a}_i \leq 10^{-28}$, we find
$$\frac{E_{\text{tot}}^i}{E^k_i} \leq 10^{-56}. \hspace{1cm} (6)$$
This means that for a given energy density distribution the initial Hubble ve-
locities must be adjusted so that the huge negative gravitational energy of the
matter is compensated by a huge positive kinetic energy to an unpre-
cedented accuracy of $10^{-54}\%$. An error in the initial velocities exceeding $10^{-54}\%$ has a
dramatic consequence: the universe either recollapses or becomes “empty” too
early. To stress the unnaturalness of this requirement one speaks of the initial
velocities problem.

**Problem 1.** How can the above consideration be made rigorous using the
Birkhoff theorem?

In general relativity the problem described can be reformulated in terms of
the cosmological parameter $\Omega (t)$. Using the definition of $\Omega (t)$ we can rewrite
Friedmann equation as
$$\Omega (t) - 1 = \frac{k}{(Ha)^2},$$ (7)
and hence
$$\Omega_i - 1 = (\Omega_0 - 1) \frac{(Ha)^2_i}{(Ha)^2} = (\Omega_0 - 1) \left( \frac{\dot{a}_0}{\dot{a}_i} \right)^2 \leq 10^{-56}. \hspace{1cm} (8)$$
Note that this relation immediately follows from (6) if we take into account that
$\Omega = |E^p| / E^k$. We infer from (3) that the cosmological parameter must initially
be extremely close to unity, corresponding to a flat universe. Therefore the
problem of initial velocities is also called the flatness problem.
**Initial perturbation problem.** One further problem we mention here for completeness is the origin of the primordial inhomogeneities needed to explain the large-scale structure of the universe. They must be initially of order $\delta \varepsilon / \varepsilon \sim 10^{-5}$ on galactic scales. This further aggravates the very difficult problem of homogeneity and isotropy, making it completely intractable. We will see later that the problem of initial perturbations has the same roots as the horizon and flatness problems and that it can also be successfully solved in inflationary cosmology. However, for the moment we put it aside and proceed with the “more easy” problems.

The above considerations clearly show that the initial conditions which led to the observed universe are very unnatural and non-generic. Of course, one can make the objection that naturalness is a question of taste and even claim that the most simple and symmetric initial conditions are “more physical.” In the absence of a quantitative measure of “naturalness” for a set of initial conditions it is very difficult to argue with this attitude. On the other hand it is hard to imagine any measure which selects the special and degenerate conditions in preference to the generic ones. In the particular case under consideration the generic conditions would mean that the initial distribution of the matter is strongly inhomogeneous with $\delta \varepsilon / \varepsilon \gtrsim 1$ everywhere or, at least, in the causally disconnected regions.

The universe is unique and we do not have the opportunity to repeat the “experiment of creation” many times. Therefore cosmological theory can claim to be a successful physical theory only if it can explain the state of the observed universe using simple physical ideas and starting with the most generic initial conditions. Otherwise it would simply amount to “cosmic archaeology,” where “cosmic history” is written on the basis of a limited number of hot big bang remnants. If we are pretentious enough to answer the question raised by Einstein, “What really interests me is whether God had any choice when he created the World,” we must be able to explain how a particular universe can be created starting with generic initial conditions. The inflationary paradigm seems to be a step in the right direction and it strongly restricts “God’s choice.”

Moreover, it makes important predictions which can be verified experimentally (observationally), thus giving cosmology the status of a physical theory.

2 Inflation: main idea

We have seen so far that the same ratio, $\dot{a}_i / \dot{a}_0$, enters both sets of independent initial conditions. The large value of this ratio determines the number of causally disconnected regions and defines the necessary accuracy of the initial velocities. If gravity was always attractive, then $\dot{a}_i / \dot{a}_0$ is necessarily larger than unity because gravity decelerates an expansion. Therefore, the conclusion $\dot{a}_i / \dot{a}_0 \gg 1$ can be avoided only if we assume that during some period of expansion gravity acted as a “repulsive” force, thus accelerating the expansion. In this case we can have $\dot{a}_i / \dot{a}_0 < 1$ and the creation of our type of universe from a single causally connected domain may become possible. A period of accelerated expansion is a
necessary condition, but whether is it also sufficient depends on the particular model where this condition is realized. With these remarks in mind we arrive at the following general definition of inflation:

Inflation is a stage of accelerated expansion of the universe when gravity acts as a repulsive force.

Figure 1 shows how the old picture of a decelerated Friedmann universe is modified by inserting a stage of cosmic acceleration. It is obvious that if we do not want to spoil the successful predictions of the standard Friedmann model, such as nucleosynthesis, inflation should begin and end sufficiently early. We will see later that the requirement of the generation of primordial fluctuations further restricts the energy scale of inflation; namely, in the simple models inflation should be over at $t_f \sim 10^{-34}$ to $10^{-36}$ s. Successful inflation must also possess a smooth graceful exit into the decelerated Friedmann stage because otherwise the homogeneity of the universe would be destroyed.

Inflation explains the origin of the Big Bang; since it accelerates the expansion, small initial velocities within a causally connected patch become very large. Furthermore, inflation can produce the whole observable universe from a small homogeneous domain even if the universe was strongly inhomogeneous outside of this domain. The reason is that in an accelerating universe there always exists an event horizon. It has size

$$r_e (t) = a (t) \int^t_{t_f} \frac{dt}{a} = a (t) \int^a_{a (t)} \frac{da}{a} = a_{\text{max}}$$

(9)

The integral converges even if $a_{\text{max}} \to \infty$ because the expansion rate $\dot{a}$ grows with $a$. The existence of an event horizon means that anything at time $t$ a distance larger than $r_e (t)$ from an observer cannot influence that observer’s future. Hence the future evolution of the region inside a ball of radius $r_e (t)$ is completely independent of the conditions outside a ball of radius $2r_e (t)$ centered at the same
place. Let us assume that at $t = t_i$ matter was distributed homogeneously and isotropically only inside a ball of radius $2r_e(t_i)$ (Fig. 2). Then an inhomogeneity propagating from outside this ball can spoil the homogeneity only in the region which was initially between the spheres of radii $r_e(t_i)$ and $2r_e(t_i)$. The region originating from the sphere of radius $r_e(t_i)$ remains homogeneous. This internal domain can be influenced only by events which happened at $t_i$ between the two spheres, where the matter was initially distributed homogeneously and isotropically.

The physical size of the homogeneous internal region increases and is equal to

$$r_h(t_f) = r_e(t_i) \frac{a_f}{a_i}$$

at the end of inflation. It is natural to compare this scale with the particle horizon size, which in an accelerated universe can be estimated as

$$r_p(t) = a(t) \int_{t_i}^{t} \frac{dt}{a} = a(t) \int_{a_i}^{a} \frac{da}{a} \sim \frac{a(t)}{a_i} r_e(t_i),$$

since the main contribution to the integral comes from $a \sim a_i$. At the end of inflation $r_p(t_f) \sim r_h(t_f)$, that is, the size of the homogeneous region, originating from a causal domain, is of order the particle horizon scale.

Thus, instead of considering a homogeneous universe in many causally disconnected regions, we can begin with a small homogeneous causal domain which inflation blows up to a very large size, preserving the homogeneity irrespective of the conditions outside this domain.

**Problem 2.** Why does the above consideration fail in a decelerating universe?

The next question is whether we can relax the restriction of homogeneity
on the initial conditions. Namely, if we begin with a strongly inhomogeneous causal domain, can inflation still produce a large homogeneous universe?

The answer to this question is positive. Let us assume that the initial energy density inhomogeneity is of order unity on scales $\sim H_i^{-1}$, that is,

$$\left(\frac{\delta\varepsilon}{\varepsilon}\right)_{t_i} \sim \frac{1}{\varepsilon} \frac{|\nabla\varepsilon|}{a_i} H_i^{-1} \sim \frac{|\nabla\varepsilon|}{\varepsilon} \frac{1}{a_i} \sim O(1), \quad (12)$$

where $\nabla$ is the spatial derivative with respect to the comoving coordinates. At $t \gg t_i$, the contribution of this inhomogeneity to the variation of the energy density within the Hubble scale $H(t)^{-1}$ can be estimated as

$$\left(\frac{\delta\varepsilon}{\varepsilon}\right)_t \sim \frac{1}{\varepsilon} \frac{|\nabla\varepsilon|}{a(t)} H(t)^{-1} \sim O(1) \frac{\dot{a}}{\dot{a}(t)}, \quad (13)$$

where we have assumed that $|\nabla\varepsilon|/\varepsilon$ does not change substantially during expansion. This assumption is supported by the analysis of the behavior of linear perturbations on scales larger than the curvature scale $H^{-1}$ (see chapters 7 and 8). It follows from (13) that if the universe undergoes a stage of acceleration, that is, $\dot{a}(t) > \dot{a}_i$ for $t > t_i$, then the contribution of a large initial inhomogeneity to the energy variation on the curvature scale disappears. A patch of size $H^{-1}$ becomes more and more homogeneous because the initial inhomogeneity is “kicked out”: the physical size of the perturbation, $\propto a$, grows faster than the curvature scale, $H^{-1} = a/\dot{a}$, while the perturbation amplitude does not change substantially. Since inhomogeneities are “devalued” within the curvature scale, the name “inflation” fairly captures the physical effect of accelerated expansion. The consideration above is far from rigorous. However, it gives the flavor of the “no-hair” theorem for an inflationary stage.

To sum up, inflation demolishes large initial inhomogeneities and produces a homogeneous, isotropic domain. It follows from (13) that if we want to avoid the situation of a large initial perturbation reentering the present horizon, $\sim H_0^{-1}$, and inducing a large inhomogeneity, we have to assume that the initial expansion rate was much smaller than the rate of expansion today, that is, $\dot{a}_i/\dot{a}_0 \ll 1$. More precisely, the CMB observations require that the variation of the energy density on the present horizon scale does not exceed $10^{-5}$. The traces of an initial large inhomogeneity will be sufficiently diluted only if $\dot{a}_i/\dot{a}_0 < 10^{-5}$.

