Theory of strong outcoupling from Bose-Einstein condensates

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We study the dynamics of a magnetically trapped Bose-Einstein condensate in the presence of an external electromagnetic field coupling trapped, untrapped and antitrapped Zeeman sublevels. For large condensates an approximate analytical solution of the coupled Gross-Pitaevskii equations is given in the regime of strong outcoupling. The theory is developed for the cases of rf-outcoupling within a hyperfine manifold of states, microwave-outcoupling connecting states in two different hyperfine manifolds, and Raman outcoupling.

I. INTRODUCTION

One of the most promising future applications of the Bose-Einstein condensates of magnetically trapped atoms is the possible realization of efficient atom lasers, whose experimental proof of principle has already been given [1]. An essential element of any atom laser is the process of outcoupling from the trapped Bose-Einstein condensate. In [1] an rf-field was used to coherently transfer atoms from the trapped state to untrapped magnetic sublevels within the same hyperfine manifold of states, enabling them to leave the trap under the influence of gravity. Another outcoupling scheme presently being explored experimentally [2] is based on Raman transitions in an external optical field.

The theory of outcoupling from Bose-Einstein condensates has been developed in a number of papers [3,4,6–8]. The theory is based on the Gross-Pitaevskii equations for the macroscopic wave functions of the atoms in the trapped and untrapped magnetic sublevels, coherently coupled by the externally imposed electromagnetic field. In [3] the coupled Gross-Pitaevskii equations for two magnetic sublevels were solved numerically in 1 dimension for a number of cases, and various dynamical regimes were identified. In [4] a detailed comparison of numerical and experimental results was given. In [5] the regime of weak outcoupling was considered and in this limit an approximate analytical solution of essentially the same theoretical model as in [3] and [4] was given. The weak outcoupling regime is defined physically by the condition that the Rabi oscillations induced by the electromagnetic fields in a certain resonance zone within the condensate are slow on the time scale on which the driven atoms leave that zone. An experimental study [6] of this regime was recently performed. In this limit the outcoupling proceeds at an approximately time-independent perturbatively calculable rate [5], and the reduced effective condensate dynamics can be described by a Markoff process, similar to the familiar Markovian description of a lossy laser mode. The opposite regime of strong outcoupling, where the atoms move very slowly on the time-scale of the Rabi cycles, has also been seen in numerical simulations [7,8] but has not yet been studied theoretically in detail. In fact, the experimental realization of an atom laser in [1] operated in this regime, which is therefore of practical as well as theoretical interest. In the present paper we give an approximate analytical solution of the coupled Gross-Pitaevskii equation in the regime of strong outcoupling. This will be done by a method similar to that of [9], [10] for the solution of the time-dependent Gross-Pitaevskii equation.

II. MODEL OF THE OUTCOUPLING PROCESS

The model of the outcoupling process we shall analyse in this paper is essentially that of ref. [3], but with allowance for the action of gravity, which is actually very important [7], at least in the case of microwave or rf-field outcoupling, for extracting the outcoupled atoms from the trap. Thus we consider a coupled set of Gross-Pitaevskii equations for the macroscopic wave function \( \psi_m \) in the different internal atomic states, e.g. Zeeman sublevels, labelled by \( m \). We shall assume that the interaction energy between the atoms is a functional of the total density of the atoms \( |\psi|^2 = \sum_m |\psi_m|^2 \) only and not dependent on the internal state \( m \). This assumption is rather well satisfied within a hyperfine manifold of fixed total spin quantum number \( F \), but we shall use it as a simple model assumption also

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for levels with different $F$-values. Then the coupled Gross-Pitaevskii equations for $rf$-outcoupling from an $F = 1$ hyperfine manifold or microwave outcoupling between two different hyperfine states take the form:

$$i\hbar \psi_m = \left( -\frac{\hbar^2 \nabla^2}{2M} + V_m(x) + U_0 |\psi|^2 \right) \psi_m + \hbar \Omega e^{i\omega t} \psi_{m+1} + \hbar \Omega e^{-i\omega t} \psi_{m-1}$$  \hspace{1cm} (2.1)$$

$M$ is the atomic mass, $U_0 = \frac{4\pi \hbar^2 \alpha}{2}$ describes the effective low-energy scattering potential with scattering length $a$. We shall restrict our discussion to the case of repulsive interaction $a > 0$.

For $rf$-outcoupling from an $F = 1$, $m_F = -1$ condensate, like the $^{23}Na$ condensate in Fig.1(a), $m = -1, 0, 1$ labels the 3 magnetic sublevels of the $F = 1$ state with trapping potentials

$$V_{\pm 1} = \pm \left( V(0) + \frac{M}{2} \omega^2 r^2 \right) + Mg(z - z_0)$$

$$V_0 = Mg(z - z_0) .$$

Here and in the following we neglect quadratic Zeeman shifts. Thus $m = 0$ and $m = 1$ refer to the untrapped and antitrapped magnetic sublevels, respectively. Here we included the gravitational potential and denote by $g\mu_\Omega = \cdot \Omega = \Omega$ is replaced by $\Omega$ is replaced by $\Omega$ is replaced by $\Omega = \mu B |B_r/|/\hbar$ is the Rabi frequency at resonance due to the magnetic $rf$-field, which is spatially independent over the condensate.

Similarly, for $rf$-outcoupling from an $F = 2$, $m_F = +2$ condensate (cf. Fig.1(b)), like the $^{87}Rb$ condensate, $m = -2, -1, 0, 1, 2$ labels the 5 magnetic sublevels of the $F = 2$ state with trapping potentials

$$V_{\pm 2} = \pm \left( V(0) + \frac{M}{2} \omega^2 r^2 \right) + Mg(z - z_0)$$

$$V_{\pm 1} = \pm \frac{1}{2} \left( V(0) + \frac{M}{2} \omega^2 r^2 \right) + Mg(z - z_0)$$

$$V_0 = Mg(z - z_0) .$$

The coupling term between magnetic sublevels in eq. (2.1) now reads

$$\frac{1}{2} \hbar \Omega \sqrt{(2 - m)(2 + m + 1)} e^{i\omega t} \psi_{m+1} + \sqrt{(2 + m)(2 - m + 1)} e^{-i\omega t} \psi_{m-1}$$

with $\Omega = g\mu_B |B_r/|/\hbar$.

For microwave outcoupling between two different hyperfine states it is sufficient, due to energy conservation, to consider only two components $m = 1, m = 2$ in eq. (2.1) which refer to the trapped and untrapped (or antitrapped) states, respectively. With the magnetic dipole matrix-element $\mu_{12}$ between the two states, the Rabi frequency is

$$\Omega = \mu_{12} |B_r/|/\hbar$$

where the magnetic microwave field is again spatially independent across the condensate. In this case the potentials $V_m$ are

$$V_1 = \frac{M}{2} \omega^2 r^2 + V(0) + Mg(z - z_0)$$

$$V_2 = -p \left( \frac{M}{2} \omega^2 r^2 + V(0) \right) + Mg(z - z_0)$$

where $p = 0$ and $p > 0$ for outcoupling to an untrapped state and an antitrapped state, respectively.

For Raman outcoupling (cf. Fig.1(c)) $m = 1, m = 2$ again refer to the trapped and untrapped (or antitrapped) state respectively, and $\Omega$ is replaced by $\Omega \cdot e^{i\omega x}$, the space-dependent effective Rabi frequency of the Raman transition, with net energy and momentum transfer to the atom of $\hbar \omega$ and $\hbar k$, respectively. In a three-level approximation for the Raman transition $\Omega = \Omega_1 \Omega_2/\delta$. Here $\Omega_1$, $\Omega_2$ are the Rabi frequencies of the two lasers with frequencies $\omega_1$, $\omega_2$ driving transitions between levels 1 and 3, and levels 2 and 3, respectively, with $\omega = \omega_1 - \omega_2$ and $k = k_1 - k_2$. $\Omega$ depends on the detuning $\delta = (E_3 - E_1)/\hbar - \omega_1$ between the level-spacing $E_3 - E_1$ and $\hbar \omega_1$, which is assumed to be large to avoid the population of the auxiliary level 3. The potentials $V_m$ in this case are as in eqs. (2.4).

