Exceptional sets for nonuniformly expanding maps

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Abstract
Given a rational map of the Riemann sphere and a subset A of its Julia set, we study the A-exceptional set, that is, the set of points whose orbit does not accumulate at A. We prove that if the topological entropy of A is less than the topological entropy of the full system then the A-exceptional set has full topological entropy. Furthermore, if the Hausdorff dimension of A is smaller than the dynamical dimension of the system then the Hausdorff dimension of the A-exceptional set is larger than or equal to the dynamical dimension, with equality in the particular case when the dynamical dimension and the Hausdorff dimension coincide.

We also discuss the case of a general conformal C¹⁺α dynamical system and, in particular, certain multimodal interval maps on their Julia set.

Keywords: topological entropy, Hausdorff dimension, exceptional sets

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1. Introduction
Consider a compact metric space (X, d) and a continuous transformation f : X → X. Let W ⊂ X be f-invariant, that is, f(W) = W. Given A ⊂ W, the A-exceptional set in W (with respect to f|W) is defined to be the set

\[ E^+_W(A) := \{ x \in W : \overline{f^n(x)} \cap A = \varnothing \}, \]
where \( \mathcal{O}_f(x) := \{ f^k(x) : k \in \mathbb{N} \cup \{0\} \} \) denotes the forward orbit of \( x \) by \( f \). In other words, \( E^+_W(f,A) \) is the set of points in \( W \) whose forward orbit does not accumulate at \( A \). In this paper we study the ‘size’ of exceptional sets in terms of their topological entropy and their Hausdorff dimension. We will consider as dynamical systems rational functions of the Riemann sphere which include those with parabolic points and critical points.

The following is our first main result stated in terms of topological entropy (we recall its definition in section 2.2).

**Theorem 1.1.** Let \( f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) be a rational function of degree \( d \geqslant 2 \) on the Riemann sphere and let \( J = J(f) \) be its Julia set.

If \( A \subset J \) satisfies \( h(f|_J,A) < h(f|_J) = \log d \), then

\[
h(f|_J,E^+_W(f,A)) = h(f|_J) = \log d.
\]

The above result will be a consequence of a corresponding statement for the entropy of a continuous shift-equivalent transformation (see proposition 5.1).

The second result in terms of the Hausdorff dimension \( \dim_\mathcal{H} \) uses canonical concepts which we briefly recall (see section 2.1 for more detail). Given a \( f \)-invariant probability measure \( \mu \), the *Hausdorff dimension* of \( \mu \) is defined by

\[
\dim_\mathcal{H} \mu := \inf \{ \dim_\mathcal{H} Y : Y \subset X \text{ and } \mu(Y) = 1 \}.
\]

The *dynamical dimension* of \( f \) is defined by

\[
\DD(f|_X) := \sup_{\mu} \dim_\mathcal{H} \mu,
\]

where the supremum is taken over all ergodic measures \( \mu \) with positive entropy. We will only consider maps where such measures do exist and where hence \( \DD \) is well defined. Note that clearly we have \( \DD(f|_X) \leqslant \dim_\mathcal{H} X \).

The following relation was established in the context of a general rational function \( f \) of degree \( \geqslant 2 \) of the Riemann sphere and \( X = J(f) \) its Julia set (see [21, chapter 12.3]) and will be fundamental for our approach. We have

\[
\DD(f|_{E^+_W(f,A)}) = h_\mathcal{D}(f|_{E^+_W(f,A)}), \quad \text{where} \quad h_\mathcal{D}(f|_{E^+_W(f,A)}) := \sup_Y \dim_\mathcal{H} Y,
\]

where the supremum is taken over all conformal expanding repellers \( Y \subset J(f) \) (we recall its definition in section 2.3), the latter number is also called the *hyperbolic dimension* of \( J(f) \).

**Theorem 1.2.** Let \( f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) be a rational function of degree \( \geqslant 2 \) on the Riemann sphere and let \( J = J(f) \) be its Julia set.

If \( A \subset J \) satisfies \( \dim_\mathcal{H} A < \DD(f|_J) \), then

\[
\dim_\mathcal{H} E^+_W(f,A) \geq \DD(f|_J).
\]

Theorem 1.2 immediately implies the following.

**Corollary 1.3.** Let \( f : \overline{\mathbb{C}} \to \overline{\mathbb{C}} \) be a rational function of degree \( \geqslant 2 \) on the Riemann sphere and let \( J = J(f) \) be its Julia set. Assume that we have

\[
\DD(f|_J) = \dim_\mathcal{H} J.
\]

Note that until recently it was unknown whether there exists a map for which \( h_\mathcal{D}(f|_J) < \dim_\mathcal{H} J \). Avila and Lyubich in [4, theorem D] show that for so-called Feigenbaum maps with periodic combinatorics whose Julia set has positive area one has \( h_\mathcal{D}(f|_J) < \dim_\mathcal{H} J = 2 \). They provide examples in [5].
If $A \subset J$ satisfies $\dim_H A < \dim_H J$ then
$$\dim_H E^+_f(J) (A) = \dim_H J.$$

We obtain an immediate conclusion in the particular case of an expansive map. For that recall that a continuous map $f: X \to X$ is expansive if there exists $\delta > 0$ such that for each pair of distinct points $x, y \in X$ there is $n \geq 1$ such that $d(f^n(x), f^n(y)) \geq \delta$. By the Bowen–Manning–McCluskey formula, in the case of a rational function $f: J(f) \to J(f)$ which is expansive equality (3) holds true (see [25, theorem 3.4]). Recall that by [9, theorem 4] a rational function of degree $\geq 2$ on its Julia set $J(f)$ is expansive (and hence (3) is true) if, and only if, $J(f)$ does not contain critical points.

Recent work by Rivera–Letelier and Shen [22] establishes (3) for a much wider class of maps. In particular they show that for a rational map of degree $\geq 2$ without neutral periodic points, and such that for each critical value $v$ of $f$ in $J(f)$ one has
$$\sum_{n=1}^{\infty} \frac{1}{|f^n'(v)|} < \infty$$
equalities (3) hold true (see [22, theorem II and section 2.1]) and corollary 1.3 applies. Note that, in particular, this is true for Collet–Eckmann maps.

Let us compare the main results with other previously known ones. Results of this sort have already a long history which starts with the Jarnik–Besicovitch theorem (see [16]), which states that the set of badly approximable numbers\footnote{Recall that a real number $x$ is badly approximable if there is a constant $C(x)$ such that for any reduced rational $p/q$ we have $|pq - x| > C(x)/q^2$.} in the interval $[0, 1]$ is 1. Observe that $x \in [0, 1]$ is badly approximable if, and only if, the forward orbit of $x$ relative to the Gauss continued fraction map $f: [0, 1] \to [0, 1]$ does not accumulate at 0, that is, if $x$ does not belong to the $\{0\}$-exceptional set of points. Here $f$ is defined by $f(x) := \lfloor 1/x \rfloor - \lfloor 1/\lfloor y \rfloor \rfloor$ if $x > 0$, where $\lfloor y \rfloor$ denotes the integer part of $y$, and $f(0) = 0$. This result is then an immediate consequence of the fact that for an expanding Markov map of the interval and any point $x_0$ the $\{x_0\}$-exceptional set has full Hausdorff dimension 1.

In analogy, in the case of $f$ being an expanding $C^2$ map of a Riemannian manifold $X$, it is known that $f$ preserves a probability measure which is equivalent to the Liouville measure [17], and hence the set of points whose forward orbit is not dense has zero measure. In particular, for every $x \in X$ the $\{x\}$-exceptional set has zero measure. However, by a result by Urbanski [24], this set is large in terms of the Hausdorff dimension. Tseng [23] strengthens this result by showing that in fact this set is a winning set in the sense of so-called Schmidt games, and hence has a full Hausdorff dimension (he also considers the case of a countable set of points $A$).

