Tiling $R^5$ by Crosses

P. Horak · V. Hromada

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Abstract An $n$-dimensional cross comprises $2n + 1$ unit cubes: the center cube and reflections in all its faces. It is well known that there is a tiling of $R^n$ by crosses for all $n$. AlBdaiwi and the first author proved that if $2n + 1$ is not a prime then there are $2^{\aleph_0}$ non-congruent regular (face-to-face) tilings of $R^n$ by crosses, while there is a unique tiling of $R^n$ by crosses for $n = 2, 3$. They conjectured that this is always the case if $2n + 1$ is a prime. To support the conjecture we prove in this paper that also for $R^5$ there is a unique regular, and no non-regular, tiling by crosses. So there is a unique tiling of $R^3$ by crosses, there are $2^{\aleph_0}$ tilings of $R^4$, but for $R^5$ there is again only one tiling by crosses. We guess that this result goes against our intuition that suggests ‘the higher the dimension of the space, the more freedom we get’.

Keywords Unit cube · $n$-Cross · Face-to-face tiling · Non-regular tiling

Tilings of $R^n$ by unit cubes go back to 1907 when Minkowski conjectured [15] that each lattice tiling of $R^n$ by unit cubes contains twins, a pair of cubes sharing a complete $n − 1$ dimensional face. This conjecture was proved by Hajós [5] in 1942.

In 1930, when Minkowski’s conjecture was still open, Keller [12] suggested that the lattice condition in the conjecture is redundant and that the nature of the problem is purely geometric, and not algebraic as assumed by Minkowski. Thus he conjectured that each tiling of $R^n$ by unit cubes contains twins. It is trivial to see that each tiling of $R^2$ by unit cubes contains twins, and it is also easy to verify it for $R^3$. However, a proof that each tiling of $R^n$, $4 \leq n \leq 6$, contains twins takes in aggregate 80 pages,
see [17]. There was no progress on Keller’s conjecture for more than 50 years. Only in 1992 Lagarias and Shor [13] constructed a tiling of $\mathbb{R}^n$, $n \geq 10$, by unit cubes with no twins. First they found such a tiling in $\mathbb{R}^{10}$, which we consider a very surprising and remarkable result. However, once one has such a tiling in hand, it is relatively easy to find it for $\mathbb{R}^n$, $n > 10$, as well. The second part supports our belief that ‘the higher the dimension of the space, the more freedom we get’. Mackey [14] proved that the Keller conjecture is false for $n = 8, 9$ as well. As to the remaining value of $n = 7$, there are only some partial results, see [2].

Since the late 1950s tilings of $\mathbb{R}^n$ by different clusters of unit cubes have been considered, see e.g. [19] and [21], many of them related to perfect error-correction codes in Lee metric (also called Manhattan metric in $\mathbb{Z}^n$). The Golomb–Welch conjecture [3] has been a main motivating power of the research in this area for the last 40 years. A perfect $e$-error-correcting Lee code over $\mathbb{Z}$ of block size $n$, denoted $PL(n, e)$, is a set $C \subset \mathbb{Z}^n$ of codewords so that each word $A \in \mathbb{Z}^n$ is at Lee distance at most $e$ from exactly one codeword in $C$. Similarly, a perfect $e$-error-correcting Lee code over $\mathbb{Z}_q$ of block size $n$, denoted $PL(n, q, e)$, is a set $C \subset \mathbb{Z}_q^n$ of codewords so that each word $A \in \mathbb{Z}_q^n$ is at Lee distance at most $e$ from exactly one codeword in $C$.

Conjecture 1 Golomb–Welch. For $n \geq 3$ and $e > 1$, there is no $PL(n, e)$ code.

Clearly, the above conjecture, if true, implies that there is no $PL(n, q, e)$ code for $n \geq 3$, $e > 1$, and $q \geq 2e + 1$. For the state of the art on the conjecture we refer the reader to [9].

In this paper we focus on tilings by $n$-crosses. An $n$-dimensional cross comprises $2n + 1$ unit cubes: the “central” one and reflections in all its faces. A tiling $L$ of $\mathbb{R}^n$ by crosses is called a $Z$-tiling if centers of all crosses in $L$ have integer coordinates. Further, $L$ is called a lattice tiling if centers of all crosses in $L$ form a lattice. A regular (also called a face-to-face) tiling is a tiling that is congruent to a $Z$-tiling; otherwise the tiling is called non-regular. We recall that two tilings $T$ and $S$ of $\mathbb{R}^n$ are congruent if there exists a linear, distance preserving bijection of $\mathbb{R}^n$ which maps $T$ on $S$. It seems that Kárteszi [11] was the first to ask whether there exists a tiling of $\mathbb{R}^3$ by crosses. Such a tiling was constructed by Freller in 1970; Korchmáros about the same time treated the case $n > 3$. Golomb and Welch showed the existence of these tilings in terms of error-correcting codes, see Sect. 3.5 in [19]. Immediately after the existence question has been answered, the enumeration of tilings has been studied. In [16] Molnár proved the following.

Theorem 2 Molnár. The number of pair-wise non-congruent lattice $Z$-tilings of $\mathbb{R}^n$ by crosses equals the number of non-isomorphic Abelian groups of order $2n + 1$.

Szabó [20] constructed a non-regular lattice tiling of $\mathbb{R}^n$ by crosses in the case when $2n + 1$ is not a prime. Using refinements of this construction it was proved in [8] that in this case there are $2^{n_0}$ non-congruent $Z$-tilings of $\mathbb{R}^n$ by crosses. In strict contrast to this result it was proved there that, for $n = 2$, and $n = 3$, there is a unique, up to a congruence, tiling of $\mathbb{R}^n$ by crosses. It is conjectured in [8], see also [1].
**Conjecture 3** If \(2n + 1\) is a prime then there exists, up to a congruence, only one \(Z\)-tiling of \(R^n\) by crosses.

It seems to us that the above conjecture, if true, would totally go against our intuition that suggests: the higher the dimension of the space, the more freedom we get; see also an above comment related to the Lagarias–Shor result on Keller’s conjecture.

To provide supporting evidence for Conjecture 3 we prove the following in this paper.

**Theorem 4** There exists, up to a congruence, a unique \(Z\)-tiling of \(R^5\) by crosses.

Although we proved Conjecture 3 only for \(n = 5\), an essential part of the proof of Theorem 4 holds for all \(n = 2 \pmod{3}\). We believe that this part will be helpful when proving this conjecture for some other values of \(n\).

Clearly, if \(\mathcal{L}\) is a \(Z\)-tiling of \(R^n\) by crosses, then centers of crosses in \(\mathcal{L}\) form a \(PL(n, 1)\) code. It is easy to check that the unique tiling of \(R^5\) by crosses is 11-periodic. Thus, as an immediate consequence of Theorem 4 we get the following.

