New Mass and Mass-Mixing Angle Relations for Pseudoscalar Mesons

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Abstract

We study the origins of the inaccuracies of Schwinger’s nonet mass, and the Sakurai mass-mixing angle, formulae for the pseudoscalar meson nonet, and suggest new versions of them, modified by the inclusion of the pseudoscalar decay constants. We use these new formulae to determine the pseudoscalar decay constants and mixing angle. The results obtained, $f_8/f_π = 1.185 \pm 0.040$, $f_9/f_π = 1.095 \pm 0.020$, $f_η/f_π = 1.085 \pm 0.025$, $f_{η'}/f_π = 1.195 \pm 0.035$, $\theta = (-21.4 \pm 1.0)^ο$, are in excellent agreement with experiment.

Key words: Schwinger’s formula, Gell-Mann–Okubo, chiral Lagrangian, pseudoscalar mesons

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1 Introduction

Schwinger’s original nonet mass formula [1] (here the symbol for the meson stands either for its mass or mass squared),

\[(4K - 3\eta - \pi)(3\eta' + \pi - 4K) = 8(K - \pi)^2,\]

and the Sakurai mass-mixing angle formula [2],

\[\tan^2 \theta = \frac{4K - 3\eta - \pi}{3\eta' + \pi - 4K},\]

both relate the masses of the isovector (\(\pi\)), isodoublet (\(K\)) and isoscalar mostly octet (\(\eta\)) and mostly singlet (\(\eta'\)) states of a meson nonet, and the nonet mixing angle (\(\theta\)). We alert the reader that, although we use notation suggestive of masses below, each formula is to be reprised in terms of mass or mass squared values in this introductory section.

The relations (1) and (2) are usually derived in the following way: For a meson nonet, the isoscalar octet-singlet mass matrix,

\[M = \begin{pmatrix} M_{88} & M_{89} \\ M_{89} & M_{99} \end{pmatrix},\]

is diagonalized by the masses of the physical \(\eta\) and \(\eta'\) states:

\[M = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta' \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix},\]

\[= \begin{pmatrix} \cos^2 \theta \eta + \sin^2 \theta \eta' & \sin \theta \cos \theta (\eta' - \eta) \\ \sin \theta \cos \theta (\eta' - \eta) & \sin^2 \theta \eta + \cos^2 \theta \eta' \end{pmatrix},\]

where \(\theta\) is the nonet mixing angle, which is determined by comparing the corresponding quadrants of the matrices (3) and (4), by any of the three following relations:

\[\tan^2 \theta = \frac{M_{88} - \eta}{\eta' - M_{88}},\]

\[\tan^2 \theta = \frac{\eta' - M_{99}}{M_{99} - \eta},\]

\[\sin 2\theta = \frac{2M_{89}}{\eta' - \eta}.\]

It is easily seen that Eqs. (5) and (6) are identical, since, due to the trace invariance of \(M\),

\[\eta + \eta' = M_{88} + M_{99},\]

and therefore, \(M_{88} - \eta = \eta' - M_{99}\), and \(\eta' - M_{88} = M_{99} - \eta\). Eliminating \(\theta\) from (5),(7), or (6),(7), with the help of \(\sin 2\theta = 2 \tan \theta/(1 + \tan^2 \theta)\), leads, respectively, to

\[(\eta - M_{88})(M_{88} - \eta') = M_{89}^2,\]

\[M_{89}^2 = \frac{\eta' - M_{99}}{M_{99} - \eta}.\]
\[(M_{99} - \eta)(\eta' - M_{99}) = M_{89}^2, \quad (10)\]

which again are identical, through (8).

