Polyadic random fields

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Abstract. The paper considers mean-square continuous, wide-sense homogeneous, and isotropic random fields taking values in a linear space of polyadics. We find a set of such fields whose values are symmetric and positive-definite dyadics, and outline a strategy for their simulation.

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1. Introduction

Stochastic continuum physics (mechanics, conductivity, electromagnetism, and coupled field) problems hinge on tensor-valued random fields (TRFs). Two types of TRFs are of particular interest: (i) fields of dependent quantities (displacement, velocity, deformation, rotation, stress,…) and (ii) fields of constitutive responses (conductivity, stiffness, permeability…); see also [16]. In the first case, one can work out restrictions on scalar correlation functions which enter the full representations of the tensor-valued correlation structure.

In the second case, there is a need to construct TRFs having a positive-definite property everywhere. This need is especially pronounced in two areas: stochastic partial differential equations and stochastic finite elements. One wants to be able to simulate TRFs as input into stochastic boundary value problems. While a theory of second-order homogeneous and isotropic TRFs has recently been formulated in [15,17], the present study provides a theoretical framework of polyads to meet this goal.

The paper starts by introducing basic concepts of polyads and polyadics and then discusses the symmetry classes for a range of different problems. Armed with this formulation, we develop second-order TRFs with homogeneous and isotropic correlation polyadics. The paper culminates in an outline of simulation strategy for rank 2 TRFs which are positive-definite. See also [28].

2. Polyads and polyadics

2.1. Notation

The symbol $d$ denotes a positive integer. As we will see later, the cases of $d = 2$ and $d = 3$ are most interesting for applications to continuum physics.

The symbol $\mathbb{R}^d$ denotes the real linear space whose elements are vectors

$$\mathbf{x} = (x_1, \ldots, x_d)^\top, \quad x_i \in \mathbb{R} \quad \text{for} \quad 1 \leq i \leq d.$$
As usual, the scalar product of two vectors is

$$\mathbf{x} \cdot \mathbf{y} = \sum_{i=1}^{d} x_i y_i,$$

and the norm of a vector $\mathbf{x} \in \mathbb{R}^d$ is

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}}.$$

The symbol $\mathbf{e}_i$, $1 \leq i \leq d$, denotes the element of $\mathbb{R}^d$ whose $j$th component is equal to the Kronecker delta, $\delta_{ij}$. We have

$$\mathbf{x} = \sum_{i=1}^{d} x_i \mathbf{e}_i, \quad \mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij},$$

where the first equality means that the set $\{ \mathbf{e}_i : 1 \leq i \leq d \}$ is a basis of the real linear space $\mathbb{R}^d$, and the second equality means that the above basis is orthonormal.

The symbol $r$ denotes a nonnegative integer. The symbol $(\mathbb{R}^d)^r$ denotes the linear space $\mathbb{R}^1$ if $r = 0$ and the Cartesian product of $r$ copies of the set $\mathbb{R}^d$ otherwise.

**Definition 1.** A function $^r\mathbb{T} : (\mathbb{R}^d)^r \rightarrow \mathbb{R}$ is called the polyadic (monadic if $r = 1$, dyadic if $r = 2$, triadic if $r = 3$, and so on), if either $r = 0$ and $^0\mathbb{T}$ is linear or $r \geq 1$ and $^r\mathbb{T}$ is $r$-linear, that is, for all integers $i$ with $1 \leq i \leq d$, for all $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{y}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r$ in $\mathbb{R}^d$, and for all $\alpha, \beta \in \mathbb{R}$ we have

$$^r\mathbb{T}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}, \alpha \mathbf{x} + \beta \mathbf{y}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r) = \alpha ^r\mathbb{T}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}, \mathbf{x}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r)$$

$$+ \beta ^r\mathbb{T}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{i-1}, \mathbf{y}, \mathbf{x}_{i+1}, \ldots, \mathbf{x}_r).$$

The number $r$ is called the rank of a polyadic. Following [3], we denote a polyadic with a capital sans serif letter with a superscript preceding it to indicate its rank, for example, $^2\mathbb{T}$ is a dyadic.

It is easy to see that the set of all polyadics is a real linear space of dimension $d^r$. Indeed, if $r = 0$, then $(\mathbb{R}^d)^r = \mathbb{R}^1$, $d^r = 1$, and a linear functional $f : \mathbb{R}^1 \rightarrow \mathbb{R}$ has the form $f(\mathbf{x}) = a_1 \mathbf{x}_1$ with $a_1 \in \mathbb{R}^1$. Otherwise, if $r \geq 1$, we define a special polyadic as follows.

**Definition 2.** A polyad (monad, dyad, triad, ...) is the polyadic $\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_r$ given by

$$\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_r(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) = (\mathbf{a}_1 \cdot \mathbf{x}_1)(\mathbf{a}_2 \cdot \mathbf{x}_2) \cdots (\mathbf{a}_r \cdot \mathbf{x}_r)$$

for all $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_r) \in (\mathbb{R}^d)^r$, where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_r$ are arbitrary vectors in $\mathbb{R}^d$.

It is easy to see that the $d^r$ polyads $\mathbf{e}_{i_1} \mathbf{e}_{i_2} \cdots \mathbf{e}_{i_r}$ with $1 \leq i_j \leq d$ for all $j$ with $1 \leq j \leq r$ form a basis in the linear space of all polyadics.

We equip this space by such a scalar product that the above basis becomes orthonormal, and call it the $r$-dot product. In particular, if $r = 1$, then the 1-dot product of two monads $\mathbf{a}$ and $\mathbf{b}$ coincides with the scalar product (1). The founder of polyadic calculus Josiah Willard Gibbs denotes the 2-dot product of two dyads by

$$\mathbf{a}_1 \mathbf{a}_2 : \mathbf{b}_1 \mathbf{b}_2 = (\mathbf{a}_1 \cdot \mathbf{b}_1)(\mathbf{a}_2 \cdot \mathbf{b}_2),$$

and the 3-dot product of two triads by

$$\mathbf{a}_1 \mathbf{a}_2 \mathbf{a}_3 : \mathbf{b}_1 \mathbf{b}_2 \mathbf{b}_3 = (\mathbf{a}_1 \cdot \mathbf{b}_1)(\mathbf{a}_2 \cdot \mathbf{b}_2)(\mathbf{a}_3 \cdot \mathbf{b}_3).$$

He introduced these notions in the paper *Elements of Vector Analysis* privately printed in New Haven in two volumes in 1881 and 1884 and reprinted in [5, pp. 17–90].

**Remark 1.** A reader familiar with tensor calculus may identify a polyad $\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_r$ with the tensor product $\mathbf{a}_1 \otimes \mathbf{a}_2 \otimes \cdots \otimes \mathbf{a}_r$ and the linear space of all polyadics with the tensor product $(\mathbb{R}^d)^{\otimes r}$ of $r$ copies of the space $\mathbb{R}^d$. In what follows, we do use the notation $(\mathbb{R}^d)^{\otimes r}$ but keep the Gibbs’ notation $\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_r$ for polyads and extend it to include the case of polyadics: $^r\mathbb{T}_1 ^r\mathbb{T}_2 = ^r\mathbb{T}_1 \otimes ^r\mathbb{T}_2$. 
Theorem 1. For \( r \geq 1 \), any \( r \)-adics can be represented as a sum of not more than \( d^{r-1} \) \( r \)-ads.

Proof. We use mathematical induction. Induction base. Any monadics \( a \) is a monad. The statement of the theorem takes the form of a trivial identity \( a = a \). Induction hypothesis. Assume that \( r \geq 2 \) and any \((r-1)\)-adics has the form

\[
r^{-1}T = \sum_{i=1}^{d^{r-2}} r^{-1}T_i,
\]

where \( r^{-1}T_i, 1 \leq i \leq d^{r-2} \) are \((r-1)\)-adics.

Induction step. Let \( rT \in (\mathbb{R}^d)^{\otimes r} \). Define the \((r-1)\)-adics \( r^{-1}T_1, \ldots, r^{-1}T_d \) by

\[
r^{-1}T_j(x_1, \ldots, x_{r-1}) = rT(x_1, \ldots, x_{r-1}, e_j), \quad 1 \leq j \leq d,
\]

and the \( r \)-adics \( rT_1, \ldots, rT_d \) by

\[
rT_j = r^{-1}T_j e_j
\]

The sum of all constructed \( r \)-adics acts by

\[
\sum_{j=1}^{d} rT_j(x_1, \ldots, x_r) = \sum_{j=1}^{d} r^{-1}T_j(x_1, \ldots, x_{r-1})(e_j \cdot x_r)
\]

\[
= rT(x_1, \ldots, x_{r-1}, \sum_{j=1}^{d} (e_j \cdot x_r) e_j)
\]

\[
= rT(x_1, \ldots, x_r),
\]

By the induction hypothesis, we have

\[
rT = \sum_{j=1}^{d} rT_j = \sum_{j=1}^{d} r^{-1}T_j e_j = \sum_{j=1}^{d} \sum_{i=1}^{d^{r-2}} r^{-1}T_i e_j.
\]

The terms of this sum are \( r \)-ads. \( \square \)

Remark 2. The representation of Theorem 1 is not unique. Later we will use other presentations.

