Two-point Functions in Affine $SL(N)$ Current Algebra

Jørgen Rasmussen

Laboratoire de Mathémathiques et Physique Théorique,
Université de Tours, Parc de Grandmont, F-37200 Tours, France

Abstract
In this letter the explicit form of general two-point functions in affine $SL(N)$ current algebra is provided for all representations, integrable or non-integrable. The weight of the conjugate field to a primary field of arbitrary weight is immediately read off.

PACS: 11.25.Hf
Keywords: Conformal field theory; affine current algebra; correlation functions
1 Introduction

Two-point functions are the simplest non-trivial correlators one may consider in (extended) conformal field theory. Nevertheless, results in the case of general representations of affine current algebras are still lacking, except for $SL(2)$ where invariance under (loop) projective transformations immediately produces the result.

The objective of the present letter is to work out explicitly the two-point functions in affine $SL(N)$ current algebra for all representations, integrable or non-integrable. The construction is based on the differential operator realization of simple Lie algebras provided in [1] and on well-known results for fundamental representations.

Besides providing us with new insight in the general structure of conformal field theory based on affine current algebra, a motivation for studying two-point functions in affine current algebra is found in the wish to understand how to generalize to higher groups the proposal by Furlan, Ganchev, Paunov and Petkova [2] for how Hamiltonian reduction of affine $SL(2)$ current algebra works at the level of correlators. A simple proof of the proposal in that case is presented in [3] based on the work [4] on correlators for degenerate (in particular admissible) representations in affine $SL(2)$ current algebra. Explicit knowledge on two-point functions may be seen as a first step in the direction of understanding that generalization.

Furthermore, an immediate application of knowing the two-point functions is to determine the weight of the conjugate (primary) field to a primary field of an arbitrary weight. This result is of importance since conjugate (or contragredient) representations are very useful in many respects. For non-integrable representations such weights are in general not known.

The remaining part of this letter is organized as follows. In Section 2 we review our differential operator realization [1] while fixing the notation. In Section 3 the construction of two-point functions in affine $SL(N)$ current algebra is provided and the conjugate weights are derived.

2 Notation

Let $g$ be a simple Lie algebra of rank $r$. $h$ is a Cartan subalgebra of $g$. The set of (positive) roots is denoted $(\Delta_+)$ $\Delta$ and the simple roots are written $\alpha_i$, $i = 1, ..., r$. $\alpha^\vee = 2\alpha/\alpha^2$ is the root dual to $\alpha$. Using the triangular decomposition

$$g = g_- \oplus h \oplus g_+$$

the raising and lowering operators are denoted $e_\alpha \in g_+$ and $f_\alpha \in g_-$, respectively, with $\alpha \in \Delta_+$, and $h_i \in h$ are the Cartan operators. In the Cartan-Weyl basis we have

$$[h_i, e_\alpha] = (\alpha_i^\vee, \alpha)e_\alpha, \quad [h_i, f_\alpha] = -(\alpha_i^\vee, \alpha)f_\alpha$$

and

$$[e_\alpha, f_\alpha] = h_\alpha = G^{ij}(\alpha_i^\vee, \alpha^\vee)h_j$$

where the metric $G_{ij}$ is related to the Cartan matrix $A_{ij}$ as

$$A_{ij} = \alpha_i^\vee \cdot \alpha_j = (\alpha_i^\vee, \alpha_j) = G_{ij}\alpha_j^2/2$$
The Dynkin labels $\Lambda_k$ of the weight $\Lambda$ are defined by

$$\Lambda = \Lambda_k \Lambda^k, \quad \Lambda_k = (\alpha^\vee_k, \Lambda) \quad (5)$$

where $\{\Lambda^k\}_{k=1,\ldots,r}$ is the set of fundamental weights satisfying

$$(\alpha^\vee, \Lambda^k) = \delta^k_i \quad (6)$$

Elements in $g_+$ may be parameterized using “triangular coordinates” denoted by $x^\alpha$, one for each positive root, thus we write general Lie algebra elements in $g_+$ as

$$g_+(x) = x^\alpha e_\alpha \in g_+ \quad (7)$$

We will understand “properly” repeated root indices as in (7) to be summed over the positive roots. Repeated Cartan indices as in (5) are also summed over. The matrix representation $C(x)$ of $g_+(x)$ in the adjoint representation is defined by

