Existence of a weak solution to the fluid-structure interaction problem in 3D

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Abstract

We study a nonlinear fluid-structure interaction problem in which the fluid is described by the three-dimensional incompressible Navier-Stokes equations, and the elastic structure is modeled by the nonlinear plate equation which includes a generalization of Kirchhoff, von Kármán and Berger plate models. The fluid and the structure are fully coupled via kinematic and dynamic boundary conditions. The existence of a weak solution is obtained by designing a hybrid approximation scheme that successfully deals with the nonlinearities of the system. We combine time-discretization and operator splitting to create two sub-problems, one piece-wise stationary for the fluid and one in the Galerkin basis for the plate. To guarantee the convergence of approximate solutions to a weak solution, a sufficient condition is given on the number of time discretization sub-intervals in every step in a form of dependence with number of the Galerkin basis functions and nonlinearity order of the plate equation.

Keywords and phrases: fluid-structure interaction, incompressible viscous fluid, nonlinear plate, three space variables, weak solution

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1 Introduction

In recent years there have been many works in the study of mathematical theory of fluid-structure interaction (FSI) problems. These problems arise from research fields like hydroelasticity, aeroelasticity, biomechanics, blood flow modeling and so on. Chambolle et. al. in [3] obtained the existence of a weak solution to the problem of viscous incompressible fluid and viscoelastic plate interaction in 3D, and Grandmont (9) studied the limiting problem when viscoelasticity coefficient tends to zero. In [17 - 21] Muha and Ćanić obtained the existence of weak solutions to several FSI problems by using the time discretization via operator splitting method. In [20] they studied the interaction in 2D case, and in [17, 21]

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the 3D cylindrical case where the structure is described by linear and nonlinear Koiter shell equations, respectively. In [18], they observed a multi-layer structure of blood vessel in 2D, and in [19], they studied the 2D model with Navier-slip condition on the fluid-structure interface, where some additional difficulties due to the loss of the trace regularity of unknowns were successfully tackled.

In this paper, we study the interaction problem of viscous incompressible fluid and an elastic structure, which is modeled by a nonlinear plate law that precisely describes certain nonlinear phenomena in plate dynamics. This plate model is an extension of the one studied by Chueshov in [5, 6] in the sense that certain higher order derivatives of displacement are allowed to be in the nonlinear term of the plate equation. In his work, Chueshov studied a plate model which is a generalization of Kirchhoff, von Kármán and Berger plate models, and got the well-posedness of the fluid-structure interaction problem where fluid is described by linearized compressible Navier-Stokes or linearized Euler equations, respectively. Unfortunately, the methods used in [5, 6] strongly rely on the linearity of the fluid equations. We need to develop a different approach for our nonlinear problem, which was inspired from the approaches for nonlinear plates and fluid-structure interaction problems. For this reason we constructed a novel hybrid approximation scheme that deals with the nonlinearities both in fluid and plate equations by using the time discretization via the operator splitting method and the Galerkin approximation for the plate. We then prove that when the number of time discretization sub-intervals in every step is sufficiently large (compared to the number of Galerkin basis functions, corresponding eigenvalues and some other parameters), we obtain the convergence of a subsequence of approximate solutions to a weak solution of the original nonlinear problem.

2 Preliminaries and main result

In this section, we will first describe the model and derive the energy equality in the classical sense. After that, we introduce the domain transformation (the LE mapping) which is used in redefining the problem on a fixed domain. At the end of the section, we derive the equation in a weak form, give the definition of weak solutions and state the main result.

2.1 Model description

Here we deal with the incompressible, viscous fluid interacting with nonlinear plate. The plate displacement is described by a scalar function \( \eta : \Gamma \to \mathbb{R} \), where \( \Gamma \subset \mathbb{R}^2 \) is a connected bounded domain with Lipschitz boundary. The fluid fills the domain between side walls, the plate and the bottom, i.e.

\[
\Omega^\eta(t) = \{(X, z) : X \in \Gamma, -1 < z < \eta(t, X)\},
\]

We will denote the graph of \( \eta \) as \( \Gamma^\eta(t) = \{(X, z) : X \in \Gamma, z = \eta(t, X)\} \) and the side wall of the domain as \( W = \{(X, z) : X \in \partial \Gamma, -1 < z < 0\} \), where the plate boundary is assumed to be fixed as \( z = 0 \) for all \( x \in \partial \Gamma \). The entire rigid part of the boundary \( \partial \Omega^\eta(t) \) will be denoted as \( \Sigma = (\Gamma \times \{-1\}) \cup W. \)

The plate displacement \( \eta(t, X) \) will be described by the nonlinear plate law:

\[
\partial_t^2 \eta + \mathcal{F}(\eta) + \Delta^2 \eta = f, \tag{1}
\]
where \( f \) is the force density in vertical direction that comes from the fluid and \( F \) is a nonlinear function corresponding to the nonlinear elastic force in various plate models (see assumptions given in [15] - [21]). The plate is considered to be clamped

\[
\eta(t,x) = 0, \quad \partial_v \eta(t,x) = 0, \quad \text{for all } x \in \partial \Gamma, t \in (0,T),
\]

where \( \nu \) is the normal vector on \( \partial \Gamma \), and supplemented with initial data

\[
\eta(0, \cdot) = \eta_0, \quad \partial_t \eta(0, \cdot) = v_0.
\]

The fluid motion is described by the Navier-Stokes equations

\[
\begin{align*}
\partial_t u + u \cdot \nabla u &= \nabla \cdot \sigma, \\
\nabla \cdot u &= 0,
\end{align*}
\]

in \( \Omega^\eta(t), \quad t \in (0,T) \).

where \( u \) is the velocity of the fluid, \( \sigma = -p I + 2 \mu D(u) \), \( \mu \) is the kinematic viscosity coefficient and \( D(u) = \frac{1}{2}(\nabla u + (\nabla u)^\top) \). The boundary condition on \( \Sigma \) for the velocity \( u \) is no-slip

\[
u \]

\[ u = 0, \quad \text{on } \Sigma, \]

and we are supplemented with the initial data

\[
u \]

\[ u(0, \cdot) = u_0. \]

The coupling between the fluid and plate is defined by two sets of boundary conditions on \( \Gamma^\eta(t) \). In the Lagrangian framework, with \( X \in \Gamma \) and \( t \in (0,T), \) they read as:

- The kinematic condition:

\[
\partial_t \eta(t, X)e_3 = u(t, X, \eta(t,X))
\]

where \( e_3 = (0,0,1) \).

- The dynamic condition:

\[
f(t, X) = -S^\eta(\sigma \nu^\eta) \cdot e_3.
\]

Here, \( \nu^\eta \) is the unit normal vector on the boundary \( \Gamma^\eta \), and \( S^\eta(t,X) \) is the Jacobian of the transformation from Eulerian to Lagrangian coordinates of the plate

\[
S^\eta(t,X) = \sqrt{1 + \partial_x \eta(t,X)^2 + \partial_y \eta(t,X)^2},
\]

with \( X = (x,y) \).

With \( \mathbf{S} \), the plate equation (1) then becomes:

\[
\partial_t^2 \eta + F(\eta) + \Delta^2 \eta = -S^\eta(\sigma \nu^\eta) \cdot e_3,
\]

The initial data given in (3) and (6) are assumed to satisfy the following compatibility conditions:

\[
u \]

\[ u_0 \in L^2(\Omega^{\eta_0})^3, \quad \nabla \cdot u_0 = 0, \quad \text{in } \Omega^{\eta_0}, \]

\[ u_0 = 0, \quad \text{on } \Sigma, \]

\[ u_0(X, \eta_0(X)) = v_0(X) \cdot e_3, \quad \text{on } \Gamma, \]

\[ \partial_n \eta_0(X) = \eta_0(X) = v_0(X) = 0, \quad \text{on } \partial \Gamma, \]

\[ \eta_0(X) > -1, \quad \text{on } \Gamma, \]

where \( \Omega^{\eta_0} = \Omega^\eta(0) \).
2.2 The energy identity for smooth solutions

In order to define the weak formulation of the problem (1)-(10), we need to choose the functional setting. Our bounds will come from the energy inequality of the approximated system, and it will dictate the regularity of solutions.

We can derive it for smooth solutions in the following way. We multiply (9) with $\partial_t \eta$ and integrate over $\Gamma$. Then we multiply the equation (4) with $u$, and integrate over $\Omega$.

We sum these two equalities, and integrate it over $(0,T)$ to obtain:

$$
E(t) + \mu |D(u)|^2_{L^2(\Omega \eta(t))} + \int_0^T (f(\eta(t)), \partial_t \eta) = E(0),
$$

where

$$
E(t) := \frac{1}{2} \left( |u(t)|^2_{L^2(\Omega \eta(t))} + ||\eta(t)||^2_{H^2(\Gamma)} + ||\partial_t \eta(t)||^2_{L^2(\Gamma)} \right)
$$

Even though the term in (11) with $F$ isn’t necessarily positive, we will still use this equality as an inspiration for the choice of function spaces in the following discussion.

2.3 LE mapping of the domain and functional setting

In order to define the problem on the fixed domain $\Omega = \{(X,z) : X \in \Gamma, -1 < z < 0\}$, we define a family of the following Lagrangian Eulerian (LE) transformations:

$$
A_{\eta}(t) : \Omega \rightarrow \Omega^0(t),
$$

$$(X,z) \mapsto (X, (z+1)\eta(t,X) + z).$$

This mapping is a bijection and its Jacobian, defined by

$$
J^0(t,X,z) := \det \nabla A_{\eta}(t,X) = 1 + \eta(t,X),
$$

is well-defined as long as $\eta(t,X) > -1$ for any $X \in \Gamma$. We also define the LE velocity

$$
w^0 := \frac{d}{dt} A_{\eta} = (z+1)\partial_t \eta e_3,
$$

In order to define the weak solution on the fixed domain $\Omega$, we need to transform the problem (1)-(10) to be defined on $\Omega$ by using $A_{\eta}$. For an arbitrary vector function $f$ defined on $\Omega^0(t)$, we define $f^\eta$ on $\Omega$ as pullback of $f$ by $A_{\eta}$, i.e

$$
f^\eta(t,X,z) := f(t,X,A_{\eta}(t,X)), \quad \text{for } (X,z) \in \Omega
$$

the gradient as the push forward by $A_{\eta}$

$$
\nabla^\eta f^\eta := (\nabla f)^0 = \nabla f^0 (\nabla A_{\eta})^{-1},
$$

with

$$
(\nabla A_{\eta})^{-1} = [e_1, e_2, \bar{A}_{\eta}], \quad \bar{A}_{\eta} = \frac{1}{\eta+1}[-(z+1)\partial_x \eta, -(z+1)\partial_y \eta, 1]^T,
$$

the symmetrized gradient as

$$
D^\eta (f^\eta) := \frac{1}{2} (\nabla^\eta f^\eta + (\nabla^\eta f^\eta)^\top),
$$
and the LE derivative as a time derivative on fixed domain $\Omega$
\[ \partial_t f_\Omega := \partial_t f + (w^\eta \cdot \nabla) f. \]  
(15)

Now we write the Navier-Stokes equations in LE formulation by using the LE mapping,
\[ \partial_t u_{\eta\Omega} + (u - w^\eta) \cdot \nabla u = \nabla \cdot \sigma, \quad \text{in } \Omega^\eta(t) \]  
(16)
where $\partial_t u_{\eta\Omega}$ and $w^\eta$ are composed with the $(A_\eta(t))^{-1}$, and the transformed divergence free condition
\[ \nabla^\eta \cdot u^\eta = 0, \quad \text{in } \Omega. \]  
(17)

From the energy equality \cite{1, 6}, the possible regularity of $\eta$ is $H^2_\gamma(\Gamma)$, and since $H^2_\gamma(\Gamma)$ is embedded into the Hölder space $C^{0,\alpha}$ for $\alpha < 1$, $\Omega^\eta(t)$ doesn’t necessarily have Lipschitz boundary. For this lower regularity boundary we define the “Lagrangian” trace operator as
\[ \gamma|_{\Gamma(t)} : C^{1,\alpha}(\Omega^\eta(t)) \rightarrow C(\Gamma), 
\gamma \mapsto \gamma(t, X, \eta(t, X)). \]

