AVERAGING PRINCIPLE AND SHAPE THEOREM FOR A GROWTH MODEL WITH MEMORY

A. DEMBO, P. GROISMAN, R. HUANG, AND V. SIDORAVICIUS

Abstract. We present a general approach to study a class of random growth models in $n$-dimensional Euclidean space. These models are designed to capture basic growth features which are expected to manifest at the mesoscopic level for several classical self-interacting processes originally defined at the microscopic scale. It includes once-reinforced random walk with strong reinforcement, origin-excited random walk, and few others, for which the set of visited vertices is expected to form a “limiting shape”. We prove an averaging principle that leads to such shape theorem. The limiting shape can be computed in terms of the invariant measure of an associated Markov chain.

1. Introduction

Random growth processes arise in great variety in a large class of physical and biological phenomena, network dynamics, etc. Starting from seminal works of Eden [14] and Hammersley and Welsh [18], a series of mathematical models have been developed to capture and understand the evolution and pattern formation of growth processes. Our motivation stems from Laplacian growth models, which are characterized by the fact that the rate at which each portion of the boundary of the domain grows is determined by the harmonic measure of the domain from some given point, which we call source. The list includes Diffusion Limited Aggregation (DLA) [40], its generalization – Dielectric Breakdown Model (DBM) [33], Hastings-Levitov process [21]; Internal DLA (IDLA) [13, 29], abelian sandpiles and rotor aggregation [31]. It also includes once-reinforced random walk with strong reinforcement (ORRW) [11], and origin-excited random walk (OERW) [28], for which the set of visited vertices is expected to form a limiting shape. For models such as DLA, DBM or Hastings-Levitov, the source is at infinity, while in models such as IDLA, the source is at the origin. Whenever the source is fixed, the process of growing in time domains is Markovian. In contrast, the latter process is non-Markovian in ORRW or excited random walks, where the source is moving and depends strongly on the last hitting point of the boundary and current shape of the domain.

In general, lattice growth models of this type are elusive, specially when the source is at infinity or when it is not fixed. A notable exception is IDLA for which Lawler, Bramson and Griffeath obtained a shape theorem (see [29]). Specifically, here particles are emitted in steps, one by one, from the source which is always located at the origin, and perform simple random walk until they visit an unvisited vertex. Each particle waits at the source until the previous one hits the external boundary, before being emitted. Gravner-Quastel [16] and Levine-Peres

2010 Mathematics Subject Classification. 60K35, 60K37, 82C22, 82C24.
Key words and phrases. Averaging principle, hydrodynamic limit, excited random walk, shape theorem.
This research was supported in part by NSF grant DMS-1613091.
generalize and relate IDLA under more general, albeit still fixed, source locations to PDE free boundary problems (a Stefan problem in [16], and an obstacle problem in [30] who also obtain analogous shape theorems for rotor-router and divisible sandpile models). An interesting variant is the Uniform IDLA, where upon hitting the boundary, the particle (source) is moved at a point chosen at random uniformly in the domain, and it is shown in [2] that the limiting shape of Uniform IDLA is the Euclidean ball.

Beyond these two examples, there is little understanding of such growth processes, despite substantial recent advances for first passage percolation. In particular, it is conjectured that for both ORRW and OERW the evolution leads to the formation of an asymptotic shape as time goes to infinity (see [27, 28]), but there is no clear vision on how to attack the problem. Recall that in ORRW the particle performs random walk on \( \mathbb{Z}^n \), but each edge (or vertex) increases its conductance by a fixed strength \( a > 0 \) after the first time it is traversed. A phase transition is expected in terms of \( a \), with a limiting shape conjectured for all \( a \) large enough. In the OERW model, the particle receives a (one-time) small drift towards the origin whenever it reaches an unvisited vertex (instead of the conductance change of the ORRW), and a shape theorem is conjectured to hold, no matter how small this positive drift is. We refer the reader to [3, 26] for background on excited random walks, and to [2, 28] for discussions on various IDLA type processes and reinforced walks, all of whom share certain similar features.

In particular, heuristically, whenever the self-interaction tends to attract the walker towards the bulk of its existing range, the boundary of the latter should change at a much slower rate than that of the walker, providing a natural setting to witness averaging.

While non-lattice isotropic models are more amenable to rigorous analysis (see [23, 34]), this typically requires having random conformal maps, hence restricted to dimension \( n = 2 \). By focusing instead on the evolutions of star-shaped domains in \( \mathbb{R}^n \), we are able to handle any \( n \geq 2 \), and mention in passing that, on the deterministic side, the works [5, 6] are close in spirit to our averaged equation (1.12).

We consider here a general random growth model in \( \mathbb{R}^n \) which is specified by two rules \( F, H \) and a scaling parameter \( \epsilon > 0 \). The rule \( F \) which is allowed to depend on the whole geometry of the domain and the position of the source, determines the (random) point at the boundary where the particle, upon starting at the prescribed position, called source, is going to hit the boundary of the domain. For example, \( F \) may be the Harmonic measure at the boundary of the domain from the source. After the particle hits the boundary, the domain grows around the hitting point with a volume increase of \( \epsilon \), followed by the particle jumping, according to the rule \( H \) to the next source position.

More precisely, fixing a small parameter \( \epsilon > 0 \), we consider evolving domains \( (D_t^\epsilon)_{t \geq 0} \) in \( \mathbb{R}^n, n \geq 2 \), which form simply-connected star-shaped compact sets (i.e. they can be parametrized by a function \( R_t^\epsilon \) defined on the sphere \( S^{n-1} \)). It is a pure jump Markov process that starts with an initial domain \( D_0^\epsilon \supseteq 0 \) and particle position \( x_0^\epsilon \) and evolves at a Poisson rate of \( \epsilon^{-1} \) by increasing the domain around randomly chosen boundary points (or equivalently, spherical angles \( \xi_t \in S^{n-1} \)). The probability density for choosing boundary points to evolve is given by the hitting kernel \( F(R_t^\epsilon, x_t^\epsilon, \cdot) \), which is a probability density on the sphere \( S^{n-1} \). After each hitting at the boundary at a point \( \xi_t \), the particle is instantaneously transported according to the specified rule \( H(R_t^\epsilon, \xi_t) \) to a point that can depend on both the domain and the last hitting position. The process \( (R_t^\epsilon, x_t^\epsilon)_{t \geq 0} \) of evolving domains in \( \mathbb{R}^n \) together with the
position of the driving particle coupled to the former is, by construction, Markov (though each marginal is in general non-Markovian). The aim is to construct a continuum simplified model of “random walk interacting with its range”, allowing for general hitting kernel and non-trivial redistribution after each interaction, while inferring whether the evolving domain has an asymptotic shape.

Figure 1. Vertex once-reinforced random walk on $\mathbb{Z}^2$ with strength parameter $a = 2$ (left), $a = 3$ (middle) and $a = 100$ (right) in a box of size 2000. The color of each vertex is proportional to the square root of its first visit time by the walk.

Figure 2. Origin-excited random walk on $\mathbb{Z}^2$ with three different excitation rules. Left: choose a coordinate with probability proportional to its absolute value and move one unit towards the origin in the chosen coordinate. Middle: move one unit towards the origin in the direction of the coordinate with largest absolute value. Right: move one unit towards the origin in each coordinate. Each site is colored according to the first visit time.

The averaging principle has been extensively studied in the theory of dynamical systems, see e.g. [4, 7, 15, 20, 24, 35, 39] and references therein. Usually one identifies a slow variable and
a fast variable. Under suitable conditions the fast variable achieves equilibrium in a time scale for which the slow variable does not evolve macroscopically. Hence, as the scale parameter $\epsilon \to 0$ one expects the slow variable to move according to a system in which the fast variable is integrated with respect to its invariant measure, which may depend on the slow variable as well. In our model, the averaging property that one expects in models such as ORRW and OERRW is explicitly shown in terms of the process $(R^\epsilon_t, x^\epsilon_t)$, where as $\epsilon \to 0$, the variable $R^\epsilon_t$ serves as the slow variable, while $x^\epsilon_t$ acts as the fast one (and though the literature on averaging is large, we found no averaging principle that fits our case, involving a Markov jump process in infinite dimensions). The averaging principle is close in spirit to hydrodynamic limits, a standard tool in the study of interacting particle systems (see [12, 25, 38] and references therein). A hydrodynamic limit is proved for a continuous version of IDLA in [16], yielding in turn a shape theorem, thanks to the scale invariance of this model (as in Lemma 3.1 below). As mentioned before, in this process particles are emitted from fixed sources. One of our goals here is to derive similar results for self-interacting random walks, where the source is clearly moving.

Under certain mild conditions on our model features (namely, the rules $F$ and $H$), we prove in Theorem 1.8 an averaging principle. It allows us to identify the limiting infinite-dimensional ODE governing the evolving domain as the slower dynamics of the pair, yielding in Theorem 1.9 the limiting shape result as a stationary solution of the limiting ODE. Then, in Theorem 1.10 we verify our assumptions for a certain class of models, and in some instances compute explicitly their limiting shape.

Let $S^{n-1}$ be the unit sphere in $\mathbb{R}^n, n \geq 2$ equipped with its Euclidean surface area measure $\sigma(\cdot)$ and for any $1 \leq p \leq \infty$ let $\|f\|_p$ denote the $L^p(S^{n-1})$ norm of $f$ with respect to $\sigma(\cdot)$. We denote by $C(S^{n-1})$ the space of strictly positive continuous functions on $S^{n-1}$, equipped with the metric induced by their $L^2$-norm.

**Definition 1.1.** A simply-connected compact set $D \subseteq \mathbb{R}^n$ is called star-shaped with respect to $0 \in D$, if the line segment connecting 0 and any $x \in \partial D$ is entirely contained in $D$.

Any star-shaped $D$ is uniquely represented by a non-negative function $r : S^{n-1} \to \mathbb{R}_+$ as

$$D = \left\{ x \in \mathbb{R}^n : x = \rho \theta, \theta \in S^{n-1}, 0 \leq \rho \leq r(\theta) \right\}.$$ 

Hereafter, by a slight abuse of notation, we identify any $r \in C(S^{n-1})$ with its graph, which encloses a star-shaped domain $D$ and denote by $\text{Leb}(r)$ the Lebesgue measure (or volume) of that domain $D$. Namely,

$$\text{Leb}(r) = n^{-1} \int_{S^{n-1}} r(\theta)^n d\sigma(\theta) = n^{-1} \|r\|_n^n.$$ 

Let $\mathcal{D}(F)$ be an open subset of $C(S^{n-1}) \times \mathbb{R}^n$ such that $\{ x : (r, x) \in \mathcal{D}(F) \}$ is non-empty for any $r \in C(S^{n-1})$. The measurable map

$$F : \mathcal{D}(F) \subseteq C(S^{n-1}) \times \mathbb{R}^n \to L^2(S^{n-1}),$$

assigns to each $(r, x) \in \mathcal{D}(F)$ an $L^2$ probability density function $F(r, x, \xi)$ with respect to $\sigma(\cdot)$. It represents the rule whereby a particle starting from $x \in \mathbb{R}^n$ chooses a point $r(\xi)\xi, \xi \in S^{n-1}$
at the boundary of the domain enclosed by \( r \), to be the center of the (small) bump we add on the domain boundary \( r \). The measurable map

\[
H(r, \xi) : C(S^{n-1}) \times S^{n-1} \to \mathbb{R}^n
\]

assigns for each \( r \in C(S^{n-1}) \) and \( \xi \in S^{n-1} \) the transported (source) location \( x = H(r, \xi) \) of a particle that hits the domain boundary \( r \) at angle \( \xi \). Assuming that \( (r', H(r, \xi)) \in D(F) \) for any \( r' \geq r \) and \( F(r, x, \cdot) \text{d}\sigma \)-a.e. \( \xi \), guarantees that a.s. the iterative composition of the rules \( H \) and \( F \) is well defined (per our dynamics (1.7)). The small bump we add is in the form of a suitable spherical approximate identity \( g_\eta(\cdot) \), as defined next.

**Definition 1.2.** A collection of continuous functions \( g_\eta : [-1, 1] \to \mathbb{R}_+ \) is called a spherical approximate identity if \( 1 \ast g_\eta = 1 \), \( \|f \ast g_\eta\|_2 \leq \|f\|_2 \) and \( \|f \ast g_\eta - f\|_2 \to 0 \) as \( \eta \to 0 \), for every \( f \in L^2(S^{n-1}) \), where (see \([9, (2.1.1)]\))

\[
(f \ast g_\eta)(z) := \frac{1}{\omega_n} \int_{S^{n-1}} f(\theta)g_\eta(\langle z, \theta \rangle) d\sigma(\theta), \quad z \in S^{n-1},
\]

\[
\omega_n = \sigma(S^{n-1}) = \frac{(2\pi)^{n/2}}{\Gamma(n/2)}
\]

is the surface area of \( S^{n-1} \), and \( \langle z, \theta \rangle \) denotes the scalar product associated with the Euclidean norm \( |\cdot| \) in \( \mathbb{R}^n \).

Utilizing \([9, Section 2.1]\) we provide in Lemma A.1 an explicit construction of such spherical approximate identity, with \( g_\eta(\langle z, \cdot \rangle) \) supported on the spherical cap of (Euclidean) radius \( 2\eta \) centered at \( z \) and \( \eta^{n-1}\|g_\eta\|_\infty \) uniformly bounded (see also Figure 3). Throughout we set the

\[
y_{r,x} := \omega_n \int_{S^{n-1}} r(\theta)^{n-1} F(r, x, \theta) d\sigma(\theta) .
\]

