ON SOME BLOCK CIPHERS AND IMPRIMITIVE GROUPS

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Abstract. The group generated by the round functions of a block ciphers is a widely
investigated problem. We identify a large class of block ciphers for which such group
is easily guaranteed to be primitive. Our class includes the AES and the SERPENT.

1. Introduction

Most block ciphers are iterated block ciphers, i.e. they are obtained by the composi-
tion of several “rounds” (or “round functions”). A round is a key-dependent permutation
of the message/cipher space. To achieve efficiency, all rounds share a similar structure.

For a given cipher, it is an interesting problem to determine the permutation group
generated by its round functions (with the key varying in the key space), since this group
might reveal weaknesses of the cipher. However, these results usually require an ad-hoc
proof (with a notable recent exception [14]).

In this paper we consider a class of block ciphers, large enough to contain some
well-known ciphers (like the AES and the SERPENT), which is such that the primitiv-
ity of the related group can be easily established by only checking some properties
of its S-Boxes. Our results may be useful to cipher designers wanting to avert group
imprimitivity, since in our context they would do it easily.

2. Preliminaries

2.1. Group theory and finite field theory. Let $G$ be a finite group acting tran-
sitively on a set $V$ and $H \leq G$ a subgroup. We write the action of an element
g \in G on an element $\alpha \in V$ as $g \alpha$. Also, $\alpha G = \{\alpha g : g \in G\}$ is the orbit of $\alpha$
and $G_\alpha = \{g \in G : \alpha g = \alpha\}$ is its stabilizer. A partition $\mathcal{B}$ of $V$ is $G$-invariant if
for any $B \in \mathcal{B}$ and $g \in G$, one has $Bg \in B$. Partition $\mathcal{B}$ is trivial if $\mathcal{B} = \{V\}$ or
$\mathcal{B} = \{\{\alpha\} : \alpha \in V\}$. If $\mathcal{B}$ is non-trivial then it is a block system for the action of $G$
on $V$ (and any $B \in \mathcal{B}$ is a block). If such a block system exists, then we say that $G$
is imprimitive in its action on $V$ (equivalently, $G$ acts imprimitively on $V$). If $G$ is
not imprimitive (and it is transitive), then we say that it is primitive. Since $G$ acts
transitively on $V$, we have then $\mathcal{B} = \{Bg : g \in G\}$.

Lemma 2.1 ([1], Theorem 1.7). Let $G$ be a finite group, acting transitively on a set $V$.
Let $\alpha \in V$. Then the blocks $B$ containing $\alpha$ are in one-to-one correspondence with the
subgroups $H$ such that with $G_\alpha < H < G$. The correspondence is given by $B = \alpha H$.

In particular, $G$ is primitive if and only if $G_\alpha$ is a maximal subgroup of $G$.

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We denote by $\text{Sym}(V)$ and $\text{Alt}(V)$, respectively, the symmetric and alternating group on $V$. When $V$ is a vector space over a finite field $\mathbb{F}_q$ with $q$ elements, we also denote by $T(V)$ the translation group $T(V) = \{ \sigma_v : v \in V \}$, where $\sigma_v : V \rightarrow V, \ w \mapsto w + v$. It is well-known that $T(V)$ is a transitive subgroup of $\text{Sym}(V)$, which is imprimitive except for the trivial case $V = \mathbb{F}_p$, with $p$ a prime. Any block system $B$ of $T(V)$ is the set of translates of a proper vector subspace $W$ of $V$, that is, $B = \{ W + v \mid v \in V \}$. We denote by $\text{AGL}(V)$ the group of all affine permutations of $V$, which is a primitive maximal subgroup of $\text{Sym}(V)$, and by $\text{GL}(V)$ the group of all linear permutations of $V$, which is a normal subgroup of $\text{AGL}(V)$.

We will need the following result from finite field theory.

**Theorem 2.2** ([8],[11]). Let $F$ be a field of characteristic two. Suppose $U \neq 0$ is an additive subgroup of $F$ which contains the inverses of each of its nonzero elements. Then $U$ is a subfield of $F$.

