CONTINUITY AND ESTIMATES OF THE LIOUVILLE HEAT KERNEL WITH APPLICATIONS TO SPECTRAL DIMENSIONS

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ABSTRACT. The Liouville Brownian motion, recently introduced by Garban, Rhodes and Vargas, is a diffusion process evolving in a planar random geometry induced by the Liouville measure $\mathcal{M}_\gamma$, formally written as $\mathcal{M}_\gamma(dz) = e^{\gamma X(z) - \gamma^2 E[X(z)^2]/2} dz$, $\gamma \in (0, 2)$, for a (massive) Gaussian free field $X$. It is an $\mathcal{M}_\gamma$-symmetric diffusion defined as the time-change of the standard two-dimensional Brownian motion by the positive continuous additive functional with Revuz measure $\mathcal{M}_\gamma$.

In this paper we provide a detailed analysis of the heat kernel $p_t(x, y)$. Specifically, we prove its joint continuity, a locally uniform sub-Gaussian upper bound of the form $p_t(x, y) \leq C_1 t^{-1/2} \log(t^{-1}) \exp(-C_2 |x - y|^2 t^{-1/2})$ for small $t$ and an on-diagonal lower bound of the form $p_t(x, x) \geq C_3 t^{-1} \log(t^{-1})^{-\eta}$ for small $t$ for $\mathcal{M}_\gamma$-a.e. $x$ with some concrete constant $\eta > 0$. As applications, we also show that the pointwise spectral dimension equals $2$ $\mathcal{M}_\gamma$-a.e. and that the global spectral dimension is also $2$.

1. INTRODUCTION

One of the main mathematical issues in 2d-Liouville quantum gravity is to construct a random geometry on a two-dimensional manifold, say $\mathbb{R}^2$ equipped with the Euclidian metric $dx^2$, which can be formally described by a Riemannian metric tensor of the form

$$e^{\gamma X(x)} \, dx^2,$$

(1.1)

where $X$ is a massive Gaussian free field on $\mathbb{R}^2$ on a probability space $(\Omega, \mathcal{A}, P)$ and $\gamma \in (0, 2)$ is a parameter. The study of Liouville quantum gravity is mainly motivated by the so-called KPZ-formula (for Knizhnik, Polyakov and Zamolodchikov), which relates some geometric quantities in a number of models in statistical physics to their formulation in a setup governed by this random geometry. In this context, by the KPZ relation the parameter $\gamma$ can be expressed in terms of the the central charge of the underlying model. We refer to [8] and to the survey article [11] for more details on this topic.
However, to give rigorous sense to the expression in (1.1) is a highly non-trivial problem. Namely, as the correlation function of the Gaussian free field $X$ exhibits a short scale logarithmically divergent behaviour, the field $X$ is not a function but a random distribution. In other words, the underlying geometry is too rough to make sense in a classical Riemannian framework, so some regularization is required. While it is not clear how to execute a regularization procedure on the level of the metric, the method is quite performing in order to construct the associated volume form. More precisely, using the theory of multiplicative chaos established by Kahane in [17] (see also [21]), by a certain cutoff-procedure one can define the associated volume measure $\mathcal{M}_\gamma$ for $\gamma \in (0, 2)$, called the Liouville measure. It can be interpreted as given by

$$M_\gamma(A) = \int_A e^{\gamma X(z) - \frac{\gamma^2}{2} \mathbb{E}[X(z)^2]} dz,$$

but this expression for $M_\gamma$ is only very formal, for $M_\gamma$ is known to be singular w.r.t. the Lebesgue measure by a result [17], (141) by Kahane (see also [21], Theorems 4.1 and 4.2). Recently, in [14] Garban, Rhodes and Vargas have constructed the natural diffusion process $(B_t)_{t \geq 0}$ associated with (1.1), which they call Liouville Brownian motion (LBM). Similar results have been simultaneously obtained in [4]. On a formal level, $B$ is the solution of the SDE

$$dB_t = e^{-\frac{\gamma}{2} X(B_t) + \frac{\gamma^2}{4} \mathbb{E}[X(B_t)^2]} dB_t,$$

where $B$ is an independent standard Brownian motion on $\mathbb{R}^2$. In view of the Dambis-Dubins-Schwarz theorem this SDE representation suggests to define the LBM $B$ as a time-change of the planar Brownian motion $B$. This has been rigorously carried out in [14], and then by general theory the LBM turns out to be symmetric w.r.t. the Liouville measure $M_\gamma$. In the companion paper [13] Garban, Rhodes and Vargas also identified the Dirichlet form associated with $B$ and they showed that the transition semigroup is absolutely continuous w.r.t. $M_\gamma$, meaning that the Liouville heat kernel $p_t(x, y)$ exists. Moreover, they observed that the intrinsic metric $d_B$ generated by that Dirichlet form is identically zero, which indicates that

$$\lim_{t \downarrow 0} t \log p_t(x, y) = -\frac{d_B(x, y)^2}{2} = 0, \quad x, y \in \mathbb{R}^2,$$

and therefore some non-Gaussian heat kernel behaviour is expected. This degeneracy of the intrinsic metric is known to occur typically for diffusions on fractals, whose heat kernels indeed satisfy the so-called sub-Gaussian estimates.

In this paper we continue the analysis of the Liouville heat kernel, which has been been initiated simultaneously and independently in [19]. As our first main result we obtain the continuity of the heat kernel.

**Theorem 1.1.** Let $\gamma \in (0, 2)$. Then, $\mathbb{P}$-a.s., the Liouville heat kernel $p_t(x, y)$ is $(0, \infty)$-valued and jointly continuous on $(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$, and in particular, the Liouville
Brownian motion $B$ is irreducible. Moreover, the associated transition semigroup

$$P_t f(x) = E_x[f(B_t)] = \int_{\mathbb{R}^2} p_t(x,y) f(y) \, M_\gamma(dy)$$

is strong Feller, i.e. it maps Borel measurable bounded functions to continuous bounded functions.

As mentioned above, due to the degeneracy of the intrinsic metric we expect anomalous off-diagonal bounds for the heat kernel to hold. Indeed, we obtain the following sub-Gaussian upper bound:

**Theorem 1.2.** Let $\gamma \in (0,2)$ and $U \subset \mathbb{R}^2$ be bounded. For any $\beta > (\gamma + 2)^2/2 > 2$, $\mathbb{P}$-a.s., there exist $C_i = C_i(X, \gamma, U, \beta)$, $i = 1, 2$, such that

$$p_t(x,y) = p_t(y,x) \leq C_1 t^{-1} \log(t^{-1}) \exp\left(-C_2 \left(\frac{|x-y|^\beta \wedge 1}{t}\right)^{1/(\beta-1)}\right) \quad (1.2)$$

for all $t \in (0, \frac{1}{2}]$, $x \in \mathbb{R}^2$ and $y \in U$.

Since $\beta > 2$, the off-diagonal part $\exp\left(-C_2 (|x-y|^\beta \wedge 1)/t\right)^{1/(\beta-1)}$ of the estimate (1.2) indicates that the process diffuses slower than the two-dimensional Brownian motion, which is why such a bound is called sub-Gaussian. We do not expect that the lower bound $(\gamma + 2)^2/2$ for the exponent $\beta$ is best possible. Unfortunately, Theorem 1.2 alone does not even exclude the possibility that $\beta$ could be taken arbitrarily close to 2, which in the case of the two-dimensional torus has been in fact disproved in a recent result [19, Theorem 5.13] by Maillard, Rhodes, Vargas and Zeitouni showing that $\beta$ satisfying (1.2) for small $t$ must be at least $2 + \gamma^2/4$.

From the conformal invariance of the planar Brownian motion $B$ it is natural to expect that the LBM $B$ as a time change of $B$ admits two-dimensional behaviour, as was observed by physicists in [1] and in a weak form proved in [22] (see Remark 1.5 below). The on-diagonal part $t^{-1} \log(t^{-1})$ in (1.2) shows a sharp upper bound in this spirit except for a logarithmic correction, and we will also prove the following on-diagonal lower bound valid for $M$-a.e. $x \in \mathbb{R}^2$, which matches (1.2) besides another logarithmic correction.

**Theorem 1.3.** Let $\gamma \in (0,2)$. Then, $\mathbb{P}$-a.s., for $M_\gamma$-a.e. $x \in \mathbb{R}^2$ there exist $C_3 = C_3(X,\gamma)$ and $t_0 = t_0(X,\gamma, x) > 0$ such that

$$p_t(x,x) \geq C_3 t^{-1} (\log(t^{-1}))^{-\eta}, \quad \forall t \in (0, t_0], \quad (1.3)$$

for some explicit constant $\eta > 0$ (for instance $\eta = 34$ would be enough).

Combining the on-diagonal estimates in Theorems 1.2 and 1.3 we can immediately identify the pointwise spectral dimension as 2.

**Corollary 1.4.** Let $\gamma \in (0,2)$. Then, $\mathbb{P}$-a.s., for $M_\gamma$-a.e. $x \in \mathbb{R}^2$,

$$\lim_{t \downarrow 0} \frac{2 \log p_t(x,x)}{-\log t} = 2.$$
Essentially from Theorem 1.2 and Corollary 1.4 we shall deduce that the global spectral dimension, that is the growth order of the Dirichlet eigenvalues of the generator on bounded domains, is also 2 (see Subsection 6.2).

**Remark 1.5.** In [22, Theorem 3.6] the following result on the spectral dimension has been proved: Almost surely w.r.t. \( P \), for any \( \alpha > 0 \) and for all \( x \in \mathbb{R}^2 \),

\[
\lim_{y \to x} \int_0^\infty e^{-\lambda t} t^\alpha p_t(x,y) \, dt < \infty, \quad \forall \lambda > 0,
\]

(1.4)

and

\[
\lim_{y \to x} \int_0^\infty e^{-\lambda t} p_t(x,y) \, dt = \infty, \quad \forall \lambda > 0.
\]

(1.5)

In [22] the left hand sides were interpreted as the integrals w.r.t. \( t \) of the on-diagonal heat kernel \( p_t(x,x) \), which was needed due to the lack of the continuity of \( p_t(x,y) \). By Theorem 1.1 this interpretation can be made rigorous now, and moreover, (1.4) follows immediately from Theorem 1.2. On the other hand, (1.5) is actually an easy consequence of the Dirichlet form theory. Indeed, by [10, Exercises 2.2.2 and 4.2.2] \( \int_0^\infty e^{-\lambda t} p_t(x,x) \, dt \) is equal to the reciprocal of the \( \lambda \)-order capacity of the singleton \( \{x\} \) w.r.t. the LBM, and this capacity is zero by [10, Lemma 6.2.4 (i)] and the fact that the same holds for the planar Brownian motion.

The proofs of our main results above are mainly based on the moment estimates for the Liouville measure \( M_\gamma \) by [17, 23] and those for the exit times of the LBM \( B \) from balls by [14], together with the general fact from time change theory that the Green operator of the LBM has exactly the same integral kernel as that of the planar Brownian motion (see (2.3) below). To turn those moment estimates into \( \mathbb{P} \)-almost sure statements, we need some Borel-Cantelli arguments which cannot provide us with control on various random constants over unbounded sets. For this reason we can expect the estimate (1.2) to hold only \textit{locally} uniformly, so that the dependence of the constants \( C_1, C_2 \) on \( U \) in Theorem 1.2 cannot be dropped. Also to remove the logarithmic corrections in (1.2) and (1.3) and the restriction to \( M_\gamma \)-a.e. points in Theorem 1.3 and Corollary 1.4 one would need to have good uniform control on the ratios of the measures of concentric balls with different radii, which we cannot hope for at this moment since no good estimate of such ratios seems to be known.

The LBM can also be constructed on other domains like the torus, the sphere or planar domains \( D \subset \mathbb{R}^2 \) equipped with a log-correlated Gaussian field like the (massive or massless) Gaussian free field (cf. [14, Section 2.9]). In fact, Theorem 1.1 has been simultaneously and independently obtained in [19] for the LBM on the torus, where thanks to the boundedness of the space one can utilize the eigenfunction expansion of the heat kernel to prove its continuity and the strong Feller property of the semigroup. On the other hand, in our case of \( \mathbb{R}^2 \) the Liouville heat kernel \( p_t(x,y) \) does not admit such an eigenfunction expansion and the proof of its continuity and the strong Feller property requires some additional arguments. Therefore, although the proofs of our results should directly transfer to the other
domains mentioned above, we have decided to work on the plane $\mathbb{R}^2$ in this paper for the sake of simplicity and in order to stress that our methods also apply to the case of unbounded domains.

In [19] Maillard, Rhodes, Vargas and Zeitouni have also obtained upper and lower estimates of the Liouville heat kernel on the torus. Their heat kernel upper bound in [19, Theorem 4.2] involves an on-diagonal part of the form $Ct^{-(1+\delta)}$ for any $\delta > 0$ and an off-diagonal part of the form $\exp\left(-C(|x-y|^\beta/t)^{1/(\beta-1)}\right)$ for any $\beta > \beta_0(\gamma)$, where $\beta_0(\gamma)$ is a constant larger than our lower bound $(\gamma + 2)^2/2$ on the exponent $\beta$ and satisfies $\lim_{\gamma \to 2} \beta_0(\gamma) = \infty$. Thus Theorem 1.2 gives a better estimate, and our on-diagonal lower bound as in Theorem 1.3 is not treated in [19]. On the other hand, their off-diagonal lower bound [19, Theorem 5.13], which implies the bound $\beta \geq 2 + \gamma^2/4$ for the exponent $\beta$ in (1.2) (in the case of the torus) as mentioned above after Theorem 1.2, is not covered by our results.

The rest of the paper is organized as follows: In Section 2 we recall the construction of the LBM [14] and we introduce the precise setup. In Section 3 we prove preliminary estimates on the volume decay of the Liouville measure and on the exit times from balls needed in the proofs. In Section 4 we show that the resolvent operators of the killed LBM have the strong Feller property, which needed in Section 5, where we prove Theorems 1.1 and 1.2. In Subsection 5.1 we show the continuity of the Dirichlet heat kernel associated with the LBM killed upon exiting a bounded domain by using its eigenfunction expansion, and in Subsection 5.2 we then deduce the continuity of the heat kernel and the strong Feller property on unbounded domains using a recent result in [16]. Finally, in Section 6 we show the on-diagonal lower bound in Theorem 1.3 and thereby identify the pointwise and global spectral dimensions.

Throughout the paper, we write $C$ for random positive constants depending on the realization of the field $X$, which may change on each appearance. Random positive constants $C_i$ will be the same through the paper. Analogously, non-random positive constants will be denoted by $c$ or $c_i$, respectively. We denote by $B(x, R) = \{y \in \mathbb{R}^2 : |x - y| < R\}$, $x \in \mathbb{R}^2$, $R > 0$, open Euclidian balls in $\mathbb{R}^2$ and for abbreviation we set $B_R := B(0, R)$. Finally, we write $\|f\|_\infty := \sup_{x \in \mathbb{R}^2} |f(x)|$ for any bounded $f$ on $\mathbb{R}^2$.