Rewriting relation (8) as

$$\Omega_0 = 1 + (\Omega_i - 1) \left(\frac{\dot{a}_i}{\dot{a}_0}\right)^2, \quad (14)$$

we see that if $|\Omega_i - 1| \sim O(1)$ then

$$\Omega_0 = 1 \quad (15)$$

to very high accuracy. This important robust prediction of inflation has a kinematical origin and it states that the total energy density of all components of
matter, irrespective of their origin, must be equal to the critical energy density today. We will see later that amplified quantum fluctuations lead to the tiny corrections to $\Omega_0 = 1$, which are of order $10^{-5}$. It is worth noting that, in contrast to a decelerating universe where $\Omega(t) \to 1$ as $t \to 0$, in an accelerating universe $\Omega(t) \to 1$ as $t \to \infty$, that is, $\Omega = 1$ is its future attractor.

**Problem 3.** Why does the consideration above fail for $\Omega_i = 0$?

## 3 How can gravity become “repulsive”?

To answer this question we recall the Friedmann equation:

$$\ddot{a} = -\frac{4\pi}{3} G(\varepsilon + 3p)a. \quad (16)$$

Obviously, if the strong energy dominance condition, $\varepsilon + 3p > 0$, is satisfied, then $\ddot{a} < 0$ and gravity decelerates the expansion. The universe can undergo a stage of accelerated expansion with $\ddot{a} > 0$ only if this condition is violated, that is, if $\varepsilon + 3p < 0$. One particular example of “matter” with broken energy dominance condition is a positive cosmological constant, for which $p_V = -\varepsilon_V$ and $\varepsilon + 3p = -2\varepsilon_V < 0$. In this case the solution of Einstein’s equations is a de Sitter universe discussed in previous Chapters. For $t \gg H^{-1}_{\Lambda}$, the de Sitter universe expands exponentially quickly, $a \propto \exp(H_{\Lambda}t)$, and the rate of expansion grows as the scale factor. The exact de Sitter solution fails to satisfy all necessary conditions for successful inflation: namely, it does not possess a smooth graceful exit into the Friedmann stage. Therefore, in realistic inflationary models, it can be utilized only as a zero order approximation. To have a graceful exit from inflation we must allow the Hubble parameter to vary in time.

Let us now determine the general conditions which must be satisfied in a successful inflationary model. Because

$$\frac{\ddot{a}}{a} = H^2 + \dot{H}, \quad (17)$$

and $\ddot{a}$ should become negative during a graceful exit, the derivative of the Hubble constant, $\dot{H}$, must obviously be negative. The ratio $|\dot{H}|/H^2$ grows toward the end of inflation and the graceful exit takes place when $|\dot{H}|$ becomes of order $H^2$. Assuming that $H^2$ changes faster than $\dot{H}$, that is, $|\dot{H}| < 2H\dot{H}$, we obtain the following generic estimate for the duration of inflation:

$$t_f \sim H_i/|\dot{H}_i|, \quad (18)$$

where $H_i$ and $\dot{H}_i$ refer to the beginning of inflation. At $t \sim t_f$ the expression on the right hand side in equation (17) changes sign and the universe begins to decelerate.
Inflation should last long enough to stretch a small domain to the scale of the observable universe. Rewriting the condition \( \dot{a_i} \dot{a_f} / \dot{a_f} \dot{a_0} < 10^{-5} \) as

\[
\frac{\dot{a_i} \dot{a_f}}{\dot{a_f} \dot{a_0}} = \frac{a_i H_i \dot{a_f}}{a_f H_f \dot{a_0}} < 10^{-5},
\]

and taking into account that \( \dot{a_f} / \dot{a_0} \) should be larger than \( 10^{28} \), we conclude that inflation is successful only if

\[
\frac{a_f}{a_i} > 10^{33} \frac{H_i}{H_f}.
\]

Let us assume that \( \left| \dot{H}_i \right| \ll H_i^2 \) and neglect the change of the Hubble parameter. Then the ratio of the scale factors can be roughly estimated as

\[
a_f / a_i \sim \exp \left( H_i t_f \right) \sim \exp \left( H_i^2 / \left| \dot{H}_i \right| \right) > 10^{33}.
\]

Hence inflation can solve the initial conditions problem only if \( t_f > 75 H_i^{-1} \), that is, it lasts longer than seventy five Hubble times (e-folds). Rewritten in terms of the initial values of the Hubble parameter and its derivative, this condition takes the form

\[
\left| \dot{H}_i \right| / H_i^2 < \frac{1}{75}.
\]

Using the Friedmann equations, we can reformulate it in terms of the bounds on the initial equation of state

\[
\frac{(\varepsilon + p)_i}{\varepsilon_i} < 10^{-2}.
\]

Thus, at the beginning of inflation the deviation from the vacuum equation of state must not exceed one percent. Therefore an exact de Sitter solution is a very good approximation for the initial stage of inflation. Inflation ends when \( \varepsilon + p \sim \varepsilon \).

**Problem 4.** Consider an exceptional case where \( \left| \dot{H} \right| \) decays at the same rate as \( H^2 \), that is, \( \dot{H} = -pH^2 \), where \( p = \text{const} \). Show that for \( p < 1 \) we have power-law inflation. This inflation has no natural graceful exit and in this sense is similar to a pure de Sitter universe.

### 4 How to realize the equation of state \( p \approx -\varepsilon \)

Thus far we have used the language of ideal hydrodynamics, which is an adequate phenomenological description of matter on large scales. Now we discuss a simple field-theoretic model where the required equation of state can be realized. The natural candidate to drive inflation is a scalar field. The name given to such
a field is the “inflaton.” We saw that the energy-momentum tensor for a scalar field can be rewritten in a form which mimics an ideal fluid. The homogeneous classical field (scalar condensate) is then characterized by energy density
\[ \varepsilon = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \tag{22} \]
and pressure
\[ p = \frac{1}{2} \dot{\varphi}^2 - V(\varphi). \tag{23} \]
We have neglected spatial derivatives here because they become negligible soon after the beginning of inflation due to the “no-hair” theorem.

**Problem 5.** Consider a massive scalar field with potential \( V = \frac{1}{2} m^2 \varphi^2 \), where \( m \ll m_{Pl} \), and determine the bound on the allowed inhomogeneity imposed by the requirement that the energy density must not exceed the Planckian value. Why does the contribution of the spatial gradients to the EMT decay more quickly than the contribution of the mass term?

It follows from equations (22) and (23) that the scalar field has the desired equation of state only if \( \dot{\varphi}^2 \ll V(\varphi) \). Because \( p = -\varepsilon + \dot{\varphi}^2 \), the deviation of the equation of state from that for the vacuum is entirely characterized by the kinetic energy, \( \dot{\varphi}^2 \), which must be much smaller than the potential energy \( V(\varphi) \). Successful realization of inflation thus requires keeping \( \dot{\varphi}^2 \) small compared to \( V(\varphi) \) during a sufficiently long time interval, or more precisely, for at least 75 e-folds. In turn this depends on the shape of the potential \( V(\varphi) \). To determine which potentials can provide us with inflation, we have to study the behavior of a homogeneous classical scalar field in an expanding universe. The equation for this field can be derived either directly from the Klein-Gordon equation or by substituting (22) and (23) into the conservation law \( \dot{\varepsilon} = -3H(\varepsilon + p) \). The result is
\[ \ddot{\varphi} + 3H \dot{\varphi} + V_{,\varphi} = 0, \tag{24} \]
where \( V_{,\varphi} \equiv \partial V / \partial \varphi \). This equation has to be supplemented by the Friedmann equation:
\[ H^2 = \frac{8\pi}{3} \left( \frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right), \tag{25} \]
where we have set \( G = 1 \) and \( k = 0 \). We first find the solutions of equations (24) and (25) for a free massive scalar field and then study the behavior of the scalar field in the case of a general potential \( V(\varphi) \).

**4.1 Simple example:** \( V = \frac{1}{2} m^2 \varphi^2 \).

Substituting \( H \) from (25) into (24), we obtain the closed form equation for \( \varphi \),
\[ \ddot{\varphi} + \sqrt{12\pi} \left( \dot{\varphi}^2 + m^2 \varphi^2 \right)^{1/2} \dot{\varphi} + m^2 \varphi = 0. \tag{26} \]
This is a nonlinear second order differential equation with no explicit time dependence. Therefore it can be reduced to a first order differential equation for \( \dot{\varphi}(\varphi) \). Taking into account that

\[
\ddot{\varphi} = \frac{d\dot{\varphi}}{d\varphi},
\]
equation (26) becomes

\[
\frac{d\dot{\varphi}}{d\varphi} = -\sqrt{\frac{12}{\pi}} \frac{\dot{\varphi}^2 + m^2 \varphi^2}{\dot{\varphi}} \left( \frac{1}{2} \right) + m^2 \varphi, \tag{27}
\]
which can be studied using the phase diagram method. The behavior of the solutions in \( \varphi - \dot{\varphi} \) plane is shown in Figure 3. The important feature of this diagram is the existence of an attractor solution to which all other solutions converge in time. One can distinguish different regions corresponding to different effective equations of state. Let us consider them in more detail. We restrict ourselves to the lower right quadrant (\( \varphi > 0, \dot{\varphi} < 0 \)); solutions in the other quadrants can easily be derived simply by taking into account the symmetry of the diagram.

**Ultra-hard equation of state.** First we study the region where \( |\dot{\varphi}| \gg m\varphi \). It describes the situation when the potential energy is small compared to the kinetic energy, so that \( \dot{\varphi}^2 \gg V \). It follows from equations (22) and (23) that in this case the equation of state is ultra-hard, \( p \approx +\varepsilon \). Neglecting \( m\varphi \) compared to \( \dot{\varphi} \) in equation (27), we obtain

\[
\frac{d\dot{\varphi}}{d\varphi} \approx \sqrt{\frac{12}{\pi}} \dot{\varphi}. \tag{28}
\]
The solution of this equation is

\[
\dot{\varphi} = C \exp \left( \sqrt{\frac{12}{\pi}} \varphi \right), \tag{29}
\]
where \( C < 0 \) is a constant of integration. In turn, solving (29) for \( \varphi(t) \) gives

\[
\varphi = \text{const} - \frac{1}{\sqrt{12\pi}} \ln t. \tag{30}
\]
Substituting this result into equation (25) and neglecting the potential term, we obtain

\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 \simeq \frac{1}{9t^2}. \]  

(31)

It immediately follows that \( a \propto t^{1/3} \) and \( \varepsilon \propto a^{-6} \) in agreement with the ultrahard equation of state. Note that the solution obtained is exact for a massless scalar field. According to (29) the derivative of the scalar field decays exponentially more quickly than the value of the scalar field itself. Therefore, the large initial value of \( |\dot{\varphi}| \) is damped within a short time interval before the field \( \varphi \) itself has changed significantly. The trajectory which begins at large \( |\dot{\varphi}| \) goes up very sharply and meets the attractor. This substantially enlarges the set of initial conditions which lead to an inflationary stage.