Abercrombie and Nair [2] proved that for a rational map on the Riemann sphere which is uniformly expanding on its Julia set for a given finite set of points $A$ satisfying some additional properties, the $A$-exceptional set has a full Hausdorff dimension (see also [1] for a precursor of this work in the case of a Markov map on the interval as well as [3] in a more abstract setting but also requiring uniform expansion of the dynamics and finiteness of the set $A$). Their method of proof (which is similarly used by Dolgopyat [10] to show theorem 1.4 stated below) is based on constructing a certain Borel measure which is supported on the set of points whose forward orbits miss certain neighbourhoods of $A$ and then the use of a mass distribution principle to estimate dimension.

Theorems 1.1 and 1.2 and corollary 1.3 generalise these results by Abercrombie and Nair in the sense that we can consider more general sets $A$ and in the sense that we can consider...
rational maps which are not uniformly expanding. They are analogues to [10, theorems 1 and 2] by Dolgopyat, which allows for a more general set $A$ but requires $f$ to be a piecewise uniformly expanding map of the interval. To the best of our knowledge, our results are the first which also apply in a nonhyperbolic setting.

Finally, note that there exists a wide range of work on so-called shrinking target problems which are somehow similar, considering instead of orbits which do not accumulate on a fixed set those orbits which do not hit a neighbourhood of a given size which shrinks with the iteration length (see, for example, Hill and Velani [13–15]).

Let us briefly outline the content of this paper and the idea of the proofs of theorems 1.1 and 1.2 (see section 5). We will choose a sequence of subsets of $J(f)$ (certain repellers) such that the dynamics inside them is expanding with all their Lyapunov exponents being close to a given number and their entropy being close to the entropy of the original system. Such repellers are provided by a construction following ideas of Katok (see theorem 3.1 in section 3). They have the property that their Lyapunov exponents and their entropies are close to the ones of an ergodic measure and their Hausdorff dimension is close to the dynamical dimension of the Julia set of whole system. Here we will also invoke the fact (2). Then we will use that (for some iterate of the map) these repellers are conjugate to a subshift of finite type and we will use the following abstract results by Dolgopyat [10] on shift spaces.

**Theorem 1.4 (10, theorem 1).** Let $\sigma : \Sigma_M^+ \rightarrow \Sigma_M^+$ be a subshift of finite type. If $B \subset \Sigma_M^+$ satisfies $h(\sigma, B) < h(\sigma)$, then $h(\sigma, E^{\sigma|\Sigma_M^+}(B)) = h(\sigma)$.

Therefore, theorem 1.4 guarantees that the entropy on a certain conjugate exceptional set in the subshift coincides with the entropy of the subshift (see section 4, where general relations for exceptional sets on subsystems are derived). To conclude the proof, it is necessary to show a relationship between topological entropy and the Hausdorff dimension inside the subrepellers, which is proven in section 2.

**Remark.** We remark that the methods in this paper extend to more general conformal $C^{\alpha}$ maps $f$ of a Riemannian manifold $X$ and a compact invariant subset $W \subset X$ studying exceptional sets in $W$ (relative to the dynamics of $f|_W$). We point out that one essential ingredient is the equality$^5$ between the hyperbolic dimension and dynamical dimension of $f|_W$ (as in (2)). Another one is the possibility of approaching any ergodic measure with positive entropy and a positive Lyapunov exponent by a certain repeller (see theorem 3.1). Then a key point is to guarantee that such repellers are contained in $W$. Whenever these facts were true, our proofs extend to this map and theorems 1.1 and 1.2 (and corollary 1.3 in case one has equality between the dynamical and Hausdorff dimension as in (3)) continue to hold true exchanging the Julia set for $W$.

For example, in [22] the authors consider the Julia set of a certain $C^3$ multimodal interval map with nonflat critical points and without neutral periodic points. We refrain from giving all the details and refer to [22] for the precise definitions. Under additional conditions, in particular on the critical points, they establish the corresponding equalities (3) for these maps. The above results apply in this context (see also [8]).

$^5$Note that in such a context we always have $h(d(f|_W) \leq D(d(f|_W) \leq \dim_W W$. Indeed, it suffices to observe that to each conformal expanding repeller $Y$ there exists an ergodic measure $\mu$ of maximal dimension $\dim_W \mu = \dim_W Y$ (e.g. [11, theorem 1]). This implies the first inequality, the second one is immediate.
2. Dimension and entropy of a \((\chi, \epsilon)\)-repeller

In this section we will derive a relationship between the Hausdorff dimension and the topological entropy for a specific type of repellers that we call \((\chi, \epsilon)\)-repellers. First, we recall briefly the dimension and entropy and some of their properties.

2.1. Hausdorff dimension

Let \(X\) be a metric space. Given a set \(Y \subset X\) and a non-negative number \(d \in \mathbb{R}\), we denote the \(d\)-dimensional Hausdorff measure of \(Y\) by
\[
\mathcal{H}_d^d(Y) := \lim_{r \to 0} \mathcal{H}_d^d(Y),
\]
where
\[
\mathcal{H}_d^d(Y) := \inf \left\{\frac{1}{r^d} \sum_{i=1}^\infty \text{diam } U_i : Y \subset \bigcup_{i=1}^\infty U_i, \text{diam } U_i < r \right\},
\]
where \(\text{diam } U_i\) denotes the diameter of \(U_i\). Observe that \(\mathcal{H}_d^d(Y)\) is monotone non-increasing in \(d\). Furthermore, if \(d \in (a, b)\) and \(\mathcal{H}_d^d(Y) < \infty\) then \(\mathcal{H}_d^d(Y) = 0\) and \(\mathcal{H}_d^d(Y) = \infty\). The unique value \(d_0\) at which \(d \mapsto \mathcal{H}_d^d(Y)\) jumps from \(\infty\) to 0 is the Hausdorff dimension of \(Y\), that is,
\[
\dim_H Y = \inf\{d \geq 0 : \mathcal{H}_d^d(Y) = 0\} = \sup\{d \geq 0 : \mathcal{H}_d^d(Y) = \infty\}.
\]
We recall some of its properties:

(H1) The Hausdorff dimension is monotone: if \(Y_1 \subset \subset Y_2 \subset X\) then \(\dim_H Y_1 \leq \dim_H Y_2\).

(H2) The Hausdorff dimension is countably stable: \(\dim_H \bigcup_{n=1}^\infty B_n = \sup \dim_H B_n\).

2.2. Topological entropy

Let us now define topological entropy. We will follow the more general approach by Bowen [6] considering the topological entropy of a general (i.e. not necessarily compact and invariant) set.