In \cite{3, 6, 16}, it was proved that by continuity we can extend this trace operator $\gamma$ to be a linear operator from $H^1(\Omega^\eta(t))$ to $H^s(\Gamma)$, $0 \leq s < 1/2$. We first define the fluid velocity space
\[ V_F := \{ u = (u_1, u_2, u_3) \in C^{1,\alpha}(\Omega^\eta(t)) : \nabla \cdot u = 0, \ u = 0 \ \text{on} \ \Sigma \} \]
and its following closure
\[ V_F := \overline{V_F}^{H^1(\Omega^\eta(t))}. \]

It is easy to have the following characterization (see \cite{3, 6}):
\[ V_F = \{ u = (u_1, u_2, u_3) \in H^1(\Omega^\eta(t)) : \nabla \cdot u = 0, \ u = 0 \ \text{on} \ \Sigma \}. \]
The transformation of the domain, $A_\eta$, isn’t necessarily Lipschitz, so the transformed velocity $u^\eta$ may not be in $H^1(\Omega)$. The transformed velocity space is defined as
\[ V_F^{\eta} := \{ u^\eta : u \in V_F \} \]
Notice that any function $u^\eta \in V_F^{\eta}$ satisfies the transformed divergence free condition \cite{17} rather than the regular divergence free condition. When the Jacobian $J = \eta + 1 > 0$, the inner product in $V_F^{\eta}$ is defined as:
\[ (f^\eta, g^\eta)_{V_F^{\eta}} := (f, g)_{H^1(\Omega^\eta(t))}. \]

Now, we define the corresponding function space for fluid velocity involving time
\[ W_F^{\eta}(0, T) := L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; V_F^{\eta}), \]
while for the structure we choose classical space
\[ W_S(0, T) := W^{1, \infty}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H^2_\gamma(\Gamma)). \]

We include the kinematic condition in the solution space
\[ W^\eta(0, T) := \{ (u^\eta, \eta) \in W_F^{\eta}(0, T) \times W_S(0, T) : (u^\eta)|_{\eta} = \partial_t \eta e_3 \} \]
and similarly for the corresponding test function space
\[ Q^\eta(0, T) := \{ (q^\eta, \psi) \in C^{1,\alpha}_L([0, T]; V_F^{\eta} \times H^2_\gamma(\Gamma)) : (q^\eta)|_{\eta} = \psi(t, z)e_3 \}. \]

Before the end of this subsection, let us impose the following assumptions on the nonlinear term $F$ of the plate equation:
(A1) Mapping $F$ is locally Lipschitz from $H_0^{2-\epsilon}(\Gamma)$ into $H^{-2}(\Gamma)$ for some $\epsilon > 0$, i.e.
\[ ||F(\eta_1) - F(\eta_2)||_{H^{-2}} \leq C_R ||\eta_1 - \eta_2||_{H^{2-\epsilon}(\Gamma)}, \quad (18) \]
for a constant $C_R > 0$, for any $||\eta_i||_{H^{2-\epsilon}(\Gamma)} \leq R$ ($i = 1, 2$).

(A2) Mapping $F$ is locally Lipschitz from $H_0^a(\Gamma)$ into $H^{-a}(\Gamma)$ for some $0 \leq a < 2$, i.e.
\[ ||F(\eta_1) - F(\eta_2)||_{H^{-a}} \leq C_R ||\eta_1 - \eta_2||_{H^a(\Gamma)}, \quad (19) \]
for a constant $C_R > 0$, for any $||\eta_i||_{H^a(\Gamma)} \leq R$ ($i = 1, 2$).

(A3) $F(\eta)$ has a potential in $H_0^2(\Gamma)$, i.e. there exists a Fréchet differentiable functional $\Pi(\eta)$ on $H_0^2(\Gamma)$ such that $\Pi'(\eta) = F(\eta)$, and there are $0 < \kappa < 1/2$ and $C^* \geq 0$, such that the following inequality holds,
\[ \kappa||\Delta \eta||_{L^2(\Gamma)}^2 + \Pi(\eta) + C^* \geq 0, \quad \text{for all } \eta \in H_0^2(\Gamma). \quad (20) \]
Moreover, for any given $\eta_0$, there is a positive constant $C(\Pi, \eta_0)$ such that
\[ ||\Pi(\eta)|| \leq C(\Pi, \eta_0) \quad (21) \]
holds for all $\eta \in H_0^2(\Gamma)$ with $||\eta||_{H^2(\Gamma)} \leq ||\eta_0||_{H^2(\Gamma)}$.

**Remark 2.1.** It was proved that in several plate models such as von Kármán, Kirchhoff and Berger plates, the nonlinear elastic force $F$ satisfies assumptions (A1)-(A3) with $a = 0$ (see \[5\] \[6\] and the reference therein). The assumption (A1) will be used to pass the convergence in the nonlinear term $F(\eta)$ when the bound of approximate solutions is obtained in $H_0^2(\Gamma)$. On the other hand the condition (A2) tells us the order of nonlinearity precisely - the function $F(\eta)$ may depend on the derivatives of $\eta$ up to order $2 + a$, and the parameter $a$ will also affect the minimal precision in time (the number of time sub-intervals) we will require for the approximate solution convergence (see inequality \[10\]). The coercivity condition \[20\] in the assumption (A3) is used to eliminate potential in the energy as it can be negative, while the constant $C(\Pi, \eta_0)$ defined in (A3) is used in bounding of the initial energy of the approximate problem in finite Galerkin basis (see section 3.1.1).

### 2.4 The weak solution formulation and main result

Before defining the weak solution for the problem \[10\] \[14\], we first calculate some terms. For any given $(u, \eta) \in W^{0}(0,T)$ and $(q, \psi) \in Q^0(0,T)$, multiplying the equation \[10\] by $q$ and integrating over $\Omega^0(t)$, the convective term can be computed in the following way:
\[ \int_{\Omega^0(t)} ((u - w^\eta) \cdot \nabla) u \cdot q = \tfrac{1}{2} \int_{\Omega^0(t)} ((u - w^\eta) \cdot \nabla) u \cdot q - \tfrac{1}{2} \int_{\Omega^0(t)} ((u - w^\eta) \cdot \nabla) q \cdot u \]
\[ + \tfrac{1}{2} \int_{\Omega^0(t)} (\nabla \cdot w^\eta) u \cdot q + \tfrac{1}{2} \int_{\Omega^0(t)} (u - w^\eta) \cdot \nu^\eta (u \cdot q), \]
in which the last term vanishes due to the kinematic coupling condition \[7\]. The diffusive part satisfies
\[ - \int_{\Omega^0(t)} (\nabla \cdot \sigma) \cdot q = 2\mu \int_{\Omega^0(t)} D(u) : D(q) - \int_{\partial \Omega^0(t)} \sigma \nu^\eta \cdot q, \]
where the last term can be expressed as
\[ \int_{\partial \Omega^0(t)} \sigma \nu^\eta \cdot q = \int_{\Gamma^0(t)} (\sigma \nu^\eta \cdot e_3) q \cdot e_3 = \int_{\Gamma} S^3(\sigma \nu^\eta) \cdot e_3 \psi. \]
The only remaining term is the one including time derivative. Since we now want to express
these integrals on the fixed domain \( \Omega \), we calculate:
\[
\int_0^T \int_{\Omega(t)} \partial_t u^n \cdot q = \int_0^T \int_{\Omega} J^n \partial_t u^n \cdot q^n
\]
\[
= - \int_0^T \int_{\Omega} \partial_t J^n u^n \cdot q^n - \int_0^T \int_{\Omega} J^n u^n \cdot \partial_t q^n - \int_0^T \int_{\Omega} J_0 u_0 \cdot q^n(0, \cdot).
\]

As in see [12, pp. 77], by a simple calculation we have
\[
\partial_t J^n = \partial_t \eta = (1 + \eta) \partial_z \left( \frac{z - \eta}{1 + \eta} + 1 \right) \partial_t \eta = J^n \nabla \cdot w^0.
\]

We finally multiply the plate equation (9) with \( \psi \) and integrate over \((0, T) \times \Gamma\), and use partial integration in time. Summing this with the fluid equation (10) multiplied with \( \eta \) and integrated over \((0, T) \times \Omega\) and using the calculation we just obtained, it leads to define:

**Definition 2.1. (Weak solution)** Under the assumptions (A1)-(A3) of \( F \), we say that \((u, \eta) \in \mathcal{W}^0(0, T)\) is a weak solution of the problem (11)-(10) defined on reference domain \( \Omega \), if the initial data \( u_0, \eta_0 \) satisfy the compatibility conditions given in (11), the following identity
\[
\frac{1}{2} \int_0^T \int_{\Omega} \left( J^n(((u - w^n) \cdot \nabla)u - (u - w^n) \cdot \nabla) \cdot q - ((u - w^n) \cdot \nabla) \cdot q \cdot u - \partial_t J^n q \cdot u \right)
\]
\[
+ \int_0^T \int_{\Omega} 2 \mu J^n D^0(u) : D^0(q) - \int_0^T \int_{\Omega} J^n u \cdot \partial_t q
\]
\[
- \int_0^T \int_{\Omega} \partial_t \eta \partial_t \psi dzdt + \int_0^T (\mathcal{F}(\eta), \psi) + \int_0^T (\Delta \eta, \Delta \psi)
\]
\[
= \int_0^T \int_{\Omega} J_0 u_0 \cdot q(0) + \int_{\Gamma} v_0 \cdot \psi(0).
\]
holds for every \((q, \psi) \in \mathcal{Q}(0, T)\).

The main result of this paper is stated as follows:

**Theorem 2.1. (Main theorem).** Assume that the nonlinear functional \( \mathcal{F} \) satisfies the conditions given in (A1)-(A3). Then, for any given initial data \( \eta_0 \in H^1_0(\Gamma), v_0 \in L^2(\Gamma) \) and \( u_0 \in L^2(\Omega^0)^3 \) that satisfies the compatibility conditions (10), there exists a weak solution \((u, \eta) \in \mathcal{W}(0, T)\) of the problem (11)-(10) in the sense of Definition 2.1 satisfying the energy inequality:
\[
E(t) + \|D(u)\|_{L^2(0, T; L^2(\Omega(t)))}^2 \leq C_0 = CE(0) + C(\Pi, \eta_0), \text{ for all } t \in (0, T)
\]
where
\[
E(t) := \frac{1}{2} \left( \|u(t)\|_{L^2(\Omega(t))}^2 + \|\eta(t)\|_{H^2(\Gamma)}^2 + \|\partial_t \eta(t)\|_{L^2(\Gamma)}^2 \right)
\]
The constants \( C(\Pi, \eta_0) \) and \( C = 1/(\frac{1}{2} - \kappa) \) come from the bound of potential function on bounded sets [21] and the coercivity estimate [20]. Moreover, this solution is defined on the time interval \((0, T)\) with \( T = \infty \), or \( T < \infty \) which is the moment when the free boundary \( \{z = \eta(t, X)\} \) touches the bottom \( \{z = -1\} \).

The remainder of the paper is organized as follows. In section 3.1 we will formulate our approximate problems. The existence of approximate solutions will be given in section 3.2, and certain properties of these solutions will be discussed in section 3.3. In section 4, we study the convergence of approximate solutions. The weak and strong convergences will be proved in sections 4.1 and 4.2, respectively. The proof of the main result will be given in section 5. In the appendix, we will present some additional calculation for the corresponding two-dimensional problem.
3 Construction of approximate solutions

In this section, by using the time discretization and Galerkin basis in \( H^2_0(\Gamma) \), we will construct approximate solutions to the original problem (1)-(10), and obtain certain uniform estimates of approximate solutions. At the end of this section, the difference between \( \partial_t \eta \) and the trace of the fluid velocity \( v \) at \( \Gamma \eta \) (which are not equal from the approximate problems) is studied, and a sufficient condition is introduced on the number of time sub-intervals \( N = T/\Delta t \) in the time discretization in order to keep the difference smaller than \( O((\Delta t)^\alpha) \) in \( L^2(\Gamma) \) for some \( \alpha > 0 \).