In the following we shall assume that the electromagnetic outcoupling fields are switched on at time $t = 0$. We shall consider the case of sudden switch-on in a time interval short compared to the final Rabi period. Before the switch-on the condensate state is assumed to be in equilibrium and described to sufficient accuracy by the Thomas-Fermi approximation, while the outcoupled states are unpopulated.
III. RABI-OSCILLATIONS IN THE CONDENSATE

For sufficiently strong external electromagnetic outcoupling fields the Rabi oscillations between the coupled internal atomic states become so fast, that the center of mass motion of the atoms can no longer follow. In this regime a combined Thomas-Fermi and Raman-Nath approximation applies, in which the kinetic energy terms in eq. (2.1) are negligible. The Rabi oscillations within a given $F$ hyperfine manifold can then be solved analytically. We consider the cases $F = 1$ and $F = 2$ in turn:

1. Rabi oscillations in an $F = 1$ manifold:

Before starting let us shift the origin of the $z$-axis into the minimum of the total external potential (trap and gravity) seen by the condensate in the $m = -1$ state via the replacement

$$z \rightarrow \tilde{z} = z + g/\bar{\omega}^2.$$ 

Furthermore let us choose the zero of the gravitational potential in $\tilde{z} = 0$ by adopting $z_0 = -g/\bar{\omega}^2$. With the notation

$$\tilde{r}^2 = x^2 + y^2 + \tilde{z}^2, \quad \tilde{V}(0) = V(\tilde{r} = 0) = V(0) + Mg^2/2\bar{\omega}^2$$

(3.1)

we then have

$$V_{-1} = \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 + \tilde{V}(0)$$
$$V_0 = Mg\tilde{z}$$
$$V_1 = -\frac{M}{2} \bar{\omega}^2 \tilde{r}^2 - \tilde{V}(0) + 2Mg\tilde{z}$$

(3.2)

(see fig.2)

Furthermore let us proceed to an interaction picture by splitting off the main frequencies according to

$$\tilde{\psi}_{-1} = e^{-i(\mu + \tilde{V}(0))t/\hbar} \tilde{\psi}_{-1}$$
$$\tilde{\psi}_0 = e^{-i(\mu + \tilde{V}(0))t/\hbar} e^{i\omega t} \tilde{\psi}_0$$
$$\tilde{\psi}_1 = e^{-i(\mu + \tilde{V}(0))t/\hbar} e^{2i\omega t} \tilde{\psi}_1.$$ 

(3.3)

Initial conditions have to be specified next. We shall assume the electromagnetic outcoupling fields are switched on at $t = 0$ in a time interval short compared to the final Rabi period. The initial condition at $t = 0$ in the Thomas-Fermi approximation \[14\], \[15\] then is

$$\tilde{\psi}_0 = 0 = \tilde{\psi}_1$$
$$\tilde{\psi}_{-1} = e^{-i(\mu + \tilde{V}(0))\mu} \sqrt{\frac{\mu - M}{2} \bar{\omega}^2 \tilde{r}^2} \Theta(\mu - M/2 \bar{\omega}^2 \tilde{r}^2)$$

where the chemical potential $\mu$ is determined by the number of atoms in the condensate via

$$N = \frac{4\pi}{15U_0} (2\mu)^{5/2}/(M\bar{\omega}^2)^{3/2}.$$ 

(3.5)

Neglecting the kinetic energy terms in a zeroth-order approximation, the coupled Gross-Pitaevskii equations reduce to a Rabi level-flopping problem, in which the local quantity $|\psi(\tilde{r}, t)|^2$ is constant in time. In the next (first-order) approximation $|\psi(\tilde{r}, t)|^2$ will be allowed to change slowly in space and time and will then be determined self-consistently. Therefore, in the present zeroth order approximation we have to treat $|\psi(\tilde{r}, t)|^2$ as constant in time, but arbitrary. In a vector and matrix notation, with

$$\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_{-1} \\ \tilde{\psi}_0 \\ \tilde{\psi}_1 \end{pmatrix}$$ 

the coupled Gross-Pitaevskii equations to zeroth order then reduce to the simple form
\[ i \dot{\psi} = \begin{pmatrix} \varepsilon & \Omega & 0 \\ \Omega & \varepsilon + \Delta & \Omega \\ 0 & \Omega & \varepsilon + 2\Delta \end{pmatrix} \psi \] (3.6)

with

\[ \hbar \varepsilon(\tilde{r}, t) = U_0|\tilde{\psi}(\tilde{r}, t)|^2 - \mu + \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 \]

\[ \hbar \Delta(\tilde{r}) = -\frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + Mg \tilde{z} - \tilde{V}(0) + \hbar \omega \] (3.7)

Its solution is recorded in the appendix, together with the corresponding solutions for the other cases considered below, in particular the case \( F = 2 \). Let us define the regime of strong outcoupling by the condition

\[ \Omega \gg |\Delta(\tilde{r})| \] (3.8)

throughout the condensate. If the rf-frequency \( \omega \) is chosen in such a way that the resonance condition \( \Delta(\tilde{r}) = 0 \) is satisfied somewhere inside the condensate, a sufficient condition for (3.8) to hold is \( \Omega \gg \mu \). Since in the regime of strong outcoupling the detuning \( \Delta \) is negligible compared to \( \Omega \), the solution of eq. (3.6) becomes very simple and reads

\[ \tilde{\psi}(\tilde{r}, t) = \frac{1}{2} \sqrt{|\psi(\tilde{r}, t)|^2} e^{-i \int_0^t d\tau [\varepsilon(\tilde{r}, \tau) + \Delta(\tilde{r})]} \cdot \begin{pmatrix} 1 + \cos \sqrt{2\Omega} t \\ -i \sqrt{2} \sin \sqrt{2\Omega} t \\ -1 + \cos \sqrt{2\Omega} t \end{pmatrix} \] (3.9)

Fig.3 shows the probabilities \( |\tilde{\psi}_m(\tilde{r}, t)|^2 / |\psi(\tilde{r}, t)|^2 \) as a function of time. Thus the Rabi oscillations in the strong outcoupling regime occur with approximately constant amplitude throughout the condensate, but with a space-dependent phase. In the next order approximation the gradient of this phase gives rise to a particle current, which in turn changes the number density \( |\psi(\tilde{r}, t)|^2 \) by particle number conservation. This will be considered in section IV.