Let \(X\) be a compact metric space. Consider a continuous map \(f : X \to X\), a set \(Y \subset X\), and a finite open cover \(\mathcal{A} = \{A_1, A_2, \ldots, A_n\}\) of \(X\). Given \(U \subset X\) we write \(U \prec \mathcal{A}\) if there is an index \(j\) so that \(U \subset A_j\), and \(U \not\prec \mathcal{A}\) otherwise. Taking \(U \subset X\) we define
\[
m_{\mathcal{A}}(U) := \begin{cases} 0 & \text{if } U \not\prec \mathcal{A}, \\
\infty & \text{if } f^k(U) \prec \mathcal{A} \quad \forall k \in \mathbb{N}, \\
\ell & \text{if } f^k(U) \prec \mathcal{A} \quad \forall k \in [0, \ldots, \ell - 1], f^{\ell}(U) \not\prec \mathcal{A}.
\end{cases}
\]
If \(\mathcal{U}\) is a countable collection of open sets, given \(d > 0\) let
\[
m(\mathcal{A}, d, \mathcal{U}) := \sum_{U \in \mathcal{U}} e^{-d m_{\mathcal{A}}(U)}.
\]
Given a set \(Y \subset X\), let
\[
m_{\mathcal{A}}(Y) := \lim_{\rho \to 0} \inf \left\{m(\mathcal{A}, d, \mathcal{U}) : Y \subset \bigcup_{U \in \mathcal{U}} U, e^{-m_{\mathcal{A}}(Y)} < \rho \text{ for every } U \in \mathcal{U}\right\}.
\]
Analogously to the Hausdorff measure, $d \mapsto m_{d,Y}(Y)$ jumps from $\infty$ to 0 at a unique critical point and we define

$$h_{\mathcal{A}}(f, Y) := \inf \{ d : m_{\mathcal{A},d}(Y) = 0 \} = \sup \{ d : m_{\mathcal{A},d}(Y) = \infty \}.$$ 

The topological entropy of $f$ on the set $Y$ is defined by

$$h(f, Y) := \sup_{\mathcal{A}} h_{\mathcal{A}}(f, Y),$$

Observe that for any finite open cover $\mathcal{A}$ of $Y$ there exists another finite open cover $\mathcal{A}'$ of $Y$ with a smaller diameter such that $h_{\mathcal{A}'}(f, Y) \geq h_{\mathcal{A}}(f, Y)$, which means that, in fact, for any $R > 0$

$$h(f, Y) = \sup \{ h_{\mathcal{A}}(f, Y) : \mathcal{A} \text{ finite open cover of } Y, \text{diam } \mathcal{A} < R \}.$$ 

When $Y = X$, we simply write $h(f) = h(f, X)$. To avoid confusion, we sometimes explicitly write $h(f, X) = h(f, Y).$

In [6, proposition 1], it is shown that in the case of a compact set $Y$ this definition is equivalent to the canonical definition of topological entropy (see, for example, [26, chapter 7]).

We recall some of its properties which are relevant in our context (see [6]).

(E1) Conjugation preserves entropy: If $f : X \to X$ and $g : Z \to Z$ are topologically conjugate, that is, there is a homeomorphism $\pi : X \to Z$ with $\pi \circ f = g \circ \pi$, then $h(f, Y) = h(g, \pi(Y))$ for every $Y \subset X$.

(E2) Entropy is invariant under iteration: $h(f, f(Y)) = h(f, Y)$.

(E3) Entropy is countably stable: $h(f, \bigcup_{i=1}^{\infty} B_i) = \sup_i h(f, B_i)$.

(E4) $h(f^m, Y) = m \cdot h(f, Y)$ for all $m \in \mathbb{N}$.

(E5) Entropy is monotone: if $Y \subset Z \subset X$ then $h(f, Y) \leq h(f, Z)$.

2.3. $(\chi, \epsilon)$-repellers

In this section let $X$ be a Riemannian manifold and $f : X \to X$ be a differentiable map. We call $f$ conformal if for each $x \in X$ we have $Df = a(x) \cdot \text{Isom}_x$, where $a(x)$ is a positive scalar and $\text{Isom}_x : T_xX \to T_{f(x)}X$ is an isometry; in this case we simply write $a(x) = |f'(x)|$. We say that a set $W \subset X$ is forward invariant if $f(W) = W$. A compact set $W \subset X$ is said to be isolated if there is an open neighbourhood $V$ of $W$ such that $f^n(x) \in V$ for all $n \geq 0$ implies $x \in W$. Given a $f$-forward invariant subset $W \subset X$ we call $f|_{W}$ expanding if there exists $n \geq 1$ such that, for all $x \in W$ we have

$$|(f^n)'(x)| > 1.$$ 

Definition 2.1. A compact $f$-forward invariant isolated expanding set $W \subset X$ of a conformal map $f$ is said to be a conformal expanding repeller.

Given numbers $\chi > 0$ and $\epsilon \in (0, \chi)$, we call a conformal expanding repeller $W \subset X$ a $(\chi, \epsilon)$-repeller if for every $x \in W$ we have

$$\limsup_{n \to \infty} \frac{1}{n} \log |(f^n)'(x)| - \chi < \epsilon.$$  

(4)

In the following, we will collect some important estimates between the Hausdorff dimension and topological entropy of $(\chi, \epsilon)$-repellers. The following estimate is in a similar spirit.
as [10, lemma 2]. The method of proof is partially inspired by [20] and [7, proof of theorem 1.2]. See also [18] for similar results. We will first prove a general result and then consider the particular case of $(\chi, \epsilon)$-repellers.

**Proposition 2.2.** Consider a Riemannian manifold $X$ and $f : X \to X$ a conformal $C^{1+\alpha}$ map. Let $W \subset X$ be a conformal expanding repellere. Let $Z \subset W$ and let $\chi > 0$ and $\epsilon \in (0, \chi)$ be numbers such that for every $x \in Z$ we have (4).

Then we have

$$\frac{h(f|_W, Z)}{\chi + \epsilon} \leq \dim_H Z \leq \frac{h(f|_W, Z)}{\chi - \epsilon}.$$ 

**Proof.** In what follows, in order to simplify the notations we avoid conceptually the unnecessary use of coordinate charts.

Given $N \in \mathbb{N}$, we define the following level sets:

$$Z_N := \left\{ x \in Z : \left| \frac{1}{n} \log |(f^n)'(x)| - \chi \right| < \epsilon \text{ for all } n \geq N \right\}.$$ 

By the hypothesis on $Z$, we have that

$$Z = \bigcup_{N \in \mathbb{N}} Z_N. \quad (5)$$

Observe that $Z_N \subset Z_N'$ for $N < N'$. Given $N \in \mathbb{N}$, for all $x \in Z_N$ and all $k \geq N$ we have

$$e^{k(\chi - \epsilon)} < |(f^k)'(x)| < e^{k(\chi + \epsilon)}. \quad (6)$$

In a sufficiently small neighbourhood $V$ of $W$ we have $|f'| \neq 0$ and hence for $\theta > 0$ there exists $R = R(\theta) > 0$ such that if $z_1, z_2 \in V$ and $d(z_1, z_2) < R$ then

$$|\log |f'(z_1)|| - \log |f'(z_2)|| < \theta. \quad (7)$$

**Step 1:** We start by showing

$$h(f|_W, Z) \leq (\chi + \epsilon) \dim_H Z. \quad (8)$$

Fix $N \in \mathbb{N}$. Fix some $\theta > 0$ and let $R = R(\theta)$ as above. We start by estimating the entropy on $Z_N$. For that we choose some finite open cover $\mathcal{A}'$ of $W$ with $\text{diam}(\mathcal{A}') \leq R$. Let $\ell = \ell(\mathcal{A}')$ denote a Lebesgue number of $\mathcal{A}'$. Let

$$r_0 = r_0(N) := \ell \min_{0 \leq k \leq N} |(f^k)'(x)|^{-1}. \quad (9)$$

We prove the following intermediate result.