$$C_{\alpha\beta}(x) = -x^\beta f_{\beta\alpha}^\beta \quad (8)$$

Now, a differential operator realization $\{\tilde{J}_a(x, \partial, \Lambda)\}$ of the simple Lie algebra $g$ generated by $\{j_a\}$ is found to be [1]

$$\tilde{E}_\alpha(x, \partial) = V^\beta_\alpha(x) \partial_\beta$$
$$\tilde{H}_i(x, \partial, \Lambda) = V^\beta_i(x) \partial_\beta + \Lambda_i$$
$$\tilde{F}_\alpha(x, \partial, \Lambda) = V^{-\beta}_\alpha(x) \partial_\beta + P^j_\alpha(x) \Lambda_j \quad (9)$$

where

$$V^\beta_\alpha(x) = [B(C(x))]^\beta_\alpha$$
$$V^\beta_i(x) = -[C(x)]^\beta_i$$
$$V^{-\beta}_\alpha(x) = [e^{C(x)}]^{-\beta}_\alpha [B(-C(x))]^\beta_\gamma$$
$$P^j_\alpha(x) = [e^{C(x)}]^j_{-\alpha} \quad (10)$$

$B$ is the generating function for the Bernoulli numbers

$$B(u) = \frac{u}{e^u - 1} = \sum_{n\geq0} \frac{B_n}{n!} u^n \quad (11)$$

whereas $\partial_\beta$ denotes partial differentiation wrt $x^\beta$. Closely related to this differential operator realization is the equivalent one $\{J_a(x, \partial, \Lambda)\}$ given by

$$E_\alpha(x, \partial, \Lambda) = -\tilde{F}_\alpha(x, \partial, \Lambda)$$
$$F_\alpha(x, \partial, \Lambda) = -\tilde{E}_\alpha(x, \partial, \Lambda)$$
$$H_i(x, \partial, \Lambda) = -\tilde{H}_i(x, \partial, \Lambda) \quad (12)$$
The matrix functions (10) are defined in terms of universal power series expansions, valid for any Lie algebra, but ones that truncate giving rise to finite polynomials of which the explicit forms depend on the Lie algebra in question.

In the case of $SL(N)$ the roots may be represented as

$$\alpha_{ij} = e_i - e_j$$

where $\{e_i\}$ is an orthonormal basis for the $N$-dimensional Euclidean space. The rank of $SL(N)$ is $r = N - 1$. The structure coefficients are given by

$$[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$$

supplemented by the symmetries

$$f^{-\alpha, -\beta, -\gamma} = -f_{\alpha, \beta, \gamma}$$
$$f_{\alpha, \beta}^{\alpha + \beta} = f_{\beta, -(\alpha + \beta)}^{-\alpha} = f_{-(\alpha + \beta), \alpha}^{-\beta}$$

Here we have introduced the abbreviations

$$e_{ij} = e_{\alpha_{ij}}, \quad f_{ij} = f_{\alpha_{ij}}$$

2.1 Affine Current Algebra

Associated to a Lie algebra is an affine Lie algebra characterized by the central extension $k$, and associated to an affine Lie algebra is an affine current algebra whose generators are conformal spin one fields and have amongst themselves the operator product expansion

$$J_a(z)J_b(w) = \frac{\kappa_{ab}}{(z-w)^2} + \frac{f_{ab}^c J_c(w)}{z-w}$$

where regular terms have been omitted. $\kappa_{ab}$ is the Cartan-Killing form of the underlying Lie algebra.

It is convenient to collect the traditional multiplet of primary fields in an affine current algebra (which generically is infinite for non-integrable representations) in a generating function for that [2, 4, 1], namely the primary field $\phi_{\Lambda}(w, x)$ which must satisfy

$$J_a(z)\phi_{\Lambda}(w, x) = -J_a(x, \partial, \Lambda)\phi_{\Lambda}(w, x)$$
$$T(z)\phi_{\Lambda}(w, x) = \frac{\Delta(\phi_{\Lambda})}{(z-w)^2}\phi_{\Lambda}(w, x) + \frac{1}{z-w}\partial\phi_{\Lambda}(w, x)$$

Here $J_a(z)$ and $T(z)$ are the affine currents and the energy-momentum tensor, respectively, whereas $J_a(x, \partial, \Lambda)$ are the differential operator realizations. $\Delta(\phi_{\Lambda})$ denotes the conformal dimension of $\phi_{\Lambda}$. The explicit construction of primary fields for general simple Lie algebra and arbitrary representation is provided in [4].