2. Liouville Brownian motion

2.1. Massive Gaussian free field. Consider a massive Gaussian free field $X$ on the whole plane $\mathbb{R}^2$, i.e. a Gaussian Hilbert space associated with the Sobolev space $H^1_m$ defined as the closure of $C_c^\infty(\mathbb{R}^2)$ w.r.t. the inner product

$$\langle f, g \rangle_m := m^2 \int_{\mathbb{R}^2} f(x) g(x) \, dx + \int_{\mathbb{R}^2} \nabla f(x) \cdot \nabla g(x) \, dx,$$
where the quantity \( m > 0 \) is called the mass. More precisely, \((X, f)_m \in \mathcal{H}_b^2\) is a family of centered Gaussian random variables on a probability space \((\Omega, \mathcal{A}, \mathbb{P})\) with
\[
\mathbb{E}[\langle X, f \rangle_m (X, g)_m] = \langle f, g \rangle_m.
\]
In other words, the covariance function is given by the massive Green function associated with the operator \( m^2 - \Delta \), which can be written as
\[
g^{(m)}(x, y) = \frac{1}{4\pi} \int_0^\infty e^{-\frac{m^2}{4u} |x-y|^2} \, du = \int_1^\infty \frac{k^{(m)}(u(x-y))}{u} \, du,
\]
where
\[
k^{(m)}(z) = \frac{1}{4\pi} \int_0^\infty e^{-\frac{m^2}{2u}|z|^2} \, du.
\]
Following [14] we now introduce an \( n \)-regularized version of \( X \). To that aim let \( (c_n)_{n \geq 1} \) be an unbounded increasing sequence with \( c_1 = 1 \) and let \( (Y_n)_{n \geq 1} \) be a family of independent centered continuous Gaussian fields on \( \mathbb{R}^2 \) with covariance
\[
\mathbb{E}[Y_n(x) Y_n(y)] = \int_{c_n}^{c_{n+1}} \frac{k^{(m)}(u(x-y))}{u} \, du =: k_n(x, y). \tag{2.1}
\]
Then, the \( n \)-regularized field is defined as
\[
X_n(x) = \sum_{k=1}^n Y_k(x),
\]
the associated random measure \( M_n = M_{n, \gamma} \) is given by
\[
M_n(dx) = \exp \left( \gamma X_n(x) - \frac{\gamma^2}{2} \mathbb{E}[X_n(x)^2] \right) \, dx.
\]
Here the factor \( \gamma \geq 0 \) is a parameter. By the classical theory of Gaussian multiplicative chaos, established in Kahane’s seminal work [17] (see also [21]), \( \mathbb{P} \)-a.s. the family \( (M_n)_n \) converges to a limiting Radon measure \( M = M_* \), called the Liouville measure, which is non-trivial if and only if \( \gamma \in [0, 2) \). Throughout the paper, we will assume from now on that \( \gamma \in [0, 2) \) is fixed.

2.2. Definition of Liouville Brownian Motion. The Liouville Brownian motion has been constructed by Garban, Rhodes and Vargas in [14] as a canonical diffusion process under the geometry induced by the field \( X \) and the measure \( M \). More precisely, they have constructed a positive continuous additive functional \( \{F_t\}_{t \geq 0} \) of the planar Brownian motion \( B \) naturally associated with the measure \( M \) and they have defined the LBM as \( B_t = B_{F_t^{-1}} \). In this subsection we briefly recall the construction.

Let \( \Omega' := C([0, \infty), \mathbb{R}^2) \) and \( \mathcal{G} \) be the Borel \( \sigma \)-field on \( \Omega' \). Further, let \( \{P_x\}_{x \in \mathbb{R}^2} \) be the family of probability measures on \((\Omega', \mathcal{G})\) such that under each \( P_x \) the coordinate process \( \{B_t\}_{t \geq 0} \) is a two-dimensional Brownian motion starting at \( x \). We denote by \( \mathcal{G}^0_t = \sigma(B_s; s \leq t), \ t \geq 0, \mathcal{G}^0_\infty = \sigma(B_s; s < \infty) \), and by \( \{\mathcal{G}_t\}_{t \in [0, \infty)} \) the minimum completed admissible filtration for \( B \) as defined e.g. in [10, Section A.2]. Moreover, let \( \{\theta_t\}_{t \geq 0} \) be the family of shift mappings on \( \Omega' \), i.e. \( B_{t+s} = B_{t} \circ \theta_s, s, t \geq 0. \) Finally,
we write \( q_t(x,y) = (2\pi t)^{-1} \exp(-|x-y|^2/2t), \) \( t \geq 0, \, x, y \in \mathbb{R}^2 \) for the Gaussian heat kernel on \( \mathbb{R}^2 \).

**Definition 2.1.** i) A process \( A = (A_t)_{t \geq 0} \) is a positive additive functional (PCAF) of \( B \) in the strict sense, if \( A_t \) is a \( \mathcal{G}_t \)-measurable, \([0, \infty]\)-valued random variable for every \( t \geq 0 \) and if there exists a set \( \Lambda \in \mathcal{G}_\infty \), called a defining set for \( A \), such that

a) for all \( x \in \mathbb{R}^2, \) \( P_x[\Lambda] = 1, \)

b) for all \( t \geq 0, \) \( \theta_t(\Lambda) \subset \Lambda, \)

c) for all \( \omega \in \Lambda, \) \( s, t \geq 0, \)

\[
A_{t+s}(\omega) = A_t(\omega) + A_s \circ \theta_t(\omega),
\]

and \( t \mapsto A_t(\omega) \) is a \([0, \infty)\)-valued continuous function with \( A_0(\omega) = 0. \)

ii) Two such functionals \( A^1 \) and \( A^2 \) are called equivalent if \( P_x[A_t^1 = A_t^2] = 1 \) for all \( t > 0, \) \( x \in \mathbb{R}^2, \) or equivalently, there exists a defining set \( \Lambda \in \mathcal{G}_\infty \) for both \( A^1 \) and \( A^2 \) such that \( A_t^1(\omega) = A_t^2(\omega) \) for all \( t \geq 0, \) \( \omega \in \Lambda. \)

iii) For any such \( A \) a Borel measure \( \mu_A \) on \( \mathbb{R}^2 \) satisfying

\[
\int_{\mathbb{R}^2} f(y) \mu_A(dy) = \lim_{t \downarrow 0} \frac{1}{t} \int_{\mathbb{R}^2} E_x \left[ \int_0^t f(B_s) \, dA_s \right] \, dx,
\]

for any non-negative Borel function \( f \) is called the Revuz measure of \( A \), which exists uniquely by general theory (see e.g. [5, Theorem A.3.5]).

For every \( n \in \mathbb{N} \) let now \( F_t^n : \Omega \times \Omega' \to [0, \infty) \) be defined as

\[
F_t^n := \int_0^t \exp \left( \gamma X_n(B_s) - \frac{\gamma^2}{2} E[X_n(B_s)^2] \right) \, ds, \quad t \geq 0,
\]

which is strictly increasing. Note that for every \( n \) the functional \( F^n \) is a PCAF of \( B \) in the strict sense with defining set \( \Omega' \) and Revuz measure \( M_n. \)

**Theorem 2.2 ([14, Theorem 2.7]).** Almost surely w.r.t. \( \mathbb{P} \) the following hold:

i) There exists a unique PCAF \( F \) whose Revuz measure is \( M. \)

ii) For all \( x \in \mathbb{R}^2, \) \( P_x\)-a.s., \( F \) is strictly increasing and satisfies \( \lim_{t \to \infty} F_t = \infty. \)

iii) For all \( x \in \mathbb{R}^2, \) \( F^n \) converges to \( F \) in \( P_x\)-probability in \( C([0, \infty), \mathbb{R}) \), equipped with the topology of uniform convergence on compact sets.

The process \( (\mathcal{B}, \{P_t\}_{t \in \mathbb{R}^2}), \) \( \mathbb{P}\)-a.s. defined by \( B_t := B_{F_t^{-1}}, \) \( t \geq 0, \) is called the (massive) Liouville Brownian motion (LBM).

Thanks to Theorem 2.2, we can apply the general theory of time changes of Markov processes to have the following properties of the LBM: first, it is a recurrent diffusion on \( \mathbb{R}^2 \) by [10, Theorems A.2.12 and 6.2.3]. Furthermore by [10, Theorem 6.2.1 (i)] (see also [14, Theorem 2.18]), the LBM is \( M \)-symmetric, i.e. its transition semigroup \( (P_t) \) given by

\[
P_t(x, A) := E_x[B_t \in A]
\]
for $t \in (0, \infty)$, $x \in \mathbb{R}^2$ and a Borel set $A \subset \mathbb{R}^2$, satisfies
\[
\int_{\mathbb{R}^2} P_t f \cdot g \, dM = \int_{\mathbb{R}^2} f \cdot P_t g \, dM
\]
for all Borel measurable functions $f, g : \mathbb{R}^2 \to [0, \infty]$. Here the Borel measurability of $P_t(\cdot, A)$ can be deduced from [14, Corollary 2.20] (or from Proposition 2.4 below).

Remark 2.3. It is claimed in [14, Corollary 2.20] that $(P_t)$ is a Feller semigroup, meaning that $P_t$ preserves the space of bounded continuous functions. Note that this is different from the notion of a Feller semigroup as for instance in [5, 10], i.e. a strongly continuous Markov semigroup on the space of continuous functions vanishing at infinity. It is not known whether $(P_t)$ is a Feller semigroup in the latter sense.

It is natural to expect that the LBM can be constructed in such a way that it depends measurably on the randomness of the field $X$. However, this measurability does not seem obvious from the construction in [14], since there the existence of the PCAF $F$ has been deduced from some general theory on the Revuz correspondence for $\mathbb{P}$-a.e. fixed realization of $M$. To overcome this issue, in the following proposition we show the pathwise convergence of $F^n$ towards $F$ which also ensures the measurability of $F$ and $B$ w.r.t. $A \otimes G^0_\infty$. The proof is given in Appendix A.

**Proposition 2.4.** There exists a set $\Lambda \in \mathcal{A} \otimes G^0_\infty$ such that the following hold:

i) For $\mathbb{P}$-a.e. $\omega \in \Omega$, $P_x[\Lambda^\omega] = 1$ for any $x \in \mathbb{R}^2$, where $\Lambda^\omega := \{\omega' \in \Omega' : (\omega, \omega') \in \Lambda\}$.

ii) For every $(\omega, \omega') \in \Lambda$ the following limits exist in $\mathbb{R}$ for all $0 < s < t$:
\[
F_{s,t}(\omega, \omega') := \lim_{n \to \infty} (F^n_s(\omega, \omega') - F^n_s(\omega, \omega')), \\
F_t(\omega, \omega') := \lim_{n \to \infty} F_{n,t}(\omega, \omega').
\]
Moreover, with $F_0(\omega, \omega') := 0$, $F_t(\omega, \omega') : [0, \infty) \to [0, \infty)$ is continuous, strictly increasing and satisfies $\lim_{t \to \infty} F_t(\omega, \omega') = \infty$.

iii) Let $t \geq 0$ and set $F_t := t$ on $\Lambda^c$. Then $F_t$ is $A \otimes G^0_\infty$-measurable.

iv) For $\mathbb{P}$-a.e. $\omega \in \Omega$, the process $(F_t(\omega, \cdot))_{t \geq 0}$ is a PCAF of $B$ in the strict sense with defining set $\Lambda^\omega$.

The previous proposition implies easily that $F$ indeed has the Revuz measure $M$. More strongly, we have the following proposition valid for any starting point $x \in \mathbb{R}^2$, which we prove for later use in a slightly more general setting in Appendix B.

**Proposition 2.5.** Almost surely w.r.t. $\mathbb{P}$ the following holds: For any $x \in \mathbb{R}^2$ and any Borel functions $\eta : [0, \infty) \to [0, \infty]$ and $f : \mathbb{R}^2 \to [0, \infty]$,
\[
E_x \left[ \int_0^\infty \eta(t) f(B_t) \, dF_t \right] = \int_0^\infty \int_{\mathbb{R}^2} \eta(t) f(y) q_t(x, y) \, M(dy),
\]
and in particular, for any $t > 0$,

$$
\int_{\mathbb{R}^2} f(y) \, M(dy) = \frac{1}{t} \int_{\mathbb{R}^2} E_x \left[ \int_0^t f(B_s) \, dF_s \right] \, dx.
$$

2.3. The Liouville Dirichlet form. By virtue of Propositions 2.4 and 2.5, we can apply the general theory of Dirichlet forms to obtain an explicit description of the Dirichlet form associated with the LBM, as it has been done in [14, 13].

Denote by $H^1(\mathbb{R}^2)$ the standard Sobolev space, that is

$$
H^1(\mathbb{R}^2) = \{ f \in L^2(\mathbb{R}^2, dx) : \nabla f \in L^2(\mathbb{R}^2, dx) \},
$$

on which we define the form

$$
\mathcal{E}(f, g) = \int_{\mathbb{R}^2} \nabla f \cdot \nabla g \, dx. \tag{2.2}
$$

Recall that $(\mathcal{E}, H^1(\mathbb{R}^2))$ is the Dirichlet form of the planar Brownian motion $B$. By $H^1_e(\mathbb{R}^2)$ we denote the extended Dirichlet space, that is the set of $dx$-equivalence classes of Borel measurable functions $f$ on $\mathbb{R}^2$ such that $\lim_{n \to \infty} f_n = f \in \mathbb{R} \, dx$-a.e. for some $(f_n)_n \subset H^1(\mathbb{R}^2)$ satisfying $\lim_{k,l \to \infty} \mathcal{E}(f_k - f_l, f_k - f_l) = 0$. By [5, Theorem 2.2.13] we have the following identification of $H^1_e(\mathbb{R}^2)$:

$$
H^1_e(\mathbb{R}^2) = \{ f \in L^2_{\text{loc}}(\mathbb{R}^2, dx) : \nabla f \in L^2(\mathbb{R}^2, dx) \}.
$$

The capacity of a set $A \subset \mathbb{R}^2$ is defined by

$$
\text{Cap}(A) = \inf_{B \subset \mathbb{R}^2 \text{open}} \inf_{f \in H^1(\mathbb{R}^2), f|_{B^c} \geq 1 \text{ dx-a.e.}} \left\{ \mathcal{E}(f, f) + \int_{\mathbb{R}^2} f^2 \, dx \right\}.
$$

A set $A \subset \mathbb{R}^2$ is called polar if $\text{Cap}(A) = 0$. We call a function $f$ quasi-continuous if for any $\varepsilon > 0$ there exists an open $U \subset \mathbb{R}^2$ with $\text{Cap}(U) < \varepsilon$ such that $f|_{\mathbb{R}^2 \setminus U}$ is real-valued and continuous. By [10, Theorem 2.1.7] any $f \in H^1_e(\mathbb{R}^2)$ admits a quasi-continuous $dx$-version $\tilde{f}$, which is unique up to polar sets by [10, Lemma 2.1.4].