3.1 Formulation of approximate problems

For any fixed \( T > 0 \) and \( N \geq 1 \), letting \( \Delta t = T/N \), we split the time interval \( [0,T] \) into \( N \) equal sub-intervals and on each sub-interval we use the Lie operator splitting, and separate the problem into two parts - the fluid and structure sub-problems. We rewrite the problem (1)-(10) as the following one:

\[
\frac{dX}{dt} = AX, \quad t \in (0,T),
\]

\[
X|_{t=0} = X^0,
\]

where \( X = (u,v,\eta)^T \), and \( v = \partial_t \eta \) and decompose \( A = A_1 + A_2 \), where \( A_1 \) and \( A_2 \) are non-trivial and correspond to these two sub-problems. Since the sub-problems are not of the same nature, from here on we proceed to define them separately.

3.1.1 The structure sub-problem (SSP) in the Galerkin basis

First, we define the biharmonic eigenvalue problem as to find non-trivial \( w \in H^2_0(\Gamma) \) and \( \xi > 0 \) such that

\[
\begin{align*}
(\Delta^2 w, \Delta f) &= \xi (w,f), \\
w(X) = \partial_n w(X) &= 0, \quad \text{on } \partial \Gamma,
\end{align*}
\]

for all \( f \in H^2_0(\Gamma) \). This eigenvalue problem has a growing unbounded sequence of eigenvalues \( 0 < \xi_1 < \xi_2 < ... \), and corresponding smooth eigenfunctions \( \{w_i\}_{i \in \mathbb{N}} \) which form a basis of \( H^2_0(\Gamma) \). The set \( \{w_i\}_{i \in \mathbb{N}} \) is called a Galerkin basis (see for example [7, Theorem 7.22]).

Denote by \( \mathcal{G}_k = \text{span}\{w_i\}_{1 \leq i \leq k} \) and the closed subspaces

\[
H^2_{0,k}(\Gamma) := (\mathcal{G}_k, \| \cdot \|_{H^2(\Gamma)})
\]

\[
L^2_{k}(\Gamma) := (\mathcal{G}_k, \| \cdot \|_{L^2(\Gamma)}).
\]

The structure sub-problem (SSP):

For any fixed \( k \geq 1 \), and \( 0 \leq n \leq N - 1 \), assume that \( v^n \in L^2_k(\Gamma) \) is given already, we define \( \eta^{n+1} \in C^1([n\Delta t, (n+1)\Delta t]; L^2_k(\Gamma) \cap C([n\Delta t, (n+1)\Delta t]; H^2_{0,k}(\Gamma)) \) the solution of the following problem inductively on \( n \),

\[
\begin{align*}
\left( \frac{\partial \eta^{n+1}(t) - v^n}{\Delta t}, \psi \right)_\Omega + (\Delta \eta^{n+1}(t), \Delta \psi(t))_\Omega + (F(\eta^{n+1}(t)), \psi(t))_\Omega &= 0, \\
\eta^{n+1}(t, X) = \partial_n \eta^{n+1}(t, X) &= 0, \quad \text{on } \partial \Gamma,
\end{align*}
\]

\[
\eta^{n+1}(n\Delta t, X) = \eta^n(n\Delta t, X)
\]

(25)
for all $\psi \in H^2_{0,k}(\Gamma)$, with $\eta^{0}_{\Delta t,k}(0,X) = \eta_{0,k}(X) := \sum_{i=1}^{k}(\eta_{0}, w_{i})w_{i}$, where we have omitted $\Delta t$ and $k$ in the subscript of notation $\eta^{n+1}$.

The approximate solution $\eta_{\Delta t,k}(t,X)$ on whole time interval $[0,T]$ is defined as the following way:

$$\eta_{\Delta t,k}(t) := \eta_{\Delta t,k}^{n+1}(t), \text{ for } t \in [n\Delta t, (n+1)\Delta t), \text{ } 0 \leq n \leq N - 1.$$  

### 3.1.2 The fluid sub-problem (FSP)

Assume that we have $\eta_{\Delta t,k}^{n+1}$ already from the problem (25), we define following average quantity:

$$\tilde{\partial}_{t}\eta_{\Delta t,k}^{n+1} := \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \partial_{t}\eta_{\Delta t,k}^{n+1} dt = \frac{\eta_{\Delta t,k}^{n+1}((n+1)\Delta t) - \eta_{\Delta t,k}^{n}(n\Delta t)}{\Delta t}$$

and the LE mapping $A^{n+1}_{\Delta t,k} = (X,(z+1)\tilde{\eta}_{\Delta t,k}^{n+1} + z))$, where $\tilde{\eta}_{\Delta t,k}^{n+1} = \eta_{\Delta t,k}^{n+1}((n+1)\Delta t)$. The discretized Jacobian and LE velocity are given as:

$$J^{n+1}_{\Delta t,k} := \text{det} \nabla A^{n+1}_{\Delta t,k} = \tilde{\eta}_{\Delta t,k}^{n+1} + 1, \text{ } w^{n+1}_{\Delta t,k} := \tilde{\partial}_{t}\eta^{n+1}_{\Delta t,k}e_{3}$$

To determine the approximate solution for the fluid part of the problem (1)-(10) in the weak form, we first discretize the time derivatives and obtain the following piece-wise stationary problem:

$$\frac{u^{n+1} - u^{n}}{\Delta t} + ((u^{n} - w^{n+1}) \cdot \nabla \eta^{n+1})u^{n+1} = \nabla \eta^{n+1} \cdot \sigma \eta^{n+1}(u^{n+1}, p^{n+1}), \text{ in } \Omega, \text{ (26)}$$

and

$$\frac{\tilde{\partial}_{t} \eta^{n+1} - v^{n+1}}{\Delta t} = -S^{n+1} \sigma \eta^{n+1}(u^{n+1}, p^{n+1}) \cdot \nu \eta^{n+1}, \text{ on } \Gamma, \text{ (27)}$$

with no-slip boundary condition on $\Sigma$, where we omitted $\Delta t, k$ in the subscript for simplicity.

For a fixed basis of Galerkin functions, we define the fluid problem function space as follows:

$$W_{k}^{n} := \{(u,v) \in \mathcal{V}_{F}^{n} \times L^{2}_{k}(\Gamma) : \nabla u \cdot v = 0, \text{ } u = v e_{3} \text{ on } \Gamma \}.$$  

For the initial data, we take

$$\nu_{0,k} = v_{0,k} := \sum_{i=1}^{k}(v_{0}, w_{i})w_{i}, \text{ } u_{0,k} = u_{0} := \tilde{u}_{0} - R_{F}(v_{0} - \nu_{0,k})$$

where $\tilde{u}_{0} = u_{0} \circ A_{\nu_{0}}$ and $R_{F}$ is a linear extension operator from $H^{s}(\Gamma), s < 1/2$, to $\mathcal{V}_{F}^{n}$ (see section 5.1 and extension of function $r_{1}$). Notice that we had to modify $u_{0}$ since the change of the initial displacement affects the compatibility conditions, the above choice makes the compatibility conditions being preserved for the approximated problem.

Omitting $\Delta t, k$ in the subscript and simplifying the notation $\nabla_{n} := \nabla \eta^{n}$, corresponding to (26)-(28), we now define the following weak form of the fluid sub-problem (FSP):
Assume that \( \tilde{\eta}^{n+1} \in H^2_0(\Gamma) \) and \( \tilde{\partial}_t \eta^{n+1} \in L^2_k(\Gamma) \) are given already, the problem is to find \((u^{n+1}, v^{n+1}) \in W_k^{\tilde{\eta}^{n+1}}\) such that

\[
\begin{align*}
\frac{1}{2} \int_\Omega J^n((u^n - w^{n+1}) \cdot \nabla^{n+1}) u^{n+1} \cdot q - ((u^n - w^{n+1}) \cdot \nabla^{n+1}) q \cdot u^{n+1} + \\
\int_\Omega J^n u^{n+1} - J^n \frac{1}{\Delta t} u^{n+1} \cdot q + \frac{1}{2} \int_\Omega J^{n+1} - J^n u^{n+1} \cdot q + \\
2 \mu \int_\Omega J^{n+1}(u^{n+1}) : D^{n+1}(q) + \int_\Gamma v^{n+1} - \tilde{\partial}_t \eta^{n+1} \frac{1}{\Delta t} \psi = 0,
\end{align*}
\]

for all \((q, \psi) \in W_k^{\tilde{\eta}^{n+1}}\).

We define piece-wise stationary solution on the whole time interval \([0, T]\):

\[
\begin{align*}
u^{\Delta t,k}(t) := v^{\Delta t,k}, & \quad v^{\Delta t,k}(t) := v^{\Delta t,k}, \quad \text{for } t \in [(n-1)\Delta t, n\Delta t), \quad 1 \leq n \leq N,
\end{align*}
\]

We will sometimes use the notation \((u^{n+1}, v^{n+1}) = FSP(\tilde{\partial}_t \eta^{n+1}, \tilde{\eta}^{n+1}, u^n)\) to denote the solution of \((29)\) and for \(N = T/\Delta t\) and \(k\), we will write the solution

\[
X_{N,k} = (u^{\Delta t,k}, \eta^{\Delta t,k}, v^{\Delta t,k})^T
\]
determined by \((25)\) and \((29)\), inductively.

**Figure 1:** The diagram of the solving procedure.

---

**Remark 3.1.** (1) We defined the structure sub-problem by using the Galerkin approximation and this way we created a hybrid approximation scheme, with fluid being piece-wise stationary and plate displacement being continuous in time (and plate displacement velocity being piece-wise continuous in time). Since these two sub-problems are communicating, we had to modify the data that structure sub-problem sends to fluid sub-problem from continuous in time to stationary, because the fluid sub-problem cannot be solved with continuous in time information. This creates a difference between functions and their averages which needs to be taken into consideration later.

(2) In the original problem \((1)-(10)\), the plate displacement velocity \(\partial_t \eta e_3\) and the trace of the fluid \(u(t, X, \eta(t, X))\) are equal due to the kinematic boundary condition \((7)\). However, in general, this is not true for approximate solutions determined from the approximate
problems. In previous work [17 - 21], their difference goes to zero by an estimate derived from the discretized energy inequalities of sub-problems (see for example [19] Proposition 4), while in the present work the situation is more complicated. We will study this difference in detail in section 3.3 by choosing a special discretization step depending on the number of Galerkin basis functions and some other parameters.

### 3.2 Solutions of (SSP), (FSP) and discrete energy estimates

We will first define appropriate version of energy for each time interval and basis of functions for \(n\Delta t \leq t \leq (n+1)\Delta t:

\[
S^0_{\Delta t,k}(t) = \frac{1}{2\Delta t} \int_{n\Delta t}^{t} ||\partial_t \eta_{\Delta t,k}^n(t)||^2_{L^2(\Gamma)} + \frac{1}{2} \Delta \eta_{\Delta t,k}^n(t) ||^2_{L^2(\Gamma)} + \Pi(\eta_{\Delta t,k}^n(t)) \\
+ \frac{1-n\Delta t}{2\Delta t} \int_{\Omega} J^{n-1} \left|\left| u_{\Delta t,k}^{n-1} \right|\right|^2 \\
F^0_{\Delta t,k}(t) = \frac{1}{2\Delta t} \int_{n\Delta t}^{t} ||v_{\Delta t,k}^n(t)||^2_{L^2(\Gamma)} + \frac{1}{2} \Delta \eta_{\Delta t,k}^{n+1}(t) ||^2_{L^2(\Gamma)} + \Pi(\eta_{\Delta t,k}^{n+1}(t)) \\
+ \frac{1-n\Delta t}{2\Delta t} \int_{\Omega} J^n \left|\left| u_{\Delta t,k}^n \right|\right|^2 \\
D^0_{\Delta t,k} = \Delta t \mu \int_{\Omega} J^n D(u_{\Delta t,k}^n) : D(u_{\Delta t,k}^n)
\]

Notice that the last term in \(S^0_{\Delta t,k}(t)\) has subscript \(n - 1\) instead of \(n\). We will omit \(\Delta t, k\) of the subscript throughout most of this section.