**Claim 2.3.** For every $\gamma > (\chi + \epsilon + \theta) \dim_H Z_N$, we have $m_{\mathcal{A}'}(Z_N) = 0$.

**Proof.** Let $D := \gamma/(\chi + \epsilon + \theta)$. Let $c = \log(\ell/2)/(\chi + \epsilon + \theta)$. Let $\zeta > 0$. As $D > \dim_H Z_N$, there is $\rho_0 > 0$ such that for all $r \in (0, \rho_0)$ we have that
\[
\inf\left\{ \sum r_i^D : Z_N \subset \bigcup_i B(x_i, r_i), r_i < r \right\} < \zeta e^\gamma.
\]

Let \( \rho_1 := \min\{\rho_0, \rho_2\} \). Then, for every \( \rho \in (0, \rho_1) \) there is \( r \in (0, \rho) \) also satisfying
\[
r < (e^\rho)^{1+\epsilon + \theta}
\]
and a cover \( U = \{ U_i \} \) of \( Z_N \) by open balls \( U_i = B(x_i, r_i), r_i < r \), so that
\[
\sum r_i^D < \zeta e^\gamma.
\]

For every \( U_i \in U \), for any \( z_1, z_2 \in U \) for all \( j \in \{0, \ldots, n_{f, \mathcal{O}}(U_i) - 1\} \) we have
\[
d(f^j(z_1), f^j(z_2)) < \text{diam } \mathcal{O} \leq R.
\]

From (7) it follows that for every \( k = 1, \ldots, n_{f, \mathcal{O}}(U_i) \) we have
\[
\left| \log |f^j(z_1)| - \log |f^j(z_2)| \right| \leq \sum_{j=0}^{k-1} \left| \log |f^j(f^j(z_1))| - \log |f^j(f^j(z_2))| \right| \leq k \theta
\]
and hence
\[
e^{-k \theta} \leq \frac{|f^j(z_1)|}{|f^j(z_2)|} \leq e^{k \theta}. \tag{10}
\]

Given \( i \), for \( x \in U_i \in U \) let \( F(x) = f^{n_{f, \mathcal{O}}(U_i)}(x) \). By the definition of \( n_{f, \mathcal{O}}(U_i) \) and of the Lebesgue number \( \ell \), for every \( U_i \in U \) it follows that \( \ell \leq \text{diam } f^{n_{f, \mathcal{O}}(U_i)}(U_i) = \text{diam } F(U_i) \). Consider \( x, y \in U_i \) such that \( \text{diam } F(U_i) = d(F(x), F(y)) \). Consider the shortest path \( \gamma : [0, 1] \to X \) linking \( x \) to \( y \), which is completely contained in \( U_i \) since \( U_i \) is a cover by balls. Thus
\[
\ell \leq d(F(x), F(y)) \leq \int_0^1 |(F \circ \gamma)'(t)| \, dt = \int_0^1 |F'(\gamma(t))| |\gamma'(t)| \, dt.
\]

Observe that \( r_i < r_0 \) implies that \( n_{f, \mathcal{O}}(U_i) > N \). Considering \( z \in U_i \cap Z_N \), with \( k = n_{f, \mathcal{O}}(U_i) > N \) in (10) and (6) we conclude
\[
\ell \leq \int_0^1 \frac{|F'(\gamma(t))|}{|F'(z)|} |\gamma'(t)| \, dt
\]
by (10) \( \leq e^{n_{f, \mathcal{O}}(U_i)\theta} \int_0^1 |(f^{n_{f, \mathcal{O}}(U_i)})'(z)| |\gamma'(t)| \, dt
\]
by (6) \( \leq e^{n_{f, \mathcal{O}}(U_i)\theta} e^{n_{f, \mathcal{O}}(U_i)\theta} \text{diam } U_i \).

Recalling the definition of \( c \) we obtain
\[
e^{-n_{f, \mathcal{O}}(U_i)} \leq (\ell^{-1} \text{diam } U_i)^{1/(\chi + \epsilon + \theta)} = \frac{1}{2} (\ell^{-1} \text{diam } U_i)^{1/(\chi + \epsilon + \theta)}. \tag{11}
\]

Since \( \text{diam } U_i < 2r < 2(e^\rho)^{1+\epsilon + \theta} \) we have \( e^{-n_{f, \mathcal{O}}(U_i)} < \rho \).
Then, we have
\[ m(\mathcal{A}', \gamma, U) = \sum_{U \in \mathcal{U}} e^{-\gamma \eta_{\mathcal{A}'(U)}} \]
by (11) \( \leq e^{-\gamma} \sum_{U \in \mathcal{U}} \left( \frac{1}{2} \text{diam } U \right)^{\gamma} \rho (\chi + \epsilon + \theta) = e^{-\gamma} \sum_{U \in \mathcal{U}} r_i^D \]
by (9) \( < e^{-\gamma} e^\gamma = \zeta. \)

In summary, for arbitrary \( \zeta > 0 \), there exists \( \rho_1 > 0 \) such that for any \( \rho \in (0, \rho_1) \) we can cover \( Z_N \) by a family of balls \( U_i \) satisfying \( e^{-\gamma \eta_{\mathcal{A}'(U)}} < \rho \) and \( \sum_{U \in \mathcal{U}} e^{-\gamma \eta_{\mathcal{A}'(U)}} < \zeta \). Thus \( m_{\mathcal{A}', \gamma}(Z_N) = 0 \) as claimed. 

By claim 2.3, for every \( \gamma > (\chi + \epsilon + \theta) \text{dim}_H Z_N \), we have \( m_{\mathcal{A}, \gamma}(Z_N) = 0 \), which implies \( h_{\mathcal{A}}(f, Z_N) \leq \gamma \). Since \( \gamma > (\chi + \epsilon + \theta) \text{dim}_H Z_N \) is arbitrary, therefore
\[ h_{\mathcal{A}}(f, Z_N) \leq (\chi + \epsilon + \theta) \text{dim}_H Z_N. \]
Thus, as \( \mathcal{A}' \) was arbitrary (but sufficiently small)
\[ h(f|_W, Z_N) \leq (\chi + \epsilon + \theta) \text{dim}_H Z_N. \]

Since \( \theta \) was arbitrary, we obtain
\[ h(f|_W, Z_N) \leq (\chi + \epsilon) \text{dim}_H Z_N. \]

Now recall that \( N \geq 1 \) was arbitrary. With (5) and countable stabilities (H2) of the Hausdorff dimension and (E3) of entropy we conclude (8) from
\[ \text{dim}_H Z = \sup_N \text{dim}_H Z_N \geq \sup_N \frac{h(f|_W, Z_N)}{\chi + \epsilon} = \frac{1}{\chi + \epsilon} \sup_N h(f|_W, Z_N) = \frac{1}{\chi + \epsilon} h(f|_W, Z). \]

This concludes step 1.

**Step 2:** We now show
\[ \text{dim}_H Z \leq \frac{h(f|_W, Z)}{\chi - \epsilon}. \tag{12} \]

Fix some \( N \in \mathbb{N} \). Fix some \( \theta \in (0, \chi - \epsilon) \) and let \( R = R(\theta) \) as above.

We start by estimating the dimension of \( Z_N \). Fix some \( \tau > 0 \) and denote
\[ D := (h(f|_W, Z_N) + \tau)(\chi - \epsilon - \theta). \]
Observe that
\[ (\chi - \epsilon - \theta)D = h(f|_W, Z_N) + \tau > h(f|_W, Z_N) = \sup \text{dim}_H (Z_N). \]

Hence, for any finite open cover \( \mathcal{A}' \) of \( W \) we have \( m_{\mathcal{A}, \chi - \epsilon - \theta}(Z_N) = 0 \). Choose some cover \( \mathcal{A}' \) with \( \text{dim}_H \mathcal{A}' \leq R \).