We note that, under the quark model inspired conditions

\[M_{88} = \frac{4K - \pi}{3}, \quad M_{99} = \frac{2K + \pi}{3}, \quad M_{89} = -\frac{2\sqrt{2}}{3}(K - \pi), \quad (11)\]

where the first of the three relations in (11) is the standard Gell-Mann–Okubo mass formula \[3\], Eqs. (9), (10) have the effectively unique solution

\[\eta = 2K - \pi, \quad \eta' = \pi, \quad \theta = \arctan \frac{1}{\sqrt{2}} \cong 35.3^\circ, \quad \text{or} \quad (12)\]

\[\eta = \pi, \quad \eta' = 2K - \pi, \quad \theta = -\arctan \sqrt{2} \cong -54.7^\circ, \quad (13)\]

which corresponds to the “ideal” structure of a meson nonet.

For all well established meson nonets, except the pseudoscalar (and, we expect, scalar) one(s), both linear and quadratic versions of Eqs. (5)-(10) are in good agreement with experiment. For example, for vector mesons, if one assumes the validity of the Gell-Mann–Okubo formula ω₈ = \((4K^* - \rho)/3\), then one obtains from Eq. (9) with the measured meson masses \[4\], \(M_{89} = -0.209 \pm 0.001 \text{ GeV}^2\) in the quadratic case, and \(-0.113 \pm 0.001 \text{ GeV}\) in the linear case. Note that this is entirely consistent with \(-0.196 \pm 0.005 \text{ GeV}^2\) and \(-0.118 \pm 0.003 \text{ GeV}\), respectively, which follow from the third element of (16). For tensor mesons, a similar comparison gives \(-0.305 \pm 0.020 \text{ GeV}^2\) vs. \(-0.287 \pm 0.015 \text{ GeV}^2\), and \(-0.105 \pm 0.008 \text{ GeV}\) vs. \(-0.104 \pm 0.005 \text{ GeV}\), respectively.

However, for the pseudoscalar nonet, one obtains from Eq. (5) with meson masses squared, \(\theta \approx -11^\circ\), in sharp disagreement with experiment, which favors the \(\eta-\eta'\) mixing angle in the vicinity of \(-20^\circ\) \[4\]. Although using linear meson masses in Eq. (5) does give \(\theta \approx -24^\circ\), in better agreement with data than its mass-squared counterpart, the value of \(M_{89}\), as given by (9), is now \(-0.165 \pm 0.004 \text{ GeV}\), vs. \(-2\sqrt{2}/3 (K - \pi) = -0.338 \pm 0.004 \text{ GeV}\). This emphasizes that neither Schwinger’s nonet mass formula nor the mass-mixing angle relations (including Sakurai’s) (5)-(7) hold for the pseudoscalar nonet.

It is well known, however, that the pseudoscalar (and, probably, scalar) mass spectrum does not follow the “ideal” structure, Eq. (11), since the mass of the pseudoscalar isoscalar singlet state is shifted up from its “ideal” value of \((2K + \pi)/3\), presumably by the instanton-induced ‘t Hooft interaction \[7\] which breaks axial U(1) symmetry \[8, 9, 10\]. However, the use of \(M_{99} = (2K + \pi)/3 + A, A \neq 0\), in Eqs. (5)-(8) will again lead to Schwinger’s formula (9), which does not hold for the pseudoscalar mesons, as just demonstrated. [In fact, the structure of this formula does not depend at all on \(M_{99}\), as seen in (9).] Therefore, instanton, as well as any other effects which may shift the mass of the pseudoscalar isoscalar singlet state, cannot constitute the explanation of the failure of Schwinger’s quartic mass and the Sakurai mass-mixing angle formulae for the pseudoscalar nonet. We believe, however, that the following analysis can resolve this problem.
2 Pseudoscalar meson mass squared matrix

Here we suggest the following form of the mass squared matrix for the pseudoscalar mesons:

\[ f^2 M^2 = \frac{1}{3} \begin{pmatrix} 4f^2_K K^2 - f_\pi^2 \pi^2 & -2\sqrt{2} f^2_K K^2 - f_\pi^2 \pi^2 \\ -2\sqrt{2} f^2_K K^2 - f_\pi^2 \pi^2 & 2f^2_K K^2 + f_\pi^2 \pi^2 + 3f^2_\eta A \end{pmatrix}, \] (14)

where \( f \)'s are the pseudoscalar decay constants defined below, and \( A \) stands for the sum of all possible contributions to the shift of the isoscalar singlet mass (from instanton effects, \( 1/N_c \)-expansion diagrams, gluon annihilation diagrams, etc.).