**Inflationary solution.** If a trajectory joins the attractor where it is flat, at \( |\varphi| \gg 1 \), then afterwards the solution describes a stage of accelerated expansion (recall that we work in Planckian units). To determine the attractor solution we assume that \( d\dot{\varphi}/d\varphi \approx 0 \) along its trajectory. It follows from (27) that

\[ \dot{\varphi}_{\text{attr}} \approx -\frac{m}{\sqrt{12\pi}}, \]  

(32)

and therefore

\[ \varphi_{\text{attr}}(t) \approx \varphi_i - \frac{m}{\sqrt{12\pi}}(t - t_i) \approx \frac{m}{\sqrt{12\pi}}(t_f - t), \]  

(33)

where \( t_i \) is the time when the trajectory joins the attractor and \( t_f \) is the moment when \( \varphi \) formally vanishes. In reality, solution (33) fails well before the field \( \varphi \) vanishes.

**Problem 6.** Calculate the corrections to the approximate attractor solution (32) and show that

\[ \dot{\varphi}_{\text{attr}} = -\frac{m}{\sqrt{12\pi}} \left( 1 - \frac{1}{2} \left( \sqrt{12\pi}\varphi \right)^{-2} + O \left( \left( \sqrt{12\pi}\varphi \right)^{-3} \right) \right). \]  

(34)

The corrections to (32) become of order the leading term when \( \varphi \sim O(1) \), that is, when the scalar field value drops to the Planckian value or, more precisely, to \( \varphi \approx 1/\sqrt{12\pi} \approx 1/6 \). Hence solution (33) is a good approximation only when the scalar field exceeds the Planckian value. This does not mean, however, that we require a theory of nonperturbative quantum gravity. Nonperturbative quantum gravity effects become relevant only if the curvature or the energy density reach the Planckian values. However, even for very large values of the scalar field they can still remain in the sub-Planckian domain. In fact, considering a massive homogeneous field with negligible kinetic energy we infer that the energy density reaches the Planckian value for \( \varphi \approx m^{-1} \). Therefore, if \( m \ll 1 \), then for \( m^{-1} > \varphi > 1 \) we can safely disregard nonperturbative quantum gravity effects.
According to (33) the scalar field decreases linearly with time after joining the attractor. During the inflationary stage

\[ p \simeq -\varepsilon + m^2/12\pi. \]

So when the potential energy density \( \sim m^2 \varphi^2 \), which dominates the total energy density, drops to \( m^2 \), inflation is over. At this time the scalar field is of order unity (in Planckian units).

Let us determine the time dependence of the scale factor during inflation. Substituting (33) into (25) and neglecting the kinetic term, we obtain a simple equation which is readily integrated to yield

\[ a(t) \simeq a_f \exp \left( -\frac{m^2}{6} (t_f - t)^2 \right) \simeq a_i \exp \left( \frac{(H_i + H(t)}{2} (t - t_i) \right), \quad (35)\]

where \( a_i \) and \( H_i \) are the initial values of the scale factor and the Hubble parameter. Note that the Hubble constant \( H(t) \simeq \sqrt{4\pi/3} m \varphi(t) \) also linearly decreases with time. It follows from (33) that inflation lasts for

\[ \Delta t \simeq t_f - t_i \simeq \sqrt{12\pi} (\varphi_i/m). \quad (36)\]

During this time interval the scale factor increases

\[ \frac{a_f}{a_i} \simeq \exp \left( 2\pi \varphi_i^2 \right) \quad (37)\]

times. The results obtained are in good agreement with the previous rough estimates (18) and (19). Inflation lasts more than 75-e-folds if the initial value of the scalar field, \( \varphi_i \), is four times larger than the Planckian value. To obtain an estimate for the largest possible increase of the scale factor during inflation, let us consider a scalar field of mass \( 10^{13} \text{ GeV} \). The maximal possible value of the scalar field for which we still remain in the sub-Planckian domain is \( \varphi_i \sim 10^6 \), and hence

\[ \left( \frac{a_f}{a_i} \right)_{\text{max}} \sim \exp \left( 10^{12} \right). \quad (38)\]

Thus, the actual duration of the inflationary stage can massively exceed the 75 e-folds needed. In this case our universe would constitute only a very tiny piece of an incredibly large homogeneous domain which originated from one causal region. The other important feature of inflation is that the Hubble constant decreases only by a factor \( 10^{-6} \), while the scale factor grows by the tremendous amount given in (38), that is,

\[ \frac{H_i}{H_f} \ll \ll \frac{a_f}{a_i}. \]

**Graceful exit and afterwards.** After the field drops below the Planckian value it begins to oscillate. To determine the attractor behavior in this regime we note that

\[ \varphi^2 + m^2 \varphi^2 = \frac{3}{4\pi} H^2 \quad (39) \]
and use the Hubble parameter $H$ and the angular variable $\theta$, defined via

$$\dot{\varphi} = \sqrt{\frac{3}{4\pi}} H \sin \theta, \quad m \varphi = \sqrt{\frac{3}{4\pi}} H \cos \theta,$$

(40)
as the new independent variables. It is convenient to replace equation (27) by a system of two first order differential equations for $H$ and $\theta$:

$$\dot{H} = -3H^2 \sin^2 \theta,$$

(41)

$$\dot{\theta} = -m - \frac{3}{2} H \sin 2\theta,$$

(42)

where a dot denotes the derivative with respect to physical time $t$. The second term on the right hand side in (42) describes oscillations with decaying amplitude, as is evident from (41). Therefore, neglecting this term we obtain

$$\theta \simeq -mt + \alpha,$$

(43)

where the constant phase $\alpha$ can be set to zero. Thus, the scalar field oscillates with frequency $\omega \simeq m$. After substituting $\theta \simeq -mt$ into (41), we obtain a readily integrated equation with solution

$$H(t) \equiv \left(\frac{\dot{a}}{a}\right) \simeq \frac{2}{3t} \left(1 - \frac{\sin(2mt)}{2mt}\right)^{-1},$$

(44)

where a constant of integration is removed by a time shift. This solution is applicable only for $mt \gg 1$. Therefore the oscillating term is small compared to unity and the expression on the right hand side in (44) can be expanded in powers of $(mt)^{-1}$. Substituting (43) and (44) into the second equation in (40), we obtain

$$\varphi(t) \simeq \frac{\cos(mt)}{\sqrt{3\pi}mt} \left(1 + \frac{\sin(2mt)}{2mt}\right) + O\left((mt)^{-3}\right).$$

(45)
The time dependence of the scale factor can easily be derived by integrating (44):

$$a(t) \propto t^{2/3} \left(1 - \frac{\cos(2mt)}{6m^2t^2} - \frac{1}{24m^2t^2} + O\left((mt)^{-3}\right)\right).$$

(46)

Thus, in the leading approximation (up to decaying oscillating corrections), the universe expands like a matter-dominated universe with zero pressure. This is not surprising because an oscillating homogeneous field can be thought of as a condensate of massive scalar particles with zero momenta. Although the oscillating corrections are completely negligible in the expressions for $a(t)$ and $H(t)$, they must nevertheless be taken into account when we calculate the curvature invariants. For example, the scalar curvature is

$$R \simeq -\frac{4}{3t^2} \left(1 + 3\cos(2mt) + O\left((mt)^{-1}\right)\right),$$

(47)
(compare to \( R = -4/3t^2 \) in a matter-dominated universe).

We have shown that inflation with a smooth graceful exit occurs naturally in models with classical massive scalar fields. If the mass is small compared to the Planck mass, the inflationary stage lasts long enough and is followed by a cold matter-dominated stage. This cold matter, consisting of heavy scalar particles, must finally be converted to radiation, baryons and leptons. We will see later that this can easily be achieved in a variety of ways.

4.2 General potential: slow-roll approximation

Equation \( (24) \) for a massive scalar field in an expanding universe coincides with the equation for a harmonic oscillator with a friction term proportional to the Hubble parameter \( H \). It is well known that a large friction damps the initial velocities and enforces a slow-roll regime in which the acceleration can be neglected compared to the friction term. Because for a general potential \( H \propto \sqrt{\varepsilon} \sim \sqrt{V} \), we expect that for large values of \( V \) the friction term can also lead to a slow-roll inflationary stage, where \( \ddot{\varphi} \) is negligible compared to \( 3H\dot{\varphi} \). Omitting the \( \ddot{\varphi} \) term and assuming that \( \dot{\varphi}^2 \ll V \), equations \( (24) \) and \( (25) \) simplify to

\[
3H\dot{\varphi} + V_{,\varphi} \simeq 0, \quad H \equiv \left( \frac{d\ln a}{dt} \right) \simeq \sqrt{\frac{8\pi}{3}} V(\varphi).
\]

(48)

Taking into account that

\[
\frac{d\ln a}{dt} = \dot{\varphi} \frac{d\ln a}{d\varphi} \simeq -\frac{V_{,\varphi}}{3H} \frac{d\ln a}{d\varphi},
\]

equations \( (48) \) give

\[
-V_{,\varphi} \frac{d\ln a}{d\varphi} \simeq 8\pi V \quad (49)
\]

and hence

\[
a (\varphi) \simeq a_i \exp \left( 8\pi \int_{\varphi_i}^{\varphi} V_{,\varphi} d\varphi \right).
\]

(50)

This approximate solution is valid only if the slow-roll conditions

\[
|\dot{\varphi}| \ll |V|, \quad |\dot{\varphi}| \ll 3H\dot{\varphi} \sim |V_{,\varphi}|,
\]

(51)

used to simplify equations \( (24) \) and \( (25) \), are satisfied. With the help of equations \( (48) \), they can easily be recast in terms of requirements on the derivatives of the potential itself:

\[
\left( \frac{V_{,\varphi}}{V} \right)^2 \ll 1, \quad \left| \frac{V_{,\varphi}}{V} \right| \ll 1.
\]

(52)

For a power-law potential, \( V = (1/n) \lambda \varphi^n \), both conditions are satisfied for \( |\varphi| \gg 1 \). In this case the scale factor changes as

\[
a (\varphi (t)) \simeq a_i \exp \left( \frac{4\pi}{n} (\varphi_i^2 - \varphi^2 (t)) \right).
\]

(53)

15
It is obvious that the bulk of the inflationary expansion takes place when the scalar field decreases by a factor of a few from its initial value. However, we are interested mainly in the last 50-70 e-folds of inflation because they determine the structure of the universe on present observable scales. The detailed picture of the expansion during these last 70 e-folds depends on the shape of the potential only within a rather narrow interval of scalar field values.