**Corollary 5** There is a \(PL(5, q, 1)\) code if and only if \(11 | q\).

As to the non-regular tilings of \(R^n\) by crosses, it was mentioned above that such a tiling exists if \(2n + 1\) is not a prime. A result of Redei [18] implies that, if \(2n + 1\) is a prime, then there is no lattice non-regular tiling of \(R^n\) by crosses. It is easy to check that a non-regular tiling of \(R^2\) by crosses does not exist. The same result for \(n = 3\) has been proved in [4]. As the other main result of this paper we will show the following.

**Theorem 6** Let \(2n + 1\) be a prime. If there is a unique \(Z\)-tiling of \(R^n\) by crosses, then there is no non-regular tiling of \(R^n\) by crosses.

Combining Theorem 4 with Theorem 6 we get the following.

**Corollary 7** There is a unique, up to a congruence, tiling of \(R^5\) by crosses, and this tiling is a \(Z\)-tiling.

Thus, there is a unique tiling of \(R^3\) by crosses, there are \(2^{n_0}\) pair-wise non-congruent \(Z\)-tilings of \(R^4\) by crosses, but for \(R^5\) there is again a unique tiling by crosses.

Also, by means of Theorem 6, it is straightforward that Conjecture 3 is equivalent to

**Conjecture 8** If \(2n + 1\) is a prime then there exists, up to a congruence, a unique tiling of \(R^n\) by crosses, and this tiling is a lattice \(Z\)-tiling.

In the next section we introduce needed notations, definitions and state some auxiliary results. Theorem 6 will be proved in Sect. 2, while Theorem 4 will be proved...
in Sect. 3. Our proof of Theorem 4 is rather long and involved. It follows ideas developed in [6] and [7], where the non-existence of some perfect error-correcting codes in Lee metric has been established. Therefore in this paper we only provide some essential ideas and techniques of the proof of Theorem 4; the interested reader can find all details in [10].

1 Preliminaries

In this section we recall some notations, notions, and results which will turn out to be useful in proving both main results of the paper, Theorem 4 and Theorem 6.

Since the problem of tilings by crosses comes originally from the area of error-correcting codes we will stick to some of its terminology. Let \( L \) be a \( \mathbb{Z} \)-tiling of \( \mathbb{R}^n \) by crosses. We will denote by \( T_L \subset \mathbb{Z}^n \) the set of centers of crosses in \( L \). The elements of \( \mathbb{Z}^n \) will be called words while the words in \( T_L \) will be called codewords. We will also say that a codeword \( W \) covers a word \( V \) if \( \rho_M(V,W) \leq 1 \). As usual \( \rho_M \) stands for the Manhattan distance of \( V = (v_1, \ldots, v_n) \) and \( W = (w_1, \ldots, w_n) \) given by

\[
\rho_M(V,W) = \sum_{i=1}^{n} |v_i - w_i|.
\]

The weight \( |V|_M \) of \( V \in \mathbb{Z}^n \) is given by \( |V|_M := \sum_{i=1}^{n} v_i = \rho_M(V,O) \), where \( O = (0,\ldots,0) \). The following simple observation will be used several times:

Claim 9 Let \( L \) be a tiling of \( \mathbb{R}^n \) by crosses. Then permuting the order of coordinates of each codeword in \( T_L \) and/or changing a sign of a coordinate for each codeword in \( T_L \) and/or adding a word \( V \in \mathbb{R}^n \) to each codeword results in a set \( T' \) which induces a tiling of \( \mathbb{R}^n \) by crosses congruent to \( L \).

If \( L \) is a tiling of \( \mathbb{R}^n \) by crosses then for each word \( V \) in \( \mathbb{Z}^n \) there is a unique codeword \( W \) so that \( \rho_M(V,W) \leq 1 \). Therefore \( T_L \) can also be seen as a decomposition (tiling) of \( \mathbb{Z}^n \) by Lee spheres \( S_{n,1} \) of radius 1 centered at \( O \), where \( S_{n,1} = \{V \in \mathbb{Z}^n, \rho_M(V,O) \leq 1\} = \{O\} \cup \{e_i, i = 1, \ldots, n\} \); and vice versa, each tiling of \( \mathbb{Z}^n \) by spheres \( S_{n,1} \) induces a tiling of \( \mathbb{R}^n \) by crosses. As usual, \( e_i = (0,\ldots,0,1,0,\ldots,0) \) where the \( i \)th coordinate is equal to 1.

In general, if \( S \) is a subset of \( \mathbb{R}^n \) (\( \mathbb{Z}^n \)), a tiling \( L \) of \( \mathbb{R}^n \) (\( \mathbb{Z}^n \)) by translations of \( S \) can be described in the form \( \{S + u, u \in U\} \), where \( u \) is a vector. Then \( L \) is a lattice tiling if \( U \) is a lattice. For the sake of simplicity we will abuse slightly the language and a subset \( U \) of \( \mathbb{R}^n \) (\( \mathbb{Z}^n \)) will be understood sometimes as a set of vectors with the obvious \( U \in U \) meaning that the vector \( u = U - O \) is in \( U \). The following theorem stated in [9] turns out to be useful when proving both main results of the paper.

Theorem 10 Let \( S \) be a subset of \( \mathbb{Z}^n \). Then there is a lattice tiling of \( \mathbb{Z}^n \) by translations of \( S \) if and only if there is an Abelian group \( G \) of order \( |S| \) and a homomorphism \( \phi : \mathbb{Z}^n \rightarrow G \) so that the restriction of \( \phi \) to \( S \) is a bijection. In addition, if \( \phi \) satisfies this condition, then the lattice tiling of \( \mathbb{Z}^n \) by translations of \( S \) is given by \( \{S + u, u \in \ker(\phi)\} \).
As an immediate consequence we get the following.

Corollary 11 Let $\phi : Z^n \to Z_{2n+1}$, the cyclic group of order $2n + 1$, be a homomorphism so that, for all $1 \leq i < j \leq n$, $\phi(e_i)$ is not an inverse element to $\phi(e_j)$, that is $\phi(e_i) \neq -\phi(e_j)$. Then $\{S_{n,1} + u, u \in \ker \phi\}$ is a lattice tiling of $Z^n$ by $S_{n,1}$.

We note that tiling of $R^n$ by crosses given in [9] and other papers is a lattice tiling. Therefore these tilings can be seen as obtained by Corollary 11.

Let $\mathcal{L}$ be a collection of crosses that tile $R^n$. We will always assume wlog that the cross $K_O$ centered at the origin belongs to $\mathcal{L}$. Then each cross $K \in \mathcal{L}$ can be seen as a translation of $K_O$ by a vector $u$. So $\mathcal{L} = \{K_O + u, u \in \mathcal{T}_\mathcal{L}\}$. For the sake of brevity we will use $K_u$ for a cross centered at a point $U = O + u$.

2 Proof of Theorem 6

In this section we provide a proof of Theorem 6. The following lemma will be the key ingredient of the proof. We recall that by Theorem 2 there is a unique lattice tiling of $R^n$ by crosses when $2n + 1$ is a prime.

Lemma 12 Let $2n + 1$ be a prime, and let $\mathcal{D}$ be a unique lattice tiling of $R^n$ by crosses. If $K$ is a cross in $\mathcal{D}$, then shifting $K$ along any axis will cause all crosses of $\mathcal{D}$ to be shifted as well.

