SPECTRA OF CONFORMAL TURBULENCE

Gregory Falkovich\textsuperscript{a} and Amihay Hanany\textsuperscript{b}

\textit{Department of Physics}  
\textit{Weizmann Institute of Science}  
\textit{Rehovot 76100, Israel}

ABSTRACT

A set of different conformal solutions corresponding to a constant flux of squared vorticity is considered. Requiring constant fluxes of all inviscid vorticity invariants (higher powers of the vorticity), we come to the conclusion that the general turbulence spectrum should be given by Kraichnan’s expression $E(k) \propto k^{-3}$. This spectrum, in particular, can be obtained as a limit of some subsequences of the conformal solutions.

PACS 47.10, 47.25C

\textsuperscript{a} e-mail: FNFAL@WEIZMANN.BITNET  
\textsuperscript{b} e-mail: FTAMI@WEIZMANN.BITNET
The problem of small-scale spectrum of two-dimensional turbulence is a peculiar problem among the variety of turbulent systems. The point is that dimensional considerations do not give a steady spectrum that corresponds to the enstrophy (squared vorticity) cascade. The spectrum obtained from dimensional analysis is \( E(k) \propto k^{-3} \) [1] which yields a logarithmic infrared divergence after substitution into the equations for the correlation functions. Since any cascade picture assumes turbulence locality, the divergence makes this spectrum rather suspicious. Kraichnan’s attempt to save the spectrum from nonlocality by introducing the slow factor \( \ln^{-1/3} k \) attains convergence only in the first order of perturbation theory [2] while the next orders reveal divergencies with higher powers of the logarithm: \( \ln^2 \) etc. The fact that the powers of the logarithm increase with the order of perturbation theory suggests that a substantial renormalization of the index occurs. No successful attempts to work out the divergencies or to show that they are cancelled are known to us. The existence of alternative predictions for the steady spectrum \( E(k) \propto k^{-4} \) by Saffman [3] and \( E(k) \propto k^{-11/3} \) by Moffatt [4], show that this is still an open problem.

A fairly new approach to the problem has recently been introduced by Polyakov [5]. He suggested to borrow a set of correlation functions from conformal field theory to satisfy a chain of equations following from Euler’s equation. Note that the conformal invariance of turbulence should be considered as a pure conjecture. It is convenient to start from the Navier-Stokes equation written for the vorticity field \( \omega(x,t) \)

\[
\frac{\partial \omega}{\partial t} + \epsilon_{\alpha\beta} \frac{\partial \psi}{\partial x_\alpha} \frac{\partial \Delta \psi}{\partial x_\beta} = \nu \Delta \omega . 
\]

(1)

Here \( \psi \) is a stream function giving the velocity field as follows: \( v_\alpha = \epsilon_{\alpha\beta} \partial_\beta \psi \). Here and below we use a shorthand notation \( \partial / \partial x_\alpha = \partial_\alpha \).

Our aim is to find a stationary set of equal time correlation functions

\[ I_n(x_1, \ldots, x_n) = \langle \omega(x_1, t) \ldots \omega(x_n, t) \rangle \ . \]

The brackets denote an average with some time independent probability distribu-
\[
\sum_{p=1}^{n} \left\langle \omega(x_{1}, t) \frac{\partial \omega(x_{p}, t)}{\partial t} \ldots \omega(x_{n}, t) \right\rangle = 0 . \tag{2}
\]

Such a stationary set is expected to exist in the inertial interval of scales, i.e., for distances that are much less than the scale of an external pump (or the scale of an initial distribution for free decay) and much larger than the viscous scale \( a \). It is possible, then, to neglect viscosity in (1) using instead a careful point splitting procedure [5] for the nonlinear term

\[
e_{\alpha \beta} \frac{\partial \psi(x)}{\partial x_{\alpha}} \frac{\partial \Delta \psi(x)}{\partial x_{\beta}} = \lim_{a \to 0} e_{\alpha \beta} \frac{\partial \psi(x + a/2)}{\partial x_{\alpha}} \frac{\partial \Delta \psi(x - a/2)}{\partial x_{\beta}}, \tag{3}
\]

where “lim” implies angle averaging. To calculate different-point pair correlators like \( \psi(x + a)\psi(x - a) \), the fusion rule of the type