Problem 7. Find the time dependence of the scale factor for the power law potential and estimate the duration of inflation.

Problem 8. Verify that for a general potential \( V \) the system of equations (24), (25) can be reduced to the following first order differential equation:

\[
\frac{dy}{dx} = -3 \left( 1 - y^2 \right) \left( 1 + \frac{V_{,\phi}}{6yV} \right),
\]

where

\[
x \equiv \sqrt{\frac{4\pi}{3}} \varphi; \quad y \equiv \sqrt{\frac{4\pi}{3}} \frac{d\varphi}{d\ln a}.
\]

Assuming that \( V_{,\phi}/V \to 0 \) as \( |\varphi| \to \infty \), draw the phase diagram and analyze the behavior of the solutions in different asymptotic regions. Consider separately the case of the exponential potential. What is the physical meaning of the solutions in the regions corresponding to \( y > 1 \)?

After the end of inflation the scalar field begins to oscillate and the universe enters the stage of deceleration. Assuming that the period of oscillation is smaller than the cosmological time, let us determine the effective equation of state. Neglecting the expansion and multiplying equation (24) by \( \varphi \), we obtain

\[
\left( \dot{\varphi}^2 \right) - \dot{\varphi}^2 + \varphi V_{,\varphi} \simeq 0.
\]

As a result of averaging over a period, the first term vanishes and hence \( \langle \dot{\varphi}^2 \rangle \simeq \langle \varphi V_{,\varphi} \rangle \). Thus, the averaged effective equation of state for an oscillating scalar field is

\[
w \equiv \frac{p}{\epsilon} \simeq \frac{\langle \varphi V_{,\varphi} \rangle - (2V)}{\langle \varphi V_{,\varphi} \rangle + (2V)}.
\]

It follows that for \( V \propto \varphi^n \) we have \( w \simeq (n-2)/(n+2) \). For an oscillating massive field \( n = 2 \) we obtain \( w \simeq 0 \) in agreement with our previous result. In the case of a quartic potential \( n = 4 \), the oscillating field mimics an ultra-relativistic fluid with \( w \simeq 1/3 \).

In fact, inflation can continue even after the end of slow-roll. Considering the potential which behaves as

\[
V \sim \ln \left( |\varphi|/\varphi_c \right)
\]

for \( 1 > |\varphi| \gg \varphi_c \) (see Fig. 4), we infer from (56) that \( w \to -1 \). This is easy to understand. In the case of a convex potential, an oscillating scalar field
spends most of the time near the potential walls where its kinetic energy is negligible and hence the main contribution to the equation of state comes from the potential term.

Problem 9. Which general conditions must a potential $V$ satisfy to provide a stage of fast oscillating inflation? How long can such inflation last and why is it not very helpful for solving the initial conditions problem?

5 Preheating and reheating

The theory of reheating is far from complete. Not only the details, but even the overall picture of inflaton decay depend crucially on the underlying particle physics theory beyond the Standard Model. Because there are so many possible extensions of the Standard Model, it does not make much sense to study the particulars of the reheating processes in each concrete model. Fortunately we are interested only in the final outcome of reheating, namely, in the possibility of obtaining a thermal Friedmann universe. Therefore, to illustrate the physical processes which could play a major role we consider only simple toy models. The relative importance of the different reheating mechanisms cannot be clarified without an underlying particle theory. However, we will show that all of them lead to the desired result.

5.1 Elementary theory

We consider an inflaton field $\varphi$ of mass $m$ coupled to a scalar field $\chi$ and a spinor field $\psi$. Their simplest interactions are described by three leg diagrams (Fig. 5), which correspond to the following terms in the Lagrangian:

$$\Delta L_{\text{int}} = -g\varphi\chi^2 - h\varphi\bar{\psi}\psi.$$  \hfill (57)

We have seen that these kinds of couplings naturally arise in gauge theories with spontaneously broken symmetry, and they are enough for our illustrative
purposes. To avoid a tachyonic instability we assume that $|g\phi|$ is smaller than the squared “bare” mass $m_\chi^2$. The decay rates of the inflaton field into $\chi\chi$- and $\bar{\psi}\psi$-pairs are determined by the coupling constants $g$ and $h$ respectively. They can easily be calculated and the corresponding results are cited in every book on particle physics:

$$
\Gamma_\chi \equiv \Gamma (\phi \to \chi\chi) = \frac{g^2}{8\pi m}, \quad \Gamma_\psi \equiv \Gamma (\phi \to \psi\bar{\psi}) = \frac{h^2 m}{8\pi}.
$$

Let us apply these results in order to calculate the decay rate of the inflaton. As we have noted, an oscillating homogeneous scalar field can be interpreted as a condensate of heavy particles of mass $m$ “at rest,” that is, their 3-momenta $k$ are equal to zero. Keeping only the leading term in (45), we have

$$
\phi (t) \simeq \Phi (t) \cos (mt),
$$

where $\Phi (t)$ is the slowly decaying amplitude of oscillations. The number density of $\varphi$-particles can be estimated as

$$
n_\varphi = \frac{\varepsilon_\varphi}{m} = \frac{1}{2m} (\dot{\varphi}^2 + m^2 \varphi^2) \simeq \frac{1}{2} m \Phi^2.
$$

This number is very large. For example, for $m \sim 10^{13}$ GeV, we have $n_\varphi \sim 10^{92}$ cm$^{-3}$ immediately after the end of inflation, when $\Phi \sim 1$ in Planckian units.

One can show that quantum corrections do not significantly modify the interactions (57) only if $g < m$ and $h < m^{1/2}$. Therefore, for $m \ll m_{Pl}$, the highest decay rate into $\chi$-particles, $\Gamma_\chi \sim m$, is much larger than the highest possible rate for the decay into fermions, $\Gamma_\psi \sim m^2$. If $g \sim m$, then the lifetime of a $\varphi$-particle is about $\Gamma_\chi^{-1} \sim m^{-1}$ and the inflaton decays after a few oscillations. Even if the coupling is not so large, the decay can still be very efficient. The reason is that the effective decay rate into bosons, $\Gamma_{\text{eff}}$, is equal to $\Gamma_\chi$, given in (58), only if the phase space of $\chi$-particles is not densely populated by previously created $\chi$-particles. Otherwise $\Gamma_{\text{eff}}$ can be made much larger by the effect of Bose condensation. This amplification of the inflaton decay is discussed in the next section.

Taking into account the expansion of the universe, the equations for the number densities of $\varphi$- and $\chi$-particles can be written as

$$
\frac{1}{a^3} \frac{d}{dt} (a^3 n_\varphi) = -\Gamma_{\text{eff}} n_\varphi; \quad \frac{1}{a^3} \frac{d}{dt} (a^3 n_\chi) = 2\Gamma_{\text{eff}} n_\varphi,
$$

(61)
where the coefficient two in the second equation arises because one \( \varphi \)–particle decays into two \( \chi \)–particles.

**Problem 10.** Substituting (60) into the first equation in (61), derive the approximate equation

\[
\ddot{\varphi} + (3H + \Gamma_{\text{eff}}) \dot{\varphi} + m^2 \varphi \simeq 0, \tag{62}
\]

which shows that the decay of the inflaton amplitude due to particle production may be roughly taken into account by introducing an extra friction term \( \Gamma_{\text{eff}} \dot{\varphi} \). Why is this equation applicable only during the oscillatory phase?

### 5.2 Narrow resonance

The domain of applicability of elementary reheating theory is limited. Bose condensation effects become important very soon after the beginning of the inflaton decay. Because the inflaton particle is “at rest,” the momenta of the two produced \( \chi \)–particles have the same magnitude \( k \) but opposite directions. If the corresponding states in the phase space of \( \chi \)–particles are already occupied, then the inflaton decay rate is enhanced by a bose factor. The inverse decay process \( \chi \chi \rightarrow \varphi \) can also take place. The rates of these processes are proportional to

\[
|\langle n_{\varphi} - 1, n_{\bf k} + 1, n_{-\bf k} + 1 | \hat{a}_{\bf k}^{\dagger} \hat{a}_{-\bf k} \hat{a}_{-\bf k}^{\dagger} \hat{a}_{\bf k} \mid n_{\varphi}, n_{\bf k}, n_{-\bf k} \rangle|^2 = (n_{\bf k} + 1)(n_{-\bf k} + 1)n_{\varphi}
\]

and

\[
|\langle n_{\varphi} + 1, n_{\bf k} - 1, n_{-\bf k} - 1 | \hat{a}_{\bf k}^{\dagger} \hat{a}_{-\bf k} \hat{a}_{-\bf k}^{\dagger} \hat{a}_{\bf k} \mid n_{\varphi}, n_{\bf k}, n_{-\bf k} \rangle|^2 = n_{\bf k}n_{-\bf k}(n_{\varphi} + 1)
\]

respectively, where \( \hat{a}_{\bf k}^{\dagger} \) are the creation and annihilation operators for \( \chi \)–particles and \( n_{\pm \bf k} \) are their occupation numbers. To avoid confusion the reader must always distinguish the occupation numbers from the number densities keeping in mind that the occupation number refers to a density per cell of volume \((2\pi)^3\) (in the Planckian units) in the phase space, while the number density is the number of particles per unit volume in the three-dimensional space. Taking into account that \( n_{\bf k} = n_{-\bf k} \equiv n_k \) and \( n_{\varphi} \gg 1 \), we infer that the number densities \( n_{\varphi} \) and \( n_{\chi} \) satisfy (61), where

\[
\Gamma_{\text{eff}} \simeq \Gamma_\chi(1 + 2n_k) \tag{63}
\]