Proof As $\mathcal{D}$ is a lattice tiling it suffices to prove the statement for the cross $K_O$.

Consider the homomorphism $\phi : Z^n \to Z_{2n+1}$ given by $\phi(e_i) = i$ for all $i = 1, \ldots, n$. Then, by Corollary 11, $\phi$ induces a lattice tiling $\mathcal{D} = \{S_{n,1} + u, u \in \ker(\phi)\}$ of $R^n$ by crosses.

Let $j$, $1 \leq j \leq n$, be fixed. We will prove that shifting the cross $K_O$ along the $j$th axis would shift all crosses in $\mathcal{D}$. We start with describing vectors $v_1, \ldots, v_n$ that form a basis of the lattice $\ker(\phi)$. Let $j^{-1}$ be the element inverse to $j$ in the multiplicative Abelian group $Z^*_{2n+1}$. For each $i = 1, \ldots, n$, $i \neq j$, we set $v_i = e_i - ij^{-1}e_j$, and $v_j = (2n + 1)e_j$. Clearly, $\phi(v_j) = 0$, that is, $v_j \in \ker(\phi)$. Indeed, for $i \neq j$, $\phi(v_i) = \phi(e_i - ij^{-1}e_j) = \phi(e_i) - ij^{-1}\phi(e_j) = (i - ij^{-1}j) \mod(2n + 1) = 0$, and $\phi(v_j) = \phi((2n + 1)e_j) = (2n + 1)j \mod(2n + 1) = 0$. Let $A$ be the matrix whose rows are vectors $v_1, \ldots, v_n$. It is easy to calculate $\det A$ as the rows and columns of $A$ can be permuted such that the resulting matrix is a lower triangular having $(2n + 1, 1, 1, \ldots, 1)$ as its diagonal entries. Therefore, $\det A = 2n + 1$, which in turn implies that $v_1, \ldots, v_n$ form a basis of the lattice $\ker(\phi)$.

Assume that the cross $K_O$ has been shifted along the $j$th axis. Then this will cause the cross $K_{v_i}$, $i = 1, \ldots, n$, to be shifted as well. Indeed, for $i \neq j$, the cross $K_{v_i}$ contains the unit cube $C_i$ centered at $v_i - e_i = -ij^{-1}e_j$ (centered at $2ne_j$ for $i = j$); that is, the center of $C_i$ lies on $j$th axis. Further, the cross $K_O$ contains the cube $C_O$ centered at $O$. Thus, when shifting $K_O$ along the $j$th axis we shift the cube $C_O$ along this axis, and this will cause the cube $C_i$ to get shifted; i.e., the cross $K_{v_i}$ will be shifted along the $j$th axis for all $i = 1, \ldots, n$. Consider now a cross $K_u$ in $\mathcal{D}$. As $\mathcal{D}$ is a lattice tiling, the above proved statement is true for any cross $K_u$. Hence:
Claim A Shifting the cross $K_u$ along the $j$th axis will cause shifting the cross $K_{u+v_i}$ for all $i$, $1 \leq i \leq n$.

With this claim in hand it is easy to provide the closing argument of our proof. Let $K_u \in \mathcal{D}$. We will prove that shifting the cross $K_O$ along the $j$th axis will cause the cross $K_u$ to be shifted as well. Since $K_u \in \mathcal{D}$, it is $u \in \ker(\phi)$, and because $v_1, \ldots, v_n$ form a basis of $\ker(\phi)$, $u$ can be written as a linear combination $u = \alpha_1 v_1 + \cdots + \alpha_n v_n$, where $\alpha_i \in \mathbb{Z}$ for all $i$. So to finish the proof it suffices to apply repeatedly Claim A.

Now we are ready to prove Theorem 6.

Proof of Theorem 6 Let $\mathcal{L} = \{K_O + u, u \in \mathcal{U}\}$ be a non-regular tiling of $\mathbb{R}^n$. Then there is $i$, $1 \leq i \leq n$, and a vector $u = (u_1, \ldots, u_n) \in \mathcal{U}$ such that $u_i$ is not an integer. Let $\alpha \in (0, 1)$ be the fractional part of $u_i$. Denote by $\mathcal{U}_\alpha$ the set of all vectors $v = (v_1, \ldots, v_n)$ in $\mathcal{U}$ such that $v_i - \lfloor v_i \rfloor = \alpha$. It is well known, see e.g. [19], that the collection of crosses $K_u, u \in \mathcal{U}_\alpha$ forms a prism $\mathcal{P}$ along the $i$th axis; i.e., if a point $X \in \mathcal{P}$ then, for all $c \in \mathbb{R}$, also the point $X + ce_i \in \mathcal{P}$. Hence, shifting all crosses $K_v, v \in \mathcal{U}_\alpha$ by any vector $w$ parallel to $e_i$, independently on other crosses in $\mathcal{L}$, results in a new tiling of $\mathbb{R}^n$ by crosses, see e.g. [19] or [20]. Moreover, if $w = (m - \alpha)e_i$, $m \in \mathbb{Z}$, then the shift results in a tiling where all crosses $K_v, v \in \mathcal{U}_\alpha$ are now centered at points with the $i$th coordinate being an integer. Repeatedly applying this procedure to other crosses that have a non-integer coordinate, we arrive at a $Z$-tiling $\mathcal{L}^*$ of $\mathbb{R}^n$ by crosses. Since we have started with a non-regular tiling $\mathcal{L}$, there is a proper subset $\mathcal{C}$ of $\mathcal{L}^*$ of crosses so that $\mathcal{C}$ comprises a prism along one of the axes.

By Lemma 12, if the lattice tiling $\mathcal{D}$ contains a prism along any axis, this prism constitutes all crosses in $\mathcal{D}$. Therefore the above tiling $\mathcal{L}^*$ is not congruent to the tiling $\mathcal{D}$. However, this contradicts our assumption that there is a unique $Z$-tiling of $\mathbb{R}^n$ by crosses. The proof of Theorem 6 is complete. □

3 Proof of Theorem 4

Let $\mathcal{L}$ be a $Z$-tiling of $\mathbb{R}^n$ by crosses, and let $\mathcal{T}_\mathcal{L} \subset \mathbb{Z}^n$ be the set of centers of crosses in $\mathcal{L}$. Since we will deal only with $Z$-tilings by crosses most of the time we will drop $Z$- and refer to $\mathcal{L}$ as a tiling of $\mathbb{R}^n$ by crosses. We use the terminology of coding theory; that is, the elements of $\mathbb{Z}^n$ will be called words and the elements of $\mathcal{T}_\mathcal{L}$ will be called codewords. In this section we describe some of the main ideas and techniques of the proof of Theorem 4.