**Figure 3.** Left: Functions \( g_\eta \) for different values of \( \eta \). Right: Function \( g_\eta(\langle z, \cdot \rangle) \) defined on the sphere \( S^2 \) with \( z = (0, 0, 1) \).
Noting that for $\xi$ of density $F(r, x, \cdot)d\sigma$
\[
\lim_{\epsilon, n \to 0} \epsilon^{-1} \mathbb{E}\left[\text{Leb}(r + \epsilon g_\eta(\langle \xi, \cdot \rangle)) - \text{Leb}(r)\right] = y_{r, x} \tag{1.3}
\]
we add at each update a bump $(\epsilon/y_{r, x}) g_\eta(\langle \xi, \cdot \rangle)$ on the current boundary $r$, so that for $\epsilon \ll 1$, the volume of $D'_t$ should grow at a nearly constant, unit rate. Using the $\epsilon$-dependent
\[
\eta(\epsilon, r, x) := \frac{1}{1/n} y_{r, x}^{-1/(n-1)}, \tag{1.4}
\]
as our spherical-scale parameter yields in view of Lemma A.1 that the bump $(\epsilon/y_{r, x}) g_\eta(\langle \xi, \cdot \rangle)$ on the boundary $r$ has about $\epsilon^{1/n}$ height (in the radial direction), uniformly in $(r, x)$. Further, this choice corresponds in the construction of Lemma A.1 to a bump supported on spherical caps of radius $2\epsilon^{1/n}$ in case of a Euclidean ball of unit surface area (namely, $r \equiv \omega_n^{-1/(n-1)}$). Clearly, when adding such $\epsilon$-dependent bumps to our boundary function, the star-shaped domain evolves by a localized bump and the new domain remains star-shaped. Specifically, fixing $\epsilon \in (0, 1]$ and starting at some $(R'_0, x'_0)$ we construct the Markov jump process $(R'_t, x'_t)_{t \geq 0}$ of jump rate $\epsilon^{-1}$ and state space $C(S^{n-1}) \times \mathbb{R}^n$, as follows. For a sequence $\{T'_i\}_{i \in \mathbb{N}}$ of auxiliary Poisson arrival times of rate $\epsilon^{-1}$, starting at $T'_0 = 0$, we freeze $(R'_t, x'_t)$ during each of the intervals $[T'_i, T'_{i+1})$, while as each $i = T'_i, i \geq 1$, conditional on the canonical filtration
\[
\mathcal{F}_{t^-} := \sigma\{R'_s, x'_s, \xi_s : s \leq t^-\},
\]
let
\[
\xi_t \overset{d}{\sim} F(R'_{t^-}, x'_{t^-}, \cdot), \tag{1.5}
\]
namely $\xi_t \in S^{n-1}$ has the density $F(R'_{t^-}, x'_{t^-}, \cdot)$ with respect to $\sigma(\cdot)$, independently of $\mathcal{F}_{t^-}$. Then, update $(R'_{t^-}, x'_{t^-})$ according to
\[
R'_t(\theta) = R'_{t^-}(\theta) + \frac{\epsilon}{y_{R'_{t^-}, x'_{t^-}}} g_\eta(\epsilon, R'_{t^-}, x'_{t^-})(\langle \xi_t, \theta \rangle), \quad \theta \in S^{n-1}, \tag{1.6}
\]
\[
x'_t = H(R'_{t^-}, \xi_t) \tag{1.7}
\]
(recall the definitions (1.2) of $y_{r, x}$ and (1.4) of $\eta(\epsilon, r, x)$). The generator $\mathcal{L}'$ of the Markov process $(R'_t, x'_t)_{t \geq 0}$ is
\[
(\mathcal{L}'f)(r, x) := \epsilon^{-1} \left[ \int_{S^{n-1}} f\left(r + \epsilon y_{r, x} g_\eta(\epsilon, r, x)(\langle \xi, \cdot \rangle), H(r, \xi)\right) F(r, x, \xi)d\sigma(\xi) - f(r, x) \right], \tag{1.8}
\]
for any $f : C(S^{n-1}) \times \mathbb{R}^n \to \mathbb{R}$ in the domain of $\mathcal{L}'$. For $(r, x) \in \mathcal{D}(F)$ and $\theta \in S^{n-1}$ let
\[
b(r, x)(\theta) := \frac{\omega_n}{y_{r, x}} F(r, x, \theta), \quad b'(r, x) := b(r, x) \ast g_\eta(\epsilon, r, x),
\]
\[
h(r, x) := \int_{S^{n-1}} H(r, \xi) F(r, x, \xi)d\sigma(\xi) - x.
\]
Considering (1.8) for $f(r, x) = r(\theta)$ the evaluation map at fixed $\theta \in S^{n-1}$ and using (1.1), we get for $(R'_t(\theta))_{t \geq 0}$ the decomposition
\[
R'_t(\theta) = R'_0(\theta) + \int_0^t b'(R'_s, x'_s)(\theta)ds + \Sigma'_t(\theta), \quad \theta \in S^{n-1}, \tag{1.9}
\]
Figure 4. The shape process \((1.6)\) on \(\mathbb{R}^2\) with \(F(r, x, \cdot)\) given by the harmonic measure on \(r\) from \(x\) and different rules \(H\). In the first row \(\epsilon = 10^{-4}\). **Left:** \(H(r, \xi) = (r(\xi) - 1)_+ \xi\) (here \(s_+\) denotes the positive part of \(s\)). **Middle:** \(H(r, \xi)\) = move one unit towards the origin from \(r(\xi)\xi\) in the direction of the coordinate with largest absolute value. **Right:** \(H(r, \xi) = \) move one unit towards the origin from \(r(\xi)\xi\) in each coordinate. In the second row \(\epsilon = 10^{-6}\). **Left:** \(H(r, \xi) = (r(\xi) - 1)_+ \xi\). **Middle:** \(H(r, \xi) = (r(\xi) - |\xi|_1)_{+} \xi\). **Right:** \(H(r, \xi) = (r(\xi) - |\xi|_2)_{+} \xi\). Third row, \(\epsilon = 10^{-6}\). **Left:** \(H(r, \xi) = \). **Middle:** \(H(r, \xi) = (1 - \frac{|\xi|_{\infty}}{10|\xi|_2})r(\xi)\xi\). **Right:** \(H(r, \xi) = (1 - \frac{|\xi|_1}{10|\xi|_2})r(\xi)\xi\). Different colors represent different times (proportional to \(t^2\)). The (linear in time) evolution of these snapshots identifies the asymptotic \(O(\sqrt{t})\) for the diameter growth. As time, hence diameter, increases, the drift gets smaller in comparison and the process starts to “feel” the different drifts, tending to different asymptotic shapes: sphere, square or diamond depending on the choice of \(H\) (similarly to what we saw for different excitation rules in Figure 2). The final time is 16 in all the pictures.

where \(\Sigma_t^{\epsilon}(\theta)\) is an \(\mathcal{F}_t\)-martingale. Similarly, taking \(f(r, x) = x \cdot \bar{e}_i, i = 1, \ldots, n\), in (1.8) yields

\[
x_t^\epsilon = x_0^\epsilon + \int_0^t e^{-1} h(R_s^\epsilon, x_s^\epsilon)ds + M_t^\epsilon,
\]  

where \(h\) is a bounded, measurable function.

\[(1.10)\]
for some $\mathbb{R}^n$-valued, $\mathcal{F}_t$-martingale $M_t^\epsilon$. For $r \in C(S^{n-1})$ let $(x_{t}^\epsilon)_{t \geq 0}$ denote the $\mathbb{R}^n$-valued Markov jump process evolving by (1.10) in the frozen domain $R_s^{\epsilon} \equiv r$. Its generator is thus

$$\mathcal{L}^\epsilon f(x) = \epsilon^{-1} \left[ \int_{S^{n-1}} f(H(r, \xi)) F(r, x, \xi) d\sigma(\xi) - f(x) \right]$$

(1.11)

for a suitable collection of functions $f : \mathbb{R}^n \to \mathbb{R}$. Consider also the deterministic dynamics $t \mapsto r_t \in L^2(S^{n-1})$ given by

$$r_t(\theta) = r_0(\theta) + \int_0^t \bar{b}(r_s)(\theta) ds , \quad \bar{b}(r)(\theta) := \int_{\mathbb{R}^n} b(r, x)(\theta) d\nu_x(r), \quad \theta \in S^{n-1}. \quad (1.12)$$

The probability measures $\nu_r$ on $\mathbb{R}^n$ for $r \in C(S^{n-1})$ will be specified in Assumption (E), with Proposition 1.6 establishing the existence and uniqueness of the solution for the infinite-dimensional ODE (1.12). For every $a \in (0, 1)$, we define the collections

$$\mathcal{A}_1(a) := \{ r \in C(S^{n-1}) : \inf_{\theta} \{ r(\theta) \} \geq a, \|r\|_2 \leq a^{-1} \} ,$$

$$\mathcal{A}(a) := \{ (r, x) \in \mathcal{D}(F) : r \in \mathcal{A}_1(a), \exists r' \in \mathcal{A}_1(a), r' \leq r \text{ such that } x \in \text{Image}(H(r', \cdot)) \}$$

and assume the following Lipschitz properties of $F, H$ and $\bar{b}$ throughout $\mathcal{A}(a)$.

**Assumption (L).** For any $a \in (0, 1)$, there exists $K = K(a)$ finite such that uniformly for $(r, x), (r', x') \in \mathcal{A}(a), z, z' \in S^{n-1}$, we have that

$$\|F(r, x, \cdot) - F(r', x', \cdot)\|_2 \leq K \left( \|r - r'\|_2 + \|x - x'\| \right), \quad (1.13)$$

$$\|H(r, z) - H(r', z')\| \leq K \left( \|r - r'\|_2 + \|z - z'\| \right), \quad (1.14)$$

$$\|\bar{b}(r) - \bar{b}(r')\|_2 \leq K \|r - r'\|_2. \quad (1.15)$$

Moreover, $F(r, x, \cdot) \in C(S^{n-1})$ for every $(r, x) \in \mathcal{D}(F)$.

Our second assumption concerns the ergodicity of the particle process in a frozen domain.

**Assumption (E).** For any $r \in C(S^{n-1})$ the process $(x_{t}^1)^{\epsilon}_{t \geq 0}$ of generator (1.11) has a unique invariant probability measure $\nu_r$, such that

$$\sup_{r \in \mathcal{A}_1(a)} \sup_{t_0 \geq 0} \mathbb{E} \left[ \frac{1}{t} \int_{t_0}^{t_0 + t} \|b(r, x^{1,\epsilon}_s) - \bar{b}(r)\|_2^2 \right] \leq \lambda(t, a), \quad (1.16)$$

where $\lambda(t, a) \to 0$ as $t \to \infty$, for any fixed $a \in (0, 1)$.

Our last assumption involves the convergence to $b(\cdot, \cdot)$ of the drift of $R_t^\epsilon$ when $\epsilon \to 0$.

**Assumption (C).** For any fixed $t \geq 0$ and $a > 0$

$$\lim_{\epsilon \to 0} \|b^\epsilon(R_t^{\epsilon, \tau^\epsilon}, x_{t+\tau^\epsilon}) - b(R_t^{\epsilon, \tau^\epsilon}, x_{t+\tau^\epsilon})\|_2 = 0, \quad \text{in probability}, \quad (1.17)$$

where $\tau^\epsilon := \inf \{ t > 0 : \|R_t^\epsilon\|_2 \geq a^{-1} \}$.

**Remark 1.3.** From Definition 1.2 we know that $\|b^\epsilon(r, x) - b(r, x)\|_2 \to 0$ as $\epsilon \to 0$, for any fixed $(r, x) \in \mathcal{D}(F)$. For Assumption (C) we need this to hold at the $\epsilon$-dependent $(R_t^{\epsilon, \tau^\epsilon}, x_{t+\tau^\epsilon})$. To this end, it suffices to bound $y_{r,x}$ of (1.2) and the RHS of (A.15) at $f = b(r, x)$, uniformly over $(r, x) = (R_t^{\epsilon, \tau^\epsilon}, x_{t+\tau^\epsilon})$. 
Equipped with these assumptions, we next state our main result.

**Theorem 1.4** (Averaging principle). Under Assumptions (L), (E) and (C), starting at \( R_0 = r_0 \in C(S^{n-1}) \), for the \( \mathcal{F}_t \)-stopping time

\[
\sigma^*(\delta) := \inf \left\{ t \geq 0 : \min_{\theta} \{ F(R_r, x_t, \theta) \} < \delta \right\}
\]

and any \( T < \infty, \iota, \delta > 0 \), we have that

\[
\lim_{\epsilon \to 0} \mathbb{P} \left( \sup_{0 \leq t \leq T_{\sigma^*(\delta)}} \| R^\epsilon_t - r_t \|_2 > \iota \right) = 0 \tag{1.19}
\]

where \( \{r_t\}_{t \geq 0} \) denotes the unique \( C(S^{n-1}) \)-solution of (1.12) (see Proposition 1.6).

**Remark 1.5.** With minor modifications of the proof, we can accommodate in Theorem 1.4 any random initial data such that \( R_0 \to r_0 \) in probability. It is crucial to have \( r_0 \) strictly positive, since the function \( b(r, x) \) blows up when \( y_{r, x} \to 0 \), hence (1.15) fails near \( r \equiv 0 \). Of course, if \( \inf \{ F(r, x, \theta) : (r, x) \in \mathcal{A}(a), \theta \in S^{n-1} \} > 0 \) for any \( a \in (0, 1) \), then we can dispense of the stopping time \( \sigma^*(\delta) \) in (1.19).

The next proposition, whose proof is deferred to the appendix, clarifies the implications of our assumptions.

**Proposition 1.6.**

(a) Conditions (1.13) and (1.14) of Assumption (L) imply that for every \( a \in (0, 1) \) there exists \( C = C(a, K) = C(a) < \infty \) such that for all \((r, x), (r', x') \in \mathcal{A}(a),\)

\[
\| b(r, x) - b(r', x') \|_2 \leq C(\| r - r' \|_2 + | x - x' |), \tag{1.20}
\]

\[
| h(r, x) - h(r', x') | \leq C(\| r - r' \|_2 + | x - x' |). \tag{1.21}
\]

(b) Condition (1.15) of Assumption (L) implies that starting at any \( r_0 \in C(S^{n-1}) \) the ODE (1.12) admits a unique \( C(S^{n-1}) \)-solution on \([0, \infty)\).

(c) To verify Assumption (E), it suffices to show that for any \( a \in (0, 1) \) there exist \( n_0(a) \in \mathbb{N}, \delta = \delta(a) > 0 \) and a probability measure \( m(\cdot) \) on \( \mathbb{R}^n \), such that the jump transition probability measure \( P_r \) of the embedded Markov chain \( \{ x_{T_0} \} \) satisfies the uniform minorisation condition

\[
\inf_{(r, x) \in \mathcal{A}(a)} \{(P_r)^{n_0}(x, \cdot)\} \geq \delta m(\cdot). \tag{1.22}
\]

Recall (1.3) that the random dynamics (1.6) has expected volume increase of \( \epsilon (1 + o(1)) \) at each Poisson jump, (irrespective of the precise choice of \( \eta(\epsilon, r, x) \to 0 \) as \( \epsilon \to 0 \)). We thus expect the following result (whose proof is also deferred to the appendix), about the linear growth of the volume of the deterministic dynamics (1.12).

**Proposition 1.7.** If the solution \( (r_t)_{t \geq 0} \) to the ODE (1.12) belongs to \( C(S^{n-1}) \) for all \( t \geq 0 \), then \( \text{Leb}(r_t) = \text{Leb}(r_0) + t \).

Under the following scaling invariance of \( F \) and \( H \), we will deduce from the averaging principle of Theorem 1.4 a shape theorem for the process \( (R_t^\ell)_{t \geq 0} \).
Assumption (I). For any scalar $c > 0$, if $(r, x) \in \mathcal{D}(F)$ then $(cr, cx) \in \mathcal{D}(F)$ and
\begin{align}
F(r, x, \cdot) &= F(cr, cx, \cdot), \\
cH(r, \cdot) &= H(cr, \cdot).
\end{align}

Definition 1.8. (a) A function $\psi \in C(S^{n-1})$ is called invariant (shape) for the ODE (1.12), if starting at $r_0 = \psi$ yields
\[ r_t = (1 + t/\text{Leb}(\psi))^{1/n} \psi, \quad t \geq 0. \]

(b) A function $\psi \in C(S^{n-1})$ is called attractive (shape) for the ODE (1.12) and a collection $C$ of initial data, if starting at any $r_0 \in C$, the solution $t \mapsto r_t \in C(S^{n-1})$ exists, with
\[ \lim_{t \to \infty} \left\| (\text{Leb}(r_0) + t)^{-1/n} r_t - \text{Leb}(\psi)^{-1/n} \psi \right\|_2 = 0. \quad (1.25) \]

In general, invariant shapes may not be unique, nor are they necessarily attractive. See Example 3.4.

Theorem 1.9 (Shape theorem). Suppose Assumption (I) holds and (1.19) applies without the stopping time $\sigma^*(\delta)$ (see Remark 1.5).