Indeed, the form of such a mass squared matrix is determined by the form of Gell-Mann–Okubo type relations among the masses of the isovector, isodoublet, and isoscalar octet and singlet states (which in our case are Eqs. (17)-(19) below), since this matrix must be equivalent to that of the form (3), which in the case we are considering is

\[ f^2 M^2 = \begin{pmatrix} f^2_\eta \eta^2 & f_8 f_9 \eta_9^2 \\ f_8 f_9 \eta_9^2 & f^2_\eta' \eta'^2 \end{pmatrix}, \] (15)

and which we assume is diagonal in the basis of the physical \( \eta \) and \( \eta' \) meson masses and decay constants:

\[ f^2 M^2 = \begin{pmatrix} f^2_\eta \eta^2 & 0 \\ 0 & f^2_\eta' \eta'^2 \end{pmatrix}. \] (16)

The equivalence of the matrices (14) and (16) is guaranteed by the validity of the following relations:

\[ f_8^2 \eta_{ss}^2 = -\frac{1}{3} m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle + 4m_s \langle \bar{s}s \rangle = \frac{4f^2_K K^2 - f_\pi^2 \pi^2}{3}, \] (17)

\[ f_8 f_9 \eta_{99}^2 = -\frac{\sqrt{2}}{3} \left[ m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle - 2m_s \langle \bar{s}s \rangle \right] = \frac{2\sqrt{2}}{3} \left( f_\pi^2 \pi^2 - f^2_K K^2 \right), \] (18)

\[ f_9^2 \eta_{99}^2 = f_9^2 A - \frac{2}{3} \left[ m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle + m_s \langle \bar{s}s \rangle \right] = f_9^2 A + \frac{2f^2_K K^2 + f_\pi^2 \pi^2}{3}, \] (19)

with

\[ K^2 \equiv \frac{(K^\pm)^2 + (K^0)^2}{2}, \] (20)

as suggested by Dmitrasinovic [10], on the basis of the (precise) Gell-Mann–Oakes-Renner formulae which relate the pseudoscalar masses and decay constants to the quark masses and condensates [11]:

\[ f_\pi^2 \pi^2 = - \left[ m_u \langle \bar{u}u \rangle + m_d \langle \bar{d}d \rangle \right], \] (21)

\[ f^2_K (K^\pm)^2 = - \left[ m_u \langle \bar{u}u \rangle + m_s \langle \bar{s}s \rangle \right], \] (22)

\[ f^2_K (K^0)^2 = - \left[ m_d \langle \bar{d}d \rangle + m_s \langle \bar{s}s \rangle \right]. \] (23)
(We ignore Dashen’s theorem violating effects [12], and only approximately take into account isospin violating effects via (20), as we are not concerned here with accuracies better than 1%).

Note that in the limit of exact nonet symmetry,

\[ f_\pi = f_K = f_8 = f_9 \equiv \bar{f}, \quad (24) \]

the mass squared matrix (14) reduces to (3) with the matrix elements defined in (11) and an additional contribution in the isoscalar singlet channel (A).

3 Modified Gell-Mann–Okubo mass formula

Here, we shall only explicitly demonstrate the validity of the modified Gell-Mann–Okubo formula (17) (the remaining relations (18) and (19) may be checked in a similar way). This formula may be obtained in standard chiral perturbation theory [13], as follows.