\[
[\psi][\psi] = [\phi] + \ldots
\]

should be used. Here we follow the notations of Ref.5 so that \([\psi]\) means the conformal class of \( \psi \), i.e. itself together with the operators \( L_{-n_1} \ldots L_{-n_k} \psi \), \( L_{-n} \) being Virasoro generators [6]. Both \( \psi \) and \( \phi \) are presumed to be taken from a set (primary fields) of some conformal field theory (in this paper we will consider the so called minimal models [6]). The primary field \( \phi \) provides the main contribution in the operator product expansion (OPE) (4) in the small-scale region, i.e. it has the smallest conformal dimension. The important thing is that the scaling indices (dimensions) of the fields are not additive so generally \( \Delta_{\phi} \neq 2 \Delta_{\psi} \). The energy density in the wave number space is expressed via \( |\psi|^2 \) and is

\[
E(k) = 2 \pi k \epsilon(k) \propto k^{4 \Delta_{\psi} + 1} , \tag{5}
\]

while the enstrophy density is \( H(k) = k^{2} E(k) \). Here \( \epsilon(k) \) is the energy density in \( k \)-space.
To choose an appropriate solution from the wealth of conformal solutions, one should impose some additional conditions that follow from the symmetries or conservation laws specific for the problem in question. According to Fjortoft’s theorem (see e.g. [7]), the vorticity is the relevant quantity in the problem of small-scale turbulence (while the energy flux determines large-scale turbulence). Following Kraichnan [1] who developed a simple and efficient (though uncontrollable, of course) closure in terms of double correlation functions, the enstrophy

\[ H_2 = \int \omega^2(x) d^2x , \]  

which is a motion integral of Euler’s equation is usually taken into account. A steady turbulence spectrum in the small-scale region should provide for a constant enstrophy flux over the scales which yields [5]

\[ \left\langle \frac{\partial \omega(x + r)}{\partial t} \omega(x) \right\rangle \propto r^0 = \text{const} . \]  

Putting \( \omega = \Delta \psi \) and

\[ \frac{\partial \omega}{\partial t} \propto (L_{-2}L_{-1}^2 - L_{-2}L_{-1}^2) \phi \]

for the time derivative, Polyakov obtained [5]

\[ (\Delta_\phi + 2) + (\Delta_\psi + 1) = 0 . \]  

As one can see, the enstrophy flux is expressed through the triple correlation function which can be expressed by the fusion rule (4) in terms of the double correlation function.

Equation (8) can be obtained also by requiring the rate of the enstrophy dissipation to remain constant while the viscosity \( \nu \) goes to zero:

\[ \frac{dH_2}{dt} = \nu \int k^2 H_k dk \propto \nu a^{-6-4\Delta_\psi} \propto \nu^{(3+\Delta_\phi+\Delta_\psi)/(\Delta_\phi-\Delta_\psi)} . \]  

The last estimate was given by using the expression for the viscous scale \( \nu \propto \)
\(a^{2\Delta_{\psi} - 2\Delta_\phi}\) that follows from the comparison of the nonlinear and the viscous terms in the Navier-Stokes equation.

Kraichnan’s dimensional approach would correspond to additive dimensions \(\Delta_\phi = 2\Delta_\psi\) giving thus \(\Delta_\psi = -1\) and \(E(k) \propto k^{-3}\). However, we have arguments that suggest that this is not a conformal solution from the set of minimal models (see Appendix).

To ensure that the conformal set of correlators is a steady solution, Polyakov imposed an extra condition requiring that rhs of (4) (which determines the time derivatives of the correlators) vanishes in the ultraviolet limit. Since

\[
\psi(z_1)\psi(z_2) = (z_1 - z_2)^{\Delta_\phi - 2\Delta_\psi} \phi(z_2) + ...
\]

as \(z_1 \to z_2\), the following inequality arises:

\[
\Delta_\phi > 2\Delta_\psi ,
\]

thus, using (8), \(\Delta_\psi < -1\).