2. Rabi oscillations in an \( F = 2 \) manifold

Let us assume the condensate is in the \( F = 2 \), \( m_F = 2 \) state. We perform an analogous sequence of steps as in the previous subsection for the case \( F = 1 \) Introducing \( \tilde{z} = z + g/\tilde{\omega}^2 \) and choosing \( z_0 = -g/\tilde{\omega}^2 \) we have in the notation (3.1)

\[ V_2 = \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \]

\[ V_1 = \frac{1}{2} \left( \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \right) + \frac{1}{2} Mg \tilde{z} \]

\[ V_0 = Mg \tilde{z} \]

\[ V_{-1} = -\frac{1}{2} \left( \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \right) + \frac{3}{2} Mg \tilde{z} \]

\[ V_{-2} = - \left( \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \right) + 2 Mg \tilde{z} \]

(3.10)

Let us split off the main frequencies according to

\[ \psi_m = e^{-i(\mu + \tilde{V}(0))t/\hbar} e^{-i(m-2) \omega t} \tilde{\psi}_m \] (3.11)

Using the fact that \( |\psi(t)|^2 = \sum_m |\psi_m(t)|^2 \) is locally conserved in the combined Thomas-Fermi and Raman-Nath approximation and left arbitrary at this stage (to be determined self-consistently later on) we obtain the coupled set of equations, in matrix notation with

\[ \vec{\psi} = \begin{pmatrix} \tilde{\psi}_2 \\ \tilde{\psi}_1 \\ \vdots \\ \tilde{\psi}_{-2} \end{pmatrix} \]
\[
\begin{pmatrix}
\varepsilon & \Omega & 0 & 0 & 0 \\
\Omega & \varepsilon + \Delta & \sqrt{\frac{3}{2}} \Omega & 0 & 0 \\
0 & \sqrt{\frac{3}{2}} \Omega & \varepsilon + 2\Delta & \sqrt{\frac{3}{2}} \Omega & 0 \\
0 & 0 & \sqrt{\frac{3}{2}} \Omega & \varepsilon + 3\Delta & \Omega \\
0 & 0 & 0 & \Omega & \varepsilon + 4\Delta
\end{pmatrix}
\]
\begin{equation}
\tilde{\psi}
\tag{3.12}
\end{equation}

where
\[
\begin{align*}
\bar{h}\varepsilon &= U_0 |\psi|^2 - \mu + M \bar{\omega}^2 \tilde{r}^2 \\
\bar{h}\Delta &= \frac{\bar{h}}{2} k^2 - (1 + p) \left( \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 - Mg\tilde{z} + \tilde{V}(0) \right) + \bar{h}\omega.
\end{align*}
\begin{equation}
\tag{3.13}
\end{equation}

The solution of (3.12) is given in the appendix. In the limit of strong outcoupling it takes the form
\[
\tilde{\psi} = \frac{1}{2} \sqrt{|\psi|^2} e^{-i \int_0^t d\tau [\varepsilon(\tilde{r},\tau) + \frac{1}{2} \Delta(\tilde{r})]} \begin{pmatrix}
\cos \Omega t + \frac{1}{2} \cos 2\Omega t + \frac{3}{4} \\
-i \sin \Omega t - \frac{1}{2} \sin 2\Omega t \\
\frac{\sqrt{2}}{4} (\cos 2\Omega t - 1) \\
i \sin \Omega t - \frac{1}{2} \sin 2\Omega t \\
- \cos \Omega t + \frac{1}{4} \cos 2\Omega t + \frac{3}{4}
\end{pmatrix}.
\begin{equation}
\tag{3.14}
\end{equation}

Fig. 4 shows the probabilities \(|\tilde{\psi}_m(\tilde{r}, t)|^2/|\psi(\tilde{r}, t)|^2\) as a function of time for this case.

3. Rabi oscillations for microwave and Raman outcoupling

For these outcoupling mechanisms the internal state of the condensate atoms is coupled to a single magnetic sublevel in another hyperfine manifold and a simple two-level Rabi oscillation results. As before we introduce \(\tilde{z} = z + g/\bar{\omega}^2\) and pick \(z_0 = -g/\bar{\omega}^2\). Then
\[
\begin{align*}
V_1 &= \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \\
V_2 &= -p \left( \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 + \tilde{V}(0) \right) + Mg(1 + p)\tilde{z}.
\end{align*}
\begin{equation}
\tag{3.15}
\end{equation}

We define \(\tilde{\psi}_m\) via
\[
\begin{align*}
\psi_1 &= e^{-i(\mu + \tilde{V}(0))t/\bar{h}} \tilde{\psi}_1 \\
\psi_2 &= e^{-i(\mu + \tilde{V}(0))t/\bar{h} + i\omega t} \tilde{\psi}_2 e^{ik\tilde{z}}
\end{align*}
\begin{equation}
\tag{3.16}
\end{equation}

where \(k \neq 0\) only in the case of Raman outcoupling. Then the equations in combined Thomas-Fermi and Raman-Nath approximation in a matrix representation with \(\tilde{\psi} = \begin{pmatrix} \tilde{\psi}_1 \\ \tilde{\psi}_2 \end{pmatrix} \) read
\[
\begin{pmatrix}
\varepsilon & \Omega \\
\Omega & \varepsilon + \Delta
\end{pmatrix}
\begin{pmatrix}
\tilde{\psi}_1 \\
\tilde{\psi}_2
\end{pmatrix}
\begin{equation}
\tag{3.17}
\end{equation}

with
\[
\begin{align*}
\bar{h}\varepsilon &= U_0 |\psi|^2 - \mu + \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 \\
\bar{h}\Delta &= \frac{\bar{h}}{2M} k^2 - (1 + p) \left( \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 - Mg\tilde{z} + \tilde{V}(0) \right) + \bar{h}\omega.
\end{align*}
\begin{equation}
\tag{3.18}
\end{equation}

Their solution in the limit of strong outcoupling \(\Omega \gg \Delta\) is
\[
\tilde{\psi}(t) = \sqrt{|\psi(0)|^2} \left( \begin{pmatrix}
\cos \Omega t \\
-i \sin \Omega t
\end{pmatrix} \right) e^{-i \int_0^t d\tau [\varepsilon(\tilde{r},\tau) + \frac{1}{2} \Delta(\tilde{r})]}.
\begin{equation}
\tag{3.19}
\end{equation}

IV. CURRENT OF ATOMS WITH OUTCOUPLING FIELDS ON

In the preceding section the kinetic energy term in the Gross-Pitaevskii equation (2.1) was neglected. No spatial transport of particles can occur in this approximation, the atoms are assumed to be at rest. The most important consequence of the kinetic energy term in eq. (2.1) is that a spatial transport of the atoms occurs. It obeys the conservation law

\[ \frac{\partial |\psi|^2}{\partial t} + \nabla \cdot j = 0 \]  

(4.1)

with the current density

\[ j = \frac{\hbar}{2Mi} \sum_m (\psi_m^* \nabla \psi_m - \psi_m \nabla \psi_m^*) \]  

(4.2)

We shall now calculate this current using the results obtained in the preceding section.

The present discussion applies to all cases 1) to 3) of sec. III. Let us begin with the case of rf-outcoupling from a \( F = 1, m_F = -1 \) condensate. The state-vector \( \tilde{\psi} \) is space-dependent due to its proportionality to \( \sqrt{|\psi(\tilde{r}, t)|^2} \) and its dependence on \( \epsilon \) and \( \Delta \) which both depend on \( \tilde{r} \). As shown in the appendix \( \tilde{\psi} \) can be written as

\[ \tilde{\psi} = |\psi| e^{i \tilde{\phi} \varphi} \]  

(4.3)

where the vector \( \varphi \) is space-dependent only via its dependence on the space-dependent detuning \( \Delta(\tilde{r}) \) and has unit norm. The phase \( \tilde{\phi} \), according to eq. (3.9), satisfies in the local rest frame of the atoms

\[ \frac{\partial \tilde{\phi}}{\partial t} = - (\epsilon(\tilde{r}, t) + \Delta(\tilde{r})) . \]  

(4.4)

Inserting (4.3) in the definition of the current density results in the expression

\[ j(\tilde{r}, t) = v(\tilde{r}, t) |\psi(\tilde{r}, t)|^2 \]  

(4.5)

with the local velocity field in the laboratory frame

\[ v(\tilde{r}, t) = \frac{\hbar}{M} \nabla \tilde{\phi} + \frac{\hbar}{2Mi} \sum_m \left( \varphi_m^* \nabla \varphi_m - \varphi_m \nabla \varphi_m^* \right) . \]  

(4.6)

As is shown in the appendix the second term on the right hand side of eq. (4.6) is given by

\[ \frac{-\sqrt{2\omega}}{2\Omega} \sin(\sqrt{2\Omega} t)(\tilde{\omega} \tilde{r} - \frac{g}{\omega} e_z) \]  

(4.7)

For strong outcoupling it is rapidly oscillating on the time-scale \( \tilde{\omega}^{-1} \) of the center of mass motion and can therefore be neglected in the following. If taken into account it would describe a small oscillatory component in the temporal evolution of the center of mass and the radius of the condensate.