Finally, a \( 2r \)-adics \( 2rT \) is called symmetric if

\[
2rT(x_1, \ldots, x_{2r}) = 2rT(x_{r+1}, \ldots, x_{2r}, x_1, \ldots, x_r),
\]

and nonnegative-definite if for all \( rU \in (\mathbb{R}^d)^{\otimes r} \) we have

\[
2rT \cdot (rU^r U) \geq 0.
\]

3. Symmetry classes

Let \( I \) be the \( d \times d \) identity matrix. The corresponding dyadic is

\[
2I = \sum_{i=1}^{d} e_i e_i,
\]

where \( \{e_i : 1 \leq i \leq d\} \) is the standard basis of the space \( \mathbb{R}^d \). The group \( O(d) \) consists of all \( d \times d \) matrices \( g \) with real-valued entries such that \( gg^T = I \). This group acts on \((\mathbb{R}^d)^{\otimes r}\) by \( g \cdot \alpha = \alpha \) for \( r = 0 \) and

\[
g \cdot \sum_{i_1 \cdots i_r} \alpha_{i_1 \cdots i_r} e_{i_1} \cdots e_{i_r} = \sum_{i_1 \cdots i_r} \alpha_{i_1 \cdots i_r} (ge_{i_1}) \cdots (ge_{i_r}).
\]

The above action is an orthogonal representation of the group \( O(d) \).

Let \( U \) be an invariant subspace of the above representation, that is, for any \( rT \in U \) we have \( g \cdot rT \in U \).
Definition 3. The orbit of a point \( rT \in U \) is the subset of \( U \) defined by
\[
O(rT) = \{ g \cdot rT : g \in O(d) \}.
\]

Definition 4. The stationary subgroup of a point \( rT \in U \) is
\[
G_{rT} = \{ g \in O(d) : g \cdot rT = rT \}.
\]

Let \( g \in O(d) \). An element \( h \in O(d) \) belongs to the group \( G_{g \cdot rT} \) if and only if \( hg \cdot rT = g \cdot rT \). Act by \( g^{-1} \) to both hands sides. We obtain \( g^{-1}hg \cdot rT = rT \), which is equivalent to \( g^{-1}hg \in G_{rT} \). This happens if and only if \( h \in G_{rT}g^{-1} \), that is, \( G_{g \cdot rT} = gG_{rT}g^{-1} \). As \( g \) runs over \( O(d) \), the point \( g \cdot rT \) runs over the orbit \( O(rT) \), and the group \( G_{g \cdot rT} \) runs over the set of all subgroups of \( O(d) \) conjugate to \( G_{rT} \).

Definition 5. Two polyadics \( rT_1 \) and \( rT_2 \) in \( U \) belong to the same orbit type or symmetry class, or isotropy class if their stationary subgroups are conjugate.

We constructed a map from \( U \) to the set of all conjugacy classes of subgroups of the group \( O(d) \). Denote by \([G]\) the conjugacy class of a group \( G \).

Definition 6. The stratum \( \Sigma_{[G]} \) is the inverse image of \([G]\) under the above constructed map:
\[
\Sigma_{[G]} = \{ rT \in U : G_{rT} \subset [G] \}.
\]

In the above definition of a symmetry class, the group \( O(d) \) can be replaced with an arbitrary compact Lie group \( G \) and the space \( U \) by a finite-dimensional real linear space where an orthogonal representation of \( G \) acts. It is well known that the number of symmetry classes is finite, see \([2,18,19]\). For a particular case of a representation of \( O(3) \) in a linear space consisting of tensors, this result is known to physicists as the Hermann theorem \([8]\). An algorithm for an effective enumeration of symmetry classes for a given finite-dimensional real orthogonal representation of the group \( O(3) \) was proposed in \([20]\). In what follows, we put \( d = 3 \).

We introduce a partial order relation on the set of the strata: \( \Sigma_{G_2} \lesssim \Sigma_{G_1} \) if and only if \( G_1 \) is conjugate to a subgroup in \( G_2 \). It turns out that the partially ordered set of strata has the greatest element, call it the generic stratum, and denote by \( \Sigma_G \). It is an open and dense subset of \( U \). The least element of the set of strata is called the minimum stratum. We denote it by \( \Sigma_{G_{N-1}} \). The linear space \( U \) is partitioned into strata:
\[
U = \Sigma_{G_0} \cup \Sigma_{G_1} \cup \cdots \cup \Sigma_{G_{N-1}}.
\]
This partition is called the isotropic stratification of \( U \). For proofs, see \([2]\).

Example 1. The group \( O(3) \) acts in \( \mathbb{R}^1 \) by \( g \cdot x = x \). There exists a unique symmetry class, \([O(3)]\), and a unique stratum \( \Sigma_{[O(3)]} = \mathbb{R}^1 \). All orbits are singletons.

For the action of the same group in \( \mathbb{R}^3 \) by matrix-vector multiplication, there are two symmetry classes. The first one is \([O(2)]\), and the orbits are centered spheres. The second one is \([O(3)]\), and the orbit is the singleton \( \{0\} \in \mathbb{R}^3 \). The generic stratum is \( \Sigma_{[O(2)]} = \mathbb{R}^3 \setminus \{0\} \), while the minimum stratum is \( \Sigma_{[O(3)]} = \{0\} \).

The action (2) on \( \mathbb{R}^3 \otimes \mathbb{R}^3 \) becomes \( g \cdot 2T = g^2Tg^{-1} \). The subspace \( U = S^2(\mathbb{R}^3) \) of symmetric dyadics is invariant under that action. The classification of symmetry classes easily follows from elementary linear algebra. If all three eigenvalues of a dyadic \( 2T \) are different, then its symmetry class is \([D_2 \times Z_2^Z] \), where \( D_2 \) is the dihedral group generated by the two \( 3 \times 3 \) diagonal matrices with elements \( -1, -1, 1 \), and \( 1, -1, 1 \), and \( Z_2^Z \) is the group of order 2 generated by the matrix \(-I\). The corresponding orbit is a prism manifold, see \([10,25,26]\) and modern description in \([12,22,27]\). If two eigenvalues of a dyadic are equal, then its symmetry class is \([O(2) \times Z_2^Z]\). The corresponding orbit is the real projective plane. Finally, if all three eigenvalues of a dyadic are equal, then its symmetry class is \([O(3)]\), and the corresponding orbit is a singleton. The generic stratum \( \Sigma_{[D_2 \times Z_2^Z]} \) is an open dense set of symmetric matrices which have three different eigenvalues. The stratum \( \Sigma_{[O(2) \times Z_2^Z]} \) is the set of symmetric matrices which have two equal eigenvalues.
eigenvalues. Finally, the minimal stratum $\Sigma_{[O(3)]}$ is the set of multiples of $I$. See many other examples in [20].

**Definition 7.** The fixed point set of a group $G$ is the set of all tensors $^r T \in U$ which are fixed by $G$:

\[ U^G = \{ ^r T \in U : g \cdot ^r T = ^r T \quad \text{for all} \quad g \in G \}. \]

In contrast with a stratum, which is a manifold, a fixed point set is a linear space. Moreover, $U^G \neq \{ ^r 0 \}$ if and only if $[G]$ is a symmetry class. In that case, $U^G$ is the biggest linear subspace of $U$, where $G$ acts trivially: $g \cdot ^r T = ^r T$ for all $^r T \in U^G$ and for all $g \in G$.

Consider a group $H$, which satisfies the following conditions: it is a subgroup of $O(3)$, $G$ is a subgroup of $H$, and $U^G$ is invariant under the action of $H$. It turns out that these conditions are equivalent to the following: $G$ is a subgroup of $H$, and $H$ is a subgroup of the normalizer of $G$:

\[ N(G) = \{ h \in O(3) : hGh^{-1} = G \}. \]

In what follows, we fix the following data: $r$, $U$, $[G]$, $U^G$, and $H$.

A possible physical interpretation of the above data may be as follows, see [1, 29]. A continuous medium occupies a compact subset $D$ of the linear space $\mathbb{R}^3$. A microstructure is attached to any material point $x \in D$. The group $G$ is the group of symmetries of the above microstructure; for simplicity we assume that the medium is homogeneous and this group is the same for all material points. At the macroscopic scale, all the details of the microstructure are lost, all what remains is the material symmetry group $G$.

On the other hand, the physical properties of the media are encoded by the physical symmetry group, the group $H$. The Curie principle states that $G \subseteq H$.

The properties of a medium are encoded in a function $^r T : D \to U^G$. At the macroscopic scale, this function is deterministic and is usually a solution of a boundary value problem for a system of partial differential equations. At the microscopic scale, this function fails to be deterministic any more and becomes stochastic. It turns out that a relevant description for such a function is a random field.

### 4. Random fields

The stochastic function $^r T : D \to U^G$ describes properties of homogeneous random medium. It is convenient to consider this function as the restriction to $D$ of another stochastic function, which maps $\mathbb{R}^3$ to $U^G$. We denote it by the same symbol: $^r T : \mathbb{R}^3 \to U^G$.