Given some \( U \in \mathcal{A}' \) with \( n = n_{f, \mathcal{A}'}(U) < \infty \), fix some point \( x \in U \cap Z_N \) and consider the sequence \( x_k = f^k(x), k = 0, \ldots, n-1 \). So, for each \( k \) there is some \( A_k \in \mathcal{A}' \) with \( x_k \in f^k(U) \subset A_k \). Denote by \( f_{x_n-1}^{x_{n-1}} \) the inverse branch \( g \) of \( f^k \) so that \( (g \circ f^k)(x_{n-1-k}) = x_{n-1-k} \). We observe the following preliminary fact.
Claim 2.4. For every $k = 0, \ldots, n - 1$ for every $x \in U$ we have

$$\text{diam} f^{-k}_{x_{k^{-1}}} (f^{n-1}(U)) \leq \left[ f^{j} (x_{n-j-1}) \right]^{-1} e^{\epsilon \theta} \cdot R.$$ 

Proof. The proof is by induction. For $k = 0$ we have $f^{n-1}(U) \subset A_{n-1} \in \mathcal{A}$ and hence $\text{diam} f^{n-1}(U) \leq R$. For $k \in \{1, \ldots, n - 1\}$, suppose the claim holds for $k - 1 = j$. Let $V_{j+1} := f^{-j-1}_{x_{n-1-j}} (f^{n-1}(U)) = f^{-j-1}_{x_{n-1-j}} (V_j)$ and observe that, in particular, $V_{j+1} \subset A_{j+1} \in \mathcal{A}$. Since for every $y, z \in A_{j+1}$, using (7), we have that $|f^{j}(y)| |f^{j}(z)| \leq e^\theta$, we can conclude

$$\text{diam} V_{j+1} \leq \sup_{y \in V_{j+1}} |f^{j}(y)|^{-1} \text{diam} V_j \leq |f^{j}(x)|^{-1} e^\theta \text{diam} V_j.$$ 

Invoking the induction hypothesis for $k = j$, we obtain

$$\text{diam} V_{j+1} \leq |f^{j}(x)|^{-1} e^\theta \cdot \left[ f^{j} \left( x_{n-j-1-j} \right) \right]^{-1} e^\theta \cdot R = \left[ f^{j+1} \left( x_{n-1-j-j+1} \right) \right]^{-1} e^{j+1} \cdot R,$$

proving the assertion for $j + 1$. This proves the claim. $\blacksquare$

Claim 2.5. $\mathcal{H}^0(Z_N) = 0$.

Proof. Let $\eta > 0$. Observe that $m_{\mathcal{A}, (\chi - \epsilon - \theta) D} (Z_N) = 0$ implies that there is $\rho > 0$ such that for every $\rho \in (0, \rho_0)$ we have that

$$\inf \left\{ m(\mathcal{A}, (\chi - \epsilon - \theta) D, U) : Z_N \subset \bigcup_{U \in \mathcal{U}} U, e^{-n_f_{\mathcal{A}}(U)} < \rho \right\} < \eta e^{-(\chi - \epsilon - \theta) D R^{-D}}.$$ 

Consider $r_1 < \min \{\rho_0, e^{-(N + 1)}\}$. Then, for every $r \in (0, r_1)$ there is $\rho \in (0, r)$ also satisfying

$$e^{\chi - \epsilon - \theta} R \cdot \rho^{\chi - \epsilon - \theta} < r. \quad (13)$$

Hence, there exists a cover $\mathcal{U} = \{U_i\}$ of $Z_N$ satisfying $e^{-n_f_{\mathcal{A}}(U_i)} < \rho$ and

$$m(\mathcal{A}, (\chi - \epsilon - \theta) D, \mathcal{U}) \leq \eta e^{-(\chi - \epsilon - \theta) D R^{-D}}. \quad (14)$$

Note that $\rho < e^{-(N + 1)}$ implies $n_{f_{\mathcal{A}}}(U_i) > N + 1$ and $f^{j}(U_i)$ lies inside an element of $\mathcal{A}$ for every $k = 0, \ldots, n_f_{\mathcal{A}}(U_i) - 1$. Consequently, with claim 2.4 for $k = n_f_{\mathcal{A}}(U_i) - 1$ and $x \in Z_N \cap U_i$ we obtain

$$\text{diam} U_i \leq \left[ f^{n_f_{\mathcal{A}}(U_i)-1} (x) \right]^{-1} e^{n_f_{\mathcal{A}}(U_i)-1} e^\theta \cdot R \leq e^{-(n_f_{\mathcal{A}}(U_i)-1)(\chi - \epsilon - \theta)} \cdot R.$$ 

Thus, since $e^{-n_f_{\mathcal{A}}(U_i)} < \rho$, we have that

$$\text{diam} U_i \leq e^{\chi - \epsilon - \theta} R \cdot e^{-n_f_{\mathcal{A}}(U_i)(\chi - \epsilon - \theta)} < e^{\chi - \epsilon - \theta} R \cdot \rho^{\chi - \epsilon - \theta}$$

by (13) $< r$. 

By (14) and the above inequality,
\[
\sum_{U_i \in \mathcal{U}} (\text{diam } U_i)^D \leq \sum_{i} \left( e^{\chi - \epsilon - \theta} R \cdot e^{-\eta r_i \cdot (U_i \chi_i - \epsilon - \theta)} \right)^D \\
= e^{\chi - \epsilon - \theta} D^D \cdot m(A_i, D(\chi - \epsilon - \theta), \mathcal{U}) < \eta.
\]

In summary, for arbitrary \( \eta > 0 \), there exists \( r_1 > 0 \) such that for every \( r \in (0, r_1) \) we can cover \( Z_N \) by \( \mathcal{U} \) such that \( \text{diam } U_i < r \) for every \( U_i \in \mathcal{U} \) and \( \sum_{U_i \in \mathcal{U}} (\text{diam } U_i)^D < \eta \). Thus, \( H^D(Z_N) = 0 \), proving the claim.

Claim 2.5 now implies immediately

\[
\dim_H Z_N \leq \frac{h(f|_W \cdot Z_N) + \tau}{\chi - \epsilon - \theta}.
\]

As \( \tau > 0 \) and \( \theta \in (0, \chi - \epsilon) \) were arbitrary, we conclude

\[
\dim_H Z_N \leq \frac{h(f|_W \cdot Z_N)}{\chi - \epsilon}.
\]

Finally, recall that \( N \) was arbitrary, by (5), (E3) and (H2), we obtain

\[
\dim_H Z = \sup_N \dim_H Z_N \leq \sup_N \frac{h(f|_W \cdot Z_N)}{\chi - \epsilon} = \frac{h(f|_W \cdot Z)}{\chi - \epsilon}.
\]

This shows (12) and finishes the proof of the proposition.

The following is now an immediate consequence of proposition 2.2.

**Corollary 2.6.** Consider a Riemannian manifold \( X \) and \( f : X \to X \) a conformal \( C^1 + \alpha \) map. Let \( W \subset X \) be a \((\chi, \epsilon)\)-repeller.

Then for every \( Z \subset W \) we have

\[
\frac{h(f|_W \cdot Z)}{\chi + \epsilon} \leq \dim_H Z \leq \frac{h(f|_W \cdot Z)}{\chi - \epsilon}.
\]

Finally, we provide some further consequences which we will need in the sequel. Given \( N \in \mathbb{N} \) let \( R \subset W \) be a compact set satisfying

\[
f^N(R) = R \quad \text{and} \quad W = \bigcup_{i=0}^{N-1} f^i(R). \tag{15}
\]

**Lemma 2.7.** \( h(f|_W) = \frac{1}{N} h(f^N|_W) \).

**Proof.** By (E3), (E2), (E4) and the \( f^N \)-invariance of \( R \) we have

\[
h(f|_W) = \max_i h(f|_W \cdot f_i(R)) = h(f|_W \cdot R) = \frac{1}{N} h(f^N|_W \cdot R) = \frac{1}{N} h(f^N|_W).
\]

This proves the lemma.

