Towards gravitating discs around stationary black holes

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Abstract

This article outlines the search for an exact general relativistic description of the exterior (vacuum) gravitational field of a rotating spheroidal black hole surrounded by a realistic axially symmetric disc of matter. The problem of multi-body stationary spacetimes is first exposed from the perspective of the relativity theory (section 1) and astrophysics (section 2), listing the basic methods employed and results obtained. Then (in section 3) basic formulas for stationary axisymmetric solutions are summarized. Sections 4 and 5 review what we have learnt with Miroslav Žáček and Tomáš Zellerin about certain static and stationary situations recently. Concluding remarks are given in section 6. Although the survey part is quite general, the list of references cannot be complete. Our main desideratum was the informative value rather than originality — novelties have been preferred, mainly reviews and those with detailed introductions.

1 The fields of multi-body systems involving black holes

The subject of self-gravitating sources around rotating black holes is interesting in several respects, relevant from the point of view of the relativity theory itself as well as in the astrophysical context.

First, due to the non-linearity of Einstein’s equations, the field of a multi-body system is a traditional challenge where one typically does not manage with a simple superposition. On the first post-Newtonian level, the “celestial mechanics” of gravitationally interacting bodies can be kept linear, but in the strong-field region the interaction may bring surprising features that have only been described in a very few cases yet (for a two-body problem — today at the centre of attention because of the expected gravitational waves from colliding compact binaries, “the current state of art is the third post-Newtonian approximation”).

The second point in which the problem embodies the essence of general relativity is the effect of inertial frame dragging due to the rotation of the sources. Contrary to the Newtonian treatment which does not discriminate between static and stationary situation, the field is now determined not only by mass-energy configuration, but also by its motion within the bodies. The inertial space can be imagined as a viscous fluid mixed by the sources. In today’s “gravitoelectromagnetic” language, gravity has not only an electric component, generated by the mass, but also a magnetic one, generated by mass currents; see [245, 103, 215, 298, 147, 79, 202, 201]

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1Even the problem of an “isolated” object is very difficult (mainly if its motion must be found as well, not mentioning the interior field) unless one makes some simple assumptions about its multipole structure; see e.g. [93, 99, 20] for a treatment of bodies with matter interior, [297] for that also valid for singular bodies such as black holes, and [40] for the case of point-like particles.

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and references therein. (The long-time effort to measure the gravitomagnetic effect of the Earth is just culminating [78]; cf. [238, 239].)

The third point is that our setting involves a black hole, perhaps the most bizarre of the new predictions of general relativity. Within and around it, the deviations from Newtonian theory become dominant, in particular the above mentioned implications of non-linearity and rotation reveal themselves prominently. Perhaps the most extreme example (of an exact space-time) involving all the three aspects is a “double Kerr(-Newman)” solution for an aligned pair of rotating ultracompact objects [96, 210, 37, 211] (cf. [176] where an approximate method for axially symmetric superpositions of rotating black holes was proposed as a starting point of computations of black hole collisions).

Relativistic spacetimes are found in three ways: by numerical solution of Einstein equations, by perturbation of previously known spacetimes, and by exact analytical solution of Einstein equations. Let us mention what the above approaches told us about stationary axisymmetric spacetimes describing rotating black holes with additional matter (ring, disc or torus).

### 1.1 Numerical solution

Although computers have been naturally employed in relativity for decades, in algebra as well as in numerics, it is only quite recently that they embarked on numerical solution of Einstein’s equations in the most interesting cases involving black holes. Teukolsky [295] lists today’s peak parallel-type hardware and software and the appropriate, hyperbolic formulation of the field equations as “three reasons why we are on the verge of important advances in the computer solution of Einstein’s equations”. Whereas analytical prospects are restricted, the present-day computational facilities can handle almost any situation, at least for certain period of time. On the other hand, it is never sure whether a given numerical solution represents a typical or just a marginal case. Due to this lack of generality, it is difficult to discern, analyse and interpret different classes of solutions within boundless ranges of possibilities. However, numerical solutions can provide explicit examples of spacetimes and processes that might otherwise remain only conjectural.

Numerical spacetimes containing a rotating black hole surrounded by an additional stationary axisymmetric source were constructed by [181] (a hole with a thin finite equatorial disc) and by [237] (a hole with a thick toroid). To mention just one particular point, [181] ended with a prolate horizon (stretching along the axis) in certain cases (when the disc was strongly counter-rotating with respect to the hole), which was not observed in [237]. The horizons have been known to inflate towards the external sources, so it would be an interesting consequence of the interplay between dragging from the hole and from the disc if it were confirmed that they can indeed become prolate, even though under extreme circumstances only.

Note, however, that the main stream of numerical relativity is focused on non-stationary problems important in generation of gravitational waves, in particular on black hole inspiral

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2The prologue to a major reference [69] (not speaking about its entire content) however convinces us that black holes are “the simplest objects” (also “the most perfect macroscopic objects”) in the universe. Cf. also [298, 115].

3Within general relativity, there is only one more ingredient that could make the situation even more dramatic: non-stationarity. The result would be a gravitational collapse or a collision of already collapsed objects. These events are currently devoted an exceptional attention as the strongest sources of gravitational waves.

4In the following, we do not review superpositions, obtained by any of the methods, of black holes with external electromagnetic fields. We refer to [34, 33, 196, 30] for test EM fields and to [40, 12, 11] for exact solutions. A more recent account can be found in [32].
collision (e.g. [296, 295, ] and a thorough review [187]).

1.2 Perturbative solution

A great deal of literature was devoted to perturbations of black-hole spacetimes and several formalisms were developed, none of which can be reviewed here. In [818], a fully explicit solution for the perturbation of the Schwarzschild metric by a (rotating) axisymmetric weakly gravitating thin equatorial ring was found by solving the perturbed Einstein equations. This direct approach is however not practicable for a rotating hole and/or for an extended external source with pressure (not mentioning more complicated situations).

The rotating case (specifically, the algebraically special vacuum case) was seized notably by Teukolsky [294] who succeeded in separating the decoupled equations for the first order perturbations of a Kerr black hole into the second-order ordinary differential equations for the scalar field (for scalar perturbations) or for the Newman-Penrose scalars constructed from the electromagnetic field tensor (for electromagnetic perturbations), from the neutrino spinor (for neutrino perturbations) or from the Weyl tensor (for gravitational perturbations). By solving the “gravitational” Teukolsky equation, the perturbative deformation of the Kerr horizon was calculated by [94]. The stationary axisymmetric Green’s function of the Teukolsky equation was provided by [195]. The most comprehensive expositions of the first-order black hole perturbation theory were given by Chandrasekhar [68, 69] in Newman-Penrose formalism. More recently, [183, 214, 213] solved the Teukolsky equation in terms of Coulomb wave functions and hypergeometric functions.

In a related method of handling the first-order perturbations of rotating fields, first devised for electromagnetic perturbations by Cohen & Kegeles and then generalized to gravitational ones [76], the perturbation components are expressed in terms of a single (Debye) potential that obeys a wavelike equation. Wald showed that this equation is just the adjoint of the Teukolsky master equation, while [160] gave a covariant extension of the method of Debye superpotentials to neutrino, electromagnetic and gravitational perturbations of all algebraically special spacetimes (see [300, 303] for summary, references and generalization). In the meantime, [77] treated the problem of perturbation of the Kerr horizon after having learnt [76] how to fix the perturbed metric from the Newman-Penrose functions. General perturbations of Schwarzschild solution were treated in this manner by [301]. Latest advance of the approach consists mainly in formulating the conservation laws for perturbations [303, 304].

Let us mention yet another approach [269], also dealing with curvature components but not restricted to algebraically special backgrounds. It has been shown, in the static case and to the linear order so far, that it can be generalized naturally to self-gravitating matter fields; in a spherically symmetric case, the connection with older, metric formalisms of Regge & Wheeler and Zerilli was elucidated.

In the charged case, the perturbation problem introduces coupling between gravitational and electromagnetic quantities, implying e.g. conversion between gravitational and electromagnetic waves. Interacting perturbations of the Reissner-Nordström black hole were studied by [32] (using the earlier formalisms of Regge & Wheeler, Zerilli and Moncrief) and by [202] (using the method of scalar potentials). A gauge invariant derivation of the basic equations of different formalisms was provided in [110], while [111] related their rotating analog to Teukolsky equation.

\footnote{In monograph [69], the author admits that “...the account, in large parts, is hardly more than an outline” at the beginning of chapter The gravitational perturbations of the Kerr black hole which has 101 pages.}
The second-order perturbation theory has also been under development since 1970s. For Schwarzschild black holes it is surveyed in [125], while the rotating case is tackled in [57].

It should be remarked that solutions describing stationary sources around black holes were not the primary aim of the black-hole perturbation strategy. Historically, attention has rather been devoted to the stability of black-hole solutions [255], to the relaxation of black holes into a stationary “no hair” state [134], involving the issue of back reaction of a perturbation propagating out of [21] or into the hole [240, 241] (cf. [144]), to non-stationary processes of astrophysical significance such as the behaviour of weakly gravitating particles or waves in black-hole backgrounds [253, 148], and, of course, to gravitational waves [23] — presently mainly to black-hole collisions as their most prominent source (e.g. [58, 59, 161]; for a survey, see [296]).

The latter is exactly an example of a problem where numerical relativity and perturbative analysis may — and should — coexist while mutually checking each other [270, 18, 199].

1.3 Exact solution

Perturbative approach is adequate in situations when the external source has only a very small effect on the field of the main body. Whenever this is not the case, the superpositions have to be described by solutions of the full, non-linear theory. At the end of Jiří Bičák’s recent survey [33], one is encouraged “not to cease in embarking upon journeys for finding them, and perhaps even more importantly, for revealing new roles of solutions already known”, because “Is there another so explicit way of how to learn more about the rich possibilities embodied in Einstein’s field equations?” For a more technical demonstration, see the canonical monograph [174] (an updated version is awaited); more recent reviews of vacuum fields can be found in [45, 48].

The only stationary (in fact static) equilibrium configuration containing more than one black hole is the Majumdar-Papapetrou case with extreme centres of the Reissner-Nordström type (e.g. [123, 250, 52, 75]), where the electrostatic repulsion exactly compensates the gravitational attraction. With rotating sources, gravitomagnetic interaction between spins (repulsive in the parallel case — see e.g. [252]) and magnetic interaction between magnetic dipole moments (repulsive in the antiparallel case) are also present. Analysis of the momentarily stationary and axisymmetric system of two identical sources was carried out by [96, 210] for the Kerr components (with mass $M$ and specific rotational angular momentum $a$) and by [37, 211] for the Kerr-Newman components (with $M$, $a$, and charge $Q$). Equilibrium was found to be only possible for the extreme values of the charges ($Q = M$). In all other cases, supporting singularities (“struts”) occur indicating that a given system of sources cannot remain stationary (or static) according to the field equations. Hence, rotating centres can only remain in equilibrium in a super-extreme case of two naked singularities (that have $Q^2 + a^2 > M^2$) [50], both the magnetic dipole-dipole and the gravitational spin-spin interactions being too weak to keep apart black holes (having $Q^2 + a^2 \leq M^2$). (On the other hand, a non-singular equilibrium is possible with two spinning Curzon particles [95]. Most recent discussions of both cases, in particular of the character of supporting singularities, can be found in [191, 43, 47].)

Accretion discs of astrophysical interest are likely to lie in the equatorial plane of the centre and unlikely to bear a considerable charge, so that neither the spin-spin nor the electromagnetic repulsion can act within them. One must rather refer to a centrifugal force resulting from the orbital motion of the material or to hoop stresses when interpreting the situation.

It turned out to be difficult to superpose a Kerr centre with an additional axisymmetric ring, disc or torus and no explicit exact solutions describing the systems of this kind have been found until now. Nevertheless, many stationary axisymmetric solutions of the (electro)vacuum
Einstein equations are known that do generalize those containing only isolated black holes. These were mostly obtained by indirect methods known as “generating techniques”. Two major approaches, developed by the end of the 1970’s — the group-theoretic and the soliton-theoretic (or inverse-scattering) techniques, work for spacetimes with two commuting symmetries. (See [170] for a review and [84, 85, 135, 206] for more detailed analysis and interrelations between different formulations, e.g. those of Kinnersley & Chitre; Maison; Belinskii & Zakharov; Harrison; Hoenselaers, Kinnersley & Xanthopoulos; Hauser & Ernst; Neugebauer; Kramer & Neugebauer; or Alekseev.) Other related methods have been proposed more recently, e.g. the simplification of the Hauser-Ernst integral equation by Sibgatullin [281], “monodromy data transform” by Alekseev [10] (also [165]), the static gravitational multipoles [128] (cf. [260]) and their superposition with stationary fields [129] by Gutsmaev & Manko, the linear transformation by Quevedo [260] or the “finite-gap” (algebraic-geometric) solutions by Korotkin & Matveev ([172] and references therein). Most of them are briefly compared in (appendix 6. of) [11]. (For other approaches, see e.g. [290, 82] or the results of Nakamura and Kyriakopoulos, referred to and worked out by [293]; cf. also [308].)