Note that the $r$-dot product induces the norm and the distance on $U^G$ in a standard way. Let $\mathcal{B}(U^G)$ be the $\sigma$-field of all Borel subsets of $U^G$, that is, the minimal $\sigma$-field that contains all open subsets of $U^G$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Definition 8.** A function $^r T(x, \omega) : \mathbb{R}^3 \times \Omega \to U^G$ is called a random field, if for any $x_0 \in \mathbb{R}^3$ the map $\Omega \to U^G, \omega \mapsto ^r T(x_0, \omega)$ is a random polyadic, that is, for any $B \in \mathcal{B}(U^G)$ the inverse image $^r T^{-1}(x_0, B)$ is an event, an element of $\mathcal{F}$.

The random field $^r T(x)$ is completely characterized by its finite-dimensional distributions, that is, the $(U^G)^n$-valued random variables

\[ (^r T(x_1), \ldots, ^r T(x_n)), \quad (3) \]

where $n$ is a positive integer, and where $x_1, \ldots, x_n$ are $n$ distinct points in $\mathbb{R}^3$. Under a shift, the map $\mathbb{R}^3 \to \mathbb{R}^3, x \mapsto x + y$, the finite-dimensional distribution (3) becomes

\[ (^r T(x_1 + y), \ldots, ^r T(x_n + y)). \quad (4) \]

**Definition 9.** A random field $^r T(x)$ is called strictly homogeneous if for any positive integer $n$, for all distinct points $x_1, \ldots, x_n \in \mathbb{R}^3$, and for all $y \in \mathbb{R}^3$, the $(U^G)^n$-valued random variables (3) and (4) are equally distributed.
What happens under the action of an element \( g \in H \)? A point \( x_i \), \( 1 \leq i \leq n \), becomes the point \( gx_i \). The random polyadic \( rT(x) \) becomes \( g \cdot rT(gx) \). The finite-dimensional distribution (3) becomes

\[
(g \cdot rT(gx_1), \ldots, g \cdot rT(gx_n)).
\] (5)

**Definition 10.** A random field \( rT(x) \) is called **strictly isotropic** if for any positive integer \( n \), for all distinct points \( x_1, \ldots, x_n \in \mathbb{R}^3 \), and for all \( g \in H \), the \((U^G)^n\)-valued random variables (3) and (5) are equally distributed.

In what follows, we suppose that a random field \( rT(x) \) is **second-order**, that is, \( \mathbb{E}[\|rT(x)\|^2] < \infty \) for all \( x \in \mathbb{R}^3 \), where the norm is induced by the \( r \)-fold product. Moreover, we suppose that the random field \( rT(x) \) is **mean-square continuous**, that is, for all \( x \in \mathbb{R}^3 \) we have

\[
\lim_{\|y-x\| \to 0} \mathbb{E}[\|rT(y) - rT(x)\|^2] = 0.
\]

It is easy to see that under a shift, the **one-point correlation polyadic**

\[
\langle rT(x) \rangle = \mathbb{E}[rT(x)]
\]

and the **two-point correlation polyadic**

\[
\langle rT(x), rT(y) \rangle = \mathbb{E}[(rT(x) - \langle rT(x) \rangle)(rT(y) - \langle rT(y) \rangle)]
\]

of a strictly homogeneous random field do not change.

**Definition 11.** A second-order random field is called **homogeneous** if its one- and two-point correlation polyadics do not change under a shift.

The one-point correlation polyadic of the random field \( g \cdot rT(gx) \) is

\[
\langle g \cdot rT(gx) \rangle = g \cdot \langle rT(x) \rangle.
\]

To calculate the two-point correlation polyadic of the above random field, define the linear operator \( \rho(g) : U^G \to U^G \) by \( \rho(g)rT = g \cdot rT \) for \( rT \in U^G \). There exists a unique linear operator

\[
\langle (\rho \otimes \rho)(g) \rangle : (U^G)^{\otimes 2} \to (U^G)^{\otimes 2}
\]

such that \( (\rho \otimes \rho)(g) rT_1 \otimes rT_2 = \rho(g) rT_1 \rho(g) rT_2 \) for all \( rT_1, rT_2 \in U^G \). The two-point correlation polyadic of the random field \( g \cdot rT(gx) \) becomes

\[
\langle g \cdot rT(gx), g \cdot rT(gy) \rangle = (\rho \otimes \rho)(g) \langle rT(gx), rT(gy) \rangle.
\]

This leads to the following definition.

**Definition 12.** A second-order random field is called **isotropic** if its correlation polyadics satisfy

\[
\langle rT(gx) \rangle = g \cdot \langle rT(x) \rangle, \quad \langle rT(gx), rT(gy) \rangle = (\rho \otimes \rho)(g) \langle rT(gx), rT(gy) \rangle.
\]

It is easy to see that if a second-order random field is strictly homogeneous and/or strictly isotropic, it is homogeneous and/or isotropic. The converse is wrong.

How to find the correlation polyadics of a homogeneous and isotropic random field? By homogeneity, the one-point correlation polyadic is a constant, say \( rT \in U^G \). By isotropy, the polyadic \( rT \) satisfies \( g \cdot rT = rT \) for all \( g \in H \). That is, the one-point correlation polyadic of a homogeneous and isotropic random field is an arbitrary \( H \)-invariant tensor of the space \( U^G \).

We give a plan for calculation the two-point correlation polyadic of a homogeneous and isotropic random field. For details, see [15,17] and the references given there. The addition of complex numbers and the multiplication of a complex number by a real one turns the set \( \mathbb{C} \) of complex numbers into a real two-dimensional linear space. Denote by \( cU^G \) the complex linear space generated by the dyads \( z \cdot rT \), with scalar-vector multiplication \( z_1(z_2 \cdot rT) = (z_1z_2) \cdot rT \). There exists a unique \( r \)-dot product on \( cU^G \) with \( (z_1 \cdot rT_1) \cdot (z_2 \cdot rT_2) = z_1z_2 \cdot rT_1 \cdot rT_2 \). The map \( U^G \to cU^G \), \( rT \mapsto 1^\ast rT \), enables to consider \( U^G \) as a subset
of \( cU^G \). Therefore, a random field \( rT(x) \) can be considered as taking values in the complex linear space \( cU^G \).

Recall that a real structure on a complex finite-dimensional linear space \( V \) is a map \( j : V \to V \) satisfying \( j(z_1v_1 + z_2v_2) = \overline{z_1}j(v_1) + \overline{z_2}j(v_2) \) and \( j^2v_1 = v_1 \) for all \( v_1, v_2 \in V \) and \( z_1, z_2 \in \mathbb{C} \).

There exists a unique real structure \( j \) on \( cU^G \) with \( j(z_1^*T_1 + z_2^*T_2) = \overline{z_1}^*T_1 + \overline{z_2}^*T_2 \). In coordinates, the real structure \( j \) maps a vector of \( cU^G \) to the vector with complex-conjugate components.

A centered \( cU^G \)-valued random field \( rT(x) \) is homogeneous if and only if

\[
\left< rT(x), j^*T(y) \right> = \int_{\mathbb{R}^3} e^{i\langle x-y, p \rangle} dF(p).
\]

(6)

Here, \( \mathbb{R}^3 \) is a copy of the space domain \( \mathbb{R}^3 \), which is called the wavenumber domain or the Fourier space, and \( F \) is a measure on the \( \sigma \)-field \( \mathcal{B}(\mathbb{R}^3) \) of Borel subsets of the Fourier space taking values in the set of Hermitian nonnegative-definite linear operators on \( cU^G \).

The map that maps a polyad \( z_1^*T_1z_2^*T_2 \) to the linear operator \( z^*T \mapsto (z^*T \cdot z_1^*T_1)z_2^*T_2 \), may be extended by linearity to an isomorphism between \( (cU^G)^{\otimes 2} \) and the linear space \( L(cU^G) \) of all linear operators in \( cU^G \). There exists a unique real structure on the complex linear space \( L(cU^G) \), call it \( j \otimes j \), that maps a linear operator \( z^*T \mapsto (z^*T \cdot z_1^*T_1)z_2^*T_2 \) to the linear operator \( z^*T \mapsto (z^*T \cdot jz_1^*T_1)jz_2^*T_2 \). In coordinates, the real structure \( j \otimes j \) maps a matrix with complex entries to the matrix with complex-conjugate entries. A Hermitian matrix is mapping to the transposed matrix.

It is easy to prove that a \( cU^G \)-valued homogeneous random field \( rT(x) \) takes values in \( U^G \) if and only if \( F(-A) = (j \otimes j)F(A) \) for all \( A \in \mathcal{B}(\mathbb{R}^3) \). The above field is isotropic if and only if \( F(gA) = (\rho \otimes \rho)(g)F(A) \) for all \( g \in H \) and \( A \in \mathcal{B}(\mathbb{R}^3) \). There is an algorithm, that constructs a group \( H \), a subspace \( \tilde{U} \) of the real linear space of all Hermitian linear operators in \( cU^G \), and an orthogonal representation \( \tilde{\rho} \) of the group \( \tilde{H} \) in the space \( \tilde{U} \), such that a \( cU^G \)-valued homogeneous random field \( rT(x) \) takes values in \( U^G \) and is isotropic if and only if the measure \( F \) takes values in the intersection of the space \( \tilde{U} \) with the set of Hermitian nonnegative-definite linear operators on \( cU^G \), and satisfies the condition \( F(gA) = \tilde{\rho}(g)F(A) \) for all \( g \in \tilde{H} \) and \( A \in \mathcal{B}(\mathbb{R}^3) \).

