Metric embeddings of Laakso graphs into Banach spaces

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Received: 23 March 2022 / Accepted: 27 July 2022
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Abstract
Let $X$ be Banach space which is not super-reflexive. Then, for each $n \geq 1$ and $\varepsilon > 0$, we exhibit metric embeddings of the Laakso graph $\mathcal{L}_n$ into $X$ with distortion less than $2 + \varepsilon$ and into $L_1[0,1]$ with distortion $4/3$. The distortion of an embedding of $\mathcal{L}_2$ (respectively, the diamond graph $D_2$) into $L_1[0,1]$ is at least $9/8$ (respectively, $5/4$).

Keywords Laakso graph · Bilipschitz embeddings · Super-reflexivity · Banach spaces

Mathematics Subject Classification 46B85 · 30L05 · 46B04 · 46B20

1 Introduction

James [5] introduced the important property of super-reflexivity: a Banach space $X$ is super-reflexive if every Banach space $Y$ which is finitely representable in $X$ is reflexive. Enflo [3] showed that super-reflexivity of $X$ is equivalent to $X$ having an equivalent uniformly convex norm.

Let us recall the definition of the diamond and Laakso graphs.
**Definition 1** The diamond graph of level 0 has two vertices joined by an edge of length 1 and is denoted by $D_0$. The *diamond graph* $D_n$ is obtained from $D_{n-1}$ in the following way. Each edge $uv$ of $D_{n-1}$ is replaced by a quadrilateral $u, a, v, b$, with edges $ua, av, vb, bu$ of length 1 (see Fig. 1).

Definition 1 was introduced in [4].

**Definition 2** The Laakso graph of level 0 has two vertices joined by an edge of length 1 and is denoted $L_0$. The *Laakso graph* $L_n$ is obtained from $L_{n-1}$ according to the following procedure. Each edge $uv \in E(L_{n-1})$ is replaced by the graph $L_1$ exhibited in Fig. 2 in which each edge has length 1.

Definition 2 was introduced in [9] based on an idea of Laakso [8].

![Fig. 1 The diamond graph $D_2$](image1)

![Fig. 2 The Laakso graphs $L_1$ and $L_2$](image2)
Let $f : (M, \rho) \rightarrow (N, \sigma)$ be a bilipschitz mapping between metric spaces. The distortion of $f$ is defined to be the infimum of $b/a$, where $a, b$ are positive constants such that

$$a \rho(x, y) \leq \sigma(f(x), f(y)) \leq b \rho(x, y) \quad (x, y \in M).$$

Bourgain [1] characterized Banach spaces which are not super-reflexive as those for which the binary trees $B_n$ of depth $n$ embed with uniformly bounded distortion. Subsequently, Johnson and Schechtman [7] characterized Banach spaces which are not super-reflexive as those for which the diamond graphs $D_n$ and the Laakso graphs $L_n$ embed with uniformly bounded distortion. The best known estimate in the literature for the distortion of embeddings of $D_n$ into arbitrary Banach spaces which are not super-reflexive, due to Pisier [13], is $2 + \varepsilon$ for every $\varepsilon > 0$, while the best known estimate for the distortion of embeddings of $D_n$ into $L_1[0, 1]$, due to Lee and Raghavendra [10], is $4/3$.

Ostrovskii and Randrianantoanina [11] constructed embeddings of the $k$-branching diamond graphs $D_{n,k}$ and the $k$-branching Laakso graphs $L_{n,k}$ into arbitrary Banach spaces which are not super-reflexive with distortion $8 + \varepsilon$. Swift [15] constructed embeddings of the family of bundle graphs generated by a finitely-branching bundle graph $G$ into Banach spaces which are not super-reflexive with distortion bounded above by a number not depending on the target space or the branching number of $G$. In particular, he proved that the finitely branching parasol graphs also embed with distortion $8 + \varepsilon$.

In the present article, we construct embeddings of $L_n$ into arbitrary Banach spaces which are not super-reflexive with distortion $2 + \varepsilon$ and into $L_1[0, 1]$ with distortion $4/3$. We also show that $L_2$ does not embed into $L_1[0, 1]$ with distortion smaller than $9/8$.

## Results

The embeddings of $L_n$ which we define depend on the following characterization of not being super-reflexive. Its negation is the characterization of super-reflexivity known as $J$-convexity.