We start with an axillary result. Let $W$ be a codeword in $\mathcal{T}_\mathcal{L}$. Then $N_k(W)$, the $k$-neighborhood of $W$, will be the set of codewords $V$ in $\mathcal{T}_\mathcal{L}$ at the distance at most $k$ from $W$, that is, $N_k(W) = \{V \in \mathcal{T}_\mathcal{L}, \rho_M(W, V) \leq k\}$. We will say that two $k$-neighborhoods $N_k(W)$ and $N_k(W')$ are equal if $\{V - W, V \in N_k(W)\} = \{V - W', V \in N_k(W')\}$; and we will say that $N_k(W)$ and $N_k(W')$ are congruent if there is a linear distance preserving transformation mapping $N_k(W)$ on $N_k(W')$. Clearly, for each codeword $W$, the neighborhoods $N_1(W)$, and $N_2(W)$ are empty sets. The following simple statement will play a key role in proving Theorem 4:
Lemma 13 A tiling $L$ of $R^n$ by crosses is a lattice tiling if and only if, for each codeword $W \in T_L$, the neighborhoods $N_5(W)$ and $N_5(O)$ are equal, and $N_5(O)$ is symmetric; that is, if $W \in N_5(O)$ then $-W \in N_5(O)$ as well.

Remark 14 The condition that $N_k(W)$ is equal to $N_k(O)$, and $N_k(O)$ is symmetric is a necessary and sufficient condition for a tiling $L$ to be a lattice one for any $k \geq 3$. We used $k = 5$, as this will be needed in what follows.

Proof The necessity of the condition is obvious. Hence we prove only the sufficiency part. To show that $L$ is a lattice tiling we will prove that, for all codewords $W$, $Z \in T_L$, $W - Z \in T_L$ as well. As $L$ is a tiling by crosses, it is not difficult to see that, for each codeword $Z \in T_L$, there is a sequence $Z_0 = O$, $Z_1$, $Z_m = Z$ of codewords in $T_L$ such that $\rho_M(Z_{i-1}, Z_i) = 3$, $i = 1, \ldots, m$. Then $Z_i \in N_5(Z_{i-1})$ and, by the assumption that $N_5(Z_{i-1})$ and $N_5(O)$ are equal and $N_5(O)$ is symmetric, we get that both $U_i := Z_i - Z_{i-1}$ and $-U_i$ belong to $N_5(O)$ for all $i = 1, \ldots, m$. Thus, as $-U_1 \in N_5(O)$, and $N_5(W)$ is equal to $N_5(O)$, we have $W - U_1 \in T_L$. With respect to our assumption that $N_5(W - U_1)$ is equal to $N_5(O)$, and since $-U_2 \in N_5(O)$, also $W - U_1 - U_2 \in T_L$. Repeatedly applying this procedure we get $W - U_1 - U_2 - \cdots - U_m \in T_L$. However, $W - U_1 - U_2 - \cdots - U_m = W - (Z_1 - O) - (Z_2 - Z_1) - \cdots - (Z_m - Z_{m-1}) = W - Z_m = W - Z$, thus $W - Z \in T_L$. The proof is complete.

As $2n + 1$ is a prime for $n = 5$, by Theorem 2, there is a unique, up to congruence, lattice $Z$-tiling of $R^5$ by crosses. Thus, to prove Theorem 4 it suffices to show that if $L$ is a tiling of $R^5$ by crosses then $L$ is a lattice tiling; thus, with respect to the above lemma it suffices to prove:

Theorem 15 Let $L$ be a tiling of $R^n$ by crosses. Then, for each $W \in T_L$, the neighborhoods $N_5(W)$ and $N_5(O)$ are equal, and $N_5(O)$ is symmetric.

We split the proof of Theorem 15 into four phases. To facilitate our discussion we introduce more notation and terminology. By a word of type $[m_1^{\alpha_1}, \ldots, m_s^{\alpha_s}]$ we mean a word having $\alpha_1$ coordinates equal to $\pm m_1, \ldots, m_s$, the other coordinates equal $0$. E.g., both words $(-2, -2, -1, -2, 0, 0)$ and $(1, 0, 2, 0, -2, 2)$ are of type $[2^3, 1^4]$. There are three types of words $V$ with its weight $|V|_M = 3$; either $V$ is of type $[3^1]$, or of type $[2^1, 1^1]$, or of type $[1^3]$. Let $Z \in N_k(W)$. Then $Z$ will be called a codeword of a type with respect to $W$ if $Z - W$ is of the given type; the number of codewords of type $[m_1^{\alpha_1}, \ldots, m_s^{\alpha_s}]$ in $N_k(W)$ will be denoted $|[m_1^{\alpha_1}, \ldots, m_s^{\alpha_s}]|_W$. If the codeword $W$ will be clear from the context, we will drop the subscript $W$. Similarly, each word $V$, $|V|_M = 4$, is either of type $[4^1]$, or $[3^1, 1^1]$, or $[2^2]$, or $[2^1, 1^2]$, or $[1^4]$. Now we are ready to describe the four phases of proving Theorem 15.

(A) First we prove a quantitative statement, which will be proved not only for $n = 5$ but for all $n = 2 \pmod{3}$. We believe that this statement might turn out to be very useful when proving Conjecture 3 for other values of $n$, where $2n + 1$ is a prime. Let $W$ be a codeword. The statement claims that the number of codewords of type
Let $L$ be a tiling of $\mathbb{R}^n$ by crosses where $n \equiv 2 \pmod{3}$ and $W$ be a codeword. Then the number of codewords of given type with respect to $W$ is $|[3^1]|_W = 0$, $|[2^1, 1^1]|_W = 2n$, and $|[1^3]|_W = \frac{2(n-2)}{3}$. Further, $|[4^1]|_W = |[2^2]|_W = 0$, $|[3^1, 1^1]|_W = 2n$, $|[2^1, 2^1]|_W = 2n(n-2)$, and $|[1^4]|_W = \frac{n(n-2)(n-3)}{3}$.

(B) We prove an analogue of Theorem 16 for the number of codewords of type $[m_{\alpha_1} \ldots m_{\alpha_s}]$, where $\sum_{i=1}^s \alpha_i m_i \leq 5$. However, we get the explicit values for the number of codewords of individual types only for $n = 5$, while for $n \equiv 2 \pmod{3}$ we get those values only as a function of the number of codewords of type $[5^1]$. We point out that this is not because the methods used are not satisfactory but for some values $n \equiv 2 \pmod{3}$, say $n = 62$, there are two (lattice) tilings of $\mathbb{Z}^n$ by crosses with different number of codewords of type $[5^1]$. We stress that for $n = 62$, the number $2n + 1 = 125$ is not a prime, hence it does not provide a counterexample to our conjecture.

(C) In this phase we prove that for any two codewords in $T_L$ their 5-neighborhoods are congruent.

(D) As the last step we show that for any two codewords in $T_L$ their 5-neighborhoods are not only congruent but the two 5-neighborhoods equal, and this joint neighborhood is symmetric, so we prove Theorem 15.

3.1 Phase A

In this subsection we prove Theorem 16. In fact we prove an extended version of the statement.