The equation of motion of \( \tilde{\phi} \) in the laboratory frame includes the kinetic energy \( M \dot{v}^2/2 = \hbar^2 (\nabla \tilde{\phi})^2/2M \) in addition to \( \Delta \) and \( \epsilon \) and reads

\[ \frac{\partial \tilde{\phi}}{\partial t} = - \left( \hbar (\nabla \tilde{\phi})^2/2M + \epsilon(\tilde{r}, t) + \Delta(\tilde{r}) \right) . \]  

(4.8)

It has to be solved together with the continuity equation

\[ \frac{\partial |\psi|^2}{\partial t} + \frac{\hbar}{M} \nabla \cdot (|\psi|^2 \nabla \tilde{\phi}) = 0 . \]  

(4.9)

Before doing this it is convenient to make a transformation to the rest-frame of the center of mass

\[ \tilde{\psi}(\tilde{r}, t) = e^{-\frac{i}{\hbar} \int_0^t \dot{\tilde{\omega}}^2(\tau) d\tau} e^{i \int_0^t \dot{\tilde{\omega}}(\tau) \cdot \tilde{r}} |\psi(x, t) \]  

(4.10)
with \( \bar{\psi}_x = |\psi_x| e^{i\phi_x} \) and

\[
x = \tilde{r} - \tilde{r}^*(t) .
\]

The continuity and phase equations in the center of mass frame read

\[
\frac{\partial |\psi_x(x,t)|^2}{\partial t} + \nabla \cdot u |\psi_x(x,t)|^2 = 0
\]

(4.12)

with

\[
u = \frac{\hbar}{M} \nabla \tilde{\phi}_x(x,t)
\]

(4.13)

and

\[
\frac{\partial \tilde{\phi}_x}{\partial t} = -\frac{\hbar}{2M} (\nabla \tilde{\phi}_x)^2 + \frac{M \tilde{r}^*}{M} \cdot x - \varepsilon (\tilde{r}^* + x, t) - \Delta (\tilde{r}^* + x) .
\]

(4.14)

With the center of mass motion \( \tilde{r}^* = -\frac{1}{2} g t^2 e_z \) and eqs. (3.7), the phase equation simplifies to

\[
\frac{\partial \tilde{\phi}_x}{\partial t} = -\frac{\hbar}{2M} (\nabla \tilde{\phi}_x)^2 - \frac{U_0}{\hbar} B + \frac{\mu + \mathcal{V}(0)}{\hbar} - \omega - \frac{M g}{\hbar} \tilde{r}^* .
\]

(4.15)

The ansatz

\[
|\psi_x|^2 = (A(t) - B(t)x^2) \Theta (A(t) - B(t)x^2)
\]

\[
\tilde{\phi}_x = \alpha(t) - \beta(t)x^2
\]

(4.16)

solves equations (4.12), (4.13), (4.15) exactly if the coefficients satisfy the differential equations

\[
\dot{B} = -\frac{10\hbar}{M} \beta B
\]

\[
\dot{\beta} = \frac{U_0}{\hbar} B - \frac{2\hbar}{M} \beta^2
\]

\[
\dot{\alpha} = \frac{U_0}{\hbar} A
\]

(4.17)

and, by the normalization condition \( N = \int d^3x |\psi_x|^2 \),

\[
A = \left( \frac{15N}{8\pi} \right)^{2/5} B^{3/5}.
\]

(4.18)

The initial conditions follow from the Thomas-Fermi wave function at \( t = 0 \). They are

\[
B(0) = \frac{M \omega^2}{2U_0}, \quad \alpha(0) = 0 = \beta(0) .
\]

(4.19)

The time-dependent Thomas-Fermi radius is related to \( B(t) \) via

\[
r_{TF}(t) = r_{TF}(0) \left( \frac{B(0)}{B(t)} \right)^{1/5}.
\]

(4.20)

Its equation of motion follows as

\[
\ddot{r}_{TF}(t) - \frac{\omega^2 r_{TF}(0)}{r_{TF}(t)} = 0 .
\]

(4.21)

It can be integrated once to give

\[
\dot{r}_{TF}^2(t) + 2\frac{\omega^2 r_{TF}^2(0)}{r_{TF}(t)} = \frac{2}{3} \omega^2 r_{TF}^2(0) .
\]

(4.22)
At large times, where \( r_{TF}(t) \gg r_{TF}(0) \), the Thomas-Fermi radius expands ballistically at a constant rate

\[
r_{TF}(\infty) = \sqrt{\frac{7}{3}} \omega r_{TF}(0). \tag{4.23}
\]

At small times it expands with acceleration \( \omega^2 r_{TF}(0) \)

\[
r_{TF}(t) \simeq r_{TF}(0) \left( 1 + \frac{1}{2} \omega^2 t^2 \right). \tag{4.24}
\]

The Thomas-Fermi radius obtained by integrating eq.(4.22) is plotted in fig.5 as a function of time.

The result for the condensate density in the laboratory frame is

\[
|\psi(t)|^2 = \frac{15N}{8\pi} \frac{r_{TF}^2(t) - \left( \tilde{r} + \frac{1}{2}gt^2 e_z \right)^2}{r_{TF}^2(t)} \Theta \left( r_{TF}(t) - \left( \tilde{r} + \frac{1}{2}gt^2 e_z \right)^2 \right). \tag{4.25}
\]

The phase of the macroscopic state vector \( \tilde{\psi} \) can also be expressed by \( r_{TF}(t) \) as

\[
\tilde{\phi}(t) = \tilde{V}(0) \frac{t}{\hbar} - \omega t - \frac{Mg}{\hbar} t \tilde{z} - \frac{\mu}{\hbar} \int_0^t d\tau \left( \frac{r_{TF}^2(\tau)}{r_{TF}^2(0)} - 1 \right) + \frac{M}{2\hbar} (\tilde{r} - \tilde{r}^*(t))^2 \frac{d}{dt} \ln \frac{r_{TF}(t)}{r_{TF}(0)} \tag{4.26}
\]

Thus under the influence of the strong electromagnetic outcoupling fields, switched on at \( t = 0 \), the whole condensate undergoes rapid Rabi oscillations through all the coupled states while freely dropping out of the trap by gravity, and at the same time expanding due to the repulsive interaction. Switching on the external electromagnetic field therefore effectively switches off the trapping potential, up to the oscillatory term \((4.7)\) whose influence is small if \( \omega \ll \Omega \), because it averages out in the Rabi oscillations through trapped, untrapped and antitrapped states of a hyperfine manifold.

The results given so far have been written down for \( rf \)-outcoupling from an \( F = 1, m_F = -1 \) condensate \((\text{III 1})\), but are easily transferred to the other cases considered in section \((\text{III})\):

**RF-Outcoupling from the \( F = 2, m_F = 2 \) state:**

The velocity field \( \mathbf{v} \) is still given by eq.(4.6), but the second term on the right hand side of that equation is changed and becomes now

\[
- \frac{\tilde{\omega}}{\Omega} \sin(\Omega t)(\tilde{\omega} \tilde{r} - \frac{g}{\tilde{\omega}} e_z). \tag{4.27}
\]

It is again rapidly oscillating on the relevant time-scale \( \tilde{\omega} \) and hence negligible. The equation of motion of the phase becomes

\[
\frac{\partial \tilde{\phi}}{\partial t} = - \left( \frac{\hbar}{2M} (\nabla \tilde{\phi})^2 + \varepsilon + \Delta \right). \tag{4.28}
\]

After inserting \( \varepsilon \) and \( \Delta \) from eq. \((3.13)\), the phase equation reduces to the same form as in the case of the \( F = 1 \) manifold and therefore the results for \( \tilde{r}^*(t), r_{TF}(t), |\psi|^2 \) and \( \tilde{\phi} \) are unchanged.