The crucial point of the soliton generating techniques is a solution of two linear differential equations (Lax pair) the integrability conditions of which are exactly the Einstein’s equations (namely the Ernst equation). The linear problem can be tackled in order to generate new solutions from the known ones: given some (“seed”) metric with two Killing vectors, it yields a different metric of the given type (the procedure is often called Bäcklund transformation in analogy with the technique used for the KdV equation). In such a manner, many known spacetimes were reproduced, but also broad families of new solutions were provided characterized by arbitrarily large sets of free parameters. However, only a very restricted number of them have been given a clear physical interpretation. Though several of these results perhaps represent a rotating black hole in an “external” gravitational field (e.g. [261, 208, 209, 51, 71, 72]), none of the latter has yet been specified to the case generated by a concrete realistic body such as ring, disc or torus (cf. the last paragraph of section IV in [190]).

Making the “random” Bäcklund transformation with some metric, it is, however, difficult to require specific physical properties of the spacetime being constructed — one rather “takes pot luck” and looks what comes out (see [260] for a hint how to overcome this); section 5.1 will illustrate what problems may arise. It would be more appropriate to express the physical boundary conditions of the Ernst equation in terms of the quantities which appear in the linear problem and then solve the latter. This leads to Riemann-Hilbert problems known from the theory of completely integrable differential equations [234, 165, 13]. Tackling the stationary axisymmetric boundary value problem with the help of the linear system, the fields of two physically relevant types of sources have been discussed: that of black holes [213, 235] and that of finite thin discs (219, 233, 14, 14 and references therein). There is some hope that the above methods could also be used to describe superpositions of both (e.g. 235).

2 Astrophysical relevance

The system of a black hole with an accretion disc is very important in astrophysical considerations. In particular, it plays the key role in models of a whole range of active galactic nuclei and of some X-ray binaries.
2.1 Galactic nuclei and X-ray binaries

Extraordinary luminosity generated within a small volume (as manifested by its rapid variability), the presence of very hard (X and gamma) radiation, high rotation speeds and broad velocity distribution of the material around indicate that there is an interacting ultracompact object in the inner regions of these sources \[262, 63, 182, 60, \] The ambient gas, present in the galactic nucleus or overflowing from the double-star primary, typically has enough angular momentum not to fall directly (almost radially) onto the object; it rather forms a disc and only gradually spirals down. The flow is strongly sheared due to a considerable non-homogeneity of the field, which enables viscous torques to heat the gas to high temperatures. (See e.g. \[253, 63, 6, 264\] for various aspects of the modern accretion disc theory. In particular, the most realistic grid of non-LTE disc models was constructed in the paper series \[136, 137, 138, 139\].) In some cases, the excess of angular momentum drives outflows through the empty regions along the rotation axis of the system. Jets of matter-energy emanate from a number of active nuclei, often at relativistic speeds, feeding giant outer lobes distinguished in radio surveys. The formation mechanism of the jets is studied by several groups using relativistic-MHD numerical simulations (e.g. \[165, 133\]). The high degree of jet collimation evidences an important role of magnetic fields \[154, 4, 93, 218\] and poses questions concerning the interaction of jets with the ambient gas and radiation. (See chapter 9 of \[315\] and \[197\] for a review.) The entire topic of active galactic nuclei is covered in \[177\] (for earlier references, see \[317\]) while that of X-ray binaries in \[193\]. Recent status of both areas can be learned from \[152\].

In spite of the observational variety of active galaxies (blazars, quasars, Seyfert galaxies, radiogalaxies, . . .), it now seems that we simply look at a similar type of source from different directions \[102\]. The discovery of jet structures accompanying several X-ray binaries in our Galaxy (the literature speaks of “microquasars” \[228, 282\]) has strengthened the conviction that the stellar-size active sources are also powered by the same accretion mechanism. Further evidence is provided by observations of compact X-ray sources in several nearby galaxies \[207\].

The suspicion of the presence of ultracompact objects has also been supported by more detailed considerations. Certain spectral features (the iron Kα line in particular) of some active galactic nuclei (a celebrated example is the Seyfert 1 galaxy MCG–6–30–15) have been interpreted as originating as close to the very centre as a few to a few dozens of Schwarzschild radii\[5\] so they may well represent data from the strongest fields ever met (see \[107\] for a review, and e.g. \[266, 267, 155, 88\] for models and interpretation issues). The iron lines have already been discovered in the X-ray spectrum of several Galactic microquasars \[224\] and references therein).

Also, the frequencies of quasi-periodic oscillations, found at a growing number of stellar-size sources, have been identified as the values characteristic for the accretion flow close to a black hole or a neutron star \[151, 230, 250, 97, 86, 286, 287, 225\]. This allows us to consider models involving orbital frequencies of “hot spots” in the inner parts of an accretion disc (e.g. \[154\]), beat of these frequencies with the centre’s spin \[257\], blobs on tilted orbits affected by Lense-Thirring precession \[87, 16, 221\], action of a secondary, intersecting the disc repeatedly while orbiting the centre \[289\], oscillations of the (thin) disc itself, perturbed off its circular
(equatorial) flow \cite{113}, or waves in tilted discs resulting from the Bardeen-Peterson effect \cite{113}. (Clearly, the processes can proceed in a symbiosis, for instance the secondary would probably actuate hot spots as well as oscillations in the disc, pulling some of its material out into a tilted orbit.) See \cite{300} for a survey and \cite{285, 87, 286, 287, 232} for discussions of the individual models in the light of new observations.

Among the sources displaying the presence of an interacting ultracompact body there is a class whose luminosity is still very low and fluctuating within a wide range. This seems to point towards the existence of a horizon, because a quasi-stationary inflow onto a neutron star should yield a relatively strong and steady output, produced mainly by the gas streams colliding with the star’s surface. With a horizon, the accretion may (perhaps occasionally) switch over to an “advection-dominated” regime when the material plumbs in without having released any significant fraction of its binding energy \cite{233, 119}. See \cite{5} for a general model of advection-dominated discs, \cite{141, 142, 179, 212, 204} for recent numerical simulations of their structure and spectrum, \cite{217, 226} for a study of jet production in such discs, and \cite{226} for a review.

The main goal of precise measurements of sources suspected of core collapse is a determination of three independent parameters — mass of the compact centre $M$, its specific angular momentum $a$, and radius $r$ where a given process happens. Describing the core field by the Kerr metric, the above three parameters appear in the formulae employed in models of the observed phenomena. These can be inverted if the necessary independent observables (at least three, in general) are really measured; see e.g. \cite{276} for a simple example of such an inference, \cite{120} for main techniques of fixing the central $M$ in galaxies (\cite{205} proposed a new one), and e.g. \cite{119, 222, 7, 200} for other prospects.

The dividing line between neutron stars and black holes being given by the Landau-Oppenheimer-Volkoff mass limit of about $M = (2 \div 2.5)M_\odot$ ($M_\odot$ is the solar mass), attention has recently been focused on determination of the rotation rate of the more massive cases. There is some evidence that highly active sources (in “hard X-state”, “radio-loud”, with strong and fast jets) host rapidly rotating holes and vice versa \cite{263, 8, 207, 217, 286, 124}. In particular, the popular microquasars GRS 1915+105 and GRO J1655-40 favour rapid central rotation, perhaps even close to the extreme value $a = M$ which separates black holes from objects without a horizon (naked singularities).\footnote{Note that GRS 1915+105 provided the first direct evidence of the disc-jet interaction \cite{100}; GRO J1655-40 is likely to have originated in a supernova explosion \cite{142}. Their central spins however remain uncertain — cf. the disjoint ranges estimated in \cite{224, 8} and \cite{113}. The spin of the central black hole in our Galaxy is discussed in \cite{224}.}

Such a rapid rotation is expected due to theoretical reasons — calculations of the effect of disc accretion on the central body \cite{314, 8}. On the other hand, a hyper-extreme object would probably be slowed down to $a = M$ by accretion \cite{56, 288, 119}. The centre (and the disc) could also lose rotational energy to the benefit of the material outflowing from the system. The significance of the Blandford-Znajek mechanism of jet fuelling \cite{11, 263} for the active galactic nuclei and X-ray binaries is still under discussion \cite{258, 122, 198, 288, 109} (while it is rather generally accepted for gamma-ray bursts \cite{186}). There also exist other extraction possibilities (see \cite{309} for a review), mainly the Punsly-Coroniti variant of Penrose mechanism \cite{259} (contrary to the Blandford-Znajek one, it does not require a magnetic field threading the horizon) which was indeed noticed in numerical simulations \cite{169}. Another efficient scenario has been proposed recently \cite{194}.

\footnote{Conversely, cf. a recently proposed gedanken experiment \cite{2} which appears to show that it is possible, in principle, to turn an extreme (Reissner-Nordström) black hole into a (Kerr-Newman) naked singularity by a radial infall of a (strongly bound, extended and electrically neutral) spinning body. In a related example, \cite{140} tried to overcharge a near-extremal Reissner-Nordström hole.
Black holes are certainly favoured by the “cosmic censorship” hypothesis which defends predictability by excluding the naked singularities as the outcome of a realistic gravitational collapse. However, this “question is still very much open” (also and references therein) whose actual significance is debatable (e.g. and references therein).

The experimental task of deciding between black holes and naked singularities thus brings us back to the fundamental problems of the theory. It should be emphasized here that Jiří Bříčák has always taught us to respect the decisive validity of observations, notwithstanding his obvious esteem for the penetrating “pure reasoning” (on the back of an envelope). Jiří often shows romantic affinity to astronomy and direct contact with nature, though majority of his results concern mathematical aspects of the theory. I heard him mentioning the need for a balance between the “right and left deviation” in one’s work. As a star on the stage of our Institute, he also alternates a glowing guitar in “Rock around the clock” and a delicate violin in Vivaldi.

2.2 Gravity of accretion discs around black holes

It is the subject of a standard university course to discuss the features of the Schwarzschild, Reissner-Nordström, Kerr and Kerr-Newman metrics (see for a review), exactly describing isolated stationary black holes (or naked singularities). Black holes expected in galactic nuclei and in X-ray binaries are not isolated, however: matter is concentrated rather than absent there. Most of the evidence for ultracompact bodies is in fact based on their interaction with the surroundings. In theoretical models, the external matter is supposed to form an accretion disc around the equatorial plane of the centre (this is justified by the approximate reflectional symmetry of the host systems and, at least in the discs’ inner parts, by Bardeen-Peterson effect; for later results, see ). Accretion is a problem governed by complicated magneto-hydrodynamics and radiative transfer which is usually tackled under the assumptions of smoothness, stationarity, axisymmetry and simple parametrization of the flow. Gravity of the disc itself is neglected, the field is thus fully determined by the centre, either described exactly by the Kerr metric, approximated by the Schwarzschild one, simulated by some pseudo-Newtonian potential, or quite reduced to a Newtonian (−M/r).

The main topic of the present paper is an inclusion of the disc gravitational field in a fully general relativistic description. treatments of the problem began with Ostriker’s equilibria of uniformly rotating, polytropic, self-gravitating slender rings. The paper opened with references to previous centuries and concluded with remarks on extragalactic radio sources and stability of rings, both of which were to boom soon. Later, polytropic self-gravitating configurations with a realistic equation of state and opacity were calculated by (cold case) and (hot case), already suggesting the possibility of a black hole present in galactic centres. incorporated viscosity (in a self-similar case). In the meantime, basic formalism for studying self-gravitating discs around black holes in a stationary axisymmetric general

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11Black holes are sometimes said to reside where God divided by zero. Regardless of whether he or she used Good coordinates or not, this is consistent with his/her being Almighty. Certain current readings of the cosmic censorship conjecture (“God abhors a naked singularity” sound somewhat disturbing, on the contrary.

12The papers must be studied in detail in order to understand precisely what was done. The topic of accretion discs is very wide and conclusions found in the literature are often hardly comparable due to different kinds of simplifications made, relating not only to the gravitational field and symmetries, but mainly to the rotation law, accretion rate, equation of state, viscosity, heat production, radiation pressure, opacity, boundary conditions (material supply), etc.
relativistic case was given by [22, 112].

Though the mass of real inner accretion discs is usually claimed not to exceed few percent of the central mass, the role of their self-gravity is not fully understood yet. Are any properties of accretion discs so sensitive to the details of the field that even a small effect of the disc itself can alter the flow significantly? It was shown by [280] that the very global structure of the disc is an example of such properties. This paper presented an extension of the disc “α-models” (both in their Newtonian version of Shakura & Sunyaev and in the relativistic version of Novikov & Thorne) which includes the effects of self-gravitation, radiation pressure and variable opacity source. The self-consistent configurations thus obtained were considerably thinner than their test counterparts, showing sharp radial decrease of the vertical dimension in the middle region. The implications of self-gravity concerning global properties of the disc (e.g. distribution of angular momentum and outer radius) were discussed and interpreted in a Newtonian analysis [6], the roles of the disc density and total mass were distinguished and specified in particular. With increasing gravity of the disc, regions occurred where free circular motion became unstable or even impossible, thus it was stated that “astrophysically relevant models of thick accretion discs must incorporate self-gravity effects in order to be self-consistent” (the same had already been indicated in [280, 316] before).

Changes in the properties of circular equatorial motion were later observed in the pseudo-Newtonian study [162] of a rotating black hole surrounded by a massive thin equatorial ring, in a relativistic analysis of the corresponding static case [24], and also in the perturbative solution [314] for a slowly rotating, weakly gravitating thin ring around a Schwarzschild black hole [313]. These changes were claimed to result in modifications of the observable characteristics, e.g. in higher temperature of the disc. The latter was confirmed by numerically constructing equilibrium structures of self-gravitating polytropic thick discs in a pseudo-Newtonian potential [12]. The accretion-disc “self-field” was also discussed in connection with the disc’s vertical structure and stability. It turns out that certain modes of instability can be damped whereas other are amplified by the disc’s gravity (see [244, 13]). The latest fully relativistic results — employing numerical spacetimes obtained in [181, 237] — include the computation of emission-line profiles from self-gravitating discs around rotating black holes; the case of light, thin finite discs was treated by [156] while that of heavy toroids by [305].