Then, as the Liouville measure $M$ is a Radon measure on $\mathbb{R}^2$ and does not charge polar sets by [14, Theorem 2.2] (or by Propositions 2.4, 2.5 and [5, Theorem 4.1.1 (i)]), the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of the LBM $B$ is a strongly local, regular Dirichlet form on $L^2(\mathbb{R}^2, M)$ which takes on the following explicit form by [10, Theorem 6.2.1]: The domain is given by

$$
\mathcal{F} = \{ u \in L^2(\mathbb{R}^2, M) : u = \tilde{f} \, M\text{-a.e. for some } f \in H^1_e(\mathbb{R}^2) \},
$$

which can be identified with $\{ f \in H^1_e(\mathbb{R}^2) : \tilde{f} \in L^2(\mathbb{R}^2, M) \}$ by [10, Lemma 6.2.1], and for $f, g \in \mathcal{F}$ the form $\mathcal{E}(f, g)$ is given by (2.2).
2.4. The killed Liouville Brownian motion. Let $U \subset \mathbb{R}^2$ be non-empty, open. We denote by $T_U := \inf\{s \geq 0 : B_s \notin U\}$ the exit time of the Brownian motion $B$ from $U$ and by $\tau_U := \inf\{s \geq 0 : B_s \notin U\}$ that of the LBM $B$. Since by definition $B_t = B_{F_s^{-1}}$, $t \geq 0$, and $F$ is a homeomorphism on $[0, \infty)$, we have $\tau_U = F_{T_U}$. Let now $B^U$ and $B^U$ denote the Brownian motion and the LBM, respectively, killed upon exiting $U$. That is, they are diffusions on the one-point compactification $U \cup \{\partial U\}$ of $U$ defined by

$$B^U_t := \begin{cases} B_t & \text{if } t < T_U, \\ \partial_U & \text{if } t \geq T_U, \end{cases} \quad B^U_t := \begin{cases} B_t & \text{if } t < \tau_U, \\ \partial_U & \text{if } t \geq \tau_U. \end{cases}$$

Then, for a bounded Borel function $f$, $t \geq 0$ and $\lambda > 0$, we write

$$P^U_t(f(x)) := E_x[f(B^U_t)] \quad \text{and} \quad R^U_t(f(x)) := E_x\left[\int_0^T e^{-\lambda t} f(B_t) dt\right]$$

for the semigroup and resolvent operators associated with the killed LBM, respectively. If $U$ is bounded, as a time-change of $B^U$ the killed LBM $B^U$ has the same integral kernel for its Green operator $G^U$ as $B^U$ (cf. Proposition B.1), namely for any non-negative Borel function $f$,

$$G^U f(x) := E_x\left[\int_0^{T_U} f(B_t) dt\right] = E_x\left[\int_0^{T_U} f(B_t) dF_t\right] = \int_U g^U(x, y) f(y) M(dy).$$  \hfill(2.3)

Here $g^U$ denotes the Euclidian Green kernel given by

$$g^U(x, y) = \int_0^\infty q^U_t(x, y) dt, \quad x, y \in \mathbb{R}^2,$$

for the jointly continuous transition density $q^U_t$ of $B^U$: $q^U_t(x, y) dy = P_x[B^U_t \in dy]$ and $q^U_t(x, y) = 0$ if $(x, y) \notin U \times U$. Finally, we recall (see e.g. [10, Example 1.5.1]) that the Green function $g_{B(x_0, R)}$ over a ball $B(x_0, R)$ is of the form

$$g_{B(x_0, R)}(x, y) = \frac{1}{\pi} \log \frac{1}{|x - y|} + \Psi_{x_0, R}(x, y),$$  \hfill(2.4)

for some bounded continuous function $\Psi_{x_0, R}$ on $B(x_0, R) \times B(x_0, R)$.

3. Preliminary estimates

3.1. Volume decay estimates. For our analysis of the Liouville heat kernel some good control on the volume of small balls under the Liouville measure is needed. An upper estimate has already been established in [14], so in the next lemma we provide a lower bound. The argument is based on some bounds on the negative moments of the mass of small balls. Such bounds have been proven in [23] in the case that the multiplicative chaos is associated with a family of Gaussian fields whose covariance function is contained a certain class of convolution kernels. Since we are not sure whether the cut-off procedure producing the approximating measures $M_n$ is covered by the results in [23] we give a comparison argument in the Lemma C.1.
Lemma 3.1. Let \( \varepsilon > 0 \), \( \alpha_1 := \frac{1}{2}(\gamma + 2)^2 \) and \( \alpha_2 := 2(1 - \frac{1}{2})^2 \). Then, for any \( R > 0 \), \( \mathbb{P} \)-a.s., there exist \( C_i = C_i(X, \gamma, R, \varepsilon) > 0 \), \( i = 4, 5 \), such that
\[
C_4 r^{\alpha_1 + \varepsilon} \leq M(B(x, r)) \leq C_5 r^{\alpha_2 - \varepsilon}, \quad \forall x \in B_R, \ R, r \in (0, 1).
\]

Proof. The upper bound is proven in [14, Theorem 2.2]. We now show the lower bound in the same manner. Since \( \alpha_1 = \inf_{q>0} (2 + \tilde{\xi}(q))/q \) with \( \tilde{\xi}(q) := 2q + \frac{q(1+q)}{2} \gamma^2 \) we may choose \( q \) such that \( (2 + \tilde{\xi}(q))/q \in (\alpha_1, \alpha_1 + \varepsilon) \). Now let \( R > 0 \) be fixed and, setting \( r_n := 2^{-n}R \) for any \( n \in \mathbb{Z} \), we define
\[
\Lambda_{R,n} := \left\{ \left( \frac{k}{2^n} R, \frac{l}{2^n} R \right) \mid k, l \in \mathbb{Z}, -2^n \leq k, l \leq 2^n \right\} \subset [-R, R]^2.
\]
Then, by Čebyšev’s inequality
\[
\mathbb{P} \left[ \inf_{x \in \Lambda_{R,n}} M(B(x, r_n)) \leq 2^{-n(\alpha_1 + \varepsilon)} \right]
\]
\[
= \mathbb{P} \left[ \sup_{x \in \Lambda_{R,n}} M(B(x, r_n))^{-q} \geq 2^{n(\alpha_1 + \varepsilon)q} \right]
\]
\[
\leq 2^{-n(\alpha_1 + \varepsilon)q} \sum_{x \in \Lambda_{R,n}} \mathbb{E} \left[ M(B(x, r_n))^{-q} \right] = 2^{-n(\alpha_1 + \varepsilon)q} 2^{2n} \mathbb{E} \left[ M(B(0, r_n))^{-q} \right]
\]
for some constant \( c = c(\gamma, R, q) > 0 \), where we used the spatial stationarity of the law of \( M \) and Lemma C.1. In particular, by our choice of \( q \),
\[
\sum_{n=1}^{\infty} \mathbb{P} \left[ \inf_{x \in \Lambda_{R,n}} M(B(x, r_n)) \leq 2^{-n(\alpha_1 + \varepsilon)} \right] < \infty,
\]
so that by the Borel-Cantelli lemma \( \mathbb{P} \)-a.s. for some \( C = C(X, \gamma, R, q) > 0 \) we have that \( M(B(x, r_n)) \geq C2^{-n(\alpha_1 + \varepsilon)} \) for all \( x \in \Lambda_{R,n} \) for all \( n \geq 1 \). Since for every \( y \in B_R \) and \( r \in (0, 1) \) we have \( B(y, r) \supset B(x, r_n) \) for some \( x \in \Lambda_{R,n} \) with \( n \) satisfying \( 2^{-n} \approx r \), the claim follows. \( \square \)

3.2. Exit time estimates. In this subsection we provide some lower estimates on the exit times from balls, which are needed in the proof of Theorems 1.1 and 1.2. More precisely, we establish estimates on the negative moments and the tail behaviour at zero of these exit times.

Proposition 3.2. For any \( q > 0 \), \( \kappa > 2 + \tilde{\xi}(q) \) with \( \tilde{\xi}(q) := 2q + \frac{q(1+q)}{2} \gamma^2 \) and any \( R \geq 1 \), \( \mathbb{P} \)-a.s., there exists a random constant \( C_6 = C_6(X, \gamma, R, \kappa, q) > 0 \) such that
\[
E_x \left[ t^{-q}_{B(x,r)} \right] \leq C_6 r^{-\kappa}, \quad \forall r \in (0, 1), \ x \in B_R.
\]

Proof. First we note that from [14, Proposition 2.12] and Fatou’s lemma, we get for all \( q > 0 \) and for all \( x \in \mathbb{R}^2 \),
\[
E E_x \left[ t^{-q}_{B(x,r)} \right] \leq c r^{-\tilde{\xi}(q)}, \quad \forall r \in (0, 1),
\]
for some \( c = c(q) > 0 \). As in the proof of Lemma 3.1 above let \( r_n := 2^{-n}R \) and \( \Lambda_{R,n} \) be defined as in (3.1) for any \( n \in \mathbb{Z} \). In the sequel we write \( E_{\mu} \) for the expectation
operator associated with the law of a Brownian motion with initial distribution \( \mu \). Let \( \mu_{x,r_n} = P_x[B_{x,r_n} \in \cdot] \) be the distribution of the LBM upon exiting \( B(x,r_n) \). Since \( B(z,r_n) \subset B(x,2r_n) \) for any \( z \in \partial B(x,r_n) \), we have \( \tau_{B(z,r_n)} \leq \tau_{B(x,2r_n)} \), \( P_z \)-a.s. Hence, using (3.2) we get

$$
\mathbb{E} E_{\tau_{B(x,2r_n)}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq \mathbb{E} E_{\tau_{B(x,2r_n)}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq c r_n^{-\xi(q)},
$$

provided that \( n \) is large enough so that \( r_n \leq 1 \). This implies

$$
\mathbb{E} E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq c r_n^{-\xi(q)}.
$$

Let now \( \kappa > 2 + \xi(q) \) and \( n_0 \) satisfying \( r_{n_0} \leq 1 \) be fixed. Then, for all \( n \geq n_0 \) we obtain by Chebychev’s inequality,

$$
P \left[ \sup_{x \in \Lambda_{R,n+1}} E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \geq r_n^{-\kappa} \right] \leq r_n^{-\kappa} \sum_{x \in \Lambda_{R,n+1}} \mathbb{E} E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq \frac{c}{2^{n(\kappa-\xi(q)-2)}},
$$

for some \( c = c(R,q,\kappa) > 0 \). Hence, by our choice of \( \kappa \),

$$
\sum_{n \geq n_0} P \left[ \sup_{x \in \Lambda_{R,n+1}} E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \geq r_n^{-\kappa} \right] < \infty
$$

and we apply the Borel-Cantelli lemma to obtain that \( P \)-a.s. for all \( n \) with \( r_n \leq 1 \) and for all \( x \in \Lambda_{R,n+1} \),

$$
E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq C r_n^{-\kappa}
$$

(3.3)

for some random constant \( C = C(X,\gamma, R, q, \kappa) > 0 \).

Now for any \( r \in (0,1] \) we choose \( n \) such that \( r_n \leq \frac{2}{3} r \leq 2 r_n \). For all \( y \in B_R \), by construction there exists an \( x \in \Lambda_{R,n+1} \) such that \( |x-y| \leq \frac{1}{2} r_n \). Furthermore, note that \( B(x,r_n) \subset B(x,2r_n) \subset B(y,r) \) and therefore \( \tau_{B(x,r_n)} < \tau_{B(x,2r_n)} \leq \tau_{B(y,r)} \), \( P_y \)-a.s. In particular, by using the strong Markov property

$$
E_y \left[ \tau_{B(y,r)}^{-q} \right] \leq E_y \left[ \tau_{B(x,2r_n)}^{-q} \right] \leq E_y \left[ \left( \tau_{B(x,2r_n)} + \tau_{B(x,r_n)} \circ \theta_{B(x,r_n)} \right)^{-q} \right] \leq E_y \left[ \left( \tau_{B(x,2r_n)} \circ \theta_{B(x,r_n)} \right)^{-q} \right] = E_y E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] = E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right],
$$

where \( \mu_{x,r_n} := P_y[B_{\tau_{B(x,2r_n)}} \in \cdot] \). Since the exit laws of the LBM and the planar Brownian motion coincide, the exact formula for the distribution of a Brownian motion upon exiting balls (see e.g. [3, Theorem II.1.17]) implies that \( \mu_{x,r} \leq c \mu_{x,r_n} \) for some explicit constant \( c > 0 \). (This can be regarded as an application of the scale-invariant elliptic Harnack inequality.) Thus, \( E_y \left[ \tau_{B(y,r)}^{-q} \right] \leq c E_{\mu_{x,r_n}} \left[ \tau_{B(x,2r_n)}^{-q} \right] \) and the claim follows from (3.3).

**Proposition 3.3.** For any \( \beta > \alpha_1 \) and any \( R \geq 1 \), \( P \)-a.s., there exist random constants \( C_i = C_i(X, \gamma, R, \beta) > 0 \), \( i = 7,8 \), such that

$$
P_x[\tau_{B(x,r)} \leq t] \leq C_7 \exp \left( -C_8 (r^\beta/t)^{1/(\beta-1)} \right), \quad \forall t > 0, x \in B_R, r \in (0,1].
$$
Proof. By [16, Theorem 7.2] (see also [15, Theorem 9.1]) it is enough to show that there exists \( \varepsilon \in (0, 1) \), \( \delta > 0 \) such that \( P_x \left[ \tau_{B(x, r)} \leq \tilde{t} \right] \leq \varepsilon \) for all \( t \in (0, \delta r^\beta) \) and all \( x \in B_R \).

Since \( \beta > \alpha_1 = \inf_{q>0} (2 + \xi(q))/q \) there exists some \( q > 0 \) such that \( \beta > (2 + \xi(q))/q \). Furthermore, for any \( \varepsilon \in (0, 1) \) let \( \delta := (\varepsilon/C_3)^{1/q} \) and let \( \kappa > 2 + \xi(q) \) be such that \( \beta > \kappa/q \). Then, the claim is immediate since

\[
P_x \left[ \tau_{B(x, r)} \leq \delta r^\beta \right] = P_x \left[ (\tau_{B(x, r)})^{-q} \geq (\delta r^\beta)^{-q} \right] \leq C_3 \delta^q r^{\beta q - \kappa} \leq \varepsilon.
\]

by \( \varepsilon \)-C\( \delta \)-v\( \lambda \)-sev's inequality and Proposition 3.2. \( \square \)

4. Strong Feller properties of the resolvent

In this section we prove that the resolvent operator of the killed LBM \( B_U^\lambda \) has the strong Feller property. We will mainly follow the arguments in [13, Theorem 3.4], where the corresponding Feller property of the original LBM \( B \) is established. The essential ingredients are a coupling lemma and the following lemma.

Lemma 4.1 ([12, Lemma 2.19]). Almost surely w.r.t. \( \mathbb{P} \), for all \( R > 0 \),

\[
\lim_{\delta \to 0} \sup_{n \geq 1} \sup_{x \in B_R} E_x [F_{t_n}^\delta] = 0.
\]

Proposition 4.2. For any non-empty open set \( U \subseteq \mathbb{R}^2 \), \( \mathbb{P}\)-a.s., for any \( \lambda > 0 \) the resolvent operator \( R^\lambda_U \) is strong Feller, i.e. it maps a Borel measurable bounded function to a continuous bounded function.