### 2.2. Vectorial Boolean functions.

Let $m \geq 1$ be a natural number. Let $A = (\mathbb{F}_2)^m$ and $A^* = A \setminus \{0\}$. Any function $F : A \rightarrow A$ is a *vectorial Boolean function* (vBf).

For any function $F : A \rightarrow A$ and any elements $a, b \in A$, $a \neq 0$, we denote

$$\delta_F(a, b) = |\{ x \in A : F(x + a) + F(x) = b \}|.$$

Let $\delta \in \mathbb{N}$. Function $F$ is called a *differentially $\delta$-uniform* function ([10]) if

$$\forall a \in A^*, \forall b \in A, \quad \delta_F(a, b) \leq \delta.$$

The smallest such $\delta$ is called the *differential uniformity* of $F$. Note that $\delta \geq 2$ for any vBf. Differentially 2-uniform mappings are called *almost perfect nonlinear*, or APN for short. If we denote by $\hat{F}_a$ the vBf which maps $x \mapsto F(x + a) + F(x)$, then $F$ is differential $\delta$-uniform if and only if $|\hat{F}_a^{-1}(b)| \leq \delta$ (for any $a$ and $b$). From now on, we shorten “differential uniformity” to “uniformity”.

Vectorial Boolean functions used as S-boxes in block ciphers must have low uniformity to prevent differential cryptanalysis (see [9],[10]). In this sense, APN functions are optimal. However, numerous experiments suggest the following conjecture

**Conjecture 2.3** (Dobbertin). If $m$ is even, no APN function is a permutation.

If this conjecture is true, then APN functions cannot be used as S-Boxes, since implementation issues require an even $m$.

Any vBf can also be regarded as a polynomial in $\mathbb{F}_{2^m}[x]$ (with degree at most $2^m - 1$). When $m$ is even, the *patched inverse* function $x^{2^m - 2}$ is a 4-uniform permutation ([10]) and was chosen as the basic S-box, with $m = 8$, in the Advanced Encryption Standard (AES) ([2]).

### 2.3. Previous results on the group generated by the round functions.

Let $C$ be any block cipher such that the plain-text space $\mathcal{M}$ coincides with the cipher space. Let $K$ be the key space. Any key $k \in K$ induces a permutation $\tau_k$ on $\mathcal{M}$. Since $\mathcal{M}$ is usually $V = (\mathbb{F}_2)^n$ for some $n \in \mathbb{N}$, we can consider $\tau_k \in \text{Sym}(V)$. We denote by $\Gamma = \Gamma(C)$ the subgroup of $\text{Sym}(V)$ generated by all the $\tau_k$’s. In literature the following properties of $\Gamma$ are considered undesirable, since they could lead to weaknesses of $C$: small cardinality, imprimitivity and intransitivity. For a detailed discussion of their consequences, see [14]. We would add that $\Gamma$ should not be a subgroup of $\text{AGL}(V)$,
otherwise it is obvious how to break the cipher. If $\Gamma$ turns out to be $\text{Alt}(V)$ or $\text{Sym}(V)$, these properties are automatically avoided. Note also that primitivity alone guarantees a non-negligible group size, but it could still be that $\Gamma$ would be weak (as for example if $\Gamma \leq \text{AGL}(V)$).

Unfortunately, the knowledge of $\Gamma(C)$ is out of reach for the most important ciphers (such as the AES, the SERPENT, the DES, the IDEA). However, researchers have been able to compute another related group. Suppose that $C$ is the composition of $l$ rounds.

Remark 2.4. Note that the division into rounds is not mathematically well-defined, but it is provided in the document describing the cipher, so this division is debatable and a cryptanalyst is allowed to modify it, if it is convenient.

Then any key $k$ would induce $l$ permutations, $\tau_{k,1}, \ldots, \tau_{k,l}$, whose composition is $\tau_k$. For any round $h$, we can consider $\Gamma_h(C)$ as the subgroup of $\text{Sym}(V)$ generated by the $\tau_{k,h}$'s (with $k$ varying in $V$). We can thus define the group $\Gamma_\infty = \Gamma_\infty(C)$ as the subgroup of $\text{Sym}(V)$ generated by all the $\Gamma_h$'s. We note the following elementary fact.