An affine transformation of a primary field is given by

$$\delta_c\phi_{\Lambda}(w, x) = \oint_w \frac{dz}{2\pi i} e^a(z)J_a(z)\phi_{\Lambda}(w, x)$$
$$= \{ \epsilon^{-\alpha}(w)V^\beta_\alpha(x)\partial_\beta + \epsilon^i(w)\left(V^\beta_i(x)\partial_\beta + \Lambda_i\right)$$
$$+ \epsilon^\alpha(w)\left(V^\beta_\alpha(x)\partial_\beta + P^i_\alpha(x)\Lambda_i\right) \} \phi_{\Lambda}(w, x)$$
and is parameterized by the $d$ ($d$ is the dimension of the underlying Lie algebra) independent infinitesimal functions $\epsilon^a(z)$.

3 Two-point Functions

Let $W_2(z, w; x, y; \Lambda, \Lambda')$ denote a general two-point function of two primary fields $\phi_\Lambda(z, x)$ and $\phi_{\Lambda'}(w, y)$. From the conformal Ward identities or projective invariance the well-known conformal property of the two-point function is found to be

$$W_2(z, w; x, y; \Lambda, \Lambda') = \frac{\delta \Delta(\phi_\Lambda) \Delta(\phi_{\Lambda'})}{(z-w)^{\Delta(\phi_\Lambda)+\Delta(\phi_{\Lambda'})}} W_2(x, y; \Lambda, \Lambda')$$

(20)

The affine Ward identity

$$\delta \epsilon W_2(z, w; x, y; \Lambda, \Lambda') = \langle \delta \epsilon \phi_\Lambda(z, x) \phi_{\Lambda'}(w, y) \rangle + \langle \phi_\Lambda(z, x) \delta \epsilon \phi_{\Lambda'}(w, y) \rangle = 0$$

(21)

may be recast (using (19)) into the following set of $d$ partial differential equations

$$\left( \tilde{J}_a(x, \partial, \Lambda) + \tilde{J}_a(y, \partial, \Lambda') \right) W_2(x, y; \Lambda, \Lambda') = 0$$

(22)

It is easily verified that only the $2r$ equations for $a = \pm \alpha_i$ are independent. By induction, this simply follows from the fact that $\{ \tilde{J}_a \}$ is a differential operator realization of a Lie algebra. It is the general solution to the equations (22) that we shall provide in the case of $SL(N)$.

The starting assumption, which we believe also to be true for more general groups, is that the affine part of $W_2$ may be expressed as a product of $r$ monomials

$$W_2(x, y; \Lambda, \Lambda') = \prod_{i=1}^r (R_i(x, y))^\mu_i(\Lambda, \Lambda')$$

(23)

where the functions $R_i(x, y)$ are common eigen-functions of the $d$ differential operators $V^\beta_a(x) \partial x^\beta + V^\beta_a(y) \partial y^\beta$. Here we mean to include also vanishing eigen-values. In the case of $SL(N)$ such eigen-functions may be obtained as follows.

First we review a well-known realization of the fundamental representations of $SL(N)$ in terms of $N$ fermionic creation and annihilation operators

$$q_j^\dagger, q_j$$

(24)

where $j = 1, ..., N$

The $\binom{N}{k}$ dimensional $k$'th fundamental representation is provided by the set of states where $k$ fermionic creation operators act on the Fermi vacuum $|0\rangle$. The highest weight vector of the $k$'th fundamental representation of weight $\Lambda^k$ is

$$|\Lambda^k\rangle = q_1^\dagger ... q_k^\dagger |0\rangle$$

(25)