Given a number density \( n_{\chi} \), let us calculate \( n_k \). A \( \varphi \)–particle “at rest” decays into two \( \chi \)–particles, both having energy \( m/2 \). Because of the interaction term (57), the effective squared mass of the \( \chi \)–particle depends on the value of the inflaton field and is equal to \( m_{\chi}^2 + 2g \varphi(t) \). Therefore the corresponding 3-momentum of the produced \( \chi \)–particle is given by

\[
k = \left( \frac{m}{2} \right)^2 - m_{\chi}^2 - 2g \varphi(t) \right)^{1/2} \tag{64}
\]
where we assume that $m^2 + 2g\phi \ll m$. The oscillating term,

$$g\phi \simeq g\Phi \cos(mt),$$

leads to a "scattering" of the momenta in phase space. If $g\Phi \ll m^2/8$, then all particles are created within a thin shell of width

$$\Delta k \simeq m \left(\frac{4g\Phi}{m^2}\right) \ll m$$

located near the radius $k_0 \simeq m/2$ (Fig. 6a). Therefore

$$n_{k=m/2} \simeq \frac{n_\chi}{(4\pi k_0^2 \Delta k) / (2\pi)^3} \simeq \frac{2\pi^2 n_\chi}{mg\Phi} = \frac{\pi^2 \Phi n_\chi}{g n_\phi}.$$  \hspace{1cm} (66)

The occupation numbers $n_k$ exceed unity, and hence, the Bose condensation effect is essential only if

$$n_\chi > \frac{g}{\pi^2 \Phi} n_\phi.$$  \hspace{1cm} (67)

Taking into account that at the end of inflation $\Phi \sim 1$, we infer that the occupation numbers begin to exceed unity as soon as the inflaton converts a fraction $g$ of its energy to $\chi$–particles. The derivation above is valid only for $g\Phi \ll m^2/8$. Therefore, if $m \sim 10^{-6}$, then at most a fraction $g \sim m^2 \sim 10^{-12}$ of the inflaton energy can be transferred to $\chi$–particles in the regime where $n_k < 1$. Thus, the elementary theory of reheating, which is applicable for $n_k < 1$, fails almost immediately after the beginning of reheating. Given the result in (66), the effective decay rate (63) becomes

$$\Gamma_{\text{eff}} \simeq \frac{g^2}{8\pi m} \left(1 + \frac{2\pi^2 \Phi n_\chi}{g n_\phi}\right),$$  \hspace{1cm} (68)
where we have used equation (58) for $\Gamma_\chi$. Substituting this expression into the second equation in (61), we obtain

$$\frac{1}{a^3} \frac{d}{dN} \left( a^3 n_\chi \right) = \frac{g^2}{2m^2} \left( 1 + \frac{2\pi^2 \Phi n_\chi}{g n_\varphi} \right) n_\varphi,$$

where $N \equiv mt/2\pi$ is the number of inflaton oscillations. Let us neglect for a moment the expansion of the universe and disregard the decrease of the inflation amplitude due to particle production. In this case $\Phi = \text{const}$ and for $n_k \gg 1$ equation (69) can be easily integrated. The result is

$$n_\chi \propto \exp \left( \frac{\pi^2 g \Phi}{m^2} N \right) \propto \exp \left( 2\pi \mu N \right),$$

where $\mu \equiv \pi g \Phi / \left( 2m^2 \right)$ is the parameter of instability.

**Problem 11.** Derive the following equation for the Fourier modes of the field $\chi$ in Minkowski space:

$$\ddot{\chi}_k + \left( k^2 + m_\chi^2 + 2g\Phi \cos mt \right) \chi_k = 0.$$  

(71)

Reduce it to the well-known Mathieu equation and, assuming that $m^2 \gg m_\chi^2 \geq 2|g\Phi|$, investigate the narrow parametric resonance. Determine the instability bands and the corresponding instability parameters. Compare the width of the first instability band with (69). Where is this band located? The minimal value of the initial amplitude of $\chi_k$ is due to vacuum fluctuations. The increase of $\chi_k$ with time can be interpreted as the production of $\chi$-particles by the external classical field $\varphi$, with $n_\chi \propto |\chi_k|^2$. Show that in the center of the first instability band,

$$n_\chi \propto \exp \left( \frac{4\pi g \Phi}{m^2} N \right),$$

(72)

where $N$ is the number of oscillations. Compare this result with (70) and explain why they are different by a numerical factor in the exponent. Thus, Bose condensation can be interpreted as a narrow parametric resonance in the first instability band, and vice versa. Give a physical interpretation of the higher order resonance bands in terms of particle production.

Using the results of this problem we can reduce the investigation of the inflaton decay due to the coupling

$$\Delta L_{\text{int}} = -\frac{1}{2} \tilde{g}^2 \varphi^2 \chi^2,$$

(73)

to the case studied above. In fact, the equation for a massless scalar field $\chi$, coupled to the inflaton $\varphi = \Phi \cos mt$, takes the form

$$\ddot{\chi}_k + \left( k^2 + g^2 \Phi^2 \cos^2 mt \right) \chi_k = 0,$$

(74)

21
which coincides with equation (71) for \( m_\chi^2 = 2g\Phi \) after substitutions \( \tilde{g}^2\Phi^2 \to 4g\Phi \) and \( m \to m/2 \). Thus, the two problems are mathematically equivalent. Using this observation and making the corresponding replacements in formula (72), we immediately find that

\[
n_\chi \propto \exp \left( \frac{\pi \tilde{g}^2\Phi^2}{4m^2} N \right). \tag{75}
\]

The condition for narrow resonance is \( \tilde{g}\Phi \ll m \) and the width of the first resonance band can be estimated from (65) as \( \Delta k \sim m \left( \tilde{g}^2\Phi^2/m^2 \right) \).

In summary, we have shown that even for a small coupling constant the elementary theory of reheating must be modified to take into account the Bose condensation effect, and that this can lead to an exponential increase of the reheating efficiency.

**Problem 12.** Taking a few concrete values for \( g \) and \( m \), compare the results of the elementary theory with those obtained for narrow parametric resonance.

So far we have neglected the expansion of the universe, the backreaction of the produced particles and their rescatterings. All these effects work to suppress the efficiency of the narrow parametric resonance. The expansion shifts the momenta of the previously created particles and takes them out of the resonance layer (Fig. 6a). Thus, the occupation numbers relevant for Bose condensation are actually smaller than what one would expect according to the naive estimate (66). If the rate of supply of newly created particles in the resonance layer is smaller than the rate of their escape, then \( n_k < 1 \) and we can use the elementary theory of reheating. The other important effect is the decrease of the amplitude \( \Phi(t) \) due to both the expansion of the universe and particle production. Because the width of the resonance layer is proportional to \( \Phi \), it becomes more and more narrow. As a result the particles can escape from this layer more easily and they do not stimulate the subsequent production of particles. The rescattering of the \( \chi^- \) particles also suppresses the resonance efficiency by removing particles from the resonance layer. Another effect is the change of the effective inflaton mass due to the newly produced \( \chi^- \) particles; this shifts the center of the resonance layer from its original location.

To conclude, narrow parametric resonance is very sensitive to the interplay of different complicating factors. It can be fully investigated only using numerical methods. From our analytical consideration we can only say that the inflaton field probably decays not as “slowly” as in the elementary theory, but not as “fast” as in the case of pure narrow parametric resonance.

### 5.3 Broad resonance

So far we have considered only the case of a small coupling constants. Quantum corrections to the Lagrangian are not very crucial if \( g < m \) and \( \tilde{g} < (m/\Phi)^{1/2} \). They can therefore be ignored when we consider inflaton decay in the strong
coupling regime: \( m > g > m^2/\Phi \) for the three-leg interaction and \( (m/\Phi)^{1/2} > \tilde{g} > m/\Phi \) for the quartic interaction (73). In this case the condition for narrow resonance is not fulfilled and we cannot use the methods above. Perturbative methods fail because the higher order diagrams, built from the elementary diagrams, give comparable contributions. Particle production can be treated only as a collective effect in which many inflaton particles participate simultaneously. We have to apply the methods of quantum field theory in an external classical background — as in Problem 1.11.

Let us consider quartic interaction (73). First, we neglect the expansion of the universe. For \( \tilde{g}\Phi \gg m \) the mode equation (see (74)):

\[
\ddot{\chi}_k + \omega^2(t) \chi_k = 0,
\]

where

\[
\omega(t) \equiv (k^2 + \tilde{g}^2 \Phi^2 \cos^2 mt)^{1/2},
\]

describes a broad parametric resonance. If the frequency \( \omega(t) \) is a slowly varying function of time or, more precisely, \( |\dot{\omega}| \ll \omega^2 \), equation (76) can be solved in the quasiclassical (WKB) approximation:

\[
\chi_k \propto \frac{1}{\sqrt{\omega}} \exp \left( \pm i \int \omega dt \right).
\]

In this case the number of particles, \( n_\chi \sim \varepsilon /\omega \), is an adiabatic invariant and is conserved. For most of the time the condition \( |\dot{\omega}| \ll \omega^2 \) is indeed fulfilled. However, every time the oscillating inflaton vanishes at \( t_j = m^{-1} (j + 1/2) \pi \), the effective mass of the \( \chi \) field, proportional to \( |\cos (mt)| \), vanishes. It is shortly before and after \( t_j \) that the adiabatic condition is strongly violated:

\[
\frac{|\dot{\omega}|}{\omega^2} = \frac{mg^2 \Phi^2 |\cos (mt)\sin (mt)|}{(k^2 + g^2 \Phi^2 \cos^2 (mt))^{3/2}} \geq 1.
\]

Considering a small time interval \( \Delta t \ll m^{-1} \) in the vicinity of \( t_j \), we can rewrite this condition as

\[
\frac{\Delta t / \Delta t_*}{(k^2 \Delta t_*^2 + (\Delta t / \Delta t_*)^2)^{3/2}} \geq 1,
\]

where

\[
\Delta t_* \simeq (\tilde{g}\Phi m)^{-1/2} = \frac{1}{m} (\tilde{g}\Phi / m)^{-1/2}.
\]

It follows that the adiabatic condition is broken only within short time intervals \( \Delta t \sim \Delta t_* \) near \( t_j \) and only for modes with

\[
k < k_* \simeq \Delta t_*^{-1} \approx m (\tilde{g}\Phi / m)^{1/2}.
\]

Therefore, we expect that \( \chi \)–particles with the corresponding momenta are created only during these time intervals. It is worth noting that the momentum of the created particle can be larger than the inflaton mass by the ratio
$$(\hat{g}\Phi/m^2)^{1/2} > 1$$; the $\chi-$particles are produced as a result of a collective process involving many inflaton particles. This is the reason why we cannot describe the broad resonance regime using the usual methods of perturbation theory.