**Microwave and Raman outcoupling:**

We give the results for Raman outcoupling. To obtain the results for microwave outcoupling from the following formulae on has to set \( k = 0 \) there. The velocity is again given by the general expression \((4.6)\), with the second term now of the form

\[
(1 + p)\frac{\tilde{\omega}}{4\Omega} \sin(2\Omega t)(\tilde{\omega} \tilde{r} - \frac{g}{\tilde{\omega}} e_z),
\]

which is again rapidly oscillating on the time-scale \( \tilde{\omega} \) and therefore negligible. The phase equation then takes the form

\[
\frac{\partial \tilde{\phi}}{\partial t} = - \left( \frac{\hbar}{2M} (\nabla \tilde{\phi})^2 + \varepsilon + \frac{\Delta}{2} \right) \tag{4.29}
\]

with
\[ \hbar \left( \epsilon + \frac{\Delta}{2} \right) = \hbar \omega/2 + \hbar U_0 |\psi|^2 - \mu + \frac{\hbar^2 k^2}{4M} + \frac{1}{2} (1 - p) \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 \]
\[ + \frac{1}{2} (1 + p) M g \ddot{z} - \frac{1}{2} (1 + p) \hat{V}(0). \] (4.30)

Let us first consider the case \( p = 1 \). There the excited state is antitrapped and has a magnetic moment precisely opposite to the trapped condensate state. The trap potential thus averages out in the Rabi oscillations. We transform to the center of mass frame with

\[ \tilde{r}^* = \frac{\hbar k}{M} - gte_z. \] (4.31)

There we can again make the ansatz (4.16) and solve for \( A(t) \), \( B(t) \) as before. Finally, transforming back to laboratory coordinates we obtain

\[ |\psi(t)|^2 = \frac{15N}{8\pi} \frac{r_{TF}^2(t) - (\tilde{r} + \frac{1}{2} e_z g t^2 - \frac{\hbar k t}{M})^2}{r_{TF}^2(t)} \] (4.32)

with the previous result (4.21, 4.22) for the equation of motion of \( r_{TF}(t) \).

In the case \( p \neq 1 \), to which we now turn, the trap potential does not average out in the Rabi oscillations and contributes to the center of mass motion and the spreading of the condensate. Now the transformation (written for \( p < 1 \) \( \tilde{r} = \tilde{r}^*(t) + x \) with

\[ \tilde{r}^*(t) = -\frac{1 + p}{1 - p} \frac{g e_z}{\tilde{\omega}^2} \left( 1 - \cos \left( \frac{1 - p}{2} \tilde{\omega} t \right) \right) + \frac{\hbar k}{M} \frac{\sin \left( \frac{1 - p}{2} \tilde{\omega} t \right)}{\sqrt{1 - \frac{p}{2}}} \] (4.33)

introduces the center of mass rest-frame. The phase equation in this frame takes the form

\[ \frac{\partial \tilde{\phi}_x}{\partial t} = -\frac{\hbar}{2M} (\nabla \tilde{\phi}_x)^2 - \frac{U_0}{\hbar} |\psi_x|^2 - \frac{1 - p}{2} \frac{M}{2} \tilde{\omega}^2 x^2 \]
\[ - \frac{1 - p}{2h} \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 - \frac{1 + p}{2h} M g \tilde{z}^* + \frac{\mu}{\hbar} - \frac{\hbar k^2}{4M} + \frac{1 + p}{2h} \tilde{V}(0) - \frac{\omega}{2} \] (4.34)

Again the continuity equation and the phase equation can be solved by an ansatz of the form (4.16). The solution for the condensate number density with the required initial conditions takes the form, in the laboratory frame,

\[ |\psi(t)|^2 = \frac{15N}{8\pi} \frac{r_{TF}^2(t) - (\tilde{r} - \tilde{r}^*(t))^2}{r_{TF}^2(t)} \Theta \left( (\tilde{r} - \tilde{r}^*(t))^2 - r_{TF}^2(t) \right) \] (4.35)

where the Thomas-Fermi radius \( r_{TF}(t) \) satisfies

\[ \dot{r}_{TF}(t) - \tilde{\omega}^2 \left( \frac{r_{TF}^2(0)}{r_{TF}(t)} - \frac{1 - p}{2} r_{TF}(t) \right) = 0. \] (4.36)

It is not influenced by the center of mass motion \( \tilde{r}^*(t) \). The phase of the state vector \( \tilde{\psi} \) expressed in terms of \( r_{TF}(t) \) is

\[ \tilde{\phi} = \left( -\frac{\hbar k^2}{4M} + \frac{1 + p}{2h} \tilde{V}(0) - \frac{\omega}{2} \right) t \]
\[ - \frac{M}{\hbar} \int_0^t d\tau \left( \frac{r_{TF}^2(0)}{r_{TF}(\tau)} - 1 \right) + \frac{M}{2h} (\tilde{r} - \tilde{r}^*(t))^2 \frac{d}{dt} \ln \frac{r_{TF}(t)}{r_{TF}(0)} \]
\[ + \frac{M}{\hbar} \tilde{r}^*(t) \cdot \tilde{r} \] (4.37)

where energy conservation of the center of mass motion led to the cancellation of some terms. Thus, for \( p < 1 \) the condensate remains bound, even in the presence of the electromagnetic outcoupling field. Its center of mass oscillates
with the frequency $\sqrt{\frac{1-p}{2} \omega}$ around the new equilibrium position at $\tilde{z} = -\frac{1+p}{1-p} \frac{g}{\omega^2}$. The Thomas-Fermi radius also oscillates, for small amplitudes with frequency $\sqrt{\frac{2}{5}(1-p)} \omega$ around its new equilibrium value (see fig.3)

$$r_{TF} = r_{TF}(0) \left( \frac{2}{1-p} \right)^{1/5}. \quad (4.38)$$

For $p > 1$ the condensate becomes antitrapped by the outcoupling field and is driven apart by the magnetic field of the trap. Its center of mass trajectory can be inferred from (4.33) by analytic continuation to $p < 1$ as

$$r^* (t) = \frac{p+1}{p-1} \frac{ge_z}{\omega^2} \left( 1 - \cosh \left( \sqrt{\frac{p-1}{2}} \tilde{\omega} t \right) \right) + \frac{\hbar k}{M} \sinh \left( \sqrt{\frac{p-1}{2} \tilde{\omega}} \right). \quad (4.39)$$

Its Thomas-Fermi radius (see fig.4) grows with acceleration $\frac{p+1}{2} \omega^2 r_{TF}(0)$ for small times $\tilde{\omega} t \ll 1$

$$r_{TF}(t) = r_{TF}(0) \left( 1 + \frac{p+1}{4} \omega^2 t^2 \right), \quad (4.40)$$

and exponentially at large times

$$r_{TF}(t) \sim \exp \left( \sqrt{\frac{p-1}{2} \tilde{\omega} t} \right) \quad (4.41)$$

provided the condensate has not yet spread beyond the parabolic part of the trapping potential.

V. OUTPUT

We assume the outcoupling field is turned off rapidly when, in a Rabi cycle nearly all atoms are in the antitrapped state. To determine their macroscopic wave function afterwards we have to solve their Gross-Pitaevskii equation again.