Self-gravitating accretion discs also appeared in other contexts, some of which are interesting from the point of view of general relativity (e.g. heavy discs around white dwarfs as a ponderable stage of a double-white-dwarf merger [130]), whereas other are not relevant for the present review (young stars [320, 321], outer parts of nuclear discs, galactic discs [279], effect on bipolar jets [117]).

Recently we have studied exact solutions for a static (Bach-Weyl) ring [64] and static thin annular disc [188] around a non-rotating black hole; some of the results are presented in section 4. We also checked the possibility to generalize them to a stationary situation [325] (section 3) when the hole and the disc were allowed to rotate and the self-gravity effects could be even more pronounced. Actually, in relativity, the kinetic (in fact any) energy also generates the field and mass-energy currents give rise to frame dragging. Whereas mass of the hole should be dominant in a real accretion system, the disc can bear much (even most) of the angular momentum, thus modifying the gravitomagnetic field of the centre significantly. This could be important for the mechanisms of rotational energy extraction.

\[13\] This only applies to the ultracompact centre (cf. e.g. [314, 231], however), not to protostellar discs [321] where even the opposite may well happen [326].
2.3 Gravitational collapse and gamma-ray bursts

We should not forget the accretion system where the mass of the disc or torus can actually be comparable to that of the central hole. It is considered in the literature as a transient state of gravitational collapse that could power the gamma-ray bursts. In a present-day relativistic astrophysics, perhaps the most popular system is a close binary of ultracompact objects. The famous Hulse-Taylor binary pulsar (PSR 1913+16) turned out to spiral in at the rate predicted by general relativistic loss of angular momentum via gravitational radiation [291], thus providing the first, indirect confirmation of the existence of gravitational waves. Nowadays late stages of evolution of such systems (inspiral, merger and “ringdown”) are studied numerically or perturbatively by a number of research groups, mainly as the most promising source of gravitational waves [290], and also as an engine suggested for the gamma-ray bursts [223]. In particular, at late stages of the neutron star–black hole binary inspiral, the neutron star tidally breaks up. Loosely speaking, if this happens outside the innermost stable orbit around the hole, the neutron debris spreads into a strongly magnetized torus (otherwise it plunges into the hole). Similar outcome is expected in case of a binary with two neutron stars (which is expected to be less frequent than the black hole–neutron star case [29]). It was pointed out in numerical analyses that the black hole–neutron torus system was liable to runaway instability (see [210] and references therein). Depending on the exact shape of the created magnetosphere, the energy release is mainly ensured by leptoni c winds from pair-creation along the axis of rotation or/and axially collimated Poynting outflow resulting from the Blandford-Znajek mechanism [186, 307]. In [307], “Long/short gamma-ray bursts are identified with suspended/hyper- accretion onto rapidly/slowly rotating black holes. . . . In long bursts, the torus is expected to radiate most of the black hole luminosity in gravitational waves. This predicts that long gamma-ray bursts are potentially the most powerful burst-sources of gravitational waves in the Universe.” (This thorough review also conjectures about connections between gamma-ray bursts and active galactic nuclei and micro-quasars.)

The black hole with a disc/torus can also arise as the outcome of a gravitational collapse of a massive rotating star. This was already mentioned in [231], while more recently in scenarios of “failed supernovae” from the cores of young stars, either isolated [203] or in a binary (this case is referred to as “hypernovae”) [243, 53]; for a review, see discussions in [307, 53].

3 Stationary axisymmetric spacetimes

It is natural first to try to include the gravitational effect of the external source in the simplest case when the system is stationary and axisymmetric. Stationary axisymmetric spacetimes are of obvious astrophysical importance: they describe the exterior of bodies like stars, galaxies or accretion discs in equilibrium. The issue has been exposed in many places (e.g. [192, 61, 22, 318, 55, 67, 103, 143, 74] or chapters 17–19 of [174]); for more information about the most important particular solutions, see [45, 33].

In the Weyl-Lewis-Papapetrou coordinates \((t, \rho, \phi, z)\) of the cylindrical type, the metric can be written as

\[
\mathrm{d}s^2 = -e^{2\nu} \mathrm{d}t^2 + \rho^2 B^2 e^{-2\nu} (\mathrm{d}\phi - \omega \mathrm{d}t)^2 + e^{2\lambda-2\nu} (\mathrm{d}\rho^2 + \mathrm{d}z^2),
\]

(1)

The quantities will be given in geometrized units in which \(c = G = 1\); the signature of the metric tensor \(g_{\mu\nu}\) is \((-+++\)). Greek indices run from 0 to 3 and Latin indices \((i, j, \ldots)\) run from 1 to 3; indices from the beginning of the Latin alphabet \((a, b, \ldots)\) represent cyclic coordinates \(t\) and \(\phi\). Partial differentiation is denoted by \(\partial\) or by a subscript comma, covariant derivative is denoted by \(\nabla\) or by a subscript semicolon.
where the unknown functions $\nu$, $B$, $\omega$ and $\lambda$ only depend on $\rho$ and $z$; $\omega$ is interpreted as the angular velocity of inertial space rotation with respect to observers at rest in spatial infinity. The metric coefficients $g_{\phi\phi}$ have invariant meaning, they can be expressed in terms of Killing fields $k^\mu = \partial x^\mu / \partial t$ and $m^\mu = \partial x^\mu / \partial \phi$:

$$g_{tt} = -e^{2\nu} + \omega^2 g_{\phi\phi} = k_k k^k,$$

$$g_{t\phi} = -\omega g_{\phi\phi} = k_i m^i,$$

$$g_{\phi\phi} = \rho^2 B^2 e^{-2\nu} = m_i m^i. \tag{4}$$

The simple result for the corresponding subdeterminant, $\det(g_{ab}) = -\rho^2 B^2$, implies

$$g^{tt} = -e^{-2\nu}, \quad g^{t\phi} = -e^{-2\nu} \omega, \quad g^{\phi\phi} = -e^{-2\nu} \omega^2 + \rho^{-2} B^{-2} e^{2\nu}. \tag{5}$$

### 3.1 Field equations

The Einstein equations read

$$\vec{\nabla} \cdot (\rho \vec{\nabla} B) = 8\pi \rho B (T_{\rho\rho} + T_{zz}), \tag{6}$$

$$\vec{\nabla} \cdot (\rho^2 B^3 e^{-4\nu} \vec{\nabla} \omega) = -16\pi B e^{2\lambda - 2\nu} T^t_{\phi}, \tag{7}$$

$$\vec{\nabla} \cdot (B \vec{\nabla} \nu) = \frac{1}{2} \rho^2 B^3 e^{-4\nu} (\vec{\nabla} \omega)^2 + 4\pi B e^{2\lambda} (2T^{tt} + e^{-2\nu} T), \tag{8}$$

where $T \equiv T^{\sigma}_{\sigma}$ and $\vec{\nabla}$ and $\vec{\nabla}$ stand for the gradient and divergence in a flat three-dimensional space with coordinates $(\rho, \phi, z)$; thus $\vec{\nabla} X = (X_\rho, 0, X_z)$ and $\vec{\nabla} \cdot \vec{X} = \rho^{-1} (X_\rho^\phi \rho + (X^z)_z)$ in the axially symmetric case. Once $B$, $\omega$ and $\nu$ are known, $\lambda$ can be integrated from equations

$$B \lambda_z - B_z + \rho (B_\rho \lambda_z + B_z \lambda_\rho - B_{\rho z} - 2 B_\nu \nu_{\rho z})$$

$$+ \frac{1}{2} \rho^3 B^3 e^{-4\nu} \omega_{\rho z} \omega_z = 8\pi \rho B T_{\rho z}; \tag{9}$$

$$2B \lambda_\rho - 2B_\rho + \rho \left[ 2B_\rho \lambda_\rho - 2B_z \lambda_z - B_{\rho \rho} + B_{zz} - 2B (\nu^2_\rho - \nu^2_z) \right]$$

$$+ \frac{1}{2} \rho^3 B^3 e^{-4\nu} (\omega^2_\rho - \omega^2_z) = 8\pi \rho B (T_{\rho\rho} - T_{zz}). \tag{10}$$

Although a suitable combination of solenoidal motions can also satisfy the assumption of axial symmetry, it is natural to expect that the axially symmetric source just rotates in the azimuthal direction, with an angular velocity $\Omega \equiv \partial \phi / \partial t$. The corresponding four-velocity is

$$u^\mu = u^t (k^t + \Omega m^t) = u^t (1, 0, \Omega, 0), \tag{11}$$

where $(u^t)^{-2} = -g_{tt} - 2\Omega g_{t\phi} - \Omega^2 g_{\phi\phi} = e^{2\nu}(1 - \hat{v}^2)$ and $\hat{v} = \rho B e^{-2\nu} (\Omega - \omega)$ is the linear speed with respect to the local zero-angular-momentum observer.

For a source made of ideal fluid, having a total mass-energy density $\epsilon$ and pressure $P$ as measured in a co-moving system, the energy-momentum tensor is $T_{\mu\nu} = (\epsilon + P) u_\mu u_\nu + P g_{\mu\nu}$.
and the equations of motion read \((\epsilon + P) a_\mu = -P_{\mu}\), where the four-acceleration \(a_\mu = u_{\mu;\nu}u^\nu\) can, for instance, be written as
\[
a_\mu = -\frac{1}{2} g_{\alpha\beta,\mu} u^\alpha u^\beta = -\frac{g_{tt,\mu} + 2\Omega g_{\phi,\mu} + \Omega^2 g_{\phi,\mu}}{2e^{2\nu}(1 - \hat{v}^2)}
\]
\[
= \frac{2e^{2\nu} g_{\phi,\mu} + 2(\Omega - \omega) g_{\phi,\mu}}{2e^{2\nu}(1 - \hat{v}^2)}.
\]
(12)

From equations (6)–(8) one then obtains
\[
\nabla \cdot (\rho \nabla B) = 16\pi \rho Be^{2\lambda - 2\nu} P,
\]
(13)
\[
\nabla \cdot (\rho^2 B^3 e^{-4\nu} \nabla \omega) = -16\pi \rho B^2 e^{2\lambda - 4\nu} (\epsilon + P) \frac{\hat{v}}{1 - \hat{v}^2},
\]
(14)
\[
\nabla \cdot (B \nabla \nu) = \frac{1}{2} \rho^2 B^3 e^{-4\nu} \nabla \omega)^2 + 4\pi Be^{2\lambda - 2\nu} \left[ (\epsilon + P) \frac{1 + \hat{v}^2}{1 - \hat{v}^2} + 2P \right],
\]
(15)

and \(\lambda\) is given by
\[
\lambda_{\rho} = \frac{B}{(\rho B)^2 + \rho^2 B^2} \left\{ B_{,\rho} + \rho B^{-1}(B^2 + B_{,\rho}^2) \right. \\
\left. + \rho \left[ (\rho B)_{,\rho} (\nu_{,\rho} - \nu_{,\rho}) + 2\rho B_{,\rho} \nu_{,\rho} \nu_{,\rho} \right] \\
- \frac{1}{4} \rho^3 B^2 e^{-4\nu} \left\{ (\rho B)_{,\rho} (\omega_{,\rho}^2 - \omega_{,\rho}^2) + 2\rho B_{,\rho} \omega_{,\rho} \omega_{,\rho} \right\} \right\},
\]
(16)
\[
\lambda_{,z} = \frac{B}{(\rho B)^2 + \rho^2 B^2} \left\{ B_{,z} + \rho \left[ B_{,z} (\nu_{,z}^2 - \nu_{,z}^2) + 2\rho B_{,z} \nu_{,z} \nu_{,z} \right] \\
- \frac{1}{4} \rho^3 B^2 e^{-4\nu} \left\{ \rho B_{,z} (\omega_{,z}^2 - \omega_{,z}^2) + 2\rho B_{,z} \omega_{,z} \omega_{,z} \right\} \right\}.
\]
(17)

The unknown metric functions are subject to boundary conditions on the horizon (if there is one), on the symmetry axis and at spatial infinity. The metric must be regular on the horizon and on the axis; in the case of an isolated source one requires asymptotic flatness. The conditions were discussed by [61, 22], for example.

### 3.2 Vacuum case and the Ernst formulation

In the following sections, we will be interested in vacuum solutions or in those with infinitely thin sources. In both cases \(P = 0\) and the equation (13) has only one solution with a satisfactory asymptotic behaviour [namely (33)], \(B = 1\). Only two of the field equations then remain, (18,19), in the form
\[
\nabla \cdot (\rho^2 e^{-4\nu} \nabla \omega) = 0,
\]
(18)
\[
\nabla^2 \nu = \frac{1}{2} \rho^2 e^{-4\nu} (\nabla \omega)^2,
\]
(19)
and the relations (16,17) reduce to

\[ \lambda,\rho = \frac{1}{4} \rho^3 e^{-4\nu}(\omega^2 - \omega_z^2), \]

(20)

\[ \lambda,\nu = 2\rho \nu,\rho z - \frac{1}{2} \rho^3 e^{-4\nu}\omega,\omega z. \]

(21)

Many other formulations of the stationary axisymmetric problem are possible, starting from a different definition of the four (or three) unknown metric functions. In the vacuum case the formulation by Ernst [104, 105] is often considered, where the field equations are translated into the (Ernst) equation

\[ -g_{tt} \nabla^2 \mathcal{E} = (\nabla \mathcal{E})^2 \]

(22)

for a complex (Ernst) potential \( \mathcal{E} \equiv -g_{tt} + i\psi \) whose imaginary part \( \psi \) is given by

\[ \psi_z = g_{tt,\rho} g_{\phi,\rho} - g_{tt,\rho} g_{\phi,\rho}, \quad \psi,\rho = g_{tt,\rho} g_{\phi,\rho} - g_{tt,\rho} g_{\phi,\rho} ; \]

(23)

\( \lambda \) is found by a line integral as above.