Standard chiral perturbation theory leads to the following expressions for the pseudoscalar meson masses and decay constants [13]:

\[
\begin{align*}
\pi^2 &= 2m_n B \left( 1 + \mu_\pi - \frac{1}{3} \mu_8 + 2m_n K_3 + (2m_n + m_s) K_4 \right), \\
K^2 &= (m_n + m_s) B \left( 1 + \frac{2}{3} \mu_8 + (m_n + m_s) K_3 + (2m_n + m_s) K_4 \right), \\
\eta_{88}^2 &= \frac{2}{3} (m_n + 2m_s) B \left( 1 + 2\mu_K - \frac{4}{3} \mu_8 + \frac{2}{3} (m_n + 2m_s) K_3 + (2m_n + m_s) K_4 \right) \\
&\quad + 2m_n B \left( -\mu_\pi + \frac{2}{3} \mu_K + \frac{1}{3} \mu_8 \right) + B (m_s - m_n)^2 K_5, \\
f_\pi &= \bar{f} \left( 1 - 2\mu_\pi - \mu_K + 2m_n K_6 + (m_n + 2m_s) K_7 \right), \\
f_K &= \bar{f} \left( 1 - \frac{3}{4} \mu_\pi - \frac{3}{2} \mu_K + \frac{3}{4} \mu_8 + (m_n + m_s) K_6 + (m_n + 2m_s) K_7 \right), \\
f_8 &= \bar{f} \left( 1 - 3\mu_K + \frac{2}{3} (m_n + 2m_s) K_6 + (m_n + 2m_s) K_7 \right),
\end{align*}
\]

(25)

where \(\mu^s\) are chiral logarithms, and the constants \(K_i\) are proper combinations of the low energy coupling constants \(L_i\). It then follows from these relations that the standard Gell-Mann–Okubo formula is broken in first nonleading order [13],

\[
\Delta_{\text{GMO}} \equiv 4K^2 - 3\eta_{88}^2 - \pi^2 = 4B \left[ m_n (\mu_\pi + \mu_8 - 2\mu_K) + 2m_s (\mu_8 - \mu_K) \right] + \ldots,
\]

(26)

where \(\ldots\) indicates higher order terms.

\(^1\)We have extracted factors absorbed into \(K_4, K_5\) and \(K_7\) in ref. [13], to make uniformly explicit the quark mass factors and the overall scale.
However, the modified Gell-Mann–Okubo formula remains valid in this order, and is violated only by second (non-leading) order SU(3)-flavor breaking effects:

\[
\begin{align*}
\Delta'_{\text{GMO}} & \equiv \frac{1}{f^2} \left( 4 f_K^2 K^2 - 3 f_8^2 \eta_{88}^2 - f_\pi^2 \pi^2 \right) \\
& = 4 B \left( m_s - m_n \right) \left[ \left( \mu_K + \frac{1}{2} \mu_8 - \frac{3}{2} \mu_\pi \right) \\
& \quad - (m_s - m_n) \left( \frac{1}{3} K_3 + \frac{3}{4} K_5 + \frac{2}{3} K_7 \right) \right] + \ldots ,
\end{align*}
\]

(27)
since the second factor in the first bracketed term on the r.h.s. of (27) must vanish in the SU(3)-flavor limit.

This analysis shows, in an essentially model independent way, that the modified Gell-Mann–Okubo formula (17) must be a more accurate relation among the octet pseudoscalar mesons than the standard one\footnote{A search for a relation of a more general form, \(4 f_K^2 K^2 = 3 f_8^2 \eta_{88}^2 + f_\pi^2 \pi^2\), which would hold in the first nonleading order of standard chiral perturbation theory, results in \(a = 2\).}, and therefore, use of the form of the mass squared matrix (14) is fully justified.\footnote{It has been suggested in the literature that the pseudoscalar decay constants should enter relations like (17)-(19) in the first rather than second power \footnote{3}. As discussed above, such relations are expected to be less accurate than ours, according to chiral perturbation theory.}

4 The Schwinger and Sakurai formulae reexamined

Starting with the mass squared matrix (14), the considerations which lead to Eqs. (5)-(10) above, will now lead, through (17)-(19), to the following two relations,

\[
\tan^2 \theta = \frac{4 f_K^2 K^2 - 3 f_8^2 \eta_{88}^2 - f_\pi^2 \pi^2}{3 f_\eta^2 \eta_{\eta}^2 + f_\pi^2 \pi^2 - 4 f_K^2 K^2} ,
\]