**Theorem A** [6, 14] $X$ is not super-reflexive if and only if, for each $m \geq 1$ and $\varepsilon > 0$, there exist $e_1, \ldots, e_m$ in the unit ball of $X$ such that, for each $1 \leq j \leq m$, we have

$$\|e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m\| \geq m - \varepsilon.$$  \hspace{1cm} (1)

**Remark 3** It follows easily from Theorem A that if $X$ is not super-reflexive then, for each $n \geq 1$ and $\varepsilon > 0$, $B_n$ embeds into $X$ with distortion $1 + \varepsilon$. This is not true, however, for $D_n$ and $L_n$ if $n \geq 2$.

We will make use of the following two consequences of Theorem A.
Lemma 4 Suppose $X$ is not super-reflexive. Let $(e_i)_{i=1}^m$ be as in Theorem A. If $\max A < \min B$ then
\[
\left\| \sum_{i \in A} e_i - \sum_{i \in B} e_i \right\| \geq |A| + |B| - \varepsilon.
\]

Proof This follows at once from (1) and the triangle inequality. \qed

Lemma 5 Suppose $X$ is not super-reflexive. Let $(e_i)_{i=1}^m$ be as in Theorem A. If $\max A < \min B$ or $\max B < \min A$ then
\[
\left\| \sum_{i \in A} \varepsilon_i e_i + \sum_{i \in B} e_i \right\| \geq |B| - \varepsilon.
\]
for all choices of signs $\varepsilon_i = \pm 1$.

Proof Let $A^+ = \{i \in A : \varepsilon_i = 1\}$ and let $A^- = \{i \in A : \varepsilon_i = -1\}$. If $|A^+| \geq |A^-|$ then
\[
\left\| \sum_{i \in A} \varepsilon_i e_i + \sum_{i \in B} e_i \right\| \geq \left\| \sum_{i \in A^+} e_i + \sum_{i \in B} e_i \right\| - |A^-|
\geq |A^+| + |B| - \varepsilon - |A^-|
\]
(by Lemma 4)
\[
\geq |B| - \varepsilon.
\]
On the other hand, if $|A^-| > |A^+|$ then
\[
\left\| \sum_{i \in A} \varepsilon_i e_i + \sum_{i \in B} e_i \right\| \geq - \left\| \sum_{i \in A^-} e_i + \sum_{i \in B} e_i \right\| - |A^+|
\geq |A^-| + |B| - \varepsilon - |A^+|
\]
(by Lemma 4)
\[
\geq 1 + |B| - \varepsilon.
\]
\qed

Theorem 6 Suppose $X$ is not super-reflexive. Then, for each $\varepsilon > 0$ and $n \geq 1$, there exists a mapping $f_n : \mathcal{L}_n \rightarrow X$ such that, for all $a, b \in \mathcal{L}_n$,
\[
\frac{1}{2} d(a, b) - \varepsilon \leq \|f_n(a) - f_n(b)\| \leq d(a, b).
\]

Proof Let $\varepsilon > 0$ be fixed. For each $n \geq 1$, select vectors $(e_i)_{i=1}^m$ satisfying Theorem A for $m = 4^n$. We define the mappings $f_n$ inductively.

We begin with the base case $n = 1$. Label the vertices of $\mathcal{L}_1$ as shown in Fig. 3.
We define \( f_1 : L_1 \rightarrow X \) as follows:

\[
\begin{align*}
  f_1(A) &= 0, \\
  f_1(T) &= e_1^1, \\
  f_1(L) &= e_1^1 + e_2^1, \\
  f_1(R) &= e_1^1 + e_3^1, \\
  f_1(B) &= e_1^1 + e_2^1 + e_3^1, \\
  f_1(U) &= e_1^1 + e_2^1 + e_3^1 + e_4^1.
\end{align*}
\]

Using Lemma 4 it is easily checked that \( f_1 \) satisfies, for all \( a, b \in L_1 \),

\[
d(a, b) - \varepsilon \leq ||f_1(a) - f_1(b)|| \leq d(a, b).
\]

For example, \( f_1(L) - f_1(R) = e_2^1 - e_3^1 \), so \( 2 - \varepsilon \leq ||f_1(L) - f_1(R)|| \leq 2 \) as required.