(a) If a function $\psi$ with $\text{Leb}(\psi) = 1$ is invariant for the ODE (1.12), then for any $c > 0$, $1 \leq T < \infty$ and $\iota > 0$,
\[ \lim_{N \to \infty} \mathbb{P}\left( \sup_{1 \leq s \leq T} \left\| \left( N(c + s) \right)^{-1/n} R_{sN}^1 \psi \right\|_2 > \iota \mid R_0^1 = (cN)^{1/n} \psi \right) = 0. \quad (1.26) \]

(b) If a function $\psi$ with $\text{Leb}(\psi) = 1$ is attractive for the ODE (1.12) and a collection $C$ of initial data, then for any $\iota > 0$ and $r_0 \in C$,
\[ \lim_{t \to \infty} \lim_{N \to \infty} \mathbb{P}\left( \left\| \left( N(\text{Leb}(r_0) + t) \right)^{-1/n} R_{tN}^1 \psi \right\|_2 > \iota \mid R_0^1 = N^{1/n} r_0 \right) = 0. \]

Our main application is a model of random growth on $\mathbb{R}^n$ motivated by the expected mesoscopic behavior of ORRW and OERW on $\mathbb{Z}^n$, where to gain regularity we consider $F$ and $H$ defined via a smoothed version of the evolving domain. Specifically, fix $\eta > 0$ and $g = g_\eta$ as in (A.10) for some probability density $\phi \in C^3([-1,1])$. Then, $\tilde{r} := r \ast g \in C^3(S^{n-1})$ for every $r \in L^2(S^{n-1})$ (see (1.1)). We set
\[ F(r, x, \theta) := \frac{\partial}{\partial n} G_{\tilde{r}}(x, y) \bigg|_{y = \tilde{r}(\theta) \theta}, \quad \theta \in S^{n-1}, \quad (1.27) \]
where $G_{\tilde{r}}(x, y)$ denotes the Green’s function of the Laplacian $-\Delta$ on star-shaped domain $D \subseteq \mathbb{R}^n$ with Dirichlet boundary conditions at $\tilde{r} = \partial D$ and $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial D$. Similarly, fix a locally Lipschitz function $\alpha : \mathbb{R}_{>0} \times S^{n-1} \to \mathbb{R}_{\geq 0}$ such that
\[ 0 \leq \alpha(\ell, z) < \ell \quad \text{on} \quad \mathbb{R}_{>0} \times S^{n-1}, \quad (1.28) \]
and set (see Section 4 for the probabilistic interpretation),
\[ H(r, z) := \alpha(\tilde{r}(z), z) z. \quad (1.29) \]

We have the following results for these rules.
Theorem 1.10 (Smoothed harmonic measure).
(a) The Averaging Principle of Theorem 1.4 holds under (1.27)-(1.29), without the stopping times \( \sigma^\epsilon(\delta) \) of (1.18).
(b) In case \( \alpha(\ell, z) = \alpha(z) \ell \), the Shape Theorem 1.9 also holds. In particular, for \( \alpha(\ell, z) = \gamma \ell \) with \( \gamma \in [0, 1) \) fixed, the centered Euclidean ball is an invariant shape; and when \( \gamma = 0 \), it is uniquely attractive.

The rest of the article is organized as follows. In Section 2 we prove Theorem 1.4 and in Section 3 we deduce the shape result, Theorem 1.9. In Section 4, we present applications of the general theorem to concrete growth models, and in particular, prove Theorem 1.10.

2. Proof of Theorem 1.4

We start with bounding the Wasserstein 2-distance between any two measures on a compact, connected Riemannian manifold, by the \( L^2 \)-distance between their densities with respect to the Riemannian measure.

Lemma 2.1. Let \( M \subseteq \mathbb{R}^n \) be a connected Riemannian manifold without boundary compactly embedded in \( \mathbb{R}^n \), equipped with its Riemannian distance \( d(\cdot, \cdot) \) and measure \( \sigma(\cdot) \). Let \( \mu, \nu \) be probability distributions on \( M \) having densities \( p, q \) respectively with respect to \( \sigma(\cdot) \), where in addition \( p(x) \geq c > 0 \) for all \( x \in M \). Then, there exists \( C = C(M, c) < \infty \) such that

\[
W_2(\mu, \nu) \leq C \| p - q \|_2,
\]

where

\[
W_2(\mu, \nu) := \inf \left\{ \left[ \mathbb{E} d(X, Y)^2 \right]^{1/2} : \text{Law}(X) = \mu, \text{Law}(Y) = \nu \right\}
\]

is the Wasserstein 2-distance between \( \mu \) and \( \nu \), and \( \| \cdot \|_2 := \| \cdot \|_{L^2(\sigma)} \).

Proof of Lemma 2.1. By [36, Theorem 1], we have the variational representation

\[
W_2(\mu, \nu) = 2 \sup_{\{f \in C^1(M) : \int_M |\nabla f|^2 \, d\mu \leq 1\}} \left| \int_M f \, d(\mu - \nu) \right|
\]

\[
\leq 2c^{-1/2} \sup_{\{f \in C^1(M)\}} \frac{\left| \int_M f \, (d\mu - d\nu) \right|}{\| \nabla f \|_2}
\]

\[
= 2c^{-1/2} \sup_{\{f \in C^1(M)\}} \frac{\left| \int_M (f - \bar{f}_M) \, (d\mu - d\nu) \right|}{\| \nabla f \|_2}
\]

\[
\leq 2c^{-1/2} \sup_{\{f \in C^1(M)\}} \frac{\| f - \bar{f}_M \|_2 \| p - q \|_2}{\| \nabla f \|_2} \leq 2c^{-1/2} c(M) \| p - q \|_2.
\]

In the last step, we have used the Poincaré inequality \( \| f - \bar{f}_M \|_2 \leq c(M) \| \nabla f \|_2 \), where \( \bar{f}_M \) denotes the \( \sigma \)-weighted average of \( f \) in \( M \) and \( c(M) \) is the Poincaré constant. \( \square \)
The proof of Theorem 1.4 is based on considering an auxiliary process in which the slow variable is frozen (this is a standard tool for proving averaging principles, see [20, 39]). Set
\[ \Delta = \Delta(\epsilon) = \epsilon \log^{1/3}(\epsilon^{-1}) \land 1. \]

Given the main process \((R^e_t, x^e_t)_{t \geq 0}\), we consider a family (indexed by \(\epsilon > 0\)) of auxiliary dynamics \((\widehat{R}^e_t, \widehat{x}^e_t)_{t \geq 0}\) defined piecewise on each time interval \([k\Delta, (k+1)\Delta)\) with \(k \in \mathbb{N}\), on the same probability space \((\Omega, \mathcal{F}, \mathbb{P})\) as the main process, as follows. Inductively for every \(k \in \mathbb{N}\), take the same Poisson clock \(\{T^e_i\}_{i \in \mathbb{N}}\) used in constructing the main process, and starting at \(\widehat{x}^e_{k\Delta} = x^e_{k\Delta}\), let \((\widehat{x}^e_t)_{t \in [k\Delta, (k+1)\Delta)}\) have the marginal distribution of the Markov jump process in the frozen domain \(R^e_{k\Delta}\) defined as in (1.11). That is, \((\widehat{x}^e_t)_{t \in [k\Delta, (k+1)\Delta)}\) jumps at each \(T^e_i \in [k\Delta, (k+1)\Delta)\), \(i \in \mathbb{N}\) in the frozen domain \(R^e_{k\Delta}\), by first using probability density \(F(R^e_{k\Delta}, \widehat{x}^e_{i\Delta}, \cdot)\) to choose a spherical angle \(\widehat{\zeta}_i\), then applying the rule \(H(R^e_{k\Delta}, \widehat{\zeta}_i)\). We further put requirement on the joint law such that at each jump, \(\widehat{\zeta}_i\) and \(\xi_i\) of (1.5) achieves within twice their Wasserstein 2-distance \(F\) on \(S^{n-1}\), where \(\mu = F(R^e_{k\Delta}, \widehat{x}^e_{i\Delta}, \cdot)\) and \(\nu = F(R^e_{k\Delta}, \widehat{x}^e_{i\Delta}, \cdot)\). Inductively the above procedure defines \((\widehat{R}^e_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\).

We then define \((\widehat{R}^e_t)_{t \geq 0}\) on \((\Omega, \mathcal{F}, \mathbb{P})\) as the dynamics driven by the ODE, with \(\widehat{R}^e_0 = R^e_0\),
\[ \widehat{R}^e_t = R^e_0 + \int_0^t b(R^e_{|s/\Delta|\Delta}, \widehat{x}^e_s)ds, \quad t \geq 0. \tag{2.2} \]

With the auxiliary processes in place, we proceed to the proof of the theorem. By Proposition 1.6(b), starting at \(r_0 \in C(S^{n-1})\), the solution \((r_t)_{t \geq 0}\) to the ODE (1.12) exists and is unique in \(C(S^{n-1})\). Fixing \(T < \infty\) and \(\epsilon \in (0, \inf_\theta r_0(\theta))\), with \(R^e_0 = r_0\) define the \(\mathcal{F}_t\)-stopping time
\[ \zeta^\epsilon(\epsilon) := \inf\{t > 0 : \|R^e_t - r_t\|_2 > \xi \}. \]

We claim that the stopped process \((R^e_{t \wedge \zeta^\epsilon(\epsilon)})_{t \in [0,T]} \in \mathcal{A}_1(a)\) for some \(a \in (0,1)\) depending only on \(\xi, r_0\) and \(T\). Indeed, since \(t \rightarrow \|r_t\|_2\) is continuous and increasing with \((r_t)_{t \geq 0}\) a-priori existing for all time, it suffices to notice that a.s. \(\|R^e_t\|_2 - \|R^e_{t-}\|_2 \leq \|R^e_t - R^e_{t-}\|_2 \leq C\epsilon^{1/n}\) for \(t = \zeta^\epsilon(\epsilon)\) and some finite \(C\) uniform (due to the uniform control on \(\eta^{-1/2}\|g\|_\infty\) per Lemma A.1). Clearly, this verifies our claim. In the rest of the proof, we only apply Assumptions (L) and (E) with Lipschitz constant \(K(a)\), resp. convergence rate \(\lambda\) in (1.16), depending on such fixed \(a\), for the stopped processes.

The proof of the following three lemmas are deferred to the end of this section. Fixing \(\delta > 0\) and set \(\tau := \zeta^\epsilon(\epsilon) \land \sigma^\epsilon(\delta)\) (see (1.18) for the latter).

**Lemma 2.2.** In the setting of Theorem 1.4, for some finite \(C = C(K(a), \delta)\) we have that
\[ \sup_{0 \leq t \leq T} \mathbb{E}\left[ |x^e_{t, \Lambda \tau} - \widehat{x}^e_{t, \Lambda \tau}|^2 \right] \leq C \epsilon. \]

**Lemma 2.3.** In the setting of Theorem 1.4, we have that
\[ \lim_{\epsilon \to 0} \mathbb{E}\left[ \sup_{0 \leq t \leq T} \|R^e_{t, \Lambda \tau} - \widehat{R}^e_{t, \Lambda \tau}\|_2^2 \right] = 0. \]
Lemma 2.4. In the setting of Theorem 1.4, we have that
\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T} \| \hat{R}_{t \wedge \tau} - r_{t \wedge \tau} \|_2^2 \right] = 0. \quad (2.3)
\]

We proceed directly to complete the proof of the theorem. By (1.9), for any \( u \leq t \) we have per \( \theta 
\]
\[(R_s^\epsilon - R_u^\epsilon)(\theta) = \int_u^t b^\epsilon(R_s^\epsilon, x_s^\epsilon)(\theta) ds + (\Sigma_s^\epsilon - \Sigma_u^\epsilon)(\theta).\]

By [10, Proposition 8.7], for the stopped martingale \( \Sigma_{s \wedge \zeta^\epsilon}(\theta) \),
\[
\mathbb{E} \left[ \sup_{s \in [u,t]} (\Sigma_{s \wedge \zeta^\epsilon}(\cdot) - \Sigma_{u \wedge \zeta^\epsilon}(\cdot))^2(\theta) \right] \leq 4\epsilon \mathbb{E} \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} b^\epsilon(R_s^\epsilon, x_s^\epsilon)^2(\theta) ds.
\]

Since spherical convolution is a contraction in \( L^2(S^{n-1}) \) (per Definition 1.2), and the Lipschitz assumption [10] holds throughout \( A(a) \), which implies that \( b(R_{s \wedge \zeta^\epsilon}(\cdot), x_{s \wedge \zeta^\epsilon}(\cdot)), s \in [0,T] \), is bounded in \( L^2 \)-norm, together with Fubini we have that for some finite \( C = C(K, \delta) \) and any \( 0 \leq u \leq t \leq T 
\]
\[
\mathbb{E} \left[ \sup_{s \in [u,t]} \| \Sigma_{s \wedge \zeta^\epsilon}(\cdot) - \Sigma_{u \wedge \zeta^\epsilon}(\cdot) \|_2^2 \right] \leq \int_{S^{n-1}} \mathbb{E} \left[ \sup_{s \in [u,t]} (\Sigma_{s \wedge \zeta^\epsilon}(\cdot) - \Sigma_{u \wedge \zeta^\epsilon}(\cdot))^2(\theta) \right] d\sigma(\theta)
\]
\[
\leq 4\epsilon \mathbb{E} \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b^\epsilon(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds \leq 4\epsilon \mathbb{E} \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds \leq C\epsilon(t-u). \quad (2.4)
\]

Consequently, by (2.2), (2.4) and Cauchy-Schwarz inequality, for any \( 0 \leq u \leq t \leq T 
\]
\[
\frac{1}{2} \mathbb{E} \| R_{t \wedge \zeta^\epsilon}(\cdot) - R_{u \wedge \zeta^\epsilon}(\cdot) \|_2^2
\]
\[
\leq \mathbb{E} \left( \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b^\epsilon(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds \right) + \mathbb{E} \| \Sigma_t^\epsilon - \Sigma_u^\epsilon \|_2^2
\]
\[
\leq \mathbb{E} \left( \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds \right) + \mathbb{E} \| \Sigma_t^\epsilon - \Sigma_u^\epsilon \|_2^2
\]
\[
\leq (t-u)\mathbb{E} \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds + C\epsilon(t-u)
\]
\[
\leq (t-u)\mathbb{E} \int_{u \wedge \zeta^\epsilon}^{t \wedge \zeta^\epsilon} \| b(R_s^\epsilon, x_s^\epsilon) \|_2^2 ds + C\epsilon(t-u)
\]
\[
\leq C(t-u)^2 + C\epsilon(t-u). \quad (2.5)
\]

This estimate will be useful later in proving all the lemmas. By Lemma 2.3 and Lemma 2.4 we have that
\[
\lim_{\epsilon \to 0} \mathbb{E} \left[ \sup_{0 \leq t \leq T \wedge \tau} \| R_t^\epsilon - r_t \|_2^2 \right] = 0.
\]
By Chebyshev’s inequality, we have that

$$\lim_{\epsilon \to 0} \mathbb{P}\left( \sup_{0 \leq t \leq T} \| R_t^\epsilon - r_t \|_2 > \epsilon \right) = 0. \quad (2.6)$$

This in turn implies that $\lim_{\epsilon \to 0} \mathbb{P}(\zeta'(t) \leq T \wedge \sigma'(\delta)) = 0$, otherwise contradicting (2.6) by the definition of $\zeta'(t)$. Therefore, we obtain (1.19) for any $t \in (0, \inf_{\theta} r_0(\theta))$. Since the LHS of (1.19) is non-increasing in the positive $t$, the conclusion extends to any $t > 0$. □

**Proof of Lemma 2.2.** Per (1.11), for each $k \in \mathbb{N}$ the auxiliary process $(\widehat{x}_t^\epsilon)_{t \in [k\Delta,(k+1)\Delta]}$ admits the decomposition

$$\widehat{x}_t^\epsilon = x_{k\Delta}^\epsilon + \epsilon^{-1} \int_{k\Delta}^t h(R_{s\Delta}, \widehat{x}_s^\epsilon) ds + \widehat{M}_{k\Delta,t}^\epsilon, \quad t \in [k\Delta, (k+1)\Delta), \quad (2.7)$$

for some $\mathbb{R}^n$-valued, $\mathcal{F}_t$-martingale $\widehat{M}_{k\Delta,t}^\epsilon$. Taking the difference of (2.7) with (1.10), we have that

$$(x_t^\epsilon - \widehat{x}_t^\epsilon) - \epsilon^{-1} \int_{k\Delta}^t [h(R_{s\Delta}, x_s^\epsilon) - h(R_{k\Delta}, \widehat{x}_s^\epsilon)] ds = M_t^\epsilon - M_{k\Delta}^\epsilon - \widehat{M}_{k\Delta,t}^\epsilon, \quad t \in [k\Delta, (k+1)\Delta), \quad (2.8)$$

a martingale in $\mathbb{R}^n$.