Conditions (8) and (10) by no means determine a single solution. The minimal model (2,21) presented by Polyakov [5] is nothing but the first example from an infinite family. The curious reader can find the first few hundred solutions in the Appendix. Polyakov’s solution corresponds to the minimal number of primary fields (in this case 10). One could, in particular, find the minimal model (5,72) that gives \(\Delta_\psi = \Delta_{(1,25)} = -7/6\) and \(E(k) \propto k^{-11/3}\) as in Moffatt’s spectrum.

Usually, when speaking about a turbulent solution carrying a constant flux, one should check that two conditions are satisfied: i) The solution should be local in \(k\)-space which means that distant scales do not interact substantially; this should be provided by the convergence of the integral determining the flux in \(k\)-space; ii) The constant that arise in this (converging) integral should be nonzero and have the correct sign to satisfy the boundary conditions in \(k\)-space, i.e., the
pumping and damping. Any solution of (8) and (10) violates both of these conditions. Inequality (10) means that $\Delta_\psi < -1$ i.e. the spectrum (5) is steeper than Kraichnan’s one (which yields an infrared logarithmic divergence) so that a power infrared divergence arises for any solution. (The correct power of the divergence can be obtained by using Lagrangian or quasi-Lagrangian variables which eliminate sweeping of small scales by larger ones [8,9]). For minimal models the three-point function $\langle \psi \psi \psi \rangle \sim \langle \phi \psi \rangle$ is equal to zero since primary fields with different dimensions ($\Delta_\psi \neq \Delta_\phi$) are orthogonal. The cases where $\psi$ appears in the operator product expansion of $[\psi][\psi]$ (like those with $\Delta_\psi = \Delta_\phi = -3/2$ which one can find in the Appendix) break parity and therefore should be excluded. The conformal solutions in question are thus fluxless. The second difficulty (zero flux) remedies to some extent the first difficulty (nonlocality) since one should not require the convergence of the integrals that are identically zero. Physically this corresponds to the fact that the spectrum of a system in thermodynamic equilibrium should not be local, unlike the cascade spectrum one. A fluxless spectrum corresponds to an equilibrium case.

But, since we are discussing a nonequilibrium situation, there still remains the question, how the spectrum carries nonzero flux from the pumping region to the viscous region of scales. Polyakov’s suggestion is that small deviations from the power law due to infrared cut-off (pumping scale) could provide nonzero values for the enstrophy flux. (It is unclear whether such a flux is local in $k$-space or it is a noncascade nonlocal solution.) The physical correlators are thus assumed to be close to the equilibrium ones in the inertial interval of scales. This is similar to what happens in two-dimensional optical turbulence described by the Nonlinear Schrödinger Equation

$$i\Psi_t + \Delta \Psi + T|\Psi|^2 = 0.$$  

For wave turbulence, the small-scale turbulent spectrum carrying constant energy flux is as follows:

$$\epsilon(k) \propto k^{\alpha-m-d},$$
where $d$ is the space dimension and $\alpha$ and $m$ are the scaling indices of the Hamiltonian coefficients (i.e. the frequency and the four-wave interaction coefficient respectively) [10]. For the NSE, $\alpha = d = 2$ and $m = 0$ so that the turbulent spectrum is $\epsilon(k) = \text{const}$ which coincides with the equilibrium equipartition. This spectrum (which is an exact steady solution) is fluxless too. Numerical simulations of the NSE show the nonequilibrium spectrum to be close to $\epsilon \approx \text{const}$, an analytical attempt to introduce a slow logarithmic factor causes some doubts [11]. The same coincidence of the equilibrium and turbulent spectra takes place for common turbulence of Langmuir and ion sound waves in plasma, the spectra carrying fluxes acquire logarithmic factors in this case [10,12]. Polyakov suggested to distort the spectra by analytical (in $x$-space) contributions. In our opinion, the main difference of his approach to hydrodynamic turbulence from the above picture of wave turbulence, namely, the assumption that the turbulent spectrum of 2d hydrodynamics should be close to an equilibrium one, is not based on solid ground. One would like to see the degeneracy that prescribes the coincidence of the turbulence spectrum with an equilibrium one.