Consider the scalar measure \( \mu(A) = trF(A) \), where \( tr \) denotes the trace. The measure \( F \) is absolutely continuous with respect to \( \mu \), and the corresponding Radon–Nikodym derivative \( f \) takes values in the convex compact set \( C \) of Hermitian nonnegative-definite operators in \( \tilde{U} \) with unit trace. The measure \( \mu \) and the function \( f \) satisfy the conditions

\[
\mu(A) = \mu(A), \quad f(gp) = \tilde{\rho}(g)f(p)
\]

for all \( g \in \tilde{H} \), \( A \in \mathcal{B}(\mathbb{R}^3) \), and \( p \in \mathbb{R}^3 \).

Let \( \mathbb{R}^3 = \Sigma_{\{H_0\}} \cup \cdots \cup \Sigma_{\{H_{N-1}\}} \) be the isotropic stratification of \( \mathbb{R}^3 \) with respect to the matrix-vector multiplication \( gp \), let \( (\mathbb{R}^3/\tilde{H})_n \), \( 0 \leq n \leq N-1 \), be the set of orbits that belong to the stratum \( \Sigma_{\{H_n\}} \), and let \( \pi_n : \Sigma_{\{H_n\}} \to (\mathbb{R}^3/\tilde{H})_n \) maps a point to its orbit. In general, the triple \( (\Sigma_{\{H_n\}}, \pi_n, (\mathbb{R}^3/\tilde{H})_n) \) is a bundle with the base \( (\mathbb{R}^3/\tilde{H})_n \). The fiber of this bundle is the orbit of \( n \)-th type, the homogeneous space \( \tilde{H}/H_n \). Fortunately, in all interesting cases, this bundle is globally trivial, that is, \( \Sigma_{\{H_n\}} = (\tilde{H}/H_n) \times (\mathbb{R}^3/\tilde{H})_n \), and \( \pi_n \) is the projection to \( (\mathbb{R}^3/\tilde{H})_n \). Let \( o_n \in \tilde{H}/H_n \) be the left coset that contains the identity matrix \( I \in \tilde{H} \). A generic point \( p \in \Sigma_{\{H_n\}} \) has coordinates \( p = (go_n, \lambda) \) with \( \lambda \in (\mathbb{R}^3/\tilde{H})_n \) and \( g \in \tilde{H} \). A \( \tilde{H} \)-invariant measure on \( \Sigma_{\{H_n\}} \) has the form

\[
d\mu_n(g o_n, \lambda) = dg d\nu_n(\lambda),
\]
where \( dg \) is the unique probabilistic \( \tilde{H} \)-invariant measure on \( \tilde{H} \) known as the *Haar measure*, and where \( d\nu_n(\lambda) \) is an arbitrary finite measure on \((\hat{\mathbb{R}^3}/\mathcal{H})_n\). Equation (6) takes the form

\[
\langle \mathbf{T}(x), \mathbf{T}(y) \rangle = \sum_{n=0}^{N-1} \int_{(\hat{\mathbb{R}^3}/\mathcal{H})_n} e^{i(g_\alpha \cdot (x-y))} \tilde{\rho}(g) \, dg \, d\nu_n(\lambda).
\]  

(7)

For \( g \in H_n \), we have \( g_\alpha = \alpha_\lambda \) and \( f(\alpha_\lambda, \lambda) = \tilde{\rho}(g)f(\alpha_\lambda, \lambda) \). It follows that \( f(\alpha_\lambda, \lambda) \) takes values in the convex compact set \( C_n = C \cap \mathcal{U}_n \), where \( \mathcal{U}_n \) is the maximal linear subspace of the space \( \mathcal{U} \) where the representation \( \tilde{\rho} \) of the group \( H_n \) acts trivially.

The next step is to calculate the inner integral in the right-hand side of Eq. (7). The function \( g \mapsto e^{i(g_\alpha \cdot (x-y))} \) is continuous on \( \tilde{H} \) and constant on the left cosets of \( \tilde{H} \) with respect to \( H_n \). By the Fine Structure Theorem [9], for any integer \( n \), \( 0 \leq n \leq N-1 \), there exists a subset \( \text{Irr}(\tilde{H})_n \) of the set \( \text{Irr}(\tilde{H}) \) of equivalence classes of irreducible orthogonal representations of \( G \), and a subset \( A_{n\rho} \) of the set \( \{1, 2, \ldots, \dim \rho\} \) such that the matrix entries \( \{\rho_{ij}(g) : \rho \in \text{Irr}(\tilde{H})_n, 1 \leq i \leq \dim \rho, j \in A_{n\rho}\} \) form an orthogonal basis in the space of complex-valued functions on \( \tilde{H} \) which are constants on the left cosets of \( \tilde{H} \) with respect to \( H_n \) and square-integrable with respect to the Haar measure. The Fourier coefficients of the above continuous function have the form \( \|\rho_{ij}(g)\|^{-2} J_{ij}(\lambda, x - y) \), where

\[
J_{ij}(\lambda, x - y) = \int_{\tilde{H}} e^{i(g_\alpha \cdot (x-y))} \rho_{ij}(g) \, dg.
\]

The Fourier series

\[
e^{i(g_\alpha \cdot (x-y))} = \sum_{\rho \in \text{Irr}(\tilde{H})_n} \sum_{i=1}^{\dim \rho} \sum_{j \in A_{n\rho}} \|\rho_{ij}(g)\|^{-2} J_{ij}(\lambda, x - y) \rho_{ij}(g)
\]  

(8)

of a continuous function converges uniformly on \( \tilde{H} \). We may substitute this series to Eq. (7) and interchange integration and summation. We obtain

\[
\langle \mathbf{T}(x), \mathbf{T}(y) \rangle = \sum_{n=0}^{N-1} \sum_{\rho \in \text{Irr}(\tilde{H})_n} \sum_{i=1}^{\dim \rho} \sum_{j \in A_{n\rho}} \|\rho_{ij}(g)\|^{-2} J_{ij}(\lambda, x - y) \int_{(\hat{\mathbb{R}^3}/\mathcal{H})_n} \rho_{ij}(g) \tilde{\rho}(g) \, dg \, d\nu_n(\lambda).
\]

There is a basis in the subspace \( \mathcal{U} \), in which the representation \( \tilde{\rho} \) breaks into irreducible components. In this basis, the integral over \( \tilde{H} \) of any matrix entry is equal either to 0 or to \( \|\rho_{ij}(g)\|^2 \). Instead of going into details, we consider an example.

**Example 2.** Put \( r = 2, U = S^2(\mathbb{R}^3) \), the linear space of symmetric dyadics, \( H = O(3) \). Then, we have \( \tilde{H} = H \) and \( \mathcal{U} \) is the linear space of symmetric linear operators over \( U \), see [15].

We determine the structure of the representation \( \tilde{\rho} \). First, following [7], we describe the representatives of equivalence classes of irreducible orthogonal representations of \( O(3) \). For a nonnegative integer \( \ell \), the representation \( \rho_\ell \) acts in the linear space \( \mathcal{H}_\ell \) of homogeneous harmonic (with null Laplacian) polynomials \( p(x) \) of degree \( \ell \) in three real variables by \( [\rho_\ell(p)](x) = p(g^{-1}x) \), while the representation \( \rho_\ell^* \) acts in the same space by \( [\rho_\ell(p)](x) = \det gp(g^{-1}x) \). Each irreducible representation is equivalent to one of the above. Note that the representations \( \rho_\ell \) with even \( \ell \) and \( \rho_\ell^* \) with odd \( \ell \) sent \(-1 \in O(3)\) to the identity operator, while the remaining representations send it to the minus identity. The Clebsch–Gordan rule tells that the tensor product of the representations with indices \( \ell_1 \) and \( \ell_2 \) is equivalent to the direct
The sum of representations with indices \( \ell, |\ell_1 - \ell_2| \leq \ell \leq \ell_1 + \ell_2 \). Later we will see how to determine which components have a star as an upper index.

The representation \( g \rightarrow g \) sends \(-I\) to \(-I\) and therefore is equivalent to \( \rho_1 \). The tensor square \( \rho_1 \otimes \rho_1 \) and all its irreducible components send \(-I\) to \(-I \otimes -I = I\). The Clebsch–Gordan rule gives \( \rho_1 \otimes \rho_1 = \rho_0 \oplus \rho'_1 \oplus \rho_2 \).

For complex representations, the orthonormal bases in the spaces of irreducible components are calculated with the help of the so-called Clebsch–Gordan coefficients which can be found in any book on quantum mechanics. For real representations, they have been calculated in [6]; we call them the Godunov–Gordienko coefficients. If the vectors \( e_{\ell}^m \) constitute an orthonormal basis in the space of the representation \( \rho_\ell \), then we have

\[
e_{\ell}^m = \sum_{m_1 = -\ell_1}^{\ell_1} \sum_{m_2 = -\ell_2}^{\ell_2} g_{\ell_1, \ell_2}^{m_1, m_2} e_{\ell_1}^{m_1} e_{\ell_2}^{m_2}.
\]

We use an algorithm for their calculations proposed in [24].