For any codeword $W$ in $T_L$ there are $2n$ words $V$ of type $[2^1]$ with respect to $W$. We recall that this means that $V - W$ is of given type. Each of them is covered by a codeword of type $[3^1]$, or by a codeword of type $[2^1, 1^1]$, with respect to $W$. On the other hand, each codeword of type $[3^1]$ and of type $[2^1, 1^1]$, with respect to $W$, covers exactly one word of type $[2^1]$ with respect to $W$. Thus we get, for each codeword $W$,

$$|[3^1]| + |[2^1, 1^1]| = 2n. \quad (1)$$

We will not repeat any longer that all codewords of given type are meant with respect to $W$.

In $\mathbb{Z}^n$ there are $2^2 \binom{n}{2}$ words $V$ of type $[1^2]$. Each of them is covered either by a codeword of type $[1^3]$, or by a codeword of type $[2^1, 1^1]$. Further, each codeword of type $[1^3]$ covers three of them while a codeword of type $[2^1, 1^1]$ covers exactly one codeword of type $[1^2]$. Hence

$$|[2^1, 1^1]| + 3|[1^3]| = 4 \binom{n}{2}. \quad (2)$$

Equations (1) and (2) are “global” equations. To get their “local” form we need to introduce some more notation. Often we will need to express the number of words,
or codewords, in a set \( A \) having their \( i \)th coordinate positive, or their \( i \)th coordinate negative. Therefore, to simplify the language, we will introduce the notion of the \( \text{signed coordinate} \) in \( \mathbb{Z}^n \). For the rest of the paper by the set of signed coordinates we will understand the set \( I = \{ +1, \ldots , +n, -1, \ldots , -n \} \). Let \( V = (v_1, \ldots , v_n) \) be a word in \( \mathbb{Z}^n \). Then the signed coordinates \( V_i \) of \( V \) are given by \( V_i = |v_i| \) and \( V_{-i} = 0 \) for \( v_i > 0 \), \( V_i = 0 \) and \( V_{-i} = |v_i| \) for \( v_i < 0 \), and \( V_i = V_{-i} = 0 \) for \( v_i = 0 \). E.g., if \( V = (2, 0, -5) \) then \( V_1 = 2 \), \( V_2 = V_{-2} = 0 \), and \( V_3 = 0 \), \( V_{-3} = 5 \). For a signed coordinate \( i \in I \), by \( |A_i| \) we will denote the number of words in \( A \) with a non-zero \( i \)th coordinate. That is, \( |A_1| \) stands for the number of words in \( A \) with the first coordinate being a positive number, while \( |A_{-3}| \) represents the number of words in \( A \) with the third coordinate being a negative number. If we need to stress that the value of the \( i \)th signed coordinate is \( m \), we will use \( |A_i^{(m)}| \) for the number of words with the \( i \)th coordinate equal \( m \). Thus, for each \( i \in I \), \(|[2^1, 1^1]_i| \) is the number of words of type \([2^1, 1^1]\) with the \( i \)th signed coordinate being non-zero, while \(|[2^1, 1^1]_{i(2)}| \) stands for the set of codewords of type \([2^1, 1^1]\) with the \( i \)th signed coordinate equal to \( 2 \).

Now we are ready to state the local form of (1) and (2). As for each \( i \in I \) there is in \( \mathbb{Z}^n \) one word \( V \) of type \([2^1]\) with \( V_i = 2 \), and \( 2(n - 1) \) words \( U \) of type \([1^2]\) with \( U_i = 1 \), we get

\[
|[3^1]_i| + |[2^1, 1^1]_{i(2)}| = 1
\]  

(3)

and

\[
|[2^1, 1^1]_i| + 2|[1^3]_i| = 2(n - 1).
\]  

(4)

Indeed, if \( A \) is a codeword of type \([3^1]\) with \( A_i = 3 \) (and then \( A_j = 0 \) for all \( j \neq i \), \( j \in I \)) then \( A \) covers a word \( V \) of type \([2^1]\) with \( V_i = 2 \). However, a codeword \( B \) of type \([2^1, 1^1]\) covers \( V \) only if \( B_i = 2 \), but does not cover it if \( B_i = 1 \). On the other hand, a codeword \( B \) of type \([2^1, 1^1]\) with \( B_i \neq 0 \) covers one word \( D \) of type \([1^2]\) with \( D_i = 1 \) regardless whether \( B_i = 2 \) or \( B_i = 1 \). Clearly, a codeword \( C \) of type \([1^3]\) with \( C_i = 1 \) covers exactly two words \( D \) of type \([1^2]\) with \( D_i = 1 \).

Now we derive identities analogous to (1)–(4) for words of weight equal to 3. As (1)–(4) have been derived in great detail, and the same type of ideas are used to prove identities (5)–(11) we will leave a part of the proofs to the reader.

In \( \mathbb{Z}^n \) there are \( 2n \) words of type \([3^1]\). Each of them is covered by a codeword of type \([3^1]\) or \([4^1]\) or \([3^1, 1^1]\), and each of those codewords covers exactly one word of type \([3^1]\). Therefore,

\[
|[3^1]| + |[4^1]| + |[3^1, 1^1]| = 2n,
\]  

(5)

and, for each \( i \in I \), we have

\[
|[3^1]_i| + |[4^1]_i| + |[3^1, 1^1]_{i(3)}| = 1.
\]  

(6)

Further, in \( \mathbb{Z}^n \) there are \( 2^3(\binom{n}{2}) \) words of type \([2^1, 1^1]\). They are covered by codewords of type \([2^1, 1^1]\), or \([3^1, 1^1]\), or \([2^2]\), or \([2^1, 1^2]\). Each codeword of type \([2^2]\), or \([2^1, 1^2]\) covers two such words, while each codeword of type \([2^1, 1^1]\), or \([3^1, 1^1]\) covers one
of them. Hence
\[
[2^1, 1^1] + [3^1, 1^1] + 2[2^2] + 2[2^1, 1^2] = 2^3 \binom{n}{2} \tag{7}
\]
The above identity has two local forms. There are \(2(n - 1)\) words \(U\) of type \([2^1, 1^1]\) with \(U_i = 2\), and \(2(n - 1)\) words \(U\) of type \([2^1, 1^1]\) with \(U_i = 1\). For each \(i \in I\) we get
\[
[2^1, 1^1]_i^{(2)} + [3^1, 1^1]_i^{(3)} + [2^2]_i + 2[2^1, 1^2]_i^{(2)} = 2(n - 1), \tag{8}
\]
and
\[
[2^1, 1^1]_i^{(1)} + [3^1, 1^1]_i^{(1)} + [2^2]_i + [2^1, 1^2]_i^{(1)} = 2(n - 1). \tag{9}
\]
Further, in \(Z^n\) there are \(2^3 \binom{n}{3}\) words of type \([1^3]\). They are covered by codewords of type \([1^3]\), or \([2^1, 1^2]\), or \([1^4]\). Each codeword of type \([1^4]\) covers four of them. Hence,
\[
|[1^3]| + [2^1, 1^2]| + 4[1^4]| = 2^3 \binom{n}{3}. \tag{10}
\]
The local form of (10) reads as follows:
\[
|[1^3]|_i + [2^1, 1^2]|_i + 3[1^4]|_i = 2^2 \binom{n - 1}{2} \tag{11}
\]
as in \(Z^n\) there are \(2^2 \binom{n-1}{2}\) words \(U\) of type \([1^3]\) with \(U_i = 1\), and each codeword \(V\) of type \([1^4]\) with \(V_i = 1\) covers three of them.