And what may be more important, Polyakov’s considerations do not take into account the presence of an infinite set of motion integrals

$$H_n = \int \omega^n(x) d^2x.$$  \hspace{1cm} (11)

The conservation of $H_n$ follows directly from the fact that the equation

$$\frac{\partial \omega}{\partial t} - e_{\alpha \beta} \frac{\partial \omega}{\partial x_{\alpha}} \frac{\partial}{\partial x_{\beta}} \delta \mathcal{H} = 0$$ \hspace{1cm} (12)

conserves the integral $\int F(\omega) \, dx \, dy$, where $F$ is an arbitrary function and the Hamiltonian $\mathcal{H}$ is an arbitrary functional of $\omega$ [not only $\mathcal{H} = \int \psi \omega \, dx \, dy$, which gives the lhs of (1)]. As one can see, even an infinite number of motion integrals does not fix the system but only its class.

7
Most authors feign indifference to the existence of the infinite number of motion integrals in 2d turbulence. Some (weak) arguments that only quadratic integrals (i.e. energy and squared vorticity) should be taken into account while considering thermodynamic equilibrium were given by Kraichnan [1,12]. However, an arbitrary turbulent pump generally produces a nonzero input of all integrals $H_n$. The theory should describe the fate of these integrals. Estimating the rate of the viscous dissipation of $H_n$ ($n \geq 2$) similarly to (9), one gets

$$\frac{dH_n}{dt} = \nu \int k^2 H_n dk \propto \nu^{\frac{(n-1)(\Delta_\psi+1)+\Delta_\phi+2}{(\Delta_\psi-\Delta_\phi)}}. \quad (13)$$

We rewrite it by the help of (8) as follows:

$$\frac{dH_n}{dt} = \nu \int k^2 H_n dk \propto \nu^{\frac{(n-2)(\Delta_\psi+1)}{(\Delta_\phi-\Delta_\psi)}}. \quad (14)$$

Formula (14) again demonstrates that $\Delta_\psi \leq -1$ while $\Delta_\phi - \Delta_\psi < 0$, otherwise the dissipation rate goes to infinity while viscosity goes to zero. For Polyakov’s solution with $\Delta_\psi < -1$, the integrals $H_n$ are not dissipated in the inviscid limit when $n > 2$. What is the fate of these integrals if they are injected? Note that it is impossible to have a local cascade of $H_2$ and a nonlocal transfer of other vorticity integrals.

One can require a constant dissipation rate of any vorticity integral. According to (13) and (14) it means that $\Delta_\psi = -1$ and we are coming back to Kraichnan’s spectrum. Indeed, this corresponds to $\psi(r) \propto |r|^2$ so that the vorticity $\omega$ has zero scaling index (maybe logarithmic). All powers of the vorticity can thus have constant fluxes in $k$-space simultaneously. Actually, Kraichnan’s spectrum equally satisfies all conservation laws.

Returning to the above set of conformal solutions, one can find subsequences (see Appendix) that give $\Delta_\psi$ which approaches the value $-1$, while the number of primary fields in the model increases. The closer $\Delta_\psi$ is to $-1$, the better the
given conformal solution satisfies the vorticity conservation laws. One can suggest to get Kraichnan’s spectrum as a limit of such subsequences, thus considering it as some weak solution. In hydrodynamics, it is physically intuitive, to account for an infinite rather than a finite number of primary fields (of topologically different field configurations). Still, some other spectra (including those of Saffman and Moffatt) could be realized for special initial conditions or as intermediate time asymptotics. Kraichnan’s spectrum seems to correspond to the most general conditions (both initial conditions in time and boundary conditions in $k$-space). This spectrum seems to be the most ergodic (e.g. isovorticity lines have the fractal dimension 2 so that they can fill the entire space [13]). It is quite natural that Kraichnan’s spectrum corresponds to an infinite number of primary fields. How this spectrum can be distorted by a slow factor to avoid the logarithmic divergence is, in our opinion, still an open problem.

Acknowledgements: Discussions with P.Wiegmann, A.Finkelstein, G.Schutz, R.Plesser, A.Schwimmer and Y.Levinson are gratefully acknowledged.

APPENDIX

In this appendix we will summarize all the conditions set by Polyakov on the solutions of a turbulence problem. Subsequently we apply these conditions to the set of minimal models and present a list of a few hundred solutions. We describe the algorithm we used to compute these solutions and give arguments why we think $\Delta_{\psi} = -1$ is not a minimal model solution. Finally we present a few solutions in a sequence that tends to $\Delta_{\psi} = -1$.