1. F = 1 manifold

For outcoupling from the $F = 1$ manifold

$$i \hbar \frac{\partial}{\partial t} \tilde{\psi}_1 = \left( -\frac{\hbar^2}{2M} \nabla^2 - \frac{M}{2} \tilde{\omega}^2 \tilde{r}^2 + 2Mg \tilde{z} - \mu - 2\tilde{V}(0) + 2\hbar \omega + U_0 |\tilde{\psi}_1|^2 \right) \tilde{\psi}_1 \quad (5.1)$$

where atoms remaining in the $m = -1, 0$ states have been neglected in the total density. This is justified if the outcoupling field is switched off at a time $T$ satisfying

$$\sqrt{2} \Omega T = (2n + 1)\pi \quad (5.2)$$

with integer $n$. To solve (5.1) we can use the same method as in section IV. The center of mass $\tilde{r}^*(t)$ for $t > T$ satisfies

$$\tilde{r}^* = \tilde{\omega} \tilde{r}^* - 2ge_z. \quad (5.3)$$

With the appropriate initial conditions at time $t = T$ it is given by

$$\tilde{r}^*(t) = \frac{2ge_z}{\tilde{\omega}^2} \left( 1 - \left( 1 + \frac{\tilde{\omega}^2 T^2}{4} \right) \cosh{\tilde{\omega}(t - T)} - \frac{\tilde{\omega} T}{2} \sinh{\tilde{\omega}(t - T)} \right), \quad (5.4)$$

i.e. the combined action of gravity and the antitrapping potential pushes the center of mass of the outcoupled condensate vertically downwards. Meanwhile its Thomas-Fermi radius keeps expanding according to the equation of motion

$$\tilde{r}_{TF}(t) - \tilde{\omega}^2 \left( \frac{r_{TF}^5(0)}{r_{TF}^4(t)} + r_{TF}(t) \right) = 0 \quad (5.5)$$
which corresponds to eq. (4.36) with \( p = 3 \). It has to be solved with initial conditions at time \( t = T, r_{TF} = r_{TF}(T) \) and

\[
\dot{r}_{TF}(T) = \sqrt{\frac{2}{3}} \frac{\omega r_{TF}(0)}{r_{TF}(T)} \left( 1 - \frac{r_{TF}^2(0)}{r_{TF}^2(T)} \right)^{1/2}
\]

(5.6)

following from the solution for \( t \leq T \) of eq. (4.21). The solution for \( r_{TF}(T) \) is plotted in fig. 8. For times \( \omega t \gg 1 \) but still small enough so that the atomic cloud has not left the parabolic part of the trap, the cloud’s center of mass coordinate and its radius both grow exponentially with the same rate \( \bar{\omega} \). The density and phase of the outcoupled wave function are then given by eqs. (4.35), (4.37), where eq. (4.37) has to be taken for \( p = 3 \). Fig. 8 gives a plot of the number density \(|\psi|^2\) in the antitrapped state. Plotted is \(|\psi|^2\) in units of \( 15N/8\pi r_{TF}^3(0) \) as a function of the radial coordinate in the center of mass frame in units of \( r_{TF}(0) \) and \( \Omega t \). In order to portray the influence of the Rabi-oscillations and the motion in the trap potential together the ratio \( \Omega/\bar{\omega} \) in this and in the following two plots has not been taken as large as would be desirable for the clear separation of time-scales we have assumed. In this plot \( \bar{\omega}/\Omega = 0.1 \), and the duration of the radio-frequency pulse is \( \bar{\omega}T = 2 \).

2. \( F = 2 \) manifold
In the limit \( \Delta/\Omega \to 0 \) a complete transfer of the occupation probability from the trapped \( m_F = 2 \) state to a single antitrapped state (cf. fig. 9) occurs at the times \( \Omega T = (2n+1)\pi \) when all atoms are in the antitrapped \( m_F = -2 \) state. We can take over the preceding analysis with minor modifications. The Gross-Pitaevskii equation of the atoms in the \( m_F = -2 \) state

\[
i\hbar \frac{\partial \tilde{\psi}_2}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 - \frac{M}{2} \bar{\omega}^2 \tilde{r}^2 + 2Mg\tilde{z} - \mu - 2\tilde{V}(0) + 4\hbar\omega + U_0|\tilde{\psi}_2|^2 \right) \tilde{\psi}_2
\]

(5.7)

takes the same form as (5.1) if there the replacement \( \hbar\omega \to 2\hbar\omega \) is made. The result for the density in the \( m_F = -2 \) state is plotted in fig. 10.

3. Microwave and Raman outcoupling
In the limit \( \Delta \ll \Omega \) all atoms in the condensate are in the antitrapped state \( m = 2 \) (for \( p > 0 \)) at times \( \Omega T = \frac{2n+1}{2}\pi \). The subsequent evolution of \( \psi_2 \) is governed by

\[
i\hbar \frac{\partial \psi_2}{\partial t} = \left( -\frac{\hbar^2}{2M} \nabla^2 - \frac{p}{2} M\bar{\omega}^2 \tilde{r}^2 + (1 + p)Mg\tilde{z} + U_0|\psi_2|^2 \right)
\]

\[
-\mu + \frac{\hbar^2 k^2}{2M} + \hbar\omega - (1 + p)\tilde{V}(0) \right) \tilde{\psi}_2.
\]

(5.8)

The center of mass satisfies the equation of motion

\[
\ddot{\tilde{r}}^* = p\bar{\omega}^2\tilde{r}^* - (1 + p)g\tilde{e}_z
\]

(5.9)

with initial conditions following from eq. (4.33). The Thomas-Fermi radius satisfies

\[
\dot{r}_{TF}(t) - \frac{\omega^2}{2} \left( \frac{r_{TF}^3(0)}{r_{TF}^3(T)} + pr_{TF}(t) \right) = 0
\]

(5.10)

with initial conditions \( r_{TF}(T) \) and

\[
\dot{r}_{TF}(T) = \bar{\omega}r_{TF}(0) \sqrt{\frac{2}{3}} \frac{\left( 1 - \frac{r_{TF}^3(0)}{r_{TF}^3(T)} \right)}{\left( \frac{r_{TF}^3(T)}{r_{TF}^3(0)} \right)^{1/2}}
\]

(5.11)

following from eq. (4.36). The solution for \( p = 1 \) is shown in fig. 8. The density \(|\psi_2|^2\) is then given in terms of \( \tilde{r}^*(t) \) and \( r_{TF}(t) \) by eq. (4.35). For the purpose of generating an output pulse with a small Thomas-Fermi radius the case \( p = 0 \) is of particular advantage. The number density in the excited state for this case is plotted in fig. 11. In this plot it is assumed that \( \Omega/\bar{\omega} = 5.91 \) and \( \Omega T = 15\pi/2 \). Finally, the phase is obtained as an expression similar to eq. (4.37), namely
\[ \phi(\vec{r}, t) = \left( -\frac{\hbar^2 k^2}{2M} + \left( 1 + p \right) \bar{V}(0) - \hbar \omega \right) \frac{t - T/2}{\hbar} \]
\[ -\frac{\mu}{\hbar} \int_0^t d\tau \left( \frac{\partial^2 \phi_{\psi}(0)}{\partial \tau^2} - 1 \right) + \frac{M}{2\hbar} (\vec{r} - \vec{r}(t))^2 \frac{d}{dt} \ln \frac{\phi_{\psi}(t)}{\phi_{\psi}(0)} \]

where \( \phi_{\psi}(\vec{r}, t) \) is the solution of eq. (5.12) for \( 0 \leq \tau \leq T \) and as solution of eq. (5.10) for \( T \leq \tau \leq t \).