In particular, an orthonormal basis in the space \( U = S^2(\mathbb{R}^3) = \rho_0 \oplus \rho_2 \) is given by

\[
2T_1 = \frac{1}{\sqrt{3}}(e_1e_1 + e_2e_2 + e_3e_3), \quad 2T_2 = \frac{1}{\sqrt{6}}(-e_1e_1 + 2e_2e_2 - e_3e_3),
\]

\[
2T_3 = \frac{1}{\sqrt{2}}(e_1e_2 + e_2e_1), \quad 2T_4 = \frac{1}{\sqrt{2}}(e_2e_3 + e_3e_2),
\]

\[
2T_5 = \frac{1}{\sqrt{2}}(-e_1e_3 - e_3e_1), \quad 2T_6 = \frac{1}{\sqrt{2}}(e_1e_1 - e_3e_3).
\]

We have \( \tilde{\rho} = S^2(\rho_0 \oplus \rho_2) \). We work with the integral over the generic stratum \( \Sigma_{[H_0]} = \Sigma_{[O(2)]} = \mathbb{R}^3 \setminus \{0\} \) first and find an orthonormal basis of the space \( \tilde{U}_0 \), where the restriction of \( \tilde{\rho} \) to \( H_0 = O(2) \) acts trivially.

The first tetrads of the above basis is 4T_1 = 2T_1^2T_1 that corresponds to the component \( \rho_0 \otimes \rho_0 \) of the representation \( \tilde{\rho} \).

The intersection of symmetric part of the component \( \rho_0 \otimes \rho_2 \oplus \rho_2 \otimes \rho_0 \) with \( \tilde{U}_0 \) is generated by the tetrads 4T_2 = \( \frac{1}{\sqrt{2}}(2T_1^2T_2 + 2T_2^2T_1) \).

Finally, the symmetric part of the tensor product \( \rho_2 \otimes \rho_2 \) is given by \( \rho_0 \oplus \rho_2 \oplus \rho_4 \). The component \( \rho_0 \) contributes by

\[
4T_3 = \frac{1}{\sqrt{5}} \sum_{i=2}^{6} 2T_i^2T_i,
\]

the component \( \rho_2 \) by

\[
4T_4 = \frac{1}{\sqrt{14}}(2^2T_2^2T_2 + 2^2T_3^2T_3 + 2^2T_4^2T_4 - 2^2T_5^2T_5 - 2^2T_6^2T_6),
\]

and the component \( \rho_4 \) by

\[
4T_5 = \frac{1}{\sqrt{70}}(6^2T_2^2T_2 - 4^2T_3^2T_3 - 4^2T_4^2T_4 + 2^2T_5^2T_5 + 2^2T_6^2T_6).
\]

In the basis (9), a generic element of the linear space \( \tilde{U}_0 \) takes the form of the 6 \times 6 matrix \( f \) with the following nonzero entries:
\[ f_{11} = \alpha_1, \]
\[ f_{12} = f_{21} = \frac{1}{\sqrt{2}} \alpha_2, \]
\[ f_{22} = \frac{1}{\sqrt{5}} \alpha_3 + \frac{\sqrt{2}}{\sqrt{7}} \alpha_4 + \frac{3\sqrt{2}}{\sqrt{35}} \alpha_5, \]
\[ f_{33} = f_{44} = \frac{1}{\sqrt{5}} \alpha_3 + \frac{1}{\sqrt{14}} \alpha_4 - \frac{2\sqrt{2}}{\sqrt{35}} \alpha_5, \]
\[ f_{55} = f_{66} = \frac{1}{\sqrt{5}} \alpha_3 - \frac{\sqrt{2}}{\sqrt{7}} \alpha_4 + \frac{1}{\sqrt{70}} \alpha_5. \]

The matrix \( f \) must be nonnegative-definite and have unit trace. Moreover, it is block diagonal and has 3 different blocks: \( \begin{pmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} \end{pmatrix} \), \( \begin{pmatrix} f_{33} & 0 \\ 0 & f_{55} \end{pmatrix} \), and \( \begin{pmatrix} f_{55} & 0 \\ 0 & f_{55} \end{pmatrix} \). Accordingly, the set of extreme points of the convex compact set \( \mathcal{C}_0 \) contains 3 connected components. The first component is an ellipse
\[ \{ f \in \tilde{U}_0 : 0 \leq f_{11} \leq 1, f_{22} = 1 - f_{11}, f_{11}f_{22} - f_{12}^2 = f_{33} = f_{55} = 0 \}. \]

The second and the third components are singletons
\[ A_1 = \{ f \in \tilde{U}_0 : f_{11} = f_{12} = f_{22} = f_{55} = 0, f_{33} = \frac{1}{2} \} \]
and
\[ A_2 = \{ f \in \tilde{U}_0 : f_{11} = f_{12} = f_{22} = f_{33} = 0, f_{55} = \frac{1}{2} \}. \]

Consider the minimal stratum \( \Sigma_{\{H_1\}} = \Sigma_{\{O(3)\}} = \{ 0 \} \). The basis of the two-dimensional space \( \tilde{U}_1 \) is \( \{ 4T_1, 4T_3 \} \). The convex compact set \( \mathcal{C}_1 \) is an interval with extreme points
\[ B_1 = \{ f \in \tilde{U}_0 : f_{11} = 1, f_{12} = f_{22} = f_{33} = f_{55} = 0 \} \]
and
\[ B_2 = \{ f \in \tilde{U}_0 : f_{11} = f_{12} = 0, f_{22} = f_{33} = f_{55} = \frac{1}{5} \}. \]

**Lemma 1.** The set \( \mathcal{C}_0 \) is the union of closed triangles \( A_1A_2B \), where \( B \) runs over the convex hull of the ellipse. If \( B \neq B' \), then the intersection of the triangles \( A_1A_2B \) and \( A_1A_2B' \) is the interval \( A_1A_2 \).

Proof. By [23, Corollary 18.5.1], \( \mathcal{C}_0 \) is the convex hull of its extreme points. It is a subset of a real space of dimension 4. By the Carathéodory Theorem [23, Theorem 17.1], any point, say \( X \), of \( \mathcal{C}_0 \) is a convex combination of 5 points. Two of them are \( A_1 \) and \( A_2 \); the other three, say \( A_3, A_4, \) and \( A_5 \), lie on the ellipse. We have \( X = \sum_{i=1}^{5} u_i A_i \). If \( u_3 = u_4 = u_5 = 0 \), then \( X \in A_1A_2 \). Otherwise, put
\[ B = \frac{1}{u_3 + u_4 + u_5} \sum_{i=3}^{5} u_i A_i. \]

By construction, \( B \in A_3A_4A_5 \), a subset of the closed ellipse, and
\[ X = \sum_{i=1}^{2} u_i A_i + (u_3 + u_4 + u_5) B \in A_1A_2B. \]

Assume that \( X \in A_1A_2B \cap A_1A_2B' \) with \( b \neq B' \). Then,
\[ X = u_1 A_1 + u_2 A_2 + u_3 B = u_1' A_1 + u_2' A_2 + u_3' B'. \]
In this case, \( u_1 = u_1' \) and \( u_2 = u_2' \). Then \( u_3 = u_3' \) and \( u_3 (B - B') = 0 \). Because \( B \neq B' \), we have \( u_3 = 0 \). Then \( X \in A_1A_2 \).
\[ \square \]
where
\[ S = \text{diag}(x, y, z) \]

Equation (7) gives
\[
\begin{array}{c}
\text{Calculations give the following nonzero elements of the } 6 \\
\times 6 \text{ symmetric matrix } \tilde{\rho}(g)A_1 \text{ that are located on and over the main diagonal:}
\end{array}
\]