To calculate the number of particles produced in a single inflaton oscillation we consider a short time interval in the vicinity of $t_j$ and approximate the cosine in (76) by a linear function. Equation (76) then takes the form

$$\frac{d^2\chi_\kappa}{d\tau^2} + (\kappa^2 + \tau^2) \chi_\kappa = 0,$$

where the dimensionless wavenumber $\kappa \equiv k/k^*$ and time $\tau \equiv (t - t_j)/\Delta t^*$ have been introduced. In terms of the new variables the adiabaticity condition is broken at $|\tau| < 1$ and only for $\kappa < 1$. It is remarkable that the coupling constant $\tilde{g}$, the mass and the amplitude of the inflaton enter explicitly only in $\kappa^2$. The adiabaticity violation is largest for $k = 0$. In this case the parameters $\tilde{g}, \Phi$ and $m$ drop from equation (83) and the amplitude $\chi_{\kappa=0}$ changes only by a numerical, parameter-independent factor as a result of passing through the nonadiabatic region at $|\tau| < 1$. Because the particle density $n$ is proportional to $|\chi|^2$, its growth from one oscillation to the next can written as

$$\left(\frac{n^{j+1}}{n^j}\right)_{k=0} = \exp (2\pi\mu_{k=0}),$$

where the instability parameter $\mu_{k=0}$ does not depend on $\tilde{g}, \Phi$ and $m$. For modes with $k \neq 0$, the parameter $\mu_{k\neq 0}$ is a function of $\kappa = k/k^*$. In this case the adiabaticity is not violated as strongly as for the $k = 0$ mode, and hence $\mu_{k\neq 0}$ is smaller than $\mu_{k=0}$. To calculate the instability parameters we have to determine the change of the amplitude $\chi$ in passing from the $\tau < -1$ region to the $\tau > 1$ region. This can be done using two independent WKB solutions of equation (83) in the asymptotic regions $|\tau| \gg 1$:

$$\chi_{\pm} = \frac{1}{(\kappa^2 + \tau^2)^{1/4}} \exp \left( \pm i \int \sqrt{\kappa^2 + \tau^2} d\tau \right) \approx |\tau|^{-\frac{1}{2} + \frac{1}{2}i\kappa^2} \exp \left( \pm i\frac{\tau^2}{2} \right). \quad (85)$$

After passing through the nonadiabatic region the mode $A_{+}\chi_{+}$ becomes a mixture of the modes $\chi_{+}$ and $\chi_{-}$, that is,

$$A_{+}\chi_{+} \rightarrow B_{+}\chi_{+} + C_{+}\chi_{-}, \quad (86)$$

where $A_{+}, B_{+}$ and $C_{+}$ are the complex constant coefficients. Similarly, for the mode $A_{-}\chi_{-}$, we have

$$A_{-}\chi_{-} \rightarrow B_{-}\chi_{-} + C_{-}\chi_{+}. \quad (87)$$

Drawing an analogy with the scattering problem for the inverse parabolic potential, we note that the mixture arises due to an overbarrier reflection of the wave. The reflection is most efficient for the waves with $k = 0$ which “touch” the top of the barrier.
The quasiclassical solution is valid in the complex plane for $|\tau| \gg 1$. Traversing the appropriate contour $\tau = |\tau| e^{i\varphi}$ in the complex plane from $\tau \ll -1$ to $\tau \gg 1$, we infer from (85), (86) and (87) that

$$B_{\pm} = \mp i e^{-\frac{\pi}{2} \kappa^2} A_{\pm}.$$  \hfill (88)

The coefficients $C_{\pm}$ are not determined in this method. To find them we use the Wronskian $W \equiv \dot{\chi}\chi^* - \chi\dot{\chi}^*$,  \hfill (89)

where $\chi$ is an arbitrary complex solution of equation (83). Taking the derivative of $W$ and using (83) to express $\ddot{\chi}$ in terms of $\dot{\chi}$, we find

$$\dot{W} = 0$$ \hfill (90)

and hence $W = \text{const}$. From this we infer that the coefficients $A, B$ and $C$ in (80) and (87) satisfy the “probability conservation” condition

$$|C_{\pm}|^2 - |B_{\pm}|^2 = |A_{\pm}|^2.$$ \hfill (91)

Substituting $B$ from (88), we obtain

$$C_{\pm} = \sqrt{1 + e^{-\pi \kappa^2}} |A_{\pm}| e^{i\alpha_{\pm}},$$ \hfill (92)

where the phases $\alpha_{\pm}$ remain undetermined.

At $|\tau| \gg 1$ the modes of field $\chi$ satisfy the harmonic oscillator equation with a slowly changing frequency $\omega \propto |\tau|$. In quantum field theory the occupation number $n_k$ in the expression for the energy of the harmonic oscillator,

$$\varepsilon_k = \omega (n_k + 1/2),$$ \hfill (93)

is interpreted as the number of particles in the corresponding mode $k$. In the adiabatic regime ($|\tau| \gg 1$) this number is conserved and it changes only when the adiabatic condition is violated. Let us consider an arbitrary initial mixture of the modes $\chi_+$ and $\chi_-$. After passing through the nonadiabatic region at $t \sim t_j$, it changes as

$$\chi^j = A_+ \chi_+ + A_- \chi_- \to \chi^{j+1} = (B_+ + C_-) \chi_+ + (B_- + C_+) \chi_-.$$ \hfill (94)

Taking into account that

$$n + \frac{1}{2} = \frac{\varepsilon}{\omega} \sim \omega |\chi|^2,$$ \hfill (95)

we see that as a result of this passage the number of particles in the mode $k$ increases

$$\left( \frac{n^{j+1} + 1/2}{n^j + 1/2} \right)_k \sim \frac{\omega |\chi^{j+1}|^2}{\omega |\chi^j|^2} \sim \frac{|B_+ + C_-|^2 + |B_- + C_+|^2}{|A_+|^2 + |A_-|^2},$$ \hfill (96)
times, where we have averaged $|\chi|^2$ over the time interval $m^{-1} > t > \omega^{-1}$. With $B$ and $C$ from (88) and (92), this expression becomes

$$\left(\frac{n^{j+1} + 1/2}{n^j + 1/2}\right)_k \simeq \left(1 + 2e^{-\pi\kappa^2}\right) + \frac{4|A_-| |A_+|}{|A_+|^2 + |A_-|^2} \cos \theta e^{-\frac{\pi\kappa^2}{2}} \sqrt{1 + e^{-\pi\kappa^2}}. \quad (97)$$

**Problem 13.** Verify the result (97) and explain the origin of the phase $\theta$.

In the vacuum initial state $n_k = 0$ but the amplitude of the field $\chi$ does not vanish because of the existence of vacuum fluctuations; we have $|A^0_+|^2 \neq 0$ and $|A^0_-|^2 = 0$. It follows from the “probability conservation” condition that

$$|A_+|^2 - |A_-|^2 = |A^0_+|^2 \quad (98)$$

at every moment of time. This means that as a result of particle production the coefficients $|A_+|^2$ and $|A_-|^2$ grow by the same amount. When $|A_+|$ becomes much larger than $|A^0_+|$ we have $|A_+| \simeq |A_-|$. Taking this into account and beginning in the vacuum state, we find from (97) that after $N \gg 1$ inflaton oscillations the particle number in mode $k$ is

$$n_k \simeq \frac{1}{2} \exp (2\pi \mu_k N), \quad (99)$$

where the instability parameter is given by

$$\mu_k \simeq \frac{1}{2\pi} \ln \left(1 + 2e^{-\pi\kappa^2} + 2 \cos \theta e^{-\frac{\pi\kappa^2}{2}} \sqrt{1 + e^{-\pi\kappa^2}}\right). \quad (100)$$

This parameter takes its maximal value

$$\mu^{\text{max}}_k = \pi^{-1} \ln \left(1 + \sqrt{2}\right) \simeq 0.28$$

for $k = 0$ and $\theta = 0$. In the interval $-\pi < \theta < \pi$ we find that $\mu_{k=0}$ is positive if $3\pi/4 > \theta > -3\pi/4$ and negative otherwise. Thus, assuming random $\theta$, we conclude that the particle number in every mode changes stochastically. However, if all $\theta$ are equally probable, then the number of particles increases three quarters of the time and therefore it also increases on average, in agreement with entropic arguments. The net instability parameter, characterizing the average growth in particle number, is obtained by skipping the $\cos \theta$-term in (100):

$$\bar{\mu}_k \simeq \frac{1}{2\pi} \ln \left(1 + 2e^{-\pi\kappa^2}\right). \quad (101)$$

With slight modifications the results above can be applied to an expanding universe. First of all we note that the expansion randomizes the phases $\theta$ and hence the effective instability parameter is given by (101). For particles with physical momenta $k < k_*/\sqrt{\pi}$, the instability parameter $\bar{\mu}_k$ can be roughly estimated by its value at the center of the instability region, $\bar{\mu}_{k=0} = \ln 3/2\pi \simeq 0.175$. To understand how the expansion can influence the efficiency of broad
resonance, it is again helpful to use the phase space picture. The particles created in the broad resonance regime occupy the entire sphere of radius $k*/\sqrt{\pi}$ in phase space (see Fig. 6b). During the passage through the nonadiabatic region the number of particles in every cell of the sphere, and hence the total number density, increases on average $\exp{(2\pi \cdot 0.175)} \simeq 3$ times. At the stage when inflaton energy is still dominant, the physical momentum of the created particle decreases in inverse proportion to the scale factor ($k \propto a^{-1}$), while the radius of the sphere shrinks more slowly, namely, as $\Phi^{1/2} \propto t^{-1/2} \propto a^{-3/4}$. As a result, the created particles move away from the boundary of the sphere towards its center where they participate in the next “act of creation,” enhancing the probability by a bose factor. Furthermore, expansion also makes broad resonance less sensitive to rescattering and backreaction effects. These two effects influence the resonance efficiency by removing those particles which are located near the boundary of the resonance sphere. Because expansion moves particles away from this region, the impact of these effects is diminished. Thus, in contrast to the narrow resonance case, expansion stabilizes broad resonance and at the beginning of reheating it can be realized in its pure form.