(28)

\[
\left( 4 f_K^2 K^2 - 3 f_8^2 \eta_{88}^2 - f_\pi^2 \pi^2 \right) \left( 3 f_\eta^2 \eta_{\eta}^2 + f_\pi^2 \pi^2 - 4 f_K^2 K^2 \right) = 8 \left( f_K^2 K^2 - f_\pi^2 \pi^2 \right)^2 ,
\]

(29)

where \(\theta\) is the mixing angle for the valence quark wavefunctions which produces the mass eigenstates from the octet-singlet basis, and which we refer to as “the Schwinger nonet mass, and Sakurai mass-mixing angle (respectively), formulae reexamined”.

In contrast to \(f_\pi\) and \(f_K\), the values of which are well established experimentally\footnote{4},

\[
\sqrt{2} f_K = 159.8 \pm 1.6 \text{ MeV}, \quad \sqrt{2} f_\pi = 130.7 \pm 0.3 \text{ MeV}, \quad \frac{f_K}{f_\pi} = 1.22 \pm 0.01
\]

(30)

the values of \(f_\eta\), \(f_{\eta'}\) and \(\theta\) are known rather poorly. We now wish to calculate these values using the relations (28),(29), and compare the results with available experimental data. It is obvious that the two relations are not enough for determining
the three unknowns. However, the additional relation, independent of (28), (29) (the trace condition for (14), (16)),
\[
f_\eta^2 \eta^2 + f_\eta' \eta'^2 = 2f_K^2 K^2 + f_\eta A,
\]
introduces yet an additional unknown, A. We therefore develop another independent relation among \(f_\eta, f_\eta', \) and \(\theta,\) as follows.

The light neutral pseudoscalar decay constants are defined by the matrix elements
\[
\langle 0 \mid \sum_q c_q^P \bar{\psi}_q(0) \gamma^\mu \gamma^5 \psi_q(0) \mid P(p) \rangle = i\delta^P f_P p^\mu, \tag{32}
\]
where \(\psi_q, q = (u, d, s)\) are the quark field operators, and we define the wave function of a neutral pseudoscalar meson \(P\) in terms of the quark basis states \(q\bar{q}\) as:
\[
\mid P \rangle = \sum_q c_q^P \mid q\bar{q} \rangle, \quad q = u, d, s, \tag{33}
\]
where for \(P = \pi^0, c_u^3 = 1/\sqrt{2} = -c_d^3, c_s^3 = 0,\) for \(P = \eta_{88}, c_u^8 = c_d^8 = 1/\sqrt{6},\) \(c_s^8 = -2/\sqrt{6}\) and for \(P = \eta_{99}, c_u^9 = c_d^9 = c_s^9 = 1/\sqrt{3}.\)

The pseudoscalar decay constants defined in (32) can now be expressed as
\[
f_P = \sum_q (c_q^P)^2 f_q\bar{q}, \tag{34}
\]
where we have introduced the auxiliary decay constants \(f_q\bar{q}\) defined as the decay constants of the \(q\bar{q}\) pseudoscalar bound states having the masses \(M(q\bar{q}).\) In the isospin limit, \(f_{u\bar{u}} = f_{d\bar{d}} = f_{u\bar{d}} = f_{\pi^0} = f_{\pi^+}.\) Using this approximation, and evaluating the appropriate matrix elements leads to the following relations:
\[
\begin{align*}
f_\eta &= \left( \frac{\cos \theta - \sqrt{2} \sin \theta}{\sqrt{3}} \right)^2 f_\pi + \left( \frac{\sin \theta + \sqrt{2} \cos \theta}{\sqrt{3}} \right)^2 f_{s\bar{s}}, \\
f_\eta' &= \left( \frac{\sin \theta + \sqrt{2} \cos \theta}{\sqrt{3}} \right)^2 f_\pi + \left( \frac{\cos \theta - \sqrt{2} \sin \theta}{\sqrt{3}} \right)^2 f_{s\bar{s}}. 
\end{align*} \tag{35, 36}
\]