Clearly, there are many solutions of (1)–(11) in natural numbers. We will prove that only one corresponds to a tiling of \(R^n\) by crosses.

Now we will prove a part of Theorem 16 that deals with types of codewords of weight 3. In addition we determine also the local values for individual types. The proof of the other part of Theorem 16 is similar in spirit but it is technically much more involved. We refer the interested reader to the complete version of the paper.

**Theorem 17** Let \(n = 2 \pmod{3}\), \(\mathcal{L}\) be a tiling of \(R^n\) by crosses, and \(W\) be a codeword. Then, the number of codewords of given type with respect to \(W\) is \(|[3^1]| = 0, [2^1, 1^1]| = 2n, and |[1^3]| = \frac{2n(n-2)}{3}\). As to the local values, for each \(i \in I\), 
\(|[2^1, 1^1]|_i = |[2^1, 1^1]|_i^{(1)} = 1, that is, |[2^1, 1^1]|_i = 2, and |[1^3]|_i = n - 2\).

**Proof** Let \(W\) be a codeword in \(\mathcal{T}_\mathcal{L}\). Clearly, then also the set \(\mathcal{T}' = \{U, U \in Z^n, U = V - W\} for some V in \mathcal{T}_\mathcal{L}\) is a tiling of \(Z^n\) by Lee spheres. Therefore, vlog we assume \(W = O\). From (3) we have \(|[2^1, 1^1]|_i^{(2)} \leq 1\), while from (4) we get \(|[2^1, 1^1]|_i\) is even, hence \(|[2^1, 1^1]|_i = |[2^1, 1^1]|_i^{(2)} \leq |[2^1, 1^1]|_i\). On the other hand, there is no \(i \in I\) with \(|[2^1, 1^1]|_i^{(2)} < |[2^1, 1^1]|_i\) as \(\sum_{i \in I} |[2^1, 1^1]|_i^{(2)} = \sum_{i \in I} |[2^1, 1^1]|_i^{(1)}\). Thus we proved the following.

**Lemma A** For each \(i \in I\), either \(|[2^1, 1^1]|_i = 0\) or \(|[2^1, 1^1]|_i = 2\). In the latter case \(|[2^1, 1^1]|_i^{(2)} = |[2^1, 1^1]|_i^{(1)} = 1\).
Now we are ready to prove that $||3^1|| = 0$. We consider two cases.

(i) Let $||3^1|| = 1$. Then, by (6), $||3^1, 1^1||(3) = 0$, and by (3), $||2^1, 1^1||(2) = 0$, which implies, by Lemma A, that $||2^1, 1^1|| = 0$. This in turn implies, see (4), $||1^3|| = n - 1$. Substituting it into (11) gives $||2^1, 1^2|| + 3||1^4|| = (n - 1)(2n - 5)$.

As we deal with the case $n = 2 \mod 3$, then $(n - 1)(2n - 5) = 2 \mod 3$ as well, and therefore $||2^1, 1^2|| = 2 \mod 3$. Subtracting (9) from (8), and using

$$||3^1, 1^1||(3) = ||2^1, 1^1||(2) = ||2^1, 1^2|| = 0,$$

we get $2||2^1, 1^2|| + ||3^1, 1^1|| = 1$. Substituting (9) from (8), and using

$$||3^1, 1^1||(3) = ||2^1, 1^1|| = 1,$$

we get $2||2^1, 1^2|| + ||3^1, 1^1|| = 3$. Adding $||2^1, 1^2||$ to both sides yields $3||2^1, 1^2|| = ||2^1, 1^2|| + ||3^1, 1^1||$. We showed above that in this case of $||3^1|| = 1$ it is $||2^1, 1^2|| = 2 \mod 3$. Therefore $||3^1, 1^1||(1) > 0$, that is, $||3^1, 1^1||(1) > ||3^1, 1^1||(3)$.

(ii) Now let $||3^1|| = 0$. By (3), we get $||2^1, 1^1||(2) = 1$, which implies, by Lemma A, that $||2^1, 1^1|| = 2$. This in turn implies, see (4), $||1^3|| = n - 2$. Substituting it into (11) gives $||2^1, 1^2|| + 3||1^4|| = (n - 2)(2n - 3)$. As $n = 2 \mod 3$, it is $(n - 2)(2n - 3) = 0 \mod 3$, and therefore $||2^1, 1^2|| = 0 \mod 3$. Subtracting (9) from (8), and using

$$||2^1, 1^1|| = ||2^1, 2^2|| = 1,$$

we get $2||2^1, 1^2|| + ||3^1, 1^1|| = 1$. Subtracting (9) from (8), and using

$$||3^1, 1^1||(3) - ||3^1, 1^1|| = ||2^1, 1^2||,$$

adding $||2^1, 1^2||$ to both sides yields $3||2^1, 1^2|| = ||2^1, 1^2|| + ||3^1, 1^1||$. As $||2^1, 1^2|| = 0 \mod 3$ in this case, we have $||3^1, 1^1|| - ||3^1, 1^1|| = 0 \mod 3$, which yields $||3^1, 1^1||(3) \geq ||3^1, 1^1||(3) \leq 1$ for all $i \in I$, see (6).

So, $||3^1|| = 1$ implies $||3^1, 1^1||(1) > ||3^1, 1^1||(3)$, while $||3^1|| = 0$ gives $||3^1, 1^1||(1) \geq ||3^1, 1^1||(3)$. However, $\sum_{i \in I} ||3^1, 1^1||(1) = \sum_{i \in I} ||3^1, 1^1||(3)$, therefore there is no $i \in I$ with $||3^1|| = 1$, that is $||3^1|| = 0$, and, for all $i \in I$,

$$||3^1, 1^1||(1) = ||3^1, 1^1||(3), \quad \text{and} \quad 3||2^1, 1^2|| = ||2^1, 1^2||.$$

Since $||3^1|| = 0$, by (1) we get $||2^1, 1^1|| = 2n$, which in turn implies, by (2), that $||1^3|| = \frac{2n(n-2)}{3}$. Further, from $||2^1, 1^1|| = 2$, we get $||1^3|| = n - 2$. The proof is complete. \qed

3.2 Phase B

In this subsection we deal with the number of codewords of individual types of weight equal to 5. First we will summarize results for all $n = 2 \mod 3$, then we concentrate on the case $n = 5$. For $n = 2 \mod 3$, all these values are expressed as a function of the number of codewords of type $5^1$. We point out that for some $n = 2 \mod 3$, there are two tilings of $R^n$ by crosses with different number of codewords of type $5^1$. Hence, unlike with codewords of weight equal to 3 or 4, the values of $||5^1||, ||4^1, 1^1||, ||3^1, 2^1||, ||3^1, 1^2||, ||2^1, 1^3||, ||2^1, 1^1||$, and $||1^5||$ do not depend only on the value of $n$ but also on a given tiling of $R^n$ by crosses.