\[
\begin{align*}
(\tilde{\rho}(g)A_1)_{22} &= \frac{2\sqrt{\pi}}{5} S_0^0(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi}}{7\sqrt{5}} S_0^2(\hat{\theta}, \hat{\phi}) - \frac{8\sqrt{\pi}}{35} S_4^0(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{23} &= \frac{\sqrt{\pi}}{7\sqrt{5}} S_1^0(\hat{\theta}, \hat{\phi}) + \frac{4\sqrt{2\pi}}{7\sqrt{15}} S_1^1(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{24} &= -\frac{\sqrt{\pi}}{7\sqrt{5}} S_1^0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{2\pi}}{7\sqrt{15}} S_1^1(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{25} &= -\frac{2\sqrt{\pi}}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{2\pi}}{7\sqrt{15}} S_2^1(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{26} &= -\frac{2\sqrt{\pi}}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{2\pi}}{7\sqrt{15}} S_2^1(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{33} &= \frac{2\sqrt{\pi}}{5} S_0^0(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{3\pi}}{105} S_0^2(\hat{\theta}, \hat{\phi}) + \frac{16\sqrt{\pi}}{21\sqrt{5}} S_4^0(\hat{\theta}, \hat{\phi}) - \frac{8\sqrt{\pi}}{21\sqrt{5}} S_4^2(\hat{\theta}, \hat{\phi}), \\
(\tilde{\rho}(g)A_1)_{34} &= -\frac{3\sqrt{\pi}}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\phi}) + \frac{8\sqrt{\pi}}{21\sqrt{5}} S_4^0(\hat{\theta}, \hat{\phi}), \\
\end{align*}
\]
\[
(\hat{\rho}(g)A_1)_{35} = \frac{-2\sqrt{3\pi}}{7\sqrt{5}} S^1_1(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{2\pi}}{21\sqrt{5}} S^1_4(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{2\pi}}{3\sqrt{35}} S^4_3(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{36} = \frac{\sqrt{3\pi}}{7\sqrt{5}} S^{-1}_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{2\pi}}{3\sqrt{35}} S^{-3}_4(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{2\pi}}{21\sqrt{5}} S^{-1}_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{44} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{3\pi}}{7\sqrt{5}} S^2_2(\hat{\theta}, \hat{\phi}) + \frac{16\sqrt{\pi}}{105} S^0_4(\hat{\theta}, \hat{\phi}) + \frac{8\sqrt{\pi}}{21\sqrt{5}} S^2_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{45} = -\frac{\sqrt{3\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{2\pi}}{3\sqrt{35}} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{2\pi}}{21\sqrt{5}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{46} = -\frac{\sqrt{3\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{2\pi}}{3\sqrt{35}} S^0_4(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{2\pi}}{21\sqrt{5}} S^3_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{55} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{105} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{3\sqrt{35}} S^4_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{56} = \frac{4\sqrt{\pi}}{3\sqrt{35}} S^{-4}_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_1)_{66} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{105} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{3\sqrt{35}} S^4_4(\hat{\theta}, \hat{\phi}).
\]

Similarly, \(A_2 = \frac{1}{\sqrt{\pi}} T_3 - \frac{\sqrt{2\pi}}{\sqrt{3}} T_4 + \frac{1}{\sqrt{10}} T_5\). Calculations give the following nonzero elements of the \(6 \times 6\) symmetric matrix \(\hat{\rho}(g)A_2\) that are located on and over the main diagonal:

\[
(\hat{\rho}(g)A_2)_{22} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi}}{35} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{23} = \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{2\pi}}{7\sqrt{15}} S^1_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{24} = \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi}}{7\sqrt{15}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{25} = \frac{4\sqrt{\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{7\sqrt{15}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{26} = \frac{4\sqrt{\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{7\sqrt{15}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{33} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{3\pi}}{7\sqrt{5}} S^2_2(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{105} S^0_4(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi}}{21\sqrt{5}} S^2_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{34} = \frac{6\sqrt{\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{21\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{3\pi}}{21\sqrt{5}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{35} = \frac{4\sqrt{3\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{21\sqrt{10}} S^0_4(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{3\sqrt{70}} S^0_0(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{36} = -\frac{2\sqrt{3\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{\pi}}{3\sqrt{70}} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{\pi}}{21\sqrt{10}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{44} = \frac{2\sqrt{\pi}}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{7\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{3\pi}}{7\sqrt{5}} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi}}{105} S^0_4(\hat{\theta}, \hat{\phi}) - \frac{2\sqrt{\pi}}{21\sqrt{5}} S^2_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\hat{\rho}(g)A_2)_{45} = \frac{2\sqrt{3\pi}}{7\sqrt{5}} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{\pi}}{3\sqrt{70}} S^0_4(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{\pi}}{21\sqrt{10}} S^0_4(\hat{\theta}, \hat{\phi}),
\]
\[
(\bar{\rho}(g)A_2)_{46} = \frac{2\sqrt{3\pi}}{7\sqrt{5}} S_2^1(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{21\sqrt{10}} S_4^1(\hat{\theta}, \hat{\varphi}) - \frac{\sqrt{\pi}}{3\sqrt{70}} S_4^3(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_2)_{55} = \frac{2\sqrt{\pi}}{5} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{4\sqrt{\pi}}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{105} S_0^2(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{3\sqrt{35}} S_4^1(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_2)_{56} = -\frac{\sqrt{\pi}}{3\sqrt{35}} S_4^{-4}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_2)_{66} = \frac{2\sqrt{\pi}}{5} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{4\sqrt{\pi}}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{105} S_0^2(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{3\sqrt{35}} S_4^1(\hat{\theta}, \hat{\varphi}).
\]

Finally,
\[
A_3(\lambda) = f_{11}(\lambda)^4 T_1 + \frac{1}{\sqrt{2}} f_{12}(\lambda)^4 T_2 + (1 - f_{11}(\lambda)) \left( \frac{1}{\sqrt{5}} T_3 + \frac{\sqrt{2}}{\sqrt{7}} T_4 + \frac{3\sqrt{2}}{\sqrt{35}} T_5 \right).
\]

Calculations give the following nonzero elements of the $6 \times 6$ symmetric matrix $\bar{\rho}(g)A_3(\lambda)$ that are located on and over the main diagonal:

\[
(\bar{\rho}(g)A_3(\lambda))_{11} = 2\sqrt{\pi}(1 - f_{22}(\lambda)) S_0^0(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{12} = \frac{\sqrt{2\pi}}{\sqrt{5}} f_{12}(\lambda) S_2^1(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{13} = \frac{\sqrt{2\pi}}{\sqrt{5}} f_{12}(\lambda) S_2^{-1}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{14} = \frac{\sqrt{2\pi}}{\sqrt{5}} f_{12}(\lambda) S_2^1(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{15} = \frac{\sqrt{2\pi}}{\sqrt{5}} f_{12}(\lambda) S_2^{-2}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{16} = \frac{\sqrt{2\pi}}{\sqrt{5}} f_{12}(\lambda) S_2^2(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{22} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{5} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{4\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_0^2(\hat{\theta}, \hat{\varphi}) + \frac{12\sqrt{\pi} f_{22}(\lambda)}{35} S_4^0(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{23} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^{-1}(\hat{\theta}, \hat{\varphi}) - \frac{6\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^{-1}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{24} = -\frac{2\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^1(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{6\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^1(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{25} = -\frac{4\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^{-2}(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{3\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^{-2}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{26} = -\frac{4\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^2(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{3\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^2(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{33} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{5} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_0^2(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{3\pi} f_{22}(\lambda)}{\sqrt{5}} S_2^2(\hat{\theta}, \hat{\varphi})
\]
\[
- \frac{8\sqrt{\pi} f_{22}(\lambda)}{35} S_4^0(\hat{\theta}, \hat{\varphi}) + \frac{4\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_4^2(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{34} = -\frac{6\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^{-2}(\hat{\theta}, \hat{\varphi}) - \frac{4\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^{-2}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{35} = -\frac{4\sqrt{3\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^1(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{5}} S_4^1(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{35}} S_4^3(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{36} = \frac{2\sqrt{3\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^{-1}(\hat{\theta}, \hat{\varphi}) - \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{35}} S_4^{-3}(\hat{\theta}, \hat{\varphi}) - \frac{\sqrt{2\pi} f_{22}(\lambda)}{7\sqrt{5}} S_4^{-1}(\hat{\theta}, \hat{\varphi}),
\]
\[
(\bar{\rho}(g)A_3(\lambda))_{44} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{5} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{3\pi} f_{22}(\lambda)}{7\sqrt{5}} S_2^0(\hat{\theta}, \hat{\varphi}).
\]
We would like to develop a strategy for simulation of a homogeneous and isotropic random field \( T \). A strategy for simulation given in Table 1. In Eq. (12), we expressed the spherical Bessel functions in terms of ordinary ones and denoted by \( \rho(g)A_3(\lambda) \). We give the answer. An \( p \) point correlation function may be calculated similarly to the case of rank 2. We substitute this expansion to Eq. (11) and obtain

\[
(\rho(g)A_3(\lambda))_{ij} = -\frac{8\sqrt{\pi} f_{22}(\lambda)}{35} S^0_1(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_1(\hat{\theta}, \hat{\phi}),
\]

\[
(\rho(g)A_3(\lambda))_{45} = -\frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_2(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{35}} S^3_4(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{5}} S^3_4(\hat{\theta}, \hat{\phi}),
\]

\[
(\rho(g)A_3(\lambda))_{46} = -\frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_2(\hat{\theta}, \hat{\phi}) + \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{35}} S^1_4(\hat{\theta}, \hat{\phi}) - \frac{\sqrt{2\pi} f_{22}(\lambda)}{\sqrt{35}} S^3_4(\hat{\theta}, \hat{\phi}),
\]

\[
(\rho(g)A_3(\lambda))_{55} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_0(\hat{\theta}, \hat{\phi})
\]

\[
+ \frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_0(\hat{\theta}, \hat{\phi}),
\]

\[
(\rho(g)A_3(\lambda))_{56} = -\frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_0(\hat{\theta}, \hat{\phi}),
\]

\[
(\rho(g)A_3(\lambda))_{66} = \frac{2\sqrt{\pi} f_{22}(\lambda)}{5} S^0_0(\hat{\theta}, \hat{\phi}) - \frac{4\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^0_2(\hat{\theta}, \hat{\phi}) + \frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_0(\hat{\theta}, \hat{\phi})
\]

\[
+ \frac{2\sqrt{\pi} f_{22}(\lambda)}{\sqrt{5}} S^2_0(\hat{\theta}, \hat{\phi}).
\]

The Fourier series (8) takes the form of the Rayleigh expansion:

\[
e^{ip \cdot (x - y)} = (2\pi)^{3/2} \sum_{\ell = 0}^{\infty} \sum_{m = -\ell}^{\ell} i^\ell J_{\ell+1/2}(\sqrt{\lambda} r) \frac{S^m_{\ell}(\theta, \phi)}{\sqrt{\lambda} r} S^m_{\ell}(\hat{\theta}, \hat{\phi}).
\]

We substitute this expansion to Eq. (11) and obtain

\[
\langle 2^T(x), 2^T(y) \rangle_{ij} = 4\pi^2 \sum_{k=1}^{3} \int J_3(\lambda, r, \theta, \phi) d\Phi_k(\lambda),
\]

where the functions \( J_{i,j,k}(\lambda, r, \theta, \phi) \) are symmetric in the first two indices and their nonzero values are given in Table 1. In Eq. (12), we expressed the spherical Bessel functions in terms of ordinary ones and denoted by \( (r, \theta, \phi) \) (resp. \( (\lambda, \hat{\theta}, \hat{\phi}) \) the spherical coordinates of the point \( x - y \) (resp. \( p \)).