Considering the generator of $(x_t^\epsilon - \widehat{x}_t^\epsilon)_{t \in [k\Delta,(k+1)\Delta]}$, we have again by [10, Proposition 8.7] that

$$\mathbb{E}|M_t^\epsilon - M_{k\Delta}^\epsilon - \widehat{M}_{k\Delta,t}^\epsilon|^2 \leq \frac{4}{\epsilon} \mathbb{E} \int_{k\Delta}^t \mathbb{E}\left[ H(R_{s\Delta}, \xi_{s\Delta}) - x_{s\Delta}^\epsilon - H(R_{k\Delta}, \widehat{\xi}_{s\Delta}) + \widehat{x}_{s\Delta}^\epsilon \right]^2 ds$$

$$\leq 8\epsilon^{-1} K^2 \mathbb{E} \int_{k\Delta}^t \left( \| R_{s\Delta}^\epsilon - R_{k\Delta}^\epsilon \|_2^2 + \mathbb{E}|\xi_{s\Delta} - \widehat{\xi}_{s\Delta}|^2 \right) ds, \quad (2.9)$$

for $k = 0, 1, ..., \lfloor T/\Delta \rfloor$, where the inner conditional expectation $\mathbb{E}$ is only over $(\xi_{s\Delta}, \widehat{\xi}_{s\Delta})$, having marginal densities $F(R_{s\Delta}^\epsilon, x_{s\Delta}^\epsilon)$ and $F(R_{k\Delta}^\epsilon, \widehat{x}_{s\Delta}^\epsilon)$, respectively, with respect to $\sigma(\cdot)$ on $\mathbb{S}^{n-1}$. In (2.9) we also used (1.14). By the coupling we chose, and Lemma 2.1 with $M = S_{n-1}$, $p = F(R_{s\Delta}^\epsilon, x_{s\Delta}^\epsilon, \cdot)$ bounded below by $\delta$, we have in (2.9)

$$\mathbb{E}|\xi_{s\Delta} - \widehat{\xi}_{s\Delta}|^2 \leq 4W_2(\xi_{s\Delta}, \widehat{\xi}_{s\Delta})^2$$

$$\leq C(\delta) \| F(R_{s\Delta}^\epsilon, x_{s\Delta}^\epsilon, \cdot) - F(R_{k\Delta}^\epsilon, \widehat{x}_{s\Delta}^\epsilon, \cdot) \|^2_2$$

$$\leq CK^2 (\| R_{s\Delta}^\epsilon - R_{k\Delta}^\epsilon \|_2^2 + \| x_{s\Delta}^\epsilon - \widehat{x}_{s\Delta}^\epsilon \|_2^2),$$

using (1.13) in the last line. Consequently, we obtain for (2.9) that

$$\mathbb{E}|M_t^\epsilon - M_{k\Delta}^\epsilon - \widehat{M}_{k\Delta,t}^\epsilon|^2 \leq C(K, \delta)\epsilon^{-1} \mathbb{E} \int_{k\Delta}^t (\| R_{s\Delta}^\epsilon - R_{k\Delta}^\epsilon \|_2^2 + \| x_{s\Delta}^\epsilon - \widehat{x}_{s\Delta}^\epsilon \|_2^2) ds$$

$$\leq C\epsilon^{-1} \Delta^3 + C\Delta^2 + C\epsilon^{-1} \mathbb{E} \int_{k\Delta}^t |x_{s\Delta}^\epsilon - \widehat{x}_{s\Delta}^\epsilon|^2 ds, \quad (2.10)$$
using (2.5) in the last line. Thus, by (2.8), Cauchy-Schwarz inequality, (2.10) and (2.5), we have for some finite $C = C(K, \delta)$ and any $t \in [k\Delta, (k + 1)\Delta)$, $k = 0, 1, \ldots, [T/\Delta]$,

$$\frac{1}{2}\mathbb{E}|x^\varepsilon_t - \tilde{x}^\varepsilon_t|^2 \\
\leq \epsilon^{-2}\mathbb{E}\left[ \int_{k\Delta}^{t+\Delta} |h(R^\varepsilon_s, x^\varepsilon_s) - h(R^\varepsilon_{k\Delta}, \tilde{x}^\varepsilon_s)| \, ds \right]^2 + \mathbb{E}|M^\varepsilon_t - M^\varepsilon_{k\Delta}\Delta - \tilde{M}^\varepsilon_{k\Delta}\Delta\Delta_t|^2 \\
\leq K^2\epsilon^{-2}\Delta\mathbb{E}\int_{k\Delta}^{t+\Delta} (\|R^\varepsilon_s - R^\varepsilon_{k\Delta}\|^2 + |x^\varepsilon_s - \tilde{x}^\varepsilon_s|^2) \, ds + C\epsilon^{-1}(\Delta^3 + \mathbb{E}\int_{k\Delta}^{t+\Delta} |x^\varepsilon_s - \tilde{x}^\varepsilon_s|^2 \, ds) \\
\leq C(\epsilon^{-2}\Delta^4 + \epsilon^{-1}\Delta^3) + C(\epsilon^{-2}\Delta + \epsilon^{-1})\mathbb{E}\int_{k\Delta}^{t} |x^\varepsilon_s - \tilde{x}^\varepsilon_s|^2 \, ds,$$

where the Lipschitz property of $h(r, x)$ follows from Proposition 1.6(a). By Gronwall’s inequality, we have that uniformly for any $t \in [0, T]$,

$$\mathbb{E}|x^\varepsilon_t - \tilde{x}^\varepsilon_t|^2 \leq C\epsilon^{-2}\Delta^4 e^{C\epsilon^{-2}\Delta^2}. \quad (2.11)$$

For our choice (2.1) of $\Delta = \Delta(\varepsilon)$, the RHS of (2.11) is bounded by $C\epsilon$ for some finite $C = C(K, \delta)$.

**Proof of Lemma 2.3.** Per (1.9) and (2.2), for any $t \geq 0$ we have that

$$R^\varepsilon_t - \hat{R}^\varepsilon_t = \int_0^t [b^\varepsilon(R^\varepsilon_s, x^\varepsilon_s) - b(R^\varepsilon_s, x^\varepsilon_s)] \, ds + \int_0^t [b(R^\varepsilon_s, x^\varepsilon_s) - b(R^\varepsilon_{t\Delta}, \tilde{x}^\varepsilon_s)] \, ds + \Sigma^\varepsilon_t.$$

Then, by Cauchy-Schwarz inequality, (1.20), (2.4), (2.5) and Lemma 2.2 we have that

$$\frac{1}{3}\mathbb{E}\sup_{0 \leq t \leq T} \|R^\varepsilon_{t\Delta} - \hat{R}^\varepsilon_{t\Delta}\|^2 \\
\leq T\mathbb{E}\int_0^T \|b^\varepsilon(R^\varepsilon_{s\Delta}, x^\varepsilon_{s\Delta}) - b(R^\varepsilon_{s\Delta}, x^\varepsilon_{s\Delta})\|^2 \, ds \\
+ T\mathbb{E}\int_0^T \|b(R^\varepsilon_{s\Delta}, x^\varepsilon_{s\Delta}) - b(R^\varepsilon_{t\Delta}, \tilde{x}^\varepsilon_{s\Delta})\|^2 \, ds + \mathbb{E}\sup_{0 \leq t \leq T} \|\Sigma^\varepsilon_t\|^2 \\
\leq \text{term (I)} + K^2T\mathbb{E}\int_0^T (\|R^\varepsilon_s - R^\varepsilon_{t\Delta}\|^2 + |x^\varepsilon_s - \tilde{x}^\varepsilon_s|^2) \, ds + CT\varepsilon \\
\leq \text{term (I)} + CT\int_0^T (s - [s/\Delta]\Delta)^2 \, ds + CT^2\varepsilon + CT\varepsilon \\
\leq \text{term (I)} + CT^2\Delta^2 + CT^2\varepsilon,$$

where term (I) converges to 0 as $\varepsilon \to 0$ by Assumption (C) and the uniform boundedness of the integrand. Thus, the whole expression tends to 0 as well.

\hfill \Box
Proof of Lemma 2.4. Per (1.16) and the fact that the event $\{k\Delta \leq \zeta^i(\ell)\}$ is measurable to the randomness of $\sigma(R^i_{k\Delta})$, we have that uniformly for $k = 0, 1, \ldots, [T/\Delta]$,

\[
\mathbb{E} \left[ 1_{\{k\Delta \leq \tau\}} \left\| \int_{k\Delta}^{(k+1)\Delta} [b(R^i_{k\Delta}, \hat{x}_s^i) - \bar{b}(R^i_{k\Delta})]ds \right\|_2^2 \right] \\
= \mathbb{E} \left[ 1_{\{k\Delta \leq \tau\}} \left\| \int_{k\Delta}^{k\Delta + \Delta/\epsilon} \epsilon [b(R^i_{k\Delta}, \hat{x}_s^i) - \bar{b}(R^i_{k\Delta})]ds \right\|_2^2 \right] \\
\leq \Delta^2 \mathbb{E} \left[ 1_{\{k\Delta \leq \zeta^i(\ell)\}} \left\| \int_{k\Delta}^{k\Delta + \Delta/\epsilon} \frac{1}{\Delta/\epsilon} [b(R^i_{k\Delta}, \hat{x}_s^i) - \bar{b}(R^i_{k\Delta})]dt \right\|_2^2 \right] \\
\leq \Delta^2 \lambda(\Delta/\epsilon, a). \tag{2.12}
\]

It then follows from (2.12), (1.15) and (2.5) that for some finite $C = C(K, \delta)$ and any $t \in [0, T]$,

\[
\frac{1}{2} \mathbb{E} \sup_{0 \leq u \leq t} \left\| \int_0^{u \wedge \tau} [b(R^i_{s/\Delta | \Delta}, \hat{x}_s^i) - \bar{b}(R^i_s)]ds \right\|_2^2 \\
\leq \mathbb{E} \sup_{0 \leq u \leq t} \left\| \int_0^{u \wedge \tau} [b(R^i_{s/\Delta | \Delta}, \hat{x}_s^i) - \bar{b}(R^i_{s/\Delta | \Delta})]ds \right\|_2^2 \\
+ \mathbb{E} \left( \int_0^{t \wedge \tau} \|\bar{b}(R|s/\Delta|\Delta) - \bar{b}(R_s)\|_2 ds \right)^2 \\
\leq \frac{t}{\Delta} \sum_{k=0}^{[t/\Delta]} \mathbb{E} 1_{\{k\Delta \leq \tau\}} \left\| \int_{k\Delta}^{(k+1)\Delta} [b(R^i_{k\Delta}, \hat{x}_s^i) - \bar{b}(R^i_{k\Delta})]ds \right\|_2^2 + C \Delta^2 \\
+ t \mathbb{E} \int_0^{t \wedge \tau} \|\bar{b}(R^i_{s/\Delta | \Delta}) - \bar{b}(R^i_s)\|_2^2 ds \\
\leq t^2 \lambda(\Delta/\epsilon, a) + C K^2 t^2 \Delta^2, \tag{2.13}
\]

where $\lambda(\Delta/\epsilon, a) \to 0$ as $\epsilon \to 0$ by (1.16) since $\Delta/\epsilon \to \infty$. We proceed to bound

\[
m^e(T) := \mathbb{E} \left[ \sup_{0 \leq u \leq T} \|\hat{R}_u \wedge \tau - r_u \wedge \tau\|_2^2 \right]
\]

via Gronwall’s inequality. Per (2.2) and (1.12), for any $t \geq 0$ we have that

\[
\hat{R}_t - r_t = \int_0^t [b(R^i_{s/\Delta | \Delta}, \hat{x}_s) - \bar{b}(R^i_s)]ds + \int_0^t [\bar{b}(R^i_s) - \bar{b}(r_s)]ds.
\]
By (1.15) and (2.13) we have that
\[
\frac{1}{2} m^\epsilon(T) \leq \mathbb{E} \sup_{0 \leq t \leq T} \left\| \int_0^{u \wedge \tau} \left[ b(R_{s/\epsilon}^\epsilon) \Delta \right] - \tilde{b}(R_s^\epsilon) ds \right\|^2 + \mathbb{E} \sup_{0 \leq u \leq T} \left\| \int_0^{T \wedge \tau} \left[ \tilde{b}(R_s^\epsilon) - \tilde{b}(r_s) \right] ds \right\|^2
\]
\[
\leq (T^2 \lambda(\Delta/\epsilon, a) + C K^2 T^2 \Delta^2) + T \mathbb{E} \int_0^{T \wedge \tau} \left\| \tilde{b}(R_s^\epsilon) - \tilde{b}(r_s) \right\|^2 ds
\]
\[
\leq (T^2 \lambda(\Delta/\epsilon, a) + C K^2 T^2 \Delta^2) + T K^2 \mathbb{E} \int_0^{T \wedge \tau} \left\| R_s^\epsilon - r_s \right\|^2 ds
\]
\[
\leq (T^2 \lambda(\Delta/\epsilon, a) + C K^2 T^2 \Delta^2) + 2T^2 K^2 \mathbb{E} \sup_{0 \leq u \leq T} \| R_{u \wedge \tau}^\epsilon - \tilde{R}_{u \wedge \tau}^\epsilon \|_2^2 + 2T K^2 \int_0^T m^\epsilon(t) dt.
\]

Gronwall’s inequality and Lemma [2.3] yield
\[
m^\epsilon(T) \leq \left( T^2 \lambda(\Delta/\epsilon, a) + C K^2 T^2 \Delta^2 + C T^2 K^2 \mathbb{E} \sup_{0 \leq u \leq T} \| R_{u \wedge \tau}^\epsilon - \tilde{R}_{u \wedge \tau}^\epsilon \|_2^2 \right) e^{C T^2 K^2}
\]
converging to zero as \( \epsilon \to 0 \), as required.

\[\square\]

3. Proof of Theorem [1.9]

The following intuitive coupling enables to transfer the Averaging Principle for the family of processes \( (R_t^\epsilon) \) as the scale parameter \( \epsilon \to 0 \) on finite time horizons, into a shape result for \( (R_t^1) \) of scale 1 as time \( t \to \infty \).

**Lemma 3.1** (coupling). Fix \( \epsilon > 0 \). Under Assumption (I), with \( (R_0^\epsilon, x_0^\epsilon) \overset{d}{=} (\epsilon^{1/n} R_0^1, \epsilon^{1/n} x_0^1) \), there exists a coupling such that
\[
(R_t^\epsilon, x_t^\epsilon) = (\epsilon^{1/n} R_t^{1/\epsilon}, \epsilon^{1/n} x_t^{1/\epsilon}), \quad t \geq 0.
\]

**Proof.** Let \( \{T_i^\epsilon\}_{i \in \mathbb{N}} \) with \( T_0^\epsilon = 0 \) denote the sequence of Poisson jump times of rate \( \epsilon^{-1} \) used in constructing \( (R_t^\epsilon, x_t^\epsilon) \) for some fixed \( \epsilon > 0 \). Set \( T_i^1 := \epsilon^{-1} T_i^\epsilon \), \( i \in \mathbb{N} \). By scaling properties of exponential distribution, \( \{T_i^1\}_{i \in \mathbb{N}} \) has the law of a sequence of Poisson arrival times of rate 1, as such we construct \( (R_t^1, x_t^1) \) for some fixed \( \epsilon > 0 \) using \( \{T_i^1\}_{i \in \mathbb{N}} \), on the same probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) as \( (R_t^\epsilon, x_t^\epsilon) \).