Fact 1. $\Gamma \leq \Gamma_\infty$.

Group $\Gamma_\infty$ is traditionally called the group generated by the round functions. Note that independent sub-keys are implicitly assumed. We collect in the following proposition some previous results on $\Gamma_\infty$.

Proposition 2.5.

- $\Gamma_\infty(\text{AES}) = \text{Alt}(V)$ \cite{16},
- $\Gamma_\infty(\text{SERPENT}) = \text{Alt}(V)$ \cite{17},
- $\Gamma_\infty(\text{DES}) = \text{Alt}(V)$ \cite{18}.

The proof of any of the results in Proposition 2.5 requires an ad-hoc proof. Recently, a generalization of some of these results have been proposed \cite{14}.

3. A CLASS OF BLOCK CIPHERS

Several definitions have been proposed for iterated block ciphers (see e.g. key-alternating block cipher in \cite{2}, or Rjindael-like ciphers in \cite{14}). We would like to define a class, large enough to include most common ciphers, yet restricted enough to have simple criteria guaranteeing the primitivity of $\Gamma_\infty$.

Let $C$ be a block cipher with $V = (F_2)^n$ and $n = ms$, $s \geq 2$. Space $V$ is a direct sum

$$V = V_1 \oplus \cdots \oplus V_s,$$

where each $V_i$ has the same dimension $m$ (over $F_2$). For any $v \in V$, we will write $v = v_1 \oplus \cdots \oplus v_s$, where $v_i \in V_i$. Also, we consider the projections $\pi_i : V \to V_i$ mapping $v \mapsto v_i$. Any $\gamma \in \text{Sym}(V)$ that acts as

$$v\gamma = v_1\gamma_1 \oplus \cdots \oplus v_s\gamma_s,$$

for some $\gamma_i \in \text{Sym}(V_i)$, is a bricklayer transformation and any $\gamma_i$ is a brick. When used in symmetric cryptography, maps $\gamma_i$’s are traditionally called $S$-boxes and map $\gamma$ is called a “parallel S-box”.

A linear (or affine) map $\lambda : V \to V$ is traditionally called a “mixing layer”, when used in composition with parallel maps.

In the following definitions we are not following established notation. We call any linear map $\lambda \in \text{GL}(V)$ a proper mixing layer if no sum of some of the
We define our class.

**Definition 3.1.** We say that $C$ is translation based (tb) if it is the composition of some rounds, such that any is of the form $\tau_{k,h} = \gamma_h \lambda_h \sigma_k$, with $k \in V$ ($\gamma_h$ and $\lambda_h$ do not depend on $k$, but they might depend on the round), where $\gamma_h$ is a bricklayer transformation and $\lambda_h$ is a linear map (but $\lambda_h$ is a proper mixing layer for at least one round).

A round when the mixing layer is proper is called a proper round.

**Remark 3.2.** A round consisting of only a translation is still acceptable, by taking $\gamma_h = \lambda_h = 1_V$ (the identity map on $V$), although obviously it is not proper.

The previous definition is similar to key-alternating block cipher (see Section 2.4.2 of [2]), although the latter is too general for our goals.

From now on, we assume $C$ is a tb cipher and that $0\gamma = 0$ (this can always be assumed). From the knowledge of block systems of $T(V)$, we immediately obtain the following.

**Fact 2.** Let $G = \Gamma_h(C)$ for any round $h$. Then $T = T(V) \subset G$. Therefore, if $G$ acts imprimitively on $M = V$, the blocks of imprimitivity are the translates of a linear subspace.

**Proof.** We show $T \subset G$. For any $k \in V$, we have $\gamma_h \lambda_h \sigma_k \in G$. By considering the zero key, we have also $\gamma_h \lambda_h \sigma_0 = \gamma_h \lambda_h \in G$. Therefore, $(\gamma_h \lambda_h)^{-1} \gamma_h \lambda_h \sigma_k = \sigma_k \in G$. \qed

**Corollary 3.3.** Let $G = \Gamma_h(C)$ for any round $h$. Then $G$ acts imprimitively if and only if there is a subspace $U < V$ ($U \neq \{0\}, V$) such that for any $v \in V$ and $u \in U$, we have

$$ (v + u)\gamma_h \lambda_h + v\gamma_h \lambda_h \in U. $$

**Proof.** $G$ is imprimitive if and only if there is a block system of type $\{v + U\}$, for some subspace $U$, $U \neq \{0\}, V$.