Proof. Let \( f : U \to \mathbb{R} \) be Borel measurable and bounded and recall that \( T_U = \inf \{ s \geq 0 : B_s \notin U \} \) denotes the exit time of the Brownian motion \( B \) from \( U \). First note that since \( \tau_U = F_{T_U} \), the resolvent operator of \( B^U \), can be written as

\[
R^\lambda_U f(x) = E_x \left[ \int_0^{\tau_U} e^{-\lambda tf(B_t)} dt \right] = E_x \left[ \int_0^{T_U} e^{-\lambda f(B_t)} dB_t \right] + E_x \left[ \int_{T_U \wedge \infty} e^{-\lambda f(B_t)} dB_t \right]
\]

=: \( N_\varepsilon(x) + R^\lambda_{\varepsilon} f(x) \), (4.1)

for any \( \varepsilon > 0 \). From Lemma 4.1 we deduce that \( N_\varepsilon(x) \leq \| f \|_\infty E_x [F_{\varepsilon}] \) converges towards zero uniformly over compact subsets of \( \mathbb{R}^2 \) as \( \varepsilon \to 0 \). Hence, it is enough to show that \( R^\lambda_{\varepsilon} f \) is continuous. By the Markov property of the Brownian motion we obtain that

\[
R^\lambda_{\varepsilon} f(x) = E_x \left[ \mathbb{1}_{\{ T_U > \varepsilon \}} \int_\varepsilon^{T_U} e^{-\lambda f(B_t)} dB_t \right]
\]

\[
= E_x \left[ \mathbb{1}_{\{ T_U > \varepsilon \}} E_{B_x} \left[ \int_0^{T_U} e^{-\lambda f(B_t)} dB_t \right] e^{-\lambda f_{\varepsilon}} \right] = E_x \left[ \mathbb{1}_{\{ T_U > \varepsilon \}} e^{-\lambda f_{\varepsilon}} R^\lambda_U f(B_x) \right].
\]

Now for any two points \( x, y \in \mathbb{R}^2 \) we use the coupling lemma [14, Lemma 2.9], which allows to construct a couple \((B^x, B^y)\) consisting of two Brownian motions
$B^x$ and $B^y$ starting at $x$ and $y$, respectively, with the property that both processes coincide after a stopping time $\tau_{xy}$. We denote by $P_{x,y}$ the law of $(B^x,B^y)$ and by $E_{x,y}$ the corresponding expectation. Then, we obtain

$$\begin{align*}
|R^{U,x}_\lambda f(x) - R^{U,x}_\lambda f(y)| &= |E_{x,y} \left[ \mathbb{1}_{\{T_0^x > \epsilon\}} e^{-\lambda F^x} R^{U}_\lambda f(B^x) - \mathbb{1}_{\{T_0^y > \epsilon\}} e^{-\lambda F^y} R^{U}_\lambda f(B^y) \right] | \\
&\leq E_{x,y} \left[ \mathbb{1}_{\{T_0^x > \epsilon\}} e^{-\lambda F^x} \left( R^{U}_\lambda f(B^x) - R^{U}_\lambda f(B^y) \right) \right] \\
&\quad + E_{x,y} \left[ \left( \mathbb{1}_{\{T_0^x > \epsilon\}} e^{-\lambda F^x} - \mathbb{1}_{\{T_0^y > \epsilon\}} e^{-\lambda F^y} \right) R^{U}_\lambda f(B^y) \right]. \quad (4.2)
\end{align*}$$

Note that on the event $\{\tau_{xy} \leq \epsilon\} \cap \{T_0^x > \epsilon\} \cap \{T_0^y > \epsilon\}$ we have $R^{U}_\lambda f(B^x) = R^{U}_\lambda f(B^y)$. Thus, the first term in (4.2) can be estimated from above by

$$E_{x,y} \left[ \mathbb{1}_{\{T_0^x > \epsilon\}} \left| R^{U}_\lambda f(B^x) - R^{U}_\lambda f(B^y) \right| \right] \leq 2\lambda^{-1} \|f\|_{\infty} E_{x,y} \left[ \mathbb{1}_{\{\tau_{xy} > \epsilon\} \cup \{T_0^y \leq \epsilon\}} \right] \leq 2\lambda^{-1} \|f\|_{\infty} \left( P_{x,y} [\tau_{xy} > \epsilon] + P_{x,y} [T_0^y \leq \epsilon] \right),$$

where we used the trivial bound $\|R^{U}_\lambda f\|_{\infty} \leq \lambda^{-1} \|f\|_{\infty}$. Since for some $r > 0$ small enough,

$$P_{x,y} \left[ T_0^y \leq \epsilon \right] \leq P_{x,y} \left[ T_{B(y,r)} \leq \epsilon \right] = P_y \left[ T_{B(y,r)} \leq \epsilon \right] \leq 2e^{-r^2/4\epsilon},$$

the second term converges to zero uniformly over $y$ in a compact set as $\epsilon \to 0$ (see e.g. [3, Proposition I.4.8]). On the other hand, for any $\epsilon > 0$ the first term converges to zero as $|x - y| \to 0$ (cf. again [14, Lemma 2.9]).

The second term in (4.2) can be estimated from above by

$$\lambda^{-1} \|f\|_{\infty} E_{x,y} \left[ \mathbb{1}_{\{T_0^x > \epsilon\}} - \mathbb{1}_{\{T_0^y > \epsilon\}} \right] e^{-\lambda F^x} + \mathbb{1}_{\{T_0^y > \epsilon\}} e^{-\lambda F^y} - e^{-\lambda F^y} \right) \right] \leq \lambda^{-1} \|f\|_{\infty} \left( P_{x,y} [T_0^y \leq \epsilon] + P_{x,y} [T_0^y \leq \epsilon] + E_{x,y} \left[ e^{-\lambda F^x} - e^{-\lambda F^y} \right] \right).$$

As above, the first two terms converge to zero uniformly over compact sets as $\epsilon \to 0$. Moreover, by the same arguments as in [13, p.14],

$$\lim \sup_{|x-y| \to 0} E_{x,y} \left[ e^{-\lambda F^x} - e^{-\lambda F^y} \right] = 0, \quad (4.3)$$

which then finishes the proof. For the sake of completeness we repeat the argument here. For any $\epsilon > 0$,

$$E_{x,y} \left[ e^{-\lambda F^x} - e^{-\lambda F^y} \right] \leq 2 P_{x,y} [\epsilon \leq \tau_{xy}] + E_{x,y} \left[ e^{-\lambda F^x} - e^{-\lambda F^y} \mathbb{1}_{\{\epsilon > \tau_{xy}\}} \right] = 2 P_{x,y} [\epsilon \leq \tau_{xy}] + E_{x,y} \left[ e^{-\lambda (F^x_{xy} + F^y - F^x_{xy})} - e^{-\lambda (F^y_{xy} + F^y - F^y_{xy})} \mathbb{1}_{\{\epsilon > \tau_{xy}\}} \right].$$
Since on the event \( \{ \varepsilon > \tau_{xy} \} \), we have \( F^x_{\varepsilon} - F^x_{\tau_{xy}} = F^y_{\varepsilon} - F^y_{\tau_{xy}} \), we further obtain that
\[
E_{x,y} \left[ e^{-\lambda F^x_{\varepsilon}} - e^{-\lambda F^y_{\varepsilon}} \right] \leq 2 P_{x,y} \left[ \varepsilon \leq \tau_{xy} \right] + E_{x,y} \left[ e^{-\lambda F^x_{\tau_{xy}}} - e^{-\lambda F^y_{\tau_{xy}}} \right] \mathbb{1}_{\{ \varepsilon > \tau_{xy} \}}
\]
\[
\leq 2 P_{x,y} \left[ \varepsilon \leq \tau_{xy} \right] + E_{x,y} \left[ \min \left( 2, \lambda |F^x_{\tau_{xy}} - F^y_{\tau_{xy}}| \right) \right]
\]
\[
\leq 2 P_{x,y} \varepsilon \leq \tau_{xy} \right] + E_{x,y} \left[ \min \left( 2, \lambda F^x_{\delta} + \lambda F^y_{\delta} \right) \right] + 2 P_{x,y} \delta < \tau_{xy},
\]
for any arbitrary \( \delta > 0 \). Thus, using the coupling lemma [14, Lemma 2.9] as above,
\[
\limsup_{y \to x} E_{x,y} \left[ e^{-\lambda F^x_{\varepsilon}} - e^{-\lambda F^y_{\varepsilon}} \right] \leq E_{x} \left[ \min \left( 2, 2\lambda F_{\delta} \right) \right].
\]
Finally, since \( \lim_{\varepsilon \to 0} E_{x,F_{\delta}} = 0 \) uniformly over \( x \) in a compact subset of \( \mathbb{R}^2 \) (see again Lemma 4.1), we obtain (4.3). \( \square \)

5. On-diagonal upper bound and continuity of the heat kernels

Recall that \( \mathcal{F} \) equipped with the norm \( \| f \|_{2}^{\mathcal{F}} := \mathcal{E}(f, f) + \| f \|_{L^2(\mathbb{R}^2, M)}^{2} \) is a Hilbert space. For any open set \( U \subset \mathbb{R}^2 \), we define \( \mathcal{F}_{U} \) to be the closure in \( (\mathcal{F}, \| \cdot \|_{\mathcal{F}}) \) of the set of all functions in \( \mathcal{F} \) that are compactly supported in \( U \). Then, it is well known that \( (\mathcal{E}, \mathcal{F}_{U}) \) is the Dirichlet form associated with the killed Liouville Brownian motion \( \mathcal{B}^{U} \) and that it is regular on \( L^2(U, M) \) (see e.g. [10, Theorems 4.4.2 and 4.4.3]). We will denote the corresponding generator on \( L^2(U, M) \) by \( \mathcal{L}_{U} \) and the associated semigroup and resolvent operators by \( (T_{t}^{U})_{t \geq 0} \) and \( (G_{\lambda}^{U})_{\lambda > 0} \), respectively.

**Theorem 5.1.** Almost surely w.r.t. \( \mathbb{P} \), for any non-empty open set \( U \subset \mathbb{R}^2 \) the following hold:

i) There exists a (unique) jointly continuous function \( p_{U}^{U} = p_{U}^{U}(x, y) : (0, \infty) \times U \times U \to [0, \infty) \) such that for all \( (t, x) \in (0, \infty) \times U \), \( P_{x}^{U}[\mathcal{B}_{t}^{U} \in dy] = p_{t}^{U}(x, y) M(dy) \), which we refer to as the Dirichlet Liouville heat kernel on \( U \).

ii) The semigroup operator \( T_{t}^{U} \) is strong Feller, i.e. it maps Borel measurable bounded functions on \( U \) to continuous bounded functions on \( U \).

iii) If \( U \) is connected, \( p_{t}^{U}(x, y) \in (0, \infty) \) for any \( (t, x, y) \in (0, \infty) \times U \times U \), and in particular, \( \mathcal{B}^{U} \) is irreducible.

From now on we will write \( p_{t}(\cdot, \cdot) \) instead of \( p_{t}^{\mathbb{R}^2}(\cdot, \cdot) \) and call it the (global) Liouville heat kernel. Note that Theorem 1.1 follows directly from Theorem 5.1 by choosing \( U = \mathbb{R}^2 \).

5.1. The heat kernel on bounded open sets. In this subsection we will prove Theorem 5.1 for a fixed non-empty, bounded, open \( U \subset \mathbb{R}^2 \). We denote by \( \| f \|_{p} \) the \( L^p(U, M) \)-norm, \( p \geq 1 \), and by \( \langle \cdot, \cdot \rangle \) the \( L^2(U, M) \)-inner product. Let \( R > 0 \) be such that \( U \subset B_{R} \).
Proposition 5.2 (Faber-Krahn-type inequality). Almost surely w.r.t. $\mathbb{P}$, there exists $C_0 = C_0(\gamma, R) > 0$ such that the smallest eigenvalue $\lambda_1(U)$ of $-\mathcal{L}_U$ satisfies

$$
\lambda_1(U) \geq \frac{C_0}{M(U) \log \left(1 + \frac{1}{M(U)}\right)}.
$$

Proof. First we recall that $\lambda_1(U)$ can be expressed by the variational formula

$$
\lambda_1(U)^{-1} = \sup \{ \langle G_U f, f \rangle, f \geq 0, \|f\|_2 = 1 \} \tag{5.1}
$$

by \cite[Theorems 1.5.4 and 4.2.6]{10} as $g^U \in L^2(U \times U, M \times M)$ similarly to (B.3). Setting $\nu := \alpha_2/2 = (1 - \frac{\gamma}{2})^2$, as $U \subset B_R$ we have for any $f \geq 0$ with $\|f\|_2 = 1$,

$$
\langle G_U f, f \rangle \leq \langle G_{B_R} f, f \rangle = \int_U \int_U \exp \left( \nu g_{B_R}(x, y) \right) M(dy) M(dx) \tag{5.2}
$$

$$
+ \int_U \int_U \frac{f(x)f(y)}{\nu} \log \left(1 + \frac{f(x)f(y)}{\nu}\right) M(dy) M(dx).
$$

Here we used the fact that $ab \leq a \log(1 + a) + e^b$ for any $a, b \geq 0$, which can be easily verified, with $a = f(x)f(y)/\nu$ and $b = \nu g_{B_R}(x, y)$. Recall the Green function over $B_R$ can be represented as in (2.4), which implies

$$
\int_U \int_U \exp(\nu g_{B_R}(x, y)) M(dy) M(dx) \leq c M(U)^2 + \int_U \int_U \frac{1}{\pi|x-y|^\nu} M(dy) M(dx).
$$

Setting $D_n := B(x, 2^{1-n}R) \setminus B(x, 2^{-n}R)$ and using Lemma 3.1 with $\varepsilon \in (0, \alpha_2 - \nu)$ we further obtain

$$
\int_U \int_U \frac{1}{|x-y|^\nu} M(dy) M(dx) \leq \int_U \int_{B(x,2R)} \frac{1}{|x-y|^\nu} M(dy) M(dx)
$$

$$
\leq \sum_{n=0}^{\infty} \int_U \int_{D_n} (2^{-n}R)^{-\nu} M(B(x,2^{-n}R)) M(dx) \leq C M(U)
$$

for some $C = C(X, \gamma, R) > 0$. Denoting $M_U = M(\cdot \cap U)/M(U)$, the second term in (5.2) becomes

$$
M(U)^2 \int_U \int_U \frac{f(x)f(y)}{\nu} \log \left(1 + \frac{f(x)f(y)}{\nu}\right) M_U(dy) M_U(dx).
$$

Set $H(s) = s^2$ and $I(s) := s \log(1 + s)$, $s \geq 0$. Then, the function $H \circ I^{-1}$ is convex and we apply Jensen’s inequality to obtain

$$
H \circ I^{-1} \left( \int_U \int_U \frac{f(x)f(y)}{\nu} \log \left(1 + \frac{f(x)f(y)}{\nu}\right) M_U(dy) M_U(dx) \right)
$$

$$
\leq \int_U \int_U \left( \frac{f(x)f(y)}{\nu} \right)^2 M_U(dy) M_U(dx) = \frac{1}{\nu^2 M(U)^2},
$$

where $f(x)f(y)/\nu > 0$ for any $x, y \in U$.
where we used that $\|f\|_2 = 1$. Hence,
\[
\int_U \int_U \frac{f(x) f(y)}{\nu} \log \left( 1 + \frac{f(x) f(y)}{\nu} \right) \mathcal{M}_U(dy) \mathcal{M}_U(dx) \leq I \circ H^{-1} \left( \frac{1}{\nu^2 \mathcal{M}(U)^2} \right) = \frac{1}{\nu \mathcal{M}(U)} \log \left( 1 + \frac{1}{\nu \mathcal{M}(U)} \right).
\]

Finally, choosing the constants large enough, we combine the above considerations to conclude that
\[
(G_U f, f) \leq C \mathcal{M}(U) + \frac{1}{\nu} \mathcal{M}(U) \log \left( 1 + \frac{1}{\nu \mathcal{M}(U)} \right) \leq C \mathcal{M}(U) \log \left( 1 + \frac{1}{\mathcal{M}(U)} \right),
\]
and by the variational formula for $\lambda_1(U)$ in (5.1) the claim follows. \(\square\)

In the next proposition we derive a Nash-type inequality from the above Faber-Krahn inequality, which implies the ultracontractivity of the semigroup $(T_t^U)$ and on-diagonal estimate on $(T_t^U)$ of the same form as stated for $p_t(\cdot, \cdot)$ in Theorem 1.2.