Starting with \( (R_0^\epsilon, x_0^\epsilon) = (\epsilon^{1/n} R_0^1, \epsilon^{1/n} x_0^1) \) on \( (\Omega, \mathcal{F}, \mathbb{P}) \), suppose we have succeeded in coupling \( (R_t^\epsilon, x_t^\epsilon) \) with \( (\epsilon^{1/n} R_{s/\epsilon}^1, \epsilon^{1/n} x_{s/\epsilon}^1) \) as in (3.1) up to time \( (T_i^\epsilon)^- \) for some \( i \in \mathbb{N} \). Then by (1.23), for any \( s \leq (T_i^\epsilon)^- \) we have
\[
F(R_s^\epsilon, x_s^\epsilon, \cdot) = F(\epsilon^{1/n} R_{s/\epsilon}^1, \epsilon^{1/n} x_{s/\epsilon}^1, \cdot) = F(R_{s/\epsilon}^1, x_{s/\epsilon}^1, \cdot),
\]
\[
y_{R_s^\epsilon, x_s^\epsilon} = \omega_n \int R_s^\epsilon(z)^{n-1} F(R_s^\epsilon, x_s^\epsilon, z) d\sigma(z)
\]
\[
= \epsilon^{n-1} \omega_n \int R_{s/\epsilon}^1(z)^{n-1} F(R_{s/\epsilon}^1, x_{s/\epsilon}^1, z) d\sigma(z) = \epsilon^{n-1} y_{R_s^\epsilon, x_s^\epsilon}^{1/\epsilon},
\]
\[
\eta(\epsilon, R_s^\epsilon, x_s^\epsilon) = \frac{\epsilon^{1/n}}{y_{R_s^\epsilon, x_s^\epsilon}^{1/(n-1)}} = y_{R_{s/\epsilon}^1, x_{s/\epsilon}^1}^{1/\epsilon} = \eta(1, R_{s/\epsilon}^1, x_{s/\epsilon}^1).
\]
The induction hypotheses and the construction (1.5), (1.6) yield at $t = T^\epsilon_i$, per $\theta$

$\xi^\epsilon_t \overset{d}{=} F(R^1_{t^-}, x^1_{t^-}, \cdot) = F(R^1_{t^-/\epsilon}, x^1_{t^-/\epsilon}, \cdot) \overset{d}{=} \xi^1_{t/\epsilon}$,

$R^1_t (\theta) = R^1_{t^-/\epsilon} (\theta) + \epsilon y R^1_{t^-/\epsilon} \frac{\partial}{\partial x} g_{\psi_R}(R^1_{t^-/\epsilon}, x^1_{t^-/\epsilon})(\xi^1_{t/\epsilon}, \theta)$

$= \epsilon^{1/n} R^1_{t^-/\epsilon}(\theta) + \epsilon^{1/n} y R^1_{t^-/\epsilon} \frac{\partial}{\partial x} g_{\psi_R}(1, R^1_{t^-/\epsilon}, x^1_{t^-/\epsilon})(\xi^1_{t/\epsilon}, \theta)$

$R^1_{t/\epsilon} (\theta) = R^1_{t^-/\epsilon} (\theta) + \epsilon y R^1_{t^-/\epsilon} \frac{\partial}{\partial x} g_{\psi_R}(1, R^1_{t^-/\epsilon}, x^1_{t^-/\epsilon})(\xi^1_{t/\epsilon}, \theta)$

By coupling the jumps of $x^\epsilon_t$ and $x^1_{t^-/\epsilon}$ at $t = T^\epsilon_i$ such that $\xi^\epsilon_t = \xi^1_{t/\epsilon}$, we deduce $R^\epsilon_t = \epsilon^{1/n} R^1_{t/\epsilon}$, $\xi^\epsilon_t = \xi^1_{t/\epsilon}$ for $t = T^\epsilon_i$, and by (1.7), (1.24) also $x^\epsilon_t = \epsilon^{1/n} x^1_{t/\epsilon}$. During $t \in [T^\epsilon_i, T^\epsilon_{i+1})$, all processes stay put, hence continuing extends to all $t \geq 0$. □

Proof of Theorem 1.9 We only prove part (b), whereas the proof of part (a) is similar. By Theorem 1.4 and Lemma 3.1, we firstly have for any $t < \infty$ and $\epsilon > 0$,

$$
\lim_{\epsilon \to 0} \mathbb{P}(\|\epsilon^{1/n} R^1_{t/\epsilon} - r_t\|_2 > \epsilon/2 \mid R^1_0 = \epsilon^{-1/n} r_0)
= \lim_{\epsilon \to 0} \mathbb{P}(\|R^\epsilon_t - r_t\|_2 > \epsilon/2 \mid R^\epsilon_0 = r_0) = 0,
$$

(3.2)

where $(r_t)_{t \geq 0}$ is the continuous solution of (1.12) with initial data $r_0 \in C \cap C(S^{n-1})$, and $\text{Leb}(r_t) = \text{Leb}(r_0) + t$. By the triangular inequality, we have that

$$
\lim_{\epsilon \to 0} \mathbb{P}
\bigg(\| (\text{Leb}(r_0) + t)^{-1/n} \epsilon^{1/n} R^1_{t/\epsilon} - \psi\|_2 > \epsilon \bigg| R^1_0 = \epsilon^{-1/n} r_0
\bigg)
\leq \lim_{\epsilon \to 0} \mathbb{P}
\bigg(\| (\text{Leb}(r_0) + t)^{-1/n} (\epsilon^{1/n} R^1_{t/\epsilon} - r_t)\|_2 > \epsilon/2 \bigg| R^1_0 = \epsilon^{-1/n} r_0
\bigg)
+ 1 \{\| (\text{Leb}(r_0) + t)^{-1/n} r_t - \psi\|_2 > \epsilon/2\}
$$

By (3.2), the first term vanishes for any $t \geq 1$, and upon taking another limit as $t \to \infty$, the second term vanishes as well by (1.25). We obtain the claims upon setting $N = \epsilon^{-1}$. □

Problem 3.2. It remains open to remove the strict positivity of initial condition in Theorem 1.4, hence to be able to take $c = 0$ in (1.26), which would correspond to a genuine shape theorem.

We have the following general characterization of invariant shapes.

Proposition 3.3. Under Assumption (I), $\psi \in C(S^{n-1})$ is invariant for the ODE (1.12) if and only if $\overline{b}(\psi) = n^{-1} \psi/\text{Leb}(\psi)$.

Proof. We prove the “only if” part, while the converse “if” direction can be checked directly. Assumption (I) implies that for any $c > 0$, $y_{cr,ex} = c^{n-1} y_{r,x}$, hence $\overline{b}(cr) = c^{-(n-1)} \overline{b}(r)$. Per Definition 1.8(a), an invariant solution $(r_t)_{t \geq 0}$ starting at $r_0 = \psi$ is such that $r_t = c_t \psi$ with $(c^n_t - 1) \text{Leb}(\psi) = t$. From the ODE (1.12) it is not hard to infer that $\overline{b}(\psi) \propto \psi$. Further, by taking derivative of (1.12) in $t$ we identify the proportional constant to be $n^{-1}/\text{Leb}(\psi)$. □

However, invariant shapes may not be unique.
Example 3.4. Consider $H \equiv 0$ (the origin) and $F(r, 0, \cdot) = r(\cdot)/\int r d\sigma \in C(S^{n-1})$. Then it is easy to check that $\bar{b}(r) = b(r, 0) = F(r, 0, \cdot)/\int F(r, 0, \cdot)r^{n-1}d\sigma = r/(n\text{Leb}(r))$. Since this choice of $F$ and $H$ satisfies Assumption (1), by Proposition 3.3, any $r \in C(S^{n-1})$ is invariant for (1.12), and not attractive except when starting from itself.

We provide sufficient condition for the centered Euclidean ball $B$ to be attractive for (1.12), where we denote henceforth by $B$ the constant 1 function on $S^{n-1}$. Unfortunately, the condition (3.3) is rather hard to check.

Proposition 3.5. Suppose the ODE (1.12) has $C^1(S^{n-1})$-solution $(r_t)_{t \geq 0}$ for any $r_0 \in C \subset C^1(S^{n-1})$, and that for any $r \in C(S^{n-1})$, it holds

$$\bar{b}(r)(\arg \max r) \leq \bar{b}(r)(\arg \min r).$$

Then $B$ is attractive for (1.12) for the collection $C$ of initial data.

Proof. Set $\text{osc}(r) = \max_{\theta} r(\theta) - \min_{\theta} r(\theta)$. Since $r_t$ is $C^1$ for all $t \geq 0$, we have that

$$\frac{d}{dt} \{ r_t(\arg \max_{\theta} r_t) \} = \bar{b}(r_t)(\arg \max_{\theta} r_t) + \frac{d}{dz} r_t(z) \bigg|_{z=\arg \max_{\theta} r_t} \cdot \frac{d}{dt} \{ \arg \max_{\theta} r_t \}$$

and similarly for $\arg \min_{\theta} r_t$. Therefore, combined with (3.3) we have that

$$\frac{d}{dt} \text{osc}(r_t) = \bar{b}(r_t)(\arg \max_{\theta} r_t) - \bar{b}(r_t)(\arg \min_{\theta} r_t) \leq 0.$$ 

Set $r_t := r_t/(\text{Leb}(r_0) + t)^{1/n}$. Then we have that $\text{osc}(r_t) = \text{osc}(r_t)/(\text{Leb}(r_0) + t)^{1/n}$ and

$$\frac{d}{dt} \text{osc}(r_t) = \frac{1}{(\text{Leb}(r_0) + t)^{1/n}} \frac{d}{dt} \text{osc}(r_t) - \frac{1}{n(\text{Leb}(r_0) + t)^{1+1/n}} \text{osc}(r_t)$$

$$\leq -\frac{1}{n(\text{Leb}(r_0) + t)} \text{osc}(r_t).$$

This yields

$$\text{osc}(r_t) \leq \text{osc}(r_0) \left( \frac{\text{Leb}(r_0) + t}{\text{Leb}(r_0)} \right)^{-1/n} \to 0,$$

as $t \to \infty$, for any $r_0 \in C$. Equivalently, for some constant $c_n$ such that $\text{Leb}(c_n B) = 1$,

$$||r_t - c_n B||_2 \leq \omega_n^{1/2} ||r_t - c_n B||_\infty \to 0.$$ 

This is exactly the definition (1.25) of attractive shapes with $\psi = \mathbb{B}$. \hfill \Box

4. Applications

In this section we consider two applications of Theorems 1.4 and 1.9, the main one being a simplified model for the growth of the range of OERW (with $F(r, x, \cdot)$ the density of harmonic measure). By Dahlberg’s theorem [8, Theorem 3 and remark], for a Lipschitz domain $D \subset \mathbb{R}^n$, harmonic measure from any point $x \in D$ is mutually absolutely continuous with respect to the $(n - 1)$-dimensional Hausdorff measure on $\partial D$, hence their Radon-Nikodym derivative which is the Poisson kernel $P(D, x, \cdot)$ exists and belongs to $L^2_{\text{loc}}(\partial D)$. If the domain is more regular, so is the Poisson kernel. Per [22, page 547], if $\partial D$ belongs to Hölder space $C^{k+1, \gamma}$ for some
4.1. **Smoothed Harmonic Measure (Proof of Theorem 1.10).** Recall the construction above (1.27) of the smoothed domain \( \tilde{r} \in C^3(S^{n-1}) \) for every \( r \in L^2(S^{n-1}) \). Due to the preceding discussion on Poisson kernels, it is clear that the regularized (as in (1.27)) Poisson kernel \( F(r, x, \cdot) \) belongs to \( C(S^{n-1}) \) for any \((r, x) \in \mathcal{D}(F)\), fulfilling part of Assumption (L).

The probabilistic meaning of the definitions of \( F \) (1.27) and \( H \) (1.29) is as follows.

If the process \((R^s, x^s)\) is defined up to time \( s \) and the state at that time is given by domain with boundary \( S^s \) and particle position \( x^s \), we wait for the next jump mark that is given by an exponential with parameter \( \epsilon^{-1} \) and we call \( t > s \) its time. To choose a point at the current boundary \( S^s \), the particle follows the law of a Brownian motion in \( \mathbb{R}^n \) with starting point \( x^s \) in the smoothed domain \( \tilde{R}^s \) = \( R^s \ast g \) until its first exit. We record its exit angle \( \xi_t \in S^{n-1} \) and define the location for the center of the new bump on the original domain by \( \tilde{R}^s(\xi_t) \xi_t \). Hence the updated domain is formed by

\[
R^s_t(\theta) = R^s_s(\theta) + \frac{\epsilon}{y_{R^s_s,x^s}} g_{y(\epsilon,R^s_s,x^s)}(\langle \xi_t, \theta \rangle).
\]

Observe that the bump is added to the original domain and not the smoothed one. Next, the particle is pushed towards the origin by a strictly positive quantity, along the radius, still in the smoothed domain \( \tilde{R}^s \), namely \( x^s_t = \alpha(\tilde{R}^s_s(\xi_t), \xi_t) \xi_t \) and there it waits for the next jump mark. Continuing in this way we define the process for every \( t > 0 \). Note that we omitted the travel time of the Brownian motion inside the smoothed domain and only deal with its exit distribution, which is without loss of generality. It is also important to notice that since \( R^s_t(\theta) \geq R^s_s(\theta) \), \( t > s \) for all \( \theta \in S^{n-1} \) implies that \( \tilde{R}^s_t(\theta) \geq \tilde{R}^s_s(\theta) \), the particle \( x^s_t \) is always contained in the smoothed domain once we assume it is the case for \((\tilde{R}^s_0, x^s_0)\).

The choice of (1.27) and (1.29) in this example is motivated by basic features of ORRW and OERW in the mesoscopic scale. The ideal choice of \( H \) to be closer to these models would be \( H(r, z, \epsilon) = (r(z) - \epsilon^{1/n}) z \). Our choice (1.29) is rather general but independent of \( \epsilon \). Further, in our continuum simplified model, it is natural to replace random walk by Brownian motion.

An advantage of our method is that, instead of Brownian motion, we can also allow the particle to follow the law of an elliptic diffusion whose generator is a uniformly elliptic second-order divergence form operator \( \mathcal{L} = -\text{div} A \nabla \), while the Green’s function used in the definition (1.27) is the one for \( \mathcal{L} \). Our proof works verbatim.

To apply Theorems 1.4 and 1.9 we rely on the following lemmas and propositions. We start by proving the Lipschitz property for the maps (1.13), (1.14), (1.15) involved in the definition of the process.

\( k \in \mathbb{N}, \gamma \in (0, 1) \), then \( P(D, x, \cdot) \in C^{k,\gamma}(\partial D) \). Since our domains are star-shaped, by an abuse of terminology we call \( F(r, x, \cdot) \) the Poisson kernel of \( r \), if it is a probability density on \( S^{n-1} \) corresponding to \( P(D, x, \cdot) \) with \( r = \partial D \) up to a change of variables.

Even for smooth domains, one cannot expect their Poisson kernel to be Lipschitz in \( L^2 \)-norm with respect to boundary perturbations as \( (1.13) \) or in any other norm. Indeed, as explained in [22], one expects \( P(D, x, \cdot) \) to be one differentiability less than the domain \( D \). However, if one forms the kernel based on a regularized domain, then the Lipschitz property can be true (as shown below in Proposition 4.2). With this in mind, we consider the following model introduced previously in Theorem 1.10 in the introduction.
Lemma 4.1. For any \( r, r' \in L^2(S^{n-1}) \), we have that
\[
\|\tilde{r} - \tilde{r}'\|_{C^3(S^{n-1})} \leq C\|r - r'\|_2,
\]
for some finite constant \( C = C(g) \) that depends on the convolution kernel \( g \).

We postpone the proof of this lemma to the appendix, as well as that of the following key proposition.

Proposition 4.2. The map \( (\tilde{r}, x) \mapsto F(r, x, \cdot) \) is locally Lipschitz from its domain to \( L^2(S^{n-1}) \), when we consider in the former space the product distance given by the \( C^{2,1/2}(S^{n-1}) \)-norm for the first variable and the Euclidean norm for the second variable.

Corollary 4.3. For every \( a \in (0, 1) \), the maps \( (r, x) \mapsto F(r, x, \cdot) \) and \( (r, x) \mapsto b(r, x) \) are both (globally) Lipschitz from \( A(a) \) to \( L^2(S^{n-1}) \).