To simulate a random field \( 2T(x) \), we apply the Rayleigh expansion (12) to the plane waves \( e^{ip \cdot x} \) and \( e^{-ip \cdot y} \) separately, see details in [14,15]. In the next section, we propose an alternative method for simulation.

### 5. A strategy for simulation

We would like to develop a strategy for simulation of a homogeneous and isotropic random field \( 2T(x) \) that takes values in the set of symmetric and positively-definite matrices. We proceed similarly to the case of rank 0. Let \( 0^T(x) \) be a centered Gaussian homogeneous and isotropic \( \mathbb{R}^3 \)-valued random field with unit variance. The random field \( 0^T(x)0T(x) \) is homogeneous, isotropic, and takes positive values. Such a field is called a field of Chi-square type, because its one-dimensional distributions are Chi-square.

Let \( 1^T(x) \) be a homogeneous and isotropic \( \mathbb{R}^3 \)-valued random field. It is necessarily centered. Its two-point correlation function may be calculated similarly to the case of rank 2. We give the answer. An equation similar to (11) has the form

\[
\langle 1^T(x), 1^T(y) \rangle = 2 \int_{O(3)} \int e^{ip \cdot (x - y)} \rho(g)A_k dg d\Phi_k(\lambda).
\]

(13)
Table 1. Functions $J_{ijk}(\lambda, r, \theta, \varphi)$ for rank 2

| $i$ | $j$ | $k$ | $J_{ijk}(\lambda, r, \theta, \varphi)$ |
|-----|-----|-----|-------------------------------------|
| 1   | 1   | 3   | $\sqrt{2}(1 - f_{22}(\lambda)) (\lambda r)^{-1/2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi)$ |
| 1   | 2   | 3   | $-f_{12}(\lambda) (5 \lambda r)^{-1/2} J_{5/2}(\lambda r) S^2_0(\theta, \varphi)$ |
| 1   | 3   | 3   | $-f_{12}(\lambda) (5 \lambda r)^{-1/2} J_{5/2}(\lambda r) S^{-1}_2(\theta, \varphi)$ |
| 1   | 4   | 3   | $-f_{12}(\lambda) (5 \lambda r)^{-1/2} J_{5/2}(\lambda r) S^{-1}_2(\theta, \varphi)$ |
| 1   | 5   | 3   | $-f_{12}(\lambda) (5 \lambda r)^{-1/2} J_{5/2}(\lambda r) S^{-2}_2(\theta, \varphi)$ |
| 1   | 6   | 3   | $-f_{12}(\lambda) (5 \lambda r)^{-1/2} J_{5/2}(\lambda r) S^2_2(\theta, \varphi)$ |

2 2 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \sqrt{2} \lambda J_{5/2}(\lambda r) S^2_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 2 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{3\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) - \frac{2\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 2 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{3\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) + \frac{6\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 3 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 3 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 3 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 4 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 4 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 4 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 5 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 5 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 5 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 6 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 6 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

2 6 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 3 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 3 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 3 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 4 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 4 2  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 4 3  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]

3 5 1  \[ \frac{1}{\sqrt{2}} \left( \sqrt{2} J_{1/2}(\lambda r) S^0_0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{2}} J_{5/2}(\lambda r) S^2_0(\theta, \varphi) \right) \]
$$\begin{array}{c|c|c|c}
 i & j & k & J_{i,j,k}(\lambda, r, \theta, \varphi) \\
 \hline
 3 & 5 & 2 & \frac{1}{\sqrt{\lambda r}} \left[ -\frac{2\sqrt{2}}{\sqrt{5}} J_{5/2}(\lambda r) S_{1/2}^1(\theta, \varphi) - J_{5/2}(\lambda r) \left( \frac{1}{2\sqrt{5}} S_{1/2}^1(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \right] + \frac{1}{\lambda r} S_{1/2}^1(\theta, \varphi) \\
 3 & 5 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{5/2}(\lambda r) S_{2}^1(\theta, \varphi) + J_{5/2}(\lambda r) \left( \frac{1}{\sqrt{2\lambda}} S_{2}^1(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \\
 3 & 6 & 1 & \frac{1}{\lambda r} \left[ -\frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{1}^1(\theta, \varphi) - J_{5/2}(\lambda r) \left( \frac{2}{3\sqrt{35}} S_{1}^3(\theta, \varphi) + \frac{1}{21\sqrt{65}} S_{4}^1(\theta, \varphi) \right) \right] + \frac{2}{\sqrt{15}} S_{1}^1(\theta, \varphi) \\
 3 & 6 & 2 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{2}^1(\theta, \varphi) - J_{5/2}(\lambda r) \left( \frac{1}{6\sqrt{35}} S_{4}^3(\theta, \varphi) + \frac{1}{21\sqrt{65}} S_{4}^1(\theta, \varphi) \right) \right] + \frac{1}{21\sqrt{65}} S_{4}^1(\theta, \varphi) \\
 3 & 6 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{5/2}(\lambda r) S_{2}^1(\theta, \varphi) - J_{5/2}(\lambda r) \left( \frac{1}{\lambda r} S_{4}^3(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{4}^1(\theta, \varphi) \right) \\
 4 & 4 & 1 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{5}} J_{1/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] + \frac{\sqrt{2}}{\lambda r} S_{1}^2(\theta, \varphi) \right] + J_{5/2}(\lambda r) \left( \frac{\sqrt{2}}{15\sqrt{5}} S_{1}^0(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \\
 4 & 4 & 2 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{5}} J_{1/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] + \frac{\sqrt{2}}{\lambda r} S_{1}^2(\theta, \varphi) \right] - J_{5/2}(\lambda r) \left( \frac{2}{\sqrt{15}} S_{1}^3(\theta, \varphi) + \frac{2}{\sqrt{15}} S_{2}^2(\theta, \varphi) \right) \\
 4 & 4 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{1/2}(\lambda r) S_{2}^0(\theta, \varphi) - \frac{\sqrt{2}}{\sqrt{5}} J_{1/2}(\lambda r) S_{2}^0(\theta, \varphi) \right] + J_{5/2}(\lambda r) \left( \frac{\sqrt{2}}{15\sqrt{5}} S_{2}^0(\theta, \varphi) + \frac{2}{\sqrt{15}} S_{2}^2(\theta, \varphi) \right) \\
 4 & 5 & 1 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] - J_{5/2}(\lambda r) \left( \frac{2}{\sqrt{15}} S_{4}^3(\theta, \varphi) + \frac{1}{21\sqrt{65}} S_{4}^1(\theta, \varphi) \right) \\
 4 & 5 & 2 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] + \frac{1}{21\sqrt{65}} S_{4}^1(\theta, \varphi) \\
 4 & 5 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) - J_{5/2}(\lambda r) \left( \frac{1}{\lambda r} S_{4}^3(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{4}^1(\theta, \varphi) \right) \\
 4 & 6 & 1 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{1/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] - J_{5/2}(\lambda r) \left( \frac{2}{\sqrt{15}} S_{1}^3(\theta, \varphi) + \frac{2}{\sqrt{85}} S_{1}^2(\theta, \varphi) \right) \\
 4 & 6 & 2 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{1/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] + \frac{2}{\sqrt{15}} S_{1}^2(\theta, \varphi) \right] - \frac{1}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \\
 4 & 6 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{1/2}(\lambda r) S_{1}^0(\theta, \varphi) + J_{5/2}(\lambda r) \left( \frac{1}{\lambda r} S_{1}^1(\theta, \varphi) \right) - \frac{1}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \\
 5 & 5 & 1 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{\sqrt{2}}{\lambda r} S_{2}^1(\theta, \varphi) \right] - J_{5/2}(\lambda r) \left( \frac{2}{\sqrt{105}} S_{2}^0(\theta, \varphi) + \frac{2}{\sqrt{85}} S_{2}^2(\theta, \varphi) \right) \\
 5 & 5 & 2 & \frac{1}{\lambda r} \left[ \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{1}{\lambda r} S_{1}^0(\theta, \varphi) \right] + J_{5/2}(\lambda r) \left( \frac{1}{105\sqrt{2}} S_{2}^0(\theta, \varphi) + \frac{1}{3\sqrt{65}} S_{2}^2(\theta, \varphi) \right) \\
 5 & 5 & 3 & \frac{\sqrt{2}(\lambda)}{\sqrt{\lambda r}} J_{5/2}(\lambda r) S_{1}^0(\theta, \varphi) + \frac{\sqrt{2}}{\sqrt{15}} J_{5/2}(\lambda r) S_{2}^1(\theta, \varphi) + J_{5/2}(\lambda r) \left( \frac{1}{\lambda r} S_{4}^3(\theta, \varphi) + \frac{1}{\sqrt{85}} S_{4}^1(\theta, \varphi) \right) \\
 \end{array}$$