**Lemma 2.8.** Consider a Riemannian manifold \( X \) and \( f : X \to X \) a conformal \( C^1 + \alpha \) map. Suppose that \( W \subset X \) is a \((\chi, \epsilon)\)-repeller of positive entropy and \( R \subset W \) a compact set satisfying \( f^N(R) = R \) and \( W = \bigcup_{i=0}^{N-1} f^i(R) \) for some \( N \geq 1 \).
Then for every $Y \subset R$ we have
\[ \dim H Y \leq \frac{h(f|_W, Y) (\chi - \epsilon)}{h(f|_W)} (\chi + \epsilon) \dim H W = \frac{h(f^N|_R, Y) (\chi - \epsilon)}{h(f^N|_R)} (\chi + \epsilon) \dim H W. \]

**Proof.** Applying corollary 2.6 we have
\[ \frac{1}{h(f|_W)} (\chi - \epsilon) \dim H W \leq 1. \]

Given $Y \subset R \subset W$, we also have
\[ \frac{h(f|_W, Y)}{\chi + \epsilon} \leq \dim H Y \]

Multiplying both inequalities, we obtain the first inequality. The equality is a consequence of lemma 2.7, property (E4) and the $f^N$-invariance of $R$. \qed

3. Expanding repellers for non-uniformly expanding maps

In order to find an approximation of ergodic quantifiers of the, in general non-expanding, maps, we follow an idea by Katok to construct suitable repellers. For a proof of the following theorem see [21, chapter 11.6] and [12, theorems 1 and 3].

**Theorem 3.1.** Consider a Riemannian manifold $X$ and $f : X \to X$ a conformal $C^{1+\alpha}$ map. Let $\mu$ be an $f$-invariant ergodic measure with positive entropy $h_\mu(f)$ and positive Lyapunov exponent $\int \chi \, d\mu = \log \|f\|$. Then for all $\epsilon > 0$ there is a compact set $W, \subset X$ such that $f|_W$ is a conformal expanding repeller satisfying:

(a) $h_\mu(f) + \epsilon \geq h(f|_W) \geq h_\mu(f) - \epsilon$.

(b) For every $f$-invariant ergodic measure $\nu$ supported in $W$ we have
\[ |\chi(\nu) - \chi(\mu)| \leq \epsilon. \]

In particular, $W$ is a $(\chi(\mu), \epsilon)$-repeller.

Moreover, there is a compact set $R, \subset W$ and some $N = N(\epsilon) \in \mathbb{N}$ such that $f^N|_R = R$, $f^N|_R$ is expanding and topologically conjugate to a topologically mixing subshift of finite type, and we have
\[ W = \bigcup_{i=0}^{N-1} f^i(R). \]

These repellers $W$ have good dimension properties as we shall see below. In particular, we can apply corollary 2.6 to them.

For the following result recall the definition of the dynamical dimension in (1).
Lemma 3.2. Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a rational function of degree $\geq 2$ on the Riemann sphere and let $J = J(f)$ be its Julia set.

Then there exist a sequence of probability measures $(\mu_n)_n$ and a sequence of positive numbers $(\epsilon_n)_n$ such that there are $(\chi(\mu_n), \epsilon_n)$-repellers $W_n = W(f, \mu_n) \subset J$ satisfying

$$\lim_{n \to \infty} \dim_{\text{H}} W_n = \text{dim}_{D}(f|_J).$$

Proof. First note that for a $f$-invariant ergodic probability measure $\mu$ of a rational function with positive Lyapunov exponent $\chi(\mu)$ we have

$$\dim_{\text{H}} \mu = \frac{h_{\mu}(f)}{\chi(\mu)}$$ (16)

([19], see also [21, chapters 8–10]).

Given $n \in \mathbb{N}$, by definition of the dynamical dimension, there is an $f$-ergodic probability measure $\mu_n$ with positive entropy (and hence positive Lyapunov exponent) such that

$$\dim_{\text{H}} \mu_n \geq \text{dim}_{D}(f|_J) - \frac{1}{n}.$$ (17)

Choose $\epsilon_n > 0$ satisfying $\epsilon_n < \chi(\mu_n)/n$. Let $W_n$ be a $(\chi(\mu_n), \epsilon_n)$-repeller provided by theorem 3.1 applied to $\mu_n$ and recall that there is $N = N(\epsilon_n)$ and $R_n \subset W_n$ such that $f^N|_{R_n}$ is expanding and conjugate to a mixing subshift of finite type. Observe that $\dim_{\text{H}} W_n = \dim_{\text{H}} R_n$. Also observe that $W_n \subset J$. Applying Bowen’s formula (see [11]) for $f^N|_{R_n}$, with $s_n = \dim_{\text{H}} R_n$ we have

$$0 = \sup_{\nu} h_{\nu}(f^N) - s_n\chi(\nu),$$

where the supremum is taken over all $f^N$-invariant measures $\nu$ supported in $R_n$. Recall that for every invariant measure $\nu$ for $f^N : R_n \to R_n$ we get an invariant measure $\tilde{\nu}$ for $f : W_n \to W_n$ by defining $\tilde{\nu} := \frac{1}{N}(\nu + f_1\nu + \ldots + f_{N-1}\nu)$ and observe that $h_{\tilde{\nu}}(f^N) = Nh_{\nu}(f)$. Further, $h(f^N|_{R_n}) = Nh(f|_{W_n})$ (lemma 2.7). By the variational principle for topological entropy (see, e.g. [21, chapter 9]), we can take $\nu$ such that $h_{\nu}(f^N) \geq Nh(f|_{W_n}) - N\epsilon_n$, which implies

$$0 \geq h(f|_{W_n}) - \epsilon_n - s_n\chi(\nu).$$

From theorem 3.1 we obtain

$$0 \geq h_{\mu_n}(f) - 2\epsilon_n - s_n(\chi(\mu_n) + \epsilon_n),$$

which implies

$$s_n \geq h_{\mu_n}(f) - 2\epsilon_n - \frac{\chi(\mu_n)}{\chi(\mu_n) + \epsilon_n}.$$ (18)

Hence, by (16), we conclude

$$s_n \geq \dim_{\text{H}} \mu_n \left(\frac{\chi(\mu_n)}{\chi(\mu_n) + \epsilon_n}\right) - \frac{2\epsilon_n}{\chi(\mu_n) + \epsilon_n}.$$
As we required $0 < e_n < \chi(n)/n$, inequalities (17) and (18) show that

$$\left( \text{DD}(f|_j) - \frac{1}{n} \right) \frac{1}{1 + 1/n} - \frac{2}{n + 1} \leq s_n = \dim_H W_n.$$  

Finally, it follows definition of hyperbolic dimension and (2) that

$$\dim_H W_n \leq \text{hD}(f|_j) = \text{DD}(f|_j).$$

Taking the limit when $n \to \infty$, we obtain

$$\lim_{n \to \infty} \dim_H W_n = \text{DD}(f|_j).$$

This proves the lemma. □

4. General properties of exceptional sets

In this section we will derive some general properties of exceptional sets. We first show that being exceptional is preserved by conjugation.

**Lemma 4.1.** If $f : X \to X$ and $g : Y \to Y$ are topologically conjugate by a homeomorphism $\pi : X \to Y$ with $g \circ \pi = \pi \circ f$, then for every $A \subset Y$ we have

$$\pi(E^+_f|_X(\pi^{-1}(A))) = E^+_g|_Y(A).$$

**Proof.** Given $y \in \pi(E^+_f|_X(\pi^{-1}(A)))$, suppose that $y \not\in E^+_g|_Y(A)$. Then there is a subsequence $(n_k)$ and $y_0 \in A$ such that $g^{n_k}(y)$ converge to $y_0$. By conjugation $f^{n_k}(\pi^{-1}(y))$ converges to $\pi^{-1}(y_0) \in \pi^{-1}(A)$, which is a contradiction. Thus, $\pi(E^+_f|_X(\pi^{-1}(A))) \subset E^+_g|_Y(A)$.