Now we have four equations, (28), (29), (35), (36), which allow us to determine the three unknowns, \(f_\eta, f_\eta', \theta,\) as well as the additional quantity introduced, namely, \(f_{s\bar{s}}.\) The solution to these four equations is
\[
\begin{align*}
\frac{f_\eta}{f_\pi} &= 1.085 \pm 0.025, \\
\frac{f_\eta'}{f_\pi} &= 1.195 \pm 0.035, \\
\frac{f_{s\bar{s}}}{f_\pi} &= 1.280 \pm 0.060, \\
\theta &= (-21.4 \pm 1.0)^\circ. \tag{37, 38, 39, 40}
\end{align*}
\]
[The \(\pi\) and \(K\) electromagnetic mass differences, and the uncertainties in the values of \(f_\pi\) and \(f_K,\) (see (30)), are taken as a measure of the uncertainties of the results.]
Before comparing the solution obtained with experiment, let us also calculate the values of \( f_8 \) and \( f_9 \) which are obtained from (35),(36) in the no-mixing case \( (\theta = 0) \):

\[
\begin{align*}
  f_8 &= \frac{1}{3} f_\pi + \frac{2}{3} f_{ss}, \\
  f_9 &= \frac{2}{3} f_\pi + \frac{1}{3} f_{ss}.
\end{align*}
\]

(41) (42)

Therefore, as follows from (39),(41),(42),

\[
\begin{align*}
  \frac{f_8}{f_\pi} &= 1.185 \pm 0.040, \\
  \frac{f_9}{f_\pi} &= 1.095 \pm 0.020.
\end{align*}
\]

(43) (44)

The \( \eta-\eta' \) mixing angle, as given in (40), is in agreement with most of experimental data which concentrate around \(-20^\circ \) \([4, 5, 6]\). Also, the values for \( f_8/f_\pi, f_9/f_\pi \) and \( \theta \) are consistent with those suggested in the literature, as we show in Table I.

| Ref.   | \( f_8/f_\pi \)   | \( f_9/f_\pi \)   | \( \theta \), deg. |
|--------|-------------------|-------------------|-------------------|
| This work | 1.185 \(\pm\) 0.040 | 1.095 \(\pm\) 0.020 | \(-21.4 \pm 1.0\) |
| [6]    | 1.11 \(\pm\) 0.06  | 1.10 \(\pm\) 0.02  | \(-16.4 \pm 1.2\) |
| [15,16]   | 1.25              | 1.04 \(\pm\) 0.04  | \(-23 \pm 3\)     |
| [17]   | 1.33 \(\pm\) 0.02  | 1.05 \(\pm\) 0.04  | \(-22 \pm 3\)     |
| [18]   | 1.12 \(\pm\) 0.14  | 1.04 \(\pm\) 0.08  | \(-18.9 \pm 2.0\) |
| [19]   | 1.38 \(\pm\) 0.22  | 1.06 \(\pm\) 0.03  | \(-22.0 \pm 3.3\) |
| [20]   | 1.254             | 1.127             | \(-19.3\)         |

Table I. Comparison of the values for \( f_8/f_\pi, f_9/f_\pi \) and \( \theta \), calculated in the paper, with the results of the papers referenced.

5 Comparison with data

We now wish to compare the values obtained above for the ratios \( f_8/f_\pi, f_9/f_\pi \), and for the \( \eta-\eta' \) mixing angle with available experimental data. We shall first consider in more detail the well-known \( \eta, \eta' \rightarrow \gamma\gamma \) decays, for which experimental data are more complete than those for other processes involving light neutral pseudoscalar mesons, and then briefly mention the \( \eta, \eta' \rightarrow \pi^+\pi^-\gamma \), and \( J/\psi \rightarrow \eta\gamma, \eta'\gamma \) decays.