Now suppose \( n \geq 2 \). We regard \( L_n \) as being obtained from \( L_1 \) by replacing each edge of \( L_1 \) by a copy of \( L_{n-1} \). Thus, \( L_n \) is composed of 6 copies of \( L_{n-1} \), labelled as \( Y, C, D, E, F \) and \( Z \) in Fig. 4.

\[\text{Fig. 3} \quad \text{The Laakso graph } L_1\]

\[\text{Fig. 4} \quad \text{The Laakso graph } L_n\]
We have labelled the vertices $A$, $T$, $L$, $R$, $B$ and $U$ of $\mathcal{L}_n$ which correspond to the vertices of $\mathcal{L}_1$. The correspondence between $\mathcal{L}_{n-1}$ and each of its copies in $\mathcal{L}_n$, namely $Y$, $C$, $D$, $E$, $F$, and $Z$, is the natural ‘downward’ correspondence in which the vertex $A$ of $\mathcal{L}_{n-1}$ is mapped to the vertices $A$, $T$, $L$, $R$, and $B$ of $\mathcal{L}_n$, respectively. Note that the vertex $T$ of $\mathcal{L}_n$ corresponds to the vertex $U$ of $Y$ and to the vertex $A$ of $C$ and $D$. There are similar correspondences for $L$, $R$ and $B$.

Let $((e^n_i)^*)_{i=1}^{4^n}$ be the coordinate functionals satisfying $(e^n_i)^*(e^n_i) = \delta_{ij}$. The mapping $f_n : \mathcal{L}_n \to X$ will be of the following form:

$$f_n(a) = \sum_{i=1}^{4^n} (e^n_i)^*(f_n(a)) e^n_i, \quad (3)$$

where $(e^n_i)^*(f_n(a)) \in \{0,1\}$ and $\text{supp}(f_n(a)) = \{ i : (e^n_i)^*(f_n(a)) = 1 \}$ has size $|\text{supp}(f_n(a))| = d(A,a)$. Note that $d(A,a)$ represents the ‘depth’ of $a$ in $\mathcal{L}_n$.

To define $f_n$ inductively, we suppose that $f_{n-1} : \mathcal{L}_{n-1} \to X$ has already been defined to be of the form (3) with $n$ replaced by $n-1$.

Let $\rho : \mathcal{L}_{n-1} \to X$ be a ‘copy’ of $f_{n-1}$ with $(e^{n-1}_i)^*_{i=1}^{4^{n-1}}$ replaced by $(e^n_i)_{i=1}^{4^n-1}$. The formal definition is as follows:

$$\rho(a) = \sum_{i=1}^{4^{n-1}} (e^{n-1}_i)^*(f_{n-1}(a)) e^n_i.$$ 

Similarly, let $\theta : \mathcal{L}_{n-1} \to X$ be a copy of $f_{n-1}$ with $(e^{n-1}_i)_{i=1}^{4^{n-1}}$ replaced by $(e^n_i)_{i=1}^{4^n-1}$.

Formally,

$$\theta(a) = \sum_{i=1}^{4^{n-1}} (e^{n-1}_i)^*(f_{n-1}(a)) e^n_{4^{n-1}+i}.$$ 

Similarly, let $\phi : \mathcal{L}_{n-1} \to X$ be a copy of $f_{n-1}$ with $(e^{n-1}_i)_{i=1}^{4^{n-1}}$ replaced by $(e^n_i)_{i=1}^{3 \cdot 4^{n-1}}$.

Formally,

$$\phi(a) = \sum_{i=1}^{4^{n-1}} (e^{n-1}_i)^*(f_{n-1}(a)) e^n_{3 \cdot 4^{n-1}+i}.$$ 

Finally, let $\sigma : \mathcal{L}_{n-1} \to X$ be a copy of $f_{n-1}$ with $(e^{n-1}_i)_{i=1}^{4^{n-1}}$ replaced by $(e^n_i)_{i=1}^{3 \cdot 4^{n-1}+1}$.

Formally,

$$\sigma(a) = \sum_{i=1}^{4^{n-1}} (e^{n-1}_i)^*(f_{n-1}(a)) e^n_{3 \cdot 4^{n-1}+i}.$$ 

Recall that $Y$, $C$, $D$, $E$, $F$ and $Z$ are ‘copies’ of $\mathcal{L}_{n-1}$. Let $W$ be any one of these copies. For $a \in W$, let $\overline{a} \in \mathcal{L}_{n-1}$ denote the element of $\mathcal{L}_{n-1}$ which corresponds to $a$.