### 3.3 Horizon

(See e.g. [61, 62, 115] for thorough accounts.) A black-hole horizon is a null hypersurface below which the spacetime is dynamical. In Weyl-Lewis-Papapetrou coordinates, it is located where \( \det(g_{ab}) = -\rho^2 B^2 \) is zero. Also vanishing there is the lapse (or redshift factor) \( \alpha \equiv e^\nu \) or the magnitude of the Killing bivector \( k_{[\kappa,m\lambda]} \); this is clear from relations

\[ -2k_{[\kappa,m\lambda]}k_{[\kappa,m\lambda]} = -\det(g_{ab}) = \alpha^2 g_{\phi\phi}, \]

(24)

because \( g_{\phi\phi} \) (as well as \( g_{\rho\rho} \)) must be regular (and positive) on the horizon. In the vacuum case \( B = 1 \), so the horizon lies on the axis (\( \rho = 0 \)). The interior of the black hole thus has to be studied in different coordinates, e.g. in spheroidal coordinates of the Boyer-Lindquist type \( (t,r,\theta,\phi) \), introduced by the transformation

\[ \rho = \sqrt{\Delta} \sin \theta, \quad z = (r - M)y, \]

(25)

where \( \Delta = (r - M)^2 - k^2 \) and \( y = \cos \theta \), \( M \) being a scale parameter (it usually represents the total mass) and \( (\pm)k \) determining where the horizon reaches up along the \( z \) axis. Sometimes an isotropic radial coordinate \( R \) is employed, given by

\[ \Delta = R^2 \left( 1 - \frac{k^2}{4R^2} \right)^2. \]

(26)

On the horizon, \( \Delta = 0 \) and both radial coordinates are constant, \( r = M + k \equiv r_H \) (hence, \( z = ky \)), \( R = k/2 \equiv R_H \) (there are more black-hole horizons in general, we mean the outermost one here).

There are three important parameters of the horizon: its surface area (proportional to the entropy of the black hole), the surface gravity (proportional to the horizon temperature) and the angular velocity relative to infinity,

\[ A = 2\pi \int_0^\pi \sqrt{(g_{\theta\theta}g_{\phi\phi})_H} \, d\theta, \quad \kappa = |\nabla e^\nu|_H, \quad \omega_H = \omega(r = r_H); \]

(27)
\(\kappa\) and \(\omega_H\) are constant all over the horizon.

For spacetimes with a symmetry lower than spherical, the question of the “true shape” of the black hole, not distorted by coordinates, is non-trivial. How is the hole influenced by rotation, electromagnetic field or an external source? The answer is obtained by representing the horizon as a two-dimensional surface in a three-dimensional Euclidean space. Following [283], the two-dimensional metric is first rewritten as

\[
ds^2 = \frac{A}{4\pi} [h^{-1}(y)dy^2 + h(y)d\phi^2],
\]

where \(h(y) = (4\pi/A)(g_{\phi\phi})_H\). An isometric embedding of the two-surface \((y,\phi)\) in \(\mathbb{E}^3\) with coordinates \((X,Y,Z)\) is then given by

\[
X = \frac{A}{4\pi} \sqrt{h} \cos \phi, \quad Y = \frac{A}{4\pi} \sqrt{h} \sin \phi, \quad Z = \frac{A}{4\pi} \int_0^y \sqrt{\frac{1}{h} \left(1 - \frac{1}{4}h_{,yy}^2\right)} dy.
\]

If its Gaussian curvature

\[
C_H = -\frac{8\pi^2}{A^2} h_{,yy}
\]

is negative anywhere, the horizon cannot be (globally) embedded in \(\mathbb{E}^3\). It was shown in [283] that rotation can flatten the horizon considerably, the Gaussian curvature of a Kerr black hole finally turns negative at the axis for \(a/M > \sqrt{3}/2\) (\(a\) denotes the angular momentum per unit \(M\)). Some of the black holes swirling in galactic nuclei may even be impossible to imagine!

### 3.4 Static case

In the static case \(\omega = 0\) and the Weyl canonical form

\[
ds^2 = -e^{2\nu} dt^2 + \rho^2 e^{-2\nu} d\phi^2 + e^{2\lambda-2\nu}(d\rho^2 + dz^2).
\]

More precisely, the metric can be written in this way provided that \(T^\rho_\rho + T^z_z = 0\) (which is fulfilled by incoherent dust, certain electromagnetic fields or certain infinitely thin sources, but not by a fluid with non-zero pressure), otherwise there remains \(B^2\) in \(g_{\phi\phi}\).

Equations (16)–(20) reduce to the Poisson equation

\[
\nabla^2 \nu \equiv \rho^{-1}\nu,\rho + \nu,\rho \nu + \nu,zz = 4\pi e^{2\lambda-2\nu}(T^\phi_\phi - T^t_t)
\]

for \(\nu\) and to relations

\[
\lambda,z - 2\rho \nu,\rho \nu,z = 8\pi \rho T_{\rho z}, \quad \lambda,\rho - \rho(\nu,\rho - \nu,z^2) = 4\pi \rho(T_{\rho \rho} - T_{zz})
\]

for \(\lambda\).

### 3.5 The axis and the equatorial plane

On the symmetry axis \((\rho = 0)\), the regularity of metric requires \(e^\lambda = B\), so

\[
ds^2 = -e^{2\nu} dt^2 + B^2 e^{-2\nu} dz^2.
\]

The formulas can also be expected to simplify in the equatorial plane \((z = 0)\) if the spacetime is reflectionally symmetric with respect to it. This is usually the case in astrophysically motivated considerations. As a special case, the solutions with stronger, cylindrical symmetry (where another Killing field, \(\partial x^\mu/\partial z\), exists) can be mentioned, where any of the planes \(z = \text{const}\) is equatorial in this sense; however, there is no place for black holes in these spacetimes.
3.6 Singularities

From astrophysical point of view, a given solution is problematic mainly if it contains physical singularities on or above the horizon, if it shows bad asymptotic properties or if it does not correspond to any realistic source. The existence of a singularity follows — according to its type (see [101, 80]) — e.g. from the divergence of scalars constructed from the metric tensor and its derivatives or from the divergence of physical (tetrad) components of the Riemann tensor. (Usually, the Kretschmann invariant $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ is checked first.) However, the spacetimes are known which do contain singularities, although all their curvature invariants vanish. It is thus difficult to verify the non-existence of singularities; the only proof of regularity at a given point is to find a (local) coordinate system in which the metric is smooth enough there.

3.7 Analytic extension and global structure

Astrophysically relevant is that part of spacetime which can communicate with spatially remote regions, i.e. the “region of outer communications” outside the horizon. General relativity is interested in the black hole interior as well — in fact even there an observer can perform physical observations (and perhaps even survive). Unfortunately, in Weyl-Lewis-Papapetrou coordinates the metric is not defined below the horizon (this would correspond to imaginary radius $\rho$) and one has to extend it there. Such a task may be solvable but numerically and hardly expectable regions may open. What is the type and shape of the singularity under the horizon? How does it change when an additional source is “switched on” above the horizon? The answer is unknown even for quite simple superpositions (on the other hand, in a highly symmetric case it may well be possible to solve the dynamical problem [178]).

Far away from the horizon, one inquires for the asymptotic behaviour of the field. That of an isolated stationary source falls to zero there in a specific manner — the spacetime is said to be asymptotically flat (see e.g. [312], chapter 11, or [26], section 3). In case of the metric (1), it must hold, for $r \to \infty$,

$$\nu = -\frac{\text{total mass}}{r} + O(r^{-2}),$$

$$B = 1 + O(r^{-2}),$$

$$\omega = \frac{2 \cdot (\text{total angular momentum})}{r^3} + O(r^{-4}),$$

$$\lambda = O(r^{-2}).$$

The courses can be expressed, in the same manner, in terms of the isotropic radial coordinate $R$ or in terms of the radius $\sqrt{\rho^2 + z^2}$, since $R = r - M + O(r^{-1})$ and $\sqrt{\rho^2 + z^2} = r - M + O(r^{-1})$ for $r \to \infty$.

3.8 Circular orbits and ergosphere

The simplest type of motion in stationary axisymmetric spacetimes is that along spatially circular orbits ($\rho = \text{const}, z = \text{const}$) with steady angular velocity $\Omega$: such a motion takes place along the symmetries, namely tangent to each individual circular orbit is a Killing field (one

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15 This is clear from an explicit form of (26), $R = \frac{1}{2}(r - M + \sqrt{\Delta})$, or $r = R + M + \frac{k^2}{4R}$. 
speaks of “quasi-Killing” trajectories); physically speaking, an observer on a circular orbit sees
time-independent field around and is thus called the stationary observer. In order that the

\[
\Omega_{\text{max}} = \omega \pm \frac{\alpha}{\sqrt{g_{\phi\phi}}};
\]

(39)
on the horizon (where \(\alpha = 0\)) the permitted range narrows down to a single value \(\omega_H\).

Calculations involving acceleration are more difficult as they also contain derivatives of the metric. The circular-orbit four-acceleration (12) has at most two non-zero components (\(a_\rho\) and \(a_z\)); \(\mu\)-component is zero if

\[
\Omega = \Omega_{\pm} = \frac{-g_{t\phi,\mu} \pm \sqrt{(g_{t\phi,\mu})^2 - g_{tt,\mu}g_{\phi\phi,\mu}}}{g_{\phi\phi,\mu}},
\]

(40)

where the upper/lower sign corresponds to a “co-rotating”/“counter-rotating” \(\mu\) orbit.

In the equatorial plane, \(a_z = 0\) due to the symmetry. If \(\Omega = \Omega_{\pm}\) (here with \(\mu = \rho\), \(a_\rho = 0\) as well and the worldline is a geodesic. Among equatorial geodesics, three particular cases are most important: the photon, the marginally bound and the marginally stable orbits. The photon orbits bound the regions where circular motion is time-like; they are given by equalities

\[
\Omega_{\pm} = \Omega_{\text{min}}\text{max}
\]

(41)

[\(\mu = \rho\) being the only non-trivial index value in (40) now]. The marginally bound orbits limit the

regions where particles on circular orbits have lower energy than is necessary for the existence

at spatial infinity; the limiting case is given by \(\gamma(\Omega = \Omega_{\pm}) = 1\), where

\[
\gamma = -u_t = -u^t(g_{tt} + \Omega g_{t\phi}) = u^t e^{2\nu} + \omega \ell \quad \text{and} \quad \ell = u_\phi = u^t g_{\phi\phi}(\Omega - \omega)
\]

(42)

are specific energy and specific azimuthal angular momentum at infinity. The marginally stable

orbits bound the regions where circular motion is stable with respect to perturbations within

the orbital plane; they satisfy \([\ell(\Omega = \Omega_{\pm})]_\rho = 0\). They play a key role in the theory of accretion

discs: the matter is “swept away” from unstable sectors; in particular, the innermost marginally

stable orbit should represent the inner disc rim. (We will see in section \(4\) that perturbations

perpendicular to the orbital plane can also be important.)

A peculiar feature of rotating fields are the dragging effects. Their history starts from

the famous Newton’s bucket experiment (if not earlier, see Doležel in this volume) and continues

notably by Mach’s ideas of relativity of motion and inertia towards Einstein and Lense &

Thirring who showed, in the early times of general relativity, that the effect is present in this

theory (see the 1913’s letter of Einstein to Mach [166]). The effect has an obvious analogy in

electromagnetism, where (electric) currents generate the magnetic component of the field. This

relationship was already noticed by Einstein (see his Prague paper of 1912 [167]) and today the

gravito-electro-magnetic approach to rotating fields prevails.

An extreme implication of dragging is the occurrence of the ergosphere in the vicinity of

ultracompact rotating objects. In this region \(\Omega_{\text{min}} > 0\), thus the stationary observer cannot

be static (at rest relative to infinity), albeit he still withstands the radial attraction. The

static-limit surface, given by \(\Omega_{\text{min}} = 0\) and hence by \(g_{tt} = 0\), also represents the set of points

\[16\] These terms may be misleading in spacetimes with large angular momentum.

\[17\] Machian inspiration keeps vivid within contemporary relativity — see, for example, the last-decade references

[149, 73, 202, 204, 163] or the whole conference [21].
from where signals emitted by static observers \((\Omega = 0)\) reach infinity with an infinite redshift. Ergospheres received considerable attention (and the name from \(\epsilon\rho\gamma\omega\)) after processes had been suggested (working only there) \([246]\) by means of which the rotational energy of the centre could be extracted (without violating the second law of black-hole dynamics that forbids the horizon area to decrease).

With the metric \([0]\), \(g_{tt} = 0\) corresponds to \(\rho B\omega = \alpha^2\) which can only be satisfied by \(\rho \geq 0\), so the static limit really lies outside the horizon in particular, \(\rho = 0\) implies \(\alpha^2 = 0\), so the static limit touches the horizon at the axis.