Taking into account that the initial volume of the resonance sphere is about

$$k^3_s \simeq m^3 (\tilde{g}\Phi_0/m)^{3/2},$$

we obtain the following estimate for the ratio of the particle number densities after $N$ inflaton oscillations:

$$\frac{n_\chi}{n_\varphi} \sim \frac{k^3_s \exp{(2\pi \hat{\mu}_k=0N)}}{m\Phi_0^2} \sim m^{1/2} \tilde{g}^{3/2} \cdot 3^N,$$  \hspace{1cm} (102)

where $\Phi_0 \sim O(1)$ is the value of the inflaton amplitude after the end of inflation. Since in the adiabatic regime the effective mass of the $\chi$-particles is of order $\tilde{g}\Phi$, where $\Phi$ decreases in inverse proportion to $N$, we also obtain an estimate for the ratio of the energy densities:

$$\frac{\varepsilon_\chi}{\varepsilon_\varphi} \sim \frac{m_\chi n_\chi}{m n_\varphi} \sim m^{-1/2} \tilde{g}^{5/2} N^{-1} 3^N.$$  \hspace{1cm} (103)

The formulae above fail when the energy density of the created particles begins to exceed the energy density stored in the inflaton field. In fact, at this time, the amplitude $\Phi(t)$ begins to decrease very quickly because of the very efficient energy transfer from the inflaton to the $\chi$-particles. Broad resonance is certainly over when $\Phi(t)$ drops to the value $\Phi_r \sim m/\tilde{g}$, and we enter the narrow resonance regime. For the coupling constant $m^{1/2} \tilde{g} > O(1)$ $m$, the number of the inflaton oscillation $N_r$ in the broad resonance regime can be roughly estimated using the condition $\varepsilon_\chi \sim \varepsilon_\varphi$:

$$N_r \sim (0.75 \text{ to } 2) \log_3 m^{-1}.$$  \hspace{1cm} (104)

As an example, if $m \simeq 10^{13}$ GeV, we have $N_r \simeq 10$ to 25 for a wide range of the coupling constants $10^{-3} > \tilde{g} > 10^{-6}$. Taking into account that the total energy
decays as $m^2 (\Phi_0/N)^2$, we obtain

$$\frac{\varepsilon_\varphi}{\varepsilon_\chi + \varepsilon_\varphi} \sim \frac{m^2 \Phi^2}{m^2 (\Phi_0/N)^2} \sim N^2 \left( \frac{m}{\tilde{g} \Phi_0} \right)^2,$$

(105)

that is, the energy still stored in the inflaton field at the end of broad resonance is only a small fraction of the total energy. In particular, for $m \approx 10^{13}$ GeV, this ratio varies in the range $10^{-6}$ to $O(1)$ depending on the coupling constant $\tilde{g}$.

**Problem 14.** Investigate inflaton decay due to the three-leg interaction $g^2 \varphi \chi^2$ in the strong coupling regime: $m > g > m^2/\Phi$.

### 5.4 Implications

It follows from the above considerations that broad parametric resonance can play a very important role in the preheating phase. During only 15 to 25 oscillations of the inflaton, it can convert most of the inflaton energy into other scalar particles. The most interesting aspect of this process is that the effective mass and the momenta of the particles produced can exceed the inflaton mass. For example, for $m \approx 10^{14}$ GeV, the effective mass $m_{\chi}^{\text{eff}} = \tilde{g} \Phi |\cos (m t)|$ can be as large as $10^{16}$ GeV. Therefore, if the $\chi$—particles are coupled to bosonic and fermionic fields heavier than the inflaton, then the inflaton may indirectly decay into these heavy particles. This brings Grand Unification scales back into play. For instance, even if inflation ends at low energy scales, preheating may rescue the GUT baryogenesis models. Another potential outcome of the above mechanism is the far-from-equilibrium production of topological defects after inflation. Obviously their numbers must not conflict with observations and this leads to cosmological bounds on admissible theories.

If, after the period of broad resonance, the slightest amount of the inflaton remained — given by (105) — it would be a cosmological disaster. Since the inflaton particles are nonrelativistic, if they were present in any substantial amount, they would soon dominate and leave us with a cold universe. Fortunately, these particles should easily decay in the subsequent narrow resonance regime or as a result of elementary particle decay. These decay channels thus become necessary ingredients of the reheating theory.

The considerations of this section do not constitute a complete theory of reheating. We have studied only elementary processes which could play a role in producing a hot Friedmann universe. The final outcome of reheating must be matter in thermal equilibrium. The particles which are produced in the preheating processes are initially in a highly non-equilibrium state. Numerical calculations show that as a result of their scatterings they quickly reach local thermal equilibrium. Parametrizing the total preheating and reheating time in terms of the inflaton oscillations number $N_T$, we obtain the following estimate for the reheating temperature:

$$T_R \sim \frac{m^{1/2}}{N_T^{1/2} N^{1/4}},$$

(106)
where \( N \) is the effective number degrees of freedom of the light fields at \( T \sim T_R \). Assuming that \( N_T \sim 10^6 \), and taking \( N \sim 10^2 \) and \( m \sim 10^{13} \) GeV, we obtain \( T_R \sim 10^{12} \) GeV. This does not mean, however, that we can ignore physics beyond this scale. As we have already pointed out, non-equilibrium preheating processes can play a nontrivial role.

Reheating is an important ingredient of inflationary cosmology. We have seen that there is no general obstacle to arranging successful reheating. A particle theory should be tested on its ability to realize reheating in combination with baryogenesis. In this way, cosmology enables us to preselect realistic particle physics theories beyond the Standard Model.

6 “Menu” of scenarios

All we need for successful inflation is a scalar condensate satisfying the slow-roll conditions. Building concrete scenarios then becomes a “technical” problem. Involving two or more scalar condensates, and assuming them to be equally relevant during inflation, extends the number of possibilities, but simultaneously diminishes the predictive power of inflation. This especially concerns cosmological perturbations, which are among the most important robust predictions of inflation. Because inflation can be falsified experimentally (or more accurately, observationally) only if it makes such predictions, we consider only simple scenarios with a single inflaton component. Fortunately all of them lead to very similar predictions which differ only slightly in the details. This makes the significance of a unique scenario, the one actually realized in nature, less important.

The situation here is very different from particle physics, where the concrete models are as important as the ideas behind them. This does not mean we do not need the correct scenario; if one day it becomes available, we will be able to verify more delicate predictions of inflation. However, even in the absence of the true scenario, we can nonetheless verify observationally the most important predictions of the stage of cosmic acceleration. The purpose of this section is to give the reader a very brief guide to the “menu of scenarios” discussed in the literature.

Inflaton candidates. The first question which naturally arises is “what is the most realistic candidate for the inflaton field?”. There are many because the only requirement is that this candidate imitates a scalar condensate in the slow-roll regime. This can be achieved by a fundamental scalar field or by a fermionic condensate described in terms of an effective scalar field. This, however, does not exhaust all possibilities. The scalar condensate can also be imitated entirely within the theory of gravity itself. Einstein gravity is only a low curvature limit of some more complicated theory whose action contains higher powers of the curvature invariants, for example,

\[
S = -\frac{1}{16\pi} \int \left( R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R^3 + \ldots \right) \sqrt{-g} dx. \tag{107}
\]

The quadratic and higher order terms can be either of fundamental origin or they
can arise as a result of vacuum polarization. The corresponding dimensional coefficients in front of these terms are likely of Planckian size. The theory with action (107) can provide us with inflation. This can easily be understood. Einstein gravity is the only metric theory in four dimensions where the equations of motion are second order. Any modification of the Einstein action introduces higher derivative terms. This means that, in addition to the gravitational waves, the gravitational field has extra degrees of freedom including, generically, a spin zero field.