**Lemma 3.1.** For given function \(v^n \in L^\infty(0,T; L^2(\Gamma))\), the solution \(\eta^{n+1}(t) = \sum_{i=1}^{k} \alpha_{i,n+1}(t)w_i\) of (SSP) satisfies the following a priori estimates:

\[
\frac{1}{2\Delta t} \int_{n\Delta t}^{t} ||\partial_t \eta^{n+1} - v^n||^2_{L^2(\Gamma)} + S^{n+1}(t) = F^n(n\Delta t), \tag{30}
\]

and

\[
\frac{t-n\Delta t}{2\Delta t} \int_{\Omega} J^n \left|\left| u_{\Delta t,k}^n \right|\right|^2 + \frac{1}{2\Delta t} \int_{n\Delta t}^{t} \left( ||\partial_t \eta^{n+1} - v^n||^2_{L^2(\Gamma)} + ||\partial_t \eta^{n+1}||^2_{L^2(\Gamma)} \right) \\
+ \n||\Delta \eta^{n+1}(t)||^2_{L^2(\Gamma)} \leq C^* + F^n(n\Delta t), \tag{31}
\]

with \(n\Delta t \leq t \leq (n+1)\Delta t\).

**Proof.** In (29), by taking \(\psi = \alpha_{i,n+1}(t)w_i\) and suming over \(i = 1, ..., k\), it follows

\[
\frac{1}{2\Delta t} \int_{n\Delta t}^{t} \left( ||\partial_t \eta^{n+1} - v^n||^2_{L^2(\Gamma)} + ||\partial_t \eta^{n+1}||^2_{L^2(\Gamma)} \right) \\
+ \frac{1}{2} \frac{d}{dt} \Pi(\eta^{n+1}(t)) = \frac{1}{2\Delta t} ||v^n||^2_{L^2(\Gamma)},
\]

by using \(\frac{d}{dt} \Pi(\eta) = (J(\eta), \partial_t \eta)\), and \(2(a - b)a = ((a - b)^2 + a^2 - b^2)\). Integrating this equality on \([n\Delta t, t]\) and adding \(\int_{\Omega} J^n \left|\left| u^n \right|\right|^2\) on both sides, we obtain (30), and using the coercivity property of potential (20), the estimate (31) follows immediately.

**Lemma 3.2.** For function \(v^n \in L^\infty(0,T; L^2(\Gamma))\), the problem (SSP) has a unique solution \(\eta^{n+1}(t) = \sum_{i=1}^{k} \alpha_{i,n+1}(t)w_i\) in \(C^1([n\Delta t, (n+1)\Delta t]; L^2(\Gamma)) \cap C([n\Delta t, (n+1)\Delta t]; H^1_0(\Gamma))\).
Proof. In the equation (25), by choosing \( \psi = w_i \) with \( i = 1, \ldots, k \), we obtain that \( \alpha_{n+1}(t) = (\alpha_{1,n+1}, \ldots, \alpha_{k,n+1})^T \) satisfies the following equation,

\[
\alpha'_{n+1}(t) + \Delta \Xi \alpha_n(t) + \Delta T F(\alpha_{n+1}(t)) = V^n,
\]

where \( \Xi = \text{diag}(\xi_1, \ldots, \xi_k) \).

By the Korn inequality (see [4, Chapter 3, Theorem 3.1], Lemma 3.3).

Proof. In the equation (25), by choosing \( \psi = w_i \) with \( i = 1, \ldots, k \), we obtain that \( \alpha_{n+1}(t) = (\alpha_{1,n+1}, \ldots, \alpha_{k,n+1})^T \) satisfies the following equation,

\[
\alpha'_{n+1}(t) + \Delta \Xi \alpha_n(t) + \Delta T F(\alpha_{n+1}(t)) = V^n,
\]

where \( \Xi = \text{diag}(\xi_1, \ldots, \xi_k) \).

To solve the above equation, let us verify that \( F(\alpha(t)) \) is a Lipschitz function with the Lipschitz constant being uniform in \( t \). For two functions \( f = \sum_{i=1}^{n} f_i(t) w_i \) and \( g = \sum_{i=1}^{n} g_i(t) w_i \) such that \( ||f||_{H^2(\Gamma)}, ||g||_{H^2(\Gamma)} \leq R \) we have:

\[
|F(f(t)) - F(g(t))| \leq C_R ||f(t) - g(t)||_{H^2(\Gamma)} \leq C_R \sum_{i=1}^{k} ||w_i||_{H^2},
\]

with \( |f(t)|_{\infty} = \max_{1 \leq i \leq k} |f_i(t)| \), where we have used \( ||w_i||_{H^2} = \sqrt{\xi_i} \) from the eigenvalue equality \( \xi_i ||w_i||^2_{H^2} = ||w_i||^2_{L^2} \). Since the solution of this equation satisfies a priori estimate (31), we can choose \( R = (C^* + \pi n(\Delta t))/c \) and obtain a uniform Lipschitz constant. Now, from the existence theory for ordinary differential equations, we obtain the unique solution \( \alpha_n \in C([n\Delta t, (n+1)\Delta t + t_0]) \) for certain \( t_0 > 0 \), and from the bound (31) it follows that \( t_0 = \Delta t \).

This finishes the proof.

Lemma 3.3. Let \( J^n \geq c > 0 \). For a given function \( \eta^{n+1} \in C([n\Delta t, (n+1)\Delta t]; L^2(\Gamma)) \cap C([n\Delta t, (n+1)\Delta t]; H^1(\Gamma)) \) there exists a unique solution \( (u^{n+1}, v^{n+1}) \in W_{\mu}^{n+1} \) of (FSP), and it satisfies following inequality:

\[
F^{n+1}((n+1)\Delta t) + \frac{1}{2} \int_{\Omega} J^n ||u^{n+1} - u^n||^2 + \frac{1}{2} ||v^{n+1} - \partial_t \eta^{n+1}||_{L^2(\Gamma)}^2 + D^{n+1} \leq S^{n+1}((n+1)\Delta t).
\]

(32)

Proof. The existence of \( (u^{n+1}, v^{n+1}) \) to (FSP) can be proved in the same way as in [20] by using the Lax-Milgram Lemma. First we want to bound the dissipation term from below by \( H^1(\Omega^{n+1}) \) norm. For this reason we map the velocity \( u^{n+1} \) back to moving domain and by the Korn inequality (see [3 Chapter 3, Theorem 3.1])

\[
||u^{n+1}||_{L^2}^2 \leq \frac{1}{\Delta t} C_{n+1} D^{n+1},
\]

where \( C_{n+1} \) is Korn’s constant that depends on the domain \( \Omega^{n+1} \). The continuity property of the functional terms (in the sense of the Lax-Milgram lemma) and the lower bounds of the remaining terms for the coercivity property are proved straightforward, so the unique solution follows.

To derive the inequality (32), letting \( q = u^{n+1} \) and \( \psi = v^{n+1} \) in [20], it follows

\[
\int_{\Omega} J^n \frac{u^{n+1} - u^n}{\Delta t} \cdot u^{n+1} + \frac{1}{2} \int_{\Omega} \frac{J^{n+1} - J^n}{\Delta t} u^{n+1} \cdot u^{n+1} = \frac{1}{2\Delta t} \int_{\Omega} (J^{n+1} ||u^{n+1}||^2 + J^n ||u^{n+1} - u^n||^2 - J^n ||u^n||^2).
\]
On the other hand, obviously we have
\[
\int_{\Gamma} \frac{v^{n+1} - \tilde{\eta}^{n+1} + \tilde{\eta}^{n+1}}{\Delta t} v^{n+1} = \frac{1}{2\Delta t} \left( \left\| v^{n+1} - \tilde{\eta}^{n+1} \right\|_{L^2(\Gamma)}^2 + \left\| v^{n+1} - \tilde{\eta}^{n+1} \right\|_{L^2(\Gamma)}^2 \right),
\]
and
\[
\left\| \tilde{\eta}^{n+1} \right\|_{L^2(\Gamma)}^2 \leq \frac{1}{\Delta t} \int_{0}^{\Delta t} \left\| \partial_t \tilde{\eta}^{n+1} \right\|_{L^2(\Gamma)}^2,
\]
from the Hölder inequality. Thus, we conclude the inequality (32) immediately.

Now we are ready to obtain the uniform bounds of the approximate solutions as follows.

**Lemma 3.4.** For a given \( \Delta t > 0 \), \( T = N\Delta t \), and \( k \in \mathbb{N} \), let \( \eta_{\Delta t,k}(t), u_{\Delta t,k}(t), v_{\Delta t,k}(t) \) be the solutions of (SSP) and (FSP) given in Lemmas 3.2 and 3.3 respectively on time interval \([0,T]\). Then, one has:

1. For all \( 0 \leq n \leq N \),
\[
F^n_{\Delta t,k}, S^n_{\Delta t,k}(t) \leq C_0 = E(0) + C(\Pi, \eta_0),
\]
where \( E(0) \) is the initial energy, and the constant \( C(\Pi, \eta_0) \) is given in (A3);
2. \( \eta_{\Delta t,k} \) is bounded in \( L^\infty(0,T; H_0^2(\Gamma)) \) uniformly with respect to \( \Delta t, k \);
3. \( \partial_t \eta_{\Delta t,k} \) is bounded in \( L^2(0,T; L^2(\Gamma)) \) uniformly with respect to \( \Delta t, k \);
4. \( v_{\Delta t,k} \) is bounded in \( L^\infty(0,T; L^2(\Gamma)) \) uniformly with respect to \( \Delta t, k \);
5. \( u_{\Delta t,k} \) is bounded in \( L^\infty(0,T; L^2(\Omega)) \) uniformly with respect to \( \Delta t, k \);
6. The following estimate
\[
\sum_{n=0}^{N-1} \left( \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} \left\| \partial_t \eta_{\Delta t,k}^{n+1} - v_{\Delta t,k}^{n+1} \right\|_{L^2(\Gamma)}^2 + \left\| v_{\Delta t,k}^{n+1} - \int_{\Omega} J^\ast_{\Delta t,k} u_{\Delta t,k}^{n+1} - u_{\Delta t,k}^{n+1} \right\|_{L^2(\Gamma)}^2 \right) + D^n_{\Delta t,k} \leq C,
\]
holds.

**Proof.** The estimates given in (1) follow from (31) and (32). The boundedness given in (2) and (3) comes from (31) and first inequality given in (1). By taking out \( \Pi(\eta^{n+1}((n+1)\Delta t)) \) in (32) from both sides, we obtain
\[
\frac{1}{2} \left( \int J^n |u_{\Delta t,k}^{n+1} - u_{\Delta t,k}^n|^2 + \int J^{n+1} |u_{\Delta t,k}^{n+1}|^2 + \left\| v_{\Delta t,k}^{n+1} - \tilde{\eta}_{\Delta t,k}^{n+1} \right\|_{L^2(\Gamma)}^2 + \left\| v_{\Delta t,k}^{n+1} \right\|_{L^2(\Gamma)}^2 \right) + D^n_{\Delta t,k}
\leq S^{n+1}((n+1)\Delta t) - \Pi(\eta_{\Delta t,k}^{n+1}((n+1)\Delta t)) \leq \frac{1}{c}(C^* + F^n),
\]
where last inequality comes from (31), \( c = (\frac{1}{\eta} - \kappa) \) with \( \kappa, C^* \) being given in the coercivity estimate from (A3). This yields the boundedness given in (4) and (5). The estimate given in (6) is obtained by summing (31) and (32) over all \( 1 \leq n \leq N \) and by telescoping.