Now we define $f_n : \mathcal{L}_n \to X$ as follows:
\begin{equation*}
\sum_{i=1}^{4^n-1} e_i^n + \theta(\overline{a}), \quad a \in C
\end{equation*}
\begin{equation*}
\sum_{i=1}^{2 \cdot 4^n-1} e_i^n + \phi(\overline{a}), \quad a \in E
\end{equation*}
\begin{equation*}
\sum_{i=1}^{3 \cdot 4^n-1} e_i^n + \sum_{i=3 \cdot 4^n-1+1}^{3 \cdot 4^n-1} e_i^n + \theta(\overline{a}), \quad a \in F
\end{equation*}
\begin{equation*}
\sum_{i=1}^{3 \cdot 4^n-1} e_i^n + \sigma(\overline{a}), \quad a \in Z.
\end{equation*}

Note that at the vertices T, L, R and B, which connect the copies of \( \mathcal{L}_{n-1} \), \( f_n \) is defined twice, but both definitions agree. Therefore, \( f_n \) is well-defined.

Now we verify (2). We begin with the right-hand inequality. If \( d(a, b) = 1 \), i.e., if \( a \) and \( b \) are adjacent vertices in \( \mathcal{L}_n \), then it is clear from the definition that \( \|f_n(a) - f_n(b)\| \leq 1 \). Since \( d \) is the shortest path metric, the right-hand inequality follows at once from the triangle inequality in \( X \).

We now turn to the left-hand inequality. If \( a \) and \( b \) belong to the same copy of \( \mathcal{L}_{n-1} \) (either \( Y, C, D, E, F \) or \( Z \)) then the left-hand inequality follows from the inductive hypothesis. So suppose that they belong to different copies. There are several cases to consider.

**Case 1.** Suppose that \( a \) is ‘above’ \( b \) in \( \mathcal{L}_n \). Then \( \text{supp}(f_n(a)) \subseteq \text{supp}(f_n(b)) \). Using Lemma 4,

\begin{equation*}
\|f_n(b) - f_n(a)\| = \left\| \sum_{i \notin \text{supp}(f_n(b)) \setminus \text{supp}(f_n(a))} e_i^n \right\| \\
\geq |\text{supp}(f_n(b))| - |\text{supp}(f_n(a))| - \varepsilon \\
= d(a, b) - \varepsilon.
\end{equation*}

**Case 2.** Suppose \( a \in C, b \in D \).

\begin{equation*}
\|f_n(a) - f_n(b)\| = \|\theta(\overline{a}) - \phi(\overline{b})\| \\
\geq |\text{supp}(\theta(\overline{a}))| + |\text{supp}(\phi(\overline{b}))| - \varepsilon
\end{equation*}

(by Lemma 4 since \( \max \text{supp}(\theta(\overline{a})) < \min \text{supp}(\phi(\overline{b})) \))

\begin{equation*}
= d(T, a) + d(T, b) - \varepsilon \\
= d(a, b) - \varepsilon.
\end{equation*}

**Case 3.** Suppose \( a \in C, b \in F \). Note that in this case \( d(a, b) \leq 2 \cdot 4^{n-1} \). Hence

\begin{equation*}
f_n(a) - f_n(b) = \theta(\overline{a}) - \theta(\overline{b}) - \sum_{i=2 \cdot 4^{n-1}+1}^{3 \cdot 4^n-1} e_i^n.
\end{equation*}

Note that \( \theta(\overline{a}) - \theta(\overline{b}) = \sum_{i \in A} \varepsilon_i e_i^n \), where \( A \subseteq \{ i : 4^{n-1} + 1 \leq i \leq 2 \cdot 4^{n-1} \} \) and \( \varepsilon_i = \pm 1 \). Hence, by Lemma 5,
Case 4. Suppose \( a \in D, b \in E \). This is similar to Case 3. Note that \( d(a, b) \leq 2 \cdot 4^{n-1} \).