**Proposition 5.3.** Almost surely w.r.t. $\mathbb{P}$, there exists a constant $C_{10} = C_{10}(X, \gamma, R)$ such that $\mathbb{P}$-a.s.
\[
\|T_t^U\|_{L^1 \to L^\infty} \leq C_{10} t^{-1} \log(t^{-1}), \quad \forall t \in (0, \frac{1}{2}],
\]
(5.3)

Here $\|A\|_{L^1 \to L^\infty}$ denotes the operator norm of a bounded linear operator $A$ mapping from $L^1(U, M)$ to $L^\infty(U, M)$.

**Proof.** Since $U \subset B_R$ and therefore $\|T_t^U\|_{L^1 \to L^\infty} \leq \|T_t^{B_R}\|_{L^1 \to L^\infty}$, it is enough to show (5.3) for $(T_t^{B_R})$. Recall that $\lambda_1(U)$ can also be expressed by the variational formula
\[
\lambda_1(U) = \inf \left\{ \frac{\mathcal{E}(f, f)}{\|f\|_2^2} : f \in \mathcal{F}_U, f \neq 0 \right\}.
\]

Hence, the Faber-Krahn inequality in Proposition 5.2 can be rewritten as
\[
\|f\|_2^2 \leq C \psi(M(U)) \mathcal{E}(f, f), \quad \forall f \in \mathcal{F}_U,
\]
(5.4)

where $\psi(s) := s \log(1 + \frac{1}{s})$. Next we will verify that
\[
\|f\|_2^2 \leq C \psi(M(\text{supp}(f))) \mathcal{E}(f, f), \quad \forall f \in \mathcal{F}_{B_R},
\]
(5.5)

where $\text{supp}(f)$ denotes the $M$-essential support of $f$ in $B_R$. First, for $f \in \mathcal{F}_{B_R}$ with $\text{supp}(f)$ compact, (5.5) follows by choosing a decreasing sequence $(U_n)_{n \geq 1}$ of open subsets of $B_R$ with $\bigcap_n U_n = \text{supp}(f)$, applying (5.4) with $U = U_n$ and letting $n \to \infty$. Next, for general $f \in \mathcal{F}_{B_R}$, as $|f| \in \mathcal{F}_{B_R}$ and $\mathcal{E}(|f|, |f|) \leq \mathcal{E}(f, f)$ we may assume $f \geq 0$. Let $(f_n)_n \in \mathcal{F}_{B_R}$ be a sequence with $\text{supp}(f_n)$ compact and $\|f_n - f\|_\mathcal{F} \to 0$, where by [10, Theorem 1.4.2 (v)] we may assume that $f_n \geq 0$ for all $n$. Then, since $f \land f_n \in \mathcal{F}_{B_R}$, $\text{supp}(f \land f_n)$ is a compact subset of $\text{supp}(f)$ and
\[
\|f - f \land f_n\|_\mathcal{F} = \| (f - f_n)^+ \|_\mathcal{F} \leq \|f - f_n\|_\mathcal{F} \to 0,
\]
we conclude (5.5) for all $f \in \mathcal{F}_{B_R}$ by letting $n \to \infty$ in (5.5) for $f \land f_n$. 


Now, by [2, Proposition 10.3] (5.5) implies that
\[ \|f\|_2^2 \leq C \psi \left( C/\|f\|_2^2 \right) \mathcal{E}(f,f) \quad \text{for all } f \in \mathcal{F}_R \text{ with } \|f\|_1 = 1. \]
In particular, for such \( f \) we have \( \theta(\|f\|_2^2) \leq \mathcal{E}(f,f) \) with \( \theta(s) := C s^2 (\log(1 + Cs))^{-1} \) and by [6, Theorem II.5] we obtain that
\[ \|T^{BR}_t\|_{L^1 \to L^\infty} \leq m(t), \quad t > 0, \] (5.6)
for a decreasing \( C^1 \)-bijection \( m \) on \((0, \infty)\) satisfying
\[ \theta(m(t)) = -m'(t), \quad \lim_{t \downarrow 0} m(t) = \infty. \] (5.7)
Set \( \Phi(s) := \int_s^\infty \theta(u)^{-1} du \). Then, \( m(t) = \Phi^{-1}(t) \) is the solution of (5.7). Moreover, for sufficiently large \( s \),
\[ \Phi(s) = \int_s^\infty \frac{C}{u^2} \log(1 + Cu) du \leq \frac{C}{s} \log(1 + Cs) =: \Psi(s). \]
Note that \( \Psi \) is strictly decreasing on \([s_0, \infty)\) for some \( s_0 > 0 \). In particular, \( \Psi^{-1}(t) \geq \Phi^{-1}(t) \) for all \( t \in (0, \Phi(s_0)) \). Finally, since for \( t \) small enough \( \Psi(t^{-1} \log(1 + \frac{1}{t})) \leq Ct \) we conclude that
\[ m(Ct) = \Phi^{-1}(Ct) \leq \Psi^{-1}(Ct) \leq C t^{-1} \log(1 + Ct^{-1}) \leq C t^{-1} \log(t^{-1}), \]
and the claim follows from (5.6).

Now we prove Theorem 5.1 for bounded open sets \( U \). Given the ultracontractivity of \((T^U_t)\) in Proposition 5.3 and the Feller property in Proposition 4.2, a general result in [7] provides the existence of a continuous kernel \( p^U_t(\cdot, \cdot) \) for \((T^U_t)\). Then, we still have to identify this kernel as the transition kernel of \( \mathcal{B}^U \).

**Proof of Theorem 5.1 for bounded \( U \).** We will divide the proof of i) into several steps.

**Step 1:** In the first step we show the existence of \( p^U_t(x,y) \) and its continuity. Let \( \lambda_n \) be the eigenvalues of \(-L_U\) in increasing order (repeated according to multiplicity) and \( \varphi_n \) be the corresponding eigenfunctions normalized such that \( \|\varphi_n\|_2 = 1 \). Then, for each \( n \), by the ultracontractivity of the semigroup \((T^U_t)\) established in Proposition 5.3, that is \( T^U_t(L^2(U,M)) \subset L^\infty(U,M) \) and \( T^U_t : L^2(U,M) \to L^\infty(U,M) \) is a bounded linear operator for any \( t > 0 \), we may choose a bounded Borel measurable version of \( \varphi_n \). Further, by the strong Feller property of the resolvents in Proposition 4.2 we have \( R^U_\lambda \varphi_n \in C_0(U) \) for any \( \lambda > 0 \), and since
\[ R^U_\lambda \varphi_n = G^U_\lambda \varphi_n = (\lambda + \lambda_n)^{-1} \varphi_n \quad \text{\( M \)-a.e.,} \] (5.8)
there exists a continuous version of \( \varphi_n \), which we still denote by \( \varphi_n \). Then by [7, Theorem 2.1.4], the series
\[ p^U_t(x,y) := \sum_{n=1}^\infty e^{-\lambda_n t} \varphi_n(x) \varphi_n(y) \] (5.9)
converges uniformly on $[s_0, \infty) \times U \times U$ for all $s_0 > 0$, from which the continuity of $p^U_t(x, y)$ follows, and this defines an integral kernel for $T^U_t$, namely for each $t > 0$ and $f \in L^2(U, M)$,

$$T^U_t f(x) = \int_U p^U_t(x, y) f(y) M(dy), \quad \text{for } M\text{-a.e. } x \in U.$$  

Step 2: In this step we show that $R^U_\lambda$ is absolutely continuous w.r.t. the Liouville measure, i.e. we establish the existence of the resolvent kernel. For any Borel set $A$ with $M(A) = 0$ we have $R^U_\lambda \mathbb{1}_A(x) = G^U_\lambda \mathbb{1}_A(x) = 0$ for $M$-a.e. $x \in U$. Since $M$ has full support, $R^U_\lambda \mathbb{1}_A(x) = 0$ on a dense subset of $U$ and since $x \mapsto R^U_\lambda \mathbb{1}_A(x)$ is continuous by the strong Feller property in Proposition 4.2, it is identically zero, and absolute continuity follows. Therefore, there exists a resolvent kernel $r^U_\lambda(\cdot, \cdot)$ such that for all bounded Borel functions $f$,

$$R^U_\lambda f(x) = \int_U r^U_\lambda(x, y) f(y) M(dy).$$  

Step 3: Next we will show that for any $x \in U$,

$$\int_0^\infty e^{-\lambda t} \left( \int_U p^U_t(x, y) f(y) M(dy) \right) dt = \int_0^\infty e^{-\lambda t} E_x[f(B^U_t)] dt, \quad (5.10)$$

for all $\lambda > 0$ and all non-negative bounded Borel functions $f$. Recall that $P^U_t f(x) = E_x[f(B^U_t)]$ denotes the transition semigroup of $B^U$. Then for any $\epsilon > 0$, since $P^U_\epsilon = T^U_\epsilon$ $M$-a.e., by the absolute continuity of $R^U_\lambda$ w.r.t. $M$ we have

$$\int_\epsilon^\infty e^{-\lambda t} P^U_t f(x) dt = e^{-\lambda \epsilon} R^U_\lambda(P^U_\epsilon f)(x) = e^{-\lambda \epsilon} R^U_\lambda(T^U_\epsilon f)(x) = e^{-\lambda \epsilon} R^U_\lambda \left( \sum_{n=1}^\infty e^{-\lambda_n \epsilon} \langle \varphi_n, f \rangle \varphi_n \right)(x) = \sum_{n=1}^\infty e^{-\langle \lambda+\lambda_n \rangle \epsilon} \frac{1}{\lambda + \lambda_n} \langle \varphi_n, f \rangle \varphi_n(x),$$

where we also used (5.8) and the uniform convergence of the series in (5.9). Setting $a_n^\epsilon = \int_\epsilon^\infty e^{-(\lambda+\lambda_n) t} dt$, we further get

$$\int_\epsilon^\infty e^{-\lambda t} P^U_t f(x) dt = \sum_{n=1}^\infty a_n^\epsilon \varphi_n(x) \langle \varphi_n, f \rangle = \lim_{N \to \infty} \sum_{n=1}^N \int_\epsilon^\infty e^{-(\lambda+\lambda_n) t} \varphi_n(x) \langle \varphi_n, f \rangle dt \quad (5.10)$$

$$= \int_\epsilon^\infty \left( \sum_{n=1}^\infty e^{-\lambda_n t} \varphi_n(x) \langle \varphi_n, f \rangle \right) dt = \int_\epsilon^\infty \left( \int_U p^U_t(x, y) f(y) M(dy) \right) e^{-\lambda t} dt,$$

and by taking the limit $\epsilon \downarrow 0$ we obtain (5.10).

Step 4: Finally, we now prove that $P^U_\epsilon(B^U_t \in dy) = p^U_t(x, y) M(dy)$ for all $(t, x) \in (0, \infty) \times U$. By the uniqueness of Laplace transforms for positive measures on $[0, \infty)$ (see e.g. [9, Section XIII.1 Theorem 1a]), we get for all non-negative bounded Borel functions $f$,

$$\int_U p^U_t(x, y) f(y) M(dy) = E_x[f(B^U_t)], \quad \text{for a.e. } t \in [0, \infty). \quad (5.11)$$
If in addition $f$ is continuous, the continuity of $p^U_t$ established in Step 1 implies that \((5.11)\) holds for all $t > 0$. Finally a density argument gives the claim. The proof of i) is now complete.

For ii), since $p^U_t$ is bounded by [7, Theorem 2.1.4], the claim is immediate from the continuity of $p^U_t$ and the fact that $M(U) < \infty$. Finally, iii) follows from [18, Proposition A.3 (2)].

\[ \square \]

5.2. The heat kernel on unbounded open sets.

**Lemma 5.4.** Almost surely w.r.t. $\mathbb{P}$, for $R \geq 1$ and $\beta > \alpha_1$, $\mathbb{P}$-a.s., there exist $C_i = C_i(X, \gamma, R, \beta)$, $i = 11, 12$, such that for any non-empty open bounded $U \subset \mathbb{R}^2$,

\[
p^U_t(x, y) = p^U_t(y, x) \leq C_{11} t^{-1} \log(t^{-1}) \exp\left(-C_{12} \left(\frac{|x - y|^{\beta} \wedge 1}{t}\right)^{1/(\beta-1)}\right)
\]

for all $t \in (0, \frac{1}{2}]$, $x \in \mathbb{R}^2$ and $y \in B_R$, where we extend the kernel $p^U(\cdot, \cdot)$ to a function on $(0, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2$ by setting $p^U_t(x, y) := 0$ for $(x, y) \in (U \times U)^c$ for any $t > 0$.

**Proof.** Since for every $n \in \mathbb{N}$ we have the on-diagonal bounds on $p^{B_n}$ in Proposition 5.3, given the exit time estimates in Proposition 3.3, the result follows from [16, Section 6].

**Remark 5.5.** The constants appearing in the upper bound in Lemma 5.4 do not depend on the set $U$. Therefore, for any $R \geq 1$ there exists $C_{13} = C_{13}(X, \gamma, R)$, also not depending on $U$, such that $p^{1/2}_{t/2}(x, y) \leq C_{13}$ for all $x \in \mathbb{R}^2$ and $y \in B_R$. In particular, by the semigroup property we have for all $t \in (\frac{1}{2}, \infty)$ and such $x$ and $y$,

\[
p^U_t(x, y) = \int_{\mathbb{R}^2} p^U_{t-1/2}(x, z) p^U_{1/2}(z, y) M(\text{d}z) \leq C_{13} < \infty.
\]

**Lemma 5.6.** Almost surely w.r.t. $\mathbb{P}$, for any increasing sequence $(U_n)$ of open subsets of $\mathbb{R}^2$ with $\bigcup_n U_n = \mathbb{R}^2$,

\[
\lim_{n \to \infty} P_x[\tau_{U_n} < t] = 0,
\]

uniformly in $(t, x)$ over compact subsets of $[0, \infty) \times \mathbb{R}^2$.