### 3.3 Estimate of \( \partial_t \eta_{\Delta t,k} - v_{\Delta t,k} \)

Since we are working with two parameters \( k \) and \( N = T/\Delta t \) in constructing the approximate solutions, we want to pass the convergence of the approximate solutions at the same time in both parameters. In order to ensure that \( \partial_t \eta_{\Delta t,k} \) and \( v_{\Delta t,k} \) converge to the same function in
\( L^2(0, T; L^2(\Gamma)) \), we want \( N = N(k) \) to be sufficiently large in every step. As a preparation, we first derive some estimates on \( \partial_t \eta_{\Delta t,k} - v_{\Delta t,k} \).

For any function \( f \in H_{0, k}^2(\Gamma) \), by using the orthogonality of \( \Delta w_1, ..., \Delta w_k \) in \( L^2(\Gamma) \), we have:

\[
||f||_{H^2_z}^2 \leq C_T ||\Delta f||_{H^2_z}^2 = C_T \sum_{i=1}^{k} (f, w_i) \Delta w_i \Delta w_i = C_T \sum_{i=1}^{k} \xi_i (f, w_i)^2.
\]

Now taking \( \psi = w_i \) in (33), we obtain the inequality

\[
\frac{1}{(\Delta t)^2} ||\partial_t \eta^{n+1}(t) - v^n, w_i||_{H^2_z(\Gamma)}^2 \leq 2(\Delta t)^2 ||\partial_t \eta^{n+1}(t), \Delta w_i||^2 + 2(\mathcal{F}(\eta^{n+1}(t)) - \mathcal{F}(0) + \mathcal{F}(0), w_i)^2
\]

\[
\leq 2\xi^2 ||\eta^{n+1}(t), w_i||^2 + 2C_R^2 (||\mathcal{F}(0)||_{H^{-a}} + ||\eta^{n+1}(t)||_{H^2})^2 ||w_i||_{H^a}^2.
\]

by using the Lipschitz continuity of \( \mathcal{F} \) given in the assumption (A2), where \( C_R \) is the Lipschitz constant given in (A2), which is uniform due to the boundedness of \( \eta_{\Delta t,k} \) given in Lemma 3.4(2).

Summing (35) over \( i = 1, ..., k \), it follows

\[
\frac{1}{(\Delta t)^2} ||\partial_t \eta^{n+1}(t) - v^n||_{H^2_z(\Gamma)}^2 \leq 2\sum_{i=1}^{k} \xi_i^2 ||\eta^{n+1}(t), w_i||^2 + 2C_R^2 (||\mathcal{F}(0)||_{H^{-a}} + ||\eta^{n+1}(t)||_{H^2})^2 \sum_{i=1}^{k} ||w_i||_{H^a}^2
\]

\[
\leq 2\xi_k ||\Delta \eta^{n+1}||_{H^2_z}^2 + 2C_R^2 (||\mathcal{F}(0)||_{H^{-a}} + ||\eta^{n+1}(t)||_{H^2})^2 \sum_{i=1}^{k} ||w_i||_{H^a}^2,
\]

by using (33).

From the interpolation inequality for the Sobolev spaces, we have

\[
||w_n||_{H^a}^2 \leq ||w_n||_{L^2(\Gamma)}^2 ||\eta||_{H^a} \leq C_T \xi_a^2,
\]

and from the uniform bound for \( \eta_{\Delta t,k} \) given in Lemma 3.4(2), we obtain

\[
\frac{1}{(\Delta t)^2} ||\partial_t \eta^{n+1}(t) - v^n||_{L^2(\Gamma)}^2 \leq 2\xi_k \frac{C_T}{c} (C^* + C_0) + 4C_R^2 C_T (||\mathcal{F}(0)||_{H^{-a}} + C_B) \sum_{i=1}^{k} \xi_i^2
\]

where \( c = (\frac{1}{2} - \kappa) \), with \( \kappa \) and \( C^* \) being the constants given in the coercivity condition of (20). Denoting by

\[
C_B := \frac{C_T}{c} (C^* + C_0)
\]

we have

\[
||\partial_t \eta^{n+1}(t) - v^n||_{L^2(\Gamma)}^2 \leq \left( 2\xi_k C_B + 4C_R^2 C_T (||\mathcal{F}(0)||_{H^{-a}} + C_B) \sum_{i=1}^{k} \xi_i^2 \right) (\Delta t)^2.
\]

In order to make the right-hand side of (39) be bounded by \((\Delta t)^{\alpha} \) for \( \alpha > 0 \), we impose the following condition for every step

\[
N(k) \geq T \left( 2\xi_k C_B + 4C_R^2 C_T (||\mathcal{F}(0)||_{H^{-a}} + C_B) \sum_{i=1}^{k} \xi_i^2 \right)^{\frac{1}{\alpha}}, \quad \alpha > 0.
\]
Remark 3.2. In appendix, for the corresponding two-dimensional problem we will derive a more precise lower bound of $N(k)$ by using the exact solutions of the one-dimensional biharmonic eigenvalue problem, see [63]-[65].

Now we are ready to have:

Lemma 3.5. Assuming that $N(k)$ satisfies (40), we have the following boundedness:

1. $\|\partial_t \eta^{n+1}(t) - v_{\Delta t,k}^n\|_{L^2(\Gamma)} \leq (\Delta t)^{2\beta}$, for all $t \in [0,\Delta t]$, where $\beta = \alpha/(1+\alpha)$;
2. $\partial_t \eta_{\Delta t,k}$ is bounded in $L^\infty(0,T;L^2(\Gamma))$ uniformly with respect to $\Delta t, k$;
3. $\nabla u_{\Delta t,k}$ is bounded in $L^2(0,T;L^2(\Omega))$ uniformly with respect to $\Delta t, k$.

Proof. From (40) and (49), the above first and second estimates follow immediately from the uniform bound of $v_{\Delta t,k}$ given in Lemma 3.4(4).

The third result is obtained in the same way as in [17] Proposition 5.3, by mapping the gradient back to the moving domain $\tilde{\Omega}$ and applying the transformed Korn’s inequality. We then use uniform bounds for $\partial_t \eta_{\Delta t,k}$ in $L^\infty(0,T;H^1_0(\Gamma))$ and $\eta_{\Delta t,k}$ in $L^\infty(0,T;H^2(\Gamma))$ to obtain the uniform Korn's constant, and since $\sum_{i=1}^N D^n_{\Delta t,k}$ is uniformly bounded from Lemma 3.4(6), we finish the proof of this lemma. \qed

Let us now estimate the difference between $\hat{\partial}_t \eta^n$ and $\partial_t \eta^n$. We first take the difference between equation (2.1) for $t$ and $\tau$–moments in $[n\Delta t, (n+1)\Delta t]$, then choose $\psi = w_i$ and sum over $i = 1, \ldots, k$. Following the calculation in (33) and (35), we have

$$\frac{1}{(\Delta t)^2} \|\partial_t \eta^n(t) - \partial_t \eta^n(\tau)\|^2_{L^2(\Gamma)} \leq 2\|\eta^n(t) - \eta^n(\tau)\|^2_{L^2(\Gamma)} + 2C_R^2 \|\eta^n(t) - \eta^n(\tau)\|^2_{H^2} \sum_{i=1}^k \|w_i\|^2_{H^2},$$

and from (33) and (36),

$$\frac{1}{(\Delta t)^2} \|\partial_t \eta^n(t) - \partial_t \eta^n(\tau)\|^2_{L^2(\Gamma)} \leq \xi_k C_R \|\eta^n(t) - \eta^n(\tau)\|^2_{L^2(\Gamma)} \left(2 + C_R^2 \sum_{i=1}^k \xi_i^{a/2}\right).$$

Since $\eta^n \in C^1([n\Delta t, (n+1)\Delta t]; L^2(\Gamma))$, we obtain

$$\frac{1}{(\Delta t)^2} \|\partial_t \eta^n(t) - \partial_t \eta^n(\tau)\|^2_{L^2(\Gamma)} \leq \xi_k C_B \left(2 + C_R^2 \sum_{i=1}^k \xi_i^{a/2}\right) |t - \tau|^2$$

From the Hölder inequality we have

$$\left| \int_{n\Delta t}^{(n+1)\Delta t} \frac{\partial_t \eta^{n+1}(t) - \partial_t \eta^{n+1}(\tau)}{\Delta t} dt \right|^2 \leq \left( \int_{n\Delta t}^{(n+1)\Delta t} \left| \partial_t \eta^{n+1}(t) - \partial_t \eta^{n+1}(\tau) \right|^2 dt \right)^{1/2}$$

Now by integrating this inequality in $L^2(\Gamma)$, and (42) over $[n\Delta t, (n+1)\Delta t]$ we obtain

$$\frac{1}{(\Delta t)^2} \|\tilde{\partial}_t \eta^n - \partial_t \eta^n\|_{L^2(\Gamma)}^2 \leq \xi_k C_B \left(2 + C_R^2 \sum_{i=1}^k \xi_i^{a/2}\right) \int_{n\Delta t}^{(n+1)\Delta t} |t - \tau|^2 dt$$

From (40), we can estimate

$$(\Delta t)^{2\beta} \geq \xi_k C_B \text{ and } \frac{1}{4CB} (\Delta t)^{2\beta} \geq C_R^2 C_T \sum_{i=1}^k \xi_i^{a/2}$$

which gives us

$$C_{R,T,B}(\Delta t) \geq \xi_k C_B \left(2 + C_R^2 C_T \sum_{i=1}^k \xi_i^{a/2}\right),$$

(46)
where
\[ C_{R,Γ,B} := \frac{2 + C^2_R C_Γ}{4C_B C^2_R C_Γ} \]

Now combining (44) and (46), we have
\[ ||\tilde{\partial}_t \eta^n - \partial_t \eta^n||_{L^2(Γ)}^2 \leq C_{R,Γ,B} 3 (∆t)^4 \beta^{+1}, \quad \beta = \alpha/(1 + \alpha). \] (47)

Using Lemma 3.4(6) and inequality (47), we obtain:
\[ \sum_{n=1}^{N-1} ||v^{n+1} - v^n||_{L^2(Γ)}^2 \leq C. \] (48)

4 Convergence of the approximate solutions

In this section, by using the estimates from section 3, we will obtain the weak convergence of the approximate solutions, and by proving some additional regularity in time that comes from the time discretization (integral equicontinuity), we will deduce the strong convergence as well.

4.1 Weak convergence

In order to prove the weak convergence of approximate solutions, we first need to prove that LE mapping defined in section 2.3 is well-defined independently from the choice of ∆t and k on the whole time interval [0, T]:

**Lemma 4.1.** Assuming that N(k) satisfies (40), we can choose T > 0 independent of ∆t and k such that for all \( t \in [0, T] \)
\[ 0 < C_{min} \leq 1 + η_{\Delta t,k}(t, X) \leq C_{max} \]

**Proof.** Using the uniform estimate of \( \partial_t η_{\Delta t,k} \) given in Lemma 3.4(3), we first obtain:
\[ ||η_{\Delta t,k}(t) - η_{\Delta t,k}(0)||_{L^2(Γ)} \leq || \int_0^t \partial_t η_{\Delta t,k}(h)dh||_{L^2(Γ)} \leq CT, \]
and from the uniform estimate of \( η_{\Delta t,k} \) in \( L^∞(0, T; H^3_0(Γ)) \) given in Lemma 3.4(2), we have:
\[ ||η_{\Delta t,k}(t) - η_{\Delta t,k}(0)||_{H^3_0(Γ)} \leq 2C. \]

By the interpolation inequality for Sobolev spaces we have:
\[ ||η_{\Delta t,k}(t) - η_{\Delta t,k}(0)||_{H^{3/2}(Γ)} \leq 2CT^{1/4}. \]

Since \( H^{3/2}(Γ) \) is imbedded into \( C(Γ) \), we can bound
\[ η_0 + 1 + 2CT^{1/4} \geq η_{\Delta t,k}(t) + 1 \geq η_0 + 1 - 2CT^{1/4}. \]

Since \( η_0 + 1 \geq C > 0 \), we can take \( T \) small enough such that
\[ η_{\Delta t,k} + 1 \geq C - 2CT^{1/4} \geq C_{min} > 0, \]
and now the upper bound \( C_{max} \) follows easily. This finishes the proof.
Notice that now on the time interval \([0, T]\) the Jacobian of the LE mapping \(A_p(t)\) is uniformly bounded from above and below by two positive constants.