5.1 \( \eta, \eta' \rightarrow \gamma\gamma \) decays

For these processes, the inclusion of both the \( \eta-\eta' \) mixing and the renormalization of the octet-singlet couplings, which leads to the predicted amplitudes

\[
\begin{align*}
  F_{\eta\gamma\gamma}(0) &= \frac{\alpha N_c}{3\sqrt{3}f_\pi} \left( \frac{f_\pi}{f_8} \cos \theta - 2\sqrt{2} \frac{f_\pi}{f_9} \sin \theta \right), \\
  F_{\eta'\gamma\gamma}(0) &= \frac{\alpha N_c}{3\sqrt{3}f_\pi} \left( \frac{f_\pi}{f_8} \sin \theta + 2\sqrt{2} \frac{f_\pi}{f_9} \cos \theta \right).
\end{align*}
\]

(45) (46)
The values of these amplitudes, as extracted from data on widths, are \[ F_{\eta\gamma\gamma}(0) = 0.024 \pm 0.001 \text{ GeV}^{-1}, \] \[ F_{\eta'\gamma\gamma}(0) = 0.031 \pm 0.001 \text{ GeV}^{-1}. \] (47)

Calculation with the help of Eqs. (30),(40),(43)-(46) yields
\[ F_{\eta\gamma\gamma}(0) = 0.025 \pm 0.001 \text{ GeV}^{-1}, \] \[ F_{\eta'\gamma\gamma}(0) = 0.030 \pm 0.001 \text{ GeV}^{-1}, \] (48)
in excellent agreement with (47).

5.2 \( \eta, \eta' \rightarrow \pi^+\pi^-\gamma \) decays

These, as well as the \( P^0 \rightarrow \gamma\gamma \), processes were extensively studied by Venugopal and Holstein \[19\] in chiral perturbation theory. The analysis of experimental data for both of these processes done in ref. \[19\] yields
\[ \frac{f_8}{f_\pi} = 1.38 \pm 0.22, \quad \frac{f_9}{f_\pi} = 1.06 \pm 0.03, \quad \theta = (-22.0 \pm 3.3)^\circ, \]
in good agreement with our Eqs. (40),(43),(44).

5.3 \( J'/\psi \rightarrow \eta\gamma, \eta'\gamma \) decays

These processes were studied by Kisselev and Petrov \[18\]. The values of \( f_8/f_\pi \), \( f_9/f_\pi \) and \( \theta \) extracted in ref. \[18\] from the experimentally measured \( P^0 \rightarrow \gamma\gamma \) and \( J'/\psi \rightarrow \eta\gamma, \eta'\gamma \) widths, as given in the last column of Table I of ref. \[18\], which corresponds to conventional mass-mixing angle relations, are
\[ \frac{f_8}{f_\pi} = 1.12 \pm 0.14, \quad \frac{f_9}{f_\pi} = 1.04 \pm 0.08, \quad \theta = (-18.9 \pm 2.0)^\circ, \]
again in good agreement with our Eqs. (40),(43),(44).

Thus, the three values of \( f_8/f_\pi \), \( f_9/f_\pi \) and \( \theta \) agree with experiment (at least, as far as the processes considered above are concerned).

5.4 Comparison with CELLO and TPC/2\( \gamma \) results

As to the remaining \( f_\eta/f_\pi \), \( f_{\eta'}/f_\pi \) ratios also calculated in the paper, the experimental values of them, as extracted from data by the CELLO \[21\] and TPC/2\( \gamma \) \[22\] collaborations, are, respectively,
\[ \frac{f_\eta}{f_\pi} = 1.12 \pm 0.12, \] (49)
\[ \frac{f_{\eta'}}{f_\pi} = 1.06 \pm 0.10, \] (50)
\[
\frac{f_\eta}{f_\pi} = 1.09 \pm 0.10, \quad (51)
\]

\[
\frac{f_\eta'}{f_\pi} = 0.93 \pm 0.09. \quad (52)
\]

While the value calculated for \(\frac{f_\eta}{f_\pi}\), Eq. (37), clearly agrees with both experimental values (49) and (51), the value calculated for \(\frac{f_\eta'}{f_\pi}\), Eq. (38), only marginally agrees with (50), and disagrees with (52) by almost 3 standard deviations.