### 3.9 Sources

In order to complete the relativistic solution, one must also describe their interior (where \(T_{\mu\nu} \neq 0\)). Finding a realistic interior solution is a remarkable achievement in general, mainly if it should match smoothly the vacuum exterior (e.g. \([180, 249, 73]\) and references therein); this is also confirmed by the history of searching for a source of the Kerr field (see \([38, 236]\) and references therein).

The problem is much simpler if the source is infinitely thin: the solution is then vacuum-type everywhere, with the energy-momentum tensor \(T_{\mu\nu} = g^{-1/2} S_{\mu\nu} \delta(z)\) found from the jump of the normal field across the source like in electrodynamics; the appropriate covariant method is known as the Israel’s formalism \([24]\). It starts from a projection of the exterior (here vacuum) metric \(g_{\mu\nu}\) onto a three-dimensional hypersurface \(S\), representing the history of the source: the projection from one (+) and the other (−) side of \(S\) yields the induced three-metrics

\[
\pm g_{\hat{A}\hat{B}} = g_{\mu\nu} \pm e^\mu_A \pm e^\nu_B, \tag{43}
\]

where \(\left\{ \pm e^\mu_A \right\}_{A=0,1,2}\) are bases tangent to \(\pm S\). The hyperplanes \(\pm S\) are however the back and the face of the same surface, so we can choose \(+ e^\mu_A = - e^\mu_A\) and only the normals \(+ n^\mu\) and \(- n^\mu\) differ. The surface density of the energy-momentum tensor \(S_{\hat{A}\hat{B}}\) reads

\[
S_{\hat{A}\hat{B}} = (8\pi)^{-1} \left( [K_{\hat{A}\hat{B}}] - g_{\hat{A}\hat{B}} [K^C_C] \right), \tag{44}
\]

where \([K_{\hat{A}\hat{B}}]\) denotes the jump of the exterior curvature across \(S\), i.e. \([K_{\hat{A}\hat{B}}] = + K_{\hat{A}\hat{B}} - K_{\hat{A}\hat{B}}\), with

\[
\pm K_{\hat{A}\hat{B}} = e^\mu_A e^\nu_B \nabla_\mu \pm n_\nu = -\pm n_\nu e^\mu_A \nabla_\mu e^\nu_B. \tag{45}
\]

The case of a thin source (a disc) in a stationary axisymmetric spacetime has notably been studied by \([184, 236]\). In a disc lying at \(z = \text{const}\) the radial pressure is zero, \(S_{\rho\rho} = 0\), and for a disc in the equatorial plane \((z = 0)\) of the reflectionally symmetric spacetime \(S_{zz} = 0\), too. Only \(S_{t\phi}\) is non-zero off the diagonal. In the Weyl-Lewis-Papapetrou coordinates the formula \((44)\) can be written as \((184, \text{chapters 3 and 4})\)

\[
S_{ab} = -\frac{\sqrt{g_{\rho\rho}}}{8\pi} \left( g_{ab} \right)_z \tag{46}
\]

\(^{18}\)In coordinates which also describe the black hole interior, two horizons and two static limits are found in general. We mean the outer horizon and the outer static limit everywhere.

\(^{19}\)The indices \(a, b, \ldots\) only run through \(t\) and \(\phi\) as anywhere.
Using (5) and (21), this yields
\begin{align}
S^t_t &= -\frac{e^{\nu - \lambda}}{8\pi} \left[ 4\nu_z (1 - \rho \nu_{,\rho}) - \rho^2 e^{-4\nu} \omega_{,z} (\omega - \rho \omega_{,\rho}) \right], \\
S^\phi_\phi &= -\frac{e^{\nu - \lambda}}{8\pi} \rho^2 e^{-4\nu} \omega_{,z}, \\
S^t_\phi &= -\frac{e^{\nu - \lambda}}{8\pi} \left[ 4\nu_z - (1 + \rho^2 e^{-4\nu} \omega^2) \omega_{,z} \right], \\
S^\phi_t &= \frac{e^{\nu - \lambda}}{8\pi} \rho \left[ 4\nu_{,\rho} \nu_{,z} - \rho e^{-4\nu} \omega_{,z} (\omega + \rho \omega_{,\rho}) \right].
\end{align}

The total mass and angular momentum of the disc are fixed by the Komar integrals that lead to
\begin{align}
M &= \frac{1}{2} \int_b^\infty g^{tc} g_{tc,\rho} \rho \, d\rho = \frac{1}{4} \int_b^\infty \frac{g_{\phi\phi}^2}{\rho} \left( \frac{g_{tt}}{g_{\phi\phi}} \right)_{,z} \rho \, d\rho, \\
J &= \frac{1}{4} \int_b^\infty g^{tc} g_{c\phi,\rho} \rho \, d\rho = \frac{1}{4} \int_b^\infty \frac{g_{\phi\phi}^2}{\rho} \omega_{,z} \rho \, d\rho,
\end{align}

where \(\rho = b > 0\) is the position of the disc inner rim. All the \(z\)-derivatives are understood to be calculated in the limit \(z \to 0^+\).

In the event that there is a black hole present in the centre of the disc, its contributions \(M_H\) and \(J_H\) need also to be included; for a given type of solution, the aggregate parameters of the spacetime can be written as the sums \(M_H + M\) and \(J_H + J\) (see e.g. [62]). The black-hole mass is related through the Smarr formula ([62], equation (6.297))
\begin{align}
M_H &= 2\omega H J_H + \frac{\kappa A}{4\pi} = 2\omega H J_H + k
\end{align}

with the angular momentum
\begin{align}
J_H &= -\frac{1}{8} \int_0^\pi (\Delta^2 e^{-4\nu} \omega_{,r})_H \sin^3 \theta \, d\theta.
\end{align}

The last step is a physical interpretation of the source. First, if
\begin{align}
D \equiv (S^\phi_\phi - S^t_t)^2 + 4S^t_t S^\phi_\phi &= \frac{e^{2\nu - 2\lambda}}{16\pi^2} (4\nu_z^2 - \rho^2 e^{-4\nu} \omega_{,z}^2)
\end{align}
is not negative, then there exists a stationary observer with the angular velocity
\begin{align}
\Omega_{iso} = (2S^\phi_\phi)^{-1} \left( S^\phi_\phi - S^t_t - \sqrt{D} \right) = \omega - \frac{2e^{4\nu} \nu_z}{\rho^2 \omega_{,z}} + \sqrt{\left( \frac{2e^{4\nu} \nu_z}{\rho^2 \omega_{,z}} \right)^2 - \frac{e^{4\nu}}{\rho^2}},
\end{align}

with respect to whom the energy-momentum tensor assumes a diagonal (“isotropic”) form
\(S^{\mu\nu} = \hat{\omega} u^\mu_{iso} u^\nu_{iso} + \hat{P} v^\mu_{iso} v^\nu_{iso}\), where \(u^\mu_{iso} = u^t_{iso}(1,0,\Omega_{iso},0)\) and \(v^\mu_{iso} = -\rho^{-1}(\ell_{iso},0,\gamma_{iso},0)\) are
the observer’s four-velocity and unit base vector in the $\phi$-direction, $\ell_{\text{iso}}$ and $\gamma_{\text{iso}}$ being the corresponding specific angular momentum and energy at infinity (see section 3.8). The observer measures the surface density
\begin{equation}
\hat{w} = \frac{\gamma_{\text{iso}}^2 S_{tt} - \ell_{\text{iso}}^2 S_{\phi\phi}}{u_{t\text{iso}}(\gamma_{\text{iso}} + \Omega_{\text{iso}}\ell_{\text{iso}})} \tag{57}
\end{equation}
and tangential pressure
\begin{equation}
\hat{P} = \frac{\rho^2 u_{t\text{iso}}^2 (S_{\phi\phi} - \Omega_{\text{iso}}^2 S_{tt})}{\gamma_{\text{iso}} + \Omega_{\text{iso}}\ell_{\text{iso}}} \tag{58}
\end{equation}
If $\hat{w} \geq \hat{P} > 0$, the energy-momentum tensor can represent two equal streams of particles moving on (accelerated) circular orbits in opposite directions at the same speed $\sqrt{\hat{P}/\hat{w}}$. This “speed of sound” was shown in [126] to be equal to the geometric mean of the local prograde and retrograde circular geodesic speeds $\hat{v}_{\pm}$ as measured by the observer $u_{\mu\text{iso}}$:
\begin{equation}
|\hat{v}_+\hat{v}_-| = \hat{P}/\hat{w}, \tag{59}
\end{equation}
where
\begin{equation}
\hat{v}_{\pm} = \frac{(v_{\text{iso}})_t + (v_{\text{iso}})_\phi \Omega_{\pm}}{(u_{t\text{iso}})_t + (u_{t\text{iso}})_\phi \Omega_{\pm}} \tag{60}
\end{equation}
and $\Omega_{\pm}$ are the angular velocities of the prograde and retrograde circular geodesics [10].

It was shown in [184] that in the absence of radial pressure the source disc can be composed of two counter-rotating streams of non-interacting particles on time-like circular geodesics, provided that the following conditions are satisfied: both geodesic frequencies (40) lie within the sub-luminal interval $(\Omega_{\text{min}}, \Omega_{\text{max}})$, the tensor (46) yields $S_{ab}X^aX^b \geq 0$ for at least one two-vector $X^a$, and $\det(S_{ab}) > 0$. The last two assumptions ensure that the superposition
\begin{equation}
S_{ab} = w_+ u_+^a u_+^b + w_- u_-^a u_-^b \tag{61}
\end{equation}
of the two dust components on circular geodesics $u_{\pm} = u_{t\pm}^t (1, \Omega_{\pm})$ comes out with positive proper surface densities of the streams
\begin{equation}
w_{\pm} = \pm \frac{S_{t\phi} - \Omega_{\pm} S_{tt}}{(u_{t\pm})^2(\Omega_+ - \Omega_-)} \tag{62}
\end{equation}
In a static case, the above formulae simplify to $\Omega_{\text{iso}} = 0$,
\begin{equation}
\dot{\hat{w}} = \frac{2w_+}{\sqrt{1 - v_+^2}} = -S_t = \frac{\epsilon^{\nu-\lambda}}{2\pi} (1 - \rho \nu, \rho) \nu, z, \quad \dot{\hat{P}} = S_{\phi\phi} = \frac{\epsilon^{\nu-\lambda}}{2\pi} \rho \nu, \rho \nu, z, \tag{63}
\end{equation}
\begin{equation}
M = \int_b^\infty \nu, z \rho d\rho, \quad J = 0, \quad M_H = k = M, \quad J_H = 0, \tag{64}
\end{equation}
\begin{equation}
\dot{\hat{v}}_{\pm} = \rho e^{-2\nu \Omega_{\pm}} = \pm \sqrt{\frac{\rho \nu, \rho}{1 - \rho \nu, \rho}} = \pm \sqrt{\frac{\hat{P}}{\hat{w}}} \tag{65}
\end{equation}
\[\text{20}\]The case of $D < 0$ (not involved here) represents discs with non-zero heat flow. Then the energy-momentum tensor must be written in a more general form $S_{\mu\nu} = \hat{w} u_{\mu\text{iso}}^\mu v_{\nu\text{iso}}^\nu + \hat{P} v_{\nu\text{iso}}^\nu v_{\text{iso}}^\nu + K(u_{\mu\text{iso}}^\mu v_{\nu\text{iso}}^\nu + u_{\nu\text{iso}}^\nu v_{\mu\text{iso}}^\mu)$ with $K = \sqrt{-D/2}$, and $\Omega_{\text{iso}} = (2S_t^t)^{-1}(S_{\phi\phi}^\phi - S_t^t) = \omega - \frac{2\nu, \rho \nu, \rho}{\rho \nu, \rho \nu, \rho}$. See [126].
Hence, the disc particles circumscribe circular geodesics in the spacetime which they generate themselves.

Now further physical requirements can be raised, namely the weak energy condition ($\hat{\omega} \geq 0$, $\hat{P} \geq -\hat{\omega}$), the dominant energy condition ($\hat{\omega} \geq 0$, $|\hat{P}| \leq \hat{\omega}$), and the non-negativity of pressure ($\hat{P} \geq 0$). Their combination yields $\hat{\omega} \geq \hat{P} \geq 0$. The requirements are very simple in the static case: if $\nu_z(z = 0^+) > 0$ (which is fulfilled for a realistic matter), then $\hat{P} \geq 0$ is equivalent to $\nu_\rho \geq 0$ which is satisfied below the Lagrangian point of zero field (where $\nu_\rho = 0$ and $\hat{v}_\perp^2 = 0$). In regions with tension ($\hat{P} < 0$), the particles are more attracted by the outer parts of the disc than by the central black hole, thus no circular geodesics exist (then hoop stresses have to be employed to interpret the disc matter); this typically happens in discs with too much matter on larger radii. The other condition $\hat{\omega} \geq \hat{P}$ is equivalent to $\hat{v}_\perp^2 \leq 1$. Hence, the above constraints are in fact ensured by the obvious condition $0 \leq \hat{v}_\perp^2 \leq 1$ which can be written explicitly as $1 \geq 1 - \rho \nu_\rho \geq 1/2$.