The conditions are:

1. $\Delta_{\psi} + \Delta_{\phi} = -3$

2. $\Delta_{\psi} < -1$

3. $\Delta_{\phi}$ must be the operator with the smallest dimension in the OPE of $[\psi]$ and $[\bar{\psi}]$. 
The minimal models are characterized by two positive co-prime integers \((p, q)\). For each minimal model \((p, q)\) there is a set of \(((p-1)(q-1))\) primary fields parameterized by two integers \((n, m)\), \(1 \leq n \leq p - 1, 1 \leq m \leq q - 1\). The spectrum of conformal dimensions is given by

\[
\Delta_{(n,m)} = \frac{(nq - mp)^2 - (q - p)^2}{4pq} .
\]  
(A.1)

From this formula we see that \(\Delta_{(n,m)} = \Delta_{(p-n,q-m)}\) and that they both correspond to the same primary field. The scaling dimension of the primary field \((n, m)\) is \(2\Delta_{(n,m)}\). From the first condition, using (A.1), we get \(\frac{p}{q}\) in terms of \(n_\psi, m_\psi, n_\phi, m_\phi\)

\[
\frac{p}{q} = \frac{n_\psi m_\psi + n_\phi m_\phi - 8 + k}{m_\psi^2 + m_\phi^2 - 2} ,
\]  
(A.2)

where \(k\) is an integer defined by

\[
k^2 = (n_\psi m_\psi + n_\phi m_\phi - 8)^2 - (n_\psi^2 + n_\phi^2 - 2)(m_\psi^2 + m_\phi^2 - 2) .
\]  
(A.3)

This defines both \(p\) and \(q\) since they are co-prime. From the third condition and the selection rules for the OPE we get that \(n_\phi, m_\phi\) are odd numbers and satisfy \(1 \leq n_\phi \leq 2n_\psi - 1, 1 \leq m_\phi \leq 2m_\psi - 1\). From the demand that \(\Delta_\phi\) be the lowest possible we get \((n_\phi q - m_\phi p) \approx 0\), this sets \(m_\phi\) to be the nearest odd to

\[
n_\phi \frac{n_\psi m_\psi - 8 \pm \sqrt{2m_\psi^2 - 16n_\psi m_\psi + 2m_\psi^2 + 60}}{n_\psi^2 - 2} .
\]  
(A.4)

An algorithm to calculate these turbulent solutions can be as follows. Given \(n_\psi, m_\psi\), one can calculate \(m_\phi\) from (A.4) and if \(k\) (eq. (A.3)) is an integer one can also find \(p\) and \(q\). Using this algorithm we got a set of solutions which are listed in the tables below. (This is a partial list, there is an infinite number of solutions.)
We also want to check if there exists a minimal model for which $\Delta_\psi = -1$ or $\Delta_\phi = -2$. We set $|n_\phi q - m_\phi p|$ to its minimal value and since $p, q$ are co-prime there exist $n_\phi, m_\phi$ such that $n_\phi q - m_\phi p = 1$ and we get, using (A.1), $-2 = \frac{1-(p-q)^2}{4pq}$ or

$$p^2 - 10pq + q^2 - 1 = 0. \quad (A.5)$$

for large $p, q$ one can neglect 1 and solve the quadratic equation to get

$$\left(\frac{q}{p}\right)_\pm = 5 \pm \sqrt{24}, \quad (A.6)$$

the two solutions correspond to the symmetry between $p$ and $q$. (This solution is irrational so in practice one can take $p, q$ co-prime such that $\frac{q}{p}$ is close to this value, one can get any accuracy by taking large $p, q$). From $\Delta_\psi = -1$ and (A.6) we have $m_\psi$ in terms of $n_\psi$ as the nearest integer to

$$m_\psi = n_\psi \frac{q}{p} + 2 \sqrt{\frac{q}{p}}. \quad (A.7)$$

The sequence defined by

$$a_{k+1} = 10a_k - a_{k-1}, \quad a_0 = 0, \quad a_1 = 1 \quad (A.8)$$

serves as a solution to (A.5) taking $p = a_k, q = a_{k+1}$ for any $k = 2, 3, \ldots$ the $k$-th term is given by