It is enough to consider a zero round key, so that

$$ (v + U)\gamma_h \lambda_h \sigma_0 = v\gamma_h \lambda_h \sigma_0 + U \quad \Longrightarrow \quad (v + U)\gamma_h \lambda_h = v\gamma_h \lambda_h + U. $$

\qed

4. Main results

We define for a vBf $f$ two new notions of non-linearity. The first is weaker than $\delta$-uniformity.

**Definition 4.1.** For any $m \geq 2$ and $\delta \geq 2$, let $A = (\mathbb{F}_2)^m$ and $f \in \text{Sym}(A)$. We say that $f$ is weakly $\delta$-uniform if for any $u \in A$, $u \neq 0$, the size of image of $f_u$ is at least

$$ |\text{Im}(f_u)| \geq \frac{2m}{\delta + 2} + 1. $$

It is trivial to prove that a $\delta$-uniform map is indeed weakly $\delta$-uniform.
Proof. Let $B = \text{Im}(\hat{f}_u)$. If $f$ is $\delta$-uniform, then $|(\hat{f}_u)^{-1}(b)| \leq \delta$, for any $b \in B$. From $A = \sqcup_{b \in B}(\hat{f}_u)^{-1}(b)$, we have

$$A = \sqcup_{b \in B}(\hat{f}_u)^{-1}(b) \implies 2^m = |A| = \sum_{b \in B}|(\hat{f}_u)^{-1}(b)| \leq \delta|B|$$

which means

$$|B| \geq \frac{2^m}{\delta} > \frac{2^m}{\delta + 2}.$$  

\[ \square \]

Remark 4.2. If a function $f$ is weakly $\delta$-uniform, with $2^r \geq \delta$ and the image $\text{Im}(\hat{f}_u)$ is contained in a subspace $W$, then the dimension of $W$ is at least $m - r$. This is the property of $f$ which will be needed in the proof of Theorem 4.4. Interestingly, if $f$ is $\delta$-uniform (as in Subsection 2.2), then the dimension of $W$ which can be guaranteed is exactly the same (and not any bigger).

Our second notion focuses on the image of vector spaces.

Definition 4.3. Let $A = (\mathbb{F}_2)^m$. We say that $f$ is $l$-anti-invariant if for any subspace $U \leq A$ such that $f(U) = U$ we have $\dim(U) < m - l$ or $U = A$.

We say that $f$ is strongly $l$-anti-invariant, if for any two subspaces $U, W \leq A$, such that $f(U) = W$, we have $\dim(U) = \dim(W) < m - l$ or $U = W = A$.

In other words, $l$-anti-invariant means that the largest subspace invariant under $f$ has codimension greater than $l$ (except for $A$ itself), while strongly $l$ anti-invariant means that the largest subspace sent by $f$ into another subspace has codimension greater than $l$ (except for $A$ itself).

We are ready for our main result (recall that $0_{\gamma} = 0$).

Theorem 4.4. Let $C$ be a tb cipher, with $\lambda_h$ a proper mixing layer, and $G = \Gamma_h(C)$. Let $1 \leq r < m/2$. If any brick of $\gamma_h$ is weakly $2^r$-uniform and strongly $r$-anti-invariant, then $G$ is primitive and hence $\Gamma_{\infty}(C)$ is primitive.

Proof. We drop the $h$-underscript in this proof and we suppose, by way of contradiction, that $G$ is imprimitive.