The other inclusion is analogous, by conjugation. □

We require the following simple fact, which we state without proof.

**Lemma 4.2.** Let $A \subset X$ be a compact set such that $f(A) = A$. If $A \subset W$ then $E^+_f|_W(A \cap W) \subset E^+_f|_W(A)$.

In order to see how exceptional sets behave with respect to iterates, for given $A \subset X$ and $N \in \mathbb{N}$ let us denote

$$A_N := \bigcup_{j=0}^{N-1} f^{-j}(A).$$  

(19)

**Lemma 4.3.** Let $W \subset X$ be a compact set such that $f(W) = W$.

If $A \subset W$ then $E^+_f|_W(A \cap W) \subset E^+_f|_W(A \cap W)$.

**Proof.** Let $x \in E^+_f|_W(A)$. Suppose that there is $y \in \bigcup_{j=0}^\infty f^j(x) \cap A_N \cap W$. Then, there are $j_0 \in \{0, \ldots, N - 1\}$ such that $y \in f^{-j_0}(A)$ and a sequence $(n_k)_{k=0}^\infty$ such that $\lim_{k \to \infty} f^{N_k}(x) = y$. By the continuity of $f$, we have that $\lim_{k \to \infty} f^{N_k + j_0}(x) = f^{j_0}(y) \in A$ and hence $\bigcup_{j=0}^\infty f^j(x) \cap A \neq \emptyset$, which is a contradiction. This proves that $E^+_f|_W(A) \subset E^+_f|_W(A \cap W)$.
Consider now \( x \in E^+_f |W(A_N \cap W) \). Suppose that there exists \( y \in \overline{\mathcal{O}_f(x)} \cap A \). Thus, there is a subsequence \( (n_k)_{k=0}^\infty \) such that \( \lim_{k \to \infty} f^{n_k}(x) = y \in A \). We can write \( n_k = N_k + r_k \), where \( 0 \leq r_k \leq N - 1 \). Then exists \( r \in \{0, \ldots, N-1\} \) such that \( (f^{N_k + r})(x) \) is a subsequence such that \( \lim_{k \to \infty} f^{N_k + r}(x) = y \in A \). By the compactness of \( W \) and because \( f^{N_k}(x) \in W \) for all \( k \), there exists a convergent subsequence \( \lim_{k \to \infty} f^{N_k}(x) = v \in W \). By the continuity of \( f \) we have that

\[
f^v = \lim_{k \to \infty} f^{N_k}(x) = \lim_{k \to \infty} f^{N_k + r}(x) = y.
\]

Thus, \( \lim_{k \to \infty} f^{N_k}(x) = v \in f^{-r}(y) \subset f^{-r}(A) \), which is a contradiction. This proves the other inclusion. \( \square \)

For the remaining results in this section, let \( W \subset X \) be a compact set such that \( f(W) = W \), let \( N \in \mathbb{N} \) and let \( R \subset W \) be a compact set satisfying

\[
f^N(R) = R \quad \text{and} \quad W = \bigcup_{i=0}^{N-1} f^i(R),
\]

and let \( A \subset W \) and let \( A_N \) be defined as in (19).

**Lemma 4.4.** For all \( i \in \{0, \ldots, N - 1\} \), we have

\[
f^i \left( E^+_f |R(A_N \cap R) \right) \subset E^+_f |f^i(R)(A_N \cap f^i(R)).
\]

**Proof.** Let \( y \in f^i \left( E^+_f |R(A_N \cap R) \right) \). Then there is \( x \in E^+_f |R(A_N \cap R) \) such that \( f^i(x) = y \). Suppose, by contradiction, that there is \( z \in \overline{\mathcal{O}_f^i(y)} \cap A_N \). Then, there are \( j \in \{0, \ldots, N - 1\} \) and a sequence \( (n_k)_{k=0}^\infty \) such that \( \lim_{k \to \infty} f^{N_k}(y) = z \in f^{-j}(A) \). By the compactness and \( f^N \)-invariance of \( R \), we have that there are \( \tilde{x} \in R \) and a subsequence \( (n_j)_{j=0}^\infty \) of \( (n_k)_{k=0}^\infty \) such that \( \lim_{j \to \infty} f^{N_k}(y) = \tilde{x} \). Note that, by the continuity of \( f^i \), follows that

\[
z = \lim_{l \to \infty} f^{N_l}(y) = f^i \left( \lim_{l \to \infty} f^{N_l}(x) \right) = f^i(\tilde{x}).
\]

In this case, \( \tilde{x} \in f^{-i}(y) \) and, since \( z \in f^{-j}(A) \), we have that \( \tilde{x} \in f^{-i+j}(A) \). If \( i + j \in \{0, \ldots, N - 1\} \), then \( \tilde{x} \in A_N \cap R \) which is a contradiction. If \( i + j \geq N \), then there are \( s, \, t \in \mathbb{N} \) such that \( i \in \{0, \ldots, N - 1\} \) and \( i + j = sN + t \). Thus, by the continuity of \( f^N \) and \( f^N \)-invariance of \( R \), it follows that

\[
\lim_{l \to \infty} f^{N_l(t+n_i)}(x) = f^{N_t}(\tilde{x}) \in f^{-i}(A) \cap R.
\]

This is again a contradiction. \( \square \)

**Lemma 4.5.** \( E^+_f |W(A_N \cap W) = \bigcup_{i=0}^{N-1} E^+_f |f^i(R)(A_N \cap f^i(R)) \).

**Proof.** Observe that \( f^N(f^i(x)) = f^i(f^N(x)) = f^i(x) \). If \( x \in E^+_f |W(A_N \cap W) \), then \( x \in f^i(R) \) for some \( i \in \{0, \ldots, N - 1\} \) and by the \( f^N \)-invariance of \( f^i(R) \) we have

\[
\overline{\mathcal{O}_f^i(f^i(R))} \cap (A_N \cap f^i(R)) = \overline{\mathcal{O}_f^i(f^i(R))} \cap A_N = \overline{\mathcal{O}_f^i(x)} \cap A_N = \emptyset.
\]

Hence,
\[ E^+_f|_W (A_N \cap W) \subset \bigcup_{i=0}^{N-1} E^+_f|_f^i(R)(A_N \cap f^i(R)). \]

On the other hand, let \( x \) be a point in the set on the right-hand side, that is, let \( x \in E^+_f|_f^i(R)(A_N \cap f^i(R)) \) for some \( i \in \{0, \ldots, N-1\} \) and, in particular, \( x \in f^i(R) \). Again, by the \( f^N \)-invariance of \( f^i(R) \), we have
\[
\overline{\sigma^i(x)} \cap A_N = \overline{\sigma^i(f^i(R))(x)} \cap A_N = \overline{\sigma^i(f^i(R))} \cap (A_N \cap f^i(R)) = \emptyset.
\]

This finishes the proof.

Finally, in this section we give a relation for the entropy of the sets \( A_N \), which we will need immediately after.

**Lemma 4.6.** If \( h(f|_W, A) < h(f|_W) \) then \( h(f^N|_R, A_N \cap R) < h(f^N|_R) \).

**Proof.** Starting from our hypothesis,
\[
h(f|_W) > h(f|_W, A)
\]
by (E2) and (15) \( h(f|_W, A_N \cap R) = h\left(f|_W, A_N \cap \bigcup_{i=0}^{N-1} f^i(R)\right)\)
by (E5) \( \geq h(f|_W, A_N \cap R) \)
by (E4) and (15) \( \frac{1}{N}h(f^N|_W, A_N \cap R) = \frac{1}{N}h(f^N|_R, A_N \cap R) \)

Hence, applying lemma 2.7 we obtain the claimed property.