\[
\|f_n(a) - f_n(b)\| = \left\| \sum_{i \in A} \varepsilon_i e_i^n - \sum_{i = 2 \cdot 4^{n-1} + 1}^{3 \cdot 4^{n-1}} e_i^n \right\| \\
\geq 4^{n-1} - \varepsilon \\
\geq \frac{1}{2} d(a, b) - \varepsilon.
\]

Note that \( \phi(a) - \phi(b) = \sum_{i \in A} \varepsilon_i e_i^n \), where \( A \subseteq \{ i : 2 \cdot 4^{n-1} + 1 \leq i \leq 3 \cdot 4^{n-1} \} \) and \( \varepsilon_i = \pm 1 \). Hence, by Lemma 5,

\[
\|f_n(a) - f_n(b)\| = \left\| \sum_{i \in A} \varepsilon_i e_i^n - \sum_{i = 2 \cdot 4^{n-1} + 1}^{3 \cdot 4^{n-1}} e_i^n \right\| \\
\geq 4^{n-1} - \varepsilon \\
\geq \frac{1}{2} d(a, b) - \varepsilon.
\]

Case 5. Suppose \( a \in E, b \in F \). This is similar to Case 2. Note that

\[
f_n(a) - f_n(b) = \left( \sum_{i = 2 \cdot 4^{n-1} + 1}^{3 \cdot 4^{n-1}} e_i^n - \phi(b) \right) - \left( \sum_{i = 2 \cdot 4^{n-1} + 1}^{3 \cdot 4^{n-1}} e_i^n - \phi(a) \right).
\]

Hence, by Lemma 4,

\[
\|f_n(a) - f_n(b)\| \geq (4^{n-1} - |\text{supp}(\theta(b))|) + (4^{n-1} - |\text{supp}(\phi(a))|) - \varepsilon \\
= d(b, B) + d(B, a) - \varepsilon \\
= d(a, b) - \varepsilon.
\]

Remark 7 The analogue of Theorem 6 for \( D_n \) is proved in [13, Theorem 13.17, (13.26)] with the same distortion of \( 2 + \varepsilon \).

We now prove a stronger result for \( X = L_1[0, 1] \).

Theorem 8 For each \( n \geq 1 \), there exists a mapping \( f_n : \mathcal{L}_n \to L_1[0, 1] \) such that, for all \( a, b \in \mathcal{L}_n \),
\[
\frac{3}{4}d(a, b) \leq \|f_n(a) - f_n(b)\|_1 \leq d(a, b).
\] (4)

The proof requires the following elementary lemma.

**Lemma 9** For \(0 \leq s, t \leq 1\),

\[
1 + \min(s + t, 2 - s - t) \leq \frac{4}{3}(1 + s + t - 2st)
\]

with equality if \(s = t = 1/2\).

**Proof** First suppose \(x := s + t \leq 1\). Then, \(\min(s + t, 2 - s - t) = x\) and \(st \leq x^2/4\). Hence

\[
\frac{4}{3}(1 + s + t - 2st) - (1 + (s + t)) \geq \frac{4}{3} \left(1 + x - \frac{x^2}{2}\right) - 1 - x
\]

\[
= \frac{1}{3} + \frac{x}{3} - \frac{2x^2}{3}
\]

\[
\geq 0.
\]

The case \(1 \leq s + t\) is similar. \(\Box\)

**Proof of Theorem 8** Each \(f_n\) will be of the following form:

\[
f_n(a) = 4^n1_{H_n(a)} \quad (a \in L_n),
\] (5)

where \(H_n(a) \subseteq [0, 1]\) and \(|H_n(a)| = 4^{-n}d(A, a)\). We begin with the base case \(n = 1\):

\[
H_1(A) = \emptyset, H_1(T) = [0, 1/4]; H_1(L) = [0, 1/2];
\]

\[
H_1(R) = [0, 1/4] \cup [1/2, 3/4]; H_1(B) = [0, 3/4]; H_1(U) = [0, 1].
\]

It is easily seen that \(f_1\) is an isometry.