**Proof.** It suffices to prove the uniform convergence in $(t, x)$ over $[0, T] \times B_R$ for any $T, R \in (0, \infty)$. By monotonicity we may assume $t = T$. Then, for any $x \in B_R$ and $n$ such that $B_{2R} \subset U_n$ we obtain by the strong Markov property

\[
P_x[\tau_{U_n} < T] = P_x[\tau_{B_{2R}} \leq \tau_{U_n} < T] = P_x[\tau_{B_{2R}} \leq \tau_{B_{2R}} + \tau_{U_n} \circ \theta_{\tau_{B_{2R}}} < T]
\]

\[
\leq P_x[\tau_{U_n} \circ \theta_{\tau_{B_{2R}}} < T] = E_x[P_{B_{2R}} \circ \theta_{\tau_{B_{2R}}} [\tau_{U_n} < T]] = P_{\mu_{0,2R}^x}[\tau_{U_n} < T],
\]

where $\mu_{0,2R}^x := P_x[B_{\tau_{B_{2R}}} \in \cdot]$ as in the proof of Proposition 3.2 above. Arguing precisely as there, from an explicit formula for the exit distribution of a Brownian motion (see [3, Theorem II.1.17]) we get that $\mu_{0,2R}^x \leq c \mu_{0,2R}$ for some explicit $c > 0$ with $\mu_{0,2R} = \mu_{0,2R}^x$. Thus $P_x[\tau_{U_n} < T] \leq c P_{\mu_{0,2R}}[\tau_{U_n} < T]$, which converges to zero as $n \to \infty$ by the dominated convergence theorem since the trajectory of $B_{[0,T]}$ is bounded and therefore contained in $U_n$ for $n$ large enough. \[ \square \]
Proof of Theorem 5.1 for unbounded $U$. i) Let $R \geq 1$ and let $f : \mathbb{R}^2 \to [0, \infty)$ be bounded and Borel measurable with $f|_{B_R^c} = 0$. Note that for any $l > k \geq R + 1$ we have $B_k \subset B_l$ and therefore $\tau_{B_k} < \tau_{B_l}$, $P_{y}$-a.s., for all $x \in B_k$. Then, by the strong Markov property we obtain for any $t > 0$ and $x \in B_k$,

$$
P_t^{B_l} f(x) = P_{t}^{B_k} f(x) + \mathbb{E}_{x} \left[ \mathbb{1}_{(\tau_{B_k} < t \wedge \tau_{B_l})} f(B_l) \right]
$$

$$
= P_{t}^{B_k} f(x) + \mathbb{E}_{x} \left[ \mathbb{1}_{(\tau_{B_k} < t \wedge \tau_{B_l})} P_{t - \tau_{B_k}}^{B_l} f(B_{\tau_{B_k}}) \right].
$$

Recall that

$$
P_{t - \tau_{B_k}}^{B_l} f(B_{\tau_{B_k}}) = \int_{B_R} f(y) P_{t - \tau_{B_k}}^{B_l} (B_{\tau_{B_k}}, y) M(dy).
$$

Now, note that by Remark 5.5

$$
\sup_n \sup_{(\frac{1}{2}, \infty) \times \mathbb{R}^2 \times \mathbb{R}^2} p^{B_n}_{\cdot \cdot}(\cdot \cdot) \leq C_{13} < \infty,
$$

Further, as $k \geq R + 1$ we have $\text{dist}(B_R, B_k) \geq 1$ and therefore Lemma 5.4 gives

$$
\sup_n \quad \sup_{(0, \frac{1}{2}) \times \partial B_k \times B_R} p^{B_n}_{\cdot \cdot}(\cdot \cdot) < \infty.
$$

Hence,

$$
P_t^{B_l} f(x) = P_{t}^{B_k} f(x) + \left( \sup_n \quad \sup_{(0, \infty) \times \partial B_k \times B_R} p^{B_n}_{\cdot \cdot}(\cdot \cdot) \right) P_{x} [\tau_{B_k} < t] \int_{B_R} f(y) M(dy),
$$

and by an approximation argument we deduce that there exists $C = C(X, \gamma, R)$ such that

$$
P_t^{B_l} (x, y) \leq P_t^{B_k} (x, y) + C P_{x} [\tau_{B_k} < t],
$$

(5.12)

for all $x \in B_k$, $y \in B_R$ and $t \in (0, \infty)$. Recall that by the results in the last subsection for all $k, l \in \mathbb{N}$ the Dirichlet heat kernels $p^{B_l}$ and $p^{B_k}$ are continuous. Therefore, (5.12) and Lemma 5.6 imply that the limit $p_t(x, y) := \lim_n p_t^{B_n}(x, y)$ exists and is continuous on $(0, \infty) \times \mathbb{R}^2 \times B_R$. Since $R \geq 1$ is arbitrary and the relation $P_{t} [B_{l} \in dy] = p_k(x, y) M(dy)$ can be obtained from that for bounded $U$ by monotone convergence, statement i) follows for the global heat kernel $p_t(x, y)$, i.e. for the case $U = \mathbb{R}^2$. For general unbounded, open $U \subset \mathbb{R}^2$, statement i) follows by similar arguments from the fact that for any $k, l \in \mathbb{N}$ with $k < l$,

$$
0 \leq p_t^{U \cap B_l}(x, y) - p_t^{U \cap B_k}(x, y) \leq p_t^{B_l}(x, y) - p_t^{B_k}(x, y), \quad x, y \in \mathbb{R}^2, t > 0.
$$

In order to see the second inequality, notice that if $(x, y) \in (B_k \times B_k)^c$ this inequality holds trivially, and for $(x, y) \in B_k \times B_k$, by the continuity of the Dirichlet heat kernels,

$$
(p_t^{B_k} - p_t^{B_k} - p_t^{U \cap B_l} + p_t^{U \cap B_k})(x, y) = \lim_{r \downarrow 0} \frac{P_{x} [B_{l} \in B(y, r), \tau_{U} \vee \tau_{B_k} \leq t < \tau_{B_l}]}{M(\{B(y, r)\})} \geq 0.
$$
ii) Let \( x \in U \) and \( t, \varepsilon > 0 \). Since
\[
\int_{\mathbb{R}^2} p_t(x, y) M(dy) = P_x[\mathcal{B}_t \in \mathbb{R}^2] = 1
\]
by \( \lim_{s \to \infty} F_s = \infty, P_x\)-a.s., we can choose \( n \in \mathbb{N} \) such that \( x \in B_n \) and
\[
\int_{B_n} p_t(x, y) M(dy) > 1 - \varepsilon.
\]
Then, by the continuity of \( p_t \) there exists \( r > 0 \) such that \( B(x, r) \subset U \) and
\[
\int_{B_n} p_t(z, y) M(dy) > 1 - \varepsilon, \quad \forall z \in B(x, r).
\]
Hence, for such \( z \),
\[
\int_{U \setminus B_n} p^U_t(z, y) M(dy) \leq \int_{B_n} p_t(z, y) M(dy) < \varepsilon.
\]
Now, for any bounded Borel function \( f : U \to \mathbb{R} \) and \( z \in B(x, r) \), writing
\[
P^U_t f(z) = \int_{U \setminus B_n} f(y) p^U_t(z, y) M(dy) + \int_{U \cap B_n} f(y) p^U_t(z, y) M(dy),
\]
we obtain
\[
|P^U_t f(x) - P^U_t f(z)| \\
\leq 2 \|f\|_\infty \varepsilon + \left( \int_{U \cap B_n} p^U_t(x, y) f(y) M(dy) - \int_{U \cap B_n} p^U_t(z, y) f(y) M(dy) \right) \\
\leq (2 \|f\|_\infty + 1) \varepsilon,
\]
provided \( |x - z| \) is sufficiently small, which proves the strong Feller property. In the last step we used the fact that, since \( 0 \leq p^U_t \leq p_t \) on \( B(x, r) \times (B_n \cap U) \) where \( p_t \) is bounded and \( p^U_t \) is continuous, the function \( z \mapsto \int_{U \cap B_n} p^U_t(z, y) f(y) M(dy) \) is continuous on \( B(x, r) \) by the dominated convergence theorem.

iii) Since \( U \) is connected, for any \( x, y \in U \) there exists a connected bounded open set \( V \subset U \) with \( x, y \in V \) and by the corresponding result for bounded sets we have \( p^U_t(x, y) \geq P_t(x, y) > 0 \) for any \( t > 0 \). \hfill \Box

Proof of Theorem 1.2. This is immediate from Lemma 5.4, since \( p_t = \lim_n p^R_t \) as shown in the above proof. \hfill \Box

6. ON-DIAGONAL LOWER BOUND AND SPECTRAL DIMENSION

In this section we prove the on-diagonal lower bound in Theorem 1.3. Indeed, we will show a more general result that also covers the Dirichlet Liouville heat kernel.

**Theorem 6.1.** Almost surely w.r.t. \( \mathbb{P} \), for any non-empty open set \( U \subseteq \mathbb{R}^2 \) the following holds: For \( M\text{-a.e.} \ x \in \mathbb{R}^2 \) there exist \( C_{14} = C_{14}(X, \gamma) > 0 \) and \( t_0 = t_0(X, \gamma, x) > 0 \) such that
\[
p^U_t(x, x) \geq C_{14} t^{-1} \left( \log(t^{-1}) \right)^{-\eta}, \quad \forall t \in (0, t_0], \tag{6.1}
\]
for some explicit constant \( \eta > 0 \).
In particular, Theorem 6.1 immediately implies Theorem 1.3 by choosing \( U = \mathbb{R}^2 \). Furthermore, we can directly deduce the pointwise spectral dimension of the Dirichlet Liouville heat kernel.

**Corollary 6.2.** Almost surely w.r.t. \( \mathbb{P} \), for any non-empty open set \( U \subseteq \mathbb{R}^2 \),

\[
\lim_{t \downarrow 0} \frac{2 \log p_t^U(x,x)}{-\log t} = 2
\]

for \( M \)-a.e. \( x \in U \).

**Proof.** This is immediate from the lower bound in Theorem 1.3 and, since \( p_t^U(x,x) \leq p_t(x,x) \), from the on-diagonal part of the upper bound in Theorem 1.2. \( \square \)

### 6.1. Proof of Theorem 6.1.

In order to show Theorem 6.1 we need further moment and tail estimates on the exit times from balls. First, we recall the representation of the expected exit time in terms of the Green kernel.

**Lemma 6.3.** For any non-empty open set \( U \subseteq \mathbb{R}^2 \) and any \( x \in U \),

\[
E_x[\tau_U] = \int_U g_U(x,y) M(dy).
\]

**Proof.** This follows immediately from Proposition B.1. \( \square \)

**Lemma 6.4.** For any \( R \geq 1 \), \( \mathbb{P} \)-a.s., there exists \( C_{15} = C_{15}(X, \gamma, R) > 0 \) such that

\[
E_x[\tau_{B(x,r)}] \leq C_{15} M(B(x,r)) \log \frac{1}{r}, \quad \forall r \in (0,1], x \in B_R.
\]

**Proof.** Recall that the Green function over \( B_{R+1} \) can be written as in (2.4) Then, as \( B(x,r) \subset B_{R+1} \) and therefore \( g_{B(x,r)} \leq g_{B_{R+1}} \), we have

\[
E_x[\tau_{B(x,r)}] = \int_{B(x,r)} g_{B(x,r)}(x,y) M(dy)
\leq \int_{B(x,r)} \left( \frac{1}{r} \log \frac{1}{|x-y|} + \Psi_{0,R+1}(x,y) \right) M(dy).
\]

Setting \( D_n := B(x,2^{1-n}r) \setminus B(x,2^{-n}r), n \geq 1 \), we obtain further that

\[
E_x[\tau_{B(x,r)}] \leq c M(B(x,r)) + c \sum_{n=1}^{\infty} (n + \log \frac{1}{r}) M(D_n),
\]

with \( c = c(R) > 0 \). On the other hand, by Lemma 3.1, for any \( \varepsilon > 0 \),

\[
(n + \log \frac{1}{r}) M\left( B(x,2^{1-n}r) \right) \leq C n 2^{-n(\alpha_2-\varepsilon)} r^{\alpha_2-\alpha_1-2\varepsilon} \leq C 2^{-n(\alpha_2-\varepsilon)/2},
\]

provided \( n \geq C \log \frac{1}{r} \) with \( C = C(X, \gamma, R) > 0 \). Hence,

\[
E_x[\tau_{B(x,r)}] \leq c M(B(x,r)) + C \sum_{n \geq C \log \frac{1}{r}} 2^{-n(\alpha_2-\varepsilon)/2} M(B(x,r))
\]

\[
+ C \log \frac{1}{r} \sum_{n < C \log \frac{1}{r}} M(D_n),
\]

where

\[
\Psi_{0,R+1}(x,y) = \int_{B_R} \left( \frac{1}{r} \log \frac{1}{|x-y|} \right) M(dy).
\]
which implies the claim. \( \square \)

**Lemma 6.5.** There exists a constant \( c_1 > 0 \) such that \( \mathbb{P} \)-a.s.
\[
E_x[\tau_{B(x,r)}] \geq c_1 M(B(x,r/2)), \quad \forall r > 0, x \in \mathbb{R}^d.
\]

**Proof.** This follows from the following scale invariance of the Gaussian Green kernel:
\[
g_{B(x,r)}(x,y) = g_{B(0,r)}(0,y-x) = g_{B(0,\lambda r)}(0,\lambda(y-x)),
\]
for any \( x,y \in \mathbb{R}^d, r > 0 \) and \( \lambda > 0 \) (cf. e.g. [10, Example 1.5.1]). Indeed, using this relation we obtain
\[
E_x[\tau_{B(x,r)}] = \int_{B(x,r)} g_{B(x,r)}(x,y) M(dy) \geq \int_{B(x,r/2)} g_{B(0,1)}(0, \frac{y-x}{r}) M(dy)
\geq c M(B(x,r/2)),
\]
which is the claim. \( \square \)

**Proposition 6.6.** For any \( R \geq 1, \mathbb{P} \)-a.s., there exists \( C_{16} = C_{16}(X, \gamma, R) > 0 \) such that
\[
P_x[\tau_{B(x,r)} \leq t] \leq 1 - C_{16} \frac{M(B(x,r/2))}{M(B(x,3r))} \log \frac{1}{r},
\]
for all \( t \leq \frac{1}{2} E_x[\tau_{B(x,r)}] \), \( r \in (0,1] \) and \( x \in B_R \).

**Proof.** Obviously, \( \tau_{B(x,r)} \leq t + \mathbb{I}_{[\tau_{B(x,r)} > t]} (\tau_{B(x,r)} - t) \) for any \( t > 0 \), so using the Markov property we get
\[
E_x[\tau_{B(x,r)}] \leq t + E_x \left[ \mathbb{I}_{[\tau_{B(x,r)} > t]} (\tau_{B(x,r)} - t) \right] = t + E_x \left[ \mathbb{I}_{[\tau_{B(x,r)} > t]} E_{B_t}[\tau_{B(x,r)}] \right]
\leq t + P_x[\tau_{B(x,r)} > t] \sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}].
\]
For \( t \leq \frac{1}{2} E_x[\tau_{B(x,r)}] \) this implies that
\[
P_x[\tau_{B(x,r)} \leq t] \leq 1 + \frac{t - E_x[\tau_{B(x,r)}]}{\sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}]} \leq 1 - \frac{1}{2} \frac{E_x[\tau_{B(x,r)}]}{\sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}]}.
\]
Finally, note that by Lemma 6.4
\[
\sup_{y \in B(x,r)} E_y[\tau_{B(x,r)}] \leq \sup_{y \in B(x,r)} E_y[\tau_{B(y,2r)}] \leq C \log \frac{1}{r} \sup_{y \in B(x,r)} M(B(y,2r))
\leq C M(B(x,3r)) \log \frac{1}{r},
\]
and the claim follows Lemma 6.5. \( \square \)

We are now in the position to show an on-diagonal lower bound on the Dirichlet heat kernel.
**Proposition 6.7.** For any $R \geq 1$, P.a.s., there exist $c_2 > 0$ and $C_{17} = C_{17}(X, \gamma, R) > 0$ such that
\[
p_t^{B(x,r)}(x,x) \geq C_{17} \left( \frac{M(B(x,r/2))}{M(B(x,3r))} \log \frac{r}{x} \right)^2 M(B(x,r))^{-1},
\]
for all $t \leq c_2 M(B(x,r/2))$, $r \in (0, \frac{1}{2})$ and $x \in B_R$.