□

### 5. Proof of the main results

We first establish a preparatory result for the entropy of a continuous transformation that can be decomposed into finite systems, each being conjugate to a subshift of finite type.

**Proposition 5.1.** Let \((W, d)\) be a compact metric space and \( f : W \to W \) a continuous transformation. Let \( R \subset W \) be a compact set satisfying \( f^N(R) = R \) and \( W = \bigcup_{i=0}^{N-1} f^i(R) \) for some \( N \geq 1 \) and suppose that \( f^N : R \to R \) is conjugate to a subshift of finite type.

Then for every compact set \( A \subset W \) satisfying \( h(f|_W, A) < h(f|_W) \) we have \( h(f|_W, E^+_f|W(A)) = h(f|_W) \).

**Proof.** By the hypothesis, there is a subshift of finite type \( \sigma : \Sigma_M^+ \to \Sigma_M^+ \) and a homeomorphism \( \pi : \Sigma_M^+ \to R \) satisfying \( \pi \circ f^N = \sigma \circ \pi \).

By the hypothesis and lemma 4.6 we have \( h(f^N|_R, A_N \cap R) < h(f^N|_R) \). By the conjugation property (E1) of entropy we have \( h(\sigma, \pi^{-1}(A_N \cap R)) < h(\sigma) \). By theorem 1.4, we have that
\[
h\left(\sigma, E^+_f|_{\Sigma_M^+}(\pi^{-1}(A_N \cap R))\right) = h(\sigma).
\]

From lemma 4.1 and property (E1), we conclude
\[ h\left( f^N | R, E_{jW}^+ | R(AN \cap R) \right) = h(f^N | R). \]

By the \( f^N \)-invariance of \( R \), properties (E2) and (E5) of entropy, lemmas 4.4, 4.5 and 2.7

\[
\begin{align*}
    h(f^N | R) & = h\left( f^N | W, E_{jW}^+ | W(AN \cap R) \right) \\
    & = h\left( f^N | W, f^i(E_{jW}^+ | W(AN \cap R)) \right) \\
    & \leq h\left( f^N | W, E_{jW}^+ | W(AN \cap f^i(R)) \right) \\
    & \leq h(f^N | R).
\end{align*}
\]

Thus, by lemma 4.5, the \( f^N \)-invariance of \( R \), (E3), and lemma 2.7, it follows

\[
\begin{align*}
    h\left( f^N | W, E_{jW}^+ | W(AN \cap W) \right) & = \max_{0 \leq i < N-1} h\left( f^N | f^i(R), E_{jW}^+ | f^i(R)(AN \cap f^i(R)) \right) \\
    & = h(f^N | R) \\
    & = Nh(f^N | W).
\end{align*}
\]

Then, by lemma 4.3 and property (E4), it follows that

\[
\begin{align*}
    h(f^N | W, E_{jW}^+ | W(A \cap W)) & = h\left( f^N | W, E_{jW}^+ | W(AN \cap W) \right) \\
    & = \frac{1}{N} h\left( f^N | W, E_{jW}^+ | W(AN \cap W) \right) \\
    & = h(f^N | W),
\end{align*}
\]

and this finishes the proof of the proposition. \( \square \)

Now we can give the proofs of theorems 1.1 and 1.2.

**Proof of theorem 1.1.** By hypothesis, \( h(f^N_j) > 0 \). By the variational principle and Ruelle’s inequality, for every \( \epsilon > 0 \) there is an ergodic measure \( \mu \) satisfying \( \chi(\mu) > 0 \) and \( h_\mu(f) \geq h(f^N_j) - \epsilon \). By theorem 3.1, there is a compact set \( W \subset J \) such that \( h(f^N_j | W) \geq h_\mu(f) - \epsilon \). If \( \epsilon \) is sufficiently small, this and our hypothesis \( h(f^N_j | A) < h(f^N_j) \) together imply \( h(f^N_j, A \cap W) < h(f^N_j | W) \). Then, by proposition 5.1, the above inequalities, and observing that \( E_{jW}^+(A \cap W) \subset E_{jW}^+(A) \), we have

\[
\begin{align*}
    h(f^N_j) & \leq h_\mu(f) + \epsilon \leq h(f^N_j | W) + 2\epsilon \\
    & = h(f^N_j | W, E_{jW}^+(A \cap W)) + 2\epsilon \\
    & \leq h(f^N_j, E_{jW}^+(A)) + 2\epsilon.
\end{align*}
\]

Since \( \epsilon \) was arbitrary, this implies the claim. \( \square \)

**Proof of theorem 1.2.** Consider the sequences \( (\mu_n)_n \), \( (\epsilon_n)_n \) and \( (W_n)_n \) from lemma 3.2. In particular, \( \epsilon_n < \chi(\mu_n) / n \) and thus

\[
\lim_{n \to \infty} \frac{\chi(\mu_n) - \epsilon_n}{\chi(\mu_n) + \epsilon_n} = 1.
\]
By hypothesis we have
\[ \dim_H A < \dim_H(f_{\mid_f^j}). \]

Hence, for \( n \) sufficiently large we have (the first inequality is simple)
\[ \dim_H(A \cap W_n) \leq \dim_H A < \frac{\lambda(\mu_n) - \epsilon_n}{\lambda(\mu_n) + \epsilon_n} \dim_H W_n \leq \dim_H(f_{\mid_f^j}). \]

Applying corollary 2.6, the above inequality, and again corollary 2.6, we obtain
\[ h(f_{\mid_f^j}|_W, A \cap W_n) \leq (\lambda(\mu_n) + \epsilon_n) \dim_H (A \cap W_n) \]
\[ < (\lambda(\mu_n) - \epsilon_n) \dim_H W_n \]
\[ \leq h(f_{\mid_f^j}|_W). \]

Hence, we can apply proposition 5.1 and obtain
\[ h(f_{\mid_f^j}|_W, E_{f_{\mid_f^j}|_W}^+(A \cap W_n)) = h(f_{\mid_f^j}|_W). \]

Together with lemma 2.8 applied to \( W = W_n \) and \( Y = E_{f_{\mid_f^j}|_W}^+(A \cap W_n) \) this implies
\[ \dim_H E_{f_{\mid_f^j}|_W}^+(A \cap W_n) \]
\[ \geq \frac{h(f_{\mid_f^j}|_W, E_{f_{\mid_f^j}|_W}^+(A \cap W_n))}{h(f_{\mid_f^j}|_W)} \frac{\lambda(\mu_n) - \epsilon_n}{\lambda(\mu_n) + \epsilon_n} \dim_H W_n \]
\[ = \frac{(\lambda(\mu_n) - \epsilon_n)}{(\lambda(\mu_n) + \epsilon_n)} \dim_H W_n. \]

Lemma 3.2 now proves that
\[ \lim_{n \to \infty} \dim_H E_{f_{\mid_f^j}|_W}^+(A \cap W_n) \geq \dim_H(f_{\mid_f^j}). \]

Observe that, by lemma 4.2 it follows that
\[ E_{f_{\mid_f^j}|_W}^+(A \cap W_n) \subset E_{f_{\mid_f^j}}^+(A). \]

Now property (H1) of the Hausdorff dimension implies
\[ \dim_H E_{f_{\mid_f^j}}^+(A) \geq \dim_H(f_{\mid_f^j}) \]
and proves the theorem. \( \square \)

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