To clarify the difference from the results referred to above, we recall that the values of \(\frac{f_\eta}{f_\pi}\) and \(\frac{f_\eta'}{f_\pi}\) are extracted by both CELLO and TPC/2\(\gamma\) from experimental data on transition form-factors \(T_{\eta(\eta')}(0, -Q^2)\), assuming that the pole mass \(\Lambda_{\eta(\eta')}\), which parametrizes their fits to the data, can be identified with \(2\sqrt{2}f_{\eta(\eta')}\). Then, these pole fits to the data are presumed to join smoothly, as \(Q^2 \to \infty\), to the perturbative QCD predictions for \(T_{\eta(\eta')}(0, -Q^2)\) \(\&\) \(\), i.e., these fits would then agree with both the QCD asymptotic form \(\sim 1/Q^2\) and its coefficient.

However, the values of both \(f_\eta\) and \(f_\eta'\) quoted by the two groups, (in MeV) \((94.0 \pm 7.1, 89.1 \pm 4.9)\) \(\&\) \(, and \((91.2 \pm 5.7, 77.8 \pm 4.9)\) \(\&\) \(, respectively, are all close to \(M(\rho)/2\pi\sqrt{2}\) \(\approx 86.5\) MeV, thus indicating a possible connection with the vector meson dominance interpretation of \(\Lambda_{\eta(\eta')}\) \(\approx M(\rho)\) in the range of \(Q^2\) investigated, which could obviate the assumptions referred to above.

On the other hand, as remarked by Klabucar and Kekez \(\&\) \(, their own bound-state calculation, as well as the model independent calculation by Gasser and Leutwyler \(\&\) \(, testing the Goldberger-Treiman relations by Scadron \(\&\) \(, and the calculation by Burden et al. \(\&\) \(, all agree that both \(f_\eta\) and \(f_\eta'\) should be noticeably larger than \(f_\pi\). Our results \(f_\eta \sim 1.1f_\pi, f_\eta' \sim 1.2f_\pi\) are in agreement with this.

It therefore seems possible that the discrepancy with other determinations may be due an inability to extract the values of \(f_\eta\) and \(f_\eta'\) from the transition form-factors \(T_{\eta(\eta')}(0, -Q^2)\) sufficiently accurately, at least in the range of \(Q^2\) investigated so far. That this may indeed be the case is indicated by the experimental value \(f_{\pi^o} = 84.1 \pm 2.8\) MeV \(\&\) \(, extracted by the same method, which is again close to \(M(\rho)/2\pi\sqrt{2}\). The central value of this \(f_{\pi^o}\), 84.1 MeV, is \(\sim 10\%\) below the well established value given in Eq. (30), 92.4\(\pm 0.2\) MeV. Such a large discrepancy cannot be explained by, e.g., small isospin violation, indicating the possibility that the extracted values for both \(f_\eta\) and \(f_\eta'\) may well have been underestimated also.

### 6 Concluding remarks

As a lagniappe, we note that Eq. (31) may be combined with our extracted values for \(f_\eta, f_\eta, f_\eta'\) to obtain the value of \(A\):

\[
A = 0.78 \pm 0.12 \text{ GeV}^2.
\]

This is consistent with the usual value 0.73 GeV\(^2\) determined by the trace condition for (14),(16) without \(f\)'s.
Let us briefly summarize the findings of this work:

i) We have found that chiral perturbation theory suggests that the natural object to study is the pseudoscalar mass squared matrix modified by the inclusion of the squared factors of the pseudoscalar decay constants, rather than the mass squared matrix itself.

ii) We have shown that this modified mass squared matrix leads to new Schwinger’s quartic mass and Sakurai mass-mixing angle relations for the pseudoscalar meson nonet.

iii) We have used these new relations to calculate the pseudoscalar decay constants and mixing angle. We have demonstrated that, except where questions may be raised regarding the reliability of the extraction of the relevant quantities from direct experimental data, the results obtained are in excellent agreement with available data.

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