Proof. Recall that a local Lipschitz function is globally Lipschitz in any compact subset of its domain. Given Proposition 4.2, the global Lipschitz property of \( F(r, x, \cdot) \) is obtained as follows. The map \( (r, x) \mapsto F(r, x, \cdot) \) is a composition of \( (r, x) \mapsto (\tilde{r}, x) \) and \( (\tilde{r}, x) \mapsto F(r, x, \cdot) \), the former globally Lipschitz per Lemma 4.1 and the latter locally Lipschitz. Now, observe that the image of the first map is a compact subset of the domain of the second one, due to the fact that, on the one hand, \( \|\tilde{r}\|_{C^3(S^{n-1})} \leq C\|r\|_{L^2(S^{n-1})} \leq C a^{-1} \) and the Arzelà-Ascoli theorem, and on the other hand the uniform compactness in \( \mathbb{R}^n \) of the image of \( H(r, \cdot) \), both throughout \( r \in A_1(a) \).

Finally, per Proposition 1.6(a) it follows that \( (r, x) \mapsto b(r, x) \) is also (globally) Lipschitz. \( \square \)

Next, we prove the Lipschitz property for \( H \).

Proposition 4.4. For every \( a \in (0, 1) \), the map \( (r, z) \mapsto H(r, z) \) is Lipschitz from \( A_1(a) \times S^{n-1} \) to \( \mathbb{R}^n \).

Proof. For any \( r, r' \in A_1(a) \), we have by the local Lipschitz property of \( \alpha(\cdot, \cdot) \) (hence Lipschitz on compact intervals) and Lemma 4.1 that
\[
|H(r, z) - H(r', z)| = |\alpha(\tilde{r}(z), z)z - \alpha(\tilde{r}'(z), z)z| \\
\leq C(a, \alpha)\|\tilde{r}(z) - \tilde{r}'(z)\| \leq C\|r - r'\|_2.
\]

Also, for any \( z, z' \in S^{n-1} \),
\[
|H(r, z) - H(r, z')| \leq \alpha(\tilde{r}(z'), z')|z - z'| + |\alpha(\tilde{r}(z), z) - \alpha(\tilde{r}(z'), z')|.
\]

For \( r \in A_1(a) \), using again Lemma 4.1 and the Lipschitz property of \( \alpha(\cdot, \cdot) \), we deduce that
\[
|H(r, z) - H(r, z')| \leq C(a, \alpha, g)|z - z'|.
\]

To show that Assumption (E) holds for this model, by Proposition 1.6(c), it is enough to prove the following uniform Doeblin condition.

Proposition 4.5. Given a Borel set \( A \subset S^{n-1} \) of positive Lebesgue measure and \( a \in (0, 1) \) there exist \( c = c(A, a, \alpha, g) > 0 \) such for every \( (r, x) \in A \), the process \( (x_1^{1,r}_t) \) (1.11) in the frozen domain \( \tilde{r} \) verifies
\[
\mathbb{P}\left(x_1^{1,r}_T / x_0^{1,r} | x_0^{1,r} = x\right) = \int_A F(r, x, \theta)d\sigma(\theta) > c. \quad (4.1)
\]
Proof. Recall that $(x_i^1)^{i=0}_r$ lives in the smoothed domain enclosed by $\tilde{r}$, which is star-shaped. This together with the fact that the image of $H(r, \cdot)$ forms a compact, connected set disjoint from the boundary $\tilde{r}$, imply that given any Borel $A \subseteq S^{n-1}$ of positive Lebesgue measure, we have that
\[
\inf_{x \in \text{Image}(H(r, \cdot))} \left\{ \int_A F(r, x, \theta) d\theta \right\} > c(\tilde{r}, A) > 0.
\]
Observe that the LHS is a continuous function of $(\tilde{r}, x)$ in a compact set (under the norm of Proposition 4.2) throughout $(r, x) \in \mathcal{A}(a)$, hence the lower bound can be taken uniform. \square

The proof simultaneously shows that we can dispense with the stopping time (1.18), for reason explained in Remark 1.5. Now we are ready to prove the last Lipschitz condition, that of $b$.

Proposition 4.6. For every $a \in (0, 1)$, the map $r \mapsto b(r)$ is Lipschitz from $\mathcal{A}_1(a)$ to $L^2(S^{n-1})$.

Proof. For fixed $r \in C(S^{n-1})$, we can project $(x_i^1)^{i=0}_r$ into a Markov process on $S^{n-1}$, since every $x \in \text{Image } (H(r, \cdot))$ is identified with a unique $x/|x| \in S^{n-1}$. By Proposition 4.5 the projected process on $S^{n-1}$ is uniformly ergodic throughout $r \in \mathcal{A}_1(a)$, with the ergodicity coefficient depending on $a$. For ease of notation, while writing $x$ instead of $x/|x|$, we envision the Markov chain having state space $S^{n-1}$ throughout this proof.

For any $r, r' \in \mathcal{A}_1(a)$, by the characterization of total variation norm of finite signed measures (cf. [17, page 124]) and (1.20), we have that for some finite $C = C(a)$,
\[
\|b(r) - b(r')\|_2 = \left\| \int b(r, x) d\nu_r(x) - \int b(r', x) d\nu_{r'}(x) \right\|_2 \\
\leq \left\| \int b(r, x) d(\nu_r - \nu_{r'})(x) \right\|_2 + \left\| b(r, x) - b(r', x) \right\|_2 d\nu_{r'}(x) \\
\leq 2\omega_n^{-1} a^{-(n-1)} \|\sup_x F(r, x, \cdot)\|_\infty \|\nu_r - \nu_{r'}\|_{TV} + C(a)\|r - r'\|_2.
\]

We are left to bound the first term. Observe that $\tilde{r} \mapsto \|\sup_x F(r, x, \cdot)\|_\infty$ is a continuous function defined on a compact set (under the norm of Proposition 4.2) throughout $r \in \mathcal{A}_1(a)$, hence uniformly bounded. To deal with $\|\nu_r - \nu_{r'}\|_{TV}$, we rely on [32, Corollary 3.1]. Denote $K_r(x, A) = \int_A F(r, x, \theta) d\sigma(\theta)$ and $\mathcal{M}_1$ the space of signed Borel measures on $S^{n-1}$ with total variation one, and we have that (per notation in [32, (2.1)])
\[
\|K_r - K_{r'}\|_{\text{op}} := \sup_{\mu \in \mathcal{M}_1} \|\mu K_r - \mu K_{r'}\|_{TV} \\
= \frac{1}{2} \sup_{\mu \in \mathcal{M}_1} \sup_{\{g \text{ measurable : } |g| \leq 1\}} \left| \int_{S^{n-1}} \int g(\theta)(F(r, x, \theta) - F(r', x, \theta)) d\mu(x) d\sigma(\theta) \right| \\
\leq \frac{1}{2} \sup_{\mu \in \mathcal{M}_1} \left\{ \omega_n^{1/2} \int \|F(r, x, \cdot) - F(r', x, \cdot)\|_2 d|\mu|(x) \right\} \\
\leq \frac{1}{2} \omega_n^{1/2} K \|r - r'\|_2.
\]
By Corollary 3.1, for some $C = C(a)$ (depending on the uniform ergodicity coefficient) we have that
\[
\|\nu_T - \nu_T\|_{TV} \leq C\|K_r - K_{r'}\|_{op} \leq C\omega_{n/2}K\|r - r'\|_2.
\]
This completes the proof. \hfill \Box

Finally, we show that Assumption (C) is verified, cf. Remark 1.3.

**Corollary 4.7.** Assumption (C) holds for this model.

**Proof.** By (A.8) and (A.9) in the proof of Proposition 4.2, we have that \(\|f\|_{C^1(S^{n-1})} \leq C(a)\), hence \(\|T_t f - f\|_2 \leq C(a)(1 - t)^{1/2}\) for \(f = F(r,x,\cdot), T_t\) of (A.12) and any \((r,x) \in A(a)\). Further, \(t_{r,x} \geq \omega_n a^{n-1}\), hence \(\eta(\epsilon, r, x) \leq c(a)\epsilon^{1/n}\) throughout \(A(a)\) (for \(c(a) := a^{-1}\omega_n^{-1/(n-1)}\) finite). Thus, by (A.15)
\[
\left\|b\left(R_{s,t}^{\epsilon}, a_{s,t}^{\epsilon}\right) - b\left(R_{s,t}^{\epsilon}, a_{s,t}^{\epsilon}\right)\right\|_2 \leq a^{-(n-1)} \sup_{(r,x) \in A(a)} \left\|F(r,x) * g_{q(\epsilon,r,x)} - F(r,x)\right\|_2 \\
\leq \sqrt{2}a^{-(n-1)}C(a)c(a)\epsilon^{1/n}
\]
converges to zero, uniformly in \(s \geq 0\), as \(\epsilon \to 0\). \hfill \Box

Having proved all these facts, we can apply Theorem 1.4 to this model without the stopping time (1.18), proving part (a) of Theorem 1.10. The following proposition, considering special cases where we have explicit descriptions, constitutes the proof of part (b) of that theorem.

**Proposition 4.8.** (a) If the function \(\alpha(\ell, z) = \alpha(\ell)\) does not separately depend on \(z\), then the centered Euclidean ball \(B\) is an invariant solution to (1.12).

(b) If the function \(\alpha(\ell, z) = \alpha(z)\ell\) depends linearly on \(\ell\), then Assumption (I) is satisfied.

(c) If \(\alpha(\ell, z) = \gamma\ell\) for some fixed number \(\gamma \in [0,1]\), then the unique invariant measure \(\nu_{\ell}\) is explicitly given by the harmonic measure from the origin in the domain enclosed by \(\gamma\tilde{r}\), for every \(r \in C(S^{n-1})\).

(d) If \(\alpha(\ell, z) \equiv 0\), then \(B\) is the unique attractive solution of (1.12).

**Proof.** (a) First note that \(B * g = B\), so regularization by \(g\) has no effect here. By rotational symmetry, the map \(x \mapsto y_{B,x}\) is constant and \(\nu_B\) equals the harmonic measure from the origin on \(\partial B\), which is the uniform measure. Thus, in this case \(\theta \mapsto B(\theta)\) is constant and \(B\) is invariant for (1.12).

(b) (1.23) is a scaling invariance property of Brownian motion itself, while (1.24) is satisfied by our choice.

(c) By the scaling invariance of Brownian motion, the harmonic measure from the origin on \(\gamma\tilde{r}\) and on \(\tilde{r}\), viewed as functions of spherical angles, are equal. Since the transition kernel of the Brownian motion from \(\gamma\tilde{r}\) to \(\tilde{r}\) is exactly given by \(F(r,x,\cdot)\), we see that the harmonic measure from the origin is the unique (due to Proposition 4.5) invariant measure for \(x^{1/r}\).

(d) We first show that \(\tilde{r}_t := r_t * g\), the regularized solution of (1.12), has as \(B\) the unique attractive solution. Then it immediately follows that \((r_t)_{t \geq 0}\) must also, since if it were not attracted to \(B\), then neither would its regularized version. To this end, recall Proposition 3.5 which gives sufficient condition for \(B\) to be attractive. Since our \(F(r,x,\cdot)\) is built from \(\tilde{r}\), we
note that if we change the condition \(\text{[3.3]}\) to

\[
\bar{b}(r)(\arg \max_{\theta} \tilde{r}) \leq \bar{b}(r)(\arg \min_{\theta} \tilde{r}),
\]

then the same proof yields the conclusion that \(\mathcal{B}\) is attractive for \((\tilde{r}_t)_{t \geq 0}\). To verify \(\text{[4.2]}\), we note since \(H \equiv 0\) (the origin), it reduces \(\bar{b}(r) = F(r, 0, \cdot)/y_{r,0}\). We only need to show that the Poisson kernel of \(\tilde{r}\) is larger at angle \(\arg \max_{\theta} \tilde{r}\) than at \(\arg \min_{\theta} \tilde{r}\). Consider two standard Brownian motions in \(\mathbb{R}^n\), one in the domain enclosed by \(\tilde{r}\), the other in the centered Euclidean ball of radius \(\arg \min_{\theta} \tilde{r}\). Couple them to move together from the origin until the first hitting time by both of the boundary of the ball, where the second Brownian motion is stopped and the first Brownian motion can continue to move till hitting the larger domain’s boundary \(\tilde{r}\). Clearly, this coupling yields that the Poisson kernel of \(\tilde{r}\) at angle \(\arg \min_{\theta} \tilde{r}\) is at least \(1/\omega_n\). An analogous coupling, between \(\tilde{r}\) and the center Euclidean ball of radius \(\arg \max_{\theta} \tilde{r}\), yields that the Poisson kernel of \(\tilde{r}\) at angle \(\arg \max_{\theta} \tilde{r}\) is at most \(1/\omega_n\). This verifies \(\text{[4.2]}\) and finishes the proof. \(\square\)

For anisotropic \(\alpha(\ell, z)\), i.e. that do not satisfy the condition of Proposition \(\text{[4.8](a)}\), one may obtain other limiting shapes as invariant solutions to the ode \(\text{(1.12)}\), such as diamond, square etc (see Figure \(\text{[4]}\), implicitly determined as in Proposition \(\text{[3.3]}\)). We can show that in the anisotropic case, the Euclidean ball is not an invariant shape for our example, in general.

**Proposition 4.9.** If \(z \mapsto \alpha(\ell, z)\) is not identically constant, then the centered Euclidean ball \(\mathcal{B}\) is not an invariant solution to \(\text{(1.12)}\).

**Proof.** Suppose for contradiction that \(\mathcal{B}\) is invariant for \(\text{(1.12)}\) in such case. Since \(F(\mathcal{B}, x, \cdot)\) are probability densities, we have that \(y_{\mathcal{B},x} = \omega_n\) independent of \(x\). Since each \(x \in \text{Image } (H(\mathcal{B}, \cdot))\) is identified with a unique \(x/|x| \in S^{n-1}\), we can project \(\nu_{\mathcal{B}}\) into a probability measure \(\tilde{\nu}_{\mathcal{B}}\) on \(S^{n-1}\), uniquely invariant for the transition kernel \(\tilde{F}(\mathcal{B}, z, \theta) := F(\mathcal{B}, \alpha(\mathcal{B}(z), z)z, \theta)\), where \(z, \theta \in S^{n-1}\).

Since we assumed that \(z \mapsto \alpha(\ell, z)\) is continuous but not rotationally symmetric, clearly \(\tilde{\nu}_{\mathcal{B}}\) is not the uniform measure on \(S^{n-1}\). But per \(\theta \in S^{n-1}\) we have that

\[
\tilde{b}(\mathcal{B})(\theta) = \omega_n^{-1} \int_{S^{n-1}} \tilde{F}(\mathcal{B}, z, \theta) d\tilde{\nu}_{\mathcal{B}}(z) = \omega_n^{-1} \tilde{\nu}_{\mathcal{B}}(\theta),
\]

the latter equality due to invariant measure property, we reach contradiction with Proposition \(\text{[3.3]}\). \(\square\)

### 4.2. Distance to particle.

We consider another natural hitting rule \(F\) that chooses a boundary point with probability “proportional to a function of the distance to the particle”. As in Subsection \(\text{[4.1]}\), we consider applying a smoothing procedure to the domain to gain regularity. Fixing some \(\eta > 0\), take \(g = g_\eta \in C^1(S^{n-1})\) of \(\text{(A.10)}\), and still denoting \(\tilde{r} := r \ast g\), let

\[
F(r, x, \theta) = \frac{\varphi(|\tilde{r}(\theta) \theta - x|)}{\int_{S^{n-1}} \varphi(|\tilde{r}(z) z - x|) d\sigma(z)}, \quad \theta \in S^{n-1},
\]

\(\text{(4.3)}\)
where \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) is \( C^1 \) and bounded away from zero. Thus, \( F(r, x, \cdot) \in C^1(S^{n-1}) \) for any \((r, x) \in \mathcal{D}(F)\), fulfilling part of Assumption (L). We take as the transportation rule

\[
H(r) = \int_{S^{n-1}} r(z) z d\sigma(z)
\]

that sends the particle to certain statistical center of the domain. As \( H(r) \) depends on the domain only, the invariant measure \( \nu_r \) is the Dirac mass at \( H(r) \), and Assumption (E) needs no verification.