whereas $e_{ij}$ and $f_{ij}$ are represented by

$$e_{ij} = q_i^\dagger q_j^\dagger, \quad i < j$$

$$f_{ij} = q_j q_i^\dagger, \quad i < j$$

(26)
A basis with a minimal set of lowering operators is then easily seen to be
\[ |\Lambda^k\rangle, \quad f_{ij}|\Lambda^k\rangle, \quad i \leq k < j \leq N \]
\[ : \quad f_{i_1p_n(1)}...f_{i_np_n(n)}|\Lambda^k\rangle, \quad i_1 < \ldots < i_n \leq k < j_1 < \ldots < j_n \leq N \]
where \( p_n \) is a permutation operator in \( n \) variables. The simplest possible choice is of course the one where all these permutations are identities. For notational reasons we will stick to that case in the following, and since we only need one basis this choice does not spoil the generality of our construction. Furthermore, it is convenient to renormalize this basis in order that all \( \binom{N}{k} \) basis elements
\[ q_{l_1}^\dagger...q_{l_k}^\dagger|0\rangle, \quad 1 \leq l_1 < \ldots < l_k \leq N \]
appear with positive sign. The corresponding (renormalized) polynomials are then
\[ b_0(x, \Lambda^k) = 1 \]
\[ b_1(x, \Lambda^k; i, j) = (-1)^{k-i} \tilde{F}_{ij}(x, \partial, \Lambda^k) = (-1)^{k-i} P^k_{\alpha_{ij}}(x), \quad i \leq k < j \leq N \]
\[ : \quad b_n(x, \Lambda^k; \{ i_j \}, \{ j_l \}) = (-1)^{n(k-(n-1)/2)} \sum_{i=1}^{n} i_i \tilde{F}_{i_1j_1}(x, \partial, \Lambda^k)...\tilde{F}_{i_{n-1}j_{n-1}}(x, \partial, \Lambda^k) \]
\[ \cdot P^k_{\alpha_{i_1}...\alpha_{i_n}j_n}(x), \quad i_1 < \ldots < i_n \leq k < j_1 < \ldots < j_n \leq N \]
Our proposal for the fundamental eigen-functions \( R_k(x, y) \) is the following
\[ R_k(x, y) = \min\{ k, N-k \} \sum_{n,n'=0} (-1)^{\sum_{i=1}^{n} (j_i-i_i)} b_n(x, \Lambda^k; \{ i_j \}, \{ j_l \}) b_{n'}(y, \Lambda^{N-k}; \{ i'_j \}, \{ j'_l \}) \]
where the second summation \( \tilde{\Sigma} \) is restricted by
\[ (\{ 1, ..., k \} \cup \{ 1, ..., N-k \} \cup \{ j_l \} \cup \{ j'_l \}) \setminus (\{ i_i \} \cup \{ i'_i \}) = \{ 1, ..., N \} \]
This restriction ensures that we only encounter terms where the corresponding bi-product of states \( (27) \) exactly "covers the single state \( q_{l_1}^\dagger...q_{l_k}^\dagger|0\rangle " \). This may be illustrated as follows. Abbreviate a term in the summation by \( b_n(x) b_{n'}(y) \) where the corresponding states are
\[ b_n(x) \sim q_{m_1}^\dagger...q_{m_k}^\dagger|0\rangle \]
\[ b_{n'}(y) \sim q_{m'_1}^\dagger...q_{m'_{N-k}}^\dagger|0\rangle \]
Such a pair of states may be represented by a diagram like in Figure 1, where the ”covering of $q_1^\dagger...q_n^\dagger|0\rangle$” is transparent.

**Figure 1**

\[
\begin{array}{cccccccc}
\bullet & \bullet & 0 & \bullet & \ldots & \bullet & \ldots & 0 & \bullet & 0 \\
0 & 0 & \bullet & 0 & \ldots & 0 & \ldots & \bullet & 0 & 0 \\
1 & 2 & 3 & 4 & \ldots & l & \ldots & \ldots & N
\end{array}
\]

A $\bullet$ in the upper line in place $l$ means that the corresponding $q_l^\dagger$ appears in the $x$-state and not in the $y$-state, and vice versa. The combined summation in (30) may be seen as a summation over all such configurations, up to the sign factor which will be accounted for in the following.

From the pictorial description it follows that

\[
\begin{align*}
\left( \tilde{E}_{ij}(x, \partial) + \tilde{E}_{ij}(y, \partial) \right) R_k(x, y) &= 0 \\
\left( \tilde{F}_{ij}(x, \partial, \Lambda^k) + \tilde{F}_{ij}(y, \partial, \Lambda^{N-k}) \right) R_k(x, y) &= 0
\end{align*}
\]

(33)

Namely, a non-vanishing action of $\epsilon_{ij}(x) + \epsilon_{ij}(y)$ produces a configuration of the form depicted in Figure 2.

**Figure 2**

\[
\begin{array}{cccccccc}
\ldots & \bullet & \ldots & 0 & \ldots \\
\ldots & \bullet & \ldots & 0 & \ldots \\
\ldots & i & \ldots & j & \ldots
\end{array}
\]

This again may be obtained in two ways; from either of the two configurations in Figure 3.