**Problem 15.** Consider a gravity theory with metric $g_{\mu\nu}$ and action

$$S = \frac{1}{16\pi} \int f(R) \sqrt{-g} dx,$$

where $f(R)$ is an arbitrary function of the scalar curvature $R$. Derive the following equations of motion:

$$\frac{\partial f}{\partial R} R^\mu_\nu - \frac{1}{2} \delta^\mu_\nu f + \left( \frac{\partial f}{\partial R} \right) ^\alpha_\mu \delta^\mu_\nu - \left( \frac{\partial f}{\partial R} \right) ^\nu_\alpha = 0.$$  \hspace{1cm} (109)

Verify that under the conformal transformation $g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = F g_{\mu\nu}$, the Ricci tensor and the scalar curvature transform as

$$R^\mu_\nu \rightarrow \tilde{R}^\mu_\nu = F^{-1} R^\mu_\nu - F^{-2} F^\gamma_\mu F^\nu_\gamma \delta^\mu_\nu + \frac{3}{2} F^{-3} F^\gamma_\mu F^\gamma_\nu,$$

$$R \rightarrow \tilde{R} = F^{-1} R - 3 F^{-2} F^\gamma_\alpha F^\gamma_\mu + \frac{3}{2} F^{-3} F^\gamma_\alpha F^\gamma_\mu.$$  \hspace{1cm} (110)

Introduce the “scalar field”

$$\varphi \equiv \sqrt{\frac{3}{16\pi}} \ln F(R),$$

and show that the equations

$$\tilde{R}^\mu_\nu - \frac{1}{2} \tilde{R} \delta^\mu_\nu = 8\pi \tilde{T}^\mu_\nu (\varphi),$$  \hspace{1cm} (113)

coincide with (109) if we set $F = \partial f / \partial R$ and take the following potential for the scalar field:

$$V(\varphi) = \frac{1}{16\pi} \frac{f - R \partial f / \partial R}{(\partial f / \partial R)^2}.$$  \hspace{1cm} (114)

**Problem 16.** Study the inflationary solutions in $R^2$-gravity:

$$S = -\frac{1}{16\pi} \int \left( R - \frac{1}{6M^2} R^2 \right) \sqrt{-g} dx.$$  \hspace{1cm} (115)

What is the physical meaning of the constant $M$?
Thus, the higher derivative gravity theory is conformally equivalent to Einstein gravity with an extra scalar field. If the scalar field potential satisfies the slow-roll conditions, then we have an inflationary solution in the conformal frame for the metric $\tilde{g}_{\mu\nu}$. One should not confuse, however, the conformal metric with the original physical metric. They generally describe manifolds with different geometries and the final results must be interpreted in terms of the original metric. In our case the use of the conformal transformation is a mathematical tool which simply allows us to reduce the problem to one we have studied before. The conformal metric is related to the physical metric $g_{\mu\nu}$ by a factor $F$, which depends on the curvature invariants; it does not change significantly during inflation. Therefore, we also have an inflationary solution in the original physical frame.

So far we have been considering inflationary solutions due to the potential of the scalar field. However, inflation can be realized even without a potential term. It can occur in Born-Infeld type theories, where the action depends nonlinearly on the kinetic energy of the scalar field. These theories do not have higher derivative terms, but they have some other peculiar properties.

**Problem 17.** Consider a scalar field with action

$$S = \int p(X, \varphi) \sqrt{-g} dx,$$

where $p$ is an arbitrary function of $\varphi$ and $X \equiv \frac{1}{2} (\partial_{\mu} \varphi \partial^{\mu} \varphi)$. Verify that the energy-momentum tensor for this field can be written in the form

$$T_{\nu}^{\mu} = (\varepsilon + p) u^{\mu} u_{\nu} - p \delta^{\mu}_{\nu},$$

where the Lagrangian $p$ plays the role of the effective pressure and

$$\varepsilon = 2X \frac{\partial p}{\partial X} - p, \quad u_{\nu} = \frac{\partial \varphi}{\sqrt{2X}}.$$

If the Lagrangian $p$ satisfies the condition $X \partial p / \partial X \ll p$ for some range of $X$ and $\varphi$, then the equation of state is $p \approx -\varepsilon$ and we have an inflationary solution. Why is inflation not satisfactory if $p$ depends only on $X$? Consider a general $p(X, \varphi)$ without an explicit potential term, that is, $p \to 0$ when $X \to 0$. Formulate the conditions which this function must satisfy to provide us with a slow-roll inflationary stage and a graceful exit. The inflationary scenario based on the nontrivial dependence of the Lagrangian on the kinetic term is called $k-$inflation.

**Scenarios.** The simplest inflationary scenarios can be subdivided into three classes. They correspond to the usual scalar field with a potential, higher derivative gravity, and $k-$inflation. The cosmological consequences of scenarios from the different classes are almost indistinguishable -- they can exactly imitate each other. Within each class, however, we can try to make further distinctions.
by addressing the questions: a) what was before inflation? and b) how does a graceful exit to a Friedmann stage occur? For our purpose it will be sufficient to consider only the simplest case of a scalar field with canonical kinetic energy. The potential can have different shapes, as shown in Fig. 7. The three cases presented correspond to the so-called old, new and chaotic inflationary scenarios. The first two names refer to their historical origins (see “Bibliography”).

**Old inflation** (see Fig. 7a) assumes that the scalar field arrives at the local minimum of the potential at \( \varphi = 0 \) as a result of a supercooling of the initially hot universe. After that the universe undergoes a stage of accelerated expansion with subsequent graceful exit via bubble nucleation. It was clear from the very beginning that this scenario could not provide a successful graceful exit because all the energy released in a bubble is concentrated in its wall and the bubbles have no chance to collide. This difficulty was avoided in the new inflationary scenario, a scenario similar to a successful model in higher derivative gravity which had previously been invented.

**New inflation** is based on a Coleman-Weinberg type potential (Fig. 7b). Because the potential is very flat and has a maximum at \( \varphi = 0 \), the scalar field escapes from the maximum not via tunneling, but due to the quantum fluctuations. It then slowly rolls towards the global minimum where the energy is released homogeneously in the whole space. Originally the pre-inflationary state of the universe was taken to be thermal so that the symmetry was restored due to thermal corrections. This was a justification for the initial conditions of the scalar field. Later on it was realized that the thermal initial state of the universe is quite unlikely, and so now the original motivation for the initial conditions in the new inflationary model seems to be false. Instead, the universe might be in a “self-reproducing” regime (for more details see section 8.5).

**Chaotic inflation** gives its name to the broadest possible class of potentials satisfying the slow roll-conditions (Fig. 7c). We have considered it in detail in the previous sections. The name chaotic is related to the possibility of having almost arbitrary initial conditions for the scalar field. To be precise, this field must initially be larger than the Planckian value but it is otherwise arbitrary. Indeed, it could have varied from one spatial region to another and, as a result,
the universe would have a very complicated global structure. It could be very
inhomogeneous on scales much larger than the present horizon and extremely
homogeneous on “small” scales corresponding to the observable domain. We
will see later that in the case of chaotic inflation, quantum fluctuations lead to
a self-reproducing universe.

Since chaotic inflation encompasses so many potentials, one might think it
worthwhile to consider special cases, for example, an exponential potential. For
an exponential potential, if the slow-roll conditions are satisfied once, they are
always satisfied. Therefore, it describes (power-law) inflation without a graceful
exit. To arrange a graceful exit we have to “damage” the potential. For two or
more scalar fields the number of options increases. Thus it is not helpful here
to go into the details of the different models.

In the absence of the underlying fundamental particle theory, one is free to
play with the potentials and invent more new scenarios. In this sense the situ-
ation has changed since the time the importance of inflation was first realized.
In fact, in the 80s many people considered inflation a useful application of the
Grand Unified Theory that was believed to be known. Besides solving the initial
conditions problem, inflation also explained why we do not have an overabun-
dance of the monopoles that are an inevitable consequence of a GUT. Either
inflation ejects all previously created monopoles, leaving less than one monopole
per present horizon volume, or the monopoles are never produced. The same
argument applies to the heavy stable particles that could be overproduced in the
state of thermal equilibrium at high temperatures. Many authors consider the
solution of the monopole and heavy particle problems to be as important as a
solution of the initial conditions problem. We would like to point out, however,
that the initial conditions problem is posed to us by nature, while the other
problems are, at present, not more than internal problems of theories beyond
the Standard Model. By solving these extra problems, inflation opens the door
to theories that would otherwise be prohibited by cosmology. Depending on
one’s attitude, this is either a useful or damning achievement of inflation.

De Sitter solution and inflation. The last point we would like to make
concerns the role of a cosmological constant and a pure de Sitter solution for
inflation. We have already said that the pure de Sitter solution cannot provide
us with a model with a graceful exit. Even the notion of expansion is not
unambiguously defined in de Sitter space. We saw in section 1.3.6 that this space
has the same symmetry group as Minkowski space. It is spatially homogeneous
and time translation invariant. Therefore any space-like surface is a hypersurface
of constant energy. To characterize an expansion we can use not only \( k = 0, \pm 1 \)
Friedmann coordinates but also, for example, “static coordinates” (see
Problem 2.7), which describe an expanding space outside the event horizon.
In all these cases the three-geometries of constant time hypersurfaces are very
different. These differences, however, simply characterize the different slicings
of the perfectly symmetrical space and there is no obvious preferable choice for
the coordinates.

It is important therefore that inflation is never realized by a pure de Sitter
solution. There must be deviations from the vacuum equation of state, which
finally determine the “hypersurface” of transition to the hot universe. The de Sitter universe is still, however, a very useful zeroth order approximation for nearly all inflationary models. In fact, the effective equation of state must satisfy the condition $\varepsilon + 3p < 0$ for at least 75 e-folds. This is generally possible only if during most of the time we have $p \approx -\varepsilon$ to a rather high accuracy. Therefore, one can use the language of constant time hypersurfaces defined in various coordinate systems in de Sitter space. Our earlier considerations show that the transition from inflation to the Friedmann universe occurs along a hypersurface of constant time in the expanding isotropic coordinates ($\eta = \text{const}$), but not along $r = \text{const}$ hypersurface in the “static coordinates.” The next question is, out of the three possible isotropic coordinate systems ($k = 0, \pm 1$), which must be used to match the de Sitter space to the Friedmann universe? Depending on the answer to this question, we obtain flat, open or closed Friedmann universes. It turns out, however, that this answer seems not to be relevant for the observable domain of the universe. In fact, if inflation lasts more than 75 e-folds, the observable part of the universe corresponds only to a tiny piece of the matched global conformal diagrams for de Sitter and Friedmann universes. This piece is located near the upper border of the conformal diagram for de Sitter space and the lower border for the flat, open or closed Friedmann universes (Fig. 8), where the difference between the hypersurfaces of constant time for flat, open or closed cases is negligibly small. After a graceful exit we obtain a very large domain of the Friedmann universe with incredibly small flatness and this domain covers all present observable scales. The global structure of the universe on scales much larger than the present horizon is not relevant for an observer — at least not for the next hundred billion years. In the next section we will see that the issue of the global structure is complicated by quantum fluctuations. These fluctuations are amplified during inflation and as a result the hypersurface of transition has “wrinkles.” The wrinkles are rather small on scales corresponding
to the observable universe but they become huge on the very large scales. Hence, globally the universe is very different from the Friedmann space and the question about the spatial curvature of the whole universe no longer makes sense. It also follows that the global properties of an exact de Sitter solution have no relevance for the real physical universe.