4 Static thin discs around non-rotating black holes

In a static spacetime rotation is excluded or it has to be compensated exactly as in the case of counter-rotating streams of matter. Within the vacuum outside of the sources, the static axisymmetric Einstein equations (32) and (33) yield the Laplace equation for $\nu$ and a simple quadrature for $\lambda$ (calculated along a vacuum path going from the axis to the given point),

$$\nabla^2 \nu = 0, \quad \lambda = \int_{\text{axis}}^\rho [\nu_\rho^2 - \nu_z^2] \, d\rho + 2 \nu_\rho \nu_z \, dz,$$

(66)

Thanks to the linearity of the Laplace equation, the solutions can simply be added to obtain fields of multiple sources. However, such a superposition may not be physically acceptable — supporting singularities ("struts") may occur in calculating $\lambda$ or it may not be possible to interpret the resulting system of sources in a reasonable way (for instance, the matter works out with negative density or pressure or moving at a superluminal speed). Consequently, even the static axisymmetric case has only afforded a few realistic superpositions. Let us refer to two of them involving a Schwarzschild black hole: the superposition with an infinite annular thin disc, obtained by Lemos & Letelier [188] by inversion of the first counter-rotating finite disc of Morgan & Morgan [229], and the one with an inverted isochrone thin disc, obtained by Klein [164]. The Lemos-Letelier solution has turned out to allow for physically satisfactory situations. We briefly summarize its properties below.

4.1 Superposition with the inverted first Morgan-Morgan disc

The inverted first Morgan-Morgan counter-rotating disc has a surface density

$$w(\rho) = \frac{2Mb}{\pi^2 \rho^4} \sqrt{\rho^2 - b^2},$$

(67)

$M$ and $b$ denoting the mass and the Weyl inner radius of the disc. The matter is thus concentrated near the rim ($w$ is maximal at $\rho = 2b/\sqrt{3} \approx 1.115b$) which is supposed to be the case in real accretion discs. The resulting spacetime is given by a sum of potentials which in Weyl coordinates read

$$\nu_{\text{Schw}} = \frac{1}{2} \ln \frac{d_1 + d_2 - 2M}{d_1 + d_2 + 2M},$$

(68)
and
\[ \nu_{\text{disc}} = -\frac{M}{\pi (\rho^2 + z^2)^{3/2}} \left[ \left( 2\rho^2 + 2z^2 - b^2 \rho^2 - 2z^2 \right) \arccot \frac{\rho - (\rho^2 - b^2 + z^2)}{2(\rho^2 + z^2)} \right. \\
\left. - (3\sigma - 3b^2 + \rho^2 + z^2) \sqrt{\frac{\rho - (\rho^2 - b^2 + z^2)}{8(\rho^2 + z^2)}} \right] , \tag{69} \]

where \( d_{1,2} = \sqrt{\rho^2 + (z \mp M)^2} \) and \( \sigma = \sqrt{(\rho^2 - b^2 + z^2)^2 + 4b^2 z^2} \); \( M \) is the Schwarzschild mass. On the axis \((\rho = 0)\), \( \nu_{\text{disc}} \) reduces to
\[ \nu_{\text{disc}} = -\frac{2M}{\pi |z|^3} \left[ (z^2 + b^2) \arctan \frac{|z|}{b} - b |z| \right] , \tag{70} \]
while in the equatorial plane \((z = 0)\)
\[ \nu_{\text{disc}}(\rho > b) = -\frac{M}{\rho} \left( 1 - \frac{b^2}{2\rho^2} \right) , \tag{71} \]
\[ \nu_{\text{disc}}(\rho < b) = -\frac{M}{\pi \rho} \left[ 2 - \frac{b^2}{\rho^2} \right] \arcsin \frac{\rho}{b} + \sqrt{\frac{b^2}{\rho^2} - 1} \right] . \tag{72} \]

In [277] we illustrated the shape of the superposed field by drawing its field-lines, namely the integral curves of the four-acceleration \( a^\mu = \nabla^\mu \nu \) of a static congruence. Then we plotted the flattening of the black hole with increasing relative mass of the disc or with decreasing inner disc radius. The explicit (static axisymmetric) form of the horizon Gaussian curvature (30),
\[ C_H = \frac{1 - 4y \nu_{\text{ext},y} - (1 - y^2)(2\nu_{\text{ext},y}^2 - \nu_{\text{ext},yy})}{4M^2 e^{2y_{\text{ext}}(y) - 4y_{\text{ext}}(1)}} \tag{73} \]
(the exterior potential \( \nu_{\text{ext}} \) is represented by \( \nu_{\text{disc}} \) in our case), implies that starting from certain (though unrealistic) values, \( C_H \) becomes negative at the axis.

The next paper [278] dealt with the motion of free test particles in the superposed fields, especially with the influence of the disc parameters on the positions of important equatorial circular geodesics (the photon, the marginally bound and the marginally stable one) which are critical for the disc accretion flows. The following questions arose: with increasing disc mass, how does the inner rim have to shift if the whole disc is to be stable permanently? Does it shift towards or away from the horizon starting from the Schwarzschild radius \( r = 6M \) where the marginally stable orbit lies in the case of a test disc \((M = 0)\)? How do other properties of the resulting spacetime evolve? The answers were given in papers [274, 275, 322].

A sequence of superposed solutions was generated (figure 4 in [274]) by gradually increasing the relative disc mass \( M/M \), while demanding that (i) all the disc matter can be interpreted as two equal counter-rotating streams of particles on stable time-like equatorial circular geodesics and that (ii) the inner disc rim is fixed at the smallest possible radius. Keeping the rim at the marginally stable orbit of the complete spacetime turned out not to be the correct recipe for maintaining the whole disc stable while changing the parameters; namely an instability first appears inside the disc, not at the rim. With increasing disc mass, the rim may first shift.
towards the horizon\textsuperscript{22} the minimal possible Schwarzschild radius \( r \) of the rim goes from 6.00 \( M \) to 3.60\( M \) with \( M \) growing from 0 to 0.42\( M \); the minimal possible circumferential radius \( R \) of the rim goes from 6.00 \( M \) to 3.86\( M \) with \( M \) growing from 0 to 0.29\( M \); and the minimal possible proper distance from the horizon \( d_\rho \) of the rim goes from 7.19\( M \) to 4.26\( M \) with \( M \) growing from 0 to 0.33\( M \). Along the generated sequence of stable discs with minimal inner radii, the radius of the marginally bound orbit below the disc was also found to decrease, whereas the photon orbit went slowly up; the horizon inflated gradually towards the external source. For \( M \) greater than the values given above, the inner radii of the sequence increase again. When the disc mass reaches 1.92\( M \), the position of the inner rim starts to be constrained by the Lagrangian point of zero field (below which free circular motion is impossible within a certain radial range); this, however, goes up steeply to infinity at \( M = 2M \), i.e. the solutions of the given type with \( M \geq 2M \) cannot be given the freely counter-rotating interpretation. The Keplerian orbital speed was checked to be subluminal everywhere within the resulting sequence of discs, reaching roughly 0.7 at its maximum.

Only the beginning of the above sequence of stable physical discs can be realistic astrophysically because the mass \( M \) of the actual accretion discs is not supposed to exceed \((0.01 \div 0.1)M\). For \( M \) between 0 and 0.0722\( M \), the inner rim of our disc sequence follows the marginally stable circular geodesic of the corresponding complete spacetime (for \( M > 0.0722M \), the rim has to lie \textit{above} the marginally stable orbit to avoid instability of the disc at larger radii). Even with such a minor increase of mass, the rim can go down considerably: from \( r = 6M \) to \( r = 4.6162M \), from \( R = 6M \) to \( R = 4.6644M \) and from \( d_\rho = 7.1914M \) to \( d_\rho = 5.4806M \).

### 4.2 Oscillations and stability of self-gravitating discs

The basic criterion for accretion-disc stability, \( \ell, \rho > 0 \), only concerns perturbations within the disc plane. Actually, the frequency \( \kappa \) (measured with respect to infinity) of a small free horizontal oscillation about a circular equatorial geodesic in a stationary axisymmetric spacetime is given by\textsuperscript{23}

\[
\kappa^2 = \frac{e^{2\nu - 2\lambda}}{(u')^2 \rho^2 B^2} \left[ \ell, \rho - (u')^3 \rho^2 B^2 \Omega, \rho \right]
\]

(\textsuperscript{275} and references therein). These oscillations (called epicyclic) have been studied in the astrophysical literature since the 1980’s, recently as the cause of low-frequency disc modes that might explain the quasi-periodic variability observed at some sources probably containing a black hole with an accretion disc (see section \textsuperscript{2.1} and the review \textsuperscript{157}; the history of frequency formulae is described in \textsuperscript{275}). \( \kappa \) is zero at radial infinity; in the Newtonian case it increases monotonically towards the centre, whereas in relativity it has a maximum at a certain distance and then goes back to zero at the marginally stable orbit. Hence, in black-hole sources, the disc

\textsuperscript{22}Such a statement is of course coordinate-dependent. Therefore, we also used some physically more relevant measures rather then the Schwarzschild radius \( r \) only — the equatorial circumferential radius \( R = pe^{-\nu(p,z=0)} = re^{-\nu_{\text{disc}}(r,\theta=\pi/2)} \) and the proper radial distance from the horizon (computed along the equatorial plane) \( d_\rho = \int_0^\infty \sqrt{1 - \lambda_{\text{disc}}/2M} \rho \, d\rho = \int_{2\lambda \rho = \lambda_{\text{disc}}(r,\theta=\pi/2)}^{\lambda_{\text{disc}}(r,\theta=\pi/2)} \sqrt{1 - 2M/r} \). Note that these transformations are non-trivial as they involve the functions \( \nu_{\text{disc}} \) and \( \lambda_{\text{disc}} \). Consequently, a given \( r \) corresponds, for different discs, to different \( R \)'s and \( d_\rho \)'s (and vice versa). However, the graphs using \( r \) and using physical radii turned out not to differ much (see the direct comparison of all the three coordinates in figure 7 of \textsuperscript{274}).

\textsuperscript{23}Note that we consider non-gravitating perturbations, i.e. the metric itself is not perturbed. It would be more sophisticated, and also much more difficult, to allow the perturbation to generate its own, self-consistent field. However, this would only have a second-order effect on oscillation frequencies.
oscillations produced near the inner rim should remain trapped in a region below this maximum rather than propagate to the outer parts of the disc [158].

In [275], perturbations in vertical direction were found to be also important, in fact, to be even more dangerous than the horizontal ones for discs with a greater mass. Perturbations perpendicular to the disc plane must be treated carefully because they take place in the vicinity of the source which is singular (infinitely thin). Although the symmetry would imply $g_{\mu\nu,z} = 0$ in the equatorial plane, there appears a jump of the field in the normal direction as is usual in case of a mass (charge) layer. For the inverted first Morgan-Morgan disc, in particular, one finds that (at $\rho > b$)

$$\lim_{z \to 0^\pm} \nu_{\text{disc},z} = \pm \frac{4M b}{\pi \rho^4} \sqrt{\rho^2 - b^2}. \quad (75)$$

Even in “small” (linear) oscillations the particle spends all of the time outside of the equatorial plane where the field is given by (75) rather than vanishing. The equation of geodesic deviation then yields

$$\Omega_\perp^2 = \Gamma_{tt,z}^z + 2\Gamma_{t\phi,z}^z \Omega + \Gamma_{\phi\phi,z}^z \Omega^2 - 4(\Gamma_{tt}^z + \Gamma_{t\phi}^z \Omega)(\Gamma_{t}^z + \Gamma_{\phi}^z \Omega) - 4(\Gamma_{t\phi}^z + \Gamma_{\phi\phi}^z \Omega)(\Gamma_{t\phi}^z + \Gamma_{\phi\phi}^z \Omega) \quad (76)$$

for the “perpendicular” frequency.

In the static case, the frequency formulae simplify to

$$\kappa^2 = \frac{e^{4\nu - 2\lambda}}{1 - \rho \nu_{,\rho}} (\nu_{,\rho} + 4 \rho \nu_{,\rho}^3 - 6 \nu_{,\rho}^2 + 3 \nu_{,\rho}/\rho), \quad (77)$$

$$\Omega_\perp^2 = \frac{e^{4\nu - 2\lambda}}{1 - \rho \nu_{,\rho}} [\nu_{,zz} - 4 \nu_{,z}^2 (1 - 2 \rho \nu_{,\rho})]. \quad (78)$$

Without the external source, $\Omega_\perp^2$ reduces to the square of the Schwarzschild orbital frequency, $\Omega^2 \approx M(M + \sqrt{\rho^2 + M^2})^{-3}$, whereas $\kappa^2$ remains different due to the pericentre precession effect.

Requiring linear stability (also) with respect to vertical perturbations, the range of physically acceptable Lemos-Letelier superpositions within the $(M, b)$-plane is narrowed further (figure 1 in [322]). The curve of the marginal vertical stability is more restrictive than that of the marginal horizontal stability for the discs with $M > 0.2296M$, it even fully excludes the discs with $M \geq 2M/7$. As the boundary of the delimited sector of physical discs (figure 2 in [322]), one obtains a sequence of stable discs whose inner rims lie right on, or very close to, the circular geodesics marginally stable with respect to both perturbations. We learned, in particular, that for the “realistic” part of this sequence, with relative mass below 0.07, the self-gravity makes the oscillations of the inner disc parts faster, and that the region of horizontal mode trapping gets somewhat smaller.

### 4.3 Redshift from observers in the disc

The redshift from a given point in the source is often mentioned as a measure of the field strength. In Weyl fields, the frequency shift between an observer moving on a circular orbit with an angular velocity $\Omega$ and an observer at rest at infinity reads

$$g \equiv \frac{f_{\infty}}{f_{\text{emit}}} = e^\nu \sqrt{1 - e^{-4\nu} \rho^2 \Omega^2}. \quad (79)$$
In our case, it is natural to compute the shift from two privileged observers within and below the disc, the static one (Ω = 0) and the geodesic one (|Ω| = 2ν√ρ/ρ+ρ−ρ), the observer co-rotates with the matter in terms of which the disc is interpreted. Figure 3 of [222] shows the curves obtained (for discs with different mass and inner radius) from the respective forms of equations (79), g = eν and g = eν√1−2ρν/ρ+ρ, by varying the emitter’s orbital radius.