The following result is a direct consequence of boundedness given in Lemma 3.4.

**Lemma 4.2.** Assuming that \(N(k)\) satisfies (11), there exist subsequences of \((u_{\Delta t,k})_{\Delta t,k}, (v_{\Delta t,k})_{\Delta t,k}\) and \((\eta_{\Delta t,k})_{\Delta t,k}\), and the functions \(v \in L^\infty(0, T; L^2(\Omega)), \eta \in W^{1,\infty}(0, T; L^2(\Gamma)) \cap L^\infty(0, T; H_0^2(\Gamma))\) and \(u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))\) such that:

\[
\begin{align*}
\eta_{\Delta t,k} & \rightharpoonup \eta \quad \text{weakly* in } L^\infty(0, T; H_0^2(\Gamma)), \\
\partial_t \eta_{\Delta t,k} & \rightharpoonup \partial_t \eta \quad \text{weakly* in } L^\infty(0, T; L^2(\Gamma)) , \\
v_{\Delta t,k} & \rightharpoonup v \quad \text{weakly* in } L^\infty(0, T; L^2(\Gamma)), \\
u_{\Delta t,k} & \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; L^2(\Omega)), \\
\nabla \eta_{\Delta t,k} & \rightharpoonup M \quad \text{weakly in } L^2(0, T; L^2(\Omega)) ,
\end{align*}
\]

as \(k \to +\infty\).

We will later prove that \(M\) is indeed the weak limit of the gradient when we introduce the approximated test functions.

### 4.2 Strong convergence

We want to prove that the fluid velocity \(u_{\Delta t,k}\), plate displacement \(\eta_{\Delta t,k}\), and the displacement velocities \(v_{\Delta t,k}\) and \(\tilde{\partial}_t \eta_{\Delta t,k}\) posses additional time regularity that comes from integral equicontinuity in time in order to prove the compactness. This approach was also used in [19] for the piece-wise linear and piece-wise stationary functions. Our case is slightly different since \(\partial_t \eta_{\Delta t,k}\) is piece-wise continuous in time with jumps at every point \(n\Delta t\). Fortunately, we will be able to estimate the change of \(\partial_t \eta_{\Delta t,k}\) on every sub-interval and control the jumps from one interval to another and this will give us the integral equicontinuity and therefore time regularity.

Define the translation in time operator as

\[ T_h f(t, \cdot) := f(t - h, \cdot), \quad h \in \mathbb{R}, \quad h \leq t \leq T. \]

**Lemma 4.3.** Assuming that \(N(k)\) satisfies (11), there exists a \(C > 0\) independent of \(\Delta t\) and \(k\) such that:

\[
\begin{align*}
\|T_h u_{\Delta t,k} - u_{\Delta t,k}\|_{L^2(h, T; L^2(\Omega))} & \leq C\sqrt{h}, \\
\|T_h v_{\Delta t,k} - v_{\Delta t,k}\|_{L^2(h, T; L^2(\Gamma))} & \leq C\sqrt{h}, \\
\|T_h \partial_t \eta_{\Delta t,k} - \partial_t \eta_{\Delta t,k}\|_{L^2(h, T; L^2(\Omega))} & \leq C\sqrt{h}, \\
\|T_h \tilde{\partial}_t \eta_{\Delta t,k} - \tilde{\partial}_t \eta_{\Delta t,k}\|_{L^2(h, T; L^2(\Gamma))} & \leq C\sqrt{h}.
\end{align*}
\]

**Proof.** First we are going to prove \(49\). If \(h = \Delta t\), then by Lemma 3.4(6), we have:

\[ \|T_h u_{\Delta t,k} - u_{\Delta t,k}\|_{L^2(h, T; L^2(\Omega))} = \Delta t \sum_{i=1}^{N-1} \|u_{i+1,\Delta t,k} - u_{i,\Delta t,k}\|_{L^2(\Gamma)} \leq C\Delta t. \]

If \(h < \Delta t\), then functions \(u_{\Delta t,k}\) and \(T_h u_{\Delta t,k}\) are equal on every interval \([n\Delta t, (n+1)\Delta t - h]\), so:

\[ \|T_h u_{\Delta t,k} - u_{\Delta t,k}\|_{L^2(h, T; L^2(\Gamma))} = h \sum_{i=1}^{N-1} \|u_{i+1,\Delta t,k} - u_{i,\Delta t,k}\|_{L^2(\Gamma)} \leq Ch. \]
When \( h \in [n\Delta t, (n+1)\Delta t) \), let \( \tilde{h} = (n+1)\Delta t - h \). By triangle inequality, we have:

\[
||T_h u_{\Delta t, k} - u_{\Delta t, k}||^2_{L^2(h,T;L^2(\Gamma))} \leq \sum_{i=1}^{n} ||T_i u_{\Delta t, k} - T_{i-1} u_{\Delta t, k}||^2_{L^2(h,T;L^2(\Gamma))} + C\tilde{h} \leq C(n\Delta t + \tilde{h}) = Ch,
\]

by previous two inequalities.

Since \( u_{\Delta t, k} \) is piece-wise stationary and satisfies inequality (48), the inequality (50) follows in the same way as above.

For (51), if \( h < \Delta t \) and \( s \in [n\Delta t + h, (n+1)\Delta t] \), by taking the difference of (25) for \( s \) and \( s - h \), as we did in (42), we have:

\[
||T_h \partial_t \eta_{\Delta t, k}(s) - \partial_t \eta_{\Delta t, k}(s)||_{L^2(\Gamma)} \leq C(\Delta t)^3 h.
\]

If \( h = \Delta t \), then by triangle inequality and (48) and Lemma 4.4.6, we have:

\[
\sum_{n=1}^{N-1} ||\partial_t \eta_{\Delta t, k}^n(t) - \partial_t \eta_{\Delta t, k}(t)||^2_{L^2(\Gamma)} \leq \sum_{n=1}^{N-1} ( ||\partial_t \eta_{\Delta t, k}^n(t) - v_{\Delta t, k}^n||^2_{L^2(\Gamma)} + ||\partial_t \eta_{\Delta t, k}(t) - v_{\Delta t, k}^{n-1}||^2_{L^2(\Gamma)} ) + \sum_{n=1}^{N-1} ||v_{\Delta t, k}^n - v_{\Delta t, k}^{n-1}||^2_{L^2(\Gamma)} \leq C.
\]

Now, using the same argument as for \( u_{\Delta t, k} \), we obtain:

\[
||T_h \partial_t \eta_{\Delta t, k}(s) - \partial_t \eta_{\Delta t, k}(s)||^2_{L^2(h,T;L^2(\Gamma))} \leq Ch.
\]

The inequality (52) follows from the bound for \( \tilde{\partial}_t \eta_{\Delta t, k} \) and \( \partial_t \eta_{\Delta t, k} \) in (43).

Now, we are ready to prove time regularity results as follows:

**Lemma 4.4.** Assuming that \( N(k) \) satisfies (40), for \( s < 1/2 \) we have:

1. \( (u_{\Delta t, k})_{\Delta t, k} \) is uniformly bounded in \( H^s(0, T; L^2(\Omega)) \cap L^2(0, T; H^{2s}(\Omega)) \);
2. \( (\partial_t \eta_{\Delta t, k})_{\Delta t, k} \) and \( (\tilde{\partial}_t \eta_{\Delta t, k})_{\Delta t, k} \) are uniformly bounded in \( H^s(0, T; L^2(\Gamma)) \cap L^2(0, T; H^{2s/3}(\Gamma)) \);
3. \( \eta_{\Delta t, k} \) is uniformly bounded in \( H^{s+1}(0, T; L^2(\Gamma)) \);
4. \( (v_{\Delta t, k})_{\Delta t} \) is uniformly bounded in \( H^s(0, T; L^2(\Gamma)) \cap L^2(0, T; H^s(\Gamma)) \);

**Proof.** To prove the first boundedness given in (1), we will prove that following intrinsic semi-norm (see [11 Chapter 7]) is finite:

\[
||u_{\Delta t, k}||^2_{H^s(0,T;L^2(\Omega))} = \int_0^T \int_0^T \frac{||u_{\Delta t, k}(t) - u_{\Delta t, k}(\tau)||^2_{L^2(\Omega)}}{|t - \tau|^{1+2s}} d\tau dt.
\]

By a change of variables \( h = t - \tau \) and using Lemma 4.3 we get that

\[
||u_{\Delta t, k}||^2_{H^s(0,T;L^2(\Omega))} = \int_{-T}^T \int_{-T}^T \frac{1}{|h|^{1+2s}} ||u_{\Delta t, k}(t - h) - u_{\Delta t, k}(t)||^2_{L^2(\Omega)} dtdh \leq \int_{-T}^T \frac{C|h|}{|h|^{1+2s}} dh.
\]

The last integral is finite for \( s < 1/2 \), so the first boundedness is proved.

From Lemma 4.3 the same follows for functions \( \tilde{\partial}_t \eta_{\Delta t, k}, v_{\Delta t, k} \) and \( \partial_t \eta_{\Delta t, k} \), so we obtain the first statements given in (2) and (4). By the uniform bound for \( \eta_{\Delta t, k} \) in \( L^2(0,T; H^2_0(\Gamma)) \cap H^{s+1}(0, T; L^2(\Gamma)) \) and the interpolation property of the Sobolev spaces (see [15] Section 1.9.4 p. 47)

\[ \eta_{\Delta t, k} \text{ is uniformly bounded in } H^{\alpha(s+1)}(0, T; H^{2-2\alpha}(\Gamma)), \quad \alpha = 1/(s+1). \]

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Now we obtain that $\eta_{\Delta t,k}$ is uniformly bounded in $H^1(0,T; H^{4s/3}(\Gamma))$, $s < 1/2$. This gives the $L^2(0,T; H^{4s/3}(\Gamma))$ regularity for $\partial_t \eta_{\Delta t,k}$ and the same follows for $\tilde{\partial}_t \eta_{\Delta t,k}$.

Before proving the second boundedness of $u_{\Delta t,k}$ given in (1), let us recall that the boundary $\Gamma^{\alpha_{l,k}}(t)$ is just $C^{0,\alpha}$, $\alpha < 1$. Denote the fluid velocity on the original domain by $\hat{u}_{\Delta t,k}(t,X,z) = u_{\Delta t,k}(t,X,A_{\Delta t,k}^{-1}(t)(X,z))$. Since
\[
\begin{align*}
 u(t,X,z) &= \hat{u}(t,X,A(t)(X,z)),
\end{align*}
\]
using the trace results (cf. [16]) we obtain
\[
||v_{\Delta t,k}||^2_{L^2(0,T; H^s(\Gamma))}, ||u_{\Delta t,k}||^2_{L^2(0,T; H^{2s}(\Omega))} \leq C ||\hat{u}_{\Delta t,k}||^2_{L^2(0,T; H^s(\Gamma^{\alpha_{l,k}}(t)))},
\]
for $s < 1/2$. The term on the right side is bounded from Lemma 3.5 and this finishes the proof.

**Remark 4.1.** Since we essentially want that the sum of equations for (FSP) and (SSP) to converge to original plate equation in weak form, it is of key importance that functions $\partial_t \eta_{\Delta t,k}$ and $v_{\Delta t,k}$ converge to the same function in $L^2(0,T; L^2(\Gamma))$. This is one of the main reasons we use the Galerkin basis. If in the equation (25) we take an arbitrary test function $\eta$, the following holds:
\[
\begin{align*}
\Delta t \int_\Omega (\partial_t \eta_{\Delta t,k} - v_{\Delta t,k}) || v_{\Delta t,k} ||^2_{L^2(\Gamma)} = 0,
\end{align*}
\]
and the same follows for $\tilde{\partial}_t \eta_{\Delta t,k}$.

From Lemma 3.5, we get
\[
\begin{align*}
\Delta t \int_\Omega (\partial_t \eta_{\Delta t,k} - v_{\Delta t,k}) || v_{\Delta t,k} ||^2_{L^2(\Gamma)} = 0,
\end{align*}
\]
and the same follows for $\tilde{\partial}_t \eta_{\Delta t,k}$.