$$a_k = \frac{(5 + \sqrt{24})^k - (5 - \sqrt{24})^k}{2\sqrt{24}} \quad (A.9)$$

Demanding that this $(p, q)$ model has $\Delta_\psi = -1$ leads to the condition that $\sqrt{4pq+1}$ is an integer or, in terms of this sequence, that

$$\frac{(5 + \sqrt{24})^{2k-1} + (5 - \sqrt{24})^{2k-1} + 14}{24} \quad (A.10)$$

is an integer squared, which is unlikely.
In the list below one can find the models with \((p, q)\) close to (A.6) and \((n_\psi, m_\psi)\) satisfying (A.7): \((13, 129), (39, 389), (109, 1082), (232, 2295)\) and \((69, 686)\) with \(-\Delta_\psi = \frac{561}{559} - \frac{391}{389}, \frac{59001}{58969} - \frac{17873}{17748}, \frac{7905}{7889}\) respectively.

REFERENCES

1. R.H. Kraichnan, *Phys. of Fluids* **10**, 1417 (1967).
2. R.H. Kraichnan, *J. Fluid Mech.* **47**, 525 (1971).
3. P.G. Saffman, *Stud. Appl. Math.* **50**, 277 (1971).
4. H.K. Moffatt, in *Advances in turbulence*, G. Comte-Bellot and J.Mathieu, eds, p.284 (Springer-Verlag 1986).
5. A. Polyakov, “Conformal Turbulence”, Preprint PUPT-1341, Bulletin Board: hep-th@xxx.lanl.gov - 9209046.
6. A. Belavin, A. Polyakov, and A. Zamolodchikov, *Nucl. Phys.* **B241**, 33 (1984).
7. M. Lesieur, *Turbulence in Fluids*, (Kluwer, London 1990).
8. R. Kraichnan, *Phys. of Fluids* **8**, 575 (1965).
9. V. Belinicher, V.Lvov, Sov. Phys. JETP **66**, 303 (1987); V.Lvov, G.Falkovich, *Phys. Rev.* **A46** (1992).
10. V. Zakharov, V. Lvov, G. Falkovich, *Kolmogorov Spectra of Turbulence* v.1 Wave Turbulence, (Springer Verlag, Heidelberg 1992).
11. V. Djachenko, A. Newell, A. Pushkarev, V. Zakharov, *Physica* **D57**, 96 (1992).
12. R. Kraichnan, *J. Fluid Mech.* **67**, 155 (1975).
13. R. Benzi, G. Paladin, A. Vulpiani, *Phys. Rev.* **A42**, 3654 (1990).
| (p,q)  | ψ   | φ   | (p,q)  | ψ   | φ   |
|--------|------|------|--------|------|------|
| n m   | Δψ  | n m  | Δφ    | n m  | Δφ  |
| (2,21) | 1 4  | 1/7  | −13/7 | (12,97) | 5 42 | −148/97 |
| (3,26) | 1 5  | 3/14 | −8/3 | (16,159) | 5 56 | 324/424 |
| (14,115) | 1 6  | 1/2  | −36/23 | (12,95) | 5 39 | 3/2 |
| (14,111) | 1 8  | 1/7  | −55/37 | (9,71) | 5 39 | 319/213 |
| (22,179) | 1 10 | 1/19 | 276/179 | (36,317) | 5 48 | 1219/951 |
| (26,223) | 1 12 | 1/7  | 372/223 | (26,205) | 5 39 | 799/533 |
| (7,62) | 1 13 | 1/9  | −54/31 | (13,113) | 5 47 | 1971/1469 |
| (6,55) | 1 14 | 1/9  | −20/11 | (35,313) | 6 58 | 387/313 |
| (34,335) | 1 16 | 1/9  | 432/67 | (43,342) | 6 47 | 3685/2561 |
| (3,25) | 2 14 | 1/7  | 8/5 | (43,347) | 6 50 | 797/573 |
| (3,26) | 2 21 | 1/7  | −22/13 | (43,422) | 6 65 | 9309/9073 |
| (11,91) | 2 14 | 3 25 | −123/77 | (59,466) | 6 47 | 20640/13747 |
| (11,87) | 2 16 | 3 23 | 476/319 | (59,487) | 6 52 | 40467/28733 |
| (11,93) | 2 20 | 3 25 | 559/341 | (83,654) | 6 47 | 135639/90477 |
| (8,67) | 3 28 | 3 25 | 109/268 | (115,906) | 6 47 | 5206/3473 |
| (15,119) | 4 32 | 1 19 | −177/119 | (23,217) | 6 62 | 705/1713 |