4.4 Singularity at the rim of the first Morgan-Morgan disc

The class of Morgan-Morgan [229] static axisymmetric solutions describes the fields of a sequence of finite thin counter-rotating discs with densities

\[ w^{(m)}(\rho \leq b) = \frac{(2m + 1)M}{2\pi b^2} \left( 1 - \frac{\rho^2}{b^2} \right)^{m-1/2} \quad (m = 1, 2, \ldots). \tag{80} \]

The zeroth (m = 0) member being clearly singular at the rim, the main attention has been devoted to the first (m = 1) member. The latter has been discussed or referred to as a prototype of a simple and physically meaningful Weyl field. In addition to the papers mentioned in previous sections, the first Morgan-Morgan disc has appeared as the limiting case of a more general class of stationary disc solutions [14, 114], while its inverted counterpart has been considered as a possible seed for stationary black hole–disc superpositions [325] (see the next section).

The first Morgan-Morgan disc has however turned out to have a curvature singularity at the rim [271]. It is caused by too steep a decrease of density (80): the gradient

\[ w^{(m)}_{,\rho} = -\frac{(4m^2 - 1)M\rho}{2\pi b^4} \left( 1 - \frac{\rho^2}{b^2} \right)^{m-3/2} \tag{81} \]

goesto infinity at ρ → b− if m = 0, 1. The singularity is inherited by the annular discs, obtained by the inversion

\[ \rho \rightarrow R = \frac{b^2\rho}{\rho^2 + z^2}, \quad z \rightarrow Z = \frac{b^2z}{\rho^2 + z^2}, \tag{82} \]

(see [322]). It was thus recommended [271] to consider “higher” (m > 1) members of the Morgan-Morgan counter-rotating family in superpositions. More accurately, even these discs have certain singularities at their rims — those calculated from higher derivatives of the metric. This can already be expected from a “Newtonian look” at densities (80): the m-th member of the family has infinite m-th derivative of density at the rim.

4.5 Inverting the m-th Morgan-Morgan counter-rotating disc

The potential for the whole family can be most easily written in oblate spheroidal coordinates x and y(= cos θ), introduced by

\[ \rho^2 = b^2(x^2 + 1)(1 - y^2), \quad z = bxy \quad (0 \leq x < \infty, 1 \geq y \geq -1): \tag{83} \]

\[ v_{\text{MM}}^{(m)} = -\frac{M}{b} \sum_{n=0}^{m} C_{2n}^{(m)} iQ_{2n}(ix)P_{2n}(y), \tag{84} \]

where

\[ C_{2n}^{(m)} = (-1)^n \frac{(4n + 1)(2m)!(2m + 1)!(m + n)!}{(n!)^2(m - n)!(2m + 2n + 1)!}, \quad (n \leq m), \tag{85} \]
\[ P_{2n}(y) \text{ and } Q_{2n}(ix) \text{ being the Legendre polynomials and Legendre functions of the second kind, respectively; note that it is often suitable to express} \]
\[ iQ_{2n}(ix) = P_{2n}(ix) \arccot x - i \sum_{k=1}^{2n} \frac{1}{k} P_{k-1}(ix) P_{2n-k}(ix). \]  
(86)

Thanks to the invariance of the Laplace equation with respect to the Kelvin transformation, one can invert finite thin discs with respect to their outer rims, to obtain annular discs (which can in principle be superposed with some central body then). In terms of the oblate coordinates, the inversion (82) is
\[
x \rightarrow X = \frac{y}{\sqrt{x^2 + 1 - y^2}}, \quad y \rightarrow Y = \frac{x}{\sqrt{x^2 + 1 - y^2}},
\]  
(87)

where \(X, Y\) are assumed to be related to \(R, Z\) in the same manner (83) as \(x, y\) are related to \(\rho, z\). The inverted counterpart of the solution \(\nu^{(m)}_{\text{MM}}(x,y)\), satisfying the Laplace equation and having the correct asymptotics \(\sim -\frac{M}{2x}\), reads
\[
\nu^{(m)}_{\text{disc}} = \frac{2^{2m+1}(m!)^2}{\pi(2m+1)!} \nu^{(m)}_{\text{MM}}(X,Y) \sqrt{x^2 + 1 - y^2}.
\]  
(88)

The inverted discs of the Morgan-Morgan class can now be superposed (for example) with a Schwarzschild black hole or they can be chosen as seeds in some procedure generating stationary generalization of such a superposition (as in the two-soliton inverse-scattering solution described in the following section).

5 Stationary thin discs around rotating black holes

In order to proceed to stationary superpositions, we employed the Belinskii-Zakharov inverse-scattering method \([27, 28]\) — one of the generating techniques which, in principle, provide very general classes of solutions for the problem with two commuting symmetries. In \([225]\), a real-two-soliton version of the method was applied to a general Weyl metric \([11]\); the metric functions of this “seed” will be denoted by a hat, i.e. \(\hat{\nu}\) and \(\hat{\lambda}\), leaving \(\nu\) and \(\lambda\) (and \(\omega\)) for the resulting potentials. In Boyer-Lindquist–type coordinates \((t,r,\theta,\phi)\), introduced by \([22]\), the obtained metric appears as
\[
\begin{align*}
\text{d}s^2 &= \frac{\Delta}{\Sigma} \left( \rho e^{\varphi} \text{d}t + S e^{-\varphi} \text{d}\phi \right)^2 + \frac{\sin^2 \theta}{\Sigma} \left( \mathcal{R} e^{\varphi} \text{d}t - \mathcal{T} e^{-\varphi} \text{d}\phi \right)^2 \\
&\quad + \mathcal{C}^2 e^{2\lambda-2\varphi} \left( \frac{\Sigma}{\Delta} \text{d}r^2 + \Sigma \text{d}\theta^2 \right).
\end{align*}
\]  
(89)

It involves two functions given by quadratures and it depends on (two) potentials describing the seed spacetime and on five independent constants; one can also choose two sign values in the derivation. If the seed is flat, one ends up with the Kerr-NUT solution. Restricting to reflectionally symmetric, asymptotically flat spacetimes containing a black hole and adjusting the coordinates in the most natural manner, the only freedom remains in the seed \((\hat{\nu}, \hat{\lambda})\) and in two constants (one puts \(\mathcal{C}^2 = 1\), among others) which we denote \(M\) and \(a\) as in the Kerr solution. The functions present in (89) then read
\[
\Delta = (r - M)^2 - k^2 = r^2 - 2Mr + a^2,
\]  
(90)
\[ \Sigma = \left( r\mathcal{P} + \frac{a^2}{k} \sinh u \right)^2 + (\mathcal{R}y + a \sinh v)^2 = \mathcal{P}\mathcal{T} + \mathcal{R}\mathcal{S}, \]  
(91)

\[ \mathcal{P} = \cosh u - (M/k) \sinh u, \]  
(92)

\[ \mathcal{R} = a \cosh v, \]  
(93)

\[ \mathcal{S} = (1 + y^2)\mathcal{R} + 2ay \sinh v - 2a\mathcal{P}e^{2\hat{\nu}}, \]  
(94)

\[ \mathcal{T} = (r^2 - a^2)\mathcal{P} + \frac{2a^2}{k}(r - M) \sinh u + 2a\mathcal{R}e^{2\hat{\nu}}, \]  
(95)

where \( k \equiv \sqrt{M^2 - a^2} \) and \( u \) and \( v \) are given by equations

\[ v,\theta = -\Delta u, \]  
\[ k \sin \theta = 2\Delta + k^2 \sin^2 \theta \[(r - M)e^{2\hat{\nu}} - (r - M)e^{2\hat{\nu}}y \], \]  
(96)

\[ v,\theta = \frac{u,\theta}{k} \sin \theta = -\frac{2}{\Delta + k^2 \sin^2 \theta} [(r - M)e^{2\hat{\nu}} \sin \theta - \Delta e^{2\hat{\nu}}y], \]  
(97)

with boundary conditions \( u(y = \pm 1) = 0, \) \( v(y = \pm 1) = \pm 2\hat{\nu}(|y| = 1) \).

A number of properties of the above metric were analyzed. The (outer) horizon is given by (the larger root of) the equation \( \Delta = 0 \). From equations of section 3.3, its area works out at

\[ A = 4\pi \left( r_H^2 e^{-2\hat{\nu}_H(y=1)} + a^2 e^{2\hat{\nu}_H(y=1)} \right) \]  
(98)

and its surface gravitation and angular velocity with respect to infinity at

\[ \kappa = \frac{4\pi k}{A} \quad \text{and} \quad \omega_H = \frac{4\pi a}{A} \cosh 2\hat{\nu}_H(y = 1). \]  
(99)

The Gaussian curvature of the horizon reduces to

\[ C_H(y = 1) = \frac{\left( r_H^2 - a^2 e^{4\hat{\nu}_H(1)} \right) \left[ 1 - 4\hat{\nu}_H,y(1) \right] - 2a^2}{(r_H^2 + a^2 e^{4\hat{\nu}_H(1)})^2} e^{2\hat{\nu}_H(1)} \]  
(100)

at the axis. This can become negative for rapid rotation, namely for

\[ a > 2M\sqrt{\frac{e^{4\hat{\nu}_H(1)} + \frac{2}{1 - e^{4\hat{\nu}_H,y(1)}}}{1 + e^{4\hat{\nu}_H(1)} + \frac{2}{1 - e^{4\hat{\nu}_H,y(1)}}}}. \]  
(101)

The static limit is located where

\[ \sqrt{\Delta\mathcal{P}} = \mathcal{R} \sin \theta. \]  
(102)

Writing the metric in the Weyl-Lewis-Papapetrou form ([]], the asymptotics of the gravitational potential \( \nu \) and of the dragging angular velocity \( \omega \),

\[ \nu = -\frac{M + \hat{M}}{r} + O(r^{-2}), \quad \omega = \frac{2a(M + 2\hat{M})}{r^3} + O(r^{-4}), \]  
(103)

tell us that \( M + \hat{M} \) is the total mass and \( (M + 2\hat{M})a \) is the total angular momentum of the solution; \( \hat{M} \) denotes the mass of the seed spacetime for which we assumed the asymptotics \( \hat{\nu} = -\hat{M}/r + O(r^{-2}) \).
In the static case \((a = 0)\), the result represents a superposition of a given seed with a Schwarzschild black hole, with \(u\) playing the role of an interaction term — \(\nu = \hat{\nu} + \nu_{\text{Schw}}\), \(\lambda = \hat{\lambda} + \lambda_{\text{Schw}} - u\). Thus one can take just the “external source” as a seed, the black hole is “supplied” by the soliton method itself. Hopefully, this applies to a general, stationary case as well. Actually, our main aim has been to learn whether the Belinskii-Zakharov technique could yield the field of a rotating black hole surrounded by an axisymmetric disc.

It seems that for small enough (though by far not negligible) values of \(a\) the solution (89)–(97) does not contain singularities on or above the horizon. However, it is quite extensive to calculate and analyse curvature invariants, even if using computer algebra. Coordinates were found in which metric is regular on the horizon (they have the meaning of the Kerr ingoing/outgoing coordinates there), but this is not enough to claim the absence of a singularity. Unfortunately, we found the horizon to be pinched rather than smooth in the equatorial plane. Indeed, the proper azimuthal circumference \(2\pi(g_{\phi\phi})^H\) of the horizon has a sharp minimum in the equatorial plane if \(a > 0\) (and \(\hat{M} > 0\)). It is to be clarified whether this pathology is physical or the Boyer-Lindquist–type coordinates just do not cover the whole manifold. Most probably, however, there appears a supporting surface between the hole and the external disc.

5.1 Stationary superposition cultivated from the inverted first Morgan-Morgan disc

Some formulas were already specified to the case of a thin equatorial source in [325], in particular, those for the calculation of the energy-momentum tensor and of integral quantities characterizing the resulting spacetime. Let us mention below several properties of the metric (89)–(97) obtained for the inverted first Morgan-Morgan disc [273], i.e. for \(\hat{\nu}\) given by (69) (with the original disc mass now denoted by \(\hat{M}\)). We saw in section 4.4 that this disc is singular at the rim. However, its other properties are quite satisfactory and the singularity is only caused by a sharp increase of density, so it is reasonable to study first this simple case and only later to embark on superpositions generated from “less-singular” seeds of inverted higher Morgan-Morgan discs (section 4.5).

On the horizon, \(\hat{\nu}_H(y) = -\frac{2\hat{M}}{\pi k^3|y|^3} \left[ (k^2 y^2 + b^2) \arctan \frac{k|y|}{b} - bk|y| \right]. \) (104)

This reduces to \(\hat{\nu}_H(y = 0) = -\frac{4\hat{M}}{3\pi b}\) in the equatorial plane. The main horizon properties (98, 99) only depend on the axial value \(\hat{\nu}_H(y = 1)\). In [273] it is shown that with growing mass or decreasing inner radius of the disc the horizon inflates, its surface gravity weakens and its rotation slows down. The presence of the external source weakens the dependence of the horizon curvature on \(a/M\). The existence of the sharp equatorial contraction, revealed by the behaviour of the proper horizon circumference, has been supported by plotting the isometric embedding.