**Remark 4.10.** Observe that it is possible to have yet another example by combining the \( F \) of this example with the \( H \) of Subsection 4.1. The proofs are similar, so we do not include them.

**Proposition 4.11.** For every \( a \in (0, 1) \), the map \((r, x) \mapsto F(r, x, \cdot)\) is Lipschitz from \( \mathcal{A}(a) \) to \( L^2(S^{n-1}) \).

**Proof.** We can prove Lipschitz property of the numerator and denominator of (4.3) separately, since the latter is bounded below. Since \( \varphi \) is \( C^1 \) it is Lipschitz on compact intervals. For any \((r, x), (r', x') \in \mathcal{A}(a)\), we have for the numerator

\[
\| \varphi(|\bar{r}(\theta) - x|) - \varphi(|\bar{r}'(\theta) - x'|) \|_2 \\
\leq C(\varphi) \| |\bar{r}(\theta) - x| - |\bar{r}'(\theta) - x'| \|_2 \\
\leq C \| (\bar{r}(\theta) - \bar{r}'(\theta)) - (x - x') \|_2 \\
\leq C \left( \| \bar{r} - \bar{r}' \|_2 + |x - x'| \right) \leq C \left( \| r - r' \|_2 + |x - x'| \right),
\]

where we used in the third line the elementary inequality \( \| a - b \| \leq |a - b| \) for \( a, b \in \mathbb{R} \). The \( L^2 \)-norm above is taken with respect to \( \theta \). The denominator is Lipschitz upon applying first Cauchy-Schwarz inequality, and then the same argument. This completes the proof. \( \square \)

Another application of Cauchy-Schwarz inequality yields that the map \( r \mapsto H(r) \) is Lipschitz from \( \mathcal{A}_1(a) \) to \( \mathbb{R}^n \), for every \( a \in (0, 1) \). We are left to prove the map \( r \mapsto \bar{b}(r) \) is Lipschitz from \( \mathcal{A}_1(a) \) to \( L^2(S^{n-1}) \). Note that from the Lipschitz property of \( F \), we can deduce the same for \( b \) by Proposition 1.6(a). Since \( \nu_r \) concentrates on a singleton, we further deduce for any \( r, r' \in \mathcal{A}_1(a) \),

\[
\| \bar{b}(r) - \bar{b}(r') \|_2 = \| b(r, H(r)) - b(r', H(r')) \|_2 \\
\leq C(a) \left( \| r - r' \|_2 + |H(r) - H(r')| \right) \leq C \| r - r' \|_2.
\]

Finally, we can verify Assumption (C) similarly to Corollary 4.7 as \( \| F(r, x, \cdot) \|_{C^1(S^{n-1})} \) is uniformly bounded for \((r, x) \in \mathcal{A}(a)\). Thus Theorem 1.4 applies to this model. If one further assumes that \( \varphi \) is a homogeneous function in the sense that \( \varphi(\lambda t) = \lambda^s \varphi(t) \) for some \( s \in \mathbb{R} \) and any scalar \( \lambda > 0 \), then Assumption (I) is satisfied and Theorem 1.9 also applies.

It is clear from (4.3) that the centered Euclidean ball \( \mathbb{B} \) is an invariant shape.
Appendix A.

Proof of Proposition 1.6. (a) Since \( b(r, x) \) is a ratio of \( F(r, x, \cdot) \) and the scalar \( y_{r,x} \), with \( F \) Lipschitz and \( y_{r,x} \geq \omega_n a^{n-1} \), it suffices to show that in addition \( (r, x) \mapsto y_{r,x} \) is Lipschitz. To this end, we note that for any \( (r, x), (r', x') \in \mathcal{A}(a) \),

\[
|y_{r,x} - y_{r',x'}| \leq \omega_n \int_{S^{n-1}} |r^{n-1}(\theta)F(r, x, \theta) - (r')^{n-1}(\theta)F(r', x', \theta)| \, d\sigma(\theta)
\]

\[
\leq \omega_n \left[ \int_{S^{n-1}} |r^{n-1} - (r')^{n-1}|(\theta)F(r, x, \theta)d\sigma(\theta) + \int_{S^{n-1}} (r')^{n-1}(\theta)|F(r, x, \theta) - F(r', x', \theta)|d\sigma(\theta) \right]
\]

\[
\leq C(a)\|r - r'\|_2\|F(r, x, \cdot)\|_2 + \omega_n\|r'\|_2^{n-1}\|F(r, x, \cdot) - F(r', x', \cdot)\|_2,
\]

by Cauchy-Schwarz inequality and the Lipschitz property of \( t^{n-1} \) on compact intervals. Thus our claim follows from (1.13) and related norm-boundedness. Similarly, for any \( (r, x), (r', x') \in \mathcal{A}(a) \), by (1.13), (1.14),

\[
|h(r, x) - h(r', x')| - |x - x'| \leq \int_{S^{n-1}} |H(r, \theta)F(r, x, \theta) - H(r', \theta)F(r', x', \theta)| \, d\sigma(\theta)
\]

\[
\leq \int_{S^{n-1}} |H(r, \theta)(F(r, x, \theta) - F(r', x', \theta))|d\sigma(\theta) + \int_{S^{n-1}} |H(r', \theta)F(r', x', \theta)d\sigma(\theta)
\]

\[
\leq \|H(\cdot, \cdot)\|_2\|F(r, x, \cdot) - F(r', x', \cdot)\|_2 + K\|r - r'\|_2 \leq C(a)(\|r - r'\|_2 + |x - x'|)
\]

as needed.

(b) Since per (1.15) \( r \mapsto \overline{b}(r) \) is Lipschitz in \( \mathcal{A}_1(a) \) for every \( a \in (0, 1) \), there exists a unique \( L^2 \)-solution to (1.12) locally in time, defined up to the first exit time by the solution of \( \mathcal{A}_1(a) \) (Theorem 7.3). By the continuity of \( F \) in \( \theta \) variable per Assumption (L), it turns out that the solution to (1.12) is continuous in \( \theta \) for each \( t \) while defined. Hence, Proposition 1.7 holds there and we can bound the growth of the \( L^2 \)-norm of the solution \( r_t \) by H"older’s inequality

\[
\|r_t\|_2 \leq \omega_n^{n/2}\|r_t\|_n \leq \omega_n^{n/2}n^{1/2}(\text{Leb}(r_0) + t)^{n/2}.
\]

Since \( r_0 \in C(S^{n-1}) \), given \( T > 0 \), we can find \( a_1 = a_1(r_0, T) \in (0, 1) \) such that the solution stays in \( \mathcal{A}_1(a_1) \) up to time \( T \) and the result follows.

(c) The minorisation (1.22) implies by standard theory of general state space Markov chains (see [37, Theorem 8]), that for any \( r \in C(S^{n-1}) \) the embedded chain \( \{x_t^{1,r}\} \) has a unique invariant measure \( \nu_r \), with the uniform on \( \mathcal{A}(a) \) convergence

\[
\sup_{(r,x) \in \mathcal{A}(a)} \{ ||P^n_r(x, \cdot) - \nu_r(\cdot)||_{\text{TV}} \} \leq (1 - \delta)^{[n/n_0]}.
\]

The proof is by coupling, which extends to the process \( (x_t^{1,r})_{t \geq 0} \) with \( x_0^{1,r} = x \) and its stationary version \( (x_t^{1,r})_{t \geq 0} \) (i.e. starting at distribution \( \nu_r \) and using the same jump times \( \{T_i\} \) for both processes). It follows that the processes coalesce at the coupling time \( T_x \) with

\[
||P_x(x_t^{1,r} \in \cdot) - \nu_r(\cdot)||_{\text{TV}} \leq P_x(x_t^{1,r} \neq x_t^{1,r}) = P(T_x > t) \leq e^{-ct},
\]
for some positive constant \(c = c(\delta, n_0) = c(a)\), any \(t \geq 0\) and all \((r, x) \in \mathcal{A}(a)\). By the triangle inequality, employing this coupling for proving (1.16), we separately bound

\[
\sup_{t_0 \geq 0} \mathbb{E} \left[ \frac{1}{t} \int_{t_0}^{t_0+t} [b(r, x^1_{s,r}) - b(r, x^0_{s,r})] ds \right]^2 \tag{A.1}
\]

and

\[
\sup_{t_0 \geq 0} \mathbb{E} \left[ \frac{1}{t} \int_{t_0}^{t_0+t} [b(r, x^1_{s,r}) - b(r)] ds \right]^2. \tag{A.2}
\]

There is no contribution to (A.1) from \(s \geq T_x\) and a-priori \(\|b(r, x)\|_2 \leq C(a) < \infty\) for all \((r, x) \in \mathcal{A}(a)\). Hence (A.1) is at most \(4C(a)^2 \mathbb{E} T_x/t \leq 4C(a)^2/(ct)\). By stationarity the expectation in (A.2) is independent of \(t_0\) and utilizing the Markov property, it equals

\[
\frac{2}{t} \int d\nu_r(x) \int_0^t \left(1 - \frac{u}{t}\right) \Delta_{r,x}(u) du, \tag{A.3}
\]

where by Fubini

\[
\Delta_{r,x}(u) := \mathbb{E}_x \left[ \int_{S^{n-1}} [b(r, x^1_{u,r})(\theta) - b(r)(\theta)] b(r, x)(\theta) d\sigma(\theta) \right].
\]

Using the preceding coupling per value of \(x\) in (A.3), we deduce that

\[
|\Delta_{r,x}(u)| \leq \Gamma_{r,x} \mathbb{P}(T_x > u),
\]

where by Cauchy-Schwarz

\[
\Gamma_{r,x} := \sup_{y, y' \in K_r} \int_{S^{n-1}} |b(r, y)(\theta) - b(r, y')(\theta)| b(r, x)(\theta) d\sigma(\theta)
\leq 2 \sup_{x' \in K_r} \|b(r, x')\|^2_2 \leq 2C(a)^2.
\]

Plugging into (A.3) this uniform bound on \(\Gamma_{r,x}\) and the uniform tail bound on \(T_x\), we deduce that the term (A.2) is at most \(4C(a)^2/(ct)\), thereby completing the proof. \(\square\)

**Proof of Proposition 1.7.** For \(C(S^{n-1})\)-solutions \((r_t)_{t \geq 0}\), (1.12) is valid in pointwise sense and we can compute

\[
\frac{d}{dt} \{\text{Leb}(r_t)\} = \frac{d}{dt} \left\{ \int_{S^{n-1}} n^{-1} r_t(\theta)^n d\sigma(\theta) \right\} = \int_{S^{n-1}} r_t(\theta)^{n-1} \frac{d}{dt} \{r_t(\theta)\} d\sigma(\theta)
\leq \int_{S^{n-1}} r_t(\theta)^{n-1} \int_{\mathbb{R}^n} b(r_t, x)(\theta)d\nu_{r_t}(x) d\sigma(\theta)
\leq \int_{\mathbb{R}^n} \left( \int_{S^{n-1}} r_t^{-1}(\theta)b(r_t, x)(\theta) d\sigma(\theta) \right) d\nu_{r_t}(x)
= \int_{\mathbb{R}^n} d\nu_{r_t}(x) = 1,
\]

yielding \(\text{Leb}(r_t) = \text{Leb}(r_0) + t\), for any \(t \geq 0\).
We can do a similar computation for the random dynamics, in particular verifying (1.3). Indeed, for any \((r, x) \in \mathcal{D}(F),\)

\[
\mathbb{E} \left[ \text{Lev}(r + \epsilon y_r^{-1}g_\eta(\langle \xi, \cdot \rangle)) - \text{Leb}(r) \right]
\]

\[
= \int_{S^{n-1}} n^{-1} \int_{S^{n-1}} [r(\theta) + \epsilon y_r^{-1}g_\eta(\langle z, \theta \rangle)]^n F(r, x, z)d\sigma(z)d\sigma(\theta) - n^{-1} \int_{S^{n-1}} r(\theta)^n d\sigma(\theta)
\]

\[
= \epsilon \int_{S^{n-1}} \int_{S^{n-1}} r(\theta)^{-n} y_r^{-1}g_\eta(\langle z, \theta \rangle) F(r, x, z)d\sigma(z)d\sigma(\theta) + o(\epsilon)
\]

\[
= \epsilon \int_{S^{n-1}} r(\theta)^{-n} [b^\prime(r, x)\theta - b(r, x)\theta] d\sigma(\theta) + \epsilon \int_{S^{n-1}} r(\theta)^{-n} b(r, x) d\sigma(\theta) + o(\epsilon).
\]

The second term gives exactly \(\epsilon\). Upon applying Cauchy-Schwarz inequality to the first term and using the \(L^2\)-approximation property (A.15) of the spherical approximate identity as \(\epsilon \to 0\), we see that the whole expression is \(\epsilon + o(\epsilon)\).

**Proof of Lemma 4.1.** Recall the definition (1.1) of spherical convolution. For any integer \(0 \leq k \leq 3\) and any \(z \in S^{n-1}\), we have that

\[
|d^k(\tilde{r} - \tilde{r}')(z)| = \left| \frac{1}{\omega_n} \int_{S^{n-1}} (r - r')\theta d^k g(\langle z, \theta \rangle) d\sigma(\theta) \right|
\]

\[
\leq \frac{1}{\omega_n} \int_{S^{n-1}} \left| (r - r')\theta d^k g(\langle z, \theta \rangle) \right| d\sigma(\theta)
\]

\[
\leq \frac{1}{\omega_n} \|r - r'\|_2 \sup_{z \in S^{n-1}} \|d^k g(\langle z, \cdot \rangle)\|_2
\]

where \(d^k\) is the \(k\)-th derivative with respect to \(z\) variable in \(S^{n-1}\). Since \(S^{n-1}\) is compact hence the supremum in the last line is finite and depends only on \(g\), the claimed bound on \(\|\tilde{r} - \tilde{r}'\|_{C^3(S^{n-1})}\) is obtained.

**Proof of Proposition 4.2.** We first show the local Lipschitz property of \(F(r, x, \cdot)\) in \(r\) variable. Let us fix \(x_0 \in \mathbb{R}^n\), and consider two open, star-shaped domains \(D, D'\) whose \(C^3\)-boundaries are \(\tilde{r} = \partial D, \tilde{r}' = \partial D'\) respectively, with \(x_0 \in D \cap D'\). We view \(D'\) as a local perturbation of \(D\) in the metric we consider. Clearly, there exists some \(\delta > 0\) such that \(B(x_0, 2\delta) \subseteq D \cap D'\). Further, it is not hard to find some \(C^3\)-diffeomorphism \(\phi : \mathbb{R}^n \to \mathbb{R}^n\) that maps \(D\) to \(D'\), taking \(\partial D\) to \(\partial D'\), is identity for \(x \in B(x_0, \delta)\), and for some finite constant \(C_1 = C_1(\delta, \|\tilde{r}\|_{C^2(S^{n-1})})\) satisfies

\[
\|\phi - \text{Id}\|_{C^1(B)} \leq C_1 \|\tilde{r} - \tilde{r}'\|_{C^{1,2}(S^{n-1})}.
\]

(A.4)

Here \(\text{Id}(x) = x\) is the identity map, whose differential is \(I_n\), the \(n\)-by-\(n\) identity matrix.