**Proof.** First note that for every $t \geq 0$ by the Cauchy-Schwarz inequality, the symmetry of the Dirichlet kernel $p_t^{B(x,r)}(x,y)$ and the Chapman-Kolmogorov equation,
\[
P_x \left[ \tau_{B(x,r)} > t \right]^2 = P_x \left[ B_t \in B(x,r), \tau_{B(x,r)} > t \right]^2 = \left( \int_{B(x,r)} p_t^{B(x,r)}(x,y) \, M(dy) \right)^2 \leq M(B(x,r)) \int_{B(x,r)} \left( p_t^{B(x,r)}(x,y) \right)^2 \, M(dy) = M(B(x,r)) \int_{B(x,r)} p_t^{B(x,r)}(x,x).
\]
On the other hand, since by Lemma 6.5 $E_x \left[ \tau_{B(x,r)} \right] \geq c_1 M(B(x,r/2))$ we have by Proposition 6.6 for $t \leq c_1 M(B(x,r/2))$,
\[
P_x \left[ \tau_{B(x,r)} > t \right] \geq C \frac{M(B(x,r/2))}{M(B(x,3r))} \log \frac{r}{x},
\]
which gives the result. \[\Box\]

**Corollary 6.8.** For any non-empty open set $U \subseteq \mathbb{R}^2$, P.a.s. (6.1) holds for any $x \in U$ for which there exist $r_0 = r_0(x)$ and $c_3 > 0$ such that
\[
M(B(x,2r)) \leq c_3 (\log \frac{1}{r})^4 M(B(x,r)), \quad \forall r \leq r_0.
\]

**Proof.** For any $x \in U$ satisfying (6.3) let $r_1(x) = r_1(x) < r_0/2$ be such that $B(x, r_1) \subset U$. Further, for $t \leq c_2 M(B(x,r_1/2))$ choose $r = r(t) \in (0, r_1)$ such that
\[
c_2 M(B(x,r/4)) \leq t \leq c_2 M(B(x,r/2)).
\]
In particular, note that by Lemma 3.1, $\log r(t) \asymp \log t$ for small $t$. Then, by Proposition 6.7,
\[
t \, p_t^U(x,x) \geq t \, p_t^{B(x,r_0)}(x,x) \geq t \, p_t^{B(x,r)}(x,x) \geq C \frac{M(B(x,r/4))}{M(B(x,r))} \left( \frac{M(B(x,r/2))}{M(B(x,3r))} \right)^2 \left( \log \frac{x}{r} \right)^{-2}.
\]
Now, by using (6.3) we have
\[
\frac{M(B(x,r/4))}{M(B(x,r))} = \frac{M(B(x,r/4))}{M(B(x,r/2))} \frac{M(B(x,r/2))}{M(B(x,3r))} \geq c \left( \log \frac{1}{r} \right)^{-8},
\]
and
\[
\frac{M(B(x,r/2))}{M(B(x,r))} \geq \frac{M(B(x,r/2))}{M(B(x,4r))} \geq c \left( \log \frac{1}{r} \right)^{-12}.
\]
Combining these estimates with (6.4) gives the result. \[\Box\]
Theorem 6.1 follows now from Corollary 6.8 and the following result, which holds in the more general setting of doubling metric space.

**Proposition 6.9.** Let $(\mathcal{X}, d)$ be a doubling metric space, i.e.

$$\sup_{x \in \mathcal{X}, r > 0} \inf \left\{ \# A \mid A \subseteq \mathcal{X}, B(x, 2r) \subseteq \bigcup_{y \in A} B(y, r) \right\} < \infty,$$

and let $\mu$ be a Borel measure on $(\mathcal{X}, d)$ with $\mu(B(x, r)) \in (0, \infty)$ for all $x \in \mathcal{X}$ and $r > 0$. Then, for $\mu$-a.e. $x \in \mathcal{X}$, there exists $r_0 = r_0(x)$ and $c_3 > 0$ such that

$$\mu(B(x, 2r)) \leq c_3 \left( \log \frac{1}{r} \right)^4 \mu(B(x, r)), \quad \forall r \leq r_0.$$

**Proof.** Fix an arbitrary $x_0 \in \mathcal{X}$. Set $B_1 = B(x_0, 1)$, $r_k := 2^{-k}$, $k \in \mathbb{N}$, and $\mu_1 := \mu(\cdot \cap B_1)$ and let

$$A_n := \left\{ x \in B_1 \mid \mu_1(B(x, r_n-1)) \geq n^2 \mu_1(B(x, r_n)) \right\}, \quad n \in \mathbb{N}.$$

For every $n$ there exists a finite set of points $\Lambda_n$ such that $B_1$ is covered by the family of balls $\{B(x, r_{n+1})\}_{x \in \Lambda_n}$. Then,

$$\int_{B_1} \frac{\mu_1(B(y, r_{n-1}))}{\mu_1(B(y, r_n))} d\mu_1(y) \leq \sum_{x \in \Lambda_n} \int_{B(x, r_{n+1})} \frac{\mu_1(B(y, r_{n-1}))}{\mu_1(B(y, r_n))} \mu_1(dy)$$

$$\leq \sum_{x \in \Lambda_n} \int_{B(x, r_{n+1})} \frac{\mu_1(B(x, r_{n-2}))}{\mu_1(B(x, r_{n+1}))} \mu_1(dy) \leq \sum_{x \in \Lambda_n} \mu_1(B(x, r_{n-2})) \leq c \mu(B_1),$$

for some $c > 0$. By using Čebyšev’s inequality, this implies

$$\sum_{n=1}^{\infty} \mu_1(A_n) \leq c \mu(B_1) \sum_{n=1}^{\infty} n^{-2} < \infty,$$

and a Borel-Cantelli argument gives that for $\mu$-a.e. $x \in B_1$ there exists $n_0(x)$ such that

$$\mu_1(B(x, r_{n-1})) \leq n^2 \mu_1(B(x, r_n)), \quad \forall n \geq n_0(x). \quad (6.5)$$

Now, for any $x \in B_1$ consider $r$ small enough such that $\mu_1(B(x, 2r)) = \mu(B(x, 2r))$ and $r_{n+1} \leq r < r_n$ for $n \geq n_0(x)$. Then, by applying (6.5) twice

$$\mu(B(x, 2r)) \leq \mu(B(x, r_{n-1})) \leq n^2(n+1)^2 \mu(B(x, r_{n+1})) \leq 4n^4 \mu(B(x, r))$$

with $n < (\log \frac{1}{r} + \log R) / \log 2$. Finally, since $x_0$ is arbitrary and $\mathcal{X}$ can be covered by countably many balls with radius 1, the claim follows.

**6.2. Global spectral dimension.** Let $U \subset \mathbb{R}^2$ be non-empty, open and bounded. Then, as above in Section 5.1, let $(\lambda_n(U))_{n \geq 1}$ be the eigenvalues of $-\mathcal{L}_U$ in increasing order (repeated according to multiplicity). Then, setting

$$Z_U(t) := \int_U p_t^U(x, x) \, M(dx) = \sum_{n=1}^{\infty} e^{-\lambda_n(U)t},$$

we obtain the following global spectral dimension from Theorem 6.2.
Corollary 6.10. Almost surely w.r.t. \( P \), for any non-empty, open and bounded \( U \subset \mathbb{R}^2 \),
\[
\lim_{t \downarrow 0} \frac{2 \log Z_U(t)}{- \log t} = 2. \tag{6.6}
\]

Proof. Again, as \( p_U^t(x,x) \leq p_t(x,x) \), the on-diagonal upper bound in Proposition 1.2 immediately implies that
\[
\limsup_{t \downarrow 0} \frac{2 \log Z_U(t)}{- \log t} \leq 2,
\]
so it remains to prove the lower bound. By Theorem 6.1 we have \( \lim_{t \downarrow 0} p_U^t(x,x) = \infty \) for \( M \)-a.e. \( x \in U \). In particular, there exists \( t_0 \in (0,1) \) such that for \( A := \{ x \in U : p_{t_0}^U(x,x) \geq 1 \} \) we have \( M(A) > 0 \). Since \( Z_U(t) \geq \int_A p_U^t(x,x) \, M(dx) \), we obtain by Jensen’s inequality
\[
\log Z_U(t) \geq \log M(A) + \int_A \log p_U^t(x,x) \, \frac{M(dx)}{M(A)} \geq \log M(A) + \int_A \log p_U^t(x,x)
\]
Then, since \( t \mapsto p_U^t(x,x) \) is decreasing, \( \log p_U^t(x,x) \geq 0 \) for all \( x \in A \) and \( t \in (0,t_0] \), so we may use Fatou’s lemma and Corollary 6.2 to conclude
\[
\liminf_{t \downarrow 0} \frac{\log Z_U(t)}{- \log t} \geq \liminf_{t \downarrow 0} \int_A \frac{\log p_U^t(x,x) \, M(dx)}{M(A)} = 1,
\]
which proves the lower bound. \( \square \)

Remark 6.11. It is not clear to the authors whether the counterpart of (6.6) for the eigenvalue counting function \( N_U(\lambda) := \# \{ n \in \mathbb{N} : \lambda_n(U) \leq \lambda \} \) also holds.

Appendix A. Proof of Proposition 2.4

The proof will be based on the following results proved in [14].

Theorem A.1. For each \( x \in \mathbb{R}^2 \), \( P \times P_x \)-a.s. the following hold:

i) For all \( t \geq 0 \), \( F_t := \lim_{n \to \infty} F_t^n \) exists in \( \mathbb{R} \).

ii) The mapping \([0,\infty) \ni t \mapsto F_t \in [0,\infty)\) is continuous, strictly increasing and satisfies \( F_0 = 0 \) and \( \lim_{t \to \infty} F_t = \infty \).

Proof. See [14, Lemma 2.8 and Proof of Theorem 2.7]. \( \square \)

We start with a preparatory lemma.

Lemma A.2. Almost surely w.r.t. \( P \), for all \( x \in \mathbb{R}^2 \),
\[
\lim_{T \to 0} \liminf_{n \to \infty} F_T^n = 0, \quad P_x \text{-a.s.}
\]
Proof. For all $x \in \mathbb{R}^2$, by Fatou’s lemma
\[ E_x \left[ \liminf_{n \to \infty} F^n_x \right] \leq \liminf_{n \to \infty} E_x[F^n_x] < \infty, \]
so using the dominated convergence theorem the claim follows immediately from Lemma 4.1.

For all $t \geq 0$ we denote by $\Lambda_t$ the set of all $(\omega, \omega') \in \Omega \times \Omega'$ such that:

i) For all $u \geq t$, $F_{t,u}(\omega, \omega') := \lim_{n \to \infty} F^n_u(\omega, \omega') - F^n_t(\omega, \omega')$ exists in $\mathbb{R}$.

ii) The mapping $[t, \infty) \ni u \mapsto F_{t,u}(\omega, \omega') \in [0, \infty)$ is continuous, strictly increasing and satisfies $F_{t,t}(\omega, \omega') = 0$ and $\lim_{u \to \infty} F_{t,u}(\omega, \omega') = \infty.$

We will write $\Lambda_t^{o} := \{ \omega' \in \Omega' : (\omega, \omega') \in \Lambda_t \}$. Note that $\Lambda_t^{o} = \theta_t^{-1}(\Lambda_t^{o})$, $t \geq 0$, since $F^n$ is a PCAF of $B$ for every $n$. Further, we have $\Lambda_t \in \mathcal{A} \otimes \mathcal{G}^{0}_{\infty}$, since all the defining properties of $\Lambda_t$ can be rephrased in terms of the values of $F^n$ for dyadic rational $s$ by virtue of the monotonicity of $F^n$. Finally, recall that by Theorem A.1 we have
\[ \mathbb{P} \times P_x[\Lambda_0] = 1 \] for all $x \in \mathbb{R}^2$.

Lemma A.3. For $\mathbb{P}$-a.e. $\omega$, $P_x[\Lambda^{o}_t] = 1$ for all $t > 0$ and $x \in \mathbb{R}^2$.

Proof. Fix any probability measure $\mu$ on $\mathbb{R}^2$ with full support. Since $\mathbb{E}P_x[\Lambda_0^{o}] = 1$ for all $x \in \mathbb{R}^2$, using Fubini’s theorem integration w.r.t. $\mu$ gives that $\mathbb{E}P_x[\Lambda_0^{o}] = 1$ with $P_{\mu} := \int_{\mathbb{R}^2} P_x \mu(dx)$. In particular, for $\mathbb{P}$-a.e. $\omega$, $P_{\mu}[(\Lambda_0^{o})^c] = 0$ and thus
\[ P_y[(\Lambda_0^{o})^c] = 0, \quad \text{for } dy\text{-a.e. } y \in \mathbb{R}^2. \] (A.1)

Now, for every such $\omega$, every $x \in \mathbb{R}^2$ and $t > 0$ we use the Markov property to obtain
\[ P_x[\Lambda_t^{o}] = P_x[\theta_t^{-1}(\Lambda_0^{o})] = E_x[\mathbb{1}_{\Lambda_0^{o}} \circ \theta_t] = E_x[P_{B_t}[\Lambda_0^{o}]] \]
\[ = \frac{1}{2\pi t} \int_{\mathbb{R}^2} P_y[\Lambda_0^{o}] e^{-|y-x|^2/2t} dy = 1, \]
where we used (A.1) in the last step. \qed

Proof of Proposition 2.4. Let
\[ \Lambda := \bigcap_{q \in \mathbb{Q} \setminus \{0\}} \Lambda_q \cap \left\{ (\omega, \omega') \in \Omega \times \Omega' : \lim_{t \to 0} \liminf_{n \to \infty} F^n_T(\omega, \omega') = 0 \right\}, \]
where $\mathbb{Q} \setminus \{0\}$ denotes the set of all positive $q \in \mathbb{Q}$. Then, i) follows immediately from Lemma A.2 and Lemma A.3.

Let now $(\omega, \omega') \in \Lambda$ be arbitrary. Then, $\omega' \in \Lambda_q^{o}$ for each $q \in \mathbb{Q} \setminus \{0\}$, so for all $s \geq q$ the limit $F_{q,s}$ exists in $\mathbb{R}$ and $s \mapsto F_{q,s}(\omega, \omega')$ continuous and strictly increasing and hence $\lim_{s \to \infty} F_{q,s}(\omega, \omega') = \infty$. Thus, for all $0 < s < t$ the limit
\[ F_{s,t}(\omega, \omega') := \lim_{n \to \infty} F^n_s(\omega, \omega') - F^n_t(\omega, \omega') \]
equalsubseteq \mathbb{R}. Hence, for any $T > 0$ and $0 < s < t \leq T$,
\[ F_{t,T}(\omega, \omega') - F_{s,T}(\omega, \omega') = \lim_{n \to \infty} \left( F^n_s(\omega, \omega') - F^n_t(\omega, \omega') \right) \leq \liminf_{n \to \infty} F^n_T(\omega, \omega'). \]
But since \( \lim_{T \to 0} \liminf_n F^n_T(\omega, \omega') = 0 \), this implies that \((F_{s,T}(\omega, \omega'))_{s \in [0,T]}\) is a Cauchy sequence as \( s \downarrow 0 \), so there exists
\[
F_T(\omega, \omega') = \lim_{s \downarrow 0} F_{s,T}(\omega, \omega').
\]

Moreover, since \( 0 \leq F_T(\omega, \omega') \leq \liminf_n F^n_T(\omega, \omega') \to 0 \) as \( T \to 0 \), we obtain \( \lim_{T \to 0} F_T(\omega, \omega') = 0 \). Thus, setting \( F_0(\omega, \omega') = 0 \), the mapping \( [0, \infty) \ni T \mapsto F_T(\omega, \omega') \) is continuous and it is also strictly increasing. To see the latter, note that for all \( 0 < s < t \), choosing \( q \in \mathbb{Q}_+ \) such that \( q \leq s \), we have
\[
F_t(\omega, \omega') - F_s(\omega, \omega') = \lim_{\varepsilon \downarrow 0} (F_{\varepsilon,t}(\omega, \omega') - F_{\varepsilon,s}(\omega, \omega'))
= \lim_{n \to \infty} \left( F^n_{\varepsilon,t}(\omega, \omega') - F^n_{\varepsilon,s}(\omega, \omega') \right) = F_{\varepsilon,t}(\omega, \omega') - F_{\varepsilon,s}(\omega, \omega') > 0,
\]
where we used in the last step that the function \( u \mapsto F_{q,u}(\omega, \omega') \) is strictly increasing on \([q, \infty)\) since \((\omega, \omega') \in \Lambda_q\). Finally, since for any \( T > 0 \) and every \( q \in \mathbb{Q}_+ \) with \( q \leq T \) we have \( F_T(\omega, \omega') \geq F_{q,T}(\omega, \omega') \) and \((\omega, \omega') \in \Lambda_q\) we obtain that \( \lim_{T \to \infty} F_T(\omega, \omega') = \infty \). This completes the proof of ii).