The only undetermined quantities in the metric (89) are the functions \(u\) and \(v\). They are fixed on the axis by the boundary conditions \(u(y = \pm 1) = 0, v(y = \pm 1) = \pm 2\hat{\nu}(|y| = 1)\). On the horizon, one easily obtains \(u_H(y) = 2\hat{\nu}_H(1) - 2\hat{\nu}_H(y), v_H(y) = 2\hat{\nu}_H(1) \cdot \text{sign} y = \mp \text{const}\). In order that the spacetime be reflectionally symmetric, the seed potential \(\hat{\nu}\) must be an even function of \(y\). Then \(u\) is also even in \(y\), whereas \(v\) is odd. Details of the courses of \(u\) and \(v\) are determined, according to the equations (96) and (97), by \(\hat{\nu}\). In our case, \(\hat{\nu} \) is continuous and negative. It is maximal on the axis, having \(\hat{\nu}_r > 0\) and \(\hat{\nu}_\theta = 0\) there [while \(\hat{\nu}_y(y = \pm 1) \geq 0\)]. In the equatorial
plane $\hat{\nu}$ is minimal; $\hat{\nu}_r$ is continuous there, being negative/positive below/above the radius $\rho = \sqrt{3^2/2} b \ (\Leftrightarrow r = M + \sqrt{k^2 + \hat{b}^2})$ where $\hat{\nu}$ assumes its global minimum of $-(2/3)^{3/2}(\hat{M}/b)$. The latitudinal gradient $\hat{\nu}_\theta$ is only zero below the inner radius of the disc, while it has a finite jump across the disc, given by (72):

$$\lim_{y \to 0^\pm} \hat{\nu}_\theta = -(r - M) \lim_{z \to 0^\pm} \hat{\nu}_z = \frac{4\hat{M}b}{\pi\Delta^2} (r - M)\sqrt{\Delta - b^2}. \quad (105)$$

At infinity, $\hat{\nu} = -\hat{M}/r + O(r^{-2})$ which implies the asymptotics

$$u = \frac{\hat{M}k\sin^2 \theta}{r^2} + O(r^{-3}), \quad v = -\frac{2\hat{M}y}{r} + O(r^{-2}). \quad (106)$$

The values of $u$ and $v$ at general locations must be computed numerically, one can only proceed further analytically in the equatorial plane. There, equations (96) and (97) yield latitudinal derivatives

$$u_\theta = -\frac{2k}{r - M} \hat{\nu}_\theta, \quad v_\theta = \frac{2\Delta}{r - M} \hat{\nu}_r. \quad (107)$$

Hence, $u$ increases in latitudinal direction from zero on the axis to a maximum in the equatorial plane; this maximum is smooth/sharp ($u_\theta$ is zero / has a finite jump there) below/above the inner disc radius. The global maximum of $u_{\max}$ and geodesic angular velocities $\Omega$ increases in latitudinal direction from zero on the axis to a maximum in the equatorial plane. Substituting from (71), (72) and (105), one comes to expressions which can be integrated to explicit formulae for $u(z = 0, \rho < b)$, $u(z = 0, \rho > b)$, $v(z = 0^\pm, \rho < b) = \pm 2\hat{\nu}_{11}(y = 1)$, $v(z = 0^\pm, \rho > b)$ (see [273]). While $u(z = 0)$ increases from the horizon to a maximum at $\rho(= \sqrt{\Delta}) = \sqrt{3^2/2} b$ and then decreases to zero at radial infinity, $v(z = 0^\pm)$ increases/decreases to zero monotonically from $\rho = b$.

Using the equatorial expressions for $u$ and $v$, one can, for example, graph the dependence of the equatorial radius of the static limit on $\hat{M}$, $b$ and $a$. We have found that the ergosphere widens both with $a/M$ and with $\hat{M}$. One can also plot the radial course of the light-like, dragging and geodesic angular velocities $\Omega_{\max}$, $\omega$ and $\Omega_\perp$.

### 5.2 Extreme limit

We assumed an under-extreme case ($k > 0$) in [223] in deriving the general metric (89)–(97). However, it can be checked where the quantities go in the extreme limit of $k \to 0$, i.e. $a \to M$. One finds that they behave as in the extreme limit of the pure Kerr solution.

The event horizon approaches $r_H = M$, its area, surface gravity and angular velocity reach the values

$$A = 8\pi M^2 \cosh 2\hat{\nu}_H, \quad \kappa = 0, \quad \omega_H = \frac{1}{2M}. \quad (109)$$
where \( \hat{\nu}_H = -\frac{4\dot{M}}{3\pi b} \) for the inverted first Morgan-Morgan disc. The Gaussian curvature of the horizon works out at

\[
C_H = -\frac{4\pi e^{2\hat{\nu}_H} \cosh^3 2\hat{\nu}_H (1 + |y|)^3 e^{4\hat{\nu}_H} + (1 - |y|)^3 e^{-4\hat{\nu}_H} - 6 \sin^2 \theta}{A \left[ 1 + (|y| \cosh 2\hat{\nu}_H + \sinh 2\hat{\nu}_H)^2 \right]^3}
\]  

(110)

and reduces to

\[
C_H(y = 1) = -\frac{1}{2M^2 \cosh 2\hat{\nu}_H} = -\frac{4\pi}{A},
\]

(111)

\[
C_H(y = 0) = \frac{3 - \cosh 4\hat{\nu}_H}{M^2 \cosh^4 2\hat{\nu}_H} e^{2\hat{\nu}_H}
\]

(112)

at the axis and in the equatorial plane.

In the extreme limit, \( u \) is seen to vanish. The equatorial behaviour of \( u \) and \( v \) is important in discussion of the singularity location \( \Sigma = 0 \). In [325] it was shown that this singularity cannot lie above the horizon (at least) if \( a/M \) is sufficiently small. For our seed (69), however, \( \Sigma(r > r_H) \) is positive in general even in the extreme limit. Actually, the first term of (91) then goes over to \( [r - M(r - M)u/k]^2 \). Using the expression for the equatorial maximum of \( u(k \to 0) \), the root \( r \) of the above term must satisfy

\[
r - M \geq \frac{M}{M - b^2/M^2} \frac{\dot{M}}{M}. \tag{113}
\]

This can only hold (for some positive \( r \)) if \( \dot{M}/M > 3b^2/M^2 \). Such a situation can be considered in principle, but it is, in fact, excluded astrophysically: \( \dot{M}/M \) is typically claimed to be much less than 1, while a realistic value of \( 3b^2/M^2 \) amounts to the order of ten.

The static-limit surface \( r_0(\theta) \) must be found numerically as equation (102) remains rather cumbersome even in the extreme limit. Regarding the limiting forms of \( u \) and \( v \), one can, however, infer the effect of the external disc on the equatorial value of \( r_0 \). It is given by

\[
(r - M)(1 - Mu/k) = M \cosh v, \quad \text{thus} \quad r - M > M \tag{114}
\]

in the extreme limit, which means that \( r_0(y = 0) \) is greater than its Kerr value \( (2M) \).

It is quite difficult to express, in terms of the parameters \( \dot{M}, b, M \) and \( a \), the contributions of the black hole and of the disc to the total mass and angular momentum of the spacetime. However, the crucial Komar formula for the angular momentum of the black hole \( J_H \), treated in equations (154)–(156) of [323], can at least be integrated in the extreme-limit case, to end up with just

\[
J_H = M^2, \quad M_H = M, \quad \mathcal{M} = \dot{M}, \quad \mathcal{J} = 2M\dot{M}. \tag{115}
\]

Another interesting feature of the \( k \to 0 \) limit of our superposition is the vanishing of the “external” gravitational field \( \hat{\nu}_r, \hat{\lambda}_r \) on the horizon [272]. This effect of expulsion of the external (stationary axisymmetric) fields from rotating (and/or charged) black holes, analogous to the Meissner effect in (super)conductors, was observed in magnetic fields before (see the recent survey [31]).
6 Concluding remarks

In sections 4 and 5, the classes of static/stationary axisymmetric superpositions were presented for a black hole with an external thin equatorial disc. They give examples of the effects that can occur if the self-gravity of the disc is not completely negligible. The sequence of stable static discs with minimal possible radii, generated in section 4.1, does not (necessarily) describe the behaviour of a real accretion disc with its mass increasing (or its inner radius changing). Actually, static annular discs need not be of the given (inverted first Morgan-Morgan) type (and need not remain within this class when parameters are changed dynamically). Also, real accretion discs should be stationary rather than static. The results reviewed in the previous sections, however, allow for various generalizations. First, instead of the first Morgan-Morgan solution, singular at the rim, one can superpose an inversion of any member of the Morgan-Morgan family of static counter-rotating\footnote{However artificial the counter-rotating interpretation may seem, counter-rotating stellar discs in galaxies have become an observational matter\cite{304}.} finite thin discs\cite{88}. Second, a different annular disc can be employed, not belonging to the Morgan-Morgan family at all (e.g. the inverted isochrone disc of Klein\cite{164}) — and possibly even a thick one (here we only mean the exterior, vacuum field). The stationary metric\cite{88}–\cite{97} applies to any vacuum static axisymmetric seed, too.

We will try to clarify the nature of the pathology which occurs in the equatorial plane of the metric\cite{88}–\cite{97}. If it corresponds to a thin layer of mass rather than to a real singularity, it might only be a consequence of the fact that the Boyer-Lindquist coordinates cut off a certain part of the manifold. It is more probable, however, that this layer represents a supporting surface which bears witness that the system’s stationarity is only artificial. What remains an open question in any case is the structure of the black-hole interior, altered by the presence of the external source. We hope to be able to answer it in future.

On a general level, one could conclude with the interview with Profs. W. B. Bonnor and R. Penrose which Jiří Bičák organized in 1968\footnote{It was exactly in 1968 that Prof. J. A. Wheeler coined the word “black hole”, but he was only interviewed by J. B. several years later \cite{31}.} about mathematics, relativity and “brain drain”\cite{30} (it was a pretty hot topic at the time when “brain strain” was to continue in Czechoslovakia, as Prof. Kuchař could confirm). In his answer to the question of in which direction to expect progress in relativity, Prof. Bonnor remarked: “It is also possible that quasars will indeed have something to do with gravitational collapse.” As I noticed in the literature, at those days this was not yet a theme for a gentleman to speak about in a decent astronomical society. Although the “collapsed-matter” interpretation of the quasi-stellar radio sources\cite{268} and the “collapsed-star” interpretation of X-ray binaries\cite{324} were both introduced by “ApJ”. (The very importance of accretion as an energy source was apparently first recognized by\cite{229}.)

Nowadays, ApJ offers “Viewing the shadow of the black hole at the Galactic center” within the next few years\cite{108}. Nevertheless, will not a (gravitational) “shine of a black hole” be delivered sooner? Gravitational radiation may prove medicinal for rheumatism or elsewhere. Will travel agencies cite papers like\cite{294, 175} when offering flights intended for “gravitational basking”? In any case, the abundance of event horizons in the last years’ NASA ADS and astro-ph libraries supports Prof. Bonnor’s tip. Today it is in fact impossible to follow, at least in some detail, but a narrow sector of the black hole research. Not within relativity, but mainly not within astrophysics. In the bibliography below, we try to recommend those papers which should be interesting both for physicists and astrophysicists, mostly avoiding pure observational and pure mathematical references.
By far not all the aspects of (“classical” general relativistic) black holes were mentioned in the survey part of this paper. The very question of their origin was only touched in connection with the cosmic censorship, with the formation of heavy discs and with the decay of perturbations. Not a single word about thermodynamics and quantum effects, about cosmology and dark matter, about primordial black holes, about economical and ecological aspects, about Tunguska impact . . .

Also, all the black holes considered in the present paper are spheroidal. Recently, toroidal holes have been investigated as another intriguing possibility, not only by theoreticians but also as a (rather non-orthodox) candidate for active galactic nuclei \[284, 254\]. Toroidal black holes can only exist provided that some of the Hawking & Ellis’ assumptions are relaxed. Gannon \[118\] proved the possibility of a smooth toroidal horizon by dropping stationarity; the period for which a toroidal black hole can persist is, however, very limited: the hole in the torus must close up before a light ray can pass through \[140\]. Toroidal horizons were really observed as a temporary stage of a numerical gravitational collapse \(293\) and references therein). The case of a toroidal horizon with a (ring) singularity outside (thus violating the predictability assumption) was obtained in \[251\]. Other possibility is to relax the asymptotic flatness and allow for a non-zero cosmological constant \(\Lambda\). A metric involving (long-term stable) higher-genus horizons in an asymptotically anti–de Sitter spacetime is quite simple; it was already obtained as a result of a gravitational collapse and generalized to charged and rotating cases (see \[168\] and references therein). Astronomical observations indicate that \(\Lambda > 0\), but it is not excluded that the stabilizing attractive effect similar to that of \(\Lambda < 0\) could be provided by a host galaxy or by a massive accretion disc around the hole (!). (For many more hints, astrophysical connotations and further references, see \[284\].)

In connection with the Weber’s putative detection of gravitational waves in 1971, R. Penrose speculated about a naked singularity in the Galactic centre \[247\], but at the end he stated: “This would remove the observational discrepancies, although it would be a radical explanation. . . In comparison black holes can now be regarded as being ‘conventional’; indeed, for this reason they must be preferred a priori in any attempt at explanation.” Our essay has been a priori ‘conventional’, preferring black holes, and of spheroidal topology. Well, Penrose then continued: “Yet nature does not always prefer conventional explanations, least of all in astronomy.”

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