Recall (1.27) that \(F(r, x_0, \theta) = \frac{\partial}{\partial y_r} G_{\mathcal{F}}(x_0, y)|_{y = \phi(\theta)\theta}\) and \(F'(r, x_0, \theta) = \frac{\partial}{\partial \rho} G_{\mathcal{F}}(x_0, y)|_{y = \phi(\theta)\theta}\). For simplicity, we also write \(\Gamma(x_0, x) = G_{\mathcal{F}}(x_0, \phi(x))\) for \(x \in \mathcal{D}\). We have that

\[
\Delta \Gamma(x_0, x) = \text{div}(D\phi(x)\nabla G_{\mathcal{F}}(x_0, \phi(x)))
\]
where $D$ is the differential operator. Subtracting $\Delta G_{\tilde{r}}$ from $\Delta \Gamma$ we have that
\[
\Delta(\Gamma - G_{\tilde{r}})(x_0, x) = \text{div}((D \phi - I_n)(x)\nabla G_{\tilde{r}}(x_0, \phi(x))) + (\Delta G_{\tilde{r}}(x_0, \phi(x)) - \Delta G_{\tilde{r}}(x_0, x))
\]
\[
= \text{div}((D \phi - I_n)(x)\nabla G_{\tilde{r}}(x_0, \phi(x))) ,
\] (A.5)
where in the second line we claimed that for any $x \in \overline{D}$,
\[
\Delta G_{\tilde{r}}(x_0, \phi(x)) - \Delta G_{\tilde{r}}(x_0, x) = 0. \quad (A.6)
\]
Indeed, since Green’s function is harmonic away from its pole, both quantities in (A.6) are zero if $x \neq x_0$. When $x = x_0$, $\Delta G_{\tilde{r}}(x_0, \phi(x)) = \delta_{x_0}(\phi(x))$ and $\Delta G_{\tilde{r}}(x_0, x) = \delta_{x_0}(x)$. But for any test function $f$, we have $(f(x), \delta_{x_0}(\phi(x))) = f(\phi^{-1}(x_0))$ and $(f(x), \delta_{x_0}(x)) = f(x_0)$, with $\phi^{-1}(x_0) = x_0$ by definition of the map $\phi$. Thus, when $x = x_0$ (A.6) also holds.

Observe now that since $\phi$ is the identity map in $B(x_0, \delta)$, $D \phi = I_n$, there is no singularity on the RHS of (A.5). With the boundary condition $(\Gamma - G_{\tilde{r}})(x_0, x) = 0$ whenever $x \in \partial D$, we have by the global Schauder estimate [19, Theorem 5.26] applied to the Poisson equation (A.5), combined with the maximum principle [19, Proposition 2.15] for the same equation, that there exists some finite $C_2 = C_2(\delta, ||\tilde{r}||_{C^{2,1/2}(S^{n-1})})$ such that
\[
||\Gamma - G_{\tilde{r}}||_{C^{2,1/2}(\overline{D})} \leq C_2 ||\text{div}((D \phi - I_n)\nabla G_{\tilde{r}}(x_0, \phi(\cdot)))||_{C^{0,1/2}(\overline{D})}.
\]
Utilizing (A.4), the above is further controlled by
\[
||\Gamma - G_{\tilde{r}}||_{C^{2,1/2}(\overline{D})} \leq C_3 ||\phi - \text{Id}||_{C^{2,1/2}(\overline{D})} \cdot ||G_{\tilde{r}}(x_0, \phi(\cdot))||_{C^{2,1/2}(\overline{D} \setminus B(x_0, \delta))} 
\]
\[
\leq C_4 ||\tilde{r} - \tilde{r}'||_{C^{2,1/2}(S^{n-1})} , \quad (A.7)
\]
where $C_4$ depends on $\delta$, $||\tilde{r}||_{C^{2,1/2}(S^{n-1})}$ and the term involving Green’s function. But the latter has sufficient regularity away from its pole up to the $C^3$-boundary, hence $C_4$ is finite. We conclude by (1.27) and (A.7) that
\[
||F(r, x_0, \cdot) - F(r', x_0, \cdot)||_{C^{1,1/2}(S^{n-1})} \leq C||\tilde{r} - \tilde{r}'||_{C^{2,1/2}(S^{n-1})} . \quad (A.8)
\]
We next turn to the local Lipschitz property of $F(r, x, \cdot)$ in $x$ variable.

Fix an open, star-shaped domain $D$ with $\tilde{r} = \partial D$ and consider two points $x, x' \in D$, such that for some $\delta > 0$ we have $\overline{B}(x, 2\delta) \cup \overline{B}(x', 2\delta) \subseteq D$, and the line segment connecting $x, x'$ is entirely contained in $D$. We view $x'$ as a local perturbation of $x$. Since the Green’s function $G(x, y) := G_{\tilde{r}}(x, y)$ has sufficient regularity away from its pole up to the $C^3$-boundary, applying the Mean Value Theorem to $G(\cdot, y)$, $\nabla_y G(\cdot, y)$ and $\nabla^2_y G(\cdot, y)$, we get for some finite constant $C$,
\[
||G(x, \cdot) - G(x', \cdot)||_{C^2(\overline{D} \setminus (B(x, \delta) \cup B(x', \delta)))} \leq C|x - x'|.
\]
This again implies by (1.27)
\[
||F(r, x, \cdot) - F(r, x', \cdot)||_{C^1(S^{n-1})} \leq C|x - x'| . \quad (A.9)
\]
Since we only require $L^2(S^{n-1})$-norm on $F$, (A.8) and (A.9) are sufficient for our purposes. $\square$
Lemma A.1. Fix \( n \geq 2 \) and a continuous probability density function \( \phi(s) \) on \([-1, 1] \), attaining its maximal value at \( s = 1 \), with \( \phi(-1) = 0 \). Then, for some \( c_\eta > 0 \) which are uniformly bounded over \( \eta \in (0, 1) \),

\[
g_\eta(t) := \frac{c_\eta}{\omega_{n-1}} \eta^{-(n-1)} \phi \left( 1 - \frac{1 - t}{\eta^2} \right), \quad t \in [-1, 1],
\]

form a spherical approximate identity as in Definition 1.2.

Proof. Note that \( g_\eta(t) \) of (A.10) is continuous, non-negative and supported on \([1 - 2\eta^2, 1] \). That is, \( g_\eta(z, \theta) \) is supported on the spherical cap \(|\theta - z| \leq 2\eta, \theta \in S^{n-1} \). By a change of variable (see [9, (2.1.8)]),

\[
(1 \ast g_\eta)(z) = \frac{\omega_{n-1}}{\omega_n} \int_{1-2\eta^2}^1 g_\eta(t)(1 - t^2) \frac{n-3}{2} dt = 1,
\]

for all \( z \in S^{n-1} \), provided

\[
c_\eta^{-1} := \frac{2^{n-2}}{\omega_n} \int_0^1 \phi(1 - 2s)s^{\frac{n-3}{2}}(1 - \eta^2 s)^{\frac{n-3}{2}} ds.
\]

Since \( \eta \mapsto c_\eta \) monotone and max(\( c_1, c_0 \)) < \( \infty \) for any density \( \phi(\cdot) \) and \( n \geq 2 \), the uniform boundedness of \( c_\eta \) follows. Next, recall [9, (2.1.8)], that for every \( z \in S^{n-1} \) and \( f \in L^2(S^{n-1}) \),

\[
(f \ast g_\eta)(z) = \frac{\omega_{n-1}}{\omega_n} \int_{1-2\eta^2}^1 g_\eta(t)T_t f(z)(1 - t^2) \frac{n-3}{2} dt,
\]

where \( \{T_t\}_{t \in [-1, 1]} \) is a family of translation operators [9, (2.1.6)] defined by

\[
T_t f(x) := \frac{(1 - t^2)^{\frac{1-n}{2}}}{\omega_{n-1}} \int_{(x,y)=t} f(y) d\ell_{x,t}(y).
\]

Here \( d\ell_{x,t} \) denotes Lebesgue measure on \( \{y \in S^{n-1} : (x, y) = t\} \). These operators satisfy

\[
\forall t, \quad \|T_t f\|_2 \leq \|f\|_2, \quad \lim_{t \to 1^{-}}\|T_t f - f\|_2 = 0,
\]

see [9, Lemma 2.1.7]. Hence, by (A.11)-(A.13) and the convexity of the norm, for all \( \eta \in (0, 1) \),

\[
\|f \ast g_\eta\|_2 \leq \frac{\omega_{n-1}}{\omega_n} \int_{1-2\eta^2}^1 g_\eta(t)\|T_t f\|_2 (1 - t^2) \frac{n-3}{2} dt \leq \|f\|_2 \tag{A.14}
\]

\[
\|f \ast g_\eta - f\|_2 \leq \frac{\omega_{n-1}}{\omega_n} \int_{1-2\eta^2}^1 g_\eta(t)\|T_t f - f\|_2 (1 - t^2) \frac{n-3}{2} dt \leq \sup_{t \in [1-2\eta^2, 1]} \{\|T_t f - f\|_2}\tag{A.15}
\]

with the RHS of (A.15) converging to zero as \( \eta \to 0 \) (see (A.13)).

**Acknowledgments.** We thank Julián Fernández Bonder and Luis Silvestre for useful conversations and Martín Arjovsky for pointing out the plausibility of Lemma 2.1 and its proof.
References

[1] H. Amann. *Ordinary differential equations*, volume 13 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1990. An introduction to nonlinear analysis, Translated from the German by Gerhard Metzen.

[2] I. Benjamini, H. Duminil-Copin, G. Kozma, and C. Lucas. Internal diffusion-limited aggregation with uniform starting points. arXiv:1707.03241.

[3] I. Benjamini and D. B. Wilson. Excited random walk. *Electron. Comm. Probab.*, 8:86–92, 2003.

[4] N. N. Bogoliubov and Y. A. Mitropolsky. *Asymptotic methods in the theory of nonlinear oscillations*. Translated from the second revised Russian edition. International Monographs on Advanced Mathematics and Physics. Hindustan Publishing Corp., Delhi, Gordon and Breach Science Publishers, New York, 1961.

[5] L. Carleson and N. Makarov. Aggregation in the plane and Loewner’s equation. *Comm. Math. Phys.*, 216(3):583–607, 2001.

[6] L. Carleson and N. Makarov. Laplacian path models. *J. Anal. Math.*, 87:103–150, 2002. Dedicated to the memory of Thomas H. Wolff.

[7] S. Cerrai. A Khasminskii type averaging principle for stochastic reaction-diffusion equations. *Ann. Appl. Probab.*, 19(3):899–948, 2009.

[8] B. E. J. Dahlberg. Estimates of harmonic measure. *Arch. Rational Mech. Anal.*, 65(3):275–288, 1977.

[9] F. Dai and Y. Xu. *Approximation theory and harmonic analysis on spheres and balls*. Springer Monographs in Mathematics. Springer, New York, 2013.

[10] R. W. R. Darling and J. R. Norris. Differential equation approximations for Markov chains. *Probab. Surv.*, 5:37–79, 2008.

[11] B. Davis. Reinforced random walk. *Probab. Theory Related Fields*, 84(2):203–229, 1990.

[12] A. De Masi and E. Presutti. *Mathematical methods for hydrodynamic limits*, volume 1501 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1991.

[13] P. Diaconis and W. Fulton. A growth model, a game, an algebra, Lagrange inversion, and characteristic classes. *Rend. Sem. Mat. Univ. Politec. Torino*, 49(1):95–119 (1993), 1991. Commutative algebra and algebraic geometry, II (Italian) (Turin, 1990).

[14] M. Eden. A two-dimensional growth process. In *Proc. 4th Berkeley Symp. Math. Statist. and Prob.*, Vol. IV, pages 223–239. Univ. California Press, Berkeley, Calif., 1961.

[15] M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, third edition, 2012. Translated from the 1979 Russian original by Joseph Szücs.

[16] J. Gravner and J. Quastel. Internal DLA and the Stefan problem. *Ann. Probab.*, 28(4):1528–1562, 2000.

[17] P. R. Halmos. *Measure theory*, volume 18 of *Graduate studies in mathematics*. Springer-Verlag, 1974.

[18] J. M. Hammersley and D. J. A. Welsh. First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. In *Proc. Internat. Res. Semin., Statist. Lab.*, *Univ. California, Berkeley, Calif.*, pages 61–110. Springer-Verlag, New York,
1965.

[19] Q. Han and F. Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, second edition, 2011.

[20] R. Z. Has’ minskii. On the principle of averaging the Itô’s stochastic differential equations. *Kybernetika (Prague)*, 4:260–279, 1968.

[21] M. B. Hastings and L. S. Levitov. Laplacian growth as one-dimensional turbulence. *Physica D: Nonlinear Phenomena*, 116(1-2):244–252, 1998.

[22] D. Jerison. Regularity of the Poisson kernel and free boundary problems. *Colloq. Math.*, 60/61(2):547–568, 1990.

[23] F. Johansson Viklund, A. Sola, and A. Turner. Small-particle limits in a regularized Laplacian random growth model. *Comm. Math. Phys.*, 334(1):331–366, 2015.

[24] Y. Kifer. Some recent advances in averaging. In *Modern dynamical systems and applications*, pages 385–403. Cambridge Univ. Press, Cambridge, 2004.

[25] C. Kipnis and C. Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

[26] E. Kosygina and M. P. W. Zerner. Excited random walks: results, methods, open problems. *Bull. Inst. Math. Acad. Sin. (N.S.)*, 8(1):105–157, 2013.

[27] G. Kozma. Problem session. *Oberwolfach Report*, 27:1552, 2007.

[28] G. Kozma. Reinforced random walk. In *European Congress of Mathematics*, pages 429–443. Eur. Math. Soc., Zürich, 2013.

[29] G. F. Lawler, M. Bramson, and D. Griffeath. Internal diffusion limited aggregation. *Ann. Probab.*, 20(4):2117–2140, 1992.

[30] L. Levine and Y. Peres. Scaling limits for internal aggregation models with multiple sources. *Journal d’Analyse Mathématique*, 111(1):151–219, 2010.

[31] L. Levine and Y. Peres. Laplacian growth, sandpiles, and scaling limits. *Bull. Amer. Math. Soc. (N.S.)*, 54(3):355–382, 2017.

[32] A. Yu. Mitrophanov. Sensitivity and convergence of uniformly ergodic Markov chains. *J. Appl. Probab.*, 42(4):1003–1014, 2005.

[33] L. Niemeyer, L. Pietronero, and H. J. Wiesmann. Fractal dimension of dielectric breakdown. *Phys. Rev. Lett.*, 52(12):1033–1036, 1984.

[34] J. Norris and A. Turner. Hastings-Levitov aggregation in the small-particle limit. *Comm. Math. Phys.*, 316(3):809–841, 2012.

[35] G. C. Papanicolaou, D. Stroock, and S. R. S. Varadhan. Martingale approach to some limit theorems. pages ii+120 pp. Duke Univ. Math. Ser., Vol. III, 1977.

[36] R. Peyre. Comparison between $W_2$ distance and $\dot{H}^{-1}$ norm and localisation of Wasserstein distance. *arXiv:1104.4631v2*.

[37] G. O. Roberts and J. S. Rosenthal. General state space Markov chains and MCMC algorithms. *Probab. Surv.*, 1:20–71, 2004.

[38] S. R. S. Varadhan. Entropy, large deviations, and scaling limits. *Comm. Pure Appl. Math.*, 66(12):1914–1932, 2013.

[39] A. Yu. Veretennikov. On an averaging principle for systems of stochastic differential equations. *Mat. Sb.*, 181(2):256–268, 1990.
[40] T. A. Witten and L. M. Sander. Diffusion-limited aggregation. *Phys. Rev. B* (3), 27(9):5686–5697, 1983.

Departments of Statistics and of Mathematics, Stanford University, California, USA

E-mail address: adembo@stanford.edu

Universidad de Buenos Aires, Buenos Aires, Argentina and NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai

E-mail address: pgroisman@dm.uba.ar

Courant Institute of Mathematical Sciences, New York, USA

E-mail address: rh138@nyu.edu

Courant Institute of Mathematical Sciences, New York and NYU-ECNU Institute of Mathematical Sciences at NYU Shanghai

E-mail address: vs1138@nyu.edu