**Theorem 18** Let $n = 2 \mod 3$, and $W$ in $T_C$. Then the number of codewords of a given type with respect to $W$ is $||4^1, 1^1|| = 2n - ||5^1||, ||3^1, 2^1|| = 2n - \frac{2n(n-2)}{3}$.
\[ |[5^1]|, |[3^1, 1^2]| = 2n(n - 3) + |[5^1]|, |[2^1, 1^3]| = n(2n - 6) + |[5^1]|, |[2^1, 1^3]| = \frac{1}{3} 2n(n - 3)(2n - 7) - |[5^1]|, \text{and} |[1^5]| = \frac{1}{5} (2^4(n^3) - n(n - 3)(3n - 8) + |[5^1]|). \]

We showed that the number of codewords in \( T_L \) of weight 3 and 4 depends only on \( n \), while the number of codewords of weight 5 depends also on the tiling \( L \). However, for \( n = 5 \) we are able to show that \( |[5^1]| = 0 \) regardless of a tiling \( L \) of \( R^n \) by crosses. Therefore we have the following.

**Theorem 19** If \( n = 5 \), then \( |[5^1]| = 0 \), \( |[4^1, 1^1]| = |[3^1, 2^1]| = 10 \), \( |[3^1, 1^2]| = |[2^1, 1^3]| = |[2^2, 1^1]| = 20 \), while \( |[1^5]| = 2 \). Moreover, if \( M \) is a word of type \([1^5]\), then \(-M\) is the other word of this type.

Proofs of Theorems 18 and 19 use similar techniques as we exhibited in Phase A. However, sometimes we needed a finer analysis than in Phase A. Thus, we had to consider double-local identities, where two signed coordinates are simultaneously taken into account.

### 3.3 Phase C

In the previous subsections we proved that, for \( n = 5 \), the 5-neighborhoods of all codewords in \( T_L \) have the same quantitative properties that do not depend on the tiling \( L \). Now we prove that they also have the same structure.

Let \( V \) be a word in \( Z^5 \). Then by \( \langle V \rangle \) we denote the collection of words comprising \( V \), and the words obtained from \( V \) by cyclic shifts of its coordinates. Hence, \( \langle (2, 1, 0, 0, 0) \rangle = \{ (2, 1, 0, 0, 0), (0, 2, 1, 0, 0), (0, 0, 2, 1, 0), (0, 0, 0, 2, 1), (1, 0, 0, 0, 2) \} \). We note that \( \langle V \rangle \) contains five words except for the case when \( V \) has all coordinates equal to the same number. Finally, we set \( \pm \langle V \rangle := \langle V \rangle \cup \langle -V \rangle \). By the canonical 5-neighborhood, or simply a canonical neighborhood, we mean the set of words \( \pm \langle (2, 1, 0, 0, 0) \rangle \cup \pm \langle (1, 0, 1, 0, -1) \rangle \cup \pm \langle (3, 0, 0, -1, 0) \rangle \cup \pm \langle (2, 0, 1, 0, 1) \rangle \cup \pm \langle (2, 0, 0, 1, -1) \rangle \cup \pm \langle (2, -1, 1, 0, 0) \rangle \cup \pm \langle (1, 1, 1, -1, 0) \rangle \cup \pm \langle (4, 0, -1, 0, 0) \rangle \cup \pm \langle (3, 0, 2, 0, 0) \rangle \cup \pm \langle (3, 0, 0, 1, 1) \rangle \cup \pm \langle (3, -1, 0, 0, -1) \rangle \cup \pm \langle (2, 0, 1, 0, 0) \rangle \cup \pm \langle (2, -2, 0, 0, 1) \rangle \cup \pm \langle (2, 0, -2, 0, -1) \rangle \cup \pm \langle (2, -1, 1, 1, 0) \rangle \cup \pm \langle (2, 0, -1, -1, 1) \rangle \cup \pm \langle (1, 1, 1, 1, 1) \rangle \). A simple inspection shows that the number of words of individual types in the canonical neighborhood coincides with the values given by Theorem 19. E.g., \( \pm \langle (2, 1, 0, 0, 0) \rangle \) is the set of ten words of type \([2^1, 1^1]\), while \( \pm \langle (1, 0, 1, 0, -1) \rangle \) comprises ten words of type \([1^5]\).

**Theorem 20** Let \( L \) be a tiling of \( R^5 \) by crosses. Then, for each codeword \( W \) in \( T_L \), the 5-neighborhood of \( W \) is congruent to the canonical one. Moreover, the 5-neighborhood of \( W \) is uniquely determined by the set of codewords of type \([2^1, 1^1]\).

**Proof** Using the same argument as at the beginning of the proof of Theorem 17, we assume wlog that \( W = O \). Two words \( U = (u_1, \ldots, u_5), V = (v_1, \ldots, v_5) \) will be called sign equivalent in the \( j \)th coordinate if \( u_jv_j > 0 \); that is, they are sign equivalent if \( u_j \neq 0 \neq v_j \), and the two non-zero values have the same sign.

Codewords of type \([1^5]\). Theorem 19 claims that \( |[1^5]| = 2 \), and the two codewords of type \([1^5]\) differ in each coordinate.

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Codewords of type $[2^1, 1^1]$. Let $B$ be the set of codewords of type $[2^1, 1^1]$, and $C$ be the set of codewords of type $[1^3]$. We know by Theorem 17 that $|B| = |C| = 10$. There are in total 10 words of type $[1^2]$ that are sign equivalent in both non-zero coordinates with $M$ and another 10 words of type $[1^2]$ that are sign equivalent in two coordinates with $\bar{M}$. Each word of type $[1^2]$ is covered by a codeword in $B \cup C$, thus also each of these 20 words is covered by a codeword in $B \cup C$. If a codeword $C$ in $C$ covers two of these 20 words, then $C$ would be sign equivalent in three coordinates with $M$ or $\bar{M}$, and the distance of $C$ to $M$ or $\bar{M}$ would be less than 3. Therefore, each codeword in $C$ covers at most one of these 20 words of type $[1^2]$. Hence, each codeword in $B$ has to cover one of these 20 words; that is, each codeword of type $[2^1, 1^1]$ is sign equivalent in both non-zero coordinates either with the codeword $M$ or with the codeword $-M$. We know by Theorem 17 that, for each $i \in I$, $|[2^1, 1^1]_i^{(2)}| = |[2^1, 1^1]_i^{(1)}| = 1$. Thus, five codewords in $B$ are sign equivalent in two coordinates with $M$, the other five with $\bar{M}$. Let $F$ be the set of the five codewords in $B$ that are sign equivalent in two coordinates with $M$. The set $F$ can be transformed onto $\langle (2, 1, 0, 0, 0) \rangle$ by suitably permuting the order of coordinates and/or possibly changing a sign of some coordinates for each codeword in $T_C$. By Claim 9 the above transformation is a congruence mapping. Therefore we may assume that $\langle (2, 1, 0, 0, 0) \rangle$ are codewords in $T_C$. In the rest of the proof we show that by having $\langle (2, 1, 0, 0, 0) \rangle \in T_C$ all the other codewords are uniquely determined, and that the 5-neighborhood of the origin is the canonical one. Trivially, $\langle (2, 1, 0, 0, 0) \rangle \in T_C$ implies that the two codewords of type $[1^3]$ are $(1, 1, 1, 1, 1)$ and $(-1, -1, -1, -1, -1)$.