VI. CONCLUSION

In this work we have studied outcoupling from magnetically trapped Bose-Einstein condensates via electromagnetic fields resonantly coupling either all the magnetic sublevels in the hyperfine manifold of the trapped state or the trapped state and a state in a different hyperfine manifold. Simplifying assumptions we made were (i) isotropic trap potentials, (ii) equally spaced magnetic sublevels, i.e. neglect of quadratic Zeeman shifts, (iii) interaction of the particles via their total density, (iv) the validity of a dynamical version of the Thomas-Fermi approximation for the condensate, in which the kinetic energy per atom is small compared to the chemical potential and (for \( \hbar = 1 \)) the Rabi-frequency, and (v) the negligibility of thermal excitations. With these assumptions we have given an analytical treatment of the problem of strong outcoupling, where everywhere in the condensate the Rabi frequency dominates over the space-dependent detuning, which is at most of the order of the chemical potential. The equations of motion derived in that limit (eqs. (4.12), (4.13), and (4.14) for rf-outcoupling, eqs. (4.24), (4.25), (4.28), and (4.31) for microwave and Raman outcoupling) could be solved exactly. For its quantitative validity our analytical treatment depends crucially on the assumptions (ii), (iii), (iv), and (v), but we expect that qualitatively our results should even be applicable if these assumptions are not well satisfied. On the other hand the method we use can be generalized to anisotropic harmonic potentials, so that assumption (i) could be relaxed, but at the cost of more tedious analytical expressions. Our method of solution expresses the densities and the phases of the time-dependent macroscopic wavefunctions in the coupled states in terms of the center of mass coordinate and the Thomas-Fermi radius of the condensate and the outcoupled state and provides the ordinary second order differential equations satisfied by these quantities.

The center of mass motion is in all cases very simple. It is, of course, independent of the interaction between the atoms and driven by the momentum transfer from the outcoupling field to the atoms in the case of the Raman outcoupling, by gravity, and by the time-averaged trapping potential seen by the atoms. The latter vanishes for resonant rf-outcoupling within the hyperfine manifold of the condensate, i.e. the center of mass in this case is freely falling after the outcoupling field is switched on. If the outcoupling field is off-resonance, and in the cases of microwave or Raman outcoupling, the time-averaged trapping potential is, in general, non-zero (cf. eq. (4.33) for the center of mass motion in this case) and leads either to a residual harmonic binding of the center of mass to the trap or to exponential repulsion from it.

The second parameter characterizing the condensate in the outcoupling fields is the time-dependent Thomas-Fermi radius. Its equation of motion (eq. (4.22)) while the outcoupling fields are on, eqs. (5.7) or (5.10) after the outcoupling fields are switched off) are still very simple and can be solved analytically up to quadratures. In figs. 6-8 the three different types of solutions are portrayed.

Fig. 6 shows the expansion of the Thomas-Fermi radius for the resonant rf-outcoupling within the hyperfine manifold of the condensate, which is the solution of eq. (4.22). An initial phase of accelerated expansion due to the mutual repulsion of the atoms is followed by a purely ballistic expansion with constant velocity given by eq. (4.23). If the outcoupling field is switched off the density of atoms in the antitrapped states enter a phase of exponential expansion. The total evolution of \( r_{TF}(t) \) for this case is shown in fig. 6 while the complete evolution of the density in the antitrapped state is seen in fig. 7 for the \( F = 1 \) manifold and in fig. 8 for the \( F = 2 \) manifold. The only difference between these two cases turns out to be the pattern of the Rabi-oscillations, seen directly in figs. 6 and 7. Fig. 6 shows the expansion of the Thomas-Fermi radius for a case where the time-averaged potential seen by the atoms is repulsive. The expansion again starts out with constant acceleration and crosses over to an exponential phase even while the outcoupling fields are on. The third possibility for the evolution of \( r_{TF}(t) \) is portrayed in fig. 8 which applies to the case where the time-averaged trapping potential seen by the atoms in the outcoupling field remains attractive. The Thomas-Fermi radius oscillates in this case around its new enlarged equilibrium value either until dissipation takes over (neglected throughout this work, which is justified if only a few oscillation cycles are considered) or until the outcoupling fields are switched off. If this is done at a moment where most atoms are in the outcoupled state, which may be untrapped...
or antitrapped, a ballistic or exponential expansion follows the oscillations, respectively. The complete evolution of the density of atoms outcoupled to an untrapped state is shown in fig.11.

It becomes clear from our treatment that external electromagnetic fields inducing rapid Rabi oscillations are excellent tools for suddenly changing or effectively turning off the trapping potential, thereby exciting the center of mass and the dilation-compression modes of the condensate. A further freedom in the possible changes of the effective trapping potential is the introduction of a constant overall detuning of the Rabi oscillations, if \( \omega \) is chosen in such a way that nowhere in the condensate the resonance condition is satisfied. Then only a small fraction of the atoms in the condensate is coupled out by each pulse of the external fields. The strong-coupling limit in this case is reached if the effective Rabi frequency, including the detuning, is large compared to the variation of the detuning across the condensate. Our treatment, in which resonance was assumed, may easily be extended to this case.

As long as the external electromagnetic field is on, the condensate is not yet coupled out, unless the antitrapped states to which it is coupled happens to be more strongly repelled by the magnetic trap than the trapped states are attracted, which is the case \( p > 1 \) in eq.(3.15). In all other cases, how many atoms are finally coupled out depends crucially on the phase \( \Omega T \) of the Rabi cycle at the moment when the electromagnetic field is switched off. In figs.8,11 we have chosen the switch-off time \( T \) in such a way that the outcoupled state has its maximum occupation probability precisely equal to or at least close to 1. A comparison of these figures, and of the formulas to which they correspond, shows that outcoupling to an untrapped state as in fig.11 rather than an antitrapped state as in figs.9,10 is highly preferable to achieve a spatially confined, narrow pulse, as the spatial width of the pulse in the antitrapped states is rapidly broadened by the repelling trap potential. However, the coherence of the outcoupled atom wave should permit a refocussing of the outcoupled pulse by the techniques developed in atom optics.

We have followed the dynamics of the atoms only in the region of space where they feel the parabolic part of the trap potential. In order to follow them further as they leave the trap and finally become freely propagating atoms it is necessary to make detailed assumptions about the trapping potential far from the center of the trap. If this can be done for a given experiment one may use semiclassical methods to solve the Schrödinger equation for the atoms [7], eventually neglecting their interaction if the atomic beam has become sufficiently dilute. This neglect is possible as soon as the \( r_{TF}^{-4} \)-term in the equation of motion for \( r_{TF} \) has become negligible. Our present results for the region inside the trapping potential provides the initial or boundary condition which is required for such a further analysis.

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**APPENDIX:**

Here we record the needed analytic expressions for the Rabi oscillations in the various cases considered in this work.

### F\(=1\) manifold:

The ansatz

\[
\tilde{\psi} = e^{-i\int_0^\tau d\tau \left[ \kappa(\tilde{r},\tau) + \Delta(\tilde{r}) \right]} |\psi(\tilde{r}, t)|\varphi(\Omega t)
\]

with \( |\varphi|^2 = 1 \) transforms eq.(3.6) into

\[
i\varphi' = \begin{pmatrix} -\delta & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & \delta \end{pmatrix} \varphi
\]

(A2)

with \( \delta = \Delta/\Omega \), where \( \varphi' \) is the derivative of \( \varphi \) with respect to \( \tau = \Omega t \). Its solution with initial condition \( \varphi_m = \delta_{m,1} \) is given by

\[
\varphi = \frac{1}{2 + \delta^2} \begin{pmatrix} 1 + (1 + \delta^2)\cos(\sqrt{2 + \delta^2} \tau) + i\delta \sqrt{2 + \delta^2} \sin(\sqrt{2 + \delta^2} \tau) \\ \delta(1 - \cos(\sqrt{2 + \delta^2} \tau)) - i\delta \sqrt{2 + \delta^2} \sin(\sqrt{2 + \delta^2} \tau) \\ -1 + \cos(\sqrt{2 + \delta^2} \tau) \end{pmatrix}.
\]

(A3)

In the present work we need \( \varphi(\Omega t) \) for \( \delta = 0 \) and
\[ (\nabla \varphi)_{\delta=0} = \frac{\partial \varphi}{\partial \delta} \big|_{\delta=0} \nabla \delta \] 