Statement iii) is clear, so it remains to show iv). Let \( \omega \in \Omega \) satisfy the property in statement i). First, for any \( t > 0 \), \( \Lambda^\omega \in \mathcal{G}_0 \subset \mathcal{G}_t \) by i), and the \( \mathcal{G}_t \)-measurability of \( F_{s,t}(\omega, \cdot)|_{\Lambda^\omega} \) implies that of \( F_t(\omega, \cdot) \). Next we show that \( \theta_t(\Lambda^\omega) \subset \Lambda^\omega \) for all \( t \geq 0 \). Since
\[
\liminf_{n \to \infty} F^n_t \circ \theta_t = \lim_{n \to \infty} F^n_{t+t} - F^n_t = F_{t,t+t} \to 0, \quad \text{as } T \to 0,
\]
it suffices to verify that \( \theta_t(\Lambda^\omega) \subset \Lambda^\omega \) for all \( q \in \mathbb{Q}_+ \). But this follows from the fact that for any \( q \in \mathbb{Q}_+ \) and \( u \geq q \),
\[
F^n_u \circ \theta_t - F^n_q \circ \theta_t = (F^n_{u+t} - F^n_t) - (F^n_{q+t} - F^n_t) = F^n_{u+t} - F^n_{q+t},
\]
converges to \( F_{q+t,u+t} \) having the required properties. Finally, a similar argument shows that \( F_{s+t} = F_t + F_s \circ \theta_t, s,t \geq 0 \). Indeed, for \( t > 0 \) we have
\[
F_{s+t} - F_t = \lim_{r \downarrow 0} F_{r,s+t} - F_{r,t} = \lim_{r \downarrow 0} \lim_{n \to \infty} \left[ F^n_{s+t} - F^n_r - F^n_t + F^n_r \right] = \lim_{n \to \infty} \left[ F^n_{s+t} - F^n_t \right] = \lim_{n \to \infty} F^n_t \circ \theta_t = F_s \circ \theta_t.
\]
Therefore \( F_t(\omega, \cdot) \) is a PCAF of \( B \) in the strict sense with defining set \( \Lambda^\omega \).

\[\square\]

\textbf{Appendix B. The Revuz Correspondence between \( M \) and \( F \)}

\textbf{Proposition B.1.} Almost surely w.r.t. \( \mathbb{P} \) the following holds: For any non-empty, open set \( U \subseteq \mathbb{R}^2 \), for all \( x \in \mathbb{R}^2 \) and all Borel measurable functions \( \eta : [0, \infty) \to [0, \infty] \) and \( f : U \to [0, \infty] \),
\[
E_x \left[ \int_0^{T_U} \eta(t) \, f(B_t^U) \, dF_t \right] = \int_0^{\infty} \int_U \eta(t) \, f(y) \, q^U_t(x,y) \, M(dy) \, dt, \quad \text{B.1}
\]
where \( q^U_t(x,y)dy = P_{z}(B_t^U \in dy) \).
Lemma B.2. Almost surely w.r.t. \( \mathbb{P} \) the following holds: For all \( x \in U, t > 0, \) and all measurable, bounded \( f : U \to [0, \infty) \) with compact support contained in \( U, \) \( \{\int_0^{t \land T_U} f(B_s^U) \, dF_s^n\}_n \) is uniformly \( P_x \)-integrable.

Proof. We shall prove that

\[
\sup_{n \geq 1} E_x \left[ \left( \int_0^{t \land T_U} f(B_s^U) \, dF_s^n \right)^2 \right] < \infty. \tag{B.2}
\]

For any bounded Borel function \( h : U \to \mathbb{R}, \) using the Markov property and the fact that \( q_t^U(x, y) \leq q_t(x, y) \) for all \( t > 0, x, y \in U, \)

\[
E_x \left[ \left( \int_0^{t \land T_U} h(B_s^U) \, ds \right)^2 \right] = \int_U \int_U h(y)h(z) \int_0^t q_s(x, y)q_{u-s}(y, z) \, du \, ds \, dz \, dy.
\]

Further, note that

\[
\int_0^t \int_s^t q_s(x, y)q_{u-s}(y, z) \, du \, ds \leq \int_0^t q_s(x, y) \, ds \int_0^t q_u(y, z) \, du \leq \frac{1}{2\pi^2} \int_0^t (y-x)^2 s^{-1} e^{-\frac{u}{2s}} \, ds \int_0^t (z-y)^2 u^{-1} e^{-\frac{t}{2u}} \, du \leq \frac{1}{2\pi^2} \left( c + \log_+ \frac{t}{y-x} \right) \left( c + \log_+ \frac{t}{z-y} \right),
\]

where \( \log_+ = \log(\cdot \vee 1). \) Let now \( R > 0 \) be such that \( \text{supp}(f) \subset B_R. \) Then, choosing \( h(y) = f(y) \exp(\gamma X_n(y) - \frac{\gamma^2}{2} \mathbb{E}[X_n(y)]), \)

\[
E_x \left[ \left( \int_0^t f(B_s^U) \, dF_s^n \right)^2 \right] = E_x \left[ \left( \int_0^t h(B_s^U) \, ds \right)^2 \right] \leq \frac{1}{2\pi^2} \int_{B_R} \int_{B_R} \left( c + \log_+ \frac{t}{y-x} \right) \left( c + \log_+ \frac{t}{z-y} \right) M_n(dz) \, M_n(dy).
\]

Setting \( D_k = B(y, 2^{1-k}R) \setminus B(y, 2^{-k}R), y \in B_R, k \in \mathbb{N}, \) by [14, Theorem 2.2]

\[
\int_{B_R} \left( c + \log_+ \frac{t}{z-y} \right) M_n(dz) \leq c M(B_R) + \sum_{k=0}^\infty \int_{D_k} \log_+ \frac{t}{(2^{-k}R)^2} M_n(dz) \leq c M(B_R) + C \sum_{k=1}^\infty (2^{1-k}R)^{\alpha_2 - \varepsilon} < \infty, \tag{B.3}
\]

uniformly in \( n \) for some constant \( C = C(X, \gamma, R, t) \) and any \( \varepsilon \in (0, \alpha_2). \) Since \( x \in B_R \) the term \( \int_{B_R} \left( c + \log_+ \frac{t}{y-x} \right) M_n(dy) \) can be treated similarly and therefore (B.2) follows.

Lemma B.3. Almost surely w.r.t. \( \mathbb{P}_x, \) for all \( x \in U, \)

\[
\int_0^{T_U} 1_{\{x\}}(B_s^U) \, dF_s = 0, \quad P_x\text{-a.s.}
\]
Proof: For $\delta > 0$ let $f_\delta$ be the continuous function on $U$ with the property that $f_\delta(x) = 1$, $f_\delta \equiv 0$ on $B(x, \delta)^c$ and decaying linearly on $B(x, \delta)$. Then, a similar computation as in the proof of Lemma B.2 above gives that for any $t > 0$,

$$E_x \left[ \int_0^{tN/tU} f(B_s^U) dF^n_s \right] \leq \frac{1}{2\pi} \int_{B(x, \delta)} \left( c + \log_+ \frac{t}{(y-x)^2} \right) M_n(dy) \leq C \delta^{\alpha_2 - \varepsilon},$$

uniformly in $n$ for some constant $C = C(X, \gamma, t)$ and any $\varepsilon$ small enough. By Lemma B.2 taking the limit $n \to \infty$ yields $E_x \left[ \int_0^{tN/tU} f(B_s^U) dF^n_s \right] \leq C \delta^{\alpha_2 - \varepsilon}$. Finally, since $t > 0$ is arbitrary, we let $\delta \to 0$ to get the claim by monotone convergence. \qed

Proof of Proposition B.1. By a density argument it suffices to consider functions $\eta$ with compact support. First note that by Fubini’s theorem we have for every $\Psi$

$$u(x,r) = \int \sum_{j=1}^\infty \eta(x) y_j^U(x,y) \, dF^n(y),$$

and we need to show that taking limits on both sides of (B.4) we obtain (B.1). Since $dF^n$ converges weakly to $dF$ the convergence of the left hand side of (B.4) follows from the uniform $P_x$-integrability of $\{\int_0^{tN/tU} \eta(t) f(B_s^U) dF^n_s\}_n$ in Lemma B.2.

For the convergence of the right hand side of (B.4), note first that we may replace $f$ in (B.1) by $f 1_{U\setminus\{x\}}$ by Lemma B.3 and the fact that $M(\{x\}) = 0$. Then, we can easily construct a sequence of continuous functions $(f_k)_k$ having compact supports in $U\setminus\{x\}$ such that pointwise $f_k \uparrow f 1_{U\setminus\{x\}}$ as $k \to \infty$. Thus, it is enough to consider the case $x \notin \text{supp}(f)$. But in this case the function $y \mapsto \int_0^{tN/tU} \eta(t) f(y) q^U_t(x,y) \, dt$ is continuous with compact support, and the desired convergence follows from the weak convergence of $M_n$ to $M$. \qed

Appendix C. Negative moments of the Liouville measure

Lemma C.1. Let $q > 0$ and set $\bar{\xi}(q) := 2q + \frac{q(1+q)}{2}\gamma^2$. Then there exists $c_4 = c_4(\gamma, q)$ such that for any $r \in (0, 1]$,

$$\sup_{n \geq 1} \mathbb{E}[M_n(B(0, r))^{-q}] \leq c_4 r^{-\bar{\xi}(q)}.$$

We remark that the law of $M_n$ is spatially stationary, so the same statement applies to any ball $B(x_0, r)$, $x_0 \in \mathbb{R}^2$ and $r \in (0, 1]$.

Proof. Recall that $X_n(x) = \sum_{j=1}^n Y_j$, where for each $j$, $Y_j$ is a centered Gaussian field with covariance kernel $k_j$ defined in (2.1). By construction we have $\sum_{j=1}^\infty k_j = g^{(m)}$ and $g^{(m)}$ can be written as

$$g^{(m)}(x,y) = \frac{1}{2\pi} \log_+ \frac{1}{|x-y|} + \Psi_m(x,y),$$

for some bounded continuous function $\Psi_m$. In particular, for a sufficiently large $T$,

$$\sum_{j=1}^\infty k_j(x,y) = g^{(m)}(x,y) \leq \frac{1}{2\pi} \log_+ \frac{T}{|x-y|} := k^0(x,y), \quad \text{if } |x-y| < 2.$$
Hence, for all $n \geq 1$ and $\varepsilon > 0$ there exists $\delta_0 > 0$ such that

$$\sum_{j=1}^{n} k_j(x, y) \leq \varepsilon + \vartheta(\delta) * k^0(x, y), \quad \forall |x - y| < 2, \delta < \delta_0,$$

where $\vartheta(x) := \frac{5}{3} (1 - |x|^{1/2})^+$ and $\vartheta(\delta) := \frac{1}{\delta^2} \vartheta(\delta)$. Then, $\vartheta$ is positive definite (see \cite{20}) and $\int_{\mathbb{R}^2} \vartheta(x) \, dx = 1$. For all $\delta > 0$ let $(X^0_\delta(x))_{x \in \mathbb{R}^2}$ be the centered Gaussian field with covariance

$$\mathbb{E}[X^0_\delta(x) X^0_\delta(y)] = \vartheta(\delta) * k^0(y - x)$$

and by \cite[Theorem 2.1]{230} the associated random measures

$$M^0_\delta(dx) = \exp\left(X^0_\delta(x) - \frac{1}{\delta^2} \mathbb{E}[X^0_\delta(x)] \right) \, dx$$

converge weakly to some measure $M^0$ as $\delta$ tends to zero. Further, for $\varepsilon > 0$ let $Z_\varepsilon \sim \mathcal{N}(0, \varepsilon)$ be independent of $(X^0_\delta(x))_{x \in \mathbb{R}^2}$. By Kahane’s convexity inequality (see \cite[Theorem 2.1]{21} or \cite{17}) we have for all bounded convex functions $F : [0, \infty) \to \mathbb{R}$, all collections of points $(y_i)_{i=1,\ldots,N} \subset B(0, 1)$ and all non-negative weights $(p_i)_{i=1,\ldots,N}, N \in \mathbb{N},$

$$\mathbb{E}\left[F\left(\sum_{i=1}^{N} p_i e^{X_\delta(y_i) - \frac{1}{\delta^2} \mathbb{E}[X_\delta(y_i)]}\right)\right] \leq \mathbb{E}\left[F\left(\sum_{i=1}^{N} p_i e^{X_\delta(y_i) - \varepsilon Z_\varepsilon - \frac{1}{\delta^2} \mathbb{E}[X_\delta(y_i)] - \frac{\varepsilon}{\delta^2}}\right)\right].$$

By choosing appropriate points $y_i$ and weights $p_i$ we deduce that for all $r \in (0, 1],$

$$\mathbb{E}\left[F\left(\int_{B(0,r)} e^{X_\delta(y) - \frac{1}{\delta^2} \mathbb{E}[X_\delta(y)]} \, dy\right)\right] \leq \mathbb{E}\left[F\left(\int_{B(0,r)} e^{X_\delta(y) - \varepsilon Z_\varepsilon - \frac{1}{\delta^2} \mathbb{E}[X_\delta(y)] - \frac{\varepsilon}{\delta^2}} \, dy\right)\right],$$

which can be rewritten as

$$\mathbb{E}\left[F\left(M_n(B(0, r))\right)\right] \leq \mathbb{E}\left[F\left(e^{\varepsilon Z_\varepsilon - \frac{\varepsilon}{\delta^2}} M^0_\delta(B(0, r))\right)\right].$$

Taking the limit $\delta \to 0$ and $\varepsilon \to 0$ afterwards, we obtain

$$\mathbb{E}\left[F\left(M_n(B(0, r))\right)\right] \leq \mathbb{E}\left[F\left(M^0(B(0, r))\right)\right].$$

Finally, we choose $F(t) = \lambda^{q-1} e^{-\lambda t}$ for $\lambda > 0$ and, since $\frac{1}{(q-1)} \int_0^\infty \lambda^{q-1} e^{-\lambda t} \, d\lambda = t^{-q},$ integration over $\lambda$ yields

$$\mathbb{E}\left[M_n(B(0, r))^{-q}\right] \leq \mathbb{E}\left[M^0(B(0, r))^{-q}\right].$$

The statement now follows from \cite[Proposition 3.7]{230}. \hfill $\square$

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