Originally we have proved the statement by a simple computer search that checked all options; the number of cases was in low hundreds. We have $\langle (2, 1, 0, 0, 0) \rangle \in T_C$, and there are four options for the remaining 5 codewords of type $[2^1, 1^1]$. It is easy to see that two of them are not viable. Next, there are four options how to choose the codewords of type $[1^3]$, and tens of ways how to choose the codewords of type $[1^4]$. When considering simultaneously codewords of types $[2^1, 1^2]$, and $[2^1, 1^1]$, it turns out that only in a single case it is possible to select these codewords in a way that all codewords chosen so far are pair-wise at distance at least 3. After that it is easy to determine the remaining types of codewords. Since we were not completely satisfied with this type of a proof, later we found a computer free proof. This new proof is not an analysis of above cases by hand; all its details are given in a complete version of this paper, see [10]. We have decided not to include this part of the proof here as the arguments used are ad hoc arguments that cannot be applied when proving Conjecture 3 for other values of $n$. 

At the end of this subsection we describe an important attribute of the canonical 5-neighborhood.

**Theorem 21** Let $U \in T_C$ be a codeword from $N_5(W)$, the 5-neighborhood of a codeword $W$. Then $2W - U$ is in $T_C$ as well and $2W - U \in N_5(W)$; that is, $N_5(W)$ is symmetric with respect to $W$. Further if $U, Z$ are codewords from $N_5(W)$, then $U + Z - W$ is a codeword as well.

**Proof** Wlog we assume $W = O$. From the previous theorem we know that the 5-neighborhood of each codeword in congruent to the canonical one. Clearly, a con-
adjacent codewords are of type $X, Y, Z$ (b) to choose codewords $T$.

Further we know by Theorem 17 that, for each codeword $U$, there are 20 codewords in $\mathcal{T}_L$ at distance 3 from $U$. We will call these codewords adjacent to $U$. Ten of the adjacent codewords are of type $[2^1, 1^1]$, and ten of them are of type $[1^3]$. The proof of the theorem is based on the following claim.

Claim D: Let $W$ be a codeword in $\mathcal{T}_L$, and let $U$ be a codeword adjacent to $W$. Further, let $S_W$ and $S_U$ be the set of codewords of type $[2^1, 1^1]$ with respect to $W$ and $U$, respectively. Then $\{Z - W, \text{where } Z \in S_W\} = \{Z - U, \text{where } Z \in S_U\}$.

Theorem 20 states that the 5-neighborhood of each codeword $W$ is uniquely determined by the codewords of type $[2^1, 1^1]$. Thus, with Claim D in hands, we know that any two adjacent codewords have the same 5-neighborhood. The rest of the proof of the theorem follows easily by induction because to each codeword $W$ there is a sequence of codewords $O = Z_0, Z_1, \ldots, Z_{m-1}, Z_m = W$ such that the codeword $Z_j$ is adjacent to the codeword $Z_{j-1}$ for all $j = 1, \ldots, m$.

Wlog we prove Claim D only for $W = O$, the origin. By Theorem 20 and Claim 9 we may assume that $N_5(O)$ is the canonical neighborhood. This implies that $\pm(1)$ is adjacent to $O$. To prove Claim D for $U$ we need to show that for each $V \in \pm(2)$ also $U + V$ is a codeword in $\mathcal{T}_L$. To do so, it suffices either:

(a) to show that $U - V$ is a codeword, or
(b) to choose codewords $X, Y, Z$ so that $Y, Z \in N_5(X)$, and $Y + Z - X = U + V$.

Indeed, as to (a), we know that for each codeword $U$ its 5-neighborhood is symmetric with respect to $U$, hence if $U - V$, $V \in \pm(2)$, is a codeword then...
also \( U + V \) is a codeword. In the case (b) consider a codeword \( X \). By Theorem 21, if there are codewords \( Y, Z \), so that \( Y, Z \in N_5(X) \), then \( Y + Z - X = U + V \) is a codeword as well.

First we choose the codeword \( U \) adjacent to the origin to be of type \([2^1, 1^1]\). Let \( U = (2, 1, 0, 0, 0) \). If \( V = (2, 1, 0, 0, 0) \) then \( U - V = (0, \ldots, 0) \) is a codeword, and hence by (a) \( U + V = (4, 2, 0, 0, 0) \) is a codeword as well. The following table provides a suitable choice for the other four codewords \( V \in \langle (2, 1, 0, 0, 0) \rangle \):

| \( V \)    | \( X \)    | \( Y \)    | \( Z \)    | \( Y + Z - X = U + V \) |
|---------|---------|---------|---------|-------------------------|
| \( (0, 2, 1, 0, 0) \) | \( (1, 0, 1, 0, -1) \) | \( (0, 3, 0, 0, -1) \) | \( (3, 0, 2, 0, 0) \) | \( (2, 3, 1, 0, 0) \) |
| \( (0, 2, 1, 0) \)    | \( (2, 1, 0, 0, 0) \)    | \( (1, 2, 0, 1, 0) \)    | \( (3, 0, 2, 0, 0) \)    | \( (2, 1, 2, 1, 0) \)    |
| \( (0, 0, 2, 1) \)    | \( (2, 1, 0, 0, 0) \)    | \( (1, 2, 0, 1, 0) \)    | \( (3, 0, 0, 1, 1) \)    | \( (2, 1, 0, 2, 1) \)    |
| \( (1, 0, 0, 0, 2) \) | \( (0, 1, 0, -1, 1) \) | \( (3, 0, 0, -1, 0) \) | \( (0, 2, 0, 0, 3) \) | \( (3, 1, 0, 0, 2) \) |

Theorem 21 guarantees that the 5-neighborhood of each codeword is symmetric. Therefore \( U + V \) is a codeword also for all \( V \in \langle (-2, -1, 0, 0, 0) \rangle \). Now let \( U' \in \langle (2, 1, 0, 0, 0) \rangle \), \( U \neq U' \). Then \( U' \) is a cyclic shift \( \sigma \) of \( U \). In order to find suitable \( X, Y, Z \) in this case, we apply the same cyclic shift \( \sigma \) to \( X, Y, Z \) given in the table. The same applies to \( U \in -\langle (2, 1, 0, 0, 0) \rangle \).

The other part of the proof, when \( U \) is a codeword of type \([1^3]\), is done in the same manner, and therefore the details are not given here. The proof is complete. \( \square \)

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