(A4)

in order to evaluate the current velocity \((h/M) \text{Im}[\varphi^* \nabla \varphi]_{\delta=0}\). Using (A3) we obtain

\[ \left( \frac{\partial \varphi}{\partial \delta} \right)_{\delta=0} = \frac{1}{2} \begin{pmatrix} i\sqrt{2} \sin(\sqrt{2}\Omega t) \\ 1 - \cos(\sqrt{2}\Omega t) \end{pmatrix} \] 

(A5)

and

\[ \text{Im}[\varphi^* \nabla \varphi]_{\delta=0} = \sqrt{\frac{2}{2\Omega}} \sin(\sqrt{2}\Omega t) \nabla \Delta \] 

(A6)

**F=2 manifold:**

The ansatz

\[ \tilde{\psi} = e^{-i \int_0^t d\tau [(\vec{r}, \tau) + 2\Delta(\vec{r})] |\psi(\vec{r}, t)| \varphi(\Omega t)} \] 

(A7)

with \(|\varphi|^2 = 1\) transforms eq.(3.12) into

\[
\begin{pmatrix} -2\delta & 1 & 0 & 0 & 0 \\ 1 & -\delta & \sqrt{\frac{3}{2}} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & \delta & 1 \\ 0 & 0 & 0 & 1 & 2\delta \end{pmatrix} \varphi = \begin{pmatrix} i\varphi' \\ \varphi \end{pmatrix}
\]

(A8)

with the same notation as in (A2). The solution with initial condition \(\varphi_m(0) = \delta_{m,2}\) is given by

\[
\varphi(\tau) = \varphi(0) + C_1 [\cos(\lambda_1 \tau) + \cos(\lambda_2 \tau) - 2] + C_2 \frac{\cos(\lambda_1 \tau) - \cos(\lambda_2 \tau)}{\lambda_1^2 - \lambda_2^2}
\]

\[ + iC_3 [\lambda_1 \sin(\lambda_1 \tau) + \lambda_2 \sin(\lambda_2 \tau)] + iC_4 \frac{\lambda_1 \sin(\lambda_1 \tau) - \lambda_2 \sin(\lambda_2 \tau)}{\lambda_1^2 - \lambda_2^2} \]

(A9)

with the characteristic dimensionless frequencies

\[
\lambda_1 = 2\sqrt{\delta^2 + 1}, \quad \lambda_2 = \sqrt{\delta^2 + 1}
\]

(A10)

and the coefficient-vectors

\[
C_1 = \frac{1}{16(\delta^2 + 1)^2} \begin{pmatrix} 8\delta^4 + 16\delta^2 + 5 \\ -6\delta^3 - 3 \\ 6\delta^2 - 1 \end{pmatrix} : C_2 = \frac{1}{16(\delta^2 + 1)} \begin{pmatrix} 24\delta^4 - 9 \\ -6\delta(8\delta^2 + 3) \\ 3\sqrt{6}(6\delta^2 + 1) \end{pmatrix} \]

\[
C_3 = \frac{1}{16(\delta^2 + 1)^2} \begin{pmatrix} 2\delta(2\delta^2 + 5) \\ 4\delta^2 - 5 \\ -3\sqrt{6}\delta \\ 3 \end{pmatrix} : C_4 = \frac{1}{16(\delta^2 + 1)} \begin{pmatrix} 6\delta(2\delta^2 - 3) \\ -9(4\delta^2 - 1) \\ 15\sqrt{6}\delta \\ -15 \end{pmatrix}
\]

(A11)

The special case \(\delta = 0\) gives \(\lambda_1 = 2, \lambda_2 = 1\) and leads to eq.(3.14). In the present work we also need \((\nabla \varphi)_{\delta=0}\) and hence \(\left( \frac{\partial \varphi}{\partial \delta} \right)_{\delta=0}\). Using (A9)-(A11) we obtain
\[
\left( \frac{\partial \varphi}{\partial \delta} \right)_{\delta=0} = \begin{pmatrix}
\frac{1}{2} (\sin(2 \Omega t) + \sin(\Omega t)) \\
-\frac{3}{4} (\cos(2 \Omega t) - 1) \\
-\frac{i \sqrt{2}}{2} (\sin(2 \Omega t) - 2 \sin(\Omega t)) \\
-\frac{1}{4} \cos(2 \Omega t) + \cos(\Omega t) - \frac{3}{4} \\
0
\end{pmatrix}
\]
(A12)

and

\[
\text{Im}[\varphi^* \nabla \varphi]_{\delta=0} = \frac{2}{\Omega} \sin(\Omega t) \nabla \Delta
\]
(A13)

Two-level case:
The ansatz

\[
\tilde{\psi} = e^{-i \int_0^\tau d\tau \left[ \epsilon(\tilde{r}, \tau) + \Delta(\tilde{r}) / 2 \right]} |\psi(\tilde{r}, \tau)| \varphi(\Omega t)
\]
(A14)

leads to

\[
\varphi(\tau) = \frac{1}{\sqrt{\delta^2 + 4}} \begin{pmatrix}
\sqrt{\delta^2 + 4} \cos(\sqrt{1 + \delta^2 / 4} \tau) + i \delta \sin(\sqrt{1 + \delta^2 / 4} \tau) \\
-2i \delta \sin(\sqrt{1 + \delta^2 / 4} \tau)
\end{pmatrix}
\]
(A15)

from which we can extract the result for

\[
\text{Im}[\varphi^* \nabla \varphi]_{\delta=0} = \frac{1}{4 \Omega} \sin(2 \Omega t) \nabla \Delta
\]
(A16)

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FIG. 1. Outcoupling schemes for radio-frequency fields (a) micro-wave fields (b) and optical fields (c)
FIG. 2. Potentials of the Zeeman sublevels of the $F = 1$ manifold
FIG. 3. Occupation probabilities of the Zeeman sublevels of the $F = 1$ manifold
FIG. 4. Occupation probabilities of the Zeeman sublevels of the $F=2$ manifold
FIG. 5. The Thomas-Fermi radius $r_{TF}(t)$ in units of $r_{TF}(0)$ as a function of $\bar{\omega}t$ during Rabi oscillations within a hyperfine-manifold.
FIG. 6. The Thomas-Fermi radius $r_{FT}(t)$ in units of $r_{FT}(0)$ as a function of $\bar{\omega}t$ during Rabi oscillations between a trapped and a weakly anti-trapped state ($p=1/2$)
FIG. 7. The Thomas-Fermi radius $r_{TF}(t)$ in units of $r_{TF}(0)$ as a function of $\bar{\omega}t$ during Rabi oscillations between a trapped and a strongly anti-trapped state ($p=3$)
FIG. 8. The Thomas-Fermi radius $r_{TF}(t)$ in units of $r_{TF}(0)$ as a function of $\bar{\omega}t$ during ($\bar{\omega}t < 2$) and after ($\bar{\omega}t > 2$) Rabi oscillations within a hyperfine manifold or between trapped and antitrapped states with opposite potentials.

FIG. 9. $|\psi_1|^2$ for outcoupling from the $F = 1$ manifold in units of $15N/8\pi r_{TF}^3(0)$ as function of the radial coordinate in the center of mass frame in units of $r_{TF}(0)$ and time in units $\Omega^{-1}$ for the choice of parameters $\bar{\omega}/\Omega = 0.1$ and $\Omega T = 20$.

FIG. 10. $|\psi_{-2}|^2$ for outcoupling from the $F = 2$ manifold plotted as in fig.9 for $\bar{\omega}/\Omega = 0.1$ and $\Omega T = 22$.

FIG. 11. $|\psi_2|^2$ for outcoupling from a trapped state to an untrapped state ($p = 0$) in a different hyperfine manifold plotted as in fig.9 for $\bar{\omega}/\Omega = 0.169$ and $\Omega T = 15\pi/2$. 
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