For \(n \geq 2\) the definition of \(f_n\) is inductive. Suppose that \(f_{n-1}\) has been defined to be of the form (5). Let \(\theta\) and \(\phi\) be identically distributed copies of the mapping \(a \mapsto H_{n-1}(a)\). Moreover, we require \(\theta\) and \(\phi\) to be stochastically independent, i.e.,

\[
|\theta(a) \cap \phi(b)| = |\theta(a)||\phi(b)| \quad (a, b \in L_{n-1}).
\]

We use \(\theta\) and \(\phi\) to define \(H_n\) as follows:
\[
H_n(a) = \begin{cases} 
\frac{1}{4}\theta(\bar{a}), & a \in Y \\
[0, 1/4] \cup \left(\frac{1}{2} + \frac{1}{4}\theta(\bar{a})\right), & a \in C \\
[0, 1/4] \cup \left(\frac{1}{2} + \frac{1}{4}\phi(\bar{a})\right), & a \in D \\
[0, 1/2] \cup \left(\frac{1}{2} + \frac{1}{4}\theta(\bar{a})\right), & a \in E \\
[0, 1/4] \cup \left(\frac{1}{2} + \frac{1}{4}\phi(\bar{a})\right) \cup \left[\frac{1}{2}, \frac{3}{4}\right], & a \in F \\
[0, 3/4] \cup \left(\frac{3}{4} + \frac{1}{4}\theta(\bar{a})\right), & a \in Z.
\end{cases}
\]

The right-hand inequality of (4) follows as in the proof of Theorem 6. For the left-hand inequality, we may assume that \(a\) and \(b\) belong to different copies of \(\mathcal{L}_{n-1}\).

**Case 1.** Suppose that \(a\) is ‘above’ \(b\) in \(\mathcal{L}_n\). Then, \(H_n(a) \subseteq H_n(b)\), so
\[
d(a, b) = 4^n(|H_n(b)| - |H_n(a)|) = ||f_n(a) - f_n(b)||_1.
\]

**Case 2.** Suppose \(a \in C, b \in D\). Then
\[
||f_n(a) - f_n(b)||_1 = 4^{n-1}(|\theta(\bar{a})| + |\phi(\bar{b})|) = d(a, T) + d(b, T) = d(a, b).
\]

**Case 3.** Suppose \(a \in C, b \in F\). Note that
\[
d(a, b) = 4^{n-1}(1 + \min(|\theta(\bar{a})| + |\phi(\bar{b})|, 2 - |\theta(\bar{a})| - |\phi(\bar{b})|)).
\]
Then,
\[
||f_n(a) - f_n(b)||_1 = 4^{n-1}(|\theta(\bar{a})| - 1|\phi(\bar{b})|) + 1)
= 4^{n-1}(|\theta(\bar{a})| + |\phi(\bar{b})| - 2|\theta(\bar{a}) \cap \phi(\bar{b})| + 1)
= 4^{n-1}(|\theta(\bar{a})| + |\phi(\bar{b})| - 2|\theta(\bar{a})||\phi(\bar{b})| + 1)
\]
(since \(\theta(\bar{a})\) and \(\phi(\bar{b})\) are independent)
\[
\geq 4^{n-1}\frac{3}{4}(1 + \min(|\theta(\bar{a})| + |\phi(\bar{b})|, 2 - |\theta(\bar{a})| - |\phi(\bar{b})|))
\]
(from Lemma 9 with \(s = |\theta(\bar{a})|\) and \(t = |\phi(\bar{b})|\))
\[
= \frac{3}{4}d(a, b).
\]

**Case 4.** Suppose \(a \in D, b \in E\). This is essentially the same as Case 3. As in Case 3, we obtain
\[
||f_n(a) - f_n(b)||_1 \geq \frac{3}{4}d(a, b).
\]
Case 5. Suppose \(a \in E, b \in F\). This is very similar to Case 2. Note that

\[
\|f_n(a) - f_n(b)\|_1 = 4^{n-1}((1 - |\vartheta(a)|) + (1 - |\phi(b)|))
= d(a, B) + d(b, B)
= d(a, b).
\]

\[\square\]

Remark 10 The analogue of Theorem 8 for \(D_n\) is proved in [10, Theorem 5.1] with the same distortion of \(4/3\).

The next result shows that the distortion of any embedding of \(L_2\) into \(L_1[0, 1]\) is at least \(9/8\).

Theorem 11 Let \(f : L_2 \to L_1[0, 1]\) satisfy

\[
d(a, b) \leq \|f(a) - f(b)\|_1 \leq cd(a, b).
\]

Then, \(c \geq 9/8\).

The proof uses the following result about hypermetric and negative type inequalities from [2].