**Figure 3**

\[
\begin{array}{cccccccc}
\ldots & 0 & \ldots & \bullet & \ldots \\
\ldots & \bullet & \ldots & 0 & \ldots \\
\ldots & i & \ldots & j & \ldots \\
\ldots & \bullet & \ldots & 0 & \ldots \\
\ldots & 0 & \ldots & \bullet & \ldots \\
\ldots & i & \ldots & j & \ldots
\end{array}
\]

However, due to the sign factor in (30) they produce the configuration in Figure 2 with opposite signs, hence the first equality in (33). The second statement in (33) is completely analogous. Note that generically the two differential operators $\tilde{F}_{ij}(x, \partial, \Lambda^k)$ and $\tilde{F}_{ij}(y, \partial, \Lambda^{N-k})$ are defined for two different fundamental representations.

We are now in a position to state the main result in this letter:
Proposition

The two-point function of the primary fields $\phi_\Lambda(z, x)$ and $\phi_{\Lambda'}(w, y)$ in affine $SL(N)$ current algebra is (up to an irrelevant normalization constant) given by

$$W_2(z, w; x, y; \Lambda, \Lambda') = \frac{\delta\Delta(\phi_\Lambda)\Delta(\phi_{\Lambda'})}{(z - w)^{\Delta(\phi_\Lambda) + \Delta(\phi_{\Lambda'})}} \prod_{i=1}^r (R_i(x, y))^\mu_i(\Lambda, \Lambda')$$

where

$$\mu_i(\Lambda, \Lambda') = \Lambda_i - \mu_i(\Lambda, \Lambda') = \Lambda'_{N-i}$$

and $R_i(x, y)$ is given by (30) and (31).

Proof

As remarked above, we only need to consider the actions of $\tilde{E}_{\alpha_j}(x, \partial) + \tilde{E}_{\alpha_j}(y, \partial)$ and $\tilde{F}_{\alpha_j}(x, \partial, \Lambda) + \tilde{F}_{\alpha_j}(y, \partial, \Lambda')$ for $j = 1, \ldots, r$. That the $r$ former operators satisfy (22) follows directly from (33). From (33) we also have

$$\left( V_\beta^\alpha(x)\partial_x^\beta + V_\beta^\alpha(y)\partial_y^\beta \right) R_i(x, y) = - \left( \tilde{P}_{-\alpha}(x) + \tilde{P}_{-\alpha}(y) \right) R_i(x, y)$$

with no summation over $i$. This implies that

$$\left( \tilde{F}_{\alpha_j}(x, \partial, \Lambda) + \tilde{F}_{\alpha_j}(y, \partial, \Lambda') \right) W_2(x, y; \Lambda, \Lambda')$$

$$= \left( \sum_{i=1}^r (\Lambda_i - \mu_i(\Lambda, \Lambda')) P_{-\alpha_j}(x) + \sum_{i=1}^r (\Lambda'_{N-i} - \mu_i(\Lambda, \Lambda')) P_{-\alpha_j}(y) \right) W_2(x, y; \Lambda, \Lambda')$$

from which we obtain (34).

From the condition on the pair $(\Lambda, \Lambda')$ in (35) it follows immediately that the conjugate weight $\Lambda^+$ to an arbitrary weight $\Lambda$, integrable or non-integrable, is given by

$$\Lambda^+ = \sum_{k=1}^r \Lambda^+_k \Lambda^k = \sum_{k=1}^r \Lambda_{N-k} \Lambda^k$$

This generalizes the well-known result for integrable representations where $\Lambda^+$ is given by minus the lowest weight in the finite-dimensional highest weight representation of $\Lambda$. We hope to come back elsewhere with a discussion on two-point functions (and conjugate weights) in affine current algebras for more general groups and supergroups. In the latter cases one may employ the recently obtained differential operator realizations of the underlying Lie superalgebras [5].

Acknowledgment

The author thanks Jens Schnittger for fruitful discussions and gratefully acknowledges the financial support from the Danish Natural Science Research Council, contract no. 9700517.
References

[1] J. Rasmussen, *Applications of Free Fields in 2D Current Algebra*, Ph.D. thesis (The Niels Bohr Institute), hep-th/9610167.
J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 502 (1997) 649

[2] P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Phys. Lett. B 267 (1991) 63;
P. Furlan, A.Ch. Ganchev, R. Paunov and V.B. Petkova, Nucl. Phys. B 394 (1993) 665;
A.Ch. Ganchev and V.B. Petkova, Phys. Lett. B 293 (1992) 56

[3] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 457 (1995) 343

[4] J.L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. B 457 (1995) 309

[5] J. Rasmussen, Nucl. Phys. B 510 (1998) 688