**Table 1. continued**
with $\Phi_1(\{0\}) = 2\Phi_1(\{0\})$. The nonzero elements of the $3 \times 3$ symmetric matrix $\tilde{\rho}(g) A_k$ that are located on and over the main diagonal have the form

$$
\begin{align*}
(\tilde{\rho}(g) A_1)_{11} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}) - \frac{\sqrt{\pi}}{\sqrt{15}} S_2^2(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_1)_{12} &= -\frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_1)_{13} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_1)_{22} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{\sqrt{15}} S_2^2(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_1)_{33} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{\sqrt{15}} S_2^2(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_2)_{11} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_2)_{12} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) + \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_2)_{13} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_2)_{22} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}), \\
(\tilde{\rho}(g) A_2)_{33} &= \frac{2\sqrt{\pi}}{3} S_0^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{\pi}}{3\sqrt{3}} S_2^0(\hat{\theta}, \hat{\varphi}) - \frac{2\sqrt{\pi}}{\sqrt{15}} S_2^2(\hat{\theta}, \hat{\varphi}).
\end{align*}
$$

(14)

Using the Rayleigh expansion, we obtain

$$
\langle v^T(x), v^T(y) \rangle_{ij} = 4\pi^2 \sum_{k=1}^{\infty} \int_0^\infty J_{ijk}(\lambda, r, \theta, \varphi) \, d\Phi_k(\lambda),
$$

(15)

where the functions $J_{ijk}(\lambda, r, \theta, \varphi)$ are symmetric in the first two indices and their nonzero values are given in Table 2.
The Rayleigh expansion applied twice gives

\[
e^{i \mathbf{p} \cdot (\mathbf{x} - \mathbf{y})} = (2\pi)^3 \sum_{\ell, \ell' = 0}^{\infty} \sum_{m = -\ell}^{\ell} \sum_{m' = -\ell'}^{\ell'} \frac{i^{\ell - \ell'}}{\sqrt{\ell! \ell'!}} J_{\ell+1/2}(\lambda r_1) J_{\ell'+1/2}(\lambda r_2) S_\ell^m(\theta, \phi) S_{\ell'}^{m'}(\theta, \phi),
\]

where \((r_1, \theta_1, \varphi_1)\) (resp. \((r_2, \theta_2, \varphi_2)\)) are the spherical coordinates of the point \(\mathbf{x}\) (resp. \(\mathbf{y}\)). We substitute this equation to (13) and use the real version of the Gaunt integral

\[
\int_{S^2} S_\ell^m(\hat{\theta}, \hat{\phi}) S_{\ell'}^{m'}(\hat{\theta}, \hat{\phi}) S_{\ell''}^{m''}(\hat{\theta}, \hat{\phi}) \, d\Omega = \sqrt{(2\ell + 1)(2\ell' + 1)} \sum_{m = 0}^{\ell} g_{\ell\ell'}^{m m'} \delta_{m, m'},
\]

see [4, 15]. We obtain

\[
\langle \mathbf{T}(\mathbf{x}), \mathbf{T}(\mathbf{y}) \rangle_{ij} = \pi^2 \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{m'=-\ell'}^{\ell'} i^{\ell - \ell'} \sqrt{(2\ell + 1)(2\ell' + 1)} C_{\ell m \ell' m'}^{\ell m'}
\]

\[
\times \int_{0}^{\infty} \frac{J_{\ell+1/2}(\lambda r_1) J_{\ell'+1/2}(\lambda r_2)}{\sqrt{\lambda r_1} \sqrt{\lambda r_2}} \, d\Phi_k(\lambda) S_\ell^m(\theta_1, \varphi_1) S_{\ell'}^{m'}(\theta_2, \varphi_2),
\]

where, for example,

\[
C_{1111}^{00} = \frac{2}{3} g_{01, 01} g_{00, 00} + \frac{1}{15} g_{20, 00} g_{02, 00} - \frac{1}{5\sqrt{3}} g_{20, 00} g_{20, 00}.
\]

The remaining coefficients are easy to restore, using Eq. (14).

The Karhunen Theorem, see [13], gives

\[
\mathbf{T}(r, \theta, \varphi) = \pi \sum_{k=1}^{\infty} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{0}^{\infty} \frac{J_{\ell+1/2}(\lambda r)}{\sqrt{\lambda r}} \, d^{1} Z^{k m}(\lambda) S_{\ell}^{m}(\theta, \varphi),
\]
where \( Z_{k\ell m} \) are centered \( \mathbb{R}^3 \)-valued random measures defined on the Borel \( \sigma \)-field \( \mathcal{B}([0, \infty)) \) that satisfy
\[
E[Z_{k\ell m}(A) Z_{k'\ell' m'}(B)] = \delta_{kk'} \delta_{\ell \ell'} \sqrt{(2\ell + 1)(2\ell' + 1)} C_{ijklm'}^{\ell m'} \Phi_k(A \cap B)
\]
for all \( A, B \in \mathcal{B}([0, \infty)) \). The random field \( T(r, \theta, \varphi) \) can be simulated by truncating the sum over \( \ell \) and the integral and applying a suitable quadrature formula.

Note that according to [21, Equation 10.7.3],
\[
J_\nu(z) \sim \frac{z^\nu}{2\nu T(\nu + 1)}, \quad \nu \neq -1, -2, \ldots.
\]
In particular,
\[
\lim_{z \to 0} \frac{J_{1/2}(z)}{\sqrt{z}} = \sqrt{2/\pi}, \quad \lim_{z \to 0} \frac{J_{5/2}(z)}{\sqrt{z}} = 0.
\]
Equation (15) and Table 2 give
\[
E[T(0) T(0)] = \frac{8\pi^{3/2}}{3} (\Phi_1([0, \infty)) + \Phi_2([0, \infty)))^2.
\]
In what follows, we suppose that the random field \( T(x) \) is Gaussian, the measures \( \Phi_1 \) and \( \Phi_2 \) are absolutely continuous with respect to the Lebesgue measure with densities \( f_1(\lambda) \) and \( f_2(\lambda) \), and
\[
\int_0^\infty (f_1(\lambda) + f_2(\lambda)) d\lambda = \frac{3}{8\pi^{3/2}}.
\]
In this case, the random vector \( T(0) \) is standard normal.

Consider the random field \( T(x) \) standard normal.

**Theorem 2.** The \( S^2(\mathbb{R}^3) \)-valued random field \( T(x) \) is homogeneous, isotropic, and positive-definite.

**Proof.** Let \( y \in \mathbb{R}^3 \) be a deterministic vector. Then we have
\[
\langle T(x) y, y \rangle = (y, T(x)) T(x) = (y, T(x))^2 \geq 0.
\]
We calculate the one-point correlation dyadic of the random field in question.
\[
\langle T(x) \rangle = E[T(x) T(x)] = \langle 1 \rangle,
\]
which does not depend on \( x \). The two-point correlation tetradic is
\[
\langle T(x), T(y) \rangle = E[T(x) T(y) - 2I] = E[T(x) T(y) - T(x) T(y) + T(x) T(y)]
\]
\[
= E[T(x) T(x)] E[T(y) T(y)] + 2 E[T(x) T(y)] E[T(x) T(y)] - 2 I
\]
\[
= 2 E[T(x) T(x)] E[T(x) T(y)]
\]
where on the second step we applied the Isserlis’ formula, see [11]. The last line does not change under shift. Under a rotation \( g \), we have
\[
2 E[T(g x) T(g y)] E[T(g x) T(g y)] = 2 (g \otimes g) E[T(x) T(y)] (g \otimes g) E[T(x) T(y)]
\]
\[
= 2 g \otimes g E[T(x) T(y)] E[T(x) T(y)].
\]
Finally, Eqs. (15) and (16) give
\[
\langle T(x), T(y) \rangle_{ij'j'} = 32\pi^4 \sum_{k,l=1}^\infty \int_0^\infty J_{ijkl}(\lambda, r_1, \theta_1, \varphi_1) J_{ij'k'l}(\lambda', r_2, \theta_2, \varphi_2) d\Phi_k(\lambda) d\Phi_l(\lambda'),
\]
where \((r_1, \theta_1, \varphi_1)\) (resp. \((r_2, \theta_2, \varphi_2)\)) are the spherical coordinates of the point \(x\) (resp. \(y\)).

Finally, we mention an alternative strategy for simulation a polyadic random field that was realized in [28]. The simulation has been performed by taking the dyadic product of two scalar random fields generated from Cauchy or Dagum correlation functions.

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