| (39,310) | 4 31 | 5 39 | 605/403 | (155,1222) | 6 47 | 28407/18941 |
| (13,105) | 4 34 | 5 41 | 40/13 | (155,1351) | 6 56 | 55047/41881 |
| (39,361) | 4 42 | 5 47 | 8575/4693 | (14,111) | 7 55 | 390/259 |
| (13,129) | 4 46 | 5 49 | 1116/559 | (14,111) | 7 56 | 390/259 |
| (21,166) | 4 48 | 5 53 | 874/581 | (14,121) | 7 64 | 1131/847 |
| (9,71) | 4 32 | 7 55 | 320/213 | (12,95) | 7 59 | 3/2 |
| (21,172) | 4 35 | 7 57 | −147/1448 | (18,175) | 7 74 | −41/21 |
| (48,425) | 7 66 | 347/340 | 7 61 | 583/340 |
| (p,q)   | $\psi$ | $\phi$ | (p,q)   | $\psi$ | $\phi$ |
|---------|-------|-------|---------|-------|-------|
| (42,331) | 7.55  | -3478 | (21,188) | 7.67  | -817  |
|         |       | 2517  |         |       | 658   |
| (27,236) | 7.65  | -689 | (18,167) | 7.70  | -580  |
|         |       | 531   |         |       | 501   |
| (63,502) | 8.63  | -378  | (22,215) | 9.94  | -489  |
|         |       | 251   |         |       | 473   |
| (71,562) | 8.63  | -30069| (26,205) | 9.71  | -801  |
|         |       | 19951 |         |       | 533   |
| (71,583) | 8.68  | -58812| (26,213) | 9.76  | -1312 |
|         |       | 41305 |         |       | 923   |
| (87,686) | 8.63  | -14948| (16,135) | 9.79  | -49   |
|         |       | 9947  |         |       | 36    |
| (87,731) | 8.70  | -29681| (25,197) | 9.71  | -1479 |
|         |       | 21199 |         |       | 969   |
| (29,246) | 8.71  | -1617 | (25,224) | 9.85  | -99   |
|         |       | 1189  |         |       | 80    |
| (29,274) | 8.81  | -4422 | (20,181) | 9.86  | -441  |
|         |       | 3973  |         |       | 362   |
| (111,874)| 8.63  | -24257| (62,489) | 9.71  | 7597  |
|         |       | 16169 |         |       | 5053  |
| (111,931)| 8.70  | -47393| (19,150) | 9.71  | -143  |
|         |       | 34447 |         |       | 95    |
| (143,1126)| 8.63  | -20786| (19,167) | 9.83  | 4107  |
|          |       | 80599 |         |       | 3173  |
| (13,107) | 8.68  | -2013 | (23,202) | 9.83  | 2920  |
|          |       | 1391  |         |       | 2323  |
| (183,1442)| 8.63  | -66043| (33,262) | 10.79 | -198  |
|          |       | 43981 |         |       | 131   |
| (61,491) | 8.66  | -43824| (33,268) | 10.83 | -98   |
|          |       | 29951 |         |       | 97    |
| (231,1822)| 8.63  | -15064| (99,953) | 10.102| -1027 |
|          |       | 10021 |         |       | 953   |
| (21,169) | 8.66  | -10  | (107,891) | 10.86| 44116 |
|          |       | 13  |         |       | 31779 |
| (33,280) | 8.71  | -15  | (123,970) | 10.79| 5076  |
|          |       | 11 |         |       | 3977  |
| (231,2027)| 8.74  | -29223| (41,381) | 10.98 | 6028  |
|          |       | 22297 |         |       | 5207  |
| (287,2266)| 8.63  | -69897| (49,386) | 10.79 | 14181 |
|          |       | 46453 |         |       | 9454  |
| (287,2515)| 8.74  | -26529| (21,169) | 10.82| 1740  |
|          |       | 20623 |         |       | 1183  |