Let $U$ be any proper subspace of $V$ s.t. $\{v + U\}_{v \in V}$ form a block system for $G$. Since $U$ is a block and $\gamma \lambda \in G$, we have $U\gamma \lambda = U + v$ for some $v \in V$. But $0_{\gamma \lambda} = 0 \in U + v$, so $v = 0$ and

\begin{equation}
U \gamma \lambda = U.
\end{equation}  

Let $I$ be the set of all $i$ s.t. $\pi_i(U) \neq 0$. Clearly, $I \neq \emptyset$. Then:

\begin{itemize}
  \item either $U \cap V_i = V_i$ for all $i \in I$,
  \item or there is $i \in I$ s.t. $U \cap V_i \neq V_i$.
\end{itemize}

In the first case, $U = \bigoplus_i V_i$, which means $U \gamma = U$. But \[(4.1)\] implies $U \lambda = U$, which is impossible since $\lambda$ is a proper mixing layer.

In the second case, we denote $W = U \gamma$ (equal to $U \lambda^{-1}$ by \[(4.1)\]) and we note that

\begin{equation}
(U \cap V_i) \gamma' = W \cap V_i,
\end{equation}

where $\gamma' = \gamma_i$ is the brick of $\gamma$ in $V_i$. By Corollary 3.3, we have that $B = \text{Im}(\hat{\gamma}_u') \subset W \cap V_i$ for any $u \in U \cap V_i$. But $\gamma'$ is weakly $2^r$-uniform, so (Remark \ref{r:1.2}) $\dim(W \cap V_i) = \dim(U \cap V_i) \geq m - r$. By \[(4.2)\], this is impossible, since $\gamma'$ is strongly $r$-anti-invariant.  

\[ \square \]
To apply our theorem to the AES, we first need a simple lemma.

**Lemma 4.5.** Let $f$ be a $vBf$. If $f^2 = 1$ and $f$ is $2r$-anti-invariant with $1 \leq r < m/2$, then $f$ is strongly $r$-anti-invariant.

**Proof.** Let $U, W$ be subspaces of codimension $l$ such that $Uf = W$. Let us consider $Z = U \cap Uf$. By standard linear algebra, $\dim(Z) \geq n - 2l$. Since $Zf = Z$ and $f$ is $2r$-anti-invariant, $l$ must be $l > r$, and so $U$ and $W$ have codimension strictly bigger than $r$. □

The first interesting consequence of our theorem is the following.

**Corollary 4.6.** Any typical round $h$ of the AES satisfies the hypotheses of Theorem 4.4. As a consequence, both $\Gamma_h(AES)$ and $\Gamma_\infty(AES)$ are primitive.

**Proof.** We first show that the mixing layer $\lambda = \lambda_h$ of a typical round of the AES is proper. Suppose $U \neq \{0\}$ is a subspace of $V$ which is invariant under $\lambda$. Suppose, without loss of generality, that $U \supseteq V_1$. Because of MixColumns [2, 3.4.3], $U$ contains the whole first column of the state. Now the action of ShiftRows [2, 3.4.2] and MixColumns on the first column shows that $U$ contains four whole columns, and considering (if the state has more than four columns) once more the action of ShiftRows and MixColumns, one sees immediately that $U = V$.

The S-box $\gamma'$ is well-known to satisfy (for any $u \neq 0$) $\text{Im}(\hat{\gamma}'_u) = 2^7 - 1 \geq 2^6 + 1$ and so it is weakly 2-uniform.

To apply the theorem we need only to show that $\gamma'$ is strongly 1-anti-invariant. Since $(\gamma')^2 = 1$, we want to apply Lemma 4.5 with $r = 1$. Indeed, $\gamma'$ is well-known to be 3-anti-invariant, since the only nonzero subspaces of $\mathbf{GF}(2^8)$ which are invariant under inversion are the subfields (Theorem 2.2), and so the largest proper one is $\mathbf{GF}(2^4)$, of codimension $4 > 3$. □

The second interesting consequence is the following.

**Corollary 4.7.** Any typical round $h$ of the SERPENT satisfies the hypotheses of Theorem 4.4. As a consequence, both $\Gamma_h(SERPENT)$ and $\Gamma_\infty(SERPENT)$ are primitive.

**Proof.** The conditions of Theorem 4.4 are satisfied with $r = 1$, as can be seen by a direct computer check on all Serpent S-boxes and on its mixing layer ([13]). □

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