Theorem B [2, Lemma 6.1.1] Let \((M, \rho)\) be a finite metric space which is isometric to a subset of \(L_1[0, 1]\). Then, for all \(k_i \in \mathbb{Z}\) \((1 \leq i \leq n)\) such that \(\sum_{i=1}^{n} k_i = 0\) (negative type inequalities) or \(\sum_{i=1}^{n} k_i = 1\) (hypermetric inequalities), we have
\[ \sum_{1 \leq i < j \leq n} k_i k_j \rho(x_i, x_j) \leq 0, \]

where \( x_1, \ldots, x_n \) are the distinct elements of \( M \).

**Proof of Theorem 11** Consider the two choices of weights for \( L_1 \) indicated in Fig. 5 (each weight is shown next to its corresponding vertex). Now define weights for \( L_2 \) by assigning the \( P \) weights to the \( C \) and \( F \) copies of \( L_1 \), the \( N \) weights to the \( D \) and \( E \) copies, and zero weights to the \( Y \) and \( Z \) copies. Let \( (k_i)_{i=1}^{30} \) be the enumeration of these weights corresponding to some enumeration of the vertices of \( L_2 \). Note that \( \sum_{i=1}^{30} k_i = 0 \). By Theorem B,

\[
72 = \sum_{i < j, k_i > 0} k_i k_j d(x_i, x_j) \\
\leq \sum_{i < j, k_i > 0} k_i k_j \|f(x_i) - f(x_j)\|_1 \\
\leq \sum_{i < j, k_i < 0} |k_i k_j| \|f(x_i) - f(x_j)\|_1 \\
\leq c \sum_{i < j, k_i < 0} |k_i k_j| d(x_i, x_j) \\
= 64c.
\]

So \( c \geq 9/8 \). \( \square \)

In a similar way, we can estimate the distortion of metric embeddings of the diamond graph \( D_2 \) into \( L_1[0, 1] \).

**Theorem 12** Let \( f : D_2 \to L_1[0, 1] \) satisfy

\[ d(a, b) \leq \|f(a) - f(b)\|_1 \leq c d(a, b). \]

Then \( c \geq 5/4 \).

**Proof** Consider the weights on \( D_1 \), denoted again \( P \) and \( N \), obtained from Fig. 5 by removing the \( A \) and \( U \) vertices of \( L_1 \). Now define weights on \( D_2 \) by assigning \( P \) to one pair of ‘opposite’ copies of \( D_1 \) and \( N \) to the other pair. Let \( (k_i)_{i=1}^{12} \) be an enumeration of these weights corresponding to some enumeration of the vertices of \( D_2 \). Note that \( \sum_{i=1}^{12} k_i = 0 \). Using Theorem B as above yields

\[
40 = \sum_{i < j, k_i > 0} k_i k_j d(x_i, x_j) \leq c \sum_{i < j, k_i < 0} |k_i k_j| d(x_i, x_j) = 32c.
\]

So \( c \geq 5/4 \). \( \square \)
Remark 13 A computer search revealed that $c = 5/4$ is the best estimate of the lower bound for the distortion of $D_2$ that can be obtained from the negative type and hypermetric inequalities of Theorem B by considering all possible choices of $k_i$ in the range $-10 \leq k_i \leq 10$, and that $c = 9/8$ is the best that can be obtained for $L_2$ by considering all possible choices of $k_i$ in the range $-1 \leq k_i \leq 1$. Actually, we could not find any embedding of $D_2$ into $L_1[0, 1]$ with distortion smaller than $4/3$, but were not able to prove that $4/3$ is optimal.

Remark 14 Since the proofs of Theorems 11 and 12 used only negative type inequalities, by [2, Theorem 6.2.2] they remain valid if $L_1[0, 1]$ is replaced by $(\ell_2^1, \| \cdot \|_2^2)$. This is a stronger result as $L_1[0, 1]$ is isometric to a subset of $(\ell_2^1, \| \cdot \|_2^2)$ (see e.g., [12, p. 20]).

Acknowledgements We thank Mikhail Ostrovskii for his comments and for drawing some references to our attention. S. J. Dilworth was supported by Simons Foundation Collaboration Grant no. 849142. Denka Kutzarova was supported by Simons Foundation Collaboration Grant no. 636954.

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