**Lemma 4.5.** Assuming that $N(k)$ satisfies (10) and $s < 1/2$, the sets $\{u_{\Delta t,k} : T/\Delta t = N \in \mathbb{N}\}$ and $\{\partial_t \eta_{\Delta t,k} : T/\Delta t = N \in \mathbb{N}\}$ are relatively compact in $L^2(0,T; H^{2s}(\Omega))$ and $L^2(0,T; H^{4s/3}(\Gamma))$, respectively, implying the following:
\[
(1) \quad u_{\Delta t,k} \to u \quad \text{in} \quad L^2(0,T; H^{2s}(\Omega)), \quad s < 1/2;
\]
\[
(2) \quad \partial_t \eta_{\Delta t,k} \to \partial_t \eta \quad \text{in} \quad L^2(0,T; H^{4s/3}(\Gamma)), \quad s < 1/2;
\]
\[
(3) \quad \tilde{\partial}_t \eta_{\Delta t,k} \to \tilde{\partial}_t \eta \quad \text{in} \quad L^2(0,T; H^{4s/3}(\Gamma)), \quad s < 1/2;
\]
\[
(4) \quad v_{\Delta t,k} \to \partial_t \eta \quad \text{in} \quad L^2(0,T; H^{2s}(\Gamma)), \quad s < 1/2.
\]

**Proof.** The only thing we need to prove is that the limiting function of $v_{\Delta t,k}$ is indeed $\partial_t \eta$. From Lemma 3.5 and from integral equicontinuity for $v_{\Delta t,k}$, we have:
\[
\begin{align*}
\int_0^T ||\partial_t \eta_{\Delta t,k} - v_{\Delta t,k}||^2_{L^2(\Gamma)} \leq \int_0^T \int_{\Delta t} ||\partial_t \eta_{\Delta t,k} - T_{\Delta t} v_{\Delta t,k}||^2_{L^2(\Gamma)} + \int_0^T ||v_{\Delta t,k} - T_{\Delta t} v_{\Delta t,k}||^2_{L^2(\Gamma)} \leq C(\Delta t)^3 + C\Delta t.
\end{align*}
\]
\[
\begin{align*}
\end{align*}
\]

Since $\eta_{\Delta t,k}$ is uniformly bounded in $W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H^2(\Gamma))$, from the continuous imbedding
\[
W^{1,\infty}(0,T; L^2(\Gamma)) \cap L^\infty(0,T; H^2(\Gamma)) \hookrightarrow C^{0,1-\alpha}([0,T], H^{2\alpha}), \quad 0 < \alpha < 1,
\]
we obtain uniform boundedness of $\eta_{\Delta t,k}$ in $C^{0,1-\alpha}([0,T], H^{2\alpha})$. Since $H^{2\alpha}$ is compactly embedded into $H^{2\alpha-\epsilon}$ for any fixed $\epsilon > 0$, and since the functions in $C^{0,1-\alpha}([0,T], H^{2\alpha}(\Gamma))$
are uniformly continuous in time on finite interval, by using the Arzela-Ascoli theorem we obtain the compactness in time as well. Therefore, we have

\[ \eta_{\Delta t,k} \to \eta \text{ in } C([0,T];H^{2s}(\Gamma)), 0 < s < 1, \]

and

\[ T_{\Delta t}\eta_{\Delta t,k} \to \eta \text{ in } C([0,T];H^{2s}(\Gamma)), 0 < s < 1, \]
as \Delta t \to 0. By the compactness we just obtained for \( \eta \), since \( \mathcal{F} \) is Lipschitz continuous from \( H^{2-\epsilon}(\Gamma) \) to \( H^{-2}(\Gamma) \) (see (18)) and since constant \( C_R \) is now uniformly bounded due to the uniform energy estimate, we have

\[ \|\mathcal{F}(\eta_{\Delta t,k}(t)) - \mathcal{F}(\eta(t))\|_{H^{-2}} \leq C\|\eta_{\Delta t,k}(t) - \eta(t)\|_{H^{2s}}, \]

so \( \mathcal{F}(\eta_{\Delta t,k}(t)) \to \mathcal{F}(\eta(t)) \) in \( H^{-2}(\Gamma) \), for all \( t \in [0,T] \).

**Corollary 4.6.** Assuming that \( N(k) \) satisfies (40), we have the following convergences for \( s < 1/2 \):

1. \( \eta_{\Delta t,k}, T_{\Delta t}\eta_{\Delta t,k} \to \eta \text{ in } C([0,T] \times \Gamma); \)
2. \( \tilde{\eta}_{\Delta t,k} \to \eta \text{ in } L^\infty(0,T;C(\Gamma)); \)
3. \( A_{\Delta t,k} \to A \text{ in } L^\infty(0,T;H^{2+s}(\Omega)); \)
4. \( w_{\Delta t,k} \to w_k \text{ in } L^2(0,T;H^{4s/3+1/2}(\Omega)^3) \text{ (since the trace of } w_{\Delta t,k} \text{ is } \tilde{\partial}_t\eta_{\Delta t,k}; \)
5. \( J_{\Delta t,k}, J_{\Delta t,k} \to J \text{ in } L^\infty(0,T;H^{2s+1/2}(\Omega)); \)
6. \( (\nabla A_{\Delta t,k})^{-1} \to (\nabla A)^{-1} \text{ in } L^\infty(0,T;H^{1+s}(\Omega)), \)

as \( k \to \infty \).

## 5 Proof of the main result

In this section, we will first construct appropriate test functions in certain spaces that will converge to the test functions of the weak solution defined in Definition 2.1. After that, we will prove the convergence of approximate solutions term by term by using the convergence results obtained in section 4. At the end of the section, we will prove that the life span of the solution is either \(+\infty\) or up to the moment of the free boundary touch the bottom. Throughout this section we assume that \( N(k) \) satisfies the condition (40).

### 5.1 Construction of test functions

Now, for given test functions \((q,\psi) \in Q^0(0,T)\), where \( \eta \) is the weak* limit of the sequence \( \eta_{\Delta t,k} \) given in Lemma 5.4, we want to construct a sequence of test functions \( q_{\Delta t,k} \) and \( \psi_{\Delta t,k} \) such that \((q_{\Delta t,k},\psi_{\Delta t,k}) \in W^\infty_k \) for all \( 1 \leq n \leq N(k) \), and \((q_{\Delta t,k},\psi_{\Delta t,k}) \to (q,\psi) \) when \( k \to \infty \) in a suitable space.

Since the original problem and the original test functions are defined on the moving domain, we start by defining the uniform domains

\[ \Omega_{\text{max}} = \bigcup_{\Delta t > 0, n \in \mathbb{N}} \Omega_{\Delta t,k}^{\infty}, \quad \Omega_{\text{min}} = \bigcap_{\Delta t > 0, n \in \mathbb{N}} \Omega_{\Delta t,k}^{\infty} \]

We define \( \chi_{\text{max}}(0,T) \) as the space of test functions \((r,\psi)\), where \( \psi \in C^1_c(0,T;H^2_0(\Gamma)) \) and \( r = r_0 + r_1 \) such that:

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1. \( r_0 \) is a smooth divergence free function with compact support in \( \Omega^0(t) \cup \Sigma \cup \Gamma^0(t) \), extended by 0 to \( \Omega_{\text{max}} \setminus \Omega^0(t) \).

2. function

\[
\mathbf{r}_1 = \begin{cases} 
(0, 0, \psi) & \text{on } \Omega_{\text{max}} \setminus \Omega_{\text{min}} \\
\text{Divergence free extension to } \Omega_{\text{min}} & (\text{see } \text{p}.127) 
\end{cases}
\]

We then define

\[
\chi^0(0, T) = \{(q, \psi) : q(t, X, z) = r(t, X, z)_{|\Omega^0(t)} \circ A_q(t), \text{ where } (r, \psi) \in \chi_{\text{max}}(0, T)\}.
\]

Notice that this function space is dense in \( Q^0(0, T) \).

For approximate solution \( X_{\Delta t, k} \) and given test functions \( (q, \psi) \in \chi^0(0, T) \), with

\[
q = r|_{\Omega^0} \circ A_q, \quad r = r_0 + r_1
\]

being the decomposition we introduced, define approximated test functions \( (q_{\Delta t, k}, \psi_{\Delta t, k}) \) in the following way:

\[
\psi_{\Delta t, k}(t, \cdot) = \psi_k(n\Delta t), \quad t \in [n\Delta t, (n + 1)\Delta t),
\]

where \( \psi_k(t) \) is a projection of \( \psi(t, \cdot) \) in \( \mathcal{V}_k = span\{w_i : 1 \leq i \leq k\} \), and

\[
q_{\Delta t, k}(t, \cdot) = r_k(n\Delta t, \cdot)|_{\Omega^0_{\Delta t, k}(t)} \circ A_{\Delta t, k}^{n+1}, \quad t \in [n\Delta t, (n + 1)\Delta t),
\]

\[
r_k = r_0 + r_{1, k}
\]

where \( r_{1, k} \) is the extension of \( (0, 0, \psi_k) \) in the same way as given in \( (53) \). Before we proceed to the convergence of approximate solutions, we first have some convergences for the test functions as follows:

**Lemma 5.1.** For every \( (q, \psi) \in \chi^0(0, T) \), we have:

\[
q_{\Delta t, k} \rightarrow q, \quad \psi_{\Delta t, k} \rightarrow \psi, \quad \nabla q_{\Delta t, k} \rightarrow \nabla q, \quad \text{uniformly on } [0, T] \times \Omega,
\]

as \( k \rightarrow \infty \).

**Proof.** For \( t \in [n\Delta t, (n + 1)\Delta t) \),

\[
q_{\Delta t, k}(t) - q(t) = r_k(n\Delta t, A_{\Delta t, k}^{n+1}) - r(t, A_q(t))
\]

\[
= \left(r_k(n\Delta t, A_{\Delta t, k}^{n+1}) - r(n\Delta t, A_{\Delta t, k}^{n+1})\right) + r(n\Delta t, A_{\Delta t, k}^{n+1}) - r(t, A_q(t)) \quad (55)
\]

Since \( r_k \rightarrow r \) when \( k \rightarrow \infty \), the first difference on the right hand side of \( (55) \) goes to 0, so we only need to study the other difference. Decompose it into

\[
r(n\Delta t, A_{\Delta t, k}^{n+1}) - r(t, A_q(t)) = r(n\Delta t, A_q(t)) - r(t, A_q(t)) + r(n\Delta t, A_{\Delta t, k}^{n+1}) - r(n\Delta t, A_q(n\Delta t)).
\]

By the mean-value theorem, there is \( \beta \in [0, 1] \) such that

\[
r(n\Delta t, A_q(t)) - r(t, A_q(t)) = \partial_t r(n\Delta t + \beta(t - n\Delta t), A_q(t))(t - n\Delta t),
\]

which converges to 0 as \( k \rightarrow \infty \), and the remaining one

\[
r(n\Delta t, A_{\Delta t, k}^{n+1}) - r(n\Delta t, A_q(n\Delta t)) \rightarrow 0, \quad \text{as } k \rightarrow \infty
\]

from the strong convergence of \( A_{\Delta t, k} \rightarrow A \). The same argument applies to deduce \( \psi_{\Delta t, k} \rightarrow \psi \) and \( \nabla q_{\Delta t, k} \rightarrow \nabla q \) as \( k \rightarrow \infty \).

\[ \square \]
Lemma 5.2. We have:
\[
\frac{\psi_{n+1,t,k} - \psi_{n,t,k}}{\Delta t} = : d\psi_{n,t,k} \rightarrow \partial_t \psi \text{ in } L^2(0,T; L^2(\Omega)), \text{ as } k \rightarrow +\infty,
\]
and
\[
\frac{q_{n+1,t,k} - q_{n,t,k}}{\Delta t} = : dq_{n,t,k} \rightarrow \partial_t q \text{ in } L^2(0,T; L^2(\Omega)), \text{ as } k \rightarrow +\infty.
\]

Proof. Obviously, we have
\[
\frac{\psi_{n+1,t,k} - \psi_{n,t,k}}{\Delta t} = \psi_t((n+1)\Delta t) - \psi_t(n\Delta t) = \partial_t \psi(n\Delta t + \beta \Delta t) - \sum_{i=k+1}^{\infty} (\partial_t \psi(n\Delta t + \beta \Delta t), w_i) w_i,
\]
for a $\beta \in (0,1)$. The last term goes to 0 when $k \rightarrow +\infty$ by using the boundedness of $\partial_t \psi$ in $L^\infty(0,T; L^2(\Gamma))$.

For the second convergence, we use similar argument as in the previous lemma to get:
\[
\frac{q_{n+1,t,k} - q_{n,t,k}}{\Delta t} = \frac{1}{\Delta t} r_k ((n+1)\Delta t, A_{n+1,t,k}^n - r_k (n\Delta t, A_n,t,k)) = \nabla r_k ((n+1)\Delta t, A_{n+1,t,k}^n - r_k (n\Delta t, A_n,t,k)) + \delta A_{n,t,k} \Delta t + \partial_t r_k ((n+1)\Delta t, A_n,t,k),
\]
with $\delta = A_n,t,k + \gamma (A_{n+1,t,k}^n - A_{n+1,t,k})$ for $\beta, \gamma \in [0,1]$. In the decomposition $r_k = r_0 + r_{1,k}$, since term $r_{1,k}$ is a simple extension of $\psi_k$, we have that $\nabla r_{1,k}$ is the extension of $\nabla \psi_k$ and $\partial_t r_{1,k}$ is the extension of $\partial_t \psi_k$. From the convergence $\psi_k \rightarrow \psi$ in $C^1_0(0,T; H_0^2(\Gamma))$, we obtain $\nabla r_k \rightarrow \nabla r$ and $\partial_t r_k \rightarrow \partial_t r$ as $k \rightarrow +\infty$. By using the convergence of $A_{n,t,k} \rightarrow A_n$, the term $\frac{A_{n+1,t,k}^n - A_{n,t,k}}{\Delta t} = w^{n+1}$ converges to $w^n$ as $k \rightarrow +\infty$, we obtain
\[
dq_{n,t,k} \rightarrow w^n \cdot \nabla r + \partial_t r = \partial_t q \text{ in } L^2(0,T; L^2(\Omega)) \text{ as } k \rightarrow +\infty.
\]

\[
\Box
\]

5.2 Passage to the limit

To define the approximate problem, for every $(q, \psi) \in \chi^n(0,T)$, by taking the sum (29) and (22) with test functions $q_{n,t,k}$ and $\psi_{n,t,k}$, we get
\[
\int_0^T \frac{1}{2} \int_\Omega T_{\Delta t} J_{\Delta t,k} \left( (T_{\Delta t} u_{\Delta t,k} - w_{\Delta t,k}) \cdot \nabla q_{\Delta t,k} \right) u_{\Delta t,k} \cdot q_{\Delta t,k} - \int_\Omega (F(\psi_{\Delta t,k}), \psi_{\Delta t,k}) + (\Delta \eta_{\Delta t,k}(t), \Delta \psi_{\Delta t,k})
\]
\[
+ \int_\Omega T_{\Delta t} J_{\Delta t,k} \partial_t u_{\Delta t,k} \cdot q_{\Delta t,k} + \frac{1}{2} \int_\Omega T_{\Delta t,k} \frac{J_{\Delta t,k} - T_{\Delta t,k}}{\Delta t} u_{\Delta t,k} \cdot q_{\Delta t,k} + (F(\psi_{\Delta t,k}), \psi_{\Delta t,k}) + (\Delta \eta_{\Delta t,k}(t), \Delta \psi_{\Delta t,k})
\]
\[
+ 2\mu \int_\Omega J_{\Delta t,k} \nabla u_{\Delta t,k} : D u_{\Delta t,k} + J_{\Delta t,k} \nabla q_{\Delta t,k} \right) dt = 0,
\]
where $u_{\Delta t,k}$ and $v_{\Delta t,k}$ are the piece-wise linear approximations of $u_{\Delta t,k}$ and $v_{\Delta t,k}$, i.e. for $t \in [n\Delta t, (n+1)\Delta t]$:
\[
u_{\Delta t,k} = u_{\Delta t,k} + \frac{u_{n+1,t,k} - u_{n,t,k}}{\Delta t} (t - n\Delta t),
\]
\[
u_{\Delta t,k} = v_{\Delta t,k} + \frac{v_{n+1,t,k} - v_{n,t,k}}{\Delta t} (t - n\Delta t).
\]
Notice that replacing the discretized time derivative with the time derivative of the piece-wise linear approximations does not change the value of the corresponding two integral terms. We rewrite the approximate problem (56) as \(\sum_{i=1}^{n} I_i = 0\), where \(I_i\) represents each integral term on the left side of (55) for \(1 \leq i \leq 8\).

We still don’t know if the limit of \(\nabla^{n+1} \tilde{u}_{\Delta t,k} \rightarrow \nabla u\). Since our space regularity on the fixed domain for \(u\) is less than \(H^1\), this limit had to be passed on the moving domain. It was proved in [17, 18] that:

\[
\nabla \tilde{u}_{\Delta t,k} \rightarrow \nabla u, \text{ weakly in } L^2(0,T;L^2(\Omega)), \text{ as } k \rightarrow \infty.
\]

Now, by using Lemma [4.3] Corollary [4.0] and Lemmas [5.1] and [5.2] we are going to prove the convergence of all terms \(I_i\).

1. Terms \(I_1\) and \(I_2\): Follows from the convergences of functions \(T_{\Delta t}J_{\Delta t,k}, u_{\Delta t,k}, w_{\Delta t,k}\) and \(q_{\Delta t,k}\) and from the weak convergence of \(\nabla^{\Delta t,k} u_{\Delta t,k}\) and convergence of \(\nabla^{\Delta t,k} q\) which follows from the convergences of \(\nabla q_{\Delta t,k}\) and \(A_{\Delta t,k}\). Notice that since \(u_{\Delta t,k}\) is integral equicontinuous in time (Lemma [4.3]), function \(T_{\Delta t}u_{\Delta t,k}\) also converges to \(u\).

2. Terms \(I_3\) and \(I_5\): Follows by partial integration on every sub-interval while being careful about left and right limits of \(q_{\Delta t,k}\) and \(\psi_{\Delta t,k}\) at points \(n \Delta t\) and by the mean-value theorem (see [20] section 7.2 or [17] section 7.2):

\[
\int_0^T \int_\Omega T_{\Delta t}J_{\Delta t,k} \partial_t \tilde{u}_{\Delta t,k} \cdot q_{\Delta t,k} \rightarrow \int_0^T \int_\Omega \partial_t J \tilde{u} q - \int_0^T \int_\Omega J_0 u_0 \partial_t q(0)
\]

and

\[
\int_0^T \int_\Gamma \tilde{v}_{\Delta t,k} \cdot \psi_{\Delta t,k} \rightarrow \int_\Gamma v_0 \psi_0 - \int_0^T \int_\Gamma \partial_t \psi_0 \cdot \psi
\]

Here we also used the strong convergence of initial data in corresponding norms.

3. Term \(I_4\): Since

\[
\frac{J_{\Delta t,k} - T_{\Delta t}J_{\Delta t,k}}{\Delta t} = \tilde{\eta}_{\Delta t,k},
\]

convergence of this term follows from the convergence of \(\tilde{\partial}_t \eta_{\Delta t,k}\).

4. Terms \(I_5, I_6\) and \(I_7\): Directly from the weak convergences of the corresponding terms and the convergence of term \(F\) in \(H^{-2}(\Gamma)\).

Thus, we have proven the first part of the Theorem 1.1. The remaining part is to study the life span of the solution. We follow the approach given in [3 pp. 397-398] (see also [20] Theorem 7.1]. For initial data \(X_0\) we solve the problem on the time interval \([0,T_0]\).

Let

\[
\lim_{t \to T_0} \min_{X \in \Gamma} \eta(t,X) + 1 = C_0 > 0
\]

Since \(\eta\) is uniformly bounded in \(W^{1,\infty}(0,T;L^2(\Gamma)) \cap L^\infty(0,T;H^2_0(\Gamma))\) which is embedded into \(C^{0,1/4}(0,T;H^{3/2}(\Gamma))\), we can choose \(T_1 > T_0\) such that

\[
||\eta_{\Delta t,k}(T_1) - \eta_{\Delta t,k}(T_0)||_{C(\Gamma)} \leq C ||\eta_{\Delta t,k}(T_1) - \eta_{\Delta t,k}(T_0)||_{H^{3/2}(\Gamma)} \leq 2C(T_1 - T_0)^{1/4},
\]

so for \(T_1 - T_0 \leq \left(\frac{C_0}{2C}\right)^4\), we can prolong the solution defined in \([0,T_0]\) to be in \([0,T_1]\) and

\[
\lim_{t \to T_1} \min_{X \in \Gamma} \eta(t,X) + 1 = C_1 > C_0/2.
\]
Now we repeat this procedure and obtain a sequence of $C_n = \lim_{t \to T_n} \eta(t, X) + 1$, and if $\lim_{n \to \infty} C_n = 0$, then we take $T = \lim_{n \to \infty} T_n$. Otherwise, there exists a $C^* > 0$ such that $C_n \geq C^*$, for all $n \in \mathbb{N}$. Now, since
\[
1 + \eta(T_{n+1}, X) \geq 1 + \eta(T_n, X) - 2C(T_{n+1} - T_n)^{1/4} \geq C^* - 2C(T_{n+1} - T_n)^{1/4},
\]
we have that
\[
T_{n+1} - T_n \geq \left( \frac{C^*}{1 + \eta(T_{n+1}, X)} \right)^4.
\]
Since $\eta$ is uniformly bounded in $C([0, T] \times \Gamma)$, we obtain that $T_{n+1} - T_n \geq C > 0$, so $\lim_{n \to \infty} T_n = \infty$. This finishes the proof of Theorem 1.1.

**Appendix - The 2D case**

The proofs that we have presented throughout the paper for the 3D model hold true for the corresponding 2D problem. We want to obtain a better understanding of the asymptotic behaviour of the sequence $N(k)$ for large $k$ by deriving a more precise sufficient condition (40). For, say $\Gamma = [-\pi/2, \pi/2]$, we can directly solve the biharmonic eigenvalue problem and obtain the following eigenvalues and eigenfunctions:
\[
w_{2n} = \cos(2nx),
\]
\[
w_{2n+1} = \sin((2n + 1)x),
\]
\[
\xi_n = 2n^4.
\]
From the eigenvalue equality we have $||\Delta w_n||^2_{L^2} = \xi_n ||w_n||^2_{L^2}$, so by the interpolation inequality of the Sobolev spaces, it follows
\[
||w_n||^2_a \leq ||w_n||^2_{L^2} - \alpha ||w_n||^2_{H^2} \leq 2C_\Gamma n^{2a}.
\]
Now we can directly calculate
\[
\sum_{i=1}^k ||w_i||^2_{H^a} \leq C_\Gamma (k + 1)^{2a+1}.
\]
The sufficient condition for $N(k)$ given in (40) now reads:
\[
N(k) \geq T \left( 4k^4C_B + 4C_\Gamma C_R^2(||F(0)||^2_{H^{-a}} + C_B)(k + 1)^{2a+1} \right)^{\frac{1+\alpha}{2+\alpha}}, \quad \alpha > 0. \quad (57)
\]
The right-hand side of (57) for large $k$ asymptotically behaves as
\[
C_1 k^{2+2\alpha}, \quad \text{for } a \leq 3/2 \quad (58)
\]
\[
C_2 k^{a+\frac{1}{2}+\alpha(a+\frac{1}{2})}, \quad \text{for } a > 3/2, \quad (59)
\]
for some constants $C_1$ and $C_2$ that can easily be obtained from (57), and specially for 1D Kirchhoff, von Kármán and Berger beams as $C_1 k^{2+2\alpha}$.

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